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COPURE SUBMODULES*

V. A. HIREMATH (Madurai)

P. M. Cohn [6] has defined the notion of a pure submodule. Subsequently C. P. Walker [21], B. Stenström [19] and D. P. Choudhury and K. Tewari [5] have introduced notions of copurity as dual to the notion of purity. We find that another notion of copurity for the category of modules using cofinitely related modules is interesting. A submodule A of a right R-module B is said to be *copure* if for every cofinitely related right R-module M, every R-homomorphism from A into M has an extension to B. Over a commutative (co-)noetherian ring R, every pure submodule of an R-module is also copure. If R is a commutative semi-local ring then every copure submodule of an R-module is pure. Hence over a commutative semi-local co-noetherian (equivalently noetherian) ring R purity and copurity are equivalent. We also prove this result for a Dedekind domain. In general neither every pure submodule of a right R-module is copure nor every copure submodule of a right R-module is pure. We compare our copurity with three copurities mentioned above. We generalize a few results for pure submodules over commutative (co-)noetherian rings to copure submodules.

Throughout this paper by a ring R we mean an associative ring with identity and by an R-module we mean a unitary right R-module. mod-R stands for the category of all right R-modules and all R-homomorphisms.

DEFINITIONS 1. (i) An R-module M is said to be *finitely embedded* [20] (or *cofinitely generated* [11]) if $E(M)=E(S_1)\oplus\ldots\oplus E(S_n)$ for some simple R-modules $S_1, \ldots, S_n$. (Here $E(X)$ denotes the injective hull of an R-module X.)

(ii) An R-module M is said to be *cocyclic* [10] if M is isomorphic to a submodule of $E(S)$ for simple R-module S.

(iii) An R-module M is said to be *cofree* [10] if M is isomorphic to $\Pi\{E(S_\alpha)$: S_α is a simple R-module, $\alpha\in I\}$ where I is some index set.

(iv) An R-module M is said to be *cofinitely related* [10] if there is an exact sequence $0\to M\to N\to K\to 0$ of R-modules with N cofinitely generated, cofree and K cofinitely generated.

(v) A ring R is said to be *right co-noetherian* [11] if every factor module of a cofinitely generated R-module is cofinitely generated.

The notion of a pure submodule was first defined by P. M. Cohn [6]. A submodule A of an R-module B is said to be *pure* if the induced sequence $0\to A\otimes_R M\xrightarrow{i^*} B\otimes_R M$ is exact for every left R-module M where $i^*=i\otimes 1_M$, i being the natural inclusion $A\to B$. Cohn has proved that A is pure in B if and only if for each $a_1, \ldots, a_n\in A$,

* This work is based on author's doctoral dissertation and was supported by the University Grants Commission and Madurai Kamaraj University.

if there exist $b_1, \ldots, b_m \in B$ and $r_{ij} \in R$ $(1 \leqq i \leqq n,\ 1 \leqq j \leqq m)$ such that $a_i = \sum_{j=1}^{m} b_j r_{ij}$ for $1 \leqq i \leqq n$, then there exist $a'_1, \ldots, a'_m \in A$ such that $a_i = \sum_{j=1}^{m} a'_j r_{ij}$ for $1 \leqq i \leqq n$.

A short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of R-modules is said to be *pure* if $\operatorname{Im} f$ is a pure submodule of B. From the above result of P. M. Cohn one can easily deduce that a short exact sequence $0 \to A \to B \to C \to 0$ of R-modules is pure if and only if every finitely related R-module is projective with respect to it. This characterization of pure short exact sequences is the basis for our dualization.

Before defining our copurity we recall the copurities defined by C. P. Walker [21], B. Stenström [19] and D. P. Choudhury and K. Tewari [5].

DEFINITIONS 2. (i) (C. P. Walker [21].) Let $\mathscr{S}$ be a collection of objects in an abelian category $\mathscr{A}$ such that $\mathscr{S}$ is closed under taking subobjects. Then subobject A of an object B in $\mathscr{A}$ is said to be *$\mathscr{S}$-copure* if for every subobject C of A such that A/C is in $\mathscr{S}$, A/C is a direct summand of B/C.

(ii) (B. Stenström [19].) A submodule A of an R-module B is said to be *copure* if for every proper cyclic left R-module M (that is, M is cyclic and not isomorphic to R as a left R-module), every R-homomorphism from A into M^* has an extension to B, where $M^* = \operatorname{Hom}_Z(M, Q/Z)$ is canonical right R-module, Q, Z being the abelian groups of rationals and integers, respectively.

(iii) (D. P. Choudhury and K. Tewari [5].) (a) An R-module M is said to be *cocyclically copresented* if both $E(M)$ and $E(M)/M$ are cocyclic.

(b) A short exact sequence E: $0 \to A \to B \to C \to 0$ of R-modules is said to be *copure relative to cocyclic* (resp. *cocyclically copresented*) *modules* if every cocyclic (resp. cocyclically copresented) module is injective with respect to E.

We now define our copurity using cofinitely related modules.

DEFINITIONS 3. (i) A short exact sequence $0 \to A \to B \to C \to 0$ of R-modules is said to be *copure* if every cofinitely related R-module is injective with respect to this sequence.

(ii) A submodule A of an R-module B is said to be *copure* in B if the canonical short exact sequence $0 \to A \to B \to B/A \to 0$ is copure.

Since every cocyclically copresented R-module is cofinitely related and every cocyclic R-module is cofinitely related if R is right co-noetherian, it follows that every copure short exact sequence of R-modules is copure relative to cocyclically copresented modules and is copure relative to cocyclic modules if R is a right co-noetherian ring.

If G is any finite cyclic group, then $G = \operatorname{Hom}_Z(G, Q/Z)$ is finite and hence cofinitely related as a Z-module as Z is a co-noetherian ring [20]. So it follows from definition that our copurity implies Stenström's copurity in the case of abelian groups.

From the definition we have the following properties of copure submodules.

PROPOSITION 4. (i) *Every direct summand of an R-module is a copure submodule.*
(ii) *Every cofinitely related, copure submodule of an R-module is a direct summand.*

PROPOSITION 5. *For any ring R, the following are equivalent.*
(i) *R is a right V-ring.*

(ii) *Every cofinitely related R-module is injective.*
(iii) *Every short exact sequence of R-modules is copure.*

PROOF. (i)⇒(ii) follows from Proposition 18 of [10].
(ii)⇒(iii) is immediate.
(iii)⇒(i). Let S be any simple left R-module. Suppose S is not injective. Let $x \in E(S) \setminus S$ and let A be a submodule of $E(S)$ maximal with respect to $S \subseteq A$ and $x \notin A$. Then $E(S)/A$ is subdirectly irreducible and hence is cofinitely generated. So A is cofinitely related. By (iii), A is a copure submodule of $E(S)$. Then, by Proposition 4(ii), A is a direct summand of $E(S)$. Since A is essential in $E(S)$ we get that $A = E(S)$ which contradicts the fact that $x \notin A$. Hence S is injective and so R is a right V-ring.
(The idea of the proof (iii)⇒(i) is due to [5].)

By the above proposition and by Theorem 3.5 of [5] it follows that for a ring R the following are equivalent.
(i) Every short exact sequence of R-modules is copure.
(ii) Every short exact sequence of R-modules is copure relative to cocyclically copresented modules.
(iii) Every short exact sequence of R-modules is copure relative to cocyclic modules.

We recall [8] that a ring R is a von Neumann regular ring if and only if every short exact sequence of R-modules is pure. From the above proposition it follows that a right V-ring is dual to a von Neumann regular ring.

REMARK 6. The converse of Proposition 4(i) is not true.

EXAMPLE. Let $R = \prod_{p \in P} Z/pZ$ where P is the set of all primes. Then R is a commutative von Neumann regular ring and hence is a V-ring by the [17]. So by the above proposition the natural short exact sequence $0 \to S \xrightarrow{i} R \to R/S \to 0$ is copure, where $S = \bigoplus_{p \in P} Z/pZ$. So S is a copure submodule of the R-module R. But S is not a direct summand of R as S is an essential ideal of R.

Next we recall the definitions of purity [3] and a proper class in the sense of MacLane [13].

DEFINITIONS 7. (i) We shall say following [3] that in the category mod-R a *purity* $\mathscr{P}$ is given if there is given a class $\mathscr{P}_m$ of monomorphisms satisfying the following axioms.
P0: Any homomorphism $\varphi: A \to B$ having a left inverse $\psi: B \to A$ belongs to $\mathscr{P}_m$,
P1: $\varphi\psi \in \mathscr{P}_m$ whenever $\varphi, \psi \in \mathscr{P}_m$,
P2: if $\varphi\psi \in \mathscr{P}_m$ and φ is a monomorphism, then $\psi \in \mathscr{P}_m$,
P3: if in the commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ (1) & & 0 & \to & K & \xrightarrow{j} & A & \xrightarrow{\sigma} & C & \to & 0 \\ & & & & \| & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \\ & & 0 & \to & K & \xrightarrow{i} & B & \xrightarrow{\tau} & D & \to & 0 \end{array}$$

with exact rows and columns, φ belongs to $\mathscr{P}_m$, then $\psi \in \mathscr{P}_m$ as well,

P4: if in (1) $i, \psi \in \mathscr{P}_m$, then $\varphi \in \mathscr{P}_m$.

If we introduce the class $\mathscr{P}_e$ of all epimorphisms σ for which the canonical embedding of $\operatorname{Ker}\sigma$ belongs to $\mathscr{P}_m$, then P3 and P4 are respectively equivalent to:

P3*: if $\tau\sigma \in \mathscr{P}_e$ and σ is an epimorphism, then $\tau \in \mathscr{P}_e$,

P4*: if $\tau, \sigma \in \mathscr{P}_e$, then $\tau\sigma \in \mathscr{P}_e$.

(ii) Let $\mathscr{F}$ be a class of short exact sequences of R-modules. Let $\mathscr{F}_m$ denote the class of all monomorphisms $f: A \to B$ such that $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is in $\mathscr{F}$ for some R-module C and an epimorphism g. Let $\mathscr{F}_e$ denote the class of all epimorphisms $g: B \to C$ such that $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is in $\mathscr{F}$ for some R-module A and a monomorphism f. The members of $\mathscr{F}_m$ and $\mathscr{F}_e$ are called $\mathscr{F}$-monomorphisms and $\mathscr{F}$-epimorphisms, respectively. $\mathscr{F}$ is said to be a *proper class* in the sense of MacLane [13] if it satisfies the following axioms:

1) $\mathscr{F}$ contains all short exact sequences isomorphic to its members,

2) for any two R-modules A, B the canonical short exact sequence $0 \to A \to A \oplus B \to B \to 0$ is in $\mathscr{F}$,

3) if $\alpha, \beta \in \mathscr{F}_m$ and $\alpha\beta$ is defined, then $\alpha\beta \in \mathscr{F}_m$,

4) if α, β are monomorphisms such that $\alpha\beta$ is defined and $\alpha\beta \in \mathscr{F}_m$, then $\beta \in \mathscr{F}_m$,

5) if $\alpha, \beta \in \mathscr{F}_e$ and $\alpha\beta$ is defined, then $\alpha\beta \in \mathscr{F}_e$,

6) if α, β are epimorphisms such that $\alpha\beta$ is defined and $\alpha\beta \in \mathscr{F}_e$, then $\alpha \in \mathscr{F}_e$.

Let $\mathscr{P}$ be a purity in mod-R and let $\mathscr{P}^*$ denote the class of all short exact sequences $0 \to A \xrightarrow{f} B \to C \to 0$ of R-modules with $f \in \mathscr{P}_m$. Then $\mathscr{P}^*$ becomes a proper class. On the other hand, if $\mathscr{F}$ is a proper class of short exact sequences of R-modules then the class $\mathscr{F}_m$ (as defined above) of all $\mathscr{F}$-monomorphisms defines a purity in mod-R.

Let $\mathscr{M}$ be the class of all cofinitely related R-modules. Then clearly, by definition, copure monomorphisms are precisely the monomorphisms i such that every M in $\mathscr{M}$ is injective with respect to i (here a monomorphism $f: A \to B$ is said to be *copure* if $\operatorname{Im} f$ is a copure submodule of B).

Then by the above remark and 1.16 of [15], we have

PROPOSITION 8. *The copure short exact sequences of R-modules form a proper class (or equivalently, copure monomorphisms of R-modules define a purity in* mod-R*).*

Also, by Theorems 10 and 12 of [1], we have

PROPOSITION 9. *Any direct sum of copure short exact sequences of R-modules is copure.*

Using standard arguments (as in [13]) one can show that the equivalence classes of copure extensions of an R-module B by an R-module A form a subgroup $\operatorname{Copext}^1_R(A, B)$ of $\operatorname{Ext}^1_R(A, B)$. (An extension $E: 0 \to B \to X \to A \to 0$ of an R-module B by an R-module A is said to be a *copure extension* of B by A if E is a copure short exact sequence.) We will now find an expression for $\operatorname{Copext}^1_R$ in terms of Hom. Let C^*_r be the collection of all cofinitely related R-modules such that any cofinitely related R-module is isomorphic to a member of C^*_r and no two distinct members of C^*_r are isomorphic. Then C^*_r is a set. We recall that if $f: A \to B$ is an R-homomorphism and X is an R-module, then the induced map $f_*: \operatorname{Ext}^1_R(X, A) \to \operatorname{Ext}^1_R(X, B)$ is defined by $f_*([E]) = [fE]$ where $[E]$ denotes the equivalence class of extensions of A by X with E as a representative.

Since copure monomorphisms define an injectively closed purity [1], we have the dual of Lemma 1 of [2].

PROPOSITION 10. *For any two R-modules A, B*

$$\mathrm{Copext}^1_R(A, B) = \bigcap\{\mathrm{Ker}\, f_*: f \in \mathrm{Hom}_R(B, M), M \in C_r^*\}.$$

We will now show that our copurity coincides with Walker's $\mathscr{S}$-copurity in some special cases.

It can be easily seen that if R is a right noetherian ring, then a submodule A of an R-module B is pure in B if and only if for every submodule C of B containing A such that C/A is finitely related, A is a direct summand of C.

We prove dually

PROPOSITION 11. *If R is a right co-noetherian ring, then a submodule B of an R-module C is copure in C if and only if for every submodule A of B such that B/A is cofinitely related, B/A is a direct summand of C/A.*

PROOF. Suppose B is a copure submodule of C and let A be a submodule of B such that B/A is cofinitely related. Since copure short exact sequences of R-modules from a proper class (cf. Proposition 8) it follows that B/A is a copure submodule of C/A. Then, by Proposition 4(ii), B/A is a direct summand of C/A.

Conversely, suppose that for every submodule A of B such that B/A is cofinitely related, B/A is a direct summand of C/A. Let $f: B \to M$ be any R-homomorphism form B into a cofinitely related R-module M and let $A = \mathrm{Ker} f$. Then f induces a monomorphism $f^*: B/A \to M$. Since R is co-noetherian and M is cofinitely related, it follows by Proposition 19 of [10] that B/A is also cofinitely related and hence, by hypothesis, B/A is a direct summand of C/A. Then f^* has an extension $g^*: C/A \to M$. If $\eta: C \to C/A$ is the canonical epimorphism then $g = g^*\eta$ extends f proving that B is copure in C.

From the above proposition it follows that our copurity is Walker's $\mathscr{S}$-copurity (cf. Definition 2(i)) for mod-R if R is a right co-noetherian ring and $\mathscr{S}$ is the class of all cofinitely related R-modules (note that $\mathscr{S}$ is closed under taking subobjects by Proposition 19 of [10]).

Purity versus copurity

We first prove that in our commutative (co-)noetherian ring R, every pure submodule of an R-module is copure. For this we recall from [22] the following definitions.

(i) Let M be an R-module. By a *linear equation* over M we mean an equation of the form $\sum_{i \in I} x_i a_i = m$ where $a_i \in R$ such that $a_i = 0$ for all but a finite number of $i \in I$ (where I is any index set) and $m \in M$. Let $\sum_{i \in I} x_i a_{ij} = m_j$, $j \in J$ (where I, J are any index sets) be any system of linear equations over M. This system is said to be *finitely solvable* if for every finite subset J^* of J, the subsystem $\sum_{i \in I} x_i a_{ij} = m_j (j \in J^*)$ is solvable in M. M is said to be *algebraically compact* if every finitely solvable system of linear equations over M admits a solution in M.

(ii) An R-module M is said to be *linearly compact* if $\{N_i\}_{i\in I}$ is a family of submodules of M and $\{a_i\}_{i\in I}$ is a family of elements of M (I being any index set) such that the family $\{a_i+N_i\}_{i\in I}$ has the finite intersection property, then the family $\{a_i+N_i\}_{i\in I}$ has nonempty intersection.

Warfield [22] has proved that over a commutative ring R, every linearly compact R-module is algebraically compact.

PROPOSITION 12. *Over a commutative (co-)noetherian ring R, every pure submodule of an R-module is also a copure submodule.*

PROOF. For this we first prove that every artinian R-module M is algebraically compact. In view of Warfield's result stated above we need only prove that M is linearly compact. Let $\{N_i\}_{i\in I}$ be a family of submodules of M and let $\{a_i\}_{i\in I}$ be a family of elements of M (I being any index set) such that the family $\{a_i+N_i\}_{i\in I}$ has the finite intersection property. Let $\mathcal{F}$ be the family of all finite intersections of the members of the family $\{N_i\}_{i\in I}$. If some member of $\mathcal{F}$ is zero, then the family $\{a_i+N_i\}_{i\in I}$ has nonempty intersection (for, if $\bigcap_{j=1}^{\gamma} N_{ij}=0$, $\bigcap_{j=1}^{\gamma}(a_{ij}+N_{ij})$ being nonempty is $a+\bigcap_{j=1}^{\gamma} N_{ij}=\{a\}$ for some $a\in M$. If $i\in I$, then $(a_i+N_i)\cap(\bigcap_{j=1}^{\gamma}(a_{ij}+N_{ij}))$ being nonempty is $\{a\}$ and hence $a\in a_i+N_i$ for each i). So let each member of $\mathcal{F}$ be nonzero. Then by minimum condition on submodules of M, $\mathcal{F}$ has a minimal element, say, $K=\bigcap_{i\in J} N_i$ for some finite subset J of I. Then $\bigcap_{i\in J}(a_i+N_i)=a+K$ for some $a\in M$. Then it easily follows from the minimality of K in $\mathcal{F}$ that $a\in\bigcap_{i\in I}(a_i+N_i)$ proving the linear (and hence the algebraic) compactness of M.

Now let A be a pure submodule of an R-module B and let C be a submodule of A such that A/C is cofinitely related. Since R is co-noetherian, A/C is artinian [10, 11] and hence is algebraic compact from above. Since A is pure in B it follows that A/C is also pure in B/C. Now A/C is pure injective by [22] and hence A/C is a direct summand of B/C. Thus A is copure in B by Proposition 11.

REMARK 13. In general every pure submodule of an R-module need not be copure.

EXAMPLE. Let V be an $\aleph_0$-dimensional vector space over the field Q of rational numbers and R be the ring of linear operators of V. Then R is a von Neumann regular ring. Let I be the two sided ideal of R of all linear operators of V of finite rank and let $\mathcal{M}$ be a maximal right ideal of R containing I. Then $S=R/\mathcal{M}$ is a simple, noninjective R-module [16] and hence $S\neq E(S)$. Let $0=x\in E(S)\setminus S$, and let T be a submodule of $E(S)$ maximal with respect to $S\subseteqq T$, $x\notin T$. Then T is cofinitely related and is not a direct summand of $E(S)$ (cf. the proof of (iii)$\Rightarrow$(i) of Proposition 5). So, by Proposition 4(ii), T is not a copure submodule of $E(S)$. Since R is a von Neumann regular ring, T is a pure submodule of $E(S)$ (actually every short exact sequence of R-modules is pure) [8].

We prove next that every copure short exact sequence of R-modules is pure if R is either a commutative semi-local ring [14] or a Dedekind domain.

PROPOSITION 14. *If R is a commutative semi-local ring, then every copure short exact sequence of R-modules is pure.*

PROOF. Since R is semi-local, by [10], mod-R has a cofinitely generated cofree cogenerator E. We first prove that if M is a finitely related R-module, then $\mathrm{Hom}_R(M, E)$ is a cofinitely related R-module. Let M be any finitely related R-module. Then we have an exact sequence $R^m \to R^n \to M \to 0$ of R-modules (for some non-negative integers m, n) which yields us the exact sequence $0 \to \mathrm{Hom}_R(M, E) \xrightarrow{f^*} \xrightarrow{f^*} E^n \to E^m$ of R-modules. Since E^n is cofinitely generated, cofree and Coker f^* (being isomorphic to a submodule of the cofinitely generated module E^m) is cofinitely generated, it follows that $\mathrm{Hom}_R(M, E)$ is cofinitely related as an R-module.

Now let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be any copure short exact sequence of R-modules and let M be any finitely related R-module. Since, from above, $\mathrm{Hom}_R(M, E)$ is cofinitely related, by the copurity of the given sequence we have the induced sequence

$$0 \to \mathrm{Hom}_R(C, \mathrm{Hom}_R(M, E)) \to \mathrm{Hom}_R(B, \mathrm{Hom}_R(M, E)) \to$$

$$\to \mathrm{Hom}_R(A, \mathrm{Hom}_R(M, E)) \to 0,$$

Then using the adjoint isomorphism of Hom and $\otimes$ and the fact that E is a cogenerator for mod-R, we get the exact sequence

$$0 \to A \otimes_R M \xrightarrow{f \otimes M} B \otimes_R M \xrightarrow{g \otimes M} C \otimes_R M \to 0.$$

Since any R-module is the direct limit of a directed system of finitely related R-modules, the purity of the sequence $0 \to A \to B \to C \to 0$ follows.

REMARK 15. Since purity implies Stenström's copurity [19] for any ring R, by the above Proposition it follows that our copurity implies Stenström's copurity for a commutative semi-local ring R.

Before proving the next proposition we recall from [12] that a submodule A of an R-module B is said to be *weakly pure* (in short *w-pure*) in B if $Ar = Br \cap A$ for each element r of R which is not a zero-divisor in R. Clearly every pure submodule of an R-module is w-pure. Over a Dedekind domain both these notions are equivalent. This follows from Steinitz's theorem [18] that every finitely generated module over a Dedekind domain is decomposable [12] and from Theorem 3 of [12] and the sufficient condition for pure submodules over a noetherian ring stated just before Proposition 11. Warfield [22] has proved that w-purity and purity are equivalent for a Prüfer domain (where he calls w-purity as RD-purity).

Before proceeding further we recall from [9] that a commutative integral domain R is said to be an *almost Dedekind domain* if $R_{\mathcal{M}}$ is a discrete valuation ring for each maximal ideal $\mathcal{M}$ of R. Clearly R is an almost Dedekind domain if and only if it is a co-noetherian Prüfer domain. Hence every Dedekind domain is an almost Dedekind domain but there exists an almost Dedekind domain which is not a Dedekind domain [9].

PROPOSITION 16. *If R is a right co-noetherian ring and A is a copure submodule of an R-module B, then $AI = BI \cap A$ for each right ideal I of R.*

PROOF. Let I be a right ideal of R. If $AI \neq BI \cap A$, there will be an $a \in BI \cap A$ such that $a \notin AI$. Let C be a submodule of A maximal with respect to $AI \subseteq C$, $a \notin C$. Then A/C is subdirectly irreducible and hence, is cofinitely generated. Since R is co-noetherian, A/C is cofinitely related [10]. Let $\eta: A \to A/C$ be the canonical map. Then, by the copurity of A in B, η has an extension $\eta': B \to A/C$. Then $\eta(a) = \eta'(a) \in \eta'(BI) \subseteq \eta'(B)I \subseteq (A/C)I = 0$ as $AI \subseteq C$. This contradicts the fact that $a \notin C$. So $AI = BI \cap A$.

COROLLARY 17. *Over a commutative (co-) noetherian ring R, every copure submodule of an R-module is a w-pure submodule.*

Since for an almost Dedekind domain purity and w-purity are equivalent [22], we have

COROLLARY 18. *Over an almost Dedekind domain R every copure short exact sequence of R-modules is pure.*

REMARK 19. In general every copure short exact sequence of R-modules need not be pure.

EXAMPLE. J. H. Cozzens [7] has constructed a ring $R = k[X, D]$ of differential polynomials in a single indeterminate X over a universal field k with a derivation D (here the multiplication is given by $aX = Xa + D(a)$, $a \in k$). The ring R has the following properties.

(i) R is a principal right ideal and a principal left ideal domain,
(ii) R is a right V-ring,
(iii) there is, up to isomorphism, a unique simple R-module and
(iv) R is not a field.

Since R is a right V-ring, by Proposition 5, the natural short exact sequence $0 \to XR \to R \to R/XR \to 0$ of R-modules is copure. Since X is not invertible in R, R/XR is not flat as an R-module and hence $0 \to XR \to R \to R/XR \to 0$ is not pure.

From Propositions 12, 14 and Corollary 18 we have

THEOREM 20. *If R is either a commutative semi-local co-noetherian (equivalently noetherian) ring or is an almost Dedekind domain, a short exact sequence of R-modules is pure if and only if it is copure.*

REMARK 21. Though copurity and purity are equivalent, by Theorem 20, for commutative semi-local noetherian rings and for almost Dedekind domains we point out that neither every Dedekind (so almost Dedekind) domain is semi-local (e.g. the ring Z of integers) nor every local (so semi-local) noetherian domain is an almost Dedekind domain (e.g. $k[X, Y]_{(X,Y)}$ where k is a field and X, Y are indeterminates, over k) (note that an almost Dedekind domain has dimension almost one [9]).

REMARK 22. Stenström [19] has proved that for a Dedekind domain purity and his notion of copurity are equivalent. From this and from Theorem 20 it follows that our copurity and Stenström's copurity are equivalent for a Dedekind domain.

COROLLARY 23. *Over a commutative (co-)noetherian ring R, every copure submodule of a flat R-module is pure.*

PROOF. Let A be a copure submodule of a flat R-module B. Then, by Proposition

16, $BI \cap A = AI$ for every ideal I of R. Since B is flat it follows from [8] that B/A is also flat. Hence A is a pure submodule of B.

By Proposition 12 every pure submodule of an R-module over a commutative (co-)noetherian ring R is copure. Below we generalize a few results for pure submodule to copure submodule when R is a commutative (co-)noetherian ring.

PROPOSITION 24. *Let R be a commutative integral domain. Then purity and w-purity are equivalent for* mod-R *if and only if R is a Prüfer domain.*

PROOF. If R is a Prüfer domain, by Warfield [22], purity and w-purity are equivalent for mod-R. Conversely, suppose that purity and w-purity are equivalent for mod-R. To prove that R is a Prüfer domain, we need only prove, by [4], that every torsion-free R-module is flat. Let $M = F/K$ be a torsion-free R-module where F is a free R-module. Then K is a w-pure submodule of F. So, by hypothesis, K is a pure submodule of F. Since F is flat it follows that M is also flat. So R is Prüfer domain.

COROLLARY 25. *Let R be a commutative (co-)noetherian integral domain. Then purity and w-purity are equivalent for mod-R if and only if R is an almost Dedekind domain.*

PROOF. Follows from the above proposition and the fact that a commutative (co-)noetherian domain is an almost Dedekind domain if and only if it is a Prüfer domain.

We now generalize the above corollary to copure submodules.

PROPOSITION 26. *Let R be a commutative (co-)noetherian domain. Then w-purity and copurity are equivalent for* mod-R *if and only if R is an almost Dedekind domain.*

PROOF. The 'if' part follows from Theorem 20 and Corollary 25.

For the 'only if' part we need only prove that R is a Prüfer domain for which it suffices to prove that every torsion free R-module is flat [4]. Let $M = F/K$ be a torsion-free R-module where F is a free R-module. Then K is a w-pure submodule of F and so, by hypothesis, K is a copure submodule of F. Then, by Proposition 16, $KI = FI \cap K$ for each ideal I of R. So M is flat by [8]. Thus R is an almost Dedekind domain.

We know [8] that if F is a flat R-module and K is a pure submodule of F then F/K is flat. We generalize this result to copure submodules.

PROPOSITION 27. *If F is a flat module over a commutative (co-)noetherian ring R and K is a copure submodule of F, then F/K is flat.*

PROOF. By Proposition 16 we have, by the copurity of K in F, $KI = FI \cap K$ for every ideal I of R. So, by [8], F/K is flat.

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PROJECTIVE CLASSES OF TORSION FREE ABELIAN GROUPS. II

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0. Introduction

A class of abelian groups is called a projective class if it is the class of all abelian groups which are projective with respect to some given class of exact sequences. Projective classes of abelian groups have been studied by several authors.

Let $\mathscr{C}$ be the category of torsion free abelian groups of finite rank — hereafter simply called "groups" — and let $\mathbf{X}=\{X_\alpha\}$ be a set of groups. In [3], the projective class $P_0C_0(\mathbf{X})$ was introduced, and a characterization of $P_0C_0(\mathbf{X})$ was given in the case where each X_α was of rank one. It follows immediately from this characterization, that $\forall G\in\mathscr{C}$, $P_0C_0(G)\big(=P_0C_0(\{G\})\big)$ is contained in $P_0C_0(\mathscr{S}_G)$, where $\mathscr{S}_G$ is the set of all rank one factors of G. In Section 1 we show that, if each rank one factor of G has idempotent type, then $P_0C_0(G)=P_0C_0(\mathscr{S}_G)$. This result, in many cases, makes $P_0C_0(G)$ easy to describe concretely.

In Section 2 a second projective class, the cotorsion-free class $\mathscr{P}\mathscr{I}(\mathbf{X})$, is explicity calculated in $\mathscr{C}$. This class was studied in [2], in the category of all abelian groups. We further show that, $\forall G\in\mathscr{C}$, $P_0C_0(G)\subseteq\mathscr{P}\mathscr{I}(G)=\mathscr{P}\mathscr{I}(\mathscr{S}_G)$.

In Section 3 the results of the previous sections are used to describe those groups G such that $P_0C_0(G)=\mathscr{P}\mathscr{I}(G)$.

Our notation is mostly that of [1]. The reader is assumed to be familiar with the basic notions of torsion free groups, including the partially ordered set of types, as described in [1], Chapter 13, and the inner and outer types, as described in [4]. The symbol $\langle\ \rangle_*$ is used to denote the pure subgroup generated by a set of subgroups and/or elements of G. A pure subgroup K of G is said to be of co-rank one if rank $(G/K)=1$. The cotypeset of G is the set of all types of torsion free rank one factors of G, i.e. cotypeset $G=\{\text{type } (G/K)|K \text{ is a pure co-rank one subgroup of } G\}$. Other notation will be introduced when it is employed.

1. A projective class in $\mathscr{C}$

Let $G\in\mathscr{C}$. In [3], the following classes are introduced:

$C_0(G)=\{A\in\mathscr{C}|\ G$ is projective with respect to all $\mathscr{C}$ exact sequences (pure exact sequences) of the form: $0\to K\to A\to A/K\to 0\}$

$P_0C_0(G)=\{Y\in\mathscr{C}|Y$ is projective with respect to all $\mathscr{C}$ exact sequences: $0\to K\to A\to A/K\to 0$, $\forall A\in C_0(G)\}$. For $\mathbf{X}=\{X_\alpha\}$, a set of groups, $C_0(\mathbf{X})$ is then defined in the natural way, $C_0(\mathbf{X})=\bigcap_\alpha C_0(X_\alpha)$, and $P_0C_0(\mathbf{X})=\{Y\in\mathscr{C}|Y$ is projective with respect to all $0\to K\to A\to A/K\to 0$, $\forall A\in\mathscr{C}_0(\mathbf{X})\}$. It is clear that $P_0C_0(\mathbf{X})$ is a projective class in $\mathscr{C}$.

Let $\mathbf{X}=\{X_\alpha\}$, where each X_α is a rank one group of type τ_α. Then $C_0(\mathbf{X})$ and $P_0C_0(\mathbf{X})$ are explicitly calculated in [3], where the following characterizations are obtained:

([3], Th. 3.1) $C_0(\mathbf{X})=\{A\in\mathscr{C}\,|\,A=D\oplus R,\ D$ divisible and R reduced, and, $\forall\alpha$, either $\mathrm{IT}(R)\geqq\tau_\alpha$ or R has no rank one factors of type $\geqq\tau_\alpha\}$.

([3]), Th. 4.3) $P_0C_0(\mathbf{X})=\{Y\in\mathscr{C}\,|\,\tau\in$ cotypeset Y implies $\tau=\bar{0}$ or $\tau=$ $=$l.u.b. $\{\tau_\alpha\,|\,\tau_\alpha\leqq\tau\}\}$. Here $\mathrm{IT}(R)$ denotes the inner type of R, $\bar{0}=$type Z, and l.u.b. means least upper bound in the partially ordered set of all types.

Let $G\in\mathscr{C}$ and let $\mathscr{S}_G$ be the set of all rank one factors of G. A consequence of Theorem 4.3 is that $P_0C_0(G)\subseteqq P_0C_0(\mathscr{S}_G)$. This is true since G clearly satisfies the criterion of Theorem 4.3, so $\{G\}\subseteqq P_0C_0(\mathscr{S}_G)$. Thus, $P_0C_0(G)\subseteqq P_0C_0[P_0C_0(\mathscr{S}_G)]=$ $=P_0C_0(\mathscr{S}_G)$.

Since $P_0C_0(\mathscr{S}_G)$ is often easy to calculate, it would be nice if $P_0C_0(G)=$ $=P_0C_0(\mathscr{S}_G)$ or if $P_0C_0(G)=P_0C_0(\mathscr{S}')$ for some suitably chosen $\mathscr{S}'\subseteqq\mathscr{S}_G$.

However, in [3], an example is given of a rank two group G such that $P_0C_0(G)\neq$ $\neq P_0C_0(\mathscr{S}')$ for any $\mathscr{S}'\subseteqq\mathscr{S}_G$ — dashing this hope in the general case. The question then naturally arises: For which G is $P_0C_0(G)=P_0C_0(\mathscr{S}')$ for some judiciously chosen $\mathscr{S}'\subseteqq\mathscr{S}_G$? In this section we obtain a partial answer to this question.

First, we recall some definitions and establish some notation. Let τ be a type and let $C\in\mathscr{C}$. Then:

(i) $C(\tau)=\{c\in C\,|$ type $c\geqq\tau\}$. Recall that $C(\tau)$ is always a pure subgroup of C.
(ii) Π is the set of all primes.
(iii) $R(C)$ is the subring of Q generated by Z and $\left\{\frac{1}{p}\,|p\in\Pi,\ pC=C\right\}$.
(iv) $\mathrm{supp}\,C=\{p\in\Pi\,|\,pC\neq C\}$.
(v) τ is called idempotent if there exists a height vector (characteristic) $h\in\tau$ such that $h(p)=0$ or $h(p)=\infty$, $\forall p\in\Pi$.
(vi) For such τ and h, let $[\tau]=\{p\in\Pi\,|\,h(p)=\infty\}$ and,
(vii) let $C_{[\tau]}$ be the localization of C at $[\tau]$, i.e. $C_{[\tau]}=Z_{[\tau]}\underset{Z}{\otimes}C$, where $Z_{[\tau]}$ is the localization of Z at the set of primes $[\tau]$. We regard $C\subseteqq C_{[\tau]}$ in the natural way.

THEOREM 1. *Let G be a group whose cotypeset contains only idempotent types. Then $P_0C_0(G)=P_0C_0(\mathscr{S}_G)$.*

PROOF. We need only show $P_0C_0(\mathscr{S}_G)\subseteqq P_0C_0(G)$ for such G. Applying P_0 reverses set inclusion, so it suffices to show $C_0(G)\subseteqq C_0(\mathscr{S}_G)$.

Let $A=D\oplus R$, D the maximal divisible subgroup of A. Since $C_0(G)$ is closed under taking direct summands, if $A\in C_0(G)$, then $R\in C_0(G)$. Moreover, in view of Theorem 3.1 of [3], $A\in C_0(\mathscr{S}_G)$ iff $R\in C_0(\mathscr{S}_G)$. Thus, to prove $C_0(G)\subseteqq C_0(\mathscr{S}_G)$, it is enough to prove that every reduced group in $C_0(G)$ is also in $C_0(\mathscr{S}_G)$.

Let $A\in C_0(G)$ be reduced. We show $A\in C_0(\mathscr{S}_G)$ by verifying the criterion of Theorem 3.1:

(∇) Let K be a pure co-rank 1 subgroup of A such that type $(A/K)\geqq\tau$ for some $\tau\in$cotypeset G. Then $\mathrm{IT}(A)\geqq\tau$.

The proof is by induction on rank A. If rank $A=1$, (∇) holds trivially.

Suppose we have verified (∇) for all reduced groups in $C_0(G)$ of rank$<$rank A.

Let K and τ be as above, and let $L \subseteq K$ be a rank one pure subgroup of A. $C_0(G)$ is, evidently, closed under torsion free factors. So, $A/L \in C_0(G)$, and type $(A/L/K/L) \geqq \tau$. By induction, $\mathrm{IT}(A/L) \geqq \tau$.

Suppose $\mathrm{IT}(A) \ngeqq \tau$. We will obtain a contradiction. First we prove that $A(\tau) \neq (0)$.

Write $R(A) \underset{\mathbb{Z}}{\otimes} G = F' \oplus G'$, where F' is a finite rank free $R(A)$ module, and $G' \in \mathscr{C}$ has no rank one factors isomorphic to $R(A)$.

Let $\sigma = \text{type } R(A)$. Since $R(A) \otimes G \to R(A) \otimes G/K \to 0$, and type $(R(A) \otimes G/K) \geqq \sigma \vee \tau > \sigma$, we must have $G' \neq (0)$. (No rank one group of type $> \sigma$ can be a homomorphic image of F'.)

It is easy to check that $A \in C_0(R(A) \otimes G)$ and, therefore, that $A \in C_0(G')$.

Furthermore, $\lambda' \in \text{cotypeset } G'$ iff $\lambda' = \sigma \vee \lambda > \sigma$, for some $\lambda \in \text{cotypeset } G$. In particular, $\tau' = \sigma \vee \tau \in \text{cotypeset } G'$. Note that $\mathrm{IT}(A/L) \geqq \tau'$, but $[\tau'] \cap \text{supp } A \neq \emptyset$.

For $n \geqq 0$, define a positive integer t_n as follows:

Let $t_0 = 1$. If $[\tau'] \cap \text{supp } A = \{p_1, \ldots, p_j\}$ is finite, for $1 \leqq n \leqq j$ let $t_n = (p_1 \ldots p_n)^n$ and for $n > j$ let $t_n = (p_1 \ldots p_j)^n$. If $[\tau'] \cap \text{supp } A = \{p_1, p_2, \ldots\}$ is infinite, for $n \geqq 1$ let $t_n = (p_1 \ldots p_n)^n$.

Choose H' a pure co-rank one subgroup of G' with type $(G'/H') = \tau'$. Let $a \in A \setminus L$ and $g' \in G' \setminus H'$. For each $n \geqq 0$, consider the following diagram:

$$\begin{array}{ccccc} & & G' & & \\ & & \downarrow \mu & & \\ & \overset{\Theta_n}{\swarrow} & G'/H' & & \\ & & \downarrow \varphi_n & & \\ A & \longrightarrow & A/L & \longrightarrow & 0. \end{array}$$

Here μ is the natural map, $\varphi_n \in \mathrm{Hom}(G'/H', A/L)$ is defined by the requirement $\varphi_n(g' + H') = 1/t_n(a + L)$, and Θ_n is the map obtained by projectivity.

For each $n \geqq 0$, localize the appropriate diagram at $[\tau']$ to obtain:

$$\begin{array}{ccccc} & & G'_{[\tau']} & & \\ & & \downarrow \mu & & \\ & \overset{\Theta_n}{\swarrow} & (G'/H')_{[\tau']} & & \\ & & \downarrow \varphi_n & & \\ A_{[\tau']} & \longrightarrow & (A/L)_{[\tau']} & \longrightarrow & 0 \end{array}$$

where we have retained the symbols μ, φ_n, Θ_n to denote the naturally induced maps.

Write $G'_{[\tau']} = F'' \oplus G''$, where F'' is a finite rank free $Z_{[\tau']}$ module and G'' has no rank one factors isomorphic to $Z_{[\tau']}$. For each $n \geqq 0$, let $\bar{\mu}, \bar{\Theta}_n$ be the restrictions of the maps μ, Θ_n to G'' and consider:

$$\begin{array}{ccccc} & & G'' & & \\ & & \downarrow \bar{\mu} & & \\ (\chi)_n & \overset{\bar{\Theta}_n}{\swarrow} & (G'/H')_{[\tau']} & & \\ & & \downarrow \varphi_n & & \\ A_{[\tau']} & \longrightarrow & (A/L)_{[\tau']} & \longrightarrow & 0. \end{array}$$

Since $(G'/H')_{[\tau']} \cong Q$, which cannot be a homomorphic image of F'', $\bar{\mu}$ is onto. Choose $g'' \in G''$ such that $\bar{\mu}(g'') = \mu(g')$. Let $H'' = \operatorname{Ker} \bar{\mu}$. Then $G'' = \langle g'', H'' \rangle_*$. ($H''$ is of co-rank 1 in G''.)

For each $n \geqq 0$, $(\chi)_n$ is a commutative diagram so:
for $n=0$: $\bar{\Theta}_0(g'') = a + l_0$, $l_0 \in L_{[\tau']}$, $\bar{\Theta}_0(H'') \subseteqq L_{[\tau']}$;
for $n>0$: $t_n \Theta_n(g'') = a + l_n$, $l_n \in L_{[\tau']}$, $\bar{\Theta}_n(H'') \subseteqq L_{[\tau']}$.

Therefore, for each $n>0$, $(\bar{\Theta}_0 - t_n \bar{\Theta}_n) \in \operatorname{Hom}(G'', L_{[\tau']})$. But $\operatorname{Hom}(G'', L_{[\tau']}) = (0)$, since each rank one factor of G'' is divisible by at least one prime $p \in \operatorname{supp} A \cap [\tau'] = \operatorname{supp} A \cap [\tau]$ and $L_{[\tau']}$ is a rank one group which can be divisible by no such prime p. (Otherwise $pL = L$ and, recalling the fact $\operatorname{IT}(A/L) \cong \tau'$, we would derive the conclusion $pA = A$ — a contradiction.)

Consequently, for each $n>0$, $\bar{\Theta}_0 = t_n \bar{\Theta}_n$. Hence, in view of the definition of the t_n's, $(0) \neq \bar{\Theta}_0(G'') \subseteqq A_{[\tau']}(\tau')$. Now if $A_{[\tau']}(\tau') \neq (0)$, then $A(\tau') \neq (0)$. ($A(\tau') = A_{[\tau']}(\tau') \cap A$.) Note that $A(\tau') = A(\tau)$, so $A(\tau) \neq (0)$.

We next claim that $A(\tau) \cap K = (0)$. Otherwise, $A/A(\tau) \cap K$ would be group in $C_0(G)$ of lower rank than A with rank one factor A/K. By induction, $\operatorname{IT}(A/A(\tau) \cap K) \geqq \tau$. But, since K is pure in A, $A(\tau) \cap K = K(\tau)$ has inner type $\geqq \tau$. For an idempotent type τ, the above facts would allow us to conclude that $\operatorname{IT}(A) \geqq \tau$ — a contradiction.

Thus, $A(\tau) \cap K = (0)$. Since co-rank $K = 1$, rank $A(\tau) = 1$. Therefore, rank $A_{[\tau']}(\tau) = 1$. Furthermore, for any $a \in A \setminus L$ we have proven that there exists an $l_0 \in L_{[\tau']}$ such that $a + l_0 = \bar{\Theta}_0(g'') \in A_{[\tau']}(\tau)$. But this means $A_{[\tau']} = \langle L_{[\tau']}, A_{[\tau']}(\tau) \rangle_*$, so rank $A_{[\tau']} = 2$. Therefore, rank $A = 2$, $A = \langle A(\tau), L \rangle_*$. ($A(\tau) \cap L \subseteqq A(\tau) \cap K = (0)$).

Choose types $\tau = \tau_1, \tau_2, \ldots, \tau_r \in \operatorname{cotypeset} G$ such that $\bigvee_{i=1}^{r} \tau_i = \operatorname{OT}(G)$, the outer type of G. Renumbering, let $\tau_1, \ldots, \tau_{r'}$, $r' \leqq r$, be those τ_i such that $A(\tau_i) \neq A$ but type $(A/K_i) \geqq \tau_i$ for some subgroup K_i of co-rank 1. Our argument up to now has established that each $A(\tau_i) \neq (0)$, $1 \leqq i \leqq r'$.

Let N be a rank one pure subgroup of A such that $N \cap A(\tau_i) = (0)$, $1 \leqq i \leqq r'$. Clearly, A/N is a rank one group of type $\geqq \tau^* = \bigvee_{i=1}^{r'} \tau_i$.

Now we can repeat our previous argument, replacing τ by τ^*, to conclude that rank $A(\tau^*) = 1$, $A = \langle A(\tau^*), N \rangle_*$.

Choose a prime q with $qA(\tau^*) \neq A(\tau^*)$ (A is reduced). Let $x \in A(\tau^*)$ be an element of q height 0 in $A(\tau^*)$, and, therefore, of q height 0 in A. Let $0 \neq y \in N$ and consider:

$$\begin{array}{ccc} & & G' \\ & & \downarrow \mu \\ & \overset{\Theta}{\swarrow} & G'/H' \\ & & \downarrow \varphi \\ A & \longrightarrow & A/\langle x + qy \rangle_* \end{array}$$

where φ is defined by $\varphi(g' + H') = y + \langle x + qy \rangle_*$ and Θ is obtained by projectivity.

We have $q\Theta(g') + \langle x + qy \rangle_* = -x + \langle x + qy \rangle_*$, so $q\Theta(g') + x = \alpha(x + qy)$, $\alpha \in Q$. Also, $\Theta(H') \subseteqq \langle x + qy \rangle_*$.

Suppose rank $\Theta(G') = 1$. Then $\Theta(H') = (0)$, in view of the requirements on

$\Theta(g')$, $\Theta(H')$. But then type $\Theta(g')\geqq\tau$, type $x\geqq\tau$, so type $\alpha(x+qy)\geqq\tau$. This can hold only if $\alpha=0$, since type $y\not\geqq\tau$. Thus, $x=-q\Theta(g')$ — a contradiction.

Consequently, we must have rank $\Theta(G')=2$. In this case, $A/A(\tau^*)\geqq\Theta(G')+A(\tau^*)/A(\tau^*)\cong\Theta(G')/\Theta(G')\cap A(\tau^*)$. Now $\Theta(G')/\Theta(G')\cap A(\tau^*)$ is a rank one factor of G' and, therefore, is divisible by some prime p with $pA\neq A$. Since rank $A/A(\tau^*)=1$, $A/A(\tau^*)$ must be divisible by p as well. By our construction, such a p is in $[\tau_i]$ for some $i\leqq r'$, so $p\in[\tau^*]$. But, if $p\in[\tau^*]$ and $p[A/A(\tau^*)]=A/A(\tau^*)$, then $pA=A$. Therefore, in this case too, we are forced into a contradiction. This final contradiction completes the proof.

2. A cotorsion-free class in $\mathscr{C}$

Following [2], if $\mathbf{X}=\{X_\alpha\}$ is a set of groups in $\mathscr{C}$ let $\mathscr{I}(\mathbf{X})=\{W\in\mathscr{C}\,|\,\mathrm{Ext}(X_\alpha,W)=(0),\ \forall X_\alpha\in\mathbf{X}\}$ and let $\mathscr{P}\mathscr{I}(\mathbf{X})=\{Y\in\mathscr{C}\,|\,\mathrm{Ext}(Y,W)=(0),\ \forall W\in\mathscr{I}(\mathbf{X})\}$. We call $\mathscr{P}\mathscr{I}(\mathbf{X})$ the cotorsion-free class of $\mathbf{X}$ in $\mathscr{C}$. ($\mathscr{P}\mathscr{I}(\mathbf{X})$ satisfies the requirements for an abstract cotorsion free class.)

It is clear that, for $Y\in\mathscr{C}$, $Y\in\mathscr{P}\mathscr{I}(\mathbf{X})$ iff Y is projective with respect to all $\mathscr{C}$ exact sequences of the form: $0\to W\to U\to V\to 0$, $W\in\mathscr{I}(\mathbf{X})$. Thus, $\mathscr{P}\mathscr{I}(\mathbf{X})$ is a projective class in $\mathscr{C}$. Clearly, $\mathbf{X}\subseteqq\mathscr{P}\mathscr{I}(\mathbf{X})$.

In [2], $\mathscr{P}\mathscr{I}(\mathbf{X})$ is investigated in the category of all abelian groups. In this section, we give a characterization of $\mathscr{P}\mathscr{I}(\mathbf{X})$ in the category $\mathscr{C}$. This characterization is employed to prove $P_0C_0(G)\subseteqq\mathscr{P}\mathscr{I}(G)$ for all $G\in\mathscr{C}$.

First, we need a little more notation. Let τ be an arbitrary type and let S be an arbitrary non-empty subset of Π. Write $\tau\underset{S}{\gtrsim}\bar{0}$ if there exists $h\in\tau$ such that $h(p)=0$ for all $p\in S$. If $\{\tau_\alpha\}$ is a set of types let:

$$\{\tau_\alpha\}^0=\{S\subseteqq\Pi\,|\,\tau_\alpha\underset{S}{\gtrsim}\bar{0},\ \forall\alpha\},$$

$$\overline{\{\tau_\alpha\}}=\{\text{types}\,\tau\,|\,\tau\underset{S}{\gtrsim}\bar{0},\ \forall S\in\{\tau_\alpha\}^0\}.$$

THEOREM 2. *Let* $\mathbf{X}=\{X_\alpha\}$ *be a set of groups in* $\mathscr{C}$. *Then, for* $Y\in\mathscr{C}$, $Y\in\mathscr{P}\mathscr{I}(\mathbf{X})$ *iff* $\mathrm{OT}(Y)\in\overline{\{\mathrm{OT}(X_\alpha)\}}$.

PROOF. Let $X=X_\alpha\in\mathbf{X}$ and $W\in\mathscr{I}(\mathbf{X})$. Choose F a free subgroup of X with X/F torsion. We have the exact sequence: $0\to\mathrm{Hom}(X,W)\to\mathrm{Hom}(F,W)\,\mathrm{Ext}(X/F,W)\to\mathrm{Ext}(X,W)=0\to 0$. Since $\mathrm{Hom}(F,W)$ is countable, so is $\mathrm{Ext}(X/F,W)$. But $\mathrm{Ext}(X/F,W)\cong\prod_{p\in\mathrm{supp}\,W}\mathrm{Ext}(T_p,W)$, where T_p is the p component of X/F. Thus, $T_p=0$ for almost all $p\in\mathrm{supp}\,W$. Furthermore, if $p\in\mathrm{supp}\,W$ and $T_p\neq 0$, then $|T_p|<\infty$ for, otherwise, $\mathrm{Ext}(T_p,W)$ would be uncountable ([1] Theorem 52.3). Therefore, $\bigoplus_{p\in\mathrm{supp}\,W}T_p$ is a finite group. This implies, $\mathrm{OT}(X)\underset{\mathrm{supp}\,W}{\sim}\bar{0}$ [4]. Since $X\in\mathbf{X}$ was arbitrary, $\mathrm{supp}\,W\in\{\mathrm{OT}(X_\alpha)\}^0$.

Conversely, if $W\in\mathscr{C}$ with $\mathrm{supp}\,W\in\{\mathrm{OT}(X_\alpha)\}^0$ then, for each $X\in\mathbf{X}$ there exists a free subgroup $F\subseteqq X$ such that X/F is a torsion group with zero p component for all $p\in\mathrm{supp}\,W$. Thus, $\mathrm{Ext}(X/F,W)=(0)$ and, consequently, $\mathrm{Ext}(X,W)=0$.

We have proved $\mathscr{I}(\mathbf{X})=\{W\in\mathscr{C}\,|\,\mathrm{supp}\,W\in\{\mathrm{OT}(X_\alpha)\}^0\}$. An analogous argument shows that $\mathscr{P}\mathscr{I}(\mathbf{X})=\{Y\in\mathscr{C}\,|\,\mathrm{OT}(Y)\in\overline{\{\mathrm{OT}(X_\alpha)\}}\}$. This completes the proof of Theorem 2.

We immediately obtain the following fact (which does not hold for general cotorsion-free classes):

COROLLARY 1. *$\mathscr{P}\mathscr{I}(\mathbf{X})$ is closed under taking factor groups in $\mathscr{C}$.*

We also have (in contrast to P_0C_0):

COROLLARY 2. *Let $G\in\mathscr{C}$. Then $\mathscr{P}\mathscr{I}(G)=\mathscr{P}\mathscr{I}(\mathscr{S}_G)$.*

PROOF. Let $\mathscr{S}_G=\{X_i\}$ and let $\tau_i=\text{type}\,X_i\,(=\text{OT}(X_i))$. Since, $\forall i$, $\tau_i\leqq\text{OT}(G)$, it follows that $\{\text{OT}(G)\}^0\subseteqq\{\tau_i\}^0$. Therefore, $\overline{\{\tau_i\}}\subseteqq\overline{\{\text{OT}(G)\}}$. On the other hand, $\text{OT}(G)=\bigvee_{j=1}^{r}\tau_{ij}$ for some finite subset $\{\tau_{i_1},\ldots,\tau_{i_r}\}\subseteqq\{\tau_i\}$. Thus, $\{\tau_i\}^0\subseteqq\{\tau_{i_1},\ldots,\tau_{i_r}\}^0=\{\text{OT}(G)\}^0$. Therefore, $\overline{\{\text{OT}(G)\}}\subseteqq\overline{\{\tau_i\}}$. Now apply Theorem 2 to obtain the result.

Another easy consequence of Theorem 2 is the following.

THEOREM 3. *Let $G\in\mathscr{C}$.* Then $P_0C_0(G)\subseteqq\mathscr{P}\mathscr{I}(G)$.

PROOF. Since $P_0C_0(G)\subseteqq P_0C_0(\mathscr{S}_G)$ and $\mathscr{P}\mathscr{I}(\mathscr{S}_G)=\mathscr{P}\mathscr{I}(G)$, it will suffice to prove that $P_0C_0(\mathbf{X})\subseteqq\mathscr{P}\mathscr{I}(\mathbf{X})$ for $\mathbf{X}$ any set of rank one groups in $\mathscr{C}$. Let $\mathbf{X}=\{X_\alpha\}$, $\tau_\alpha=\text{type}\,X_\alpha$, and let $Y\in P_0C_0(\mathbf{X})$. Any projective class in $\mathscr{C}$ must contain all free groups of finite rank and must be closed under taking direct summands and finite direct sums. Therefore we can, without loss of generality, establish the result by proving $Y\in\mathscr{P}\mathscr{I}(\mathbf{X})$ under the additional assumption that Y has no free factors.

Choose $\tau_1,\ldots,\tau_r\in\text{cotypeset}\,Y$ such that $\tau=\text{OT}(Y)=\bigvee_{i=1}^{r}\tau_i$. Since $Y\in P_0C_0(\mathbf{X})$, each $\tau_i=\text{l.u.b.}\,\{\tau_\alpha|\tau_\alpha\leqq\tau_i\}$. It follows that $\tau=\text{l.u.b.}\,\{\tau_\alpha|\tau_\alpha\leqq\tau\}$.

Let $S\in\{\tau_\alpha|\tau_\alpha\leqq\tau\}^0$. We must have $\tau\not\sim\bar{0}$. Otherwise, if $h\in\tau$, define a height vector h' by: $h'(p)=0$ if $p\in S$, $h'(p)=h(p)$ if $p\notin S$. Let τ' be the type with $h'\in\tau'$. Then $\tau'<\tau$, but τ' is still an upper bound for $\{\tau_\alpha|\tau_\alpha\leqq\tau\}$ — a contradiction. Thus, $\tau\in\overline{\{\tau_\alpha|\tau_\alpha\leqq\tau\}}\subseteqq\overline{\{\tau_\alpha\}}$, and, by Theorem 2, $Y\in\mathscr{P}\mathscr{I}(\mathbf{X})$.

3. Equality of classes

In this section we characterize those groups G for which $P_0C_0(G)=\mathscr{P}\mathscr{I}(G)$. Let $\text{cotypeset}^*\,G=\{\tau\in\text{cotypeset}\,G|\tau>\bar{0}\}$.

THEOREM 4. *Let $G\in\mathscr{C}$. Then $P_0C_0(G)=\mathscr{P}\mathscr{I}(G)$ iff there exists an idempotent type τ with $[\tau]$ finite such that* $\text{cotypeset}^*\,G=\{\text{types}\,\tau'|\bar{0}<\tau'\leqq\tau\}$.

REMARK. In this case $\tau=\text{OT}(G)$ and cotypeset G is necessarily finite.

PROOF. Suppose $P_0C_0(G)=\mathscr{P}\mathscr{I}(G)$, and, without loss of generality, that G is not free. Since $P_0C_0(G)\subseteqq P_0C_0(\mathscr{S}_G)\subseteqq\mathscr{P}\mathscr{I}(\mathscr{S}_G)=\mathscr{P}\mathscr{I}(G)$, necessarily $P_0C_0(\mathscr{S}_G)=\mathscr{P}\mathscr{I}(\mathscr{S}_G)$.

Let $\text{cotypeset}^*\,G=\{\tau_i\}$. In view of Theorem 2 and Theorem 4.3 of [3], $\mathscr{P}\mathscr{I}(\mathscr{S}_G)$ iff $\mathscr{L}\{\tau_i\}=\overline{\{\tau_i\}}$, where $\mathscr{L}\{\tau_i\}=\{\text{types}\,\tau''|\tau''=\text{l.u.b.}\,T$ for some $T\subseteqq\{\tau_i\}\}$. (Here, of course, we are only considering those subsets $T\subseteqq\{\tau_i\}$ such that l.u.b. T exists in

the partially ordered set of all types. We define $\bar{0}$ to be the least upper bound of $T=\emptyset$.)

Choose $h_i \in \tau_i$ and let $S_i=\{p\in\Pi | h_i(p)>0\}$. Each S_i must be a finite set of primes. For otherwise, if some S_{i_0} were infinite, we could write S_{i_0} as a countable disjoint union of countable subsets, say $S_{i_0}=\bigcup_{j=1}^{\infty} V_j$. Let f_j be the characteristic function of V_j, and let σ_j be the type with $f_j\in\sigma_j$. For each j, $\sigma_j\in\{\overline{\tau_{i_0}}\}\subseteq\{\overline{\tau_i}\}$. Thus, for each j, $\sigma_j\in\mathscr{L}\{\tau_i\}$, as well.

Since $\sigma_j>\bar{0}$, there must exist, for each j, a type $\tau_{i(j)}\in\{\tau_i\}$ with $\bar{0}<\tau_{i(j)}\leqq\sigma_j$. Choose a pure subgroup $K_j\subseteqq G$, of co-rank 1, such that type $G/K_j=\tau_{i(j)}$. For any positive integer m, define $K^m=\bigcap_{j=1}^{m} K_j$. Then K^m is a pure subgroup of G, and we have the natural embedding $0\to G/K^m\to\bigoplus_{j=1}^{m} G/K_j$. Now it is easy to check that $\bigoplus_{j=1}^{m} G/K_j$ has no rank one factor of type $\geqq\tau_{i(m+1)}$. But G has a rank one factor of type $\tau_{i(m+1)}$. Thus, for all m, $K^m\neq(0)$. This contradicts the fact that G has finite rank. So each S_i is finite. Therefore, each τ_i is an idempotent type with $[\tau_i]$ finite. Pick $\tau_{i_1},\ldots,\tau_{i_r}\in\{\tau_i\}$ such that $\tau=\mathrm{OT}(G)=\bigvee_{j=1}^{r}\tau_{ij}$. Then τ is an idempotent type, $[\tau]$ is finite, and cotypeset $G\subseteqq\{\tau'|\tau'\leqq\tau\}$.

Finally, let $[\tau]=\{p_1\ldots p_n\}$ and let δ_l, $1\leqq l\leqq n$, be the type of the height vector h_l, where h_l is defined by: $h_l(p_l)=\infty$; $h_l(p)=0$, $p\neq p_l$. Then, each $\delta_l\in\{\overline{\tau_i}\}$, so each $\delta_l\in\mathscr{L}\{\tau_i\}$. But, since $\bar{0}$ is the only type less than any δ_l, each δ_l must actually be in $\{\tau_i\}$. This means, for each l, there exists a homomorphism Θ_l such that $G\xrightarrow{\Theta_l} Z_{-p_l}\to 0$, where Z_{-p_l} is the subring of Q generated by Z and $\frac{1}{p_l}$. For $S=\{p_{l_1},\ldots,p_{l_t}\}$ any non-empty subset of $[\tau]$, let Z_{-S} be the subring of Q generated by Z and $\left\{\frac{1}{p}|p\in S\right\}$ and let Θ_S: $\bigoplus_{p_l\in S} Z_{-p_l}\to Z_{-S}\to 0$ be the natural map. It is easy to check that $\Theta_S\circ$ $\circ(\bigoplus_{p_l\in S}\Theta_l)$: $G\to Z_{-S}\to 0$. This proves that cotypeset* $G=\{\tau'|\bar{0}<\tau'\leqq\tau\}$.

Conversely, if G is free, the condition on cotypeset* G holds vacuously and $P_0C_0(G)=\mathscr{P}\mathscr{I}(G)=\mathscr{F}$, the class of finite rank free groups.

Assume that G is a non free group such that cotypeset* $G=\{\tau'|0<\tau'\leqq\tau\}$ for some idempotent type τ with $[\tau]$ finite. Then, by Theorem 1, $P_0C_0(G)=P_0C_0(\mathscr{S}_G)$. Moreover, it is immediate that $\mathscr{L}(\text{cotypeset* } G)=\overline{(\text{cotypeset* } G)}$, so $P_0C_0(\mathscr{S}_G)=$ $=\mathscr{P}\mathscr{I}(\mathscr{S}_G)$. Since $\mathscr{P}\mathscr{I}(\mathscr{S}_G)=\mathscr{P}\mathscr{I}(G)$ always holds, $P_0C_0(G)=\mathscr{P}\mathscr{I}(G)$.

EXAMPLE. Let G be a rank two group such that $\mathscr{S}_G=\{Z_p|p\in\Pi\}$. (It is not difficult to construct such a G.) Then every rank one factor of G has idempotent type, but G fails to satisfy the condition of Theorem 4. Applying Theorem 1 and Theorem 4.3 of [3], $P_0C_0(G)=P_0C_0(\mathscr{S}_G)=\{Y\in\mathscr{C}|$ every rank one factor of Y is isomorphic to either Z, Q, or Z_p for some $p\}$. Note that $\mathrm{OT}(G)=$type Q, so $\{\mathrm{OT}(G)\}^0=\emptyset$. Therefore, $\overline{\{\mathrm{OT}(G)\}}$ is the set of all types and, by Theorem 2, $\mathscr{P}\mathscr{I}(G)=\mathscr{C}$.

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MAINTENANCE OF OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF DIFFERENTIAL INEQUALITIES UNDER THE EFFECT OF ADVANCED AND MIXED ARGUMENT

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0. Introduction

Consider the nonlinear differential inequality

$$y(t)\left[(a(t)y'(t))'+p(t)f(y(\tau(t)))\right] \leqq 0, \quad t \geqq t_0 \tag{N}$$

and the corresponding differential equation

$$(a(t)y'(t))'+p(t)f(y(\tau(t))) = 0, \quad t \geqq t_0 \tag{E}$$

under the assumptions listed below:

a) $\tau \in C[[t_0, \infty), R]$ *and* $\lim_{t\to\infty} \tau(t) = \infty$;

b) $a \in C[[t_0, \infty), (0, \infty)]$ *and* $p \in C[[t_0, \infty), [0, \infty)]$;

c) *f is a nondecreasing function on R and $uf(u) > 0$ for $u \neq 0$.*

The following notions will be used in the sequel. A continuous real-valued function F on $[T_0, \infty)$ *eventually* has some property if there exists a real number $T_1 \geqq T_0$ such that F has this property on $[T_1, \infty)$. The function F on $[T_0, \infty)$ is called *nonoscillatory* if it is eventually of constant sign. Otherwise, it is called *oscillatory*.

By a solution of (N) (respectively (E)) we mean a function $u \in C^2[[t_u, \infty), R]$ which satisfies (N) (resp. (E)) for all $t \geqq t_u$. The continuity of the real-valued functions f on R and p and τ on $[t_0, \infty)$ as well as sufficient smoothness for the existence of solutions of (N) (resp. (E)) on an infinite subinterval of $[t_0, \infty)$ will be assumed. In what follows the term "solution" is always used only for such solutions $y(t)$ of (N) (resp. (E)), which are defined for all large t.

Inequality (N) (resp. (E)) is called *oscillatory* if its every solution is oscillatory; otherwise it is called nonoscillatory. Inequality (N) (resp. equation (E)) has a *property* ($\tilde{\text{O}}$) if its every solution is oscillatory or monotonically tends to zero as $t \to \infty$.

The crucial point in the oscillation theory of (N) and (E), in the case of retarded argument $\tau(t) \leqq t$, is the recent result of the present author and Grammatikopoulos [6], which asserts that (N) is oscillatory (has a property ($\tilde{\text{O}}$)) iff (E) is oscillatory (has a property ($\tilde{\text{O}}$)). Obviously, the proof of this result can be extended to cover the most general case of the general deviating argument τ satisfying a).

One of the most important problems in the theory of differential equations with deviating argument is the *problem of maintenance of some property* (P) under the effect of deviation of argument. Shortly, under which conditions on a, p, f and τ has equa-

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tion (E) the same property (P) as the corresponding ordinary differential equation:

$$\text{(OE)} \qquad (a(t)y'(t))' + p(t)f(y(t)) = 0, \quad t \geqq t_0.$$

This problem was posed by V. N. Shevelo [14] and was answered partly by Kulenović and Grammatikopoulos [6] in the case of retarded argument $\tau(t) \leqq t$, where (P) is "oscillation of (E)" or "specific asymptotic behavior of (E)". As pointed out in [9], the case of advanced argument $\tau(t) \geqq t$, which was not widely investigated in the literature, is remarkably different from the case of retarded argument $\tau(t) \leqq t$. The purpose of this paper is to proceed in the direction of establishing the conditions which provide the maintenance of oscillation and specific asymptotic behavior of (E) and (N) in the cases of the advanced and mixed type argument.

The results of this paper extend some recent results obtained by Grammatikopoulos [2], Kamenev [3], Kiguradze [4] and Kusano and Onose [9], as well as the well known results of Atkinson [1], Moore and Nehari [11] and Nehari [12].

1. Preliminaries

In what follows we will use the functions $R(t)$ and $\varrho(t)$ defined by

$$R(t) = \int_{t_0}^{t} \frac{ds}{a(s)}, \quad t \geqq t_0$$

and

$$\varrho(t) = \int_{t}^{\infty} \frac{ds}{a(s)}, \quad t \geqq t_0 \quad \text{if} \quad \int_{t_0}^{\infty} \frac{ds}{a(s)} < \infty.$$

Further we will make use of a very simple fixed point theorem essentially due to Knäster:

FIXED POINT THEOREM ([10], p. 65). *Let $\leqq$ be a partial ordering with field A, and suppose that every $B \subseteq A$ has a least upper bound. Suppose that F maps A into A such that for all x, y in A, $x \leqq y$ implies $Fx \leqq Fy$. Then $Fx = x$ for some $x \in A$.*

2. Sufficient conditions for the oscillatory and asymptotic behavior

In this section we present some results concerning the oscillation and property ($\tilde{O}$) for inequality (N) and equation (E).

THEOREM 2.1. *Consider the differential inequality* (N) *subject to the conditions*

(i) $\tau()) \geqq t$ *for all large* $t \geqq t_0$

and

(ii) $\tau(t)$ *is nondecreasing for all large* $t \geqq t_0$.

If there exists a function $g \in C^2[[t_0, \infty), (0, \infty)]$ *such that* $g'(t) \leqq 0$ *and* $(a(t)g'(t))' \geqq 0$ *for every* $t \geqq t_0$, *then under the conditions*

$$(C_1) \qquad \int^{\infty} g(\tau(t))p(t)\,dt = \infty$$

and

$$(C_2) \qquad \int^{\infty} \frac{1}{g(\tau(t))a(t)} \int_{t_0}^{t} g(\tau(s))p(s)\,ds\,dt = \infty$$

the differential inequality (N) *has property* ($\tilde{O}$).

PROOF. Let y be a nonoscillatory solution of (N). Then y is eventually of constant sign and in both cases of $y>0$ and $y<0$ we have eventually

$$(2.1) \qquad (a(t)y'(t))'/f(y(\tau(t))) \leqq -p(t)$$

which shows that y' is eventually of constant sign. Thus, only one of the following two possibilities can occur:

1) $yy'>0$ eventually
2) $yy'<0$ eventually.

Suppose that 1) *holds.* Differentiating the function

$$V(t) = a(t)y'(t)g(\tau(t))/f(y(t)), \quad t \geqq t_1 \geqq t_0$$

and using (2.1) we have for every $t \geqq t_1$

$$V'(t) \leqq -g(\tau(t))p(t)f(y(\tau(t)))/f(y(t)) + a(t)y'(t)\tau'(t)g'(\tau(t))/f(y(t)) -$$

$$-g(\tau(t))f'(y(t))\frac{a(t)(y'(t))^2}{(f(y(t)))^2} \leqq -g(\tau(t))p(t).$$

Integrating this inequality from t_1 to t we get by (C_1)

$$V(t) \leqq V(t_1) - \int_{t_1}^{t} g(\tau(s))p(s)\,ds < 0,$$

for all large t which is an immediate contradiction.

Suppose that 2) *holds.* We consider only the case where $y>0$ (the case $y<0$ can be treated in a similar way). So we have $\lim_{t\to\infty} y(t) = c \in [0, \infty)$. Suppose that $c>0$. Differentiating the function

$$(2.2) \qquad U(t) = a(t)y'(t)g(\tau(t))$$

and using (2.1) we get eventually

$$U'(t) \leqq -g(\tau(t))p(t)f(y(\tau(t))) + a(t)y'(t)g'(\tau(t))\tau'(t),$$

which by integration leads to

$$U(t) \leqq U(t_2) + \int_{t_2}^{t} a(s)y'(s)\tau'(s)g'(\tau(s))\,ds - \int_{t_2}^{t} g(\tau(s))p(s)f(y(\tau(s)))\,ds.$$

An integration by parts in the second integral yields

(2.3)
$$U(t) \leqq U(t_2) + \int_{t_2}^{t} a(s)y'(s)g'(\tau(s))\tau'(s)\,ds - f(y(\tau(t)))\int_{t_2}^{t} g(\tau(s))p(s)\,ds +$$
$$+ \int_{t_2}^{t} f'(y(\tau(s)))y'(\tau(s))\tau'(s)\int_{t_2}^{s} g(\tau(u))p(u)\,du\,ds.$$

Using (i) and the fact that $a(t)y'(t)$ is a nondecreasing function we obtain

(2.4)
$$a(\tau(t))y'(\tau(t)) \leqq a(t)y'(t),$$

which by the second Bonnet's theorem implies

$$\int_{t_2}^{t} a(s)y'(s)g'(\tau(s))\tau'(s)\,ds \leqq \int_{t_2}^{t} a(\tau(s))y'(\tau(s))g'(\tau(s))\tau'(s)\,ds =$$
$$= a(\tau(t_2))g'(\tau(t_2))(y(\tilde{\tau}(t)) - y(\tau(t_2))) \leqq ca(\tau(t_2))g'(\tau(t_2)) < \infty,$$

where $\tilde{\tau}(t) \in [\tau(t_2), \tau(t)]$. Taking into account this relation and using the conditions of the theorem we get from (2.3)

$$U(t) \leqq -\frac{f(c)}{2}\int_{t_2}^{t} g(\tau(s))p(s)\,ds$$

eventually, which yields

$$y(t) \leqq y(t_2) - \frac{f(c)}{2}\int_{t_2}^{t}\frac{1}{g(\tau(s))a(s)}\int_{t_2}^{t} g(\tau(u))p(u)\,du\,ds.$$

By (C_2) we get $\lim_{t\to\infty} y(t) = -\infty$ which is an immediate contradiction. Thus, $c=0$ and $\lim_{t\to\infty} y(t)=0$ monotonically.

THEOREM 2.2. *Suppose that the assumptions of Theorem* 2.1 *hold where* (C_2) *is replaced by*

(C_3)
$$\int^{\infty} \frac{dt}{g(\tau(t))a(t)} = \infty$$

and the additional condition

(C_4)
$$\int_{0}^{\pm\varepsilon} \frac{du}{f(u)} < \infty, \quad \varepsilon > 0$$

is satisfied. Then (N) *is oscillatory.*

PROOF. Suppose that y is a nonoscillatory solution. As in the proof of Theorem 2.1, only one of two possible cases can occur.

Case 1 can be treated in a similar way as the corresponding one in Theorem 2.1 in order to obtain a contradiction.

Suppose that *Case* 2 holds. Since (C_1) and (C_3) are stronger than (C_1) and (C_2) we conclude that $y(t)\downarrow 0$ as $t\to\infty$. We consider the case where $y>0$, since the case $y<0$ can be treated in a similar way. Differentiating the function

$$W(t) = a(t)y'(t)g(\tau(t))/f(y(\tau(t)))$$

and using relations (2.1) and (2.4) we obtain eventually

$$W'(t) \leqq -g(\tau(t))p(t) - \frac{W(t)}{f(y(\tau(t)))}\frac{d}{dt}(f(y(\tau(t))))+$$

$$+a(\tau(t))y'(\tau(t))\tau'(t)g'(\tau(t))/f(y(\tau(t))),$$

which by integration yields:

$$W(t) \leqq W(t_3) - \int_{t_3}^{t} g(\tau(s))p(s)\,ds + \int_{t_3}^{t} W(s)\left[\frac{df(y(\tau(s)))}{f(y(\tau(s)))}\right]+$$

$$+\int_{t_3}^{t} a(\tau(s))y'(\tau(s))\tau'(s)g'(\tau(s))/f(y(\tau(s)))\,ds, \quad \text{for large } t.$$

Applying the second Bonnet's theorem on the third integral in this relation and using (C_4) we get

$$\int_{t_3}^{t} a(\tau(s))y'(\tau(s))\tau'(s)g'(\tau(s))/f(y(\tau(s)))\,ds \leqq K < \infty,$$

which by (C_3) implies

$$W(t) \leqq -1 + \int_{t_3}^{t} W(s)\left[-\frac{df(y(\tau(s)))}{f(y(\tau(s)))}\right], \quad \text{for large } t.$$

Using the well-known Bellman—Gronwal's inequality we get eventually

$$W(t) \leqq -f(y(\tau(t_3)))/f(y(\tau(t))),$$

which yields eventually

$$a(t)y'(t)g(\tau(t)) \leqq -f(y(\tau(t_3))) \quad \text{and} \quad y'(t) \leqq -\frac{f(y(\tau(t_3)))}{g(\tau(t))a(t)}.$$

Integrating this relation from t_3 to t and letting $t\to\infty$ we get $\lim_{t\to\infty} y(t) = -\infty$ which is an immediate contradiction.

REMARK 2.1. It is easy to show that Theorems 2.1 and 2.2 hold for the more general inequality

$$y(t)((a(t)b(y(t))y'(t))' + p(t)f(y(\tau(t)))) \leqq 0, \quad t \geqq t_0$$

where $b\in C[R, (0, \infty)]$ and the condition (C_4) is replaced by the condition

$$\int_0^{\pm\varepsilon} \frac{b(u)}{f(u)}\,du < \infty, \quad \varepsilon > 0.$$

The following example shows that the assumptions of Theorem 2.1 do not imply the oscillation of (N).

EXAMPLE 2.1. The inequality

$$(t^3 y'(t))' + \tau^{\alpha}(t)(y(\tau(t)))^{\alpha} = 0, \quad t \geqq 1, \quad \alpha \in (0, \infty)$$

has a nonoscillatory solution $y(t)=\frac{1}{t}$. Obviously all conditions of Theorem 2.1 are satisfied with $g(t)=1$.

REMARK 2.2. Theorem 2.2 is an extension of some results of Grammatikopoulos [2] and Kamenev [3]. Moreover, Theorem 2.2 for $g(t)=\varrho(t)$ is the improvement in the point of touch, of a recent result of Kusano and Onose [9, Theorem 4], since the class of strongly sublinear functions, considered by Kusano and Onose, is obviously the subclass of the functions which satisfy condition (C_4).

It is worth noting (see [6]) that the global oscillation (i.e. oscillation of all solutions of (N)) is implied by global oscillation of the corresponding ordinary differential equation (OE) in the case $\int^{\infty} \frac{dt}{a(t)} = \infty$.

We note that in some interesting cases conditions (C_1) and (C_2) and (C_1) and (C_3) are equivalent to the corresponding conditions, where $\tau(t)=t$. For instance if there exists a constant $K>0$ such that

$$g(\tau(t)) \geqq Kg(t), \quad \text{for large } t$$

or there exist a constant $\beta \geqq 1$ and a function $h\colon [1, \infty) \to (0, \infty)$ such that $\tau(t)/t \leqq \beta$ and $g(\beta t) \geqq h(\beta) g(t)$ for large t, we have the above mentioned equivalence.

Conditions (C_1), (C_2) and (C_3) where $\tau(t)=t$, will be denoted by $(\tilde{C}_1)$, $(\tilde{C}_2)$ and $(\tilde{C}_3)$ respectively.

3. The optimal choice of g

In this part we discuss the optimal choice of the function g in the conditions of Theorems 2.1—2.2. For this we introduce the following classes of functions.

Consider the class of positive, differentiable functions g on $[t_0, \infty)$. We say that g is of *type* A (B) if for every $t \geqq t_0$, g is nondecreasing (nonincreasing) and ag' is nonincreasing (nondecreasing).

It is easy to get some "a priori" bounds for the functions of types A and B. Obviously such bounds depend on the value of the integral $\int_{t_0}^{\infty} dt/a(t)$. Thus, we have

g is of type A

$$g(t) \leqq K_1 R(t), \quad t \geqq t_0 \quad \text{if} \quad \int_{t_0}^{\infty} dt/a(t) = \infty$$

$$g(t) \leqq K_2, \quad t \geqq t_0 \quad \text{if} \quad \int_{t_0}^{\infty} dt/a(t) < \infty,$$

and

$$g \text{ is of type B}$$

$$g(t) \leqq K_3, \quad t \geqq t_0 \quad \text{if} \quad \int_{t_0}^{\infty} dt/a(t) = \infty$$

$$g(t) \leqq K_4 \varrho(t), \quad t \geqq t_0 \quad \text{if} \quad \int_{t_0}^{\infty} dt/a(t) < \infty.$$

From these "a priori" bounds we get some optimal choices of the function g with respect to the condition (C_1) or $(\tilde{C}_1)$ as follows:

$$\text{if } g \text{ is of type A then the optimal values are } g(t) = \begin{cases} R(t), & \int^{\infty} dt/a(t) = \infty \\ 1, & \int^{\infty} dt/a(t) < \infty \end{cases}$$

$$\text{if } g \text{ is of type B then the optimal values are } g(t) = \begin{cases} 1, & \int^{\infty} dt/a(t) = \infty \\ \varrho(t), & \int^{\infty} dt/a(t) < \infty, \end{cases}$$

which means that for these values of g (C_1) and $(\tilde{C}_1)$ are the *weakest* ones on the classes of functions A and B, respectively. Now, we will show that for these optimal values of g the sufficient conditions (C_1) and (C_2) or (C_1) and (C_3) or $(\tilde{C}_1)$ and $(\tilde{C}_2)$ for oscillatory and asymptotic behavior of (N) become necessary and sufficient conditions for such behavior. If g is of type A, then condition (C_3) or $(\tilde{C}_3)$ becomes the *weakest* one (i.e. *optimal* in the above sense) for $g(t)=1$.

THEOREM 3.1. *Consider the differential inequality* (N) *subject to the condition* (ii) *and*

$$\int^{\infty} p(t)\, dt = \infty. \tag{3.1}$$

Then inequality (N) *has porperty* $(\tilde{O})$ *iff*

$$\int^{\infty} 1/a(t) \int_{t_0}^{t} p(s)\, ds\, dt = \infty. \tag{3.2}$$

PROOF. The sufficient part follows immediately from Theorem 2.1 in [7] taking for g the optimal value $g(t)=1$.

In order to prove the necessary part suppose that (3.2) is not satisfied. Then there exist $\omega_1 > \omega \geqq t_0$ such that $\tau(t) \geqq \omega$ for $t \geqq \omega_1$ and

$$f(2M) \int_{\omega}^{\infty} 1/a(t) \int_{t_0}^{t} p(s)\, ds\, dt \leqq M,$$

where $M>0$ is an arbitrary constant. Let X be the set of all nonincreasing functions x defined on $[\omega, \infty)$ and such that $M \leqq x(t) \leqq 2M$ for every $t \geqq \omega$. X is considered

as endowed with the usual point-wise ordering $\leqq$:

$$x_1 \leqq x_2 \Leftrightarrow (\forall t \geqq \omega)\ x_1(t) \leqq x_2(t).$$

It is easy to verify that for every $A \subseteq X$, $\sup A \in X$. We consider the mapping $\mathscr{F}$ on X defined as follows:

$$(\mathscr{F}x)(t) = \begin{cases} 2M + \int_{\omega_1}^{\infty} \frac{1}{a(s)} \int_{t_0}^{s} p(u) f(x(\tau(u)))\, ds\, ds, & t \in [\omega, \omega_1) \\ M + \int_{t}^{\infty} \frac{1}{a(s)} \int_{t_0}^{s} p(u) f(x(\tau(u)))\, du\, ds, & t \geqq \omega_1. \end{cases}$$

Obviously, $\mathscr{F}$ satisfies all conditions of the fixed point theorem and consequently there exists $z \in X$ such that $\mathscr{F}z = z$ for every $t \geqq \omega$. It is obvious now that z is a solution of (E) with the property $\lim_{t\to\infty} z(t) = M$, which completes the proof.

REMARK 3.1. From the proof of the above theorem it follows that if condition (3.2) fails then (E) (and so (N)) has infinitely many nonoscillatory solutions and, actually, for every $M \in R \setminus \{0\}$, there exists a solution y_M of (E) (which depends on M) such that $\lim_{t\to\infty} y_M(t) = M$.

Now we give the corresponding result for another optimal value of g.

THEOREM 3.2. *Consider the differential inequality* (N) *subject to the condition* (ii) *and*

$$\int^{\infty} dt/a(t) < \infty. \tag{3.3}$$

Then, (N) *has property* (Õ) *iff*

$$\int^{\infty} \varrho(t) p(t)\, dt = \infty. \tag{3.4}$$

PROOF. The sufficiency of (3.4) follows directly from Theorem 2.1 in [7] taking $g(t) = \varrho(t)$.

Suppose that (3.4) fails. Then, there exist $\omega_1 > \omega \geqq t_0$ such that $\tau(t) \geqq t_0$ for $t \geqq \omega$ and $\tau(t) \geqq \omega$ for $t \geqq \omega_1$ and

$$f(2K) \int_{\omega}^{\infty} \varrho(t) p(t)\, dt \leqq K,$$

for an arbitrary constant $K > 0$. Let X be the set of all nonincreasing functions x on $[\omega, \infty)$, with the property $K \leqq x(t) \leqq 2K$, for every $t \geqq \omega$. X is subject to the point-wise ordering $\leqq$ as in the proof of Theorem 3.1. We consider the mapping F on X defined as follows:

$$(Fx)(t) = \begin{cases} K + \int_{\omega_1}^{\infty} \varrho(s) p(s) f(x[\tau(s)])\, ds, & t \in [\omega, \omega_1) \\ K + \varrho(t) \int_{\omega_1}^{t} (sp) f(x[\tau(s)])\, ds + \int_{t}^{\infty} \varrho(s) p(s) f(x[\tau(s)])\, ds, & t \geqq \omega_1. \end{cases}$$

It is easy to see that all conditions of the fixed point theorem are satisfied, which implies that there exists $z \in X$ such that $Fz = z$. Clearly z is the nonoscillatory solution of (E), which completes the proof. Now, we introduce the condition

$$(C_5) \qquad \int_{\pm\varepsilon}^{\pm\infty} du/f(u) < \infty, \quad \varepsilon > 0,$$

which will be used in the sequel.

THEOREM 3.3. *Consider the differential inequality* (N) *subject to the conditions* (i), (ii), (C_5) *and*

$$(3.5) \qquad \int^{\infty} \frac{dt}{a(t)R(t)} \int_{t_0}^{t} R(s)p(s)\,ds\,dt = \infty.$$

Then, (N) *has property* $(\tilde{O})$ *iff*

$$(3.6) \qquad \int^{\infty} R(t)p(t)\,dt = \infty.$$

PROOF. The sufficiency follows immediately from Theorem 2.3 in [7] taking $g(t) = R(t)$.

In order to prove the necessity suppose that (3.6) does not hold. Then, there exist $\omega_1 > \omega \geqq t_0$ such that $\tau(t) \geqq \omega$ for $t \geqq \omega_1$ and

$$f(2M) \int_{\omega}^{\infty} R(t)p(t)\,dt \leqq M$$

where $M > 0$ is an arbitrary constant. Let X be the set of all nondecreasing functions x defined on $[\omega, \infty)$ and such that $M \leqq x(t) \leqq 2M$ for every $t \geqq \omega$. X is considered as endowed with the usual point-wise ordering $\leqq$ as it was defined in the proof of Theorem 3.1. We also consider the mapping $\mathscr{H}$ on X defined as follows:

$$(\mathscr{H}x)(t) = \begin{cases} M + R(\omega_1) \int_{\omega_1}^{\infty} p(s) f\big(x(\tau(s))\big)\,ds. & t \in [\omega, \omega_1) \\ M + R(t) \int_{t}^{\infty} p(s) f\big(x(\tau(s))\big)\,ds + \int_{\omega_1}^{t} R(s)p(s) f\big(x(\tau(s))\big)\,ds, & t \geqq \omega_1. \end{cases}$$

It is easy to verify that the fixed point theorem can be applied to the mapping $\mathscr{H}$ on X. So, there exists $y \in X$ such that $\mathscr{H}y = y$ and $\lim_{t\to\infty} y(t) \in (M, 2M]$.

THEOREM 3.4. *Consider the differential inequality* (N) *subject to the conditions* (i), (ii), (C_5) *and*

$$(3.7) \qquad \int^{\infty} dt/a(t) = \infty.$$

Then (N) *is oscillatory iff* (3.6) *holds.*

PROOF. It follows immediately from [7, Corollary 2.3] (sufficiency) and from Theorem 3.3. (necessity).

REMARK 3.2. From the proof of Theorem 3.3 it follows that if (3.6) fails, then (N) has infinitely many nonoscillatory solutions $y(t)$ with the property $\lim_{t\to\infty} y(t)=M$. Obviously because of (C_5) these solutions are linearly independent.

Taking $f(u)=u^{2n+1}$, $n=1, 2, \ldots$ in Theorem 3.4 we get an extension of the well-known result of Atkinson [1] and Moore and Nehari [11, Theorem 1]. Moreover, since the class of superlinear functions f is obviously the subclass of the class of functions f which satisfy (C_5), Theorem 3.4 is the generalization and improvement of some results of Kusano and Onose [9, Theorem 7] and the extension of [12, Theorem 1]. As it was shown in [6] the corresponding result does not hold in the case of retarded argument $\tau(t)\leqq t$. As it was pointed out in [9] there is a remarkable difference between the oscillatory character of advanced differential equations and that property of retarded differential equations.

In the case when condition (3.3) holds we mention the following result of Kusano and Onose [9, Theorem 3].

THEOREM 3.5. *Consider the differential equation* (E) *subject to the conditions* (i) *and* (3.3). *Suppose that* (E) *is strongly superlinear. Then* (E) *is oscillatory iff*

$$\int^{\infty} p(t)\left|f\left(c\varrho(\tau(t))\right)\right| dt = \infty, \quad \text{for all} \quad c \neq 0. \tag{3.8}$$

Now, we are able to formulate two *comparison* results concerning the oscillation of (E) and the corresponding ordinary differential equation (OE) in the case when (3.3) holds. These results, which are in some sense unusual, roughly speaking show that *if* (E) *is oscillatory, then* (OE) *is also oscillatory*. Usually, all known comparison results hold in the converse way.

THEOREM 3.6. *Consider the differential equations* (OE) *and* (E) *subject to the conditions* (3.3) *and* (C_4). *If* (i) *holds, then*

(E) *is oscillatory*$\Rightarrow$(OE) *is oscillatory.*

PROOF. Let y be a nonoscillatory solution of (OE). Then by [6] and [7] $\int^{\infty} \varrho(t)p(t)dt<\infty$, which by the necessity part of Theorem 3.2 implies that (E) is nonoscillatory, a contradiction.

THEOREM 3.7. *Consider the differential equations* (E) *and* (OE) *subject to the conditions* (3.3) *and suppose that* f *is strongly superlinear* (i.e. *there exists a number* $q>1$ *such that* $f(u)/|u|^q \operatorname{sgn} u$ *is nondecreasing for* $u>0$ *and nonincreasing for* $u<0$). *If* (i) *holds, then*

(E) *is oscillatory*$\Rightarrow$(OE) *is oscillatory.*

PROOF. Suppose that (E) is oscillatory. Let y be a nonoscillatory solution of (OE). Then by Theorem 3.5 we have $\int^{\infty} p(t)\left|f\left(c\varrho(t)\right)\right|dt<\infty$. Taking into account

the monotonicity of f and the fact that $\varrho(\tau(t)) \leqq \varrho(t)$ we conclude that

$$\int^{\infty} p(t)\left|f(c\varrho(\tau(t)))\right| dt < \infty,$$

which by Theorem 3.5 implies that (E) is nonoscillatory, a contradiction.

We conclude this section by giving a characterization for the oscillation of (E).

THEOREM 3.8. *Consider the differential equation* (E) *subject to the conditions* (i), (ii), (C_5) *and* (3.7). *Then* (E) *is oscillatory iff*

$$\int_{t_0}^{\infty} \frac{1}{a(t)} \int_{t}^{\infty} p(u)\, du\, dt = \infty. \tag{3.9}$$

PROOF. The necessity of (3.9) follows immediately from the result of Philos [13].

Suppose that (3.9) holds. Let y be a nonoscillatory solution of (E). Without loss of generality we may suppose that $y>0$ eventually, which by (3.7) implies that $y' \leqq 0$ eventually. Thus, there exists $T \geqq t_0$ such that $y(t)>0$, $y'(t) \leqq 0$ and $y'(\tau(t)) \leqq 0$ for every $t \geqq T$. Differentiating the function

$$w(t) = a(t)y'(t)/f(y(\tau(t))),$$

we obtain, for every $t \geqq T$

$$w'(t) = (a(t)y'(t))'/f(y(\tau(t))) -$$

$$-a(t)y'(t)\tau'(t)y'(\tau(t))f'(y(\tau(t)))/(f(y(\tau(t))))^2 \leqq -p(t)$$

which, by integration from T to $t \geqq T$, yields

$$w(t) - w(T) \leqq -\int_{T}^{t} p(s)\, ds$$

and

$$w(T) \geqq \int_{T}^{t} p(s)\, ds.$$

This relation implies

$$w(t) \geqq \int_{t}^{\infty} p(s)\, ds \quad \text{for every} \quad t \geqq T.$$

Taking into account condition (i) we obtain

$$y'(t)/f(y(t)) \geqq y'(t)/f(y(\tau(t))) \geqq \frac{1}{a(t)} \int_{t}^{\infty} p(s)\, ds,$$

which, by integration from T to $t \geqq T$ implies:

$$\int_{y(T)}^{y(t)} du/f(u) \geqq \int_{T}^{t} \frac{1}{a(s)} \int_{s}^{\infty} p(u)\, du\, ds,$$

which is, by (C_5) and (3.8), an immediate contradiction.

We observe that when (3.7) holds then (3.6) is equivalent to (3.9).

4. Maintenance of oscillatory and asymptotic behaviour

In this part we list some results on the maintenance of asymptotic and oscillatory behaviour of all solutions of (N) and (E). These results are immediate consequences of results of the preceding part and will be given without proofs.

THEOREM 4.1. *Consider the differential equation* (E) *subject to the conditions* (ii) *and* (3.1). *Then*

(E) *has a property* $(\tilde{O}) \Leftrightarrow$ (OE) *has a property* $(\tilde{O})$.

THEOREM 4.2. *Consider the differential equation* (E) *subject to the conditions* (ii) *and* (3.3). *Then*

(E) *has a property* $(\tilde{O}) \Leftrightarrow$ (OE) *has a property* $(\tilde{O})$.

THEOREM 4.3. *Consider the differential equation* (E) *subject to the conditions* (i), (ii), (3.5) *and* (C_5). *Then*

(E) *has a property* $(\tilde{O}) \Leftrightarrow$ (OE) *has a property* $(\tilde{O})$.

THEOREM 4.4. *Consider the differential equation* (E) *subject to the conditions* (i), (ii), (3.7) *and* (C_5). *Then*

(E) *is oscillatory* $\Leftrightarrow$ (OE) *is oscillatory*.

In conclusion we note that obviously all results of this paper hold for the more general inequality of the form

$$y(t)\left[\left(a(t)y'(t)\right)' + F\left(t, y\langle h(t)\rangle\right)\right] \leqq 0, \quad t \geqq t_0$$

and the corresponding equation

$$\left(a(t)y'(t)\right)' + F\left(t, y\langle h(t)\rangle\right) = 0, \quad t \geqq t_0,$$

where $y\langle h(t)\rangle = \left(y(h_1(t)), \ldots, y(h_r(t))\right)$, $h=(h_1, \ldots, h_r)$ and F satisfy some appropriate conditions as in [13]. Such kind of extensions are just of technical nature and can be done using the same methods as those presented in this paper.

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HEREDITARY RADICALS AND QUASIRADICALS IN UNIVERSAL ALGEBRAS

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Dedicated to Dr. habil. Hans-Jürgen Hoehnke

Introduction

The theory of radicals, first developed for rings, has since been extended to various other algebraic structures. There is also a general radical theory for arbitrary universal algebras, which goes back to H. J. Hoehnke [7]. A *radical* in the sense of Hoehnke assigns to each algebra $\mathfrak{A}$ of a given class (e.g., a variety) a congruence relation $R(\mathfrak{A})$ so that

(R1) $f(R(\mathfrak{A})) \subseteq R(f(\mathfrak{A}))$ for any homomorphism f from $\mathfrak{A}$, and

(R2) $R(\mathfrak{A}/R(\mathfrak{A})) = \mathbf{0}$.

Specializing this notion to rings, however, one arrives at radicals which are more general than the classical Kuroš—Amitsur radicals. A direct generalization of the latter to universal algebras has been given by R. Mlitz [15], [16].

In the present paper we consider radicals in the sense of Hoehnke which are *hereditary* with respect to subalgebras:

(R3) $R(\mathfrak{A}') = R(\mathfrak{A}) \cap \mathfrak{A}' \times \mathfrak{A}'$ for subalgebras $\mathfrak{A}'$ of $\mathfrak{A}$.

In case of rings these radicals prove to be radicals in the sense of Kuroš—Amitsur, too. Note that hereditary radicals are not the same as the usually considered radicals with hereditary radical class (for hereditariness properties of radical or semisimple classes see, e.g., Stewart [19], Wiegandt [20], Leavitt [11]).

Heredity is a very strong property (in case of rings it is called strong heredity), which enables to give a full description of all hereditary radicals in arbitrary varieties of algebras. Keeping the strong condition of heredity we can, on the other hand, weaken the notion of radical: applying the same methods we get a description of all hereditary quasiradicals (i.e. "radicals" without (R2)) in universal algebras.

The approach presented here is — in some sense — a generalization of Maranda's description of hereditary (quasi-)radicals in the variety ***Mod S*** of unitary right modules over a ring S [13]. Maranda assigns to every (quasi-)radical R the set M of all (right-)ideals B of S such that S/B is radical. Conversely, assigning to a set M of right ideals ($=S$-submodules) of S the mapping R defined by $R(\mathfrak{A}) = \{a \in \mathfrak{A} \mid \ker(1_S \mapsto a) \in M\}$. Maranda gets all hereditary (quasi-)radicals starting with sets M of specified properties (here $(1_S \mapsto a)$ denotes the uniquely determined S-module-homomorphism $S \to \mathfrak{A}$ which sends the unit $1_S \in S$ to $a \in \mathfrak{A}$, $\mathfrak{A} \in$ ***Mod*** S).

In general, there is no 1—1-correspondence between subalgebras and congruence relations (kernels of homomorphisms). This leads to the distinction between *(quasi-) radicals* and 0-*(quasi-)radicals* (the latter assign to each algebra $\mathfrak{A}$ a subalgebra $r(\mathfrak{A})$). Because of heredity, the algebras generated by one or two elements determine

the 0-(quasi-)radicals or (quasi-)radicals, resp., at all. Such one- or two-generated algebras can be represented as homomorphic images (or, equivalently, as congruences) of F_1 or F_2, the *free algebras* (in a given variety) with one or two generators, resp. In fact, this was done by Maranda since S can be considered as the free S-module with one generator (cf. 8.4). Now in the general setting, instead of S, we shall use F_1 or F_2, and the above sets M of right-ideals (in case of ***Mod*** S) correspond to subsets (so called Q-sets or R-sets) of Con F_1 or Con F_2 (the congruence lattices of F_1 and F_2).

The present paper consists of 7 sections. In §1 we give some preliminary definitions and notations. In §2 we introduce an important technical tool for our investigations (namely Q-sets and R-sets). §3 consists of the main result of this paper: *Every hereditary radical or quasiradical in a variety* $\mathscr{V}$ *can be uniquely characterized in terms of a subset of* Con F_2. In particular, (quasi-)radicals can be presented by quantifier-free formulas of the first order predicate calculus provided that F_2 is finite. In §4 semisimple and radical classes of hereditary radicals are investigated and characterized. As an example, hereditary (quasi-)radicals in the variety of S-systems are considered in §5, and all radicals are listed for a concrete semigroup S.

Throughout the last three sections we suppose that our variety admits a constant fundamental operation whose value 0 is a subalgebra. §6 deals with hereditary 0-quasiradicals and characterizes them in terms of subsets of Con F_1. Hereditary 0-quasiradicals are uniquely determined by their "radical" classes, which are also characterized. In §7, hereditary 0-radicals are considered. If there is a "satisfactory" connection between congruences and 0-classes ("normal subalgebras"), then hereditary 0-radicals can be characterized analogously to 0-quasiradicals. Finally, §8 specializes the results of §6, §7 to modules and rings. For unitary left (or right) modules over a ring S, specialization of our results yields essentially those of Maranda [13]. We also give an explicit classification of hereditary (quasi-)radicals for abelian groups (i.e. $\mathbf{Z}$-modules) by means of ideals in the lattice $\langle \mathbf{N}$; g.c.d., l.c.m.$\rangle$. In the variety of rings, hereditary radicals are A-radicals in the sense of Gardner [3], i.e., they are already determined by a radical defined on the additive groups of the rings. Therefore the characterization of hereditary radicals in the variety of abelian groups provides at the same time a characterization of hereditary ring radicals, too. For associative rings, however, one gets another characterization.

The investigation of hereditary radicals in universal algebras was suggested to the author by H. J. Hoehnke more than 10 years ago. Most of the results of the present paper have already been obtained in 1970/71. Encouragement by L. Márki and R. Wiegandt during the author's stay in Budapest in 1981 finally led to the publication of these results. The author thanks J. H. Hoehnke, L. Márki, R. Mlitz and R. Wiegandt for stimulating discussions and many useful hints. The author also thanks the referee for many useful critical remarks, which led to an improved version of the manuscript.

§1. Preliminaries

1.1. Let $\mathscr{V}$ be a *variety* (equational class) of universal algebras. $\mathscr{V}$ is fixed throughout the next sections. For a given algebra $\mathfrak{A}=\langle A; (f_i)_{i\in I}\rangle\in\mathscr{V}$ let Ω_n be the set of all n-ary *fundamental operations* of $\mathfrak{A}$ and let $\Omega=\bigcup_{n=0}^{\infty}\Omega_n$. We write $x\in\mathfrak{A}$ if x is an element of the base set A of $\mathfrak{A}$. The property of $\mathfrak{A}'$ being a *subalgebra*

of $\mathfrak{A}$ will be denoted by $\mathfrak{A}' \cong \mathfrak{A}$. Con $\mathfrak{A}$ denotes the *congruence lattice* of $\mathfrak{A} \in \mathscr{V}$. The *trivial* congruences are $\mathbf{0} = \mathbf{0}_A = \{(a, a) | a \in A\}$ and $\mathbf{1} = \mathbf{1}_A = A \times A$.

1.2. We recall some well-known notations: For a map $f\colon A \to B$ we have

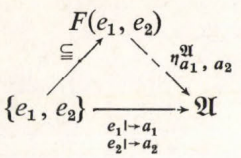
$$\ker f = \{(a, b) | f(a) = f(b)\} \quad (\textit{kernel of } f),$$

$$f(A) = \operatorname{Im} f = \{f(a) | a \in A\} \quad (\textit{image of } f),$$

$$f(\theta) = \{(f(a), f(b)) | (a, b) \in \theta\} \quad \text{for} \quad \theta \subseteq A \times A.$$

For a congruence $\theta \in \operatorname{Con} \mathfrak{A}$ ($\mathfrak{A} \in \mathscr{V}$), $[a]_\theta$ denotes the *equivalence class* of θ containing $a \in A$ (the index θ will not be mentioned if it is clear from the context), $\lambda_\theta\colon \mathfrak{A} \to \mathfrak{A}/\theta\colon a \mapsto [a]_\theta$ denotes the *natural homomorphism* onto the *factor algebra* $\mathfrak{A}/\theta$.

1.3. For a set X, let $F_{\mathscr{V}}(X)$ — for short $F(X)$ — be the $\mathscr{V}$*-free algebra* in $\mathscr{V}$ generated by X, in particular let $F_1 = F(e)$ and $F_2 = F(e_1, e_2)$ be the $\mathscr{V}$-free algebras generated by the elements e and e_1, e_2, respectively. Let $\eta^{\mathfrak{A}}_{a_1, a_2}$ be the unique homomorphism from F_2 into an algebra $\mathfrak{A} \in \mathscr{V}$ which makes the following diagramm commutative:



The kernel $\ker \eta^{\mathfrak{A}}_{a_1, a_2}$ will be denoted by $\theta^{\mathfrak{A}}(a_1, a_2)$ or sometimes by the more impressive notation $\ker_{\mathfrak{A}}(e_1 \mapsto a_1, e_2 \mapsto a_2)$ (the index $\mathfrak{A}$ will be deleted if $\mathfrak{A}$ is clear from the context). Defining $\alpha(a_1, a_2) := \eta^{\mathfrak{A}}_{a_1, a_2}(\alpha)$, the elements $\alpha \in F(e_1, e_2)$ — notation $\alpha = \alpha(e_1, e_2)$ — can be interpreted in every $\mathfrak{A} \in \mathscr{V}$ (in particular in F_2 itself) as binary term functions $\alpha\colon A^2 \to A\colon (a_1, a_2) \mapsto \alpha(a_1, a_2)$. Then $\theta(a_1, a_2)$ can be presented as

$$\theta^{\mathfrak{A}}(a_1, a_2) = \ker_{\mathfrak{A}}(e_1 \mapsto a_1, e_2 \mapsto a_2) =$$

$$= \{(\alpha(e_1, e_2), \beta(e_1, e_2)) \in F_2^2 | \alpha(a_1, a_2) = \beta(a_1, a_2)\}.$$

1.4. Note that $\operatorname{Im} \eta^{\mathfrak{A}}_{a_1, a_2}$ is exactly the subalgebra $\langle a_1, a_2 \rangle_{\mathfrak{A}}$ of $\mathfrak{A}$ generated by a_1 and a_2. Thus

$$\langle a_1, a_2 \rangle_{\mathfrak{A}} = \eta^{\mathfrak{A}}_{a_1, a_2}(F_2) \cong F(e_1, e_2)/\theta^{\mathfrak{A}}(a_1, a_2).$$

§2. Q-sets and R-sets

Hereditary (quasi-)radicals will be characterized by so-called Q-sets and R-sets, which are subsets of Con F_2. In this section we introduce these notions by defining operations on Con F_2 under which Q-sets and R-sets have to be closed.

2.1. Let θ_{id} be the congruence of $F(e_1, e_2)$ generated by identifying e_1 and e_2:

$$\theta_{\mathrm{id}} = \ker_{F(e)}(e_1 \mapsto e, e_2 \mapsto e) = \{(\alpha(e_i, e_j), \alpha(e_k, e_l)) | \alpha \in F_2;\ i, j, k, l \in \{1, 2\}\}.$$

We have $F(e) \cong F(e_1, e_2)/\theta_{\mathrm{id}}$ (cf. 1.4).

2.2. For $\theta\in\mathrm{Con}\,F_2$, let θ^- be defined by "interchanging" the roles of e_1 and e_2:

$$\theta^- = \ker_{F_2/\theta}(e_1\mapsto[e_2]_\theta,\ e_2\mapsto[e_1]_\theta) =$$

$$= \{(\alpha(e_1, e_2), \beta(e_1, e_2))\in F_2^2 \,|\, (\alpha(e_2, e_1), \beta(e_2, e_1))\in\theta\}.$$

2.3. Let $\theta_1, \theta_2\in\mathrm{Con}\,F_2$. We embed θ_1 and θ_2 into $F_3=F(e_1, e_2, e_3)$ as follows (θ_2 is considered as a congruence of $F(e_2, e_3)$, e_2 and e_3 playing the roles of e_1 and e_2):

$$\theta_1^* = \eta_{e_1,e_2}^{F_3}(\theta_1) = \{(\alpha(e_1, e_2), \beta(e_1, e_2))\in F_3^2 \,|\, (\alpha(e_1, e_2), \beta(e_1, e_2))\in\theta_1\},$$

$$\theta_2^* = \eta_{e_2,e_3}^{F_3}(\theta_2) = \{(\alpha(e_2, e_3), \beta(e_2, e_3))\in F_3^2 \,|\, (\alpha(e_1, e_2), \beta(e_1, e_2))\in\theta_2\}.$$

Let θ^* be the congruence of F_3 generated by $\theta_1^*\cup\theta_2^*$. We define $\theta_1\triangledown\theta_2\in\mathrm{Con}\,F_2$ by

$$\theta_1\triangledown\theta_2 = \ker_{F_3/\theta^*}(e_1\mapsto[e_1]_{\theta^*},\ e_2\mapsto[e_3]_{\theta^*}) = \boldsymbol{\theta}^{F_3/\theta^*}([e_1]_{\theta^*}, [e_3]_{\theta^*}).$$

2.4. Let $f\in\Omega_n$, $n\geqq 1$, and let $\theta_1, \ldots, \theta_n\in\mathrm{Con}\,F_2$. Now consider θ_i as an element of $\mathrm{Con}\,F(e_{1i}, e_{2i})$ and embed $\theta_1, \ldots, \theta_n$ into $\mathfrak{A}:=F(e_{11}, e_{21}, \ldots, e_{1i}, e_{2i}, \ldots, e_{1n}, e_{2n})$ (all e_{ji} are to be different). One gets

$$\theta_i^* := \eta_{e_{1i},e_{2i}}^{\mathfrak{A}}(\theta_i) = (\alpha(e_{1i}, e_{2i}), \beta(e_{1i}, e_{2i}))\in\mathfrak{A}^2 \,|\, (\alpha(e_1, e_2), \beta(e_1, e_2))\in\theta_i\},$$

$$i = 1, \ldots, n.$$

Let θ^* be the congruence of $\mathfrak{A}$ generated by $\bigcup_{i=1}^{n}\theta_i$. We define $f[\theta_1, \ldots, \theta_n]\in\mathrm{Con}\,F_2$ by

$$f[\theta_1, \ldots, \theta_n] = \ker_{\mathfrak{A}/\theta^*}(e_1\mapsto[a_1]_{\theta^*},\ e_2\mapsto[a_2]_{\theta^*}) = \boldsymbol{\theta}^{\mathfrak{A}/\theta^*}([a_1]_{\theta^*}, [a_2]_{\theta^*}),$$

where $a_1=f(e_{11}, \ldots, e_{1n})$ and $a_2=f(e_{21}, \ldots, e_{2n})$.

If f is unary ($n=1$), then $f[\theta]$ can be given more explicitly as follows:

$$f[\theta] = \ker_{F_2/\theta}(e_1\mapsto[f(e_1)]_\theta,\ e_2\mapsto[f(e_2)]_\theta) =$$

$$= \{(\alpha(e_1, e_2), \beta(e_1, e_2)\in F_2^2 \,|\, (\alpha(f(e_1), f(e_2)), \beta(f(e_1), f(e_2)))\in\theta\}.$$

2.5. For $L\subseteqq\mathrm{Con}\,F_2$ and $\theta\in\mathrm{Con}\,F_2$ we define

$$(L|\theta) := \{(\alpha, \beta)\in F_2^2 \,|\, \ker_{F_2/\theta}(e_1\mapsto[\alpha]_\theta,\ e_2\mapsto[\beta]_\theta)\in L\}.$$

For short, let

$$(\theta\colon\ (\alpha, \beta)) := \ker_{F_2/\theta}(e_1\mapsto[\alpha]_\theta,\ e_2\mapsto[\beta]_\theta)$$

(in generalization of a ring-theoretic notation). Thus

$$(L|\theta) = \{(\alpha, \beta)\in F_2^2 \,|\, (\theta\colon\ (\alpha, \beta))\in L\}.$$

We collect some properties of the above defined notions in the following proposition, the straightforward proof of which easily follows from the definitions and will be omitted.

2.6. PROPOSITION. *Let* $\mathfrak{A}\in\mathscr{V}$, $\theta, \theta'\in\operatorname{Con} F_2$, $a, a_i, a_i'\in\mathfrak{A}$ $(i\in\{1, \ldots, n\})$, $f\in\Omega_n$ $(n\geqq 1)$, $\alpha=\alpha(e_1, e_2)\in F_2$ *and* $\beta=\beta(e_1, e_2)\in F_2$. *Then*

a) $\theta_{\mathrm{id}}\subseteqq\boldsymbol{\theta}^{\mathfrak{A}}(a, a)$, $(\boldsymbol{\theta}^{\mathfrak{A}}(a_1, a_2))^- = \boldsymbol{\theta}^{\mathfrak{A}}(a_2, a_1)$,

b) $\boldsymbol{\theta}^{\mathfrak{A}}(a_1, a_2)\vee\boldsymbol{\theta}^{\mathfrak{A}}(a_2, a_3)\subseteqq\boldsymbol{\theta}^{\mathfrak{A}}(a_1, a_3)$,

c) $f[\boldsymbol{\theta}^{\mathfrak{A}}(a_1, a_1'), \ldots, \boldsymbol{\theta}^{\mathfrak{A}}(a_n, a_n')]\subseteqq\boldsymbol{\theta}^{\mathfrak{A}}\big(f(a_1, \ldots, a_n), f(a_1', \ldots, a_n')\big)$,

d) $(\theta\colon(\alpha, \beta)) = (\theta\colon(\beta, \alpha))^-$,

e) $(\mathbf{0}\colon(\alpha, \beta)) = \boldsymbol{\theta}^{F_2}(\alpha, \beta)$, $(\mathbf{1}\colon(\alpha, \beta)) = \mathbf{1}$, $(\alpha, \beta)\in\theta\Leftrightarrow\theta_{\mathrm{id}}\subseteqq(\theta\colon(\alpha, \beta))$,

f) $(\boldsymbol{\theta}^{\mathfrak{A}}(a_1, a_2)\colon(\alpha, \beta)) = \boldsymbol{\theta}^{\mathfrak{A}}(\alpha(a_1, a_2), \beta(a_1, a_2))$,

g) $\boldsymbol{\theta}^{F_2/\theta}([e_1]_\theta, [e_2]_\theta) = \theta = (\theta\colon(e_1, e_2))$,

h) $\theta\subseteqq\theta'\Rightarrow(\theta\colon(\alpha, \beta))\subseteqq(\theta'\colon(\alpha, \beta))$.

2.7. DEFINITION. A subset $L\subseteqq\operatorname{Con} F(e_1, e_2)$ is called a *Q-set*, if L satisfies the following conditions:

(L0): (L0a) $\theta_{\mathrm{id}}\in L$,

(L0b) $\theta\in L\Rightarrow\theta^-\in L$,

(L0c) $\theta_1, \theta_2\in L\Rightarrow\theta_1\vee\theta_2\in L$;

(L1): $\theta\in L$, $\theta\subseteqq\theta'\Rightarrow\theta'\in L$ (for $\theta'\in\operatorname{Con} F_2$);

(L2): $\theta_1, \ldots, \theta_n\in L\Rightarrow f[\theta_1, \ldots, \theta_n]\in L$ (for $f\in\Omega_n$, $n\geqq 1$).

L is called an *R-set*, if it, in addition, satisfies the following condition:

(L3) $(L|\theta)\in L\Rightarrow\theta\in L$ (for $\theta\in\operatorname{Con} F_2$).

2.8. REMARKS. a) Conditions (L0)—(L3) are chosen in such a way that the relation $\{(a, b)\in\mathfrak{A}^2|\ker(e_1\mapsto a, e_2\mapsto b)\in L\}$ (which, for each $\mathfrak{A}$, we shall assign to L in §3) becomes an equivalence by (L0) and (L1), a congruence by (L1) and (L2) and a radical by (L3).

b) Note that for condition (L3) the set $(L|\theta)$ of pairs has to be a congruence relation. We shall see later (cf. 3.4b) that $(L|\theta)$ is always a congruence for Q-sets L.

c) While (L1) works "upwards" in the lattice $\operatorname{Con} F_2$, condition (L3) goes "downwards", since $\theta\subseteqq(L|\theta)$ for Q-sets L (due to 2.6e and (L0a), (L1)). Moreover, if $\theta'\subseteqq\theta$ and if $L\subseteqq\operatorname{Con} F_2$ satisfies (L1), then $(L|\theta')\subseteqq(L|\theta)$ by 2.6h and 2.5. Therefore, in order to check property (L3) for L, it suffices to check $(L|\theta)\notin L$ only for all $\theta\in T$ of a "covering" subset T of $\operatorname{Con} F_2\setminus L$ satisfying $\forall\theta'\in\operatorname{Con} F_2\setminus L\,\exists\theta\in T\colon\theta'\subseteqq\theta$. In fact, $(L|\theta')\in L$ would imply $(L|\theta)\in L$ by (L1).

§3. Characterization of hereditary quasiradicals and radicals

3.1. DEFINITIONS. Let $\mathscr{V}$ be a variety and let R be a mapping which assigns to each $\mathfrak{A}\in\mathscr{V}$ a congruence $R(\mathfrak{A})\in\operatorname{Con}\mathfrak{A}$. R is called a *quasiradical* (e.g. [7, p. 375]) *in* $\mathscr{V}$ if

(R1) $g(R(\mathfrak{A}))\subseteq R(g(\mathfrak{A}))$ for every homomorphism g from $\mathfrak{A}\in\mathscr{V}$.

R is called a *radical* (in the sense of Hoehnke, [7, p. 356]) if it satisfies in addition

(R2) $R(\mathfrak{A}/R(\mathfrak{A}))=\mathbf{0}$ for all $\mathfrak{A}\in\mathscr{V}$.

A (quasi-)radical R is said to be *hereditary* if we have

(R3) $\mathfrak{A}'\leqq\mathfrak{A}\Rightarrow R(\mathfrak{A}')=R(\mathfrak{A})\cap\mathfrak{A}'\times\mathfrak{A}'$

for all $\mathfrak{A},\mathfrak{A}'\in\mathscr{V}$. Let $\operatorname{Qrad}_{\mathscr{V}}$ and $\operatorname{Rad}_{\mathscr{V}}$ be the sets of all quasiradicals and radicals in $\mathscr{V}$, respectively.

REMARK. A hereditary quasiradical R satisfies

(R1)′ $g(R(\mathfrak{A}))\subseteq R(\mathfrak{B})$

for all homomorphisms $g\colon\mathfrak{A}\to\mathfrak{B}$ $(\mathfrak{A},\mathfrak{B}\in\mathscr{V})$.

Now we are going to define two operations $\boldsymbol{R}$ and $\boldsymbol{L}$ which will establish a connection between the subsets of $\operatorname{Con}F_2$ and the mappings $R\colon A\mapsto R(A)$.

3.2. DEFINITIONS and REMARKS. Let $L\subseteq\operatorname{Con}F_2$, and $R\colon\mathfrak{A}\mapsto R(\mathfrak{A})$ be a mapping defined on $\mathscr{V}$ such that $R(\mathfrak{A})\subseteq\mathfrak{A}\times\mathfrak{A}$ is a binary relation. Then we define a mapping $\boldsymbol{RL}\colon\mathfrak{A}\mapsto\boldsymbol{RL}(\mathfrak{A})\subseteq\mathfrak{A}\times\mathfrak{A}$ and a subset $\boldsymbol{LR}\subseteq\operatorname{Con}F_2$ as follows:

(1)
$$RL(\mathfrak{A}):=\{(a,b)\in\mathfrak{A}^2\mid\ker_{\mathfrak{A}}(e_1\mapsto a,\ e_2\mapsto b)\in L\}=$$
$$=\{(a,b)\in\mathfrak{A}^2\mid\theta^{\mathfrak{A}}(a,b)\in L\}\quad\text{for}\quad\mathfrak{A}\in\mathscr{V},$$

(2)
$$\boldsymbol{LR}:=\{\ker\lambda\in\operatorname{Con}F_2\mid(\lambda(e_1),\lambda(e_2))\in R(\lambda(F_2)),\ \lambda\text{ homomorphism from }F_2\}=$$
$$=\{\theta\in\operatorname{Con}F_2\mid([e_1]_\theta,[e_2]_\theta)\in R(F(e_1,e_2)/\theta)\}.$$

Suppose now $R(\mathfrak{A})\in\operatorname{Con}\mathfrak{A}$ (e.g. in case $R\in\operatorname{Qrad}_{\mathscr{V}}$) then $\boldsymbol{LR}$ can also be defined as follows:

(2)′
$$\boldsymbol{LR}=\{\theta\in\operatorname{Con}F_2\mid\theta_{\mathrm{id}}\subseteq H_R(\theta)\},$$

where $H_R(\theta)$ is the complete preimage of $R(F_2/\theta)$ in F_2, i.e.,

$$H_R(\theta):=\{(\alpha,\beta)\in F_2^2\mid([\alpha]_\theta,[\beta]_\theta)\in R(F_2/\theta)\}.$$

a) *We have* $L=\boldsymbol{LRL}$ *for all* $L\subseteq\operatorname{Con}F_2$ (what directly follows from the definitions).

b) *Condition* $R=\boldsymbol{RLR}$ *is equivalent to heredity of the mapping* R, i.e. to the property

$$\forall\mathfrak{A}\in\mathscr{V}:R(\mathfrak{A})=\{(a,b)\in\mathfrak{A}^2\mid(a,b)\in R(\langle a,b\rangle_{\mathfrak{A}})\}.$$

In fact, $R=\boldsymbol{RLR}$ implies $\boldsymbol{RLR}(\mathfrak{B})=\{(a,b)|\theta^{\mathfrak{B}}(a,b)\in\boldsymbol{LR}\}=\boldsymbol{RLR}(\mathfrak{A})\cap\mathfrak{B}\times\mathfrak{B}$ for $\mathfrak{B}\leqq\mathfrak{A}$, since $\theta^{\mathfrak{B}}(a,b)=\theta^{\mathfrak{A}}(a,b)$ for $a,b\in\mathfrak{B}$ (cf. 1.4, $\langle a,b\rangle_{\mathfrak{A}}=\langle a,b\rangle_{\mathfrak{B}}$). Conversely, for hereditary R we have $(a,b)\in R(\mathfrak{A})\Leftrightarrow(a,b)\in R(\mathfrak{A})\cap(\langle a,b\rangle_{\mathfrak{A}})^2=R(\langle a,b\rangle_{\mathfrak{A}})\Leftrightarrow$ $\Leftrightarrow(\lambda(e_1),\lambda(e_2))\in R(\lambda(F_2))$ for $\lambda=\eta^{\mathfrak{A}}_{a,b}$ (cf. 1.4)$\Leftrightarrow\ker\lambda\in\boldsymbol{LR}\Leftrightarrow(a,b)\in\boldsymbol{RLR}(\mathfrak{A})$. □

It turns out that Q-sets (R-sets) and hereditary quasiradicals (radicals) correspond to each other under $\boldsymbol{R}$ and $\boldsymbol{L}$. We have:

3.3. CHARACTERIZATION THEOREM. *Let $L\subseteqq\operatorname{Con}F_2$. Then the following conditions are equivalent:*

(i) *L is a Q-set (R-set, resp.),*
(ii) *there exists a hereditary quasiradical (radical, resp.) R in $\mathscr{V}$ such that $L=\boldsymbol{LR}$,*
(iii) *$\boldsymbol{RL}$ is a hereditary quasiradical (radical, resp.) in $\mathscr{V}$,*
(iv) *$\boldsymbol{RL}$ is a quasiradical (radical, resp.) in $\mathscr{V}$.*

Let R: $\mathfrak{A}\mapsto R(\mathfrak{A})$ be a mapping defined on $\mathscr{V}$ such that $R(\mathfrak{A})\subseteqq\mathfrak{A}\times\mathfrak{A}$. Then the following conditions are equivalent:

(i)′ *R is a hereditary quasiradical (radical, resp.),*
(ii)′ *there exists a Q-set (R-set, resp.) $L\subseteqq\operatorname{Con}F_2$ such that $R=\boldsymbol{RL}$,*
(iii)′ *$R=\boldsymbol{RLR}$ and $\boldsymbol{LR}$ is a Q-set (R-set, resp.).*

PROOF. We are going to prove (i)⇒(iv)⇒(iii)⇒(ii)⇒(i) in case of quasiradicals:

(i)⇒(iv). Let L be a Q-set (see 2.7). Then $\boldsymbol{RL}$ is a quasiradical. In fact, $\boldsymbol{RL}(\mathfrak{A})=$ $=\{(a,b)|\theta^{\mathfrak{A}}(a,b)\in L\}$ is a congruence relation because $\boldsymbol{RL}(\mathfrak{A})$ is reflexive, symmetric and transitive by (L0), (L1), (2.6a) and (2.6b), and $\boldsymbol{RL}(\mathfrak{A})$ is a congruence by (L2), (L1) and (2.6c). Moreover, $\boldsymbol{RL}$ fulfils 3.1 (R1) since $\theta^{\mathfrak{A}}(a,b)\subseteqq\theta^{\mathfrak{B}}(g(a),g(b))$ for $a,b\in\mathfrak{A}$ and any surjective homomorphism g: $\mathfrak{A}\to\mathfrak{B}$, i.e. (by (L1)), $g(\boldsymbol{RL}(\mathfrak{A}))\subseteqq$ $\subseteqq\boldsymbol{RL}(\mathfrak{B})$.

(iv)⇒(iii) follows from $\boldsymbol{RL}=\boldsymbol{RLRL}$ (3.2a) and 3.2b.

(iii)⇒(ii) trivially follows from $L=\boldsymbol{LRL}$.

(ii)⇒(i). Let $L=\boldsymbol{LR}$ for a hereditary quasiradical R. Then $R=\boldsymbol{RL}$ because of 3.2b. We show L is a Q-set: Condition (L0a) trivially holds (see e.g. 3.2(2)′). Condition (L0b) is fulfilled because $([e_1]_\theta,[e_2]_\theta)\in R(F_2/\theta)$ implies $([e_2]_{\theta^-},[e_1]_{\theta^-})\in R(F_2/\theta^-)$ (use 2.2 and (R1)). To show condition (L0c) we use the notations given in 2.3: let $\theta_1,\theta_2\in\boldsymbol{LR}$, i.e., $([e_1]_{\theta_i},[e_2]_{\theta_i})\in R(F_2/\theta_i)$ $(i=1,2)$. This implies (by 3.1(R1)′, cf. 2.3): $([e_1]_{\theta^*},[e_2]_{\theta^*})\in R(F_3/\theta^*)$ and $([e_2]_{\theta^*},[e_3]_{\theta^*})\in R(F_3/\theta^*)$. Thus $([e_1],[e_3])\in R(F_3/\theta^*)=$ $=\boldsymbol{RL}(F_3/\theta^*)$, i.e., $\theta_1\triangledown\theta_2=\boldsymbol{\theta}^{F_3/\theta^*}([e_1],[e_3])\in L$ and we are done in showing (L0c).

Now, L fulfils (L1). In fact, let $\theta\in L$, $\theta\subseteqq\theta'$. Then, by (R1), $([e_1]_\theta,[e_2]_\theta)\in R(F_2/\theta)$ implies $\varphi([e_1]_\theta,[e_2]_\theta)=([e_1]_{\theta'},[e_2]_{\theta'})\in R(F_2/\theta')$ for the canonical mapping φ: $F_2/\theta\to$ $\to F_2/\theta'$. Consequently $\theta'\in\boldsymbol{LR}=L$.

L fulfils (L2). Let $\theta_1,\ldots,\theta_n\in L$, $f\in\Omega_n$ $(n\geqq1)$ and $\mathfrak{A}_i=F_2/\theta_i$, thus $([e_1]_{\theta_i},$ $[e_2]_{\theta_i})\in R(\mathfrak{A}_i)$, $1\leqq i\leqq n$. Using the notations given in 2.4, this implies $([e_{1i}]_{\theta^*},$ $[e_{2i}]_{\theta^*})\in R(\mathfrak{A}/\theta^*)$ by 3.1(R1)′. $R(\mathfrak{A}/\theta^*)$ is a congruence, therefore $([a_1]_{\theta^*},[a_2]_{\theta^*})\in$ $\in R(\mathfrak{A}/\theta^*)$ for $a_1=f(e_{11},\ldots,e_{1n})$, $a_2=f(e_{21},\ldots,e_{2n})$. Thus (note 2.4, 3.2(1) and $R=\boldsymbol{RL}$), $f[\theta_1,\ldots,\theta_n]=\boldsymbol{\theta}^{\mathfrak{A}/\theta^*}([a_1],[a_2])\in L$ and we are done.

It remains to prove the above equivalences in case of radicals. Because every radical (R-set, resp.) is also a quasiradical (Q-set, resp.) we have to show for hereditary quasiradicals R only that $L=\boldsymbol{LR}$ satisfies (L3) if and only if $R=\boldsymbol{RL}$ satisfies

(R3). This will be done in the next lemma (3.4c). Therefore the first part of 3.3 is proved. Concerning the second part,

(i)′⇒(iii)′ follows from (ii)⇒(i) and 3.2b,
(iii)′⇒(ii)′ is trivial,
(ii)′⇒(i)′ follows from (i)⇒(iii).

Therefore all conditions (i)′—(iii)′ are equivalent, too. □

3.4. LEMMA. *Let $L\subseteq \mathrm{Con}\, F_2$ be a Q-set and $R=\boldsymbol{R}L$ the corresponding hereditary quasiradical* (cf. 3.3). *Then we have:*

a) $\boldsymbol{\theta}^{\mathfrak{A}/R(\mathfrak{A})}([a]_{R(\mathfrak{A})}, [b]_{R(\mathfrak{A})}) = \big(L|\boldsymbol{\theta}^{\mathfrak{A}}(a, b)\big)$ for $a, b\in\mathfrak{A}\in\mathscr{V}$,

b) $(L|\theta)=H_R(\theta)$ *for* $\theta\in\mathrm{Con}\, F_2$ (cf. 3.2),

c) *L is an R-set if and only if R is a hereditary radical.*

PROOF. a) $\boldsymbol{\theta}^{\mathfrak{A}/R(\mathfrak{A})}([a], [b]) =$

$$= {}_{(1.3)}\big\{(\alpha(e_1, e_2), \beta(e_1, e_2))\in F_2^2 \big| \alpha([a], [b]) = \beta([a], [b]),\ \text{i.e.}\ [\alpha(a, b)] = [\beta(a, b)]\big\} =$$

$$= \big\{(\alpha(e_1, e_2), \beta(e_1, e_2))\big|(\alpha(a, b), \beta(a, b))\in R(\mathfrak{A}) = \boldsymbol{R}L(\mathfrak{A})\big\} =$$

$$= {}_{(3.2(1))}\big\{(\alpha(e_1, e_2), \beta(e_1, e_2))\big|\boldsymbol{\theta}^{\mathfrak{A}}(\alpha(a, b), \beta(a, b))\in L\big\} =$$

$$= {}_{(2.6f)}\big\{(\alpha(e_1, e_2), \beta(e_1, e_2))\big|(\boldsymbol{\theta}^{\mathfrak{A}}(a, b)\colon (\alpha, \beta))\in L\big\} = \big(L\big|\boldsymbol{\theta}^{\mathfrak{A}}(a, b)\big).$$

b) follows from a) and (2.6g) because $\boldsymbol{\theta}^{\mathfrak{A}/R(\mathfrak{A})}([a], [b])=H_R(\theta)$ in case $\mathfrak{A}=F_2/\theta$, $a=[e_1]_\theta$, $b=[e_2]_\theta$.

c) Assume $L=\boldsymbol{L}R$ is an R-set. Then $R=\boldsymbol{R}L$ is a hereditary quasiradical by 3.3 and satisfies (R3). In fact,

$$([a]_{R(\mathfrak{A})}, [b]_{R(\mathfrak{A})})\in R\big(\mathfrak{A}/R(\mathfrak{A})\big) = \boldsymbol{R}L\big(\mathfrak{A}/R(\mathfrak{A})\big) \Rightarrow_{(3.2(1))}$$

$$\boldsymbol{\theta}^{\mathfrak{A}/R(\mathfrak{A})}([a], [b]) \underset{\text{a)}}{=} \big(L|\boldsymbol{\theta}^{\mathfrak{A}}(a, b)\big)\in L \underset{(\text{L3})}{\Rightarrow} \boldsymbol{\theta}^{\mathfrak{A}}(a, b)\in L \Rightarrow$$

$$\Rightarrow (a, b)\in \boldsymbol{R}L(\mathfrak{A}) = R(\mathfrak{A}),\quad \text{i.e.,}\quad R\big(\mathfrak{A}/R(\mathfrak{A})\big) = \boldsymbol{0}.$$

Conversely, if R is a hereditary radical, then the Q-set $L=\boldsymbol{L}R$ also satisfies (L3). In fact, let $(L|\theta)\in L$ for $\theta\in\mathrm{Con}\, F_2$. Then $\theta=\boldsymbol{\theta}^{\mathfrak{A}}(a, b)$ for appropriate $a, b\in\mathfrak{A}\in\mathscr{V}$ (cf. 2.6g) and $\big(L|\boldsymbol{\theta}^{\mathfrak{A}}(a, b)\big)\underset{\text{a)}}{=}\boldsymbol{\theta}^{\mathfrak{A}/R(\mathfrak{A})}([a], [b])\in L$ implies $([a], [b])\in R\big(\mathfrak{A}/R(\mathfrak{A})\big)=\boldsymbol{0}$ (since $R=\boldsymbol{R}L$). Consequently, $(a, b)\in R(\mathfrak{A})=\boldsymbol{R}L(\mathfrak{A})$, i.e., $\theta=\boldsymbol{\theta}^{\mathfrak{A}}(a, b)\in L$. □

3.5. For a given (Q- or) R-set L, the corresponding (quasi-)radical $R=\boldsymbol{R}L$ can be expressed more explicitly. Let $L'=\{\theta_i|i\in I\}$ be a subset of L with the property $\forall\theta\in L\exists i\in I\colon \theta_i\subseteq\theta$. Then $R(\mathfrak{A})=\{(a, b)\in\mathfrak{A}^2|\exists i\in I\colon \theta_i\subseteq\boldsymbol{\theta}^{\mathfrak{A}}(a, b)\}$ (cf. 3.2(1), 2.7(L1)). But this gives (cf. 1.3)

$$R(\mathfrak{A}) = \big\{(a, b)\in\mathfrak{A}^2|\exists i\in I\forall(\alpha(e_1, e_2), \beta(e_1, e_2))\in\theta_i\colon \alpha(a, b) = \beta(a, b)\big\}.$$

This description can be very useful in many special cases; in particular, $R(\mathfrak{A})$ is presented by a quantifier-free first order formula provided that F_2 is finite (see also 5.2).

§4. Semisimple and radical classes

For technical reasons we have to distinguish the class $\mathscr{V}^0$ of all 1-element algebras in the variety $\mathscr{V}$. Every radical R in a variety $\mathscr{V}$ is determined by its *semisimple class* (cf. e.g. [7, Korollar 25])

$$\mathscr{S} = \mathscr{S}(R) := \{\mathfrak{A} \in \mathscr{V} \setminus \mathscr{V}^0 \mid R(\mathfrak{A}) = \mathbf{0}\}$$

via $R(\mathfrak{A}) = (\boldsymbol{R}\mathscr{S})(\mathfrak{A})$, where

$$\boldsymbol{R}\mathscr{S}(\mathfrak{A}) := \bigcap \{\theta \in \operatorname{Con} \mathfrak{A} \mid \mathfrak{A}/\theta \in \mathscr{S}\}.$$

We have $R = \boldsymbol{R}\mathscr{S}R$ for $R \in \operatorname{Rad}_{\mathscr{V}}$.

On the other hand, the radical is not, in general, determined by its *radical class*

$$\mathscr{R}(R) := \{\mathfrak{A} \in \mathscr{V} \mid R(\mathfrak{A}) = \mathbf{1}\}.$$

(The algebras in $\mathscr{V}^0$ are considered to be radical algebras.) Sometimes, however, $\mathscr{R}(R)$ uniquely determines R, e.g. in case of Kuroš—Amitsur radicals for groups or rings.

It is natural to ask how one can characterize semisimple and radical classes of hereditary radicals in $\mathscr{V}$. In this section we shall treat this question. The answer will be given in terms of R-sets (it is an open problem to give a more satisfactory answer in terms of the elements of the semisimple or radical class only). We begin with a necessary condition for $\mathscr{S}R$:

4.1. PROPOSITION. *Let R be a hereditary radical in $\mathscr{V}$. Then $\mathscr{S} = \mathscr{S}R$ satisfies the following conditions:*

(i) *$\mathscr{S}$ is closed under subdirect products,*

(ii) *$\mathscr{S}$ is hereditary* (i.e. closed under taking subalgebras),

(iii) *$\mathfrak{A} \in \mathscr{S}$ if (and, by* (ii), *only if) $(a, b) \notin (\boldsymbol{R}\mathscr{S})(\langle a, b\rangle_{\mathfrak{A}})$ for all $a, b \in \mathfrak{A}$, $a \neq b$. Equivalently, $\mathfrak{A} \in \mathscr{S}$ if and only if for all $a \neq b \in \mathfrak{A}$ there exists a homomorphism φ from $\langle a, b\rangle_{\mathfrak{A}}$ into an element of $\mathscr{S}$ such that $(a, b) \notin \ker \varphi$.*

REMARKS. Property (i) is known to characterize completely the semisimple classes of (arbitrary) radicals in $\mathscr{V}$, in particular, (i) implies $\mathscr{S} = \mathscr{S}\boldsymbol{R}\mathscr{S}$ (see e.g. [7, Kor. 25]). Condition (ii) provides "*half-heredity*", namely it implies $R(\mathfrak{A}') \subseteq R(\mathfrak{A})$ for $\mathfrak{A}' \leqq \mathfrak{A}$, which in turn gives $\boldsymbol{RLR}(\mathfrak{A}) \subseteq R(\mathfrak{A})$ (as it can be easily seen by checking the definitions). We also note that (iii) implies the following condition (provided that (i) and therefore $\mathscr{S} = \mathscr{S}\boldsymbol{R}\mathscr{S}$ holds):

(iv) *$\mathfrak{A} \in \mathscr{S}$ if and only if every 2-generated subalgebra of $\mathfrak{A}$ (i.e. generated by two elements) belongs to $\mathscr{S}$.*

The straightforward proof of 4.1 is omitted. □

4.2. For a given class $\mathscr{S} \subseteq \mathscr{V}$ we define

$$\boldsymbol{L}\mathscr{S} := \{\theta \in \operatorname{Con} F_2 \mid (e_1, e_2) \in \theta' \text{ for all } \theta' \in \operatorname{Con} F_2 \text{ with } \theta \subseteq \theta' \text{ and } F_2/\theta' \in \mathscr{S}\}.$$

In view of $\boldsymbol{R}\mathscr{S}(F_2/\theta) = \bigcap \{\theta'/\theta \mid F_2/\theta' \in \mathscr{S}\}$ we have:

$$\boldsymbol{L}\mathscr{S} = \{\theta \in \operatorname{Con} F_2 \mid ([e_1]_\theta, [e_2]_\theta) \in \boldsymbol{R}\mathscr{S}(F_2/\theta)\} = \boldsymbol{LR}\mathscr{S} = \{\theta \in \operatorname{Con} F_2 \mid \theta_{\mathrm{id}} \subseteq H_{R\mathscr{S}}(\theta)\}$$

(cf. 3.2(2), (2)′).

4.3. THEOREM. *Let $\mathscr{S} \subseteq \mathscr{V}$. The following conditions are equivalent:*
(a) *$\mathscr{S}$ is the semisimple class of a hereditary radical in $\mathscr{V}$,*
(b) *$\mathscr{S}$ satisfies* 4.1(i)—(iii) *and $\boldsymbol{L}\mathscr{S}$ is an* R*-set* (cf. 4.2, 2.7),
(c) *$\mathscr{S}$ satisfies* 4.1(i)—(iii) *and $\boldsymbol{L}\mathscr{S}$ satisfies* (L0) *and* (L2),
(d) *$\mathscr{S}$ satisfies* 4.1(i)—(iii) *and $\boldsymbol{RL}\mathscr{S} \supseteq \boldsymbol{R}\mathscr{S}$.*
Moreover, under these assumptions we have $\boldsymbol{R}\mathscr{S} = \boldsymbol{RL}\mathscr{S}$, $\mathscr{S} = \mathscr{S}\boldsymbol{RL}\mathscr{S}$ and $\boldsymbol{L}\mathscr{S} = \boldsymbol{LR}\mathscr{S}$.

PROOF. (a)⇒(b). The first part of (b) follows from 4.1. Now, $\boldsymbol{L}\mathscr{S} =_{(4.2)} \boldsymbol{LR}\mathscr{S}$ is an R-set because $\boldsymbol{R}\mathscr{S}$ is hereditary and 3.3(i)′⇒(iii)′.

(b)⇒(a). We shall show $\mathscr{S} = \mathscr{S}\boldsymbol{RL}\mathscr{S}$, then we are done because $\boldsymbol{RL}\mathscr{S}$ is a hereditary radical by 3.3(ii)′⇒(i)′. We have

$$(\boldsymbol{RL}\mathscr{S})(\mathfrak{A}) = \{(a, b) \in \mathfrak{A}^2 \mid \boldsymbol{\theta}^{\mathfrak{A}}(a, b) \in \boldsymbol{L}\mathscr{S}\} =$$

$$=_{(4.2)} \{(a, b) \in \mathfrak{A}^2 \mid ([e_1], [e_2]) \in \boldsymbol{R}\mathscr{S}(F_2/\boldsymbol{\theta}^{\mathfrak{A}}(a, b))\} = \{(a, b) \mid (a, b) \in \boldsymbol{R}\mathscr{S}(\langle a, b\rangle_{\mathfrak{A}})\}.$$

This, however, implies

$$\mathscr{S}\boldsymbol{RL}\mathscr{S} = \{\mathfrak{A} \in \mathscr{V} \mid \boldsymbol{RL}\mathscr{S}(\mathfrak{A}) = \boldsymbol{0}\} = \{\mathfrak{A} \in \mathscr{V} \mid \forall a \neq b \in \mathfrak{A}\colon (a, b) \notin \boldsymbol{R}\mathscr{S}(\langle a, b\rangle_{\mathfrak{A}}) =$$

$$=_{(4.1(iii))} \{\mathfrak{A} \in \mathscr{V} \mid \mathfrak{A} \in \mathscr{S}\} = \mathscr{S}.$$

(a)⇔(d). If $\mathscr{S}$ satisfies (a), $\boldsymbol{R}\mathscr{S}$ is hereditary, thus $\boldsymbol{RL}\mathscr{S} = \boldsymbol{RLR}\mathscr{S} = \boldsymbol{R}\mathscr{S}$ by 4.2 and 3.3(iii)′. Conversely, if $\mathscr{S}$ satisfies (d), then $\boldsymbol{R}\mathscr{S} \subseteq \boldsymbol{RL}\mathscr{S} = \boldsymbol{RLR}\mathscr{S} \subseteq \boldsymbol{R}\mathscr{S}$ (see remark to 4.1). By 3.2b), $\boldsymbol{RLR}\mathscr{S} = \boldsymbol{R}\mathscr{S}$ is hereditary, i.e., $\mathscr{S} = \mathscr{S}\boldsymbol{R}\mathscr{S}$ satisfies (a).

In order to prove 4.3 it remains to show (c)⇒(b) since (b)⇒(c) is trivial. Thus assume (c). One directly derives from definition 4.2 that (L1) holds for $\boldsymbol{L}\mathscr{S}$, therefore $\boldsymbol{L}\mathscr{S}$ is a Q-set. Then $(\boldsymbol{L}\mathscr{S} \mid \theta) = H_{\boldsymbol{RL}\mathscr{S}}(\theta)$ by 3.4b. As above one gets $\boldsymbol{RL}\mathscr{S} = \boldsymbol{RLR}\mathscr{S} \subseteq \boldsymbol{R}\mathscr{S}$ (by 4.2 and the remark to 4.1). Due to 4.1(i), $\mathscr{S}$ is the semisimple class of the radical $\boldsymbol{R}\mathscr{S}$, therefore $H_{\boldsymbol{R}\mathscr{S}}$ is a closure operator as shown e.g. in [7; Satz 38, p. 372]. Now, let $(\boldsymbol{L}\mathscr{S} \mid \theta) = H_{\boldsymbol{RL}\mathscr{S}}(\theta)$ be an element of $\boldsymbol{L}\mathscr{S}$. Then, by definition 4.2,

$$\theta_{\mathrm{id}} \subseteq H_{\boldsymbol{R}\mathscr{S}}(H_{\boldsymbol{RL}\mathscr{S}}(\theta)) \subseteq H_{\boldsymbol{R}\mathscr{S}}(H_{\boldsymbol{R}\mathscr{S}}(\theta)) = H_{\boldsymbol{R}\mathscr{S}}(\theta),$$

which gives $\theta \in \boldsymbol{L}\mathscr{S}$, i.e., $\boldsymbol{L}\mathscr{S}$ satisfies (L3), too; therefore it is an R-set and (b) is proved. □

Now what can be said about the radical classes of hereditary radicals? How to characterize them and how to recognize whether different hereditary radicals have the same radical class? We are going to investigate these questions and begin with proving some necessary conditions.

4.4. PROPOSITION. *Let R be a hereditary radical in $\mathscr{V}$. Then the radical class $\mathscr{K} = \mathscr{R}(R)$ satisfies the following conditions:*
(i) *$\mathscr{K}$ is homomorphically closed and $\mathscr{V}^{\circ} \subseteq \mathscr{K}$,*
(ii) *$\mathscr{K}$ is hereditary* (cf. 4.1(ii)),
(iii) *$\mathfrak{A} \in \mathscr{K}$ if and only if every 2-generated subalgebra of $\mathfrak{A}$ belongs to $\mathscr{K}$.*

The proof is straightforward and will be omitted. □

REMARK. 4.4(i) characterizes radical classes of arbitrary radicals [9; p. 205(9.1)].

4.5. For a radical R in $\mathscr{V}$, let $\langle R\rangle$ be the *least hereditary radical containing* R (i.e., $R(\mathfrak{A})\subseteq\langle R\rangle(\mathfrak{A})$ for all $\mathfrak{A}\in\mathscr{V}$). For $L\subseteq\operatorname{Con}F_2$, let $\langle L\rangle$ be the *least* R-*set containing* L. This R-set $\langle L\rangle$ exists because the intersection of R-sets can easily be shown to be also an R-set.

4.6. PROPOSITION. *Let* $\mathscr{K}\subseteq\mathscr{V}$ *be a class of algebras satisfying* 4.4(i)—(iii) *and* $L_0:=\{\theta\in\operatorname{Con}F_2\mid F_2/\theta\in\mathscr{K}\}$. *Moreover, let* $R=\boldsymbol{R}(\mathscr{V}\setminus\mathscr{K})$ *be the radical with semisimple class* $\mathscr{V}\setminus\mathscr{K}$. *Then* $\langle R\rangle=\boldsymbol{R}\langle\boldsymbol{L}R\rangle=\boldsymbol{R}\langle L_0\rangle$ *and* $\langle R\rangle$ *is the least hereditary radical the radical class of which contains* $\mathscr{K}$.

PROOF. Due to 4.4(i) (and the definitions), we have $R(\mathfrak{A})=\mathbf{0}$ for $\mathfrak{A}\notin\mathscr{K}$ and $R(\mathfrak{A})=\mathbf{1}$ for $\mathfrak{A}\in\mathscr{K}$, i.e. $\mathscr{R}(R)=\mathscr{K}$. Therefore R is the least radical with $\mathscr{R}(R)\supseteq\mathscr{K}$ and $\langle R\rangle$ is the hereditary radical claimed in the proposition.

At first we show $\boldsymbol{RLR}\supseteq R$. In fact, $R(\mathfrak{A})\neq\mathbf{0}$ gives $R(\mathfrak{A})=\mathbf{1}$ and, by 4.4(ii), $R(\mathfrak{A}')=\mathbf{1}$ for all 2-generated $\mathfrak{A}'\leqq\mathfrak{A}$. Thus $\boldsymbol{\theta}^{\mathfrak{A}}(a,b)\in\boldsymbol{LR}$ for all $a,b\in\mathfrak{A}$, i.e., $(\boldsymbol{RLR})(\mathfrak{A})=\mathbf{1}$. Therefore $\boldsymbol{R}\langle\boldsymbol{LR}\rangle\supseteq\boldsymbol{RLR}\supseteq R$ is a hereditary radical containing R, consequently $\langle R\rangle\subseteq\boldsymbol{R}\langle\boldsymbol{LR}\rangle$. Conversely,

$$\langle R\rangle=\bigcap\{R'\in\operatorname{Rad}_{\mathscr{V}}\mid R'\ \text{ hereditary},\ \ R\subseteq R'\}=$$

$$=_{(3.3)}\bigcap\{\boldsymbol{R}L'\mid L'\ \text{ is an R-set},\ \ R\subseteq\boldsymbol{R}L'\}=$$

$$=_{(3.2(1))}\boldsymbol{R}(\bigcap\{L'\mid L'\ \text{ is an R-set},\ \ R\subseteq\boldsymbol{R}L'\})\supseteq$$

$$\supseteq\boldsymbol{R}(\bigcap\{L'\mid L'\ \text{ is an R-set},\ \ \boldsymbol{LR}\subseteq\boldsymbol{LR}L'=L'\})=\boldsymbol{R}\langle\boldsymbol{LR}\rangle,$$

i.e., $\langle R\rangle=\boldsymbol{R}\langle\boldsymbol{LR}\rangle$. It remains to prove $\langle\boldsymbol{LR}\rangle=\langle L_0\rangle$. Indeed,

$$\boldsymbol{LR}=\{\theta\in\operatorname{Con}F_2\mid([e_1]_\theta,[e_2]_\theta)\in R(F_2/\theta)\}=\{\theta\mid R(F_2/\theta)=\mathbf{1}\}=\{\theta\mid F_2/\theta\in\mathscr{K}\}=L_0.\ \square$$

4.7. THEOREM. *Let* $\mathscr{K}\subseteq\mathscr{V}$ *and* $L_0=\{\theta\in\operatorname{Con}F_2\mid F_2/\theta\in\mathscr{K}\}$. *Then the following conditions are equivalent:*

(a) $\mathscr{K}$ *is the radical class for (at least one) hereditary radical in* $\mathscr{V}$,

(b) $\mathscr{K}$ *satisfies* 4.4(i)—(iii) *and* $(\langle L_0\rangle\mid\theta)=\mathbf{1}$ *implies* $\theta\in L_0$ (*for* $\theta\in\operatorname{Con}F_2$),

(c) $\mathscr{K}=\mathscr{R}(\boldsymbol{R}\langle L_0\rangle)$ (cf. 4.5, 3.2(1)).

PROOF. (c)$\Rightarrow$(a) is trivial and (a)$\Rightarrow$(c) follows from 4.6 (in view of 4.4).

(c)$\Rightarrow$(b). The first part of (b) follows from 4.4. Moreover, we have

$$(\langle L_0\rangle\mid\theta)=\mathbf{1}\Leftrightarrow H_{\boldsymbol{R}\langle L_0\rangle}(\theta)=\mathbf{1}\quad(\text{cf. 3.4b})\Leftrightarrow$$

$$\Leftrightarrow\boldsymbol{R}\langle L_0\rangle(F_2/\theta)=\mathbf{1}\underset{(c)}{\Leftrightarrow}F_2/\theta\in\mathscr{K}\Leftrightarrow\theta\in L_0,$$

(b)$\Rightarrow$(c). Assuming (b), for $\mathfrak{A}\in\mathscr{V}$ we have to show $\boldsymbol{R}\langle L_0\rangle(\mathfrak{A})=\mathbf{1}\Rightarrow\mathfrak{A}\in\mathscr{K}$ (the opposite implication directly follows from 4.6). $\boldsymbol{R}\langle L_0\rangle(\mathfrak{A})=\mathbf{1}$ implies $\boldsymbol{R}\langle L_0\rangle(\mathfrak{A}')=\mathbf{1}$ for all 2-generated subalgebras $\mathfrak{A}'$ of $\mathfrak{A}$. Such an $\mathfrak{A}'=\langle a,b\rangle_{\mathfrak{A}}$ is isomorphic to F_2/θ where $\theta=\boldsymbol{\theta}^{\mathfrak{A}}(a,b)$ (cf. 1.4), therefore (as above) $H_{\boldsymbol{R}\langle L_0\rangle}(\theta)=(\langle L_0\rangle\mid\theta)=\mathbf{1}$. By (b), we get $\theta\in L_0$, i.e., $\mathfrak{A}'\cong F_2/\theta\in\mathscr{K}$ for every 2-generated subalgebra $\mathfrak{A}'$. Due to 4.4(iii), we have $\mathfrak{A}\in\mathscr{K}$. $\square$

4.8. PROPOSITION. *Let* R *be a hereditary radical in* $\mathscr{V}$ *and* $L_0=\{\theta\in\operatorname{Con}F_2\mid(\boldsymbol{LR}\mid\theta)=\mathbf{1}\}$. *Then* $R'=\boldsymbol{R}\langle L_0\rangle$ *is the least hereditary radical which has the same*

radical class as R. Two hereditary radicals R_1 and R_2 in $\mathscr{V}$ have the same radical class if and only if

$$(\boldsymbol{L}R_1|\theta) = \mathbf{1} \Leftrightarrow (\boldsymbol{L}R_2|\theta) = \mathbf{1}$$

for all $\theta \in \operatorname{Con} F_2$.

PROOF. By 4.6 and 4.7 we are done with the following observation: $F_2/\theta \in \mathscr{K} = \mathscr{R}(R) \Leftrightarrow H_R(\theta) = \mathbf{1} \Leftrightarrow (\boldsymbol{L}R|\theta) = \mathbf{1}$ $(\Leftrightarrow \theta \in L_0)$, (cf. 3.4b, note $R = \boldsymbol{R}\boldsymbol{L}R$). □

Finally we can ask the question how to describe $\langle R \rangle$, the least hereditary radical containing R (a special case was treated in 4.6). The next proposition provides an answer if R already fulfils some properties of a hereditary radical.

4.9. PROPOSITION. *Let R be a radical in $\mathscr{V}$, the semisimple class $\mathscr{S} = \mathscr{S}R$ of which fulfils* 4.1(iii). *Then*

$$\langle R \rangle = \boldsymbol{R}\langle \boldsymbol{L}R \rangle = \boldsymbol{R}\langle \boldsymbol{L}\mathscr{S} \rangle \quad \text{(cf. 4.2).}$$

PROOF. Let $R' = \boldsymbol{R}\langle \boldsymbol{L}R \rangle$. We have

$$R'(\mathfrak{A}) \supseteq \boldsymbol{R}\boldsymbol{L}R(\mathfrak{A}) = \{(a, b) | \boldsymbol{\theta}^{\mathfrak{A}}(a, b) \in \boldsymbol{L}R\} = \{(a, b) | (a, b) \in R(\langle a, b \rangle_{\mathfrak{A}})\}.$$

Thus, $R'(\mathfrak{A}) = \mathbf{0}$ implies $R(\mathfrak{A}) = \mathbf{0}$ via 4.1(iii), i.e., $\mathscr{S}R' \subseteq \mathscr{S}R$, which gives $R \subseteq R'$. Let R'' be a hereditary radical containing R. Then $R'' = \boldsymbol{R}L'' \supseteq R$ for some R-set L'' and we get $L'' = \boldsymbol{L}R'' \supseteq \boldsymbol{L}R$. Thus $L'' \supseteq \langle \boldsymbol{L}R \rangle$ and $R'' = \boldsymbol{R}L'' \supseteq \boldsymbol{R}\langle \boldsymbol{L}R \rangle = R'$. Consequently, $R' = \langle R \rangle$. □

4.10. PROBLEM. It seems that the radical class $\mathscr{R}(R)$ of a hereditary radical R does not uniquely determine the radical R. The author, however, does not know an example of two hereditary radicals with the same radical classes.

Problem: Is there a variety $\mathscr{V}$ and R such that $R \neq R'$ (for notations see 4.8)?

For 0-quasiradicals see 6.6.

§5. Hereditary (quasi-)radicals of S-systems

This section shall serve as an example to demonstrate the results of the preceding sections in case of a very special variety — the variety of S-systems.

Let S be a fixed semigroup with unit 1. A *(unitary) S-system* (cf. [6], [8]) is an algebra $\langle M; (s)_{s \in S} \rangle$ such that for each $s \in S$ there is a unary fundamental operation

$$s\colon M \to M\colon m \mapsto ms$$

satisfying $m1 = m$ and $(ms)s' = m(ss')$ for all $m \in M$, $s, s' \in S$ (ss' denotes the multiplication in S).

Let $\mathscr{V}$ be the variety of all unitary S-systems. We are going to describe all hereditary (quasi-)radicals in $\mathscr{V}$ using Theorem 3.3. At first we must therefore investigate the structure of $F(e_1, e_2)$ and its congruences.

5.1. The $\mathscr{V}$-freely generated S-system on 2 generators is simply the disjoint union of two copies of S:

$$F_2 = F(e_1, e_2) = \{e_1 s | s \in S\} \cup \{e_2 s | s \in S\},$$

where the fundamental operations are naturally defined by $(e_i s)s' = e_i(ss')$ $(i = 1, 2;$ $s, s' \in S)$.

Note that for $m_1, m_2 \in \mathfrak{A} \in \mathscr{V}$, the congruence $\boldsymbol{\theta}^{\mathfrak{A}}(m_1, m_2)$ (see 1.3) looks as follows:

$$\boldsymbol{\theta}^{\mathfrak{A}}(m_1 m_2) = \ker_{\mathfrak{A}}(e_1 \mapsto m_1, e_2 \mapsto m_2) = \{(e_i s, e_j s') \in F_2^2 \mid m_i s = m_j s' \ (i, j \in \{1, 2\})\}.$$

5.2. REMARK. If S is finite, F_2 and Con F_2 are finite, too. Therefore the number of hereditary quasiradicals is also finite. Moreover, every hereditary (quasi-)radical R in $\mathscr{V}$ can be described by a finite quantifier-free first order formula (cf. 3.5). In fact, let $R = \boldsymbol{R}L$ for a Q-set L (cf. 3.3) and let $L' = \{\theta_1, \ldots, \theta_t\}$ be the set of all minimal congruences in L. Then

$$R(\mathfrak{A}) = \{(m_1, m_2) \in \mathfrak{A}^2 \mid \theta_1 \subseteqq \boldsymbol{\theta}^{\mathfrak{A}}(m_1, m_2) \text{ or} \ldots \text{or } \theta_t \subseteqq \boldsymbol{\theta}^{\mathfrak{A}}(m_1, m_2)\}$$

for $\mathfrak{A} \in \mathscr{V}$, or, more explicitly (see 3.5):

$$R(\mathfrak{A}) = \left\{(m_1, m_2) \in \mathfrak{A}^2 \,\middle|\, \bigvee_{k=1}^{t} \bigwedge_{\substack{(e_i s, e_j s') \in \theta_k \\ i, j \in \{1, 2\}}} (m_i s = m_j s')\right\}.$$

5.3. Now it is not difficult to describe explicitly the operations θ_{id}, θ^-, $\theta \triangledown \theta'$, $s[\theta]$, $(\theta\colon (\alpha, \beta))$ and $(L|\theta)$ for $\theta, \theta' \in \mathrm{Con}\, F_2$ (e.g., $s[\theta] = \{(e_i s', e_j s'') \mid (e_i ss', e_j ss'') \in \theta\}$), see 2.1—2.5, but we shall not go into technical details. Then, according to 2.7, it is quite clear what Q-sets and R-sets are, and these in turn characterize hereditary quasiradicals and radicals for S-systems. A concrete example will be given below.

REMARK. J. K. Luedeman [12] investigates torsion theories for S-systems via so-called left quotient filters in S. There seems to be some connections (not clarified yet) to our characterization of hereditary radicals via R-sets, in particular in case of 0-(quasi-)radicals (cf. §6, §7) of S-systems.

5.4. As an *example,* from now on let $S = \langle \{1, a, b\}; \cdot \rangle$ be the following semigroup (1 is the "unit", b is a "zero"):

$\cdot$	1	a	b
1	1	a	b
a	a	1	b
b	b	b	b

$F(e) = \{e1, ea, eb\} \cong S$ has 3 congruences (we describe them by the corresponding partitions of $F(e)$):

$$\theta_0'(e) = \{\{e1\}, \{ea\}, \{eb\}\} = \mathbf{0},$$

$$\theta_a'(e) = \{\{e1, ea\}, \{eb\}\},$$

$$\theta_1'(e) = \{\{e1, ea, eb\}\} = \mathbf{1}.$$

5.5. *There are* 21 *congruence relations in* $F(e_1, e_2)$, namely 9, 9, 2 and 1 of type (1), (2), (3) and (4), respectively:

(1) $\theta_{xy}=\theta'_x(e_1)\cup\theta'_y(e_2)$, $x, y\in\{0, a, 1\}$;

where $\theta'_x(e_i)$ is equal to the partition $\theta'_x(e)$ (see 5.4) when e is substituted by e_i;

(2) $\theta^1_{xy}=\langle\theta_{xy}\cup\{e_1b, e_2b\}\rangle$=the congruence generated by θ_{xy} and identifying e_1b and e_2b $(x, y\in\{0, a, 1\})$;

(3) $\theta^0_{\mathrm{id}}=\{\{e_11, e_21\}, \{e_1a, e_2a\}, \{e_1b, e_2b\}\}=\theta_{\mathrm{id}}$ (cf. (2.1)),
$\theta^a_{\mathrm{id}}=\{\{e_11, e_1a, e_21, e_2a\}, \{e_1b, e_2b\}\}$;

(4) $\theta^0=\{\{e_11, e_2a\}, \{e_1a, e_21\}, \{e_1b, e_2b\}\}$.

It is an easy but tedious task to see that these relations are congruences and that there are no others. The congruence lattice Con F_2 is given in Fig. 1.

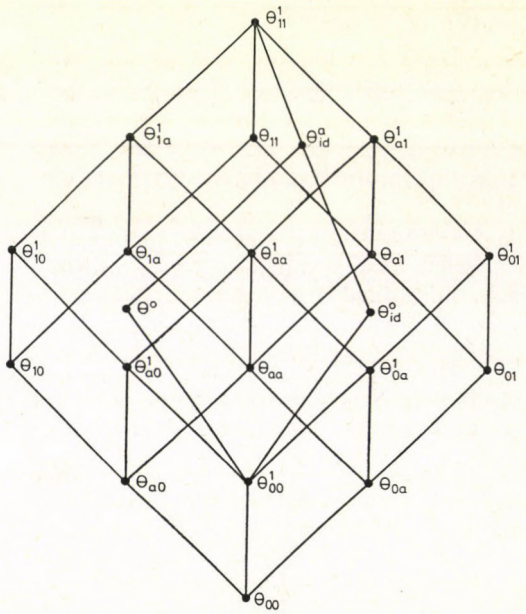


Fig. 1

5.6. Let us compute explicitly the operations on Con F_2 given in 5.3:

(i) $\theta_{\mathrm{id}}=\theta^0_{\mathrm{id}}$,

(ii) $\theta^-_{xy}=\theta_{yx}$, $(\theta^1_{xy})^-=\theta^1_{yx}$, $(\theta^z_{\mathrm{id}})^-=\theta^z_{\mathrm{id}}$, $(\theta^0)^-=\theta^0$,

(iii) $\theta_{xy}\triangledown\theta_{x'y'}=\theta_{xy'}$, $\theta_{xy}\triangledown\theta^1_{x'y'}=\theta_{xy'}$, $\theta_{xy}\triangledown\theta^z_{\mathrm{id}}=\theta_{x,(y+z)}$,
$\theta_{xy}\triangledown\theta^0=\theta_{xy}$, $\theta^1_{xy}\triangledown\theta^1_{x'y'}=\theta^1_{xy'}$, $\theta^1_{xy}\triangledown\theta^z_{\mathrm{id}}=\theta^1_{x,(y+z)}$,
$\theta^1_{xy}\triangledown\theta^0=\theta^1_{xy}$, $\theta^z_{\mathrm{id}}\triangledown\theta^{z'}_{\mathrm{id}}=\theta^{z+z'}_{\mathrm{id}}$, $\theta^0_{\mathrm{id}}\triangledown\theta^0=\theta^0$,
$\theta^a_{\mathrm{id}}\triangledown\theta^0=\theta^a_{\mathrm{id}}$, $\theta^0\triangledown\theta^0=\theta^0_{\mathrm{id}}$,

where $x, y \in \{0, a, 1\}$, $z \in \{0, a\}$ and $x+y=\max(x, y)$ with respect to the ordering $0<a<1$. The operations $\theta' \triangledown \theta$ not listed here can be obtained from (ii) via $\theta' \triangledown \theta = (\theta^- \triangledown (\theta')^-)^-$.

(iv) $s[\theta]=\theta$ for $\theta \in \operatorname{Con} F_2$ and $s \in \{1, a\}$,

$$b[\theta] = \begin{cases} \theta_{11} & \text{for} \quad \theta = \theta_{xy} \\ \theta^1_{11} & \text{otherwise.} \end{cases}$$

5.7. With 5.6 we are ready to find all Q-sets (see 2.7). By the symmetry of 5.6(ii) combined with 2.7(L0b) and by 5.6(iii) combined with 2.7(L0c), the Q-sets L must be "generated" (i.e., using (L1) and (L0a)) by θ^0 and/or by elements θ lying on the vertical middle axis of Fig. 1, in notation: $L=\langle\theta\rangle_Q$. Thus *there exist exactly the following* 10 Q-sets:

$$L_1 = \{\theta \mid \theta \supseteq \theta_{id}\} = \{\theta^0_{id}, \theta^a_{id}, \theta^1_{11}\}, \quad L_1' = L_1 \cup \{\theta^0\},$$

$$L_2 = \{\theta \mid \theta \supseteq \theta_{11}\} \cup L_1 = \langle\theta_{11}\rangle_Q, \quad L_2' = L_2 \cup \{\theta^0\},$$

$$L_3 = \{\theta \mid \theta \supseteq \theta^1_{aa}\} \cup L_1 = \langle\theta^1_{aa}\rangle_Q, \quad L_3' = L_3 \cup \{\theta^0\},$$

$$L_4 = \{\theta \mid \theta \supseteq \theta_{aa}\} \cup L_1 = \langle\theta_{aa}\rangle_Q, \quad L_4' = L_4 \cup \{\theta^0\},$$

$$L_5 = \{\theta \mid \theta \supseteq \theta^1_{00}\} = \langle\theta^1_{00}\rangle_Q,$$

$$L_6 = \{\theta \mid \theta \supseteq \theta_{00}\} = \langle\theta_{00}\rangle_Q = \operatorname{Con} F_2.$$

5.8. *The hereditary quasiradicals* $R_i = \boldsymbol{R}L_i$ *corresponding to the* Q-sets *described in* 5.7 *can be characterized as follows* (we use the characterization described in 5.2), $\mathfrak{A} \in \mathscr{V}$:

$$R_1(\mathfrak{A}) = \{(m_1, m_2) \mid m_1 = m_2\} = \mathbf{0} \quad \text{(trivial radical)},$$

$$R_2(\mathfrak{A}) = \{(m_1, m_2) \mid m_1 = m_2 \vee (m_1 = m_1 b \wedge m_2 = m_2 b)\},$$

$$R_3(\mathfrak{A}) = \{(m_1, m_2) \mid m_1 = m_2 \vee (m_1 = m_1 a \wedge m_2 = m_2 a \wedge m_1 b = m_2 b)\},$$

$$R_4(\mathfrak{A}) = \{(m_1, m_2) \mid m_1 = m_2 \vee (m_1 = m_1 a \wedge m_2 = m_2 a)\},$$

$$R_5(\mathfrak{A}) = \{(m_1, m_2) \mid m_1 b = m_2 b\},$$

$$R_6(\mathfrak{A}) = \mathfrak{A} \times \mathfrak{A} = \mathbf{1} \quad \text{(trivial radical)},$$

$$R_1'(\mathfrak{A}) = \{(m_1, m_2) \mid m_1 = m_2 \vee m_1 = m_2 a\},$$

$$R_2'(\mathfrak{A}) = \{(m_1, m_2) \mid m_1 = m_2 \vee m_1 = m_2 a \vee (m_1 = m_1 b \wedge m_2 = m_2 b)\},$$

$$R_3'(\mathfrak{A}) = \{(m_1, m_2) \mid m_1 = m_2 \vee m_1 = m_2 a \vee (m_1 = m_1 a \wedge m_2 = m_2 a \wedge m_1 b = m_2 b)\},$$

$$R_4'(\mathfrak{A}) = \{(m_1, m_2) \mid m_1 = m_2 \vee m_1 = m_2 a \vee (m_1 = m_1 a \wedge m_2 = m_2 a)\}.$$

5.9. It turns out that *all quasiradicals in* 5.8, *except* R_3' *and* R_4', *are radicals, too.* This can be shown by proving that the corresponding Q-sets are (or are not) R-sets. But we can also directly show the radical property 3.1(R2). In fact, e.g. for R_4 we have

$$([m_1]_{R_4(\mathfrak{A})}, [m_2]_{R_4(\mathfrak{A})}) \in R_4(\mathfrak{A}/R_4(\mathfrak{A})) \Rightarrow ([m_1] = [m_1 a] \wedge [m_2] = [m_2 a]) \vee$$

$$\vee [m_1] = [m_2] \Rightarrow ((m_1, m_1 a) \in R_4(\mathfrak{A}) \wedge (m_2, m_2 a) \in R_4(\mathfrak{A})) \vee [m_1] = [m_2] \Rightarrow$$

$$\Rightarrow (m_1 = m_1 a \wedge m_2 = m_2 a) \vee [m_1] = [m_2] \Rightarrow (m_1, m_2) \in R_4(\mathfrak{A}) \Rightarrow$$

$$\Rightarrow [m_1] = [m_2], \quad \text{i.e.,} \quad R_4(\mathfrak{A}/R_4(\mathfrak{A})) = \mathbf{0}.$$

Analogously one gets $R(\mathfrak{A}/R(\mathfrak{A}))=\mathbf{0}$ for $R \in \{R_i | 1 \leqq i \leqq 6\} \cup \{R_1', R_2'\}$. However, R_3' and R_4' are not radicals as shown by the following example. Let $\mathfrak{A}$ be the S-system generated by m_1 and m_2 such that $m_1 b = m_2 b$ (see Fig. 2). Then $(m_1, m_2) \notin R(\mathfrak{A})$ but $([m_1], [m_2]) \in R(\mathfrak{A}/R(\mathfrak{A}))$ for $R \in \{R_3', R_4'\}$, i.e., $R(\mathfrak{A}/R(\mathfrak{A})) \neq \mathbf{0}$.

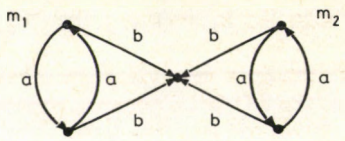


Fig. 2

Fig. 3 shows the *lattice* (with respect to inclusion) *of all hereditary quasiradicals* in the variety of all S-systems where S is the semigroup given in 5.4 (the radicals are marked with full dots).

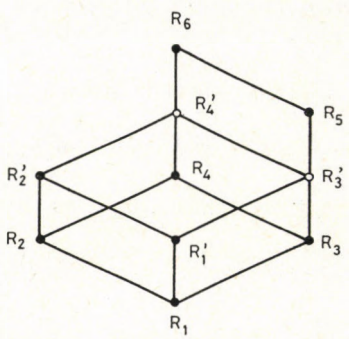


Fig. 3

§6. 0-quasiradicals and their characterization

The ideas and methods used for the characterization of (quasi-)radicals also apply to the characterization of 0-quasiradicals (the case of 0-radicals is treated in the next section). Definitions, results and proofs for 0-quasiradicals are similar to those of quasiradicals. Therefore, in this section, we develop the corresponding notions and results very briefly.

6.1. From now on let $\mathscr{V}_0$ be a variety of universal algebras such that there is a constant fundamental operation whose value, say 0, is a subalgebra. To avoid some technical modifications we shall assume that there are no other constants except 0 (i.e., $|\Omega_0|=1$). Let Sub $\mathfrak{A}$ be the lattice of all subalgebras of $\mathfrak{A}\in\mathscr{V}_0$. Note that $0\in\mathfrak{A}'$ for each $\mathfrak{A}'\in\mathrm{Sub}\,\mathfrak{A}$.

6.2. DEFINITIONS. A map r: $\mathfrak{A}\mapsto r(\mathfrak{A})$ which assigns to each $\mathfrak{A}\in\mathscr{V}_0$ a subalgebra $r(\mathfrak{A})\in\mathrm{Sub}\,\mathfrak{A}$ is called 0-*quasiradical* (cf. [10]) *in* $\mathscr{V}_0$ if

(R°1) $\quad g(r(\mathfrak{A}))\subseteq r(g(\mathfrak{A}))$ for every homomorphism g from $\mathfrak{A}\in\mathscr{V}_0$.

A 0-quasiradical r is called *hereditary* if for all $\mathfrak{A}'$, $\mathfrak{A}\in\mathscr{V}_0$:

(R°3) $\quad \mathfrak{A}'\leqq\mathfrak{A}\Rightarrow r(\mathfrak{A}')=r(\mathfrak{A})\cap\mathfrak{A}'$;

in particular we have $r(r(\mathfrak{A}))=r(\mathfrak{A})$.

REMARK. A hereditary 0-quasiradical also satisfies

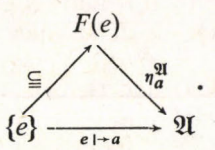
(R°1)′ $\quad g(r(\mathfrak{A}))\subseteq r(\mathfrak{B})$ for every homomorphism g: $\mathfrak{A}\to\mathfrak{B}$.

We shall characterize hereditary 0-quasiradicals by means of subsets of Con $F(e)$ — so-called Q°-sets which we are going to define now.

6.3. DEFINITIONS and NOTATIONS. In $F_1=F(e)$ (the $\mathscr{V}_0$-free algebra on one generator), we define analogously to 1.3

$$\tau^{\mathfrak{A}}(a) := \ker\eta_a^{\mathfrak{A}} = \{(\alpha(e),\beta(e))\in F_1^2\,|\,\alpha(a)=\beta(a)\}.$$

where $\eta_a^{\mathfrak{A}}$ is the unique homomorphism which makes the following diagramm commutative:



$F_1/\tau^{\mathfrak{A}}(a)$ is isomorphic to $\langle a\rangle_{\mathfrak{A}}$ — the subalgebra of $\mathfrak{A}$ generated by a.

Let $f\in\Omega_n$ ($n\geqq 1$) and $\tau_1,\ldots,\tau_n\in\mathrm{Con}\,F_1$. We embed τ_i into $\mathfrak{A}=F(e_1,\ldots,e_n)$ (considering τ_i as an element of Con $F(e_i)$) as follows:

$$\tau_i^* = \eta_{e_i}^{\mathfrak{A}}(\tau_i) = \{(\alpha(e_i),\beta(e_i))\in\mathfrak{A}^2\,|\,(\alpha(e),\beta(e))\in\tau_i\}.$$

Let τ^* be the congruence of $\mathfrak{A}$ generated by $\bigcup_{i=1}^{n}\tau_i^*$. We define $f[\tau_1,\ldots,\tau_n]\in\mathrm{Con}\,F_1$ by

$$f[\tau_1,\ldots,\tau_n] = \ker\left(F_1\xrightarrow[\eta_a^{\mathfrak{A}}]{}\mathfrak{A}\xrightarrow[\lambda_{\tau^*}]{}\mathfrak{A}/\tau^*\right) = \tau^{\mathfrak{A}/\tau^*}([a]_{\tau^*})$$

where $a=f(e_1,\ldots,e_n)$. Now, a subset $l\subseteq\mathrm{Con}\,F_1$ is called a *Q°-set* if it satisfies

(L°0) $\quad \mathbf{1}\in l$,

(L°1) $\quad \tau\in l,\ \tau\subseteq\tau'\Rightarrow\tau'\in l$ (for $\tau'\in\mathrm{Con}\,F_1$),

(L°2) $\quad \tau_1,\ldots,\tau_n\in l\Rightarrow f[\tau_1,\ldots,\tau_n]\in l$ (for $f\in\Omega_n$, $n\geqq 1$).

REMARK. If $|\Omega_0|>1$ then (L°0) must be modified into $\tau^{\mathfrak{A}}(c)\in l$ for every $c\in\Omega_0$.

6.4. DEFINITIONS and REMARKS. Let $l\subseteq \operatorname{Con} F_1$ and let r: $\mathfrak{A}\mapsto r(\mathfrak{A})$ be a map defined on $\mathscr{V}_0$ with $r(\mathfrak{A})\subseteq\mathfrak{A}$. We define a subset $\boldsymbol{lr}\subseteq\operatorname{Con} F_1$ and a map $\boldsymbol{rl}$: $\mathfrak{A}\mapsto\boldsymbol{rl}(\mathfrak{A})$ as follows:

(1) $$\boldsymbol{rl}(\mathfrak{A}) := \{a\in\mathfrak{A}\mid\tau^{\mathfrak{A}}(a)\in l\},$$

(2) $$\boldsymbol{lr} := \{\tau\in\operatorname{Con}F(e)\mid[e]_\tau\in r(F(e)/\tau)\}.$$

If $r(\mathfrak{A})\in\operatorname{Sub}\mathfrak{A}$ for $\mathfrak{A}\in\mathscr{V}_0$ then we have

$$\boldsymbol{lr} = \{\tau\mid r(F(e)/\tau) = F(e)/\tau\}.$$

a) $\boldsymbol{lrl}=l$ *holds always* (follows from the definitions).

b) *A function* r *with* $r(\mathfrak{A})\in\operatorname{Sub}\mathfrak{A}$ *is hereditary* (equivalently, r fulfils $r(\mathfrak{A})=\{a\in\mathfrak{A}\mid r(\langle a\rangle_{\mathfrak{A}})=\langle a\rangle_{\mathfrak{A}}\}$) *if and only if* $\boldsymbol{rlr}=r$. In fact, $r=\boldsymbol{rlr}$ implies $r(\mathfrak{B})=\boldsymbol{rlr}(\mathfrak{B})=\{a\in\mathfrak{B}\mid\tau^{\mathfrak{B}}(a)\in\boldsymbol{lr}\}=\boldsymbol{rlr}(\mathfrak{A})\cap\mathfrak{B}=r(\mathfrak{A})\cap\mathfrak{B}$ for $\mathfrak{B}\leqq\mathfrak{A}$ since $\tau^{\mathfrak{B}}(a)=\tau^{\mathfrak{A}}(a)$ for $a\in\mathfrak{B}$ (cf. 6.3). Conversely, for hereditary r we get $\boldsymbol{rlr}(\mathfrak{A})=\{a\in\mathfrak{A}\mid\tau^{\mathfrak{A}}(a)\in\boldsymbol{lr}\}=\{a\in\mathfrak{A}\mid r(F_1/\tau^{\mathfrak{A}}(a))=F_1/\tau^{\mathfrak{A}}(a)\}=\{a\in\mathfrak{A}\mid r(\langle a\rangle_{\mathfrak{A}})=\langle a\rangle_{\mathfrak{A}}\}=r(\mathfrak{A})$. □

Now we are ready to formulate the characterization theorem for hereditary 0-quasiradicals.

6.5. THEOREM. *Let* r *and* l *be as in* 6.4.

(a) r *is a hereditary* 0*-quasiradical in* $\mathscr{V}_0$ *if and only if* $\boldsymbol{lr}$ *is a* Q°*-set and* $r=\boldsymbol{rlr}$.

(b) l *is a* Q°*-set if and only if* $\boldsymbol{rl}$ *is a hereditary* 0*-quasiradical.*

PROOF. (b) Let l be a Q°-set, then $\boldsymbol{rl}$ fulfils 6.2(R°1) due to 6.3(L°1). Moreover, $\boldsymbol{rl}(\mathfrak{A})$ is a subalgebra of $\mathfrak{A}$. In fact, $0\in\boldsymbol{rl}(\mathfrak{A})$ since $\tau^{\mathfrak{A}}(0)=\mathbf{1}\in l$, and for $a_1, \ldots, a_n\in\boldsymbol{rl}(\mathfrak{A})$, $f\in\Omega_n$, we have $\tau^{\mathfrak{A}}(a_1), \ldots, \tau^{\mathfrak{A}}(a_n)\in l$, consequently $f[\tau^{\mathfrak{A}}(a_1), \ldots, \tau^{\mathfrak{A}}(a_n)]\subseteq\tau^{\mathfrak{A}}(f(a_1, \ldots, a_n))\in l$ (due to (L°1) and (L°2)), i.e., $f(a_1, \ldots, a_n)\in\boldsymbol{rl}(\mathfrak{A})$. Finally, $\boldsymbol{rl}$ is hereditary because of $\boldsymbol{rl}=\boldsymbol{rlrl}$ and 6.4b). Conversely, if $\boldsymbol{rl}$ is a hereditary 0-quasiradical, then $l=\boldsymbol{lrl}$ is a Q°-set by (a).

(a) Let r be a hereditary 0-quasiradical. Then $r=\boldsymbol{rlr}$ by 6.4b). We show that $\boldsymbol{lr}$ is a Q°-set. Condition (L°0) is satisfied since $\mathbf{1}=\tau^{F(e)}(0)\in\boldsymbol{lr}$ due to $0\in r(F(e))$. Condition (L°1) easily follows from 6.2(R°1). To show (L°2), let $\tau_1, \ldots, \tau_n\in\boldsymbol{lr}$ and $f\in\Omega_n$ ($n\geqq 1$). With the notations given in 6.3 one gets $a_i:=[e_i]_{\tau^*}\in r(\mathfrak{A}/\tau^*)$ by (R°1)′, thus $[a]_{\tau^*}=f(a_1, \ldots, a_n)\in r(\mathfrak{A}/\tau^*)=\boldsymbol{rlr}(\mathfrak{A}/\tau^*)$. Therefore $f[\tau_1, \ldots, \tau_n]=\tau^{\mathfrak{A}/\tau^*}([a]_{\tau^*})\in\boldsymbol{lr}$. The converse implication of (a) directly follows from the above proof of (b). □

For a 0-quasiradical r in $\mathscr{V}_0$, the set

$$\mathscr{R}(r) = \{\mathfrak{A}\in\mathscr{V}_0\mid r(\mathfrak{A}) = \mathfrak{A}\}$$

is called the *radical class* of r, the elements of which are said to be *radical algebras*. The following theorem characterizes such radical classes. Moreover, it turns out, that hereditary 0-quasiradicals are completely determined by their radical classes.

6.6. THEOREM. (A) *A class* $\mathscr{K}\subseteq\mathscr{V}_0$ *is the radical class of a hereditary* 0*-quasiradical if and only if* $\mathscr{K}$ *satisfies the following conditions:*

(i) *$\mathscr{K}$ is homomorphically closed,*
(ii) *$\mathscr{K}$ is hereditary,*
(iii) *the union of radical subalgebras is again radical.*

(B) *If $\mathscr{K}$ is the radical class of a hereditary* 0*-quasiradical r, then r is uniquely defined by*

$$r(\mathfrak{A}) = \{a \in \mathfrak{A} \mid \langle a \rangle_{\mathfrak{A}} \in \mathscr{K}\}.$$

(Different hereditary 0*-quasiradicals have different radical classes.) Moreover, $r(\mathfrak{A})$ is the greatest radical subalgebra of $\mathfrak{A} \in \mathscr{V}_0$.*

REMARK. For $\mathscr{K}$ as in (B), $l = \{\tau \in \operatorname{Con} F_1 \mid F_1/\tau \in \mathscr{K}\}$ is a Q°-set.

PROOF. (B) follows directly from heredity (cf. also 6.4b)), in particular, for $\mathfrak{A}' \in \mathscr{K}$, $\mathfrak{A}' \leqq \mathfrak{A}$, we have $\mathfrak{A}' = r(\mathfrak{A}') = r(\mathfrak{A}) \cap \mathfrak{A}' \subseteq r(\mathfrak{A})$.

To prove (A), let $\mathscr{K} = \mathscr{R}(r)$ for a hereditary 0-quasiradical r. Then conditions (i)—(iii) are easily checked ((iii) follows from (B)). Conversely, if $\mathscr{K}$ satisfies (i)—(iii), then $r(\mathfrak{A}) := \{a \in \mathfrak{A} \mid \langle a \rangle_{\mathfrak{A}} \in \mathscr{K}\}$ defines a hereditary 0-quasiradical; in fact, $r(\mathfrak{A})$ is a subalgebra of $\mathfrak{A}$ due to (ii) and (iii); 6.2(R°1) follows from (i) and (R°3) is obvious (or see 6.4b)). □

§7. 0-radicals

7.1. In order to define a radical property "$R(\mathfrak{A}/R(\mathfrak{A})) = \mathbf{0}$" for 0-quasiradicals in $\mathscr{V}_0$ one needs a connection between subalgebras and congruences. For $\theta \in \operatorname{Con} \mathfrak{A}$ and $\mathfrak{B} \in \operatorname{Sub} \mathfrak{A}$, $\mathfrak{A} \in \mathscr{V}_0$, we define

$$\theta^0 = \{x \in \mathfrak{A} \mid (x, 0) \in \theta\} \quad (= [0]_\theta, \textit{ the 0-class of } \theta),$$

$$\overline{\mathfrak{B}} = \bigcap \{\theta \in \operatorname{Con} \mathfrak{A} \mid \mathfrak{B} \subseteq \theta^0\} \quad \textit{(the congruence "generated" by } \mathfrak{B}).$$

We have a very good situation e.g. for abelian groups or modules ($\overline{\theta^0} = \theta$ and $\overline{\mathfrak{B}}^0 = \mathfrak{B}$ for $\mathfrak{B} \leqq \mathfrak{A}$) or for rings ($\overline{\theta^0} = \theta$ and $\overline{\mathfrak{B}}^0 = \mathfrak{B}$ for ideals $\mathfrak{B}$ of $\mathfrak{A}$) but, clearly, these properties do not hold in general.

7.2. DEFINITIONS. A 0-quasiradical r in $\mathscr{V}_0$ (cf. 6.2) is called 0*-radical in $\mathscr{V}_0$* if it satisfies

(R°2) $$r(\mathfrak{A}/\bar{r}(\mathfrak{A})) = \{0\} \text{ for all } \mathfrak{A} \in \mathscr{V}_0,$$

where $\bar{r}(\mathfrak{A}) := \overline{r(\mathfrak{A})}$ is the congruence "generated" by $r(\mathfrak{A})$. We remark that a 0-quasiradical r is a 0-radical if and only if $\bar{r}$ is a radical in $\mathscr{V}_0$.

A 0-(quasi-)radical is called 0*-(quasi-)radical* if it satisfies

(∗) $$(\bar{r}(\mathfrak{A}))^0 = r(\mathfrak{A}).$$

We shall see that there is a satisfactory characterization of hereditary 0-radicals only if the congruence $\bar{r}(\mathfrak{A})$ defines the 0-radical $r(\mathfrak{A})$ uniquely by means of (∗), i.e., if $r(\mathfrak{A})$ is a "normal subalgebra" of $\mathfrak{A}$.

REMARK. (∗) trivially holds e.g. in the variety of abelian groups and also in any variety of S-systems (cf. §5) with zero.

Having 6.5 in mind, now it is natural to ask which conditions a Q°-set l has to satisfy in order to define not only a 0-quasiradical rl (via 6.4(1)) but also a 0-radical or a 0*-radical. The following definitions and results shall prepare the answer.

7.3. DEFINITIONS. A Q°-set $l \subseteq \mathrm{Con}\, F_1$ is called *Q^*-set*, if rl satisfies 7.2(∗) (i.e., is a 0*-quasiradical). A Q°-set (or Q*-set) is called *R°-set* (or *R^*-set*) if the following condition holds:

(L°3) $$(l|\tau) \in l \Rightarrow \tau \in l \quad \text{for} \quad \tau \in \mathrm{Con}\, F_1,$$

where

$$(l|\tau) := \{(\alpha(e), \beta(e)) \in F(e)^2 \,|\, (\alpha(a), \beta(a)) \in \overline{rl(\mathfrak{A})}\} = \tau^{\mathfrak{A}/\overline{rl(\mathfrak{A})}}([a])$$

with $\mathfrak{A} = F(e)/\tau$, $a = [e]_\tau$.

The definition of a Q*-set l is not very satisfactory because it does not use properties of l only. In the most important varieties, however, the property 7.2(∗) of being the 0-class of a congruence can be expressed by closure properties with respect to fixed term functions. E.g., in varieties of groups or rings we have: A subgroup (or subring) $\mathfrak{B} \leqq \mathfrak{A}$ is a normal subgroup (or an ideal, resp.) if and only if $f(b, x) \in \mathfrak{B}$ and $f(x, b) \in \mathfrak{B}$ for all $x \in \mathfrak{A}$, $b \in \mathfrak{B}$ where $f(x, y) = x^{-1}yx$ in case of groups and $f(x, y) = xy$ in case of rings. The following proposition covers all these cases and can be used for a "satisfactory" characterization of Q*-sets.

7.4. PROPOSITION. *Let $\mathscr{V}_0$ be a variety with* 0, *$f \in \Omega_n$ and $i \in \{1, \ldots, n\}$. A hereditary 0-quasiradical r in $\mathscr{V}_0$ satisfies*

(1)* $$\forall \mathfrak{A} \in \mathscr{V}_0 \forall a_1, \ldots, a_i \in r(\mathfrak{A}) \forall a_{i+1}, \ldots, a_n \in \mathfrak{A}\colon\ f(a_1, \ldots, a_i, \ldots, a_n) \in r(\mathfrak{A})$$

if and only if the corresponding Q°-set $l = \mathbf{l}r$ satisfies

(2)* $$\forall \tau_1, \ldots, \tau_i \in l\colon\ f[\tau_1, \ldots, \tau_i, \mathbf{0}, \ldots, \mathbf{0}] \in l.$$

COROLLARY. *In case of rings, Q*-sets are characterized by* 6.3(L°0)—(L°2) *and*

(L°2)* $\forall \tau \in l\colon f[\tau, 0] \in l$ *and* $f[0, \tau] \in l$ *where $f \in \Omega_2$ is the ring multiplication.*

PROOF. Let (2)* be fulfilled. Then $a_j \in rl(\mathfrak{A}) = r(\mathfrak{A})$ imply $\tau^{\mathfrak{A}}(a_j) \in l$ for $j \in \{1, \ldots, i\}$. But

$$f[\tau^{\mathfrak{A}}(a_1), \ldots, \tau^{\mathfrak{A}}(a_i), \mathbf{0}, \ldots, \mathbf{0}] \subseteq f[\tau^{\mathfrak{A}}(a_1), \ldots, \tau^{\mathfrak{A}}(a_n)] \subseteq \tau^{\mathfrak{A}}(f(a_1, \ldots, a_n))$$

for arbitrary $a_1, \ldots, a_n \in \mathfrak{A}$ (cf. 2.6c), thus $\tau^{\mathfrak{A}}(f(a_1, \ldots, a_n)) \in l$ due to (L°1) and (2)*, consequently $f(a_1, \ldots, a_n) \in rl(\mathfrak{A}) = r(\mathfrak{A})$, i.e., (1)* is satisfied. Conversely, assume $r = rl$ satisfies (1)* for a Q°-set l. We use the notations given in 6.3: if $\tau_j \in l$ ($j \in \{1, \ldots, i\}$), then $[e]_{\tau_j} \in rl(F_1/\tau_j)$, thus $[e_j]_{\tau^*} \in rl(\mathfrak{A}/\tau^*)$ by 6.2(R°1)′. Hence

$$[a]_{\tau^*} = [f(e_1, \ldots, e_n)]_{\tau^*} = f([e_1], \ldots, [e_n]) \in rl(\mathfrak{A}/\tau^*)$$

by (1)*. Thus

$$f[\tau_1, \ldots, \tau_i, \tau_{i+1}, \ldots, \tau_n] = \tau^{\mathfrak{A}/\tau^*}([a]_{\tau^*}) \in l,$$

in particular for $\tau_{i+1} = \ldots = \tau_n = \mathbf{0}$. □

7.5. LEMMA. *Let l be a Q^*-set and $\tau \in \mathrm{Con}\, F_1$. For $\alpha(e) \in F_1$ define*

$$(\tau\colon \alpha(e)) = \tau^{F_1/\tau}([\alpha(e)]_\tau) = \{(\beta_1(e), \beta_2(e)) \,|\, (\beta_1(\alpha(e)), \beta_2(\alpha(e))) \in \tau\}.$$

Then

$$(l|\tau)^0 = \{\alpha(e)\in F_1 | (\tau\colon \alpha(e))\in l\}.$$

REMARKS. We have $(\tau\colon \alpha(e))^0 = \{\beta(e)|\beta(\alpha(e))\in\tau^0\}$ and $(\tau^{\mathfrak{A}}(a)\colon \alpha(e)) = \tau^{\mathfrak{A}}(\alpha(a))$ for every $a\in\mathfrak{A}\in\mathscr{V}_0$. In the variety of rings, $(\tau\colon \alpha(e))^0$ is the so-called Tschirnhaus-*transformation* of the ideal τ^0 with α (cf. [18; §122, p. 468]).

PROOF. Let $\mathfrak{A}$ and a be as in 7.3. Then

$$(l|\tau)^0 = \{\alpha(e)|\alpha(a)\in\overline{rl(\mathfrak{A})}^0 =_{(*)} rl(\mathfrak{A})\} = \{\alpha(e)|\tau^{\mathfrak{A}}(\alpha(a))\in l\} =$$
$$= \{\alpha(e)|(\tau\colon \alpha(e))\in l\}. \quad \square$$

7.6. For a Q°-set l, the 0-quasiradical rl satisfies $\overline{rl(\mathfrak{A}')}\subseteq\mathfrak{A}'^2\cap\overline{rl(\mathfrak{A})}$ for $\mathfrak{A}'\leqq\mathfrak{A}$ (due to the heredity of rl). The following condition claims equality for single-generated $\mathfrak{A}'$:

$(**)$ $\qquad \overline{rl(\langle a\rangle_{\mathfrak{A}})} = \langle a\rangle_{\mathfrak{A}}^2\cap\overline{rl(\mathfrak{A})}$ *for all* $a\in\mathfrak{A}\in\mathscr{V}_0$.

This condition is equivalent to

$(**)'$ $\qquad \tau^{\mathfrak{A}/\overline{rl(\mathfrak{A})}}([a]) = (l|\tau^{\mathfrak{A}}(a))$ *for all* $a\in\mathfrak{A}\in\mathscr{V}_0$,

since the left and right side of $(**)'$ are equal to

$$\{(\alpha(e),\beta(e))\in F_1^2 | (\alpha(a),\beta(a))\in\overline{rl(\mathfrak{A})}\} \quad \text{and} \quad \{(\alpha(e),\beta(e))\in F_1^2 | (\alpha(a),\beta(a))\in\overline{rl(\langle a\rangle_{\mathfrak{A}})}\},$$

respectively (cf. 7.3, 6.3).

The next lemma will show that this condition holds under various other (stronger) conditions. For the theorem below, however, we shall need $(**)'$ only.

7.7. LEMMA. *Condition* 7.6$(**)$ *is satisfied if e.g.*

(i) *l is a Q^*-set and $\overline{\tau^0}=\tau$ for all $\tau\in\mathrm{Con}\,F_1$, in particular for varieties $\mathscr{V}_0$ of abelian groups, modules or rings, or if*

(ii) *$\mathscr{V}_0$ is a variety of S-systems with* 0 (cf. §5).

PROOF. (i) $\overline{rl(\mathfrak{A})}\cap\mathfrak{A}'^2$ is a congruence on $\mathfrak{A}'$ for $\mathfrak{A}'\leqq\mathfrak{A}$. Therefore

$$\overline{(rl(\mathfrak{A})\cap\mathfrak{A}'^2)^0} = \overline{rl(\mathfrak{A})}^0\cap\mathfrak{A}' =_{(7.2(*))} rl(\mathfrak{A})\cap\mathfrak{A}' =_{(6.5b)} rl(\mathfrak{A}') = \overline{rl(\mathfrak{A}')}^0.$$

But this implies $(**)$ since $\overline{\tau^0}=\tau$.

(ii) Condition $(**)$ can be checked directly using the definitions. $\square$

REMARK. Case 7.7(ii) can be generalized to all varieties $\mathscr{V}_0$ in which the property $(a_1,a_2)\in\overline{\mathfrak{A}'}$ for $\mathfrak{A}'\leqq\mathfrak{A}\in\mathscr{V}_0$ can be expressed by a quantifier-free first order formula in which besides term equations only predicates $f(a_1,a_2)\in\mathfrak{A}'$ can occur where f is some term function.

Now we are ready to characterize hereditary 0-radicals.

7.8. THEOREM. (a) *If r is a hereditary 0^*-radical in $\mathscr{V}_0$, then $\boldsymbol{l}r$ is an R^*-set and $r=\boldsymbol{rl}r$.*

(b) *If $l\subseteq\mathrm{Con}\,F_1$ is an R°-set (R^*-set, resp.) satisfying* 7.6$(**)$, *then $\boldsymbol{rl}$ is a hereditary* 0*-radical (0*-radical, resp.) and $l=\boldsymbol{lrl}$.*

REMARK. A comparision with 6.5 shows that the characterization of 0-quasiradicals went off more smoothly than those of 0-radicals. However, Theorem 7.8 gives a 1—1-correspondence between 0^*-radicals and R^*-sets, if, e.g., $\overline{\tau^0}=\tau$ for all $\tau\in\mathrm{Con}\,F_1$ (because every R^*-set then satisfies $(**)$, cf. 7.7). Moreover, 7.8 provides a 1—1-correspondence between 0-radicals and R°-sets if, e.g., in addition $\overline{\mathfrak{B}^0}=\mathfrak{B}$ for $\mathfrak{B}\leqq\mathfrak{A}$ holds in $\mathscr{V}_0$ (because $(*)$ is then always fulfilled).

PROOF OF 7.8. (a) Let r be a hereditary 0^*-radical. Then r is in particular a 0-quasiradical, thus $r=rlr$ and lr is a Q°-set by 6.5(a). Moreover, lr is a Q^*-set due to Definition 7.3. We show 7.3(L°3). Let $\tau\in\mathrm{Con}\,F_1$, $\mathfrak{A}=F_1/\tau$, $a=[e]_\tau$. Then

$$(lr\,|\,\tau)\in lr \Rightarrow_{(7.3)} \tau^{\mathfrak{A}/\bar{r}(A)}([a]) = \tau^{\mathfrak{A}/\overline{rlr}(\mathfrak{A})}([a])\in lr \Rightarrow$$

$$\Rightarrow_{(6.4(1))} [a]_{\bar{r}(\mathfrak{A})}\in rlr\big(\mathfrak{A}/\bar{r}(\mathfrak{A})\big) = r\big(\mathfrak{A}/\bar{r}(\mathfrak{A})\big) = \{[0]_{\bar{r}(\mathfrak{A})}\} \Rightarrow$$

$$\Rightarrow a\in\big(\bar{r}(\mathfrak{A})\big)^0 =_{(7.2(*))} r(\mathfrak{A}) \Rightarrow_{(6.4(1))} \tau = \tau^{\mathfrak{A}}(a)\in lr.$$

(b) Due to 6.5(b) it remains to show 7.2(R°2) for rl where l is a Q°-set satisfying $7.6(**)$. In fact, let $a\in\mathfrak{A}\in\mathscr{V}_0$. Then

$$[a]\in rl\big(\mathfrak{A}/\overline{rl(\mathfrak{A})}\big) \Rightarrow_{(6.4(1))} \tau^{\mathfrak{A}/\overline{rl(\mathfrak{A})}}([a])\in l \Rightarrow$$

$$\Rightarrow_{(7.6(**)')} \big(l\,|\,\tau^{\mathfrak{A}}(a)\big)\in l \Rightarrow_{(L^\circ 3)} \tau^{\mathfrak{A}}(a)\in l \Rightarrow a\in rl(\mathfrak{A}) \Rightarrow [a]=[0].$$

Thus $rl\big(\mathfrak{A}/\overline{rl(\mathfrak{A})}\big)=\{[0]\}$. □

§8. Hereditary (quasi-)radicals of abelian groups and rings

8.1. Let $\boldsymbol{Ab}$ be the variety of *all abelian groups*. In view of the 1—1-correspondence between subgroups and congruences *there is no distinction between (quasi-) radicals, 0-(quasi-)radicals and 0^*-(quasi-)radicals.*

We use the following notations: For $F(e)$ in $\boldsymbol{Ab}$ we take $\mathbf{Z}$ (generated by $e=1$). A subgroup $U\leqq\mathbf{Z}$ is generated by a uniquely determined element $u\in\mathbf{N}=\{0, 1, 2, \ldots\}$; the corresponding congruence $\overline{U}$ is denoted by $\varkappa(u)$ $(\varkappa(u)^0=[0]_{\varkappa(u)}=\langle u\rangle)$. Thus $\langle\mathbf{N};|\rangle\cong\langle\mathrm{Con}\,F(e);\supseteqq\rangle$ (where $u|v:\Leftrightarrow u$ is a divisor of v) and $\varkappa$ is the isomorphism. Therefore we can describe subsets of $\mathrm{Con}\,F(e)$ by means of subsets of $\mathbf{N}$. One can imagine that this simplifies considerably the description of Q-sets and hereditary (quasi-)radicals. Reformulating the notions defined in §6 we get the following:

$$\tau^G(a) = \varkappa\big(\mathrm{ord}\,(a)\big) \quad \text{for} \quad a\in G\in\boldsymbol{Ab},$$

where

$$\mathrm{ord}\,(a) = \begin{cases} 0 & \text{if} \quad na\neq 0 \qquad\qquad \text{for all} \quad n>0,\\ \min\{n\,|\,na=0,\ n>0\}, & \text{otherwise;}\end{cases}$$

$$\varkappa(v)\subseteqq\varkappa(u)\Leftrightarrow u\,|\,v, \quad \omega_-[\varkappa(u)]=\varkappa(u), \quad \omega_+[\varkappa(u),\varkappa(v)]=\varkappa\big(\mathrm{l.c.m.}\,(u,v)\big)$$

for $u, v\in\mathbf{N}$ and the fundamental operations $\omega_-\in\Omega_1$: $x\mapsto -x$, $\omega_+\in\Omega_2$: $(x, y)\mapsto x+y$ (see 6.3). Moreover, we have

$$(\varkappa(u)\colon z) = \varkappa\big(u/\mathrm{g.c.d.}\,(u, z)\big)$$

(since $(\varkappa(u)\colon z)^0=\{w\,|\,u|wz\}$, cf. 7.5) for $u\in\mathbf{N}$, $z\in\mathbf{Z}$; in particular $\varkappa(p)\in l$ implies $\varkappa(p^m)\subseteq(l|\varkappa(p^{m+1}))$ for a prime number p and a Q°-set (=Q*-set) $l\subseteq\mathrm{Con}\,F(e)$, $m\geqq 1$ (since $(\varkappa(p^{m+1})\colon p^m)=\varkappa(p)\in l$, thus $p^m\in(l|\varkappa(p^{m+1}))^0$, cf. 7.5). These observations lead to the following theorem:

8.2. THEOREM. *For a subset* $l\subseteq\mathbf{N}$, $\boldsymbol{rl}(G)=\{a\in G|\,\mathrm{ord}\,(a)\in l\}$ *defines a hereditary* (0-)*quasiradical or radical, respectively, in* $\boldsymbol{Ab}$ *if and only if* l *satisfies*

(L°0) $1\in l$,
(L°1) $a\in l$ *and* $b\mid a\Rightarrow b\in l$,
(L°2) $a, b\in l\Rightarrow$l.c.m. $(a, b)\in l$,

or, respectively, in addition

(L°3) $\qquad p\in l\Rightarrow p^m\in l$ *for prime numbers* p *and for* $m\in\mathbf{N}$.

Conversely, if r *is a hereditary* (0-)*quasiradical (radical, resp.) then*

$$\boldsymbol{lr}=\{n\in\mathbf{N}\,|\,r(\mathbf{Z}_n)=\mathbf{Z}_n\}$$

(*where* $\mathbf{Z}_n=\mathbf{Z}/\varkappa(n)$) *fulfils* (L°0)—(L°2) ((L°3), *resp.*) *and* $r=\boldsymbol{rlr}$.

PROOF. The proof follows immediately from 8.1, 6.4, 6.5 and 7.8. The only thing we have to be sure is the equivalence of 8.2(L°3) (for $l\subseteq\mathbf{N}$) and 7.3(L°3) (where l has to be substituted by $\varkappa(l)=\{\varkappa(u)\mid u\in l\}$).

In fact, the implications $p\in l\Rightarrow\varkappa(p)\in\varkappa(l)\Rightarrow\varkappa(p^m)\in\varkappa(l)$ (by (L°1), 8.1 and induction on m)$\Rightarrow p^m\in l$ show the above conditions to be necessary. Conversely, let 8.2(L°3) be fulfilled, let $\varkappa(w):=(\varkappa(l)|\varkappa(u))\in\varkappa(l)$ for some $u\in\mathbf{N}$ and let $\varkappa(u)\notin\varkappa(l)$. We have to find a contradiction. By 8.2(L°3) and (L°2) there also exists a prime number $p|u$ with $p\notin l$. Then $p\nmid w$ (otherwise $\varkappa(w)\subseteq\varkappa(p)\in\varkappa(l)$ by (L°1)) and $p|(u/\text{g.c.d.}\,(u, w))$. Therefore $(u/\text{g.c.d.}\,(u, w))\notin l$ (by (L°1)), i.e., $(\varkappa(u)\colon w)\notin\varkappa(l)$ (cf. 8.1). Hence (cf. 7.5) $w\notin(\varkappa(l)|\varkappa(u))^0=\varkappa(w)^0$, a contradiction. □

8.3. In view of (L°3), hereditary radicals r in $\boldsymbol{Ab}$ are uniquely defined by a subset P' of $P=\{p|p$ prime or $p=1\}$ or by $P'=\{0\}\cup P$ as follows: $r=r_{P'}:=\boldsymbol{rl}$ (cf. 8.2), where

$$l=l(P'):=\Big\{\prod_{i\in I}p_i^{\alpha_i}\Big|\ I\text{ finite},\ p_i\in P',\ \alpha_i\in\mathbf{N}\Big\}.$$

Thus every hereditary radical is the direct sum of its prime components:

$$r_{P'}(G)=\sum_{p\in P'}r_p(G),\quad G\in\boldsymbol{Ab},$$

where

$$r_p(G)=\{a\in G|\,\exists n\in\mathbf{N}\colon\ \mathrm{ord}\,(a)=p^n\}.$$

(REMARK. This is the converse of a result of S. Dickson [1] that the sum of "hom-orthogonal" families of torsion subfunctors is again torsion.)

For $P'=P$ we get the *classical torsion radical* in $\boldsymbol{Ab}$:

$$r_P(G)=\{a\in G|\,\exists n\in\mathbf{N}\setminus\{0\}\colon\ na=0\}.$$

The *trivial radicals* can be obtained from $P'=\{1\}$ and $P'=P\cup\{0\}$.

8.4. REMARKS. a) The hereditary (0-)(quasi-)radicals in the variety ***Mod S*** of unitary left-modules over a ring S (with unit) can be treated analogously to the variety ***Ab***. In this case, Theorem 3.3 (or, equivalently, 6.5 and 7.8) lead to the results of J. M. Maranda [13; Proposition 4, p. 119], where (L°2) for "+" and the module operations is equivalent to Maranda's conditions (G2) and (G3) and

$$(\mathrm{L}°3): \qquad \{s\in S \mid (t\colon s)\in L\}\in L \Rightarrow t\in L$$

(L is a Q°-set of left ideals of S, t is a left ideal of S) is equivalent to Maranda's condition (G4) [13; Proposition 3, p. 119].

If S is a commutative principal ideal ring with no proper zero divisors then we get for ***Mod S*** the same characterization as in case of ***Ab*** = ***Mod Z***.

b) *In every variety of not only commutative semigroups with unit (monoids) or groups there do not exist non-trivial hereditary* 0*-quasiradicals* (one can prove, e.g., $\omega[\tau,\tau]=\mathbf{0}$ for $\tau\in\mathrm{Con}\,F(e)$, $\tau\neq\mathbf{1}$, where ω denotes the operation of multiplication; therefore only trivial Q°-sets exist). The results (for monoids) also follow from the fact that every monoid can be embedded into a congruence-free monoid as observed in [14; Theorem 3] (since then $F(e)$ is either radical or semisimple, consequently 1-generated subalgebras are all either radical or semisimple, i.e., the hereditary radical is trivial).

8.5. Let now $\mathscr{V}$ be the variety ***Ring*** of all rings (not necessarily associative) or the variety ***aRing*** of all associative rings. The notion of a radical for universal algebras in the sense of Hoehnke (cf. 3.1) is — when specialized to rings — much more general than the classical notion of radical in the sense of Kuroš—Amitsur (see e.g. [2]). Since congruences in rings are uniquely determined by their 0-classes, i.e., by ideals, there is *no distinction between radicals and* 0*-radicals* (7.2) *in* $\mathscr{V}$. Moreover, because of 6.6(B), the radical classes of hereditary 0*-radicals are also radical classes in the sense of Kuroš—Amitsur. (But we have to choose 0*-radicals and not 0-radicals in order that $r(\mathfrak{A})$ be an ideal and not only a subring.)

Consequently, *hereditary* 0*-*radicals in* $\mathscr{V}$ *are precisely the Kuroš—Amitsur radicals which are hereditary with respect to subrings* (in ring theory these radicals are called *strongly hereditary*).

Thus, *Theorem* 7.8 *(and* 7.7, 7.4*) will give a complete description of strongly hereditary Kuroš—Amitsur radicals via* R^**-sets* (i.e., subsets of $F(e)$ satisfying 6.3 (L°0)—(L°2), 7.4(L°2)* and 7.3(L°3)). To get a more explicit characterization of such R^*-sets it remains to reformulate the conditions to be satisfied in terms of ideals (or their parameters) of $F(e)=\{\sum_{i=1}\alpha_i e^i \mid \alpha_i\neq 0$ for finitely many $i\}$. This, however, would exceed the aim of this paper.

But for $\mathscr{V}=$ ***Ring***, we shall see that the determination of R^*-sets can be avoided because the whole problem can be traced back to the variety ***Ab***.

8.6. Let r be a hereditary radical (or, equivalently, a 0*-radical) in ***Ring***. Then the semisimple class of r is hereditary (4.1(ii)). Therefore we can apply a result of B. J. Gardner [5; Corollary 2.5] (cf. [4; Theorem 2.4]) which gives that the radical class $\mathscr{R}(r)$ of r is a so-called *A-radical* [3], i.e., $\mathscr{R}(r)=\{\mathfrak{A}\in\mathbf{Ring}\mid\mathfrak{A}^+\in\mathscr{R}(r^+)\}$, where $\mathfrak{A}^+$ is the additive group of the ring $\mathfrak{A}$ and r^+ is the radical in ***Ab*** defined as the restriction of r to all zero rings. Thus hereditary radicals in ***Ring*** are in 1—1-correspondence with hereditary radicals in ***Ab***: $r(\mathfrak{A})=r^+(\mathfrak{A}^+)$. From the results of Gardner it follows

that $r^+(\mathfrak{A}^+)$ is an ideal in every ring $\mathfrak{B}$ with $\mathfrak{B}^+=\mathfrak{A}^+$. This property can also be directly seen from 8.2. In fact, ord (ax) and ord (xa) are divisors of ord (a) for any elements a, x of a ring $\mathfrak{A}=\langle A;\ +,\ \cdot\rangle$, thus $rl(\mathfrak{A}^+)=\{a\in\mathfrak{A}|\text{ord }(a)\in l\}$ is an ideal by (L°1).

With Theorem 8.2 we already know all hereditary radicals in ***Ab***. Due to the above observations, the following proposition is obvious:

8.7. PROPOSITION. *Theorem* 8.2 *remains valid in case of radicals if* ***Ab*** *is substituted by* ***Ring***. □

It is not clear whether the same holds for quasiradicals, too. But in this case, of course, one can use Theorem 6.5 and 7.4, in order to get a characterization of 0*-quasiradicals (or, equivalently, quasiradicals) in ***Ring*** or ***aRing***.

With 8.7 and 8.2 we have a complete description of all strongly hereditary radicals in ***Ring*** in the sense of Kuroš—Amitsur.

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RELATIONS BETWEEN RATIONAL AND SPLINE APPROXIMATIONS

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1. Introduction

In this paper we study some relations between the best rational and spline approximations of real functions and their derivatives defined on a compact interval in C and L_p metrics. Our main results and some corollaries are formulated in Section 2. The proofs of these results together with some auxiliary statements are given in Section 3. We first give the necessary notations and recall some well-known facts.

Denote by R_n the set of all rational functions of degree at most n, i.e. $q\in R_n$ if $q=\frac{p_1}{p_2}$ where p_1 and p_2 are polynomials of degree at most n with real coefficients. By $S(k, n, [a, b])$ we shall denote the set of all piecewise polynomial functions (spline functions with defect) of degree at most k with $n+1$ free knots on $[a, b]$, i.e. $\varphi\in S(k, n, [a, b])$ if there exist $n+1$ points $\{x_i\}_{i=0}^n$, $a=x_0<x_1<\ldots<x_n=b$, such that in each interval (x_{i-1}, x_i), φ is an algebraic polynomial of degree at most k. We shall assume that φ is continuous either on the right or on the left at each knot x_i. Then $\tilde{S}(k, n, [a, b])=S(k, n, [a, b])\cap C^{k-1}_{[a,b]}$ is the set of all spline functions of degree at most k with $n+1$ free knots on $[a, b]$.

Denote by $R_n(f)_{L_p}=R_n(f)_{L_p[a,b]}$ and $E_n^k(f)_{L_p}=E_n^k(f)_{L_p[a,b]}$ the best approximations of the function $f\in L_p[a, b]$, $1\leqq p\leqq\infty$, by means of the elements of R_n and $S(k, n, [a, b])$, respectively, in L_p metrics, i.e.

$$R_n(f)_{L_p}=R_n(f)_{L_p[a,b]}=\inf_{q\in R_n}\|f-q\|_{L_p[a,b]},$$

$$E_n^k(f)_{L_p}=E_n^k(f)_{L_p[a,b]}=\inf_{\varphi\in S(k,n,[a,b])}\|f-\varphi\|_{L_p[a,b]},$$

$$\|g\|_{L_p[a,b]}=\Big(\int_a^b|g(x)|^p\,dx\Big)^{1/p}\quad\text{for}\quad 1\leqq p<\infty\quad\text{and}\quad\|g\|_{L_\infty[a,b]}=\operatorname*{Vrai\,sup}_{x\in[a,b]}|g(x)|.$$

By $R_n(f)_C=R_n(f)_{C[a,b]}$ and $E_n^k(f)_C=E_n^k(f)_{C[a,b]}$ we denote the best uniform approximations of $f\in C_{[a,b]}$ by means of the elements of R_n and $S(k, n, [a, b])$, respectively.

Throughout the paper $C, C_1, C_2, \ldots$ denote absolute positive constants and $C(\ldots), C_1(\ldots), C_2(\ldots), \ldots$ denote positive constants depending only on the corresponding parameters. For $1\leqq p\leqq\infty$ we denote by p' the conjugate number of p, i.e. $\frac{1}{p}+\frac{1}{p'}=1$, when $1\leqq p<\infty$ and $p'=1$, when $p=\infty$.

The following results are well-known:

THEOREM A [1]. *Let* $f\in C_{[0,1]}$. *Then*

$$E_n^0(f)_C \leqq 2^7 \frac{\sum_{k=0}^{n} R_k(f)_C}{n+1} \quad \text{for} \quad n \geqq 1.$$

THEOREM B [2]. *Let* $f\in C^r_{[0,1]}$, $r\geqq 1$. *Then*

$$R_n(f)_C \leqq C(r) \frac{\sum_{k=0}^{n} R_k(f^{(r)})_C}{p^{r+1}} \quad \text{for} \quad n \geqq r.$$

Yu. A. Brudnyi [3] obtained some results, which are closely connected to ours. V. A. Popov [4] obtained connections between the rational uniform approximation of functions and the polynomial approximation of their derivatives in L_p metric, see also [14].

In this paper we shall give detailed proofs of some results announced in [5].

2. Main results

In this section we shall formulate our main results and discuss some corollaries.

THEOREM 1. *Let f be absolutely continuous on $[a, b]$, $f'\in L_p$, $1<p\leqq\infty$, $\alpha>0$ and $k\geqq 0$. Then*

$$R_n(f)_C \leqq C(p,k,\alpha)(b-a)^{1/p'} \frac{\sum_{\nu=1}^{n} \nu^{\alpha-1} E_\nu^k(f')_{L_p}}{n^{\alpha+1}} \quad \text{for} \quad n \geqq k+1,$$

where $C(p,k,\alpha)=C(k,\alpha)(p')^{4(\alpha+3)}$ *and*

$$R_n(f)_C \leqq C(p,\alpha)(b-a)^{1/p'} \frac{\sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} R_\nu(f')_{L_p}}{n^{\alpha+1}} \quad \text{for} \quad n \geqq 1,$$

where $C(p,\alpha)=C(\alpha)(p')^{4(\alpha+3)}$.

THEOREM 2. *Let f be absolutely continuous on $[a, b]$, $f'\in L_p$, $1<p\leqq\infty$, $\alpha>0$ and $r>\alpha$. Then*

$$E_n^r(f)_C \leqq C(p,r,\alpha)(b-a)^{1/p'} \frac{\sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} R_\nu(f')_{L_p}}{n^{\alpha+1}} \quad \text{for} \quad n \geqq 1,$$

where $C(p,r,\alpha)=\dfrac{C(r)(p')^{r+2}}{(r-\alpha)^{r+1}}$.

THEOREM 3. *Let f be absolutely continuous on $[a, b]$, $1\leqq p<\infty$, $\alpha>0$ and $k\geqq 0$. Then*

$$R_n(f)_{L_p} \leqq C(p,k,\alpha)(b-a)^{1/p} \frac{\sum_{\nu=1}^{n} \nu^{\alpha-1} E_\nu^k(f')_{L_1}}{n^{\alpha+1}} \quad \text{for} \quad n \geqq k+1,$$

where $C(p, k, \alpha)=C(k, \alpha)p^{7\alpha+22}$ *and*

$$R_n(f)_{L_p} \leqq C(p, \alpha)(b-a)^{1/p} \frac{\sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} R_\nu(f')_{L_1}}{n^{\alpha+1}} \quad \text{for} \quad n \geqq 1,$$

where $C(p, \alpha)=C(\alpha)p^{7\alpha+22}$.

THEOREM 4. *Let f be absolutely continuous on $[a, b]$, $1 \leqq p < \infty$ and $0<\alpha<r$. Then*

$$E_n^r(f)_{L_p} \leqq C(p, r, \alpha)(b-a)^{1/p} \frac{\sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} R_\nu(f')_{L_1}}{n^{\alpha+1}} \quad \text{for} \quad n \geqq 1,$$

where $C(p, r, \alpha)=\dfrac{C(r)p^{r+1}}{(r-\alpha)^{r+1}}$.

The following theorem is based on Theorems 1 and 4.

THEOREM 5. *Let f be defined on $[a, b]$ such that $f^{(r)}$ $(r \geqq 1)$ is absolutely continuous. Let $\alpha>0$ and $k \geqq 0$. Then*

$$R_n(f)_C \leqq C(r, k, \alpha)(b-a)^r \frac{\sum_{\nu=1}^{n}{}' \nu^{\alpha-1} E_\nu^k(f^{(r+1)})_{L_1}}{n^{r+1+\alpha}} \quad \text{for} \quad n \geqq r+k+1,$$

and

$$R_n(f)_C \leqq C(r, \alpha)(b-a)^r \frac{\sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} R_\nu(f^{(r+1)})_{L_1}}{n^{r+1+\alpha}} \quad \text{for} \quad n \geqq r+1.$$

THEOREM 6. *Let f be defined on $[a, b]$ such that $f^{(r)}$ $(r \geqq 1)$ is absolutely continuous. Let $\mu \geqq 1$ and $0<\alpha<\mu$. Then*

$$E_n^{r+\mu}(f)_C \leqq C(r, \mu, \alpha)(b-a)^r \frac{\sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} R_\nu(f^{(r+1)})_{L_1}}{n^{r+1+\alpha}} \quad \text{for} \quad n \geqq 1$$

where $C(r, \mu, \alpha)=\dfrac{C(r, \mu)}{(\mu-\alpha)^{\mu+1}}$.

Theorem 3 follows from Theorem 4 and the following statement, proved in [6] (see also [5]):

THEOREM 7. *Let $1 \leqq p<q \leqq \infty$, $f \in L_q[a, b]$, $\alpha>0$ and $k \geqq 0$. Then*

$$R_n(f)_{L_p} \leqq C(p, q, k, \alpha)(b-a)^{\frac{1}{p}-\frac{1}{q}} \frac{\sum_{\nu=1}^{n}{}' \nu^{\alpha-1} E_\nu^k(f)_{L_q}}{n^\alpha} \quad \text{for} \quad n \geqq k,$$

where $C(p, q, k, \alpha)=\dfrac{C(k, \alpha)}{\left(\frac{1}{p}-\frac{1}{q}\right)^{6(\alpha+2)}}$.

An estimate in the opposite direction is also true (see [6]):

THEOREM 8. *Let* $1 \leqq p < q \leqq \infty$, $f \in L_q[a, b]$, $r \geqq 0$ *and* $0 < \alpha < r+1$. *Then*

$$E_n^r(f)_{L_p} \leqq C(p, q, r, \alpha)(b-a)^{\frac{1}{p}-\frac{1}{q}} \frac{\sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} R_\nu(f)_{L_q}}{n^\alpha} \quad \text{for} \quad n \geqq 1,$$

where

$$C(p, q, r, \alpha) = \frac{C(r)}{\left(\frac{1}{p}-\frac{1}{q}\right)(r+1-\alpha)^{r+1}}.$$

In what follows we shall formulate some corollaries. To this end we shall make use of the following notations:

$$R(\lambda, p) = \{f \in L_p[0, 1]: R_n(f)_{L_p} = O(n^{-\lambda})\},$$
$$S_k(\lambda, p) = \{f \in L_p[0, 1]: E_n^{k-1}(f)_{L_p} = O(n^{-\lambda})\},$$
$$R^r(\lambda, p) = \{f \in W_p^r[0, 1]: R_n(f^{(r)})_{L_p} = O(n^{-\lambda})\},$$
$$S_k^r(\lambda, p) = \{f \in W_p^r[0, 1]: E_n^{k-1}(f^{(r)})_{L_p} = O(n^{-\lambda})\},$$

where $W_p^r[0, 1]$ is the set of all functions f such that $f^{(r-1)}$ is absolutely continuous on $[0, 1]$ and $f^{(r)} \in L_p[0, 1]$.

COROLLARY 1. *Let* $k \geqq 1$, $\lambda > 0$ *and* $m > r+\lambda$. *Then the inclusions*

(1) $\quad S_k^r(\lambda, p) \subset R(r+\lambda, q), \quad R^r(\lambda, p) \subset R(r+\lambda, q), \quad R^r(\lambda, p) \subset S_m(r+\lambda, q)$

hold true in the following situations:

i) $r = 1, \quad p > 1, \quad q = \infty;$

ii) $r = 1, \quad p = 1, \quad q < \infty;$

iii) $r \geqq 2, \quad p = 1, \quad q = \infty.$

An analogous corollary follows from Theorems 7 and 8, see [6].

It is clear that inclusions (1) are not true in the situation $r=1, p=1, q=1$. On the other hand, E. P. Dolženko [7], [8] proved that, if $R_n(f)_C = O(n^{-1-\lambda})$, $\lambda > 0$, then f is absolutely continuous and $R_n(f')_{L_1} = O(n^{-\lambda})$. Therefore for every $\varepsilon > 0$ and $\lambda > 0$ we have

$$R^1(\lambda, 1+\varepsilon) \subset R(1+\lambda, \infty) \subset R^1(\lambda, 1)$$

and here $\varepsilon > 0$ can not be omitted.

Theorems 1—6 imply also the following result, proved by Ju. A. Brudnyi [3]:

COROLLARY 2. *If* $f \in W_p^r[0, 1]$, $1 \leqq p \leqq \infty$, *then*

$$R_n(f)_{L_q} = O\left(\frac{\omega_k\left(f^{(r)}; \frac{1}{n}\right)_{L_p}}{n^r}\right)$$

for a given $k \geqq 1$ *and*

i) $r = 1, \quad p > 1, \quad q = \infty;$

ii) $r = 1, \quad p = 1, \quad q < \infty;$

iii) $r \geqq 2, \quad p = 1, \quad q = \infty,$

where $\omega_k(g; \delta)_{L_p}$ *is the k-th modulus of continuity of g in* L_p:

$$\omega_k(g; \delta)_{L_p} = \sup_{0<h\leqq\delta} \left(\int_0^{1-kh} |\Delta_h^k g(x)|^p \, dx\right)^{1/p}, \quad \Delta_h^k g(x) = \sum_{\nu=0}^{k} (-1)^{k+\nu} \binom{k}{\nu} g(x+\nu h).$$

V. A. Popov [4] obtained the following connections between the rational uniform and the polynomial L_p approximations of the functions and their derivatives, see also [14]: if $f \in W_p^r[0, 1]$, then

$$R_{n+r}(f)_C = O\left(\frac{E_n(f^{(r)})_{L_p}}{n^r}\right)$$

in the situations: i) $r=1$, $p>1$ and ii) $r \geqq 2$, $p=1$, where $E_n(g)_{L_p}$ is the best L_p approximation of g by means of algebraic polynomials of degree at most n. This result implies Corollary 2, cases i) and iii).

REMARK 1. The right-hand sides of our estimates in Theorems 1—8 involve rational or piecewise polinomyal functions as tools for approximation. From our proofs it will become clear that these tools can be replaced by more general ones. Namely, it is sufficient that the approximating functions and their derivatives up to a certain order are monotone in $O(n^{-1})$ subintervals. However, such a general formulation is not used here, since other approximation tools that produce a better order of approximation are not known to us.

REMARK 2. It is well-known [9] that there is no substantial difference between piecewise polynomial functions and spline functions as for the order of approximation that both classes produce. Since the piecewise polynomial functions appear to us as a simpler tool for approximation, we prefer to use them instead of spline functions.

REMARK 3. Clearly the estimates in Theorems 1—8 are exact with respect to the order of approximation. Theorems 1 and 3 do not hold true for $p=1$ and $p=\infty$, respectively, since $R_n(f)_C$ may tend to zero as slow as we want in the class of absolutely continuous functions f.

3. Proof of the main results

In this section we shall prove the results formulated in Section 2. For this purpose we shall use auxiliary theorems.

THEOREM 9. *Let f be absolutely continuous on* $[a, b]$, $f' \in L_p[a, b]$, $1 < p \leqq \infty$, *and assume that there are points* $a = x_0 < x_1 < \ldots < x_l = b$, $l \geqq 1$, *and rational functions*

$q_i \in R_{k_i}$, $k_i \geqq 0$ $(i=0, \ldots, l-1)$ *such that*

$$\|f-q_i\|_{C[x_i, x_{i+1}]} \leqq \varepsilon_1, \quad i = 0, 1, \ldots, l-1; \ \varepsilon_1 > 0. \tag{2}$$

Let $\varepsilon_2 > 0$. *Then there is a rational function* q *of degree* $\leqq 2 \sum_{i=0}^{l-1} k_i + Dp' l \ln^2 \left(e + \frac{\|f'\|_{L_p} (b-a)^{1/p'}}{l\varepsilon_2} \right)$ *such that*

$$\|f-q\|_{C[a,b]} \leqq \varepsilon_1 + 5\varepsilon_2,$$

where $D>1$ *is an absolute constant,* $\|g\|_{C[a,b]} = \sup_{x \in [a,b]} |g(x)|$.

PROOF. Since the function $x \ln^2 \left(e + \frac{1}{x} \right)$ is increasing on $(0, \infty)$ it is clear that if Theorem 9 is true for $l=2^s$, $s=0, 1, \ldots$, then it is true with some other constant D for all $l=1, 2, \ldots$. Assume that $l=2^s$, s an integer and $1<p<\infty$. We need the following lemmas, where we use the notations introduced above.

LEMMA 1. *There exists a constant* $B>1$ *such that for every positive* α, β *and* γ *there exists a rational function* σ *of degree at most* $B \ln \left(e + \frac{1}{\gamma} \right) \ln \left(e + \frac{\beta}{\alpha} \right)$ *such that*

$$|\sigma(x)| = \gamma^{3/2} \quad \text{for} \quad x \in (-\infty, -\alpha] \cap [-\beta, 0],$$

$$|1-\sigma(x)| \leqq \gamma^{3/2} \quad \text{for} \quad x \in [0, \beta] \cap [\alpha, \infty),$$

$$0 \leqq \sigma(x) \leqq 1 \quad \text{for} \quad x \in (-\infty, \infty).$$

This lemma follows directly from the results of A. A. Gončar [10], see also [11], Lemma 5.

Denote $\Delta_i = [x_i, x_{i+1}]$ for $i=0, 1, \ldots, l-1$.

LEMMA 2. *There are rational functions* $\{p_i\}_{i=0}^{l-1}$ *such that*

$$\|f-p_i\|_{C(\Delta_i)} \leqq \varepsilon_1 + \varepsilon_2,$$

$$\|p_i - f(x_i)\|_{C(-\infty, \infty)} \leqq (V_{\Delta_i} f)^{3/2} \varepsilon_2^{-1/2}, \quad p_i \in R_{2k_i}, \quad i = 0, 1, \ldots, l-1,$$

where $V_{\Delta_i} f$ *is the total variation of* f *on the interval* Δ_i.

PROOF. Let $0 \leqq i \leqq l-1$. If $V_{\Delta_i} f = 0$, then it is clear that the rational function $p_i \equiv f(x_i)$ satisfies the requirements of Lemma 2.

Let $V_{\Delta_i} f > 0$. Consider the function (see J. Szabados [12])

$$p_i(x) = f(x_i) + \frac{q_i(x) - f(x_i)}{1 + \eta_i (q_i(x) - f(x_i))^2},$$

where $\eta_i = \frac{\varepsilon_2}{4 (V_{\Delta_i} f)^3}$. Obviously $p_i \in R_{2k_i}$. We have

$$\|p_i - f(x_i)\|_{C(-\infty, \infty)} = \left\| \frac{q_i(x) - f(x_i)}{1 + \eta_i (q_i(x) - f(x_i))^2} \right\|_{C(-\infty, \infty)} \leqq \frac{1}{2\sqrt{\eta_i}} = (V_{\Delta_i} f)^{3/2} \varepsilon_2^{-1/2}.$$

It remains to estimate $\|f-p_i\|_{C(\Delta_i)}$. If

$$\|q_i-f(x_i)\|_{C(\Delta_i)} > 2\|f-f(x_i)\|_{C(\Delta_i)},$$

then

$$\|f-f(x_i)\|_{C(\Delta_i)} < \|q_i-f(x_i)\|_{C(\Delta_i)} - \|f-f(x_i)\|_{C(\Delta_i)} \leqq \|f-q_i\|_{C(\Delta_i)} \leqq \varepsilon_1$$

and therefore the rational function $\bar{p}_i \equiv f(x_i)$ satisfies the requirements of Lemma 2. Assume that $\|q_i-f(x_i)\|_{C(\Delta_i)} \leqq 2\|f-f(x_i)\|_{C(\Delta_i)}$. Then using (2) we obtain

$$\|f-p_i\|_{C(\Delta_i)} = \left\|f-f(x_i)-\frac{q_i-f(x_i)}{1+\eta_i(q_i-f(x_i))^2}\right\|_{C(\Delta_i)} \leqq$$

$$\leqq \|f-q_i\|_{C(\Delta_i)} + \eta_i\|f-f(x_i)\|_{C(\Delta_i)}\|q_i-f(x_i)\|^2_{C(\Delta_i)} \leqq$$

$$\leqq \|f-q_i\|_{C(\Delta_i)} + 4\eta_i\|f-f(x_i)\|_{C(\Delta_i)} \leqq \varepsilon_1 + 4\eta_i(V_{\Delta_i}f)^3 = \varepsilon_1+\varepsilon_2.$$

This establishes the lemma.

We shall prove Theorem 9 by several applications of the following lemma. Denote

$$N(m,[x,y]) = Bp'\sum_{\nu=0}^{m} 2^\nu \ln^2(e+2^m 4^{-\nu}\|f'\|_{L_p[x,y]}(y-x)^{1/p'}\varepsilon_2^{-1}),$$

where $B>1$ is the constant from Lemma 1; f, p and ε_2 are from the assumptions of Theorem 9; m, x and y are parameters.

LEMMA 3. *Let* $0 \leqq m \leqq s-1$ *and assume that for each* i, $0 \leqq i \leqq l-2^m$, *there is a rational function* p_i^* *of degree* $\leqq 2\sum_{\nu=i}^{i+2^m-1} k_\nu + N(m,[x_i, x_{i+2^m}])$ *such that*

$$\|f-p_i^*\|_{C(x_i, x_{i+2^m}]} \leqq \varepsilon_1 + \varphi(m)\varepsilon_2, \tag{3}$$

where $\varphi(m) \geqq 1$, $\varphi(m)$ *depends only on* m, *and*

$$\|p_i^*-f(x)\|_{C(-\infty,\infty)} \leqq (V_{x_i}^{x_{i+2^m}}f)^{3/2}\varepsilon_2^{-1/2}. \tag{4}$$

Then for each i, $0 \leqq i \leqq l-2^{m+1}$, *there exists a rational function* q_i^* *of degree* $\leqq 2\sum_{\nu=i}^{i+2^{m+1}-1} k_\nu + N(m+1,[x_i, x_{i+2^{m+1}}])$ *such that*

$$\|f-q_i^*\|_{C[x_i, x_{i+2^{m+1}}]} \leqq \varepsilon_1 + \left(\varphi(m)+\frac{4}{2^{m+1}}\right)\varepsilon_2 \tag{5}$$

and

$$\|q_i^*-f(x)\|_{C(-\infty,\infty)} = (V_{x_i}^{x_{i+2^{m+1}}}f)^{3/2}\varepsilon_2^{-1/2}. \tag{6}$$

PROOF. We shall use some ideas from [13], [14] and [4]. Obviously, if $V_{x_i}^{x_{i+2^{m+1}}}f \leqq \varepsilon_2$, then the rational function $q_i^* \equiv f(x_i)$ satisfies (5), (6) and $q_i^* \in R_0$.

Let $V_{x_i}^{x_{i+2^{m+1}}}f > \varepsilon_2$. Denote $z_0 = x_i$, $z_1 = x_{i+2^m}$, $z_2 = x_{i+2^{m+1}}$, $M_0 = V_{z_0}^{z_1}f$, $M_1 = V_{z_1}^{z_2}f$, $M = V_{z_0}^{z_2}f = M_0+M_1$, $\Delta = [z_0, z_2]$, $\eta = \left(\frac{\varepsilon_2}{2^{m+2}\|f'\|_{L_p(\Delta)}}\right)^{p'}$,

$\Delta_0 = [z_0, z_1+\eta]$, $\Delta_1 = [z_1-\eta, z_2]$. We need an estimate of the modulus of continuity of the function f on Δ:

$$\omega(f, \Delta; \delta)_C = \sup_{\substack{x', x'' \in \Delta \\ |x'-x''| \leqq \delta}} |f(x')-f(x'')|.$$

By Hölder's inequality we obtain

$$|f(x')-f(x'')| = \left| \int_{x'}^{x''} f'(x)\,dx \right| \leqq \|f'\|_{L_p[x', x'']} |x'-x''|^{1/p'}$$

for $x', x'' \in \Delta$. Consequently

$$\omega(f, \Delta; \delta)_C \leqq \|f'\|_{L_p(\Delta)} \delta^{1/p'} \quad \text{for} \quad \delta \geqq 0. \tag{7}$$

Let α_j $(j=0, 1)$ be the increasing linear function mapping the interval Δ_j onto $[z_j, z_{j+1}]$. Clearly

$$\|\alpha_j(x)-x\|_{C(\Delta_j)} = \eta, \quad j = 0, 1. \tag{8}$$

Put $p_0(x)=p_i^*(\alpha_0(x))$ and $p_1(x)=p_{i+2^m}^*(\alpha_1(x))$. By (4) we have

$$\|p_j-f(z_j)\|_{C(-\infty, \infty)} \leqq (V_{z_j}^{z_{j+1}} f)^{3/2} \varepsilon_2^{-1/2} = M_j^{3/2} \varepsilon_2^{-1/2}, \quad j = 0, 1. \tag{9}$$

Using (3), (7) and (8), we obtain

$$\|f-p_j\|_{C(\Delta_j \cap \Delta)} \leqq \|f(x)-f(\alpha_j(x))\|_{C(\Delta_j \cap \Delta)} + \|f(\alpha_j(x))-p_j(x)\|_{C(\Delta_j \cap \Delta)} \leqq$$

$$\leqq \omega(f, \Delta; \eta)_C + \varepsilon_1 + \varphi(m)\varepsilon_2 \leqq \|f'\|_{L_p(\Delta)} \eta^{1/p'} + \varepsilon_1 + \varphi(m)\varepsilon_2.$$

Hence

$$\|f-p_j\|_{C(\Delta_j \cap \Delta)} \leqq \varepsilon_1 + \left(\varphi(m) + \frac{1}{2^{m+1}}\right)\varepsilon_2, \quad j = 0, 1. \tag{10}$$

By the assumption of Lemma 3 and the definitions of p_0, p_1 it follows that the rational function p_0 is of degree $\leqq 2 \sum_{\nu=i}^{i+2^m-1} k_\nu + N(m, [z_0, z_1])$ and the function p_1 is of degree $\leqq 2 \sum_{\nu=i+2^m}^{i+2^{m+1}-1} k_\nu + N(m, [z_1, z_2])$.

Consider the rational function

$$q_i^*(x) = p_0(x) + \sigma(x-z_1)(p_1(x)-p_0(x)),$$

where σ is the rational function of Lemma 1 with $\alpha=\eta$, $\beta=|\Delta|$ and $\gamma=\dfrac{\varepsilon_2}{2^{m+1}M}$. The rational function σ is of degree

$$\leqq B \ln(e+2^{m+1}M\varepsilon_2^{-1}) \ln\left(e+\frac{|\Delta|}{\eta}\right) \leqq$$

$$\leqq Bp' \ln(e+2^{m+1}M\varepsilon_2^{-1}) \ln(e+2^{m+1}\|f'\|_{L_p(\Delta)}|\Delta|^{1/p'}\varepsilon_2^{-1}).$$

Since $M=V_{z_0}^{z_1} f=\|f'\|_{L(\Delta)} \leqq \|f'\|_{L_p(\Delta)} |\Delta|^{1/p'}$ then σ is of degree $\leqq Bp' \ln^2(e+2^{m+1}$.

$\cdot \|f'\|_{L_p(\Delta)}|\Delta|^{1/p'}\varepsilon_2^{-1})$. Then the rational function q_i^* is of degree

$$\leqq 2\sum_{\nu=i}^{i+2^m-1} k_\nu + N(m,[z_0,z_1]) + 2\sum_{\nu=i+2^m}^{i+2^{m+1}-1} k_\nu + N(m,[z_1,z_2]) +$$

$$+ Bp' \ln^2(e+2^{m+1}\|f'\|_{L_p(\Delta)}|\Delta|^{1/p'}\varepsilon_2^{-1}).$$

We shall estimate the degree of q_i^*, using the following inequalities:

$$\ln^2(e+x_1)+\ln^2(e+x_2) \leqq 2\ln^2\left(e+\frac{x_1+x_2}{2}\right) \quad (x_1, x_2 \geqq 0),$$

i.e. $\ln^2(e+x)$ is a convex function on $[0, \infty)$,

$$\sum_j x_j^{1/p} y_j^{1/p'} \leqq \left(\sum_j x_j\right)^{1/p}\left(\sum_j y_j\right)^{1/p'} \quad (x_j, y_j \geqq 0), \tag{11}$$

i.e. $x^{1/p}y^{1/p'}$ is a convex function on the set $\{(x, y): x, y \geqq 0\}$. Inequality (11) implies

$$\|f'\|_{L_p[z_0,z_1]}(z_1-z_0)^{1/p'} + \|f'\|_{L_p[z_1,z_2]}(z_2-z_1)^{1/p'} \leqq \|f'\|_{L_p[z_0,z_2]}(z_2-z_0)^{1/p'}.$$

We have

$$N(m,[z_0,z_1]) + N(m,[z_1,z_2]) + Bp'\ln^2(e+2^{m+1}\|f'\|_{L_p(\Delta)}|\Delta|^{1/p'}\varepsilon_2^{-1}) =$$

$$= Bp'\sum_{\nu=0}^{m} 2^\nu \left\{\ln^2\left(e+2^m 4^{-\nu}\|f'\|_{L_p[z_0,z_1]}(z_1-z_0)^{1/p'}\varepsilon_2^{-1}\right) + \right.$$

$$\left. + \ln^2\left(e+2^m 4^{-\nu}\|f'\|_{L_p[z_1,z_2]}(z_2-z_1)^{1/p'}\varepsilon_2^{-1}\right)\right\} + Bp'\ln^2(e+2^{m+1}\|f'\|_{L_p(\Delta)}|\Delta|^{1/p'}\varepsilon_2^{-1}) \leqq$$

$$\leqq Bp'\sum_{\nu=0}^{m} 2^{\nu+1}\ln^2\left(e+\frac{2^m 4^{-\nu}}{2}\left(\|f'\|_{L_p[z_0,z_1]}(z_1-z_0)^{1/p'} + \|f'\|_{L_p[z_1,z_2]}(z_2-z_1)^{1/p'}\right)\varepsilon_2^{-1}\right) +$$

$$+ Bp'\ln^2(e+2^{m+1}\|f'\|_{L_p(\Delta)}|\Delta|^{1/p'}\varepsilon_2^{-1}) \leqq$$

$$\leqq Bp'\sum_{\nu=0}^{m} 2^{\nu+1}\ln^2(e+2^{m+1}4^{-(\nu+1)}\|f'\|_{L_p(\Delta)}|\Delta|^{1/p'}\varepsilon_2^{-1}) +$$

$$+ Bp'\ln^2(e+2^{m+1}\|f'\|_{L_p(\Delta)}|\Delta|^{1/p'}\varepsilon_2^{-1}) =$$

$$= Bp'\sum_{\mu=0}^{m+1} 2^\mu \ln^2(e+2^{m+1}4^{-\mu}\|f'\|_{L_p(\Delta)}|\Delta|^{1/p'}\varepsilon_2^{-1}) = N(m+1,\Delta).$$

Consequently, the rational function q_i^* is of degree $\leqq 2\sum_{\nu=i}^{i+2^{m+1}-1} k_\nu + N(m+1, \Delta)$, as required.

Now we estimate $\|q_i^*-f(z_0)\|_{C(-\infty,\infty)}$. Using (9) and the inequality $M>\varepsilon_2$, we obtain for each $x\in(-\infty, \infty)$

$$|q_i^*(x)-f(z_0)| \leqq (1-\sigma(x-z_1))|p_0(x)-f(z_0)|+\sigma(x-z_1)|p_1(x)-f(z_0)| \leqq$$

$$\leqq (1-\sigma(x-z_1))\|p_0-f(z_0)\|_{C(-\infty,\infty)}+\sigma(x-z_1)\big(\|p_1-f(z_1)\|_{C(-\infty,\infty)}+|f(z_1)-f(z_0)|\big) \leqq$$

$$\leqq (1-\sigma(x-z_1))M_1^{3/2}\varepsilon_2^{-1/2}+\sigma(x-z_1)(M_2^{3/2}\varepsilon_2^{-1/2}+M_1) \leqq$$

$$\leqq (1-\sigma(x-z_1))M^{3/2}\varepsilon_2^{-1/2}+\sigma(x-z_1)M^{3/2}\varepsilon_2^{-1/2} = M^{3/2}\varepsilon_2^{-1/2}.$$

Hence

$$\|q_i^*-f(z_0)\|_{C(-\infty,\infty)} \leqq (V_{z_0}^{z_2}f)^{3/2}\varepsilon_2^{-1/2}.$$

The inequality (6) is proved.

It remains to estimate $\|f-q_i^*\|_{C(\Delta)}$. Using (9), (10) and the properties of the function σ from Lemma 1, we obtain:

i) If $x\in(-\infty, z_1-\eta]\cap\Delta$, then

$$|f(x)-q_i^*(x)| \leqq$$

$$\leqq |f(x)-p_0(x)|+\sigma(x-z_1)\big(\|p_1-f(z_1)\|_{C(-\infty,\infty)}+\|p_0-f(z_0)\|_{C(-\infty,\infty)}+|f(z_1)-f(z_0)|\big) \leqq$$

$$\leqq \varepsilon_1+\left(\varphi(m)+\frac{1}{2^{m+1}}\right)\varepsilon_2+\frac{(2M^{3/2}\varepsilon_2^{-1/2}+M)\varepsilon_2^{3/2}}{(2^{m+1})^{3/2}M^{3/2}} \leqq \varepsilon_1+\left(\varphi(m)+\frac{4}{2^{m+1}}\right)\varepsilon_2.$$

ii) If $x\in[z_1-\eta, z_1+\eta]\cap\Delta$, then

$$|f(x)-q_i^*(x)| \leqq (1-\sigma(x-z_1))|f(x)-p_0(x)|+\sigma(x-z_1)|f(x)-p_1(x)| \leqq$$

$$\leqq \varepsilon_1+\left(\varphi(m)+\frac{1}{2^{m+1}}\right)\varepsilon_2.$$

iii) If $x\in\Delta\cap[z_1+\eta, \infty)$, then

$$|f(x)-q_i^*(x)| \leqq |f(x)-p_1(x)|+(1-\sigma(x-z_1))\big(\|p_1-f(z_1)\|_{C(-\infty,\infty)}+$$

$$+\|p_0-f(z_0)\|_{C(-\infty,\infty)}+|f(z_1)-f(z_0)|\big) \leqq \varepsilon_1+\left(\varphi(m)+\frac{4}{2^{m+1}}\right)\varepsilon_2.$$

Consequently, we have

$$\|f-q_i^*\|_{C(\Delta)} \leqq \varepsilon_1+\left(\varphi(m)+\frac{4}{2^{m+1}}\right)\varepsilon_2,$$

i.e. the estimate (5) holds true. Lemma 3 is proved.

COMPLETION OF THE PROOF OF THEOREM 9. Starting from Lemma 2 and applying Lemma 3 s times, we obtain that there is a rational function q of degree $\leqq 2\sum_{\nu=0}^{2^s-1}k_\nu+N(s,[a,b])$ such that

$$\|f-q\|_{C[a,b]} \leqq \varepsilon_1+\left(1+\sum_{m=1}^{s}\frac{4}{2^m}\right)\varepsilon_2 < \varepsilon_1+5\varepsilon_2.$$

It remains to prove that

$$N(s,[a,b]) \leqq Dp'l\ln^2\left(e+\frac{\|f'\|_{L_p[a,b]}(b-a)^{1/p'}}{l\varepsilon_2}\right).$$

By the definition of $N(s,[a,b])$ we have

$$N(s,[a,b]) = Bp'\sum_{\nu=0}^{s}2^\nu\ln^2(e+2^s4^{-\nu}\|f'\|_{L_p[a,b]}(b-a)^{1/p'}\varepsilon_2^{-1}) \leqq$$

$$\leqq Bp'\sum_{\nu=0}^{s}2^\nu\ln^2\left(\left(e+\frac{\|f'\|_{L_p[a,b]}(b-a)^{1/p'}}{2^s\varepsilon_2}\right)4^{s-\nu}\right) \leqq$$

$$\leqq Bp'\sum_{\nu=0}^{s}2^\nu\ln^2\left(e+\frac{\|f'\|_{L_p[a,b]}(b-a)^{1/p'}}{2^s\varepsilon_2}\right)+$$

$$+2Bp'\sum_{\nu=0}^{s}2^\nu\ln\left(e+\frac{\|f'\|_{L_p[a,b]}(b-a)^{1/p'}}{2^s\varepsilon_2}\right)(\ln 4)\left(s+1+\frac{1}{\ln 2}-\nu\right)+$$

$$+Bp'\sum_{\nu=0}^{s}2^\nu(\ln 4)^2\left(s+1+\frac{2}{\ln 2}-\nu\right)^2.$$

Since the functions $2^x\left(s+1+\frac{1}{\ln 2}-x\right)$ and $2^x\left(s+1+\frac{2}{\ln 2}-x\right)^2$ are increasing on $[0,s+1]$ we have

$$\sum_{\nu=0}^{s}2^\nu\left(s+1+\frac{1}{\ln 2}-\nu\right) < \int_0^{s+1}2^x\left(s+1+\frac{1}{\ln 2}-x\right)dx < \frac{4}{(\ln 2)}2^s,$$

$$\sum_{\nu=0}^{s}2^\nu\left(s+1+\frac{2}{\ln 2}-\nu\right)^2 < \int_0^{s+1}2^x\left(s+1+\frac{2}{\ln 2}-x\right)dx < \frac{16}{(\ln 2)^3}2^s.$$

Therefore

$$N(s,[a,b]) \leqq Dp'2^s\ln^2\left(e+\frac{\|f'\|_{L_p[a,b]}(b-a)^{1/p'}}{2^s\varepsilon_2}\right) =$$

$$= Dp'l\ln^2\left(e+\frac{\|f'\|_{L_p[a,b]}(b-a)^{1/p'}}{l\varepsilon_2}\right).$$

Theorem 9 is proved.

Theorem 9 directly implies the following statement:

THEOREM C (V. A. Popov [4]). *Let f be absolutely continuous on $[a,b]$ and $f'\in L_p[a,b]$, $1<p\leqq\infty$. Then*

$$R_n(f)_C \leqq C_1p'\frac{\|f'\|_{L_p[a,b]}(b-a)^{1/p'}}{n} \quad \textit{for} \quad n\geqq 1.$$

PROOF. Let $n \geqq 1$. If $n \leqq Dp' \ln^2(e+1)$, where $D>1$ is the constant from Theorem 9, then

$$R_n(f)_C \leqq V_a^b f = \|f'\|_{L_p[a,b]} \leqq \frac{Dp' \ln^2(e+1)\|f'\|_{L_p[a,b]}(b-a)^{1/p'}}{n},$$

i.e. in this case Theorem C holds true.

Let $n > Dp' \ln^2(e+1)$. Choose l so that

$$Dp' l \ln^2(e+1) < n \leqq 2Dp' l \ln^2(e+1).$$

Clearly there is a step-function $\varphi \in S(0, l, [a, b])$ such that

$$\|f-\varphi\|_{C[a,b]} \leqq \frac{V_a^b f}{l} = \frac{\|f'\|_{L_1[a,b]}}{l} \leqq \frac{\|f'\|_{L_p[a,b]}(b-a)^{1/p'}}{l}.$$

By Theorem 9 with $\varepsilon_1=\varepsilon_2=\dfrac{\|f'\|_{L_p[a,b]}(b-a)^{1/p'}}{l}$ there is a rational function q of degree $\leqq Dp' l \ln^2(e+1) \leqq n$ such that

$$\|f-q\|_{C[a,b]} \leqq 6\frac{\|f'\|_{L_p[a,b]}(b-a)^{1/p'}}{l} \leqq \frac{12Dp' \ln^2(e+1)\|f'\|_{L_p[a,b]}(b-a)^{1/p'}}{n}.$$

Theorem C is proved.

THEOREM 10. *Let f be defined on $[a, b]$, $f' \in L_p[a, b]$, $1<p\leqq\infty$ and let $f', f'', \ldots, f^{(r)}$ $(r \geqq 1)$ be monotone on (a, b). Then*

$$R_n(f)_C \leqq C_1(p, r)\frac{\|f'\|_{L_p[a,b]}(b-a)^{1/p'}}{n^{r+1}} \quad \textit{for} \quad n \geqq 1,$$

where $C_1(p, r)=C_1(r)(p')^{4(r+1)}$.

PROOF. Suppose that $[a, b]=[0, 1]$, $\|f'\|_{L_p[0,1]}=1$, $1<p<\infty$ and $f^{(r)} \in C_{(0,1)}$. Clearly, if Theorem 10 is true in this case then it is true in the general case.

Denote $x_i=\dfrac{1}{2^i}$, $z_i=1-\dfrac{1}{2^i}$, $i=1, 2, \ldots$. We shall estimate $|f^{(r)}(x_i)|$ and $|f^{(r)}(z_i)|$, $i=1, 2, \ldots$. Since f' is monotone on $(0, 1)$ and $0<x_i\leqq\dfrac{1}{2}$ we have

$$1 = \int_0^1 |f'(x)|^p\,dx \geqq |f(x_i)|^p x_i$$

and therefore

$$|f'(x_i)| \leqq 2^{i/p}, \quad i = 1, 2, \ldots. \tag{11}$$

Similarly

$$|f'(z_i)| \leqq 2^{i/p}, \quad i = 1, 2, \ldots, \tag{12}$$

Since f'' is monotone on $(0, 1)$ the function f' is either convex or concave on $(0, 1)$. Then it is clear that

$$|f''(x_i)| \leqq \max\left\{\frac{|f'(x_i)-f'(x_{i+1})|}{x_i-x_{i+1}}, \frac{|f'(x_i)-f'(x_{i-1})|}{x_{i-1}-x_i}\right\} \quad (i \geqq 2),$$

$$|f''(x_1)| \leqq \max\left\{\frac{|f'(x_1)-f'(x_2)|}{x_1-x_2}, \frac{|f'(z_1)-f'(z_2)|}{z_2-z_1}\right\}.$$

Thus by (11) and (12) we have

$$|f''(x_i)| \leqq \frac{2^{(i+1)/p}+2^{i/p}}{2^{-i-1}} < 2^3 2^{i\left(1+\frac{1}{p}\right)} \quad (i \geqq 1). \tag{13}$$

Similarly

$$|f''(z_i)| \leqq 2^3 2^{i\left(1+\frac{1}{p}\right)} \quad (i \geqq 1). \tag{14}$$

Using (13) and (14) we obtain in a similar way

$$\begin{cases} |f'''(x_i)| \leqq 2^7 2^{i\left(2+\frac{1}{p}\right)}, \\ |f'''(z_i)| \leqq 2^7 2^{i\left(2+\frac{1}{p}\right)} \quad \text{for} \quad i \geqq 1, \\ \quad \vdots \\ |f^{(r)}(x_i)| \leqq 2^{\sum_{\nu=3}^{r+1}\nu} 2^{i\left(r-1+\frac{1}{p}\right)} < 2^{(r+1)^2} 2^{i\left(r-\frac{1}{p'}\right)}, \\ |f^{(r)}(z_i)| \leqq 2^{\sum_{\nu=3}^{r+1}\nu} 2^{i\left(r-1+\frac{1}{p}\right)} < 2^{(n+1)^2} 2^{i\left(r-\frac{1}{p'}\right)} \quad \text{for} \quad i \geqq 1. \end{cases} \tag{15}$$

Let $n \geqq 1$. Clearly, if $n \leqq A = D^2(r+1)^6(p')^4 e^{18}$ where $D > 1$ is the constant from Theorem 9 then

$$R_n(f)_C \leqq V_0^1 f = \|f'\|_{L_1[0,1]} \leqq \|f'\|_{L_p[0,1]} \leqq \frac{(D^2(r+1)^6(p')^4 e^{18})^{r+1}}{n^{r+1}}$$

and therefore

$$R_n(f)_C \leqq \frac{C_2(r)(p')^{4(r+1)}}{n^{r+1}} \quad (1 \leqq n \leqq A). \tag{16}$$

Let $n > A = e^{18}D^2(r+1)^6(p')^4$. Choose s to be an integer such that

$$\left(\frac{16r(r+1)p'}{n \ln 2}\right)^{(r+1)p'} \leqq \frac{1}{2^s} \leqq 2\left(\frac{16r(r+1)p'}{n \ln 2}\right)^{(r+1)p'}, \quad s \geqq 2.$$

Denote $\Delta_i = [x_{i+1}, x_i]$, $\Delta_i^* = [z_i, z_{i+1}]$, $i = 1, 2, \ldots, s$,

$$\Delta_{s+1} = \left[0, \frac{1}{2^{s+1}}\right], \quad \Delta_{s+1}^* = \left[1-\frac{1}{2^{s+1}}, 1\right].$$

Put

$$k_i = \left[\frac{n \ln 2}{16(r+1)p' 2^{\frac{i}{(r+1)p'}}}\right]^*, \quad \tilde{k}_i = k_i, \quad i = 1, 2, \ldots, s.$$

Further we shall use the following statement proved by V. A. Popov [14]: if f is defined on $[a, b]$ and $V_a^b f^{(r)} < \infty$, $r \geqq 1$, then

$$R_n(f)_C \leqq C(r) \frac{(V_a^b f^{(r)})(b-a)^r}{n^{r+1}} \quad \text{for} \quad n \geqq r. \tag{17}$$

From the choice of s and n it follows that $k_i = k_i^* \geqq r$. Then by (17) and (15) we have

$$R_{k_i}(f)_{C(\Delta_i)} \leqq C(r) \frac{(V_{\Delta_i} f^{(r)})|\Delta_i|^r}{k_i^{r+1}} \leqq C(r) \frac{(|f^{(r)}(x_i)| + |f^{(r)}(x_{i-1})|)|\Delta_i|^r}{k_i^{r+1}} \leqq$$

$$\leqq C(r) \frac{2 \cdot 2^{(r+1)^2} 2^{i\left(r-\frac{1}{p'}\right)} (2^{-i})^r}{\left(\frac{n \ln 2}{32(r+1)p' 2^{\frac{i}{(r+1)p'}}}\right)^{r+1}} = \frac{C_3(r)(p')^{r+1}}{n^{r+1}}, \quad i = 1, 2, \ldots, s.$$

Hence there is a rational function $q_i \in R_{k_i}$ such that

$$\|f - q_i\|_{C(\Delta_i)} \leqq \frac{C_3(r)(p')^{r+1}}{n^{r+1}}, \quad i = 1, 2, \ldots, s. \tag{18}$$

Similarly for every i, $1 \leqq i \leqq s$, there is a rational function $q_i^* \in R_{k_i^*}$ such that

$$\|f - q_i^*\|_{C(\Delta_i^*)} \leqq \frac{C_3(r)(p')^{r+1}}{n^{r+1}}. \tag{19}$$

Put $q_{s+1}(x) = f(x_{s+1})$ and $q_{s+1}^*(x) = f(z_{s+1})$. Then we have

$$\|f - q_{s+1}\|_{C(\Delta_{s+1})} \leqq V_{\Delta_{s+1}} f = \|f'\|_{L_1(\Delta_{s+1})} \leqq \|f'\|_{L_p(\Delta_{s+1})} |\Delta_{s+1}|^{1/p'} =$$

$$= \left(\frac{1}{2^{s+1}}\right)^{1/p'} \leqq \left(\frac{16r(r+1)p'}{n \ln 2}\right)^{r+1}, \quad q_{s+1} \in R_0$$

and therefore

$$\|f - q_{s+1}\|_{C(\Delta_{s+1})} \leqq \frac{C_4(r)(p')^{r+1}}{n^{r+1}}, \quad q_{s+1} \in R_0. \tag{20}$$

Similarly

$$\|f - q^*\|_{C(\Delta_{s+1}^*)} \leqq \frac{C_4(r)(p')^{r+1}}{n^{r+1}}, \quad q_{s+1}^* \in R_0. \tag{21}$$

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* $[x]$ denotes the integral part of x.

In addition

$$\sum_{i=1}^{s} k_i + \sum_{i=1}^{s} k_i^* \leqq 2 \sum_{i=1}^{s} \frac{n \ln 2}{16(r+1)p' 2^{\frac{i}{(r+1)p'}}} \leqq$$

$$\leqq \frac{n \ln 2}{8(r+1)p'} \sum_{i=0}^{\infty} \left(\frac{1}{2^{\frac{1}{(r+1)p'}}} \right)^i \leqq \frac{2n \ln 2}{8(r+1)p'(e^{\frac{\ln 2}{(r+1)p'}} - 1)} \leqq \frac{n}{4},$$

i.e.

$$\sum_{i=1}^{s} k_i + \sum_{i=1}^{s} k_i^* \leqq \frac{n}{4}. \tag{22}$$

By Theorem 9 with $\varepsilon_1 = \frac{(C_3(r)+C_4(r))(p')^{4(r+1)}}{n^{r+1}}$, $\varepsilon_2 = \frac{1}{n^{r+1}}$ and (18)—(21) there is a rational function q such that

$$\|f-q\|_{C[0,1]} \leqq \frac{(C_3(r)+C_4(r))(p')^{4(r+1)}}{n^{r+1}} + \frac{5}{n^{r+1}} \leqq \frac{C_5(r)(p')^{4(r+1)}}{n^{r+1}}$$

and q is of degree

$$\leqq 2 \sum_{i=1}^{s} k_i + 2 \sum_{i=1}^{s} k_i^* + Dp' 2s \ln^2 \left(e + \frac{1}{\frac{1}{n^{r+1}} \cdot 2s} \right) \leqq$$

$$\leqq \frac{n}{2} + \frac{2D(r+1)(p')^2 \ln n}{\ln 2} \ln^2 (e + n^{r+1}) \leqq$$

$$\leqq \frac{n}{2} + 16D(r+1)^3 (p')^2 \ln^3 n \leqq \frac{n}{2} + 16D(r+1)^3 (p')^2 \sqrt{n} \leqq n.$$

We used the inequality $\ln^3 n \leqq \sqrt{n}$ for $n \geqq e^{18}$, the assumption $n \geqq A$ and the choice of s.

Consequently,

$$R_n(f)_C \leqq \frac{C_5(r)(p')^{4(r+1)}}{n^{r+1}} \quad (n > A). \tag{23}$$

The estimates (16) and (23) prove Theorem 10.

THEOREM 11. *Let f be absolutely continuous on $[a, b]$, $f' \in L_p[a, b]$, $1 < p \leqq \infty$, and assume that there are points $\{x_i\}_{i=0}^{m}$, $\alpha = x_0 < x_1 < \ldots < x_m = b$, $m \geqq 1$ so that $f', f'', \ldots, f^{(r)}$ $(r \geqq 1)$ are monotone on (x_{i-1}, x_i) for $i = 1, 2, \ldots, m$. Then*

$$R_n(f)_C \leqq C_2(p, r) \frac{\|f'\|_{L_p[a,b]} (b-a)^{1/p'} m^r}{n^{r+1}} \quad \text{for} \quad n \geqq 1,$$

where $C_2(p, r) = C_2(r)(p')^{4(r+1)}$.

PROOF. Let $1 \leqq n \leqq A = e^{10} D^2 (r+1)^4 (p')^2 m$, where $D > 1$ is the constant from Theorem 9. Then by Theorem C

$$R_n(f)_C = C_1 p' \frac{\|f'\|_{L_p[a,b]} (b-a)^{1/p'}}{n} \leqq \frac{C_1 p' (D^2 (r+1)^2 (p')^2 e^{10} m)^r \|f'\|_{L_p[a,b]} (b-a)^{1/p'}}{n^{r+1}}$$

and therefore

$$R_n(f)_C \leqq C_3(r) (p')^{2(r+1)} \frac{\|f'\|_{L_p[a,b]} (b-a)^{1/p'} m^r}{n^{r+1}} \quad \text{for} \quad 1 \leqq n \leqq A. \tag{24}$$

If $\|f'\|_{L_p[a,b]} = 0$, then, obviously, Theorem 11 is true.
Let $\|f'\|_{L_p[a,b]} > 0$ and $n > A$. Put

$$k_i = \left[\frac{n}{8} \left(\frac{\|f'\|_{L_p(\Delta_i)} |\Delta_i|^{1/p'}}{\|f'\|_{L_p[a,b]} (b-a)^{1/p'} m^r} \right)^{1/(r+1)} + 1 \right], \quad \Delta_i = [x_{i-1}, x_i], \quad i = 1, 2, \ldots, m.$$

Using Theorem 10 for the interval Δ_i, $1 \leqq i \leqq m$, we obtain

$$R_{k_i}(f)_{C(\Delta_i)} \leqq C_1(p, r) \frac{\|f'\|_{L_p(\Delta_i)} |\Delta_i|^{1/p}}{k_i^{r+1}} \leqq$$

$$\leqq 8^{r+1} C_1(p, r) \frac{\|f'\|_{L_p[a,b]} (b-a)^{1/p'} m^r}{n^{r+1}}, \quad C_1(p, r) = C_1(r) (p')^{4(r+1)}$$

and therefore for each i, $1 \leqq i \leqq m$, there is a rational function $q_i \in R_{k_i}$ such that

$$\|f - q_i\|_{C(\Delta_i)} \leqq 8^{r+1} C_1(p, r) \frac{\|f'\|_{L_p[a,b]} (b-a)^{1/p'} m^r}{n^{r+1}}. \tag{25}$$

By Theorem 9 with

$$\varepsilon_1 = 8^{r+1} C_1(p, r) \frac{\|f'\|_{L_p[a,b]} (b-a)^{1/p'} m^r}{n^{r+1}}, \quad \varepsilon_2 = \frac{\|f'\|_{L_p[a,b]} (b-a)^{1/p'} m^r}{n^{r+1}}, \quad l = m$$

and (25) there is a rational function q such that

$$\|f - q\|_{C[a,b]} \leqq \left(8^{r+1} C_1(r) (p')^{4(r+1)} + 5\right) \frac{\|f'\|_{L_p[a,b]} (b-a)^{1/p'} m^r}{n^{r+1}} \tag{26}$$

and q is of degree $\leqq 2 \sum_{i=1}^{m} k_i + Dp'm \ln^2 \left(e + \left(\frac{n}{m} \right)^{r+1} \right)$.

Using the choice of k_i and inequalities of type (11), we obtain

$$2 \sum_{i=1}^{m} k_i \leqq 2 \sum_{i=1}^{m} \left\{ \frac{n}{8} \left(\frac{\|f'\|_{L_p(\Delta_i)} |\Delta_i|^{1/p'}}{\|f'\|_{L_p[a,b]} (b-a)^{1/p'} m^r} \right)^{1/(r+1)} + 1 \right\} \leqq$$

$$\leqq 2m + \frac{n}{(4\|f'\|_{L_p[a,b]} (b-a)^{1/p'})^{1/(r+1)} m^{r/(r+1)}} \sum_{i=1}^{m} (\|f'\|_{L_p(\Delta_i)} |\Delta_i|^{1/p'})^{1/(r+1)} \leqq$$

$$\leqq 2m + \frac{n}{4(\|f'\|_{L_p[a,b]} (b-a)^{1/p'})^{1/(r+1)} m^{r/(r+1)}} \left(\sum_{i=1}^{m} \|f'\|_{L_p(\Delta_i)} |\Delta_i|^{1/p'} \right)^{1/(r+1)} m^{r/(r+1)} \leqq$$

$$\leqq 2m + \frac{n}{4} \leqq \frac{n}{2}.$$

On the other hand, since $\ln^2 x \leqq \sqrt{x}$ for $x \geqq e^{10}$ and $n > A$ we have

$$Dp'm \ln^2 \left(e + \left(\frac{n}{m} \right)^{r+1} \right) \leqq Dp'm \ln^2 \left(\left(\frac{n}{m} \right)^{2(r+1)} \right) \leqq$$

$$\leqq 4D(r+1)^2 p'm \ln^2 \frac{n}{m} \leqq 4D(r+1)^2 p' \sqrt{mn} \leqq \frac{n}{2}.$$

Thus $q \in R_n$ and by (26) we obtain

$$R_n(f)_C \leqq \left(8^{r+1} C_1(r)(p')^{4(r+1)} + 5\right) \frac{\|f'\|_{L_p[a,b]} (b-a)^{1/p'} m^r}{n^{r+1}} \quad \text{for} \quad n > A. \tag{27}$$

Estimates (24) and (27) prove Theorem 11.

PROOF OF THEOREM 1. Let f be absolutely continuous on $[a, b]$, $f' \in L_p[a, b]$, $1 < p \leqq \infty$, $\alpha > 0$ and $k \geqq 0$. For each $i \geqq 0$ there is a function $\varphi_i \in S(, 2^i, [a, b])$ such that

$$\|f' - \varphi_i\|_{L_p[a,b]} = E^k_{2^i}(f')_{L_p}. \tag{28}$$

Put $\Phi_i(x) = \int \varphi_i(x)dx$, $i = 0, 1, \ldots$ and $r = [\alpha] + 2$.

Let $s \geqq 0$ be an integer. For each i, $1 \leqq i \leqq s$, we choose a natural number N_i such that

$$2^{\frac{s(\alpha+1)+i(r-\alpha)}{r+1}} \leqq N_i \leqq 2 \cdot 2^{\frac{s(\alpha+1)+i(r-\alpha)}{r+1}}. \tag{29}$$

Put $N_0 = k+1$, $N_{s+1} = 2^s$, $N = \sum_{i=0}^{s+1} N_i$. Clearly

$$R_N(f)_C \leqq R_{N_{s+1}}(f - \Phi_s)_C + R_{N_s}(\Phi_s - \Phi_{s-1})_C + R_{N_{s-1}}(\Phi_{s-1} - \Phi_{s-2})_C + \ldots$$
$$\ldots + R_{N_1}(\Phi_1 - \Phi_0)_C + R_{N_0}(\Phi_0)_C. \tag{30}$$

Consider the function $\psi_i = \Phi_i - \Phi_{i-1}$, $1 \leqq i \leqq s$. Obviously $\psi_i' = \varphi_i - \varphi_{i-1} \in S(k, 3 \cdot 2^{i-1}, [a, b])$ and therefore there are at most $m_i + 1 = 3 \cdot 2^{i-1}(1+1+2+\ldots+k)+1 < 3 \cdot 2^{i-1}(k+1)^2+1$ points $\{x_\nu^{(i)}\}_{\nu=0}^{m_i}$, $a = x_0^{(i)} < x_1^{(i)} < \ldots < x_{m_i}^{(i)} = b$, so that the functions $\psi_i', \psi_i'', \ldots, \psi_i^{(r)}$ are monotone on each interval $(x_{\nu-1}^{(i)}, x_\nu^{(i)})$. By Theorem 11 we have

$$R_{N_i}(\Phi_i - \Phi_{i-1})_C \leqq C_2(p, r) \frac{\|\varphi_i - \varphi_{i-1}\|_{L_p[a,b]} (b-a)^{1/p'} m_i^r}{N_i^{r+1}},$$

where $C_2(p, r) = C_2(r)(p')^{4(r+1)}$. From this estimate, using (28) and (29), we obtain

$$R_{N_i}(\Phi_i - \Phi_{i-1})_C \leqq C_3(p, r, k)(b-a)^{1/p'} \frac{E^k_{2^{i-1}}(f')_{L_p} 2^{(i-1)\alpha}}{2^{s(\alpha+1)}} \quad \text{for} \quad i = 1, 2, \ldots, s, \tag{31}$$

where $C_3(p, r, k) = C_3(r, k)(p')^{4(r+1)}$. By Theorem C and (25) we have

$$R_{N_{s+1}}(f - \Phi_s)_C \leqq C_1 p' \frac{\|f' - \varphi_s\|_{L_p[a,b]} (b-a)^{1/p'}}{2^s} = C_1 p' (b-a)^{1/p'} \frac{E^k_{2^s}(f')_{L_p} 2^{s\alpha}}{2^{s(\alpha+1)}}. \tag{32}$$

Obviously

(33) $$R_{N_0}(\Phi_0) = 0.$$

Combining (30) with (31)—(33), we obtain

$$R_N(f)_C \leqq C_4(r, k)(p')^{4(r+1)}(b-a)^{1/p'} \frac{\sum_{i=0}^{s} 2^{i\alpha} E_{2^i}^k(f')_{L_p}}{2^{s(\alpha+1)}},$$

where

$$N = \sum_{i=0}^{s+1} N_i \leqq k+1+\sum_{i=1}^{s} 2 \cdot 2^{\frac{s(\alpha+1)+i(r-\alpha)}{r+1}} + 2^s < \frac{9(r+1)(k+1)}{r-\alpha} 2^s.$$

Consequently, for each $s \geqq 0$ there is a natural number $N \leqq 9(\alpha+3)(k+1)2^s$ such that

(34) $$R_N(f)_C \leqq C_5(k, \alpha)(p')^{4(\alpha+3)}(b-a)^{1/p'} \frac{\sum_{i=0}^{s} 2^{i\alpha} E_{2^i}^k(f')_{L_p}}{2^{s(\alpha+1)}}.$$

Let $n \geqq k+1$. If $n \leqq A = 9(\alpha+3)(k+1)$, then

$$R_n(f)_C \leqq R_0(f-\Phi_0)_C + R_{k+1}(\Phi_0)_C = R_0(f-\Phi)_C \leqq V_a^b(f-\Phi_0) =$$

$$= \|f'-\varphi_0\|_{L_1[a,b]} \leqq (b-a)^{1/p'} \|f'-\varphi_0\|_{L_p[a,b]} = (b-a)^{1/p'} E_1^k(f')_{L_p} \leqq$$

$$\leqq (9(\alpha+3)(k+1))^{\alpha+1}(b-a)^{1/p'} \frac{\sum_{\nu=1}^{n} \nu^{\alpha-1} E_\nu^k(f')_{L_p}}{n^{\alpha+1}},$$

i.e.

(35) $$R_n(f)_C \leqq C_6(k, \alpha)(b-a)^{1/p'} \frac{\sum_{\nu=1}^{n} \nu^{\alpha-1} E_\nu^k(f')_{L_p}}{n^{\alpha+1}}$$

for $k+1 \leqq n \leqq A$.

Let $n > A$. Choose an integer s such that $A \cdot 2^s \leqq n < A \cdot 2^{s+1}$. By (34) we obtain

$$R_n(f)_C \leqq R_N(f)_C \leqq 2^\alpha C_5(k, \alpha)(p')^{4(\alpha+3)}(b-a)^{1/p'} \frac{E_1^k(f')_{L_p} + \sum_{i=1}^{s} 2^{(i-1)\alpha} E_{2^i}^k(f')_{L_p}}{2^{s(\alpha+1)}} \leqq$$

$$\leqq 2^\alpha (2A)^{\alpha+1} C_5(k, \alpha)(p')^{4(\alpha+3)}(b-a)^{1/p'} n^{-\alpha-1} \{E_1^k(f')_{L_p} +$$

$$+ \sum_{i=1}^{s} 2^{(i-1)(\alpha-1)} \left(\sum_{\nu=2^{i-1}+1}^{2^i} E_\nu^k(f')_{L_p} \right) \}.$$

Hence

(36) $$R_n(f)_C \leqq C_7(k, \alpha)(p')^{4(\alpha+3)}(b-a)^{1/p'} \frac{\sum_{\nu=1}^{n} \nu^{\alpha-1} E_\nu^k(f')_{L_p}}{n^{\alpha+1}} \quad (n > A).$$

The estimates (35) and (36) prove the first estimate in Theorem 1. The proof of the second estimate in Theorem 1 is similar to that of the first estimate and is omitted. Theorem 1 is proved.

The proofs of Theorems 2 and 4 are similar to that of Theorem 1 and are based on the following statements:

THEOREM 12. *Let f be absolutely continuous on $[a, b]$, $f' \in L_p[a, b]$, $1<p\leqq\infty$ and assume that there are points $\{x_i\}_{i=0}^m$, $a=x_0<x_1<\ldots<x_m=b$, $m\geqq 1$, so that $f', f'', \ldots, f^{(r)}$ $(r\geqq 1)$ are monotone on (x_{i-1}, x_i) for $i=1, 2, \ldots, m$. Then*

$$E_n^r(f)_C \leqq C_3(p, r)\frac{\|f'\|_{L_p[a,b]}(b-a)^{1/p'}m^r}{n^{r+1}} \quad \text{for} \quad n\geqq 1,$$

where $C_3(p, r)=C_3(r)(p')^{r+2}$.

THEOREM 13. *Let f be absolutely continuous on $[a, b]$ and assume that there are points $\{x_i\}_{i=0}^m$, $a=x_0<x_1<\ldots<x_m=b$, $m\geqq 1$, so that $f', f'', \ldots, f^{(r)}$ $(r\geqq 1)$ are monotone on (x_{i-1}, x_i) for $i=1, 2, \ldots, m$. Let $1\leqq p<\infty$. Then*

$$E_n^r(f)_{L_p} \leqq C_4(p, r)\frac{\|f'\|_{L_1[a,b]}(b-a)^{1/p}m^r}{n^{r+1}} \quad \text{for} \quad n\geqq 1,$$

where $C_4(p, r)=C_4(r)p^{r+1}$.

We shall prove Theorem 13. Theorem 12 can be proved in a similar way; see also the proof of Theorem 11.

PROOF OF THEOREM 13 IN THE CASE $m=1$. Let f be absolutely continuous on $[a, b], f', f'', \ldots, f^{(r)}$ $(r\geqq 1)$ monotone on (a, b) and $1\leqq p<\infty$. Consider the situation when $[a, b]=[0, 1]$, $\|f'\|_{L_1[0,1]}=1$ and $f^{(r)}\in C_{(0,1)}$. Clearly, if Theorem 13 with $m=1$ is true in this case then it is true with $m=1$ in the general case.

If $1\leqq n\leqq A=\dfrac{16^2(r+1)^2p}{(\ln 2)^2}$, then

$$E_n^r(f)_{L_p} \leqq \frac{V_0^1 f}{n} = \frac{\|f'\|_{L_1[0,1]}}{n} \leqq \frac{A^r}{n^{r+1}}$$

i.e.

$$E_n^r(f)_{L_p} \leqq \frac{C_5(r)p^r}{n^{r+1}} \quad (1\leqq n\leqq A). \tag{37}$$

Let $n>A$. We shall use the following statement [15]: If f is defined on $[a, b]$, $f^{(r-1)}$ $(r\geqq 1)$ is absolutely continuous on $[a, b]$ and $V_a^b f^{(r)}<\infty$, then

$$E_n^r(f)_C \leqq C(r)\frac{(V_a^b f^{(r)})(b-a)^r}{n^{r+1}} \quad (n\geqq 1). \tag{38}$$

Denote $x_i=\dfrac{1}{2^i}$, $z_i=1-\dfrac{1}{2^i}$, $i=1, 2, \ldots$. By the proof of Theorem 10 (see (15) with $p=1$) we have

$$\begin{cases} |f^{(r)}(x_i)| \leqq 2^{(r+1)^2}2^{ir}, \\ |f^{(r)}(z_i)| \leqq 2^{(r+1)^2}2^{ir}, \end{cases} \quad i=1, 2, \ldots. \tag{39}$$

Choose $s \geqq 0$ to be an integer so that

$$\left(\frac{p}{n}\right)^{(r+1)p} \leqq \frac{1}{2^s} \leqq 2\left(\frac{p}{n}\right)^{(r+1)p}. \tag{40}$$

Denote $\Delta_i=[x_{i+1}, x_i]$, $\Delta_i^*=[z_i, z_{i+1}]$, $i=1, 2, \ldots, s$,

$$\Delta_{s+1} = \left[0, \frac{1}{2^{s+1}}\right], \quad \Delta_{s+1}^* = \left[1-\frac{1}{2^{s+1}}, 1\right].$$

Put $k_i = \left[\dfrac{(\ln 2)\, n}{16(r+1)p2^{\frac{i}{2(r+1)p}}}\right]$ and $k_i^* = k_i$, $i = 1, 2, \ldots, s$. From the condition $n>A$ and (40) it follows that $k_i=k_i^*\geqq 1$, $i=1, 2, \ldots, s$. Using (38), (39) and the choice of k_i we obtain

$$E_{k_i}^r(f)_{L_p(\Delta_i)} \leqq |\Delta_i|^{1/p} E_{k_i}^r(f)_{C(\Delta_i)} \leqq C(r)\frac{(V_{\Delta_i} f^{(r)})|\Delta_i|^{r+\frac{1}{p}}}{k_i^{r+1}} \leqq$$

$$\leqq C(r)\frac{(|f^{(r)}(x_i)|+|f^{(r)}(x_{i+1})|)|\Delta_i|^{r+\frac{1}{p}}}{k_i^{r+1}} \leqq C(r)\frac{2^{(r+1)^2}(2^{ir}+2^{(i+1)r})2^{-(i+1)\left(r+\frac{1}{p}\right)}}{\left(\dfrac{(\ln 2)\, n}{32(r+1)p2^{\frac{i}{2(r+1)p}}}\right)^{r+1}}$$

and therefore

$$E_{k_i}^r(f)_{L_p(\Delta_i)} \leqq \frac{C_6(r)\, p^{r+1}}{n^{r+1}2^{i/(2p)}} \quad (i = 1, 2, \ldots, s). \tag{41}$$

Similarly

$$E_{k_i^*}^r(f)_{L_p(\Delta_i^*)} \leqq \frac{C_6(r)\, p^{r+1}}{n^{r+1}2^{i/(2p)}} \quad (i = 1, 2, \ldots, s). \tag{42}$$

Obviously

$$E_1^r(f)_{L_p(\Delta_{s+1})} \leqq |\Delta_{s+1}|^{1/p}\|f'\|_{L_1(\Delta_{s+1})} \leqq |\Delta_{s+1}|^{1/p} = \left(\frac{1}{2^{s+1}}\right)^{1/p} \leqq \frac{p^{r+1}}{n^{r+1}}, \tag{43}$$

$$E_1(f)_{L_p(\Delta_{s+1}^*)} \geqq \frac{p^{r+1}}{n^{r+1}}. \tag{44}$$

In addition we have

$$\sum_{i=1}^{s} k_i + \sum_{i=1}^{s} k_i^* + 2 \leqq 2\sum_{i=1}^{s} \frac{(\ln 2)\, n}{16(r+1)p2^{\frac{i}{2(r+1)p}}} + 2 \leqq \tag{45}$$

$$\leqq 2 + \frac{2(\ln 2)\, n}{16(r+1)p\left(1-2^{-\frac{1}{2(r+1)p}}\right)} \leqq n.$$

Then using (41)—(45) we obtain

$$E_n^2(f)_{L_p[0,1]} \leqq \left\{E_1^r(f)^p_{L_p(\Delta_{s+1})} + \sum_{i=1}^{s} E_{k_i}^r(f)^p_{L_p(\Delta_i)} + \sum_{i=1}^{s} E_{k_i^*}^r(f)^p_{L_p(\Delta_i^*)} + E_1^r(f)^p_{L_p(\Delta_{s+1}^*)}\right\}^{1/p} \leqq$$

$$\leqq \left\{2\left(\frac{p^{r+1}}{n^{r+1}}\right)^p + 2\sum_{i=1}^{s}\left(\frac{C_6(r)\,p^{r+1}}{n^{r+1}2^{i/(2p)}}\right)^p\right\}^{1/p} \leqq \frac{C_7(r)\,p^{r+1}}{n^{r+1}},$$

i.e.

$$E_n^r(f)_{L_p} \leqq \frac{C_7(r)\,p^{r+1}}{n^{r+1}} \quad (n > A). \tag{46}$$

The estimates (37) and (46) prove Theorem 13 in the case $m=1$.

PROOF OF THEOREM 13 IN THE GENERAL CASE. If $1 \leqq n \leqq 2m$ then

$$E_n^r(f)_{L_p} \leqq (b-a)^{1/p} E_n^r(f)_C \leqq \frac{(b-a)^{1/p} V_a^b f}{n} = \frac{\|f'\|_{L_1[a,b]}(b-a)^{1/p}}{n} \leqq$$

$$\leqq \frac{2^r \|f'\|_{L_1[a,b]}(b-a)^{1/p} m^r}{n^{r+1}}.$$

If $\|f'\|_{L_1[a,b]}=0$, then Theorem 13 is obviously true.
Let $n>2m$ and $\|f'\|_{L_1[a,b]}>0$. Put

$$k_i = \left[\frac{n}{2}\left(\frac{\|f'\|_{L_1(\Delta_i)}}{\|f'\|_{L_1[a,b]}m^r}\right)^{1/(r+1)} + 1\right], \quad \Delta_i = [x_{i-1}, x_i], \quad i = 1, 2, \ldots, m.$$

Then by Theorem 13 with $m=1$ we have

$$E_{k_i}^r(f)_{L_p(\Delta_i)} \leqq C_8(r)\,p^{r+1}\frac{\|f'\|_{L_1(\Delta_i)}|\Delta_i|^{1/p}}{k_i^{r+1}} \leqq \tag{47}$$

$$\leqq 2^r C_8(r)\,p^{r+1}\frac{\|f'\|_{L_1[a,b]}|\Delta_i|^{1/p}m^r}{n^{r+1}}, \quad i = 1, 2, \ldots, m.$$

Using an inequality of the type (11) we obtain

$$\sum_{i=1}^{m} k_i \leqq m + \frac{n}{2\|f'\|^{1/(r+1)}_{L_1[a,b]}\,m^{r/(r+1)}}\sum_{i=1}^{m}\|f'\|^{1/(r+1)}_{L_1(\Delta_i)} \leqq n. \tag{48}$$

By (47) and (48) we have

$$E_n^r(f)_{L_p} \leqq \left(\sum_{i=1}^{m} E_{k_i}^r(f)^p_{L_p(\Delta_i)}\right)^{1/p} \leqq \left\{\sum_{i=1}^{m}\left(2^r C_8(r)\,p^{r+1}\frac{\|f'\|_{L_1[a,b]}|\Delta_i|^{1/p}m^r}{n^{r+1}}\right)^{1/p}\right\} \leqq$$

$$\leqq C_9(r)\,p^{r+1}\frac{\|f'\|_{L_1[a,b]}(b-a)^{1/p}m^r}{n^{r+1}}.$$

Theorem 13 is proved.

PROOF OF THEOREM 3. We shall prove the second estimate in Theorem 3. If $1 \leqq n \leqq [\alpha]+3$ then

$$R_n(f)_{L_p} \leqq (b-a)^{1/p} R_1(f)_C \leqq (b-a)^{1/p} R_0(f')_{L_1} \leqq \tag{49}$$

$$\leqq (\alpha+3)^{\alpha+1}(b-a)^{1/p} \frac{\sum_{\nu=1}^{n} (\nu+1)^{\alpha-1} R_\nu(f')_{L_1}}{n^{\alpha+1}}.$$

Let $n > [\alpha]+3$. Denote $\alpha_1 = \alpha+1$, $p_1 = p$ and $q_1 = 2p$. By Theorem 7 with $p = p_1$, $q = q_1$, $k = [\alpha]+3$ and $\alpha = \alpha_1$ we have

$$R_n(f)_{L_p} \leqq C_1(\alpha) p^{6(\alpha+3)} (b-a)^{1/(2p)} \frac{\sum_{m=1}^{n} m^\alpha E_m^{[\alpha]+3}(f)_{L_{2p}}}{n^{\alpha+1}}. \tag{50}$$

By Theorem 4 with $r = [\alpha]+3$, $\alpha = \alpha_1$ and $p = q_1$ we have

$$E_m^{[\alpha]+3}(f)_{L_{2p}} \leqq C_2(\alpha) p^{\alpha+4} (b-a)^{1/(2p)} \frac{\sum_{\nu=0}^{m} (\nu+1)^\alpha R_\nu(f')_{L_1}}{m^{\alpha+2}} \quad (m \geqq 1). \tag{51}$$

Combining (50) with (51) we obtain

$$R_n(f)_{L_p} \leqq C_3(\alpha) p^{7\alpha+22} (b-a)^{1/p} n^{-\alpha-1} \left\{ \sum_{m=1}^{n} m^{-2} \left(\sum_{\nu=0}^{m} (\nu+1)^\alpha R_\nu(f')_{L_1} \right) \right\}.$$

Changing the order of the summation we get

$$R_n(f)_{L_p} \leqq C_3(\alpha) p^{7\alpha+22} (b-a)^{1/p} n^{-\alpha-1} \left\{ \sum_{\nu=1}^{n} (\nu+1)^\alpha \left(\sum_{m=\nu}^{n} m^{-2} \right) R_\nu(f')_{L_1} + \right.$$

$$\left. + \left(\sum_{m=1}^{n} m^{-2} \right) R_0(f')_{L_1} \right\}.$$

Hence

$$R_n(f)_{L_p} \leqq C_4(\alpha) p^{7\alpha+22} (b-a)^{1/p} \frac{\sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} R_\nu(f')_{L_1}}{n^{\alpha+1}} \quad (n \geqq A). \tag{52}$$

(49) and (52) prove the second estimate in Theorem 3. The first one can be proved in a similar way.

The proof of Theorem 5 is similar to that of Theorem 3 and is based on Theorems 1 and 4. Theorem 6 follows by Theorem 4. Corollary 1 follows by Theorems 1—6 immediately. Corollary 2 follows by Theorems 1, 3 and 5 and the statement: If $f \in L_p[a, b]$, $1 \leqq p \leqq \infty$ and $k \geqq 1$, then $E_n^{k-1}(f)_{L_p} = O\left(\omega_k\left(f; \frac{1}{n}\right)_{L_p}\right)$, see [13].

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A CATEGORICAL APPROACH TO LARGEST AND SMALLEST CONNECTEDNESSES AND DISCONNECTEDNESSES

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§0. Introduction

Motivated by the existence of largest and smallest proper connectednesses and disconnectednesses in the categories of topological spaces **(Top)** and undirected graphs with loops **(Graph)**, we look for necessary and sufficient conditions for the existence of such classes in a general categorical setting. This leads us to consider certain classes of simple objects.

We start by giving the necessary theory of radical classes (connectednesses) and semisimple classes (disconnectednesses). In the next section we define certain subclasses of the class of all simple objects. These classes are then used in the third section to obtain largest and smallest radical and semisimple classes. Lastly we generalize the results of Wiegandt [8] concerning the hereditariness and cohereditariness of connectednesses and disconnectednesses.

K will always be an arbitrary category and **E** and **M** will denote subclasses of the class of **K**-epimorphisms and **K**-monomorphisms respectively. Throughout this paper we shall assume that both **E** and **M** are closed under composition with isomorphisms and that the class of all **K**-identities is contained in $\mathbf{E} \cap \mathbf{M}$. Note that these two assumptions imply that the class of all **K**-isomorphisms is contained in $\mathbf{E} \cap \mathbf{M}$. $A \to B \in \mathbf{E}$ will mean there is a morphism $\alpha\colon A \to B$ in K with $\alpha \in \mathbf{E}$. $A \to B \in \mathbf{M}$ will have a similar meaning. Let **T** be the class of all trivial objects in **K** (cf. [2]), i.e. **T** is that class of objects which satisfies the following conditions:

(T1) If there exists a constant epimorphism $A \to B$, then $B \in \mathbf{T}$.

(T2) If there exists a constant monomorphism $B \to A$, then $B \in \mathbf{T}$.

(T3) If $B \in \mathbf{T}$, then every morphism $A \to B$ and $B \to C$ is constant.

§1. Radical theory

A class of objects **R** in **K** is an **(E, M)**-*radical class* if it satisfies the following two conditions:

(R1) If $A \in \mathbf{R}$ and $A \to B \in \mathbf{E}$, $B \notin \mathbf{T}$, then there exists an $I \to B \in \mathbf{M}$, $I \notin \mathbf{T}$, with $I \in \mathbf{R}$.

(R2) If for all $A \to B \in \mathbf{E}$, $B \notin \mathbf{T}$, there is an $I \to B \in \mathbf{M}$, $I \notin \mathbf{T}$ and $I \in \mathbf{R}$, then $A \in \mathbf{R}$ must hold.

If **R** only satisfies condition (R1), then **R** is called an **(E, M)**-*coregular* class.

A class of objects **S** in **K** is an **(E, M)**-*semisimple class* if it satisfies the following two conditions:

(S1) If $A \in \mathbf{S}$ and $B \to A \in \mathbf{M}$, $B \notin \mathbf{T}$, then there exists a $B \to I \in \mathbf{E}$, $I \notin \mathbf{T}$, with $I \in \mathbf{S}$.

(S2) If for all $B \to A \in \mathbf{M}$, $B \notin \mathbf{T}$, there is a $B \to I \in \mathbf{E}$, $I \notin \mathbf{T}$ with $I \in \mathbf{S}$, then $A \in \mathbf{S}$ must hold.

If **S** only satisfies condition (S1), we say **S** is an **(E, M)**-*regular class*.

A class of objects **B** is **M**-*hereditary* (**E**-*cohereditary*) if $A \to B \in \mathbf{M}$ ($B \to A \in \mathbf{E}$) and $B \in \mathbf{B}$ implies $A \in \mathbf{B}$ must hold. Obviously, if **B** is **M**-hereditary (**E**-cohereditary) then **B** is **(E, M)**-regular (**(E, M)**-coregular). We define two operators $\mathbf{R}_\mathbf{E}$ and $\mathbf{S}_\mathbf{M}$, acting on a class of objects **B**, as follows:

$$\mathbf{R}_\mathbf{E}\mathbf{B} = \{A \mid \text{there is no } A \to B \in \mathbf{E},\ B \notin \mathbf{T},\ \text{with } B \in \mathbf{B}\}$$

and

$$\mathbf{S}_\mathbf{M}\mathbf{B} = \{A \mid \text{there is no } B \to A \in \mathbf{M},\ B \notin \mathbf{T},\ \text{with } B \in \mathbf{B}\}.$$

Among others, the following can be proved for these operators where **B** and **B′** are classes of objects:

1.1. PROPOSITION. (1)(a) **B** *satisfies* (R1) *iff* $B \subseteq \mathbf{R}_\mathbf{E}\mathbf{S}_\mathbf{M}\mathbf{B}$ *and* **B** *satisfies* (R2) *iff* $\mathbf{R}_\mathbf{E}\mathbf{S}_\mathbf{M}\mathbf{B} \subseteq \mathbf{B}$. *Hence* **B** *is an* **(E, M)**-*radical class iff* $\mathbf{B} = \mathbf{R}_\mathbf{E}\mathbf{S}_\mathbf{M}\mathbf{B}$.

(b) *If* $\mathbf{B} \subseteq \mathbf{B}'$, *then* $\mathbf{R}_\mathbf{E}\mathbf{B}' \subseteq \mathbf{R}_\mathbf{E}\mathbf{B}$.

(c) *If* **E** *is closed under composition and* **B** *is an* **(E, M)**-*regular class, then* $\mathbf{R}_\mathbf{E}\mathbf{B}$ *is an* **(E, M)**-*radical class.*

(d) *If* **B** *is an* **(E, M)**-*semisimple class, then* $\mathbf{R}_\mathbf{E}\mathbf{B}$ *is an* **(E, M)**-*radical class.*

(e) $\mathbf{R}_\mathbf{E}\mathbf{T} = \mathrm{Ob}\,\mathbf{K}$ *(class of all the objects in* **K***) and* $\mathbf{R}_\mathbf{E}\,\mathrm{Ob}\,\mathbf{K} = \mathbf{T}$.

(f) $\mathbf{B} \cap \mathbf{R}_\mathbf{E}\mathbf{B} \subseteq \mathbf{T}$ *and the equality holds if* **B** *is an* **(E, M)**-*semisimple class.*

(2)(a) **B** *satisfies* (S1) *iff* $\mathbf{B} \subseteq \mathbf{S}_\mathbf{M}\mathbf{R}_\mathbf{E}\mathbf{B}$ *and* **B** *satisfies* (S2) *iff* $\mathbf{S}_\mathbf{M}\mathbf{R}_\mathbf{E}\mathbf{B} \subseteq \mathbf{B}$. *Hence* **B** *is an* **(E, M)**-*semisimple class iff* $\mathbf{B} = \mathbf{S}_\mathbf{M}\mathbf{R}_\mathbf{E}\mathbf{B}$.

(b) *If* $\mathbf{B} \subseteq \mathbf{B}'$, *then* $\mathbf{S}_\mathbf{M}\mathbf{B}' \subseteq \mathbf{S}_\mathbf{M}\mathbf{B}$.

(c) *If* **M** *is closed under composition and* **B** *is* **(E, M)**-*coregular, then* $\mathbf{S}_\mathbf{M}\mathbf{B}$ *is an* **(E, M)**-*semisimple class.*

(d) *If* **B** *is an* **(E, M)**-*radical class, then* $\mathbf{S}_\mathbf{M}\mathbf{B}$ *is an* **(E, M)**-*semisimple class.*

(e) $\mathbf{S}_\mathbf{M}\mathbf{T} = \mathrm{Ob}\,\mathbf{K}$ *and* $\mathbf{S}_\mathbf{M}\,\mathrm{Ob}\,\mathbf{K} = \mathbf{T}$.

(f) $\mathbf{B} \cap \mathbf{S}_\mathbf{M}\mathbf{B} \subseteq \mathbf{T}$ *and the equality holds if* **B** *is an* **(E, M)**-*radical class.*

As far as the hereditariness and cohereditariness of $\mathbf{R}_\mathbf{E}\mathbf{B}$ and $\mathbf{S}_\mathbf{M}\,\mathbf{B}$ are concerned, the following can be found in [2] or [8]:

1.2. PROPOSITION. 1(a) $\mathbf{R}_\mathbf{E}\mathbf{B}$ *is* **M**-*hereditary iff* **B** *satisfies condition* (H) *where* (H): *If* $I \to A \in \mathbf{M}$ *and* $I \to B \in \mathbf{E}$, $B \notin \mathbf{T}$ *with* $B \in \mathbf{B}$, *then there exists an* $A \to C \in \mathbf{E}$ *with* $C \in \mathbf{B}$, $C \notin \mathbf{T}$.

(b) $\mathbf{S}_\mathbf{M}\mathbf{B}$ *is* **M**-*hereditary iff* **B** *satisfies condition* (F) *where* (F): *If* $I \to A \in \mathbf{M}$ *and* $B \to I \in \mathbf{M}$, $B \in \mathbf{B}$ *with* $B \notin \mathbf{T}$, *then there exists* $C \to A \in \mathbf{M}$ *with* $C \in \mathbf{B}$ *and* $C \notin \mathbf{T}$.

2(a) $\mathbf{R}_\mathbf{E}\mathbf{B}$ *is* **E**-*cohereditary iff* **B** *satisfies condition* (F^*) *where* (F^*): *If* $A \to I \in \mathbf{E}$ *and* $I \to B \in \mathbf{E}$, $B \in \mathbf{B}$ *with* $B \notin \mathbf{T}$ *then there exists an* $A \to C \in \mathbf{E}$ *with* $C \in \mathbf{B}$ *and* $C \notin \mathbf{T}$.

(b) $\mathbf{S}_\mathbf{M}\mathbf{B}$ *is* **E**-*cohereditary iff* **B** *satisfies condition* (H^*) *where:* (H^*) *If* $A \to I \in \mathbf{E}$ *and* $B \to I \in \mathbf{M}$, $B \in \mathbf{B}$ *and* $B \notin \mathbf{T}$, *then there exists a* $C \to A \in \mathbf{M}$ *with* $C \in \mathbf{B}$ *and* $C \notin \mathbf{T}$.

Obviously if **M** is closed under composition, then condition (F) in 2.1(b) is trivially satisfied. Likewise, if **E** is closed under composition, then condition (F^*) in 2.2(a) is trivially satisfied.

1.3. PROPOSITION. *For each* $i \in I$, *I some index set, let* $\mathbf{A}_i$ *be a class of objects in* $\mathbf{K}$. *Then*

$$\mathbf{R}_E\{\bigcup_{i\in I} \mathbf{A}_i\} = \bigcap_{i\in I} \mathbf{R}_E\mathbf{A}_i \tag{1}$$

$$\mathbf{S}_M\{\bigcup_{i\in I} \mathbf{A}_i\} = \bigcap_{i\in I} \mathbf{S}_M\mathbf{A}_i. \tag{2}$$

1.4. PROPOSITION. *Suppose* $\mathbf{M}$ *is closed under composition. If there exists an* $(\mathbf{E}, \mathbf{M})$*-coregular class* $\mathbf{A} \not\subseteq \mathbf{T}$ *such that* $\mathbf{A} \subseteq \mathbf{R}$ *for all* $(\mathbf{E}, \mathbf{M})$*-radical classes* $\mathbf{R} \neq \mathbf{T}$, *then* $\mathbf{K}$ *has a largest proper (i.e.* $\neq \mathrm{Ob}\,\mathbf{K}$*)* $(\mathbf{E}, \mathbf{M})$*-semisimple class viz.* $\mathbf{S}_M\mathbf{A}$. *Hence* $\mathbf{R}_E\mathbf{S}_M\mathbf{A}$ *is the smallest nontrivial (i.e.* $\neq \mathbf{T}$*)* $(\mathbf{E}, \mathbf{M})$*-radical class in* $\mathbf{K}$.

PROOF. By 1.1.2(e) $\mathbf{S}_M\mathbf{A}$ is an $(\mathbf{E}, \mathbf{M})$-semisimple class. If $\mathbf{P} \neq \mathrm{Ob}\,\mathbf{K}$ is any $(\mathbf{E}, \mathbf{M})$-semisimple class in $\mathbf{K}$, then $\mathbf{A} \subseteq \mathbf{R}_E\mathbf{P}$ by our hypothesis and 1.1.1(d). Hence $\mathbf{P} = \mathbf{S}_M\mathbf{R}_E\mathbf{P} \subseteq \mathbf{S}_M\mathbf{A}$ by 1.1.2(a) and (b). Lastly, $\mathbf{S}_M\mathbf{A} \neq \mathrm{Ob}\,\mathbf{K}$ because $\mathbf{A} \not\subseteq \mathbf{T}$.

1.4*. PROPOSITION. *Suppose* $\mathbf{E}$ *is closed under composition. If there exists an* $(\mathbf{E}, \mathbf{M})$*-regular class* $\mathbf{A} \not\subseteq \mathbf{T}$ *such that* $\mathbf{A} \subseteq \mathbf{S}$ *for all* $(\mathbf{E}, \mathbf{M})$*-semisimple classes* $\mathbf{S} \neq \mathbf{T}$, *then* $\mathbf{K}$ *has a largest proper* $(\mathbf{E}, \mathbf{M})$*-radical class viz.* $\mathbf{R}_E\mathbf{A}$. *Hence* $\mathbf{S}_M\mathbf{R}_E\mathbf{A}$ *is the smallest non-trivial* $(\mathbf{E}, \mathbf{M})$*-semisimple class in* $\mathbf{K}$.

1.5. PROPOSITION. *Suppose* $\mathbf{K}$ *has a largest proper* $(\mathbf{E}, \mathbf{M})$*-semisimple class* $\mathbf{P}$. *Then there exists an* $(\mathbf{E}, \mathbf{M})$*-coregular class* $\mathbf{A} \not\subseteq \mathbf{T}$ *such that* $\mathbf{A} \subseteq \mathbf{R}$ *for all* $(\mathbf{E}, \mathbf{M})$*-radical classes* $\mathbf{A} \neq \mathbf{T}$.

PROOF. By 1.1.2(a) the equality $\mathbf{P} = \mathbf{S}_M\mathbf{R}_E\mathbf{P}$ holds. Let $\mathbf{A} = \mathbf{R}_E\mathbf{P}$. Then $\mathbf{A}$ is $(\mathbf{E}, \mathbf{M})$-coregular, $\mathbf{A} \not\subseteq \mathbf{T}$ and $\mathbf{A} \subseteq \mathbf{R}$ for all $(\mathbf{E}, \mathbf{M})$-radical classes $\mathbf{R} \neq \mathbf{T}$.

1.5*. PROPOSITION. *Suppose* $\mathbf{K}$ *has a largest proper* $(\mathbf{E}, \mathbf{M})$*-radical class* $\mathbf{R}$. *Then there exists an* $(\mathbf{E}, \mathbf{M})$*-regular class* $\mathbf{A} \not\subseteq \mathbf{T}$ *such that* $\mathbf{A} \subseteq \mathbf{S}$ *for all* $(\mathbf{E}, \mathbf{M})$*-semisimple classes* $\mathbf{S} \neq \mathbf{T}$.

Combining 2.4 and 2.5 and 2.4* and 2.5* we have:

1.6. THEOREM. *Suppose* $\mathbf{M}$ *is closed under composition. Then* $\mathbf{K}$ *has a largest proper* $(\mathbf{E}, \mathbf{M})$*-semisimple class (and thus also a smallest proper* $(\mathbf{E}, \mathbf{M})$*-radical class) iff there exists an* $(\mathbf{E}, \mathbf{M})$*-coregular class* $\mathbf{A} \not\subseteq \mathbf{T}$ *such that* $\mathbf{A} \subseteq \mathbf{R}$ *for all* $(\mathbf{E}, \mathbf{M})$*-radical classes* $\mathbf{R} \neq \mathbf{T}$.

1.6*. THEOREM. *Suppose* $\mathbf{E}$ *is closed in* $\mathbf{K}$. *Then* $\mathbf{K}$ *has a largest proper* $(\mathbf{E}, \mathbf{M})$*-radical class (and thus also a smallest proper* $(\mathbf{E}, \mathbf{M})$*-semisimple class) iff there exists an* $(\mathbf{E}, \mathbf{M})$*-regular class* $\mathbf{A} \not\subseteq \mathbf{T}$ *such that* $\mathbf{A} \subseteq \mathbf{S}$ *for all* $(\mathbf{E}, \mathbf{M})$*-semisimple classes* $\mathbf{S} \neq \mathbf{T}$.

Whenever we apply our theory in concrete categories, we will always let $\mathbf{E}$ be the class of all normal epimorphisms in the categories of associative rings **(Rng)**, alternative rings **(Arng)**, groups **(Grp)** and R-Modules **(R-Mod)** and the class of epimorphisms in the categories **Top** and **Graph**. $\mathbf{M}$ will be the class of all normal monomorphisms in **Rng**, **Arng**, **Grp** and **R-Mod** and the class of extremal monomorphisms in **Top** and **Graph**. In the algebraic categories mentioned above, $\mathbf{T}$ consists of all the one element algebras, in **Top** $\mathbf{T}$ consists of all the trivial topological space (i.e. those spaces with at most one point) and in **Graph** $\mathbf{T}$ consists of all the one vertex graphs.

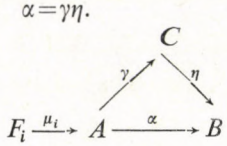
In **Rng**, **Arng**, **Grp** and **R-Mod**, the (**E**, **M**)-radical theory yields the usual Kurosh—Amitsur radical theory. If $\overline{\mathbf{M}}$ is the class of all accessible monomorphisms in these categories, it is well known that the (**E**, **M**)-radical classes coincide with the (**E**, $\overline{\mathbf{M}}$)-radical classes. This coincidence also holds for (**E**, **M**)-semisimple classes and (**E**, $\overline{\mathbf{M}}$)-semisimple classes. The benefit of this characterization being the fact that $\overline{\mathbf{M}}$ is closed under composition (transitive) while **M** is not (except in **R-Mod**). Because there is no largest proper semisimple class in the categories **Rng**, **Arng** and **Grp**, there does not exist an (**E**, $\overline{\mathbf{M}}$)-coregular class $\mathbf{A} \subsetneqq \mathbf{T}$ such that **A** is contained in every proper radical class (cf. 1.6). Hence there is no (**E**, **M**)-coregular (or **E**-cohereditary) class **A** with the above-mentioned properties in these categories. Likewise, because there is no largest proper radical class in these categories, there does not exist an (**E**, **M**)-regular (or **M**-hereditary) class $\mathbf{A} \subsetneqq \mathbf{T}$ such that $\mathbf{A} \subseteq \mathbf{S}$ for all semisimple classes $\mathbf{S} \neq \mathbf{T}$ (cf. 1.6*).

In both the categories **Top** and **Graph**, we have a largest proper connectedness (radical class) and a largest proper disconnectedness (semisimple class). To see what they are, we first need the following:

§2. Partitions of the class of simple objects

The concepts *fiber*, **K** *has fibers*, *cokernels*, **K** *has cokernels*, *chained sink* and **K** *is right precise*, has been defined in [6]. For their properties see [6] and [7].

Let $\alpha\colon A \to B$ be any morphism in **K** and let $(F_i \xrightarrow{\mu_i} A)_I$ be the sink of all the fibers of α. Let $\alpha : A \to C$ be the cokernel of this sink. By definition of a cokernel there exists a unique $\eta\colon C \to B$ such that $\alpha = \gamma\eta$.



2.1. DEFINITION. **K** is a *strong right precise* category if for every morphism α in **K**, the η, as constructed above, is a monomorphism.

Obviously, if **K** is strong right precise, then it is also right precise. We will sometimes require the following conditions:

(A1) **K** has fibers.

(A2) **K** has cokernels.

(A3) **K** is a strong right precise category.

(A4) Every epimorphism in **K** is chained (i.e. if $\alpha\beta$ is constant where α is an epimorphism, then β must be constant).

(A5) For all $A \in \mathrm{Ob}\,\mathbf{K}$, there exist morphisms $A \to T$ and $T' \to A$ for some T and T' in **T**.

If **K** satisfies (A1), (A3) and (A4), then every morphism has an (extremal epi, mono)-factorization and $\alpha\colon A \to B$ is a monomorphism iff whenever $\mu\colon F \to A$ is a fiber of α, then $F \in \mathbf{T}$ (or equivalently, μ a constant morphism) must hold.

2.2. DEFINITION. An object A in $\mathbf{K}$ is $\mathbf{M}$-*simple* if $A \notin \mathbf{T}$ and whenever $\mu\colon I \to A \in \mathbf{M}$, $I \notin \mathbf{T}$, then $(I, \mu) = (A, 1_A)$ must hold.

By $\mathbf{S}$ we shall denote the class of all $\mathbf{M}$-simple objects in $\mathbf{K}$.

2.3. PROPERTIES OF $\mathbf{S}$. (1) $\mathbf{S}$ is an abstract class (i.e. if $A \in \mathbf{S}$ and $\alpha\colon A \to B$ is an isomorphism, the $B \in \mathbf{S}$ must hold).

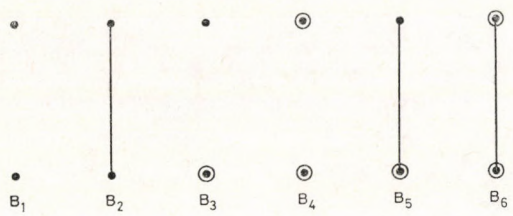
(2) If $\mathbf{R}$ is an $(\mathbf{E}, \mathbf{M})$-radical class, then every $\mathbf{M}$-simple object must declare itself with respect to $\mathbf{R}$, i.e., if A is $\mathbf{M}$-simple, then either $A \in \mathbf{R}$ or $A \in \mathbf{S_M R}$ must hold.

(3) $\mathbf{S}'$ is $(\mathbf{E}, \mathbf{M})$-regular for any $\mathbf{S}' \subseteq \mathbf{S}$.

(4) If $\mathbf{K}$ satisfies conditions (A1), (A2), (A3) and (A5) and $\mathbf{M}$ is the class of all fibers in $\mathbf{K}$, then $\mathbf{A}$ is $\mathbf{M}$-simple iff whenever $\alpha\colon A \to B$ exists, then α must be constant or a monomorphism. (This condition is the definition of a simple object by Klein [5] in a pointed category.)

The $\mathbf{M}$-simple objects in **Rng** and **Arng**, **Grp** and **R-Mod** are the same as the $\overline{\mathbf{M}}$-simple objects which is the simple rings, simple groups and simple $\mathbf{R}$-modules respectively.

In **Top** we have three non-homeomorphic $\mathbf{M}$-simple objects viz the three two point spaces. By D_2, I_2 and C_2 we shall denote the two point discrete space, the two point indiscrete space and the two point connected space (Sierpinski space), respectively. The $\mathbf{M}$-simple objects in **Graph** are the two vertex graphs. The six non-isomorphic two vertex graphs will be denoted by



2.4. DEFINITION. An object A in $\mathbf{K}$ is $\mathbf{M}$-*simply indiscrete* if

(1) $A \in \mathbf{S}$.

(2) For all $X \in \mathrm{Ob}\,\mathbf{K}$, $X \notin \mathbf{T}$, there exists an epimorphism $X \to A$.

(3) Whenever $\alpha\colon A \to B$, $B \in \mathbf{S}$, exists, then α is either constant or an isomorphism.

By $\mathbf{I}_2$ we shall denote the class of all $\mathbf{M}$-simply indiscrete objects.

Among others, the following can be proved for the class $\mathbf{I}_2$:

(1) $\mathbf{I}_2$ is an abstract class.

(2) Any two objects in $\mathbf{I}_2$ are isomorphic.

(3) If $A \in \mathbf{I}_2$ and $\beta\colon A \to B$ and $\alpha\colon B \to A$ exist with both α and β non-constant and α monomorphism, then both α and β are isomorphisms.

(4) If $\mathbf{K}$ satisfies condition (A4) and $A \in \mathbf{I}_2$ with $\alpha\colon A \to B$ and $\beta\colon B \to A$ both non-constant morphisms, then α and β must be isomorphisms.

(5) If $\mathbf{K}$ satisfies condition (A4) and $A \in \mathbf{I}_2$ with $\beta\colon A \to B$ a non-constant epimorphism, then $B \in \mathbf{I}_2$ must hold. Hence $\mathbf{I}_2 \cup \mathbf{T}$ is $\mathbf{E}$-cohereditary and thus also $(\mathbf{E}, \mathbf{M})$-coregular.

(6) $\mathbf{I}_2 \cup \mathbf{T}$ is $\mathbf{M}$-hereditary and thus also $(\mathbf{E}, \mathbf{M})$-regular.

2.5 DEFINITION. An object A in $\mathbf{K}$ is $\mathbf{M}$-*simply discrete* if

(1) $A \in \mathbf{S}$.

(2) For all $X \in \mathrm{Ob}\,\mathbf{K}$, $X \notin \mathbf{T}$, there exists a monomorphism $A \to X$.

(3) Whenever $\alpha\colon B \to A$, $B \in \mathbf{S}$, exists, then α is an isomorphism or α must be constant.

The class of all $\mathbf{M}$-simply discrete objects will be denoted by $\mathbf{D}_2$.

Some of the properties of the class $\mathbf{D}_2$ are listed below:

(1) $\mathbf{D}_2$ is an abstract class.

(2) Any two objects in $\mathbf{D}_2$ are isomorphic.

(3) If $A \in \mathbf{D}_2$ and both $\beta\colon A \to B$ and $\alpha\colon B \to A$ are non-constant with α a monomorphism, then both α and β must be isomorphisms.

(4) If $\mathbf{K}$ satisfies condition (A4) and both $\alpha\colon A \to B$ and $\beta\colon B \to A$ are non-constant with α an epimorphism and $A \in \mathbf{D}_2$, then both α and β must be isomorphisms.

(5) Let $A \in \mathbf{D}_2$ and $\beta\colon B \to A$ a non-constant monomorphism. Then $B \in \mathbf{D}_2$ must hold. Hence $\mathbf{D}_2 \cup \mathbf{T}$ is Mono $\mathbf{K}$ — hereditary and, in particular, thus also $(\mathbf{E}, \mathbf{M})$-regular.

(6) If $\mathbf{I}_2 \cap \mathbf{D}_2 \neq \emptyset$, then $\mathbf{I}_2 = \mathbf{D}_2$. Furthermore, if $\mathbf{I}_2 = \mathbf{D}_2 \neq \emptyset$, then $\mathbf{I}_2 = \mathbf{D}_2 = \mathbf{S}$. Thus, if $\mathbf{I}_2 \cap \mathbf{D}_2 \neq \emptyset$, then $\mathbf{I}_2 = \mathbf{D}_2 = \mathbf{S}$.

We need two more subclasses of $\mathbf{S}$. They are

$$\mathbf{C}_2 = \{A \in \mathbf{S} - (\mathbf{I}_2 \cup \mathbf{D}_2) \mid \text{if } B \text{ is any object, } I \to B \in \mathbf{M} \text{ and if there is an epimorphism } I \to A, \text{ then there exists an epimorphism } B \to A\}$$

and $\mathbf{B}_2 = (\mathbf{S} - \mathbf{I}_2) \cap \mathbf{R}_{\mathrm{Epi}}\mathbf{C}_2$ where Epi is the class of all epimorphisms in $\mathbf{K}$.

Obviously $\mathbf{C}_2 \cap \mathbf{D}_2 = \mathbf{C}_2 \cap \mathbf{I}_2 = \mathbf{B}_2 \cap \mathbf{I}_2 = \mathbf{B}_2 \cap \mathbf{C}_2 = \emptyset$. Both $\mathbf{C}_2$ and $\mathbf{B}_2$ are abstract classes. If $\mathbf{K}$ has an $(\mathbf{E}, \mathbf{M})$-factorization structure and $\mathbf{C}_2 \neq \emptyset$, then $\mathbf{B}_2 \cap \mathbf{D}_2 = \emptyset$ and if $\mathbf{C}_2 = \mathbf{S} - (\mathbf{I}_2 \cup \mathbf{D}_2)$, then $\mathbf{B}_2 = \emptyset$.

Furthermore, $\mathbf{I}_2 \subseteq \mathbf{R}_{\mathrm{Epi}}\mathbf{C}_2$ which yields $\mathbf{I}_2 \cup \mathbf{B}_2 = \mathbf{S} \cap \mathbf{R}_{\mathrm{Epi}}\mathbf{C}_2$. Hence $\mathbf{B}_2 = \emptyset$ iff $\mathbf{I}_2 = \mathbf{S} \cap \mathbf{R}_{\mathrm{Epi}}\mathbf{C}_2$.

In **Rng**, **Arng** and **Grp**, the class $\mathbf{D}_2$ is empty and the existence of $\mathbf{M}$-simply indiscrete objects in these categories seems highly unlikely. In **Top** we have $\mathbf{I}_2 = \{I_2\}$, $\mathbf{D}_2 = \{D_2\}$, $\mathbf{C}_2 = \{C_2\}$ and $\mathbf{B}_2 = \emptyset$. In **Graph**, $\mathbf{I}_2 = \{B_6\}$, $\mathbf{D}_2 = \{B_1\}$, $\mathbf{C}_2 = \{B_5\}$ and $\mathbf{B}_2 = \{B_4\}$.

An apparent discrepancy concerning the class $\mathbf{I}_2$ (and also the class $\mathbf{D}_2$) seems to exist between the categories **Top** and **Graph**. In **Top** $\mathbf{I}_2$ consists of the two point space with the least possible structure while as in **Graph** $\mathbf{I}_2$ consists of the two vertex graph with the most possible structure. It has been pointed out to the author by J. Heidema that, in a categorical sense, it seems more natural to describe a topological space in terms of filter (or net) convergence. In such a case, a function between two topological spaces has to preserve convergence to make it continuous (i.e. to be a morphism in the category **Top**). This approach corresponds with most other categories where a function has to preserve structure in order to be a morphism in that specific category. By doing this, i.e. describing the topology in terms of filter (or net) convergence, the indiscrete space $\mathbf{I}_2$ is maximally rich with structure which annihilates the above discrepancy between **Top** and **Graph**.

§3. Applications to radical theory

In this and the next section we have to impose a factorization structure on **K**. We are going to assume that **K** is an (**E**, **M**)-category (cf. [4]). Hence both **E** and **M** are closed under composition and each morphism $\alpha: A \to B$ has a unique (up to isomorphism) factorization as $A \xrightarrow{\alpha} B = A \xrightarrow{\tau} C \xrightarrow{\nu} B$ with $\tau \in \mathbf{E}$ and $\nu \in \mathbf{M}$.

Neither of the categories **Rng**, **Arng** and **Grp** has a (normal epi, normal mono)-factorization structure. Hence, the theory developed in this and the next section, cannot be applied to the usual Kurosh—Amitsur radical theory in these categories.

By a simple object we shall always mean an **M**-simple object. A last condition we sometimes need is the existence of inverse images. More specifically

(A6) **K** has inverse (**E**, **M**)-images, i.e. if $\alpha: A \to B \in \mathbf{E}$ and $\nu: M \to B \in \mathbf{M}$, then there exists a $\varrho: P \to A \in \mathbf{M}$ and a $\tau: P \to M \in \mathbf{E}$ such that $\tau\nu$ is the (**E**, **M**)-factorization of $\varrho\alpha$.

$$\begin{array}{ccc} P & \xrightarrow{\tau} & M \\ {\scriptstyle \varrho}\downarrow & & \downarrow{\scriptstyle \nu} \\ A & \xrightarrow[\alpha]{} & B \end{array}$$

3.1. PROPOSITION. *Every epimorphism in* **K** *is chained.*

PROOF. Let $\alpha: A \to B$ be an epimorphism with $\beta: B \to C$ such that $\alpha\beta$ is constant. Consider the (**E**, **M**)-factorization of β, say $B \xrightarrow{\beta} C = B \xrightarrow{\tau} D \xrightarrow{\nu} C$. Hence $\alpha\tau\nu = \alpha\beta$ which is constant and because ν is a monomorphism, $\alpha\tau$ is a constant epimorphism which yields $D \in \mathbf{T}$. Hence $\beta = \tau\nu$ is constant.

Because of the (**E**, **M**)-factorization structure on **K**, whenever $\alpha: A \to B$ is a non-constant morphism with B a simple object then $\alpha \in \mathbf{E}$ must hold. Hence, $\mathbf{R}_{\mathrm{Epi}}\mathbf{C}_2 = \mathbf{R}_{\mathbf{E}}\mathbf{C}_2$, $\mathbf{C}_2 = \{A \in \mathbf{S} - (\mathbf{I}_2 \cup \mathbf{D}_2) |$ whenever $I \to A \in \mathbf{E}$ where $I \to B \in \mathbf{M}$ then there is an $B \to A \in \mathbf{E}\}$ and if $\mathbf{I}_2 \neq \emptyset$, then $\mathbf{R}_{\mathbf{E}}\mathbf{I}_2 = \mathbf{T}$ (or $\mathrm{Ob}\,\mathbf{K} = \mathbf{S}_{\mathbf{M}}\mathbf{R}_{\mathbf{E}}\mathbf{I}_2$).

3.2. PROPOSITION. *If* **K** *satisfies condition* (A6), *then* $\mathbf{S} \cup \mathbf{T}$ *is* **E**-*cohereditary*.

PROOF. Let $\alpha: A \to B \in \mathbf{E}$ with $A \in \mathbf{S} \cup \mathbf{T}$. If $A \in \mathbf{T}$ or if α is constant, then $B \in \mathbf{T}$ follows. Suppose thus $A \in \mathbf{S}$ and α is non-constant. Let $\nu: M \to B \in \mathbf{M}$ with inverse image $\varrho: P \to A \in \mathbf{M}$. Thus there exists a $\tau: P \to M \in \mathbf{E}$ such that $\tau\nu$ is the (**E**, **M**)-factorization of $\varrho\alpha$. Because $A \in \mathbf{S}$, either ϱ is constant or ϱ is an isomorphism. If ϱ is constant, then $M \in \mathbf{T}$ must hold. On the other hand, if ϱ is an isomorphism, then $\tau\nu = \varrho\alpha \in \mathbf{E}$ which implies $\nu \in \mathbf{E} \cap \mathbf{M}$.

Thus ν an isomorphism follows.

3.3. THEOREM. *If* $\mathbf{I}_2 \neq \emptyset$, *then* **K** *has a largest proper* (**E**, **M**)-*semisimple class viz* $\mathbf{S}_{\mathbf{M}}\mathbf{I}_2$ *and a smallest proper* (**E**, **M**)-*radical class viz* $\mathbf{R}_{\mathbf{E}}\mathbf{S}_{\mathbf{M}}\mathbf{I}_2$.

PROOF. By Theorem 1.6 it is sufficient to establish the existence of an (**E**, **M**)-coregular class $\mathbf{A} \not\subseteq \mathbf{T}$ which is contained in every (**E**, **M**)-radical class $\mathbf{R} \neq \mathbf{T}$. Let $\mathbf{A} = \mathbf{I}_2 \cup \mathbf{T}$. Then $\mathbf{A} \not\subseteq \mathbf{T}$ because $\mathbf{I}_2 \neq \emptyset$. Furthermore **A** is (**E**, **M**)-coregular as we have seen in the previous section. Let $\mathbf{R} \neq \mathbf{T}$ be an (**E**, **M**)-radical class. Let $B \in \mathbf{R}$, $B \notin \mathbf{T}$ and let $A \in \mathbf{A}$. If $A \in \mathbf{T}$, then obviously $A \in \mathbf{R}$. If $A \in \mathbf{I}_2$, then, by defini-

tion, there is an $\alpha: B \to A \in \mathbf{E}$. Because $\mathbf{R}$ is cohereditary, $A \in \mathbf{R}$ follows. Hence $\mathbf{A} \subseteq \mathbf{R}$. Lastly, $\mathbf{S_M A} = \mathbf{S_M}(\mathbf{I_2} \cup \mathbf{T}) = \mathbf{S_M I_2} \cap \mathbf{S_M T} = \mathbf{S_M I_2} \cap \mathrm{Ob}\,\mathbf{K} = \mathbf{S_M I_2}$.

Whether the converse of the above theorem is true (i.e. if $\mathbf{K}$ has a largest non-trivial $(\mathbf{E}, \mathbf{M})$-semisimple class, then $\mathbf{I_2} \neq \emptyset$) is still an open question.

Because $\mathbf{I_2} \subseteq \mathbf{R_E C_2}$, we have $\mathbf{S_M R_E C_2} \subseteq \mathbf{S_M I_2}$. In certain categories these two classes coincide. We now investigate under what conditions we have the equality. Firstly we prove

3.4. PROPOSITION. $\mathbf{R_E C_2}$ *is* $\mathbf{M}$*-hereditary.*

PROOF. Let $\mu: I \to A \in \mathbf{M}$, $A \in \mathbf{R_E C_2}$. If $I \in \mathbf{R_E C_2}$, then there is an $\alpha: I \to B \in \mathbf{E}$, $B \in \mathbf{C_2}$, $B \notin \mathbf{T}$. By the definition of $\mathbf{C_2}$, the existence of a morphism $\beta: A \to B \in \mathbf{E}$ follows. This however contradicts $A \in \mathbf{R_E C_2}$. Hence $I \in \mathbf{R_E C_2}$ must hold.

Before we prove the next proposition we need the following two remarks:

(1) As we have seen in the previous section, $\mathbf{I_2} \subseteq \mathbf{S} \cap \mathbf{R_E C_2}$ and the equality holds iff $\mathbf{B_2} = \emptyset$. Furthermore, $\mathbf{D_2} \cap \mathbf{R_E C_2} = \emptyset$ and if $\mathbf{C_2} = \mathbf{S} - (\mathbf{I_2} \cup \mathbf{D_2})$, then $\mathbf{B_2} = \emptyset$.

(2) Suppose $\mathbf{K}$ satisfies condition (A6) and $\mathbf{D_2} \neq \emptyset$. Let $A \in \mathrm{Ob}\,\mathbf{K}$, $A \notin \mathbf{T}$. Then there is a morphism $\mu: S \to A \in \mathbf{M}$, $S \in \mathbf{S}$. Indeed, let $B \in \mathbf{D_2}$. By definition of $\mathbf{D_2}$, there is a monomorphism $\beta: B \to A$. Consider the $(\mathbf{E}, \mathbf{M})$-factorization of β, say $B \xrightarrow{\beta} A = B \xrightarrow{\tau} S \xrightarrow{\mu} A$. Then $\mu: S \to A \in \mathbf{M}$ and $S \in \mathbf{S}$ follows from 3.2.

3.5. PROPOSITION. *If* $\mathbf{S_M I_2} = \mathbf{S_M R_E C_2}$, *then* $\mathbf{B_2} = \emptyset$. *Conversely, if* $\mathbf{K}$ *satisfies condition* (A6) *and* $\mathbf{D_2} \neq \emptyset$, *then* $\mathbf{S_M I_2} = \mathbf{S_M R_E C_2}$ *if* $\mathbf{B_2} = \emptyset$. *Hence, in a category which satisfies these conditions,* $\mathbf{B_2} = \emptyset$ *iff* $\mathbf{S_M I_2} = \mathbf{S_M R_E C_2}$.

PROOF. Suppose $\mathbf{S_M I_2} = \mathbf{S_M R_E C_2}$ and let $A \in \mathbf{B_2} = (\mathbf{S} - \mathbf{I_2}) \cap \mathbf{R_E C_2}$. Then $A \in \mathbf{S}$, $A \in \mathbf{I_2}$ and $A \in \mathbf{R_E C_2}$. Hence $A \in \mathbf{S_M R_E C_2} = \mathbf{S_M I_2}$. Thus there is a $\mu: I \to A \in \mathbf{M}$, $I \in \mathbf{I_2}$. By definition of $\mathbf{I_2}$, μ must be an isomorphism. Because $\mathbf{I_2}$ is an abstract class, $A \in \mathbf{I_2}$ follows. This, however, contradicts $A \in \mathbf{I_2}$. Hence $\mathbf{B_2} = \emptyset$.

For the converse we only have to show $\mathbf{S_M I_2} \subseteq \mathbf{S_M R_E C_2}$. Let $\mu: I \to A \in \mathbf{M}$, $I \notin \mathbf{T}$ with $A \in \mathbf{S_M I_2}$. Using remark (2) above, there is a $\tau: S \to I \in \mathbf{M}$, $S \in \mathbf{S}$. Because $\tau\mu: S \to A \in \mathbf{M}$, $S \notin \mathbf{I_2}$. Because $\mathbf{B_2} = \emptyset$, from remark (1) above, the equality $\mathbf{I_2} = \mathbf{S} \cap \mathbf{R_E C_2}$ follows. Hence $S \notin \mathbf{R_E C_2}$ and because $\mathbf{R_E C_2}$ is $\mathbf{M}$-hereditary, $I \notin \mathbf{R_E C_2}$ follows which proves the proposition.

This concludes our diversion and we continue our search for largest radical-classes.

3.6. THEOREM. *If* $\mathbf{D_2} \neq \emptyset$, *then* $\mathbf{K}$ *has a largest proper* $(\mathbf{E}, \mathbf{M})$*-radical class viz* $\mathbf{R_E D_2}$ *and hence a smallest proper* $(\mathbf{E}, \mathbf{M})$*-semisimple class viz* $\mathbf{S_M R_E D_2}$.

PROOF. By Theorem 1.6* it is sufficient to establish the existence of an $(\mathbf{E}, \mathbf{M})$-regular class $\mathbf{A} \not\subseteq \mathbf{T}$ which is contained in every $(\mathbf{E}, \mathbf{M})$-semisimple class $\mathbf{P} \neq \mathbf{T}$. Let $\mathbf{A} = \mathbf{D_2} \cup \mathbf{T}$. Then $\mathbf{A} \not\subseteq \mathbf{T}$ because $\mathbf{D_2} \neq \emptyset$ and $\mathbf{A}$ is $(\mathbf{E}, \mathbf{M})$-regular. Let $\mathbf{P}$ be any $(\mathbf{E}, \mathbf{M})$-semisimple class such that $\mathbf{P} \neq \mathbf{T}$. Then there is a $B \in \mathbf{P}$ such that $B \notin \mathbf{T}$. Let $A \in \mathbf{A}$. If $A \in \mathbf{T}$ then obviously $A \in \mathbf{P}$ must hold. If $A \in \mathbf{D_2}$, by definition, there is a monomorphism $\mu: A \to B$. Consider the $(\mathbf{E}, \mathbf{M})$-factorization of μ, say $A \xrightarrow{\mu} B = A \xrightarrow{\tau} D \xrightarrow{\nu} B$. Because μ is non-constant, $D \notin \mathbf{T}$ and because $\mathbf{P}$ is $\mathbf{M}$-hereditary, $D \in \mathbf{P}$ must hold. Hence $A \notin \mathbf{R_E P}$. Because every simple object must declare itself

with respect to every radical class, $A \in \mathbf{S_M R_E P} = \mathbf{P}$ follows. Lastly, $\mathbf{R_E A} = \mathbf{R_E}(\mathbf{D_2} \cup \mathbf{T}) = \mathbf{R_E D_2} \cap \mathbf{R_E T} = \mathbf{R_E D_2} \cap \mathrm{Ob}\ \mathbf{K} = \mathbf{R_E D_2}$.

Whether the converse of the above theorem is true (i.e. if $\mathbf{K}$ has a largest non-trivial $(\mathbf{E}, \mathbf{M})$-radical class, then $\mathbf{D_2} \neq \emptyset$) is still an open question.

3.7. PROPOSITION. *Suppose* $\mathbf{K}$ *satisfies condition* (A6), $\mathbf{D_2} \neq \emptyset$ *and* $\mathbf{B_2} = \emptyset$. *Let* $\mathbf{P}$ *be an* $(\mathbf{E}, \mathbf{M})$*-semisimple class such that* $\mathbf{C_2} \subseteq \mathbf{P}$ *and* $\mathbf{I_2} \cap \mathbf{P} = \emptyset$. *Then* $\mathbf{P} = \mathbf{S_M R_E C_2}$.

PROOF. Obviously $\mathbf{S_M R_E C_2} \subseteq \mathbf{S_M R_E P} = \mathbf{P}$. Conversely, let $A \in \mathbf{P}$ and let $\mu\colon I \to A \in \mathbf{M}$, $I \notin \mathbf{T}$. By Remark 2 preceding 3.5, there is a $\tau\colon J \to I \in \mathbf{M}$, $J \in \mathbf{S}$. Hence $\tau\mu\colon J \to A \in \mathbf{M}$ and because $\mathbf{P}$ is $\mathbf{M}$-hereditary, $J \in \mathbf{P}$ follows. If $I \in \mathbf{R_E C_2}$, then $J \in \mathbf{R_E C_2} \cap \mathbf{S} = \mathbf{I_2}$ because $\mathbf{B_2} = \emptyset$. Hence $J \in \mathbf{I_2} \cap \mathbf{P} = \emptyset$ which is not possible. Thus $I \notin \mathbf{R_E C_2}$ which yields $A \in \mathbf{S_M R_E C_2}$.

To conclude this section, we investigate what subclasses of $\mathbf{S}$ yield radical and semisimple classes when the operators $\mathbf{R_E}$ and $\mathbf{S_M}$ are applied to them respectively. As we have seen in Section 2, $\mathbf{S}'$ is $(\mathbf{E}, \mathbf{M})$-regular for any $\mathbf{S}' \subseteq \mathbf{S}$. Thus $\mathbf{R_E S}'$ is an $(\mathbf{E}, \mathbf{M})$-radical class for all $\mathbf{S}' \subseteq \mathbf{S}$. Note that because $R_\mathbf{E} \mathbf{D} = \mathbf{T}$ for all $(\mathbf{E}, \mathbf{M})$-regular classes $\mathbf{D}$ such that $\mathbf{I_2} \subseteq \mathbf{D}$, the only non-trivial $(\mathbf{E}, \mathbf{M})$-radical classes are obtained by at most those subclasses $\mathbf{S}'$ of $\mathbf{S}$ for which $\mathbf{I_2} \not\subseteq \mathbf{S}'$. For the operator $\mathbf{S_M}$ we have

3.8. PROPOSITION. *Suppose* $\mathbf{K}$ *satisfies condition* (A6). *Let* $\mathbf{S}'$ *be an abstract subclass of* $\mathbf{S}$. *Then the following are equivalent:*

(1) $\mathbf{S_M S}'$ *is an* $(\mathbf{E}, \mathbf{M})$*-semisimple class.*
(2) $\mathbf{S}' \subseteq \mathbf{R_E S_M S}'$.
(3) $\mathbf{S}'$ *is* $(\mathbf{E}, \mathbf{M})$*-coregular.*
(4) $\mathbf{S}' \cup \mathbf{T}$ *is* $\mathbf{E}$*-cohereditary.*

PROOF. The equivalence (2)$\Leftrightarrow$(3) holds for any class $\mathbf{S}'$ and the implications (2)$\Rightarrow$(1) and (4)$\Rightarrow$(3) are immediate. To see that (1)$\Rightarrow$(2), let $A \in \mathbf{S}'$. Then either $A \in \mathbf{S_M S}'$ or $A \in \mathbf{R_E S_M S}'$ must hold. Obviously A is not in $\mathbf{S}' \cap \mathbf{S_M S}'$. Thus $A \in \mathbf{R_E S_M S}'$ follows. Lastly we show (3)$\Rightarrow$(4). Let $A \in \mathbf{S}' \cup \mathbf{T}$ and let $\alpha\colon A \to B \in \mathbf{E}$. If $A \in \mathbf{T}$, then $B \in \mathbf{T}$. Suppose thus $A \notin \mathbf{T}$ and α is non-constant. Because $\mathbf{S}'$ is $(\mathbf{E}, \mathbf{M})$-coregular, there is a $\mu\colon I \to B \in \mathbf{M}$, $I \notin \mathbf{T}$ with $I \in \mathbf{S}'$. Because $B \in \mathbf{S}$, μ must be an isomorphism. Hence $B \in \mathbf{S}'$ because $\mathbf{S}'$ is an abstract class.

Because $\mathbf{I_2} \subseteq \mathbf{D}$ for every $\mathbf{E}$-cohereditary class $\mathbf{D} \not\subseteq \mathbf{T}$, we have that if $\mathbf{S}' \subseteq \mathbf{S}$ such that $\mathbf{I_2} \subseteq \mathbf{S}'$, then $\mathbf{S}' \cup \mathbf{T}$ is not $\mathbf{E}$-cohereditary. Thus, if $\mathbf{K}$ satisfies condition (A6) and $\mathbf{S}'$ is an abstract subclass of $\mathbf{S}$ such that $\mathbf{S_M S}'$ is an $(\mathbf{E}, \mathbf{M})$-semisimple class, then $\mathbf{I_2} \subseteq \mathbf{S}'$ must hold. Hence the only candidates among the subclasses of $\mathbf{S}$ which yield $(\mathbf{E}, \mathbf{M})$-semisimple classes when the operator $\mathbf{S_M}$ is applied on them are those who at least contain $\mathbf{I_2}$.

Another interesting $(\mathbf{E}, \mathbf{M})$-semisimple class is $\mathbf{S_M}(\mathbf{S} - \mathbf{D_2})$. Suppose $\mathbf{K}$ satisfies condition (A6). Then the class $(\mathbf{S} - \mathbf{D_2}) \cup \mathbf{T}$ is $\mathbf{E}$-cohereditary. Indeed, if $\alpha\colon A \to B \in \mathbf{E}$ and $A \in \mathbf{S} - \mathbf{D_2}$, then $B \in \mathbf{S}$. If $B \in \mathbf{D_2}$, then $A \in \mathbf{D_2}$ follows which is not possible. Hence $B \in \mathbf{S} - \mathbf{D_2}$. Thus $\mathbf{S_M}[(\mathbf{S} - \mathbf{D_2}) \cup \mathbf{T}] = \mathbf{S_M}(\mathbf{S} - \mathbf{D_2})$ is an $(\mathbf{E}, \mathbf{M})$-semisimple class. Note that $\mathbf{S_M}(\mathbf{S} - \mathbf{D_2}) = \{A \in \mathrm{Ob}\ \mathbf{K} |$ whenever $I \to A \in \mathbf{M}$ with $I \in \mathbf{S}$, then $I \in \mathbf{D_2}$ must hold$\}$.

Applying the theory of this section to concrete categories we have:

In **Top** (cf. [1]):

$$\mathbf{S_M I_2} = \mathbf{S_M R_E C_2} = \{\text{all } T_0\text{-spaces}\}$$

is the largest disconnectedness (semisimple class) and

$$\mathbf{R_E S_M I_2} = \mathbf{R_E S_M R_E C_2} = \mathbf{R_E C_2} = \{\text{all indiscrete spaces}\}$$

is the smallest connectedness (radical class).

$\mathbf{R_E D_2} = \{\text{all connected spaces}\}$ is the largest connectedness and $\mathbf{S_M R_E D_2} = \{\text{all totally disconnected spaces}\}$ is the smallest disconnectedness.

In this category it can be shown that the disconnectedness $\mathbf{S_M(S - D_2)} = \{\text{all } T_1\text{-spaces}\}$ is the second largest disconnectedness and that every disconnectedness not equal to the class of all topological space or the class of all T_0-spaces is contained in $\mathbf{S_M(S - D_2)}$. Note that $\mathbf{R_E S_M(S - D_2)} = \mathbf{R_E}$ {all T_1-spaces} = {all absolutely connected spaces}.

In **Graph** (cf. [3]):

$\mathbf{S_M I_2} = \{A |$ if $x, y \in A$ are different vertices with loops, then x and y are not connected$\}$

is the largest disconnectedness and

$\mathbf{R_E S_M I_2} = \{$all connected graphs with loops and all trivial graphs$\}$

is the smallest connectedness. Note that, because $\mathbf{B_2} = \{B_4\} \neq \emptyset$, $\mathbf{S_M I_2} \neq \mathbf{S_M R_E C_2} = \{$all graphs having at most one loop but may possess edges$\}$.

$\mathbf{R_E D_2} = \{A |$ if A is non-trivial then it contains at least one edge or loop$\}$

is the largest connectedness and

$\mathbf{S_M R_E D_2} = \{A |$ if A is non-trivial then A has no edges or loops$\}$

is the smallest disconnectedness.

§4. Hereditariness and cohereditariness

Because **E** is closed under composition, $\mathbf{R_E D}$ is an **E**-cohereditary (**E M**)-radical class for any (**E**, **M**)-regular class **D**. Likewise, because **M** is closed under composition, $\mathbf{S_M B}$ is an **M**-hereditary (**E**, **M**)-semisimple class for any (**E**, **M**)-coregular class **B**. (Conditions (F) and (F*) of Proposition 1.2 are trivially satisfied.)

4.1. PROPOSITION. *Suppose* **K** *satisfies condition* (A6) *and both* $\mathbf{I_2}$ *and* $\mathbf{D_2}$ *are non-empty. Let* $\mathbf{D} \subsetneqq \mathbf{T}$ *be an* **E***-cohereditary class. Then* **D** *satisfies condition* (H*) (cf. 1.2) *iff* $\mathbf{D_2} \subseteq \mathbf{D}$.

PROOF. Suppose **D** satisfies condition (H*). Because $\mathbf{I_2}$ is contained in every **E**-cohereditary class which does not consist only of trivial spaces, $\mathbf{I_2} \subseteq \mathbf{D}$ must hold. Let $\mathbf{B} \in \mathbf{D_2}$. By definition of $\mathbf{I_2}$, there is a β: $B \to I \in \mathbf{E}$, $I \in \mathbf{I_2}$. Now 1_I: $I \to I \in \mathbf{M}$, $I \notin \mathbf{T}$ and $I \in \mathbf{I_2} \subseteq \mathbf{D}$. By condition (H*) there is a τ: $J \to B \in \mathbf{M}$, $J \notin \mathbf{T}$ with $J \in \mathbf{D}$. Because B is simple, τ must be an isomorphism. Hence $B \in \mathbf{D}$ follows. Conversely, suppose $\mathbf{D_2} \subseteq \mathbf{D}$. Then $\mathbf{S} \subseteq \mathbf{D}$. Indeed, if $A \in \mathbf{S}$ and $B \in \mathbf{D_2} \subseteq \mathbf{D}$, then there is a monomor-

phism α: $B \to A$. Because α is non-constant, $\alpha \in \mathbf{E}$ must hold. Hence $A \in \mathbf{D}$ follows. Let β: $A \to B \in \mathbf{E}$ with μ: $I \to B \in \mathbf{M}$, $I \notin \mathbf{T}$, $I \in \mathbf{D}$. Using a remark which precedes 3.5, we have a ξ: $S \to A \in \mathbf{M}$, $S \in \mathbf{S}$. Then $S \notin \mathbf{T}$ and $S \in \mathbf{D}$ which shows that **D** satisfies condition (H*).

4.2. REMARKS. Suppose **K** satisfies condition (A6) and both $\mathbf{I}_2$ and $\mathbf{D}_2$ are non-empty.

(1) If **D** is an **E**-cohereditary class, then $\mathbf{S}_M\mathbf{D}$ is an **E**-cohereditary (**E**, **M**)-semisimple class iff **D** satisfies condition (H*), i.e. iff $\mathbf{D}_2 \subseteq \mathbf{D}$.

In particular, if **D** is an (**E**, **M**)-semisimple class, then **D** is **E**-cohereditary iff $\mathbf{R}_E\mathbf{D}$ satisfies condition (H*), i.e. iff $\mathbf{D}_2 \subseteq \mathbf{R}_E\mathbf{D}$.

(2) Let **D** be any proper **E**-cohereditary (**E**, **M**)-semisimple class. From (1) above, $\mathbf{D}_2 \subseteq \mathbf{R}_E\mathbf{D}$. In the proof of 3.6 it has been shown that $\mathbf{D}_2 \subseteq \mathbf{P}$ for any (**E**, **M**)-semisimple class $\mathbf{P} \neq \mathbf{T}$. In particular, we then have $\mathbf{D}_2 \subseteq \mathbf{S}_M\mathbf{R}_E\mathbf{D} \cap \mathbf{R}_E\mathbf{D} = \mathbf{D} \cap \mathbf{R}_E\mathbf{D} = \mathbf{T}$. This, however, is not possible. Hence $\mathbf{D} = \mathbf{T}$ or $\mathbf{D} = \mathrm{Ob}\,\mathbf{K}$.

By the above we see that, if $\mathbf{D}_2 \neq \emptyset$, the **E**-cohereditary (**E**, **M**)-semisimple classes are only the trivial ones, viz. **T** and Ob **K**. Let us now turn our attention to **M**-hereditary (**E**, **M**)-radical classes. So far we have come across one viz. $\mathbf{R}_E\mathbf{C}_2$. Note that $\mathbf{C}_2$ satisfies condition (H).

4.3. PROPOSITION. *Suppose* **K** *satisfies conditions* (A6) *and* $\mathbf{C}_2$, $\mathbf{D}_2$ *and* $\mathbf{I}_2$ *are non-empty. Let* **D** *be an* **M***-hereditary class. If either of the following conditions hold*

(1) $\mathbf{I}_2 \subseteq \mathbf{D}$ *or*

(2) $\mathbf{C}_2 \subseteq \mathbf{D}$ *and* $\mathbf{B}_2 \cap \mathbf{D} = \emptyset$, *then* **D** *satisfies condition* (H).

PROOF. Suppose $\mathbf{I}_2 \subseteq \mathbf{D}$. Using the definition of $\mathbf{I}_2$ it is easily seen that **D** satisfies condition (H) in this case. Suppose $\mathbf{I}_2 \subseteq \mathbf{D}$, $\mathbf{C}_2 \subseteq \mathbf{D}$ and $\mathbf{B}_2 \cap \mathbf{D} = \emptyset$. Because $\mathbf{I}_2 \subseteq \mathbf{D}$, $\mathbf{I}_2 \cap \mathbf{D} = \emptyset$ follows. Let μ: $I \to A \in \mathbf{M}$ and α: $I \to B$ with $B \notin \mathbf{T}$ and $B \in \mathbf{D}$. Using a remark in the previous section, there is a τ: $J \to B \in \mathbf{M}$, $J \in \mathbf{S}$. Because **D** is **M**-hereditary, $J \in \mathbf{D}$ must hold. If $B \in \mathbf{R}_E\mathbf{C}_2$, then $J \in \mathbf{R}_E\mathbf{C}_2 \cap \mathbf{S} = \mathbf{I}_1 \cup \mathbf{B}_2$. This is not possible because both $\mathbf{D} \cap \mathbf{I}_2$ and $\mathbf{D} \cap \mathbf{B}_2$ are empty. Hence $\mathbf{B} \in \mathbf{R}_E\mathbf{C}_2$. Thus there is an α: $B \to C \in \mathbf{E}$, $C \in \mathbf{C}_2$. The definition of $\mathbf{C}_2$ then yields a μ: $A \to C \in \mathbf{E}n$ which shows that **D** satisfies conditions (H).

4.4. REMARKS. Suppose **K** satisfies condition (A6) and $\mathbf{I}_2$, $\mathbf{D}_2$ and $\mathbf{C}_2$ are all non-empty. Let **D** be any (**E**, **M**)-radical class, $\mathbf{D} \neq \mathbf{T}$.

(1) $\mathbf{D} = \mathbf{R}_E\mathbf{S}_M\mathbf{D}$ is **M**-hereditary iff $\mathbf{S}_M\mathbf{D}$ satisfies condition (H) which is implied by either (1) $\mathbf{I}_2 \subseteq \mathbf{S}_M\mathbf{D}$ or (2) $\mathbf{C}_2 \subseteq \mathbf{S}_M\mathbf{D}$ and $\mathbf{B}_2 \cap \mathbf{S}_M\mathbf{D} = \emptyset$.

(2) If $\mathbf{C}_2 \subseteq \mathbf{S}_M\mathbf{D}$ and $\mathbf{B}_2 \cap \mathbf{S}_M\mathbf{D} = \emptyset$, then $\mathbf{D} = \mathbf{R}_E\mathbf{C}_2$. Indeed, it trivially follows that $\mathbf{D} \subseteq \mathbf{R}_E\mathbf{C}_2$. Let $A \in \mathbf{R}_E\mathbf{C}_2$. Note that $\mathbf{B}_2 \cap \mathbf{S}_M\mathbf{D} = \emptyset$ implies $\mathbf{B}_2 \subseteq \mathbf{D}$. Let α: $A \to B \in \mathbf{E}$, $B \notin \mathbf{T}$. Then there is a τ: $J \to B \in \mathbf{M}$, $J \in \mathbf{S}$. If $J \in \mathbf{I}_2$, then J and hence $A \in \mathbf{D}$ follows. (Remember $\mathbf{I}_2$ is contained in any non-trivial (**E**, **M**)-radical class.) If $J \notin \mathbf{I}_2$, then $J \in (\mathbf{S} - \mathbf{I}_2) \cap \mathbf{R}_E\mathbf{C}_2 = \mathbf{B}_2 \subseteq \mathbf{D}$ because $\mathbf{R}_E\mathbf{C}_2$ is **E**-cohereditary and **M**-hereditary. Thus $A \in \mathbf{D}$ follows.

In certain categories, e.g. **Top** and **Graph**, it can be shown (cf. [8]) that if $\mathbf{D} \not\subseteq \mathbf{T}$ is an **M**-hereditary class which satisfies condition (H), then

(1) $\mathbf{I}_2 \cap \mathbf{D} = \emptyset$ implies $\mathbf{B}_2 \cap \mathbf{D} = \emptyset$ and

(2) $\mathbf{I}_2 \subseteq \mathbf{D}$ or $\mathbf{C}_2 \subseteq \mathbf{D}$.

Thus, in such categories, for any $(\mathbf{E}, \mathbf{M})$-radical class $\mathbf{D} \neq \mathrm{Ob}\,\mathbf{K}$, the following are equivalent:

(1) $\mathbf{D} = \mathbf{R}_E \mathbf{S}_M \mathbf{D}$ is $\mathbf{M}$-hereditary.

(2) $\mathbf{S}_M \mathbf{D}$ satisfies condition (H).

(3) (i) $\mathbf{I}_2 \subseteqq \mathbf{S}_M \mathbf{D}$ or
(ii) $\mathbf{C}_2 \subseteqq \mathbf{S}_M \mathbf{D}$ and $\mathbf{I}_2 \cap \mathbf{S}_M \mathbf{D} = \emptyset$ implies $\mathbf{B}_2 \cap \mathbf{S}_M \mathbf{D} = \emptyset$.

(4) $\mathbf{D} = \mathbf{T}$ or $\mathbf{D} = \mathbf{R}_E \mathbf{C}_2$.

Hence the only $\mathbf{M}$-hereditary $(\mathbf{E}, \mathbf{M})$-radical classes are $\mathbf{T}$, $\mathbf{R}_E \mathbf{C}_2$ and $\mathrm{Ob}\,\mathbf{K}$.

Applying the above theory to the category **Top**, we see that there are no proper continuously closed disconnectednesses and that there is only one proper hereditary connectedness viz. $\mathbf{R}_E \mathbf{C}_2 = \{\text{all indiscrete spaces}\}$ (cf. [1] or [8]).

In the category **Graph**, there is no proper homomorphically closed disconnectedness and only one proper hereditary connectedness viz. $\mathbf{R}_E \mathbf{C}_2 = \{A \mid$ if A is non-trivial, then each vertex of A has a loop$\}$ (cf. [8]).

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A BOUND OF k FOR TILING BY (k, n) CROSSES AND SEMI-CROSSES

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Let k, n be positive integers. The union of the translates of a closed n-dimensional unit cube whose edges are parallel to the unit coordinate vectors $e_1, \ldots, e_n$ and whose centers are ie_j, $|i| \leqq k$, $1 \leqq j \leqq n$ is called a (k, n) cross. This (k, n) cross consists of $2kn+1$ cubes; a central cube together with $2n$ arms of length k.

A system of translates of a (k, n) cross whose union is the n-dimensional Euclidean space and whose interiors are disjoint is an integer lattice tiling if every translation vector has integer coordinates and if the translation vectors form a lattice.

The union of the translates of the preceding cube whose centers are ie_j, $0 \leqq i \leqq k$, $1 \leqq j \leqq n$ which consists of $kn+1$ cubes formed of n arms of length k attached to a central cube is called a (k, n) semi-cross. We define an integer lattice tiling by (k, n) semi-crosses in a similar manner as in the case of crosses.

A number of authors examined the problem: for which integers k, n there exists a tiling of the n-dimensional Euclidean space by (k, n) crosses and semi-crosses respectively. (See [1]—[12].)

The following results are known about the bound of k.

If there exists an integer tiling by (k, n) crosses and if $n \geqq 2$, then $k \leqq 2n-2$ ([6], Theorem 3.2). We should remark that here it is not assumed that the translation vectors form a lattice.

If there exists an integer lattice tiling by (k, n) semicrosses and if $n \geqq 3$, then $k \leqq 2n-1$ ([2], Theorem 2.2).

In this paper we shall show that about half of the preceding bounds are suitable. In the case of semi-crosses it is fairly good since if p is an odd prime, then there exists an integer lattice tiling by $(p-1, p+1)$ semi-crosses that is there are infinitely many examples with $n-2=k$ ([6]).

THEOREM 1. *If* $n \geqq 3$, *then* $n \geqq k+1$ *for every integer lattice tiling by* (k, n) *crosses.*

PROOF. Assume that there exists an integer lattice tiling by (k, n) crosses. If there exists an integer lattice tiling by (k, n) crosses, then there exists an abelian group G of order $2kn+1$ such that G is represented in the form

$$(1) \qquad \begin{cases} G = \{0, g_1, 2g_1, \ldots, kg_1, -g_1, -2g_1, \ldots, -kg_1, \ldots, \\ \qquad g_n, 2g_n, \ldots, kg_n, -g_n, -2g_n, \ldots, -kg_n\}, \end{cases}$$

that is every g of $G \setminus \{0\}$ is uniquely represented in the form ig_j, $1 \leqq |i| \leqq k$, $1 \leqq j \leqq n$. A proof of this statement can be found in [2], Theorem 1.1.

First we shall show that for each integer i, $2 \leqq i \leqq n$, there exist integers x_i, y_i such that $k+1 \leqq x_i \leqq 2n-1$, $|y_i| \leqq k$ and $x_i g_1 + y_i g_i = 0$.

For simplicity we assume that $i=2$. Consider the elements $a_1 g_1 + a_2 g_2$, $0 \leqq a_1 \leqq 2n-1$, $0 \leqq a_2 \leqq k$. The number of these elements, $2n(k+1)$, is greater than the order of G, hence there exist (b_1, b_2) and (c_1, c_2) such that $(b_1, b_2) \neq (c_1, c_2)$, $0 \leqq b_1, c_1 \leqq 2n-1$, $0 \leqq b_2, c_2 \leqq k$ and $b_1 g_1 + b_2 g_2 = c_1 g_1 + c_2 g_2$. Assume that $b_1 \geqq c_1$ and let $d_1 = b_1 - c_1$, $d_2 = b_2 - c_2$. Then $d_1 g_1 + d_2 g_2 = 0$, where $(d_1, d_2) \neq (0, 0)$ and $0 \leqq d_1 \leqq 2n-1$, $|d_2| \leqq k$. If $0 \leqq d_1 \leqq k$, then $d_1 g_1 = -d_2 g_2$ which violates (1). Thus there exist $n-1$ pairs of integers x_i, y_i, $2 \leqq i \leqq n$ such that $k+1 \leqq x_i \leqq 2n-1$, $|y_i| \leqq k$ and $x_i g_1 + y_i g_i = 0$.

Assume that the x_i's are distinct for $2 \leqq i \leqq n$. Since $\{k+1, k+2, \ldots, 2n-1\} \supseteq \{x_2, x_3, \ldots, x_n\}$ so $2n-k-1 \geqq n-1$ that is $n \geqq k$.

Assume that there exist i, j, $i \neq j$, $2 \leqq i, j \leqq n$ such that $x_i = x_j$. In virtue of equations $x_i g_1 + y_i g_i = 0$ and $x_j g_1 + y_j g_j = 0$ we have $y_i g_i = y_j g_j$. This does not violate (1) only if $y_i = y_j = 0$ since $|y_i|, |y_j| \leqq k$. Then $x_i g_1 = 0$. By (1) the elements

$$-kg_1, -(k-1)g_1, \ldots, -g_1, 0, g_1, 2g_1, \ldots, kg_1$$

are distinct so the order of g_1 is at least $2k+1$. The number x_i is divisible by the order of g_1 hence $x_i \geqq 2k+1$. But $2n-1 \geqq x_i$ that is $n \geqq k+1$.

We have proved $n \geqq k$. Now we shall prove that $n \geqq k+1$. Since $(2n-1) \cdot (k+1) = 2nk+2n-k-1 \geqq 2nk+n-1 \geqq 2nk+2 > 2nk+1$ so we can prove that for each integer i, $2 \leqq i \leqq n$, there exist integers x_i, y_i such that $k+1 \leqq x_i \leqq 2n-2$, $|y_i| \leqq k$ and $x_i g_1 + y_i g_i = 0$.

If the x_i's are distinct for $2 \leqq i \leqq n$, then from $\{k+1, k+2, \ldots, 2n-2\} \supseteq \{x_2, x_3, \ldots, x_n\}$ it follows that $2n-2-k \geqq n-1$ that is $n \geqq k+1$.

If there exist i, j, $i \neq j$, $2 \leqq i, j \leqq n$ such that $x_i = x_j$, then $2n-2 \geqq x_i \geqq 2k+1$ that is $n > k+1$.

THEOREM 2. *If $n \geqq 3$, then $n \geqq k$ for every integer lattice tiling by (k, n) semi-crosses.*

We should remark that the proof of Theorem 2 gives $n-1 \geqq k$ for each even $n \geqq 4$.

PROOF. Assume that there exists an integer lattice tiling by (k, n) semi-crosses. Then there exists an abelian group G of order $kn+1$ such that G is represented in the form

$$G = \{0, g_1, 2g_1, \ldots, kg_1, \ldots, g_n, 2g_n, \ldots, kg_n\} \tag{2}$$

that is every g of $G \setminus \{0\}$ is uniquely represented in the form ig_j, $1 \leqq i \leqq k$, $1 \leqq j \leqq n$. A proof of this statement can be seen in [2], Theorem 1.1.

If $k \geqq n-1$, then for each integer i, j, $1 \leqq i, j \leqq n$, $i \neq j$ there exist integers x_{ij}, x_{ji} such that $1 \leqq x_{ij}, x_{ji} \leqq n-1$ and $x_{ij} g_i + x_{ji} g_j = 0$. Furthermore $\{x_{ij}: 1 \leqq j \leqq n, j \neq i\} = \{1, 2, \ldots, n-1\}$ for each i, $1 \leqq i \leqq n$. A proof of this statement can be seen under the assumption $k \geqq n$ in [2], Lemma 2.1. The reader can readily verify that the proof is correct for $k \geqq n-1$.

Now we shall show that there are i, j, $1 \leqq i, j \leqq n$, $i \neq j$ such that $2x_{ij} \leqq n-1$ and $2x_{ji} \leqq n+1$.

Indeed, let n be an odd number. Since the number of x_{ij} for which $1 \leqq x_{ij} \leqq (n-1)/2$ equals $n(n-1)/2$ and the number of x_{ji} for which $(n+1)/2+1 \leqq x_{ji} \leqq n-1$ equals $n[(n-1)/2+1]$, hence there are i, j such that $x_{ij} \leqq (n-1)/2$ and $x_{ji} \leqq (n+1)/2$. Let n be an even number and assume that there do not exist i, j such that $x_{ij} \leqq n/2-1$ and $x_{ji} \leqq n/2$.

Since the number of x_{ij} for which $1 \leqq x_{ij} \leqq n/2-1$ equals $n(n/2-1)$ and the number of x_{ji} for which $n/2+1 \leqq x_{ji} \leqq n-1$ equals $n(n/2-1)$ hence there are i, j such that $x_{ij}=x_{ji}=n/2$. Now $n/2$ is prime to $|G|=kn+1$, hence $0=g_i+g_j$ as well. But then for that t which satisfies $x_{it}=1$ we have $0=g_i+x_{ti}g_t$, $x_{ti}g_t=g_j$, a contradiction, as $1 \leqq x_{ti} \leqq n-1 \leqq k$.

Finally, since $2x_{ij} \leqq n-1$, hence there is a t, $1 \leqq t \leqq n, j \neq t$ such that $2x_{ij}=x_{it}$. Then $2x_{ij}g_i+2x_{ji}g_j=x_{it}g_i+x_{ti}g_t$ and $2x_{ji}g_j=x_{ti}g_t$, where $1 \leqq x_{ti} \leqq n-1 \leqq k$ and $j \neq t$. Thus $n+1 \geqq 2x_{ji} > k$.

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AN EQUICONVERGENCE THEOREM FOR THE SCHRÖDINGER OPERATOR

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In the last years a new and fruitful method was developed by V. A. Il'in for the investigation of non-selfadjoint differential operators. By the aid of this method, in this paper we shall prove two results. In the first theorem, under general conditions, upper and lower estimates will be given for the sum of squares of an orthonormal system consisting of eigenfunctions of the one-dimensional Schrödinger operator. This theorem extends some previous results of I. Joó [3], [4]; the improvement of his results is made possible by the ideas of the papers [5], [6].

Using the upper estimate from Theorem 1, a general equiconvergence theorem will be proved, which contains, as special cases, a theorem of B. M. Levitan [7] and a result of I. Joó [4]. Our theorem is more general than that of Levitan in two aspects: the potential function is not assumed to be real and there are no boundary conditions. The theorem of Joó is the special case of our result when the eigenvalues are real and nonnegative. In the proof of Theorem 2, a crucial role is played by Lemma 2 which enables us to deal e.g. with arbitrary real eigenvalues.

We shall frequently use the following formula of E. C. Titchmarsh [1]: if u is an eigenfunction of the operator $Lu=-u''+qu$ with some eigenvalue $\lambda\in\mathbf{C}$, then

$$u(x+t)+u(x-t) = 2u(x)\cos\sqrt{\lambda}\,t+\int_{x-t}^{x+t} q(\xi)u(\xi)\frac{\sin\sqrt{\lambda}\,(t-|x-\xi|)}{\sqrt{n}}\,d\xi$$

whenever $u(x+t)$ and $u(x-t)$ are defined.

Throughout this paper let G denote an arbitrary open interval on the real line, $q\colon G\to\mathbf{C}$ a locally integrable function and $(u_k)\subset L^2(G)$ a complete orthonormal system of eigenfunctions of the formal Schrödinger operator $Lu=-u''+qu$. (We recall that a function $0\not\equiv u\colon G\to\mathbf{C}$ is called an eigenfunction of the operator L with some eigenvalue $\lambda\in\mathbf{C}$ if u and u' are absolutely continuous on every compact subinterval of G and if the equation

$$-u''(x)+q(x)u(x) = \lambda u(x)$$

holds for almost all $x\in G$.) Denoting by λ_k the eigenvalue of u_k, we shall also assume that there exists a constant $B_0>0$ such that for any u_k,

(1) $$\text{either}\quad \left|\operatorname{Im}\sqrt{\lambda_k}\right|\leqq B_0 \quad\text{or}\quad \left|\sqrt{\lambda_k}\right|\leqq B_0\left|\operatorname{Im}\sqrt{\lambda_k}\right|.$$

We remark that condition (1) is automatically satisfied if all the eigenvalues λ_k are real.

Under these conditions we shall prove the following result:

THEOREM 1. a) *Given any compact interval* $K \subset G$, *there exists a number* $R>0$ *such that*

$$\sup_{\mu>0} \sum_{|\mu-|\mathrm{Re}\sqrt{\lambda_k}||\leq 1} \left\{\|u_k\|_{L^\infty(K)} \operatorname{ch}\left(R \operatorname{Im}\sqrt{\lambda_k}\right)\right\}^2 < \infty.$$

b) *Given any compact interval* $K \subset G$, *there exist numbers* $M, R>0$ *such that*

$$\inf_{\mu>0} \inf_{x\in K} \sum_{|\mu-|\mathrm{Re}\sqrt{\lambda_k}||\leq M} \left|u_k(x) \operatorname{ch}\left(R \operatorname{Im}\sqrt{\lambda_k}\right)\right|^2 > 0.$$

Let us now give an other locally integrable function $\hat{q}$: $G \to \mathbf{C}$, and a complete orthonormal system $(\hat{u}_k) \subset L^2(G)$ of eigenfunctions of the formal Schrödinger operator $\hat{L}u = -u'' + \hat{q}u$ such that — denoting by $\hat{\lambda}_k$ the eigenvalue of $\hat{u}_k$ — there exists a constant $\hat{B}_0>0$ such that for any $\hat{u}_k$,

$$\text{(2)} \qquad \text{either} \quad \left|\mathrm{Im}\sqrt{\hat{\lambda}_k}\right| \leq \hat{B}_0 \quad \text{or} \quad \left|\sqrt{\hat{\lambda}_k}\right| \leq \hat{B}_0 \left|\mathrm{Im}\sqrt{\hat{\lambda}_k}\right|.$$

Let us introduce for any $f \in L^2(G)$, $\mu>0$ and $x \in G$ the notations

$$\sigma_\mu(f, x) \equiv \sum_{|\mathrm{Re}\sqrt{\lambda_k}|<\mu} \langle f, u_k\rangle u_k(x), \quad \hat{\sigma}_\mu(f, x) \equiv \sum_{|\mathrm{Re}\sqrt{\hat{\lambda}_k}|<\mu} \langle f, \hat{u}_k\rangle \hat{u}_k(x).$$

Under these conditions, using also Theorem 1, we shall prove the following equiconvergence theorem:

THEOREM 2. *Given any compact interval* $K \subset G$, *for all* $f \in L^2(G)$ *the following limit relation is true:*

$$\lim_{\mu\to\infty} \sup_{x\in K} |\sigma_\mu(f, x) - \hat{\sigma}_\mu(f, x)| = 0.$$

REMARKS. Theorem 1 extends some previous results of I. Joó [3], [4]. He considered the problem under the stronger condition

$$\sup_k \left|\mathrm{Im}\sqrt{\lambda_k}\right| < \infty$$

instead of (1).

Theorem 2 generalizes a result of B. M. Levitan. He proved the equiconvergence theorem for the case when the functions q, $\hat{q}$ are real, with quite different methods [7].

For the sake of brevity, we shall denote in the sequel by μ_k an arbitrary square root of λ_k and we put $\varrho_k \equiv \mathrm{Re}\,\mu_k$, $v_k \equiv \mathrm{Im}\,\mu_k$.

First we prove two lemmas from which Theorem 1 will follow easily.

LEMMA 1. *Under the conditions of Theorem* 1, *for any compact interval* $K \subset G$ *and for any* $B>0$,

$$\sup_{\mu>0} \sum_{\substack{|\mu-|\varrho_k||\leq 1 \\ |v_k|\leq B}} \|u_k\|^2_{L^\infty(K)} < \infty.$$

PROOF. Given $K=[a, b] \subset G$, fix a positive number A satisfying the following conditions:

$$\text{(3)} \qquad \text{for any} \quad z, w \in \mathbf{C} \quad \text{with} \quad |\mathrm{Im}\, z| \leq 1, \ |\mathrm{Im}\, w| \leq 1,$$

$|\sin z|<A$, $|\cos z|<A$, $|\sin z|\leqq A|z|$ and $|\cos z-\cos w|\leqq A|z-w|$,

(4) $$\|u_k\|_{L^\infty(K)} \leqq A\|u_k\|_{L^2(K)} \quad \text{whenever} \quad |v_k| \leqq B.$$

(Condition (4) can be fulfilled in view of a theorem of Joó [2]). Now we put for brevity $Q\equiv\|q\|_{L^1(K)}$ and

(5) $$R \equiv \min\left(B^{-1}, 4^{-1}(b-a),\ 2^{-1}A^{-2}(B+1+Q)^{-1}(b-a)^{-0,5}\right).$$

Fixing $a\leqq x\leqq 2^{-1}(a+b)$ and $\mu>0$ arbitrarily, introduce the function $w\colon G\to\mathbf{R}$ by the formula

$$w(y) \equiv \begin{cases} -0,5+2\cos\mu(y-x) & \text{if} \quad x\leqq y\leqq x+R, \\ -0,5 & \text{if} \quad x+R<y\leqq x+2R, \\ 0 & \text{otherwise.} \end{cases}$$

One can see easily that

(6) $$\|w\|^2_{L^2(G)} \leqq 7R.$$

For any u_k and $0\leqq t\leqq R$, using the Titchmarsh-formula,

$$u_k(x) = 2u_k(x+t)\cos\mu_k t - u_k(x+2t) + \int_x^{x+2t} q(\xi)u_k(\xi)\frac{\sin\mu_k(t-|x+t-\xi|)}{\mu_k}\,d\xi.$$

Integrating both sides by t from 0 to R we obtain

$$Ru_k(x) = \int_0^R [2u_k(x+t)\cos\mu t - u_k(x+2t)]\,dt + \int_0^R 2u_k(x+t)(\cos\mu_k t-\cos\mu t)\,dt +$$

$$+\int_0^R\int_x^{x+2t} q(\xi)u_k(\xi)\frac{\sin\mu_k(t-|x+t-\xi|)}{\mu_k}\,d\xi\,dt \equiv I_1+I_2+I_3.$$

Here $I_1=\langle u_k, w\rangle$ is the k-th Fourier coefficient of w. Moreover, in case $|\mu-|\varrho_k||\leqq 1$ and $|v_k|\leqq B$ we have in view of (3)—(5)

$$|I_2| \leqq A(B+1)R^2\|u_k\|_{L^\infty(K)}, \quad |I_3| \leqq AQR^2\|u_k\|_{L^\infty(K)},$$

$$|I_2+I_3| \leqq A(B+1+Q)R^2\|u_k\|_{L^\infty(K)} \leqq$$

$$\leqq A^2(B+1+Q)R^2\|u_k\|_{L^2(K)} \leqq 2^{-1}(b-a)^{-0,5}R\|u_k\|_{L^2(K)}.$$

Hence

$$|Ru_k(x)| \leqq |\langle u_k, w\rangle| + 0,5(b-a)^{-0,5}R\|u_k\|_{L^2(K)}$$

and

$$|Ru_k(x)|^2 \leqq 2|\langle u_k, w\rangle|^2 + 0,5(b-a)^{-1}(R\|u_k\|_{L^2(K)})^2.$$

Let now $I\subset\{k\in\mathbf{N}\colon |\mu-|\varrho_k||\leqq 1$ and $|v_k|\leqq B\}$ be an arbitrary finite set, then, using also (6) and the Bessel-inequality,

$$\sum_{k\in I}|Ru_k(x)|^2 \leqq 14R + 0,5(b-a)^{-1}\sum_{k\in I}(R\|u_k\|_{L^2(K)})^2.$$

This is true for all $a \leqq x \leqq 0{,}5(a+b)$ but a similar argument proves it for all $0{,}5(a+b) \leqq x \leqq b$, too. Therefore, integrating both sides by x on K, we obtain

$$\sum_{k \in I} (R\|u_k\|_{L^2(K)})^2 \leqq 14R(b-a) + 0{,}5 \sum_{k \in I} (R\|u_k\|_{L^2(K)})^2,$$

whence, using also (4),

$$\sum_{k \in I} \|u_k\|^2_{L^2(K)} \leqq 28(b-a)R^{-1}, \quad \sum_{k \in I} \|u_k\|^2_{L^\infty(K)} \leqq 28(b-a)A^2R^{-1}.$$

Taking into account that I was chosen arbitrarily, hence Lemma 1 follows.

LEMMA 2. *Under the conditions of Theorem* 1, *given any compact interval* $K \subset G$, *there exist numbers* $B, R > 0$ *such that*

$$\sum_{|\nu_k| > B} (\|u_k\|_{L^\infty(K)} \operatorname{ch}(\nu_k R))^2 < \infty.$$

PROOF. By an elementary compactness argument, it suffices to show that each point $x \in G$ is contained in a compact interval $K' \subset G$ with the above property. Let now $K = [a, b] \subset G$ be an arbitrary compact interval such that, putting $R = 0{,}25(b-a)$ and $K_R \equiv [a-R, b+R]$, the following conditions be satisfied:

$$\text{(7)} \qquad \begin{cases} K_R \subset G, \quad \operatorname{sh}(BR) \geqq 4, \quad \operatorname{cth}(BR) \leqq 2, \quad B \geqq B_0 \quad \text{and} \\ Q \equiv \|q\|_{L^1(K_R)} \leqq \min(20^{-1}B, 1440^{-1}R^{-1}). \end{cases}$$

Obviously, each point of G is contained in a compact interval K of this type: it suffices to choose K to be sufficiently small and then choose B to be sufficiently large. We shall prove the lemma for such intervals. Let us introduce the following notations for $\xi \in K_R$:

$$d(\xi) \equiv \max(\xi - (a-R), (b+R) - \xi), \quad v_k(\xi) \equiv u_k(\xi) \operatorname{ch}(\nu_k d(\xi)).$$

Obviously, $d(\xi) \geqq R$ if $\xi \in K$. We shall repeatedly use the following inequalities in the sequel:

$$\operatorname{sh}|\operatorname{Im} z| \leqq |\sin z| \leqq \operatorname{ch}(\operatorname{Im} z), \quad \operatorname{sh}|\operatorname{Im} z| \leqq |\cos z| \leqq \operatorname{ch}(\operatorname{Im} z)$$

for any $z \in \mathbf{C}$. Using them and the properties (7), we obtain for any $x \in K$ by the Titchmarsh-formula

$$|v_k(x)| \leqq |\operatorname{cth} \nu_k d(x)|\,|u_k(x) \cos \nu_k d(x)| \leqq$$

$$\leqq |\operatorname{cth} \nu_k d(x)| \{|u_k(x - d(x))| + |u_k(x + d(x))| + Q|\mu_k|^{-1}\|v_k\|_{L^\infty(K_R)}\} \leqq$$

$$\leqq |\operatorname{cth} \nu_k d(x)| \{2\|v_k\|_{L^\infty(K_R \setminus K)} + (\operatorname{ch} \nu_k R)^{-1}\|v_k\|_{L^\infty(K)} + Q|\mu_k|^{-1}\|v_k\|_{L^\infty(K_R)}\} \leqq$$

$$\leqq 4\|v_k\|_{L^\infty(K_R \setminus K)} + 0{,}25\|v_k\|_{L^\infty(K)} + 0{,}1\|v_k\|_{L^\infty(K_R)} \leqq 5\|v_k\|_{L^\infty(K_R \setminus K)} + 0{,}5\|v_k\|_{L^\infty(K)}$$

whenever $|\nu_k| \geqq B$. We assume henceforth that $|\nu_k| \geqq B$, thus

$$\|v_k\|_{L^\infty(K)} \leqq 5\|v_k\|_{L^\infty(K_R \setminus K)} + 0{,}5\|v_k\|_{L^\infty(K)}, \quad \|v_k\|_{L^\infty(K)} \leqq 10\|v_k\|_{L^\infty(K_R \setminus K)},$$

and

$$\text{(8)} \qquad \|v_k\|_{L^\infty(K_R)} \leqq 10\|v_k\|_{L^\infty(K_R \setminus K)} \quad (|\nu_k| \geqq B).$$

Let now $a-R \leqq x \leqq a$ and $R \leqq t \leqq 2R$ be arbitrary. Multiplying the Titchmarsh-formula by $\operatorname{ch} v_k d(x)$ and using the inequality $\operatorname{ch}\alpha \operatorname{ch}\beta \leqq \operatorname{ch}(\alpha+\beta)$ $(\alpha, \beta \geqq 0)$,

$$|v_k(x)| \leqq |v_k(x+2t)| + 2|v_k(x+t)| + Q|\mu_k|^{-1}\|v_k\|_{L^\infty(K_R)}.$$

Integrating by t from R to $2R$,

$$R|v_k(x)| \leqq \int_R^{2R} |v_k(x+2t)|\,dt + 2\int_R^{2R} |v_k(x+t)|\,dt + RQ|\mu_k|^{-1}\|v_k\|_{L^\infty(K_R)};$$

applying the Cauchy—Schwarz inequality to the integrals and dividing by R, we obtain

$$|v_k(x)| \leqq 3R^{-0,5}\|v_k\|_{L^2(K)} + Q|\mu_k|^{-1}\|v_k\|_{L^\infty(K_R)}.$$

This is true for all $a-R \leqq x \leqq a$ and analogously also for all $b \leqq x \leqq b+R$. Hence, using (7), (8) and $|v_k| \geqq B$,

$$\|v_k\|_{L^\infty(K_R \setminus K)} \leqq 3R^{-0,5}\|v_k\|_{L^2(K)} + Q|\mu_k|^{-1}\|v_k\|_{L^\infty(K_R)},$$

$$\|v_k\|_{L^\infty(K_R \setminus K)} \leqq 6R^{-0,5}\|v_k\|_{L^2(K)}$$

and finally

$$\|v_k\|_{L^\infty(K_R)} \leqq 60R^{-0,5}\|v_k\|_{L^2(K)} \quad (|v_k| \geqq B). \tag{9}$$

Furthermore, taking into account that $\xi \in K$ and $|v_k| \geqq B$ imply $|\operatorname{sh} v_k d(\xi)| \geqq 0{,}5 \operatorname{ch} v_k d(\xi)$, (9) yields

$$\|v_k\|_{L^\infty(K_R)} \leqq 120R^{-0,5}\Big(\int_K |u_k(\xi) \operatorname{sh} v_k\, d(\xi)|^2\, d\xi\Big)^{0,5} \tag{10}$$

whenever $|v_k| \geqq B$.

Fix now $x \in K$ arbitrarily and define $w\colon G \to \mathbf{R}$ as

$$w(y) \equiv \begin{cases} 1 & \text{if } |y-x| \leqq d(x), \\ 0 & \text{otherwise.} \end{cases}$$

Applying the Titchmarsh-formula for any u_k with $|v_k| > B$,

$$\langle u_k, w\rangle = \int_0^{d(x)} u_k(x)\cos\mu_k t\,dt + \int_0^{d(x)}\int_{x-t}^{x+t} q(\xi)u_k(\xi)\frac{\sin\mu_k(t-|x-\xi|)}{\mu_k}\,d\xi\,dt,$$

whence in view of (7), (10) and $d(x) \leqq 3R$

$$\left|u_k(x)\frac{\operatorname{sh} v_k d(x)}{\mu_k}\right| \leqq \left|u_k(x)\frac{\sin\mu_k d(x)}{\mu_k}\right| \leqq$$

$$\leqq |\langle u_k w\rangle| + 360QR^{0,5}\left(\int_K \left|u_k(\xi)\frac{\operatorname{sh} v_k d(\xi)}{\mu_k}\right|^2 d\xi\right)^{0,5},$$

$$\left|u_k(x)\frac{\operatorname{sh} v_k d(x)}{\mu_k}\right|^2 \leqq 2|\langle u_k, w\rangle|^2 + 0{,}5(720Q)^2 R\int_K \left|u_k(\xi)\frac{\operatorname{sh} v_k d(\xi)}{\mu_k}\right|^2 d\xi,$$

$$\left|u_k(x)\frac{\operatorname{sh} v_k d(x)}{\mu_k}\right|^2 \leqq 2|\langle u_k, w\rangle|^2 + 8^{-1}R^{-1}\int_K \left|u_k(\xi)\frac{\operatorname{sh} v_k d(\xi)}{\mu_k}\right|^2 d\xi.$$

Let now $I \subset \{k \in \mathbf{N}: |\nu_k| > B\}$ be an arbitrary finite set. Then, using the Bessel inequality and the obvious estimate

$$\|w\|^2_{L^2(G)} = 2\,d(x) \leqq 6R,$$

$$\sum_{k\in I} \left| u_k(x) \frac{\operatorname{sh} \nu_k\, d(x)}{\mu_k} \right|^2 \leqq 12R + 8^{-1}R^{-1} \sum_{k\in I} \int_K \left| u_k(\xi) \frac{\operatorname{sh} \nu_k\, d(\xi)}{\mu_k} \right|^2 d\xi.$$

Integrating both sides by x on K (the length of which is $4R$),

$$\sum_{k\in I} \int_K \left| u_k(x) \frac{\operatorname{sh} \nu_k\, d(x)}{\mu_k} \right|^2 dx \leqq 48R^2 + 0{,}5 \sum_{k\in I} \int_K \left| u_k(x) \frac{\operatorname{sh} \nu_k\, d(x)}{\mu_k} \right|^2 dx,$$

$$\sum_{k\in I} \int_K \left| u_k(x) \frac{\operatorname{sh} \nu_k\, d(x)}{\mu_k} \right|^2 dx \leqq 96R^2,$$

whence, using (10) and the arbitrariness of I,

$$\sum_{|\nu_k|>B} |\mu_k|^{-2} \|v_k\|^2_{L^\infty(K_R)} \leqq 120^2 96R.$$

Hence Lemma 2 follows. Indeed, (1), (7) and $|\nu_k| > B$ imply $|\mu_k| \leqq B|\nu_k|$. Furthermore $|v_k(x)| \geqq |u_k(x)| \operatorname{ch} \nu_k R$ if $x \in K$. Therefore

$$\sum_{|\nu_k|>B} \left(\|u_k\|_{L^\infty(K)} \frac{\operatorname{sh} \nu_k R}{\nu_k R} \right)^2 \leqq 120^2 96 B^2 R^{-1}.$$

But $\operatorname{ch} \alpha \geqq \alpha \cdot \operatorname{ch}(0{,}5\,\alpha)$ if $\alpha > 0$, therefore

$$\sum_{|\nu_k|>B} \left(\|u_k\|_{L^\infty(K)} \operatorname{ch}(0{,}5\,\nu_k R) \right)^2 \leqq 120^2 96 B^2 R^{-1},$$

and Lemma 2 is proved (with $0{,}5R$ instead of R).

Part a) of Theorem 1 follows from Lemmas 1 and 2 easily. Indeed, it suffices to choose the numbers B, R to the compact interval $K \subset G$ according to Lemma 2 and then apply Lemma 1 with this B.

Using the already proved part of Theorem 1, we now prove Part b) of this theorem. Given $K = [a, b] \subset G$ arbitrarily, let us fix two positive numbers R, C satisfying the following conditions:

$$K_R \equiv [a - R, b + R] \subset G, \tag{11}$$

$$\sup_{\mu>0} \sum_{|\mu - |\varrho_k||\leqq 1} (\|u_k\|_{L^\infty(K_R)} \operatorname{ch} \nu_k R)^2 < C, \tag{12}$$

and, putting $Q \equiv \|q\|_{L^1(K_R)}$,

$$64Q^2 R(R+1)^2 C < 1. \tag{13}$$

This choice is possible in view of Part a) of Theorem 1. Fixing now $x \in K$ and $\mu > 0$ arbitrarily, let us introduce the function $w: G \to \mathbf{R}$ by the formula

$$w(y) \equiv \begin{cases} \cos \mu(y - x) & \text{if } |y - x| \leqq R, \\ 0 & \text{otherwise.} \end{cases}$$

One can see easily that $w \in L^2(G)$ and

$$\|w\|^2_{L^2(G)} = R + \frac{\sin 2\mu R}{2\mu} \geqq R - (2\mu)^{-1}. \tag{14}$$

Applying again the Titchmarsh formula, we can write

$$\langle u_k, w\rangle = u_k(x) \int_0^R 2\cos\mu t \cos\mu_k t\, dt +$$

$$+ \int_{x-R}^{x+R} q(\xi)\, u_k(\xi) \int_{|x-\xi|}^{R} \cos\mu t \frac{\sin\mu_k(t-|x-\xi|)}{\mu_k}\, dt\, d\xi \equiv w_k' + w_k''.$$

For any $M>0$ it follows from the Parseval formula that

$$\sum_{|\mu-|\varrho_k||\leqq M} |w_k'|^2 \geqq 2^{-1}\|w\|^2_{L^2(G)} - \sum_{|\mu-|\varrho_k||>M} |w_k'|^2 - \sum_{k=1}^{\infty} |w_k''|^2. \tag{15}$$

Now we estimate the sums occurring in (15). First of all, obviously

$$\sum_{|\mu-|\varrho_k||\leqq M} |w_k'|^2 \leqq 4R^2 \sum_{|\mu-|\varrho_k||\leqq M} |u_k(x) \operatorname{ch} v_k R|^2. \tag{16}$$

Furthermore, in case $|\mu-|\varrho_k||>M$ we have

$$\left|\int_0^R 2\cos\mu t\cos\mu_k t\, dt\right| = \left|\frac{\sin(\mu+\mu_k)R}{\mu+\mu_k} + \frac{\sin(\mu-\mu_k)R}{\mu-\mu_k}\right| \leqq 2|\mu-|\varrho_k||^{-1} \operatorname{ch} v_k R;$$

therefore, using also (12),

$$\sum_{|\mu-|\varrho_k||>M} |w_k'|^2 \leqq \sum_{|\mu-|\varrho_k||>M} 4|\mu-|\varrho_k||^{-2}|u_k(x)\operatorname{ch} v_k R|^2 \leqq$$

$$\leqq 4\sum_{i=0}^{\infty}(M+i)^{-2} \sum_{M+i<|\mu-|\varrho_k||\leqq M+i+1} |u_k(x)\operatorname{ch} v_k R|^2 \leqq 8C\sum_{i=0}^{\infty}(M+i)^{-2} \leqq 8CM^{-1},$$

i.e.

$$\sum_{|\mu-|\varrho_k||>M} |w_k'|^2 \leqq 8CM^{-1}. \tag{17}$$

One can see easily that for any $0\leqq s\leqq R$,

$$\left|\frac{\sin\mu_k s}{\mu_k}\right| \leqq \frac{2(R+1)}{1+|\mu_k|} \operatorname{ch} v_k R.$$

Indeed, in case $|\mu_k|\leqq R^{-1}$ (using that $|\sin z|\leqq 2|z|$ if $|z|\leqq 1$),

$$\left|\frac{\sin\mu_k s}{\mu_k}\right| \leqq 2R \leqq 2\frac{R+1}{1+|\mu_k|} \leqq \frac{2(R+1)}{1+|\mu_k|} \operatorname{ch} v_k R,$$

while in case $|\mu_k|>R^{-1}$

$$\left|\frac{\sin\mu_k s}{\mu_k}\right| \leqq \frac{\operatorname{ch} v_k R}{|\mu_k|} \leqq \frac{R+1}{1+|\mu_k|} \operatorname{ch} v_k R.$$

Hence

$$|w_k''| \leqq Q\|u_k\|_{L^\infty(K_R)} 2R(R+1)(1+|\mu_k|)^{-1} \operatorname{ch} \nu_k R$$

and

$$\sum_{k=1}^{\infty} |w_k''|^2 \leqq 4Q^2R^2(R+1)^2 \sum_{k=1}^{\infty} (1+|\mu_k|)^{-2}(\|u_k\|_{L^\infty(K_R)} \operatorname{ch} \nu_k R)^2 \leqq$$

$$\leqq 4Q^2R^2(R+1)^2 \sum_{i=1}^{\infty} i^{-2} \sum_{i-1 \leqq |\varrho_k| < i} (\|u_k\|_{L^\infty(K_R)} \operatorname{ch} \nu_k R)^2 \leqq 8Q^2R^2(R+1)^2C$$

i.e.

$$\sum_{k=1}^{\infty} |w_k''|^2 \leqq 8Q^2R^2(R+1)^2C. \tag{18}$$

It follows from properties (13)—(18) that

$$4R^2 \sum_{|\mu-|\varrho_k|| \leqq M} |u_k(x) \operatorname{ch} \nu_k R|^2 \geqq 2^{-1}R - (4\mu)^{-1} - 8CM^{-1} - 8^{-1}R,$$

whence

$$\sum_{|\mu-|\varrho_k|| \leqq M} |u_k(x) \operatorname{ch} \nu_k R|^2 \geqq 8^{-1}R^{-1}(0{,}75 - (2\mu R)^{-1} - 16CR^{-1}M^{-1}).$$

Now fix $M \equiv R^{-1} \max(2, 64C)$, then $16CR^{-1}M^{-1} \leqq 0{,}25$ and $(2\mu R)^{-1} \leqq 0{,}25$ for all $\mu \geqq M$. Therefore

$$\sum_{|\mu-|\varrho_k|| \leqq M} |u_k(x) \operatorname{ch} \nu_k R|^2 \geqq (32R)^{-1}$$

for all $x \in K$ and $\mu \geqq M$. Hence Part b) of Theorem 1 follows.

REMARK. In the proof of Part a) of Theorem 1 we did not use all the conditions of the theorem: our considerations remain valid if (u_k) is not a complete orthonormal system but only a Bessel-system of eigenfunctions of the operator L.

Let us now turn to the proof of Theorem 2. First we prove some preliminary estimates.

LEMMA 3. *Given any $R>0$, under the conditions of Theorem 2, there exists a constant $C(R)$ such that, setting*

$$\delta(\mu, |\varrho_k|) = \begin{cases} 1 & \text{if } \mu > |\varrho_k|, \\ 0{,}5 & \text{if } \mu = |\varrho_k|, \\ 0 & \text{if } \mu < |\varrho_k| \end{cases}$$

for all $k \in \mathbf{N}$ and $\mu > 0$, we have

$$\left| 2\pi^{-1} \int_0^R \frac{\sin \mu t \cos \mu_k t}{t}\, dt - \delta(\mu, |\varrho_k|) \right| \leqq \frac{C(R) \operatorname{ch} \nu_k R}{2 + |\mu - |\varrho_k||}.$$

PROOF. We recall that

$$\int_0^{\infty} \frac{\sin t}{t}\, dt = 2^{-1}\pi \tag{19}$$

and

$$\left|\int_x^\infty \frac{\sin t}{t}\,dt\right| < 6(1+x)^{-1} \quad \text{for all} \quad x \geqq 0. \tag{20}$$

((20) can be shown by integration by parts). Setting for brevity $\varrho_k=\varrho$ and $\nu_k=\nu$, we can write

(21)

$$2\pi^{-1}\int_0^R \frac{\sin \mu t \cos \mu_k t}{t}\,dt = 2\pi^{-1}\int_0^R \frac{\sin \mu t}{t}(\cos \varrho t \operatorname{ch} \nu t - i \sin \varrho t \operatorname{sh} \nu t)\,dt =$$

$$= 2\pi^{-1}\int_0^\infty \frac{\sin \mu t \cos \varrho t}{t}\,dt - 2\pi^{-1}\int_R^\infty \frac{\sin \mu t \cos \varrho t}{t}\,dt +$$

$$+2\pi^{-1}\int_0^R \sin \mu t \cos \varrho t \frac{\operatorname{ch} \nu t - 1}{t}\,dt - 2i\pi^{-1}\int_0^R \sin \mu t \sin \varrho t \frac{\operatorname{sh} \nu t}{t}\,dt \equiv I_1 + I_2 + I_3 + I_4.$$

Here, using also (19),

$$I_1 = \pi^{-1}\int_0^\infty \frac{\sin(\mu+\varrho)t + \sin(\mu-\varrho)t}{t}\,dt = \tag{22}$$

$$= \pi^{-1}(\operatorname{sgn}(\mu+\varrho) + \operatorname{sgn}(\mu-\varrho))\int_0^\infty \frac{\sin t}{t}\,dt = \delta(\mu, |\varrho|)$$

and

$$I_2 = -\pi^{-1}\int_R^\infty \frac{\sin(\mu+\varrho)t + \sin(\mu-\varrho)t}{t}\,dt =$$

$$= -\pi^{-1}\left(\operatorname{sgn}(\mu+\varrho)\int_{|\mu+\varrho|R}^\infty \frac{\sin t}{t}\,dt + \operatorname{sgn}(\mu-\varrho)\int_{|\mu-\varrho|R}^\infty \frac{\sin t}{t}\,dt\right)$$

whence in view of (20)

$$|I_2| \leqq \pi^{-1}(6(1+|\mu+\varrho|R)^{-1} + 6(1+|\mu-\varrho|R)^{-1}) \leqq 4(1+|\mu-|\varrho||R)^{-1}. \tag{23}$$

Considering now the quantities I_3, I_4, we have obviously

$$|I_3| \leqq 2\pi^{-1}\int_0^R \frac{\operatorname{ch} \nu t - 1}{t}\,dt \leqq 2\pi^{-1}\int_0^{|\nu R|} \frac{\operatorname{ch} x - 1}{x}\,dx \leqq \tag{24}$$

$$\leqq 2\pi^{-1}\int_0^{|\nu R|} \operatorname{sh} x\,dx = 2\pi^{-1}(\operatorname{ch} \nu R - 1)$$

and

$$|I_4| \leqq 2\pi^{-1} \int_0^R \left| \frac{\operatorname{sh} vt}{t} \right| dt = 2\pi^{-1} \int_0^{|vR|} \frac{\operatorname{sh} x}{x} dx \leqq \tag{25}$$

$$\leqq 2\pi^{-1} \int_0^{|vR|} \operatorname{ch} x \, dx = 2\pi^{-1} |\operatorname{sh} vR| \leqq 2\pi^{-1} \operatorname{ch} vR.$$

On the other hand, in case $\mu \neq |\varrho|$ we have

$$I_3 = \pi^{-1} \int_0^R (\sin(\mu+\varrho)t + \sin(\mu-\varrho)t) \frac{\operatorname{ch} vt - 1}{t} dt =$$

$$= \pi^{-1} \left[\left(\frac{-\cos(\mu+\varrho)t}{\mu+\varrho} + \frac{-\cos(\mu-\varrho)t}{\mu-\varrho} \right) \frac{\operatorname{ch} vt - 1}{t} \right]_0^R +$$

$$+ \pi^{-1} \int_0^R \left(\frac{\cos(\mu+\varrho)t}{\mu+\varrho} + \frac{\cos(\mu-\varrho)t}{\mu-\varrho} \right) \left(\frac{\operatorname{ch} vt - 1}{t} \right)' dt$$

and

$$I_4 = i\pi^{-1} \int_0^R (\cos(\mu+\varrho)t - \cos(\mu-\varrho)t) \frac{\operatorname{sh} vt}{t} dt =$$

$$= i\pi^{-1} \left[\left(\frac{\sin(\mu+\varrho)t}{\mu+\varrho} - \frac{\sin(\mu-\varrho)t}{\mu-\varrho} \right) \frac{\operatorname{sh} vt}{t} \right]_0^R -$$

$$- i\pi^{-1} \int_0^R \left(\frac{\sin(\mu+\varrho)t}{\mu+\varrho} - \frac{\sin(\mu-\varrho)t}{\mu-\varrho} \right) \left(\frac{\operatorname{sh} vt}{t} \right)' dt.$$

Taking into account that the functions

$$\left(\frac{\operatorname{ch} vt - 1}{t} \right)', \quad \left(\frac{\operatorname{sh} vt}{t} \right)'$$

do not change sign, hence we obtain

$$|I_3| \leqq 2\pi^{-1} |\mu - |\varrho||^{-1} \left(\frac{\operatorname{ch} vR - 1}{R} + \left| \int_0^R \left(\frac{\operatorname{ch} vt - 1}{t} \right)' dt \right| \right) \leqq \tag{26}$$

$$\leqq 4\pi^{-1} R^{-1} |\mu - |\varrho||^{-1} (\operatorname{ch} vR - 1)$$

and

$$|I_4| \leqq 2\pi^{-1} |\mu - |\varrho||^{-1} \left(\left| \frac{\operatorname{sh} vR}{R} - v \right| + \left| \int_0^R \left(\frac{\operatorname{sh} vt}{t} \right)' dt \right| \right) \leqq \tag{27}$$

$$\leqq 4\pi^{-1} R^{-1} |\mu - |\varrho|| \, |\operatorname{sh} vR - vR|.$$

Now the Lemma follows from relations (21)—(27).

LEMMA 4. *Given any* $R>0$, *there exists a constant* $C(R)$ *such that for all* $k\in\mathbf{N}$, $\mu>0$ *and* $0\leqq r\leqq R$,

$$\left|\int_r^R \frac{\sin \mu t}{\pi t}\,\frac{\sin \mu_k(t-r)}{\mu_k}\,dt\right| \leqq C(R)\,\mathrm{ch}\,\nu_k R(1+|\varrho_k|)^{-0,75}.$$

PROOF. Obviously

$$\left|\int_r^R \frac{\sin \mu t}{\pi t}\,\frac{\sin \mu_k(t-r)}{\mu_k}\,dt\right| \leqq \int_0^R \left|\frac{\sin \mu_k x}{\pi \mu_k x}\right| dx.$$

Furthermore, in case of $|\varrho_k|\leqq R^{-1}$ using the estimate

$$\left|\frac{\sin z}{z}\right| \leqq 2\,\mathrm{ch}\,(\mathrm{Im}\,z),$$

we have

$$\int_0^R \left|\frac{\sin \mu_k x}{\pi \mu_k x}\right| dx \leqq R\,\mathrm{ch}\,\nu_k R,$$

and if $|\varrho_k|>R^{-1}$ then

$$\int_0^R \left|\frac{\sin \mu_k x}{\pi \mu_k x}\right| dx \leqq \int_0^{|\varrho_k|^{-1}} \left|\frac{\sin \mu_k x}{\pi \mu_k x}\right| dx + \int_{|\varrho_k|^{-1}}^R \left|\frac{\sin \mu_k x}{\pi \mu_k x}\right| dx \leqq$$

$$\leqq \frac{\mathrm{ch}\,\nu_k R}{|\varrho_k|} + \int_{|\varrho_k|^{-1}}^R \frac{\mathrm{ch}\,\nu_k R}{|\varrho_k| x}\,dx = \frac{\mathrm{ch}\,\nu_k R}{|\varrho_k|}(1+\ln R+\ln|\varrho_k|).$$

The Lemma follows from these two inequalities.

Now we turn to the proof of Theorem 2. Let $K=[a, b]\subset G$ be an arbitrary compact interval. In view of Theorem 1 we can fix a number $R>0$ such that $K_R\equiv$ $\equiv[a-R, b+R]\subset G$ and for some $D<\infty$

$$\sup_{\mu>0} \sum_{|\mu-|\varrho_k||\leqq 1} \{\|u_k\|_{L^\infty(K_R)}\,\mathrm{ch}\,\nu_k R\}^2 < D. \tag{28}$$

Let us now fix a constant $C=C(R)$ in view of Lemmas 1 and 2 such that for any $\mu>0$ and $k\in\mathbf{N}$,

$$\left|2\pi^{-1}\int_0^R \frac{\sin \mu t\cos \mu_k t}{t}\,dt - \delta(\mu, |\varrho_k|)\right| \leqq \frac{C\,\mathrm{ch}\,\nu_k R}{2+|\mu-|\varrho_k||} \tag{29}$$

and

$$\sup_{0\leqq r\leqq R}\left|\int_r^R \frac{\sin \mu t}{\pi t}\,\frac{\sin \mu_k(t-r)}{\mu_k}\,dt\right| \leqq \frac{C\,\mathrm{ch}\,\nu_k R}{(1+|\varrho_k|)^{0,75}}. \tag{30}$$

Fixing $\mu>0$ and $x\in K$ arbitrarily, let us introduce the function $w: G\to\mathbf{R}$ by the formula

$$w(y)\equiv\begin{cases}\dfrac{\sin \mu(y-x)}{\pi(y-x)} & \text{if } |y-x|\leqq R,\\ 0 & \text{if } |y-x|>R.\end{cases}$$

Furthermore, for $f \in L^2(G)$ let us set

$$S_\mu(f, x) \equiv \int_{x-R}^{x+R} \frac{\sin \mu(y-x)}{\pi(y-x)} f(y)\, dy.$$

It is well-known [8] that for all continuously differentiable functions $f \in L^2(G)$,

$$\lim_{\mu \to 0} \sup_{x \in K} |f(x) - S_\mu(f, x)| = 0. \tag{31}$$

Applying the Titchmarsh formula,

$$\langle u_k, w \rangle = u_k(x) 2\pi^{-1} \int_0^R \frac{\sin \mu t \cos \mu_k t}{t}\, dt + \tag{32}$$

$$+ \int_{x-R}^{x+R} q(\xi) u_k(\xi) \int_{|x-\xi|}^{R} \frac{\sin \mu t \sin \mu_k (t - |x-\xi|)}{\pi t \mu_k}\, dt\, d\xi.$$

Being $w \in L^2(G)$ and $(u_k) \subset L^2(G)$ a complete orthonormal system we have for any $f \in L^2(G)$

$$\langle f, w \rangle = \sum_{k=1}^{\infty} \langle f, u_k \rangle \langle u_k, w \rangle. \tag{33}$$

Putting $Q \equiv \|q\|_{L^1(K_R)} (< \infty)$, (32), (29) and (30) yield

$$\left|\langle u_k, w \rangle - u_k(x) \delta(\mu, |\varrho_k|)\right| \leqq C |u_k(x)| \operatorname{ch} \nu_k R (2 + |\mu - |\varrho_k||)^{-1} +$$

$$+ CQ \|u_k\|_{L^\infty(K_R)} \operatorname{ch} \nu_k R (1 + |\varrho_k|)^{-0,75}.$$

Using (33), the definitions of $\delta(\mu, |\varrho_k|)$, $\sigma_\mu(f, x)$ and the equality

$$\langle f, w \rangle = S_\mu(f, x),$$

we obtain

$$|S_\mu(f, x) - \sigma_\mu(f, x)| \leqq 0{,}5 \sum_{|\varrho_k| = \mu} |\langle f, \mu_k \rangle u_k(x)| + \tag{34}$$

$$+ C \sum_{k=1}^{\infty} |\langle f, u_k \rangle| |u_k(x)| \operatorname{ch} \nu_k R (2 + |\mu - |\varrho_k||)^{-1} +$$

$$+ CQ \sum_{k=1}^{\infty} |\langle f, u_k \rangle| \|u_k\|_{L^\infty(K_R)} \operatorname{ch} \nu_k R (1 + |\varrho_k|)^{-0,75}.$$

(We remark that if the sum defining $\sigma_\mu(f, x)$ turns out to be infinity then we have to show that this sum makes sense. This follows from (28) and the Bessel inequality, after having used the Cauchy—Schwarz inequality for the sum defining $\sigma_\mu(f, x)$.)

Here, applying Parseval's formula and the estimate (28),

$$\sum_{k=1}^{\infty} |\langle f, u_k\rangle|^2 = \|f\|^2_{L^2(G)}, \tag{35}$$

(36)

$$\sum_{k=1}^{\infty} |u_k(x) \operatorname{ch} \nu_k R|^2 (2+|\mu-|\varrho_k||)^{-2} \leqq \sum_{i=0}^{\infty} \sum_{i \leqq |\varrho_k| < i+1} |u_k(x) \operatorname{ch} \nu_k R|^2 (1+|\mu-i|)^{-2} \leqq$$

$$\leqq D \sum_{i=0}^{\infty} (1+|\mu-i|)^{-2} \leqq 2D \sum_{i=1}^{\infty} i^{-2} \leqq 4D,$$

$$\sum_{k=1}^{\infty} (\|u_k\|_{L^\infty(K_R)} \operatorname{ch} \nu_k R)^2 (1+|\varrho_k|)^{-1,5} \leqq \tag{37}$$

$$\leqq \sum_{i=1}^{\infty} \sum_{i-1 \leqq |\varrho_k| < i} (\|u_k\|_{L^\infty(K_R)} \operatorname{ch} \nu_k R)^2 i^{-1,5} \leqq D \sum_{i=1}^{\infty} i^{-1,5} \leqq 3D.$$

Finally, we can conclude from (34), using the Cauchy—Schwarz inequality and the estimates (28), (35), (36), (37) that

$$|S_\mu(f, x) - \sigma_\mu(f, x)| \leqq \|f\|_{L^2(G)} D^{0,5}(0,5 + 2C + 3^{0,5}CQ).$$

Here $\mu > 0$, $x \in K$ and $f \in L^2(G)$ are arbitrary. Summing up our considerations, there exists a constant E such that for all $f \in L^2(G)$,

$$\sup_{\mu>0} \sup_{x \in K} |S_\mu(f, x) - \sigma_\mu(f, x)| \leqq E \|f\|_{L^2(G)}. \tag{38}$$

Let now $f \in L^2(G)$ and $\varepsilon > 0$ be arbitrary and choose a finite linear combination $P \equiv c_1 u_1 + \ldots + c_m u_m$ of the eigenfunctions u_k with

$$\|f - P\|_{L^2(G)} \leqq 2^{-1} E^{-1} \varepsilon. \tag{39}$$

Being P continuously differentiable, by (31) there exists a constant $N = N(\varepsilon)$ such that

$$\sup_{x \in K} |P(x) - S_\mu(P, x)| < 2^{-1}\varepsilon \quad \text{whenever} \quad \mu > N. \tag{40}$$

Using (38), (39), (40), we have for all $x \in K$ and $\mu > \max(N, m)$

$$|\sigma_\mu(f, x) - S_\mu(f, x)| \leqq |P(x) - S_\mu(P, x)| +$$

$$+ |\sigma_\mu(f - P, x) - S_\mu(f - P, x)| \leqq 2^{-1}\varepsilon + E\|f - P\|_{L^2(G)} \leqq \varepsilon$$

i.e.

$$\lim_{\mu \to \infty} \sup_{x \in K} |\sigma_\mu(f, x) - S_\mu(f, x)| = 0. \tag{41}$$

Analogously we have

$$\lim_{\mu \to \infty} \sup_{x \in K} |\hat{\sigma}_\mu(f, x) - S_\mu(f, x)| = 0. \tag{42}$$

(41) and (42) yield

$$\lim_{\mu \to \infty} \sup_{x \in K} |\sigma_\mu(f, x) - \hat{\sigma}_\mu(f, x)| = 0$$

and Theorem 2 is proved.

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MAL'CEV CONDITIONS FOR HORN SENTENCES WITH CONGRUENCE PERMUTABILITY

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1. Introduction

From the general description of properties of varieties characterizable by Mal'cev conditions (cf. Taylor [12] and Neumann [10]) it follows (Jónsson [9, Theorem 2.16]) that the satisfaction of a Horn sentence in the congruence lattices of n-permutable varieties can be characterized by Mal'cev conditions. Yet, no concrete Mal'cev conditions have been known in general (for exceptions cf. [1, 2, 3]). The aim of the present note is to give an algorithm which associates a suitable concrete Mal'cev condition with any (lattice) Horn sentence.

By a (universal lattice) Horn sentence we mean a first order formula of the form

$$\chi\colon\ (\forall\alpha_0)\ldots(\forall\alpha_s)\left(\bigwedge_{i=1}^{k}(p_i\leqq q_i)\Rightarrow p\leqq q\right)$$

where p_i, q_i, p, and q are lattice terms over the set $S=\{\alpha_0, \alpha_1, \ldots, \alpha_s\}$ of variables, and $k\geqq 0$. Let χ of the above form be fixed throughout, and let us fix an integer $n\geqq 2$, too. A variety $\mathbf{V}$ of algebras is said to be n-permutable if $\alpha\vee\beta=\alpha\circ\beta\circ\alpha\circ\beta\circ\alpha\circ\ldots$ (with n factors) holds for congruences of its members.

Given a variety $\mathbf{V}$, the class of lattices embeddable into congruence lattices of its members will be denoted by $\mathbf{SCV}=\mathbf{S}\{\mathrm{Con}\,(A)\colon A\in\mathbf{V}\}=\{L\colon L\leqq\mathrm{Con}\,(A)$ for some $A\in\mathbf{V}\}$. The quest for our Mal'cev conditions can be motivated by the following.

PROPOSITION 1.1 (cf. Hutchinson [6] for varieties of modules). *For any n-permutable variety* $\mathbf{V}$, $\mathbf{SCV}$ *is a quasivariety, i.e., a class of lattices definable by a set of Horn sentences.*

In [8] rings are classified according to lattice identities being satisfied in congruence lattices of the corresponding modules. This classification is based on (strong) Mal'cev conditions supplied by the Wille—Pixley algorithm [11, 13]. Hutchinson [7] has shown that this classification can be properly refined if we use Horn sentences rather than only identities, which gives another motivation for our investigations.

2. Proof of Proposition 1.1

Let Σ be the set of all Horn formulae that hold in $\mathbf{CV}$ or, equivalently, in $\mathbf{SCV}$, and let Σ hold in a lattice L, too. We have to show that $L\in\mathbf{SCV}$. Choose a well-ordering of L, i.e., $L=\{a_\alpha\colon \alpha<\gamma\}$ and over the set $X=\{x_\alpha\colon \alpha<\gamma\}$ of variables let D be the operation table (or diagram statement) of L. I.e., D is the conjunction of all

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inequalities of the forms $x_\delta \leqq x_\varepsilon$, $x_\mu \wedge x_\nu \leqq x_\eta$, $x_\zeta \leqq x_\varrho \vee x_\xi$ that hold in L under the substitution $X \to L$, $x_\alpha \mapsto a_\alpha$ $(\alpha < \gamma)$. Note that D holds for a system $\{b_\alpha : \alpha < \gamma\}$ of elements in a lattice A iff $L \to A$, $a_\alpha \mapsto b_\alpha$ is a homomorphism. Now let $a_\delta \not\leqq a_\varepsilon$ in L and consider the universally quantified generalized Horn sentence $D \Rightarrow x_\delta \leqq x_\varepsilon$. Then $D \Rightarrow x_\delta \leqq x_\varepsilon$ does not hold in L and we claim that it does not hold in **SCV** either. Suppose indirectly that $D \Rightarrow x_\delta \leqq x_\varepsilon$ still hold in **SCV** and, following the method of Taylor's [12], consider the similarity type $\tau = \sigma \cup \{r_\alpha : \alpha < \gamma\}$ where σ is the similarity type of **V** and r_α are binary relation symbols. Let Δ_1 be the set of all identities holding in **V** and let Δ_2 be the set of first order τ-formulae that express that r_α are congruences and satisfy D. (This is possible. If, e.g., $a_\zeta \leqq a_\varrho \vee a_\xi$ holds in L then consider the formula $(\forall y_0)(\forall y_n)(y_0 r_\zeta y_n \Rightarrow (\exists y_1) \dots (\exists y_{n-1})(y_0 r_\varrho y_1 \& y_1 r_\xi y_2 \& y_2 r_\varrho y_3 \& \& y_3 r_\xi y_4 \& \dots y_n))$, cf. also Taylor [12].) Since $D \Rightarrow x_\delta \leqq x_\varepsilon$ holds in **CV**, we have that $(\forall y)(\forall z)\ (y r_\delta z \Rightarrow y r_\varepsilon z)$ is a consequence of $\Delta_1 \cup \Delta_2$. By the compactness theorem it is a consequence of $\Delta_1 \cup \Delta_2'$ for some finite $\Delta_2' \subseteq \Delta_2$, too. Therefore there exists a finite part D' of D (i.e., D' is only a finite conjunction of some members occurring in D) such that $D' \Rightarrow x_\delta \leqq x_\varepsilon$ holds in **CV**. But $D' \Rightarrow x_\delta \leqq x_\varepsilon$ is a Horn sentence. Hence it holds in L, which is a contradiction (choose the substitution $x_\alpha \mapsto a_\alpha$).

Therefore $D \Rightarrow x_\delta \leqq x_\varepsilon$ does not hold in **CV**, whence there is an algebra $A_{\delta,\varepsilon} \in \mathbf{V}$ and a lattice homomorphism $\varphi_{\delta,\varepsilon} : L \to \mathrm{Con}\,(A_{\delta,\varepsilon})$ for which $\varphi_{\delta,\varepsilon}(a_\delta) \not\leqq \varphi_{\delta,\varepsilon}(a_\varepsilon)$. Let Γ denote the set $\{(\delta, \varepsilon) : a_\delta \not\leqq a_\varepsilon \text{ in } L,\ \delta, \varepsilon < \gamma\}$, let $\prod_{(\delta,\varepsilon)\in\Gamma} A_{\delta,\varepsilon} = A \in \mathbf{V}$, and consider the map $\varphi : L \to \mathrm{Con}\,(A)$, $a_\alpha \mapsto \varphi(a_\alpha) = \{(f, g) \in A^2 : f(\delta, \varepsilon)\varphi_{\delta,\varepsilon}(a_\alpha) g(\delta, \varepsilon)$ for all $(\delta, \varepsilon) \in \Gamma\}$. We claim that φ is an embedding. It is easy to see that φ preserves the meet. Since $\varphi(a) \vee \varphi(b) \leqq \varphi(a \vee b)$ $(a, b \in L)$ follows from φ being monotone, the converse inequality has to be shown. Suppose $(f, g) \in \varphi(a \vee b)$. Then, for all $(\delta, \varepsilon) \in \Gamma$, $(f(\delta, \varepsilon), g(\delta, \varepsilon)) \in \varphi_{\delta,\varepsilon}(a \vee b) = \varphi_{\delta,\varepsilon}(a) \vee \varphi_{\delta,\varepsilon}(b)$. By n-permutability we can choose elements $h^0_{\delta,\varepsilon}, \dots, h^n_{\delta,\varepsilon} \in A_{\delta,\varepsilon}$ such that $f(\delta, \varepsilon) = h^0_{\delta,\varepsilon}$, $g(\delta, \varepsilon) = h^n_{\delta,\varepsilon}$, $(h^i_{\delta,\varepsilon}, h^{i+1}_{\delta,\varepsilon}) \in \varphi_{\delta,\varepsilon}(a)$ for $i < n$, i even, and $(h^i_{\delta,\varepsilon}, h^{i+1}_{\delta,\varepsilon}) \in \varphi_{\delta,\varepsilon}(b)$ for $i < n$, i odd. Defining $h^0, h^1, \dots, h^n \in A$ by $h^i(\delta, \varepsilon) = h^i_{\delta,\varepsilon}$ we obtain $f = h^0$, $g = h^n$, $(h^i, h^{i+1}) \in \varphi(a)$ for i even, and $(h^i, h^{i+1}) \in \varphi(b)$ for i odd. Thus $(f, g) \in \varphi(a) \vee \varphi(b)$, and φ is a homomorphism. Finally, for $(\delta, \varepsilon) \in \Gamma$, $\varphi_{\delta,\varepsilon}(a_\delta) \not\leqq \varphi_{\delta,\varepsilon}(a_\varepsilon)$ implies $\varphi(a_\delta) \not\leqq \varphi(a_\varepsilon)$, whence φ is injective. The proof, which uses only that **V** is axiomatizable, n-permutable, and closed under products, is complete.

3. The Mal'cev condition and its necessity

If we presented the main theorem before having proved anything from it, it could seem too complicated at the first sight. Hence the proof of its necessity part will preceed the main theorem. However, the necessary notations and concepts will be emphasized in this section.

To simplify our forthcoming notations, if $Q = \{z_0, z_1, \dots, z_s\}$ is a set (or a system) and d is an $(s+1)$-ary term over Q then let $d(Q)$, $d(y : y \in Q)$, and $d(z_i : i \leqq s)$ stand for $d(z_0, z_1, \dots, z_s)$. If no (well-)ordering of Q is given then any *fixed* one can be considered. Therefore our Mal'cev conditions will be unique only modulo the choices of these orderings, but we shall get appropriate conditions at all choices.

First for any lattice term $d = d(\alpha : \alpha \in S)$ we define a graph $G(d) = G_n(d)$ associated with d. The edges of $G(d)$ will be coloured by the variables $\alpha \in S$, and two distinguished vertices, the so-called left and right endpoints, will have special roles.

In figures these endpoints will be always placed on the left-hand side and on the right-hand side, respectively. An α-coloured edge connecting the vertices x and y will be often denoted by (x, α, y). Before defining $G(d)$ we introduce two kinds of operations for graphs. We obtain the *parallel connection* of graphs G_1 and G_2 by taking disjoint copies of G_1 and G_2 and identifying their left (right, resp.) endpoints (Figure 1). By

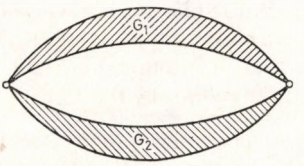


Fig. 1

taking disjoint graphs $H_1, H_2, \ldots, H_n$ such that $H_i \cong G_1$ for i odd and $H_i \cong G_2$ for i even, and identifying the right endpoint of H_i and the left endpoint of H_{i+1} for $i = 1, 2, \ldots, n-1$ we obtain the *serial connection* (of length n) of the graphs G_1 and G_2. (The left endpoint of H_1 and the right one of H_n are the endpoints of the serial connection, cf. Figure 2.)



Fig. 2

Now, if d is a variable then let $G(d)$ be the graph consisting of two vertices (the endpoints) and a single d-coloured edge connecting them. Let $G(d_1 \wedge d_2)$ be the parallel connection of $G(d_1)$ and $G(d_2)$ while $G(d_1 \vee d_2)$ is defined to be the serial connection (of length n) of $G(d_1)$ and $G(d_2)$.

For an algebra A, a lattice term $d(\alpha\colon \alpha \in S)$, $a_0, a_1 \in A$ and congruences $\mathrm{con}(\alpha)$ $(\alpha \in S)$ of A we say that a_0 and a_1 *can be connected by* $G(d)$ *in* A if there is a map φ (referred to as *connecting map*) from the vertex set of $G(d)$ into A such that a_0 and a_1 are the images of the left and right endpoint, respectively, and for any $\alpha \in S$ and α-coloured edge (x, α, y) we have $(\varphi(x), \varphi(y)) \in \mathrm{con}(\alpha)$. If it is necessary we can emphasize that the *colour* α *is represented by the congruence* $\mathrm{con}(\alpha)$.

The following statement follows by a straightforward induction from definitions, therefore its proof will be omitted.

PROPOSITION 3.1. *Consider an algebra A with n-permutable congruences, $a_0, a_1 \in A$, a lattice term $d(\alpha\colon \alpha \in S)$ and congruences $\mathrm{con}(\alpha)$ $(\alpha \in S)$ of A. Then $(a_0, a_1) \in d(\mathrm{con}(\alpha)\colon \alpha \in S)$ if and only if a_0 and a_1 can be connected by $G(d)$ in A.*

We also need the following.

PROPOSITION 3.2. *Let $\mathbf{V}$ be an n-permutable variety, $a, b \in A \in \mathbf{V}$, and let $\mathrm{con}(a, b)$ be the congruence of A generated by (a, b). Then, for $x, y \in A$, $(x, y) \in \mathrm{con}(a, b)$ if and only if there are translations (i.e., unary algebraic functions) $t_0, \ldots, t_{n-2}$ of A such that $x = t_0(a)$, $y = t_{n-2}(b)$, and $t_i(b) = t_{i+1}(a)$ for all i, $0 \leqq i < n-2$.*

PROOF. It is easy to check that the relation $\Psi=\{(x, y)\in A^2$: $t(a)=x$ and $t(b)=y$ for some translation $t\}$ is reflexive and compatible. By applying a theorem of Hagemann [4] (cf. also the Remark in [5]) we obtain that $\Phi=\Psi\circ\Psi\circ\ldots\circ\Psi$ ($n-1$ factors) is already a congruence. Now $(a, b)\in\Phi\subseteq\mathrm{con}(a, b)$ completes the proof.

In order to formulate an evident corollary to Proposition 3.2 precisely *we define two functions,* v: $\{1, 2, 3, \ldots\}\to\{1, 2, 3, \ldots\}$ and k: $\{(u, i)$: $1\leqq i\leqq v(u)\}\to$ $\to\{1, 2, 3, \ldots\}$, via induction. Put $v(1)=k(1, 1)=1$ and assume that $v(u)$, $k(u, 1)$, $k(u, 2)$, ..., $k(u, v(u))$ have already been defined. Then let $v(u+1)=v(u)+1+$ $+v(u)+1+v(u)+1+\ldots$ (with n summands), $k(u+1, i)=k(u, j)$ for $1\leqq j\leqq v(u)$ and $j\equiv i \bmod (v(u)+1)$, and $k(u+1, i)=u+1$ for $i\equiv 0 \bmod (v(u)+1)$.

COROLLARY 3.3. *Given an n-permutable variety* $\mathbf{V}$, $A\in\mathbf{V}$, $(x, y)\in A^2$ *and* $H=\{(a_1, b_1), \ldots, (a_u, b_u)\}\subseteq A^2$, *we have* $(x, y)\in\mathrm{con}(H)=\bigvee\{\mathrm{con}(a_i, b_i)$: $1\leqq i\leqq u\}$ *if and only if there are translations* t_{ij} $(1\leqq i\leqq v(u)$, $0\leqq j\leqq 2)$ *such that* $x=t_{10}(a_{k(u,1)})$, $y=t_{v(u),n-2}(b_{k(u,v(u))})$, $t_{i,n-2}(b_{k(u,i)})=t_{i+1,0}(a_{k(u,i+1)})$ *for* $1\leqq i<v(u)$, *and* $t_{ij}(b_{k(u,i)})=$ $=t_{i,j+1}(a_{k(u,i)})$ *for* $1\leqq i\leqq v(u)$ *and* $0\leqq j<n-2$.

Now let us fix an n-permutable variety $\mathbf{V}$ of similarity type σ and consider the Horn sentence χ from the Introduction. Following the customary way of finding Mal'cev conditions we intend to define suitable congruences of a free $\mathbf{V}$-algebra for which the premise of χ holds.

Continue the list $p_1\leqq q_1$, ..., $p_k\leqq q_k$ cyclicly by repeating its elements, i.e., for $k<i<\omega$ p_i and q_i are defined to be p_j and q_j, respectively, provided $i\equiv j \bmod (k)$. For a lattice term $d=d(\alpha$: $\alpha\in S)$ let $V(d)$, $W(d)$ and $E(d)$ denote the *vertex* set of $G(d)$, the set of inner vertices of $G(d)$ (vertex set *without* the endpoints), and the *edge* set of $G(d)$, respectively. Note that $0\leqq|W(d)|=|V(d)|-2$ and $E(d)\subseteq V(d)\times S\times$ $\times V(d)$.

Now let $X_0=Y_0=V(p)$, with $x_0, x_1\in V(p)$ the endpoints, and put $A_0=F_{\mathbf{V}}(X_0)$, the free $\mathbf{V}$-algebra generated by X_0. For $\alpha\in S$ let $\mathrm{con}_0(\alpha)$ be the congruence of A_0 generated by $\{(x, y)$: (x, α, y) is an edge of $G(p)\}$. By Proposition 3.1 we have $(x_0, x_1)\in p(\mathrm{con}_0(\alpha)$: $\alpha\in S)$. To make the premise of χ hold we shall improve this construction in countably many steps.

Suppose $A_{i-1}=F_{\mathbf{V}}(X_{i-1})$ and $\mathrm{con}_{i-1}(\alpha)$ $(\alpha\in S)$ have already been defined for some $1\leqq i<\omega$. For a lattice term $d(\alpha$: $\alpha\in S)$ let $\mathrm{con}_{i-1}(d)$ stand for $d(\mathrm{con}_{i-1}(\alpha)$: $\alpha\in S)$. Put $Y_i=\{i\}\times N\times\mathrm{con}_{i-1}(p_i)\times W(q_i)$ (where N is the set of natural numbers) and $X_i=X_{i-1}\cup Y_i=X_{i-1}\dot\cup Y_i=Y_0\dot\cup\ldots\dot\cup Y_{i-1}\dot\cup Y_i$ (where $\dot\cup$ stands for disjoint union). Note that $\{i\}$ and N are for making Y_i disjoint from X_{i-1} and for making Y_i infinite (if not empty). For $\alpha\in S$ let $\mathrm{con}_i(\alpha)$ be the congruence of A_i generated by $\mathrm{con}_{i-1}(\alpha)\cup\{((i, j, a, b, x), (i, j, a, b, y))$: $j\in N$, $(a, b)\in\mathrm{con}_{i-1}(p_i)$, $x, y\in W(q_i)$, and $(x, \alpha, y)\in E(q_i)\}\cup\{(a, (i, j, a, b, z))$: $j\in N$, $(a, b)\in\mathrm{con}_{i-1}(p_i)$, and (left endpoint, α, $Z)\in E(q_i)\}\cup\{((i, j, a, b, z), b)$: $j\in N$, $(a, b)\in\mathrm{con}_{i-1}(p_i)$, and $(z, \alpha$, right endpoint$)\in E(q_i)\}$ where (i, j, a, b, x) stands for $(i, j, (a, b), x)\in Y_i$. Observe that, by Proposition 3.1, $\mathrm{con}_{i-1}(p_i)\subseteq q_i(\mathrm{con}_i(\alpha)$: $\alpha\in S)=\mathrm{con}_i(q_i)$.

Finally, let X and $A=F_{\mathbf{V}}(X)$ be the directed unions of X_i and A_i $(i<\omega)$, respectively. Then, for $\alpha\in S$, $\mathrm{con}(\alpha)=\bigcup\{\mathrm{con}_i(\alpha)$: $i<\omega\}$ is a congruence. From Proposition 3.1 it follows that $d(\mathrm{con}(\alpha)$: $\alpha\in S)=\bigcup\{d(\mathrm{con}_i(\alpha)$: $\alpha\in S)$: $i<\omega\}$, for any lattice term d. Hence we obtain that, for $1\leqq j\leqq k$ and $(a, b)\in p_j(\mathrm{con}(\alpha)$: $\alpha\in S)$, $(a, b)\in p_j(\mathrm{con}_{i-1}(\alpha)$: $\alpha\in S)=\mathrm{con}_{i-1}(p_j)=\mathrm{con}_{i-1}(p_i)\subseteq\mathrm{con}_i(q_i)=\mathrm{con}_i(q_j)=$

$=q_j(\mathrm{con}_i(\alpha)\colon \alpha\in S)\subseteq q_j(\mathrm{con}(\alpha)\colon \alpha\in S)$ for some $i<\omega$, $i\equiv j \bmod (k)$. I.e., the premise equations of χ hold for $\mathrm{con}(\alpha)$, $\alpha\in S$.

Suppose that χ holds in **CV**. Then we have $(x_0, x_1)\in p(\mathrm{con}(\alpha)\colon \alpha\in S)\subseteq q(\mathrm{con}(\alpha)\colon \alpha\in S)$, from which we will derive our Mal'cev condition.

For a cardinal $\varkappa>0$ and a lattice term $d(\alpha\colon \alpha\in S)$ let $G(\varkappa\times d)$ denote the disjoint union of $\varkappa$ copies of $G(d)$. Let $E(\varkappa\times d)$ and $V(\varkappa\times d)$ stand for the edge set and for the vertex set of $G(\varkappa\times d)$, respectively. The graph $G(\varkappa\times d)$ has $2\varkappa$ endpoints. However, $\varkappa$ pairs of endpoints will have special roles, the two components c_0, c_1 of any such pair (c_0, c_1) are the left and right endpoints, respectively, of the same copy of $G(d)$. Let us also agree that $E(d_1)\mathring{\cup}E(\varkappa\times d_2)$ stands for the edge set of $G(d_1)\mathring{\cup}\mathring{\cup}G(\varkappa\times d_2)$, etc. The set of inner (not endpoint) vertices of $G(\varkappa\times d)$ is denoted by $W(\varkappa\times d)$.

Now from $(x_0, x_1)\in q(\mathrm{con}(\alpha)\colon \alpha\in S)$ we obtain that x_0 and x_1 can be connected by $G(q)$ in A, using a connecting map $\varphi\colon V(q)\to A$ and representing a colour α by $\mathrm{con}(\alpha)$. By finiteness, there exists an m, $2\leqq m<\omega$, such that $\varphi(V(q))\subseteq A_m$ and each colour $\alpha\in S$ is represented by $\mathrm{con}_m(\alpha)$.

Consider the map $\varrho\colon V(p)\mathring{\cup}\mathring{\bigcup}\{V(\varkappa_i\times q_i)\colon 1\leqq i<\omega\}\to A$ where $\varkappa_i=|N|\cdot|\mathrm{con}_{i-1}(p_i)|$, $\varrho(x)=(i, j, a, b, x)\in Y_i$ if x is an inner vertex of the $(j, (a, b))$-th copy of $G(q_i)$, $\varrho(x)=a$ ($=b$, resp.) if x is the left (right, resp.) endpoint in the $(j, (a, b))$-th copy of $G(q_i)$ for some i, j, and $\varrho(x)=x$ if $x\in V(p)$. Observe that the restriction of ϱ to $V(p)\mathring{\cup}\mathring{\bigcup}\{W(\varkappa_i\times q_i)\colon 1\leqq i<\omega\}$ is injective and maps onto X. Further, $\mathrm{con}_i(\alpha)$ is generated by $\{(\varrho(a), \varrho(b))\colon (a, \alpha, b)\in E(p)\mathring{\cup}E(\varkappa_1\times q_1)\mathring{\cup}\ldots\mathring{\cup}E(\varkappa_i\times q_i)\}$.

Since $G(q)$ is finite, there are finite cardinals $u_1', u_2', \ldots, u_{m-1}', u_m$ and a restriction ψ of ϱ such that for any $(f, \alpha, g)\in E(q)$ $(\varphi(f), \varphi(g))$ is in the congruence of A_m generated by $\{(\psi(x), \psi(y))\colon (x, \alpha, y)\in E(p)\mathring{\cup}E(u_1'\times q_1)\mathring{\cup}\ldots\mathring{\cup}E(u_{m-1}'\times q_{m-1})\mathring{\cup}\mathring{\cup}E(u_m\times q_m)\}$ and for any $f\in V(q)$ $\varphi(f)$ is in the subalgebra generated by $\psi(V(p)\mathring{\cup}\mathring{\cup}W(u_1'\times q_1)\mathring{\cup}\ldots\mathring{\cup}W(u_{m-1}'\times q_{m-1}\mathring{\cup}W(u_m\times q_m))$. If (a, b) is a pair of endpoints in $G(u_i'\times q_i)$, for $1\leqq i\leqq m$ (where $u_i'=u_m$ in case of $i=m$), then $(\psi(a), \psi(b))\in\mathrm{con}_{i-1}(p_i)$. Hence, by Proposition 3.1, there is a (multiple connecting) map $\eta\colon V(u_1'\times p_1)\mathring{\cup}\ldots\mathring{\cup}V(u_{m-1}'\times p_{m-1})\mathring{\cup}V(u_m\times p_m)\to A_{m-1}$, representing the colour of any $(a, \alpha, b)\in E(u_i'\times p_i)$ by $\mathrm{con}_{i-1}(\alpha)$, such that ψ and η are "*compatible*". I.e., for any $1\leqq i\leqq m$ and $1\leqq j\leqq u_i'$, if (a, b) and (c, d) are the j-th pairs of endpoints in $G(u_i'\times p_i)$ and $G(u_i'\times q_i)$, respectively (i.e., they are taken from the j-th copies of $G(p_i)$ and $G(q_i)$), then $\eta(a)=\psi(c)$ and $\eta(b)=\psi(d)$.

Similarly, there are $u_{m-1}\geqq u_{m-1}'$, $u_1''\geqq u_1'$, $\ldots$, $u_{m-2}''\geqq u_{m-2}'$ and we can extend ψ and η such that, besides the previous properties, for any $(f, \alpha, g)\in E(u_m\times p_m)$ $(\eta(f), \eta(g))$ belongs to the congruence generated by $\{(\psi(a), \psi(b))\colon (a, \alpha, b)\in E(p)\mathring{\cup}\mathring{\cup}E(u_1''\times q_1)\mathring{\cup}\ldots\mathring{\cup}E(u_{m-2}''\times q_{m-2})\mathring{\cup}E(u_{m-1}\times q_{m-1})\}$ and for any $f\in V(u_m\times p_m)$ $\eta(f)$ is in the subalgebra generated by $\psi(V(p)\mathring{\cup}W(u_1''\times q_1)\mathring{\cup}\ldots\mathring{\cup}W(u_{m-2}''\times q_{m-2})\mathring{\cup}\mathring{\cup}W(u_{m-1}\times q_{m-1}))$. And so on, finally we obtain $u_1, u_2, \ldots, u_m$ such that, besides the earlier properties, for $1\leqq i\leqq m$, $(f, \alpha, g)\in E(u_i\times p_i)$ and $h\in V(u_i\times p_i)$ the congruence generated by $\{(\psi(a), \psi(b))\colon (a, \alpha, b)\in E(p)\mathring{\cup}E(u_1\times q_1)\mathring{\cup}\ldots\mathring{\cup}E(u_{i-1}\times q_{i-1})\}$ contains $(\eta(f), \eta(g))$ and $\eta(h)$ is in the subalgebra generated by $\psi(V(p)\mathring{\cup}W(u_1\times q_1)\mathring{\cup}\ldots\ldots\mathring{\cup}W(u_{i-1}\times q_{i-1}))$.

Without loss of these properties any u_i can be enlarged. Indeed, we can define ψ and η for the (u_i+1)-th copy of $V(q_i)$ and $V(p_i)$ as follows. Let $\eta(V(p_i))=\{x_0\}\subseteq \subseteq X_0$, $\psi(\text{left endpoint})=\psi(\text{right endpoint})=\psi(x_0)=x_0$, and $\psi(a)=(i, j, x_0, x_0, a)$ if $a\in W(q_i)$ and $j\in N$ is the smallest index not occurring before.

Therefore $u_1=u_2=\ldots=u_m=m$ can be assumed (to enlarge m it suffices to enlarge $u_{m+1}=0$) and we obtain:

$$\psi\colon\ V(p)\mathbin{\mathring{\cup}}V(m\times q_1)\mathbin{\mathring{\cup}}\ldots\mathbin{\mathring{\cup}}V(m\times q_m)\to A_m,$$

$$\eta\colon\ V(m\times p_1)\mathbin{\mathring{\cup}}\ldots\mathbin{\mathring{\cup}}V(m\times p_m)\to A_{m-1},$$

ψ restricted to $V(p)\mathbin{\mathring{\cup}}W(m\times q_1)\mathbin{\mathring{\cup}}\ldots\mathbin{\mathring{\cup}}W(m\times q_m)$ is injective, ψ and η are compatible, $\psi(x_0)=x_0$, $\psi(x_1)=x_1$, $\eta\left(V(m\times p_i)\right)$ is a subset of the subalgebra of A_{i-1} generated by $\psi\left(V(p)\mathbin{\mathring{\cup}}W(m\times q_1)\mathbin{\mathring{\cup}}\ldots\mathbin{\mathring{\cup}}W(m\times q_{i-1})\right)$, $\varphi\left(V(q)\right)$ is in the subalgebra of A_m generated by $\psi\left(V(p)\mathbin{\mathring{\cup}}W(m\times q_1)\mathbin{\mathring{\cup}}\ldots\mathbin{\mathring{\cup}}W(m\times q_m)\right)$, for $(f, \alpha, g)\in E(m\times p_i)$ $\left(\eta(f), \eta(g)\right)$ belongs to the congruence generated by $\{(\psi(a), \psi(b))\colon (a, \alpha, b)\in \in E(p)\mathbin{\mathring{\cup}}E(m\times q_1)\mathbin{\mathring{\cup}}\ldots\mathbin{\mathring{\cup}}E(m\times q_{i-1})\}$, for $(f, \alpha, g)\in E(q)$ $\left(\varphi(f), \varphi(g)\right)$ belongs to the congruence generated by $\{(\psi(a), \psi(b))\colon (a, \alpha, b)\in E(p)\mathbin{\mathring{\cup}}E(m\times q_1)\mathbin{\mathring{\cup}}\ldots \ldots\mathbin{\mathring{\cup}}E(m\times q_m)\}$, $\psi\left(W(m\times q_i)\right)\subseteq Y_i$, and ψ restricted to $V(p)=X_0$ is the identity map.

Now any element of $A=F_V(X)$ can be written in the form $f(x\colon x\in X)$ where f is a σ-term over X, i.e. it is an element of the absolute free σ-algebra $F_\sigma(X)$. Hence we can consider appropriate maps (not homomorphisms!) $\bar\varphi\colon V(q)\to F_\sigma(X)$, $\bar\psi\colon V(p)\mathbin{\mathring{\cup}}V(m\times q_1)\mathbin{\mathring{\cup}}\ldots\mathbin{\mathring{\cup}}V(m\times q_m)\to F_\sigma(X)$, and $\bar\eta\colon V(m\times p_1)\mathbin{\mathring{\cup}}\ldots\mathbin{\mathring{\cup}}V(m\times p_m)\to \to F_\sigma(X)$ such that $\varphi(a)=\bar\varphi(a)(x\colon x\in X)$, $\psi(b)=\bar\psi(b)(x\colon x\in X)$, and $\eta(c)= =\bar\eta(c)(x\colon x\in X)$ hold in A for any admissible choice of a, b, and c. The right hand sides of these equations will be abbreviated by $a(X)$, $b(X)$, and $c(X)$, respectively.

Let $X_0', X_1', \ldots, X_m'$ stand for $X_0=V(p)=\psi(V(p))$, $\psi(V(p)\mathbin{\mathring{\cup}}W(m\times q_1))$, $\ldots$ $\ldots, \psi\left(V(p)\mathbin{\mathring{\cup}}W(m\times q_1)\mathbin{\mathring{\cup}}\ldots\mathbin{\mathring{\cup}}W(m\times q_m)\right)$, respectively. We have seen that, for $1\leqq i\leqq m$,

$$(1)\qquad \begin{cases}\bar\psi\left(V(m_i\times q_i)\setminus W(m_i\times q_i)\right)\subseteq F_\sigma(X_{i-1}'),\\ \bar\eta\left(V(m\times p_i)\right)\subseteq F_\sigma(X_{i-1}') \quad\text{and}\quad \bar\varphi\left(V(q)\right)\subseteq F_\sigma(X_m').\end{cases}$$

Further, for $1\leqq i\leqq m$,

$$(2)\qquad \begin{cases}\text{if}\quad (f, \alpha, g)\in E(m\times p_i)\quad\text{then}\quad \left(f(X_{i-1}'), g(X_{i-1}')\right)\\ \text{belongs to the congruence of}\quad A_{i-1}\quad\text{generated by}\\ \{(a(X_{i-1}'), b(X_{i-1}'))\colon (a, \alpha, b)\in E(p)\mathbin{\mathring{\cup}}E(m\times q_1)\mathbin{\mathring{\cup}}\ldots\mathbin{\mathring{\cup}}E(m\times q_{i-1})\}.\end{cases}$$

Similarly,

$$(2')\qquad \begin{cases}\text{if}\quad (f, \alpha, g)\in E(q)\quad\text{then}\quad \left(f(X_m'), g(X_m')\right)\quad\text{belongs to}\\ \text{the congruence of}\quad A_m\quad\text{generated by}\\ \{(a(X_m'), b(X_m'))\colon (a, \alpha, b)\in E(p)\mathbin{\mathring{\cup}}E(m\times q_1)\mathbin{\mathring{\cup}}\ldots\mathbin{\mathring{\cup}}E(m\times q_m)\}.\end{cases}$$

Now we intend to introduce a condition which is similar to a strong Mal'cev (cf. Jónsson [9] or Taylor [12]) condition. Suppose we have two graphs G_1, G_2 with

edge sets $E_i \subseteq V(G_i) \times S \times V(G_i)$, and $\pi\colon V(G_1) \to F_\sigma(Y)$, $\tau\colon V(G_2) \to F_\sigma(Y)$ are maps for some set Y. The condition to be introduced will be denoted by $T(E_1, \pi, E_2, \tau, Y)$ and we want, e.g., (2) to be equivalent to $T(E(p) \mathring{\cup} E(m \times q_1) \mathring{\cup} \dots \mathring{\cup} E(m \times q_{i-1}), \bar{\psi}, E(m \times p_i), \bar{\eta}, X'_{i-1})$. For $\alpha \in S$ let E_α stand for $\{(a, \alpha, b)\colon (a, \alpha, b) \in E_1\}$ and let, say, $E_\alpha = \{(a_i, \alpha, b_i)\colon 1 \leqq i \leqq u\}$. Let z be a variable symbol, $z \notin Y$, and consider an edge $(f, \alpha, g) \in E_2$. Then $T_{f,\alpha,g}$ is defined to be the following condition:

"There are terms $t_{ij}(z, Y) = t_{ij}(z; x\colon x \in Y)$, for $1 \leqq i \leqq v(u)$ and $0 \leqq j \leqq n-2$, such that the identities

$$\tau(f)(x\colon x \in Y) = t_{10}(\pi(a_{k(u,1)})(x\colon x \in Y);\ x\colon x \in Y)$$

$$\text{or shortly } \tau(f)(Y) = t_{10}(\pi(a_{k(u,1)})(Y), Y),$$

$$\tau(g)(Y) = t_{v(u),n-2}(\pi(b_{k(u,v(u))})(Y), Y), t_{i,n-2}(\pi(b_{k(u,i)})(Y), Y) =$$

$$= t_{i+1,0}(\pi(a_{k(u,i+1)})(Y), Y) \quad \text{for} \quad 1 \leqq i < v(u),$$

and

$$t_{ij}(\pi(b_{k(u,i)})(Y), Y) = t_{i,j+1}(\pi(a_{k(u,i)})(Y), Y) \quad \text{for} \quad 1 \leqq i \leqq v(u)$$

and $0 \leqq j < n-2$ hold."

Now let $T(E_1, \pi, E_2, \tau, Y)$ *be the conjunction of all* $T_{f,\alpha,g}$ where $\alpha \in S$ and $(f, \alpha, g) \in E_2$.

Since any translation of A_{i-1} is of the form $t(z, X_{i-1})$ for some σ-term t and identities holding for the free generators of A_{i-1} also hold in $\mathbf{V}$, from Corollary 3.3 and (2) we can conclude that $T(E(p) \mathring{\cup} E(m \times q_1) \mathring{\cup} \dots \mathring{\cup} E(m \times q_{i-1}), \bar{\psi}, E(m \times p_i), \bar{\eta}, X_{i-1})$ holds in $\mathbf{V}$. By identifying any variable $y \in X_{i-1} \setminus X'_{i-1}$ with $x_0 \in X_0$ we obtain that, for $1 \leqq i \leqq m$,

(3) $T(E(p) \mathring{\cup} E(m \times q_1) \mathring{\cup} \dots \mathring{\cup} E(m \times q_{i-1}), \bar{\psi}, E(m \times p_i), \bar{\eta}, X'_{i-1})$ holds in $\mathbf{V}$.

Similarly, from (2′) we obtain that

(3′) $T(E(p) \mathring{\cup} E(m \times q_1) \mathring{\cup} \dots \mathring{\cup} E(m \times q_m), \bar{\psi}, E(q), \bar{\varphi}, X'_m)$ holds in $\mathbf{V}$.

Conversely, from Corollary 3.3 it is easy to check that

(4) $$\begin{cases} \text{if} \quad T(E_1, \pi, E_2, \tau, Y) \quad \text{holds in} \quad \mathbf{V}, \quad \text{and} \quad \zeta\colon F_\sigma(Y) \to A \\ \text{is a homomorphism then for any} \quad (f, \alpha, g,) \in E_2 \\ (\zeta(\tau(f)), \zeta(\tau(g))) \quad \text{belongs to the congruence of} \quad A \\ \text{generated by} \quad \{(\zeta(\pi(a)), \zeta(\pi(b)))\colon (a, \alpha, b) \in E_1\}, \end{cases}$$

Now we can define our Mal'cev condition. (We shall write X_i, $W(m \times q_i)$, φ, ... instead of X'_i, $\psi(W(m \times q_i))$, $\bar{\varphi}$, ...).

A variety $\mathbf{V}$ *of type* σ *is said to satisfy the (strong Mal'cev) condition* $U(n, \chi, m)$ if and only if

"Considering the pairwise disjoint sets $V(p)$, $W(m \times q_1)$, ..., $W(m \times q_m)$ and denoting $V(p)$ and $V(p) \mathring{\cup} W(m \times q_1) \mathring{\cup} \dots \mathring{\cup} W(m \times q_i)$ by X_0 and X_i, respectively, there are maps

$\varphi\colon V(q) \to F_\sigma(X_m)$,

$\psi\colon V(p) \mathring{\cup} V(m \times q_1) \mathring{\cup} \dots \mathring{\cup} V(m \times q_m) \to F_\sigma(X_m)$, and

η: $V(m\times p_1)\,\dot\cup\ldots\dot\cup\, V(m\times p_m)\to F_\sigma(X_{m-1})$ such that the following five conditions hold:

(a) For $1\leqq i\leqq m$ $\psi(V(m\times q_i)\setminus W(m\times q_i))$ and $\eta\,(V(m\times p_i))$ are subsets of $F_\sigma(X_{i-1})$, and ψ restricted to X_m is the identity map;

(b) For $1\leqq i\leqq m$ $T(E(p)\,\dot\cup\, E(m\times q_1)\,\dot\cup\ldots\dot\cup\, E(m\times q_{i-1}), \psi, E(m\times p_i), \eta, X_{i-1})$ holds in **V**;

(c) $T(E(p)\,\dot\cup\, E(m\times q_1)\,\dot\cup\ldots\dot\cup\, E(m\times q_m), \psi, E(q), \varphi, X_m)$ holds in **V**;

(d) η and ψ are compatible in the sense that for any $1\leqq i, j\leqq m$, if (c_0, c_1) and (e_0, e_1) are the j-th pairs of endpoints in $G(m\times p_i)$ and $G(m\times q_i)$, respectively, then the "endpoint" identities $\eta(c_0)(x\colon x\in X_{i-1})=\psi(e_0)(x\colon x\in X_{i-1})$ and $\eta(c_1)(x\colon x\in X_{i-1})=\psi(e_1)(x\colon x\in X_{i-1})$ hold in **V**; and

(e) φ and ψ are compatible in the sense that for the endpoints x_0, x_1 and f_0, f_1 of $G(p)$ and $G(q)$, respectively, the identities $\varphi(f_i)(x\colon x\in X_m)=\psi(x_i)=x_i$, $i=0, 1$, hold in **V**."

We have shown that if χ holds in **CV** then there exists an m, $2\leqq m<\omega$, such that the (strong Mal'cev) condition $U(n, \chi, m)$ holds in **V**.

4. Main theorem and its sufficiency

THEOREM. *For any n-permutable variety* **V**, *a lattice Horn sentence χ is satisfied in the congruence lattices of members of* **V** *if and only if $U(n, \chi, m)$ holds in* **V** *for some m, $2\leqq m<\omega$.*

To prove the sufficiency assume $U(n, \chi, m)$. Let con (α), $\alpha\in S$, be congruences of an algebra $B\in\mathbf{V}$ such that $p_i(\text{con }(\alpha)\colon \alpha\in S)\leqq q_i(\text{con }(\alpha)\colon \alpha\in S)$ for all $i\geqq 1$. Let $(a_0, a_1)\in p(\text{con }(\alpha)\colon \alpha\in S)$. We have to show that $(a_0, a_1)\in q(\text{con }(\alpha)\colon \alpha\in S)$ as well.

For a map γ, γx will stand for $\gamma(x)$. Via induction we define homomorphisms $\zeta_i\colon F_\sigma(X_i)\to B$ for $i=0, 1, \ldots, m$ such that ζ_i is an extension of ζ_{i-1}, $\zeta_i\psi$ and $\zeta_i\eta$ are (multiple) connecting maps* (i.e. if (a, α, b) and (c, α, d) are edges in the corresponding domains then $(\zeta_i\psi a, \zeta_i\psi b)\in\text{con }(\alpha)$ and $(\zeta_i\eta c, \zeta_i\eta d)\in\text{con }(\alpha)$), and for the endpoints x_0, x_1 of $V(p)$ $\zeta_i\psi x_0=a_0$ and $\zeta_i\psi x_1=a_1$ hold.

By Proposition 3.1, a_0 and a_1 can be connected by $G(p)$ in B. Choose a connecting map $\varrho\colon V(p)=X_0\to B$ and extend it to a homomorphism $\zeta_0\colon F_\sigma(X_0)\to B$. Then $\zeta_0\psi=\varrho_0$ is clearly a connecting map. Since $T(E(p), \psi, E(m\times p_1), \eta, X_0)$ holds in **V**, (4) applies and we have that $\{(\zeta_0\psi a, \zeta_0\psi b)\colon (a, \alpha, b)\in E(p)\}$ generates a congruence, say α_0, which collapses $\zeta_0\eta f$ and $\zeta_0\eta g$ for any $(f, \alpha, g)\in E(m\times p_1)$. But $\alpha_0\subseteq\alpha$ follows from $\zeta_0\psi$ being connecting, whence $\zeta_0\eta$ is (multiple) connecting, too.

Now assume that $\zeta_0\subseteq\zeta_1\subseteq\ldots\subseteq\zeta_{i-1}$ $(0<i\leqq m)$ have been defined appropriately. Since $\zeta_{i-1}\eta$ is a (multiple) connecting map, Proposition 3.1 yields that $(\zeta_{i-1}\eta f, \zeta_{i-1}\eta g)\in p_i(\text{con }(\alpha)\colon \alpha\in S)\subseteq q_i(\text{con }(\alpha)\colon \alpha\in S)$ holds for any pair (f, g) of endpoints in $G(m\times p_i)$. Therefore, by Proposition 3.1 again, there is a (multiple) connecting map $\varrho_i\colon V(m\times q_i)\to B$ such that ϱ_i and $\zeta_{i-1}\eta$ (restricted to $G(m\times p_i)$) are compatible. Now consider the map $X_i\to B$, $x\mapsto\zeta_{i-1}x$ if $x\in X_{i-1}$ and $x\mapsto\varrho_i x$ if $x\in W(m\times q_i)$,

* In this context ψ and η will be restricted to $V(p)\,\dot\cup\, V(m\times q_1)\,\dot\cup\ldots\dot\cup\, V(m\times q_i)$ and, if $i<m$, to $V(m\times p_1)\,\dot\cup\ldots\dot\cup\, V(m\times p_{i+1})$, respectively.

and extend it to a homomorphism ζ_i: $F_\sigma(X_i) \to B$. Clearly ζ_i extends ζ_{i-1}. Hence to show that $\zeta_i\psi$ is a connecting map it suffices to check that ϱ_i and the restriction of $\zeta_i\psi$ to $V(m\times q_i)$ coincide. If $x\in W(m\times q_i)$ then $\zeta_i\psi x=\zeta_i x=\varrho_i x$ while, by (d), for the j-th pairs (c_0, c_1) and (e_0, e_1) of endpoints in $G(m\times p_i)$ and $G(m\times q_i)$ we have $\zeta_i\psi e_0=\zeta_{i-1}\psi e_0=\zeta_{i-1}\eta c_0=\varrho_i e_0$, and the same for e_1. Since $\zeta_i\eta$ and $\zeta_{i-1}\eta$ coincide on $V(m\times p_1)\,\mathring{\cup}\ldots\mathring{\cup}\,V(m\times p_i)$; it suffices to show that (if $i<m$) $\zeta_i\eta$ restricted to $V(m\times p_{i+1})$ is connecting. Since $T(E(p)\,\mathring{\cup}\,E(m\times q_1)\,\mathring{\cup}\ldots\mathring{\cup}\,E(m\times q_i), \psi, E(m\times p_{i+1}), \eta, X_i)$ holds in $\mathbf{V}$, (4) applies and we have that the congruence α_i generated by $\{(\zeta_i\psi a, \zeta_i\psi b)\colon (a, \alpha, b)\in E(p)\,\mathring{\cup}\,E(m\times q_1)\,\mathring{\cup}\ldots\mathring{\cup}\,E(m\times q_i)\}$ collapses $\zeta_i\eta f$ and $\zeta_i\eta g$ for any $(f, \alpha, g)\in E(m\times p_{i+1})$. But we have $\alpha_i\subseteqq\alpha$ from $\zeta_i\psi$ being a connecting map, whence $\zeta_i\eta$ (restricted to $V(m\times p_{i+1})$) is a connecting map either.

Now put $\zeta=\zeta_m$ and denote α_m the congruence generated by $\{(\zeta\psi a, \zeta\psi b)\colon (a, \alpha, b)\in E(p)\,\mathring{\cup}\,E(m\times q_1)\,\mathring{\cup}\ldots\mathring{\cup}\,E(m\times q_m)\}$. From (c) and (4) we obtain that $\zeta\varphi$: $V(q)\to B$ is a connecting map which represents α by α_m. Since $\zeta\psi$ is also a connecting map, $\alpha_m\subseteqq\mathrm{con}\,(\alpha)$. Hence $\zeta\varphi$ represents α by con (α) as well. From Proposition 3.1 it follows that $(\zeta\varphi f_0, \zeta\varphi f_1)\in q((\mathrm{con}\,(\alpha)\colon \alpha\in S)$. Now by making use of (e) we have $(\zeta\varphi f_0, \zeta\varphi f_1)=(\zeta\psi x_0, \zeta\psi x_1)=(a_0, a_1)$, which completes the proof of the sufficiency.

5. Concluding remarks

The usual form of a strong Mal'cev condition is (roughly saying, for precise definition cf., e.g., Jónsson [9]) the following: "There are certain finitary terms $f_0(X)$, ..., $f_s(X)$ for which certain prescribed identities hold". If any prescribed identity is of the form $f_i(\mu(x)\colon x\in X)=f_j(\nu(x)\colon x\in X)$ or $f_i(\mu(x)\colon x\in X)=y$ for some maps μ: $X\to X$ and ν: $X\to X$ and $y\in X$ then the strong Mal'cev condition will be called linear. While the Wille–Pixley algorithm [11, 13] yields linear Mal'cev conditions, ours does not. (However, the $U(n, \chi, m)$ in Theorem is a strong Mal'cev condition.)

Now we intend to point out that our algorithm cannot be improved so that $U(n, \chi, m)$ in Theorem be linear in general.

The reason for this is the following. For a ring R with 1 let $\mathbf{M}(R)$ denote the variety of unitary left R-modules. Hutchinson [7] has produced two rings, R_1 and R_2, such that though $\mathbf{CM}(R_1)$ and $\mathbf{CM}(R_2)$ satisfy the same lattice identities, they satisfy different sets of Horn sentences. But, as it is implicit in [8], if $\mathbf{CM}(R_1)$ and $\mathbf{CM}(R_2)$ satisfy the same lattice identities then $\mathbf{M}(R_1)$ and $\mathbf{M}(R_2)$ satisfy the same linear strong Mal'cev conditions.

The condition $(\exists m)\,(U(n, \chi, m))$ in our Theorem does not seem to be a Mal'cev condition yet. To make it a real Mal'cev condition we have to prescribe appropriate linear orders of the occurring edge sets. Let the orderings of $E(p)$, $E(q)$, $E(p_i)$, and $E(q_i)$, $1\leqq i\leqq k$, be fixed arbitrarily. The linear orders of $E^m=E(p)\,\mathring{\cup}\,E(m\times q_1)\,\mathring{\cup}\ldots$ $\ldots\,\mathring{\cup}\,E(m\times q_m)$ and ${}^mE=E(m\times p_1)\,\mathring{\cup}\ldots\mathring{\cup}\,E(m\times p_m)$ will be defined via induction. Since $E^0=E(p)$ and ${}^0E=\emptyset$, their orderings are unique. Having the linear orders of E^m and ${}^mE$ defined put $E^{m+1}=E^m+E(q_1)+\ldots+E(q_m)+E(q_{m+1})+\ldots+E(q_{m+1})$ and ${}^{m+1}E={}^mE+E(p_1)+\ldots+E(p_m)+E(p_{m+1})+\ldots+E(p_{m+1})$. Here $+$ stands for the ordinal sum, whence, e.g., any element of E^m precedes all the elements of $E^{m+1}\setminus E^m$. Further, let us also agree that if E_1 has a fixed linear ordering in the definition of $T(E_1, \pi, E_2, \tau, Y)$ then $(a_i, \alpha, b_i)\in E_\alpha$ precedes $(a_{i+1}, \alpha, b_{i+1})$ for $i=1, 2, \ldots, u-1$.

Now $(\exists m)\,(U(n, \chi, m))$ has become a Mal'cev condition. To show that $U(n, \chi, m)$ implies $U(n, \chi, m+1)$ (in any variety and for all $m \geqq 2$) indeed, we can repeat some earlier terms, consider some projections, and introduce irrelevant variables. The long technical details will be omitted.

Added in proof (April 5, 1984). Recently it has appeared that Proposition 1.1 is a special case of Herrmann and Poguntke's [14, Thm. 6]. Yet, our approach is different from [14].

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MULTIPLICATIVE FUNCTIONS WITH REGULARITY PROPERTIES. IV

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1. Introduction

This paper is a continuation of [1], [2], [3]. The notations $\mathcal{M}^*$, $\mathcal{L}$ used in [3] will be preserved here. Let $\chi_T(n)$ or $\chi(n; T)$ be a multiplicative character mod T.

We would like to determine all $f, g \in \mathcal{M}^*$ that satisfy

$$\sum_{n=1}^{\infty} \frac{|g(n+K)-f(n)|}{n} < \infty. \tag{1.1}$$

This problem was solved completely in [3] if $K=2$ or $K=$odd (see Theorem 4). Recalling previous results we may assume that $f(p)=0$ and $g(p)=0$ if $p|K$, and $f(p)\neq 0$, $g(p)\neq 0$ if $(p, K)=1$ ([3], Theorem 1).

We shall give now the solutions for every K under the additional condition

$$|g(n)| \geqq 1, \ |f(n)| \geqq 1 \text{ if } (n, K)=1, \quad g(n)=f(n)=0 \quad \text{if} \quad (n, K)>1. \tag{1.2}$$

It is obvious that $f(n)=g(n)=n^s$ satisfies (1.1) if and only if $\operatorname{Re} s<1$. Furthermore, if F and G are completely multiplicative functions such that

$$G(n+K) = F(n) \quad (n \in \mathcal{N}), \tag{1.3}$$

then $(f, g)=(F, G)$ is a solution. If (1.3) holds, then by $f(n)=n^sF(n)$, $g(n)=n^sG(n)$ the relation (1.1) is satisfied. I hope that the inverse assertion is true, namely that from (1.1) it follows that $f(n)=n^sF(n)$, $g(n)=n^sG(n)$, where (1.3) holds, or $f\in\mathcal{L}$ and $g\in\mathcal{L}$.

THEOREM 1. *Let* $F, G\in\mathcal{M}^*$ *satisfy* (1.3). *Assume that* $F(1)\neq 0$ *and* $F(n)=0$ *if* $(n, K)>1$. *Then the following assertions hold:*

a) $F(n)=G(n)=\chi_K(n)$ *is a solution, where* χ_K *is an arbitrary multiplicative character* mod K,

b) *there is no other solution if* K *is odd or* $K\equiv 0 \pmod 4$,

c) *if* $K=2R$, $(R, 2)=1$, *then all further solutions have the form*

$$F(n) = \chi(n; 8)\Psi_R(n); \quad G(n) = \chi(n; 4)F(n),$$

where $\Psi_R(n)$ *is an arbitrary character* mod R, $\chi(n; 4)$ *is the nonprincipal character* mod 4, *and* $\chi(n; 8)$ *is the character* mod 8 *defined by the relations*

$$\chi_8(n) = \begin{cases} 1 & n \equiv \pm 1 \pmod 8 \\ -1 & n \equiv \pm 3 \pmod 8 \end{cases} \quad \text{if} \quad R \equiv 1 \pmod 4,$$

$$\chi_8(n) = \begin{cases} 1 & n \equiv 1,3 \pmod 8 \\ -1 & n \equiv 5,7 \pmod 8 \end{cases} \quad \text{if} \quad R \equiv -1 \pmod 4.$$

We shall prove this assertion in Section 5. Now we remark only that (1.3) implies $F(n)\neq 0$, $G(n)\neq 0$ if $(n, K)=1$, except the trivial solution $F(n)=0$. To prove this assertion we have to repeat the argumentation used in the proof of Theorem 1, [3] under the condition: "COND: $f(n)=0$ if and only if $g(n+K)=0$".

In Section 2 we shall prove the following

THEOREM 2. *If U and V are arbitrary complex valued completely additive functions defined on the set of integers coprime to K such that*

$$\lim_{x\to\infty}\frac{1}{x}\sum_{\substack{n<x\\(n,K)=1}}|V(n+K)-U(n)|=0, \tag{1.4}$$

then $U(n)=V(n)$ for every n, $(n, K)=1$.

Furthermore, we have

THEOREM 3. *If U is an arbitrary complex valued completely additive function defined on the set of integers coprime to K such that*

$$\lim_{x\to\infty}\frac{1}{x}\sum_{\substack{n<x\\(n,K)=1}}|U(n+K)-U(n)|=0, \tag{1.5}$$

then $U(n)$ is a constant multiple of $\log n$ on the set $(n, K)=1$.

We shall not prove Theorem 3, since we could do it similarly as in [4].

Theorems 2 and 3 imply that $|F(n)|=|G(n)|=1$ for $(n, K)=1$, if (1.3) holds and F is not identically zero. Indeed, as it was observed, $F(n)\neq 0$, $G(n)\neq 0$ for $(n, K)=1$, therefore the completely additive functions $U(n)$, $V(n)$ are well-defined by $|F(n)|=e^{U(n)}$, $|G(n)|=e^{V(n)}$, and satisfy the relation $V(n+K)-U(n)=0$ $(\forall n, (n, K)=1)$. From Theorem 2 we get that $V(n)=U(n)$ $(\forall n, (n, K)=1)$, consequently the conditions of Theorem 3 are satisfied. Then $U(n)=V(n)=\sigma\log n$ if $(n, K)=1$, which by (1.3) implies that $\sigma=0$. This means that $|F(n)|=|G(n)|=1$ if $(n, K)=1$.

THEOREM 4. *Let us assume that $f, g\in\mathcal{M}^*$ and that* (1.1) *and* (1.2) *hold. Then $f(n)=n^s F(n)$, $g(n)=n^s G(n)$,* Re $s<1$, *and for F, G the relation* (1.3) *holds.*

We shall prove Theorem 4 in Section 4. Here we mention only that it is enough to prove it under the condition $|f(n)|=|g(n)|=1$ for $(n, k)=1$. Let us define $U(n)$, $V(n)$ by $|f(n)|=e^{U(n)}$, $|g(n)|=e^{V(n)}$. $U(n)$ and $V(n)$ are completely additive functions. Since they are nonnegative,

$$|V(n+K)-U(n)|<|e^{V(n+K)}-e^{U(n)}|=\big||g(n+K)|-|f(n)|\big|<|g(n+K)-f(n)|,$$

and so, from (1.3) we get that

$$\sum_{\substack{n\\(n,k)=1}}\frac{1}{n}|V(n+K)-U(n)|<\infty.$$

Thus by (1.4), and Theorems 2 and 3 the relation $U(n)=V(n)=\sigma\log n$ $(\sigma\geqq 0)$ holds. Let $f(n)=n^{\sigma}f_1(n)$, $g(n)=n^{\sigma}g_1(n)$, $|f_1(n)|=|g_1(n)|=1$ if $(n, K)=1$. Since

$$\sum \frac{\big||g(n+K)|-|f(n)|\big|}{n}<\infty,$$

it follows that $\sigma<1$. So we get immediately that

$$\sum_n \frac{|g_1(n+K)-f_1(n)|}{n}<\infty.$$

The following assertion is the main lemma in the proof of Theorem 4.

THEOREM 5. *Let U and V be completely additive real valued functions defined on the set of integers coprime to K satisfying the condition*

$$\frac{1}{x}\sum_{\substack{n\leqq x\\(n,K)=1}}\big\|V(n+K)-U(n)\big\|\to 0 \quad (x\to\infty). \tag{1.5}$$

Here $\|z\|$ denotes the distance of z to the nearest integer.

Then $e^{2\pi i(V(n)-U(n))}=\chi_{K^2}(n)$ if $(n, K)=1$, where χ_{K^2} is a suitable character mod K^2.

2. Proof of Theorem 2

Let $H(n)=V(n)-U(n)$. Let $\varepsilon(n)=V(n+K)-U(n)$. For arbitrary positive integers $m, p(>2)$ such that $(mp, K)=1$, put

$$\delta_p(m)=H(p)+H(m)+H(m+K)+\ldots+H\big(m+(p-2)K\big).$$

Starting from the relation

$$\varepsilon(n)+H(p)=V(pn+pK)-U(pn)=$$

$$=\big[V(pn+pK)-U\big(pn+(p-1)K\big)\big]+\big[U\big(pn+(p-1)K\big)-V\big(pn+(p-1)K\big)\big]+$$

$$+\big[V\big(pn+(p-1)K\big)-U(pn)\big]=\ldots$$

$$\ldots=\varepsilon\big(pn+(p-1)K\big)+\ldots+\varepsilon(pn)-H\big(pn+(p-1)K\big)-\ldots-H(pn+K),$$

taking into account (1.5), we get immediately that

$$\frac{1}{x}\sum_{\substack{n<x\\(n,K)=1}}\big|\delta_p(pn+K)\big|\to 0,$$

i.e. that

$$\frac{1}{x}\sum_{\substack{m<x\\m\equiv K(\mathrm{mod}\ p)\\(m,K)=1}}|\delta_p(m)|\to 0. \tag{2.1}$$

Let now the integers p, q, r be chosen so that $p>q$, $r=p-q+1$, $(pqr, K)=1$, and let m be an integer for which the relations

$$m \equiv K \pmod{p}, \quad m \equiv K \pmod{q}, \tag{2.2}$$

$$m+(q-2)K \equiv 0 \pmod{r}, \tag{2.3}$$

$$(m, K) = 1 \tag{2.4}$$

hold.

If p, q, r, m are so chosen then

$$\delta_p(m) = H(p)+\big[H(m)+\ldots+H\big(m+(q-2)K\big)\big]+\big[H\big(m+(q-1)K\big)+\ldots \tag{2.5}$$

$$\ldots+H\big(m+(p-2)K\big)\big] = H(p)+\big[\delta_q(m)-H(q)\big]+\big[\delta_r\big(m+(q-1)K\big)-H(r)\big] =$$

$$= \big(H(p)-H(q)-H(r)\big)+\delta_q(m)+\delta_r\big(m+(q-1)K\big).$$

If there exists at least one m for which the conditions (2.2), (2.3), (2.4) hold, then the density of the solutions in m are positive, and so from (2.5) and (1.5) we get immediately that

$$H(p) = H(q)+H(r). \tag{2.6}$$

If $(pqr, K)=1$ and there exists an m satisfying (2.2), (2.3), then there exists some for which $(m, K)=1$ holds additionally.

Let r be given, and let $q=\eta r+1$, $p=(1+\eta)r$, $m=pqv+K$, η, v be positive integers. The conditions under (2.2) hold, (2.3) is equivalent to

$$pqv+(q-1)K = 0 \pmod{r}.$$

This relation holds, since $q-1\equiv 0 \pmod{r}$, $p\equiv 0 \pmod{r}$. Finally we assume that $(pqr, K)=\big(r(\eta r+1)(1+\eta), K\big)=1$. So, from (2.6) we get that $H\big((1+\eta)r\big)=$ $=H(\eta r+1)+H(r)$, i.e. that

$$H(1+\eta) = H(1+r\eta) \tag{2.7}$$

if

$$\big(r(\eta r+1)(1+\eta), K\big) = 1. \tag{2.8}$$

Let now $\eta=\lambda K$. Then (2.8) holds, if $(r, K)=1$. Consequently

$$H(1+\lambda K) = H(1+r\lambda K) \tag{2.9}$$

if $(r, k) = 1$.

We shall use the following obvious assertion: if λ and μ are integers such that $(\lambda, K)=(\mu, K)=1$, then there exist r and s coprime to K such that $r\lambda=s\mu$. This implies immediately that

$$H(1+\lambda K) = H(1+\mu K) \quad \text{if} \quad (\lambda, K) = (\mu, K). \tag{2.10}$$

Let now $(v, K)=1$, $v>1$. Then $(1+\lambda K)^v=1+\lambda_v K$, $\lambda_v\equiv v\lambda \pmod{K}$. Consequently $(\lambda_v, K)=(v\lambda, K)=(\lambda, K)$, and so by (2.10),

$$vH(1+\lambda K) = H(1+\lambda_v k) = H(1+\lambda K), \quad H(1+\lambda K) = 0.$$

So we proved that $H(1+\lambda K)=0$ for every $\lambda \geqq 1$. Let now P be coprime to K. Then $P^{\varphi(K)} \equiv 1 \pmod{K}$, and so

$$\varphi(K)H(P) = H(P^{\varphi(K)}) = 0,$$

i.e. $H(P)=0$. By this the proof has been finished.

3. Proof of Theorem 5

Let $H(n)=V(n)-U(n)$, $\varepsilon(n)=V(n+K)-U(n)$, as they were defined in Section 2. Let $E(n)$ be the nearest integer to $\varepsilon(n)$, $\tilde{\varepsilon}(n)=\varepsilon(n)-E(n)$. Starting from the formula

$$\varepsilon(n)-\left[\varepsilon(pn)+\ldots+\varepsilon(pn+(p-1)K)\right] = -\delta_p(pn+K),$$

and observing that

$$\|\delta_p(pn+K)\| \leqq \|\tilde{\varepsilon}(n)\|+\|\tilde{\varepsilon}(pn)\|+\ldots+\left\|\tilde{\varepsilon}(pn+(p-1)K)\right\|,$$

from (1.5) we deduce that

$$\frac{1}{x} \sum_{\substack{m \leqq x \\ m \equiv K \pmod{p} \\ (m,K)=1}} \|\delta_p(m)\| \to 0.$$

Arguing now as in the previous section, we deduce that $\|H(p)-H(q)-H(r)\|=0$, if p, q, r satisfy the conditions stated there. Hence we deduce that

$$H(1+\lambda K) = H(1+\mu K) \pmod{1} \quad \text{if} \quad (\lambda, K) = (\mu, K). \tag{3.1}$$

Thus we can deduce almost immediately that the fractional part of $H(n)$ is periodic with the period K^2. Indeed, let $A=B \pmod{K^2}$, $(A, K)=1$. Then, there exists a θ, $A\theta \equiv 1 \pmod{K^2}$, i.e. $B\theta \equiv 1 \pmod{K^2}$. It is obvious that $H(A\theta) \equiv H(B\theta) \pmod{1}$, and so H is periodic (mod 1) with the period K^2. Then $e^{2\pi i H(n)}=\alpha(n)$ has to be a character, as it is wellknown.

4. Proof of Theorem 4

As it was mentioned it is enough to consider the assertion under the condition $|f(n)|=|g(n)|=1$, $(n, K)=1$. Let $f(n)=e^{2\pi i U(n)}$, $g(n)=e^{2\pi i V(n)}$. From (1.1) we get

$$\sum_{\substack{n \\ (n,K)=1}} \frac{\|V(n+K)-U(n)\|}{n} \ll \sum_{\substack{n \\ (n,K)=1}} \frac{|g(n+K)-f(n)|}{n} < \infty$$

and the finiteness of the left hand side implies (1.5). Consequently, by Theorem 5 we get $g(n)=\chi_{K^2}(n)f(n)$. So we have

$$\sum_{(n,K)=1} \frac{1}{n} |\chi_{K^2}(n+K)f(n+K)-f(n)| < \infty. \tag{4.1}$$

Let

$$E(n) := \prod_{j=1}^{k} \bar{\chi}_{K^2}(n+jK), \quad (n, K) = 1. \tag{4.2}$$

First we observe that $E(n)$ is periodic with K, $E(n)=C(\iota)$ if $n\equiv\iota \pmod K$. From (4.1) we deduce that

$$\sum_{n\equiv\iota \pmod K} \frac{1}{n} |f(n+K^2)-C(\iota)f(n)| < \infty, \quad (K, \iota) = 1.$$

Let h be an integer for which $C(\iota)^h=1$ holds for every ι. $h=\varphi(K^2)$ is a suitable integer. Let $\xi=C(\iota)$. Since

$$\sum_{j=1}^{h} \xi^{h-j}(f(n+jK^2)-f(n+(j-1)K^2)) = f(n+hK^2)-f(n),$$

we have

$$\sum_n \frac{1}{n} |f(n+hK^2)-f(n)| < \infty. \tag{4.3}$$

In other words

$$\sum_n \frac{1}{n} |(E^B-I)f(n)| < \infty, \tag{4.4}$$

holds with $B=hK^2$. Let $h=\varphi(K^2)$, $B=B_1B_2$, where B_1 contains all prime powers in B that are coprime to B. Let $n=mB_1$. Since $(E^B-I)f(mB_1)=f(B_1)(E^{B_2}-I)f(m)$, and $f(B_1)\neq 0$, then there exists a B that contains only prime factors of K such that (4.4) holds. From Theorem 3 in [3] we get $f(n)=\chi_B(n)n^{i\tau}$. So we have proved that

$$\sum_n \frac{|\chi_{K^2}(n+K)\chi_B(n+K)(n+K)^{i\tau}-\chi_B(n)n^{i\tau}|}{n} < \infty.$$

Observing that $(n+K)^{i\tau}=n^{i\tau}+O(1/n)$, we get

$$\sum_n \frac{|\chi_{K^2}(n+K)\chi_B(n+K)-\chi_B(n)|}{n} < \infty. \tag{4.5}$$

Since the functions $F(n)=\chi_B(n)$, $G(n)=\chi_B(n)\chi_{K^2}(n)$ are periodic with period $[K^2, B]$, $G(n+K)=F(n)$ for every n. The proof is complete.

5. Proof of Theorem 1

It is obvious that the functions defined in a) and c) satisfy (1.3). We have to prove that there are no other solutions. The case K is odd has been considered in [3] (Theorem 4). Let us assume that K is even. As it was mentioned, $|F(n)|=|G(n)|=1$ for $(n, K)=1$. Since the conditions of Theorem 4 hold, we get $F(n)=\chi_B(n)$, $G(n)=\chi_{K^2}(n)F(n)$. Since all the prime factors of B are contained in K, we may assume that $B=K^\alpha$, $\alpha\geqq 2$ being a suitable positive integer. Let now $F(n)$ and $G(n)$ be defined for negative n by $\chi_B(n)$ and $\chi_{K^2}(n)\chi_B(n)$.

Then

$$F(-N) = \chi_{K^\alpha}(-N) = \chi_{K^\alpha}(tK^\alpha - N) =$$

$$= F(tK^\alpha - N) = G(tK^\alpha - N + K) = \chi_{K^2}(tK^\alpha - N + K)\chi_{K^\alpha}(-N + K) = G(-N + K).$$

That is F and G satisfy (1.3) for negative values, too. Consequently

$$\chi_{K^2}(N+K)F(N+K) = F(N) \tag{5.1}$$

holds for every integer N.

By putting $N+K=1$, we have

$$F(1-K) = 1, \quad F(K-1) = F(-1). \tag{5.2}$$

From $N+K=-1$ we get

$$\chi_{K^2}(-1)F(-1) = F(-K-1) = F(-1)F(K+1), \quad \chi_{K^2}(-1) = F(K+1). \tag{5.3}$$

Consider first the case $\chi_{K^2}(-1)=+1$. From (5.3) we get $F(K+1)=1$. By putting $N=1$ into (5.1) we get $\chi_{K^2}(1+K)=1$. Since $(1+K)^v \equiv 1+vK \pmod{K^2}$, we have $\chi_{K^2}(1+vK)=1$ $(v=0, 1, \ldots)$. But this implies that χ_{K^2} is periodic with period K. Indeed, let $A \equiv B \pmod K$, $(A, K)=1$. Let A^{-1} be defined by $AA^{-1}=1 \pmod K$. Since $BA^{-1} \equiv 1 \pmod K$, therefore $1=\chi_{K^2}(BA^{-1})=\chi_{K^2}(B)\chi_{K^2}(A)$, i.e. $\chi_{K^2}(A)=\chi_{K^2}(B)$. Now we put $N=1, K+1, 2K+1, \ldots$ into (5.1) and deduce that $1=F(1)=F(1+K)=F(1+2K)=\ldots$. This implies that F is a character mod K. But then from (5.1) we get $\chi_{K^2}(n)=1$ for every n, $(n, K)=1$, i.e. χ_{K^2} is the principal character. We get the solutions stated in a).

Consider now the case $\chi_{K^2}(-1)=-1$. If the pair χ_{K^2}, F satisfies (5.1), then the same holds for $(\chi_{K^2}^2, F^2)$. But $\chi_{K^2}^2(-1)=1$, and as it was shown under this assumption we have a solution only if $\chi_{K^2}^2$ is the principal character. We have proved that if (χ_{K^2}, F) is a solution then χ_{K^2} is a quadratic character i.e. assumes only real values. We have $F(K+1)=-1$ (see (5.3)). By putting $N=1$, we get $\chi_{K^2}(K+1)=-1$. Since $(K+1)^2 \equiv 2K+1 \pmod{K^2}$, we get $\chi_{K^2}(2K+1)=+1$. Since $(2K+1)^v \equiv \equiv 2vK+1 \pmod{K^2}$, we get $\chi_{K^2}(2vK+1)=1$ $(v=1, 2, \ldots)$. This implies that χ_{K^2} is a character mod $2K$.

Let $(n, K)=1$. Then n is odd. Since $(n+K)(1+K)=n+K(n+1)+K^2 \equiv n \pmod{2K}$, therefore

$$\chi_{K^2}(n+K)\chi_{K^2}(1+K) = \chi_{K^2}(n), \quad \text{and so} \quad \chi_{K^2}(n+K) = -\chi_{K^2}(n)$$

for every n. So we have $\chi_{K^2}(n+2K)F(n+2K)=F(n+K)=\chi_{K^2}(n+K)F(n)$, i.e. $F(n+2K)=-F(n)$, i.e. $F(n+4K)=F(n)$. Consequently F is a character mod $4K$. Now we have

$$F((m+K)(n+K)) = (-\chi_{K^2}(m))(-\chi_{K^2}(n))F(mn) = \chi(mn)F(mn).$$

The left hand side is $F(mn+(m+n)K+K^2)$. If $4|K$, then $K^2 \equiv 0 \bmod 4K$, and this is equal to $F(mn+(m+n)K)$. Consequently for $m+n \equiv 0 \pmod 4$ we get $F(mn)=\chi_{K^2}(mn)F(mn)$, which leads to $\chi_{K^2}(-1)=1$. This case has been considered earlier.

Let now $K=2R$, $(R, 2)=1$. Then

$$F(n) = \chi(n; 8)\Psi_R(n), \quad \chi_{K^2}(n) = \chi(n; 4)\chi(n; R).$$

From (5.1) we get

$$\chi(n+K;\,4)\chi(n+K;\,R)\chi(n+K;\,8)\Psi_R(n+K) = \chi(n;\,8)\Psi_R(n). \tag{5.4}$$

We have $\Psi_R(n+K)=\Psi_R(n)$. From (5.4) we deduce that $\chi(n;\,R)$ is constant for $(n, R)=1$, and so $\chi(n;\,R)=1$ if $(n, R)=1$. Furthermore, $\chi(n+K;\,4)=-\chi(n;\,4)$, consequently

$$-\chi(n;\,4)\chi(n+2R;\,8) = \chi(n;\,8). \tag{5.5}$$

(5.5) holds if and only if $\chi(n;\,8)$ satisfies the requirements stated in c).

The proof has been completed.

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ON *S*-INTEGRAL SOLUTIONS OF THE EQUATION $y^m = f(x)$

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1. Introduction

Let $\mathbf{K}$ be an algebraic number field with ring of integers $O_{\mathbf{K}}$, and let $f(x) \in O_{\mathbf{K}}[x]$. We assume throughout that $f(x)$ has the representations

$$f(x) = a_0 x^N + \ldots + a_N = \prod_{i=1}^{n} (x - \alpha_i)^{r_i}$$

with $a_0 \neq 0$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. Let $\mathbf{K}_0 = \mathbf{K}(\alpha_1, \ldots, \alpha_n)$ and let $\mathbf{L}$ be any finite extension of $\mathbf{K}$ with ring of integers $O_{\mathbf{L}}$ and $[\mathbf{L}: \mathbf{Q}] = l$. Further let $m \geqq 2$ be a natural number, and put

$$t_i = m/(m, r_i), \quad i = 1, \ldots, n.$$

Improving a well-known theorem of C. L. Siegel [6], W. J. LeVeque [5] showed that then the equation

$$f(x) = y^m \tag{1}$$

has only finitely many solutions x, y in $O_{\mathbf{L}}$, unless $\{t_1, \ldots, t_n\}$ is a permutation of one of the n-tuples

$$\{t, 1, \ldots, 1\}, \quad t \geqq 1, \tag{a}$$

$$\{2, 2, 1, \ldots, 1\} \tag{b}$$

when (1) may have infinitely many solutions. In particular, if $m \geqq 3$ and $f(x)$ possesses at least two simple zeros, or if $m = 2$ and $f(x)$ possesses at least three simple zeros, then the equation (1) has only finitely many solutions in $O_{\mathbf{L}}$.

LeVeque's theorem is ineffective. A. Baker [1] obtained the first general effective results for equations of the form (1) by showing that if $\mathbf{K} = \mathbf{L} = \mathbf{Q}$ then, in the important special cases mentioned above, (1) has only finitely many solutions and all these can be effectively determined.

In fact Baker [1] gave explicit bounds for all the solutions of (1). Baker's estimate was considerably improved by Sprindžuk [7], [8], [9]. Later, Trelina [10] generalized some of Sprindžuk's results to the S-integral solutions of (1) in $\mathbf{L}$.

The purpose of this paper is to make LeVeque's theorem effective and to extend Trelina's theorem to the case of any permutation $\{t_1, \ldots, t_n\}$ different from (a) and (b).

In order to be able to state our result we introduce some notations. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ ($s \geqq 0$) be distinct prime ideals in $\mathbf{L}$, let $P = \max_i N(\mathfrak{p}_i)$ (with $P = 1$ if $s = 0$) and let S denote the set of all valuations[1] of $\mathbf{L}$ corresponding to $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$. Let $O_{\mathbf{L},S}$ denote the ring of S-integers of $\mathbf{L}$.

[1] We adopt the terminology of Borevich and Shafarevich [2]. We recall that an element α of $\mathbf{L}$ is said to be S-integral if $v(\alpha) \geqq 0$ for all $v \notin S$.

THEOREM. *Suppose that* $\{t_1, \ldots, t_n\}$ *is not a permutation of the n-tuples* (a) *and* (b). *Then all solutions* $(x, y) \in O_{\mathbf{L},s}^2$ *of the equation* (1) *satisfy*[2]

$$\max\{H(x), H(y)\} < \exp\exp\{CP^2(s+1)^3\},$$

where C is an effectively computable constant depending only on $\mathbf{L}$, $f(x)$ *and m.*

From our theorem it follows the estimate $P > C'(\log\log H)^{1/5}$, $H = \max\{H(x), H(y)\}$ for the maximum norm P of the prime ideal divisors of the denominators of an arbitrary solution x, y in $\mathbf{L}$ of (1), where $C' = C'(f(x), \mathbf{L}, m)$ is effectively computable.

In [4] and [11] K. Győry, R. Tijdeman and M. Voorhoeve used the above mentioned results of Baker and LeVeque to obtain finiteness results concerning exponential diophantine equations. Their main result in [11] is ineffective because of the ineffective character of LeVeque's theorem. One of our motivations in writing this paper was to make the results of [11] effective. This application of our theorem above will be published in a separate paper.

2. Proof

Let r be the number of fundamental units in $\mathbf{L}$, and let $R_{\mathbf{L}}$ be the regulator of $\mathbf{L}$. For any $\alpha \in \mathbf{L}$, $\alpha^{(1)}, \ldots, \alpha^{(l)}$ will denote the conjugates of α (over $\mathbf{Q}$).

LEMMA 1. *Let* α *be a non-zero element in* $\mathbf{L}$ *with* $|N_{\mathbf{L}/\mathbf{Q}}(\alpha)| = M$ *and let* v *be a positive integer. There exists a unit* $\varepsilon \in O_{\mathbf{L}}$ *such that*

$$\left|\log|M^{-1/l}(\alpha\varepsilon^v)^{(i)}|\right| \leqq c_0 \frac{rv}{2} R_{\mathbf{L}}, \quad i = 1, \ldots, l$$

where $c_0 = (6rl^2/\log r)^r$ *if* $r > 1$ *and* $c_0 = 1$ *otherwise.*

PROOF. See [3].

In Lemma 2 below we shall suppose that $\mathbf{L} = \mathbf{K}$. Let $[\mathbf{K}: \mathbf{Q}] = k$, and $F(x) \in O_{\mathbf{K}}[x]$. We suppose that

$$F(X) = d_0 \prod_{i=1}^{n} (x - \omega_i)^{s_i},$$

where $\omega_i \neq \omega_j$ for $i \neq j$. Further, let

$$G(t) = \prod_{i=1}^{n} (t - d_0\omega_i)^{s_i}.$$

Denote G_i the minimal defining polynomial of $d_0\omega_i$ over $\mathbf{K}$. Let $[\mathbf{K}(\omega_i): \mathbf{K}] = u_i$, $u = u_1u_2m(m-1)\varphi(m)$, $u' = 4u_1u_2u_3$ and $n_1 = \min\{n, m-1\}$. Let $D_{\mathbf{K}}$ be the discriminant of $\mathbf{K}$. Further, let D_i denote the absolute value of the discriminant of $d_0\omega_i$ over $\mathbf{Q}$, and let $R_i = R(G', G_i)$ the resultant of G' and G_i. Under the above assumptions, L. A. Trelina [10] proved the following theorem.

[2] By the height $H(\alpha)$ of an algebraic number α we mean, as usual, the maximum of the absolute values of the relatively prime integer coefficients in its minimal defining polynomial.

LEMMA 2 (L. A. Trelina [10]). *If* $m\geqq 3$ *and* $s_1=s_2=1$, *then all solutions in S-integers* x, y *of the equation* $Ay^m=F(x)$ *satisfy*

$$\max\{H(x), H(y)\} < H^{k_0}(\omega_1)\exp\left\{(k_1(k+s))^{50u(k+s)}C_1|N_{\mathbf{K}/\mathbf{Q}}(A)|^{6u^2(1/u_1+1/u_2)(k+s)}P^{2uk}\right\},$$

where

$$C_1 = \left|D_{\mathbf{K}}^{1/2(m+3)}(D_1D_2)^{1/2mk+2}|N_{\mathbf{K}/\mathbf{Q}}(d_0^{n_1}(R_1R_2)^{m-1})^{1/u_1+1/u_2}|\right|^{6u^2(k+s)}$$

and k_0, k_1 *are effectively computable constants depending only on* k, n *and* m.

If $m=2$ *and* $s_1=s_2=s_3=1$, *then all S-integral solutions* x, y *satisfy*

$$\max\{H(x), H(y)\} < H^{k_2}(\omega_1)\exp\left\{(k_3(k+s)^{65u'(k+s)}C_2|N_{\mathbf{K}/\mathbf{Q}}(A)|^{32u'^2(k+s)}P^{3u'k}\right\},$$

where

$$C_2 = |D_{\mathbf{K}}^3(D_1D_2D_3)^{k+3}N_{\mathbf{K}/\mathbf{Q}}(d_0R_1R_2R_3)^{32u'^2(k+s)}|$$

and k_2, k_3 *are effectively computable constants depending only on* k *and* n.

PROOF. See [10].

Proof of the Theorem

We shall prove our theorem by combining LeVeque's argument with Lemma 2. We may suppose without loss of generality that $\mathbf{K}_0\subseteqq\mathbf{L}$. In this section we abbreviate $O_{\mathbf{L}}$ to O, and $N_{\mathbf{L}/\mathbf{Q}}(\alpha)$ to $N(\alpha)$. Choose $\beta\in O$ in such a way that $\beta\alpha_1, \ldots, \beta\alpha_n$ and β^N/a_0 belong to O (e.g., let $\beta=\alpha_0$). Put $a=\beta^N/a_0$ and $\beta_i=\beta\alpha_i$ for $i=1, \ldots, n$. Let x, y be an arbitrary but fixed solution of (1) in $O_{L,S}$. Then we have

$$ay^m = \prod_{i=1}^{n}(\beta x-\beta_i)^{r_i}.$$

Using brackets to indicate principal ideals in $\mathbf{L}$, we get

$$[a][y]^m = \prod_{i=1}^{n}[\beta x-\beta_i]^{r_i}, \tag{2}$$

where $[x]=\mathfrak{X}\mathfrak{p}_1^{-a_1}\ldots\mathfrak{p}_s^{-a_s}$ and $[y]=\mathfrak{W}\mathfrak{p}_1^{-b_1}\ldots\mathfrak{p}_s^{-b_s}$ with integral ideals $\mathfrak{X}$ and $\mathfrak{W}$ ($\subseteqq O$) in $\mathbf{L}$ and nonnegative integers $a_1, b_1, \ldots, a_s, b_s$.

It is known that there exist ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ such that $N(\mathfrak{q}_i)<c_1$ with an effectively computable $c_1=c_1(\mathbf{L})$ and $\mathfrak{p}_i\mathfrak{q}_i=[f_i]$ is principal ($f_i\in O$) for $i=1, \ldots, s$. Then we have

$$(\mathfrak{p}_1\mathfrak{q}_1)^{a_1+b_1}\ldots(\mathfrak{p}_s\mathfrak{q}_s)^{a_s+b_s} = [f_0],$$

where $f_0\in 0$.

Let x', y' and f be defined by $x'=xf_0^m$, $y'=yf_0^N$, and $f_0^m=f$. In view of (2) this implies

$$[a][y']^m = \prod_{i=1}^{n}[\beta x'-\beta_i f]^{r_i}, \tag{3}$$

where $x', y', f\in O$. Put $\tau=\prod_{1\leqq i<j\leqq n}[\beta_i-\beta_j]^N$. Now for $i\neq j$,

$$[\beta x'-\beta_i f, \beta x'-\beta_j f]|[\beta_i-\beta_j][f],$$

so that the g. c. d. of any two factors on the right hand side of (3) is a divisor of the ideal $\tau([f]^N$. We now fix i, and write $t_i=t$, $r_i=r$. Hence we can write

$$[\beta x'-\beta_i f]^r = \mathfrak{h}_1\mathfrak{h}_2\mathfrak{s}^m,$$

where $\mathfrak{h}_1 | [a]\tau^m$ and $\mathfrak{h}_2 | [f]^{Nm}$ for some ideal $\mathfrak{s}$ in O depending on x', f and i.

Let $\mathfrak{p}$ be a prime ideal dividing $\mathfrak{s}$ and such that

$$N(\mathfrak{p}) > \max\{c_1, N([a]\tau^m), P\} = c_2.$$

Then if $\mathfrak{p}^{t'} \| \mathfrak{s}^m$, t' must be a multiple of m and of r, and hence of their l. c. m. rt, so that we can write $\mathfrak{s}^m=\mathfrak{s}_1^{rt}\cdot\mathfrak{s}_2$, where $\mathfrak{s}_2$ is divisible only by powers of prime ideals $\mathfrak{p}$ with $N(\mathfrak{p})\leqq c_2$.

The ideal $\mathfrak{s}_2$, in turn, can be written as a product $\mathfrak{s}_3^{rt}\mathfrak{s}_4$, where $\mathfrak{s}_4$ is such that every prime power factor has an exponent less than rt; and similarly $\mathfrak{h}_2=\mathfrak{h}_3^{rt}\cdot\mathfrak{h}_4$.

We can see that

$$[\beta x'-\beta_i f]^r = \mathfrak{h}_1(\mathfrak{h}_3\mathfrak{s}_1\mathfrak{s}_3)^{rt}\mathfrak{h}_4\mathfrak{s}_4$$

or

$$[\beta x'-\beta_i f]^r = \mathfrak{s}_5^{rt}\mathfrak{s}_6$$

so that

$$[\beta x'-\beta_i f] = \mathfrak{s}_5^t\mathfrak{s}_7. \tag{4}$$

Further it is easy to see that

$$N(\mathfrak{s}_7) \leqq N(\mathfrak{s}_4\mathfrak{h}_1\mathfrak{h}_4) < \exp\{c_3(s+1)P^2\} = c_4,$$

where c_3 is an effectively computable constant depending only on $\mathbf{L}$, $f(x)$ and m.

Now let $\mathfrak{z}$ run over a fixed system of representatives of the various ideal classes in O; we can assume that $N(\mathfrak{z})<c_1$ with the above c_1. Each $\mathfrak{s}_5$ occurring in (4) is equivalent to some $\mathfrak{z}$, so that there are γ and δ_0 in O such that $[\gamma]\mathfrak{s}_5=[\delta_0]\mathfrak{z}$.

Let $[\gamma, \delta_0]=\mathfrak{n}$, $[\gamma]=\mathfrak{n}\mathfrak{d}$, $[\delta_0]=\mathfrak{n}\mathfrak{g}$. Then $\mathfrak{d}\mathfrak{s}_5=\mathfrak{g}\mathfrak{z}$, whence $\mathfrak{d}|\mathfrak{z}$, so that $N(\mathfrak{d})<c_1$. Since $\mathfrak{s}_5\sim\mathfrak{z}$, it follows that $\mathfrak{d}\sim\mathfrak{g}$. There is an $\mathfrak{h}$ in O with $N(\mathfrak{h})<c_1$ for which $\mathfrak{d}\mathfrak{h}=[\zeta]$ is principal. Then

$$\mathfrak{h}\mathfrak{d}\mathfrak{s}_5 = \mathfrak{h}\mathfrak{g}\mathfrak{z}, \quad [\zeta]\mathfrak{s}_5 = [\xi]\mathfrak{z} \quad (\xi\in O)$$

and $N([\zeta])<c_1^2$. This shows that γ can be chosen to be one of the finitely many different integers ζ required to generate the various ideals $\mathfrak{d}\mathfrak{h}$. This together with (4) yields

$$[\beta x'-\beta_i f][\gamma]^t = \mathfrak{s}_7[\gamma]^t\mathfrak{s}_5^t$$

or

$$[\beta x'-\beta_i f][\gamma]^t = \mathfrak{s}_7[\delta_0]^t\mathfrak{z}^t.$$

Clearly $\mathfrak{s}_7\mathfrak{z}^t=[\vartheta]$ is principal and $|N(\vartheta)|<c_4c_1^m$, moreover

$$(\beta x'-\beta_i f)\gamma^t = \varepsilon\vartheta\delta_0^t, \tag{5}$$

where $\varepsilon\in O$ is a unit. Thus, using Lemma 1 we can see that there are γ_1 and ϑ_1 in O and units η_1, $\eta_2\in O$ such that $\gamma_1=\eta_1\gamma$, $\vartheta_1=\eta_2\vartheta$, $H(\gamma_1)<c_5$ with an effective $c_5=c_5(\mathbf{L})$ and

$$H(\vartheta_1) < \exp\{c_6(s+1)P^2\},$$

with an effectively computable constant c_6 depending only on $\mathbf{L}$, $f(x)$ and m.

Then by (5) we have

$$\gamma_1^t(\beta x'-\beta_i f)=\varepsilon_0\vartheta_1\delta_0^t,$$

where ε_0 is a unit in O ($\gamma_1, \varepsilon_0, \vartheta_1, \delta_0$ depending on f, x', i).

By Lemma 1 we can write $\varepsilon_0=\varepsilon_1\varepsilon_2^t$, where ε_1, $\varepsilon_2\in O$ are units and $H(\varepsilon_1)<c_7$ with an effective $c_7=c_7(\mathbf{L}, m)$. So from (5) we get

(6a) $$\gamma_{1i}^{t_i}(\beta x'-\beta_i f)=\varepsilon_{1i}\vartheta_{1i}\delta_i^{t_i},$$

and

(6b) $$\gamma_{1j}^{t_j}(\beta x'-\beta_j f)=\varepsilon_{1j}\vartheta_{1j}\delta_j^{t_j},$$

where $\delta_k=\varepsilon_{2k}\cdot\delta_{0k}$ for $k=1, \ldots, n$ and $i\neq j$. In view of (6a) and (6b) this implies

(7) $$\lambda_1 z^{t_i}=\lambda_2 w^{t_j}+\lambda_3 f,$$

where $z=\delta_i$, $w=\delta_j$, $\lambda_1=\varepsilon_{1i}\vartheta_{1i}\gamma_{1j}^{t_j}$, $\lambda_2=\varepsilon_{1j}\vartheta_{1j}\gamma_{1i}^{t_i}$, and $\lambda_3=(\beta_j-\beta_i)\gamma_{1i}^{t_i}\gamma_{1j}^{t_j}$. It is easily seen that

$$\max_i\{H(\lambda_i)\}<\exp\{c_8(s+1)P^2\},$$

where c_8 is an effectively computable constant depending only on $f(x)$, $\mathbf{L}$ and m.

We now distinguish the following cases:

(A) There exist $i\neq j$ such that $t_i\geqq 3$ and $t_j\geqq 2$;

(B) There exist $i_1<i_2<i_3$ such that $t_{i_1}=t_{i_2}=t_{i_3}=2$.

Consider first the case (A). Since $[f]=[f_1]^{\mu_1}\ldots[f_s]^{\mu_s}$, so let us write $[f]$ in the form $[f_{ij}][g_{ij}]^{t_it_j}$, where $[f_{ij}]=[f_1]^{\nu_1}\ldots[f_s]^{\nu_s}$ with $\nu_l\equiv\mu_l \pmod{t_it_j}$ and $0\leqq\nu_l<t_it_j$.

Clearly

$$|N(f_{ij})|<c_2^{2(s+1)m^2}<\exp\{c_9(s+1)P\},$$

where c_9 is an effectively computable constant depending only on $f(x)$, $\mathbf{L}$ and m. So by Lemma 1 we can write $f=f'_{ij}g'^{t_it_j}_{ij}$, where f'_{ij}, $g'_{ij}\in O$ and

$$H(f'_{ij})<\exp\{c_{10}(s+1)P\}$$

with an effectively computable constant c_{10} depending only on $f(x)$, $\mathbf{L}$ and m. Therefore from (7) we get

(8) $$\lambda_1(z/g'^{t_j}_{ij})^{t_i}=\lambda_2(w/g'^{t_i}_{ij})^{t_j}+\lambda_3 f'_{ij}.$$

Let S' be the set of valuations of $\mathbf{L}$ corresponding to the prime ideals dividing $\mathfrak{p}_1, \ldots, \mathfrak{p}_s, \mathfrak{q}_1, \ldots, \mathfrak{q}_s$. The maximum norm of these prime ideals does not exceed $c_{11}P$, where $c_{11}=\max\{1, c_1\}$. Further, putting $s'=|S'|$, we have $s'\leqq s(1+c_1)$. Clearly $z/g'^{t_j}_{ij}$ and $w/g'^{t_i}_{ij}$ are S'-integers in $\mathbf{L}$. Apply now Lemma 2 to (8). Let

$$F(x)=\lambda_2+x^{t_j}+\lambda_3 f'_{ij}=\lambda_2\prod_{i=1}^{t_j}(x-x_i), \quad G(t)=\prod_{i=1}^{t_j}(t-\lambda_2 x_i),$$

D_i the absolute value of the discriminant of $\lambda_2 x_i$ over $\mathbf{Q}$ and R_i the resultant of G' and the minimal defining polynomial of $\lambda_2 x_i$ over $\mathbf{L}$. Then

$$H_i = H(x_i) < \exp\{c_{12}(s+1)^2 P^2\} \quad (i = 1, \ldots, t_j),$$

$$|N_{\mathbf{L}/\mathbf{Q}}(\lambda_1)| \leqq H^l(\lambda_1) < \exp\{c_{13}(s+1)P^2\},$$

$$D_i \leqq (lH(\lambda_2 x_i))^{2l-1} < \exp\{c_{14}(s+1)^2 P^2\},$$

and

$$|N_{\mathbf{L}/\mathbf{Q}}(R_i)| < \exp\{c_{15}(s+1)^2 P^2\},$$

where $c_{12}, c_{13}, c_{14}, c_{15}$ are effectively computable constants depending only on $\mathbf{L}$, $f(x)$ and m. So, applying Lemma 2 we get

$$H(z/g_{ij'}^{'t_j}) < \exp\exp\{c_{16}(s+1)^3 P^2\},$$

which implies

$$H(\delta_i^{t_i}/f) < \exp\exp\{c_{17}(s+1)^3 P^2\},$$

where c_{17} is an effectively computable constant depending only on $\mathbf{L}$, $f(x)$ and m.

From (6a) we get

$$x = x'/f = (\varepsilon_{1i}\vartheta_{1i}(\delta_i^{t_i}/f)\gamma_{1i}^{-t_i} + \beta_i)\beta^{-1}.$$

Finally we obtain

$$H(x) < \exp\exp\{c_{18}(s+1)^3 P^2\},$$

where c_{18} denotes an effectively computable constant depending only on $\mathbf{L}$, $f(x)$ and m.

Let us return to the case (B). We can assume without loss of generality that $i_1=1$, $i_2=2$, $i_3=3$. Then we can write

$$\gamma_{1i}^2(\beta x' - \beta_i f) = \varepsilon_{1i}\vartheta_{1i}\delta_i^2 \quad (i = 1, 2, 3)$$

or

$$\gamma_0^2 \prod_{i=1}^{3} (\beta x' - \beta_i f) = o\vartheta_0 \varrho^2, \tag{9}$$

where $\gamma_0 = \gamma_{11}\gamma_{12}\gamma_{13}$, $\sigma = \varepsilon_{11}\varepsilon_{12}\varepsilon_{13}$, $\varrho = \delta_1\delta_2\delta_3$ and $\vartheta_0 = \vartheta_{11}\vartheta_{12}\vartheta_{13}$. We can assume again that $f = f'g^2$, where

$$H(f') < \exp\{c_9(s+1)P\}.$$

We obtain from (9)

$$\gamma_0^2 \prod_{i=1}^{3} (\beta x'/f - \beta_i) = \sigma\vartheta_0 f'^{-3}(\varrho/g^3)^2. \tag{10}$$

Using Lemma 2 we have again

$$H(x) < \exp\exp\{c_{20}(s+1)^3 P^2\},$$

where c_{20} is an effectively computable constant depending only on $\mathbf{L}$, $f(x)$ and m. The proof is like that of case (A).

Finally, from (1) we get

$$H(y) < \exp\exp\{c_{20}(s+1)^3 P^2\},$$

where c_{20} denotes an effectively computable constant which depends only on $\mathbf{L}$, $f(x)$ and m. This completes the proof.

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ÜBER DIE MITTEL VON ORTHOGONALEN FUNKTIONEN

K. TANDORI (Szeged), Mitglied der Akademie

1.

In folgenden bezeichnet $\lambda=\{\lambda_k\}_1^\infty$ eine nichtabnehmende Folge von positiven Zahlen mit

$$\lambda_0 = \lambda_1 = 1 \quad \text{und} \quad \lim_{k\to\infty} \lambda_k = \infty.$$

Die Klasse der orthonormierten Systeme $\varphi=\{\varphi_k(x)\}_1^\infty$ im Grundintervall $(0, 1)$ bezeichnen wir mit Ω.

In dieser Arbeit werden wir die Konvergenzverhältnisse der Mittel

$$\frac{a_1\varphi_1(x)+\ldots+a_n\varphi_n(x)}{\lambda_n} \quad (n = 1, 2, \ldots) \tag{1}$$

betrachten, wobei $\varphi\in\Omega$ und $a=\{a_k\}_1^\infty$ eine reelle Zahlenfolge sind.

Die sich ergebenden Resultate sind in einem beliebigen, endlichen und nichtatomischen Maßraum gültig; nur einfachkeitshalber betrachten wir das Intervall (0,1) mit den Lebesgueschen Maß.

Um die Resultate formulieren zu können, sind gewisse Bezeichnungen und Ergebnisse vorauszuschicken.

Für eine reelle Zahlenfolge a setzen wir

$$\|a\| = \sup_{\varphi\in\Omega} \int_0^1 \sup_n \left\{\left(\sum_{k=1}^n a_k\varphi_k(x)\right)^2 dx\right\}^{1/2} \quad (\leqq\infty),$$

$$\|a;\lambda;N\| = \sup_{\varphi\in\Omega}\left\{\int_0^1 \sup_{n\geqq N}\left(\frac{1}{\lambda_n}\sum_{k=1}^n a_k\varphi_k(x)\right)^2 dx\right\}^{1/2} \quad (\leqq\infty)\ (N = 1, 2, \ldots),$$

$$\|a;\lambda\| = \|a;\lambda;1\|.$$

Die folgenden Sätze sind bekannt [2].

SATZ A. *Es sei a eine Zahlenfolge. Ist* $\|a\|<\infty$, *so konvergiert die Reihe*

$$\sum_{k=1}^\infty a_k\varphi_k(x) \tag{2}$$

bei jedem $\varphi\in\Omega$ *in* $(0, 1)$ *fast überall. Ist aber* $\|a\|=\infty$, *so gibt es ein* $\Phi\in\Omega$ *derart, daß die Reihe*

$$\sum_{k=1}^\infty a_k\Phi_k(x) \tag{3}$$

in $(0, 1)$ *fast überall divergiert.*

SATZ B. *Ist* $|a_k| \leqq |b_k|$ $(k=1, 2, \ldots)$, *so gilt* $\|a\| \leqq \|b\|$.

SATZ C. *Für jede Folge a gilt die Abschätzung*

$$\|a\| \leqq c_1 \left(a_1^2 + \sum_{k=2}^{\infty} a_k^2 \log^2 k\right)^{1/2}.$$

Im Falle $|a_k| \geqq |a_{k+1}|$ $(k=1, 2, \ldots)$, *gilt noch*

$$\|a\| \geqq c_2 \left(a_1^2 + \sum_{k=2}^{\infty} a_k^2 \log^2 k\right)^{1/2}.$$

($c_1, c_2, \ldots$ bezeichnen positive Konstanten.)

Es sei $\mu = \{\mu_k\}_1^\infty$ eine Folge von ganzen Zahlen mit $0 = \mu_1 < \ldots < \mu_k < \ldots$.

Mit $T(\mu)$ bezeichnen wir das folgenderweise definierte Summationsverfahren: die Reihe $\sum_{k=1}^{\infty} u_k$ ist $T(\mu)$-summierbar, wenn die Folge der μ_k-ten Partialsummen dieser Reihe konvergiert.

Es sei T ein Summationsverfahren. Für eine Folge a und für ein $\varphi \in \Omega$ bezeichnen wir die n-te Partialsumme, bzw. das n-te T-Mittel der Reihe (2) mit $s_n(a; \varphi; x)$, bzw. mit $t_n(a; \varphi; x)$. Wir sagen, daß das Summationsverfahren T mit der Methode $T(\mu)$ äquivalent ist, wenn die Konvergenz in (0, 1) fast überall der Folge $\{s_{\mu_k}(a; \varphi; x)\}_1^\infty$ und die Konvergenz in (0, 1) fast überall der Folge $\{t_n(a; \varphi; x)\}_1^\infty$ für jedes $a \in l^2$ und für jedes $\varphi \in \Omega$ äquivalent ist.

Für eine Folge a setzen wir

$$A_k(\mu) = \left\{\sum_{l=\mu_k+1}^{\mu_{k+1}} a_l^2\right\}^{1/2} \quad (k = 1, 2, \ldots).$$

SATZ D. *Es sei T ein reguläres Toeplitzsches Summationsverfahren, welches mit der Methode* $T(\mu)$ *äquivalent ist. Gilt* $\|A(\mu)\| < \infty$, *so ist die Reihe* (2) *für jedes* $\varphi \in \Omega$ *in* (0, 1) *fast überall T-summierbar. Gilt aber* $\|A(\mu)\| = \infty$, *so gibt es ein* $\Phi \in \Omega$ *derart, daß die Reihe* (3) *in* (0, 1) *fast überall nicht T-summierbar ist.*

SATZ E. *Für die Folge* λ *gibt es eine Folge* $\nu = \nu(\lambda) = \{\nu_k(\lambda)\}_1^\infty$ *von ganzen Zahlen mit* $0 = \nu_1(\lambda) < \ldots < \nu_k(\lambda) < \ldots$ *derart, daß das Riesz-Verfahren* (R, λ) *mit der Methode* $T(\nu)$ *äquivalent ist.*

Der Satz E stellt eine Verschärfung eines bekannten Satzes (s.z.B. [1], S. 139—140) dar, den wir im Punkt 3 beweisen werden.

Aus den Sätzen D und E folgt unmittelbar:

SATZ F. *Gilt* $\|A(\nu)\| < \infty$ *für eine Folge a, so ist die Reihe* (2) *für jedes* $\varphi \in \Omega$ *in* (0, 1) *fast überall* (R, λ)*-summierbar. Gilt aber* $\|A(\nu)\| = \infty$, *so gibt es ein* $\Phi \in \Omega$ *derart, daß die Reihe* (3) *in* (0, 1) *fast überall nicht* (R, λ)*-summierbar ist.*

Für eine Zahlenfolge a und für eine Folge $h = \{h_k\}_1^\infty$ der von 0 verschiedenen Zahlen setzen wir:

$$\frac{a}{h} = \left\{\frac{a_k}{h_k}\right\}_1^\infty, \quad A_k\left(\frac{1}{h}; \nu\right) = \left\{\sum_{l=\nu_k+1}^{\nu_{k+1}} \frac{a_l^2}{h_l^2}\right\}^{1/2} \quad (k = 1, 2, \ldots).$$

Wir werden die folgenden Sätze beweisen.

SATZ I. *Gilt* $\|a;\lambda\|<\infty$ *für die Folge a, so ist*

$$\lim_{n\to\infty}\frac{a_1\varphi_1(x)+\ldots+a_n\varphi_n(x)}{\lambda_n}=0 \tag{4}$$

bei jedem $\varphi\in\Omega$ *in* $(0,1)$ *fast überall. Gilt aber* $\|a;\lambda\|=\infty$, *so gibt es ein* $\Phi\in\Omega$ *derart, daß*

$$\overline{\lim_{n\to\infty}}\frac{|a_1\Phi_1(x)+\ldots+a_n\Phi_n(x)|}{\lambda_n}=\infty \tag{5}$$

in $(0,1)$ *fast überall besteht.*

SATZ II. *Aus* $\left\|\frac{a}{\lambda}\right\|<\infty$ *folgt* $\|a;\lambda\|<\infty$ *für jede Folge a.*

SATZ III. *Es sei a eine Zahlenfolge. Gibt es eine monoton wachsende Folge h von positiven Zahlen mit* $\left\|\frac{a}{\lambda h}\right\|=\infty$ *und* $\left\|A\left(\frac{1}{\lambda h};\ v\right)\right\|<\infty$, *so ist* $\|a;\lambda\|=\infty$.

SATZ IV. *Wir nehmen an, daß für* λ $\log^2 k\leqq c_3\log^{2-\varepsilon}\nu_k(\lambda)$ $(k=2,3,\ldots)$ *mit einem* $\varepsilon>0$ *besteht. Ist die Folge* $\left\{\frac{a_k^2}{\lambda_k^2}\right\}_1^\infty$ *monoton nichtwachsend, und gilt weiterhin*

$$\sum_{k=2}^{\infty}\frac{a_k^2}{\lambda_k^2}\log^2 k=\infty, \tag{6}$$

so gibt es ein $\Phi\in\Omega$ *derart, daß* (5) *in* $(0,1)$ *fast überall besteht.*

BEMERKUNGEN. 1. Aus dem Satz I bekommen wir unmittelbar auch die folgenden Behauptungen. Gilt (4) bei jedem $\varphi\in\Omega$ in $(0,1)$ fast überall, so ist

$$\sup_n\frac{|a_1\varphi_1(x)+\ldots+a_n\varphi_n(x)|}{\lambda_n}\in L^2(0,1)$$

für jedes $\varphi\in\Omega$. Gilt aber (4) für ein $\varphi\in\Omega$ in $(0,1)$ nicht fast überall, so gibt es ein $\Phi\in\Omega$ derart, daß (5) in $(0,1)$ fast überall besteht.

2. Auf Grund der Sätze A und I ist die Behauptung des Satzes II mit der folgenden Behauptung äquivalent: Konvergiert die Reihe

$$\sum_{k=1}^{\infty}\frac{a_k}{\lambda_k}\varphi_k(x) \tag{7}$$

bei jedem $\varphi\in\Omega$ in $(0,1)$ fast überall, so gilt (4) bei jedem $\varphi\in\Omega$ in $(0,1)$ fast überall.

So folgt der Satz II unmittelbar durch Anwendung des wohlbekannten Kroneckerschen Lemmas, das folgenderweise lautet.

KRONECKERSCHES LEMMA. *Konvergiert die Reihe*

$$\sum_{k=1}^{\infty}\frac{u_k}{\lambda_k}, \tag{8}$$

so gilt

$$\lim_{n\to\infty}\frac{u_1+\ldots+u_n}{\lambda_n}=0. \tag{9}$$

3. Es ist bekannt, daß die Umkehrung der Implikation in diesem Lemma im allgemeinen nicht richtig ist. D.h. aus (9) folgt die Konvergenz der Reihe (8) im allgemeinen nicht, oder m.a.W. aus der Divergenz der Folge $\left\{\frac{u_1+\ldots+u_n}{\lambda_n}\right\}_1^\infty$ folgt die Divergenz von (8) nicht immer, was das Beispiel $u_k=1$, $\lambda_n=k$ $(k=1, 2, \ldots)$ zeigt.

Man kann noch zeigen, daß die umgekehrte Implikation im Falle nicht richtig ist, wenn $u_k=a_k\varphi_k(x)$ $(k=1, 2, \ldots)$ ist, wobei a eine Zahlenfolge bezeichnet, $\varphi\in\Omega$ ist, und die Divergenz der Reihe (7) in (0, 1) fast überall erfordert wird.

Es seien nähmlich $\lambda_k=k$ $(k=1, 2, \ldots)$ und

$$a_k=\begin{cases}2^n/\sqrt{n\log^3 n}, & k=2^n;\ n=2, 3, \ldots,\\ 0 & \text{sonst.}\end{cases}$$

Dann gelten

$$\sum_{k=1}^{\infty}\frac{a_k^2}{\lambda_k^2}=\sum_{n=2}^{\infty}\frac{1}{n\log^3 n}<\infty, \tag{10}$$

und

$$\sum_{n=2}^{\infty}\frac{a_{2^n}^2}{\lambda_{2^n}^2}\log^2 n=\sum_{n=2}^{\infty}\frac{1}{n\log n}=\infty; \tag{11}$$

weiterhin gilt

$$\frac{a_{2^n}^2}{\lambda_{2^n}^2}\geqq\frac{a_{2^{n+1}}^2}{\lambda_{2^{n+1}}^2}\quad(n=2, 3, \ldots). \tag{12}$$

Durch Anwendung der Sätze A, C und auf Grund von (11), (12) folgt $\left\|\left\{\frac{a_{2^n}}{\lambda_{2^n}}\right\}_2^\infty\right\|=\infty$, woraus sich auf Grund der Definition von $\|\cdot\|$ $\left\|\frac{a}{\lambda}\right\|=\infty$ ergibt. Daraus erhalten wir durch Anwendung des Satzes A, daß ein $\varphi\in\Omega$ derart existiert, daß die Reihe (7) in (0, 1) fast überall divergiert.

Anderseits gilt

$$\sum_{n=2}^{\infty}\int_0^1\left(\frac{1}{\lambda_{2^n}}\sum_{k=1}^{2^n}a_k\varphi_k(x)\right)^2dx=\sum_{n=2}^{\infty}\frac{1}{\lambda_{2^n}^2}\sum_{k=1}^{2^n}a_k^2=$$

$$=\sum_{k=1}^{\infty}a_k^2\sum_{2^n\geqq k}\frac{1}{\lambda_{2^n}^2}=\sum_{k=1}^{\infty}a_k^2\sum_{2^n\geqq k}\frac{1}{2^{2n}}\leqq c_4\sum_{k=1}^{\infty}\frac{a_k^2}{k^2}=c_4\sum_{k=1}^{\infty}\frac{a_k^2}{\lambda_k^2}<\infty$$

für jedes $\varphi\in\Omega$, auf Grund von (10). Daraus folgt

$$\lim_{n\to\infty}\frac{1}{\lambda_{2^n}}(a_1\varphi_1(x)+\ldots+a_{2^n}\varphi_{2^n}(x))=0 \tag{13}$$

bei jedem $\varphi\in\Omega$ in (0, 1) fast überall. Da auf Grund der Definition von a

$$\frac{|a_1\varphi_1(x)+\ldots+a_k\varphi_k(x)|}{\lambda_k}\leqq\frac{|a_1\varphi_1(x)+\ldots+a_{2^n}\varphi_{2^n}(x)|}{2^n}$$

$$(x\in(0,1);\ 2^n<k<2^{n+1};\ n=2,3,\ldots)$$

gilt, folgt (4) bei jedem $\varphi\in\Omega$ in (0, 1) fast überall aus (13).

Für die Richtigkeit der umgekehrten Implikation im Kroneckerschen Lemma kann man eine einfache Bedingung geben.

HILFSSATZ I. *Ist die Reihe* (8) *divergent, und ist sie* (R,λ)*-summierbar, so divergiert die Folge* $\left\{\frac{u_1+\ldots+u_n}{\lambda_n}\right\}_1^\infty$ *auch.*

Der Hilfssatz I ergibt sich leicht aus der folgenden Gleichung:

$$\frac{1}{\lambda_n}\sum_{k=1}^{n}u_k=\sum_{k=1}^{n}\frac{\lambda_k}{\lambda_n}\frac{u_k}{\lambda_k}=-\sum_{k=1}^{n}\left(1-\frac{\lambda_k}{\lambda_n}\right)\frac{u_k}{\lambda_k}+\sum_{k=1}^{n}\frac{u_k}{\lambda_k},$$

da hier $\sum_{k=1}^{n}\left(1-\frac{\lambda_k}{\lambda_n}\right)\frac{u_l}{\lambda_l}$ das n-te (R,λ)-Mittel der Reihe (8) ist.

Aus dem Hilfssatz I folgt unmittelbar die folgende Behauptung. Ist die Reihe (7) für ein $\varphi\in\Omega$ in (0, 1) fast überall divergent, und ist diese Reihe für dieses φ in (0, 1) fast überall (R,λ)-summierbar, so ist die Folge (1) in (0, 1) fast überall divergent.

Auf Grund des Satzes I ergibt sich sogar, daß unter diesen Bedingungen auch ein solches $\Phi\in\Omega$ existiert, daß (5) in (0, 1) fast überall gilt.

Aus dem Hilfssatz I erhalten wir auf Grund der Sätze A, F und I auch folgendes:

SATZ III′. *Gelten* $\left\|\frac{a}{\lambda}\right\|=\infty$ *und* $\left\|A\left(\frac{1}{\lambda};\ v\right)\right\|<\infty$ *für die Folge* a*, so ist* $\|a;\lambda\|=\infty$.

5. Satz III ist die Verschärfung des Satzes III′; nach dem Satz B folgt nämlich $\left\|\frac{a}{\lambda}\right\|=\infty$ aus $\left\|\frac{a}{\lambda h}\right\|=\infty$, wegen $\left\|\frac{a}{\lambda}\right\|\geqq h,\left\|\frac{a}{\lambda h}\right\|$.

Durch Anwendung des Hilfssatzes I und Satzes I bekommen wir unmittelbar auch die folgende Behauptung. Gibt es eine nichtabnehmende Folge h von positiven Zahlen derart, daß die Reihe

$$\sum_{k=1}^{\infty}\frac{a_k}{\lambda_k h_k}\varphi_k(x)$$

für ein gewißes $\varphi\in\Omega$ in (0, 1) fast überall divergiert und diese Reihe in (0, 1) fast überall (R,λ)-summierbar ist, so gibt es ein $\Phi\in\Omega$ derart, daß (5) in (0, 1) fast überall besteht.

6. Satz IV ist die Verallgemeinerung eines bekannten Satzes [3].

2.

Zum Beweis der Sätze sind wir weitere Hilfssätze vorauszuschicken.

Für eine Zahlenfolge a und für die ganzen Zahlen M_1, M_2, N_1, N_2 $(1 \leqq M_1 \leqq M_2, 1 \leqq N_1 \leqq N_2)$ setzen wir

$$a(M_1, M_2) = \{0, \ldots, 0, a_{M_1}, \ldots, a_{M_2}, 0, \ldots\}, \quad a(M_1, \infty) = \{0, \ldots, 0, a_{M_1}, \ldots\},$$

$$\|a(M_1, M_2); \lambda; N_1, N_2\| = \sup_{\varphi \in \Omega} \left\{ \int_0^1 \sup_{N_1 \leqq n \leqq N_2} \left(\frac{1}{\lambda_n} \sum_{k=M_1}^{\min(n, M_2)} a_k \varphi_k(x) \right)^2 dx \right\}^{1/2}.$$

HILFSSATZ II. *Es seien a eine Zahlenfolge und $\overline{M}_1, \overline{M}_2, \overline{N}_1, \overline{N}_2$ ganze Zahlen $(1 \leqq \overline{M}_1 \leqq \overline{M}_2, 1 \leqq \overline{N}_1 \leqq \overline{N}_2)$. Dann gibt es ein orthonormiertes System von Treppenfunktionen $\psi_k(\overline{M}_1, \overline{M}_2, \overline{N}_1, \overline{N}_2; x)$ $(k = \overline{M}_1, \ldots, \overline{M}_2)$ in $(0, 1)$ und eine einfache Menge*[1] *$E(\overline{M}_1, \overline{M}_2, \overline{N}_1, \overline{N}_2)$ $(\subseteq (0, 1))$ derart, daß* $\operatorname{mes} E(\overline{M}_1, \overline{M}_2, \overline{N}_1, \overline{N}_2) \geqq 1/2$ *und*

$$\sup_{N_1 \leqq n \leqq N_2} \frac{1}{\lambda_n} \left| \sum_{k=\overline{M}_1}^{\min(n, \overline{M}_2)} a_k \psi_k(\overline{M}_1, \overline{M}_2, \overline{N}_1, \overline{N}_2; x) \right| \geqq c_5 \|a(\overline{M}_1, \overline{M}_2; \lambda; \overline{N}_1, \overline{N}_2\|$$

$(x \in E(\overline{M}_1, \overline{M}_2, \overline{N}_1, \overline{N}_2))$ *erfüllt sind.*

BEWEIS DES HILFSSATZES II. Ohne Beschränkung der Allgemeinheit können wir $\|a(\overline{M}_1, \overline{M}_2; \lambda; \overline{N}_1, \overline{N}_2\| > 0$ voraussetzen. Nach der Definition von $\|a(\overline{M}_1, \overline{M}_2); \lambda; \overline{N}_1, \overline{N}_2\|$ gibt es ein orthonormiertes System von Funktionen $\varphi_k(x)$ $(k = \overline{M}_1, \ldots, \overline{M}_2)$ in $(0, 1)$ mit

$$\int_0^1 \sup_{N_1 \leqq n \leqq N_2} \left(\frac{1}{\lambda_n} \sum_{k=M_1}^{\min(n, \overline{M}_2)} a_k \varphi_k(x) \right)^2 dx > \frac{1}{2^2} \|a(\overline{M}_1, \overline{M}_2); \lambda; \overline{N}_1, \overline{N}_2\|^2. \tag{14}$$

Für ein $\eta > 0$ seien $\chi_k(x)$ $(k = \overline{M}_1, \ldots, \overline{M}_2)$ Treppenfunktionen in $(0, 1)$ mit

$$\int_0^1 (\varphi_k(x) - \chi_k(x))^2 dx < \eta \quad (k = \overline{M}_1, \ldots, \overline{M}_2).$$

Aus (14) folgt

$$\int_0^1 \sup_{N_1 \leqq n \leqq N_2} \left(\frac{1}{\lambda_n} \sum_{k=M_1}^{\min(n, \overline{M}_2)} a_k \chi_k(x) \right)^2 dx > \frac{1}{2^2} \|a(\overline{M}_1, \overline{M}_2,); \lambda; \overline{N}_1, \overline{N}_2\|^2, \tag{15}$$

wenn die Zahl η genügend klein ist. Wir setzen

$$\alpha_{kl} = \int_0^1 \chi_k(x) \chi_l(x) dx \quad (k, l = \overline{M}_1, \ldots, \overline{M}_2).$$

Es sei I_{kl} $(k, l = \overline{M}_1, \ldots, \overline{M}_2; k \neq l)$ eine Einteilung des Intervalls $\left(\frac{1}{2}, 1\right)$ in paarweise disjunkte Intervalle mit gleicher Länge $\operatorname{mes} I_{kl} = ((\overline{M}_2 - \overline{M}_1 + 1)(\overline{M}_2 - \overline{M}_1))^{-1}$. Dann

[1] Eine Menge heißt einfach, wenn sie die Vereinigung endlich vieler Intervalle ist.

setzen wir

$$\bar{\chi}_k(x) = \begin{cases} \sqrt{2}\,\chi_k(2x), & x \in \left(0, \frac{1}{2}\right), \\ \sqrt{\frac{|\alpha_{kl}|}{2\,\mathrm{mes}\, I_{kl}}}, & x \in I_{kl};\ l = \overline{M}_1, \ldots, \overline{M}_2;\ l \neq k, \\ -\sqrt{\frac{|\alpha_{kl}|}{2\,\mathrm{mes}\, I_{kl}}}\ \mathrm{sign}\,\alpha_{kl}, & x \in I_{lk};\ l = \overline{M}_1, \ldots, \overline{M}_2;\ l \neq k, \\ 0 & \text{sonst} \end{cases}$$

$(k = \overline{M}_1, \ldots, \overline{M}_2)$. Es ist klar, daß diese Funktionen Treppenfunktionen sind, die ein orthogonales System in (0, 1) bilden; weiterhin folgt aus (15)

$$(16) \qquad \int_0^1 \sup_{N_1 \leqq n \leqq N_2} \left(\frac{1}{\lambda_n} \sum_{k=\overline{M}_1}^{\min(n, \overline{M}_2)} a_k \bar{\chi}_k(x) \right)^2 dx > \frac{1}{2^2} \|a(\overline{M}_1, \overline{M}_2); \lambda; \overline{N}_1, \overline{N}_2\|^2.$$

Wir setzen

$$\psi_k(x) = \bar{\chi}_k(x) \Big/ \left\{ \int_0^1 \bar{\chi}_k^2(t)\, dt \right\}^{1/2} \quad (k = \overline{M}_1, \ldots, \overline{M}_2).$$

Diese Funktionen sind Treppenfunktionen und bilden ein orthonormiertes System im (0, 1); weiterhin folgt aus (16)

$$(17) \qquad a^2 = \int_0^1 \sup_{N_1 \leqq n \leqq N_2} \left(\frac{1}{\lambda_n} \sum_{k=\overline{M}_1}^{\min(n\, \overline{M}_2)} a_k \psi_k(x) \right)^2 dx > \frac{1}{2^2} \|a(\overline{M}_1, \overline{M}_2); \lambda; \overline{N}_1, \overline{N}_2\|^2$$

für genügend kleines η.

Wir setzen $a'_k = 2\dfrac{a_k}{a}$ $(k = \overline{M}_1, \ldots, \overline{M}_2)$ und

$$F(x) = \left\{ \sup_{N_1 \leqq n \leqq N_2} \left(\frac{1}{\lambda_n} \sum_{k=\overline{M}_1}^{\min(n, \overline{M}_2)} a'_k \psi_k(x) \right)^2 \right\}^{1/2}.$$

Aus (17) ergibt sich:

$$(18) \qquad \int_0^1 F^2(x)\, dx = 2.$$

Da $F(x)$ eine Treppenfunktion ist, gibt es eine Einteilung des Intervalls (0, 1) in paarweise disjunkte Intervalle $I_r = (a_r, b_r)$ $(r = 1, \ldots, \varrho)$ derart, daß $F(x)$ in jedem I_r konstant ist; den Wert von $F(x)$ in I_r bezeichnen wir mit w_r $(r = 1, \ldots, \varrho)$. Es seien $1 \leqq r_1 < \ldots < r_\sigma \leqq \varrho$ diejenigen Indizes r, für welche $w_r \geqq 1$ gilt; die übrigen Indizes r $(1 \leqq r \leqq \varrho)$ bezeichnen wir mit $1 \leqq s_1 < \ldots < s_{\varrho-\sigma} \leqq \varrho$. Aus (18) folgt

$$(19) \qquad b = \sum_{p=1}^{\sigma} w_{r_p}^2\, \mathrm{mes}\, I_{r_p} = \sum_{r=1}^{\varrho} w_r^2\, \mathrm{mes}\, I_r - \sum_{q=1}^{\varrho-\sigma} w_{s_q}^2\, \mathrm{mes}\, I_{s_q} \geqq 2 - \sum_{q-1}^{\varrho-\sigma} \mathrm{mes}\, I_{s_q} \geqq 1.$$

Es seien $J'_p = (a'_p, b'_p)$ $(p = 1, \ldots, \sigma)$ paarweise disjunkte Intervalle in $(0, b)$ mit $\mathrm{mes}\, J'_p = w_{r_p}^2\, \mathrm{mes}\, J_{r_p}$ $(p = 1, \ldots, \sigma)$ und $J''_q = (a''_q, b''_q)$ $(q = 1, \ldots, \varrho - \sigma)$ paarweise

disjunkte Intervalle in $(b, 2)$ mit $\operatorname{mes} J_q'' = \operatorname{mes} J_{s_q}$ $(s=1, \ldots, \varrho-\sigma)$. Wir setzen

$$\bar{\psi}_k(\overline{M}_1, \overline{M}_2, \overline{N}_1, \overline{N}_2; x) = \begin{cases} \dfrac{1}{w_{r_p}} \psi_k\left(\dfrac{x-a_p'}{w_{r_p}^2} + a_{r_p}\right), & x \in J_p'; \quad p = 1, \ldots, \sigma, \\ \psi_k(x - a_{s_q}), & x \in J_q''; \quad q = 1, \ldots, \varrho - \sigma \end{cases}$$

$(k=\overline{M}_1, \ldots, \overline{M}_2)$ und $\bar{E}(\overline{M}_1, \overline{M}_2, \overline{N}_1, \overline{N}_2) = \bigcup_{p=1}^{\sigma} J_p'$. Es sei endlich

$$\psi_k(\overline{M}_1, \overline{M}_2, \overline{N}_1, \overline{N}_2; x) = \sqrt{2}\bar{\psi}_k(\overline{M}_1, \overline{M}_2, \overline{N}_1, \overline{N}_2; 2x) \quad (x \in (0, 1);\ k = \overline{M}_1, \ldots, \overline{M}_2),$$

und $E(\overline{M}_1, \overline{M}_2, \overline{N}_1, \overline{N}_2)$ bezeichne die Menge, die aus $\bar{E}(\overline{M}_1, \overline{M}_2, \overline{N}_1, \overline{N}_2)$ unter der Transformation $y=x/2$ entsteht.

Es ist klar, daß diese Funktionen Treppenfunktionen sind, in (0, 1) ein orthonormiertes System bilden, $\operatorname{mes} E(\overline{M}_1, \overline{M}_2, \overline{N}_1, \overline{N}_2) \geqq 1/2$ besteht, weiterhin

$$\sup_{N_1 \leqq n \leqq N_2} \left(\frac{1}{\lambda_n} \sum_{k=\overline{M}_1}^{\min(n, \overline{M}_2)} a_k' \psi_k(\overline{M}_1, \overline{M}_2, \overline{N}_1, \overline{N}_2; x) \right)^2 = 1 \quad (x \in E(\overline{M}_1, \overline{M}_2, \overline{N}_1, \overline{N}_2))$$

gilt. Aus (17) und (18) folgt

$$\sup_{N_1 \leqq n \leqq N_2} \left(\frac{1}{\lambda_n} \sum_{k=\overline{M}_1}^{\min(n, \overline{M}_2)} a_k \psi_k(\overline{M}_1, \overline{M}_2, \overline{N}_1, \overline{N}_2; x) \right)^2 =$$

$$= \frac{a^2}{4} > \frac{1}{2^4} \|a(\overline{M}_1, \overline{M}_2);\ \lambda;\ \overline{N}_1, \overline{N}_2\|$$

für $x \in E(\overline{M}_1, \overline{M}_2, \overline{N}_1, \overline{N}_2)$.

Damit haben wir Hilfssatz II bewiesen.

Auf Grund der Definition von $\|\cdot; \lambda; N\|$ gilt $\|a; \lambda; N\| \geqq \|a; \lambda; N+1\|$ $(N=1, 2, \ldots)$; so existiert $\lim_{N\to\infty} \|a; \lambda; N\|$ $(\leqq \infty)$ für jede Folge a, und es ist $\lim_{N\to\infty} \|a; \lambda; N\| \geqq 0$.

HILFSSATZ III. *Gilt* $\lim_{N\to\infty} \|a; \lambda; N\| = 0$ *für die Folge a, so gilt* (4) *bei jedem* $\varphi \in \Omega$ *in* (0, 1) *fast überall. Gilt aber* $\lim_{N\to\infty} \|a; \lambda; N\| > 0$, *so gibt es ein* $\Phi \in \Omega$ *derart, daß* (5) *in* (0, 1) *fast überall besteht.*

BEWEIS DES HILFSSATZES III. a) Wir nehmen $\lim_{N\to\infty} \|a; \lambda; N\| = 0$ an. Es seien $\varphi \in \Omega$ und

$$F_N(x) = \sup_{n \geqq N} \frac{1}{\lambda_n} |a_1 \varphi_1(x) + \ldots + a_n \varphi_n(x)| \quad (N = 1, 2, \ldots).$$

Dann gelten

(20) $$F_N(x) \geqq F_{N+1}(x) \quad (x \in (0, 1);\ N = 1, \ldots) 2,$$

und

(21) $$\int_0^1 F_N^2(x)\, dx \leqq \|a; \lambda; N\|^2 \to 0 \quad (N \to \infty).$$

Aus (20) und (21) erhalten wir, daß (4) für dieses System φ in (0, 1) fast überall besteht.

b) Es seien $\lim_{N\to\infty} \|a; \lambda; N\| > 0$ und ϱ eine positive Zahl mit $\lim_{N\to\infty} \|a; \lambda; N\| > \varrho$. Wir zeigen, daß für jede positive ganze Zahl M eine positive ganze Zahl $N_0 = N_0(M)$ derart existiert, daß

$$\|a(M, \infty); \lambda; N\| > \frac{\varrho}{2} \quad (N \geqq N_0(M); M = 1, 2, \ldots) \tag{22}$$

gilt. Auf Grund der Definition von $\|\cdot; \lambda; N\|$ gibt es ein $\varphi(N) \in \Omega$ mit

$$\left\{\int_0^1 \sup_{n \geqq N} \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} a_k \varphi_k(N; x)\right)^2 dx\right\}^{1/2} > \varrho \quad (N = 1, 2, \ldots),$$

woraus

(23)

$$\left\{\int_0^1 \sup_{n \geqq N} \left(\frac{1}{\lambda_n} \sum_{k=M}^{n} a_k \varphi_k(N; x)\right)^2 dx\right\}^{1/2} > \varrho - \left\{\int_0^1 \sup_{n \geqq N} \left(\frac{1}{\lambda_n} \sum_{k=1}^{M-1} a_k \varphi_k(N; x)\right)^2 dx\right\}^{1/2}$$

folgt. Ist $N \geqq N_0(M)$, so ist

$$\left\{\int_0^1 \sup_{n \geqq N} \left(\frac{1}{\lambda_n} \sum_{k=1}^{M-1} a_k \varphi_k(N; x)\right)^2 dx\right\}^{1/2} \leqq \frac{1}{\lambda_N} \sum_{k=1}^{M-1} |a_k| < \frac{\varrho}{2},$$

und so bekommen wir aus (23)

$$\left\{\int_0^1 \sup_{n \geqq N} \left(\frac{1}{\lambda_n} \sum_{k=M}^{n} a_k \varphi_k(N; x)\right)^2 dx\right\}^{1/2} > \frac{\varrho}{2} \quad (N \geqq N_0(M); M = 1, 2, \ldots),$$

woraus sich (22) ergibt.

Dann zeigen wir, daß es für beliebige positive ganze Zahlen $M, N_0 (\geqq M)$ positive ganze Zahlen N_1, N_2 $(N_0 \leqq N_1 \leqq N_2)$ gibt, derart, daß

(24)

$$\|a(M, N_2); \lambda; N_1, N_2\| > \frac{\varrho}{2} \quad (M = 1, 2, \ldots; N_0 = M, M+1, \ldots; N_0 \leqq N_1 \leqq N_2)$$

besteht.

Es seien nämlich $M, N_0 (\geqq M)$ positive ganze Zahlen. Auf Grund von (22) gibt es eine ganze Zahl $N_1 (\geqq \max(N_0, N_0(M)))$ mit

$$\|a(M, \infty); \lambda; N_1\| > \frac{\varrho}{2}. \tag{25}$$

Auf Grund der Definitionen von $\|\cdot; \lambda; \overline{N}\|$ und $\|\cdot; \lambda; \overline{N}_1, \overline{N}_2\|$ gilt

$$\|a(M, \infty); \lambda; N_1, N\| \nearrow \|a(M, \infty); \lambda; N_1\| \quad (N_1 \leqq N \nearrow \infty).$$

Daraus und aus (25) folgt

$$\|a(M, \infty); \lambda; N_1, N_2\| > \frac{\varrho}{2} \tag{26}$$

für genügend großes $N_2(\geqq N_1)$. Da auf Grund der Definition von $\|\cdot\,; \lambda; \overline{N}_1, \overline{N}_2\|$

$$\|a(M, \infty); \lambda; N_1, N_2\| = \|a(M, N_2); \lambda; N_1, N_2\|$$

ist, folgt aus (26) (24).

Durch vollständige Induktion definieren wir zwei Folgen $\{N_m\}_0^\infty$, $\{N'_m\}_1^\infty$ von ganzen Zahlen, eine Folge $\{E_m\}_1^\infty$ von einfachen Teilmengen von (0, 1) und ein orthonormiertes System $\Phi = \{\Phi_k(x)\}_1^\infty$ von Treppenfunktionen in (0, 1) derart, daß die Mengen E_m $(m=1, 2, \ldots)$ stochastisch unabhängig und für jedes $m(=1, 2, \ldots)$ die folgenden Bedingungen erfüllt sind.

(i) Es ist

$$N_{m-1} \leqq N'_m \leqq N_m.$$

(ii) Es gilt

$$\operatorname{mes} E_m \geqq 1/2m.$$

(iii) Es besteht

$$\sup_{N'_m \leqq n \leqq N_m} \frac{1}{\lambda_n} |a_{N_{m-1}+1} \Phi_{N_{m-1}+1}(x) + \ldots + a_n \Phi_n(x)| \geqq c_5 \frac{\varrho}{2} \sqrt{m} \quad (x \in E_m).$$

(iv) Weiterhin ist

$$\frac{1}{\lambda_n} |a_1 \Phi_1(x) + \ldots + a_{N_{m-1}} \Phi_{N_{m-1}}(x)| \leqq c_5 \frac{\varrho}{2} \quad (x \in (0, 1);\ n = N'_m, N'_m + 1, \ldots).$$

Es seien $N_0=0$ und $N'_1=1$. Auf Grund von Voraussetzung gibt es eine ganze Zahl $N_1(\geqq 1)$ mit

$$\|a(1, N_1); \lambda; 1, N_1\| = \|a(1, \infty); \lambda; 1, N_1\| > \frac{\varrho}{2}.$$

Wir wenden den Hilfssatz II im Falle $\overline{M}_1=1$, $\overline{M}_2=N_1$, $\overline{N}_1=1$, $\overline{N}_2=N_1$ und setzen

$$\Phi_k(x) = \psi_k(1, N_1, 1, N_1; x) \quad (k = 1, \ldots, N_1), \quad E_1 = E(1, N_1, 1, N_1).$$

Diese Funktionen sind Treppenfunktionen, sie bilden ein orthonormiertes System in (0, 1), die Menge E_1 ist einfach, weiterhin sind (i), (ii), (iii), (iv) im Falle $m=1$ erfüllt.

Es sei m_0 eine positive ganze Zahl. Wir nehmen an, daß die Indizes $N_0, \ldots, M_{m_0}$, $N'_1, \ldots, N'_{m_0}$, die Mengen $E_m(\subseteq(0, 1))$ $(m=1, \ldots, m_0)$ und die Funktionen $\Phi_k(x)$ $(k=1, \ldots, N_{m_0})$ schon derart definiert sind, daß diese Funktionen Treppenfunktionen sind, und ein orthonormiertes System in (0, 1) bilden, diese Mengen stochastisch unabhängig sind, weiterhin (i), (ii), (iii), (iv) im Falle $m=1, \ldots, m_0$ erfüllt sind.

Es sei $N_0(\geqq N_{m_0})$ eine ganze Zahl mit

$$\frac{1}{\lambda_n} |a_1 \Phi_1(x) + \ldots + a_{N_{m_0}} \Phi_{N_{m_0}}(x)| \leqq c_5 \frac{\varrho}{2} \quad (x \in (0, 1);\ n = N_0, N_0 + 1, \ldots). \tag{27}$$

Auf Grund von (24) gibt es ganze Zahlen N'_{m_0+1}, N_{m_0+1} $(N_{m_0} \leqq N_0 \leqq N'_{m_0+1} \leqq N_{m_0+1})$ mit

$$\|a(N_{m_0} + 1, N_{m_0+1}); \lambda; N'_{m_0+1}, N_{m_0+1}\| > \frac{\varrho}{2}.$$

Wir wenden den Hilfssatz II im Falle $\overline{M}_1=N_{m_0+1}$, $\overline{M}_2=N_{m_0+1}$, $\overline{N}_1=N'_{m_0+1}$, $\overline{N}_2=N_{m_0+1}$ an. Nach dem Hilfssatz II gelten

(28)
$$\sup_{N'_{m_0+1}\leqq n\leqq N_{m_0+1}} \frac{1}{\lambda_n}\left|\sum_{k=N_{m_0}+1}^{n} a_k\psi_k(N_{m_0}+1, N_{m_0+1}, N'_{m_0+1}, N_{m_0+1}; x)\right| \leqq c_5\frac{\varrho}{2}$$
$$(x\in E(N_{m_0}+1, N_{m_0+1}, N'_{m_0+1}, N_{m_0+1})),$$

(29) $$\operatorname{mes} E(N_{m_0}+1, N_{m_0+1}, N'_{m_0+1}, N_{m_0+1}) \geqq 1/2.$$

Wir setzen

$$\bar{\Phi}_k(x)=\begin{cases}\sqrt{m}\,\psi_k(N_{m_0}+1, N_{m_0+1}, N'_{m_0+1}, N_{m_0+1}; mx), & 0<x<1/m,\\ 0 & \text{sonst}\end{cases}$$

$(k=N_{m_0}+1, \ldots, N_{m_0+1})$; und wir bezeichnen mit $\bar{E}_{m_0+1}$ die Menge, die unter der Transformation $y=x/m$ aus $E(N_{m_0}+1, N_{m_0+1}, N'_{m_0+1}, N_{m_0+1})$ hervorgeht. Aus (28), (29) bekommen wir

(30) $$\sup_{N'_{m_0+1}\leqq n\leqq N_{m_0+1}} \frac{1}{\lambda_n}\left|\sum_{k=N_{m_0}+1}^{n} a_k\bar{\Phi}_k(x)\right| \leqq c_5\frac{\varrho}{2}\sqrt{m} \quad (x\in\bar{E}_{m_0+1}),$$

(31) $$\operatorname{mes}\bar{E}_{m_0+1} \geqq 1/2m.$$

Da die Funktionen $\Phi_k(x)$ $(k=1, \ldots, N_{m_0})$ Treppenfunktionen und die Mengen E_m $(m=1, \ldots, m_0)$ einfach sind, kann man das Intervall $(0, 1)$ in paarweise disjunkte Intervalle I_r $(r=1, \ldots, \varrho)$ derart einteilen, daß jede Funktion $\Phi_k(x)$ $(k=1, \ldots, N_{m_0})$ konstant in jedem I_r und jede Menge E_m $(m=1, \ldots, m_0)$ die Vereinigung gewisser I_r sind. Die zwei Hälfte von I_r bezeichnen wir mit I'_r, I''_r $(r=1, \ldots, \varrho)$. Für eine in $(0, 1)$ definierte Funktion $f(x)$ und für ein Intervall $I=(a, b)$ $(\subseteq(0, 1))$ setzen wir

$$f(I; x)=\begin{cases} f\left(\dfrac{x-a}{b-a}\right), & a<x<b,\\ 0 & \text{sonst},\end{cases}$$

und für eine Menge H $(\subseteq(0, 1))$ sei $H(I)$ die Menge, die aus H unter der Transformation $y=(b-a)x+a$ hervorgeht. Wir setzen

$$\Phi_k(x)=\sum_{r=1}^{\varrho}\bar{\Phi}_k(I'_r; x)-\sum_{r=1}^{\varrho}\bar{\Phi}_k(I''_r; x) \quad (k=N_{m_0}+1, \ldots, N_{m_0+1}),$$

$$E_{m_0+1}=\left(\bigcup_{r=1}^{\varrho}\bar{E}_{m_0+1}(I'_r)\right)\cup\left(\bigcup_{r=1}^{\varrho}\bar{E}_{m_0+1}(I''_r)\right).$$

Es ist klar, daß die Funktionen $\Phi_k(x)$ $(k=N_{m_0}+1, \ldots, N_{m_0+1})$ Treppenfunktionen sind, die Funktionen $\Phi_k(x)$ $(k=1, \ldots, N_{m_0+1})$ in $(0, 1)$ ein orthonormiertes System bilden, die Menge E_{m_0+1} einfach ist und die Mengen E_m $(m=1, \ldots, m_0+1)$ stochastisch unabhängig sind. Auf Grund der Definition von N'_{m_0+1}, N_{m_0+1} ist (i) im Falle $m=m_0+1$ erfüllt. Weiterhin folgt aus (27), (30) und (31), daß (ii), (iii), (iv) im Falle $m=m_0+1$ erfüllt sind. Die Indexfolgen $\{N_m\}_0^\infty$, $\{N'_m\}_1^\infty$, die Mengenfolge $\{E_m\}_1^\infty$ und das System Φ mit den erwähnten Eigenschaften erhalten wir also durch Induktion.

Aus (iii) und (iv) folgt

$$(32)\qquad \sup_{1\leqq n\leqq N_m}\frac{1}{\lambda_n}\left|\sum_{k=1}^{n}a_k\Phi_k(x)\right|\geqq\sup_{N'_m\leqq n\leqq N}\frac{1}{\lambda_n}\left|\sum_{k=1}^{n}a_k\Phi_k(x)\right|\geqq$$

$$\geqq\sup_{N'_m\leqq n\leqq N_m}\frac{1}{\lambda_n}\left|\sum_{k=N_{m-1}+1}^{n}a_k\Phi_k(x)\right|-\sup_{N'_m\leqq n\leqq N_m}\frac{1}{\lambda_n}\left|\sum_{k=1}^{N_{m-1}}a_k\Phi_k(x)\right|\geqq$$

$$\geqq c_5\frac{\varrho}{2}(\sqrt{m}-1)\quad(x\in E_m;\ m=1,2,\ldots).$$

Da die Mengen E_m $(m=1, 2, \ldots)$ stochastisch unabhängig sind, folgt aus (ii)

$$\operatorname{mes}(\overline{\lim_{m\to\infty}}E_m)=1.$$

Daraus und aus (32) folgt, daß für dieses Funktionensystem Φ (5) in (0, 1) fast überall gilt.

Damit haben wir Hilfssatz III beweisen.

HILFSSATZ IV. *Gilt* $|a_k|\leqq|b_k|$ $(k=1, 2, \ldots)$, *so ist* $\|a;\lambda\|\leqq\|b;\lambda\|$.

BEWEIS DER HILFSSATZES IV. Es sei $\varepsilon>0$ beliebig. Auf Grund der Definition von $\|\cdot;\lambda\|$ gibt es ein $\varphi\in\Omega$ mit

$$(33)\qquad \int_0^1\sup_n\left(\frac{1}{\lambda_n}\sum_{k=1}^{n}a_k\varphi_k(x)\right)^2dx\geqq\|a;\lambda\|^2-\varepsilon.^{2}$$

Wir setzen

$$\psi_k(x)=\begin{cases}\sqrt{2}\,\dfrac{a_k}{b_k}\,\varphi_k(2x), & x\in(0,1/2),\\[2ex] \sqrt{\left(1-\dfrac{a_k^2}{b_k^2}\right)}\,2\varphi_k(2x-1), & x\in(1/2,1)\end{cases}$$

$(k=1, 2, \ldots)$.[3] Es ist klar, daß $\psi\in\Omega$. Auf Grund von Definition der $\|a;\lambda\|$ und von (33) gilt

$$\|b;\lambda\|^2\geqq\int_0^1\sup_n\left(\frac{1}{\lambda_n}\sum_{k=1}^{n}b_k\psi_k(x)\right)^2dx\geqq\int_0^{1/2}\sup_n\left(\frac{1}{\lambda_n}\sum_{k=1}^{n}b_k\psi_k(x)\right)^2dx=$$

$$=2\int_0^{1/2}\sup_n\left(\frac{1}{\lambda_n}\sum_{k=1}^{n}a_k\varphi_k(2x)\right)^2dx=\int_0^1\sup_n\left(\frac{1}{\lambda_n}\sum_{k=1}^{n}a_k\varphi_k(x)\right)^2dx\geqq\|a;\lambda\|^2-\varepsilon.$$

Da $\varepsilon>0$ beliebig ist, folgt die Behauptung des Hilfssatzes IV.

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[2] Ist $\|a;\lambda\|=\infty$, so soll man den Beweis anstatt $\|a;\lambda\|^2-\varepsilon$ mit einer beliebigen positiven Zahl M weiterführen.

[3] Im Falle $b_k=0$ soll man unter $\dfrac{a_k}{b_k}$ 0 verstehen.

3. Beweis der Sätze

BEWEIS DES SATZES E. Durch vollständige Induktion können wir die Indexfolge $\nu=\nu(\lambda)=\{\nu_k(\lambda)\}_1^\infty$ mit folgenden Eigenschaften definieren:

$$\nu_1(\lambda) = 0,$$

und für jede positive ganze Zahl k sei $\nu_{k+1}(\lambda)$ die kleinste ganze Zahl, mit

$$2 \leqq \frac{\lambda_{\nu_{k+1}(\lambda)}}{\lambda_{\nu_k(\lambda)}} \leqq 4, \quad \text{venn es eine ganze Zahl } l > \nu_k(\lambda) \text{ mit } 2 < \frac{\lambda_l}{\lambda_{\nu_k(\lambda)}} \leqq 4 \text{ gibt,}$$

bzw.

$$\nu_{k+1}(\lambda)=\nu_k(\lambda)+1, \quad \text{wenn } \lambda_{\nu_k(\lambda)+1}>4\lambda_{\nu_k(\lambda)} \text{ ist.}$$

Es sei bemerkt, daß aus der Definition von ν

$$\frac{\lambda_{\nu_{k+1}(\lambda)}}{\lambda_{\nu_k(\lambda)}} \geqq 2 \quad (k = 2, 3, \ldots) \tag{34}$$

folgt.

Wir zeigen, daß für diese Indexfolge $\{\nu_k(\lambda)\}_1^\infty$ die Bedingungen des Satzes E erfüllt sind.

Es seien $a\in l^2$ und $\varphi\in\Omega$. Es sei

$$r_n(a;\varphi;x) = \sum_{k=1}^{n}\left(1-\frac{\lambda_k}{\lambda_n}\right)a_k\varphi_k(x)$$

das n-te (R,λ)-Mittel der Reihe (2). Dann folgt aus (34):

$$\sum_{k=2}^{\infty}\int_0^1 \left(s_{\nu_k(\lambda)}(a;\varphi;x)-r_{\nu_k(\lambda)}(a;\varphi;x)\right)^2 dx = \sum_{k=2}^{\infty}\frac{1}{\lambda_{\nu_k(\lambda)}^2}\sum_{l=1}^{\nu_k(\lambda)} a_l^2\lambda_l^2 =$$

$$= \sum_{l=1}^{\infty} a_l^2\lambda_l^2 \sum_{\substack{k \\ \nu_k(\lambda)\geqq l}} \frac{1}{\lambda_{\nu_k(\lambda)}^2} \leqq c_6 \sum_{l=1}^{\infty} a_l^2 < \infty,$$

woraus sich

$$\lim_{k\to\infty}\left(s_{\nu_k(\lambda)}(a;\varphi;x)-r_{\nu_k(\lambda)}(a;\varphi;x)\right) = 0 \tag{35}$$

in (0, 1) fast überall ergibt.

Ist also die Reihe (2) in (0, 1) fast überall (R,λ)-summierbar, so konvergiert die Folge $\{s_{\nu_k(\lambda)}(a;\varphi;x)\}_1^\infty$ in (0, 1) fast überall.

Wir nehmen an, daß die Folge $\{s_{\nu_k(\lambda)}(a;\varphi;x)\}_r^\infty$ in (0, 1) fast überall konvergiert. Nach (35) existiert dann $\lim_{k\to\infty} r_{\nu_k(\lambda)}(a;\varphi;x)$ in (0, 1) fast überall. Es sei Z die Menge der ganzen Zahlen $k(\geqq 2)$, für welche $\nu_{k+1}(\lambda)>\nu_k(\lambda)+1$. Für ein $k\in Z$ setzen wir

$$\delta_k(x) = \max_{\nu_k(\lambda)<\sigma<\nu_{k+1}(\lambda)} |r_\sigma(a;\varphi;x)-r_{\nu_k(\lambda)}(a;\varphi;x)|.$$

Durch einfache Rechnung ergibt sich

$$\delta_k(x) \leqq \sum_{s=\nu_k(\lambda)}^{\nu_{k+1}(\lambda)-1} |r_{s+1}(a;\varphi;x)-r_s(a;\varphi;x)| = \tag{36}$$

$$= \sum_{s=\nu_k(\lambda)}^{\nu_{k+1}(\lambda)-1} \sqrt{\frac{\lambda_{s+1}-\lambda_s}{\lambda_s}} \sqrt{\frac{\lambda_s}{\lambda_{s+1}-\lambda_s}} |r_{s+1}(a;\varphi;x)-r_s(a;\varphi;x)| \leqq$$

$$\leqq \left\{\sum_{s=\nu_k(\lambda)}^{\nu_{k+1}(\lambda)-1} \frac{\lambda_{s+1}-\lambda_s}{\lambda_s}\right\}^{1/2} \left\{\sum_{k=\nu_k(\lambda)}^{\nu_{k+1}(\lambda)-1} \frac{\lambda_s}{\lambda_{s+1}-\lambda_s}\left(r_{s+1}(a;\varphi;x)-r_s(a;\varphi;x)\right)^2\right\}^{1/2}.$$

Da auf Grund der Definition von ν und Z

$$\sum_{s=\nu_k(\lambda)}^{\nu_{k+1}(\lambda)-1} \frac{\lambda_{s+1}-\lambda_s}{\lambda_s} \leqq \frac{1}{\lambda_{\nu_k(\lambda)}} \sum_{s=\nu_k(\lambda)}^{\nu_{k+1}(\lambda)-1} (\lambda_{s+1}-\lambda_s) \leqq \frac{\lambda_{\nu_{k+1}(\lambda)}}{\lambda_{\nu_k(\lambda)}} \leqq 4$$

gilt, folgt aus (36)

$$\delta_k(x) \leqq 2\left\{\sum_{s=\nu_k(\lambda)}^{\nu_{k+1}(\lambda)-1} \frac{\lambda_s}{\lambda_{s+1}-\lambda_s}\left(r_{s+1}(a;\varphi;x)-r_s(a;\varphi;x)\right)^2\right\}^{1/2} \quad (k\in Z).$$

Somit gilt

$$\sum_{k\in Z}\int_0^1 \delta_k^2(x)\,dx \leqq 4\sum_{k\in Z}\left(\sum_{s=\nu_k(\lambda)}^{\nu_{k+1}(\lambda)-1} \frac{\lambda_s}{\lambda_{s+1}-\lambda_s}\left(\sum_{l=1}^{s}\left(\frac{1}{\lambda_{s+1}}-\frac{1}{\lambda_s}\right)^2 \lambda_l^2 a_l^2\right)\right) \leqq \tag{37}$$

$$\leqq 4\sum_{k=2}^{\infty}\left(\sum_{s=\nu_k(\lambda)}^{\nu_{k+1}(\lambda)-1} \frac{\lambda_s}{\lambda_{s+1}-\lambda_s}\left(\sum_{l=1}^{s}\left(\frac{1}{\lambda_{s+1}}-\frac{1}{\lambda_s}\right)^2 \lambda_l^2 a_l^2\right)\right) \leqq$$

$$\leqq 4\sum_{s=1}^{\infty}\frac{\lambda_{s+1}-\lambda_s}{\lambda_s\lambda_{s+1}^2}\left(\sum_{l=1}^{s} a_l^2\lambda_l^2\right) = 4\sum_{l=1}^{\infty} a_l^2\lambda_l^2\left(\sum_{s=l}^{\infty}\frac{\lambda_{s+1}-\lambda_s}{\lambda_s\lambda_{s+1}^2}\right).$$

Da

$$\frac{\lambda_{s+1}-\lambda_s}{\lambda_s\lambda_{s+1}^2} = \frac{(\lambda_{s+1}-\lambda_s)(\lambda_{s+1}+\lambda_s)}{\lambda_s\lambda_{s+1}^2(\lambda_{s+1}+\lambda_s)} \leqq \frac{\lambda_{s+1}^2-\lambda_s^2}{\lambda_s^2\lambda_{s+1}^2} = \frac{1}{\lambda_s^2}-\frac{1}{\lambda_{s+1}^2}$$

ist, folgt aus (37)

$$\sum_{k\in Z}\int_0^1 \delta_k^2(x)\,dx \leqq 4\sum_{l=1}^{\infty} a_l^2\lambda_l^2\left(\sum_{s=l}^{\infty}\left(\frac{1}{\lambda_s^2}-\frac{1}{\lambda_{s+1}^2}\right)\right) \leqq 4\sum_{l=1}^{\infty} a_l^2 < \infty,$$

woraus sich $\lim_{k\to\infty;\, k\in Z} \delta_k(x)=0$ in $(0, 1)$ fast überall ergibt. Da im Falle $\nu_k(\lambda)<n<\nu_{k+1}(\lambda)$ $(k\in Z)$

$$|r_n(a;\varphi;x)-r_{\nu_k(\lambda)}(a;\varphi;x)| \leqq \delta_k(x) \quad (x\in(0,1))$$

gilt, folgt nach dem Obigen, daß $\lim_{n\to\infty} r_n(a;\varphi;x)$ in $(0, 1)$ fast überall existiert.

Damit haben wir Satz E bewiesen.

BEWEIS DES SATZES I. Satz I folgt unmittelbar aus dem Hilfssatz III.

BEWEIS DES SATZES III. Es sei also h eine monoton wachsende Folge von positiven Zahlen mit (i) $\left\|\frac{a}{\lambda h}\right\|=\infty$, (ii) $\left\|A\left(\frac{1}{\lambda h};\ \gamma\right)\right\|<\infty$. Aus dem Satz A folgt, daß ein $\varphi\in\Omega$ derart existiert, daß die Reihe

$$\sum_{l=1}^{\infty}\frac{a_l}{\lambda_l h_l}\varphi_l(x)$$

in (0, 1) fast überall divergiert; wegen (ii) ist aber diese Reihe auf Grund des Satzes F in (0, 1) fast überall (R, λ)-summierbar, und so folgt durch Anwendung des Hilfssatzes I, daß

$$\left\{\frac{1}{\lambda_n}\sum_{l=1}^{n}\frac{a_l}{h_l}\varphi_l(x)\right\}_1^{\infty}$$

in (0, 1) fast überall divergiert. Nach dem Hilfssatz III folgt daraus $\left\|\frac{a}{h};\ \lambda\right\|=\infty$, und nach Hilfssatz IV ergibt sich endlich $\|a;\lambda\|=\infty$, weil $\left|\frac{a_l}{h_l}\right|\leqq\frac{1}{h_1}|a_l|$ $(l=1, 2, \ldots)$ und somit $\left\|\frac{a}{h};\ \lambda\right\|\leqq\left\|\frac{1}{h_1}\{a_l\}_1^{\infty};\ \lambda\right\|=\frac{1}{h_1}\|a;\lambda\|$ bestehen.

Damit haben wir auch Satz III bewiesen.

BEWEIS DES SATZES IV. Wir können ohne Beschränkung der Allgemeinheit annehmen, daß

$$|a_2|\geqq 2\lambda_2 \tag{38}$$

ist und in (6) log den Logarithmus mit dem Basis 2 bedeutet. Es sei

$$h_k=\left\{\left(\sum_{l=2}^{k}\frac{a_l^2}{\lambda_l^2}\log^2 l\right)\log\left(\sum_{l=2}^{k}\frac{a_l^2}{\lambda_l^2}\log^2 l\right)\right\}^{1/2}\quad(k=2, 3, \ldots),\quad h_1=1.$$

Auf Grund von (38) ist $h=\{h_k\}_1^{\infty}$ eine monoton wachsende Folge von positiven Zahlen. Durch einfache Rechnung ergibt sich:

$$\sum_{k=1}^{\infty}\frac{a_k^2}{\lambda_k^2 h_k^2}\log^2 k=\sum_{k=2}^{\infty}\frac{a_k^2}{\lambda_k^2 h_k^2}\log^2 k\geqq \tag{39}$$

$$\geqq c_7\sum_{k=3}^{\infty}\frac{a_k^2}{\lambda_k^2}\log^2 k\frac{1}{\left(\sum_{l=2}^{k-1}\frac{a_l^2}{\lambda_l^2}\log^2 l\right)\log\left(\sum_{l=2}^{k-1}\frac{a_l^2}{\lambda_l^2}\log^2 l\right)}\geqq c_7\int_{\frac{a_2^2}{\lambda_2^2}}^{\infty}\frac{dx}{x\log x}=\infty,$$

$$\sum_{k=1}^{\infty}\frac{a_k^2}{\lambda_k^2 h_k^2}\log^{2-\varepsilon}k=\sum_{k=2}^{\infty}\frac{a_k^2}{\lambda_k^2}\log^2 k\frac{1}{\left(\sum_{l=2}^{k}\frac{a_l^2}{\lambda_l^2}\log^2 l\right)\log\left(\sum_{l=2}^{k}\frac{a_l^2}{\lambda_l^2}\log^2 l\right)\log^{\varepsilon}k}\leqq \tag{40}$$

$$\leqq c_8\sum_{k=2}^{\infty}\frac{a_k^2}{\lambda_k^2}\log^2 k\frac{1}{\left(\sum_{l=2}^{k}\frac{a_l^2}{\lambda_l^2}\log^2 l\right)\log^{1+\varepsilon}\left(\sum_{l=2}^{k}\frac{a_l^2}{\lambda_l^2}\log^2 l\right)}\leqq c_8\int_{\frac{a_1^2}{\lambda_1^2}}^{\infty}\frac{dx}{x\log^{1+\varepsilon}x}<\infty,$$

da

$$\log\left(\sum_{l=2}^{k}\frac{a_l^2}{\lambda_l^2}\log^2 l\right)\leqq c_9\log k \quad (k=2,3,\ldots)$$

wegen $\frac{a_l^2}{\lambda_l^2}\geqq\frac{a_{l+1}}{\lambda_{l+1}^2}$ $(l=1,2,\ldots)$ sich ergibt. Auf Grund der Voraussetzung bezüglich λ erhalten wir

$$(41)\qquad \sum_{k=2}^{\infty}A_k^2\left(\frac{1}{\lambda h};\ v\right)\log^2 k\leqq c_3\sum_{k=2}^{\infty}\left(\sum_{l=v_k(\lambda)+1}^{v_{k+1}(\lambda)}\frac{a_l^2}{\lambda_l^2h_l^2}\right)\log^{2-\varepsilon}v_k(\lambda)\leqq$$

$$\leqq c_3\sum_{l=v_2(\lambda)+1}^{\infty}\frac{a_l^2}{\lambda_l^2h_l^2}\log^{2-\varepsilon}l<\infty,$$

wegen (40). Aus (39), bzw. aus (41) folgt $\left\|\frac{a}{\lambda h}\right\|=\infty$, bzw. $\left\|A\left(\frac{1}{\lambda h};\ v\right)\right\|<\infty$ durch Anwendung der Sätze A und B. Aus Satz III bekommen wir $\|a;\lambda\|=\infty$, woraus sich durch Anwendung des Satzes I die Behauptung ergibt.

Damit haben wir Satz IV bewiesen.

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METRIZATION AND LIAPUNOV FUNCTIONS. II

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We carry on our investigations concerning metrics of Liapunov type [4], [5], [6].

Let (X, d) be a metric space and let $T_t\colon X\to X$, $t\in\mathbf{R}$ be a dynamical system on X. Consider a point $x\in X$. As usual, the α- and ω-limit sets of the trajectory through x are denoted by $\alpha(x)$ and $\omega(x)$, respectively.

DEFINITION. We say that the trajectory through x is (strictly) monotonous, or, more precisely, it is (strictly) monotonous with respect to the metric d if the following conditions are satisfied:

(i) the function $f_x\colon \mathbf{R}\to\mathbf{R}^+$, $t\to d(\alpha(x), T_t x)$ is (strictly) increasing.

(ii) the function $g_x\colon \mathbf{R}\to\mathbf{R}^+$, $t\to d(T_t x, \omega(x))$ is (strictly) decreasing.

LEMMA. *Assume that the trajectory through x is monotonous with respect to the metric d. Then, for* $y_1, y_2\in\alpha(x)$, $z_1, z_2\in\omega(x)$

$$d(\alpha(x), z_1) = d(\alpha(x), z_2); \quad d(y_1, \omega(x)) = d(y_2, \omega(x)).$$

PROOF. We may restrict ourselves to the proof of the first relation. There exist sequences $\{t_n\}_1^\infty$, $\{\tau_n\}_1^\infty$ such that $t_1<\tau_1<t_2<\tau_2<\dots$, $t_n\to\infty$ and $T_{t_n}x\to z_1$, $T_{\tau_n}x\to z_2$ as $n\to\infty$. The desired result is implied by $d(\alpha(x), T_{t_n}x)\leqq d(\alpha(x), T_{\tau_n}x)\leqq d(\alpha(x), T_{t_{n+1}}x)$. Thus the lemma is proved.

Assume that M is a non-void closed asymptotically stable set. Let $A(M)$ be its region of attraction. Assume that for any $x\in A(M)\setminus M$, the trajectory through x is monotonous with respect to the metric d. It is clear that d is not necessarily a metric of Liapunov type, i.e. $d(x, M)$ is not necessarily a Liapunov function with respect to M. The converse is not true, either. This is shown by the following

EXAMPLE. Let X be $\mathbf{R}^2$, d be the usual euclidean metric. The points $(0, 0)\in X$ and $(0, 1)\in X$ are denoted by C and P, respectively. Let $S=\{x\in X\,|\,d(x, C)=1\}$, $M=\{x\in X\,|\,d(x, C)\leqq 1\}$. It is easy to define a dynamical system with the following properties: for $x\in X\setminus M$, $\alpha(x)=\emptyset$, $\omega(x)=P$; M consists of equilibrium points, $A(M)=X$, d is a metric of Liapunov type. It follows directly that the first positive prolongational limit set [1] of P, $J^+(P)=S$. Assume that for $x\in X\setminus M$, the trajectory through x is monotonous. A simple argument similar to the one used in the proof of the Lemma shows that for $y\in S$, $d(y, P)=d(P, P)=0$, a contradiction.

From now on, let X be the two-dimensional euclidean unit sphere S_2 with the usual spherical metric d. Let $T_t\colon S_2\to S_2$ be a Morse—Smale [9] dynamical system. We investigate the question of existence of a new metric ϱ defined on S_2 such that any trajectory is monotonous with respect to the new metric ϱ. We require that d and ϱ should be topologically equivalent.

We conjecture that the answer is affirmative. Nevertheless, we can prove only a weaker result. We are not able to get rid of the hypothesis that the periodic orbits of our Morse—Smale dynamical system are — in some rather complicated technical sense — isolated.

To formulate our result, we need a definition first. Throughout this paper, a trajectory is said to be a separatrix if it connects a saddle point with a critical point of other type. The separatrices as edges and the critical points as vertices form a graph, the graph of separatrices.

THEOREM 1. *Let T_t be a Morse—Smale dynamical system on S_2. Assume that*

(1) *any saddle point is connected at least with three other critical points.*

Further, assume that for any two distinct periodic orbits γ_1 and γ_2, one of the following conditions is fulfilled:

(2)′ *the annulus bounded by γ_1 and γ_2 contains no critical points,*

(2)″ *any curve on S_2 cutting both γ_1 and γ_2, cuts the graph of separatrices at least three times.*

Then there exists a new metric ϱ on S_2, topologically equivalent to d, such that any trajectory is monotonous with respect to the new metric ϱ. In fact, any trajectory except critical points and periodic orbits will be strictly monotonous with respect to ϱ.

Before entering into details, we summarize the proof in advance. The periodic orbits and the graph of separatrices divide S_2 into a finite number of components. The closures of these components are called to be elementary cells. First, we define the desired metric on the elementary cells. We do this by defining model cells. A model cell τ is a metric space (S_τ, d_τ), represented as a surface — the model surface —, and a dynamical system Φ_τ, the model flow, defined on S_τ. Although trajectories do not determine dynamical systems uniquely, we define model flows by their trajectories: any flow having the prescribed trajectories will do. Moreover, the trajectories of the model flow Φ_τ will not be given explicitly but in case of $\tau=\mathscr{D}_k, \mathscr{E}$. In the remaining cases, for lack of space, we restrict ourselves to the formulation of some conditions to be fulfilled by Φ_τ. Nevertheless, in Theorem 2, under fairly general conditions, we give a method of defining flows whose trajectories are monotonous.

The model surfaces are inbedded into $\mathbf{R}^3$ or $\mathbf{R}^5$. In case of $\tau=\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}_k, \mathscr{J}$, d_τ is defined as the restriction of the euclidean metric: $d_\tau=\Delta_{\text{eukl}}$. In the other cases, d_τ is defined by geodesics, i.e. the distance between two points of S_τ is defined as the infimum of euclidean length of arcs connecting each other: $d_\tau=\Delta_{\text{geod}}$. Any trajectory of a model flow will be monotonous with respect to the metric of the corresponding model surface.

If an elementary cell is of type τ, then there exists a trajectory preserving homeomorphism mapping it onto the model cell τ. In constructing the desired metric on the elementary cells, the corresponding homeomorphisms play a crucial part. Continuing our metrization process, we "stick" the elementary cells together. In this final part of the proof the metric properties of the model cells as well as the Jordan curve theorem are extensively used.

We remark that our classification of the elementary cells is a modified version of the usual one [3], [8]. As we make use of the metrical and not only of the topological

properties of the model cells, it may occur that two different model cells are topologically equivalent. On the other hand, let us observe that our definition of separatrices deviates slightly from the usual one [3], [8].

The case when the presence of periodic orbits is not permitted, was settled in [6]. The investigation of the present, more general case requires some new ideas.

The boundary of a set $C \subset X$ is denoted by ∂C.

PROOF OF THEOREM 1. The proof consists of several steps. If there are neither periodic orbits nor saddle points, the theorem can be proved directly.

Step 1. *Classification of the elementary cells.* The periodic orbits and the graph of separatrices divide S_2 into a finite number of components. The closures of these components are called to be elementary cells. We denote them by $C_1, C_2, \ldots, C_N$, $N \in \mathbf{N}$, $N \geqq 2$. The graph of separatrices is denoted by Σ. Omitting the isolated points of Σ, we obtain a new graph Σ'. The components of Σ' are denoted by $\Sigma_1, \Sigma_2, \ldots, \Sigma_M$. For $m=1, 2, \ldots, M$ we define the set $E_m = \cup \{C_i | \partial C_i \subset \Sigma_m\}$. By conditions (1) and (2)'', ∂E_m is made up of a finite number of circles of the graph Σ. Any such circle Γ consists of an even number of separatrices and of the same number of critical points. Γ is a component of ∂E_m. On the other hand, Γ is a curve on S_2.

We say that an elementary cell is of type $\mathscr{A}$ if it contains an isolated point of the graph of separatrices. It is clear that the boundary of an elementary cell of type $\mathscr{A}$ is made up of a periodic orbit.

Consider an elementary cell T whose boundary is made up of two periodic orbits. T is called to be of type $\mathscr{B}$ or $\mathscr{C}$ accordingly as the two periodic orbits lying on ∂T have the same orientation or not.

Consider an elementary cell T whose boundary contains a critical point as well as a periodic orbit. By conditions (1), (2)' and (2)'', ∂T consists of the periodic orbit and of a component Γ of ∂E_m, for some $m \in \{1, 2, \ldots, M\}$. If Γ contains $2k$ critical points, we say that T is of type $\mathscr{D}_k$, $k=1, 2, 3, \ldots$.

Assume that ∂T contains no periodic orbits. Then $T \subset E_m$, for some $m \in \{1, 2, \ldots, M\}$. It is well-known [3], that T has three or four critical points, an attractive one and a repulsive one as well as one or two saddle points. If $T \cap \partial E_m = \emptyset$, we say that T is of type $\mathscr{E}$ or $\mathscr{F}$ accordingly as it has three or four critical points. If $T \cap \partial E_m \neq \emptyset$, then, by condition (2)'', $T \cap \partial E_m$ is contained in a component Γ of ∂E_m. Assume that Γ contains $2k$ critical points. We say that the elementary cell T containing four critical points is of type $\mathscr{G}_k$ ($k=2, 3, \ldots$), $\mathscr{H}_k$ ($k=1, 2, \ldots$) or $\mathscr{I}$ accordingly $T \cap \Gamma$ contains three, two or one critical points. On the other hand, the elementary cell T containing three critical points is said to be of type $\mathscr{J}$ or $\mathscr{K}$ accordingly as $T \cap \Gamma$ contains two or one critical points. Let us remark that T being of type $\mathscr{J}$ implies $T = E_m$.

Step 2. *Description of the model cells.* The upper hemisphere of radius one is denoted by $H^+(1)$:

$$H^+(1) = \left\{(\cos\vartheta \cos\varphi, \cos\vartheta \sin\varphi, \sin\vartheta) \in \mathbf{R}^3 \,|\, 0 \leqq \vartheta \leqq \frac{\pi}{2},\ 0 \leqq \varphi \leqq 2\pi\right\}.$$

If $S \subset \mathbf{R}^n$, $\lambda \in \mathbf{R}$, λS is defined as $\{\lambda s \in \mathbf{R}^n | s \in S\}$. $\lambda H^+(1)$ is denoted by $H^+(\lambda)$.

Let m_0 be the positive constant specified in [6]. We may assume that $m_0 > 2$.

Let $k^*=\max\{k\in\mathbf{N}|$ there exists an elementary cell of type $\mathscr{D}_k\}$. Let L be a constant satisfying

$$L>2R_{k^*} \quad \text{where} \quad R_{k^*}=\frac{\pi}{2}\Big/\sin\frac{\pi}{2k^*}. \tag{3}$$

The model cell $\mathscr{A}=(S_{\mathscr{A}}, d_{\mathscr{A}}, \Phi_{\mathscr{A}})$, $S_{\mathscr{A}}=H^+(L)$, $d_{\mathscr{A}}=\Delta_{\text{eukl}}$, $\Phi_{\mathscr{A}}$ has the following properties: $P=(0, 0, L)$ is a critical point, $\gamma=\partial S_{\mathscr{A}}$ is a periodic orbit. For $x\in S_{\mathscr{A}}\setminus(\gamma\cup P)$, $\alpha(x)=P$, $\omega(x)=\gamma$, any trajectory is monotonous.

The model cell $\mathscr{B}=(S_{\mathscr{B}}, d_{\mathscr{B}}, \Phi_{\mathscr{B}})$, $S_{\mathscr{B}}=\{(L\cos\varphi, L\sin\varphi, z)\in\mathbf{R}^3|0\leqq\varphi\leqq 2\pi, 0\leqq z\leqq 1\}$, $d_{\mathscr{B}}=\Delta_{\text{eukl}}$, $\Phi_{\mathscr{B}}$ has the following properties: $\gamma_1=\{(x, y, z)\in S_{\mathscr{B}}|z=0\}$, $\gamma_2=\{(x, y, z)\in S_{\mathscr{B}}|z=1\}$ are periodic orbits having the same orientation. For $x\in S_{\mathscr{B}}\setminus(\gamma_1\cup\gamma_2)$, $\alpha(x)=\gamma_1$, $\omega(x)=\gamma_2$, any trajectory is monotonous.

The model cell $\mathscr{C}=(S_{\mathscr{C}}, d_{\mathscr{C}}, \Phi_{\mathscr{C}})$, $S_{\mathscr{C}}=S_{\mathscr{B}}$, $d_{\mathscr{C}}=d_{\mathscr{B}}$, $\Phi_{\mathscr{C}}$ has the following properties: γ_1 and γ_2 are periodic orbits having opposite orientation. For $x\in S_{\mathscr{C}}\setminus(\gamma_1\cup\gamma_2)$, $\alpha(x)=\gamma_1$, $\omega(x)=\gamma_2$, any trajectory is monotonous.

The model cell $\mathscr{D}_k=(S_{\mathscr{D}_k}, d_{\mathscr{D}_k}, \Phi_{\mathscr{D}_k})$, $k=1, 2, \ldots$,

$$S_{\mathscr{D}_k}=\{((1-\tau^2)R_k(\cos t, \sin t, 0, 0, 0)+L\tau(0, 0, \cos t, \sin t, 0))\in\mathbf{R}^5|$$

$$|0\leqq t\leqq 2\pi, 0\leqq\tau\leqq 1\}\cup\{(0, 0, L\cos t, L\sin t, z)\in\mathbf{R}^5|0\leqq t\leqq 2\pi,\ 0\leqq z\leqq 1\}$$

where $R_k=\frac{\pi}{2}\Big/\sin\frac{\pi}{2k}$. $\partial S_{\mathscr{D}_k}=\gamma\cup\Gamma_{\mathscr{D}_k}$ where

$$\gamma=\{(0, 0, L\cos\omega, L\sin\omega, 1)\in\mathbf{R}^5|0\leqq\omega\leqq 2\pi\};$$

$$\Gamma_{\mathscr{D}_k}=\{R_k(\cos t, \sin t, 0, 0, 0)\in\mathbf{R}^5|0\leqq t\leqq 2\pi\}.$$

$d_{\mathscr{D}_k}=\Delta_{\text{eukl}}$. Let us observe that $S_{\mathscr{D}_k}$ is homeomorphic to $S_{\mathscr{B}}$. We define the trajectories of $\Phi_{\mathscr{D}_k}$: γ is a periodic orbit. $\Gamma_{\mathscr{D}_k}$ contains $2k$ critical points,

$$P_l=\left(R_k\cos\frac{l}{k}\pi,\ R_k\sin\frac{l}{k}\pi, 0, 0, 0\right),\quad l=0, 1, \ldots, 2k-1$$

and the same number of trajectories connecting P_l with P_{l+1}, $l=0, 1, \ldots, 2k-1$, $P_{2k}=P_0$.

$P_0, P_2, P_4, \ldots, P_{2k-2}$ is a source and for $x\in S_{\mathscr{D}_k}\setminus\Gamma_{\mathscr{D}_k}$, $\omega(x)=\gamma$. For $l=1, 3, 5, \ldots, 2k-1$, there is no other trajectory issuing from P_l but Γ_0^l, where, for $l=0, 1, \ldots, 2k-1$, Γ_0^l is defined as

$$\left\{\left((1-\tau^2)R_k\left(\cos\frac{l}{k}\pi, \sin\frac{l}{k}\pi, 0, 0, 0\right)+L\tau\left(0, 0, \cos\frac{l}{k}\pi, \sin\frac{l}{k}\pi, 0\right)\right)\middle|0<\tau\leqq 1\right\}\cup$$

$$\cup\left\{\left(0, 0, L\cos\left(\frac{l}{k}\pi+\varphi\right), L\sin\left(\frac{l}{k}\pi+\varphi\right), 1-e^{-\varphi}\right)\middle|0\leqq\varphi<\infty\right\}.$$

For $l=0, 2, 4, \ldots, 2k-2$, $s\in\left(-\frac{\pi}{k}, \frac{\pi}{k}\right)\setminus\{0\}$ we define

$$\Gamma_s^l = \left\{\left(\left(1-\left(\frac{\varphi}{s}\right)^2\right)R_k\left(\cos\left(\frac{l}{k}\pi+\varphi\right), \sin\left(\frac{l}{k}\pi+\varphi\right), 0, 0, 0\right)+\right.$$

$$\left.+L\frac{\varphi}{s}\left(0, 0, \cos\left(\frac{l}{k}\pi+\varphi\right), \sin\left(\frac{l}{k}\pi+\varphi\right), 0\right)\right)\Big| 0<|\varphi|\leqq|s|,$$

$$\operatorname{sgn} s=\operatorname{sgn}\varphi\}\cup\left\{\left(0, 0, L\cos\left(\frac{l}{k}\pi+\varphi+s\right),\right.\right.$$

$$\left.\left.L\sin\left(\frac{l}{k}\pi+\varphi+s\right), 1-e^{-\varphi}\right)\Big| 0\leqq\varphi<\infty\right\}.$$

Γ_s^l is a trajectory issuing from P_l, $l=0, 2, 4, \ldots, 2k-2$.

We prove that any trajectory is monotonous. By a continuity argument, we may restrict ourselves to the trajectories

$$\Gamma_s^l, \quad s\in\left(-\frac{\pi}{k}, \frac{\pi}{k}\right)\setminus\{0\}; \quad l=0, 2, 4, \ldots, 2k-2.$$

Using symmetry, it is enough to consider the trajectories Γ_s^0, $s\in(0, \pi/k)$. Redefining Γ_s^0 as

$$\Gamma_s^0(\varphi)=\left\{\left(\left(1-\left(\frac{\varphi}{s}\right)^2\right)R_k(\cos\varphi, \sin\varphi, 0, 0, 0)+L\frac{\varphi}{s}(0, 0, \cos\varphi, \sin\varphi, 0)\right)\right.$$

$$\left.\Big| 0<\varphi\leqq s\right\}\cup\{(0, 0, L\cos\varphi, L\sin\varphi, 1-e^{-\varphi+s})|s\leqq\varphi<\infty\}$$

we have to show that the functions

$$\varphi\to d_{\mathscr{D}_k}(P_0, \Gamma_s^0(\varphi)); \quad \varphi\to d_{\mathscr{D}_k}(\Gamma_s^0(\varphi), \gamma)$$

are monotonous in $0<\varphi<\infty$, for $s\in(0, \pi/k)$.

A direct computation shows that

$$d^2_{\mathscr{D}_k}(P_0, \Gamma_s^0(\varphi))=\begin{cases} R_k^2\left[\left(1-\left(\frac{\varphi}{s}\right)^2\right)^2-2\left(1-\left(\frac{\varphi}{s}\right)^2\right)\cos\varphi+1\right]+L^2\left(\frac{\varphi}{s}\right)^2, & 0<\varphi\leqq s \\ R_k^2+L^2+(1-e^{-\varphi+s})^2, & s\leqq\varphi<\infty. \end{cases}$$

For $0<\varphi<s$,

$$\frac{d}{d\varphi}d^2_{\mathscr{D}_k}(P_0, \Gamma_s^0(\varphi))=R_k^2\left[-4\frac{\varphi}{s^2}\left(1-\left(\frac{\varphi}{s}\right)^2\right)+\right.$$

$$\left.+4\frac{\varphi}{s^2}\cos\varphi+2\left(1-\left(\frac{\varphi}{s}\right)^2\right)\sin\varphi\right]+2L^2\frac{\varphi}{s^2}\geqq\frac{\varphi}{s^2}(-8R_k^2+2L^2)>0,$$

using inequality (3). For $s<\varphi<\infty$, $\frac{d}{d\varphi}d^2_{\mathscr{D}_k}(P_0, \Gamma_s^0(\varphi))=2e^{-\varphi+s}(1-e^{-\varphi+s})>0$.

On the other hand,

$$d^2_{\mathscr{G}_k}(\Gamma^0_s(\varphi), \gamma) =$$

$$= \left\{ \begin{array}{ll} \inf\limits_{\omega\in[0,2\pi]} \left\{ R_k^2 \left(1-\left(\frac{\varphi}{s}\right)^2\right)^2 + L^2\left(\left(\frac{\varphi}{s}\right)^2 - 2\frac{\varphi}{s}\cos(\varphi-\omega)+1\right)+1\right\}, & 0<\varphi\leqq s \\ \inf\limits_{\omega\in[0,2\pi]} \{L^2(2-2\cos(\varphi-\omega))+(1-e^{-\varphi+s})^2\}, & s\leqq\varphi<\infty \end{array}\right\} =$$

$$= \left\{ \begin{array}{ll} R_k^2\left(1-\left(\frac{\varphi}{s}\right)^2\right)^2 + L^2\left(1-\frac{\varphi}{s}\right)^2+1, & 0<\varphi\leqq s \\ (1-e^{-\varphi+s})^2, & s\leqq\varphi<\infty \end{array}\right.$$

which is monotonously decreasing.

The model cell $\mathscr{E}=(S_{\mathscr{E}}, d_{\mathscr{E}}, \Phi_{\mathscr{E}})$,

$$S_{\mathscr{E}} = \{(\cos t, \sin t, z)\in\mathbf{R}^3 | 0\leqq t\leqq 2\pi,\ 0\leqq z\leqq 1\}\cup$$

$$\cup\{(r\cos t, r\sin t, 1+m_0(1-r))\in\mathbf{R}^3 | 0\leqq t\leqq 2\pi,\ 0\leqq r\leqq 1\},$$

$d_{\mathscr{E}}=\Delta_{\text{geod}}$. $\Phi_{\mathscr{E}}$ has the following properties: $P=(1,0,0)$, $Q=(0,0,m_0+1)$, $R=(-1,0,0)$ are critical points. P is a source and Q is a sink. $\partial S_{\mathscr{E}}$ consists of P and of R and of two trajectories issuing from P and tending toward R. The trajectory

$$\Gamma_0 = \{(-1,0,z) | 0<z\leqq 1\}\cup\{(-r,0,1+m_0(1-r)) | 0<r\leqq 1\}$$

issues from R and tends toward Q. For $x\in S_{\mathscr{E}}\setminus(\Gamma_0\cup\partial S_{\mathscr{E}})$, $\alpha(x)=P$, $\omega(x)=Q$, any trajectory is monotonous. The explicit definition of $\Phi_{\mathscr{E}}$ can be found in Lemma 2 of [6].

The model cell $\mathscr{F}=(S_{\mathscr{F}}, d_{\mathscr{F}}, \Phi_{\mathscr{F}})$, $S_{\mathscr{F}}=H^+(2)$, $d_{\mathscr{F}}=\Delta_{\text{geod}}\cdot\Phi_{\mathscr{F}}$ has the following properties: $P=(2,0,0)$ is a source, $Q=(-2,0,0)$ is a sink. There are two other critical points $R_1=(0,2,0)$, $R_2=(0,-2,0)$ as well. $\partial S_{\mathscr{F}}$ consists of these four critical points and of four other trajectories issuing from P, P, R_1, R_2 and tending toward R_1, R_2, Q, Q respectively. For $x\in S_{\mathscr{F}}\setminus\partial S_{\mathscr{F}}$, $\alpha(x)=P$, $\omega(x)=Q$, any trajectory is monotonous.

The model cell $\mathscr{G}_k=(S_{\mathscr{G}_k}, d_{\mathscr{G}_k}, \Phi_{\mathscr{G}_k})$, $k=2,3,\ldots$, $S_{\mathscr{G}_k}=H^+(\tilde{\tilde{R}}_k)\cup\tilde{\tilde{Y}}_k$ where $\tilde{\tilde{R}}_k=\left(2L+2R_k\sin\frac{\pi}{k}\right)\Big/2\pi$,

$$\tilde{\tilde{Y}}_k = \left\{(\tilde{\tilde{R}}_k\cos\psi, \tilde{\tilde{R}}_k\sin\psi, z)\in\mathbf{R}^3 | |\psi|\leqq\tilde{\tilde{\theta}}_k,\ 0\geqq z\geqq R_k\cos\frac{\pi}{k}-(R_k^2-(\tilde{\tilde{R}}\psi)^2)^{1/2}\right\},$$

$\tilde{\tilde{\theta}}_k=R_k\sin\frac{\pi}{k}\Big/\tilde{\tilde{R}_k}$, R_k is already defined. $d_{\mathscr{G}_k}=\Delta_{\text{geod}}$. $\Phi_{\mathscr{G}_k}$ has the following properties: $P=\left(\tilde{\tilde{R}}_k, 0, R_k\cos\frac{\pi}{k}-R_k\right)$ is a source, $Q=(-\tilde{\tilde{R}}_k, 0, 0)$ is a sink. There are two other critical points,

$$R_1 = (\tilde{\tilde{R}}_k\cos\tilde{\tilde{\theta}}_k, \tilde{\tilde{R}}_k\sin\tilde{\tilde{\theta}}_k, 0),\ R_2 = (\tilde{\tilde{R}}_k\cos\tilde{\tilde{\theta}}_k, -\tilde{\tilde{R}}_k\sin\tilde{\tilde{\theta}}_k, 0)$$

as well. $\partial S_{\mathscr{G}_k}$ consists of these four critical points and of four other trajectories issuing from P, P, R_1, R_2 and tending toward R_1, R_2, Q, Q respectively. For $x \in S_{\mathscr{G}_k} \setminus \partial S_{\mathscr{G}_k}$, $\alpha(x)=P$, $\omega(x)=Q$, any trajectory is monotonous.

The model cell $\mathscr{H}_k=(S_{\mathscr{H}_k}, d_{\mathscr{H}_k}, \Phi_{\mathscr{H}_k})$, $k=1, 2, \ldots$, $S_{\mathscr{H}_k}=H^+(\tilde{R}) \cup \tilde{Y}_k$ where $\tilde{R}=(2\pi+2L)/2\pi$,

$$\tilde{Y}_k = \left\{(\tilde{R}\cos\psi, \tilde{R}\sin\psi, z) \in \mathbf{R}^3 \middle| |\psi| \leqq \tilde{\theta}, 0 \geqq z \geqq R_k \cos\frac{\pi}{2k} - \left(R_k^2 - (\tilde{R}\psi)^2\right)^{1/2}\right\},$$

$\tilde{\theta}=\pi/2\tilde{R}$, R_k is already defined. $d_{\mathscr{H}_k}=\Delta_{\text{geod}}$. $\Phi_{\mathscr{H}_k}$ has the following properties: $P=(\tilde{R}\cos\tilde{\theta}, \tilde{R}\sin\tilde{\theta}, 0)$ is a source, $Q=(-\tilde{R}\cos\tilde{\theta}, -\tilde{R}\sin\tilde{\theta}, 0)$ is a sink. There are two other critical points $R_1=(\tilde{R}\cos\tilde{\theta}, -\tilde{R}\sin\tilde{\theta}, 0)$, $R_2=(-\tilde{R}\cos\tilde{\theta}, \tilde{R}\sin\tilde{\theta}, 0)$ as well. $\partial S_{\mathscr{H}_k}$ consists of these four critical points and of four other trajectories issuing from P, P, R_1, R_2 and tending toward R_1, R_2, Q, Q respectively. For $x \in S_{\mathscr{H}_k} \setminus \partial S_{\mathscr{H}_k}$, $\alpha(x)=P$, $\omega(x)=Q$, any trajectory is monotonous.

The model cell $\mathscr{I}=(S_{\mathscr{I}}, d_{\mathscr{I}}, \Phi_{\mathscr{I}})$, $S_{\mathscr{I}}=H^+(\tilde{R})$, $\tilde{R}$ is already defined, $d_{\mathscr{I}}=\Delta_{\text{geod}}$, $\Phi_{\mathscr{I}}$ has the following properties: $P=(\tilde{R}, 0, 0)$ is a source, $Q=(-\tilde{R}, 0, 0)$ is a sink. There are two other critical points $R_1=(-\tilde{R}\cos 2\tilde{\theta}, \tilde{R}\sin 2\tilde{\theta}, 0)$, $R_2=(-\tilde{R}\cos 2\tilde{\theta}, -\tilde{R}\sin 2\tilde{\theta}, 0)$ as well, $\tilde{\theta}$ is already defined. $\partial S_{\mathscr{I}}$ consists of these four critical points and of four other trajectories issuing from P, P, R_1, R_2 and tending toward R_1, R_2, Q, Q respectively. For $x \in S_{\mathscr{I}} \setminus \partial S_{\mathscr{I}}$, $\alpha(x)=P$, $\omega(x)=Q$, any trajectory is monotonous.

The model cell $\mathscr{J}=(S_{\mathscr{J}}, d_{\mathscr{J}}, \Phi_{\mathscr{J}})$, $S_{\mathscr{J}}=\frac{\pi}{2} S_{\mathscr{E}}$, $d_{\mathscr{J}}=\Delta_{\text{eukl}}$, $\tilde{\Gamma} \subset S_{\mathscr{J}}$ is a trajectory of $\Phi_{\mathscr{J}}$ if and only if $\frac{2}{\pi}\tilde{\Gamma} \subset S_{\mathscr{E}}$ is a trajectory of $\Phi_{\mathscr{E}}$. For details, see [6]. A direct computation shows that $m_0>2$ implies any trajectory of $\Phi_{\mathscr{J}}$ is monotonous.

The model cell $\mathscr{K}=(S_{\mathscr{K}}, d_{\mathscr{K}}, \Phi_{\mathscr{K}})$, $S_{\mathscr{K}}=\frac{L}{\pi} S_{\mathscr{E}}$, $d_{\mathscr{K}}=\Delta_{\text{geod}}$, $\tilde{\Gamma} \subset S_{\mathscr{K}}$ is a trajectory of $\Phi_{\mathscr{K}}$ if and only if $\frac{\pi}{L}\tilde{\Gamma} \subset S_{\mathscr{E}}$ is a trajectory of $\Phi_{\mathscr{E}}$. It is clear that any trajectory is monotonous.

Step 3. *Metrization of the elementary cells.* We state some metric properties of the model cells.

(4) Let $\gamma \subset \partial S_\tau$ be a periodic orbit of the model cell τ. Then (γ, d_τ) is isometric to

$$\left(\{(L\cos t, L\sin t) \in \mathbf{R}^2 | 0 \leqq t \leqq 2\pi\}, \Delta_{\text{eukl}}\right).$$

(5) Let P, Q, $P \neq Q$ be critical points of the model cell τ. Assume that P and Q are connected by a trajectory lying on ∂S_τ. Then $d_\tau(P, Q)=\pi$ or $d_\tau(P, Q)=L$ and — except the cases detailed in (6a), (6b), (6c) — $(\{$the trajectory connecting P and $Q\}$, $d_\tau)$ is isometric to $(\{t \in \mathbf{R} | 0<t<\pi\}, \Delta_{\text{eukl}})$ or to $(\{t \in \mathbf{R} | 0<t<L\}, \Delta_{\text{eukl}})$, respectively.

(6a) $$\left(\left\{R_k(\cos t, \sin t, 0, 0, 0) \in \Gamma_{\mathscr{D}_k} | 0 \leqq t \leqq \frac{\pi}{k}\right\}, d_{\mathscr{D}_k}\right)$$

is isometric to $(\tilde{Y}_k \cap \partial S_{\mathscr{H}_k}, d_{\mathscr{H}_k})$, $k = 1, 2, \ldots$.

(6b) $$\left(\left\{R_k(\cos t, \sin t, 0, 0, 0) \in \Gamma_{\mathscr{D}_k} | 0 \leqq t \leqq \frac{2\pi}{k}\right\}, d_{\mathscr{D}_k}\right)$$

is isometric to $(\tilde{\tilde{Y}}_k \cap \partial S_{\mathscr{G}_k}, d_{\mathscr{G}_k})$ $k = 2, 3, \ldots$.

(6c) $(\Gamma_{\mathscr{D}_1}, d_{\mathscr{D}_1})$ is isometric to $(\partial S_{\mathscr{J}}, d_{\mathscr{J}})$.

The verification of properties (4), (5), (6) requires a step by step reconsideration of the definitions of model cells. Some elementary calculation, especially in proving (6a) and (6b) is needed as well.

If the elementary cell C_i is of type τ, then there exists [3] a trajectory preserving homeomorphisms h_i mapping C_i onto the model cell τ. We do not require the order of points of trajectories described by the increasing time to be preserved. For $i=1, 2, \ldots, N$, we define a metric d_i on the elementary cell C_i. For $x, y \in C_i$, let $d_i(x, y) = d_\tau(h_i(x), h_i(y))$.

We may define h_i on ∂C_i at first, then extend it to the entire C_i [8]. Therefore, in virtue of properties (4), (5), (6), $h_i: C_i \to S_\tau$ can be chosen to ensure

(7) $$d_i(x, y) = d_j(x, y), \quad x, y \in C_i \cap C_j, \quad i, j = 1, 2, \ldots, N.$$

Step 4. *Global metrization.* For $m=1, 2, \ldots, M$, let $D_m = E_m \cup \{C_i | C_i \cap E_m \neq \emptyset\}$. Since $D_m \setminus E_m$ consists of a finite number of pairwise disjoint elementary cells of type $\mathscr{D}_k$, ∂D_m is made up of a finite number of periodic orbits. We define a metric ϱ_m on D_m as follows: for $x, y \in D_m$, let

$$\varrho_m(x, y) = \inf\left\{\sum_{k=0}^{n} d_{i_k}(a_k, a_{k+1}) | n \in \mathbf{N},\right.$$

$$\left. a_0 = x, a_{n+1} = y, C_{i_k} \subset D_m;\ a_k, a_{k+1} \in C_{i_k}\ (k = 0, 1, \ldots, n)\right\}.$$

It is clear that for $x, y, z \in D_m$, $\varrho_m(x, y) = \varrho_m(y, x)$ and $\varrho_m(x, z) \leqq \varrho_m(x, y) + \varrho_m(y, z)$. We prove that ϱ_m is an extension of d_i, i.e.

(8) $$d_i(x, y) = \varrho_m(x, y) \quad \text{if} \quad x, y \in C_i \subset D_m.$$

If $E_m = C_i$ for some $i \in \{1, 2, \ldots, N\}$, then C_i is of type $\mathscr{J}$ and (8) follows directly from (7).

Thus, we may assume that E_m contains at least two elementary cells. In order to prove (8), we define a graph G_m. As usual, the set of vertices and the set of edges of G_m are denoted by $V(G_m)$ and $E(G_m)$, respectively.

$G_m = \{x \in \partial C_i | C_i \subset E_m\}$, $V(G_m) = \{P \in G_m | P$ is a critical point$\} \setminus \{P \in \partial E_m | P$ is the source or the sink of an elementary cell of type $\mathscr{G}_k$, $k=2, 3, \ldots\}$. $E(G_m) =$ {the components of $G_m \setminus V(G_m)$}.

The following results are obtained by a brief examination of model cells:

(9) Let $C_i \subset E_m$, $x, y \in \partial C_i$. Assume that for an $e \in E(G_m)$, $x \in e$, $y \notin e$. Then there exists a $P \in V(G_m) \cap \partial C_i$ such that $d_i(x, y) = d_i(x, P) + d_i(P, y)$.

In fact, P is one of the endpoints of the edge e.

(10) Let $C_i \subset E_m$, $P, Q \in V(G_m) \cap C_i$, $P \neq Q$. Then $d_i(P, Q) \cong \pi$. Moreover, $P \in \partial E_m$, $Q \notin \partial E_m$ imply $d_i(P, Q) \cong L$.

Now we are in a position to prove (8). Let us suppose, on the contrary, that there exists an elementary cell $\tilde{C} \subset D_m$ and there exist a finite sequence of elementary cells $\{C_k^*\}_0^n \subset D_m$ and a finite sequence of points $\{a_k^*\}_0^{n+1} \subset D_m$ such that

$$(*) \qquad \tilde{d}(a_0^*, a_{n+1}^*) > \sum_{k=0}^{n} d_k^*(a_k^*, a_{k+1}^*)$$

where

$$a_0^*, a_{n+1}^* \in \tilde{C}, \quad a_k^*, a_{k+1}^* \in C_k^*, \quad k = 0, 1, \ldots, n.$$

Without loss of generality we may assume that $C_k^* \neq C_{k+1}^*$, $k = 0, 1, \ldots, n$. If $a_0^* \notin G_m$, then $\tilde{C} = C_0^*$ and $(*)$ changes to

$$\tilde{d}(a_0^*, a_{n+1}^*) > \tilde{d}(a_0^*, a_1^*) + \sum_{k=1}^{n} d_k^*(a_k^*, a_{k+1}^*).$$

Since $\tilde{d}(a_0^*, a_1^*) + \tilde{d}(a_1^*, a_{n+1}^*) \cong \tilde{d}(a_0^*, a_{n+1}^*)$ it follows that $\tilde{d}(a_1^*, a_{n+1}^*) > \sum_{k=1}^{n} d_k^*(a_k^*, a_{k+1}^*)$.

Thus, our indirect hypothesis can be reformulated as

$$(**) \qquad \tilde{d}(a_0^*, a_{n+1}^*) > \sum_{k=0}^{n} d_k^*(a_k^*, a_{k+1}^*)$$

where $a_0^*, a_{n+1}^* \in \tilde{C}$; $a_k^*, a_{k+1}^* \in G_m \cap C_k^*$, $k = 0, 1, \ldots, n$. In virtue of property (7), it may be assumed without loss of generality that for $q = 1, \ldots, n$, $a_q^* \in e \in E(G_m)$ implies $a_{q-1}^*, a_{q+1}^* \notin e$. Let $a_q^* \in e \in E(G_m)$, $q \in \{1, \ldots, n\}$. We distinguish two cases according as both C_{q-1}^* and C_q^* are of type $\mathscr{D}_k$ or not.

In the first case, using condition (2)'', $C_{q-1}^* = C_q^*$. By (7) and by the triangle inequality we have

$$d_{q-1}^*(a_{q-1}^*, a_q^*) + d_q^*(a_q^*, a_{q+1}^*) = d_q^*(a_{q-1}^*, a_q^*) + d_q^*(a_q^*, a_{q+1}^*) \cong d_q^*(a_{q-1}^*, a_{q+1}^*).$$

In the second case assuming e.g. that C_q^* is not of type $\mathscr{D}_k$, using (7), (9) and the triangle inequality we obtain

$$d_{q-1}^*(a_{q-1}^*, a_q^*) + d_q^*(a_q^*, a_{q+1}^*) = d_{q-1}^*(a_{q-1}^*, a_q^*) +$$

$$+ d_q^*(a_q^*, P) + d_q^*(P, a_{q+1}^*) = d_{q-1}^*(a_{q-1}^*, a_q^*) +$$

$$+ d_{q-1}^*(a_q^*, P) + d_q^*(P, a_{q+1}^*) \cong d_{q-1}^*(a_{q-1}^*, P) + d_q^*(P, a_{q+1}^*)$$

for some $P \in V(G_m) \cap C_{q-1}^* \cap C_q^*$.

Therefore, our indirect hypothesis can be reformulated as

$$(***) \qquad \tilde{d}(a_0^*, a_{n+1}^*) > \sum_{k=0}^{n} d_k^*(a_k^*, a_{k+1}^*)$$

where $a_0^*, a_{n+1}^* \in \tilde{C}$; $a_k^*, a_{k+1}^* \in G_m \cap C_k^*$, $k = 0, 1, \ldots, n$; $a_k^* \in V(G_m)$, $k = 1, 2, \ldots, n$. We have to show that $(***)$ leads to a contradiction.

If $\tilde{C}$ is of type $\mathscr{D}_k$, then $\tilde{d}(a_0^*, a_{n+1}^*) \leqq 2R_k$. Therefore, (3) and (10) imply $a_k^* \in \tilde{C} \cap V(G_m)$, $k=1, \ldots, n$. Using (7), $(***)$ changes to $\tilde{d}(a_0^*, a_{n+1}^*) > \sum_{k=0}^{n} \tilde{d}(a_k^*, a_{k+1}^*)$, a contradiction.

If $\tilde{C} \subset E_m$, we have $\tilde{d}(a_0^*, a_{n+1}^*) \leqq L + \pi$. Moreover, $\tilde{C}$ being of type $\mathscr{E}$ implies $\tilde{d}(a_0^*, a_{n+1}^*) \leqq \pi$, $\tilde{C}$ being of type $\mathscr{F}$ implies $\tilde{d}(a_0^*, a_{n+1}^*) \leqq 2\pi$. Arguing similarly as above, (7), (9) and (10) lead to a contradiction. This completes the proof of (8).

It is now possible for us to complete the proof of the theorem. For the sake of brevity, the elementary cells of type $\mathscr{A}, \mathscr{B}, \mathscr{C}$ shall be denoted by $D_{M+1}, \ldots, D_K$. It is clear that $\bigcup_{m=1}^{K} D_m = S_2$. For $m=M+1, \ldots, K$, we define $\varrho_m = d_i$ if $D_m = C_i$. Using the remark following the definition of D_m for $m=1, 2, \ldots, M$, (7) and (8) imply

$$\varrho_m(x, y) = \varrho_n(x, y), \quad x, y \in D_m \cap D_n, \quad n, m = 1, 2, \ldots, K. \tag{11}$$

We define an abstract graph H. The vertices of the graph H are the sets D_m, $m=1, 2, \ldots, K$. D_n and D_m $(n \neq m)$ are connected by an edge in H if and only if $D_n \cap D_m \neq \emptyset$. Applying the Jordan curve theorem for our two-dimensional unit sphere S_2, we arrive at the conclusion that H is a tree in the sense of graph theory. Thus, for $n \neq m$ there is a unique path of the graph H connecting D_n and D_m i.e. there is a unique sequence $\{D_k^{n,m}\}_{k=0}^{N(n,m)+1}$ such that

$$D_0^{n,m} = D_n, \quad D_{N(n,m)+1}^{n,m} = D_m, \quad D_k^{n,m} \cap D_{k+1}^{n,m} \neq \emptyset, \quad k = 0, 1, \ldots, N(n, m),$$

$$D_k^{n,m} \cap D_l^{n,m} = \emptyset, \quad |k-l| > 1, \quad k, l = 0, 1, \ldots, N(n, m)+1.$$

For $x, y \in D_m$, $m=1, 2, \ldots, K$, in view of (11), let $\varrho(x, y) = \varrho_m(x, y)$.
For $x \in D_n$, $y \in D_m$, $n \neq m$, $n, m = 1, 2, \ldots, K$ let

$$\varrho(x, y) = \inf \Big\{ \sum_{k=0}^{N(n,m)} \varrho_k^{n,m}(a_k, a_{k+1}) \,|\, a_0 = x, a_{N(n,m)+1} = y,$$

$$a_k, a_{k+1} \in D_k^{n,m}, \quad k = 0, 1, \ldots, N(n, m) \Big\}.$$

We state that ϱ is a metric on S_2. This follows via induction from Theorem 4 of [2].

It is clear that the new metric ϱ is topologically equivalent to the usual spherical metric d. Since for $x, y \in C_i$, $i=1, 2, \ldots, N$, $\varrho(x, y) = d_i(x, y)$, it is obvious that any trajectory is monotonous with respect to the new metric ϱ. Thus the theorem is proved.

From now on, let (M, d) be a metric space, M being a compact differentiable manifold, d being a metric which is topologically equivalent to a Riemannian one. The boundary of M is denoted by ∂M. Using partition of the unity, with the aid of local conditions on (M, d) it is possible to construct a dynamical system whose trajectories are monotonous. A result of this kind is the following theorem.

THEOREM 2. *Assume that the previous conditions on (M, d) are fulfilled. Let $p, q \in M$, $p \neq q$. Assume that for $x \in M \setminus \{p, q\}$ there exists a cone $C_x \subset T_x M$ satisfying*

(i) $v, w \in C_x$, $\lambda, \mu > 0$ *imply* $\lambda v + \mu w \in C_x$;

(ii) $0 \notin C_x$;

(iii) *for* $x \in \partial M, C_x \subset T_x(\partial M)$;

(iv) *for any differentiable curve* $\gamma\colon (-\varepsilon, \varepsilon) \to M$, $\gamma(0)=x$, $\dot{\gamma}(0) \in C_x$ *there exists a* $\delta=\delta(\gamma)>0$ *such that the functions* $t \to d(p, \gamma(t))$ *and* $t \to d(\gamma(t), q)$ *are strictly increasing and strictly decreasing for* $t \in (-\delta, \delta)$, *respectively.*

Further, assume that there exist an open covering $\{U_i\}_{i=1}^n$ *of* M *and a family* $\{X_i\}_{i=1}^n$ *of local vector fields* $X_i\colon U_i \to TM$ *with the following properties:*

(v) *for* $x \in U_i \setminus \{p, q\}$, $X_i(x) \in C_x$, $i=1, \ldots, n$;

(vi) *for* $p \in U_i$: $X_i(p)=0$; *for* $q \in U_i$: $X_i(q)=0$.

Then there exists a dynamical system on M *such that for* $x \in M \setminus \{p, q\}$, $\alpha(x)=p$, $\omega(x)=q$ *and the trajectory through* x *is strictly monotonous.*

PROOF. Let $\{f_i\}_{i=1}^n$ be a differentiable partition of the unity [7] subordinated to the open covering $\{U_i\}_{i=1}^n$. For $x \in M$ let $X(x)=\sum_{i=1}^{n} f_i(x) X_i(x)$, where X is a vector field on M. By (i), (iii), (v), X defines a dynamical system $T_t\colon M \to M$, $t \in \mathbf{R}$. Clearly there are no critical points but p and q. Pick a point $x \in M \setminus \{p, q\}$ and consider the trajectory through x. It follows from (i), (iv), (v) that

(12) the function $f_x\colon \mathbf{R} \to \mathbf{R}^+$, $t \to d(p, T_t x)$ is strictly increasing,

(13) the function $g_x\colon \mathbf{R} \to \mathbf{R}^+$, $t \to d(T_t x, q)$ is strictly decreasing.

We have to prove that $\alpha(x)=p$, $\omega(x)=q$.

The compactness of M implies $\omega(x) \neq \emptyset$. By a continuity argument similar to the one used in the proof of the Lemma, (13) implies that for $z_1, z_2 \in \omega(x)$, $d(z_1, q)=d(z_2, q)$. It is well-known [1] that $\omega(x)$ consists of entire trajectories.

On the other hand, for $z \in \omega(x)$, the function $g_z\colon \mathbf{R} \to \mathbf{R}^+$, $t \to d(T_t z, q)$ is strictly decreasing. Therefore, $\omega(x)$ consists of critical points. (12) implies $d(p, \omega(x))>0$, therefore $\omega(x)=q$, which was to be proved.

Similarly, $\alpha(x)=p$.

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ON SUMS OF SEQUENCES OF INTEGERS. II

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1. Throughout this paper, we use the following notation: $c_1, c_2, \ldots, N_0, N_1, \ldots$ denote positive absolute constants. $\theta_1, \theta_2, \ldots$ are real numbers such that $|\theta_i| \leqq 1$ for all i. We write $e^x = \exp(x)$ and $e^{2\pi i\alpha} = e(\alpha)$. The distance from α to the nearest integer is denoted by $\|\alpha\|$ so that $\|\alpha\| = \min(\alpha - [\alpha], [\alpha] + 1 - \alpha)$. We put $\min\left(A, \frac{1}{0}\right) = A$. We denote the least prime factor of n by $p(n)$, while the greatest prime factor of n is denoted by $P(n)$.

2. In this series, we study the arithmetic nature of the numbers of the form $a+b$ where a, b are taken from "dense" sequences of integers. In Part I (see [1]) we showed the existence of a, b such that $P(a+b)$ is "small" (in terms of $a+b$). Here our goal is first to generalize the results of Part I. In fact, Sections 3—6 will be devoted to the proof of the following theorem:

THEOREM 1. *Let N be an integer satisfying $N > N_0$, y a real number satisfying*

$$1 \leqq y \leqq \sqrt{N}, \tag{1}$$

and let $\mathscr{A}, \mathscr{B}, \mathscr{K}$ be sequences of integers such that

(i)

$$N/2 < a \leqq N \quad \text{for} \quad a \in \mathscr{A}, \tag{2}$$

$$N/2 < b \leqq N \quad \text{for} \quad b \in \mathscr{B} \tag{3}$$

and

$$2N/y < k \leqq 4N/y \quad \text{for} \quad k \in \mathscr{K}. \tag{4}$$

(ii) *Writing*

$$A = \sum_{a \in \mathscr{A}} 1, \quad B = \sum_{b \in \mathscr{B}} 1 \quad \text{and} \quad K = \sum_{k \in \mathscr{K}} 1,$$

we have

$$ABKy^2 > 10^6 N^3 \log N. \tag{5}$$

(iii) *If $k \in \mathscr{K}$ then*

$$p(k) > \left(\frac{N \log N}{K}\right)^{1/2}. \tag{6}$$

Then there exist integers a, b, k such that $a \in \mathscr{A}$, $b \in \mathscr{B}$, $k \in \mathscr{K}$ and

$$k \mid (a+b). \tag{7}$$

In Section 7, we derive from this theorem that if $\mathscr{A}, \mathscr{B}$ are "dense" sequences of integers then there exist a, b such that $a \in \mathscr{A}$, $b \in \mathscr{B}$ and $P(a+b)$ is "large" (in terms

of $a+b$). Finally, we show in Section 8 that the lower bound for $P(a+b)$ obtained in this way can be improved by using a direct method, based on the large sieve.

3. Put $Q=\left(\frac{kN}{\log N}\right)^{1/2}$ so that in view of (1) and (4), for large N we have

$$Q \leqq \left(\frac{\frac{4N}{y}N}{\log N}\right)^{1/2} \leqq \left(\frac{4N^2}{\log N}\right)^{1/2} < N. \tag{8}$$

Furthermore, put

$$U = [8N/y]+1, \quad d_n = \sum_{\substack{mk=n \\ m \leqq y \\ k \in \mathscr{K}}} 1,$$

$$S_x(\alpha) = \sum_{n \leqq x} d_n e(n\alpha) \quad (\text{for} \quad 0 \leqq x \leqq 2N), \quad S(\alpha) = S_{2N}(\alpha) = \sum_{n=1}^{2N} d_n e(n\alpha),$$

$$S = S(0) = \sum_{n=1}^{2N} d_n, \quad U(\alpha) = \sum_{n=0}^{U-1} e(n\alpha),$$

$$S(\alpha)U(\alpha) = \sum_{n=1}^{2N+U-1} v_n e(n\alpha) \quad (\text{so that} \quad v_n = \sum_{n-U<j \leqq n} d_j),$$

$$F(\alpha) = \sum_{a \in \mathscr{A}} e(a\alpha), \quad G(\alpha) = \sum_{b \in \mathscr{B}} e(b\alpha)$$

and (in view of (2) and (3))

$$H(\alpha) = F(\alpha)G(\alpha) = \sum_{\substack{a \in \mathscr{A} \\ b \in \mathscr{B}}} e\big((a+b)\alpha\big) = \sum_{n=N+1}^{2N} h_n e(n\alpha)$$

$$(\text{so that} \quad h_n = \sum_{\substack{a+b=n \\ a \in \mathscr{A} \\ b \in \mathscr{B}}} 1).$$

We start out from the integral

$$J = \int_0^1 F(\alpha)G(\alpha)S(-\alpha)\,d\alpha =$$

$$= \int_0^1 H(\alpha)S(-\alpha)\,d\alpha = \int_0^1 \sum_{n=N+1}^{2N} \sum_{m=1}^{2N} h_n d_m e\big((n-m)\alpha\big)\,d\alpha = \sum_{n=N+1}^{2N} h_n d_n.$$

Obviously, $d_n>0$ implies the existence of an integer k such that $k|n$, while $h_n>0$ implies that n can be written in the form $a+b=n$ ($a\in\mathscr{A}$, $b\in\mathscr{B}$). Thus in order to prove the solvability of (7), it is sufficient to show that

$$J = \sum_{n=N+1}^{2N} h_n d_n > 0. \tag{9}$$

In order to prove this (by using the Hardy—Littlewood method), we need some lemmas.

4. In this section, we assert some preliminary lemmas.

LEMMA 1. *If L is a positive integer, α a real number then we have*

$$\left|\sum_{n=0}^{L-1} e(n\alpha) - L\right| < 4L^2|\alpha|.$$

This is identical with Lemma 2 in [1].

LEMMA 2. *For arbitrary real numbers α, M we have*

$$\left|\sum_{1\leqq m\leqq M} e(m\alpha)\right| \leqq \min\left(M, \frac{1}{2\|\alpha\|}\right).$$

See e.g. [3], p. 9.

LEMMA 3. *If α, V are real numbers and a, q, f are integers such that $q>0, (a,q)=1$ and $\left|\alpha-\frac{a}{q}\right|\leqq\frac{1}{q^2}$ then we have*

$$\sum_{m=f+1}^{f+q} \min\left(V, \frac{1}{2\|\alpha m\|}\right) \leqq 6V + q\log q.$$

See e.g. [3], p. 23.

LEMMA 4. *If α, M, V are real numbers and a, q are integers such that $M\geqq 1$, $q>0$, $(a,q)=1$ and $\left|\alpha-\frac{a}{q}\right|\leqq\frac{1}{q^2}$ then we have*

$$\sum_{m\leqq M} \min\left(V, \frac{1}{2\|\alpha m\|}\right) \leqq \left(\frac{M}{q}+1\right)(6V+q\log q) = 6\frac{MV}{q} + 6V + (M+q)\log q.$$

This is identical with Lemma 5 in [1].

5. In this section, we estimate S, $S(\alpha)$ and v_n.

LEMMA 5. *We have $S\leqq Ky$.*

PROOF.

$$S = \sum_{n=1}^{2N} d_n = \sum_{n=1}^{2N} \sum_{\substack{mk=n \\ m\leqq y \\ k\in\mathscr{K}}} 1 = \sum_{m\leqq y} \sum_{\substack{k\leqq N/m \\ k\in\mathscr{K}}} 1 \leqq \sum_{m\leqq y} \sum_{k\in\mathscr{K}} 1 = K[y] \leqq Ky.$$

LEMMA 6. *If $1\leqq u\leqq 2N$ and a, q are integers such that $2\leqq q\leqq N/Q$ and $(a,q)=1$ then we have $|S_u(a/q)|\leqq Kq$.*

PROOF. We have

$$S_u(a/q) = \sum_{n\leqq u} d_n e(na/q) = \sum_{b=1}^{q} \Big(\sum_{\substack{n\leqq u \\ na\equiv b(\mathrm{mod}\, q)}} d_n\Big) e(b/q). \tag{10}$$

Here the inner sum can be rewritten in the following form:

$$(11)\qquad \sum_{\substack{n\leqq u\\ na\equiv b(\mathrm{mod}\,q)}} d_n = \sum_{\substack{n\leqq u\\ na\equiv b(\mathrm{mod}\,q)}} \sum_{\substack{mk=n\\ m\leqq y\\ k\in\mathscr{K}}} 1 =$$

$$= \sum_{\substack{mk\leqq u\\ mka\equiv b(\mathrm{mod}\,q)\\ m\leqq y\\ k\in\mathscr{K}}} 1 = \sum_{\substack{k\leqq u\\ k\in\mathscr{K}}} \sum_{\substack{m\leqq u/k\\ mka\equiv b(\mathrm{mod}\,q)\\ m\leqq y}} 1 = \sum_{\substack{k\leqq u/y\\ k\in\mathscr{K}}} \sum_{\substack{m\leqq y\\ mka\equiv b(\mathrm{mod}\,q)}} 1 + \sum_{\substack{u/y<k\leqq u\\ k\in\mathscr{K}}} \sum_{\substack{m\leqq u/k\\ mka\equiv b(\mathrm{mod}\,q)}} 1.$$

By (6), we have

$$p(k) > \left(\frac{N\log N}{K}\right)^{1/2} = \frac{N}{Q} \geqq q.$$

This inequality and $(a, q)=1$ imply that $(ka, q)=1$ hence

$$\sum_{\substack{m\leqq y\\ mka\equiv b(\mathrm{mod}\,q)}} 1 = \frac{y}{q}+\theta_1 \quad \text{and} \quad \sum_{\substack{m\leqq u/k\\ mka\equiv b(\mathrm{mod}\,q)}} 1 = \frac{u}{kq}+\theta_2$$

(where $\theta_1=\theta_1(a, b, k)$, $\theta_2=\theta_2(a, b, k)$). Thus we obtain from (10) and (11) that

$$S_u(a/q) = \sum_{b=1}^{q}\Big(\sum_{\substack{n\leqq u\\ na\equiv b(\mathrm{mod}\,q)}} d_n\Big)e(b/q) =$$

$$= \sum_{b=1}^{q}\left(\sum_{\substack{k\leqq u/y\\ k\in\mathscr{K}}}\left(\frac{y}{q}+\theta_1\right)+\sum_{\substack{u/y<k\leqq u\\ k\in\mathscr{K}}}\left(\frac{u}{kq}+\theta_2\right)\right)e(b/q) =$$

$$= \sum_{b=1}^{q}\left\{\left(\sum_{\substack{k\leqq u/y\\ k\in\mathscr{K}}}\frac{y}{q}+\sum_{\substack{u/y<k\leqq u\\ k\in\mathscr{K}}}\frac{u}{kq}\right)+\theta_3\sum_{\substack{k\leqq u\\ k\in\mathscr{K}}}1\right\}e(b/q) =$$

$$= \left(\sum_{\substack{k\leqq u/y\\ k\in\mathscr{K}}}\frac{y}{q}+\sum_{\substack{u/y<k\leqq u\\ k\in\mathscr{K}}}\frac{u}{kq}\right)\sum_{b=1}^{q}e(b/q)+\sum_{b=1}^{q}\Big(\theta_4\sum_{k\in\mathscr{K}}1\Big) = \theta_5 qK \leqq qK$$

(where $\theta_3, \theta_4, \theta_5, \theta_6$ depend on a, b, k) since by $q\geqq 2$, $\sum_{b=1}^{q} e(b/q)=0$.

LEMMA 7. *If α is a real number and a, q are integers such that $2\leqq q\leqq N/Q$, $(a, q)=1$ and $\left|\alpha-\frac{a}{q}\right|<\frac{1}{qQ}$ then we have*

$$|S(\alpha)| < 14\,(KN\log N)^{1/2}.$$

PROOF. We write $\beta=\alpha-\frac{a}{q}$ so that $|\beta|=\left|\alpha-\frac{a}{q}\right|<\frac{1}{qQ}$. Then by using Lemma 6 and the well-known inequality $|1-e(\beta)|\leqq 2\pi|\beta|$, we obtain by partial summation

that

$$|S(\alpha)| = \left|\sum_{n=1}^{2N}\left(S_n(a/q)-S_{n-1}(a/q)\right)e(n\beta)\right| =$$

$$= \left|\sum_{n=1}^{2N} S_n(a/q)\left(e(n\beta)-e((n+1)\beta)\right)+S_n(a/q)e\left((2N+1)\beta\right)\right| \leqq$$

$$\leqq \sum_{n=1}^{2N} Kq|1-e(\beta)|+Kq \leqq Kq(2N2\pi|\beta|+1) <$$

$$< Kq\left(2N2\pi\frac{1}{qQ}+1\right) = 4\pi\frac{KN}{Q}+Kq \leqq 4\pi\frac{KN}{Q}+\frac{KN}{Q} < 14KN/Q =$$

$$= 14KN\left(\frac{\log N}{KN}\right)^{1/2} = 14(KN\log N)^{1/2}.$$

LEMMA 8. *If α is a real number and a, q are integers such that $N/Q<q\leqq Q$, $(a,q)=1$ and $\left|\alpha-\frac{a}{q}\right|\leqq\frac{1}{q^2}$ then we have*

$$|S(\alpha)| < 3(KN\log N)^{1/2}.$$

PROOF. By using Lemmas 2 and 4, and with respect to (1), (4), (5) and (8), we obtain that

$$|S(\alpha)| = \left|\sum_{n=1}^{2N} d_n e(n\alpha)\right| = \left|\sum_{\substack{mk\leqq 2N\\ m\leqq y\\ k\in\mathcal{K}}} e(mk\alpha)\right| =$$

$$= \left|\sum_{k\in\mathcal{K}}\left(\sum_{m\leqq\min(2N/k,\,y)} e(mk\alpha)\right)\right| \leqq \sum_{k\in\mathcal{K}}\left|\sum_{m\leqq\min(2N/k,\,y)} e(mk\alpha)\right| \leqq$$

$$\leqq \sum_{k\leqq 4N/y}\min\left(\min(2N/k,\,y),\,\frac{1}{2\|k\alpha\|}\right) \leqq \sum_{k\leqq 4N/y}\min\left(y,\,\frac{1}{2\|k\alpha\|}\right) \leqq$$

$$\leqq 6\frac{\frac{4N}{y}y}{q}+6y+\left(\frac{4N}{y}+q\right)\log q <$$

$$< 24\frac{N}{N/Q}+6\sqrt{N}+\left(\frac{4N}{10^3N^{3/2}(\log N)^{1/2}/(ABK)^{1/2}}+Q\right)\log Q <$$

$$< 24Q+6\frac{Q(\log N)^{1/2}}{K^{1/2}}+\left(\frac{1}{100}\left(\frac{ABK}{N\log N}\right)^{1/2}+Q\right)\log N <$$

$$< \left(\frac{1}{100}\left(\frac{ABK}{N\log N}\right)^{1/2}+2\left(\frac{KN}{\log N}\right)^{1/2}\right)\log N = 2(KN\log N)^{1/2}\left(\frac{1}{200}\frac{(AB)^{1/2}}{N}+1\right) <$$

$$< 2(KN\log N)^{1/2}\left(\frac{1}{200}\frac{(N\cdot N)^{1/2}}{N}+1\right) < 3(KN\log N)^{1/2}.$$

LEMMA 9. *If*

$$\frac{1}{Q} < \alpha < 1-\frac{1}{Q} \tag{12}$$

then we have

$$|S(\alpha)| < 14\,(KN\log N)^{1/2}. \tag{13}$$

PROOF. By Dirichlet's theorem, there exist integers a, q such that $1 \leqq q \leqq Q$, $(a, q)=1$ and $\left|\alpha-\frac{a}{q}\right| < \frac{1}{qQ}\left(\leqq \frac{1}{q^2}\right)$.

(12) implies that $q>1$. If $2 \leqq q \leqq N/Q$ then (13) is a consequence of Lemma 7 while if $N/Q<q\leqq Q$ then (13) holds by Lemma 8.

LEMMA 10. *If n is a positive integer satisfying $U \leqq n \leqq 2N$ then we have $v_n \geqq K$.*

PROOF. For $U \leqq n \leqq 2N$ we have

$$v_n = \sum_{j=n-U+1}^{n} d_j = \sum_{j=n-U+1}^{n} \sum_{\substack{mk=j \\ m\leqq y \\ k\in\mathscr{K}}} 1 = \sum_{\substack{n-U<mk\leqq n \\ m\leqq y \\ k\in\mathscr{K}}} 1 = \sum_{k\in\mathscr{K}} \sum_{\substack{\frac{n-U}{k}<m\leqq\frac{n}{k} \\ m\leqq y}} 1 =$$

$$= \sum_{k\in\mathscr{K}} \sum_{\frac{n-U}{k}<m\leqq\frac{n}{k}} 1 \geqq \sum_{k\in\mathscr{K}} \left(\frac{U}{k}-1\right) \geqq$$

$$\geqq \sum_{k\in\mathscr{K}} \left(\frac{U}{k}-\frac{U}{8N/y}\right) \geqq \sum_{k\in\mathscr{K}} \left(\frac{U}{k}-\frac{U}{2k}\right) = \frac{U}{2}\sum_{k\in\mathscr{K}} \frac{1}{k} \geqq$$

$$\geqq \frac{U}{2}\sum_{k\in\mathscr{K}} \frac{1}{4N/y} = \frac{Uy}{8N}\sum_{k\in\mathscr{K}} 1 = \frac{Uy}{8N}K > K$$

since for $k\in\mathscr{K}$ and $n\leqq 2N$, $\frac{n}{k} < \frac{2N}{2N/y} = y$.

6. In this section, we complete the proof of the theorem. By using (1), Lemmas 1, 5, 9, Cauchy's inequality and Parseval's formula, we obtain for large N that

$$\left|J-\frac{1}{U}\int_0^1 F(\alpha)G(\alpha)U(-\alpha)S(-\alpha)\,d\alpha\right| = \left|\int_{-1/Q}^{+1/Q} F(\alpha)G(\alpha)S(-\alpha)\left(1-\frac{U(-\alpha)}{U}\right)d\alpha + \right.$$

$$\left. + \int_{+1/Q}^{1-1/Q} F(\alpha)G(\alpha)S(-\alpha)\left(1-\frac{U(-\alpha)}{U}\right)d\alpha\right| \leqq$$

$$\leqq \int_{-1/Q}^{+1/Q} |F(\alpha)|\,|G(\alpha)|\,|S(-\alpha)|\left|\frac{U-U(-\alpha)}{U}\right|d\alpha +$$

$$+ \int_{+1/Q}^{1-1/Q} |F(\alpha)|\,|G(\alpha)|\,|S(-\alpha)|\left(1+\left|\frac{U(-\alpha)}{U}\right|\right)d\alpha \leqq \int_{-1/Q}^{+1/Q} |F(\alpha)|\,|G(\alpha)|\,S\,\frac{4U^2|\alpha|}{U}\,d\alpha +$$

$$+\int_{+1/Q}^{1-1/Q}|F(\alpha)||G(\alpha)|\Big(\max_{1/Q<\beta<1-1/Q}|S(\beta)|\Big)2\,d\alpha \leqq \int_{-1/Q}^{+1/Q}|F(\alpha)||G(\alpha)|\,Ky4U\frac{1}{Q}\,d\alpha+$$

$$+\int_{+1/Q}^{1-1/Q}|F(\alpha)||G(\alpha)|\,14(KN\log N)^{1/2}2\,d\alpha \leqq$$

$$\leqq \left(4\frac{KyU}{Q}+28(KN\log N)^{1/2}\right)\int_0^1|F(\alpha)||G(\alpha)|\,d\alpha <$$

$$< \left(4\frac{Ky9N/y}{\left(\frac{KN}{\log N}\right)^{1/2}}+28(KN\log N)^{1/2}\right)\left\{\left(\int_0^1|F(\alpha)|^2\,d\alpha\right)\left(\int_0^1|G(\alpha)|^2\,d\alpha\right)\right\}^{1/2} =$$

$$= 64(KN\log N)^{1/2}(AB)^{1/2} = 64(ABKN\log N)^{1/2}.$$

Furthermore, by (2), (3) and Lemma 10 and since $h_n \geqq 0$ and $v_n \geqq 0$, we have

$$(15)\quad \int_0^1 F(\alpha)G(\alpha)U(-\alpha)S(-\alpha)\,d\alpha = \int_0^1\left(\sum_{n=N+1}^{2N}h_n e(n\alpha)\right)\left(\sum_{n=1}^{2N+U-1}v_n e(-n\alpha)\right)d\alpha =$$

$$= \sum_{n=N+1}^{2N}h_n v_n \geqq \sum_{n=N+1}^{2N}h_n K = K\sum_{n=N+1}^{2N}h_n =$$

$$= K\sum_{\substack{N<a+b\leqq 2N\\ a\in\mathscr{A}\\ b\in\mathscr{B}}}1 = K\Big(\sum_{a\in\mathscr{A}}1\Big)\Big(\sum_{b\in\mathscr{B}}1\Big) = ABK.$$

(14) and (15) yield with respect to (5) that

$$|J| \geqq \frac{1}{U}\left|\int_0^1 F(\alpha)G(\alpha)U(-\alpha)S(-\alpha)\,d\alpha\right| - 64(ABKN\log N)^{1/2} \geqq$$

$$\geqq \frac{ABK}{U} - 64(ABKN\log N)^{1/2} > \frac{ABK}{9N/y} - 64(ABKN\log N)^{1/2} =$$

$$= \frac{1}{9}(ABKN\log N)^{1/2}\left(\left(\frac{ABKy^2}{N^3\log N}\right)^{1/2} - 576\right) > \frac{1}{9}(ABKN\log N)^{1/2}(10^3-576) > 0$$

which proves (9) and this completes the proof of Theorem 1.

7. In this section, we derive the following corollary from Theorem 1.

COROLLARY 1. *Let* $N > N_1$ *and let* $\mathscr{A}, \mathscr{B}$ *be sequences of integers such that*

$$N/2 < a \leqq N \quad (\textit{for } a\in\mathscr{A}), \qquad N/2 < b \leqq N \quad (\textit{for } b\in\mathscr{B})$$

and

$$(16)\qquad AB \geqq 10^6 N^{3/2}(\log N)^2.$$

Then there exist integers a, b such that $a\in\mathcal{A}$, $b\in\mathcal{B}$ and

$$P(a+b) > 10^{-6}\frac{AB}{N(\log N)^2}.$$

Note that in the special case $\mathcal{A}=\mathcal{B}$, an old result of P. Erdős and P. Turán (see [2]) yields the much weaker estimate

$$\max_{a,b} P(a+b) \gg \log|A|;$$

on the other hand, this estimate holds without a condition of type (16).

PROOF. Put $y=10^6\frac{N^2(\log N)^2}{AB}$ and let $\mathcal{K}$ denote the set of prime numbers such that $2N/y<k\leqq 4N/y$. In order to show that Theorem 1 can be applied with this y, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{K}$, we have to show that (1), (5) and (6) hold.

Obviously, we have

$$y = 10^6\frac{N^2(\log N)^2}{AB} \geqq 10^6\frac{N^2(\log N)^2}{N^2} = 10^6(\log N)^2 > 1 \tag{17}$$

and in view of (16),

$$y = 10^6\frac{N^2(\log N)^2}{AB} \leqq 10^6\frac{N^2(\log N)^2}{10^6N^{3/2}(\log N)^2} = N^{1/2}. \tag{18}$$

(17) and (18) yield (1).

By the prime number theorem, for large N we have

$$K = \sum_{2N/y<p\leqq 4N/y} 1 > \frac{N/y}{\log N/y} > \frac{N}{y\log N}. \tag{19}$$

Thus with respect to (16), we have

$$ABKy^2 > AB\frac{N}{y\log N}y^2 = AB\frac{N}{\log N}y = AB\frac{N}{\log N}10^6\frac{N^2(\log N)^2}{AB} = 10^6N^3\log N$$

which proves (5).

Finally, in view of (19), $k\in\mathcal{K}$ implies that

$$p(k) = k > 2N/y = \frac{2N^{1/2}K^{1/2}}{y(\log N)^{1/2}}\left(\frac{N\log N}{K}\right)^{1/2} > \frac{2N^{1/2}\left(\frac{N}{y\log N}\right)^{1/2}}{y(\log N)^{1/2}}\left(\frac{N\log N}{K}\right)^{1/2} =$$

$$= \frac{2N}{y^{3/2}\log N}\left(\frac{N\log N}{K}\right)^{1/2} \geqq \frac{2N}{(N^{1/2})^{3/2}\log N}\left(\frac{N\log N}{K}\right)^{1/2} =$$

$$= \frac{2N^{1/4}}{\log N}\left(\frac{N\log N}{K}\right)^{1/2} > \left(\frac{N\log N}{K}\right)^{1/2}$$

which proves that also (6) holds, so that in fact, Theorem 1 can be applied.

By using Theorem 1, we obtain the existence of integers a, b, k such that $a\in\mathscr{A}$, $b\in\mathscr{B}$, $k\in\mathscr{K}$ and $k|(a+b)$. But this implies that

$$P(a+b)\geqq P(k)=k>\frac{2N}{y}=\frac{2N}{10^6\dfrac{N^2(\log N)^2}{AB}}>10^{-6}\frac{AB}{N(\log N)^2}$$

which completes the proof of Corollary 1.

8. If $A\gg N$, $B\gg N$, then Corollary 1 yields the existence of a, b such that $a\in\mathscr{A}$, $b\in\mathscr{B}$ and

$$P(a+b)\gg\frac{N}{(\log N)^2}. \tag{20}$$

We conjecture that this assertion is true also with N on the right hand side; unfortunately, we have not been able to show this. However, by using a different method (based on the large sieve), we can show that the right hand side of (20) can be replaced by $\dfrac{N}{\log N}$; furthermore, for small AB, the estimate obtained in this way is much better than the one in Corollary 1.

THEOREM 2. *If* $N>N_2$, $\mathscr{A}\subset\{1, 2, \ldots, N\}$, $\mathscr{B}\subset\{1, 2, \ldots, N\}$ *and*

$$AB>100\,N(\log N)^2 \tag{21}$$

(*where* $A=\sum_{a\in\mathscr{A}}1$, $B=\sum_{b\in\mathscr{B}}1$) *then there exist integers* a, b *such that* $a\in\mathscr{A}$, $b\in\mathscr{B}$ *and* $P(a+b)>\dfrac{1}{16}\dfrac{(AB)^{1/2}}{\log N}$.

PROOF. We use the following form of the large sieve:

LEMMA 11. *Let* $\mathscr{N}$ *be a set of* Z *integers in the interval* $[M+1, M+N]$. *For prime* p *let* $\omega(p)$ *denote the number of residue classes modulo* p *that contain no element of* $\mathscr{N}$. *Then for* $Q\geqq 1$

$$Z\leqq\frac{N+2Q^2}{L}$$

where

$$L=\sum_{q\leqq Q}\mu^2(q)\prod_{p|q}\frac{\omega(p)}{p-\omega(p)}.$$

See [4], p. 25.

In order to prove Theorem 2, put

$$Q=\frac{1}{8}\frac{(AB)^{1/2}}{\log N},$$

and assume indirectly that

$$P(a+b)\leqq Q/2 \tag{22}$$

(for all $a\in\mathscr{A}$ and $b\in\mathscr{B}$) which implies that

$$p\nmid(a+b) \quad \text{for} \quad Q/2<p\leqq Q, \quad a\in\mathscr{A}, \quad b\in\mathscr{B}. \tag{23}$$

For $Q/2<p\leqq Q$, let $\varphi(p)$ and $\psi(p)$ denote the number of residue classes modulo p that contain no element of $\mathscr{A}$ and $\mathscr{B}$, respectively. If $\mathscr{A}$ contains an element of the residue class represented by the integer r, then by (23), $\mathscr{B}$ must not intersect the residue class represented by $p-r$. Thus we have $\psi(p)\geqq p-\varphi(p)$ hence

$$\varphi(p)+\psi(p)\geqq p. \tag{24}$$

By using Lemma 11 with $\mathscr{A}$ and $\mathscr{B}$, respectively, in place of $\mathscr{N}$, we obtain that

(25)

$$A\leqq\frac{N+2Q^2}{\sum\limits_{q\leqq Q}\mu^2(q)\prod\limits_{p\mid q}\frac{\varphi(p)}{p-\varphi(p)}}\leqq\frac{N+2Q^2}{\sum\limits_{Q/2<p\leqq Q}\mu^2(p)\frac{\varphi(p)}{p-\varphi(p)}}=\frac{N+2Q^2}{\sum\limits_{Q/2<p\leqq Q}\frac{\varphi(p)}{p-\varphi(p)}}$$

and in view of (24),

(26)

$$B\leqq\frac{N+2Q^2}{\sum\limits_{q\leqq Q}\mu^2(q)\prod\limits_{p\mid q}\frac{\psi(p)}{p-\psi(p)}}\leqq\frac{N+2Q^2}{\sum\limits_{Q/2<p\leqq Q}\mu^2(p)\frac{\psi(p)}{p-\psi(p)}}\leqq\frac{N+2Q^2}{\sum\limits_{Q/2<p\leqq Q}\frac{p-\varphi(p)}{\varphi(p)}}.$$

By using Cauchy's inequality and with respect to (21), we obtain from (25) and (26) that for large N (so that also Q is large by (21)),

$$AB\leqq\frac{(N+2Q^2)^2}{\sum\limits_{Q/2<p\leqq Q}\frac{\varphi(p)}{p-\varphi(p)}\sum\limits_{Q/2<p\leqq Q}\frac{p-\varphi(p)}{\varphi(p)}}\leqq$$

$$\leqq\frac{(N+2Q^2)^2}{\left(\sum\limits_{Q/2<p\leqq Q}\left(\frac{\varphi(p)}{p-\varphi(p)}\right)^{1/2}\left(\frac{p-\varphi(p)}{\varphi(p)}\right)^{1/2}\right)^2}=$$

$$=\frac{(N+2Q^2)^2}{\left(\sum\limits_{Q/2<p\leqq Q}1\right)^2}<\frac{(N+2Q^2)^2}{\frac{1}{5}\frac{Q^2}{(\log Q)^2}}<5Q^2\left(\frac{N}{Q^2}+2\right)^2(\log Q)^2=$$

$$=5\frac{1}{64}\frac{AB}{(\log N)^2}\left(\frac{64N(\log N)^2}{AB}+2\right)^2\left(\log\frac{1}{8}\frac{(AB)^{1/2}}{\log N}\right)^2<$$

$$<\frac{5}{64}\frac{AB}{(\log N)^2}\left(\frac{64\,N(\log N)^2}{100\,N(\log N)^2}+2\right)^2(\log N)^2<\frac{5}{64}AB3^2<AB,$$

and this contradiction shows that the indirect assumption (22) cannot hold which completes the proof of Theorem 2.

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ON COMPOSITION CLOSED FUNCTION CLASSES

EVA LOWEN-COLEBUNDERS (Brussel)

Composition closed function classes were introduced by Á. Császár in [3] and provide on a given set X a characterization of function classes that are the trace on X of the collection of continuous functions $C(Y)$ on some topological space Y in which X is a dense subset.

If, in addition, a topology on X is given then it is required that X is a subspace of Y. In [3] and [4] it is shown that (1) for a topological space X and a function class φ there exists a completely regular topological space Y containing X as a dense subspace and such that $C(Y)|_X=\varphi$ if and only if φ is composition closed and $Z(\varphi)$ is a closed base in X. The problem remained to find a characterization when Y is not necessarily completely regular (Problem 2 in [4]).

In this paper we give two partial answers to this problem in the class of spaces satisfying a weak separation condition P, namely that two points x and y have the same collection of convergent filters whenever there exists a filter converging to both x and y. P is satisfied by a large class of spaces, e.g. by all Hausdorff spaces and by all regular topological spaces. We are able to drop the complete regularity in (1) and replace it by P if Y is permitted to be a convergence space instead of a topological space. If, on the contrary, Y must be topological then we still are able to considerably weaken the condition of complete regularity. The solutions are obtained by giving a Cauchy space formulation of the problem.

0. Preliminaries

For definitions and notations on function classes we refer to [4], for notions on convergence spaces to [6] or [9] and for the theory of Cauchy spaces to [7] or [11]. We recall some of the fundamental definitions.

Let X be a set and $F(X)$ the collection of filters on X. A *Cauchy structure* $\mathscr{C}$ on X is a subset of $F(X)$ subject to

(1) the filter $\dot{x}$ generated by $\{x\}$ belongs to $\mathscr{C}$ for all $x\in X$,

(2) if a filter belongs to $\mathscr{C}$ then so does every finer filter,

(3) if $\mathscr{F}$ and $\mathscr{G}$ belong to $\mathscr{C}$ and $\mathscr{F}\vee\mathscr{G}$ exists then $\mathscr{F}\cap\mathscr{G}$ belongs to $\mathscr{C}$.

The space $(X,\mathscr{C})$ is a *Cauchy space*. $\mathscr{C}$ *generates a convergence structure* q *on* X as follows: $\mathscr{F}$ q-converges to x if and only if $\mathscr{F}\cap\dot{x}\in\mathscr{C}$. Then $\mathscr{C}$ is said to be *compatible* with q. A convergence structure q has a compatible Cauchy structure if and only if (X,q) satisfies condition P mentioned above [7]. (X,q) has P if and only if it has the reciprocal (initial) convergence structure generated by the quotient map to its Hausdorff identification. In this

paper we call spaces satisfying P *reciprocal spaces*. On a Cauchy space $(X, \mathscr{C})$ the *closure* of a subset A is defined in q and denoted by $\mathrm{cl}_q A$. The set A is *dense* if $\mathrm{cl}_q A = X$. For a filter $\mathscr{F}$ on X let $\mathrm{cl}_q \mathscr{F} = [\{\mathrm{cl}_q F | F \in \mathscr{F}\}]$ and the *adherence* of $\mathscr{F}$ is denoted by $\alpha_q \mathscr{F}$. Every uniform space in the sense of [2] generates a Cauchy space, namely the collection of its Cauchy filters. A Cauchy structure is called *uniformizable* if it is the collection of Cauchy filters in some uniform space [10]. A map $f: (X, \mathscr{C}) \to (Y, \mathscr{D})$ between two Cauchy spaces is *Cauchy continuous* if for every $\mathscr{F} \in \mathscr{C}$ we have that $f(\mathscr{F})$, the filter generated by $\{f(F) \mid F \in \mathscr{F}\}$, belongs to $\mathscr{D}$.

R is equipped with its usual uniformity and with the Cauchy structure generated by it.

Let $\Gamma(X, \mathscr{C})$ be the collection of all real valued Cauchy continuous maps on $(X, \mathscr{C})$. Cauchy continuous maps are continuous on the generated convergence structure.

Let $C(X, q)$ be the collection of all real valued continuous maps. Let $\mu(X, \mathscr{C})$ be the weak topology of $\Gamma(X, \mathscr{C})$ i.e. the coarsest topology making all maps in $\Gamma(X, \mathscr{C})$ continuous [5]. When $\mathscr{C}$ and $\mathscr{D}$ are Cauchy structures on the same set then $\mathscr{C} \leqq \mathscr{D}$ ($\mathscr{C}$ is *coarser*, $\mathscr{D}$ is *finer*) if $\mathscr{D} \subset \mathscr{C}$. When A is a subset of $(X, \mathscr{C})$ then $\mathscr{C}_A = \{\mathscr{F} \in F(A) | [\mathscr{F}] \in \mathscr{C}\}$ is the Cauchy structure induced on A. Then $(A, \mathscr{C}_A)$ is a *Cauchy subspace* of $(X, \mathscr{C})$. A Cauchy space is *complete* if every Cauchy filter converges (in the generated convergence structure). $(Y, \mathscr{D})$ is a *completion* of $(X, \mathscr{C})$ if $(Y, \mathscr{D})$ is complete, $(X, \mathscr{C})$ is a Cauchy subspace and X is dense in Y. X is *strictly dense* in Y if $\mathrm{cl}\ X = Y$ and whenever $\psi \in F(Y)$ converges to y, there is a $\mathscr{G}$ converging to y such that $X \in \mathscr{G}$ and $\psi \supset \mathrm{cl}\, \mathscr{G}$. If $(Y, \mathscr{D})$ is a completion of $(X, \mathscr{C})$ and X is strictly dense in Y then $(Y, \mathscr{D})$ is a *strict completion* [5].

1. Completions

In this paper no separation properties are implicitly assumed. However, most of the theory on Cauchy spaces developed in the literature deals with Hausdorff spaces. For instance the construction of the Kowalsky completion given in [9] or [11] only applies to spaces satisfying certain separation properties. We shall modify the construction to obtain a completion in the non-Hausdorff case.

Let $(X, \mathscr{C})$ be a Cauchy space. Two filters $\mathscr{F}$ and $\mathscr{G}$ in $\mathscr{C}$ are *equivalent* if $\mathscr{F} \cap \mathscr{G} \in \mathscr{C}$. Let Z be the collection of all equivalence classes of non-convergent filters in $\mathscr{C}$ and let $Y = X \cup Z$. Let Σ_Z be the set of all maps $\sigma: Z \to F(X)$ such that $\sigma(z) \in z$ for $z \in Z$. For each $\sigma \in \Sigma_Z$ and $A \subset X$ let $A^\sigma = A \cup \{z | A \in \sigma(z)\}$ and for $\mathscr{F} \in F(X)$ let $\mathscr{F}^\sigma$ be the filter on Y generated by $\{F^\sigma | F \in \mathscr{F}\}$. A filter ψ on Y converges to some $x \in X$ if for every $\sigma \in \Sigma_Z$ there is a filter $\mathscr{F}$ on X converging to x such that $\mathscr{F}^\sigma \subset \psi$. The filter ψ converges to some $z \in Z$ if for every $\sigma \in \Sigma_Z$ there is a filter $\mathscr{F} \in z$ such that $\mathscr{F}^\sigma \subset \psi$. Let p be the convergence structure on Y thus defined. If $\mathscr{F}^\sigma \vee \mathscr{G}^\sigma$ exists for some $\sigma \in \Sigma_Z$ then $\mathscr{F} \vee \mathscr{G}$ exists. It follows that (Y, p) is reciprocal. Hence $\mathscr{D} = \{\psi \in F(Y) | \psi$ is p-convergent$\}$ is a compatible Cauchy structure for (Y, p). By straightforward verification the following result can be shown.

THEOREM (1.1). *$(Y, \mathscr{D})$ is a strict completion of $(X, \mathscr{C})$.*

The proof of the theorem below is analogous to the one of the extension theorem shown in [11] for a more restrictive class of completions.

THEOREM (1.2). *Every strict completion has the* **R***-extension property, i.e. every real valued Cauchy continuous map can be continuously extended to the completion in a unique way.*

The previous theorem remains true when **R** is replaced by any regular Hausdorff complete Cauchy space.

2. Function classes on a set

The results of this section can essentially be found in [3] and [4]. We shall present them in the Cauchy space setting since we need this formulation in the next sections.

If φ is a function class on a set X let $\mathscr{C}_\varphi$ be the *weak Cauchy structure:* $\mathscr{F}\in\mathscr{C}_\varphi$ if and only if $f(\mathscr{F})$ is **R**-convergent for every $f\in\varphi$. $\mathscr{C}_\varphi$ is the coarsest Cauchy structure making all the maps in φ Cauchy continuous. $\mathscr{C}_\varphi$ is a uniformizable Cauchy structure, it is the collection of Cauchy filters of the weak uniformity $\mathscr{U}_\varphi$ induced by φ. In the terminology of [3] $\mathscr{C}_\varphi$ is the collection of all φ-filters.

LEMMA 2.1. *If* $(X,\mathscr{C})$ *is a Cauchy space and* $\varphi=\Gamma(X,\mathscr{C})$ *then* $\Gamma(X,\mathscr{C}_\varphi)=\Gamma(X,\mathscr{C})$. *If the cardinality of* X *is nonmeasurable, then* $\mathscr{C}_\varphi$ *is the finest uniformizable Cauchy structure on* X *coarser than* $\mathscr{C}$ *and* $\mathscr{C}_\varphi$ *is a reflection of* $\mathscr{C}$ *in the category of uniformizable Cauchy spaces with nonmeasurable cardinality.*

PROOF. Clearly $\mathscr{C}_\varphi\leqq\mathscr{C}$. Hence $\Gamma(X,\mathscr{C}_\varphi)\subset\Gamma(X,\mathscr{C})$. If $f\in\Gamma(X,\mathscr{C})$ and $\mathscr{F}\in\mathscr{C}_\varphi$ then by definition $f(\mathscr{F})$ is **R**-convergent. So we have $f\in\Gamma(X,\mathscr{C}_\varphi)$.

Suppose X has nonmeasurable cardinality and $\mathscr{D}$ is a uniformizable Cauchy structure, $\mathscr{D}\leqq\mathscr{C}$. Let $\mathscr{F}\in\mathscr{C}_\varphi$ and $f\in\Gamma(X,\mathscr{D})$. Then $f\in\Gamma(X,\mathscr{C})=\Gamma(X,\mathscr{C}_\varphi)$ and $f(\mathscr{F})$ is **R**-convergent. But then $\mathscr{F}\in\mathscr{D}$ since $\mathscr{D}$ is uniformizable [10]. Therefore $\mathscr{D}\leqq\mathscr{C}_\varphi$. Let $(Y,\mathscr{E})$ be any uniformizable Cauchy space then it is the weak Cauchy structure of $\Gamma(Y,\mathscr{E})$ [10] and this implies that $(X,\mathscr{C})$ and $(X,\mathscr{C}_\varphi)$ have the same collection of Cauchy continuous maps to $(Y,\mathscr{E})$. It follows that $\mathscr{C}_\varphi$ is a reflection of $\mathscr{C}$.

THEOREM 2.3. *For any set* X *and* φ *a function class on* X *the following properties are equivalent:*

(1) *There exists a Cauchy structure* $\mathscr{C}$ *on* X *such that* $\Gamma(X,\mathscr{C})=\varphi$.

(2) $\Gamma(X,\mathscr{C}_\varphi)=\varphi$.

(3) φ *is composition closed.*

(4) *There is a completely regular topological space* Y *containing* X *as a dense subset and such that* $C(Y)|_X=\varphi$.

(5) *There is a reciprocal convergence space* Y *containing* X *as a strictly dense subset and such that* $C(Y)|_X=\varphi$.

PROOF. (1)⇒(2) follows from the previous lemma.

(2)⇒(3). $\Gamma(X,\mathscr{C}_\varphi)=\varphi$ means that φ is filter closed [3] and hence φ is composition closed [3].

(3)⇒(4). The topological space constructed in the proof of Theorem 4 in [4] is completely regular.

(4)⇒(5). Every dense subset of a topological space is strictly dense and every regular topological space is reciprocal.

(5)⇒(1). If Y is reciprocal then $\mathscr{D}=\{\psi\in F(Y)|\psi$ is Y-convergent$\}$ is a compatible structure inducing a Cauchy structure $\mathscr{C}$ on X. Then $(Y,\mathscr{D})$ is a strict completion of $(X,\mathscr{C})$. It follows from Theorem 1.2 that $\Gamma(X,\mathscr{C})\subset\varphi$. On the other hand the restriction to X of every continuous map on Y is Cauchy continuous on $(X,\mathscr{C})$. It follows that $\Gamma(X,\mathscr{C})=\varphi$.

COROLLARY 2.4. *If there exists a Cauchy structure $\mathscr{C}$ such that $\Gamma(X,\mathscr{C})=\varphi$ then $\mathscr{C}_\varphi$ is the coarsest structure with this property.*

3. Function classes on a convergence space

In this section a first partial solution to Problem 2 in [4] is presented. Given a function class φ on a topological space (X,q) we give necessary and sufficient conditions for the existence of a reciprocal convergence space Y containing X as a strictly dense subspace and such that $C(Y)|_X=\varphi$. Since the extension Y is in general not topological we may as well start with a convergence space (X,q). Of course the results of this section also apply to the particular case that X is topological. Let φ be a function class on a convergence space (X,q). $\mathscr{C}_\varphi$ is defined as in § 2. Let q_φ be the weak topology induced by φ, i.e. the coarsest topology on X making all maps in φ continuous. Then q_φ also is the convergence structure generated by $\mathscr{C}_\varphi$.

Another natural Cauchy structure associated with (X,q) and φ is the following

$$\mathscr{D}_\varphi=\{\mathscr{F}\in\mathscr{C}_\varphi|\forall x\in X \text{ if } \mathscr{F}\ q_\varphi\text{-converges to } x\in X \text{ then either } \mathscr{F}\ q\text{-converges to } x \text{ or } x\notin\alpha_q\mathscr{F}\}.$$

An easy verification shows that $\mathscr{D}_\varphi$ is a Cauchy structure and that $\mathscr{C}_\varphi\leqq\mathscr{D}_\varphi$.

LEMMA 3.1. *$\mathscr{D}_\varphi$ is compatible with (X,q) if and only if (X,q) is reciprocal and $\varphi\subset C(X,q)$.*

PROOF. Suppose (X,q) is reciprocal and $\varphi\subset C(X,q)$. If $\mathscr{F}$ q-converges to x then so does $\mathscr{F}\cap\dot{x}$. Since $q_\varphi\leqq q$ we have $\mathscr{F}\cap\dot{x}\in\mathscr{C}_\varphi$. If $\mathscr{F}\cap\dot{x}$ q_φ-converges to some $y\in X$ and if $y\in\alpha_q(\mathscr{F}\cap\dot{x})$ then x and y have the same q-convergent filters. Hence $\mathscr{F}\cap\dot{x}$ q-converges to y and $\mathscr{F}\cap\dot{x}\in\mathscr{D}_\varphi$. Conversely suppose $\mathscr{F}\cap\dot{x}\in\mathscr{D}_\varphi$. Then $\mathscr{F}$ q_φ-converges to x and since $x\in\alpha_q(\mathscr{F}\cap\dot{x})$ we have that $\mathscr{F}$ q-converges to x. To prove the other implication suppose $\mathscr{D}_\varphi$ is compatible with (X,q). Hence (X,q) is reciprocal [7] and $\varphi\subset\Gamma(X,\mathscr{C}_\varphi)\subset\Gamma(X,\mathscr{D}_\varphi)\subset C(X,q)$.

THEOREM 3.2. *For any function class φ on a convergence space (X,q) the following properties are equivalent:*

(1) *There exists a compatible Cauchy structure $\mathscr{C}$ on (X,q) such that $\Gamma(X,\mathscr{C})=\varphi$.*

(2) *There is a reciprocal convergence space (Y,p) containing (X,q) as a strictly dense subspace and such that $C(Y)|_X=\varphi$.*

PROOF. (1)⇒(2). Let $\mathscr{C}$ be a compatible Cauchy structure with $\Gamma(X,\mathscr{C})=\varphi$. Let $(Y,\mathscr{D})$ be the strict completion of Theorem 1.1 and let p be the generated convergence structure on Y. Clearly $\mathscr{C}(Y)|_X=\varphi$.

(2)⇒(1). Analogous to the proof of (5)⇒(1) in Theorem 2.3.

THEOREM 3.3. *For any function class φ on a convergence space (X, q) the following properties are equivalent:*

(1) *There is a reciprocal convergence space (Y, p) containing (X, q) as a strictly dense subspace and such that $C(Y)|_X=\varphi$.*

(2) *$\mathscr{D}_\varphi$ is compatible with (X, q) and $\Gamma(X, \mathscr{D}_\varphi)=\varphi$.*

(3) (i) *(X, q) is reciprocal and $\varphi \subset C(X, q)$.*

(ii) *φ is composition closed.*

(iii) *$q_\varphi=\mu(X, \mathscr{D}_\varphi)$.*

PROOF. (1)⇒(2). From Theorem 3.2 there is a compatible Cauchy structure $\mathscr{C}$ such that $\Gamma(X, \mathscr{C})=\varphi$. Then clearly (X, q) is reciprocal and $\varphi \subset C(X, q)$. From Lemma 3.1 we have that $\mathscr{D}_\varphi$ is compatible. Next we show that $\mathscr{C} \cong \mathscr{D}_\varphi$.

Let $\mathscr{F} \in \mathscr{C}$. Either there is an $x \in X$ with $\mathscr{F} \cap \dot{x} \in \mathscr{C}$. Then $\mathscr{F}$ q-converges to x and $\mathscr{F} \in \mathscr{D}_\varphi$. Or for every $x \in X$ we have $\mathscr{F} \cap \dot{x} \notin \mathscr{C}$. Since $\Gamma(X, \mathscr{C})=\varphi$ we have $\mathscr{C}_\varphi \cong \mathscr{C}$ and so $\mathscr{F} \in \mathscr{C}_\varphi$. If $\mathscr{F}$ is q_φ-convergent to some $x \in X$ then clearly $x \notin \alpha_q \mathscr{F}$. It follows that $\mathscr{F} \in \mathscr{D}_\varphi$. Since $\mathscr{C}_\varphi \cong \mathscr{D}_\varphi \cong \mathscr{C}$ we have $\varphi=\Gamma(X, \mathscr{C}) \supset \Gamma(X, \mathscr{D}_\varphi) \supset$ $\supset \Gamma(X, \mathscr{C}_\varphi) \supset \varphi$.

(2)⇒(3). Suppose $\mathscr{D}_\varphi$ is compatible and $\Gamma(X, \mathscr{D}_\varphi)=\varphi$. (i) follows from Lemma 3.1 and (ii) from Theorem 2.3. Since $\Gamma(X, \mathscr{D}_\varphi)=\varphi$ these function classes determine the same weak topology.

(3)⇒(1). φ is composition closed and so we have $\varphi=\Gamma(X, \mathscr{C}_\varphi) \subset \Gamma(X, \mathscr{D}_\varphi)$. If $f \in \Gamma(X, \mathscr{D}_\varphi)$ then f is continuous on $\mu(X, \mathscr{D}_\varphi)=q_\varphi$. Let $\mathscr{F} \in \mathscr{C}_\varphi$ then either $\mathscr{F}$ is not q_φ-convergent and then $\mathscr{F} \in \mathscr{D}_\varphi$ and $f(\mathscr{F})$ **R**-converges. Or there is an $x \in X$ and $\mathscr{F}$ q_φ-converges to x. By the continuity of f we have that $f(\mathscr{F})$ **R**-converges. Hence $f \in \Gamma(X, \mathscr{C}_\varphi)=\varphi$. So we have $\Gamma(X, \mathscr{D}_\varphi)=\varphi$. The compatibility of $\mathscr{D}_\varphi$ follows from Lemma 3.1. Assertion (1) then follows from Theorem 3.2.

COROLLARY 3.4. *If there exists a Cauchy structure $\mathscr{C}$ compatible with (X, q) and such that $\Gamma(X, \mathscr{C})=\varphi$ then $\mathscr{D}_\varphi$ is the coarsest structure with these properties.*

Theorem 3.3 solves Problem 2 of [4] in the class of reciprocal convergence spaces. In general the construction of (Y, p) as a completion of $(X, \mathscr{D}_\varphi)$ does not provide a topological extension if (X, q) is topological. In order to prove a theorem about topological extensions in Section 4, we first need some preliminaryd efinitions and results about a class of spaces strictly larger than the class of completely regular spaces.

For any convergence space (Y, p) let ωp be the finest completely regular topology on Y coarser than p. A convergence space (Y, p) is *ω-regular* if $\mathrm{cl}_{\omega p} \mathscr{F}$ converges to $y \in Y$ whenever $\mathscr{F}$ converges to y. This property has been very useful for the characterization of c-embedded spaces [12]. For topological spaces ω-regularity lies strictly between complete regularity and regularity [1]. If $X \subset Y$ then (Y, p) is *relatively ω-regular with respect to X* if $(\mathrm{cl}_{\omega p} \mathscr{F}) \vee X$ converges if it exists and if $\mathscr{F}$ converges. A Cauchy space $(X, \mathscr{C})$ is *μ-regular* if $\mathrm{cl}_{\mu(X, \mathscr{C})} \mathscr{F} \in \mathscr{C}$ whenever $\mathscr{F} \in \mathscr{C}$ [5]. If (X, q) is a convergence space and φ is a function class on it then (X, q) is *φ-regular* if $\mathrm{cl}_{q_\varphi} \mathscr{F}$ q-converges to x whenever $\mathscr{F}$ q-converges to x. Let $\mathscr{E}_\varphi=\{\mathscr{F} \in \mathscr{C}_\varphi | \forall x \in X$ if $\mathscr{F}$ q_φ-converges to x then $\mathscr{F}$ q-converges to $x\}$.

LEMMA 3.5. *If (X, q) is φ-regular and reciprocal and $\varphi \subset C(X, q)$ then $\mathscr{E}_\varphi$ is a μ-regular Cauchy structure compatible with (X, q).*

PROOF. We first show that $\mathscr{E}_\varphi$ is a Cauchy structure. If $\dot{x}$ q_φ-converges to y then we have $\dot{x} \supset \mathrm{cl}_{q_\varphi} \dot{y}$ since q_φ is uniformizable. Since (X, q) is φ-regular this implies that $\dot{x}$ q-converges to y. Hence $\dot{x} \in \mathscr{E}_\varphi$. If $\mathscr{F} \in \mathscr{E}_\varphi$ and $\mathscr{G} \supset \mathscr{F}$ then $\mathscr{F}$ and $\mathscr{G}$ both belong to $\mathscr{C}_\varphi$. Since $\mathscr{C}_\varphi$ is compatible with q_φ we have that $\mathscr{F}$ q_φ-converges to some x whenever $\mathscr{G}$ q_φ-converges to x. Hence $\mathscr{G} \in \mathscr{E}_\varphi$. If $\mathscr{F} \in \mathscr{E}_\varphi$ and $\mathscr{G} \in \mathscr{E}_\varphi$ and $\mathscr{F} \vee \mathscr{G}$ exists then clearly $\mathscr{F} \cap \mathscr{G} \in \mathscr{E}_\varphi$.

To see that $\mathscr{E}_\varphi$ is compatible suppose that $\mathscr{F}$ q-converges to $x \in X$. Since $\varphi \subset C(X, q)$ we have $\mathscr{F} \cap \dot{x} \in \mathscr{C}_\varphi$. If $\mathscr{F} \cap \dot{x}$ q_φ-converges to y then $\dot{y} \supset \mathrm{cl}_{q_\varphi}(\mathscr{F} \cap \dot{x})$ and from the φ-regularity we have that $\dot{y}$ q-converges to x. Since (X, q) is reciprocal $\mathscr{F} \cap \dot{x}$ q-converges to y and $\mathscr{F} \cap \dot{x} \in \mathscr{E}_\varphi$. If $\mathscr{F} \cap \dot{x} \in \mathscr{E}_\varphi$ then clearly $\mathscr{F}$ q-converges to x. Next we show that $\mathscr{E}_\varphi$ is μ-regular. Let $\mathscr{F} \in \mathscr{E}_\varphi$. Either there is an $x \in X$ such that $\mathscr{F}$ q_φ-converges to x. Then $\mathscr{F}$ and $\mathrm{cl}_{q_\varphi} \mathscr{F}$ q-converge to x. Since $\mathscr{E}_\varphi$ is compatible $\mathrm{cl}_{q_\varphi} \mathscr{F} \in \mathscr{E}_\varphi$ and since $\mu(X, \mathscr{E}_\varphi) \geqq q_\varphi$ also $\mathrm{cl}_{\mu(X, \mathscr{E}_\varphi)} \mathscr{F} \in \mathscr{E}_\varphi$. Or $\mathscr{F}$ is not q_φ-convergent. Since $\mathscr{F} \in \mathscr{C}_\varphi$ we also have $\mathrm{cl}_{q_\varphi} \mathscr{F} \in \mathscr{C}_\varphi$ and $\mathrm{cl}_{q_\varphi} \mathscr{F}$ does not q_φ-converge. It follows that $\mathrm{cl}_{\mu(X, \mathscr{E}_\varphi)} \mathscr{F} \in \mathscr{E}_\varphi$.

THEOREM 3.6. *For any function class φ on a convergence space (X, q) the following properties are equivalent:*

(1) *There is a μ-regular compatible Cauchy structure $\mathscr{C}$ on (X, q) such that $\Gamma(X, \mathscr{C}) = \varphi$.*

(2) *There exists a reciprocal convergence space (Y, p) relatively ω-regular with respect to X, containing X as a strictly dense subspace and such that $C(Y)|_X = \varphi$.*

PROOF. (1)$\Rightarrow$(2). Using Theorems 1.1 and 3.2, the only property left to show is the relative ω-regularity. Let ψ be a filter on (Y, p) converging to y and suppose $(\mathrm{cl}_{\omega p} \psi) \vee X$ exists. Let $\mathscr{F} \in F(X)$ such that $[\mathscr{F}]$ converges to y and $\mathrm{cl}_p \mathscr{F} \subset \psi$. Then from the μ-regularity we have $\mathrm{cl}_{q_\varphi} \mathscr{F} \in \mathscr{C}$. Since $\mathrm{cl}_{q_\varphi} \mathscr{F} = (\mathrm{cl}_{\omega p}[\mathscr{F}]) \vee X$ we have $\mathrm{cl}_{q_\varphi} \mathscr{F} \subset (\mathrm{cl}_{\omega p} \psi) \vee X$ and so $(\mathrm{cl}_{\omega p} \psi) \vee X$ converges in (Y, p).

(2)$\Rightarrow$(1). We use the same proof as for (5)$\Rightarrow$(1) in Theorem 2.3.

It remains to be shown that the Cauchy structure $\mathscr{C}$ is μ-regular. Let $\mathscr{F} \in \mathscr{C}$. Then $[\mathscr{F}]$ p-converges and $\mathrm{cl}_{\omega p}[\mathscr{F}] \vee X = \mathrm{cl}_{q_\varphi} \mathscr{F}$. Hence $\mathrm{cl}_{q_\varphi} \mathscr{F} \in \mathscr{C}$ and since $\varphi = \Gamma(X, \mathscr{C})$ we have that $\mathscr{C}$ is μ-regular.

THEOREM 3.7. *For any function class φ on a convergence space (X, q) the following properties are equivalent:*

(1) *There is a reciprocal convergence space (Y, p), relatively ω-regular with respect to X, containing X as a strictly dense subspace and such that $C(Y)|_X = \varphi$.*

(2) *$\mathscr{E}_\varphi$ is a μ-regular Cauchy structure, compatible with (X, q) and $\Gamma(X, \mathscr{E}_\varphi) = \varphi$.*

(3) (i) *(X, q) is reciprocal, φ-regular and $\varphi \subset C(X, q)$.*

(ii) *φ is composition closed.*

(iii) *$q_\varphi = \mu(X, \mathscr{E}_\varphi)$.*

PROOF. (1)$\Rightarrow$(2). From Theorem 3.6 there is a μ-regular compatible Cauchy structure $(X, \mathscr{C})$ such that $\Gamma(X, \mathscr{C}) = \varphi$. Then all the conditions of Lemma 3.5 are fulfilled. Hence $\mathscr{E}_\varphi$ is μ-regular and compatible. Next we show that $\mathscr{C} \cong \mathscr{E}_\varphi$. Let $\mathscr{F} \in \mathscr{C}$. Then either $\mathscr{F}$ q-converges and then $\mathscr{F} \in \mathscr{E}_\varphi$, or $\mathscr{F}$ is not q-convergent. Since $\mathscr{C}$ is μ-regular and $\Gamma(X, \mathscr{C}) = \varphi$ we have $\mathrm{cl}_{q_\varphi} \mathscr{F} \in \mathscr{C}$. Since $\mathscr{C}$ is compatible $\dot{x} \not\geqq \mathrm{cl}_{q_\varphi} \mathscr{F}$ for every $x \in X$. Hence $\mathscr{F}$ is not q_φ-convergent. Since $\mathscr{C}_\varphi \cong \mathscr{C}$ we finally have $\mathscr{F} \in \mathscr{E}_\varphi$. It follows that $\varphi = \Gamma(X, \mathscr{C}) \supset \Gamma(X, \mathscr{E}_\varphi) \supset \Gamma(X, \mathscr{C}_\varphi) \supset \varphi$ and so $\Gamma(X, \mathscr{E}_\varphi) = \varphi$.

(2)⇒(3). Suppose $\mathscr{E}_\varphi$ is μ-regular, compatible and $\Gamma(X, \mathscr{E}_\varphi)=\varphi$. Then (i) clearly is fulfilled, (ii) follows from Theorem 2.3 and (iii) from $\Gamma(X, \mathscr{E}_\varphi)=\varphi$.

(3)⇒(1). From (i) and Lemma 3.5 we have that $\mathscr{E}_\varphi$ is a μ-regular structure compatible with (X, q). Since φ is composition closed we have $\varphi=\Gamma(X, \mathscr{C}_\varphi)\subset\Gamma(X, \mathscr{E}_\varphi)$. Let $f\in\Gamma(X, \mathscr{E}_\varphi)$ then f is $\mu(X, \mathscr{E}_\varphi)$-continuous and by (iii) it is q_φ-continuous. Let $\mathscr{F}\in\mathscr{C}_\varphi$, then either $\mathscr{F}$ is not q_φ-convergent and then $\mathscr{F}\in\mathscr{E}_\varphi$ and $f(\mathscr{F})$ **R**-converges. Or $\mathscr{F}$ q_φ-converges to some $x\in X$. Then by the q_φ-continuity $f(\mathscr{F})$ **R**-converges. Hence $f\in\Gamma(X, \mathscr{C}_\varphi)$ and $\Gamma(X, \mathscr{E}_\varphi)=\varphi$. Assertion (1) then follows from Theorem 3.6.

COROLLARY 3.8. *If there exists a μ-regular Cauchy structure $\mathscr{C}$ compatible with (X, q) and such that $\Gamma(X, \mathscr{C})=\varphi$ then $\mathscr{E}_\varphi$ is the coarsest structure with these properties.*

4. Function classes on a topological space

DEFINITION 4.1. *A Cauchy space $(X, \mathscr{C})$ is topological if it is a Cauchy subspace of a reciprocal topological space* (considered as a complete Cauchy space).

LEMMA 4.2. *A Cauchy space $(X, \mathscr{C})$ is topological if and only if every equivalence class has a minimum element with an open base.*

PROOF. Suppose $(X, \mathscr{C})$ is a Cauchy subspace of a reciprocal topological space (Y, p). We may assume that X is a dense subspace of (Y, p). For $\mathscr{F}\in\mathscr{C}$ there is a $y\in Y$ such that $[\mathscr{F}]$ converges to y. The filter $\mathscr{V}(y)|_X$ is the minimum element in the class of $\mathscr{F}$ and it has an open base. To prove the converse suppose $(X, \mathscr{C})$ has minima with an open base. Let $(Y, \mathscr{D})$ be the completion of Theorem 1.1. Then the induced convergence space (Y, p) is reciprocal and $(X, \mathscr{C})$ is a subspace. We show that (Y, p) is topological. Let $\lambda: Z\to F(X)$ such that $\lambda(z)=\mathscr{M}_z$ where $\mathscr{M}_z$ is the minimum element in z. Then $\lambda\in\Sigma_Z$. A filter ψ converges to $z\in Z$ (to $x\in X$) if and only if $\mathscr{M}_z^\lambda\subset\psi$ $(\mathscr{V}(x)^\lambda\subset\psi)$. If $A\subset X$ is open then so is A^λ which proves the assertion.

LEMMA 4.3. *If φ is a function class on a reciprocal φ-regular topological space (X, q) such that $\varphi\subset C(X, q)$ then $\mathscr{E}_\varphi$ is topological.*

PROOF. Let $\mathscr{G}\in\mathscr{E}_\varphi$, $\langle\mathscr{G}\rangle$ its equivalence class in $\mathscr{E}_\varphi$ and let $\mathscr{F}=\bigcap\{\mathscr{H}\,|\,\mathscr{H}\in\langle\mathscr{G}\rangle\}$. The filters in $\langle\mathscr{G}\rangle$ are also equivalent in $\mathscr{C}_\varphi$, hence $\mathscr{F}\in\mathscr{C}_\varphi$. If $\mathscr{F}$ is not q_φ-convergent then $\mathscr{F}\in\mathscr{E}_\varphi$. If $\mathscr{F}$ q_φ-converges to $x\in X$ then $\mathscr{H}$ q-converges to x for every $\mathscr{H}\in\langle\mathscr{G}\rangle$ and so $\mathscr{F}$ q-converges to x. Hence $\mathscr{F}\in\mathscr{E}_\varphi$. Next we show that $\mathscr{F}$ has an open base. If $\mathscr{F}$ does not q_φ-converge, if $\mathscr{H}\in\mathscr{C}_\varphi$ and $\mathscr{H}$ is equivalent with $\mathscr{F}$ in $\mathscr{C}_\varphi$ then for every $x\in X$ we have $(\mathscr{H}\cap\mathscr{F})\cap\dot{x}=\mathscr{H}\cap(\mathscr{F}\cap\dot{x})\notin\mathscr{C}_\varphi$. Hence $\mathscr{H}\cap\mathscr{F}\in\mathscr{E}_\varphi$ and $\mathscr{H}\supset\mathscr{F}$. So $\mathscr{F}$ is the minimum of an equivalence class of $\mathscr{C}_\varphi$ and it has a q_φ-open base and then also a q-open base. If $\mathscr{F}$ does q_φ-converge to some $x\in X$ then we saw that $\mathscr{F}$ q-converges to x. It follows that $\mathscr{F}=\mathscr{V}_q(x)$ and $\mathscr{F}$ has an open base.

THEOREM 4.4. *For any function class φ on a topological space (X, q) the following properties are equivalent:*

(1) *There is a reciprocal topological space (Y, p), relatively ω-regular with respect to X, containing X as a dense subspace and such that $C(Y)|_X=\varphi$.*

(2) (i) (X, q) *is reciprocal,* φ*-regular and* $\varphi \subset C(X, q)$.
(ii) φ *is composition closed.*
(iii) $q_\varphi = \mu(X, \mathscr{E}_\varphi)$.

PROOF. (1)⇒(2) follows immediately from Theorem 3.7 since a dense subset of a topological space is always strictly dense.

(2)⇒(1). If (2) holds then by Theorem 3.7 and Lemma 4.3 $\mathscr{E}_\varphi$ is a topological μ-regular structure compatible with (X, q) and such that $\Gamma(X, \mathscr{E}_\varphi) = \varphi$. From the proof of Lemma 4.2 it follows that the completion of Theorem 1.1 is topological.

REMARK 4.5. The previous theorem gives a solution to Problem 2 in [4] in the class of reciprocal spaces that are relatively ω-regular with respect to X. This class is larger than the class of completely regular spaces, it contains all ω-regular topological spaces and many examples are known of ω-regular spaces that are not completely regular. See e.g. [1]. The condition of relatively ω-regularity for Y can be dropped if topological space (X, q) is regular and locally compact.

LEMMA 4.6. *If* φ *is a function class on a locally compact regular topological space* (X, q) *and* $\varphi \subset C(X, q)$ *then* $\mathscr{D}_\varphi$ *is topological.*

PROOF. Let $\mathscr{G} \in \mathscr{D}_\varphi$, $\langle \mathscr{G} \rangle$ its equivalence class in $\mathscr{D}_\varphi$ and $\mathscr{F} = \cap \{\mathscr{H} | \mathscr{H} \in \langle \mathscr{G} \rangle\}$. Either $\mathscr{G}$ q-converges to some $x \in X$ and then $\mathscr{H}$ q-converges to x for every $\mathscr{H} \in \langle \mathscr{G} \rangle$. It follows that $\mathscr{F}$ q-converges to x and $\mathscr{F} \in \mathscr{D}_\varphi$. Or $\mathscr{G}$ does not q-converge and then $\mathscr{H}$ is not q-convergent for every $\mathscr{H} \in \langle \mathscr{G} \rangle$. Since the filters in $\langle \mathscr{G} \rangle$ are also equivalent in $\mathscr{C}_\varphi$ we have $\mathscr{F} \in \mathscr{C}_\varphi$. Suppose that $\mathscr{F}$ q_φ-converges then we show that $\alpha_q \mathscr{F} = \emptyset$. If $\mathscr{W}$ is an ultrafilter $\mathscr{W} \supset \mathscr{F}$ and $\mathscr{W}$ q-converges then $\mathscr{W}$ contains a compact set K. It follows from [8, Prop. 2] that there is an ultrafilter $\mathscr{U} \in \langle \mathscr{G} \rangle$ containing K. This is a contradiction. So $\alpha_q \mathscr{F} = \emptyset$ and $\mathscr{F} \in \mathscr{D}_\varphi$. Next we show that $\mathscr{F}$ has a q-open base. In the first case that $\mathscr{G}$ converges to x we clearly have $\mathscr{F} = \mathscr{V}_q(x)$ which has an open base. In the second case that $\mathscr{G}$ does not q-converge and $\mathscr{F}$ is not q_φ-convergent then $\langle \mathscr{G} \rangle$ in $\mathscr{D}_\varphi$ is equal to $\langle \mathscr{G} \rangle$ in $\mathscr{C}_\varphi$ and then $\mathscr{F}$ has a q_φ-open base and therefore also a q-open base. In the third case $\mathscr{G}$ does not q-converge and $\mathscr{F}$ q_φ--converges. Let $\mathscr{L} = \{O | O q\text{-open}, O \in \mathscr{F}\}$. If $\mathscr{F}$ q_φ-converges to y then $\mathscr{F} \supset \mathscr{L} \supset \supset \mathscr{V}_{q_\varphi}(y)$, hence $\mathscr{L} \in \mathscr{C}_\varphi$. If $\mathscr{L}$ q_φ converges to z then so does $\mathscr{F}$ and we know that $z \notin \alpha_q \mathscr{F}$. Since (X, q) is regular we have $z \notin \alpha_q \mathscr{L}$. So $\mathscr{L} \in \mathscr{D}_\varphi$ and we have $\mathscr{L} = \mathscr{F}$.

THEOREM 4.7. *For any function class* φ *on a locally compact and regular topological space* (X, q) *the following properties are equivalent:*

(1) *There is a reciprocal topological space* (Y, p) *containing* (X, q) *as a dense subspace and such that* $C(Y)|_X = \varphi$.

(2) (i) $\varphi \subset C(X, q)$.
(ii) φ *is composition closed.*
(iii) $q_\varphi = \mu(X, \mathscr{D}_\varphi)$.

PROOF. (1)⇒(2) follows from Theorem 3.3. (2)⇒(1) Theorem 3.3, Lemma 4.6, Lemma 4.2.

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ON A CHARACTERIZATION OF SOME ARITHMETICAL MULTIPLICATIVE FUNCTIONS

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1. Introduction

An arithmetical function $f(n)$ is *multiplicative* (resp. *additive*), if $f(mn)=f(m)\cdot f(n)$ (resp. $f(mn)=f(m)+f(n)$) for any pair m, n of relatively prime natural numbers, and *completely multiplicative* (resp. *completely additive*), if the above equality holds for any pair m, n.

In 1946, P. Erdös [1] proved that if an additive function $F(n)$ satisfies one of the two conditions:

i) $F(n+1)\geqq F(n)$ for any $n\in N$,

ii) $\lim_{n\to\infty}\{F(n+1)-F(n)\}=0$,

then there exists some constant c such that $F(n)=c(\log n)$, $n\in N^*$.

Since then, many results have been obtained about this kind of problem. On the other hand, there are not so many results concerning the similar problem for multiplicative functions (see, for instance, references in [2] or [3]).

Here we shall give the proofs of the following results (see [5]):

THEOREM. *Let $f(n)$ be a multiplicative function. Suppose that there exists a positive, non-decreasing function $g(x)$ satisfying*

$$\text{I)} \qquad \lim_{x\to\infty} g(x)=\infty \quad \textit{and} \quad g(2x)=O(g(x)),$$

$$\text{II)} \qquad \lim_{x\to\infty} g(x)^{-1}\sum_{n\leqq x}|f(n+1)-f(n)|=0,$$

$$\text{III)} \qquad \limsup_{x\to\infty} g(x)^{-1}\Big|\sum_{n\leqq x} f(n)\Big|>0.$$

Then, $f(n)$ is completely multiplicative, and there exists $\lambda\geqq -1$, such that $|f(n)|=n^{\lambda}$.

We obtain from this theorem:

COROLLARY. *Let $f(n)$ be a multiplicative function and $g(x)$ be a positive, non-decreasing function satisfying* II) *of the above theorem. Suppose that $g(x)$ satisfies the following* I′) *and* III′) *instead of* I) *and* III) *of our theorem:*

I′) $\lim_{x\to\infty} g(x)=\infty$, $\lim_{x\to\infty} g(dx)/g(x)=h(d)$ *exists and is finite for any* $d\in N^*$,

III′) $\lim_{x\to\infty} g(x)^{-1}\sum_{n\leqq x} f(n)=\mu$ *exists and is finite and* $\neq 0$. *Then* $f(n)=n^{\lambda}$ *with some* $\lambda\geqq -1$.

We shall prove some lemmas in the next section, and prove the Theorem and the Corollary in the third and fourth section respectively.

We give here some additional remarks.

1. There are many functions $g(x)$ which satisfy condition I'). For example, x^α $(\alpha>0)$, $x^\alpha(\log x)^\beta$ $(\alpha>0)$, $(\log x)^\beta$ $(\beta>0)$, $\pi(x)$, etc.

2. It is to be noticed that there exist multiplicative functions $f(n)$ which satisfy II) and III) for some $g(x)$ but do not satisfy III'). For example, we take $f(n)=n^{it}$ $(t\in R^*)$ and $g(x)=x^{1/2}$. Then it is easy to see that

$$\lim_{x\to\infty} x^{-1/2}\sum_{n\leqq x}|f(n+1)-f(n)| = 0,$$

but

$$\lim_{x\to\infty} x^{-1/2}\Big|\sum_{n\leqq x} f(n)\Big| = \infty,$$

and $\lim_{x\to\infty} x^{-1/2}\sum_{n\leqq x} f(n)$ does not exist.

3. In our Corollary, it is shown that $f(n)$ must be n^λ; so the function $g(x)$ is also almost determined. In fact, we have

$$\sum_{n\leqq x} f(n)\sim\begin{cases}\dfrac{1}{1+\lambda}x^{1+\lambda} & \text{if} \quad \lambda>-1,\\ \log x & \text{if} \quad \lambda=-1,\end{cases}$$

therefore $g(x)$ must be

$$g(x)\sim C_1 x^{1+\lambda} \quad \text{or} \quad g(x)\sim C_2\log x,$$

because $g(x)$ satisfies III').

2. Some lemmas

LEMMA 1. *Suppose $t(n)$ is a positive, non-decreasing multiplicative function. Then $t(n)=n^\lambda$ for some $\lambda\geqq 0$.*

PROOF. Since $\log(t(n))$ is additive and non-decreasing, the above result of Erdös [1] gives Lemma 1.

LEMMA 2. *Let $f(n)$ be a multiplicative function and $g(x)$ be a positive, non-decreasing function satisfying conditions* I) *and* II) *of our Theorem. Then we have, for any $d\in N^*$,*

$$\sum_{\substack{n\leqq x\\ d|n}} f(n)-\frac{1}{d}\sum_{n\leqq x} f(n) = o(g(x)).$$

PROOF. We begin by proving that

$$f([x]) = o(g(x)). \tag{2.1}$$

In fact, for $x\geqq 2$,

$$f([x])-f(1)\leqq\Big|\sum_{n\leqq x-1}\{f(n+1)-f(n)\}\Big|\leqq\sum_{n\leqq x}|f(n+1)-f(n)| = o(g(x)).$$

So $f([x])=f(1)+o(g(x))=o(g(x))$, since $\lim_{x\to\infty} g(x)=\infty$. Now we remark that, for $d\geqq 2$, $d\in N$:

$$\frac{1}{d}-\left(\left\{\frac{n}{d}\right\}-\left\{\frac{n-1}{d}\right\}\right)=\begin{cases}1 & \text{if } d|n,\\ 0 & \text{otherwise,}\end{cases}$$

where $\{z\}$ denotes the fractional part of z. Making use of this relation, we have

$$\sum_{\substack{n\leqq x\\ d|n}} f(n)=\sum_{n\leqq x}\left(\frac{1}{d}-\left(\left\{\frac{n}{d}\right\}-\left\{\frac{n-1}{d}\right\}\right)\right)f(n)=$$

$$=\frac{1}{d}\sum_{n\leqq x} f(n)-\sum_{n\leqq x}\left(\left\{\frac{n}{d}\right\}-\left\{\frac{n-1}{d}\right\}\right)f(n)=$$

$$=\frac{1}{d}\sum_{n\leqq x} f(n)-\left\{\frac{[x]}{d}\right\}f([x])+\sum_{n\leqq x-1}\left\{\frac{n}{d}\right\}(f(n+1)-f(n)).$$

Therefore

$$\left|\sum_{\substack{n\leqq x\\ d|n}} f(n)-\frac{1}{d}\sum_{n\leqq x} f(n)\right|\leqq |f([x])|+\sum_{n\leqq x}|f(n+1)-f(n)|=o(g(x)),$$

by (2.1) and II).

From now on, we shall use the notation

$$F(x)=\Omega(g(x)) \quad \text{for} \quad \limsup_{x\to\infty}\frac{|F(x)|}{g(x)}>0.$$

LEMMA 3. *Let* $a\in N^*$ *and* $b\in Z$.

a) *If* $f(n)$ *and* $g(x)$ *satisfy conditions* I) *and* II) *of the Theorem, then*

$$\sum_{\substack{n\leqq x\\ an+b\in N}} |f(an+b+1)-f(an+b)|=o(g(x)).$$

b) If $f(n)$ *and* $g(x)$ *satisfy conditions* I), II) *and* III) *of the Theorem, then*

$$\left|\sum_{n\leqq x} f(an+1)\right|=\Omega(g(x)).$$

PROOF OF a). Here we notice that, from the condition $g(2x)=O(g(x))$, we can deduce:

$$g(dx)=O(g(x)) \quad \text{for any} \quad d\in N^*, \tag{2.2}$$

where the "large O" depends on d. Now, we have for x large enough

$$g(x)^{-1}\sum_{\substack{n\leqq x\\ an+b\in N}} |f(an+b+1)-f(an+b)|\leqq g(x)^{-1}\sum_{n\leqq(a+1)x} |f(n+1)-f(n)|\leqq$$

$$\leqq\frac{g((a+1)x)}{g(x)}\left\{g((a+1)x)^{-1}\sum_{n\leqq(a+1)x}|f(n+1)-f(n)|\right\}=o(1).$$

PROOF OF b).

$$(2.3)\qquad g(x)^{-1}\Big|\sum_{n\leqq x} f(an+1)\Big| = g(x)^{-1}\Big|\sum_{n\leqq x}\{(f(an+1)-f(an))+f(an)\}\Big| \geqq$$

$$\geqq \Big|g(x)^{-1}\Big|\sum_{n\leqq x} f(an)\Big| - g(x)^{-1}\Big|\sum_{n\leqq x}(f(an+1)-f(an))\Big|\Big|.$$

We have

$$g(x)^{-1}\Big|\sum_{n\leqq x}(f(an+1)-f(an))\Big| \leqq g(x)^{-1}\sum_{n\leqq x}|f(an+1)-f(an)| = o(1)$$

by a). So (2.3) yields

$$(2.4)\qquad \limsup_{x\to\infty} g(x)^{-1}\Big|\sum_{n\leqq x} f(an+1)\Big| \geqq \limsup_{x\to\infty} g(x)^{-1}\Big|\sum_{n\leqq x} f(an)\Big|.$$

On the other hand, Lemma 2, where d and x are replaced by a and ax, gives

$$\Big|\sum_{\substack{m\leqq ax\\ a|m}} f(m) - \frac{1}{a}\sum_{m\leqq ax} f(m)\Big| = o(g(ax)) = o(g(x))$$

by (2.2), that is

$$\sum_{n\leqq x} f(an) = \frac{1}{a}\sum_{n\leqq ax} f(n) + o(g(x)).$$

Thus

$$g(x)^{-1}\Big|\sum_{n\leqq x} f(an)\Big| = \frac{1}{a}\frac{g(ax)}{g(x)}\frac{1}{g(ax)}\Big|\sum_{n\leqq ax} f(n)\Big| + o(1).$$

Since $g(x)$ is non-decreasing, $g(ax)/g(x)\geqq 1$. Therefore,

$$\limsup_{x\to\infty} g(x)^{-1}\Big|\sum_{n\leqq x} f(an)\Big| \geqq \frac{1}{a}\limsup_{x\to\infty}\frac{1}{g(x)}\Big|\sum_{n\leqq x} f(n)\Big|.$$

So (2.4) and III) give the desired result.

3. Proof of the theorem

LEMMA 4. *Suppose $f(n)$ and $g(x)$ satisfy the assumptions* I), II) *and* III) *of our Theorem. Then $f(n)\neq 0$ for any $n\in N^*$.*

PROOF. Let p be a prime number and $r\in N^*$. Then,

$$(3.1)\qquad \Big|f(p^r)\sum_{\substack{n\leqq xp^{-r}\\ (n,p)=1}} f(n) - \Big(1-\frac{1}{p}\Big)p^{-r}\sum_{n\leqq x} f(n)\Big| =$$

$$= \Big|\sum_{\substack{n\leqq x\\ p^r|n}} f(n) - \sum_{\substack{n\leqq x\\ p^{r+1}|n}} f(n) - p^{-r}\sum_{n\leqq x} f(n) + p^{-r-1}\sum_{n\leqq x} f(n)\Big| \leqq$$

$$\leqq \Big|\sum_{\substack{n\leqq x\\ p^r|n}} f(n) - p^{-r}\sum_{n\leqq x} f(n)\Big| + \Big|\sum_{\substack{n\leqq x\\ p^{r+1}|n}} f(n) - p^{-(r+1)}\sum_{n\leqq x} f(n)\Big| = o(g(x))$$

by Lemma 2. This, with condition III), yields

$$f(p^r) \sum_{\substack{n \leqq x p^{-r} \\ (n,p)=1}} f(n) = \Omega(g(x)).$$

Thus $f(p^r)$ cannot be equal to 0 for any p and any r.

LEMMA 5. *Under the assumptions of Lemma* 4, $f(n)$ *is completely multiplicative.*

PROOF. We shall prove this lemma by the same method as in [4].
Let q be an even positive integer, k an integer $\geqq 1$, and set

$$S_k(q) = \sum_{r=0}^{k-1} q^r.$$

It is easy to verify that, for $k \geqq 2$,

$$q^k m + S_k(q) - 1 = q\{q^{k-1} m + S_{k-1}(q)\}$$

and q and $\{q^{k-1}m + S_{k-1}(q)\}$ are relatively prime for any $m \in N^*$. So, for $k \geqq 2$,

$$f(q^k m + S_k(q)) - f(q^k m + S_k(q) - 1) =$$

$$= f(q^k m + S_k(q)) - f(q) f(q^{k-1} m + S_{k-1}(q))$$

and Lemma 3-a) shows that

$$\sum_{m \leqq x} \left| f(q^k m + S_k(q)) - f(q) f(q^{k-1} m + S_{k-1}(q)) \right| = o(g(x)). \tag{3.2}$$

Making use of this relation, we can prove that, for $k \geqq 2$,

$$\sum_{m \leqq x} \left| f(q^k m + S_k(q)) - f(q)^{k-1} f(qm) \right| = o(g(x)). \tag{3.3}$$

In fact

$$\sum_{m \leqq x} \left| f(q^k m + S_k(q)) - f(q)^{k-1} f(qm) \right| \leqq \sum_{m \leqq x} \left| f(q^k m + S_k(q)) - f(q)^{k-1} f(qm+1) \right| +$$

$$+ \sum_{m \leqq x} |f(q)^{k-1}| |f(qm+1) - f(qm)|,$$

where we have

$$\sum_{m \leqq x} \left| f(q^k m + S_k(q)) - f(q)^{k-1} f(qm+1) \right| =$$

$$= \sum_{m \leqq x} \left| \sum_{j=2}^{k} f(q)^{k-j} \{ f(q^j m + S_j(q)) - f(q) f(q^{j-1} m + S_{j-1}(q)) \} \right| \leqq$$

$$\leqq \sum_{m \leqq x} \sum_{j=2}^{k} |f(q)|^{k-j} \left| f(q^j m + S_j(q)) - f(q) f(q^{j-1} m + S_{j-1}(q)) \right| \leqq$$

$$\leqq \left\{ \max_{2 \leqq j \leqq k} |f(q)|^{k-j} \right\} \sum_{j=2}^{k} \sum_{m \leqq x} \left| f(q^j m + S_j(q)) - f(q) f(q^{j-1} m + S_{j-1}(q)) \right| = o(g(x))$$

Then Lemma 2, where x is replaced by dx, gives

$$f(d)\,g(x)\,M(x)-\frac{1}{d}\,g(dx)\,M(dx)=o\big(g(dx)\big),$$

so that, by I') and III'), $\mu\left\{f(d)-\dfrac{h(d)}{d}\right\}=0$. So $f(d)=\dfrac{h(d)}{d}=d^{\varkappa-1}$ by Lemma 6.

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LANDAU-TYPE INEQUALITIES FOR FUNCTIONS DEFINED ON A BOUNDED INTERVAL. APPLICATION TO FUNCTIONS OF SEVERAL VARIABLES

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1. Inequalities on a bounded interval

Let $a_1, \ldots, a_d$ be d points in $[-1, 1]$, $p \geqq d$, $\sigma \in [0, 1]$ and $g \in C^p([-1, 1])$ be a function vanishing at $a_1, \ldots, a_d$. We prove that one can give an upper bound for the uniform norm of the quotient of g by $[|x-a_1| \ldots |x-a_d|]^{1-\sigma}$, depending only on g, d and σ but not on the positions of the a_i's in $[-1, 1]$.

1.1. Introduction and notation

We denote by $\mathbf{N}^*$ the set of strictly positive integers.

Let K be a bounded interval. For any $p \in \mathbf{N}^*$ and $\alpha \in [0, 1]$ satisfying $p+\alpha \geqq 1$, we denote by $C^{p,\alpha}(K)$ the set of restrictions to K of real-valued functions defined on $\mathbf{R}$, possessing a derivative $f^{(p)}$, continuous if $\alpha = 0$ and satisfying a Lipschitz condition with exponent α if $\alpha \neq 0$.

For every continuous function g we set: $\|g\|_K = \sup_{x \in K} |g(x)|$, and for every function $g \in C^{p,\alpha}(K)$ we set

$$M_{p,\alpha,K}(g) = \begin{cases} \|g^{(p)}\|_K & \text{if } \alpha = 0 \\ \sup\limits_{x,y \in K} |x-y|^{-\alpha} |g^{(p)}(x) - g^{(p)}(y)| & \text{if } \alpha \neq 0. \end{cases}$$

Let K be a bounded interval of length l. Let $a_1, \ldots, a_d$ be a sequence of (not necessarily distinct) points in K, m be the function defined by $m(x) = (x-a_1) \ldots (x-a_d)$ and c be a given positive constant.

The aim of this first part is to establish

THEOREM 1. *For any $p \in \mathbf{N}^*$ such that $p \geqq d$ and $\alpha \in [0, 1]$, there exist two positive constants A and B (depending on c, d, p, α; but independent of $l, a_1, \ldots, a_d$) such that: for every continuous function f such that $fm \in C^{p,\alpha}(K)$, for every function $u \in C^{p,\alpha}(K)$ satisfying $|u(x)| > c$ $(x \in K)$ and for every $\sigma \in [0, 1[$, we have either*

$$\|f|m|^\sigma\|_K \leqq A(2/l)^{d(1-\sigma)} \|fmu\|_K$$

or

$$\|f|m|^\sigma\|_K \leqq B \|fmu\|^{1-[d(1-\sigma)/(p+\alpha)[} \left(M_{p,\alpha,K}(fmu)\right)^{d(1-\sigma)/(p+\alpha)}.$$

and thus show that there exist a constant C_6 and a sequence $(P_n)_{n>p}$ of polynomials satisfying $P_n \in H_{n-d}$ such that, for $n>p$, $\|f-P_n\|_I \leqq C_6 n^{d-p-\alpha} M_{p,\alpha,I}(g)$ and $\|P_n\|_I \leqq (4n)^d \|Q_n\|_I$. Therefore we have

$$\|f\|_I \leqq \|f-P_n\|_I + \|P_n\|_I \leqq C_6 n^{d-p-\alpha} M_{p,\alpha,I}(g) + (4n)^d \|Q_n\|_I$$

and

$$\|Q_n\|_I \leqq \|g-Q_n\|_I + \|g\|_I \leqq C_4 n^{-p-\alpha} M_{p,\alpha,I}(g) + \|g\|_I;$$

which yields

$$\|f\|_I \leqq C_6 n^{d-p-\alpha} M_{p,\alpha,I}(g) + (4n)^d [C_4 n^{-p-\alpha} M_{p,\alpha,I}(g) + \|g\|_I] \leqq$$

$$\leqq C n^d [n^{-p-\alpha} M_{p,\alpha,I}(g) + \|g\|_I]$$

where $C = \operatorname{Max}(C_6 + 4^d C_4, 4^d)$.

2nd *case:* $p+\alpha \leqq d$. The hypotheses of the theorem then imply $p=d, \alpha=0$. Let $a \in I$ and $h(x)=(x-a)f(x)$ $(x \in I)$. If $h \in C^{p,0}(I)$ we have

$$f(x) = \int_0^1 h'(a+t(x-a))\,dt \quad \text{and} \quad f^{(k)}(x) = \int_0^1 t^k h^{(k+1)}(a+t(x-a))\,dt \quad (k<p)$$

therefore

$$\|f^{(k)}\|_I \leqq \frac{1}{k+1} \|h^{(k+1)}\|_I, \tag{5}$$

which implies

$$\|f\|_I \leqq \|[(x-a_1)f]'\|_I \leqq \frac{1}{2} \|[(x-a_1)(x-a_2)f]''\|_I \leqq \ldots \leqq \frac{1}{d!} \|(fm)^{(d)}\|_I,$$

whence inequality (4) easily follows. (We note that the second inequality of Theorem 1 is now established in the case $p=d, \alpha=0$.)

We suppose now $\sigma \neq 0$. For every x such that $|m(x)| \geqq n^{-d}$ we have

$$|m(x)f(x)| = |m(x)|^\sigma |f(x)|\,|m(x)|^{1-\sigma} \geqq n^{-d(1-\sigma)} |m(x)|^\sigma |f(x)|$$

thus

$$|m(x)|^\sigma |f(x)| \leqq n^{d(1-\sigma)} \|g\|_I.$$

For every x such that $|m(x)| \leqq n^{-d}$ we have

$$|m(x)|^\sigma |f(x)| < n^{-d\sigma} \|f\|_I$$

and using (4) in the case $\sigma=0$

$$|m(x)|^\sigma |f(x)| \leqq C n^{d(1-\sigma)} [n^{-p-\alpha} M_{p,\alpha,I}(g) + \|g\|_I].$$

Given that $C>1$ it follows that

$$\|f\,|m|^\sigma\|_I \leqq C n^{d(1-\sigma)} [n^{-p-\alpha} M_{p,\alpha,I}(g) + \|g\|_I],$$

which proves Lemma 5.

1.4. Proof of Theorem 1

First, we note that an affine change of variable allows us to prove the theorem only in the case $K=I$, $l=2$. If $f\equiv 0$ the statement is trivial. If $f\not\equiv 0$ we have $\|fmu\|_I\neq \neq 0$. We distinguish two cases.

1st *case*: $(M_{p,\alpha,I}(fmu)/\|fmu\|_I)^{1/p+\alpha}\leqq p$. Then

$$M_{p,\alpha,I}(fmu)\leqq p^{p+\alpha}\|fmu\|_I$$

and inequality (3) with $n=p+1$ gives $\|f|m|^\sigma\|_I\leqq A\|fmu\|_I$.

2nd *case*: $(M_{p,\alpha,I}(fmu)/\|fmu\|_I)^{1/p+\alpha}>p$. Let n be an integer satisfying

$$(M_{p,\alpha,I}(fmu)/\|fmu\|_I)^{1/p+\alpha}\leqq n<(M_{p,\alpha,I}(fmu)/\|fmu\|_I)^{1/p+\alpha}+1.$$

Then we have

$$n<2(M_{p,\alpha,I}(fmu)/\|fmu\|_I)^{1/p+\alpha}.$$

On the other hand

$$n^{-p-\alpha}M_{p,\alpha,I}(fmu)\leqq\|fmu\|_I.$$

Substituting in (3) yields

$$\|f|m|^\sigma\|_I\leqq C2^{d(1-\sigma)}(M_{p,\alpha,I}(fmu)/\|fmu\|_I)^{d(1-\sigma)/(p+\alpha)}2\|fmu\|_I=$$
$$=B(M_{p,\alpha,I}(fmu))^{d(1-\sigma)/(p+\alpha)}\|fmu\|_I^{1-[d(1-\sigma)/(p+\alpha)]}.$$

Theorem 1 is now proved.

2. Inequalities in several variables

In the second part of this article, we shall state results analogous to Theorem 1, for three special cases, for functions of several variables. The process consists in covering the sets where the functions are defined by segments on which we shall apply the results of the first part.

Let F be a closed bounded subset of $\mathbf{R}^N$. Here we denote by $C^p(F)$ the set of restrictions to F of functions belonging to $C^p(\mathbf{R}^N)$. For any subset E of $\mathbf{R}^N$ and any real function f defined on E we put $\|f\|_E=\sup_{x\in E}|f(x)|$.

To make things clearer, we shall take $\sigma=0$ though results could be stated as above when $\sigma\in[0, 1[$.

2.1. The first example

PROPOSITION 1. *Let* $p\geqq 1$, $F=I\times I$, $w(x, y)=y-x^2$. *There exist two constants* A_1 *and* B_1 *such that, for any* $f\in C^p(F)$ *either*

$$\|f\|_F\leqq A_1\|fw\|_F$$

or

$$\|f\|_F\leqq B_1\|fw\|_F^{1-1/p}\left\|\left(\frac{\partial}{\partial y}\right)^p(fw)\right\|_F^{1/p}.$$

PROOF. 1. For given x and y satisfying $x^2+y^2\leqq 1$ we set $S(x, y)=\{(x, y, z) |z\in I\}$; Putting $x^2+x^2=a^2$ we have $w(x, y, z)=a^2-z^2=(a-z)(a+z)$ with $a\in I$. Applying Lemma 5 on the segment $S(x, y)$ shows that there exists a constant C_{11} such that for every $f\in C^p(F)$ and any $n>p$

$$\|f\|_{S(x,y)} \leqq C_{11} n^2 \left[n^{-p} \left\| \left(\frac{\partial}{\partial z} \right)^p (fw) \right\|_{S(x,y)} + \|fw\|_{S(x,y)} \right].$$

The constant C_{11} being independent of x and y, we can replace $S(x, y)$ by $F_1=\{(x, y, z)|x^2+y^2\leqq 1,\ z\in I\}$ in the last estimate.

2. We set $F_2=\{(x, y, z)\in F||z|\geqq 1/2\}$. We shall use cylindrical coordinates by putting $x=r\cos\theta$, $y=r\sin\theta$ ($r>0$, $\theta\in[0, 2\pi[$).

For given θ and b satisfying $|b|\geqq 1/2$, let $S(\theta, b)$ be the segment $\{(r\cos\theta, r\sin\theta, b)|r>0\}\cap F$. The length of $S(\theta, b)$ varies between 1 and $\sqrt{2}$. We put $g(r, \theta, z)=f(r\cos\theta, r\sin\theta, z)$. On $S(\theta, b)$, $w(x, y, z)=r^2-b^2=(r-b)(r+b)=$ $=(r-b)u(r)$ with $u(r)>b\geqq 1/2$. Applying Lemma 5 on $S(\theta, b)$ shows that there exists a constant C_{12} such that for every function $r\to g(r, \theta, z)$ and any $n>p$

$$\|g\|_{S(\theta,b)} \leqq C_{12} n \left[n^{-p} \left\| \left(\frac{\partial}{\partial r} \right)^p [(r^2-b^2)g] \right\|_{S(\theta,b)} + \|(r^2-b^2)g\|_{S(\theta,b)} \right].$$

But $\dfrac{\partial}{\partial r}=\cos\theta\dfrac{\partial}{\partial x}\theta+\sin\theta\dfrac{\partial}{\partial y}$; so there exists a constant C_{13} independent of θ and b such that

$$\|f\|_{S(\theta,b)} \leqq C_{13} n \left[n^{-p} \sum_{|s|=p} \|D^s(fw)\|_{S(\theta,b)} + \|fw\|_{S(\theta,b)} \right]$$

is satisfied, and one can replace $S(\theta, b)$ by F_2 in this estimate.

3. Let $F_3=F\setminus(F_1\cup F_2)$. If $(x, y, z)\in F_3$, $w(x, y, z)=x^2+y^2-z^2>3/4$ whence $\|f\|_{F_3}\leqq 4/3\|fw\|_{F_3}$.

Estimates on F_1, F_2 and F_3 together give

$$\|f\|_F \leqq C_{14} n^2 \left[n^{-p} \sum_{|s|=p} \|D^s(fw)\|_F + \|fw\|_F \right],$$

for some constant C_{14}.

We complete the proof as in Theorem 1.

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A NOTE ON S-CLOSED SPACES

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1. Introduction

In [6], O. Njåstad has first introduced and investigated the concept of α-sets in a topological space. The collection of all α-sets with respect to a topology on a set, is again a topology on that set, which is nothing but the finest topology in the equivalence class of all topologies on that set having the same semi-open sets as the given topology [2]. In [1], Cameron has first introduced the concept of I-compact spaces and has established that I-compact spaces are precisely the S-closed extremely disconnected spaces. In this note, we improve upon some results of T. Noiri [11] using α-sets and I-compact spaces. In the preliminary chapter, we state and prove some results about semi-open sets and α-sets, which we need in Sections 3 and 4. In § 3, we shall discuss the restrictions of semi-continuous, irresolute and semi-homeomorphic functions to an α-set. Also we shall establish that the inverse image of an S-closed space under an almost open, pre-semi-open bijection is S-closed. In § 4, we shall characterize I-compact spaces in terms of semiopen sets. As an improvement of some results of T. Noiri [11], we shall establish the following: (i) An irresolute image of a relative I-compact subset is relatively S-closed. (ii) Every semi-continuous mapping from an I-compact space into a Hausdorff space is almost closed. (iii) Every semi-continuous bijection from an I-compact space onto a Hausdorff space is an irresolute.

2. Preliminaries

Let (X, τ) be a topological space and let A be a subset of X. We will denote the relative topology on A by $\tau|A$. The closure and interior of A will be denoted by $C(A)$ (or $C_X(A)$) and $I(A)$ (or $I_X(A)$) respectively. A is said to be *regular open (regular closed)* if $A=I(C(A))$ $(A=C(I(A)))$. The space (X, τ) is said to be *extremely disconnected* if the closure of every open set is open. A subset A of X is said to be *semi-open*, [5], if there exists an open set G such that $G \subset A \subset C(G)$. The family of all semi-open sets will be denoted by $SO(X, \tau)$ or $SO(X)$. The complement of a semi-open set is *semi-closed*. The intersection of all semi-closed sets containing A is called the *semi-closure of A*, [2], and is denoted by $sC(A)$ (or $sC_X(A)$). A subset A of X is an *α-set*, [6], if $A \subset A^+$ where $A^+=I(C(I(A)))$. If τ^α is the family of all α-sets, then $\tau \subset \tau^\alpha \subset SO(X, \tau)$, τ^α is a topology on X, and $SO(X, \tau)=SO(X, \tau^\alpha)$. If $T(X)$ is the lattice of topologies on a set of points X, for $\tau \in T(X)$, if $[\tau]$ is the equivalence class of topologies which yield the same semi-open sets as τ, then there is a finest element of $[\tau]$, denoted by $F(\tau)$ [2]. By Proposition 4 of [6] and Theorem 2 of [4], $F(\tau)=\tau^\alpha$. Hereafter in this paper, we shall denote the space (X, τ) simply by X and the space $(X, F(\tau))$ by X^*. If $A \subset X$, $C^*(A)$ and $I^*(A)$ will denote the closure and

COROLLARY 3.7. *Let $f: X \to Y$ be a semi-homeomorphism and G be open in X^*. If G is connected in X, then $f(G)$ is connected in Y.*

PROOF. Since connectedness is a semi-topological property [3, Theorem 2.12], the proof follows from Theorem 3.6.

The following theorem is a generalization of Lemmas 2.14 and 2.15 of [3].

THEOREM 3.8. *In a space X, if A is semi-open, then A is connected in X if and only if A is connected in X^*.*

PROOF. If $A \in SO(X, \tau)$, by Lemma 2.5, $SO(A, \tau|A) = SO(A, F(\tau)|A)$. If $i: X \to X^*$ is the identity function, then $i_A: A \to i(A) = A$ is a semi-homeomorphism. Hence, if A is connected in X, $i_A(A) = A$ is connected in X^*. Similarly we can prove the converse.

A function $f: X \to Y$ is *almost open*, [18], if $f^{-1}(C_Y(V)) \subset C_X(f^{-1}(V))$ for each open set V in Y.

LEMMA 3.9. *A function $f: X \to Y$ is almost open if and only if $f^{-1}(C_Y(V)) \subset C_X(f^{-1}(V))$ for each semi-open set V in Y.*

PROOF. Suppose f is almost open. If V is semi-open in Y, then there exists an open set G in Y such that $G \subset V \subset C_Y(G)$. Now $f^{-1}(C_Y(V)) = f^{-1}(C_Y(G)) \subset C_X(f^{-1}(G)) \subset C_X(f^{-1}(V))$. Since open sets are semi-open, the converse is immediate.

THEOREM 3.10. *If a function $f: X \to Y$ is an almost open, pre-semi-open bijection, and Y is S-closed, then X is S-closed.*

PROOF. If $\{V_\beta | \beta \in \nabla\}$ is a semi-open cover of X, then $\{f(V_\beta) | \beta \in \nabla\}$ is a semi-open cover of Y. Since Y is S-closed, $Y = \cup \{C_Y(f(V_\beta)) | \beta \in \nabla_0\}$ for some finite subfamily ∇_0 of ∇. Then $X = \cup \{f^{-1}(C_Y(f(V_\beta))) | \beta \in \nabla_0\} = \cup \{C_X(V_\beta) | \beta \in \nabla_0\}$, which implies that X is S-closed.

COROLLARY 3.11. *If a function $f: X \to Y$ is an almost open, pre-semi-open bijection and G is S-closed relative to Y, then $f^{-1}(G)$ is S-closed relative to X.*

A space X is *quasi-H-closed (QHC)*, [14], if every open cover of X has a finite subfamily whose closures cover X. The following theorem, the proof of which is similar to Theorem 3.10, is a modification of Theorem 2.6 of [9].

THEOREM 3.12. *If a function $f: X \to Y$ is an almost open, semi-open bijection and Y is S-closed, then X is QHC.*

4. I-compact spaces

A space X is *I-compact*, [1], if every cover of X by regular closed sets has a finite subfamily whose interiors cover X. A subset A of a space X is *I-compact relative to X*, if every cover of A by regular closed subsets of X has a finite subfamily whose interiors cover A. In [1], it was established that I-compact spaces are precisely the S-closed spaces which are extremely disconnected and the regular closed subsets of an I-compact space X are I-compact relative to X. In this section, we shall give one charac-

terization of I-compact spaces in terms of semi-open sets and generalize some results of T. Noiri [11].

THEOREM 4.1. *A space X is I-compact if and only if every semi-open cover of X has a finite subfamily, the interior of whose closures cover X.*

PROOF. If $\{V_\beta|\beta\in\nabla\}$ is a semi-open cover of the I-compact space X, then $\{C(V_\beta)|\beta\in\nabla\}$ is a regular closed cover of X. Therefore, $X=\cup\{I(C(V_\beta))|\beta\in\nabla_0\}$ for some finite subfamily ∇_0 of ∇. Since regular closed sets are semi-open the converse is immediate.

COROLLARY 4.2. *A subset A of a space X is I-compact relative to X if and only if every cover of A by semi-open sets of X has a finite subfamily, the interiors of whose closures cover A.*

In a space X, it is easy to see that, if A is I-compact relative to X, then A is S-closed relative to X but not the converse. In an extremely disconnected space X, if A is S-closed relative to X, then A is I-compact relative to X. Since it is so, the following theorem is more general than that of Theorem 4.2 of [11].

THEOREM 4.3. *If a function f: $X\to Y$ is irresolute and $A\subset X$ is I-compact relative to X, then $f(A)$ is S-closed relative to Y.*

PROOF. Let $\{V_\beta|\beta\in\nabla\}$ be a cover of $f(A)$ where each V_β is semi-open in Y. Then $\{f^{-1}(V_\beta)|\beta\in\nabla\}$ is a cover of A by semi-open sets of X. Since A is I-compact relative to X, $A\subset\cup\{I_X(C_X(f^{-1}(V_\beta)))|\beta\in\nabla_0\}$ for some finite subfamily ∇_0 of ∇, and so by Corollary 2.2 and Lemma 1.1 of [4], $A\subset\cup\{sC_X(f^{-1}(V_\beta))|\beta\in\nabla_0\}$. Since f is irresolute, $f(A)\subset\cup\{sC_Y(V_\beta)|\beta\in\nabla_0\}\subset\cup\{C_Y(V_\beta)|\beta\in\nabla_0\}$, which implies that $f(A)$ is S-closed relative to X.

COROLLARY 4.4. *If a function f: $X\to Y$ is irresolute, Y is extremely disconnected and $A\subset X$ is I-compact relative to X, then $f(A)$ is I-compact relative to Y.*

COROLLARY 4.5. *If a function f: $X\to Y$ is semi-open and semi-continuous, Y is extremely disconnected, and $A\subset X$ is I-compact relative to X, then $f(A)$ is I-compact relative to Y.*

PROOF. By Theorem 1.14 of [12], f is irresolute and hence the proof follows from Corollary 4.4.

A function f: $X\to Y$ is said to be *semi-closed*, [8], (*almost closed* [15]) if $f(F)$ is semi-closed (closed) for every closed (regular closed) subset F of X. The following theorem is a generalization of Theorem 4.7 of [11].

THEOREM 4.6. *Let f: $X\to Y$ be a semi-closed almost open surjection and $f^{-1}(y)$ be I-compact relative to X for each $y\in Y$. If G is S-closed relative to Y, then $f^{-1}(G)$ is S-closed relative to X.*

PROOF. Let $\{V_\beta|\beta\in\nabla\}$ be a cover of $f^{-1}(G)$ where each V_β is semi-open in X. For each $y\in G$, since $f^{-1}(y)$ is I-compact relative to X, there exists a finite subfamily $\nabla(y)$ of ∇ such that $f^{-1}(y)\subset\cup\{I_X(C_X(V_\beta))|\beta\in\nabla(y)\}$. Since f is a semi-closed surjection, there exists a semi-open set G_y in Y such that $y\in G_y$ and $f^{-1}(G_y)\subset$ $\subset\cup\{I_X(C_X(V_\beta))|\beta\in\nabla(y)\}$ [8, Theorem 5]. Then $\{G_y|y\in G\}$ is a semi-open cover

of G and hence, $G \subset \bigcup_{i=1}^{n} \{C_Y(G_{y_i})\}$ for some finite positive integer n. Then, $f^{-1}(G) \subset$ $\subset \bigcup_{i=1}^{n} f^{-1}(C_Y(G_{y_i})) \subset \bigcup_{i=1}^{n} \{C_X(f^{-1}(G_{y_i}))\}$ by Lemma 3.9. Hence $f^{-1}(G) \subset$ $\subset \cup \{C_X(V_\beta) | \beta \in \nabla(y_i), 1 \leqq i \leqq n\}$, which implies that $f^{-1}(G)$ is S-closed relative to X.

COROLLARY 4.7. *If a function f: $X \to Y$ is a semi-closed, almost open surjection, $f^{-1}(y)$ is I-compact relative to X for each $y \in Y$ and Y is S-closed, then X is S-closed.*

Since every S-closed Hausdorff space is extremely disconnected [16], every S-closed Hausdorff space is an I-compact space but an I-compact space need not be a Hausdorff space (e.g., an uncountable set with the cocountable topology). With this in mind, we have the following theorem which is a generalization of Theorem 5.3 of [11].

THEOREM 4.8. *If a function f: $X \to Y$ is semi-continuous, X is I-compact and Y s Hausdorff, then f is almost closed.*

PROOF. If R is regular closed in X, then R is open and S-closed relative to X and so, R is an S-closed subspace of X. Since f_R: $R \to f(R)$ is semi-continuous, $f(R)$ is H-closed in Y [17, Theorem 3.2]. Since Y is Hausdorff, $f(R)$ is closed in Y and hence f is almost closed.

The following corollary shows that the condition "S-closed Hausdorff" on the space in Corollary 5.5 of [11], can be replaced by "I-compact".

COROLLARY 4.9. *If X is I-compact, Y is Hausdorff and f: $X \to Y$ is a semi-continuous bijection, then f is irresolute and Y is I-compact.*

PROOF. Let V be semi-open in Y. Since f is semi-continuous, $f^{-1}(I_Y(V))$ is semi-open in X. If $F = C_X(f^{-1}(I_Y(V)))$, then F is regular closed in X and $f^{-1}(I_Y(V)) \subset F$. By Theorem 4.8, $f(F)$ is closed and so, $C_Y(V) = C_Y(I_Y(V)) \subset f(F)$ which implies that $f^{-1}(C_Y(V)) \subset F$. Then $f^{-1}(I_Y(V)) \subset f^{-1}(V) \subset f^{-1}(C_Y(V)) \subset$ $\subset F = C_X(f^{-1}(I_Y(V)))$ which implies that $f^{-1}(V)$ is semi-open in X [5, Theorem 3]. Therefore, f is irresolute and hence, Y is I-compact.

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CHOVER'S LAW OF THE ITERATED LOGARITHM AND WEAK CONVERGENCE

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1. Introduction

Let $(X_n, n \geqq 1)$ be a sequence of independent random variables (r.v.) with a common distribution function (d.f.) F. Assume that $EX_1 = 0$ whenever it exists. Define $H(x) = \int_{-x}^{x} y^2 dF(y)$. When F is symmetric stable with exponent α, $0 < \alpha < 2$, J. Chover [1] proved that

$$P\left\{\limsup_{n\to\infty} |n^{-1/\alpha} S_n|^{\frac{1}{\log\log n}} = e^{1/\alpha}\right\} = 1. \tag{1}$$

We call (1) as Chover's law of the iterated logarithm. When F is standard normal R. P. Pakshirajan and R. Vasudeva [5] observed that

$$P\left\{\limsup_{n\to\infty} |n^{-1/2} S_n|^{\frac{1}{\log\log n}} = 1\right\} = 1.$$

In this paper we assume that there exists a sequence (B_n) of normalizing constants such that $(B_n^{-1} S_n)$ converges weakly to a non degenerate r.v., say Y. We define $T_n = |B_n^{-1} S_n|^{\frac{1}{\log\log n}}$, $n \geqq 3$. When Y is stable with exponent α, $0 < \alpha < 2$, we show that lim sup $T_n = e^{1/\alpha}$ almost surely (a.s.). When the limit r.v. Y is normal then we show that lim sup T_n is a number in $[1, e^{1/2}]$ a.s. Thus we note that as soon as $\{B_n^{-1} S_n\}$ converges weakly to a non degenerate r.v., lim sup T_n is a number in $[1, \infty)$ a.s. (i.e. Chover's form of the law of the iterated logarithm holds). In Remark 1 we present a d.f. which is not in the domain of attraction of any stable law and show that Chover's form of the law of the iterated logarithm holds here. We recall Feller [4] who shows that weak convergence of normalised partial sums is neither necessary nor sufficient for the classical type of law of the iterated logarithm to hold. In Remark 2 we note a characterization between weak convergence of $\{B_n^{-1} S_n\}$ and the Chover's form of the law of the iterated logarithm.

2. Notations and preliminaries

Denote by G_α a stable d.f. with characteristic exponent α, $0 < \alpha \leqq 2$. If there exists a sequence (B_n) of normalising constants such that, for all $x \in (-\infty, \infty)$,

$$\lim_{n\to\infty} P(S_n \leqq x B_n) = G_\alpha(x),$$

then say that $F \in D(\alpha)$, $0 < \alpha \leqq 2$. Let c, with or without a suffix, stand for a positive constant and let i.o. mean 'infinitely often'.

THEOREM 1. (a) *A sequence* (a_n) *of positive constants exists such that* $\limsup (a_n^{-1} S_n)=1$ *a.s. if and only if for some constant* $\lambda>0$,

$$\int_{\lambda}^{\infty} \frac{dH(x)}{H(x)\log\log x} < \infty. \tag{2}$$

Further a_n *can be taken as* $B_n\sqrt{2\log\log B_n}$, *where* B_n *is the largest root of the equation* $B_n^2 = nH(B_n)$.

(b) *In case the integral in* (2) *is divergent then for any sequence* (a_n) *of positive constants* $P(|S_n|>a_n \text{ i.o.})=0$ *or* 1 *according as*

$$P(|X_n| > a_n \text{ i.o.}) = 0 \quad \text{or} \quad 1.$$

For proof see W. Feller [4].

NOTE. When $F\in D(2)$ the sequence (B_n) defined in Theorem 1 is the appropriate normalizing constant for weak convergence also.

3.

THEOREM 2. *Let* $F\in D(\alpha)$, $0<\alpha\leqq 2$. *Then* $\limsup T_n$ *is a number in* $[1, \infty)$ *a.s. Further, for any* $q\in[1, \infty)$ *there corresponds a d.f.* $F\in D(\alpha)$, $0<\alpha\leqq 2$, *such that* $\limsup T_n = q$ *a.s.*

PROOF. In order to establish that $\limsup T_n$ is a number in $[1, \infty)$ we first show that for any $\varepsilon>0$,

$$P(T_n \geqq e^{-\varepsilon} \text{ i.o.}) = 1, \tag{3}$$

whatever be the d.f. F $(F\in D(\alpha))$. Equivalently we establish that

$$P(|S_n| \geqq B_n(\log n)^{-\varepsilon} \text{ i.o.}) = 1.$$

We have

$$P(|S_n| \geqq B_n(\log n)^{-\varepsilon} \text{ i.o.}) = \lim_{n\to\infty} P\Big(\bigcup_{k=n}^{\infty} (|S_k| \geqq B_k(\log k)^{-\varepsilon})\Big) \geqq$$

$$\geqq \lim_{n\to\infty} P(|S_n| \geqq B_n(\log n)^{-\varepsilon}).$$

Since $F\in D(\alpha)$, there exists an integer N such that for all $n\geqq N$,

$$P(|S_n| \geqq B_n(\log n)^{-\varepsilon}) \geqq \frac{1-G_\alpha(1)+G_\alpha(-1)}{2} > 0.$$

Consequently,

$$P(|S_n| \geqq B_n(\log n)^{-\varepsilon} \text{ i.o.}) \geqq \frac{1-G_\alpha(1)+G_\alpha(-1)}{2} > 0. \tag{4}$$

In view of Kolmogorov's zero-one law (4) implies (3). We have just shown that $\limsup T_n \geqq 1$ a.s. In order to claim that $\limsup T_n$ is a number in $[1, \infty)$ it now remains for us to show that $\limsup T_n < \infty$ whenever $F\in D(\alpha)$, $0<\alpha\leqq 2$. As a

first step, when $F\in D(\alpha)$, $0<\alpha<2$, we establish that

$$P(\limsup T_n = e^{1/\alpha}) = 1. \tag{5}$$

Equivalently, we claim that for any $\varepsilon>0$,

$$P\left(|S_n| \geqq B_n(\log n)^{\frac{1+\varepsilon}{\alpha}} \text{ i.o.}\right) = 0 \tag{6}$$

and

$$P\left(|S_n| \geqq B_n(\log n)^{\frac{1-\varepsilon}{\alpha}} \text{ i.o.}\right) = 1. \tag{7}$$

By Feller [2], (6) and (7) hold if $P\left(|X_n|\geqq B_n(\log n)^{\frac{1+\varepsilon}{\alpha}} \text{ i.o.}\right)=0$ and $P\left(|X_n|\geqq\right.$ $\left.\geqq B_n(\log n)^{\frac{1-\varepsilon}{\alpha}} \text{ i.o.}\right)=1$, respectively. Since $F\in D(\alpha)$, $0<\alpha<2$, the above probabilities are immediate by an appeal to the Borel—Cantelli lemma. In order to show that $\limsup T_n<\infty$ a.s. it now remains for us to prove that $\limsup T_n<\infty$ whenever $F\in D(2)$.

We establish this by showing that $\limsup T_n$ is a number in $[1, e^{1/2}]$ a.s. whenever $F\in D(2)$.

For a d.f. $F\in D(2)$ let H satisfy (2). Then we know that

$$P\left\{\limsup \frac{|S_n|}{B_n\sqrt{2\log\log B_n}} = 1\right\} = 1.$$

Equivalently, we know for any $\varepsilon>0$,

$$P\left(|S_n| \geqq (1+\varepsilon)B_n\sqrt{2\log\log B_n} \text{ i.o.}\right) = 0 \tag{8}$$

and

$$P\left((1-\varepsilon)B_n\sqrt{2\log\log B_n} \leqq |S_n| \leqq (1+\varepsilon)B_n\sqrt{2\log\log B_n} \text{ i.o.}\right) = 1. \tag{9}$$

Observing that for any $\varepsilon>0$ we have

$$\left(|S_n| \geqq B_n(\log n)^{\varepsilon}\right) \subset \left(|S_n| \geqq (1+\varepsilon)B_n\sqrt{2\log\log B_n}\right)$$

and

$$\left(B_n(\log n)^{-\varepsilon} \leqq |S_n| \leqq B_n(\log n)^{\varepsilon}\right) \supset$$

$$\supset \left((1-\varepsilon)B_n\sqrt{2\log\log B_n} \leqq |S_n| \leqq (1+\varepsilon)B_n\sqrt{2\log\log B_n}\right)$$

we have

$$P\left(|S_n| \geqq B_n(\log n)^{\varepsilon} \text{ i.o.}\right) = 0 \tag{10}$$

and

$$P\left(B_n(\log n)^{-\varepsilon} \leqq |S_n| \leqq B_n(\log n)^{\varepsilon} \text{ i.o.}\right) = 1. \tag{11}$$

(10) and (11) together imply that

$$P(\limsup T_n = 1) = 1 \tag{12}$$

when H satisfies (2).

For any $F\in D(2)$ with the integral in (2) divergent, we show that lim sup T_n is a number $\leqq e^{1/2}$ a.s. Since lim sup T_n is degenerate, it is enough if we show that lim sup $T_n\leqq e^{1/2}$ a.s.

For an $F\in D(2)$, by Feller [3], page 303, we have

$$\lim_{x\to\infty}\frac{x^2(1-F(x)+F(-x))}{H(x)}=0.$$

Consequently, for any given $\varepsilon>0$, there exists an integer N such that for all $n\geqq N$,

$$P\{|X_n|\geqq B_n(\log n)^{\frac{1+\varepsilon}{2}}\}\leqq\frac{cH\left(B_n(\log n)^{\frac{1+\varepsilon}{2}}\right)}{B_n^2(\log n)^{1+\varepsilon}}\leqq\frac{c_1H(B_n)}{B_n^2(\log n)^{1+\varepsilon/3}}.$$

The fact that $F\in D(2)$ implies that $\frac{nH(B_n)}{B_n^2}\to 1$ as $n\to\infty$. We therefore have

$$P\{|X_n|\geqq B_n(\log n)^{\frac{1+\varepsilon}{2}}\}\leqq\frac{c_2}{n(\log n)^{1+\varepsilon/3}}.$$

By Borel—Cantelli lemma and by Theorem 1 we have

$$P\{|S_n|\geqq B_n(\log n)^{\frac{1+\varepsilon}{2}}\text{ i.o.}\}=0$$

or

$$P(\limsup T_n\leqq e^{1/2})=1. \tag{13}$$

(5), (12) and (13) together enable us to conclude that lim sup T_n is a number in $[1,\infty)$ whenever $F\in D(\alpha)$, $0<\alpha\leqq 2$.

We now proceed to show that for any $q\in[1,\infty)$ there corresponds an $F\in D(\alpha)$, for some $\alpha\in(0,2]$, such that lim sup $T_n=q$ a.s. For $q\in(e^{1/2},\infty)$ we have already observed in (5) that all d.f.'s, $F\in D((\log q)^{-1})$ have lim sup $T_n=q$ a.s. Also, for $q=1$ we have established in (12) that all d.f.'s $F\in D(2)$ with H satisfying (2) have lim sup $T_n=1$ a.s. In order to complete the proof of the theorem it now remains for us to show that for each $q\in(1,e^{1/2}]$ there corresponds a d.f. F $(F\in D(2))$ such that lim sup $T_n=q$ a.s.

Define the density function f of a d.f. F symmetric about zero, by

$$f(x)=\begin{cases}0 & \text{if } 0\leqq|x|<e\\ \dfrac{c\exp(\log|x|)^p}{|x^3|(\log|x|)^{1-p}} & \text{if } |x|\geqq e,\ 0<p<1,\end{cases} \tag{14}$$

where c is such that $\int_{-\infty}^{\infty}f(x)\,dx=1$. Observe that the function $H(x)\sim$ $\sim\frac{2c}{p}\exp((\log x)^p)$, which is slowly varying as $x\to\infty$. Consequently, by Feller [3], Theorem 1a, page 544, $F\in D(2)$. The sequence (B_n) of normalizing constants, on simplification, becomes $\sqrt{n}\exp\left(\frac{1}{2}(\log\sqrt{n})^p\right)$. We now show that lim sup $T_n=$

$=e^{p/2}$ a.s. This follows once we prove that for any $\varepsilon>0$,

$$P\{|S_n| \geqq B_n(\log n)^{\frac{p+\varepsilon}{2}} \text{ i.o.}\} = 0 \tag{15}$$

and

$$P\{|S_n| \geqq B_n(\log n)^{\frac{p-\varepsilon}{2}} \text{ i.o.}\} = 1. \tag{16}$$

Here we have for any $\lambda>e$,

$$\int_{\lambda}^{\infty} \frac{dH(x)}{H(x)\log\log x} = 2c \int_{\lambda}^{\infty} \frac{dx}{x(\log x)^{1-p}\log\log x} = \infty.$$

By Theorem 1(b), (15) and (16) are established once we show that

$$P\{|X_n| \geqq B_n(\log n)^{\frac{p+\varepsilon}{2}} \text{ i.o.}\} = 0$$

and

$$P\{|X_n| \geqq B_n(\log n)^{\frac{p-\varepsilon}{2}} \text{ i.o.}\} = 1.$$

As the function $f(x)$ is regularly varying at ∞, by Feller [3], page 273, we have

$$P(|X_1| \geqq x) = \int_{|y|\geqq x} f(y)\,dy \sim \frac{2c\exp((\log x)^p)}{x^2(\log x)^{1-p}}.$$

Consequently, there exists an integer N such that for all $n \geqq N$,

$$P\{|X_n| \geqq B_n(\log n)^{\frac{p+\varepsilon}{2}}\} \leqq \frac{c_1}{n(\log n)^{1+\varepsilon/2}}$$

and

$$P\{|X_n| \geqq B_n(\log n)^{\frac{p-\varepsilon}{2}}\} \geqq \frac{c_2}{n(\log n)^{1-\varepsilon/2}}.$$

By Borel—Cantelli lemma (15) and (16) are now immediate.

Corresponding to any $q\in(1, e^{1/2})$ we have produced an $F\in D(2)$ such that $\limsup T_n = q$ a.s. Below we give an example of a d.f. $F\in D(2)$ for which $\limsup T_n = e^{1/2}$ a.s. We define the density function f_1 of a d.f. F_1 by

$$f_1(x) = \begin{cases} 0 & \text{if } 0 \leqq |x| < e^2 \\ \dfrac{c_0 \exp\left(\dfrac{\log|x|}{\log\log|x|}\right)}{|x|^3 \log\log|x|}\left(1-\dfrac{1}{\log\log|x|}\right) & \text{if } |x| \geqq e^2, \end{cases}$$

where c_0 is found such that $\int_{-\infty}^{\infty} f_1(x)dx = 1$. On simplification one can show that $H(x) \sim 2c_0 \exp\left(\dfrac{\log x}{\log\log x}\right)$, which is slowly varying at ∞. Hence $F_1 \in D(2)$. Here

one can show that the integral in (2) is ∞. Proceeding as in the previous example one can show that $\limsup T_n = e^{1/2}$ a.s. The details are omitted.

REMARK 1. Theorem 2 establishes a characterization between the weak limit of $\{B_n^{-1}S_n\}$ and $\limsup T_n$. Assuming that $\{B_n^{-1}S_n\}$ converges weakly to a stable r.v. with exponent α, $0<\alpha\leqq 2$, by Theorem 2 we have $\limsup T_n = e^{1/\alpha}$ when $0<\alpha<2$ and $\limsup T_n = p$ for some $p\in[1, e^{1/2}]$ when $\alpha=2$. On the other hand, if one assumes that the weak limit of $\{B_n^{-1}S_n\}$ exists and $\limsup T_n = p$ a.s., then again by Theorem 2 it is clear that the weak limit is a stable r.v., with exponent $(\log p)^{-1}$ if $p\in(e^{1/2}, \infty)$ and is normal if $p\in[1, e^{1/2}]$. Further, when $p\in(1, e^{1/2}]$, it is clear that the r.v. X_1 has infinite variance.

REMARK 2. In Remark 1 we have assumed that the sequence $\{B_n^{-1}S_n\}$ converges weakly and noticed a characterization between the weak limit and $\limsup T_n$. One may rise the question, whether it is necessary at all to assume the weak convergence of $\{B_n^{-1}S_n\}$ for the above characterization to hold. We answer this affirmatively by presenting an example for which $\limsup T_n = e^{1/\sqrt{2}}$ a.s. but the weak convergence of $\{B_n^{-1}S_n\}$ fails for whatever sequence (B_n) of normalizing constants is considered. Even though the construction of such examples is quite simple, we have presented one here, for the sake of completeness.

EXAMPLE. Let F be a d.f. defined by

$$F(-x) = 1 - F(x) = \begin{cases} 1/2 & \text{if } 0 < x \leqq 1 \\ \dfrac{x^{-\sqrt{2}}}{2}\left(1 + \dfrac{1}{12}\sin(\log x)\right) & \text{if } x \geqq 1. \end{cases}$$

Let (X_n) be a sequence of independent r.v.'s with the common d.f. F defined above. Define $B_n = n^{1/\sqrt{2}}$ and $T_n = |B_n^{-1}S_n|^{\frac{1}{\log\log n}}$, $n\geqq 3$. Notice that

$$\frac{11}{12}x^{-\sqrt{2}} \leqq 1 - F(x) + F(-x) \leqq \frac{13}{12}x^{-\sqrt{2}}$$

for all $x>0$. Consequently we have for any $\varepsilon>0$,

$$P\{|X_n| \geqq B_n(\log n)^{\frac{1-\varepsilon}{\sqrt{2}}}\} \geqq \frac{c_1}{n(\log n)^{1-\varepsilon}}$$

and

$$P\{|X_n| \geqq B_n(\log n)^{\frac{1+\varepsilon}{\sqrt{2}}}\} \leqq \frac{c_2}{n(\log n)^{1+\varepsilon}},$$

for all $n\geqq N$ (a positive integer). By Borel—Cantelli lemma and by Feller [2] we can readily conclude that

$$P\{|S_n| \geqq B_n(\log n)^{\frac{1-\varepsilon}{\sqrt{2}}} \text{ i.o.}\} = 1$$

and

$$P\{|S_n| \geqq B_n(\log n)^{\frac{1+\varepsilon}{\sqrt{2}}} \text{ i.o.}\} = 0$$

or

$$P(\limsup T_n = e^{\frac{1}{\sqrt{2}}}) = 1.$$

Proceeding on the lines of Zolotarev and Koralyuk [6], one can show that the sequence $(B_n^{-1}S_n\}$ does not converge weakly (to a non-degenerate r.v.) for any choice of the sequence (B_n) of positive constants.

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ON THE DOMAIN OF CONVERGENCE OF SERIES IN POLYNOMIALS ORTHOGONAL WITH RESPECT TO GENERAL WEIGHT FUNCTIONS ON THE WHOLE REAL LINE

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1. Introduction

The analogue of the Cauchy—Hadamard formula giving the domain of convergence of a Jacobi series in terms of its coefficients is well known ([10], Chapter IX). In [6], E. Hille proved the following theorem giving the domain of convergence of the Hermite series:

THEOREM 1. *Let $\{h_n\}$ be the sequence of orthonormal Hermite polynomials, $\{a_n\}$ a sequence of complex numbers and*

$$d_1 = \liminf_{n\to\infty} \frac{1}{\sqrt{2n+1}} \log \frac{1}{|a_n|} > 0. \tag{1}$$

Then the series $\sum a_n h_n(z)$ converges uniformly and absolutely on compact subsets of the strip $|\operatorname{Im} z| < d_1$ and diverges at each point outside the closure of this strip.

In this note, I shall prove the analogue of the above theorem for series in polynomials orthonormal with respect to weights which are much more general than the Hermite weight; in particular, when the weight is of the form $\exp(-|x|^\alpha)$, $\alpha \geqq 2$. E. Hille's proof depends upon the knowledge of precise asymptotics of the Hermite polynomials. We have to rely upon some rather crude estimates. As a result, our theorem will not be as complete as that of Hille's.

2. Main results

We consider weights of the form $w_Q(x) = \exp(-Q(x))$, where
1) Q is even, convex and in $C^2(0, \infty)$,
2) Q'' is increasing,
3) we have

$$1 \leqq C_1 \leqq \frac{xQ''(x)}{Q'(x)} \leqq C_2 < \infty \tag{2}$$

for some constants C_1 and C_2.

In particular, the weights $\exp(-|x|^\alpha)$ are allowed if $\alpha \geqq 2$. Let q_n be the least positive solution of the equation $xQ'(x) = n$:

$$q_n Q'(q_n) = n. \tag{3}$$

Our theorem can now be stated as follows:

THEOREM 2. *Let* $\{P_n(w_Q^2, x)\}$ *be the system of polynomials orthonormal with respect to the weight function* w_Q^2. *Let* $\{a_n\}$ *be a sequence of complex numbers and*

$$d = \liminf_{n\to\infty} \frac{q_n}{n} \log \frac{1}{|a_n|} > 0. \tag{4}$$

Then there are constants C_3 *and* C_4 *depending upon* Q *alone with the property that the series* $\sum a_n P_n(z)$

(a) *converges uniformly and absolutely on compact subsets of the strip*

$$|\mathrm{Im}\ z| < C_3 d,$$

(b) *diverges at each point outside the strip* $|\mathrm{Im}\ z| \leqq C_4 d$.

COROLLARY 3. *If* z *is a non-real complex number and* C_3 *and* C_4 *as in Theorem* 2, *we have*

$$\frac{1}{C_4} |\mathrm{Im}\ z| \leqq \liminf_{n\to\infty} \frac{q_n}{n} \log |P_n(z)| \leqq \limsup_{n\to\infty} \frac{q_n}{n} \log |P_n(z)| \leqq \frac{1}{C_3} |\mathrm{Im}\ z|.$$

REMARKS. In view of this corollary, if we can sharpen Theorem 2 so as to get $C_3 = C_4$ (i.e. to make the strip of convergence coincident with the strip whose complement is the set of divergence) we would have the asymptotics for $\{P_n\}$ in the complex domain. Thus, in a sense, Theorem 2 is the best possible without having the asymptotics; which seem to be very hard to obtain. In the case of the weights $\exp(-|x|^\alpha)$, $\alpha > 1$, such asymptotics were recently announced by Rahmanov [9]. Unfortunately, no proof was published. In any case, our theorem neither includes nor is included by the theorem of Rahmanov.

COROLLARY 4. *Let* $1 \leqq p \leqq \infty$, $w_Q f \in L^p(R)$. *Put*

$$\varepsilon_n(Q, p, f) = \min \|w_Q(f - P)\|_p$$

where the minimum is taken over all polynomials P *of degree at most* n. *If*

$$D = \liminf_{n\to\infty} \frac{q_n}{n} \log \frac{1}{\varepsilon_n(Q, p, f)} > 0$$

then f *has an extension as an analytic function in the strip* $|\mathrm{Im}\ z| < C_3 D$.

REMARKS. If f is a bounded analytic function in the strip $|\mathrm{Im}\ z| < W$, Dzrbasyan [1] has proved that the quantity D defined in Corollary 4 is at least W in the case $p = \infty$, and $Q(x) = |x|^\alpha$, $\alpha > 1$. While the complete converse of the theorem of Dzrbasyan is clearly not true, Corollary 4 provides a partial converse. We believe that the converse of Corollary 4 should be true.

3. Proofs

PROOF OF THEOREM 2. Let $0 < \mu < d$ be arbitrary and a be an arbitrary real number. We have for large values of k:

$$I_k = \sum_{n=k}^{\infty} |a_n| |P_n^{(k)}(a)| \leqq w_Q^{-1}(a) \sum_{n=k}^{\infty} \frac{1}{\exp\left(\mu \frac{n}{q_n}\right)} \|w_Q P_n^{(k)}\|_\infty.$$

Expanding $P_n^{(k)}$ in terms of P_j's, using Schwarz inequality and an estimate on the Christoffel functions obtained by Freud [3]; or using the Nikolskii-type inequalities (cf. [6]), we see that

$$I_k \leqq C_5 w_Q^{-1}(a) \sum_{n=k}^{\infty} \frac{(k+4)!\, q_n^{k+4}}{\mu^{k+4} n^{k+4}} \left(\frac{n-k}{q_{n-k}}\right)^{1/2} \|w_Q P_n^{(k)}\|_2 .$$

We then use the Markov—Bernstein-type inequalities (cf. [4]) to get

(5)

$$I_k \leqq C_5 C_6^k w_Q^{-1}(a) \sum_{n=k}^{\infty} \frac{(k+4)!\, q_n^{k+4}}{\mu^{k+4} n^{k+4}} \left(\frac{n-k}{q_{n-k}}\right)^{1/2} \frac{n(n-1)\ldots(n-k+1)}{q_n q_{n-1}\cdots q_{n-k+1}} \|w_Q P_n\|_2 \leqq$$

$$\leqq C_7 C_6^k w_Q^{-1}(a) \frac{(k+4)!}{\mu^{k+4}} \sum_{n=k}^{\infty} \left(\frac{q_n}{n}\right)^3 \leqq C_8 C_6^k w_Q^{-1}(a) \frac{(k+4)!}{\mu^{k+4}}$$

$\left(\text{since } \frac{q_n}{n} \text{ is decreasing}\right)$. In the last inequality, we used (2) to see that

$$q_n^2 \leqq C_2 \frac{q_n Q'(q_n)}{Q''(q_n)} \leqq \frac{C_2}{Q''(q_1)} n \leqq C_9 n.$$

Here, and throughout this paper, C_1, C_2, etc. denote constants depending upon Q alone.

Thus, from (5), the double series

$$\sum_{k=0}^{\infty} \left[\frac{1}{k!} \sum_{n=k}^{\infty} |a_n P_n^{(k)}(a)|\right] |z-a|^k$$

converges uniformly on compact subsets of the disc $|z-a|<C_6^{-1}\mu$. Rearranging the terms of this series, we see that $\sum a_n P_n(z)$ converges uniformly and absolutely on compact subsets of the disc $|z-a|<C_6^{-1}\mu$, hence of $|z-a|<C_6^{-1}d$ since $0<\mu<d$ was arbitrary and finally, of $|\mathrm{Im}\, z|<C_6^{-1}d$ since a was an arbitrary real number. This proves part (a) with $C_3=C_6^{-1}$. Note that in the case of Hermite weight we may choose $C_6=\sqrt{2}$, thus giving us the convergence part of Hille's theorem.

To prove (b), we need some notation. We denote the zeroes of P_n by x_{kn};

$$x_{n,n} < x_{n-1,n} < \ldots < x_{kn} < \ldots < x_{1n} = X_n$$

(cf. [2]). The symbols λ_{kn} will denote the Christoffel numbers:

$$\lambda_{kn} = \left[\sum_{m=0}^{n-1} P_m^2(x_{kn})\right]^{-1}. \tag{6}$$

The leading coefficient of P_n will be denoted by $\gamma_n (\gamma_n>0)$. We have:

$$\max_{1\leqq m\leqq n+1} \frac{\gamma_{m-1}}{\gamma_m} \leqq C_{10} q_n \tag{7}$$

(cf. [3]) and

$$C_{11} q_n \leqq X_{n+1} \leqq C_{12} q_n \tag{8}$$

(cf. [5]).

Let T_n be the Chebyshev polynomial of degree n:

$$T_n(z) = \mathrm{Re}\left(z+\sqrt{z^2-1}\right)^n. \tag{9}$$

Now (cf. [2])

$$T_{n-1}\left(\frac{z}{X_n}\right) = \frac{\gamma_{n-1}}{\gamma_n} P_n(z) \sum_{k=1}^{n} \frac{\lambda_{kn} P_{n-1}(x_{kn})}{z-x_{kn}} T_{n-1}\left(\frac{x_{kn}}{X_n}\right). \tag{10}$$

Hence, using the fact that $|T_{n-1}(x)| \leqq 1$ on $[-1, 1]$ and (7);

$$\left|T_{n-1}\left(\frac{z}{X_n}\right)\right| \leqq \frac{C_{10} q_n |P_n(z)|}{|\mathrm{Im}\, z|} \left[\sum_{k=1}^{n} \lambda_{kn} P_{n-1}^2(x_{kn}) \sum_{k=1}^{n} \lambda_{kn}\right]^{1/2}.$$

Thus,

$$|P_n(z)| \geqq C_{13} \frac{|\mathrm{Im}\, z|}{q_n} \left|T_{n-1}\left(\frac{z}{X_n}\right)\right|. \tag{11}$$

Now, for sufficiently large n $(n \geqq N(z))$;

$$\left[\frac{z}{X_n}+i\sqrt{1-\frac{z^2}{X_n^2}}\right]^{n-1} = \left[\frac{z}{X_n}+i+O\left(\frac{z^2}{X_n^2}\right)\right]^{n-1} =$$

$$= (i)^{n-1}\left[1+\frac{(1+\varepsilon_n) z}{iX_n}\right]^{X_n \frac{n-1}{n} \frac{n}{X_n}}$$

where $\varepsilon_n \to 0$ as $n \to \infty$.

Hence, a straightforward estimate shows, with (9) and (11) that

$$|P_n(z)| \geqq C_{13}(z) \exp\left(\frac{1}{2} \frac{n}{X_n} |\mathrm{Im}\, z|\right). \tag{12}$$

We, therefore, observe that if

$$|\mathrm{Im}\, z| > \liminf_{n\to\infty} \frac{2X_n}{n} \log \frac{1}{|a_n|} \tag{13}$$

then the series $\sum a_n P_n(z)$ contains infinitely many terms with absolute value greater than 1. This along with (8) proves Theorem 2(b).

PROOF OF COROLLARY 3. The estimate on lim sup follows from the fact that for every $\varepsilon > 0$, the series

$$\sum \exp\left(-\frac{n}{q_n} \frac{1}{C_3} |\mathrm{Im}\, z|(1+\varepsilon)\right) P_n(z)$$

is convergent and hence the sequence of its terms is bounded. If the lower estimate on the lim inf were false, there would be a sequence $\{n_k\}$ of integers and numbers a, b $(0<a<b<1)$ such that

$$\frac{q_{n_k}}{n_k} \log |P_{n_k}(z)| < \frac{a}{C_4} |\mathrm{Im}\, z| < \frac{b}{C_4} |\mathrm{Im}\, z| < \frac{|\mathrm{Im}\, z|}{C_4}.$$

Then the series

$$\sum \exp\left(-\frac{bn_k}{C_4 q_{n_k}}|\operatorname{Im} z|\right) P_{n_k}(z)$$

converges, a contradiction to Theorem 2(b).

PROOF OF COROLLARY 4. The case $p=2$ follows from Parseval's identity and Theorem 2(a). The result can then be extended to other values of p using Nikolskii-type inequalities (cf. [7]). The argument for this is standard, made in greater detail in [8] and hence ommitted here.

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ON χ_{mub}-LATTICES AND CONVEX SUBSTRUCTURES OF LATTICES AND SEMILATTICES

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1. Introduction

In a poset $(P, \leqq)$, which is a lattice, $a \vee b = \text{lub}\ \{a, b\}$ and $a \wedge b = \text{glb}\ \{a, b\}$ for every two elements $a, b \in P$. In fact, lub picks up a single element from the set ub $\{a, b\}$ of upper bounds of a and b, and moreover, this element is the member of the set of minimal upper bounds mub $\{a, b\}$ of a and b. Similarly, glb picks up a single element from the set of maximal lower bounds mlb $\{a, b\}$ of a and b. The picking up is impossible by means of lub (glb) if lub $\{a, b\}$ (glb $\{a, b\}$) does not exist for some pair a, b of elements of P. If $\text{mub}\ \{a, b\} \neq \emptyset \neq \text{mlb}\ \{a, b\}$ for every two elements $a, b \in P$, an algebra $(P, \vee, \wedge)$ can be constructed for describing the poset $(P, \leqq)$ where $a \vee b$ is a freely chosen single element of mub $\{a, b\}$ and $a \wedge b$ that of mlb $\{a, b\}$. When lub $\{a, b\}$ exists, then $\text{lub}\ \{a, b\} = \text{mub}\ \{a, b\}$, and thus lub $\{a, b\}$ is the only possible choice for $a \vee b$. Similarly, when glb $\{a, b\}$ exists, it is the only choice for $a \wedge b$. Hence, if $(P, \leqq)$ is a lattice, the corresponding algebra $(P, \vee, \wedge)$ is a lattice, too, and otherwise $(P, \vee, \wedge)$ is a generalized lattice called a χ-lattice. χ-lattices are constructed and characterized in [5] and some properties of distributive and modular χ-lattices are given in [6]. The purpose of this paper is to present a further generalization of lattices called a χ_{mub}-lattice. χ_{mub}-lattices provide a tool for handling the algebra of convex substructures of lattices and semilattices. Convex sublattices of lattices are considered e.g. by Koh in [3] and [4] and by Fried and Schmidt in [1]. The connection between prime substructures of χ-lattices and distributivity will also be considered. The papers [5] and [6] are not necessary for reading this paper.

2. χ_{mub}-posets and χ_{mub}-lattices

Let $(P, \leqq)$ be a poset. Trivially, $\text{mub}\ \{a, b\} \subset \text{ub}\ \{a, b\}$ and $\text{mlb}\ \{a, b\} \subset \text{lb}\ \{a, b\}$ for every two elements $a, b \in P$. A poset $(P, \leqq)$ is called a χ-poset if $\text{mub}\ \{a, b\} \neq \emptyset$ and $\text{mlb}\ \{a, b\} \neq \emptyset$ for every two elements $a, b \in P$. $(P, \leqq)$ is called a χ_{mub}-poset if $\text{mub}\ \{a, b\} \neq \emptyset$ for all pairs $a, b \in P$ and $\text{mlb}\ \{a, b\} \neq \emptyset$ whenever $\text{lb}\ \{a, b\} \neq \emptyset$; a χ_{mlb}-poset is defined dually. Thus in a χ_{mub}-poset $(P, \leqq)$ mlb $\{a, b\}$ can also be an empty set. Trivially, every χ-poset is a χ_{mub}-poset as well as a χ_{mlb}-poset. χ is a choice-function choosing a single element from $\text{mub}\ \{a, b\} \neq \emptyset$ as well as from $\text{mlb}\ \{a, b\} \neq \emptyset$. The chosen element $\chi(\text{mub}\ \{a, b\})$ is denoted by $a \vee b$ and $\chi(\text{mlb}\ \{a, b\})$ by $a \wedge b$. After choice χ the elements $a \vee b$ and $a \wedge b$ are fixed. Because $\text{mub}\ \{a, b\} = \text{mub}\ \{b, a\}$ and $\text{mlb}\ \{a, b\} = \text{mlb}\ \{b, a\}$ the choice χ is independent of the order of the elements a and b, whence $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$. On the other hand, choice χ is not assumed to be consequential, i.e. although

mub $\{a, b\}=$mub $\{c, d\}\neq\emptyset$ for some elements $a, b, c, d\in P$, where $c\neq a, b$, $a\vee b$ and $c\vee d$ need not be equal; an analogy holds for $a\wedge b$ and $c\wedge d$. Thus choice χ depends on elements a and b only. Note that if $(P, \leqq)$ is a finite χ-poset, then there is a least element $0\in P$ as well as a greatest element $1\in P$. Moreover, in a finite χ_{mub}-poset $(P, \leqq)$ $1\in P$ and in a finite χ_{mlb}-poset $(P, \leqq)$ $0\in P$.

If $(P, \leqq)$ is a χ-poset, then a χ-lattice $(P, \vee, \wedge)$ is an algebra with operations $\vee$ and $\wedge$, where $a\vee b=\chi(\text{mub}\ \{a, b\})$ and $a\wedge b=\chi(\text{mlb}\ \{a, b\})$. Properties of $\vee$ and $\wedge$ in a χ-lattice are given in Theorem 1 of [5].

In a χ_{mub}-poset $(P, \leqq)$ the choice χ is done as reported above when possible. Because mub $\{a, b\}\neq\emptyset$ for all pairs $a, b\in P$, $a\vee b=\chi(\text{mub}\ \{a, b\})$, and $a\wedge b=$ $=\chi(\text{mlb}\ \{a, b\})$ when mlb $\{a, b\}\neq\emptyset$. The open case is mlb $\{a, b\}=\emptyset$, and we define that then $a\wedge b=b\wedge a=\chi(\text{mub}\ (a, b\})=a\vee b$.

THEOREM 1. *Let $(P, \leqq)$ be a χ_{mub}-poset. Then the χ_{mub}-lattice $(P, \vee, \wedge)$ is an algebra with operations $\vee$ and $\wedge$, where $a\vee b=\chi(\text{mub}\ \{a, b\})$ and $a\wedge b=\chi(\text{mlb}\ \{a, b\})$ when* mlb $\{a, b\}\neq\emptyset$, *otherwise $a\wedge b=a\vee b$, satisfying the following properties for all $a, b, c\in P$:*

(1) $a\wedge a=a$; (1′) $a\vee a=a$;
(2) $a\wedge b=b\wedge a$; (2′) $a\vee b=b\vee a$;
(3) $a\wedge b=a\Leftrightarrow a\vee b=b$;
(3′) *if* $a\neq b$, $c\vee a=a$ *and* $c\vee b=b$, *then* $a\vee b\neq a\wedge b$;
(4) $(a\wedge b)\wedge((a\vee b)\vee c)=a\wedge b$;
(4′) *if* $a\wedge b=c$, *or if* $a\wedge b\neq c$ *and* $c\wedge(a\wedge b)\neq c\vee(a\wedge b)$ *then* $(a\vee b)\vee((a\wedge b)\wedge c)=a\forall b$;
(5) *if* $a\wedge c=b\wedge c=c\vee(a\wedge b)=c$, *then* $c=a\wedge b$;
(5′) *if* $a\vee c=b\vee c=c\wedge(a\vee b)=c$, *then* $c=a\vee b$.

PROOF. The binary operations $\vee$ and $\wedge$ are well defined because of the properties of $(P, \leqq)$ and χ, and hence it remains to show the validity of (1)—(5) and (1′)—(5′).

(1): $a\wedge a=\chi(\text{mlb}\ \{a, a\})=\chi(a)=a$; (1′) is proved similarly.

(2) and (2′) hold because the choice χ is independent of the order of the elements a and b.

(3): If $a\vee b=b$, then $b\in$mub $\{a, b\}$, and thus $a\leqq b$ in $(P, \leqq)$. Hence mlb $\{a, b\}=a$, and consequently $a\wedge b=a$. This proves $\Leftarrow$. The proof for $\Rightarrow$ is similar when mlb $\{a, b\}\neq\emptyset$. So let mlb $\{a, b\}=\emptyset$, and thus $a=a\wedge b=a\vee b$, whence $a\in$mub $\{a, b\}$ and $a\geqq b$ in $(P, \leqq)$. But then $b\in$mlb $\{a, b\}\neq\emptyset$, which is a contradiction. Hence $\Rightarrow$ holds also in this case and (3) is valid.

(3′): Because $c\vee a=a$ and $b\vee c=b$, $c\in$lb $\{a, b\}$, and thus mlb $\{a, b\}\neq\emptyset$. If $a\neq b$, then mub $\{a, b\}\cap$mlb $\{a, b\}=\emptyset$, and so $a\vee b\neq a\wedge b$.

(4) The element $a\vee b$ is an upper bound for a and b, and further, $(a\vee b)\vee c$ is an upper bound for $a\vee b$ and c. Thus $(a\vee b)\vee c\in$ub $\{a, b\}$. Further, $a\wedge b\in$ $\in$mlb $\{a, b\}$ or $a\wedge b=a\vee b$. By combining these observations we obtain $a\wedge b\leqq$ $\leqq(a\vee b)\vee c$ in $(P, \leqq)$, whence mlb $\{a\wedge b, (a\vee b)\vee c\}=a\wedge b$. Thus the only possible choice for $(a\wedge b)\wedge((a\vee b)\vee c)$ is $a\wedge b$, and (4) follows.

(4′): If $a\wedge b=c$, (4′) holds obviously. If $c\neq a\wedge b$ and $(a\wedge b)\wedge c\neq(a\wedge b)\vee c$, then mlb $\{a\wedge b, c\}\neq\emptyset$ and $(a\wedge b)\wedge c\leqq a\vee b$ in $(P, \leqq)$. Accordingly, $(a\vee b)\vee$ $\vee((a\wedge b)\wedge c)=a\vee b$, and (4′) follows.

(5): Because $a\wedge c=b\wedge c=c$, $c\in \mathrm{lb}\ \{a, b\}$, and so also $\mathrm{mlb}\ \{a, b\}\neq\emptyset$. The equation $c\vee(a\wedge b)=c$ implies $a\wedge b\leqq c$. Because $a\wedge b$ is a maximal lower bound of a and b, c a lower bound of a and b, and $a\wedge b\leqq c$, the maximality of $a\wedge b$ requires that $a\wedge b=c$. (5′) is proved analogously. This completes the proof.

In particular, the property (3) shows that $a\leqq b\leftrightarrow a\wedge b=a\leftrightarrow a\vee b=b$ in a χ_{mub}-poset $(P, \leqq)$. As easily seen, every finite join-semilattice is a finite χ_{mub}-poset and a finite χ_{mub}-lattice, too. We say that a χ_{mub}-lattice is derived from a χ_{mub}-poset. As easily seen, one may derive several different χ_{mub}-lattices from a single χ_{mub}-poset. The following theorem shows how a χ_{mub}-poset can be reconstructed from a χ_{mub}-lattice.

THEOREM 2. *If $(P, \vee, \wedge)$ is an algebra with binary operations $\vee$ and $\wedge$ satisfying* (1)—(5) *and* (1′)—(5′), *i.e. a χ_{mub}-lattice, then $a\vee b=b\leftrightarrow a\leqq b$ determines a partial order on P and $(P, \leqq)$ is a χ_{mub}-poset. Moreover, if χ_{mub}-poset $(P, \leqq)$ determines a χ_{mub}-lattice, and if it determines further a χ_{mub}-poset $(P, \geqslant)$, then $(P, \leqq)=(P, \geqslant)$.*

PROOF. Let $(P, \vee, \wedge)$ be a χ_{mub}-lattice and $a\leqq b$ $(\leftrightarrow a\vee b=b)$ a binary relation on P. We show first that $\leqq$ is a partial order on P. $a\leqq a$ for all $a\in P$ because of (1′). Let $a\leqq b$ and $b\leqq a$, whence $a\vee b=b$ and $b\vee a=a$. (2′) implies now $a\vee b=$ $=b\vee a$, and so $a=b$. Let $a\leqq b$ and $b\leqq c$. Then $a\vee b=b$ and $b\vee c=c$. Now, $a\vee b=b$ and (3) imply that $a\wedge b=a$. Further, $a\wedge c=a\wedge(b\vee c)=(a\wedge b)\wedge((a\vee b)\vee c)$ which equals $a\wedge b$ according to (4), and so $a\wedge c=a$. Because of (3), we obtain $a\vee c=c$, and thus, finally, $a\leqq c$. Hence $\leqq$ is a partial order on P.

Next we show that $(P, \leqq)$ is a χ_{mub}-poset, i.e. $\mathrm{mub}\ \{a, b\}\neq\emptyset$ for all $a, b\in P$ and $\mathrm{mlb}\ \{a, b\}\neq\emptyset$ if $\mathrm{lb}\ \{a, b\}\neq\emptyset$. In fact we show that $a\vee b\in\mathrm{mub}\ \{a, b\}$ from which $\mathrm{mub}\ \{a, b\}\neq\emptyset$ follows. Obviously $a\vee b\in\mathrm{ub}\ \{a, b\}$, and assume that there is an element $c\in\mathrm{ub}\ \{a, b\}$ such that $c\leqq a\vee b$. Then $c=a\vee c=b\vee c=c\wedge(a\vee b)$, and according to (5′), $c=a\vee b$. Hence $a\vee b$ is a minimal upper bound of a and b, and so $a\vee b\in\mathrm{mub}\ \{a, b\}$. If $a=b$, then $\mathrm{mlb}\ \{a, b\}=a\neq\emptyset$. Hence we assume that $a\neq b$ and $a\vee b\neq a\wedge b$; (3′) contradicts the assumption that $a\vee b=a\wedge b$ and $\mathrm{lb}\ \{a, b\}\neq\emptyset$ for $a\neq b$. By (4) we obtain now $a\vee(a\wedge b)=(a\vee a)\vee((a\wedge a)\wedge b)=$ $=a\vee a=a$, whence $a\leqq a\wedge b$ in $(P, \leqq)$. Similarly we see that $b\leqq a\wedge b$. Thus $\mathrm{lb}\ \{a, b\}\neq\emptyset$. The relation $a\wedge b\in\mathrm{mlb}\ \{a, b\}$ follows now from (5).

For showing $(P, \leqq)=(P, \geqslant)$ one needs to observe that $a\leqq b\leftrightarrow a\vee b=b\leftrightarrow a\leqslant b$. This completes the proof.

Note that $\vee$ and $\wedge$ are not associative. Moreover, $a\leqq b$ and $c\leqq d$ need not imply $a\vee c\leqq b\vee d$ or $a\wedge c\leqq b\wedge d$. Consequently, e.g. the relations $a\wedge(b\vee c)\geqq$ $\geqq(a\wedge b)\vee(a\wedge c)$ and $a\vee(b\wedge c)\leqq(a\vee b)\wedge(a\vee c)$ need not hold in a χ_{mub}-lattice.

The dual forms of Theorems 1 and 2 hold for χ_{mlb}-posets and χ_{mlb}-lattices. In a χ-lattice, (3) and (3′) are not needed and (4′) is substituted briefly by $(a\vee b)\vee$ $\vee((a\wedge b)\wedge c)=a\vee b$.

In Theorem 2 we saw that if $(P, \vee, \wedge)$ is derived from a χ_{mub}-poset $(P, \leqq)$ and $(P, \geqslant)$ from $(P, \vee, \wedge)$, then $(P, \leqq)=(P, \geqslant)$. On the other hand, if $(P, \leqq)$ is derived from a given χ_{mub}-lattice $(P, \vee, \wedge)$ and $(P, \smile, \frown)$ from $(P, \leqq)$, then the equality $(P, \vee, \wedge)=(P, \smile, \frown)$ need not hold. This missing equality can be constructed as follows: Let $(P, \leqq)$ be a χ_{mub}-poset. As reported previously, we make choices in order to be able to derive a χ_{mub}-lattice $(P, \vee, \wedge)$ from $(P, \leqq)$. After making all necessary choices we write them down and denote them by symbol X. Then the pair $((P, \leqq), X)$ determines a χ_{mub}-lattice $(P, \vee, \wedge)$ and the choices X control the operations $\vee$ and $\wedge$: $X(\mathrm{mub}\ \{a, b\})=a\vee b$ and $X(\mathrm{mlb}\ \{a, b\})=a\wedge b$, when $\mathrm{mlb}\ \{a, b\}\neq\emptyset$. When

$(P, \geqslant)$ is derived from $(P, \vee, \wedge)$, then $(P, \cong)=(P, \geqslant)$, $a \vee b \in \operatorname{mub}\{a, b\}$ and $a \wedge b \in \operatorname{mlb}\{a, b\}$ (if $\operatorname{mlb}\{a, b\} \neq \emptyset$) in $(P, \geqslant)$ for all $a, b \in P$. Thus $\vee$ and $\wedge$ reconstruct the choices X in $(P, \geqslant)$, and hence $(P, \vee, \wedge)$ transmits even the identity $((P, \cong), X)=((P, \geqslant), \mathscr{X})$, where $\mathscr{X}$ denotes the choices determined by $\vee$ and $\wedge$ in $(P, \geqslant)$. Let now $(P, \vee, \wedge)$ be a given χ_{mub}-lattice, $((P, \cong), X)$ the pair of a χ_{mub}-poset and a set of choices in $(P, \cong)$ determined by $(P, \vee, \wedge)$, and $(P, \smile, \frown)$ a χ_{mub}-lattice determined by $((P, \cong), X)$. Then $(P, \vee, \wedge)=(P, \smile, \frown)$ as one can easily prove, and the missing equality is obtained. In particular, a lattice can always be interpreted as a pair $((P, \cong), X)$ with a fixed set X of choices.

3. Some special χ_{mub}-lattices

In this section we will consider some special χ_{mub}-lattices rising by a natural way from the general theory of lattices. The χ_{mub}-lattices of this section are special ones because $\operatorname{mub}\{a, b\}=\operatorname{lub}\{a, b\}$ for all pairs a, b of elements and $\operatorname{mlb}\{a, b\}=$ $=\operatorname{glb}\{a, b\}$ whenever $\operatorname{lb}\{a, b\} \neq \emptyset$, and we will call such χ_{mub}-lattices χ_{lub}-lattices. For example, convex sublattices of a lattice constitute a lattice only by considering the empty set as a convex sublattice, but the non-empty convex sublattices of a lattice constitute a χ_{lub}-lattice considered in this section. Note that in a χ_{lub}-lattice $\vee$ is associative and $a \cong b$, $c \cong d$ imply $a \vee c \cong b \vee d$, but unfortunately, $a \wedge c \cong b \wedge d$ need not hold and $\wedge$ need not be associative. We call a χ_{lub}-lattice $(P, \vee, \wedge)$ distributive if D_1 and D_2 below hold, and modular if M_1 and M_2 below hold for all three elements $a, b, c \in P$:

$$\mathrm{D}_1: \quad a \vee (b \wedge c) \cong (a \vee b) \wedge (a \vee c);$$

$$\mathrm{D}_2: \quad a \wedge (b \vee c) \leqq (a \wedge b) \vee (a \wedge c);$$

$$\mathrm{M}_1: \quad a \vee \big(b \wedge (a \vee c)\big) \cong (a \vee b) \wedge (a \vee c);$$

$$\mathrm{M}_2: \quad a \wedge \big(b \vee (a \wedge c)\big) \leqq (a \wedge b) \vee (a \wedge c).$$

It is an exercise to construct examples (finite trees) where only $>$ of D_1 and $<$ of D_2 hold for some elements. One can also prove (see [6]) that a distributive χ_{lub}-lattice is also modular.

Let L be a lattice and A and B its two convex sublattices. As well known, $A \vee B=\{x \mid a_1 \wedge b_1 \leqq x \leqq a_2 \vee b_2$ for some $a_1, a_2 \in A$ and $b_1, b_2 \in B\}$ and $A \wedge B=$ $=A \cap B$ if $A \cap B \neq \emptyset$. In the following we denote by $\operatorname{Csub}(L)$ the χ_{lub}-lattice of all non-empty convex sublattices of a lattice L, where $A \wedge B=A \cap B$ if $A \cap B \neq \emptyset$ and otherwise $A \wedge B=A \vee B$. Special convex sublattices of L are its ideals and dual ideals.

THEOREM 3. *A lattice L is distributive if and only if the χ_{lub}-lattice* $\operatorname{Csub}(L)$ *is distributive.*

PROOF. Let $\operatorname{Csub}(L)$ be distributive. If L is not distributive, then it contains convex sublattices, which are ideals, not satisfying D_1 and D_2, whence $\operatorname{Csub}(L)$ is not distributive. This contradiction implies the distributivity of L.

Conversely, let L be distributive and A, B, C three arbitrary elements of Csub (L). When $x \in A \wedge (B \vee C)$ we obtain two cases: in 1° $A \cap (B \vee C) \neq \emptyset$ and in 2° $A \cap (B \vee C) = \emptyset$.

2°: Because $A \cap (B \vee C) =$, then also $A \cap B = \emptyset = A \cap C$, and thus $A \wedge B = A \vee B$ and $A \wedge C = A \vee C$. On the other hand, $A \wedge (B \vee C) = A \vee (B \vee C)$ and by using the properties of $\vee$, we obtain $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$.

1°: The relation $x \in A \wedge (B \vee C)$ implies $x \in A$ and $x \in B \vee C$. Thus $b_1 \wedge c_1 \leqq x \leqq b_2 \vee c_2$ for some elements $b_1, b_2 \in B$ and $c_1, c_2 \in C$. According to the distributivity of L, $x = x \wedge (b_2 \vee c_2) = (x \wedge b_2) \vee (x \wedge c_2)$. If $A \cap B \neq \emptyset$, then $x \wedge b_2 \in A \cap B = A \wedge B$, and if $A \cap B = \emptyset$, then $x \wedge b_2 \in A \vee B = A \wedge B$; similarly, $x \wedge c_2 \in A \wedge C$. Now $(x \wedge b_2) \vee (x \wedge c_2) \in (A \wedge B) \vee (A \wedge C)$, whence $x \in (A \wedge B) \vee (A \wedge C)$, and D_2 follows.

Let $x \in (A \vee B) \wedge (A \vee C)$. Because $A \subset A \vee B$ and $A \subset A \vee C$, $(A \vee B) \cap (A \vee C) \neq \emptyset$, and so $x \in A \vee B$ and $x \in A \vee C$. The relation $x \in A \vee B$ implies $a_1 \wedge b_1 \leqq x \leqq a_2 \vee b_2$ and that of $x \in A \vee C$, $a_1' \wedge c_1 \leqq x \leqq a_2' \vee c_2$. By combining $a_1 \wedge b_1 \leqq x$ and $a_1' \wedge c_1 \leqq x$ we obtain $(a_1 \wedge a_1') \wedge (b_1 \wedge c_1) \leqq x$. On the other hand, $x \leqq a_2 \vee b_2$ and $x \leqq a_2' \vee c_2$ imply that $x \leqq (a_2 \vee a_2') \vee b_2$ and $x \leqq (a_2 \vee a_2') \vee c_2$. We denote $a_2 \vee a_2'$ briefly by a_3. Now, by using the distributivity of L we obtain $x \leqq (a_3 \vee b_2) \wedge (a_3 \vee c_2) = a_3 \vee (b_2 \wedge c_2)$. Thus $(a_1 \wedge a_1') \wedge (b_1 \wedge c_1) \leqq x \leqq a_3 \vee (b_2 \wedge c_2)$ and this implies that $x \in A \vee (B \wedge C)$, where $b_1 \wedge c_1, b_2 \wedge c_2 \in B \wedge C$ when $B \cap C \neq \emptyset$ and when $B \cap C = \emptyset$. Hence also D_1 holds, and the distributivity of Csub (L) follows.

The proof of the following theorem is an analogy of the preceding one, whence it is omitted.

THEOREM 4. *A lattice L is modular if and only if the χ_{lub}-lattice* Csub (L) *is modular.*

Also some join-semilattices are χ_{lub}-lattices, for example all finite join-semilattices are χ_{lub}-lattices. In the following we will consider such join-semilattices. A join-semilattice S is a tree if every two non-comparable elements have no common lower bound in S.

LEMMA. *A join-semilattice S which is a tree is also a distributive χ_{lub}-lattice.*

PROOF. Let $a, b, c \in S$. The element $a \wedge (b \vee c)$ equals a or $b \vee c$ or $a \vee (b \vee c)$. In the first two cases a and $b \vee c$ are comparable: in the first case $a \leqq b \vee c$ and in the second $b \vee c \leqq a$. In the third case a and $b \vee c$ are non-comparable. When $b \vee c \leqq a$, then $b, c \leqq a$, and thus $a \wedge b = b$, $a \wedge c = a$ and $(a \wedge b) \vee (a \wedge c) = b \vee c \geqq a \wedge (b \vee c)$. When $a \leqq b \vee c$, then both $b < a$ and $c < a$ cannot hold, and thus either $b \wedge a \geqq a$ or $c \wedge a \geqq a$. Hence $a \wedge (b \vee c) = a \leqq (a \wedge b) \vee (a \wedge c)$. When a and $b \vee c$ are non-comparable, then a is non-comparable with b and c, whence $a \wedge b = a \vee b$ and $a \wedge c =$

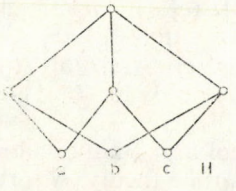


Fig. 1

$=a\vee c$. Thus $a\wedge(b\vee c)=a\vee(b\vee c)\leqq(a\vee b)\vee(a\vee c)=(a\wedge b)\vee(a\wedge c)$. Accordingly, D_2 holds in the χ_{lub}-lattice S. The proof for showing the validity of D_1 is essentially similar, and hence it is omitted.

In fact, one can show that every median semilattice (a tree is a median semilattice) is a distributive χ_{lub}-lattice; median semilattices are considered e.g. by Sholander in [8]. The class of distributive χ_{lub}-lattices is wider than that of median semilattices as one can see by means of the χ_{lub}-lattice H of Figure 1: H is distributive but the elements a, b and c have no median.

A subset I of a χ_{lub}-lattice H is an ideal of H if $a, b\in I$ implies $a\vee b\in I$, and if $x\leqq a$ and $a\in I$ imply $x\in I$. If I is an ideal, then clearly $a, b\in I$ implies also $a\wedge b\in I$. As well known, the join of two ideals I and J is $I\vee J=\{x|x\leqq i\vee j, i\in I \text{ and } j\in J\}$ and $I\wedge J=I\cap J$ if $I\cap J\neq\emptyset$. The non-empty ideals of H constitute a χ_{lub}-lattice $I(H)$, where $I\wedge J=I\cap J$ if $I\cap J\neq\emptyset$ and otherwise $I\wedge J=I\vee J$. Now we can prove

THEOREM 5. *Let H be a χ_{lub}-lattice. H is distributive if and only if $I(H)$ is distributive, and H is modular if and only if $I(H)$ is modular.*

PROOF. In the proof we will treat only the distributive case; the proof for the modular case is analogous and hence omitted.

Assume first that H is distributive and A, B, C are three arbitrary ideals of H. When $x\in A\wedge(B\vee C)$, two subcases arise: 1° $A\cap(B\vee C)\neq\emptyset$ and 2° $A\cap(B\vee C)=\emptyset$. The proof for 2° is the same as that of 2° in the proof of Theorem 3, and hence we concentrate on 1°. When $x\in A\wedge(B\vee C)$, then $x\in A$ and $x\in B\vee C$. The relation $x\in B\vee C$ implies that $x\leqq b\vee c$ for $b\in B$ and $c\in C$. According to the distributivity of H we obtain $x=x\wedge x=x\wedge(b\vee c)\leqq(x\wedge b)\vee(x\wedge c)$. If $A\cap B=\emptyset=A\cap C$, then $x\wedge b\in A\wedge B$, $x\wedge c\in A\wedge C$ and $x\in(A\wedge B)\vee(A\wedge C)$. Assume that $A\cap B\neq\emptyset\neq A\cap C$, and let $b'\in A\cap B$, $c'\in A\cap C$. Then $x\leqq x\vee b'\vee c'\leqq(b\vee b')\vee(c\vee c')$, where $x\vee b'\vee c'=x''\in A$, $b\vee b'=b''\in B$ and $c\vee c'=c''\in C$. Moreover, because $b'\leqq x'', b''$, glb $\{x'', b''\}=b''\wedge x''$ and $b''\wedge x''\in A\wedge B$; similarly $c''\wedge x''\in A\wedge C$. By using the distributivity of H we obtain $x\leqq x''=x''\wedge(b''\vee c'')\leqq(x''\wedge b'')\vee(x''\wedge c'')\in(A\wedge B)\vee(A\wedge C)$. If $A\cap C=\emptyset$, only b' is used, and if $A\cap B=\emptyset$, only c' is used in the preceding proof. Thus D_2 holds for $I(H)$.

Let $x\in(A\vee B)\wedge(A\vee C)$. Because of A, $(A\vee B)\cap(A\vee C)\neq\emptyset$, and so $x\in A\vee B$, $A\vee C$. Consequently, $x\leqq a'\vee b'$, $a''\vee c'$. If $B\cap C\neq\emptyset$, an element $k\in B\cap C$, and clearly $x\leqq(a'\vee a'')\vee b'\vee k$, $(a'\vee a'')\vee c'\vee k$. Because $a'\vee a''\in A$, $b'\vee k\in B$ and $c'\vee k\in C$, we can briefly write $x\leqq a\vee b, a\vee c$. By using the distributivity of H we obtain $x\leqq(a\vee b)\wedge(a\vee c)\leqq a\vee(b\wedge c)$, where $b\wedge c\in B\wedge C$ because of k (if $B\cap C=\emptyset$, then $b\wedge c\in B\vee C=B\wedge C$). Hence also D_1 holds for $I(H)$, and the distributivity of $I(H)$ follows.

Conversely, let $I(H)$ be distributive. Clearly $(a]=\{x|x\leqq a\}$ is an ideal of H, $(a\vee b]=(a]\vee(b]$ and $(b\wedge c]=(b]\wedge(c]$ if $b\wedge c=$ glb $\{b\ c\}$. If $b\wedge c=b\vee c$, then lb $\{b, c\}=\emptyset$, whence $(b]\cap(c]=\emptyset$ and so $(b]\wedge(c]=(b]\vee(c]=(b\vee c]=(b\wedge c]$. Accordingly, $(a\wedge(b\vee c)]=(a]\wedge((b]\vee(c])\leqq((a]\wedge(b])\vee((a]\wedge(c])=((a\wedge b)\vee(a\wedge c)]$. Thus $a\wedge(b\vee c)\leqq(a\wedge b)\vee(a\wedge c)$, and D_2 holds for H. The validity of D_1 for H is proved similarly and hence omitted.

As seen above, the concept of a χ_{lub}-lattice allows one to extend and generalize several concepts and results of lattice theory. A further study about extensions and generalizations will be the topic of a subsequent paper. We close this section by

THEOREM 6. *Let S be a tree join-semilattice. The convex sublattices* Csub (S) *of the χ_{lub}-lattice S constitute a distributive χ_{lub}-lattice.*

The elements of Csub (S) are convex subtrees of S. In Theorem 4 of [7] it is shown that these constitute a median semilattice, which is a distributive χ_{lub}-lattice as stated above.

4. On distributive χ-lattices

In this section we present some results about properties of the ideal lattices $I(H)$ of a finite distributive χ-poset $(P, \leqq)$. These results show how the situation changes more complicate when lub is substituted properly by mub and glb by mlb. As in [6], a χ-poset $(P, \leqq)$ is called distributive if every χ-lattice $H=(P, \vee, \wedge)$ derived from $(P, \leqq)$ is distributive (i.e. satisfies D_1 and D_2 for all $a, b, c \in H$). A non-empty subset I of H is an ideal of H, if $a, b \in I$ implies $a \vee b \in I$, and if $x \leqq a$ and $a \in I$ imply $x \in I$. If I and J are ideals of H, then also $I \cap J$ is an ideal of H, and because $a \wedge b$ exists for every pair a, b of H, $I \cap J$ is always non-empty. Thus the ideals of H constitute a lattice $I(H)$. The dual ideals of H are defined dually. Note that $(a]$ need not be an ideal in every H derived from $(P, \leqq)$.

An ideal I of H is prime if $a \wedge b \in I$ implies $a \in I$ or $b \in I$. A non-empty set $I \subset P$ is an ideal of the χ-poset $(P, \leqq)$, if I is an ideal of some χ-lattice H derived from $(P, \leqq)$; the concept of a dual ideal of a χ-poset is defined similarly.

THEOREM 6. *Let $(P, \leqq)$ be a finite distributive χ-poset, I its ideal and D its dual ideal such that $I \cap D = \emptyset$. Then there exists a χ-lattice H derived from $(P, \leqq)$ such that J is a prime ideal of H, $I \subset J$, $D \cap J = \emptyset$, I is an ideal of H and D a dual ideal of H.*

PROOF. Because I is an ideal of $(P, \leqq)$, there is a χ-lattice H_1 derived from $(P, \leqq)$, where I is an ideal. Similarly, there is an H_2, where D is a dual ideal. Because P is finite, $I=(i]$ for some $i \in P$. Indeed, if there are two greatest elements in I, say a and b, then $a \vee b \in I$ and $a, b < a \vee b$, which is a contradiction. Similarly, $D=[d)= \{x | x \geqq d\}$ for some $d \in P$. Let now p be a maximal element of P such that $(i] \subset (p]$ and $(p] \cap [d) = \emptyset$. We derive a χ-lattice H_3 from $(P, \leqq)$ as follows:

— if $a, b \in D$, then $a \wedge b$ and $a \vee b$ are the same as in H_2;

— if $a, b \in I$, then $a \wedge b$ and $a \vee b$ are the same as in H_1;

— if $a, b \leqq p$ and $a, b \notin I$, then $a \vee b$ is chosen such that $a \vee b \leqq p$ and $a \wedge b$ is chosen freely;

— otherwise $a \wedge b$ and $a \vee b$ are chosen freely.

Clearly $(i]$ and $(p]$ are ideals of H_3 and $[d)$ a dual ideal of H_3. Assume that $a \wedge b \in (p]$ in H_3 but $a, b \notin (p]$. According to the distributivity of H_3, $p \vee (a \wedge b) \geqq (p \vee a) \wedge (p \vee b)$, and because $(p]$ is an ideal, $p \vee (a \wedge b) \in (p]$, and thus $(p \vee a) \wedge (p \vee b) \in (p]$, too. If $p \vee a, p \vee b \in (d]$, $(p \vee a) \wedge (p \vee b)$ also belongs to $[d)$, and a contradiction is obtained with $(p] \cap [d) = \emptyset$. If $p \vee a \notin [d)$, $(p]$ can be substituted by $(p \vee a]$ and another χ-lattice H_3' can be derived from $(P, \leqq)$ such that $(i] \subset (p \vee a]$ and $(p \vee a] \cap [d) = \emptyset$. But his contradicts the maximality of p or otherwise $p \vee a = p$, which contradicts the assumption $a \notin (p]$. Hence the first assumption $a, b \notin (p]$ is false, and thus $a \wedge b \in (p]$ implies $a \in (p[$ or $b \in (p]$. This shows that $(p]$ is a prime ideal of H_3, and H_3 and $(p]$ have the properties asserted in the theorem.

Unfortunately, we were not able to show the distributivity of a finite χ-poset $(P, \geqslant)$ by means of the condition of Theorem 6. As an easy corollary we can write

COROLLARY 1. *Let $(P, \geqq)$ be a finite distributive χ-poset. For every two disjoint elements a and b of P there is a χ-lattice H derived from $(P, \geqq)$ such that J is a prime ideal of H, $a \in J$ and $b \notin J$, or $a \notin J$ and $b \in J$.*

We call J a prime ideal of $(P, \geqq)$ if it is a prime ideal of some H derived from $(P, \geqq)$. Because every ideal of a finite χ-lattice is of the type $(a]$, we can write another corollary

COROLLARY 2. *Let $(P, \geqq)$ be a finite distributive χ-poset and I its ideal. Then I is the intersection of all prime ideals J of $(P, \geqq)$ containing I.*

PROOF. Let $I=(a]$ and $\cap\{J \mid J$ is prime and $I \subset J\}$ contain an element $b \notin (a]$. Clearly $[b)$ is a dual ideal of $(P, \geqq)$ and $(a] \cap [b) = \emptyset$. According to Theorem 6 there is a prime ideal K of $(P, \geqq)$ containing $(a]$ but not b. Hence $b \notin \cap\{J \mid J$ is prime and $I \subset J\}$, and thus $I = \cap\{J \mid J$ is prime and $I \subset J\}$.

Note that every intersection of prime ideals of a finite distributive χ-poset $(P, \geqq)$ need not be an ideal of $(P, \geqq)$. The preceding results on finite χ-posets can be proved also for finite distributive χ_{mub}-posets by taking into account that $a \wedge b$ contains always two alternatives.

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POLYNOMIAL FUNCTIONS (mod m)

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1. Introduction

In [6] A. J. Kempner provided an extensive and detailed account of that subclass of the m^m functions on the ring $\mathbf{Z}/m\mathbf{Z}$ to itself whose members can be expressed as polynomials. Among other things there appears in that work a quite lengthy discussion of the magnitude of this subclass (see Theorem V). The author seems, however, to have overlooked the possibility of a simpler and more explicit formula. One of the main objects of this note is to prove such a formula by completely elementary means, the consequences of which are presented below in Theorems 2.1 and 2.2.

The method of the current paper also permits an extension of its results to functions of several variables and to that class of functions which permute the elements of $\mathbf{Z}/m\mathbf{Z}$. (See the theorems of Sections 3 and 4.) In the latter case we have used a device suggested in [4].

It should also be mentioned that L. E. Dickson [2, p. 21], Rédei and Szele [7], and L. Carlitz [1] have discussed related questions, but they did not specifically address the problems considered here.

2. Polynomials (mod p^n)

Let $\tau(m)$ denote the number of functions on $\mathbf{Z}/m\mathbf{Z}$ which can be realized as polynomials. It is an easy consequence of the Chinese remainder theorem that $\tau(m)$ is multiplicative, so that only $\tau(p^n)$ for each prime p and exponent n need be evaluated.

Extensive use will be made of the function $v(s)$ representing the power of p dividing $s!$; we recall that

$$v(s) = \sum_{r=1}^{\infty} \left[\frac{s}{p^r}\right].$$

As a kind of quadratic analogue to $v(s)$, the function

$$\delta(s) = \frac{1}{2} \sum_{r=1}^{\infty} p^r \left[\frac{s}{p^r}\right]\left(\left[\frac{s}{p^r}\right]+1\right) \tag{2.1}$$

will also play a significant role. The number $N=N(n)$ will always denote the greatest integer such that $v(N)<n$; that is, N is the unique integer such that $p^n|(N+1)!$, but $p^n \nmid N!$. Clearly $N \equiv -1 \pmod{p}$.

It will be far more convenient to write polynomials in terms of factorial powers $x^{(j)}=x(x-1)\ldots(x-j+1)$. Thus, in view of the definition of N, every polynomial

function can be expressed in the form

$$f(x) = \sum_{j=0}^{N} a_j x^{(j)}. \tag{2.2}$$

The set of all such polynomials, some of which will be equal as functions, forms an additive group G of order $p^{n(N+1)}$. Let now H be the subgroup of G consisting of all members

$$h(x) = \sum_{j=0}^{N} h_j x^{(j)}$$

for which $h(x) \equiv 0 \pmod{p^n}$ for all x. Then the order of the quotient group G/H is the number of distinct functions represented in (2.2).

We employ now the difference operator Δ, where $\Delta h(x) = h(x+1) - h(x)$, and recall that $\Delta^j x^{(i)} = i(i-1)\ldots(i-j+1)x^{(i-j)}$. Thus $\Delta^j h(0) \equiv h_j j! \equiv 0 \pmod{p^n}$, which implies $h_j \equiv 0 \pmod{p^{n-v(j)}}$ for each $j \geqq 0$. Hence H is composed of exactly those polynomials $h(x)$ for which $h_j = tp^{n-v(j)}$ $(t = 0, 1, \ldots, p^{v(j)} - 1)$. We get therefore that H has order $p^{\sum_{j=0}^{N} v(j)}$ and that G/H has order

$$\tau(p^n) = p^{n(N+1) - \sum_{j=0}^{N} v(j)}. \tag{2.3}$$

As an immediate consequence of these remarks we may state the following theorem.

THEOREM 2.1. *The distinct polynomial functions* $(\mathrm{mod}\ p^n)$ *are exactly those represented in* (2.2), *where each* a_j *runs through a complete residue system* $(\mathrm{mod}\ p^{n-v(j)})$.

PROOF. Since the expressions $p^{n-v(j)}x^{(j)}$ generate the subgroup H, each a_j in (2.2) can be reduced $(\mathrm{mod}\ p^{n-v(j)})$.

We now calculate $\tau(p^n)$ explicitly.

THEOREM 2.2. *For every prime power* p^n

$$\tau(p^n) = p^{(N+1)(n-v(N))+\delta(N)} \tag{2.4}$$

where $\delta(N)$ *is determined by* (2.1).

PROOF. First we evaluate the sum in (2.3). If $N = \left[\frac{N}{p^r}\right] p^r + R$, then

$$\sum_{j=0}^{N} \left[\frac{j}{p^r}\right] = \left(1+2+\ldots+\left(\left[\frac{N}{p^r}\right]-1\right)\right) p^r + (R+1)\left[\frac{N}{p^r}\right] =$$

$$= \frac{1}{2} p^r \left(\left[\frac{N}{p^r}\right]-1\right)\left[\frac{N}{p^r}\right] + \left(N - \left[\frac{N}{p^r}\right] p^r + 1\right)\left[\frac{N}{p^r}\right] =$$

$$= (N+1)\left[\frac{N}{p^r}\right] - \frac{1}{2} p^r \left[\frac{N}{p^r}\right]\left(\left[\frac{N}{p^r}\right]+1\right);$$

hence

$$\sum_{j=0}^{N} v(j) = \sum_{r=0}^{N} \sum_{r=1}^{\infty} \left[\frac{j}{p^r}\right] = (N+1)v(N) - \delta(N).$$

Formula (2.4) now derives from (2.3).

For instance, it is of interest to apply the theorem to the class of rings for which $n \leqq p$. In this case $N=np-1$, $v(N)=n-1$ and $\delta(N)=\frac{1}{2}pn(n-1)$, so that we may deduce the following formula of Kempner.

COROLLARY 2.3. *If* $n \leqq p$, *then* $\tau(p^n)=p^{\frac{1}{2}pn(n+1)}$.

As for the general case, the calculations are made easier by an estimation of N in terms of n. In [5] Kempner derives an algorithm for determining N without trial, but the following procedure is shorter and more practical in most examples.

First write n in p-adic form

$$n = b_0 + b_1 p + \ldots + b_k p^k \quad (0 \leqq b_i < p).$$

Then

$$np - n = -b_0 + (b_0 - b_1)p + \ldots + (b_{k-1} - b_k)p^k + b_k p^{k+1}$$

and a short computation shows that $v(np-n)=n-Q$, where Q is the number of negative terms in the sequence $-b_0, b_0-b_1, b_1-b_2, \ldots, b_{k-1}-b_k$. Thus $np-n \leqq \leqq N \leqq np-n+(Q-1)p$, so that, since $N \equiv -1 \pmod p$, we have

$$N = \left[\frac{np-n}{p}\right] p + ip - 1 \quad (1 \leqq i \leqq Q).$$

For example, if $p=5$ and $n=72=2+4\cdot 5+2\cdot 5^2$, then $Q=2$ and N is 289 or 294. In fact, $N=294$, $v(294)=71$, $\delta(294)=10,580$ and by (2.4) $\tau(5^{72})=5^{10,875}$.

We can also extend Corollary 2.3 to the case where $p<n<2p$. Here $Q=1$, $N=np-1$ and a similar calculation leads to

$$\tau(p^n) = p^{\frac{1}{2}p(n^2-n+2p)}. \tag{2.5}$$

3. Functions of several variables

In this section we continue with a discussion of functions of two or more variables. The number of polynomial functions on $\mathbf{Z}/m\mathbf{Z}$ in k variables will be denoted by $\tau_k(m)$. Again, $\tau_k(m)$ is clearly multiplicative in m.

It is not difficult to show, using partial rather than ordinary differences, that the analogue of Theorem 2.1 holds. That is, every polynomial function (mod p^n) in k variables possesses a unique representation in the form

$$f(x_1, \ldots, x_k) = \sum a(i_1, \ldots, i_k) x_1^{(i_1)} \ldots x_k^{(i_k)},$$

where $0 \leqq a(i_1, \ldots, i_k) < p^{n-(v(i_1)+\ldots+v(i_k))}$ and the sum is over the set $S_k(n)$ of all k-tuples $(i_1, \ldots, i_k)$ such that $v(i_1)+\ldots+v(i_k)<n$. Thus the number of such func-

tions is the product (over $S_k(n)$)

$$\tau_k(p^n) = \prod p^{n-(v(i_1)+\ldots+v(i_k))}. \tag{3.1}$$

We will now evaluate $\tau_k(p^n)$ explicitly, but only for the case $n \leqq p$. The number of elements in $S_1(n)$ is np. By induction on k, it is then easy to verify that the order of $S_k(n)$ is

$$|S_k(n)| = p^k \binom{n+k-1}{k}. \tag{3.2}$$

Next we calculate the sum $T_k(n) = \sum(v(i_1)+\ldots+v(i_k))$ over $S_k(n)$. Fixing i_1 at 0,1, ..., $np-1$ and summing over $S_k(n)$, $S_k(n-1)$, ..., we arrive at the recursion formula

$$T_{k+1}(n) = p^{k+1} \sum_{i=0}^{n-1} i\binom{n+k-i-1}{k} + p \sum_{i=1}^{n} T_k(i).$$

Summation by parts on the first of the sums, together with the initial formula $T_1(n) = \sum_{i=0}^{np-1} v(i) = \frac{1}{2} pn(n-1)$, then leads ultimately (by induction on k) to

$$T_{k+1}(n) = (k+1)p^{k+1}\binom{n+k}{k+2}.$$

From (3.1) and (3.2) we obtain as a consequence the following generalization of Corollary 2.3.

THEOREM 3.1. *If* $n \leqq p$, *then*

$$\tau_k(p^n) = p^{\binom{k+n}{k+1} p^k}.$$

It does not appear feasible to calculate $\tau_k(p^n)$ for $n > p$.

4. Permutations (mod m)

We finish with a brief look at those functions which permute the elements of $\mathbf{Z}/m\mathbf{Z}$ and are expressible as polynomials. Here we let $\omega(m)$ denote the number of such functions and mention once more that only $\omega(p^n)$ need be considered.

It is a direct consequence of Theorem 123 of [3] that $f(x)$ in (2.2) permutes the elements of $\mathbf{Z}/p^n\mathbf{Z}$ if and only if it permutes the elements of $\mathbf{Z}/p\mathbf{Z}$ and $f'(a) \not\equiv 0 \pmod p$ for every integer a. Thus, since every permutation of $\mathbf{Z}/p\mathbf{Z}$ is a polynomial, we can, according to Theorem 2.1, choose $a_0, a_1, \ldots, a_{p-1}$ in a total of $p!\, p^{p(n-1)}$ ways.

Next we evaluate $f'(x)$ in (2.2) and reduce (mod p). If $j \geqq 2p$, then for every $x = a$, $(x^{(j)})' \equiv 0 \pmod p$. If $p \leqq j < 2p$, then, since $x^{(p)} \equiv x^p - x \pmod p$,

$$(x^{(j)})' \equiv ((x^p - x)(x-p)\ldots(x-j+1))' \equiv -x^{(j-p)} \pmod p$$

for each $x = a$. Hence

$$f'(x) \equiv \sum_{j=0}^{p-1} a_j (x^{(j)})' - \sum_{j=p}^{2p-1} a_j x^{(j-p)} \pmod p,$$

so, by taking $x=0, 1, 2, \ldots$, we find that each a_j $(p \leqq j<2p)$ may be chosen in $p-1$ ways in order to satisfy $f'(a) \not\equiv 0 \pmod{p}$. Therefore we have a total of $(p-1)^p p^{p(n-2)}$ choices for $a_p, \ldots, a_{2p-1}$ in (2.2). The values for a_j $(j \geqq 2p)$ are arbitrary, within the limits set by Theorem 2.1. Thus

$$\omega(p^n) = p!(p-1)^p p^{p(2n-3)} \prod_{j=2p}^{N} p^{n-v(j)} = p!(p-1)^p p^{-2p} \prod_{j=0}^{N} p^{n-v(j)} =$$

$$= p!(p-1)^p p^{-2p} \tau(p^n).$$

From Theorem 2.2 we get then the next formula.

THEOREM 4.1. *The number of polynomial functions which permute the elements of* $\mathbf{Z}/p^n\mathbf{Z}$ $(n>1)$ is

$$\omega(p^n) = p!(p-1)^p p^{(N+1)(n-v(N))+\delta(N)-2p},$$

where $\delta(N)$ *is defined by* (2.1) *and* N *is the greatest integer such that* $v(N)<n$.

In particular, Corollary 2.3 implies

$$\omega(p^n) = p!(p-1)^p p^{\frac{1}{2}p(n^2+n-4)}$$

whenever $1<n \leqq p$, and (2.5) yields a corresponding result for $p<n<2p$. Other suitable restrictions on n presumably produce similar formulas.

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ON LOCAL PROPERTIES OF k-PARAMETER SEMIFLOWS OF NONSINGULAR POINT TRANSFORMATIONS

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1. Introduction

In this paper Krengel's result [3] concerning local ergodic properties of one-parameter measurable semiflows of nonsingular point transformations will be generalized to k-parameter semiflows, k being a positive integer.

Let $R_0=[0, \infty)$ and $R_+=(0, \infty)$. Throughout this paper, $(\theta_t: t\in R_0^k)$ will be a k-parameter measurable semiflow of nonsingular point transformations on a σ-finite measure space $(\Omega, \mathscr{F}, \mu)$. Thus each θ_t is a measurable point transformation of Ω into itself such that $\mu(E)=0$ implies $\mu(\theta_t^{-1}E)=0$, and the transformation $(t, \omega)\to\theta_t\omega$ of $R_0^k\times\Omega$ into Ω is measurable. $(K_t: t\in R_0^k)$ will be the k-parameter semigroup of operators on the space of all measurable functions f on $(\Omega, \mathscr{F}, \mu)$, defined by $K_tf(\omega)=f(\theta_t\omega)$. (We do not distinguish between two functions f and g provided that $f=g$ almost everywhere on Ω.) As is well-known, $(K_t: t\in R_0^k)$ induces a k-parameter semigroup $(T_t: t\in R_0^k)$ of contraction operators on $L_1(\Omega)=L_1(\Omega, \mathscr{F}, \mu)$ by the relation $\int(T_tf)g\,d\mu=\int f(K_tg)\,d\mu$ for $f\in L_1(\Omega)$ and $g\in L_\infty(\Omega)$.

Without assuming the separability condition on $L_1(\Omega)$ we shall first prove that $(T_t: t\in R_0^k)$ is strongly continuous on R_+^k, and then that strong-$\lim_{t\to 0} T_t=I$, I being the identity operator, if and only if $\mu(E)>0$ implies $\mu(\theta_t^{-1}E)>0$ for some $t\in R_+^k$. In case $k=1$, this was proved by Krengel [3] (see also Lin [4]). Our proof uses techniques in [6] and [7]. Second, we shall prove an $L_1(\Omega)$ individual local ergodic theorem for the semiflow $(\theta_t: t\in R_0^k)$ under certain additional hypotheses.

2. Results

THEOREM 1. (a) *The semigroup* $(T_t: t\in R_0^k)$ *is strongly continuous on* R_+^k, i.e. *for each* $f\in L_1(\Omega)$ *and each* $t\in R_+^k$ *we have* $\lim_{s\to t}\|T_sf-T_tf\|_1=0$.

(b) strong-$\lim_{t\to 0} T_t=I$ *if and only if* $\mu(E)>0$ *implies* $\mu(\theta_t^{-1}E)>0$ *for some* $t\in R_+^k$.

PROOF. By the Radon—Nikodym theorem we may assume that $\mu(\Omega)<\infty$. If $E\in\mathscr{F}$, define

$$\lambda(E)=\int_{R_0^k}\exp\left(-\sum_{i=1}^k t_i\right)\mu(\theta_t^{-1}E)\,dt$$

where $t=(t_1, \ldots, t_k)$. Then $\lambda=f_0\,d\mu$ for some $0\leqq f_0\in L_1(\Omega)$, and $T_tf_0\leqq$

$\leqq \exp\left(\sum_{i=1}^{k} t_i\right) f_0$. Letting $\Omega_0=\{f_0=0\}$ and $\Omega_+=\{f_0>0\}$, we have $\mu(\theta_t^{-1}\Omega_0)=0$ for $t\in R_+^k$, and using the measurability condition of the semiflow and the Riesz convexity theorem we see that the restriction of $(T_t\colon t\in R_0^k)$ to $L_1(\Omega_+)$ is strongly continuous on R_+^k (cf. e.g. [7], p. 53). Hence (a) follows. For the proof of (b) see Lin [4], p. 26 (cf. also [1]).

THEOREM 2. *Assume that* $(\theta_t\colon t\in R_0^k)$ *satisfies the additional conditions:*
(i) $\mu(E)>0$ *implies* $\mu(\theta_t^{-1}E)>0$ *for some* $t\in R_+^k$.
(ii) *For every* $t\in R_0^k$,

$$\sup_{0<\mu(E)<\infty} \mu(\theta_t^{-1}E)/\mu(E) < \infty.$$

(iii) $$\sup_{0<\mu(E)<\infty} \mu(E)^{-1} \int_{[0,1]^k} \mu(\theta_t^{-1}E)\, dt < \infty.$$

Let $(E_\alpha\colon 0<\alpha<\infty)$ *be a one-parameter family of Lebesgue measurable subsets of* R_0^k, *with* $|E_\alpha|>0$ *where* $|E_\alpha|$ *denotes the Lebesgue measure of* E_α, *such that* $\alpha<\beta$ *implies* $E_\alpha\subset E_\beta$ *and also such that* $\lim_{\alpha\to 0} E_\alpha=0$ (i.e. $\lim_{\alpha\to 0} x_\alpha=0$ *if* $x_\alpha\in E_\alpha$ *for each* $\alpha>0$). *Suppose for each* $\alpha>0$ *there corresponds a closed cell* P_α *in* R_0^k *(i.e.* P_α *has the form* $P_\alpha=[a_1, b_1]\times\ldots\times[a_k, b_k]$) *such that* $E_\alpha\subset P_\alpha$ *and*

$$\sup_{0<\alpha<\infty} |P_\alpha|/|E_\alpha| < \infty.$$

Then for every $f\in L_1(\Omega)$

$$\lim_{\alpha\to 0} |E_\alpha|^{-1} \int_{E_\alpha} f(\theta_t\omega)\, dt = f(\omega) \quad a.e.\ on \quad \Omega.$$

To prove the above theorem we need a lemma due to Fava [2].

LEMMA (Fava). *Let* $(E_\alpha\colon 0<\alpha<\infty)$ *be as in Theorem* 2. *If* $g\in L_1(R_0^k)$, *define*

$$Ug(t) = \sup_{0<\alpha<\infty} |E_\alpha|^{-1} \int_{E_\alpha} |g(t+s)|\, ds \quad (t\in R_0^k).$$

Then there exists a constant $C>0$, *independent of* $g\in L_1(R_0^k)$, *such that for each* $g\in L_1(R_0^k)$ *and all* $\beta>0$

$$|\{t\colon Ug(t)>\beta\}| \leqq \frac{C}{\beta}\|g\|_1.$$

PROOF OF THEOREM 2. First we remark that conditions (ii) and (iii) in the theorem are equivalent to saying that $(K_t\colon t\in R_0^k)$ is a semigroup of bounded operators on $L_1(\Omega)$, strongly integrable with respect to the Lebesgue measure over every finite interval in R_0^k. Hence there exists $a=(a_1, \ldots, a_k)\in R_+^k$ such that for each $f\in L_1(\Omega)$ the function $t\mapsto \exp\left(-\sum_{i=1}^{k} a_i t_i\right) K_t f$ of R_0^k into $L_1(\Omega)$ is strongly integrable with respect to the Lebesgue measure. Define

$$Kf = \int_{R_0^k} \exp\left(-\sum_{i=1}^{k} a_i t_i\right) K_t f\, dt \quad (f\in L_1(\Omega)).$$

Then K is bounded, because K is positive. Let $u=K^*1$ $(\in L_\infty(\Omega))$. Then $u>0$ a.e. on Ω, because condition (i) in the theorem implies that for every $E\in\mathscr{F}$ with $0<\mu(E)<\infty$

$$\int_E u\,d\mu = \int K1_E\,d\mu = \int_{R_0^k} \exp\left(-\sum_{i=1}^k a_i t_i\right)\mu(\theta_t^{-1}E)\,dt > 0.$$

Similarly, $K_t^* u \leqq \exp\left(\sum_{i=1}^k a_i t_i\right) u$ $(t=(t_1, \ldots, t_k)\in R_0^k)$. Therefore $(K_t\colon t\in R_0^k)$ is a semigroup of operators on $L_1(\Omega, ud\mu)$ satisfying $\|K_t\|_1 \leqq \exp\left(\sum_{i=1}^k t_i a_i\right)$. Further, as in the proof of Theorem 1(b), strong-$\lim_{t\to 0} K_t = I$ on $L_1(\Omega, ud\mu)$. Hence, by a standard argument (cf. e.g. Lemma 1.2 in Terrell [8]), there exists a dense subset M of $L_1(\Omega, ud\mu)$ such that for each $f\in M$

$$\lim_{\alpha\to 0} |E_\alpha|^{-1} \int_{E_\alpha} f(\theta_t\omega)\,dt = f(\omega) \quad \text{a.e. on} \quad \Omega. \tag{1}$$

Since $L_1(\Omega, \mu)\subset L_1(\Omega, ud\mu)$, it is now enough to show that (1) holds for every $f\in L_1(\Omega, ud\mu)$. To do this, however, we may and will assume without loss of generality that $(K_t\colon t\in R_0^k)$ itself is a semigroup of operators on $L_1(\Omega, \mu)$ such that strong-$\lim_{t\to 0} K_t = I$ on $L_1(\Omega, \mu)$. (In fact, considering the measure $ud\mu$ instead of μ, we have $(K_t\colon t\in R_0^k)$ satisfying the required properties.)

If $0\leqq f\in L_1(\Omega, \mu)$, define

$$Mf(\omega) = \limsup_{\alpha\to 0} |E_\alpha|^{-1} \int_{E_\alpha} f(\theta_t\omega)\,dt \quad (\omega\in\Omega).$$

We shall prove that there exists a constant $\gamma>0$, independent of $0\leqq f\in L_1(\Omega, \mu)$, such that for every $\beta>0$

$$\mu(\{\omega\colon Mf(\omega) > \beta\}) \leqq \frac{\gamma}{\beta}\|f\|_1. \tag{2}$$

To do this, fix $0\leqq f\in L_1(\Omega, \mu)$ and $\beta>0$. Let $F\in\mathscr{F}$, $0<\mu(F)<\infty$, and $Mf(\omega)>\beta$ for all $\omega\in F$. Since $\lim_{t\to 0}\|K_t 1_F - 1_F\|_1 = 0$, we can choose $0<\delta\leqq 1$ so that

$$t\in[0, \delta]^k \quad \text{implies} \quad \mu(\theta_t^{-1}F) = \|K_t 1_F\|_1 > 2^{-1}\mu(F). \tag{3}$$

Then define two functions $\tilde{F}$ and $\tilde{F}^*$ on $R_0^k\times\Omega$ by the relations

$$\tilde{F}(t, \omega) = \begin{cases} f(\theta_t\omega) & \text{if} \quad t\in[0, \delta]^k \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\tilde{F}^*(t, \omega) = \sup_{E_\alpha\subset[0, \delta/2]^k} |E_\alpha|^{-1} \int_{E_\alpha} \tilde{F}(t+s, \omega)\,ds.$$

We see that for almost all $(t, \omega)\in[0, \delta/2]^k\times\Omega$ with respect to the product measure $\tilde{F}^*(t, \omega)\leqq Mf(\theta_t\omega)$. Thus, for almost all $t\in[0, \delta/2]^k$ with respect to the Lebesgue measure

$$\theta_t^{-1}F\subset\theta_t^{-1}\{\omega\colon\ Mf(\omega)>\beta\}\subset\{\omega\colon\ \tilde{F}^*(t, \omega)>\beta\}. \tag{4}$$

Write $E^*=\{(t, \omega)\colon\ \tilde{F}^*(t, \omega)>\beta\}$, $E_t^*=\{\omega\colon\ (t, \omega)\in E^*\}$ and $E_\omega^*=\{t\colon\ (t, \omega)\in E^*\}$. Then, by the Lemma and Fubini's theorem,

$$\|1_{E^*}\|_1=\int_\Omega|E_\omega^*|\,d\mu(\omega)\leqq\int_\Omega\left[\frac{C}{\beta}\int_{R_0^k}\tilde{F}(t, \omega)\,dt\right]d\mu(\omega)=$$

$$=\frac{C}{\beta}\int_\Omega\int_{[0, \delta]^k}f(\theta_t\omega)\,dt\,d\mu(\omega)=\frac{C}{\beta}\int_{[0, \delta]^k}\|K_tf\|_1\,dt\leqq\frac{C}{\beta}\delta^kD\|f\|_1,$$

where we put $D=\sup\{\|K_t\|_1\colon\ t\in[0, 1]^k\}$ $(<\infty)$.

On the other hand, by (3) and (4),

$$\|1_{E^*}\|_1=\int_{R_0^k}\mu(E_t^*)\,dt\geqq\int_{[0, \delta/2]^k}\mu(\theta_t^{-1}F)\,dt\geqq(\delta/2)^k\mu(F)/2.$$

Therefore $\mu(F)\leqq2^{k+1}\frac{C}{\beta}D\|f\|_1$, and this proves (2) with $\gamma=2^{k+1}CD$.

To complete the proof of the theorem, let $f\in L_1(\Omega, \mu)$ and choose $g_n\in\mathcal{M}$, $n=1, 2, \ldots$, so that $\lim_n\|f-g_n\|_1=0$, where we may assume that $\lim_n g_n(\omega)=f(\omega)$ a.e. on Ω. Since $\lim_{\alpha\to0}|E_\alpha|^{-1}\int_{E_\alpha}g_n(\theta_t\omega)dt=g_n(\omega)$ a.e. on Ω, it follows that

$$\limsup_{\alpha\to0}\left||E_\alpha|^{-1}\int_{E_\alpha}f(\theta_t\omega)\,dt-f(\omega)\right|\leqq$$

$$\leqq M(|f-g_n|)(\omega)+|f(\omega)-g_n(\omega)|\quad\text{a.e. on}\quad\Omega,$$

and (2) implies that $\lim_n\mu(\{\omega\colon\ M(|f-g_n|)(\omega)>\beta\})=0$ for every $\beta>0$. This completes the proof.

REMARK 1. By a similar argument (cf. Theorem 4 in [7]), we see that *for every* $f\in L_\infty(\Omega)$, $\lim_{\alpha\to0}|E_\alpha|^{-1}\int_{E_\alpha}f(\theta_t\omega)dt=f(\omega)$ a.e. *on* Ω; thus in Theorem 2, $f\in L_1(\Omega)$ can be replaced by $f\in L_1(\Omega)+L_\infty(\Omega)$.

REMARK 2. It would be interesting to know whether Theorem 2 holds when the semiflow $(\theta_t\colon\ t\in R_0^k)$ is replaced by an arbitrary k-parameter semigroup $(T_t\colon\ t\in R_0^k)$ of positive contraction operators on $L_1(\Omega)$ satisfying strong-$\lim_{t\to0}T_t=I$. In case E_α shrinks to zero *nicely*, this is known to hold. See e.g. [8]. See also [1] and [5], where differentiation of additive processes with respect to k-parameter semigroups of positive operators is discussed.

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A FAMILY OF FOURTH ORDER ITERATIVE METHODS FOR SOLVING NONLINEAR OPERATOR EQUATIONS

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In this paper a one parameter family of fourth order methods for the iterative solution of nonlinear operator equations of the form

$$P(x) = 0 \tag{1}$$

where P maps the Banach space X into a Banach space Y, is described. For non-zero values of the parameter, each application of each family member requires no explicit evaluations of derivatives. The convergence of these methods is proved under several assumptions, and a numerical example is given.

I. A one-parameter family of fourth order iterative methods

In order to obtain solutions of (1), we consider a family of iterative methods defined by the formula

$$x_{n+1} = w_n - [3I - \Gamma_n(P(w_n, x_n) + P(v_n, w_n))]\,\Gamma_n P(w_n) \tag{2}$$

where

$$w_n = x_n - \Gamma_n P(x_n); \quad \Gamma_n = [P(x_n, v_n)]^{-1}; \quad v_n = x_n - cP(x_n);$$

$P(x, y)$ is the first divided difference of $P(x)$ [1], c is a real parameter, I is the unit operator.

We now prove a convergence theorem for (2).

THEOREM. *Let us assume that the following conditions are satisfied:*

1) *For the initial point* x_0 *the operator*

$$\Gamma_0 = [P(x_0, x_0 - cP(x_0))]^{-1}$$

exists and $\|\Gamma_0\| \leqq B_0$;

2) $$\|P(x_0)\| \leqq \eta_0;$$

3) $$\|P(x, y, z)\| \leqq M; \quad \|P(x, y)\| \leqq N;$$

$$\|P(x, y, z) - P(z, \tilde{y}, y)\| \leqq L_1\|x - z\| + L_2\|y - \tilde{y}\| + L_3\|z - y\|$$

in $\bar{G}_0$ *defined by* $\{x: \|x - x_0\| \leqq r_0 + |c|\eta_0\}$, *where*

$$r_0 = \frac{k_0 B_0 \eta_0}{1 - c_0 l_0^2 h_0^3};$$

4) $$h_0 = B_0^2 M\eta_0; \quad k_0 h_0(2+|c|N) < 1;$$

$$k_0 = 1+g_0 h_0+g_0(1+g_0)h_0^2; \quad g_0 = 1+\frac{|c|}{B_0};$$

5) $$s_0^2 = \frac{l_0^2 h_0^3}{[1-k_0 h_0(2+|c|N)]^2} = c_0^2 l_0^2 h_0^3 < 1; \quad c_0 = \frac{1}{1-k_0 h_0(2+|c|N)};$$

$$l_0^2 = g_0\left[\frac{L_3+g_0 L_2+k_0 L_1}{B_0 M^2}+(1+3g_0+g_0^2)+2g_0 h_0(1+g_0)+g_0 h_0^2(1+2g_0+g_0^2)\right].$$

Then the equation (1) *has a solution* x^* *in* $G_0=\{x: \|x-x_0\|\leqq r_0\}$; *the sequence* $\{x_n\}$ *defined by* (2) *converges to* x^*, *and the rate of convergence is given by the following inequality:*

(3) $$\|x_n-x^*\| \leqq k_0\left(\frac{1}{c_0}\right)^n (s_0^2)^{\frac{4^n-1}{3}} \frac{B_0\eta_0}{1-c_0 l_0^2 h_0^3}.$$

PROOF. This theorem is proved by induction. We first show that conditions like 1), 2), 3), 4), 5) are fulfilled at x_1, and by induction at all x_n.

From the conditions of the theorem and using the Newton interpolating formula [1], we obtain:

(4) $$\|x_1-x_0\| \leqq k_0 B_0\eta_0.$$

Condition 1. We have

$$\|\Gamma_0[P(x_0, v_0)-P(x_1, v_1)]\| = \|-\Gamma_0[P(x_1, v_1, x_0)(x_1-x_0)+$$
$$+P(v_1, x_0, v_0)(I-cP(x_1, x_0))(x_1-x_0)]\| \leqq k_0 h_0(2+|c|N) < 1.$$

By Banach's lemma, the operator

$$K_0 = I-\Gamma_0[P(x_0, v_0)-P(x_1, v_1)]$$

has an inverse, and

$$\|K_0^{-1}\| \leqq \frac{1}{1-k_0 h_0(2+|c|N)}.$$

Also, $\Gamma_1=K_0^{-1}\Gamma_0$. Therefore

(5) $$\|\Gamma_1\| \leqq \frac{B_0}{1-k_0 h_0(2+|c|N)} = B_1$$

and Condition 1 holds at x_1.

Condition 2. Using the Newton interpolating formula, we have:

(6) $$P(x_1) = P(w_0)+P(w_0, x_0)(x_1-w_0)+P(x_1, w_0, x_0)(x_1-x_0)(x_1-w_0).$$

Taking norms and using the conditions of the theorem and (2), we get:

(7) $$\|P(x_1)\| \leqq \left[\frac{L_3+g_0 L_2+k_0 L_1}{B_0 M^2}+(1+3g_0+g_0^2)+\right.$$
$$\left.+2g_0 h_0(1+g_0)+g_0 h_0^2(1+2g_0+g_0^2)\right] g_0 h_0^3\eta_0 = l_0^2 h_0^3\eta_0 = \eta_1$$

i.e. Condition 2 at x_1 is satisfied.

Condition 4. Now, from (5) and (7) we may show

(8)
$$h_1 = B_1^2 M\eta_1 = \frac{B_0^2}{[1-k_0h_0(2+|c|N)]^2} Ml_0^2h_0^3\eta_0 = \frac{l_0^2h_0^3}{[1-k_0h_0(2+|c|N)]^2} h_0 < h_0$$

from which it is easy to see that $k_1h_1(2+|c|N)<1$.

Thus Condition 4 at the point x_1 is verified.

Condition 5. With direct calculation we can prove that Condition 5 at x_1 is satisfied.

Condition 3. We consider next the ball

$$\bar{G}_1 = \left\{x: \|x-x_1\| \leqq \frac{k_1B_1\eta_1}{1-c_1l_1^2h_1^3} + |c|\eta_1\right\}$$

and we note that the following inequality holds for any $x\in\bar{G}_1$:

$$\|x-x_0\| \leqq \|x-x_1\| + \|x_1-x_0\| \leqq$$

$$\leqq \frac{k_1B_1\eta_1}{1-c_1l_1^2h_1^3} + k_0B_0\eta_0 + |c|\eta_1 \leqq \frac{k_0B_0\eta_0}{1-c_0l_0^2h_0^3} + |c|\eta_0.$$

Hence, $\bar{G}_1\subseteq\bar{G}_0$ i.e. Condition 3 at x_1 is verified.

Now it can be shown by induction on n that the sequence $\{x_n\}$ is welldefined and that conditions 1) to 5) at the point x_n are verified for all n. We obtain also the inequality

(9)
$$\eta_n \leqq \left(\frac{1}{c_0}\right)^{2n} (s_0^2)^{\frac{4^n-1}{3}} \eta_0$$

from which we deduce

(10)
$$\|x_{n+p}-x_n\| \leqq \sum_{i=n}^{n+p-1} \|x_{i+1}-x_i\| \leqq k_0\left(\frac{1}{c_0}\right)^n (s_0^2)^{\frac{4^n-1}{3}} B_0\eta_0\left\{1+\frac{1}{c_0}(s_0^{2/3})^{4^{n+1}-4^n}+\right.$$

$$\left.+\ldots+\frac{1}{c_0^{p-1}}(s_0^{2/3})^{4^{n+p-1}-4^n}\right\} \leqq k_0\left(\frac{1}{c_0}\right)^n (s_0^2)^{\frac{4^n-1}{3}} \frac{B_0\eta_0}{1-c_0l_0^2h_0^3}.$$

Thus, $\{x_n\}$ is a Cauchy sequence in G_0; hence, $\lim\limits_{n\to\infty} x_n = x^*$ exists, $x^*\in G_0$ and the error bound (3) holds.

To show that $P(x^*)=0$, observe that

$$\|P(x_n)\| \leqq \eta_n \leqq \left(\frac{1}{c_0}\right)^{2n} (s_0^2)^{\frac{4^n-1}{3}} \eta_0.$$

It follows that

$$\lim_{n\to\infty} \|P(x_n)\| = 0.$$

By the continuity of P and $x_n \to x^*$ when $n \to \infty$, we have

$$\lim_{n\to\infty} P(x_n) = P(x^*) = 0$$

and x^* is a solution of (1).

Thus the theorem is completely proved.

REMARK 1. With $c=0$, (2) reduces to

$$x_{n+1} = w_n - [3I - \Gamma_n(P(w_n, x_n) + P(x_n, w_n))]\Gamma_n P(w_n) \tag{11}$$

where

$$w_n = x_n - \Gamma_n P(x_n); \quad \Gamma_n = [P'(x_n)]^{-1}.$$

This method is a generalization of King's method [2] for solving nonlinear operator equations.

REMARK 2. It is interesting to compare (2) with fourth order iterative methods [3], [4] and [5]. The advantage of (2) is the fact that it does not require any derivatives of P, and only one inverse operator evaluation is needed at each stage, namely $\Gamma_n = [P(x_n, v_n)]^{-1}$.

REMARK 3. For solving equations in one dimension, the family of fourth order methods (2) is of the form:

$$x_{n+1} = w_n - c\,\frac{f(x_n)f(v_n) + f(w_n)[f(x_n)+f(v_n)]}{f(x_n)-f(v_n)}\,\frac{f(w_n)}{f(v_n)} \tag{12}$$

where

$$w_n = x_n - \frac{cf^2(x_n)}{f(x_n)-f(v_n)}; \quad v_n = x_n - cf(x_n); \quad c \neq 0.$$

For non-zero values of the parameter c, each application of (12) will normally require three function evaluations per iteration.

Our comparison with the methods [6], [7], [8] and [9] shows that the fourth order methods (12) will be superior.

If $c=0$, from (11), we obtain King's method [2] in one dimension:

$$x_{n+1} = w_n - \frac{f(x_n)+2f(w_n)}{f(x_n)}\,\frac{f(w_n)}{f'(x_n)} \tag{13}$$

where

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This method requires no more than two function and one derivative evaluations per iteration.

II. Numerical illustration

We now give a numerical example, using (12), which not only illustrates the method practically but also serves to check the validity of the theoretical results we have derived. We consider the following equation:

$$f(x) \equiv x^3 - 2x - 5 = 0. \tag{14}$$

The correct value of the root to 13 decimal places is [10]: $x^* = 2.0945514815423$.

We solve this equation by the Newton—Raphson method and by (12) with $c = 0.1$, taking $x_0 = 2$ as our first approximation. The results are set out in the following table:

The Newton-Raphson method	The method (12) with $c = 0.1$
$x_1 = 2.1$	$x_1 = 2.094\,550\,7$
$x_2 = 2.094\,568\,1$	
$x_3 = 2.094\,551\,4$	

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THE GAUSS, CODAZZI AND KÜHNE EQUATIONS OF RIEMANN—OTSUKI SPACES

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Introduction

In this paper using the author's former results we shall discuss the connection between curvature tensors of Riemann—Otsuki (R—O_n) spaces and its subspace (see [3]). Consider the Riemann—Otsuki space and its subspace S_m $(m<n)$. The subspace is determined as usual by $x^i=x^i(u^1, \ldots, u^m)$. We use the notation $\xi^i_\alpha := \partial x^i/\partial u^\alpha$ where $\operatorname{rang} \|\partial x^i/\partial u^\alpha\| = m$.[1] We define the covariant differential of a tensor using three kinds of indices as in [1]:

$$\overset{*}{D}T^{i\alpha\mu}_{j\beta\nu} := \overset{*}{\nabla}_\varkappa T^{i\alpha\mu}_{j\beta\nu}\, du^\varkappa = P^i_a P^b_j P^\alpha_\varepsilon P^\eta_\beta P^\mu_\sigma P^\tau_\nu T^{a\varepsilon\sigma}_{b\eta\tau|\varkappa}\, du^\varkappa \tag{0.1}$$

where

$$T^{a\varepsilon\sigma}_{b\eta\tau|\varkappa} := \partial_\varkappa T^{a\varepsilon\sigma}_{b\eta\tau} + {}'\Gamma_{s\ \varkappa}^{\ a} T^{s\varepsilon\sigma}_{b\eta\tau} + {}'\overset{*}{\Gamma}_{\gamma\ \varkappa}^{\ \varepsilon} T^{a\gamma\sigma}_{b\eta\tau} + {}'\Lambda_{\varrho\ \varkappa}^{\ \sigma} T^{a\varepsilon\varrho}_{b\eta\tau} - \tag{0.2}$$

$$- {}''\Gamma_{b\ \varkappa}^{\ s} T^{a\varepsilon\sigma}_{s\eta\tau} - {}''\overset{*}{\Gamma}_{\eta\ \varkappa}^{\ \gamma} T^{a\varepsilon\sigma}_{b\gamma\tau} - {}''\Lambda_{\tau\ \varkappa}^{\ \varrho} T^{a\varepsilon\sigma}_{b\eta\varrho}.$$

The basic elements of R—O_n are the symmetric metric tensor g_{ij} $(\det\|g_{ij}\| \neq 0)$ and the tensor P^i_j $(\det\|P^i_j\| \neq 0)$ satisfying $g_{ij}P^i_a = g_{ia}P^i_j$. The coefficients ${}''\Gamma_{j\ k}^{\ i}$ are constructed as in the Riemann geometry and ${}'\Gamma_{j\ k}^{\ i}$ satisfies the Otsuki's relation

$$\partial_k P^s_r + P^i_r\, {}''\Gamma_{i\ k}^{\ s} - {}'\Gamma_{r\ k}^{\ i} P^s_i = 0. \tag{0.3}$$

The tensors P^α_β and P^μ_ν are projections of the tensor P^i_j onto S_m, and S_{n-m} resp. The connection coefficients ${}'\overset{*}{\Gamma}_{\beta\ \gamma}^{\ \alpha}$, ${}''\overset{*}{\Gamma}_{\beta\ \gamma}^{\ \alpha}$, ${}'\Lambda_{\nu\ \varkappa}^{\ \mu}$ and ${}''\Lambda_{\nu\ \varkappa}^{\ \mu}$ are constructed so that the subspace S_m and its orthogonal space S_{n-m} are Riemann—Otsuki spaces R—O_m and R—O_{n-m}, resp. Using the fact that a tensor P^i_j determines its inverse tensor Q^i_j it was proved in [1] that there are tensors Q^α_β and Q^μ_ν which are projections of the tensor Q^i_j onto S_m and S_{n-m}, resp. so that

$$P^i_j Q^j_s = \delta^i_s; \quad P^\alpha_\beta Q^\beta_\gamma = \delta^\alpha_\gamma; \quad P^\mu_\nu Q^\nu_\sigma = \delta^\mu_\sigma. \tag{0.4}$$

It is known that the tangential vectors ξ^i_α determine the vectors ξ^α_i of the tangential space T_m of S_m and the vectors $\underset{\mu}{N}{}^i$ and $\underset{\mu}{N}_i$[2] orthogonal to S_m so that

$$\xi^i_\alpha \underset{\mu}{N}_i = 0; \quad \xi^i_\alpha \xi^\beta_i = \delta^\beta_\alpha; \quad \xi^i_\alpha \xi^\alpha_j + \underset{\mu}{N}{}^i \underset{\mu}{N}_j = \delta^i_j; \quad \underset{\mu}{N}{}^i \underset{\nu}{N}_i = \delta^\mu_\nu. \tag{0.5}$$

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[1] In this paper Latin indices run from 1 to n, Greek indices $\alpha, \beta, \ldots, \lambda$ run from 1 to m, but $\mu, \nu, \ldots, \omega$ run from $(m+1)$ to n.

[2] In what follows the indices running from $(m+1)$ to n will be co- or contravariant as the original indices were.

The definition of the eigen-vetors is given, as usual, by the relation $P_j^i V_i = \tau V_j$, $\tau = \tau(x)$. The eigen relations are related always to the Latin indices. In [1] it was proved that in the above subspaces the eigen properties

$$\text{(0.6)} \qquad \text{a)} \quad P_j^i \overset{\varepsilon}{\xi}{}_i^\alpha = P_\beta^\alpha \overset{\varepsilon}{\xi}{}_j^\beta \qquad \text{b)} \quad P_j^i N_{\mu i} = P_\nu^\mu N_j$$

hold. Using the notations $M_j^i := P_a^i P_j^a$; $M_\beta^\alpha := P_\gamma^\alpha P_\beta^\gamma$; $M_\nu^\mu := P_\sigma^\mu P_\nu^\sigma$, we obtain from (0.6) that

$$\text{(0.7)} \qquad \text{a)} \quad M_j^i \overset{\varepsilon}{\xi}{}_i^\alpha = M_\beta^\alpha \overset{\varepsilon}{\xi}{}_j^\beta \qquad \text{b)} \quad M_j^i N_{\mu i} = M_\nu^\mu N_j .$$

One can see that the relations (0.6) and (0.7) are essentially of the similar form according to P_j^i or M_j^i, resp. From the relation $P_j^i g_{is} = P_s^i g_{ij}$ and the possibility of conversion between indices it follows that the above relations hold equally for co- and contravariant indices.

The equations listed in the title of this paper appear in (2.5), (2.6), (2.9) and (2.10).

§ 1. The derivation $''\overset{*}{\nabla}$ and its application for the vectors $\overset{\varepsilon}{\xi}{}_\alpha^i$ and N_μ^i

It is known that in the theory of the Otsuki spaces it is possible to define a covariant derivative with respect to only one of the connections. So

$$\text{(1.1)} \qquad \text{a)} \quad ''\overset{*}{\nabla}_\varkappa T_{j\beta\nu}^{i\alpha\mu} := P_a^i P_j^b P_\varepsilon^\alpha P_\beta^\eta P_\sigma^\mu P_\nu^\tau T_{b\eta\tau||\varkappa}^{a\varepsilon\sigma}$$

$$\text{b)} \quad T_{b\eta\tau||\varkappa}^{a\varepsilon\sigma} := \partial_\varkappa T_{b\eta\tau}^{a\varepsilon\sigma} + ''\Gamma_{s\ \varkappa}^{\ a} T_{b\eta\tau}^{s\varepsilon\sigma} + ''\overset{*}{\Gamma}_{\gamma\ \varkappa}^{\ \varepsilon} T_{b\eta\tau}^{a\gamma\sigma} + ''\Lambda_{\varrho\ \varkappa}^{\ \sigma} T_{b\eta\tau}^{a\varepsilon\varrho} -$$

$$- ''\Gamma_{b\ \varkappa}^{\ s} T_{s\eta\tau}^{a\varepsilon\sigma} - ''\overset{*}{\Gamma}_{\eta\ \varkappa}^{\ \gamma} T_{b\gamma\tau}^{a\varepsilon\sigma} - ''\Lambda_{\tau\ \varkappa}^{\ \varrho} T_{b\eta\varrho}^{a\varepsilon\sigma}.$$

In particular

$$''\overset{*}{\nabla}_\varkappa N_\mu^i = ''\overset{*}{\nabla}_\varkappa N_\mu^i = P_a^i P_\mu^\nu (\partial_\varkappa N_\nu^a + ''\Gamma_{s\ \varkappa}^{\ a} N_\nu^s - ''\Lambda_{\nu\ \varkappa}^{\ \tau} N_\tau^a).$$

Applying (1.1a) for the vector $\overset{\varepsilon}{\xi}{}_\alpha^i$ of a tangent space T_m of S_m we get

$$\text{(1.2)} \qquad ''\overset{*}{\nabla}_\gamma \overset{\varepsilon}{\xi}{}_\alpha^i = P_a^i P_\alpha^\beta (\partial_\gamma \overset{\varepsilon}{\xi}{}_\beta^a + ''\Gamma_{s\ \gamma}^{\ a} \overset{\varepsilon}{\xi}{}_\beta^s - ''\overset{*}{\Gamma}_{\beta\ \gamma}^{\ \eta} \overset{\varepsilon}{\xi}{}_\eta^a).$$

The coefficients of the connection $''\overset{*}{\Gamma}$ were defined by

$$\text{(1.3)} \qquad ''\overset{*}{\Gamma}_{\beta\ \gamma}^{\ \eta} = ''\Gamma_{j\ k}^{\ i} \overset{\varepsilon}{\xi}{}_i^\eta \overset{\varepsilon}{\xi}{}_\beta^j \overset{\varepsilon}{\xi}{}_\gamma^k + \overset{\varepsilon}{\xi}{}_{\beta\gamma}^i \overset{\varepsilon}{\xi}{}_i^\eta$$

(see [3] (19)). Substituting this in (1.2) and using the relation (0.5) for $\overset{\varepsilon}{\xi}{}_i^\eta \overset{\varepsilon}{\xi}{}_\eta^a$ we get

$$\text{(1.4)} \qquad ''\overset{*}{\nabla}_\gamma \overset{\varepsilon}{\xi}{}_\alpha^i = P_a^i P_\alpha^\beta (''\Gamma_{\beta\ \gamma}^{\ s} + \overset{\varepsilon}{\xi}{}_{\beta\gamma}^s) N_{\mu s} N_\mu^a = H_{\gamma\alpha}^i .$$

Applying the eigen property (0.6b) for $P_a^i N_\mu^a$ we can write

$$\text{(1.5)} \qquad ''\overset{*}{\nabla}_\gamma \overset{\varepsilon}{\xi}{}_\alpha^i = [P_\alpha^\beta (''\Gamma_{\beta\ \gamma}^{\ s} + \overset{\varepsilon}{\xi}{}_{\beta\gamma}^s) N_{\mu s} P_\mu^\nu] N_\nu^i .$$

Using the notation

$$H_{\gamma}{}^{\nu}{}_{\alpha} := P_{\alpha}^{\beta}({}''\Gamma_{\beta}{}^{s}{}_{\gamma}+\xi^{s}_{\beta\gamma})\underset{\mu}{N}_{s}P_{\mu}^{\nu}$$

we get from (1.4) and (1.5)

$$H_{\gamma}{}^{i}{}_{\alpha} = H_{\gamma}{}^{\nu}{}_{\alpha}\underset{\nu}{N}^{i}. \tag{1.6}$$

This means that $H^{i}_{\gamma\alpha}$, i.e. the covariant derivative of the vector ξ^{i}_{α} is orthogonal to T_m. It is not difficult to show that $H^{i}_{\gamma\alpha}$ is not symmetric in the lower indices γ, α. Namely the relation

$$H^{i}_{\gamma\alpha}-H^{i}_{\alpha\gamma} = \left\{[P_{\alpha}^{\beta}({}''\Gamma_{\beta}{}^{s}{}_{\gamma}+\xi^{s}_{\beta\gamma})-P_{\gamma}^{\beta}({}''\Gamma_{\beta}{}^{s}{}_{\alpha}+\xi^{s}_{\beta\alpha})]\underset{\mu}{N}_{s}P_{\mu}^{\nu}\right\}\underset{\nu}{N}^{i}$$

does not vanish in general. In the special case when $P^{i}_{j}=\varrho\delta^{i}_{j}$, $\varrho=\varrho(x)$, or ξ^{i}_{α} are eigen vectors $\left(P^{i}_{j}\xi^{j}_{\alpha}=\varrho\xi^{i}_{\alpha},\ \varrho=\varrho(x)\right)$ we have $P^{\alpha}_{\beta}=\varrho\delta^{\alpha}_{\beta}$, hence the coefficients of the connection ${}''\Gamma_{j}{}^{i}{}_{k}$ are symmetric in the lower indices.

Further consider the quantity ${}''\overset{*}{\nabla}_{\gamma}{}''\overset{*}{\nabla}_{\alpha}\xi^{i}_{\beta}$. The eigen property follows from (0.6b) applied to $P^{i}_{j}\underset{\nu}{N}^{j}$, (1.6), (1.1a and b). Thus we get

$${}''\overset{*}{\nabla}_{\gamma}{}''\overset{*}{\nabla}_{\alpha}\xi^{i}_{\beta} = {}''\overset{*}{\nabla}_{\gamma}(H^{\nu}_{\alpha\beta}\underset{\nu}{N}^{i}) = P_{\nu}^{\mu}P_{\alpha}^{\varepsilon}P_{\beta}^{\eta}(H^{\nu}_{\varepsilon\eta||\gamma})\underset{\mu}{N}^{i}+H^{\nu}_{\varepsilon\eta}(N^{j}_{||\gamma})P^{i}_{j}P_{\alpha}^{\varepsilon}P_{\beta}^{\eta}.$$

Since according to (0.4) $H^{\nu}_{\varepsilon\eta}=\delta^{\nu}_{\mu}H^{\mu}_{\varepsilon\eta}=P^{\nu}_{\varrho}Q^{\varrho}_{\mu}H^{\mu}_{\varepsilon\eta}$ using (1.1a) we finally get that

$${}''\overset{*}{\nabla}_{\gamma}{}''\overset{*}{\nabla}_{\alpha}\xi^{i}_{\beta} = ({}''\overset{*}{\nabla}_{\gamma}H^{\mu}_{\alpha\beta})\underset{\mu}{N}^{i}+({}''\overset{*}{\nabla}_{\gamma}\underset{\varrho}{N}^{i})H^{\mu}_{\varepsilon\eta}P_{\alpha}^{\varepsilon}P_{\beta}^{\eta}Q^{\varrho}_{\mu}. \tag{1.7}$$

The deviation of this derivation of $H^{\nu}_{\alpha\beta}\underset{\nu}{N}^{i}$ from the known Leibniz rule consists in multiplication by the tensor P in the second term. From (1.7) it follows that

$${}''\overset{*}{\nabla}_{[\gamma}{}''\overset{*}{\nabla}_{\alpha]}\xi^{i}_{\beta} = ({}''\overset{*}{\nabla}_{[\gamma}H^{\nu}_{\alpha]\beta})\underset{\nu}{N}^{i}+P^{\varepsilon}_{[\alpha}({}''\overset{*}{\nabla}_{\gamma]}\underset{\nu}{N}^{i})Q^{\nu}_{\varrho}P^{\eta}_{\beta}H^{\varrho}_{\varepsilon\eta} = {}''\overset{*}{\nabla}_{[\gamma}H^{i}_{\alpha]\beta}.^{3} \tag{1.8}$$

This relation will be used in the following section.

Now we apply (1.1a) to the vector $\underset{\nu}{N}^{i}$ orthogonal to S_m, substitute ${}''\Lambda^{\sigma}_{\varrho\gamma}=$ $=({}''\Gamma_{a}{}^{s}{}_{k}\underset{\sigma}{N}_{s}\xi^{k}_{\gamma}-\partial_{\gamma}\underset{\sigma}{N}_{a})\underset{\varrho}{N}^{a}$ (see [1], (2.3)), and use the relation $(\partial_{\gamma}\underset{\sigma}{N}_{r})\underset{\varrho}{N}^{r}=-\underset{\sigma}{N}_{r}(\partial_{\gamma}\underset{\varrho}{N}^{r})$. Applying (0.5) to $\underset{\tau}{N}_{s}\underset{\tau}{N}^{a}$ we get

$${}''\overset{*}{\nabla}_{\gamma}\underset{\nu}{N}^{i} = P^{i}_{a}P^{\varrho}_{\nu}(\partial_{\gamma}\underset{\varrho}{N}^{a}+{}''\Gamma_{s}{}^{a}{}_{\gamma}\underset{\varrho}{N}^{s}-{}''\Lambda_{\varrho}{}^{\sigma}{}_{\gamma}\underset{\sigma}{N}^{a}) = P^{i}_{a}P^{\varrho}_{\nu}[{}''\Gamma_{r}{}^{\alpha}{}_{\gamma}\underset{\varrho}{N}^{r}+(\partial_{\gamma}\underset{\varrho}{N}^{r})\xi^{\alpha}_{r}]\xi^{a}_{\alpha}.$$

According to the eigen property $P^{i}_{a}\xi^{a}_{\alpha}=P^{\beta}_{\alpha}\xi^{i}_{\beta}$ it follows that

$${}''\overset{*}{\nabla}_{\gamma}\underset{\nu}{N}^{i} = P_{\alpha}^{\beta}P^{\varrho}_{\nu}[{}''\Gamma_{a}{}^{r}{}_{\gamma}\underset{\varrho}{N}^{r}+(\partial_{\gamma}\underset{\varrho}{N}^{r})\xi^{\alpha}_{r}]\xi^{i}_{\beta} := L^{i}_{\gamma\nu}. \tag{1.9}$$

In what follows we shall use the notation

$$L^{\beta}_{\gamma\nu} := P_{\alpha}^{\beta}P^{\varrho}_{\nu}[{}''\Gamma_{r}{}^{\alpha}{}_{\gamma}\underset{\varrho}{N}^{r}+(\partial_{\gamma}\underset{\varrho}{N}^{r})\xi^{\alpha}_{r}]. \tag{1.10}$$

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[3] The Schouten's notation was used.

Finally we get that

$$L^i_{\gamma\nu} = L^\beta_{\gamma\nu}\xi^i_\beta. \tag{1.11}$$

This means that $L^i_{\gamma\nu}$ belong to T_m.

Now we observe $''\overset{*}{\nabla}_\alpha\,''\overset{*}{\nabla}_\gamma \underset{\nu}{N}{}^i$. Using (1.11) we have that $''\overset{*}{\nabla}_\alpha\,''\overset{*}{\nabla}_\gamma \underset{\nu}{N}{}^i = ''\overset{*}{\nabla}_\alpha(\xi^i_\beta L^\beta_{\gamma\nu})$. Applying (1.1a and b) after some calculations and according to (1.9) it follows

$$''\overset{*}{\nabla}_\alpha L^i_{\gamma\nu} = P^i_a P^\varepsilon_\gamma P^\mu_\nu(\xi^a_{\beta||\alpha}L^\beta_{\varepsilon\mu} + \xi^a_\beta L^\beta_{\varepsilon\mu||\alpha}).$$

As above we use that $L^\beta_{\varepsilon\mu} = \delta^\beta_\lambda L^\lambda_{\varepsilon\mu} = P^\beta_\eta Q^\eta_\lambda L^\lambda_{\varepsilon\mu}$. According to (1.1a) and the eigen property $P^i_a\xi^a_\beta = P^\alpha_\beta\xi^i_\alpha$ following from (0.6a) we have

$$''\overset{*}{\nabla}_\alpha L^i_{\gamma\nu} = (''\overset{*}{\nabla}_\alpha\xi^i_\eta)P^\varepsilon_\gamma P^\mu_\nu L^\lambda_{\varepsilon\mu}Q^\eta_\lambda + (''\overset{*}{\nabla}_\alpha L^\eta_{\gamma\nu})\xi^i_\eta.$$

From this relation, (1.9), (1.4) and (1.6) it follows that

$$''\overset{*}{\nabla}_{[\alpha}\,''\overset{*}{\nabla}_{\gamma]}\underset{\nu}{N}{}^i = P^\varepsilon_{[\gamma}H^\varrho_{\alpha]\eta}\underset{\varrho}{N}{}^i P^\mu_\nu L^\varkappa_{\varepsilon\mu}Q^\eta_\varkappa + (''\overset{*}{\nabla}_{[\alpha}L^\varkappa_{\gamma]\nu})\xi^i_\varkappa. \tag{1.12}$$

§ 2. The Ricci formula of the derivation $''\overset{*}{\nabla}$ and its applications

In [2] the author has given the Ricci formula of basic covariant derivation and in [5] the same formula was given with respect to the Otsuki derivation denoted by ",". Here we compute the Ricci formula with respect to the derivation $''\overset{*}{\nabla}$ applied on the vectors ξ^i_α and $\underset{\nu}{N}{}^i$. Applying (1.1a) to (1.2) we have

$$''\overset{*}{\nabla}_\gamma(''\overset{*}{\nabla}_\alpha\xi^i_\beta) = P^i_a P^\varepsilon_\beta P^\eta_\alpha[(''\overset{*}{\nabla}_\eta\xi^a_\varepsilon)_{||\gamma}].$$

We know that the coefficients of connections of the observed space and its subspace S_m and the tensors P^i_j and P^α_β resp. satisfy the Otsuki relation (0.3). Hence it is possible to eliminate the partial derivatives $\partial_k P^a_r$ and $\partial_\gamma P^\varkappa_\varepsilon$. So we get

$$''\overset{*}{\nabla}_\gamma(''\overset{*}{\nabla}_\alpha\xi^i_\beta) = P^i_a P^\varepsilon_\beta P^\eta_\alpha\{P^a_r{}'\Gamma^{\ r}_{s\ k}\xi^k_\gamma P^\varkappa_\varepsilon\xi^s_{\varkappa||\eta} + P^a_r P^{\varkappa\prime}_\lambda\overset{*}{\Gamma}{}^{\lambda\ \ r}_{\varepsilon\ \gamma}\xi^r_{\varkappa||\eta} -$$

$$-P^a_r\,''\overset{*}{\Gamma}{}^{\lambda\ \ r}_{\varkappa\ \gamma}P^\varkappa_\varepsilon\xi^r_{\lambda||\eta} + P^a_r P^\varkappa_\varepsilon\partial_\gamma\xi^r_{\varkappa||\eta} - ''\overset{*}{\Gamma}{}^{\varkappa\ \ r}_{\eta\ \gamma}P^a_r P^\lambda_\varepsilon\xi^r_{\lambda||\varkappa} - ''\overset{*}{\Gamma}{}^{\lambda\ \ r}_{\varepsilon\ \gamma}P^a_r P^\varkappa_\lambda\xi^r_{\varkappa||\eta}\}.$$

In the Otsuki's theory the covariant derivative of the Kronecker-δ symbol is not zero, i.e. $\delta^\lambda_{\varepsilon|\gamma} = {}'\overset{*}{\Gamma}{}^{\lambda\ \ r}_{\varepsilon\ \gamma} - ''\overset{*}{\Gamma}{}^{\lambda\ \ r}_{\varepsilon\ \gamma}$ and the same is true for $\delta^r_{s|k}$. Substituting this and using (1.1b) we get

$$''\overset{*}{\nabla}_\gamma(''\overset{*}{\nabla}_\alpha\xi^i_\beta) = -M^i_r P^\varepsilon_\beta P^\eta_\alpha P^\varkappa_\lambda\delta^\lambda_{\varepsilon|\gamma}\xi^r_{\varkappa||\eta} - M^i_r M^\varkappa_\beta P^\eta_\alpha[\delta^r_{s|k}\xi^k_\gamma\xi^s_{\varkappa||\eta} - \xi^r_{\varkappa||\eta||\gamma}], \tag{2.1}$$

where $M^i_r := P^i_a P^a_r$ and $M^\varkappa_\beta := P^\varkappa_\varepsilon P^\varepsilon_\beta$. It is known that $\xi^r_{\varkappa\|\eta\|\gamma} = \xi^r_{\varkappa\|[\eta\|\gamma]} + \xi^r_{\varkappa\|(\eta\|\gamma)}$. Since the covariant derivative denoted by "$\|$" has the properties of the derivative at an affine space we have

$$\xi^r_{\varkappa\|[\eta\|\gamma]} = ''R_s{}^r{}_{\gamma\eta}\xi^s_\varkappa - ''\overset{*}{R}{}^{\lambda\ \ r}_{\varkappa\ \gamma\eta}\xi^r_\lambda \tag{2.2}$$

where

$$''R_s{}^r{}_{\gamma\eta} := \partial_\gamma ''\Gamma_s{}^r{}_\eta - \partial_\eta ''\Gamma_s{}^r{}_\gamma + ''\Gamma_t{}^r{}_\gamma ''\Gamma_s{}^t{}_\eta - ''\Gamma_t{}^r{}_\eta ''\Gamma_s{}^t{}_\gamma,$$

$$''\overset{*}{R}_\lambda{}^\varkappa{}_{\gamma\eta} := \partial_\gamma ''\overset{*}{\Gamma}_\lambda{}^\varkappa{}_\eta - \partial_\eta ''\overset{*}{\Gamma}_\lambda{}^\varkappa{}_\gamma - ''\overset{*}{\Gamma}_\alpha{}^\varkappa{}_\gamma ''\overset{*}{\Gamma}_\lambda{}^\alpha{}_\eta + ''\overset{*}{\Gamma}_\alpha{}^\varkappa{}_\eta ''\overset{*}{\Gamma}_\lambda{}^\alpha{}_\gamma.$$

In (2.2) it was used that the coefficients of the connection $''\overset{*}{\Gamma}_\eta{}^\alpha{}_\gamma$ are symmetric in the lower indices (see [3]). Substituting (2.2) in (2.1), using the definition of the derivative denoted by ",", the change of the indices α, γ and the skew-symmetry in γ, η of the tensors $''R_s{}^r{}_{\gamma\eta}$ and $''\overset{*}{R}_\lambda{}^\varkappa{}_{\gamma\eta}$ we get

$$(2.3)\qquad ''\overset{*}{\nabla}_{[\gamma}\,''\overset{*}{\nabla}_{\alpha]}\xi^i_\beta = -M^i_r\delta^\varkappa_{\beta,[\gamma}P^\eta_{\alpha]}\xi^r_{\varkappa\|\eta} - M^i_r M^\varkappa_\beta[\delta^r_{s|[\gamma}P^\eta_{\alpha]}\xi^s_{\varkappa\|\eta} -$$

$$-\xi^r_{\varkappa\|(\eta\|[\gamma)}P^\eta_{\alpha]} + ''R_s{}^r{}_{\eta[\gamma}P^\eta_{\alpha]}\xi^s_\varkappa - ''\overset{*}{R}_\varkappa{}^\lambda{}_{\eta[\gamma}P^\eta_{\alpha]}\xi^r_\lambda].$$

Now we use (2.3) and (1.8). Substituting the expression of the right side from (1.8) in the left side of (2.3) using (1.9) and (1.11) we get

$$(2.4)\qquad (''\overset{*}{\nabla}_{[\gamma}H^\nu_{\alpha]\beta})\underset{\nu}{N}{}^i + P^\varepsilon_{[\alpha}L^\lambda_{\gamma]\nu}\xi^i_\lambda Q^\nu_\varrho P^\eta_\beta H^\varrho_{\varepsilon\eta} = -M^i_r\delta^\varkappa_{\beta,[\gamma}P^\eta_{\alpha]}\xi^r_{\varkappa\|\eta} -$$

$$-M^i_r M^\varkappa_\beta\left[\delta^r_{s|[\gamma}P^\eta_{\alpha]}\xi^s_{\varkappa\|\eta} - \xi^r_{\varkappa\|(\eta\|[\gamma)}P^\eta_{\alpha]} + ''R_s{}^r{}_{\eta[\gamma}P^\eta_{\alpha]}\xi^s_\varkappa - ''\overset{*}{R}_\varkappa{}^\lambda{}_{\eta[\gamma}P^\eta_{\alpha]}\xi^r_\lambda\right].$$

Multiplying this by ξ^δ_i and using that the vectors ξ^α_i and $\underset{\mu}{N}{}^i$ are orthogonal, i.e. (0.5) holds we get an expression, from which by $M^i_r\xi^\alpha_i = M^\alpha_\lambda\xi^\lambda_r$ (following from (0.7a), and (0.5)) after contraction by $\tilde{M}^\varepsilon_\delta$, $\tilde{M}^{\beta 4}_\varkappa$ we get

$$(2.5)\qquad ''\overset{*}{R}_\varkappa{}^\varepsilon{}_{\eta[\gamma}P^\eta_{\alpha]} = P^\beta_{[\alpha}L^\delta_{\gamma]\nu}\tilde{M}^\varepsilon_\delta Q^\nu_\varrho Q^\eta_\varkappa H^\varrho_{\beta\eta} - \xi^\varepsilon_r[\tilde{M}^\beta_\varkappa\delta^\lambda_{\beta,[\gamma}P^\eta_{\alpha]}\xi^r_{\lambda\|\eta} +$$

$$+\delta^r_{s|[\gamma}P^\eta_{\alpha]}\xi^s_{\varkappa\|\eta} - \xi^r_{\varkappa\|(\eta\|[\gamma)}P^\eta_{\alpha]} + ''R_s{}^r{}_{\eta[\gamma}P^\eta_{\alpha]}\xi^s_\varkappa].$$

This is the *Gauss' equation* of our space.

In the case when the normal general connection is the classical affine connection ($P^i_j=\delta^i_j$) T. Otsuki [6] obtained a formula generalizing the Gauss equation in the theory of subspaces of a Riemann geometry (see [7]). The equation (2.5) is equivalent with (4.4) in [6] if $P^i_j=\delta^i_j$.

Now we multiply (2.4) by $\underset{\mu}{N}_i$. Substituting $M^i_r\underset{\mu}{N}_i = M^\mu_\varrho\underset{\varrho}{N}_r$, which follows from (0.7b), and using (0.5) we get

(2.6)

$$''\overset{*}{\nabla}_{[\gamma}H^\mu_{\alpha]\beta} = -\underset{\varrho}{N}_r M^\mu_\varrho[\delta^\varkappa_{\beta,[\gamma}P^\eta_{\alpha]}\xi^r_{\varkappa\|\eta} + M^\varkappa_\beta(-\xi^r_{\varkappa\|(\eta\|[\gamma)}P^\eta_{\alpha]} + ''R_a{}^r{}_{\eta[\gamma}P^\eta_{\alpha]}\xi^a_\varkappa + \delta^r_{s|[\gamma}P^\eta_{\alpha]}\xi^s_{\varkappa\|\eta})].$$

This is the analogy of the *first Codazzi equation* of a Riemann space in our case.

In what follows we calculate $''\overset{*}{\nabla}_\alpha\,''\overset{*}{\nabla}_\gamma\underset{\nu}{N}{}^i$. We use similar calculations as above. Applying (1.1a) to $\underset{\nu}{N}{}^i$ twice and using that (0.3) is satisfied in R—O_n and S_{n-m}, we get

$$''\overset{*}{\nabla}_\alpha\,''\overset{*}{\nabla}_\gamma\underset{\nu}{N}{}^i = P^\varepsilon_\gamma P^\sigma_\nu P^i_b P^b_c\left[-P^\tau_\varrho\delta^\varrho_{\sigma|\alpha}\underset{\tau}{N}{}^c_{\|\varepsilon} + P^\tau_\sigma\underset{\tau}{N}{}^c_{\|\varepsilon\|\alpha} - \delta^c_{k|\alpha}P^\tau_\sigma\underset{\tau}{N}{}^k_{\|\varepsilon}\right].$$

[4] The inverse tensors of the tensors M^i_j and M^α_β are denoted by $\tilde{M}^i_j := Q^i_\alpha Q^\alpha_j$ and $\tilde{M}^\alpha_\beta := Q^\alpha_\varrho Q^\varrho_\beta$ where $M^i_j\tilde{M}^j_s = \delta^i_s$, $M^\alpha_\beta\tilde{M}^\beta_\gamma = \delta^\alpha_\gamma$.

Using that $N^c_{\tau||\varepsilon||\alpha}=N^c_{\tau||(\varepsilon||\alpha)}+N^c_{\tau||[\varepsilon||\alpha]}$ and that the derivation "$||$" has analogous properties as the derivation in affin spaces we get

$$(2.7)\qquad {}''\overset{*}{\nabla}_{[\alpha}\,{}''\overset{*}{\nabla}_{\gamma]}N^i_\nu = M^i_c\{M^\tau_\nu[N^c_{\tau||(\varepsilon||[\alpha)}P^\varepsilon_{\gamma]}-\delta^c_{k|[\alpha}P^\varepsilon_{\gamma]}N^k_{\tau||\varepsilon}-{}''R_s{}^c{}_{\varepsilon[\alpha}P^\varepsilon_{\gamma]}N^s_\tau+$$
$$+{}''\tilde{R}_\tau{}^\varrho{}_{\varepsilon[\alpha}P^\varepsilon_{\gamma]}N^c_\varrho]-P^\sigma_\nu P^\tau_\varrho\delta^\varrho_{\sigma|[\alpha}P^\varepsilon_{\gamma]}N^c_{\tau||\varepsilon}\}$$

where

$${}''\tilde{R}_\tau{}^\varrho{}_{\alpha\varepsilon}=\partial_\varepsilon{}''\Lambda_\tau{}^\varrho{}_\alpha-\partial_\alpha{}''\Lambda_\tau{}^\varrho{}_\varepsilon-{}''\Lambda_\tau{}^\sigma{}_\varepsilon{}''\Lambda_\sigma{}^\varrho{}_\alpha+{}''\Lambda_\tau{}^\sigma{}_\alpha{}''\Lambda_\sigma{}^\varrho{}_\varepsilon.$$

Using (2.7) and (1.12) we have

$$(2.8)\qquad \xi^i_\varkappa\,{}''\overset{*}{\nabla}_{[\alpha}L^\varkappa_{\gamma]\nu}+P^\eta_{[\gamma}H^\varrho_{\alpha]\varepsilon}N^i_\varrho P^\mu_\nu L^\varkappa_{\eta\mu}Q^\varepsilon_\varkappa=-M^i_cM^\omega_\nu N^k_{\omega||\eta}\delta^c_{k|[\alpha}P^\eta_{\gamma]}+$$
$$+M^i_cM^\omega_\nu[N^c_{\omega||(\eta||[\alpha)}P^\eta_{\gamma]}-{}''R_s{}^c{}_{\eta[\alpha}P^\eta_{\gamma]}N^s_\omega+{}''\tilde{R}_\omega{}^\varrho{}_{\eta[\alpha}P^\eta_{\gamma]}N^c_\varrho]-M^i_c\delta^\tau_{\nu,[\alpha}P^\varepsilon_{\gamma]}N^c_{\tau||\varepsilon}.$$

Multiplying this relation by ξ^ε_i and using the relation $M^i_b\xi^\varepsilon_i=M^\varepsilon_\varkappa\xi^\varkappa_b$, following from (0.7a), and (0.5), we get that

$$(2.9)\qquad {}''\overset{*}{\nabla}_{[\alpha}L^\varepsilon_{\gamma]\nu}=M^\varepsilon_\lambda\xi^\lambda_c[M^\omega_\nu(N^k_{\omega||\eta}\delta^c_{k|[\alpha}P^\eta_{\gamma]}+N^c_{\omega||(\eta||[\alpha)}P^\eta_{\gamma]}-$$
$$-{}''R_s{}^c{}_{\eta[\alpha}P^\eta_{\gamma]}N^s_\omega)-\delta^\omega_{\nu,[\alpha}P^\eta_{\gamma]}N^c_{\omega||\eta}].$$

This is the analogue of the *second Codazzi equation.*

Multiplication of (2.8) by N^i_μ according to (0.5) and $M^i_cN^\nu_i=M^\nu_\mu N^c_\nu$ implies the relation

$$(2.10)\qquad P^\eta_{[\gamma}H^\mu_{\alpha]\varkappa}P^\sigma_\nu L^\lambda_{\eta\sigma}Q^\varkappa_\lambda=N^c_\varrho M^\mu_\varrho[M^\omega_\nu(-N^k_{\omega||\eta}\delta^c_{k|[\alpha}P^\eta_{\gamma]}+N^c_{\omega||(\eta||[\alpha)}P^\eta_{\gamma]})-$$
$$-\delta^\omega_{\nu,[\alpha}P^\eta_{\gamma]}N^c_{\omega||\eta}]-M^\mu_\varrho M^\omega_\nu({}''R_s{}^c{}_{\eta[\alpha}P^\eta_{\gamma]}N^s_\omega N^c_\varrho+{}''\tilde{R}_\omega{}^\varrho{}_{\eta[\alpha}P^\eta_{\gamma]}).$$

This is the analogue of the *Kühne equation.*

If $P^i_j=\delta^i_j$, i.e. the regular general connection reduces to the simple affine connection, it is not difficult to see that the above equations turn into the suitable formulae of the Riemannian geometry.

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DECOMPOSABLE KAEHLER MANIFOLDS

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1. Introduction

We consider an $n\ (=2m>2)$ real dimensional Kaehlerian manifold K_n with local coordinates $\{x^i\}$ where the indices $h, i, j, k, \ldots$ run over the range $\{1, 2, \ldots, n\}$. Then the positive definite Riemannian metric g_{ji} and the complex structure J_i^h satisfy the following equations:

$$J_j^a J_a^i = \delta^i{}_j, \quad J_j^h{}_{,k} = 0, \quad g_{ij,k} = 0, \tag{1.1}$$

where a comma followed by an index denotes the operator of covariant differentiation with respect to the Riemannian connection constructed from g_{ji}. We shall denote by $R_{kji}{}^h$, R_{ji} and R the curvature tensor, the Ricci tensor and the scalar curvature respectively.

It is well known that

$$J_{ji} \overset{\text{def}}{=} J_j^a g_{ai} \tag{1.2}$$

and

$$S_{ji} \overset{\text{def}}{=} J_j^a R_{ai} \tag{1.3}$$

are skew symmetric.

Sinha [5] and Sinha and Singh [6] have defined the following tensors:

$$H_{kjih} = R_{kjih} + \frac{1}{n+4}\{R_{ki}g_{jh} - R_{ji}g_{kh} + g_{ki}R_{jh} - g_{ji}R_{kh} + \\ + S_{ki}J_{jh} - S_{ji}J_{kh} + J_{ki}S_{jh} - J_{ji}S_{kh} + 2S_{kj}J_{ih} + 2J_{kj}S_{ih}\}, \tag{1.4}$$

$$L_{kjih} = R_{kjih} - \frac{1}{n+4}\{g_{ji}R_{kh} - g_{ki}R_{jh} + J_{ji}S_{kh} - J_{ki}S_{jh} - 2J_{kj}S_{ih}\}, \tag{1.5}$$

$$M_{kjih} = R_{kjih} - \frac{1}{n+4}\{R_{kh}g_{ji} - R_{jh}g_{ki} + S_{kh}J_{ji} - S_{jh}J_{ki} - 2S_{ih}J_{kj}\} + \\ + \frac{R}{(n+2)(n+4)}\{g_{ji}g_{kh} - g_{ki}g_{jh} + J_{ji}J_{kh} - J_{ki}J_{jh} - 2J_{kj}J_{ih}\}, \tag{1.6}$$

$$N_{kjih} = R_{kjih} - \frac{1}{n+4}\{R_{ji}g_{kh} - R_{ki}g_{jh} + S_{ji}J_{kh} - S_{ki}J_{jh} - 2S_{kj}J_{ih}\} + \\ + \frac{R}{(n+2)(n+4)}\{g_{ji}g_{kh} - g_{ki}g_{jh} + J_{ji}J_{kh} - J_{ki}J_{jh} - 2J_{kj}J_{ih}\}, \tag{1.7}$$

which are called H-conhormonic curvature tensor, H-conhormonic* curvature tensor, H-conformal* curvature tensor and H-conformal** curvature tensor, respectively.

Moreover, Tachibana [8] has given the following tensors:

$$P_{kjih} = R_{kjih} + \frac{1}{n+2}\{R_{ki}g_{jh} - R_{ji}g_{kh} + S_{ki}J_{jh} - S_{ji}J_{kh} + 2S_{kj}J_{ih}\}, \tag{1.8}$$

$$U_{kjih} = R_{kjih} + \frac{R}{n(n+2)}\{g_{ki}g_{jh} - g_{ji}g_{kh} + J_{ki}J_{jh} - J_{ji}J_{kh} + 2J_{kj}J_{ih}\}, \tag{1.9}$$

$$\begin{aligned} B_{kjih} = R_{kjih} &+ \frac{1}{n+4}\{R_{ki}g_{jh} - R_{ji}g_{kh} + g_{ki}R_{jh} - g_{ji}R_{kh} + \\ &+ S_{ki}J_{jh} - S_{ji}J_{kh} + J_{ki}S_{jh} - J_{ji}S_{kh} + 2S_{kj}J_{ih} + \\ &+ 2J_{kj}S_{ih}\} - \frac{R}{(n+2)(n+4)}\{g_{ki}g_{jh} - g_{ji}g_{kh} + J_{ki}J_{jh} - J_{ji}J_{kh} + 2J_{kj}J_{ih}\}, \end{aligned} \tag{1.10}$$

which are called H-projective curvature tensor, H-concircular curvature tensor[1] and Bochner curvature tensor respectively.

A tensor T_{hijk} is said to be K-curvature-like [8] if it satisfies

$$T_{hijk} = -T_{ihjk} = -T_{hikj}, \tag{1.11}$$

$$T_{hijk} + T_{ijhk} + T_{jhik} = 0, \tag{1.12}$$

and

$$T_{hija}J_k^a = -T_{hiak}J_j^a. \tag{1.13}$$

It turns out that the tensors given by (1.4), (1.5), (1.6), (1.7), (1.8), (1.9) and (1.10) are K-curvature like.

2. Decomposable Kaehler manifolds

A Kaehler manifold K_n is said to be decomposable if it can be expressed as a product $K_r \times K_{n-r}$ of two Kaehler manifolds. This occurs locally when there is a coordinate system such that the metric of K_n is of the form

$$dS^2 = g_{j_1 i_1}\,dx^{j_1}\,dx^{i_1} + g_{j_2 i_2}\,dx^{j_2}\,dx^{i_2}, \tag{2.0}$$

where $g_{j_1 i_1}$ are functions of $x^1, x^2, \ldots, x^r$ only, and $g_{j_2 i_2}$ are functions of x^{r+1}, $x^{r+2}, \ldots, x^n$ only. The indices $i_1, j_1, \ldots$ and $i_2, j_2, \ldots$ range over $1, 2, \ldots, r$ and $r+1, r+2, \ldots, n$, respectively. The two sums on the right of (2.0) give the metrices of K_r and K_{n-r}, respectively, and K_r and K_{n-r} are called decomposition manifolds of K_n. It can be easily verified that the Christoffel symbols and the curvature tensor in K_n are zero unless all indices belong to the same range $1, 2, \ldots, r$ or $r+1, r+2, \ldots, n$

[1] Hasegawa [1].

and every component with indices from only one range, say $1, 2, \ldots, r$ is equal to the component of the corresponding symbols or curvature tensor for K_r.

We shall have the following definitions.

DEFINITION 2.1. A Kaehler manifold K_n satisfying

$$R_{kjih,l} = 0, \tag{2.1a}$$

$$H_{kjih,l} = 0, \tag{2.1b}$$

$$L_{kjih,l} = 0, \tag{2.1c}$$

$$M_{kjih,l} = 0, \tag{2.1d}$$

$$N_{kjih,l} = 0, \tag{2.1e}$$

and

$$B_{kjih,l} = 0 \tag{2.1f}$$

are called Kaehler symmetric in the sense of Cartan, H-conharmonic symmetric, H-conharmonic* symmetric H-conformal* symmetric, H-conformal** symmetric and Bochner symmetric (or with parallel Bochner curvature tensor), respectively.

DEFINITION 2.2. The Kaehler manifolds characterised by

$$R_{kjih,l} = a_l R_{kjih}, \tag{2.2a}$$

$$P_{kjih,l} = a_l P_{kjih}, \tag{2.2b}$$

$$U_{kjih,l} = a_l U_{kjih}, \tag{2.2c}$$

$$H_{kjih,l} = a_l H_{kjih}, \tag{2.2d}$$

$$L_{kjih,l} = a_l L_{kjih}, \tag{2.2e}$$

$$M_{kjih,l} = a_l M_{kjih} \tag{2.2f}$$

and

$$N_{kjih,l} = a_l N_{kjih}, \tag{2.2g}$$

for some non zero vector field a_l, are called Kaehler recurrent, H-projective recurrent, H-concircular recurrent, H-conharmonic recurrent, H-conharmonic* recurrent, H-conformal* recurrent and H-conformal** recurrent, respectively.

3. Some properties of decomposable Kaehler symmetric manifolds

THEOREM 3.1. *If a H-conharmonic (or H-conharmonic* or H-conformal* or H-conformal**) symmetric Kaehler manifold is expressed as a product* $K_r \times K_{n-r}$ *(r and n−r both even) for some r, then* K_r *and* K_{n-r} *are both Kaehler symmetric in the sense of Cartan.*

PROOF. From (1.4) we have

$$H_{k_1j_2i_1h_2} = \frac{1}{n+4}\{R_{k_1i_1}g_{j_2h_2} + S_{k_1i_1}J_{j_2h_2}\}.$$

In view of (2.1b) the above relation reduces to

$$R_{k_1i_1,l_1}g_{j_2h_2} + J_{j_2h_2}S_{k_1i_1,l_1} = 0. \tag{3.1}$$

Transvecting this equation with $g^{j_2h_2}$ we get

$$R_{k_1i_1,l_1} = 0 \tag{3.2}$$

and consequently

$$S_{k_1i_1,l_1} = 0. \tag{3.3}$$

Similarly, one can have

$$R_{j_2h_2,l_2} = 0 \tag{3.4}$$

and consequently

$$S_{j_2h_2,l_2} = 0. \tag{3.5}$$

Using relations (3.2), (3.3), (3.4), (3.5) and (2.1b) we obtain $R_{k_1j_1i_1h_1,l_1}=0$, i.e. K_r is Kaehler symmetric in the sense of Cartan and $R_{k_2j_2i_2h_2,l_2}=0$, i.e. K_{n-r} is Kaehler symmetric in the sense of Cartan. This proves the theorem.

THEOREM 3.2. *If a Bochner symmetric Kaehler manifold (i.e. a Kaehler manifold with parallel Bochner curvature tensor) is expressed as a product $K_r \times K_{n-r}$ (r and $n-r$ both even) for some r, then K_r and K_{n-r} are both Kaehler symmetric in the sense of Cartan.*

PROOF. From (1.10) we have

$$B_{k_1j_2i_1h_2} = \frac{1}{n+4}\{R_{k_1i_1}g_{j_2h_2} + g_{k_1i_1}R_{j_2h_2} + S_{k_1i_1}J_{j_2h_2} + \tag{3.6}$$

$$+ J_{k_1i_1}S_{j_2h_2}\} - \frac{R}{(n+2)(n+4)}\{g_{k_1i_1}g_{j_2h_2} + J_{k_1i_1}J_{j_2h_2}\}.$$

In view of (2.1f) the above relation reduces to

$$R_{k_1i_1,l_1}g_{j_2h_2} + S_{k_1i_1,l_1}J_{j_2h_2} - \frac{\underset{(1)}{R},_{l_1}}{n+2}\{g_{k_1i_1}g_{j_2h_2} + J_{k_1i_1}J_{j_2h_2}\} = 0, \tag{3.7}$$

where

$$\underset{(1)}{R} = R_{k_1i_1}g^{k_1i_1}.$$

Transvecting (3.7) with $g^{j_2h_2}$ we get

$$R_{k_1i_1,l_1} = \frac{\underset{(1)}{R},_{l_1}}{n+2}g_{k_1i_1}, \tag{3.8}$$

which gives

$$\underset{(1)}{R}, l_1 = 0, \tag{3.9}$$

since $n+2-r\neq 0$. Using (3.9) in (3.8) we obtain

$$R_{k_1 i_1, l_1} = 0 \tag{3.10}$$

and consequently

$$S_{k_1 i_1, l_1} = 0. \tag{3.11}$$

Hence, in view of (3.9), (3.10), (3.11) and (1.6) we get $R_{k_1 j_1 i_1 h_1, l_1}=0$, i.e. K_r is Kaehler symmetric in the sense of Cartan. Similarly, one can get $R_{k_2 j_2 i_2 h_2, l_2}=0$, i.e. K_{n-r} is Kaehler symmetric in the sense of Cartan. This completes the proof.

4. Some properties of decomposable Kaehler recurrent manifolds

THEOREM 4.1. *If a H-conharmonic (or H-conharmonic*) recurrent Kaehler manifold is expressed as a product $K_r \times K_{n-r}$ (r and $n-r$ both even) for some r, then K_r and K_{n-r} are both Kaehler recurrent.*

PROOF. Equation (2.2d) gives

$$H_{k_1 j_1 i_1 h_1, l_1} = a_1 H_{k_1 j_1 i_1 h_1} \tag{4.1}$$

and

$$H_{k_2 j_2 i_2 h_2, l_2} = a_{l_2} H_{k_2 j_2 i_2 h_2}. \tag{4.2}$$

Transvecting (4.1) with $g^{k_1 h_1}$ and after some simplification we obtain

$$(n-r)(R_{j_1 i_1, l_1} - a_{l_1} R_{j_1 i_1}) = \underset{(1)}{R}, l_1\, g_{j_1 i_1} - a_{l_1} \underset{(1)}{R} g_{j_1 i_1}. \tag{4.3}$$

Transvecting the above equation with $g^{j_1 i_1}$ we get

$$\underset{(1)}{R}, l_1 = a_{l_1} \underset{(1)}{R}. \tag{4.4}$$

From (4.3) and (4.4) we obtain

$$R_{j_1 i_1, l_1} = a_{l_1} R_{j_1 i_1} \tag{4.5}$$

and consequently

$$S_{j_1 i_1, l_1} = a_{l_1} S_{j_1 i_1}. \tag{4.6}$$

Equations (4.5), (4.6) and (4.1) give $R_{k_1 j_1 i_1 h_1, l_1}=a_{l_1} R_{k_1 j_1 i_1 h_1}$, i.e. K_r is Kaehler recurrent. Similarly, proceeding with (4.2) one gets $R_{k_2 j_2 i_2 h_2, l_2}=a_{l_2} R_{k_2 j_2 i_2 h_2}$, i.e. K_{n-r} is Kaehler recurrent. The following theorem is known [1].

THEOREM 4.2. *The necessary and sufficient condition for a Kaehler manifold to be H-projective recurrent is that $U_{kjih,l}=a_l U_{kjih}$ holds, for some non-zero vector a_l.*

Now we prove

THEOREM 4.3. *If a H-conformal* (or H-conformal**) recurrent Kaehler manifold is expressed as a product $K_r \times K_{n-r}$ (r and $n-r$ both even) for some r, then K_r and K_{n-r} are both H-projective recurrent.*

PROOF. Assume that the Kaehler manifold satisfying (2.2f) decomposes locally into $K_r \times K_{n-r}$. Then we have

$$M_{k_1 j_1 i_1 h_1, l_1} = a_{l_1} M_{k_1 j_1 i_1 h_1} \tag{4.7}$$

and

$$M_{k_2 j_2 i_2 h_2, l_2} = a_{l_2} M_{k_2 j_2 i_2 h_2}. \tag{4.8}$$

Transvecting (4.7) with $g^{k_1 h_1}$ and after some simplification with the help of identities

$$S_{jh} J_{ki} g^{kh} = -R_{ji}, \quad J_{jh} J_{ki} g^{hk} = -g_{ji}, \tag{4.9}$$

we get

$$(n+2)(R_{j_1 i_1, l_1} - a_{l_1} R_{j_1 i_1}) = \frac{n-r}{n+2} (\underset{(1)}{R}, l_1 - a_{l_1} \underset{(1)}{R}) + \tag{4.10}$$

$$+ g_{j_1 i_1} + \frac{r+2}{n+2} g_{j_1 i_1} \underset{(2)}{R} a_{l_1}.$$

Multiplying (4.10) by $g^{j_1 i_1}$ and summing for j_1 and i_1 we obtain

$$(n+2)(\underset{(1)}{R}, l_1 - a_{l_1} \underset{(1)}{R}) = \frac{r(n-r)}{n+2} (\underset{(1)}{R}, l_1 - a_{l_1} \underset{(1)}{R}) + \frac{r(r+2)}{n+2} \underset{(2)}{R} a_{l_1}. \tag{4.11}$$

Substituting (4.11) in (4.10), we have

$$R_{j_1 i_1, l_1} - a_{l_1} R_{j_1 i_1} = \frac{1}{r} (\underset{(1)}{R}, l_1 - a_{l_1} \underset{(1)}{R}) g_{j_1 i_1} \tag{4.12}$$

and consequently

$$S_{j_1 i_1, l_1} - a_{l_1} S_{j_1 i_1} = \frac{1}{r} (\underset{(1)}{R}, l_1 - a_{l_1} \underset{(1)}{R}) J_{j_1 i_1}. \tag{4.13}$$

Furthermore, substituting (4.11), (4.12) and (4.13) in (4.7), after some lengthy but straightforward computations, we obtain

$$R_{k_1 j_1 i_1 h_1, l_1} - \frac{\underset{(1)}{R}, l_1}{r(r+2)} \{g_{j_1 i_1} g_{k_1 h_1} - g_{k_1 i_1} g_{j_1 h_1} + J_{j_1 i_1} J_{k_1 h_1} - \tag{4.14}$$

$$- J_{k_1 i_1} J_{j_1 h_1} - 2 J_{k_1 j_1} J_{i_1 h_1}\} = a_{l_1} \Big[R_{k_1 j_1 i_1 h_1} - \frac{\underset{(1)}{R}}{r(r+2)} \{g_{j_1 i_1} g_{k_1 h_1} - g_{k_1 i_1} g_{j_1 h_1} +$$

$$+ J_{j_1 i_1} J_{k_1 h_1} - J_{k_1 i_1} J_{j_1 h_1} - 2 J_{k_1 j_1} J_{i_1 h_1}\}\Big],$$

i.e.

$$U_{k_1 j_1 i_1 h_1, l_1} = a_{l_1} U_{k_1 j_1 i_1 h_1}. \tag{4.15}$$

Similarly, starting from (4.8) we can obtain

(4.16)
$$(n+2)(R_{j_2i_2,l_2}-a_{l_2}R_{j_2i_2}) = \frac{r}{n+2}(\underset{(2)}{R}, l_2-a_{l_2}\underset{(2)}{R})g_{j_2i_2}-\frac{n-r+2}{n+2}g_{j_2i_2}\underset{(1)}{R}a_{l_2},$$

(4.17)
$$(n+2)(\underset{(2)}{R}, l_2-a_{l_2}\underset{(2)}{R}) = \frac{(n-r)r}{n+2}(\underset{(2)}{R}, l_1-a_{l_1}\underset{(2)}{R})+\frac{(n-r)(n-r+2)}{n+2}\underset{(1)}{R}a_{l_1},$$

and subsequently

(4.18)
$$R_{j_2i_2,l_2}-a_{l_2}R_{j_2i_2} = \frac{1}{n-r}(\underset{(2)}{R}, l_2-a_{l_2}\underset{(2)}{R})\,g_{j_2i_2},$$

(4.19)
$$S_{j_2i_2,l_2}-a_{l_2}S_{j_2i_2} = \frac{1}{n-r}(\underset{(2)}{R}, l_2-a_{l_2}\underset{(2)}{R})\,J_{j_2i_2}$$

and

(4.20)
$$U_{k_2j_2i_2h_2,l_2} = a_{l_2}U_{k_2j_2i_2h_2}.$$

The statement follows in view of relations (4.15), (4.20) and Theorem 4.2.

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ON PERTURBATION OF OPERATORS WITH COMPLEMENTED RANGE

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§ 1. Introduction

In [1] Campbell and Faulkner studied conditions under which the closure of the range of a bounded linear operator on a Banach space is complemented in the space. Among their results is the following one concerning the perturbation of such operators.

THEOREM 1 [1, p. 126]. *Let X be a Banach space having a basis $\{x_n\}_{n=1}^{\infty}$ for which $0<\inf_n \|x_n\| \leqq \sup_n \|x_n\|<+\infty$, and let $A: X\to X$ be an operator bounded below on X and having complemented range in X. If $K: X\to X$ is an operator for which $\sum_{n=1}^{\infty}\|Kx_n\|<+\infty$, then $\overline{\operatorname{ran}(A+K)}$ is complemented in X.*

The purpose of this paper is to prove a result of the same type but of significantly greater generality; namely, that Theorem 1 remains valid if X and Y are arbitrary Banach spaces, $A: X\to Y$ is bounded below on X with a complemented range in Y, and the perturbing operator $K: X\to Y$ is compact. Moreover, we show that the range of $A+K$ is actually closed in Y (thereby strengthening the conclusion of Theorem 1 as well). This basic result (Theorem 2) is proved in §2 as a culmination of a sequence of more specialized results of a similar nature. In §3 we consider the possibility of generalizing Theorem 2 still further to the case where A is no longer assumed to be bounded below on X (but still has closed and complemented range). We mention in §3 two such extensions which follow easily from our previous work in §2 and prove a result (Theorem 3) which indicates a sense in which these generalizations are the best possible.

Throughout the paper the term "operator" denotes a continuous linear transformation, and the symbols "$\operatorname{ran}(T)$", "$\overline{\operatorname{ran}(T)}$", and "$\ker T$" denote, respectively, the range, the closure of the range, and the kernel of an operator T. If T is an operator on X and N is a subspace of X, then $T|_N$ denotes the operator on N which is obtained by restricting T to N. Any unexplained notation and definitions used without reference are as in [2].

§ 2. A perturbation theorem

In this section we state and prove the main result of the paper, Theorem 2. It is helpful to proceed in stages, first proving certain special cases and then combining these at the end to obtain a proof of the general case.

PROPOSITION 1. *Let X and Y be Banach spaces and $A: X \to Y$ an operator which is bounded below on X and for which* $\operatorname{ran}(A)=Y$. *If $S: X \to Y$ is a compact operator, then* $\operatorname{ran}(A+S)$ *is closed and complemented in Y.*

PROOF. Let I_X denote the identity operator on X. By assumption A is invertible, so let $A^{-1}: Y \to X$ denote the inverse operator. Since $S: X \to Y$ compact the operator $A^{-1}S$ is also compact, and by the Fredholm Alternative (e.g., [2, p. 609—610]) it follows that $\operatorname{ran}(I_X+A^{-1}S)$ is a closed subspace of finite codimension in X. Using the fact that A is bounded below on X we then have $\operatorname{ran}(A+S)=A(\operatorname{ran}(I_X+A^{-1}S))$, and that this last is closed and of finite codimension in Y. It follows trivially that $\operatorname{ran}(A+S)$ is complemented in Y.

Our proof that Proposition 1 remains valid whenever $\operatorname{ran}(A)$ is complemented in Y is based on the following special case.

PROPOSITION 2. *Let X and Y be Banach spaces and $A: X \to Y$ an operator which is bounded below on X and whose range is complemented in Y. If $S: X \to Y$ is a compact operator whose range is contained in a subspace of Y complementary to* $\operatorname{ran}(A)$, *then* $\operatorname{ran}(A+S)$ *is both closed and complemented in Y.*

PROOF. We show, first of all, that $\operatorname{ran}(A+S)$ is closed in Y.

By assumption there is a projection P of Y onto $\operatorname{ran} A$ so that the complementary projection $Q=I_Y-P$ projects Y onto a subspace N of Y that contains $\operatorname{ran}(S)$. Let $y_0 \in \overline{\operatorname{ran}(A+S)}$. Then there is a sequence $\{x_n\}_{n=1}^{\infty}$ in X for which $\lim_{n\to\infty}(A+S)(x_n)=y_0$. Since $\operatorname{ran}(A)=\operatorname{ran}(P)$ and $\operatorname{ran}(S) \subset \operatorname{ran}(Q)$ it follows that $\lim_{n\to\infty} Ax_n=Py_0$ and $\lim_{n\to\infty} Sx_n=Qy_0$. But A is invertible from X onto $\operatorname{ran}(A)$, so if A^{-1} denotes the inverse of this operator then $\lim_{n\to\infty} x_n=A^{-1}(Py_0)$, implying $\lim_{n\to\infty} Sx_n=SA^{-1}Py_0$. By the above we then have $Qy_0=SA^{-1}Py_0$, and $(A+S)(A^{-1}Py_0)=Py_0+SA^{-1}Py_0=Py_0+Qy_0=y_0$. That is, $y_0 \in \operatorname{ran}(A+S)$, so $\operatorname{ran}(A+S)$ is closed.

In order to show that $\operatorname{ran}(A+S)$ is complemented in Y we first get an alternate description of this set.

CLAIM. $\operatorname{ran}(A+S)=\ker(Q-SA^{-1}P)$.

To see this is so, note first that the above argument shows that if $y_0 \in \operatorname{ran}(A+S)$ then $Qy_0=SA^{-1}Py_0$. That is, $\operatorname{ran}(A+S) \subset \ker(Q-SA^{-1}P)$, and to establish the claim we need only show the reverse inclusion.

Choose any $y_0 \in \ker(Q-SA^{-1}P)$. Since $\operatorname{ran}(A+S)=\overline{\operatorname{ran}(A+S)}=[\ker(A^*+S^*)]_\perp$ (see, e.g., [2, p. 479]), to show $y_0 \in \operatorname{ran}(A+S)$ we need only show that $\langle y_0, f\rangle=0$ whenever $f \in \ker(A^*+S^*)$.

If $f \in \ker(A^*+S^*)$ then $A^*f=-S^*f$, and since $y_0 \in \ker(Q-SA^{-1}P)$ we have $y_0=Py_0+Qy_0=Py_0+SA^{-1}Py_0$, implying

$$\langle y_0, f\rangle = \langle Py_0+SA^{-1}Py_0, f\rangle = \langle y_0, P^*f+(A^{-1}P)^*S^*f\rangle = \langle y_0, P^*f-(A^{-1}P)^*A^*f\rangle.$$

Now if $x\in X$ and $g\in X^*$ then

$$\langle (A^{-1}P)^*A^*g, x\rangle = \langle (A^{-1}P)^*A^*g, Px+Qx\rangle =$$

$$= \langle A^*g, (A^{-1}P)Px\rangle + \langle A^*g, (A^{-1}P)Qx\rangle = \langle A^*g, A^{-1}Px\rangle = \langle g, Px\rangle = \langle P^*g, x\rangle.$$

That is, $(A^{-1}P)^*A^*=P^*$, so our equation above becomes

$$\langle y_0, f\rangle = \langle y_0, P^*f-(A^{-1}P)^*A^*f\rangle = \langle y_0, P^*f-P^*f\rangle = 0,$$

i.e. $y_0\in\ker(A^*+S^*)_\perp = \text{ran}\,(A+S)$, and by our previous comments $\text{ran}\,(A+S) = \ker(A-SA^{-1}P)$.

In order to finish the proof we now show that the operator $P+SA^{-1}P$ is a projection of Y onto $\text{ran}\,(A+S)$.

According to our argument above, $\text{ran}\,(A+S)=\ker(Q-SA^{-1}P)$. Therefore if $y\in\text{ran}\,(A+S)$, $Qy=SA^{-1}Py$, and $y=Py+Qy=Py+SA^{-1}Py=(P+SA^{-1}P)y$, i.e. $\text{ran}\,(A+S)\subset\text{ran}\,(P+SA^{-1}P)$. On the other hand, if $y\in\text{ran}\,(P+SA^{-1}P)$ then $y=Px+SA^{-1}Px$ for some $x\in X$, so since $\text{ran}\,(A)=\text{ran}\,(P)$ and $\text{ran}\,(S)\subset\text{ran}\,(Q)$ we have $(Q-SA^{-1}P)y=(Q-SA^{-1}P)(Px+SA^{-1}Px)=SA^{-1}Px-SA^{-1}Px=0$, or $y\in\ker(Q-SA^{-1}P)$. But this last is $\text{ran}\,(A+S)$ by the claim proved earlier, so we see that $\text{ran}\,(P+SA^{-1}P)$ is actually equal to $\text{ran}\,(A+S)$. Since $\text{ran}\,(S)\subset\text{ran}\,(Q)$ it is immediate that $(P+SA^{-1}P)^2=P+SA^{-1}P$, establishing that $P+SA^{-1}P$ is a projection of Y onto $\text{ran}\,(P+SA^{-1}P)$, and since we have just shown this to be $\text{ran}\,(A+S)$ the proof is complete.

Using Propositions 1 and 2 we can now prove the main result of the paper. The proof itself is interesting in that it establishes another perturbation result which will be discussed in § 3.

THEOREM 2. *Let X and Y be Banach spaces and $A:X\to Y$ an operator which is bounded below on X and whose range is complemented in Y. If $S:X\to Y$ is any compact operator then* $\text{ran}\,(A+S)$ *is closed and complemented in Y.*

PROOF. Let P be a projection of Y onto $\text{ran}\,(A)$ and let $Q=I_X-P$ be the complementary projection. The operator $PS:X\to Y$ is compact and has range in $\text{ran}\,(P)$. Thus since A is bounded below on X and $\text{ran}\,(A)=\text{ran}\,(P)$ we see by Proposition 1 that the operator $A+PS$ has its range closed and complemented in the space $\text{ran}\,(A)$, a closed and complemented subspace of Y. Therefore $\text{ran}\,(A+PS)$ is closed and complemented in Y also.

Now let $M=\ker(A+PS)$, a finite dimensional subspace of X (by the same argument as was used in the proof of Proposition 1 since PS is compact with range in $\text{ran}\,(A)$ and A is bounded below). Since M is finite dimensional in X it is complemented there, and if N is any complementary subspace then we can write $X=M\oplus N$ where the operator $A+PS$ is bounded below on N and $\text{ran}\,(A+PS)|_N=\text{ran}\,(A+PS)$.

Set $A_1=(A+PS)|_N$, an operator from N to Y, and $S_1=QS|_N$, a compact operator from N into Y whose range is contained in $\text{ran}\,(Q)$, and hence in a subspace complementary to that of $\text{ran}\,(A_1)$. Since A_1 is bounded below on N and has range complemented in Y, an application of Proposition 2 shows that $\text{ran}\,(A_1+S_1)$ is closed and complemented in Y. If E and F denote the complementary projections

of X onto N and M, respectively, then

$$\operatorname{ran}(A+S) = \operatorname{ran}[(A+PS)+QS] = \operatorname{ran}([(A+PS)+QS]E+[(A+PS)+QS]F) =$$

$$= \{[(A+PS)+QS]n+[(A+PS)+QS]m \mid n\in N,\ m\in M\} = \operatorname{ran}[(A+PS)+QS]|_N \oplus$$

$$\oplus \operatorname{ran}[(A+PS)+QS]|_M = \operatorname{ran}(A_1+S_1)\oplus \operatorname{ran}(QS)|_M .$$

Since we have just seen that ran (A_1+S_1) is closed and complemented in Y and since ran $(QS)|_M$ is a finite dimensional subspace of Y, it is well known [e.g., [1, p. 126]) that their sum, which is ran $(A+S)$ according to the above, has the same properties. The proof is complete.

§ 3. Generalizations

As Campbell and Faulkner remark in connection with their Theorem 1 ([1, p. 126]), it is necessary to assume some fairly strong property of the operator A (e.g., A bounded below) in order to obtain perturbation results of the type given in § 2. However we can note the following somewhat more general version of Theorem 2 whose proof is exactly the argument given in the proof of Theorem 2 for the operator $A+PS$ there, and will not be repeated here.

COROLLARY 1. *Let X and Y be Banach spaces and let $A: X\to Y$ have a finite dimensional kernel and a closed complemented range. If $S: X\to Y$ is a compact operator then* ran $(A+S)$ *is closed and complemented in Y.*

Another result of this which is, in a sense, complementary to Corollary 1, but which fails to have quite as strong a conclusion, is the following:

PROPOSITION 3. *Let X and Y be Banach spaces and let $A: X\to Y$ be an operator whose range is closed and of finite codimension in Y. If $S: X\to Y$ is a compact operator, then* $\overline{\operatorname{ran}(A+S)}$ *is of finite codimension, and hence complemented, in Y.*

PROOF. It is well known [2, p. 479] that $\overline{\operatorname{ran}(A+S)} = \ker(A^*+S^*)_\perp$, so we need only show $\ker(A^*+S^*)$ is finite dimensional.

Since ran (A) is of finite codimension in Y we can write $Y = \operatorname{ran}(A)\oplus M$, for M some finite dimensional space, and hence $Y^* = \operatorname{ran}(A)^\perp \oplus M^\perp = \ker(A^*)\oplus M^\perp$. By assumption ran (A) is closed, implying ran (A^*) is closed [2, p. 487] and so we easily see that $A^*|_{M^\perp}$ is bounded below on $M^\perp$. Using the fact that S^* is compact and our previous appeal to the Fredholm Alternative we once more get that the operator $(A^*+S^*)|_{M^\perp}$ has a finite dimensional kernel in $M^\perp$. If A^*+S^* had an infinite dimensional kernel, say K, in Y^*, then since $M^\perp$ is of finite codimension in Y^* we would have $M^\perp \cap K$ is infinite dimensional in $M^\perp$, a contradiction to the above.

Therefore $\ker(A^*+S^*)$ is finite dimensional and the conclusion follows.

NOTE. In particular, Proposition 3 implies that whenever $A: X\to Y$ is *onto* Y and $S: X\to Y$ is any compact operator then $\overline{\operatorname{ran}(A+S)}$ is complemented in Y.

These last two results say, then, that if $A: X\to Y$ is an operator with closed range and *either*

1) $\ker(A)$ is finite dimensional, or
2) $\operatorname{ran}(A)$ is of finite codimension,

then $\overline{\operatorname{ran}(A+S)}$ is complemented in Y for every compact operator $S: X \to Y$. It is natural to ask whether further generalizations along these lines are possible. The following result shows the essential impossibility of such a program.

THEOREM 3. *Let X and Y be separable Banach spaces and $A: X \to Y$ an operator for which* $\ker(A)$ *is complemented and infinite dimensional in X and* $\operatorname{ran}(A)$ *is closed, complemented, and of infinite codimension in Y. Then* $\overline{\operatorname{ran}(A+S)}$ *is complemented in Y for every compact operator $S: X \to Y$ only if* $\operatorname{ran}(A)$ *has a complementary subspace isomorphic to a Hilbert space.*

PROOF. By assumption $X = \ker(A) \oplus M$ and $Y = \operatorname{ran}(A) \oplus N$, where M and N are infinite dimensional. Let P denote the projection of X onto $\ker(A)$ whose kernel is M.

Now suppose N is *not* isomorphic to Hilbert space. Then N has a subspace N_1 which is not complemented in N [3] and hence not in Y either. Since Y is assumed to be separable, let $\{n_i\}_{i=1}^{\infty}$ be any normalized sequence in N_1 whose linear span is dense in N_1. Similarly $\ker(A)$ is separable and infinite dimensional, so there exists a normalized sequence $\{k_i\}_{i=1}^{\infty}$ in $\ker(A)$ whose linear span is dense in $\ker(A)$ and for which there is a biorthogonal sequence $\{k_i^*\}_{i=1}^{\infty}$ in $\ker(A)^*$ such that $\langle k_i^*, k_j \rangle = \delta_{ij}$ for all i and j and for which $\sup \|k_i^*\| < +\infty$ (see, e.g., [4]). Let $T: \ker(A) \to N_1$ be defined by

$$T(k) = \sum_{i=1}^{\infty} \frac{1}{2^i} \langle k_i^*, k \rangle n_i.$$

Clearly T is compact and $\overline{\operatorname{ran}(T)} = N_1$.

Now let $S = T \circ P: X \to Y$, so S is also compact and $\overline{\operatorname{ran}(S)} = N_1$. Since $N_1 \subset N$ and $Y = \operatorname{ran}(A) \oplus N$ one easily verifies that $\overline{\operatorname{ran}(A+S)} = \operatorname{ran}(A) \oplus \overline{\operatorname{ran}(S)} = \operatorname{ran}(A) \oplus N_1$. If this last were complemented in Y, then N_1 would be also, a contradiction. Hence we see that if N is not isomorphic to a Hilbert space then $\overline{\operatorname{ran}(A+S)}$ will not be complemented for every compact operator $S: X \to Y$, and it appears that these results are, in a sense, the best ones of this type and generality.

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THE SECOND CENTRE OF A WREATH PRODUCT OF GROUPS

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In this note we shall suppose that A and B are non trivial groups such that B acts in a faithful but not necessarily transitive way, on a set Ω. We shall assume that all B-orbits Ω_i, $i\in I$ have length greater than 1. As usual B_ω is the stabiliser in B of $\omega\in\Omega$. We shall denote by C_i the normal closure in B of B_ω; this subgroup depends on the orbit Ω_i of ω.

We shall write A Wr B for the unrestricted wreath product of A and B corresponding to the mentioned action of B on Ω. The main definitions and results on wreath products we use can be found in [1] and [2]. By $A^\wedge$ we shall denote the base group of A Wr B and, more generally, $S^\wedge$ will represent the subgroup of $A^\wedge$ associated to S, subgroup of A. Finally, if no confusion is induced, we shall identify B with its natural isomorphic copy in A Wr B.

The aim of this note is the determination of the structure of the second centre $Z_2(A \text{ Wr } B)$. As an initial remark let us remember that the centre of A Wr B is given by all pairs $(f, 1)$ such that $f\colon \Omega\to Z(A)$ is constant on each B-orbit Ω_i (see [2], Corollary 3.4 for the regular case).

LEMMA 1. *Under the hypothesis assumed at the beginning for the groups A and B, one has:*

(i) $Z_2(A \text{ Wr } B)\cap A^\wedge\subseteq Z(A)^\wedge$.

(ii) *An element* $(f, 1)$ *in* $A^\wedge$ *is in* $Z_2(A \text{ Wr } B)$ *if and only if* $f(\omega)\in Z(A)$ *for all* $\omega\in\Omega$ *and for each* $b\in B$*; the element* $z_f(b)=f(\omega)^{-1}f(\omega b^{-1})$ *is independent of* $\omega\in\Omega_i$.

(iii) *Given* $(f, 1)\in Z_2(A \text{ Wr } B)\cap A^\wedge$, *the function* z_f *of* B *into* $Z(A)^I$ *given by* $z_f(b)_i=f(\omega)^{-1}f(\omega b^{-1})$ $(\omega\in\Omega_i)$ *is a homomorphism whose kernel contains* C, *the intersection of all* C_i.

(iv) $Z_2(A \text{ Wr } B)\cap A^\wedge$ *is isomorphic to the cartesian product* $\prod_{i\in I}[Z(A)\times \times \text{Hom}(B/C_i, Z(A))]$.

PROOF. (i) Take $(f, 1)\in Z_2(A \text{ Wr } B)\cap A^\wedge$. For every $(g, 1)$ one has that the commutator $[(f, 1), (g, 1)]$ belongs to the centre of A Wr B, therefore, for each $i\in I$,

$$f(\omega)^{-1}g(\omega)^{-1}f(\omega)g(\omega)=f(\delta)^{-1}g(\delta)^{-1}f(\delta)g(\delta)$$

for every pair ω, δ in Ω_i. Let us fix $a\in A$ and $\omega\in\Omega$, and consider the function $g_\omega\colon \Omega\to A$ by given $g_\omega(\delta)=1$ for all $\delta\neq\omega$ and $g_\omega(\omega)=a$; then $f(\omega)^{-1}a^{-1}f(\omega)a=1$ for every $\omega\in\Omega$ and $a\in A$ (in fact, take δ in the B-orbit of ω but $\delta\neq\omega$). Therefore $f(\omega)$ is in $Z(A)$ for every ω in Ω, and so $(f, 1)$ is in $Z(A)^\wedge$.

(ii) If $(f, 1)$ is in $Z_2(A \operatorname{Wr} B) \cap A\hat{}$, for every b in B one has

$$[(f, 1), (e, b)] = (h, 1) \in Z(A \operatorname{Wr} B),$$

and therefore $h(\omega) = f(\omega)^{-1} f(\omega b^{-1})$ is constant on each B-orbit Ω_i; moreover using (i), $f(\omega)$ is in $Z(A)$ for each ω in Ω. Conversely, let $f\colon \Omega \to A$ be such that $f(\omega) \in Z(A)$ for every ω in Ω and such that for every b in B, $f(\omega)^{-1} f(\omega b^{-1})$ is constant on each B-orbit Ω_i. Then, for any $(g, c) \in A \operatorname{Wr} B$ one has that the commutator $[(f, 1), (g, c)] = [(p, 1)]$ is such that

$$p(\omega) = f(\omega)^{-1} g(\omega c^{-1})^{-1} f(\omega c^{-1}) g(\omega c^{-1}) = f(\omega)^{-1} f(\omega c^{-1})$$

is independent of ω in Ω_i for each i; therefore $(p, 1)$ is in $Z(A \operatorname{Wr} B)$.

(iii) By the above argument, each element $(f, 1)$ in the second centre of $A \operatorname{Wr} B$, each b in B and each i in I determine an element $z_f(b)_i = f(\omega)^{-1} f(\omega b^{-1})$ in $Z(A)$ which is independent of $\omega \in \Omega_i$. Take $(f, 1)$ in $Z_2(A \operatorname{Wr} B) \cap A\hat{}$; given b, c in B one has

$$\begin{aligned} z_f(b)_i z_f(c)_i &= f(\omega)^{-1} f(\omega b^{-1}) f(\omega)^{-1} f(\omega c^{-1}) = \\ &= f(\omega c^{-1})^{-1} f(\omega c^{-1} b^{-1}) f(\omega)^{-1} f(\omega c^{-1}) = \\ &= f(\omega)^{-1} f(\omega (bc)^{-1}) = z_f(bc)_i. \end{aligned}$$

So z_f is a homomorphism of B to $Z(A)^I$. Moreover, if b is in B and stabilizes ω in Ω_i, then $z_f(b)_i = 1$, therefore b is contained in the kernel of $\pi_i z_f$, where π_i denotes the i-th projection of $Z(A)^I$ onto $Z(A)$; one has:

$$C = \bigcap_{i \in I} C_i \subseteq \bigcap_{i \in I} \ker \pi_i z_f = \ker z_f.$$

(iv) Using (iii), we can say that for each $(f, 1)$ in the intersection of $Z_2(A \operatorname{Wr} B)$ and $A\hat{}$ there is a unique homomorphism $z_{f,i}\colon B/C_i \to Z(A)$ such that $z_{f,i} \sigma_i = \pi_i z_f$, where $\sigma_i\colon B \to B/C_i$ is the natural projection. Once a complete system of representatives of the B-orbits in Ω: $\Delta = \{\omega_i \in \Omega_i | i \in I\}$ is fixed, one can consider the function

$$F_\Delta\colon Z_2(A \operatorname{Wr} B) \cap A\hat{} \to \prod_{i \in I} [Z(A) \times \operatorname{Hom}(B/C_i, Z(A))]$$

given by $F_\Delta(f, 1)_i = (f(\omega_i), z_{f,i})$ which is easily proved to be an injective homomorphism. Let us prove that F_Δ is onto. Fix any pair (a_i, z_i), a_i in $Z(A)$ and z_i a homomorphism from B/C_i to $Z(A)$. Consider the function $f\colon \Omega \to A$ given by $f(\omega) = f(\omega_i) z_i(bC_i) = a_i z_i(bC_i)$ inside the B-orbit Ω_i and such that $b \in B$ and $\omega = \omega_i b^{-1}$.

First of all, f is well defined because if $\omega_i b^{-1} = \omega_i c^{-1}$ then $b^{-1} c \in B_{\omega_i} \subseteq C_i$. It is clear that $f(\omega) \in Z(A)$ for every ω in Ω. On the other hand, given d in B and ω in Ω_i with $\omega = \omega_i b^{-1}$,

$$f(\omega)^{-1} f(\omega d^{-1}) = z_i(bC_i)^{-1} f(\omega_i)^{-1} f(\omega_i) z_i(dbC_i) = z_i(dC_i)$$

is independent of ω in Ω_i. Therefore $(f, 1)$ is in $Z_2(A \operatorname{Wr} B)$. It is easily proved that the i-th component of $F_\Delta(f, 1)$ is (a_i, z_i) so F_Δ is an isomorphism.

NOTE 2. The isomorphism F_Δ in the proof of (iv) depends on the selection of Δ. Take another system of representatives Σ. For each i in I there is a b_i in B such that $\omega_i b_i = \delta_i$ in Σ. Take b in B^I whose i-th component is b_i. If one considers the restriction of the action of B on each orbit Ω_i, one can define for each i the corresponding wreath product $A \operatorname{Wr}_i B$ and then the cartesian product $G = \prod_{i \in I} A \operatorname{Wr}_i B$. The

mapping $A \operatorname{Wr} B \to G$ which associates to the pair (f, c) the element of G whose i-th component is (f_i, c) where f_i denotes the restriction of f to Ω_i is an injective homomorphism wich permits to identify $Z_2(A \operatorname{Wr} B) \cap A^{\hat{}}$ with a certain subgroup of G. This subgroup is invariant under the action of any element of G of the form $(e_i, c_i)_{i \in I}$ where e_i: $\Omega_i \to A$ is the constant function $e_i(\omega)=1$. It can be proved that $F_A = F_\Sigma H$ where H is the automorphism of $Z_2(A \operatorname{Wr} B) \cap A^{\hat{}}$ induced by the element of G $(e_i, b_i)_{i \in I}$ with $\omega_i b_i = \delta_i$.

LEMMA 3. *If A is 2-elementary abelian and all the orbits of B have length 2, one has $Z_2(A \operatorname{Wr} B) = A \operatorname{Wr} B$.*

PROOF. By our hypothesis, $|\Omega_i|=2$; we can write $\Omega_i=\{1, 2\}$. Let (f, b) and (g, c) be two elements in $A \operatorname{Wr} B$. Then $[(f, b), (g, c)]=(h, 1)$ where

$$h(\omega) = f(\omega b^{-1}) g(\omega b^{-1} c^{-1})^{-1} f(\omega b^{-1} c^{-1}) g(\omega c^{-1}).$$

As B is necessarily 2-elementary abelian, in A and in B every element coincides with its inverse, and so

$$h(\omega) = f(\omega b) f(\omega bc) g(\omega c) g(\omega bc).$$

Considering all the possible actions of b and c on $\{1, 2\}$ it follows that $h(1)=h(2)$ for every b and c, so $(h, 1)$ is in the centre of $A \operatorname{Wr} B$ and so this group must be nilpotent of class 2.

LEMMA 4. *If A is not 2-elementary abelian or all the orbits of B have length greater than 2, then $Z_2(A \operatorname{Wr} B) \subsetneqq A^{\hat{}}$.*

PROOF. *Case 1:* A is not abelian. Suppose that (f, b) is in $Z_2(A \operatorname{Wr} B)$ and $b \neq 1$. As the action of B is faithful, there is an ω in Ω such that $\omega b \neq \omega$. Take a in A but not in $Z(A)$ and consider the element $(g, 1)$ in $A \operatorname{Wr} B$ given by

$$g(\omega) = a, \quad g(\delta) = 1 \quad \text{if} \quad \delta \neq \omega.$$

One has: $[(f, b), (g, 1)]=(h, 1) \in Z(A \operatorname{Wr} B)$ and

$$h(\delta) = f(\delta b^{-1}) g(\delta b^{-1})^{-1} f(\delta b^{-1}) g(\delta) \in Z(A)$$

for all δ in Ω. In particular, if $\delta=\omega$ one reaches the contradiction $h(\omega)=a$.

Case 2: A is abelian and all the orbits of B have at least three elements. Take as before, (f, b) in the second centre and such that $b \neq 1$; for every $(g, 1)$ in $A^{\hat{}}$ one has

$$[(f, b), (g, 1)] = (h, 1) \in Z(A \operatorname{Wr} B)$$

where $h(\omega)=g(\omega b^{-1})^{-1} g(\omega)$ $(\omega \in \Omega)$ because A is abelian. Take $\omega \in \Omega$ such that $\omega b \neq \omega$ and choose δ in the B-orbit of ω, different from ω and ωb. There are two possibilities:

(a) $\delta b = \delta$; in this case, take g: $\Omega \to A$ such that $g(\omega b^{-1}) \neq g(\omega)$, then $h(\omega) = g(\omega b^{-1})^{-1} g(\omega) \neq 1$ and $h(\delta)=g(\delta b^{-1})^{-1} g(\delta)=1$ and h is not constant in the B-orbit of ω.

(b) $\delta b \neq \delta$; in this case, take g: $\Omega \to A$ such that $g(\omega b^{-1})=g(\omega)$ and $g(\delta b^{-1}) \neq g(\delta)$. The same contradiction is derived: $h(\omega)=1$ and $h(\delta) \neq 1$. So, in both cases, $b=1$.

Case 3: A is abelian but it is not 2-elementary abelian. As usual, suppose (f, b) in $Z_2(A \operatorname{Wr} B)$ and $b \neq 1$. For every $(g, 1)$ in $A\hat{}$ one has $[(f, b), (g, 1)] = (h, 1) \in Z(A \operatorname{Wr} B)$ and consider the following two possibilities.

(a) b moves some letter ω such that the $\langle b \rangle$-orbit of ω has at least three elements: proceed as in Case 2.

(b) The $\langle b \rangle$-orbits have one or two elements. In this case $|b| = 2$. Denoting by $\{1, 2\}$ one of these orbits with two elements, one has:

$$h(1) = g(1b^{-1})g(1) = g(2)^{-1}g(1),$$

$$h(2) = g(2b^{-1})g(2) = g(1)^{-1}g(2).$$

Take $g: \Omega \to A$ such that $g(1) = 1$ and $g(2) \neq g(2)^{-1}$ (this is possible because A is not 2-elementary abelian); then derive the contradiction $h(1) \neq h(2)$.

As consequence of 1—4, one obtains the main result of this note:

THEOREM 5. *Given A and B non trivial groups such that B is a transitive permutation group on Ω, one has:*

(i) $Z_2(A \operatorname{Wr} B) = A \operatorname{Wr} B$ *if and only if A is 2-elementary abelian and B is cyclic of order 2.*

(ii) *In the remaining cases, $Z_2(A \operatorname{Wr} B)$ is the set of all pairs $(f, 1)$ where $f(\omega) \in Z(A)$ for all ω in Ω and $f(\omega)^{-1} f(\omega b^{-1})$ is independent of ω for all b in B.*

Moreover for each ω in Ω there is an isomorphims F_ω from $Z_2(A \operatorname{Wr} B)$ onto $Z(A) \times \operatorname{Hom}\big(B/C, Z(A)\big)$ given by $F_\omega(f, 1) = \big(f(\omega), z_f\big)$ where $z_f(bC) = f(\omega)^{-1} \times f(\omega b^{-1})$. Here C is the normal closure in B of the stabilizer of any ω in Ω. Finally, given ω and δ in Ω, one has $F_\omega = F_\delta H$, where H is the automorphism of $Z_2(A \operatorname{Wr} B)$ induced by (e, b) with $\omega b = \delta$.

Particular cases. If B is primitive on Ω but not regular, one has $C = B$ and all the isomorphisms F_ω are equal; moreover $Z_2(A \operatorname{Wr} B) = Z(A \operatorname{Wr} B)$. This is the case of the monomial groups (see [3]) except for the case B cyclic of order 2 and A 2-elementary abelian.

If A and B are finite of coprime orders, $\operatorname{Hom}\big(B/C, Z(A)\big)$ is trivial and as before, $Z_2(A \operatorname{Wr} B)$ is equal to $Z(A \operatorname{Wr} B)$.

As a curiosity one can note that every group G can be subnormally embedded in a group X such that $Z(G)$ is isomorphic to $Z(X) = Z_\infty(X)$; in fact, take $X = G \operatorname{Wr} H$ with H primitive and non regular.

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SYNTOPOGENOUS SPACES WITH PREORDER. II (CONTINUITY)

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In [9] a series of notices on preordered syntopogenous spaces had been started. The present part of this series deals with a common generalization of two conditions mentioned below, which were studied independently in preordered topological and uniform spaces, respectively.

Green [8], McCartan [10] and Priestley ([12], p. 508) examined preordered topological spaces $(E, \tau, \leqq)$, in which

$(*)$ $\qquad V \subset E$ is τ-open $\Rightarrow$ $i(V)$ and $d(V)$ are τ-open.

Nachbin ([11], Theorem 10) and Redfield ([13], pp. 652, 457) considered preordered uniform spaces $(E, \mathcal{U}, \leqq)$ such that

$(**)$ for $U \in \mathcal{U}$, there exist $U_1, U_2 \in \mathcal{U}$ with $G \circ U_1 \subset U \circ G$ and $U_2 \circ G \subset G \circ U$

(where G is the graph of the preorder $\leqq$).

0. Introduction

Let E be a set. A *preorder* $\leqq$ on E is a reflexive and transitive relation on E. $(E, \leqq)$ is called a *preordered space*. The preorder $\leqq$ is *linear* iff $x, y \in E$ implies $x \leqq y$ or $y \leqq x$. A subset X is said to be *increasing (decreasing)* in a preordered space $(E, \leqq)$ iff $x \in X$, $x \leqq y$ $(y \leqq x)$ imply $y \in X$. It is *convex* iff $x, z \in X$, $x \leqq y \leqq z$ imply $y \in X$. For an arbitrary $X \subset E$, $i(X) = \{y \in E: x \leqq y, x \in X\}$, $d(X) = \{y \in E: y \leqq x \in X\}$ and $c(X) = i(X) \cap d(X)$, respectively, are smallest of all increasing, decreasing and convex sets containing X in $(E, \leqq)$. For the points $x \in X$, $i(x)$ and $d(x)$ will be written instead of $i(\{x\})$ and $d(\{x\})$. The *graph* of the preorder $\leqq$ is defined by $G(\leqq) = \{(x, y) \in E \times E: x \leqq y\}$. If $V, W \subset E \times E$, we define $V \circ W = \{(x, y) \in E \times E: \exists z \in E, (x, z) \in V, (z, y) \in W\}$. A mapping f of the preordered space $(E, \leqq)$ into a preordered space $(E', \leqq')$ is called *preorder preserving (inversing)* iff $x, y \in E$, $x \leqq y$ imply $f(x) \leqq' f(y)$ $(f(y) \leqq' f(x))$. The *product* of the preordered spaces $(E_j, \leqq_j)$ $(j \in J)$ is a preordered space $(E, \leqq)$, where $E = \mathop{\mathbf{X}}_{j \in J} E_j$, and $(x_j) \leqq (y_j)$ iff $x_j \leqq_j y_j$ for each $j \in J$.

A triplet $(E, \mathcal{S}, \leqq)$ consisting of a set E, a syntopogenous structure $\mathcal{S}$ (see [5]) and a preorder $\leqq$ on E is said to be a *preordered syntopogenous space*. The graph of the *preorder generated by the syntopogenous structure* $\mathcal{S}$ is

$$G(\mathcal{S}) = \cap \{G(\prec): \prec \in \mathcal{S}\},$$

where

$$G(<) = \{(x, y) \in E \times E : x \not< E - y\}$$

(see [2], 3.1). The preordered syntopogenous space $(E, \mathscr{S}, \leqq)$ (or the syntopogenous structure $\mathscr{S}$ on $(E, \leqq)$) is called *increasing (decreasing)* iff $G(\leqq) \subset G(\mathscr{S})$ $(G(\leqq)^{-1} \subset \subset G(\mathscr{S}))$. For an arbitrary $(E, \mathscr{S}, \leqq)$, among the increasing (decreasing) syntopogenous structures coarser than $\mathscr{S}$ there exists a finest one on $(E, \leqq)$, and it is denoted by $\mathscr{S}^u$ $(\mathscr{S}^l)$ (cf. [3], p. 74).

Let us consider the family $\mathscr{E}_c$ of elementary topogenous orders defined by

$$\mathscr{E}_c = \{<_{X,C} : X \subset C \subset E, C \text{ is convex}\},$$

and let ^{a} denote an arbitrary elementary operation (cf. [5], pp. 42 and 69). The preordered syntopogenous space $(E, \mathscr{S}, \leqq)$ is said to be *weakly ^{a}-convex* iff, for any $< \in \mathscr{S}$, there exists a family $\mathscr{E} \subset \mathscr{E}_c$ such that $\{<\} < \mathscr{E}^{ta} < \mathscr{S}$. $(E, \mathscr{S}, \leqq)$ is called *^{a}-convex* iff $\mathscr{S} \sim (\mathscr{S}^u \vee \mathscr{S}^l)^a$ (see [9], cf. [2]—[4]).

1. Continuous preordered syntopogenous spaces

A preordered syntopogenous space $(E, \mathscr{S}, \leqq)$ will be said to be *continuous* iff, for every $< \in \mathscr{S}$, there exists $<_1 \in \mathscr{S}$ such that $A < B$ implies $i(A) <_1 i(B)$ and $d(A) <_1 d(B)$.

REMARKS. Any syntopogenous space $[E, \mathscr{S}]$ can be regarded as a preordered syntopogenous space $(E, \mathscr{S}, =)$, where $=$ is the relation of the equality on E. Such a space is always continuous.

Let us observe that in order to show the continuity of a preordered syntopogenous space $(E, \mathscr{S}, \leqq)$, it is sufficient to verify that, for $< \in \mathscr{S}$, there are orders $<', <'' \in \mathscr{S}$ such that $A < B$ implies $i(A) <' i(B)$ and $d(A) <'' d(B)$ (see axiom (S_1) of [5], p. 58).

As to the term of a "continuous" preordered syntopogenous space, let us recall that a mapping f of the syntopogenous space $[E, \mathscr{S}]$ into the syntopogenous space $[E', \mathscr{S}']$ is $(\mathscr{S}, \mathscr{S}')$-continuous iff the following condition is satisfied:

(1) For $<' \in \mathscr{S}'$ there is $< \in \mathscr{S}$ such that $A <' B$ implies $f^{-1}(A) < f^{-1}(B)$.

Further let us consider a set-valued mapping f: $E \to 2^{E'}$, and, as it is usual, define, for every $A \subset E'$

$$f^{-1}(A) = \{x \in E : f(x) \cap A \neq \emptyset\}. \tag{2}$$

Let f be called $(\mathscr{S}, \mathscr{S}')$-*continuous* iff it fulfils (1). In a preordered space we have two mappings F, G: $E \to 2^E$ determined by $F(x) = d(x)$ and $G(x) = i(x)$ for $x \in E$. Then obviously

$$F^{-1}(A) = i(A) \quad \text{and} \quad G^{-1}(A) = d(A) \quad \text{for} \quad A \subset E,$$

therefore $(E, \mathscr{S}, \leqq)$ is continuous iff the set-valued mappings F and G are $(\mathscr{S}, \mathscr{S})$-continuous. □

First of all we give a summary of the classical preordered spaces having a continuous syntopogenous associated.

(1.1) THEOREM. *Let $(E, \mathscr{T}, \leqq)$ be a preordered topological space and τ be the classical topology associated with $\mathscr{T}$. $(E, \mathscr{T}, \leqq)$ is continuous iff $i(V)$ and $d(V)$ are τ-open for any τ-open set $V \subset E$* (i.e. $(E, \tau, \leqq)$ is *continuous* in the sense of McCartan [10]).

PROOF. If $(E, \mathscr{T}, \leqq)$ is continuous, and V is τ-open, then $V \prec V$, where $\mathscr{T} = \{\prec\}$. From here $i(V) \prec i(V)$ and $d(V) \prec d(V)$, therefore $i(V)$ and $d(V)$ are also τ-open. Conversely, if $i(V)$ and $d(V)$ are τ-open for each τ-open set V, then $A \prec B$ implies $A \subset V \subset B$, where V is τ-open, consequently $i(A) \subset i(V) \subset i(B)$ and $d(A) \subset d(V) \subset d(B)$ give $i(A) \prec i(B)$ and $d(A) \prec d(B)$. □

REMARK. McCartan [10] defined the notion of an *anti-continuous* preordered (classical) topological space $(E, \tau, \leqq)$ so that $i(F)$ and $d(F)$ are τ-closed for any τ-closed set $F \subset E$. This is another special case of the "syntopogenous" continuity, because with the notations of (1.1), $(E, \tau, \leqq)$ is anti-continuous iff $(E, \mathscr{T}^c, \leqq)$ is continuous. □

(1.2) THEOREM. *Denote by δ the quasi-proximity associated with the topogenous structure $\mathscr{T}$ on the preordered space $(E, \leqq)$. Then a necessary and sufficient condition for $(E, \mathscr{T}, \leqq)$ to be continuous is that $i(A) \delta X$ (or $d(A) \delta X$) implies $A \delta X$ for any decreasing (or increasing) set X in $(E, \leqq)$.*

PROOF. Suppose that $(E, \mathscr{T}, \leqq)$ is continuous, and e.g. $A \bar{\delta} X$ for a decreasing set X in $(E, \leqq)$. Then, for $\mathscr{T} = \{\prec\}$, we have $A \prec E - X$, where $E - X$ is increasing, thus $i(A) \prec i(E - X) = E - X$, i.e. $i(A) \bar{\delta} X$. Conversely, if the condition given for δ is satisfied, then $A \prec B$ means $A \bar{\delta} E - B \supset E - i(B)$. $E - i(B)$ is decreasing, therefore $i(A) \bar{\delta} E - i(B)$, that is $i(A) \prec i(B)$. In the same manner we obtain $d(A) \prec d(B)$, too, so that $(E, \mathscr{T}, \leqq)$ is continuous. □

(1.3) THEOREM. *Let $(E, \mathscr{S}, \leqq)$ be a preordered biperfect syntopological space, and $\mathscr{U}$ be the quasi-uniformity associated with $\mathscr{S}$. $(E, \mathscr{S}, \leqq)$ is continuous iff, for every $U \in \mathscr{U}$, there exists $U_1 \in \mathscr{U}$ such that $G \circ U_1 \subset U \circ G$ and $G^{-1} \circ U_1 \subset U \circ G^{-1}$, where $G = G(\leqq)$.*

PROOF. Suppose $\prec$, $\prec_1 \in \mathscr{S}$ and let $U, U_1 \in \mathscr{U}$ be associated with $\prec$ and $\prec_1$ respectively. We show that the following conditions are equivalent:

$$A \prec B \Rightarrow i(A) \prec_1 i(B) \quad \text{for any} \quad A, B \subset E; \tag{1.3.1}$$

$$G \circ U_1 \subset U \circ G. \tag{1.3.2}$$

(1.3.1)⇒(1.3.2). Assume $(x, y) \in G \circ U_1$. Then $x \prec U(x)$, therefore $i(x) \prec_1 \prec_1 i(U(x))$. There exists $x_0 \in E$ such that $x \leqq x_0$ and $(x_0, y) \in U_1$. $x_0 \in i(x)$, thus $y \in i(U(x))$. This means that a point $y_0 \in E$ can be found such that $(x, y_0) \in U$ and $y_0 \leqq y$, that is $(x, y) \in U \circ G$.

(1.3.2)⇒(1.3.1). Let us suppose indirectly that $A \prec B$, but $i(A) \prec_1 i(B)$ is not true. Then there exist $x \in i(A)$ and $y \in E - i(B)$, for which $x \prec_1 E - y$ does not hold, that is $(x, y) \in U_1$. $x_0 \leqq x$ for some $x_0 \subset A$, thus $(x_0, y) \in G \circ U_1 \subset U \circ G$. Then there

is $y_0 \in E$, for which $(x_0, y_0) \in U$, $y_0 \leqq y$. We have $x_0 \in A < B$, therefore $y_0 \in B$, consequently $y \in i(B)$, which contradicts the choice of y.

One can complete the proof with the similar verification of the equivalence of the following properties:

(1.3.3) $$A < B \Rightarrow d(A) <_1 d(B) \quad \text{for any} \quad A, B \subset E;$$

(1.3.4) $$G^{-1} \circ U_1 \subset U \circ G^{-1}. \quad \square$$

(1.4) COROLLARY. *If under the conditions of* (1.3) $\mathscr{S}$ *is symmetrical (in other words, if* $\mathscr{U}$ *is a uniformity), then* $(E, \mathscr{S}, \leqq)$ *is continuous iff, for every* $U \in \mathscr{U}$, *there exists* $U_1 \in \mathscr{U}$ *such that* $G \circ U_1 \subset U \circ G$ *and* $U_1 \circ G \subset G \circ U$.

PROOF. $U = U^{-1}$ for every $U \in \mathscr{U}$, hence $G^{-1} \circ U_1 \subset U \circ G^{-1}$ is equivalent to $U_1 \circ G \subset G \circ U$. $\square$

(1.5) EXAMPLE. Let $\mathscr{S}$ denote one of the syntopogenous structures $\mathscr{I}$, $\mathscr{I}^c$, $\mathscr{I}^t$, $\mathscr{I}^{ct}$, $\mathscr{I}^{tp}$, $\mathscr{I}^{ctp}$, $\mathscr{I}^s$, $\mathscr{I}^{st}$, $\mathscr{H} = \mathscr{I}^{sb}$ and $\mathscr{H}^{tp} = \mathscr{I}^{stp}$ on the naturally ordered real line $(R, \leqq)$ (see [5], (7.12) and p. 89). Then $(R, \mathscr{S}, \leqq)$ *is continuous.*

Since $\mathscr{I} \sim \mathscr{I} \vee \mathscr{O}_R$, where the indiscrete syntopogenous structure $\mathscr{O}_R$ of R is increasing, $\mathscr{I}$ is ^{i}-convex, thus each one of the spaces listed above is ^{i}-, or ^{p}-convex. Hence the statement concerning their continuity can be deduced from a more general theorem:

(1.6) THEOREM. *Any* ^{i}- *or* ^{p}-*convex linearly preordered syntopogenous space is continuous.*

PROOF. First of all let us consider an increasing set I and a decreasing one D in a linearly preordered space $(E, \leqq)$.

(1.6.1) If $I \cap D \neq \emptyset$, then $i(I \cap D) = I$ and $d(I \cap D) = D$. In fact, $I \cap D \subset I$ implies $i(I \cap D) \subset i(I) = I$. Conversely, if $x \in I$ and $y \in I \cap D$, then $y \leqq x$ implies $x \in i(I \cap D)$, or $x \leqq y$ gives $x \in D$, therefore $x \in I \cap D \subset i(I \cap D)$. Thus $I \subset i(I \cap D)$ is also true. The other equality is analogous.

Let $(E, \mathscr{S}, \leqq)$ be ^{i}-convex. Then $\mathscr{S} \sim \mathscr{S}_1 \vee \mathscr{S}_2$, where $\mathscr{S}_1$ is increasing, $\mathscr{S}_2$ is decreasing on $(E, \leqq)$ (cf. [9], (2.4)). For $< \in \mathscr{S}$, there exist $<_1 \in \mathscr{S}_1$ and $<_2 \in \mathscr{S}_2$ such that $< \subset (<_1 \cup <_2)^q$. Suppose $<_1' \in \mathscr{S}_1$, $<_1 \subset <_1'^3$, $<_2' \in \mathscr{S}_2$, $<_2 \subset <_2'^3$, finally $<_1' \cup <_2' \subset <' \in \mathscr{S}$. If $\emptyset \neq A < B$, then $A \subset \bigcup_{j=1}^{n} (X_j \cap Y_j)$, $\bigcup_{j=1}^{n} (U_j \cap V_j) \subset B$, where $X_j <_1 U_j$, $Y_j <_2 V_j$ and $X_j \cap Y_j \neq \emptyset$. Assume $X_j <_1' X_j' <_1' U_j' <_1' U_j$ and $Y_j <_2' Y_j' <_2' V_j' <_2' V_j$ $(1 \leqq j \leqq n)$. Then by [9], (1.2) there are increasing (decreasing) sets C_j, D_j (F_j, G_j) such that $X_j \subset C_j \subset X_j'$, $U_j' \subset D_j \subset U_j$ $(Y_j \subset F_j \subset Y_j'$, $V_j' \subset G_j \subset V_j)$, so that

$$i(A) \subset \bigcup_{j=1}^{n} i(C_j \cap F_j) = \bigcup_{j=1}^{n} C_j <' \bigcup_{j=1}^{n} D_j = \bigcup_{j=1}^{n} i(D_j \cap G_j) \subset i(B),$$

and

$$d(A) \subset \bigcup_{j=1}^{n} d(C_j \cap F_j) = \bigcup_{j=1}^{n} F_j <' \bigcup_{j=1}^{n} G_j = \bigcup_{j=1}^{n} d(D_j \cap G_j) \subset d(B),$$

hence $i(A) <' i(B)$ and $d(A) <' d(B)$.

Let $(E, \mathscr{S}, \leqq)$ be ^{p}-convex. Then $\mathscr{S} \sim (\mathscr{S}_1 \vee \mathscr{S}_2)^p$, where $\mathscr{S}_1$ is increasing, $\mathscr{S}_2$ is decreasing on $(E, \leqq)$. For $\prec \in \mathscr{S}$, put $\prec_1 \in \mathscr{S}_1$, $\prec_2 \in \mathscr{S}_2$ such that $\prec \subset (\prec_1 \cup \prec_2)^{qp}$, further $\prec_1' \in \mathscr{S}_1$, $\prec_1 \subset \prec_1'^3$, $\prec_2' \in \mathscr{S}_2$, $\prec_2 \subset \prec_2'^3$, $\prec' \in \mathscr{S}$, $\prec_1' \cup \prec_2' \subset \prec'$, and $\prec'' \in \mathscr{S}$, $\prec'^p \subset \prec''$ (see [9], (2.3)). If $\emptyset \neq A \prec B$, then, for every $x \in A$, we have $x \in X \cap Y$, $U \cap V \subset B$, where $X \prec_1 U$, $Y \prec_2 V$. Then repeating the above reasoning, we obtain $i(x) \prec' i(B)$, $d(x) \prec' d(B)$, thus

$$i(A) = \bigcup_{x \in A} i(x) \prec'' i(B) \quad \text{and} \quad d(A) = \bigcup_{x \in A} d(x) \prec'' d(B). \quad \square$$

(1.7) EXAMPLE. It is worth observing that, in general, there is no implication between the convexity and the continuity of a preordered syntopogenous space.

Denote by Q the set of the rational numbers, and let us consider the order $\leqq_0$ defined on the real line R as follows:

$$x \leqq_0 y \Leftrightarrow x \leqq y, \quad \text{if} \quad x, y \in Q, \quad x = y \quad \text{otherwise.}$$

Putting $\mathscr{T} = \mathscr{H}^{tp}$, $(R, \mathscr{T}, \leqq_0)$ *is* ^{p}*-convex, but it is not continuous.* In fact, for every $p \in R$, the set (p, ∞) $\big((-\infty, p)\big)$ is increasing (decreasing) and open in $(R, \mathscr{T}, \leqq_0)$, thus $\mathscr{I}^{ctp} \prec \mathscr{T}^u \prec \mathscr{T}$ and $\mathscr{I}^{tp} \prec \mathscr{T}^l \prec \mathscr{T}$ (see e.g. [3], 3.3). From here

$$\mathscr{T} = \mathscr{I}^{stp} = (\mathscr{I}^c \vee \mathscr{I})^{tp} = (\mathscr{I}^{ctp} \vee \mathscr{I}^{tp})^{tp} \sim (\mathscr{I}^{ctp} \vee \mathscr{I}^{tp})^p \prec (\mathscr{T}^u \vee \mathscr{T}^l)^p \prec \mathscr{T},$$

that is $\mathscr{T} \sim (\mathscr{T}^u \vee \mathscr{T}^l)^p$, i.e. $(R, \mathscr{T}, \leqq_0)$ is ^{p}-convex. On the other hand, the interval $(0, 1)$ is $\mathscr{T}$-open, but $i\big((0, 1)\big) = (0, 1) \cup \big([1, \infty) \cap Q\big)$ and $d\big((0, 1)\big) = (0, 1) \cup \cup \big((-\infty, 0] \cap Q\big)$ are not, therefore $(R, \mathscr{T}, \leqq_0)$ is not continuous.

Conversely, let $\mathscr{T}_0$ denote the topology on the naturally ordered real line $(R, \leqq)$, in which the neighbourhood base of a point $x \in R$ is $\{\{x\} \cup \big((p, q) \cap Q\big) : p, q \in R, p < x < q\}$. Then obviously $(R, \mathscr{T}_0, \leqq)$ *is not locally convex,* at the same time *it is continuous.* Indeed, if V is $\mathscr{T}_0$-open, and $x \in i(V)$ is arbitrary, then there exists $y \in V$ with $y \leqq x$. For some $p, q \in R$, we have $p < y < q$ and $W = \{y\} \cup \big((p, q) \cap Q\big) \subset V$. Using that Q is dense in R, we obtain $x \in (p, \infty) = i(W) \subset i(V)$. Since (p, ∞) is $\mathscr{T}_0$-open, $i(V)$ is also $\mathscr{T}_0$-open. The verification of that $d(V)$ is $\mathscr{T}_0$-open is analogous. $\square$

It is usual to define a *topological lattice* as a pair $[E, \tau]$ consisting of a lattice E and a (classical) topology τ such that the lattice operations $\wedge$ and $\vee$ are $(\tau \times \tau, \tau)$-continuous. Priestley ([12], p. 509) noted that in this case $(E, \tau, \leqq)$ has the property $(*)$, where $\leqq$ is the natural order of E defined for $x, y \in E$ by

$$x \leqq y \Leftrightarrow x \wedge y = x, \tag{1.8.1}$$

or equivalently

$$x \leqq y \Leftrightarrow x \vee y = y. \tag{1.8.2}$$

The syntopogenous structure corresponding to $\tau \times \tau$ is $(\mathscr{T} \times \mathscr{T})^{tp} \sim (\mathscr{T} \times \mathscr{T})^p$, where the topology $\mathscr{T}$ is associated with τ. Denoting by $\mathscr{D}_E$ the discrete syntopogenous structure $\{\subset\}$ of E, which is the finest one on E, we can generalize and render more strict the observation of Priestley as follows:

(1.9) THEOREM. *Suppose that $\leqq$ is the natural order of the lattice E, and $\mathscr{S} \sim \mathscr{S}^a$ is a syntopogenous structure on E, where $^a = ^i$, ^{p} or ^{b}. If the lattice operations $\wedge$ and $\vee$ are $((\mathscr{D}_E \times \mathscr{S})^a, \mathscr{S})$-continuous, then $(E, \mathscr{S}, \leqq)$ is also continuous.*

PROOF. Let pr: $E\times E\to E$ be the projection of the product onto its second component, and put $f(x,y)=x\wedge y$ and $g(x,y)=x\vee y$ for $x, y\in E$. Then for each $A\subset E$

$$i(A)=\mathrm{pr}\left(f^{-1}(A)\right) \quad \text{and} \quad d(A)=\mathrm{pr}\left(g^{-1}(A)\right). \tag{1.9.1}$$

In fact, $x\in\mathrm{pr}\left(f^{-1}(A)\right)$ implies $(a,x)\in f^{-1}(A)$ for some $a\in E$, hence $a\wedge x\in A$. Since $a\wedge x\leqq x$, we get $x\in i(A)$. Conversely, if $x\in i(A)$, then $a\leqq x$ for some $a\in A$, which means $a\wedge x=a$. From here $(a,x)\in f^{-1}(A)$, and $x\in\mathrm{pr}\left(f^{-1}(A)\right)$. The other equality is completely similar.

Suppose ${}^a=^i$. If $<\in\mathscr{S}$, then there exists $<_1\in\mathscr{S}$ such that $A<B$ implies

$$f^{-1}(A)\subset\bigcup_{j=1}^{n}(A_j\times A_j') \quad \text{and} \quad \bigcup_{j=1}^{n}(B_j\times B_j')\subset f^{-1}(B),$$

where $A_j\subset B_j$ and $A_j'<_1 B_j'$ for $1\leqq j\leqq n$ (see [5], (6.4), (10.1) and (11.10)). Without loss of generality we can assume $A\neq\emptyset$ and $A_j\neq\emptyset$ $(1\leqq j\leqq n)$, thus

$$i(A)=\mathrm{pr}\left(f^{-1}(A)\right)\subset\mathrm{pr}\left(\bigcup_{j=1}^{n}(A_j\times A_j')\right)=\bigcup_{j=1}^{n}A_j'<_1\bigcup_{j=1}^{n}B_j'=$$

$$=\mathrm{pr}\left(\bigcup_{j=1}^{n}(B_j\times B_j')\right)\subset\mathrm{pr}\left(f^{-1}(B)\right)=i(B),$$

so that $i(A)<_1 i(B)$. Writing d and g instead of i and f, $d(A)<_1 d(B)$ can be obtained.

Put ${}^a=^b$. For $<\in\mathscr{S}$, there is $<_1\in\mathscr{S}$ such that $A<B$, $(a,x)\in f^{-1}(A)$ imply the existence of sets A', A'', B', B'', for which $(a,x)\in A'\times A''$, $B'\times B''\subset f^{-1}(B)$, $A'\subset B'$, $A''<_1 B''$. Finally suppose $<_1^p\subset<_2\in\mathscr{S}$. If $x\in i(A)=\mathrm{pr}\left(f^{-1}(A)\right)$, then $(a,x)\in f^{-1}(A)$ for some $a\in E$, therefore choosing the sets A', A'', B', B'' mentioned above, we have $a\in A'\subset B'$ and $x\in A''<_1 B''=\mathrm{pr}(B'\times B'')\subset\mathrm{pr}\left(f^{-1}(B)\right)=i(B)$, that is $x<_1 i(B)$ for any $x\in i(A)$. From here $i(A)<_2 i(B)$. The proof of $d(A)<_2 d(B)$ is dual.

If ${}^a=^b$, then $(\mathscr{D}_E\mathbf{X}\mathscr{S})^b=(\mathscr{D}_E^b\mathbf{X}\mathscr{S}^b)^p\sim(\mathscr{D}_E\mathbf{X}\mathscr{S})^p$, and $\mathscr{S}\sim\mathscr{S}^p$, thus we have to do the verification for ${}^a=^p$ again (cf. [5], (11.25)). □

2. Characterizations of continuous spaces

i: $2^E\to 2^E$ and d: $2^E\to 2^E$ are *monotone mappings* in the sense that $A\subset B\subset E$ implies $i(A)\subset i(B)$ and $d(A)\subset d(B)$, therefore we can construct two syntopogenous structures denoted by $i(\mathscr{S})$ and $d(\mathscr{S})$ for every preordered syntopogenous space $(E,\mathscr{S},\leqq)$. This construction for arbitrary monotone mapings and syntopogenous spaces is due to Császár [6].

(2.1) LEMMA. *Let $(E,\leqq)$ be a preordered space. For any semi-topogenous order $<$ on E define the orders $<'$ and $<''$ on E in the following way:*

(2.1.1) *$A<'B$ ($A<''B$) iff there exist sets $X, Y\subset E$ such that $X<Y$ and $A\subset i(X)$, $i(Y)\subset B$ ($A\subset d(X)$, $d(Y)\subset B$).*

Then $<'$ and $<''$ are also semi-topogenous orders on E. Put

$$i(<)=<'^{q} \quad \text{and} \quad d(<)=<''^{q}. \tag{2.1.2}$$

If $\mathscr{S}$ is a syntopogenous structure on E, then

$$i(\mathscr{S})=\{i(<):\ <\in\mathscr{S}\} \quad \text{and} \quad d(\mathscr{S})=\{d(<):\ <\in\mathscr{S}\} \tag{2.1.3}$$

are syntopogenous structures on E, too.

PROOF. [6], (2.3), (3.1) and (3.19). □

Let us introduce the notation $c(\mathscr{S})=i(\mathscr{S})\vee d(\mathscr{S})$ for an arbitrary preordered syntopogenous space $(E, \mathscr{S}, \leqq)$.

(2.2) REMARKS. Let τ be the classical topology associated with the topology $\mathscr{T}$ on the preordered space $(E, \leqq)$. Then the systems $\mathfrak{S}_i=\{i(V):\ V \text{ is } \tau\text{-open}\}$, $\mathfrak{S}_d=\{d(V):\ V \text{ is } \tau\text{-open}\}$ and $\mathfrak{S}_c=\mathfrak{S}_i\cup\mathfrak{S}_d$ are the subbases of the open sets of three classical topologies denoted by τ_i, τ_d, τ_c. It is clear that $\tau_c=\sup\{\tau_i, \tau_d\}$, and these classical topologies are associated with $i(\mathscr{T})^p$, $d(\mathscr{T})^p$ and $c(\mathscr{T})^{tp}\sim c(\mathscr{T})^p$ (cf. [5], (8.100)). τ_c was studied in detail by Green [8] and McCartan [10].

Let us consider the quasi-proximity δ associated with a topogenous structure $\mathscr{T}$ on the preordered space $(E, \leqq)$. Then $i(\mathscr{T})$ and $d(\mathscr{T})$ are associated with the quasi-proximities δ_i and δ_d respectively, which can be defined as follows:

$$\begin{cases} A\bar{\delta}_i B \quad \text{iff} \quad A\subset\bigcap_{j=1}^{n} i(X_j), \quad B\subset\bigcup_{j=1}^{n} Y_j, \quad \text{where} \\ X_j\delta Y_j, \quad \text{and} \quad Y_j \quad \text{is decreasing} \quad (1\leqq j\leqq n). \end{cases} \tag{2.2.1}$$

$$\begin{cases} A\bar{\delta}_d B \quad \text{iff} \quad A\subset\bigcap_{j=1}^{n} d(X_j), \quad B\subset\bigcup_{j=1}^{n} Y_j, \quad \text{where} \\ X_j\delta Y_j, \quad \text{and} \quad Y_j \quad \text{is increasing} \quad (1\leqq j\leqq n). \end{cases} \tag{2.2.2}$$

$\delta_c=\sup\{\delta_i, \delta_d\}$ is associated with $c(\mathscr{T})^t\sim c(\mathscr{T})$.

In fact, if $\mathscr{T}=\{<\}$ and $Ai(<)E-B$, then $A=\bigcup_{h=1}^{m} A_h$, $E-B=\bigcap_{j=1}^{n} B'_j$, where $A_h\subset i(X_{hj})$, $i(Y'_{hj})\subset B'_j$ and $X_{hj}<Y'_{hj}$ $(1\leqq h\leqq m,\ 1\leqq j\leqq n)$ (see [5], (3.7)). Then

$$Y'_j=i\left(\bigcup_{h=1}^{m} Y'_{hj}\right)=\bigcup_{h=1}^{m} i(Y'_{hj})\subset B'_j \quad \text{and} \quad X_j=\bigcup_{h=1}^{m} X_{hj}<\bigcup_{h=1}^{m} Y'_{hj}\subset Y'_j,$$

where Y'_j is increasing, so that $X_j\bar{\delta}E-Y'_j=Y_j$, and Y_j is decreasing. We have

$$A=\bigcup_{h=1}^{m} A_h\subset\bigcup_{h=1}^{m} i(X_{hj})=i\left(\bigcup_{h=1}^{m} X_{hj}\right)=i(X_j)$$

for any $1\leqq j\leqq n$, hence $A\subset\bigcap_{j=1}^{n} i(X_j)$. Finally

$$B=\bigcup_{j=1}^{n}(E-B'_j)\subset\bigcup_{j=1}^{n}(E-Y'_j)=\bigcup_{j=1}^{n} Y_j.$$

Conversely, assume $A\bar{\delta}_l B$. Then $X_j < E - Y_j = B'_j = i(B'_j)$, and

$$A \subset \bigcap_{j=1}^{n} i(X_j) i(<) \bigcap_{j=1}^{n} i(B'_j) = \bigcap_{j=1}^{n} B'_j \subset E - B,$$

that is $Ai(<)E-B$. The proof of (2.2.2) is analogous.

Let $(E, \mathcal{S}, \leqq)$ be a preordered biperfect syntopological space, $\mathcal{U}$ be the quasi-uniformity associated with $\mathcal{S}$, and put $G=G(\leqq)$. If $< \in \mathcal{S}$, and $U=U_<$, then denote by U_i and U_d the reflexive relations corresponding to $i(<)^b$ and $d(<)^b$ respectively. Then

(2.2.3) U_i (U_d) is the largest of all subsets V of $E \times E$ such that $G \circ V \subset U \circ G$ ($G^{-1} \circ V \subset U \circ G^{-1}$).

$i(\mathcal{S})^b$, $d(\mathcal{S})^b$ and $c(\mathcal{S})^b$ are associated with $\mathcal{U}_i = \{U_i : U \in \mathcal{U}\}$, $\mathcal{U}_d = \{U_d : U \in \mathcal{U}\}$ and $\mathcal{U}_c = \sup\{\mathcal{U}_i, \mathcal{U}_d\}$ (cf. [5], p. 95 and (8.99)).

For the sake of the verification of (2.2.3), firstly we show $G \circ U_i \subset U \circ G$. Suppose $(x, y) \in G \circ U_i$. Then $x \leqq t$ and $(t, y) \in U_i$ for some $t \in E$. Since $x < U(x)$, we have $t \in i(x) i(<) i(U(x))$. $ti(<)E-y$ cannot be true, hence $y \in i(U(x))$, that is $(x, v) \in U$ and $v \leqq y$ for a suitable $v \in E$. This means $(x, y) \in U \circ G$. After this assume $V \subset E \times E$, $G \circ V \subset U \circ G$ and $(x, y) \in V - U_i$. Then $xi(<)E-y$. There exist sets $A, B \subset E$ such that $A < B$, $x \in i(A)$ and $i(B) \subset E-y$. $a \leqq x$ for some $a \in A$, therefore $(a, y) \in G \circ V$, hence we can find $t \in E$ so that $(a, t) \in U$ and $t \leqq y$. Then $t \in B$, thus $y \in i(B)$, but this is impossible. Therefore $V \subset U_i$, which was to be proven. The case of U_d is similar. □

(2.3) LEMMA. *For any preordered syntopogenous space $(E, \mathcal{S}, \leqq)$, $i(\mathcal{S})$ is increasing, $d(\mathcal{S})$ is decreasing on $(E, \leqq)$. We have $\mathcal{S}^u < i(\mathcal{S})$ and $\mathcal{S}^l < d(\mathcal{S})$.*

PROOF. If $X \subset E$, then $i(X)$ ($d(X)$) is increasing (decreasing). The intersection and the union of increasing (decreasing) sets are also increasing (decreasing), therefore $Ai(<)B$ ($Ad(<)B$) implies $A \subset C \subset B$ for some increasing (decreasing) set C (see (2.1)). Thus $i(\mathcal{S})$ is increasing, $d(\mathcal{S})$ is decreasing by [9], (1.2). If $< \in \mathcal{S}^u$, then look for $<_1 \in \mathcal{S}^u$ such that $< \subset <_1^3$ and choose $<_2 \in \mathcal{S}$ so that $<_1 \subset <_2$. Then $A < B$ implies $A <_1 C <_1 D <_1 B$, and $A \subset i(A) \subset C <_2 D \subset i(D) \subset B$ by [9], (1.2), because $\mathcal{S}^u$ is increasing. This shows $< \subset i(<_2)$, and in general, $\mathcal{S}^u < i(\mathcal{S})$. The proof of $\mathcal{S}^l < d(\mathcal{S})$ is dual. □

Now we can characterize the continuous preordered syntopogenous spaces as follows:

(2.4) THEOREM. *The following statements are equivalent for every preordered syntopogenous space $(E, \mathcal{S}, \leqq)$:*

(2.4.1) *$(E, \mathcal{S}, \leqq)$ is continuous.*

(2.4.2) *$i(\mathcal{S}) < \mathcal{S}$ and $d(\mathcal{S}) < \mathcal{S}$.*

(2.4.3) *$c(\mathcal{S}) < \mathcal{S}$.*

(2.4.4) *$c(\mathcal{S})^a < \mathcal{S}$ for any elementary operation ^{a} such that $\mathcal{S}^a \sim \mathcal{S}$.*

(2.4.5) *$i(\mathcal{S}) \sim \mathcal{S}^u$ and $d(\mathcal{S}) \sim \mathcal{S}^l$.*

PROOF. (2.4.1)⇔(2.4.2). If $(E, \mathscr{S}, \leqq)$ is continuous, then, for $\prec \in \mathscr{S}$, there exists $\prec_1 \in \mathscr{S}$ such that $A \prec B$ implies $i(A) \prec_1 i(B)$ and $d(A) \prec_1 d(B)$. This means $i(\prec) \subset \prec_1^q = \prec_1$ and $d(\prec) \subset \prec_1^q = \prec_1$ by (1.2). Conversely, if $i(\mathscr{S}) < \mathscr{S}$ and $d(\mathscr{S}) < \mathscr{S}$, then, for $\prec \in \mathscr{S}$, there are $\prec_1 \in \mathscr{S}$ and $\prec_2 \in \mathscr{S}$ such that $i(\prec) \subset \prec_1$ and $d(\prec) \subset \prec_2$. Suppose $\prec' \in \mathscr{S}$, $\prec_1 \cup \prec_2 \subset \prec'$. In this case $A \prec B$ implies $i(A) i(\prec) i(B)$ and $d(A) d(\prec) d(B)$, thus $i(A) \prec' i(B)$ and $d(A) \prec' d(B)$.

(2.4.2)⇔(2.4.5). $\mathscr{S}^u$ ($\mathscr{S}^l$) is the finest one of all increasing (decreasing) syntopogenous structures on $(E, \leqq)$ coarser than $\mathscr{S}$, therefore the equivalence of the statements is the consequence of (2.3).

(2.4.2)⇔(2.4.3) is obvious.

(2.4.3)⇔(2.4.4). Since $c(\mathscr{S}) < c(\mathscr{S})^a$, $c(\mathscr{S})^a < \mathscr{S}$ implies $c(\mathscr{S}) < \mathscr{S}$. If $\mathscr{S}^a \sim \mathscr{S}$, then from $c(\mathscr{S}) < \mathscr{S}$ the inequality $c(\mathscr{S})^a < \mathscr{S}^a \sim \mathscr{S}$ follows. □

REMARK. In view of (2.4.1)⇔(2.4.4) we can deduce the condition of the continuity of a preordered topological, topogenous or biperfect syntopological space in the associated "classical" space. These spaces have the properties described in (1.1)—(1.3) iff, with the notations of (2.2), τ_c, δ_c, $\mathscr{U}_c$ is coarser than τ, δ or $\mathscr{U}$. (Cf. [8], Theorem 1, and [10], Theorem 3.)

In order to give another characterization of the continuous preordered syntopogenous spaces, let us consider two operations defined on the set of the bounded real functions of a preordered space.

(2.5) LEMMA. *Let f be a bounded real function on the preordered space $(E, \leqq)$. Define the functions $f_\bullet$ and $f^\bullet$ at $x \in E$ as follows:*

$$f_\bullet(x) = \inf \{f(y) : y \in E,\ x \leqq y\}$$

and

$$f^\bullet(x) = \inf \{f(y) : y \in E,\ y \leqq x\}.$$

Then $f_\bullet$ and $f^\bullet$ are also bounded.

(2.5.1) *$f_\bullet(f^\bullet)$ is preorder preserving (inversing), and $f_\bullet \leqq f$, $f^\bullet \leqq f$.*

(2.5.2) *If g is a preorder preserving (inversing) function on $(E, \leqq)$ such that $g \leqq f$, then $g \leqq f_\bullet$ ($g \leqq f^\bullet$).*

PROOF. It is obvious that the bounds of f are that of $f_\bullet$ and $f^\bullet$.

(2.5.1): Suppose $x \leqq y$. Then $i(y) \subset i(x)$ implies $f(i(y)) \subset f(i(x))$, therefore $f_\bullet(x) \leqq f_\bullet(y)$. In the same manner $f^\bullet(y) \leqq f^\bullet(x)$. Because of $x \leqq x$ we have $f_\bullet(x) \leqq f(x)$ and $f^\bullet(x) \leqq f(x)$.

(2.5.2): Let g be e.g. a preorder preserving function on $(E, \leqq)$ such that $g \leqq f$. Then, for each $\varepsilon > 0$, there exists $y \in E$ with the properties $x \leqq y$ and $f(y) - f_\bullet(x) < \varepsilon$. This implies $g(x) \leqq f_\bullet(x)$. □

Let us introduce the notations $\varphi_\bullet = \{f_\bullet : f \in \varphi\}$, $\varphi^\bullet = \{f^\bullet : f \in \varphi\}$ for any functional family φ, and $\Phi_\bullet = \{\varphi_\bullet : \varphi \in \Phi\}$, $\Phi^\bullet = \{\varphi^\bullet : \varphi \in \Phi\}$ for any functional structure Φ on $(E, \leqq)$.

(2.6) THEOREM. *Let $(E, \mathscr{S}, \leqq)$ be a preordered syntopogenous space, and let Φ denote a saturated ordering structure on E such that $\mathscr{S} \sim \mathscr{S}_\Phi$* (cf. [5], ch. 12; [9], (1.11)). *Then $d(\mathscr{S}) \sim \mathscr{S}_{\Phi_\bullet}$ and $i(\mathscr{S}) \sim \mathscr{S}_{\Phi^\bullet}$.*

PROOF. The verification will be detailed only for $d(\mathscr{S})$. First of all we prove the following inequalities for an arbitrary function f and $\varepsilon>0$:

$$(2.6.1) \qquad d\big(f^{-1}(<_\varepsilon)\big) \subset f_\bullet^{-1}(<_\varepsilon), \quad f_\bullet^{-1}(<_\varepsilon) \subset d\big(f^{-1}(<_{\varepsilon/2})\big).$$

Put $Af^{-1}(<_\varepsilon)B$. Then, disregarding the trivial cases of $A=E$, or $B=\emptyset$, we have $A\subset f^{-1}\big((-\infty, p]\big)$ and $f^{-1}\big((-\infty, p+\varepsilon)\big)\subset B$ for some $p\in R$. If $x\in d(A)$, then $f(a)\leqq p$, $x\leqq a$, where $a\in A$. Then $f_\bullet(x)\leqq p$, that is $x\in f_\bullet^{-1}\big((-\infty, p]\big)$. If $x\in f_\bullet^{-1}\big((-\infty, p+\varepsilon)\big)$, then $f_\bullet(x)<p+\varepsilon$, thus there exists $b\in E$ such that $x\leqq b$ and $f(b)<p+\varepsilon$. Hence $b\in B$, so that $x\in d(B)$. We obtained $d(A)\subset f_\bullet^{-1}\big((-\infty, p]\big)$ and $f_\bullet^{-1}\big((-\infty, p+\varepsilon)\big)\subset d(B)$, i.e. $d(A)f_\bullet^{-1}(<_\varepsilon)d(B)$. This shows $d\big(f^{-1}(<_\varepsilon)\big)\subset \subset f_\bullet^{-1}(<_\varepsilon)^q=f_\bullet^{-1}(<_\varepsilon)$. Conversely, suppose $Af_\bullet^{-1}(<_\varepsilon)B$ $(A\neq E, B\neq\emptyset)$. Then $A\subset f_\bullet^{-1}\big((-\infty, p]\big)$, $f_\bullet^{-1}\big((-\infty, p+\varepsilon)\big)\subset B$ for some $p\in R$. Define $X=f^{-1}\big((-\infty, p+\varepsilon/2]\big)$ and $Y=f^{-1}\big((-\infty, p+\varepsilon)\big)$. $x\in f_\bullet^{-1}\big((-\infty, p]\big)$ implies $f(y)\leqq p+\varepsilon/2$ for some $y\in E$ such that $x\leqq y$, hence $x\in d(X)$. If $x\in d(Y)$, and $x\leqq y\in Y$, then $f_\bullet(x)\leqq \leqq f(y)<p+\varepsilon$, i.e. $x\in f_\bullet^{-1}\big((-\infty, p+\varepsilon)\big)$. Thus we get $A\subset d(X)$ and $d(Y)\subset B$. Since $Xf^{-1}(<_{\varepsilon/2})Y$, $Ad\big(f^{-1}(<_{\varepsilon/2})B$.

Suppose $<\in\mathscr{S}$. Then there exists $\varphi\in\Phi$ such that $<\subset <_{\varphi,\varepsilon}$ for some $\varepsilon>0$. If $X<Y$, then $Xf^{-1}(<_\varepsilon)Y$ for a suitable $f\in\varphi$ ([5], (12.14)), therefore $d(X)d\big(f^{-1}(<_\varepsilon)\big)d(Y)$, and this implies $d(X)f_\bullet^{-1}(<_\varepsilon)d(Y)$ by (2.6.1). Hence $d(<)\subset \subset \big(\bigcup_{f\in\varphi} f_\bullet^{-1}(<_\varepsilon)\big)^q=<_{\varphi_\bullet,\varepsilon}$.

On the other hand, assume $\varphi\in\Phi$, and choose an order $<\in\mathscr{S}$ such that $<_{\varphi,\varepsilon/2}\subset <$. Then $f_\bullet^{-1}(<_\varepsilon)\subset d\big(f^{-1}(<_{\varepsilon/2})\big)\subset d(<)$ for any $f\in\varphi$, thus $<_{\varphi_\bullet,\varepsilon}= =\big(\bigcup_{f\in\varphi} f_\bullet^{-1}(<_\varepsilon)\big)^q\subset d(<)^q=d(<)$.

For the verification of the equivalence $i(\mathscr{S})\sim\mathscr{S}_{\Phi^\bullet}$ we have to prove

$$(2.6.2) \qquad i\big(f^{-1}(<_\varepsilon)\big) \subset f^{\bullet-1}(<_\varepsilon), \quad f^{\bullet-1}(<_\varepsilon) \subset i\big(f^{-1}(<_{\varepsilon/2})\big)$$

similarly to (2.6.1), then the further part of the proof is analogous. □

Following the terminology of [5], ch. 12, a functional family φ is said to be $(\mathscr{S}, \mathscr{I})$-*continuous* iff, for every $\varepsilon>0$, there exists $<\in\mathscr{S}$ such that $f^{-1}(<_\varepsilon)\subset <$ for each $f\in\varphi$.

(2.7) THEOREM. *In order that the preordered syntopogenous space* $(E, \mathscr{S}, \leqq)$ *be continuous, it is necessary and sufficient that* $\varphi^\bullet$ *and* $\varphi_\bullet$ *be* $(\mathscr{S}, \mathscr{I})$*-continuous for any* $(\mathscr{S}, \mathscr{I})$*-continuous functional family* φ *on* E.

PROOF. If the $(\mathscr{S}, \mathscr{I})$-continuous functional families satisfy this condition, then, with the notations of (2.6), $i(\mathscr{S})\sim\mathscr{S}_{\Phi^\bullet}<\mathscr{S}$ and $d(\mathscr{S})\sim\mathscr{S}_{\Phi_\bullet}<\mathscr{S}$ (cf. [5], (12.33)), i.e. the space is continuous by (2.4). Conversely, if $(E, \mathscr{S}, \leqq)$ is continuous, and φ is an $(\mathscr{S}, \mathscr{I})$-continuous functional family on E, then, for an arbitrary $\varepsilon>0$, choose $<\in\mathscr{S}$ such that $f^{-1}(<_{\varepsilon/2})\subset <$, provided $f\in\varphi$. If $<_1\in\mathscr{S}$ has the properties $i(<)\subset <_1$, $d(<)\subset <_1$, then $f^{\bullet-1}(<_\varepsilon)\subset i\big(f^{-1}(<_{\varepsilon/2})\big)\subset i(<)\subset <_1$, and similarly $f_\bullet^{-1}(<_\varepsilon)\subset d\big(f^{-1}(<_{\varepsilon/2})\big)\subset d(<)\subset <_1$ for any $f\in\varphi$ (see (2.6.1)—(2.6.2)), hence $\varphi^\bullet$ and $\varphi_\bullet$ are $(\mathscr{S}, \mathscr{I})$-continuous, too. □

We can deduce immediately from (2.7) and [5], (12.32):

(2.8) COROLLARY. *The preordered topogenous space* $(E, \mathscr{T}, \leqq)$ *is continuous iff* $f^\bullet$ *and* $f_\bullet$ *are* $(\mathscr{T}, \mathscr{I})$*-continuous for every* $(\mathscr{T}, \mathscr{I})$*-continuous function* f *on* E.

3. Properties of continuous spaces

First of all we shall generalize a result, which was proved by Green for topologies only ([8], Theorem 1, cf. [10], Theorem 3(ii)).

(3.1) LEMMA. *Let $(E, \mathscr{S}, \leqq)$ be a preordered syntopogenous space, ^{a} be an elementary operation, and $\mathscr{S}_1$ be a weakly ^{a}-convex syntopogenous structure on $(E, \leqq)$ such that $\mathscr{S}_1 < \mathscr{S}$. Then $\mathscr{S}_1 < c(\mathscr{S})^a$.*

PROOF. Suppose $\prec_1 \in \mathscr{S}_1$. There exists $\mathscr{E} \subset \mathscr{E}_c$ such that $\{\prec_1\} < \mathscr{E}^{ta} < \mathscr{S}_1 < \mathscr{S}$, thus one can find an order $\prec \in \mathscr{S}$, for which $(\bigcup \mathscr{E})^{qa} \subset \prec$. We prove $\prec_1 \subset (i(\prec) \cup d(\prec))^{qa}$. In fact, if $\prec_{X,C} \in \mathscr{E}$, then C is convex and $X \prec C$. $i(X) i(\prec) i(C)$, $d(X) d(\prec) d(C)$, $X \subset i(X) \cap d(X)$ and $C = i(C) \cap d(C)$, hence $X (i(\prec) \cup d(\prec))^q C$. From this $\prec_{X,C} \subset (i(\prec) \cup d(\prec))^q$, and in general $\bigcup \mathscr{E} \subset (i(\prec) \cup d(\prec))^q$. Finally $\prec_1 \subset (\bigcup \mathscr{E})^{qa} \subset (i(\prec) \cup d(\prec))^{qqa} = (i(\prec) \cup d(\prec))^{qa}$, i.e. $\mathscr{S}_1 < (i(\mathscr{S}) \vee d(\mathscr{S}))^a = c(\mathscr{S})^a$. ∎

In [9], (2.8) we showed that any ^{a}-convex space is weakly ^{a}-convex. In continuous spaces the converse is also true:

(3.2) THEOREM. *For an arbitrary elementary operation ^{a}, any continuous, weakly ^{a}-convex preordered syntopogenous space is ^{a}-convex.*

PROOF. If $(E, \mathscr{S}, \leqq)$ is weakly ^{a}-convex, then $\mathscr{S}^a \sim \mathscr{S}$ by [9], (2.3). If it is also continuous, then owing to (3.1) and (2.4.4) we have $\mathscr{S} < c(\mathscr{S})^a < \mathscr{S}$, thus $\mathscr{S} \sim (\mathscr{S})^a = (i(\mathscr{S}) \vee d(\mathscr{S}))^a \sim (\mathscr{S}^u \vee \mathscr{S}^l)^a$ by (2.4.5). □

As a generalization of Green's another result ([8], Theorem 1, cf. [10], Theorem 3(i)), let us consider the next one:

(3.3) THEOREM. *Let ^{a} be an elementary operation, and $(E, \mathscr{S}, \leqq)$ be a continuous preordered syntopogenous space such that $\mathscr{S}^a \sim \mathscr{S}$. Then the ^{a}-convex space $(E, c(\mathscr{S})^a, \leqq)$ is also continuous.*

REMARK. In view of (2.4.5), this fact can be drafted so that the *^{a}-convex epireflection* of a continuous preordered syntopogenous space $(E, \mathscr{S}, \leqq)$ with $\mathscr{S}^a \sim \mathscr{S}$ is also continuous (see [9], (4.4)—(4.5)).

PROOF OF THE THEOREM. $c(\mathscr{S})$ is ^{i}-convex by (2.3), thus $c(\mathscr{S})^a$ is ^{a}-convex (cf. [9], (2.4)). Under the conditions of the theorem we have $c(\mathscr{S})^a < \mathscr{S}$ (see (2.4.4)), therefore $c(c(\mathscr{S}^a)) < c(\mathscr{S}) < c(\mathscr{S})^a$, hence $(E, c(\mathscr{S})^a, \leqq)$ is continuous by (2.4.3). □

From the point of view of an open question belonging to the topic of the convex spaces (see [9], (5.3)), it is important to know whether a given preordered syntopogenous space $(E, \mathscr{S}, \leqq)$ satisfies the conditions

(3.4) $$\mathscr{S}^{up} \sim \mathscr{S}^{pu} \quad \text{and} \quad \mathscr{S}^{lp} \sim \mathscr{S}^{pl}.$$

(3.5) THEOREM. *In a continuous preordered syntopogenous space $(E, \mathscr{S}, \leqq)$ the equivalences of* (3.4) *are always true.*

We need the following lemma:

(3.6) LEMMA. *For any semi-topogenous order $<$ on the preordered space $(E, \leqq)$, we have $i(<^p) \subset i(<)^p$ and $d(<^p) \subset d(<)^p$.*

PROOF. With the notation of (2.1.1), $(<^p)' = <'^p$ by [6], (2.18). Thus $i(<^p) = (<^p)'^q = <'^{pq} \subset <'^{qp} = i(<)^p$. The verification of the other inequality is dual. □

PROOF OF THE THEOREM. $\mathscr{S}^{up} \prec \mathscr{S}^{pu}$ and $\mathscr{S}^{lp} \prec \mathscr{S}^{pl}$ are always true (see [9], (1.7)). Conversely, $\mathscr{S}^{pu} \prec i(\mathscr{S}^p) \prec i(\mathscr{S})^p \sim \mathscr{S}^{up}$ and $\mathscr{S}^{pl} \prec d(\mathscr{S}^p) \prec d(\mathscr{S})^p \sim \mathscr{S}^{lp}$ by (2.3), (3.6) and (2.4.5). □

Further on it will be examined: how can the continuity of a preordered syntopogenous space $(E, \mathscr{S}, \leqq)$ be hereditary for $(E, \mathscr{S}^k, \leqq)$, where $^{k} = {}^{p}, {}^{b}$ or ^{t}?

(3.7) LEMMA. *For any preordered syntopogenous space $(E, \mathscr{S}, \leqq)$, we have $c(\mathscr{S}^p) \prec c(\mathscr{S})^p$ and $c(\mathscr{S}^t) \sim c(\mathscr{S})^t$.*

PROOF. By (3.6)

$$i(\mathscr{S}^p) \prec i(\mathscr{S})^p \quad \text{and} \quad d(\mathscr{S}^p) \prec d(\mathscr{S})^p, \tag{3.7.1}$$

therefore

$$c(\mathscr{S}^p) =$$

$$= i(\mathscr{S}^p) \vee d(\mathscr{S}^p) \prec i(\mathscr{S})^p \vee d(\mathscr{S})^p \prec \big(i(\mathscr{S})^p \vee d(\mathscr{S})^p\big)^p = \big(i(\mathscr{S}) \vee d(\mathscr{S})\big)^p = c(\mathscr{S})^p$$

(see [5], (8.99)).

After this we prove

$$i(\mathscr{S}^t) = i(\mathscr{S})^t \quad \text{and} \quad d(\mathscr{S}^t) = d(\mathscr{S})^t. \tag{3.7.2}$$

Put $\mathscr{S}^t = \{<_0\}$. $i(<) \subset i(<_0)$ for any $< \in \mathscr{S}$, therefore $i(\mathscr{S})^t \prec i(\mathscr{S}^t)$. If $Ai(<_0)B$, then $A = \bigcup_{h=1}^{m} A_h$, $\bigcap_{j=1}^{n} B_j = B$, where $A_h \subset i(X_{hj})$, $i(Y_{hj}) \subset B_j$ and $X_{hj} <_0 Y_{hj}$ $(1 \leqq h \leqq m,\ 1 \leqq j \leqq n)$. Then there exists $<_{hj} \in \mathscr{S}$ such that $X_{hj} <_{hj} Y_{hj}$, and one can select $< \in \mathscr{S}$, for which $\bigcup \{<_{hj}: 1 \leqq h \leqq m,\ 1 \leqq j \leqq n\} \subset <$. We have $X_{hj} < Y_{hj}$, so that $Ai(<)B$, consequently $i(\mathscr{S}^t) \prec i(\mathscr{S})^t$. In view of that $i(\mathscr{S}^t)$ and $i(\mathscr{S})^t$ are simple syntopogenous structures, $i(\mathscr{S}^t) = i(\mathscr{S})^t$. The proof of the other equality of (3.7.2) is analogous. Finally we obtain

$$c(\mathscr{S}^t) = i(\mathscr{S}^t) \vee d(\mathscr{S}^t) = i(\mathscr{S})^t \vee d(\mathscr{S})^t \sim \big(i(\mathscr{S}) \vee d(\mathscr{S})\big)^t = c(\mathscr{S})^t$$

(cf. [5], (8.101)). □

(3.8) THEOREM. *If the preordered syntopogenous space $(E, \mathscr{S}, \leqq)$ is continuous, then $(E, \mathscr{S}^p, \leqq)$, $(E, \mathscr{S}^t, \leqq)$ and $(E, \mathscr{S}^{tp}, \leqq)$ are also continuous.*

PROOF. By (3.7) from $c(\mathscr{S}) \prec \mathscr{S}$ we can deduce $c(\mathscr{S}^k) \prec c(\mathscr{S})^k \prec \mathscr{S}^k$ for $^{k} = {}^{p}, {}^{t}$ or ^{tp}. □

For the operation ^{b} the statement corresponding to (3.8) is not always true.

(3.9) EXAMPLE. Suppose E is the interval $(-2, 2)$, and $\leqq$ is the natural order of E. Define the sets $A_{m,n}$, $B_{m,n}$ on E for any positive integer m, n as follows:

$$A_{m,n} = (-1-1/m, -1) \cup \{0\} \cup (1, 1+1/n),$$

$$B_{m,n} = (-1-1/m, 1+1/n).$$

Then $\mathfrak{S}=\{\emptyset, A_{m,n}, B_{m,n}: m, n=1, 2, \ldots\}$ is the system of the open sets for a topology $\mathscr{T}=\{<\}$ on E. It is easy to show that $(E, \mathscr{T}, \leqq)$ *is continuous, but* $(E, \mathscr{T}^b, \leqq)$ *is not.* In fact, $i(A_{m,n})=i(B_{m,n})=B_{m,1}$, and $d(A_{m,n})=d(B_{m,n})=B_{1,n}$. On the other hand, we have $\{0\}<^b\{0\}$, because $A_{n,n}<A_{n,n}$ and $\bigcap_{n\geqq 1} A_{n,n}=\{0\}$. However, $i(0)<^b i(0)$ and $d(0)<^b d(0)$ fail to hold. Indeed, putting $x=1$ and $y=-1$, we have $x\in i(0)$, $y\in E-i(0)$, but $x<E-y$ is not true, because, for any $S\in\mathfrak{S}$, $x\in S$ iff $y\in S$. In the same manner $y\in d(0)$, $x\in E-d(0)$, but $y<E-x$ is false. □

As the following two examples show, the statements of (3.8) cannot be conversed.

(3.10) EXAMPLE. Let $(R, \leqq)$ be the naturally ordered real line, and define $H_n=(-\infty, -n]\cup[n, \infty)$ for any natural number n. Let us consider the symmetrical topogenous structure $\mathscr{T}=\{<\}$ on R, where $A<B$ iff $A\subset B$, and there exists n such that $A\cap H_n\neq\emptyset$ implies $H_n\subset B$. In [9], (1.10) it was proved that $\mathscr{T}^p=\mathscr{D}_R$, therefore $(R, \mathscr{T}^p, \leqq)$ *is continuous* by (2.4.3). But $\mathscr{T}^{up}\not\prec\mathscr{T}^{pu}$ and $\mathscr{T}^{lp}\not\prec\mathscr{T}^{pl}$ (see [9], (1.10)), hence (3.5) shows that $(R, \mathscr{T}, \leqq)$ *cannot be continuous.* □

(3.11) EXAMPLE. Let R, Q, I, N denote the set of the real, rational, irrational and natural numbers, respectively. Assume $I_n=(n, \infty)\cap I$ for any natural number n, and let $\leqq$ be the natural order of R. Putting $U_\varepsilon=\{(x, y)\in R\times R: |x-y|<\varepsilon\}$, define an order $<_{\varepsilon,n}$ on R for any $\varepsilon>0$ and $n\in N$ so that

$$A<_{\varepsilon,n}B \text{ iff } A\subset B,\ U_\varepsilon(A)\cap Q\subset B, \text{ and } A\cap I_n\neq\emptyset \text{ implies } (n-\varepsilon, \infty)\subset B.$$

It is easy to prove that $<_{\varepsilon,n}$ is a biperfect topogenous order on R associated with the reflexive relation

$$U_{\varepsilon,n} = D\cup\big(U_\varepsilon\cap(R\times Q)\big)\cup\big(I_n\times(n-\varepsilon, \infty)\big),$$

where D is the diagonal of $R\times R$.

We have $<_{\delta,m}\mathbf{U}<_{\varepsilon,n}\subset<_{\gamma,k}$ for $\gamma=\min\{\delta, \varepsilon\}$ and $k=\max\{m, n\}$, besides $<_{\varepsilon,n}\subset<^2_{\varepsilon/2,n}$. (In order to see the last inequality, it is sufficient to prove that $A<_{\varepsilon,n}B$ implies

$$A<_{\varepsilon/2,n}\big(A\cup(U_{\varepsilon/2}(A)\cap Q)\cup C\big)<_{\varepsilon/2,n}B,$$

where $C=\emptyset$ for $A\cap I_n=\emptyset$, and $C=(n-\varepsilon/2, \infty)$ for $A\cap I_n\neq\emptyset$). These give that $\mathscr{S}=\{<_{\varepsilon,n}: \varepsilon>0, n\in N\}$ is a biperfect syntopology on R.

It will be verified that $(R, \mathscr{S}^t, \leqq)$ *is continuous, but* $(R, \mathscr{S}, \leqq)$ *is not.*

In fact, suppose $A<_{\varepsilon,n}B$ for some $\varepsilon>0$, $n\in N$. $A\subset B$ implies $i(A)\subset i(B)$. If A has no lower bound, then $i(A)=R$, thus $i(A)<_{\varepsilon,n}i(B)$ is evident. If A has a lower bound, then $p=\inf A$ exists, and, as it will be shown, we have $(p-\varepsilon, \infty)\subset i(B)$. Indeed, assume $x\in(p-\varepsilon, \infty)$. If there exists $a\in A$ with $a\leqq x$, then $x\in i(A)\subset i(B)$. If $x\leqq a$ for any $a\in A$, then $x\geqq p$. In this case we can find $q\in Q$ and $a_0\in A$ such that $p-\varepsilon<q\leqq x\leqq p\leqq a_0<q+\varepsilon$, thus $|a_0-q|<\varepsilon$, and $q\in U_\varepsilon(A)\cap Q\subset B$, hence

$x\in i(B)$. $U_\varepsilon\big(i(A)\big)\cap Q\subset i(B)$ is also true, since $x\in U_\varepsilon(i(A))$ implies $a\leqq y$, $|y-x|<\varepsilon$ for some $a\in A$, $y\in R$, thus $p-\varepsilon\leqq a-\varepsilon\leqq y-\varepsilon<x$, therefore $x\in(p-\varepsilon,\infty)\subset i(B)$. If $m\in N$, for which $p<m$, then $(m-\varepsilon,\infty)\subset(p-\varepsilon,\infty)\subset i(B)$. Summing up, $i(A)<<_{\varepsilon,m}i(B)$. If A has no upper bound, then $d(A)=R$, hence $d(A)<_{\varepsilon,n}d(B)$ is trivial. If $A\subset(-\infty,m]$, then $d(A)\subset(-\infty,m]$, therefore $d(A)\cap I_m=\emptyset$. $U_\varepsilon\big(d(A)\big)\cap Q\subset$ $\subset d(B)$ also holds. If $x\in U_\varepsilon\big(d(A)\big)\cap Q$, i.e. $x\in Q$, there is $y\in R$ such that $y\leqq a$ for some $a\in A$, and $|y-x|<\varepsilon$, then $x\leqq a$ or $a\leqq x$. In the first case $x\in d(A)\subset$ $\subset d(B)$, and in the second one $y\leqq a\leqq x<y+\varepsilon$ gives $x\in U_\varepsilon(A)\cap Q\subset B\subset d(B)$. Thus $d(A)<_{\varepsilon,m}d(B)$.

However, let us consider two arbitrary elements $<_{\delta,m}$ and $<_{\varepsilon,n}$ of $\mathscr{S}$. Let us choose $x\in Q$ such that $n<x$. Then $x<_{\delta,m}(-\infty,x+\delta)$, but $d(x)\cap I_n\neq\emptyset$ and $(n-\varepsilon,\infty)\not\subset(-\infty,x+\delta)=d\big((-\infty,x+\delta)\big)$ show that $d(x)<_{\varepsilon,n}d\big((-\infty,x+\delta)\big)$ fails to be true, hence $(R,\mathscr{S},\leqq)$ cannot be continuous. □

(3.12) THEOREM. *Let $(E,\mathscr{S},\leqq)$ be a totally bounded preordered syntopogenous space. If $(E,\mathscr{S}^t,\leqq)$ is continuous, then $(E,\mathscr{S},\leqq)$ is also continuous.*

PROOF. First of all we deduce that in this case $i(\mathscr{S})$ and $d(\mathscr{S})$ are also totally bounded. Suppose $<\in\mathscr{S}$, and consider a system $\mathfrak{S}=\{S_1,\ldots,S_n\}$ of the subsets of E such that $A<B$ implies $A\subset S_h<_1S_j\subset B$ for some $S_h,S_j\in\mathfrak{S}$, and for a given $<_1\in\mathscr{S}$ (see [5], (19.2), (17.4)). Then the finite family $\mathscr{E}=\{<_{i(S_h),i(S_j)}\colon S_h<_1S_j\}$ of elementary topogenous orders has the property $\{i(<)\}<\mathscr{E}^t<i(\mathscr{S})$, and this means that $i(\mathscr{S})$ is totally bounded (cf. [5], pp. 277, 278 and 339). The case of $d(\mathscr{S})$ is similar. Because of [5], (19.7) $c(\mathscr{S})$ is also totally bounded, therefore if $(E,\mathscr{S}^t,\leqq)$ is continuous, then $c(\mathscr{S})^t\sim c(\mathscr{S}^t)<\mathscr{S}^t$ by (3.7) and (2.4.3), thus $c(\mathscr{S})<\mathscr{S}$ issues from [5], (19.39). □

In the final part of the paper it will be studied whether the forming of the infimum, supremum, inverse image and product of continuous preordered syntopogenous spaces preserves this property.

If $\{S_j\colon j\in J\neq\emptyset\}$ is a family of syntopogenous structures on the set E, then among the syntopogenous structures coarser than each $\mathscr{S}_j$ $(j\in J)$ there exists a finer one, which is called the *infimum* of the structures $\mathscr{S}_j$, and it is denoted by $\bigwedge_{j\in J}\mathscr{S}_j$ (see [1]).

(3.13) LEMMA. *Let $\{\mathscr{S}_j\colon j\in J\}$ be a non-empty family of syntopogenous structures on the preordered space $(E,\leqq)$, and denote by r one of the operations i, d or c. Then*

$$r\Big(\bigwedge_{j\in J}\mathscr{S}_j\Big)<\bigwedge_{j\in J}r(\mathscr{S}_j)\quad\text{and}\quad\bigvee_{j\in J}r(\mathscr{S}_j)<r\Big(\bigvee_{j\in J}\mathscr{S}_j\Big).$$

PROOF. If $\mathscr{S}$ and $\mathscr{S}'$ are syntopogenous structures on $(E,\leqq)$ such that $\mathscr{S}<\mathscr{S}'$, then $i(\mathscr{S})<i(\mathscr{S}')$ and $d(\mathscr{S})<d(\mathscr{S}')$ are clear, therefore $c(\mathscr{S})<c(\mathscr{S}')$ is also true. Because of

$$r\Big(\bigwedge_{j\in J}\mathscr{S}_j\Big)<r(\mathscr{S}_j)\quad\text{and}\quad r(\mathscr{S}_j)<r\Big(\bigvee_{j\in J}\mathscr{S}_j\Big)$$

for any $j\in J$, we have

$$r\Big(\bigwedge_{j\in J}\mathscr{S}_j\Big)<\bigwedge_{j\in J}r(\mathscr{S}_j)\quad\text{and}\quad\bigvee_{j\in J}r(\mathscr{S}_j)<r\Big(\bigvee_{j\in J}\mathscr{S}_j\Big).\ \square$$

(3.14) THEOREM. *If* $\{\mathscr{S}_j: j\in J\neq\emptyset\}$ *is a family of syntopogenous structures on the preordered space* $(E, \leqq)$ *such that* $(E, \mathscr{S}_j, \leqq)$ *is continuous for each* $j\in J$, *then* $\left(E, \bigwedge_{j\in J}\mathscr{S}_j, \leqq\right)$ *is also continuous.*

PROOF. $c(\mathscr{S}_j)<\mathscr{S}_j$ for any $j\in J$, therefore by (3.13) $c\left(\bigwedge_{j\in J}\mathscr{S}_j\right)<\bigwedge_{j\in J}c(\mathscr{S}_j)<$ $<\bigwedge_{j\in J}\mathscr{S}_j$ (cf. (2.4.3)). □

(3.15) COROLLARY. *In every preordered syntopogenous space* $(E, \mathscr{S}, \leqq)$, *among those syntopogenous structures* $\mathscr{S}'$ *finer than* $\mathscr{S}$, *for which* $(E, \mathscr{S}', \leqq)$ *is continuous, there exists a coarsest one.*

PROOF. Because of (3.14) it is sufficient to look for at least one structure $\mathscr{S}'$ such that $\mathscr{S}<\mathscr{S}'$, and $(E, \mathscr{S}', \leqq)$ is continuous. Such a structure is the discrete syntopogenous structure $\mathscr{D}_E=\{\subset\}$ of E by (2.4.3). □

In general, I have neither a positive result nor a counterexample in respect of the statement corresponding to (3.14) for $\left(E, \bigvee_{j\in J}\mathscr{S}_j, \leqq\right)$. Yet, in a particular case this question can be answered:

(3.16) LEMMA. *Let* $\{\mathscr{S}_j: j\in J\neq\emptyset\}$ *be a family of syntopogenous structures on the preordered space* $(E, \leqq)$. *Suppose that for any natural number* n, *and any index* $j_t\in J$ $(t=1, 2, \ldots, n)$ *such that* $j_{t_1}\neq j_{t_2}$, *if* $t_1\neq t_2$, $<_t\in\mathscr{S}_{j_t}$, $X_t<_t Y_t$ *then* $\bigcap_{t=1}^{n} Y_t\neq\emptyset$ *imply*

$$\bigcap_{t=1}^{n} i(X_t)\subset i\left(\bigcap_{t=1}^{n} Y_t\right) \quad \text{and} \quad \bigcap_{t=1}^{n} d(X_t)\subset d\left(\bigcap_{t=1}^{n} Y_t\right).$$

Then $r\left(\bigvee_{j\in J}\mathscr{S}_j\right)\sim\bigvee_{j\in J}r(\mathscr{S}_j)$ *for* $r=i, d$ *or* c. *If* $(E, \mathscr{S}_j, \leqq)$ *is continuous for each* $j\in J$, *then so is* $\left(E, \bigvee_{j\in J}\mathscr{S}_j, \leqq\right)$, *too.*

PROOF. In view of (3.13) we have to verify only $r\left(\bigvee_{j\in J}\mathscr{S}_j\right)<\bigvee_{j\in J}r(\mathscr{S}_j)$. If $<\in\bigvee_{j\in J}\mathscr{S}_j$, then $<=(\bigcup\mathscr{E})^q$, where $\mathscr{E}$ is a finite subfamily of $\bigcup_{j\in J}\mathscr{S}_j$ (see [5], (8.89)). Choose those indices $j_t\in J$ $(t=1, 2, \ldots, n)$, for which $\mathscr{E}\cap\mathscr{S}_{j_t}\neq\emptyset$, and select $<_t\in\mathscr{S}_{j_t}$ such that $<'\in\mathscr{E}\cap\mathscr{S}_{j_t}$ implies $<'\subset<_t$. Then $<\subset\left(\bigcup_{t=1}^{n}<_t\right)^q$. Assume $<_t'\in\mathscr{S}_{j_t}$, $<_t\subset<_t'^2$. If $A<B$, one can find a natural number m, and sets $A_{ht}, B_{ht}\subset E$ such that

$$A_{ht}<_t B_{ht} \quad \text{for} \quad 1\leqq h\leqq m, 1\leqq t\leqq n, \quad A=\bigcup_{h=1}^{m}\bigcap_{t=1}^{n}A_{ht}, \quad \bigcup_{h=1}^{m}\bigcap_{t=1}^{n}B_{ht}=B$$

(cf. [5], (3.26)). The case of $A=\emptyset$ is trivial, thus without loss of generality, we can assume $\bigcap_{t=1}^{n}B_{ht}\neq\emptyset$. Now supposing $A_{ht}<_t'C_{ht}<_t'B_{ht}$, we have

$$i(A)\subset\bigcup_{h=1}^{m}\bigcap_{t=1}^{n}i(A_{ht}) \quad \text{and} \quad \bigcup_{h=1}^{m}\bigcap_{t=1}^{n}i(C_{ht})\subset\bigcup_{h=1}^{m}i\left(\bigcap_{t=1}^{n}B_{ht}\right)=i\left(\bigcup_{h=1}^{m}\bigcap_{t=1}^{n}B_{ht}\right)=i(B).$$

Owing to $i(A_{ht})i(\prec_t')i(C_{ht})$, we obtain $i(A)\left(\bigcup_{t=1}^{n} i(\prec_t')\right)^q i(B)$. This shows

$$i(\prec) \subset \left(\bigcup_{t=1}^{n} i(\prec_t')\right)^{qq} = \left(\bigcup_{t=1}^{n} i(\prec_t')\right)^{q} \in \bigvee_{j=J} i(\mathscr{S}_j),$$

so that, in general $i\left(\bigvee_{j\in J}\mathscr{S}_j\right) < \bigvee_{j\in J} i(\mathscr{S}_j)$. The proof of $d\left(\bigvee_{j\in J}\mathscr{S}_j\right) < \bigvee_{j\in J} d(\mathscr{S}_j)$ is similar. Finally

$$c\left(\bigvee_{j\in J}\mathscr{S}_j\right) = i\left(\bigvee_{j\in J}\mathscr{S}_j\right) \vee d\left(\bigvee_{j\in J}\mathscr{S}_j\right) < \left(\bigvee_{j\in J} i(\mathscr{S}_j)\right)\vee\left(\bigvee_{j\in J} d(\mathscr{S}_j)\right) =$$

$$= \bigvee_{j\in J}\left(i(\mathscr{S}_j)\vee d(\mathscr{S}_j)\right) = \bigvee_{j\in J} c(\mathscr{S}_j).$$

If each of the spaces $(E, \mathscr{S}_j, \leqq)$ is continuous, then

$$c\left(\bigvee_{j\in J}\mathscr{S}_j\right) \sim \bigvee_{j\in J} c(\mathscr{S}_j) < \bigvee_{j\in J}\mathscr{S}_j,$$

thus $\left(E, \bigvee_{j\in J}\mathscr{S}_j, \leqq\right)$ is also continuous by (2.4.3). □

(3.17) THEOREM. *Let f be a preorder preserving mapping of a preordered space $(E, \leqq)$ into the preordered syntopogenous space $(E', \mathscr{S}', \leqq')$, and denote by r one of the operations i, d or c.*

(3.17.1) *If $\prec \in \mathscr{S}'$, $X \prec Y$, $f^{-1}(Y) \neq \emptyset$ imply $f^{-1}(i(X)) \subset i(f^{-1}(Y))$ and $f^{-1}(d(X)) \subset d(f^{-1}(Y))$, then $r(f^{-1}(\mathscr{S}')) < f^{-1}(r(\mathscr{S}'))$.*

(3.17.2) *If $(E', \mathscr{S}', \leqq')$ is continuous, then $f^{-1}(r(\mathscr{S}')) < r(f^{-1}(\mathscr{S}'))$.*

(3.17.3) *If under the condition of* (3.17.1) *$(E', \mathscr{S}', \leqq')$ is continuous, then $(E, f^{-1}(\mathscr{S}'), \leqq)$ is also continuous, and $r(f^{-1}(\mathscr{S}')) \sim f^{-1}(r(\mathscr{S}'))$.*

PROOF. (3.17.1): Since f is preorder preserving, it satisfies $i(f^{-1}(X)) \subset f^{-1}(i(X))$ and $d(f^{-1}(X)) \subset f^{-1}(d(X))$ for any $X \subset E'$. Suppose $\prec \in \mathscr{S}'$, $\prec_1 \in \mathscr{S}'$, $\prec \subset \prec_1^2$. We show $i(f^{-1}(\prec)) \subset f^{-1}(i(\prec_1))$ and $d(f^{-1}(\prec)) \subset f^{-1}(d(\prec_1))$. In fact, if $\emptyset \neq A f^{-1}(\prec) B$, then there are sets $A', B' \subset E'$ with $A' \prec B'$, $A \subset f^{-1}(A')$, $f^{-1}(B') \subset B$. Assume $A' \prec_1 C' \prec_1 B'$, then by our conditions $i(A) \subset i(f^{-1}(A')) \subset f^{-1}(i(A'))$ and $f^{-1}(i(C')) \subset i(f^{-1}(B')) \subset i(B)$. Because of $i(A')i(\prec_1)i(C')$, we have $i(A) f^{-1}(i(\prec_1)) i(B)$. Thus

$$i(f^{-1}(\prec)) \subset f^{-1}(i(\prec_1))^q = f^{-1}(i(\prec_1)^q) = f^{-1}(i(\prec_1)).$$

The proof of $d(f^{-1}(\prec)) \subset f^{-1}(d(\prec_1))$ is analogous. Hence $i(f^{-1}(\mathscr{S}')) < f^{-1}(i(\mathscr{S}'))$ and $d(f^{-1}(\mathscr{S}')) < f^{-1}(d(\mathscr{S}'))$. From these it follows that

$$c(f^{-1}(\mathscr{S}')) = i(f^{-1}(\mathscr{S}')) \vee d(f^{-1}(\mathscr{S}')) < f^{-1}(i(\mathscr{S}')) \vee f^{-1}(d(\mathscr{S}')) =$$

$$= f^{-1}(i(\mathscr{S}') \vee d(\mathscr{S}')) = f^{-1}(c(\mathscr{S}'))$$

(cf. [5], (9.10)).

(3.17.2): If $(E', \mathscr{S}', \leqq')$ is continuous, then $i(\mathscr{S}') < \mathscr{S}'$ and $d(\mathscr{S}') < \mathscr{S}'$, therefore $f^{-1}(i(\mathscr{S}')) < f^{-1}(\mathscr{S}')$ and $f^{-1}(d(\mathscr{S}')) < f^{-1}(\mathscr{S}')$. $f^{-1}(i(\mathscr{S}'))$ is increas-

ing, $f^{-1}(d(\mathscr{S}'))$ is decreasing on $(E, \leqq)$ (see (2.3) and [9], (1.1.4)), hence by (2.3) $f^{-1}(i(\mathscr{S}')) < f^{-1}(\mathscr{S}')^u < i(f^{-1}(\mathscr{S}'))$ and $f^{-1}(d(\mathscr{S}')) < f^{-1}(\mathscr{S}')^l < d(f^{-1}(\mathscr{S}'))$. Therefore

$$f^{-1}(c(\mathscr{S}')) = f^{-1}(i(\mathscr{S}') \vee d(\mathscr{S}')) = f^{-1}(i(\mathscr{S}')) \vee f^{-1}(d(\mathscr{S}')) <$$
$$< i(f^{-1}(\mathscr{S}')) \vee d(f^{-1}(\mathscr{S}')) = c(f^{-1}(\mathscr{S}')).$$

(3.17.3): If $(E', \mathscr{S}', \leqq')$ is continuous and satisfies the condition of (3.17.1), then $c(\mathscr{S}') < \mathscr{S}'$, and in view of (3.17.1)—(3.17.2), we have $r(f^{-1}(\mathscr{S}')) \sim f^{-1}(r(\mathscr{S}'))$ for $r=i$, d or c, consequently $c(f^{-1}(\mathscr{S}')) \sim f^{-1}(c(\mathscr{S}')) < f^{-1}(\mathscr{S}')$, i.e. $(E, f^{-1}(\mathscr{S}'), \leqq)$ is also continuous. □

Let $(E, \mathscr{S}, \leqq)$ be a preordered syntopogenous space, and $\emptyset \neq E_0 \subset E$. Denote by $\leqq|E_0$ the preorder defined on E_0, the graph of which is $G(\leqq|E_0) = G(\leqq) \cap (E_0 \times E_0)$. Following the terminology of Priestley ([12], ch. 3), $(E_0, \mathscr{S}|E_0, \leqq|E_0)$ will be called a *preorder subspace* of $(E, \mathscr{S}, \leqq)$ iff $(\mathscr{S}|E_0)^u \sim \mathscr{S}^u|E_0$ and $(\mathscr{S}|E_0)^l \sim \mathscr{S}^l|E_0$.

(3.18) COROLLARY (cf. [10], p. 527). *Let $(E, \mathscr{S}, \leqq)$ be a continuous preordered syntopogenous space, and $\emptyset \neq E_0 \subset E$ such that $< \in \mathscr{S}$, $X < Y$, $Y \cap E_0 \neq \emptyset$ imply $i(X) \cap E_0 \subset i(Y \cap E_0)$ and $d(X) \cap E_0 \subset d(Y \cap E_0)$. Then $(E_0, \mathscr{S}|E_0, \leqq|E_0)$ is also continuous, and it is a preorder subspace of $(E, \mathscr{S}, \leqq)$.*

PROOF. Let f denote the canonical injection of E_0 into E. Then f is preorder preserving, and it satisfies the conditions of (3.17.3), hence $(E_0, \mathscr{S}|E_0, \leqq|E_0)$ is continuous. At the same time we have

$$(\mathscr{S}|E_0)^u \sim i(\mathscr{S}|E_0) \sim i(\mathscr{S})|E_0 \sim \mathscr{S}^u|E_0$$

and

$$(\mathscr{S}|E_0)^l \sim d(\mathscr{S}|E_0) \sim d(\mathscr{S})|E_0 \sim \mathscr{S}^l|E_0$$

by (2.4.5) and (3.17.3). □

(3.19) THEOREM. *Let $(E_j, \mathscr{S}_j, \leqq_j)$ be a preordered syntopogenous space for any $j \in J \neq \emptyset$, and $\leqq$ be the product preorder on $E = \bigtimes_{j \in J} E_j$. Then*

(3.19.1) $r(\bigtimes_{j \in J} \mathscr{S}_j) \sim \bigtimes_{j \in J} r(\mathscr{S}_j)$ *for* $r=i$, d *or* c.

(3.19.2) *Supposing* $\mathscr{S}_j = \mathscr{S}_j^k$ $(j \in J)$, *where* $^k = {}^i, {}^p, {}^b, {}^t$ *or* ^{tp}, $(E, (\bigtimes_{j \in J} \mathscr{S}_j)^k, \leqq)$ *is continuous if and only if $(E_j, \mathscr{S}_j, \leqq_j)$ is continuous for each $j \in J$.*

REMARK. Considering the cases of $^k = {}^{tp}$, ^{t} and ^{b}, this theorem can be applied immediately for the study of the continuity of the product of preordered "classical" topological, (quasi-)proximity and (quasi-)uniform spaces (cf. (1.1)—(1.3), (2.2) and [5], p. 134—139).

PROOF OF THE THEOREM. (3.19.1): Assume $\emptyset \neq J_0 \subset J$ and $H_j \subset E_j$ for $j \in J_0$. Then

$$\text{(3.19.3)} \qquad \begin{cases} i(\bigcap_{j \in J_0} \mathrm{pr}_j^{-1}(H_j)) = \bigcap_{j \in J_0} \mathrm{pr}_j^{-1}(i(H_j)), & \text{and} \\ d(\bigcap_{j \in J_0} \mathrm{pr}_j^{-1}(H_j)) = \bigcap_{j \in J_0} \mathrm{pr}_j^{-1}(d(H_j)), \end{cases}$$

where $\mathrm{pr}_j\colon E\to E_j$ is the j-th projection. In fact, pr_j is preorder preserving for every $j\in J$, therefore

$$i\Big(\bigcap_{j\in J_0}\mathrm{pr}_j^{-1}(H_j)\Big)\subset\bigcap_{j\in J_0}i\big(\mathrm{pr}_j^{-1}(H_j)\big)\subset\bigcap_{j\in J_0}\mathrm{pr}_j^{-1}\big(i(H_j)\big)$$

is obvious. Conversely, if $x=(x_j)\in\bigcap_{j\in J_0}\mathrm{pr}_j^{-1}\big(i(H_j)\big)$, then $x_j\in i(H_j)$ for any $j\in J_0$, i.e. there exists $h_j\in H_j$ with $h_j\leqq_j x_j$ $(j\in J_0)$. Put $y=(y_j)\in E$, for which $y_j=h_j$ for $j\in J_0$, and $y_j=x_j$ for $j\notin J_0$. Then $y\leqq x$, and $\mathrm{pr}_j(y)\in H_j$ for $j\in J_0$, therefore $y\in\bigcap_{j\in J_0}\mathrm{pr}_j^{-1}(H_j)$, and $x\in i\big(\bigcap_{j\in J_0}\mathrm{pr}_j^{-1}(H_j)\big)$. The proof of the other equality is analogous. As a particular case of (3.19.3), for $J_0=\{j\}$, $H\subset E_j$, we have

(3.19.4) $\quad i\big(\mathrm{pr}_j^{-1}(H)\big)=\mathrm{pr}_j^{-1}\big(i(H)\big)$ and $d\big(\mathrm{pr}_j^{-1}(H)\big)=\mathrm{pr}_j^{-1}\big(d(H)\big)$.

After this we verify

(3.19.5) $\quad r\big(\mathrm{pr}_j^{-1}(\mathscr{S}_j)\big)\sim\mathrm{pr}_j^{-1}\big(r(\mathscr{S}_j)\big)$ for $j\in J$ and $r=i,d$ or c.

Indeed, by (3.19.4) the spaces $(E_j,\mathscr{S}_j,\leqq_j)$ and the mappings pr_j satisfy the condition of (3.17.1), therefore $r\big(\mathrm{pr}_j^{-1}(\mathscr{S}_j)\big)\prec\mathrm{pr}_j^{-1}\big(r(\mathscr{S}_j)\big)$ for any $j\in J$. Conversely, suppose $\prec\in\mathscr{S}_j$. Then with the notation of (2.1.1) $\mathrm{pr}_j^{-1}\big(i(\prec)\big)=\mathrm{pr}_j^{-1}(\prec'^q)=$ $=\mathrm{pr}_j^{-1}(\prec')^q$. Suppose $A\,\mathrm{pr}_j^{-1}(\prec')B$. In this case $A\subset\mathrm{pr}_j^{-1}(A_0)$, $\mathrm{pr}_j^{-1}(B_0)\subset B$, where $A_0\prec' B_0$. This means $A_0\subset i(X)$, $i(Y)\subset B_0$ for some $X\prec Y$. From here $A\subset\mathrm{pr}_j^{-1}\big(i(X)\big)=i\big(\mathrm{pr}_j^{-1}(X)\big)$ and $i\big(\mathrm{pr}_j^{-1}(Y)\big)=\mathrm{pr}_j^{-1}\big(i(Y)\big)\subset B$, thus owing to $\mathrm{pr}_j^{-1}(X)\mathrm{pr}_j^{-1}(\prec)\mathrm{pr}_j^{-1}(Y)$, we get $Ai\big(\mathrm{pr}_j^{-1}(\prec)\big)B$. This yields $\mathrm{pr}_j^{-1}(\prec')\subset$ $\subset i\big(\mathrm{pr}_j^{-1}(\prec)\big)$, so that

$$\mathrm{pr}_j^{-1}\big(i(\prec)\big)=\mathrm{pr}_j^{-1}(\prec')^q\subset i\big(\mathrm{pr}_j^{-1}(\prec)\big)^q=i\big((\mathrm{pr}_j^{-1}(\prec)\big),$$

hence, in general, $\mathrm{pr}_j^{-1}\big(i(\mathscr{S}_j)\big)\prec i\big(\mathrm{pr}_j^{-1}(\mathscr{S}_j)\big)$. The verification of the inequality $\mathrm{pr}_j^{-1}\big(d(\mathscr{S}_j)\big)\prec d\big(\mathrm{pr}_j^{-1}(\mathscr{S}_j)\big)$ is analogous. Finally

$$\mathrm{pr}_j^{-1}\big(c(\mathscr{S}_j)\big)=\mathrm{pr}_j^{-1}\big(i(\mathscr{S}_j)\vee d(\mathscr{S}_j)\big)=\mathrm{pr}_j^{-1}\big(i(\mathscr{S}_j)\big)\vee\mathrm{pr}_j^{-1}\big(d(\mathscr{S}_j)\big)\prec$$

$$\prec i\big(\mathrm{pr}_j^{-1}(\mathscr{S}_j)\big)\vee d\big(\mathrm{pr}_j^{-1}(\mathscr{S}_j)\big)=c\big(\mathrm{pr}_j^{-1}(\mathscr{S}_j)\big)$$

by [5], (9.10).

Further let us observe that the structures $\mathrm{pr}_j^{-1}(\mathscr{S}_j)$ $(j\in J)$ satisfy the condition of (3.16) (see (3.19.3)), thus

(3.19.6) $\quad r\Big(\bigvee_{j\in J}\mathrm{pr}_j^{-1}(\mathscr{S}_j)\Big)\sim\bigvee_{j\in J}r\big(\mathrm{pr}_j^{-1}(\mathscr{S}_j)\big)$ for $r=i,d$ or c.

Comparing (3.19.6) with (3.19.5), we obtain

$$r\Big(\bigtimes_{j\in J}\mathscr{S}_j\Big)\sim\bigtimes_{j\in J}r(\mathscr{S}_j)\quad(r=i,d\quad\text{or}\quad c),$$

which was to be proved.

(3.19.2): If each one of the spaces $(E_j,\mathscr{S}_j,\leqq_j)$ $(j\in J)$ is continuous, then $c(\mathscr{S}_j)\prec\mathscr{S}_j$ for every $j\in J$, therefore by (3.19.1) $c\big(\bigtimes_{j\in J}\mathscr{S}_j\big)\sim\bigtimes_{j\in J}c(\mathscr{S}_j)\prec\bigtimes_{j\in J}\mathscr{S}_j$ (see [5], (11.20)), hence the space $\big(E,\bigtimes_{j\in J}\mathscr{S}_j,\leqq\big)$ is also continuous. For the proof of the continuity of $\big(E,(\bigtimes_{j\in J}\mathscr{S}_j)^k,\leqq\big)$ (3.8) can be used even if $^k=^b$ by [5], (11.25).

Conversely, if $\left(E, \underset{j\in J}{\mathbf{X}} \mathscr{S}_j)^k, \leqq\right)$ is continuous, then

$$\underset{j\in J}{\mathbf{X}} c(\mathscr{S}_j) \sim c\left(\underset{j\in J}{\mathbf{X}} \mathscr{S}_j\right) < c\left(\left(\underset{j\in J}{\mathbf{X}} \mathscr{S}_j\right)^k\right) < \left(\underset{j\in J}{\mathbf{X}} \mathscr{S}_j\right)^k$$

(cf. (2.4)), therefore $\mathrm{pr}_j^{-1}(c(\mathscr{S}_j)) < \left(\underset{j\in J}{\mathbf{X}} \mathscr{S}_j\right)^k$ for any $j\in J$. Assuming $<\in c(\mathscr{S}_j)$, an order $<'\in\left(\underset{j\in J}{\mathbf{X}} \mathscr{S}_j\right)^k$ can be found with $\mathrm{pr}_j^{-1}(<)\subset <'$. Then an element $<_j$ of $\mathscr{S}_j$ can be selected such that $A, B\subset E$, $A<'B$ implies $\mathrm{pr}_j(A)<_j \mathrm{pr}_j(B)$ (see lemmas (6.1) and (6.2) of [7]). Now if $X<Y$, then $\mathrm{pr}_j^{-1}(X)<' \mathrm{pr}_j^{-1}(Y)$, thus $X= \mathrm{pr}_j(\mathrm{pr}_j^{-1}(X))<_j \mathrm{pr}_j(\mathrm{pr}_j^{-1}(Y))=Y$, that is $<\subset <_j$. We got $c(\mathscr{S}_j)<\mathscr{S}_j$, hence $(E_j, \mathscr{S}_j, \leqq_j)$ is continuous by (2.4) $(j\in J)$. □

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SYNTOPOGENOUS SPACES WITH PREORDER. III (SEPARATION)

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After the study of the (weakly) convex [7] and continuous [8] spaces, in the third part of our series the notion of a *T_i-preordered syntopogenous space* will be introduced for $i=0, 2$, and the definition of Burgess—Fitzpatrick [2] will be adopted for $i=1$.

In Section 1 we shall examine the most important properties of these spaces, and in Section 2 a generalization of the quotient space corresponding to the separative partition of a syntopogenous space will be given. Using this construction, we shall show that the category of the T_0-ordered syntopogenous spaces is epi-reflective in the category of all preordered syntopogenous spaces and continuous preorder preserving mappings.

0. Introduction

A *preordered space* is a pair $(E, \leqq)$ consisting of a set E, and a reflexive and transitive relation $\leqq$ (so-called *preorder*) on E. An *order* $\leqq$ is a preorder such that $x\leqq y$, $y\leqq x$ imply $x=y$. A preorder $\leqq$ is *linear* iff, for $x, y\in E$, $x\leqq y$ or $y\leqq x$. The *graph* of the preorder $\leqq$ is

$$G(\leqq) = \{(x, y)\in E\times E\colon\ x \leqq y\}.$$

A set $X\subset E$ is called *increasing (decreasing)* iff $x\in X$, $x\leqq y$ $(y\leqq x)$ imply $y\in X$. X is *convex* iff $x, z\in E$, $x\leqq y\leqq z$ imply $y\in X$. For an arbitrary $X\subset E$, the sets $i(X)=\{y\in E\colon\ x\leqq y,\ x\in X\}$, $d(X)=\{y\in E\colon\ y\leqq x\in X\}$ and $c(X)=i(X)\cap d(X)$ are the smallest of all increasing, decreasing and convex sets respectively, which contain X. A mapping f of the preordered space $(E, \leqq)$ into a preordered space $(E', \leqq')$ is said to be *preorder preserving (inversing)*, if $x, y\in E$, $x\leqq y$ imply $f(x)\leqq' f(y)$ $\big(f(y)\leqq' f(x)\big)$. The *product* of the preordered spaces $(E_j, \leqq_j)$ $(j\in J\neq\emptyset)$ is a preordered space $(E, \leqq)$, where $E=\bigtimes_{j\in J} E_j$, and $(x_j)\leqq(y_j)$ iff $x_j\leqq_j y_j$ for every index $j\in J$.

As a common generalization of topological, proximity and uniform spaces, the notion of a *syntopogenous space* was introduced by Á. Császár. In respect of the terminology and the notations this paper follows his monograph [4].

A syntopogenous space $[E, \mathscr{S}]$ equipped with a preorder $\leqq$ is called a *preordered syntopogenous space*, and it is denoted by $(E, \mathscr{S}, \leqq)$ (see [1]—[3], [7]—[8]). For a semi-topogenous order $\prec$, and, for an order family $\mathscr{A}$ on a preordered space $(E, \leqq)$, define

$$G(\prec) = \{(x, y)\in E\times E\colon\ x \prec E-y \quad \text{is false}\}$$

and

$$G(\mathscr{A}) = \cap \{G(<): <\in \mathscr{A}\}.$$

It is obvious that, for $\mathscr{A}^{tb} = \{<_0\}$, we have

(0.1) $$(x, y) \in G(\mathscr{A}) \quad \text{iff} \quad x <_0 E - y \quad \text{is false.}$$

If $\mathscr{A} = \mathscr{S}$ is a syntopogenous structure, then $G(\mathscr{S})$ is the graph of a preorder on E (cf. [1], 3.1).

The preordered syntopogenous space $(E, \mathscr{S}, \leqq)$ (or shortly $\mathscr{S}$) is said to be *increasing (decreasing)* iff

$$G(\leqq) \subset G(\mathscr{S}) \quad (G(\leqq)^{-1} \subset G(\mathscr{S}))$$

(see [7], cf. [1]—[2]).

The *upper (lower) syntopogenous structure* of a preordered syntopogenous space $(E, \mathscr{S}, \leqq)$ will be denoted by $\mathscr{S}^u$ ($\mathscr{S}^l$); it is the finest of all increasing (decreasing) syntopogenous structures coarser than $\mathscr{S}$ on $(E, \leqq)$ (see [2]).

Suppose $\mathscr{E}_c = \{<_{X,C}: X \subset C \subset E,\ C \text{ is convex}\}$ (cf. [4], p. 42), and let ^{a} be an *elementary operation* (see [4], p. 69). $(E, \mathscr{S}, \leqq)$ is called *weakly ^{a}-convex* iff, for every $< \in \mathscr{S}$, there exists $\mathscr{E} \subset \mathscr{E}_c$ such that $\{<\} \prec \mathscr{E}^{ta} \prec \mathscr{S}$ (see [7]). $(E, \mathscr{S}, \leqq)$ is said to be *^{a}-convex* iff $\mathscr{S} \sim (\mathscr{S}^u \vee \mathscr{S}^l)^a$ (see [7], cf. [1]—[2]). $(E, \mathscr{S}, \leqq)$ is *symmetrizable* iff there exists a symmetrical ^{i}-convex syntopogenous structure $\mathscr{S}_0$ on $(E, \leqq)$ such that $\mathscr{S}_0 \prec \mathscr{S} \prec \mathscr{S}_0^p$ (see [7]). We say that $(E, \mathscr{S}, \leqq)$ is *continuous* iff, for each $< \in \mathscr{S}$, there is $<_1 \in \mathscr{S}$, for which $A < B$ implies $i(A) <_1 i(B)$ and $d(A) <_1 d(B)$ (see [8]).

1. T_0-, T_1- and T_2-preordered syntopogenous spaces

If $<_1$, $<_2$ are semi-topogenous orders, and $\mathscr{A}$, $\mathscr{B}$ are order families on the set E, then we define, for $A, B \subset E$,

$$A(<_1 <_2)B \quad \text{iff} \quad A <_1 X <_2 B \quad \text{for some} \quad X \subset E,$$

and

$$\mathscr{A}\mathscr{B} = \{<_1 <_2: <_1 \in \mathscr{A},\ <_2 \in \mathscr{B}\}.$$

Then $\mathscr{A}\mathscr{B}$ is also an order family coarser than $\mathscr{A}$ and $\mathscr{B}$ (cf. [5], (2.19)).

Let $(E, \mathscr{S}, \leqq)$ be a preordered syntopogenous space, and put $G = G(\leqq)$. Then

(1.0) $$G \subset G(\mathscr{S}^u) \cap G(\mathscr{S}^{lc}),$$

because

(1.1.1) $$G \subset G(\mathscr{S}^u)$$

and

(1.1.2) $$G \subset G(\mathscr{S}^{lc})$$

by $G = (G^{-1})^{-1} \subset G(\mathscr{S}^l)^{-1} = G(\mathscr{S}^{lc})$ (see [1], 3.2). The order family $\mathscr{S}^u\mathscr{S}^{lc}$ is coarser than $\mathscr{S}^u$, therefore $G(\mathscr{S}^u) \subset G(\mathscr{S}^u\mathscr{S}^{lc})$, hence

(1.2) $$G \subset G(\mathscr{S}^u\mathscr{S}^{lc}).$$

By prescribing the inclusion in the conversed direction in formulas (1.0), (1.1.1)—(1.1.2) and (1.2), we obtain three conditions, namely

(T_0) $$G(\mathscr{S}^u)\cap G(\mathscr{S}^{lc})\subset G;$$

(T_1) $$G(\mathscr{S}^u)\subset G \quad \textit{and} \quad G(\mathscr{S}^{lc})\subset G$$

(see [2]), and

(T_2) $$G(\mathscr{S}^u\mathscr{S}^{lc})\subset G.$$

It is obvious that $(T_2)\Rightarrow(T_1)\Rightarrow(T_0)$ by [1], 3.3(i). The (pre)ordered syntopogenous space $(E, \mathscr{S}, \leqq)$ will be called T_i-*(pre)ordered* iff it satisfies axiom (T_i) $(i=0, 1, 2)$.

(1.3) REMARKS. (1.3.1) By (1.0)—(1.2) it is clear that in the axioms (T_0)—(T_2) the sign $=$ can be written instead of $\subset$.

(1.3.2) It is also obvious that

$(T_0)\Leftrightarrow x, y\in E, x\not\leqq y$ imply $x<'E-y$ for some $<'\in\mathscr{S}^u$, or $y<''E-x$ for some $<''\in\mathscr{S}^l$.

$(T_1)\Leftrightarrow x, y\in E, x\not\leqq y$ imply $x<'E-y$ and $y<''E-x$ for some $<'\in\mathscr{S}^u$ and $<''\in\mathscr{S}^l$.

$(T_2)\Leftrightarrow x, y\in E, x\not\leqq y$ imply $x<'X<''^cE-y$ for some $<'\in\mathscr{S}^u$, $<''\in\mathscr{S}^l$, and $X\subset E$.

(1.3.3) Let $(E, \mathscr{T}, \leqq)$ be a preordered topological space, and τ be the associated "classical" topology. Then $(E, \mathscr{T}, \leqq)$ is T_0-preordered iff, for $x, y\in E$, $x\not\leqq y$, there exists a τ-open set V such that V is increasing, $x\in V$ and $y\notin V$, or V is decreasing, $y\in V$ and $x\notin V$. Further $(E, \mathscr{T}, \leqq)$ is T_1- or T_2-preordered iff $(E, \tau, \leqq)$ satisfies the corresponding separation axiom in McCartan's "stronger" sense [9].

(1.3.4) A discretely ordered syntopogenous space $(E, \mathscr{S}, =)$ is T_i-ordered iff $[E, \mathscr{S}]$ is a T_i-space in the sense of [4], Ch. 14. □

Further on some theorems of [4], Ch. 14 will be generalized (see (1.4)—(1.12)).

Let us consider the biperfect topologies $\mathscr{U}_E$ and $\mathscr{L}_E$ on a preordered space $(E, \leqq)$, which are defined by

$$\mathscr{U}_E=\{<'\},\quad \mathscr{L}_E=\{<''\},\quad \text{where}\quad A<'B\ \ (A<''B)\ \text{ iff}$$

$$A\subset X\subset B \text{ for a suitable increasing (decreasing) } X\subset E.$$

(see [1], 3.5 and [7], (1.2)).

(1.4) THEOREM. *For any preordered syntopogenous space* $(E, \mathscr{S}, \leqq)$, *we have*

$$(T_0)\Leftrightarrow(\mathscr{S}^u\vee\mathscr{S}^{lc})^{tb}=\mathscr{U}_E$$

$$(T_1)\Leftrightarrow\mathscr{S}^{utb}=\mathscr{U}_E \quad \textit{and} \quad \mathscr{S}^{ltb}=\mathscr{L}_E$$

$$(T_2)\Leftrightarrow(\mathscr{S}^u\mathscr{S}^{lc})^{tb}=\mathscr{U}_E.$$

PROOF. Recalling $G(\mathscr{U}_E)=G(\leqq)$ and $G(\mathscr{L}_E)=G(\leqq)^{-1}$ (see [1], 3.5; [7], (1.2)), these statements are immediate consequences of (1.3.1), (0.1) and [1], 3.3. □

(1.5) THEOREM. *Any symmetrical* T_0*-preordered syntopogenous space is* T_2*-preordered.*

PROOF. By [7], (1.8) $\mathscr{S}^{lc} \sim \mathscr{S}^{u}$, and $\mathscr{S}^{u} < \mathscr{S}^{u2} < \mathscr{S}^{u}\mathscr{S}^{u} < \mathscr{S}^{u}$, provided $\mathscr{S}$ is symmetrical. Therefore $(\mathscr{S}^{u}\mathscr{S}^{lc})^{tb} = (\mathscr{S}^{u}\mathscr{S}^{u})^{tb} = \mathscr{S}^{utb} = (\mathscr{S}^{u} \vee \mathscr{S}^{u})^{tb} = (\mathscr{S}^{u} \vee \mathscr{S}^{lc})^{tb} = \mathscr{U}_E$. □

(1.6) LEMMA. *If $(E, \mathscr{S}, \leqq)$ is a T_i-preordered syntopogenous space and $\mathscr{S} < \mathscr{S}'$, then $(E, \mathscr{S}', \leqq)$ is also T_i-preordered* $(i=0, 1, 2)$.

PROOF. $\mathscr{S}^{u} < \mathscr{S}'^{u}$, $\mathscr{S}^{l} < \mathscr{S}'^{l}$, thus e.g. (1.3.2) can be applied. □

(1.7) THEOREM. *$(E, \mathscr{S}, \leqq)$ is T_i-preordered iff so is also $(E, \mathscr{S}^{t}, \leqq)$* $(i=0, 1, 2)$.

PROOF. Because of $\mathscr{S}^{ut} \sim \mathscr{S}^{tu}$ and $\mathscr{S}^{lt} \sim \mathscr{S}^{tl}$ (see [7], (1.9)) we have $(\mathscr{S}^{u} \vee \mathscr{S}^{lc})^{tb} = (\mathscr{S}^{ut} \vee \mathscr{S}^{lct})^{tb} = (\mathscr{S}^{tu} \vee \mathscr{S}^{tlc})^{tb}$. $\mathscr{S}^{utb} = \mathscr{S}^{uttb} = \mathscr{S}^{tutb}$ and $\mathscr{S}^{ltb} = \mathscr{S}^{lttb} = \mathscr{S}^{tltb}$. Finally $(\mathscr{S}^{u}\mathscr{S}^{lc})^{tb} = (\mathscr{S}^{u}\mathscr{S}^{lc})^{ttb} = (\mathscr{S}^{ut}\mathscr{S}^{lct})^{tb} = (\mathscr{S}^{tu}\mathscr{S}^{tlc})^{tb}$. In view of these equalities (1.4) can be used. □

(1.8) EXAMPLES. (1.8.1) The well-known decreasing ordered syntopological space $(R, \mathscr{I}, \leqq)$ of [7], (1.4) in T_0-ordered, but it is not T_1-ordered. A large number of examples for T_1-, but non-T_2-ordered spaces can be obtained using the interval topology, in which the sets of the form $i(x) \cap d(y)$ $(x \leqq y)$ are the subbase of the closed sets (see e.g. [9], ex. 2).

(1.8.2) (1.7) shows that the T_i-preorderedness of a space $(E, \mathscr{S}, \leqq)$ depends only on $(E, \mathscr{S}^{t}, \leqq)$. This cannot be stated, if we write ^{p} instead of ^{t}.

Let $(R, \leqq)$ be the naturally ordered real line, and $\mathscr{T} = \{<\}$ be the symmetrical topogenous structure of [7], (1.10), which is defined with the help of the sets $H_n = (-\infty, n] \cup [n, \infty)$ for all natural numbers n so that $A < B$ is equivalent to $A \subset B$, $A \cap H_n = \emptyset$ or $H_n \subset B$ for some n. Then $\mathscr{T}^{pu} \sim \mathscr{U}_R$, $\mathscr{T}^{pl} \sim \mathscr{L}_R$ imply that $(R, \mathscr{T}^{p}, \leqq)$ *is T_2-ordered,* however $(R, \mathscr{T}, \leqq)$ *is not T_0-ordered,* because $\mathscr{T}^{u} = \mathscr{T}^{l}$ is the indiscrete syntopogenous structure of R (see [7], (1.10)). □

(1.9) THEOREM. *Let f_j be a mapping of the set E into the T_i-preordered syntopogenous space $(E_j, \mathscr{S}_j, \leqq_j)$ for every $j \in J \neq \emptyset$. Suppose $\mathscr{S} = \bigvee_{j \in J} f_j^{-1}(\mathscr{S}_j)$, and $x \leqq y$ iff $f_j(x) \leqq_j f_j(y)$ for each $j \in J$ $(x, y \in E)$. Then the preordered syntopogenous space $(E, \mathscr{S}, \leqq)$ is also T_i-preordered* $(i=0, 1, 2)$.

PROOF. It is clear that $\mathscr{S}$ is a syntopogenous structure, and $\leqq$ is a preorder on E. f_j is preorder preserving for every $j \in J$, therefore $f_j^{-1}(\mathscr{S}_j^{u})$ is increasing on $(E, \leqq)$ ([7], (1.1.4)). $f_j^{-1}(\mathscr{S}_j^{u}) < f_j^{-1}(\mathscr{S}_j) < \mathscr{S}$ implies $f_j^{-1}(\mathscr{S}_j^{u}) < \mathscr{S}^{u}$ $(j \in J)$. In the same way $f_j^{-1}(\mathscr{S}_j^{l}) < \mathscr{S}^{l}$ $(j \in J)$. If $x, y \in E$, $x \not\leqq y$, then $f_j(x) \not\leqq_j f_j(y)$ for at least one $j \in J$. For $<_j' \in \mathscr{S}_j^{u}$ $(<_j'' \in \mathscr{S}_j^{l})$ there exists $<' \in \mathscr{S}^{u}$ $(<'' \in \mathscr{S}^{l})$ such that $f_j^{-1}(<_j') \subset <'$ $\left(f_j^{-1}(<_j'') \subset <''\right)$. If $f_j(x) <_j' E_j - f_j(y)$ $\left(f_j(y) <_j'' E_j - f_j(x)\right)$, then

$$x \in f_j^{-1}(f_j(x)) <' f_j^{-1}(E_j - f_j(y)) = E - f_j^{-1}(f_j(y)) \subset E - y.$$

Similarly $f_j(y) <_j'' E_j - f_j(x)$ implies $y <'' E - x$. Finally, if $f_j(x) <_j' C <_j''^{c} E_j - f_j(y)$, then $x <' f_j^{-1}(C) <''^{c} E - y$. This completes the proof for (1.3.2). □

(1.10) COROLLARY. *If $\{\mathscr{S}_j : j \in J \neq \emptyset\}$ is a family of syntopogenous structures on the preordered space $(E, \leqq)$ such that $(E, \mathscr{S}_j, \leqq)$ is T_i-preordered for $j \in J$, then the space $\left(E, \bigvee_{j \in J} \mathscr{S}_j, \leqq\right)$ is also T_i-preordered* $(i=0, 1, 2)$.

PROOF. Let f_j be the identity of E for any $j\in J$. □

(1.11) COROLLARY. *The product of an arbitrary non-empty family of T_i-preordered syntopogenous spaces is also T_i-preordered* $(i=0, 1, 2)$.

PROOF. Let f_j be the projection of the product onto its j-th component, and $\leqq$ be the product preorder (cf. [4], (11.4)). □

(1.12) THEOREM. *In order that $(E, \mathscr{S}, \leqq)$ be T_0-preordered, it is necessary and sufficient that one of the following conditions be satisfied for any $x, y\in E$, $x\not\leqq y$:*

(1.12.1) *there exists a bounded preorder preserving $(\mathscr{S}, \mathscr{I})$-continuous function f on E such that $f(y)<f(x)$;*

(1.12.2) *there exists a bounded preorder inversing $(\mathscr{S}, \mathscr{I})$-continuous function f on E such that $f(x)<f(y)$.*

$(E, \mathscr{S}, \leqq)$ is T_1-preordered iff both (1.12.1) *and* (1.12.2) *are satisfied for every $x, y\in E$, $x\not\leqq y$.*

PROOF. It will be verified that $(1.12.1)\Leftrightarrow(x, y)\notin G(\mathscr{S}^{lc})$, and $(1.12.2)\Leftrightarrow(x, y)\notin G(\mathscr{S}^{u})$.

In fact, if $(x, y)\notin G(\mathscr{S}^{lc})$, then $x<^{c}E-y$, that is $y<E-x$ for some $<\in\mathscr{S}^{l}$. From [4], (12.41) it follows that $f(E)\subset[0, 1]$, $f(y)=0$, $f(x)=1$ for an $(\mathscr{S}^{l}, \mathscr{I})$-continuous function f, which is always preorder preserving by [7], (1.6). Conversely, if (1.12.1) is satisfied, then $f(y)<_{\varepsilon}R-f(x)$ for some $\varepsilon>0$ is obvious. $f^{-1}(\mathscr{I})$ is decreasing (see [7], (1.4), (1.1.4)), and $f^{-1}(\mathscr{I})<\mathscr{S}$ implies $f^{-1}(\mathscr{I})<\mathscr{S}^{l}$, thus there is $<\in\mathscr{S}^{l}$, for which $y<E-x$, i.e. $(y, x)\notin G(\mathscr{S}^{l})$, or equivalently $(x, y)\notin G(\mathscr{S}^{lc})$. The other equivalence is analogous (instead of [7], (1.1.4) we need to use [7], (1.1.5)). □

(1.13) COROLLARY (cf. [10], p. 52—53, [7], (4.8)). *Let $(E, \mathscr{S}, \leqq)$ be a symmetrizable preordered syntopogenous space. Then the following conditions are equivalent:*

(1.13.1) *For $x, y\in E$, $x\not\leqq y$, there exists a bounded preorder preserving $(\mathscr{S}, \mathscr{I}^{s})$-continuous function f such that $f(y)<f(x)$.*

(1.13.2) *There exists a symmetrical ^{i}-convex syntopogenous structure $\mathscr{S}_0$ on $(E, \leqq)$ such that $\mathscr{S}_0<\mathscr{S}<\mathscr{S}_0^{p}$ and $(E, \mathscr{S}_0, \leqq)$ is T_0-preordered.*

If $\mathscr{S}$ is a syntopology, then one can write $\mathscr{S}_0^{p}\sim\mathscr{S}$, and $\mathscr{H}=\mathscr{I}^{sb}$ instead of $\mathscr{I}^{s}$.

PROOF. (1.13.1)⇒(1.13.2). Let us denote by Φ the set of all $(\mathscr{S}, \mathscr{I}^{s})$-continuous ordering families of preorder preserving functions of $(E, \leqq)$. Then with $\mathscr{S}_0=\mathscr{S}_{\Phi}^{s}$, we have $\mathscr{S}_0<\mathscr{S}<\mathscr{S}_0^{p}$, and $\mathscr{S}_0$ is the finest one among those symmetrical ^{i}-convex syntopogenous structures, which are coarser than $\mathscr{S}$ (see [7], (4.8); [4], Ch. 12). If f is a function described in (1.13.1), then $f^{-1}(\mathscr{I}^{s})<\mathscr{S}$, and, because of that $f^{-1}(\mathscr{I}^{s})$ is symmetrical and ^{i}-convex ([4], (9.7); [7], (2.9)), $f^{-1}(\mathscr{I}^{s})<\mathscr{S}_0$. In view of that in this way f is $(\mathscr{S}_0, \mathscr{I}^{s})$-, and *a fortiori* $(\mathscr{S}_0, \mathscr{I})$-continuous, (1.12.1) can be used for the proof of the T_0-preorderedness of $(E, \mathscr{S}_0, \leqq)$.

(1.13.2)$\Rightarrow$(1.13.1). $(E, \mathscr{S}_0, \leqq)$ is T_1-preordered (see (1.5)), hence $x \not\leqq y$ implies $f(y)<f(x)$ for a bounded preorder preserving $(\mathscr{S}_0, \mathscr{I})$-continuous function f by (1.12). f is $(\mathscr{S}_0, \mathscr{I}^s)$-, and $(\mathscr{S}, \mathscr{I}^s)$-continuous.

If $\mathscr{S}$ is a syntopology, then $\mathscr{S}_0^p < \mathscr{S}^p < \mathscr{S}_0^{pp} = \mathscr{S}_0^p$ gives $\mathscr{S}_0^p \sim \mathscr{S}$, and f is $(\mathscr{S}, \mathscr{I}^s)$-continuous iff it is $(\mathscr{S}, \mathscr{H})$-continuous by $\mathscr{H} = \mathscr{I}^{sp}$ (cf. [4], (8.57)). □

We can complete [7], (4.8) as follows:

(1.14) THEOREM (cf. [10], Th. 9; [1], 8.2). *Let $(E, \mathscr{T}, \leqq)$ be a preordered topological space, and τ the classical topology associated with $\mathscr{T}$. Then the following statements are equivalent:*

(1.14.1) *$(E, \tau, \leqq)$ is uniformizable in the sense of Nachbin.*

(1.14.2) *$\mathscr{T} = \mathscr{S}^{tp}$ for a symmetrical syntopology $\mathscr{S}$ such that $(E, \mathscr{S}, \leqq)$ is ^{p}-convex and T_0-preordered.*

(1.14.3) *$(E, \mathscr{T}, \leqq)$ is symmetrizable, and $x \not\leqq y$ implies $f(y)<f(x)$ for a suitable bounded preorder preserving $(\mathscr{T}, \mathscr{H}^{tp})$-continuous function f.*

PROOF. (1.14.2)$\Leftrightarrow$(1.14.3). If (1.14.2) is satisfied, then $\mathscr{S} \sim (\mathscr{S}^u \vee \mathscr{S}^l)^p = (\mathscr{S}^u \vee \mathscr{S}^{cl})^p \sim (\mathscr{S}^u \vee \mathscr{S}^{uc})^p \sim \mathscr{S}^{usp}$ (see [7], (1.8)). Suppose $\mathscr{S}_0 = \mathscr{S}^{ust}$. Then $\mathscr{S}_0$ is symmetrical and ^{i}-convex by [6], (4.1.2). $\mathscr{S}_0^p = \mathscr{S}^{ustp} = \mathscr{S}^{usptp} = \mathscr{S}^{tp} = \mathscr{T}$ (see [4], (8.50)), therefore $\mathscr{T}$ is symmetrizable. $\mathscr{S}^u < \mathscr{S}_0$ implies $\mathscr{S}^u < \mathscr{S}_0^u$, hence $G(\mathscr{S}_0^u) \subset G(\mathscr{S}^u) \subset G$ (cf. (1.5), and [1], 3.3), i.e. $(E, \mathscr{S}_0, \leqq)$ is T_0-preordered, and we get (1.14.3) by (1.13).

Conversely, assume (1.14.3). Then $\mathscr{S}_0 < \mathscr{T} < \mathscr{S}_0^p$ for a symmetrical ^{i}-convex syntopogenous structure $\mathscr{S}_0$ such that $(E, \mathscr{S}_0, \leqq)$ is T_0-preordered. $\mathscr{S}_0^t = \mathscr{S}^t$ for a totally bounded symmetrical syntopology $\mathscr{S}$ on E by [4], (19.38). Owing to [7], (5.4) $\mathscr{S}$ is ^{i}-convex (and *a fortiori* ^{p}-convex). Because of (1.7) $(E, \mathscr{S}, \leqq)$ is T_0-preordered, finally $\mathscr{S}^{tp} = \mathscr{S}_0^{tp} < \mathscr{T}^{tp} = \mathscr{T} < \mathscr{S}_0^{ptp} = \mathscr{S}_0^{tp} = \mathscr{S}^{tp}$, i.e. $\mathscr{T} = \mathscr{S}^{tp}$.

(1.14.1)$\Leftrightarrow$(1.14.3). Nachbin's definition of a uniformizable preordered topological space $(E, \tau, \leqq)$ can be transformed into the following equivalent form (cf. [10], pp. 52—53):

1) If V is a neighbourhood of $x \in E$, then there are continuous real-valued functions f, g such that $f(g)$ is preorder preserving (inversing), $f(E)$, $g(E) \subset [0, 1]$, $f(x) = g(x) = 0$, and $\max\{f(y), g(y)\} = 1$ for $y \in E - V$. (In fact, consider $f' = 1 - g$ and $g' = 1 - f$.)

2) If $x, y \in E$, $x \not\leqq y$, then there exists a bounded, continuous and preorder preserving function f with $f(y)<f(x)$. (Indeed, if f' is an arbitrary continuous, preorder preserving function such that $f'(y)<f'(x)$, then putting

$$f(z) = \max\{\min\{f'(x), f'(z)\}, f'(y)\},$$

we get $f'(y) \leqq f(z) \leqq f'(x)$ for any $z \in E$, and f satisfies Condition 2).)

Thus the proof is clear by [7], (4.8), and (1.13). □

Finally we study the connection existing between the separation properties of $(E, \mathscr{S}, \leqq)$ and $[E, \mathscr{S}]$.

(1.15) THEOREM. *Let* $[E, \mathscr{S}]$ *be a syntopogenous space, and* $\leqq$ *be an order on* E.

(1.15.1) *If* $(E, \mathscr{S}, \leqq)$ *is* T_j*-ordered, then* $[E, \mathscr{S}]$ *is a* T_j*-space* $(j=0, 1, 2)$.

(1.15.2) *If* $[E, \mathscr{S}]$ *is a* T_j*-space,* $\leqq$ *is linear, and* $\mathscr{S}$ *is* ^{i}*- or* ^{p}*-convex, then* $(E, \mathscr{S}, \leqq)$ *is* T_j*-ordered* $(j=0, 1, 2)$.

The statement of (1.15.2) was proved by Burgess and Fitzpatrick for $j=1$ (see [2], 5.5).

PROOF. (1.15.1) is clear by $\mathscr{S}^u < \mathscr{S}$, $\mathscr{S}^l < \mathscr{S}$ and (1.3.2), because in our case $x \neq y$ implies $x \not\leqq y$ or $y \not\leqq x$.

(1.15.2). Suppose $[E, \mathscr{S}]$ is T_0, and $(x, y) \in G(\mathscr{S}^u) \cap G(\mathscr{S}^{lc})$. If $x=y$, then $(x, y) \in G$. If $x \neq y$, then there exists $\prec \in \mathscr{S}$ such that $x \prec E-y$ or $y \prec E-x$. If $\mathscr{S}$ is either ^{i}- or ^{p}-convex, we have $\mathscr{S} < (\mathscr{S}^u \vee \mathscr{S}^l)^b$. For $\prec' \in \mathscr{S}^u$, $\prec'' \in \mathscr{S}^l$, $\prec \subset (\prec' \cup \prec'')^{qb} = (\prec' \cup \prec'')^b$, we have at least one of the following inequalities:

$$x \prec' E-y, \tag{1}$$

$$x \prec'' E-y, \tag{2}$$

$$y \prec' E-x, \tag{3}$$

$$y \prec'' E-x. \tag{4}$$

Because of the choice of (x, y), (1) and (4) are impossible. Thus $d(x) \subset E-y$ or $i(y) \subset E-x$ issues from (2) or (3), therefore $y \not\leqq x$. On the basis of the linearity of $\leqq$, we get $(x, y) \in G$, that is $(E, \mathscr{S}, \leqq)$ is T_0-ordered.

Let now $[E, \mathscr{S}]$ be T_2, and assume $(x, y) \in G(\mathscr{S}^u \mathscr{S}^{lc})$ is such that $x \not\leqq y$. Then $y \leqq x$ and $x \neq y$. There exists $\prec \in \mathscr{S}$ with $x \prec C \prec^c E-y$. In view of $\mathscr{S} < (\mathscr{S}^u \vee \mathscr{S}^l)^p$, one can find $\prec_1 \in \mathscr{S}^u$, $\prec_2 \in \mathscr{S}^l$ such that $\prec \subset (\prec_1 \cup \prec_2)^{qp}$, and $\prec' \in \mathscr{S}^u$, $\prec'' \in \mathscr{S}^l$, for which $\prec_1 \subset \prec'^2$, $\prec_2 \subset \prec''^2$. We can select sets C_1, C_2, C_1', C_2' such that $x \prec_1 C_1$, $x \prec_2 C_2$, $y \prec_1 C_1'$, $y \prec_2 C_2'$, $C_1 \cap C_2 \subset C$ and $C_1' \cap C_2' \subset E-C$. Suppose $x \prec' X_1 \prec' C_1$, $x \prec'' X_2 \prec'' C_2$, $y \prec' Y_1 \prec' C_1'$ and $y \prec'' Y_2 \prec'' C_2'$. From [7], (1.2) it follows that $X_1 \subset I \subset C_1$, $X_2 \subset D \subset C_2$, $Y_1 \subset I' \subset C_1'$ and $Y_2 \subset D' \subset C_2'$, where I and I' are increasing, D and D' are decreasing. We have $x \prec' I$, $x \prec'' D$, $y \prec' I'$, $y \prec'' D'$, $I \cap D \subset C$ and $I' \cap D' \subset E-C$. Now $I \cap D' = \emptyset$ is impossible, because in this case $x \prec' I \subset E-D' \prec^c E-y$, which contradicts the choice of (x, y). Therefore there exists $z \in I \cap D'$. Moreover we get $x \in I'$ and $y \in D$ from $y \leqq x$. Owing to the linearity of $\leqq$, we have to distinguish three different cases only, namely:

$$z \leqq y \leqq x, \tag{1}$$

$$y \leqq z \leqq x, \tag{2}$$

$$y \leqq x \leqq z. \tag{3}$$

It will be seen that each one of the conditions (1)—(3) contains a contradiction. In fact; (1): $z \leqq y$ implies $y \in I$, therefore $y \in I \cap D \subset C \subset E-y$. (2): $y \leqq z$ implies $z \in I'$, and $z \leqq x$ gives $z \in D$, thus $z \in I \cap D \cap I' \cap D' \subset C \cap (E-C) = \emptyset$. (3): $x \leqq z$ implies $x \in D'$, consequently $x \in I' \cap D' \subset E-C \subset E-x$. Hence our initial condition cannot be satisfied, i.e. $G(\mathscr{S}^u \mathscr{S}^{lc}) \subset G$, so that $(E, \mathscr{S}, \leqq)$ is T_2-ordered. □

2. T_0-ordered epi-reflection

Let $(E, \mathscr{S}, \leqq)$ be a preordered syntopogenous space. For the sake of simplicity, let us introduce the notation $H=G(\mathscr{S}^u)\cap G(\mathscr{S}^{lc})$. It is clear that H is the graph of a preorder, and $H\cap H^{-1}$ is that of an equivalence $\sim$ on E. Let us denote by x_* the $\sim$-class containing x, i.e. $x_*=\{y\in E\colon x\sim y\}$. If $\leqq$ is the relation of the equality (the so-called *discrete order* of E), then $\mathscr{S}^u\sim\mathscr{S}^l\sim\mathscr{S}$, therefore $H=G(\mathscr{S})\cap G(\mathscr{S}^c)$, thus $H=H^{-1}$, and x_* is a cellule of the *separative partition* of the space in question (see [4], ch. 14).

Returning to the general case, put $E_*=\{x_*\colon x\in E\}$. Let us observe that $x\sim u$, $y\sim v$, $(x,y)\in H$ imply $(u,v)\in H$. (In fact, $(u,x)\in H$, $(x,y)\in H$, $(y,v)\in H\Rightarrow(u,v)\in H$.) Hence a relation $\leqq_*$ can be defined on E_* as follows: $x_*\leqq_* y_*$ iff $(x,y)\in H$. Further on let us consider the canonical mapping R of E onto E_*, which is determined by $R(x)=x_*$ $(x\in E)$. Among those syntopogenous structures $\mathscr{S}'$ on E_*, for which R is $(\mathscr{S},\mathscr{S}')$-continuous, there exists a finest one, and it will be denoted by $\mathscr{S}_*$. We have

$$\mathscr{S}_*=\bigvee\{\mathscr{S}'\colon R^{-1}(\mathscr{S}')<\mathscr{S}\}$$

(see [4], (9.24)).

In the rest of this article we assume that the reader is familiar with the notion of a mapping *compatible* with a syntopogenous structure and the definition of the *image* of a syntopogenous structure (see [4], pp. 105—110).

(2.1). LEMMA. *The relation $\leqq_*$ is an order on E_*. The mapping R is preorder preserving and compatible with $\mathscr{S}^u$ and $\mathscr{S}^l$. In the ordered syntopogenous space $(E_*,\mathscr{S}_*,\leqq_*)$ the equivalences $(\mathscr{S}_*)^u\sim R(\mathscr{S}^u)$ and $(\mathscr{S}_*)^l\sim R(\mathscr{S}^l)$ hold.*

PROOF. H is the graph of a preorder, thus owing to the definition, $\leqq_*$ is also a preorder. If $x_*\leqq_* y_*$ and $y_*\leqq_* x_*$, then $(x,y),(y,x)\in H$, thus $x\sim y$, i.e. $x_*=y_*$.

If $x\leqq y$, then we have $(x,y)\in H$ by (1.0), so that $R(x)=x_*\leqq_* y_*=R(y)$.

Now suppose $<\in\mathscr{S}^u$ or $<\in\mathscr{S}^l$, and $A<B$. Then $x\in R^{-1}(R(A))$ implies $R(a)=R(x)$ for some $a\in A$. This means $a\sim x$, hence $(a,x)\in H\subset G(\mathscr{S}^u)$ and $(x,a)\in H\subset G(\mathscr{S}^{lc})$. This gives that $a<E-x$ is false, therefore $x\in B$. Summing up, $R^{-1}(R(A))\subset B$, so that R is compatible with $\mathscr{S}^u$ and $\mathscr{S}^l$.

R is $(\mathscr{S},\mathscr{S}_*)$-continuous, thus $R^{-1}((\mathscr{S}_*)^u)<R^{-1}(\mathscr{S}_*)<\mathscr{S}$. By [7], (1.1.4) $R^{-1}((\mathscr{S}_*)^u)$ is increasing on $(E,\leqq)$, hence $R^{-1}((\mathscr{S}_*)^u)<\mathscr{S}^u$. Thus $(\mathscr{S}_*)^u<R(\mathscr{S}^u)$ issues from [4], (9.29). In the same way, we obtain $(\mathscr{S}_*)^l<R(\mathscr{S}^l)$. On the other hand, $R^{-1}(R(\mathscr{S}^u))\sim\mathscr{S}^u<\mathscr{S}$, $R^{-1}(R(\mathscr{S}^l))\sim\mathscr{S}^l<\mathscr{S}$ by [4], (9.29), therefore $R(\mathscr{S}^u)<\mathscr{S}_*$ and $R(\mathscr{S}^l)<\mathscr{S}_*$. We show that $R(\mathscr{S}^u)$ is increasing, $R(\mathscr{S}^l)$ is decreasing on $(E_*,\leqq_*)$, and these yield $R(\mathscr{S}^u)<(\mathscr{S}_*)^u$ and $R(\mathscr{S}^l)<(\mathscr{S}_*)^l$. In fact, suppose $<\in\mathscr{S}^u$ and $x_*R(<)E_*-y_*$. By definition the relations

$$x\in R^{-1}(x_*)<R^{-1}(E_*-y_*)=E-R^{-1}(y_*)\subset E-y$$

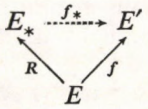
imply $x<E-y$, hence $(x,y)\notin G(\mathscr{S}^u)$, consequently $(x,y)\notin H$, thus $(x_*,y_*)\notin G(\leqq_*)$. We got $G(\leqq_*)\subset G(R(\mathscr{S}^u))$, which was to be proved. The verification of $G(\leqq_*)^{-1}\subset G(R(\mathscr{S}^l))$ is analogous. □

(2.2) THEOREM. *For every preordered syntopogenous space* $(E, \mathscr{S}, \leqq)$,

(2.2.1) $(E_*, \mathscr{S}_*, \leqq_*)$ *is* T_0*-ordered.*

(2.2.2) R *is a continuous preorder preserving surjection of* $(E, \mathscr{S}, \leqq)$ *onto* $(E_*, \mathscr{S}_*, \leqq_*)$.

(2.2.3) *If* f *is a continuous preorder preserving mapping of* $(E, \mathscr{S}, \leqq)$ *into a* T_0*-ordered syntopogenous space* $(E', \mathscr{S}', \leqq')$, *then there exists a unique continuous order preserving mapping* f_* *of* $(E_*, \mathscr{S}_*, \leqq_*)$ *into* $(E', \mathscr{S}', \leqq')$ *such that the diagram*



is commutative (i.e. $f=f_*\circ R$).

PROOF. (2.2.1) The space is ordered by (2.1). Suppose $x_*, y_*\in E_*$, $x_* \not\leqq_* y_*$. Then $(x,y)\notin H$, i.e. $(x,y)\notin G(\mathscr{S}^u)$, or $(x,y)\notin G(\mathscr{S}^{lc})$ (or equivalently $(y,x)\notin G(\mathscr{S}^l)$). In the first case we have $x<E-y$ for some $<\in\mathscr{S}^u$. Assume $<_1\in\mathscr{S}^u$, $<\subset<_1^3$, then $x<_1A<_1B<_1E-y$ for suitable $A, B\subset E$. R is compatible with $\mathscr{S}^u$, therefore $R^{-1}(x_*)=R^{-1}(R(x))\subset A<_1B\subset R^{-1}(R(B))\subset E-y$, thus $x_*R(<_1)R(B)\subset E_*-y_*$. Since $R(\mathscr{S}^u)\sim(\mathscr{S}_*)^u$, we have $(x_*,y_*)\notin G((\mathscr{S}_*)^u)$. In the other case $(y,x)\notin G(\mathscr{S}^l)$ implies similarly $y_*R(<_1)E_*-x_*$ for some $<_1\in\mathscr{S}^l$, and because of $R(\mathscr{S}^l)\sim(\mathscr{S}_*)^l$, we obtain $(x_*,y_*)\notin G((\mathscr{S}_*)^{lc})$.

(2.2.2) is clear.

(2.2.3) Under the conditions of this statement, $x, y\in E$, $x\sim y$ implies $f(x)=f(y)$. Indeed, if $f(x)\neq f(y)$, then $f(x)\not\leqq' f(y)$, or $f(y)\not\leqq' f(x)$. From the T_0-orderedness of $(E', \mathscr{S}', \not\leqq')$ it follows simply that one of the inequalities $xf^{-1}(<')E-y$, $yf^{-1}(<'')(<'')E-x$, $yf^{-1}(<')E-x$, or $xf^{-1}(<'')E-y$ is true for a suitable $<'\in\mathscr{S}'^u$ or $<''\in\mathscr{S}'^l$. By [7], (1.1.4) from $f^{-1}(\mathscr{S}'^u)<f^{-1}(\mathscr{S}')<\mathscr{S}$, and $f^{-1}(\mathscr{S}'^l)<f^{-1}(\mathscr{S}')<\mathscr{S}$ we get $f^{-1}(\mathscr{S}'^u)<\mathscr{S}^u$ and $f^{-1}(\mathscr{S}'^l)<\mathscr{S}^l$, thus $(x,y)\notin H$ or $(y,x)\notin H$, which means $x\not\sim y$.

This shows that it is possible to define a mapping $f_*: E_*\to E'$ with $f_*(x_*)=f(x)$ for $x\in E$, so that $f_*\circ R=f$. Since R is a surjection, this property determines f_* uniquely. f_* is order preserving, because $x, y\in E$, $f_*(x_*)\not\leqq' f_*(y_*)$ means $f(x)\not\leqq' f(y)$, and in this case, as we already showed, $(x,y)\notin H$, i.e. $x_*\not\leqq_* y_*$. Finally f_* is $(\mathscr{S}_*, \mathscr{S}')$-continuous, because $R^{-1}(f_*^{-1}(\mathscr{S}'))=f^{-1}(\mathscr{S}')<\mathscr{S}$ implies $f_*^{-1}(\mathscr{S}')<\mathscr{S}_*$. □

In the terminology of the categories (2.2) means that the category of the T_0-ordered syntopogenous spaces is an epi-reflective subcategory of the category of all preordered syntopogenous spaces, therefore $(E_*, \mathscr{S}_*, \leqq_*)$ will be called the T_0-*ordered epi-reflection of* $(E, \mathscr{S}, \leqq)$ (cf. [6]).

REMARKS. Every preordered space $(E, \leqq)$ can be regarded as a preordered syntopogenous space $(E, \mathscr{D}, \leqq)$, where $\mathscr{D}$ is the discrete syntopogenous structure $\{\subset\}$

of E. Then $\mathscr{D}^u=\mathscr{U}_E$, $\mathscr{D}^l=\mathscr{L}_E$, and $H=G(\mathscr{U}_E)\cap G(\mathscr{L}_E^c)=G(\mathscr{U}_E)$ (see [7], (1.2)). From this $x\sim y$ iff $x\leqq y$ and $y\leqq x$. It is obvious that $\mathscr{D}_*$ is equivalent to the discrete syntopogenous structure of E_*, thus $(E_*, \mathscr{D}_*, \leqq_*)$ is an ordered space topologized similarly to $(E, \mathscr{D}, \leqq)$. As a corollary of (2.2), we can easily verify that the category of the ordered spaces (with order preserving mappings) is epi-reflective in the category of the preordered spaces and preorder preserving mappings.

On the other hand, considering a syntopogenous space $(E, \mathscr{S}, =)$ preordered by the relation of the equality, the reader can easily prove that $=_*$ is also the equality relation on E_*, and $[E_*, \mathscr{S}_*]$ is isomorphic to the quotient space corresponding to the separative partition of $[E, \mathscr{S}]$. This implies that the category of the separated syntopogenous spaces (with continuous mappings) is epi-reflective in the category of all syntopogenous spaces and continuous mappings (see, for topological spaces, [6], p. 105 (2)).

Two preordered syntopogenous spaces $(E, \mathscr{S}, \leqq)$ and $(E', \mathscr{S}', \leqq')$ will be said to be *isomorphic*, if there is a one-to-one mapping $f\colon E\to E'$ such that both f and $g=f^{-1}$ are continuous and preorder preserving.

(2.3) THEOREM. *A preordered syntopogenous space $(E, \mathscr{S}, \leqq)$ is T_0-ordered iff it is isomorphic to $(E_*, \mathscr{S}_*, \leqq_*)$.*

PROOF. T_0-orderedness is invariant obviously with respect to isomorphisms. Conversely, if $(E, \mathscr{S}, \leqq)$ is T_0-ordered, then there exists a continuous order preserving mapping id_* of $(E_*, \mathscr{S}_*, \leqq_*)$ into $(E, \mathscr{S}, \leqq)$ such that $\mathrm{id}=\mathrm{id}_*\circ R$, where id denotes the identity of E (see (2.2)). Since R is a surjection, id_* is the inverse of R. □

Further on we shall study properties hereditary for the T_0-ordered epi-reflection.

(2.4) LEMMA. *Suppose $\mathscr{S}\sim\mathscr{S}^k$ for an arbitrary ordinary operation ^{k} ([4], p. 74). Then $\mathscr{S}_*\sim(\mathscr{S}_*)^k$. Disregarding isomorphisms, the T_0-ordered epi-reflection of a preordered topological, topogenous, perfect, biperfect or symmetrical syntopogenous space is also of this kind.*

PROOF. With the above notations $R^{-1}((\mathscr{S}_*)^k)=R^{-1}(\mathscr{S}_*)^k<\mathscr{S}^k\sim\mathscr{S}$, therefore $(\mathscr{S}_*)^k<\mathscr{S}_*$, i.e. $\mathscr{S}_*\sim(\mathscr{S}_*)^k$. It is evident that the space $(E_*, \mathscr{S}_*, \leqq_*)$ is isomorphic to $(E_*, (\mathscr{S}_*)^k, \leqq_*)$, provided $\mathscr{S}_*\sim(\mathscr{S}_*)^k$. □

(2.5) REMARK. It is worth constructing the T_0-ordered epi-reflection of a preordered topological space $(E, \mathscr{T}, \leqq)$ so that this construction be applicable for "classical" spaces, too. For that purpose let us denote by $U(x)$ $(L(x))$ the intersection of all increasing (decreasing) $\mathscr{T}$-open sets containing $x\in E$. It is easy to see that $(x, y)\in H$ is equivalent to $U(y)\subset U(x)$ and $L(x)\subset L(y)$. Consequently, for $x, y\in E$, $x_*=y_*$ iff $U(x)=U(y)$ and $L(x)=L(y)$, moreover $x_*\leqq_* y_*$ iff $U(y)\subset U(x)$ and $L(x)\subset L(y)$. We have $\mathscr{T}_*\sim(\mathscr{T}_*)^{tp}$, and the latter is the finest of all topologies $\mathscr{T}'$ on E_*, for which R is $(\mathscr{T}, \mathscr{T}')$-continuous. Hence $V_*\subset E_*$ is $(\mathscr{T}_*)^{tp}$-open iff $R^{-1}(V_*)$ is $\mathscr{T}$-open, in particular, V_* is increasing (decreasing) and open in the space $(E_*, (\mathscr{T}_*)^{tp}, \leqq_*)$ if and only if $R^{-1}(V_*)$ is of the same type in $(E, \mathscr{T}, \leqq)$. □

(2.6) THEOREM. *Let* $(E, \mathscr{S}, \leqq)$ *be a preordered syntopogenous space, and* ^{a} *be an elementary operation. The following conditions are equivalent:*

(2.6.1) $(E, \mathscr{S}, \leqq)$ *is* ^{a}*-convex.*

(2.6.2) R *is compatible with* $\mathscr{S}$*, and* $(E_*, \mathscr{S}_*, \leqq_*)$ *is* ^{a}*-convex.*

PROOF. If $(E, \mathscr{S}, \leqq)$ is ^{a}-convex, then $\mathscr{S} \sim (\mathscr{S}^u \vee \mathscr{S}^l)^a$, therefore R is compatible with $\mathscr{S}$, and $\mathscr{S}_* \sim R(\mathscr{S}) \sim R\big((\mathscr{S}^u \vee \mathscr{S}^l)^a\big) \sim \big(R(\mathscr{S}^u) \vee R(\mathscr{S}^l)\big)^a \sim \big((\mathscr{S}_*)^u \vee (\mathscr{S}_*)^l\big)^a$ by (2.1), and [4], (9.34), (9.33). Conversely, (2.6.2)$\Rightarrow$(2.6.1) on the basis of [7], (2.9), because $R^{-1}(\mathscr{S}_*) \sim \mathscr{S}$ (see [4], (9.29)). □

(2.7) THEOREM. *The following conditions are equivalent for every preordered syntopogenous space* $(E, \mathscr{S}, \leqq)$:

(2.7.1) $(E, \mathscr{S}, \leqq)$ *is symmetrizable.*

(2.7.2) R *is compatible with* $\mathscr{S}$*, and* $(E_*, \mathscr{S}_*, \leqq_*)$ *is symmetrizable.*

PROOF. If $(E, \mathscr{S}, \leqq)$ is symmetrizable, then $\mathscr{S}_0 \prec \mathscr{S} \prec \mathscr{S}_0^p$ for a suitable ^{i}-convex symmetrical syntopogenous structure $\mathscr{S}_0$ on $(E, \leqq)$. $\mathscr{S}_0^u \prec \mathscr{S}^u$, $\mathscr{S}_0^l \prec \mathscr{S}^l$, therefore R is compatible with $\mathscr{S}_0 \sim \mathscr{S}_0^u \vee \mathscr{S}_0^l$ by (2.1) and [4], (9.32), (9.34). Thus R is compatible with $\mathscr{S}_0^p$, and *a fortiori* with $\mathscr{S}$ ([4], (9.33), (9.32)). We have $\mathscr{S}' = R(\mathscr{S}_0)^s = R(\mathscr{S}_0) \sim R(\mathscr{S}_0^u \vee \mathscr{S}_0^l) \sim R(\mathscr{S}_0^u) \vee R(\mathscr{S}_0^l)$, where $R(\mathscr{S}_0^u)$ is increasing, $R(\mathscr{S}_0^l)$ is decreasing by [4], (9.32), [7], (1.1.1), and lemma (2.1) of the present paper. Since in this way $\mathscr{S}'$ is symmetrical and ^{i}-convex, $(E_*, \mathscr{S}_*, \leqq_*)$ is symmetrizable, namely $\mathscr{S}' = R(\mathscr{S}_0) \prec R(\mathscr{S}) \sim \mathscr{S}_*$ and $\mathscr{S}_* \sim R(\mathscr{S}) \prec R(\mathscr{S}_0^p) \sim R(\mathscr{S}_0)^p = \mathscr{S}'^p$.

Conversely, if $(E_*, \mathscr{S}_*, \leqq_*)$ is symmetrizable, and R is compatible with $\mathscr{S}$, then $\mathscr{S}' \prec \mathscr{S}_* \prec \mathscr{S}'^p$ for a symmetrical ^{i}-convex syntopogenous structure $\mathscr{S}'$ on $(E_*, \leqq_*)$, thus $R^{-1}(\mathscr{S}') \prec R^{-1}(\mathscr{S}_*) \prec R^{-1}(\mathscr{S}'^p) = R^{-1}(\mathscr{S}')^p$. $R^{-1}(\mathscr{S}')$ is symmetrical and ^{i}-convex, and $R^{-1}(\mathscr{S}_*) \sim \mathscr{S}$ (see [7], (2.9), [4], (9.7) and (9.29)). Hence $(E, \mathscr{S}, \leqq)$ is symmetrizable. □

(2.8) THEOREM. *If* $(E, \mathscr{S}, \leqq)$ *is continuous, then so is* $(E_*, \mathscr{S}_*, \leqq_*)$*, too.*

PROOF. $R^{-1}(\mathscr{S}_*) \prec \mathscr{S}$, therefore if $<' \in \mathscr{S}_*$, we can select $< \in \mathscr{S}$ such that $R^{-1}(<') \sqsubset <$. Suppose $<_1 \in \mathscr{S}^u$, $i(<) \sqsubset <_1$ (see [8], (2.4)), $<_2 \in \mathscr{S}^u$, $<_1 \sqsubset <_2^2$. R is compatible with $\mathscr{S}^u$, and $R(<_2) \in R(\mathscr{S}^u) \sim (\mathscr{S}_*)^u \prec \mathscr{S}_*$ by (2.1), hence there is $<'' \in \mathscr{S}_*$, for which $R(<_2) \sqsubset <''$. We prove $A', B' \subset E_*$, $A' <' B' \Rightarrow i(A') <'' i(B')$. In fact, $A' <' B'$ implies $R^{-1}(A') < R^{-1}(B')$. Then

$$R^{-1}(A') \subset i\big(R^{-1}(A')\big) i(<) i\big(R^{-1}(B')\big) \subset R^{-1}\big(i(B')\big),$$

because R is preorder preserving. Owing to $R^{-1}(A') <_1 R^{-1}\big(i(B')\big)$, a set $D \subset E$ can be found with $R^{-1}(A') <_2 D <_2 R^{-1}\big(i(B')\big)$. We show $R^{-1}\big(i(A')\big) \subset D$. Indeed, $R(x) \in i(A')$ implies that $R(a) \in A'$, $R(a) \leqq_* R(x)$ for some $a \in E$. Then $(a, x) \in H$, therefore $a \in R^{-1}(A') <_2 D \subset E - x$ fails to be true, hence $x \in D$. Now $R^{-1}\big(i(A')\big) \subset D <_2 R^{-1}\big(i(B')\big)$ implies $i(A') R(<_2) i(B')$, thus $i(A') <'' i(B')$.

A similar argument can be applied to prove that, for $<' \in \mathscr{S}_*$, there exists $<'' \in \mathscr{S}_*$ with $A' <' B' \Rightarrow d(A') <'' d(B')$. □

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ON THE EQUICONVERGENCE OF EXPANSIONS BY RIESZ BASES FORMED BY EIGENFUNCTIONS OF A LINEAR DIFFERENTIAL OPERATOR OF ORDER $2n$

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The theory of non-selfadjoint differential operators has great importance in several applications. Developing a fruitful method of V. A. Il'in [1], in [2] I. Joó described a new method which makes possible to avoid the use of the basic solution in such investigations. Using this method, a general equiconvergence theorem was proved in [9] for the Schrödinger operator, having generalized the results of the papers [5], [6], [7] and [8]. This theorem concerns the expansions by Riesz bases formed by eigenfunctions of higher order of the Schrödinger operator (but seems to be new also for the case of orthonormal bases). The existence of such Riesz bases was proved for a wide class of operators (also of higher order) by V. P. Mikhailov [10] and G. M. Keselman [11].

Later on, the method of [2] was generalized for the case of differential operators of higher order in [4], [12] and [13]. These generalizations are based on a new formula which extends the well-known mean-value formula of E. C. Titchmarsh for more general operators.

Using the method and the results of these papers, we shall prove a general equiconvergence theorem for linear differential operators of higher order. In this theorem the eigenvalues of the Riesz basis may be arbitrary complex numbers. Our result is new also for orthonormal bases.

1. On the eigenfunctions of the operator $Lu=u^{(2n)}+qu$

Let G be an arbitrary open interval on the real line, $q\in L^2_{\mathrm{loc}}(G)$ an arbitrary complex function, $n\in\mathbf{N}$ and consider the linear differential operator

$$Lu = u^{(2n)}+qu \tag{1}$$

of order $2n$. Given a complex number λ, a function $u\colon G\to\mathbf{C}$, $u\not\equiv 0$ is called an eigenfunction of the operator L with the eigenvalue λ if u is locally absolutely continuous together with its first $2n-1$ derivatives on G and if $Lu=\lambda u$ a.e. on G.

Given any $\lambda\in\mathbf{C}$, let us denote by $\mu_1, \dots, \mu_{2n}$ the $2n$-th roots of λ such that

$$\operatorname{Re}\mu_1 \geqq \operatorname{Re}\mu_2 \geqq \dots \geqq \operatorname{Re}\mu_{2n}. \tag{2}$$

(We put $\mu_1=\dots=\mu_{2n}=0$ if $\lambda=0$.) We shall write μ instead of μ_n. Obviously, there exists a number $C=C_n$ such that for any $\lambda\in\mathbf{C}$,

$$\begin{cases} \operatorname{Re}\mu_p \geqq 0 \quad \text{if} \quad p\leqq n, \quad \operatorname{Re}\mu_p \leqq 0 \quad \text{if} \quad p>n, \\ |\operatorname{Re}\mu_p| \geqq C|\mu| \quad \text{if} \quad p\neq n,\ n+1, \\ \operatorname{Re}(\mu_p-\mu_q) \geqq C|\mu| \quad \text{if} \quad p+2\leqq q. \end{cases} \tag{3}$$

Let us denote by $f_k(\mu)$ the elementary symmetric polynomial of degree k of the variables $e^{\mu_1}, \ldots, e^{\mu_{2n}}$ with leading coefficient $(-1)^{n-k}$ $(k=0, \ldots, 2n)$.

Throughout this section let us fix an arbitrary eigenfunction u of the operator L with some eigenvalue λ. It follows from Theorem 1 of [4] and from the remarks in [15] that

$$-f_n(\mu t)u(x) = \sum_{k=1}^{n} f_{n-k}(\mu t)[u(x-kt)+u(x+kt)] + \int_{x-nt}^{x+nt} F(\mu, x, t, \xi)q(\xi)u(\xi)\,d\xi \tag{4}$$

whenever $x-nt\in G$ and $x+nt\in G$, where

$$F(\mu, x, t, \xi) := \begin{cases} \sum_{p=1}^{2n} \frac{\mu_p}{2n\lambda} \sum_{k=l}^{n} f_{n-k}(\mu t)\, e^{\mu_p(x+kt-\xi)} \\ \text{if} \quad x+lt-t \leqq \xi < x+lt, \quad l = 1, \ldots, n, \\ \sum_{p=1}^{2n} \frac{-\mu_p}{2n\lambda} \sum_{k=1-l}^{n} f_{n-k}(\mu t)\, e^{\mu_p(x-kt-\xi)} \\ \text{if} \quad x+lt-t \leqq \xi < x+lt, \quad l = 0, \ldots, 1-n. \end{cases} \tag{5}$$

(The explicit form of the functions f_k was obtained by I. Joó in [15].) Let us further introduce the notations

$$\begin{cases} g_k(\mu) := e^{-(\mu_1+\ldots+\mu_{n-1})} f_k(\mu), \\ G(\mu, x, t, \xi) := e^{-(\mu_1+\ldots-\mu_{n-1})t} F(\mu, x, t, \xi), \\ h_k(\mu) := e^{-\mu} g_k(\mu), \\ H(\mu, x, t, \xi) := e^{-\mu t} G(\mu, x, t, \xi). \end{cases} \tag{6}$$

Then (4) can also be written in the following forms:

$$-g_n(\mu t)u(x) = \sum_{k=1}^{n} g_{n-k}(\mu t)[u(x-kt)+u(x+kt)] + \int_{x-nt}^{x+nt} G(\mu, x, t, \xi)q(\xi)u(\xi)\,d\xi, \tag{7}$$

and

$$-h_n(\mu t)u(x) = \sum_{k=1}^{n} h_{n-k}(\mu t)[u(x-kt)+u(x+kt)] + \int_{x-nt}^{x+nt} H(\mu, x, t, \xi)q(\xi)u(\xi)\,d\xi. \tag{8}$$

We shall need the following simple consequences of (2) and (3):

LEMMA. 1. *There exists a constant* $C>0$ *such that for any* $\lambda\in\mathbf{C}$ *and for any* $t\geqq 0$,

$$(9)\qquad \begin{cases} |g_n(\mu t)-2\operatorname{ch}\mu t|\leqq Ce^{-\operatorname{Re}\mu_{n-1}t}, \\ |1+g_{n-1}(\mu t)|\leqq Ce^{\operatorname{Re}(\mu-\mu_{n-1})t}, \\ |g_{n-k}(\mu t)|\leqq Ce^{-\operatorname{Re}\mu_{n-1}t} \quad (k\geqq 2), \\ \left|\frac{d}{dt}g_{n-1}(\mu t)\right|\leqq C|\mu|e^{\operatorname{Re}(\mu-\mu_{n-1})t}, \\ \left|\frac{d}{dt}g_{n-k}(\mu t)\right|\leqq C|\mu|e^{-\operatorname{Re}\mu_{n-1}t} \quad (k\geqq 2), \end{cases}$$

$$(10)\qquad \begin{cases} |h_n(\mu t)-1|\leqq Ce^{-2\operatorname{Re}\mu t}, \\ |h_{n-1}(\mu t)|\leqq Ce^{-\operatorname{Re}\mu t}, \\ |h_{n-k}(\mu t)|\leqq Ce^{-\operatorname{Re}\mu_{n-1}t} \quad (k\geqq 2), \\ \left|\frac{d}{dt}h_{n-1}(\mu t)\right|\leqq C|\mu|e^{-\operatorname{Re}\mu t}, \\ \left|\frac{d}{dt}h_{n-k}(\mu t)\right|\leqq C|\mu|e^{-\operatorname{Re}\mu_{n-1}t} \quad (k\geqq 2). \quad \square \end{cases}$$

LEMMA 2. *Given any* $B>0$, *there exist the numbers* $A, C>0$ *such that*

$$(11)\qquad |g_n(\mu^1 t)-g_n(\mu^2 t)|\leqq At$$

whenever $|\mu^1-\mu^2|$, $\operatorname{Re}\mu^1, \operatorname{Re}\mu^2\leqq B$, $|\operatorname{Im}\mu^1|, |\operatorname{Im}\mu^2|\geqq C$ *and* $0\leqq t\leqq B$.

PROOF. Let us fix $C>0$ so large that the above conditions imply the inequalities

$$\operatorname{Re}\mu^1<\operatorname{Re}\mu^1_{n-1}, \quad \operatorname{Re}\mu^2<\operatorname{Re}\mu^2_{n-1}, \quad (\operatorname{Im}\mu^1)(\operatorname{Im}\mu^2)>0.$$

Then we can choose $n-1$ from the $2n$-th roots of unity $\omega_1, \ldots, \omega_{2n}$, say $\omega_1, \ldots, \omega_{n-1}$, such that

$$g_n(\mu^1 t)=e^{-\mu^1(\omega_1+\ldots+\omega_{n-1})t}f_n(\mu^1 t),$$

$$g_n(\mu^2 t)=e^{-\mu^2(\omega_1+\ldots+\omega_{n-1})t}f_n(\mu^2 t).$$

In view of the definition of f_n, $g_n(\mu^1 t)-g_n(\mu^2 t)$ is the sum of $\binom{2n}{n}$ expressions of type

$$e^{\mu^1(\omega_{i_1}+\ldots+\omega_{i_n}-\omega_1-\ldots-\omega_{n-1})t}[1-e^{(\mu^2-\mu^1)(\omega_{i_1}+\ldots+\omega_{i_n}-\omega_1-\ldots-\omega_{n-1})t}]$$

$$(1\leqq i_1<\ldots<i_n\leqq 2n).$$

Here

$$\operatorname{Re}\mu^1(\omega_{i_1}+\ldots+\omega_{i_n}-\omega_1-\ldots-\omega_{n-1})t\leqq\operatorname{Re}\mu t\leqq B^2,$$

$$|(\mu^2-\mu^1)(\omega_{i_1}+\ldots+\omega_{i_n}-\omega_1-\ldots-\omega_{n-1})t|\leqq(2n-1)B^2;$$

therefore the first (resp. the second) factor of the above product can be estimated by C_1 (resp. $C_1 t$) with some constant C_1. Hence the expression on the left hand side of

(11) can be estimated by $\binom{2n}{n} C_1^2 t$ and the Lemma is true with

$$A := \binom{2n}{n} C_1^2. \quad \square$$

LEMMA 3. a) *For any fixed* $\lambda \in \mathbf{C}$, $x \in \mathbf{R}$ *and* $t \geqq 0$ *the functions* $G(\mu, x, t, \cdot)$, $H(\mu, x, t, \cdot)$: $[x-nt, x+nt] \to \mathbf{C}$ *are continuous and even with respect to* x.

b) *There exists a constant* $C>0$ *such that for any* $\lambda \in \mathbf{C}$, $t \geqq 0$ *and* $x-nt \leqq \xi \leqq x+nt$,

$$|G(\mu, x, t, \xi)| \leqq \begin{cases} C|\mu|^{1-2n} e^{\operatorname{Re}\mu(t-|x-\xi|)} & \text{if} \quad |x-\xi| \leqq t, \\ C|\mu|^{1-2n} & \text{if} \quad t \leqq |x-\xi| \leqq nt, \end{cases} \tag{12}$$

$$|D_4 G(\mu, x, t, \xi)| \leqq \begin{cases} C|\mu|^{2-2n} e^{\operatorname{Re}\mu(t-|x-\xi|)} & \text{if} \quad |x-\xi| \leqq t, \\ C|\mu|^{2-2n} & \text{if} \quad t \leqq |x-\xi| \leqq nt, \end{cases}$$

$$|H(\mu, x, t, \xi)| \leqq C|\mu|^{1-2n}, \quad |D_4 H(\mu, x, t, \xi)| \leqq C|\mu|^{2-2n}. \tag{13}$$

PROOF. a) follows immediately from the definitions. In view of a) it suffices to prove b) for $\xi \geqq x$. Assume $x+lt-t \leqq \xi \leqq x+lt$, $1 \leqq l \leqq n$.

Let us introduce for brevity the notation $f_k^p(\mu)$ for the elementary symmetric polynomial of degree k of the variables $e^{\mu_1}, \ldots, e^{\mu_{p-1}}, e^{\mu_{p+1}}, \ldots, e^{\mu_{2n}}$ with leading coefficient $(-1)^{n-k}$. We show that

$$G(\mu, x, t, \xi) = \sum_{p=1}^{2n} \frac{\mu_p}{2n\lambda} e^{-(\mu_1+\ldots+\mu_{n-1})t} f_{n-l}^p(\mu t) e^{\mu_p(x+lt-\xi)}.$$

Indeed, one can see easily that for any fixed $1 \leqq p \leqq 2n$

$$\sum_{k=l}^{n} e^{(k-l)\mu_p t} f_{n-k}(\mu t) = f_{n-l}^p(\mu t).$$

Substituting this into the above formula, we obtain the original definition of $G(\mu, x, t, \xi)$. In case $l=1$, each term of the sum can be estimated by

$$C|\mu|^{1-2n} e^{\operatorname{Re}\mu(t-|x-\xi|)}.$$

This can be seen easily, using (2), (3) and the relation $x \leqq \xi \leqq x+t$. We have to consider the cases $p \leqq n$ and $p > n$ separately. In case $l \geqq 2$, using again (2), (3) and the inequalities $x+lt-t \leqq \xi \leqq x+lt$, considering the cases $1 \leqq p \leqq n-l$, $n-l < p \leqq n$, $p > n$ separately, each term can be estimated by $C|\mu|^{1-2n}$; hence the first estimate of (12) is proved.

The quantity $D_4 G(\mu, x, t, \xi)$ can be estimated similarly, using the derivative of the above formula. Finally, (13) is an easy consequence of (12) and the definition of $H(\mu, x, t, \xi)$. $\square$

The following assertion is in close connection with the Proposition proved in [3]:

LEMMA 4. *Let* $[a-R, a+R]$ *be a compact interval such that* $[a-nR, a+nR] \subset G$. *Then there exists a constant* C *such that for any eigenfunction* u *of the operator* L, *introducing the function* v: $[a-R, a+R] \to \mathbf{C}$,

$$v(x) := u(x) e^{\mu(R-|a-x|)},$$

we have

$$\|v\|_{L^\infty(a-R, a+R)} \leqq C \|u\|_{L^\infty(a-nR, a+nR)}.$$

PROOF. It suffices to show the above inequality for eigenfunctions having sufficiently large eigenvalues. For any $x\in[a-R, a+R]$, putting $t:=R-|a-x|$, it follows from (7), (9), (12) and the Hölder inequality that

$$v(x) \leqq C \sum_{k=-n}^{n} |u(x+kt)| + C|\mu|^{1-2n}\|v\|_{L^\infty(a-R,a+R)} + \\ + C|\mu|^{1-2n}\|u\|_{L^\infty(a-nR,a+nR)},$$

where C is a constant independent of u and x. Hence in case $|\mu|\geqq 1$ we obtain with some other absolute constant C' the inequality

$$\|v\|_{L^\infty(a-R,a+R)} \leqq C'\|u\|_{L^\infty(a-nR,a+nR)} + C|\mu|^{1-2n}\|v\|_{L^\infty(a-R,a+R)}.$$

Hence the desired estimate follows for any u with $|\mu|\geqq 1$ and $C|\mu|^{1-2n}\leqq 1/2$, and the Lemma is proved. □

COROLLARY. *Let* $[a-R, a+R]$ *be a compact interval such that* $[a-2nR, a+2nR]\subset G$. *Then there exists a constant* C *such that for any eigenfunction* u,

$$\|u\|_{L^\infty(a-R,a+R)} e^{\operatorname{Re}\mu R} \leqq C\|u\|_{L^\infty(a-2nR,a+2nR)}. \tag{14}$$

PROOF. Apply Lemma 4 to the interval $[a-2R, a+2R]$, and take into account that

$$\|u\|_{L^\infty(a-R,a+R)} e^{\operatorname{Re}\mu R} \leqq \|v\|_{L^\infty(a-2R,a+2R)}. \quad \square$$

LEMMA 5. *Let us given two arbitrary compact intervals* $K\subset G$ *and* $K'\subset \operatorname{int} K$. *Then there exists a constant* C *such that for any eigenfunction* u,

$$\|u\|_{L^\infty(K')} \leqq C\|u\|_{L^2(K)}. \tag{15}$$

PROOF. It suffices to show that each point $y\in\operatorname{int} K$ has a neighbourhood K_y such that with some constant C_y,

$$\|u\|_{L^\infty(K_y)} \leqq C_y\|u\|_{L^2(K)} \tag{16}$$

for any eigenfunction u. Indeed, hence (15) follows by an easy compactness argument.

Fixing $y\in\operatorname{int} K$ arbitrarily, let us choose an arbitrary compact neighbourhood $K_y:=[a-R, a+R]$ of y such that $[a-5nR, a+5nR]\subset G$. By (10) we may fix a number $A>0$ so as to satisfy the condition

$$\operatorname{Re}\mu \geqq A \quad \text{and} \quad t\geqq R \quad \text{imply} \quad |h_n(\mu t)| \geqq 2^{-1}, \tag{17}$$

and then by (9) another number $B>0$ such that

$$\operatorname{Re}\mu \leqq A, \quad |\mu| \geqq B \quad \text{and} \quad t\geqq R \quad \text{imply} \quad |g_{n-1}(\mu t)| \geqq 2^{-1}. \tag{18}$$

We distinguish three cases.

a) $\operatorname{Re}\mu\geqq A$. Applying (8) for any $x\in K_y$, $R\leqq t\leqq 2R$, and using (10), (13), (17), the relation $\|q\|_{L^2(K)}<\infty$, and the Hölder inequality, we obtain

$$|u(x)| \leqq \sum_{k=1}^{n} C\big(|u(x-kt)| + |u(x+kt)|\big) + C\|u\|_{L^2(K)}$$

uniformly in $x\in K_y$.

b) $\operatorname{Re}\mu<A$ and $|\mu|\geqq B$. Now applying (7) (replacing x by $x+t$) for any $x\in K_y$ and $R\leqq t\leqq 2R$, using (9), (12), (18), $\|q\|_{L^2(K)}<\infty$ and the Hölder inequality, we obtain

$$|u(x)| \leqq C\sum_{k=1}^{n-1}|u(x-kt)|+C\sum_{k=1}^{n+1}|u(x+kt)|+C\|u\|_{L^2(K)}$$

uniformly in $x\in K_y$.

c) $|\mu|<B$. Applying (4) (with $x+nt$ instead of x) for any $x\in K_y$ and $R\leqq t\leqq 2R$, in view of the continuity of the functions f_k and $|f_0|\equiv 1$, we obtain

$$|u(x)| \leqq C\sum_{k=1}^{2n}|u(x+kt)|+C\|u\|_{L^2(K)}$$

uniformly in $x\in K_y$.

Now the proof can be completed in all cases as follows. Let us integrate the inequality by t from R to $2R$, divide by R and apply the Hölder inequality. Then

$$|u(x)| \leqq C\|u\|_{L^2(K)}$$

uniformly in $x\in K_y$. Hence (16) follows, and the Lemma is proved. □

2. Bessel-systems of eigenfunctions

As in the preceding section, let us given an arbitrary open interval $G\subset\mathbf{R}$, an arbitrary complex function $q\in L^2_{\mathrm{loc}}(G)$, a natural number n, and consider the differential operator $Lu=u^{(2n)}+qu$ of order $2n$.

Furthermore let us given an arbitrary Bessel-system $(u_r)\subset L^2(G)$ consisting of eigenfunctions of the operator L. We recall that a system $(u_r)\subset L^2(G)$ is called a Bessel-system if there exists a constant C such that

$$\sum_r |\langle u_r, w\rangle|^2 \leqq C\|w\|^2_{L^2(G)} \quad \text{for all} \quad w\in L^2(G). \tag{19}$$

Let us denote the eigenvalue of u_r by λ_r and introduce the notations μ_{rp}, $\mu_r=\mu_{rn}$ as in the preceding section. The purpose of this section is to prove the following result:

THEOREM 1. *Given any compact interval $K\subset G$, there exists a number $R>0$ such that*

$$\sup_{\mu\geqq 1}\sum_{|\mu-|\operatorname{Im}\mu_r||\leqq 1}(\|u_r\|_{L^\infty(K)}e^{\operatorname{Re}\mu_r R})^2<\infty. \tag{20}$$ □

For brevity we shall use the notations

$$\varrho_r := \operatorname{Re}\mu_r (\geqq 0), \quad \nu_r := \operatorname{Im}\mu_r.$$

Let us now given a compact interval $K=[a,b]\subset G$ arbitrarily.

LEMMA 6. *There exists a number $A>0$ such that*

$$\sup_{\mu\geqq 1}\sum_{\substack{|\mu-|\nu_r||\leqq 1\\ |\nu_r|\geqq A\varrho_r\geqq A^2}}\|u_r\|^2_{L^2(K)}<\infty.$$

PROOF. Introducing the notation

$$I_\mu(A) := \{r: |\mu - |v_r|| \leqq 1, \quad |v_r| \geqq A\varrho_r \geqq A^2\},$$

we shall prove that

$$\sum_{r \in I_\mu(A)} \|u_r\|^2_{L^2(K)} = O(1) \quad (A \to \infty),$$

uniformly in $\mu \geqq 1$. We shall consider only the case $n \geqq 2$; the case $n=1$ is contained in [9].

Let us choose $R>0$ such that $[a-2nR, b+2nR] \subset G$. Fixing $\mu \geqq 1$ and $x \in K$ arbitrarily, multiplying (7) by $\cos \mu t$, integrating by t from R to $2R$ and integrating by parts on the right side, for any $r \in I_\mu(A)$ we obtain

$$-\int_R^{2R} \cos \mu t g_n(\mu_r t)\, dt u_r(x) =$$

$$= \sum_{k=1}^{n} g_{n-k}(2\mu_r R) \int_R^{2R} \cos \mu t [u_r(x-kt) + u_r(x+kt)]\, dt -$$

$$- \sum_{k=1}^{n} \int_R^{2R} \frac{d}{dt} g_{n-k}(\mu_r t) \int_R^t \cos \mu\xi [u_r(x-k\xi) + u_r(x+k\xi)]\, d\xi\, dt +$$

$$+ \int_R^{2R} \cos \mu t G(\mu_r, x, t, x+nt) \int_{x-nt}^{x+nt} q(\tau) u_r(\tau)\, d\tau\, dt -$$

$$- \int_R^{2R} \cos \mu t \int_{x-nt}^{x+nt} D_4 G(\mu_r, x, t, \tau) \int_{x-nt}^{\tau} q(\xi) u_r(\xi)\, d\xi\, d\tau\, dt.$$

Introducing the functions $w^{(1)}_{\mu, x, k, t}, w^{(2)}_{x, t, \tau} \in L^2(G)$,

$$w^{(1)}_{\mu, x, k, t}(\eta) := k^{-1} \cos \mu k^{-1}(\eta - x) \chi_{\{kR \leqq |x-\eta| \leqq kt\}}(\eta),$$

$$w^{(2)}_{x, t, \tau}(\xi) := \overline{q(\xi)} \chi_{\{x-nt \leqq \xi \leqq \tau\}}(\xi)$$

and applying (9), (12), we obtain uniformly in $\mu \geqq 1$, $r \in I_\mu(A)$ and $x \in K$ the estimate

$$\left| \int_R^{2R} 2 \cos \mu t \operatorname{ch} \mu_r t\, dt + o(1) \right| |u_r(x)| \leqq \tag{21}$$

$$\leqq \sum_{k=1}^{n} O(1) |\langle u_r, w^{(1)}_{\mu, x, k, 2R} \rangle| + \sum_{k=1}^{n} o(1) \int_R^{2R} |\langle u_r, w^{(1)}_{\mu, x, k, t} \rangle|\, dt +$$

$$+ O(\mu_r^{1-2n} e^{2\varrho_r R}) \int_R^{2R} |\langle u_r, w^{(2)}_{x, t, 2R} \rangle|\, dt + O(\mu_r^{2-2n} e^{2\varrho_r R}) \int_R^{2R} \int_{x-nt}^{x+nt} |\langle u_r, w^{(2)}_{x, t, \tau} \rangle|\, d\tau\, dt$$

(remark that now $r \in I_\mu(A)$ implies $\frac{d}{dt} g_{n-1}(\mu_r t) = o(1)$). Computing the integral on

the left side, one can see easily that

$$\left|\int_R^{2R} 2\cos\mu t \operatorname{ch}\mu_r t\,dt\right| \geqq [1-o(1)]\frac{e^{2\varrho_r R}}{2\varrho_r}. \tag{22}$$

Now multiplying (21) by $2\varrho_r e^{-2\varrho_r R}$, applying (22), taking the square of both sides, using the Hölder inequality and finally taking into account that $\varrho_r^2\mu_r^{4-4n}=o(1)$ because of $n\geqq 2$, we obtain the following estimate:

$$\begin{aligned}[1-o(1)]|u_r(x)|^2 \leqq \sum_{k=1}^{n} o(1)\,|\langle u_r, w^{(1)}_{\mu,x,k,2R}\rangle|^2 + \\ + \sum_{k=1}^{n} o(1)\int_R^{2R} |\langle u_r, w^{(1)}_{\mu,x,k,t}\rangle|^2\,dt + o(1)\int_R^{2R} |\langle u_r, w^{(2)}_{x,t,2R}\rangle|^2\,dt + \\ + o(1)\int_R^{2R}\int_{x-nt}^{x+nt} \langle u_r, w^{(2)}_{x,t,\tau}\rangle|^2\,d\tau\,dt.\end{aligned}$$

If we sum up these inequalities for $r\in I_\mu(A)$, apply the Bessel inequality (19) and take into account the estimates

$$\begin{cases}\|w^{(1)}_{\mu,x,k,t}\|^2_{L^2(G)} \leqq 2R = O(1), \\ \|w^{(2)}_{x,t,\tau}\|^2_{L^2(G)} \leqq \|q\|^2_{L^2(a-2nR,\,b+2nR)} = O(1),\end{cases} \tag{23}$$

then we obtain

$$[1-o(1)]\sum_{r\in I_\mu(A)} |u_r(x)|^2 = o(1)$$

uniformly in $\mu\geqq 1$ and $x\in K$. Integrating by x on K, the lemma follows. □

LEMMA 7. *Given any* $A>0$, *there exists a constant* $B>0$ *such that*

$$\sum_{\substack{|v_r|\leqq A\varrho_r \\ \varrho_r\geqq B}} \|u_r\|^2_{L^2(K)} < \infty.$$

PROOF. Introduce for $B>0$ the notation

$$I(B) := \{r:\ \varrho_r \geqq B,\ |v_r| \leqq A\varrho_r\},$$

and choose $R>0$ so as to satisfy the condition $[a-2nR, b+2nR]\subset G$. Fixing $x\in K$ arbitrarily, integrate (8) by t from R to $2R$ and integrate by parts,

$$\begin{aligned}-\int_R^{2R} h_n(\mu_r t)\,dtu_r(x) = \sum_{k=1}^{n} h_{n-k}(2\mu_r R)\int_R^{2R}[u_r(x-kt)+u_r(x+kt)]\,dt - \\ -\sum_{k=1}^{n}\int_R^{2R}\frac{d}{dt}h_{n-k}(\mu_r t)\int_R^{t}[u_r(x-k\xi)+u_r(x+k\xi)]\,d\xi\,dt + \\ +\int_R^{2R} H(\mu_r, x, t, x+nt)\int_{x-nt}^{x+nt} q(\tau)u_r(\tau)\,d\tau\,dt - \\ -\int_R^{2R}\int_{x-nt}^{x+nt} D_4H(\mu_r, x, t, \tau)\int_{x-nt}^{\tau} q(\xi)u_r(\xi)\,d\xi\,d\tau\,dt.\end{aligned}$$

Introducing the functions $w^{(1)}_{0,x,k,t}$, $w^{(2)}_{x,t,\tau}\in L^2(G)$ by the same formulas as in the

preceding Lemma, and using (10), (13), we obtain the inequality

$$[R+o(1)]|u_r(x)| \leqq \sum_{k=1}^{n} o(1)|\langle u_r, w^{(1)}_{x,k,2R}\rangle| + \\ + \sum_{k=1}^{n} o(1) \int_R^{2R} |\langle u_r, w^{(1)}_{x,k,t}\rangle|\, dt + \int_R^{2R} o(1)|\langle u_r, w^{(2)}_{x,t,x+nt}\rangle|\, dt + \\ + \int_R^{2R} \int_{x-nt}^{x+nt} O(1)|\langle u_r, w^{(2)}_{x,t,\tau}\rangle|\, d\tau\, dt$$

for $B\to\infty$ uniformly in $r\in I(B)$ and $x\in K$ (here we used that now we have $\frac{d}{dt}h_{n-1}(\mu_r t)=o(1)$ since $r\in I(B)$).

Taking squares, applying the Hölder inequality, summing for $r\in I(B)$, using (23) and the Bessel inequality (19), we obtain uniformly in $x\in K$

$$[R^2+o(1)] \sum_{r\in I(B)} (u_r(x)|^2 = O(1).$$

Integrating it by x on K, the lemma follows. □

LEMMA 8. *Given $B>0$ arbitrarily, there exists a constant $D>0$ such that*

$$\sup_{\mu\geqq 1} \sum_{\substack{|\mu-|\nu_r||\leqq 1 \\ \varrho_r\leqq B,\ |\nu_r|\geqq D}} \|u_r\|^2_{L^2(K)} < \infty.$$

PROOF. Fix $R>0$ with $4R\leqq b-a$ and $[a-4nR, b+4nR]\subset G$. Fix also $\mu\geqq 1$ and $x\in K$ arbitrarily (we assume by symmetry that $a\leqq x\leqq 2^{-1}(a+b)$). Apply (7) to any $r\in I_\mu(D)$ and $R\leqq t\leqq 2R$ where

$$I_\mu(D) := \{r:\ \varrho_r \leqq B,\ |\mu-|\nu_r|| \leqq 1,\ |\nu_r| \geqq D\}$$

(with $x+t$ instead of x) and integrate by parts,

$$-\int_R^{2R} g_{n-1}(\mu_r t)\, dt\, u_r(x) = \int_R^{2R} g_n(i\mu t)\, u_r(x+t)\, dt + \int_R^{2R} [g_n(\mu_r t)-g_n(i\mu t)]\, u_r(x+t)\, dt + \\ + g_{n-1}(2\mu_r R) \int_R^{2R} u_r(x+2t)\, dt - \int_R^{2R} \frac{d}{dt} g_{n-1}(\mu_r t) \int_R^t u_r(x+2\xi)\, d\xi\, dt + \\ + \sum_{k=2}^{n} g_{n-k}(2\mu_r R) \int_R^{2R} u_r(x+t-kt)+u_r(x+t+kt)\, dt - \\ - \sum_{k=2}^{n} \int_R^{2R} \frac{d}{dt} g_{n-k}(\mu_r t) \int_R^t u_r(x+\xi-k\xi)+u_r(x+\xi+k\xi)\, d\xi\, dt + \\ + \int_R^{2R} G(\mu_r, x+t, t, x+t+nt) \int_{x+t-nt}^{x+t+nt} q(\xi)\, u_r(\xi)\, d\xi\, dt - \\ - \int_R^{2R} \int_{x+t-nt}^{x+t+nt} D_4 G(\mu_r, x+t, t, \xi) \int_{x+t-nt}^{\xi} q(\tau) u_r(\tau)\, d\tau\, d\xi\, dt.$$

(We assume that $\nu_r>0$. In the contrary case we take $g_n(-i\mu t)$ instead of $g_n(i\mu t)$.)

Let us introduce the functions $w^{(3)}_{\mu,x}, w^{(4)}_{x,k,t} \in L^2(G)$,

$$w^{(3)}_{\mu,x}(\eta) := \overline{g_n(i\mu(\eta-x))}\chi_{\{x+R\leqq\eta\leqq x+2R\}}(\eta),$$

$$w^{(4)}_{x,1,t}(\eta) := 2^{-1}\chi_{\{x+2R\leqq\eta\leqq x+2t\}}(\eta),$$

$$w^{(4)}_{x,k,t}(\eta) := (k-1)^{-1}\chi_{\{x-(k-1)t\leqq\eta\leqq x-(k-1)R\}}(\eta)+$$

$$+(k+1)^{-1}\chi_{\{x+(k+1)R\leqq\eta\leqq x+(k+1)t\}}(\eta) \quad (k\geqq 2)$$

and consider also the functions $w^{(2)}_{x,t,\tau}$ defined in Lemma 6. Using (9), (11), (12) and the condition $\varrho_r \leqq B$, we obtain the inequality

$$[R+o(1)]|u_r(x)| \leqq |\langle u_r, w^{(3)}_{\mu,x}\rangle| + C_0 R^{3/2}\|u_r\|_{L^2(K)} +$$

$$+\sum_{k=1}^{n} C|\langle u_r, w^{(4)}_{x,k,2R}\rangle| + C\int_R^{2R} |\langle u_r, w^{(4)}_{x,k,t}\rangle|\,dt + C\int_R^{2R} |\langle u_r, w^{(2)}_{x+t,t,x+t+nt}\rangle|\,dt +$$

$$+C\int_R^{2R}\int_{x+t-nt}^{x+t+nt} |\langle u_r, w^{(2)}_{x+t,t,\xi}\rangle|\,d\xi\,dt \quad (D\to\infty);$$

here (and in the sequel) we denote by C (resp. C_0) diverse constants independent of the choice of $\mu\geqq 1$, $r\in I_\mu(D)$, $x\in K$ (resp. $\mu\geqq 1$, $r\in I_\mu(D)$, $x\in K$ and R).

Taking the square of these inequalities, summing up for an arbitrary finite index set $I\subset I_\mu(D)$, using the Bessel inequality and the estimates

$$\|w^{(2)}_{x,t,\tau}\|_{L^2(G)}, \quad \|w^{(3)}_{\mu,x}\|_{L^2(G)}, \quad \|w^{(4)}_{x,k,t}\|_{L^2(G)} \leqq C,$$

we obtain uniformly in $\mu\geqq 1$, $I\subset I_\mu(D)$ that

$$[R^2+o(1)]\sum_{r\in I}\|u_r\|^2_{L^2(K)} \leqq C + C_0 R^3 \sum_{r\in I}\|u_r\|^2_{L^2(K)}$$

after integrating by x on K. Let us now choose R at the beginning of the proof so as to satisfy also the condition $C_0 R\leqq 2^{-1}$. Then in view of the finiteness of I,

$$[2^{-1}R^2+o(1)]\sum_{r\in I}\|u_r\|^2_{L^2(K)} \leqq C$$

uniformly in μ and I. Hence

$$\sum_{r\in I_\mu(D)}\|u_r\|^2_{L^2(K)} = O(1)$$

uniformly in $\mu\geqq 1$. □

LEMMA 9. *For any* $B, D>0$, *we have*

$$\sum_{\substack{\varrho_r\leqq B\\ |\gamma_r|\leqq D}}\|u_r\|^2_{L^2(K)} < \infty.$$

PROOF. More general results were proved in [12] and [13]. □

PROOF OF THEOREM 1. Given any compact interval $K \subset G$, it follows from Lemmas 6—9 that

$$\sup_{\mu \geqq 1} \sum_{|\mu - |\gamma_r|| \leqq 1} \|u_r\|^2_{L^2(K)} < \infty.$$

Hence an application of the Corollary of Lemma 4 and Lemma 5 yields that each point $x \in G$ has a neighbourhood K such that for some $R > 0$ the relation (20) holds. Now Theorem 1 follows by a simple compactness argument. □

3. An equiconvergence theorem

Let us given, as before, an open interval G on the real line, an arbitrary complex function $q \in L^2_{\mathrm{loc}}(G)$, a natural number n and an arbitrary Riesz basis $(u_r) \subset L^2(G)$ consisting of eigenfunctions of the operator $Lu = u^{(2n)} + qu$. As in the preceding section, we shall denote by μ_r a $2n$-th root of the eigenvalue λ_r of u_r for which $\operatorname{Re} \mu_r \geqq 0$ and $|\operatorname{Re} \mu_r|$ is minimal. We put further $\varrho_r := \operatorname{Re} \mu_r$ and $\nu_r := \operatorname{Im} \mu_r$.

Let us denote by $(v_r) \subset L^2(G)$ the dual system of (u_r), then — as it is well-known — for any $f, w \in L^2(G)$,

$$\langle f, w \rangle = \sum_r \langle f, v_r \rangle \langle u_r, w \rangle, \tag{24}$$

$$\sum_r |\langle f, v_r \rangle|^2 \leqq C \|f\|^2_{L^2(G)}, \tag{25}$$

$$\sum_r |\langle u_r, w \rangle|^2 \leqq C \|w\|^2_{L^2(G)}, \tag{26}$$

where C is a constant independent of f and w. In particular, (u_r) is a Bessel-system.

For any $f \in L^2(G)$, $\mu > 0$ and $x \in G$, introducing the notations

$$\sigma_\mu(f, x) := \sum_{|\nu_r| < \mu} \langle f, v_r \rangle u_r(x), \tag{27}$$

$$S_\mu(f, x, R) := \int_{x-R}^{x+R} \frac{\sin \mu(y-x)}{\pi(y-x)} f(y)\, dy \tag{28}$$

(in (28) we assume also that $R > 0$ and $x - R, x + R \in G$). We shall prove the following result:

THEOREM 2. *Given any compact interval $K \subset G$, there exists a positive number R_0 such that for any $0 < R \leqq R_0$ and $f \in L^2(G)$,*

$$\lim_{\mu \to \infty} \sup_{x \in K} |S_\mu(f, x, R) - \sigma_\mu(f, x)| = 0.$$

Let us now given another complex function $q \in L^2_{\mathrm{loc}}(G)$ on the same interval G, another natural number $\hat{n}$ and another Riesz basis $(\hat{u}_r) \subset L^2(G)$ consisting of eigenfunctions of the operator $\hat{L}u = u^{(2n)} + \hat{q}u$. Then the following statement is an immediate consequence of Theorem 2

THEOREM 3. *Given any compact interval* $K \subset G$, *we have*

$$\lim_{\mu \to \infty} \sup_{x \in K} |\sigma_\mu(f, x) - \hat{\sigma}_\mu(f, x)| = 0$$

for any function $f \in L^2(G)$, *where* $\hat{\sigma}_\mu(f, x)$ *is defined similarly as* $\sigma_\mu(f, x)$.

To prove Theorem 2, we need the following assertions:

LEMMA 10. *There exist functions* $f_1(\mu, R)$, $f_0(\mu, R, t)$, $f_k(\mu, R, t)$ $(k=2, \ldots, n)$, $F(u, \mu, x, R, t)$ *such that for any eigenfunction* u *of the operator* L, *using the notations of Section* 1,

$$f_1(\mu, R)[u(x-t)+u(x+t)-2u(x)\operatorname{ch}\mu t] =$$

$$= f_0(\mu, R, t)u(x) + \sum_{k=2}^{n} f_k(\mu, R, t)[u(x-kR)+u(x+kR)] + F(u, \mu, x, R, t)$$

whenever $x-nR, x+nR, x-t, x+t \in G$.

Furthermore, there exists a constant $C>0$ *independent of* R *such that for any* $R>0$, $x-nR, x+nR \in G$ *and* $0 < t \leqq R$,

$$|t^{-1} f_0(\mu, R, t)| \leqq C|\mu| e^{\operatorname{Re}[2\mu R + \mu_{n-1}(t-2R)]},$$

$$|t^{-1} f_k(\mu, R, t)| \leqq C|\mu| e^{\operatorname{Re}\mu_{n-1}(t-2R)} \quad (k \geqq 2),$$

$$\int_0^R |t^{-1} F(u, \mu, x, R, t)|\, dt \leqq C \frac{\ln(1+|\mu|)}{1+|\mu|} e^{\operatorname{Re}\mu R} \|q\|_{L^1(x-nR, x+nR)} \|u\|_{L^\infty(x-nR, x+nR)}.$$

Finally, to any $R>0$ *there exist positive constants* A *and* ε *(depending on* R*) such that*

$$\int_{R/2}^{R} |f_1(\mu, r)|\, dr > \varepsilon$$

whenever $|\mu| \geqq A$.

PROOF. In case $q \equiv 0$ the formula follows from the relation

$$\begin{vmatrix} u(x) & u(x-t)+u(x+t) & u(x-2R)+u(x+2R) & \ldots & u(x-nR)+u(x+nR) \\ 1 & 2\operatorname{ch}\mu_1 t & 2\operatorname{ch} 2\mu_1 R & & 2\operatorname{ch} n\mu_n R \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 2\operatorname{ch}\mu_n t & 2\operatorname{ch} 2\mu_n R & \ldots & 2\operatorname{ch} n\mu_n R \end{vmatrix} \equiv 0$$

after dividing by $e^{\operatorname{Re}(n\mu_1 + \ldots + 3\mu_{n-2} + 2\mu_{n-1})R}$. (In the case $\lambda \neq 0$ the first row is the linear combination of the others, the case $\lambda = 0$ is obvious.) Hence one can easily show also the estimates of the functions f_k.

In the general case we introduce the function

$$v(y) := u(y) + \int_x^y \sum_{p=1}^{2n} \frac{\mu_p}{2n\lambda} e^{\mu_p(y-\xi)} q(\xi) u(\xi)\, d\xi.$$

This is an eigenfunction (with the same eigenvalue as u) of the operator $Lu = u^{(2n)}$ (see the proof of Theorem 1 in [4]). Therefore we can apply the already proved part

of this Lemma to the function v. Now the functions f_k remain unchanged and an explicit form of $F(u, \mu, x, R, t)$ is obtained. A slight modification of the arguments used in Lemma 3 of this paper and in Lemma 7 of [9] provides the required estimate for the function F.

COROLLARY. *To any* $R>0$ *there exist* $A, C>0$ *such that for any* $x-nR$, $x+nR\in G$, *in case* $|\mu|\geqq A$

$$\int_0^R t^{-1}[u(x-t)+u(x+t)-2u(x)\operatorname{ch}\mu t]\,dt \leqq \tag{29}$$

$$\leqq C(1+|\mu|)^{-3/4}\|u\|_{L^\infty(x-nR,\,x+nR)}e^{2\operatorname{Re}\mu R}.$$

Now we turn to the proof of Theorem 2. Given $K=[a, b]\subset G$ arbitrarily, by Theorem 1 we can fix a number $R_0>0$ with the properties

$$K_0 := [a-nR_0,\, b+nR_0] \subset G, \tag{30}$$

$$\sup_{\mu\geqq 1}\sum_{|\mu-|\nu_r||\leqq 1}(\|u_r\|_{L^\infty(K_0)}e^{2\varrho_r R_0})^2 < \infty, \tag{31}$$

and let

$$0 < R \leqq R_0 \tag{32}$$

be arbitrary. First we show the existence of a constant C such that for any $\mu>0$ and $x\in K$,

$$\sum_r |\langle u_r, w\rangle - \delta(\mu, |\nu_r|)u_r(x)|^2 \leqq C, \tag{33}$$

where

$$w\in L^2(G), \quad w(y) := \frac{\sin\mu(y-x)}{\pi(y-x)}\chi_{\{|y-x|\leqq R\}}(y) \tag{34}$$

and

$$\delta(\mu, |\nu_r|) := \begin{cases} 1 & \text{if } \mu > |\nu_r|, \\ 1/2 & \text{if } \mu = |\nu_r|, \\ 0 & \text{if } \mu < |\nu_r|. \end{cases} \tag{35}$$

We recall that for any r,

$$\left|\frac{2}{\pi}\int\frac{\sin\mu t\operatorname{ch}\mu_r t}{t}\,dt - \delta(\mu, |\nu_r|)\right| \leqq C\frac{e^{\operatorname{Re}\mu_r R}}{2+|\mu-|\nu_r||} \tag{36}$$

(see [9], Lemma 6); here and in the sequel we denote by C absolute constants (not necessarily the same in different places) which do not depend on $\mu>0$, r and $x\in K$.

Consider first the case $|\mu_r| \geqq A$. By (29), (30), (32) and (36) we have

$$\langle u_r, w\rangle - \delta(\mu, |v_r|)\, u_r(x) = \tag{37}$$

$$= \left| \int_0^R \frac{\sin \mu t}{\pi t} [u_r(x-t) + u_r(x+t)]\, dt - \delta(\mu, |v_r|)\, u_r(x) \right| =$$

$$= \left| \int_0^R \frac{\sin \mu t}{\pi t} [u_r(x-t) + u_r(x+t) - 2u_r(x) \operatorname{ch} \mu_r t]\, dt \right| +$$

$$+ \int_0^R \frac{\sin \mu t}{\pi t} 2 \operatorname{ch} \mu_r t - \delta(\mu, |v_r|) \big| |u_r(x)| \leqq$$

$$\leqq C(1+|\mu_r|)^{-3/4} \|u_r\|_{L^\infty(K_0)} e^{2\varrho_r R_0} + C\big(2+|\mu - |v_r|\big|\big)^{-1} |u_r(x)| e^{\varrho_r R_0},$$

for any $\mu > 0$ and $x \in K$. Hence, using (36) and (31),

$$\sum_{|\mu_r| \geqq A} \big|\langle u_r, w\rangle - \delta(\mu, |v_r|)\, u_r(x)\big|^2 \leqq \tag{38}$$

$$\leqq C \sum_{|\mu_r| \geqq A} \left\{(1+|\mu_r|)^{-3/2} + \big(2+|\mu - |v_r|\big|\big)^{-2}\right\} \left(\|u_r\|_{L^\infty(K_0)} \operatorname{ch} \varrho_r R_0\right)^2 \leqq$$

$$\leqq C \sum_{i=1}^{\infty} \left(i^{-3/2} + (1+|\mu - i|)^{-2}\right) \sum_{i-1 \leqq |v_r| < i} \left(\|u_r\|_{L^\infty(K_0)} \operatorname{ch} \varrho_r R_0\right)^2 \leqq C \sum_{i=1}^{\infty} (i^{-3/2} + 2i^{-2}) = C.$$

Consider now the case $|\mu_r| < A$. For any $\mu > 0$ and $x \in K$, integrating by parts

$$\langle u_r, w\rangle = \int_0^R \frac{\sin \mu t}{\pi t}\, dt\, [u_r(x+R) + u_r(x-R)] -$$

$$- \int_0^R \int_0^t \frac{\sin \mu \tau}{\pi \tau}\, d\tau\, [u_r'(x+t) - u_r'(x-t)]\, dt,$$

whence, taking into account that the improper integral $\int_0^\infty \frac{\sin x}{x}\, dx$ is convergent, we obtain

$$|\langle u_r, w\rangle| \leqq C \|u_r\|_{L^\infty(K_0)} + C \|u_r'\|_{L^\infty(K_0)},$$

and hence by Theorem 2 of [4] and by the property $|\mu_r| < A$,

$$|\langle u_r, w\rangle| \leqq C \|u_r\|_{L^\infty(K_0)}.$$

Therefore

$$\big|\langle u_r, w\rangle - \delta(\mu, |v_r|)\, u_r(x)\big| \leqq C \|u_r\|_{L^\infty(K_0)}.$$

Using again (31) and the property $|\mu_r| < A$,

$$\sum_{|\mu_r| < A} \big|\langle u_r, w\rangle - \delta(\mu, |v_r|)\, u_r(x)\big|^2 \leqq C. \tag{39}$$

Now (33) follows from (38) and (39).

For any $f\in L^2G$, $x\in K$ and $\mu>0$, applying (24), (27), (28), (34) and (35),

$$|S_\mu(f, x, R)-\sigma_\mu(f, x)| \leqq 2^{-1} \sum_{|v_r|=\mu} \langle f, v_r\rangle u_r(x) +$$

$$+\Big|\sum_r \langle f, v_r\rangle(\langle u_r, w\rangle-\delta(\mu, |v_r|)u_r(x))\Big|;$$

using the Cauchy—Schwarz inequality, (25) and (33), we obtain for any $f\in L^2(G)$, uniformly in $\mu>0$ and $x\in K$

$$|S_\mu(f, x, R)-\sigma_\mu(f, x)| \leqq C\|f\|_{L^2(G)}. \tag{40}$$

On the other hand, it is well-known (cf. [14]) that for any continuously differentiable function $f\in L^2(G)$ (in particular for all u_r)

$$\lim_{\mu\to\infty} \sup_{x\in K} |S_\mu(f, x, R)-f(x)| = 0. \tag{41}$$

Taking into account that $\sigma_\mu(u_r, x)=u_r(x)$ if $\mu>|v_r|$ and that $(u_r)\subset L^2(G)$ is a complete system, hence Theorem 2 (and also Theorem 3) follows.

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LAKUNÄRE SPLINE-FUNKTION UND DAS CAUCHY-PROBLEM

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1. Einleitung, Problemstellung

Betrachten wir das Cauchy-Problem:

$$y'' = f(x, y, y') \quad (x \in [0, 1]), \qquad y(0) = y_0, \quad y'(0) = y_0',$$

u. zw. seien die folgenden Bedingungen erfüllt:

1. $f[x, y(x), y'(x)] \in C^r[0,1]$, wo r eine nichtnegative ganze Zahl ist.

2. $|f^{(q)}(x, y_1, y_1') - f^{(q)}(x, y_2, y_2')| \leqq L\{|y_1 - y_2| + |y_1' - y_2'|\}$ für $(q = 0, 1, 2, \ldots, r)$,
wo $0 \leqq x \leqq 1$ und y_1, y_2, y_1', y_2' beliebege Variablen sind.

T. Fawzy hat in [1] die nachstehenden Näherungswerte konstruiert und für die Differenzen zwischen den Näherungswerten $\bar{y}_k^{(q)}$ und den „echten" Werten $y^{(q)}(x_k) = y_k^{(q)}$ $(q = 0, 1, 2, \ldots, r)$ Abschätzungen gegeben.

DEFINITION (T. Fawzy).

$$\bar{y}_0 = y_0, \quad \bar{y}_0' = y_0', \quad \bar{y}_0^{(2+q)} = f^{(q)}(x_0, y_0, y_0') \quad (q = 0, 1, 2, \ldots, r),$$

$$\bar{y}_{k+1} = \bar{y}_k + \bar{y}_k' h + \int_{x_k}^{x_{k+1}} \int_{x_k}^{t} f[u, y_k^*(u), y_k^{**\prime}(u)]\, du\, dt,$$

$$\bar{y}_{k+1}' = \bar{y}_k' + \int_{x_k}^{x_{k+1}} f[t, y_k^*(t), y_k^{**\prime}(t)]\, dt,$$

$$\bar{y}_{k+1}^{(2+q)} = f^{(q)}(x_{k+1}, \bar{y}_{k+1}, \bar{y}_{k+1}') \quad (q = 0, 1, 2, \ldots, r; \quad k = 0, 1, 2, \ldots, m-1)$$

wo $x_k = \dfrac{k}{m}$, $x_m = 1$, $h = \dfrac{1}{m}$ und

$$y_k^*(x) = \sum_{j=0}^{r+2} \frac{\bar{y}_k^{(j)}}{j!} (x - x_k)^j, \quad x_k \leqq x \leqq x_{k+1},$$

$$y_k^{*\prime}(x) = \sum_{j=0}^{r+1} \frac{\bar{y}_k^{(j+1)}}{j!} (x - x_k)^j, \quad y_k^{**\prime}(x) = \bar{y}_k' + \int_{x_k}^{x} f[t, y_k^*(t), y_k^{*\prime}(t)]\, dt.$$

T. Fawzy hat den folgenden Satz bewiesen:

$$|y_{k+1}^{(j)} - \bar{y}_{k+1}^{(j)}| \leqq c_j \omega_{r+2}(h) h^{r+2} \quad (j = 0, 1, 2, \ldots, r+2, \quad k = 0, 1, 2, \ldots, m-1)$$

wo c_j eine Konstante und

$$\omega_{r+2}(h) = \sup_{|x-x_1|\leq h} |y^{(r+2)}(x) - y^{(r+2)}(x_1)|$$

bedeuten.

In dieser Arbeit konstruieren wir — die angeführten Näherungswerte für $r=3$ verwendend — die approximativen lakunären Spline-Funktionen $\bar{S}_\Delta(x)\in C[0,1]$ und $\bar{G}_\Delta(x)\in C[0,1]$, welche im ersten und im letzten Intervall aus Polynomen sechsten, sonst fünften Grades bestehen, und folgende Bedingungen erfüllen:

(1.1.a) $$\bar{S}_k(x_k) = \bar{y}_k \quad (k = 0, 1, 2, \ldots, m-1),$$

(1.1.b) $$\bar{S}_k^{(q)}(x_k) = \bar{y}_k^{(q)} \quad (q = 2, 3;\ k = 0, 1, 2, \ldots, m-1),$$

(1.1.c) $$\bar{S}_{m-1}(x_m) = \bar{y}_m,$$

(1.1.d) $$\bar{S}_{m-1}^{(q)}(x_m) = \bar{y}_m^{(q)} \quad (q = 2, 3),$$

(1.2.a) $$\bar{S}_k(x_{k+1}) = \bar{S}_{k+1}(x_{k+1}) = \bar{y}_{k+1} \quad (k = 0, 1, 2, \ldots, m-2),$$

(1.2.b) $$\bar{S}_k^{(q)}(x_{k+1}) = \bar{S}_{k+1}^{(q)}(x_{k+1}) = \bar{y}_{k+1}^{(q)} \quad (q=2, 3;\quad k = 0, 1, 2, \ldots, m-2),$$

(1.3) $$\bar{S}_0'(x_0) = \bar{y}_0', \quad \bar{S}_{m-1}'(x_m) = \bar{y}_m',$$

(2.1.a) $$\bar{G}_k(x_k) = \bar{y}_k \quad (k = 0, 1, 2, \ldots, m-1),$$

(2.1.b) $$\bar{G}_k^{(q)}(x_k) = \bar{y}_k^{(q)} \quad (q = 2, 4;\quad k = 0, 1, 2, \ldots, m-1),$$

(2.1.c) $$\bar{G}_{m-1}(x_m) = \bar{y}_m,$$

(2.1.d) $$\bar{G}_{m-1}^{(q)}(x_m) = \bar{y}_m^{(q)} \quad (q = 2, 4),$$

(2.2.a) $$\bar{G}_k(x_{k+1}) = \bar{G}_{k+1}(x_{k+1}) = \bar{y}_{k+1} \quad (k = 0, 1, 2, \ldots, m-2),$$

(2.2.b) $$\bar{G}_k^{(q)}(x_{k+1}) = \bar{G}_{k+1}^{(q)}(x_{k+1}) = \bar{y}_{k+1}^{(q)} \quad (q = 2, 4;\quad k = 0, 1, 2, \ldots, m-2),$$

(2.3) $$\bar{G}_0'(x_0) = \bar{y}_0', \quad \bar{G}_{m-1}'(x_m) = \bar{y}_m',$$

wo $x_0 = 0$, $x_k = k\dfrac{1}{m}$, $h = \dfrac{1}{m}$ sind.

Mit diesen Spline-Funktionen approximieren wir die Lösung des Cauchy-Problems $y(x)$ und ihre Ableitungen bis zum 5. Grad. Außerdem wird die Ordnung der Approximation untersucht.

2. Existenz und Unizität

a) *Der Fall* (0, 2, 3).

SATZ 2.1. *Es sei:*

$$\bar{S}_\Delta(x) := \begin{cases} \bar{S}_0(x), & \text{wenn} \quad x_0 \leqq x \leqq x_1 \\ \bar{S}_k(x), & \text{wenn} \quad x_k \leqq x \leqq x_{k+1} \quad (k = 1, 2, \ldots, m-2) \\ \bar{S}_{m-1}(x), & \text{wenn} \quad x_{m-1} \leqq x \leqq x_m \end{cases}$$

wo

$$\bar{S}_0(x) = \bar{y}_0 + \bar{y}_0'(x-x_0) + \frac{\bar{y}_0''}{2}(x-x_0)^2 + \frac{\bar{y}_0'''}{6}(x-x_0)^3 +$$

$$+ \bar{a}_{0,4}(x-x_0)^4 + \bar{a}_{0,5}(x-x_0)^5 + \bar{a}_{0,6}(x-x_0)^6,$$

$$\bar{S}_k(x) = \bar{y}_k + \bar{a}_{k,1}(x-x_k) + \frac{\bar{y}_k''}{2}(x-x_k)^2 + \frac{\bar{y}_k'''}{6}(x-x_k)^3 +$$

$$+ \bar{a}_{k,4}(x-x_k)^4 + \bar{a}_{k,5}(x-x_k)^5,$$

$$\bar{S}_{m-1}(x) = \bar{y}_{m-1} + \bar{a}_{m-1,1}(x-x_{m-1}) + \frac{\bar{y}_{m-1}''}{2}(x-x_{m-1})^2 + \frac{\bar{y}_{m-1}'''}{6}(x-x_{m-1})^3 +$$

$$+ \bar{a}_{m-1,4}(x-x_{m-1})^4 + \bar{a}_{m-1,5}(x-x_{m-1})^5 + \bar{a}_{m-1,6}(x-x_{m-1})^6.$$

Durch die Bedingungen (1.1), (1.2) *und* (1.3) *sind die Koeffizienten des Polynoms eindeutig bestimmt.*

BEWEIS. Der Beweis ergibt sich aus dem Satz 2.1 in [2]. Für die Koeffizienten bekommen wir (laut [2]):

$$\bar{a}_{0,4} = \frac{5}{h^4}\left\{\bar{y}_1 - \bar{y}_0 - \bar{y}_0' h - \frac{\bar{y}_0''}{2}h^2 - \frac{\bar{y}_0'''}{6}h^3\right\} - \frac{1}{2h^2}\{\bar{y}_1'' - \bar{y}_0'' - \bar{y}_0''' h\} + \frac{1}{12h}\{\bar{y}_1''' - \bar{y}_0'''\},$$

$$\bar{a}_{0,5} = -\frac{6}{h^5}\left\{\bar{y}_1 - \bar{y}_0 - \bar{y}_0' h - \frac{\bar{y}_0''}{2}h^2 - \frac{\bar{y}_0'''}{6}h^3\right\} + \frac{4}{5h^3}\{\bar{y}_1'' - \bar{y}_0'' - \bar{y}_0''' h\} - \frac{3}{20h^2}\{\bar{y}_1''' - \bar{y}_0'''\},$$

$$\bar{a}_{0,6} = \frac{2}{h^6}\left\{\bar{y}_1 - \bar{y}_0 - \bar{y}_0' h - \frac{\bar{y}_0''}{2}h^2 - \frac{\bar{y}_0'''}{6}h^3\right\} - \frac{3}{10h^4}\{\bar{y}_1'' - \bar{y}_0'' - \bar{y}_0''' h\} + \frac{1}{15h^3}\{\bar{y}_1''' - \bar{y}_0'''\},$$

$$\bar{a}_{k,1} = \frac{1}{h}\left\{\bar{y}_{k+1} - \bar{y}_k - \frac{\bar{y}_k''}{2}h^2 - \frac{\bar{y}_k'''}{6}h^3\right\} - \frac{3}{20}h\{\bar{y}_{k+1}'' - \bar{y}_k'' - \bar{y}_k''' h\} + \frac{1}{30}h^2\{\bar{y}_{k+1}''' - \bar{y}_k'''\},$$

$$\bar{a}_{k,4} = \frac{1}{4h^2}\{\bar{y}_{k+1}'' - \bar{y}_k'' - \bar{y}_k''' h\} - \frac{1}{12h}\{\bar{y}_{k+1}''' - \bar{y}_k'''\},$$

$$\bar{a}_{k,5} = -\frac{1}{10h^3}\{\bar{y}_{k+1}'' - \bar{y}_k'' - \bar{y}_k''' h\} + \frac{1}{20h^2}\{\bar{y}_{k+1}''' - \bar{y}_k'''\},$$

$$\bar{a}_{m-1,1} = \frac{2}{h}\left\{\bar{y}_m - \bar{y}_{m-1} - \frac{\bar{y}_{m-1}''}{2}h^2 - \frac{\bar{y}_{m-1}'''}{6}h^3\right\} - \left\{\bar{y}_m' - \bar{y}_{m-1}'' h - \frac{\bar{y}_{m-1}'''}{2}h^2\right\} +$$

$$+ \frac{1}{5}h\{\bar{y}_m'' - \bar{y}_{m-1}'' - \bar{y}_{m-1}''' h\} - \frac{1}{60}h^2\{\bar{y}_m''' - \bar{y}_{m-1}'''\},$$

$$\bar{a}_{m-1,4} = -\frac{5}{h^4}\left\{\bar{y}_m - \bar{y}_{m-1} - \frac{\bar{y}_{m-1}''}{2}h^2 - \frac{\bar{y}_{m-1}'''}{6}h^3\right\} +$$

$$+ \frac{5}{h^3}\left\{\bar{y}_m' - \bar{y}_{m-1}'' h - \frac{\bar{y}_{m-1}'''}{2}h^2\right\} - \frac{3}{2h^2}\{\bar{y}_m'' - \bar{y}_{m-1}'' - \bar{y}_{m-1}''' h\} + \frac{1}{6h}\{\bar{y}_m''' - \bar{y}_{m-1}'''\},$$

$$\bar{a}_{m-1,5}=\frac{6}{h^5}\left\{\bar{y}_m-\bar{y}_{m-1}-\frac{\bar{y}''_{m-1}}{2}h^2-\frac{\bar{y}'''_{m-1}}{6}h^3\right\}-\frac{6}{h^4}\left\{\bar{y}'_m-\bar{y}''_{m-1}h-\frac{\bar{y}'''_{m-1}}{2}h^2\right\}+$$

$$+\frac{2}{h^3}\{\bar{y}''_m-\bar{y}''_{m-1}-\bar{y}'''_{m-1}h\}-\frac{1}{4h^2}\{\bar{y}'''_m-\bar{y}'''_{m-1}\},$$

$$\bar{a}_{m-1,6}=-\frac{2}{h^6}\left\{\bar{y}_m-\bar{y}_{m-1}-\frac{\bar{y}''_{m-1}}{2}h^2-\frac{\bar{y}'''_{m-1}}{6}h^3\right\}+$$

$$+\frac{2}{h^5}\left\{\bar{y}'_m-\bar{y}''_{m-1}h-\frac{\bar{y}'''_{m-1}}{2}h^2\right\}-\frac{7}{10h^4}\{\bar{y}''_m-\bar{y}''_{m-1}-\bar{y}'''_{m-1}h\}+\frac{1}{10h^3}\{\bar{y}'''_m-\bar{y}'''_{m-1}\}.$$

b) *Der fall* (0, 2, 4).

SATZ 2.2. *Es sei:*

$$\bar{G}_4(x):=\begin{cases}\bar{G}_0(x), & \text{wenn} \quad x_0\leqq x\leqq x_1\\ \bar{G}_k(x), & \text{wenn} \quad x_k\leqq x\leqq x_{k+1}\\ \bar{G}_{m-1}(x), & \text{wenn} \quad x_{m-1}\leqq x\leqq x_m\end{cases}\quad (k=1,2,\ldots,m-2),$$

wo

$$\bar{G}_0(x)=\bar{y}_0+\bar{y}'_0(x-x_0)+\frac{\bar{y}''_0}{2}(x-x_0)^2+\bar{b}_{0,3}(x-x_0)^3+$$

$$+\frac{\bar{y}^{(4)}_0}{24}(x-x_0)^4+\bar{b}_{0,5}(x-x_0)^5+\bar{b}_{0,6}(x-x_0)^6,$$

$$\bar{G}_k(x)=\bar{y}_k+\bar{b}_{k,1}(x-x_k)+\frac{\bar{y}''_k}{2}(x-x_k)^2+\bar{b}_{k,3}(x-x_k)^3+$$

$$+\frac{\bar{y}^{(4)}_k}{24}(x-x_k)^4+\bar{b}_{k,5}(x-x_k)^5,$$

$$\bar{G}_{m-1}(x)=\bar{y}_{m-1}+\bar{b}_{m-1,1}(x-x_{m-1})+\frac{\bar{y}''_{m-1}}{2}(x-x_{m-1})^2+$$

$$+\bar{b}_{m-1,3}(x-x_{m-1})^3+\frac{\bar{y}^{(4)}_{m-1}}{24}(x-x_{m-1})^4+\bar{b}_{m-1,5}(x-x_{m-1})^5+\bar{b}_{m-1,6}(x-x_{m-1})^6.$$

Durch die Bedingungen (2.1), (2.2) *und* (2.3) *sind die Koeffizienten des Polynoms eindeutig bestimmt.*

BEWEIS. Der Beweis ergibt sich aus dem Satz 2.2 in [2]. Für die Koeffizienten bekommen wir (laut [2]):

$$\bar{b}_{0,3}=\frac{5}{3h^3}\left\{\bar{y}_1-\bar{y}_0-\bar{y}'_0h-\frac{\bar{y}''_0}{2}h^2-\frac{\bar{y}^{(4)}_0}{24}h^4\right\}-$$

$$-\frac{1}{9h}\left\{\bar{y}''_1-\bar{y}''_0-\frac{\bar{y}^{(4)}_0}{2}h^2\right\}-\frac{1}{216}h\{\bar{y}^{(4)}_1-\bar{y}^{(4)}_0\},$$

$$\bar{b}_{0,5} = -\frac{1}{h^5}\left\{\bar{y}_1 - \bar{y}_0 - \bar{y}_0' h - \frac{\bar{y}_0''}{2} h^2 - \frac{\bar{y}_0^{(4)}}{24} h^4\right\} +$$

$$+ \frac{1}{6h^3}\left\{\bar{y}_1'' - \bar{y}_0'' - \frac{\bar{y}_0^{(4)}}{2} h^2\right\} - \frac{1}{90h}\{\bar{y}_1^{(4)} - \bar{y}_0^{(4)}\},$$

$$\bar{b}_{0,6} = \frac{1}{3h^6}\left\{\bar{y}_1 - \bar{y}_0 - \bar{y}_0' h - \frac{\bar{y}_0''}{2} h^2 - \frac{\bar{y}_0^{(4)}}{24} h^4\right\} -$$

$$- \frac{1}{18h^4}\left\{\bar{y}_1'' - \bar{y}_0'' - \frac{\bar{y}_0^{(4)}}{2} h^2\right\} + \frac{7}{1080h^2}\{\bar{y}_1^{(4)} - \bar{y}_0^{(4)}\},$$

$$\bar{b}_{k,1} = \frac{1}{h}\left\{\bar{y}_{k+1} - \bar{y}_k - \frac{\bar{y}_k''}{2} h^2 - \frac{\bar{y}_k^{(4)}}{24} h^4\right\} -$$

$$- \frac{h}{6}\left\{\bar{y}_{k+1}'' - \bar{y}_k'' - \frac{\bar{y}_k^{(4)}}{2} h^2\right\} + \frac{7}{360} h^3 \{\bar{y}_{k+1}^{(4)} - \bar{y}_k^{(4)}\},$$

$$\bar{b}_{k,3} = \frac{1}{6h}\left\{\bar{y}_{k+1}'' - \bar{y}_k'' - \frac{\bar{y}_k^{(4)}}{2} h^2\right\} - \frac{1}{36} h\{\bar{y}_{k+1}^{(4)} - \bar{y}_k^{(4)}\},$$

$$\bar{b}_{k,5} = \frac{1}{120h}\{\bar{y}_{k+1}^{(4)} - \bar{y}_k^{(4)}\},$$

$$\bar{b}_{m-1,1} = \frac{2}{h}\left\{\bar{y}_m - \bar{y}_{m-1} - \frac{\bar{y}_{m-1}''}{2} h^2 - \frac{\bar{y}_{m-1}^{(4)}}{24} h^4\right\} - \left\{\bar{y}_m' - \bar{y}_{m-1}'' h - \frac{\bar{y}_{m-1}^{(4)}}{6} h^3\right\} +$$

$$+ \frac{h}{6}\left\{\bar{y}_m'' - \bar{y}_{m-1}'' - \frac{\bar{y}_{m-1}^{(4)}}{2} h^2\right\} - \frac{h^3}{360}\{\bar{y}_m^{(4)} - \bar{y}_{m-1}^{(4)}\},$$

$$\bar{b}_{m-1,3} = -\frac{5}{3h^3}\left\{\bar{y}_m - \bar{y}_{m-1} - \frac{\bar{y}_{m-1}''}{2} h^2 - \frac{\bar{y}_{m-1}^{(4)}}{24} h^4\right\} +$$

$$+ \frac{5}{3h^2}\left\{\bar{y}_m' - \bar{y}_{m-1}'' h - \frac{\bar{y}_{m-1}^{(4)}}{6} h^3\right\} - \frac{7}{18h}\left\{\bar{y}_m'' - \bar{y}_{m-1}'' - \frac{\bar{y}_{m-1}^{(4)}}{2} h^2\right\} + \frac{h}{108}\{\bar{y}_m^{(4)} - \bar{y}_{m-1}^{(4)}\},$$

$$\bar{b}_{m-1,5} = \frac{1}{h^5}\left\{\bar{y}_m - \bar{y}_{m-1} - \frac{\bar{y}_{m-1}''}{2} h^2 - \frac{\bar{y}_{m-1}^{(4)}}{24} h^4\right\} -$$

$$- \frac{1}{h^4}\left\{\bar{y}_m' - \bar{y}_{m-1}'' h - \frac{\bar{y}_{m-1}^{(4)}}{6} h^3\right\} + \frac{1}{3h^3}\left\{\bar{y}_m'' - \bar{y}_{m-1}'' - \frac{\bar{y}_{m-1}^{(4)}}{2} h^2\right\} - \frac{1}{72h}\{\bar{y}_m^{(4)} - \bar{y}_{m-1}^{(4)}\},$$

$$\bar{b}_{m-1,6} = \frac{1}{3h^6}\left\{\bar{y}_m - \bar{y}_{m-1} - \frac{\bar{y}_{m-1}''}{2} h^2 - \frac{\bar{y}_{m-1}^{(4)}}{24} h^4\right\} +$$

$$+ \frac{1}{3h^5}\left\{\bar{y}_m' - \bar{y}_{m-1}'' h - \frac{\bar{y}_{m-1}^{(4)}}{6} h^3\right\} - \frac{1}{9h^4}\left\{\bar{y}_m'' - \bar{y}_{m-1}'' - \frac{\bar{y}_{m-1}^{(4)}}{2} h^2\right\} + \frac{1}{135h^2}\{\bar{y}_m^{(4)} - \bar{y}_{m-1}^{(4)}\}.$$

3. Probleme der Konvergenz

SATZ 3.1. *Es seien* $\bar{S}_\Delta(x)$ *die vorher konstruierte approximative lakunäre Spline-Funktion und* $S_\Delta(x)$ *die in* [2] *konstruierte lakunäre „echte" Spline-Funktion. Dann gelten*

$$|S_0^{(q)}(x)-\bar{S}_0^{(q)}(x)| \leqq K_q\omega_5(h)h^{7-q} \quad \textit{für} \quad q=0,1,2,3,4,5,6,$$

$$|S_k^{(q)}(x)-\bar{S}_k^{(q)}(x)| \leqq C_q\omega_5(h)h^{5-q} \quad \textit{für} \quad q=0,1;\quad k=1,2,\ldots,m-2,$$

$$|S_k^{(q)}(x)-\bar{S}_k^{(q)}(x)| \leqq C_q\omega_5(h)h^{7-q} \quad \textit{für} \quad q=2,3,4,5;\quad k=1,2,\ldots,m-2,$$

$$|S_{m-1}^{(q)}(x)-\bar{S}_{m-1}^{(q)}(x)| \leqq M_q\omega_5(h)h^{5-q} \quad \textit{für} \quad q=0,1,2,3,4,5,$$

wo K_q, C_q *und* M_q *Konstanten und*

$$\omega_5(h)=\sup_{|x-x_1|\leqq h}|y^{(5)}(x)-y^{(5)}(x_1)|$$

bedeuten.

SATZ 3.2. *Es seien* $\bar{G}_\Delta(x)$ *die vorher konstruierte approximative lakunäre Spline-Funktion und* $G_\Delta(x)$ *die in* [2] *konstruierte lakunäre „echte" Spline-Funktion. Dann gelten*

$$|G_0^{(q)}(x)-\bar{G}_0^{(q)}(x)| \leqq K_q\omega_5(h)h^{7-q} \quad \textit{für} \quad q=0,1,2,\ldots,6,$$

$$|G_k^{(2q)}(x)-\bar{G}_k^{(2q)}(x)| \leqq C_{2q}\omega_5(h)h^5 \quad \textit{für} \quad q=0,1,2;\quad k=1,2,\ldots,m-2,$$

$$|G_k^{(2q+1)}(x)-\bar{G}_k^{(2q+1)}(x)| \leqq C_{2q+1}\omega_5(h)h^4 \quad \textit{für} \quad q=0,1,2;\quad k=1,2,\ldots,m-2,$$

$$|G_{m-1}^{(q)}(x)-\bar{G}_{m-1}^{(q)}(x)| \leqq M_q\omega_5(h)h^{5-q} \quad \textit{für} \quad q=0,1,2,3,4,5.$$

BEWEIS DES SATZES 3.1. Es sei $x_0 \leqq x \leqq x_1$. Aus den Lemmata 2.2.3 und 2.2.1 in [1] bekommen wir für $k=0$:

$$|y_1-\bar{y}_1| \leqq C_7\omega_5(h)h^7, \quad |y_1'-\bar{y}_1'| \leqq C_2\omega_5(h)h^6,$$

$$|y_1^{(q)}-\bar{y}_1^{(q)}| = |f^{(q)}(x_1,y_1,y_1')-f^{(q)}(x_1,\bar{y}_1,\bar{y}_1')| \leqq$$

$$\leqq K\{|y_1-\bar{y}_1|+|y_1'-\bar{y}_1'|\} \leqq K\omega_5(h)h^6 \quad \text{für} \quad q=2,3,4,5.$$

Mit Hilfe dieser Abschätzungen ergeben sich die nachstehenden Ungleichungen:

$$|a_{0,4}-\bar{a}_{0,4}| \leqq L_4h^3\omega_5(h), \quad |a_{0,5}-\bar{a}_{0,5}| \leqq L_5h^2\omega_5(h), \quad |a_{0,6}-\bar{a}_{0,6}| \leqq L_6h\omega_5(h).$$

Durch Anwendung dieser Ungleichungen bekommen wir leicht den Satz 3.1 für $S_0(x)$.

Der Beweis der anderen Behauptung kann ähnlich geführt werden.

BEWEIS DES SATZES 3.2. Man kann diesen Satz ebenso beweisen, wie den Satz 3.1.

Mit Hilfe dieser Sätze zeigen wir, daß die Funktion $y(x)$ und ihre Ableitungen bis zum 5. Grad mit den Spline-Funktionen $\bar{S}_\Delta(x)$ und $\bar{G}_\Delta(x)$ ebenso gut approximierbar sind, wie mit Hilfe der Spline Funktionen $S_\Delta(x)$ und $G_\Delta(x)$.

SATZ 3.3. *Es sei $y(x)$ die Lösung des Cauchy-Problems, ferner sei $\bar{S}_\Delta(x)$ die vorher konstruierte approximative lakunäre Spline-Funktion. Dann gilt*

$$|y^{(q)}(x)-\bar{S}_\Delta^{(q)}(x)| \leqq K_q\omega_5(h)h^{5-q} \quad \text{für} \quad q=0,1,2,3,4,5.$$

SATZ 3.4. *Es seien $y(x)$ die Lösung des Cauchy-Problems und $\bar{G}_\Delta(x)$ die vorher konstruierte approximative lakunäre Spline-Funktion. Dann gilt*

$$|y^{(q)}(x)-\bar{G}_\Delta^{(q)}(x)| \leqq M_q\omega_5(h)h^{5-q} \quad \text{für} \quad q=0,1,2,3,4,5.$$

BEWEIS DES SATZES 3.3. Wir haben

$$|y^{(q)}(x)-\bar{S}_\Delta^{(q)}(x)| \leqq |y^{(q)}(x)-S_\Delta^{(q)}(x)|+|S_\Delta^{(q)}(x)-\bar{S}_\Delta^{(q)}(x)|,$$

und so ergibt sich Satz 3.3 aus den Sätzen 3.1 dieser Arbeit und 3.1 in [2].

BEWEIS DES SATZES 3.4. Der Beweis kann ähnlich wie beim Satz 3.3 geführt werden.

Es gelten noch die folgenden Sätze:

SATZ 3.5. *Es sei $\bar{S}_\Delta(x)$ die vorher konstruierte approximative lakunäre Spline-Funktion. Dann gilt*

$$|\bar{S}_\Delta''(x)-f[x,\bar{S}_\Delta(x),\bar{S}_\Delta'(x)]| \leqq C\omega_5(h)h^3.$$

SATZ 3.6. *Es sei $\bar{G}_\Delta(x)$ die vorher konstruierte approximative lakunäre Spline-Funktion. Dann gilt*

$$|\bar{G}_\Delta''(x)-f[x,\bar{G}_\Delta(x),\bar{G}_\Delta'(x)]| \leqq C_1\omega_5(h)h^3.$$

BEWEIS DES SATZES 3.5.

$$|\bar{S}_\Delta''(x)-f[x,\bar{S}_\Delta(x),\bar{S}_\Delta'(x)]| \leqq |\bar{S}_\Delta''(x)-y''(x)|+$$

$$+|f[x,y(x),y'(x)]-f[x,\bar{S}_\Delta(x),\bar{S}_\Delta'(x)]| \leqq$$

$$\leqq K_2\omega_5(h)h^3+L\{|y(x)-\bar{S}_\Delta(x)|+|y'(x)-\bar{S}_\Delta'(x)|\} \leqq C\omega_5(h)h^3.$$

BEWEIS DES SATZES 3.6. Man führt den Beweis ähnlich wie bei dem Satz 3.5.

Endlich möchte ich Herrn Prof. Dr. J. Balázs für seine wertvolle Hilfe [3] meinen Dank ausdrücken.

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A NOTE ON THE DISTRIBUTION OF PRIMES IN SHORT INTERVALS

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1. We know from the works of Koch [4] and Littlewood [5], that on the Riemann Hypothesis (RH) we have

$$\Delta(x) := \psi(x) - x =: \sum_{p^m \leq x} \log p - x = O(\sqrt{x} \log^2 x)$$

and

$$\Delta(x) = \Omega_{\pm}(\sqrt{x} \log_3 x)$$

resp., where $\log_\nu x$ denotes the ν-fold iterated logarithmic function. The gap between the above two estimates disappears already if we consider the mean-value of $\Delta(x)$. It is well known that (assuming RH)

$$E(x) := \frac{1}{x} \int_1^x \Delta(t)\, dt = O(\sqrt{x}) \quad \text{and} \quad E(x) = \Omega_{\pm}(\sqrt{x}).$$

Cramér [1] showed in 1922 that (on RH)

$$\frac{1}{x} \int_1^x \Delta^2(t)\, dt = O(x)$$

which implies (on RH)

$$D(x) := \frac{1}{x} \int_1^x |\Delta(t)|\, dt = O(\sqrt{x}).$$

What concerns the lower estimate of $D(x)$, Knapowski [3] showed in 1961 that

$$D(x) > \sqrt{x} \exp\left(-c_1 \frac{\log x}{\log_2 x} \log_3 x\right).$$

The author proved recently [7] (without any hypothesis)

$$D(x) > \frac{\sqrt{x}}{22\,000} \quad \text{for} \quad x > 2.$$

(In an earlier paper [6] a similar estimate was proved for $x > x_0$, an ineffective contant.) This means that we know the precise order of magnitude of $D(x)$, if we suppose RH. Since the result of Koch can be formulated as

$$\int_{x-1}^x |\Delta(t)|\, dt = O(\sqrt{x} \log^2 x),$$

it is natural to ask, what can we assert on

$$D\left(x-\frac{x}{H}, x\right)=\frac{1}{(x/H)}\int_{x-x/H}^{x}|\Delta(t)|\,dt \quad (1\leqq H\leqq x).$$

The aim of the present note is to show the following generalization of the results of Cramér and Koch.

THEOREM. *Assume RH. Then for* $1\leqq H\leqq x$

$$\frac{1}{(x/H)}\int_{x-x/H}^{x}|\Delta^2(t)|\,dt\ll x\log^4(H+1).$$

COROLLARY. *Assume RH. Then for* $1\leqq H\leqq x$

$$D(x-x/H, x)\ll\sqrt{x}\log^2(H+1).$$

2. A similar problem is to consider the mean value of $\Delta(x)$ in short intervals, i.e. the function

$$E\left(x-\frac{x}{H}, x\right):=\frac{1}{(x/H)}\int_{x-x/H}^{x}\Delta(t)\,dt.$$

Supposing RH, Jurkat [2] proved the following generalization of Littlewood's result:

$$E\left(x-\frac{x}{\log_2 x}, x\right)=\Omega_{\pm}(\sqrt{x}\log_3 x).$$

Without using any unproved hypothesis, the author showed that

$$E\left(x-\frac{x}{H}, x\right)=\Omega_{\pm}(\sqrt{x}\log(H+1)) \quad \left(1\leqq H\leqq\frac{\log_2 x}{\log_3^2 x}\right).$$

This is a generalization of (a slightly weaker form of) Jurkat's result and of the easy estimate $E(x)=\Omega_{\pm}(\sqrt{x})$, mentioned earlier. The above result shows that for $H\leqq\log_2 x/\log_3^2 x$ our Corollary cannot be true with an exponent smaller than 1. Probably our Theorem and Corollary are optimal for $H\leqq\log_2 x/\log_3^2 x$ (on RH), because probably

$$E\left(x-\frac{x}{H}, x\right)=\Omega_{\pm}(\sqrt{x}\log^2(H+1)\left(1\leqq H\leqq\frac{\log_2 x}{\log_3^2 x}\right).$$

The Ω-type estimate for $E(x-x/H, x)$ mentioned earlier can be proved relatively easily, if one assumes RH, or even without any condition, but only in an ineffective form. It is much more complicated, but possible to prove this in an effective form, using Turán's method. To these questions we shall return later on.

Finally we mention that Montgomery conjectures (based on the linear independence of the imaginary parts of the zeta-zeros in the upper half-plane), that

$$\overline{\lim_{x\to\infty}} \frac{\Delta(x)}{\sqrt{x}\log_3^2 x} = \pm\frac{1}{2\pi}$$

which is very probably true, but hopeless to prove at present.

3. To prove our Theorem we start with the shortened prime number formula of Riemann—Von Mangoldt:

$$\Delta(t) = -2\sum_{0<\gamma<x} \frac{t^{1/2+i\gamma}}{\varrho} + O(\log^2 x) \quad (t \leqq x)$$

where $\varrho = 1/2+i\gamma$ runs through non-trivial zeros of $\zeta(s)$. This implies

$$\Delta^2(t) = 4\sum_{0<\gamma,\gamma'<x} \frac{t^{1+i(\gamma-\gamma')}}{\varrho\bar{\varrho}'} + O(\sqrt{x}\log^4 x).$$

This in order to show our Theorem, it is sufficient to prove

$$\frac{1}{(x/H)}\sum_{\gamma,\gamma'>0}\frac{1}{\gamma\gamma'}\min\left\{\frac{x}{H}x, \frac{x^2}{|2+i(\gamma-\gamma')|}\right\} \ll x\log^4 H_1$$

where $H_1 = H+1$. This can be further reduced to the proof of

$$\sum_{0<\gamma\leqq\gamma'}\frac{1}{\gamma\gamma'}\min\left\{1, \frac{H}{\gamma'-\gamma}\right\} \ll \log^4 H_1.$$

We shall estimate this sum by splitting it in five parts, as follows:

$$\Sigma_1 := \sum_{0<\gamma\leqq H}\frac{1}{\gamma}\sum_{\gamma\leqq\gamma'\leqq\gamma+H}\frac{1}{\gamma'} \leqq \left(\sum_{0<\gamma\leqq 2H}\frac{1}{\gamma}\right)^2 \ll \log^4 H_1,$$

$$\Sigma_2 := \sum_{0<\gamma\leqq H}\frac{1}{\gamma}\sum_{\gamma'>\gamma+H}\frac{H}{\gamma'(\gamma'-\gamma)} \leqq \sum_{0<\gamma\leqq H}\frac{1}{\gamma}\sum_{\gamma'>H}\frac{H}{(\gamma'^2/2)} \ll \log^3 H_1,$$

$$\Sigma_3 := \sum_{\gamma>H}\frac{1}{\gamma}\sum_{\gamma\leqq\gamma'\leqq\gamma+H}\frac{1}{\gamma'} \ll \sum_{\gamma>H}\frac{1}{\gamma}\frac{H\log\gamma}{\gamma} \ll \log^2 H_1,$$

$$\Sigma_4 := \sum_{\gamma>H}\frac{1}{\gamma}\sum_{\gamma+H<\gamma'\leqq 2\gamma}\frac{H}{\gamma'(\gamma'-\gamma)} \ll \sum_{\gamma>H}\frac{H}{\gamma^2}\sum_{1\leqq n\leqq[\gamma/H]}\frac{H\log\gamma}{nH} \ll \log^3 H_1,$$

$$\Sigma_5 := \sum_{\gamma>H}\frac{1}{\gamma}\sum_{\gamma'>2\gamma}\frac{H}{\gamma'(\gamma'-\gamma)} \ll \sum_{\gamma>H}\frac{H}{\gamma}\frac{\log\gamma}{\gamma} \ll \log^2 H_1,$$

Q.E.D.

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ON SUMS OF SEQUENCES OF INTEGERS. III

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1.

Throughout this paper, we use the following notation: $c_1, c_2, \ldots, N_0, N_1, \ldots$ denote positive absolute constants. We write $e^x=\exp(x)$ and $e^{2\pi i\alpha}=e(\alpha)$. The distance from α to the nearest integer is denoted by $\|\alpha\|$ so that $\|\alpha\|=\min(\alpha-[\alpha], [\alpha]+1-\alpha)$. We put $\min\left(A, \frac{1}{0}\right)=A$. We denote the least prime factor of n by $p(n)$, while the greatest prime factor of n is denoted by $P(n)$. $\tau(n)$ denotes the divisor function: $\tau(n)=\sum_{d|n} 1$.

2.

In this series, we study the arithmetic nature of the numbers of form $a+b$ where a, b are taken from "dense" sequences of integers. In Part I (see [1]), we showed the existence of a, b such that $P(a+b)$ is "small". In Part II (see [2]), we proved the existence of a, b such that $P(a+b)$ is "large", and we generalized our problems by showing that if $\mathscr{A}, \mathscr{B}, \mathscr{K}$ are "dense" sequences of integers which satisfy certain further assumptions then there exist $a\in\mathscr{A}, b\in\mathscr{B}, k\in\mathscr{K}$ such that $k|(a+b)$. However, in all these cases, all the sequences $\mathscr{A}, \mathscr{B}$ and, mostly, $\mathscr{K}$ are dense; in fact, our methods must be modified in order to be able to treat a problem when $\mathscr{K}$ is "scarce". The most interesting problem of this type is when $\mathscr{K}$ is the set of the squares of the integers (of a certain interval). Here our goal is to study this problem; in fact, we are going to study the even more difficult problem when $\mathscr{K}$ is the set of the squares of the primes. We prove the following theorem:

THEOREM. *There exist absolute constants N_0, c_1, c_2 satisfying the following conditions:*

Let $N>N_0$, and let $\mathscr{A}, \mathscr{B}$ be sequences of integers such that

(1) $$\frac{N}{2} < a \leqq N \quad \textit{for} \quad a\in\mathscr{A},$$

(2) $$\frac{N}{2} < b \leqq N \quad \textit{for} \quad b\in\mathscr{B}.$$

Put

$$A=\sum_{a\in\mathscr{A}} 1, \quad B=\sum_{b\in\mathscr{B}} 1.$$

If

(3) $$AB > c_1 N^{24/13} (\log N)^{42/13},$$

then there exist integers a, b and a prime number p such that $a \in \mathcal{A}$, $b \in \mathcal{B}$,

$$p^2 | (a+b) \tag{4}$$

and

$$p^2 > c_2 \frac{(AB)^{5/2}}{N^4 (\log N)^7}. \tag{5}$$

Furthermore, this assertion holds with

$$c_1 = 2^2 \cdot 10^6 \cdot c_3^2, \quad c_2 = 2^{-4} \cdot 10^{-15} \cdot c_3^{-5} \tag{6}$$

where

$$c_3 \geqq 1 \tag{7}$$

denotes a constant satisfying (17) *in Lemma* 5.

Note that for $A \gg N$, $B \gg N$, our theorem yields the existence of a, b, p satisfying (4) and

$$p^2 \gg \frac{(N^2)^{5/2}}{N^4 (\log N)^7} = \frac{N}{(\log N)^7}.$$

This shows that (5) in our theorem is best possible apart from the factor $(\log N)^7$ in the denominator on the right hand side.

The proof is based on a modified form of the method worked out in [1] and [2].

3.

Put

$$y = c_4 \frac{N^5 (\log N)^7}{(AB)^{5/2}}$$

where

$$c_4 = c_1^{5/2} = 2/c_2 = 2^5 \cdot 10^{15} \cdot c_3^5 \tag{8}$$

(here c_3 is the constant in (7)). Then for large N we have

$$y = c_4 \frac{N^5 (\log N)^7}{(AB)^{5/2}} \geqq c_4 \frac{N^5 (\log N)^7}{(N \cdot N)^{5/2}} = c_4 (\log N)^7, \tag{9}$$

and by (3) and (8),

$$y = c_4 \frac{N^5 (\log N)^7}{(AB)^{5/2}} < c_4 \frac{N^5 (\log N)^7}{(c_1 N^{24/13} (\log N)^{42/13})^{5/2}} = \frac{N^{5/13}}{(\log N)^{14/13}}. \tag{10}$$

Furthermore, let $\mathcal{K}$ denote the set of the numbers of the form p^2 where p is a prime number and

$$\frac{2N}{y} < p^2 \leqq \frac{4N}{y},$$

i.e.,

$$\sqrt{\frac{2N}{y}} < p \leqq \sqrt{\frac{4N}{y}}.$$

Put

$$K = \sum_{k \in \mathscr{K}} 1.$$

Then by (10) and the prime number theorem, for $N \to +\infty$ we have

$$K = \sum_{\sqrt{\frac{2N}{y}} < p \leqq \sqrt{\frac{4N}{y}}} 1 = (1+o(1)) \frac{(4N/y)^{1/2} - (2N/y)^{1/2}}{\log (4N/y)^{1/2}} = 2(2-\sqrt{2}+o(1)) \frac{(N/y)^{1/2}}{\log 4N/y},$$

so that for large N,

$$K > \frac{N^{1/2}}{y^{1/2} \log N} \tag{11}$$

and

$$K < \frac{6}{5} \frac{N^{1/2}}{y^{1/2} \log \frac{N}{N^{5/13}/(\log N)^{14/13}}} < \frac{6}{5} \cdot \frac{13}{8} \frac{N^{1/2}}{y^{1/2} \log N} < 2 \frac{N^{1/2}}{y^{1/2} \log N}. \tag{12}$$

Put

$$Q = \frac{N}{(\log N)^{7/5} y^{4/5}},$$

so that $Q < N$. Furthermore, let

$$U = [8N/y]+1, \quad d_n = \sum_{\substack{mp^2=n \\ m \leqq y \\ p^2 \in \mathscr{K}}} 1,$$

$$S_x(\alpha) = \sum_{n \leqq x} d_n e(n\alpha) \quad (\text{for } 0 \leqq x \leqq 2N), \quad S(\alpha) = S_{2N}(\alpha) = \sum_{n=1}^{2N} d_n e(n\alpha),$$

$$S = S(0) = \sum_{n=1}^{2N} d_n, \quad U(\alpha) = \sum_{n=0}^{U-1} e(n\alpha),$$

$$S(\alpha)U(\alpha) = \sum_{n=1}^{2N+U-1} v_n e(n\alpha) \quad (\text{so that } v_n = \sum_{n-U < j \leqq n} d_j),$$

$$F(\alpha) = \sum_{a \in \mathscr{A}} e(a\alpha), \quad G(\alpha) = \sum_{b \in \mathscr{B}} e(b\alpha),$$

and (in view of (1) and (2))

$$H(\alpha) = F(\alpha)G(\alpha) = \sum_{\substack{a \in \mathscr{A} \\ b \in \mathscr{B}}} e((a+b)\alpha) = \sum_{n=N+1}^{2N} h_n e(n\alpha)$$

$$(\text{so that } h_n = \sum_{\substack{a+b=n \\ a \in \mathscr{A} \\ b \in \mathscr{B}}} 1).$$

We start out from the integral

$$J=\int_0^1 F(\alpha)G(\alpha)S(-\alpha)d\alpha=\int_0^1 H(\alpha)S(-\alpha)d\alpha=$$

$$=\int_0^1 \sum_{n=N+1}^{2N}\sum_{m=1}^{2N} h_n d_m e((n-m)\alpha)d\alpha=\sum_{n=N+1}^{2N} h_n d_n.$$

Obviously, $d_n>0$ implies the existence of a prime p such that

$$p^2|n \tag{13}$$

and $p^2\in\mathscr{K}$ so that

$$p^2>\frac{2N}{y}=\frac{2}{c_4}\frac{(AB)^{5/2}}{N^4(\log N)^7}=c_2\frac{(AB)^{5/2}}{N^4(\log N)^7}. \tag{14}$$

On the other hand, $h_n>0$ implies that n can be written in the form

$$a+b=n \quad (a\in\mathscr{A},\ b\in\mathscr{B}). \tag{15}$$

(13), (14) and (15) yield (4) and (5). Thus in order to prove the existence of a, b, p of the desired properties, it is sufficient to show that

$$J=\sum_{n=N+1}^{2N} h_n d_n>0. \tag{16}$$

In order to prove this (by using the Hardy—Littlewood method), we need some lemmas.

4.

In this section, we assert some preliminary lemmas.

LEMMA 1. *If L is a positive integer, α a real number then we have*

$$\left|\sum_{n=0}^{L-1} e(n\alpha)-L\right|<4L^2|\alpha|.$$

This is identical with Lemma 2 in [1].

LEMMA 2. *For arbitrary real numbers α, M we have*

$$\left|\sum_{1\leqq m\leqq M} e(m\alpha)\right|\leqq\min\left(M,\frac{1}{2\|\alpha\|}\right).$$

See e.g. [3], p. 9.

The most important new tool in the proof of our theorem is Lemma 4 which will be derived from Lemma 3 and which will be used in order to estimate the contribution of the "minor arcs" ("supplementary intervals"); it plays the same role in the proof as Lemma 4 does in [2] in the proof of Theorem 1. In fact, a weaker estimate could

be derived also from Lemma 4 in [2], but we should loose a factor $\exp\left(c\frac{\log N}{\log\log N}\right)$ in (5) in this way (while (5) is best possible apart from a factor $(\log N)^c$).

LEMMA 3. *If R is a positive real number and a, q, f are integers such that $q>0$ and $(a,q)=1$ then we have*

$$\sum_{x=f+1}^{f+q}\left(\min\left(R,\frac{1}{2\left\|x\frac{a}{q}\right\|}\right)\right)^2<2R^2+3qR.$$

PROOF. We have

$$\sum_{x=f+1}^{f+q}\left(\min\left(R,\frac{1}{2\left\|x\frac{a}{q}\right\|}\right)\right)^2=\sum_{b=0}^{q-1}\left(\min\left(R,\frac{1}{2\|b/q\|}\right)\right)^2\leqq$$

$$\leqq 2\sum_{0\leqq b\leqq q/2R}\left(\min\left(R,\frac{1}{2\|b/q\|}\right)\right)^2+2\sum_{q/2R<b\leqq q/2}\left(\min\left(R,\frac{1}{2\|b/q\|}\right)\right)^2\leqq$$

$$\leqq 2\sum_{0\leqq b\leqq q/2R}R^2+2\sum_{q/2R<b\leqq q/2}\frac{q^2}{4b^2}\leqq 2\left(1+\frac{q}{2R}\right)R^2+\sum_{q/2R<b\leqq q/2}\frac{q^2}{b(b+1)}<$$

$$<2R^2+qR+q^2\sum_{b=[q/2R]+1}^{+\infty}\left(\frac{1}{b}-\frac{1}{b+1}\right)=$$

$$=2R^2+qR+q^2\cdot\frac{1}{[q/2R]+1}<2R^2+qR+q^2\frac{1}{q/2R}=2R^2+3qR.$$

LEMMA 4. *If R, V are positive real numbers and a, q are integers such that $V\geqq 1$, $q>0$ and $(a,q)=1$ then we have*

$$\sum_{0<x\leqq V}\left(\min\left(R,\frac{1}{2\left\|x\frac{a}{q}\right\|}\right)\right)^2<\left(\frac{V}{q}+1\right)(2R^2+3qR)\left(=2\frac{VR^2}{q}+2R^2+3VR+3qR\right).$$

PROOF. By Lemma 3, we have

$$\sum_{0<x\leqq V}\left(\min\left(R,\frac{1}{2\left\|x\frac{a}{q}\right\|}\right)\right)^2\leqq\sum_{k=1}^{[V/q]+1}\sum_{x=(k-1)q+1}^{kq}\left(\min\left(R,\frac{1}{2\left\|x\frac{a}{q}\right\|}\right)\right)^2\leqq$$

$$\leqq\sum_{k=1}^{[V/q]+1}(2R^2+3qR)=\left(\left[\frac{V}{q}\right]+1\right)(2R^2+3qR)<\left(\frac{V}{q}+1\right)(2R^2+3qR).$$

LEMMA 5. *For $x\geqq 2$ we have*

$$\sum_{n\leqq x}(\tau(n))^2<c_3x(\log x)^3. \tag{17}$$

See e.g. [4], p. 26.

5.

In this section, we estimates S, $S(\alpha)$ and v_n.

LEMMA 6. *We have* $S \leqq 2\dfrac{(Ny)^{1/2}}{\log N}$.

PROOF. The inequality

$$S \leqq Ky \tag{18}$$

holds by Lemma 5 in [2] (note that here $\mathscr{K}$, K, y are defined differently; however, (18) can be proved in the same way as in [2]). (12) and (18) yield that

$$S \leqq Ky < 2\frac{N^{1/2}}{y^{1/2}\log N}y = 2\frac{(Ny)^{1/2}}{\log N}.$$

LEMMA 7. *If* $1 \leqq u \leqq 2N$ *and* a, q *are integers such that* $2 \leqq q \leqq N/Q$ *and* $(a, q)=1$, *then we have* $|S_u(a/q)| \leqq Kq$.

This is identical with Lemma 6 in [2]. Again, here $\mathscr{K}$, K, Q are defined differently; however, the lemma can be proved in the same way. In fact, there is just one critical point in the proof: namely, the proof of Lemma 6 in [2] was based on the fact that

$$p(k) > N/Q (\geqq q) \quad \text{for} \quad k \in \mathscr{K}. \tag{19}$$

We have to show that this holds also for $\mathscr{K}$ and Q defined above.

If $k \in \mathscr{K}$ then k can be written in the form $k=p^2$ where $\sqrt{2N/y} < p$ so that in view of (3) and (8),

$$p(k) > \sqrt{2N/y} = (\log N)^{7/5} y^{4/5} (2N(\log N)^{-14/5} y^{-13/5})^{1/2} =$$

$$= \frac{N}{Q}\{2N(\log N)^{-14/5} c_4^{-13/5} N^{-13} (\log N)^{-91/5} (AB)^{13/2}\}^{1/2} >$$

$$> \frac{N}{Q}\{2c_4^{-13/5} N^{-12} (\log N)^{-21} (c_1 N^{24/13} (\log N)^{42/13})^{13/2}\}^{1/2} = \frac{N}{Q} 2^{1/2} > \frac{N}{Q}$$

which proves (19).

LEMMA 8. *If* α *is a real number and* a, q *are integers such that* $2 \leqq q \leqq N/Q$, $(a, q)=1$ *and* $\left|\alpha - \dfrac{a}{q}\right| < \dfrac{1}{qQ}$ *then we have*

$$|S(\alpha)| < 28N^{1/2}(\log N)^{2/5} y^{3/10}.$$

PROOF. The inequality

$$|S(\alpha)| \leqq 14KN/Q \tag{20}$$

can be derived (by partial summation) from Lemma 7 in the same way as the same inequality in the proof of Lemma 7 in [2] from Lemma 6. (12) and (20) yield that

$$|S(\alpha)| < 14KN/Q = 14K(\log N)^{7/5}y^{4/5} <$$

$$< 14\cdot 2\frac{N^{1/2}}{y^{1/2}\log N}(\log N)^{7/5}y^{4/5} = 28N^{1/2}(\log N)^{2/5}y^{3/10}.$$

The crucial point of the proof is the next lemma whose proof is based on Lemma 4 (in the same way as the proof of Lemma 8 in [2] is based on Lemma 4).

LEMMA 9. *If* $1\leqq u\leqq 2N$ *and* a, q *are integers such that* $N/Q<q\leqq Q$ *and* $(a, q)=1$ *then we have*

$$|S_u(a/q)| \leqq 4c_3 N^{1/2}(\log N)^{2/5}y^{3/10}.$$

PROOF. By Cauchy's inequality and Lemma 2 we have

$$|S_u(a/q)| = \Big|\sum_{n\leqq u} d_n e(na/q)\Big| = \Big|\sum_{\substack{mp^2\leqq u\\ m\leqq y\\ p^2\in\mathscr{X}}} e(mp^2a/q)\Big| = \tag{21}$$

$$= \Big|\sum_{m\leqq y}\sum_{2N/y<p^2\leqq\min(4N/y,\,u/m)} e(mp^2a/q)\Big| \leqq \sum_{m\leqq y}\Big|\sum_{2N/y<p^2\leqq\min(4N/y,\,u/m)} e(mp^2a/q)\Big| \leqq$$

$$\leqq y^{1/2}\Big(\sum_{m\leqq y}\Big|\sum_{2N/y<p^2\leqq\min(4N/y,\,u/m)} e(mp^2a/q)\Big|^2\Big)^{1/2} =$$

$$= y^{1/2}\Big(\sum_{m\leqq y}\sum_{2N/y<p_1^2,p_2^2\leqq\min(4N/y,\,u/m)} e\big(m(p_1^2-p_2^2)a/q\big)\Big)^{1/2} =$$

$$= y^{1/2}\Big(\sum_{|n|<2N/y}\sum_{\substack{2N/y<p_1^2,p_2^2\leqq 4N/y\\ p_1^2-p_2^2=n}}\sum_{m\leqq\min\left(y,\,\frac{u}{p_1^2},\,\frac{u}{p_2^2}\right)} e(mna/q)\Big)^{1/2} \leqq$$

$$\leqq y^{1/2}\Big(\sum_{|n|<2N/y}\sum_{\substack{2N/y<p_1^2,p_2^2\leqq 4N/y\\ p_1^2-p_2^2=n}}\Big|\sum_{m\leqq\min\left(y,\,\frac{u}{p_1^2},\,\frac{u}{p_2^2}\right)} e(mna/q)\Big|\Big)^{1/2} \leqq$$

$$\leqq y^{1/2}\Big(\sum_{|n|<2N/y}\sum_{\substack{p_1^2,p_2^2\leqq 4N/y\\ p_1^2-p_2^2=n}}\min\left(\min\left(y,\,\frac{u}{p_1^2},\,\frac{u}{p_2^2}\right),\,\frac{1}{2\|na/q\|}\right)\Big)^{1/2} \leqq$$

$$\leqq y^{1/2}\Big(\sum_{|n|<2N/y}\Big(\sum_{\substack{p_1^2,p_2^2\leqq 4N/y\\ p_1^2-p_2^2=n}}1\Big)\min\left(y,\,\frac{1}{2\|na/q\|}\right)\Big)^{1/2} \leqq$$

$$\leqq y^{1/2}\Big(y(4N/y)^{1/2}+2\sum_{0<n<2N/y}\Big(\sum_{\substack{0<u,v\\ u^2-v^2=n}}1\Big)\min\left(y,\,\frac{1}{2\|na/q\|}\right)\Big)^{1/2}.$$

If

$$u^2-v^2 = (u-v)(u+v) = n(>0) \quad \text{and} \quad 0<u, v \tag{22}$$

then $d=u+v|n$. On the other hand, if $d|n$ then there exists at most one pair u, of integers satisfying $d=u+v$ and (22) (in fact, the equations $d=u+v$, $\frac{n}{d}=u-$ determine u, v uniquely). Thus for $n>0$ we have

$$\sum_{\substack{0<u,v\\ u^2-v^2=n}} 1 \leqq \tau(n). \tag{23}$$

By using again Cauchy's inequality and Lemmas 4 and 5, and in view of (9), (10) and (23), we obtain from (21) that

$$|S_u(a/q)| \leqq y^{1/2}\left((4Ny)^{1/2}+2\sum_{0<n<2N/y}\tau(n)\min\left(y,\frac{1}{2\|na/q\|}\right)\right)^{1/2} \leqq$$

$$\leqq y^{1/2}\left\{(4Ny)^{1/2}+2\left(\left(\sum_{0<n<2N/y}(\tau(n))^2\right)\left(\sum_{0<n<2N/y}\left(\min\left(y,\frac{1}{2\|na/q\|}\right)\right)^2\right)\right)^{1/2}\right\}^{1/2} <$$

$$< y^{1/2}\left\{2(Ny)^{1/2}+2\left(c_3\frac{2N}{y}\left(\log\frac{2N}{y}\right)^3\left(2\frac{\frac{2N}{y}y^2}{q}+2y^2+3\frac{2N}{y}y+3qy\right)\right)^{1/2}\right\}^{1/2} <$$

$$< y^{1/2}\left\{2(Ny)^{1/2}+2\left(c_3\frac{2N}{y}(\log N)^3\left(4\frac{Ny}{N/Q}+2y^2+6N+3Qy\right)\right)^{1/2}\right\}^{1/2} <$$

$$< y^{1/4}\left\{2N^{1/2}y+(8c_3N(\log N)^3(7Qy+N+6N))^{1/2}\right\}^{1/2} =$$

$$= y^{1/4}\left\{2N^{1/2}y+\left(56c_3N^2(\log N)^{8/5}y^{1/5}\left(1+\left(\frac{(\log N)^7}{y}\right)^{1/5}\right)\right)^{1/2}\right\}^{1/2} <$$

$$< y^{1/4}\left\{2N+(56c_3N^2(\log N)^{8/5}y^{1/5}(1+c_4^{1/5}))^{1/2}\right\}^{1/2} <$$

$$< y^{1/4}\left\{\left(\frac{1}{10}+1\right)(56c_3N^2(\log N)^{8/5}y^{1/5}(1+1))^{1/2}\right\}^{1/2} =$$

$$= N^{1/2}(\log N)^{2/5}y^{3/10}\left(\frac{121\cdot 56\cdot 2}{100}c_3\right)^{1/4} < 4c_3N^{1/2}(\log N)^{2/5}y^{3/10}.$$

LEMMA 10. *If α is a real number and a, q are integers such that $N/Q<q\leqq Q$, $(a, q)=1$ and $\left|\alpha-\frac{a}{q}\right|<\frac{1}{qQ}$ then we have*

$$|S(\alpha)| < 56c_3N^{1/2}(\log N)^{2/5}y^{3/10}.$$

PROOF. We write $\beta=\alpha-\frac{a}{q}$ so that

$$|\beta|=\left|\alpha-\frac{a}{q}\right|<\frac{1}{qQ}\leqq\frac{1}{\frac{N}{Q}\cdot Q}=\frac{1}{N}.$$

The by using Lemma 9 and the well-known inequality $|1-e(\beta)|\leqq 2\pi|\beta|$, we obtain by partial summation that

$$|S(\alpha)|=|\sum_{n=1}^{2N}(S_n(a/q)-S_{n-1}(a/q))e(n\beta)|=$$

$$=|\sum_{n=1}^{2N}S_n(a/q)(e(n\beta)-e((n+1)\beta))+S_N(a/q)e((2N+1)\beta)|\leqq$$

$$\leqq\sum_{n=1}^{2N}|S_n(a/q)||1-e(\beta)|+|S_N(a/q)|\leqq 4c_3N^{1/2}(\log N)^{2/5}y^{3/10}\left(\sum_{n=1}^{2N}2\pi\frac{1}{N}\cdot+1\right)\leqq$$

$$\leqq 56c_3N^{1/2}(\log N)^{2/5}y^{3/10}.$$

LEMMA 11. *If*

$$\frac{1}{Q}<\alpha<1-\frac{1}{Q} \tag{24}$$

then we have

$$|S(\alpha)|<56c_3N^{1/2}(\log N)^{215}y^{3/10}. \tag{25}$$

PROOF. By Dirichlet's theorem, there exist integers a, q such that $1\leqq q\leqq Q$, $(a, q)=1$ and

$$\left|\alpha-\frac{a}{q}\right|<\frac{1}{qQ}.$$

(24) implies that $q>1$. If $2\leqq q\leqq N/Q$ then in view of (7), (25) is a consequence of Lemma 8, while if $N/Q<q\leqq Q$ then (25) holds by Lemma 10.

LEMMA 12. *If n is a positive integer satisfying $U\leqq n\leqq 2N$ then we have*

$$v_n>\frac{N^{1/2}}{y^{1/2}\log N}. \tag{26}$$

PROOF. The inequality

$$v_n\geqq K \tag{27}$$

holds by Lemma 10 in [2]. (11) and (27) yield (26).

6.

In this section, we complete the proof of the theorem. By using (7), (10), Lemmas 1, 6, 11, Cauchy's inequality and Parseval's formula, we obtain for large N that

$$\left|J-\frac{1}{U}\int_0^1 F(\alpha)G(\alpha)S(-\alpha)U(-\alpha)\,d\alpha\right| = \tag{28}$$

$$= \left|\int_{-1/Q}^{+1/Q} F(\alpha)G(\alpha)S(-\alpha)\left(1-\frac{U(-\alpha)}{U}\right)d\alpha+\right.$$

$$\left.+\int_{+1/Q}^{1-1/Q} F(\alpha)G(\alpha)S(-\alpha)\left(1-\frac{U(-\alpha)}{U}\right)d\alpha\right| \leqq$$

$$\leqq \int_{-1/Q}^{+1/Q} |F(\alpha)|\,|G(\alpha)|\,|S(-\alpha)|\left|\frac{U-U(\alpha)}{U}\right|d\alpha+$$

$$+\int_{+1/Q}^{1-1/Q} |F(\alpha)|\,|G(\alpha)|\,|S(-\alpha)|\left(1+\left|\frac{U(-\alpha)}{U}\right|\right)d\alpha \leqq$$

$$\leqq \int_{-1/Q}^{+1/Q} F(\alpha)|\,|G(\alpha)|\,S\frac{4U^2|\alpha|}{U}\,d\alpha+$$

$$+\int_{+1/Q}^{1-1/Q} |F(\alpha)|\,|G(\alpha)|\,\Big(\max_{1/Q<\beta<1-1/Q}|S(\beta)|\Big)\,2d\alpha \leqq$$

$$\leqq \int_{-1/Q}^{+1/Q} |F(\alpha)|\,|G(\alpha)|\,2\frac{(Ny)^{1/2}}{\log N}\cdot 4U\cdot\frac{1}{Q}\,d\alpha+$$

$$+\int_{+1/Q}^{1-1/Q} |F(\alpha)|\,|G(\alpha)|\,56c_3N^{1/2}(\log N)^{2/5}y^{3/10}\,2d\alpha <$$

$$< \left(8\frac{(Ny)^{1/2}}{\log N}\,9\,\frac{N}{y}\,\frac{(\log N)^{7/5}y^{4/5}}{N}+112c_3N^{1/2}(\log N)^{2/5}y^{3/10}\right)\int_0^1 |F(\alpha)|\,|G(\alpha)|\,d\alpha \leqq$$

$$\leqq (72+112c_3)\,N^{1/2}(\log N)^{2/5}y^{3/10}\left\{\left(\int_0^1 |F(\alpha)|^2\,d\alpha\right)\left(\int_0^1 |G(\alpha)|^2 d\alpha\right)\right\}^{1/2} <$$

$$< 200c_3\,N^{1/2}(\log N)^{2/5}y^{3/10}(AB)^{1/2}.$$

Furthermore, by (1), (2) and Lemma 12, and since $h_n \geqq 0$ and $v_n \geqq 0$, we have

$$(29)\quad \int_0^1 F(\alpha)G(\alpha)S(-\alpha)U(-\alpha)\,d\alpha = \int_0^1 \Big(\sum_{n=N+1}^{2N} h_n e(n\alpha)\Big)\Big(\sum_{n=1}^{2N+U-1} v_n e(-n\alpha)\Big)\,d\alpha =$$

$$= \sum_{n=N+1}^{2N} h_n v_n \geqq \sum_{n=N+1}^{2N} h_n \frac{N^{1/2}}{y^{1/2}\log N} = \frac{N^{1/2}}{y^{1/2}\log N} \sum_{n=N+1}^{2N} h_n =$$

$$= \frac{N^{1/2}}{y^{1/2}\log N} \sum_{\substack{N<a+b\leqq 2N \\ a\in\mathscr{A},\, b\in\mathscr{B}}} 1 = \frac{N^{1/2}}{y^{1/2}\log N}\Big(\sum_{a\in\mathscr{A}}\Big)\Big(\sum_{b\in\mathscr{B}} 1\Big) = \frac{N^{1/2}AB}{y^{1/2}\log N}.$$

(28) and (29) yield with respect to (7), (8) and (10), that

$$|J| \geqq \frac{1}{U}\Big|\int_0^1 F(\alpha)\,G(\alpha)\,U(-\alpha)\,S(-\alpha)\,d\alpha\Big| - 200c_3 N^{1/2}(\log N)^{2/5}\,y^{3/10}\,(AB)^{1/2} \geqq$$

$$\geqq \frac{1}{U}\frac{N^{1/2}AB}{y^{1/2}\log N} - 200c_3 N^{1/2}(\log N)^{2/5}\,y^{3/10}(AB)^{1/2} >$$

$$> \frac{1}{9N/y}\frac{N^{1/2}AB}{y^{1/2}\log N} - 200c_3 N^{1/2}(\log N)^{2/5}\,y^{3/10}(AB)^{1/2} =$$

$$= \frac{1}{9}N^{1/2}(\log N)^{2/5}\,y^{3/10}(AB)^{1/2}\left(\frac{(AB)^{1/2}\,y^{1/5}}{N(\log N)^{7/5}} - 1800c_3\right) =$$

$$= \frac{1}{9}N^{1/2}(\log N)^{2/5}\,y^{3/10}(AB)^{1/2}(c_4^{1/5} - 1800c_3) =$$

$$\geqq \frac{1}{9}N^{1/2}(\log N)^{2/5}\,y^{3/10}(AB)^{1/2}(2000c_3 - 1800c_3) =$$

$$= 200c_3 N^{1/2}(\log N)^{2/5}\,y^{3/10}(AB)^{1/2} > 0$$

which proves (16) and this completes the proof of our theorem.

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A SIMPLE PROOF FOR AN INEQUALITY

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In [1] the inequalities

$$(1)\quad \left((1-x)^{-\alpha-1}(1+x)^{-\beta-1}\right)^{(2n)} > 0 \quad (\alpha, \beta \geqq -1;\quad -1 < x < 1;\quad n = 0, 1, 2, \ldots),$$

$$(2)\quad \left(x^{-\alpha-1}e^{x}\right)^{(2n)} > 0 \quad (\alpha \geqq -1;\ x > 0;\ n = 0, 1, 2, \ldots)$$

were proved (cf. [1] and [2 p. 319]). These inequalities played an important role in the proof of some extremal problems investigated in [1] and [4]. C. Balázs [4] gave a new proof for (1) and (2) using the integral representation of the gamma function. In the same paper [4] she gave an elementary proof for a result given in the book of S. Karlin and W. J. Studden [3] and proved some new results of similar nature using the estimates (1) and (2).

The aim of this note is to give another, more elementary proof for (1) and (2).

First we prove (1). Using

$$(3)\quad (1+x)^{\gamma} = \sum_{k=0}^{\infty} \frac{\gamma(\gamma-1)\ldots(\gamma-k+1)}{k!} x^k \quad (|x| < 1,\ \gamma \text{ real};\ [2, \text{p. } 244]),$$

and taking into consideration $(1+x)^{-\beta} = 2^{-\beta}\left(1+(-1)\frac{1-x}{2}\right)^{-\beta}$, we obtain

$$(1-x)^{-\alpha}(1+x)^{-\beta} = 2^{-\alpha-\beta} \sum_{k=0}^{\infty} \frac{\beta(\beta+1)\ldots(\beta+k-1)}{k!} \left(\frac{1-x}{2}\right)^{k-\alpha},$$

and hence

(4)

$$\left((1-x)^{-\alpha}(1+x)^{-\beta}\right)^{(n)} = \frac{(-1)^n}{2^{\beta}(1-x)^{\alpha+n}} \sum_{k=0}^{\infty} \frac{\beta\ldots(\beta+k-1)(k-\alpha)\ldots(k-\alpha-n+1)}{k!} \times$$

$$\times \left(\frac{1-x}{2}\right)^{k} = \frac{(-1)^n}{2^{\beta}(1-x)^{\alpha+n}} \sum_{k=0}^{\infty} C_k \left(\frac{1-x}{2}\right)^{k}.$$

Now suppose that n is even, further $\alpha > 0$ and $\beta > 0$ are not integers. It is easy to see that $C_k < 0$ implies $k-\alpha-n+1 < 0 < k-\alpha$, i.e. $\alpha < k < \alpha+n-1$, further

$$C_k < 0 \quad \text{for} \quad k = [\alpha]+2j-1 \quad \left(j = 1, 2, \ldots, \frac{n}{2}\right).$$

It is enough to prove that

$$\sum_{k=[\alpha]}^{[\alpha]+n} C_k \left(\frac{1-x}{2}\right)^k > 0 \quad (-1 < x < 1). \tag{5}$$

To this end it is enough to remark that

$$C_{[\alpha]} \left(\frac{1-x}{2}\right)^{[\alpha]} > 0, \quad C_{[\alpha]+n} \left(\frac{1-x}{2}\right)^{[\alpha]+n} > 0,$$

further for some s $([\alpha] \leqq s \leqq [\alpha]+n)$;

$$|C_s| \left(\frac{1-x}{2}\right)^s \leqq \ldots \leqq |C_{[\alpha]}| \left(\frac{1-x}{2}\right)^{[\alpha]}; \quad |C_s| \left(\frac{1-x}{2}\right)^s \leqq \ldots \leqq |C_{[\alpha]+n}| \left(\frac{1-x}{2}\right)^{[\alpha]+n}.$$

This follows from the fact that

$$\frac{|C_k| \left(\frac{1-x}{2}\right)^k}{|C_{k-1}| \left(\frac{1-x}{2}\right)^{k-1}} = \frac{1-x}{2k} \left|\frac{(\beta+k-1)(k-\alpha)}{k-\alpha-n}\right|$$

monotone increases in k in the interval $[\alpha] < k \leqq [\alpha]+n$. (Indeed, it is easy to see that

$$\frac{|C_{k-1}| \left(\frac{1-x}{2}\right)^{k-1}}{|C_k| \left(\frac{1-x}{2}\right)^k} = \frac{2}{1-x} \left(\frac{n}{1-\frac{\alpha}{k}} - k\right) \frac{1}{\beta+k-1}$$

is decreasing in the interval $[\alpha] < k \leqq [\alpha]+n$.) Thus (1) is proved for non-integer α and β. If α and β are integers, then we approximate them by non-integer α and β, thus the estimate (1) follows. The proof of (2) is analogous, only instead of (3) we use

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{6}$$

and hence

$$(x^{-\alpha} e^x)^{(n)} = \sum_{k=0}^{\infty} \frac{(k-\alpha)\ldots(k-\alpha-n+1)}{k!} x^{k-\alpha-n} = \tag{7}$$

$$= \frac{1}{x^{\alpha+n}} \sum_{k=0}^{\infty} \frac{(k-\alpha)\ldots(k-\alpha-n+1)}{k!} x^k = \frac{1}{x^{\alpha+n}} \sum_{k=0}^{\infty} C_k x^k.$$

Now suppose that n is even, further $\alpha > 0$ is not integer. It is easy to see that $C_k < 0$ implies $k-\alpha-n+1 < 0 < k-\alpha$, i.e. $\alpha < k < \alpha+n-1$, further

$$C_k < 0 \quad \text{for} \quad k = [\alpha]+2j-1 \quad \left(j = 1, 2, \ldots, \frac{n}{2}\right).$$

It is enough to prove that

$$\sum_{k=[\alpha]}^{[\alpha]+n} C_k x^k > 0 \quad (x > 0). \tag{8}$$

To this end it is enough to remark that

$$C_{[\alpha]} x^{[\alpha]} > 0, \quad C_{[\alpha]+n} x^{[\alpha]+n} > 0,$$

further for some s $([\alpha] \leqq s \leqq [\alpha]+n)$:

$$|C_s| x^s \leqq \ldots \leqq |C_{[\alpha]}| x^{[\alpha]}; \quad |C_s| x^s \leqq \ldots \leqq |C_{[\alpha]+n}| x^{[\alpha]+n}.$$

This follows from the fact that

$$\frac{|C_k| x^k}{|C_{k-1}| x^{k-1}} = (k-\alpha)\, x$$

increases in k in the interval $[\alpha] < k \leqq [\alpha]+n$. (Indeed, it is easy to see that

$$\frac{|C_{k-1}| x^{k-1}}{|C_k| x^k} = \frac{1}{x} \frac{1}{k-\alpha}$$

is monotone decreasing in the interval $[\alpha] < k \leqq [\alpha]+n$.) Thus (2) is proved.

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DIFFERENTIABLE RESTRICTIONS OF CONTINUOUS FUNCTIONS

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1.

The surprising theorem stating that any continuous function has a differentiable restriction was proved by Bruckner, Ceder and Weiss in [1]. More exactly, they showed that if $P \subset \mathbf{R}$ is perfect and $f\colon P \to \mathbf{R}$ is continuous, then there is a perfect subset $Q \subset P$ such that the finite or infinite derivative

$$(f|_Q)'(x) = \lim_{y \to x,\, y \in Q} \frac{f(y)-f(x)}{y-x}$$

exists at each point of Q. The infinite derivatives cannot be excluded: it is easy to construct a continuous function f on a perfect set P such that $f'(x) = +\infty$ holds at each point of P and then $(f|_Q)'(x) = +\infty$ holds also for every perfect subset Q and $x \in Q$.

Our aim is to show that, if $\lambda(P) > 0$ (λ denotes the Lebesgue measure) then f has a differentiable restriction with *finite* derivatives. Moreover, in this case the perfect subset Q can be selected in a special way: either we can suppose $(f|_Q)'(x) \equiv 0$, or we can choose Q to be large. In the sequel by the term "differentiable" we shall always mean "differentiable with finite derivative".

THEOREM 1. *Let $P \subset \mathbf{R}$ be perfect, $\lambda(P) > 0$ and let $f\colon P \to \mathbf{R}$ be continuous. Then one of the following assertions is true.*

(1) *There is a non-empty perfect subset $Q \subset P$ such that $(f|_Q)'(x) = 0$ for every $x \in Q$.*

(2) *There is an everywhere dense and open (relative to P) subset $U \subset P$ such that f is differentiable a.e. in U. In addition, for every $\varepsilon > 0$ there is a perfect subset $Q \subset P$ such that $\lambda(P-Q) < \varepsilon$ and $f|_Q$ is differentiable at each point of Q.*

LEMMA 2. *Let f be continuous on the perfect set $P \subset \mathbf{R}$ and let A denote the set of points $x \in P$ for which there is a sequence $x_n \in P - \{x\}$, $x_n \to x$ such that $\lim_{n\to\infty} \frac{f(x_n)-f(x)}{x_n - x} = 0$. Suppose that A is everywhere dense in P. Then (1) holds.*

PROOF. Let $p(n)$ denote the greatest integer k such that $2^k | n$. Then $0 \leqq p(n) < n$ for every $n = 1, 2, \ldots$ and for each non-negative integer k there are infinitely many natural n with $p(n) = k$.

Let $x_0 \in A$ be fixed. We put $\delta_1 = 1$ and choose a point $x_1 \in A$ with $0 < |x_0 - x_1| \leqq 1$ and

$$\left| \frac{f(x_1)-f(x_0)}{x_1 - x_0} \right| < 1.$$

Let $n \geqq 2$ be given and suppose that the points $x_0, x_1, \ldots, x_{n-1} \in A$ and the positive numbers $\delta_1, \ldots, \delta_{n-1}$ have been defined such that

$$\left|\frac{f(x_i)-f(x_j)}{x_i-x_j}\right| < \frac{1}{k}$$

whenever $0 \leqq i < j < n$, $1 \leqq k < n$ and $|x_i - x_j| \leqq \delta_k$. Then we put

$$\delta_n = \frac{1}{2} \min \{|x_i - x_j|;\ 0 \leqq i < j < n\}.$$

There is an $\eta > 0$ such that $|x - x_{p(n)}| < \eta$ implies $|x - x_i| > \delta_k$ whenever $0 \leqq i < n$, $i \neq p(n)$, $1 \leqq k < n$ and $|x_{p(n)} - x_i| > \delta_k$. Since f is continuous, we can select $0 < \eta' < \eta$ such that

$$\left|\frac{f(x)-f(x_i)}{x-x_i}\right| < \frac{1}{k}$$

for $0 \leqq i < n$, $i \neq p(n)$, $1 \leqq k < n$, $0 < |x - x_i| \leqq \delta_k$, $|x - x_{p(n)}| < \eta'$.

Now we choose a point $x_n \in A$ such that

$$0 < |x_n - x_{p(n)}| < \min\left(\eta', \delta_n, \frac{1}{n}\right) \tag{3}$$

and

$$\left|\frac{f(x_n)-f(x_{p(n)})}{x_n-x_{p(n)}}\right| < \frac{1}{n}.$$

This is possible since $x_{p(n)} \in A$, A is dense in P and f is continuous.

In this way we have defined the sequences $S = \{x_n\}_{n=0}^{\infty} \subset A$ and $\{\delta_n\}_{n=1}^{\infty} \subset \mathbf{R}^+$ such that $x, y \in S$ and $0 < |x - y| \leqq \delta_n$ imply $\left|\frac{f(x)-f(y)}{x-y}\right| < \frac{1}{n}$ $(n = 1, 2, \ldots)$. Let Q be the closure of S. It follows from (3) that S is dense in itself and hence Q is perfect. By the continuity of f, $x, y \in Q$ and $0 < |x - y| < \delta_n$ imply $\left|\frac{f(x)-f(y)}{x-y}\right| \leqq \frac{1}{n}$ and hence $(f|_Q)'(x) = 0$ for each $x \in Q$ and the lemma is proved.

Proof of Theorem 1. Let A denote the set defined in Lemma 2. If A is dense in a portion of P then, applying Lemma 2 for that portion, we get (1). So we can suppose that A is nowhere dense in P. Let $U = P - \operatorname{cl} A$ ($\operatorname{cl} A$ denotes the closure of A), then U is dense and open in P. We show that f is a.e. differentiable in U. Let G denote the graph of the function f restricted to U. Since $U \cap A = \emptyset$, the contingent of G at any point $x \in G$ does not contain a half-plane (see [6], pp. 262 and 263). It follows then, that there is a subset $H_1 \subset G$ of length zero such that G has a unique tangent at each point of $G - H_1$ ([6], ch. IX, Theorem 3.6, p. 266). Now let H_2 denote the set of points $x \in G - H_1$ at which the tangent of G is parallel to the y-axis. Let K_i denote the projection of H_i on the x-axis $(i = 1, 2)$, then f is differentiable at every point of $U - (K_1 \cup K_2)$. We have $\lambda(K_2) = 0$ by [6], Ch. IX, Theorem 3.7 (p. 267) and $\lambda(K_1) = 0$ since H_1 is of length zero. Thus f is differentiable a.e. in U which is the first half of (2).

Now let H denote the class of subsets $B\subset P$ satisfying the following conditions: B is measurable, $\lambda(B)>0$ and for every $\varepsilon>0$ there is a perfect subset $Q\subset B$ such that $\lambda(B-Q)<\varepsilon$ and $f|_Q$ is differentiable at each point of Q. In order to finish the proof we have to show that either (1) holds or $P\in H$.

Let $H_1\subset H$ be a maximal disjoint subfamily of H. Since $\lambda(B)>0$ for every $B\in H$, H_1 is (at most) countable; let $H_1=\{B_n\}_{n=1}^{\infty}$ and put $B=\bigcup_{n=1}^{\infty} B_n$. We show $B\in H$. Let $\varepsilon>0$ be given and let $\{Q_n\}_1^{\infty}$ be a sequence of perfect sets such that $Q_n\subset B_n$, $f|_{Q_n}$ is differentiable on Q_n $(n=1, 2, ...)$ and $\lambda\left(\bigcup_1^{\infty}(B_n-Q_n)\right)<\varepsilon/2$. Then for every sequence of indices $\{n_k\}_{k=-\infty}^{+\infty}$ the set $Q_0=\bigcup_{k=-\infty}^{\infty}\bigcup_{n=1}^{n_k} Q_n\cap[k, k+1]$ is closed and $f|_{Q_0}$ is differentiable on Q_0. In addition, if we choose n_k large enough, then $\lambda(B-Q_0)<\varepsilon$ will also hold. Then we can take a perfect $Q\subset Q_0$ with $\lambda(B-Q)<\varepsilon$ showing $B\in H$.

Now assuming that (1) is not true, we shall prove $\lambda(P-B)=0$. Suppose $\lambda(P-B)>0$, then there is a perfect subset $P_1\subset P-B$ such that any portion of P_1 is of positive measure. Since (1) does not hold for $f|_{P_1}$ hence, by our preceding argument, there exists a dense open subset $U_1\subset P_1$ such that $f|_{P_1}$ is a.e. differentiable in U_1. This easily implies $U_1\in H$ which contradicts the maximality of H_1. Hence $\lambda(P-B)=0$, $P\in H$ and this completes the proof of Theorem 1.

2.

In this section we prove some consequences of Theorem 1. First of all, applying Lusin's theorem, we immediately obtain that for any measurable function f: $H\to\mathbf{R}$ defined on the measurable set H with $\lambda(H)>0$, there exists a perfect subset $Q\subset H$ such that $f|_Q$ is differentiable. Moreover, as we shall prove, we can find Q such that $f|_Q$ is *infinitely* derivable, i.e. $(f|_Q)^{(n)}$ eixsts on Q for every $n=1, 2, ...$

THEOREM 3. *Let f: $H\to\mathbf{R}$ be a measurable function defined on the measurable set $H\subset\mathbf{R}$ with $\lambda(H)>0$. Then either*

(4) *there exists a perfect subset $Q\subset H$ such that $f|_Q$ is infinitely differentiable on Q and $(f|_Q)^{(n)}\equiv 0$ for n sufficiently large, or*

(5) *for every $\varepsilon>0$ there exists a perfect subset $Q\subset H$ such that $\lambda(H-Q)<\varepsilon$ and $f|_Q$ is infinitely differentiable on Q.*

PROOF. Suppose that (4) is not true and let $0<\varepsilon<\lambda(H)$ be given. Let $P\subset H$ be a perfect set such that $\lambda(H-P)<\varepsilon/8$ and $f|_P$ is continuous. Since (1) implies (4) therefore, by Theorem 1, (2) holds tru for $f|_P$ and thus we can find a perfect set $Q_1\subset P$ such that $\lambda(P-Q_1)<\varepsilon/8$ and $f|_{Q_1}$ is differentiable on Q_1. Suppose that $n\geqq 1$ and the perfect subset $Q_n\subset H$ is defined such that $\lambda(H-Q_n)<\left(\frac{3}{4}-\frac{1}{2^n}\right)\varepsilon$ and $f|_{Q_n}$ is n times differentiable on Q_n. We put $g(x)=(f|_{Q_n})^{(n)}(x)$ $(x\in Q_n)$. Then g is Baire 1 on Q_n (see [5], Lemma 5.5.1, p. 208) and hence there is a perfect subset $P_n\subset Q_n$ such that $\lambda(Q_n-P_n)<\frac{1}{2^{n+2}}\varepsilon$ and $g|_{P_n}$ is continuous. Applying Theorem 1 to $g|_{P_n}$

we can see that (2) holds true (because (1) would imply (4) again) and hence there is a perfect subset $Q_{n+1}\subset P_n$ with $\lambda(P_n-Q_{n+1})<\frac{1}{2^{n+2}}\varepsilon$ such that $g|_{Q_{n+1}}$ is differentiable (i.e. $f|_{Q_{n+1}}$ is $(n+1)$ times differentiable) on Q_{n+1}. In this way we have defined the sequence of perfect sets Q_n. Let $Q_0=\bigcap_{n=1}^{\infty} Q_n$, then Q_0 is closed and $\lambda(H-Q_0)\leqq \leqq\frac{3}{4}\varepsilon$. Choosing a perfect subset $Q\subset Q_0$ with $\lambda(Q)=\lambda(Q_0)$ we can easily check that (5) is satisfied which completes the proof.

The following corollaries are immediate consequences of Theorems 1 and 3 taking into consideration the simple fact that if $Q\subset H$, $f|_Q$ is differentiable and $x\in Q$ is a density point of Q then f is approximately differentiable in x.

COROLLARY 4. *Let f be measurable on the measurable set $H\subset\mathbf{R}$ with $\lambda(H)>0$. Then either there is a perfect $Q\subset H$ such that $(f|_Q)'(x)=0$ for every $x\in Q$, or f is a.e. approximately differentiable in H.*

COROLLARY 5. *Let f be measurable on the measurable set $H\subset\mathbf{R}$ with $\lambda(H)>0$. Then either there is a perfect set $Q\subset H$ such that $f|_Q$ is infinitely differentiable on Q and $(f|_Q)^{(n)}\equiv 0$ for n large enough, or f is a.e. infinitely approximately differentiable in H.*

3.

If the set Q is not connected, then the condition $(f|_Q)'\equiv 0$ does not imply that f is constant on Q. This raises the question whether it is possible to sharpen Theorem 1 by substituting (1) by the stronger statement that $f|_Q$ is constant.

In this section we show that the answer is generally negative, but we can prove something more than Theorem 1 in the case when P is an interval.

THEOREM 6. *Let P be a nowhere dense perfect set with $\lambda(P)>0$. Then there exists a continuous function $f\colon P\to\mathbf{R}$ which is one-to-one on P and approximately differentiable at no density point of P. (This implies that there is no perfect set $Q\subset P$ such that $f|_Q$ is constant and there is no perfect set $Q\subset P$ such that $\lambda(Q)>0$ and $f|_Q$ is differentiable.)*

Our proof is based on the following

LEMMA 7. *Let $P\subset[a,b]$ be a nowhere dense, closed set. Then the family F of those continuous functions $f\colon[a,b]\to\mathbf{R}$ which are one-to-one on P is a residual subset of the space $C[a,b]$.*

PROOF. First we show that F is dense in $C[a,b]$. Let $f\in C[a,b]$ and $\varepsilon>0$ be given. We shall construct a function $g\in F$ with $|g-f|<\varepsilon$. Let $\delta>0$ be such that $x,y\in[a,b]$ and $|x-y|<\delta$ imply $|f(x)-f(y)|<\varepsilon/4$ and let the points $a=x_0<\ldots<x_1<\ldots<x_n=b$ be chosen according to $x_i-x_{i-1}<\delta$ $(i=1,\ldots,n)$ and $x_i\notin P$ $(i=1,\ldots,n-1)$. We can find non-constant linear functions l_i such that

$$Z_i=l_i(P\cap[x_{i-1},x_i])\subset\left[f(x_i)-\frac{\varepsilon}{2},\ f(x_i)+\frac{\varepsilon}{2}\right]\quad(i=1,\ldots,n)$$

and $Z_i \cap Z_j = \emptyset\ (0 \leqq i < j \leqq n)$. Now we put $g(x) = l_i(x)$ $(x \in P \cap [x_{i-1}, x_i],\ i = 1, \ldots, n)$ and extend g to $[a, b]$ continuously such that

$$f(x_i) - \frac{\varepsilon}{2} \leqq g(x) \leqq f(x_i) + \frac{\varepsilon}{2} \quad (x \in [x_{i-1}, x_i], \quad i = 1, \ldots, n).$$

(Since $|f(x_i) - f(x_{i-1})| < \frac{\varepsilon}{4}$, this is possible.) Then $g \in F$ and for $x \in [x_{i-1}, x_i]$ we have

$$|g(x) - f(x)| \leqq |g(x) - f(x_i)| + |f(x_i) - f(x)| \leqq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} < \varepsilon.$$

Let $H_k = \{f \in C[a, b]$; there exist $x, y \in P$ with $|x - y| \geqq \frac{1}{k}$ and $f(x) = f(y)\}$. It is easy to see that H_k is closed in $C[a, b]$ for every $k = 1, 2, \ldots$, and $F = C[a, b] - \bigcup_{k=1}^{\infty} H_k$. Therefore F is a dense G_δ set in $C[a, b]$ and, a fortiori, F is residual.

PROOF OF THEOREM 6. We can suppose that P is bounded (otherwise we split P into bunded parts); let $P \subset [a, b]$. By a theorem of Jarník [3], the family G of those continuous functions on $[a, b]$ which are nowhere approximately differentiable is residual in $C[a, b]$. By Lemma 7 F is residual, too and hence so is $F \cap G$. It is obvious that for any function $f \in F \cap G$, $f|_P$ satisfies the requirements of Theorem 6.

Now we consider continuous functions defined on an interval. Our next proposition shows that in this case a part of the assertion of Theorem 1 can be sharpened.

PROPOSITION 8. *If $f \in C[a, b]$ then either there exists a perfect set $Q \subset [a, b]$ such that $f|_Q$ is constant, or there exists a dense open set $U \subset [a, b]$ such that f is a.e. differentiable in U.*

In fact, if f is not constant on any perfect set then f takes every value on a countable set. By a theorem of K. Padmavally, this implies that there is a dense, open subset $U \subset [a, b]$ such that f is monotonic in every component of U ([4], see also [2]), and hence f is a.e. differentiable in U.

The other part of the assertion of Theorem 1 states that either (1) holds or there are big perfect sets on which f is differentiable. We conclude with proving that (1) cannot be susbtituted here by the condition that $f|_Q$ is constant, even for continuous functions defined on an interval.

PROPOSITION 9. *For every $\varepsilon > 0$ there is a function $g \in C[a, b]$ such that g takes each of its values on a countable set and the set of points at which g is approximately differentiable is of measure smaller than ε.*

Indeed, let $P \subset [a, b]$ be a nowhere dense perfect set with $\lambda(P) > b - a - \varepsilon$ and let $f\colon P \to \mathbf{R}$ be a continuous function satisfying the conditions of Theorem 6. Let $g(x) = f(x)$ for $x \in P$ and let g be defined linearly in the intervals contiguous to P. It is easy to see that g satisfies the requirements of Proposition 9.

Proposition 8 implies that if $f \in C[a, b]$ takes every value on a countable set, then there is a perfect Q such that $f|_Q$ is differentiable and $\lambda(Q) > 0$. However, by Proposition 9, it is possible that $\lambda(Q) < \varepsilon$ holds for all these Q.

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A NOTE ON THE PROOF OF THE MAIN THEOREM IN „STRONG SUBBAND-SEPARATING EXTENSIONS OF ORTHODOX SEMIGROUPS"

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In this short note we call the attention to an error in the proof of a lemma in [1] which does not influence the validity of the only theorem applying it.

In the proof of Lemma 3.1 in [1] it is tacitly utilized that $s \neq s'$ for each pair of elements s, s' which are inverses of each other. Consider the additive group of integers and the congruence mod 2 on it. Clearly, this is a strong subband-separating congruence but there is no integer k congruent to 1 mod 2 such that $2k=0$. This shows that Lemma 3.1 in [1] is false in general. All we can prove by the method used in the original proof is the following:

LEMMA 3.1. *Let T be an orthodox semigroup with band of idempotents B. Suppose $\varkappa$ to be a $\bar{B}$-separating congruence on T. Denote $T/\varkappa$ by S. Then there exists a cross-section $\{u_s: s \in S,\ u_s \varkappa = s\}$ of the $\varkappa$-classes with the properties that* (i) $u_s \in S_B$ *for every* $s \in S$, (ii) u_e *is the greatest idempotent in the $\varkappa$-class e for each idempotent e in S,* (iii) $u_s \sim u_{\bar{s}}$ *provided $s, \bar{s}$ are elements in S with $s \sim \bar{s}$ and* (iv) $u_s u_{s'} = u_{ss'}$ *whenever s is a non-idempotent element in S such that s is not an inverse of itself and s' is an inverse of s.*

The only properties of the cross-section $\{u_s: s \in S,\ u_s \varkappa = s\}$ essentially utilized in the proof of Theorem 3.2, the main result of [1] are (i)—(iii). That means that by replacing $u_{s'}$ with the inverse u'_s of u_s with $u'_s \varkappa = s'$ and, of course, by making several obvious changes in the text according to this the proof of Theorem 3.2 becomes correct.

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NOTE ON MY PAPER "A SIMPLE PROOF FOR VON NEUMANN'S MINIMAX THEOREM"

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At present the proofs for the minimax principles avoiding Brower's fixed point theorem are essentially of two types. One method is based on the application of some separating theorems (cf. Parthasaraty [9] and Balakrishnan [10]). Recently a different treatment was developed in [2]—[7] which can be called the method of level sets, and which turns out to be more farreaching in several aspects than the previous one (as it is pointedout in the work of L. L. Stachó [5]).

The question naturally arises: what is the deeper connection between the two approaches?

One of the aims of this paper is to point out how the finite intersection property of the level sets follows from the Hahn—Banach theorem. Theorem 1 below is essentially a consequence of [5, Proposition 3] because convex (resp. concave) functions are automatically continuous in the interior of a straight line segment in a vector space. However our proof is more direct and bypasses the technical difficulty arising from the [5, Proposition 3] should be applied to $(X \setminus \operatorname{ext} X) \times (Y \setminus \operatorname{ext} Y)$ instead of $X \times Y$ immediately.

1. Let E and F be arbitrary vector spaces, and let $X \subset E$ and $Y \subset F$ be arbitrary convex sets. Let $f(x, y)$ be a real function on $X \times Y$ which is concave in x for any fixed $y \in Y$ and convex in y for any fixed $x \in X$. Denote $H_y^{(c)} = \{x \in X: f(x, y) \geqq c\}$ $(y \in Y)$ and $\mathscr{C}$ the set of real numbers c for which $H_y^{(c)} \neq \emptyset$ whenever $y \in Y$. Denote $c^* = \sup \mathscr{C}$.

THEOREM 1. *For any $c < c^*$ the system of sets $\{H_y^{(c)}, y \in Y\}$ has the finite intersection property.*

PROOF. First we show that for any $y_1, y_2 \in Y$ and $c < c^*$ we have $H_{y_1}^{(c)} \cap H_{y_2}^{(c)} \neq \emptyset$. If $c^* = -\infty$ then the statement of Theorem 1 is trivial, hence we may assume $c^* > -\infty$. Suppose $H_{y_1}^{(c)} \cap H_{y_2}^{(c)} = \emptyset$ for some $y_1, y_2 \in Y$ and $c < c^*$. Then there exists $\lambda \in [0, 1]$ such that $H_{y_\lambda}^{(c)} = \{x: f(x, y) = c\}$ for $y_\lambda \overset{\text{def}}{=} \lambda y_1 + (1-\lambda) y_2$. Indeed, suppose the contrary, i.e. for every $\lambda \in [0, 1]$ there exists $x \in X$ with $f(x, y_\lambda) > c$. Then, because of the convexity of f in y we have: for every $\lambda \in [0, 1]$ there exists $x \in X$ such that

$$\lambda f(x, y_1) + (1-\lambda) f(x, y_2) > c$$

i.e.

$$\big((f(x, y_1) - c, f(x, y_2) - c), (\lambda, 1-\lambda)\big) > 0.$$

Then, according to a classical separating theorem for disjoint convex sets in R^2 we obtain: there exist $x_1, x_2, \ldots, x_n \in X$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{R}_+$ with $\Sigma\lambda_i = 1$ such that

$$\sum_{i=1}^{n} \lambda_i (f(x_i, y_1) - c) \geqq 0, \quad \sum_{i=1}^{n} \lambda_i (f(x_i, y_2) - c) \geqq 0.$$

Since f is concave in x we obtain

$$f(x_\lambda, y_1) \geqq c, \quad f(x_\lambda, y_2) \geqq c \quad \left(x_\lambda \overset{\text{def}}{=} \sum \lambda_i x_i\right).$$

This means that $H_{y_1}^{(c)} \cap H_{y_2}^{(c)} \neq \emptyset$ which contradicts our indirect assumption. So we have proved the existence of $y_\lambda \in Y$ with the property $H_{y_\lambda}^{(c)} = \{x \in X: f(x, y_\lambda) = c\}$. Hence $c \geqq c^*$ follows and this contradicts $c < c^*$. We proved that $H_{y_1}^{(c)} \cap H_{y_2}^{(c)} \neq \emptyset$ if $y_1, y_2 \in Y$ and $c < c^*$. Suppose $\bigcap_{i=1}^{n} H_{y_i}^{(c)} \neq \emptyset$ if $n \leqq N$, $y_i \in Y$ and $c < c^*$. Let $y_1, y_2, \ldots, y_{N+1} \in Y$ and $c < c^*$ be arbitrary. We prove that $\bigcap_{1}^{N+1} H_{y_i}^{(c)} \neq \emptyset$. We know that $\tilde{X} \overset{\text{def}}{=} \bigcap_{1}^{N-1} H_{y_i}^{(c)} \neq \emptyset$ and the set $\tilde{X}$ is convex. We can repeat the proof above for $\tilde{f} = f|_{\tilde{X} \times Y}$ and $\tilde{H}_{y_N}^{(c)} = H_{y_N}^{(c)} \cap \tilde{X}$, $\tilde{H}_{y_{N+1}}^{(c)} = H_{y_{N+1}}^{(c)} \cap \tilde{X}$, and Theorem 1 follows by induction on N.

THEOREM 2. *Let X, Y be arbitrary sets and let $f: X \times Y \to \mathbf{R}$ be an arbitrary function. Then*

$$\left(B \overset{\text{def}}{=}\right) \sup_{x \in X} \inf_{y \in Y} f(x, y) = \inf_{y \in Y} \sup_{x \in X} f(x, y) \left(\overset{\text{def}}{=} J\right)$$

if and only if for every $c < c^*$ $\left(H^{(c)} \overset{\text{def}}{=}\right) \bigcap_{y \in Y} H_y^{(c)} \neq \emptyset$. In this case $B = J = c^*$.

PROOF. It is easy to see that $B \leqq J \leqq c^*$. Indeed $B \leqq J$ is trivial. If $c > c^*$ then there exists $y \in Y$ such that $f(x, y) < c$ for every $x \in X$, i.e. $J \leqq c$. This proves the inequality $J \leqq c^*$. Now let $c < c^*$ and $x_c \in H^{(c)}$. Then $f(x_c, y) \geqq c$ for every $y \in Y$, i.e. $B \geqq c$. Hence $B = J = c^*$ follows.

Now suppose $B = J$. Let $c < c^*$ and $\varepsilon > 0$ such that $c + \varepsilon < c^*$. This means that for every $y \in Y$ there exists $x \in X$ such that $f(x, y) \geqq c + \varepsilon$, i.e. $J \geqq c + \varepsilon$. This means (using $B = J$) that there exists $x \in X$ such that $f(x, y) \geqq c$ for every $y \in Y$, i.e. $\bigcap_{y \in Y} H_y^{(c)} \neq \emptyset$.

REMARK. It is easy to see that for an arbitrary function f we have $J = c^*$. $J \leqq c^*$ is proved above. Now we prove $J \geqq c^*$. Let $c < c^*$. Then for every $y \in Y$ there exists $x \in X$ such that $f(x, y) \geqq c$, hence $J \geqq c$ and the desired inequality $J > c^*$ follows.

From Theorems 1 and 2 we obtain immediately

THEOREM 3. *Suppose the conditions of Theorem 1 are fulfilled and E is a topological vector space, $X \subset E$ is a compact subset, further $f(x, y)$ is upper semi continuous in x for every fixed $y \in Y$. Then we have*

$$\inf_{y \in Y} \max_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).$$

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NOTE ON A PROBLEM OF Q. I. RAHMAN AND P. TURÁN

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1.

Q. I. Rahman and P. Turán began to investigate the following problem in 1972. Let

$$v(z) = \prod_{j=1}^{n} (z-\xi_j) = \sum_{k=0}^{n} B_k z^k \quad (B_k \in \mathbf{C}), \tag{1.1}$$

a polynomial of degree n, and

$$u(z) = \sum_{k=0}^{t} A_k z^k = A_0 \prod_{j=1}^{t} \left(1-\frac{z}{\eta_j}\right) \quad (A_k \in \mathbf{C}) \tag{1.2}$$

a polynomial of degree $t<n$. Consider the rational function

$$f(z)=\frac{u(z)}{v(z)} \tag{1.3}$$

of degree n, and suppose the normalizations

$$f(0) = 1 \tag{1.4}$$

and

$$0 < |\xi_1| \leqq \ldots \leqq |\xi_{n-1}| \leqq \xi_n = 1. \tag{1.5}$$

Then we are seeking for the best possible lower estimate of the integral mean

$$I^p(f, r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\varphi})|^p \, d\varphi \tag{1.6}$$

taken on the circle with radius r, assuming always the natural condition

$$(0 <)r < |\xi_1|. \tag{1.7}$$

Q. I. Rahman and P. Turán, based on heuristic arguments, conjectured that there is an extremal function which gives the least possible integral mean for all p and r considered, and this extremal function should be $\frac{1}{1-z^n}$. The conjecture is undecided at the moment. Rahman and Turán reached the following result.

THEOREM (Q. I. Rahman and P. Turán [1]). *If* $1<p\leqq 2$ *and the notations and normalizations* (1.1)—(1.7) *are used, then we have with* $q=\frac{p}{p-1}$ (the usual dual of p)

$$I^p(f,r)\geqq\left\{1+\frac{r^{qn}}{\left(\sum_{k=0}^{n-1}\binom{n}{k}^p r^{pk}\right)^{q/p}}\right\}^{p/q}. \tag{1.8}$$

For large r, (1.8) is obviously far from being optimal, but when $r\to 0$, it is asymptotically

$$\{1+r^{qn}(1+o(1))\}^{p-1}=1+(p-1)r^{qn}(1+o(1)). \tag{1.9}$$

On the other hand

$$\left(\frac{1}{1-z^n}\right)^{p/2}=(1+\sum_{l=1}^{\infty}z^{nl})^{p/2}=\sum_{k=0}^{\infty}\binom{p/2}{k}(\sum_{l=1}^{\infty}z^{nl})^k=$$

$$=1+\sum_{m=1}^{\infty}\left\{\sum_{k=1}^{m}\binom{p/2}{k}\binom{m-1}{k-1}\right\}z^{mn}\quad(|z|<2^{-1/n})$$

and so by the Parseval formula (for $r<2^{-\frac{1}{n}}$)

$$I^p\left(\frac{1}{1-z^n},r\right)=\frac{1}{2\pi}\int_0^{2\pi}\left|\left(\frac{1}{1-r^ne^{in\varphi}}\right)^{p/2}\right|^2 d\varphi=\sum_{m=0}^{\infty}\left\{\sum_{k=1}^{m}\binom{p/2}{k}\binom{m-1}{k-1}\right\}^2 r^{2mn}. \tag{1.10}$$

Hence we have the asymptotical expansion

$$I^p\left(\frac{1}{1-z^n},r\right)=1+\binom{p/2}{1}^2 r^{2n}+O(r^{4n})=1+\frac{p^2}{4}r^{2n}(1+o(1)), \tag{1.11}$$

which is equal to (1.9) in the central case $p=2$. When $1<p<2$, Rahman and Turán use the Young—Hausdorff inequality, and therefore they do not obtain a sharp asymptotic in this case.

A further reason to believe in the truth of the conjecture is an observation of Q. I. Rahman, namely

THEOREM (Q. I. Rahman [2]). *Supposing* (1.1)—(1.7) *we have*

$$\max_{0\leqq\varphi\leqq 2\pi}|f(re^{i\varphi})|\geqq\max_{|z|=r}\left|\frac{1}{1-z^n}\right|=\frac{1}{1-r^n}. \tag{1.12}$$

In other words, if $p=\infty$, then the conjecture is true.

On the other hand, if $p=1$ or $2<p<\infty$, there are no known non-trivial results similar to (1.8).

The following result shows that the conjecture is true for small values of r, for arbitrary $p\geqq 1$.

THEOREM 1. *For* $1 \leqq p < \infty$ *and for any* r *and* f *satisfying* (1.1)—(1.7), *there exists a radius* $r_0(p, f) > 0$ *such that for all* $r < r_0(p, f)$ *we have*

$$I^p(f, r) \geqq I^p\left(\frac{1}{1-z^n}, r\right). \tag{1.13}$$

If $f(z) \neq \frac{1}{1-z^n}$ *then there exists a radius* $r_1(p,f) > 0$ *such that for all* $r < r_1(p, f)$ *we have*

$$I^p(f, r) > I^p\left(\frac{1}{1-z^n}, r\right). \tag{1.14}$$

Furthermore, if there is a root of the denominator of f *which has absolute value less than* 1, *then we can take*

$$r_1(p,f) = r_1(n, p, |B_0|) \quad (< |\xi_1|). \tag{1.15}$$

COROLLARY. *If for some* p, *there exists an extremal function such that for all* $r < |\xi_1|$ *it has minimal* p^{th} *integral mean, then this function is* $\frac{1}{1-z^n}$.

2.

Rahman and Turán also investigated the important special case when f is a logarithmic derivative of a polynomial of degree n. Their result is stronger than the immeadiate consequence of (1.8) which would depend also on $|f(0)|$.

THEOREM (Q. I. Rahman and P. Turán [1]). *If* $v(z)$ *is a polynomial satisfying* (1.1) *and* (1.5), $1 < p \leqq 2$ *and* r *is chosen according to* (1.7), *then*

$$I^p\left(\frac{v'}{v}, r\right) \geqq \frac{n^p r^{p(n-1)}}{1 + r^p + \sum_{k=2}^{n-1} \binom{n}{k}^p r^{kp}}. \tag{2.1}$$

Here the estimate is asymptotically

$$n^p r^{p(n-1)}(1 + o(1)) \tag{2.2}$$

as $r \to 0$, and for $v(z) = z^n - 1$

$$I^p\left(\frac{nz^{n-1}}{z^n-1}, r\right) = n^p r^{p(n-1)} \sum_{k=0}^{\infty} \binom{-p/2}{k} r^{2kn} = n^p r^{p(n-1)}(1 + o(1)), \tag{2.3}$$

which shows that (2.1) is asymptotically[1] sharp for any p in the interval $1 < p \leqq 2$.

[1] Here we regard only the main term, whence this is not determined by a normalizatiton, as it was in the previous section.

Besides, in the case $p=\infty$ the expected theorem can be proved.

THEOREM (Q. I. Rahman [2]). *If $v(z)$ is a polynomial subject to* (1.1) *and* (1.5) *and r satisfies* (1.7), *then*

$$\max_{|z|=r}\left|\frac{v'(z)}{v(z)}\right| \leqq \max_{|z|=r}\left|\frac{nz^{n-1}}{z^n-1}\right| = \frac{nr^{n-1}}{1-r^n}. \tag{2.4}$$

Now, the extremality of z^n-1 in a small neighbourhood of zero can be proved in this problem, too.

THEOREM 2. *For any $1\leqq p<\infty$ and any polynomial $v(z)$ subject to* (1.1) *and* (1.5), *there exists a radius $r_2(p, v)>0$ such that for all $r<r_2(p, v)$ we have*

$$I^p\left(\frac{v'}{v}, r\right) \leqq I^p\left(\frac{nz^{n-1}}{z^n-1}, r\right). \tag{2.5}$$

If $v(z)\neq z^n-1$, then there exists a radius $r_3(p, v)>0$, such that for all $r<r_3(p, v)$

$$I^p\left(\frac{v'}{v}, r\right) > I^p\left(\frac{nz^{n-1}}{z^n-1}, r\right). \tag{2.6}$$

3.

PROOF OF THEOREM 1. If the Taylor expansion of f around $z=0$ is

$$f(z) = \sum_{\nu=0}^{\infty} g(\nu)z^\nu = 1+\sum_{\nu=1}^{\infty} g(\nu)z^\nu, \tag{3.1}$$

then for $r<r(f)$ we have

$$\sum_{\nu=1}^{\infty} |g(\nu)|r^\nu < 1. \tag{3.2}$$

Since

$$f(z)^{p/2} = \left(1+\sum_{\nu=1}^{\infty} g(\nu)z^\nu\right)^{p/2} = \sum_{k=0}^{\infty}\binom{p/2}{k}\left(\sum_{\nu=1}^{\infty} g(\nu)z^\nu\right)^k =$$

$$= 1+\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m}\binom{p/2}{k}\sum_{\substack{l_j\geqq 1\\ l_1+\ldots+l_k=m}} g(l_1)\ldots g(l_k)\right)z^m,$$

hence with the notation

$$h(m) = \sum_{k=1}^{m}\binom{p/2}{k}\sum_{\substack{l_j\geqq 1\\ l_1+\ldots+l_k=m}} g(l_1)\ldots g(l_k) \tag{3.4}$$

we have

$$I^p(f, r) = 1+\sum_{m=1}^{\infty}|h(m)|^2 r^{2m}. \tag{3.5}$$

Thus, if there is a coefficient among $h(1), \ldots, h(n-1)$ different from zero, then (1.10) and (3.5) immediately lead to (1.14).

In the opposite case

$$h(j) = 0 \quad (j = 1, \ldots, n-1), \tag{3.6}$$

(3.4) yields

$$g(j) = 0 \quad (j = 1, \ldots, n-1). \tag{3.7}$$

(1.1), (1.2), (1.3) and (3.1) give

$$\Big(\sum_{\nu=0}^{\infty} g(\nu) z^{\nu}\Big)\Big(\sum_{k=0}^{n} B_k z^k\Big) = \sum_{k=0}^{t} A_k z^k$$

which leads to

$$\sum_{\nu=l}^{n+l} g(\nu) B_{n+l-\nu} = 0 \quad (l = 0, 1, 2, \ldots) \tag{3.8}$$

that is

$$g(n+l) = \frac{-1}{B_0} \sum_{\nu=l}^{n+l-1} g(\nu) B_{n+l-\nu}. \tag{3.9}$$

So if (3.6) and (3.7) hold, then by (1.1)

$$g(n) = \frac{-1}{B_0} g(0) B_n = \frac{(-1)^{n+1}}{\xi_1 \ldots \xi_n}. \tag{3.10}$$

If we take $m=n$ in (3.4) then (3.7), (3.10), (1.10) and (3.5) yield (1.14) if

$$|\xi_1| < 1. \tag{3.11}$$

If

$$|\xi_1| = 1 \tag{3.12}$$

then (3.5) and (1.10) give (1.14) in every such case when for some j with $n+1 \leqq j \leqq 2n-1$, $h(j) \neq 0$. So we are entitled to assume further on, that besides (3.6), (3.7) and (3.12)

$$h(j) = 0 \quad (j = n+1, \ldots, 2n-1)$$

holds, too. Since from (3.4) we get

$$g(j) = \frac{1}{\binom{p/2}{1}} \Big\{ h(j) - \sum_{k=2}^{j} \binom{p/2}{k} \sum_{\substack{l_1 \geqq 1 \\ l_1 + \ldots l_k = j}} g(l_1) \ldots g(l_k) \Big\},$$

we have

$$g(j) = 0 \quad (j = n+1, \ldots, 2n-1) \tag{3.13}$$

too, since in the inner sum all terms contain at least one factor $g(l)$ with $1 \leqq l \leqq n-1$, and this factor is zero according to (3.7). So we have to prove (1.13)—(1.14) only when (3.7), (3.12) and (3.13) are valid. By (3.8), (3.9) and (3.10) here

$$g(2n) = \frac{-1}{B_0} g(n) B_n = g^2(n)$$

and

$$B_1 = B_2 = \ldots = B_{n-1} = 0, \tag{3.14}$$

further

$$g(ln) = g^l(n) \quad (l = 1, 2, \ldots) \tag{3.15}$$

and

$$g(j) = 0 \quad (j \neq ln). \tag{3.16}$$

This means that

$$f(z) = 1 + \sum_{l=1}^{\infty} g^l(n) z^{ln} = \frac{1}{1 - g(n) z^n}, \tag{3.17}$$

and so by $v(1)=0$ (c.f. (1.5)[2])

$$f(z) = \frac{1}{1 - z^n}. \tag{3.18}$$

In order to prove (1.15) we have to give our argumentation a quantitative form. Suppose that

$$|\xi_1| < 1, \tag{3.19}$$

and denote

$$\delta = \max \{|g(1)|, \ldots, |g(n-1)|\}. \tag{3.20}$$

Since from (1.1) and (1.5)

$$|B_k| \leqq \binom{n}{k}, \tag{3.21}$$

from (3.9) we obtain with the choice $l=0$

$$|g(n)| \geqq \frac{1}{\left| \prod_{j=1}^{n} \xi_j \right|} (1 - 2^{n-1} \delta) = \frac{1 - 2^{n-1}\delta}{|B_0|}. \tag{3.22}$$

On the other hand, by (3.4) and (3.20)

$$|h(m)| \geqq \left| \binom{p/2}{1} g(m) \right| - \sum_{k=2}^{m} \left| \binom{p/2}{k} \right| \binom{m-1}{k-1} \left(\max_{1 \leqq l < m} |g(l)| \right)^k. \tag{3.23}$$

So if for some $1 \leqq l < n$

$$|g(l)| = \delta, \tag{3.24}$$

then by (3.20) and (3.23)

$$|h(l)| \geqq \frac{p}{2} \delta - \sum_{k=2}^{l} \left| \binom{p/2}{k} \right| \binom{l-1}{k-1} \delta^k. \tag{3.25}$$

---

[2] For the sake of absolute uniqueness we used $\ldots \leqq \xi_n = 1$ in (1.5) instead of the normalization $\ldots \leqq |\xi_n| = 1$ used in [1].

Since for any $1 \leqq p < \infty$ and for $k \geqq 2$

$$\left|\binom{p/2}{k}\right| = \left|\prod_{j=1}^{k} \frac{p+2-2j}{2j}\right| \leqq \left(\frac{p}{2}\right)^{p/2+1}, \tag{3.26}$$

(3.23) implies

$$|h(m)| \geqq \frac{p}{2}|g(m)| - \left(\frac{p}{2}\right)^{p/2+1} 2^{m-1} \max_{\substack{1 \leqq l < n \\ k \geqq 2}} |g(l)|^k \tag{3.27}$$

and thus

$$|g(m)| \leqq \frac{2}{p}|h(m)| + \left(\frac{p}{2}\right)^{p/2} 2^{m-1} \max_{\substack{1 \leqq l < m \\ k \geqq 2}} |g(l)|^k. \tag{3.28}$$

Taking in account (3.26) we derive from (1.10) the estimate

$$I^p\left(\frac{1}{1-z^n}, r\right) \leqq 1 + \left(\frac{p}{2}\right)^2 r^{2n} + \sum_{m=2}^{\infty} \left\{\frac{p}{2} + 2^{m-1}\left(\frac{p}{2}\right)^{p/2+1}\right\}^2 r^{2mn} \leqq \tag{3.29}$$

$$\leqq 1 + \frac{p^2}{4} r^{2n} + \left(\frac{p}{2}\right)^{p+2} \left\{\frac{1}{4} \max_{p \geqq 1} \left(\frac{p}{2}\right)^{-p/2} + \frac{1}{2}\right\}^2 \sum_{m=2}^{\infty} (4r^{2n})^m.$$

Now for any

$$r < r_4(|B_0|, n, p) = \min\left\{\frac{1}{4}, \left(\frac{1-|B_0|}{20\left(\frac{p}{2}\right)^p}\right)^{1/2n}\right\} \tag{3.30}$$

this leads to

$$I^p\left(\frac{1}{1-z^n}, r\right) \leqq 1 + \frac{p^2}{4} r^{2n} + \left(\frac{p}{2}\right)^{p+2} \left(\frac{2+\sqrt{2}}{4}\right)^2 \frac{4}{3} 16 r^{4n} < \tag{3.31}$$

$$< 1 + \frac{p^2}{4} r^{2n} + 20\left(\frac{p}{2}\right)^{p+2} r^{4n} \leqq 1 + \frac{p^2}{4}\{1 + (1-|B_0|)\} r^{2n}.$$

Using (3.9), (3.20) and (3.21) we get by induction

$$|g(n+l)| \leqq \frac{1}{|B_0|} 2^n \max_{l \leqq \nu \leqq n+l-1} g(\nu)| \leqq \ldots \leqq \left(\frac{2^n}{|B_0|}\right)^{l+1} \max(1, \delta), \tag{3.32}$$

and so (3.2) is valid for every

$$r < \frac{1}{2\max(1, \delta)} \frac{|B_0|}{2^n}. \tag{3.33}$$

Now we distinguish three cases as follows:

$$\begin{cases} \textit{Case} \quad \text{I:} & 0 \leqq \delta \leqq \delta_0 = \min\left(\dfrac{1-|B_0|}{2^{n+1}}, \left(\dfrac{2}{p}\right)^{p/2}\right). \\ \textit{Case} \quad \text{II:} & \delta_0 < \delta \leqq 1. \\ \textit{Case} \quad \text{III:} & 1 < \delta. \end{cases} \tag{3.34}$$

In Cases I and II (3.33) means the condition

$$r < r_5(|B_0|, n) = \frac{|B_0|}{2^{n+1}} \quad (< |\xi_1|), \tag{3.35}$$

and so for these values of r (3.3) and so (3.5) holds. In Case I we get from (3.22) and (3.27)

$$|h(n)| \geqq \frac{p}{2}\frac{1-\delta_0 2^n}{|B_0|} - \left(\frac{p}{2}\right)^{p/2+1} 2^{n-1}\delta_0^2 = \tag{3.36}$$

$$= \frac{p}{2}\frac{1}{|B_0|}\left\{1-\delta_0 2^n\left(\left(\frac{p}{2}\right)^{p/2}\delta_0+1\right)\right\} \geqq \frac{p}{2}\left\{1+\frac{1-|B_0|}{2}\right\}.$$

Collecting (3.5), (3.31) and (3.36) we obtain

$$I^p(f, r) \geqq 1+\frac{p^2}{4}\left\{1+\frac{1-|B_0|}{2}\right\}^2 r^{2n} > 1+\frac{p^2}{4}\{1+(1-|B_0|)\}r^{2n} \geqq I^p\left(\frac{1}{1-z^n}, r\right)$$

when $r<r_4$ and $r<r_5$.

In Case II we can find an index $1 \leqq m < n$ for which[3]

$$|g(m)| \geqq \delta_0\left\{\left(\frac{p}{2}\right)^{p/2+1} 2^n\right\}^{m+1-n},$$

but

$$|g(v)| < \delta_0\left\{\left(\frac{p}{2}\right)^{p/2+1} 2^n\right\}^{v+1-n} \qquad (v = 1, \ldots, m-1).$$

For this m (3.27) gives

$$|h(m)| \geqq \frac{p}{2}\delta_0\left(\left(\frac{p}{2}\right)^{p/2+1} 2^n\right)^{m+1-n} - \left(\frac{p}{2}\right)^{p/2+1} 2^{n-1}\delta_0^2\left(\left(\frac{p}{2}\right)^{p/2+1} 2^n\right)^{m-n} \geqq \tag{3.37}$$

$$\geqq \frac{p}{4}\delta_0\left(\left(\frac{p}{2}\right)^{p/2+1} 2^n\right)^{-n} = pr_6(|B_0|, n, p) = pr_6,$$

say, and so for $r<r_6$ we get by (3.5)

$$I^p(f, r) \geqq 1+(pr_6)^2 r^{2m} > 1+p^2 r^{2m+2}.$$

Considering the condition (3.30) we may use (3.31) to obtain the required inequality

Now we begin with the case $1<\delta$. Regarding the lower estimation of $h(m)$ we are in an easier position as before. But this is compensated by the fact that we can not guarantee the validity of (3.3) and so that of (3.5) within a fixed circle, since in (3.33) δ can be large. This is caused not by a weak method of computation, but the fact that in this case the numerator of $f(z)$ can be zero in the close neighbourhood of zero, and so for general p (3.3) might be divergent.

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[3] Since $n=1$ is a trivial case, we suppose $n \geqq 2$ and so $\left(\frac{p}{2}\right)^{p/2+1} 2^n > 1$ for $p \geqq 1$.

Similarly to Case II, we find an index $1 \leqq m < n$ for which

$$|g(m)| \geqq \left\{\left(\frac{p}{2}\right)^{p/2+1} 2^n\right\}^{m+1-n},$$

but

$$|g(v)| < \left\{\left(\frac{p}{2}\right)^{p/2+1} 2^n\right\}^{v+1-n} \qquad (v = 1, \ldots, m-1),$$

and by (3.27) we get

(3.38)
$$|h(m)| \geqq \frac{p}{2}\left\{\left(\frac{p}{2}\right)^{p/2+1} 2^n\right\}^{m+1-n} - \left(\frac{p}{2}\right)^{p/2+1} 2^{n-2}\left\{\left(\frac{p}{2}\right)^{p/2+1} 2^n\right\}^{2m-2n} >$$

$$> \frac{p}{4}\left\{\left(\frac{p}{2}\right)^{p/2+1} 2^n\right\}^{-n}.$$

In the following lines we restrict ourselves to those values of r for which

(3.39) $$r < r_7(|B_0|, n, p) = \min\left\{r_4(|B_0|, n, p), \frac{1}{3^n}\left(\left(\frac{p}{2}\right)^{p/2+1} 2^n\right)^{-n}\right\},$$

and so automatically we have (3.31) and its consequence,

(3.40) $$I^p\left(\frac{1}{1-z^n}, r\right) < 1 + \frac{p^2}{2} r^{2n}.$$

Now, if for some r, (3.3) and hence (3.5) is valid, we get from (3.5) and (3.38) the estimate

(3.41) $$I^p(f, r) \geqq 1 + \frac{p^2}{16} 3^{2n} r_7^2 r^{2n-2}.$$

If we order the roots η_j of $u(z)$ according to nondecreasing order of modulus, and introduce the notation

$$\eta = |\eta_1| = \min_{1 \leqq j \leqq t} |\eta_j|,$$

then f has no singularity and even no zeros if $r < |\xi_1|$ and

$$r < \min\{\eta, r_7\}.$$

Thus for any such r we have both (3.41) and (3.40), which gives (1.15). It follows that we are ready if $\eta \geqq r_7$. If $\eta < r_7$, then we have (3.41) for all $r < \eta$, and even for $r = \eta$ by continuity. Taking into account that (1.6) is a monotonically increasing function of r, we have for any r in the interval

(3.42) $$\eta < r \leqq \min\{3\eta, r_7\},$$

by an application of (3.41) for $r=\eta$ and by (3.40), that

$$I^p(f, r) \geqq I^p(f, \eta) > 1+\frac{p^2}{16} 3^{2n} r_7^2 \eta^{2n-2} > 1+\frac{p^2}{2} r_7^2 (3\eta)^{2n-2} \geqq \tag{3.43}$$

$$\geqq 1+\frac{p^2}{2} r^{2n} > I^p\left(\frac{1}{1-z^n}, r\right).$$

So if $3\eta \geqq r_7$, then we have (1.15) for all r satisfying (3.39), and so we have completed the proof of Case III. Finally, if $3\eta < r_7$, then for any r with $3\eta \leqq r < r_7$ with the notation $f(z)=\left(1-\frac{z}{\eta_1}\right) F(z)$ we have

(3.44)

$$I^p(f, r) = \frac{1}{2\pi} \int_0^{2\pi} \left|\left(1-\frac{re^{i\varphi}}{\eta_1}\right) F(re^{i\varphi})\right|^p d\varphi > \min_{0 \leqq \varphi \leqq 2\pi} \left|1-\frac{re^{i\varphi}}{\eta_1}\right|^p |F(0)| > 2^p,$$

since F is analytic at every point where f is, $F(0)=f(0)=1$, and so for $r<r_7$, $I^p(F, r)$ is also an increasing function of r. Now (3.40) and (3.44) give (1.15) again, so we can colect the results of Cases I, II and III to obtain the final expression

$$r_1(|B_0|, n, p) = \min \{r_4, r_5, r_6, r_7\}.$$

4.

PROOF OF THEOREM 2. From (1.1) we get for all r satisfying (1.7) the formula

$$\frac{v'(z)}{v(z)} = \sum_{j=1}^{n} \frac{1}{z-\xi_j} = \sum_{j=1}^{n} \frac{-\frac{1}{\xi_j}}{1-\frac{z}{\xi_j}} = -\sum_{\nu=0}^{\infty} \left\{\sum_{j=1}^{n} \left(\frac{1}{\xi_j}\right)^{\nu+1}\right\} z^\nu. \tag{4.1}$$

Denoting

$$z_j = \frac{1}{\xi_j} \tag{4.2}$$

we have

$$|z_1| \geqq \ldots \geqq |z_{n-1}| \geqq z_n = 1. \tag{4.3}$$

Let

$$e(\nu) = \sum_{j=1}^{n} z_j^\nu. \tag{4.4}$$

Then we have

$$\frac{v'(z)}{v(z)} = -\sum_{\nu=0}^{\infty} e(\nu+1) z^\nu. \tag{4.5}$$

If

$$m = \min \{\nu \geqq 1: e(\nu) \neq 0\}, \tag{4.6}$$

then it is clear again, that for $m \leqq r-1$ and for

$$r < r(v) \tag{4.7}$$

we can estimate by the leading term trivially and get

$$I^p\left(\frac{v'}{v}, r\right) \geqq \frac{1}{2}|e(m)|^p r^{(m-1)p} > 2n^p r^{(n-1)p} > I^p\left(\frac{nz^{n-1}}{z^n-1}, r\right). \tag{4.8}$$

So we can suppose, that

$$e(1) = \ldots = e(n-1) = 0. \tag{4.9}$$

But in this case the function $\frac{v'(z)}{z^{n-1}v(z)}$ is analytic at 0, and thus $\frac{v'(z)}{z^{n-1}}$ is analytic at 0 too. According to (1.1) this means that $v'(z)=nB_n z^{n-1}=nz^{n-1}$, and $B_{n-1}=\ldots=$ $=B_1=0$, i.e. $v(z)=B_0+z^n$. Since $\xi_n=1$ is a root of v, from this we get $v(z)=z^n-1$. Q. E. D.

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METRIZATION AND LIAPUNOV FUNCTIONS. III

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The aim of this paper is to investigate attractor-repeller systems concerning the existence of metrics of Liapunov type.

The concept of metrics of Liapunov type was suggested by Wilson [12, p. 426]: "...we can think of the Liapunov function as the choice of a particular metric in which asymptotic stability is satisfied. Such a function (or metric) defines a set of neighbourhoods for A which has a countable basis...". The precise definition as well as local theorems were given in [3], [5]. Reconsidering the cellular decomposition method from metric point of view, it has been shown [4], [6], that the results can be generalised for some types of two-dimensional Morse—Smale dynamical systems. In this paper we consider general Morse—Smale dynamical systems as well as attractor-repeller pairs in metric spaces.

Let (X, d) be a compact closed manifold. (d is a Riemannian metric on X.) Let T: $\mathbf{R}\times X\to X$ be a Morse—Smale dynamical system [11] on X.

THEOREM 1. *Under the previous assumptions, there exists a new metric ϱ, topologically equivalent to d such that any trajectory — except periodic ones — is strictly monotonous* [6] *with respect to the new metric ϱ.*

To be more precise, we prove Theorem 1 in a reformulated version as Theorem 1′.

We need some notation first. The critical points and the periodic orbits of our Morse—Smale dynamical system are denoted by $P_1, P_2, \ldots, P_N$ and $\Gamma_1, \Gamma_2, \ldots, \Gamma_M$, respectively. The corresponding stable (attracting) and unstable (repelling) manifolds are denoted by $\mathscr{S}(P_i)$, $\mathscr{U}(P_i)$ and $\mathscr{S}(\Gamma_j)$, $\mathscr{U}(\Gamma_j)$, respectively.

The starting point of the reformulation as well as of the proof of Theorem 1′ is the existence of a global Liapunov function:

LEMMA 1 [10]. *There exists a continuous function V: $X\to\mathbf{R}$ such that for $x\in X\setminus(\Gamma_1\cup\Gamma_2\cup\ldots\cup\Gamma_M)$, V is strictly decreasing along the trajectory through x.*

Clearly V is constant on periodic orbits, let us say $V|_{\Gamma_j}=c_j$, $j=1, 2, \ldots, M$. $\mathscr{S}(P_i)\cup\mathscr{U}(P_i)$ and $\mathscr{S}(\Gamma_j)\cup\mathscr{U}(\Gamma_j)$ are denoted by $\mathscr{M}(P_i)$ and $\mathscr{M}(\Gamma_j)$, respectively.

Now we can state Theorem 1 in its final form:

THEOREM 1′. *Under the previous assumptions, there exists a new metric ϱ on X topologically equivalent to d such that*

(i) $x\in\mathscr{M}(P_i)$ *implies* $\varrho(x, P_i) = |V(x)-V(P_i)|$, $i = 1, 2, \ldots, N$,

(ii) $x\in\mathscr{M}(\Gamma_j)$ *implies* $\varrho(x, \Gamma_j) = |V(x)-c_j|$, $j = 1, 2, \ldots, M$.

PROOF. The proof is based on Urysohn's metrization theorem. We construct the desired metric ϱ.

Step 1. We can choose a positive constant δ_0 so that the neighbourhoods $\mathcal{N}(\Gamma_j)=\{w\in X \mid d(w,\Gamma_j)\leqq\delta_0\}$ of the periodic orbits Γ_j, $j=1,2,\ldots,M$ do not overlap. We may assume that $P_i\notin\bigcup_{j=1}^{M}\mathcal{N}(\Gamma_j)$, $i=1,2,\ldots,N$. Let $r_j:\mathcal{N}(\Gamma_j)\to\Gamma_j$ be a retraction of $\mathcal{N}(\Gamma_j)$ onto Γ_j.

By a compactness argument, there exists a positive constant ε_0 such that

(1) $$x\in\mathcal{M}(P_i)\cap\mathcal{N}(\Gamma_j) \quad \text{implies} \quad |V(x)-V(P_i)|\geqq\varepsilon_0,$$

(2) $$x\in\mathcal{M}(\Gamma_j)\setminus\mathcal{N}(\Gamma_j) \quad \text{implies} \quad |V(x)-c_j|\geqq\varepsilon_0.$$

For $j=1,2,\ldots,M$ we construct a sequence $\{g_n^{(j)}\}_{n\in\mathbf{N}}$ of continuous functions $g_n^{(j)}: X\to\mathbf{R}$ such that

(3) for $x\notin\mathcal{N}(\Gamma_j)$; $g_n^{(j)}(x)=0$,

(4) for $x\in\mathcal{N}(\Gamma_j)$; $0\leqq g_n^{(j)}(x)\leqq\varepsilon_0/2^n$,

(5) for $x,y\in\Gamma_j$; $g_n^{(j)}(x)=g_n^{(j)}(y)$, $n\in\mathbf{N}$ imply $x=y$,

(6) for $x\in\mathcal{M}(\Gamma_j)\cap\mathcal{N}(\Gamma_j)$ it holds

$$|g_n^{(j)}(x)-g_n^{(j)}(r_j(x))|\leqq|V(x)-c_j|.$$

In fact, let $\{h_n^{(j)}\}_{n\in\mathbf{N}}$ be a sequence of continuous functions $h_n^{(j)}:\Gamma_j\to\mathbf{R}$ separating the points of Γ_j. We may assume that for each $x\in\Gamma_j$, $0\leqq h_n^{(j)}(x)\leqq\varepsilon_0/2^n$.

On the other hand, there exists a strictly increasing continuous function $\alpha:[0,\delta_0]\to\mathbf{R}^+$ such that $x\in\mathcal{M}(\Gamma_j)\cap\mathcal{N}(\Gamma_j)$ implies $\alpha(d(x,\Gamma_j))\leqq|V(x)-c_j|$.

Clearly there exists a constant k, $0<k<1$, and there exists a continuous function $\beta:[0,\delta_0]\to\mathbf{R}^+$ so as to satisfy $\beta(0)=0$, $\beta(\delta_0)=1$ and for $t\in[0,\delta_0]$, $k\cdot\beta(t)\leqq\leqq\alpha(t)$.

Now we are in a position to define the function $g_n^{(j)}$. For $j=1,2,\ldots,M$, $n\in\mathbf{N}$ let

$$g_n^{(j)}(x)=\begin{cases}k(1-\beta(d(x,\Gamma_j)))h_n^{(j)}(r_j(x)) & \text{if } x\in\mathcal{N}(\Gamma_j)\\ 0 & \text{if } x\notin\mathcal{N}(\Gamma_j).\end{cases}$$

It is easy to see that $\{g_n^{(j)}\}_{n\in\mathbf{N}}$ is a sequence of continuous functions satisfying (3), (4), (5), (6).

Step 2. Applying similar arguments, we can construct a sequence $\{g_n\}_{n\in\mathbf{N}}$ of continuous functions $g_n: X\to\mathbf{R}$ such that

(7) for $x\in X$; $0\leqq g_n(x)\leqq\varepsilon_0/2^n$,

(8) for $x,y\in X\setminus(\Gamma_1\cup\Gamma_2\cup\ldots\cup\Gamma_M)$; $g_n(x)=g_n(y)$, $n\in\mathbf{N}$ imply $x=y$,

(9) for $x\in\mathcal{M}(P_i)$ it holds $|V(x)-V(P_i)|\geqq|g_n(x)-g_n(P_i)|$,

(10) for $x\in\mathcal{M}(\Gamma_j)$, $|V(x)-c_j|\geqq|g_n(x)|$ holds.

Let us remark that (10) implies $g_n|_{\Gamma_j}=0$.

Step 3. The sequence $\{f_n\}_{n\in\mathbf{N}}$ of continuous functions $f_n\colon X\to\mathbf{R}$ is defined as

$$f_n=\begin{cases} g_k^{(j)} & \text{if} \quad n=j+k(M+1);\quad j=1,2,\ldots,M;\quad k\in\mathbf{N} \\ g_k & \text{if} \quad n=k(M+1);\quad k\in\mathbf{N}. \end{cases}$$

For $x, y\in X$, let

$$\varrho(x,y)=\sup_{n\in\mathbf{N}}\{|V(x)-V(y)|,\ |f_n(x)-f_n(y)|\}.$$

In view of (5) and (8), $\{f_n\}_{n\in\mathbf{N}}$ separates the points of X. Consequently, using (3), (4) and (7), it follows that ϱ is a metric on X, topologically equivalent to d. In order to prove (i) and (ii), let us observe first that (3), (4) and (7) imply

$$|f_n(x)-f_n(y)|\leqq\varepsilon_0;\quad \text{for}\quad x,y\in X. \tag{11}$$

Consequently, we obtain from (1) that $x\in\mathcal{M}(P_i)\cap\mathcal{N}(\Gamma_j)$ implies

$$\varrho(x,P_i)=\sup_{n\in\mathbf{N}}\{|V(x)-V(P_i)|,\quad |f_n(x)-f_n(P_i)|\}=$$

$$=\sup\{|V(x)-V(P_i)|,\varepsilon_0\}=|V(x)-V(P_i)|.$$

Similarly, by virtue of (3) and (9); $x\in\mathcal{M}(P_i)\setminus\bigcup_{j=1}^{M}\mathcal{N}(\Gamma_j)$ implies

$$\varrho(x,P_i)=\sup_{n\in\mathbf{N}}\{|V(x)-V(P_i)|,\ |f_n(x)-f_n(P_i)|\}=$$

$$=\sup_{n\in\mathbf{N}}\{|V(x)-V(P_i)|,\ |g_n(x)-g_n(P_i)|\}=|V(x)-V(P_i)|.$$

Thus, we have arrived at the conclusion that (i) holds.

Consider now a point $x\in\mathcal{M}(\Gamma_j)$. For each $z\in\Gamma_j$, clearly $\varrho(x,z)\geqq|V(x)-c_j|$. Therefore $\varrho(x,\Gamma_j)\geqq|V(x)-c_j|$.

On the other hand, in view of (2) and (11), $x\in\mathcal{M}(\Gamma_j)\setminus\mathcal{N}(\Gamma_j)$ implies

$$\varrho(x,z)=\sup_{n\in\mathbf{N}}\{|V(x)-V(z)|,|f_n(x)-f_n(z)|\}=\sup\{|V(x)-c_j|,\varepsilon_0\}=|V(x)-c_j|$$

if $z\in\Gamma_j$. Consequently, $\varrho(x,\Gamma_j)=|V(x)-c_j|$.

Similarly, by (3), (6) and (10), $x\in\mathcal{M}(\Gamma_j)\cap\mathcal{N}(\Gamma_j)$ implies

$$\varrho\big(x,r_j(x)\big)=\sup_{n\in\mathbf{N}}\big\{|V(x)-V(r_j(x))|,\ |f_n(x)-f_n(r_j(x))|\big\}=$$

$$=\sup_{n\in\mathbf{N}}\{|V(x)-c_j|,\ |g_n^{(j)}(x)-g_n^{(j)}(r_j(x))|,\ |g_n(x)-g_n(r_j(x))|\}=|V(x)-c_j|.$$

Since $r_j(x)\in\Gamma_j$, we obtain $\varrho(x,\Gamma_j)\leqq|V(x)-c_j|$.

Thus, we have arrived at the conclusion that (ii) holds. Finally, let us remark that Theorem 1 gives an affirmative answer to a conjecture in [6]. Q. E. D.

From now on, let (X,d) be a compact metric space, and let M and N be non-void compact subsets of X. Let $T\colon\mathbf{R}\times X\to X$ be a dynamical system on X. Assume that M is an asymptotically stable invariant set. Its region of attraction is denoted by $A(M)$. Assume that N is a negatively asymptotically stable invariant set. Its region

of repulsion is denoted by $R(N)$. As usual, we say that (M, N) is an attractor-repeller pair if $A(M)=X\setminus N$ and $R(N)=X\setminus M$.

DEFINITION. Let ϱ be a metric defined on X. We say that ϱ is a metric of Liapunov type with respect to the attractor-repeller pair (M, N) if ϱ is topologically equivalent to d and, for $x\in X\setminus(M\cup N)$, $\varrho(x, M)$ and $\varrho(x, N)$ are strictly decreasing resp. increasing along the trajectory through x.

The existence of metrics of Liapunov type is based on the existence of appropriate Liapunov functions.

LEMMA 2. *Assume that (M, N) is an attractor-repeller pair. Then there exists a constant $c_0>0$ and there exists a continuous function $V\colon X\to\mathbf{R}^+$ satisfying the following conditions:*

(12) $V^{-1}(0)=M,\ V^{-1}(c_0)=N.$

(13) for $x\in X\setminus(M\cup N)$, V *is strictly decreasing along the trajectory through* x.

(14) for $x\in X$; $d(x, M)\leqq V(x)\leqq c_0-d(x, N)$.

PROOF. $\sup\{d(x, y)|x\in M, y\in N\}$ is denoted by Δ. Fix a real number c_0 satisfying $c_0>2(\Delta+1)$.

By a theorem of [1, p. 213], there exists a continuous function $\Phi\colon X\to[0, 1]$ such that $\Phi^{-1}(0)=M$, $\Phi^{-1}(1)=N$ and, for $x\in X\setminus(M\cup N)$, Φ is strictly decreasing along the trajectory through x.

Applying a compactness argument, there exist strictly increasing functions $\alpha\colon \mathbf{R}^+\to\mathbf{R}^+$, $\beta\colon \mathbf{R}^+\to\mathbf{R}^+$ such that $\alpha(0)=0$, $\beta(0)=0$ and, for $x\in X$,

$$\alpha\big(d(x, M)\big)\leqq\Phi(x)\leqq 1-\beta\big(d(x, N)\big).$$

Without loss of generality, we may assume that

$$\alpha(\Delta+1)=1,\quad \beta(\Delta+1)=1.$$

Choose a strictly increasing continuous function $k\colon[0, 1]\to[0, c_0]$ satisfying $k(0)=0$, $k(1)=c_0$ and, for $s\in[0, 1]$

$$\alpha^{-1}(s)\leqq k(s)\leqq c_0-\beta^{-1}(1-s).$$

The existence of such a function k is guaranteed by the inequality

$$\alpha^{-1}(s)+\beta^{-1}(1-s)<2(\Delta+1).$$

For $x\in X$, we define $V(x)=k\big(\Phi(x)\big)$. It is easy to see that V satisfies conditions (12), (13), (14). Q.E.D.

LEMMA 3. *Assume that (M, N) is an attractor-repeller pair and $f\colon X\to\mathbf{R}^+$ is a continuous function for which $f^{-1}(0)\supset M\cup N$ holds. Then there exists a constant $c_0>0$ and there exist continuous functions $V, W\colon X\to[0, c_0]$ satisfying the following conditions:*

(15) $$V^{-1}(0)=W^{-1}(c_0)=M;\quad V^{-1}(c_0)=W^{-1}(0)=N.$$

(16) *for* $x \in X \setminus (M \cup N)$, V *and* W *are strictly decreasing resp. increasing along the trajectory through* x.

(17) $V(x)+W(x) \leqq c_0+f(x)$ *for* $x \in X$.

PROOF. The proof is similar to that of the previous Lemma. Q. E. D.

THEOREM 2. *Assume that* (M, N) *is an attractor-repeller pair. Then there exists a metric of Liapunov type.*

PROOF. The proof is an easy application of Lemma 2. In fact, the desired metric ϱ can be given as follows: for $x, y \in X$, let $\varrho(x, y)=\max\{d(x, y), |V(x)-V(y)|\}$. (14) implies $\varrho(x, M)=V(x)$; $\varrho(x, N)=c_0-V(x)$. Therefore, in view of (13), ϱ is a metric of Liapunov type. We remark that $\varrho|_M=d|_M$, $\varrho|_N=d|_N$. Q. E. D.

We carry on our investigation concerning attractor-repeller pairs. With the notation already adopted, let $P_M(x)$ denote the set of nearest points, i.e.

$$P_M(x) = \{y \in M | \varrho(x, M) = \varrho(x, y)\}.$$

Evidently $P_M(x)$ is a non-void compact subset of M. In case of $P_M(x)$ being a singleton, let $P_M(x)$ be denote this element too.

We want to modify the construction of our metric of Liapunov type to ensure that $P_M(x)$ and $P_N(x)$ be singletons for every $x \in X$.

To motivate the following theorem, let us recall a result of Kuratowsky [2, p. 219]: A closed subset M of a metric space X is a retract of X if and only if there exists a metrization of X such that for every $x \in X$, $P_M(x)$ is a singleton. Therefore, we have to suppose that M is a retract of X. Similarly, we have to suppose that N is a retract of X. Unfortunately, we are able to prove Theorem 3 only on condition (*) below.

THEOREM 3. *Assume that* (M, N) *is an attractor-repeller pair. Further assume that there exist retractions* $r_M: X \to M$, $r_N: X \to N$ *satisfying the following condition:*

$$(*) \qquad \begin{cases} \text{for} \quad x \in M; & r_M(r_N(x)) = x \\ \text{for} \quad x \in N; & r_N(r_M(x)) = x. \end{cases}$$

Then there exists a metric of Liapunov type such that $P_M(x)=r_M(x)$, $P_N(x)=r_N(x)$ *for* $x \in X$.

PROOF. Let $f: X \to \mathbf{R}^+$ be defined as

$$f(x) = \max\{d(r_M(x), r_M(r_N(x))), d(r_N(x), r_N(r_M(x)))\}.$$

Condition (*) implies that the function f satisfies the conditions of Lemma 3. Therefore, there exists a constant $c_0>0$ and continuous functions $V, W: X \to [0, c_0]$ satisfying (15), (16), (17).

Let $\{g_n\}_{n \in \mathbf{N}}$ be a sequence of continuous functions $g_n: X \to \mathbf{R}$ separating the points of $X \setminus (M \cup N)$, i.e. $g_n(x)=g_n(y)$, $x, y \in X \setminus (M \cup N)$, $n=0, 1, 2, \ldots$ imply $x=y$. The function g_n can be chosen to satisfy

(18) $$0 \leqq g_n(x) \leqq 2^{-n} \min\{V(x), W(x)\}, \quad x \in X, \quad n \in \mathbf{N}.$$

Now we are in a position to define the desired metric ϱ. The construction of ϱ is modelled after the usual proof of Urysohn's metrization theorem [9].

For $x, y \in X$, let

$$\varrho(x, y) = \sup_{n \in \mathbf{N}} \{|g_n(x) - g_n(y)|,$$

$$|V(x) - V(y)| + d(r_M(x), r_M(y)), |W(x) - W(y)| + d(r_N(x), r_N(y))\}.$$

It is a lengthy but straightforward task to prove that ϱ is a metric topologically equivalent to d. We remark that $\varrho|_M \cong d|_M$, $\varrho|_N \cong d|_N$.

For $y \in M$, using (15), (18), we obtain

$$\varrho(x, y) = \max\{V(x) + d(r_M(x), y), c_0 - W(x) + d(r_N(x), r_N(y))\}.$$

It is obvious, that $\varrho(x, M) = \inf\{\varrho(x, y) | y \in M\} \cong V(x)$. On the other hand, in view of (17), the infimum is attained if and only if $y = r_M(x)$. Thus, $\varrho(x, M) = V(x)$ and $P_M(x) = r_M(x)$.

Similarly, $\varrho(x, N) = W(x)$ and $P_N(x) = r_N(x)$.

Since $\varrho(x, M) = V(x)$ and $\varrho(x, N) = W(x)$, (16) implies that ϱ is of Liapunov type. Q. E. D.

EXAMPLE 1. Let X be the two-dimensional torus T_2. Let M and N be disjoint closed differentiable curves on T_2. Assume that $T_2 \setminus M$ and $T_2 \setminus N$ are connected Then Condition $(*)$ is satisfied.

EXAMPLE 2. The method used in proving Theorem 3 can be applied for some types of attractor-repeller systems as well.

Let X be a compact two-dimensional manifold. Let T be a Morse—Smale dynamical system on X. Assume that T has no periodic orbits. The sinks and the sources of T are denoted by $M_1, M_2, \ldots, M_K$ and $N_1, N_2, \ldots, N_L$, respectively.

The same method used in the proof of Lemma 3 guarantees the existence of continuous functions

$$V_k, W_l: X \to [0, 1], \quad k = 1, 2, \ldots, K; \quad l = 1, 2, \ldots, L,$$

satisfying the following conditions:

$$\begin{cases} V_k^{-1}(0) = M_k, & V_k^{-1}(1) = X \setminus A(M_k), \\ W_l^{-1}(0) = N_l, & W_l^{-1}(1) = X \setminus R(N_l). \end{cases} \tag{19}$$

(20) V_k and W_l are Liapunov functions with respect to M_k and N_l, respectively.

(21) for $x \in X$, $V_k(x) + W_l(x) \geqq 1$, $k = 1, 2, \ldots, K$; $l = 1, 2, \ldots, L$.

Choose a sequence $\{g_n\}_{n \in \mathbf{N}}$ of continuous functions $g_n: X \to \mathbf{R}$ separating the points of $X \setminus \{M_1, M_2, \ldots, M_K, N_1, N_2, \ldots, N_L\}$ and satisfying

(22) $0 \leqq g_n(x) \leqq 2^{-n} \min\{V_k(x), W_l(x) | \quad k = 1, 2, \ldots, K; \ l = 1, 2, \ldots, L\}$.

Without loss of generality, we may assume that for each $n\in\mathbf{N}$, g_n has the following properties as well:

(23) If S is a saddle point and e is a separatrix issuing from S and tending toward M_{k_0}, then $x\in e$ implies $|g_n(x)-g_n(S)|\leqq 1-V_{k_0}(x)$.

(24) If S is a saddle point and e is a separatrix tending toward S and issuing from N_{l_0}, then $x\in e$ implies $|g_n(x)-g_n(S)|\leqq 1-W_{l_0}(x)$.

Let the metric ϱ be defined as

$$\varrho(x,y)=\sup\{|g_n(x)-g_n(y)|,\ |V_k(x)-V_k(y)|,$$

$$|W_l(x)-W_l(y)|\,|n\in\mathbf{N};\quad k=1,2,\ldots,K;\quad l=1,2,\ldots,L\}.$$

It is easy to check that ϱ is a metric of Liapunov type. (We have to make use of the fact that there is no junction between saddle points [11].)

EXAMPLE 3. Instead of formulation of further results, we restrict ourselves to give an example of the global version of condition $(*)$.

From now on, the two-dimensional sphere with N holes $(N\geqq 1)$ is denoted by X_N. Thus, ∂X_N, the boundary of X_N, can be written as the disjoint union of closed differentiable curves $\gamma_1,\gamma_2,\ldots,\gamma_N$.

Assume that there is a retraction $r\colon X_N\to\gamma_1$ such that $r|_{\gamma_i}\colon\gamma_i\to\gamma_1$ is a homeomorphism, $i=1,2,\ldots,N$. Then N is an even number. This follows easily from Hirsch's famous proof [8] of the nonretractibility of a cell onto its boundary.

On the other hand, if N is an even number, it is not hard to show by induction that there exist retractions $r_i\colon X_N\to\gamma_i$ such that for $x\in\gamma_i$, $r_i(r_j(x))=x$, $i,j=1,2,\ldots,N$. Moreover, r_i can be taken for deformation retraction.

Concluding this paper, we generalise Theorem 14. 13. 1 of Hartman [7], who investigated the case when (X,d) is supposed to be a finite-dimensional manifold. (Correspondingly, his new metric ϱ is a Riemannian one.)

Let (X,d) be a metric space, $T\colon\mathbf{R}\times X\to X$ be a dynamical system on X. Assume that $x_0\in X$ is a uniformly asymptotically stable equilibrium point. Consequently [1], $A(x_0)\setminus\{x_0\}$ is parallelizable, i.e. it has a section S with $\tau\colon A(x_0)\setminus\{x_0\}\to\mathbf{R}$ continuous. For $x\in A(x_0)\setminus\{x_0\}$, $\tau(x)$ is defined by $T_{-\tau(x)}x\in S$.

Assume that x_0 is a uniform attractor. Further, assume that x_0 is an equi-attractor [1]. Let us remark that uniform attractivity does not imply equi-attractivity and conversely. On the other hand, asymptotic stability implies uniform attractivity and equi-attractivity provided that X is locally compact.

Without loss of generality, we may assume that

(25) $$\sup\{d(x,y)|x,y\in S\}\leqq 2.$$

For $x,y\in A(x_0)$, let the function $h^d_{x,y}\colon\mathbf{R}\to\mathbf{R}^+$ be defined by $h^d_{x,y}(t)=d(T_tx,T_ty)$. Clearly $h^d_{x,y}(t)$ tends to zero as t tends to infinity.

THEOREM 4. *On the previous conditions, there exists a new metric ϱ defined on $A(x_0)$, topologically equivalent to d and restricted to $A(x_0)$, such that for $x,y\in A(x_0)$, $x\neq y$, the function $h^{\varrho}_{x,y}$ is strictly decreasing.*

PROOF. Let f: $\mathbf{R}\to\mathbf{R}$ be a continuous strictly decreasing function satisfying

$$\lim_{t\to\infty} f(t) = 0. \tag{26}$$

Let g: $\mathbf{R}\to\mathbf{R}$ be a continuous strictly increasing function satisfying

$$\lim_{t\to\infty} g(t) = 1. \tag{27}$$

f and g can be chosen so that the inequalities

$$g(t)-g(\tau) \geqq f(\tau)-f(t), \quad t, \tau\in\mathbf{R}, \quad t \geqq \tau, \tag{28}$$

$$g(t+h)-g(\tau+h) < g(t)-g(\tau), \quad t, \tau\in\mathbf{R}, \quad t \geqq \tau, \quad h > 0 \tag{29}$$

hold. For example, let $f(t)=\exp(-t)$, $g(t)=1-\exp(-t)$.

For $x, y\in A(x_0)\setminus\{x_0\}$, let

$$\varrho(x, y) = d(T_{-\tau(x)}x, T_{-\tau(y)}y) f(\max\{\tau(x), \tau(y)\})+|g(\tau(x))-g(\tau(y))|.$$

For $x\in A(x_0)\setminus\{x_0\}$, let $\varrho(x, x_0)=\varrho(x_0, x)=1-g(\tau(x))$. Finally, let $\varrho(x_0, x_0)=0$.

By definition, $\varrho(x, y)=\varrho(y, x)\geqq 0$. Clearly $\varrho(x, y)=0$ implies $x=y$. To prove the triangle inequality, we distinguish two cases according as $x, y, z\in A(x_0)\setminus\{x_0\}$ or $x, y\in A(x_0)\setminus\{x_0\}$, $z=x_0$.

Firstly, when $x, y, z\in A(x_0)\setminus\{x_0\}$, then $x=T_t p$, $y=T_\tau q$, $z=T_s r$ for some $p, q, r\in S$, $t, \tau, s\in\mathbf{R}$. Without loss of generality, we may assume that $t\geqq\tau\geqq s$. Since $f(t)\leqq f(\tau)\leqq f(s)$ and $g(t)\geqq g(\tau)\geqq g(s)$ we obtain

$$\varrho(x, y) = d(p, q)f(t)+g(t)-g(\tau),$$

$$\varrho(y, z) = d(q, r)f(\tau)+g(\tau)-g(s),$$

$$\varrho(x, z) = d(p, r)f(t)+g(t)-g(s).$$

It is obvious that $\varrho(x, y)+\varrho(y, z)\geqq\varrho(x, z)$ and $\varrho(x, z)+\varrho(y, z)\leqq\varrho(x, y)$. $\varrho(x, z)+\varrho(x, y)\geqq\varrho(y, z)$ follows from (25) and (28). In fact,

$$\varrho(x, z)+\varrho(x, y) = \big(d(p, q)+d(p, r)\big)f(t)+2\big(g(t)-g(\tau)\big)+g(\tau)-g(s) \geqq$$

$$\geqq d(q, r)f(t)+d(q, r)\big(f(\tau)-f(t)\big)+g(\tau)-g(s) = \varrho(y, z).$$

Secondly, when $x, y\in A(x_0)\setminus\{x_0\}$, $z=x_0$, then the triangle inequality is implied by (25), (26), (27), (28). Thus, ϱ is a metric on $A(x_0)$.

In order to prove that ϱ and d restricted to $A(x_0)$ are topologically equivalent, we have to show that for every sequence $\{x_n\}_{n=1}^{\infty}\subset A(x_0)$ and for $x\in A(x_0)$

$$\varrho(x_n, x)\to 0 \text{ if and only if } d(x_n, x)\to 0. \tag{30}$$

If $x\neq x_0$, (30) is a simple corollary of the continuity of f, g, τ, T as well as of the monotonicity of g.

If $x=x_0$, (30) follows from (27) and from the uniform and equi-attractivity of x_0.

Finally, we have to show that for $x, y \in A(x_0)$, $x \neq y$, the function $h^{\varrho}_{x,y}$ is strictly decreasing. Since f is decreasing, it is an inmediate corollary of (29). Q. E. D.

REMARK. A brief examination of Example 1 of [6] shows that Theorem 4 does not remain valid if x_0 is substituted by a uniformly and equi-attracting asymptotically stable set M.

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ON HERMITE—FEJÉR TYPE INTERPOLATION

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1. Introduction

We give a necessary and sufficient condition for the convergence of the Hermite—Fejér operator based on the roots of orthogonal polynomials generated by the weight $|x|^{2\alpha+1}(1-x^2)^\beta$ $(-1<\alpha, \beta)$. Further we investigate the convergence of quasi-Hermite—Fejér type operators.

2. Notations and preliminary results

2.1. Let us consider the system of nodes $X=\{x_{k,n}\}_{k=1}^{n}{}_{n=1}^{\infty}$ where $-1\leqq x_{n,n}<$ $<x_{n-1,n}<\ldots<x_{1,n}\leqq 1$ $(n=1, 2, \ldots)$. For $f\in C[-1, 1]$ there exists a unique, so called Hermite—Fejér interpolatory polynomial $H_n(f, X, A, x)$ of degree $\leqq 2n-1$ such that (sometimes omitting the superfluous notations)

$$H_n(f, X, A, x_k)=f(x_k), \quad H_n'(f, X, A, x_k)=a_k \quad (k=1, 2, \ldots, n)$$

where $a_k=a_{k,n}$ are arbitrary real numbers, $A=\{a_{kn}\}_{k=1}^{n}{}_{n=1}^{\infty}$. The following representation is well-known:

$$H_n(f, X, A, x)=\sum_{k=1}^{n} f(x_k)\, v_k(x)\, l_k^2(x)+\sum_{k=1}^{n} a_k(x-x_k)\, l_k^2(x)$$

where $l_k(x)=l_{k,n}(X, x)$ are the fundamental polynomials of Lagrange interpolation and $v_k(x)=v_{k,n}(X, x)$ are appropriate linear functions $(k=1, 2, \ldots, n)$.

If $a_{k,n}=0$ for every k, n then we use the notation $H_n(f, X, 0, x)$ or $H_n(f, X, x)$.

The case $X^{(\alpha,\beta)}=\{x_{k,n}^{(\alpha,\beta)}\}_{k=1}^{n}{}_{n=1}^{\infty}$ $(\alpha, \beta>-1)$ i.e. when the nodes are the corresponding roots of the n-th Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ was investigated by several authors (see e.g. [5], [4], [8]). In [5] G. Szegő proved

THEOREM A. *The relation*

$$\lim_{n+\infty} \|H_n(f, X^{(\alpha,\beta)}, x)-f(x)\|_{[-1+\varepsilon,1]}=0 \tag{2.1}$$

holds for any $f\in C[-1, 1]$ *if* $-1\leqq\alpha<0$, $\beta\geqq -1$ ($\|\cdot\|_{[a,b]}$ *is the usual maximum norm on* $[a, b]$, $\varepsilon>0$ *is fixed).*

If $\alpha\geqq 0$ convergence holds only for certain functions f from $C[-1, 1]$. In [8] P. Vértesi proved

THEOREM B. *Suppose* $\alpha\in[p-1, p)$, *where* p *is an integer,* $\beta\geqq -1$. *Then for any* $f\in C[-1, 1]$ (2.1) *is valid if and only if*

$$\lim_{n\to\infty} H_n(f, X^{(\alpha,\beta)}, 1) = f(1),$$

moreover if $\alpha\geqq 1$

$$[H_n(f, X^{(\alpha,\beta)}, x)]^{(r)}_{x=1} = o(n^{2r}) \quad (r = 1, 2, \ldots, p-1).$$

Similar characterisation was given on $[0, \infty)$ by Barbara Háy [1] when X corresponds to the roots of the Laguerre polynomials.

2.2. Let us consider the weight function

$$p(\alpha, \beta;\ x) = |x|^{2\alpha+1}(1-x^2)^{\beta} \quad (\alpha, \beta > -1,\ |x| \leqq 1)$$

and denote the corresponding orthogonal system by $\{R_n^{(\alpha,\beta)}(x)\}_{n=1}^{\infty}$. One can prove (see Lascenov [2])

$$\begin{cases} R_{2n}^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(1-2x^2) \\ R_{2n+1}^{(\alpha,\beta)}(x) = xP_n^{(\alpha+1,\beta)}(1-2x^2) \end{cases} \quad (n = 0, 1, \ldots), \tag{2.2}$$

from where for the roots $\{y_{k,n}^{(\alpha,\beta)}\}_{k,n}=Y$ of $\{R_n^{(\alpha,\beta)}(x)\}$ we have

$$y_{k,n}^{(\alpha,\beta)} = -y_{-k,n}^{(\alpha,\beta)} = \begin{cases} \sin\dfrac{1}{2}\vartheta_{k,m}^{(\alpha,\beta)} & (n = 2m;\ k = 1, 2, \ldots, m), \\ \sin\dfrac{1}{2}\vartheta_{k,m}^{(\alpha+1,\beta)} & (n = 2m+1;\ k = 0, 1, 2, \ldots, m) \end{cases} \tag{2.3}$$

where $x_{k,n}^{(\alpha,\beta)}=\cos\vartheta_{k,n}^{(\alpha,\beta)}$ $(k=1, 2, \ldots, n)$ and $\vartheta_{0,n}^{(\alpha,\beta)}=0$.

For the corresponding Hermite—Fejér step parabolas Vértesi proved in [7]

THEOREM C. *The necessary and sufficient condition that for any* $f\in C[-1, 1]$

$$\lim_{n\to\infty} \|H_n(f, Y, x)-f(x)\| = 0 \tag{2.4}$$

is that the relations $-1<\alpha, \beta<0$ *and* $\beta-\alpha\leqq 0.5$ *hold.*

The aim of this paper is to give a Theorem B like characterization of the convergence for the node system Y. We shall prove

THEOREM 2.1. *Let us use the notations* $p=[2\alpha]+1$, $P=p+1-(-1)^n$, $q=[\beta]+1$, $H_n(f, x)=H_n(f, Y, x)$. *Then for any fixed* $f\in C[-1, 1]$ (2.4) *is valid if and only if*

$$H_{2n}(f, 0) \to f(0), \tag{2.5}$$

$$H_n^{(r)}(f, 0) = o(n^r) \quad \left(r = 1, 2, \ldots, P-1;\ \text{if}\ \ \alpha \geqq \frac{1}{2}\right), \tag{2.6}$$

$$\begin{cases} H_n(f, 1) \to f(1), \\ H_n(f, -1) \to f(-1); \end{cases} \tag{2.7}$$

moreover, if $\beta\geqq 1$, *then*

$$\begin{cases} H_n^{(r)}(f, 1) = o(n^{2r}), \\ H_n^{(r)}(f, -1) = o(n^{2r}) \end{cases} \quad (r = 1, 2, \ldots, q-1). \tag{2.8}$$

REMARK. From the proof of Theorem 2.1 it follows that if $\alpha<0$ then

$$\|f(x)-H_n(f,x)\|_{[-1+\varepsilon,1-\varepsilon]}\to 0 \quad (\varepsilon>0 \text{ is fixed}). \tag{2.9}$$

In the proofs we follow the general line of the proof of Theorem B (see [8]). Here I should like to express my thanks to Professor Vértesi who called my attention to this problem.

3. The $H_{n,p,q}$ process

To prove Theorem 2.1 we need the following process which is interesting in itself. Consider the uniquely determined polynomial $H_{n,p,q}(x)=H_{n,p,q}(f, Y, A, B, C, D, x)$ of degree $\leqq 2n+p+2q-1$ ($p, q\geqq 0$ integers, $f\in C$) such that

$$H_{n,p,q}(y_k)=f(y_k) \qquad \left(k=\pm 1, \pm 2, \ldots, \pm m;\ m=\left[\frac{n}{2}\right]\right), \tag{3.1}$$

$$H'_{n,p,q}(y_k)=a_k \tag{3.2}$$

$$H^{(r)}_{n,p,q}(0)=b_r \quad (r=0, 1, \ldots, P-1;\ b_0=f(0)) \tag{3.3}$$

(if n is even and $p=0$ we omit this condition),

$$H^{(r)}_{n,p,q}(1)=c_r \qquad (r=0, 1, \ldots, q-1;\ c_0=f(1),\ d_0=(f-1)) \tag{3.4}$$

$$H^{(r)}_{n,p,q}(-1)=d_r \tag{3.5}$$

(if $q=0$ we omit the last two conditions), $B=\{b_{rn}\}_{r=0}^{P-1}{}_{n=1}^{\infty}$, $C=\{c_{rn}\}_{r=0}^{q-1}{}_{n=1}^{\infty}$ and $D=\{d_{rn}\}_{r=0}^{q-1}{}_{n=1}^{\infty}$. Obviously $H_{n,0,0}(f,x)=H_n(f,x)$ and if $(p,q)=(0,1)$ we get the quasi-Hermite—Fejér polynomial.

If we write $H_{n,p,q}(x)$ in the form

$$H_{n,p,q}(x)=\sum_{|k|=1}^{m} f(y_k)F_k(x)+\sum_{|k|=1}^{m} a_k A_k(x)+\sum_{r=0}^{P-1} b_r B_r(x)+\sum_{r=0}^{q-1} c_r C_r(x)+\sum_{r=0}^{q-1} d_r D_r(x) \tag{3.6}$$

then evidently

$$F_k(x)=\left(\frac{x^2-1}{y_k^2-1}\right)^q\left(\frac{x}{y_k}\right)^p w_k(x)l_k^2(x), \tag{3.7}$$

$$A_k(x)=\left(\frac{x^2-1}{y_k^2-1}\right)^q\left(\frac{x}{y_k}\right)^p (x-y_k)l_k^2(x), \tag{3.8}$$

$$B_r(x)=\frac{P_m^2(1-2x^2)}{P_m^2(1)}(1-x^2)^q\sum_{j=0}^{P-1}\frac{x^j}{j!}b_{jr}, \tag{3.9}$$

$$C_r(x)=\frac{P_m^2(1-2x^2)}{P_m^2(-1)}x^p\left(\frac{1+x}{2}\right)^q\sum_{j=0}^{q-1}\frac{(x-1)^j}{j!}c_{jr}, \tag{3.10}$$

$$D_r(x)=\frac{P_m^2(1-2x^2)}{P_m^2(-1)}(-x)^p\left(\frac{1-x}{2}\right)^q\sum_{j=0}^{q-1}\frac{(x+1)^j}{j!}d_{jr}, \tag{3.11}$$

where

$$l_{k,n}(x)=\begin{cases}-\dfrac{P_m(1-2x^2)}{4y_kP'_m(x_{|k|})(x-y_k)} & (n=2m,\quad k=\pm1,\ldots,\pm m)\\[2ex] -\dfrac{P_m(1-2x^2)x}{4y_k^2P'_m(x_{|k|})(x-y_k)} & (n=2m+1,\quad k=\pm1,\ldots,\pm m)\end{cases} \tag{3.12}$$

and

$$w_k(x)=1-(x-y_k)\left[2l'_k(y_k)+2\frac{qy_k}{y_k^2-1}+\frac{p}{y_k}\right]. \tag{3.13}$$

(Here and later on if n is even then the corresponding parameters of $P_m(x)$, $P'_m(x)$, $x_{k,n}$ are (α,β); for odd n they are $(\alpha+1,\beta)$.) The numbers b_{jr}, c_{jr} and d_{jr} will be determined later, using the requirements

$$B_r^{(i)}(0)=\delta_{ir}\quad(0\leqq r,\ i\leqq P-1), \tag{3.14}$$

$$C_r^{(i)}(1)=D_r^{(i)}(-1)=\delta_{ir}\quad(0\leqq r,\ i\leqq q-1). \tag{3.15}$$

For the operator $H_{n,p,q}$ we state the following

THEOREM 3.1. *If* $\beta>-1$, $\alpha\geqq-\frac{1}{2}$ *then for any* $f\in C$

$$\lim_{n\to\infty}\|H_{n,p,q}(f,Y,A,B,C,D,x)-f(x)\|=0$$

if

$$\begin{cases}a_k=o\left(\min\left(\dfrac{n}{\log n},\ n^{p-2\alpha},\ n^{2q-2\beta}\right)\right) & (k=\pm1,\pm2,\ldots,\pm m)\\ b_r=o(n^r)\quad(r=0,1,\ldots,P-1),\\ \left.\begin{aligned}c_r&=o(n^{2r})\\ d_r&=o(n^{2r})\end{aligned}\right\}\quad(r=0,1,\ldots,q-1).\end{cases} \tag{3.17}$$

4. Proofs

4.1. Using [5] we have

$$\frac{c_1}{n}\leqq\vartheta_{k+1,n}-\vartheta_{k,n}\leqq\frac{c_2}{n}\quad(k=0,1,\ldots,n) \tag{4.1}$$

with $0<c_1=c_1(\alpha,\beta)$, $c_2=c_2(\alpha,\beta)$, $\vartheta_0=0$, $\vartheta_{n+1}=\pi$. Further

$$P_n^{(\alpha,\beta)}(x)=(-1)^nP_n^{(\beta,\alpha)}(-x),\ P_n^{(\alpha,\beta)}(1)=\binom{n+\alpha}{n}\sim n^\alpha, \tag{4.2}$$

$$|P_n^{(\alpha,\beta)}(\cos\vartheta)|=\begin{cases}O(\vartheta^{-\alpha-1/2}n^{-1/2}) & \text{if}\quad\dfrac{c_3}{n}\leqq\vartheta\leqq\pi-\varepsilon,\\[2ex] O(n^\alpha) & \text{if}\quad 0\leqq\vartheta\leqq\dfrac{c_3}{n},\end{cases} \tag{4.3}$$

$$|P_n^{(\alpha,\beta)'}(x_k)| \sim \begin{cases} k^{-\alpha-3/2} n^{\alpha+2} & (0 < \vartheta_k \leqq \pi-\varepsilon), \\ (n-k+1)^{-\beta-3/2} n^{\beta+2} & (\varepsilon \leqq \vartheta_k \leqq \pi), \end{cases} \tag{4.4}$$

$$P_n^{(\alpha,\beta)'}(x) = \frac{1}{2}(n+\alpha+\beta+1) P_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{4.5}$$

$$\frac{P_n^{(\alpha,\beta)''}(x_k)}{P_n^{(\alpha,\beta)'}(x_k)} = \frac{\alpha-\beta+(\alpha+\beta+2)x_k}{1-x_k^2} \tag{4.6}$$

(see [5], (4.1.3), (4.1.1), (7.32.5), (8.9.2), (8.9.9), (4.21.7) and (14.5.1)).

4.2. To prove our theorems we need several lemmas. Using these relations, (2.3) and (3.12) we easily get the following estimation:

LEMMA 4.1. *If* $k \neq j$ *where* $|x-y_j| = \min_k |x-y_k|$, $x = \sin\frac{\vartheta}{2} > 0$, $\varepsilon > 0$ *then*

A) *if* $0 \leqq \vartheta \leqq \pi-\varepsilon$

$$l_k^2(x) = \begin{cases} O\left(\left|\frac{k}{j}\right|^{2\alpha+1} \frac{1}{(k-j)^2}\right) & \text{if } 0 < \vartheta_k < \pi-\varepsilon, \\ O\left(n^{2\alpha-2\beta-2} \frac{(n-k+1)^{2\beta+3}}{j^{2\alpha+1}} \frac{1}{(k-j)^2}\right) & \text{if } \varepsilon \leqq \vartheta_k < \pi, \end{cases} \tag{4.7}$$

B) *if* $\varepsilon \leqq \vartheta \leqq \pi$

$$l_k^2(x) = \begin{cases} O\left(n^{2\beta-2\alpha} \frac{|k|^{2\alpha+1}}{(n+1-j)^{2\beta+1}} \frac{1}{(j-k)^2}\right) & \text{if } 0 < \vartheta_k < \pi-\varepsilon, \\ O\left(\frac{1}{(j-k)^2} \frac{(n+1-k)^{2\beta+3}}{(n+1-j)^{2\beta+1}} \frac{1}{(2n+2-j-k)^2}\right) & \text{if } \varepsilon \leqq \vartheta_k < \pi. \end{cases} \tag{4.8}$$

One can deduce from [6], (4.1) by a rather long calculation using only the estimations of Lemma 4.1, the following

LEMMA 4.2. *We have*

$$\sum_k |v_k(x)| l_k^2(x) = \begin{cases} O(1+n^{2\alpha}) & \text{if } |x| < \varepsilon, \\ O(1) & \text{if } \varepsilon \leqq |x| \leqq 1-\varepsilon, \\ O\left(1+n^{2\beta-2\alpha-1} \sum_{k=1}^{n} k^{2\alpha}\right) & \text{if } 1-\varepsilon < |x| \leqq 1, \end{cases} \tag{4.9}$$

$$\sum_k |x-y_k| l_k^2(x) = \begin{cases} O\left(\max\left(\frac{\log n}{n}, n^{2\alpha}\right)\right) & \text{if } |x| < \varepsilon, \\ O\left(\frac{\log n}{n}\right) & \text{if } \varepsilon \leqq |x| \leqq 1-\varepsilon, \\ O\left(\max\left(\frac{\log n}{n}, n^{2\beta}\right)\right) & \text{if } 1-\varepsilon < |x| \leqq 1. \end{cases} \tag{4.10}$$

Let $\omega(x) = P_m(1-2x^2)$, $\Omega(x) = \omega^2(x)$ and $\Delta(x) = f(x) - H_n(x)$.

LEMMA 4.3.

$$\left[\frac{\Omega(x)}{\Omega(1)}\right]^{(s)}_{x=\pm 1} \sim n^{2s}, \quad \frac{\Omega^{(s)}(0)}{\Omega(0)} \begin{cases} =0 & \text{if } s \text{ is odd,} \\ \sim n^s & \text{if } s \text{ is even.} \end{cases} \tag{4.11}$$

PROOF OF LEMMA 4.3. With induction we get

$$\omega^{(2k)}(x) = \sum_{j=0}^{k} \gamma_{2k,j} x^{2j} P_m^{(k+j)}(1-2x^2)(-1)^{k+j}, \tag{4.12}$$

$$\omega^{(2k+1)}(x) = \sum_{j=0}^{k} \gamma_{2k+1,j} x^{2j+1} P_m^{(k+j+1)}(1-2x^2)(-1)^{k+j+1} \tag{4.13}$$

where $\gamma_{k,j}>0$ are independent of m. By [5], (4.1.1) and (4.21.7),

$$\frac{P_n^{(k)}(1)}{P_n(1)} = \frac{n^{2k}}{2^k(\alpha+1)(\alpha+2)\ldots(\alpha+k)}(1+o(1)). \tag{4.14}$$

Using the relation

$$[\omega^2(x)]^{(s)} = \sum_{j=0}^{s} \binom{s}{j} [\omega(x)]^{(j)} [\omega(x)]^{(s-j)}$$

and (4.12), (4.13), (4.14) we get our statement.

LEMMA 4.4. A) *If* $0 \leqq \alpha \leqq \beta$ *then*

$$\frac{\Omega(x)}{\Omega(0)}(1-x^2)^q = O(1).$$

B) *If* $0 \leqq \beta \leqq \alpha$ *then*

$$\frac{\Omega(x)}{\Omega(1)} x^P = O(1).$$

PROOF OF LEMMA 4.4. It is simple consequence of (4.3).

4.3. Proof of Theorem 3.1. To prove our theorem it is sufficient to show that

$$\sum_k |F_k(x)| = O(1), \tag{4.15}$$

$$\sum_k |A_k(x)| = O(1) \max\left(\frac{\log n}{n},\ n^{2\alpha-P},\ n^{2\beta-2q}\right), \tag{4.16}$$

$$|B_r(x)| = O(n^{-r}) \quad (r = 0, 1, \ldots, P-1), \tag{4.17}$$

$$\left.\begin{aligned} |C_r(x)| &= O(n^{-2r}) \\ |D_r(x)| &= O(n^{-2r}) \end{aligned}\right\} \quad (r = 0, 1, \ldots, q-1) \tag{4.18, 4.19}$$

(see [8], 3.3). But using Lemma 4.1, (4.1) and (2.3) we get in the case of $0 \leqq \vartheta \leqq \pi - \varepsilon$

$$\left(\frac{x^2-1}{y_k^2-1}\right)^q \left(\frac{x}{y_k}\right)^p l_k^2(x) = \tag{4.20}$$

$$= \begin{cases} O\left(\left|\frac{k}{j}\right|^{2\alpha-p+1} \frac{1}{(k-j)^2}\right) & \text{if} \quad 0 < \vartheta_k < \pi - \varepsilon \\ O\left(n^{2\alpha-p-2(\beta-q)-2} \frac{(n-k+1)^{2(\beta-q)+3}}{j^{2\alpha-p+1}} \frac{1}{(k-j)^2}\right) & \text{if} \quad \varepsilon \leqq \vartheta_k < \pi. \end{cases}$$

Similarly if $\varepsilon \leqq \vartheta \leqq \pi$

(4.21)

$$\left(\frac{x^2-1}{y_k^2-1}\right)^q \left(\frac{x}{y_k}\right)^p l_k^2(x) =$$

$$= \begin{cases} O\left(n^{2(\beta-q)-2(\alpha-p/2)} \frac{|k|^{2(\alpha-p/2)+1}}{(n+1-j)^{2(\beta-q)+1}} \frac{1}{(j-k)^2}\right) & \text{if} \quad 0 < \vartheta_k \leqq \pi - \varepsilon, \\ O\left(\frac{(n+1-k)^{2(\beta-q)+3}}{(n+1-j)^{2(\beta-q)+1}} \frac{1}{(k-j)^2}\right) & \text{if} \quad \varepsilon \leqq \vartheta_k < \pi. \end{cases}$$

By (3.13) and (4.6) it follows

$$w_k(x) = 1 + \frac{x - y_k}{y_k} \left[2 \frac{\left(\alpha - \frac{p}{2}\right) - (\beta - q) + \left(\left(\alpha - \frac{p}{2}\right) + (\beta - q) + 2\right) x_{|k|}}{1 + x_{|k|}} - 1 \right] \tag{4.22}$$

so

$$w_{k,p,q}^{(\alpha,\beta)}(x) \sim v_k^{(\alpha-p/2,\beta-q)}(x). \tag{4.23}$$

Hence Lemma 4.2, (4.20), (4.21) and (4.23) give (4.15) and (4.16). To obtain (4.18) we first prove

$$|c_{jr}| = \begin{cases} 0 & \text{if} \quad 0 \leqq j < r \\ O(n^{2(j-r)}) & \text{if} \quad r \leqq j \leqq q-1 \end{cases} \tag{4.24}$$

for $0 \leqq r \leqq p-1$.

Indeed, by (3.15), for any fix r $(0 \leqq r \leqq q-1)$

$$C_r^{(s)}(1) = \sum_{j=0}^{s} \binom{s}{j} \left(\sum_{i=0}^{q-1} \frac{(x-1)^i}{i!} c_{ir} \right)_{x=1}^{(j)} \left(\frac{\Omega(x)}{\Omega(1)} x^p \left(\frac{1+x}{2} \right)^q \right)_{x=1}^{(s-j)} =$$

$$= \sum_{j=0}^{s} \binom{s}{j} c_{jr} \sum_{t=0}^{s-j} \binom{s-j}{t} \left(\frac{\Omega(x)}{\Omega(1)} \right)_{x=1}^{(t)} \left(x^p \left(\frac{1+x}{2} \right)^q \right)_{x=1}^{(s-j-t)} = \sum_{j=0}^{s} a(j,s) c_{jr}.$$

By Lemma 4.3 we get $|a(j,s)| \sim n^{2(s-j)}$. By these relations, using the above recur-

rence relations, we immediately obtain (4.24). Now

$$|C_r(x)| = O\left(\frac{P_m^2(\cos\vartheta)}{P_m^2(-1)}\left|\sin\frac{\vartheta}{2}\right|^P \sum_{j=r}^{q-1}\left(\cos^2\frac{\vartheta}{2}\right)^j n^{2j-2r}\right) =$$

$$= \frac{O(1)}{n^{2r}}\frac{P_m^2(\cos\vartheta)}{P_m^2(-1)}\left|\sin\frac{\vartheta}{2}\right|^P \sum_{j=r}^{q-1}\left|n\cos\frac{\vartheta}{2}\right|^{2j}$$

from where we obtain (4.18) considering the cases $\left|n\cos\frac{\vartheta}{2}\right| \geqq 1$ and $\leqq 1$ separately.

In a similar way we may get (4.17) and (4.19), too.

4.4. Proof of Theorem 2.1. First we verify that (2.4) implies the conditions of Theorem 2.1. (2.5) and (2.7) are obvious. To obtain (2.8) we use an estimation of Stečkin [3] which says that for any polynomial $p_m(x)$ of degree $\leqq m$

$$|p_m'(x)| \leqq O(m^2)\,\mathscr{W}\left(p_m, \frac{1}{m}\right) \quad (x\in[-1,1])$$

where $\mathscr{W}$ is the modulus of continuity. From this using the Markov inequality, we obtain

$$|p_m^{(r)}(x)| = O(m^{2r})\mathscr{W}\left(p_m, \frac{1}{m}\right) = O(m^{2r})\left[\mathscr{W}\left(p_m - f, \frac{1}{m}\right) + \mathscr{W}\left(f, \frac{1}{m}\right)\right] =$$

$$= O(m^{2r})\left[2\|p_m - f\| + \mathscr{W}\left(f, \frac{1}{m}\right)\right] \quad (f\in C[-1,1]).$$

Now if $p_m(x) = H_n(f,x)$ we obtain (2.8). If we apply another estimation of Stečkin [3]

$$|p_m'(x)| \leqq \frac{m}{2\sqrt{1-x^2}}\mathscr{W}\left(p_m, 2\sin\frac{\pi}{2m}\right) \quad (|x|<1)$$

and the Bernstein inequality, we get (2.6). So (2.5)—(2.8) are necessary.

Now we turn to the proof of the sufficiency. We have to deal only with the case when $\max(\alpha,\beta)\geqq 0$.

A) First suppose $\alpha \geqq -\frac{1}{2}$. With the conditions of Theorem 2.1 $\|H_{n,p,q}(x) - f(x)\| \to 0$ so it is enough to prove that $\|H_{n,p,q}(x) - H_n(x)\| \to 0$. In the following we choose A, B, C and D for $H_{n,p,q}$ such that $a_k \equiv 0$ and the values of B, C and D do not depend on p, q but satisfy (3.17). Then it is simple to verify the formulae

$$H_{n,t+1,s}(x) - H_{n,t,s}(x) = \frac{\Omega(x)}{\Omega(0)}\frac{x^{t+\delta}}{(t+\delta)!}(1-x^2)^s\left(b_{t+\delta} - H_{n,t,s}^{(t+\delta)}(0)\right) \tag{4.25}$$

$$(t = 0, 1, \ldots, p-1;\ s = 0, \ldots, q),$$

$$H_{n,t,s+1}(x) - H_{n,t,s}(x) = \frac{\Omega(x)}{\Omega(1)}x^{t+\delta}\left(\frac{x^2-1}{2}\right)^s\frac{1}{s!}(u_s x + v_s) \tag{4.26}$$

$$(t = 0, 1, \ldots, p;\ s = 0, 1, \ldots, q-1)$$

where $\delta=1-(-1)^n$ and

$$\begin{cases} u_s=\dfrac{1}{2}\{[c_s-H^{(s)}_{n,t,s}(1)]-(-1)^{t+s}[d_s-H^{(s)}_{n,t,s}(-1)]\} \\ v_s=\dfrac{1}{2}\{[c_s-H^{(s)}_{n,t,s}(1)]+(-1)^{t+s}[d_s-H^{(s)}_{n,t,s}(-1)]\}. \end{cases} \tag{4.27}$$

B) Let n be even and $\alpha\leqq\beta$. Then by (4.26), (4.27) and (2.7)

$$u_0=\frac{1}{2}\{\Delta(1)-\Delta(-1)\}=o(1), \quad v_0=\frac{1}{2}\{\Delta(1)+\Delta(-1)\}=o(1)$$

and

$$H_{n,0,1}(x)=H_n(x)+\frac{\Omega(x)}{\Omega(1)}(u_0x+v_0)$$

so according to Lemma 4.3 and (2.8)

$$(H_{n,0.1}(x))^{(r)}_{x=\pm1}=(H_n(x))^{(r)}_{x=\pm1}+\left(\frac{\Omega(x)}{\Omega(1)}(u_0x+v_0)\right)^{(r)}_{x=\pm1}=o(n^{2r}) \quad (r=0,1,\ldots,q-1).$$

Hence if we shoose c_r as $H^{(r)}_{n,0,1}(1)$ and d_r as $H^{(r)}_{n,0,1}(-1)$ $(r=1,2,\ldots,q-1)$ then (3.17) fulfils and by induction we get

$$H_{n,0,q}(x)-H_n(x)=\frac{\Omega(x)}{\Omega(1)}(u_0x+v_0)=o(1). \tag{4.28}$$

If $\alpha<0$ then we are ready. If $\alpha\geqq0$ then by (4.25) and (4.28) we obtain

$$H_{n,1,q}(x)=H_n(x)+\frac{\Omega(x)}{\Omega(1)}(u_0x+v_0)-\frac{\Omega(x)}{\Omega(0)}(1-x^2)^q\Delta(0)-v_0\frac{\Omega(x)}{\Omega(1)}(1-x^2)^q.$$

Applying again Lemma 4.3 we get $(r=1,2,\ldots,p-1)$

$$H^{(r)}_{n,1,q}(0)=o(n^r)+\frac{\Omega(0)}{\Omega(1)}o(n^r)+o(n^r)+\frac{\Omega(0)}{\Omega(1)}o(n^r)=o(n^r).$$

So if we choose b_r as $H^{(r)}_{n,1,q}(0)$ then $b_r=o(n^r)$ and $H_{n,p,q}(x)=H_{n,1,q}(x)$ i.e. by Lemma 4.4

$$H_{n,p,q}(x)-H(x)=\frac{\Omega(x)}{\Omega(1)}[(u_0x+v_0)-v_0(1-x^2)^q]-\frac{\Omega(x)}{\Omega(0)}(1-x^2)^q\Delta(0)=o(1).$$

C) Let n be even and $\beta\leqq\alpha$. Now we apply (4.25) first:

$$H_{n,1,0}(x)=H_n(x)+\frac{\Omega(x)}{\Omega(0)}\Delta(0).$$

By (2.5), (2.6) and Lemma 4.3 we get

$$H^{(r)}_{n,1,0}(0)=H^{(r)}_n(0)+o(n^r)=o(n^r) \quad (r=1,2,\ldots,p-1).$$

This means that if we choose b_r as $H^{(r)}_{n,1,0}(0)$ then $b_r=o(n^r)$ and

$$H_{n,p,0}(x)-H_n(x)=\frac{\Omega(x)}{\Omega(0)}\Delta(0).$$

If $\beta<0$ then we are ready. If $\beta\geqq 0$ then by (4.27), (2.5) and (2.8)

$$u_0=\frac{1}{2}\left\{\left[\Delta(1)-\frac{\Omega(1)}{\Omega(0)}\Delta(0)\right]-(-1)^p\left[\Delta(-1)-\frac{\Omega(1)}{\Omega(0)}\Delta(0)\right]\right\}=o(1),$$

$$v_0=\frac{1}{2}\left\{\left[\Delta(1)-\frac{\Omega(1)}{\Omega(0)}\Delta(0)\right]+(-1)^p\left[\Delta(-1)-\frac{\Omega(1)}{\Omega(0)}\Delta(0)\right]\right\}=o(1),$$

so applying (4.26) we obtain

$$H_{n,p,1}(x)=H_n(x)+\frac{\Omega(x)}{\Omega(0)}\Delta(0)+\frac{\Omega(x)}{\Omega(1)}x^p(u_0x+v_0).$$

Hence again by Lemma 4.3 and (2.8)

$$[H^{(r)}_{n,p,1}(x)]_{x=\pm 1}=o(n^{2r})+\frac{\Omega(1)}{\Omega(0)}O(n^{2r})o(1)+O(n^{2r})o(1)=o(n^{2r}).$$

By the appropriate selection of c_r and d_r it follows using Lemma 4.4:

$$H_{n,p,q}(x)-H_n(x)=\frac{\Omega(x)}{\Omega(0)}\Delta(0)+\frac{\Omega(x)}{\Omega(1)}x^p(u_0x+v_0)=o(1).$$

D) If n is odd then by (4.25) we obtain

$$H_{n,1,0}(x)=H_n(x)+\frac{\Omega(x)}{\Omega(0)}\frac{x^2}{2}(b_2-H_n''(0)).$$

Now we suppose $b_2=H_n''(0)$ and generally $b_r=H_n^{(r)}(0)$ $(r=2,\ldots,P-1)$. So we have got $H_{n,p,0}(x)=H_n(x)$. If $\beta<0$ we are ready. Let us suppose that $p\geqq 0$. Now we apply (4.26)

$$H_{n,p,1}(x)=H_n(x)+\frac{\Omega(x)}{\Omega(1)}x^{p+2}(u_0x+v_0)$$

where

$$u_0=\frac{1}{2}[\Delta(1)-(-1)^p\Delta(-1)]=o(1),\quad v_0=\frac{1}{2}[\Delta(1)+(-1)^p\Delta(-1)]=o(1).$$

Using again Lemma 4.3 it follows

$$[H_{n,p,1}(x)]^{(r)}_{x=\pm 1}=o(n^{2r})\quad (r=1,2,\ldots,q-1)$$

so we may choose c_r and d_r so that v_r and u_r be 0 if $r=1,2,\ldots,q-1$. Hence

$$H_{n,p,q}(x)=H(x)+\frac{\Omega(x)}{\Omega(1)}x^{p+2}(u_0x+v_0).$$

By Lemma 4.4 this means that $H_{n,p,q}(x)-H_n(x)=o(1)$.

E) Now we deal with the case when $-1<\alpha<-\frac{1}{2}$ and $\beta\geqq 0$. In this case it is enough to prove the convergence only if $\frac{1}{2}\leqq|x|\leqq 1$ (see (2.9)).

First let n be even. Then we define a new operator

$$H_{n,q}(f,x)=H_{n,q}(x)=\sum_{|k|=1}^{m} f(y_k)F_k(x)+\sum_{r=0}^{q-1}\left(c_r C_r(x)+d_r D_r(x)\right)$$

where

$$F_k(x)=\left(\frac{x^2-1}{y_k^2-1}\right)^q \frac{y_k}{x}w_k(x)l_k^2(x),\quad C_r(x)=\frac{\Omega(x)}{\Omega(1)}\left(\frac{1+x}{2}\right)^q\frac{1}{x}\sum_{j=0}^{q-1}\frac{(x-1)^j}{j!}c_{jr},$$

$$D_r(x)=\frac{\Omega(x)}{\Omega(1)}\left(\frac{1-x}{2}\right)^q\left(\frac{-1}{x}\right)\sum_{j=0}^{q-1}\frac{(x+1)^j}{j!}d_{jr}.$$

Here $w_k(x)$ is a linear function determined by the relations $F_k(y_k)=1$, $F_k'(y_k)=0$ and for C_r and D_r it is true that $C_r^{(s)}(1)=D_r^{(s)}(-1)=\delta_{rs}$ $(s=0,1,\ldots,q-1)$. Evidently $H_{n,q}(x)$ plays the role of the operator "$H_{n,-1,q}(x)$". Analogously to Theorem 3.1 one can prove the following

LEMMA 4.5. *If* $c_r=o(n^{2r})$ *and* $d_r=o(n^{2r})$ *then for any* $f\in C$

$$\lim_{n\to\infty}\left(H_{n,q}(f,x)-f(x)\right)=0 \quad \textit{if} \quad \frac{1}{2}\leqq|x|\leqq 1.$$

Obviously

$$x\left(H_{n,q}(x)-H_n(x)\right)=\Omega(x)Q(x) \tag{4.29}$$

where $Q(x)$ is a polynomial with $\deg Q\leqq 2q-1$. Hence

$$\Delta(1)=\Omega(1)Q(1),\quad \Delta(-1)=-\Omega(1)Q(-1) \tag{4.30}$$

and

$$\left.\begin{aligned} c_r-H_n^{(r)}(1)&=\left[\frac{\Omega(x)}{x}Q(x)\right]_{x=1}^{(r)}\\ d_r-H_n^{(r)}(-1)&=\left[\frac{\Omega(x)}{x}Q(x)\right]_{x=-1}^{(r)}\end{aligned}\right\}\quad (r=1,2,\ldots,q-1). \tag{4.31}$$

For $r=1$ we get (using (4.30))

$$Q'(1)\Omega(1)=c_1-H_n'(1)-Q(1)\left(\frac{\Omega(x)}{x}\right)'_{x=1}=c_1+o(n^2)+o(n^{-2\beta})O(n^{2+2\beta})=c_1+o(n^2)$$

i.e. we can choose c_1 so that $c_1=o(n^2)$ and $Q'(1)=0$. By induction one can prove that with the appropriate choice of c_r and d_r, (3.17) and $Q^{(r)}(\pm 1)=0$ $(r=1,2,\ldots,q-1)$ hold. Let us take the following Hermite interpolating polynomials G_1 and G_2:

$$\left.\begin{aligned} G_1^{(r)}(1)&=\delta_{r,0},\ G_1^{(r)}(-1)=0\\ G_2^{(r)}(1)&=0,\ G_2^{(r)}(-1)=\delta_{r,0}\end{aligned}\right\}\quad (r=0,1,\ldots,q-1).$$

As it is well known, G_1 and G_2 exist and are unique. Thus we have got the following representation

$$Q(x) = \frac{1}{\Omega(1)}[\Delta(1)G_1(x) - \Delta(-1)G_2(x)]$$

i.e.

$$H_{n,q}(x) - H_n(x) = \frac{\Omega(x)}{\Omega(1)}\frac{1}{x}[\Delta(1)G_1(x) - \Delta(-1)G_2(x)].$$

By Lemma 4.4 we got the convergence if $|x| \geqq \frac{1}{2}$.

F) The remaining case, i.e. when n is odd and $-1 < \alpha < -\frac{1}{2}$, $\beta \geqq 0$ can be treated similarly to case E.

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ON THE CHARACTERIZATION OF COMPLEX-VALUED MULTIPLICATIVE FUNCTIONS. I

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I. If f is a real-valued monotonic, multiplicative function then $f(n)=n^k$, if $f(m)\neq 0$ for any $m\in\mathbf{N}$ [1]. It is easy to prove that this theorem is true on an arbitrary set A having upper density one $\left(\text{i.e. } \overline{\lim}\frac{1}{N}\sum_{a\in A, a\leqq N} 1=1\right)$.

Let us consider the case when f is a complex-valued multiplicative function. If we assume that $|f|$ is monotonic, then it is obvious that $|f(n)|=n^k$, namely $|f|$ is a real-valued multiplicative function. Then the following theorem is true.

THEOREM. *Let f denote a complex-valued multiplicative function. If at least two of the functions* $\mathrm{Re}\, f$, $\mathrm{Im}\, f$ *and* $|f|$ *are montonic on a set A having upper density one, then $f(n)=n^k$, except the cases* a) $f(a)=0$ *for all* $a\in A$, $a\geqq a_0$ *and* b) $\mathrm{Im}\, f\equiv 0$. *(In this case $f(p^k)=p^k$ or $-p^k$ is possible for any prime p.)*

A simple example $\mathrm{Im}\, f\equiv 0$ shows that it is not enough to assume the monotonicity of $\mathrm{Im}\, f$ to have $f(n)=n^k$. Another simple example shows, that it is not enough to assume the monotonicity of $\mathrm{Re}\, f$ to have $f(n)=n^k$. Namely let us define for any $\nu\in\mathbf{N}$

$$f(p^\nu)=\begin{cases} p_0^{\nu k}e^{i\varphi_0}/\cos\varphi_0 & \text{if} \quad p=p_0 \\ p^{\nu k} & \text{if} \quad p\neq p_0, \end{cases}$$

where p_0 is an arbitrary fixed prime. One can easily see that $\mathrm{Re}\, f=n^k$, but $f(n)\neq n^k$. P. Erdős conjectures that the monotonicity of $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$, resp. implies $f(n)=n^k$ excluding the mentioned counterexamples.

II. To prove the theorem first we introduce the notations $f_1:=\mathrm{Re}\, f$ and $f_2:=\mathrm{Im}\, f$. Further $f_l\geqq 0$ denotes that $f_l(n)\geqq 0$ for all $n\in\mathbf{N}$ ($l=1$ or $l=2$).

PROPOSITION 1. *Let $k, c_1, \ldots, c_\nu$, be finitely many fixed natural numbers. There exists an $n\in\mathbf{N}$ such that $(n,k)=1$, $nc_1, nc_2, \ldots, nc_\nu\in A$.*

PROOF. In the opposite case we get for arbitrary x at least $\frac{\varphi(k)x}{\nu}$ natural numbers n in the interval $[1, kx]$ such that $(n,k)=1$ and with an $i_0\in[1,\nu]$, $nc_{i_0}\notin A$, a contradiction, since A is a set having upper density one. □

This clearly implies

COROLLARY 1. *For any $c_1, \ldots, c_\nu$ there exists a set $\{t_m\}_{m=1}^{\infty}$, such that for any $i\neq j$ $(t_i, t_j)=1$ and $t_m c_1, \ldots, t_m c_\nu\in A$ and $\prod_{m\in M} t_m\in A$ for any finite non-empty set $M\subset\mathbf{N}$.*

PROPOSITION 2. *Let B be a set and suppose $\{b_i\}_{i=1}^{\infty} \subset B$ where for any $i \neq j$ $(b_i, b_j)=1$ and $\prod_{i \in M} b_i \in B$ for any finite non-empty set $M \subset \mathbf{N}$. Then the following assertions hold:*

(i) *If $f_1(a) \geqq 0$ ($(f_1(a) \leqq 0)$ for all $a \in B$, $a \geqq a_0$, then $\lim_{i \to \infty} \operatorname{arc} f(b_i)=0$.*

(ii) *If $f_2 \geqq 0$ ($f_2 \leqq 0$) for all $a \in B, a \geqq a_0$, then $(\{\operatorname{arc} f(b_i)\}_{i=1}^{\infty})' \subset \{0, \pi\}$, where $(M)'$ denotes the set of cluster points of M.*

PROOF. (i) If φ is the cluster point of $\operatorname{arc} f(b_i)$, then $k\varphi$ is a cluster point of the set $\operatorname{arc} \prod_{j=1}^{k} f(b_{i_j})$, thus for any natural number k

$$-\frac{\pi}{2} \leqq k\varphi \leqq \frac{\pi}{2} (\operatorname{mod} 2\pi) \quad \left(\frac{\pi}{2} \leqq k\varphi \leqq \frac{3\pi}{2} (\operatorname{mod} 2\pi)\right),$$

in wiew of $f_1 \geqq 0$ ($f_1 \leqq 0$), consequently only $\varphi \equiv 0 \bmod 2\pi$ is possible.

(ii) The proof is similar to (i). □

LEMMA 1. *Let $l=1$ or $l=2$. If $f_l(a) \geqq 0$ $(f_l(a) \leqq 0)$ for all $a \in A$, $a \geqq a_0$, then $f_l \geqq 0$ $(f_l \leqq 0)$.*

PROOF. Let us assume that there exists an s_0 such that $f_l(s_0)<0$ $(f_l(s_0)>0)$. Using Proposition 1, there exists a set $\{b_i\}_{i=0}^{\infty}$, such that $b_0=s_0$, $(b_i, b_j)=1$ if $0 \leqq i<j$, moreover $\prod_{j \in M} b_j \in A$, $s_0 \prod_{j \in M} b_j \in A$ for any finite non-empty set $M \subset \mathbf{N}$ and $\lim_{i \to \infty} \operatorname{arc} f(b_i)=0$. In case $l=1$ this follows immediately from Propositions 1 and 2. In case $l=2$ Proposition 2 ($B:=A$) gives a sequence (b_i) with the property $(\{\operatorname{arc} f(b_i)\}_{i=1}^{\infty})' \subset \{0, \pi\}$. If $0 \in (\{\operatorname{arc} f(b_i)\}_{i=1}^{\infty})'$, then we have a subsequence (b_{i_j}) with $\lim_{j \to \infty} \operatorname{arc} f(b_{i_j})=0$ and then we work with the sequence (b_{i_j}) instead of the original (b_i). If $\lim_{i \to \infty} \operatorname{arc} f(b_i)=\pi$, we work with the sequence $(b_{2i-1} b_{2i})$ instead of the original (b_i). Thus we get $\lim_{i \to \infty} \operatorname{arc} f(s_0 b_i)=\operatorname{arc} f(s_0)$, a contradiction. □

Lemma 1 implies

COROLLARY 2. *If f_l ($l=1$ or $l=2$, resp.) is monotonic on A, then $f_l \geqq 0$ or $f_l \leqq 0$.*

COROLLARY 3. *If $l=1$ or $l=2$, A is a set having upper density one and $f_l(a) \geqq 0$ $(f_l(a) \leqq 0)$ for all $a \in A$, $a \geqq a_0$ (for example if f_1 or f_2 is monotonic on A), then in case $l=1$*

$$\lim_{i \to \infty} \operatorname{arc} f(c_i) = 0$$

and in the case $l=2$

$$\lim_{i \to \infty} \operatorname{arc} f(c_i) = 0 \quad \text{or} \quad f_2 \equiv 0,$$

whenever (c_i) is a sequence of pairwise coprime numbers.

This follows for $l=1$ from Lemma 1 and Proposition 2 (with $B:=\mathbf{N}$).

For $l=2$ Lemma 1 and Proposition 2 ($B:=\mathbf{N}$) imply $f_2 \geqq 0$ or $f_2 \leqq 0$ and $(\{\arg f(c_i)_{i=1}^{\infty})' \subset \{0, \pi\}$ for all sets $\{c_i\}_{i=1}^{\infty}$, where $(c_i, c_j)=1$ for $i \neq j$. Now let us assume the existence of an x_0 with $f_2(x_0) \neq 0$ and a subsequence $\{c_{i_\nu}\}_{\nu=1}^{\infty}$ with $\lim_{\nu \to \infty} \arg f(c_{i_\nu}) = \pi$. Then as $(c_i, c_j)=1$ for $i \neq j$, we have $(x_0, c_i)=1$ for $i \geqq i_0$. Thus

$$\lim_{\nu \to \infty} \arg f(x_0 c_{i_\nu}) = \pi + \arg f(x_0)$$

which implies $\operatorname{sgn} f_2(x_0 c_{i_\nu}) = -\operatorname{sgn} f_2(x_0)$ for $\nu \geqq \nu_0$, a contradiction. □

LEMMA 2. *If f_1 is monotonic on A, then $f_1 \geqq 0$.*

PROOF. If there exists an $a \in A$ such that $f_1(a) > 0$, then using the monotonicity of f_1 on A and Lemma 1, we have $f_1 \geqq 0$. Thus we shall prove the existence of an $a \in A$ with $f(a) > 0$.

Using Proposition 1, there exist $a_i \in A$ $(i=1, 2, \ldots, 9)$ pairwise coprime numbers such that $\prod_{j \in M} a_j \in A$ for any $M \subseteq \{1, 2, \ldots, 9\}$.

If

$$\frac{\pi}{2} \leqq \arg f(a_j) \leqq \frac{3\pi}{2} \pmod{2\pi} \quad (1 \leqq j \leqq 9),$$

then we can easily verify that

$$\frac{\pi}{2} \leqq \arg f(a_j a_k) \leqq \frac{3\pi}{2} \pmod{2\pi} \quad (1 \leqq j < k \leqq 9),$$

$$\frac{\pi}{2} \leqq \arg f(a_j a_k a_s) \leqq \frac{3\pi}{2} \pmod{2\pi} \quad (1 \leqq j < k < s \leqq 9),$$

$$\frac{\pi}{2} \leqq \arg f(a_j a_k a_s a_t) \leqq \frac{3\pi}{2} \pmod{2\pi} \quad (1 \leqq j < k < s < t \leqq 9)$$

is not possible at the same time.

(We have either

a) $\geqq 4$ elements with $\arg f(a) = \dfrac{\pi}{2}$

b) $\geqq 4$ elements with $\arg f(a) = \dfrac{3\pi}{2}$

or

c) $\geqq 3$ elements with $\dfrac{\pi}{2} < \arg f(a) < \dfrac{3\pi}{2}$.

Cases a) and b) can be settled trivially. In case c) we can distinguish further

c) (i) there are a', a'' with

$$\frac{\pi}{2} < \arg f(a') \leqq \pi \leqq \arg f(a'') < \frac{3\pi}{2},$$

c) (ii) there are a', a'', a''' with

$$\frac{\pi}{2} < \operatorname{arc} f(a') \leqq \operatorname{arc} f(a'') \leqq \operatorname{arc} f(a''') \leqq \pi,$$

c) (iii) there are a', a'', a''' with

$$\pi \leqq \operatorname{arc} f(a') \leqq \operatorname{arc} f(a'') \leqq \operatorname{arc} f(a''') < \frac{3\pi}{2},$$

and simple considerations yield the assertion.) □

III. In the proof of the theorem we shall distinguish the following cases:

1. *Both $|f|$ and f_1 are monotonic on A.*

a) *Both $|f|$ and f_1 are monotonically increasing on A.* Since $|f|$ is monotonic, taking into account the functions excluded by a), we have $|f(n)|=n^k$. Using Corollary 3, $\lim_{i\to\infty} \operatorname{arc} f(c_i)=0$, whenever (c_i) is a sequence of pairwise coprime numbers. Further Lemma 2 gives $f_1 \geqq 0$, i.e. $\cos \operatorname{arc} f(x_0) \geqq 0$. We prove that $\operatorname{arc} f \equiv 0$, i.e. f is a real-valued non-negative function. Let us assume that there exists an x_0 such that $\operatorname{arc} f(x_0) \neq 0$. Let ε be a positive number satisfying $(1+\varepsilon)^k < \dfrac{1}{\cos \operatorname{arc} f(x_0)}$ (k is defined by $|f(n)|=n^k$). We can choose strictly increasing sequences $\{p_i\}_{i=1}^{\infty}$, $\{q_i\}_{i=1}^{\infty}$, $\{s_i\}_{i=1}^{\infty} \subset \mathbf{N}$ such that p_i, s_j, q_t, x_0 are pairwise coprime numbers for any $i, j, t \in \mathbf{N}$, p_i is a prime for any $i \in \mathbf{N}$ and $\lim_{i\to\infty} \dfrac{s_i}{q_i} = 1+\varepsilon$. If i is sufficiently large we have a prime $P_i \in \left[\dfrac{p_i}{x_0}, \dfrac{p_i}{x_0}\dfrac{s_i}{q_i}\right]$. So $q_i p_i < x_0 q_i P_i < p_i s_i$. Let us assume first $A=\mathbf{N}$. We have

$$x_0^k q_i^k P_i^k < p_i^k s_i^k \tag{1}$$

and

$$f_1(q_i p_i) = q_i^k p_i^k \cos \operatorname{arc} f(q_i p_i) < f_1(x_0 q_i P_i) = x_0^k q_i^k P_i^k \cos \operatorname{arc} f(x_0 q_i P_i) < \tag{2}$$
$$< p_i^k s_i^k \cos \operatorname{arc} f(x_0 q_i P_i),$$

since f_1 is monotonically increasing and $|f(n)|=n^k$. Dividing by $q_i^k p_i^k$ we obtain

$$\cos \operatorname{arc} f(q_i p_i) < \frac{s_i^k}{q_i^k} \cos \operatorname{arc} f(x_0 q_i P_i). \tag{3}$$

In wiew of Corollary 3 we have

$$\lim_{i\to\infty} \operatorname{arc} f(p_i) = \lim_{i\to\infty} \operatorname{arc} f(q_i) = \lim_{i\to\infty} \operatorname{arc} f(P_i) = 0. \tag{4}$$

So for $i \to \infty$

$$\lim_{i\to\infty} \cos \operatorname{arc} f(q_i p_i) = 1 \leqq \lim_{i\to\infty} \frac{s_i^k}{q_i^k} \cos \operatorname{arc} f(x_0 q_i P_i) = (1+\varepsilon)^k \cos \operatorname{arc} f(x_0), \tag{5}$$

which contradicts the definition of ε.

If $A \neq \mathbf{N}$, it is possible that $q_i p_i$, $x_0 q_i P_i$, $p_i s_i$ are not in A. But using Proposition 1 we can find a set $\{t_{im}\}_{m=1}^{\infty}$ for any fixed $i \in \mathbf{N}$ such that $q_i p_i t_{im}$, $x_0 q_i P_i t_{im}$, $p_i s_i t_{im} \in A$ and $(t_{im_1}, t_{im_2})=1$ if $m_1 \neq m_2$, $(t_{im}, p_i s_i q_i x_0 P_i)=1$ for any i, $m \in \mathbf{N}$. So we get similarly to (2)

$$t_{im}^k q_i^k p_i^k \cos \operatorname{arc} f(t_{im} q_i p_i) < t_{im}^k p_i^k s_i^k \cos \operatorname{arc} f(t_{im} x_0 q_i P_i).$$

Dividing by $t_{im}^k q_i^k p_i^k$,

$$\cos \operatorname{arc} f(t_{im} q_i p_i) \leqq \frac{s_i^k}{q_i^k} \cos \operatorname{arc} f(t_{im} x_0 q_i P_i).$$

By Corollary 3 $\lim\limits_{m \to \infty} \operatorname{arc} f(t_{im})=0$ for any fixed $i \in \mathbf{N}$. So if $m \to \infty$, we get (5) again. □

b) *Both $|f|$ and f_1 are monotonically decreasing on A.* We shall prove $\operatorname{arc} f \equiv 0$ similarly to Case a). Let us assume that there exists an x_0 with $\operatorname{arc} f(x_0) \neq 0$. As in Case a) we can find analogous sequences $\{p_i'\}_{i=1}^{\infty}$, $\{q_i'\}_{i=1}^{\infty}$, $\{s_i'\}_{i=1}^{\infty}$, $\{t_{im}'\}_{i=1}^{\infty}{}_{m=1}^{\infty}$, $\{P_i'\}_{i=1}^{\infty}$, with

$$P_i' \in \left[\frac{p_i' s_i'}{x_0 q_i'}, \frac{p_i'}{x_0}\right], \quad \text{where} \quad \lim_{i \to \infty} \frac{s_i'}{q_i'} = \frac{1}{1+\varepsilon}.$$

This leads to a contradiction similarly to Case a).

The cases

c) *$|f|$ is monotonically decreasing and f_1 is monotonically increasing on A,*

d) *$|f|$ is monotonically increasing and f_1 is monotonically decreasing on A*

lead to Cases a) and b), resp., considering the function $\frac{1}{f}$.

2) *$|f|$ and f_2 are monotonic on A.* In this case we get the contradiction similarly to 1), if we exclude the cases mentioned in the theorem.

3) *Both f_1 and f_2 are monotonic on A.*

a) *Both f_1 and f_2 are monotonically increasing on A.* Using Lemma 1 and Lemma 2, $f_1 > 0$, further $f_2 \geqq 0$ or $f_2 \leqq 0$.

(i) If $f_1 \geqq 0$ anf $f_2 \geqq 0$, then $|f|$ is monotonically increasing too, so we use 1/a).

(ii) If $f_1 \geqq 0$ and $f_2 \leqq 0$, then

$$-\frac{\pi}{2} \leqq \operatorname{arc} f(n) \leqq 0 \pmod{2\pi}$$

and $\operatorname{arc} f|_A$ is monotonically increasing. Thus for any $x_0 \in \mathbf{N}$ there exists an $m \in A$, $m > x_0$ such that $m x_0 \in A$ too and $(m, x_0)=1$. Then

$$\operatorname{arc} f(m) \leqq \operatorname{arc} f(m x_0) = \operatorname{arc} f(m) + \operatorname{arc} f(x_0),$$

consequently $\operatorname{arc} f(x_0)=0$. So f is real-valued, which was excluded in the formulation of the theorem.

b) *Both f_1 and f_2 are monotonically decreasing on A.* Similarly to a) we have two cases.

(i) If $f_1 \geqq 0$ and $f_2 \geqq 0$, then $|f|$ is monotonically decreasing too, so we have the case 1b).

(ii) If $f_1 \gneqq 0$ and $f_2 \lneqq 0$, then

$$-\frac{\pi}{2} \leqq \arg f(n) \leqq 0 \pmod{2\pi}$$

and $\arg f|_A$ is monotonically decreasing. By Corollary 3 and Proposition 1, for all sets $\{c_i\}_{i=1}^{\infty} \subset A$ which are sets of pairwise coprime numbers, $\lim_{i \to \infty} \arg f(c_i) = 0$. This implies $f_2 \equiv 0$, which was excluded in the formulation of the theorem. The other cases lead to 2a) and 2b), resp., considering $\frac{1}{f}$. □

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ZEROS OF SCHROEDINGER EIGENFUNCTIONS AT POTENTIAL SINGULARITIES

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Let Ω be a domain in $\mathbf{R}^N$ $(N\geqq 3)$, $0\in\Omega$ and let $q\in L^1_{\mathrm{loc}}(\Omega)$ denote a positive valued function. Recently Alimov and Joó [1], [2] achieved interesting results concerning the summability of eigenfunction expansions of the Schroedinger operator L defined as a self-adjoint extension of L_0: $C_0^\infty(\Omega)\ni f\mapsto -\Delta f+qf$ under the hypothesis $q=q_0+q_1$ where $q_0\in L^2(\Omega)$ is a radially symmetric positive function, $q(x)\leqq\mathrm{const}\cdot|x|^{-2}$ and $q_1\in L^\infty(\Omega)$. Their considerations are based upon Titchmarsh's mean value formula whose proof requires, as far we know (cf. [3]), the Gauss—Green theorem and hence the smoothness of the eigenfunctions of L. Nevertheless, the above conditions on q seem not too restrictive from the view points of applications e.g. in quantum chemistry, except for the assumption on the singularity of q_0. The aim of this short note is to point out that a bit stronger singularity of the potential q at 0 entails the vanishing of all the Schroedinger eigenfunctions at 0 even in a much more general setting. This fact may have interest in itself from the view points of quantum chemistry [4].

PROPOSITION. *Suppose $u\in C(\Omega)$ is a function satisfying the equation $-\Delta u+qu=\lambda u$ in distribution sense for some λ. Then $u(0)=0$ unless* $\varlimsup_{r\downarrow 0} r^{-N+2}\int_{|x|<r} q(x)\,dx<\infty$.

PROOF. Let $u(0)\neq 0$. Then we may assume without loss of generality $u(0)=1$ and we may choose $r_0>0$ such that $\{x\colon |x|<r_0\}\subset\Omega$ and $\sup_{|x|<r_0}|u(x)-1|\leqq 1/2$. Let us fix $r\in(0, r_0)$ arbitrarily and consider the function $\psi_0(r)\equiv\operatorname{sgn}\left(\tau-\frac{3}{4}r\right)\chi(\tau)$ where χ denotes the characteristic function of the interval $\left(\frac{r}{2}, r\right)$. Observe that for a suitable sequence $\psi_n\in C_0^\infty\left(\frac{r}{2}, r\right)$, $|\psi_n|\leqq 1$ $(n=1, 2, \ldots)$ we have $\psi_n(\tau)\to\psi_0(\tau)$ $(n\to\infty)$ whenever $\tau\in[0, \infty)$. Now define the functions $\varphi_n\colon [0, \infty)\to\mathbf{R}$ and $f_n\colon \Omega\to\mathbf{R}$ by

$$\varphi_n(\varrho)\equiv\int_\varrho^\infty\int_\xi^\infty \psi_n(\tau)\,d\tau\,d\xi \quad\text{and}\quad f_n(x)\equiv\varphi_n(|x|)\quad (n=0, 1, 2, \ldots).$$

Clearly, $f_n\in C_0^\infty(\Omega)$ $(n=1, 2, \ldots)$ and

$$\Delta f_n(x)=\varphi_n''(|x|)+\frac{N-1}{|x|}\varphi_n'(|x|)\to\Delta f_0(x)\quad (n\to\infty)$$

while

$$|\Delta f_n(x)| \leqq 1+\frac{N-1}{|x|}\left||x|-\frac{3}{4}r\right| \leqq \frac{N+1}{2} \quad \text{if} \quad \frac{r}{2} \leqq |x| \leqq r$$

and $\Delta f_n(x)=0$ elsewhere. Thus

$$\int_\Omega u[-\Delta f_n-\lambda f_n] \to \int_\Omega u[-\Delta f_0-\lambda f_0]$$

and

$$\int_\Omega quf_n \to \int_\Omega quf_0 \quad (n \to \infty).$$

By hypothesis,

$$\int_\Omega u(-\Delta f+qf) = \lambda \int_\Omega uf$$

for all $f \in C_0^\infty(\Omega)$, in particular for $f=f_1, f_2, \ldots$. By passing to the limit, hence we obtain

$$\int_\Omega u(\Delta f_0+\lambda f_0) = \int_\Omega quf_0. \tag{1}$$

However,

$$\left|\int_\Omega u(\Delta f_0+\lambda f_0)\right| \leqq \left(\frac{N+1}{2}+|\lambda|\right) \int_{r/2<|x|<r} |u(x)|\,dx \leqq \left(\frac{N+1}{2}+|\lambda|\right)\frac{3}{2}\omega_N r^N$$

where ω_N denotes the volume of the unit ball in $\mathbf{R}^N$. On the other hand,

$$\operatorname{Re}\int_\Omega quf_0 = \int_\Omega q(\operatorname{Re} u)f_0 \geqq \int_{|x|<r/2} q(x)\frac{1}{2}\frac{r^2}{16}\,dx$$

since $q, f_0 \geqq 0$ and $f_0(x)=r^2/16$ whenever $|x|<r/2$. Therefore, by (1), $r^2 \cdot \int_{|x|<r/2} q \leqq$ $\leqq cr^N$ where the constant c does not depend on $r(<r_0)$. This completes the proof.

COROLLARY. *If* $q(x) \geqq c|x|^{-(2+\varepsilon)}$ $(x \in \Omega)$ *for some* $c, \varepsilon > 0$ *then every continuous solution* u *of the Schroedinger equation* $-\Delta u+qu=\lambda u$ *vanishes at* 0.

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SOME THEOREMS ON BEST L_1-APPROXIMATION OF CONTINUOUS FUNCTIONS

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In the last years the problems of unicity of best L_1-approximation of continuous functions have been widely investigated. The development of L_1-approximation theory was inspired by the fact that the unicity of best L_1-approximation does not impose strong restrictions on the approximating subspaces such as the uniqueness of Chebyshev approximation. Indeed, by the famous theorem of A. Haar, only the Haar subspaces can guarantee the unicity of best Chebyshev approximation of each continuous function. This excludes many important families, for example spline functions. Furthermore, Mairhuber's result on the absence of real Haar subspaces of functions of several variables shows that there is no possibility to achieve unicity of Chebyshev approximation in the multivariate case. Since the analogue of Haar's theorem is not true in case of L_1-approximation of continuous functions, the family of unicity subspaces is wider in this case. A remarkable illustration of this phenomenon was given by Galkin [2], Strauss [10], Carroll and Braess [1], Sommer [8] and others who verified the unicity of best L_1-approximation of real continuous functions by different families of spline functions. In [6] we studied the unicity of multivariate L_1-approximation and gave some examples of unicity subspaces of functions of several variables.

In the present note we make an attempt to unify the study of best L_1-approximation considering vector-valued functions on convex bodies in $\mathbf{R}^n$ ($n \geqq 1$). Applying our main result (Theorem 3) we shall obtain some known, as well as new results. In particular, Theorem 3 implies the uniqueness of best L_1-approximation of continuous complex-valued functions by different families of complex splines. This extends the results of [1], [2], [8], [10] to the complex case.

Let us start with some notations and definitions. Let $K \subset \mathbf{R}^n$ ($n \geqq 1$) be a convex body, i.e. K is a convex compact subset of $\mathbf{R}^n$ with nonempty interior. By $C^m(K)$ ($m \geqq 1$) we denote the linear space of continuous functions $f\colon K \to \mathbf{R}^m$, i.e., $C^m(K) = \{f=(f_1, \ldots, f_m)\colon f_i$ is real and continuous on K for each $1 \leqq i \leqq m\}$. Let $W = \{w\colon w$ is measurable, bounded and positive on $K\}$ be the set of weight functions. Then taking an arbitrary $w \in W$ we denote by $C_w^m(K)$ the linear space $C^m(K)$ endowed with the weighted L_1-norm

$$\|f\|_{w,m} = \int_K |f|_m w dx \quad (f \in C^m(K)). \tag{1}$$

Here $|\cdot|_m$ denotes the Euclidean norm in $\mathbf{R}^m$. Let U be a finite dimensional subspace of $C^m(K)$. We shall study the approximation of functions $f \in C_w^m(K)$ by elements of U in the norm (1). $u_0 \in U$ is called a best $L_{w,m}$-approximation of $f \in C_w^m(K)$ if $\|f-u_0\|_{w,m} \leqq \|f-u\|_{w,m}$ for each $u \in U$. If 0 is a best $L_{w,m}$-approximation of f out of U then we say that f is $L_{w,m}$-orthogonal to U. The subspace U is called a unicity

subspace of $C_w^m(K)$ if each $f\in C_w^m(K)$ possesses a unique best $L_{w,m}$-approximation out of U. In what follows $\langle .,. \rangle$ will denote the inner product in $\mathbf{R}^m$; for $f\in C_w^m(K)$ we set $Z(f)=\{x\in K: f(x)=0\}$,

$$\operatorname{sign}_m f = \begin{cases} f/|f|_m, & \text{on } K\setminus Z(f) \\ 0, & \text{on } Z(f). \end{cases}$$

REMARK 1. In the case when $m=1$ we shall omit the index m, i.e. $C^1(K)=C(K)$, $C_w^1(K)=C_w(K)$, $|\cdot|_1=|\cdot|$, $\operatorname{sign}_1 f=\operatorname{sign} f$, $\|\cdot\|_{w,1}=\|\cdot\|_w$, etc.

First of all we shall give a characterization theorem, which is an extension of a known criterion to the vector-valued case (see Singer, [7], p. 46). For the sake of completeness we include the proof, which is based on standard variational arguments.

THEOREM 1. *Let U be a finite dimensional subspace of $C^m(K)$, $w\in W$, $f\in C_w^m(K)$. Then the following statements are equivalent:*

(i) *f is $L_{w,m}$-orthogonal to U;*

(ii) *we have for any $u\in U$*

$$\left| \int_{K\setminus Z(f)} \langle \operatorname{sign}_m f, u\rangle w\,dx \right| \leqq \int_{Z(f)} |u|_m w\,dx. \tag{2}$$

PROOF. (i)$\Rightarrow$(ii). Consider an arbitrary $u\in U$ and $t>0$. Since

$$\frac{1}{t}(\|f+tu\|_{w,m}-\|f\|_{w,m})=\frac{1}{t}\int_{K\setminus Z(f)}(|f+tu|_m-|f|_m)w\,dx+$$

$$+\int_{Z(f)}|u|_m w\,dx=\int_{K\setminus Z(f)}\frac{2\langle f,u\rangle+t|u|_m^2}{|f+tu|_m+|f|_m}w\,dx+\int_{Z(f)}|u|_m w\,dx,$$

we obtain by the Lebesgue dominated convergence theorem

$$\lim_{t\to+0}\frac{1}{t}(\|f+tu\|_{w,m}-\|f\|_{w,m})=\int_{K\setminus Z(f)}\langle \operatorname{sign}_m f,u\rangle w\,dx+\int_{Z(f)}|u|_m w\,dx. \tag{3}$$

It follows from (i) that the quantity on the left hand of (3) is nonnegative for any $u\in U$. This immediately implies (ii).

(ii)$\Rightarrow$(i). We have for any $u\in U$

$$\|f\|_{w,m}=\int_{K\setminus Z(f)}|f|_m w\,dx=\int_{K\setminus Z(f)}\langle \operatorname{sign}_m f,f\rangle w\,dx=$$

$$=\int_{K\setminus Z(f)}\langle \operatorname{sign}_m f,f-u\rangle w\,dx+\int_{K\setminus Z(f)}\langle \operatorname{sign}_m f,u\rangle w\,dx\leqq$$

$$\leqq\int_{K\setminus Z(f)}|f-u|_m w\,dx+\int_{Z(f)}|u|_m w\,dx=\|f-u\|_{w,m}.$$

The proof of the theorem is complete.

Our next theorem characterizes the unicity subspaces of $C_w^m(K)$. This characterization is based on inner properties of approximating subspaces. In the real case this result was given by Strauss [12]. For the complex functions a similar statement was

proved in an earlier paper of Havinson [3]. Our theorem generalizes the above mentioned results for the vector-valued functions.

For $U\subset C^m(K)$ we set $U^*=\{u^*\in C^m(K)$: there exists $u'\in U$ such that $u^*(x)=$ $=u'(x)$ or $-u'(x)$ for any $x\in K\}$.

THEOREM 2. *Let $w\in W$ and let U be a finite dimensional subspace of $C^m(K)$. Then the following statements are equivalent:*

(i) *U is a unicity subspace of $C^m_w(K)$;*

(ii) *the only element of U^* which is $L_{w,m}$-orthogonal to U is zero.*

PROOF. (i)$\Rightarrow$(ii). Assume the contrary. Then by Theorem 1 there exists $u^*\in U^*\setminus\{0\}$ such that

$$\left|\int_{K\setminus Z(u^*)}\langle \operatorname{sign}_m u^*,\, u\rangle\, w\,dx\right| \leqq \int_{Z(u^*)} |u|_m\, w\,dx \tag{4}$$

for any $u\in U$. By definition of U^* there exists $u'\in U\setminus\{0\}$ such that $u^*(x)=\pm u'(x)$ for each $x\in K$. Set $\bar u=u^*+\varepsilon u'$, where $0<\varepsilon<1$. Evidently $Z(\bar u)=Z(u')=Z(u^*)$. Moreover, since $u'(x)=\gamma(x)u^*(x)$, where $\gamma(x)\in\mathbf{R}$, $|\gamma(x)|=1$, we have for any $x\in K\setminus Z(\bar u)$

$$\operatorname{sign}_m \bar u = \frac{\bar u}{|\bar u|_m} = \frac{u^*+\varepsilon u'}{|u^*+\varepsilon u'|_m} = \frac{(1+\varepsilon\gamma)u^*}{|1+\varepsilon\gamma|\,|u^*|_m} = \operatorname{sign}_m u^*.$$

Thus it follows from (4) and Theorem 1 that $\bar u$ is $L_{w,m}$-orthogonal to U, i.e. $u^*\in C^m_w(K)$ has two distinct best $L_{w,m}$-approximations, 0 and $-\varepsilon u'\in U\setminus\{0\}$.

(ii)$\Rightarrow$(i). Assume again the contrary. Then there exists $f\in C^m_w(K)$ and $u_1, u_2\in U$, $u_1\neq u_2$ such that u_1 and u_2 are best $L_{w,m}$-approximations of f. Since $(u_1+u_2)/2$ is also a best $L_{w,m}$-approximation we have

$$2\int_K \left|f-\frac{u_1+u_2}{2}\right|_m w\,dx = \int_K (|f-u_1|_m+|f-u_2|_m)\,w\,dx.$$

Thus by continuity of the functions f, u_1 and u_2 we obtain that

$$|(f-u_1)+(f-u_2)|_m = |f-u_1|_m+|f-u_2|_m \tag{5}$$

for any $x\in K$. Therefore

$$\langle f-u_1, f-u_2\rangle = |f-u_1|_m\,|f-u_2|_m \quad (x\in K). \tag{6}$$

Set $f^*=f-\frac{u_1+u_2}{2}\in C^m_w(K)$, $u=u_1-u_2\in U\setminus\{0\}$. By (5), $Z(f^*)\subset Z(u)$. Consider an arbitrary $\bar x\in K\setminus Z(u)$. (6) implies that at $\bar x$, $(f-u_1)=\alpha(f-u_2)$ or $(f-u_2)=$ $=\alpha(f-u_1)$, where $\alpha\in\mathbf{R}$, $\alpha\geqq 0$. Moreover, since $\bar x\in K\setminus Z(u)$, $\alpha\neq 1$. Assume, e.g., that $(f-u_1)=\alpha(f-u_2)$. Then $f=\frac{u_1-\alpha u_2}{1-\alpha}$ and we have by easy calculations

$$f^* = \frac{1}{2}\{(f-u_1)+(f-u_2)\} = \frac{1}{2}(f-u_2)(1+\alpha) = \frac{1+\alpha}{2(1-\alpha)}(u_1-u_2),$$

i.e.

$$f^* = c(u_1-u_2) \quad (c\in\mathbf{R},\ c\neq 0) \tag{7}$$

at $\bar{x}\in K\setminus Z(u)$. The relation $(f-u_2)=\alpha(f-u_1)$ also leads to (7). Thus (7) holds for any $x\in K\setminus Z(u)$ with a constant c depending on x. This implies that

$$\operatorname{sign}_m f^* = \pm \operatorname{sign}_m (u_1-u_2) = \pm \operatorname{sign}_m u \tag{8}$$

for any $x\in K\setminus Z(u)$. Set $u^*=|u|_m \operatorname{sign}_m f^*$. Since u is continuous and $\operatorname{sign}_m f^*$ has discontinuities only at $Z(f^*)\subset Z(u)$, $u^*\in C^m(K)$. Moreover (8) implies that for any $x\in K\setminus Z(u)$

$$u^* = |u|_m \operatorname{sign}_m f^* = \pm |u|_m \operatorname{sign}_m u = \pm u.$$

Hence $u^*=\pm u$ for each $x\in K$, where $u\in U\setminus\{0\}$. Therefore $u^*\in U^*\setminus\{0\}$. By our construction f^* is $L_{w,m}$-orthogonal to U, hence (2) implies that

$$\left| \int_{K\setminus Z(f^*)} \langle \operatorname{sign}_m f^*, u\rangle w dx \right| \leqq \int_{Z(f^*)} |u|_m w dx \quad (u\in U). \tag{9}$$

Furthermore, for any $x\in K\setminus Z(u)=K\setminus Z(u^*)$

$$\operatorname{sign}_m u^* = \frac{u^*}{|u^*|_m} = \frac{|u|_m \operatorname{sign}_m f^*}{|u|_m} = \operatorname{sign}_m f^* \tag{10}$$

holds. Thus (9), (10) and relation $Z(f^*)\subset Z(u)=Z(u^*)$ imply that

$$\left| \int_{K\setminus Z(u^*)} \langle \operatorname{sign}_m u^*, u\rangle w dx \right| = \left| \int_{K\setminus Z(u^*)} \langle \operatorname{sign}_m f^*, u\rangle w dx \right| \leqq$$

$$\leqq \left| \int_{K\setminus Z(f^*)} \langle \operatorname{sign}_m f^*, u\rangle w dx \right| + \left| \int_{Z(u^*)\setminus Z(f^*)} \langle \operatorname{sign}_m f^*, u\rangle w dx \right| \leqq$$

$$\leqq \int_{Z(f^*)} |u|_m w dx + \int_{Z(u^*)\setminus Z(f^*)} |u|_m w dx = \int_{Z(u^*)} |u|_m w dx$$

for each $u\in U$. Finally, it follows from Theorem 1 that $u^*\in U^*\setminus\{0\}$ is $L_{w,m}$-orthogonal to U, a contradiction.

Theorem 2 shows that the property of U to be a unicity subspace of $C_w^m(K)$ is closely connected with the weight w. Now we are interested in such U, which are unicity subspaces of $C_w^m(K)$ for each $w\in W$. This property of U is already independent of the weight.

Set $m=1$ and let U be a finite dimensional subspace of $C(K)$.

DEFINITION. We say that U satisfies the Property A (is an A subspace of $C(K)$) if for any $u^*\in U^*\setminus\{0\}$ there exists $u'\in U\setminus\{0\}$ such that

a) $u'=0$ a.e. on $Z(u^*)$

b) $u' \operatorname{sign} u^* = |u'|$ on $K\setminus Z(u^*)$.

In the particular case when $n=1$ and elements of U have finite number of separated zeros the notion of A subspace was introduced by DeVore and Strauss (see [11]). A further investigation of Property A was given by Sommer [8], [9]. Let us give some important examples of A subspaces.

EXAMPLE 1. Let $n=1$, $K=[a, b]$ and let U be a Haar subspace of $C[a, b]$. Then by the properties of Haar subspaces we can easily prove that U is an A subspace of $C[a. b]$.

EXAMPLE 2. Set $n=1$, $K=[a, b]$; $U=S_{p,k}=\{\sum_{i=0}^{p} a_i x^i+\sum_{i=1}^{k} a_{p+i}(x-x_i)_+^p$, where $a<x_1<\ldots<x_k<b$ are fixed and $a_i\in\mathbf{R}$, $0\leqq i\leqq p+k$, are arbitrary$\}$; $S_{p,k}$ is the set of splines of order p with k fixed knots. Then $S_{p,k}$ is an A subspace of $C[a, b]$. (For the proof see Sommer [9].)

EXAMPLE 3. Let $n=1$, $K=[a, b]$, $a=x_0<x_1<\ldots<x_s<x_{s+1}=b$ and let U_i be a Haar subspace of $C[x_i, x_{i+1}]$, $0\leqq i\leqq s$. Then set $U=\{u\in C[a, b]$: $u|_{[x_i, x_{i+1}]}\in U_i$ for each $0\leqq i\leqq s\}$. This is the so called continuously composed Haar space. Then U is an A subspace of $C[a, b]$ (see Sommer [9]).

The above examples show that in case when $n=1$ the Haar subspaces and certain families of spline functions satisfy the A property.

Now we shall give an example of an A subspace of functions of several variables.

EXAMPLE 4. Let $n\in\mathbf{N}$ and let K be a convex body in $\mathbf{R}^n$. Let U^0 be the set of homogeneous linear functions. Then U^0 is an A subspace of $C(K)$. The proof of this fact is essentially contained in [6].

Let U_i, $1\leqq i\leqq m$, be finite dimensional subspaces of $C(K)$. Then $U_1\times\ldots\times U_m$ will denote the Cartesian product of these subspaces, i.e. $U_1\times\ldots\times U_m=$ $=\{u=(u_1, \ldots, u_m)$: $u_i\in U_i$ for each $1\leqq i\leqq m\}$. Evidently, $U_1\times\ldots\times U_m$ is a finite dimensional subspace of $C^m(K)$.

We shall prove now our main theorem.

THEOREM 3. *Let $n, m\in\mathbf{N}$ and let U_i $(1\leqq i\leqq m)$ be A subspaces of $C(K)$. Then for any weight $w\in W$ the subspace $U=U_1\times\ldots\times U_m$ is a unicity subspace of $C_w^m(K)$.*

REMARK 2. In the particular case when $n=m=1$, $w\equiv 1$ this theorem was verified by Strauss [11].

PROOF. Assume that for some $w^*\in W$ the subspace U is not a unicity subspace of $C_{w^*}^m(K)$. Then applying Theorems 1 and 2 we can find a $u^*\in U^*\setminus\{0\}$ such that

$$\left|\int_{K\setminus Z(u^*)} \langle \operatorname{sign}_m u^*, u\rangle w^* dx\right| \leqq \int_{Z(u^*)} |u|_m w^* dx \tag{11}$$

for any $u\in U$. Since $u^*=(u_1^*, \ldots, u_m^*)\in U^*\setminus\{0\}$, there exists $u=(u_1, \ldots, u_m)\in U\setminus\{0\}$ such that $u^*(x)=\pm u(x)$ for any $x\in K$. Therefore $u_i^*(x)=\pm u_i(x)$ for any $x\in K$ and $1\leqq i\leqq m$. But $u_i\in U_i$, hence $u_i^*\in U_i^*$ $(1\leqq i\leqq m)$. Moreover, for some $1\leqq j\leqq m$, $u_j^*\in U_j^*\setminus\{0\}$. By the A property of U_j^* there exists $u'\in U_j\setminus\{0\}$ such that $u'=0$ a.e. on $Z(u_j^*)$ and $|u'|=u'$ sign u_j^* on $K\setminus Z(u_j^*)$. Furthermore (11) implies that

$$\left|\int_{K\setminus Z(u^*)} \frac{u_j^*}{|u^*|_m} u_j w^* dx\right| \leqq \int_{Z(u^*)} |u_j| w^* dx \tag{12}$$

for any $u_j\in U_j$. Setting $u_j=u'$ in (12) and using that $Z(u^*)\subset Z(u_j^*)$ we have

$$0=\int_{K\setminus Z(u^*)} \frac{u_j^*}{|u^*|_m} u' w^* dx=\int_{K\setminus Z(u_j^*)} \frac{|u_j^*|}{|u^*|_m} |u'| w^* dx.$$

Therefore $|u_j^*||u'|w^*=0$ a.e. on $K\setminus Z(u_j^*)$, i.e. $u'=0$ a.e. on $K\setminus Z(u_j^*)$. But u' is also zero a.e. on $Z(u_j^*)$, hence we obtain that u' is identically zero, a contradiction. The theorem is proved.

This theorem implies a number of interesting corollaries. E.g., using Example 1 we obtain that a Cartesian product of Haar subspaces of $C[a, b]$ is a unicity subspace of $C_w^m[a, b]$. This Jackson—Krein type theorem was proved in [5] by a different method. In a special case (complex functions) it was verified earlier by Kripke and Rivlin [4]. Moreover, we can extend the results on unicity of best L_1-approximation by real splines given in [1, 2], [8—10] to the case of approximation by Cartesian products of families of spline functions. In particular, this implies the unicity of best L_1-approximation of complex valued continuous functions by certain families of complex splines. Let us give a typical result of this kind.

Let $\tilde{C}[a, b]$ be the space of complex valued continuous functions on $[a, b]$ endowed with the L_w-norm on $[a, b]$. ($w \in W$ is arbitrary.) Set $\tilde{S}_{p,k} = \{\sum_{i=0}^{p} c_i x^i + \sum_{i=1}^{k} c_{p+i}(x-x_i)_+^p$, where $a < x_1 < \ldots < x_k < b$ are fixed and $c_i \in \mathbf{C}$, $0 \leqq i \leqq p+k$, are arbitrary complex numbers$\}$. This is the set of complex splines of order p with k fixed knots. Then we have the following

COROLLARY 1. *Each $f \in \tilde{C}[a, b]$ possesses a unique best L_w-approximation out of $\tilde{S}_{p,k}$.*

PROOF. Evidently, this approximation problem may be considered as approximation in $C_w^2[a, b]$ by elements of $S_{p,k} \times S_{p,k}$. Since $S_{p,k}$ is an A subspace of $C[a, b]$. Theorem 3 immediately implies the needed statement.

By Theorem 3 and Example 4 we obtain that $\underbrace{U^0 \times \ldots \times U^0}_{m}$ is a unicity subspace of $C_w^m(K)$, where K is a convex body in $\mathbf{R}^n$ and $w \in W$ is arbitrary. For $m=1$, $w \equiv 1$ this result was given in [6].

EXAMPLE 5. Set $m=1, n=2, w \equiv 1, I=[0, 1]$ and consider the space $C_1(I^2)$. It is proved in [6] that the product of a 2-dimensional and a k-dimensional Haar subspace is a unicity subspace of $C_1(I^2)$. For example,

$$P_{1,k-1} = \Big\{ \sum_{i=0}^{1} \sum_{j=0}^{k-1} a_{ij} x^i y^j : \; a_{ij} \in \mathbf{R} \Big\},$$

the set of algebraic polynomials of degree 1 and $k-1$ in x and y, resp., is a unicity subspace of $C_1(I^2)$. On the other hand, it can be easily shown that $P_{1,k-1}$ is not an A subspace of $C_1(I^2)$.

By Theorem 3, the A subspaces are unicity subspaces of $C_w(K)$ for any $w \in W$, while Example 5 shows that in general for a given weight $w \in W$, A subspaces are not the only unicity subspaces of $C_w(K)$. This leads us to the following problem: is it true, that only A subspaces of $C(K)$ have the property to be unicity subspaces of $C_w(K)$ for each $w \in W$? Now we shall verify that the above conjecture is true in case of approximation by one dimensional subspaces.

THEOREM 4. *Let $\varphi \in C^m(K)$, where K is a convex body in $\mathbf{R}^n$. Set $U = \{c\varphi : c \in \mathbf{R}\}$. Then the following statements are equivalent:*

(i) *U is a unicity subspace of $C_w^m(K)$ for each $w \in W$;*

(ii) *$K \setminus Z(\varphi)$ is connected;*

(iii) *$U^* = U$.*

PROOF. (i)⇒(ii). Assume the contrary, i.e. (i) holds and $K\setminus Z(\varphi)$ is disconnected. Then $K\setminus Z(\varphi)=A\cup B$, where $A\cap B=\emptyset$ and A and B are open (in K) nonempty subsets of K. Since the Lebesgue measures of A and B are positive and φ does not vanish on A and B, $a=\int_A |\varphi|_m dx>0$ and $b=\int_B |\varphi|_m dx>0$. Set

$$\bar{w}=\begin{cases} 1/a, & \text{on } A \\ 1/b, & \text{on } B \\ 1, & \text{on } Z(\varphi), \end{cases} \qquad \varphi^*=\begin{cases} \varphi, & \text{on } A \\ -\varphi, & \text{on } B \\ 0, & \text{on } Z(\varphi). \end{cases} \tag{13}$$

Clearly, $\bar{w}\in W$. Furthermore, $\varphi^*(x)=\pm\varphi(x)$ for any $x\in K$. Let us prove that $\varphi^*\in C^m(K)$. Let $x_0\in K$ and $x_i\to x_0$, $x_i\in K$. If e.g. $x_0\in A$, then using that A is open in K we get that $x_i\in A$ for $i\geqq i_0$. Hence (13) and the continuity of φ imply that φ^* is continuous at x_0. Assume now that $x_0\in Z(\varphi)$, i.e. $\varphi^*(x_0)=0$. Since $|\varphi^*(x_i)|_m=|\varphi(x_i)|_m$, the continuity of φ at $x_0\in Z(\varphi)$ implies that $\varphi^*(x_i)\to 0$. Thus $\varphi^*\in C^m(K)$ and therefore $\varphi^*\in U^*\setminus\{0\}$. Further, using (13) we have

$$\int_{K\setminus Z(\varphi^*)} \langle \operatorname{sign}_m \varphi^*, \varphi\rangle \bar{w}dx = \frac{1}{a}\int_A |\varphi|_m dx - \frac{1}{b}\int_B |\varphi|_m dx = 0.$$

Hence it follows from Theorem 1 that φ^* is $L_{\bar{w},m}$-orthogonal to U. Since $\varphi^*\in U^*\setminus\{0\}$ this contradicts Theorem 2.

(ii)⇒(iii). Take an arbitrary $u^*\in U^*\setminus\{0\}$. Then $u^*(x)=c_0\gamma(x)\varphi(x)$, where $c_0\in\mathbf{R}\setminus\{0\}$, $\gamma(x)=\pm 1$ $(x\in K)$. Consider an arbitrary $x_0\in K\setminus Z(\varphi)$. Set $\varphi=(\varphi_1, \varphi_2, \ldots, \varphi_m)$, where $\varphi_i\in C(K)$ $(1\leqq i\leqq m)$. Since $Z(\varphi)=\bigcap_{i=1}^m Z(\varphi_i)$, $x_0\in K\setminus Z(\varphi_j)$ for some $1\leqq j\leqq m$. Setting $u^*=(u_1^*, u_2^*, \ldots, u_m^*)$ we have $u_j^*(x)=c_0\gamma(x)\varphi_j(x)$. But $\varphi_j(x_0)\neq 0$, hence $\gamma(x)=u_j^*(x)/(c_0\varphi_j(x))$ in a neighbourhood of x_0. Thus γ is continuous at x_0, i.e. γ is continuous on $K\setminus Z(\varphi)$. Moreover, $\gamma=1$ or -1 on $K\setminus Z(\varphi)$ where $K\setminus Z(\varphi)$ is connected. Then for any $x\in K\setminus Z(\varphi)$, $\gamma(x)=\gamma_0$, where $\gamma_0=1$ or -1, i.e. $u^*=c_0\gamma_0\varphi\in U$.

(iii)⇒(i). Take arbitrary $w\in W$, $u^*\in U^*\setminus\{0\}$. Since $U^*=U$, u^* is not $L_{w,m}$-orthogonal to U. Hence Theorem 2 yields that U is a unicity subspace of $C_w^m(K)$.

The proof of the theorem is complete.

COROLLARY 2. *Let* $m=1$, $U\subset C(K)$, $\dim U=1$. *Then the following statements are equivalent:*

(i) *U is a unicity subapce of $C_w(K)$ for each $w\in W$;*

(ii) *U is an A subspace of $C(K)$.*

PROOF. (i)⇒(ii) follows from (i)⇒(iii) of Theorem 4. (ii)⇒(i) is a consequence of Theorem 3.

We now apply Theorem 2 to the study of best simultaneous L_1-approximation of continuous functions. Let $f_1, f_2, \ldots, f_m\in C(K)$ and let U be a finite-dimensional subspace of $C(K)$. In [13] Carroll and McLaughlin studied the question of uniqueness of the element $\bar{u}\in U$ for which

$$\int_K \sum_{i=1}^m |f_i-\bar{u}|dx = \inf_{u\in U}\int_K \sum_{i=1}^m |f_i-u|dx. \tag{14}$$

This problem of best simultaneous L_1-approximation was considered in [13] in case when $K=[a, b]$ and U is the set of algebraic polynomials of order at most k. It was shown by the authors that (14) has a unique solution if m is odd.

We now present a different approach to the question of best simultaneous L_1-approximation. For given $f_1, f_2, \ldots, f_m \in C(K)$ we are looking for the solution of

$$\int_K \Big(\sum_{i=1}^m (f_i-\bar{u})^2\Big)^{1/2} dx = \inf_{u\in U} \int_K \Big(\sum_{i=1}^m (f_i-u)^2\Big)^{1/2} dx \quad (\bar{u}\in U). \tag{15}$$

The existence of a solution of (15) follows by standard compactness arguments. Moreover, it can be proved that if U is an A subspace of $C(K)$, then for any $m\geqq 1$ and $f_1, \ldots, f_m \in C(K)$ the solution of (15) is unique.

For a given $U\in C(K)$ we set $U_{(m)}=\{(u, \ldots, u)\in C^m(K): u\in U\}$. Then $U_{(m)}$ is a finite dimensional subspace of $C^m(K)$ and the approximation problem (15) is equivalent to the $L_{1,m}$-approximation of $f=(f_1, f_2, \ldots, f_m)\in C_1^m(K)$ by elements of $U_{(m)}$ (here $w\equiv 1$).

We have the following

THEOREM 5. *Let U be an A subspace of $C(K)$. Then for any $m\geqq 1$, $U_{(m)}$ is a unicity subspace of $C_1^m(K)$.*

This theorem immediately implies the needed statement.

COROLLARY 3. *Let U be an A subspace of $C(K)$. Then for any $m\geqq 1$ and $f_1, \ldots, f_m \in C(K)$ there exists a unique solution of* (15).

PROOF. Assume that the theorem is false. Then by Theorem 2 there exists $u_m^*\in U_{(m)}^*\setminus\{0\}$ which is $L_{1,m}$-orthogonal to $U_{(m)}$, i.e. by Theorem 1

$$\Big|\int_{K\setminus Z(u_m^*)} \langle \operatorname{sign}_m u_m^*, u^{(m)}\rangle dx\Big| \leqq \int_{Z(u_m^*)} |u^{(m)}|_m dx \tag{16}$$

for any $u^{(m)}\in U_{(m)}$. Furthermore, $u_m^*(x)=\gamma(x)u_0^{(m)}(x)$ for some $u_0^{(m)}\in U_{(m)}\setminus\{0\}$, where $\gamma(x)=\pm 1$ $(x\in K)$, $u_0^{(m)}=(u, \ldots, u)$ $(u\in U\setminus\{0\})$, $u_m^*=(\bar{u}_1, \ldots, \bar{u}_m)$. Therefore $\bar{u}_i(x)=\gamma(x)u(x)$, $1\leqq i\leqq m$, hence $\bar{u}_1=\bar{u}_2=\ldots=\bar{u}_m=u^*\in U^*\setminus\{0\}$. Property A of U implies that for given $u^*\in U^*\setminus\{0\}$ there exists $u'\in U\setminus\{0\}$ such that

$$\begin{cases} u' = 0 \quad \text{a.e.} & \text{on} \quad Z(u^*) \\ u' \operatorname{sign} u^* = |u'| & \text{on} \quad K\setminus Z(u^*). \end{cases} \tag{17}$$

Set $u_m'=(u', \ldots, u')\in U_{(m)}$. Since $Z(u_m^*)=Z(u^*)$ we have by (16) and (17)

$$0 = \int_{K\setminus Z(u^*)} \langle \operatorname{sign}_m u_m^*, u_m'\rangle dx = \int_{K\setminus Z(u^*)} \sum_{i=1}^m \frac{\bar{u}_i}{|u_m^*|_m} u' dx =$$

$$= \int_{K\setminus Z(u^*)} \sqrt{m}\, u' \operatorname{sign} u^* dx = \sqrt{m} \int_{K\setminus Z(u^*)} |u'| dx.$$

Thus $u'=0$ a.e. on $K\setminus Z(u^*)$. This and (17) imply that u' is the zero function, a contradiction.

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