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ON THE APPROXIMATE SOLUTIONS OF NON-LINEAR FUNCTIONAL EQUATIONS UNDER MILD DIFFERENTIABILITY CONDITIONS

I. K. ARGYROS (Lawton)

Introduction. The present note concerns the examination of Newton's method under assumptions different than those of L.V. Kantorovich [3] and M. Altman [1], [2].

In [3] the Fréchet-differential must have a continuous inverse. The examination of the existence of this inverse and the estimate of its norm presents the greatest difficulty for the application of Kantorovich's method.

In [2] the above difficulty is eliminated. However one of the assumptions made is that the norm of the second Fréchet-differential must be bounded. The computation of such a norm is a difficult task in general.

One can refer to [4], [5] and the references there for a further study on Newton's method.

Here we generalize the above methods under the assumption that the Fréchet-differential is only Hölder continuous on some closed sphere $S(x_0, r)$ centered at the initial guess x_0 and of radius $r > 0$. Some interesting examples are provided where our method can be applied whereas the two mentioned above cannot.

Let X be a Banach space and let $F(x)$, $x \in X$ be a nonlinear continuous functional defined on $S(x_0, r)$.

Consider the nonlinear functional equation

$$F(x) = 0. \tag{1}$$

We suppose that $F(x)$ is a Fréchet-differentiable on $S(x_0, r)$ and denote by $f(x) = F'(x)$ the Fréchet-differential of $F(x)$.

Setting $f_0 = f(x_0) = F'(x_0)$ for some $y \in X$, we introduce, as in [2], the iteration

$$x_1 = x_0 - \frac{F(x_0)}{f_0(y)} y, \quad x_{n+1} = x_n - \frac{F(x_n)}{f_0(y)} y, \quad n = 0, 1, 2, \ldots, \tag{2}$$

to solve (1).

We will need the following:

DEFINITION. Assume that F is Fréchet-differentiable and $F'(x)$ is the first Fréchet-differential at a point x. We recall that $F'(x) \in L(X, \mathbf{R})$, the space of bounded linear operators from X to $\mathbf{R}$. We say that the Fréchet-differential is Hölder continuous over a domain R if for some $c > 0$, $p \in [0.1]$, and all $x, y \in R$

$$|F'(x) - F'(y)| \leqq c\|x - y\|^p. \tag{3}$$

In this case we say that $F'(x) \in H_R(c,p)$.

We include the following lemma for completeness [3].

LEMMA. *Let $F : X \to \mathbf{R}$ and $\tilde{D} \subseteqq X$. Assume $\tilde{D}$ is open and that $F'(x) \in \in H_{\tilde{D}_0}(c,p)$ for some convex $\tilde{D}_0 \subseteqq \tilde{D}$. Then for all $x, y \in \tilde{D}_0$*

$$|F(x) - F(y) - F'(x)(x - y)| \leqq \frac{c}{p+1}\|x - y\|^{p+1}. \tag{4}$$

THEOREM. *Suppose:*

(a) *that there exists $x_0 \in X$ and numbers D, B, r such that*

$$|F(x_0)| \leqq D, \quad \frac{\|y\|}{|f_0(y)|} \leqq B, \tag{5}$$

$$0 < r < \left[\frac{1}{(p+1)Bc}\right]^{\frac{1}{p}} \tag{6}$$

and

$$Bcr^{p+1} - r + DB \leqq 0; \tag{7}$$

(b) *the linear operator $F'(x) \in H_R(c,p)$ where $R = S(x_0, r)$.*

Then the sequence defined by (2) converges to a solution $x^* \in S(x_0, r)$ of (1).

Moreover, the following estimate holds:

$$\|x^* - x_n\| \leqq \frac{(Bcr^p)^n}{1 - Bcr^p} DB. \tag{8}$$

PROOF. By (2) we obtain

$$f_0(x_0 - x_{n+1}) = F(x_n); \tag{9}$$

Since,

$$f_0(x_0 - x_{n+1}) = F(x_n) - F(x_{n-1}) - f_0(x_n - x_{n-1}), \tag{10}$$

using (2), (9) and (10)

$$\|x_{n+1} - x_n\| = \frac{\|y\|}{|f_0(y)|} \left|F(x_n) - F(x_{n-1}) - F'(x_0)(x_n - x_{n-1})\right|.$$

By (4) and (5)

$$\|x_{n+1}-x_n\| \leqq B\cdot c\cdot r^p\cdot \|x_n-x_{n-1}\|. \tag{11}$$

Therefore,

$$\|x_{n+q}-x_n\| \leqq [(Bcr^p)^q+(Bcr^p)^{q-1}+\cdots+(Bcr^p)]\|x_n-x_{n-1}\| \leqq \tag{12}$$

$$\leqq \frac{1-(Bcr^p)^q}{1-(Bcr^p)}(Bcr^p)^n\|x_1-x_0\| \leqq \frac{1-(Bcr^p)^q}{1-(Bcr^p)}(Bcr^p)^n DB.$$

By the choice of r the right hand side of (12) shows that $\{x_n\}$ is a Cauchy sequence in a Banach spaces X and as such it converges to an element $x^* \in X$.

Letting $q\to\infty$ in (12) we obtain (8), from which it also follows by the choice of r that $x_n \in S(x_0,r)$, $n=0,1,2,\dots$.

Note that $x^*\in S(x_0,r)$ by (8) and the fact that $S(x_0,r)$ is a closed ball. Finally by (9) it follows immediately that x^* is a solution of (1).

That completes the proof of theorem.

REMARKS. (a) The real function g defined by

$$g(r)=Bcr^{p+1}-r+DB$$

is such that $g(0)=DB>0$ and

$$g'(r)=(p+1)Bcr^p-1<0.$$

If (6) was not satisfied then $g(r)>0$ for all $r\in[0,+\infty]$.

(b) The condition

$$0<r<\left[\frac{1}{BC}\right]^{\frac{1}{p}}$$

is sufficient for the convergence to zero of the right hand side of (12).

(c) In practice r will be chosen to be the minimum positive number satisfying (5) and (6) in order to minimize the error estimate (8).

(d) For $p=1$, Theorem 2 in [1] follows as special case of the above theorem.

EXAMPLE 1. Consider the function G defined on $[0,b]$ by

$$G(t)=\frac{2}{3}t^{3/2}+t-3$$

for some $b>0$.

Let $\|\ \|$ denote the max norm on $\mathbf{R}$, then

$$\|G''(t)\|=\max_{t\in[0,b]}\left|\frac{1}{2}t^{-1/2}\right|=\infty,$$

which implies that the basic hypothesis on $\|G''(t)\|$ in [2] and [3] for the application of Newton's method is not satisfied for finding a solution of the equation

$$G(t) = 0. \tag{13}$$

However, it can easily be seen that $G'(t)$ is Hölder continuous on $[0, b]$ with $c = 1$ and $p = \frac{1}{2}$. Therefore, under the assumptions of the theorem, iteration (2) will converge to a solution t^* of (13).

A more interesting nontrivial application is given by the following example. However it concerns only Newton's iteration

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n). \tag{14}$$

to solve the nonlinear equation $F(x) = 0$ in X. Note that we do not pursue the goal of providing sufficient conditions for the convergence of (14), since this has already been done in [6].

EXAMPLE 2. Consider the differential equation

$$x'' + x^{1+p} = 0, \quad p \in [0,1], \quad x(0) = x(1) = 0.$$

We divide the interval $[0, 1]$ into n subintervals and we set $h = \frac{1}{h}$. Let $\{v_k\}$ be the points of subdivision with

$$0 = v_0 < v_1 < \cdots < v_n = 1.$$

A standard approximation for the second derivate is given by

$$x_i'' = \frac{x_{i-1} - 2x_i + x_{i+1}}{h^2}, \quad x_i = x(v_i), \quad i = 1, 2, \ldots, n-1.$$

Take $x_0 = x_n = 0$ and define the operator $F : \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ by

$$F(x) = H(x) + h^2 \varphi(x),$$

$$H = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad \varphi(x) = \begin{bmatrix} x_1^{1+p} \\ x_2^{1+p} \\ \vdots \\ x_{n-1}^{1+p} \end{bmatrix}, \quad \text{and } x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix}.$$

Then

$$F'(x) = H + h^2(p+1) \begin{bmatrix} x_1^p & & & 0 \\ & x_2^p & & \\ & & \ddots & \\ 0 & & & x_{n-1}^p \end{bmatrix}.$$

Newton's method cannot be applied to the equation $F(x) = 0$.

We may not be able to evaluate the second Fréchet-derivative since it would involve the evaluation of quantities of the form x_i^{-p} and they may not exists.

Let $x \in \mathbf{R}^{n-1}$, $H \in \mathbf{R}^{n-1} \times \mathbf{R}^{n-1}$ and define the norms of x and H by

$$\|x\| = \max_{1 \leqq j \leqq n-1} |x_j|, \quad \|H\| = \max_{1 \leqq j \leqq n-1} \sum_{k=1}^{n-1} |h_{jk}|.$$

For all $x, z \in \mathbf{R}^{n-1}$ for which $|x_i| > 0$, $|z_i| > 0$, $i = 1, 2, \ldots, n-1$ we obtain, for $p = \frac{1}{2}$ say,

$$\|F'(x) - F'(z)\| = \left\|\operatorname{diag}\left\{\left(1 + \frac{1}{2}\right) h^2 \left(x_j^{1/2} - z_j^{1/2}\right)\right\}\right\| =$$

$$= \frac{3}{2} h^2 \max_{1 \leqq j \leqq n-1} \left|x_j^{1/2} - z_j^{1/2}\right| \leqq \frac{3}{2} h^2 \left[\max |x_j - z_j|\right]^{1/2} = \frac{3}{2} h^2 \|x - z\|^{1/2}.$$

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DEPARTMENT OF MATHEMATICS
CAMERON UNIVERSITY
LAWTON, OK 73505
USA

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THE DEGREE OF APPROXIMATION OF DIFFERENTIABLE FUNCTIONS BY HERMITE INTERPOLATION POLYNOMIALS

R. SAKAI (Aichi Nishikamo)

1. Introduction

We denote the zeros of the Chebyshev polynomial $T_n(x) = \cos nt$, $x = \cos t$, by

$$S_n : x_k = \cos\theta_k, \quad \theta_k = (2k-1)\pi/(2n), \quad k = 1, 2, \ldots, n.$$

Let $f \in C[-1,1]$, and let $L_n[f;x]$ be the Lagrange interpolatory polynomial corresponding to the abscissas S_n. If $f(x)$ has the modulus of continuity $w(\varepsilon) = o(|\log(\varepsilon)|^{-1})$, then $L_n[f;x] \to f(x)$, $-1 \leqq x \leqq 1$ (see [5, p. 337]). On the other hand, if we consider the Hermite–Fejér interpolatory polynomial $H_{2n-1}[f;x]$ of degree $2n-1$ such that

$$H_{2n-1}[f;x_k] = f(x_k), \quad H'_{2n-1}[f;x_k] = 0, \quad k = 1, 2, \ldots, n,$$

then we have

$$|f(x) - H_{2n-1}[f;x]| =$$

$$= O(1)[(T_n^2(x)/n)\sum_{k=1}^{n}\{w(f;(1-x^2)^{1/2}/k) + w(f;1/k^2)\} + w(f;|T_n(x)|/n)]$$

for any continuous function f on $[-1,1]$ (see [1]).

In this paper we consider an interpolation problem of the smoother functions. We can show the following.

THEOREM. *If $f \in C^p[-1,1]$ we have an interpolatory polynomial $L_{p,n}[f;x]$ of degree $n(p+1)-1$ such that*

$$L_{p,n}^{(k)}[f;x_i] = f^{(k)}(x_i), \quad i = 1, 2, \ldots, n, \quad k = 0, 1, \ldots, p,$$

and

$$\|f(\cdot) - L_{p,n}[f;\cdot]\| = O(1)\{\log(n)\}n^{-p}w\left(f^{(p)};n^{-1}\right),$$

where $\|f\|$ is the maximum norm on $[-1,1]$, and $w(f;t)$ is the modulus of continuity of f.

2. Preliminaries and proof of the theorem

Let p be a fixed integer, and let $f \in C^p[-1,1]$. Define

$$H_{rin}(x) = [T_n(x)/\{T_n'(x_i)(x-x_i)\}]^{p+1} \sum_{j=r}^{p} A_{rin}(j)(x-x_i)^j;$$

$$H_{rin}^{(k)}(x_q) = \delta_{rk}\delta_{iq}, \quad r,k = 0,1,\dots,p, \quad i,q = 1,2,\dots,n,$$

where $A_{rin}(j)$ are the coefficients depending on r, i and n, and $\delta_{rk} = 1$ if $r = k$, $= 0$ if $r \neq k$.

Now, we define an Hermite intepolatory polynomial by

$$L[f;x] = L_{p,n}[f;x] = \sum_{i=1}^{n}\sum_{k=0}^{p} f^{(k)}(x_i)H_{kin}(x)$$

which is uniquely defined for each $f \in C[-1,1]$, and is of degree at most $n(p+1)-1$. To prove our theorem we need the Gopenganz–Malozemov–Teliakovskii theorem (see e.g. [2]) as follows.

LEMMA 1. *For each $f \in C^p[-1,1]$ we have a polynomial P_n of degree n such that*

(1) $$|f^{(k)}(x) - P_n^{(k)}(x)| = O(1)\{\Delta_n(x)\}^{p-k}w(f^{(p)};\Delta_n(x)), \quad k = 0,1,\dots,p,$$

where $\Delta_n(x) = n^{-1}\{(1-x^2)^{1/2} + n^{-1}\}$.

The following lemma is concerned with the Lebesgue function of the operator $L_{p,n}[f]$.

LEMMA 2. *We have*

(2) $$|H_{rin}(x)| = O(1)\begin{cases} (X_i/n)^r q^{-1} & \text{if } |\theta-\theta_i| \sim q/n, \\ (X_i/n)^r & \text{if } |\theta-\theta_i| < 1/n, \end{cases}$$

where $X_i = \sin\theta_i$, $i = 1,2,\dots,n$, $r = 1,2,\dots,n$, *and* $A_n \sim B_n$ *means* $C_1 \leqq A_n/B_n \leqq C_2$, $n = 1,2,\dots$ *for some positive constants* C_1, *and* C_2.

PROOF. By [5, (7.32.10)] we have

$$|T_n^{(k)}(x_i)| = O(1)(n/X_i)^k, \quad i = 1,2,\dots,n, \quad k = 1,2,\dots,$$

thus by the induction concerning j we see

$$|A_{rin}(j)| = O(1)(X_i/n)^{r-j},$$
$$j = r, r+1,\dots,p, \quad r = 0,1,\dots,p, \quad i = 1,2,\dots,n.$$

If $|\theta - \theta_i| \sim q/n$ then

$$1/|x - x_i| = O(1)\{n/(qX_i)\},$$

thus we have

$$\left| [T_n(x)/\{T_n'(x_i)(x - x_i)\}]^{p+1} A_{rin}(j)(x - x_i)^j \right| =$$

$$= O(1)\begin{cases} (X_i/n)^r q^{-1} & \text{if} \quad |\theta - \theta_i| \sim q/n, \\ (X_i/n)^r & \text{if} \quad |\theta - \theta_i| < 1/n. \end{cases}$$

Consequently we have (2). □

PROOF OF THEOREM 1. Let $f \in C^p[-1,1]$ and let P_n satisfy (1). By Lemmas 1 and 2 we have

$$|L_{p,n}[f;x] - f(x)| = |L_{p,n}[f - P_n; x] + P_n(x) - f(x)| =$$

$$= O(1)\sum_{q=1}^{n}\sum_{k=0}^{p} n^{k-p}w(f^{(p)}; n^{-1})n^{-k}q^{-1} = O(1)\{\log(n)\}n^{-p}w(f^{(p)}; n^{-1}). \ \square$$

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DEPARTMENT OF MATHEMATICS
AICHI PREFECTURAL
KAMOGAOKA SENIOR HIGH SCHOOL
OTASHIRO 1137, FUJIOKA-CHO INO
AICHI NISHIKAMO 470-04
JAPAN

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A NOTE ON DOMINATED SPACES

C. R. BORGES (Davis)

A topological space X is said to be (compactly) *dominated* by the family $\mathcal{K} = \{K_\alpha\}_{\alpha\in\Lambda}$ of (compact) subsets of X provided that $A \subset X$ is closed iff A has a closed intersection with every element of some subcollection $\mathcal{K}_1$ of $\mathcal{K}$ which covers A. X is said to have the *weak topology* over the family $\mathcal{S} = \{S_\alpha\}_{\alpha\in\Delta}$ of closed subsets of X provided that $A \subset X$ is closed iff A has a closed intersection with each $S_\alpha \in \mathcal{S}$. A family $\mathcal{C} = \{C_\alpha\}_{\alpha\in\Gamma}$ of subsets of X is said to be (hereditarily) closure-preserving provided that, for any $\Gamma_1 \subset \Gamma$ (and $D_\alpha \subset C_\alpha$, $\bigcup_{\alpha\in\Gamma_1} D_\alpha^- = (\bigcup_{\alpha\in\Gamma_1} D_\alpha)^-$). $\bigcup_{\alpha\in\Gamma_1} C_\alpha = (\bigcup_{\alpha\in\Gamma_1} C_\alpha)^-$. It is clear that locally finite collections of subsets of a space X are hereditarily closure-preserving. Example 2 shows that closure-preserving collections may fail to be hereditarily closure-preserving. (An interesting study of hereditarily closure-preserving collections of sets appears in [1].)

Theorem 2.10 of [2] claims that a space X is dominated by a closed covering $\{A_\alpha\}_{\alpha\in\Lambda}$ iff the natural map $q : \bigvee_{\alpha\in\Lambda} A_\alpha \to X$ from the disjoint topological union of all the A_α (precisely, $\bigvee_{\alpha\in\Lambda} A_\alpha = \bigcup_{\alpha\in\Lambda} A_\alpha \times \{\alpha\}$), is a closed continuous map. Unfortunately, this result is false, as the following simple example shows.

Example 1. Let $I = [0,1]$ be the closed unit interval with the topology inherited from the real line. For each n, let $A_n = [0, \frac{1}{n}]$. Clearly, X is dominated by $\{A_n | n \in \omega\}$ but the natural map $q : \bigvee_{n\in\omega} A_n \to X$ is not closed; for example, letting $A = \{(\frac{1}{n}, n) \,|n = 1,2,\dots\}$, we get that A is a closed subset of $\bigvee_{n\in\omega} A_n$ but $q(A) = \{\frac{1}{n} | n = 1,2,\dots\}$ is not a closed subset of I.

It is well-known that if K is a CW-complex of Whitehead (i.e. K is a simplicial complex with the weak topology over the family $\{s_\alpha\}_{\alpha\in\Lambda}$ of closed simplexes in K) then K is dominated by $\{s_\alpha\}_{\alpha\in\Lambda}$. However, it is still not always true that the natural map $q : \bigvee_{\alpha\in\Lambda} s_\alpha \to K$ is a closed continuous map, as the following example shows.

Example 2. Let K be the CW-complex with (distinct) vertices ν_n, $n \in \omega$, whose closed simplexes are $s_n = \langle \nu_1, \dots, \nu_n\rangle^-$, for $n \in \omega$. Pick a sequence $\{x_n\}$ in the open 1-simplex $\langle \nu_1, \nu_2\rangle$ which converges to ν_1. Then

$A = \{(x_n, n) | n = 1, 2, \dots\}$ is a closed subset of $\bigvee_{n\in\omega} s_n = \bigcup_{n\in\omega} s_n \times \{n\}$, but the natural map $q : \bigvee_{n\in\omega} s_n \to K$ maps A to the non-closed subset $\{x_n | n \in \omega\}$ of K. Furthermore, K does not have a hereditarily closure-preserving cover by compact spaces (we thank the referee for this simple argument): Assume $\mathcal{C} = \{C_\alpha\}_{\alpha\in\Gamma}$ is a hereditarily closure-preserving cover of K by compact spaces. Then each point of $\langle\nu_1, \nu_2\rangle$ belongs to infinitely many C_α's (because K is not locally compact at any point of $\langle\nu_1, \nu_2\rangle$). Again, pick a sequence $\{x_n\}$ in $\langle\nu_1, \nu_2\rangle$ which converges to ν_1. Then there is a sequence $\{\alpha_n\} \subset \Gamma$ such that $\alpha_n \neq \alpha_m$ if $n \neq m$ and $x_n \in C_{\alpha_n}$. This shows that $\mathcal{C}$ is not hereditarily closure-preserving.

Example 2 shows that Corollaries 2.12 and 3.6 of [2] are false. Later, we will give correct versions of these results.

In light of the preceding examples, the following results are essentially best possible and quite useful.

PROPOSITION 3. *Let $\mathcal{A} = \{A_\alpha\}_{\alpha\in\Lambda}$ be a closed cover of a space X. Then*

(a) *X has the weak topology over $\mathcal{A}$ iff the natural map $q : \bigvee_{\alpha\in\Lambda} A_\alpha \to X$ is a quotient map.*

(b) *X is dominated by $\mathcal{A}$ iff $\mathcal{A}$ is closure-preserving and, for each $\mathcal{C} \subset \mathcal{A}$, $\cup\mathcal{C}$ has the weak topology over $\mathcal{C}$.*

(c) *X is dominated by $\mathcal{A}$ iff the natural map $q : \bigvee_{\alpha\in\Lambda} A_\alpha \to X$ satisfies the following condition: For each $\Gamma \subset \Lambda$, $q(\bigvee_{\alpha\in\Gamma} A_\alpha)$ is a closed subset of X and $q| \bigvee_{\alpha\in\Gamma} A_\alpha : \bigvee_{\alpha\in\Gamma} \to \bigcup_{\alpha\in\Gamma} A_\alpha$ is a quotient map.*

(d) *If $\mathcal{A}$ is hereditarily closure-preserving then X is dominated by $\mathcal{A}$.*

PROOF. Part (a) is well-known (see Theorem VI. 8.5 of [3]). Part (b) follows immediately from the pertinent definitions. Part (c) is a restatement of part (b).

Part (d). Let A be a subset of X and $\{A_\alpha\}_{\alpha\in\Delta}$ a subfamily of $\mathcal{A}$ which covers A such that $A \cap A_\alpha$ is closed in A_α, for each $\alpha \in \Delta$. Then, $\{A \cap A_\alpha\}_{\alpha\in\Delta}$ is closure-preserving, which implies that

$$A = \bigcup_{\alpha\in\Delta} A \cap A_\alpha = \bigcup_{\alpha\in\Delta} (A \cap A_\alpha)^- = \left(\bigcup_{\alpha\in\Delta} A \cap A_\alpha\right)^- = A.$$

This proves that A is closed, which completes the proof.

It is noteworthy that Proposition 3(b) cannot be weakened to "X is dominated by $\mathcal{A}$ iff $\mathcal{A}$ is closure-preserving and X has the weak topology over $\mathcal{A}$", as the following example shows.

EXAMPLE 4. Let I be the space of Example 1. For $n = 2, 3, \dots$, let $A_n = \{0\} \cup [\frac{1}{n}, 1]$; let $A_1 = I$. Clearly, $\{A_n\}_{n\in\omega}$ is closure-preserving, and I

has the wak topology over $\{A_n\}_{n\in\omega}$. However, $\{A_n\}_{n\in\omega}$ does not dominate I, since the set $A =]0,1]$ has a closed intersection with A_n, for $n = 2, 3, \ldots$, and $A \subset \bigcup_{n=2}^{\infty} A_n$, but A is not closed.

The following well-known example further illustrates the subtleties of the concepts in Proposition 3.

EXAMPLE 5. Let Ω be the space of countable ordinals with the order topology. For each $\alpha \in \Omega$, let $A_\alpha = \{\beta \in \Omega | \beta \leqq \alpha\}$. It is well-known and easily seen that X has the weak topology over $\mathcal{A} = \{A_\alpha\}_{\alpha\in\Omega}$. By Theorem 8.2 of [4], X is not dominated by $\mathcal{A}$ (because each A_α is paracompact but X is not paracompact). No subover of $\mathcal{A}$ is closure-preserving!

THEOREM 6. *Let $\mathcal{A} = \{A_\alpha\}_{\alpha\in\Lambda}$ be a closed cover of a space X. The natural map $q : \bigvee_{\alpha\in\Lambda} A_\alpha \to X$ is a closed continuous map iff $\mathcal{A}$ is hereditarily closure-preserving.*

PROOF. The "only if" part is obvious. The "if" part is trivial.

The following result corrects Corollary 2.12 of [2].

PROPOSITION 7. *A space X is a closed continuous image of a disjoint topological union of compact spaces iff X has a hereditarily closure-preserving cover by compact subspaces.*

PROOF. Immediate from Theorem 6.

LEMMA 8. *Let $f : X \to Y$ be a closed continuous map from X into Y. If $\{A_\alpha\}_{\alpha\in\Lambda}$ is a hereditarily closure-preserving collection of subsets of X then $\{f(A_\alpha)\}_{\alpha\in\Lambda}$ is a hereditarily closure-preserving collection of subsets of Y.*

PROOF. Let $\Lambda_1 \subset \Lambda$ and pick $B_\alpha \subset f(A_\alpha)$, for each $\alpha \in \Lambda_1$. Next, for each $\alpha \in \Lambda_1$, pick $C_\alpha \subset A_\alpha$ such that $f(C_\alpha) = B_\alpha$. Since, by hypothesis, $\bigcup_{\alpha\in\Lambda_1} C_\alpha^- = \left(\bigcup_{\alpha\in\Lambda_1} C_\alpha\right)^-$ and f is closed continuous (equivalently, $f(A^-) = \overline{f(A)}$, for any subset A of X), we get that

$$\bigcup_{\alpha\in\Lambda_1} B_a^- = \bigcup_{\alpha\in\Lambda_1} \overline{f(C_\alpha)} = \bigcup_{\alpha\in\Lambda_1} f(C_\alpha^-) = f\left(\overline{\bigcup_{\alpha\in\Lambda_1} C_\alpha}\right) = \\ = \overline{f\left(\bigcup_{\alpha\in\Lambda_1} C_\alpha\right)} = \overline{\bigcup_{\alpha\in\Lambda_1} B_\alpha}.$$

This proves that $\{f(A_\alpha)\}_{\alpha\in\Lambda}$ is hereditarily closure-preserving, which completes the proof.

The following result corrects Corollary 3.6 of [2].

THEOREM 9. *A space Y has a hereditarily closure-preserving cover by compact sets iff Y is the closed continuous image of a locally compact paracompact space.*

PROOF. *The "only if" part* follows immediately from Theorem 6.

The "if" part. Let X be a locally compact paracompact space and $f : X \to Y$ be a closed continuous map onto Y. Let $\mathcal{U}$ be an open cover of X such that, for each $U \in \mathcal{U}$, U^- is a compact subspace of X. Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a locally finite closed refinement of $\mathcal{U}$. Then, by Lemma 8, $\{f(A_\alpha)\}_{\alpha\in\Lambda}$ is a hereditarily closure-preserving cover of Y by compact sets. This completes the proof.

Our last result yields a correct proof of Theorem 3.3(a) of [2].

THEOREM 10. *Let $f : X \to Y$ be a closed continuous function onto Y. If X is dominated by $\{X_\alpha\}_{\alpha\in\Lambda}$ then Y is dominated by $\{f(X_\alpha)\}_{\alpha\in\Lambda}$.*

PROOF. First note that each $f(X_\alpha)$ is a closed subset of Y. Now, let $B \subset \bigcup_{\alpha\in\Gamma} f(X_\alpha)$, $\Gamma \subset \Lambda$, such that $B \cap f(X_\alpha)$ is closed, for each $\alpha \in \Gamma$. Then $f^{-1}(B \cap f(X_\alpha))$ is closed in X, for each $\alpha \in \Gamma$; therefore $f^{-1}(B) \cap X_\alpha = f^{-1}(B \cap f(X_\alpha)) \cap X_\alpha$ is closed in X_α, for each $\alpha \in \Gamma$. Let $A = f^{-1}(B) \cap (\bigcup_{\alpha\in\Gamma} X_\alpha)$. Then A is closed in X (because $A \cap X_\alpha = f^{-1}(B) \cap X_\alpha$, for each $\alpha \in \Lambda$) and $f(A) = B \cap f(\bigcup_{\alpha\in\Gamma} X_\alpha) = \bigcup_{\alpha\in\Gamma}(B \cap f(X_\alpha)) = B$, which shows that B is closed and completes the proof.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
DAVIS, CALIFORNIA 95616
USA

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ATTRACTORS OF SYSTEMS CLOSE TO AUTONOMOUS ONES HAVING A STABLE LIMIT CYCLE

N. V. MINH and T. V. NHUNG (Hanoi)

1. Introduction. M. Farkas [2] has obtained useful explicit estimates for the radius of attractivity of systems close to periodic ones by using quadratic forms as Lyapunov functions. One of his main assumptions is that periodic unperturbed systems have a uniform asymptotically stable nonconstant periodic solution. This condition is needed for applying Yoshizawa's theorem [11, p. 134] to get the estimates mentioned above. The case of systems close to autonomous ones having an asymptotically stable equilibrium state is also considered by Farkas [4]. As interesting illustrations, Farkas' results are applied to some important second order nonlinear differential equations, e.g. Duffing equation [3] and van der Pol equation in case time tends to $-\infty$ [4].

In this paper we consider the case in which unperturbed systems are assumed to be autonomous and to have an asymptotically, orbitally stable nonconstant periodic solution (a stable limit cycle), e.g. van der Pol equation in case time tends to $+\infty$. Unfortunately, Farkas' estimates in [2] are inapplicable to this case, because now the graph of the periodic solution is not a uniform asymptotically stable set. However, if we note that the orbital stability of the closed path of the periodic solution in $\mathbf{R}^n$, say Γ, is equivalent to the stability of the cylinder $\mathbf{R}\times\Gamma$, then $\mathbf{R}\times\Gamma$ is a uniform asymptotically stable set of the autonomous unperturbed system. Therefore, by applying Lyapunov functions and the theorems due to Yoshizawa and La Salle respectively, we can get the inequalities characterizing a uniform asymptotically stable invariant set around the cylinder $\mathbf{R}\times\Gamma$ (but not around the graph of the periodic solution as in Farkas' case, [2]!) and its region of attractivity. To use Farkas' idea of construction of Lyapunov functions in the quadratic forms [2, 4], we shall introduce a local coordinate system [5, 10] into a small "tube" around Γ.

2. Let us consider the autonomous system

$$\dot{x} = g(x) \tag{2.1}$$

where $\cdot = d/dt$, $t \in \mathbf{R}$, $x = \mathrm{col}(x_1, \dots, x_n) \in \Omega \subset \mathbf{R}^n$, Ω is some open region of $\mathbf{R}^n$, $g : \Omega \to \mathbf{R}^n$ is smooth enough, e.g. $g \in C^2[\Omega, \mathbf{R}^n]$. Assume further that the system (2.1) has a nonconstant periodic solution $x = p(t)$ of (least) period $\tau > 0$ such that its path Γ lies in Ω, and $n-1$ characteristic multipliers of the variational system

$$\dot{z} = g'(p(t))z \tag{2.2}$$

are in modulus less than one: $|\lambda_i| < 1, i = 1, \ldots, n-1$ ($\lambda_n = 1$). Under these conditions it is well-known by the theorem of Andronow and Witt (see, e.g., [7, 11]) that the solution $x = p(t)$ is then asymptotically, orbitally stable with asymptotic phase. Some important generalizations of this theorem can be seen in Hale [5], Hale and Stokes [6], Yoshizawa and Kato [12] and Aulbach [1].

In conjunction with the system (2.1) let us consider the following "neighbouring" system:

$$\dot{x} = f(t, x) \tag{2.3}$$

where $t \in \mathbf{R}$, $x \in \Omega$, $f \in C^0[\mathbf{R} \times \Omega, \mathbf{R}^n]$ and $f'_x \in C^0\left[\mathbf{R} \times \Omega, \mathbf{R}^{n^2}\right]$. Suppose that for any compact set $Q \subset \Omega$ there exists an $\eta > 0$ such that

$$\|f(t, x) - g(x)\| < \eta, \quad (t, x) \in \mathbf{R} \times Q \tag{2.4}$$

where $\|.\|$ is the Euclidean norm.

Let a sufficiently small ρ_1-neighbourhood $U(\Gamma, \rho_1)$ of Γ be taken such that its closure $\overline{U}(\Gamma, \rho_1) \subset \Omega$ and a local coordinate system $(\theta, y_1, \ldots, y_{n-1})$ (see, e.g., Hale [5] or Pliss [10]) can be introduced into the tube $U(\Gamma, \rho_1)$ instead of the old $(x_1, \ldots, x_n)$. The new coordinates are related to the old ones by the formula

$$x = p(\theta) + \phi(\theta) y \tag{2.5}$$

where $y = \mathrm{col}(y_1, \ldots, y_{n-1})$ and ϕ is an $n \times (n-1)$ dimensional matrix.

Differentiating (2.5) with respect to t and solving the system of equations thus obtained for $\dot{\theta}$ and $\dot{y}$, from (2.1) we get the following new system of differential equations:

$$\begin{cases} \dot{\theta} = a(\theta, y), \\ \dot{y} = b(\theta, y) \end{cases} \tag{2.6}$$

where $a(\theta, y) = 1 + g_1(\theta, y)$, $b(\theta, y) = D(\theta) y + g_2(\theta, y)$,

$$D(\theta) = \phi^T(\theta) \left[-\frac{d\phi(\theta)}{d\theta} + \frac{\partial g(p(\theta))}{\partial x} \phi(\theta) \right],$$

T means transpose, and $g_1(\theta, y)$, $g_2(\theta, y)$ are continuous in θ, y, τ-periodic in θ, have continuous first derivatives with respect to y, and

$$\begin{cases} |g_1(\theta, y)| = O(\|y\|) \quad \text{as} \quad \|y\| \to 0, \\ g_2(\theta, 0) = 0, \ \partial g_2(\theta, 0)/\partial y = 0 \end{cases} \tag{2.7}$$

(see Hale [5], p. 219).

By (2.5), the periodic solution $x = p(t)$ of the system (2.1) is transformed into the solution $\theta = t$, $y = 0$ of the system (2.6). By our assumptions, it is possible to find a sufficiently small $\rho_2 > 0$ such that in the domain $\theta \in \mathbf{R}$, $\|y\| < \rho_2$ we have

$$(2.8) \qquad 1/2 < \dot{\theta} = a(\theta, y) < 2.$$

It follows from (2.8) that the map taking t to $\theta(t)$ has an inverse $t : \mathbf{R} \to \mathbf{R}$, $t = t(\theta)$, and for $\theta \in \mathbf{R}$, $\|y\| < \rho_2$

$$(2.9) \qquad 1/2 < dt/d\theta = 1/a < 2$$

(see Hale [5], p. 221–222).

Let us consider the variational system of $\dot{y} = b(\theta(t), y)$ with respect to the solution $\theta = t$, $y = 0$ of the system (2.6), i.e.

$$(2.10) \qquad \dot{u} = D(t)u$$

which is clearly a linear system of order $n - 1$ having a continuous coefficient matrix τ-periodic in t. From our assumption that $|\lambda_i| < 1$, $i = 1, \dots, n - 1$, and Lemma 2.1 in Hale [5], p. 220, it follows that the characteristic exponents $\beta_1, \dots, \beta_{n-1}$ of the system (2.10) are $\beta_i = (\log \lambda_i)/\tau$, $i = 1, \dots, n - 1$, so $\max\limits_i \operatorname{Re} \beta_i = -\beta < 0$. By Floquet's theory the periodic linear system (2.10) is reducible, i.e. we can find a continuously differentiable, regular, τ-periodic matrix function $S(t)$ such that the transformation $v = S(t)u$ carries (2.10) into the linear system

$$(2.11) \qquad \dot{v} = Bv \quad (v \in \mathbf{R}^{n-1})$$

with constant coefficients where by our assumptions all the eigenvalues of B, namely $\beta_1, \dots \beta_{n-1}$, have negative real parts. For (2.11) it is possible to find a positive definite quadratic form (with constant coefficients)

$$V(v) = v^T A v = \sum_{i,j=1}^{n-1} a_{ij} v_i v_j$$

such that its derivative with respect to (2.11) is negative definite

$$(2.12) \qquad \dot{V}_{(2.11)}(v) \leqq -\beta \|v\|^2, \quad v \in \mathbf{R}^{n-1}.$$

The form $\overline{V}(t, u) = V(S(t)u) = u^T(S^T(t)A\, S(t))u$ is clearly a Lyapunov function for (2.10). Putting

$$W(\theta, y) = \overline{V}(t(\theta), y) = y^T \left(S^T A S\right) y, \quad S = S(t(\theta)),$$

we are going to show that $W(\theta, y)$ is a Lyapunov function for (2.6) in a sufficiently small neighbourhood of the line $\theta \in \mathbf{R}$, $y = 0$ in the (θ, y)-space.

Taking into account (2.9) and the estimates

$$\left|\frac{\partial W}{\partial \theta}\right| \leqq 2\left|\frac{\partial W}{\partial t}\right| \leqq 4\overline{C}C'\|A\|\,\|y\|^2 \quad (\theta \in \mathbf{R}, \|y\| < \rho_2), \tag{2.13}$$

$$\|\mathrm{grad}_y W\| \leqq 2\|A\|(\overline{C})^2\|y\| \quad (\theta \in \mathbf{R}, \|y\| < \rho_2), \tag{2.14}$$

where $\overline{C} := \max\limits_{t\in[0,\tau)} \|S(t)\|$, $C' := \max\limits_{t\in[0.\tau)} \|S'(t)\|$,

$$|g_1| \leqq K\|y\|, \quad K = \mathrm{const}\,(\theta \in \mathbf{R}, \|y\| < \rho_3), \tag{2.15}$$

$$\|g_2\| \leqq M\|y\|^2, \quad M = \mathrm{const}\,(\theta \in \mathbf{R}, \|y\| < \rho_3), \tag{2.16}$$

we get

$$\begin{aligned} \dot{W}_{(2.6)}(\theta, y) &= \frac{\partial W}{\partial \theta} a(\theta, y) + \big(\mathrm{grad}_y W, b(\theta, y)\big) = \\ &= \frac{\partial W}{\partial \theta} + \big(\mathrm{grad}_y W, D(\theta)y\big) + \frac{\partial W}{\partial \theta} g_1 + \big(\mathrm{grad}_y W, g_2\big) = \\ &= \dot{V}_{(2.11)}(S(t)y) \cdot \frac{dt}{d\theta} + \frac{\partial W}{\partial \theta} g_1 + \big(\mathrm{grad}_y W, g_2\big) \leqq \\ &\leqq \|y\|^2 \left[-\frac{\beta\lambda}{2} + 2\overline{C}\|A\| \cdot \|y\|(2C'K + M\overline{C})\right] \end{aligned} \tag{2.17}$$

for all $\theta \in \mathbf{R}$ and $\|y\| < \min(\rho_2, \rho_3)$ where $\lambda := \min\limits_{t\in[0,\tau)} \lambda_s(t) > 0$, $\lambda_s(t)$ denotes the least eigenvalue of the τ-periodic positive definite matrix function $S^T(t)S(t)$. Therefore

$$\dot{W}_{(2.6)}(\theta, y) < 0 \tag{2.18}$$

in the domain $\theta \in \mathbf{R}$ and

$$\|y\| < \min\left(\rho_2, \rho_3, \frac{\beta\lambda}{4\overline{C}\|A\|(2C'K + M\overline{C})}\right).$$

3. In this last part, by using the stationary Lyapunov function $\overline{W}(t, \theta, y) \equiv W(\theta, y)$, $t \in \mathbf{R}$, where W is constructed above, and the theorems of Yoshizawa and La Salle, we shall construct a uniform asymptotically stable invariant set for (2.3) in $\mathbf{R} \times \mathbf{R}^n$ (as $t \to +\infty$) containing the cylinder $\mathbf{R} \times \Gamma$, and its region of attractivity for η small.

Suppose that the system corresponding to (2.3) in the local coordinate system is of the form

$$\begin{cases} \dot{\theta} = f_1(t, \theta, y), \\ \dot{y} = f_2(t, \theta, y) \end{cases} \tag{3.1}$$

where $f_1 : \mathbf{R} \times \mathbf{R} \times \{\|y\| < \rho_2\} \to \mathbf{R}$ $f_2 : \mathbf{R} \times \mathbf{R} \times \{\|y\| < \rho_2\} \to \mathbf{R}^{n-1}$ (see the explicit form of f_1 and f_2 in Hale [5], p. 233). By our assumption (2.4), for each set N of the form $N = \mathbf{R} \times N_y$, where N_y is a closed set contained in $\{\|y\| < \rho_2\}$, there exists an $\eta_1 > 0$ such that

$$(3.2) \qquad \begin{cases} |f_1(t,\theta,y) - a(\theta,y)| < \eta_1, \\ \|f_2(t,\theta,y) - b(\theta,y)\| < \eta_1 \end{cases}$$

for all $(t,\theta,y) \in \mathbf{R} \times \mathbf{R} \times N_y$.

Taking the derivative of the Lyapunov function W with respect to the system (3.1) we get

$$(3.3) \qquad \dot{W}_{(3.1)}(\theta,y) = \frac{\partial W}{\partial \theta} f_1 + \left(\operatorname{grad}_y W, f_2\right) =$$
$$= \frac{\partial W}{\partial \theta} a + \left(\operatorname{grad}_y W, b\right) + \frac{\partial W}{\partial \theta}(f_1 - a) +$$
$$+ \left(\operatorname{grad}_y W, f_2 - b\right) = \dot{W}_{(2.6)}(\theta,y) + \delta(\theta,y)$$

where

$$(3.4) \qquad \delta(\theta,y) = \frac{\partial W}{\partial \theta}(f_1 - a) + \left(\operatorname{grad}_y W, f_2 - b\right) =$$
$$= \frac{\partial W}{\partial t} \cdot \frac{(f_1 - a)}{a} + \left(\operatorname{grad}_y W, f_2 - b\right).$$

As in (2.13), we have the estimate

$$(3.5) \qquad \left|\frac{\partial W}{\partial t}(\theta,y)\right| \leqq 2\overline{C}C'\|A\|\,\|y\|^2 \quad (\theta \in \mathbf{R},\ \|y\| < \rho_2).$$

From (2.9), (2.14), (3.2), (3.4) and (3.5) it follows that

$$(3.6) \qquad |\delta(\theta,y)| < 2\eta_1\overline{C}\|A\|\,\|y\|\left(2C'\|y\| + \overline{C}\right)$$

for every $\theta \in \mathbf{R}$ and $\|y\| < \rho_2/2$ where η_1 is the positive constant corresponding to the set $N = \mathbf{R} \times \{\|y\| \leqq \rho_2/2\}$.

Thus, by (2.17), (3.3) and (3.6) we get

$$\dot{W}_{(3.1)}(\theta,y) < \|y\|^2\left[-\frac{\beta\lambda}{2} + 2\overline{C}\|A\|\,\|y\|\left(2C'K + M\overline{C}\right)\right] +$$
$$+ 2\eta_1\overline{C}\|A\|\,\|y\|\left(2C'\|y\| + \overline{C}\right)$$

for all $\theta \in \mathbf{R}$ and $\|y\| < \min(\rho_2/2, \rho_3)$, so

$$\dot{W}_{(3.1)}(\theta,y) < -\frac{\beta\lambda\|y\|^2}{4} + 3\eta_1(\overline{C})^2\|A\|\,\|y\|$$

for all $\theta \in \mathbf{R}$ and

$$\|y\| < d_2 := \min\left(\rho_2/2, \rho_3, \frac{\overline{C}}{4C'}, \frac{\beta\lambda}{8\overline{C}\|A\|\left(2C'K + M\overline{C}\right)}\right).$$

Therefore

$$\dot{W}_{(3.1)}(\theta, y) < 0 \tag{3.7}$$

in the domain

$$\theta \in \mathbf{R}, \quad d_1 < \|y\| < d_2 \tag{3.8}$$

where $d_1 := 12\eta_1(\overline{C})^2\|A\|/(\beta\lambda)$. The set of y's satisfying condition (3.8) is not empty if $d_1 < d_2$, i.e. if

$$0 < \eta_1 < \eta_0 \tag{3.9}$$

where

$$\eta_0 := \frac{\beta\lambda}{12(\overline{C})^2\|A\|}\min\left(\rho_2/2, \rho_3, \frac{\overline{C}}{4C'}, \frac{\beta\lambda}{8\overline{C}\|A\|\left(2C'K + M\overline{C}\right)}\right).$$

Let us denote the least and the largest eigenvalue of the τ-periodic positive definite matrix $S^T(t)AS(t)$ by $\lambda_1(t)$ and $\lambda_2(t)$, respectively, and let

$$\alpha_1 := \min_{\theta\in\mathbf{R}, \|y\|=d_2} W(\theta, y), \qquad \alpha_2 := \max_{\theta\in\mathbf{R}, \|y\|=d_1} W(\theta, y).$$

Then it is easy to see that

$$\lambda_1 := \min_{t\in[0,\tau)} \lambda_1(t) > 0, \qquad \lambda_2 := \max_{t\in[0,\tau)} \lambda_2(t) > 0,$$

and $\alpha_1 = \lambda_1 d_2^2$, $\alpha_2 = \lambda_2 d_1^2$. Let us denote

$$\begin{aligned} A_{\eta_1} &= \{(\theta, y) \in \mathbf{R} \times \mathbf{R}^{n-1} : W(\theta, y) \leqq \alpha_2\}, \\ B &= \{(\theta, y) \in \mathbf{R} \times \mathbf{R}^{n-1} : W(\theta, y) < \alpha_1\}. \end{aligned}$$

Now we are in a position to formulate the following

THEOREM. *Suppose that all conditions mentioned before are satisfied and η_1 is such that*

$$0 < \eta_1 < (\lambda_1/\lambda_2)^{1/2}\eta_0. \tag{3.10}$$

Then the set $\mathbf{R} \times A_{\eta_1}$ is a uniform asymptotically stable invariant set of (3.1) *(as $t \to +\infty$) and its region of attractivity contains the set $\mathbf{R} \times B$. Returning* *to the original variables* $x_1, \dots, x_n$ *(see* (2.5)*), from the sets* $\mathbf{R} \times A_{\eta_1}$ *and* $\mathbf{R} \times B$ *we get, respectively, a uniform asymptotically stable invariant cylindrical set* $\tilde{A}_\eta$ *in* $\mathbf{R} \times \mathbf{R}^n$ *for the system* (2.3) *around the cylinder* $\mathbf{R} \times \Gamma$ *and a cylindrical set* $\tilde{B}$ *contained in the domain of attractivity of* $\tilde{A}_\eta$*.*

To prove our theorem let us first note that $\lambda_1 \leqq \lambda_2$, hence (3.10) implies (3.9) and $\alpha_2 < \alpha_1$, thus $A_{\eta_1} \subset B$. Then $B - A_{\eta_1}$ is contained in the domain defined by (3.8), so (3.7) holds in $B - A_{\eta_1}$. After that, to establish the uniform asymptotic stability of the set $\mathbf{R} \times A_{\eta_1}$, we can use the stationary Lyapunov function $\overline{W}(t, \theta, y) \equiv W(\theta, y)$ for $t \in \mathbf{R}$ and the proof of Yoshizawa's theorem [11, p. 134].

Remarks. 1. $\mathbf{R} \times \Gamma \subset \tilde{A}_\eta$ and $\tilde{A}_\eta \to \mathbf{R} \times \Gamma$ as $\eta \to 0$.

2. Unlike in Farkas' case, we can only construct a uniform asymptotically stable invariant set $\tilde{A}_\eta$ around the cylinder $\mathbf{R} \times \Gamma$, but not around the graph of the periodic solution $x = p(t)$ of (2.1).

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FACULTY OF MATHEMATICS AND MECHANICS
UNIVERSITY OF HANOI
KHOA TOAN, DAI HOC TONG HOP HANOI,
VIETNAM

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ON THE MEANS OF THE ARGUMENT OF THE RIEMANN ZETA-FUNCTION ON THE CRITICAL LINE

L. GROENEWALD (Bloemfontein)

1. Let $\zeta(s)$ denote the Riemann zeta–function and put

$$\pi S(t) = \Delta_L \arg \zeta(s)$$

where Δ_L denotes the variation in the argument of $\zeta(s)$ along the polygonal line L extending from 2 to $2 + it$ and then to $\frac{1}{2} + it$. Since $\arg\zeta(2) = 0$, we can express $S(t)$ in the form $\pi S(t) = \arg\zeta\left(\frac{1}{2} + it\right)$ provided the argument is defined by continuous variation along L ([1], p. 98).

In [2] Ghosh proved for $k = 1$ and k an even number that

$$(1)\qquad \int_T^{T+H} |S(t)|^k\, dt \sim \frac{2^k}{\sqrt{\pi}} \Gamma\left(\frac{k+1}{2}\right)\left(\frac{1}{2\pi}\right)^k H(\log\log T)^{k/2}, \quad T \to \infty$$

with an error term which holds uniformly in $k \ll (\log\log T)^{1/6}$.

Ghosh's main theorem in [2] on sign changes of $S(t)$ in the interval $(T, T + H)$ is deduced from these latter estimates. For recent conditional results on sign changes of $S(t)$, see [3].

Ghosh [2] mentions without proof that the asymptotic relation (1) can be extended to all integral values of k. It is the aim of this paper to prove Ghosh's claim.

THEOREM. *Let H be a function of T such that $T^\alpha \leqq H(T) \leqq T$, where $\frac{1}{2} < \alpha \leqq 1$ for all $T \geqq 1$. Then, for any positive integer k*

$$\int_T^{T+H} |S(t)|^k\, dt \sim \frac{2^k}{\sqrt{\pi}} \Gamma\left(\frac{k+1}{2}\right)\left(\frac{1}{2\pi}\right)^k H(\log\log T)^{k/2}, \quad T \to \infty.$$

2. We shall need the following:

LEMMA 2.1.

$$\int_0^\infty \frac{1}{u^2} \sum_{j=1}^\infty \frac{(-1)^{j+1}(2u)^{2j}(2k+2j)!}{(2j)!\,(k+j)!}\, du = 2^{2k+1} k! \sqrt{\pi}, \quad k = 0, 1, 2, \ldots.$$

Proof. Since

$$\frac{(2k+2j)!}{(2j)!(k+j)!}=$$
$$=\frac{(2k+2j)(2k+2j-2)\dots(2j+2)(2j)!(2k+2j-1)(2k+2j-3)\dots(2j+1)}{(k+j)(k+j-1)\dots(j+1)(2j)!j!}=$$
$$=\frac{2^k(2k+2j-1)(2k+2j-3)\dots(2j+1)}{j!}$$

if $k \geqq 1$, it follows, on substituting z for $2u$, that the integral above can be written as

$$2^{k+1}\int_0^\infty \frac{1}{z^2}F_{2k-1}(z)\,dz$$

where

(2) $$F_{-1}(z)=\sum_{j=1}^{\infty}(-1)^{j+1}\frac{z^{2j}}{j!}=1-e^{-z^2}$$

and

$$F_{2k-1}(z)=\sum_{j=1}^{\infty}(-1)^{j+1}\frac{z^{2j}}{j!}(2k+2j-1)(2k+2j-3)\dots(2j+1),\quad k\geqq 1.$$

Note that

(3) $$F_{2k+1}(z)=\frac{1}{z^{2k}}\frac{d}{dz}\left(z^{2k+1}F_{2k-1}(z)\right)\quad\text{if}\quad k\geqq 0.$$

Every $F_{2k-1}(z)$ can be written in the form

(4) $$F_{2k-1}(z)=\sum_{i=0}^{k}a_{ki}z^iF_{-1}^{(i)}(z)$$

where the a_{ki} are constants. Indeed, (4) is obvious if $k=0$. If (4) holds for $k=n$, then

$$F_{2n+1}(z)=\frac{1}{z^{2n}}\frac{d}{dz}\left(z^{2n+1}\sum_{i=0}^{n}a_{ni}z^iF_{-1}^{(i)}(z)\right)=$$
$$=\sum_{i=0}^{n}(2n+1+i)a_{ni}z^iF_{-1}^{(i)}(z)+\sum_{i=0}^{n}a_{ni}z^{i+1}F_{-1}^{(i+1)}(z)$$

so that (4) holds for $k = n + 1$. It follows by induction that (4) holds for every k.

If $i \geqq 1$, then $F_{-1}^{(i)}(z)$ can be written as $P_i(z)e^{-z^2}$ for some polynomial $P_i(z)$. Therefore, it follows from (4), that for $k \geqq 0$,

$$\lim_{z\to 0} \frac{F_{2k-1}(z)}{z} = \lim_{z\to\infty} \frac{F_{2k-1}(z)}{z} = 0.$$

Consequently, by (3) integration by parts yields for $k \geqq 1$

$$A_k := \int_0^\infty \frac{1}{z^2} F_{2k-1}(z)dz = \int_0^\infty \frac{1}{z^{2k}} \frac{d}{dz}\left(z^{2k-1}F_{2k-3}(z)\right)dz = 2k\int_0^\infty \frac{1}{z^2}F_{2k-3}(z)dz.$$

We iterate the identity $A_k = 2kA_{k-1}$ for $k = 1, 2, \dots$ to show that

$$A_k = 2^k k! A_0 = 2^k k! \int_0^\infty \frac{F_{-1}(z)}{z^2}dz = 2^k k! \int_0^\infty \frac{1-e^{-z^2}}{z^2}dz.$$

The result follows on noting that

$$\int_0^\infty \frac{1}{z^2}\left(1-e^{-z^2}\right)dz = 2\int_0^\infty e^{-z^2}dz = \sqrt{\pi}.$$

3. Proof of the Theorem. Write $W(t) = 2\pi(\log\log T)^{-\frac{1}{2}}S(t)$. If we put $f(T) = (\log\log\log T)^{\frac{1}{2}}$, it follows from Ghosh [2] that

$$(5)\qquad \int_T^{T+H} |W(t)|^{2j}\,dt = \frac{(2j)!}{j!}H + O\left(\frac{H}{(\log\log T)^{\frac{1}{4}}}\right),$$

uniformly in $1 \leqq j \leqq f(T)$. In view of (1), it suffices to show that for fixed $k \geqq 1$

$$(6)\qquad \int_T^{T+H} |W(t)|^{2k+1}\,dt = \frac{2^{2k+1}}{\sqrt{\pi}}k!H + o_k(H), \quad T\to\infty.$$

Following Ghosh [2], we note that

$$|F| = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin|F|u}{u}\right)^2 du$$

so that the left hand side of (6) can be written as

$$(7)\qquad \frac{2}{\pi}\int_T^{T+H}|W(t)|^{2k}\int_0^{\lambda}\left(\frac{\sin|W(t)|u}{u}\right)^2 du+O\left(\frac{1}{\lambda}\int_T^{T+H}|W(t)|^{2k}\,dt\right)$$

for every $\lambda>0$.

Let $N=N(T)$ be such that $N(T)\to\infty$ as $T\to\infty$ and $N(T)+$ $+k<f(T)$ for all T sufficiently large. Put $2\lambda^3=N$. Since $2\sin^2 x=$ $=\sum_{j=1}^{\infty}(-1)^{j+1}(2x)^{2j}/(2j)!$, we can write

$$\sin^2|W(t)|u=\frac{1}{2}\sum_{j=1}^{N}\frac{(-1)^{j+1}(2|W(t)|u)^{2j}}{(2j)!}+O\left(\frac{(2|W(t)|u)^{2N+2}}{(2N+2)!}\right)$$

and (7) becomes

$$(8)\qquad \frac{1}{\pi}\int_0^{\lambda}\frac{1}{u^2}\int_T^{T+H}|W(t)|^{2k}\sum_{j=1}^{N}\frac{(-1)^{j+1}(2u)^{2j}}{(2j)!}|W(t)|^{2j}\,dt\,du+$$

$$+O\left(\frac{4^N}{(2N+2)!}\int_0^{\lambda}u^{2N}\int_T^{T+H}|W(t)|^{2(N+1+k)}dt\right)+o(H).$$

By (5), the main term in (8) can be written as

$$\frac{H}{\pi}\int_0^{\lambda}\frac{1}{u^2}\sum_{j=1}^{N}\frac{(-1)^{j+1}(2u)^{2j}}{(2j)!}\,\frac{(2k+2j)!}{(k+j)!}\,du+o_k(H)=$$

$$=\frac{H}{\pi}\int_0^{\lambda}\frac{1}{u^2}\sum_{j=1}^{\infty}a_j(u)\,du+O\left(H\int_0^{\infty}\frac{1}{u^2}\left|\sum_{j=N+1}^{\infty}a_j(u)\right|du\right)+o_k(H)$$

where $a_j(u)$ is the jth term under summation.

By Lemma 2.1, the above can be written as

$$\frac{2^{2k+1}}{\sqrt{\pi}}k!H+o_k(H),\quad T\to\infty.$$

It remains to estimate the error term in (8). By (5), this is

$$\ll\frac{4^N}{(2N+2)!}\,\frac{\lambda^{2N+1}}{2N+1}\,\frac{(2N+2+2k)!}{(N+1+k)!}H\ll\frac{\lambda^{3N}}{(2N)!}H\ll\left(\frac{\lambda^3}{N}\right)^N H=o(H).$$

The proof of the theorem is complete.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF THE ORANGE FREE STATE
BLOEMFONTEIN
9300
REPUBLIC OF SOUTH AFRICA

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WHICH TRIANGULAR NUMBERS ARE PRODUCTS OF THREE CONSECUTIVE INTEGERS?

S. P. MOHANTY (Kanpur)

Introduction

Mordell [2] has shown that all integer solutions of the equation $y(y+1) = x(x+1)(x+2)$ are $x = 0, -1, -2$, $y = 0, -1$; $x = 1$, $y = 2, -3$; $x = 5$, $y = 14, -15$. To find all tetrahedral numbers which are triangular, Avanesov [1] solved the equation $3y(y+1) = x(x+1)(x+2)$ and obtained all positive integer solutions given by $x = 1$, $y = 1$; $x = 3$, $y = 4$; $x = 8$, $y = 15$; $x = 20$, $y = 55$; and $x = 34$, $y = 119$. In this paper we try to solve the diophantine equation $y(y+1) = 2x(x+1)(x+2)$ in order to get all triangular numbers which are products of three consecutive integers. The result is contained in the following theorem.

THEOREM 1. *Let the n^{th} triangular number $\frac{n(n+1)}{2}$ be denoted by T_n. Then T_3, T_{15}, T_{20}, T_{44}, T_{608}, and T_{22736} are the only triangular numbers that are products of three consecutive integers.*

PROOF. We consider the diophantine equation

$$y(y+1) = 2x(x+1)(x+2), \tag{1}$$

where x and y are positive integers. Substituting $Y = 2y+1$, $X = 2x+2$ in (1) we get

$$Y^2 = X^3 - 4X + 1, \quad X \geqq 4,\ Y \geqq 3. \tag{2}$$

From Delone and Fadeev's "The theory of irrationalities of the third degree" we note the following facts in $Q(\theta)$ given by

$$f(\theta) = \theta^3 - 4\theta + 1 = 0. \tag{3}$$

(i) The integers in $Q(\theta)$ are $a + b\theta + c\theta^2$, where a, b, c are rational integers.

(ii) The class number $h = 1$ and hence unique factorization exists in $Q(\theta)$.

(iii) The discriminant $D(\theta)$ being $229 > 0$, $f(\theta) = 0$ has three real roots. Hence there are two fundamental units.

Using Billevich's algorithm we find the two fundamental units to be θ and $\theta - 2$. Since $\frac{1}{\theta} = -\theta^2 + 4$, $-\theta^2 + 4$ and $\theta - 2$ are taken as the fundamental units for simplifying the calculations.

Equation (2) can be written as

$$Y^2 = (X - \theta)(X^2 + \theta X + \theta^2 - 4). \tag{4}$$

Let π be a common prime factor of $X - \theta$ and $X^2 + \theta X + \theta^2 - 4$. Then $X \equiv \theta (\bmod \pi)$ and hence $3\theta^2 - 4 \equiv 0 (\bmod \pi)$. Since $|N(3\theta^2 - 4)| = 229$ is a prime, $3\theta^2 - 4$ is the only possible common prime divisor and we have

$$X - \theta = \pm(3\theta^2 - 4)^n \varepsilon^p \eta^q (a + b\theta + c\theta^2)^2, \tag{5}$$

where $\varepsilon = \theta - 2$, $\eta = -\theta^2 + 4$ and $n, p, q \in \{0, 1\}$ as the other powers can be absorbed in the square term.

Taking norm on equation (5) with $n = 1$ we see that $Y^2 = X^3 - 4X + 1 = 229Z^2$ which is clearly impossible. Hence

$$X - \theta = \pm \varepsilon^p \eta^q (a + b\theta + c\theta^2)^2 \tag{6}$$

has four possibilities $(p, q) = (0,0), (1,0), (0,1), (1,1)$. We consider each case separately.

Case 1: $(p, q) = (0, 0)$. Using (3) and expanding the right hand side of (6) we get

$$a^2 - 2bc = \pm X, \tag{7}$$

$$2ab + 8bc - c^2 = \mp 1, \tag{8}$$

$$b^2 + 4c^2 + 2ac = 0. \tag{9}$$

From (7) a is even as X is even. From (8) c is odd. From (9) b is even. Substituting $a = 2a_1$, $b = 2b_1$ in (8) and taking congruence mod 4, we see that the positive sign on the right hand side is impossible. Hence,

$$a^2 - 2bc = X, \tag{10}$$

$$2ab + 8bc - c^2 = -1. \tag{11}$$

From (9) we get $\left(\frac{b}{2}\right)^2 = -c\left(c + \frac{a}{2}\right)$. Since $\left(\frac{a}{2}, c\right) = 1$ implies $\left(c, \frac{a}{2} + c\right) = 1$, we take $-c = u^2$, $\frac{a}{2} + c = v^2$ or $c = u^2$, $\frac{a}{2} + c = -v^2$ and $b = \pm 2uv$. Then (11) yields $\pm 8uv(v^2 - u^2) - u^4 = -1$ or $\pm 8uv(u^2 - v^2) - u^4 = -1$.

In either case u divides the left hand side whence $u = \pm 1$. Hence $\pm 8v(v^2 - 1) = 0$. Either $v = 0$ or $v = \pm 1$. Taking $v = 0$, $u = \pm 1$ we get $c = \pm 1$, $a = \mp 2$, $b = 0$. Again $v = \pm 1$ and $u = \pm 1$ yield $c = -1$, $a = 4$ or $c = 1$, $a = -4$; $b = \pm 2$. Hence $(a, b, c) = (2, 0, -1), (-2, 0, 1), (4, 2, -1), (4, -2, -1), (-4, 2, 1), (-4, -2, 1)$. Then $X = a^2 - 2bc$ implies $X = 4, 12, 20$ or $x = 1, 5, 9$. Correspondingly $y = 3$, 20, and 44.

Case 2: $(p, q) = (0, 1)$. Using (3) and expanding the right hand side of $X - \theta = \pm(4 - \theta^2)(a + b\theta + c\theta^2)^2$ we get

$$4a^2 - c^2 + 2ab = \pm X, \tag{12}$$

$$b^2 + 4c^2 + 2ac = \mp 1, \tag{13}$$

$$a^2 - 2bc = 0. \tag{14}$$

From (12), (13) and (14) it is clear that a, b, c are even, odd and even, respectively. Therefore negative sign is not possible on the right hand side of (13). Hence we have

$$b^2 + 4c^2 + 2ac = 1, \tag{15}$$

$$4a^2 - c^2 + 2ab = -X. \tag{16}$$

Using (14), (15) and (16) we see that

(i) $a \neq 0, c \neq 0$.

(ii) b and c have same sign while a and c have opposite sign.

(iii) if (a, b, c) is a solution so is $(-a, -b, -c)$ and they yield the same value for X.

Hence without loss of generality we may assume b and c to be positive. Therefore a is negative. Since $a^2 = 2c \cdot b$ and $(2c, b) = 1$, take $2c = u^2$, $b = v^2$ and $a = -uv$, where u and v are of same sign. Substituting now the value of a, b, c in terms of u and v in (15) we get

$$u^4 + v^4 - u^3 v = 1, \quad u \neq 0. \tag{17}$$

Taking u and v to be positive and writing (17) in two different ways as $u^3(u - v) + v^4 = 1$ and $u^4 + v(v^3 - u^3) = 1$ we see that neither $u > v$ nor $v > u$. Again $u = v$ is impossible because u is even and v is odd. If u and v are both negative, then setting $u = -u_1$, $v = -v_1$ in (17) we obtain $u_1^4 + v_1^4 - u_1^3 v_1 = 1$ with u_1, v_1 positive which is the same equation as (17). Thus, (17) has no integral solutions $u \neq 0$.

Case 3: $(p, q) = (1, 1)$. We have $X - \theta = \pm(\theta - 2)(4 - \theta^2)(a + b\theta + c\theta^2)^2$ or

$$\theta X - \theta^2 = \pm(\theta - 2)(a + b\theta + c\theta^2)^2. \tag{18}$$

Expanding the right hand side of (18) and using $\theta^3 - 4\theta + 1 = 0$ we get

$$a^2 + 4b^2 + 18c^2 - 4ab - 18bc + 8ac = \pm X, \tag{19}$$

$$-2b^2 - 9c^2 + 2ab + 8bc - 4ac = \mp 1, \tag{20}$$

$$2a^2 + b^2 + 4c^2 - 4bc + 2ac = 0. \tag{21}$$

Since a is even, c is odd and b is even, the positive sign on the right hand side of (20) is impossible by congruence modulo 4. So we have

$$a^2 - 2bc = X - 2, \tag{22}$$

$$(23) \qquad 4a^2 - c^2 + 2ab = -1,$$

$$(24) \qquad (b-2c)^2 = -2a(a+c).$$

Using the fact that $a+c$ is odd we have $a=0$ if and only if $b=2c$. In this case $(a,b,c)=(0,2,1)$ or $(0,-2,-1)$ and $X=-2$, i.e., $x=-2$. Suppose $a \neq 0$ i.e., $b \neq 2c$. Then from (23), $(a,c)=1$. Now $(b-2c)^2 = -2a(a+c)$ with $(a, a+c) = 1$ yielding

$$-2a = u^2,\ a+c = v^2,\ b-2c = \pm uv$$

or

$$2a = u^2,\ a+c = -v^2,\ b-2c = \pm uv.$$

Substituting the values of a, b, c in (23) we get

$$(u^2 \pm 2uv)^2 + 8u^2v^2 + 4v^4 = 4,$$

which is impossible for $u \neq 0$ and $v \neq 0$.

Case 4: $(p,q) = (1,0)$. Expanding the right hand side of $X - \theta = \pm(\theta - 2)(a + b\theta + c\theta^2)^2$ and equating the coefficients of like powers as before we get

$$(25) \qquad -2a^2 - b^2 - 4c^2 + 4bc - 2ac = \pm X,$$

$$(26) \qquad a^2 + 4b^2 + 18c^2 - 4ab - 18bc + 8ac = \mp 1,$$

$$(27) \qquad -2b^2 - 9c^2 + 2ab + 8bc - 4ac = 0.$$

We see that b is even, a is odd and c is even from (25), (26) and (27), respectively. Taking congruence mod 4 in (26) negative sign on the right hand side of (26) is ruled out. Therefore, we have

$$(28) \qquad a^2 - 2bc = 1,$$

$$(29) \qquad -4a^2 + c^2 - 2ab = -2X$$

and

$$(30) \qquad c_1^2 = (2c_1 - b_1)(-a + 2b_1 - 4c_1), \quad \text{where} \quad b = 2b_1 \text{ and } c = 2c_1.$$

From (28) we see that b and c are of the same sign. Since (a,b,c) and $(-a,-b,-c)$ appear as solutions we can take b and c to be both positive. Since $(2c_1 - b_1, -a + 2b_1 - 4c_1) = 1$ we have

$$(31) \qquad 2c_1 - b_1 = u^2, \quad -a + 2b_1 - 4c_1 = v^2, \quad c_1 = \pm uv$$

or

$$(32) \qquad b_1 - 2c_1 = u^2, \quad a - 2b_1 + 4c_1 = v^2, \quad c_1 = \pm uv.$$

We note that u and v have opposite sign if $c_1 = -uv$ and u and v are of same sign if $c_1 = uv$.

Using (31) and (32) with $c_1 = -uv$ or uv the equation (28) reduces to

$$4u^4 + v^4 - 12u^2v^2 + 8u^3v = 1 \tag{33}$$

and

$$4u^4 + v^4 - 12u^2v^2 - 8u^3v = 1. \tag{34}$$

Since equation (34) is obtainable from (33) by taking $u = -u$, $v = v$ or $u = u$, $v = -v$ it is enough to consider the equation (33). We note that if (u, v) is a solution of (33) then so is $(-u, -v)$.

The diophantine equation (33) can be written as

$$u^2(3v + u)(v - u) = \frac{v^2 + 1}{2} \cdot \frac{v^2 - 1}{2}.$$

If $v^2 = 1$, then $u^2(3v + u)(v - u) = 0$, whence $u = 0$ or $u = v$ or $u = -3v$. Then we have $(u, v) = (0, 1)$, $(0, -1)$, $(1, 1)$, $(-1, -1)$, $(-3, 1)$, $(3, -1)$. If $u^2 = 1$, then $(u, v) = (1, 1)$, $(1, 3)$, $(-1, -1)$, $(-1, -3)$ are also solutions. Suppose $u^2 > 1$ and $v^2 > 1$. Now u^2 divides one of $\frac{v^2+1}{2}$ and $\frac{v^2-1}{2}$ but not both. Again writing $4u^4 + v^4 - 12u^2v^2 + 8u^3v = 1$ as $(2u^2 + 2uv)^2 + v^2(v^2 - 16u^2) = 1$ we see that $v^2 \geqq 16u^2$ is impossible. Therefore $v^2 < 16u^2$.

If $u^2 | \frac{v^2-1}{2}$, then $\frac{v^2-1}{2u^2}$ is positive integer, less than $\frac{16u^2-1}{2u^2} = 8 - \frac{1}{2u^2}$. Hence $\frac{v^2-1}{2u^2} = 1, 2, 3, \ldots, 7$ or $v^2 = 2u^2 + 1, 4u^2 + 1, \ldots, 14u^2 + 1$.

We consider $(3v + u)(v - u) = \frac{v^2-1}{2u^2} \cdot \frac{v^2+1}{2}$ for $\frac{v^2-1}{2u^2} = 1, 2, \ldots, 7$. For example, when $\frac{v^2-1}{2u^2} = 3$ our equation $3v^2 - 2uv - u^2 = \frac{v^2-1}{2u^2} \cdot \frac{v^2+1}{2}$ becomes $3(6u^2 + 1) - 2uv - u^2 = 3(3u^2 + 1)$, or $v = 4u$, a contradiction. If we take $\frac{v^2-1}{2u^2} = 4$, then we have $3(8u^2 + 1) - 2uv - u^2 = 4(4u^2 + 1)$. On simplification we get $7u^2 - 2uv - 1 = 0$ or $v = \frac{7u^2-1}{2u}$. Then $\left(\frac{7u^2-1}{2u}\right)^2 = v^2 = 8u^2 + 1$ yields $(17u^2 - 1)(u^2 - 1) = 0$, whence $u = \pm 1$ and $v = \pm 3$. We solve $3v^2 - 2uv - u^2 = \frac{v^2-1}{2u^2} \cdot \frac{v^2+1}{2}$ as above for every value of v^2 as listed above. Similarly, if $u^2 | \frac{v^2+1}{2}$ then $\frac{v^2+1}{2u^2}$ is a positive integer $\leqq 8$. We solve $3v^2 - 2uv - u^2 = \frac{v^2+1}{2u^2} \cdot \frac{v^2-1}{2}$ for $\frac{v^2+1}{2u^2} = 1, 2, \ldots, 8$. We do not get any new solution for (u, v). Hence all solutions (u, v) are as above. Thus the positive integral solutions for $4u^4 + v^4 - 12u^2v^2 + 8u^3v = 1$ and $4u^4 + v^4 - 12u^2v^2 - 8u^3v = 1$ are given by $(u, v) = (1, 1)$, $(1, 3)$ and $(3, 1)$ respectively. They in turn give $(a, b, c) = (-3, 2, 2)$, $(-11, 10, 6)$ and $(19, 30, 5)$. Substituting these values in (29), we get $X = 10$, 114, 1274. Hence this case gives $x = 4$, 56 and 636. Corresponding to $x = 4, 56$ and 636 we have $y = 15$, 608, and 22736. We get three more triangular numbers T_{15}, T_{608} and T_{22736}. Thus the theorem is established.

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DEPARTMENT OF MATHEMATICS
I. I. T. KANPUR
208016 INDIA

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ON DARBOUX FUNCTIONS IN HONORARY BAIRE CLASS TWO

I. POKORNÝ (Košice)

1. Introduction

In [5] R. J. O'Malley introduced and developed the idea of selective differentiation theory.

In [1], Bagemihl and Piranian defined a function g as honorary Baire class two if there exists a $\mathcal{B}_1$ (Baire class one) function h such that the set $\{x : h(x) \neq g(x)\}$ is at most countable. See also [3], [9].

We know that the class of selective derivatives is a proper subclass of the class of Darboux functions in honorary Baire class two (see [5, Theorem 11] and [6, Proposition 3]). Hence it is interesting to investigate this class, because this class plays the same role for the selective derivatives as the class $\mathcal{DB}_1$ for the derivatives.

Our main results are the following.

1) Every $\mathcal{DHB}_2$ function is pointwise discontinuous.

2) For every $f \in \mathcal{DHB}_2$ there exists a $g \in \mathcal{B}_1$ such that the points of continuity of f and g coincide and $\{x : f(x) \neq g(x)\}$ is countable (i.e. at most countable).

3) The maximal additive class for $\mathcal{DHB}_2$ is the class of all constant functions.

4) $\mathcal{DHB}_2$ is not closed under the uniform convergence.

The last two results show that, as for the maximal additive class and uniform convergence, the class $\mathcal{DHB}_2$ behaves similarly to the class $\mathcal{D}$. On the other hand, it is well-known that the maximal additive class for $\mathcal{DB}_1$ is the class of continuous functions, and $\mathcal{DB}_1$ is closed under uniform limits ([2], pp 14, 15).

In [7], T. Radakovic proved that the maximal additive class for $\mathcal{D}$ (but not $\mathcal{HB}_2$) is the class of constant functions.

In [8], J. Smítal proved that the class $\mathcal{DB}_2$ is not closed under uniform limits. In his proof, functions from $\mathcal{B}_2 \setminus \mathcal{HB}_2$ are used in an essential way.

Our approach is different.

2. Preliminaries

Throughout this article, the functions under consideration are usually real valued functions defined on the closed interval $I = \langle 0, 1 \rangle$.

The class of all Darboux functions on I is denoted by $\mathcal{D}$.

Let $\mathcal{G}$ be a class of functions defined on an interval I. A subclass $\mathcal{F}$ of $\mathcal{G}$ is called the maximal additive class for $\mathcal{G}$ provided $\mathcal{F}$ is the set of all functions in $\mathcal{G}$ such that $f+g\in\mathcal{G}$ whenever $f\in\mathcal{F}$ and $g\in\mathcal{G}$.

Further, $[x,y]$ will denote the closed interval having endpoints x and y regardless of whether $x<y$ or $y<x$.

We frequently refer to certain other classes of functions: the Baire class α functions, the honorary Baire class two functions and the continuous functions. We denote these classes by $\mathcal{B}_\alpha$, $\mathcal{HB}_2$ and $\mathcal{C}$, respectively.

Let f be a function. We denote the set of points of discontinuity, resp. continuity by D_f, resp. C_f.

We say that f is pointwise discontinuous if C_f is a dense set in I.

Let x be a point of I. By the cluster set of f at x, denoted by $C(f,x)$, we mean the set of numbers y such that there exists a sequence $x_n\to x$ such that $x_n\neq x$ and $f(x_n)\to y$. The one-sided cluster sets $C(f,x,+)$ and $C(f,x,-)$ are defined in the obvious way.

It A is a set, then $\operatorname{int} A$, $\operatorname{cl} A$ and A' denote the interior, closure and the set of accumulation points of the set A, respectively.

3. $\mathcal{DHB}_2$ functions and continuity points

LEMMA 1. *Let $f\in\mathcal{D}$ and let g be a function such that the set* $A = \{x : f(x)\neq g(x)\}$ *is countable. Then* $C_g\subset C_f\setminus A$ *and* $A\cup D_f\subset D_g$.

PROOF. Let $x_0\in C_g$ be fixed. Let $\varepsilon>0$ be given and let $\delta>0$ be such that $|g(x)-g(x_0)|<\varepsilon$ for $|x-x_0|<\delta$. Therefore, by assumption, $|f(x)-g(x_0)|<\varepsilon$ holds for every $x\in(x_0-\delta\,,\;x_0+\delta)$ apart from a countable set. Since f is Darboux, this implies that $|f(x)-g(x_0)|\le\varepsilon$ holds for every $x\in(x_0-\delta\,,\;x_0+\delta)$. This obviously implies that $f(x_0)=g(x_0)$ and $x_0\in C_f$. Hence we obtain $C_g\subset C_f\setminus A$ and, taking the complements, $A\cup D_f\subset D_g$. □

COROLLARY 2. *Let $f,g\in\mathcal{D}$. If the set $A=\{x : f(x)\neq g(x)\}$ is countable, then $A\subset D_f=D_g$.*

PROOF. By Lemma 1, $A\cup D_f\subset D_g$ and $A\cup D_g\subset D_f$ from which the assertion follows. □

THEOREM 3. *Each $f\in\mathcal{DHB}_2$ is a pointwise discontinuous function.*

PROOF. Let $f\in\mathcal{DHB}_2$ and let $g\in\mathcal{B}_1$ be such that $\{x : f(x)\neq g(x)\}$ is countable. Since $g\in\mathcal{B}_1$, C_g is everywhere dense. By Lemma 1, $C_g\subset C_f$ and hence C_f is also everywhere dense. □

REMARK. Theorem 8 in [4] follows immediately from this theorem, because each selective derivative belongs to the class $\mathcal{DHB}_2$.

Our next aim is to prove that for each function $f\in\mathcal{DHB}_2$ there is a function $g\in\mathcal{B}_1$ for which the set $\{x : f(x)\neq g(x)\}$ is countable and $D_f=D_g$.

LEMMA 4. *Let $f \in \mathcal{DHB}_2$. Then there is a function $h \in \mathcal{B}_1$ with the following properties:*

1) *The set $\{x : f(x) \neq h(x)\}$ is countable;*
2) *For each $x \in I : h(x) \in C(f, x)$.*

PROOF. If $f \in \mathcal{DB}_1$ then we can take $h \equiv f$, because $f(x) \in C(f, x)$ for each $f \in \mathcal{D}$.

Let $f \in \mathcal{DHB}_2 \setminus \mathcal{B}_1$. Then there is a function $g \in \mathcal{B}_1$ for which the set $A = \{x : f(x) \neq g(x)\}$ is countable (by the definition of the class $\mathcal{HB}_2$).

Define

$$h(x) = \begin{cases} g(x); & \text{if } \ g(x) \in C(f, x), \\ t \in C(f, x); & \text{otherwise,} \end{cases}$$

where $|t - g(x)| = \operatorname{dist}(g(x), C(f, x))$.

We prove that h has all the required properties.

From the definition of h it follows that for each $x \in I$ we have $h(x) \in \in C(f, x)$ and that the set $\{x : f(x) \neq h(x)\}$, being a subset of A, is countable.

To prove that $h \in \mathcal{B}_1$ we proceed as follows.

We prove that for each non-empty perfect set $P \subset I$ the restriction of h to P has a point of continuity.

Let P be a non-empty perfect set in I. Since $g \in \mathcal{B}_1$, there is a point $x \in P \setminus A$ at which $g|P$ is continuous, because the set of continuity points of $g|P$ is of second category in P and A is countable. Then we have $f(x) = = g(x) = h(x)$.

We show that $h|P$ is also continuous at x.

Let $\varepsilon > 0$ be given, let $J = \langle g(x) - \varepsilon, g(x) + \varepsilon \rangle$, and let $\delta > 0$ be such that $g(y) \in J$ holds for every $y \in P \cap (x - \delta, x + \delta)$. Let $y \in P \cap (x - \delta, x + \delta)$ be fixed. Since every portion of P is of cardinality of the continuum and A is countable, there is a sequence $y_n \in P \cap (x - \delta, x + \delta) \setminus A$, $y_n \to y$, $y_n \neq y$. For every n we have $f(y_n) = g(y_n) \in J$ and hence we can select a subsequence such that $f(y_{n_k}) \to z \in J$. This shows that $C(f, y) \cap J \neq \emptyset$. Since $f \in \mathcal{D}$, $C(f, y)$ is an interval. As $g(y) \in J$, it follows from the definition of h that $h(y) \in J$. Therefore $h(y) \in J$ holds for every $y \in P \cap (x - \delta, x + \delta)$ and hence $h|P$ is continuous at x. □

THEOREM 5. *For every $f \in \mathcal{DHB}_2$ there is $h \in \mathcal{B}_1$ such that the set $\{x : f(x) \neq h(x)\}$ is countable and $C_f = C_h$.*

PROOF. Let $f \in \mathcal{DHB}_2$ be given, and let h be the function defined in Lemma 4. Since $h(x) \in C(f, x)$ holds everywhere, it follows that $C_f \subset C_h$. On the other hand, by Lemma 1, we have $C_h \subset C_f$ and hence $C_f = C_h$. □

4. On the maximal additive class of $\mathcal{DHB}_2$

In this section we prove that the maximal additive class of $\mathcal{DHB}_2$ is the class of all constant functions.

LEMMA 6. *Let P be a non-empty, bounded and nowhere dense perfect subset of R. Let $a = \min P$ and $b = \max P$. Let $c, d \in R$, $c < d$. Then there is a function $g : \langle a, b\rangle \to \langle c, d\rangle$ in the class $\mathcal{HB}_2 \setminus (\mathcal{B}_1 \cup \mathcal{D})$ such that*

1) *the cluster set $C(g, x) = \langle c, d\rangle$, for each $x \in P$;*

2) *the set $\{x \in \langle a, b\rangle : g(x) = (c + d)/2\} = \emptyset$.*

PROOF. Let all assumptions of this lemma be satisfied. Let $e = (c+d)/2$. We decompose the class of all contiguous intervals of P (on the interval $\langle a, b\rangle$) into two classes $\mathcal{A}$ and $\mathcal{B}$ with the following property: For each two elements of one of these classes there is an element of the other class which is located between them.

I. Let $(u, v) \in \mathcal{A}$ and let $A_{u,v}$ be an arbitrary subset of $\langle u, v\rangle$ such that $\{u, v\} = A_{u,v} \cap A'_{u,v}$. Then we can define a function g on the interval $\langle u, v\rangle$ with the following properties:

(I.1) the function g is continuous on (u, v),

(I.2) the range of $g_{|(u,v)\setminus A_{u,v}} = (e, d)$,

(I.3) $C(g, u, +) = C(g, v, -) = \langle e, d\rangle$,

(I.4) $g_{|A_{u,v}} \equiv d$.

II. Let $(u, v) \in \mathcal{B}$ and let $B_{u,v}$ be an arbitrary subset of $\langle u, v\rangle$ such that $\{u, v\} = B_{u,v} \cap B'_{u,v}$. Then we can define a function g on the interval $\langle u, v\rangle$ with the following properties:

(II.1) the function g is continuous on (u, v),

(II.2) the range of $g_{|(u,v)\setminus B_{u,v}} = (c, e)$,

(II.3) $C(g, u, +) = C(g, v, -) = \langle c, e\rangle$,

(II.4) $g_{|B_{u,v}} \equiv c$.

III. At the points of $P \setminus \cup B_{u,v}$ we define g by $g(x) \equiv d$.

We show that this function g has all the required properties.

1) From the definition of g we have that

a) $g : \langle a, b\rangle \to \langle c, d\rangle$,

b) $\{x \in \langle a, b\rangle : g(x) = e\} = \emptyset$, where $e = (c + d)/2$.

2) Since each $x \in P$ is a limit point of elements of the class $\mathcal{A}$ and a limit point of elements of the class $\mathcal{B}$, we have

$$C(g, x) = \langle c, d\rangle \quad \text{for each} \quad x \in P\,.$$

(Properties (I.3) and (II.3).)

3) From properties (I.4) and (II.4) it follows that the function $g|P$ does not have a point of continuity and therefore $g \notin \mathcal{B}_1$.

4) The classes $\mathcal{A}$ and $\mathcal{B}$ are non-empty. Since for $x \in (u, v) \in \mathcal{A}$ and $y \in (u', v') \in \mathcal{B}$ we have $g(y) < e < g(x)$ and $g(z) \neq e$ for each $z \in [x, y]$ (Property 1.b), necessarily $g \notin \mathcal{D}$.

5) We show that $g \in \mathcal{HB}_2$.

Let

$$h(x) = \begin{cases} g(x), & \text{if} \quad x \notin P, \\ d, & \text{if} \quad x \in P. \end{cases}$$

Then h has the following properties:

a) the function $h \in \mathcal{B}_1$, because $C_h = \langle a, b \rangle \setminus P$ and $h|P \equiv d$. (For each perfect set Q the restriction $h|Q$ has a point of continuity.),

b) the set $\{x \in \langle a, b \rangle : g(x) \neq h(x)\}$ is a set of endpoints of contiguous intervals in class $\mathcal{B}$, which is a countable set. Then by the definition of the class $\mathcal{HB}_2$ we have $g \in \mathcal{HB}_2$. □

THEOREM 7. *Let h be a nonconstant continuous function on $I = \langle 0, 1 \rangle$. Then there is a function $g \in \mathcal{HB}_2 \setminus \mathcal{D}$ such that $f = g + h \in \mathcal{DHB}_2$.*

PROOF. Let $m = \min\{h(x) : x \in I\}$ and $M = \max\{h(x) : x \in I\}$; since h is nonconstant, $m < M$. We may assume that $m = 0$ and $M = 1$. We prove first that there is a non-empty perfect set P such that h is strictly monotonic on P. Let $a_0, b_0 \in I$ be such that $h(a_0) - 0$ and $h(b_0) = 1$. We may assume, without loss of generality, that $a_0 < b_0$. Let $x_r = \min\{x \in \in \langle a_0, b_0 \rangle : h(x) = r\}$ for each $r \in \langle 0, 1 \rangle$. It is easy to check that h is strictly increasing on the set $Q = \{x_r : r \in \langle 0, 1 \rangle\}$, and that Q is uncountable and G_δ. Therefore we can select a non-empty, perfect and nowhere dense subset $P \subset Q$. Now we apply Lemma 6 with this perfect set P and with $c = -1$, $d = 1$. Let g denote the function constructed in the proof of Lemma 6. The function g is defined on $\langle a, b \rangle$, where $a = \min P$ and $b = \max P$. We extend g to I by defining $g(x) = g(a)$ for $x \in \langle 0, a)$ and $g(x) = g(b)$ for $x \in (b, 1\rangle$. It is easy to see that $g \in \mathcal{HB}_2 \setminus \mathcal{D}$.

Let $f = g + h$, then $f \in \mathcal{HB}_2$ since $g \in \mathcal{HB}_2$ and h is continuous. We shall prove that $f \in \mathcal{D}$. Since f is continuous on the intervals $\langle 0, a \rangle$ and $\langle b, 1 \rangle$, it is enough to show that f is Darboux on $\langle a, b \rangle$.

Let L_r (L_ℓ) denote the set of right (left) endpoints of the intervals contiguous to P. First we prove that

$$\text{(A)} \qquad f(\langle x, y \rangle) \supset [f(x), f(y)]$$

whenever $x < y$, $x \in P \setminus L_\ell$ and $y \in P \setminus L_r$. Let (u, v) be an interval contiguous to P and suppose that (u, v) belongs to the class $\mathcal{A}$. Then $C(g, u, +) = \langle 0, 1 \rangle$ and, as both g and h are continuous in (u, v) and h is continuous at u, it follows that $f((u, v)) \supset (h(u), h(u) + 1)$. Similarly, if (u, v) belongs to the class $\mathcal{B}$ then $f((u, v)) \supset (h(u) - 1, h(u))$. Since $x \in P \setminus L_\ell$, every right hand side neighbourhood of x contains elements of both classes $\mathcal{A}$ and $\mathcal{B}$, and hence

$$f((x, y)) \supset (h(x) - 1, h(x)) \cup (h(x), h(x) + 1)\,.$$

We also have $h(x) \in f((x, y))$. Indeed, if $(u, v) \subset (x, y)$ is an element of the class $\mathcal{B}$ then $h(u) - 1 < h(x) < h(u)$, since h is strictly increasing on P. Therefore we have

$$f((x, y)) \supset (h(x) - 1, h(x) + 1)\,.$$

Similar argument shows that

$$f((x,y)) \supset (h(y)-1, h(y)+1).$$

Since $0 \le h(x) < h(y) \le 1$,

$$(h(x)-1, h(x)+1) \cup (h(y)-1, h(y)+1) = (h(x)-1, h(y)+1),$$

and hence $f((x,y)) \supset (h(x)-1, h(y)+1)$. Now, $|g| \le 1$ implies $f(x), f(y) \in \in \langle h(x)-1, h(y)+1\rangle$ which proves (A).

Let $a \le x < y \le b$ be arbitrary. If $x \in P \setminus L_\ell$ then let $x' = x$. If $x \notin P \setminus L_\ell$ then let $x' \in L_r$ be such that $(x,x') \cap P = \emptyset$. Similarly, we put $y' = y$ if $y \in P \setminus L_r$, and if $y \notin P \setminus L_r$ then we take $y' \in L_\ell$ such that $(y', y) \cap P = \emptyset$. It is easy to check that $f((x,x')) \supset \operatorname{int}[f(x), f(x')]$ and $f((y',y)) \supset \operatorname{int}[f(y'), f(y)]$. Since, by (A), $f([x',y']) \supset [f(x'), f(y')]$, we have $f(\langle x,y\rangle) \supset [f(x), f(y)]$ and this proves the Darboux property of f. □

It is well-known that the maximal additive class for $\mathcal{DB}_1$ is $\mathcal{C}$. (Viz. Theorem 3.2 on p. 14 in [2].)

In [7], Radakovic proved that the maximal additive class for $\mathcal{D}$ is the class of all constant functions. The same holds for the class $\mathcal{DHB}_2$, too.

COROLLARY 8. *The maximal additive class for $\mathcal{DHB}_2$ is the class of all constant functions.*

PROOF. Let h be a constant function. Then trivially $h+g \in \mathcal{DHB}_2$ for each $g \in \mathcal{DHB}_2$. Let $h \in \mathcal{DHB}_2$ be a discontinuous function. Let x_0 be a point of discontinuity of h, and suppose h is discontinuous from the right at x_0. Choose $y_0 \ne h(x_0)$ in the interval $C(h, x_0, +)$. Define g by

$$g(x) = \begin{cases} -h(x), & \text{if } x \in (x_0, 1\rangle, \\ -y_0, & \text{if } x \in \langle 0, x_0\rangle. \end{cases}$$

It is easy to verify that $g \in \mathcal{DHB}_2$. But $h+g$ vanishes for $x \in (x_0, 1\rangle$, and $h(x_0) + g(x_0) \ne 0$, so $h+g$ does not have the Darboux property. Let h be a nonconstant continuous function. Then $-h$ is a nonconstant continuous function, too. By Theorem 7 there is $f \in \mathcal{HB}_2 \setminus \mathcal{D}$ for $-h$ such that $g = f - h \in \mathcal{DHB}_2$. But $h + g = f \notin \mathcal{D}$. □

5. On the uniform convergence in $\mathcal{DHB}_2$

In this section we prove that the class $\mathcal{DHB}_2$ is not closed under the uniform convergence.

THEOREM 9. *There is a sequence of $\mathcal{DHB}_2$ functions such that $f_n \rightrightarrows f \notin \mathcal{D}$.*

PROOF. Let C be the well-known Cantor set on the interval I. We use Lemma 6 and the notation of its proof, where $P = C$, $c = -1$ and $d = 1$.

Our function f will be the function g (from Lemma 6) and the functions f_n will be the following modifications of g ($n = 1, 2, \dots$):

1) Let $(u, v) \in \mathcal{A}$, let $\{\langle x_i, y_i\rangle : i = 1, 2, \dots\}$ be a sequence of disjoint closed subintervals of (u, v) such that $A_{u,v} \cap \langle x_i, y_i\rangle = \emptyset$, let $u, v \in \{x_i : i = = 1, 2, \dots\}'$ and let $g(z_i) \to 0$, where $z_i = (x_i + y_i)/2$ for $i = 1, 2, \dots$. Let

$$f_n(x) = \begin{cases} g(x), & \text{if} \quad x \in (u, v) \setminus \bigcup_{i=1}^{\infty} \langle x_i, y_i\rangle, \\ g(x) - h_{n,i}(x), & \text{if} \quad x \in \langle x_i, y_i\rangle \text{ for some natural } i, \end{cases}$$

where

$$h_{n,i}(x) = \begin{cases} (x - x_i)/(n(z_i - x_i)), & \text{for} \quad x \in \langle x_i, z_i\rangle, \\ (y_i - x)/(n(y_i - z_i)), & \text{for} \quad x \in (z_i, y_i\rangle, \end{cases} \qquad i = 1, 2, \dots .$$

2) Let $(u, v) \in \mathcal{B}$. This case is analogous to the case 1). The difference between these cases is the sign of $h_{n,i}$ in the definition of f_n.

3) $f_n(x) = g(x)$ otherwise.

We show that these functions f and f_n have all the required properties.

a) Of course, $|f(x) - f_n(x)| \le 1/n$ for each $x \in I$ and therefore $f_n \rightrightarrows f$.

b) The functions $f, f_n \in \mathcal{HB}_2$, for $n = 1, 2, \dots$. Indeed, let

$$F(x) = \begin{cases} f(x), & \text{if} \quad x \notin C, \\ 1, & \text{if} \quad x \in C, \end{cases}$$

and

$$F_n(x) = \begin{cases} f_n(x), & \text{if} \quad x \notin C, \\ 1, & \text{if} \quad x \in C, \end{cases}$$

for $n = 1, 2, \dots$.

The set $\{x : f(x) \ne F(x)\}$ and the sets $\{x : f_n(x) \ne F_n(x)\}$ are countable, because they are subsets of the set of endpoints of elements of $\mathcal{B}$. The functions F_n and F are obviously Baire 1.

c) For each $n = 1, 2, \cdots : f_n \in \mathcal{D}$. Since f_n takes the value zero in every interval contiguous to P, it is easy to verify that f_n is Darboux.

d) Finally, $f \notin \mathcal{D}$ follows from Lemma 6. □

We finish this paper with the following problem:

Problem. What is the maximal multiplicative class for $\mathcal{DHB}_2$?

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KATEDRA RIADENIA VÝROBNÝCH PROCESOV
BANÍCKEJ FAKULTY TECHNICKEJ UNIVERZITY
B. NĚMCOVEJ Č. 3.
042 00 KOŠICE
CZECHOSLOVAKIA

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INTEGRABLE p-ALMOST TANGENT MANIFOLDS AND TANGENT BUNDLES OF p^1-VELOCITIES

M. DE LEÓN (Madrid),
ISABEL MÉNDEZ and M. SALGADO (Santiago de Compostela)

1. Introduction

An almost tangent structure on a $2n$-dimensional manifold N is a tensor field J of type (1.1) of rank n such that $J^2 = 0$ (N is said to be an almost tangent manifold). Also, an almost tangent structure J may be interpreted as a type of G-structure, where G is some Lie subgroup of $Gl(2n, R)$. Almost tangent structures were introduced by Clark and Bruckheimer [2] and Eliopoulos [6] around 1960 and have been studied by several authors (see [1], [3], [4], [9], [13]).

As it is well-known the tangent boundle TM of any manifold M carries a canonical integrable almost tangent structure. Moreover, any integrable almost tangent structure is locally equivalent to this canonical almost tangent structure. But not every integrable almost tangent manifold N is globally isomorphic to the tangent bundle TM of a manifold M. Recently, Crampin and Thompson [4] proved that an integrable almost tangent manifold N which defines a fibration (that is, the space of leaves M of the foliation defined by the integrable distribution $V = \operatorname{Im} J$) with certain additional hyphotheses is an affine bundle modelled on TM.

In [10], we have introduced and studied a new type of geometric structures (called *p-almost tangent structures*) which are a natural generalization of almost tangent structures. A p-almost tangent structure consists of a p-tuple of tensor fields $(J_1, \dots, J_p)$ of type (1.1) on a $(p+1)n$-dimensional manifold N satisfying some compatibility conditions (N is said to be a *p-almost tangent manifold*). The tangent bundle T^1_pM of p^1-velocities of any n-dimensional manifold M carries an integrable canonical p-almost tangent structure (hence the name). In [10] we have proved that any integrable p-almost tangent manifold N is locally equivalent to the canonical p-almost tangent structure on T^1_pM.

In this paper we consider the global problem of equivalence. Then we consider an integrable p-almost tangent manifold N which defines a fibration (that is, the space of leaves M of the foliation defined by the integrable distribution $V = (\operatorname{Im} J_1) \oplus \dots \oplus (\operatorname{Im} J_p)$) have the structure of differentiable manifold. In such a case, under certain hypotheses on the leaves of the foliations defined by the integrable distributions V, $V_a = \operatorname{Im} J_a$, $1 \leqq a \leqq p$, we prove that *N is an affine bundle modelled on T^1_pM*. Obviously, when $p = 1$, we reobtain the result of Crampin and Thompson.

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2. The tangent bundle of p^1-velocities

Let M be an n-dimensional manifold. By T_p^1M we denote the *tangent bundle of* p^1*-velocities of* M, that is, the manifold of all 1-jets of mappings from R^p to M at the origin $0 \in R^p$ (see [5], [11]). The manifold T_p^1M is locally characterized as follows: if (x^i) is a coordinate system on M then the coordinates $(x^i, y_1^i, \ldots, y_p^i)$ on T_p^1M are defined by

$$x^i(j_0^1\sigma) = x^i(\sigma(0)),$$

$$y_a^i(j_0^1\sigma) = \big(\partial(x^i \circ \sigma)/\partial t^a\big)|_{t=0}, \quad 1 \leqq i \leqq n, \quad 1 \leqq a \leqq p,$$

where $j_0^1\sigma$ is the 1-jet at $0 \in R^p$ of the map $\sigma\colon R^p \to M$ and $t = (t^1, \ldots, t^p) \in \in R^p$. Clearly, T_p^1M is a manifold of dimension $(p+1)n$. We denote by $\pi\colon T_p^1M \to M$ the canonical projection given by $\pi(j_0^1\sigma) = \sigma(0)$.

Remark. When $p = 1$, then T_p^1M is the tangent bundle TM of M.

Next, we shall prove that $\pi\colon T_p^1M \to M$ has the structure of vector bundle with standard fibre the vector space R^{pn}. To do this, we proceed as follows. We have a canonical diffeomorphism

$$\Lambda\colon T_p^1M \to TM \oplus \overset{p}{\ldots} \oplus TM$$

of T_p^1M with the Whitney sum of TM with itself p times; Λ is given by

$$\Lambda(j_0^1\sigma) = (j_0^1\sigma_1, \ldots, j_0^1\sigma_p),$$

where $\sigma_a\colon R \to M$ is the curve on M defined by

$$\sigma_a(t) = \sigma(0, \ldots, t, \ldots, 0),$$

with t placed at the a^{th} position. Then each element $u \in (T_p^1M)_x = \pi^{-1}(x)$, $x \in M$ may be identified, via Λ, with a p-tuple $(u_1, \ldots, u_p)$ of tangent vectors $u_a \in T_xM$, $1 \leqq a \leqq p$. If we now define

$$u + v = (u_1 + v_1, \ldots, u_p + v_p), \quad \lambda u = (\lambda u_1, \ldots, \lambda u_p),$$

where $u = (u_1, \ldots, u_p)$, $v = (v_1, \ldots, v_p) \in (T_p^1M)_x$, $\lambda \in R$, then it is easy to prove that $\pi\colon T_p^1M \to M$ is a vector bundle over M, isomorphic, as vector bundles, with the Withney sum of TM with itself p times.

Now, if $u \in T_xM$, $x \in M$, we may define a vertical tangent vector $u^{(a)}$ to T_p^1M at a point $y = (y_1, \dots, y_p) \in (T_p^1M)_x$, for each a, $1 \leqq a \leqq p$, by setting

$$u^{(a)} = \text{the tangent vector at } t=0 \text{ to the curve } t \to (y_1, \dots, y_a + tu, \dots, y_p)).$$

Locally, if $u = u^i(\partial/\partial x^i)$ then we have $u^{(a)} = u^i(\partial/\partial y_a^i)$.

Next, we may define p tensor fields $J_1, \dots, J_p$ of type (1.1) on T_p^1M as follows:

$$(J_a)_y X = (T\pi(y)X)^{(a)}, \quad 1 \leqq a \leqq p.$$

We locally have

$$J_a = (\partial/\partial y_a^i) \oplus (dx^i), \quad 1 \leqq a \leqq p. \tag{2.1}$$

From (2.1) we deduce the following properties:

$$J_a J_b = J_b J_a = 0, \tag{2.2}$$

$$\operatorname{rank}(J_a) = n, \tag{2.3}$$

$$\operatorname{Im} J_a \cap (\underset{b \neq a}{+} \operatorname{Im} J_b) = 0 \quad \text{for all} \quad a. \tag{2.4}$$

Moreover, if we put $V_a = \operatorname{Im} J_a$, it is easy to prove that the (a)-vertical lift mapping

$$u \in T_xM \to u^{(a)} \in V_y, \quad y \in (T_p^1M)_x \quad \text{for each} \quad x \in M,$$

is a linear isomorphism.

3. p-almost tangent structures

Bearing in mind the geometric structure of the tangent bundle of p^1-velocities T_p^1M of an n-dimensional manifold M, we have introduced in [10] the following definition.

Definition 3.1. Let N be a $(p+1)n$-dimensional manifold endowed with p tensor fields $(J_1 \dots, J_p)$ of type (1.1) satisfying (2.2), (2.3) and (2.4). Then $(J_1, \dots, J_p)$ is said to be a p-almost tangent structure on N and $(N, (J1, \dots, J_p))$ is said to be a p-almost tangent manifold.

Remark. When $p = 1$, then a 1-almost tangent structure is an almost tangent structure.

If we put $V_a = \operatorname{Im} J_a$, $1 \leqq a \leqq p$, then V_a is an n-dimensional distribution on N. Therefore,

$$V = \bigoplus_{a=1}^{p} V_a$$

is a pn-dimensional distribution on N. In [10] we have interpreted a p-almost tangent structure as a type of G-structure. We briefly recall this definition and its relation to the tensorial one.

Let x be a point of N. Then V_x is a pn-dimensional subspace of T_xN. Choose a complement H_x in T_xN to V_x and let $\{e^i\}$ be a basis of H_x. Then $\{e_i, e_1^i = (J_1)_x e^i, \dots, e_p^i = (J_p)_x e^i\}$ is a frame at x (called an *adapted* frame). If $\{\bar{e}^i, \bar{e}_1^i, \dots, \bar{e}_p^i\}$ is another such frame, where $\{\bar{e}^i\}$ is a basis for a different complement to V_x, then there are $n \times n$ matrices $A, A_1, \dots, A_p$, with $A \in Gl(n, R)$, such that

$$\bar{e}^i = A_j^i e^j + (A_1)_j^i e_1^j + \dots + (A_p)_j^i e_p^j,$$

and hence

$$\bar{e}_a^i = A_j^i e_a^j, \qquad 1 \leqq a \leqq p.$$

The two frames are therefore related by the $(p+1)n \times (p+1)n$ matrix

$$\begin{pmatrix} A & 0 & \dots & 0 \\ A_1 & A & \dots & 0 \\ \dots & \dots & \dots & \dots \\ A_p & 0 & \dots & A \end{pmatrix}.$$

The set of such matrices is a Lie subgroup G of $Gl((p+1)n, R)$ and the set of adapted frames at all points of N defines a G-structure on N.

Conversely, let $B_G(N)$ be a G-structure on N. Since the group G may be described as the invariance group of the matrices

$$(J_1)_0 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ I & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \dots, (J_p)_0 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ I & 0 & \dots & 0 \end{pmatrix},$$

where I is the $n \times n$ identity matrix, the tensor field J_a, $1 \leqq a \leqq p$, may be defined as the tensor field of type (1.1) on N which has the matrix representation $(J_a)_0$ at any point.

The fundamental problem of the theory of G-structures is to decide whether a given G-structure is equivalent to the standard flat G-structure on $R^{(p+1)n}$. In [10] we have proved the following theorem.

THEOREM. *A p-almost tangent structure $(J_1, \dots, J_p)$ on N is intregrable if and only if $\{J_a, J_b\} = 0$, $1 \leqq a, b \leqq p$, where $\{J_a, J_b\}$ is a tensor field of type* (1.2) *on n given by*

$$\{J_a, J_b\}(X, Y) = [J_aX, J_bY] = J_a[X, J_bY] - J_b[J_aX, Y].$$

To end this section, we establish (for an integrable p-almost tangent structure on N) the existence of a symmetric linear connection $\overline{V}$ on N

with respect to which the covariant derivatives $\overline{V}J_a$ are zero, for any a, $1 \leqq a \leqq p$. In fact, this follows from the general theory of G-structures, since if $(J_1, \ldots, J_p)$ is integrable, then the first structure tensor of the G-structure vanishes (see [7]).

4. Integrable p-almost tangent structures which define fibrations

Let $(J_1, \ldots, J_p)$ be an integrable p-almost tangent structure on a $(p+1)n$-dimensional manifold N. Then the distributions $V, V_1, \ldots, V_p$ are involutive. Therefore $V, V_1, \ldots, V_p$ define $p+1$ foliations such that each leaf of V is foliated by the leaves of V_a, $1 \leqq a \leqq p$; in fact, each leaf of V is locally a product of p leaves of the foliations defined by $V_1, \ldots, V_p$. Now, we define an equivalence relation on N as follows: two points of N are equivalent if they lie on the same leaf of the foliation defined by V. We say that $(J_1, \ldots, J_p)$ define a fibration if the quotient of N by this equivalence relation (that is, the space of leaves) has the structure of a differentiable manifold. This will be the case if for every leaf one can find an embedded local submanifold of N of dimension n through a point of the leaf which intersects each leaf which it does in only one point. In this case, the space of leaves M is a pn-dimensional manifold and the canonical projection $\pi \colon N \to M$ is a surjective submersion (that is, M is a quotient manifold of N). Then $\pi \colon N \to M$ is a fibred manifold and

$$V_y = T_y(\pi^{-1}(x)), \quad y \in N, \quad x = \pi(y),$$

for each point $y \in N$.

Example. The canonical p-almost tangent structure on the tangent bundle T^1_pM of p^1-velocities of any manifold M is integrable and defines a fibration.

Bearing in mind the example above, we may define the (a)-vertical lift of tangent vectors on M to N, $1 \leqq a \leqq p$, when N is an integrable p-almost tangent manifold which defines a fibration.

If $u \in T_xM$ and $y \in \pi^{-1}(x)$ we define $u^{(a)} \in T_yN$ by $u^{(a)} = (J_a)_y(\overline{u})$, where $\overline{u} \in T_yN$ and $T\pi(u) = \overline{u}$. Since $\operatorname{Ker}\{T\pi \colon T_yN \to T_xM\} = V_y$ and $(J_a)_yV_y = 0$, then $u^{(a)}$ is well-defined. Moreover, $u^{(a)} \in (V_a)_y$, and the map $u \to u^{(a)}$ is a linear isomorphism of T_xM with $(V_a)_y$. If X is a vector field on M, we may define its (a)-vertical lift on N given by $X^{(a)} = J_a\overline{X}$, where $\overline{X}$ is any vector field on N which is π-related to X. Clearly, $X^{(a)} \in V_a$, $1 \leqq a \leqq p$.

Proposition 4.1. *Let X, Y be two vector fields on M. Then we have:*

$$[X^{(a)}, Y^{(b)}] = 0, \tag{1}$$

$$L_{X^{(a)}}J_b = 0 \tag{2}$$

for every a, b, $1 \leqq a,\ b \leqq p$.

PROOF. (1) Let $\overline{X}, \overline{Y}$ be vector fields on N π-related to X, Y. Then

$$\left[X^{(a)}, Y^{(b)}\right] = [J_a\overline{X}, J_b\overline{Y}] = J_a\left[\overline{X}, J_b\overline{Y}\right] + J_b\left[J_a\overline{X}, \overline{Y}\right] =$$

$$= J_a\left[\overline{X}, Y^{(b)}\right] + J + b\left[X^{(a)}, \overline{Y}\right].$$

But $\overline{X}$ is π-related to X and $Y^{(b)}$ is π-related to 0; thus $T\pi[\overline{X}, Y^{(b)}] = [T\pi\overline{X}, T\pi Y^{(b)}] = 0$ and similarly $T\pi[X^{(a)}, \overline{Y}] = 0$. Then $[\overline{X}, Y^{(b)}], [X^{(a)}, \overline{Y}] \in V$. So $[X^{(a)}, Y^{(b)}] = 0$.

(2) For any vector field Z on N we have

$$(L_{X^{(a)}}J_b)\,Z = \left[X^{(a)}, J_bZ\right] - J_b\left[X^{(a)}, Z\right].$$

Now, supppose that $Z = Y^{(c)}$ for some vector field Y on M. Then both terms on the right-hand side vanish by part (1). Moreover, if Z is π-related to a vector field Y on M, that is, $Z = \overline{Y}$, then we have

$$(L_{X^{(a)}}J_b)\overline{Y} = \left[X^{(a)}, J_b\overline{Y}\right] - J_b\left[X^{(a)}, \overline{Y}\right] =$$

$$= \left[X^{(a)}, Y^{(b)}\right] - J_b\left[X^{(a)}, \overline{Y}\right] = -J_b\left[X^{(a)}, \overline{Y}\right].$$

But as was proved above, $[X^{(a)}, \overline{Y}] \in V$. This ends the proof. □

Now, let ∇ be a symmetric linear connection on N such that $\nabla J_a = 0$, $1 \leqq a \leqq p$. Then we have

PROPOSITION 4.2. ∇ *induces by restriction a connection on each leaf of* $V, V_1, \ldots, V_p$ *which is flat.*

PROOF. In fact, for any vector fields X, Y on M we have

$$\nabla_{X^{(a)}}Y^{(b)} = \nabla_{X^{(a)}}\left(J_b\overline{Y}\right) = J_b\left(\nabla_{X^{(a)}}\overline{Y}\right) = J_b\left(\nabla_{\overline{Y}}X^{(a)} + \left[X^{(a)}, \overline{Y}\right]\right) =$$

$$= J_b\left(\nabla_{\overline{Y}}X^{(a)}\right) = \nabla_{\overline{Y}}\left(J_bX^{(a)}\right) = 0,$$

$\overline{Y}$ is any vector field on $N\pi$-related to Y. This establishes the result. □

Before proceeding further, let us recall some well-known definitions and properties of affine bundles (a beautiful and brief exposition about this subject can be found in [4]).

DEFINITION 4.1. An affine bundle consists of a fibred manifold $\pi\colon A \to M$ and a vector bundle $\tau\colon E \to M$, together with a morphism $\varrho\colon Ax_ME \to A$ of fibred manifolds over id_M, such that for each $x \in M$,

$$\varrho_x\colon \pi^{-1}(x) \times \tau^{-1}(x) \to \pi^{-1}(x)$$

is a free transitive action of the vector space $\tau^{-1}(x)$ on $\pi^{-1}(x)$. So, each fibre $\pi^{-1}(x)$ of the affine bundle $\pi\colon A \to M$ is an affine space modelled on the vector space $\tau^{-1}(x)$. We say that the affine bundle $\pi\colon A \to M$ is modelled on the vector bundle $\tau\colon E \to M$.

We have the following result (see [4]).

PROPOSITION 4.3. *Let $\pi : A \to M$ be an affine bundle modelled via a morphism ϱ of fibred manifolds, on the vector bundle $\tau : E \to M$. Then $\pi : A \to M$ is a fibre bundle with standard fibre; the standard fibre F of E regarded as an affine space, and with structure group the group of affine automorphisms of V.*

REMARK. Let $\tau : E \to M$ be a vector bundle. Then one may form an affine bundle with the same total space E by taking $\varrho : Ex_M E \to M$ to be the additive action of $\tau^{-1}(x)$ on itself, for each $x \in M$. This affine bundle will be denoted by AE.

Next, we prove our main theorem.

THEOREM. *Let $(N,(J_1,\ldots,J_p))$ be an integrable p-almost tangent structure which defines a fibration $\pi : N \to M$. Let ∇ be any symmetric linear connection on N such that $\nabla J_a = 0$, $1 \leqq a \leqq p$, and suppose that with respect to the flat connection induced on it by ∇, each leaf of the foliations defined by $V, V_1, \ldots, V_p$ is geodesically complete. Suppose further that each leaf of the foliation defined by V (that is, the fibres of $\pi : N \to M$) is simply connected. Then N is an affine bundle modelled on $T^1_p M$.*

PROOF. We shall define a morphism

$$\varrho : N_{x_M} T^1_p M \to N$$

of fibred manifolds over id_M such that for each $x \in M$.

$$\varrho_x : \pi^{-1}(x) \times (T^1_p M)_x \to \pi^{-1}(x)$$

is a free, transitive action of the vector space

$$(T^1_p M)_x = \bigoplus_{p \text{ times}} (T_x M)$$

on $\pi^1(x)$. To do this, we proceed as follows. For any

$$u = (u_1, \ldots, u_p), \quad u_a \in T_x M, \quad 1 \leqq a \leqq p,$$

we may define p vertical vector fields U_a, $1 \leqq a \leqq p$, on $\pi^{-1}(x)$ given by

$$U_a(y) = \left((u_a)^{(a)}\right)_y,$$

for every $y \in \pi^{-1}(x)$. Then $\nabla_{U_a} U_b = 0$, $1 \leqq a, b \leqq p$. Particularly, $\nabla_{U_a} U_a = 0$, and therefore U_a is a geodesic field for every a, $1 \leqq a \leqq p$. Consequently, U_a is a complete vector field on $\pi^{-1}(x)$, that is, it generates a one-parameter group

$$\phi_{U_a} : R \times \pi^{-1}(x) \to \pi^{-1}(x).$$

Let $t \to \phi_{U_a}(t, y)$ be the integral curve of U_a such that $\phi_{U_a}(0, y) = y$. We define ϱ by

$$\varrho_x(y, u) = \phi_{U_p}(1, \ldots, \phi_{U_3}(1, \phi_{U_2}(1, \phi_{U_1}(1, y))), \ldots,),$$

where $u = (u_1, \ldots, u_p)$. Now, we shall prove that ϱ_x defines an action which is transitive and free. First, for any $u = (u_1, \ldots, u_p)$, $v = (v_1, \ldots, v_p) \in$ $\in (T^1_p M)_x$, the corresponding vector fields U_a, V_b on $\pi^{-1}(x)$ satisfy $[U_a, V_b] =$ $= 0$ (by Proposition 4.1). Thus their one-parameter groups commmute:

$$(4.1) \qquad (\phi_{U_a})(s, \phi_{V_b}(t, y)) = (\phi_{V_b}(t, \phi_{U_a}(s, y))).$$

Furthermore, we know that if two complete vector fields commute then the composition of their one-parameter groups is a one-parameter group whose generator is their sum. So, we have

$$(4.2) \qquad (\phi_{U_a})(t, \phi_{V_b}(t, y)) = (\phi_{V_b})(t, \phi_{U_a}(t, y)) = \phi_{U_a + V_b}(t, y).$$

Since $u + v = (u_1 + v_1, \ldots, u_p + v_p) \in (T^1_p M)_x$, a simple computation using (4.2) shows that

$$\varrho_x(\varrho_x(y, u), v) = \varrho_x(\varrho_x(y, v), u) = \varrho_x(y, u + v).$$

Then ϱ_x define an action of $(T^1_p M)_x$ on $\pi^{-1}(x)$. Next, we shall prove that this action is transitive. Let (,) be any scalar product on $T_x M$. We define a Riemannian metric on each leaf of the foliation V_a, $1 \leqq a \leqq p$, as follows:

$$(4.3) \qquad g_a(U_a, V_a) = (u_a, v_a).$$

(Let us remark that the vector fields $U_a, V_a, \ldots$ span the distribution V_a and are tangent to each leaf of the foliation defined by V_a.) From Proposition 4.2 and (4.3) we deduce that the vector fields U_a, V_a are covariant constant and have constant inner product. Then ∇ is the Riemannian connection for g_a. Therefore, each leaf of the foliation defined by V_a is a geodesically complete Riemannian manifold. Now, since each leaf of the foliation defined by V is a local product of p leaves of the foliations defined by $V_1, \ldots, V_p$ we deduce, by the Hopf–Rinow theorem, that any two points of $\pi^{-1}(x)$ may be joined by a piecewise differentiable curve γ with a finite number of geodesic arcs $\{\gamma_1, \ldots, \gamma_q\}$ in such a way that γ_r, $1 \leqq r \leqq q$, is a geodesic arc on a leaf of the foliation defined by the distributions V_a, for some a. We may suppose that $\gamma(0) = y$ and $\gamma(1) = z$. Moreover, from (4.1) and (4.2) one can find an element $u = (u_1, \ldots, u_p) \in T_p^1 M_x$ such that

$$\gamma(0) = (u_1)^{(1)}, \quad \text{and} \quad z = \phi_{U_p}(1, \ldots, \phi_{U_2}(1, \phi_{U_1}(1, y)), \ldots).$$

Consequently, we have

$$z = \varrho_x(y, (u_1, \ldots, u_p)).$$

Finally, we prove that the action ϱ_x is free. Let $\Gamma(y)$ be the isotropy group of $y \in \pi^{-1}(x)$ under the action of $(T_p^1 M)_x$, that is

$$\Gamma(y) = \left\{ u = (u_1, \ldots, u_p) \in (T_p^1 M)_x / \varrho_x(y, u) = y \right\}.$$

From the definition of ϱ_x, one can easily prove that the following diagram

$$\begin{array}{ccc} T_x M \oplus \overset{p}{\cdots} \oplus T_x M & \xrightarrow{\quad \varrho_x \quad} & \pi^{-1}(x) \\ \psi \big\downarrow & \nearrow_{\exp_y} & \\ T_y(\pi^{-1}(x)) & & \end{array}$$

is commutative, where $\exp_y$ denotes the exponential map of ∇ restricted to $\pi^{-1}(x)$ and ψ is the linear isomorphism given by

$$\psi(u) = \psi(u_1, \ldots, u_p) = (u_1)^{(1)} + \cdots + (u_p)^{(p)}.$$

Since $\exp_y$ is a local diffeomorphism, then so is ϱ_x. Therefore

$$\Gamma(y) = (\varrho_x)^{-1}(y)$$

must be a discrete (additive) subgroup of $(T_p^1 M)_x$. Then the elements of $\Gamma(y)$ are integer linear combinations of some k linearly independent vectors $\alpha_1, \ldots, \alpha_k$, where $1 \leqq k \leqq pn$. So we have

$$(T_p^1 M) / \Gamma(y) \cong \left(R^k \times R^{pn-k} \right) / Z^k \cong T^k \times R^{pn-k},$$

where T^k is a k-torus. But, since $(T_p^1 M)_x$ acts transitively on $\pi^{-1}(x)$, then $\pi^{-1}(x)$ is diffeomorphic to the coset space $(T_p^1 M)_x / \Gamma(y)$. Thus, if $\Gamma(y)$ is non-trivial, then $\pi^{-1}(x)$ is diffeomorphic to $T^k \times R^{pn-k}$, which is not simply connected. Consequently, $\Gamma(y)$ must be trivial and then the action is free. □

COROLLARY 4.1. *If* $(N,(J_1,\ldots,J_p))$ *verifies all the hypotheses of the theorem and in addition* $\pi : N \to M$ *admits a global section (fot instance, if* M *is paracompact), then* N *is isomorphic (as a vector bundle) to* TM. *This isomorphism depends on the choice of section.* □

COROLLARY 4.2. *If* $(N,(J_1,\ldots,J_p))$ *verifies all the hypotheses of the theorem except the hypothesis that the leaves of the foliation defined by* V *are sympty connected and this leaves assumed to be mutually homeomorphic, then* T^1_pM *is a covering space of* N *and the leaves of* V *are of the form* $T^k\times R^{pn-k}$, *where* T^k *is a* k*-dimensional torus,* $0 \leqq k \leqq pn$. *Moreover, if it is assumed that the leaves of* V *are compact, then* T^1_pM *is a covering space of* N *and the fibres are diffeomorphic to* T^{pn}.

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UNIDAD DE MATEMÁTICAS
CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS
SERRANO, 123, 28006 MADRID, SPAIN

DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA
FACULTAD DE MATEMÁTICAS
UNIVERSIDAD DE SANTIAGO DE COMPOSTELA
SANTIAGO DE COMPOSTELA, SPAIN

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ON THE EQUICONVERGENCE OF THE RIESZ MEANS WITH EXACT ORDER

N. H. LOI (Hanoi)

Developing a fruitful method of V. A. Il'in [1], members of his school proved some equiconvergence theorems with exact order ([2], [3], [5] and [6]). In this paper we shall prove an equiconvergence of the Riesz means with exact order for functions with given integral modulus of continuity. The theorem of Š. A. Alimov and I. Joó [2] is only a special case of our result when $s = 0$.

Let $\omega(t)$ be a continuous function on $[0,\infty)$ satisfying the following conditions:

(i) $\omega(0) = 0, \omega(t) > 0$ if $t > 0$;

(ii) $\omega(2t) \leqq C\omega(t)$;

(iii) $\omega(t)$ is not decreasing;

(iv) $\omega(t)/t$ is not increasing.

Denote by $H_1^\omega[0,1] = H_1^\omega$ the set of those functions $f \in L_1[0,1]$ for which the integral modulus of continuity

$$\omega_1(f,\delta) := \sup_{|h|\leqq\delta} \int_0^{1-h} |f(x+h) - f(x)|\, dx$$

satisfies the condition $\omega_1(f,\delta) \leqq C\omega(\delta)$.

Define

$$\| f \|_\omega := \| f \|_{L_1[0,1]} + \sup_{\delta>0} \frac{\omega_1(f,\delta)}{\omega(\delta)}.$$

We consider the Schrödinger operators

$$Lu := -u'' + q(x)\,u(x), \quad \hat{L}u := -u'' + \hat{q}(x)\,u(x).$$

where $q(x)$, $\hat{q}(x) \in L_p[0,1]$ $(p > 1)$ are arbitrary real functions. Let $\{u_k\}_{k=1}^\infty$ and $\{\hat{u}_k\}_{k=1}^\infty$ be complete orthonormal systems of eigenfunctions of the corresponding operators in $L_2[0,1]$; further denote $\{\lambda_k\}_{k=1}^\infty$ and $\{\hat{\lambda}_k\}_{k=1}^\infty$ the positive eigenvalues $(0 \leqq \lambda_1 \leqq \lambda_2 \leqq \ldots, 0 \leqq \hat{\lambda}_1 \leqq \hat{\lambda}_2 \leqq \ldots)$. For brevity we use the notation $\mu_k := \sqrt{\lambda_k}$.

For any $f \in L_1[0,1]$, $\mu > 0$, $s \in [0, \frac{1}{2})$ consider the partial sums of the s^{th} Riesz means of the spectral expansion of f:

$$\sigma_\mu^s(f,x) := \sum_{\mu_k < \mu} (f, u_k)\, u_k(x) \left(1 - \frac{\mu_k^2}{\mu^2}\right)^s,$$

$$\hat{\sigma}_\mu^s(f,x) := \sum_{\hat{\mu}_k < \mu} (f, \hat{u}_k)\, \hat{u}_k(x) \left(1 - \frac{\hat{\mu}_k^2}{\mu^2}\right)^s.$$

The aim of the present paper is to prove the following

THEOREM. *Given any compact subset $K \subset (0,1)$, for any $f \in H_1^\omega[0,1]$, $x \in K$, $\mu \geq 1$, $s \in [0, \frac{1}{2})$ we have*

$$\sigma_\mu^s(f,x) - \hat{\sigma}_\mu^s(f,x) = O\left(\omega\left(\frac{1}{\mu}\right)\right) \cdot \mu^{-s}. \tag{1}$$

The order of (1) *cannot be improved in the sense that $o\left(\omega\left(\frac{1}{\mu}\right)\right)$ cannot be written on the right hand side of* (1).

We recall some well-known results which are necessary for our proof:

$$|u_k(x)| \leqq C \quad (0 \leqq x \leqq 1, \quad k = 1,2,\dots) \tag{2}$$

(cf. [3]);

$$|(f, u_k)| \leqq C(q) \parallel f \parallel_\omega \omega\left(\frac{1}{\mu_k}\right) \quad (f \in H_1^\omega[0,1], \quad k = 1,2,\dots) \tag{3}$$

and

$$\sum_{k=1}^{\infty} \frac{\omega\left(\frac{1}{\mu_k}\right)}{1 + (\mu - \mu_k)^2} \leqq C\omega\left(\frac{1}{\mu}\right) \quad (\mu \geq 1) \tag{4}$$

(cf. [2]);

$$\left| \int_0^R g_0(u_k, \mu_k, x, t) t^{-s-\frac{1}{2}} J_{s+\frac{1}{2}}(\mu t)\, dt \right| \leqq C \min\left\{ \frac{1}{\mu^{\frac{3}{2}}}, \frac{\mu^{\frac{1}{2}}}{\mu_k^2} \right\}. \tag{5}$$

where

$$g_0(u_k,\mu_k,x,t) := \int\limits_{x-t}^{x+t} \frac{\sin\mu_k(t-|x-\xi|)}{\mu_k}\, q(\xi)u_k(\xi)\,d\xi$$

and

$$\left|\mathcal{D}_{R_0} K^{\mu}_{\mu_k}(R)\right| \leqq \frac{C(K,s)}{1+(\mu-\mu_k)^2}, \tag{6}$$

where

$$K^{\mu}_{\mu_k}(R) := \mu^{\frac{1}{2}} \int\limits_{R}^{\infty} t^{-s-\frac{1}{2}} J_{s+\frac{1}{2}}(\mu t)\cos\mu_k t\,dt;$$

$K \subset (0,1)$ is an arbitrary compact subset, $0 < R_0 < \frac{1}{2}\operatorname{dist}(K, \partial(0,1))$,

$$\mathcal{D}_{R_0} g := \frac{2}{R_0} \int\limits_{\frac{R_0}{2}}^{R_0} g(R)\,dR$$

(cf. [8]).

The proof of our theorem is based on some lemmas. Introduce the notations

$$\alpha_\mu(f,x) := \sum_{k=1}^{\infty} f_k u_k(x) \mathcal{D}_{R_0} K^{\mu}_{\mu_k}(R) \tag{7}$$

and

$$\beta_\mu(f,x) := \sum_{k=1}^{\infty} \mathcal{D}_{R_0}\left[\int\limits_0^R g_0(u_k,\mu_k,x,t) t^{-s-\frac{1}{2}} J_{s+\frac{1}{2}}(\mu t)\,dt\right] f_k, \tag{8}$$

where

$$f_k := (f, u_k). \tag{9}$$

Lemma 1. *For any* $f \in H_1^{\omega}[0,1]$ *and* $x \in K$,

$$|\alpha_\mu(f,x)| \leqq C(K,q,s)\, \| f \|_\omega\, \omega\left(\frac{1}{\mu}\right) \quad (\mu \geq 1). \tag{10}$$

Proof. Applying (2), (3), (6) and (9) we have

$$|\alpha_\mu(f,x)| \leqq C(K,q,s)\, \| f \|_\omega \sum_{k=1}^{\infty} \omega\left(\frac{1}{\mu_k}\right) \frac{1}{1+(\mu-\mu_k)^2}.$$

Hence, using (4), the estimate (10) follows at once. Lemma 1 is proved. □

LEMMA 2. *For any $f \in H_1^\omega[0,1]$ and $x \in K$,*

$$|\beta_\mu(f,x)| \leqq C \parallel f \parallel_\omega \omega\left(\frac{1}{\mu}\right) \quad (\mu \geq 1). \tag{11}$$

PROOF. Using (5) and (3) we get

$$|\beta_\mu(f,x)| \leqq C \parallel f \parallel_\omega \sum_{k=1}^{\infty} \omega\left(\frac{1}{\mu_k}\right) \min\left\{\frac{1}{\mu^{\frac{3}{2}}}, \frac{\mu^{\frac{1}{2}}}{\mu_k^2}\right\}.$$

Hence (11) follows by the method used in the proof of Lemma 2 in [2]. Lemma 2 is proved. □

Now we return to the proof of our theorem.

Given any compact $K \subset (0,1)$ denote R an arbitrary number from the interval $(0, \operatorname{dist}(K, \partial(0,1)))$. Now fix $x \in K$ arbitrarily and define the function $W_R^s : (0,1) \to \mathbf{R}$ by

$$W_R^s(x+t) := \begin{cases} a(s)\mu^{\frac{1}{2}-s}|t|^{-s-\frac{1}{2}} J_{s+\frac{1}{2}}(\mu|t|) & \text{if} \quad |t| \leqq R, \\ 0 & \text{otherwise}, \end{cases} \tag{12}$$

where

$$a(s) := 2^s(2\pi)^{-\frac{1}{2}}\Gamma(s+1).$$

PROOF OF THE THEOREM. We consider the Fourier coefficients of the function $W_R^s(x+t)$ with respect to the system $\{u_k\}$. An easy calculation shows

$$W_k^s = (u_k, W_R^s) = \int_{x-R}^{x+R} W_R^s(|x-y|)\, u_k(y)\, dy =$$

$$= \int_0^R W_R^s(t)\, [u_k(x-t) + u_k(x+t)]\, dt.$$

Applying the Titchmarsh formula [9], we obtain

$$u_k(x+t) + u_k(x-t) = 2u_k(x)\cos\mu_k t + \int_{x-t}^{x+t} q(\xi)u_k(\xi)\, \frac{\sin\mu_k(t-|x-\xi|)}{\mu_k}\, d\xi$$

and using the integral transformation $\int_0^R = \int_0^\infty - \int_R^\infty$ we get

$$W_k^s = 2u_k(x)\int_0^R W_R^s(t)\cos\mu_k t\, dt + \int_0^R W_R^s(t)\, g_0(u_k, \mu_k, x, t)\, dt = \tag{13}$$

$$= u_k(x)2a(s)\mu^{-s}\left\{\mu^{\frac{1}{2}}\int_0^\infty t^{-s-\frac{1}{2}}J_{s+\frac{1}{2}}(\mu t)\cos\mu_k t\,dt - K^{\mu}_{\mu_k}(R)\right\}+$$

$$+a(s)\mu^{-s}\left\{\mu^{\frac{1}{2}}\int_0^R t^{-s-\frac{1}{2}}J_{s+\frac{1}{2}}(\mu t)\,g_0(u_k,\mu_k,x,t)\,dt\right\}.$$

It is well-known (cf. [10, p. 107, (34)]) that

$$\text{(14)}\qquad 2a(s)\mu^{\frac{1}{2}-s}\int_0^\infty t^{-s-\frac{1}{2}}J_{s+\frac{1}{2}}(\mu t)\cos\mu_k t\,dt = \delta^{\mu}_{\mu_k}\left(1-\frac{\mu_k^2}{\mu^2}\right)^s,$$

where

$$\delta^{\mu}_{\mu_k} := \begin{cases} 1 & \text{if } \mu_k < \mu, \\ 0 & \text{if } \mu_k > \mu. \end{cases}$$

Substituting (14) into (13) we obtain

$$W_k^s = \delta^{\mu}_{\mu_k}u_k(x)\left(1-\frac{\mu_k^2}{\mu^2}\right)^s - 2a(s)\mu^{-s}u_k(x)K^{\mu}_{\mu_k}(R)+$$

$$+a(s)\mu^{-s}\left[\mu^{\frac{1}{2}}\int_0^R t^{-s-\frac{1}{2}}J_{s+\frac{1}{2}}(\mu t)\,g_0(u_k,\mu_k,x,t)\,dt\right].$$

Since for any fixed $x \in K$ and $\mu > 0$, $W_R^s(|x-y|)$ as function of y belongs to $L_2[0,1]$, we have the following equality in $L_2[0,1]$-convergence in y:

$$W_R^s(|x-y|) - \sum_{\mu_k<\mu} u_k(k)u_k(y)\left(1-\frac{\mu_k^2}{\mu^2}\right)^s =$$

$$= -2a(s)\mu^{-s}\sum_{k=1}^{\infty} u_k(x)\,u_k(y)K^{\mu}_{\mu_k}(R)+$$

$$+a(s)\mu^{-s}\sum_{k=1}^{\infty}\left[\mu^{\frac{1}{2}}\int_0^R t^{-s-\frac{1}{2}}J_{s+\frac{1}{2}}(\mu t)g_0(u_k,\mu_k,x,t)\,dt\right]u_k(y).$$

Apply the operation $\mathcal{D}_{R_0}$ term by term on both sides of the last equality to get

$$\text{(15)}\qquad \mathcal{D}_{R_0}W_R^s(|x-y|) - \sum_{\mu_k<\mu} u_k(x)\,u_k(y)\left(1-\frac{\mu_k^2}{\mu^2}\right)^s =$$

$$= -2a(s)\mu^{-s} \sum_{k=1}^{\infty} u_k(x)\, u_k(y) \mathcal{D}_{R_0} K^{\mu}_{\mu_k}(R) +$$

$$+2a(s)\mu^{-s} \sum_{k=1}^{\infty} \mathcal{D}_{R_0} \left[\mu^{\frac{1}{2}} \int_0^R t^{-s-\frac{1}{2}} J_{s+\frac{1}{2}}(\mu t)\, g_0(u_k, \mu_k, x, t)\, dt \right] u_k(y).$$

It is easy to prove that after multiplication of both sides of (15) by any $f \in H_1^{\omega}[0,1]$, one can integrate the resulting equality term by term over $[0,1]$ in y. Introducing the notations

$$S_1 := -2a(s)\mu^{-s} \sum_{k=1}^{\infty} u_k(x) f_k \mathcal{D}_{R_0} K^{\mu}_{\mu_k}(R),$$

$$S_2 := a(s)\mu^{-s} \sum_{k=1}^{\infty} \mathcal{D}_{R_0} \left[\mu^{\frac{1}{2}} \int_0^R t^{-s-\frac{1}{2}} J_{s+\frac{1}{2}}(\mu t)\, g_0(u_k, \mu_k, x, t)\, dt \right] f_k$$

and taking into consideration (2), (10) and (11) we have the following estimates:

$$|S_i| \leqq C(K,q,s) \parallel f \parallel_{\omega} \omega\left(\frac{1}{\mu}\right) \cdot \mu^{-s} \quad (\mu \geq 1, \quad i = 1,2).$$

Therefore

$$\left| \int_0^1 \mathcal{D}_{R_0} W_R^s(|x-y|)\, f(y)\, dy - \sigma_{\mu}^s(f,x) \right| \leqq C(K,q,s) \parallel f \parallel_{\omega} \omega\left(\frac{1}{\mu}\right) \cdot \mu^{-s},$$

similarly

$$\left| \int_0^1 \mathcal{D}_{R_0} W_R^s(|x-y|)\, f(y)\, dy - \hat{\sigma}_{\mu}^s(f,x) \right| \leqq C(K,q,s) \parallel f \parallel_{\omega} \omega\left(\frac{1}{\mu}\right) \cdot \mu^{-s}.$$

After this preparation we obtain (1) by the triangle inequality.

Now we have to prove that the estimate (1) is not refinable in the sense that $o\left(\omega\left(\frac{1}{\mu}\right)\right)$ can not be written on the right hand side of (1). This was proved in [2] for the case $s = 0$.

Theorem is proved. □

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DEPARTMENT OF MATH. ANALYSIS,
HANOI PEDAGOGIC UNIVERSITY II
HANOI

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INTERPOLATION BETWEEN DYADIC HARDY SPACES $\mathbf{H}^p$: THE COMPLEX METHOD

V. ECHANDIA (Caracas)

Preliminaries

1.1. *Dyadic Hardy spaces* $\mathbf{H}^p$. In what follows L^p will denote $L^p[0,1]$, $0 < p < \infty$. In this section, following [2], we shall introduce the dyadic Hardy spaces.

DEFINITION 1. For each $f \in L^1$, let

$$E_n f = S_{2^n} f \quad (n \in \mathbf{N}),$$

where $S_{2^n} f$ is the 2^nth partial sum of the Walsh–Fourier series of f. The dyadic maximal function $E^* f$ is defined by

$$E^* f = \sup_{n \in \mathbf{N}} |E_n f| \quad (f \in L^1).$$

DEFINITION 2. For $f \in L^1$, set

$$\tilde{E}_0 f = |E_0 f|,$$

and for $n \in \mathbf{P}$, also define

$$\tilde{E}_n f = \sup_{0 < m \leqq n} (|E_n(f)| + |E_m(f \cdot r_m)|)$$

where r_m is the m-th Rademacher function.

For each $f \in L^1$ set

$$\tilde{E} f = \sup_{n \in \mathbf{N}} \tilde{E}_n f.$$

It is easy to see that

$$E^* f \leqq \tilde{E} f \leqq 3 E^* f. \tag{1}$$

The following lemma is proved in [2].

LEMMA 1. *If $f \in L^1$ has zero mean and we define $f_n = E_n f$ $(n \in \mathbf{N})$, and for $k \in \mathbf{Z}$ we put*

$$f^{(k)} = \sum_{n=0}^{\infty} \chi\{2^k < \tilde{E}_n f \leqq 2^{k+1}\}(f_{n+1} - f_n),$$

then we have

$$E^*(f^{(k)}) \leqq 2^{k+2}\chi\{\tilde{E}f > 2^k\}$$

and

$$f = \sum_{k=-\infty}^{\infty} f^{(k)}$$

a.e. on $[0,1]$. *Moreover, if* $E^*f \in L^1$, *then this series converges to* f *in* L^1 *norm.* ($\chi\{\dots\}$ *denotes the characteristic function of the set* $\{\dots\}$.)

The above series will be called the canonical decomposition of f.

DEFINITION 3. The dyadic Hardy spaces are defined as

$$\mathbf{H}^p := \left\{ f \in L^1 : \|f\|_{\mathbf{H}^p} := \left(\int_0^1 (E^*f)^p \right)^{1/p} < \infty \right\} \quad (0 < p < \infty).$$

Notice that the set of dyadic step functions L is dense in $\mathbf{H}^p$ $(0 < p < \infty)$. For a detailed study of these spaces see [2].

1.2. *The complex method of interpolation.* In this section we define the interpolation spaces $\overline{A}_{[\theta]}$, in the same way as in [1].

Given a couple $\overline{A} = (A_0, A_1)$ of Banach spaces, we shall consider the space $\mathcal{F} = \mathcal{F}(\overline{A})$ of all functions f with values in $\sum(\overline{A})$, which are bounded and continuous in the strip $S = \{z \in \mathbf{C} : 0 \leqq \operatorname{Re} z \leqq 1\}$ and analytic in the open strip $S_0 = \{z \in \mathbf{C} : 0 \leqq \operatorname{Re} z < 1\}$. Moreover, the functions $t \to f(j+it)$ $(j = 0, 1$ $i = \sqrt{-1})$ are continuous from the real line into A_j and tend to zero as $|t| \to \infty$. $\mathcal{F}(\overline{A})$ is a vector space. We provide $\mathcal{F}$ with the norm

$$\|f\|_{\mathcal{F}} = \max(\sup \|f(it)\|_{A_0}, \sup \|f(1+it)\|_{A_1}).$$

(The supremum is taken over all real numbers t.)

We have the following result.

LEMMA 2. *The space* $\mathcal{F}$ *is a Banach space.*

For a proof see [1].

DEFINITION 4. Given a couple $\overline{A} = (A_0, A_1)$ of Banach spaces and $0 < \theta < 1$, the space $\overline{A}_{[\theta]}$ is defined as

$$\overline{A}_{[\theta]} = \left\{ a \in \sum(\overline{A}) : a = f(\theta), \quad \text{for some} \quad f \in \mathcal{F}(\overline{A}) \right\}.$$

The space $\overline{A}_{[\theta]}$ is a Banach space with the norm

$$\|a\|_{[\theta]} = \inf\{\|f\|_{\mathcal{F}} : f(\theta) = a, \quad f \in \mathcal{F}\}.$$

(See [1].)

For $\overline{A}_{[\theta]}$ we have the following

PROPOSITION 1. *The space* $\overline{A}_{[\theta]}$ *is an interpolation space with respect to* $\overline{A}$.

For a proof and for a detailed study see [1].

2. Characterization of intermediate spaces between $\mathbf{H}^{p_0}$ and $\mathbf{H}^{p_1}$ $(1 \leqq p_j < \infty,\ j = 0, 1)$

In this section we shall use the idea of [2], to prove the following result.

THEOREM. *Assume that* $p_0 \geqq 1$, $p_1 \geqq 1$ *and* $0 < \theta < 1$. *Then*

$$(\mathbf{H}^{p_0}, \mathbf{H}^{p_1})_{[\theta]} = \mathbf{H}^p \quad \textit{(equivalent norms)},$$

if

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

PROOF. It is sufficient to prove that there exist constants $c_1, c_2 > 0$ such that

$$c_1 \|a\|_{\mathbf{H}^p} \leqq \|a\|_{[\theta]} \leqq c_2 \|a\|_{\mathbf{H}^p},$$

for all functions $a \in L$.

For any $a \in L$ define

$$a_z = \sum_{k=-\infty}^{\infty} 2^{kp/p_z} a^{(k)} 2^{-k}, \quad z \in S$$

where $a = \sum_{k=-\infty}^{\infty} a^{(k)}$ is the canonical decomposition of a, and

$$\frac{1}{p_z} = \frac{1-z}{p_0} + \frac{z}{p_1}, \quad z \in S.$$

Clearly, $a_\theta = a$. Moreover there exists a $c_1 > 0$ such that, for $j = 0, 1$,

$$\|a_{j+iy}\|_{\mathbf{H}^{p_j}} \leqq c_1 \|a\|_{\mathbf{H}^p}^{p/p_j} \quad (y \in \mathbf{R}).$$

In fact, by Lemma 1 we have that

$$E^*(a_z) \leqq \sum_{k=-\infty}^{\infty} \left|2^{kp/p_z}\right| 2^{-k} E^*(a^{(k)}) \leqq 4 \sum_{k=-\infty}^{\infty} 2^{kp/p_j} \chi\left\{\tilde{E}a > 2^k\right\}$$

where $z = j + iy$. Applying Abel's transformation we obtain that

$$E^*(a_{j+iy}) \leqq 4 \sum_{k=-\infty}^{\infty} 2^{kp/p_j} \chi\left\{\tilde{E}a > 2^k\right\} =$$

$$= \frac{4}{2^{p/p_j} - 1} \sum_{k=-\infty}^{\infty} \left(2^{(k+1)p/p_j} - 2^{kp/p_j}\right) \chi\left\{\tilde{E}a > 2^k\right\} =$$

$$= \frac{4 \cdot 2^{p/p_j}}{2^{p/p_j} - 1} \sum_{k=-\infty}^{\infty} 2^{kp/p_j} \chi\left\{2^k < \tilde{E}a \leqq 2^{k+1}\right\}.$$

We can conclude from (1) that

$$\|a_{j+iy}\|_{\mathbf{H}^{p_j}} = \|E^*(a_{j+iy})\|_{p_j} \leqq \gamma_p \|\tilde{E}(a)\|_p^{p/p_j} \leqq 3\gamma_p \|E^*(a)\|_p^{p/p_j} = c_1 \|a\|_{\mathbf{H}^p}^{p/p_j}$$

where $\gamma_p > 0$ depends only on p.

Now let $\varepsilon > 0$ and define

$$f(z) = a_z \cdot \exp(\varepsilon z^2 - \varepsilon\theta^2), \quad \text{for all} \quad z \in S_0.$$

Assuming that $\|a\|_{\mathbf{H}_p} = 1$, we have that $f(\theta) = a$, $f \in \mathcal{F}$, and $\|f\|_{\mathcal{F}} \leqq c_1 e^{\varepsilon}$. We conclude that

$$\|a\|_{[\theta]} \leqq c_1 e^{\varepsilon}, \quad \text{for all} \quad \varepsilon > 0.$$

Hence $\|a\|_{[\theta]} \leqq c_1 \|a\|_{\mathbf{H}^p}$.

The converse inequality follows from the relation (see [3])

$$\|a\|_{\mathbf{H}^p} = \sup\left\{|\langle a, b\rangle| : \|b\|_{(\mathbf{H}^p)^*} = 1;\ b \in L\right\}$$

where

$$< a, b >= \begin{cases} \int_0^1 a \cdot b, & \text{if} \quad p > 1 \\ \lim_{m\to\infty} \int_0^1 E_m(a) E_m(b), & \text{if} \quad p = 1 \end{cases}$$

and $(\mathbf{H}^p)^*$ stands for the dual space of $\mathbf{H}^p$. In fact, given $b \in L$, $\varepsilon > 0$ and $1 > \delta > 0$ put

$$g(z) = b_z \cdot \exp(\varepsilon z^2 - \varepsilon\theta^2) \quad \text{for} \quad z \in S.$$

Pick an $h \in \mathcal{F}(\overline{A})$ such that $h(\theta) = a$ and

$$\|h\|_{\mathcal{F}} \leqq \|a\|_{[\theta]} + \delta.$$

If we define

$$F(z) = \langle h(z), g(z)\rangle \quad \text{for} \quad z \in S,$$

then applying Hölder's or Fefferman's inequality and supposing that $\|a\|_{[\theta]} = 1$ and $\|b\|_{\mathbf{H}^{p'}} = 1$, we get for $y \in \mathbf{R}$

$$\|F(iy)\| \leqq \|h(iy)\|_{\mathbf{H}^{p_0}} \|g(iy)\|_{\mathbf{H}^{p'_0}} \leqq c_0 e^{\varepsilon} \|h(iy)\|_{\mathbf{H}^{p_0}} \leqq$$

$$\leqq c_0 e^{\varepsilon} \|h\|_{\mathcal{F}} \leqq (\|a\|_{[\theta]} + \delta) c_0 e^{\varepsilon} \leqq 2 c_0 e^{\varepsilon},$$

where

$$\frac{1}{p_0} + \frac{1}{p'_0} = 1.$$

Similarly, we obtain that there exists a constant $c_2 > 0$, with

$$|F(1+iy)| \leqq c_2 e^{\varepsilon} \quad (y \in \mathbf{R}).$$

Since $h \in \mathcal{F}(\overline{A})$ and $g \in L$, the function F is holomorphic on S_0 and continuous on S. Consequently, the three lines theorem implies

$$| < a, b > | = |F(\theta)| \leqq \tilde{c}_2 (e^{\varepsilon})^{\theta} \cdot (e^{\varepsilon})^{(1-\theta)} \leqq \tilde{c}_2 e^{\varepsilon}.$$

Hence we obtain that

$$\|a\|_{\mathbf{H}^p} \leqq \tilde{c}_2 e^{\varepsilon}, \quad \text{for all} \quad \varepsilon > 0.$$

Therefore

$$\|a\|_{\mathbf{H}^p} \leqq \tilde{c}_2 \|a\|_{[\theta]}.$$

The proof is complete.

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UNIVERSIDAD CENTRAL DE VENEZUELA
FACULTAD DE CIENCIAS
DEPARTAMENTO DE MATEMATICAS
APTDO 40 645
CARACAS 1040–A, VENEZUELA

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WEIGHTED SIMULTANEOUS APPROXIMATION BY ALGEBRAIC PROJECTION OPERATORS

E. VAN WICKEREN (Aachen)

1. Introduction

The present note originates in work on simultaneous approximation by projection operators, given in [6] for trigonometric approximation. Here the corresponding algebraic problems are considered in the frame of appropriate weighted Sobolev spaces. In fact, the direct estimates are established is terms of the error of best weighted approximation which then turns out to be sharp.

Let $C = C[-1,1]$ be the space of functions, continuous on the compact interval $[-1,1]$, and let $[C]$ be the space of linear bounded operators of C into itself, endowed with the sup-norm $\|\cdot\|_C$ and operator norm $\|\cdot\|_{[C]}$, respectively. An operator $L_n \in [C]$ is called a polynomial projection operator on $\mathcal{P}_n$, the set of algebraic polynomials of degree at most $n \geqq 0$, if

$$(1.1) \qquad L_n f \in \mathcal{P}_n \quad (f \in C), \quad L_n p_n = p_n \quad (p_n \in \mathcal{P}_n).$$

In terms of the error of best approximation

$$(1.2) \qquad E_n(f) := \inf \{\|f - p_n\|_C : p_n \in \mathcal{P}_n\}$$

it is well-known that

$$\|L_n f - f\|_C \leqq K \|L_n\|_{[C]} E_n(f) \quad (f \in C).$$

It is the purpose of this paper to give an analogous result for the remainder of the simultaneous approximation of f by $L_n f$, i.e., for the r-th derivative $(f - L_n f)^{(r)}(x)$, as well as to discuss the sharpness of the relevant estimates. It turns out that one has to consider the weighted Sobolev space C^r_φ of functions $f \in C$, which are r-times differentiable on $(-1,1)$ such that $\varphi^r f^{(r)} \in C$, $\varphi(x) := \sqrt{1-x^2}$ ($C^0_\varphi := C[-1,1]$). In terms of the error of best weighted approximation

$$(1.3) \qquad E_n^{\varphi^r}(f) := \inf \{\|\varphi^r (f - p_n)\|_C : p_n \in \mathcal{P}_n\}$$

the main result is given by $(n \geqq r)$

$$(1.4) \qquad \|\varphi^r (f - L_n f)^{(r)}\|_C \leqq K \|L_n\|_{[C]}\, E_{n-r}^{\varphi^r}(f^{(r)}) \quad (f \in C^r_\varphi),$$

which is quite analogous to the estimates in the periodic case (see [6], also Theorems 2.6,7). This can then be extended to error bounds for $\|\varphi^i(f - L_n f)^{(i)}\|_C$ simultaneously for each $0 \leqq i \leqq r$ given in Corollary 2.5. Essential for the proofs are some facts on weighted approximation, recently published in [4], which allow to strengthen the result on simultaneous approximation by the polynomial of best approximation, given in [5] (see Lemma 2.2).

Concerning the sharpness of (1.4) only those L_n are considered which are optimal in the sense of the theorem of Harsiladze–Lozinski, i.e., the norm $\|L_n\|_{[C]}$ behaves like $\log n$. The proof is based on a quantitative extension of the uniform boundedness principle (see [3] and the literature cited there) and is reduced to arguments concerned with trigonometric approximation.

2. Direct estimates

Let us first recall some facts on algebraic best approximation in the weighted Sobolev space C^r_{φ}, given in [4, Theorems 2.1.1, 7.2.1, 7.3.1].

LEMMA 2.1. *Let $f \in C^r_\varphi$ and let $p_n \in \mathcal{P}_n$ denote its polynomial of best approximation. Then*

$$\|f - p_n\|_C \leqq K n^{-r} \|\varphi^r f^{(r)}\|_C, \tag{2.1}$$

$$\|\varphi^r p_n^{(r)}\|_C \leqq K \|\varphi^r f^{(r)}\|_C. \tag{2.2}$$

The following lemma is the key to derive (1.4) and improves the result given in [5].

LEMMA 2.2. *Let $f \in C^r_\varphi$, and let $p_n \in \mathcal{P}_n$ denote its polynomial of best approximation (with regard to (1.2)). Then for $n \geqq r$*

$$\|f - p_n\|_C \leqq K n^{-r} E^{\varphi^r}_{n-r}(f^{(r)}), \tag{2.3}$$

$$\|\varphi^r (f - p_n)^{(r)}\|_C \leqq K E^{\varphi^r}_{n-r}(f^{(r)}). \tag{2.4}$$

PROOF. Since $\mathcal{P}_n$ is finite dimensional and $\varphi^r f^{(r)} \in C$, there exists $Q_{n-r} \in \mathcal{P}_{n-r}$ with (cf. (1.3))

$$\|\varphi^r (f^{(r)} - Q_{n-r})\|_C = E^{\varphi^r}_{n-r}(f^{(r)}).$$

Then

$$q_n(x) := \int_0^x \cdots \int_0^{u_{r-1}} Q_{n-r}(u_r)\,du_r \dots du_1$$

belongs to $\mathcal{P}_n$ with $q_n^{(r)} = Q_{n-r}$. Setting $F := f - q_n$, one therefore has

(2.5) $$\|\varphi^r F^{(r)}\|_C = E_{n-r}^{\varphi^r}(f^{(r)}).$$

Let $t_n \in \mathcal{P}_n$ denote the polynomial of best approximation of F, i.e., $\|F - t_n\|_C = E_n(F)$. In view of (2.2) one obtains

(2.6) $$\|\varphi^r t_n^{(r)}\|_C \leqq K\|\varphi^r F^{(r)}\|_C.$$

Now $q_n \in \mathcal{P}_n$ so that

$$\|f - q_n - t_n\|_C = E_n(F) = E_n(f - q_n) = E_n(f),$$

thus $p_n = q_n + t_n$, since p_n is unique. This implies by (2.1,5)

$$\|f - p_n\|_C = \|F - t_n\|_C \leqq Kn^{-r}\|\varphi^r F^{(r)}\|_C = Kn^{-r}E_{n-r}^{\varphi^r}(f^{(r)}),$$

thus (2.3). Moreover, (2.4) is a consequence of (2.5, 6) and

$$\|\varphi^r(f-p_n)^{(r)}\|_C \leqq \|\varphi^r F^{(r)}\|_C + \|\varphi^r t_n^{(r)}\|_C \leqq K\|\varphi^r F^{(r)}\|_C = KE_{n-r}^{\varphi^r}(f^{(r)}). \quad \square$$

Now, we are in the position to establish the main result.

THEOREM 2.3. *Let $L_n \in [C]$ be polynomial projection operators on $\mathcal{P}_n$. Then* (1.4) *holds true for each $r \geqq 0$.*

PROOF. Let $p_n \in \mathcal{P}_n$ denote the polynomial of best approximation of $f \in C_\varphi^r$. In view of (1.1), (2.3,4) one obtains

(2.7) $$\|\varphi^r(f - L_n f)^{(r)}\|_C \leqq \|\varphi^r(f - p_n)^{(r)}\|_C + \|\varphi^r L_n^{(r)}(f - p_n)\|_C \leqq$$
$$\leqq KE_{n-r}^{\varphi^r}(f^{(r)}) + Kn^r\|L_n(f - p_n)\|_C \leqq$$
$$\leqq KE_{n-r}^{\varphi^r}(f^{(r)}) + Kn^r\|L_n\|_{[C]}n^{-r}E_{n-r}^{\varphi^r}(f^{(r)}),$$

upon applying the Bernstein-inequality

(2.8) $$\|\varphi^r q_n^{(r)}\|_C \leqq Kn^r\|q_n\|_C \quad (q_n \in \mathcal{P}_n). \quad \square$$

To extend Theorem 2.3 to error bounds for $\|\varphi^i(f - L_n f)^{(i)}\|_C$, $0 \leqq i \leqq r$, let us first establish

LEMMA 2.4. *Let $0 \leqq i \leqq r$. Then $C_\varphi^r \subset C_\varphi^i$ and*

$$E_{n-r}^{\varphi^i}(f^{(i)}) \leqq Kn^{-(r-i)} E_{n-r}^{\varphi^r}(f^{(r)}) \quad (f \in C_\varphi^r). \tag{2.9}$$

PROOF. To show $C_\varphi^r \subset C_\varphi^i$ it is enough to consider $i = r - 1 \geqq 1$. For $f \in C_\varphi^r$ and $0 \leqq x < 1$ one has

$$\varphi^{r-1}(x) f^{(r-1)}(x) = \int_0^x \varphi^{r-1}(x) f^{(r)}(t)dt + \varphi^{r-1}(x) f^{(r-1)}(0).$$

Since $\varphi^{r-1}(x) \leqq \varphi^{r-1}(t)$ for $0 \leqq t \leqq x$ the integrand is bounded by $\|\varphi^r f^{(r)}\|_C / \varphi(t)$ and converges to zero for $x \to 1-$ and fixed t. Thus by dominated convergence one obtains

$$\lim_{x \to 1-} \varphi^{r-1}(x) f^{(r-1)}(x) = 0.$$

Similarly, this result is valid for $x \to -1+$ so that $\varphi^{r-1} f^{(r-1)} \in C[-1, 1]$, thus $f \in C_\varphi^{r-1}$. To establish (2.9) it is again enough to consider $i = r - 1 \geqq 0$. Then an iterative application of (8.2.1), (6.2.6), (6.1.1) (with weight $w = \varphi^{r-1}$) of [4] yields the Jackson-type inequality

$$E_{n-r+1}^{\varphi^{r-1}}(f^{(r-1)}) \leqq Kn^{-1} \|\varphi^r f^{(r)}\|_C \quad (f \in C_\varphi^r).$$

Now let $f \in C_\varphi^r$ be fixed and let q_n as in the proof of Lemma 2.2. Then $q_n^{(r-1)} \in \mathcal{P}_{n-r+1}$ so that

$$E_{n-r+1}^{\varphi^{r-1}}(f^{(r-1)}) = E_{n-r+1}^{\varphi^{r-1}}(f^{(r-1)} - q_n^{(r-1)}) \leqq$$

$$\leqq Kn^{-1} \|\varphi^r (f^{(r)} - q_n^{(r)})\|_C = Kn^{-1} E_{n-r}^{\varphi^r}(f^{(r)}),$$

hence (2.9). □

The following corollary is an immediate consequence of Theorem 2.3 applied to $0 \leqq i \leqq r$ and Lemma 2.4.

COROLLARY 2.5. *Let $L_n \in [C]$ be polynomial projection operators on $\mathcal{P}_n$. Then $(n \geqq r)$*

$$\|\varphi^i (f - L_n f)^{(i)}\|_C \leqq Kn^{-(r-i)} \|L_n\|_{[C]} E_{n-r}^{\varphi^r}(f^{(r)}) \quad (f \in C_\varphi^r), \tag{2.10}$$

simultaneously for each $0 \leqq i \leqq r$.

Let us remark that the proofs still work if, instead of (1.1), for some fixed $q \in \mathbf{N}$

$$L_n f \in \mathcal{P}_{qn} \quad (f \in C), \qquad L_n p_n = p_n \quad (p_n \in \mathcal{P}_n). \tag{2.11}$$

Such an operator is given by the algebraic version of the de la Vallée Poussin means. To be more precise, let $C_{2\pi}$ be the space of functions, 2π-periodic and continuous on $\mathbf{R}$, endowed with the sup-norm $\|\cdot\|_C$, and let $C_{2\pi}^+$ be the subspace of even functions. The latter one is isometric to C via the transformation $Uf(x) := f(\cos x)$, $f \in C$. The de la Vallée Poussin means are defined by $(g \in C_{2\pi})$

$$V_n := \frac{1}{n}\sum_{k=n}^{2n-1} S_k, \quad S_k g(x) := \sum_{j=-k}^{k} e^{ijx}\frac{1}{2\pi}\int_{-\pi}^{\pi} g(u)e^{-iju}\,du.$$

It is well-known, that $V_n g \in C_{2\pi}^+$ for $g \in C_{2\pi}^+$ and

$$V_n g \in \Pi_{2n-1} \quad (g \in C), \qquad V_n t_n = t_n \quad (t_n \in \Pi_n), \tag{2.12}$$

$$\|V_n\|_{[C_{2\pi}]} \leqq 3, \tag{2.13}$$

where Π_n is the set of trigonometric polynomials of order at most n. Then $W_n := U^{-1}V_n U$ fulfills (2.11) so that one obtains

$$\|\varphi^r(f - W_n f)^{(r)}\|_C \leqq K E_{n-r}^{\varphi^r}(f^{(r)}) \quad (f \in C_\varphi^r). \tag{2.14}$$

Let us also mention the analogous result in the trigonometric case itself. Denote by $C_{2\pi}^r$ the space of functions $g \in C_{2\pi}$ which are r-times continuously differentiable on $\mathbf{R}$.

THEOREM 2.6. *Let $M_n \in [C_{2\pi}]$ be such that*

$$M_n g \in \Pi_n \quad (g \in C_{2\pi}), \qquad M_n t_n = t_n \quad (t_n \in \Pi_n). \tag{2.15}$$

Then with $\tilde{E}_n(g) := \inf\{\|g - t_n\|_C : t_n \in \Pi_n\}$

$$\|(M_n g - g)^{(i)}\|_C \leqq K n^{-(r-i)}\|M_n\|_{[C_{2\pi}]}\tilde{E}_n(g^{(r)}) \quad (g \in C_{2\pi}^r), \tag{2.16}$$

simultaneously for each $0 \leqq i \leqq r$.

The proof works parallel to (2.7) applying the inequalities in [8, 5.6(27), 8.4(60)] instead of Lemma 2.2,4. One may also compare (2.16), $i = r$ with

$$\|(M_n g - g)^{(r)}\|_C \leqq K\left[\tilde{E}_n(g^{(r)}) + \tilde{E}_n(g)\|M_n^{(r)}\|_{[C_{2\pi}]}\right],$$

given in [6] (for a similar treatment in C see [7]).

Let us conclude this section with the analogon of Theorem 2.6 to even functions.

THEOREM 2.7. *Let* $M_n \in [C_{2\pi}^+]$ *be such that*

$$(2.17)\qquad M_n g \in \Pi_n^+ := \Pi_n \cap C_{2\pi}^+ \quad (g \in C_{2\pi}^+), \qquad M_n t_n = t_n \quad (t_n \in \Pi_n^+).$$

Then for $g \in C_{2\pi}^+ \cap C_{2\pi}^r$

$$(2.18)\qquad \|(M_n g - g)^{(i)}\|_C \leqq K n^{-(r-i)} \|M_n\|_{[C_{2\pi}^+]} \tilde{E}_n(g^{(r)}),$$

simultaneously for each $0 \leqq i \leqq r$.

The only difference in the proof is the observation that for even function $g \in C_{2\pi}^+$ the polynomial of best approximation is also even, i.e.,

$$(2.19)\qquad \tilde{E}_n(g) = E_n^+(g) := \inf\left\{\|g - t_n\|_C : t_n \in \Pi_n^+\right\}.$$

3. The sharpness

The sharpness of the estimate (1.4) can be established for those L_n, the norm of which behave like $\log n$. To this end, let $\varepsilon = \{\varepsilon_n\}$ be a positive decreasing nullsequence satisfying

$$(3.1)\qquad \varepsilon_n = O(\varepsilon_{2n}).$$

THEOREM 3.1. *Let* $L_n \in [C]$ *be polynomial projection operators on* $\mathcal{P}_n$. *Then for each* ε *subject to* (3.1) *there exists a counterexample* $f_\varepsilon \in C_\varphi^r$ *such that*

$$(3.2)\qquad E_{n-r}^{\varphi^r}(f_\varepsilon^{(r)}) = O(\varepsilon_n),$$

$$(3.3)\qquad \|\varphi^r(f_\varepsilon - L_n f_\varepsilon)^{(r)}\|_C \neq o(\varepsilon_n \log n).$$

This result follows as an application of the subsequent quantitative extension of the uniform boundedness principle (see [3] and the literature cited there). Let X be a Banach space with norm $\|\cdot\|_X$ and X^* the space of sublinear, bounded functionals on X.

THEOREM 3.2. *Let* ψ_n *be a decreasing nullsequence and* $\sigma_n > 0$. *Suppose that for* $U_n, R_n \in X^*$ *there are elements* $h_n \in X$ *satisfying* $(m, n \in \mathbf{N})$

$$(3.4)\qquad \|h_n\|_X \leqq K,$$

$$(3.5)\qquad |U_m h_n| \leqq K \min\{1, \sigma_m/\psi_n\},$$

$$(3.6)\qquad |R_n h_n| \neq o(1).$$

Then for each $0 < \alpha < 1$ *there exists* $f_\alpha \in X$ *with*

$$(3.7)\qquad |U_n f_\alpha| = O(\sigma_n^\alpha),$$

$$(3.8)\qquad |R_n f_\alpha| \neq o(\psi_n^\alpha).$$

The proof of Theorem 3.1 is now divided via the following lemmas.

LEMMA 3.3. *C^r_φ is a Banach space under the norm*

$$\|f\|_{\varphi,r} := \sum_{j=0}^{r-1} |f^{(j)}(0)| + \|\varphi^r f^{(r)}\|_C. \tag{3.9}$$

Moreover, for $f \in C^r_\varphi$

$$\|f\|_C \leqq \frac{\pi}{2}\|f\|_{\varphi,r}. \tag{3.10}$$

PROOF. The first assertion may be shown as usual, applying the representation

$$f(x) = \sum_{j=0}^{r-1} \frac{x^j}{j!} f^{(j)}(0) + \frac{1}{(r-1)!}\int_0^x (x-u)^{r-1} f^{(r)}(u)\,du \quad (|x|<1).$$

To obtain (3.10) set $\mu_x(u) := (x-u)/\varphi(u)$. Then

$$|f(x)| \leqq \sum_{j=0}^{r-1} |f^{(j)}(0)| + \left| \int_0^x \frac{|\mu_x(u)|^{r-1}}{\varphi(u)} \varphi^r(u) |f^{(r)}(u)|\,du \right| \leqq \frac{\pi}{2}\|f\|_{\varphi,r} \quad (|x|<1)$$

since $|\mu_x(u)| \leqq 1$ and $\left|\int_0^x du/\varphi(u)\right| \leqq \pi/2$. □

In view of (1.3), (2.8), (3.10) the functionals $(n \geqq r)$

$$U_n f = E^{\varphi^r}_{n-r}(f^{(r)}), \quad R_n f = \|\varphi^r (f - L_{2n} f)^{(r)}\|_C / \log(2n) \tag{3.11}$$

belong to X^* where X is the Banach space C^r_φ. To construct the test elements h_n, some results on trigonometric approximation are needed.

LEMMA 3.4. *For the partial sums S_{2n} one has the inequality*

$$\|g - S_{2n} g\|_C \leqq 4\tilde{E}_n(g - S_{2n} g) \quad (g \in C_{2\pi}). \tag{3.12}$$

Moreover, there exists $r_n \in \Pi^+_{4n}$ such that $(c_0 > 0)$

$$\|r_n\|_C \leqq 3, \quad \|r_n - S_{2n} r_n\|_C \geqq c_0 \log(2n). \tag{3.13}$$

PROOF. Let I be the identity operator. In view of (2.12,13)

$$\|(I - V_n) g\|_C \leqq 4\tilde{E}_n(g),$$

thus (3.12) since $I - S_{2n} = (I - V_n)(I - S_{2n})$. To establish (3.13) note first that the Dirichlet kernel is even, thus

$$(3.14) \qquad \begin{cases} S_{2n}(g(-u))(x) = S_{2n}g(-x) \quad (g \in C_{2\pi}), \\ \qquad V_{2n}g \in \Pi_{4n}^{+} \quad (g \in C_{2\pi}^{+}). \end{cases}$$

Since the norm of the functional $S_{2n}g(0)$ behaves like $\log(2n)$ there exists $h_n \in C_{2\pi}$ with

$$\|h_n\|_C \leqq 1, \quad |S_{2n}h_n(0)| \geqq c_0 \log(2n) + 3.$$

Setting $r_n := V_{2n}g_n$, $g_n(x) := (h_n(x) + h_n(-x))/2$ it follows that $r_n \in \Pi_{4n}^{+}$ with $\|r_n\|_C \leqq 3$ and

$$\|r_n - S_{2n}r_n\|_C \geqq |S_{2n}r_n(0)| - |r_n(0)| \geqq c_0 \log(2n)$$

in view of (3.14) and

$$S_{2n}r_n(0) = V_{2n}S_n g_n(0) - S_{2n}g_n(0) = S_{2n}h_n(0). \quad \square$$

For $t \in \mathbf{R}$ let $T_t \in [C_{2\pi}]$ be the translation operator $T_t g(x) := g(x+t)$ which is an isometry.

LEMMA 3.5. *Suppose that for $t \in \mathbf{R}$ there are functions $h_t \in C_{2\pi}^{+}$ satisfying*

$$(3.15) \qquad h_t = h_{-t},$$

$$(3.16) \qquad \lim_{s \to t} \|h_s - h_t\|_C = 0.$$

Let $r_{nt} \in \Pi_n$ denote the polynomial of best approximation of h_t. Then

$$(3.17) \qquad s_n(x) := \frac{1}{\pi} \int_{-\pi}^{\pi} T_t r_{nt}(x)\, dt \in \Pi_n^{+}.$$

PROOF. Since the operator of best approximation is continuous, it follows that

$$\lim_{s \to t} \|r_{ns} - r_{nt}\|_C = 0$$

by (3.16). Let $a_{kn}(t)$ be the coefficients of r_{nt}, i.e., $r_{nt}(x) = \sum_{k=-n}^{n} a_{kn}(t) e^{ikx}$. Then $a_{kn}(t)$ are continuous in t since for fixed $n \in \mathbf{N}$

$$\left\| \sum_{k=-n}^{n} \alpha_k e^{ikx} \right\|_C$$

is a norm on $\mathbf{R}^{2n+1}$, equivalent to $\max|\alpha_k|$. Therefore the integral in (3.17) exists and

$$s_n(x) = \sum_{k=-n}^{n} \frac{1}{\pi} \int_{-\pi}^{\pi} a_{kn}(t) e^{ikt} dt\, e^{ikx} \in \Pi_n.$$

In view of (2.19) the polynomials r_{nt} are even and satisfy $r_{n,-t} = r_{nt}$ by (3.15) and the uniqueness of r_{nt}. Therefore $T_t r_{nt}(-x) = T_{-t} r_{n,-t}(x)$ so that s_n is even, too. □

LEMMA 3.6. *If $M_n \in [C_{2\pi}^+]$ satisfies* (2.17), *then*

$$\|g - S_{2n} g\|_C \leqq 8 \max_{t \in \mathbf{R}} E_n^+ \left(T_t^+ g - M_{2n} T_t^+ g\right) \quad (g \in C_{2\pi}^+) \tag{3.18}$$

with $T_t^+ := (T_t + T_{-t})/2 \in [C_{2\pi}^+]$.

PROOF. For fixed $g \in C_{2\pi}^+$ set $h_t := T_t^+ g - M_{2n} T_t^+ g \in C_{2\pi}^+$. Obviously, (3.15,16) follow since $\|T_s g - T_t g\|_C$ converges to zero for $s \to t$, and M_{2n} is linear and bounded. Let $r_{nt} \in \Pi_n^+$ be the polynomial of best approximation of h_t, and let s_n be defined as in (3.17). Then the Faber–Marcinkiewicz–Berman identity (cf. [2, p. 214])

$$(I - S_{2n}) g(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} T_t (I - M_{2n}) T_t^+ g(x)\, dt \tag{3.19}$$

and (3.12) imply the estimate

$$\|g - S_{2n} g\|_C \leqq 4 \tilde{E}_n (g - S_{2n} g) \leqq 4 \|g - S_{2n} g - s_n\|_C =$$

$$= 4 \left\| \frac{1}{\pi} \int_{-\pi}^{\pi} T_t (h_t - r_{nt})\, dt \right\|_C \leqq 8 \sup_{t \in \mathbf{R}} \|h_t - r_{nt}\|_C = 8 \sup_{t \in \mathbf{R}} E_n^+ (h_t).$$

Since $h_{t+2\pi} = h_t$, the supremum is attained, and (3.18) follows. □

LEMMA 3.7. *If $L_n \in [C]$ satisfies* (1.1), *there exists $p_n \in \mathcal{P}_{4n}$ such that* ($c_1 > 0$)

$$\|p_n\|_C \leqq 3, \tag{3.20}$$

$$\|\varphi^r (p_n - L_{2n} p_n)^{(r)}\|_C \geqq c_1 n^r \log(2n). \tag{3.21}$$

PROOF. Since $U\mathcal{P}_n = \Pi_n^+$, the operator $M_n = U L_n U^{-1} \in [C_{2\pi}^+]$ satisfies (2.17). Now let $r_n \in \Pi_{4n}^+$ be the polynomials, satisfying (3.13). Then Lemma 3.6 implies that there exists $t_n \in \mathbf{R}$ such that

$$8 E_n^+ \left(T_{t_n}^+ r_n - M_{2n} T_{t_n}^+ r_n\right) \geqq c_0 \log(2n).$$

With $p_n := U^{-1}T_{t_n}^{+}r_n \in \mathcal{P}_{4n}$ this yields

$$8E_n(p_n - L_{2n}p_n) \geqq c_0 \log(2n)$$

since $E_n^+(Uf) = E_n(f)$. Thus (3.20,21) follow by (2.1) and (3.13). □

PROOF OF THEOREM 3.1. For the functionals (3.11) the conditions (3.4–6) have to be verified for

$$h_n(x) := n^{-r}\left\{p_n(x) - \sum_{j=0}^{r-1} \frac{x^j}{j!} p_n^{(j)}(0)\right\},$$

where p_n is given via Lemma 3.7. One obtains (3.4) in view of (2.8), (3.20) and

$$\|h_n\|_{\varphi,r} = n^{-r}\left\|\varphi^r p_n^{(r)}\right\|_C \leqq K.$$

Moreover, $U_m h_n = 0$ for $m \geqq 4n$ since $h_n^{(r)} \in \mathcal{P}_{4n-r}$. If $m < 4n$, then $\varepsilon_n \leqq K\varepsilon_{4n} \leqq K\varepsilon_m$ by (3.1) so that

$$U_m h_n \leqq \|h_n\|_{\varphi,r} \leqq K \leqq K\varepsilon_m^2 / \varepsilon_{2n}^2,$$

thus (3.5) with $\sigma_n = \varepsilon_n^2$, $\psi_n = \varepsilon_{2n}^2$. Since

$$h_n - L_{2n}h_n = n^{-r}(p_n - L_{2n}p_n)$$

it follows that $R_n h_n \geqq c_1$ by (3.21) and therefore (3.6). The assertion of Theorem 3.2 with $\alpha = 1/2$ then yields (3.2,3). □

Let us mention that one may deduce the sharpness of Theorem 2.6,7 in a similarly way.

THEOREM 3.8. (i) *Let* $M_n \in [C_{2\pi}]$ *satisfy* (2.15). *Then for each* ε *subject to* (3.1) *there exists* $g_\varepsilon \in C_{2\pi}^r$ *such that*

$$\tilde{E}_n(g_\varepsilon^{(r)}) = O(\varepsilon_n), \quad \|(M_n g - g)^{(r)}\|_C \neq o(\varepsilon_n \log n). \tag{3.21}$$

(ii) *If* $M_n \in [C_{2\pi}^+]$ *satisfies* (2.17), *then for each* ε *subject to* (3.1) *there exists* $g_\varepsilon \in C_{2\pi}^+ \cap C_{2\pi}^r$ *with* (3.21).

Note that the proof of (ii) (and similarly for (i)) may be simplified in view of (3.19) and ($D^r g := g^{(r)}$)

$$g^{(r)}(x) - S_{2n}^{(r)}g(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} T_t\left(D^r - M_{2n}^{(r)}\right) T_t^+ g(x)\, dx,$$

$$\|S_{2n}^{(r)}\|_{[C_{2\pi}]} \geqq c_0 n^r \log n$$

(for the latter inequality see [1]).

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DER RWTH AACHEN
D–5100 AACHEN
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BOUNDS FOR EXTENDED LIPSCHITZ CONSTANTS

J. ANGELOS, E. KAUFMAN, Jr., T. LENKER (Mount Pleasant)
and
M. S. HENRY (Norfolk)

1. Introduction

Let X be a closed subset of $I = [-1, 1]$ with cardinality at least $n + 2$, and suppose $f \in C[X]$, the space of continuous real-valued functions on X endowed with the uniform norm $\|\cdot\|$. Denote the set of all polynomials of degree n or less by Π_n, and let $B_n(f)$ be the best uniform approximation to f from Π_n. The global (classical) Lipschitz constant is defined to be

$$\lambda_n(f) = \sup\{\|B_n(f) - B_n(g)\|/\|f - g\| : g \in C[X],\ f \neq g\}, \tag{1.1}$$

and the local Lipschitz constant is

$$\hat{\lambda}_n(f) = \lim_{\delta\to 0+} \sup\{\|B_n(f) - B_n(g)\|/\|f - g\| : g \in C[X],\ 0 < \|f - g\| \leqq \delta\}. \tag{1.2}$$

Global and local Lipschitz constants have been the subject of several recent papers [1, 2, 3], and figure prominently in the current paper.

For $f \in C[X]$, let

$$e_n(f)(x) = f(x) - B_n(f)(x), \quad x \in X. \tag{1.3}$$

Then the extremal set of the error function $e_n(f)$ is

$$E_n(f) = \{x \in X : |e_n(f)(x)| = \|e_n(f)\|\}. \tag{1.4}$$

An alternant of the error function is any set

$$X_n = \{x_0, x_1, \dots, x_{n+1}\} \subseteqq E_n(f)$$

with $x_0 < x_1 < \dots < x_{n+1}$ for which $e_n(f)(x_i) = \gamma(-1)^i\|e_n(f)\|$, $i = 0, 1, \dots, n + 1$, where $\gamma = \operatorname{sgn} e_n(f)(x_0)$.

When the cardinality $|E_n(f)|$ of $E_n(f)$ is $n + 2$, then the local Lipschitz constant can be explicitly displayed and is equal to the norm of a certain "derivative" of the best approximation operator B_n, [1]. In contrast, even when $|E_n(f)| = n+2$, precise estimates of the global Lipschitz constant have proved to be somewhat elusive. To facilitate the investigation of the behavior of the global Lipschitz constant, the authors and A. Kroó [3] introduced the extended global Lipschitz constant (EGLC), a constant of interest in its own right. Specifically, the EGLC is defined to be

$$(1.5)\qquad G_n(f)=\sup\{\lambda_n(h): h\in C[X],\ E_n(h)=E_n(f),\ \text{and}\ \operatorname{sgn} e_n(h)(x)=\gamma\operatorname{sgn} e_n(f)(x),\ x\in E_n(f),\ \text{where}\ \gamma=+1\ \text{or}\ -1\}.$$

It is clear from (1.5) that $\lambda_n(f)\leqq G_n(f)$. Of particular interest is the relationship between $G_n(f)$ and the classical strong unicity constant given in Theorem 2 below. First, if $f\in C[X]$, then the strong unicity constant is defined as

$$(1.6)\qquad M_n(f)=\sup\{\|p-B_n(f)\|/(\|f-p\|-\|f-B_n(f)\|): p\in\Pi_n,\ p\neq B_n(f)\}.$$

THEOREM 1. [3]. *For any* $f\in C[I]$,

$$(1.7)\qquad M_n(f)\leqq G_n(f)\leqq 2M_n(f).$$

It can be shown [13, 3 (Lemma 1)] that any two functions possessing the same extremal set and sign orientation generate the same strong unicity constant. The upper bound in (1.7) follows from this observation and the well-known inequality [7, p. 82], $\lambda_n(f)\leqq 2M_n(f)$ for $f\in C[X]$. The proof of the lower bound in (1.7) is somewhat technical and is given in [3].

A rather natural and equally interesting companion to the extended global Lipschitz constant can be defined. Specifically, the extended local Lipschitz constant (ELLC) is defined to be

$$(1.8)\qquad L_n(f)=\inf\{\lambda_n(h): h\in C[X],\quad E_n(h)=E_n(f),\quad \text{and}\quad \operatorname{sgn} e_n(h)(x)=\gamma\operatorname{sgn} e_n(f)(x),\quad x\in E_n(f),\ \text{where}\ \gamma=+1\ \text{or}\ -1\}.$$

From (1.8) it is clear that $L_n(f)\leqq\lambda_n(f)$. The main objective of the remainder of this paper is to establish the ELLC analogue to Theorem 1.

2. Lemmas

The definitions of both the EGLC and ELLC can be simplified. In particular, the modified form of the ELLC displayed in the lemma below will be used throughout the remainder of the paper.

LEMMMA 1 [2]. *For* $f\in C[X]$, $f\not\equiv 0$, *suppose* $E_n(f)=X_n=\{x_0,x_1,\dots,x_{n+1}\}$. *Then*

$$(2.1)\qquad G_n(f)=\sup\{\lambda_n(h): E_n(h)=X_n\ \text{and}\ h(x_i)=(-1)^i,\ i=0,1,\dots,n+1\},$$

and

(2.2)
$$L_n(f) = \inf\{\lambda_n(h) : E_n(h) = X_n \text{ and } h(x_i) = (-1)^i,\ i = 0, 1, \dots, n+1\}.$$

PROOF. We first observe for $h \in C[X]$ that $\lambda_n(h - B_n(h)) = \lambda_n(h)$. Therefore we may assume in (1.5) and (1.8) that $B_n(h) \equiv 0$, which in turn implies that

$$e_n(h)(x_i) = h(x_i) = (\operatorname{sgn} e_n(h)(x_0))(-1)^i\|e_n(h)\| = \\ = (\operatorname{sgn} h(x_0))(-1)^i\|h\|,\quad i = 0, 1, \dots, n+1.$$

It can also be shown that $\lambda_n(\alpha h) = \lambda_n(h)$ for $\alpha \neq 0$. Thus, without loss of generality the requirement that $\operatorname{sgn} e_n(h)(x) = \gamma \operatorname{sgn} e_n(f)(x)$, $x \in E_n(f)$, in (1.5) and (1.8) can be replaced by $h(x_i) = (-1)^i$, $i = 0, 1, \dots, n+1$, whenever $E_n(f) = X_n$. □

The next theorem is the main theorem of this paper.

THEOREM 2. *Suppose $f \in C[X]$, and suppose that $E_n(f) = X_n = \{x_0, x_1, \dots, x_{n+1}\}$. Then*

$$\hat{\lambda}_n(f) \leqq L_n(f) \leqq 6 + 4\hat{\lambda}_n(f). \tag{2.3}$$

If $X = X_n$ in Theorem 2, then we actually have $\hat{\lambda}_n(f) = L_n(f) = \lambda_n(f)$, [1]. Thus hereafter we assume $X - X_n$ is nonempty (note that this implies $f \not\equiv 0$). In this setting the proof of Theorem 2 depends on a series of sometimes technical lemmas and will follow the statements and proofs of these lemmas. Before proceeding, it is worth emphasizing that the strong unicity and local Lipschitz constants do not depend on f when $E_n(f) = X_n$, but rather only on X_n [1, Theorem 2]. In this case the notation $M_n(X_n)$ and $\hat{\lambda}_n(X_n)$ is employed.

LEMMA 2. *Let $X_n = \{x_0, x_1, \dots, x_{n+1}\}$. For δ sufficiently small choose ℓ large enough to insure that $(x_{i+1} - 1/\ell) - (x_i + 1/\ell) = x_{i+1} - x_i - 2/\ell \geqq \delta > 0$, $i = 0, 1, \dots, n$. For any $\overline{g} \in C[X]$ with error function $e_n(\overline{g})$ and alternant $\{y_0, y_1, \dots, y_{n+1}\}$ satisfying $y_i \in (x_i - 1/\ell, x_i + 1/\ell) \cap X$, $i = 0, 1, \dots, n+1$, there exists a constant μ depending only on X_n such that for any $g \in C[X]$,*

$$\|B_n(g) - B_n(\overline{g})\| \leqq \mu\|g - \overline{g}\|. \tag{2.4}$$

PROOF. Clearly $y_{i+1} - y_i \geqq \delta$, $i = 0, 1, \dots, n$. Thus the error function $e_n(\overline{g})$ has an alternant with separation greater than or equal to δ. Let $F_\delta \subseteqq C[X]$ be the subset of $C[X]$ such that if $f \in F_\delta$, then $e_n(f)$ has an alternant with separation greater than or equal to δ. Then the arguments of Dunham [8, Theorem 2] with X replacing I imply that there exists a constant μ such that for every $f \in F_\delta$ and $g \in C[X]$,

$$\|B_n(g) - B_n(f)\| \leqq \mu\|g - f\|.$$

Since $\overline{g} \in F_\delta$, (2.4) is established. □

Inequality (2.4) is essentially a uniform Lipschitz constant result for changing f. The interested reader is referred to the survey papers [4, 9] for a discussion of other uniform Lipschitz constant results.

Prior to stating the next lemma we define a set U_ℓ and a function h_ℓ, both of which will be utilized in several of the proofs that follow.

First, let

$$d_0 = \begin{cases} x_0 + 1 & \text{if } x_0 > -1 \\ 2 & \text{if } x_0 = -1, \end{cases}$$

and

$$d_{n+1} = \begin{cases} 1 - x_{n+1} & \text{if } x_{n+1} < 1 \\ 2 & \text{if } x_{n+1} = 1. \end{cases}$$

Then let $\overline{d} = \min\{d_0, d_{n+1}, (1/3)\min\{x_{i+1} - x_i;\ i = 0, 1, \dots, n\}\}$. Now let $\overline{\ell}_0 = [1/\overline{d}] + 1$, and for $\ell \geqq \overline{\ell}_0$, define

$$(2.5) \qquad U_\ell = \Big(\bigcup_{i=0}^{n+1} (x_i - 1/\ell, x_i + 1/\ell)\Big) \cap X.$$

By definition, $\overline{d} \leqq (1/3)\min\{x_{i+1} - x_i;\ i = 0, \dots, n\}$, and $1/\ell < \overline{d}$. Therefore, the intervals $(x_i - 1/\ell, x_i + 1/\ell)$, $i = 0, 1, \dots, n+1$, are disjoint. Since by assumption $X - X_n$ is nonempty, there exists an $\ell_0 \geqq \overline{\ell}_0$ such that for $\ell \geqq \ell_0$, $X - U_\ell$ is nonempty. Hereafter, we assume that $\ell \geqq \ell_0$. Define $h_\ell \in C[X]$ by $h_\ell(-1) = 0$ if $-1 \notin X_n$, $h_\ell(1) = 0$ if $1 \notin X_n$, $h_\ell(x_i \pm 1/\ell) = 0$, and $h_\ell(x_i) = (-1)^i$, $i = 0, 1, \dots, n+1$; let h_ℓ be linear between the points where h_ℓ has just been defined.

Because of the manner in which h_ℓ has been constructed, $B_n(h_\ell) \equiv 0$ and $E_n(h_\ell) = X_n$. Thus h_ℓ is one of the functions considered in Lemma 1. Let $H_\ell = \{g \in C[X]; g - B_n(g)$ possesses no alternant $\{y_0, y_1, \dots, y_{n+1}\}$ with $y_i \in (x_i - 1/\ell, x_i + 1/\ell) \cap X$ and $\operatorname{sgn}\ (g - B_n(g))(y_i) = (-1)^i$, $i = 0, 1, \dots, n+1\}$. This set will be utilized in subsequent arguments.

LEMMA 3. *For $\ell \geqq \ell_0$, let $\beta_\ell = \inf\{\|g - h_\ell\| : g \in H_\ell\}$. Then there exists a $\overline{g}_\ell \in C[X]$ and $\overline{x}_\ell \in X$ with $\|\overline{g}_\ell - h_\ell\| < \beta_\ell$, $|(\overline{g}_\ell - B_n(\overline{g}_\ell))(\overline{x}_\ell)| \geqq \|\overline{g}_\ell - B_n(\overline{g}_\ell)\| - 1/\ell$, and either $\overline{x}_\ell \in X - U_\ell$ or $\overline{x}_\ell \in (x_i - 1/\ell, x_i + 1/\ell) \cap X$ for some i and* $\operatorname{sgn}\ (\overline{g}_\ell - B_n(\overline{g}_\ell))(\overline{x}_\ell) = (-1)^{i+1}$.

PROOF. We first show that $\beta_\ell > 0$. For suppose that $\beta_\ell = 0$. Then there exists a sequence $\{g_j\}_{j=0}^{\infty} \subseteqq H_\ell$ such that $\lim_{j\to\infty} \|g_j - h_\ell\| = 0$. This in turn implies that $\lim_{j\to\infty} \|B_n(g_j) - B_n(h_\ell)\| = 0$. Thus $\lim_{j\to\infty} \|B_n(g_j) - g_j\| = \|h_\ell - B_n(h_\ell)\| = 1$. In this case we assert that for j sufficiently large $g_j - B_n(g_j)$ must possess an alternant $\{y_0, \dots, y_{n+1}\}$ with $y_i \in (x_i - 1/\ell, x_i + 1/\ell) \cap X$ and with $\operatorname{sgn}(g_j - B_n(g_j))(y_i) = \operatorname{sgn}\ h_\ell(x_i) = (-1)^i$, $i = 0, 1, \dots, n+1$, a contradiction of the definition of H_ℓ. This assertion follows because $g_j - B_n(g_j)$ must have some alternant and $B_n(h_\ell) = h_\ell = 0$ on $X - U_\ell$ (thus no point from the alternant for $g_j - B_n(g_j)$ can be in $X - U_\ell$ since $g_j - B_n(g_j)$ is too small there), and because the sign of $g_j - B_n(g_j)$ at a point of the alternant in $(x_i - 1/\ell, x_i + 1/\ell) \cap X$ must be $(-1)^i$, so no two consecutive points from the alternant for $g_j - B_n(g_j)$ can lie in the same $(x_i - 1/\ell, x_i + 1/\ell) \cap X$.

Now let $\{g_j\}_{j=0}^{\infty} \subseteqq H_\ell$ and $\{x^{(j)}\}_{j=0}^{\infty} \subseteqq X$ satisfy $\|g_j - h_\ell\| \downarrow \beta_\ell > 0, x^{(j)} \in E_n(g_j)$, and either $x^{(j)} \in X - U_\ell$ or $x^{(j)} \in (x_i - 1/\ell, x_i + 1/\ell) \cap X$ for some i and $\operatorname{sgn}(g_j - B_n(g_j))(x^{(j)}) = (-1)^{i+1}$. Let $\lambda_j = j\beta_\ell/((j+1)\|g_j - h_\ell\|)$. Clearly $0 < \lambda_j < 1$ for all j. Let $\tilde{g}_j = \lambda_j g_j + (1-\lambda_j)h_\ell$. Then $\|\tilde{g}_j - h_\ell\| < \beta_\ell$. Now $\lambda_j \to 1$ as $j \to +\infty$, and hence

$$\lim_{j\to\infty} \|\tilde{g}_j - g_j\| = 0. \tag{2.6}$$

Therefore

$$\begin{aligned}
&\|\tilde{g}_j - B_n(\tilde{g}_j)\| - |(\tilde{g}_j - B_n(\tilde{g}_j))(x^{(j)})| = \\
&= \|\tilde{g}_j - B_n(\tilde{g}_j)\| - \|g_j - B_n(g_j)\| + \|g_j - B_n(g_j)\| - \\
&- |(g_j - B_n(g_j))(x^{(j)})| + |(g_j - B_n(g_j))(x^{(j)})| - |(\tilde{g}_j - B_n(\tilde{g}_j))(x^{(j)})| \leqq \\
&\leqq \|\tilde{g}_j - g_j + B_n(g_j) - B_n(\tilde{g}_j)\| + 0 + |g_j(x^{(j)}) - \tilde{g}_j(x^{(j)}) + B_n(\tilde{g}_j)(x^{(j)}) - B_n(g_j)(x^{(j)})| \leqq \\
&\leqq 2\|\tilde{g}_j - g_j\| + 2\|B_n(\tilde{g}_j) - B_n(g_j)\|.
\end{aligned} \tag{2.7}$$

Inequality (2.7) and Lemma 2 (with $\ell \geqq \ell_0$) now imply that

$$\|\tilde{g}_j - B_n(\tilde{g}_j)\| - |(\tilde{g}_j - B_n(\tilde{g}_j))(x^{(j)})| \leqq 2(1+\mu)\|\tilde{g}_j - g_j\|. \tag{2.8}$$

From (2.6) and (2.8) we may now infer for j sufficiently large that

$$|(\tilde{g}_j - B_n(\tilde{g}_j))(x^{(j)})| \geqq \|\tilde{g}_j - B_n(\tilde{g}_j)\| - 1/\ell. \tag{2.9}$$

Also either $x^{(j)} \in X - U_\ell$, or $x^{(j)} \in (x_i - 1/\ell, x_i + 1/\ell) \cap X$ for some i and

$$(g_j - B_n(g_j))(x^{(j)}) = (-1)^{i+1}\|g_j - B_n(g_j)\|. \tag{2.10}$$

In the latter case, assume there exists a $\delta > 0$ such that for all j sufficiently large $\|g_j - B_n(g_j)\| \geqq \delta$. (This fact will be established in the next lemma.) Then from (2.6) we also have $\lim_{j\to\infty} \|B_n(\tilde{g}_j) - B_n(g_j)\| = 0$.

Therefore, for j sufficiently large (2.10) implies $\operatorname{sgn}(\tilde{g}_j - B_n(\tilde{g}_j))(x^{(j)}) = \operatorname{sgn}(g_j - B_n(g_j))(x^{(j)}) = (-1)^{i+1}$. Thus in either case, for j sufficiently large $\tilde{g}_j$ and $x^{(j)}$ will serve as $\overline{g}_\ell$ and $\overline{x}_\ell$, completing the proof. □

LEMMA 4. *Let* $\{g_j\}_{j=0}^{\infty} \subseteqq H_\ell$ *be such that* $\|g_j - h_\ell\| \downarrow \beta_\ell$. *Then there exists a* $\delta > 0$ *such that* $\|g_j - B_n(g_j)\| \geqq \delta$ *for all* j *sufficiently large.*

PROOF. Assume the conclusion of Lemma 4 is not true. Then without loss of generality we may assume that

$$\lim_{j\to\infty} \|g_j - B_n(g_j)\| = 0. \tag{2.11}$$

Therefore for j sufficiently large, $\|B_n(g_j)\| \leqq 2\|g_j\| \leqq 2(\beta_\ell + 2)$. This implies by going to subsequences if necessary that $\lim_{j\to\infty} B_n(g_j) = P \in \Pi_n$, and hence (2.11) implies that $\lim_{j\to\infty} g_j = P$. But then $\beta_\ell = \lim_{j\to\infty} \|h_\ell - g_j\| = \|h_\ell - P\|$. Thus $\beta_\ell \geqq 1$, for otherwise $\|h_\ell - P\| < \|h_\ell - B_n(h_\ell)\|$, which is not possible.

We now construct a $g \in H_\ell$ satisfying $\|g - h_\ell\| < 1$. Such a construction would contradict the definition of β_ℓ. We first select any point $x^* \in X - U_\ell$. To illustrate how to proceed in the construction it is sufficient to assume that $x^* \in [x_{i^*} + 1/\ell, x_{i^*+1} - 1/\ell]$ for some i^*, $1 \leqq i^* \leqq n$. The cases where $x^* \leqq x_0 - 1/\ell$, $x^* \geqq x_{n+1} + 1/\ell$, or $x^* \in [x_0 + 1/\ell, x_1 - 1/\ell]$ are similar. Define g as follows:

$$g(x) = \begin{cases} (1/2)h_\ell(x), & x \in [-1, x_{i^*} - 1/\ell] \cup [x_{i^*+1}, 1] \\ (1/2)h_\ell(x_{i^*}), & x = x^* \\ \text{linear on } [x_{i^*} - 1/\ell, x^*] & \text{and on } [x^*, x_{i^*+1}]. \end{cases}$$

From the definitions of g and h_ℓ, it is clear that $\max\{|h_\ell(x) - g(x)| : x \in [-1, x_{i^*} - 1/\ell] \cup [x_{i^*+1}, 1]\} = 1/2$. For $x \in [x_{i^*} - 1/\ell, x_{i^*+1}]$, it can be shown that $|h_\ell(x) - g(x)| < 1$. Therefore, $\|h_\ell - g\| < 1$. On the other hand, g is constructed to insure that $B_n(g) \equiv 0$. It is clear that $E_n(g) = X_n \cup \{x^*\} - \{x_{i^*}\}$. Since $x^* \in [x_{i^*} + 1/\ell, x_{i^*+1} - 1/\ell]$, g possesses no alternant $\{y_0, y_1, \ldots, y_{n+1}\}$ with $y_i \in (x_i - 1/\ell, x_i + 1/\ell) \cap X$ and $\operatorname{sgn}(g - B_n(g))(y_i) = (-1)^i$, $i = 0, \ldots, n+1$. Thus we have constructed a $g \in H_\ell$ such that $\|h_\ell - g\| < \beta_\ell$. This contradicts the definition of β_ℓ. Therefore, the proof of Lemma 4 is complete. □

Earlier it was noted that $\hat\lambda_n(f) = \hat\lambda_n(X_n)$ when $E_n(f) = X_n$. Let $\{q_i\}_{i=0}^{n+1} \subseteqq \Pi_n$ be determined by

$$q_i(x_j) = (-1)^i, \quad i = 0, 1, \ldots, n+1; \quad i \neq j, \quad j = 0, 1, \ldots, n+1. \tag{2.12}$$

The explicit representation for $\hat\lambda_n(X_n)$ [1, Theorem 2] mentioned in Section 1 is then

$$\hat\lambda_n(X_n) = \left\| \sum_{i=0}^{n+1} |q_i| / (1 + |q_i(x_i)|) \right\|. \tag{2.13}$$

Both (2.12) and (2.13) are used in the next lemma, a lemma which is stated without proof in [2].

LEMMA 5. *If* $g \in C[X]$ *is such that* $g - B_n(g)$ *possesses an alternant* $\{y_0, y_1, \dots, y_{n+1}\}$ *with* $y_i \in (x_i - 1/\ell, x_i + 1/\ell) \cap X$ *for* $i = 0, 1, \dots, n+1$ *and* $\operatorname{sgn}(g - B_n(g))(y_i) = (-1)^i$, $i = 0, 1, \dots, n+1$, *then there is a constant* K *independent of* g *and* ℓ *such that*

$$\frac{\|B_n(g) - B_n(h_\ell)\|}{\|g - h_\ell\|} \leqq (1 + K/\ell)\hat{\lambda}_n(X_n). \tag{2.14}$$

PROOF. For $h \in C[X]$ with alternant X_n, it can be shown [1, Lemma 1] that

$$B_n(h) = \sum_{j=0}^{n+1} \frac{(-1)^{j+1} h(x_j)}{1 + |q_j(x_j)|} q_j.$$

Thus

$$B_n(h_\ell) - B_n(g) = \sum_{j=0}^{n+1} \frac{(-1)^{j+1}[h_\ell(x_j) - B_n(g)(x_j) - (g - B_n(g))(y_j)]q_j}{1 + |q_j(x_j)|} +$$

$$+ \sum_{j=0}^{n+1} \frac{(-1)^{j+1}(g - B_n(g))(y_j)}{1 + |q_j(x_j)|} q_j.$$

But using [1, Lemma 2],

$$\sum_{j=0}^{n+1} \frac{(-1)^{j+1}(g - B_n(g))(y_j)}{1 + |q_j(x_j)|} q_j = -\|g - B_n(g)\| \sum_{j=0}^{n+1} \frac{q_j}{1 + |q_j(x_j)|} \equiv 0.$$

Let $t_j = h_\ell(x_j) - g(y_j) + B_n(g)(y_j) - B_n(g)(x_j)$. Then

$$B_n(h_\ell) - B_n(g) = \sum_{j=0}^{n+1} \frac{(-1)^j t_j}{1 + |q_j(x_j)|} q_j. \tag{2.15}$$

We now claim there is a function $R(j, g)$ with

$$|R(j, g)| \leqq \|g - h_\ell\| \cdot K/\ell \tag{2.16}$$

for some constant K independent of j, ℓ, and g such that

$$(h_\ell - g)(y_j) + R(j, g) \leqq t_j \leqq (h_\ell - g)(x_j) \text{ if } (h_\ell - B_n(h_\ell))(x_j) > 0, \tag{2.17}$$

and

$$(h_\ell - g)(x_j) \leqq t_j \leqq (h_\ell - g)(y_j) + R(j, g) \text{ if } (h_\ell - B_n(h_\ell))(x_j) < 0. \tag{2.18}$$

Since the proof of (2.17) is very similar to a proof appearing in [1, expressions (4.7) through (4.16)], we focus our attention on establishing (2.18). In this case j is odd, so that

$$-(g - B_n(g))(y_j) \geqq -(g - B_n(g))(x_j).$$

Thus

$$t_j = h_\ell(x_j) - g(x_j) + (g - B_n(g))(x_j) - (g - B_n(g))(y_j) \geqq h_\ell(x_j) - g(x_j).$$

Also $(h_\ell - B_n(h_\ell))(x_j) \leqq (h_\ell - B_n(h_\ell))(y_j)$, so

$$\begin{aligned} t_j &= (h_\ell - g)(y_j) + (h_\ell - B_n(h_\ell))(x_j) - (h_\ell - B_n(h_\ell))(y_j) + \\ &+ (B_n(h_\ell) - B_n(g))(x_j) - (B_n(h_\ell) - B_n(g))(y_j) \leqq \\ &\leqq (h_\ell - g)(y_j) + (B_n(h_\ell) - B_n(g))(x_j) - (B_n(h_\ell) - B_n(g))(y_j) = \\ &= (h_\ell - g)(y_j) + R(j,g), \end{aligned}$$

where

$$R(j,g) = (B_n(h_\ell) - B_n(g))(x_j) - (B_n(h_\ell) - B_n(g))(y_j). \tag{2.19}$$

We have established (2.18).

From (2.19),

$$|R(j,g)| = |x_j - y_j| \cdot |(B_n(h_\ell) - B_n(g))'(\xi)|,$$

where ξ is between x_j and y_j. Hence by Markoff's inequality [7, p. 91],

$$|R(j,g)| \leqq (1/\ell)\|(B_n(h_\ell) - B_n(g))'\| \leqq (n^2/\ell)\|B_n(h_\ell) - B_n(g)\|.$$

Now Lemma 2 implies that

$$|R(j,g)| \leqq (n^2/\ell)\mu\|h_\ell - g\| \equiv (K/\ell)\|h_\ell - g\|, \tag{2.20}$$

which establishes (2.16). From (2.16) and either (2.17) or (2.18) we see that

$$|t_j| \leqq (1 + K/\ell)\|g - h_\ell\|. \tag{2.21}$$

Utilizing (2.21) in (2.15) yields

$$\frac{\|B_n(g) - B_n(h_\ell)\|}{\|g - h_\ell\|} \leqq (1 + K/\ell) \left\| \sum_{j=0}^{n+1} \frac{|q_j|}{1 + |q_j(x_j)|} \right\|,$$

and thus (2.13) now implies (2.14). □

The last lemma of this section provides a useful lower bound for $\|g - h_\ell\|$ for functions $g \in H_\ell$.

LEMMA 6. *Let* $g \in H_\ell$. *Then* $\|g - h_\ell\| \geqq \frac{1-1/\ell}{2(1+\hat{\lambda}_n(X_n)(1+K/\ell))}$, *where as in* (2.20), $K = n^2\mu$.

PROOF. From Lemma 3, there is a $\overline{g}_\ell$ and $\overline{x}_\ell$ with $\|\overline{g}_\ell - h_\ell\| < \beta_\ell$,

$$(2.22) \qquad |(\overline{g}_\ell - B_n(\overline{g}_\ell))(\overline{x}_\ell)| \geqq \|\overline{g}_\ell - B_n(\overline{g}_\ell)\| - 1/\ell,$$

and either $\overline{x}_\ell \in X - U_\ell$ or $\overline{x}_\ell \in (x_i - 1/\ell, x_i + 1/\ell) \cap X$ for some i and

$$(2.23) \qquad \operatorname{sgn}(\overline{g}_\ell - B_n(\overline{g}_\ell))(\overline{x}_\ell) = (-1)^{i+1}.$$

We note that $\|\overline{g}_\ell - h_\ell\| < \beta_\ell$ implies that $\overline{g}_\ell - B_n(\overline{g}_\ell)$ has an alternant $\{y_0, y_1, \dots, y_{n+1}\}$ with $y_i \in (x_i - 1/\ell, x_i + 1/\ell) \cap X$ and with $\operatorname{sgn}(\overline{g}_\ell - B_n(\overline{g}_\ell))(y_i) = (-1)^i$, $i = 0, 1, \dots, n+1$. Now

$$\|\overline{g}_\ell - B_n(\overline{g}_\ell)\| \geqq |(\overline{g}_\ell - B_n(\overline{g}_\ell))(x_0)| \geqq |h_\ell(x_0) - B_n(h_\ell)(x_0)| - |\overline{g}_\ell(x_0) - h_\ell(x_0)| -$$
$$- |B_n(h_\ell)(x_0) - B_n(\overline{g}_\ell)(x_0)| \geqq 1 - \|\overline{g}_\ell - h_\ell\| - \|B_n(\overline{g}_\ell) - B_n(h_\ell)\|.$$

The the conclusion of Lemma 5 implies that

$$(2.24) \qquad \|\overline{g}_\ell - B_n(\overline{g}_\ell)\| \geqq 1 - \|\overline{g}_\ell - h_\ell\|(1 + (1 + K/\ell)\hat{\lambda}_n(X_n)).$$

Now if $\overline{x}_\ell \in X - U_\ell$, then by definition, $h_\ell(\overline{x}_\ell) = 0$. Therefore from Lemma 5

$$(2.25) \qquad |(\overline{g}_\ell - B_n(\overline{g}_\ell))(\overline{x}_\ell)| \leqq |\overline{g}_\ell(x_\ell)| + |B_n(\overline{g}_\ell)(\overline{x}_\ell)| =$$
$$= |(\overline{g}_\ell - h_\ell)(\overline{x}_\ell)| + |(B_n(\overline{g}_\ell) - B_n(h_\ell))(\overline{x}_\ell)| \leqq \|\overline{g}_\ell - h_\ell\| + \|B_n(\overline{g}_\ell) - B_n(h_\ell)\| \leqq$$
$$\leqq \|\overline{g}_\ell - h_\ell\|(1 + (1 + K/\ell)\hat{\lambda}_n(X_n)).$$

On the other hand, suppose $\overline{x}_\ell \in (x_i - 1/\ell, x_i + 1/\ell) \cap X$ for some i, $i = 0, 1, \dots, n+1$. Without loss of generality we may assume that i in (2.23) is even, so that $(\overline{g}_\ell - B_n(\overline{g}_\ell))(\overline{x}_\ell) < 0$ and $h_\ell(x_i) - B_n(h_\ell)(x_i) = h_\ell(x_i) = 1$. Now $h_\ell(\overline{x}_\ell) \geqq 0$. Thus if $\overline{g}_\ell(\overline{x}_\ell) < 0$, then $|\overline{g}_\ell(\overline{x}_\ell)| \leqq |\overline{g}_\ell(\overline{x}_\ell) - h_\ell(\overline{x}_\ell)|$. Therefore

$$|(\overline{g}_\ell - B_n(\overline{g}_\ell))(\overline{x}_\ell)| \leqq |\overline{g}_\ell(\overline{x}_\ell)| + |B_n(\overline{g}_\ell)(\overline{x}_\ell)| \leqq$$
$$\leqq |\overline{g}_\ell(\overline{x}_\ell) - h_\ell(\overline{x}_\ell)| + |B_n(\overline{g}_\ell)(\overline{x}_\ell) - B_n(h_\ell)(\overline{x}_\ell)| \leqq$$
$$\leqq \|\overline{g}_\ell - h_\ell\| + \|B_n(\overline{g}_\ell) - B_n(h_\ell)\|.$$

Consequently when $\overline{g}_\ell(\overline{x}_\ell) < 0$, we again obtain (2.25). If $\overline{g}_\ell(\overline{x}_\ell) \geqq 0$, then (2.23) implies that $\overline{g}_\ell(\overline{x}_\ell) < B_n(\overline{g}_\ell)(\overline{x}_\ell)$. Therefore

$$|(\overline{g}_\ell - B_n(\overline{g}_\ell))(\overline{x}_\ell)| \leqq |B_n(\overline{g}_\ell)(\overline{x}_\ell)| \leqq |(\overline{g}_\ell - h_\ell)(\overline{x}_\ell)| + |(B_n(\overline{g}_\ell) - B_n(h_\ell))(\overline{x}_\ell)|.$$

Thus in this last case we also obtain (2.25). Now (2.22) implies in all of the above cases that

$$(2.26) \qquad \|\overline{g}_\ell - B_n(\overline{g}_\ell)\| \leqq (1 + (1 + K/\ell)\hat{\lambda}_n(X_n))\|\overline{g}_\ell - h_\ell\| + 1/\ell.$$

By utilizing (2.24) and (2.26) we see that

$$1-(1+(1+K/\ell)\hat{\lambda}_n(X_n))\|\overline{g}_\ell - h_\ell\| \leqq (1+(1+K/\ell)\hat{\lambda}_n(X_n))\|\overline{g}_\ell - h_\ell\| + 1/\ell.$$

From this inequality we obtain

$$\|\overline{g}_\ell - h_\ell\| \geqq \frac{1-1/\ell}{2(1+(1+K/\ell)\hat{\lambda}_n(X_n))}. \tag{2.27}$$

Now if $g \in H_\ell$, then $\|g - h_\ell\| \geqq \beta_\ell > \|\overline{g}_\ell - h_\ell\|$. This inequality and (2.27) imply the conclusion of Lemma 6. □

3. Theorem

We are finally in a position to prove Theorem 2. The conclusions of Lemmas 5 and 6 will play prominent roles in the proof of the Theorem.

PROOF OF THEOREM 2. For fixed $\ell \geqq \ell_0$, let $g \in C[X]$ satisfy $\|g-h_\ell\| \neq 0$. If

$$\|g - h_\ell\| < \frac{1-1/\ell}{2(1+(1+K/\ell)\hat{\lambda}_n(X_n))},$$

then the contrapositive of Lemma 6 implies that $g - B_n(g)$ has an alternant $\{y_0, y_1, \dots, y_{n+1}\}$, where $y_i \in (x_i - 1/\ell, x_i + 1/\ell) \cap X$ and $\operatorname{sgn}(g - B_n(g))(y_i) = (-1)^i$, $i = 0, 1, \dots, n+1$. In this case Lemma 5 implies that (2.14) holds. Now assume that

$$\|g - h_\ell\| \geqq \frac{1-1/\ell}{2(1+(1+K/\ell)\hat{\lambda}_n(X_n))}. \tag{3.1}$$

We observe that

$$\|g - B_n(g)\| \leqq \|g\| \leqq \|g - h_\ell\| + \|h_\ell\| = \|g - h_\ell\| + 1.$$

Therefore

$$\|B_n(g) - B_n(h_\ell)\| \leqq \|B_n(g) - g\| + \|g - h_\ell\| + \|h_\ell\| \leqq 2(\|g - h_\ell\| + 1). \tag{3.2}$$

For $\ell \geqq \ell_0$, (3.1) and (3.2) imply that

(3.3)
$$\frac{\|B_n(g) - B_n(h_\ell)\|}{\|g - h_\ell\|} \leqq 2\left(1 + \frac{1}{\|g - h_\ell\|}\right) \leqq 2\left(1 + \frac{2(1+(1+K/\ell)\hat{\lambda}_n(X_n))}{1-1/\ell}\right).$$

Thus from (2.14) and (3.3) we have that

(3.4)
$$\lambda_n(h_\ell) \leqq \max\left\{(1+K/\ell)\hat{\lambda}_n(X_n), 2\left(1+\frac{2(1+(1+K/\ell)\hat{\lambda}_n(X_n))}{1-1/\ell}\right)\right\}.$$

The definition of h_ℓ, (2.2), and (3.4) combine to imply that

(3.5)
$$L_n(f) \leqq \max\left\{(1+K/\ell)\hat{\lambda}_n(X_n), 2\left(1+\frac{2(1+(1+K/\ell)\hat{\lambda}_n(X_n))}{1-1/\ell}\right)\right\}.$$

Letting $\ell \to \infty$ in (3.5) yields

$$L_n(f) \leqq 6 + 4\hat{\lambda}_n(X_n). \tag{3.6}$$

To establish the lower bound, let

$$g \in V \equiv \{h \in C[X] : E_n(h) = X_n \text{ and } h(x_i) = (-1)^i,\ i = 0, 1, \dots, n+1\}.$$

Clearly for all $g \in V$, $\hat{\lambda}_n(g) \leqq \lambda_n(g)$. But since for any $g \in V$, $E_n(g) = X_n$, $\hat{\lambda}_n(g) = \hat{\lambda}_n(X_n)$. Thus $\hat{\lambda}_n(X_n) \leqq \lambda_n(g)$ for all $g \in V$. Therefore

$$\hat{\lambda}_n(X_n) \leqq \inf\{\lambda_n(g) : g \in V\} = L_n(f). \tag{3.7}$$

Inequalities (3.6) and (3.7) imply the conclusion (2.3) of Theorem 2. □

COROLLARY 1. *Let $f \in C[X]$, and suppose that $E_n(f) = X_n$. Then*

$$\hat{\lambda}_n(X_n) \leqq L_n(f) \leqq 10\hat{\lambda}_n(X_n). \tag{3.8}$$

PROOF. Inequality (3.8) follows immediately from (2.3) and the observation that $\hat{\lambda}_n(X_n) \geqq 1$. □

We conclude this paper with some other observations. If X is dense in I, the results of Theorem 2 and Corollary 1 can be sharpened. In particular, inequality (2.3) can be replaced by the inequality

$$\hat{\lambda}_n(X_n) \leqq L_n(f) \leqq 4 + 2\hat{\lambda}_n(X_n). \tag{3.9}$$

The proof of (3.9) uses much of the machinery developed in Section 2, as well as some constructions that depend on X being dense in an interval.

We also note that although the first inequality in (3.8) can be an equality (as in the case where $X = X_n$), it can also be a strict inequality. To see this, observe that if $n = 1$ and $X_n = \{-1, 0, 1\}$, then direct computation using (2.12) and (2.13) gives $\hat{\lambda}_n(X_n) = 3/2$, but if $X = [-1, 1]$ then for every $n \geqq 1$ we have $\lambda_n(h) \geqq 2$ for all $h \in C[-1,1]$, which implies $L_n(f) \geqq 2$ from (1.8). The statement $\lambda_n(h) \geqq 2$ for all $h \in C[-1,1]$, $n \geqq 1$, follows from the fact that $\|B_n(h) - B_n(g_m)\|/\|h - g_m\|$ can be made arbitrarily close to 2 by choosing m large, where $g_m(-1 + i/m) = i + (-1)^{n-i}m$ for $i = 0, \dots, n$, $g_m(1) = m$, and g_m is linear in between these points; note that $E_n(g_m) = \{-1, -1 + 1/m, \dots, -1 + n/m, 1\}$, $B_n(g_m)(x) = m(x+1)$, $\|B_n(g_m)\| = 2m$, and $\|g_m\| = m + n$. It can also be derived from the results in [5].

Let $f(x) = e^x$, $x \in I$. Then it can be shown [3, 10, 11] that $|E_n(f)| = n + 2$ and that $\hat{\lambda}_n(X_n)/M_n(X_n) = O\left(\frac{\log(n+1)}{n+1}\right)$. Thus Theorems 1 and 2 imply that $\lim\limits_{n\to\infty} \frac{L_n(f)}{G_n(f)} = 0$ when $f(x) = e^x$. Hence in an asymptotic sense, the ELLC and EGLC can be very different. It would be of interest to find functions $f \in C[X] - \Pi_n$ for which either $\left\{\frac{\lambda_n(f)}{L_n(f)}\right\}_{n=0}^{\infty}$ is bounded above by a constant not depending on n, or for which $\left\{\frac{G_n(f)}{\lambda_n(f)}\right\}_{n=0}^{\infty}$ is bounded above by a constant not depending on n.

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DEPARTMENT OF MATHEMATICS
CENTRAL MICHIGAN UNIVERSITY
MOUNT PLEASANT, MICHIGAN 48859
USA

DEPARTMENT OF MATHEMATICS AND STATISTICS
OLD DOMINION UNIVERSITY
NORFOLK, VIRGINIA 23529
USA

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STRONG NEGATIVE PARTITION RELATIONS BELOW THE CONTINUUM

S. SHELAH (Jerusalem)[1]

0. Introduction

DEFINITION 1. If λ is a cardinal, $\mathrm{Pr}^+(\lambda)$ means that there is a function $c\colon [\lambda]^2 \to \lambda$ such that if $1 \leqq n < \omega$ and the sets $\{\zeta_\alpha^0, \dots, \zeta_\alpha^{n-1}\}$ are disjoint for $\alpha < \lambda$ and $\zeta_\alpha^0 < \dots < \zeta_\alpha^{n-1}$ then for every $h\colon n \times n \to \lambda$ there are $\alpha < \beta$ such that $c(\zeta_\alpha^i, \zeta_\beta^j) = h(i,j)$ for $i, j < n$.

DEFINITION 2. $\mathrm{Pr}(\lambda)$ is the same but only for every $h\colon n \times n \to \lambda$ with h constant, i.e. $h(i,j) = \gamma$ for $i, j < n$.

LEMMA 1. *If λ is regular, not strong limit, then* $\mathrm{Pr}(\lambda)$ *implies* $\mathrm{Pr}^+(\lambda)$.

PROOF. We use the idea in the proof of the Engelking–Karlowitz theorem. Assume that $\mu < \lambda$ and $2^\mu \geqq \lambda$. Let $\{A_\alpha\colon \alpha < \lambda\}$ be different subsets of μ. Assume that c^- witnesses $\mathrm{Pr}(\lambda)$. Put $G = \{\langle w, g\rangle : w \in [\mu]^{<\omega}, g\colon P(w)^2 \to \lambda\}$. Clearly, $|G| = \lambda$, so we can enumerate it as $\{\langle w_\alpha, g_\alpha\rangle : \alpha < < \lambda\}$. Now put $c(\alpha,\beta) = g_\gamma(A_\alpha \cap w_\gamma, A_\beta \cap w_\gamma)$, where $\gamma = c^-(\alpha,\beta)$.

Assume that $\{\zeta_\alpha^i\colon i < n, \alpha < \lambda\}$ are given as in Definition 1, $h\colon n{\times}n \to \lambda$. For $\alpha < \lambda$, $i < j < n$, pick $\gamma_\alpha^{i,j} \in A_{\zeta_\alpha^i} \triangle A_{\zeta_\alpha^j}$, and let $w^\alpha = \{\gamma_\alpha^{i,j}\colon i < j < n\}$. As $w^\alpha \subseteqq \mu < \lambda$, we may assume that there exist w, $B_i \subseteqq w$ $(i < n)$, such that $w^\alpha = w$, $A_{\zeta_\alpha^i} \cap w = B_i$ for $\alpha < \lambda$. Let $g\colon P(w)^2 \to \lambda$ be a function satisfying $g(B_i, B_j) = h(i,j)$. There is a $\gamma < \lambda$ with $\langle w, g\rangle = \langle w_\gamma, g_\gamma\rangle$, and by $\mathrm{Pr}(\lambda)$ there are $\alpha < \beta < \lambda$ such that if $i < j < n$, then $c^-(\zeta_\alpha^i, \zeta_\beta^j) = \gamma$. But then $c(\zeta_\alpha^i, \zeta_\beta^j) = g_\gamma(A_{\zeta_\alpha^i} \cap w_\gamma, A_{\zeta_\beta^j} \cap w_\gamma) = g(B_i, B_j) = h(i,j)$, and we are done.

We now state the main result of this paper. We remind the reader that $S \subsetneqq \lambda$ is a non-reflecting stationary set if it is stationary and $S \cap \alpha$ is non-stationary in α for every limit $\alpha < \lambda$.

THEOREM. $\mathrm{Pr}(\lambda)$ *holds whenever there exists a nonreflecting stationary set S in λ with $\mathrm{cf}(\alpha) > \omega_1$ for every $\alpha \in S$.*

This work is continued in [10] (see also [11]).

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1. Construction of the coloring

For $\alpha < \lambda$ limit let $C_\alpha \subseteqq \alpha$ be a closed unbounded set of order type $\mathrm{cf}(\alpha)$ disjoint from S. For $\alpha = \beta + 1$ we let $C_\alpha = \{\beta\}$. For $0 < \alpha < \beta < \lambda$ let $\gamma(\beta, \alpha) = \min(C_\beta - \alpha)$. Obviously, $\alpha \leqq \gamma(\beta, \alpha) < \beta$. We now define $\gamma_\ell(\beta, \alpha)$ for $\ell \leqq k(\beta, \alpha)$ as follows: $\gamma_0(\beta, \alpha) = \beta$, $\gamma_{\ell+1}(\beta, \alpha) = \gamma(\gamma_\ell(\beta, \alpha), \alpha)$. If $\gamma_\ell(\beta, \alpha) = \alpha$ then we terminate the definition and put $k = k(\beta, \alpha) = \ell$. Clearly, $\alpha = \gamma_k(\beta, \alpha) < \ldots < \gamma_0(\beta, \alpha) = \beta$. The string $\varrho(\beta, \alpha) = \langle \gamma_0(\beta, \alpha), \ldots, \gamma_k(\beta, \alpha) \rangle$ is the Todorcevic walk from β to α.

Fix a decomposition $S = \bigcup\{S^\gamma : \gamma < \lambda\}$ into stationary sets (possible, by Solovay's theorem). Let $H\colon \lambda \to \omega_1$ be a mapping such that for every $i < \omega_1$ the set $S_i = S \cap H^{-1}(\{i\})$ is stationary in λ. Let $\omega_1 = \bigcup\{R_n : n < \omega\}$ be a partition into stationary sets. For $0 < \alpha < \beta < \lambda$ we let

$$w_1(\beta, \alpha) = \{p > k/2\colon \text{ for every } q < k/2, H(\gamma_p) > H(\gamma_q)\}$$

and $p_1 = \min(w_1)$. Here and in several cases later, we omit (β, α) after w_1, p_1, k etc. if it is obvious what we are speaking of. We now define

$$w_2 = \left\{ q < \frac{k}{2}\colon \text{ for every } \frac{k}{2} < p \leqq k, p \notin w_1 \text{ implies } H(\gamma_q) > H(\gamma_p) \right\}.$$

Let p_2 be such that $\min\{H(\gamma_q) : q \in w_2\} \in R_{p_2}$. Now if $0 \leqq p_1 - p_2 \leqq k$ and $\gamma_{p_1 - p_2}(\beta, \alpha) \in S^\gamma$ we put $c(\beta, \alpha) = \gamma$ otherwise $c(\beta, \alpha)$ is chosen arbitrarily.

2. Preliminaries

DEFINITION 3. If $s_1 = \langle s_1(0), \ldots, s_1(t_1) \rangle$, $s_2 = \langle s_2(0), \ldots, s_2(t_2) \rangle$ are strings, their *concatenation* $s_1 \wedge s_2$ is $\langle s_1(0), \ldots s_1(t_1 - 1), s_2(0), \ldots, s_2(t_2) \rangle$.

The reason why we are removing the border element is that in our applications $s_1(t_1) = s_2(0)$ holds, so we only remove an immediate repetition.

LEMMA 2. *If $\delta \in S$, $\beta > \delta$ then there exists a $\chi(\beta, \delta) < \delta$ such that for every α with $\chi(\beta, \delta) \leqq \alpha < \delta$, $\varrho(\beta, \delta)$ is an initial segment of $\varrho(\beta, \alpha)$. Moreover, $\varrho(\beta, \alpha) = \varrho(\beta, \delta) \wedge \varrho(\delta, \alpha)$.*

PROOF. If $\alpha < \delta$ is large enough, $\gamma(\beta, \alpha) = \gamma(\beta, \delta)$. Therefore, if $\alpha \geqq \chi(\gamma(\beta, \delta), \delta)$ also holds, the statement is true. We get, therefore, a proof by induction on β.

LEMMA 3. *If $A, B \in [\lambda]^\lambda$, $k < \omega$, then there exist $\alpha \in A$, $\beta \in B$, $\alpha < \beta$ with $k(\beta, \alpha) > k$.*

PROOF. We define $C_0 = A'$, and by induction, $C_{i+1} = (S \cap C_i)'$. Pick $\gamma_k \in C_k \cap S$, then $\beta \in B$ with $\beta > \gamma_k$, $\chi_k = \chi(\beta, \gamma_k)$. If γ_{i+1}, χ_{i+1} are found, pick $\gamma_i \in S \cap C_i$ with $\chi_{i+1} < \gamma_i < \gamma_{i+1}$ and χ_i with $\chi_i > \chi(\gamma_{i+1}, \gamma_i)$,

$\chi_{i+1} < \chi_i < \gamma_i$. Given γ_0, χ_0 let $\alpha \in A$ satisfy $\chi_0 < \alpha < \gamma_0$, then by Lemma 2, for $\ell \leqq k$ there exists an $m \leqq k(\beta, \alpha)$ such that $\gamma_m(\beta, \alpha) = \gamma_\ell$, so $k(\beta, \alpha) > k$.

DEFINITION 4. $\varrho_H(\beta, \alpha) = \langle H(\gamma_\ell(\beta, \alpha)) \colon \ell \leqq k(\beta, \alpha) \rangle$. If $\sigma \in \omega_1^{<\omega}$, i.e. is a finite string of countable ordinals, then for $i < \omega_1$ σ^i is the following string $|\sigma^i| = |\sigma|$, and

$$\sigma^i(\ell) = \begin{cases} \sigma(\ell) & \text{if } \sigma(\ell) < i, \\ \omega_1 & \text{if } \sigma(\ell) \geqq i. \end{cases}$$

DEFINITION 5. If $T \subseteqq \lambda$, $\delta < \lambda$, $R \subseteqq \omega_1$ stationary, then $U(\delta, T, R)$ denotes the set of those $\varrho \in (\omega_1+1)^{<\omega} - \omega_1^{<\omega}$ such that for every $i < \omega_1$ there exists a $\beta > \delta$, $\beta \in T$ with $\varrho_H(\beta, \delta)^i = \varrho$ and $\min\{\varrho_H(\ell) : \varrho^i(\ell) = \omega_1\} \in R$. $\varrho \in U(\delta, T, R, \chi)$ denotes that β even satisfies $\chi(\beta, \delta) < \chi$.

LEMMA 4. *If $T \in [\lambda]^\lambda$, then there is a $\delta(T) < \lambda$ such that for $\delta(T) \leqq \delta < \lambda$, $U(\delta, T, R) \neq \emptyset$. If* $\mathrm{cf}(\delta) > \omega_1$, *then there is a $\chi < \delta$ such that $U(\delta, T, R, \chi) \neq \emptyset$.*

PROOF. For $i < \omega_1$ we let $A_i = \{\delta < \lambda\colon$ if $\beta > \delta, \beta \in T$, then $i \notin \varrho_H(\beta, \delta)\}$.

CLAIM. $|A_i| < \lambda$ *for* $i < \omega_1$.

PROOF OF CLAIM. Suppose that $|A_i| = \lambda$ for some $i < \omega_1$ and select a $\delta \in S_i \cap A_i'$, $\beta \in T$ with $\beta > \delta$. Choose an $\alpha \in A_i$, $\chi(\beta, \delta) < \alpha < \delta$. Then $\delta \in \varrho(\beta, \alpha)$, and $i = H(\delta) \in \varrho_H(\beta, \alpha)$, a contradiction.

Now we define $\delta(T)$ with $\cup\{A_i : i < \omega_1\} \subseteqq \delta(T)$. Assume that $\delta(T) \leqq \delta < \lambda$. For every $i < \omega_1$, there is a $\beta_i > \delta$, $\beta_i \in T$ such that $i \in \varrho_H(\beta_i, \delta)$.

Consider $\{\varrho_H(\beta_i, \delta) : i \in R\}$. There exist a stationary $R_1 \subseteqq R$ and a $k < \omega$ such that for $i \in R_1$, $|\varrho_i| = k$, where $\varrho_i = \varrho_H(\beta_i, \delta)$. We even assume that for every $\ell < k$ either for every $i \in R_1$ $\varrho_i(\ell) < i$ or for every $i \in R_1$ $\varrho_i(\ell) \geqq i$. Applying Fodor's theorem we can find a stationary $R_2 \subseteqq R_1$ and an $\eta \in (\omega_1 + 1)^{<\omega} - \omega_1^{<\omega}$ such that $\varrho_H(\beta_i, \delta)^i = \eta$ $(i \in R_2)$. For $i \in R_2$, $\min\{\varrho^i(\ell) : \eta(\ell) = \omega_1\} = i \in R_2 \subseteqq R$, so $\eta \in U(\delta, T, R)$.

If $\mathrm{cf}(\delta) > \omega_1$, $\{\chi(\beta_i, \delta) : i \in R_2\}$ is bounded below δ, so $\eta \in U(\delta, T, R, \lambda)$, if $\lambda > \lambda(\beta_i, \delta)$ $(i \in R_2)$.

DEFINITION 6. If $T \subseteqq \lambda$, $\delta < \lambda$, then $L(\delta, T)$ consists of those $\varrho \in (\omega_1 + 1)^{<\omega} - \omega_1^{<\omega}$ for which for every $\alpha < \delta$, and large enough $i < \omega_1$ there is a $\beta \in T$, $\alpha < \beta < \delta$ such that $\varrho_H(\delta, \beta)^i = \varrho$. For $T \in [\lambda]^\lambda$ we let $C(T) = \bigcap\{(S_i \cap T')' : i < \omega_1\}$.

Obviously, $C(T)$ is closed unbounded in

LEMMA 5. *If $\delta \in C(T)$,* $\mathrm{cf}(\delta) \geqq \omega_1$, *then $L(\delta, T) \neq \emptyset$.*

PROOF. *Case 1:* $\mathrm{cf}(\delta) = \omega_1$. Let $\{\delta_i : i < \omega_1\}$ converge to δ. For $i < \omega_1$ pick an $\alpha_i \in S_i \cap T'$, $\delta_i < \alpha_i < \delta$ (possible, as $\delta \in C(T)$). Now choose $\beta_i \in T$, $\delta_i < \beta_i < \alpha_i$ with $\chi(\delta, \alpha_i) < \beta_i$. Then $i = H(\alpha_i) \in \varrho_H(\delta, \beta_i)$.

As in Lemma 4, there is a stationary $X \subseteqq \omega_1$ and a $\varrho \in (\omega_1+1)^{<\omega} - \omega_1^{<\omega}$ such that $\varrho_H(\delta,\beta_i)^i = \varrho$ $(i \in X)$, so $\varrho \in L(\delta, T)$.

Case 2: $\mathrm{cf}(\delta) > \omega_1$. Let $\{\delta_\alpha : \alpha < \mathrm{cf}(\delta)\}$ converge to δ. For $\alpha < \mathrm{cf}(\delta)$, $i < \omega_1$, pick $\beta_i^\alpha \in T$ with $\delta_\alpha \leqq \beta_i^\alpha < \delta$ as in Case 1. For $\alpha < \mathrm{cf}(\delta)$, there is a $\varrho^\alpha \in (\omega_1+1)^{<\omega} - \omega_1^{<\omega}$ such that there exist an $X^\alpha \in [\omega_1]^{\omega_1}$ with $\varrho_H(\delta,\beta_i^\alpha)^i = \varrho^\alpha$ for $i \in X^\alpha$. There is a ϱ with $\varrho^\alpha = \varrho$ for $\mathrm{cf}(\delta)$ many α's. Clearly, $\varrho \in L(\delta, T)$.

3. Proof of the theorem

Assume that the sets $\{\zeta_\alpha^0, \dots, \zeta_\alpha^{n-1}\}$ are disjoint $(\alpha < \lambda, n < \omega)$. We may assume that $\alpha < \zeta_\alpha^0 < \zeta_\alpha^1 < \dots < \zeta_\alpha^{n-1}$. There is a closed unbounded set $C \subseteqq \lambda$ such that if $\alpha < \delta$, $\delta \in C$, then $\zeta_\alpha^{n-1} < \delta$.

For $\delta \in S \cap C$, as $\mathrm{cf}(\delta) > \omega_1$, there are $\{\nu_\ell^\delta : \ell < n\}$ such that $\sup\{\alpha < \delta : \varrho_H(\delta, \zeta_\alpha^\ell) = \nu_\ell^\delta\} = \delta$. For a stationary $T_1 \subseteqq S \cap C$, $\nu_\ell^\delta = \nu_\ell$ $(\delta \in T_1)$. By Lemma 5, for $\delta \in S \cap C(T_1)$, $L(\delta, T_1) \neq \emptyset$, so there is a stationary $T_2 \subseteqq S \cap C(T_1)$, and $\tau \in (\omega_1+1)^{<\omega} - \omega_1^{<\omega}$ such that $\tau \in L(\delta, T_1)$ for $\delta \in T_2$. We put $\ell^* = \min\{\ell : \tau(\ell) = \omega_1\}$. Again, by Lemma 5, for $\delta \in S \cap C(T_2)$, $L(\delta, T_2) \neq \emptyset$, so there is a stationary $T_3 \subseteqq S^\gamma \cap C(T_2)$, and ϱ with $\varrho \in L(\delta, T_2)$ $(\delta \in T_3)$.

Since $\lambda > \omega_1$, there is a stationary $T^1 \subseteqq S$ and $\{\nu^\ell : \ell < n\}$ such that $\varrho_H(\zeta_\delta^\ell, \delta) = \nu^\ell$ $(\delta \in T^1)$. By Fodor's theorem, there is a $T^2 \subseteqq T^1$, and $\chi^2 < \lambda$, with $\chi(\zeta_\delta^\ell, \delta) < \chi^2$ for $\delta \in T^2$. By Lemma 4, if $\delta \in S - \delta(T^2)$, then there is a $\chi < \delta$ such that $U(\delta, T^2, R_{\ell^*+|\varrho|}, \chi) \neq \emptyset$, so there are $\eta, \chi^3 > \chi^2$, and $T^3 \subseteqq S - \delta(T^2)$ stationary with $\eta \in U(\delta, T^2, R_{\ell^*+|\varrho|}, \chi^3)$ $(\delta \in T^3)$.

We now apply Lemma 2 with $A = T_3 - (\chi^3+1)$, $B = T^3$ to get a $\beta_3 \in T_3 - (\chi^3+1)$, and $\beta^3 \in T^3$ such that $\beta^3 > \beta_3$ and

$$k(\beta^3, \beta_3) > \max\{|\nu_\ell| : \ell < n\} + |\tau| + |\varrho| + |\eta| + \max\{|\nu^\ell| : \ell < n\}.$$

Choose $i_0 < \omega_1$ which is larger than every countable ordinal in $\varrho_H(\beta^3, \beta_3)$, η, ν^ℓ, ν_ℓ $(\ell < n)$. Since $\varrho \in L(\beta_3, T_2)$, there is a $\beta_2 \in T_2$ with $\chi^3 < \beta_2 < \beta_3$, $\chi(\beta^3, \beta_3) < \beta_2$ such that $\varrho_H(\beta_3, \beta_2)^{i_0} = \varrho$. Pick a χ_2 with $\chi^3 < \chi_2 < \beta_2$, $\chi(\beta^3, \beta_3) < \chi_2$ such that $\chi(\beta_3, \beta_2) < \chi_2$.

Next fix an $i_1 < \omega_1$ which is larger than the ordinals in $\varrho_H(\beta_3, \beta_2)$ and i_0. Then, as $\beta^3 \in T^3$ and $\eta \in U(\beta^3, T^2, R_{\ell^*+|\varrho|}, \chi^3)$, there exists a $\beta \in T^2$, $\beta > \beta^3$ with $\varrho_H(\beta, \beta^3)^{i_1} = \eta$ and $\chi(\beta, \beta^3) < \chi^3$. Since $\beta \in T^2$ we have $\varrho_H(\zeta_\beta^\ell, \beta) = \nu^\ell$ and $\chi(\zeta_b^\ell, \beta) < \chi^3$ $(\ell < n)$.

Finally, choose $i_2 < \omega_1$ which is larger than the countable ordinals in $\varrho_H(\beta, \beta^3)$ and i and use $\tau \in L(\beta_2, T_1)$ to find $\beta_1 \in T_1$ with $\chi_2 < \beta_1 < \beta_2$, $\varrho_H(\beta_2, \beta_1)^{i_2} = \tau$. Also, fix $\chi_1 > \chi(\beta_2, \beta_1)$, $\chi_2 < \chi_1 < \beta_1$. Since $\beta_1 \in T_1$, there is an α, $\chi_1 < \alpha < \beta_1$, such that for $\ell < n$, $\varrho_H(\beta_1, \zeta_\alpha^\ell) = \nu_\ell$.

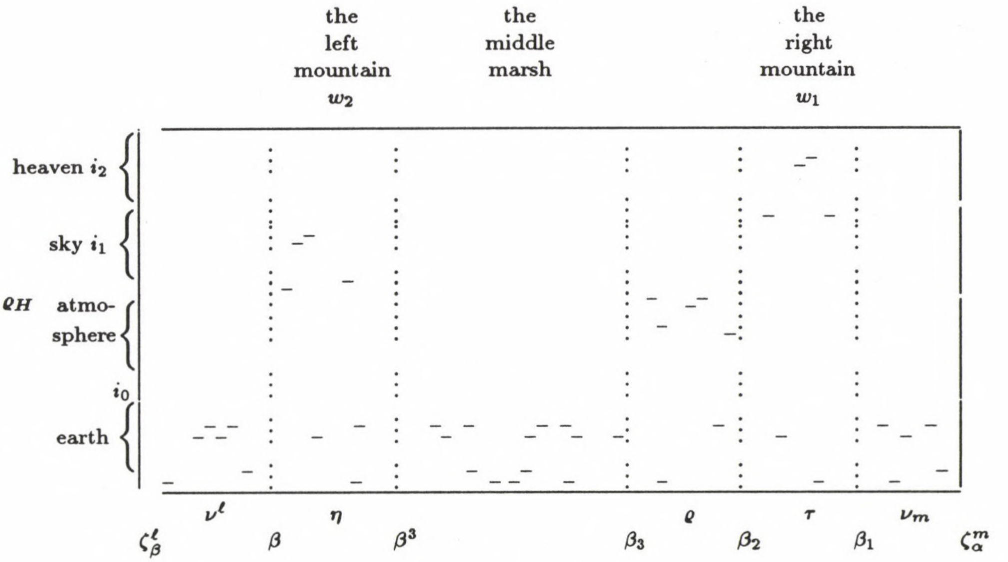

Fig. 1. The sequence $\varrho_H(\zeta^\ell_\beta, \zeta^m_\alpha)$.

Now, by Lemma 2, as $\alpha < \zeta^m_\alpha$,

$$\chi(\zeta^\ell_\beta, \beta) < \chi^3 < \chi_2 < \chi_1 < \alpha \text{ implies } \varrho(\zeta^\ell_\beta, \zeta^m_\alpha) = \varrho(\zeta^\ell_\beta, \beta) \wedge \varrho(\beta, \zeta^m_\alpha);$$
$$\chi(\beta, \beta^3) < \chi^3 < \alpha \text{ implies } \varrho(\beta, \zeta^m_\alpha) = \varrho(\beta, \beta^3) \wedge \varrho(\beta^3, \zeta^m_\alpha);$$
$$\chi(\beta^3, \beta_3) < \chi_2 < \alpha \text{ implies } \varrho(\beta^3, \zeta^m_\alpha) = \varrho(\beta^3, \beta_3) \wedge \varrho(\beta_3, \zeta^m_\alpha);$$
$$\chi(\beta_3, \beta_2) < \chi_2 < \alpha \text{ implies } \varrho(\beta_3, \zeta^m_\alpha) = \varrho(\beta_3, \beta_2) \wedge \varrho(\beta_2, \zeta^m_\alpha);$$
$$\chi(\beta_2, \beta_1) < \chi_1 < \alpha \text{ implies } \varrho(\beta_2, \zeta^m_\alpha) = \varrho(\beta_2, \beta_1) \wedge \varrho(\beta_1, \zeta^m_\alpha),$$

i.e.

$$\varrho(\zeta^\ell_\beta, \zeta^m_\alpha) = \varrho(\zeta^\ell_0, \beta) \wedge \varrho(\beta, \beta^3) \wedge \varrho(\beta^3, \beta_3) \wedge \varrho(\beta_3, \beta_2) \wedge \varrho(\beta_2, \beta_1) \wedge \varrho(\beta_1, \zeta^m_\alpha).$$

A similar identity holds for ϱ_H.

Now it is obvious that the middle, i.e. the $k(\zeta^\ell_\beta, \zeta^m_\alpha)/2$-th element of the string lies in the $\varrho(\beta^3, \beta_3)$ portion — selected to be so long for this purpose. By the respective selections of i_1, i_2 the largest ϱ_H value of the first half of the string is at least i_1 but less than i_2. It follows that $w_1(\zeta^\ell_\beta, \zeta^m_\alpha)$ consists of those indices p in the $\varrho(\beta_2, \beta_1)$ portion where $\varrho_H(\beta_2, \beta_1)(p) \geqq i_2$, so, in particular, $p_1 = s + |\varrho| + \ell^*$ where $s = |\varrho(\zeta^\ell_\beta, \beta_3)|$. $w_2(\zeta^\ell_\beta, \zeta^m_\alpha)$ then consists of those indices q in the $\varrho(\beta, \beta^3)$ portion where $\varrho_H(\beta, \beta^3)(q) \geqq i_1$. By the choices of η and $\varrho(\beta, \beta^3)$ we have that the minimum of $\{H(\gamma_q) : q \in w_2\}$ is in $R_{\ell^*+|\varrho|}$, i.e. $p_2 = \ell^* + |\varrho|$. From this, $\gamma_{p_1-p_2} = \gamma_s = \beta_3 \in S^\gamma$, so $c(\zeta^\ell_\beta, \zeta^m_\alpha) = \gamma$, as required.

4. Corollaries

COROLLARY. *If $\kappa > \omega_1$ is regular, then*
(a) $\mathrm{Pr}^+(\kappa^+)$ *holds;*
(b) κ^+*-c.c.-ness is not a productive property of Boolean algebras;*
(c) *there is a* κ^+*-separable not* κ^+*-Lindelöf Hausdorff-space;*
(d) *there is a* κ^+*-Lindelöf not* κ^+*-separable Hausdorff-space.*

PROOF. (a) From the Theorem and Lemma 1.
(b) See [6].
(c)–(d) See [1].

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THE HEBREW UNIVERSITY
INSTITUTE OF MATHEMATICS
JERUSALEM, ISRAEL

RUTGERS UNIVERSITY
DEPARTMENT OF MATHEMATICS
NEW BRUNSWICK, NJ
U.S.A.

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ϕ-ORTHOGONALLY ADDITIVE MAPPINGS. I

Gy. SZABÓ (Debrecen)

1. Introduction

The representation of orthogonally additive functionals on the Hilbert-space $L^2[0,1]$ has been studied by Pinsker [2]. In the recent decades several other authors have dealt with the same problem on vector spaces using some kind of bilinear forms for defining orthogonality. Thus in [8], Vajzović considered the A-orthogonality $\perp^A$ on a (real or complex) Hilbert-space H, i.e. $x \perp^A y \iff \langle Ax, y\rangle = 0$, where $A\colon H \to H$ is a continuous selfadjoint operator. He gave the general form of the continuous A-orthogonally additive functionals. Also, Fochi [1] has studied the same question for non-continuous functionals, proving the odd solutions to be additive, while the even ones to be quadratic. Sundaresan and Kapoor [7] defined the T-orthogonality $\perp^T$ on a real Hausdorff topological vector space E with the aid of a (non-continuous) linear mapping $T\colon E \to E^*$, by $x \perp^T y \iff Tx(y) = 0$. They described in (almost) full details the class of all continuous T-orthogonally additive functionals.

A different approach was given by Rätz [3] for arbitrary orthogonally additive mappings from a real inner product space X with the ordinary orthogonality, into an abelian group Y. Later in [4], he turned these results into a more general context, namely for ϕ-orthogonality $\perp^\phi$ on a vector space X over a euclidean ordered field, i.e. $x \perp^\phi y \iff \phi(x,y) = 0$, where ϕ is a non-isotropic bilinear functional.

In the present work we offer a common generalization of the above mentioned results into three directions:

1) we allow vector spaces over a quite arbitrary field rather than over **R** or **C**;

2) we use orthogonality based on an arbitrary sesquilinear form with respect to an automorphism of the field;

3) we study arbitrary orthogonally additive mappings with values in an abelian group.

This is the first part of our investigations in which we consider the case of symmetric orthogonality. The non-symmetric case and other related topics will be dealt with in some forthcoming papers. Here we can apply the abstract theory of orthogonally additive mappings developed in [5], thus we use the same notation and terminology. Namely, throughout the paper, Φ will denote a field of char $\Phi \neq 2$, X a Φ-vector space with $\dim_\Phi X \geqq 2$ and

$(Y, +)$ an abelian group. Also $\mathcal{P}$ or $\operatorname{lin} V$ stand for the family of all 2-dimensional linear subspaces of X or the linear hull of $V \subset X$, respectively. Assuming that $\perp$ is a binary relation (called *orthogonality*) on X, for $P \in \mathcal{P}$ let $\perp_P$ denote the set of all $(u, v) \in \perp$ such that $\operatorname{lin}\{u, v\} = P$ (the set of all *orthogonal bases* in P). The mappings A, Q and $F \colon X \to Y$ are said to be *additive, quadratic* or *orthogonally additive* ($\perp$*-additive*), if they satisfy the equations:

$$\begin{aligned} A(x+y) &= A(x) + A(y), && x, y \in X, \\ Q(x+y) + Q(x-y) &= 2Q(x) + 2Q(y), && x, y \in X, \text{ or} \\ F(x+y) &= F(x) + F(y), && x, y \in X,\ x \perp y, \end{aligned}$$

respectively. We shall use the notation:

$$\begin{aligned} \operatorname{Hom}(X, Y) &= \{A \colon X \to Y \mid A \text{ is additive}\}, \\ \operatorname{Quad}(X, Y) &= \{Q \colon X \to Y \mid Q \text{ is quadratic}\}, \\ \operatorname{Hom}_\perp(X, Y) &= \{F \colon X \to Y \mid F \text{ is } \perp\text{-additive}\}, \\ (\mathrm{o})\operatorname{Hom}_\perp(X, Y) &= \{D \colon X \to Y \mid D \text{ is odd and } \perp\text{-additive}\}, \\ (\mathrm{e})\operatorname{Hom}_\perp(X, Y) &= \{E \colon X \to Y \mid E \text{ is even and } \perp\text{-additive}\}. \end{aligned}$$

Finally, $\mathbf{R}$ is the real line, $\mathbf{C}$ is the complex field, 0 denotes the scalar zero, the zero vector as well as the identity element of the group Y. The actual meaning of 0 always will be clear from the context. The sign $\mathbf{O}$ stands for the constant zero mapping.

Now we remind the reader of some useful concepts and results known for an abstract orthogonality $\perp$ on a Φ-vector space X:

DEFINITION 1.1 ([5], Definition 1.2). a) We say $P \in \mathcal{P}$ to be a $\perp$*-normal plane,* if there are $(u_i, v_i) \in \perp_P$ $(i = 1, 2)$ with

$$\bigcap_{i=1}^{2} (\operatorname{lin}\{u_i\} \cup \operatorname{lin}\{v_i\}) = \{0\}.$$

The subfamily of all $\perp$-normal planes in $\mathcal{P}$ will be denoted by $\mathcal{P}_n$.

b) The vector $x \in X$ is said to be a

- τ_0*-element,* if $x \in \operatorname{lin}\{u, v\}$ for some $u, v \in X$ such that ($x \perp u$ or $u \perp x$) and ($x \perp v$ or $v \perp x$);
- τ_1*-element,* if it is contained in a $\perp$-normal plane: $x \in P \in \mathcal{P}_n$;
- τ*-element,* if it is a τ_0- or τ_1-element.

Let X_0, X_1 or X_τ denote the set of all τ_0-, τ_1- or τ-elements in X, respectively.

c) We consider the following subfamilies in $\mathcal{P}$:

$$\begin{aligned} \mathcal{P}_0 &= \{P \in \mathcal{P} \mid \perp_P \cap (X_0 \times X_0) \neq \emptyset\}, \\ \mathcal{P}_1 &= \{P \in \mathcal{P} \mid \perp_P \cap (X_0 \times X_1 \cup X_1 \times X_0) \neq \emptyset\}, \\ \mathcal{P}_s &= \{P \in \mathcal{P} \mid \perp_P \cap (X_\tau \times X_\tau) \neq \emptyset\}, \\ \mathcal{P}'_s &= \mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_n. \end{aligned}$$

Clearly, $\mathcal{P}'_s \subset \mathcal{P}_s$.

d) Finally, consider the axioms

(O3) $x, y \in X$, $x \perp y$, $\alpha, \beta \in \Phi \Longrightarrow \alpha x \perp \beta y$ (homogeneity);

(O8) $\mathcal{P} = \mathcal{P}_s$;

(O8') $\mathcal{P} = \mathcal{P}'_s$.

Obviously, (O8')$\Longrightarrow$(O8).

THEOREM 1.2 ([5], Theorem 2.7). *If* $(X, \perp)$ *satisfies axioms* (O3) *and* (O8) *(or even more* (O8')*), then*

i) (o)$\mathrm{Hom}_\perp(X, Y) = \mathrm{Hom}(X, Y)$;

ii) (e)$\mathrm{Hom}_\perp(X, Y) \subset \mathrm{Quad}(X, Y)$;

iii) $\mathrm{Hom}_\perp(X, Y) = \mathrm{Hom}(X, Y) \Longleftrightarrow$ (e)$\mathrm{Hom}_\perp(X, Y) = \{\mathbf{0}\}$.

2. The symmetric ϕ-orthogonality

In this section we examine the properties of a ϕ-orthogonality relation showing that it satisfies axioms (O3) and (O8') under some natural assumptions.

DEFINITION 2.1. Consider a sesquilinear functional $\phi\colon X \times X \to \Phi$ with respect to an automorphism $\bar{\cdot}\colon \Phi \to \Phi$. Now define the *$\phi$-orthogonality* relation $\perp^\phi$ on X by

$$\perp^\phi = \{(x, y) \in X \times X \mid \phi(x, y) = 0\}.$$

A vector $z \in X$ is said to be *isotropic,* if $\phi(z, z) = 0$. It will be fundamental in the sequel the condition

($*$) there exist vectors $u_0, v_0 \in X$ such that $\phi(u_0, u_0) \neq 0 \neq \phi(v_0, v_0)$ and $\phi(u_0, v_0) = 0$.

LEMMA 2.2. *Assume that the automorphism of* Φ *is involutory, i.e.* $\bar{\bar{\alpha}} = \alpha$ *for all* $\alpha \in \Phi$. *If* $V \subset X$ *is a linear subspace such that the ϕ-orthogonality on* V *is symmetric and there is a non-isotropic vector* $t \in V$, *then*

$$\phi(y, x) = \gamma \overline{\phi(x, y)}, \qquad x, y \in V,$$

where $\gamma = \phi(t, t)/\overline{\phi(t, t)}$ *and so* $\gamma\bar{\gamma} = 1$. *Thus for any couple* $x, y \in V$ *with non-isotropic* y, $\phi(x, x)/\phi(y, y) = \gamma\overline{\phi(x, x)}/[\gamma\overline{\phi(y, y)}] = \overline{\phi(x, x)/\phi(y, y)}$ *is a fix element of* Φ *with respect to its automorphism. Moreover, in the particular case of* $\bar{\cdot} = \mathrm{id}_\Phi$, *i.e. if* ϕ *is bilinear, then it is symmetric on* V.

PROOF. Let $x, y \in V$ be arbitrary vectors and $\xi = \phi(x, t)/\phi(t, t)$ and $\eta = \phi(y, t)/\phi(t, t)$. Then for $u = x - \xi t$ and $v = y - \eta t$, we have

$$\phi(u, t) = \phi(x, t) - \xi\phi(t, t) = 0 \text{ and } \phi(v, t) = \phi(y, t) - \eta\phi(t, t) = 0,$$

whence, regarding the symmetry of $\perp^{\phi}$, $\phi(t,u) = 0 = \phi(t,v)$. Now for $\zeta = \phi(u,v)/\phi(t,t)$, it follows that

$$\phi(u - \zeta t, v + t) = \phi(u,v) - \zeta\phi(t,t) = 0$$

and so again by the symmetry of $\perp^{\phi}$,

$$\phi(v,u) - \bar{\zeta}\phi(t,t) = \phi(v + t, u - \zeta t) = 0,$$

i.e. $\phi(v,u) = \bar{\zeta}\phi(t,t) = \gamma\overline{\phi(u,v)}$. Finally,

$$\phi(y,x) = \phi(v + \eta t, u + \xi t) = \phi(v,u) + \eta\bar{\xi}\phi(t,t) =$$

$$= \gamma\overline{(\phi(u,v) + \xi\bar{\eta}\phi(t,t))} = \gamma\overline{\phi(u + \xi t, v + \eta t)} = \gamma\overline{\phi(x,y)}.$$

LEMMA 2.3. *If the ϕ-orthogonality on X is symmetric and condition $(*)$ is satisfied, then the automorphism of Φ is involutory.*

PROOF. Let $\alpha \in \Phi\backslash\{0\}$ be arbitrarily fixed, $\beta = \phi(\alpha u_0, \alpha u_0)/\phi(v_0, v_0)$ and $\gamma = \phi(v_0,v_0)/\overline{\phi(v_0,v_0)}$, where $u_0, v_0 \in X$ are defined by $(*)$. Then we have

$$\phi(\alpha u_0 - \beta v_0, \alpha u_0 + v_0) = \phi(\alpha u_0, \alpha u_0) - \beta\phi(v_0,v_0) = 0,$$

whence, by the symmetry of $\perp^{\phi}$,

$$\phi(\alpha u_0, \alpha u_0) - \bar{\beta}\phi(v_0,v_0) = \phi(\alpha u_0 + v_0, \alpha u_0 - \beta v_0) = 0,$$

i.e. $\phi(\alpha u_0, \alpha u_0) = \bar{\beta}\phi(v_0,v_0) = \gamma\overline{\phi(\alpha u_0, \alpha u_0)}$. Now, using this equality also for $\alpha = 1$, we have

$$\gamma\overline{\bar{\alpha}\bar{\bar{\alpha}}\phi(u_0,u_0)} = \gamma\overline{\alpha\bar{\alpha}\phi(u_0,u_0)} = \gamma\overline{\phi(\alpha u_0, \alpha u_0)} =$$

$$= \phi(\alpha u_0, \alpha u_0) = \alpha\bar{\alpha}\phi(u_0,u_0) = \alpha\bar{\alpha}\gamma\overline{\phi(u_0,u_0)},$$

i.e. $\bar{\bar{\alpha}} = \alpha$.

Now defining for $x \in X$ the linear functional $\phi_x \colon X \to \Phi$ by $\phi_x(t) = \phi(t,x)$, we can present a more familiar condition instead of $(*)$ in terms of a subspace of the conjugate space:

$$X^*_{\phi} = \{\phi_x \mid x \in X\} \subset X^*.$$

PROPOSITION 2.4. *If the ϕ-orthogonality on X is symmetric, then the following assertions are equivalent:*

i) *Condition $(*)$ holds true;*

ii) *There exist $x, y, z \in X$ with $\phi(x,x) \neq 0$, $\phi(z,x) = 0$ and $\phi(z,y) \neq 0$;*

iii) *There is a non-isotropic vector in X and $\dim X^*_{\phi} \geqq 2$.*

PROOF. i)$\Longrightarrow$iii): Obviously $\phi_{v_0} \neq 0$ and $\phi_{u_0} \notin \operatorname{lin}\{\phi_{v_0}\}$.

iii)$\Longrightarrow$ii): Choose a non-isotropic vector $x \in X$. Since $\dim X^*_\phi \geqq 2$, there is $y \in X$ such that $\phi_y \in X^*_\phi \backslash \mathrm{lin}\{\phi_z\}$. On the contrary, suppose that for any $t \in X$, $\phi(t,x) = 0$ implies $\phi(t,y) = 0$, or equivalently, $\phi(t_1,x) = \phi(t_2,x)$ makes $\phi(t_1,y) = \phi(t_2,y)$ whenever $t_1, t_2 \in X$. Let now $\lambda = \phi(x,y)/\phi(x,x)$ and $t \in X$ be arbitrarily fixed. Then for $\mu = \phi(t,x)/\phi(x,x)$, we have $\phi(t,x) = \phi(\mu x, x)$ and so

$$\phi(t,y) = \phi(\mu x, y) = \mu\phi(x,y) = \mu\lambda\phi(x,x) = \lambda\phi(\mu x, x) = \lambda\phi(t,x).$$

This means that $\phi_y = \lambda\phi_x$, which is a contradiction.

ii)$\Longrightarrow$i): There may occur exactly the possibilities below:

a) $\phi(z,z) \neq 0$: Then let $u_0 = x$ and $v_0 = z$.

b) $\phi(z,z) = 0$: Then we have to deal with the following cases:

b/1) $\phi(x,y) \neq 0$: Then

b/1/i) *either* $\phi(y,y) \neq 0$: Let $\alpha = \phi(x,y)/\phi(z,y)$ and define $u_0 = x - \alpha z$, $v_0 = y$. Then $\phi(u_0,u_0) = \phi(x,x) \neq 0 \neq \phi(y,y) = \phi(v_0,v_0)$ and $\phi(u_0,v_0) = \phi(x,y) - \alpha\phi(z,y) = 0$.

b/1/ii) *or* $\phi(y,y) = 0$: Let $\alpha = \phi(x,y)/\phi(z,y)$ and $u_0 = x - \alpha z$. For non-isotropic $y+z$ let $v_0 = y+z$. Then $\phi(u_0,u_0) = \phi(x,x) \neq 0 \neq \phi(v_0,v_0)$ and $\phi(u_0,v_0) = \phi(x,y) - \alpha\phi(z,y) = 0$. Now suppose that $y+z$ is isotropic. Then $\bar{\cdot} \neq \mathrm{id}_\phi$, since otherwise, by Lemma 2.2, ϕ would be symmetric and so $\phi(y+z,y+z) = 2\phi(y,z) \neq 0$. Thus choosing $\beta \in \Phi$ with $\bar\beta \neq \beta$, we can define $v_0 = y + \beta z$. The only thing to show is $\phi(v_0,v_0) = \phi(y,\beta z) + \phi(\beta z, y) = (\bar\beta - \beta)\phi(y,z) \neq 0$.

b/2) $\phi(x,y) = 0$: Then

b/2/i) *either* $\phi(y,y) \neq 0$: Let $u_0 = x$, $v_0 = y$.

b/2/ii) *or* $\phi(y,y) = 0$: Let $u_0 = x$ and v_0 be chosen according to the same process as described in case b/1/ii).

PROPOSITION 2.5. *Suppose that $\Phi \neq GF(3)$ and the ϕ-orthogonality on X is symmetric. If $u,v \in X$ are such that $\phi(u,v) \neq 0 \neq \phi(v,v)$ and $\phi(u,v) = 0$, then u and v are linearly independent and $P = \mathrm{lin}(u,v)$ is a $\perp^\phi$-normal plane. In particular, u and v are τ_1-elements.*

PROOF. For $v = \lambda u$ we would have

$$0 \neq \phi(v,v) = \phi(\lambda u, v) = \lambda\phi(u,v) = 0.$$

Similarly, $u \neq \mu v$, i.e. $P = \mathrm{lin}\{u,v\} \in \mathcal{P}$ and $(u,v) \in \perp^\phi{}_P$.

Next we show the existence of a ϕ-orthogonal base $(x,y) \in \perp^\phi{}_P$ such that $x,y \notin \mathrm{lin}\{u\} \cup \mathrm{lin}\{v\}$. Since $\perp^\phi$ is homogeneous, it suffices to look for x and y in the form $x = \alpha u + v$, $y = u - \beta v$ with $\alpha, \beta \neq 0$. Then these x and y are linearly independent if

$$\alpha\beta \neq -1, \tag{2.1}$$

and $x \perp^{\phi} y$ if

$$\alpha = \bar{\beta}\phi(v,v)/\phi(u,u), \tag{2.2}$$

which is obtained from the condition $\phi(\alpha u+v, u-\beta v) = \alpha\phi(u,u)-\bar{\beta}\phi(v,v) = 0$. Substituting (2.2) into (2.1) we reduced the problem to looking for a solution of the inequalities

$$\bar{\beta}\beta \neq -\phi(u,u)/\phi(v,v), \qquad \beta \neq 0,$$

which is always solvable if $\Phi \neq GF(3)$. This means that P is a ϕ-normal plane with $(u_1,v_1) = (u,v)$ and $(u_2,v_2) = (x,y)$.

PROPOSITION 2.6. *Suppose that $\Phi \neq GF(3)$ and the ϕ-orthogonality on X is symmetric while condition $(*)$ holds true. Then every non-isotropic vector $t \in X$ is a τ_1-element; namely, there is a non-isotropic $u \in X$ with $\phi(u,t) = 0$.*

PROOF. We are to deal with the three cases below:

a) $\phi(u_0,u_0)\phi(t,t) \neq \phi(u_0,t)\phi(t,u_0)$: Then for $\beta = \phi(u_0,t)/\phi(t,t)$, $u = u_0 - \beta t$, $v = t$, we have

$$\phi(u,u) = \phi(u_0-\beta t, u_0-\beta t) =$$

$$= \phi(u_0,u_0) - \beta\phi(t,u_0) - \bar{\beta}\phi(u_0,t) + \beta\bar{\beta}\phi(t,t) =$$

$$= \phi(u_0,u_0) - \frac{\phi(u_0,t)\phi(t,u_0)}{\phi(t,t)} \neq 0 \neq \phi(t,t) = \phi(v,v)$$

and $\phi(u,v) = \phi(u_0,t)-\beta\phi(t,t) = 0$. Thus by Proposition 2.5, $t \in \operatorname{lin}\{u,v\} \in \mathcal{P}_n$, i.e. via the definition, t is a τ_1-element.

b) $\phi(v_0,v_0)\phi(t,t) \neq \phi(v_0,t)\phi(t,v_0)$: See case a).

c) $\phi(u_0,u_0)\phi(t,t) = \phi(u_0,t)\phi(t,u_0)$, $\phi(v_0,v_0)\phi(t,t) = \phi(v_0,t)\phi(t,v_0)$: It follows immediately that $\phi(u_0,t)$, $\phi(t,u_0)$, $\phi(v_0,t)$, $\phi(t,v_0) \neq 0$. Let $\beta = \phi(u_0,t)/\phi(v_0,t)$, $u = u_0 - \beta v_0$, $v = t$. Then

$$\phi(u,u) = \phi(u_0-\beta v_0, u_0-\beta v_0) =$$

$$= \phi(u_0,u_0) - \beta\phi(v_0,u_0) - \bar{\beta}\phi(u_0,v_0) + \beta\bar{\beta}\phi(v_0,v_0) =$$

$$= \phi(u_0,u_0) + \frac{\phi(u_0,t)\overline{\phi(u_0,t)}}{\phi(v_0,t)\overline{\phi(v_0,t)}}\phi(v_0,v_0) = \phi(u_0,u_0) + \frac{\phi(u_0,t)\phi(t,u_0)}{\phi(v_0,t)\phi(t,v_0)}\phi(v_0,v_0) =$$

$$= 2\phi(u_0,u_0) \neq 0 \neq \phi(t,t) = \phi(v,v).$$

Also we have $\phi(u,v) = \phi(u_0,t) - \beta\phi(v_0,t) = 0$, and so by Proposition 2.5 $t \in \operatorname{lin}\{u,v\} \in \mathcal{P}_n$, i.e. t is a τ_1-element as well.

Proposition 2.7. *Suppose that $\Phi \neq GF(3)$ and the ϕ-orthogonality on X is symmetric while condition $(*)$ holds true. If $P \in \mathcal{P}$ is such that every $z \in P$ is isotropic, then ϕ is identically zero on P, and so $P \in \mathcal{P}_0$.*

Proof. Let $x, y \in P$ be arbitrarily fixed. Then $\phi(x,y) + \phi(y,x) = \overline{}$ $= \phi(x+y, x+y) = 0$, i.e. $\phi(x,y) = -\phi(y,x) = -\gamma\overline{\phi(x,y)}$. Now, if $\bar{\cdot} = \mathrm{id}_\Phi$, then $\phi(x,y) = -\phi(x,y)$, i.e. $\phi(x,y) = 0$. Otherwise, choose a scalar $\alpha \in \Phi$ with $\alpha \neq \bar{\alpha}$ and take αx for x:

$$\alpha\phi(x,y) = \phi(\alpha x, y) = -\gamma\overline{\phi(\alpha x, y)} =$$
$$= -\gamma\overline{\alpha\phi(x,y)} = \bar{\alpha}(-\gamma\overline{\phi(x,y)}) = \bar{\alpha}\phi(x,y).$$

Then clearly $\phi(x,y) = 0$ again.

Theorem 2.8. *Assume that $\Phi \neq GF(3)$. If the ϕ-orthogonality on X is symmetric and condition $(*)$ holds true, then $\perp^\phi$ satisfies axioms (O3) and (O8').*

Proof. The validity of (O3) is obvious. Now we are going to show (O8'). For this reason, let $P \in \mathcal{P}$ be arbitrarily fixed. Then there may occur exactly the possibilities as follows:

a) *Every $z \in P$ is isotropic:* Then Proposition 2.7 implies $P \in \mathcal{P}_0$.

b) *There exists non-isotropic $v \in P$:* Then by Proposition 2.6, v is a τ_1-element. Also, for a fixed $x \in P \backslash \mathrm{lin}\{v\}$, we define $u = x - [\phi(x,v)/\phi(v,v)]v \in P$. Then clearly $(u,v) \in \perp^\phi{}_P$ and

b/1) *either* $\phi(u,u) = 0$, when u is a τ_0-element and so $(u,v) \in \perp^\phi{}_P \cap (X_0 \times X_1)$, i.e. $P \in \mathcal{P}_1$,

b/2) *or* $\phi(u,u) \neq 0$, when by Proposition 2.5, $P = \mathrm{lin}\{u,v\} \in \mathcal{P}_n$.

Corollary 2.9. *Assume that $\Phi \neq GF(3)$. If the ϕ-orthogonality on X is symmetric and condition $(*)$ holds true, then*

i) (o)$\mathrm{Hom}_{\perp^\phi}(X,Y) = \mathrm{Hom}(X,Y)$;

ii) (e)$\mathrm{Hom}_{\perp^\phi}(X,Y) \subset \mathrm{Quad}(X,Y)$;

iii) $\mathrm{Hom}_{\perp^\phi}(X,Y) = \mathrm{Hom}(X,Y) \iff$ (e)$\mathrm{Hom}_{\perp^\phi}(X,Y) = \{\mathbf{O}\}$.

Remark 2.10. The condition $\Phi \neq GF(3)$ cannot be omitted from the previous statements. To check this, let $\Phi = GF(3) = \{-1, 0, 1\}$, $X = \Phi^2$ and define $\phi\colon X \times X \to \Phi$ by

$$\phi((\xi_1;\xi_2),(\eta_1;\eta_2)) = \xi_1\eta_1 - \xi_2\eta_2, \qquad (\xi_1;\xi_2),(\eta_1;\eta_2) \in X.$$

Then for $u_0 = (1;0)$ and $v_0 = (0;1)$, we have $\phi(u_0,u_0) = 1 \neq 0 \neq -1 = \phi(v_0,v_0)$, $\phi(u_0,v_0) = 0$, however $\mathcal{P} = \{X\}$ and

$$\perp^\phi{}_X = \{(\lambda u_0, \mu v_0), (\mu v_0, \lambda u_0) \mid \lambda, \mu \in \Phi\backslash\{0\}\},$$

showing $X \notin \mathcal{P}_0 \cup \mathcal{P}_n = \mathcal{P}'_s$. Actually, (e)$\mathrm{Hom}_{\perp^\phi}(X,Y) \subset \mathrm{Quad}(X,Y)$ holds no longer in general. E.g., define $E\colon X \to \mathbf{R}$ to be even and satisfying

$$E(1;0) = 1, \ \ E(0;1) = -1, \ \ E(1;1) = 0, \ \ E(1;-1) = 0.$$

Then it can be shown easily that $E \in (\mathrm{e})\mathrm{Hom}_{\perp\phi}(X, \mathbf{R})$, however

$$E((1;0)+(1;1))+E((1;0)-(1;1))=$$
$$=E(-1;1)+E(0;-1)=0+(-1)=-1\neq$$
$$\neq 2=2\cdot 1+2\cdot 0=2E(1;0)+2E(1;1).$$

3. Even solutions

The previous section has left open the question how to select the even solutions from among the quadratic functions. Now we answer this question under the following assumptions on the field Φ:

Throughout this section, using the notations $\Omega=\{\alpha\in\Phi \mid \alpha=\bar{\alpha}\}$, $\Omega_+=\{\mu\bar{\mu} \mid \mu\in\Phi\}$ and $\Omega_-=-\Omega_+$, we assume that

$$\Omega_+ + \Omega_+ \subset \Omega_+;\quad \Omega=\Omega_-\cup\Omega_+;\quad \Omega_+=\{\omega^2 \mid \omega\in\Omega_+\}.$$

These conditions are motivated by the natural properties of the complex field $\mathbf{C}$, but they are valid e.g. for the subfield of the algebraic complex numbers, too. More generally, starting from a *euclidean* ordered field Ω, i.e. an ordered field in which every nonnegative element has a square root, it is quite evident that the cartesian product $\Phi=\Omega\times\Omega$ turns into a field of the above type just as $\mathbf{C}=\mathbf{R}\times\mathbf{R}$. In each example given till now, the particular automorphism should be chosen to be the usual conjugation. Notice that the first condition excludes the possibility of $\Phi=GF(3)$. However, any euclidean ordered field or fields having only square elements, meet all of the conditions with the identical automorphism. For more information see e.g. [6].

Also, further on, the ϕ-orthogonality on X is supposed to be symmetric and satisfying condition $(*)$. Then by Lemmas 2.2 and 2.3, there is a scalar $\gamma\in\Phi$ such that $\gamma\bar{\gamma}=1$ and $\phi(y,x)=\gamma\overline{\phi(x,y)}$ for all $x,y\in X$.

LEMMA 3.1. *There is a sesquilinear and Hermite-symmetric functional* $\phi_0\colon X\times X\to\Phi$ *such that* $\perp^{\phi_0}=\perp^{\phi}$.

PROOF. Since $\phi(v_0,v_0)\overline{\phi(v_0,v_0)}\in\Omega_+$, we have $\omega_0\in\Omega_+$ with $\omega_0^2=$ $=\phi(v_0,v_0)\cdot\overline{\phi(v_0,v_0)}$. Hence for $\chi=\phi(v_0,v_0)/\omega_0$, it follows that $\chi\bar{\chi}=1$ and $\chi^2=\gamma$. Let define $\phi_0\colon X\times X\to\Phi$ by

$$\phi_0(x,y)=\bar{\chi}\phi(x,y),\qquad x,y\in X.$$

Clearly, ϕ_0 is sesquilinear and the Hermite-symmetry can be verified as follows:

$$\phi_0(y,x)=\bar{\chi}\phi(y,x)=\bar{\chi}\gamma\overline{\phi(x,y)}=\bar{\chi}\chi^2\overline{\phi(x,y)}=\chi\overline{\phi(x,y)}=$$

$$= \overline{\chi\phi(x,y)} = \overline{\phi_0(x,y)}.$$

Finally, $\perp^{\phi_0} = \perp^{\phi}$ is trivial from the definiton.

LEMMA 3.2. *If ϕ is Hermite-symmetric and $E \in$ (e)Hom$_{\perp^\phi}(X,Y)$, then for each non-isotropic $t \in X$*

$$E(\tau t) = E(t), \qquad \tau \in \Phi, \quad \tau\bar{\tau} = 1.$$

PROOF. Let $t \in X$ be non-isotropic. By Proposition 2.6, one can choose a non-isotropic $u \in X$ with $\phi(t,u) = 0$. Then $\phi(t,t)/\phi(u,u) \in \Omega$ and so it is equal to $\pm\mu\bar{\mu}$ for some $\mu \in \Phi$, i.e. $\phi(t,t) = \pm\phi(\mu u, \mu u)$. This implies that either $\phi(t + \mu u, t - \mu u) = 0$, whence

$$E(t) = E\left(\frac{t+\mu u}{2} + \frac{t-\mu u}{2}\right) = E\left(\frac{t+\mu u}{2}\right) + E\left(\frac{t-\mu u}{2}\right) =$$

$$= E\left(\frac{t+\mu u}{2}\right) + \left(-\frac{t-\mu u}{2}\right) = E\left(\frac{t+\mu u}{2} - \frac{t-\mu u}{2}\right) = E(\mu u),$$

or $\phi(t + \mu u, t + \mu u) = 0$, whence

$$E(t) + E(\mu u) = E(t + \mu u) = E\left(\frac{t+\mu u}{2}\right) + E\left(\frac{t+\mu u}{2}\right) =$$

$$= E\left(\frac{t+\mu u}{2}\right) + E\left(\frac{-t-\mu u}{2}\right) = E(0) = 0.$$

Now applying this for τt ($\tau \in \Phi$, $\tau\bar{\tau} = 1$), we have by the above argument that either

$$E(\tau t) = E(\mu u) \quad \text{or} \quad E(\tau t) = -E(\mu u).$$

This means that in both cases $E(\tau t) = E(t)$.

COROLLARY 3.3. *If ϕ is Hermite-symmetric, then for any $E \in$ $\in$ (e)Hom$_{\perp^\phi}(X,Y)$ we have*

$$E(x) = E(y), \qquad x, y \in X, \ \phi(x,x) = \phi(y,y).$$

PROOF. If $\phi(x,y) = 0$, then $\phi(x+y, x-y) = 0$ and so $E(x) = E(y)$. Otherwise, we can choose $\omega \in \Omega_+$ such that $\omega^2 = \phi(x,y)\overline{\phi(x,y)}$. Then for $\eta = \phi(x,y)/\omega$ we have $\eta\bar{\eta} = 1$ and $\phi(x + \eta y, x - \eta y) = 0$. Thus Lemma 3.2 implies that $E(x) = E(\eta y) = E(y)$.

THEOREM 3.4. *Suppose that ϕ is Hermite-symmetric. Then $E \in$ $\in$ (e)Hom$_{\perp^\phi}(X,Y)$ if and only if*

$$E(x) = a(\phi(x,x)), \qquad x \in X,$$

for some $a \in \mathrm{Hom}(\Omega, Y)$.

PROOF. By Corollary 3.3, $E(x)$ depends only on $\phi(x,x)$:

$$E(x) = a(\phi(x,x)), \qquad x \in X$$

for some $a\colon \Omega \to Y$. Now we have only to show that a is additive. For this reason, first observe that choosing any $u, v \in X$ with $\phi(u,u) \neq 0 \neq \phi(v,v)$, $\phi(u,v) = 0$, then for all $\alpha, \beta \in \Omega_+$ and $\lambda = \phi(\alpha u, \alpha u)$, $\mu = \phi(\beta v, \beta v)$, we have

$$a(\lambda+\mu) = a(\phi(\alpha u, \alpha u) + \phi(\beta v, \beta v)) = a(\phi(\alpha u + \beta v, \alpha u + \beta v)) =$$

$$= E(\alpha u + \beta v) = E(\alpha u) + E(\beta v) = a(\phi(\alpha u, \alpha u)) + a(\phi(\beta v, \beta v)) = a(\lambda) + a(\mu).$$

Since for the vectors $u_0, v_0 \in X$ given by $(*)$, $\phi(u_0,u_0), \phi(v_0,v_0) \in \Omega$, there are exactly the following possibilities:

a) $\phi(u_0,u_0), \phi(v_0,v_0) \in \Omega_+$: Then for each $\lambda, \mu \in \Omega_+$ there exist $\alpha, \beta \in \in \Omega_+$ such that $\alpha^2 = \lambda/\phi(u_0,u_0)$ and $\beta^2 = \mu/\phi(v_0,v_0)$. Thus by the above observation

$$a(\lambda+\mu) = a(\lambda) + a(\mu), \qquad \lambda, \mu \in \Omega_+$$

follows, i.e. a is additive on Ω_+, and choosing a to be odd, it is additive on the whole Ω.

b) $\phi(u_0,u_0), \phi(v_0,v_0) \in \Omega_-$: See case a).

c) $\phi(u_0,u_0) \in \Omega_+$, $\phi(v_0,v_0) \in \Omega_-$: Then for each $\lambda, \mu \in \Omega_+$ there exist $\alpha, \beta \in \Omega_+$ such that $\alpha^2 = \lambda/\phi(u_0,u_0)$ and $\beta^2 = -\mu/\phi(v_0,v_0)$. Thus referring again to the above observation, we have

$$a(\lambda-\mu) = a(\lambda) + a(-\mu), \qquad \lambda, \mu \in \Omega_+.$$

Now letting $\lambda = \mu$, $a(-\mu) = -a(\mu)$ follows, i.e. a is an odd function. Finally, for any $\varrho, \sigma \in \Omega_+$, defining $\lambda = \varrho + \sigma$, $\mu = \sigma \in \Omega_+$, we obtain

$$a(\varrho+\sigma) = a(\lambda) = a(\lambda-\mu) + a(\mu) = a(\varrho) + a(\sigma).$$

This means that a is additive on Ω_+, and because of its oddness, $a \in \in \mathrm{Hom}(\Omega, Y)$.

d) $\phi(u_0,u_0) \in \Omega_-$, $\phi(v_0,v_0) \in \Omega_+$: See case c).

COROLLARY 3.5. *Under the general assumptions on the field* Φ *and on the orthogonality* $\perp^\phi$ *at the beginning of this section, we have*

$$\text{(e)}\mathrm{Hom}_{\perp^\phi}(X,Y) = \{a \circ \Delta^\phi \mid a \in \mathrm{Hom}(\Phi, Y)\},$$

where $\Delta^\phi(x) = \phi(x,x)$ *for all* $x \in X$.

PROOF. This follows immediately from Lemma 3.1 and Theorem 3.4 above.

REMARK 3.6. Now the result of Vajzović [8], Theorems 1, 2, Fochi [1], Theorems 1, 3 and Corollaries 1, 2, Sundaresan–Kapoor [7], Theorems 2, 3, Rätz [3], Theorem 9, Corollary 10 and [4], Theorem 3.8 c) can be derived in an obvious way from our theory, actually from Proposition 2.4 and Corollaries 2.9, 3.5 above.

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INSTITUTE OF MATHEMATICS
L. KOSSUTH UNIVERSITY
EGYETEM TÉR 1
H–4010 DEBRECEN, PF. 12
HUNGARY

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REPRESENTATION OF COMPLEX NUMBERS IN NUMBER SYSTEMS

B. KOVÁCS (Debrecen)[1]

Introduction

Let $\mathbf{R}$ be an integral domain (with unit element), $\alpha \in \mathbf{R}$ and $\mathcal{N} = \{k_1, \dots, k_n\}$ a finite subset of the set of rational integers $\mathbf{Z}$. $\{\alpha, \mathcal{N}\}$ is called a number system in $\mathbf{R}$ if every $\gamma \in \mathbf{R}$ can be uniquely written in the form

$$(1.1)\quad \gamma = a_0 + a_1\alpha + \ldots + a_k\alpha^k, \quad a_i \in \mathcal{N} \quad (0 \leqq i \leqq k), \quad a_k \neq 0 \text{ if } k \neq 0.$$

If $\mathcal{N} = \{0, 1, \dots, n\}$ then the number system $\{\alpha, \mathcal{N}\}$ is called a canonical number system.

This concept is a natural generalization of negative base number systems in $\mathbf{Z}$ considered by several authors. The canonical number systems were completely described by Kátai and Szabó [1], Kátai and Kovács [2], [3], if $\mathbf{R}$ is the ring of integers of a quadratic number field. Kovács [4] gave a necessary and sufficient condition for the existence of canonical number systems in $\mathbf{R}$. It is proved in Pethő and Kovács [5] that for any $q < -1$, $q \in \mathbf{Z}$, $\{\alpha, \mathcal{N}\}$ is a number system in $\mathbf{Z}$ with infinitely many $\mathcal{N} \subset \mathbf{Z}$. In [6] Pethő and Kovács characterized all those integral domains which have number systems and gave necessary and sufficient conditions for $\{\alpha, \mathcal{N}\}$ to be a number system in an order θ. Furthermore they characterized effectively the base of all canonical number systems of θ and computed the representatives of all classes of bases of canonical number systems in rings of integers of some totally real cubic fields.

In [1] Kátai and Szabó proved that if $\{\alpha, \mathcal{N}\}$ is a canonical number system in the ring of Gaussian integers, then any complex number γ can be written in the form

$$(1.2)\quad \gamma = a_k\alpha^k + a_{k-1}\alpha^{k-1} + \ldots + a_0 + a_{-1}\alpha^{-1} + \ldots, \quad a_i \in \mathcal{N} \quad (i = k, k-1, \dots).$$

This result was extended for the ring of integers of imaginary quadratic fields in Kátai and Kovács [3]. In connection with this Daróczy and Kátai proved that for every complex number α, $|\alpha| > 1$, there exists a set $\{0, 1, \dots, n\} = \mathcal{N}$ such that any complex number γ is representable in the form (1.2) ([7]).

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In this paper we first give a necessary and sufficient condition for a number system $\{\alpha, \mathcal{N}\}$ that any complex number γ can be written in the form (1.2). Using this theorem we describe a family of number systems with the property above, further we prove that every number system in the ring of integers of any cubic imaginary field has this property.

Results

In the sequal **R** will denote an integral domain of characteristic 0, **Z** the ring of integers, **Q** the field of rationals. If α is an algebraic integer over **Q**, $\mathbf{Z}[\alpha]$ denotes the subring of $\mathbf{Q}(\alpha)$, generated by **Z** and α.

If $\{\alpha, \mathcal{N}\}$ is a number system in $\mathbf{Z}[\beta]$ and

$$\gamma = a_0 + a_1\alpha + \ldots a_k\alpha^k, \qquad a_i \in \mathcal{N} \quad (0 \leqq i \leqq k), \quad a_k \neq 0 \text{ if } k \neq 0$$

then the exponent k is denoted by $L(\gamma, \alpha)$. With this notation we have

THEOREM 1. *Let $\{\alpha, \mathcal{N}\}$ be a number system in $\mathbf{Z}[\beta]$, (β is an algebraic integer over* **Q***). A real or complex number γ can be written in the form* (1.2) *— according as α is real or non-real — if and only if there exist sequences $\gamma(k)$, $\delta(k)$ with the following properties:*

1. *$\gamma \cdot \alpha^k = \gamma(k) + \delta(k)$ for every positive integer k,*
2. *$\gamma(k) \in \mathbf{Z}[\beta]$ and $L(\gamma(k), \alpha) \leqq k + c_1$ where c_1 is an appropriate constant which does not depend on k,*
3. *$\delta(k)/\alpha^k \to 0$ if $k \to \infty$.*

This theorem is rather general because if an integral domain **R** of characteristic 0 has a number system then $\mathbf{R} \cong \mathbf{Z}[\alpha]$, where α is an algebraic element over **Q** (see Theorem 1, [6]). Using Theorem 1 we prove

THEOREM 2. *Let $\{\alpha, \mathcal{N}\}$ be a number system in $\mathbf{Z}[\beta]$, where β is an algebraic integer of degree $n \geqq 1$ over* **Q** *and let us suppose that $|\alpha| \leqq |\alpha^{(1)}|$ for every conjugate of α over* **Q***. Then every complex number z has a representation in the form* (1.2) *if α is not real and every real number r has a representation in the form* (1.2) *if α is a real.*

From this result one can deduce the already mentioned results of Kátai and Szabó [1] and Kátai and Kovács [3], moreover in our case this theorem is stronger than the result of Daróczy and Kátai [7].

Finally, with the aid of Theorem 1 and Theorem 2 we get

THEOREM 3. *Let α be a non-real algebraic integer of degree* 3 *(over* **Q***). If $\{\alpha, \mathcal{N}\}$ is a canonical number system in a $\mathbf{Z}[\beta]$ then every complex number γ has a representation in the form* (1.2).

Proofs

In order to prove our theorems we need three lemmas.

LEMMA 1. *If $\{\alpha, \mathcal{N}\}$ is a number system in $\mathbf{Z}[\beta]$, where β is an algebraic integer over $\mathbf{Q}$, then $|\alpha^{(i)}| > 1$ holds for every conjugate of α.*

PROOF. This is one of the statements of Theorem 3 in [6].

LEMMA 2. *Let β be an algebraic integer over $\mathbf{Q}$ of degree $n \geqq 1$ and let $\{\alpha, \mathcal{N}\}$ be a number system in $\mathbf{Z}[\beta]$. Then there exist effectively computable constants $c_1(\alpha, \mathcal{N})$, $c_2(\alpha, \mathcal{N})$ depending only on α and $\mathcal{N}$ such that*

$$\max_{1 \leqq i \leqq n} \frac{\log |\gamma^{(i)}|}{\log |\alpha^{(i)}|} + c_1(\alpha, \mathcal{N}) \leqq L(\gamma, \alpha) \leqq \max_{1 \leqq i \leqq n} \frac{\log |\gamma^{(i)}|}{\log |\alpha^{(i)}|} + c_2(\alpha, \mathcal{N})$$

where $\gamma^{(i)}$ and $\alpha^{(i)}$ are the i-th conjugates of γ and α, respectively.

PROOF. See [8].

LEMMA 3. *Let α be an algebraic integer over $\mathbf{Q}$. If $\alpha^{(i)} \geqq -1$ holds for some real conjugate of α then $\{\alpha, \mathcal{N}\}$ is not a canonical number system in $\mathbf{Z}[\alpha]$.*

PROOF. See Lemma 6 in [6].

PROOF OF THEOREM 1. First let us assume that γ can be written as

(2.1)
$$\gamma = a_N \alpha^N + \ldots + a_1 \alpha + a_0 + a_{-1} \alpha^{-1} + \ldots, \quad a_i \in \mathcal{N} \quad (i = N, N-1, \ldots).$$

For every positive integer k let

$$\gamma(k) = a_N \alpha^{N+k} + a_{N-1} \alpha^{N+k-1} + \ldots + a_{-k+1} \alpha + a_{-k} \text{ and } \delta(k) = \alpha^k \cdot \sum_{i=-k-1}^{-\infty} a_i \alpha^i.$$

It is easy to verify that these sequences $\gamma(k)$, $\delta(k)$ satisfy the conditions of our theorem because $|\alpha| > 1$ by Lemma 1.

Of course, we may assume that $\gamma \neq 0$.

Let us now suppose that for a $\gamma \neq 0$ there exist sequences $\gamma(k)$, $\delta(k)$ with properties 1, 2, 3. Let $N(k) = L(\gamma(k), \alpha) - k$ and

$$\gamma(k) = b_{L(\gamma(k),\alpha)} \alpha^{L(\gamma(k),\alpha)} + \ldots + b_1 \alpha + b_0, \quad b_i \in \mathcal{N} \text{ and } b_{L(\gamma(k),\alpha)} \neq 0.$$

We write

$$z(k) = \gamma(k)/\alpha^k = b_{L(\gamma(k),\alpha)} \alpha^{N(k)} + \ldots + b_{k+1} \alpha + b_k + b_{k-1} \alpha^{-1} + \ldots + b_0 \alpha^{-k}.$$

Since, by assumption, $L(\gamma(k), \alpha) - k$ is bounded above and $N(k)$ is bounded from below because of $\gamma(k)/\alpha^k \to \gamma \neq 0$ ($|\alpha| > 1$), hence there exists an infinite set $S_{N(k)}$ of those indices k for which $k_1, k_2 \in S_{N(k)}$ implies $N(k_1) = N(k_2)$.

Let $C_{N(k)}$ be such a value (in $\mathcal{N}$) for which $C_{N(k)} = b_{L(\gamma(k),\alpha)}$ where $k \in S_{N(k)}$.

Consider now the set of those k's in $S_{N(k)}$ for which $b_{N(k)-1} = C_{N(k)-1}$ holds infinitely many times.

Let this index set be denoted by $S_{N(k)-1}$. Repeating this argument we get a monotone index set, all of which have infinitely many elements, and a chain $C_{N(k)}, C_{N(k)-1}, \dots$ $(C_j \in \mathcal{N})$.

Let $W = C_{N(k)}\alpha^{N(k)} + \dots + C_1\alpha + C_0 + C_{-1}\alpha^{-1} + \dots$.

Let furthermore $k(r) \in S_{N(k)-r+1}$, $k(1) < k(2) < \dots$. Then $\lim z(k(r)) = W$, but $\lim z(k) = \gamma$ because of $\lim \delta(k)/\alpha^k = 0$ and $\gamma = (\gamma \cdot \alpha^k)/\alpha^k = (\gamma(k) + \delta(k))/\alpha^k$, and so

$$\gamma = C_{N(k)}\alpha^{N(k)} + \dots + C_1\alpha + C_0 + C_{-1}\alpha^{-1} + \dots, \quad C_j \in \mathcal{N}.$$

This completes the proof of our theorem.

PROOF OF THEOREM 2. Let γ be a real number if α is real and a complex number otherwise. Of course we can suppose that $\gamma \neq 0$.

Let $\mathcal{L} = \{A + \beta\alpha \mid A, B \in \mathbf{Z}\}$.

i) If α is a non-real complex number, then $\mathcal{L}$ is a lattice in the complex plane. For every positive integer k, let $\gamma_k = A_k + B_k\alpha$ be one of the lattice points of that fundamental parallelogram of $\mathcal{L}$ which contains the number $\gamma \cdot \alpha^k$. One can readily verify that for every k

$$(3.1) \qquad |\gamma \cdot \alpha^k - \gamma_k| < c_1, \quad |A_k| < c_2 \cdot |\alpha^k| \quad \text{and} \quad |B_k| < c_3 \cdot |\alpha^k|$$

hold with suitable constants c_1, c_2, c_3 not depending on k.

ii) If α is a real algebraic integer with degree $\geqq 2$, then $\mathcal{L}$ is a dense set and so it is easy to see that for every positive integer k we can choose a $\gamma_k = A_k + B_k\alpha$ such that γ_k satisfies (3.1).

iii) If α is a rational number then the existence of a sequence γ_k with the property (3.1) is also evident.

In the sequel let γ_k be as above.

From (3.1) we can simply deduce that

$$(3.2) \qquad |\gamma_k^{(i)}| < c_4|\alpha|^k$$

holds for every positive integer k and for every conjugate $\gamma_k^{(i)}$ of γ_k with an appropriate constant c_4 which does not depend on k.

From (3.2) we get

$$(3.3) \qquad \log|\gamma_k^{(i)}| < \log c_4 + k \cdot \log|\alpha|$$

and so

$$\max_{1\leqq i\leqq n} \frac{\log|\gamma_k^{(i)}|}{\log|\alpha^{(i)}|} \leqq \frac{\log c_4 + k\cdot\log|\alpha|}{\log|\alpha|} \leqq k + c_5 \tag{3.4}$$

where the constant c_5 does not depend on k.

Further because of Lemma 1 $|\alpha| > 1$ holds, consequently

$$\delta_k/\alpha^k \to 0 \text{ if } k\to\infty \tag{3.5}$$

where $\delta_k = \gamma\cdot\alpha^k - \gamma_k$.

(3.1), (3.4) and (3.5) show that the sequences γ_k, δ_k defined above satisfy the conditions of Theorem 1. This proves the theorem.

Proof of Theorem 3. The case of $\gamma = 0$ is trivial, so we assume that $\gamma \neq 0$.

Let $\alpha^{(1)}$ be the real conjugate of α, $\alpha^{(2)} = \alpha$ and $\alpha^{(3)} = \bar{\alpha}$.

a) We begin our proof with the case $\arg(\alpha^{(2)}) \neq (2m\pi)/n$ ($m, n \in \mathbf{Z}$, $n \neq 0$). For every positive integer k let

$$B_k = \left[\frac{|\gamma(\alpha^{(2)})^k|}{||\alpha^{(1)}| + \alpha^{(2)}|}\right] \tag{4.1}$$

where [ ] and | | denote the integer part and the absolute value, respectively.

By Lemma 1, $|\alpha^{(i)}| > 1$ ($i = 1,2,3$). Further $\gamma \neq 0$, and so if k is large enough then

$$(1/B_k)|\gamma\cdot(\alpha^{(2)})^k| = ||\alpha^{(1)}| + \alpha^{(2)}| + c(k)/B_k \tag{4.2}$$

where $c(k) \geqq 0$ and bounded from above.

Since $\arg(\alpha^{(2)}) \neq (2m\pi)/n$ ($m, n \in \mathbf{Z}$, $n \neq 0$), the set $\{\arg(\alpha^{(2)})^k | 0 < < k \in \mathbf{Z}\}$ mod 2π is dense. Consequently, we can choose an infinite sequence $k(1) < k(2) < \ldots$ of positive integers such that

$$\arg(\gamma\cdot(\alpha^{(2)})^{k(i)}) \to \arg(|\alpha^{(1)}| + \alpha^{(2)}) \quad \text{if} \quad k(i)\to\infty. \tag{4.3}$$

From (4.2) and (4.3), it follows that

$$(1/B_{k(i)})\cdot\gamma\cdot(\alpha^{(2)})^{k(i)} = |\alpha^{(1)}| + \alpha^{(2)} + \delta_{k(i)} \tag{4.4}$$

such that $\delta_{k(i)} \to 0$ if $i\to\infty$. And so

$$\begin{aligned}\gamma\cdot(\alpha^{(2)})^{k(i)} &= B_{k(i)}|\alpha^{(1)}| + B_{k(i)}\alpha^{(2)} + B_{k(i)}\delta_{k(i)} = \\ &= \left[B_{k(i)}|\alpha^{(1)}|\right] + B_{k(i)}\alpha^{(2)} + B_{k(i)}\delta_{k(i)} + r_{k(i)}\end{aligned} \tag{4.5}$$

where $0 \leqq r_{k(i)} < 1$.

Let $A_{k(i)} = [B_{k(i)}|\alpha^{(1)}|]$ and $\partial_{k(i)} = B_{k(i)}\delta_{k(i)} + r_{k(i)}$.

By (4.4) and (4.5) we can deduce that

$$\partial_{k(i)}/B_{k(i)} = \delta_{k(i)} + r_{k(i)}/B_{k(i)} \to 0 \quad \text{if} \quad i \to \infty. \tag{4.6}$$

Because of (4.1) $B_{k(i)} = c_1|(\alpha^{(2)})^{k(i)}|$, where c_1 is bounded and so by (4.6) we get

$$\partial_{k(i)}/(\alpha^{(2)})^{k(i)} \to 0 \quad \text{if} \quad i \to \infty. \tag{4.7}$$

Now we shall prove that

$$L(A_{k(i)} + B_{k(i)}\alpha^{(2)}, \alpha^{(2)}) \leqq k(i) + c_2, \tag{4.8}$$

where c_2 is a constant not depending on $k(i)$.

It is evident that for every $k(i)$

$$\frac{\log|A_{k(i)} + B_{k(i)}\alpha^{(2)}|}{\log|\alpha^{(2)}|} = \frac{\log|A_{k(i)} + B_{k(i)}\alpha^{(3)}|}{\log|\alpha^{(3)}|}. \tag{4.9}$$

We shall prove that the following inequality holds for every $k(i)$

$$\frac{\log|A_{k(i)} + B_{k(i)}\alpha^{(1)}|}{\log|\alpha^{(1)}|} \leqq \frac{\log|A_{k(i)} + B_{k(i)}\alpha^{(2)}|}{\log|\alpha^{(2)}|} + \omega \tag{4.10}$$

where w is a constant not depending on $k(i)$.

(4.10) holds if and only if

$$|A_{k(i)} + B_{k(i)}\alpha^{(1)}| \leqq |A_{k(i)} + B_{k(i)}\alpha^{(2)}|^u \cdot v \tag{4.11}$$

where $u = (\log|\alpha^{(1)}|)/(\log|\alpha^{(2)}|)$ and $v = \||\alpha^{(1)}|^{\omega}$. Since $|\alpha^{(1)}| = -\alpha^{(1)}$ by Lemma 3 and

$$A_{k(i)} + B_{k(i)}\alpha^{(1)} = \left[\beta_{k(i)}|\alpha^{(1)}|\right] + \beta_{k(i)}\alpha^{(1)},$$

hence the left hand side of (4.11) is bounded from above.

But $|\alpha^{(1)}| > 1$ $(i = 1, 2, 3)$, and so $u > 0$. From this it follows immediately

$$(1/B_{k(i)})^u|A_{k(i)} \to B_{k(i)}\alpha^{(1)}| \to 0, \quad \text{if} \quad i \to \infty. \tag{4.12}$$

But $A_{k(i)}/B_{k(i)} \to |\alpha^{(1)}|$ if $i \to \infty$ and $\alpha^{(2)} \neq \alpha^{(1)}$, consequently

$$|(A_{k(i)}/B_{k(i)} + \alpha^{(2)}|^u \neq 0 \tag{4.13}$$

and it is bounded.

From (4.12) and (4.13) we get that if v is large enough, then

$$(1/B_{k(i)})^u |A_{k(i)} + B_{k(i)}\alpha^{(1)}| \leqq |(A_{k(i)}/B_{k(i)}) + \alpha^{(2)}|^u \cdot v \tag{4.14}$$

holds for every $k(i)$.

Because of $|\alpha^{(1)}| > 1$ and by definition we can choose v such that (4.14) holds. Since (4.14) holds if and only if (4.11) holds, consequently (4.10) also holds.

By $A_{k(i)} = [B_{k(i)}|\alpha^{(1)}|]$ and $B_{k(i)} = c_1|\alpha^{(2)}|^{k(i)}$ we have

$$|A_{k(i)} + B_{k(i)}\alpha^{(2)}| \leqq c_6|\alpha^{(2)}|^{k(i)}, \tag{4.15}$$

where the constant c_6 does not depend on $k(i)$, and this means that

$$\frac{\log|A_{k(i)} + B_{k(i)}\alpha^{(2)}|}{\log|\alpha^{(2)}|} \leqq \frac{\log c_6 + k(i)\log|\alpha^{(2)}|}{\log|\alpha^{(2)}|} \leqq c_7 + k(i) \tag{4.16}$$

where c_7 is a constant which does not depend on $k(i)$.

(4.8) follows immediately from (4.9), (4.10) and (4.16). By (4.7) and (4.8) the sequences $\gamma_{k(i)} = A_{k(i)} + B_{k(i)}\alpha^{(2)}$ and $\partial_{k(i)}$ satisfy the conditions of Theorem 1. Consequently, in the case under consideration the proof of our theorem is complete.

b) Let now $\arg(\alpha^{(2)}) = (2m\pi)/n$ ($m, n \in \mathbf{Z}$, $n \geqq 3$). It is easy to see that $\alpha^{(2)}/\alpha^{(3)}$ is a root of unity of degree 3 or 6 because $\alpha^{(2)}$ and $\alpha^{(3)}$ are conjugate elements of degree 3. We can readily verify by this statement that $(\alpha^{(1)})^3 = r_1$ and $(\alpha^{(2)})^3 = r_2 = (\alpha^{(3)})^3 = r_3$, where r_1 and $r_2 = r_3$ are real algebraic numbers of degree 1 or 3. But the latest case is impossible because r_1, r_2 and r_3 are conjugate elements and $r_2 = r_3$. Consequently, r_1 is a rational number. Since α is an algebraic integer, hence $\alpha^3 = n$ where $n \in \mathbf{Z}$.

But if α is a root of the polynomial $x^3 + n$ where $n < 0$, then α has a positive conjugate, and so $\{\alpha, \mathcal{N}\}$ can not be a number system in $\mathbf{Z}[\beta]$ by Lemma 3.

If $n = 1$, then $|\alpha| = 1$ and so $\{\alpha, \mathcal{N}\}$ also does not form a number system in $\mathbf{Z}[\beta]$ (see Lemma 1).

If $n > 1$, then all the conjugates of α have the same absolute value, consequently we can apply Theorem 2 to complete the proof of the theorem in Case b). Thus the theorem is proved.

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INSTITUTE OF MATHEMATICS
L. KOSSUTH UNIVERSITY
H–4010 DEBRECEN, PF. 12
HUNGARY

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MULTIPLICATIVE FUNCTIONS WITH SMALL INCREMENTS. III

K.-H. INDLEKOFER (Paderborn)
and
I. KÁTAI[1] (Budapest), member of the Academy

1. Let Ω be the set of all arithmetical functions having complex values. Sometimes a function $f \in \Omega$ is considered as an infinite-dimensional vector, the nth coordinate of which is $f(n)$. We write $\mathbf{f} = (f(1), f(2), \dots)$. Let $\mathbf{x} = (x_1, x_2, \dots)$ be a general element of Ω. The operators I, E, Δ, Δ_B $(\Omega \to \Omega)$ are defined according to the following rules: the nth coordinate of $I\mathbf{x}$, $E\mathbf{x}$, $\Delta\mathbf{X}$, $\Delta_B\mathbf{x}$ are x_n, x_{n+1}, $x_{n+1} - x_n$, $x_{n+B} - x_n$, respectively. Let $\Delta^k = (E-I)^k$, $\Delta_B^k = (E^B - I)^k$. If $P \in \mathbf{C}[z]$ is a polynomial, $P(z) = a_0 + a_1 z + \dots + a_k z^k$, then the nth coordinate of $P(E)\mathbf{x}$ equals

$$a_0 x_n + a_1 x_{n+1} + \dots + a_k x_{n+k}.$$

Let $\alpha \geqq 1$ be a constant, $\varrho\colon [1, \infty) \to [1, \infty)$ a slowly varying function, i.e. such that

$$\lim_{x \to \infty} \max_{\frac{x}{2} \leqq y \leqq x} \left| \frac{\varrho(y)}{\varrho(x)} - 1 \right| = 0. \tag{1.1}$$

Let $\Omega_{\alpha,\varrho}$ $(\subseteqq \Omega)$ denote the set of those $\mathbf{x} \in \Omega$ for which

$$\sup_{x \geqq 1} \frac{1}{x \varrho(x)^\alpha} \sum_{n \leqq x} |x_n|^\alpha \tag{1.2}$$

is finite.

It is clear that Ω_α, ϱ is a linear space, i.e. for $\mathbf{f}, \mathbf{g} \in \Omega_{\alpha,\varrho}$, $c_1, c_2 \in \mathbf{C}$ we have $c_1\mathbf{f} + c_2\mathbf{g} \in \Omega_{\alpha,\varrho}$.

Let $\mathcal{M}$ (resp. $\mathcal{M}^*$) denote the set of complex-valued multiplicative (completely multiplicative) functions. Let $\mathcal{L}_{\alpha,\varrho} = \mathcal{M} \cap \Omega_{\alpha,\varrho}$, $\mathcal{L}^*_{\alpha,\varrho} = \mathcal{M}^* \cap \Omega_{\alpha,\varrho}$.

In our preceding paper [1] we proved that if $f \in \mathcal{M}$ and $\Delta_K \mathbf{f} \in \Omega_{\alpha,\varrho}$ holds for some $K \in \mathbf{N}$, then either $f \in \mathcal{L}_{\alpha,\varrho}$ or $f(n) = n^s u(n)$, where $0 \leqq \operatorname{Re} s \leqq 1$ and $u(n + K) = u(n)$ for every $n \in \mathbf{N}$.

Our purpose is to prove the following

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THEOREM. *If* $f \in \mathcal{M}$, $P \in \mathbf{C}[z]$, $P \neq 0$, $k = \deg P$, *and*

$$P(E)\mathbf{f} \in \Omega_{\alpha,\varrho} \tag{1.3}$$

then either $\mathbf{f} \in \mathcal{L}_{\alpha,\varrho}$ *or* $f(n) = n^s u(n)$, *where* $0 \leqq \operatorname{Re} s \leqq k$ *and*

$$P(E)\mathbf{u} = 0. \tag{1.4}$$

REMARK. We shall not determine the solutions of (1.4). From the proof of the theorem it will follow that there exists an integer B such that $u(n) = \chi_B(n)$ whenever $(B, n) = 1$, and χ_B is a suitable character mod B.

2. Notations. For an $n \in \mathbf{N}$ let $p(n)$ be the smallest prime factor of n. For a prime p and an integer n let $\ell_p(n)$ be the exponent of p in n, i.e. $p^{\ell_p(n)} \| n$. For an arbitrary sequence $\mathbf{x}$, $L(x_n, \ldots, x_{n+k})$ or $L_j(x_n, \ldots, x_{n+k})$ denote fixed linear combinations of the variables $x_n, \ldots, x_{n+k}$. For a $k \in \mathbf{N}$ let $\chi_{0,k}(n)$ be the principal character mod k.

3. Let $f \in \mathcal{M}$ and $\mathcal{A}_f = \mathcal{A}$ be the set of those polynomials $P \in \mathbf{C}[z]$ for which $P(E)\mathbf{f} \in \Omega_{\alpha,\varrho}$. Assume that $\mathcal{A}$ contains a nonzero element. Then $P_1, P_2 \in \mathcal{A}$ imply that $c_1P_1 + c_2P_2 \in \mathcal{A}$, furthermore, if $P(z) \in \mathcal{A}$ then $zP(z) \in \mathcal{A}$. Thus, if $P \in \mathcal{A}$ and $Q \in \mathbf{C}[z]$, then $QP \in \mathcal{A}$. Hence we get that $\mathcal{A}$ is an ideal.

Observe furthermore that if $zQ(z) \in \mathcal{A}$, then $Q(z) \in \mathcal{A}$ as well.

The ideal $\mathcal{A}$ is generated by its least degree monic element P_1. All the other elements $P \in \mathcal{A}$ can be written as $P = Q \cdot P_1$, $Q \in \mathbf{C}[z]$.

It is enough to prove the Theorem in the case when P is the generator element of $\mathcal{A}$.

Let P be the generating element of $\mathcal{A}$, $k = \deg P$. If $k = 0$, then $f \in \mathcal{L}_{\alpha,\varrho}$. We may assume from now on that $k \geqq 1$. If $P(0) = 0$, then $P(z) = zQ(z) \in \mathcal{A}$, and $Q \in \mathcal{A}$. This cannot occur, since P was assumed to be the generator element.

Let $\Theta_1, \ldots, \Theta_k$ be the roots of P, $P(z) = \prod_{j=1}^{k} (z - \Theta_j)$. Let $m \geqq 1$ be an integer,

$$Q_m(z) := \prod_{j=1}^{k} (z - \Theta_i^m) = b_0 + b_1 z + \ldots + b_k z^k, \quad b_k = 1.$$

$P(z)$ is a divisor of $Q_m(z^m)$, so $Q_m(E^m)\mathbf{f} \in \Omega_{\alpha,\varrho}$. Then

$$\sum_{n \leqq x} |Q_m(E^m) f(mn)|^\alpha \ll x\varrho(x)^\alpha. \tag{3.1}$$

Let

$$Y_n = Q_m(E^m) f(mn), \qquad Z_n = f(m) Q_m(E) f(n),$$

$$\Delta(n) = Y_n - Z_n = \sum_{j=0}^{k} b_j \{f(m(n+j)) - f(m)f(n+j)\}. \tag{3.2}$$

Since $\mathbf{Y} = (Y_1, Y_2, \dots) \in \Omega_{\alpha,\varrho}$, therefore $P(E)\mathbf{Y} \in \Omega_{\alpha,\varrho}$. Since $P(E)\mathbf{f} \in \Omega_{\alpha,\varrho}$, therefore

$$P(E)\mathbf{Z} = f(m)Q_m(E)P(E)\mathbf{f} \in \Omega_{\alpha,\varrho},$$

consequently

$$P(E)\mathbf{\Delta} \in \Omega_{\alpha,\varrho}, \qquad \mathbf{\Delta} = (\Delta(1), \Delta(2), \dots). \tag{3.3}$$

Let $m = p^a$, where p is a prime larger than $2k+2$. Observe that

$$P(E)\Delta(n) = b_0 (f(mn) - f(m)f(n)) P(0) \quad \text{if} \quad p|n. \tag{3.4}$$

Let n be running over the integers $n = p^b\nu$, where $b \geqq 1$, p^b is fixed and ν is coprime to p.

Then, from (3.3), (3.4) we infer that

$$|f(p^{a+b}) - f(p^a)f(p^b)|^\alpha \sum_{\substack{\nu \leqq x \\ (\nu,p)=1}} |f(\nu)|^\alpha \ll x\varrho^\alpha(x). \tag{3.5}$$

Let

$$S_p(x) = \sum_{\substack{\nu \leqq x \\ (\nu,p)=1}} |f(\nu)|^\alpha. \tag{3.6}$$

Now we prove that

$$\overline{\lim} \frac{S_p(x)}{x\varrho^\alpha(x)} = \infty \qquad (x \to \infty) \tag{3.7}$$

which will imply that

$$f(p^{a+b}) = f(p^a)f(p^b) \tag{3.8}$$

for every $a, b \in \mathbf{N}$.

Assume that (3.7) is not true, i.e.

$$S_p(x) \ll x\varrho(x)^\alpha. \tag{3.9}$$

Let us choose a large constant c. Then

$$\sum_{\substack{n \leqq x \\ \ell_p(n) \leqq c}} |f(n)|^\alpha \ll x\varrho(x)^\alpha \tag{3.10}$$

holds as well.

For an n satisfying $\ell_p(n) > c$ we consider

$$t_n := f(n) - \frac{1}{P(0)} P(E)f(n).$$

It is clear that t_n is a linear combination of $f(n+1), \dots, f(n+k)$, $t_n = c_1 f(n+1) + \dots + c_k f(n+k)$ with suitable constants $c_1, \dots, c_k$, furthermore $\ell_p(n+j) \leqq c$ for every $j = 1, \dots, k$. Thus $\sum_{\substack{n \leqq x \\ \ell_p(n) > c}} |t_n|^\alpha \ll x\varrho^\alpha(x)$, which by our assumption $P(E)\mathbf{f} \in \Omega_{\alpha,\varrho}$ gives that

$$\sum_{\substack{\ell_p(n) > c \\ n \leqq x}} |f(n)|^\alpha \ll x\varrho(x)^\alpha$$

and so by (3.10) we get $\mathbf{f} \in \mathcal{L}_{\alpha,\varrho}$. This contradicts the minimality of P in $\mathcal{A}$.

We proved the following

LEMMA 1. *Let $P(E)\mathbf{f} \in \Omega_{\alpha,\varrho}$ with some polynomial $P(z)$ of degree k. Let P be the smallest degree polynomial with this property. Assume that $f \in \mathcal{M}$ and $k \geqq 1$. Then $f(mn) = f(m)f(n)$ whenever $p(m) > 2k+2$ or $p(n) > 2k+2$.*

Assume now that m is such an integer for which $p(m) > 2k+2$. Then $\Delta(n) = 0$ identically. Consequently $Y_n = Z_n$, and from (3.1) we obtain

$$|f(m)|^\alpha \sum_{n \leqq x} |Q_m(E)|f(n)|^\alpha \ll x\varrho(x)^\alpha. \tag{3.11}$$

(3.11) implies that either $f(m) = 0$ or $Q_m(z) \in \mathcal{A}$. Assume that $f(m) \neq 0$. Since $Q_m(z) \in \mathcal{A}$, $\deg Q_m(z) = k$, therefore it is a minimal degree monic element of $\mathcal{A}$, so $P(z) = Q_m(z)$, consequently

$$\{\Theta_1, \dots, \Theta_k\} = \{\Theta_1^m, \dots, \Theta_k^m\}. \tag{3.12}$$

From (3.12) we infer that $\{\Theta_1, \dots, \Theta_k\} = \{\Theta_1^{m^r}, \dots, \Theta_k^{m^r}\}$ holds for every $r = 1, 2, \dots$. Since $\Theta_j \neq 0$, therefore $|\Theta_j| = 1$ for every root Θ_j. Let $\varphi_j = \frac{\arg \Theta_j}{2\pi}$. If φ_j were an irrational number for some j, then all the numbers $\Theta_j^{m^r}$ would be pairwise distinct, which cannot occur. Consequently φ_j $(j = 1, \dots, k)$ are rational numbers. Let $\varphi_j = \frac{a_j}{B}$ with $(a_1, \dots, a_k, B) = 1$, $B > 0$. Then $\Theta_j^B = 1$ for every j, i.e. Θ_j are Bth roots of unity. Since the multiplicity of the occurrence of some root of unity in the system $\{\Theta_1, \dots, \Theta_k\}$ is at most k, therefore $P(z)$ is a divisor of $(z^B - 1)^k$ and so

$$(E^B - I)^k \mathbf{f} \in \Omega_{\alpha,\varrho}. \tag{3.13}$$

We deduced the relation (3.13) under the assumption that there exists $m \in \mathbf{N}$ with $p(m) > 2k+2$, $f(m) \neq 0$. We shall prove now that this is true, whenever $P(E)\mathbf{f} \in \Omega_{\alpha,\varrho}$, $\mathbf{f} \notin \mathcal{L}_{\alpha,\varrho}$. Indeed, if $f(m) = 0$ were satisfied for every such m, then $f(n) \leqq 0$ could occur at most in the case when n is composed of primes less than $2k+2+1$. Let $a_1 < a_2 < \dots$ be the whole sequence of such integers. It was proved by G. Pólya that $a_{\nu+1} - a_\nu \to \infty$ as $\nu \to \infty$. By this we get

$$f(a_\nu) = \frac{1}{P(0)} P(E) f(a_\nu)$$

if a_ν is a large element. Hence we get

$$\sum_{n \leqq x} |f(n)|^\alpha = \sum_{a_\nu \leqq x} |f(a_\nu)|^\alpha \ll 1 + \frac{1}{|P(0)|^\alpha} \sum_{a_\nu \leqq x} |P(E) f(a_\nu)|^\alpha \ll x \varrho^\alpha(x),$$

i.e. $f \in \mathcal{L}_{\alpha,\varrho}$. This is a contradiction.

So we proved

LEMMA 2. *Assume that $f \in \mathcal{M}$, $f \notin \mathcal{L}_{\alpha,\varrho}$ and there exists a polynomial P of degree k such that $P(E)\mathbf{f} \in \Omega_{\alpha,\varrho}$. Assume that P is a minimal degree polynomial with this property. Then there exists a suitable integer B such that $P(z) | (z^B - 1)^k$, and so $(E^B - I)^k \mathbf{f} \in \Omega_{\alpha,\varrho}$.*

4. Assume that the conditions of our theorem hold; furthermore let k be minimal, $k \geqq 1$. This implies that $\mathbf{f} \notin \mathcal{L}_{\alpha,\varrho}$. If the assertion of Lemma 2 is true with B, then it is true with Br $(r = 1, 2, \dots)$ as well. Therefore we may assume that all the primes up to $2k+2$ divide B. Let us assume this. Let

$$f^*(n) = \chi_{0,B}(n) f(n). \tag{4.1}$$

It is clear that

$$(E^B - I)^k \mathbf{f}^* \in \Omega_{\alpha,\varrho}; \tag{4.2}$$

furthermore $f^* \in \mathcal{M}^*$.

Since $\chi_{0,B}(n) = 1$ for $(n, B) = 1$, therefore $f(n) = f^*(n)$ whenever $(n, B) = 1$. We want to prove that $f^* \notin \mathcal{L}_{\alpha,\varrho}$. This will follow from

LEMMA 3. *If there exists an integer D such that*

$$\sum_{\substack{n \leqq x \\ (n,D)=1}} |f(n)|^\alpha \ll x \varrho^\alpha(x), \tag{4.3}$$

then $f \in \mathcal{L}_{\alpha,\varrho}$.

PROOF. For an arbitrary n let $a(n)$ be the product of the prime factors of n composed from the prime divisors of $[D, B]$, and let $b(n)$ be defined by $n = a(n) b(n)$. Let H be an arbitrary large but fixed integer.

From (4.3) we get

$$\sum_{\substack{n \leqq x \\ a(n) \leqq H}} |f(n)|^\alpha \ll x\varrho^\alpha(x) \qquad (x \to \infty). \tag{4.4}$$

Let $p_1, \dots, p_r$ be the set of the prime divisors of $[D, B]$. Let $B = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, $\alpha_j \geqq 0$. Let $\beta_1, \dots, \beta_r$ and S be large positive integers. For an arbitrary $n \in \mathbf{N}$ let $d_\ell := n + \ell B$ $(\ell = 0, \dots, S-1)$. Then the cardinality of d_ℓ satisfying $p_j^{\beta_j+\alpha_j} | d_\ell$ is at most $s/p_j^{\beta_j} + 1$. Assume that $\beta_1, \dots, \beta_r$, S are so large that

$$S\left(\frac{1}{p_1^{\beta_1}} + \dots + \frac{1}{p_r^{\beta_r}}\right) + r < [S/k + 1]$$

holds. Then there exists an integer $s_n \in [0, S-k)$ for which

$$\ell_{p_j}(n + (s_n + \nu)B) \leqq \beta_j + \alpha_j \qquad (j = 1, \dots, r;\ \nu = 0, \dots, k)$$

holds. Assume that H is so large that $\Pi p_j^{\beta_j+\alpha_j} \leqq H$.

Let $Q(z) = (z^B - 1)^k$. It is clear that

$$|f(n)| \leqq |Q(E)f(n)| + L_1(|f(n+B)|, \dots, |f(n+kB)|).$$

Iterating this inequality, we get that

$$|f(n)| \leqq c_1 \sum_{\ell=0}^{s_n-1} |Q(E)f(n+\ell B)| + c_2 \sum_{\ell=s_n}^{s_n+k} |f(n+\ell B)|,$$

with suitable constants c_1, c_2, which may depend only on S. By using the Hölder inequality, hence we deduce that

$$|f(n)|^\alpha \leqq c_3 \sum_{\ell=0}^{s_n-1} |Q(E)f(n+\ell B)|^\alpha + c_4 \sum_{\ell=s_n}^{s_n+k} |f(n+\ell b)|^\alpha.$$

It is important that $a(n + \ell B) \leqq H$ is satisfied for the integers occurring in the last sum on the right hand side. Summing up for n, taking into account (4.4) and $Q(E)\mathbf{f} \in \Omega_{\alpha,\varrho}$, we get our assertion immediately.

COROLLARY. *We have* $f^* \notin \mathcal{L}_{\alpha,\varrho}$.

5. Assume that B contains all the primes up to $2k+2$, $f^* \in \mathcal{M}^*$, $f^*(p) = 0$ if $p|B$, furthermore that

$$(E^B - I)^k f^* \in \Omega_{\alpha,\varrho}, \tag{5.1}$$

(5.2) $$f^* \notin \mathcal{L}_{\alpha,\varrho}.$$

From these conditions we shall deduce that $f^*(n) = n^s \nu(n)$, $0 \leqq \operatorname{Re} s \leqq \leqq k$,

(5.3) $$(E^B - I)\nu = 0 \quad \text{identically.}$$

We shall use induction on k. The case $k = 1$ was treated in [1]. We shall assume that the assertion is proved for $k-1$ instead of k. We may assume furthermore that the condition is not true for $k-1$ instead of k.

Let

$$H(n) := (E^B - I)^{k-1} f^*(n).$$

Let q be a fixed positive integer coprime to B, $q > 1$. From (5.1) we have

(5.4) $$\sum_{\substack{n \leqq x \\ (n,B)=1}} \max_{0 \leqq \ell \leqq K} |H(n+\ell B) - H(n)|^\alpha \ll x\varrho^\alpha(x) \qquad (x \to \infty)$$

for every fixed K. Let $h = (q-1)(k-1)$, and let $\beta_0, \dots, \beta_h$ be the coefficients of the polynomial $(1 + z + \dots + z^{q-1})^{k-1}$,

$$(1 + z + \dots + z^{q-1})^{k-1} = \beta_0 + \dots + \beta_h z^h.$$

It is clear that $\beta_0 + \dots + \beta_h = q^{k-1}$, furthermore that

(5.5)

$$\left(E^{Bq} - I\right)^{k-1} f^*(qn) = \left(I + E^B + \dots + E^{B(q-1)}\right)^{k-1} \left(E^B - I\right)^{k-1} f^*(qn) =$$

$$= \sum_{j=0}^{h} \beta_j H(qn + jB).$$

Let $(n, B) = 1$. The left hand side of (5.5) is $f^*(q)H(n)$. Let K be a large constant, ℓ_n any integer, $0 \leqq \ell_n \leqq K$. From (5.4) we get that

(5.6) $$H(qn + \ell_n B) = \frac{f^*(q)}{q^{k-1}} H(n) + \varepsilon_{n,\ell_n},$$

where

(5.7) $$\sum_{\substack{n \leqq x \\ (n,B)=1}} |\varepsilon_{n,\ell_n}|^\alpha \ll x\varrho^\alpha(x).$$

Let

(5.8) $$E(x) := \sum_{\substack{n \leqq x \\ (n,B)=1}} |H(n)|^\alpha.$$

For an integer N let $a(N) \in \{0, 1, \dots, q-1\}$ be the integer for which $N - a(N)B$ is a multiple of q. Let N_1 be defined by the equation $N = qN_1 + a(N)B$. It is clear that

$$\frac{N}{q} - B \leqq N_1 \leqq \frac{N}{q}. \tag{5.9}$$

Some fixed integer M plays the role of N_1 for q distinct values of N, namely for $qM + \ell B$ $(\ell = 0, 1, \dots, q-1)$.

From (5.6) we obtain (for $N \geqq qB$, $(N, B) = 1$)

$$H(N) = \frac{f^*(q)}{q^{k-1}} H(N_1) + \varepsilon_{N_1, a(N)}. \tag{5.10}$$

Let $\Theta = \Theta_q = \left| \frac{f^*(q)}{q^{k-1}} \right|$. From (5.10) we get that

$$|H(N)| = \Theta |H(N_1)| + \varrho_{N_1, a(N)}, \quad |\varrho_{N_1, a(N)}| \leqq |\varepsilon_{N_1, a(N)}|. \tag{5.11}$$

If c and d are positive numbers, then

$$|c^\alpha - d^\alpha| = \alpha \left| \int u^{\alpha-1} du \right| \leqq \alpha |c - d| (c^{\alpha-1} + d^{\alpha-1}). \tag{5.12}$$

Furthermore, the Hölder inequality gives that

$$\sum_{n=1}^{x} |u_n| |\nu_n|^{\alpha-1} \leqq \left(\sum |u_n|^\alpha \right)^{1/\alpha} \left(\sum |\nu_n|^\alpha \right)^{\frac{\alpha-1}{\alpha}} \tag{5.13}$$

is true for all complex numbers $u_1, \dots, u_x, \nu_1, \dots, \nu_x$. Thus for positive $c_1, \dots, c_x$, $d_1, \dots, d_x$ we obtain

$$\left| \sum_{i=1}^{x} c_i^\alpha - \sum_{i=1}^{x} d_i^\alpha \right| \leqq \alpha \sum |c_i - d_i| \left(c_i^{\alpha-1} + d_i^{\alpha-1} \right) \leqq \tag{5.14}$$

$$\leqq \alpha \left(\sum |c_i - d_i|^\alpha \right)^{1/\alpha} \left\{ \left(\sum c_i^\alpha \right)^{(\alpha-1)/\alpha} + \left(\sum d_i^\alpha \right)^{(\alpha-1)/\alpha} \right\}.$$

We shall apply this inequality with

$$c_N = |H(N)|, \qquad d_N = \Theta |H(N_1)|.$$

Taking into account (5.7) we get rapidly that

$$E(x) - \Theta^\alpha q E\left(\frac{x}{q} \right) \leqq c x^{1/\alpha} \varrho(x) E(x)^{(\alpha-1)/\alpha}. \tag{5.15}$$

with a suitable positive c.

Similarly, summing up for every such N for which $N_1 \leqq y$ holds, we obtain

$$\Theta^\alpha qE(y) - E(qy+qB) \leqq cy^{1/\alpha}\varrho(y)E(qy+qB)^{(\alpha-1)/\alpha}. \tag{5.16}$$

Let us assume first that there exists a q, $(q,B) = 1$ for which $\Theta = \Theta_q < 1$, i.e. $|f^*(q)| < q^{k-1}$. From this assumption we shall deduce that $(E^B - I)^{k-1}f^* \in \Omega_{\alpha,\varrho}$ contrary to our hypothesis that k was the least number satisfying (5.1).

Let $e(x) = \frac{E(x)}{x\varrho^\alpha(x)}$, and let q be such an integer for which $(q,B) = 1$, $\Theta_q < 1$. Assume that $\overline{\lim}\, e(x) = \infty$. From (5.15) we get that

$$e(x)x\varrho^\alpha(x) \leqq \Theta^\alpha q\frac{x}{q}\varrho^\alpha\left(\frac{x}{q}\right)e\left(\frac{x}{q}\right) + cx\varrho^\alpha(x)e(x)^{\frac{\alpha-1}{\alpha}},$$

and after dividing by $x\varrho^\alpha(x)$ and taking into account that $\frac{\varrho(x/q)}{\varrho(x)} \to 1$ as $x \to \infty$, we obtain that

$$e(x) - ce(x)^{\frac{\alpha-1}{\alpha}} \leqq \Theta^\alpha(1+\varepsilon)e\left(\frac{x}{q}\right) \tag{5.17}$$

is valid for each large x. Here $\varepsilon > 0$ is an arbitrary constant. Let us choose it so that $\Theta^\alpha(1+\varepsilon) < 1-\varepsilon$. Then,

$$e(x) - ce(x)^{\frac{\alpha-1}{\alpha}} \leqq (1-\varepsilon)e\left(\frac{x}{q}\right) \tag{5.18}$$

holds for every large x. From (5.18) we deduce that $e(x)$ is bounded in $[1,\infty)$. Indeed, let Y be a large value which is taken on by $e(y)$ at the point x, so that $e(y) \leqq Y$ whenever $y \leqq x$. From (5.18) we obtain that

$$Y - cY^{(\alpha-1)/\alpha} \leqq (1-\varepsilon)Y,$$

and so $\varepsilon Y \leqq cY^{(\alpha-1)/\alpha}$, $Y^{1/\alpha} \leqq c/\varepsilon$. Y is bounded. From now on we may assume that $|f^*(n)| \geqq n^{k-1}$ holds for every n, $(n,B) = 1$. On the other hand, it is easy to see that $|f^*(n)| \leqq n^k$ if $(n,B) = 1$. Indeed, $\varrho^\alpha(x) = O(x^\varepsilon)$ is true for every $\varepsilon > 0$. From (5.1) we get that

$$|(E^B - I)^{j-1}f^*(n)| \leqq \sum_{\nu<n}|(E^B-I)^j f^*(\nu)| + O(1) \quad (j = 1,\dots,k) \tag{5.19}$$

and

$$\sum_{\nu<n}|(E^B-I)^k f^*(\nu)| \ll n^{1+\varepsilon}. \tag{5.20}$$

Hence we get that $(E^B - I)^{k-1} f^*(\nu) = O(\nu^{1+\varepsilon})$, and by (5.19) that $(E^B - I)^{k-2} f^*(n) = O(n^{2+\varepsilon})$. Repeating this argument, we get that

$$|f^*(n)| \leqq C_\varepsilon n^{k+\varepsilon} \tag{5.21}$$

holds for every $\varepsilon > 0$ with a suitable positive constant C_ε whenever $(n, B) = 1$. Let us write now $n = q^t$ into (5.21). Since $f^*(q^t) = f^*(q)^t$, we obtain that

$$|f^*(q)| \leqq C_\varepsilon^{1/t} q^{k+\varepsilon}.$$

Setting $t \to \infty$, we get $|f^*(q)| \leqq q^k$.

Let now q be fixed, $(q, B) = 1$, $q > 1$, $\Theta = \Theta_q = \left|\frac{f^*(q)}{q^{k-1}}\right|$, and let $\eta_q = \eta$ be defined by $\Theta = q^\eta$. Then $0 \leqq \eta \leqq 1$. We shall prove now that for every $\varepsilon > 0$,

$$\overline{\lim_{x\to\infty}} \frac{e(x)}{x^{\eta\alpha+\varepsilon}} < \infty, \qquad \overline{\lim_{x\to\infty}} \frac{e(x)}{x^{\eta\alpha-\varepsilon}} = \infty. \tag{5.22}$$

This will imply that $|f^*(n)| = n^{k-1+\eta}$ for every n coprime to B, and that $\eta = \eta_n =$ constant.

First we prove the first assertion in (5.22). Let ε, ε_1 be small positive numbers, and let x_0 be so large that (5.17) is true with ε_1 instead of ε, for every $x > x_0$. Then

$$e(x) - ce(x)^{(\alpha-1)/\alpha} \leqq q^{\eta\alpha}(1+\varepsilon_1) e\left(\frac{x}{q}\right), \quad \text{if} \quad x > x_0. \tag{5.23}$$

Let $s(x) = \frac{e(x)}{x^{\eta\alpha+\varepsilon}}$. From (5.23) we obtain

$$s(x) x^{\eta\alpha+\varepsilon} - c x^{(\eta\alpha+\varepsilon)\frac{\alpha-1}{\alpha}} s(x)^{\frac{\alpha-1}{\alpha}} \leqq q^{\eta\alpha}(1+\varepsilon_1)\left(\frac{x}{q}\right)^{\eta\alpha+\varepsilon} s\left(\frac{x}{q}\right),$$

and after dividing by $x^{\eta\alpha+\varepsilon}$,

$$s(x) - c x^{-\frac{1}{\alpha}(\eta\alpha+\varepsilon)} s(x)^{\frac{\alpha-1}{\alpha}} \leqq q^{-\varepsilon}(1+\varepsilon_1) s\left(\frac{x}{q}\right). \tag{5.24}$$

Let ε_1 be so small that $q^{-\varepsilon}(1+\varepsilon_1) < 1 - \varepsilon_1$, say. Repeating the argument used earlier, we deduce immediately that $s(x)$ is bounded.

Let us prove the second assertion in (5.22). If $\eta = 0$ then this follows from the assumption $E(x) \neq O(x\varrho^\alpha(x))$. Let $\eta > 0$, $\varepsilon > 0$ be fixed. Let Y_0 and x_0 be large values such that $e(x_0) \geqq Y_0$. Starting from (5.16), dividing by $y\varrho(y)^\alpha$ we obtain

$$q^{1+\eta\alpha} e(y) \leqq \left(q + \frac{qB}{y}\right) e(qy+qB) \frac{\varrho^\alpha(qy+qB)}{\varrho^\alpha(y)} + \tag{5.25}$$

$$+c\frac{\varrho(qy+qB)^{\alpha-1}}{\varrho(y)^{\alpha-1}}e(qy+qB)^{(\alpha-1)/\alpha}\left(q+\frac{qB}{y}\right)^{(\alpha-1)/\alpha}$$

and for every large y

$$q^{1+\eta\alpha}e(y)\leqq q\left(1+\frac{B}{y}\right)(1+\varepsilon_1)e(qy+qB)+ \tag{5.26}$$

$$+cq\left(1+\frac{B}{y}\right)^{(\alpha-1)/\alpha}(1+\varepsilon_1)e(qy+qB)^{(\alpha-1)/\alpha}.$$

Substitute now $y=x_0$. From (5.26) we obtain that

$$e(qx_0+qB)\geqq q^{\eta\alpha-\varepsilon}e(x_0),$$

assuming that x_0 was so chosen for which x_0 and $e(x_0)$ were large enough. Let now $x_1=qx_0+qB$, $x_{\nu+1}=qx_\nu+qB$ $(\nu=1,2,\dots)$. Then $e(x_{\nu+1})\geqq \geqq q^{\eta\alpha-\varepsilon}e(x_\nu)$, and so $e(x_\nu)\geqq (q^\nu)^{\eta\alpha-\varepsilon}$. Observe that $x_\nu/q^\nu x_0$ is bounded. This proves the second assertion.

Consequently, $f^*(n)=n^{k-1+\eta}t(n)$, where $0\leqq\eta\leqq 1$, $t\in\mathcal{M}^*$, $|t(n)|=1$ for $(n,B)=1$ and $|t(n)|=0$ for $(n,B)>1$.

Since

$$\Delta_B^k f^*(n)=\sum_{l=0}^{k}(-1)^{k-l}\binom{k}{l}(n+lB)^{k-1+\eta}t(n+lB)=$$

$$=\left(\Delta_B^k t(n)\right)n^{k-1+\eta}+O(n^{k-2+\eta}),$$

therefore

$$|\Delta_B^k t(n)|\leqq\frac{|\Delta_B^k f^*(n)|}{n^{k-1+\eta}}+\frac{c}{n}.$$

Hence, by (5.1), and $\varrho(x)\ll x^\varepsilon$, $k\geqq 2$ we obtain that

$$\sum_{(n,B)=1}\frac{|\Delta_B^k t(n)|}{n}<\infty. \tag{5.27}$$

In [2] it was proved that $t(n)=n^{i\tau}\chi_B(n)$, with some real number τ and a suitable character mod B. (See Theorems 2 and 3.)

6. Now we finish the proof of our theorem. Starting from the conditions (1.3) and $f\notin\mathcal{L}_{\alpha,\varrho}$ we deduced that there exist positive integers ℓ,B, $1\leqq\ell\leqq \leqq k$, such that the function $f^*(n)=\chi_{0,B}(n)f(n)\in\mathcal{M}^*$, $(E^B-I)^\ell\mathbf{f}^*\in\Omega_{\alpha,\varrho}$ and $(E^B-I)^{\ell-1}\mathbf{f}^*\notin\Omega_{\alpha,\varrho}$ and $f^*(n)=n^{\ell-1+\eta+i\tau}\chi_B(n)$, with some real number τ, $0 \leqq \eta \leqq 1$. Let now $u(n)$ be defined by $f(n) = n^s u(n)$, $s = \ell - 1 + \eta + i\tau$. Let the coefficients of $P(z)$ be $a_0, \dots, a_k$, $P(z) = a_0 + \dots + a_k z^k$,

$$S(n) = \sum_{j=0}^{k} a_j u(n+j) = P(E)u(n).$$

We shall prove that (1.4) is true. Assume the contrary: there exists an $n_0 \in \mathrm{N}$ for which $S(n_0) \neq 0$. For an arbitrary n let $b(n)$ be the maximal divisor of n which is coprime to B, and let $a(n)$ be defined by $n = a(n)b(n)$. Let now $n_1 < n_2 < \dots$ be the sequence of those integers for which

$$b(n_j + \ell) \equiv b(n_0 + \ell) \mod B, \qquad a(n_j + \ell) = a(n_0 + \ell) \qquad (\ell = 0, \dots, k).$$

It is obvious that $S(n_j) = S(n_0)$ and $\{n_j\}$ has a positive density. Furthermore,

$$P(E)f(n) = \sum_{j=0}^{k} a_j u(n+j)(n+j)^s = n^s S(n) +$$

$$+ \sum_{j=0}^{k} a_j u(n+j)((n+j)^s - n^s).$$

Since $(n+j)^s - n^s = O(n^{\sigma-1})$, $\sigma = \ell - 1 + \eta$, and $u(n+j)$ are bounded on the sequence $\{n_t\}$, therefore $(t \geqq t_0,\ A > 0)$ $|P(E)f(n_t)| \geqq A n_t^{\sigma}$. This contradicts (1.3) if $\sigma > 0$.

Let us consider the case $\sigma = 0$. Then $\ell = 1$, $\eta = 0$. Consequently $|f(n)| = |u(n)| = 1$ for $(n, B) = 1$. By using Lemma 3, we obtain $\mathbf{f} \in \Omega_{\alpha,\varrho}$ which is a contradiction.

The proof of the theorem is complete.

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UNIVERSITY OF PADERBORN
D–4790 PADERBORN
WARBURGER STRASSE 100
BRD

COMPUTER CENTER OF
EÖTVÖS LORÁND UNIVERSITY
BUDAPEST, H–1117
BOGDÁNFY ÚT 10/B
HUNGARY

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ON HIGHER ORDER HERMITE–FEJÉR INTERPOLATION IN WEIGHTED L_p-METRIC

J. SZABADOS* (Budapest) and A. K. VARMA (Gainesville)

Let

$$(1)\qquad (1>)x_1 > x_2 > \ldots > x_n(> -1), \qquad x_k = \cos\theta_k \quad (k = 1, \ldots, n)$$

be the roots of the ultraspherical Jacobi polynomials $P_n^{(\alpha)}(x)$ $(\alpha > -1)$ normalized such that $P_n^{(\alpha)}(1) = \binom{n+\alpha}{n}$. For an arbitrary continuous function $f(x) \in C[-1,1]$ and integer $m \geqq 1$, consider the mth order Hermite–Fejér interpolating polynomial $H_{nm}(f,x)$ defined by

$$H_{nm}^{(j)}(f, x_k) = \delta_{0j} f(x_k) \qquad (k = 1, \ldots, n;\ j = 0, 1, \ldots, m-1).$$

$H_{nm}(f,x)$ is a uniquely determined polynomial of degree at most $mn - 1$.

The case m even has been extensively investigated by P. Vértesi [7, 8]. (Actually, he considered the procedure under more general conditions.) His main results restricted to our particular situation state that for $m = 2, 4, \ldots$,

(a) if $\max\left(-\frac{1}{2} - \frac{2}{m}, -1\right) < \alpha < -\frac{1}{2} + \frac{1}{m}$ then $H_{nm}(f,x)$ converges uniformly in $[-1,1]$;

(b) if $-\frac{1}{2} + \frac{1}{m} \leqq \alpha$, $a > -1$, $0 < p < \frac{4(a+1)}{m(2\alpha+1)-2}$ then

$$\lim_{n\to\infty} \int_{-1}^{1} |f(x) - H_{nm}(f,x)|^p (1-x^2)^a \, dx = 0$$

for all $f(x) \in C[-1,1]$.

We also note that for the special case $m = 2$, P. Vértesi and Y. Xu [9] gave an error estimate for the mean convergence.

Our purpose here is to settle the corresponding problems for $m = 1, 3, \ldots$. (At this point we mention that for $m = 1$ (i.e. Lagrange interpolation) the problem has been completely solved by P. Nevai [1], [2].) Although in stating our Theorem 1 we will not restrict ourselves to odd m's, this case will be of main interest because of the above quoted results of P. Vértesi.

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In case m odd we cannot expect uniform convergence (see P. Vértesi [7, Theorem 2.7]). In fact, we proved in [4] that for $m = 3$, there is no uniform convergence for *any* system of nodes. This justifies that we turn to investigating mean convergence.

Let us introduce the notation

$$\|f\|_{p,a} = \left(\int_{-1}^{1} |f(x)|^p (1-x^2)^a dx \right)^{1/p} \qquad (p > 0, \ a > -1)$$

for an arbitrary $f \in C[-1,1]$, and let $\omega(f,h)$ be the ordinary modulus of continuity of $f(x)$.

THEOREM 1. *We have for $m = 1,2,\dots$, $a > -1$ and $f \in C[-1,1]$*

$$\|f(x) - H_{nm}(f,x)\|_{p,a} = O\left(\omega\left(f, \frac{1}{n}\right)\right)$$

provided one of the following two conditions holds:

(i) $\max\left(-\frac{1}{2} - \frac{2}{m}, -1\right) < \alpha \leqq -\frac{1}{2}$, $p > 0$;

(ii) $\alpha > -\frac{1}{2}$, $0 < p < \frac{4(a+1)}{m(2\alpha+1)}$.

PROOF. From the notion of Hermite interpolation it follows that there exist numbers e_{ik} such that with

(2)
$$h_{jk}(x) = \frac{\ell_k(x)^m}{j!} \sum_{i=j}^{m-1} e_{i-j,k}(x-x_k)^i \qquad (j = 0,1,\dots,m-1; \ k = 1,\dots,n)$$

($\ell_k(x)$ are the fundamental polynomials of Lagrange interpolation based on the roots (1)) we have

$$p(x) = \sum_{j=0}^{m-1} \sum_{k=1}^{n} p^{(j)}(x_k) h_{jk}(x) \tag{3}$$

for any polynomial of degree at most $nm - 1$, and

$$H_{nm}(f,x) = \sum_{k=1}^{n} f(x_k) h_{0k}(x). \tag{4}$$

Here

$$|e_{ik}| = O\left(\left(\frac{n}{\sin\theta_k}\right)^i \frac{1}{(n\sin\theta_k)^{i-2[i/2]}}\right) \qquad (i = 0,1,\dots; \ k = 1,\dots,n) \tag{5}$$

(see P. Vértesi [7, Lemma 3.11]).

Now let $p(x)$ be the polynomial of best approximation to $f(x)$ of degree at most $nm-1$; then by Jackson's theorem

$$\max_{|x|\leqq 1}|f(x)-p(x)|=O\left(\omega\Big(f,\frac{1}{n}\Big)\right), \tag{6}$$

and by a well-known result of S. B. Steckin (see A. F. Timan [6], p. 252)

$$|p^{(j)}(x)|=O\left(\omega\Big(f,\frac{1}{n}\Big)n^j\right)\min\left(n^j,(1-x^2)^{-j/2}\right)\quad (|x|\leqq 1,\ j=0,1,\dots). \tag{7}$$

Thus we obtain by (2)–(4)

$$p(x)-H_{nm}(p,x)=\sum_{j=1}^{m-1}\sum_{k=1}^{n}p^{(j)}(x_k)h_{jk}(x)=\sum_{i=0}^{m-1}\sum_{k=1}^{n}\alpha_{ik}(x-x_k)^i\ell_k(x)^m$$

where by (5) and (7)

$$\alpha_{ik}=\sum_{j=1}^{m-1}e_{i-j,k}p^{(j)}(x_k)=O\left(\omega\Big(f,\frac{1}{n}\Big)\right)\sum_{j=1}^{m-1}\left(\frac{n}{\sin\theta_k}\right)^{i-j}\left(\frac{n}{\sin\theta_k}\right)^{j}= \tag{8}$$

$$=O\left(\omega\Big(f,\frac{1}{n}\Big)\right)\left(\frac{n}{\sin\theta_k}\right)^{i}\qquad (i=0,\dots,m-1;\ k=1,\dots,n)$$

with the understanding that $e_{i-j,k}=0$ if $i<j$. Hence and by (6) and (2)

$$f(x)-H_{nm}(f,x)=f(x)-p(x)+p(x)-H_{nm}(p,x)+H_{nm}(p-f,x)= \tag{9}$$

$$=O\left(\omega\Big(f,\frac{1}{n}\Big)\right)+\sum_{k=1}^{n}\sum_{i=0}^{m-1}\alpha_{ik}(x-x_k)^i\ell_k(x)^m+$$

$$+\sum_{k=1}^{n}[p(x_k)-f(x_k)]\sum_{i=0}^{m-1}e_{ik}(x-x_k)^i\ell_k(x)^m=$$

$$=O\left(\omega\Big(f,\frac{1}{n}\Big)\right)+\sum_{k=1}^{n}\sum_{i=0}^{m-1}\beta_{ik}(x-x_k)^i\ell_k(x)^m$$

where by (8), (5) and (6)

$$\beta_{ik}=\alpha_{ik}+e_{ik}[p(x_k)-f(x_k)]=O\left(\omega\Big(f,\frac{1}{n}\Big)\right)\left(\frac{n}{\sin\theta_k}\right)^{i} \tag{10}$$

$$(i = 0, \dots, m-1;\ k = 1, \dots, n).$$

Now using the estimates

$$\text{(11)}\qquad P_n^{(\alpha)}(x) = O(\Delta_n(x)^{-\alpha-1/2} n^{-1/2}) \qquad (\alpha > -1,\ |x| \leqq 1)$$

(where $\Delta_n(x) = \sqrt{1-x^2} + 1/n$),

$$\theta_k \sim \frac{k\pi}{n} \qquad (k = 1, \dots, n),$$

and

$$\text{(12)}\qquad P_n^{(\alpha)\prime}(x_k) \sim n^{1/2} \sin^{-\alpha-3/2} \theta_k \qquad (k = 1, \dots, n)$$

(cf. G. Szegö [5], (7.32.5), (8.9.1) and (8.9.2)), as well as the notation

$$|\theta - \theta_j| = \min_{1 \leqq k \leqq n} |\theta - \theta_k|,$$

we obtain from (10) and $(\alpha + 1/2)m + 2 \geqq 0$ (see condition (i) in Theorem 1)

$$\text{(13)}\quad \left|\sum_{k=1}^{n} \beta_{ik}(x-x_k)^i \ell_k(x)^m\right| = |P_n^{(\alpha)}(x)|^m \sum_{k \neq j} \frac{|\beta_{ik}|}{|P_n^{(\alpha)\prime}(x_k)|^m |x-x_k|^{m-i}} +$$

$$+O(\beta_{ij}|x-x_j|^i) = O\left(n^{-m}\Delta_n(x)^{-(\alpha+1/2)m}\omega\Big(f, \frac{1}{n}\Big)\right) \sum_{k \neq j} \left(\frac{n|x-x_k|}{\sin\theta_k}\right)^i \cdot$$

$$\cdot \frac{\sin^{(\alpha+3/2)m}\theta_k}{|x-x_k|^m} + O\left(\omega\Big(f, \frac{1}{n}\Big)\right) = O\left(n^{-2}\Delta_n(x)^{-(\alpha+1/2)m}\omega\Big(f, \frac{1}{n}\Big)\right) \cdot$$

$$\cdot \frac{\sin^{(\alpha+1/2)m+2}\theta_k}{\sin^2\frac{\theta-\theta_k}{2}\sin^2\frac{\theta+\theta_k}{2}} + O\left(\omega\Big(f, \frac{1}{n}\Big)\right) = O\left(n^{-2}\Delta_n(x)^{-(\alpha+1/2)m}\omega\Big(f, \frac{1}{n}\Big)\right) \cdot$$

$$\cdot \sum_{\sin\theta_k < \sin\theta} \frac{\sin^{(\alpha+1/2)m+2}\theta_k}{(\theta-\theta_k)^2} + O\left(n^{-2}\Delta_n(x)^{-(\alpha+1/2)m}\omega\Big(f, \frac{1}{n}\Big)\right) \cdot$$

$$\cdot \sum_{\sin\theta_k > \sin\theta} \frac{\sin^{(\alpha+1/2)m}\theta_k}{(\theta-\theta_k)^2} + O\left(\omega\Big(f, \frac{1}{n}\Big)\right) = O\left(\omega\Big(f, \frac{1}{n}\Big)\right) \sum_{\substack{k=1\\k\neq j}}^{n} \frac{1}{(k-j)^2} +$$

$$+\begin{cases} \sum\limits_{\substack{k=1\\k\neq j}}^{n} \frac{1}{(j-k)^2} & \text{if } \alpha \leqq -1/2 \\ \Delta_n(x)^{-(\alpha+1/2)m} \sum\limits_{\substack{k=1\\k\neq j}}^{n} \frac{1}{(k-j)^2} & \text{if } \alpha \geqq -1/2 \end{cases} =$$

$$=O\left(\omega\left(f,\frac{1}{n}\right)\right)\cdot\begin{cases}O(1) & \text{if } \alpha \leqq -1/2 \\ (1-x^2)^{-m(\alpha+1/2)/2} & \text{if } \alpha \geqq -1/2\end{cases} \quad (0 \leqq i \leqq m-2, \ |x| \leqq 1).$$

Thus in case $\alpha \leqq -\frac{1}{2}$ the quantity (13) is of the required order even in the uniform norm. When $\alpha > -\frac{1}{2}$, then by condition (ii)

$$\int_{-1}^{1}\left|\sum_{k=1}^{n}\beta_{ik}(x-x_k)^i\ell_k(x)^m\right|^p(1-x^2)^a dx = O\left(\omega\left(f,\frac{1}{n}\right)^p\right)\int_{-1}^{1}(1-x^2)^{a-\frac{pm}{4}(2\alpha+1)}dx =$$

$$= O\left(\omega\left(f,\frac{1}{n}\right)^p\right) \quad (0 \leqq i \leqq m-2, \ m \geqq 2).$$

All that remained to estimate is
(14)
$$A_n(x) = \sum_{k=1}^{n}\beta_{m-1,k}(x-x_k)^{m-1}\ell_k(x)^m = P_n^{(\alpha)}(x)^{m-1}n^{\frac{m-1}{2}}\omega\left(f,\frac{1}{n}\right)\sum_{k=1}^{n}\gamma_k\ell_k(x)$$

where by (10) and (12)
(15)
$$\gamma_k = \frac{\beta_{m-1,k}}{P_n^{(\alpha)\prime}(x_k)^{m-1}n^{\frac{m-1}{2}}\omega\left(f,\frac{1}{n}\right)} = O\left(\sin^{\frac{m-1}{2}(2\alpha+1)}\theta_k\right) \quad (k=1,\dots,n).$$

Here
$$\gamma_k = O\left(n^{\frac{1-m}{2}(2\alpha+1)}\right) \qquad \left(k=1,\dots,n; \ \alpha \leqq -\frac{1}{2}\right),$$

whence and by (11)

$$|A_n(x)| = O\left(|P_n^{(\alpha)}(x)|^{m-1}\omega\left(f,\frac{1}{n}\right)n^{\alpha(1-m)}\right)\sum_{k=1}^{n}|\ell_k(x)| =$$

$$= O\left(\omega\left(f,\frac{1}{n}\right)\log n\right) \qquad \left(\alpha \leqq -\frac{1}{2}, \ 1-x^2 \leqq \frac{c}{n^2}\right)$$

(see G. Szegő [5, the proof of Theorem 14.4]). Therefore

$$\int_{1-x^2\leqq\frac{c}{n^2}}|A_n(x)|^p(1-x^2)^a dx = O\left(\omega\left(f,\frac{1}{n}\right)^p\log^p n\right)\int_{1-x^2\leqq\frac{c}{n^2}}(1-x^2)^a dx =$$

$$= O\left(\omega\left(f,\frac{1}{n}\right)^p\right)\frac{\log^p n}{n^{2a+2}} = O\left(\omega\left(f,\frac{1}{n}\right)^p\right) \qquad \left(\alpha \leqq -\frac{1}{2}\right).$$

This shows that instead of estimating (14) we can estimate the quantity

$$B_n(x) = (1-x^2)^{(1-m)(2\alpha+1)/4}\omega\left(f, \frac{1}{n}\right) \sum_{k=1}^{n} \gamma_k \ell_k(x) \tag{16}$$

obtained from (14) by using the estimate

$$|P_n^{(\alpha)}(x)| = O\left((1-x^2)^{-(2\alpha+1)/4} n^{-1/2}\right) \qquad (|x| < 1)$$

valid for $\alpha \geqq -1/2$ (see (11)). Here we apply the following special case of a more general theorem of P. Nevai [2, Theorem 1]:

Let $\alpha > -1$, $0 < p < \infty$, $b > -1$ and c an arbitrary real number. If

$$b + cp > -1, \quad b > \frac{2\alpha+1}{4}p - 1 \quad \text{and} \quad c > -\frac{2\alpha+5}{4} \tag{17}$$

then

$$\sup_{n \geqq 1} \|H_{n,1}\left((1-x^2)^c f(x), \cdot\right)\|_{p,b} \leqq \text{const}\|f\|_\infty \tag{18}$$

for every bounded function $f(x)$, with some constant independent of f.

Now apply this with

$$b = a - \frac{2\alpha+1}{4}(m-1)p, \qquad c = \frac{2\alpha+1}{4}(m-1) \tag{19}$$

and

$$f(x) = \begin{cases} \gamma_k (1-x_k^2)^{(1-m)(2\alpha+1)/4} & \text{if } x = x_k \ (k = 1, \dots, n) \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

Then by (i)–(ii) of Theorem 1, conditions (17) are satisfied; moreover, by (15), $\|f\|_\infty = O(1)$. Thus we obtain by (16), (19) and (18)

$$\|B_n(x)\|_{p,a} = \omega\left(f, \frac{1}{n}\right) \left\| \sum_{k=1}^{n} \gamma_k \ell_k(x) \right\|_{p,b} = \omega\left(f, \frac{1}{n}\right) \left\| H_{n,1}((1-x^2)^c f(x), \cdot) \right\|_{p,b} =$$

$$= O\left(\omega\left(f, \frac{1}{n}\right)\right),$$

which proves Theorem 1.

We now prove that the restriction (ii) in Theorem 1 cannot be essentially loosened.

THEOREM 2. *Let $m \geqq 1$ be an odd number, $\alpha > -1/2$, $a > -1$ and assume that*

$$p > \frac{4(a+1)}{m(2\alpha+1)}. \tag{21}$$

Then

$$\lim_{n\to\infty} \sup_{\|f\|_C \leqq 1} \|H_{nm}(f,x)\|_{p,a} = \infty.$$

PROOF. Using (3.11) from R. Sakai and P. Vértesi [3], we obtain with a certain $k_0 = k_0(m)$

$$e_{m-1,k} \sim \left(\frac{n}{\sin\theta_k}\right)^{m-1} \quad (k_0 \leqq k \leqq n-k_0). \tag{22}$$

Now choose an $f(x) \in C[-1,1]$ such that $\|f\|_C = 1$ and

$$f(x_k) = \begin{cases} 0 & \text{if } 1 \leqq k \leqq k_0 \text{ or } n-k_0 < k \leqq n \\ (-1)^k & \text{if } k_0 \leqq k \leqq n-k_0. \end{cases}$$

Then we obtain from (4), (2), (22) and (5)

$$H_{nm}(f,1) = \sum_{k=k_0}^{n-k_0} |\ell_k(1)|^m \Big|\sum_{i=0}^{m-1} e_{ik}(1-x_k)^i\Big| \geqq$$

$$\geqq \sum_{k=k_0}^{n-k_0} |\ell_k(1)|^m \left\{ e_{m-1,k}(1-x_k)^{m-1} - \sum_{i=0}^{m-2} |e_{ik}|(1-x_k)^i \right\} \geqq$$

$$\geqq \sum_{k=k_0}^{n-k_0} |\ell_k(1)|^m \left\{ \frac{1}{2} b_{\frac{m-1}{2}} (n \sin\theta_k)^{m-1} - O\left((n\sin\theta_k)^{m-3}\right) \right\} \geqq$$

$$\geqq C_m \sum_{k=k_0}^{n-k_0} |\ell_k(1)|^m (n\sin\theta_k)^{m-1}$$

with some $c_m > 0$, if only k_0 is chosen large enough (independently of n). Hence and by (11), (12)

$$H_{nm}(f,1) \geqq c'_m \sum_{k=k_0}^{n-k_0} \frac{n^{\alpha m} \sin^{(\alpha+\frac{3}{2})m} \theta_k}{n^{m/2}} (n\sin\theta_k)^{m-1} \geqq$$

$$\geqq c'_m n^{m(\alpha+\frac{1}{2})-1} \sum_{k=k_0}^{[n/2]} \sin^{m(\alpha+\frac{1}{2})-1} \theta_k \geqq c''_m n^{m(\alpha+\frac{1}{2})},$$

since $\sin\theta_k \sim k/n$ $(k \leqq n/2)$. Applying Lemma 5 from Nevai [1], we obtain by (21)

$$\|H_{n,m}(f,x)\|_{p,a} \geqq cn^{-\frac{2(a+1)}{p}} |H_{n,m}(f,1)| \geqq c'n^{m(\alpha+\frac{1}{2})-\frac{2(a+1)}{p}} \to \infty$$

as $n \to \infty$.

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
H–1364 BUDAPEST, P.O.B. 127
HUNGARY

UNIVERSITY OF FLORIDA
DEPARTMENT OF MATHEMATICS
GAINESVILLE, FLORIDA 32611
U.S.A.

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GENERAL ABSOLUTES OF TOPOLOGICAL SPACES

Á. CSÁSZÁR (Budapest), member of the Academy

0. Introduction. This paper is a continuation of [2]. Terminology and notation are, unless explicitly mentioned, taken from there; however, many of the definitions and results are recalled below.

A *Ponomarev absolute* of a topological space is an extremally disconnected (= the closure of an open set is open) (briefly: EDC) space of which the given space is the image under an ultraperfect map ([7] for T_2-spaces, [11] for the general case). A map f is *ultraperfect* iff it is continuous, closed, irreducible, compact, and separated ($x_1 \neq x_2$, $f(x_1) = f(x_2)$ imply that x_1 and x_2 have disjoint neighbourhoods) (in [2], separatedness is not included in the definition). An *Iliadis absolute* of a space is a regular EDC space of which the given space is the image under a ϑ-perfect map ([5] for T_2-spaces, [2] for the general case). A map f is *ϑ-perfect* iff it is ϑ-continuous ($f(x) \in V$, V open imply that there is an open U with $x \in U$, $f(\bar{U}) \subset \bar{V}$), closed, irreducible, compact, and separated (without separatedness in [2]).

A Ponomarev absolute of X can be constructed (see [8]) as follows. Let UX denote the set of all maximal open filters in X, equipped with the topology for which the sets

$$s(H) = \{\mathfrak{s} \in UX : H \in \mathfrak{s}\} \qquad (H \subset X \text{ open}) \tag{1}$$

constitute a base. UX is a compact T_2-space, and the sets (1) are clopen in UX. Now take the product space $X \times UX$ and its subspace PX on the subset

$$\alpha X = \{(X, \mathfrak{s}) \in X \times UX : \mathfrak{s} \to x \text{ in } X\}. \tag{2}$$

Then PX is EDC and the map

$$k_X : \alpha X \to X, \qquad k_X(x, \mathfrak{s}) = x \tag{3}$$

is ultraperfect from PX onto X.

In order to obtain an Iliadis absolute a similar construction can be applied (see [2]). We take $X \times UX$ equipped with the product of the indiscrete topology on X and the above topology on UX, and the subspace EX on the set αX. Then EX is regular, EDC, and $k_X : EX \to X$ is ϑ-perfect.

If X is EDC then $k_X : PX \to X$ is a homeomorphism, and the same holds for $k_X : EX \to X$ if X is regular and EDC.

Now PX and EX are essentially the unique Ponomarev and Iliadis absolutes of X, respectively. More precisely, if $f\colon Z \to X$ is ϑ-perfect, then there is a unique map $f^*\colon \alpha Z \to \alpha X$ that is continuous from PZ to EX and satisfies

$$f \circ k_Z = k_X \circ f^*; \tag{4}$$

$f^*\colon EZ \to EX$ is a homeomorphism. Thus, if Z is regular and EDC, then $f^* \circ k_Z^{-1}\colon Z \to EX$ is a homeomorphism such that $f = k_X \circ (f^* \circ k_Z^{-1})$.

If f is ultraperfect then, by [8], $f^*\colon PZ \to PX$ is a homeomorphism. Consequently, if in addition Z is EDC, then $f^* \circ k_Z^{-1}\colon Z \to PX$ is a homeomorphism satisfying $f = k_X \circ (f^* \circ k_Z^{-1})$.

The main purpose of this paper is to study a generalization of the concepts of Ponomarev and Iliadis absolutes, and to illustrate this generalization by a concrete special case.

1. Absolutes of regular spaces. For regular spaces, the Ponomarev and Iliadis absolutes coincide:

THEOREM 1.1. *For a topological space X, the following statements are equivalent:*

(a) *X is regular,*

(b) *$PX = EX$,*

(c) *PX is regular.*

PROOF. (a)$\Longrightarrow$(b): $k_X\colon EX \to X$ is ϑ-perfect. As a ϑ-continuous map to a regular space is continuous, k_X is ultraperfect as well. Hence there is a homeomorphism $h\colon EX \to PX$ such that $k_X = k_X \circ h$. Since $h^{-1}\colon PX \to EX$ and $\mathrm{id}_{\alpha X}\colon PX \to EX$ are both continuous and

$$\mathrm{id}_X \circ k_X = k_X \circ h^{-1} = k_X \circ \mathrm{id}_{\alpha X},$$

necessarily $h^{-1} = \mathrm{id}_{\alpha X}$ so that $PX = EX$.

(b)$\Longrightarrow$(c): obvious.

(c)$\Longrightarrow$(a): see Lemma 1.2. □

LEMMA 1.2. *If $f\colon Y \to Z$ is ultraperfect and Y is regular then Z is regular, too.*

PROOF. Let V be an open neighbourhood of $z \in Z$. Since $f^{-1}(z)$ is compact and $f^{-1}(V)$ is open in the regular space Y, there is an open set $U \subset Y$ such that

$$f^{-1}(z) \subset U \subset \bar{U} \subset f^{-1}(V).$$

Then

$$z \in Z - f(Y - U) \subset f(U) \subset f(\bar{U}) \subset V,$$

and $Z - f(Y - U)$ is open, $f(\bar{U})$ is closed in Z. □

REMARK 1.3. The proof remains valid if f is continuous, surjective, closed, and compact (i.e. perfect according to the usual terminology), cf. [3], 3.7.20.

2. The categories rTop and δTop. It is well-known that the regular open (briefly: r-open) subsets of a topological space X constitute a base for a coarser topology. Thus we obtain a space rX (RX in [2]), the *semi-regularization* of X. The terminology is motivated by the fact that rX is always *semi-regular* (i.e. the r-open sets in rX constitute a base for rX) because X and rX have the same r-open subsets; in fact, for any open subset $G \subset X$, we have $\mathrm{cl}_X G = \mathrm{cl}_{rX} G$ (and dually $\mathrm{int}_X F = \mathrm{int}_{rX} F$ for any closed set $F \subset X$). Hence $\mathrm{cl}_Y G = \mathrm{cl}_X G$ for any set $G \subset X$ open in X, $\mathrm{int}_Y F = \mathrm{int}_X F$ for any set $F \subset X$ closed in X, and for any space Y lying between X and rX (i.e. having the same underlying set and a topology finer than that of rX and coarser than that of X); in this case X and Y contain the same r-open sets, consequently $rY = rX$.

A regular space is obviously semi-regular. Observe that, in an EDC space, r-open sets coincide with clopen sets, thus an EDC space is semi-regular iff it is regular. Conversely:

EXAMPLE 2.1 (cf. [10], p. 100). Let $Y = \{p\} \cup Q$ where Q is the unit square $(0,1) \times (0,1)$ and $p \notin Q$. For $z \in Q$, let the Euclidean plane neighbourhoods constitute a neighbourhood base in Y, while the neighbourhood filter of p is generated by the filter base composed of the sets

$$(2.1.1) \qquad V_\varepsilon = \{p\} \cup \left(\left(0, \frac{1}{2}\right) \times (0, \varepsilon)\right) \qquad (\varepsilon > 0).$$

The space is clearly T_2, the sets V_ε are r-open, but V_1 does not contain any closed neighbourhood of p; hence Y is semi-regular without being regular. □

The fact that $rrX = rX$ makes plausible the conjecture that the semi-regular spaces constitute a bireflective subcategory in **Top**, rX being the reflection of X. However, this is not true because there exist a semi-regular space Y and a closed subspace $X \subset Y$ that is not semi-regular; then the embedding $f\colon X \to Y$ is continuous without $f\colon rX \to rY$ being continuous (see [3], 2.7.6). The example below produces a similar phenomenon with a bijective map f:

EXAMPLE 2.2. Let Y be the space in 2.1 and X be a space with the same underlying set and the same neighbourhoods of p, but, for $z = (a, b) \in Q$, let a neighbourhood base be composed of the sets

$$(a - \varepsilon, a] \times (b - \varepsilon, b + \varepsilon) \subset Q \qquad (\varepsilon > 0).$$

Then id: $X \to Y$ is continuous but the r-open set $V_1 \subset Y$ is not open in rX, i.e. it is not a union of r-open sets in X. In fact, one of the members of this union, say G, would contain p and then a set $V_\varepsilon \subset G$. But $\mathrm{cl}_X V_\varepsilon =$

$= \{p\} \cup \left((0,\frac{1}{2}] \times (0,\varepsilon]\right) \subset \mathrm{cl}_X G$, and any point $\left(\frac{1}{2}, y\right) \in \mathrm{cl}_X V_\varepsilon$ $(0 < y < \varepsilon)$ belongs to $\mathrm{int}_X\, \mathrm{cl}_X V_\varepsilon \subset \mathrm{int}_X\, \mathrm{cl}_X G$. Thus G cannot be r-open in X, and $\mathrm{id}\colon rX \to rY$ fails to be continuous. □

However, the character of rX as a reflection can be saved if we replace the category **Top** by another one. For this purpose, let us say that a map $f\colon X \to Y$ is *r-continuous* (R-map in [1]) iff $f^{-1}(G)$ is r-open in X whenever G is r-open in Y. It is said to be *δ-continuous* [6] iff $f^{-1}(G)$ is a union of r-open sets in X whenever G is r-open in Y. We also recall that f is said to be *almost continuous* [9] iff $f^{-1}(G)$ is open in X whenever G is r-open in Y.

LEMMA 2.3 ([4]). *$f\colon X \to Y$ is δ-continuous iff $f\colon rX \to rY$ is continuous, and almost continuous iff $f\colon X \to rY$ is continuous.* □

LEMMA 2.4. *The following implications hold for any map:*

$$\begin{array}{c} \text{continuous} \\ \Downarrow \\ r\text{-continuous} \Rightarrow \delta\text{-continuous} \Rightarrow \text{almost continuous} \Rightarrow \vartheta\text{-continuous.} \end{array}$$

PROOF. Only the last implication is not obvious. (Cf. [9], Remark 3.3.) Let $f\colon X \to Y$ be almost continuous, $x \in X$, $V \subset Y$ an open neighbourhood of $f(x)$. Then $U = f^{-1}(\mathrm{int}\,\bar{V})$ is an open neighbourhood of x, and $\bar{U} \subset \subset f^{-1}(\bar{V})$ because $\bar{V} \subset Y$ is r-closed and $f^{-1}(\bar{V})$ is closed. Thus f is ϑ-continuous. □

None of the above implications can be reversed.

EXAMPLE 2.5. Let $X = Y = \mathbf{R}$, and let X be equipped with the Sorgenfrey topology, Y with the Euclidean one, $f\colon X \to Y = \mathrm{id}_{\mathbf{R}}$. Then f is (continuous and) δ-continuous because X is regular, hence every open set is a union of r-open sets. However, the interval $(0,1)$ is r-open in Y without being so in X. Thus f is not r-continuous. □

In 2.2, id is continuous without being δ-continuous. If X is not semi-regular, then $\mathrm{id}\colon rX \to X$ is r-continuous without being continuous. In [9], Example 2.3, a ϑ-continuous map $f\colon X \to Y$ is defined that is not almost continuous; however, Y is not T_1 in this example. In the following one, X can be chosen to be T_2:

EXAMPLE 2.6. Let X be a semi-regular, non-regular space. Then $k_X\colon EX \to X$ is ϑ-continuous. We show that it is not almost continuous.

By 1.1, we have $PX \neq EX$. Thus there is a set open in PX but not in EX. We can choose this set in the form $U_0 = (G \times s(H)) \cap \alpha X$ where H is open and G is r-open in X (since X is semi-regular). Then U_0 is not a union of sets of the form $(X \times s(H_i)) \cap \alpha X$, H_i open in X. If $k_X^{-1}(G) = (G \times UX) \cap \alpha X$ were a union of sets of the above form, then the same would hold for U_0 (because $s(H_i) \cap s(H) = s(H_i \cap H)$). Hence $k_X\colon EX \to X$ is not almost continuous. □

In the opposite sense, we can say:

LEMMA 2.7. *An almost continuous map to an EDC space is r-continuous.*

PROOF. If $f\colon X \to Y$ is almost continuous, Y is EDC, and $G \subset Y$ is r-open, then it is clopen and r-closed, hence $f^{-1}(G)$ is clopen and r-open in X. □

LEMMA 2.8. $\mathrm{id}_X\colon X \to rX$ *and* $\mathrm{id}_X\colon rX \to X$ *are both r-continuous.* □

LEMMA 2.9. *If* $f\colon X \to Y$ *and* $g\colon Y \to Z$ *are both r-continuous or δ-continuous, then so is* $g \circ f\colon X \to Z$. □

LEMMA 2.10. *If* $f\colon X \to Y$ *is r-continuous or δ-continuous, then so is* $f\colon rX \to Y$, *too.* □

PROOF. 2.8. □

By 2.9, we obtain two categories r**Top** and δ**Top** with the topological spaces as objects and the r-continuous or δ-continuous maps as morphisms, respectively. Now 2.8 and 2.10 furnish:

THEOREM 2.11. *The semi-regular spaces constitute a bireflective subcategory with the reflection rX of X in any of the following categories:*

$$r\mathbf{Top},\quad \delta\mathbf{Top},\quad r\mathbf{Top}\cap\mathbf{Top},\quad \delta\mathbf{Top}\cap\mathbf{Top}.$$

PROOF. For the two last mentioned categories, observe that an almost continuous map to a semi-regular space is continuous. □

3. General absolutes. Let us call *absolute* of a topological space every EDC space of which the given space is the image under a ϑ-perfect map. Thus the Ponomarev and Iliadis absolutes are special cases of this general concept.

Our purpose is to find all possible absolutes of a given space.

LEMMA 3.1. *If X is an EDC space, then every space lying between X and rX is EDC.*

PROOF. Let Y be a space lying between X and rX. If G is open in Y, it is open in X, hence $\mathrm{cl}_X G = \mathrm{cl}_Y G$. Now $\mathrm{cl}_X G$ is clopen in X, hence r-open, so that it is r-open in Y. □

THEOREM 3.2. *If Y is a space lying between PX and EX then Y is EDC and* $k_X\colon Y \to X$ *is ϑ-perfect. Therefore Y is an absolute of X.*

PROOF. By [2], 6.3, $EX = rPX$. Hence Y is EDC by 3.1. $k_X\colon Y \to X$ is ϑ-continuous and separated because so is $k_X\colon EX \to X$, and it is closed, irreducible, and compact because so is $k_X\colon PX \to X$. □

The following theorem says that the converse is essentially true:

THEOREM 3.3. *Let Z be an EDC space and $f\colon Z \to X$ ϑ-perfect. Then there exists a unique space Y lying between PX and EX and a unique homeomorphism $h\colon Z \to Y$ such that $f = k_X \circ h$.*

PROOF. There is a homeomorphism $f^*\colon EZ \to EX$ such that $f \circ k_Z = k_X \circ f^*$. Since Z is EDC, $k_Z\colon PZ \to Z$ is a homeomorphism, and $h = f^* \circ k_Z^{-1}\colon Z \to EX$ is bijective and continuous. Define Y to have the underlying set αX and the quotient topology with respect to h; then $h\colon Z \to Y$ is a homeomorphism. Since $h\colon Z \to EX$ is continuous, the topology of Y is finer than that of EX.

Now let $F \subset \alpha X$ be closed in Y, $(x, \mathfrak{s}) \in \alpha X - F$. By

$$k_X = f \circ k_Z \circ f^{*-1} = f \circ h^{-1},$$

$k_X\colon Y \to X$ is closed and compact. Now $k_X^{-1}(x)$ is compact in Y and T_2 in EX (it is homeomorphic to a subspace of UX), consequently the topologies of Y and EX coincide on the subspace $k_X^{-1}(x)$. Therefore there is an open set $H \subset X$ such that

$$\mathfrak{s} \in s(H), \qquad (X \times s(H)) \cap F \cap k_X^{-1}(x) = \emptyset. \tag{3.3.1}$$

As $s(H)$ is closed in UX, $(X \times s(H)) \cap F$ is closed in Y and $k_X((X \times s(H)) \cap F)$ is closed in X. By (3.3.1), x does not belong to the latter set, and there is an open set $G \subset X$ such that

$$x \in G, \qquad G \cap k_X((X \times s(H)) \cap F) = \emptyset,$$

hence $(G \times s(H)) \cap \alpha X$ is a PX-neighbourhood of $(x, \mathfrak{s})$ disjoint from F. Thus F is closed in PX, and the topology of Y is coarser than that of PX.

The uniqueness statement can be formulated more precisely as follows:

($*$) *If Z is EDC, $f\colon Z \to X$ ϑ-perfect, $h'\colon Z \to EX$ continuous, and $f = k_X \circ h'$, then necessarily $h' = h$ (constructed above). Hence, if h' is a homeomorphism from Z onto a space Y' over αX having a topology finer than that of EX, then $Y' = Y$ (constructed above).*

In fact, the map $h' \circ k_Z\colon PZ \to EX$ is continuous and satisfies $f \circ k_Z = k_X \circ (h' \circ k_Z)$. Therefore

$$h' \circ k_Z = f^*, \qquad h' = f^* \circ k_Z^{-1} = h. \qquad \square$$

4. The absolute RX. We illustrate the above theory by a special case.

For a topological space X, let us denote by RX the set αX equipped with the subspace topology of the product $rX \times UX$. Then RX lies between PX and EX, and it is an absolute of X according to 3.2.

LEMMA 4.1. *$k_X\colon RX \to X$ is almost continuous.* $\square$

The map $k_X\colon RX \to X$ is not always δ-continuous (see 4.3).

LEMMA 4.2. *$RX = PX$ iff X is semi-regular.*

PROOF. If X is semi-regular, then $rX = X$ and $RX = PX$. Conversely, let $G \subset X$ be open but not a union of r-open sets. Then $(G \times UX) \cap \alpha X$ is open in PX but not in RX.

In fact, assume

$$(G \times UX) \cap \alpha X = \bigcup_{i \in I} ((G_i \times s(H_i)) \cap \alpha X),$$

where $G_i \subset X$ is r-open, $H_i \subset X$ is open. Let $x_0 \in G$. Then $\mathfrak{s} \in UX$, $\mathfrak{s} \to x_0$ imply $(x_0, \mathfrak{s}) \in \alpha X$, hence $(x_0, \mathfrak{s}) \in G_i \times s(H_i)$ for some i, i.e. $x_0 \in G_i$, $H_i \in \mathfrak{s}$. By [2], 2.2, there is a finite subset $I_0 \subset I$ such that $\bigcup_{i \in I_0} \bar{H}_i$ is a neighbourhood of x_0 in X. Therefore $V = \bigcap_{i \in I_0} G_i \cap \operatorname{int} \bigcup_{i \in I_0} \bar{H}_i$ is an r-open neighbourhood of x_0, and $V \subset \bigcup_{i \in I_0} (G_i \cap \bar{H}_i) \subset G$. In fact, $G_i \cap \bar{H}_i \subset G$ for each $i \in I$, because $x \in G_i \cap \bar{H}_i$ implies the existence of $\mathfrak{s} \in UX$ such that $H_i \in \mathfrak{s}$, $\mathfrak{s} \to x$, whence

$$(x, \mathfrak{s}) \in (G_i \times s(H_i)) \cap \alpha X \subset (G \times UX) \cap \alpha X,$$

so that $x \in G$. Now $x_0 \in V \subset G$ contradicts the choice of G. □

LEMMA 4.3. *If X is semi-regular but non-regular, then $k_X \colon RX \to X$ is not δ-continuous.*

PROOF. By 4.2, $RX = PX$ and, by 1.1, $PX \neq EX$. Let the r-open set $G \subset X$ and the open set $H \subset X$ be chosen such that $(G \times s(H)) \cap \alpha X$ is not open in $EX = rPX$ ([2], 6.3) i.e. not a union of r-open sets in PX. Then $(G \times UX) \cap \alpha X$ is not a union of r-open sets in PX either, because $(X \times s(H)) \cap \alpha X$ is clopen in PX. Hence $k_X^{-1}(G)$ is not a union of r-open sets in $RX = PX$. □

From 4.2, we can obtain spaces satisfying $RX \neq PX$. Our next purpose is to construct a T_2-space such that $EX \neq RX \neq PX$.

LEMMA 4.4. *Let Y be a T_2-space, $G_0 \subset Y$ r-open, $x_0 \in G_0$, $X \supset Y$ a space such that the neighbourhoods of $y \in Y$ constitute a neighbourhood base for y in X, and let the trace $\mathfrak{s}(p)$ in Y of the neighbourhood filter of any $p \in X - Y$ fulfil the following conditions:*

(a) *$\mathfrak{s}(p)$ does not have a cluster point in Y,*

(b) *$p \neq q$, $p, q \in X - Y$ imply that $\mathfrak{s}(p)$ and $\mathfrak{s}(q)$ contain disjoint elements,*

(c) *$G_0 \notin \mathfrak{s}(p)$,*

(d) *if $G \subset Y$ is open and $x_0 \in \bar{G}$ then there is a $p \in X - Y$ such that G intersects every element of $\mathfrak{s}(p)$.*

Then, if the sets $\{p\} \cup S$, $S \in \mathfrak{s}(p)$ constitute a neighbourhood base of $p \in X - Y$, X is a T_2-space such that $RX \neq EX$.

PROOF. X is T_2 by (a) and (b). The set G_0 is r-open in X, too, since $x \in \mathrm{cl}_X G_0 - G_0$ implies either $x = y \in Y$ and then the open neighbourhoods of y in Y (open in X) are not contained in $\mathrm{cl}_Y G_0$ and not in $\mathrm{cl}_X G_0$ either, or $x = p \in X - Y$ and then no open element of $\mathfrak{s}(p)$ can be contained in $\mathrm{cl}_X G_0$ since then it would be included in $\mathrm{cl}_Y G_0$ and in G_0 (r-open in Y) in contradiction with (c).

Thus $U = (G_0 \times UX) \cap \alpha X$ is open in RX. Let $\mathfrak{s}_0 \in UX$, $\mathfrak{s}_0 \to x_0 \in G_0$. We show that $(x_0, \mathfrak{s}_0)$ does not lie in the interior of U in EX. In fact, a neighbourhood base of this point is composed of the sets $(X \times s(V)) \cap \alpha X$ where $V \in \mathfrak{s}_0$ is open in X. Now $G = V \cap G_0 \in \mathfrak{s}_0$ and $x_0 \in \mathrm{cl}_X G$ follow from $\mathfrak{s}_0 \to x_0$, whence $x_0 \in \mathrm{cl}_Y G$. Choose p according to (d); then $p \in \mathrm{cl}_X G \subset \mathrm{cl}_X V$, so that there is an $\mathfrak{s} \in UX$ such that $V \in \mathfrak{s}$, $\mathfrak{s} \to p$, and

$$(p, \mathfrak{s}) \in (X \times s(V)) \cap \alpha X, \qquad (p, \mathfrak{s}) \notin (G_0 \times UX) \cap \alpha X = U. \qquad \square$$

EXAMPLE 4.5. There exists a space Y fulfilling the conditions in 4.4 such that X is not semi-regular. Then, by 4.4 and 4.2, X is T_2 and $EX \neq RX \neq$ $\neq PX$.

Let $Y = \mathbf{Q}$ be equipped with the topology inherited from the Euclidean topology of $\mathbf{R}$. Put $G_0 = (-1,1) \cap \mathbf{Q}$, $x_0 = 0$. Consider a well-ordering of the open subsets of Y in the type γ where γ is the initial ordinal of 2^ω; choose G_0 to be the 0th element in this well-ordering. Select $y_0 \in (-1,1) - \mathbf{Q}$, $z_0 \in (1,2) - \mathbf{Q}$.

Suppose y_ξ and z_ξ are defined for $\xi < \alpha$ $(< \gamma)$. Let H_α be an open subset of $\mathbf{R}$ such that $G_\alpha = H_\alpha \cap \mathbf{Q}$, and $y_\alpha \in H_\alpha - \mathbf{Q}$ be chosen distinct from all y_ξ and z_ξ $(\xi < \alpha)$. If $G_\alpha \subset G_0$, let $z_\alpha \in (1,2) - \mathbf{Q}$ be distinct from all y_ξ and z_ξ; if $G_\alpha - G_0 \neq \emptyset$, let $z_\alpha \in (H_\alpha - (-1,1)) - \mathbf{Q}$ again be distinct from all y_ξ and z_ξ previously chosen. Let $X \supset Y$ be chosen such that $|X - Y| = 2^\omega$, $X - Y = \{p_\xi : \xi < \gamma\}$, and, for $p_\xi \in X - Y$, define $\mathfrak{s}(p_\xi)$ to be the filter in Y generated by the sets

$$((y_\xi - \varepsilon, y_\xi + \varepsilon) \cup (z_\xi - \varepsilon, z_\xi + \varepsilon)) \cap \mathbf{Q} \qquad (\varepsilon > 0).$$

Then (a), (b), (c) are clearly true. If $G \subset Y$ is open, say, $G = G_\alpha$ (and $0 \in \mathrm{cl}_Y G_\alpha$), then $p_\alpha \in X - Y$ fulfils (d). X is not semi-regular because, if $G_\alpha - G_0 \neq \emptyset$, then $p_\alpha \in \mathrm{cl}_X G_\alpha - G_\alpha$ is interior to $\mathrm{cl}_X G_\alpha$ so that G_α is not r-open and Y is not a union of r-open sets. $\square$

It is well-known that αX is closed in $X \times UX$ while it is dense in $IX \times UX$ where IX is the underlying set of X equipped with the indiscrete topology ([2], 2.1). In this respect, RX is similar to PX:

THEOREM 4.6. *αX is closed in $rX \times UX$.*

PROOF. For $(x_0, \mathfrak{s}_0) \in (rX \times UX) - \alpha X$, choose an open $G_0 \subset X$ such that $x_0 \in G_0 \notin \mathfrak{s}_0$. Then there is an open $S_0 \subset X$ satisfying $S_0 \in \mathfrak{s}_0$,

$G_0 \cap S_0 = \emptyset$. Hence $H_0 = \operatorname{int} \bar{G}_0$ is r-open in X and $H_0 \cap S_0 = \emptyset$, $x_0 \in H_0$, so that $H_0 \times s(S_0)$ is a neighbourhood of $(x_0, \mathfrak{s}_0)$ in $rX \times UX$ not intersecting αX. In fact, $(x, \mathfrak{s}) \in (H_0 \times s(S_0)) \cap \alpha X$ would imply $x \in \bar{S}_0$, $x \notin H_0$: a contradiction. □

Thus RX is a closed subspace of a semi-regular space (namely of $rX \times \times UX$). Unfortunately, this statement does not contain any restriction on the quality of RX:

LEMMA 4.7. *Every topological space is homeomorphic to a closed subspace of a suitable semi-regular space.*

PROOF. For a space X, let $Y = X \times [0, +\infty)$. Let the points (x, y), $y > 0$ be isolated in Y, and let a base in Y be composed of the corresponding singletons and the sets

$$B(f) = \{(x, y) \colon \ x \in X, \ \ 0 \leqq y < f(x)\}$$

where $f \colon X \to [0, +\infty)$ is a function such that

$$Z(f) = \{x \in X \colon \ \ f(x) = 0\}$$

is closed in X. This is in fact a base as $B(f) \cap B(g) = B(h)$ for $h = \min(f, g)$, $Z(h) = Z(f) \cup Z(g)$. The set $X^* = X \times \{0\}$ is closed in Y and the subspace topology on X^* coincides with that of X (more precisely, $\operatorname{pr}_X | X^*$ is a homeomorphism) because

$$\operatorname{pr}_X(B(f) \cap X^*) = X - Z(f).$$

The singletons in $Y - X^*$ are clopen, and the sets $B(f)$ are r-open as well. In fact,

$$\overline{B(f)} = B(f) \cup (F \times \{0\})$$

where $F = \operatorname{cl}_X(X - Z(f))$, and $x \in F \cap Z(f)$ implies that every neighbourhood $B(g)$ of $(x, 0)$ contains points (x, y) satisfying $y > 0$, not belonging to $\overline{B(f)}$. □

It would be interesting to know a non-trivial subclass of topological spaces that contains all spaces RX. Semi-regular spaces do not do; in fact, if RX is semi-regular then it is regular and $RX = EX$ (which fails to hold in general).

Similarly, the fact that $k_X \colon RX \to X$ is almost continuous does not characterize RX: $RX \neq PX$ can happen and $k_X \colon PX \to X$ is (almost) continuous. However, it is not difficult to see that RX has a kind of extremal character with respect to this property. For this purpose, let us call *almost ultraperfect* an almost continuous ϑ-perfect map.

THEOREM 4.8. *If Y is a space lying between PX and RX then it is EDC and $k_X: Y \to X$ is almost ultraperfect. Conversely, let Z be an EDC space and $f: Z \to X$ be almost ultraperfect. Then there is a homeomorphism $h: Z \to Y$ onto a space Y lying between PX and RX such that $f = k_X \circ h$. Z and f uniquely determine h and Y.*

PROOF. The first statement is obvious by 3.2 and 4.1. If Z is EDC and $f: Z \to X$ is almost ultraperfect, then, by 3.3, there is a homeomorphism $h: Z \to Y$ onto a space lying between PX and EX such that $f = k_X \circ h$. If $G \subset X$ is r-open, then $k_X^{-1}(G) = (G \times UX) \cap \alpha X$ is open in Y and so is $(X \times s(H)) \cap \alpha X$ for any open set $H \subset X$. Thus the topology of Y is finer than that of RX. The uniqueness statement can be formulated more precisely similarly to $(*)$ given in the proof of 3.3: if $h': Z \to EX$ is continuous and fulfils $f = k_X \circ h'$ then $h' = h$. □

COROLLARY 4.9. *The space $Y = RX$ and the map $k = k_X$ have the following properties:*

(a) *Y is EDC,*

(b) *$k: Y \to X$ is an almost ultraperfect map,*

(c) *whenever Z is EDC and $f: Z \to X$ is almost ultraperfect, there exists a bijective and continuous map $g: Z \to Y$ such that $f = k \circ g$.*

Conversely, if Y and k satisfy (a), (b), (c), *then there is a homeomorphism $h: Y \to RX$ such that $k = k_X \circ h$.*

PROOF. The first part follows from 4.1 and 4.8. Conversely, if Y and k satisfy (a), (b), (c), then by 4.8 there is a homeomorphism $h: Y \to Y'$ onto a space Y' lying between PX and RX such that $k = k_X \circ h$. Applying (c) for $Z = RX$ and $f = k_X$, we obtain a bijective and continuous map $g: RX \to Y$ such that $k_X = k \circ g$. Now Y' is EDC, $k_X = k \circ h^{-1}: Y' \to X$ is almost ultraperfect and $k_X = k_X \circ h \circ g: Y' \to X$ where $h \circ g: Y' \to EX$ is continuous, while $k_X = k_X \circ \mathrm{id}_{\alpha X}: Y' \to X$, $\mathrm{id}_{\alpha X}: Y' \to EX$ is continuous as well. Hence, by $(*)$ in 3.3, $h \circ g = \mathrm{id}_{\alpha X}: \alpha X \to \alpha X$, and $h \circ g = \mathrm{id}_{\alpha X}: RX \to Y'$ is continuous, showing $Y' = RX$. □

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF ANALYSIS
BUDAPEST, MÚZEUM KRT. 6–8
H–1088

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ON SOME PROBLEMS OF I. JOÓ

A. BOGMÉR, M. HORVÁTH and A. SÖVEGJÁRTÓ (Budapest)

In [3] Joó raised the following problem. Let $1 < q < 2$ and consider the expansion of the number 1 of the form

$$(1) \qquad 1 = \sum_{i=1}^{\infty} q^{-n_i}$$

where $\{n_i\}$ is a subsequence of $\{1, 2, 3, \dots\}$. For fixed q such an expansion is not necessarily unique, so the problem of unicity or that of finding the number of solutions of (1) arises. On the other hand we can investigate the problem of finding an expansion (1) for a fixed q, satisfying

$$(2) \qquad \sup(n_{i+1} - n_i) = \infty.$$

Both questions are investigated in the papers [3], [4], [5], [7]. While preparing these publications, I. Joó raised (among others) the following two questions:

(A) Does there exist an expansion (1) satisfying (2) for every $1 < q < \frac{1+\sqrt{5}}{2}$?

(B) Does the following statement hold for every $1 < q < 2$: there exists an expansion (1) satisfying (2) if and only if there exist $2^{\aleph_0}$ many different expansions?

In this paper we give negative answers to both problems. Our considerations have number-theoretic character, so we start with recalling some known facts and notions from algebraic number theory. A number $\alpha \in \mathbf{C}$ is called algebraic if it is the zero of a polynomial with entire (or rational) coefficients. If the polynomial is irreducible over the field $\mathbf{Q}$ of rationals, then its other zeros are called the conjugates of α; we denote them by $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_s$. If α is the zero of a polynomial with entire coefficients and the leading coefficient is 1 then we call α an algebraic integer. The Pisot numbers ([1], [6]) are algebraic integers α satisfying

$$(3) \qquad \alpha > 1, \quad |\alpha_i| < 1, \quad 2 \leqq i \leqq s.$$

We shall prove the following

THEOREM. *Let* $1 < q < 2$ *and*

$$(4) \qquad 1 = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{q^n}, \quad \varepsilon_n = 0 \text{ or } 1.$$

If q is a Pisot number, then the numbers

$$x_k := \varepsilon_k + \frac{\varepsilon_{k+1}}{q} + \frac{\varepsilon_{k+2}}{q^2} + \dots$$

give only finitely many different values.

This theorem answers negatively the problems (A) and (B). Indeed, we can first see easily that there are Pisot numbers less than $\frac{1+\sqrt{5}}{2}$; for example the real zero of the polynomial $q^3 - q^2 - 1$ is such a number. By the theorem the x_k are bounded from below, $x_k \geqq \delta > 0$, but then there exists $t = t(\delta) > 0$ independent of k such that among $\varepsilon_k, \varepsilon_{k+1}, \dots, \varepsilon_{k+t}$ there must be a digit 1, hence with the notation of (1)

$$\sup(n_{i+1} - n_i) \leqq t < \infty.$$

So (A) is answered. In [1] the authors proved that for all $1 < q < \frac{1+\sqrt{5}}{2}$ there exist $2^{\aleph_0}$ different expansions (1) of 1, so the answer for (B) is also negative.

PROOF OF THE THEOREM. The numbers $x_k = q^k \left(1 - \sum_{n=1}^{k-1} \frac{\varepsilon_n}{q^n}\right)$ are algebraic and are contained in the field extension $\mathbf{Q}(q)$ of $\mathbf{Q}$, $x_k \in \mathbf{Q}(q)$. We shall prove that the numbers x_k and all their conjugates have a common upper bound and the x_k are algebraic integers. In this case all x_k are the zeros of polynomials with entire coefficients whose order and coefficients have a bound independent of k, hence the set $\{x_k\}$ is indeed finite.

Let the number q have s conjugates $q_1 = q, q_2, \dots, q_s$. Since q is a Pisot number, we have

$$|q_2|, \dots, |q_s| < 1.$$

As it is known ([2], p. 42–43), there are s monomorphisms

$$\sigma_i \colon \mathbf{Q}(q) \to \mathbf{C}, \qquad i = 1, \dots, s$$

and σ_i satisfies $\sigma_i(q) = q_i$. We know further that if $y \in \mathbf{Q}(q)$ then $y = \sigma_1(y)$, $\sigma_2(y), \dots, \sigma_s(y)$ run over the conjugates of y (may be with multiplicity). By definition, x_k and x_{k+1} are linked by the relation

$$x_{k+1} = q(x_k - \varepsilon_k) \tag{5}$$

and $x_1 = q$. Since the product of two algebraic integers is an algebraic integer ([2], p. 47), we get by induction on k that x_k are algebraic integers. Applying σ_i to the recursion (5) we get

$$x_{k+1,i} = q_i(x_{k,i} - \varepsilon_k)$$

and consequently

$$|x_{k+1,i}| \leqq |q_i|(1 + |x_{k,i}|), \qquad i \geqq 2. \tag{6}$$

Let now

$$\delta := \max_{i \geqq 2} |q_i| < 1, \qquad M_k := \max_{i \geqq 2} |x_{k,i}|$$

then (6) implies $M_{k+1} \leqq \delta(M_k + 1)$, whence we get by induction that

$$M_{k+1} \leqq \delta^k M_1 + \delta^k + \delta^{k-1} + \ldots + \delta$$

and then

$$M_{k+1} \leqq M_1 + \frac{\delta}{1-\delta}.$$

So the conjugates of x_k are indeed bounded. On the other hand the sequence x_k itself is obviously bounded:

$$x_k \leqq 1 + q^{-1} + q^{-2} + \ldots = \frac{q}{q-1}.$$

By the above arguments we see that $\{x_k\}$ is indeed a finite set, so the proof is complete.

REMARK. The Pisot numbers form a closed subset of $(1, \infty)$, see [1], hence there exists a least Pisot-number $q_0 > 1$. So the following modification of (A) remained open:

PROBLEM. Does there exist an expansion (1) satisfying (2) for every $1 < q < q_0$?

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EÖTVÖS LORÁND UNIVERSITY
CHAIR FOR ANALYSIS
MÚZEUM KRT. 6–8
H–1088 BUDAPEST, HUNGARY

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SOME SATURATION THEOREMS FOR CLASSICAL ORTHOGONAL EXPANSIONS. II

M. HORVÁTH (Budapest)

The famous theorem of G. Alexits [6] states that the Fejér means of the Fourier expansion of a continuous and 2π-periodic function f converge uniformly to f in the order $O\left(\frac{1}{n}\right)$ if and only if the trigonometric conjugate $\tilde{f}$ of f belongs to the Lip 1 class, i.e. $\tilde{f}$ is absolutely continuous and $\tilde{f}' \in L^\infty$. It was I. Joó who initiated the extension of this theorem for classical orthogonal expansions. He obtained Alexits type results for Hermite expansions ([7], [8], [9]) and one of the implications of the Alexits theorem for Laguerre expansions in [7]. In [8] he also derived a saturation theorem for the Abel–Poisson means of Hermite expansions. A. Bogmér [10] proved an Alexits type theorem for Jacobi expansions. In [11] we gave another Alexits type theorem and a saturation theorem in the Jacobi case.

In what follows we obtain similar results for Laguerre expansions of non-negative parameter. In all these investigations the norm estimates of the Abel–Poisson means and of the conjugate function are essential; see Stein and Muckenhoupt [2] and Muckenhoupt [3], [4], [5]. We shall modify these results in order to adapt them for our purposes; see later.

Let $\alpha > -1$ and define the weight

$$u_\alpha(x); = x^\alpha e^{-x} \qquad (x > 0).$$

The normed Laguerre polynomials $\ell_n^{(\alpha)}$ of order α are defined by

$$\int_0^\infty \ell_n^{(\alpha)} \ell_k^{(\alpha)} u_\alpha = \delta_{n,k}. \tag{1}$$

The connection with the notation $L_n^{(\alpha)}$, used by Szegő [1] is

$$L_n^{(\alpha)} = (-1)^n \sqrt{\Gamma(\alpha+1)\binom{n+\alpha}{n}}\,\ell_n^{(\alpha)}.$$

We shall need the differentiation formulas

$$[\ell_n^{(\alpha)}]' = \sqrt{n}\,\ell_{n-1}^{(\alpha+1)} \tag{2}$$

and

$$[u_{\alpha+1}\ell_{n-1}^{(\alpha+1)}]' = -\sqrt{n}\,u_\alpha \ell_n^{(\alpha)}. \tag{3}$$

Remark that (2) is explicitly given in [1] in terms of $L_n^{(\alpha)}$ and (3) follows from the Rodrigues formula ([1], (5.1.5))

$$u_\alpha L_n^{(\alpha)} = \frac{1}{n!}[u_{\alpha+n}]^{(n)}$$

which implies that

$$\frac{1}{n}\left[u_{\alpha+1}L_{n-1}^{(\alpha+1)}\right]' = u_\alpha L_n^{(\alpha)}$$

and this, in turn, implies (3).

Consider a function f defined on $(0,\infty)$. Its Laguerre–Fourier series (if exists) is defined by

$$(4)\qquad f \sim \sum_{k=0}^{\infty} a_k \ell_k^{(\alpha)}, \qquad a_k := \int_0^\infty f\ell_k^{(\alpha)} u_\alpha.$$

Let $1 \leqq p \leqq \infty$ and define the weighted spaces

$$L^p(\sqrt{u_\alpha}) := \left\{ f : \|f\sqrt{u_\alpha}\|_p := \left(\int_0^\infty |f\sqrt{u_\alpha}|^p\right)^{\frac{1}{p}} < \infty \right\},$$

$$L^p_{u_\alpha} := \left\{ f : \|f\|_{p,\alpha} := \left(\int_0^\infty |f|^p u_\alpha\right)^{\frac{1}{p}} < \infty \right\}, \quad \text{if } 1 \leqq p < \infty,$$

$$L^\infty_{u_\alpha} := L^\infty(0,\infty); \quad \|f\|_{\infty,\alpha} := \|f\|_\infty.$$

If $\alpha \geqq 0$ then $\sqrt{u_\alpha}\ell_k^{(\alpha)} \in L^1(0,\infty) \cap L^\infty(0,\infty)$ hence the Fourier series of any $f \in L^p(\sqrt{u_\alpha})$ exists. If $-1 < \alpha < 0$ then $\ell_k^{(\alpha)} \in L^p(\sqrt{u_\alpha})$ if and only if $p < -\frac{2}{\alpha}$, consequently, using the Hölder inequality we see that the Fourier series exists for all $f \in L^p(\sqrt{u_\alpha})$ if and only if

$$\frac{2}{2+\alpha} < p \leqq \infty.$$

Denote by $\sigma_n f$ and $R_n f$ the Fejér and Riesz means of parameter $\frac{1}{2}$ of the expansion of f, resp.:

$$\sigma_n f := \sum_{k=0}^{n}\left(1 - \frac{k}{n+1}\right) a_k \ell_k^{(\alpha)}, \quad R_n f := \sum_{k=0}^{n}\left(1 - \frac{\sqrt{k}}{\sqrt{n+1}}\right) a_k \ell_k^{(\alpha)}.$$

LEMMA 1. *The Riesz means R_n are uniformly bounded in the $L^p(\sqrt{u_\alpha})$ norm. In other words, let $1 \leqq p \leqq \infty$ if $\alpha > 0$ and $\frac{2}{2+\alpha} < p < -\frac{2}{\alpha}$, if $-1 < \alpha \leqq 0$. Then for $f \in L^p(\sqrt{u_\alpha})$ we have*

$$\|\sqrt{u_\alpha} R_n f\|_p \leqq c(p) \|\sqrt{u_\alpha} f\|_p \tag{5}$$

with a constant $c(p) > 0$ independent of f and n.

PROOF. As Poiani proved in [15], p. 11, the estimate

$$\|\sqrt{u_\alpha} \sigma_n f\|_p \leqq c(p) \|\sqrt{u_\alpha} f\|_p \tag{6}$$

holds for $1 \leqq p \leqq \frac{7}{4}$ in case $\alpha > 0$ and for $\frac{2}{2+\alpha} < p < -\frac{2}{\alpha}$ if $-1 < \alpha \leqq 0$. Now if $\alpha > 0$, we can extend (6) from $p = 1$ to $p = \infty$ since

$$\|\sqrt{u_\alpha}\sigma_n f\|_\infty = \sup_{\|\sqrt{u_\alpha} g\|_1 \leqq 1} \int_0^\infty \sigma_n(f) g u_\alpha = \sup_{\|\sqrt{u_\alpha} g\|_1 \leqq 1} \int_0^\infty \sigma_n(g) f u_\alpha \leqq$$

$$\leqq \|\sqrt{u_\alpha} f\|_\infty \sup_{\|\sqrt{u_\alpha} g\|_1 \leqq 1} \|\sqrt{u_\alpha}\sigma_n g\|_1 \leqq c \|\sqrt{u_\alpha} f\|_\infty$$

and for $1 < p < \infty$ the same result follows from the Marcinkiewicz interpolation theorem. Denote

$$S_k := S_k(f) := \sum_{j=0}^{k} a_j \ell_j^{(k)}$$

the k-th partial sum operator, then

$$R_n f = \sum_{k=0}^{n} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{n+1}} S_k = \sum_{k=0}^{n} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{n+1}} ((k+1)\sigma_k - k\sigma_{k-1}) =$$

$$= \sqrt{n+1}(\sqrt{n+1} - \sqrt{n})\sigma_n + \sum_{k=0}^{n} \sigma_k (k+1) \frac{2\sqrt{k+1} - \sqrt{k} - \sqrt{k+2}}{\sqrt{n+1}}.$$

Using the trivial estimates

$$\sqrt{n+1} - \sqrt{n} = O\left(\frac{1}{\sqrt{n+1}}\right), \qquad 2\sqrt{k+1} - \sqrt{k} - \sqrt{k+2} = O\left(\frac{1}{(k+1)^{\frac{3}{2}}}\right)$$

we get that

$$\|\sqrt{u}_\alpha R_n f\|_p \leqq c \|\sqrt{u_\alpha}\sigma_n f\|_p + c \sum_{k=0}^{n} \frac{\sqrt{k+1}}{\sqrt{n+1}} \|\sqrt{u_\alpha}\sigma_k f\|_p \leqq c \|\sqrt{u_\alpha} f\|_p$$

which proves Lemma 1.

REMARK. It is shown in [16], p. 222 that the Fejér means (and in general any (C,j)-means, $j \in \mathbf{N}$) are not bounded in $L^p_{u_\alpha}$ norm unless $p=2$ (when $\|f\|_{2,\alpha} = \|f\sqrt{u_\alpha}\|_2$). The norm estimates given in [3], [5] for the Abel–Poisson means and the conjugate function are proved for the $L^p_{u_\alpha}$ norm; that is why we give first their $L^p(\sqrt{u_\alpha})$-variant. We mention that references concerning the boundedness of Cesàro means of some expansions can be found in [17]. We shall need the following

PROPOSITION. *Let $\gamma \geqq 0$, $\beta > 0$ and consider the system*

$$\Phi := \{x^{n+\gamma}e^{-\beta x} : n \in \mathbf{N}, x > 0\}.$$

a) *Φ is complete in $L^p(0,\infty)$, $1 \leqq p \leqq \infty$.*
b) *The linear hull of Φ is dense in $L^p(0,\infty)$, $1 \leqq p < \infty$.*

PROOF. a) We shall use some ideas of Stone [19], p. 74–79, see also [20], p. 131–132. Suppose that the function $f \in L^p(0,\infty)$ satisfies

$$\int_0^\infty f(x)x^{n+\gamma}e^{-\beta x}dx = 0, \qquad n \in \mathbf{N}.$$

Define the function

$$g(x) := e^{-\frac{\beta}{4}x}\int_0^x f(t)t^\gamma e^{-\frac{\beta}{4}t}dt.$$

Since $t^\gamma e^{-\frac{\beta}{4}t} \in L^q(0,\infty)$, $\frac{1}{p}+\frac{1}{q}=1$, hence

$$|g(x)| \leqq ce^{-\frac{\beta}{4}x}$$

and then $g \in L^2(0,\infty)$. On the other hand

$$\int_0^\infty g(x)x^n e^{-\frac{\beta}{2}x}dx = \int_0^\infty \left(\int_0^x f(t)t^\gamma e^{-\frac{\beta}{4}t}dt\right) x^n e^{-\frac{3}{4}\beta x}dx =$$

$$= \int_0^\infty f(x)x^\gamma e^{-\frac{\beta}{4}x}\int_x^\infty t^n e^{-\frac{3}{4}\beta t}dtdx = \int_0^\infty f(x)x^\gamma p_n(x)e^{-\beta x}dx$$

where the polynomial $p_n(x)$ is defined by

$$\int_x^\infty t^n e^{-\frac{3}{4}\beta t}dt = p_n(x)e^{-\frac{3}{4}\beta x}.$$

Since the polynomials multiplied by $e^{-\frac{x}{2}}$, the square root of the Laguerre weight of parameter 0, are dense in $L^2(0,\infty)$ (see [1], Theorem 5.7.1), we get that $g(x)=0$ a.e. and then $f(x)=0$ a.e.

b) Denote $V=V(\Phi)$ the closed linear hull of Φ in $L^p(0,\infty)$. Suppose indirectly that there exists $f\in L^p(0,\infty)$, $f\notin V$. By $p\neq\infty$ there exists a function $g\in L^q(0,\infty)$, $\frac{1}{p}+\frac{1}{q}=1$ so that

$$\int_0^\infty fg=1,\quad \int_0^\infty hg=0,\qquad g\in V.$$

But this contradicts a), so $V=L^p(0,\infty)$. The proof is complete.

The Poisson kernel for Laguerre expansion is given by

$$K(r,y,z)=\sum_{n=0}^\infty \ell_n^{(\alpha)}(y)\ell_n^{(\alpha)}(z)r^n,\qquad x,y>0,\ 0\leqq r<1.$$

It is known ([1]) the Mehler type formula

$$K(r,y,z)=\frac{1}{1-r}e^{-(y+z)\frac{r}{1-r}}\frac{J_\alpha\left(i\frac{2\sqrt{yzr}}{1-r}\right)}{i^\alpha(yzr)^{\frac{\alpha}{2}}}. \tag{7}$$

Introduce the notation

$$a\asymp b$$

for $a,b>0$; this means that there exist positive constants c,C which may depend only on α and p but not on other quantities so that

$$ca\leqq b\leqq Ca.$$

Using (7) we can easily obtain (see [3]) that

$$K(r,y,z)\asymp\begin{cases}\frac{1}{(1-r)^{\alpha+1}}e^{-(y+z)\frac{r}{1-r}} & \text{if } z\leqq\frac{(1-r)^2}{4yr}\\ \frac{1}{\sqrt{1-r}}\frac{\exp\left\{-(y+z)\frac{r}{1-r}+2\frac{\sqrt{yzr}}{1-r}\right\}}{(yzr)^{\frac{\alpha}{2}+\frac{1}{4}}} & \text{if } z>\frac{(1-r)^2}{4yr}.\end{cases} \tag{8}$$

Consequently for fixed r and $y>0$ we have

$$\sqrt{u_\alpha}(z)K(r,y,z)\in L^1(0,\infty)\cap L^\infty(0,\infty)$$

and hence for $1\leqq p\leqq\infty$, $f\in L^p(\sqrt{u_\alpha})$ the Poisson integral of f, defined to be

$$g(r,y):=\int_0^\infty K(r,y,z)f(z)u_\alpha(z)dz \tag{9}$$

exists. To prove the $L^p(\sqrt{u_\alpha})$-boundedness of $g(r,y)$ we need the following variant of [3], Corollary 1.

LEMMA 2. *Let I be a finite or infinite interval, $d\mu$ an absolute continuous (positive) measure on I. Let $L(y,z) \geqq 0$ be a function for which $z \mapsto \mu'(z)L(y,z)$ is monotone increasing for $z \leqq y$, decreasing for $z \geqq y$ and*

$$\mu'(y) \int_I L(y,z)d\mu(z) \leqq B \qquad (y \in I). \tag{10}$$

Define further

$$g(y) := \int_I K(y,z)f(z)\mu'^2(z)dz$$

where the kernel function $K(y,z)$ is measurable and satisfies

$$|K(y,z)| \leqq L(y,z).$$

Then we have

$$\mu'(y)|g(y)| \leqq B(\mu' f)^*(y). \tag{11}$$

Here

$$F^*(y) := \sup_{y \in J \subset I} \frac{1}{|J|} \int_J |F|$$

denotes the Hardy–Littlewood maximal function of F ([13]), where the supremum runs over the closed segments J containing y.

PROOF. a) Suppose first that $\mu'(z)L(y,z)$, as a function of z, is a step-function of the form

$$\mu'(z)L(y,z) = \sum a_i \chi_{(y_i', y_i'')}(z)$$

where

$$a_i \geqq 0, \quad y_i' \leqq y_i'', \quad y_i', y_i'' \in I, \qquad \forall i.$$

Then we have

$$\mu'(y)|g(y)| \leqq \mu'(y) \int_I L(y,z)|f(z)|\mu'^2(z)dz =$$

$$= \mu'(y) \sum a_i \int_{y_i'}^{y_i''} |f| d\mu \leqq \mu'(y)(f\mu')^*(y) \sum a_i(y_i'' - y_i') =$$

$$= \mu'(y)(f\mu')^*(y) \int_I L(y,z)d\mu(z) \leqq B(f\mu')^*(y).$$

b) In the general case we can give a sequence

$$\varphi_1 \leqq \dots \leqq \varphi_n \leqq \varphi_{n+1} \leqq \dots$$

of stepfunctions of the form given in a) which converge a.e. to $\mu'(z)L(y,z)$. Using twice the Beppo–Levi theorem we obtain

$$\mu'(y)|g(y)| \leqq \mu'(y) \int_I L(y,z)|f(z)|\mu'^2(z)dz = \mu'(y) \lim_{n\to\infty} \int_I \varphi_n(z)|f(z)|d\mu(z) \leqq$$

$$\leqq \mu'(y)(f\mu')^*(y) \lim_{n\to\infty} \int_I \varphi_n(z)dz = \mu'(y)(f\mu')^*(y) \int_I L(y,z)d\mu(z) \leqq B(f\mu')^*(y)$$

as we asserted.

LEMMA 3. *Suppose* $\alpha \geqq 0$. *Then*

$$\sqrt{u_\alpha}(y) \int_0^\infty K(r,y,z)\sqrt{u_\alpha}(z)dz \leqq c \tag{12}$$

where c *is independent of* r *and* y.

PROOF. Denote by $H(r,y,z)$ the function on the right hand side of (8); we have to prove that

$$\sqrt{u_\alpha}(y) \int_0^\infty H(r,y,z)\sqrt{u_\alpha}(z)dz \leqq c. \tag{13}$$

Let

$$I_1 := y^{\frac{\alpha}{2}} e^{-\frac{y}{2}} \int_0^{\frac{(1-r)^2}{4ry}} -\frac{z^{\frac{\alpha}{2}}}{(1-r)^{\alpha+1}} \exp\left\{-\frac{z}{2} - (y+z)\frac{r}{1-r}\right\} dz,$$

$$I_2 := y^{\frac{\alpha}{2}} e^{-\frac{y}{2}} \int_{\frac{(1-r)^2}{4ry}}^\infty \frac{z^{\frac{\alpha}{2}}}{\sqrt{1-r}} \exp\left\{-\frac{z}{2} + \frac{-yr + 2\sqrt{yrz} - zr}{1-r}\right\} dz;$$

it is enough to show the boundedness of I_1 and I_2. Consider first I_1. We distinguish some cases.

Case a: $r \leqq \frac{1}{2}$. Then we have by $\alpha \geqq 0$

$$I_1 \leqq c y^{\frac{\alpha}{2}} e^{-\frac{y}{2}} \int_0^\infty z^{\frac{\alpha}{2}} e^{-\frac{z}{2}} dz \leqq c.$$

Case b: $r \geqq \frac{1}{2}, \frac{1-r}{4y} \geqq 1$. Then y is bounded by $\frac{1-r}{4}$ hence

$$I_1 \leqq cy^{\frac{\alpha}{2}}(1-r)^{-\alpha-1}\int_0^{\frac{(1-r)^2}{4ry}} z^{\frac{\alpha}{2}}e^{-z\frac{r}{1-r}}dz.$$

Substituting $u = z\frac{r}{1-r}$ we get

$$I_1 \leqq cy^{\frac{\alpha}{2}}(1-r)^{-\alpha-1}\frac{1-r}{r}\int_0^{\frac{1-r}{4y}} u^{\frac{\alpha}{2}}e^{-u}du\cdot\left(\frac{1-r}{r}\right)^{\frac{\alpha}{2}} \leqq$$

$$\leqq c\left(\frac{y}{1-r}\right)^{\frac{\alpha}{2}}\int_0^{\infty} u^{\frac{\alpha}{2}}e^{-u}du \leqq c.$$

Case c: $r \geqq \frac{1}{2}, \frac{1-r}{4y} \leqq 1$. Then, repeating the arguments of Case b, we can write

$$I_1 \leqq c\left(\frac{y}{1-r}\right)^{\frac{\alpha}{2}}e^{-\frac{y}{2}}\int_0^{\frac{1-r}{4y}} u^{\frac{\alpha}{2}}e^{-u}du \leqq c\left(\frac{y}{1-r}\right)^{\frac{\alpha}{2}}\left(\frac{1-r}{4y}\right)^{\frac{\alpha}{2}}\int_0^{\infty} e^{-u}du \leqq c.$$

Now consider I_2. The exponent figuring in I_2 can be written in the form

$$-\frac{y}{2}-\frac{z}{2}+\frac{-yr+2\sqrt{yrz}-zr}{1-r} = -\frac{(\sqrt{yr}-\sqrt{z})^2+(\sqrt{y}-\sqrt{rz})^2}{2(1-r)}$$

and hence applying the substitution $z = u^2$ we get

$$I_2 = (1-r)^{-\frac{1}{2}}r^{-\frac{\alpha}{2}-\frac{1}{4}}y^{-\frac{1}{4}}\int_{\frac{(1-r)^2}{4ry}}^{\infty} z^{-\frac{1}{4}}\exp\left\{-\frac{(\sqrt{yr}-\sqrt{z})^2+(\sqrt{y}-\sqrt{rz})^2}{2(1-r)}\right\}dz =$$

$$= 2(1-r)^{-\frac{1}{2}}r^{-\frac{\alpha}{2}-\frac{1}{4}}y^{-\frac{1}{4}}\int_{\frac{1-r}{2\sqrt{ry}}}^{\infty} u^{\frac{1}{2}}\exp\left\{-\frac{(\sqrt{yr}-u)^2+(u\sqrt{r}-\sqrt{y})^2}{2(1-r)}\right\}du.$$

We shall use the following estimate. If $\gamma > 1$ and $x \geqq \sqrt{\gamma - 1}$ or if $\gamma \leqq 1$ and $x > 0$, then

$$\int_x^{\infty} y^{\gamma}e^{-y^2}dy \leqq x^{\gamma-1}e^{-x^2}. \tag{14}$$

Indeed, equality holds in (14) for $x=\infty$, and differentiating both sides the converse inequality holds. Return to the estimate of I_2.

Case a: $r \leqq \frac{1}{2}$. Since $r \leqq \frac{1}{2}$ implies $(\sqrt{ry}-u)^2+(u\sqrt{r}-\sqrt{y})^2 \geqq c(u^2+y)$ hence by (14)

$$I_2 \leqq cr^{-\frac{\alpha}{2}-\frac{1}{4}}y^{-\frac{1}{4}}\int\limits_{\frac{1-r}{2\sqrt{ry}}}^{\infty} u^{\frac{1}{2}}e^{-c(u^2+y)}du \leqq$$

$$\leqq cr^{-\frac{\alpha}{2}-\frac{1}{4}}y^{-\frac{1}{4}}\frac{(ry)^{\frac{1}{4}}}{\sqrt{1-r}}\exp\left\{-c\left(\frac{(1-r)^2}{4ry}+y\right)\right\} \leqq$$

$$\leqq cr^{-\frac{\alpha}{2}}\exp\left\{-c\left(\frac{(1-r)^2}{4ry}+y\right)\right\} \leqq cr^{-\frac{\alpha}{2}}e^{-\frac{c}{r}} \leqq c.$$

Case b: $r \geqq \frac{1}{2}$, $\frac{1-r}{y} \leqq 1$. Then the substitution $v=\frac{u}{\sqrt{1-r}}$ gives

$$I_2 \leqq c(1-r)^{-\frac{1}{2}}y^{-\frac{1}{4}}\int\limits_{\frac{1-r}{2\sqrt{ry}}}^{\infty} u^{\frac{1}{2}}\exp\left\{-\frac{(\sqrt{ry}-u)^2+(\sqrt{y}-\sqrt{r}u)^2}{2(1-r)}\right\}du \leqq$$

$$\leqq cy^{-\frac{1}{4}}\int\limits_{\frac{1}{2}\sqrt{\frac{1-r}{ry}}}^{\infty} v^{\frac{1}{2}}(1-r)^{\frac{1}{4}}\exp\left\{-\frac{1}{2}\left(\sqrt{\frac{ry}{1-r}}-v\right)^2-\frac{1}{2}\left(v\sqrt{r}-\sqrt{\frac{y}{1-r}}\right)^2\right\}dv \leqq$$

$$\leqq c\left(\frac{1-r}{y}\right)^{\frac{1}{4}}\int\limits_0^{\infty} v^{\frac{1}{2}}\exp\left\{-\frac{1}{2}\left(v-\sqrt{\frac{ry}{1-r}}\right)^2\right\}dv.$$

Now

$$\left(\frac{1-r}{y}\right)^{\frac{1}{4}}\int\limits_{2\sqrt{\frac{ry}{1-r}}}^{\infty} v^{\frac{1}{2}}\exp\left\{-\frac{1}{2}\left(v-\sqrt{\frac{ry}{1-r}}\right)^2\right\}dv \leqq$$

$$\leqq c\int\limits_{2\sqrt{\frac{ry}{1-r}}}^{\infty}\left(v-\sqrt{\frac{ry}{1-r}}\right)^{\frac{1}{2}}\exp\left\{-\frac{1}{2}\left(v-\sqrt{\frac{ry}{1-r}}\right)^2\right\}dv \leqq c\int\limits_0^{\infty} v^{\frac{1}{2}}e^{-\frac{v^2}{2}}dv \leqq c$$

and

$$\left(\frac{1-r}{y}\right)^{\frac{1}{4}}\int\limits_0^{2\sqrt{\frac{ry}{1-r}}} v^{\frac{1}{2}}\exp\left\{-\frac{1}{2}\left(v-\sqrt{\frac{ry}{1-r}}\right)^2\right\}dv =$$

$$= \left(\frac{1-r}{y}\right)^{\frac{1}{4}} \int\limits_0^{\sqrt{\frac{ry}{1-r}}} \left[\left(\sqrt{\frac{ry}{1-r}} - v\right)^{\frac{1}{2}} + \left(\sqrt{\frac{ry}{1-r}} + v\right)^{\frac{1}{2}}\right] e^{-\frac{v^2}{2}} dv \leqq$$

$$\leqq \left(\frac{1-r}{y}\right)^{\frac{1}{4}} \int\limits_0^{\sqrt{\frac{ry}{1-r}}} 2\left(2\sqrt{\frac{ry}{1-r}}\right)^{\frac{1}{2}} e^{-\frac{v^2}{2}} dv \leqq c \int\limits_0^{\infty} e^{-\frac{v^2}{2}} dv \leqq c.$$

Case c: $r \geqq \frac{1}{2}$, $\frac{1-r}{y} \geqq 1$. As we have seen in Case b

$$I_2 \leqq c \left(\frac{1-r}{y}\right)^{\frac{1}{4}} \int\limits_{\frac{1}{2}\sqrt{\frac{1-r}{ry}}}^{\infty} v^{\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(v - \sqrt{\frac{ry}{1-r}}\right)^2\right\} dv \leqq$$

$$\leqq c \left(\frac{1-r}{y}\right)^{\frac{1}{4}} \int\limits_{\frac{1}{2}\sqrt{\frac{1-r}{ry}}}^{\infty} |v-1|^{\frac{1}{2}} \exp\left\{-\frac{1}{2}(v-1)^2\right\} dv \leqq c \left(\frac{1-r}{y}\right)^{\frac{1}{4}} \left(\frac{ry}{1-r}\right)^{\frac{1}{4}} \leqq c.$$

Lemma 3 is proved.

Define the function $L(r,y,z)$ by

$$\sqrt{u_\alpha}(z) L(r,y,z) := \begin{cases} \sup\limits_{0 \leqq z' \leqq z} \sqrt{u_\alpha}(z') H(r,y,z') & \text{if } z \leqq y \\ \sup\limits_{z \leqq z'} \sqrt{u_\alpha}(z') H(r,y,z') & \text{if } z > y. \end{cases}$$

Obviously $K(r,y,z) \leqq cL(r,y,z)$ and the function $z \mapsto \sqrt{u_\alpha}(z)L(r,y,z)$ increases for $z \leqq y$ and decreases for $z \geqq y$. We assert that the third requirement given in Lemma 2 also fulfils:

Lemma 4. *Let $\alpha \geqq 0$, then*

$$\sqrt{u_\alpha}(y) \int\limits_0^{\infty} L(r,y,z) \sqrt{u_\alpha}(z) dz \leqq c. \tag{15}$$

Proof. By Lemma 3 we can restrict the integration to the union of the segments in which $L(r,y,z) > H(r,y,z)$. Of course, in these segments $L(r,y,z)\sqrt{u_\alpha}(z)$ will be constant. Define

$$z_0 := \alpha \frac{1-r}{1+r}$$

and in case $y > \frac{1-r^2}{2r}$ let

$$\sqrt{z_{1,2}} := 2\frac{\sqrt{yr} \pm \sqrt{yr - \frac{1-r^2}{2}}}{1+r}.$$

The investigation of the sign of the derivative $\frac{\partial}{\partial z}[\sqrt{u_\alpha}(z)H(r,y,z)]$ easily gives the following statements. For $0 \leqq z \leqq \min\left\{z_0, \frac{(1-r)^2}{4ry}\right\}$ the function $\sqrt{u_\alpha}(z)H(r,y,z)$ increases and in case $z_0 < \frac{(1-r)^2}{4ry}$ it decreases in $z \in \in \left[z_0, \frac{(1-r)^2}{4ry}\right]$; in case $y \leqq \frac{1-r^2}{2r}$ it decreases in $\left[\frac{(1-r)^2}{4ry}, \infty\right)$ and in case $y > \frac{1-r^2}{2r}$ it decreases in $\left[\frac{(1-r)^2}{4ry}, z_1\right]$, increases in $[z_1, z_2]$ and decreases in $[z_2, \infty)$.

Remark that

$$\frac{(1-r)^2}{4ry} \leqq z_1 \leqq y. \tag{16}$$

Indeed, $z_1 \leqq y$ follows from $\frac{2\sqrt{r}}{1+r} \leqq 1$ and $\frac{1-r}{2\sqrt{ry}} \leqq \sqrt{z_1}$ can be proved as follows

$$\frac{1-r}{2\sqrt{yr}} \leqq 2\frac{\sqrt{yr} - \sqrt{yr - \frac{1-r^2}{2}}}{1+r}, \quad \frac{1-r^2}{4} \leqq yr - \sqrt{yr\left(yr - \frac{1-r^2}{2}\right)},$$

$$yr\left(yr - \frac{1-r^2}{2}\right) \leqq (yr)^2 - yr\frac{1-r^2}{2} + \left(\frac{1-r^2}{4}\right)^2.$$

Finally remark that

$$H\left(r, y, \frac{(1-r)^2}{4ry}+\right) \asymp H\left(r, y, \frac{(1-r)^2}{4ry}-\right) \tag{17}$$

and the implicit constants do not depend on r and y. Investigate two cases, denoted by A and B, namely $y \leqq \frac{1-r^2}{2r}$ and $y \geqq \frac{1-r^2}{2r}$. Consider first the case A.

$$\frac{(1-r)^2}{4ry} \leqq y \leqq \frac{1-r^2}{2r}, \quad \frac{(1-r)^2}{4ry} \leqq z_0. \tag{A_{11}}$$

Then $\sqrt{u_\alpha}(z)H(r,y,z)$ increases for $z \leqq \frac{(1-r)^2}{4ry}$ and decreases for $z \geqq \frac{(1-r)^2}{4ry}$. Consequently

$$\sqrt{u_\alpha}(z)L(r,y,z) = \begin{cases} \sqrt{u_\alpha}(z)H(r,y,z) & \text{if } 0 < z \leqq \frac{(1-r)^2}{4ry} \\ \sqrt{u_\alpha}\left(\frac{(1-r)^2}{4ry}\right) H\left(r, y, \frac{(1-r)^2}{4ry}\right) & \text{if } \frac{(1-r)^2}{4ry} < z \leqq y \\ \sqrt{u_\alpha}(z)H(r,y,z) & \text{if } y < z \end{cases}$$

and by Lemma 3 we have only to show that

$$\sqrt{u_\alpha}(y)\left(y-\frac{(1-r)^2}{4ry}\right)\sqrt{u_\alpha}\left(\frac{(1-r)^2}{4ry}\right)H\left(r,y,\frac{(1-r)^2}{4ry}\right)\leqq c.$$

Putting here the definition of u_α and H we have to prove that

$$y^{\frac{\alpha}{2}}e^{-\frac{y}{2}}\left(y-\frac{(1-r)^2}{4ry}\right)\left(\frac{(1-r)^2}{4ry}\right)^{\frac{\alpha}{2}}e^{-\frac{(1-r)^2}{8ry}}\frac{1}{\sqrt{1-r}}\left(yr\frac{(1-r)^2}{4ry}\right)^{-\frac{\alpha}{2}-\frac{1}{4}}\cdot$$

$$\cdot\exp\left\{\frac{-yr+2\sqrt{yr\frac{(1-r)^2}{4yr}}-r\frac{(1-r)^2}{4yr}}{1-r}\right\}\leqq$$

$$\leqq cye^{-\frac{y}{2}}\frac{1}{1-r}r^{-\frac{\alpha}{2}}\exp\left\{\frac{-yr+1-r-\frac{(1-r)^2}{4y}}{1-r}\right\}\leqq$$

$$\leqq c\frac{y}{1-r}\exp\left\{-y\left(\frac{1}{2}+\frac{r}{1-r}\right)\right\}$$

is bounded. But this is true since in case $r\leqq\frac{1}{2}$ it is bounded by $cye^{-\frac{y}{2}}\leqq c$, and in case $r\geqq\frac{1}{2}$ by

$$c\frac{y}{1-r}e^{-\frac{1}{2}\frac{y}{1-r}}\leqq c.$$

$$(\mathrm{A}_{12})\qquad z_0\leqq\frac{(1-r)^2}{4ry}\leqq y\leqq\frac{1-r^2}{2r}.$$

In this case $\sqrt{u_\alpha}(z)H(r,y,z)$ increases in $[0,z_0]$, decreases in $\left[z_0,\frac{(1-r)^2}{4ry}\right]$ and in $\left[\frac{(1-r)^2}{4ry},\infty\right)$. Taking (17) into account we have to show that

$$\sqrt{u_\alpha}(y)(y-z_0)\sqrt{u_\alpha}(z_0)H(r,y,z_0)\leqq c.$$

Using $z_0\leqq c(1-r)$ we get

$$\sqrt{u_\alpha}(y)(y-z_0)\sqrt{u_\alpha}(z_0)H(r,y,z_0)\leqq$$

$$\leqq cy^{\frac{\alpha}{2}+1}e^{-\frac{y}{2}}(1-r)^{\frac{\alpha}{2}}\frac{1}{(1-r)^{\alpha+1}}e^{-\frac{ry}{1-r}}=c\left(\frac{y}{1-r}\right)^{\frac{\alpha}{2}+1}e^{-y\left(\frac{1}{2}+\frac{r}{1-r}\right)}\leqq c$$

by the same reasoning as in (A_{11}).

$$(\mathrm{A}_{21})\qquad z_0\leqq y\leqq\frac{(1-r)^2}{4ry}.$$

Now we have to show again that

$$\sqrt{u_\alpha}(y)(y-z_0)\sqrt{u_\alpha}(z_0)H(r,y,z_0) \leqq c,$$

which can be proved as in (A_{12}).

$$(A_{22}) \qquad y \leqq z_0 \leqq \frac{(1-r)^2}{4ry}.$$

In this case we have to verify that

$$\sqrt{u_\alpha}(y)(z_0-y)\sqrt{u_\alpha}(z_0)H(r,y,z_0) \leqq c.$$

Using that $z_0 \leqq c(1-r)$ we get

$$\sqrt{u_\alpha}(y)(z_0-y)\sqrt{u_\alpha}(z_0)H(r,y,z_0) \leqq$$

$$\leqq c(1-r)^{\frac{\alpha}{2}+1}(1-r)^{\frac{\alpha}{2}}\frac{1}{(1-r)^{\alpha+1}}e^{-y\frac{r}{1-r}} \leqq ce^{-y\frac{r}{1-r}} \leqq c.$$

$$(A_{23}) \qquad y \leqq \frac{(1-r)^2}{4ry} \leqq z_0.$$

Now the inequality to be proved is

$$\sqrt{u_\alpha}(y)\left(\frac{(1-r)^2}{4ry}-y\right)\sqrt{u_\alpha}\left(\frac{(1-r)^2}{4ry}\right)H\left(r,y,\frac{(1-r)^2}{4ry}\right) \leqq c.$$

Since $y \leqq z_0 \leqq c(1-r)$, the left hand side can be estimated by

$$c(1-r)^{\frac{\alpha}{2}+1}(1-r)^{\frac{\alpha}{2}}\frac{1}{(1-r)^{\alpha+1}} \leqq c.$$

So (15) is proved in case A. Take now the case B.

$$(B_{11}) \qquad \frac{(1-r)^2}{4ry} \leqq z_0, \qquad \frac{(1-r)^2}{4ry} \leqq z_1 \leqq y \leqq z_2.$$

Then the function $\sqrt{u_\alpha}(z)H(r,y,z)$ increases in $\left[0,\frac{(1-r)^2}{4ry}\right]$, decreases in $\left[\frac{(1-r)^2}{4ry},z_1\right]$, increases in $[z_1,z_2]$ and decreases in $[z_2,\infty)$, so we have to prove that

a) $\sqrt{u_\alpha}(y)\left(y-\frac{(1-r)^2}{4ry}\right)\sqrt{u_\alpha}\left(\frac{(1-r)^2}{4ry}\right)H\left(r,y,\frac{(1-r)^2}{4ry}\right) \leqq c$

and

b) $\sqrt{u_\alpha}(y)(z_2-y)\sqrt{u_\alpha}(z_2)H(r,y,z_2) \leqq c.$

Since $y \geqq \frac{1-r^2}{2r}$ implies that $\frac{(1-r)^2}{4ry} \leqq c(1-r)$, hence a) becomes

$$cy^{\frac{\alpha}{2}+1}e^{-\frac{y}{2}}(1-r)^{\frac{\alpha}{2}}\frac{1}{(1-r)^{\alpha+1}}e^{-y\frac{r}{1-r}} = c\left(\frac{y}{1-r}\right)^{\frac{\alpha}{2}+1} e^{-y\left(\frac{1}{2}+\frac{r}{1-r}\right)} \leqq c$$

and from $z_2 \asymp yr$ we see that b) can be estimated by

$$cy^{\frac{\alpha}{2}+1}e^{-\frac{y}{2}}r(yr)^{\frac{\alpha}{2}}e^{-\frac{z_2}{2}}\frac{(ry)^{-\alpha-\frac{1}{2}}}{\sqrt{1-r}}\exp\left\{\frac{-ry+2\sqrt{ryz_2}-rz_2}{1-r}\right\} =$$

$$= c\sqrt{\frac{y}{1-r}}r^{-\frac{\alpha}{2}+\frac{1}{2}}\exp\left\{-\frac{(\sqrt{ry}-\sqrt{z_2})^2+(\sqrt{y}-\sqrt{rz_2})^2}{2(1-r)}\right\}.$$

Now $y \leqq z_2 \leqq cyr$ implies $0 < c \leqq r$ hence the term $r^{-\frac{\alpha}{2}+\frac{1}{2}}$ can be omitted. The estimate

$$\sqrt{\frac{y}{1-r}}\exp\left\{-\frac{(\sqrt{ry}-\sqrt{z_2})^2}{2(1-r)}\right\} \leqq c$$

obviously holds for $y \leqq \frac{1-r^2}{r}$ and if $y \geqq \frac{1-r^2}{r}$ then $yr - \frac{1-r^2}{2} \geqq y\frac{r}{2} \geqq cy$, so

$$\sqrt{z_2}-\sqrt{ry} \geqq \sqrt{ry}\left(\frac{2}{1+r}-1\right)+\frac{2}{1+r}\sqrt{ry-\frac{1-r^2}{2}} \geqq c\sqrt{y}$$

and then

$$\sqrt{\frac{y}{1-r}}\exp\left\{-\frac{(\sqrt{ry}-\sqrt{z_2})^2}{2(1-r)}\right\} \leqq \sqrt{\frac{y}{1-r}}e^{-c\frac{y}{1-r}} \leqq c.$$

$$(\mathrm{B}_{12}) \qquad \frac{(1-r)^2}{4ry} \leqq z_0, \quad \frac{(1-r)^2}{4ry} \leqq z_1 \leqq z_2 \leqq y.$$

We have to prove that

a) $\sqrt{u_\alpha}(y)\left(y-\frac{(1-r)^2}{4ry}\right)\sqrt{u_\alpha}\left(\frac{(1-r)^2}{4ry}\right)H\left(r,y,\frac{(1-r)^2}{4ry}\right) \leqq c$,

b) $\sqrt{u_\alpha}(y)(y-z_2)\sqrt{u_\alpha}(z_2)H(r,y,z_2) \leqq c$.

Now a) was proved in (B_{11}); b) becomes by $\frac{1}{r} \leqq c\frac{y}{1-r}$

$$r^{-\frac{\alpha+1}{2}}\sqrt{\frac{y}{1-r}}\exp\left\{-\frac{(\sqrt{ry}-\sqrt{z_2})^2+(\sqrt{y}-\sqrt{rz_2})^2}{2(1-r)}\right\} \leqq$$

$$\leqq c\left(\frac{y}{1-r}\right)^{\frac{\alpha}{2}+1}\exp\left\{-\frac{(\sqrt{ry}-\sqrt{z_2})^2+(\sqrt{y}-\sqrt{rz_2})^2}{2(1-r)}\right\} \leqq c.$$

If $r \leqq r_0 < 1$ and r_0 is small enough then $(\sqrt{y}-\sqrt{rz_2})^2 \asymp y$ and then

$$\left(\frac{y}{1-r}\right)^{\frac{\alpha}{2}+1} \exp\left\{-\frac{(\sqrt{y}-\sqrt{rz_2})^2}{2(1-r)}\right\} \leqq cy^{\frac{\alpha}{2}+1}e^{-cy} \leqq c$$

and if $r_0 \leqq r$ then

$$\left(\frac{y}{1-r}\right)^{\frac{\alpha}{2}+1} \exp\left\{-\frac{(\sqrt{ry}-\sqrt{z_2})^2+(\sqrt{y}-\sqrt{rz_2})^2}{2(1-r)}\right\} \leqq c$$

follows as in (B_{11}).

(B_{21}) $$z_0 \leqq \frac{(1-r)^2}{4ry} \leqq z_1 \leqq y \leqq z_2.$$

Then we need

a) $\sqrt{u_\alpha}(y)(y-z_0)\sqrt{u_\alpha}(z_0)H(r,y,z_0) \leqq c$,
b) $\sqrt{u_\alpha}(y)(z_2-y)\sqrt{u_\alpha}(z_2)H(r,y,z_2) \leqq c$.

Now b) can be proved as in (B_{11}) and a) as in (A_{12}).

(B_{22}) $$z_0 \leqq \frac{(1-r)^2}{4ry} \leqq z_1 \leqq z_2 \leqq y.$$

Then we need again

a) $\sqrt{u_\alpha}(y)(y-z_0)\sqrt{u_\alpha}(z_0)H(r,y,z_0) \leqq c$

proved in (A_{12}) and

b) $\sqrt{u_\alpha}(y)(y-z_2)\sqrt{u_\alpha}(z_2)H(r,y,z_2) \leqq c$.

Since $\frac{(1-r)^2}{4ry} \leqq z_2 \leqq cry$ implies that $\frac{1}{r} \leqq c\frac{y}{1-r}$, b) follows just like in (B_{12}).

Lemma 4 is completely proved.

Introduce the function

$$U(x,r) := \frac{x \exp\left\{\frac{x^2}{4}\log r\right\}}{2\sqrt{\pi}r(-\log r)^{\frac{3}{2}}},$$

then ([3])

(18) $$\int_0^1 U(x,r)r^n dr = e^{-\sqrt{n}x}, \qquad x>0,\ n \in \mathbf{N}.$$

Define the alternate Poisson integral of f by

$$f(x,y) := \int_0^1 U(x,r)g(r,y)dr \tag{19}$$

then

$$f(x,y) = \int_0^\infty \left(\int_0^1 U(x,r)K(r,y,z)dr \right) f(z)u_\alpha(z)dz.$$

It follows from (18) that if f has the expansion $f \sim \sum_{k=0}^{\infty} a_k l_k^{(\alpha)}$ then

$$f(x,y) \sim \sum_{k=0}^{\infty} a_k e^{-\sqrt{k}x} l_k^{(\alpha)}(y). \tag{20}$$

THEOREM 1. *Let* $\alpha > 0$, $1 \leqq p \leqq \infty$ *and* $f \in L^p(\sqrt{u_\alpha})$. *Then*

a) $\sqrt{u_\alpha}(y) \sup_{x>0} |f(x,y)| \leqq c(\sqrt{u_\alpha} f)^*(y)$ *a.e.*,

b) $\|\sqrt{u_\alpha}(y)[f(x,y) - f(y)]\|_p \to 0 \quad (x \to 0+), \quad p \neq 1, \infty,$

c) $\lim_{x \to 0+} f(x,y) = f(y)$ *a.e.*, $p \neq \infty$,

d) $\|\sqrt{u_\alpha}(y) \sup_{x>0} |f(x,y)|\|_p \leqq c(p) \|\sqrt{u_\alpha} f\|_p, \quad p \neq 1, \infty.$

PROOF. a) Define the function

$$\hat{L}(x,y,z) = \int_0^1 L(r,y,z)U(x,r)dr;$$

then

$$\int_0^1 K(r,y,z)U(x,r)dr \leqq c\hat{L}(x,y,z);$$

further the function $z \mapsto \sqrt{u_\alpha}(z)\hat{L}(x,y,z)$ increases for $z \leqq y$, decreases for $z \geqq y$ and

$$\sqrt{u_\alpha}(y) \int_0^\infty \hat{L}(x,y,z)\sqrt{u_\alpha}(z)dz =$$

$$= \int_0^1 U(x,r)\sqrt{u_\alpha}(y) \int_0^\infty L(r,y,z)\sqrt{u_\alpha}(z)dzdr \leqq c \int_0^1 U(x,r)dr = c$$

hence a) follows from Lemma 2.

b), c), d). We know that the set of polynomials is dense in $L^p(\sqrt{u_\alpha})$, $1 \leqq p < \infty$. Taking (19) into account, b) follows from a) and d) by the Banach–Steinhaus theorem, and c) follows from the Banach–Steinhaus type theorem related to the convergence in measure ([14]). Finally d) follows from the estimate

$$\|(\sqrt{u_\alpha}f)^*\|_p \leqq c(p)\|\sqrt{u_\alpha}f\|_p, \qquad 1 < p \leqq \infty$$

see in [13]. Theorem 1 is proved.

In what follows , following Muckenhoupt [5], we shall investigate the conjugate function. Let

$$q(x,y,z) := \sqrt{y}\int_0^1 \frac{\partial}{\partial y}\left[\frac{\exp\left\{\frac{x^2}{4\log r} - \frac{r(y+z)}{1-r}\right\} J_\alpha\left(i\frac{2\sqrt{ryz}}{1-r}\right)}{r(1-r)\sqrt{-\log r}\, i^\alpha (ryz)^{\frac{\alpha}{2}}}\right] dr. \tag{21}$$

It is not hard to see that

$$\int_0^\infty q(x,y,z) l_{n-1}^{(\alpha+1)}(y) u_{\alpha+\frac{1}{2}}(y)dz = e^{-\sqrt{n}x} l_n^{(\alpha)}(z) \tag{22}$$

and

$$q(x,y,z) = \sqrt{y}\sum_{n=1}^\infty e^{-\sqrt{n}x} l_{n-1}^{(\alpha+1)}(y) l_n^{(\alpha)}(z) \tag{23}$$

(meant pointwise). Further it is proved in [5] that

$$|q(x,y,z)| \leqq c = c(y,z,\alpha) \text{ for } y,z > 0 \text{ and } x \geqq a > 0, \tag{24}$$

consequently the conjugate Poisson integral

$$\tilde{f}(x,y) := \int_0^\infty q(x,y,z) f(z) u_\alpha(z) dz \tag{25}$$

exists for all $x, y > 0$.

REMARK. The norm estimate of $f(x,y)$ was proved by majorizing it by the maximal function operator. To prove norm estimate for $\tilde{f}(x,y)$ we have to decompose it into two parts one of which is majorized by the maximal function and the other one by the maximal Hilbert transform. Let $1 \leqq p < \infty$ and $f \in L^p(\mathbf{R})$. The maximal Hilbert transform of f is defined by

$$H^* f(x) := \frac{1}{\pi}\sup_{\varepsilon > 0}\left|\int_{|x-t|>\varepsilon} \frac{f(t)}{x-t}\,dt\right|, \qquad x \in \mathbf{R}.$$

It is known ([13], p. 133) that

$$f \in L^1(\mathbf{R}) \Rightarrow |(H^*f > \lambda)| \leqq \frac{c}{\lambda}\|f\|_1, \tag{26}$$

$$f \in L^p(\mathbf{R}), \quad 1 < p < \infty \Rightarrow \|H^*f\|_p \leqq c(p)\|f\|_p. \tag{27}$$

We shall use the following result from [4]:

LEMMA 5 ([4]). *If $K(z) = -K(-z)$ and if $zK(z)$ (defined as 0 for $z = 0$) has total variation V on $[0, m]$ then*

$$\sup_{0<a<b\leqq m}\left|\int\limits_{a\leqq|z|\leqq b} f(y-z)K(z)dz\right| \leqq V \sup_{0<a<b\leqq m}\left|\int\limits_{a\leqq|z|\leqq b}\frac{f(y-z)}{z}dz\right|.$$

Let now I be an arbitrary (finite or infinite) interval and let $w > 0$, $w \in L^1(I)$. As in [4], we say that a partition

$$I = \bigcup_{n\in\mathbf{Z}} I_n$$

of I into disjoint segments I_n has property A if for all $n \in \mathbf{Z}$

a) I_n stands left to I_{n+1},

b) $|I_n| \leqq 2|I_{n+1}|$, $|I_n| \leqq 2|I_{n-1}|$,

c) $\sup_{I_n} w / \inf_{I_n} w \leqq B < \infty$.

We need the following modification of Lemma 3 of [4].

LEMMA 6. *Let I be an interval, $w > 0$ a weight and (I_n) a partition of I having property A. Let a function f be defined in I and denote*

$$g(y) := \sup_{a,b}{}^*\left|\int\limits_{a\leqq|z|\leqq b}\frac{f(y-z)}{z}dz\right|$$

where $\sup^*$ *runs over the pairs* $0 < a < b \leqq \frac{|I_n|}{2}$, *where n is defined by $y \in I_n$. Then*

a) $f \in L^1(w, I) \Rightarrow |\{y \in I : w(y)g(y) > \lambda\}| \leqq \frac{c}{\lambda}\|fw\|_1$,

b) $f \in L^p(w, I), 1 < p < \infty \Rightarrow \|wg\|_p \leqq c(p)\|wf\|_p$.

PROOF. Denote

$$E_\lambda := \{y \in I : w(y)g(y) > \lambda\}, \quad J_n := I_{n-1} \cup I_n \cup I_{n+1}, \quad f_n := f\chi_{J_n}.$$

It follows from the definition of the property A that for $y \in I_n$

$$w(y)g(y) = \sup_{0<a<b\leqq\frac{|I_n|}{2}} \left| \int_{a\leqq|z|\leqq b} \frac{f_n(y-z)}{z} dz \right| w(y) \leqq$$

$$\leqq c \sup_{0<a<b\leqq\frac{|I_n|}{2}} \int_{a\leqq|z|\leqq b} \frac{|f_n(y-z)|w(y-z)}{z} dz \leqq$$

$$\leqq c \sup_{0<a} \int_{a\leqq|z|} \frac{|f_n(y-z)|w(y-z)}{z} dz \leqq cH^*(f_n w)(y).$$

Hence for $f \in L^1(w,I)$ we have

$$|E_\lambda| = \sum_{n\in\mathbf{Z}} |E_\lambda \cap I_n| \leqq \sum_{n\in\mathbf{Z}} \left| \left(H^*(f_n w) > \frac{\lambda}{c} \right) \right| \leqq \sum_{n\in\mathbf{Z}} \frac{c}{\lambda} \|f_n w\|_1 \leqq \frac{c}{\lambda} \|fw\|_1$$

and for $f \in L^p(w,I)$, $1 < p < \infty$

$$\|wg\|_p^p = \sum_{n\in\mathbf{Z}} \int_{I_n} |wg|^p \leqq c \sum_{n\in\mathbf{Z}} \int_{I_n} [H^*(f_n w)]^p \leqq$$

$$\leqq c \sum_{n\in\mathbf{Z}} \int_I [H^*(f_n w)]^p \leqq c \sum_{n\in\mathbf{Z}} \|f_n w\|_p^p \leqq c\|fw\|_p^p.$$

LEMMA 7. *Let $\alpha \geqq 0$, $x, y > 0$. Then there exists a partition*

(28) $$q(x,y,z) = j(x,y,z) + u_\alpha^{-1}(z)k(x,y,z)$$

satisfying the following properties:

a) $|j(x,y,z)| \leqq cJ(y,z)$ *where the function* $z \mapsto \sqrt{u_\alpha}(z)J(y,z)$ *increases for* $z \leqq y$, *decreases for* $z \geqq y$ *and*

(29) $$\sqrt{u_\alpha}(y) \int_0^\infty J(y,z)\sqrt{u_\alpha}(z)dz \leqq c;$$

b) $k(x,y,z) = 0$ *if* $|y-z| > m := \min\left\{\frac{1}{4}, \frac{y}{4}\right\}$,

$$k(x,y,y+h) = -k(x,y,y-h), \quad V((y-z)k(x,y,z)) \leqq c$$

(here V denotes the total variation).

PROOF. It is shown in the Lemma of [5] that there exists a decomposition of the form (28), where k satisfies b) and $|j(x,y,z)| \leqq cn(y,z)$ if $n(y,z)$ is defined as follows. For $0 < y \leqq 1$ let

$$n(y,z) = \begin{cases} y^{-\alpha-1} & \text{if } 0 < z \leqq \frac{3}{4}y \\ y^{-\alpha-1}\log\frac{y}{|y-z|} & \text{if } \frac{3}{4}y < z \leqq \frac{5}{4}y \\ y^{\frac{1}{2}}z^{-\alpha-\frac{3}{2}} & \text{if } \frac{5}{4}y < z \leqq 2 \\ y^{\frac{1}{2}} & \text{if } 2 < z \end{cases}$$

and for $y > 1$

$$n(y,z) = \begin{cases} y^{-\frac{1}{2}} & \text{if } 0 < z \leqq \min\left\{\alpha+2, \frac{3}{4}y\right\} \\ y^{-\frac{1}{2}}z^{-\alpha-\frac{1}{2}}e^{z} & \text{if } \alpha+2 < z \leqq \frac{3}{4}y \\ y^{-\alpha-1}e^{z}\left(1+\frac{y}{8(y-z)^{\frac{3}{2}}}\right) & \text{if } \frac{3}{4}y < z \leqq y-\frac{1}{4} \\ y^{-\alpha}e^{y}(1-\log|y-z|) & \text{if } y-\frac{1}{4} < z \leqq y+\frac{1}{4} \\ y^{-\alpha}e^{y} & \text{if } y+\frac{1}{4} < z. \end{cases}$$

Remark that

$$n(y,z+) \asymp n(y,z-), \qquad y,z > 0.$$

We distinguish two cases: $0 < y \leqq 1$, denoted by A and $y > 1$, denoted by B.

$$(\mathrm{A}_1) \qquad 0 < z \leqq \frac{3}{4}y.$$

Then $\sqrt{u_\alpha}n = z^{\alpha/2}e^{-z/2}y^{-\alpha-1}$. It increases in case $z \leqq \alpha$ and decreases for $z \geqq \alpha$ so let

$$\sqrt{u_\alpha}J := \begin{cases} z^{\alpha/2}e^{-z/2}y^{-\alpha-1} & \text{if } 0 < z \leqq \min\left\{\alpha, \frac{3}{4}y\right\} \\ \alpha^{\alpha/2}e^{-\alpha/2}y^{-\alpha-1} & \text{if } \alpha < z \leqq \frac{3}{4}y. \end{cases}$$

Here (29) is obvious.

$$(\mathrm{A}_2) \qquad \frac{3}{4}y < z \leqq y.$$

Then

$$\sqrt{u_\alpha}n = z^{\alpha/2}e^{-z/2}y^{-\alpha-1}\log\frac{y}{y-z} \asymp y^{-\frac{\alpha}{2}-1}\log\frac{y}{y-z}$$

hence we can define for large c

$$\sqrt{u_\alpha} J = c y^{-\frac{\alpha}{2}-1} \log \frac{y}{y-z}.$$

Now

$$\sqrt{u_\alpha}(y) \int_{\frac{3}{4}y}^{y} J(y,z)\sqrt{u_\alpha}(z)dz \leqq \frac{c}{y} \int_{\frac{3}{4}y}^{y} [\log y - \log(y-z)]dz =$$

$$= \frac{1}{4}\left(\log y + 1 - \log\frac{y}{4}\right) = \frac{1}{4}(1+\log 4).$$

(A$_3$) $$2 < z, \quad \alpha \leqq 2.$$

Then let

$$\sqrt{u_\alpha} J := y^{1/2} z^{\alpha/2} e^{-z/2}.$$

(A$_4$) $$2 < z, \quad \alpha > 2,$$

$$\sqrt{u_\alpha} J := \begin{cases} y^{1/2}\alpha^{\alpha/2}e^{-\alpha/2} & \text{if } 2 < z \leqq \alpha \\ y^{1/2} z^{\alpha/2} e^{-z/2} & \text{if } \alpha < z. \end{cases}$$

In both cases (29) fulfils and $\sqrt{u_\alpha}(2)J(y,2+) \asymp y^{1/2}$.

(A$_5$) $$\frac{5}{4}y < z \leqq 2.$$

Then $\sqrt{u_\alpha} n \asymp y^{\frac{1}{2}} z^{-\frac{\alpha+3}{2}}$, so for large c define

$$\sqrt{u_\alpha} J := c y^{\frac{1}{2}} z^{-\frac{\alpha+3}{2}}$$

and

$$y^{\frac{\alpha+1}{2}} \int_{\frac{5}{4}y}^{2} z^{-\frac{\alpha+3}{2}} dz \leqq c$$

proves (29). Finally $\sqrt{u_\alpha}\left(\frac{5}{4}y\right) J\left(y, \frac{5}{4}y+\right) \asymp y^{-\frac{\alpha}{2}-1}$.

(A$_6$) $$y < z \leqq \frac{5}{4}y.$$

In this case $\sqrt{u_\alpha} n \asymp y^{-\frac{\alpha}{2}-1} \log \frac{y}{z-y}$ hence we can set

$$\sqrt{u_\alpha} J := c y^{-\frac{\alpha}{2}-1} \log \frac{y}{z-y}$$

for large c. Now (29) follows as in (A_2). The case $y \leqq 1$ being ready, investigate the case $y > 1$. We shall use the estimate

$$\int_x^\infty z^{\frac{\alpha}{2}} e^{-\frac{z}{2}} dz \leqq 4x^{\frac{\alpha}{2}} e^{-\frac{x}{2}} \qquad (x \geqq 2\alpha).$$

Indeed, for $x = \infty$ equality holds and the derivatives of both sides fulfil the converse inequality. We also get that

$$\int_x^\infty z^{\frac{\alpha}{2}} e^{-\frac{z}{2}} dz \leqq c(\alpha) x^{\frac{\alpha}{2}} e^{-\frac{x}{2}} \qquad (x \geqq 1). \tag{30}$$

(B_1) $$0 < z \leqq \min\left(\alpha + 2, \frac{3}{4}y\right).$$

Then $\sqrt{u_\alpha} n \asymp y^{-1/2} z^{\alpha/2}$, so let

$$\sqrt{u_\alpha} J := c y^{-\frac{1}{2}} z^{\frac{\alpha}{2}}$$

for large c. By $\sqrt{u_\alpha}(y) \int_0^{\frac{3}{4}y} y^{-\frac{1}{2}} z^{\frac{\alpha}{2}} dz \leqq c y^{\frac{1}{2}} e^{-\frac{y}{2}} \leqq c$ (29) fulfils.

(B_2) $$\alpha + 2 < z \leqq \frac{3}{4} y.$$

Then $\sqrt{u_\alpha} n \asymp y^{-\frac{1}{2}} z^{-\frac{\alpha+1}{2}} e^{\frac{z}{2}}$, hence we can define

$$\sqrt{u_\alpha} J := c y^{-\frac{1}{2}} z^{-\frac{\alpha+2}{2}} e^{\frac{z}{2}}$$

and

$$y^{\frac{\alpha}{2}} e^{-\frac{y}{2}} y^{-\frac{1}{2}} \int_{\alpha+2}^{\frac{3}{4}y} z^{-\frac{\alpha+1}{2}} e^{\frac{z}{2}} dz \leqq y^{\frac{\alpha-1}{2}} e^{-\frac{y}{8}} \int_{\alpha+2}^{\frac{3}{4}y} z^{-\frac{\alpha+1}{2}} dz$$

which is $\leqq c y^{\frac{\alpha-1}{2}} y^{-\frac{\alpha-1}{2}} = c$ for $\alpha \neq 1$ and $\leqq c y \log y \, e^{-\frac{y}{8}} \leqq c$ for $\alpha = 1$.

(B_3) $$\frac{3}{4} y < z \leqq y - \frac{1}{4}.$$

Now

$$\sqrt{u_\alpha} n = e^{\frac{z}{2}} y^{-\alpha-1} z^{\frac{\alpha}{2}} \left(1 + \frac{y}{8(y-z)^{3/2}}\right) \leqq c y^{-\frac{\alpha}{2}-1} e^{\frac{y}{2}} \left(1 + \frac{y}{8(y-z)^{3/2}}\right).$$

So let

$$\sqrt{u_\alpha}J := ce^{\frac{y}{2}}y^{-\frac{\alpha}{2}-1}\left(1+\frac{y}{8(y-z)^{3/2}}\right)$$

and (29) follows from

$$y^{-1}\int_{\frac{3}{4}y}^{y-\frac{1}{4}}\left(1+\frac{y}{8(y-z)^{3/2}}\right)dz \leqq c\left(1+\int_{\frac{3}{4}y}^{y-\frac{1}{4}}(y-z)^{-\frac{3}{2}}dz\right) \leqq c.$$

(B$_4$)
$$y-\frac{1}{4} < z \leqq y.$$

Then

$$\sqrt{u_\alpha}n \asymp e^{\frac{y}{2}}y^{-\frac{\alpha}{2}}(1-\log(y-z))$$

and since

$$\sqrt{u_\alpha}\left(y-\frac{1}{4}\right)J\left(y, y-\frac{1}{4}\right) \asymp y^{-\frac{\alpha}{2}}e^{\frac{y}{2}}$$

hence we can write

$$\sqrt{u_\alpha}J := cy^{-\frac{\alpha}{2}}e^{\frac{y}{2}}(1-\log(y-z))$$

and (29) is obvious.

(B$_5$)
$$y+\frac{1}{4} < z.$$

Then $\sqrt{u_\alpha}n = z^{\alpha/2}e^{-z/2}e^{y}y^{-\alpha}$. Now in case $\alpha \leqq y+\frac{1}{4}$ let

$$\sqrt{u_\alpha}J := z^{\frac{\alpha}{2}}e^{-\frac{z}{2}}e^{y}y^{-\alpha}$$

and in case $y+\frac{1}{4} < \alpha$ let

$$\sqrt{u_\alpha}J := \begin{cases} z^{\frac{\alpha}{2}}e^{-\frac{z}{2}}e^{y}y^{-\alpha}\left(\asymp z^{\frac{\alpha}{2}}e^{-\frac{z}{2}}\right) & \text{if } \alpha < z \\ \alpha^{\frac{\alpha}{2}}e^{-\frac{\alpha}{2}}e^{y}y^{-\alpha}(\asymp 1) & \text{if } y+\frac{1}{4} < z \leqq \alpha. \end{cases}$$

Now in case $\alpha \leqq y+\frac{1}{4}$ the integral condition (29) follows from (30) and in case $\alpha > y+\frac{1}{4}$ it is trivial. In both cases we have

$$\sqrt{u_\alpha}\left(y+\frac{1}{4}\right)J\left(y, y+\frac{1}{4}+\right) \asymp y^{-\frac{\alpha}{2}}e^{\frac{y}{2}}.$$

(B$_6$)
$$y < z \leqq y+\frac{1}{4}.$$

Then $\sqrt{u_\alpha}n \asymp e^{y/2}y^{-\alpha/2}(1-\log(z-y))$, hence we define

$$\sqrt{u_\alpha}J := cy^{-\frac{\alpha}{2}}e^{\frac{y}{2}}(1-\log(z-y))$$

and the integral condition is obvious. The proof of Lemma 7 is complete.

THEOREM 2. *Suppose* $\alpha > 0$. *Then*

a) $f\sqrt{u_\alpha} \in L^1(0,\infty) \Rightarrow \left|\{y : \sqrt{u_\alpha}(y)\sup_{x>0}|\tilde{f}(x,y)| > \lambda\}\right| \leqq$ $\leqq \frac{c}{\lambda}\|\sqrt{u_\alpha}f\|_1$,

b) $f\sqrt{u_\alpha} \in L^p(0,\infty)$, $1 < p < \infty \Rightarrow \|\sqrt{u_\alpha}(y)\sup_{x>0}|\tilde{f}(x,y)|\|_p \leqq$ $\leqq c(p)\|\sqrt{u_\alpha}f\|_p$,

c) $f\sqrt{u_\alpha} \in L^p(0,\infty)$, $1 \leqq p < \infty$ *implies that the limit*

$$\tilde{f}(y) := \lim_{x\to 0+}\tilde{f}(x,y) \tag{31}$$

exists for a.e. $y > 0$,

d) $f\sqrt{u_\alpha} \in L^p(0,\infty)$, $1 < p < \infty \Rightarrow \|\sqrt{u_\alpha}\tilde{f}\|_p \leqq c(p)\|\sqrt{u_\alpha}f\|_p$ *and* $\lim_{x\to 0+}\|\sqrt{u_\alpha}(y)[\tilde{f}(y) - \tilde{f}(x,y)]\|_p = 0$.

e) *If* $f\sqrt{u_\alpha} \in L^p$, $1 < p < \infty$ *and* $f(y) \sim \sum_{k=0}^{\infty} a_k l_k^{(\alpha)}(y)$ *then*

$$\tilde{f}(y) \sim \sum_{k=1}^{\infty} a_k\sqrt{y}\ell_{k-1}^{(\alpha+1)}(y), \quad \tilde{f}(x,y) \sim \sum_{k=1}^{\infty} a_k e^{-\sqrt{k}x}\sqrt{y}l_{k-1}^{(\alpha+1)}(y) \tag{32}$$

(this means that $y^{-1/2}\tilde{f}(y) \in L^p(\sqrt{u}_{\alpha+1})$ *has the expansion*

$$\sum_{k=1}^{\infty} a_k l_{k-1}^{(\alpha+1)}(y), \quad a_k = \int_0^\infty \tilde{f}(y)y^{\frac{1}{2}}\ell_{k-1}^{\alpha+1}(y)u_\alpha(y)dy).$$

PROOF. By Lemma 7

$$\tilde{f}(x,y) = \int_0^\infty j(x,y,z)f(z)u_\alpha(z)dz + \int_0^\infty k(x,y,z)f(z)dz =:$$

$$=: T_1(f,x,y) + T_2(f,x,y).$$

From Lemma 2 we see that a) and b) hold when replacing $\tilde{f}(x,y)$ by $T_1(f,x,y)$. On the other hand define the partition

$$(0,\infty) = \bigcup_{n\in\mathbf{Z}} I_n, \quad I_n := \begin{cases} [n, n+1] & \text{if} \quad n \geqq 1 \\ [2^{n-1}, 2^n] & \text{if} \quad n \leqq 0; \end{cases}$$

this partition has property A with respect to the weight $w := \sqrt{u_\alpha}$. By Lemma 5 we have

$$|T_2(f,x,y)| \leqq c \sup_{0<a<b\leqq m}\left|\int_{a\leqq|z|\leqq b}\frac{f(y-z)}{z}dz\right| \leqq c \sup_{a,b}{}^*\left|\int_{a\leqq|z|\leqq b}\frac{f(y-z)}{z}dz\right|$$

since $y \in I_n$ implies $m = \min\left(\frac{1}{4}, \frac{y}{4}\right) \leqq \frac{|I_n|}{2}$. Now Lemma 6 states that a) and b) hold with $T_2(f, x, y)$ instead of $\tilde{f}(x, y)$. So a) and b) are proved. The statement c) holds if f is a polynomial. Since the polynomials are dense in $L^p(\sqrt{u_\alpha})$, $1 \leqq p < \infty$ hence c) follows from a) and b) by the Banach theorem mentioned in proving Theorem 1. The statement d) is an immediate corollary of b), c) and the Banach–Steinhaus theorem. Finally e) is easy to check for polynomials; in general (32) follows by Proposition b).

Now we prove an Alexits type theorem.

THEOREM 3. *Let $\alpha > 0$, $1 < p < \infty$ and $f \in L^p(\sqrt{u_\alpha})$. The following statements are equivalent:*

$$\|\sqrt{u_\alpha}(f - R_n f)\|_p = O\left(\frac{1}{\sqrt{n}}\right), \tag{33}$$

$$\begin{cases} \tilde{f} \quad \text{is locally absolutely continuous and} \\ \left[u_{\alpha+\frac{1}{2}}\tilde{f}\right]' u_\alpha^{-1} \in L^p(\sqrt{u_\alpha}), \quad \lim_{x\to 0+} u_{\alpha+\frac{1}{2}}(x)\tilde{f}(x) = 0. \end{cases} \tag{34}$$

REMARK. The implication (34)⇒(33) is essentially stated in [7] for Fejér means. The proof of the converse implication does not work for Fejér means because the corresponding variant of the Alexits Lemma does not hold. That is why we use Riesz means instead of Fejér means (see [8] for more details).

PROOF. (33)⇒(34). By the Alexits Lemma (33) is equivalent to

$$\|\sqrt{u_\alpha}R_n(\textstyle\sum \sqrt{k}a_k\ell_k^{(\alpha)})\|_p = O(1)$$

(see [8]). This last estimate implies the existence of a function $g \in L^p(\sqrt{u_\alpha})$ such that

$$g \sim \sum \sqrt{k}a_k\ell_k^{(\alpha)}.$$

From (3) it follows that

$$\left[u_{\alpha+\frac{1}{2}}R_n\tilde{f}\right]' = -u_\alpha R_n g.$$

This can be rewritten as

$$\int_x^\infty u_\alpha R_n g = u_{\alpha+\frac{1}{2}}(x)R_n\tilde{f}(x) \tag{35}$$

since both sides tend to zero as $x \to \infty$. From Lemma 1 it follows that

$$\left|\int_x^\infty u_\alpha(g - R_n g)\right| \leqq \|\sqrt{u_\alpha}(g - R_n g)\|_p \|\sqrt{u_\alpha}\|_q \to 0 \quad (n \to \infty);$$

hence the uniform limit

$$\int_x^\infty u_\alpha g = \lim_{n\to\infty} \int_x^\infty u_\alpha R_n g = \lim_{n\to\infty} u_{\alpha+\frac{1}{2}}(x) R_n \tilde{f}(x) \tag{36}$$

exists. Again by Lemma 1 (used with $\alpha+1$ instead of α) we get that $\sqrt{u_\alpha} R_n \tilde{f}$ tends to $\sqrt{u_\alpha}\tilde{f}$ in $L^p(0,\infty)$, hence

$$\int_x^\infty u_\alpha g = \lim_{n\to\infty} u_{\alpha+\frac{1}{2}}(x) R_n \tilde{f}(x) = u_{\alpha+\frac{1}{2}}(x)\tilde{f}(x)$$

which proves (34).

(34)⇒(33). Let $g := \left[u_{\alpha+\frac{1}{2}}\tilde{f}\right]' u_\alpha^{-1} \in L^p(\sqrt{u_\alpha})$ and compute its coefficients by (2):

$$b_k = \int_0^\infty g \ell_k^{(\alpha)} u_\alpha = \int_0^\infty \left[u_{\alpha+\frac{1}{2}}\tilde{f}\right]' \ell_k^{(\alpha)} =$$

$$= \lim_{x\to\infty} u_{\alpha+\frac{1}{2}}(x)\tilde{f}(x)\ell_k^{(\alpha)}(x) - \sqrt{n}\int_0^\infty \tilde{f}\ell_{k-1}^{(\alpha+1)} u_{\alpha+\frac{1}{2}}.$$

It is not hard to see that $g \in L^p(\sqrt{u_\alpha})$ implies

$$\lim_{x\to\infty} u_{\alpha+\frac{1}{2}}(x)\tilde{f}(x)x^k = 0, \qquad k = 0,1,2,\ldots. \tag{37}$$

Indeed, for $x \geqq 1$ we have by (30)

$$\left|u_{\alpha+\frac{1}{2}}(x)\tilde{f}(x) + c_0\right| = \left|\int_x^\infty \left[u_{\alpha+\frac{1}{2}}\tilde{f}\right]'\right| \leqq$$

$$\leqq \|\sqrt{u_\alpha} g\|_p \left(\int_x^\infty u_\alpha^{\frac{q}{2}}\right)^{\frac{1}{q}} \leqq c\|\sqrt{u_\alpha} g\|_p \sqrt{u_\alpha}(x)$$

and this is compatible with $\sqrt{u_\alpha}\tilde{f} \in L^p(0,\infty)$ only in case $c_0 = 0$ and then for $x \to \infty$

$$x^k |u_{\alpha+\frac{1}{2}}(x)\tilde{f}(x)| = x^k \left|\int_x^\infty [u_{\alpha+\frac{1}{2}}\tilde{f}]'\right| \leqq c x^k \sqrt{u_\alpha}(x) \to 0.$$

It follows from (37) that

$$g \sim -\sum \sqrt{k} u_k \ell_k^{(\alpha)}$$

and then Lemma 1 implies

$$\|\sqrt{u_\alpha} R_n g\|_p = O(1)$$

which is equivalent to (33) as we mentioned above. The proof is complete.

THEOREM 4. *Let $\alpha > 0$, $1 < p < \infty$ and $f \in L^p(\sqrt{u_\alpha})$. Then*

$$\|\sqrt{u_\alpha}(y)[f(x,y) - f(y)]\|_p = o(x) \quad (x \to 0+) \Leftrightarrow f = 0, \tag{38}$$

$$\|\sqrt{u_\alpha}(y)[f(x,y) - f(y)]\|_p = O(x) \Leftrightarrow \tag{39}$$

$$\Leftrightarrow [u_{\alpha+\frac{1}{2}}\tilde{f}]' u_\alpha^{-1} \in L^p(\sqrt{u_\alpha}), \quad \lim_{x\to 0+} u_{\alpha+\frac{1}{2}}(x)\tilde{f}(x) = 0.$$

PROOF. The operators $T_x f(y) := f(x,y)$, $x > 0$, $T_0 f(y) = f(y)$ have the semigroup property

$$T_{x_1} T_{x_2} f = T_{x_1+x_2} f. \tag{40}$$

Indeed

$$T_{x_1} T_{x_2} f(y) = \int_0^\infty K(x_1, y, z) u_\alpha(z) T_{x_2} f(z) dz =$$

$$= \int_0^\infty K(x_1, y, z) u_\alpha(z) \int_0^\infty K(x_2, z, t) f(t) u_\alpha(t) dt dz =$$

$$= \int_0^\infty f(t) u_\alpha(t) \int_0^\infty K(x_1, y, z) K(x_2, z, t) u_\alpha(z) dz dt$$

and

$$\int_0^\infty K(x_1, y, z) K(x_2, z, t) u_\alpha(z) dz =$$

$$= \int_0^\infty \left\{ \sum_{n=0}^\infty e^{-\sqrt{n} x_1} \ell_n^{(\alpha)}(y) \ell_n^{(\alpha)}(z) \right\} \left\{ \sum_{k=0}^\infty e^{-\sqrt{k} x_2} \ell_k^{(\alpha)}(z) \ell_k^{(\alpha)}(t) \right\} u_\alpha(z) dz =$$

$$= \sum_{n=0}^\infty e^{-\sqrt{n}(x_1+x_2)} \ell_n^{(\alpha)}(y) \ell_n^{(\alpha)}(t) = K(x_1 + x_2, y, t)$$

which proves (40). The continuity of this semigroup is proved in Theorem 1. It is known [12] that the saturation class of an operator semigroup is the domain of its infinitesimal generator and the saturation order is $O(x)$, $x > 0$. Hence all we have to prove is that the domain $D(A)$ of the infinitesimal generator A of the semigroup $\{T_x : x \geqq 0\}$ consists of the functions $f \in L^p(\sqrt{u_\alpha})$ satisfying (34). Denote $D_1(A)$ the set of these f. As we have seen in proving Theorem 3,

$$D_1(A) = \{f : \exists g \in L^p(\sqrt{u_\alpha}), g \sim \Sigma k a_k \ell_k^{(\alpha)}\} =: D_2(A).$$

We shall prove $D(A) = D_2(A)$. Let first $f \in D(A)$. By definition

$$\left\| \left[Af - \frac{T_x f - f}{x} \right] \sqrt{u_\alpha} \right\|_p \to 0 \qquad (x \to 0+);$$

hence

$$\int_0^\infty A(f)\ell_k^{(\alpha)} u_\alpha = \lim_{x\to 0+} \int_0^\infty \frac{T_x f - f}{x} \ell_k^{(\alpha)} u_\alpha = \lim_{x\to 0+} \frac{e^{-\sqrt{k}x} - 1}{x} a_k = -\sqrt{k} a_k,$$

$$Af \sim -\sum k a_k \ell_k^{(\alpha)}$$

and then $f \in D_2(A)$. Conversely suppose $f \in D_2(A)$. We know that

$$A(R_n f) = -R_n g.$$

Since $\|\sqrt{u_\alpha}(R_n f - f)\|_p \to 0$, $\|\sqrt{u_\alpha}(A(R_n(f) + g)\|_p \to 0$ and A is closed ([12]), hence $Af = -g$, $f \in D(A)$. Theorem 4 is proved.

In this final section of the present paper we prove another Alexits and Abel–Poisson type saturation theorems.

THEOREM 5. *Let $\alpha \geqq 0$, $1 < p < \infty$ and $f \in L^p(\sqrt{u_\alpha})$. The following statements are equivalent:*

(41) $$\|\sqrt{u_\alpha}(\tilde{f} - R_n \tilde{f})\|_p = O\left(\frac{1}{\sqrt{n}}\right),$$

(42) f *is locally absolutely continuous and* $f' y^{1/2} \in L^p(\sqrt{u_\alpha})$.

Here $R_n \tilde{f}$ denotes the Riesz means of the series

$$y^{1/2} \sum a_k \ell_{k-1}^{(k+1)}(y).$$

PROOF. As in Theorem 3 we see that (41) is equivalent to the existence of a function $g \in L^p(\sqrt{u_\alpha})$ having the expansion

$$g(y) \sim y^{1/2} \sum \sqrt{k} a_k \ell_{k-1}^{(\alpha+1)}(y).$$

It follows from (2) that

$$\int_1^x y^{-\frac{1}{2}} R_n g(y)dy = R_n f(x) - R_n f(1). \tag{43}$$

We know that

$$\left|\int_1^x y^{-\frac{1}{2}}(g(y) - R_n g(y))dy\right| \leqq \|\sqrt{u_\alpha}(g - R_n g)\|_p \left(\int_1^x [y^{-\frac{\alpha+1}{2}} e^{\frac{y}{2}}]^q dy\right)^{\frac{1}{q}};$$

here the second term can be estimated by

$$c\left(\int_1^x y^{-\frac{\alpha+1}{2}q} dy\right)^{\frac{1}{q}} \leqq c\left(x^{\frac{1}{q}-\frac{\alpha+1}{2}} + 1\right)$$

for $x < 2$ and by

$$x^{-\frac{\alpha+1}{2}}\left(\int_{x/2}^x [e^{\frac{y}{2}}]^q dy\right)^{1/q} + \left(\int_1^{x/2} e^{\frac{y}{2}q} dy\right)^{1/q} \leqq cx^{-\frac{\alpha+1}{2}} e^{x/2} + ce^{x/4} \leqq cx^{-\frac{\alpha+1}{2}} e^{x/2}$$

for $x > 2$. It follows from Lemma 1 and from the Proposition that

$$\int_1^x y^{-\frac{1}{2}} R_n g(y)dy \to \int_1^x y^{-\frac{1}{2}} g(y)dy \qquad (x > 0),$$

and that $\sqrt{u_\alpha} R_n f$ converges to $\sqrt{u_\alpha} f$ in $L^p(0, \infty)$. Taking a subsequence n_k we can suppose that $R_{n_k} f(x) \to f(x)$ a.e. By (43) the series $R_{n_k} f(1)$ converges to a constant C_0; taking the limit $k \to \infty$ (43) becomes

$$\int_1^x y^{-\frac{1}{2}} g(y)dy = f(x) - C_0$$

which proves (42). Conversely suppose (42) and prove (41). Let $g = f' y^{1/2} \in$ $\in L^p(\sqrt{u_\alpha})$. Then

$$g(y) \sim y^{\frac{1}{2}} \sum_{k=0} b_k \ell_k^{(\alpha+1)}$$

where, by (3)

$$b_k = \int_0^\infty g(y) y^{\frac{1}{2}} \ell_k^{(\alpha+1)}(y) u_\alpha(y) dy = \int_0^\infty f' u_{\alpha+1} \ell_k^{(\alpha+1)} =$$

$$= \lim_{x\to\infty} f(x)u_{\alpha+1}(x)\ell_k^{(\alpha+1)}(x) - \lim_{x\to 0} f(x)u_{\alpha+1}(x)\ell_k^{(\alpha+1)}(x) + \sqrt{k+1}a_{k+1}.$$

We shall show that $f'y^{1/2} \in L^p(\sqrt{u_\alpha})$ implies

$$\text{(44)} \qquad \lim_{x\to 0} f(x)u_{\alpha+1}(x) = \lim_{x\to\infty} f(x)u_{\alpha+1}(x)x^k = 0 \quad (k = 0, 1, \ldots).$$

Indeed, we can suppose $f(1) = 0$ and then

$$f(x) = \int_1^x gy^{-\frac{1}{2}}$$

hence

$$|f(x)u_{\alpha+1}(x)| \leqq c\|g\sqrt{u_\alpha}\|_p u_{\alpha+1}(x)x^{\frac{1}{q}-\frac{\alpha+1}{2}} \leqq cx^{\frac{1}{q}+\frac{\alpha+1}{2}} \to 0 \qquad (x \to 0)$$

for $x < 2$ and

$$x^k|f(x)|u_{\alpha+1}(x) \leqq c\|g\sqrt{u_\alpha}\|_p u_{\alpha+1}(x)x^{k-\frac{\alpha+1}{2}}e^{x/2} \leqq$$

$$\leqq ce^{-x/2}x^{k+\frac{\alpha+1}{2}} \to 0 \qquad (x \to \infty)$$

for $x > 2$. The statement (44) being proved we obtain that

$$g(y) \sim y^{\frac{1}{2}} \sum_{k=1}^{\infty} a_k\sqrt{k}\ell_{k-1}^{(\alpha+1)}(y)$$

and this implies (41). Theorem 5 is proved.

THEOREM 6. *Let* $\alpha \geqq 0$, $1 < p < \infty$ *and* $f \in L^p(\sqrt{u_\alpha})$. *Then*

a) $\|\sqrt{u_\alpha}[\tilde{f} - \tilde{f}(x,\cdot)]\|_p = o(x)$ $(x \to 0+) \Leftrightarrow f = c$.

b) $\|\sqrt{u_\alpha}[\tilde{f} - \tilde{f}(x,\cdot)]\|_p = O(x) \Leftrightarrow f$ *is locally absolutely continuous and* $f'y^{1/2} \in L^p(\sqrt{u_\alpha})$.

PROOF. Consider the operators

$$T_x \colon L^p(\sqrt{u_\alpha}) \to L^p(\sqrt{u_\alpha}),$$

$$T_x f(y) := \int_0^\infty M(x,y,z)f(z)\sqrt{yz}u_\alpha(z)dz \qquad (x > 0),$$

$$T_0 f := f, \quad M(x,y,z) := \int_0^1 U(x,r)K(r,y,z)dr$$

where $K(r,y,z)$ is the Abel–Poisson kernel corresponding to the weight $u_{\alpha+1}$:

$$K(r,y,z)=\sum_{n=0}^{\infty} r^n \ell_n^{(\alpha+1)}(y)\ell_n^{(\alpha+1)}(z).$$

Now

$$\int_0^\infty T_x f(y)\sqrt{y}\ell_k^{(\alpha+1)}(y)u_\alpha(y)dy=$$

$$=\int_0^\infty\int_0^\infty M(x,y,z)f(z)\sqrt{z}u_\alpha(z)dz\ell_k^{(\alpha+1)}(y)u_{\alpha+1}(y)dy=$$

$$=\int_0^\infty f(z)\sqrt{z}u_\alpha(z)\int_0^\infty M(x,y,z)\ell_k^{(\alpha+1)}(y)u_{\alpha+1}(y)dydz=$$

$$=\int_0^\infty f(z)\sqrt{z}u_\alpha(z)e^{-\sqrt{k}x}\ell_k^{(\alpha+1)}(z)dz$$

which shows that

$$T_x\tilde{f}(y)=\tilde{f}(x,y),\qquad x\geqq 0. \tag{45}$$

The semigroup property for the system $\{T_x : x\geqq 0\}$ can be proved the same way as (40). Now Theorem 1 states the continuity of this semigroup (with $\alpha+1$ instead of α and $y^{-1/2}f(y)$ instead of $f(y)$).

Denote by A the infinitesimal generator of this semigroup; then its saturation class is $D(A)$ and the saturation order is $O(x)$. This implies that a) and b) will follow if we show that

(46) $\tilde{f}\in D(A)\Leftrightarrow f$ is locally absolutely continuous and $f'y^{\frac{1}{2}}\in L^p(\sqrt{u_\alpha})$.

Taking Theorem 5 into account, we have to prove that

$$\tilde{f}\in D(A)\Leftrightarrow \exists g\in L^p(\sqrt{u_\alpha}),\quad g(y)\sim y^{\frac{1}{2}}\sum\sqrt{k}a_k\ell_{k-1}^{(\alpha+1)}(y). \tag{47}$$

Let first $\tilde{f}\in D(A)$. This means that the $L^p(\sqrt{u_\alpha})$-limit

$$A\tilde{f}=\lim_{x\to 0+}\frac{T_x\tilde{f}-\tilde{f}}{x}$$

exists. Now it follows from (45) and (32) that

$$\int_0^\infty A\tilde{f}(y)y^{\frac{1}{2}}\ell_{k-1}^{(\alpha+1)}(y)u_\alpha(y)dy=$$

$$= \lim_{x\to 0+} \int_0^\infty \frac{T_x\tilde f(y) - \tilde f(y)}{x} y^{\frac12} \ell_{k-1}^{(\alpha+1)}(y) u_\alpha(y)dy = \lim_{x\to 0+} \frac{e^{-\sqrt{k}x}-1}{x} a_k = -\sqrt{k} a_k$$

hence

$$A\tilde f(y) \sim -y^{\frac12} \sum \sqrt{k} a_k \ell_{k-1}^{(\alpha+1)}(y)$$

which proves the "only if" part of (47). To see the "if" part, take $g \in \in L^p(\sqrt{u_\alpha})$ with the expansion

$$g(y) \sim -y^{\frac12} \sum \sqrt{k} a_k \ell_{k-1}^{(\alpha+1)}(y).$$

We can check from the definition of A that

$$A(R_n\tilde f) = R_n g.$$

It follows from Lemma 1 that

$$\|\sqrt{u_\alpha}(R_n\tilde f - \tilde f)\|_p \to 0, \quad \|\sqrt{u_\alpha}(R_n g - g)\|_p \to 0 \quad (n \to \infty).$$

Since the operator A is closed, we get that $\tilde f \in D(A)$ and $A\tilde f = g$. Theorem 6 is proved.

REMARK 1. Theorems 3 and 5 hold also for $p = \infty$. We give briefly the needed modifications in the proofs. In proving (33)⇒(34) we showed that (33) implies the existence of a function $g \in L^\infty(\sqrt{u_\alpha})$ having the expansion

$$g \sim \sum \sqrt{k} a_k \ell_k^{(\alpha)}.$$

In particular the 0-th coefficient vanishes, i.e.

$$\int_0^\infty g u_\alpha = 0.$$

Now

$$(48) \qquad \left| \int_x^\infty u_\alpha g \right| \leqq \begin{cases} \|\sqrt{u_\alpha} g\|_\infty \int_x^\infty \sqrt{u_\alpha} \leqq c\sqrt{u_\alpha}(x) & \text{if} \quad x > 1 \\ \|\sqrt{u_\alpha} g\|_\infty \int_0^x \sqrt{u_\alpha} \leqq c x^{\frac{\alpha}{2}+1} & \text{if} \quad x < 1 \end{cases}$$

which implies that

$$\frac{1}{u_{\alpha+\frac12}(x)} \int_x^\infty u_\alpha g \in L^\infty(\sqrt{u_\alpha}).$$

Compute the coefficients of this function by the aid of (48):

$$\int_0^\infty \frac{1}{u_{\alpha+\frac{1}{2}}(x)}\left(\int_x^\infty u_\alpha g\right)\sqrt{x}\,\ell_{k-1}^{(\alpha+1)}(x)u_\alpha(x)dx = \int_0^\infty\left(\int_x^\infty u_\alpha g\right)\ell_{k-1}^{(\alpha+1)}(x)dx =$$

$$= \left[\left(\int_x^\infty u_\alpha g\right)\frac{\ell_k^{(\alpha)}(x)}{\sqrt{k}}\right]_{x=0}^\infty + \frac{1}{\sqrt{k}}\int_0^\infty u_\alpha(x)g(x)\ell_k^{(\alpha)}(x)dx = a_k.$$

So we have

$$\frac{1}{u_{\alpha+\frac{1}{2}}(x)}\int_x^\infty u_\alpha g = \tilde{f}(x) \in L^\infty(\sqrt{u_\alpha})$$

and $\lim_{x\to 0} \tilde{f}(x)u_{\alpha+\frac{1}{2}}(x) = 0$ follows again from (48). Analogously, in proving (41)⇒(42) we have a function $g \in L^\infty(\sqrt{u_\alpha})$ with

$$g(y) \sim y^{\frac{1}{2}}\sum \sqrt{k}a_k\ell_{k-1}^{(\alpha+1)}(y).$$

Now

(49)
$$\left|\int_1^x y^{-\frac{1}{2}}g(y)dy\right| \leqq \|\sqrt{u_\alpha}g\|_\infty \int_1^x y^{-\frac{\alpha+1}{2}}e^{\frac{y}{2}}dy \leqq \begin{cases} c\left(1+x^{1-\frac{\alpha+1}{2}}\right) & \text{if } x<2 \\ cx^{-\frac{\alpha+1}{2}}e^{x/2} & \text{if } x>2 \end{cases}$$

implies that

$$\int_1^x y^{-\frac{1}{2}}g(y)dy \in L^\infty(\sqrt{u_\alpha})$$

and the coefficients are, by (49)

$$\int_0^\infty\left(\int_1^x y^{-\frac{1}{2}}g(y)dy\right)\ell_k^{(\alpha)}(x)u_\alpha(x)dx =$$

$$= -\frac{1}{\sqrt{k}}\left[\left(\int_1^x y^{-\frac{1}{2}}g(y)dy\right)\ell_{k-1}^{(\alpha+1)}(x)u_{\alpha+1}(x)\right]_{x=0}^\infty +$$

$$+\frac{1}{\sqrt{k}}\int_0^\infty x^{-\frac{1}{2}}g(x)\ell_{k-1}^{(\alpha+1)}(x)u_{\alpha+1}(x)dx = a_k \qquad (k \geqq 1)$$

and then Proposition a) implies $f(x) = \int_1^x y^{-\frac{1}{2}} g(y)dy + c$. The proof of the converse implications remains the same.

REMARK 2. During the preparation of this paper we raised the following problem. Do there exist orthogonal systems, different from the classical ones, for which an Alexits type theorem holds? Recently I. Joó answered this in the positive sense proving an Alexits theorem for the Walsh system; see [18].

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EÖTVÖS LORÁND UNIVERSITY
CHAIR FOR ANALYSIS
MÚZEUM KRT. 6–8
H–1088 BUDAPEST, HUNGARY

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ORTHONORMAL SYSTEMS ON VILENKIN GROUPS

Gy. GÁT (Nyíregyháza)

1. Introduction. Let $m := (m_k, k \in \mathbf{N} := \{0, 1, \dots\})$ be a sequence such that $N \ni m_k \geqq 2$ $(k \in \mathbf{N})$. Denote by G_m the direct product of discrete cyclic groups $Z_{m_k} := \{0, 1, \dots, m_k - 1\}$ $(k \in \mathbf{N})$. Thus G_m is a compact Abelian group. The direct product μ of the measures $\mu_k(\{j\}) := 1/m_k$ $(j \in Z_{m_k}, k \in \mathbf{N})$ is a Haar measure on G_m, $\mu(G_m) = 1$. If $M_0 := 1$, $M_{k+1} := m_k M_k$ $(k \in \mathbf{N})$, then every $n \in \mathbf{N}$ can be uniquely expressed as $n = \sum_{i=0}^{\infty} n_i M_i$ $(n_k \in Z_{m_k},\ k \in \mathbf{N})$. Denote $r_k(x) := \exp(2\pi i x_k / m_k)$ $(x = (x_0, x_1, \dots) \in G_m,\ k \in \mathbf{N})$ and $\psi_n := \prod_{k=0}^{\infty} r_k^{n_k}$ $(n \in \mathbf{N})$. If $x, y \in G_m$, $n, s \in \mathbf{N}$ and

$$n \oplus s := \sum_{k=0}^{\infty} ((n_k + s_k) \mod m_k) M_k,$$

then $\psi_n(x + y) = \psi_n(x)\psi_n(y)$, $\overline{\psi}_n = 1/\psi_n$ and $\psi_{n \oplus s} = \psi_n \psi_s$. It is known that the system $(\psi_n,\ n \in \mathbf{N})$ is the character system of G_m and also that it is orthonormal and complete. Let $x \in G_m$ and denote

$$I_n(x) := \{y \in G_m : y_0 = x_0, \dots, y_{n-1} = x_{n-1}\},\ I_n := I_n(0)\ (I_0(x) := G_m).$$

Denote by $\mathcal{A}_n$ the σ-algebra generated by the system $\{I_n(z) : z \in G_m\}$ and by E_n the conditional expectation operator with respect to $\mathcal{A}_n$. Suppose that there are given functions α_j^k $(j, k \in \mathbf{N})$ on G_m such that α_j^k is $\mathcal{A}_j$-measurable and $|\alpha_j^k| = 1$, $\alpha_0^k = \alpha_j^0 = \alpha_j^k(0) = 1$ $(j, k \in \mathbf{N})$. If $j, n \in \mathbf{N}$, then let $j(n) := \sum_{i=j}^{\infty} n_i M_i$, $\alpha_n := \prod_{j=0}^{\infty} \alpha_j^{j(n)}$ and $\chi_n := \psi_n \alpha_n$.

This paper deals with the system $(\chi_n : n \in \mathbf{N})$. It is obvious that $\chi_n(x + y) \neq \chi_n(x)\chi_n(y)$ $(x, y \in G_m,\ n \in \mathbf{N})$ in general, i.e. χ_n is not a character of G_m and similarly $\chi_{n \oplus m} \neq \chi_n \chi_m$ $(n, m \in \mathbf{N})$. The systems $(\chi_n : n \in \mathbf{N})$ and $(\psi_n : n \in \mathbf{N}) = \hat{G}_m$ differ. A good property of $(\chi_n : n \in \mathbf{N})$ which enables us to use the techniques known in Vilenkin system theory is that if $y \in I_k$, $n < M_{k+1}$, then

$$\chi_n(x + y) = \chi_n(x)\chi_n(y) \qquad (x \in G_m,\ n \in \mathbf{N}).$$

2. Results on $(\chi_n : n \in \mathbf{N})$. THEOREM 1. *The system $(\chi_n : n \in \mathbf{N})$ is orthonormal and complete in $L(G_m)$.*

Let $n, s \in \mathbf{N}$ and

$$K_{n,s}(x,y) := \sum_{k=0}^{n-1} \chi_{k+s}(x)\overline{\chi}_{k+s}(y) \quad (x, y \in G_m).$$

We need the following lemma very often. This lemma is the base of several results.

LEMMA 2.

$$K_{M_t,pM_t}(x+y,y) = \begin{cases} 0 & (x \notin I_t) \\ M_t\alpha_{pM_t}(x+y)\overline{\alpha}_{pM_t}(y) & (x \in I_t) \end{cases} \quad (p, t \in \mathbf{N}).$$

Denote by $D_n(x,y) := K_{n,0}(x,y)$ $(x, y \in G_m,\ n \in \mathbf{N})$ the Dirichlet kernels. The following corollary is one of the basic and most often used results in the theory of generalized Vilenkin systems.

COROLLARY 3.

$$D_{M_t}(x,y) = \begin{cases} 0 & (x - y \notin I_t) \\ M_t & (x - y \in I_t) \end{cases} \quad (t \in \mathbf{N}).$$

The following proposition is the third basic result which is used all over the rest of this paper.

PROPOSITION 4. *If $n \geqq M_k$ $(n, k \in \mathbf{N})$, $y \in G_m$, then*

$$\int_{I_k} \chi_n(x+y)\,d\mu(x) = 0.$$

Let $f \in L^p(G_m)$ $(1 \leqq p \leqq \infty)$. Denote

$$\omega_n^{(p)}(f) := \sup_{h \in I_n} \|f(\cdot + h) - f(\cdot)\|_p \quad (n \in \mathbf{N})$$

the L^p modulus of continuity of f on $L^p(G_m)$, and let

$$\hat{f}(n) := \int_{G_m} f\overline{\chi}_n, \quad S_nf := \sum_{k=0}^{n} \hat{f}(k)\chi_k \quad (n \in \mathbf{N},\ f \in L(G_m)).$$

THEOREM 5. *If $n \geqq M_k$ $(n, k \in \mathbf{N})$, $f \in L(G_m)$, then*

$$|\hat{f}(n)| \leqq \omega_k^{(1)}(f).$$

THEOREM 6. *If $f \in L^2(G_m)$, $k \in \mathbf{N}$, then*

$$\left(\sum_{n=M_k}^{\infty} |\hat{f}(n)|^2\right)^{\frac{1}{2}} \leqq \frac{1}{\sqrt{2}}\omega_k^{(2)}(f).$$

The following theorem gives an upper bound for the Lebesgue constant L_n.

THEOREM 7. *We have*

$$L_n := \sup_{y\in G_m} \int_{G_m} |D_n(x,y)|d\mu(x) \leqq \sum_{i=0}^{\infty} n_i \quad (n \in \mathbf{N}).$$

Let

$$E_n^{(p)}(f) := \inf_{\{a_k\}} \left\| f - \sum_{k=0}^{n-1} a_k\chi_k \right\|_p \quad (1 \leqq p \leqq \infty,\ a_k \in C,\ k,n \in \mathbf{N},\ f \in L^p(G_m)).$$

The following theorem is a generalization of the well-known Efimov's theorem on the best approximating Vilenkin polynomial.

THEOREM 8. *We have*

$$E_{M_n}^{(p)}(f) \leqq \omega_n^{(p)}(f) \leqq 2E_{M_n}^{(p)}(f) \ \ (1 \leqq p \leqq \infty,\ n \in \mathbf{N},\ f \in L^p(G_m)).$$

Next we give a generalization of Zantlesov's convergence theorem with respect to the generalized system. Corollaries 10 and 11 show that a certain convergence condition on the L^2 and L^∞ moduli of continuity, resp., imply the absolute convergence of S_nf with respect to every system discussed in this paper, not only to the original Vilenkin system.

THEOREM 9. *Let $f \in L^p(G_m)$, $1 < p \leqq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 \leqq \beta \leqq q$, $-1 < \gamma < 0$. Put $\Theta = 0$ if $\beta \neq 1$ and $\Theta = 1$ if $\beta = 1$. If*

$$Q := \sum_{k=0}^{\infty} M^{\gamma+1-\frac{\beta}{q}} m_k^{\beta} (\ln m_k)^{\Theta} (\omega_k^{(p)}(f))^{\beta} < \infty,$$

then

$$\sum_{k=1}^{\infty} |\hat{f}(k)|^{\beta} k^{\gamma} < C_{p,\beta} Q.$$

COROLLARY 10. *If $f \in L^2(G_m)$ and $\sum_{k=0}^{\infty} M_k^{\frac{1}{2}} m_k \ln m_k \omega_k^{(2)}(f) < \infty$, then S_nf absolutely converges.*

COROLLARY 11. *If $f \in C(G_m)$ and $\sum_{k=0}^{\infty} M_k^{\frac{1}{2}} m_k \ln m_k \omega_k^{(\infty)}(f) < \infty$, then S_nf absolutely converges as $n \to \infty$*

$$(\omega_k^{(\infty)}(f) := \sup_{h\in I_k} \sup_{x\in G_m} |f(x+h) - f(x)| \ \ (k \in \mathbf{N},\ f \in C(G_m))).$$

LEMMA 12. *Let $y \in G_m$, $0 < j \in \mathbf{N}$, $n \in \mathbf{N}$ and $x \notin I_j(y)$ be fixed. Then $D_n(x,t)\overline{\chi}_n(x)\chi_n(t)$ is constant as t ranges over $I_j(y)$.*

This lemma is needed in the proof of the following theorem.

THEOREM 13. *Let $f \in L^p(G_m)$, $1 < p < \infty$, $1 \leqq n \in \mathbf{N}$ and* $\sup m < \infty$. *Then there exists a constant A_p depending only on p such that $\|S_n f\|_p \leqq \leqq A_p \|f\|_p$. Moreover $\|S_n f\|_p = \|f\|_p O(p)$ $(n \to \infty)$.*

3. Proofs. Theorem 1 can be proved in the following way. If $n = \sum_{i=0}^{\infty} n_i M_i < S = \sum_{i=0}^{\infty} S_i M_i (\in \mathbf{N})$ and $k := \max\{j \in \mathbf{N} : n_j \neq S_j\}$, then $\chi_x \overline{\chi}_n = \Phi r_k^{S_k} \overline{r}_k^{n_k}$ where Φ is $\mathcal{A}_k$-measurable. Hence

$$\int_{G_m} \chi_S \overline{\chi}_n d\mu = E_0(\chi_S \overline{\chi}_n) = E_0(E_k(\chi_S \overline{\chi}_n)) = E_0(\Phi E_k(r_k^{S_k} \overline{r}_k^{n_k})) = 0,$$

because $n_k \neq S_k$. The completeness of the system $(\chi_n : n \in \mathbf{N})$ can be proved by Corollary 3 and the method used in the case of $\alpha_j^k = 1$ $(j, k \in \mathbf{N})$, [3].

The proof of Lemma 2 in the case of $x \in I_t$ is trivial. If $x \notin I_t$, then $x \in I_\ell \setminus I_{\ell+1}$ for some $\ell = 0, 1, \dots, t-1$. Thus

$$K_{M_t, pM_t}(x+y, y) = \Phi_{p,t}(x,y) \sum_{\ell=0}^{m_\ell - 1} r_\ell^j(x) = 0.$$

Corollary 3 is a simple consequence of Lemma 2.

PROOF OF PROPOSITION 4. Let $S := \max\{j \in \mathbf{N} : n_j \neq 0\}$.

$$E_k(\chi_n(\cdot + y)) = E_k(E_S(\chi_n(\cdot + y))) = E_k(\Phi E_S(r_S(\cdot + y))) = 0,$$

where Φ is $\mathcal{A}_S$-measurable. Theorems 5, 6, 7, 8 can be proved by similar techniques usual in the theory in the case of $\alpha_j^k = 1$ $(j, k \in \mathbf{N})$, [1] and by means of Theorem 1, Lemma 2, Corollary 3 and Proposition 4.

PROOF OF THEOREM 9. Let $k \in \mathbf{N}$, $y \in I_k$ and $F(x) := f(x+y) - f(x)$ $(x \in G_m)$. Thus if $n = M_k, M_k + 1, \dots, M_{k+1} - 1$, then

$$\hat{F}(n) = \int_{G_m} f(x) \overline{\chi}_n(x-y) d\mu(x) - \hat{f}(n) = \psi_n(y) \int_{G_m} f(x) \overline{\psi}_n(x) \alpha_n(x-y) d\mu(x) - \hat{f}(n),$$

$$\alpha_n(x-y) = \prod_{j=0}^{k} \alpha_j^{j(n)}(x-y) = \prod_{j=0}^{k} \alpha_j^{j(n)}(x) = \alpha_n(x).$$

This implies that $\hat{F}(n) = (\psi_n(y) - 1)\hat{f}(n)$. The Hausdorff–Young inequality gives

$$\left(\sum_{n=jM_k}^{(1+j)M_k - 1} (|\hat{f}(n)|\,|\psi_n(y) - 1|)^q \right)^{\frac{1}{q}} = \left(\sum_{n=jM_k}^{(1+j)M_k - 1} (|\hat{F}(n)|^q) \right)^{\frac{1}{q}} \leqq$$

$$\leqq \left(\sum_{n=0}^{\infty}|\hat{F}(n)|^q\right)^{\frac{1}{q}} \leqq \|F\|_p \leqq \omega_k^{(p)}(f) \quad (j=1,\dots,m_k-1).$$

The rest of the proof is as Zantlesov's proof in the case of $\alpha_j^k = 1$ $(j,k \in \mathbf{N})$, [6].

Corollaries 10, 11 follow from Theorem 9.

PROOF OF LEMMA 12. We have

$$K_{n,qM_\ell}(y+x,x) = K_{n-n_jM_j,n_jM_j+qM_\ell}(y+x,x)$$
$$(y \notin I_j,\ x \in G_m,\ n,q \in \mathbf{N},\ M_j \leqq n < M_{j+1},\ j < \ell).$$

$x - t \notin I_j$ and

$$D_n(x,t)\overline{\chi}_n(x)\chi_n(t) = K_{n,0}(x,t)\overline{\chi}_n(x)\chi_n(t) =$$
$$= K_{n-n_jM_j,n_jM_j}(x,t)\overline{\chi}_n(x)\chi_n(t).$$

This completes the proof of Lemma 12.

Theorem 13 can be proved by the method of Gosselin [3] used in the case of $\alpha_j^k = 1$ $(j,k \in \mathbf{N})$. The main difference between the proofs is that (21) of [3] is proved by Lemma 12.

4. Application. An arithmetical function g is called even mod k if $g((n,k)) = g(n)$ for each $n \in p := \mathbf{N} \setminus \{0\}$. The set of these functions is denoted by $\mathcal{B}_k$. $\mathcal{B} := \bigcup_{k \in P} \mathcal{B}_k$ is the set of even arithmetical functions. The limit $M(g) := \lim n^{-1} \sum_{j \leqq n} g(j)$, if it exists, is called the mean value of g. The upper limit $\overline{M}(g) := \overline{\lim}\, n^{-1} \sum_{j \leqq n} g(j)$ gives rise to a semi-norm

$$\|g\|_p := (\overline{M}(|g|^p))^{\frac{1}{p}} \quad (1 \leqq p < \infty).$$

The closure of $\mathcal{B}$ with respect to $\|\cdot\|_p$ is the set of $\mathcal{B}^p$-almost-even arithmetical functions [4]. The Ramanujan function C_r is defined by

$$C_r(n) := \sum_{\substack{a=1 \\ (a,r)=1}}^{r} \exp(2\pi i a n/r).$$

It is known that if $g \in \mathcal{B}^2$ ($g \in \mathcal{B}^1$ bounded) and $M(g\overline{c}_r) = 0$ for each $r \in P$, then $\|g\|_2 = 0$ ($\|g\|_1 = 0$), [5]. The techniques of this paper enable us to prove that if $g \in \mathcal{B}^p$ $(1 \leqq p < \infty)$ and $M(g\overline{C}_r) = 0$ for every $r \in P$, then $\|g\|_p = 0$, [2].

If $g \in \mathcal{B}^p$ $(1 \leqq p < \infty)$ and $\hat{g}(r) := \varphi^{-1}(r)M(g\overline{C}_r)$ $(r \in \varphi$, φ is the Euler function), then $L_S g := \sum_{j|S!} \hat{g}(r)C_r$ $(S \in P)$ $\|\cdot\|_p$ converges to g, [2].

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BESSENYEI GYÖRGY COLLEGE
DEPARTMENT OF MATHEMATICS
NYÍREGYHÁZA, P.O.BOX 166, H–4401
HUNGARY

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ON THE DISTRIBUTION OF THE SET $\{\sum_{i=1}^{n} \varepsilon_i q^i : \varepsilon_i \in \{0,1\}, n \in \mathbf{N}\}$

I. JOÓ (Budapest)

Let $1 < q < \sqrt{2}$ be arbitrary fixed and

$$H := \Big\{ \sum_{i=1}^{n} \varepsilon_i q^{2(n-i)} : \varepsilon_i \in \{0,1\}, n = 1,2,\dots \Big\} =$$
$$= \{y_n(q)\} = \{y_n\} \nearrow \infty \ (n \to \infty).$$

We shall prove the following

THEOREM. *If $y_{n+1} - y_n \to 0$ $(n \to \infty)$ then there exists an expansion $1 = \sum_{i=1}^{\infty} q^{-n_i}$ such that $\sup_i (n_{i+1} - n_i) = \infty$.*

In [1] it is proved that if q is a Pisot number, then there is no such expansion of 1, further it is well known that the smallest non-zero Pisot number is between 1 and $\sqrt{2}$. Hence we obtain

COROLLARY. *For any Pisot number $1 < q < \sqrt{2}$, $y_{n+1} - y_n \nrightarrow 0$, $n \to \infty$.*

For the proof of the Theorem we need the following

LEMMA. *Let $1 < q < \sqrt{2}$ be any fixed number and let $N \in \mathbf{N}$ be arbitrary. Then there exists an expansion $1 = \sum q^{-n_i}$ such that $\sup(n_{i+1} - n_i) > 2N$, whenever $y_{n+1} - y_n \to 0$ as $n \to \infty$.*

PROOF. Let $0 < x < 1$ and expand the numbers x and $(1-x)/q$ by the system (q^{-2n}). If we have N consecutive zeros at the same places in these expansions, then adding these expansions we get a desired expansion of 1. Let $x = \sum_{i=1}^{n} \varepsilon_i q^{-2i}$. We have to find such values ε_i', for which $\sum_{i=1}^{n} \varepsilon_i' q^{-2i} < \frac{1-x}{q} < \sum_{i=1}^{n} \varepsilon_i' q^{-2i} + q^{-2(n+N)}$, i.e.

$$(1) \qquad 0 < q^{2n} - \sum_{i=1}^{n} \varepsilon_i q^{2(n-i)} - q \sum_{i=1}^{n} \varepsilon_i' q^{2(n-i)} < q^{-2N}.$$

Let $\varepsilon := 10^{-1} \cdot q^{-2N}$. We show first that for every n there exist $\varepsilon_1, \dots, \varepsilon_n \in \{0,1\}$ such that

$$(A) \qquad 1 < q^{2n} - \sum_{i=1}^{n} \varepsilon_i q^{2(n-i)} < (q^2-1)^{-1} + 1 = \frac{q^2}{q^2-1}.$$

Indeed, expand q^{2n} by the system $q^{2(n-i)}, q^{2(n-2)}, \dots$. If we "cut" such an expansion at non-negative exponents, then the error is smaller than $q^{-2}+$ $+q^{-4}+\dots=(q^2-1)^{-1}$ i.e.

$$0 \le q^{2n} - \sum_{i=1}^{n} \hat{\varepsilon}_i q^{2(n-i)} < (q^2-1)^{-1}.$$

If the difference is larger than 1 then we are ready, if not, then consider the largest i with $\hat{\varepsilon}_i = 1$, and replace the corresponding term $q^{2(n-i)}$ by the non-negative part of its expansion in terms of smaller exponents. Then the error resulting from the modification is $< (q^2-1)^{-1}$, hence the total error is $< 1+(q^2-1)^{-1}$. If this error is > 1 then we are ready, if not, then continue this process (replace the smallest exponent by the non-negative part of its expansion in terms of the smaller exponents).

If there is no such a step when the error is > 1, then $\varepsilon_n = 1$ and we omit $\varepsilon_n q^0$ and arrive to an expansion with an error between 1 and 2. Statement (A) is proved. Multiplying (A) by q^{2k} we get: for every k and $n \ge k$ there exist $\varepsilon_1, \dots, \varepsilon_n \in \{0,1\}$ such that

$$\text{(B)} \qquad q^{2k} < q^{2n} - \sum_{i=1}^{n} \varepsilon_i q^{2(n-i)} < \frac{q^{2(k+1)}}{q^2-1}.$$

Choose $k = k(\varepsilon)$ so that $y_n > q^{2k-1} - 1$ implies $y_{n+1} - y_n < \varepsilon$.

Taking (B) into account there exist $\varepsilon_1, \dots, \varepsilon_n$ such that

$$q^{-1}\left(q^{2n} - \sum_{i=1}^{n} \varepsilon_i q^{2(n-i)}\right) \in \left(q^{2k-1}, \frac{q^{2k+1}}{q^2-1}\right)$$

and then for $n > n(\varepsilon, q)$ there exist $\varepsilon_1', \dots, \varepsilon_n'$ such that

$$\sum_{i=1}^{n} \varepsilon_i' q^{2(n-i)} + \varepsilon < \frac{q^{2n} - \sum_{i=1}^{n} q^{2(n-i)}}{q} < \sum_{i=1}^{n} \varepsilon_i' q^{2(n-i)} + 2\varepsilon$$

which means that (1) is fulfilled. The Lemma is proved. □

PROOF OF THE THEOREM. Let $k = k(q)$ be such that

$$\text{(2)} \qquad (1+q)(q^{-2k} + q^{-(2k+2)} + \dots) < 1 + q^{-2} + q^{-4} + \dots .$$

We use induction. Suppose there exists a segment I_N such that for $x \in I_N$ the numbers x, $(1-x)/q$ have the expansions

$$x = \sum_{j=1}^{s} \varepsilon_j q^{-2j} + \dots, \quad (1-x)/q = \sum_{j=1}^{s} \varepsilon_j' q^{-2j} + \dots$$

where there are $1, 2, \dots, N-1, N+k$ consecutive 0's at the same places and $N+k$ zeros at the end. Suppose I_N is the maximal segment with this property for fixed (ε_j) and (ε'_j). Let $\tilde{I}_N$ be the maximal segment, where the last k zero coefficients are omitted. We extend the sequences (ε_j), (ε'_j) to an index n such that

$$(3) \qquad 0 < q^{2n} - \sum_{i=1}^{n} \varepsilon_i q^{2(n-i)} - q \cdot \sum_{i=1}^{n} \varepsilon'_i q^{2(n-i)} < q^{-2(N+k+1)}$$

be fulfilled. Let

$$Q := q^{2n} - \sum_{j=1}^{s-k} \varepsilon_j q^{2(n-j)} - q \cdot \sum_{j=1}^{s-k} \varepsilon'_j q^{2(n-j)}.$$

For any $x \in I_N$ we have

$$\sum_{j=1}^{s-k} \varepsilon_j q^{-2j} < x < \sum_{j=1}^{s-k} \varepsilon_j q^{-2j} + q^{-2(s+1)} + q^{-2(s+2)} + \dots,$$

$$\sum_{j=1}^{s-k} \varepsilon'_j q^{-2j} < \frac{1-x}{q} < \sum_{j=1}^{s-k} \varepsilon_j q^{-2j} + q^{-2(s+1)} + q^{-2(s+2)} + \dots.$$

Multiplying these inequalities by q^{2n} resp. q^{2n+1} and adding them we obtain

$$0 < q^{2n} - \sum_{j=1}^{s-k} \varepsilon_j q^{2(n-j)} - q \cdot \sum_{j=1}^{s-k} \varepsilon'_j q^{2(n-j)} = Q <$$

$$< (q+1)(q^{2(n-s-1)} + q^{2(n-s-2)} + \dots) < q^{2(n-s+k-1)} + q^{2(n-s+k-2)} + \dots.$$

(We have used (2).) This means that we can expand Q by the system $q^{2(n-s+k-1)}$, $q^{2(n-s+k-2)}, \dots$, hence by the idea used in the proof of (A), expanding Q instead of q^{2n} we get: for every $n > s$ there exist $\varepsilon_{s-k+1}, \dots, \varepsilon_n$ such that

$$(\text{A'}) \qquad 1 < Q - \sum_{j=s-k+1}^{n} \varepsilon_j q^{2(n-j)} < q^2/(q^2-1).$$

Let $\varepsilon := 10^{-1} q^{-2(N+k+1)}$ and $d_0 = d_0(\varepsilon)$ be such that $y_n \geq d_0$ implies $y_{n+1} - y_n < \varepsilon$. Let $\ell = \ell(q, \varepsilon)$ be such that $d_0 < q^{2\ell-1} - 1$. Multiplying (A') by $q^{2\ell}$ we obtain for another n (for $n + \ell$ in place of n)

$$q^{-1}\left(q^{2n} - \sum_{i=1}^{n} \varepsilon_i q^{2(n-i)} - q \cdot \sum_{i=1}^{s-k} \varepsilon'_i q^{2(n-i)}\right) \in \left(q^{2\ell-1}, \frac{q^{2\ell+1}}{q^2-1}\right).$$

On the other hand, according to the assumptions of the Theorem, for sufficiently large n the points of the set

$$A_{n-s+k}=\left\{\sum_{i=s-k+1}^{n}\varepsilon_i' q^{2(n-i)}\right\}$$

fill the interval $\left(q^{2\ell-1},\frac{q^{2\ell+1}}{q^2-1}\right)$ with an error $<\varepsilon$ (i.e. the distance between these points is $<\varepsilon$). Instead of (3) we can ensure

$$\text{(3')}\quad 10^{-1}q^{-2(N+k+1)}<q^{2n}-\sum_{i=1}^{n}\varepsilon_i q^{2(n-i)}-q\cdot\sum_{i=1}^{n}\varepsilon_i' q^{2(n-i)}<5^{-1}q^{-2(N+k+1)}.$$

Hence we can finish the induction in the following way. If we start from $s=$ $=s_N$, then $s_{N+1}=n+N+k+1$, $\varepsilon_j=\varepsilon_j'=0\ (n<j\le n+N+k+1)$ and I_{N+1}, $\tilde{I}_{N+1}$ are maximal intervals for which the expansions of x and $(1-x)/q$ start with $\sum^{s_{N+1}}\varepsilon_i q^{-2i}$ and $\sum^{s_{N+1}}\varepsilon_i' q^{-2i}$ resp. $\sum^{s_{N+1}-k}\varepsilon_i q^{-2i}$, $\sum^{s_{N+1}-k}\varepsilon_i' q^{-2i}$. Obviously, $\tilde{I}_{N+1}\subset\tilde{I}_N$. The remaining difficulty is the fact that the intervals $\tilde{I}_N$ are open. Consider the following statements:

a) among $\varepsilon_{s-k+1},\dots,\varepsilon_{n+N+k+1}$ there exist 0 and 1 too,
b) the same holds for $\varepsilon_{s-k+1}',\dots,\varepsilon_{n+N+k+1}'$.

If a) and b) hold then both endpoints of the maximal interval move in the direction of the interior of the interval, i.e. $\overline{\tilde{I}_{N+1}}\subset\tilde{I}_N$ and hence $\bigcap_N\tilde{I}_N\neq$ $\neq\emptyset$. The statement b) is trivial. If a) does not hold, we can ensure the occurrence of a new term 1 in the following way: we consider $\tilde{Q}:=Q\cdot$ $\cdot q^{2r}>q^2/(q^2-1)$ in place of Q and expand this number by the system $q^{2(n-s+k-1+r)},q^{2(n-s+k-2+r)},\dots$. In this case we set $s_{N+1}=n+r+N+k+1$ and let $I_{N+1},\tilde{I}_{N+1}$ be the maximal intervals for which the expansion of x and $(1-x)/q$ can be extended with $\sum^{s_{N+1}-k}\varepsilon_i q^{-2i}$ and $\sum^{s_{N+1}-k}\varepsilon_i' q^{-2i}$. □

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EÖTVÖS LORÁND UNIVERSITY
MATHEMATICAL INSTITUTE
MÚZEUM KRT. 6–8
H–1088 BUDAPEST, HUNGARY

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ON L^1-CONVERGENCE OF WALSH–FOURIER SERIES. II

F. MÓRICZ (Szeged)

1. Introduction. We consider the Walsh orthonormal system $\{w_k(x) : k = 0, 1, \dots\}$ defined on the interval $[0, 1)$ in the Paley enumeration (see, e.g. [1, p. 60]). Our goal is to study the L^1-convergence behavior of the Walsh–Fourier series

$$\sum_{k=0}^{\infty} a_k w_k(x), \qquad a_k := \int_0^1 f(x) w_k(x) dx, \tag{1}$$

of an integrable function $f(x)$, in sign $f \in L^1(0,1)$. In this note, integral is meant in the Lebesgue sense.

2. Previous results. We denote by

$$s_n(f, x) := \sum_{k=0}^{n} a_k w_k(x) \qquad (n = 0, 1, \dots)$$

the partial sums of the series (1). Concerning pointwise convergence, in [4] we proved the following.

THEOREM A. *If $f \in L^1(0,1)$ and the condition*

$$\lim_{\lambda \downarrow 1} \limsup_{n\to\infty} \sum_{k=n}^{[\lambda n]} |\Delta^m a_k| = 0 \tag{2}$$

is satisfied for $m = 1$ or 2, then

$$\lim_{n\to\infty} s_n(f, x) = f(x) \quad a.e.$$

and

$$\lim_{n\to\infty} \int_0^1 |s_n(f, x) - f(x)|^r dx = 0 \quad \textit{for} \quad 0 < r < 1/m.$$

Here and in the sequel, we use the notations

$$\Delta^1 a_k := \Delta a_k = a_k - a_{k+1},$$

$$\Delta^2 a_k := \Delta(\Delta a_k) = a_k - 2a_{k+1} + a_{k+2} \ (k = 0, 1, \dots).$$

Furthermore, $[\cdot]$ denotes the integral part.

In order to conclude the convergence of the series (1) in L^1-norm, we need a slightly stronger condition than (2). Namely, in [5] we proved the following.

THEOREM B. *If $f \in L^1(0,1)$ and for some $p > 1$*

$$\lim_{\lambda \downarrow 1} \limsup_{n\to\infty} \sum_{k=n}^{[\lambda n]} k^{p-1} |\Delta a_k|^p = 0, \tag{3}$$

then

$$\lim_{n\to\infty} \int_0^1 |s_n(f,x) - f(x)| dx = 0 \tag{4}$$

if and only if

$$\lim_{n\to\infty} a_n \int_0^1 |D_n(x)| dx = 0. \tag{5}$$

Here

$$D_n(x) := \sum_{k=0}^{n} w_k(x) \qquad (n = 0, 1, \dots)$$

is the *Walsh–Dirichlet kernel.* As is known [2],

$$\int_0^1 |D_n(x)| dx = O(\ln n).$$

Thus, under condition (3),

$$\lim_{n\to\infty} a_n \ln n = 0$$

is a sufficient condition for the L^1-convergence of the series (1).

The *Tauberian condition of Hardy–Karamata kind* expressed in (3) is well-known in the literature. Since the fulfillment of (3) for some $p > 0$ implies its fulfillment for any $\tilde{p}$, $0 < \tilde{p} < p$, we may always assume that $1 < p \leqq 2$ in (3).

3. Main result. If condition (3) is satisfied, then

$$H(\lambda) := \limsup_{n\to\infty} \sum_{k=n}^{[\lambda n]} k^{p-1}|\Delta a_k|^p \tag{6}$$

is finite for some $\lambda > 1$. The converse is not true in general. However, (6) must be finite for all $\lambda > 1$ if it is finite for some $\lambda > 1$. This follows from the inequality $H(\lambda^2) \leqq 2H(\lambda)$, which can easily be proved. In fact, for any $n \geqq 0$ we have

$$[\lambda^2 n] - [\lambda[\lambda n]] \leqq [\lambda + 1]$$

and recall that $f \in L^1(0,1)$ implies

$$\lim_{k\to\infty} a_k = 0. \tag{7}$$

Now we improve Theorem B as follows.

THEOREM 1. *If $f \in L^1(0,1)$ and $H(\lambda)$ defined in condition (6) is finite for some $\lambda > 1$ and $p > 1$, then conditions (4) and (5) are equivalent.*

4. Auxiliary results. In [6] we proved the following *Sidon type inequality.*

LEMMA A. *For every $1 < p \leqq 2$, sequence $\{a_k\}$ of real numbers, and integer $n \geqq 0$, we have*

$$\int_0^1 \left|\sum_{k=0}^n a_k D_k(x)\right| dx \leqq \frac{2p}{p-1}(n+1)^{1/q}\left(\sum_{k=0}^n |a_k|^p\right)^{1/p} \quad \left(\frac{1}{p} + \frac{1}{q} = 1\right). \tag{8}$$

Unfortunately, this inequality is not enough to prove Theorem 1. Therefore, we prove a modified version.

LEMMA 1. *For every $0 < \gamma < 1$, $1 < p \leqq 2$, sequence $\{a_k\}$ of real numbers, and integer $n \geqq 0$, we have*

$$\int_\gamma^1 \left|\sum_{k=0}^n a_k D_k(x)\right| dx \leqq \frac{2p}{p-1}\gamma^{-1/q}\left(\sum_{k=0}^n |a_k|^p\right)^{1/p}. \tag{9}$$

Clearly, (9) is superior to (8) in the case when $\gamma = \gamma_n$ and $(n+1)\gamma_n$ is bounded from below.

PROOF OF LEMMA 1. It follows in great lines that of [6, Lemma 1], with the warning that $n+1$ should stand in place of n there. Taking into account [6, formulas (3.6)–(3.9)] we arrive at

$$I := \int_\gamma^1 \left|\sum_{k=0}^n a_k D_k(x)\right| dx \leqq$$

$$\leqq \left(\sum_{k=0}^{n}|a_k|^p\right)^{1/p}\sum_{j=0}^{m}2^j\left(\int_{\gamma}^{2^{-j}}|r_j(x)h(x)|^p dx\right)^{1/p},$$

where m is defined by the condition $2^m \leqq n+1 < 2^{m+1}$, and

$$h(x) := \operatorname{sign}\sum_{k=0}^{n} a_k D_k(x).$$

Now assume $2^{-j_o-1} \leqq \gamma < 2^{-j_0}$ with some $j_0 \geqq 0$. Then

$$\left(\int_{\gamma}^{2^{-j}}|r_j(x)h(x)|^p dx\right)^{1/p} = \begin{cases}(2^{-j}-\gamma)^{1/p} \leqq 2^{-j/p} & \text{if } 0 \leqq j \leqq j_0,\\ 0 & \text{if } j > j_0.\end{cases}$$

Consequently,

$$I \leqq \left(\sum_{k=0}^{n}|a_k|^p\right)^{1/p}\sum_{j=0}^{j_0}2^{j/q},$$

whence (9) follows through a simple computation. In fact, observing that the auxiliary function $z(t) = t(1-2^{-t})^{-1}$ is increasing for $t \geqq 0$ and $z(1) = 2$, it follows immediately that

$$\sum_{j=0}^{j_0}2^{j/q} < \frac{2^{(j_0+1)/q}}{2^{1/q}-1} < \frac{\gamma^{-1/q}}{1-2^{-1/q}} \leqq 2q\gamma^{-1/q}.$$

Next, we consider the so-called generalized *de la Vallée–Poussin means* defined by

$$(10) \qquad \tau_n(f,\lambda,x) := \frac{1}{\lambda_n - n + 1}\sum_{j=n}^{\lambda_n} s_j(f,x),$$

where $\lambda > 1$ and $\lambda_n = [\lambda n]$ $(n = 0, 1, \dots)$. The following lemma is an easy consequence of a result by Morgenthaler [3] on the $(C,1)$-summability of Walsh–Fourier series (see also [5]).

LEMMA 2. *If $f \in L^1(0,1)$ and $\lambda > 1$, then*

$$\lim_{n\to\infty}\int_0^1 |\tau_n(f,\lambda,x) - f(x)|dx = 0.$$

5. Proof of Theorem 1. *Sufficiency.* We assume that (5) is satisfied and prove

$$\lim_{\lambda\downarrow 1}\limsup_{n\to\infty}\int_0^1 |\tau_n(f,\lambda,x)-s_n(f,x)|\,dx=0. \tag{11}$$

Clearly, (11) implies (4) via Lemma 2.

To this effect, we use the representation

$$\tau_n(f,\lambda,x)-s_n(f,x)=\frac{1}{\lambda_n-n+1}\sum_{j=n+1}^{\lambda_n}\sum_{k=n+1}^{j}a_k w_k(x)$$

(cf. (10)) and split the integral in (11) into two parts: one extended over $(0,1/\gamma_n)$ and the other over $(1/\gamma_n,1)$, where $\gamma_n:=\lambda_n-n+1$.

First we apply a trivial estimate to obtain

$$|\tau_n(f,\lambda,x)-s_n(f,x)|\leqq\frac{1}{\gamma_n}\sum_{j=n+1}^{\lambda_n}\sum_{k=n+1}^{j}|a_k|=\frac{1}{\gamma_n}\sum_{k=n+1}^{\lambda_n}(\lambda_n-k+1)|a_k|\leqq\sum_{k=n+1}^{\lambda_n}|a_k|.$$

By (7),

$$J_1:=\int_0^{1/\gamma_n}|\tau_n(f,\lambda,x)-s_n(f,x)|\,dx\leqq \tag{12}$$

$$\leqq\frac{1}{\gamma_n}\sum_{k=n+1}^{\lambda_n}|a_k|\to 0\quad\text{as}\quad n\to\infty.$$

Second, by summation by parts, we get

$$\tau_n(f,\lambda,x)-s_n(f,x)= \tag{13}$$

$$=\frac{1}{\gamma_n}\sum_{j=n+1}^{\lambda_n}\Big(-a_nD_n(x)+\sum_{k=n}^{j-1}D_k(x)\Delta a_k+a_jD_j(x)\Big),$$

whence

$$J_2:=\int_{1/\gamma_n}^{1}|\tau_n(f,\lambda,x)-s_n(f,x)|\,dx\leqq$$

$$\leqq\int_{1/\gamma_n}^{1}|a_nD_n(x)|\,dx+\frac{1}{\gamma_n}\int_{1/\gamma_n}^{1}\Big|\sum_{j=n+1}^{\lambda_n}\sum_{k=n}^{j-1}D_k(x)\Delta a_k\Big|\,dx+$$

$$+\frac{1}{\gamma_n}\int_{1/\gamma_n}^{1}\left|\sum_{j=n+1}^{\lambda_n} a_j D_j(x)\right|dx =: J_{21}+J_{22}+J_{23}, \text{ say.}$$

By Lemma 1,

$$J_{23} \leqq \frac{2p}{(p-1)\gamma_n}\gamma_n^{1/q}\left(\sum_{j=n+1}^{\lambda_n}|a_j|^p\right)^{1/p} = \tag{14}$$

$$=\frac{2p}{p-1}\left(\frac{1}{\gamma_n}\sum_{j=n+1}^{\lambda_n}|a_j|^p\right)^{1/p} \to 0 \text{ as } n\to\infty,$$

owing to (7) again.

We interchange the summations with respect to j and k, then apply Lemma 1 to obtain

$$J_{22}=\frac{1}{\gamma_n}\int_{1/\gamma_n}^{1}\left|\sum_{k=n}^{\lambda_n-1}(\lambda_n-k)D_k(x)\Delta a_k\right|dx \leqq \tag{15}$$

$$\leqq \frac{2p}{(p-1)\gamma_n}\gamma_n^{1/q}\left(\sum_{k=n}^{\lambda_n-1}(\lambda_n-k)^p|\Delta a_k|^p\right)^{1/p} \leqq$$

$$\leqq \frac{2p}{p-1}(\lambda-1)^{1/q}\left(n^{p-1}\sum_{k=n}^{\lambda_n-1}|\Delta a_k|^p\right)^{1/p},$$

whence, by (6),

$$\lim_{\lambda\downarrow 1}\limsup_{n\to\infty} J_{22}=0. \tag{16}$$

Finally, by (5),

$$\lim_{n\to\infty} J_{21}=0. \tag{17}$$

Combining (12)–(17) yields (11) to be proved.

Necessity. This time we assume the fulfillment of (4). Then, by Lemma 2, for any $\lambda>1$,

$$\lim_{n\to\infty}\int_0^1|\tau_n(f,\lambda,x)-s_n(f,x)|dx=0. \tag{18}$$

Using the notations introduced in the sufficiency part, we can write that

$$\int_0^1 |\tau_n(f,\lambda,x) - s_n(f,x)|dx \geqq J_{21} - J_1 - J_{22} - J_{23}.$$

On the basis of (12), (14), (16), and (18), we conclude that

$$\lim_{\lambda \downarrow 1} \limsup_{n\to\infty} \int_{1/\gamma_n}^1 |a_n D_n(x)|dx = 0. \tag{19}$$

Since

$$\lim_{n\to\infty} \frac{n+1}{\gamma_n} = \frac{1}{\lambda - 1},$$

by (7), we have for every $\lambda > 1$,

$$\int_0^{1/\gamma_n} |a_n D_n(x)|dx \leqq \frac{(n+1)|a_n|}{\gamma_n} \to 0 \text{ as } n \to \infty. \tag{20}$$

Obviously, (19) and (20) imply (5) to be proved.

6. Concluding remarks. Analysing the proof of Theorem 1 (see especially (15)), we can achieve the following more general result.

THEOREM 2. *If $f \in L^1(0,1)$ and for some $p > 1$ and $\lambda > 1$,*

$$\limsup_{n\to\infty} \sum_{k=n}^{\lambda_n - 1} \left(\frac{\lambda_n - k}{\lambda_n - n + 1}\right)^p k^{p-1} |\Delta a_k|^p$$

is finite, then conditions (4) *and* (5) *are equivalent.*

Note added in proof (July 11, 1991). After having submitted the manuscript, it came to the author's knowledge that Stanojević [7] had announced an analogous result on the L'-convergence of trigonometric Fourier series.

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UNIVERSITY OF SZEGED
BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
H–6720 SZEGED, HUNGARY

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ON THE NUMBER OF PRIME FACTORS OF $\varphi(\varphi(n))$

I. KÁTAI* (Budapest), member of the Academy

1. Let $\varphi(n)$ be the Euler-totient function, $\sigma(n)$ the sum of positive divisors of n, $\omega(n)$ the number of distinct prime divisors of n, and $\Omega(n)$ the number of prime divisors of n counted them with multiplicity. Let $\varphi_2(n) = \varphi(\varphi(n))$, and in general $\varphi_{k+1}(n) = \varphi(\varphi_k(n))$. Similarly, $\sigma_2(n) = \sigma(\sigma(n))$, $\sigma_{k+1}(n) = \sigma(\sigma_k(n))$.

Our purpose in this paper is to prove the following

THEOREM 1. *We have*

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \leqq x \;\middle|\; \frac{\omega(\varphi_2(n)) - \frac{1}{6}(\log\log n)^3}{\frac{1}{\sqrt{10}}(\log\log n)^{5/2}} < y \right\} = \Phi(y),$$

for every real number y, where Φ is the standard Gaussian law.

Earlier, P. Erdős and C. Pommerance [1] and M. Ram Murty and V. Kumar Murty [7], [8] proved that

$$\frac{\omega(\varphi(n)) - \frac{1}{2}(\log\log n)^2}{\frac{1}{\sqrt{10}}(\log\log n)^{3/2}} \tag{1.1}$$

and the author [2] that

$$\frac{\omega(\sigma(p+1)) - \frac{1}{2}(\log\log p)^2}{\frac{1}{\sqrt{3}}(\log\log p)^{3/2}}$$

are distributed according to the standard Gaussian law.

M. Ram Murty and N. Saradha [9] proved the existence of the limit distribution of (1.1) by using only the Eratosthenian sieve.

2. Let A, A^*, A_s be the set of additive, completely additive and strongly additive functions, respectively. The letters $c, c_1, c_2, \ldots$ will denote suitable positive constants, not necessarily the same at every occurrence. We shall

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use the following abbreviations: $x_1 = \log x$, $x_{k+1} = \log x_k$ $(k = 1, 2, \dots)$. The letters $p, p_1, p_2, \dots, q, q_1, q_2, \dots, P, P_1, P_2, \dots, Q, Q_1, Q_2, \dots$, will denote primes. $P(n)$ and $p(n)$ denote the largest and the smallest prime divisor of n, respectively. $\pi(x, k, \ell)$ is the number of primes p up to x satisfying $p \equiv \ell \pmod{k}$.

The main idea of the proof is to approximate $\omega(\varphi_2(n))$ by an additive function. Hence, by using the Bombieri–Vinogradov mean-value theorem, some sieve results and Kubilius theory for the distribution of additive functions, we shall get our theorem.

LEMMA 1 (Bombieri–Vinogradov). *We have*

$$\sum_{k \leqq \sqrt{x}/(\log x)^A} \max_{(\ell,k)=1} \max_{z \leqq x} \left| \pi(z, k, \ell) - \frac{\operatorname{li} z}{\varphi(k)} \right| \ll \frac{x}{(\log x)^B},$$

where A and B are arbitrary positive constants satisfying the inequality $A \geqq$ $\geqq 4B + 40$ (see [6]*).*

LEMMA 2. *Let $\Psi(x, y)$ be the number of integers $n \leqq x$ satisfying the condition $P(n) \leqq y$. Then*

$$\Psi(x, y) < c_1 x \exp\left(-c \frac{\log x}{\log y}\right)$$

uniformly for all $y \leqq x$.

For the proof see [3].

LEMMA 3. *We have*

$$\pi(x, k, \ell) < \frac{3x}{\varphi(k) \log x/k},$$

if $\ell \leqq k < x$ and $(k, \ell) = 1$.

For the proof see Halberstam–Richert [4], Theorem 3.8.

LEMMA 4. *The number of solutions of the equation $p - \ell = aq$ in prime variables p and q, where p runs in the range $\ell < p \leqq x$, is less than*

$$\frac{cx}{\varphi(a) \log^2(x/a)}$$

for every positive integer a. The constant c is an absolute one.

See [4], Theorem 2.3.

LEMMA 5. *The number of solutions of the equation $p - \ell = A\gamma$ where p runs over the primes in the range $[\ell, x]$ and γ over the integers satisfying $p(\gamma) \geqq y$ is less than*

$$\frac{cx}{\varphi(A)(\log x)(\log y)}$$

uniformly if $A \leqq x^{3/4}$, $\ell < y < x$.

See [4].

LEMMA 6. *Let*

$$\sigma(x, k, \ell) := \sum_{\substack{k \leqq p < x \\ p \equiv \ell \,(\mathrm{mod}\, k)}} p^{-1}.$$

Then

$$\sigma(x, k, \ell) < c\frac{x_2}{\varphi(k)}$$

if $\ell \leqq k < x$ and $(\ell, k) = 1$.

PROOF. This is an immediate consequence of Lemma 3. Since $\pi(k \cdot 2^t, k, \ell) < c\frac{k \cdot 2^t}{\varphi(k)t}$, therefore $\sum 1/p$ for the primes in $[k \cdot 2^{t-1}, k \cdot 2^t]$ is less than $c\frac{1}{\varphi(k)t}$, if $t \geqq 1$. Summing up for t up to $2^t \leqq k$, we have

$$\sigma(k^2, k, \ell) < c\frac{\log\log k}{\varphi(k)}.$$

In the range $x \geqq k^2$, the inequality in Lemma 3 can be replaced by $\pi(x, k, \ell) < \frac{6x}{\varphi(k)\log x}$. This gives rapidly that

$$\sigma(x, k, \ell) - \sigma(k^2, k, \ell) < C_1\frac{x_2}{\varphi(k)}.$$

LEMMA 7. *Let $\mathcal{R}$ be a set of primes Q with the property that*

$$\#\{Q \in [y, 2y]\} < c_1\frac{y}{(\log y)^A}$$

holds for every $y \geqq 2$. Here $A \geqq 2$ is a constant. Let $\mathcal{P}_z$ be the set of those primes P for which there exists at least one $Q \in \mathcal{R}$, $Q > z$, such that $Q|P-1$. Then

$$S_{x,z} := \#\{P \leqq x | P \in \mathcal{P}_z\} \leqq c_2\left(\frac{\pi(x)}{x_1^{A-1}} + \frac{\pi(x)}{(\log z)^{A-1}}\right).$$

PROOF. It is clear that

$$S_{x,z} \leqq \sum_{\substack{Q \geqq z \\ Q \in \mathcal{R}}} \pi(x, Q, \ell).$$

Since $\pi(x,Q,\ell) < c\frac{x}{Qx_1}$ if $Q < \sqrt{x}$, and $\leqq x/Q$ if $Q < x$, therefore

$$S_{x,z} \leqq c\frac{x}{x_1}\Bigg(\sum_{\substack{z<Q<\sqrt{x}\\ Q\in\mathcal{R}}} 1/Q\Bigg) + x\sum_{\substack{\sqrt{x}<Q<x\\ Q\in\mathcal{R}}}\frac{1}{Q}.$$

By using the assumption for the number of primes of $\mathcal{R}$ in intervals of type $[M,2M]$ we get the assertion of our lemma immediately.

As an immediate consequence, we have

LEMMA 8. *Assume that the conditions of Lemma 7 are satisfied. Then the number of integers $n \leqq x$ having a divisor $P \in \mathcal{P}_z$ is less than $\frac{cxx_2}{(\log z)^{A-1}}$.*

LEMMA 9 (Turán–Kubilius inequality). *If $f \in A_s$, then*

$$\sum_{n\leqq x}(f(n) - A_x)^2 \leqq cxB_x,$$

where

$$A_x = \sum_{p\leqq x}\frac{f(p)}{p}, \qquad B_x^2 = \sum_{p\leqq x}\frac{f^2(p)}{p}$$

and c is an absolute constant. [5]

LEMMA 10. *Let $x > 100$. Then the number of primes p up to x satisfying $\omega(p-1) \geqq 2k$ is less than $c\frac{(x_2+O(1))^k}{k!}\cdot\frac{x}{x_1}$. Especially, the number of primes $p \leqq x$ satisfying $\omega(p-1) \geqq 15\log\log p$ is less than $O\left(\frac{x}{x_1^{11}}\right)$.*

PROOF. If $\omega(p-1) \geqq 2k$, then the product d of the first k smallest distinct prime divisors of $p-1$ is less than $\sqrt{x}$. Thus the number of primes p with $\omega(p-1) \geqq 2k$ is less than

$$\sum_{\substack{d\leqq\sqrt{x}\\ \omega(d)=k}} \pi(x,d,1)|\mu(d)|.$$

By using Lemma 3, and that

$$\sum_{d\leqq\sqrt{x}}\frac{1}{\varphi(d)} \leqq \frac{1}{k!}\left(\sum_{p\leqq\sqrt{x}}\frac{1}{p}\right)^k,$$

the first assertion follows rapidly. The second assertion is an immediate consequence of the first one and the Stirling formula for $k!$.

LEMMA 11. *If p runs over the set of primes, then*

$$\sum_{p \leqq x} \omega(p-1) = \text{li } x \cdot \log\log x + O(\text{li } x),$$

$$\sum_{p \leqq n} \omega^2(p-1) = \text{li } x \cdot (\log\log x)^2 + O(x_2 \text{li } x)$$

$$\sum_{p \leqq n} (\omega(p-1) - \log\log p)^2 \ll x_2 \text{li } x.$$

Lemma 11 can be proved by routine application of the Bombieri–Vinogradov mean value theorem.

3. Proof of the theorem. It is clear that $d|n$ implies $\varphi(d)|\varphi(n)$ and $\omega(d) \leqq \omega(n)$. Consequently $\varphi_2(d)|\varphi_2(n)$, and $\omega(\varphi_2(d)) \leqq \omega(\varphi_2(n))$. Let J be an interval, and let $\omega(n|J)$ denote the number of distinct prime divisors of n belonging to J. If $J = [y, \infty]$ then we simply write $\omega(n|y)$ instead of $\omega(n|J)$. Furthermore, let $\omega_z(n)$ denote the number of prime divisors of n which are not greater than z.

Let us consider the integers $n \leqq x$. For an n let $n = A(n)B(n)$, where $A(n)$ and $B(n)$ are defined such that $P(A(n)) \leqq x_1$, $p(B(n)) > x_1$. Observe that for $A(n) < \exp(x_2^2)$,

$$\omega\left(\varphi_2(A(n))\right) \leqq c\frac{\log \varphi_2(A(n))}{\log\log \varphi_2(A(n))} \leqq cx_2^2/x_3$$

and that the cardinality of $n \leqq x$ satisfying $A(n) \geqq \exp(x_2^2)$ is $O(x/x_2)$. Indeed, let us count the integers n with some fixed $A(n) = A$. All these integers can be written as $\gamma A(\leqq x)$, where γ runs over the integers $1 \leqq \gamma \leqq$ $\leqq x/A$, $p(\gamma) > x_1$. So, by using known sieve results, this is less than

$$c\frac{x}{A}\prod_{q<x_1}\left(1-\frac{1}{q}\right) \leqq \frac{c_1 x}{Ax_2}$$

if $A \leqq x/x_1$. If $A > x/x_1$, then only $\gamma = 1$ can occur. Now we consider the sum $\sum \frac{1}{A}$ extended for those A for which $\exp(x_2^2) \leqq A < x$, $P(A) < x_1$ is satisfied. By using Lemma 2 we can get easily that this sum is bounded as $x \to \infty$. Thus, for a non-exceptional n,

$$\omega(\varphi_2(n)) = \omega(\varphi_2(B(n)) + O(x_2^2/x_3)$$

holds. The number of integers $n \leqq x$ for which there is a $q \geqq x_1$ such that $q^2|n$ is less than x/q^2.

Summing up for $x_1 \leqq q$, we have that

$$x\sum \frac{1}{q^2} \leqq x/x_1.$$

Thus for all but at most $O(x/x_1)$ integers $n \leqq x$, $B(n)$ is a square free number.

Let us estimate now $\omega_{x_2^4}(\varphi_2(B(n)))$. We shall prove that this is less than $O(x_2^2 x_5)$ for all but $o(x)$ integers $n \leqq x$. Since the total number of primes $p \leqq x_2^2$ is less than $O(x_2^2/x_3)$, it is enough to estimate $\omega(\varphi_2(B(n))|J)$, where $J = [x_2^2, x_2^4]$.

Let us consider the sum

$$\sum := \sum_{n \leqq x} \omega(\varphi_2(B(n))|J),$$

where J is an arbitrary interval $\subseteqq [x_2^2, x]$.

If $q|\varphi_2(n)$, then either $q^2|\varphi(n)$ or there exists a prime $Q \equiv 1 \pmod q$ such that $Q|\varphi(n)$. In the second case either $Q^2|n$ or there exist a prime $P \equiv 1 \pmod Q$ such that $P|n$. Let us fix a $q \in J$.

The contribution of the second case to the sum $\sum$ is less than

$$\ll x \sum_{Q \equiv 1(q)} \sum_{p \equiv 1(Q)} \frac{1}{p} + x \sum_{Q \equiv 1(q)} \frac{1}{Q^2}.$$

By using Lemma 6, this is less than

$$\frac{x}{q} + x \sum_{Q \equiv 1(q)} \frac{c x_2}{Q} \leqq \frac{c_1 x x_2^2}{q}.$$

Let us consider the first case. If $q^2|\varphi(n)$, then either $q^2|n$, or there exist distinct primes P_1, P_2 such that $P_1 \equiv 1 \pmod q$, $P_2 \equiv 1 \pmod q$, $P_1 P_2|n$. Thus the contribution of the first case is less than

$$\frac{x}{q^2} + x \sum_{P_1, P_2 \equiv 1 \pmod q} \frac{1}{P_1} \cdot \frac{1}{P_2} \ll \frac{x}{q^2} + x \left(\sum_{P \equiv 1(q)} \frac{1}{p} \right)^2 \leqq \frac{x x_2^2}{q^2}.$$

Summing up for $q \in J$, we have

$$\sum_{n \leqq x} \omega(\varphi_2(B(n))|J) \ll x x_2^2 \left(\sum_{g \in J} 1/q \right)^2. \tag{3.1}$$

Especially, for the choice $J = [x_2^2, x_2^4]$, $\sum 1/q = O(1)$, thus the right hand side is $O(x x_2^2)$, consequently our assertion is true.

Thus, for all but $o(x)$ integers $n \leqq x$, we have

$$\omega(\varphi_2(n)) = \omega(\varphi_2(B(n))|x_2^4) + O(x_2^2 x_5). \tag{3.2}$$

We shall prove that

$$(3.3)\qquad \omega(\varphi_2(B(n))|x_2^4) = \sum_{\substack{p|n\\ p>x_1}} \sum_{Q|p-1} \omega(Q-1|x_2^4) + O(x_2^2 x_5)$$

for all but $o(x)$ of the integers $n \leqq x$. Let

$$(3.4)\qquad f(n) := \sum_{\substack{p|n\\ p>x_1}} \sum_{Q|p-1} \omega(Q-1|x_2^4) - \omega(\varphi_2(B(n))|x_4).$$

Assume that $B(n) = P_1 P_2 \dots P_t$ is a square free number. In $\varphi_2(B(n))$ every $q > x_2^4$, $q|\varphi_2(B(n))$ is counted once.

If $q|\varphi_2(B(n))$ then either $q^2|\varphi(B(n))$ or there exists a prime Q, $Q|\varphi(B(n))$, such that $Q \equiv 1 \pmod q$.

Let B_x be the set of integers $n \leqq x$ for which there exists a $q > x_2^4$, $q^2|\varphi(B(n))$. If $n \in B_x$, then either $q^2|n$ or there exists a prime divisor P of n such that $q^2|P-1$ or a couple of primes P_1, P_2, such that $P_1P_2|n$, $P_1 \equiv 1 \pmod q$, $P_2 \equiv 1 \pmod q$. Thus

$$\operatorname{card}(B_x) \leqq x \sum_{P\equiv 1(q^2)} \frac{1}{p} + x \sum_{q>x_2^4} \sum_{P_1,P_2\equiv 1(q)} \frac{1}{P_1P_2} \ll x x_2^2 \sum_{q>x_2^4} 1/q^2 = O(x/x_2^2).$$

Assume now that $n \notin B_x$. Let C_x be the set of those integers $n \leqq x$ for which there exists Q, $Q > x_2^4$ which divides $P_i - 1$ and $P_j - 1$ $(i \neq j)$ (where $P_iP_j|B(n)$). It is clear that

$$\operatorname{card}(C_x) \leqq x \sum_{Q>x_2^4} \sum_{\substack{P_1\equiv 1(Q)\\ P_2\equiv 1(Q)}} \frac{1}{P_1P_2} \ll x x_2^2 \sum_{Q>x_2^4} \frac{1}{Q^2} = O(x/x_3).$$

Let now D_x be the set of those integers $n \leqq x$ for which $B(n)$ is square-free, $n \notin B_x \cup D_x$. Let us consider $f(n)$ ((3.4)) for $n \in D_x$. In the double sum some $q, q|\varphi_2(B(n))$ is counted only once, if there exists no more than one Q such that $Q \equiv 1 \pmod q$. But this q is counted in $\omega(\varphi_2(B(n))|x_4)$ as well. So, the multiplicity of some q, $q|\varphi_2(B(n))$ occurring on the right hand side of (3.4) is not greater than the occurrence of $Q_i \neq Q_j$, P_u, P_v such that $q|Q_i - 1$, $q|Q_j - 1$; $P_u \equiv 1(Q_i)$, $P_v \equiv 1(Q_j)$. Here $P_u = P_v$ is not excluded. Thus, by applying Lemma 6,

$$\sum_{\substack{n\leqq x\\ n\in D_x}} f(n) \leqq x \sum_{q>x_2^4} \sum_{\substack{Q_1\equiv 1(q)\\ Q_2\equiv 1(q)}} \sum_{\substack{P_1\equiv 1(Q_1)\\ P_2\equiv 1(Q_2)}} \frac{1}{P_1P_2} + x \sum_{q>x_2^4} \sum_{\substack{Q_1\equiv 1(q)\\ Q_2\equiv 1(q)}} \sum_{p\equiv 1(Q_1Q_2)} \frac{1}{p} \ll x x_2^2.$$

Now we shall substitute each $\omega(Q-1|x_2^4)$ by $(Q-1|[x_2^4, Q^{1/16}])$ on the right hand side of (3.3). The error is $O(1)$ for every Q, the total error is less than

$$\ll \sum_{p|n} \omega(p-1).$$

Averaging this for $n \leqq x$,

$$\sum_{n\leqq x}\sum_{p|n}\omega(p-1) \leqq x\sum_{p|n}\frac{\omega(p-1)}{p} \ll xx_2^2,$$

from which we get that the error is less than $O(x_2^2 x_5)$ for all but $O(x/x_5)$ integers $n \leqq x$.

Let us consider the sum

$$\ell_n := \sum_{\substack{p|n\\ x^{1/16}<p<x}} \sum_{Q|p-1} \omega(Q-1|[x_2^4, Q^{1/16}]).$$

Let R be the set of those primes Q for which $Q > x_2^4$ and $\omega(Q-1) \geqq \geqq 15\log\log p$. Then, by Lemma 10

$$\#\{Q \in [y,2y] | Q \in R\} < c_1 y/(\log y)^{11}$$

and by Lemma 7,

$$\#\{P \in [y,2y] | P \in P_z\} < c_2\frac{\pi(y)}{(\log z)^{10}}.$$

The number of integers $n \leqq x$ for which there exists $p \in S_{x,z}$, $p|n$, $x^{1/16} < < p < x$ is less than

$$x\sum_{x^{1/16}<p<x} 1/p = O(x/x_3^9).$$

If n has a prime divisor $p \in S_{x,z}$; $p > x^{1/16}$ then

$$\ell_n \leqq 15 \sum_{\substack{p|n\\ p>x^{1/16}}} \sum_{Q|p-1} \log\log Q = T_n.$$

Averaging the right hand side, we get

$$\sum_{n\leqq x} T_n \leqq x \sum_{x^{1/16}<p<x} \frac{1}{p} \sum_{Q|p-1} \log\log Q \leqq xx_2 \sum_{x^{1/16}<p<x} \frac{\omega(p-1)}{p}.$$

But

$$\sum_{x^{1/16}<p<x} \frac{\omega(p-1)}{p} \ll x_2,$$

which comes from the estimation $\sum_{p\leqq x} \omega(p-1) \ll \frac{x}{x_1} x_2$ (see Lemma 11), and so $\sum_{n\leqq x} T_n \ll x x_2^2$.

Collecting our inequalities we conclude

$$\omega(\varphi_2(n)) = \sum_{\substack{p|n \\ x_1<p<x^{1/16}}} \sum_{Q|p-1} \omega(Q-1|[x_2^4, Q^{1/16}]) + O(x_2^2 x_3).$$

Let us consider now

$$b_n = \sum_{\substack{p|n \\ x_1<p<x^{1/16}}} \sum_{\substack{Q|p-1 \\ Q>p^{1/16}}} \omega(Q-1|[x_2^4, Q^{1/16}]).$$

We split b_n into two parts, $b_n = b_n^{(1)} + b_n^{(2)}$, where in $b_n^{(1)}$ we sum over those pairs (p, Q) for which $\omega(Q-1) < 15 \log\log Q$, and in $b_n^{(2)}$ over the others. Since for every p at most 16 distinct Q occur, therefore

$$b_n^{(1)} \leqq c_1 \sum_{p|n} \log\log p$$

and

$$\sum b_n^{(1)} \ll x \sum \frac{\log\log p}{p} \ll x \cdot x_2^2.$$

Furthermore

$$\sum_{n\leqq x} b_n^{(2)} \leqq x \sum_{p<x^{1/16}} \sum_{\substack{Q|p-1 \\ p^{1/16}<Q}} \omega(Q-1) \leqq x \sum_{Q} \omega(Q-1) \sum_{\substack{Q<p<Q^{16} \\ p\equiv 1 \,(\mathrm{mod}\, Q)}} 1/p \ll$$

$$\ll x \sum \frac{\omega(Q-1)(\log\log Q)}{Q}$$

where Q is summed only over those Q for which $\omega(Q-1) > 15 \log\log Q$ is satisfied. Since for every y, the number of such Q in $[y, 2y]$ is less than $\pi(y)/(\log y)^5$ and $\omega(Q-1)\log\log Q \leqq c \log y$, therefore

$$\sum \frac{\omega(Q-1)\log\log Q}{Q} \ll 1.$$

So we have

$$(3.5) \qquad \omega(\varphi_2(n)) = \sum_{\substack{p|n \\ x_1<p<x^{1/16}}} \sum_{\substack{Q|p-1 \\ Q<p^{1/16}}} \omega(Q-1|[x_2^4, Q^{1/16}]) + O(x_2^2 x_3),$$

for all but at most $o(x)$ integers $n \leqq x$.

Let now $u(p) = u_x(p)$ be defined as

$$u_x(p) := \sum_{\substack{Q|p-1 \\ Q<p^{1/16}}} \omega\left(Q-1|[x_2^4, Q^{1/16}]\right) - \frac{1}{2}(\log\log p)^2$$

if $x_1 < p < x^{1/16}$, and let $u_x(p) = 0$ if $p \leqq x_1$ or $p > x^{1/16}$. We shall consider $u_x(n)$ as a strongly additive function. Similarly, let

$$v_x(p) := \begin{cases} \frac{1}{2}(\log\log p)^2 & \text{if } x_1 < p < x^{1/16} \\ 0 & \text{otherwise} \end{cases}$$

and let $v_x(n)$ be a strongly additive function. Thus,

$$(3.6) \qquad \omega(\varphi_2(n)) = v_x(n) + u_x(n) + O(x_2^2 x_3)$$

holds for all but $o(x)$ integers $n \leqq x$.

4. Completion of the proof. We can see easily that after normalizing, $v_x(n)$ is distributed in limit according to the Gaussian law. Let us consider

$$t_x(n) := \frac{v_x(n)}{x_2^2}.$$

Then $t_x(p)$ is bounded on the set of primes, furthermore

$$A_x := \sum_{p<x} \frac{t_x(p)}{p} = \frac{1}{2\cdot 3}x_2 + O(1), \quad B_x^2 := \sum_{p\leqq x} \frac{t_x^2(p)}{p} = \frac{1}{20}x_2 + O(1)$$

as easy to calculate them. Thus, by the well-known Erdős–Kac theorem

$$\frac{1}{x}\#\left\{n \leqq x \;\middle|\; \frac{t_x(n) - A_x}{B_x} < y\right\} \to \Phi(y) \qquad (x \to \infty)$$

for every real number y. Since $B_x \to \infty$ and Φ is a continuous function, therefore we may substitute A_x by $\frac{1}{6}x_2$, and B_x by $\frac{1}{\sqrt{20}}\sqrt{x_2}$. After doing this and multiplying by x_2^2, we have that

$$(4.1) \qquad \lim_x \frac{1}{x}\#\left\{n \leqq x \;\middle|\; \frac{v_x(n) - \frac{1}{6}x_2^3}{\frac{1}{\sqrt{20}}x_2^{5/2}} < y\right\} = \Phi(y).$$

Finally, we shall prove that for all but $o(x)$ integers $n \leqq x$, $u_x(n)$ is bounded by a function of x growing as slowly as $o(x_2^{5/2})$. This can be done by the routine application of Turán–Kubilius inequality and the Bombieri–Vinogradov mean-value theorem.

Starting from the inequality,

$$\sum_{n \leqq x}\left(u_x(n) - \sum_{p \leqq x}\frac{u_x(p)}{p}\right)^2 \leqq cx\sum_{p \leqq x}\frac{u_x^2(p)}{p}, \tag{4.2}$$

we shall estimate the quantities

$$A_x := \sum_{p \leqq x}\frac{u_x(p)}{p}; \quad B_x = \sum_{p \leqq x}\frac{u_x^2(p)}{p}. \tag{4.3}$$

For this reason, we shall estimate

$$a(\omega) := \sum u_x(p), \qquad d(\omega) = \sum u_x^2(p),$$

where in these sums p runs over the set of primes belonging to the interval $J(\omega) = [\omega, \omega']$, $\omega' = \omega(\log\omega)^{10}$. Assume that $e^{x_2^2} \leqq \omega < \omega' < x^{1/16}$. Let us write $u_x(p)$ as

$$(u_x(p) =) - \frac{1}{2}(\log\log p)^2 + t_1(p) + t_2(p)$$

where

$$t_1(p) = \sum_{\substack{Q \mid p-1 \\ x_2^4 < Q < \omega^{1/16}}} \omega(Q-1 \| [x_2^4, Q^{1/16}]),$$

$$t_2(p) = \sum_{\substack{Q \mid p-1 \\ \omega^{1/16} \leqq Q < p^{1/16}}} \omega(Q-1 \| [x_2^4, Q^{1/16}]).$$

Then

$$a(\omega) = \sum_{\omega \leqq p < \omega'} -\frac{1}{2}(\log\log p)^2 + \sum t_1(p) + \sum t_2(p) = \tag{4.4}$$

$$= a_1(\omega) + a_2(\omega) + a_3(\omega).$$

We have $a_3(\omega) \geqq 0$ and

$$a_3(\omega) \leqq \sum_{\omega^{1/16} \leqq Q < \omega'^{1/16}} \omega(Q-1)(\pi(\omega', Q, 1) - \pi(\omega, Q, 1)).$$

Choosing a large B, $B=50$, say, in Lemma 1 and observing that $\omega(Q-1)\ll\ll\log Q$, we get

$$(4.5)\qquad a_3(\omega)\leqq(\operatorname{li}\omega'-\operatorname{li}\omega)\sum_{\omega^{1/16}<Q<\omega'^{1/16}}\frac{\omega(Q-1)}{Q-1}+O\left(-\frac{\operatorname{li}\omega}{(\log\omega)^{48}}\right).$$

Furthermore,

$$a_2(\omega)=\sum_{x_2^4<Q<\omega^{1/16}}\omega(Q-1|[x_2^4,Q^{1/16}])(\pi(\omega',Q,1)-\pi(\omega,Q,1))$$

and by Lemma 1, choosing $B=50$, we get

$$(4.6)\quad a_2(\omega)=(\operatorname{li}\omega'-\operatorname{li}\omega)\sum_{x_2^4<Q<\omega^{1/16}}\frac{\omega(Q-1|[x_2^4,Q^{1/16}])}{Q-1}+O\left(\frac{\operatorname{li}\omega}{(\log\omega)^{48}}\right).$$

Let

$$(4.7)\qquad S(\omega):=\sum_{x_2^4\leqq Q<\omega^{1/16}}\frac{\omega(Q-1|[x_2^4,Q^{1/16}])}{Q-1}.$$

Since $\omega(Q-1|[x_2^4,Q^{1/16}])=\omega(Q-1)+O(1)$, by using Lemma 1, after partial summation we have

$$(4.8)\qquad S(\omega)=\frac{1}{2}(\log\log\omega)^2+O(\log\log\omega)+O(x_4^2).$$

Thus

$$(4.9)\qquad a_2(\omega)=\frac{1}{2}(\log\log o)^2(\operatorname{li}\omega'-\operatorname{li}\omega)O\left(\frac{|\omega'-\omega|}{\log\omega}(\log\log\omega+x_2^4)\right).$$

Since $(\log\log\omega')^2-(\log\log\omega)^2\leqq 1$, therefore, by the prime number theorem,

$$\sum_{\omega\leqq p<\omega'}-\frac{1}{2}(\log\log p)^2=-\frac{1}{2}(\log\log\omega)^2(\operatorname{li}\omega'-\operatorname{li}\omega)+O\left(\frac{|\omega'-\omega|}{\log\omega}\right).$$

Collecting our results, we have

$$(4.10)\qquad a(\omega)=O(\operatorname{li}\omega'-\operatorname{li}\omega)(\log\log\omega+O(x_4^2)).$$

Hence we can get easily that

$$(4.10)\qquad A_x=O(x_2^2).$$

To do this we have to split the summation interval $[x_1, x]$ of p into intervals of type $[\omega, \omega']$ and use the relation (4.10). The contribution of the terms $\omega < e^{x_2^2}$ can be estimated roughly, the denominator p in $\sum_{p \in J(\omega)} \frac{u_x(p)}{p}$ can be substituted by ω at the expense of the total error $O(x_2^2)$.

We can estimate B_x^2 similarly. We split the interval $[x_1, x]$ into subintervals of type $[\omega, \omega']$ as earlier. Thus we have

$$B_x^2 \leqq \sum_{\omega} \frac{1}{\omega} d(\omega) + O(x_2^2),$$

where on the right hand side we consider only those ω for which $e^{x_2^2} \leqq \omega$ holds. To estimate $d(\omega)$, first we observe

$$u_x(p)^2 \leqq 2\left(t_1(p) - \frac{1}{2}(\log\log p)^2\right)^2 + 2t_2^2(p),$$

whence we have

$$d(\omega) \leqq 2(\Sigma_1 - \Sigma_2 + \Sigma_3) + \Sigma_4,$$

where

$$\Sigma_1 = \sum_p t_1^2(p), \quad \Sigma_2 = \sum_p (\log\log p)^2 t_1(p),$$

$$\Sigma_3 = \frac{1}{4} \sum (\log\log p)^4, \quad \Sigma_4 = \sum_p t_2^2(p).$$

Since $(\log\log p)^2$, $(\log\log p)^4$ are very slowly growing in $J(\omega)$, therefore

$$\Sigma_3 = \frac{1}{4}(\log\log\omega)^4(\mathrm{li}\,\omega' - \mathrm{li}\,\omega) + O\left(\frac{|\omega' - \omega|}{(\log\omega)^5}\right),$$

$$\Sigma_2 = (\log\log\omega)^2 \left(1 + O\left(\frac{1}{(\log\omega)^{1/2}}\right)\right) a_2(\omega).$$

To estimate Σ_1, we observe that

$$\Sigma_1 = \sum_{Q_1, Q_2} \omega(Q-1|[x_2^4, Q^{1/16}]) \cdot \omega(Q_2 - 1|[x_2^4, Q^{1/16}]) \cdot L_{Q_1,Q_2},$$

where

$$L_{Q_1,Q_2} = \pi(\omega', [Q_1, Q_2], 1) - \pi(\omega, [Q_1, Q_2], 1).$$

By Lemma 1, we get

$$\Sigma_1 = (\mathrm{li}\,\omega' - \mathrm{li}\,\omega)\Big(\sum \frac{\omega^2(Q-1|[x_2^4, Q^{1/16}])}{Q-1} + s^2(\omega) +$$

$$+O\Big(\sum \frac{\omega^2(Q-1)}{Q^2}\Big)\Big) + O\left(\omega/(\log\omega)^{48}\right).$$

Furthermore, we have

$$\Sigma_4 \ll \sum \omega(Q_1-1)\omega(Q_2-1)(\pi(\omega',[Q_1,Q_2],1)-\pi(\omega,[Q_1,Q_2],1)),$$

where Q_1, Q_2 run over the primes of the interval $[\omega^{1/16},\omega'^{1/16}]$, independently. It is clear that $\Sigma_4 \ll (\mathrm{li}\,\omega' - \mathrm{li}\,\omega)/(\log\omega)$, say. Collecting our inequalities, taking into account (4.8), (4.9) we infer

$$d(\omega) \ll (\log\log\omega)^3(\mathrm{li}\,\omega' - \mathrm{li}\,\omega)\sum \frac{\omega^2(Q-1|[x_2^4, Q^{1/16}])}{Q-1}+$$

$$+O((\log\log\omega)^3(\mathrm{li}\,\omega' - \mathrm{li}\,\omega)) + O\Big(\frac{|\omega-\omega'|}{\log\omega}\Big).$$

By using Lemma 11, we get

$$d(\omega) \ll (\log\log\omega)^3(\mathrm{li}\,\omega' - \mathrm{li}\,\omega).$$

Now, summing up for the intervals $J(\omega)$, we conclude that

$$B_x^2 \ll \sum_{p<x} \frac{(\log\log p)^2}{p} + O(x_2^4) \ll x_2^4. \tag{4.11}$$

Thus, by (4.2) we have, for all but $O(x|x_4^2)$ integers $n \leqq x$, the inequality $|u_x(n)| \leqq Cx_2^2x_4$ holds true.

By this the proof of our theorem is finished.

5. Remarks. By this method we can prove that

$$\lim_x \frac{1}{x}\#\left\{n \leqq x \;\middle|\; \frac{f(g(n)) - \frac{1}{6}x_2^3}{\frac{1}{\sqrt{20}}x_2^{5/2}} < y\right\} = \Phi(y)$$

for any choice of $f(n) = \omega(n)$, $f(n) = \Omega(n)$, $g(n) = \sigma(\varphi(n))$, $\varphi(\sigma(n))$, $\sigma(\sigma(n))$, $\varphi(\varphi(n))$.

We hope that by a refinement of this method we can prove that

$$\frac{\omega(\varphi_k(n)) - c_kx_2^{k+1}}{d_kx_2^{k+1/2}}$$

is distributed in limit according to the standard Gaussian law, for every fixed k.

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INTERVAL FILLING SEQUENCES AND COMPLETELY ADDITIVE FUNCTIONS

J. C. PARNAMI (Chandigarh)

1. Introduction

A sequence $\{\lambda_n\}$ of reals with $\lambda_n > \lambda_{n+1} > 0$ $(n \in \mathbf{N})$ and $\sum_{n=1}^{\infty} \lambda_n = L < \infty$ is said to be interval filling if every number $x \in [0, L]$ can be written as $x = \sum_{n=1}^{\infty} \varepsilon_n \lambda_n$ with $\varepsilon_n = 0$ or 1. For example, $\{1/q^n\}$ is interval filling iff $1 < q \leqq 2$ (see [1]).

A function $F\colon [0, L] \to \mathbf{R}$ is said to be completely additive with respect to an interval filling sequence $\{\lambda_n\}$ if

$$F\left(\sum_{n=1}^{\infty} \varepsilon_n \lambda_n\right) = \sum_{n=1}^{\infty} \varepsilon_n F(\lambda_n)$$

for every sequence $\{\varepsilon_n\}$ in $\{0, 1\}$.

In [1] Daróczy, Járai and Kátai proved that for $1 < q \leqq q(2)$, a completely additive function F with respect to $\{1/q^n\}$ is of the type $F(x) = cx$ for all $x \in [0, L]$ where $L = \sum_{n=1}^{\infty} 1/q^n = 1/(q-1)$ and $q(k)$ denotes the root of the equation $L - 1 = 1/q^k$ lying between 1 and 2.

In this paper an attempt is made to determine interval filling sequences for which every completely additive function is linear. In the process it has been possible to extend the result of Daróczy, Járai and Kátai to all q in $(1, 2]$.

2. Interval filling sequences

Definition 1. A sequence $\{\lambda_n\}$ with $\lambda_n > \lambda_{n+1} > 0$ $(n \in \mathbf{N})$ and $\sum_{n=1}^{\infty} \lambda_n = L < \infty$ is said to be interval filling if every number $x \in [0, L]$ can be written as

$$x = \sum_{n=1}^{\infty} \varepsilon_n \lambda_n, \qquad \varepsilon_n = 0 \text{ or } 1. \tag{2.1}$$

Interval filling sequences have another characterization as given in Satz 2.1 of [1], namely:

A sequence $\{\lambda_n\}$ with $\lambda_n > \lambda_{n+1} > 0$ $(n \in \mathbf{N})$ and $L = \sum_{n=1}^{\infty} \lambda_n < \infty$ is interval filling iff

$$\lambda_n \leqq \sum_{j=n+1}^{\infty} \lambda_j \text{ for all } n. \tag{2.2}$$

We write down some immediate consequences of this result, which will be useful in our investigation.

Corollary 2.1. *If $\{\lambda_n\}$ is an interval filling sequence and m is a natural number, then $\{\lambda_n\}_{n \geqq m}$ is also interval filling.*

Corollary 2.2. *If $\{\lambda_n\}$ is an interval filling sequence and $1 \leqq n_1 < < n_2 < \cdots < n_r$ is a finite sequence of natural numbers, then a number x can be written as*

$$x = \lambda_{n_1} + \lambda_{n_2} + \cdots + \lambda_{n_r} + \sum_{n > n_r} \varepsilon_n \lambda_n, \quad \varepsilon_n = 0 \text{ or } 1$$

iff

$$0 \leqq x - \lambda_{n_1} - \lambda_{n_2} - \cdots - \lambda_{n_r} \leqq \sum_{n > n_r} \lambda_n.$$

Corollary 2.3. *Let $\{\lambda_n\}$, $n_1, n_2, \ldots, n_r$ be as in Corollary 2.2, then $x = \lambda_{n_1} + \cdots + \lambda_{n_r}$ has a representation*

$$x = \lambda_{n_1} + \cdots + \lambda_{n_{r-1}} + \sum_{n > n_r} \varepsilon_n \lambda_n, \quad \varepsilon_n = 0 \text{ or } 1.$$

3. About the numbers $q(k)$

For a natural number k, the equation

$$q^{k+1} - 2q^k + q - 1 = 0 \tag{3.1}$$

has a unique root lying between 1 and 2 [see 1]. We denote it by $q(k)$.

Proposition 3.1. a) *The sequence $\{q(k)\}$ is strictly monotone and converges to 2.*

b) *For $q(k) < q \leqq q(k+1)$, we have*

$$1/q^{k+1} \leqq \sum_{j=1}^{\infty} 1/q^j - 1 < 1/q^k. \tag{3.2}$$

PROOF. a) Let $2 > q \geqq q(k+1)$, then $q^{k+2} - 2q^{k+1} - 1 \geqq 0$ and so

$$(q^{k+1} - 2q^k + q - 1) = (q^{k+2} - 2q^{k+1} + q - 1)/q + (q-1)^2/q > 0.$$

Hence $q > q(k)$ and in particular $q(k+1) > q(k)$. By the equation (3.1) for $q(k)$, we have

$$0 < (2 - q(k))/(q(k) - 1) = (1/q(k))^k \leqq (1/q(1))^k \to 0 \text{ as } k \to \infty.$$

Hence $\{q(k)\}$ converges to 2.

b) For $q(k) < q \leqq q(k+1)$, we have

$$q^{k+1} - 2q^k + q - 1 > 0 \quad \text{and} \quad q^{k+2} - 2q^{k+1} + q - 1 \leqq 0.$$

These inequalities can be rewritten as $1/q^{k+1} \leqq (2-q)/(q-1) < 1/q^k$ i.e.

$$1/q^{k+1} \leqq \sum_{j=1}^{\infty} 1/q^j - 1 < 1/q^k.$$

4. Some special interval filling sequences and unambiguous numbers

For a fixed natural number k, we denote by Λ_k the set of interval filling sequences $\{\lambda_n\}$ satisfying the property that

$$\lambda_{n+k+1} \leqq \sum_{j=1}^{\infty} \lambda_{n+j} - \lambda_n < \lambda_{n+k} \tag{4.1}$$

for every natural number n. By Proposition 2.1, Part b) it follows that $\{1/q^n\}$ is in Λ_k whenever $q(k) < q \leqq q(k+1)$. By Corollary 2.1, it follows that for any sequence $\{\lambda_n\}$ in Λ_k and a natural number m, the subsequence $\{\lambda_n\}_{n \geqq m}$ is also in Λ_k.

PROPOSITION 4.1. *For any interval filling sequence $\{\lambda_n\}$ in Λ_k, we have, for any natural n,*

$$\lambda_n < 2\lambda_{n+1}, \tag{4.2}$$

$$\lambda_{n+1} + \cdots + \lambda_{n+k} < \lambda_n, \tag{4.3}$$

$$\sum_{j=2}^{\infty} \lambda_{n+j} < \lambda_n, \tag{4.4}$$

$$\lambda_{n+1}+\lambda_{n+k+2}<\lambda_n. \tag{4.5}$$

PROOF. By (4.1), we have

$$\sum_{j=1}^{\infty}\lambda_{n+1+j}-\lambda_{n+1}<\lambda_{n+k+1}\leqq\sum_{j=1}^{\infty}\lambda_{n+j}-\lambda_n$$

and therefore $-\lambda_{n+1}<\lambda_{n+1}-\lambda_n$ i.e. $\lambda_n<2\lambda_{n+1}$. This proves (4.2).

By (4.1), we obtain on using (2.2)

$$\sum_{j=1}^{\infty}\lambda_{n+j}-\lambda_n<\lambda_{n+k}\leqq\sum_{j=1}^{\infty}\lambda_{n+k+j}$$

i.e. $\sum_{j=1}^{k}\lambda_{n+j}-\lambda_n<0$. This proves (4.3).

Now to prove (4.4). By (4.1), we have

$$\sum_{j=1}^{\infty}\lambda_{n+j}-\lambda_n<\lambda_{n+k}\leqq\lambda_{n+1}$$

and so $\sum_{j=2}^{\infty}\lambda_{n+j}-\lambda_n<0$.

Finally to prove (4.5), we obtain from (4.1)

$$\lambda_{n+k+2}\leqq\sum_{j=1}^{\infty}\lambda_{n+1+j}-\lambda_{n+1}<\lambda_n-\lambda_{n+1}$$

on using (4.4), and so

$$\lambda_{n+k+2}\lambda_{n+1}<\lambda_n.$$

DEFINITION 2. For a given interval filling sequence $\{\lambda_n\}$, a number $x\in$ $\in[0,L]$, $L=\sum_{n=1}^{\infty}\lambda_n$, is said to be unambiguous if there is a unique representation of x as $\sum_{n=1}^{\infty}\varepsilon_n\lambda_n$, $\varepsilon_n=0$ or 1; otherwise we say that x is ambiguous.

Now we prove some results about unambiguous numbers which will be useful in our investigation.

PROPOSITION 4.2. *Let $\{\lambda_n\}$ be an interval filling sequence in Λ_k. Suppose that a number*

$$x = \lambda_n + \lambda_{n+1} + \cdots + \lambda_{n+t} + \sum_{m \geqq n+t+2} \varepsilon_m \lambda_m, \quad \varepsilon_m = 0 \text{ or } 1$$

$t \geqq 0$, $n > 1$, is unambiguous relative to the sequence $\{\lambda_n\}$. Then $t \leqq k-1$ and $x - \lambda_n$ is unambiguous.

PROOF. First we claim that $x < \lambda_{n=1}$. Suppose on the contrary, that $x \geqq \lambda_{n-1}$. Then $0 \leqq x - \lambda_{n-1} < x \leqq \sum_{j=n}^{\infty} \lambda_j$ and by Corollary 2.2, x has a representation of the type $x = \lambda_{n-1} + \sum_{j=n}^{\infty} \varepsilon_j \lambda_j$, $\varepsilon_j = 0$ or 1. This is impossible as x is unambiguous. Hence $x < \lambda_{n-1}$.

Now we assert that $x > \lambda_n + \cdots + \lambda_{n+t-1} + \sum_{j=1}^{\infty} \lambda_{n+t+j}$. Suppose otherwise, then $0 \leqq x - \lambda_n - \cdots - \lambda_{n+t-1} \leqq \sum_{j=1}^{\infty} \lambda_{n+t+j}$ and by Corollary 2.2, x has a representation of the type $x = \lambda_n + \cdots + \lambda_{n+t-1} + \sum_{j=1}^{\infty} \varepsilon_j \lambda_{n+t+j}$, with $\varepsilon_j = 0$ or 1. This is impossible as x is unambiguous.

By the above considerations, we have

$$\lambda_n + \cdots + \lambda_{n+t-1} + \sum_{j=1}^{\infty} \lambda_{n+t+j} < \lambda_{n-1}$$

i.e. $\sum_{j=n}^{\infty} \lambda_j - \lambda_{n+t} < \lambda_{n-1}$. On using (4.1), we obtain that

$$\lambda_{n+k} \leqq \sum_{j=n}^{\infty} \lambda_j - \lambda_{n-1} < \lambda_{n+t},$$

which implies that $t < k$ i.e. $t \leqq k-1$.

Now suppose that

$$x - \lambda_n = \sum_{m=1}^{\infty} \eta_m \lambda_m, \quad \eta_m = 0 \text{ or } 1.$$

Since $x < \lambda_{n-1}$, therefore $x - \lambda_n < \lambda_{n-1} - \lambda_n < \lambda_n$ on using (4.2). Thus we have $\eta_m = 0$ for $m \leqq n$ and

$$x = \lambda_n + \sum_{m \geqq n} \eta_m \lambda_m \equiv \lambda_n + \cdots + \lambda_{n+t} + \sum_{m \geqq n+t+2} \varepsilon_m \lambda_m$$

as expressions, because x is unambiguous. Hence $x-\lambda_n$ has unique representation of the type (2.1), namely

$$x-\lambda_n=\lambda_{n+1}+\dots+\lambda_{n+t}+\sum_{m\geqq n+t+2}\varepsilon_m\lambda_m.$$

This proves that $x-\lambda_n$ is unambiguous.

PROPOSITION 4.3. *Let $\{\lambda_n\}$ be an interval filling sequence in Λ_k and $x=$ $=\lambda_1+\dots+\lambda_t+\sum_{m\geqq t+2}\varepsilon_m\lambda_m$ be unambiguous with respect to $\{\lambda_n\}$. Then $y=x-\lambda_1-\dots-\lambda_t$ is unambiguous.*

PROOF. Suppose that $y=\sum_{n=1}^{\infty}\eta_n\lambda_n$, $\eta_n=0$ or 1. Since

$$y=\sum_{m\geqq t+2}\varepsilon_m\lambda_m\leqq\sum_{m\geqq t+2}\lambda_m<\lambda_t$$

on using (4.4), therefore $\eta_n=0$ for all $n\leqq t$ and hence $x=\lambda_1+\dots+\lambda_t+$ $+\sum_{n\geqq t+1}\eta_n\lambda_n$. Since x is unambiguous, therefore $\sum_{n\geqq t+1}\eta_n\lambda_n$ and $\sum_{m\geqq t+2}\varepsilon_m\lambda_m$ are the same representations. Thus y has a unique representation of the type (2.1) i.e. y is unambiguous.

PROPOSITION 4.4. *Let $\{\lambda_n\}$ be an interval filling sequence in Λ_k and $x=$ $=\lambda_n+\lambda_{n+u}+\sum_{m\geqq n+u+1}\varepsilon_m\lambda_m$, $\varepsilon_m=0$ or 1 be unambiguous. Then we have $u\leqq k+1$.*

PROOF. Since x is unambiguous, therefore so is

$$L-x=\lambda_1+\dots+\lambda_{n-1}+\lambda_{n+1}+\dots+\lambda_{n+u-1}+\sum_{m\geqq n+u+1}(1-\varepsilon_m)\lambda_m.$$

By Proposition 4.3,

$$\lambda_{n+1}+\dots+\lambda_{n+u-1}+\sum_{m\geqq n+u+1}(1-\varepsilon_m)\lambda_m$$

is unambiguous and by Proposition 4.2 we have $u-2\leqq k-1$ i.e. $u\leqq k+1$.

PROPOSITION 4.5. *Let $\{\lambda_n\}$ be an interval filling sequence in Λ_k. For a number $x\in(C,\lambda_1)$, $x=\sum_{i=1}^{\infty}\lambda_{n_i}$ with $n_{i+1}>n_i$ for all i to be unambiguous with respect to $\{\lambda_n\}$, it is necessary that $n_{i+1}\leqq n_i+k+1\leqq n_{i+k}$ for all i.*

PROOF. Suppose that $x\in(0,\lambda_1)$, $x=\sum_{i=1}^{\infty}\lambda_{n_i}$ with $n_{i+1}>n_i$ for all i is unambiguous. Then $n_1>1$ and by repeated application of Proposition 4.2, for every fixed j, $\sum_{i \geqq j} \lambda_{n_i}$ is unambiguous. By Proposition 4.4, we have $n_{j+1} - n_j \leqq k + 1$. Since $n_{i+1} - n_i \geqq 1$, therefore $n_{j+k} - n_j \geqq k$ and equality holds iff $n_{j+i} = n_{j+i-1} + 1$ for $1 \leqq i \leqq k$, which is not possible in view of Proposition 4.2. Thus we have $n_{j+k} \geqq n_j + k + 1$.

NOTE. If $\lambda_1 \leqq x \leqq L - \lambda_1 = \sum_{j \geqq 2} \lambda_j$ then by Corollary 2.2, x is ambiguous. Moreover a number y lying between $L - \lambda_1$ and L is unambiguous iff $x = L - y$ is unambiguous and $0 < x < \lambda_1$. So the condition $x \in (0, \lambda_1)$ in Proposition 4.5 is virtually not a restriction.

PROPOSITION 4.6. *Let $\{\lambda_n\}$ be an interval filling sequence in Λ_k. Suppose that $\xi \in (0, L)$, $\xi = \sum_{n=1}^{\infty} \varepsilon_n \lambda_n$, $\varepsilon_n = 0$ or 1, is unambiguous with respect to $\{\lambda_n\}$. Then there exists a natural number N with the following properties:*

i) *For every $m \geqq N$, $\xi_m = \sum_{n \geqq m} \varepsilon_n \lambda_n$ is unambiguous.*

ii) *For every $m \geqq N$, at least one of $\varepsilon_{m+1}, \ldots, \varepsilon_{m+k+1}$ is 1.*

PROOF. Let $P = \{n : \varepsilon_n = 1\}$ and $Q = \{n : \varepsilon_n = 0\}$. Since $0 < \xi < L$ therefore P and Q are both non-empty. Since ξ is unambiguous, therefore by Corollary 2.3, P is infinite. Find a natural number $M \in Q$ and a natural number $N \in P$ such that $N > M$ and $N \geqq 3$. By Proposition 4.2 and 4.3, $\xi_N = \sum_{n \geqq N} \varepsilon_n \lambda_n$ is unambiguous, moreover $\xi_N \leqq \sum_{n \geqq 3} \lambda_n < \lambda_1$ on using (4.4). Again using Proposition 4.2, we obtain that ξ_m is unambiguous for every $m \geqq N$. This proves (i).

Now let $m \geqq N$. Find maximal $n_1 \leqq m$ such that $\varepsilon_{n_1} = 1$ and least $n_2 > m$ such that $\varepsilon_{n_2} = 1$. Then $\varepsilon_j = 0$ for all j satisfying $n_1 + 1 \leqq j \leqq n_2 - 1$. By Proposition 4.5, $n_2 \leqq n_1 + k + 1 \leqq m + k + 1$, also $n_2 \geqq m + 1$ and $\varepsilon_{n_2} = 1$. This proves (ii).

5. Completely additive functions

Let $\{\lambda_n\}$ be a given interval filling sequence and $L = \sum_{n=1}^{\infty} \lambda_n$.

DEFINITION 3. We call a function $F \colon [0, L] \to \mathbf{R}$ completely additive if for every sequence $\{\varepsilon_n\}$ in $\{0, 1\}$ we have

$$F\left(\sum_{n=1}^{\infty} \varepsilon_n \lambda_n\right) = \sum_{n=1}^{\infty} \varepsilon_n F(\lambda_n). \tag{5.1}$$

In this section we find some interval filling sequences for which every completely additive function is linear.

THEOREM 5.1. *Let $1 < q \leqq 2$ and $F\colon [0,L] \to \mathbf{R}$ be completely additive with respect to the interval filling sequence $\{1/q^n\}$. Then there exists a constant c such that $F(x) = cx$ for all x in $[0,L]$. We shall use the following:*

LEMMA 5.1. *Let k be a fixed natural number and $\{\lambda_n\}$ be an interval filling sequence in Λ_k. Suppose that $F\colon [0,L] \to \mathbf{R}$ is a completely additive function with respect to $\{\lambda_n\}$, satisfying $F(L) = 0$, $F \not\equiv 0$. Then there exists a natural number N such that*

$$2F(\lambda_n) < -F(\lambda_1) - \cdots - F(\lambda_{n-1}) \tag{5.2}$$

for all $n \geqq N$.

PROOF. Consider $P = \{n \in \mathbf{N} : a_n = F(\lambda_n) > 0\}$ and $\xi = \sum_{n\in P} \lambda_n$. If $P = \emptyset$, then $a_n \leqq 0$ for all n and since $\sum_{n=1}^{\infty} a_n = 0$, therefore $a_n = 0$ for all n and $F \equiv 0$. If $P = \mathbf{N}$ then we would have $F(L) > 0$. Hence $P \neq \emptyset$, $\mathbf{N}$ i.e. $\xi \in (0,L)$. By Satz 3.2 of [1], ξ is unambiguous. Let N be as in Proposition 4.6. Fix any $j \geqq N$, then by Proposition 4.4

$$\eta_j = \lambda_j + \sum_{\substack{m\in P\\ m\geqq j+k+2}} \lambda_m \tag{5.3}$$

is ambiguous. Since ξ is unambiguous, therefore by Proposition 4.2, $\eta_j - \lambda_j$ is unambiguous and in view of Corollary 2.1, we have $\eta_j - \lambda_j < \lambda_{j+k+1}$. Hence

$$\eta_j < \lambda_j + \lambda_{j+k+1} < \lambda_{j-1} \tag{5.4}$$

on using (4.5). Since η_j is ambiguous and $\eta_j - \lambda_j$ is unambiguous, therefore it follows from (5.3) and (5.4) that

$$\eta_j \leqq \sum_{n\geqq j+1} \lambda_n. \tag{5.5}$$

Moreover, we have

$$\eta_j > \lambda_j > \lambda_{j+1} + \cdots + \lambda_{j+k} \tag{5.6}$$

on applying (4.3). By using Corollary 2.2, we obtain from (5.5) and (5.6) that

$$\eta_j = \lambda_{j+1} + \cdots + \lambda_{j+k} + \sum_{m>j+k} \varepsilon_m \lambda_m \tag{5.7}$$

for suitable ε_m's in $\{0,1\}$. Since F is completely additive, therefore it follows from (5.3) and (5.7) that

$$a_j + \sum_{\substack{m \geqq j+k+2 \\ m \in P}} a_m = a_{j+1} + \cdots + a_{j+k} + \varepsilon_{j+k+1} a_{j+k+1} + \sum_{m \geqq j+k+2} \varepsilon_m a_m$$

and so

$$a_j \leqq a_{j+1} + \cdots + a_{j+k} + \varepsilon_{j+k+1} a_{j+k+1}. \tag{5.8}$$

Thus, we have

$$\begin{cases} a_j \leqq a_{j+1} + \cdots + a_{j+k} + a_{j+k+1} & \text{if } j+k+1 \in P, \\ a_j \leqq a_{j+1} + \cdots + a_{j+k} & \text{if } j+k+1 \notin P. \end{cases} \tag{5.9}$$

In both the cases, we have

$$a_j \leqq a_{j+1} + \cdots + a_{j_1} \tag{5.10}$$

where j_1 is the largest integer in P such that $j+1 \leqq j_1 \leqq j+k+1$. (Existence of such a j_1 is guaranteed by Proposition 4.6.) Set $j_0 = j$, and define j_{i+1} to be the largest integer in P such that $j_i + 1 \leqq j_{i+1} \leqq j_i + k + 1$. By (5.10) we obtain

$$a_{j_i} \leqq a_{j_i+1} + \cdots + a_{j_{i+1}}$$

and adding over all $i = 0, 1, 2, \ldots$, we get

$$a_{j_0} + a_{j_1} + a_{j_2} + \cdots \leqq a_{j_0+1} + a_{j_0+2} + \cdots = -(a_1 + a_2 + \cdots + a_{j_0})$$

and therefore $a_j = a_{j_0} < -(a_1 + a_2 + \cdots + a_j)$ i.e. $2a_j < -a_1 - a_2 - \cdots - a_{j-1}$.

Corollary 5.1. *Let F be a completely additive function with respect to an interval filling sequence $\{\lambda_n\}$ in Λ_k such that $F(L) = 0$. Then $F \equiv 0$.*

Proof. Suppose that $F \not\equiv 0$. By Lemma 5.1, there exists a natural number N such that

$$2F(\lambda_n) < -F(\lambda_1) - \cdots - F(\lambda_{n-1}) \text{ for all } n \geqq N. \tag{5.11}$$

Replacing F by $-F$, we obtain that there is a natural number M such that

$$2(-F(\lambda_n)) < -(-F(\lambda_1)) - \cdots - (-F(\lambda_{n-1})) \text{ for all } n \geqq M$$

i.e.

$$2F(\lambda_n) > -F(\lambda_1) - \cdots - F(\lambda_{n-1}) \text{ for all } n \geqq M. \tag{5.12}$$

(5.11) and (5.12) contradict each other if we take $n \geqq \operatorname{Max}(M, N)$. Hence $F \equiv 0$.

COROLLARY 5.2. *Let F be a completely additive function with respect to an interval filling sequence $\{\lambda_n\}$ in Λ_k. Then there exists a constant c such that*

$$F(x) = cx \text{ for all } x \text{ in the domain of } F.$$

PROOF. Define $\hat{F}\colon [0, L] \to R$ by $\hat{F}(x) = F(x) - F(L)\frac{x}{L}$, $L = \sum\limits_{n=1}^{\infty} \lambda_n$. Then $\hat{F}$ is completely additive with $\hat{F}(L) = 0$ and by Corollary 5.1, $\hat{F} \equiv 0$ i.e. $F(x) = cx$ for all $x \in [0, L]$, where $c = \frac{F(L)}{L}$.

PROOF OF THEOREM 5.1. For $1 < q \leqq q(1)$, the result has been proved in [1, Korollar 3.1]. For a natural number k and $q(k) < q \leqq q(k+1)$, the sequence $\{1/q^n\}$ is in Λ_k and the result follows by Corollary 5.2. By Proposition 3.1, Part a)

$$(1, 2) = \bigcup_{k \geqq 1} (q(k), q(k+1)] \cup (1, q(1)],$$

therefore it only remains to prove the theorem in case $q = 2$.

Now let F be a completely additive function with respect to interval filling sequence $\{2^{-n}\}$. Since $2^{-n} = \sum\limits_{m=n+1}^{\infty} 2^{-m}$ for all $n \geqq 1$, therefore

$$F(2^{-n}) = \sum_{m=n+1}^{\infty} F(2^{-m}) \quad \text{for all} \quad n \geqq 1. \tag{5.13}$$

Changing n to $n+1$ we obtain

$$F(2^{-(n+1)}) = \sum_{m=n+2}^{\infty} F(2^{-m}). \tag{5.14}$$

Subtracting (5.14) from (5.13) we obtain that

$$F(2^{-n}) = 2F(2^{-(n+1)}) \quad \text{for all} \quad n \geqq 1$$

and by induction we have

$$F(2^{-n}) = 2^{-(n-1)} F(1/2).$$

Hence for $\varepsilon_n \in \{0, 1\}$

$$F\left(\sum_{n=1}^{\infty} \varepsilon_n 2^{-n}\right) = \sum_{n=1}^{\infty} \varepsilon_n F(2^{-n}) = 2F(1/2) \sum_{n=1}^{\infty} \varepsilon_n 2^{-n} = c\left(\sum_{n=1}^{\infty} \varepsilon_n 2^{-n}\right)$$

where $c = 2F(1/2)$. This completes the proof.

6. Interval filling sequences for which every completely additive function is linear

We denote by Λ to be the set of interval filling sequences $\{\lambda_n\}$ such that every completely additive function with respect to $\{\lambda_n\}$ is linear. We have

(6.1) $$\Lambda_k \subset \Lambda \text{ for every natural number } k$$

(Corollary 5.1),

(6.2) $$\{1/q^n\} \in \Lambda \text{ for every } q \text{ in } (1,2]$$

(Theorem 5.1),

(6.3) Any plentiful interval filling sequence (i.e. an interval filling sequence $\{\lambda_n\}$ such that every number between 0 and $L = \sum_{n=1}^{\infty} \lambda_n$ is ambiguous) is in Λ (Satz 3.1 of [1]).

Now we describe a property of Λ.

THEOREM 6.1. *If an interval filling sequence has a subsequence which is in Λ, then the original sequence is in Λ.*

PROOF. Let $\{\lambda_n\}$ be an interval filling sequence and $\{\mu_n\}$ be a subsequence of $\{\lambda_n\}$ which lies in Λ. Set $L = \sum_{n=1}^{\infty} \lambda_n$ and $L_1 = \sum_{n=1}^{\infty} \mu_n$. Let $F\colon [0,L] \to \mathbf{R}$ be completely additive with respect to $\{\lambda_n\}$. Define $F_1\colon [0,L_1] \to \mathbf{R}$ by setting $F_1(x) = F(x)$ for all $x \in [0,L_1]$. Then F_1 is completely additive with respect to $\{\mu_n\}$ and hence there exists a constant c such that $F_1(x) = cx$ for all $x \in [0,L_1]$. Since $\lambda_n \to 0$ as $n \to \infty$, therefore there exists an integer $N : \lambda_n \leqq L_1$ for all $n \geqq N$ and so $F(\lambda_n) = F_1(\lambda_n) = c\lambda_n$ for all $n \geqq N$. In case $N > 1$, we have $\lambda_{N-1} \leqq \sum_{n \geqq N} \lambda_n$ and so $\lambda_{N-1} = \sum_{n \geqq N} \varepsilon_n \lambda_n$, $\varepsilon_n \in \{0,1\}$ and in turn

$$F(\lambda_{N-1}) = F\left(\sum_{n \geqq N} \varepsilon_n \lambda_n\right) = \sum_{n \geqq N} \varepsilon_n F(\lambda_n) =$$
$$= \sum_{n \geqq N} \varepsilon_n c \lambda_n = c\left(\sum_{n \geqq N} \varepsilon_n \lambda_n\right) = c\lambda_{N-1}.$$

We conclude that $F(\lambda_n) = c\lambda_n$ for all $n \geqq 1$ and consequently for any $x = \sum_{n=1}^{\infty} \eta_n \lambda_n$, $\eta_n \in \{0,1\}$ we have

$$F(x) = F\left(\sum_{n=1}^{\infty} \eta_n \lambda_n\right) = \sum_{n=1}^{\infty} \eta_n F(\lambda_n) = \sum_{n=1}^{\infty} \eta_n c \lambda_n = cx.$$

This proves that $F(x) = cx$ for all $x \in [0, L]$.

Added in proof (July 18, 1991). Theorem (5.1) has also been proved independently by T. Szabó, *Publ. Math. (Debrecen)*, **36** (1989/90). In the meantime Z. Daróczy, I. Kátai and T. Szabó, *Arch. Math. (Basel)*, **54** (1990), have extended the result to an arbitrary interval filling sequence.

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DEPARTMENT OF MATHEMATICS
PENJAB UNIVERSITY
CHANDIGARH–160014
INDIA

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ON THE UNIFORM APPROXIMATION BY GENERALIZED BERNSTEIN-MEANS

R. GÜNTTNER (Osnabrück)

1. Let $S_n[g]$ denote the trigonometric polynomial of degree at most n interpolating the function $g \in C_{2\pi}$ at $m = 2n+1$ equidistant nodes

$$(1) \qquad t_i^{(m)} = \tau + \frac{2i\pi}{m}, \quad S_n[g](t_i^{(m)}) = g(t_i^{(m)}), \quad i = 0, \pm 1, \pm 2, \dots .$$

Let us focus attention on the generalized Bernstein-means

$$(2) \qquad B_{kn}[g](t) = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} S_n[g]\left(t + \frac{k-2j}{m}\pi\right), \quad k = 0, 1, 2, \dots ,$$

which for $k = 1$ and $k = 2$ were first introduced by S. N. Bernstein. If not otherwise stated we take $m > k$ to ensure that the arguments of $S_n[g]$ in (2) lie within a period of length 2π. $B_{kn}[g]$ can also be written in the form

$$(3) \qquad B_{kn}[g](t) = \sum_{i=-n}^{n} g(t_i^{(m)}) \cdot s_i^{(m)}(t)$$

for certain functions $s_i^{(m)} \in C_{2\pi}$ (cf. [10]).

$C_{2\pi}$ is made into a normed linear space by setting

$$\|g\| := \sup_t |g(t)|.$$

The norm or Lebesgue constant $\|B_{kn}\|$ of the bounded linear operator B_{kn} can be determined as the norm of the so called Lebesgue function

$$(4) \qquad B_{kn}(t) := \sup_{\|g\|=1} |B_{kn}[g](t)| = \sum_{i=-n}^{n} |s_i^{(m)}(t)|.$$

It is known that (see [2] or [4])

$$(5) \qquad |B_{0n}[g](t) - g(t)| \leqq \frac{1}{2}(1 + B_{0n}(t)) \cdot \omega(g, h),$$

where $\omega(g, \cdot)$ denotes the modulus of continuity of g and $h = 2\pi/m$. More generally Kiš and Névai [20] investigated

$$(6) \qquad |B_{kn}[g](t) - g(t)| \leqq \tilde{M}_{kn}(t) \cdot \omega(g, h), \qquad g \in C_{2\pi},$$

$\tilde{M}_{kn}(t)$ being optimal (for fixed k and n). Numerical evaluations of

$$c_{kn} := \|\tilde{M}_{kn}\| \tag{7}$$

are given in [10] and [7], for example we have

$$c_{1n} = 1 + \frac{1}{2m}\left(\operatorname{cosec}\frac{\pi}{2m} - 1\right) \leqq 1 + \frac{1}{\pi}, \qquad c_{2n} = \frac{5}{4}. \tag{8}$$

Of course (6) and (7) imply that

$$\|B_{kn}[g] - g\| \leqq c_{kn} \cdot \omega(g, h). \tag{9}$$

From a result of Gavriljuk (cf. [5], proof of Theorem 2) we derive that

$$|B_{1n}[g](t) - g(t)| \leqq \frac{1}{2}(1 + B_{1n}(t)) \cdot \omega\left(g, \frac{3}{2}h\right). \tag{10}$$

More generally we shall prove the following

THEOREM 1. *For $k = 0, 1, 2, \ldots$ we have*

$$|B_{kn}[g](t) - g(t)| \leqq \frac{1}{2}(1 + B_{kn}(t)) \cdot \omega\left(g, \frac{k+2}{2}h\right). \tag{11}$$

This estimation in some sense seems to be natural. First for every fixed $t \neq t_i^{(m)}$ it is easy to construct a function $g_t \in C_{2\pi}$, $\|g_t\| = 1$, $g_t(t) = -1$, $g_t \not\equiv$ const., such that $B_{kn}[g_t](t) - g_t(t) = 1 + B_{kn}(t)$. Considering $2 \geqq \omega(g_t, (k+2)h/2)$ it follows that

$$|B_{kn}[g_t](t) - g_t(t)| \geqq \frac{1}{2}(1 + B_{kn}(t)) \cdot \omega\left(g_t, \frac{k+2}{2}h\right), \tag{12}$$

which means that (11) is optimal (for fixed k and n), $t \neq t_i^{(m)}$. Furthermore as a corollary of Theorem 1 we note that

$$\|B_{kn}[g] - g\| \leqq \frac{1}{2}(1 + \|B_{kn}\|) \cdot \omega\left(g, \frac{k+2}{2}h\right). \tag{13}$$

Now comparing this with the estimate given by (9) we prove that (13) is even asymptotically optimal which seems not to be true for (9).

THEOREM 2. *Let k be fixed. For every $\varepsilon > 0$ there exists a function $g_\varepsilon \in C_{2\pi}$ and an infinite sequence $n_1(\varepsilon), n_2(\varepsilon), \ldots$ such that*

$$\|B_{kn}[g_\varepsilon] - g_\varepsilon\| > \frac{1-\varepsilon}{2}(1 + \|B_{kn}\|) \cdot \omega\left(g_\varepsilon, \frac{k+2}{2}h\right), \quad n = n_1, n_2, \ldots. \tag{14}$$

2. Proofs. We omit the superscript (m). To prove Theorem 1 for arbitrary $g \in C_{2\pi}$ it is sufficient to focus our attention on the interval $t_0 \leqq t \leqq t_0 + h/2$. This is an easy consequence of the facts that if $g_h(t) := := g(t+h)$ then likewise $B_{kn}[g_h](t) = B_{kn}[g](t+h)$, and if $f(t_0 - t) := g(t_0 + t)$ then $B_{kn}[f](t_0 - t) = B_{kn}[g](t_0 + t)$. The proof is based upon two lemmas due to Kiš and Névai [10]. Setting $v = (k+1)/2$, k odd, we obtain

LEMMA 1. *We have*

$$s_i(t) \geqq 0, \ -v \leqq i \leqq v, \ (-1)^{i+v} \cdot s_i(t) \geqq 0, \ v < |i| \leqq n \ (t_0 \leqq t \leqq t_0 + h/2).$$

Of course we have from (2) and (3)

$$\sum_{i=-n}^{n} s_i(t) \equiv 1, \tag{15}$$

thus

$$B_{kn}[g](t) - g(t) = \sum_{i=-n}^{-v} + \sum_{i=-v+1}^{v-1} + \sum_{i=v}^{n} [g(t_i) - g(t)] \cdot s_i(t).$$

Now we apply Abel's transformation to the first and the last sum to obtain

$$B_{kn}[g](t) - g(t) = \sum_{i=-n}^{-v-1} [g(t_i) - g(t_{i+1})] \cdot \sigma_i(t) + [g(t_{-v} - g(t)] \cdot \sigma_{-v}(t) +$$

$$+ \sum_{i=-v+1}^{v-1} [g(t_i) - g(t)] \cdot s_i(t) + \sum_{i=v+1}^{n} [g(t_i) - g(t_{i-1})] \cdot \sigma_i(t) + [g(t_v) - g(t)] \cdot \sigma_v(t),$$

where

$$\sigma_i(t) := \begin{cases} \sum_{j=i}^{n} s_j(t) & \text{for } i = 1, 2, \dots, n, \\ \sum_{j=-n}^{i} s_j(t) & \text{for } i = -n, -n+1, \dots, 0. \end{cases} \tag{16}$$

But looking at the largest difference in the arguments we find

$$|g(t_{-v}) - g(t)| \leqq \omega\left(g, \frac{2v+1}{2} h\right),$$

thus

$$|B_{kn}[g](t) - g(t)| \leqq \omega\left(g, \frac{2v+1}{2} h\right) \cdot \left\{ |\sigma_{-v}(t)| + \sum_{i=-v+1}^{v-1} |s_i(t)| + |\sigma_v(t)| + \right. \tag{17}$$

$$\left. + \left(\sum_{i=-n}^{-v-1} + \sum_{i=v+1}^{n} |\sigma_i(t)| \right) \right\}.$$

LEMMA 2. *Putting $\sigma_i(t)$ instead of $s_i(t)$ in Lemma 1, the corresponding statements remain true.*

Lemma 2 together with (15) and (16) allows us to conclude that

$$|\sigma_{-v}(t)| + \sum_{i=-v+1}^{v-1} |s_i(t)| + |\sigma_v(t)| \equiv 1, \tag{18}$$

and

$$\sum_{i=-n}^{-v-1} |\sigma_i(t)| + \sum_{i=v+1}^{n} |\sigma_i(t)| = [-s_{-v-1}(t) - s_{-v-3}(t) - s_{-v-5}(t) - \dots] + \tag{19}$$

$$+[-s_{v+1}(t) - s_{v+3}(t) - s_{v+5}(t) - \dots] = \sum_{\substack{i=-n \\ s_i \leqq 0}}^{n} (-s_i(t)) =$$

$$= \frac{1}{2}\left[\sum_{\substack{i=-n \\ s_i \leqq 0}}^{n} (-s_i(t)) + \sum_{\substack{i=-n \\ s_i > 0}}^{n} s_i(t)\right] + \frac{1}{2}\left[\sum_{\substack{i=-n \\ s_i \leqq 0}}^{n} (-s_i(t)) - \sum_{\substack{i=-n \\ s_i > 0}}^{n} s_i(t)\right] =$$

$$= \frac{1}{2}\sum_{i=-n}^{n} |s_i(t)| - \frac{1}{2}\sum_{i=-n}^{n} s_i(t) = \frac{1}{2}B_{kn}(t) - \frac{1}{2}.$$

From (17), (18), (19) and $v = (k+1)/2$, we have proved Theorem 1, k odd. The case k even can be proved quite similarly, the 'non-alternating part' now consisting of (cf. [10])

$$s_{-\frac{k}{2}}(t) \geqq 0, \quad s_{-\frac{k}{2}+1}(t) \geqq 0, \dots, s_{\frac{k}{2}}(t) \geqq 0, \quad s_{\frac{k}{2}+1}(t) \geqq 0,$$

and the largest difference in the arguments that must be taken into account now being

$$|g\left(t_{\frac{k}{2}+1}\right) - g(t)| \leqq \omega\left(g, \left(\frac{k}{2}+1\right)h\right).$$

To prove Theorem 2 we only have to consider the functionals

$$r_n(g) := \|B_{kn}[g] - g\| \,/\, (\|B_{kn}\| + 1), \qquad q_n(g) := \frac{1}{2}\omega\left(g, \frac{k+2}{2}h\right),$$

which both fulfil the properties of a seminorm p with norm $\|p\| = 1$ in the Banach space $C_{2\pi}$. In particular this means for p that

$$p(g) \geqq 0, \quad p(\alpha \cdot g) = |\alpha| \cdot p(g), \quad p(f+g) \leqq p(f) + p(g),$$

$$\|p\| = \inf\{M | p(g) \leqq M \cdot \|g\|, g \in C_{2\pi}\} = 1.$$

Now Theorem 2 is an easy consequence of the following lemma proved in [8].

LEMMA 3. *Let r_n, q_n be seminorms, $\|r_n\| = \|q_n\| = 1$, defined on a Banach space X and satisfying $r_n(g) \leqq q_n(g)$, $g \in X$. If $q_n(g) \to 0$ $(n \to \infty)$, for every fixed $g \in X$, then given $\varepsilon > 0$ there exists an infinite sequence $n_1, n_2, n_3, \dots$ and an element $\tilde{g} \in X$ such that $q_{n_k}(\tilde{g}) > 0$ and*

$$r_n(\tilde{g}) > (1-\varepsilon) \cdot q_n(\tilde{g}) \qquad (n = n_1, n_2, n_3, \dots).$$

3. Remarks. Theorems 1 and 2 remain valid for discrete operators (3) defined by symmetric kernel functions satisfying the analogue of (15), Lemma 1 and Lemma 2. This is discussed in further details in [9] with emphasis on the case $k = 0$.

The norms $\|B_{0n}\|$ of the trigonometric interpolation operator $B_{0n} = S_n$ at $m = 2n+1$ equidistant nodes are well known, see [3]:

$$\|B_{0n}\| = \frac{1}{m}\left[1 + 2\sum_{i=0}^{n-1} \operatorname{cosec}\left(\frac{2i+1}{m}\frac{\pi}{2}\right)\right] = \frac{1}{m}\sum_{i=0}^{m-1} \cot\left(\frac{2i+1}{m}\frac{\pi}{4}\right).$$

These numbers coincide with the norms $\lambda_{m-1}(T)$ of the algebraic interpolation operator at the Chebyshev nodes T. For the asymptotic expansion of the norms see Günttner [6]. Bernstein [1] has shown that $\|B_{1n}\| < 4/\pi$.

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FB 6 MATHEMATIK
UNIVERSITÄT OSNABRÜCK
POSTFACH 4469
FRG

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A PÁL-TYPE LACUNARY INTERPOLATION PROBLEM

M. R. AKHLAGHI (Prestonsburg)

1. Introduction. For a fixed positive integer $n > 2$, let $w_n(x)$ be a polynomial of degree n with real distinct zeros in $[-1,1]$. If m is any non-negative integer less than n, let

$$(1.1)\quad X_{n,m} := \{x_{k,j} : \omega_n^{(j)}(x_{k,j}) = 0, \quad k = 1,\dots,n-j, \quad j = 0,1,\dots,m\}.$$

We first consider the interpolation problem of finding a polynomial $P(x)$ of degree $(m+1)(n-\frac{m}{2})-1$ such that

$$(1.2)\qquad P^{(j)}(x_{k,j}) = \alpha_{k,j}, \quad k = 1,\dots,n-j, \quad j = 0,1,\dots,m,$$

where $\alpha_{k,j}$'s are arbitrary real numbers. This interpolation problem, which may be called the problem of $(0;1;\dots;m)$ interpolation, is singular for any $m \geqq 1$, i.e., for any positive integer m, $1 \leqq m \leqq n-1$, there exists no unique polynomial $P(x)$ of degree $(m+1)(n-\frac{m}{2})-1$ satisfying (1.2) on the set of nodes $X_{n,m}$. For if $P(x)$ is such a polynomial, then $P(x)+c\omega_n(x)$, for any constant $c \neq 0$, is another such polynomial. To insure the regularity we consider the modified $(0;1;,\dots;m)$ interpolation problem, and add the condition

$$(1.3)\qquad P'(-1) = \alpha_0$$

to (1.2) where α_0 is an arbitrary real number. We shall call this lacunary interpolation problem, the problem of modified $(0;1;\dots;m)$ interpolation.

The case $m = 1$ was studied by Pál [6], where he used the condition $P(a) = 0$, $a \neq x_{k,0}$, $k = 0,1,\dots,n$, instead of (1.3). Eneduanya [1] has proved some convergence results for the case $m = 1$, using conditions (1.2) and (1.3) on $X_{n,1}$, with

$$(1.4)\qquad \omega_n(x) = \Pi_n(x) = -n(n-1)\int_{-1}^{x} P_{n-1}(t)dt = (1-x^2)P'_{n-1}(x),$$

where $P_n(x)$ is the Legendre polynomial of degree n with normalization $P_n(1) = 1$. Eneduanya [2] and Szili [8] have also investigated $(0;1)$ problem for $\omega_n(x) = T_n(x)$ and $H_n(x)$, respectively.

In this paper we study the problem of modified (0; 1; 2) interpolation on $X_{n,2}$, for $\omega_n(x) = \Pi_n(x)$. Section 2 deals with the statements of the main results and some preliminaries. In Section 3 we prove the regularity of this problem and in Section 4 we obtain the fundamental polynomials. Section 5 is devoted to the convergence problem.

2. Preliminaries and main results. Let $x_k = x_{k,0}$, $1 \leqq k \leqq n$; $\xi_k = x_{k,1}$, $1 \leqq k \leqq n-1$, and $x_{k,2}$, $k = 1, \dots, n-2$, be the zeros of $\Pi_n(x)$, $\Pi'_n(x)$, and $\Pi''_n(x)$, respectively. The following relations are valid:

(2.1)
$$-1 = x_1 < \xi_1 < x_2 < \xi_2 < \cdots < \xi_{n-2} < x_{n-1} < \xi_{n-1} < x_n = 1, \quad n = 2, 3, \dots .$$

It is known that the polynomials $P_{n-1}(x)$ and $\Pi_n(x)$ satisfy the differential equations

$$(1 - x^2)P''_{n-1}(x) - 2xP'_{n-1}(x) + n(n-1)P_{n-1}(x) = 0 \tag{2.2}$$

and

$$(1 - x^2)\Pi''_n(x) + n(n-1)\Pi_n(x) = 0 \tag{2.3}$$

respectively. (2.3) leads to

$$x_{k,2} = x_{k+1}, \qquad k = 1, 2, \dots, n-2. \tag{2.4}$$

Let $\ell_k(x)$ and $\ell^*_k(x)$ denote the fudamental polynomials of Lagrange interpolation such that

$$\begin{cases} \ell_k(x_j) = \delta_{kj} = \begin{cases} 0 & k \neq j \\ 1 & k = j \end{cases} & (k, j = 1, \dots, n) \\ \ell^*_k(\xi_j) = \delta_{kj} & (k, j = 1, \dots, n-1). \end{cases} \tag{2.5}$$

These polynomials can be represented as

$$\ell_k(x) = \frac{\Pi_n(x)}{(x - x_k)\Pi'_n(x_k)} \tag{2.6}$$

and

$$\ell^*_k(x) = \frac{\Pi'_n(x)}{(x - \xi_k)\Pi''_n(\xi_k)} = \frac{P_{n-1}(x)}{(x - \xi_k)P'_{n-1}(\xi_k)}. \tag{2.6a}$$

We recall that

$$\begin{cases} P_{n-1}(1) = 1 = (-1)^{n-1}P_{n-1}(-1), \\ P'_{n-1}(1) = \frac{n(n-1)}{2} = (-1)^n P'_{n-1}(-1), \\ P''_{n-1}(1) = \frac{(n+1)n(n-1)(n-2)}{8} = (-1)^{n-1}P''_{n-1}(-1). \end{cases} \tag{2.7}$$

We shall require the following:

$$(2.8)\qquad \begin{cases} \Pi_n'(1) = -n(n-1) = (-1)^{n+1}\Pi_n'(-1) \\ \Pi_n''(1) = -\frac{n^2(n-1)^2}{2} = (-1)^n\Pi_n''(-1). \end{cases}$$

From $(x-x_k)\ell_k(x) = \frac{\Pi_n(x)}{\Pi_n'(x_k)}$, on differentiating once and twice we get

$$(2.9)\qquad \ell_1'(1) = \frac{1}{2}(-1)^{n+1} = -\ell_n'(-1),$$

$$(2.10)\qquad \ell_1'(-1) = -\frac{n(n-1)}{4} = -\ell_n'(1)$$

respectively. The known orthogonal property

$$(2.11)\qquad \int_{-1}^{1} P_k(x)P_j(x)dx = \frac{2}{2k+1}\delta_{k,j}$$

and the known identities

$$\frac{P_{n-1}'(x)}{x-x_k} = \frac{-1}{P_{n-1}(x_k)}\sum_{j=2}^{n-1}\frac{2j-1}{2j(j-1)}P_{j-1}'(x_k)P_{j-1}'(x), \qquad 2 \leqq k \leqq n-1$$

and

$$\frac{P_{n-1}(x)}{x-\xi_k} = \frac{1}{\Pi_n(\xi_k)}\sum_{j=1}^{n-1}(2j-1)P_{j-1}(\xi_k)P_{j-1}(x), \qquad 1 \leqq k < n-1$$

lead to

$$(2.12)\qquad \int_{-1}^{1}\frac{P_{n-1}(t)P_{n-1}'(t)}{t-x_k}dt = 0, \qquad 2 \leqq k \leqq n-1$$

and

$$(2.13)\qquad \int_{-1}^{1}\frac{P_{n-1}(t)P_{n-1}'(t)}{t-\xi_k}dt = \frac{2}{1-\xi_k^2}, \qquad 1 \leqq k \leqq n-1.$$

Our main results are:

THEOREM 1. *The modified* $(0;1;2)$ *interpolation on* $X_{n,2}$ *with* $\omega_n(x) = \Pi_n(x)$ *is regular.*

If we denote the fundamental polynomials of modified $(0;1;2)$ interpolation by $L_{k,0}(x)$, $L_{k,1}(x)$ and $L_{k,2}(x)$, then we shall prove

THEOREM 2. *The fundamental polynomials of modified* $(0;1;2)$ *interpolation are given by*

$$L_{k,0}(x) = A_{n-k+1}(x) + \Pi_n(x) g_k(x), \qquad 1 \leqq k \leqq n, \tag{2.14}$$

where $A_k(x)$, $1 \leqq k \leqq n$ *are the explicit formulae of the fundamental polynomials in the paper of Eneduanya* [1], *and*

$$\begin{cases} g_1(x) = \int\limits_{-1}^{x} \frac{\Pi_n'(x)}{n^2(n-1)^2}\left(\frac{1-\ell_1(x)+(x+1)\ell_1'(-1)}{(1+x)^2} + \frac{n(n-1)}{2}\frac{1-\ell_1(x)}{1+x}\right), \\ g_k(x) = \int\limits_{-1}^{x} \frac{\Pi_n'(x)}{\Pi'^2(x_k)}\left(\frac{1-\ell_k(x)}{(x-x_k)^2} - \frac{n(n-1)}{3(1-x_k^2)}\ell_k(x)\right), \quad 2 \leqq k \leqq n-1, \\ g_n(x) = \int\limits_{-1}^{x} \frac{\Pi_n'(x)}{n^2(n-1)^2}\left(\frac{1-\ell_n(x)+(x+1)\ell_n'(1)}{(1-x)^2} + \frac{n(n-1)}{2}\frac{1-\ell_n(x)}{1-x}\right), \end{cases} \tag{2.15}$$

$$L_{k,1}(x) = \frac{\Pi_n(x)}{(1-\xi_k^2)P_{n-1}'^3(\xi_k)} \int\limits_{-1}^{x} \frac{P_{n-1}(t)P_{n-1}'(t)}{t-\xi_k}dt, \qquad 1 \leqq k \leqq n-1 \tag{2.16}$$

and

$$L_{k,2}(x) = \frac{(1-x_k^2)\Pi_n(x)}{2n^2(n-1)^2 P_{n-1}^3(x_k)} \int\limits_{-1}^{x} \frac{P_{n-1}(t)P_{n-1}'(t)}{t-x_k}dt, \qquad 2 \leqq k \leqq n-1. \tag{2.17}$$

For $f \in C^{(r)}([-1,1])$, $r \geqq 2$, set

$$Q_{3n-3}(x;f) := \sum_{k=1}^{n} f(x_k)L_{k,0}(x) + \sum_{k=1}^{n-1} f'(\xi_k)L_{k,1}(x) + \tag{2.18}$$

$$+ \sum_{k=2}^{n-1} f''(x_k)L_{k,2}(x) + f'(-1)\frac{\Pi_n(x)}{\Pi_n'(-1)}.$$

We shall prove

THEOREM 3. *If $f \in C^{(r)}([-1,1])$, $r \geqq 2$, then for every $x \in [-1,1]$ and $n \geqq \frac{4}{3}(r+2)$ we have*

$$(2.20) \qquad |f(x) - Q_n(x;f)| = O(1)n^{\frac{5}{2}-r}\log n\,\omega(\frac{1}{n};f^{(r)}),$$

where $\omega(\cdot;f^{(r)})$ is the modulus of continuity of $f^{(r)}(x)$.

REMARK. For $r=2$, the Theorem 3 implies convergence only if

$$\sqrt{n}\log n\,\omega(\frac{1}{n};f'') = o(1).$$

This relation obviously holds if for example $f'' \in \mathrm{Lip}\,\alpha$, $\frac{1}{2} < \alpha \leqq 1$.

3. Proof of Theorem 1. Set

$$Q(x) = Q_{3n-3}(x) = \Pi_n(x)q(x), \qquad \deg q(x) \leqq 2n-3.$$

We shall show that $Q(x) \equiv 0$ is the only polynomial of degree $3n-3$ satisfying (1.2) and (1.3) with

$$\alpha_0 = 0 \quad \text{and} \quad \alpha_{k,j} = Q^{(j)}(x_{k,j}) = 0, \quad k = 1,\dots,n-j, \quad j = 0,1,2.$$

$Q(x)$ satisfies $Q(x_{k,0}) = Q(x_k) = 0$. $Q'(x_{k,1}) = Q'(\xi_k) = 0$, $1 \leqq k \leqq n-1$ and $Q''(x_{k,2}) = Q''(x_k) = 0$, $1 \leqq k \leqq n-2$ implies $q'(\xi_k) = 0$, $1 \leqq k \leqq n-1$ and $q'(x_k) = 0$, $2 \leqq k \leqq n-1$ respectively. Hence $q'(x) = cP_{n-1}(x)P'_{n-1}(x)$. But $\deg q(x) \leqq 2n-3$, therefore, $c = 0$ and hence $q(x) = c_1$ for some constant c_1. Using (1.3) we get $c_1 = 0$. Therefore $Q(x) \equiv 0$ and this completes the proof of Theorem 1. □

4. Explicit formulae for $\{L_{k,0}(x)\}_{k=1}^{n}$, $\{L_{k,1}(x)\}_{k=1}^{n-1}$ and $\{L_{k,2}(x)\}_{k=2}^{n-1}$. Let us denote the fundamental polynomials of the modified (0;1;2) interpolation problem by $\{L_{k,0}(x)\}_{k=1}^{n}$, $\{L_{k,1}(x)\}_{k=1}^{n-1}$ and $\{L_{k,2}(x)\}_{k=2}^{n-1}$ respectively. Every polynomial $P(x)$ of degree $3n-3$ has a representation of the form

$$(4.1) \qquad P(x) = \sum_{k=1}^{n} P(x_k)L_{k,0}(x) + \sum_{k=1}^{n-1} P'(\xi_k)L_{k,1}(x) + \\ + \sum_{k=2}^{n-1} P''(x_k)L_{k,2}(x) + P'(-1)\frac{\Pi_n(x)}{\Pi'_n(-1)}.$$

PROOF OF THEOREM 2. (i) The fundamental polynomials $L_{k,0}(x)$, $1 \leqq k \leqq n$ are determined by the condition

$$(4.2) \qquad \begin{cases} L_{k,0}(x_j) = \delta_{k,j}, & j = 1,\dots,n \\ L'_{k,0}(\xi_j) = 0, & j = 1,\dots,n-1 \\ L''_{k,0}(x_j) = 0, & j = 2,\dots,n-1 \\ L'_{k,0}(-1) = 0. \end{cases}$$

Set

$$(4.3)\qquad L_{k,0}(x) = A_{n-k+1}(x) + \Pi_n(x) r_k(x), \qquad \deg r_k(x) \leqq 2n-3$$

where $A_k(x)$, $1 \leqq k \leqq n$ are the explicit formulae of the fundamental polynomials given in [1]. (These are used with suitable corrections, since there are some misprints in the text in [1]. It may be remarked that our notations are slightly different from his. Thus while we are listing nodes as $-1 = x_1 < x_2 < \ldots < x_{n-1} < x_n =!$, the nodes he uses are numbered in the reverse order.) These polynomials satisfy

$$(4.4)\qquad \begin{cases} A_{n-k+1}(x_j) = \delta_{kj}, & k,j = 1,\ldots,n, \\ A'_{n-k+1}(\xi_j) = 0, & k = 1,\ldots,n, \quad j = 0,1,\ldots,n-1, \\ A'_{n-k+1}(-1) = 0, & k = 1,\ldots,n. \end{cases}$$

$L_{k,0}(x)$ satisfies the conditions (4.2), if

$$(4.5)\qquad r'_k(\xi_j) = 0, \qquad j = 1,\ldots,n-1,$$

$$(4.6)\qquad r'_k(x_j) = -\frac{A''_{n-k+1}(x_j)}{2\Pi'_n(x_j)}, \qquad j = 2,\ldots,n-1,$$

$$(4.7)\qquad r_k(-1) = 0.$$

For $k = n$, we see from [1], that

$$A''_1(x_j) = \frac{-2\Pi'^2_n(x_j)}{(1-x_j)^2\Pi'^2_n(1)}\left(1 + \frac{n(n-1)}{2}(1-x_j)\right),$$

for $2 \leqq k \leqq n-1$

$$A''_{n-k+1}(x_j) = \begin{cases} \frac{-2\Pi'^2_n(x_j)}{(x_k-x_j)^2\Pi'^2_n(x_k)}, & j \neq k \\ \frac{2n(n-1)}{3(1-x_k^2)}, & j = k \end{cases},$$

and, for $k = 1$, we obtain

$$A''_n(x_j) = \frac{-2\Pi'^2_n(x_j)}{(1+x_j)^2\Pi'^2_n(-1)}\left(1 + \frac{n(n-1)}{2}(1+x_j)\right).$$

From (4.6) it follows that

$$(4.8)\qquad r'_k(x_j) = \begin{cases} \frac{\Pi'_n(x_j)}{n^2(n-1)^2}\left(\frac{1}{(1+x_j)^2} + \frac{n(n-1)}{2}\frac{1}{1+x_j}\right), & k = 1, \\ \begin{cases} \frac{\Pi'_n(x_j)}{(x_k-x_j)^2\Pi'^2_n(x_k)}, & k \neq j \\ \frac{-n(n-1)}{3(1-x_k^2)\Pi'_n(x_k)}, & k = j \end{cases} & k = 2,\ldots,n-1 \\ \frac{\Pi'_n(x_j)}{n^2(n-1)^2}\left(\frac{1}{(1-x_j)^2} + \frac{n(n-1)}{2}\frac{1}{1-x_j}\right), & k = n. \end{cases}$$

From using (4.5) and (4.8) we get (2.15). From (2.15) and (4.7), the formula (2.14) for $L_{k,0}(x)$, $1 \leqq k \leqq n$, is now evident.

(ii) Fundamental polynomials $L_{k,1}(x)$, $1 \leqq k \leqq n-1$ are determined by the conditions

$$(4.9)\qquad \begin{cases} L_{k,1}(x_j) = 0, & j = 1, \dots, n, \\ L'_{k,1}(\xi_j) = \delta_{k,j}, & j = 1, \dots, n-1, \\ L''_{k,1}(x_j) = 0, & j = 2, \dots, n-1, \\ L'_{k,1}(-1) = 0. \end{cases}$$

Set, for $1 \leqq k \leqq n-1$,

$$(4.10)\qquad L_{k,1}(x) = \Pi_n(x) s_k(x), \qquad \deg s_k(x) \leqq 2n-3.$$

$L_{k,1}(x)$ satisfies the conditions (4.9), if

$$(4.11)\qquad s'_k(\xi_j) = \frac{\delta_{kj}}{\Pi_n(\xi_k)}, \qquad j = 1, \dots, n-1,$$

$$(4.12)\qquad s'_k(x_j) = 0, \qquad j = 2, \dots, n-1,$$

$$(4.13)\qquad s_k(-1) = 0.$$

From (4.11), (4.12) it follows that

$$s'_k(x) = \frac{\Pi'_n(x) P'_{n-1}(x)}{(x-\xi_k)\Pi''_n(\xi_k) P'_{n-1}(\xi_k) \Pi_n(\xi_k)} = \frac{1}{(1-\xi_k^2) P'^3_{n-1}(\xi_k)} \cdot \frac{P_{n-1}(x) P'_{n-1}(x)}{x - \xi_k}.$$

Using (4.13), we get

$$(4.14)\qquad s_k(x) = \frac{1}{(1-\xi_k^2) P'^3_{n-1}(\xi_k)} \int_{-1}^{x} \frac{P_{n-1}(t) P'_{n-1}(t)}{t - \xi_k}\, dt.$$

From (4.10) and (4.14) we get the formula (2.16) for $L_{k,1}(x)$, $1 \leqq k \leqq n-1$. Proof of (2.17) is very similar to the proof of (2.16). We omit the details.

5. Some estimates. To find some estimates for the fundamental polynomials we need the following facts (see [4], [1]):

$$(5.1)\qquad |P_n(x)| \leqq 1,$$

$$(5.2)\qquad |\Pi_n(x)| \leqq \sqrt{\frac{2n}{\pi}},$$

$$(5.3)\qquad \begin{cases} c_1\frac{k}{n} \leqq \sqrt{1-x_k^2} \leqq c_2\frac{k}{n}, & 2 \leqq k \leqq [\frac{n}{2}] \\ c_1\frac{n-k}{n} \leqq \sqrt{1-x_k^2} \leqq c_2\frac{n-k}{n}, & [\frac{n}{2}] < k \leqq n-1, \end{cases}$$

$$(5.4)\qquad |P_{n-1}(x_k)| \geqq \begin{cases} \frac{1}{\sqrt{8\pi k}}, & 2 \leqq k \leqq [\frac{n}{2}] \\ \frac{1}{\sqrt{8\pi(n-k)}}, & [\frac{n}{2}] < k \leqq n-1, \end{cases}$$

$$(5.5)\qquad \ell_k^2(x) \leqq \sum_{k=1}^{n} \ell_k^2(x) \leqq 1, \qquad x \in [-1,1],$$

$$(5.6)\qquad \sum_{k=1}^{n-1} \int_{-1}^{x} \ell_k^{*2}(t)dt \leqq \sum_{k=1}^{n-1} \int_{-1}^{1} \ell_k^{*2}(t)dt \leqq 2,$$

$$(5.7)\qquad |\Pi_n(\xi_k)| \geqq c\sqrt{k}, \qquad 1 \leqq k \leqq n-1,$$

$$(5.8)\qquad \sum_{k=2}^{n-1} |A_k(x)| = O(n),$$

where c_1, c_2 and c are positive constants independent of n and k. Further [5]

$$(5.9)\qquad \sum_{k=1}^{n-1} |\ell_k^*(x)| = O(n).$$

From Theorem 7.3.1 of Szegő [7] one may conclude that

$$(1-\xi_k^2)P_{n-1}'^2(\xi_k) \geqq P_{n-1}'^2(0) > \frac{n}{4}.$$

We now estimate $\lambda_0(x) = \sum_{k=2}^{n-1} |L_{k,0}(x)|$, $\lambda_1(x) = \sum_{k=1}^{n-1} |L_{k,1}(x)|$ and $\lambda_2(x) = \sum_{k=2}^{n-1} |L_{k,2}(x)|$.

LEMMA 5.1. *For* $-1 \leqq x \leqq 1$ *we have*

$$\lambda_0(x) = O(n^2\sqrt{n}\log n). \tag{5.11}$$

PROOF. It is enough to prove this inequality for $x \neq x_k$, $k = 1, \dots, n$ since (2.14) implies $\lambda_0(x_k) = 1$, $k = 2, \dots, n-1$ and $\lambda_0(\pm 1) = 0$. From (2.14), (2.15) and (5.8), for $2 < k < n-1$, we have

$$\lambda_0(x) \leqq O(n) + |\Pi_n(x)| \sum_{k=2}^{n-1} (I_{k,1}(x) + I_{k,2}(x)), \tag{5.12}$$

where

$$I_{k,1}(x) = \left| \int_{-1}^{x} \frac{\Pi_n'(t)}{\Pi_n'^2(x_k)} \frac{1-\ell_k(t)}{(t-x_k)^2}\, dt \right|$$

and

$$I_{k,2}(x) = \frac{n(n-1)}{3(1-x_k^2)} \left| \int_{-1}^{x} \frac{\Pi_n'(t)}{\Pi_n'^2(x_k)} \ell_k(t)\, dt \right|.$$

From (1.4) and (5.5), for $-1 < x < x_k$, we have

$$I_{k,1}(x) \leqq \frac{2}{n(n-1)P_{n-1}^2(x_k)} \int_{-1}^{x} \frac{1}{(t-x_k)^2}\, dt =$$

$$= \frac{2}{n(n-1)P_{n-1}^2(x_k)} \left(\frac{1}{x_k - x} - \frac{1}{1+x_k} \right)$$

and hence, for $-1 < x < x_k$, we get

$$I_{k,1}(x) \leqq \frac{1}{n(n-1)P_{n-1}^2(x_k)} \frac{1}{x_k - x}.$$

For $x_k < x < 1$, since $I_{k,1}(1) = 0$ by (2.11), and since $\int_x^1 \frac{1}{(t-x_k)^2} dt < \frac{1}{x-x_k}$ we obtain the same estimate as above with $x_k - x$ instead of $x - x_k$. Hence for $-1 \leqq x \leqq 1$, we have

$$I_{k,1}(x) \leqq \frac{2}{n(n-1)P_{n-1}^2(x_k)} \frac{1}{|x - x_k|}. \tag{5.13}$$

On using (1.4), (5.1) and (5.5), for $-1 \leqq x \leqq 1$, we also get

$$I_{k,2}(x) \leqq \frac{1}{(1-x_k^2)P_{n-1}^2(x_k)}. \tag{5.14}$$

Therefore, using (5.12), (5.13) and (5.14) we see that

$$\lambda_0(x) \leqq O(n) + \frac{2}{n(n-1)} \sum_{k=2}^{n-1} \frac{|\Pi_n(x)|}{|x - x_k| P_{n-1}^2(x_k)} + |\Pi_n(x)| \sum_{k=2}^{n-1} \frac{1}{(1 - x_k^2) P_{n-1}^2(x_k)}.$$

Moreover (1.4), (2.6), (5.2), (5.3) and (5.4) show that

$$\lambda_0(x) \leqq O(n) + 2 \sum_{k=2}^{n-1} \left| \frac{\ell_k(x)}{P_{n-1}(x_k)} \right| + O(n^2 \sqrt{n} \log n). \tag{5.15}$$

The Schwarz inequality and relations (5.4) and (5.5) imply that

$$\sum_{k=2}^{n-1} \left| \frac{\ell_k(x)}{P_{n-1}(x_k)} \right| \leqq \left(\sum_{k=2}^{n-1} \frac{1}{P_{n-1}^2(x_k)} \right)^{1/2} = O(n)$$

and hence, from (5.15), we get (5.11). □

LEMMA 5.2. *For $-1 \leqq x \leqq 1$, we have*

$$\lambda_1(x) = O(n). \tag{5.16}$$

PROOF. We first estimate $\int_{-1}^{x} \frac{P_{n-1}(t) P_{n-1}'(t)}{t - \xi_k} \, dt$. By partial integration, for $-1 < x < \xi_k$, we have

$$\int_{-1}^{x} \frac{P_{n-1}(t) P_{n-1}'(t)}{t - \xi_k} dt = \frac{P_{n-1}^2(t)}{t - \xi_k} \bigg|_{-1}^{x} - \int_{-1}^{x} \frac{P_{n-1}(t)(t - \xi_k) - P_{n-1}(t)}{(t - \xi_k)^2} P_{n-1}(t) dt$$

so that (2.7) yields

$$\int_{-1}^{x} \frac{P_{n-1}(t) P_{n-1}'(t)}{t - \xi_k} \, dt = \frac{P_{n-1}^2(x)}{2(x - \xi_k)} + \frac{1}{2(1 + \xi_k)} + \frac{1}{2} \int_{-1}^{x} \frac{P_{n-1}^2(t)}{(t - \xi_k)^2} \, dt. \tag{5.17}$$

From (2.6a) and (5.6), we have

$$\int_{-1}^{x} \frac{P_{n-1}^2(t)}{(t - \xi_k)^2} \, dt = P_{n-1}'^2(\xi_k) \int_{-1}^{x} \ell_k^{*2} \, dt \leqq 2 P_{n-1}'^2(\xi_k) \tag{5.18}$$

so that in view of (2.6a) (5.17), (5.18), for $-1 < x < \xi_k$, we obtain

$$\left| \int_{-1}^{x} \frac{P_{n-1}(t) P_{n-1}'(t)}{(t - \xi_k)} \, dt \right| \leqq \frac{1}{2} |P_{n-1}'(\xi_k) \ell_k^*(x)| + \frac{1}{2(1 + \xi_k)} + P_{n-1}'^2(\xi_k).$$

For $\xi_k < x < 1$, using (2.13), we have

$$\int_{-1}^{x} \frac{P_{n-1}(t)P'_{n-1}(t)}{t-\xi_k}\,dt = \frac{2}{1-\xi_k^2} - \int_{x}^{1} \frac{P_{n-1}(t)P'_{n-1}(t)}{t-\xi_k}\,dt$$

where the last integral does not exceed $\frac{1}{2}|P'_{n-1}(\xi_k)\ell_k^*(x)| + \frac{1}{2(1-\xi_k)} + P'^2_{n-1}(\xi_k)$. Hence for all $x \neq \xi_k$,

$$\text{(5.19)}\qquad \Big|\int_{-1}^{x} \frac{P_{n-1}(t)P'_{n-1}(t)}{t-\xi_k}\,dt\Big| \leqq \frac{3}{1-\xi_k^2} + \frac{1}{2}|P'_{n-1}(\xi_k)\ell_k^*(x)| + P'^2_{n-1}(\xi_k).$$

Applying (2.16) and (5.19), for $-1 \leqq x \leqq 1$, we obtain

$$|L_{k,1}(x)| \leqq 3\frac{|\Pi_n(x)|}{(1-\xi_k^2)P'^2_{n-1}(\xi_k)|\Pi_n(\xi_k)|} + \left|\frac{\Pi_n(x)}{\Pi_n(\xi_k)}\right| + \frac{1}{2}\frac{|\Pi_n(x)\ell_k^*(x)|}{(1-\xi_k^2)P'^2_{n-1}(\xi_k)}.$$

Therefore we obtain (5.16), for $-1 \leqq x \leqq 1$, from (5.2), (5.7), (5.9) and (5.10). □

LEMMA 5.3. *For $-1 \leqq x \leqq 1$, we have*

$$\text{(5.20)}\qquad \lambda_2(x) = O\Big(\frac{1}{\sqrt{n}}\Big).$$

PROOF. It is sufficient to verify (5.20) only in the case when $x \neq x_k$, $k = 2, \dots, n-1$. Let $-1 < x < x_k$, we first estimate

$$I_k(x) = \int_{-1}^{x} \frac{P_{n-1}(t)P'_{n-1}(t)}{t-x_k}\,dt.$$

By partial integration, we get

$$2I_k(x) = \frac{P^2_{n-1}(x)}{x-x_k} + \frac{1}{1+x_k} + \int_{-1}^{x} \frac{P^2_{n-1}(t)}{(t-x_k)^2}\,dt.$$

The absolute value of the last term does not exceed

$$\int_{-1}^{x} \frac{1}{(t-x_k)^2}\,dt = \frac{1}{x_k-x} - \frac{1}{1+x_k}.$$

Therefore

$$|I_k(x)| \leqq \frac{1}{x_k-x}.$$

For $x_k < x < 1$, by (2.12), we have

$$I_k(x) = -\int_x^1 \frac{P_{n-1}(t)P'_{n-1}(t)}{t - x_k}\,dt$$

and the integral, on the right, does not exceed $\frac{1}{|x-x_k|}$. Hence for all $x \neq x_k$,

$$|I_k(x)| \leqq \frac{1}{|x - x_k|}. \tag{5.21}$$

Using (2.17), (5.21) and (2.6), we obtain

$$|L_{k,2}(x)| \leqq \frac{(1 - x_k^2)|\ell_k(x)|}{n(n-1)P_{n-1}^2(x_k)}.$$

Therefore

$$\lambda_2(x) = \sum_{k=2}^{n-1} |L_{k,2}(x)| \leqq \frac{1}{2n(n-1)} \sum_{k=2}^{n-1} \frac{|\ell_k(x)|}{P_{n-1}^2(x_k)}.$$

Applying Schwarz inequality and relations (5.4) and (5.5), we get

$$\lambda_2(x) \leqq \frac{1}{2n(n-1)} \Big(\sum_{k=2}^{n-1} \ell_k^2(x)\Big)^{1/2} \Big(\sum_{k=2}^{n-1} \frac{1}{P_{n-1}^2(x_k)}\Big)^{1/2} \leqq$$

$$\leqq \frac{1}{2n(n-1)} O(n^{3/2}) = O\Big(\frac{1}{\sqrt{n}}\Big),$$

which completes the proof of Lemma 5.3. □

Proof of Theorem 3. If $f(x) \in C^{(r)}[-1,1]$, then by a result of Gopengaus [3], there exists a polynomial $G_m(x;f)$ of degree $m \geqq 4r+5$, such that for all $x \in [-1,1]$

(5.22)

$$|f^{(s)}(x) - G_m^{(s)}(x;f)| = O(1)\Big(\frac{\sqrt{1-x^2}}{m}\Big)^{r-s} \omega\Big(\frac{\sqrt{1-x^2}}{m}; f^{(r)}\Big), \quad s = 0,1,\ldots,r,$$

where $\omega(\cdot; f^{(r)})$ is the modulus of continuity of the function $f^{(r)}(x)$. From (5.22), we see that

$$f(1) - G_{3n-3}(1;f) = f(-1) - G_{3n-3}(-1;f) = f'(-1) - G'_{3n-3}(-1;f) = 0.$$

Therefore, for $r \geqq 2$ and $3n - 3 \geqq 4r + 5$, using (2.18) we conclude that

$$|f(x) - Q(x;f)| \leqq |f(x) - G_{3n-3}(x;f)| + \sum_{k=2}^{n-1} |G_{3n-3}(x_k;f) - f(x_k)||L_{k,0}(x)| +$$

$$+\sum_{k=1}^{n-1}|G'_{3n-3}(\xi_k;f)-f'(\xi_k)||L_{k,1}(x)|+\sum_{k=2}^{n-1}|G''_{3n-3}(x_k;f)-f''(x_k)||L_{k,2}(x)|,$$

for $1 \leqq x \leqq 1$. Using (5.11), (5.16), (5.20) and (5.22) we see that

$$|f(x)-Q(x;f)|=O(1)\Big(\frac{\sqrt{1-x^2}}{3n-3}\Big)^r\omega\Big(\frac{\sqrt{1-x^2}}{3n-3};f^{(r)}\Big)+$$

$$+O(1)\frac{\sqrt{n}\log n}{n^{r-2}}\omega\Big(\frac{\sqrt{1-x^2}}{3n-3};f^{(r)}\Big)+O(1)\frac{1}{n^{r-2}}\omega\Big(\frac{\sqrt{1-x^2}}{3n-3};f^{(r)}\Big)+$$

$$+O(1)\frac{1}{n^{r-\frac{3}{2}}}\omega\Big(\frac{\sqrt{1-x^2}}{3n-3};f^{(r)}\Big)=O(1)\frac{\sqrt{n}\log n}{n^{r-2}}\omega\Big(\frac{\sqrt{1-x^2}}{3n-3};f^{(r)}\Big),$$

for $n \geqq \frac{4r+8}{3}$. Since $\omega(x;f)$ is a non-decreasing function we obtain (2.20). Thus Theorem 3 is proved. □

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DIVISION OF PHYSICAL SCIENCES
AND RELATED TECHNOLOGIES
PRESTONSBURG COMMUNITY COLLEGE
UNIVERSITY OF KENTUCKY
PRESTONSBURG, KY 41653
U.S.A.

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GRADED RADICALS OF GRADED RINGS

M. BEATTIE (Sackville) and P. STEWART (Halifax)

Let G be a group and λ a radical property in the category of associative rings. Using the generalized smash product of [1], we introduce a method for defining a corresponding radical property λ_{ref} in the category of associative G-graded rings and grade-preserving ring homomorphisms. We investigate the properties of these new radicals and compare them with graded radicals which have been previously studied.

For $\lambda = J$, the Jacobson radical, λ_{ref} is the usual graded Jacobson radical. (See for example [2], [7].) If λ is the prime radical, then for G finite and R a G-graded ring, $\lambda_{\text{ref}}(R)$ is the graded prime radical of [3], i.e. the intersection of the graded prime ideals of R. However, this intersection of graded ideals may be properly contained in $\lambda_{\text{ref}}(R)$ for G infinite. If λ is the strongly prime radical, then λ_{ref} is the graded strongly prime radical of [8] for G finite, but again may properly contain this ideal for G infinite. We also discuss the cases of λ equal to the Levitzski, Brown–McCoy and von Neumann regular radicals, and compare λ_{ref} to suitable intersections of graded ideals.

1. Preliminaries and definition of the reflected radical

Let G be a group with identity e. A ring R is called G-graded if $R = \bigoplus_{g\in G} R_g$, and $R_gR_h \subseteqq R_{gh}$ for all $g,h \in G$. The elements of R_g are called the homogeneous elements of grade g. If $r \in R$, r_g denotes the gth homogeneous component of r. If $R_gR_h = R_{gh}$ for all $g,h \in G$, then R is called strongly graded. A left R-module M is G-graded if $M = \bigoplus_{g\in G} M_g$, and $R_gM_h \subseteqq M_{gh}$ for all $g,h \in G$. Ideals of R are called G-graded if they are graded left and right submodules of R. (Ideal will always mean two-sided ideal.) A graded ideal P of R is called a graded prime ideal of R if $IJ \subseteqq P$ for graded ideals I,J of R implies that $I \subseteqq P$ or $J \subseteqq P$. For I any ideal of a graded ring R, I_G will denote the largest graded ideal of R contained in I, i.e. I_G is the graded ideal generated by the homogeneous elements of R contained in I.

For R a G-graded ring, we define the associative ring $R\#G^*$ to be the left R-module $\oplus Rp_g$, $g \in G$, with multiplication defined by $(rp_g)(sp_h) = rs_{gh^{-1}}\varrho_h$ [1]. If R has an identity, 1, and G is finite, then $R\#G^*$ is also a ring with identity, namely $\sum_{g\in G} p_g$ where we write p_g for $1p_g$. The group G

acts as a group of automorphisms on the right of $R\#G^*$ by $(rp_h)^g = rp_{hg}$.

If J is a graded ideal of R, then we define $J\#G^*$ to be all finite sums of elements xp_g, $x \in J$, $g \in G$. $J\#G^*$ is an ideal of $R\#G^*$ invariant under the action of G.

If I is an ideal of $R\#G^*$, define ideals I_R and $I^\downarrow$ of R by

$$I_R = \{r : r \in R, rp_g \in I \text{ for all } g \in G\},$$

and

$$I^\downarrow = (I_R)_G,$$

i.e., $I^\downarrow$ is the largest graded ideal in I_R. Note that $I^\downarrow$ contains any graded ideal L such that $L\#G^* \subseteq I$, for if K is a graded ideal of R with $K\#G^* \subseteq I$, then $K \subseteq I_R$, and therefore $K \subseteq (I_R)_G = I^\downarrow$. If P is a prime ideal of $R\#G^*$, $P^\downarrow$ is a graded prime of R since if $IJ \subseteq P^\downarrow$ where I and J are graded ideals, $(I\#G^*)(J\#G^*) \subseteq P$. Note that if J is a graded ideal of R, then $(J\#G^*)^\downarrow = J$. If R has an identity, then $x = \sum_{i=1}^{t} r_i p_{g_i} \in I$ implies $xp_{g_i} = r_i p_{g_i} \in I$ for each i. Also $rp_g \in I$ for all $g \in G$ implies that $p_{hg}(rp_g) = r_h p_g \in I$ for all $g, h \in G$, and therefore I_R is a graded ideal of R, i.e. $I_R = I^\downarrow$. Thus if I is invariant under the action of G on $R\#G^*$, $I = I^\downarrow\#G^*$.

The next lemma will show that this equality holds for any ring R if I is a G-invariant intersection of prime ideals or a radical.

A graded ring R without identity may be embedded in a graded ring R^1 with identity in the following way. Let $R^1 = R \times Z$, with addition and multiplication defined by

$$(r,n) + (s,m) = (r+s, n+m) \quad \text{and} \quad (r,n)(s,m) = (rs + mr + ns, nm)$$

for $r, s \in R$ and $n, m \in Z$. Now define

$$(R^1)_e = \{(r,n) : r \in R_e, n \in Z\}$$

and

$$(R^1)_g = \{(r,0) : r \in R_g\}$$

for g different from e. $R \cong R \times \{0\}$ is a graded ideal of R^1, and $R\#G^*$ is an ideal of $R^1\#G^*$.

Throughout this paper, for R a G-graded ring not necessarily with identity, R^1 will be used to denote the G-graded ring with identity containing R as a graded ideal as constructed above.

LEMMA 1.1. *Let R be a G-graded ring and I an ideal of $R\#G^*$. Suppose that I is either an intersection of prime ideals of $R\#G^*$ or $\lambda(R\#G^*)$ for λ a radical in the category of associative rings. Then I_R is a graded ideal of R and if I is G-invariant, $I = I^\downarrow\#G^*$.*

PROOF. Embed R in a graded ring R^1 with identity as described above. Then $R\#G^* \subseteq R^1\#G^*$. Suppose that an ideal I of $R\#G^*$ is also an ideal of $R^1\#G^*$. Then by the discussion above, if I is G-invariant, $I = I^\downarrow\#G^*$.

But if I is an intersection of prime ideals of $R\#G^*$, then by Andrunakievic's Lemma [4, Lemma 61], I is an ideal of $R^1\#G^*$. Also if $I = \lambda(R\#G^*)$ for some radical λ, then I is an ideal of $R^1\#G^*$ by [4, Theorem 47]. □

We now define the reflected radical. Recall that a nonempty class λ of associative rings is a radical class if

(i) λ is homomorphically closed;

(ii) if A/B and B are in λ, then A is in λ;

(iii) if I_α, $\alpha \in \Delta$, is an ascending chain of ideals of A with each I_α in λ, then $\bigcup_\alpha I_\alpha$ is in λ.

We denote by $\lambda(A)$ the largest ideal of A which is in λ. Recall that a radical λ is called hereditary if $\lambda(I) = \lambda(A) \cap I$ for any ideal I of A.

Now let F be the functor from the category of associative G-graded rings to the category of associative rings such that $F(R) = R\#G^*$ and for f a grade-preserving ring homomorphism from R to S, $F(f) : R\#G^* \to S\#G^*$ is defined by $F(f)(rp_g) = f(r)p_g$. The functor F is exact and preserves unions of ascending chains of ideals. Thus we have the following:

PROPOSITION 1.2. *If λ is a radical class in the category of associative rings, then*

$$\lambda_{\text{ref}} = \{R\colon R \text{ is a } G\text{-graded ring with } R\#G^* \in \lambda\}$$

is a radical class of G-graded rings.

PROOF. This follows directly from the above discussion or see [5, Theorem 1]. □

PROPOSITION 1.3. *If λ is a radical in the category of associative rings, then for R a G-graded ring, $\lambda_{\text{ref}}(R) = (\lambda(R\#G^*))^{\downarrow}$, and thus $\lambda_{\text{ref}}(R)\#G^* = = \lambda(R\#G^*)$.*

2. The reflected Jacobson, prime and strongly prime radicals

In this section, we discuss λ_{ref} for three radicals λ for which a definition of a graded version of λ already exists, namely for λ the Jacobson, prime or strongly prime radical, and compare the reflected radical to the existing graded versions of these radicals.

2.1. The reflected Jacobson radical. Recall that a (graded) left R-module M is (graded) irreducible if $RM = M$, and (0) and M are the only (graded) submodules of M. The graded Jacobson radical of R, $J_G(R)$ has been defined as the set of elements of R which annihilate all G-graded irreducible left (or all graded irreducible right) R-modules. (Equivalent definitions and a discussion of the graded Jacobson radical may be found in [2] or [7].) In [1], it is shown that for R a G-graded ring with identity,

$J_G(R)\#G^* = J(R\#G^*)$, so that $J_G(R) = J_{\mathrm{ref}}(R)$. A modified version of the argument in [1] will yield the same result for a G-graded ring R, not necessarily with identity.

We show first that every irreducible left $R\#G^*$-module is a graded irreducible left R-module and vice versa.

First note that any G-graded left R-module M has a left $R\#G^*$-module structure via

$$(1) \qquad (rp_g)m = rm_g.$$

Let M be an irreducible G-graded R-module, and write M' for M with the $R\#G^*$-module structure above. Since $RM = M$, $(R\#G^*)M' = M'$. Let L' be an $R\#G^*$-submodule of M' and let $x = \sum_{i=1}^{t} x_{g_i}$ be a nonzero element of L'. Then $(R\#G^*)x = \sum_{i=1}^{t} Rx_{g_i}$. Since M is an irreducible G-graded R-module, the submodule $\{m : m \in M, Rm_g = (0)\} = (0)$; thus $\sum_{i=1}^{t} Rx_{g_i}$, as a nonzero G-graded R-submodule of M, must equal M. Therefore $L' = M'$, and M' is irreducible.

Let M be an irreducible left $R\#G^*$-module. For each $g \in G$, let $M_g'' = \sum_{h\in G} (R_{gh^{-1}}p_h)\, M$. Since $(R\#G^*)M = M$, M is the sum of the M_g'' and we must show that this sum is direct. Suppose $x \in M_g'' \cap M_h''$, with g and h different elements of G. Since $(Rp_s)M_t'' = (0)$ for s different from t, $(R\#G^*)x = (0)$. But since M is irreducible, the submodule $\{m : m \in M, (R\#G^*)m = (0)\} = (0)$, and thus $x = 0$.

Define a left R-module structure on $M'' = \bigoplus_{g\in G} M_g''$ by

$$(2) \qquad rx = (rp_g)x$$

for $r \in R$, $x \in M_g''$. As in [1], it is easy to verify that M'' is a G-graded left R-module, and since $(R\#G^*)M = M$, $RM'' = M''$. A little checking shows that the left $R\#G^*$-module structure defined by (1), when applied to M'', will agree with the original $R\#G^*$-module structure on M. Thus, M'' is irreducible.

Again, it is straightforward to check that if we start with an irreducible G-graded R-module M and apply (1) and then (2), the resulting G-graded R-module structure is that of the original.

Thus we have the following.

PROPOSITION 2.1. *The categories of irreducible left $R\#G^*$-modules and irreducible left G-graded R-modules are isomorphic.*

We can now see that $J_{\mathrm{ref}} = J_G$.

PROPOSITION 2.2. *For R a G-graded ring, $J_{\mathrm{ref}}(R) = (J(R\#G^*))^{\downarrow} = J_G(R)$.*

PROOF. Suppose $r \in (J(R\#G^*))^{\downarrow}$, r homogeneous of grade g. To show that $r \in J_G(R)$, we show that r annihilates M, for M any irreducible G-graded left R-module. But since $rp_h \in J(R\#G^*)$ for all $h \in G$, $(rp_h)M' = (0)$ for all $h \in G$, and thus $rM = (0)$. Therefore $J_{\mathrm{ref}} \subseteqq J_G(R)$.

To complete the proof, we show that $J_G(R)\#G^* \subseteqq J(R\#G^*)$. Let $rp_g \in J_G(R)\#G^*$; since $J_G(R)$ is graded, we may assume r is homogeneous. Let M be an irreducible left $R\#G^*$-module. Then M'' is an irreducible G-graded left R-module so $rM'' = (0)$ and $rM''_h = (0)$ for all $h \in G$. Since $(R\#G^*)M = M$,

$$(rp_g)M = (rp_g)\sum_{f,h\in G} R_{hf^{-1}}p_f M \subseteqq rM''_g = (0)$$

and rp_g annihilates M. □

2.2. The reflected prime radical. We now consider $\lambda = N$, the prime radical. Recall that for a ring A, $N(A)$ is the intersection of the prime ideals of A and contains every nilpotent ideal of A. In [3], the ideal $N_G(R)$ is defined to be the intersection of the graded prime ideals of R, for G finite and R a G-graded ring with identity. Let us denote by $N_G(R)$ the intersection of the graded primes of R for any group G and G-graded ring R.

THEOREM 2.3. (i) *$N_G(R) \subseteqq N_{\mathrm{ref}}(R)$.*
(ii) *If G is finite, $N_G(R) = N_{\mathrm{ref}}(R)$.*
(iii) *If G is infinite, the inclusion in* (i) *may be proper.*

PROOF. If P is a prime ideal of $R\#G^*$, then $P^{\downarrow}$ is a graded prime of R and thus $N_G(R)\#G^* \subseteqq N(R\#G^*)$ so that $N_G(R) \subseteqq N_{\mathrm{ref}}(R)$.

Now suppose G is finite. If R has an identity, then (ii) follows from [3, Theorem 5.3]. Otherwise recall that the prime radical is a hereditary radical so that

$N(R\#G^*) = N(R^1\#G^*) \cap R\#G^*$ since N is hereditary
$= (N_G(R^1)\#G^*) \cap R\#G^*$ by [3, Theorem 5.3]
$= (N(R^1)_G\#G^*) \cap R\#G^*$ by [3, Lemma 5.1] which holds for all groups G
$= (N(R^1)_G \cap R)\#G^*$
$= (N(R^1) \cap R)_G\#G^*$ since R is a graded ideal of R^1
$= N(R)_G\#G^*$ since N is hereditary.

The fact that the inclusion may be proper for infinite G follows from the next example. □

EXAMPLE 2.4. Let k be a field and $R = k[t]$, the polynomial ring graded by $G = Z$ in the usual way. Since (0) is a graded prime ideal, $N_G(R) = (0)$. Let I be the principal left ideal $(R\#G^*)tp_0$ of $R\#G^*$. Then $I^2 = (0)$,

$J = I + I(R\#G^*)$ is a nilpotent two-sided ideal of $R\#G^*$, and therefore $N(R\#G^*) = N_{\text{ref}}(R)\#G^*$ is nonzero. □

2.3. The reflected strongly prime radical. A third example of a radical for which a graded version has been defined is the strongly prime radical. Recall that if I is an ideal of a ring A, a (right) insulator for I is a finite subset $F \subseteqq I$ such that if $Fa = 0$ for $a \in A$, then $a = 0$. The ring A is said to be (right) strongly prime if every nonzero (two-sided) ideal of A contains an insulator. An ideal P is called strongly prime if A/P is a strongly prime ring. The strongly prime radical of A is

$$s(A) = \cap\{P: \ P \text{ is a strongly prime ideal of } A\}.$$

If R is a G-graded ring, then R is said to be (right) graded strongly prime if each nonzero graded ideal of R contains an insulator [8]. The following definition is also from [8]:

DEFINITION 2.5. The graded strongly prime radical of a G-graded ring R is defined to be

$$s_G(R) = \cap\{P: \ P \text{ is a graded strongly prime ideal of } R\}.$$

From [8, Corollary 1], $s_G(R) = (s(R))_G$.

THEOREM 2.6. *For R a G-graded ring, the graded strongly prime radical defined above is related to the reflected radical s_{ref} in the following way.*

(i) *For all G, $s_G(R) \subseteqq s_{\text{ref}}(R)$.*

(ii) *If G is finite, $s_G(R) = s_{\text{ref}}(R)$.*

(iii) *For G infinite, the inclusion in* (i) *may be proper.*

PROOF. (i) To prove the required inclusion, we show that $s_G(R)\#G^* \subseteqq s(R\#G^*) = s_{\text{ref}}(R)\#G^*$. Let P be a strongly prime ideal of $R\#G^*$. It suffices to show that $P^{\downarrow}$ is graded strongly prime in R, since then $s_G(R)\#G^* \subseteqq P^{\downarrow}\#G^* \subseteqq P$ for all strongly prime ideals P of $R\#G^*$.

Suppose that $P^{\downarrow}$ is properly contained in I where I is a graded ideal of R. Then $I\#G^*$ is an ideal of $R\#G^*$ and $I\#G^*$ is not contained in P, so that $(I\#G^* + P)/P$ contains an insulator F and we may assume that $F = \{a_1 p_{g_1} + P, \dots, a_n p_{g_n} + P\}$ where $a_1, \dots, a_n$ are homogeneous elements of I. We will show that $\{a_1 + P^{\downarrow}, \dots, a_n + P^{\downarrow}\}$ is an insulator in $I/P^{\downarrow}$. Assume that for some $r \in R$, $a_i r \in P^{\downarrow}$ for all $i = 1, \dots, n$. Since $P^{\downarrow}$ is graded and the a_i are homogeneous, $a_i r_g \in P^{\downarrow}$ for all $i = 1, \dots, n$ and all homogeneous components r_g of r. It follows that $a_i r_g p_h \in P$ for all $i = 1, \dots, n$ and all $g, h \in G$, and therefore $(a_i p_{g_i})(r p_h) \in P$ for all $i = 1, \dots, n$ and all $h \in G$. Since F is an insulator, $r p_h \in P$ for all $h \in G$. Thus $r \in P^{\downarrow}$ and the proof of (i) is complete.

Now assume that G is finite and Q is a graded strongly prime ideal of R. Using Zorn's lemma, we may choose P maximal in the set of ideals

I of $R\#G^*$ containing $Q\#G^*$, and such that $I/Q\#G^*$ does not contain an insulator in $R\#G^*/Q\#G^*$. By [9, p. 1101], P is a strongly prime ideal of $R\#G^*$.

We wish to show that $P^{\downarrow} = Q$. Suppose $P^{\downarrow}$ properly contains Q. Let $\{a_1 + Q, \dots, a_k + Q\}$ be an insulator with $a_1, \dots, a_k \in P^{\downarrow}$, and let $F = \{a_i p_g + Q\#G^*\colon\ i = 1, \dots, k,\ g \in G\}$. If $(a_i p_g) \sum_{j=1}^{t} b_j p_{g_j} \in Q\#G^*$ for all $i = 1, \dots, k$ and all $g \in G$, then by summing over g, we see that $\sum_{j=1}^{t} a_i b_j p_{g_j} \in Q\#G^*$ for all $i = 1, \dots, k$. Thus, $a_i b_j \in Q$ for all $i = 1, \dots, k$, $j = 1, \dots, t$ so that $b_1, \dots, b_t \in Q$. It follows that F is an insulator in $P/Q\#G^*$, contradicting our choice of P; therefore $P^{\downarrow} = Q$. By Lemma 1.1 and the fact that P is strongly prime, we see that

$$s(R\#G^*) = s(R\#G^*)^{\downarrow}\#G^* \subseteqq P^{\downarrow}\#G^* = Q\#G^*.$$

Intersecting over all graded strongly prime ideals Q, we obtain $s(R\#G^*) \subseteqq$ $\subseteqq s_G(R)\#G^*$. Thus for G finite, $s_G = s_{\mathrm{ref}}$.

The last statement follows from Example 2.8. □

LEMMA 2.7. *Let R be a strongly graded ring with* 1, *G an infinite group. Then if I is an ideal of $R\#G^*$ containing some p_g, $I = R\#G^*$.*

PROOF. Let h be any element of G. Since R is strongly graded, there exist $x_i \in R_{hg^{-1}}$, $y_i \in R_{gh^{-1}}$, $i = 1, \dots, t$, such that $\sum_{i=1}^{t} x_i y_i = 1$. But then $p_g \in I$ implies $p_h = \sum_{i=1}^{t} (x_i p_g)(y_i p_h) \in I$. □

EXAMPLE 2.8. Let R be a strongly graded ring with identity and G an infinite group. By Lemma 2.7, if I is an ideal of $R\#G^*$ containing any p_g then I is all of $R\#G^*$.

Let P be a strongly prime ideal in $R\#G^*$. Since $R\#G^*/P$ has a finite insulator but the p_g, $g \in G$, are an infinite set of mutually orthogonal idempotents, $p_h \in P$ for some $h \in G$. Thus $P = R\#G^*$, $s(R\#G^*) = R\#G^*$ and $s_{ref}(R) = R$. However, since maximal graded ideals are graded strongly prime [8], $s_G(R)$ is not R. □

3. More examples of reflected radicals

In this final section we discuss the reflected Levitzki, Brown–McCoy and von Neumann regular radicals.

3.1. The reflected Levitzki radical. Recall that an ideal I of a ring A is called locally nilpotent if every finitely generated subring of I is nilpotent.

The Levitzki radical of A, $L(A)$, is the intersection of the prime ideals P of A such that A/P has no nonzero locally nilpotent ideals. Equivalently, $L(A)$ is the union of the locally nilpotent ideals of A [4, Chapter 6].

DEFINITION 3.1. For R a graded ring, $L_G(R)$ is the intersection of the graded prime ideals P of R such that R/P has no nonzero graded locally nilpotent ideals.

PROPOSITION 3.2. *For R a G-graded ring, $L_G(R) = (L(R))_G$.*

PROOF. If P is a prime ideal of R, then it is easy to see that P_G is a graded prime ideal of R. Furthermore, if R/P has no nonzero locally nilpotent ideals, then R/P_G has no nonzero locally nilpotent graded ideals. For if I is a locally nilpotent graded ideal in R/P_G, then $(I+P)/P$ is a nonzero locally nilpotent ideal in R/P. Thus $L_G(R) \subseteqq (L(R))_G$.

Conversely, since $L(R)$, and hence $(L(R))_G$, is a locally nilpotent ideal, $(L(R))_G \subseteqq Q$ for all graded prime ideals Q such that R/Q has no nonzero locally nilpotent graded ideals. Thus $L(R)_G \subseteqq L_G(R)$. □

We now compare L_G and L_{ref}.

THEOREM 3.3. (i) *For any group G, $L_G(R) \subseteqq L_{\text{ref}}(R)$.*

(ii) *If G is locally finite, $L_G(R) = L_{\text{ref}}(R)$.*

(iii) *For infinite G, the inclusion in* (i) *may be proper.*

PROOF. Let P be a prime ideal of $R\#G^*$ such that $R\#G^*/P$ has no nonzero locally nilpotent ideals. Then, $P^{\downarrow} = P_R$ is a graded prime ideal of R, and we show that $R/P^{\downarrow}$ has no nonzero locally nilpotent graded ideals.

Let I be a graded ideal containing $P^{\downarrow}$ such that $I/P^{\downarrow}$ is locally nilpotent. We will show that the ideal $(I\#G^* + P)/P$ is a locally nilpotent ideal of $R\#G^*/P$. Let $W = \left\{ \sum_{i=1}^{n} a_{ij}p_{g_i} : \ j = 1, \dots, m \right\}$ be a finite subset of $I\#G^*$. The set $\{(a_{ij})_g : i = 1, \dots, n, \ j = 1, \dots, m, \ g \in G\}$ is a finite subset of I and so the subring S it generates satisfies $S^k \subseteqq P^{\downarrow}$ for some positive integer k. Thus, if T is the subring of $R\#G^*$ generated by W, then $T^k \subseteqq S^k\#G^* \subseteqq P^{\downarrow}\#G^* \subseteqq P$. It follows that $(I\#G^*+P)/P$ is a locally nilpotent ideal and so $I\#G^* \subseteqq P$. Thus $I \subseteqq P^{\downarrow}$ and hence $L_G(R) \subseteqq P^{\downarrow}$. This completes the proof that $L_G(R)\#G^* \subseteqq L(R\#G^*)$ so that $L_G(R) \subseteqq L_{\text{ref}}(R)$.

To prove (ii), we show that $L_{\text{ref}}(R)$ is locally nilpotent and then the statement follows from Proposition 3.2. Let $W = \{b_1, \dots, b_s\}$ be a finite subset of $L_{\text{ref}}(R)$. The subring generated by W is contained in the subring S generated by the homogeneous components of the elements of W; call this set $V = \{a_1, \dots, a_n\}$. Let H be the (finite) subgroup of G generated by elements h of G such that $a_i \in R_h$ for some $a_i \in V$. The finite set $\{a_i p_h : \ i = 1, \dots, n, \ h \in H\}$ is in $L(R\#G^*)$, and hence the subring T it generates is nilpotent, say $T^m = 0$. Now if $c_1, \dots, c_m$ are (not necessarily distinct) elements of V with $c_i \in R_{h_i}$, then $c_1 \dots c_m p_{g_m} = c_1 p_{g_1} c_2 p_{g_2} \dots c_m p_{g_m} \in T^m$ where g_m can be any element of H and the g_i are defined inductively by

$g_{i-1} = h_i g_i$. Since $T^m = 0$, $c_1 \dots c_m = 0$; thus the subring S of L_{ref} is nilpotent.

Example 2.4 shows that the containment $L_G \subseteqq L_{\text{ref}}$ may be proper. For here, $R = k[t]$ has no proper locally nilpotent graded ideals, so that $L_G(R) = (0)$ although $L(R\#G^*) \supseteqq N(R\#G^*)$ is nonzero. □

3.2. The reflected Brown–McCoy radical. Recall that $\mathcal{G}(A)$, the Brown–McCoy radical of a ring A, is the intersection of the ideals M of A such that A/M is a simple ring with identity.

DEFINITION 3.4. For R a G-graded ring, define $\mathcal{G}_G(R)$ to be the intersection of the graded ideals of R such that R/M is a graded simple ring with identity.

PROPOSITION 3.5. *For all G-graded rings R, $\mathcal{G}(R)_G \subseteqq \mathcal{G}_G(R)$, and this containment may be proper.*

PROOF. Let M be a graded ideal of R such that R/M is a graded simple ring with identity $e + M$. We wish to show that $\mathcal{G}(R)_G \subseteqq M$ for all such M.

Suppose not. Then $\mathcal{G}(R)_G + M = R$ and $e = x + m$ for some $x \in \mathcal{G}(R)_G$, $m \in M$. Also R/M has an identity so we may choose $Q = N + M$, a maximal proper ideal of R/M. Then $\mathcal{G}(R) \subseteqq Q$ and hence $e \in Q$. This is impossible since Q was a proper ideal of R/M.

The example following [2, Lemma 12] shows that the inclusion may be proper; here R is a commutative ring with 1 so $\mathcal{G}(R) = J(R)$ and $\mathcal{G}_G(R) = J_G(R)$. □

THEOREM 3.6. (i) *For all G, $\mathcal{G}_G(R) \subseteqq \mathcal{G}_{\text{ref}}(R)$.*

(ii) *If G is finite, $\mathcal{G}_G(R) = \mathcal{G}_{\text{ref}}(R)$.*

(iii) *The inclusion in* (i) *may be proper.*

PROOF. Assume first that R has an identity 1. To prove (i), we show that $\mathcal{G}_G(R)\#G^* \subseteqq \mathcal{G}(R\#G^*) = \mathcal{G}_{\text{ref}}(R)\#G^*$.

Let M be an ideal of $R\#G^*$ such that $R\#G^*/M$ is a simple ring with identity $w + M$. We will now show that $R/M^{\downarrow}$ is a graded simple ring with identity and it will then follow that $\mathcal{G}_G(R)\#G^* \subseteqq M^{\downarrow}\#G^* \subseteqq M$ for all such M.

Suppose there is a graded ideal T of R which properly contains $M^{\downarrow}$. Then $T\#G^*$ is not contained in M and $T\#G^* + M = R\#G^*$. Therefore there exist $a_i \in T$, $g_i \in G$, $m \in M$ such that $\sum_{i=1}^{t} a_i p_{g_i} + m = w$. Since $wp_{g_k} - p_{g_k} \in M$, $a_k p_{g_k} - p_{g_k} \in M$, we have

$$p_{g_k}(a_k p_{g_k} - p_{g_k}) = (a_k)_e p_{g_k} - p_{g_k} \in M.$$

Therefore $[(a_k)_e - 1]p_{g_k} \in M$ for $k = 1, \dots, t$ and if we let $\kappa = \prod_{k=1}^{t} [(a_k)_e - 1]$, then $\kappa p_{g_k} \in M$ for $k = 1, \dots, t$. Since $w + M$ is the identity in $R\#G^*/M$,

$wp_h - p_h \in M$ for all h but if $h \notin \{g_1, \dots, g_t\}$, $wp_h = mp_h \in M$ and so $p_h \in M$. Thus $\kappa p_g \in M$ for all $g \in G$; therefore $\kappa \in M^{\downarrow} \subset T$.

By the definition of κ, $\kappa = (-1)^t + \gamma$ where $\gamma \in T$. Therefore $T = R$, $M^{\downarrow}$ is a maximal graded ideal of R as required and $\mathcal{G}_G(R)\#G^* \subseteqq \mathcal{G}(R\#G^*)$ for R a ring with identity.

Now suppose that R does not have an identity and embed R in R^1 as usual. Suppose that M is a maximal graded ideal of R^1. Then $(R+M)/M$ is a graded ideal of R^1/M and so is either (0) or R^1/M. If $(R+M)/M = (0)$, then $\mathcal{G}_G(R) \subseteqq R \subseteqq M$. If $(R+M)/M = R^1/M$, then since $R^1/M = (R+M)/M \cong R/(R \cap M)$, $R \cap M$ is a maximal graded ideal of R, and $\mathcal{G}_G(R) \subseteqq R \cap M$. Hence in either case, $\mathcal{G}_G(R) \subseteqq M$ for all such M and $\mathcal{G}_G(R) \subseteqq \mathcal{G}_G(R^1)$.

Therefore we have

$$\begin{aligned}\mathcal{G}_G(R)\#G^* &= (\mathcal{G}_G(R)\#G^*) \cap (R\#G^*)\\ &\subseteqq (\mathcal{G}_G(R^1)\#G^*) \cap (R\#G^*) \text{ by the above argument}\\ &\subseteqq \mathcal{G}(R^1\#G^*) \cap (R\#G^*) \text{ since } R^1 \text{ has an identity}\\ &= \mathcal{G}(R\#G^*) \text{ since } \mathcal{G} \text{ is hereditary [4, p. 125].}\end{aligned}$$

To see that this inclusion may be proper, let k be a field, $\langle x \rangle$ the infinite cyclic group and $R = k\langle x \rangle$ the group ring. R is strongly Z-graded and so, since $s(A) \subseteqq \mathcal{G}(A)$ for all rings A, by Example 2.8, $\mathcal{G}_{\text{ref}}(R)\#Z^* = \mathcal{G}(R\#Z^*) = R\#Z^*$. However, because (0) is a maximal graded ideal, $\mathcal{G}_G(R) = (0)$. Therefore $\mathcal{G}_G(R)$ is properly contained in $\mathcal{G}_{\text{ref}}(R)$, and statements (i) and (iii) are proved.

Now suppose that G is finite and let I be a graded ideal of R such that R/I is a simple graded ring with identity. Then $I\#G^*$ is an ideal of $R\#G^*$, and since $R\#G^*/I\#G^* \cong (R/I)\#G^*$ has a 1, we may choose an ideal M of $R\#G^*$ maximal in the set of ideals containing $I\#G^*$. Since $I\#G^*$ is invariant under the action of G, $I\#G^* \subseteqq N = \bigcap_{g\in G} M^g$, where M^g is the image of M under the automorphism $g \in G$. Therefore $(I\#G^*)_R = I \subseteqq N_R$. By the maximality of I, $I = N_R$, and so by Lemma 1.1, $N = I\#G^*$. Since $R\#G^*/M^g$ is a simple ring with 1 for all g, $\mathcal{G}(R\#G^*) \subseteqq N = I\#G^*$. Intersecting over all maximal graded ideals I, we obtain $\mathcal{G}(R\#G^*) \subseteqq \mathcal{G}_G(R)\#G^*$, and thus $\mathcal{G}_G(R) = \mathcal{G}_{\text{ref}}(R)$. □

3.3. The reflected von Neumann regular radical. Recall that, for any ring A, the regular radical of A, $r(A)$, is the unique largest von Neumann regular ideal of A, where an ideal I of A is regular if and only if every finitely generated right (left) ideal of I is generated by an idempotent [6, Theorem 1.1]).

DEFINITION 3.7. For R a G-graded ring, let $r_G(R)$ be the unique largest graded von Neumann regular ideal of R. Clearly $r_G(R) = r(R)_G$.

LEMMA 3.8. *Let R be a G-graded ring with identity. If $x_1, \dots, x_n$ are homogeneous elements of R of degree $g_1, \dots, g_n$ respectively and $Rx_1 + \dots + Rx_n = Ru$ for some idempotent u, then for each $g \in G$, there is an idempotent $v = v(g) \in R\#G^*$ such that $(x_1 + \dots + x_n)p_g R\#G^* = v(R\#G^*)$.*

PROOF. Direct calculation shows that $(xp_g)(b_1 p_{g_1 g} + \dots + b_n p_{g_n g})(xp_g) = xp_g$ where $x = x_1 + \dots + x_n$ and $b_1, \dots, b_n$ are such that $b_1 x_1 + \dots + b_n x_n = u$. Then $v = (xp_g)(b_1 p_{g_1 g} + \dots + b_n p_{g_n g})$ is the required idempotent. □

THEOREM 3.9. (i) *For all G, $r_G(R) \subseteqq r_{\mathrm{ref}}(R)$.*

(ii) *$R\#G^*$ is a von Neumann regular ring if and only if for all $g \in G$, $x \in R_g$, there is a $y \in R_{g^{-1}}$ such that $xyx = x$. Thus, even for finite G, the inclusion in* (i) *may be proper.*

PROOF. To prove (i), we assume first that R has a 1.

Let $F = \{u_1, \dots, u_k\}$ be a finite set of elements of $r_G(R)\#G^*$ and let I be the right ideal generated by F. Then, since $1p_g = p_g$ is in $R\#G^*$, we may assume that the elements of F are of the form $(x_1 + \dots + x_n)p_g$ where the x_i are homogeneous elements of $r_G(R)$.

From Lemma 3.8, we see that $u_1(R\#G^*) = v_1(R\#G^*)$ for some idempotent v_1, and $(u_2 - v_1 u_2)R\#G^* = w_2 R\#G^*$ for some idempotent w_2. Moreover, since $v_1 w_2(R\#G^*) = 0$, $v_1 w_2 = 0$. Therefore v_1 and $v_2 = w_2 - w_2 v_1$ are orthogonal idempotents and $u_1(R\#G^*) + u_2(R\#G^*) = (v_1 + v_2)R\#G^*$. Since $v_1 + v_2$ is an idempotent, we can repeat the argument with u_3 and $v_1 + v_2$. Continuing, we obtain $I = w(R\#G^*)$ for some idempotent w. Thus $r_G(R)\#G^*$ is a regular ideal of $R\#G^*$ and so $r_G(R)\#G^* \subseteqq r(R\#G^*)$ and $r_G(R) \subseteqq r_{\mathrm{ref}}(R)$.

If R does not have an identity, embed R in R^1, and argue as in the proof of Theorem 2.3, using the fact that r is a hereditary radical and $r_G(R^1) = r(R^1)_G$.

Now assume that $R\#G^*$ is regular. Then for each $g \in G$, $r \in R_g$, there is a $z = xp_{g^2} \in R\#G^*$ such that $rp_g = (rp_g)xp_{g^2}(rp_g) = rx_{g^{-1}}rp_g$ and so $r = rx_{g^{-1}}r$.

To prove the converse, we show that the subring T of $R^1\#G^*$ which is generated by $R\#G^*$ and $\{p_g : g \in G\}$ is regular, and then use the fact that every two-sided ideal in a regular ring is regular. Let H be a finite set of elements of G, $w = \sum_{h \in H} p_h$, and let S be the subring of T generated by $w(R\#G^*)w$ and $\{p_h : h \in H\}$. Then by [6, Lemma 1.6], S is regular if and only if for each $g, h \in H$ and for each $x \in p_g S p_h$, there is a $y \in p_h S p_g$ such that $xyx = x$. But it is easily checked that the condition in (ii) then guarantees S is regular. Since T is the union of such subrings S, T is regular and thus so is $R\#G^*$.

The last example shows that the inclusion (i) may be proper. □

EXAMPLE 3.10. Let $R = Z_2[X]/(X^2)$ be $G = Z/2Z$ graded by $R_0 = \{0, 1\}$ and $R_1 = \{0, x+1\}$ where $x = X + (X^2)$. It follows from Theorem 3.9

(ii) that $R\#G^*$ is regular, so that $r_{\mathrm{ref}}(R) = R$, but since R has only one proper ideal, namely the nilpotent principal ideal generated by x, $r(R) = (0)$ so that $r(R)_G = r_G(R) = (0)$. □

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
MOUNT ALLISON UNIVERSITY
SACKVILLE, N.B., CANADA EOA 3CO

DEPARTMENT OF MATHEMATICS, STATISTICS
AND COMPUTING SCIENCE
DALHOUSIE UNIVERSITY
HALIFAX, N.S., CANADA B3H 3J5

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COMMUTATIVITY RESULTS FOR PERIODIC RINGS

H. ABU-KHUZAM (Beirut)

A theorem of Herstein [8] states that a ring R which satisfies the identity $(xy)^n = x^n y^n$, where n is a fixed positive integer greater than 1, must have nil commutator ideal. In [1], the author proved that if n is a fixed positive integer greater than 1, and R is an $n(n-1)$-torsion-free ring with identity such that $(xy)^n = x^n y^n$ for all x, y in R, then R is commutative. In [7], Gupta proved that if R is a semiprime ring satisfying $(xy)^2 - x^2y^2 \in Z$ for all x, y in R, where Z is the center of R, then R is commutative. Recently [3], it was proved that a semiprime ring R such that for each x in R there exists a positive integer $n = n(x) > 1$ such that $(xy)^n - x^n y^n \in Z$ and $(x^2y)^n - x^{2n}y^n \in Z$ for all y in R, then R is commutative. In this direction we prove Theorem 1 and Theorem 2 below.

R is called periodic if for every x in R, there exists distinct positive integers $m = m(x)$, $n = n(x)$ such that $x^m = x^n$. By a theorem of Chacron (see [6, Theorem 1]), R is periodic if and only if for each $x \in R$, there exists a positive integer $k = k(x)$ and a polynomial $f(\lambda) = f_x(\lambda)$ with integer coefficients such that $x^k = x^{k+1}f(x)$.

Throughout this note, R is an associative ring, Z denotes the center of R, N denotes the set of nilpotent elements of R, and $[x, y]$ denotes the commutator $xy - yx$.

We start with the following lemmas. Lemma 1 is well known, Lemma 2 is proved in [5], Lemma 3 is proved in [4], and Lemma 4 is a result proved in [2].

LEMMA 1. *If* $[x,[x,y]] = 0$, *then* $[x^k, y] = kx^{k-1}[x,y]$ *for all integers* $k \geqq 1$.

LEMMA 2. *If R is a periodic ring, then R has each of the following properties:*

(a) *For each $x \in R$, some power of x is idempotent.*

(b) *For each $x \in R$, there exists an integer $k = k(x)$ such that $x - x^k$ is nilpotent.*

(c) *If $f\colon R \to R^*$ is an epimorphism, then $f(N)$ coincides with the set of nilpotent elements of R^*.*

(d) *If N is central, then R is commutative (Herstein).*

LEMMA 3. *Let R be a periodic ring. If N is commutative, then the commutator ideal of R is nil, and N forms an ideal of R.*

Lemma 4. *Let R be a periodic ring such that N is commutative. Suppose that for each x in R and a in N, there exists an integer $n = n(x,a) \geqq 1$ such that $[x^n,[x^n,a]] = 0$ and $[x^{n+1},[x^{n+1},a]] = 0$. Then R is commutative.*

Now we will state and prove our first theorem.

Theorem 1. *Let n be a positive integer and let R be an $n(n+1)$-torsion-free periodic ring such that $(xy)^n - y^n x^n \in Z$ and $(xy)^{n+1} - y^{n+1}x^{n+1} \in Z$. If N is commutative, then R is commutative.*

Proof. By Lemma 3, the set N of nilpotent elements of R is an ideal of R, and since N is commutative, we have

$$N^2 \subseteqq Z. \tag{1}$$

Let e be an idempotent element of R, and let x be any element in R. From the hypothesis

$$(e(e+ex-exe))^n - (e+ex-exe)^n e^n \in Z$$

thus

$$(e+ex-exe) - (e+ex-exe)e \in Z$$

and hence $(ex - exe) \in Z$. This implies that $e(ex-exe) = (ex-exe)e$ and hence $ex = exe$. Similarly, $xe = exe$. Thus $ex = xe$, and

(2) the idempotent elements of R are central.

Let x and y be any two elements of R. Then by the hypothesis,

$$(xy)^n - y^n x^n = z_1 \in Z \quad \text{and} \quad (yx)^n - x^n y^n = z_2 \in Z. \tag{3}$$

Now $(xy)^n x = x(yx)^n$ and using (3), this implies that $(y^n x^n + z_1)x = x(x^n y^n + z_2)$. So $x^{n+1}y^n - y^n x^{n+1} = (z_1 - z_2)x$. Thus,

$$[x^{n+1},[x^{n+1},y^n]] = 0 \quad \text{for all} \quad x, y \text{ in } R. \tag{4}$$

Let $a \in N$, put $y = a+1$ in (4), and use the fact that $N^2 \subseteqq Z$ in (1) to get that $n[x^{n+1},[x^{n+1},a]] = 0$. Since R is n-torsion-free, this implies that

$$[x^{n+1},[x^{n+1},a]] = 0 \quad \text{for all} \quad x \in R,\ a \in N. \tag{5}$$

Repeating the above process from (3) using the hypothesis $(xy)^{n+1} - y^{n+1}x^{n+1} \in Z$ we get

$$[x^{n+2},[x^{n+2},a]] = 0 \quad \text{for all} \quad x \in R,\ a \in N. \tag{6}$$

Now, using (5), (6), and Lemma 4, we see that R must be commutative. This completes the proof of Theorem 1.

The following example shows that the analogue of Theorem 1 is not true if the condition "$(xy)^n - y^n x^n \in Z$ and $(xy)^{n+1} - y^{n+1}x^{n+1} \in Z$" is replaced by the condition "$(xy)^n - x^n y^n \in Z$ and $(xy)^{n+1} - x^{n+1}y^{n+1} \in Z$".

EXAMPLE. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in GF(3) \right\}$. Clearly, R is periodic since it is finite, and the set of nilpotent elements N is commutative. It is easy to verify that $(xy)^4 = x^4y^4$ and $(xy)^5 = x^5y^5$. R is also (4) (5)-torsion-free but not commutative.

In Theorem 2 below, we prove that if only the condition "$(xy)^{n+1} - y^{n+1}x^{n+1} \in Z$" is replaced by the condition "$(xy)^{n+1} - x^{n+1}y^{n+1} \in Z$" in Theorem 1, then the result still holds. In preparation for the proof of Theorem 2, we need to prove the following lemma.

LEMMA 5. *Let R be a ring with characteristic $q \neq 0$ and let n be a positive integer. Let $f\colon R \to R^*$ be an epimorphism. If R is n-torsion-free, then R^* is n-torsion-free.*

PROOF. Let d be the greatest common divisor of q and n. This implies that $q = k_1 d$ and $n = k_2 d$ for some positive integers k_1 and k_2. If $d \neq 1$, then Char $R = q \neq k_1$, and hence there exists an element $y \in R$ such that $k_1 y \neq 0$. Now

$$n(k_1 y) = (k_2 d)k_1 y = k_2 q y = 0.$$

This contradicts the hypothesis that R is n-torsion-free. So $d = 1$ and $(q, n) = 1$. Since $f\colon R \to R^*$ is an epimorphism, then for each $x^* \in R^*$ there exists an element $x \in R$ such that $x^* = f(x)$. Now

$$qx^* = qf(x) = f(qx) = f(0) = 0 \quad \text{for all} \quad x^* \in R^*.$$

So Char $R^* = q'$, where q' divides q. Hence $(q', n) = 1$ since $(q, n) = 1$. This implies that $rq' + sn = 1$ for some integers r and s. If $ny^* = 0$ for some $y^* \in R^*$, then

$$y^* = (rq' + sn)y^* = r(q'y^*) + s(ny^*) = 0.$$

So R^* is n-torsion-free.

THEOREM 2. *Let n be a positive integer and let R be an $n(n+1)$-torsion-free periodic ring such that $(xy)^n - y^n x^n \in Z$ and $(xy)^{n+1} - x^{n+1}y^{n+1} \in Z$. If N is commutative, then R is commutative.*

PROOF. As in Theorem 1, since N is a commutative ideal, we have

$$N^2 \subseteqq Z. \tag{7}$$

Also, since $(xy)^n - y^n x^n \in R$ and R is n-torsion-free, the proofs of (2) and (5) in Theorem 1 still hold, and so

(8) the idempotents of R are central,

and

$$(9) \qquad [x^{n+1},[x^{n+1},a]]=0 \quad \text{for all} \quad x\in R,\ a\in N.$$

R is isomorphic to a subdirect sum of subdirectly irreducible rings R_α. Since R_α is a homomorphic image of R, it is easy to verify that

(10) each R_α satisfies all the hypotheses of R except possibly that R_α may not be $n(n+1)$-torsion-free.

We now distinguish two cases.

Case 1: R_α does not have an identity. Then, since R_α is periodic, Lemma 2(a) implies that for each $x_\alpha \in R_\alpha$, there exists a positive integer $t=t(x_\alpha)$ such that x_α^t is idempotent. By (10), the proof of (8) holds for R_α, and x_α^t is a central idempotent. But R_α is subdirectly irreducible and has no identity in this case. So $x_\alpha^t=0$ and R_α is a nil ring. This implies that R_α is commutative since the set of nilpotent elements of R_α is commutative from (10).

Case 2: R_α has an identity element 1_α. Since R_α is periodic, $(2.1_\alpha)^i = (2.1_\alpha)^j$ for distinct positive integers i and j, and hence Char $R_\alpha = q_\alpha \neq 0$. So by Lemma 5, R_α is $n(n+1)$-torsion-free. This implies, using (10), that

(11) R_α satisfies all the hypotheses of R, and thus we may assume that R is subdirectly irreducible with identity 1.

Again as in Case 1, for each $x\in R$, there exists a positive integer $t=t(x)$ such that x^t is a central idempotent. Using (11), we have $x^t=0$ or $x^t=1$. Thus,

(12) every element of R is either nilpotent or invertible.

Let x and y be any two elements of R. Then by the hypothesis,

$$(13) \qquad (xy)^{n+1}-x^{n+1}y^{n+1}=z\in Z \quad \text{and} \quad (yx)^{n+1}-y^{n+1}x^{n+1}=z'\in Z.$$

Now $(xy)^{n+1}x=x(yx)^{n+1}$ and using (13), this implies that $(x^{n+1}y^{n+1}+z)x = x(y^{n+1}x^{n+1}+z')$. So $x^{n+1}y^{n+1}x - xy^{n+1}x^{n+1} = (z'-z)x$. Thus,

$$(14) \qquad x(x^{n+1}y^{n+1}x-xy^{n+1}x^{n+1}) = (x^{n+1}y^{n+1}x-xy^{n+1}x^{n+1})x.$$

If x is invertible, then (14) implies that $[x,[x^n,y^{n+1}]]=0$ and hence,

$$(15) \qquad [x^n,[x^n,y^{n+1}]]=0, \quad \text{where } x \text{ is invertible and} \quad y\in R.$$

If x is nilpotent, then since N is commutative and the commutator ideal is nil, we have,

$$(16) \qquad [x^n,[x^n,y^{n+1}]]=0 \quad \text{where } x \text{ is nilpotent and} \quad y\in R.$$

Now, using (12), (15), and (16) we have,

(17) $$[x^n,[x^n,y^{n+1}]]=0 \quad \text{for all} \quad x,y \quad \text{in} \quad R.$$

Let $a \in N$, put $y=a+1$ in (17), and use the fact that $N^2 \subseteqq Z$ in (7) to get that $(n+1)[x^n,[x^n,a]]=0$. Since R is $(n+1)$-torsion-free, this implies that

(18) $$[x^n,[x^n,a]]=0 \quad \text{for all} \quad x \in R,\ a \in N.$$

Now, using (9), (18), and Lemma 4, we see that R must be commutative. This completes the proof of Theorem 2.

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DEPARTMENT OF MATHEMATICS
AMERICAN UNIVERSITY OF BEIRUT
BEIRUT
LEBANON

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UMKEHRSÄTZE FÜR RIESZ-VERFAHREN ZUR SUMMIERUNG VON DOPPELREIHEN

S. BARON (Ramat-Gan) und H. TIETZ (Stuttgart)

1. Einleitung

Es seien p und q zwei nichtnegative reelle Zahlen, $\lambda = \{\lambda_k\}$ und $\mu = \{\mu_\ell\}$ zwei streng monoton gegen ∞ strebende Folgen nichtnegativer Zahlen. Bei vorgegebener Doppelreihe

$$(1.1) \qquad \sum_{k,\ell=0}^{\infty} u_{k\ell}$$

mit komplexen Gliedern und der Teilsummenfolge $\{s_{mn}\}$ mit $s_{mn} := \sum_{k,\ell=0}^{m,n} u_{k\ell}$ sei für alle $x, y > 0$

$$(1.2) \qquad R(x,y) := \frac{1}{x^p y^q} \sum_{\lambda_k,\mu_\ell<x,y} (x-\lambda_k)^p (y-\mu_\ell)^q u_{k\ell}$$

und für alle $m, n = 0, 1, \ldots$

$$(1.3) \qquad R_{mn} := \frac{1}{\lambda_{m+1}^p \mu_{n+1}^q} \sum_{k,\ell=0}^{m,n} (\lambda_{m+1}-\lambda_k)^p (\mu_{n+1}-\mu_\ell)^q u_{k\ell}.$$

Wir verwenden schon jetzt die Bezeichnungen aus Abschnitt 2. Die Reihe (1.1) heißt *beschränkt* R-*summierbar zum Wert* σ, kurz bR-$\sum u_{k\ell} = \sigma$, wenn gilt $R(x,y) = O(1) \wedge R(x,y) \to \sigma$ für $x, y \to \infty$; sie heißt *beschränkt* R*-*summierbar zum Wert* σ, kurz bR*-$\sum u_{k\ell} = \sigma$, wenn gilt $R_{mn} = O(1) \wedge$ $\wedge R_{mn} \to \sigma$ für $m, n \to \infty$. Die Reihe (1.1) heißt *absolut* R-*summierbar zum Wert* σ, kurz aR-$\sum u_{k\ell} = \sigma$, wenn gilt $R(x,y) = \Omega(1) \wedge R(x,y) \to \sigma$ für $x, y \to \infty$; sie heißt *absolut* R*-*summierbar zum Wert* σ, kurz aR*-$\sum u_{k\ell} = \sigma$, wenn gilt $R_{mn} = \Omega(1) \wedge R_{mn} \to \sigma$ für $m, n \to \infty$.

Für $p = q = 1$ ist durch (1.3) das Verfahren der bewichteten Mittel definiert, das für $\lambda = \{k\}$ und $\mu = \{\ell\}$ gerade das $(C,1,1)$-Mittel

$$(1.4) \qquad \frac{1}{(m+1)(n+1)} \sum_{k,\ell=0}^{m,n} s_{k\ell}$$

der Folge $\{s_{mn}\}$ ergibt.

Außer einem "high indices theorem" für R von Mears [8], auf das wir in Abschnitt 4 zurückkommen, und einigen Resultaten von Burljaĭ [3, 4] für bewichtete Mittel, wurden für Riesz-Verfahren unseres Wissens Umkehrsätze nur für die speziellen $(C,1,1)$-Mittel (1.4), meist in allgemeinerem Rahmen, behandelt. Neben Knopp [6] und Meyer-König [9] (sowie den bei diesen Autoren genannten Arbeiten) sind hier zum Beispiel noch Agnew [1], Topuriya [19], Čelidze [5], Obrechkoff [13] und Slepenčuk [15, 16, 17] zu nennen.

Ausgangspunkt unserer Untersuchungen ist der folgende Umkehrsatz von Knopp [6], S. 575–578, für beschränkte $(C,1,1)$-Summierbarkeit.

Satz K. *Aus* b$(C,1,1)$-$\sum u_{k\ell} = \sigma$ *folgt* b-$\sum u_{k\ell} = \sigma$, *wenn die folgenden zwei Bedingungen erfüllt sind:*

$$\sum_{k=1}^{m} k \sum_{\ell=0}^{n} u_{k\ell} = o_b(m+1), \tag{1.5}$$

$$\sum_{\ell=1}^{n} \ell \sum_{k=0}^{m} u_{k\ell} = o_b(n+1). \tag{1.6}$$

In Abschnitt 3 wird Satz K für beschränkte und für absolute R*-Summierbarkeit verallgemeinert. Durch Spezialisierung ergeben sich außer Satz K Resultate von Young [20] und Obrechkoff [13]. In Abschnitt 4 beweisen wir ein "high indices theorem" für beschränkte und für absolute R-Summierbarkeit. Unsere Methoden sind neben beschränkter und absoluter Summierbarkeit auch auf andere Summierbarkeitsbegriffe für Doppelfolgen anwendbar. Wir werden darauf allerdings nicht näher eingehen.

2. Bezeichnungen

Wenn nichts Besonderes gesagt ist, sollen alle Indizes von 0 an laufen. Terme mit einem negativen Index sind gleich 0 zu setzen.

Ist $\{x_n\}$ eine Folge komplexer Zahlen, so sei $\overline{\Delta} x_n := x_n - x_{n-1}$ für alle n. Ist $\{y_n\}$ eine weitere Folge komplexer Zahlen mit $y_n \neq 0$ für alle n, so bedeute $x_n = o(y_n)$, $x_n = O(y_n)$ und $x_n = \Omega(y_n)$ beziehentlich $x_n/y_n \to 0$, $\sup |x_n/y_n| < \infty$ und $\sum |\overline{\Delta}(x_n/y_n)| < \infty$.

Ist $\{x_{mn}\}$ eine Doppelfolge komplexer Zahlen, so sei $\overline{\Delta}_m x_{mn} := x_{mn} - x_{m-1,n}$, $\overline{\Delta}_n x_{mn} := x_{mn} - x_{m,n-1}$ und $\overline{\Delta}_{mn} x_{mn} := \overline{\Delta}_m(\overline{\Delta}_n x_{mn}) = \overline{\Delta}_n(\overline{\Delta}_m x_{mn})$. Ist $\{y_{mn}\}$ eine weitere Doppelfolge komplexer Zahlen mit $y_{mn} \neq 0$ für alle m, n, so bedeute $x_{mn} = o(y_{mn})$, $x_{mn} = O(y_{mn})$, $x_{mn} = o_b(y_{mn})$ und $x_{mn} = \Omega(y_{mn})$ beziehentlich $x_{mn}/y_{mn} \to 0$ für $m, n \to \infty$ (im Pringsheimschen Sinne), $\sup |x_{mn}/y_{mn}| < \infty$, $x_{mn} = o(y_{mn}) \wedge x_{mn} = O(y_{mn})$ und $\sum |\overline{\Delta}_{mn}(x_{mn}/y_{mn})| < \infty$.

Für die Reihe (1.1) bedeute b-$\sum u_{k\ell} = \sigma$ so viel wie $s_{mn} = O(1) \wedge s_{mn} \to \sigma$ und bedeute a-$\sum u_{k\ell} = \sigma$ dasselbe wie $s_{mn} = \Omega(1) \wedge s_{mn} \to \sigma$.

Auch für jede auf $(0,\infty)\times(0,\infty)$ definierte Funktion f ist $f(x,y)\to\sigma$ für $x,y\to\infty$ im üblichen ("Pringsheimschen") Sinne gemeint, bedeutet $f(x,y)=O(1)$ dasselbe wie $\sup|f(x,y)|<\infty$ und $f(x,y)=\Omega(1)$, daß für jede Wahl der Indexfolgen $\{x_m\}$, $\{y_n\}$ und mit $t_{mn}:=f(x_m,y_n)$ gilt $t_{mn}=\Omega(1)$.

3. Umkehrsätze für R*

Wenn nichts Besonderes gesagt ist, sollen die Zahlen p und q immer ganz sein. Aus (1.3) ergibt sich dann durch Anwendung der binomischen Formel

$$R_{mn}=\sum_{r,s=0}^{p,q}\binom{p}{r}\binom{q}{s}(-1)^{r+s}\frac{1}{\lambda_{m+1}^{r}\mu_{n+1}^{s}}\sum_{k,\ell=0}^{m,n}\lambda_k^r\mu_\ell^s u_{k\ell}.$$

Spalten wir hier den Term für $(r,s)=(0,0)$ ab, so erhalten wir für die Teilsummen der Reihe (1.1) die Gleichung

$$s_{mn}=R_{mn}-\sum_{\substack{r,s=0\\(r,s)\neq(0,0)}}^{p,q}\binom{p}{r}\binom{q}{s}(-1)^{r+s}\frac{1}{\lambda_{m+1}^{p}\mu_{n+1}^{q}}\sum_{k,\ell=0}^{m,n}\lambda_k^r\mu_\ell^s u_{k\ell},$$

aus der man (bei Teil b) wegen der absoluten Permanenz von R*) folgenden Hilfssatz abliest.

HILFSSATZ 3.1. a) *Aus* bR*-$\sum u_{k\ell}=\sigma$ *folgt* b-$\sum u_{k\ell}=\sigma$, *wenn für alle* $(r,s)\in\{0,\dots,p\}\times\{0,\dots,q\}\setminus\{(0,0)\}$ *gilt*

$$\sum_{k,\ell=0}^{m,n}\lambda_k^r\mu_\ell^s u_{k\ell}=o_b(\lambda_{m+1}^r\mu_{n+1}^s). \tag{3.1}$$

b) *Aus* aR*-$\sum u_{k\ell}=\sigma$ *folgt* a-$\sum u_{k\ell}=\sigma$, *wenn für alle* $(r,s)\in$ $\in\{0,\dots,p\}\times\{0,\dots,q\}\setminus\{(0,0)\}$ *gilt*

$$\sum_{k,\ell=0}^{m,n}\lambda_k^r\mu_\ell^s u_{k\ell}=\Omega(\lambda_{m+1}^r\mu_{n+1}^s). \tag{3.2}$$

Mit Hilfssatz 3.1 läßt sich folgender Umkehrsatz für R* beweisen.

SATZ 3.2. a) *Aus* bR*-$\sum u_{k\ell}=\sigma$ *folgt* b-$\sum u_{k\ell}=\sigma$, *wenn für jedes* $r\in\{1,\dots,p\}$ *und jedes* $s\in\{1,\dots,q\}$ *die folgenden zwei Bedingungen erfüllt sind:*

$$\sum_{k=0}^{m}\lambda_k^r\sum_{\ell=0}^{n}u_{k\ell}=o_b(\lambda_{m+1}^r), \tag{3.3}$$

$$(3.4)\qquad \sum_{\ell=0}^{n} \mu_\ell^s \sum_{k=0}^{m} u_{k\ell} = o_b(\mu_{n+1}^s).$$

b) *Aus* aR*-$\sum u_{k\ell} = \sigma$ *folgt* a-$\sum u_{k\ell} = \sigma$, *wenn für jedes* $r \in \{1, \dots, p\}$ *und jedes* $s \in \{1, \dots, q\}$ *die folgenden zwei Bedingungen erfüllt sind:*

$$(3.5)\qquad \sum_{k=0}^{m} \lambda_k^r \sum_{\ell=0}^{n} u_{k\ell} = \Omega(\lambda_{m+1}^r),$$

$$(3.6)\qquad \sum_{\ell=0}^{n} \mu_\ell^s \sum_{k=0}^{m} u_{k\ell} = \Omega(\mu_{n+1}^s).$$

BEWEIS. a) Nach Hilfssatz 3.1 genügt es zu zeigen, daß (3.1) erfüllt ist. Für die Fälle $r > 0 \wedge s = 0$ und $r = 0 \wedge s > 0$ ist dies (3.3) bzw. (3.4). Also bleibt zu zeigen, daß (3.1) auch für $(r, s) \in \{1, \dots, p\} \times \{1, \dots, q\}$ gilt. Dazu sei

$$(3.7)\qquad \eta_{mn} := \lambda_{m+1}^{-r} \sum_{k=0}^{m} \lambda_k^r \sum_{\ell=0}^{n} u_{k\ell},$$

nach (3.3) also $\eta_{mn} = o_b(1)$. Da die Folge μ monoton gegen ∞ strebt, ergibt sich hieraus

$$(3.8)\qquad \mu_{n+1}^{-s} \sum_{\nu=0}^{n} (\mu_{\nu+1}^s - \mu_\nu^s)\eta_{m\nu} = o_b(1),$$

also auch

$$(3.9)\qquad \eta_{mn} - \mu_{n+1}^{-s} \sum_{\nu=0}^{n} (\mu_{\nu+1}^s - \mu_\nu^s)\eta_{m\nu} = o_b(1).$$

Die mit $\lambda_{m+1}^r \mu_{n+1}^s$ multiplizierte linke Seite in (3.9) ist aber gerade

$$\mu_{n+1}^s \sum_{k=0}^{m} \lambda_k^r \sum_{\ell=0}^{n} u_{k\ell} - \sum_{\nu=0}^{n} (\mu_{\nu+1}^s - \mu_\nu^s) \sum_{k=0}^{m} \lambda_k^r \sum_{\ell=0}^{\nu} u_{k\ell} =$$

$$= \sum_{k=0}^{m} \lambda_k^r \left\{ \mu_{n+1}^s \sum_{\ell=0}^{n} u_{k\ell} - \sum_{\ell=0}^{n} (\mu_{n+1}^s - \mu_\ell^s) u_{k\ell} \right\} = \sum_{k,\ell=0}^{m,n} \lambda_k^r \mu_\ell^s u_{k\ell},$$

so daß (3.1) erfüllt ist.

b) In Analogie zum Beweis von a) ist nur zu zeigen, daß (3.2) für $(r, s) \in \{1, \dots, p\} \times \{1, \dots, q\}$ gilt. Mit η_{mn} aus (3.7) ist wegen (3.5) zunächst $\eta_{mn} = \Omega(1)$, und hieraus folgt in Analogie zu (3.8) wegen der absoluten

Permanenz der bewichteten Mittel (vgl. Mohanty [11], Lemma 4, oder [2], Korollar 17.1) jetzt

$$\mu_{n+1}^{-s} \sum_{\nu=0}^{n} (\mu_{\nu+1}^{s} - \mu_{\nu}^{s}) \eta_{m\nu} = \Omega(1),$$

wenn man $\eta_{m0} = \Omega(1)$ beachtet. Damit ergibt sich der Rest des Beweises wie bei a).

Wegen der Monotonieeigenschaften von R* (vgl. Mears [8] und Obrechkoff [12]) ist Satz 3.2 auch anwendbar, wenn p und q nicht ganz sind. Ist etwa p nicht ganz, so muß man nur (3.3) und (3.5) für jedes $r \in \{1, \ldots, [p+1]\}$ fordern. Entsprechend ist zu verfahren, wenn q nicht ganz ist.

Für $R^* = (C, 1, 1)$ ergibt Satz 3.2.a) gerade den Satz K, während Satz 3.2.b) folgende Verallgemeinerung eines Resultats von Obrechkoff [13], Satz 4, liefert.

KOROLLAR 3.3. *Aus* a$(C,1,1)$-$\sum u_{k\ell} \doteq \sigma$ *folgt* a-$\sum u_{k\ell} = \sigma$, *wenn die folgenden zwei Bedingungen erfüllt sind:*

$$\sum_{k=1}^{m} k \sum_{\ell=0}^{n} u_{k\ell} = \Omega(m+1), \tag{3.10}$$

$$\sum_{\ell=1}^{n} \ell \sum_{k=0}^{m} u_{k\ell} = \Omega(n+1). \tag{3.11}$$

Durch (3.3) und (3.4) bzw. (3.5) und (3.6) sind jeweils $p+q$ Bedingungen gegeben. Die im folgenden Satz angegebenen stärkeren Umkehrbedingungen haben den Vorteil, von p und q unabhängig zu sein und damit für jedes Verfahren R* zu gelten.

SATZ 3.4. a) *Aus* bR*-$\sum u_{k\ell} = \sigma$ *folgt* b-$\sum u_{k\ell} = \sigma$, *wenn die folgenden zwei Bedingungen erfüllt sind:*

$$\lambda_k \sum_{\ell=0}^{n} u_{k\ell} = o_b(\overline{\Delta}\lambda_k), \tag{3.12}$$

$$\mu_\ell \sum_{k=0}^{m} u_{k\ell} = o_b(\overline{\Delta}\mu_\ell). \tag{3.13}$$

b) *Aus* aR*-$\sum u_{k\ell} = \sigma$ *folgt* a-$\sum u_{k\ell} = \sigma$, *wenn die Folgenden drei Bedingungen erfüllt sind:*

$$\lambda_k = \Omega(\lambda_{k+1}), \quad \mu_\ell = \Omega(\mu_{\ell+1}), \tag{3.14}$$

$$(3.15)\qquad \lambda_k \sum_{\ell=0}^{n} u_{k\ell} = \Omega(\overline{\Delta}\lambda_k),$$

$$(3.16)\qquad \mu_\ell \sum_{k=0}^{m} u_{k\ell} = \Omega(\overline{\Delta}\mu_\ell).$$

BEWEIS. a) Wir verwenden Satz 3.2 und zeigen, daß aus (3.12) für jedes $r > 0$ die Bedingung (3.3) folgt. Sei also $r > 0$ und

$$L_m := \sum_{k=0}^{m} (\overline{\Delta}\lambda_k)\lambda_k^{r-1}.$$

Für η_{mn} aus (3.7) erhalten wir dann

$$(3.17)\qquad \eta_{mn} = \left\{ L_m^{-1} \sum_{k=0}^{m} (\overline{\Delta}\lambda_k)\lambda_k^{r-1} \cdot \frac{\lambda_k}{\overline{\Delta}\lambda_k} \sum_{\ell=0}^{n} u_{k\ell} \right\} \cdot L_m \lambda_{m+1}^{-r},$$

wobei der Ausdruck in der geschweiften Klammer wegen (3.12) und $L_m \to \infty$ von der Form $o_b(1)$ ist, und wegen der Monotonie der Folge λ noch $L_m\lambda_{m+1}^{-r} = O(1)$ gilt. Damit ist (3.3) gezeigt. Analog folgt aus (3.13) für jedes $s > 0$ die Bedingung (3.4).

b) Wieder verwenden wir Satz 3.2 und zeigen, daß aus (3.15) und dem ersten Teil von (3.14) für jedes $r > 0$ die Bedingung (3.5) folgt. Jetzt ist in (3.17) der Ausdruck in der geschweiften Klammer wegen (3.15) und der absoluten Permanenz der bewichteten Mittel von der Form $\Omega(1)$, und wegen des ersten Teils von (3.14) gilt $L_m\lambda_{m+1}^{-r} = \Omega(1)$ nach einem Resultat von Pati [14], Lemma 2 (vgl. [18], Hilfssatz 5.3). Damit ergibt sich $\eta_{mn} = \Omega(1)$ aus dem nachfolgenden Hilfssatz 3.5. Analog folgt aus (3.16) und dem zweiten Teil von (3.14) für jedes $s > 0$ die Bedingung (3.6).

HILFSSATZ 3.5. *Aus $x_{mn} = \Omega(1)$ und $y_m = \Omega(1)$ folgt $x_{mn}y_m = \Omega(1)$.*

BEWEIS. Es ist

$$\overline{\Delta}_{mn}(x_{mn}y_m) = (\overline{\Delta}_n x_{mn})\overline{\Delta} y_m + (\overline{\Delta}_{mn} x_{mn}) y_{m-1},$$

und da $y_{m-1} = O(1)$ aus $y_m = \Omega(1)$ folgt, genügt es, noch

$$\sum_{n=0}^{\infty} |\overline{\Delta}_n x_{mn}| = O(1) \quad \text{für} \quad m \to \infty$$

zu zeigen. Dies folgt aber wegen $x_{mn} = \Omega(1)$ aus

$$\sum_{n=0}^{\infty} |\overline{\Delta}_n x_{mn}| = \sum_{n=0}^{\infty} \left| \sum_{k=0}^{m} \overline{\Delta}_k(\overline{\Delta}_n x_{kn}) \right| \leqq \sum_{m,n=0}^{\infty} |\overline{\Delta}_{mn} x_{mn}|.$$

Für $R^* = (C, 1, 1)$ ergibt Satz 3.4.a) ein Ergebnis von Young [20], Abschnitt 17, während Satz 3.4.b) folgende Verallgemeinerung eines Resultats von Obrechkoff [13], Satz 5, liefert.

KOROLLAR 3.6. *Aus* a$(C,1,1)$-$\sum u_{k\ell} = \sigma$ *folgt* a-$\sum u_{k\ell} = \sigma$, *wenn die folgenden zwei Bedingungen erfüllt sind:*

$$k\sum_{\ell=0}^{n} u_{k\ell} = \Omega(1), \tag{3.18}$$

$$\ell\sum_{k=0}^{m} u_{k\ell} = \Omega(1). \tag{3.19}$$

Da aus bR-$\sum u_{k\ell} = \sigma$ immer bR*-$\sum u_{k\ell} = \sigma$ und aus aR-$\sum u_{k\ell} = \sigma$ immer aR*-$\sum u_{k\ell} = \sigma$ folgt, darf in Hilfssatz 3.1 sowie in Satz 3.2 und Satz 3.4 jeweils R* durch R ersetzt werden.

4. Ein "high indices theorem"

In diesem Abschnitt beweisen wir ein "high indices theorem" für das Verfahren R und übertragen dabei eine Beweismethode von Minakshisundaram [10] (vgl. auch [18]) von Einfachfolgen auf Doppelfolgen.

SATZ 4.1. *Sind die Bedingungen*

$$\liminf \frac{\lambda_{m+1}}{\lambda_m} > 1 \qquad \textit{und} \qquad \liminf \frac{\mu_{n+1}}{\mu_n} > 1$$

erfüllt, so gilt:
a) *Aus* bR-$\sum u_{k\ell} = \sigma$ *folgt* b-$\sum u_{k\ell} = \sigma$.
b) *Aus* aR-$\sum u_{k\ell} = \sigma$ *folgt* a-$\sum u_{k\ell} = \sigma$.

BEWEIS. Wir zeigen zunächst, daß

$$\sum_{k,\ell=0}^{\infty} u_{k\ell} = \sigma \tag{4.1}$$

gilt. Dazu wählen wir $p+1$ Zahlen $r_1, \dots, r_{p+1}$ mit

$$1 < r_1 < \dots < r_{p+1} < \liminf(\lambda_{m+1}/\lambda_m) \tag{4.2}$$

und $q+1$ Zahlen $s_1, \dots, s_{q+1}$ mit

$$1 < s_1 < \dots < s_{q+1} < \liminf(\mu_{n+1}/\mu_n). \tag{4.3}$$

Ferner wählen wir m_0 mit $\lambda_{m+1}/\lambda_m > r_{p+1}$ für alle $m > m_0$ und n_0 mit $\mu_{n+1}/\mu_n > s_{q+1}$ für alle $n > n_0$. Dann gilt $\lambda_m < \lambda_m r_1 < \dots < \lambda_m r_{p+1} < \lambda_{m+1}$ für alle $m > m_0$, $\mu_n < \mu_n s_1 < \dots < \mu_n s_{q+1} < \mu_{n+1}$ für alle $n > n_0$, und wir erhalten mit (1.2) für alle $i = 1, \dots, p+1$, alle $j = 1, \dots, q+1$, alle $m > m_0$ und alle $n > n_0$ das lineare Gleichungssystem

$$(4.4) \qquad r_i^p s_j^q R(\lambda_m r_i, \mu_n s_j) = \sum_{\alpha,\beta=0}^{p,q} d_{\alpha\beta} R^{(p-\alpha,q-\beta)}(\lambda_m, \mu_n)$$

mit

$$d_{\alpha\beta} := \binom{p}{\alpha}\binom{q}{\beta}(r_i - 1)^\alpha (s_j - 1)^\beta$$

und

$$R^{(p-\alpha,q-\beta)}(\lambda_m, \mu_n) := \frac{1}{\lambda_m^{p-\alpha}\mu_n^{q-\beta}} \sum_{k,\ell=0}^{m,n} (\lambda_m - \lambda_k)^{p-\alpha}(\mu_n - \mu_\ell)^{q-\beta} u_{k\ell}.$$

Um das zu sehen, wende man im Ausdruck für $R(\lambda_m r_i, \mu_n s_j)$ auf $[(\lambda_m r_i - -\lambda_m) + (\lambda_m - \lambda_k)]^p$ die binomische Formel an und verfahre entsprechend mit $\mu_n s_j - \mu_\ell$. Damit haben wir bei festen $m > m_0$, $n > n_0$ ein lineares Gleichungssystem für die Unbekannten $R^{(p-\alpha,q-\beta)}(\lambda_m, \mu_n)$ mit $\alpha = 0, \dots, p$ und $\beta = 0, \dots, q$, das wir nach $R^{(0,0)}(\lambda_m, \mu_n) = s_{mn}$ auflösen wollen. Eine elementare Rechnung zeigt, daß die Koeffizientendeterminante dieses Systems den Wert

$$(4.5) \qquad D_r^{q+1} D_s^{p+1} \prod_{\alpha,\beta=0}^{p,q} \binom{p}{\alpha}^{q+1} \binom{q}{\beta}^{p+1}$$

hat, wobei D_r und D_s die Vandermondeschen Determinanten der Zahlen $r_1, \dots, r_{p+1}$ bzw. $s_1, \dots, s_{q+1}$ sind. Insbesondere ist die Koeffizientendeterminante des Systems also von 0 verschieden, und es gibt somit komplexe Zahlen c_{ij} $(i = 1, \dots, p+1;\ j = 1, \dots, q+1)$ mit

$$(4.6) \qquad s_{mn} = \sum_{i,j=1}^{p+1,q+1} c_{ij} r_i^p s_j^q R(\lambda_m r_i, \mu_n s_j) \quad \text{für} \quad m > m_0,\ n > n_0.$$

Hieraus liest man, da R beschränkt permanent ist, (4.1) ab.

Für a) ist jetzt noch zu zeigen:

$$(4.7) \qquad s_{mn} = O(1) \quad (m \to \infty) \quad \text{für alle} \quad n \in \{0, \dots, n_0\},$$

$$(4.8) \qquad s_{mn} = O(1) \quad (n \to \infty) \quad \text{für alle} \quad m \in \{0, \dots, m_0\}.$$

Wir beweisen (4.7): Es sei $n \in \{0, \dots, n_0\}$ fest und $\mu_n > 0$. Zu den $p+1$ Zahlen $r_1, \dots, r_{p+1}$ mit (4.2) wählen wir jetzt $q+1$ Zahlen $\sigma_1, \dots, \sigma_{q+1}$ mit

$$(4.9) \qquad \mu_n < \mu_n\sigma_1 < \dots < \mu_n\sigma_{q+1} < \mu_{n+1}.$$

Damit läuft, mit σ_j an Stelle von s_j, formal alles wie oben. Es gibt also komplexe Zahlen γ_{ij} ($i = 1, \dots, p+1$; $j = 1, \dots, q+1$), mit denen in Analogie zu (4.6) jetzt

$$s_{mn} = \sum_{i,j=1}^{p+1,q+1} \gamma_{ij} r_i^p \sigma_j^q R(\lambda_m r_i, \mu_n \sigma_j) \quad \text{für} \quad m > m_0 \tag{4.10}$$

gilt. Hieraus liest man (4.7) ab. Ist $n = 0$ und $\mu_0 = 0$, so gilt mit den $p+1$ Zahlen $r_1, \dots, r_{p+1}$ mit (4.2) für alle $i = 1, \dots, p+1$ und alle $m > m_0$ jetzt

$$r_i^p R\left(\lambda_m r_i, \frac{\mu_1}{2}\right) = \sum_{\alpha=0}^{p} \binom{p}{\alpha} (r_i - 1)^\alpha \frac{1}{\lambda_m^{p-\alpha}} \sum_{k=0}^{m} (\lambda_m - \lambda_k)^{p-\alpha} u_{k0},$$

und hieraus folgt $s_{m0} = O(1)$ wie im Falle des Riesz-Verfahrens zur Limitierung von Einfachfolgen (vgl. Minakshisundaram [10] und [18]). Die Behauptung (4.8) wird wie (4.7) bewiesen.

Für b) ist jetzt noch zu zeigen:

$$\sum_{m=m_0}^{\infty} |\overline{\Delta}_{mn} s_{mn}| < \infty \quad \text{für alle} \quad n \in \{0, \dots, n_0\}, \tag{4.11}$$

$$\sum_{n=n_0}^{\infty} |\overline{\Delta}_{mn} s_{mn}| < \infty \quad \text{für alle} \quad m \in \{0, \dots, m_0\}, \tag{4.12}$$

Um (4.11) zu beweisen, geht man wie beim Beweis von (4.7) vor und (4.12) beweist man wie (4.8).

Der erste Teil des Beweises von Satz 4.1 liefert das in der Einleitung erwähnte "high indices theorem" von Mears [8], Theorem XII. Auch zwei dazu ähnliche Ergebnisse von Mears [8], Theorems X und XI, lassen sich mit unserer Methode beweisen.

Daß man in Satz 4.1 das Verfahren R durch R* ersetzen darf, ist nicht zu erwarten, da das "high indices theorem", wie Kuttner [7] gezeigt hat, schon für das "unstetige" Riesz-Verfahren zur Summierung von Einfachfolgen nicht uneingeschränkt gilt.

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DEPARTMENT OF MATHEMATICS
AND COMPUTER SCIENCE
BAR-ILAN UNIVERSITY
RAMAT-GAN 52900
ISRAEL

UNIVERSITÄT STUTTGART
MATHEMATISCHES INSTITUT A
D-7000 STUTTGART-80
BUNDESREPUBLIK DEUTSCHLAND

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THE SPACE OF DENSITY CONTINUOUS FUNCTIONS

K. CIESIELSKI (Morgantown) and L. LARSON (Louisville)

We denote by $\mathbf{R}_d$ the set of real numbers, $\mathbf{R}$, endowed with the density topology. A function $f\colon \mathbf{R}_d \to \mathbf{R}_d$ is said to be *density continuous,* if it is continuous with respect to the topology on $\mathbf{R}_d$ in both the domain and range. The set of density continuous functions has been studied in several limited ways. Bruckner [1] and Niewiarowski [3] have studied density continuous functions which are homeomorphisms under the standard topology on $\mathbf{R}$. Ostaszewski has investigated the local behavior of density continuous functions [4] and has investigated their behavior as a semigroup [5].

In this paper, we consider the composition of the set of density continuous functions. The structure of this set seems to be quite complicated. Ostaszewski [5] has noted that it is not closed under uniform convergence. In Example 2 we show that it is not a vector space. Corollary 3 shows that each real-analytic function is density continuous, but Example 1 is a C^∞ function which is not density continuous. It is not difficult to construct a density continuous function which is not continuous. On the other hand, every density continuous function must be approximately continuous.

In what follows, the right (left) unilateral derivatives of a function f are represented as f^+ (f^-). The Lebesgue measure of a set A is denoted by $|A|$ and the Lebesuge density (right, left Lebesgue density) of A at a point x is written as $d(A,x)(d^+(A,x),d^-(A,x))$. The set of functions which are infinitely differentiable on $\mathbf{R}$ is written as C^∞. Finally, if A and B are two sets such that $\sup A \leqq \inf B$, then we write $A \ll B$.

Before stating the main result, we first present the following lemma.

LEMMA 1. *Suppose I is a compact interval and $f\colon I \to \mathbf{R}$. If there exist numbers α and β such that*

$$(1) \qquad 0<\alpha<\frac{f(x)-f(y)}{x-y}<\beta<\infty, \text{ for all } x,y\in I,\ \ x\neq y,$$

then f is density continuous on I.

PROOF. From (1) it is easy to see that f is strictly increasing and continuous on I. If $g=f^{-1}$, then it follows from (1) that

$$(2) \qquad 0<\frac{1}{\beta}<\frac{g(u)-g(v)}{u-v}<\frac{1}{\alpha}, \text{ for all } u,v\in f(I),\ \ u\neq v.$$

The right-hand inequality in (2) implies that g is a Lipschitz function on $f(I)$ and hence g is absolutely continuous and g' is bounded above a.e. The left-hand inequality in (2) shows that g' is bounded away from 0 on $f(I)$ a.e. Now a result of Bruckner [1, Corollary 1] shows that g preserves density points. This implies the density continuity of f.

THEOREM 1. *If I is an open interval and $f\colon I \to \mathbf{R}$ is convex, then f is density continuous.*

PROOF. Fix a point $a \in I$. It will be shown that f is right density continuous at a. To do this, we lose no generality in supposing that $f(a) = = a = 0$, because the translation of a density continuous function is obviously density continuous.

According to [6, Theorem 10.11], there exists a nondecreasing function $h\colon I \to \mathbf{R}$ such that

$$f(x) = \int_0^x h(t)dt, \text{ for all } x \in I. \tag{3}$$

Because of this, it is easy to see that there must exist a real number $b > 0$ such that f is monotone on $[0,b]$. We may assume that f is strictly monotone on $[0,b]$ because if it is not, f must be constant on some right neighborhood of 0, and right density continuity at 0 follows at once. With this assumption, f is a homeomorphism from $[0,b]$ onto $f([0,b])$. Denote $g = (f|_{[0,b]})^{-1}$.

There are now two cases to consider, depending upon whether f is strictly increasing or strictly decreasing on $[0,b]$.

Assume first that f is strictly decreasing on $[0,b]$. Then by (3), $h < 0$ on $[0,b)$. There is no generality lost in assuming $h(b) < 0$. If $0 \leqq x < y \leqq b$, then considering the average value of h on (x,y) and recalling that h is nondecreasing, it is obvious that

$$0 > h(b) \geqq \frac{\int_x^y h}{y-x} = \frac{f(y)-f(x)}{y-x} \geqq h(0).$$

This implies

$$0 < -h(b) < \frac{(-f(y)) - (-f(x))}{y-x} < -h(0) < \infty, \text{ for all } x,y \in [0,b].$$

($h(0)$ is finite because h is monotone on a neighborhood of 0.) Lemma 1 now shows that $-f$ is density continuous on $[0,b]$. Since density continuity is easily shown to be preserved under constant multiplication, it follows that f is density continuous on $[0,b]$ and therefore right density continuous at 0.

Next, assume that f is strictly increasing on $(0,b)$ and that $I_n = [a_n, b_n]$ is a sequence of disjoint intervals from $(0, f(b))$ such that I_n decreases to 0 and

$$\frac{\left|\bigcup_{n=1}^{\infty} I_n \cap (0,t)\right|}{t} > \varrho > 0, \text{ for all } t \in (0, f(b)). \tag{4}$$

Let $S = \bigcup_{n=1}^{\infty} I_n$, $J_n = g(I_n)$ and $G_n = (b_{n+1}, a_n)$. From (4), it follows that

$$\frac{\left|\bigcup_{k=n}^{\infty} I_k\right|}{\left|\bigcup_{k=n-1}^{\infty} G_k\right|} > \frac{\varrho}{1-\varrho}, \text{ for all } n > 1. \tag{5}$$

Before proceeding with the proof, we make the following useful observations. From (3) and the assumption that f is increasing we see that $h > 0$ on $(0,b)$. Let A and B be intervals contained in $(0,b)$ such that $A \ll B$. Then because h is nondecreasing,

$$\frac{|f(A)|}{|A|} = \frac{\int_A h}{|A|} \leqq \sup_{t\in A} h(t) \leqq \inf_{t\in B} h(t) \leqq \frac{\int_B h}{|B|} = \frac{|f(B)|}{|B|}.$$

This implies the statement

$$|g(C)| \geqq |g(D)| \frac{|C|}{|D|} \tag{6}$$

for all intervals C and D from $(0, f(b))$ such that $C \ll D$, and this estimate immediately extends to the case when C, D are finite unions of disjoint intervals.

We define an infinite partition S_n of S as follows. Let $\alpha_1 = a_1$. By (5), there exists an $\alpha_2' < \alpha_1$ such that

$$\frac{|(\alpha_2', \alpha_1) \cap S|}{|G_1|} = \frac{\varrho}{1-\varrho}.$$

Let $\alpha_2 = \min\{\alpha_2', a_2\}$. Assume that α_k has been chosen for $k = 1, 2, \dots, n-1$ so that either $\alpha_k \geqq a_k$ or $\alpha_k < a_k$ and

$$\frac{|(\alpha_k, \alpha_{k-1}) \cap S|}{|G_{k-1}|} = \frac{\varrho}{1-\varrho},$$

and equality holds if $\alpha_k < a_k$. Choose $\alpha'_n < \alpha_{n-1}$ such that

$$\frac{|(\alpha'_n, \alpha_{n-1}) \cap S|}{|G_{n-1}|} = \frac{\varrho}{1-\varrho}.$$

To see that such a choice is possible, there are two cases to consider, depending on α_{n-1}. If $\alpha_{n-1} = a_{n-1}$, it can be seen immediately from (5). In case $\alpha_{n-1} < a_{n-1}$, let

$$m = \max\{k < n : \alpha_k = a_k\}.$$

Then $|(\alpha_k, \alpha_{k-1}) \cap S| = \varrho|G_{k-1}|/(1-\varrho)$ for $m+1 \leqq k \leqq n-1$ so that

$$|(\alpha_{n-1}, \alpha_m) \cap S| = \frac{\varrho}{1-\varrho} \sum_{k=m}^{n-1} |G_{k-1}|. \tag{7}$$

According to (5), there is a $t < \alpha_{n-1}$ such that

$$|(t, \alpha_m) \cap S| = \frac{\varrho}{1-\varrho} \sum_{k=m}^{n} |G_{k-1}|. \tag{8}$$

Subtracting (7) from (8) gives

$$|(t, \alpha_{n-1}) \cap S| = \frac{\varrho}{1-\varrho}|G_{n-1}|.$$

We set $\alpha'_n = t$ in this case. Then let $\alpha_n = \min\{\alpha'_n, a_n\}$. Define $S_n = [\alpha_{n+1}, \alpha_n) \cap S$. From the choice of $\alpha_n \leqq a_n$, and the fact that $a_n \notin S_n$, we see $\sup S_n \leqq b_{n+1}$. So $S_n \ll G_n = (b_{n+1}, a_n)$ and

$$\frac{|S_n|}{|G_n|} \geqq \frac{\varrho}{1-\varrho}.$$

Finally, we use (6) and the preceding inequality to see

$$\frac{\left|g\left(\bigcup_{n=1}^{\infty} S_n\right)\right|}{\left|g\left(\bigcup_{n=1}^{\infty} G_n\right)\right|} = \frac{\sum_{n=1}^{\infty} |g(S_n)|}{\sum_{n=1}^{\infty} |g(G_n)|} \geqq \frac{\sum_{n=1}^{\infty} |g(G_n)|\frac{|S_n|}{|G_n|}}{\sum_{n=1}^{\infty} |g(G_n)|} \geqq \frac{\varrho}{1-\varrho}.$$

Hence,

$$\frac{\left|g\left(\bigcup_{n=1}^{\infty} S_n\right)\right|}{|g((0, a_1))|} \geqq \varrho.$$

Because ϱ can be made as close to 1 as desired, we see that f is right density continuous at 0.

Similar arguments show that f is left density continuous at every point of I. This completes the proof of the theorem.

COROLLARY 1. *If* $g\colon [a,b] \to \mathbf{R}$ *is convex on* (a,b) *and* $\{g^+(a), g^-(b)\} \subset \subset \mathbf{R}$, *then* g *is density continuous.*

PROOF. Define

$$f(x) = \begin{cases} g^+(a)(x-a) + g(a) & \text{if } x < a, \\ g(x) & \text{if } a \leqq x \leqq b, \\ g^-(b)(x-b) + g(b) & \text{if } x > b \end{cases}$$

and apply Theorem 1.

By using $g = -f$ in Theorem 1 and Corollary 1 we arrive at the following corollary.

COROLLARY 2. *If* g *is concave downward on an open interval* I, *then* g *is density continuous on* I. *Further, if* g *is concave downward on the interval* $[a,b]$ *with both* $g^+(a)$ *and* $g^-(b)$ *finite, then* g *is density continuous on* $[a,b]$.

Ostaszewski [5, Question 4] asked whether polynomials are density continuous. The following corollary provides an affirmative answer to this question.

COROLLARY 3. *Real analytic functions are density continuous.*

PROOF. If f is real analytic, then f' is finite everywhere and f'' has only a finite number of zeroes in every interval, so applications of Corollaries 1 and 2 suffice to establish this corollary.

COROLLARY 4. *If* $f(x) = x^\alpha$ *for* $\alpha \in \mathbf{R}$, *then* f *is density continuous on its domain.*

PROOF. If $\alpha \leqq 0$, then this follows directly from Theorem 1. If $\alpha \geqq 1$, then this corollary is a consequence of Corollary 1.

Suppose $0 < \alpha < 1$. It is clear that Theorem 1 implies f is density continuous on $\mathrm{Dom}(f) \setminus \{0\}$. So, it must be shown that f is density continuous at 0.

Let $h > 0$ and suppose $A \subset (0,h)$. Then, we use the fact that $(f^{-1})'$ is an increasing function to see

$$\frac{|f^{-1}(A)|}{f^{-1}(h)} = \frac{1}{h^{1/\alpha}} \int_A \frac{x^{1/\alpha)-1}}{\alpha} \geqq \frac{1}{h^{1/\alpha}} \int_0^{|A|} \frac{x^{(1/\alpha)-1}}{\alpha} = \frac{|A|^{1/\alpha}}{h^{1/\alpha}} = (|A|/h)^{1/\alpha}.$$

It follows from this inequality that f is right density continuous at 0. A similar argument holds from the left.

EXAMPLE 1. There is a function $f \in C^\infty$ which is not density continuous.

Choose any sequence of disjoint intervals $J_n = [a_n, b_n] \subset [0,1]$ decreasing to 0 such that

$$d^+\left(\bigcup_{n=1}^{\infty} J_n, 0\right) = 0 \tag{9}$$

and let h be a C^∞ function satisfying

(10) $\quad h(0)=0,\ \ h(1)=1,$ and $h^{(n)}(0)=h^{(n)}(1)=0,$ for all $n\in\mathbf{N}$.

(An example of such a function is

$$h(x)=\varrho\int_0^x \exp(-1/t^2-1/(t-1)^2)dt,$$

for suitable ϱ.) Let

(11) $$\alpha_n=\max\{|h^{(k)}(x)|: 0\leqq k\leqq n \text{ and } 0\leqq x\leqq 1\}\geqq 1,$$

(12) $$h_n(x)=\begin{cases}0 & \text{if } x<a_n,\\ \frac{a_n(b_n-a_n)^n}{\alpha_n}h\left(\frac{x-a_n}{b_n-a_n}\right) & \text{if } x\in J_n,\\ \frac{a_n(b_n-a_n)^n}{\alpha_n} & \text{if } x>b_n\end{cases}$$

and

$$f(x)=\sum_{n=1}^{\infty}h_n(x).$$

From the choice of h, we see that $h_n\in C^\infty$ for each n. Obviously, using (9) and (11), it follows that

(13) $$\sum_{n=1}^{\infty}\frac{a_n(b_n-a_n)^n}{\alpha_n}\leqq\sum_{n=1}^{\infty}(b_n-a_n)<\infty,$$

so that f exists everywhere. Moreover, because the J_n are pairwise disjoint, it follows that f is infinitely differentiable on $\mathbf{R}\setminus 0$ and continuous on $\mathbf{R}$.

To prove that $f^{(k+1)}(0)$ exists and equals 0, let us assume that $f^{(k)}(0)=0$ and choose $a_n\leqq s<a_{n-1}$ for some $n>k$. Then it follows from (11) and (12) that

$$\frac{f^{(k)}(s)-f^{(k)}(0)}{s-0}=\begin{cases}\frac{1}{s}\sum\limits_{i=n}^{\infty}h_i(s)\leqq\sum_{i=n}^{\infty}(b_j-a_j)^j<b_n & \text{if } k=0,\\ \frac{1}{s}h_n^{(k)}(s)\leqq\frac{a_n(b_n-a_n)^n-\alpha_k}{s\alpha_n(b_n-a_n)^k}\leqq b_n-a_n<b_n & \text{if } k>0.\end{cases}$$

Since $s\to 0$ implies $b_n\to 0$, this shows $f^{(k+1)}(0)=0$. Therefore, f is a C^∞ function.

But, f cannot be density continuous because of (9) and the fact that

$$f\left(\mathbf{R}\setminus\bigcup_{n=1}^{\infty}J_n\right)$$

is countable.

EXAMPLE 2. There is a continuous, density continuous function $f\colon \mathbf{R} \to \mathbf{R}$ such that $f(x) + x$ is not density continuous.

To construct such a function, we first choose two differentiable functions h_1 and h_2 satisfying:

(i) $0 < h_1 < h_2$ on $(0, \infty)$;

(ii) $h_1(x) = h_2(x) = x$ for $x \leqq 0$; and,

(iii) $1/2 < h_1'(x) < 1 < h_2'(x) < 2$ when $x > 0$.

Let a_n and b_n be any two sequences converging to 0 such that $1 = b_1 > a_1 > b_2 > a_2 > \ldots$, and both

$$\frac{h_2(b_n) - h_1(a_n)}{b_n - a_n} = 2 \quad \text{and} \quad \frac{h_1(a_n) - h_2(b_{n+1})}{a_n - b_{n+1}} = 1/2. \tag{14}$$

Define a piecewise linear function f_0 by letting $f_0(a_n) = h_1(a_n)$, $f_0(b_n) = h_2(b_n)$ and $f_0(x) = x + f_0(b_1) - b_1$ when $x > 1$ and $f_0(x) = x$ when $x \leqq 0$. The function f_0 is easily seen to be continuous because h_1 and h_2 are continuous and have value 0 at 0. Equation (14) implies

$$\frac{1}{2} \leqq \frac{f_0(b) - f_0(a)}{b - a} \leqq 2, \text{ for all } a, b \in (0, \infty).$$

It follows from Lemma 1 that f must be density continuous.

Denote $A(1/2) = \bigcup_{n=1}^{\infty} [b_{n+1}, a_n]$ and $A(2) = \bigcup_{n=1}^{\infty} [a_n b_n]$. Either

$$(-\infty, 0] \cup A(1/2) \quad \text{or} \quad (-\infty, 0) \cup A(2)$$

has positive upper density at 0. Without loss of generality we assume that it is the former. Then $f_1(x) = f_0(x) - x/2$ is constant on each component of $A(1/2)$. But this implies that $|f_1(A(1/2))| = 0$ and $A(1/2) = f_1^{-1}(f_1(A(1/2)))$ has positive density at 0. Therefore, f_1 is not density continuous at 0. So, it is enough to define $f(x) = -2f_0(x)$ to obtain the desired function.

We note that the f in Example 2 can actually be constructed as a C^∞ function by a method analogous to the construction in Example 1.

This example answers questions posed by Ostaszewski [5, Questions 5 and 6].

We wish to thank Krzysztof Ostaszewski for bringing to our attention several of the questions we have considered here.

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DEPARTMENT OF MATHEMATICS
METHODIST COLLEGE
FAYETTEVILLE, NC 28311
USA

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LOUISVILLE
LOUISVILLE, KY 40292
USA

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CONVOLUTION RINGS OF MULTIPLICATIONS OF AN ABELIAN GROUP

J. R. CLAY (Tucson)

1. Introduction

For an abelian group $(A,+)$, the group of left and right distributive multiplications, Mult A, and the group of left distributive multiplications, $\mathrm{Mult}_L A$, have been a source of interesting abelian groups [1, 2, 4, 5, 6, 7, 8, 9, 10, 14, 15, 16, 17, 18], and were first suggested for study by Baer [7]. Regarding the related question as to whether Mult A, or $\mathrm{Mult}_L A$, could themselves be the additive group of interesting rings, it is natural to take motivation or direction from the ring of $\mathcal{L}_1$ functions from the reals $\mathbf{R}$ to $\mathbf{R}$ with respect to convolution $*$, where

$$f*g(x)=\int_{-\infty}^{+\infty} f(x-t)g(t)dt.$$

For an arbitrary but fixed finite subset $X \subseteqq A$, this convolution operation motivates the following two operations for $\mathrm{Mult}_L A$ and/or Mult A.

$$\alpha\cdot\beta(a,b)=\sum_{x\in X}\alpha(x,\beta(a,b)); \tag{1}$$

$$\alpha\cdot\beta(a,b)=\sum_{x\in X}\alpha(a,\beta(x,b)). \tag{2}$$

In addition to these two operations being closed binary operations, it is straightforward, but tedious, to show that they are i) associative, ii) left distributive over $+$, and iii) right distributive over $+$. In short, if $\cdot$ is operation (1) or (2), then $(\mathrm{Mult}_L A,+,\cdot)$ is an associative ring with subring $(\mathrm{Mult}\,A,+,\cdot)$.

Since the structure $\mathrm{Mult}_L A \cong \mathrm{Map}(A,\mathrm{End}\,A)$, the group of all mappings from A to the endomorphisms of A, End A [2], it is considerably easier to study the rings on $\mathrm{Mult}_L A$. We have $\mathrm{Mult}\,A \cong \mathrm{Hom}(A,\mathrm{End}\,A)$, the homomorphisms from A to End A [7], but the rings on Mult A will not be considered much in this work.

Numerous interesting and amusing properties will be exhibited for the rings on the $\mathrm{Mult}_L A$ with operations (1) and (2). It will also be shown that operations (1) and (2) are but special cases of a more general and powerful construction method. This more general method will provide ways of making rings on R-modules M, and some unusual R-modules will be used to illustrate this method. See Theorem 1 and the examples following it.

The elements of Mult A are the left and right distributive multiplications on A. That is, mappings $\alpha\colon A \times A \to A$ such that $\alpha(a, b+c) = \alpha(a,b)+$ $+\alpha(a,c)$ and $\alpha(a+b,c) = \alpha(a,c) + \alpha(b,c)$ for all $a,b,c \in A$. The elements of $\mathrm{Mult}_L A$ are the left distributive multiplications on A, so they are the mappings $\alpha\colon A \times A \to A$ such that $\alpha(a,b+c) = \alpha(a,b) + \alpha(a,c)$ for all $a,b,c \in A$.

Most of our results will be about $\mathrm{Mult}_L A$, with operation (1). We note in Examples 5 that $\mathrm{Mult}_L A$, with operation (2), is an opposite ring. Hence, the results relative to $\mathrm{Mult}_L A$, and operation (1), should have companion results, like Propositions 3 and 4, and like Proposition 14 with its Corollary 15. This is somewhat surprising if one only takes a superficial look at the definitions of operations (1) and (2).

2. Rings from modules, and applications

Examples 4 and 5 below show that the multiplications (1) and (2) for $\mathrm{Mult}_L A$, or Mult A, are special cases of the more general case described in

THEOREM 1. *Let R be a ring with left R-module M. Fix an $f \in$ $\in \mathrm{Hom}_R(M,R)$, and define $\cdot = \cdot_f$ on M by $a \cdot_f b = f(a)b$. Then $(M,+,\cdot_f)$ is a ring.*

The proof is direct.

Our subsequent work will be centered about the following five types of examples.

EXAMPLES 1. For an R-module M, suppose we have something like a bilinear map $\langle\, ,\rangle\colon M \times M \to R$, but really, all we require is a) $\langle\, ,\rangle\colon M\times$ $\times M \to R$; b) $\langle\, ,\rangle(a,x+y) = \langle\, ,\rangle(a,x) + \langle\, ,\rangle(a,y)$; and c) $\langle\, ,\rangle(a,rx) = r\langle\, ,\rangle(a,x)$, for all $a,x,y \in M$ and for all $r \in R$. Define $F_{\langle a,\cdot\rangle}\colon M \to R$ by $F_{\langle a,\cdot\rangle}(b) =$ $= \langle\, ,\rangle(a,b) = \langle a,b\rangle$. Then each $F_{\langle a,\cdot\rangle} \in \mathrm{Hom}_R(M,R)$. Let $\cdot_a$ be the multiplication on M defined via Theorem 1. So $(M,+,\cdot_a)$ is a ring. For $c \in M$, let $I_c = \{rc \mid r \in \in R\}$. Then I_c is a left ideal, and I_c is an ideal if $\langle a,c\rangle = 0$.

For $R = \mathbf{R}$, the field of real numbers, and $M = \mathbf{R}^n$, the n-dimensional vector space over $\mathbf{R}$, let $\langle\, ,\rangle$ be the usual inner product. For a fixed $a =$ $= (a_1, \ldots, a_n)$, then $x \cdot y = (a_1x_1 + \ldots + a_nx_n)y$. For $c = (c_1, \ldots, c_n) \in M$, I_c is an ideal if and only if $\langle a,c\rangle = 0$.

If I_c is such an ideal, then $M/I_c \cong \mathbf{R}$. For $n = 2$, it is interesting to determine the identity in M/I_c and the isomorphism.

EXAMPLES 2. These are really special cases of Examples 1, but we single them out because of their unifying effect and because they are related also to

Examples 3 and Examples 4. Let $M = R^{(S)}$ be the free R-module on the set S [13]. For a finite subset $X \subseteqq S$, define $F_X : M \to R$ by $F_X(a) = \sum_{x \in X} a_x$. Then $F_X \in \mathrm{Hom}_R(M, R)$, and defines a ring $(M, +, \cdot_X)$ with $a \cdot_X b = \left(\sum_{x \in X} a_x\right) b$. The map F_X is an epimorphism.

EXAMPLES 3. Let $(Y, \mathcal{A}, \mu)$ be a measure space. (The terminology and notation used here will be influenced by that of Hewitt and Stromberg [11].) Fix an $X \in \mathcal{A}$. Let M_L be the family of all functions $\alpha\colon Y \times \mathbf{R} \to \mathbf{R}$ which satisfy:

a) for each $b \in \mathbf{R}$, $\alpha(\cdot, b) \in \mathcal{L}_1(Y, \mathcal{A}, \mu)$, i.e.,

$$\int_Y \alpha(y, b) d\mu(y)$$

exists;

b) for $f \in \mathcal{L}_1(Y, \mathcal{A}, \mu)$ and $a \in Y$,

$$\int_Y \alpha(a, f(y)) d\mu(y) = \alpha\Big(a, \int_Y f(y) d\mu(y)\Big);$$

c) for each $y \in Y$, $\alpha(y, \cdot) \in \mathrm{Hom}_R(\mathbf{R}, \mathbf{R})$, i.e.,

$$\alpha(y, sa + tb) = s\alpha(y, a) + t\alpha(y, b)$$

for all $a, b, s, t \in \mathbf{R}$.

Now $(\mathbf{R}^{Y \times \mathbf{R}}, +)$ is an abelian group, and M_L is a subgroup. It is direct to see that M_L is an $\mathbf{R}$-module. Define $F_X\colon M_L \to \mathbf{R}$ by $F_X(\alpha) = \int_X \alpha(x, 1) d\mu(x)$. Then $F_X \in \mathrm{Hom}_R(M_L, \mathbf{R})$ and defines via Theorem 1, a ring $(M_L, +, \cdot_X)$ where for $\alpha, \beta \in M_L$, one gets

$$\alpha \cdot \beta(a, b) = \int_X \alpha(x, \beta(a, b)) d\mu(x).$$

One should be assured that there are nontrivial M_L's.

Let $Y = X = \{0, 1, 2, \dots\}$ with $\mu(\{i\}) = 1$ for each $i \in Y$. If $\alpha(i, b) = b/i!$, then $\alpha \in M_L$.

Let $Y = X = [0, 1]$ with the usual Riemann integral. If $\phi\colon [0, 1] \to \mathbf{R}$ is continuous, and $l_m(a) = ma$, then $\alpha(x, a) = \phi(x) l_m(a)$ defines an element of M_L.

Let $(Y, \mathcal{A}, P)$ denote a probability space. Fix $X \in \mathcal{A}$, and let $f\colon Y \to \mathbf{R}$ be a random variable with finite expectation. Define $\alpha\colon Y \times \mathbf{R} \to \mathbf{R}$ by $\alpha(y, a) = f(y)a$. Then $\alpha \in M_L$. Here,

$$P(X)^{-1} F_X(\alpha) = P(X)^{-1} \int_X \alpha(x, 1) dP(x)$$

is exactly the conditional expectation of the random variable $f = \alpha(\cdot, 1)$ given X [12, p. 338].

PROPOSITION 2. *Suppose X has a binary operation $+$ and let $M = \{\alpha \in M_L \mid$ for each $a \in \mathbf{R}$, $\alpha(x+y,a) = \alpha(x,a) + \alpha(y,a)$ for all $x, y \in X\}$. Then M is a left ideal of M_L.*

PROOF. It is direct to see that $(M, +)$ is a subgroup of $(M_L, +)$. For $\alpha \in M$ and $\gamma \in M_L$, we get

$$\gamma \cdot \alpha(x+y,a) = \int_X \gamma(t, \alpha(x+y,a))\, d\mu(t) = \int_X \gamma(t, \alpha(x,a) + \alpha(y,a))\, d\mu(t) =$$

$$= \int_X \gamma(t, \alpha(x,a))\, d\mu(t) + \int_X \gamma(t, \alpha(y,a))\, d\mu(t) = \gamma \cdot \alpha(x,a) + \gamma \cdot \alpha(y,a).$$

So $\gamma \cdot \alpha \in M$.

EXAMPLES 4. We now consider $\mathrm{Mult}_L A$, or $\mathrm{Mult}\, A$, for an abelian group $(A, +)$. Fix a finite subset $X \subseteqq A$. For operation (1), we have

$$\alpha \cdot \beta(a,b) = \sum_{x \in X} \alpha(x, \beta(a,b)) = \Big(\Big(\sum_{x \in X} \alpha_x\Big) \circ \beta\Big)(a,b)$$

where $\alpha_x(c) = \alpha(x,c)$.

Now define $F_X \colon \mathrm{Mult}_L A \to \mathrm{End}\, A$ by $F_X(\alpha) = \sum_{x \in X} \alpha_x$. Now $\mathrm{Mult}_L A$, or $\mathrm{Mult}\, A$, is an $\mathrm{End}\, A$-module and $F_X \in \mathrm{Hom}_{\mathrm{End}\, A}(\mathrm{Mult}_L A, \mathrm{End}\, A)$ is an epimorphism. Thus, Theorem 1 shows that $(\mathrm{Mult}_L A, +, \cdot)$ is a ring with subring $(\mathrm{Mult}\, A, +, \cdot)$ if $\cdot$ is defined by 1).

PROPOSITION 3. $(\mathrm{Mult}\, A, +, \cdot)$ *is a left ideal in* $(\mathrm{Mult}_L A, +, \cdot)$.

PROOF. For $\alpha \in \mathrm{Mult}_L A$ and $\mu \in \mathrm{Mult}\, A$, we get

$$\alpha \cdot \mu(a+b, c) = \sum_{x \in X} \alpha(x, \mu(a+b, c)) =$$

$$= \sum_{x \in X} \alpha(x, \mu(a,c)) + \sum_{x \in X} \alpha(x, \mu(b,c)) = \alpha \cdot \mu(a,c) + \alpha \cdot \mu(b,c).$$

Thus, $\alpha \cdot \mu \in \mathrm{Mult}_L A$.

EXAMPLES 5. Consider operation (2) for $\mathrm{Mult}_L A$ and $\mathrm{Mult}\, A$. Then $\alpha * \beta(a,b) = \sum_{x \in X} \alpha(a, \beta(x,b)) = \alpha\Big(a, \big(\sum_{x \in X} \beta_x\big)(a)\Big)$. Let $\mathcal{E}(A)$ be the opposite ring of $\mathrm{End}\, A$. Then $\mathrm{Mult}_L A$ and $\mathrm{Mult}\, A$ are $\mathcal{E}(A)$-modules, where $f * \alpha(a,b) = \alpha(a, f(b))$. The $F_X \colon \mathrm{Mult}_L A \to \mathcal{E}(A)$ defined in Examples 4 is also in $\mathrm{Hom}_{\mathcal{E}(A)}(\mathrm{Mult}_L A, \mathcal{E}(A))$. Define $*$ on $\mathrm{Mult}_L A$ by $\alpha * \beta = F_X(\beta) * \alpha$, the opposite ring of $\mathrm{Mult}_L A$ defined by F_X via Theorem 1. That is, from Theorem 1, we would have $\beta \cdot \alpha = F_X(\beta) * \alpha$, and the opposite ring is $\alpha * \beta = \beta \cdot \alpha = F_X(\beta) * \alpha$.

PROPOSITION 4. $(\mathrm{Mult}\,A,+,*)$ *is a right ideal in* $(\mathrm{Mult}_L A,+,*)$.

PROOF. For $\alpha \in \mathrm{Mult}_L A$ and $\mu \in \mathrm{Mult}\,A$, we get $\mu * \alpha(a+b,c) = = \sum_{x\in X} \mu(a+b,\alpha(x,c)) = \sum_{x\in X} \mu(a,\alpha(x,c)) + \sum_{x\in X} \mu(b,\alpha(x,c)) = \mu * \alpha(a,c) + +\mu * \alpha(b,c)$, so $\mu * \alpha \in \mathrm{Mult}\,A$.

As far as the construction of rings $(M,+,\cdot)$ via Theorem1, only the elements of the image of f are involved, and the image of f is a subring R' of R, and certainly M is an R'-module. So, there is no loss in assuming that $f \in \mathrm{Hom}_R(M,R)$ is an epimorphism.

We have $(M,+)$ as an M-module and also an R-module. Let $\mathrm{Ann}_M M = = \{a \in M \mid ax = 0 \text{ for each } x \in M\}$ and $\mathrm{Ann}_R M = \{r \in R \mid rx = 0 \text{ for each } x \in M\}$.

THEOREM 5. *Let M be a left faithful R-module. Fix $f \in \mathrm{Hom}_R(M,R)$. Then the kernel of f is* $\ker f = \mathrm{Ann}_M M$.

PROOF. It is direct to see that $\ker f \subseteqq \mathrm{Ann}_M M$. For $a \in \mathrm{Ann}_M M$, we have $a \cdot b = 0$ for each $b \in M$, so $f(a)b = 0$ for each $b \in M$, thus $f(a) \in \mathrm{Ann}_R M$. This means that $f(\mathrm{Ann}_M M) \subseteqq \mathrm{Ann}_R M$. Since M is a faithful R-module, $\mathrm{Ann}_R M = \{0\}$, so $f(a) = 0$ and $\mathrm{Ann}_M M \subseteqq \ker f$.

REMARK. Examples 3, 4, and 5, have the modules as unitary and faithful. Many cases from Examples 1 and 2 are also unitary and faithful.

THEOREM 6. *Let M be a faithful R-module and let $f \in \mathrm{Hom}_R(M,R)$ be an epimorphism. Then*

$$\frac{M}{\mathrm{Ann}_M M} \cong R.$$

PROOF. $f(a \cdot b) = f(f(a)b) = f(a)f(b)$. Now apply Theorem 5.

COROLLARY 7. *Let $M = \mathrm{Mult}_L A$. Then $M/\mathrm{Ann}_M M \cong \mathrm{End}\,A$, and* $\mathrm{Ann}_M M = \left\{\alpha \in M \mid \sum_{x\in X} \alpha_x = 0\right\}$.

PROOF. As seen in Examples 4, $F_X \in \mathrm{Hom}_{\mathrm{End}\,A}(\mathrm{Mult}_L A, \mathrm{End}\,A)$ is an epimorphism. Now apply Theorem 6.

Let R be a ring with identity 1, and suppose M is a unitary left R-module with epimorphism $f \in \mathrm{Hom}_R(M,R)$. If $f(e) = 1$, then $e \cdot b = f(e)b = b$, so, $f(e) = 1$ means that e is a left identity. For a left identity e, we define $R_e = \{a \in M \mid ae = a\}$ and $B_e = \{ae \mid a \in M\}$.

PROPOSITION 8. $R_e = B_e$.

PROOF. For $ae \in B_e$, $(ae)e = a(ee) = ae$, so $B_e \subseteqq R_e$. If $a \in R_e$, then $ae = a$. But $ae \in B_e$, hence $R_e \subseteqq B_e$.

Theorem 9. *$(B_e,+,\cdot)$ is a subring of $(M,+,\cdot)$.*

Proof. Define $\psi_e\colon M \to B_e$ by $\psi_e(a) = ae$. It is direct to see that ψ_e is a group epimorphism. Now $\psi_e(ab) = (ab)e = (a(eb))e = ((ae)b)e = (ae)(be)$. Thus, ψ_e is a ring epimorphism.

Theorem 10. *$(B_e,+,\cdot) \cong (R,+,\cdot)$.*

Proof. We have $B_e \cong M/\ker\psi_e$. By Theorem 4, $M/\mathrm{Ann}_M M \cong R$. We now proceed to show that $\ker\psi_e = \mathrm{Ann}_M M$. For $a \in \ker\psi_e$, $0 = \psi_e(a) = a\cdot e$, so for $b \in M$, $a\cdot b = a\cdot(e\cdot b) = (a\cdot e)\cdot b = 0$. Thus $\ker\psi_e \subseteqq \mathrm{Ann}_M M$. The reverse inclusion is trivial.

Corollary 11. *If e and e' are left identities, then the subrings B_e and $B_{e'}$ are isomorphic.*

Proof. As an alternate to the obvious proof, let $\psi_{e',e} = \psi_e \mid B_{e'}$, the restriction of ψ_e to $B_{e'}$. Then $\psi_{e',e}$ is an isomorphism.

Proposition 12. *If e and e' are left identities, then $B_e = B_{e'}$ if and only if $e = e'$.*

Proof. If $B_e = B_{e'}$ for left identities e and e', then $ae = be'$ implies $(ae)e' = (be')e'$, or $ae' = be'$. So $ae = ae'$. This being true for each $a \in M$, we get $ee = ee'$, or $e = e'$.

Proposition 13. *For a ring R with identity 1, let M be the ring on $R^{(S)}$ of Examples 2 for a fixed finite subset $X \subseteqq S$. Then M has at least $|R|^{|X|-1}$ subrings each isomorphic to R.*

Proof. To make $1 = F_X(a) = \sum\limits_{x\in X} a_x$, we can choose $|X|-1$ of the a_x's arbitrarily, and the $|X|$th one suitably.

Proposition 14. *For an abelian group A and a finite subset $X \subseteqq A$, consider the ring $(\mathrm{Mult}_L A,+,\cdot)$ from Examples 4. The ring $\mathrm{Mult}_L A$ has $|\mathrm{End}\,A|^{|X|-1}$ left identities, and at least $|\mathrm{End}\,A|^{|X|-1}$ subrings isomorphic to $\mathrm{End}\,A$.*

Proof. To make $1 = F_X(\alpha) = \sum\limits_{x\in X} \alpha_x$, we proceed as in the proof of Proposition 13.

Corollary 15. *For an abelian group A and a finite subset $X \subseteqq A$, consider a ring $(\mathrm{Mult}_L A,+,*)$ of Examples 5. This ring has $|\mathrm{End}\,A|^{|\overline{X}|-1}$ right identities, and at least this number of subrings isomorphic to $\mathrm{End}\,A$.*

One of the remarkable consequences of studying Mult A is that there are nontrivial abelian groups $(A,+)$ for which Mult $A = \{0\}$. Such groups are called *nil groups* [7]. This will not happen for $\mathrm{Mult}_L A$, since $\alpha_1(a,b) = b$

defines $\alpha_1 \in \mathrm{Mult}_L A$, as does $\alpha_0(a,b) = 0$. Further, for any subset $S \subseteqq A \setminus \{0\}$,

$$\alpha_S(a,b) = \begin{cases} 0, & \text{if } a \notin S; \\ b, & \text{if } a \in S, \end{cases}$$

defines $\alpha_S \in \mathrm{Mult}_L A$ [3]. Thus, $|\mathrm{Mult}_L A| \geqq 2^{|A|-1} + 1$. It is unknown if there is a group, abelian or nonabelian, of order greater than 2, for which these are the only elements of $\mathrm{Mult}_L A$, i.e., are there any "nil groups" for $\mathrm{Mult}_L A$?

THEOREM 16. *For an abelian group $(A,+)$, if $|\mathrm{Mult}_L A| > 1$, then $(\mathrm{Mult}_L A, +)$ is not a nil group. In particular, $\mathrm{Mult}_L A$ is not a torsion divisible group.*

PROOF. $\mathrm{Mult}_L A \cong \mathrm{Map}(A, \mathrm{End}\, A)$, so $A \neq \{0\}$. So there is a finite $X \subseteqq A$ with $X \neq \emptyset$. The multiplications $\cdot = \cdot_X$ defined in Examples 4 are not trivial. Thus, $\mathrm{Mult}_L A$ is not a nil group. Torsion divisible groups are nil groups [7, Theorem 71.1].

COROLLARY 17. *If $(A,+)$ is a notrivial abelian group, then $\mathrm{Mult}_L A$ is not a nil group.*

THEOREM 18. *Let $\phi \in S_A$ where S_A denotes the group of permutations on A. Suppose $Y = \phi(X)$, where $X \subseteqq A$ is a finite subset, and consider the multiplications $\cdot_X$ and $\cdot_Y$ defined as in Examples 4. The map $\Phi_\phi \colon \mathrm{Mult}_L A \to \mathrm{Mult}_L A$ defined by $\Phi_\phi(\alpha) = \alpha^\phi$, where $\alpha^\phi(a,b) = \alpha(\phi(a), b)$, is an isomorphism from $(\mathrm{Mult}_L A, +, \cdot_Y)$ onto $(\mathrm{Mult}_L A, +, \cdot_X)$.*

PROOF. Certainly each $\alpha^\phi \in \mathrm{Mult}_L A$, and $\alpha^\phi_a = \alpha_{\phi(a)}$. Φ_ϕ is easily seen to be a group homomorphism. $\Phi_\phi(\alpha^{\phi^{-1}}) = \alpha$, so Φ_ϕ is surjective. If $\alpha^\phi = 0$, then $\alpha(\phi(a), b) = 0$ for all $a, b \in A$, making $\alpha = 0$. Thus Φ_ϕ is injective.

Consider $\Phi_\phi(\alpha \cdot_Y \beta) = (\alpha \cdot_Y \beta)^\phi$ and $\Phi_\phi(\alpha) \cdot_X \Phi_\phi(\beta) = \alpha^\phi \cdot_X \beta^\phi$. For any finite $T \subseteqq A$, $(\alpha \cdot_T \beta)(c,d) = (\sum_{t\in T} \alpha_t) \circ \beta_c(d)$. So $(\alpha \cdot_T \beta)_c = (\sum_{t \in T} \alpha_t) \circ \beta_c$. So, $(\alpha \cdot_Y \beta)^\phi_a = (\alpha \cdot_Y \beta)_{\phi(a)} = (\sum_{y \in Y} \alpha_y) \circ \beta_{\phi(a)}$, and $(\alpha^\phi \cdot_X \beta^\phi)_a = (\sum_{x \in X} \alpha^\phi_x) \circ \beta_{\phi(a)} = (\sum_{y \in Y} \alpha_y) \circ \beta_{\phi(a)}$. Thus, for each $a \in A$,

$$(\alpha \cdot_Y \beta)^\phi_a = (\alpha^\phi \cdot_X \beta^\phi)_a,$$

hence $(\alpha \cdot_Y \beta)^\phi = \alpha^\phi \cdot_X \beta^\phi$. This means Φ_ϕ is also a ring isomorphism.

COROLLARY 19. *For $|X| = |Y|$, $(\mathrm{Mult}_L A, +, \cdot_X) \cong (\mathrm{Mult}_L A, +, \cdot_Y)$. If $|\mathrm{End}\, A| < \infty$, then $(\mathrm{Mult}_L A, +, \cdot_X) \cong (\mathrm{Mult}_L A, +, \cdot_Y)$ if and only if $|X| = |Y|$.*

PROOF. If $|X| = |Y|$, then there is a permutation $\phi \in S_A$ such that $\phi(X) = Y$. If $|\mathrm{End}\, A| < \infty$, and $(\mathrm{Mult}_L A, +, \cdot_X) \cong (\mathrm{Mult}_L A, +, \cdot_Y)$, then each has the same number of left identities, so by Proposition 14, $|X| = |Y|$.

REMARK. It is not known if $|\mathrm{End}\, A| < \infty$ in Corollary 19 is needed.

For $\phi, \lambda \in S_A$, with $\phi(X) = Y$ and $\lambda(Y) = Z$, then $\Phi_\phi \circ \Phi_\lambda = \Phi_{\lambda\circ\phi}$. We then have

THEOREM 20. *Fix an abelian group A. The following describes two categories $\mathcal{F}(A)$ and $\mathcal{M}(A)$. The objects of $\mathcal{F}(A)$ are the finite subsets of A, and the objects of $\mathcal{M}(A)$ are the rings $\mathcal{R}(X) = (\mathrm{Mult}_L A, +, \cdot_X)$ where $\cdot_X = \cdot$ is defined in Examples 4. Morphisms in $\mathcal{F}(A)$ are $\hom(X,Y) = \{\phi \in S_A \mid \phi(X) = Y\}$, and morphisms in $\mathcal{M}(A)$ are just the ring homomorphisms. Define Φ by $\Phi(X) = \mathcal{R}(X)$ and $\Phi(\phi) = \Phi_\phi$ of Theorem 18. Then Φ is a contravariant functor* [13].

NOTE. For a finite $X \subseteqq A$, $1_A \in \hom(X, X)$ is the identity morphism for X, where $1_A \in S_A$ is the identity permutation.

The proof of the theorem is easy and shows no new techniques.

The finite subsets of an abelian group $(A, +)$ form a boolean algebra with respect to $\cup$ and $\cap$. For finite subsets Y and Z, and $X = Y \cup Z$, we have for the multiplications of Example 4,

$$\alpha \cdot_X \beta = \alpha \cdot_Y \beta + \alpha \cdot_Z \beta - \alpha \cdot_{Y\cap Z} \beta,$$

for arbitrary $\alpha, \beta \in \mathrm{Mult}_L A$. Thus $\cdot_X = \cdot_Y + \cdot_Z - \cdot_{Y\cap Z}$. This leads to

THEOREM 21. *The objects $\mathcal{R}(X)$ of the category $\mathcal{M}(A)$ form a boolean algebra where $\mathcal{R}(Y) \bigvee \mathcal{R}(Z) = \mathcal{R}(Y \cup Z)$ and $\mathcal{R}(Y) \bigwedge \mathcal{R}(Z) = \mathcal{R}(Y \cap Z)$.*

Certainly $\mathrm{Ann}_M M$ is an ideal of M. We now construct further ideals of the examples defined in Examples 2, 3, and 4. For the rings defined in Examples 2, 3, and 4, a set X plays a role in the definition of the product. For a suitable subset T, there is a left ideal $I(T)$. For $R_1 = R^{(S)}$ and $T \subseteqq S$, let $I(T) = \{a \in R^{(S)} \mid a_t = 0 \text{ for each } t \in T\}$. For $R_1 = \mathrm{Mult}_L A$ and $T \subseteqq A$, let $I(T) = \{\alpha \in \mathrm{Mult}_L A \mid \alpha(t, \cdot) = 0 \text{ for each } t \in T\}$. And for $R_1 = M_L$ and $T \in \mathcal{A}$, let $I(T) = \{\alpha \in M_L \mid \alpha(t, \cdot) = 0 \text{ for each } t \in T\}$. The following theorem shows why we use the notation $I(T)$ for all three cases.

THEOREM 22. *Let $R_1 \in \{R^{(S)}, M_L, \mathrm{Mult}_L A\}$, and consider the corresponding $I(T)$ as defined above. Then:*

1) *$I(T)$ is a left ideal.*

2) *As groups, $R_1^+ = I(T)^+ \oplus I(T^c)^+$, where T^c denotes the complement of T in S, in Y, or in A, as is appropriate.*

3) *In each case, for the appropriate X, if $X \subseteqq T$, then $I(T)$ is an ideal in R_1, and $R_1/I(T) \cong I(T^c)$.*

4a) *If $I(T)$ is an ideal and $R_1 \neq M_L$, then $X \subseteqq T$.*

4b) *Suppose $R_1 = M_L$ and $\mu(X) < \infty$. Then $I(T)$ is an ideal if and only if $\mu(X \cap T^c) = 0$.*

5a) *Suppose $R_1 \neq M_L$. Then $I(T_1) \subseteqq I(T_2)$ if and only if $T_2 \subseteqq T_1$.*

5b) *Suppose* $R_1 = M_L$ *and* $\mu(X) < \infty$. *Then* $I(T_1) \subsetneqq I(T_2)$ *if and only if* $T_2 \subsetneqq T_1$.

PROOF. We shall sketch the proof for $R_1 = M_L$. The other two cases have proofs very similar, but simpler.

For 1), take $\alpha, \beta \in I(T)$, and $t \in T$. Then $(\alpha - \beta)(t,b) = \alpha(t,b)- -\beta(t,b) = 0 - 0 = 0$, so $\alpha - \beta \in I(T)$. For $\gamma \in R_1 = M_L$, and $t \in T$, $(\gamma \cdot \alpha)(t,b) = \int_X \gamma(x, \alpha(t,b))d\mu(x) = \int_X \gamma(x,0)d\mu(x) = \int_X 0 d\mu(x) = 0$. Hence, $\gamma \cdot \alpha \in I(T)$, and so $I(T)$ is a left ideal.

For 2), let $\alpha \in R_1 = M_L$ and define

$$\alpha_T(t,b) = \begin{cases} 0, & \text{if } t \in T; \\ \alpha(t,a), & \text{if } t \notin T, \end{cases}$$

and

$$\alpha'(t,b) = \begin{cases} \alpha(t,a), & \text{if } t \in T; \\ 0, & \text{if } t \notin T. \end{cases}$$

Certainly $\alpha = \alpha' + \alpha_T$, and $\alpha' \in I(T^c)$, and $\alpha_T \in I(T)$, if $\alpha', \alpha_T \in M_L$. If one is in M_L, the other is also, and we shall shortly demonstrate that $\alpha_T \in M_L$. Assuming $\alpha_T \in M_L$, we certainly have $M_L^+ = I(T)^+ + I(T^c)^+$, and if $\beta \in I(T) \cap I(T^c)$, then $\beta = 0$. So we need only to show that $\alpha_T \in M_L$.

Take $b \in \mathbf{R}$. Then $\int_Y \alpha_T(y,b)d\mu(y) = \int_T \alpha(y,b)d\mu(y)$ exists, since $\alpha \in M_L$. For $f \in \mathcal{L}_1(Y, \mathcal{A}, \mu)$ and $a \in Y$, if $a \in T$, then $\int_Y \alpha_T(a, f(y))d\mu(y) = 0 = = \alpha_T(a, \int_Y f(y)d\mu(y))$. If $a \notin T$, then

$$\int_Y \alpha_T(a, f(y))d\mu(y) = \int_Y \alpha(a, f(y))d\mu(y) =$$

$$= \alpha\Big(a, \int_Y f(y)d\mu(y)\Big) = \alpha_T\Big(a, \int_Y f(y)d\mu(y)\Big),$$

since $\alpha \in M_L$. Finally, for $y \in T$, $\alpha_T(y, sa + tb) = 0 = s\alpha_T(y,a) + t\alpha_T(y,b)$, and for $y \notin T$, $\alpha_T(y, sa+tb) = \alpha(y, sa+tb) = s\alpha(y,a)+t\alpha(y,b) = s\alpha_T(y,a)+ +t\alpha_T(y,b)$. So, $\alpha_T \in M_L$ as promised.

We also assume $X \subsetneqq T$ for 3). Take $\alpha \in I(T)$ and $\gamma \in R_1 = M_L$. Then for $t \in T$, $(\alpha \cdot \gamma)(t,b) = \int_X \alpha(x, \gamma(t,b))d\mu(x) = \int_X 0 d\mu(x) = 0$, so $\alpha \cdot \gamma \in I(T)$, and $I(T)$ is an ideal. As groups, from 2), we have $R_1^+/I(T)^+ \cong I(T^c)^+$. The map $\alpha \mapsto \alpha'$ is certainly a group epimorphism. We will show now that $(\alpha \cdot \beta)' = \alpha' \cdot \beta'$, thus completing the proof of 3).

For $t \in T$, $(\alpha' \cdot \beta')(t,b) = \int_X \alpha'(x,\beta'(t,b))d\mu(x) = \int_X \alpha(x,\beta(t,b))d\mu(x) =$ $= (\alpha \cdot \beta)(t,b)$. For $t \notin T$,

$$(\alpha' \cdot \beta')(t,b) = \int_X \alpha'(x,\beta'(t,b))d\mu(x) =$$

$$= \int_x \alpha(x,0)d\mu(x) = \int_X 0 d\mu(x) = 0,$$

and $(\alpha \cdot \beta)'(t,b) = 0$, also. Thus $(\alpha \cdot \beta)' = \alpha' \cdot \beta'$.

For 4b), we also assume $\mu(X) < \infty$. If $\mu(X \cap T^c) = 0$, $\alpha \in I(T)$, $\gamma \in M_L$, and $t \in T$, then

$$(\alpha \cdot \gamma)(t,b) = \int_X \alpha(x,\gamma(t,b))d\mu(x) =$$

$$= \int_{X\cap T} \alpha(x,\gamma(t,b))d\mu(x) + \int_{X\cap T^c} \alpha(x,\gamma(t,b))d\mu(x) = \int_{X\cap T} 0 d\mu(x) + 0 = 0,$$

since $\mu(X \cap T^c) = 0$. Hence $I(T)$ is an ideal.

For the converse, we assume $I(T)$ is an ideal. If $\alpha \in I(T)$, $\gamma \in M_L$ and $t \in T$, then

$$0 = (\alpha \cdot \gamma)(t,b) = \int_X \alpha(x,\gamma(t,b))d\mu(x) =$$

$$= \int_{X\cap T} \alpha(x,\gamma(t,b))d\mu(x) + \int_{X\cap T^c} \alpha(x,\gamma(t,b))d\mu(x) = \int_{X\cap T^c} \alpha(x,\gamma(t,b))d\mu(x) = (\dagger).$$

Define α by

$$\alpha(t,b) = \begin{cases} b, & \text{if } t \in X \cap T^c; \\ 0, & \text{otherwise.} \end{cases}$$

Choose a γ and a b so that $\gamma(t,b) \neq 0$. If $\alpha \in M_L$, then $\alpha \in I(T)$, and

$$(\dagger) = \int_{X\cap T^c} \gamma(t,b)d\mu(x) = \gamma(t,b)\mu(X \cap T^c).$$

Since $\gamma(t,b) \neq 0$, we have $\mu(X \cap T^c) = 0$.

Let us now show that $\alpha \in M_L$. For a), $\int_Y \alpha(y,b)d\mu(y) = \int_{X\cap T^c} b d\mu(y) =$ $= b\mu(X \cap T^c)$ exists. For b), take $f \in \mathcal{L}_1(Y,\mathcal{A},\mu)$. Then for $a \in X \cap T^c$, $\int_Y \alpha(a,f(y))d\mu(y) = \int_Y f(y)d\mu(y) = \alpha(a, \int_Y f(y)d\mu(y))$. For $a \notin X \cap T^c$,

$\int\limits_Y \alpha(a, f(y))d\mu(y) = 0 = \alpha(a, \int\limits_Y f(y)d\mu(y))$. Finally, for c), let $y \in X \cap \cap T^c$. Then $\alpha(y, sa+tb) = sa+tb = s\alpha(y,a)+t\alpha(y,b)$. For $y \notin X \cap T^c$, $\alpha(y, sa+tb) = 0 = s\cdot 0 + t\cdot 0 = s\alpha(y,a)+t\alpha(y,b)$. Hence, $\alpha \in M_L$ as promised.

Finally, for 5b), we suppose that $I(T_1) \subseteqq I(T_2)$, and that there is a $t_2 \in T_2 \setminus T_1$. Define α by $\alpha(t_2, b) = b$ and $\alpha(t,b) = 0$ if $t_2 \neq t$. Then, if $\alpha \in M_L$, we have $\alpha \in I(T_1)$ but $\alpha \notin I(T_2)$, a contradiction. So we need only see that $\alpha \in M_L$.

It is direct to see that b) and c) requirements for being in M_L are satisfied. For a), let $b \in \mathbf{R}$. Then

$$\int\limits_Y \alpha(y,b)d\mu(y) = \begin{cases} 0, & \text{if } \{t_2\} \notin \mathcal{A}; \\ b\mu(\{t_2\}), & \text{if } \{t_2\} \in \mathcal{A}. \end{cases}$$

So $\alpha \in M_L$.

The converse is trivial.

COROLLARY 23. a) *Suppose* $R_1 \neq M_L$. *Then* $I(T_1) \subset I(T_2)$ *if and only if* $T_2 \subset T_1$.

b) *Suppose* $R_1 = M_L$ *and* $\mu(X) < \infty$. *Then* $I(T_1) \subset I(T_2)$ *if and only if* $T_2 \subset T_1$.

PROOF. Suppose $I(T_1) \subset I(T_2)$. Then $I(T_1) \subseteqq I(T_2)$. From the theorem, $T_2 \subseteqq T_1$. If $T_2 = T_1$, then $T_1 \subseteqq T_2$ and the theorem gives us that $I(T_2) \subseteqq \subseteqq I(T_1)$, which cannot be. Conversely, suppose $T_2 \subset T_1$. Then $T_2 \subseteqq T_1$ and so $I(T_1) \subseteqq I(T_2)$. If $I(T_1) = I(T_2)$, then $I(T_2) \subseteqq I(T_1)$, which forces $T_1 \subseteqq T_2$.

COROLLARY 24. *For the appropriate case for* R_1, *assume that* S, A, *or* Y *is infinite. Then neither the descending chain condition (d.c.c.) for left ideals nor the d.c.c. for ideals holds.*

PROOF. For the ideal case, there is an infinite chain

$$X \subset T_1 \subset T_2 \subset \cdots \subset T_n \subset \cdots .$$

So

$$I(X) \supset I(T_1) \supset I(T_2) \supset \cdots \supset I(T_n) \supset \cdots .$$

The definition of M_L depends upon a measure space $(Y, \mathcal{A}, \mu)$, and this point could be emphasized by writing $M_L(Y, \mathcal{A}, \mu)$, if necessary, for M_L. For a μ-measurable set $T \in \mathcal{A}$, one gets the measure space $(T, \mathcal{A}_T, \mu_T)$ where the σ-algebra $\mathcal{A}_T = \{F \in \mathcal{A} \mid F \subseteqq T\}$, and $\mu_T = \mu \mid \mathcal{A}_T$ [11, 11.22, 11.37, 12.31].

THEOREM 25. *Let* $R_1 = M_L(Y, \mathcal{A}, \mu)$ *and fix* $X, T \in \mathcal{A}$ *with* $X \subseteqq T$, *and let* X *define the multiplication in the* M_L*'s. Then*

$$\frac{M_L(Y, \mathcal{A}, \mu)}{I(T)} \cong M_L(T, \mathcal{A}_T, \mu_T).$$

PROOF. From Theorem 22, we have $M_L(Y,\mathcal{A},\mu)/I(T) \cong I(T^c)$, so we shall show that $I(T^c) \cong M_L(T,\mathcal{A}_T,\mu_T)$. Define $\Lambda\colon I(T^c) \to M_L(T,\mathcal{A}_T,\mu_T)$ by $\Lambda(\alpha) = \alpha \mid T \times \mathbf{R} = \alpha^*$, the restriction of α to $T \times \mathbf{R}$. Certainly, $\Lambda(\alpha+\beta) = \Lambda(\alpha)+\Lambda(b)$, or $(\alpha+\beta)^* = \alpha^*+\beta^*$. If $\alpha^* = 0$, then $\alpha^*(t,\cdot) = 0$ for each $t \in T$, so $\alpha(t,\cdot) = 0$ for each $t \in T$. Since $\alpha \in I(T^c)$, then $\alpha(t,\cdot) = 0$ for each $t \in T^c$, making $\alpha(t,\cdot) = 0$ for each $t \in T \cup T^c = Y$. So $\alpha = 0$ and Λ is injective.

Take any $\alpha^* \in M_L(T,\mathcal{A}_T,\mu_T)$, and define

$$\alpha(y,b) = \begin{cases} \alpha^*(y,b), & \text{if } y \in T; \\ 0, & \text{if } y \in T^c. \end{cases}$$

If $\alpha \in M_L(Y,\mathcal{A},\mu)$, then $\alpha \in I(T^c)$ and $\Lambda(\alpha) = \alpha^*$, making Λ surjective.

To see that $\alpha \in M_L(Y,\mathcal{A},\mu)$, take $b \in \mathbf{R}$, and note that $\int\limits_Y \alpha(y,b)d\mu(y) = = \int\limits_T \alpha(y,b)d\mu(y) + \int\limits_{T^c} \alpha(y,b)d\mu(y) = \int\limits_T \alpha^*(y,b)d\mu_T(y)$.

For $f \in \mathcal{L}_1(Y,\mathcal{A},\mu)$ and $a \in Y$, we have $\int\limits_Y \alpha(a,f(y))d\mu(y) = 0 = = \alpha(a,\int\limits_Y f(y)d\mu(y))$, if $a \in T^c$, and if $a \in T$, then $\int\limits_Y \alpha(a,f(y))d\mu(y) = = \int\limits_Y \alpha^*(a,f(y))d\mu(y) = \int\limits_Y f(y)\alpha^*(a,1)d\mu(y) = [\int\limits_Y f(y)d\mu(y)]\alpha^*(a,1) = = \alpha^*(a,\int\limits_Y f(y)d\mu(y)) = \alpha(a,\int\limits_Y f(y)d\mu(y))$.

Certainly $\alpha(y,sa+tb) = s\alpha(y,a)+t\alpha(y,b)$ if $y \in T^c$, and $\alpha(y,sa+tb) = = \alpha^*(y,sa+tb) = s\alpha^*(y,a)+t\alpha^*(y,b) = s\alpha(y,a)+t\alpha(y,b)$, if $y \in T$. In summary, $\alpha \in M_L(Y,\mathcal{A},\mu)$, and to see that Λ is a ring isomorphism, one needs only now to see that $(\alpha\cdot\beta)^* = \alpha^*\cdot\beta^*$.

Now $\alpha^*\cdot\beta^*(a,b) = \int\limits_X \alpha^*(x,\beta^*(a,b))d\mu_T(x) = \int\limits_X \alpha(x,\beta(a,b))d\mu(x) = = (\alpha\cdot\beta)(a,b) = (\alpha\cdot\beta)^*(a,b)$, for each $(a,b) \in T \times \mathbf{R}$. Hence $(\alpha\cdot\beta)^* = \alpha^*\cdot\beta^*$. This completes the proof of Theorem 25.

Recall that $R^{(S)}$ can be thought of as all functions $r\colon S \to R$ with finite support [13]. The proof of Theorem 25 can be easily modified to give a proof of

COROLLARY 26. *For $R_1 = R^{(S)}$ and $X \subsetneqq T \subsetneqq S$, we have $R^{(S)}/I(T) \cong \cong R^{(T)}$.*

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ARIZONA
TUCSON, ARIZONA 85721
USA

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ON THE UNIQUE EXISTENCE OF ALMOST PERIODIC SOLUTIONS OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

K. WANG (Changchun)*

This paper deals with the uniqueness and existence of almost periodic solutions of Volterra integro-differential equations of the form

$$x'(t) = A(t)x(t) + \int_{-\infty}^{t} C(s-t)x(s)ds + f(t), \tag{1}$$

and

$$x'(t) = A(t)x(t) + \int_{-\infty}^{t} D(s-t, x(s))ds + r(t, x(t)), \tag{2}$$

where A, C are continuous matrices; D, f, r are continuous n-dimensional vectors; and $A(t+T) = A(t)$, $T \geqq 0$.

The existence and uniqueness of almost periodic solutions of Volterra integro-differential equations have been studied by many authors, see [1–4]. Using the technique of [5], we present some new unique existence criteria for (1) and (2).

If $x = (x_1, x_2, \dots, x_n) \in R^n$, $A = (a_{ij})$ is an $n \times n$ matrix, then define

$$|x| = \sum_{i=1}^{n} |x_i|, \quad |A| = \sum_{i,j=1}^{n} |a_{ij}|.$$

Let AP denote the set of almost periodic functions, define $\|g\| = \sup_{t \in R} |g(t)|$, for $g \in \mathrm{AP}$. The space $(\mathrm{AP}, \|\cdot\|)$ is a Banach space.

DEFINITION 1. A matrix $A(t)$ is said to be noncritical with respect to AP if the only solution in AP of the equation $x' = A(t)x$ is the zero solution $x = 0$.

LEMMA 1 [5]. *If $A(t+T) = A(t)$, $T > 0$, then the equation*

$$x' = A(t)x + f(t) \tag{3}$$

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has a solution Kf in AP *for every $f \in$* AP *if and only if $A(t)$ is noncritical with respect to* AP.

LEMMA 2 [5]. *If $A(t+T) = A(t)$, and A is noncritical with respect to* AP *then Kf is the only solution of (3) in* AP, *and Kf is continuous and linear in f, and there is a constant*

$$k = T \sup_{0 \leqq s,t \leqq T} \left| \left(X^{-1}(t+T,t) - I\right)^{-1} X(t,t+s) \right|$$

such that $\|Kf\| \leqq k\|f\|$, where I is the unit matrix, $X(t,s)$, $X(s,s) = I$, is the principal matrix solution of $x' = A(t)x$.

LEMMA 3. *If $A(t+T) = A(t)$, then $A(t)$ is noncritical with respect to* AP *if and only if all characteristic exponents of $x' = A(t)x$ have nonzero real parts.*

LEMMA 4. *If $G \overset{\text{def}}{=} \int_{-\infty}^{0} |C(u)|du < +\infty$, then the function*

$$(Qg)(t) \overset{\text{def}}{=} \int_{-\infty}^{t} C(s-t)g(s)ds$$

is almost periodic, for any $g \in$ AP.

PROOF. Suppose $\{a_k\}$ is a sequence in R. Since $g \in$ AP, so there is a continuous function $g^*(t)$ and a subsequence $\{a_i\} \subset \{a_k\}$ such that $\{g(t+a_i)\}$ uniformly converges to $g^*(t)$ on R, that is, for any given $\varepsilon > 0$, there is a $N > 0$ such that

$$|g(t+a_i) - g^*(t)| < \varepsilon, \quad \text{for} \quad t \in R \quad \text{and} \quad i \geqq N.$$

Since

$$(Qg)(t+a_k) = \int_{-\infty}^{t+a_k} C(s-t-a_k)g(s)ds = \int_{-\infty}^{t} C(s-t)g(s+a_k)ds,$$

therefore,

$$|(Qg)(t+a_i) - (Qg^*)(t)| \leqq \int_{-\infty}^{t} |C(s-t)||g(s+a_i) - g^*(s)|ds \leqq$$

$$\leqq \varepsilon \int_{-\infty}^{0} |C(u)|du, \text{ for } t \in R \text{ and } i \geqq N.$$

This implies that $\{(Qg)(t+a_i)\}$ uniformly converges to $(Qg^*)(t)$ on R, thus $(Qg)(t)$ is almost periodic.

THEOREM 1. *If*

1° $A(t)$ *is noncritical with respect to* AP;

2° $kG < 1$;

then (1) *has one and only one almost periodic solution for every* $f \in$ AP.

PROOF. Suppose $f \in$ AP is given, we define maps $P, Q\colon \mathrm{AP} \to \mathrm{AP}$ by the following way

$$(Qg)(t) = \int_{-\infty}^{t} C(s-t)g(s)ds, \quad \text{for} \quad g \in \mathrm{AP},$$

and $Pg = K(Qg + f)$. It is easy to see that P, Q are well defined and continuous in g, and $\|Qg\| \leqq G\|g\|$.

Take $M > 0$ so large that $(1 - kG)M > k$, if $f \neq 0$. Let

$$S = \{g \in \mathrm{AP}\colon \|g\| \leqq N\|f\|\}.$$

Then for $g \in S$,

$$\|Pg\| = \|K(Qg + f)\| \leqq k(\|Qg\| + \|f\|) \leqq kG\|g\| = \\ = k\|f\| \leqq MkG\|f\| + k\|f\| \leqq M\|f\|.$$

Therefore P is a map of S into itself.

If $g, h \in S$, then

$$\|Pg - Ph\| = \|K(Qg + f) - K(Qh + f)\| \leqq \|KQg - KQh\| \leqq \\ \leqq k\|Q(g - h)\| \leqq kG\|g - h\|.$$

From condition 2°, the map P is a contraction of S. The contraction principle implies there is a unique fixed point g^* of P on S, that is,

$$\frac{d}{dt}g^*(t) = A(t)g^*(t) + (Qg^* + f)(t) = A(t)g^*(t) + \int_{-\infty}^{t} C(s-t)g^*(s)ds + f(t),$$

and $g^*(t)$ is an almost periodic solution of (1). If there is another almost periodic solution $h^*(t)$ of (1), take $M > 0$ so large that $\|h^*\| \leqq M\|f\|$, then h^* is a fixed point of P on S, from the uniqueness of fixed point of P on $S, h^* = g^*$. This implies the uniqueness of almost periodic solution of (1).

If $f = 0$, let

$$S = \{g \in \mathrm{AP}\colon \|g\| \leqq M\}.$$

The remaining argument proceeds as in case $f \neq 0$.

COROLLARY 1. *If A is a real constant matrix, and*
1° *all characteristic roots of A have nonzero real parts;*
2° $\int_{-\infty}^{0} |C(u)|du < \left(T\left|(e^{-AT}-I)^{-1}\right|e^{|A|T}\right)^{-1}$;
then equation (1) has one and only one almost periodic solution.

COROLLARY 2. *If $n=1$ and*
1° $\int_0^T A(s)ds \neq 0$;
2° $\int_{-\infty}^{0} |C(u)|du < T^{-1}\left|1-e^{-\int_0^T A(s)ds}\right| e^{\inf_{0\leqq s,t\leqq T}\int_t^{t+s} A(z)dz}$;
then equation (1) has one and only one almost periodic solution.

COROLLARY 3. *If*
1° $A(t)A(s)=A(s)A(t)$ *for all $t,s\in R$, and all characteristic roots of the matrix $\int_0^T A(z)dz$ have nonzero real parts;*

$$2^\circ \quad \int_{-\infty}^{0} |C(u)|du < \left(T\left|\left(e^{-\int_0^T A(z)dz}-I\right)^{-1}\right|e^{\|A\|T}\right)^{-1};$$

where $\|A\| = \sup_{0\leqq t\leqq T}|A(t)|$, then equation (1) has one and only one almost periodic solution.

Now let us consider the more complicated nonlinear Volterra integro-differential equation

$$x'(t)=A(t)x(t)+\int_{-\infty}^{t} D(s-t,x(s))ds+r(t,x(t)), \tag{2}$$

where $D\colon(-\infty,0]\times R^n\to R^n$ and $r\colon R\times R^n\to R^n$ are continuous functions, $D(\cdot,0)\equiv 0$, from $g\in \mathrm{AP}$ it follows that $\int_{-\infty}^{t} D(s-t,g(s))ds$ is continuous on R, moreover there are real constants $c>0$, $L\geqq 0$ and a continuous function $C_1\colon(-\infty,0]\to[0,\infty)$ such that

$$|D(u,x)-D(u,y)|\leqq C_1(u)|x-y|,\quad |r(t,x)|\leqq c,$$

$$|r(t,x)-r(t,y)|\leqq L|x-y|$$

for all $u\leqq 0$, $x,y\in R^n$, $t\in R$.

LEMMA 5. *If*

$$F \stackrel{\text{def}}{=} \int_{-\infty}^{0} |C_1(u)|ds < \infty,$$

then the function

$$(Qg)(t) \stackrel{\text{def}}{=} \int_{-\infty}^{t} D(s-t, g(s))ds$$

is almost periodicfor any $g \in \mathrm{AP}$.

PROOF. Suppose $\{a_k\}$ is a sequence in R. Since $g \in \mathrm{AP}$, so there is a continuous function $g^*(t)$ and a subsequence $\{a_i\} \subset \{a_k\}$ such that $\{g(t+a_i)\}$ uniformly converges to $g^*(t)$ on R, that is, for any given $\varepsilon > 0$, there is an $N > 0$ such that

$$|g(t+a_i) - g^*(t)| < \varepsilon, \quad \text{for } t \in R \text{ and } i \geqq N.$$

Since

$$(Qg)(t+a_k) = \int_{-\infty}^{t+a_k} D(s-t-a_k, g(s))ds = \int_{-\infty}^{t} D(s-t, g(s+a_k))ds,$$

therefore,

$$|(Qg)(t+a_i) - (Qg^*)(t)| \leqq \int_{-\infty}^{t} |D(s-t, g(s+a_i)) - D(s-t, g^*(s))|ds \leqq$$

$$\leqq \int_{-\infty}^{t} C_1(s-t)|g(s+a_i) - g^*(s)|ds \leqq \varepsilon \int_{-\infty}^{0} |C_1(u)|du, \quad \text{for } t \in R \text{ and } i \geqq N.$$

This implies that $\{(Qg)(t+a_i)\}$ uniformly converges to $(Qg^*)(t)$ on R. The lemma is proved.

THEOREM 2. *If*
1° $r(t, g(t))$ *is almost periodic for any* $g \in \mathrm{AP}$;
2° A *is noncritical with respect to* AP;
3° $kL < 1$, *and* $\int_{-\infty}^{0} |C_1(u)|du < k^{-1} - L$;
then (2) *has one and only one almost periodic solution.*

PROOF. Let

$$F = \int_{-\infty}^{0} |C_1(u)|du.$$

It is easy to see that $kF < 1$. Take $M > 0$ so large that $(1 - kF)M > k$. Define maps $P, Q \colon \mathrm{AP} \to \mathrm{AP}$ by the following way

$$(Qg)(t) \int_{-\infty}^{t} D(s-t, g(s))ds, \quad \text{for } g \in \mathrm{AP},$$

and

$$(Pg)(t) = K((Qg)(\cdot) + r(\cdot, g(\cdot))).$$

It is easy to see that P, Q are well defined and continuous.

Let

$$S = \{g \in \mathrm{AP} \colon \|g\| \leqq Mc\}.$$

Then, for $g \in S$, we have

$$\|Pg\| = \|K((Qg)(\cdot) + r(\cdot, g(\cdot)))\| \leqq$$

$$\leqq k \int_{-\infty}^{t} C_1(s-t)|g(s)|ds + kc \leqq kFMc + kc < Mc.$$

Therefore, P is a map of S into itself.

Suppose $g, h \in S$, then

$$\|Pg - Ph\| = \|K((Qg)(\cdot) + r(\cdot, g(\cdot))) - K((Qh)(\cdot) + r(\cdot, h(\cdot)))\| \leqq$$

$$\leqq \|KQg - KQh\| + \|Kr(\cdot, g(\cdot)) - Kr(\cdot, h(\cdot))\| \leqq k\|Qg - Qh\| + kL\|g - h\| \leqq$$

$$\leqq k \int_{-\infty}^{t} |D(s-t, g(s)) - D(s-t, h(s))|ds + kL\|g - h\| \leqq$$

$$\leqq k \int_{-\infty}^{t} C_1(s-t)|g(s) - h(s)|ds + kL\|g - h\| \leqq kF\|g - h\| + kL\|g - h\|.$$

From 3°, $kF + kL < 1$, and P is a contraction on S. The contraction principle implies there is a unique fixed point $g^* \in S$ of P, that is,

$$\frac{d}{dt} g^*(t) = A(t)g^*(t) + (Qg^*)(t) + r(t, g^*(t)) =$$

$$= A(t)g^*(t) + \int_{-\infty}^{t} D(s-t, g^*(s))ds + r(t, g^*(t)).$$

Therefore, $g^*(t)$ is an almost periodic solution of (2). The uniqueness of almost periodic solution of (2) can be proved by the same way as in the proof of Theorem 1.

COROLLARY 4. *If A is a real constant matrix and*
$1°$ *all characteristic roots of A have nonzero real parts;*
$2°$ $\left(T\left|(e^{-AT}-I)^{-1}\right|e^{|A|T}\right)^{-1} > L+F$;
then equation (2) *has one and only one almost periodic solution.*

COROLLARY 5. *If $n=1$, and*
$1°$ $\int_0^T A(s)ds \neq 0$;

$$2° \ F+L < T^{-1}\left|1-e^{-\int_0^T A(z)dz}\right| e^{\inf_{0\leqq s,t\leqq T}\int_t^{t+s} A(z)dz};$$

then equation (2) *has one and only one almost periodic solution.*

COROLLARY 6. *If*
$1°$ $A(t)A(s)=A(s)A(t)$ *for all $t,s\in R$, and all characteristic roots of the matrix $\int_0^T A(z)dz$ have nonzero real parts;*

$$2° \ F+L < \left(T\left|\left(e^{-\int_0^T A(z)dz}-I\right)^{-1}\right|e^{\|A\|T}\right)^{-1};$$

then equation (2) *has one and only one almost periodic solution.*

COROLLARY 7. *If $n=1$ and*
$1°$ $A(t)\equiv A\neq 0$, *where A is a constant;*
$2°$ $F+L<|A|$;
then equation (2) *has one and only one almost periodic solution.*

PROOF. If $A>0$, by Corollary 5 we have,

$$k \leqq T(1-e^{-AT})^{-1} \overset{\text{def}}{=} K(T),$$

where T is any positive constant.

Since

$$e^{AT}=1+AT+\frac{1}{2}(AT)^2+\ldots>1+AT, \quad \frac{d}{dt}K(T)=\frac{1-(1+AT)e^{-AT}}{(1-e^{-AT})^2}>0,$$

and

$$k \leqq \lim_{T\to 0} K(T) = A^{-1}.$$

If $A<0$, we have

$$k \leqq T(e^{-AT}-1)^{-1}e^{-AT} = T(1-e^{AT})^{-1}.$$

The rest of the argument proceeds as in case $A>0$.

EXAMPLE 1. The equation

$$x'(t) = 3x(t) + \int_{-\infty}^{t} e^{s-t}x(s)ds + \arctan(\sin t + \cos \pi t + x(t))$$

has one and only one almost periodic solution.

REMARK. For each $f \in \mathrm{AP}$, there is a corresponding Fourier series

$$f \sim \sum_{k=0}^{\infty} a_k e^{i\lambda_k t}$$

with frequencies λ_k in R and coefficients a_k in C^n. The requirement $\lambda_k \geqq \geqq q > 0$ for $k = 1, 2, \cdots$, is needed in [4], while in this paper we do not need such kind of conditions at all.

EXAMPLE 2. The equation

$$x'(t) = 5x(t) + \int_{-\infty}^{t} \frac{x(s)ds}{1 + (t-s)^2} + \sum_{k=1}^{\infty} \frac{1}{2^k} \cos \sqrt{\frac{1}{k}}\, t \tag{4}$$

has one and only one almost periodic solution.

PROOF. We have $A = 5$, $F = \frac{1}{2}\pi$, $L = 0$. By Corollary 7, this example is obvious. But, since $\inf \left\{\sqrt{\frac{1}{k}}\right\} = 0$, it is difficult to determine the unique existence of almost periodic solutions for (4) by the results in [4].

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DEPARTMENT OF MATHEMATICS
NORTHEAST NORMAL UNIVERSITY
CHANGCHUN, JILIN
PEOPLE'S REPUBLIC OF CHINA

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ON THE RIEMANNIAN CURVATURE OF A TWISTOR SPACE

J. DAVIDOV and O. MUSKAROV (Sofia)*

§ **1. Introduction.** The twistor space of an oriented Riemannian 4-manifold M is the 2-sphere bundle Z on M consisting of the unit (-1)-eigenvectors of the Hodge star operator acting on $\wedge^2 TM$. The 6-manifold Z admits a natural 1-parameter family of pseudo-Riemannian metrics h_t, $t \neq 0$. For $t > 0$, these metrics are definite and have been studied by Friedrich and Kurke [4] in connection with the classification of self-dual Einstein 4-manifolds with positive scalar curvature. In [3], Friedrich and Grunewald have given the geometric conditions on M ensuring that h_t, $t > 0$, is an Einstein metric. In the case $t < 0$, h_t is indefinite and has been studied by Vitter in [10] where local formulas for the curvature and Ricci forms have been obtained. K. Sekigawa [8] has considered the metrics h_t, $t > 0$, on the twistor space of an oriented Riemannian $2n$-manifold.

The main purpose of this paper is to give a coordinate-free formula for the sectional curvature of the pseudo-Riemannian manifold (Z, h_t) in terms of the curvature of M. This is achieved by means of the O'Neill formulas [6] for Riemannian submersions. As applications we discuss the Ricci curvature of (Z, h_t) and the holomorphic sectional curvatures with respect to the almost complex structures on Z introduced by Atiyah, Hitchin and Singer [1] and Eells and Salamon [2], respectively.

§ **2. Preliminaries.** Let M be an oriented Riemannian 4-manifold with metric g. Then g induces a metric on the bundle of 2-vectors $\wedge^2 TM$ by the formula

$$g(A_1 \wedge A_2, A_3 \wedge A_4) = \frac{1}{2} \det\left(g(A_i, A_j)\right).$$

The Riemannian connection of M determines a connection of the vector bundle $\wedge^2 TM$ (both denoted by ∇) and the respective curvatures are related by

$$R(A \wedge B)(C \wedge D) = R(A, B)C \wedge D + C \wedge R(A, B)D$$

for $A, B, C, D \in \mathcal{X}(M)$; $\mathcal{X}(M)$ stands for the Lie algebra of smooth vector fields on M. (For the curvature tensor R of M we adopt the following

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definition: $R(A,B) = \nabla_{[A,B]} - [\nabla_A, \nabla_B]$.) The curvature operator $\mathcal{R}$ is the self-adjoint endomorphism of $\wedge^2 TM$ defined by

$$g(\mathcal{R}(A \wedge B), C \wedge D) = g(R(A,B)C, D)$$

for all $A,B,C,D \in \mathcal{X}(M)$. The Hodge star operator defines an endomorphism $*$ of $\wedge^2 TM$ with $*^2 = \mathrm{Id}$. Hence

$$\wedge^2 TM = \wedge^2_+ TM \oplus \wedge^2_- TM$$

where $\wedge^2_\pm TM$ are the subbundles of $\wedge^2 TM$ corresponding to the (± 1)-eigenvectors of $*$. Let (E_1, E_2, E_3, E_4) be a local oriented orthonormal frame of TM. Set

$$(2.1) \qquad \begin{cases} s_1 = E_1 \wedge E_2 - E_3 \wedge E_4, & \bar{s}_1 = E_1 \wedge E_2 + E_3 \wedge E_4, \\ s_2 = E_1 \wedge E_3 - E_4 \wedge E_2, & \bar{s}_2 = E_1 \wedge E_3 + E_4 \wedge E_2, \\ s_3 = E_1 \wedge E_4 - E_2 \wedge E_3, & \bar{s}_3 = E_1 \wedge E_4 + E_2 \wedge E_3. \end{cases}$$

Then (s_1, s_2, s_3) (resp. $(\bar{s}_1, \bar{s}_2, \bar{s}_3)$) is a local oriented orthonormal frame of $\wedge^2_- TM$ (resp. $\wedge^2_+ TM$). The matrix of $\mathcal{R}$ with respect to the frame $(\bar{s}_i, s_i)$ of $\wedge^2 TM$ has the form

$$\mathcal{R} = \begin{bmatrix} A & B \\ {}^tB & C \end{bmatrix}$$

where the 3×3 matrices A and C are symmetric and have equal traces. Let $\mathcal{B}$, $\mathcal{W}_+$ and $\mathcal{W}_-$ be the endomorphisms of $\wedge^2 TM$ with matrices

$$\mathcal{B} = \begin{bmatrix} 0 & B \\ {}^tB & 0 \end{bmatrix}, \quad \mathcal{W}_+ = \begin{bmatrix} A - \lambda I & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{W}_- = \begin{bmatrix} 0 & 0 \\ 0 & C - \lambda I \end{bmatrix}$$

where $\lambda = \frac{1}{3}\mathrm{Trace}\, C$ and I is the unit 3×3 matrix. Then $\mathcal{R} = \lambda \mathrm{Id} + \mathcal{B} + \mathcal{W}_+ + \mathcal{W}_-$ is the irreducible decomposition of $\mathcal{R}$ under the action of SO(4) found by Singer and Thorpe [9]. Note that $\lambda = 1/6$ scalar curvature; $\lambda \mathrm{Id} + \mathcal{B}$ and $\mathcal{W} = \mathcal{W}_+ + \mathcal{W}_-$ represent the Ricci tensor and the Weyl conformal tensor, respectively. The manifold M is called self-dual (anti-self-dual) if $\mathcal{W}_- = 0$ ($\mathcal{W}_+ = 0$). It is Einstein exactly when $\mathcal{B} = 0$.

The twistor space of M is the submanifold Z of $\wedge^2_- TM$ consisting of all unit vectors. The Riemannian connection ∇ of M gives rise to a splitting $TZ = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle of Z into horizontal and vertical components. More precisely, let $\pi \colon \wedge^2_- TM \to M$ be the natural projection. By definition, the vertical space at $\sigma \in Z$ is

$$\mathcal{V}_\sigma = \{V \in T_\sigma Z / \pi_*(V) = 0\}$$

($T_\sigma Z$ is always considered as a subspace of $T_\sigma(\wedge^2_- TM)$.) Note that $\mathcal{V}_\sigma$ consists of those vectors of $T_\sigma Z$ which are tangent to the fibre $Z_p = \pi^{-1}(p) \cap Z$,

$p = \pi(\sigma)$, of Z through the point σ. Since Z_p is the unit sphere in the vector space $\wedge^2_- T_pM$, $\mathcal{V}_\sigma$ is the orthogonal complement of σ in $\wedge^2_- T_pM$.

Let s be a local section of Z such that $s(p) = \sigma$. Since s has constant length, $\nabla_A s \in \mathcal{V}_\sigma$ for all $A \in T_pM$. Given $A \in T_pM$, the vector

$$A^h = s_* A - \nabla_A s \in T_\sigma Z$$

depends only on p and σ. By definition the horizontal space at σ is

$$\mathcal{H}_\sigma = \{A^h / A \in T_pM\}.$$

Note that the map $A \to A^h$ is an isomorphism between T_pM and $\mathcal{H}_\sigma$.

Each point $\sigma \in Z$ defines a complex structure S on T_pM, $p = \pi(\sigma)$, by

$$g(SA, B) = 2g(\sigma, A \wedge B), \quad A, B \in T_pM. \tag{2.2}$$

Note that S is compatible with the metric g and the opposite orientation of M at p. The 2-vector 2σ is dual to the fundamental 2-form of S.

Denote by $\times$ the usual vector product in the oriented 3-dimensional vector space $\wedge^2_- T_pM$, $p \in M$. Then it is easily checked that

$$g(R(a)b, c) = -g(\mathcal{R}(b \times c), a) \tag{2.3}$$

for $a \in \wedge^2 T_pM$, $b, c \in \wedge^2_- T_pM$ and

$$g(\sigma \times V, A \wedge SB) = g(\sigma \times V, SA \wedge B) = -g(V, A \wedge B) \tag{2.4}$$

for $V \in \mathcal{V}_\sigma$, $A, B \in T_pM$.

Following [1] and [2] define two almost complex structures J_1 and J_2 on Z by

$$J_n V = (-1)^n \sigma \times V \quad \text{for} \quad V \in \mathcal{V}_\sigma,$$

$$J_n A^h = (SA)^h \quad \text{for} \quad A \in T_pM, \ p = \pi(\sigma).$$

It is well-known ([1]) that J_1 is integrable (i.e. comes from a complex structure on Z) iff M is self-dual. Unlike J_1, the almost complex structure J_2 is never integrable [2].

As in [4] define a pseudo-Riemannian metric h_t on Z by

$$h_t = \pi^* g + t g^v$$

where $t \neq 0$, g is the metric of M and g^v is the restriction of the metric of $\wedge^2 TM$ on the vertical distribution $\mathcal{V}$. Then h_t is a pseudo-Hermitian metric with respect to the almost complex structures J_1 and J_2.

§ **3. The sectional curvature of a twistor space.** In this section we derive an explicit formula for the sectional curvature of the pseudo-Riemannian manifold (Z, h_t). We shall use the O'Neill formulas for the Riemannian submersion $\pi\colon (Z,h_t) \to (M,g)$. Following [6] denote by T and A the tensor fields on Z defined by

$$T(E,F) = \mathcal{H}D_{\mathcal{V}E}{}^{\mathcal{V}F} + \mathcal{V}D_{\mathcal{V}E}{}^{\mathcal{H}F}, \quad A(E,F) = \mathcal{V}D_{\mathcal{H}E}{}^{\mathcal{H}F} + \mathcal{H}D_{\mathcal{H}E}{}^{\mathcal{V}F}$$

where $D(=D_t)$ is the Levi–Civita connection of (Z,h_t) and $\mathcal{H}$ (resp. $\mathcal{V}$) denote the horizontal (resp. vertical) component. Since the fibres of the Riemannian submersion $\pi\colon (Z,h_t) \to (M,g)$ are totally geodesic submanifolds of (Z,h_t), it follows that $T \equiv 0$.

Now we obtain some useful formulas which will be needed later. Let (U,x_1,x_2,x_3,x_4) be a local coordinate system of M and let (E_1,E_2,E_3,E_4) be an oriented orthonormal frame of TM on U. If (s_1,s_2,s_3) is the local frame of $\wedge^2_- TM$ defined by (2.1) then $\tilde{x}_i = x_i \circ \pi$, $y_j(\sigma) = g(\sigma,(s_j\circ\pi)(\sigma))$, $1 \leqq i \leqq 4$, $1 \leqq j \leqq 3$, are local coordinates of $\wedge^2_- TM$ on $\pi^{-1}(U)$. For each vector field

$$X = \sum_{i=1}^{4} X^i \frac{\partial}{\partial x_i}$$

on U the horizontal lift X^h of X on $\pi^{-1}(U)$ is given by

$$X^h = \sum_{i=1}^{4}(X^i\circ\pi)\frac{\partial}{\partial\tilde{x}_i} - \sum_{j,k=1}^{3} y_j\,(g(\nabla_X s_j, s_k)\circ\pi)\,\frac{\partial}{\partial y_k}. \tag{3.1}$$

Hence

$$[X^h,Y^h] - [X,Y]^h = \sum_{j,k=1}^{3} y_j\,(g\,(R(X\wedge Y)s_j,s_k)\circ\pi)\,\frac{\partial}{\partial y_k} \tag{3.2}$$

for all $X,Y \in \mathcal{X}(U)$. Let $\sigma \in Z$ and $\pi(\sigma) = p$. Using the standard identification $T_\sigma(\wedge^2_- T_pM) \cong \wedge^2_- T_pM$ this formula can be written as

$$[X^h,Y^h]_\sigma - [X,Y]^h_\sigma = R_p(X\wedge Y)\sigma. \tag{3.3}$$

LEMMA 3.1. *If $X,Y \in \mathcal{X}(M)$ and V is a vertical vector field on Z then*

$$\left(D_{X^h}Y^h\right)_\sigma = (\nabla_X Y)^h_\sigma + \frac{1}{2}R(X\wedge Y)\sigma, \tag{3.4}$$

$$(D_V X^h)_\sigma = \mathcal{H}(D_{X^h}V)_\sigma = \frac{t}{2}(R_p(\sigma\times V)X)^h_\sigma \tag{3.5}$$

for all $\sigma \in Z$.

PROOF. The equality (3.4) follows from (3.3) using the standard formula for the Levi–Civita connection in terms of inner products and Lie brackets.

To prove (3.5) note that $D_V X^h$ is a horizontal vector field since $T = 0$. On the other hand $[V, X^h]$ is a vertical vector field, hence $D_V X^h = \mathcal{H} D_{X^h} V$. Then

$$h_t(D_V X^h, Y^h) = h_t\left(D_{X^h} V, Y^h\right) = -h_t\left(V, D_{X^h} Y^h\right)$$

and (3.5) follows from (3.4) and (2.3).

Denote by $\nabla\mathcal{R}$ the covariant derivative of $\mathcal{R}$ on the vector bundle $\mathrm{End}(\wedge^2 TM)$.

LEMMA 3.2. *If $V \in \mathcal{V}_\sigma$ and $X, Y \in \mathcal{X}(M)$ then*

$$2h_t\left((D_{X^h}A)(X^h, Y^h)_\sigma, V\right) = -\mathrm{tg}\left((\nabla_{X_p}\mathcal{R})(X \wedge Y), \sigma \times V\right)$$

where $p = \pi(\sigma)$.

PROOF. Let s be a local section of Z such that $s(p) = \sigma$ and $(\nabla s)_p = 0$. First we shall prove that if W is a vertical vector field on Z then

$$\mathcal{V} D_{X^h} W = \nabla_X (W \circ s) \tag{3.6}$$

where $W \circ s$ is considered as a section of $\wedge^2_- TM$. In the local coordinates of $\wedge^2_- TM$ introduced above,

$$W = \sum_{j=1}^{3} f_j \frac{\partial}{\partial y_j} \quad \text{with} \quad \sum_{j=1}^{3} f_j y_j = 0.$$

Then

$$\mathcal{V} D_{X^h} W = [X^h, W] = \sum_{j=1}^{3}\left(f_j\left[X^h, \frac{\partial}{\partial y_j}\right] + X^h(f_j)\frac{\partial}{\partial y_j}\right).$$

It follows from (3.1) that

$$\left[X^h, \frac{\partial}{\partial y_j}\right] = \sum_{k=1}^{3}\left(g(\nabla_X s_j, s_k) \circ \pi\right)\frac{\partial}{\partial y_k}.$$

Considering $\mathcal{V}(D_{X^h} W)_\sigma$ as an element of $\wedge^2_- T_p M$ gives

$$\mathcal{V}(D_{X^h} W)_\sigma = \sum_{k=1}^{3}\left(\sum_{j=1}^{3} f_j(\sigma) g_p(\nabla_X s_j, s_k) + X_p(f_k \circ s)\right) s_k(p) =$$

$$= \sum_{j=1}^{3}\left(f_j(\sigma)\nabla_{X_p} s_j + X_p(f_j \circ s)\right) s_j(p) = \nabla_{X_p}(W \circ s)$$

since

$$W \circ s = \sum_{j=1}^{3} (f_j \circ s) s_j.$$

Now, to prove the lemma, note that $2A(X^h, Y^h) = \mathcal{V}[X^h, Y^h]$ (c.f. [6]). Extending V to a section of $\wedge^2_- TM$ one gets by (3.3) and (3.6) that

$$\begin{aligned} 2h_t\left(D_{X^h} A(X^h, Y^h), V\right) &= \mathrm{tg}(\nabla_X R(X \wedge Y)s, V) = \\ &= tX(g(R(X \wedge Y)s, V)) - \mathrm{tg}(R(X \wedge Y)s, \nabla_X V) = \\ &= -tX(g(s \times V, \mathcal{R}(X \wedge Y))) + \mathrm{tg}(s \times \nabla_X V, \mathcal{R}(X \wedge Y)) = \\ &= -\mathrm{tg}(s \times V, \nabla_X \mathcal{R}(X \wedge Y)) + \mathrm{tg}(s \times \nabla_X V - \nabla_X (s \times V), \mathcal{R}(X \wedge Y)). \end{aligned}$$

Since

$$\nabla_{X_p}(s \times V) = \nabla_{X_p} s \times V + s(p) \times \nabla_{X_p} V = s(p) \times \nabla_{X_p} V$$

one obtains

$$2h_t\left(D_{X^h} A(X^h, Y^h), V\right)_\sigma = -\mathrm{tg}\left(\sigma \times V, \nabla_{X_p} \mathcal{R}(X \wedge Y)\right).$$

On the other hand by (3.4) and (2.3) one gets

$$\begin{aligned} 2h_t\left(A\left(D_{X^h} Y^h, Y^h\right)_\sigma + A\left(X^h, D_{X^h} Y^h\right)_\sigma, V\right) &= \\ = -\mathrm{tg}\left(\sigma \times V, \mathcal{R}\left(\nabla_{X_p}(X \wedge Y)\right)\right) \end{aligned}$$

and the lemma is proved.

Lemma 3.3. *If $V, W \in \mathcal{V}_\sigma$ and $X, Y \in T_pM$, $p = \pi(\sigma)$, then*

$$h_t\left(A(X^h, V), A(Y^h, W)\right) = \frac{t^2}{4} g(R(\sigma \times V)X, R(\sigma \times W)Y).$$

Proof. Let (E_1, E_2, E_3, E_4) be a local oriented orthonormal frame of TM near the point p. Then by (3.5) one has

$$\begin{aligned} h_t\left(A(X^h, V), A(Y^h, W)\right) &= h_t(\mathcal{H} D_{X^h} V, \mathcal{H} D_{Y^h} W) = \\ &= \sum_{i=1}^{4} h_t\left(D_{X^h} V, E_i^h\right) h_t\left(D_{Y^h} W, E_i^h\right) = \\ &= \frac{t^2}{4} \sum_{i=1}^{4} g(R(\sigma \times V)X, E_i)\, g(R(\sigma \times W)Y, E_i) = \\ &= \frac{t^2}{4} g(R(\sigma \times V)X, R(\sigma \times W)Y). \end{aligned}$$

LEMMA 3.4. *If* $V, W \in \mathcal{V}_\sigma$ *and* $X, Y \in T_pM$, $p = \pi(\sigma)$, *then*

$$h_t((D_V A)(X^h, Y^h), W) = -\mathrm{tg}(\mathcal{R}(\sigma), X \wedge Y) g(\sigma \times V, W) -$$
$$-\frac{t^2}{4}(g(R(\sigma \times V)X, R(\sigma \times W)Y) + g(R(\sigma \times W)X, R(\sigma \times V)Y)).$$

PROOF. First we prove that

$$D_V A(X^h, Y^h) = -\frac{1}{2} g(\mathcal{R}(\sigma), X \wedge Y)(\sigma \times V) \tag{3.7}$$

for all $X, Y \in \mathcal{X}(M)$.

Let (s_1, s_2, s_3) be a local frame of $\wedge^2_- TM$ defined by (2.1) such that $s_1(p) = \sigma$. Set

$$U = \left(1 - y_3^2\right)^{-1/2} \left(-y_2 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial y_2}\right).$$

Then

$$J_1 U = \left(1 - y_3^2\right)^{-1/2} \left(y_1 y_3 \frac{\partial}{\partial y_1} + y_2 y_3 \frac{\partial}{\partial y_2} - \left(1 - y_3^2\right) \frac{\partial}{\partial y_3}\right)$$

and $(U, J_1 U)$ is a g-orthonormal frame of the vertical distribution $\mathcal{V}$ on a neighbourhood of the point σ. It is enough to check (3.7) for $V = U_\sigma$ and $V = J_1 U_\sigma$. Since $D_U U$ and $D_U J_1 U$ are vertical vector fields and $[U, J_1 U]_\sigma = 0$ it follows from the standard formula for the Levi–Civita connection that $(D_U U)_\sigma = (D_U J_1 U)_\sigma = 0$. Hence

$$2D_{U_\sigma} A(X^h, Y^h) = U_\sigma \left(g([X^h, Y^h], U)\right) U_\sigma + U_\sigma \left(g([X^h, Y^h], J_1 U)\right) J_1 U_\sigma.$$

A direct computation using (3.2) shows that

$$2D_{U_\sigma} A(X^h, Y^h) = g_p \left(R(X \wedge Y) s_2, s_3\right) s_3(p)$$

since $y_1(\sigma) = 1$, $y_2(\sigma) = y_3(\sigma) = 0$. Now (3.7) follows from (2.3). A similar reasoning yields (3.7) for $V = J_1 U_\sigma$.

To prove the lemma note that

$$h_t(A(D_V X^h, Y^h), W) = -h_t(A(Y^h, D_V X^h), W) =$$
$$= h_t(D_V X^h, \mathcal{H} D_{Y^h} W) = h_t(\mathcal{H} D_{X^h} V, \mathcal{H} D_{Y^h} W) = h_t(A(X^h, V), A(Y^h, W)).$$

Similarly

$$h_t(A(X^h, D_V Y^h), W) = -h_t(A(Y^h, V), A(X^h, W))$$

and the lemma follows from (3.7) and Lemma 3.3.

Denote by R_Z the Riemannian curvature tensor of the twistor space (Z, h_t). Combining Lemmas 3.1–3.4 and the O'Neill formulas [6] we obtain the following:

PROPOSITION 3.5. *Let $E, F \in T_\sigma Z$ and $X = \pi_* E$, $Y = \pi_* F$, $V = \mathcal{V}E$, $W = \mathcal{V}F$. Then*

$$h_t(R_Z(E \wedge F)E, F) = g(R(X \wedge Y)X, Y) - t g((\nabla_X \mathcal{R})(X \wedge Y), \sigma \times W) +$$
$$+ t g((\nabla_Y \mathcal{R})(X \wedge Y), \sigma \times V) - 3t g(\mathcal{R}(\sigma), X \wedge Y) g(\sigma \times V, W) -$$
$$-t^2 g(R(\sigma \times V)X, R(\sigma \times W)Y) + \frac{t^2}{4}\|R(\sigma \times W)X + R(\sigma \times V)Y\|^2 -$$
$$-\frac{3t}{4}\|R(X \wedge Y)\sigma\|^2 + t\left(\|V\|^2\|W\|^2 - g(V, W)^2\right).$$

In the case when M is self-dual and Einstein this formula takes an apparently simple form.

COROLLARY 3.6. *Let M be a self-dual Einstein manifold with scalar curvature s. Then*

$$h_t(R_Z(E \wedge F)E, F) = g(R(X \wedge Y)X, Y) - \frac{ts}{2} g(\sigma, X \wedge Y) g(\sigma \times V, W) -$$
$$-(1/2)(ts/12)^2 g(X, Y) g(V, W) + 3(ts/12)^2 g(X \wedge Y, V \times W) +$$
$$+(ts/24)^2\left(\|X\|^2\|W\|^2 + \|Y\|^2\|V\|^2\right) -$$
$$-6t(s/24)^2\left(\|X \wedge Y\|^2 - 2g(\sigma, X \wedge Y)^2\right) +$$
$$+t\left(\|V\|^2\|W\|^2 - g(V, W)^2\right).$$

PROOF. In this case $\mathcal{R} = (s/6)\mathrm{Id} + \mathcal{W}_+$. Since $\mathcal{W}_+$ maps $\wedge^2 TM$ into $\wedge^2_+ TM$ and ∇ preserves $\wedge^2_+ TM$ one gets

$$g((\nabla_X \mathcal{R})(X \wedge Y), \sigma \times W) = 0. \tag{3.8}$$

Now we shall show that

$$g(R(\sigma \times V)X, R(\sigma \times W)Y) = (s/12)^2 (g(X, Y) g(V, W) - 2g(X \wedge Y, V \times W)). \tag{3.9}$$

Recall that each $\sigma \in Z$ defines a complex structure S_σ on T_pM, $p = \pi(\sigma)$ via (2.2). It is easy to check that for $\sigma, \tau \in Z$ with $\pi(\sigma) = \pi(\tau)$ one has

$$S_\sigma \circ S_\tau = -g(\sigma, \tau)\mathrm{Id} - S_{\sigma \times \tau} \qquad (S_0 \equiv 0).$$

To prove (3.9) we may assume that $\|V\| = \|W\| = 1$. Then by (2.4) one has

$$g(R(\sigma \times V)X, R(\sigma \times W)Y) = (s/6) g(\sigma \times V, X \wedge R(\sigma \times W)Y) =$$
$$= (s/12) g(S_{\sigma \times V} X, R(\sigma \times W)Y) = (s^2/72) g(\sigma \times W, Y \wedge S_{\sigma \times V} X) =$$
$$= -(s/12)^2 g(S_{\sigma \times V} S_{\sigma \times W} Y, X) =$$

$$= (s/12)^2(g(X,Y)g(V,W) - 2g(X \wedge Y, V \times W)).$$

Let $U \in \mathcal{V}_\sigma$ and $\|U\| = 1$. Then

$$\|R(X \wedge Y)\sigma\|^2 = g(R(X \wedge Y)\sigma, U)^2 + g(R(X \wedge Y)\sigma, \sigma \times U)^2 =$$
$$= (s/6)^2 \left(g(X \wedge Y, \sigma \times U)^2 + g(X \wedge Y, U)^2\right).$$

Since the projection of $X \wedge Y$ on $\mathcal{V}_\sigma$ is $\frac{1}{2}(X \wedge Y - S_\sigma X \wedge S_\sigma Y)$ one obtains

$$\|R(X \wedge Y)\sigma\|^2 = 2(s/12)^2 \left(\|X \wedge Y\|^2 - 2g(\sigma, X \wedge Y)^2\right). \tag{3.10}$$

Now the corollary follows from Proposition 3.5 and formulas (3.8)–(3.10).

§ 4. **The Ricci curvature of a twistor space.** Let M be an oriented Riemannian 4-manifold with Ricci tensor c_M. Denote by $\mathcal{R}_-$ the restriction of the curvature operator $\mathcal{R}\colon \wedge^2TM \to \wedge^2TM$ on $\wedge^2_-TM$.

PROPOSITION 4.1. *Let c_Z be the Ricci tensor of the twistor space (Z, h_t). If $E \in T_\sigma Z$, $X = \pi_*E$ and $V = \mathcal{V}E$ then*

$$c_Z(E,E) = c_M(X,X) + t\,\mathrm{Trace}(A \to (\nabla_A R)(\sigma \times V, X)) +$$
$$+(t^2/4)\|\mathcal{R}(\sigma \times V)\|^2 - (t/2)\,\|i_X \circ \mathcal{R}_-\|_p^2 + (t/2)\,\|(i_X \circ \mathcal{R})(\sigma)\|^2 + \|V\|^2$$

where $i_X\colon \wedge^2TM \to TM$ is the interior product.

PROOF. Let (E_1, E_2, E_3, E_4) be an oriented orthonormal basis of T_pM, $p = \pi(\sigma)$, and U a g-unit vertical vector at σ. Then $(E_1^h, E_2^h, E_3^h, E_4^h, U, \sigma\times U)$ is an h_t-orthogonal basis of $T_\sigma Z$ and Proposition 3.5 gives:

$$c_Z(E,E) = c_M(X,X) + t\,\mathrm{Trace}(A \to (\nabla_A R)(\sigma \times V, X)) + \tag{4.1}$$
$$+(t^2/4)\sum_{i=1}^{4}\|R(\sigma \times V)E_i\|^2 - (3t/4)\sum_{i=1}^{4}\|R(X \wedge E_i)\sigma\|^2 + (t/4)(\|R(U)X\|^2 +$$
$$+\|R(\sigma \times U)X\|^2 + \|V\|^2.$$

Further one has

$$\sum_{i=1}^{4}\|R(\sigma \times V)E_i\|^2 = 2\sum_{i<j} g(\mathcal{R}(\sigma \times V), E_i \wedge E_j) = \|\mathcal{R}(\sigma \times V)\|^2. \tag{4.2}$$

Since $R(X \wedge E_i)\sigma$ is a vertical vector at σ it follows that

$$\sum_{i=1}^{4}\|R(X \wedge E_i)\sigma\|^2 = \sum_{i=1}^{4}(g(R(U)X, E_i)^2 + g(R(\sigma \times U)X, E_i)^2 = \tag{4.3}$$
$$= (\|R(U)X\|^2 + \|R(\sigma \times U)X\|^2) = \|i_X \circ \mathcal{R}_-\|^2 - \|(i_X \circ \mathcal{R})(\sigma)\|^2.$$

Now the proposition is a consequence of (4.1)–(4.3).

COROLLARY 4.2. *The scalar curvature s_Z of the twistor space (Z, h_t) is given by*

$$s_Z(\sigma) = s_M(p) + (t/4)\left(\|\mathcal{R}(\sigma)\|^2 - \|\mathcal{R}_-\|_p^2\right) + 2/t$$

where $p = \pi(\sigma)$ and s_M is the scalar curvature of M.

PROOF. Since

$$\sum_{k=1}^{4} \|(i_{E_k} \circ \mathcal{R})(\tau)\|^2 = \sum_{j,k=1}^{4} g\left((i_{E_k} \circ \mathcal{R})(\tau), E_j\right)^2 = \|\mathcal{R}(\tau)\|^2$$

for each $\tau \in \wedge_-^2 TM$, the result is a direct consequence of Proposition 4.1.

COROLLARY 4.3. *Let M be a self-dual Einstein 4-manifold with scalar curvature s. Then the Ricci tensor c_Z and the scalar curvature s_Z of (Z, h_t) are given by*

$$c_Z(E, E) = \left(s/4 - t(s/12)^2\right)\|X\|^2 + \left(1 + (ts/12)^2\right)\|V\|^2,$$

$$s_Z = 2/t + s - (t/72)s^2$$

where $X = \pi_* E$, $V = \mathcal{V}E$.

PROOF. These formulas follow from Proposition 4.1 and Corollary 4.2 since

$$\mathcal{R} = (s/6)\mathrm{Id} + \mathcal{W}_+, \quad \mathcal{R}_- = (s/6)\mathrm{Id}$$

and

$$g((\nabla_Y R)(W, X), Y) = g((\nabla_Y \mathcal{R})(X \wedge Y), W) = 0$$

for $X, Y \in \mathcal{X}(M)$ and $W \in \mathcal{V}$.

As an application of Proposition 4.1 we prove the following

PROPOSITION 4.4. *The pseudo-Riemannian manifold (Z, h_t) is Einstein if and only if M is a self-dual Einstein manifold with scalar curvature $s = 6/t$ or $s = 12/t$.*

PROOF. Suppose that (Z, h_t) is Einstein. Then by Proposition 4.1 one gets

$$t\,\|(i_X \circ \mathcal{R})(\sigma)\|^2 = c_M(X, X) - (s_Z/6)\|X\|^2, \tag{4.4}$$

$$t^2\|\mathcal{R}(\sigma)\|^2 = (2t/3)s_Z - 4 \tag{4.5}$$

for each $\sigma \in Z$, $X \in T_pM$, $p = \pi(\sigma)$. Let (E_1, E_2, E_3, E_4) be an oriented orthonormal basis of T_pM and $(\bar{s}_i, s_i)$ the basis of $\wedge^2 T_pM$ defined by (2.1). Then (4.4) is equivalent to the identity

$$t\sum_{i=1}^{4} g(\mathcal{R}(\sigma), E_i \wedge E_j) g(\mathcal{R}(\sigma), E_i \wedge E_k) = \sum_{i=1}^{4} g(\mathcal{R}(E_i \wedge E_j), E_i \wedge E_k) - (s_Z/6)\delta_{jk},$$

which implies that

$$\text{(4.6)} \qquad \operatorname{tg}(\mathcal{R}(\sigma), s_j) g(\mathcal{R}(\sigma), \bar{s}_k) = g(\mathcal{R}(s_j), \bar{s}_k)$$

for $\sigma \in Z$, $1 \leqq j, k \leqq 4$. For a fixed j, take a point σ such that $g(\mathcal{R}(\sigma), s_j) = 0$. Then (4.6) gives $g(\mathcal{R}(s_j), \bar{s}_k) = 0$ for $1 \leqq k \leqq 4$. Hence M is an Einstein manifold. Now $\mathcal{R}(\sigma) \in \wedge^2_- T_pM$ and by (2.2) one has

$$\|i_X \circ \mathcal{R})(\sigma)\|^2 = \sum_{i=1}^{4} g(\mathcal{R}(\sigma), X \wedge E_i)^2 = \|\mathcal{R}(\sigma)\|^2 \|X\|^2/4.$$

This together with (4.4) and (4.5) implies

$$\text{(4.7)} \qquad \|\mathcal{R}(\sigma)\|^2 = (st - 4)/2t^2$$

for each $\sigma \in Z$. Since M is Einstein, there exists a basis (E_1, E_2, E_3, E_4) of T_pM such that $g(\mathcal{R}(s_i), s_j) = \delta_{ij} r_i$, $1 \leqq i, j \leqq 3$, for some constants r_i [9]. Then (4.7) gives

$$r_1^2 = r_2^2 = r_3^2 = (st - 4)/2t^2.$$

Since $r_1 + r_2 + r_3 = s/2$ and $(s/2)^2 \neq (st - 4)/2t^2$ one concludes that $r_1 = r_2 = r_3 = s/6$. Therefore M is self-dual and $s^2/36 = (st - 4)/2t^2$. The last equation shows that $st = 6$ or $st = 12$.

The "if" part of the proposition follows at once from Corollary 4.3.

REMARKS. 1. Proposition 4.4 is due to Friedrich and Grunewald [3] for $t > 0$.

2. A complete, connected self-dual Einstein 4-manifold with positive scalar curvature is isometric to the sphere S^4 or the complex projective space $\mathbf{CP}^2$ with their standard metrics [4], [5] (cf. also [7]). In the case of negative scalar curvature a complete classification is not available and the only known examples are quotients of the unit ball in $\mathbf{C}^2$ with the metric of constant negative curvature or the Bergman metric [10].

3. (Z, h_t, J_1) is a Kähler–Einstein manifold iff M is self-dual, Einstein and $s = 12/t$ (cf. [4] for $t > 0$ and [10] for $t < 0$).

§ **5. The holomorphic sectional curvature of a twistor space.** One can compute the holomorphic sectional curvature H_n of the almost Hermitian manifold (Z, h_t, J_n), $n = 1, 2$ by means of Proposition 3.5. The respective formula simplifies significantly when the base M of Z is self-dual and Einstein. More precisely, by Corollary 3.6 and (2.4) one gets the following:

PROPOSITION 5.1. *Let M be a self-dual Einstein manifold with sectional curvature K and scalar curvature s. Let $E \in T_\sigma Z$ be an h_t-unit vector and S the complex structure on T_pM, $p = \pi(\sigma)$, defined by σ. Then*

$$H_n(E) = K(X, SX)\|X\|^4 + t\|V\|^4 + (2(st/24)^2(3(-1)^n + 1) +$$

$$+(-1)^{n+1}(st/4))\|X\|^2\|V\|^2$$

where $X = \pi_* E$ *and* $V = \mathcal{V}E$.

Now we describe the twistor spaces of constant holomorphic sectional curvature.

PROPOSITION 5.2. *The almost Hermitian manifold* (Z, h_t, J_1) *has a constant holomorphic sectional curvature* $\mathcal{X}$ *if and only if* M *is of constant sectional curvature* $\mathcal{X} = 1/t$.

The holomorphic sectional curvature of (Z, h_t, J_2) *is never constant.*

PROOF. Assume that $H_n \equiv \mathcal{X}$. By Proposition 3.5 it follows that for every $\sigma \in Z$ and $X \in T_pM$, $p = \pi(\sigma)$, $\|X\| = 1$, one has

$$\mathcal{X} = g(R(X, SX)X, SX) - (3t/4)\|R(X \wedge SX)\sigma\|^2 \tag{5.1}$$

where S is the complex structure on T_pM defined by σ. Let s_1, s_2, s_3 be the local sections of Z given by (2.1) and $\sigma = \sum_{i=1}^{3} \lambda_i s_i$, $\sum_{i=1}^{3} \lambda_i^2 = 1$. Denote by S_i the complex structure on T_pM determined by $s_i(p)$. Set

$$a_{ij} = g(\mathcal{R}(s_i), X \wedge S_j X), \quad b_{ij} = g(\mathcal{R}(X \wedge S_i X), X \wedge S_j X).$$

Then

$$\|R(X \wedge SX)\sigma\|^2 = \sum_{i=1}^{3} g\left(\mathcal{R}(\sigma \times s_i), X \wedge SX\right)^2 =$$

$$= \sum_{i=1}^{3}\left(\sum_{j=1}^{3} \lambda_j a_{ij}\right)^2 - \left(\sum_{i,j=1}^{3} \lambda_i \lambda_j a_{ij}\right)^2$$

and

$$g(R(X, SX)X, SX) = \sum_{i,j=1}^{3} \lambda_i \lambda_j b_{ij}.$$

Varying $(\lambda_1, \lambda_2, \lambda_3)$ over the unit sphere S^3 one gets from (5.1)

$$a_{ii} - (3t/4)\sum_{k=1}^{3} b_{ki} + (3t/4)b_{ii}^2 = \mathcal{X},$$

$$a_{ii} + a_{jj} - (3t/4)\sum_{k=1}^{3}\left(b_{ki}^2 + b_{kj}^2\right) + 3tb_{ij}^2 + (3t/2)b_{ii}b_{jj} = 2\mathcal{X},$$

$$a_{ij} + a_{ji} - (3t/2)\sum_{k=1}^{3} b_{ki}b_{kj} + 3tb_{ii}b_{ij} = 0$$

for $1 \leqq i \neq j \leqq 3$. These identities imply $b_{ii} = b_{jj}$ and $b_{ij} = 0$ for $i \neq j$, i.e.

$$g(\mathcal{R}(X \wedge S_iX), X \wedge S_iX) = g(\mathcal{R}(X \wedge S_jX), X \wedge S_jX),$$

$$g(\mathcal{R}(X \wedge S_iX), X \wedge S_jX) = 0, \qquad i \neq j.$$

Now varying X over the unit sphere of T_pM gives

$$g(\mathcal{R}(s_i), s_j) = \delta_{ij} g(\mathcal{R}(s_1), s_1),$$

$$g(\mathcal{R}(\bar{s}_i), \bar{s}_j) = \delta_{ij} g(\mathcal{R}(\bar{s}_1), \bar{s}_1),$$

$$g(\mathcal{R}(s_i), \bar{s}_j) = 0, \qquad 1 \leqq i, j \leqq 3.$$

This together with the identity $a_{11} = \mathcal{X}$ shows that M is of constant sectional curvature $\mathcal{X}$. Now by Proposition 5.1 one has

(5.2)
$$\mathcal{X} = \mathcal{X}\|X\|^4 + t\|V\|^4 = \left((\mathcal{X}^2t^2/2)(3(-1)^n + 1) + 3(-1)^{n+1}\mathcal{X}t\right)\|X\|^2\|V\|^2$$

for all $X \in \mathcal{X}(M)$ and $V \in \mathcal{V}$ with $\|X\|^2 + t\|V\|^2 = 1$. For $n = 1$ (5.2) is equivalent to $t = 1/\mathcal{X}$, while for $n = 2$, (5.2) is impossible. Thus the proposition is proved.

Assume that M is complete and simply connected. If $t > 0$, M is the sphere $S^4_{1/\sqrt{\mathcal{X}}}$ and it is well-known that the twistor space Z is $\mathbf{CP}^3$ with a multiple of the Fubini–Study metric. If $t < 0$, M is the unit 4-ball and the twistor space Z is an open subset of $\mathbf{CP}^3$. The precise description of Z and the indefinite metric h_t is given in [10, p. 119].

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INSTITUTE OF MATHEMATICS
BULGARIAN ACADEMY OF SCIENCES
UL. "ACAD. G. BONTCHEV"
1090–SOFIA
BULGARIA

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ON THE UNIQUENESS OF THE EXPANSIONS $1 = \sum q^{-n_i}$

P. ERDŐS, member of the Academy, M. HORVÁTH and I. JOÓ (Budapest)

Consider a number $1 < q < 2$ and take an expansion

$$1 = \sum_{n=1}^{\infty} \varepsilon_n / q^n, \qquad \varepsilon_n = \begin{cases} 0 \\ 1 \end{cases} . \tag{1}$$

Such an expansion is not unique in general. There exist two particular expansion algorithms, the greedy and the lazy algorithm. The digits of the greedy resp. lazy algorithm are defined inductively as follows:

$$\varepsilon_n(x) := \begin{cases} 1 & \text{if } \sum_{i=1}^{n-1} \frac{\varepsilon_i(x)}{q^i} + \frac{1}{q^n} \leqq x \\ 0 & \text{if } \sum_{i=1}^{n-1} \frac{\varepsilon_i(x)}{q^i} + \frac{1}{q^n} > x, \end{cases} \tag{2}$$

$$\tilde{\varepsilon}_n(x) := \begin{cases} 1 & \text{if } \sum_{i=1}^{n-1} \frac{\tilde{\varepsilon}_i(x)}{q^i} + \frac{1}{q^{n+1}} + \frac{1}{q^{n+2}} + \ldots < x \\ 0 & \text{if } \sum_{i=1}^{n-1} \frac{\tilde{\varepsilon}_i(x)}{q^i} + \frac{1}{q^{n+1}} + \frac{1}{q^{n+2}} + \ldots \geqq x. \end{cases} \tag{3}$$

In this paper we shall investigate the unicity of the expansions of 1 and the boundedness of the series formed by consecutive 0 or 1 digits in (1). First we prove

THEOREM 1 (uniqueness) 1. *For* $1 < q < A := \frac{1+\sqrt{5}}{2}$ *there exist* $2^{\aleph_0}$ *expansions* (1) *of* 1.

2. *There exist (at least) countably many* $1 < q < 2$ *for which* 1 *has precisely countably many expansions.*

3. *There exist* $2^{\aleph_0}$ *many* q *for which the expansion of* 1 *is unique.*

4. *The following expansions are unique:*

$$1 = q^{-1} + q^{-2} + \ldots + q^{-k} + \sum_{i=1}^{\infty} q^{-n_i}, \tag{4}$$

where $k \geqq 2$, $2 \leqq n_1 - k \leqq k$, $1 \leqq n_{i+1} - n_i \leqq k$, $n_{i+k-1} - n_i \geqq k$. *In other words, the expansion starts with* k *consecutive* 1*'s and in the further digits* *there do not exist k consecutive 0's or 1's. Conversely, if an expansion* (4) *is unique then*

(4') $k \geqq 2, \quad 2 \leqq n_1 - k \leqq k, \quad 1 \leqq n_{i+1} - n_i \leqq k+1, \quad n_{i+k} - n_i \geqq k+1.$

PROOF 1. The number A is a solution of $x^2 = x + 1$ hence $1 < q < A$ means that

$$q^{-n} < q^{-n-2} + q^{-n-3} + \dots, \quad n \in \mathbf{N}. \tag{5}$$

This implies that for some $k : q^{-n} < q^{-n-2} + \dots + q^{-n-k}$. Take a sequence n_j with $n_{j+1} - n_j > k$ then the sequence $\{q^{-n} : n \neq n_j\} = \{\lambda_1 > \lambda_2 > \dots\}$ satisfies

$$\lambda_n < \lambda_{n+1} + \lambda_{n+2} + \dots \tag{6}$$

and hence the subsums of $\sum_{n=1}^{\infty} \lambda_n$ run over the segment $\left[0, \sum_1^{\infty} \lambda_n\right]$. If n_1 is large enough then $\sum_{n=1}^{\infty} \lambda_n > 1 + \sum_{j=1}^{\infty} q^{-n_j}$. This implies that for any subsum $\sum_{j=1}^{\infty} \varepsilon_j / q^{n_j}$, $\varepsilon_j = \begin{cases} 0 \\ 1 \end{cases}$ there exist $\delta_n = \begin{cases} 0 \\ 1 \end{cases}$ satisfying $\sum_{j=1}^{\infty} \varepsilon_j / q^{n_j} + \sum_{n=1}^{\infty} \delta_n \lambda_n = 1$ so the desired $2^{\aleph_0}$ expansions are constructed.

2. Consider first the case $q = A$. It has precisely the following expansions:

$$\begin{aligned}
1 &= q^{-2} + q^{-3} + q^{-4} + \dots, \\
1 &= q^{-1} + q^{-2}, \\
1 &= q^{-1} + q^{-4} + q^{-5} + q^{-6} + \dots, \\
1 &= q^{-1} + q^{-3} + q^{-4}, \\
1 &= q^{-1} + q^{-3} + q^{-6} + q^{-7} + q^{-8} + \dots, \\
1 &= q^{-1} + q^{-3} + q^{-5} + q^{-6}, \\
1 &= q^{-1} + q^{-3} + q^{-5} + q^{-8} + q^{-9} + q^{-10} + \dots, \\
&\dots\dots\dots \\
1 &= q^{-1} + q^{-3} + q^{-5} + \dots.
\end{aligned}$$

It is easy to see that $q = A$ satisfies these expansions. Consider an expansion (1) of 1 with $q = A$. If $\varepsilon_1 = 0$ then the only possibility is $1 = q^{-2} + q^{-3} + q^{-4} + \dots$ since all the other terms must be used. If $\varepsilon_1 = \varepsilon_2 = 1$ then we must have $1 = q^{-1} + q^{-2}$. If $\varepsilon_1 = 1, \varepsilon_2 = \varepsilon_3 = 0$ then by $1 = q^{-1} + q^{-2} = q^{-1} + q^{-4} + q^{-5} + q^{-6} + \dots$ we see that the only possibility is $1 = q^{-1} + q^{-4} + q^{-5} + \dots$.

If $\varepsilon_1 = 1$, $\varepsilon_2 = 0$, $\varepsilon_3 = 1$, $\varepsilon_4 = \varepsilon_5 = 0$ then $1 = q^{-1} + q^{-3} + q^{-6} + q^{-7} + q^{-8} + \ldots$. We can continue in this way the discussion with the digit sequences 101011, 1010100 etc. Finally, there remains the sequence 10101010 ... which corresponds to the expansion $1 = q^{-1} + q^{-3} + q^{-5} + q^{-7} + \ldots$. Now take another q satisfying $1 = q^{-1} + q^{-2} + \ldots + q^{-k}$ with some $k \geqq 3$. For different values k the values q are also different and we have $q > A$, consequently

$$q^{-n} > q^{-n-2} + q^{-n-3} + q^{-n-4} + \ldots \tag{7}$$

for all n. Using this property we can prove as above that the only expansions of 1 with this q are

$$\begin{aligned}
&1 = q^{-1} + \ldots + q^{-k},\\
&1 = q^{-1} + \ldots + q^{-k+1} + q^{-k-1} + \ldots + q^{-2k},\\
&1 = q^{-1} + \ldots + q^{-k+1} + q^{-k-1} + \ldots + q^{-2k+1} + q^{-2k-1} + \ldots + q^{-3k},\\
&\ldots\ldots\ldots\\
&1 = \sum_{\substack{n \geqq 1\\ k \nmid n}} q^{-n}.
\end{aligned}$$

For example, the first $k-1$ digits must be 1 because

$$q^{-1} + \ldots + q^{-k+2} + q^{-k} + q^{-k-1} + q^{-k-2} + \ldots < 1 = q^{-1} + \ldots + q^{-k}.$$

If $\varepsilon_1 = \ldots = \varepsilon_{k-1} = 1$ and $\varepsilon_k = 0$ then we must have $\varepsilon_{k+1} = \ldots = \varepsilon_{2k-1} = 1$ because by (7)

$$q^{-1} + \ldots + q^{-k+1} + q^{-k-1} + \ldots + q^{-2k+2} + q^{-2k} + q^{-2k-1} + q^{-2k-2} + \ldots <$$
$$< 1 = q^{-1} + \ldots + q^{-k+1} + q^{-k-1} + \ldots + q^{-2k}, \quad \text{and so on.}$$

3. As we proved in Parts 1 and 2, the unique expansions may occur only for $q > A$, hence $1 > q^{-1} + q^{-2}$. Let k be the number satisfying

$$q^{-1} + \ldots + q^{-k} < 1 < q^{-1} + \ldots + q^{-k} + q^{-k-1}.$$

Equality can not occur since the finite expansions are never unique. Since the first k digits can not be changed, we must have

$$q^{-1} + \ldots + q^{-k+1} + q^{-k-1} + q^{-k-2} + \ldots < 1. \tag{8}$$

So $\varepsilon_1 = \ldots = \varepsilon_k = 1$ is ensured. Suppose that $\varepsilon_{k+1} = \ldots = \varepsilon_{2k} = 0$. This means that

$$q^{-1} + \ldots + q^{-k} + q^{-2k} > 1. \tag{9}$$

But (8) and (9) are in contradiction. Indeed, let q_1 be defined by $1 = q_1^{-1} + \ldots + q_1^{-k+1} + q_1^{-k-1} + q_1^{-k-2} + \ldots$, then

$$1-(q_1^{-1}+\ldots+q_1^{-k}+q_1^{-2k})=-q_1^{-k}+(q_1^{-k-1}+\ldots+q_1^{-2k+1}+q_1^{-2k-1}+\ldots)=$$
$$=-q_1^{-k}[1-(q_1^{-1}+\ldots+q_1^{-k+1}+q_1^{-k-1}+\ldots)]=0,$$

$1 = q_1^{-1} + \ldots + q_1^{-k} + q_1^{-2k}$ and so (8) implies $q > q_1$, further (9) implies $q < q_1$. This proves that between $\varepsilon_{k+2}, \ldots, \varepsilon_{2k}$ there exists a digit 1, i.e. if we denote the expansion by $1 = q^{-1}+\ldots+q^{-k}+\sum_{i=1}^{\infty} q^{-n_i}$ then $2 \leqq n_1 - k \leqq k$ is proved.

Next we show that there are no $k+1$ consecutive 0 or 1 digits. Indeed, suppose that $1 = q^{-1}+\ldots+q^{-k}+\sum_{i\leqq j} q^{-n_i}+0\cdot q^{-n_j-1}+\ldots+0\cdot q^{-n_j-k-1}+\ldots$. Since q^{-n_i} can not be omitted,

$$1 > q^{-1}+\ldots+q^{-k}+\sum_{i<j} q^{-n_i}+q^{-n_j-1}+q^{-n_j-2}+\ldots;$$

ε_{n_j+k+1} can not be substituted by 1, hence

$$1 < q^{-1}+\ldots+q^{-k}+\sum_{i\leqq j} q^{-n_i}+q^{-n_j-k-1}.$$

Subtracting the inequalities we get

$$q^{-n_j} > q^{-n_j-1}+\ldots+q^{-n_j-k}+q^{-n_j-k-2}+\ldots$$

i.e. $1 > q^{-1}+\ldots+q^{-k}+q^{-k-2}+q^{-k-3}+\ldots$ in contradiction with the expansion $1 = q^{-1}+\ldots+q^{-k}+\sum_{i=1}^{\infty} q^{-n_i}$. Analogously, if there are $k+1$ consecutive 1 digits, i.e.

$$1 = q^{-1}+\ldots+q^{-k}+\sum_{i\leqq j} q^{-n_i}+q^{-n_j-1}+\ldots+q^{-n_j-k}+\sum_{i\geqq j+k+1} q^{-n_i}$$

and $\varepsilon_{n_j-1}=0$ then $1 < q^{-1}+\ldots+q^{-k}+\sum_{i<j} q^{-n_i}+q^{-n_j-1}$,

$$1>q^{-1}+\ldots+q^{-k}+\sum_{i\leqq j} q^{-n_i}+q^{-n_j-1}+\ldots+q^{-n_j-k+1}+q^{-n_j-k-1}+q^{-n_j-k-2}+\ldots$$

and subtraction gives a contradiction. Conversely, consider an expansion (4) with no k consecutive 0's or 1's in the digits $\varepsilon_{k+1}=0, \varepsilon_{k+2}, \ldots$. Then q^{-k}

can not be omitted since it is larger than the sum of the subsequent not used members:

$$\sum_{n\geqq k+1}(1-\varepsilon_n)q^{-n}\leqq q^{-k-1}+\ldots+q^{-2k+1}+q^{-2k-1}+\ldots+q^{-3k+1}+q^{-3k-1}+\ldots<q^{-k}$$

because $1>q^{-1}+\ldots+q^{-k}$ implies $1>q^{-1}+\ldots+q^{-k+1}+q^{-k-1}+\ldots+$ $+q^{-2k+1}+q^{-2k-1}+\ldots$. By the same argument, no 1 can be changed with 0 in the expansion (4). On the other hand, no 0 can be changed by 1: if $\varepsilon_{n_j-1}=0$, then q^{-n_j+1} is larger than the sum $\sum\limits_{i\geqq j}q^{-n_i}$, because

$$\sum_{i\geqq j}q^{-n_i}\leqq(q^{-n_j}+\ldots+q^{-n_j-k+2})\cdot(1+q^{-k}+q^{-2k}+\ldots)=$$

$$=q^{-n_j+1}(q^{-1}+\ldots+q^{-k+1})(1+q^{-k}+q^{-2k}+\ldots)<q^{-n_j+1}.$$

So the uniqueness is proved.

4. We need two lemmas.

LEMMA 1. *Let* $\varepsilon_1,\ldots,\varepsilon_n\in\{0,1\}$, $\varepsilon_1+\ldots+\varepsilon_n\geqq 1$ *be given and consider the interval* I *of all values* q *for which the expansion of* 1 *begins with* $\varepsilon_1,\ldots,\varepsilon_n$ *and contains further* 1*'s and* 0*'s, i.e.*

$$\sum_{i=1}^{n}\varepsilon_iq^{-i}<1<\sum_{i=1}^{n}\varepsilon_iq^{-1}+q^{-n-1}+q^{-n-2}+\ldots$$

and the subinterval $J\subset I$ *described by*

$$\sum_{i=1}^{n}\varepsilon_iq^{-i}+q^{-n-1}<1<\sum_{i=1}^{n}\varepsilon_iq^{-i}+q^{-n-2}+q^{-n-3}+\ldots,$$

for this q *the expansion of* 1 *can start with the digits* $\varepsilon_1,\ldots,\varepsilon_n,0$ *and also with* $\varepsilon_1,\ldots,\varepsilon_n,1$. *Now if* $I\subset(1+\delta,2-\delta)$ *for some* $\delta>0$, *then* $|I|\leqq C(\delta)|J|$, *where* $C(\delta)>0$ *is independent of* n *and* $\varepsilon_1,\ldots,\varepsilon_n$.

PROOF. Define q_1,q_2,q_1^*,q_2^* by the relations

$$1=\sum_{i=1}^{n}\varepsilon_iq_1^{-i},\quad 1=\sum_{i=1}^{n}\varepsilon_iq_2^{-i}+q_2^{-n-1}+q_2^{-n-2}+\ldots,$$

$$1=\sum_{i=1}^{n}\varepsilon_iq_1^{*-i}+q_1^{*-n-1},\quad 1=\sum_{i=1}^{n}\varepsilon_iq_2^{*-i}+q_2^{*-n-2}+q_2^{*-n-3}+\ldots.$$

Then we have obviously $I=(q_1,q_2)$, $J=(q_1^*,q_2^*)$, $q_1<q_1^*<q_2^*<q_2$. The inequality $1+\delta<q_1$ means the existence of $k=k(q)\in\mathbf{N}$ such that the

digits $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k$ can not vanish at the same time. Consequently for some $1 \leqq j \leqq k$ we have

$$q_2^{-n-1} + q_2^{-n-2} + \dots = \sum_{i=1}^{n} \varepsilon_i (q_1^{-i} - q_2^{-i}) \geqq q_1^{-j} - q_2^{-j} \geqq \frac{q_2 - q_1}{q_1 q_2^j}$$

i.e.

$$q_2 - q_1 \leqq 2q_2^j q_2^{-n-1}(1 - q_2^{-1})^{-1} \leqq 2q_2^{-n+k}(q_2 - 1)^{-1} \leqq C(\delta) q_2^{-n}. \tag{10}$$

On the other hand,

$$\sum_{i=1}^{n} \varepsilon_i (q_1^{*-i} - q_2^{*-i}) + (q_1^{*-n-1} - q_2^{*-n-1}) =$$

$$= -q_2^{*-n-1} + q_2^{*-n-2} + q_2^{*-n-3} + \dots = q_2^{*-n-1}[-1 + (q_2 - 1)^{-1}] \geqq C(\delta) q_2^{*-n-1}$$

and hence

$$C(\delta) q_2^{*-n-1} \leqq \sum_{i=1}^{n} \varepsilon_i (q_1^{*-i} - q_2^{*-i}) = (q_1^{*-1} - q_2^{*-1})[1 + (q_1^{*-1} + q_2^{*-1}) +$$

$$+ \dots + (q_1^{*-n+1} + q_1^{*-n+2} q_2^{*-1} + \dots + q_2^{*-n+1})] \leqq$$

$$\leqq (q_1^{*-1} - q_2^{*-1})[1 + 2q_1^{*-1} + 3q_1^{*-2} + \dots + n q_1^{*-n+1}] \leqq C(\delta)(q_1^{*-1} - q_2^{*-1}),$$

$$q_2^* - q_1^* \geqq C(\delta) q_2^{*-n}. \tag{11}$$

The estimates (10) and (11) prove Lemma 1.

LEMMA 2. *Let $\delta > 0$, $\eta > 0$ and let I be an interval satisfying*

$$\sum_{i=1}^{n} \varepsilon_i q^{-i} < 1 < \sum_{i=1}^{n} \varepsilon_i q^{-1} + q^{-n-1} + q^{-n-2} + \dots \quad (q \in I). \tag{12}$$

If $(1+\delta, 2-\delta) \supset I$ then there exists a system $I_1, I_2, \dots$ of disjoint subintervals of I such that

a) $\bigcup_{j=1}^{\infty} I_j$ *is dense in I,*

b) $\left| I \setminus \bigcup_{j=1}^{\infty} I_j \right| < \eta$,

c) *for all* I_j *there exist two different continuations* $\varepsilon_{n+1}, \dots, \varepsilon_{n+k}$ *and* $\varepsilon'_{n+1}, \dots, \varepsilon'_{n+k}$ *of* $\varepsilon_1, \dots, \varepsilon_n$ *(here* k *and the* ε*'s may depend on* j*) such that for all* $q \in I_j$

$$\sum_{i=1}^{n+k} \varepsilon_i q^{-i} < 1 < \sum_{i=1}^{n+k} \varepsilon_i q^{-i} + q^{-n-k-1} + q^{-n-k-2} + \dots, \tag{13}$$

(14)

$$\sum_{i=1}^{n} \varepsilon_i q^{-i} + \sum_{i=n+1}^{n+k} \varepsilon'_i q^{-i} < 1 < \sum_{i=1}^{n} \varepsilon_i q^{-i} + \sum_{i=n+1}^{n+k} \varepsilon'_i q^{-i} + q^{-n-k-1} + q^{-n-k-2} + \dots.$$

PROOF. Choose a number $N \in \mathbf{N}$ and consider the intervals of all q satisfying

$$\sum_{i=1}^{n+N} \varepsilon_i q^{-i} < 1 < \sum_{i=1}^{n+N} \varepsilon_i q^{-i} + q^{-n-N-1} + q^{-n-N-2} + \dots$$

where $\varepsilon_{n+1}, \dots, \varepsilon_{n+N}$ is any (fixed) continuation of the digits $\varepsilon_1, \dots, \varepsilon_n$. The 2^N intervals so constructed cover I and if $N > N(\delta)$ then the length of these intervals is bounded by $C(\delta) q_1^{-n-N}$. Applying Lemma 1 for these 2^N intervals we get a system of intervals $\{J_1, \dots, J_{2^N}\} = \mathcal{J}_N$ such that $|J_\ell| \geqq C(\delta) q_1^{-n-N}$ (the constants $C(\delta)$ may be different at different occurrences). Every subinterval of I of length $C(\delta) q_1^{-n-N}$ has an intersection of measure $\geqq C(\delta) q_1^{-n-N}$ with $\cup \mathcal{J}_N$ and in every J_ℓ there exist two different expansions of 1 starting with $\varepsilon_1, \dots, \varepsilon_{n+N}, 0$ and $\varepsilon_1, \dots, \varepsilon_{n+N}, 1$. If we repeat the above construction with $2N$, $3N$, ... instead of N, we get the systems $\mathcal{J}_{2N}, \mathcal{J}_{3N}, \dots$. By the Lebesgue density theorem $\left| I \setminus \bigcup_{i=1}^{\infty} \cup \mathcal{J}_{iN} \right| = 0$ whence, for large j the finite interval system $\bigcup_{i \leqq j} \mathcal{J}_{iN}$ satisfies the conditions b) and c) of Lemma 2. In order to ensure a) it is enough to show the following fact: For any interval $J \subset (1 + \delta, 2 - \delta)$ for which (12) holds for all $q \in J$, there exists a subinterval $K \subset J$ and two different continuations $\varepsilon_{n+1}, \dots, \varepsilon_{n+k}$ and $\varepsilon'_{n+1}, \dots, \varepsilon'_{n+k}$ such that (13) and (14) hold for all $q \in K$. But this is easy: take $q \in J$ such that $1 = \sum_{i=1}^{\infty} \varepsilon_i / q^i$ contains infinitely many 0 and 1 digits (only countable q are so excluded) and take a large k with $\varepsilon_{n+k} = 1$. Then $\sum_{i=1}^{n+k-1} \varepsilon_i q^{-i} + q^{-n-k} < 1$. Define q_1 and q_2 by

$$\sum_{i=1}^{n+k-1} \varepsilon_i q_1^{-i} + q_1^{-n-k} = 1 = \sum_{i=1}^{n+k-1} \varepsilon_i q_2^{-i} + q_2^{-n-k-1} + q_2^{-n-k-2} + \dots,$$

then $q_1 < q < q_2$ and in the interval $K = (q_1, q_2)$ the expansion of 1 can be started with $\varepsilon_1, \dots, \varepsilon_{n+k-1}, 0$ and $\varepsilon_1, \dots, \varepsilon_{n+k-1}, 1$. As we have seen in the proof of Lemma 1, we have $q_2 - q_1 \leqq C(\delta) q_2^{-n-k} \leqq C(\delta)(1+\delta)^{-n-k}$, so for large k we have $K \subset J$. Lemma 2 is proved.

We return to the proof of Theorem 1, Part 4. It is enough to prove that for any $\delta > 0$ in the segment $(1+\sqrt{5})/2 =: A < q < 2$ for a.e. q and for every q except for a set of first category there are $2^{\aleph_0}$ expansions of 1. Let $n \in \mathrm{N}$ be fixed and apply Lemma 2 with $\eta = 2^{-N}$, $n = 1$, $\varepsilon_1 = 1$, $I = (A, 2-\delta)$. Denote $A_1 := \bigcup_{j=1}^{\infty} I_j$, then A_1 is open and dense in I further $|I \backslash A_1| < 2^{-N}$. For every interval I_j apply again Lemma 2 with one expansion $\varepsilon_1, \dots, \varepsilon_{n+k}$; we get the intervals I_{j,j_1}; then for all I_{j,j_1} we apply Lemma 2 with the other expansion $\varepsilon_1, \dots, \varepsilon_n, \varepsilon'_{n+1}, \dots, \varepsilon'_{n+k}$ to obtain the system I_{j,j_1,j_2}. Denote $A_2 := \bigcup_{j,j_1,j_2} I_{j,j_1,j_2}$ then A_2 is open, dense in I and we can ensure $|I \backslash A_2| < 2^{-N} + 2^{-N-1}$, further in every interval I_{j,j_1,j_2} there exist four different beginnings of the expansion of 1, common for all $q \in I_{j,j_1,j_2}$. In the third step Lemma 2 applies for I_{j,j_1,j_2} with the first expansion, for all I_{j,j_1,j_2,j_3} with the second one, for all $I_{j,j_1,j_2,j_3,j_4,j_5}$ with the fourth one, further define A_3 as the union of all the intervals $I_{j,j_1,j_2,j_3,j_4,j_5,j_6}$. Continuing this process we obtain the open and dense sets A_n with $|I \backslash A_n| < 2^{-N} + 2^{-N-1} + \dots + 2^{-N-n+1}$. By the construction for every q in the set $A := \bigcap_{n=1}^{\infty} A_n$, 1 has $2^{\aleph_0}$ many expansions further $I \backslash A$ is of first category and $|I \backslash A| < 2^{-N+1}$. Since N can be arbitrarily large, Part 4 is proved. The proof of Theorem 1 is complete.

REMARK. In Part 3 we formulated a necessary and another sufficient condition for the uniqueness. The sufficient condition does not contain all unique expansions as the following example shows:

$$1 = q^{-1} + q^{-2} + q^{-4} + q^{-5} + q^{-7} + q^{-9} + q^{-11} + \dots. \tag{15}$$

This is a unique expansion. Indeed, q^{-2} can not be omitted because $1 > q^{-1} + q^{-4} + q^{-6} + q^{-8} + \dots$; ε_3 can not be substituted by 1 since $1 > q^{-1} + q^{-2} + q^{-4} + q^{-6} + q^{-8} + \dots$; $\varepsilon_5, \varepsilon_7, \varepsilon_9, \dots$ can not be omitted, $\varepsilon_6, \varepsilon_8, \varepsilon_{10}, \dots$ can not be changed because $1 > q^{-1} + q^{-3} + q^{-5} + q^{-7} + \dots$ holds by $1 > q^{-1} + q^{-2}$. So (15) is unique indeed. On the other hand the necessary condition is not sufficient as the following example shows: in the expansion $1 = q^{-1} + q^{-2} + q^{-4} + q^{-7} + q^{-9} + q^{-11} + q^{-13} + \dots$, q^{-4} can be omitted, because $1 < q^{-1} + q^{-2} + q^{-4} + q^{-6} + \dots$ hence

$$0 < 1 - q^{-1} - q^{-2} < q^{-5} + q^{-6} + q^{-7} + q^{-9} + q^{-11} + q^{-13} + \dots.$$

The following questions arise.

PROBLEM 1. Determine the unique expansions.

PROBLEM 2. Does there exist q such that 1 has precisely two (or n) expansions? Describe them.

PROBLEM 3. Do there exist precisely $\aleph_0$ numbers q for which 1 has precisely $\aleph_0$ expansions? Characterize these numbers.

In what follows we consider the problem of the boundedness of the length of 0-sequences in the expansions of 1. Remark that quantitative and qualitative results on this topic are published in [2–5].

THEOREM 2. 1. *If the expansion* (1) *is unique then its zero sequences are bounded.*

2. *For* $1 < q < (1+\sqrt{5})/2$ *there exists an expansion* (1) *where the zero sequences are bounded.*

PROOF. 1. Suppose that $\varepsilon_n = 1$, $\varepsilon_{n+1} = \ldots = \varepsilon_{n+k} = 0$ is a unique expansion. Since q^{-n} can not be omitted, we must have $q^{-n} > q^{-n-1} + \ldots + q^{-n-k}$,

$$1 > q^{-1} + q^{-2} + \ldots + q^{-k} = \frac{1-q^{-k}}{q-1}, \quad q^{-k} > 2 - q.$$

This can not hold for infinitely many k since for large k, q must be close to 2.

2. For $q = \frac{1+\sqrt{5}}{2}$ such an expansion is $1 = q^{-1} + q^{-3} + q^{-5} + \ldots$. If $q < (1+\sqrt{5})/2$ then $1 < q^{-2} + q^{-3} + q^{-5} + \ldots$ and hence for some large r, $1 < q^{-2} + \ldots + q^{-r}$. On the other hand for large r we have $1 > q^{-1} + q^{-r-1} + q^{-2r-1} + \ldots$. Now let $x := 1 - (q^{-1} + q^{-r-1} + q^{-2r-1} + \ldots)$, then $0 < x < q^{-2} + \ldots + q^{-r} + q^{-r-2} + \ldots + q^{-2r} + \ldots$. The members on the right hand side form a sequence $\lambda_1 > \lambda_2 > \ldots > 0$ satisfying $\lambda_n < \lambda_{n+1} + \lambda_{n+2} + \ldots$ for all n. Consequently the sums $\sum \varepsilon_n \lambda_n$, $\varepsilon_n = \begin{cases} 0 \\ 1 \end{cases}$ fill in the segment $[0, \sum \lambda_n]$; in particular $x = \sum \varepsilon_n \lambda_n$ and then $1 = q^{-1} + q^{-r-1} + q^{-2r-1} + \ldots + \sum \varepsilon_n \lambda_n$ is the desired expansion of 1.

Finally we formulate some open problems. Prove or disprove:

PROBLEM 4. If $1 = \sum q^{-n_i}$ and $\sup(n_{i+1} - n_i) = \infty$ then there exist $2^{\aleph_0}$ expansions of 1.

PROBLEM 5. Conversely, if there exist $2^{\aleph_0}$ expansions then there is an expansion with $\sup(n_{i+1} - n_i) = \infty$.

PROBLEM 6. If there exist precisely $\aleph_0$ expansions of 1 then for any expansion $1 = \sum q^{-n_i}$ (with infinitely many 1 digits) $\sup(n_{i+1} - n_i) < \infty$.

PROBLEM 7. If there exist precisely $\aleph_0$ expansions of 1 then there is a finite expansion (with finitely many 1 digits).

PROBLEM 8. There exists $1 < q < 2$ which has $2^{\aleph_0}$ expansions and has a finite expansion.

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MATHEMATICAL INSTITUTE OF THE
HUNGARIAN ACADEMY OF SCIENCES
REÁLTANODA U. 13–15
H–1053 BUDAPEST

EÖTVÖS LORÁND UNIVERSITY
MATHEMATICAL INSTITUTE
MÚZEUM KRT. 6–8
H–1088 BUDAPEST

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MULTIPLICATIVE FUNCTIONS WITH REGULARITY PROPERTIES. VI

I. KÁTAI* (Budapest), member of the Academy

Dedicated to F. Schipp on his fiftieth birthday

1. Recently A. Hildebrand proved the following theorem [1].

There exists a positive constant c with the following property. If $g \in M^*$ (the set of completely multiplicative functions), $|g(n)| = 1$ for every $n \in \mathbf{N}$, and $|g(p) - 1| \leqq c$ for every prime p, then either $g(n) = 1$ identically, or

$$\liminf \frac{1}{x} \sum_{n \leqq x} |g(n+1) - g(n)| > 0.$$

Our purpose in this short paper is to prove the following

THEOREM 1 *There exist positive constants $\beta(\leqq 1/2)$ and δ with the following property. If $g \in M^*$ and $|g(n)| = 1$ for every $n \in \mathbf{N}$, furthermore*

$$\limsup_x \sum_{x^\beta < p < x} \frac{|g(p) - 1|}{p} < \delta \tag{1.1}$$

and

$$\liminf_x \frac{1}{x} \sum_{\frac{x}{2} \leqq n \leqq x} |g(n+1) - g(n)| = 0, \tag{1.2}$$

then $g(n) = 1$.

REMARK To prove the theorem we shall use some ideas due to Hildebrand [1], and apply a theorem of Halász on the existence of the mean value of multiplicative functions, furthermore some sieve results.

2. Proof of the theorem, first case. Assume that $g \in M^*$ and $|g(n)| = 1$ holds for every $n \in \mathbf{N}$. In [2] Hildebrand proved that

$$\sum_p \frac{|1 - g(p)p^{i\tau}|^2}{p} < \infty \tag{2.1}$$

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implies that

$$\frac{2}{x}\sum_{\frac{x}{2}\leqq n<x}\overline{g(n+1)}g(n)\to\prod\Phi_p, \tag{2.2}$$

where

$$\Phi_p=1-\frac{2}{p}+2\left(1-\frac{1}{p}\right)\mathrm{Re}\frac{g(p)p^{i\alpha}}{p-g(p)p^{i\alpha}}.$$

(1.2) and (2.2) together imply that $\Phi_p = 1$ holds for each prime p, i.e. $g(p)=p^{-iT}$. If $T=0$ we get the function $g(n)\equiv 1$. Assume that $\tau\neq 0$. We shall show that in this case (1.1) cannot be satisfied if δ is small enough and $\beta\leqq 1/2$. Indeed, by using the prime number theorem,

$$\sum_{\sqrt{x}\leqq p<x}\frac{|1-p^{-i\tau}|}{p}=2\sum\frac{|\sin\frac{\tau}{2}\log p|}{p}=\int_{\frac{\tau}{4}x_1}^{\frac{\tau}{2}x_1}\frac{|\sin\lambda|}{\lambda}d\lambda+o_x(1),$$

$x_1=\log x$. Since the limit superior of

$$\int_{y/2}^{y}\frac{|\sin\lambda|}{\lambda}d\lambda$$

is bounded below by an absolute positive constant, therefore (1.1) cannot hold if δ is small.

From now on, we may assume that for every $\tau\in R$,

$$\sum_p\frac{|1-g(p)p^{i\tau}|^2}{p}=\infty. \tag{2.1}$$

But in this case, by Halász' theorem,

$$\sum_{n\leqq x}g(n)=o(x),$$

which implies easily that

$$L(x):=\sum_{n\leqq x}\frac{g(n)}{n}=o(\log x). \tag{2.2}$$

Let us consider the sum

$$L(x|m):=\sum_{\substack{n\leqq x\\(n,m)=1}}\frac{g(n)}{n} \tag{2.3}$$

for every m in $[1,x]$. Then, by the Moebius formula,

$$L(x|m) = \sum_{n\leqq x} \frac{g(n)}{n} \sum_{d|(n,m)} \mu(d) = \sum_{d|m} \mu(d) \sum_{k\leqq x/d} \frac{g(d)g(k)}{kd} = \tag{2.4}$$

$$= \sum_{d|m} \frac{\mu(d)g(d)}{d} L\left(\frac{x}{d}\right) = \prod_{p|m}\left(1 - \frac{g(p)}{p}\right) L(x) + \sigma_{m,x},$$

where

$$\sigma_{m,x} = \sum_{d|m} \frac{\mu(d)g(d)}{d}\left(L\left(\frac{x}{d}\right) - L(x)\right),$$

and so

$$|\sigma_{m,x}| \leqq 2\sum_{d|m} \frac{\log d}{d}. \tag{2.5}$$

Doing the same for the function $g(n) \equiv 1$, we have

$$\sum_{\substack{n\leqq x\\ (n,m)=1}} \frac{1}{n} = \frac{\varphi(m)}{m} \log x + \tau_{m,x} \tag{2.6}$$

where

$$|\tau_{m,x}| \leqq 2\sum_{d|m} \frac{\log d}{d} L. \tag{2.7}$$

Let us consider now the sum

$$T(y) := \sum_{(u,v)=1} \frac{|g(u) - g(v)|}{uv}. \tag{2.8}$$

It is clear that

$$\sum_{u\leqq y} \frac{1}{u}\left|g(u) \sum_{\substack{v\leqq y\\ (v,u)=1}} 1/v - L(y|u)\right| \leqq T(y),$$

and so, by (2.4)–(2.7) we get that

$$\sum_{u} \frac{1}{u}\left|g(u)\frac{\varphi(u)}{u}\log y - \prod_{p|u}\left(1 - \frac{g(p)}{p}\right) L(y)\right| \leqq \tag{2.9}$$

$$\leqq T(y) + 4\sum_{u\leqq y} \frac{1}{u} \sum_{d|u} \frac{\log d}{d}.$$

The second sum on the right hand side has the order $O(\log y)$. Since $L(y) = o(\log y)$, and

$$\left|\prod_{p|u}\left(1-\frac{g(p)}{p}\right)\right| \leqq \prod_{p|u}\left(1+\frac{1}{p}\right),$$

therefore, from (2.9) we obtain that for every positive $\varepsilon > 0$ and for every large $y > y_0(\varepsilon)$,

$$(\log y)\sum_{n\leqq y}\frac{\varphi(u)}{u^2} \leqq T(y) + \varepsilon(\log y)^2 L.$$

It is known that

$$\sum_{n\leqq y}\frac{\varphi(u)}{u^2} = A\log y + O(1)$$

with some absolute constant $A(>0)$. Thus we have

$$T(y) > (A-2\varepsilon)(\log y)^2 \quad \text{if} \quad y > y_1(\varepsilon). \tag{2.10}$$

3. Let $p(n)$ and $P(n)$ be the smallest and the largest prime factor of n, resp. Let $N_\beta(x|u,v)$ denote the number of solutions of $Qv - Ru = 1$ in integers Q, R satisfying the conditions $Ru \in \left(\frac{x}{2}, x\right]$, $P(Q) > x^\beta$, $P(R) > x^\beta$.

One can deduce from sieve results that

$$N_\beta(x|u,v) > c_\beta\frac{x}{(\log x)^2 u\cdot v} \quad \text{if} \quad x > x_0(\beta) \tag{3.1}$$

with some positive constant c_β, whenever β is small enough, and u, v are coprime integers satisfying the conditions $1 \leqq u, v \leqq x^\beta$.

Let us consider the sum

$$\sum_{\substack{1\leqq u,v\leqq x^\beta\\(u,v)=1}} |g(n)-g(v)|N_\beta(x|u,v). \tag{3.2}$$

From (3.1) we get that

$$S \geqq c_\beta\frac{x}{\log^2 x}T(x_\beta). \tag{3.3}$$

Now we want to give an upper estimation for S in terms of

$$U(x) := \sum_{\frac{x}{2}\leqq n\leqq x}|\Delta g(n)|.$$

We can observe that in (3.2) $|g(u) - g(v)|$ occurs as many times as many solutions the equation $Qv - Ru = 1$ has. Let $n = Ru$, $n+1 = Qv$. It is clear that some n (and $n+1$) can be represented as Ru (and Qv) at most once. Furthermore,

$$(3.4) \qquad |g(u) - g(v)| = |g(u)\overline{g(v)} - 1| = |\bar{g}(R)g(Q)g(n)\overline{g(n+1)} - 1| \leqq$$
$$\leqq |g(n)\overline{g(n+1)} - 1| + |\overline{g(R)}g(Q) - 1| \leqq |\Delta g(n)| + |g(Q) - 1| + |g(R) - 1|.$$

Let $A(n)$ be the product of the prime factors larger than x^β. Let

$$V = \sum_{\frac{x}{2} \leqq n \leqq x+1} |g(A(n)) - 1|.$$

From (3.4) we obtain that

$$(3.5) \qquad S \leqq U(x) + 2V.$$

The contribution of the integers n for which $A(n)$ is not a square-free number is small, $\ll x/\log x$, say. Thus

$$V \leqq 2xH + c_\beta \chi/\log x, \qquad H := \sum_{1<m\leqq x} \frac{|g(m)-1|}{m} L,$$

where m runs over those square-free integers greater than 1, the prime factors of which belong to $[x^\beta, x]$. Let $t(p) = g(p) - 1$, and let t be extended as a multiplicative function. Since

$$g(m) - 1 = \prod_{g|m} (1 + t(q)) - 1 = \sum_{\substack{d|m \\ d>1}} t(d),$$

therefore

$$H \leqq \sum_{d>1} \frac{|t(d)|}{d} \prod_{x^\beta \leqq q \leqq x} \left(1 + \frac{1}{q}\right) \leqq \frac{2}{\beta}\left\{ \prod_{x^\beta \leqq p \leqq x} \left(1 + \frac{|t(p)|}{p}\right) - 1 \right\} \leqq$$
$$\leqq \frac{2}{\beta}\left\{\exp\left(\sum \frac{|t(p)|}{p}\right) - 1\right\} \leqq \frac{2}{\beta}(e^\delta - 1).$$

So we have

$$(3.6) \qquad S \leqq \frac{8}{\beta}(e^\delta - 1)x + 2c_\beta x/\log x + U(x).$$

From (2.10), (3.3), (3.6) we have

$$(3.7) \qquad c_\beta \beta^2 (A - 2\varepsilon) \leqq \frac{8}{\beta}(e^\delta - 1) + \frac{2c_\beta}{\log x} + \frac{U(x)}{x}$$

for each large x. If δ is chosen so that $c_\beta \beta^2 (A - 2\varepsilon) > \frac{8}{\beta}(e^\delta - 1)$, then $\frac{U(x)}{x}$ has to be bounded below by a positive constant.

By this the proof is finished.

4. THEOREM 2. *Let $f,g \in M$, $|f(n)| = |g(n)| = 1$ for every n, furthermore*

$$\liminf_{x} \frac{1}{x} \sum_{n \leqq x} |g(n+1) - f(n)| = 0. \tag{4.1}$$

Then $f(n) = g(n)$ for every n, and $f \in M^*$.

This leads immediately to the following generalization of Theorem 1.

THEOREM 1'. *There exist positive constants β $(\leqq 1/2)$ and δ with the following property. If $f,g \in M$, $|g(n)| = 1$ and $|f(n)| = 1$ for every $n \in \mathbf{N}$, furthermore (1.1) and (4.1) hold true, then $f(n) = g(n) = 1$ for every $n \in \mathbf{N}$.*

PROOF OF THEOREM 2. If (4.1) holds true, then there exist sequences $x_\nu \to \infty$, $\varepsilon_\nu \to 0$ such that

$$\sum_{n \leqq x_\nu} |g(n+1) - f(n)| \leqq \varepsilon_\nu x_\nu. \tag{4.2}$$

Let

$$r(n) := \frac{g(n+1)}{f(n)}, \qquad H(n) := \frac{g(n)}{f(n)}, \qquad D := \frac{g(4)}{g(2)f(2)}.$$

Let us observe that

$$\bar{H}(16k+11)r(16k+10)r(16k+11) = \frac{g(16k+12)}{f(16k+10)} = Dr(8k+5). \tag{4.3}$$

From (4.2) we have

$$\sum_{n \leqq x_n u} |r(n) - 1| \leqq \varepsilon_\nu x_\nu$$

and so by (4.3) we deduce that

$$\sum_{16k+11 \leqq x_\nu} |DH(16k+11) - 1| < c\varepsilon_\nu x_\nu \tag{4.4}$$

where c is an absolute positive constant. (4.4) can be written as

$$\sum_{\substack{n \leqq x_\nu \\ n \equiv 11 \pmod{16}}} \left| H(n) - \frac{1}{D} \right| < c\varepsilon_\nu x_\nu. \tag{4.5}$$

Let us choose now some odd m, and substitute n by nm. Then we have

$$\sum_{\substack{n \leqq x_\nu/m \\ (n,m)=1 \\ nm \equiv 11 \pmod{16}}} |H(m)H(n) - 1/D| < c\varepsilon_\nu x_\nu. \tag{4.6}$$

Since $|H(m)-1| = |(H(m)-1)H(n)| \leqq \left|H(m)H(n)-\frac{1}{D}\right| + \left|H(n)-\frac{1}{D}\right|$, from (4.6) we obtain that

$$\sum_{\substack{n \leqq x_\nu/m \\ nm \equiv 11 \pmod{16} \\ (n,m)=1}} |H(m)-1| \leqq c\varepsilon_\nu x_\nu,$$

which implies that $H(m)=1$. So we proved that $H(m)=1$ holds for every odd m. Then, from (4.5) we get that $D=1$. Let $m=1+2\ell$, $(\ell,2)=1$. By the triangle inequality,

$$|1-\overline{H(2)}| = |g(m)-\overline{H(2)}f(m)| = |g(m)-f(m-1)+f(2)f(\ell)-$$
$$-f(2)g(\ell+1)+f(2)g(\ell+1)-\overline{g(2)}f(2)f(m)| \leqq |g(m)-f(m-1)|+$$
$$+|g(\ell+1)-f(\ell)|+|g(m+1)-f(m)|.$$

Summing up for odd ℓ's up to $2\ell \leqq x_\nu$, we can deduce that $f(2)=g(2)$. This together with $D=1$, gives that $g(4)=g(2)^2$. Since

$$\frac{g(2n)}{f(2(n-1))} = r(2n-1)r(2n-2),$$

therefore

$$(4.7) \qquad \sum_{2n \leqq x_\nu} \left|\frac{g(2n)}{f(2(n-1))}-1\right| < c\varepsilon_n u x_n.$$

Let $s \geqq 1$, $n=2^s k$, $(k,2)=1$. Then

$$(4.8) \qquad \frac{g(2n)}{f(2(n-1))} = \frac{g(2^{s+1})}{g(2^s)f(2)} r(n-1),$$

and so by (4.2) and (4.7) we get easily that

$$(4.9) \qquad g(2^{s+1}) = g(2^s)f(2) = g(2^s)g(2).$$

Similarly summing up the summands of (4.4) only for the integers $n=1+2^s k$, we get

$$(4.10) \qquad f(2^{s+1}) = f(2^s)g(2) = f(2^s)f(2).$$

Consequently $f(2^s)=f(2)^s=g(2)^s=g(2^s)$.

So $f(n)=g(n)$ holds for every $n \in \mathbf{N}$.

It remains to prove that $f \in M^*$. This is easily seen from

$$(4.11) \qquad \sum_{n \leqq x_\nu} |\Delta f(n)| \leqq \varepsilon_\nu x_\nu.$$

Let $m > 1$ be arbitrary, $\Delta_m f(n) = f(n+m) - f(n)$. From (4.11) we have

$$\sum_{n \leqq x_\nu/2m} |U(m,n)| < c\varepsilon_\nu x_\nu \tag{4.12}$$

where

$$U(m,n) = (f(m(n+1)) - f(m)f(n+1)) - (f(mn) - f(m)f(n)). \tag{4.13}$$

Let P be an arbitrary prime, $m = P$, and let n in (4.12) run over only the integers satisfying $P^\alpha \| n$. Since the set of these integers n has a positive density, and $|U(P,n)| = |f(P^{\alpha+1}) - f(P)f(P^\alpha)|$ for them, we obtain that $f(P^{\alpha+1}) = f(P)f(P^\alpha)$.

This completes the proof of our theorem.

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UNIVERSITY COMPUTER CENTER
BUDAPEST, BOGDÁNFY U. 10/B
H–1117

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A HAAR-TYPE THEORY OF BEST UNIFORM APPROXIMATION WITH CONSTRAINTS

A. KROÓ[1,2] (Budapest) and D. SCHMIDT[2] (Rochester)

Introduction

The literature of approximation theory contains numerous results concerning best approximation under a variety of constraints. These results mainly concentrate on investigating problems in L_∞-norm, see, e.g., papers by Chalmers [1] and Chalmers and Taylor [2]. Since in most cases existence can be easily established the truly interesting question consists in studying the uniqueness of best approximants. In [1] Chalmers introduced a general method of investigating uniqueness of best approximations with constraints, which provided a unified approach to the problem. However, this approach essentially provided only *sufficient* conditions for uniqueness of best constrained approximation. In this paper we shall be concerned with developing a *Haar-type* theory for constrained L_∞-approximation, that gives *necessary* and *sufficient* conditions for uniqueness. In this sense our paper is closer in spirit to a recent paper by Pinkus and Strauss [8] where necessary and sufficient conditions for uniqueness were given in the special case of best L_∞-approximation with coefficient constraints. Our goal is to provide similar characterizations of uniqueness in the general setting of linear constraints. In order to impose the constraints we shall use linear operators, instead of linear functionals used in [1]. This approach, while preserving generality of constraints, will provide us with a convenient tool leading to technically simple characteristics of uniqueness.

Let us recall now the classical theorem of A. Haar [5]. Let U_n be an n-dimensional subspace of $C[a,b]$, the space of real-valued continuous functions with L_∞-norm. Then the Haar theorem states that every $f \in C[a,b]$ possesses a unique best approximant out of U_n if and only if each $p \in U_n \setminus \{0\}$ has at most $n-1$ distinct zeros in $[a,b]$. (Throughout this paper subspaces U_n satisfying the above property will be called Haar spaces.)

Let $L\colon U_n \to C(K)$ be a linear operator mapping U_n into $C(K)$, where K is a finite union of intervals and points in $\mathbf{R}$. For given $v,u \in C(K)$ such that $v < u$ on K set $\tilde{U}_n(v,u) = \{p \in U_n : v \leqq Lp \leqq u, x \in K\}$. We

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[2] Prepared while both authors were visiting Old Dominion University, Norfolk, Virginia 23529.

shall say that Int $\tilde{U}_n(v,u) \neq \emptyset$ if there exists $\tilde{p} \in U_n$ satisfying $v < L\tilde{p} < u$ $(x \in K)$. Now we consider the problem of approximating in norm $\|g\| = \max_{a \leqq x \leqq b} |g(x)| \quad (g \in C[a,b])$ by elements of $\tilde{U}_n(v,u)$. We say that $p_0 \in \tilde{U}_n(v,u)$ is a best approximant of $f \in C[a,b]$ if $\|f-p_0\| = \inf\{\|f-p\|: p \in \tilde{U}_n(v,u)\}$. Throughout the paper we shall study best approximation on the interval $[a,b]$ and impose restrictions on the compact set K. Uniqueness of best approximation by elements of $\tilde{U}_n(v,u)$ depends, of course, on U_n, L and v and u. In order to get simple and elegant descriptions of unicity we shall be interested in *boundary independent* uniqueness, i.e., uniqueness for *every* v, u. This leads us to the following central question.

PROBLEM A. Given $U_n \subset C[a,b]$ and $L\colon U_n \to C(K)$ find a necessary and sufficient condition so that for every $f \in C[a,b]$ and $v, u \in C(K)$ with Int $\tilde{U}_n(v,u) \neq \emptyset$ the best approximant of f from $\tilde{U}_n(v,u)$ is unique.

We shall give a complete solution of Problem A and, in particular show that corresponding subspaces U_n and operators L are somewhat rare. It turns out that requiring uniqueness for every continuous boundary is too restrictive. On the other hand working with *smooth* C'-boundaries leads to much more meaningful results. Therefore we also investigate the next

PROBLEM B. Given $U_n \subset C[a,b]$ and $L\colon U_n \to C'(K)$ find a necessary and sufficient condition so that for every $f \in C[a,b]$ and $v, u \in C'(K)$ with Int $\tilde{U}_n(v,u) \neq \emptyset$ the best approximant of f in $\tilde{U}_n(v,u)$ is unique.

As it was mentioned above the variety of subspaces U_n and operators L providing positive solution to Problem B is essentially wider than for Problem A.

Our method of solving Problems A and B will be based on the notion of extremal sets, which is frequently used in the literature. A set of at most $n+1$ points $x_1, \dots, x_r \in [a,b]$ $(1 \leqq r \leqq n+1)$ is called an extremal set for U_n if there exist nonzero numbers $c_1, \dots, c_r$ so that

$$\sum_{i=1}^{r} c_i p(x_i) = 0, \qquad p \in U_n. \tag{1}$$

Using this notion one can easily give the following equivalent form of the Haar Theorem: $U_n \subset C[a,b]$ is a Haar subspace if and only if no nontrivial element of U_n vanishes on an extremal set of U_n. Moreover, U_n is Haar if and only if every extremal set of U_n consists of exactly $n+1$ points.

We shall give similar solutions to Problems A and B using the notion of *L-extremal sets,* which is a natural extension of extremal sets defined by (1).

Our paper is organized in the following way. Section 1 contains the general uniqueness theory, i.e., solutions to Problems A and B and related results. In the next section we include the complete theory of strong uniqueness related to Problems A and B. The final part contains different applications of our results. We shall show how the theory can be applied in order to distinguish the "good" and "bad" subspaces and operators. Since our results provide not only sufficient, but also necessary conditions of uniqueness, we shall be able to characterize completely those spaces of lacunary algebraic polynomials which satisfy the requirements of Problem B with $L = D^k$ (differentiation operator). Furthermore, we shall consider Problem B for operator $L = D - \alpha I$ ($\alpha \in \mathbf{R}$, I is identity operator) and show that algebraic polynomials of degree $\leqq n-1$ provide a positive solution to Problem B if and only if $|\alpha| \leqq \frac{n-1}{2}$. It will be also shown that subspaces of rational functions with a fixed denominator, in general, fail to satisfy requirements of Problem B for $L = D$, if the degree of denominator is at least two. With regard to Problem A we shall give a precise constructive description of spaces U_n providing uniqueness and show that they are very scarce unless int $K = 0$.

1. Uniqueness of best constrained approximation

In order to obtain solutions to the problems outlined in the introduction we shall need some standard characterizations of best approximations from $\tilde{U}_n(v,u)$. For any $f \in C[a,b]$ we denote $E(f) = \{x \in [a,b] : |f(x)| = \|f\| = \max_{a \leqq x \leqq b} |f(x)|\}$, while for $g \in C(K)$, $Z(g) = \{x \in K : g(x) = 0\}$.

THEOREM 1.1. *Suppose that for given $v, u \in C(K)$ and $L\colon U_n \to C(K)$, Int $\tilde{U}_n(v,u) \neq \emptyset$. Then $p_0 \in \tilde{U}_n(v,u)$ is a best approximant to $f \in C[a,b]$ from $\tilde{U}_n(v,u)$ if and only if for every $p \in U_n$ satisfying $Lp \leqq 0$ on $Z(u - Lp_0)$ and $Lp \geqq 0$ on $Z(v - Lp_0)$ we have*

$$\min_{x \in E(f-p_0)} (f - p_0)p \leqq 0. \tag{2}$$

PROOF. *Sufficiency.* For any $p \in \tilde{U}_n(v,u)$ we have $L(p - p_0) \leqq 0$ on $Z(u - Lp_0)$ and $L(p - p_0) \geqq 0$ on $Z(v - Lp_0)$, and thus by (2)

$$\min_{x \in E(f-p_0)} (f - p_0)(p - p_0) = (f - p_0)(p - p_0)(\tilde{x}) \leqq 0$$

for some $\tilde{x} \in E(f - p_0)$. Now

$$\|f - p_0\| = (f - p_0)\operatorname{sgn}(f - p_0)(\tilde{x}) =$$
$$= (f - p)\operatorname{sgn}(f - p_0)(\tilde{x}) + (p - p_0)\operatorname{sgn}(f - p_0)(\tilde{x}) \leqq \|f - p\|.$$

Necessity. Suppose p_0 is a best approximant to f from $\tilde{U}_n(v,u)$, and assume that for $p \in U_n$ $Lp < 0$ on $Z(u - Lp_0)$, $Lp > 0$ on $Z(v - Lp_0)$ but (2) fails. Then $\operatorname{sgn} p = \operatorname{sgn}(f - p_0)$ on $E(f - p_0)$ and for $t > 0$ sufficiently small $p_0 + tp \in \tilde{U}_n(v,u)$ and

$$\|f - (p_0 + tp)\| = \|(f - p_0) - tp\| < \|f - p_0\|,$$

a contradiction. Thus (2) should hold if $Lp < 0$ on $Z(u - Lp_0)$, $Lp > 0$ on $Z(v - Lp_0)$. Let now $p \in U_n$ be such that $Lp \leqq 0$ on $Z(u - Lp_0)$, $Lp \geqq 0$ on $Z(v - Lp_0)$ and choose $\tilde{p} \in U_n$ satisfying $v < L\tilde{p} < u$ on K. (This choice is possible because Int $\tilde{U}_n(v, u) \neq \emptyset$.) Then for every $t > 0$ $L(p + t(\tilde{p} - p_0)) < 0$ on $Z(u - Lp_0)$ and $L(p + t(\tilde{p} - p_0)) > 0$ on $Z(v - Lp_0)$ and applying (2) to $p + t(\tilde{p} - p_0)$ and letting $t \to 0^+$ we obtain the needed statement. □

THEOREM 1.2. *Suppose that* Int $\tilde{U}_n(v, u) \neq \emptyset$. *Then* $p_0 \in \tilde{U}_n(v, u)$ *is a best approximant to* $f \in C[a, b]$ *from* $\tilde{U}_n(v, u)$ *if and only if there exist points* $x_1, \ldots, x_s \in E(f - p_0)$ $(s \geqq 1)$, $y_1, \ldots, y_m \in Z(u - Lp_0)$, $y_{m+1}, \ldots, y_r \in Z(v - Lp_0)$ *with* $s + r \leqq n + 1$, *and constants* $c_1, \ldots, c_s$, $d_1, \ldots, d_r$, *where* $\operatorname{sgn} c_i = \operatorname{sgn}(f - p_0)(x_i)$ $(1 \leqq i \leqq s)$, $d_i < 0$ $(1 \leqq i \leqq m)$ *and* $d_i > 0$ $(m + 1 \leqq i \leqq r)$ *such that for every* $p \in U_n$

$$\sum_{i=1}^{s} c_i p(x_i) + \sum_{i=1}^{r} d_i (Lp)(y_i) = 0. \tag{3}$$

PROOF. *Sufficiency.* We may assume that $\sum_{i=1}^{s} |c_i| = 1$. Let $p \in \tilde{U}_n(v, u)$, i.e., $v \leqq Lp \leqq u$. Then by (3)

$$\|f - p_0\| = \sum_{i=1}^{s} c_i (f - p_0)(x_i) = \sum_{i=1}^{s} c_i f(x_i) + \sum_{i=1}^{r} d_i (Lp_0)(y_i) =$$

$$= \sum_{i=1}^{s} c_i f(x_i) + \sum_{i=1}^{m} d_i u(y_i) + \sum_{i=m+1}^{r} d_i v(y_i) \leqq$$

$$\leqq \sum_{i=1}^{s} c_i f(x_i) + \sum_{i=1}^{r} d_i (Lp)(y_i) = \sum_{i=1}^{s} c_i (f - p)(x_i) \leqq \|f - p\|.$$

Necessity. Assume that p_0 is a best approximant of f from $\tilde{U}_n(v, u)$. Consider a basis $\{p_1, \ldots, p_n\}$ in U_n and let $P \subset R^n$ be given by

$$P = \{((f - p_0)p_i(x))_{i=1}^n : x \in E(f - p_0)\} \cup$$

$$\cup \{(-(Lp_i)(x))_{i=1}^n : x \in Z(u - Lp_0)\} \cup \{((Lp_i)(x))_{i=1}^n : x \in Z(v - Lp_0)\}.$$

Denote by Q the convex hull of P. If $\bar{0} \notin Q$ then using that Q is closed there exists a $\bar{t} = (t_i)_{i=1}^n \in R^n$ such that $\langle \bar{t}, \bar{h} \rangle > 0$ for every $\bar{h} \in Q$. Then for $p^* = \sum_{i=1}^{n} t_i p_i$ we have $Lp^* < 0$ on $Z(u - Lp_0)$, $Lp^* > 0$ on $Z(v - Lp_0)$ and $(f - p_0)p^* > 0$ on $E(f - p_0)$, contradicting Theorem 1.1. Thus $\bar{0} \in Q$ and Caratheodory's Theorem yields the existence of proper x_i-s, y_i-s, c_i-s and d_i-s for which (3) holds, except possibly for condition $s \geqq 1$. But if $s = 0$

then (3) fails to hold for $p = \tilde{p} - p_0$, where $v < L\tilde{p} < u$ (Int $\tilde{U}_n(v,u) \neq \emptyset$). Thus $s \geqq 1$. □

Now we shall give an extension of the notion of extremal sets given in the introduction (see (1)), for the case of constrained approximation.

DEFINITION 1. Let $U_n \subset C[a,b]$ and $L\colon U_n \to C(K)$ be as above. Then the set of points $(\{x_i\}_{i=1}^s, \{y_i\}_{i=1}^r)$, where $x_i \in [a,b]$, $1 \leqq i \leqq s$; $y_i \in K$, $1 \leqq i \leqq r$; $s \geqq 1$, $r \geqq 0$ and $r + s \leqq n + 1$, is called an *L-extremal set* for U_n if there exist nonzero constants $\{c_i\}_{i=1}^s$, $\{d_i\}_{i=1}^r$ such that

$$(4) \qquad \sum_{i=1}^{s} c_i p(x_i) + \sum_{i=1}^{r} d_i (Lp)(y_i) = 0, \qquad p \in U_n.$$

Moreover, we call an L-extremal set *nondegenerate* if $\dim L(U_n)\mid_{\{y_i\}_{i=1}^r} = r$.

REMARK. Nondegeneracy of the L-extremal set essentially means that (4) can not hold for a proper subset of the extremal set which does not contain x_i-s. This, in turn, is equivalent to saying that linear functionals $\delta_{y_i} L$, $1 \leqq i \leqq r$, are linearly independent on U_n (δ_{y_i} denotes the point evaluation functional related to y_i). Furthermore, if $(\{x_i\}_{i=1}^s, \{y_i\}_{i=1}^r)$ is a degenerate L-extremal set, then for some $\ell_1, \ldots \ell_r$ (not all of them zero) we have on U_n: $\sum_{i=1}^{r} \ell_i \delta_{y_i} L = 0$. Thus by (4) for every $t \in R$

$$(5) \qquad \sum_{i=1}^{s} c_i \delta_{x_i} + \sum_{i=1}^{r} (d_i - t\ell_i) \delta_{y_i} L = 0$$

on U_n. Choosing $t = d_j/\ell_j$ ($\ell_j \neq 0$) we drop out at least one term in (5). Thus repeating this process we shall obtain a nondegenerate L-extremal subset of the original L-extremal set.

DEFINITION 2. We say that $p \in U_n$ L-vanishes on the L-extremal set $(\{x_i\}_{i=1}^s, \{y_i\}_{i=1}^r)$ for U_n if $p(x_i) = 0$ $(1 \leqq i \leqq s)$ and $(Lp)(y_i) = 0$ $(1 \leqq i \leqq r)$.

As we have seen in the Introduction the Haar property, which is necessary and sufficient for uniqueness of unconstrained Chebyshev approximation, is equivalent to the requirement that no element of the subspace vanishes on an extremal set of this subspace. Now we give an analogous description of uniqueness of constrained approximation, thus providing a complete solution to Problem A.

THEOREM 1.3. *Let $U_n \subset C[a,b]$; $L\colon U_n \to C(K)$ be a linear operator. Then in order that for every $f \in C[a,b]$ and $v, u \in C(K)$ with* Int $\tilde{U}_n(v,u) \neq \emptyset$ *the best approximant of f in $\tilde{U}_n(v,u)$ be unique it is necessary and sufficient that no $p \in U_n \setminus \{0\}$ L-vanishes on a nondegenerate L-extremal set for U_n.*

PROOF. *Sufficiency.* Assume that for some $v, u \in C(K)$ with Int $\tilde{U}_n(v,u) \neq \emptyset$ and $f \in C[a,b]$ there are two distinct best approximants

$p_1, p_2 \in \tilde{U}_n(v,u)$ for f. Then $(p_1+p_2)/2$ is also a best approximant, and setting for a $g \in C[a,b]$, $E_+(g) = \{x \in [a,b] : g(x) = \|g\|\}$, $E_-(g) = \{x \in \in [a,b] : g(x) = -\|g\|\}$ we have

$$E_+\Big(f - \frac{p_1+p_2}{2}\Big) \subseteqq E_+(f-p_1) \cap E_+(f-p_2) \subseteqq Z(p_1-p_2),$$

$$E_-\Big(f - \frac{p_1+p_2}{2}\Big) \subseteqq E_-(f-p_1) \cap E_-(f-p_2) \subseteqq Z(p_1-p_2),$$

$$Z\Big(u - L\Big(\frac{p_1+p_2}{2}\Big)\Big) \subseteqq Z(u-Lp_1) \cap Z(u-Lp_2) \subseteqq Z(L(p_1-p_2)),$$

$$Z\Big(v - L\Big(\frac{p_1+p_2}{2}\Big)\Big) \subseteqq Z(v-Lp_1) \cap Z(v-Lp_2) \subseteqq Z(L(p_1-p_2)).$$

By Theorem 1.2 applied to $p_0 = (p_1+p_2)/2$, $E(f-p_0)$ and $Z(u-Lp_0) \cup \cup Z(v-Lp_0)$ contain an L-extremal set for U_n, and $p_1 - p_2 \in U_n \setminus \{0\}$ L-vanishes on this L-extremal set. By the remark made after Definition 1 an L-extremal set contains a nondegenerate L-extremal set completing the proof of sufficiency in Theorem 1.3.

Necessity. Assume that some $p^* \in U_n \setminus \{0\}$ L-vanishes on a nondegenerate L-extremal set $(\{x_i\}_{i=1}^r, \{y_i\}_{i=1}^s)$ satisfying (4), that is $p^*(x_i) = 0$, $1 \leqq \leqq i \leqq s$, $(Lp^*)(y_i) = 0$, $1 \leqq i \leqq r$. We may assume that $\|p^*\| = 1$. Evidently, we can construct $f \in C[a,b]$ so that $f(x_i) = \operatorname{sgn} c_i$ $(1 \leqq i \leqq s)$ and $|f| \leqq \leqq 1 - |p^*|$ on $[a,b]$. Then $\|f - tp^*\| = 1$ and $E(f-tp^*) = E(f) \supseteqq \{x_1, \dots, x_s\}$ for all $|t| \leqq 1$.

Assume that $d_i < 0$ $(1 \leqq i \leqq m)$ and $d_i > 0$ $(m+1 \leqq i \leqq r)$. The nondegeneracy of the L-extremal set implies that $(L\tilde{p})(y_i) = -1$ $(1 \leqq i \leqq m)$, $(L\tilde{p})(y_i) = 1$ $(m+1 \leqq i \leqq r)$ for some $\tilde{p} \in U_n$. Set $v = \min(-|Lp^*|, L\tilde{p} - 1)$, $u = \max(|Lp^*|, L\tilde{p}+1)$, $v, u \in C(K)$. Since $v < L\tilde{p} < u$, Int $\tilde{U}_n(v,u) \neq \emptyset$. Set $p_t = tp^* \in U_n$ $(|t| \leqq 1)$. Then $u(y_i) = 0 = (Lp_t)(y_i)$ $(1 \leqq i \leqq m)$; $v(y_i) = 0 = (Lp_t)(y_i)$ $(m+1 \leqq i \leqq r)$ and, evidently, $p_t \in \tilde{U}_n(v,u)$. Moreover, by Theorem 1.2 p_t is best approximant for f for $|t| \leqq 1$. □

Since the condition of Theorem 1.3 characterizes C-boundary independent uniqueness of constrained approximation related to L we introduce the following natural notion.

DEFINITION 3. Let $U_n \subset C[a,b]$, $L\colon U_n \to C(K)$. Then U_n is called *L-Haar* if no $p \in U_n \setminus \{0\}$ L-vanishes on a nondegenerate L-extremal set for U_n.

It turns out that the L-Haar property can be characterized without involving the notion of L-extremal sets. In fact, it can be reduced to the study of Haar property. Let us mention that L-Haar spaces, are in particular Haar spaces, since any extremal set for U_n is also a nondegenerate L-extremal set. For $A \subseteqq K$ denote $G_A = \{p \in U_n : Lp = 0 \text{ on } A\}$.

THEOREM 1.4. *The following statements are equivalent:*

a) U_n *is an L-Haar space;*

b) *every L-extremal set for* U_n *contains* $n+1$ *points;*

c) G_A *is a Haar space for every* $A \subseteqq K$*;*

d) G_{S_k} *is a Haar space for every* $S_k = \{y_1, \dots, y_k\} \subseteqq K$ $(0 \leqq k \leqq n)$ *such that* $\dim LU_n|_{S_k} = k$.

PROOF. a)⇒b). Assume that there is an L-extremal set $(\{x_i\}_{i=1}^s, \{y_i\}_{i=1}^r)$ for U_n with $s+r \leqq n$. Then the matrix of linear system $p(x_i) = 0$, $1 \leqq i \leqq s$; $(Lp)(y_i) = 0$ $(1 \leqq i \leqq r,\ p \in U_n)$ has rank less than $r+s \leqq n$, i.e., the system has a nontrivial solution, contradicting the L-Haar property.

b)⇒c). Assume that c) fails, that is G_A is not Haar for some $A \subseteqq K$. Let $\dim G_A = k$, where $1 \leqq k < n$. (If $k = n$ then $G_A = U_n$ is not Haar, yielding that U_n possesses an extremal, and thus L-extremal, set of fewer than $n+1$ points.) Let V be a complementary subspace of G_A in U_n. Then $\dim V = \dim L(U_n)|_A = n-k$. Let $V = \operatorname{span}[g_1, \dots, g_{n-k}]$ and choose $y_1, \dots, y_{n-k} \in A$ so that $\det[(Lg_i)(y_j)]_{i,j=1}^{n-k} \neq 0$. Since G_A is not Haar there exists an extremal set $\{x_i\}_{i=1}^s$ for G_A with $s \leqq k$, i.e., for some $c_1, \dots, c_s \neq 0$

$$\sum_{i=1}^{s} c_i p(x_i) = 0, \qquad p \in G_A. \tag{6}$$

We can find $d_1, \dots, d_{n-k}$ so that

$$\sum_{i=1}^{n-k} d_i (Lg_j)(y_i) = -\sum_{i=1}^{s} c_i g_j(x_i), \quad 1 \leqq j \leqq n-k. \tag{7}$$

Obviously, by (6) and (7) $(\{x_i\}_{i=1}^s, \{y_i\}_{i=1}^{n-k})$ is an L-extremal set for U_n, where $s+n-k \leqq n$.

c)⇒d) is trivial.

d)⇒a). Assume that a) fails. Then some $p \in U_n \setminus \{0\}$ L-vanishes on a nondegenerate L-extremal set $(\{x_i\}_{i=1}^s, \{y_i\}_{i=1}^r)$ satisfying (4), where $\dim L(U_n)|_{\{y_i\}_{i=1}^r} = r$. Set $S_r = \{y_i\}_{i=1}^r$. Then $p \in G_{S_r}$, and by (4) for every $g \in G_{S_r}$, $\sum_{i=1}^{s} c_i g(x_i) = 0$. Evidently, $1 \leqq \dim G_{S_r} = n-r$ and $s \leqq (n-r)+1$. Hence $\{x_i\}_{i=1}^s$ is an extremal set for G_{S_r}, while $p \in G_{S_r} \setminus \{0\}$ vanishes on x_i, $1 \leqq i \leqq s$. Thus G_{S_r} is not a Haar space. □

Statements c), d) of Theorem 1.4 show that the study of uniqueness of constrained approximation for C-boundaries can be reduced to investigating the Haar properties of certain subspaces. This observation leads to an interesting result concerning constraints given by linear functionals.

Let $U_n \subset C[a,b]$, $\varrho_1, \dots, \varrho_r \in U_n^*$, and $\bar{a} = \{a_i\}_{i=1}^r, \bar{b} = \{b_i\}_{i=1}^r \in R^r$ be such that $\bar{a} < \bar{b}$ (i.e. $a_i < b_i$, $1 \leqq i \leqq r$). Set $\tilde{U}_n(\bar{a}, \bar{b}) = \{p \in U_n :$

$a_i \leqq \varrho_i(p) \leqq b_i,\ 1 \leqq i \leqq r\}$. Evidently, $\tilde{U}_n(\bar{a},\bar{b}) = \tilde{U}_n(v,u)$, where $\tilde{U}_n(v,u)$ is defined as above by $L\colon U_n \to C(K)$ with $K = \{1,2,\dots,r\}$, $v(i) = a_i$, $u(i) = b_i$, $(Lp)(i) = \varrho_i(p)$ $(1 \leqq i \leqq r)$. Furthermore, for $A = \{s_1,\dots,s_m\} \subseteqq \{1,2,\dots,r\} = K$, $G_A = \{p \in U_n : Lp = 0 \text{ on } A\} = \bigcap_{1\leqq j\leqq m} \operatorname{Ker} \varrho_{s_j}$ $(G_\emptyset = U_n)$. Thus Theorem 1.4 c) implies the following.

COROLLARY 1.5. *Let $U_n \subset C[a,b]$, $\varrho_i \in U_n^*$ $(1 \leqq i \leqq r)$. Then in order that for every $f \in C[a,b]$ and $\bar{a},\bar{b} \in R^r$ with* Int $\tilde{U}_n(\bar{a},\bar{b}) \neq \emptyset$ *the best approximant of f from $\tilde{U}_n(\bar{a},\bar{b})$ be unique it is necessary and sufficient that U_n and $\bigcap_{1\leqq j\leqq m}$* Ker ϱ_{s_j} *be Haar spaces for every $\{s_1,\dots,s_m\} \subseteqq \{1,2,\dots,r\}$.*

In the special case of approximation with coefficient constraints when $U_n = \operatorname{span}[p_1,\dots,p_n]$ and for $p = \sum_{i=1}^{n} d_i p_i \in U_n$, $\varrho_i(p) = d_i$ $(i \in J \subset \subset \{1,2,\dots,n\})$ the above statement is due to Pinkus and Strauss [8].

Now we turn our attention to Problem B raised in the introduction. To this end we assume that L is a linear operator mapping $U_n \subset C[a,b]$ into $C'(K)$.

DEFINITION 4. We say that $p \in U_n$ *L'-vanishes* on an L-extremal set $(\{x_i\}_{i=1}^{s}, \{y_i\}_{i=1}^{r})$ for U_n if $p(x_i) = 0$ $(1 \leqq i \leqq s)$, $(Lp)(y_i) = 0$ $(1 \leqq i \leqq r)$ and $(Lp)'(y_i) = 0$ whenever $y_i \in \operatorname{Int} K$.

Our next result gives an answer to the Problem B.

THEOREM 1.6. *In order that for every $f \in C[a,b]$ and $v,u \in C'(K)$ with* Int $\tilde{U}_n(v,u) \neq \emptyset$ *the best approximant of f in $\tilde{U}_n(v,u)$ be unique it is necessary and sufficient that no $p \in U_n \setminus \{0\}$ L'-vanishes on a nondegenerate L-extremal set for U_n.*

PROOF. *Sufficiency* follows by the same argument used in proof of Theorem 1.3. However, we also have to observe here, that if p_1, p_2 are best approximants of f from $\tilde{U}_n(v,u)$ then

$$Z\Big(u - L\Big(\frac{p_1+p_2}{2}\Big)\Big) \cap \operatorname{Int} K \subset Z(L'(p_1-p_2)),\quad Z\Big(v - L\Big(\frac{p_1+p_2}{2}\Big)\Big) \cap \operatorname{Int} K \subset \subset Z(L'(p_1-p_2)).$$

Necessity. Again we follow the lines of the proof of Theorem 1.3. In particular, we consider the same $f \in C[a,b]$ and $p^*, \tilde{p} \in U_n$, and construct suitable $v,u \in C'(K)$.

Since $(L\tilde{p})(y_i) = -1$ $(1 \leqq i \leqq m)$ we can choose closed disjoint intervals $[\alpha_i,\beta_i] \subset K$, $1 \leqq i \leqq m$, such that $y_i \in [\alpha_i,\beta_i]$; $L\tilde{p} < 0$ on $[\alpha_i,\beta_i]$; $\alpha_i < \beta_i$ if $y_i \in \overline{\operatorname{Int} K}$ and $y_i \in (\alpha_i,\beta_i)$ if $y_i \in \operatorname{Int} K$ $(1 \leqq i \leqq m)$. Define u on $[\alpha_i,\beta_i]$ by

$$u(y) = (\operatorname{sgn}(y-y_i)) \int_{y_i}^{y} [|(Lp^*)'|(t) + (t-y_i)^2]\,dt, \quad 1 \leqq i \leqq m.$$

Evidently, $u \in C'(\bigcup_{i=1}^{m}[\alpha_i,\beta_i])$, $Lp^* \leqq u$ on $\bigcup_{i=1}^{m}[\alpha_i,\beta_i]$ and $(Lp^*)(y_i) = u(y_i) = 0$ $(1 \leqq i \leqq m)$. Furthermore, $0 < u(\alpha_i)$ unless $\alpha_i = y_i$ and $u(\beta_i) > 0$ unless $\beta_i = y_i$.

Now we can extend u to K so that $u \in C'(K)$ and $u > 0$ on $K \setminus \setminus(\bigcup_{i=1}^{m}[\alpha_i,\beta_i])$. Since $u > 0$ for $y \in K \setminus \{y_i\}_{i=1}^{m}$ and $Lp^* \leqq u$ in a neighborhood of $\{y_i\}_{i=1}^{m}$ (relative to K), it follows that $L(\delta p^*) \leqq u$ on K for $\delta > 0$ small enough. Similarly, $L(\delta\tilde{p}) < u$ for δ small enough. We can repeat this construction for v, yielding a pair of functions $v, u \in C'(K)$ with $\operatorname{Int} \tilde{U}_n(v,u) \neq \emptyset$ satisfying $v \leqq L(\delta p^*) \leqq u$ $(0 < \delta \leqq \delta_0)$ and such that $L(\delta p^*)(y_i) = u(y_i) = 0$, $1 \leqq i \leqq m$; $L(\delta p^*)(y_i) = v(y_i) = 0$, $m+1 \leqq i \leqq r$. Hence δp^* $(0 < \delta \leqq \delta_0)$ is a best approximant of f in $\tilde{U}_n(v,u)$. □

In view of Theorem 1.6 it is natural to introduce the following

Definition 5. Let $U_n \subset C[a,b]$; $L\colon U_n \to C'(K)$. Then U_n is called *L'-Haar* if no $p \in U_n \setminus \{0\}$ can L'-vanish on a nondegenerate L-extremal set for U_n.

Theorem 1.4 provides some useful criteria for a subspace to be L-Haar. Unfortunately, there do not appear to be corresponding criteria for L'-Haar spaces; however, we give a useful necessary condition for a subspace to be L'-Haar. For $A \subseteqq K$, define $G'_A = \{p \in U_n : Lp = 0 \text{ on } A \text{ and } (Lp)' = 0 \text{ on } A \cap \operatorname{Int} K\}$. Note that $G'_A \subseteqq G_A$.

Corollary 1.7. *If, for some $A \subseteqq K$, G_A is not a Haar space and $G'_A = G_A$, then U_n is not an L'-Haar space.*

Proof. The proof carries on as in the proof of b)⇒c) in Theorem 1.4. We choose $\{y_i\}_{i=1}^{n-k} \subset A$ as in b)⇒c), and since G_A is not a Haar space we choose an extremal set $\{x_i\}_{i=1}^{s}$ for G_A on which some $p \in G_A \setminus \{0\}$ vanishes. As in b)⇒c), $(\{x_i\}_{i=1}^{s}, \{y_i\}_{i=1}^{r})$ is an L-extremal set for U_n, and since $p \in G_A = G'_A$, p L'-vanishes on this L-extremal set. Hence, U_n is not L'-Haar.

Remark. If $A \subseteqq BdyK$, then $G'_A = G_A$. It follows immediately from Corollary 1.7 that if U_n is L'-Haar, then G_A is Haar for all $A \subseteqq BdyK$. In particular, we see that if U_n is L'-Haar, then $U_n = G_\emptyset$ is Haar.

The results of this section lead to the conclusion that L-Haar and L'-Haar properties are necessary and sufficient for uniqueness of constrained approximation with C- and C'-boundaries, respectively. Let us note that the development above does not require that the underlying topological space on which approximation is conducted, be an interval $[a,b]$. We can replace it by any Hausdorff compact set, however L-Haar and L'-Haar spaces necessarily satisfy the Haar property, yielding (Mairhuber [7]) that the compact set should be homeomorphic to the circle or a subset of it.

On the other hand replacing K by a circle leads to a slight difference in definition of periodic L'-Haar spaces, because K has no boundary in this case and thus in Definition 4 we have to require that $(Lp)'(y_i) = 0$ for every $1 \leqq i \leqq r$. The rest of notations and results given above extend to the case when $[a,b]$ is replaced by any compact set or K is replaced by a circle.

2. Strong uniqueness of best constrained approximation

In this section we develop the theory of strong uniqueness for L-Haar and L'-Haar spaces. Let us recall the corresponding definition. If $f \in C[a,b]$, $K \subset C[a,b]$ and p_0 is the unique best approximant of f from K, then we say that p_0 is strongly unique of order γ $(0 < \gamma \leqq 1)$ if there exists a positive constant c depending only on f and K so that for every $p \in K$ satisfying $\|f - p\| \leqq \|f - p_0\| + 1$ we have

$$\|p_0 - p\| \leqq c(\|f-p\| - \|f - p_0\|)^{\gamma}. \tag{8}$$

In case when $\gamma = 1$ we simply say that p_0 is strongly unique. Since L-Haar and L'-Haar properties are characteristic for uniqueness with C- and C'-boundaries it is natural to raise the question of strong uniqueness for L- and L'-Haar spaces. Our first result here asserts that L-Haar property is sufficient for strong uniqueness for constrained approximation.

THEOREM 2.1. *Let U_n be an L-Haar space. Then for every $u, v \in C[a,b]$ with* Int $\tilde{U}_n(v,u) \neq \emptyset$ *and $f \in C[a,b]$, the best approximant to f from $\tilde{U}_n(v,u)$ is strongly unique.*

PROOF. Assume that U_n is L-Haar and let $p_0 \in \tilde{U}_n(v,u)$ be the best approximant of $f \in C[a,b]$, where $v, u \in C[a,b]$ and Int $\tilde{U}_n(v,u) \neq \emptyset$. Consider the L-extremal set $(\{x_i\}_{i=1}^s, \{y_i\}_{i=1}^r)$ and numbers $\{c_i\}_{i=1}^s, \{d_i\}_{i=1}^r$ as in Theorem 1.2 for which (3) holds $\left(\sum_{i=1}^s |c_i| = 1\right)$. Since U_n is L-Haar

$$N(p) = \max_{1 \leqq i \leqq s} |p(x_i)| + \max_{1 \leqq i \leqq r} |(Lp)(y_i)| \qquad (p \in U_n)$$

is a norm on U_n.

Let $p_1 \in \tilde{U}_n(v,u)$ be such that $\|f - p_1\| = \|f - p_0\| + \varepsilon$ with some $\varepsilon > 0$. Since $(f - p_0)(x_i) = (\operatorname{sgn} c_i)\|f - p_0\|$ $(1 \leqq i \leqq s)$ it follows that

$$(\operatorname{sgn} c_i)(p_0 - p_1)(x_i) \leqq \varepsilon \quad (1 \leqq i \leqq s). \tag{9}$$

Furthermore, $y_1, \dots, y_m \in Z(u - Lp_0)$, $y_{m+1}, \dots, y_r \in Z(v - Lp_0)$ yield

$$L(p_0-p_1)(y_i) \geqq 0 \quad (1 \leqq i \leqq m), \quad L(p_0-p_1)(y_i) \leqq 0 \quad (m+1 \leqq i \leqq r). \tag{10}$$

Thus by (3) and (9) applied to $p^* = p_0 - p_1$

$$0 = \sum_{i=1}^s c_i p^*(x_i) + \sum_{i=1}^r d_i(Lp^*)(y_i) \leqq \varepsilon + \sum_{i=1}^r d_i(Lp^*)(y_i). \tag{11}$$

Moreover, (10) yields that $d_i(Lp^*)(y_i) \leqq 0$ for every $1 \leqq i \leqq r$. Taking also into account (11) we have

$$(12) \qquad |(Lp^*)(y_i)| \leqq M_1\varepsilon \quad (1 \leqq i \leqq r),$$

where $M_1 = \max\{1/|d_i|, 1 \leqq i \leqq r\}$. Using (9) and (11) we obtain for every $1 \leqq j \leqq s$

$$c_j p^*(x_j) = -\sum_{i=1, i\neq j}^{s} c_i p^*(x_i) - \sum_{i=1}^{r} d_i (Lp^*)(y_i) \geqq$$

$$\geqq -\sum_{i=1, i\neq j}^{s} c_i p^*(x_i) \geqq -\varepsilon \sum_{i=1, i\neq j}^{s} |c_i| \geqq -\varepsilon.$$

Combining this with (9) yields

$$(13) \qquad |p^*(x_i)| \leqq M_2\varepsilon \quad (1 \leqq i \leqq s),$$

with $M_2 = \max\{1/|c_i|, 1 \leqq i \leqq s\}$. Hence by (12) and (13) $N(p^*) \leqq \leqq (M_1 + M_2)\varepsilon$, and by equivalence of norms in finite dimensional spaces

$$\|p_0 - p_1\| = \|p^*\| \leqq M_3 N(p^*) \leqq c\varepsilon = c\{\|f - p_1\| - \|f - p_0\|\}. \qquad \square$$

In the special case of coefficient constrained approximation, the above result was proven by Pinkus and Strauss [8]. (Strong uniqueness of constrained approximation was also studied by Chalmers and Taylor [3].) Fletcher and Roulier [4] showed that strong uniqueness fails in case of monotone polynomial approximation, although uniqueness is known to hold in this situation. This turns out to be an example of L'-Haar space for which strong uniqueness fails. Our next result shows that strong uniqueness fails for every L'-Haar space which does not satisfy the L-Haar property; in fact, strong uniqueness of arbitrarily small degree γ fails to hold.

THEOREM 2.2. *Assume that U_n is an L'-Haar space which does not satisfy the L-Haar property, and let $0 < \gamma \leqq 1$ be arbitrary. Then there exist $v, u \in \in C'(K)$ with* Int $\tilde{U}_n(v,u) \neq \emptyset$ *and $f \in C[a,b]$ such that its best approximant from $\tilde{U}_n(v,u)$ is not strongly unique of order γ.*

PROOF. Let $p^* \in U_n \setminus \{0\}$ be such that it L-vanishes on a nondegenerate L-extremal set $(\{x_i\}_{i=1}^r, \{y_i\}_{i=1}^s)$ satisfying (4) and let f be chosen as in the proof of necessity in Theorem 1.3. We may assume that $|(Lp^*)| \leqq 1$ on Int K, and $d_i < 0$ $(1 \leqq i \leqq m)$, $d_i > 0$ $(m+1 \leqq i \leqq r)$ in (4). For a given $\alpha > 0$ we can construct $v, u \in C'(K)$ such that $u(y) = |y - y_i|^{1+\alpha}$ in a neighborhood of y_i if $1 \leqq i \leqq m$, $v(y) = -|y - y_i|^{1+\alpha}$ in a neighborhood of y_i if $m+1 \leqq i \leqq r$, and $u > 0$, $v < 0$ for $y \neq y_i$, $1 \leqq i \leqq r$ (neighborhoods are relative to K). As usual, nondegeneracy of the L-extremal set implies existence of $\tilde{p} \in U_n$ such that $(L\tilde{p})(y_i) = -1$ $(1 \leqq i \leqq m)$, $(L\tilde{p})(y_i) = 1$ $(m+1 \leqq i \leqq r)$. Choose $A > 2\alpha(1+\alpha)^{-\frac{1+\alpha}{\alpha}}$. We claim that for $\varepsilon > 0$ small enough $\varepsilon p^* + A\varepsilon^{\frac{\alpha+1}{\alpha}}\tilde{p} \in \tilde{U}_n(v,u)$. Assume that, on the contrary, there exist $\varepsilon_k \downarrow 0$ and $t_k \in K$ such that, say

$$\varepsilon_k(Lp^*)(t_k) + A\varepsilon_k^{\frac{\alpha+1}{\alpha}}(L\tilde{p})(t_k) > u(t_k) \qquad (k = 1, 2, \dots). \tag{14}$$

Without loss of generality, $t_k \to y_j (k \to \infty)$ for some $1 \leqq j \leqq m$. Then for k large enough $u(t_k) = |y_j - t_k|^{1+\alpha}$, $(L\tilde{p})(t_k) < -1/2$, and, in addition,

$$|(Lp^*)(t_k)| = |(Lp^*)(t_k) - (Lp^*)(y_j)| \leqq |t_k - y_j|.$$

Thus using (14) we have

$$|y_j - t_k|^{1+\alpha} < \varepsilon_k|t_k - y_j| - \frac{A\varepsilon_k^{\frac{\alpha+1}{\alpha}}}{2},$$

i.e.,

$$\frac{A}{2}\varepsilon_k^{\frac{\alpha+1}{\alpha}} < \varepsilon_k|t_k - y_j| - |y_j - t_k|^{1+\alpha} \leqq \max_{h \geqq 0}(\varepsilon_k h - h^{1+\alpha}) = \varepsilon_k^{\frac{\alpha+1}{\alpha}}\alpha(1+\alpha)^{-\frac{\alpha+1}{\alpha}}.$$

But this, obviously, contradicts our choice of A. Thus $p_\varepsilon = \varepsilon p^* + A\varepsilon^{\frac{\alpha+1}{\alpha}}\tilde{p} \in \tilde{U}_n(v,u)$ for $\varepsilon > 0$ small enough. Moreover, Int $\tilde{U}_n(v,u) \neq \emptyset$ and 0 is the unique best approximant of f in $\tilde{U}_n(v,u)$ (by Theorem 1.2 and L'-Haar property of U_n). On the other hand $\|p_\varepsilon\| \geqq \varepsilon\|p^*\| - A\varepsilon^{\frac{\alpha+1}{\alpha}}\|\tilde{p}\| \geqq c_1\varepsilon$ $(0 < \varepsilon \leqq \varepsilon_0)$, while by construction of f $(|f| \leqq 1 - |p^*|)$

$$\|f - p_\varepsilon\| \leqq \|f - \varepsilon p^*\| + A\varepsilon^{\frac{\alpha+1}{\alpha}}\|\tilde{p}\| \leqq \|f\| + c_2\varepsilon^{\frac{\alpha+1}{\alpha}}.$$

Since the choice of $\alpha > 0$ is arbitrary the statement of the theorem follows.

REMARK. It can be easily seen from the proof of Theorem 2.2 that we can choose u and v such that $v', u' \in \text{Lip}\ \alpha$ $(\alpha > 0)$, while the degree of strong uniqueness of the proper $f \in C[a,b]$ can not be larger than $\frac{\alpha}{\alpha+1}$. This indicates that for $v, u \in C^2(K)$ $(\alpha = 1)$ strong uniqueness of degree $\frac{1}{2}$ might hold. Our next theorem provides this result.

THEOREM 2.3. *Let $U_n \subseteqq C[a,b]$ be an L'-Haar space with $L\colon U_n \to C^2(K)$, and let $v, u \in C^2(K)$ be such that* Int $\tilde{U}_n(v,u) \neq \emptyset$. *Then for every $f \in C[a,b]$ its best approximant in $\tilde{U}_n(v,u)$ is strongly unique of degree $\frac{1}{2}$.*

PROOF. Throughout the proof we shall denote by $M_1, M_2, \dots$, positive constants depending only on f and $\tilde{U}_n(v,u)$. Let $p_0 \in \tilde{U}_n(v,u)$ be the best approximant of f and consider the corresponding L-extremal set $(\{x_i\}_{i=1}^s, \{y_i\}_{i=1}^r)$ for which (3) holds. For an arbitrary $p_1 \in \tilde{U}_n(v,u)$ such that $\|f - p_1\| = \|f - p_0\| + \varepsilon$ with some $0 < \varepsilon \leqq 1$ set $p^* = p_0 - p_1$. Then as in the proof of Theorem 2.1

$$(15) \qquad |p^*(x_i)| \leqq M_2\varepsilon \quad (1 \leqq i \leqq s), \quad |(Lp^*)(y_i)| \leqq M_1\varepsilon \quad (1 \leqq i \leqq r)$$

(see (12) and (13)). Let $y_j \in \operatorname{Int} K$, where without loss of generality we may assume $y_j \in Z(u - Lp_0)$. Set $u_1 = Lp_0 - u$. Since $u_1, Lp^* \in C^2(K)$ and $\|p^*\| \leqq 2\|f - p_0\| + 1$ we have on Int K $|u_1''|, |(Lp^*)''| \leqq M_3$. Set $M_4 = \max\{M_1, M_2, M_3\}$, $M_5 = \min\{\operatorname{dist}(y_i, Bdy K) : y_i \in \operatorname{Int} K\}$.

Now we claim that

$$(16) \qquad |(Lp^*)'(y_j)| \leqq M_4\Big(2M_5 + \frac{1}{M_5}\Big)\sqrt{\varepsilon}.$$

Assume to the contrary that

$$(17) \qquad |(Lp^*)'(y_j)| = \xi(Lp^*)'(y_j) > M_4\Big(2M_5 + \frac{1}{M_5}\Big)\sqrt{\varepsilon} \quad (\xi = \pm 1).$$

Using $Lp^* \geqq u_1$ we have by (15)

$$(18) \qquad (Lp^*)(x) - (Lp^*)(y_j) \geqq u_1(x) - M_4\varepsilon \quad (x \in K).$$

Set $x_\varepsilon = y_j - \xi M_5\sqrt{\varepsilon}$, where every point between y_j and x_ε belongs to K. For some $\eta \in \operatorname{Int} K$ between x_ε and y_j we have

$$(19) \qquad (Lp^*)(x_\varepsilon) - (Lp^*)(y_j) = (Lp^*)'(\eta)(x_\varepsilon - y_j) = -\xi M_5\sqrt{\varepsilon}(Lp^*)'(\eta).$$

Furthermore,

$$|(Lp^*)'(\eta) - (Lp^*)'(y_j)| \leqq M_4|\eta - y_j| \leqq M_4M_5\sqrt{\varepsilon}.$$

Hence (17) yields that $\operatorname{sgn}(Lp^*)'(\eta) = \operatorname{sgn}(Lp^*)'(y_j) = \xi$. Therefore (18) and (19) imply

$$|(Lp^*)'(\eta)| = -\frac{1}{M_5\sqrt{\varepsilon}}((Lp^*)(x_\varepsilon) - (Lp^*)(y_j)) \leqq \frac{M_4\varepsilon - u_1(x_\varepsilon)}{M_5\sqrt{\varepsilon}}.$$

On the other hand $u_1(y_j) = u_1'(y_j) = 0$ hence $|u_1(x_\varepsilon)| \leqq M_4|x_\varepsilon - y_j|^2 \leqq M_4M_5^2\varepsilon$. Applying this in the last estimate we obtain

$$|(Lp^*)'(\eta)| \leqq \frac{M_4\varepsilon + M_4M_5^2\varepsilon}{M_5\sqrt{\varepsilon}} = M_4\Big(M_5 + \frac{1}{M_5}\Big)\sqrt{\varepsilon}.$$

Finally, this implies that

$$|(Lp^*)'(y_j)| \leqq |(Lp^*)'(\eta)| + |(Lp^*)'(y_j) - (Lp^*)'(\eta)| \leqq$$
$$\leqq M_4\Big(M_5 + \frac{1}{M_5}\Big)\sqrt{\varepsilon} + M_4|y_j - \eta| \leqq$$
$$\leqq M_4\Big(M_5 + \frac{1}{M_5}\Big)\sqrt{\varepsilon} + M_4M_5\sqrt{\varepsilon} = M_4\Big(2M_5 + \frac{1}{M_5}\Big)\sqrt{\varepsilon},$$

contradicting (17). Hence (16) holds, implying that for every $y_j \in \operatorname{Int} K$

$$(20) \qquad |(Lp^*)'(y_j)| \leqq M_6\sqrt{\varepsilon}.$$

By the L'-Haar property of U_n no element of U_n can L'-vanish on the L-extremal set of U_n, i.e.,

$$\tilde{N}(p) = \max_{1 \leqq i \leqq s} |p(x_i)| + \max_{1 \leqq i \leqq r} |(Lp^*)(y_i)| + \max_{y_i \in \operatorname{Int} K} |(Lp^*)'(y_i)|$$

is a norm on U_n. By (15) and (20) we have $\tilde{N}(p^*) \leqq (M_1 + M_2 + M_6)\sqrt{\varepsilon}$, hence by equivalence of norms in finite-dimensional spaces

$$\|p_0 - p_1\| = \|p^*\| \leqq M_7\tilde{N}(p^*) \leqq M_8\sqrt{\varepsilon} = M_8(\|f - p_1\| - \|f - p_0\|)^{\frac{1}{2}}. \qquad \square$$

Let us conclude this section by noting that in the special case of monotone polynomial approximation Theorem 2.3 is verified in [10].

3. Applications

We give several applications of our theory in Section 1. The first results given in 3.1 primarily follow from Theorem 1.4 and demonstrate that L-Haar spaces are rather scarce. On the other hand, applications of Theorem 1.6 and Corollary 1 in 3.2 and the existing literature show that L'-Haar spaces are not so rare. In the case of restricted derivative approximation (that is, $L = D^k$ where $k \geqq 1$ and $K = [-1, 1]$), Roulier and Taylor [9] proved that the space π_{n-1} of polynomials of degree $n - 1$ or less is L'-Haar (see also [6, p. 127]). In 3.2, we shall completely determine those lacunary polynomial spaces that are L'-Haar in this context. Furthermore, the negative result of 3.1 clarifies the necessity of using smooth boundary functions in the constraints. Further in 3.3, we consider the differential operator $L = D - \alpha I$ where $K = [a, b] = [-1, 1]$ and α is constant. We shall find that π_{n-1} is L'-Haar precisely when $|\alpha| \leqq (n - 1)/2$. Finally, in 3.4, we examine rational function spaces with the operator $L = D$ and $K = [-1, 1]$. We shall see that introducing quadratic denominators these spaces can alter the L'-Haar property.

3.1. Some negative results concerning L-Haar spaces. Throughout this section, $K = [a, b]$. We say that a linear operator $L\colon S \to C[a, b]$ is a k-*Rolle operator* ($k \geqq 0$) if whenever $f \in S$ and f has $k + 1$ distinct zeros $x_1 < \ldots < x_{k+1}$ in $[a, b]$, we have that Lf has a zero in $[x_1, x_{k+1}]$. Evidently, $D^k\colon C^k[a, b] \to C[a, b]$ is a k-Rolle operator. We shall give a wider class of differential operators that are k-Rolle.

THEOREM 3.1. *Let $L\colon U \to C[a,b]$ be a nontrivial k-Rolle ($k \geqq 0$) operator where U is a finite dimensional subspace of $C[a,b]$. If $\dim U \geqq k+2$, then U cannot be L-Haar.*

PROOF. Let $n = \dim U \geqq k+2$. Choose $\tilde{p} \in U$ where $L\tilde{p} \neq 0$ and an open interval $(\alpha,\beta) \subsetneqq [a,b]$ on which $L\tilde{p}$ never vanishes. Now select $n-1$ points $x_1 < \ldots < x_{n-1}$ in (α,β) and find $p \in U \setminus \{0\}$ so that $p(x_i) = 0$ ($1 \leqq i \leqq n-1$). Since $n-1 \geqq k+1$ and L is a k-Rolle operator, $Lp(y) = 0$ for some $y \in [x_1, x_{n-1}] \subsetneqq (\alpha,\beta)$. Since $L\tilde{p}(y) \neq 0$, $\dim G_{\{y\}} = n-1$. But $p \in G_{\{y\}} \setminus \{0\}$ and has $n-1$ zeros. So $G_{\{y\}}$ is not a Haar space and by Theorem 1.4, U is not L-Haar. □

We shall use the next lemma both to demonstrate a family of k-Rolle operators and to establish positive results in 3.3 and 3.4.

LEMMA 3.2. *Let $L\colon C'[a,b] \to C[a,b]$ be given by $L = D + \alpha(x)I$ where $\alpha \in C[a,b]$, and let $a \leqq x < y \leqq b$. If $f \in C'[a,b]$ and $f(x) = f(y) = 0$, then $Lf \equiv 0$ on $[x,y]$ or Lf changes sign in (x,y).*

PROOF. Let $A(t) = \int_a^t \alpha(s)ds$. Then

$$\frac{d}{dt}(e^{A(t)}f(t)) = e^{A(t)}(Lf)(t)$$

so that

$$\int_x^y e^{A(t)}(Lf)(t)dt = e^{A(y)}f(y) - e^{A(x)}f(x) = 0,$$

and the conclusion follows readily. □

It follows that the operator L in Lemma 3.2 is 1-Rolle. If $L\colon C^k[a,b] \to$ $\to C[a,b]$ is given by

$$L = (D + \alpha_k(x)I)\ldots(D + \alpha_1(x)I) \tag{21}$$

where $\alpha_i \in C^{k-1}[a,b]$ ($1 \leqq i \leqq k$), repeated applications of Lemma 3.2 show that L is k-Rolle. Moreover, Ker L has dimension k so that the restriction of L to any space of dimension $k+2$ or greater is nontrivial. We thus have the following

COROLLARY 3.3. *Let $L\colon C^k[a,b] \to C[a,b]$ be given by* (21). *Then there are no L-Haar spaces of dimension $k+2$ or greater.*

REMARK. For restricted range approximation, the operator is the identity operator I. A consequence of Theorem 3.1 is that there are no I-Haar spaces of dimension 2 or greater, since I is 0-Rolle. The situation is even worse as there are no I'-Haar spaces of dimension 2 or greater. Let U_n be a subspace of $C'[a,b]$ of dimension $n \geqq 2$. Then $G_{\{\alpha\}}$ is not Haar since it is nontrivial and all of its elements vanish at a. By Corollary 1.7, U_n is not I'-Haar. We note that for restricted range approximation by polynomials uniqueness of best approximations requires the additional condition that the function being approximated also satisfies the constraint (see [9]). Our observation shows that this condition is essential.

REMARK. In the case of approximation with restrictions on the derivatives of order $0 \leqq k_1 < \ldots < k_l$, we take K to be the union of l disjoint copies of $[a, b]$ and Lp represents $D^{k_i}p$ on the i-th copy of $[a, b]$. The methods above show that there are no L-Haar spaces of dimension k_1 or greater, and when $k_1 = 0$ there are no L'-Haar spaces of dimension 2 or greater.

As was noted at the end of Section 1, our theory has direct analogs in the periodic cases. Specifically, if a and b were identified (or, equivalently, $[a, b]$ were replaced with a circle), we would restrict our attention to $C^*[a, b] = \{f \in C[a, b]\colon f(a) = f(b)\}$. We have the following

THEOREM 3.4. *Let $L\colon U \to C(K)$ be a nontrivial operator where U is a finite dimensional subspace of $C^*[a, b]$. If $\dim U \geqq 2$, then U cannot be L-Haar.*

PROOF. Suppose $\dim U = n \geqq 2$ and U is L-Haar. Choose $y \in [a, b]$ so that $(Lq)(y) \neq 0$ for some $q \in U$. By the periodic analog of Theorem 1.4, U would be an n-dimensional Haar space in $C^*[a, b]$ and $G_{\{y\}}$ would be an $(n-1)$-dimensional Haar space in $C^*[a, b]$. But it is well known that nontrivial periodic Haar spaces can only have odd dimension and we reach a contradiction. □

We next consider two brief examples to show that Corollary 3.3 is sharp.

EXAMPLE 1. We take $L = D^k$ and note that π_k is a $(k+1)$-dimensional D^k-Haar subspace of $C[a, b]$. On π_k, D^k reduces to a linear functional, and by Corollary 1.5 that π_k is D^k-Haar follows from π_k and $\pi_{k-1} = \operatorname{Ker} D^k$ being Haar spaces.

EXAMPLE 2. As an example that does not reduce to a linear functional, take $L = D^k$ and $U_{k+1} = \operatorname{span}\{x, x^2, \ldots, x^{k+1}\}$ as a subspace of $C[a, b]$ where $0 < a < b$ and $\frac{a}{b} > \frac{k}{k+1}$. To check that U_{k+1} is D^k-Haar, Theorem 1.4 (d) and $D^k U_{k+1} = \pi_1$ having dimension 2 imply that we need only check that U_{k+1}, $G_{\{\alpha\}}$, and $G_{\{\alpha,\beta\}}$ are Haar spaces for $\alpha, \beta \in [a, b]$. Clearly, U_{k+1} is Haar.

For $p(x) = \sum_{i=1}^{k+1} c_i x^i$, $D^k p(x) = c_{k+1}(k+1)!x + c_k k!$. For distinct $\alpha, \beta \in [a, b]$,

$$G_{\{\alpha,\beta\}} = \operatorname{span}\{x, \ldots, x^{k-1}\}$$

is Haar. For $\alpha \in [a, b]$, $G_{\{\alpha\}} = \operatorname{span}\{x, \ldots, x^{k-1}, x^{k+1} - (k+1)\alpha x^k\}$. To see that this space is Haar suppose $q \in G_{\{\alpha\}} \setminus \{0\}$ has k distinct zeros in $[a, b]$.

Then $q \notin \operatorname{span}\{x, \dots, x^{k-1}\}$ and we may assume that

$$q(x) = x^{k+1} - (k+1)\alpha x^k + \sum_{i=1}^{k-1} c_i x^i.$$

Letting $Z_1, \dots, Z_k$ be the zeros of q in $[a, b]$ (0 is the other zero), we have that

$$(k+1)\alpha = Z_1 + \dots + Z_k.$$

Since $\alpha, Z_1, \dots, Z_k \in [a, b]$, we have $(k+1)a < kb$ and $\frac{a}{b} < \frac{k}{k+1}$, a contradiction. Thus $G_{\{\alpha\}}$ is Haar. Hence, U_{k+1} is D^k-Haar.

3.2. Lacunary polynomials. Let $P_n = \operatorname{span}\{1 = x^{k_1}, x^{k_2}, \dots, x^{k_n} = x^N\}$ where $0 = k_1 < k_2 < \dots < k_n = N$. We take $K = [a, b] = [-1, 1]$ and $L = D^k$. We assume that $k \leqq N - 1$ so that L is not a linear functional over P_n.

THEOREM 3.5. *P_n is L'-Haar with $L = D^k$ $(1 \leqq k \leqq N-1)$ if and only if $k_{i+1} - k_i$ is odd $(1 \leqq i \leqq n-1)$ and either $x^k \notin P_n$ or $x^k, x^{k+1} \in P_n$.*

Before proving Theorem 3.5, we note that Corollary 1.7 implies that L'-Haar spaces are Haar spaces. The condition that each $k_{i+1} - k_i$ is odd is equivalent to P_n being Haar on $[-1, 1]$ (see [6, p. 132]).

The proof of sufficiency uses Birkhoff interpolation. We refer the reader to Chapter 1 of the text [6] for the appropriate terminology involving interpolation matrices and regularity theorems.

PROOF OF THEOREM 3.5. *Sufficiency.* Assume that each $k_{i+1} - k_i$ is odd and that either $x^k \notin P_n$ or $x^k, x^{k+1} \in P_n$. Let $(\{x_i\}_{i=1}^s, \{y_i\}_{i=1}^r)$ be a nondegenerate D^k-extremal set for P_n where $s \geqq 1$,

$$\sum_{i=1}^{s} c_i p(x_i) + \sum_{i=1}^{r} d_i p^{(k)}(y_i) = 0 \qquad (p \in P_n) \tag{22}$$

with all c_i and d_i nonzero, and $D^k P_n$ has dimension r on $\{y_1, \dots, y_r\}$. Let $r' = \#\{i : 1 \leqq i \leqq r \text{ and } y_i \in (-1, 1)\}$, $r'' = r - r'$, and $l = s + 2r' + r'' + N - n$.

Consider the Birkhoff interpolation problem of finding a polynomial $p \in \in \pi_\ell$ satisfying the $\ell + 1$ conditions

$$\begin{cases} \text{a) } p^{(r)}(0) = 0, & 1 \leqq r \leqq N-1,\ r \neq k_i,\ 2 \leqq i \leqq n-1, \\ \text{b) } p(x_i) = 0, & 1 \leqq i \leqq s, \\ \text{c) } p^{(k)}(y_i) = 0, & 1 \leqq i \leqq r, \\ \text{d) } p^{(k+1)}(y_i) = 0, & 1 \leqq i \leqq r, \quad |y_i| < 1. \end{cases} \tag{23}$$

We first note that conditions (23a) and (23cd) do not overlap. If $y_i = 0$ for some $1 \leqq i \leqq r$, then $x^k \in P_n$. Otherwise, $\dim D^k P_n|_{\{y_1, \dots, y_r\}} \leqq r - 1$

which contradicts the nondegeneracy of the D^k-extremal set at hand. By hypothesis, $x^{k+1} \in P_n$. So (23a) does not impose conditions on $p^{(k)}(0)$, $p^{(k+1)}(0)$.

Let E be the interpolation matrix for (23). (E has $\ell+1$ columns indexed from 0 to ℓ.) Since each $k_{i+1}-k_i$ is odd and the conditions (23a) and (23cd) do not overlap, E has no odd supported sequences of "ones".

We now establish that $\ell \geqq N$ and that E satisfies the Pólya condition (that is, the number of "ones" in columns indexed 0 to j is at least $j+1$ for $0 \leqq j \leqq \ell$). Let E' be the matrix formed by augmenting E with infinitely many zero columns. The number of "ones" in the 0-indexed column of E' is $s \geqq 1$. Let j be the smallest index for which the number of "ones" in columns indexed 0 to j of E' is less than $j+1$. Evidently, $j \geqq 1$, the j-indexed column of E' contains only "zeros", the columns indexed 0 to $(j-1)$ of E' contain j "ones", and the matrix E'' consisting of the columns indexed 0 to $(j-1)$ of E' satisfies the Pólya condition and has no odd supported sequences. It suffices to prove that $j > N$. In this case, all $\ell+1$ "ones" in E' are in columns indexed 0 to $(j-1)$ so that $\ell+1=j$. Thus $\ell = j-1 \geqq N$ and $E=E''$. Now assume that $j \leqq N$. By the Atkinson–Sharma–Ferguson Theorem [6, p. 10], E'' is order regular so that there is a unique polynomial $p \in \pi_{j-1}$ satisfying

$$(24)\qquad \begin{cases} \text{a) } p^{(r)}(0)=0, & 1 \leqq r \leqq j-1,\ r \neq k_i,\ 2 \leqq i \leqq r-1, \\ \text{b) } p(x_i)=c_i, & 1 \leqq i \leqq s, \\ \text{and if } k \leqq j-1, & \\ \text{c) } p^{(k)}(y_i)=0, & 1 \leqq i \leqq r, \\ \text{d) } p^{(k+1)}(y_i)=0, & |y_i|<1,\ 1 \leqq i \leqq r. \end{cases}$$

Since $p^{(r)} \equiv 0$ if $r \geqq j$, p satisfies all of the conditions (23a) and (23c). Since $j-1 \leqq N$, (23a) implies that $p \in P_n$ and (24b) and (23c) contradict (22). Thus $j > N$, hence $\ell \geqq N$ and E satisfies the Pólya condition.

We thus have that E is order regular, and thus if $p \in \pi_\ell$ satisfies (23) then $p=0$. Finally, if $p \in P_n$ and p L'-vanishes on the extremal set $(\{x_i\}_{i=1}^s, \{y_i\}_{i=1}^r)$, then $p \in \pi_\ell$ and satisfies (23) and therefore $p=0$. By Theorem 1.6, P_n is L'-Haar.

Necessity. Assume that P_n is L'-Haar. Then since P_n is Haar, each $k_{i+1}-k_i$ is odd. Suppose that $x^k \in P_n$ and $x^{k+1} \notin P_n$. Consider $G_{\{0\}} = \{p \in P_n : p^{(k)}(0)=0\} = \operatorname{span}\{x^{k_i} : 1 \leqq i \leqq n,\ k_i \neq k\}$. Now there are two consecutive powers in $G_{\{0\}}$ with differences of their exponents being even. Thus $G_{\{0\}}$ is not Haar. Moreover, $G'_{\{0\}} = \{p \in G_{\{0\}} : p^{(k+1)}(0)=0\} = G_{\{0\}}$ since $x^{k+1} \notin P_n$. By Corollary 1.7, P_n is not L'-Haar. □

3.3. The operator $L = D - \alpha I$. In the literature on constrained approximation, constraints involving derivatives have involved the operator D^k. However, other differential operators can certainly come into play. In this section, we consider $U_n = \pi_{n-1}$ $(n \geqq 4)$ and $L = D - \alpha I$ (α real) with $[a,b] = K = [-1,1]$. From 3.1 and 3.2, π_{n-1} is D'-Haar but not I'-Haar. We shall see that D is the dominant operator precisely when $|\alpha| \leqq (n-1)/2$.

THEOREM 3.6. *π_{n-1} is L'-Haar if and only if $|\alpha| \leqq (n-1)/2$.*

PROOF. *Necessity.* Suppose that $|\alpha| > (n-1)/2$. Let $q(x) = \prod_{i=1}^{n-1}(x-x_i) \in \pi_{n-1}$ where $-1 < x_1 < \ldots < x_{n-1} < 1$. We can choose $x_1, \ldots, x_{n-1}$ so that

$$\frac{q'(\operatorname{sgn}\alpha)}{q(\operatorname{sgn}\alpha)} = \sum_{i=1}^{n-1} \frac{1}{\operatorname{sgn}\alpha - x_i} = \alpha \tag{25}$$

and thus $(Lq)(\operatorname{sgn}\alpha) = 0$. Now $G_{\{\operatorname{sgn}\alpha\}} = \{p \in \pi_{n-1} : (Lp)(\operatorname{sgn}\alpha) = 0\}$ has dimension $n-1$ and is not Haar since $q \in G_{\{\operatorname{sgn}\alpha\}}$ has $n-1$ zeros in $[-1,1]$. Since $\{\operatorname{sgn}\alpha\} \subsetneqq \{-1,1\}$, Corollary 1.7 implies that π_{n-1} is not L'-Haar.

Before proving sufficiency, we establish a lemma.

LEMMA 3.7. i) *If $\alpha \geqq -m/2$ and $p \in \pi_m \setminus \{0\}$ has m zeros in $(-1,1]$ with at least one zero in $(-1,1)$, then $(Lp)(-1) \neq 0$.*

ii) *If $\alpha \leqq m/2$ and $p \in \pi_m \setminus \{0\}$ has m zeros in $[-1,1)$ with at least one zero in $(-1,1)$, then $(Lp)(1) \neq 0$.*

iii) *If $|\alpha| \leqq m/2$ and $p \in \pi_m \setminus \{0\}$ has $m-1$ zeros in $(-1,1)$, then $(Lp)(1) \neq 0$ or $(Lp)(-1) \neq 0$.*

PROOF. For i), write $p(x) = c\prod_{i=1}^{m}(x-z_i)$ where $c \neq 0$, each $z_i \in (-1,1]$, and some $z_i \in (-1,1)$. Then

$$\frac{p'(-1)}{p(-1)} = \sum_{i=1}^{m} \frac{1}{-1-z_i} < -\frac{m}{2} \leqq \alpha$$

so $(Lp)(-1) \neq 0$. The proof of ii) is similar.

For iii) suppose $\deg p = m-1$ and write $p(x) = c\prod_{i=1}^{m-1}(x-z_i)$ where $c \neq 0$ and each $z_i \in (-1,1)$. Then

$$\frac{p'(-1)}{p(-1)} = \sum_{i=1}^{m-1} \frac{1}{-1-z_i} < 0 < \sum_{i=1}^{m-1} \frac{1}{1-z_i} = \frac{p'(1)}{p(1)}$$

and the conclusion holds. Suppose now that $\deg p = m$ and write

$$p(x) = c(x-z)\prod_{i=1}^{m-1}(x-z_i)$$

where $c \neq 0$ and each $z_i \in (-1,1)$. If $z \in [-1,1]$, then i) or ii) yields the conclusion. Without loss of generality, $z < -1$. If $p'(-1)/p(-1) = p'(1)/p(1)$, then

$$\frac{p'(-1)}{p(-1)} = \frac{1}{-1-z} + \sum_{i=1}^{m-1} \frac{1}{-1-z_i} = \alpha \tag{26}$$

and

$$\frac{p'(1)}{p(1)} = \frac{1}{1-z} + \sum_{i=1}^{m-1} \frac{1}{1-z_i} = \alpha. \tag{27}$$

Equating the expressions (26) and (27) and using $z < -1$, we see that $z = -\sqrt{1+1/A}$ where

$$A = \sum_{i=1}^{m-1} \frac{1}{1-z_i^2} = \sum_{i=1}^{m-1} \Big(\frac{1+z_i}{1-z_i}\Big) \frac{1}{(1+z_i)^2}.$$

Substituting into (26) and (27) and averaging the resulting expressions yields

$$\alpha = \sqrt{A^2+A} + \sum_{i=1}^{m-1} \frac{z_i}{1-z_i^2} = \sqrt{A^2+A} - A + \sum_{i=1}^{m-1} \frac{1}{1-z_i}.$$

Letting $B = A + \sqrt{A^2+A}$, we see that $B > 2A > \frac{2}{1-z_i^2} > \frac{1}{1+z_i}$ $(1 \leqq i \leqq n-2)$. Thus we have that

$$\alpha - \frac{m}{2} = \sqrt{A^2+A} - A + \sum_{i=1}^{m-1} \frac{1}{1-z_i} - \frac{m}{2} =$$

$$= \frac{A}{A+\sqrt{A^2+1}} - \frac{1}{2} + \sum_{i=1}^{m-1} \Big(\frac{1}{1-z_i} - \frac{1}{2}\Big) =$$

$$= \frac{1}{2}\Big(\frac{A-\sqrt{A^2+A}}{A+\sqrt{A^2+A}} + \sum_{i=1}^{m-1} \frac{1+z_i}{1-z_i}\Big) = \frac{1}{2}\Big(\sum_{i=1}^{m-1} \frac{1+z_i}{1-z_i} - \frac{A}{B^2}\Big) =$$

$$= \frac{1}{2B^2}\Big(\sum_{i=1}^{m-1} B^2 \frac{1+z_i}{1-z_i} - \sum_{i=1}^{m-1} \Big(\frac{1+z_i}{1-z_i}\Big) \frac{1}{(1+z_i)^2}\Big) =$$

$$= \frac{1}{2B^2} \sum_{i=1}^{m-1} \frac{1+z_i}{1-z_i} \Big(B^2 - \frac{1}{(1+z_i)^2}\Big) > 0.$$

This is a contradiction since $|\alpha| \leqq m/2$.

Sufficiency. Assume that $|\alpha| \leqq (n-1)/2$. Let $(\{x_i\}_{i=1}^s, \{y_i\}_{i=1}^r)$ be an L-extremal set for π_{n-1} where $s \geqq 1$ and

$$\sum_{i=1}^{s} c_i p(x_i) + \sum_{i=1}^{r} d_i (Lp)(y_i) = 0 \quad (p \in \pi_{n-1}) \tag{28}$$

with all c_i and d_i nonzero. Let $r' = \#\{i : 1 \leqq i \leqq r \text{ and } |y_i| < 1\}$ and $r'' = r - r'$. (Evidently, $r'' = 0, 1$, or 2.) We may assume that $|y_i| < 1$ for $1 \leqq i \leqq r'$.

We prove that $s+2r'+r'' > n$. Assume, to the contrary, that $s+2r'+r'' \leqq \leqq n$. Consider the interpolation problem of finding $p \in \pi_{n-1}$ so that

$$\begin{cases} \text{a) } p(x_i) = 0, & 1 \leqq i \leqq s, \\ \text{b) } p(y_i) = p'(y_i) = 0, & 1 \leqq i \leqq r', \\ \text{c) } (Lp)(y_i) = 0, & r'+1 \leqq i \leqq r. \end{cases} \tag{29}$$

In case some x_i and y_i $(i = r'+1)$ coincide, we remove them from a) and c) and put them in b). Further, by inserting additional points x_i and removing others, we may assume that $s+2r'+r'' = n$ and that no y_i coincides with an x_i. (This may change s and r', but we can still insist on $|y_i| = 1$ for $r'+1 \leqq i \leqq r$.) We claim that if $s + 2r' + (r - r') = n$, the x_i's are different from the y_i's, and $|y_i| = 1$ for $r'+1 \leqq i \leqq r$, then (29) has only the trivial solution. Then an appropriate choice of evaluations in a nonhomogeneous counterpart of (29) would contradict (28).

There are three cases. If $r - r' = 0$, then (29) is a Hermite problem and our claim is obvious. If $r - r' = 1$, suppose that $p \in \pi_{n-1} \setminus \{0\}$ is a solution of (29) with $y_r = 1$. Then p has $n-1$ zeros in $[-1, 1)$ counting multiplicities up to order 2, and since $n \geqq 4$, at least one of these zeros is in $(-1, 1)$. By Lemma 3.7 ii), $(Lp)(1) \neq 0$, a contradiction. If $r - r' = 2$, suppose that $p \in \pi_{n-1} \setminus \{0\}$ is a solution of (29). By Lemma 3.7 iii), $(Lp)(1) \neq 0$ or $(Lp)(-1) \neq 0$, a contradiction. Thus the claim is established.

We conclude that $s + 2r' + r'' > n$. To complete the proof of sufficiency, suppose that $p \in \pi_{n-1}$ L'-vanishes on the extremal set $(\{x_i\}_{i=1}^s, \{y_i\}_{i=1}^r)$. Let $x_1 < \ldots < x_s$. For $1 \leqq i \leqq s-1$, Lemma 3.2 implies that Lp has a sign change, say ξ, in (x_i, x_{i+1}). If $\xi = y_j$ for some j, then $(Lp)(y_j) = (Lp)'(y_j) = (Lp)''(y_j) = 0$ since Lp changes sign at y_j. In any case, we have that Lp has at least n zeros counting multiplicites since $s+2r'+r'' > n$. Since $Lp \in \pi_{n-1}$, $Lp = 0$ and therefore $p = 0$ (because $s \geqq 1$). Thus no $p \in \pi_{n-1} \setminus \{0\}$ L'-vanishes on an L-extremal set for π_{n-1}, and by Theorem 1.6, π_{n-1} is L'-Haar. □

REMARK. When $n = 2$ or 3, π_{n-1} is L'-Haar if and only if $|\alpha| < (n-1)/2$. In these cases, when $|\alpha| = (n-1)/2$ $G_{\{1\}}$ and $G_{\{-1,1\}}$ fail to be Haar spaces, respectively.

The situation is much simpler in the periodic case. It was noted at the end of Section 1 that if K is a circle then in order that a function L'-vanish on an L-extremal set $(\{x_i\}_{i=1}^{s}, \{y_i\}_{i=1}^{r})$ we require that $(Lp)'(y_i) = 0$ for all $1 \leqq i \leqq r$ in addition to the other conditions. Specifically, let $U_n = \operatorname{span}\{1, \cos x, \sin x, \ldots, \cos kx, \sin kx\}$ be taken as a subspace of $C^*[0, 2\pi] = \{f \in C[0, 2\pi] : f(0) = f(2\pi)\}$ and $L = D - \alpha I$. For an L-extremal set, proving that $s + 2r > n$ follows as in the case for $r'' = 0$ in the proof of Theorem 3.6. The proof that no $p \in U_n \setminus \{0\}$ L'-vanishes on an L-extremal set for U_n follows exactly as in Theorem 3.6. Thus we have

THEOREM 3.8. *Let $L = D - \alpha I$. Then $U_n = \operatorname{span}\{1, \cos x, \sin x, \ldots, \cos kx, \sin kx\}$ is L'-Haar (in $C^*[0, 2\pi]$) for all real α.*

3.4. Rational spaces. With $[a, b] = K = [-1, 1]$ and $L = D$, the space π_{n-1} is D'-Haar (see Theorem 3.6 or [6]). In this section we consder the space $U_n = \frac{1}{\omega}\pi_{n-1}$ where ω is a fixed polynomial and $\omega(x) > 0$ for all $x \in [-1, 1]$. We find that if ω is linear, then $\frac{1}{\omega}\pi_{n-1}$ is D'-Haar. However, even using a quadratic denominator can destroy the D'-Haar property.

THEOREM 3.9. *If $\omega(x) = (x - 1 + \varepsilon)^2 + \varepsilon^2$ where $0 < \varepsilon < 2/(n-1)$, then $\frac{1}{\omega}\pi_{n-1}$ is not D'-Haar.*

PROOF. Choose $-1 < x_1 < \ldots < x_{n-1} < 1$ so that

$$\frac{q'(1)}{q(1)} = \sum_{i=1}^{n-1} \frac{1}{1 - x_i} = \frac{1}{\varepsilon} = \frac{\omega'(1)}{\omega(1)}$$

where $q(x) = \prod_{i=1}^{n-1} (x - x_i)$. Then $(q/\omega)'(1) = 0$ and q/ω has $n - 1$ zeros in $(-1, 1)$. Now $G_{\{1\}} = \left\{p/\omega \in \frac{1}{\omega}\pi_{n-1} : (p/\omega)'(1) = 0\right\}$ has dimension $n - 1$ and is not Haar since $q \in G_{\{1\}}$. Corollary 1.7 then implies that $\frac{1}{\omega}\pi_{n-1}$ is not D'-Haar. □

THEOREM 3.10. *If $\omega \in \pi_1$ and $\omega > 0$ on $[-1, 1]$, then $\frac{1}{\omega}\pi_{n-1}$ is a D'-Haar space ($n \geqq 1$).*

PROOF. We may assume that $\omega(x) = x - \gamma$, where $\gamma < -1$. Furthermore $\frac{1}{\omega}\pi_{n-1}$ is D'-Haar if and only if π_{n-1} is L'-Haar with $L = D - \frac{\omega'}{\omega}I = D - \frac{1}{x-\gamma}I$, thus it suffices to show that π_{n-1} is L'-Haar. Let $n \geqq 3$. Let $(\{x_i\}_{i=1}^{s}, \{y_i\}_{i=1}^{r})$ be an L-extremal set for $\pi_{n-1} (s \geqq 1)$ and set $r' = \#\{i : 1 \leqq i \leqq r$ and $|y_i| < 1\}$. We claim that $s + 2r' + (r - r') > n$. Assume, to the contrary that $s + 2r' + (r - r') \leqq n$, and consider the interpolation problem (29) for $p \in \pi_{n-1}$. We can again assume without loss of generality that $s + 2r' + (r - r') = n$, $y_i \neq x_j$, and $|y_i| = 1$ for $r' + 1 \leqq i \leqq r$. As in Theorem 3.5, we need only show that (29) has only the trivial solution. Suppose then that $p \in \pi_{n-1} \setminus \{0\}$ satisfies (29) and further that $r - r' \geqq 1$.

Case 1. $r - r' = 1$. Then by (27) p has $n-1$ zeros $z_1, \dots, z_{n-1} \in [-1,1]$ counting with multiplicities up to order 2 ($|y_r| = 1, z_i \neq y_r, 1 \leqq i \leqq n-1$) and

$$\frac{p'}{p}(y_r) = \sum_{i=1}^{n-1} \frac{1}{y_r - z_i} = \frac{1}{y_r - \gamma}. \tag{31}$$

If $y_r = -1$, then $0 > \sum_{i=1}^{n-1} \frac{1}{y_r - z_i}$, contradicting (31). If $y_r = 1$, then

$$\frac{1}{2} > \frac{1}{1-\gamma} = \sum_{i=1}^{n-1} \frac{1}{1-z_i} \geqq \frac{n-1}{2},$$

contradicting again (31).

Case 2. $r - r' = 2$. Then by (27) p has $n-2$ zeros $z_1, \dots, z_{n-2} \in (-1,1)$ (counting with multiplicities up to 2) and for $y_r = \pm 1$, $\frac{p'}{p}(y_r) = \frac{1}{y_r - \gamma}$ (note that $p(-1)p(1) \neq 0$). If $p \in \pi_{n-2}$ or p has an additional zero in $(-1,1)$ then we obtain a contradiction as in Case 1. Let p have an extra zero z_0 outside of $[-1,1]$. Then

$$\frac{1}{y_r - z_0} + \sum_{i=1}^{n-2} \frac{1}{y_r - z_i} = \frac{1}{y_r - \gamma} \quad (y_r = \pm 1). \tag{32}$$

If $z_0 > 1$ then for $y_r = -1$ right and left sides of (32) have different signs. If $z_0 < -1$ then setting $y_r = 1$ in (32) we have

$$\frac{1}{2} > \frac{1}{1-\gamma} = \frac{1}{1-z_0} + \sum_{i=1}^{n-2} \frac{1}{1-z_i} > \sum_{i=1}^{n-2} \frac{1}{1-z_i} > \frac{n-2}{2},$$

a contradiction.

Thus $s + 2r' + (r - r') > n$. Suppose now that $p \in \pi_{n-1}$ L'-vanishes on the L-extremal set $(\{x_i\}_{i=1}^s, \{y_i\}_{i=1}^r)$. Applying Lemma 3.2 to $L = D - \frac{1}{x-\gamma} I$ implies that Lp has a sign change ξ_i in (x_i, x_{i+1}) for every $1 \leqq i \leqq s-1$. If $\xi_i = y_j$ for some j then $(Lp)(y_j) = (Lp)'(y_j) = (Lp)''(y_j) = 0$ since Lp changes sign at y_j. Thus counting with multiplicities we obtain at least n zeros of Lp (because $s + 2r' + (r - r') > n$), i.e., $p'(x)(x-\gamma) - p(x) \in \pi_{n-1}$ has at least n zeros. This yields, that $p'(x)(x-\gamma) - p(x) \equiv 0$, i.e., $p \equiv c(x-\gamma)$. But since $s \geqq 1$, p has a zero in $[-1,1]$, a contradiction. This completes the proof for $n \geqq 3$. If $n = 1$ the claim of Theorem follows from the fact that $s \geqq 1$ for any D-extremal set $(\{x_i\}_{i=1}^s, \{y_i\}_{i=1}^r)$, while $p \in \frac{1}{\omega}\pi_0$, $p \equiv \frac{c}{x-\gamma}$ does not vanish. If $n = 2$ and $p \in \frac{1}{\omega}\pi_1$ is not a constant function, then p' does not vanish. Thus $p \in \frac{1}{\omega}\pi_1 \setminus \{0\}$ can not D'-vanish on a D-extremal set if $r \geqq 1$. On the other hand if $r = 0$ then, obviously, $s \geqq 2$ and no $p \in \frac{1}{\omega}\pi_1 \setminus \{0\}$ can have two zeros. □

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MATHEMATICAL INSTITUTE OF THE
HUNGARIAN ACADEMY OF SCIENCES
H–1364 BUDAPEST, P.O.B. 127

DEPARTMENT OF MATHEMATICS
OAKLAND UNIVERSITY
ROCHESTER, MI 48063
USA

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ÜBER DIE BANACH-EIGENSCHAFT VON MATRIZEN

K. TANDORI (Szeged), Mitglied der Akademie

1. Die Klasse von allen orthonormierten Systemen $\varphi = \{\varphi_k(x)\}_0^\infty$ im Intervall $(0,1)$ bezeichnen wir mit Ω. (Die folgenden Betrachtungen bleiben auch für die in einem endlichen und nichtatomischen Maßraum orthonormierten Systeme gültig, nur einfachkeitshalber betrachten wir den Fall $(0,1)$ mit dem gewöhnlichen Lebesgueschen Maß.)

Es sei $B = \|b_{n,k}\|_{n,k=0}^\infty$ eine Matrix mit

$$\sum_{k=0}^\infty b_{n,k}^2 \to 0 \qquad (n \to \infty).$$

Für jedes System $\varphi \in \Omega$ existieren die Summen

$$b_n(\varphi; x) = \sum_{k=0}^\infty b_{n,k}\varphi_k(x) \qquad (n = 0, 1, \dots)$$

in der Metrik von $L^2(0,1)$.

Man sagt, daß die Matrix B die Banach-Eigenschaft ($B \in \mathrm{BE}$) besitzt, wenn für jedes $\varphi \in \Omega$

$$\lim_{n\to\infty} b_n(\varphi; x) = 0$$

in $(0,1)$ fast überall gilt.

Verf. und F. Móricz [1], [3] haben eine notwendige und hinreichende Bedingung dafür angegeben, daß $B \in \mathrm{BE}$ ist. Diese Bedingung lautet folgenderweise.

Satz A. $B \in \mathrm{BE}$ *gilt dann und nur dann, wenn*

$$\|B\| = \sup_{\varphi\in\Omega}\left\{\int_0^1 (\sup_n b_n^2(\varphi; x))dx\right\}^{1/2} < \infty.$$

2. Diese Bedingung ist notwendig und hinreichend; leider ist es in speziällen Fällen schwer zu entscheiden, ob sie für eine Matrix B erfüllt ist.

In dieser Note werden wir hinreichende, aber brauchbarere Bedingungen für die Banach-Eigenschaft angeben.

Für positive ganze Zahlen $M, N (M \leqq N)$ setzen wir

$$\|B; M, N\| = \sup_{\varphi \in \Omega} \left\{ \int_0^1 \left(\sup_{M \leqq n \leqq N} b_n^2(\varphi; x) \right) dx \right\}^{1/2}.$$

Man kann leicht zeigen, daß für beliebige Matrizen B, $\overline{B}$

$$\|B + \overline{B}; M, N\| \leqq \|B; M, N\| + \|\overline{B}; M, N\|$$

gilt. (Hier ist $B + \overline{B} = \|b_{n,k} + \overline{b}_{n,k}\|_{n,k=0}^{\infty}$.)

Erstens beweisen wir den folgenden Satz.

SATZ 1. *Es sei* $\{N_\nu\}_1^\infty$ *eine monoton wachsende Folge von positiven ganzen Zahlen. Gilt*

$$\sum_{\nu=1}^{\infty} \|B; N_\nu, N_{\nu+1} - 1\|^2 < \infty, \tag{1}$$

so ist $B \in \mathrm{BE}$.

BEWEIS. Es sei $B^{(\nu)} = \|b_{n,k}^{(\nu)}\|_{n,k=0}^{\infty}$, wobei

$$b_{n,k}^{(\nu)} = b_{N_\nu,k} \qquad (n, k = 0, 1, \dots),$$

und $B_\nu = B - B^{(\nu)}$ $(\nu = 1, 2, \dots)$. Dann gilt

$$\|B_\nu; N_\nu, N_{\nu+1} - 1\| \leqq \|B; N_\nu, N_{\nu+1} - 1\| + \|B^{(\nu)}; N_\nu, N_{\nu+1} - 1\| = \tag{2}$$

$$= \|B; N_\nu, N_{\nu+1} - 1\| + \left\{ \sum_{k=0}^{\infty} b_{N_\nu,k}^2 \right\}^{1/2} \qquad (\nu = 1, 2, \dots).$$

Es sei $\varphi \in \Omega$. Es ist klar, daß

$$\sum_{k=0}^{\infty} b_{N_\nu,k}^2 = \int_0^1 b_{N_\nu}^2(\varphi; x)\,dx \leqq \|B; N_\nu, N_{\nu+1} - 1\|^2 \quad (\nu = 1, 2, \dots) \tag{3}$$

ist. Aus (1) und (3) erhalten wir

$$\sum_{\nu=1}^{\infty} \int_0^1 b_{N_\nu}^2(\varphi; x)\,dx < \infty,$$

und so folgt, daß die Reihe

$$\sum_{\nu=1}^{\infty} b_{N_\nu}^2(\varphi; x)$$

in $(0,1)$ fast überall konvergiert, woraus sich ergibt, daß fast überall in $(0,1)$

$$\lim_{\nu\to\infty} b_{N_\nu}(\varphi; x) = 0. \tag{4}$$

Es sei

$$\delta_\nu(x) = \max_{N_\nu < n < N_{\nu+1}} |b_n(\varphi; x) - b_{N_\nu}(\varphi; x)| \qquad (\nu = 1, 2, \dots).$$

Offensichtlich gilt

$$\int_0^1 \delta_\nu^2(x)dx \leqq \|B_\nu; N_\nu, N_{\nu+1} - 1\|^2 \qquad (\nu = 1, 2, \dots).$$

Daraus und aus (1), (2), und (3) erhalten wir

$$\sum_{\nu=1}^{\infty} \int_0^1 \delta_\nu^2(x)dx < \infty.$$

So folgt, daß die Reihe

$$\sum_{\nu=1}^{\infty} \delta_\nu^2(x)$$

in $(0,1)$ fast überall konvergiert, und daher

$$\lim_{\nu\to\infty} \delta_\nu(x) = 0$$

in $(0,1)$ fast überall besteht. Daraus und aus (4) ergibt sich $B \in \mathrm{BE}$.

Da

$$\|B; N_\nu, N_{\nu+1} - 1\| \leqq \|B_\nu; N_\nu, N_{\nu+1} - 1\| + \|B^{(\nu)}; N_\nu, N_{\nu+1} - 1\| =$$

$$= \|B_\nu; N_\nu, N_{\nu+1} - 1\| + \left\{\sum_{k=0}^{\infty} b_{N_\nu,k}^2\right\}^{1/2} \qquad (\nu = 1, 2, \dots)$$

ist, folgt aus Satz I unmittelbar:

SATZ II. *Wenn*

$$\sum_{\nu=1}^{\infty}\Big\{\sum_{k=0}^{\infty} b^2_{N_\nu,k} + \|B_\nu; N_\nu, N_{\nu+1}-1\|^2\Big\} < \infty$$

ist, so gilt $B \in \mathrm{BE}$.

Um eine brauchbarere Bedingung zu bekommen, müssen wir $\|B_\nu; N_\nu, N_{\nu+1}-1\|$ mit den Zahlen $b_{n,k}$ abschätzen. In folgenden werden wir zwei einfachen Abschätzungen angeben.

ABSCHÄTZUNG I. *Es gilt*

$$\|B_\nu; N_\nu, N_{\nu+1}-1\| \leqq \sum_{n=N_\nu+1}^{N_{\nu+1}-1}\Big\{\sum_{k=0}^{\infty}(b_{n,k}-b_{n-1,k})^2\Big\}^{1/2} \quad (\nu = 1,2,\dots).$$

BEWEIS. Es sei $\varphi \in \Omega$. Dann gilt für ein beliebiges n_0 ($N_\nu < n_0 < N_{\nu+1}$)

$$|b_{n_0}(\varphi;x) - b_{N_\nu}(\varphi;x)| = \Big|\sum_{n=N_\nu+1}^{n_0}(b_n(\varphi;x)-b_{n-1}(\varphi;x))\Big| \leqq$$

$$\leqq \sum_{n=N_\nu+1}^{N_{\nu+1}-1}|b_n(\varphi;x)-b_{n-1}(\varphi;x)|,$$

und so ist

$$\sup_{N_\nu<n<N_{\nu+1}}|b_n(\varphi;x)-b_{N_\nu}(\varphi;x)| \leqq \sum_{n=N_\nu+1}^{N_{\nu+1}-1}|b_n(\varphi;x)-b_{n-1}(\varphi;x)|.$$

Daraus folgt

$$\Big\{\int_0^1\big(\sup_{N_\nu<n<N_{\nu+1}}|b_n(\varphi;x)-b_{N_\nu}(\varphi;x)|\big)^2\,dx\Big\}^{1/2} \leqq$$

$$\leqq \sum_{n=N_\nu+1}^{N_{\nu+1}}\Big\{\int_0^1(b_n(\varphi;x)-b_{n-1}(\varphi;x))^2\,dx\Big\}^{1/2} =$$

$$= \sum_{n=N_\nu+1}^{N_{\nu+1}}\Big\{\sum_{k=0}^{\infty}(b_{n,k}-b_{n-1,k})^2\Big\}^{1/2}.$$

Da diese Ungleichung für jedes $\varphi \in \Omega$ besteht, ergibt sich unsere Abschätzung.

ABSCHÄTZUNG II. *Es gilt*

$$\|B_\nu; N_\nu, N_{\nu+1}-1\| \leqq$$

$$\leqq \sum_{0<s<\log(N_{\nu+1}-N_\nu)} \sum_{0\leqq \ell<\frac{N_{\nu+1}-N_\nu}{2^s}-1} \left\{\sum_{k=0}^{\infty}\left(b_{N_\nu+(\ell+1)2^s-1,k} - b_{N_\nu+\ell 2^s,k}\right)^2\right\}^{1/2}$$

$$(\nu = 1,2,\dots).$$

($\log \alpha$ bezeichnet den Logarithmus zur Basis 2.)

BEWEIS. Es sei $\varphi \in \Omega$. Für jedes n_0 $(N_\nu < n_0 < N_{\nu+1})$ gilt

$$n_0 = 2^{\nu_1} + \dots + 2^{\nu_r} + N_\nu$$

mit nichtnegativen ganzen Zahlen $\nu_1, \dots, \nu_r$ $(0 \leqq \nu_r < \dots < \nu_1 \leqq \log(N_{\nu+1}-$ $-N_\nu))$. So gilt

$$|b_{n_0}(\varphi;x) - b_{N_\nu}(\varphi;x)| = |(b_{2^{\nu_1}+\dots+2^{\nu_r}+N_\nu}(\varphi;x)-$$

$$-b_{2^{\nu_1}+\dots+2^{\nu_{r-1}}+N_\nu}(\varphi;x)) + \dots + (b_{2^{\nu_1}+N_\nu}(\varphi;x) - b_{N_\nu}(\varphi;x))| \leqq$$

$$\leqq \sum_{0<s<\log(N_{\nu+1}-N_\nu)} \sum_{0\leqq \ell<\frac{N_{\nu+1}-N_\nu}{2^s}-1} |b_{N_\nu+(\ell+1)2^s-1}(\varphi;x) - b_{N_\nu+\ell 2^s}(\varphi;x)|,$$

woraus sich

$$\left\{\int_0^1 \sup_{N_\nu<n<N_{\nu+1}} (b_n(\varphi;x) - b_{N_\nu}(\varphi;x))^2\,dx\right\}^{1/2} \leqq$$

$$\leqq \sum_{0<s<\log(N_{\nu+1}-N_\nu)} \sum_{0\leqq \ell<\frac{N_{\nu+1}-N_\nu}{2^s}-1} \left\{\int_0^1 (b_{N_\nu+(\ell+1)2^s-1}(\varphi;x)-\right.$$

$$\left.-b_{N_\nu+\ell 2^s}(\varphi;x))^2\,dx\right\}^{1/2} =$$

$$= \sum_{0<s<\log(N_{\nu+1}-N_\nu)} \sum_{0\leqq \ell<\frac{N_{\nu+1}-N_\nu}{2^s}-1} \left\{\sum_{k=0}^{\infty}\left(b_{N_\nu+(\ell+1)2^s-1,k} - b_{N_\nu+\ell 2^s,k}\right)^2\right\}^{1/2}$$

ergibt. Da diese Ungleichung für jedes $\varphi \in \Omega$ besteht, folgt unsere Abschätzung.

Aus den Sätzen I–II und aus den Abschätzungen I–II erhalten wir die folgenden Kriterien.

Satz III. *Gilt*

$$\sum_{\nu=1}^{\infty}\left\{\sum_{k=0}^{\infty} b_{N_\nu,k}^2 + \left(\sum_{n=N_\nu+1}^{N_{\nu+1}-1}\left\{\sum_{k=0}^{\infty}(b_{n,k}-b_{n-1,k})^2\right\}^{1/2}\right)^2\right\} < \infty,$$

so ist $B \in \mathrm{BE}$.

Satz IV. *Gilt*

$$\sum_{\nu=1}^{\infty}\left\{\sum_{k=0}^{\infty} b_{N_\nu,k}^2 + \right.$$

$$\left. + \sum_{0<s<\log(N_{\nu+1}-N_\nu)} \sum_{0\leqq \ell < \frac{N_{\nu+1}-N}{2^s}-1} \left(\sum_{k=0}^{\infty}(b_{N_\nu+(\ell+1)2^s-1,k} - b_{N_\nu+\ell 2^s,k})^2\right)^{1/2}\right\}^2,$$

so gilt $B \in \mathrm{BE}$.

3. Anwendungen. A. Es sei $\lambda = \{\lambda_n\}_0^\infty$ eine monoton nichtabnehmende Folge von positiven Zahlen mit

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n^2} < \infty. \tag{5}$$

Es sei weiterhin

$$b_{n,k} = \begin{cases} \frac{1}{\lambda_n}\left(1 - \frac{k}{n+1}\right), & k = 0,\dots,n, \\ 0, & k = n+1, n+2, \dots. \end{cases}$$

Wir wünschen zu zeigen, daß $B = \|b_{n,k}\|_{n,k=0}^{\infty} \in \mathrm{BE}$.

Es sei $\varphi \in \Omega$. Jetzt ist

$$b_n(\varphi; x) = \frac{\sigma_n(x)}{\lambda_n} \qquad (n = 0, 1, \dots),$$

wobei

$$\sigma_n(x) = \sum_{k=0}^{n}\left(1 - \frac{k}{n+1}\right)\varphi_k(x) \qquad (n = 0, 1, \dots).$$

Es sei

$$b_{n,k}^* = \begin{cases} \frac{1}{\lambda_{2^m}}\left(1 - \frac{k}{n+1}\right), & 2^m \leqq n < 2^{m+1} \ (m = 0, 1, \dots),\ k = 0, \dots, n, \\ 0, & 2^m \leqq n < 2^{m+1} \ (m = 0, 1, \dots),\ k = n+1, \dots. \end{cases}$$

Offensichtlich genügt es zu zeigen, daß $B^* = \|b_{n,k}^*\|_{m,k=0}^{\infty} \in \mathrm{BE}$ gilt.

Aus (5) folgt

$$\sum_{m=0}^{\infty} \frac{2^m}{\lambda_{2^m}^2} < \infty. \tag{6}$$

Wir werden den Satz III mit der Folge $N_\nu = 2^\nu$ ($\nu = 1, 2, \ldots$) anwenden.

Jetzt gilt auf Grund von (6)

$$\sum_{\nu=1}^{\infty} \Big\{ \sum_{k=0}^{\infty} (b^*_{N_\nu,k})^2 + \Big(\sum_{n=N_\nu+1}^{N_{\nu+1}-1} \Big\{ \sum_{k=0}^{n} (b^*_{n,k} - b^*_{n-1,k})^2 \Big\}^{1/2} \Big)^2 \Big\} =$$

$$= \sum_{\nu=1}^{\infty} \frac{1}{\lambda_{2^\nu}^2} \Big\{ \Big(\sum_{k=0}^{2^\nu} \Big(1 - \frac{k}{2^\nu + 1}\Big)^2 + \Big(\sum_{n=2^\nu+1}^{2^{\nu+1}} \Big(\sum_{k=0}^{n} \frac{k^2}{n^2(n-1)^2} \Big)^{1/2} \Big)^2 \Big) \Big\} \leqq$$

$$\leqq c_1 \sum_{\nu=1}^{\infty} \frac{2^\nu}{\lambda_{2^\nu}^2} < \infty.$$

Auf Grund des Satzes III bekommen wir $B^* \in \mathrm{BE}$.

Das bedeutet, daß für jedes System $\varphi \in \Omega$

$$\sum_{k=0}^{n} \Big(1 - \frac{k}{n+1}\Big) \varphi_k(x) = o_x(\lambda_n)$$

fast überall in $(0,1)$, wenn für die Folge λ (5) erfüllt ist. (In [2] haben wir gezeigt, daß diese Abschätzung genau ist.)

B. Es sei $\lambda = \{\lambda_n\}_0^\infty$ eine monoton nichtabnehmende Folge von positiven Zahlen mit

$$\sum_{n=1}^{\infty} \frac{\log^2 n}{\lambda_n^2} < \infty. \tag{7}$$

Es sei

$$b_{n,k} = \begin{cases} \frac{1}{\lambda_n}, & k = 0, \ldots, n, \\ 0, & k = n+1, \ldots . \end{cases}$$

Wir wünschen zu zeigen, daß $B \in \mathrm{BE}$.

Für ein System $\varphi \in \Omega$ ist jetzt

$$b_n(\varphi; x) = \frac{1}{\lambda_n} \sum_{k=1}^{n} \varphi_k(x) \qquad (n = 0, 1, \ldots).$$

Es sei

$$b^*_{n,k} = \begin{cases} \frac{1}{\lambda_{2^m}}, & 2^m \leqq n < 2^{m+1} \ (m = 0, 1, \ldots), \ k = 0, \ldots, n, \\ 0, & 2^m \leqq n < 2^{m+1} \ (m = 0, 1, \ldots), \ k = n+1, \ldots . \end{cases}$$

Offensichtlich genügt es $B^* = \|b^*_{n,k}\|^{\infty}_{n,k=0} \in \mathrm{BE}$ zu zeigen. Aus (7) folgt

$$\sum_{m=1}^{\infty} \frac{m^2 2^m}{\lambda^2_{2^m}} < \infty. \tag{8}$$

Wir werden den Satz IV mit der Folge $N_\nu = 2^\nu$ $(\nu = 1, 2, \dots)$ anwenden.

Jetzt gilt auf Grund von (8)

$$\sum_{\nu=1}^{\infty} \left\{ \sum_{k=0}^{\infty} (b^*_{N_\nu,k})^2 + \right.$$

$$\left. + \left(\sum_{0<s<\log(N_{\nu+1}-N_\nu)} \sum_{0\leqq \ell < \frac{N_{\nu+1}-N_\nu}{2^s}-1} \left\{ \sum_{k=0}^{\infty} (b^*_{N_\nu+(\ell+1)2^s-1,k} - b^*_{N_\nu+\ell 2^s,k})^2 \right\}^{1/2} \right)^2 \leqq \right.$$

$$\leqq \sum_{\nu=1}^{\infty} \left. \frac{2^\nu+1}{\lambda^2_{2^\nu}} + \frac{1}{\lambda^2_{2^\nu}} \nu^2 2^\nu \right\} < \infty.$$

Auf Grund des Satzes IV erhalten wir $B^* \in \mathrm{BE}$.

Das bedeutet, daß für jedes System $\varphi \in \Omega$

$$\sum_{k=0}^{n} \varphi_k(x) = o_x(\log n)$$

in $(0, 1)$ fast überall, wenn für die Folge λ (7) erfüllt wird. (In [2] haben wir gezeigt, daß diese Abschätzung genau ist.)

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JÓZSEF ATTILA UNVERSITÄT
BOLYAI INSTITUT
H-6720 SZEGED
ARADI VÉRTANÚK TERE 1

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ON AN ASSERTION OF RIEMANN CONCERNING THE DISTRIBUTION OF PRIME NUMBERS

J. PINTZ* (Budapest)

1. Riemann [8] stated in his famous memoir in 1859 without proof for the number of primes $\leqq x$ the inequality

$$\pi(x) < \mathrm{Li}\,(x) \overset{\mathrm{def}}{=} \int_0^x \frac{dt}{\log t} \quad (x > 2) \tag{1.1}$$

or more precisely he wrote the following lines (with the notation $\pi(x) = F(x)$): "Thus the known approximation $F(x) = \mathrm{Li}\,(x)$ is correct only to an order of magnitude of $x^{1/2}$ and gives a value which is somewhat too large, because the nonperiodic terms in the expression of $F(x)$ are, except for quantities which remain bounded as x increases,

$$\mathrm{Li}\,(x) - \frac{1}{2}\mathrm{Li}\,(x^{1/2}) - \frac{1}{3}\mathrm{Li}\,\left(x^{1/3}\right) - \frac{1}{5}\mathrm{Li}\,\left(x^{1/5}\right) + \frac{1}{6}\mathrm{Li}\,\left(x^{1/6}\right) - \frac{1}{7}\mathrm{Li}\,\left(x^{1/7}\right) \dots .$$

In fact the comparison of $\mathrm{Li}\,(x)$ with the number of primes less than x which was undertaken by Gauss and Goldschmidt and which was pursued up to $x =$ three million shows that the number of primes is already less than $\mathrm{Li}\,(x)$ in the first hundred thousand and that the difference, with minor fluctuations, increases gradually as x increases."

The assertion of Riemann was the starting point for a number of interesting and deep investigations. So it was proved e.g. by E. Schmidt [9] in 1903 that (1.1) implies the truth of the famous Riemann hypothesis on the zeros of the zetafunction. Riemann's assertion seemed to be supported by the calculation of D. N. Lehmer [4] who showed its validity for $x \leqq 10^7$. But Littlewood [5] disproved it in 1914, i.e. in the same year, showing that $\pi(x) - \mathrm{Li}\,(x)$ changes sign infinitely often as $x \to \infty$.

Later even the number $V(Y)$ of sign changes of $\pi(x) - \mathrm{Li}\,(x)$ in the interval $[2, Y]$ could be estimated from below using Turán's method. Thus S. Knapowski [2] proved for $V(Y)$ the lower bound $c \log\log Y$ in 1961 and this was improved by Knapowski–Turán [3] in 1974 to $c(\varepsilon) \log^{1/4-\varepsilon} Y$. The present author [6] improved this estimate to $c \log Y / (\log\log Y)^3$. Later Kaczorowski (Acta Arith. 45(1985), 65–74) improved this result further to

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$c \log Y$. This shows on the one hand that Riemann's assertion is very far from being true.

On the other hand we may quote (with minor changes) some lines from the book of Ingham [1]:

"The above remarks relate only to individual values of x. But the inequality $\pi(x) < \mathrm{Li}(x)$ and Riemann's formula acquire some significance when considered from the point of view of averages, at any rate if the Riemann hypothesis is true. Thus (assuming the Riemann hypothesis in what follows)... we can show that

$$\int_2^X (\pi(x) - \mathrm{Li}(x))\, dx < 0 \quad (X > X_0), \tag{1.2}$$

so that $\pi(x) - \mathrm{Li}(x)$ is 'negative on the average'."

But one can show by standard methods that the inequality (1.2) is true if and only if the Riemann hypothesis is true. (We may note here that in the book of Prachar [7] on p. 260 the truth of the formula (1.2) is mentioned but the words 'under the Riemann hypothesis' are unfortunately missing.)

The above assertion suggests that to decide the weaker version of (1.1), i.e. the assertion '$\pi(x) - \mathrm{Li}(x)$ is negative on the average' is hopeless at present. This is really the case if we use the most direct interpretation (1.2) (which, under the Riemann hypothesis is probably true even for every $X > 2$). The aim of the present work is to show at the same time that it is possible to find a relatively simple type of averaging procedure for which the assertion '$\pi(x) - \mathrm{Li}(x)$ is negative on the average' is true without any unproved hypothesis. This will show that the assertion (1.1) of Riemann is by far not so wrong as indicated earlier. So we can assert rightly that in a precisely formulated sense $\pi(x) - \mathrm{Li}(x)$ is negative on the average.

2. We shall prove the following

THEOREM. *For $y > c_1$ the inequality*

$$\int_1^\infty (\pi(x) - \mathrm{Li}(x)) \exp\left(-\frac{\log^2 x}{y}\right) dx < -\frac{c_2}{y} e^{\frac{9}{16}y} \tag{2.1}$$

where c_1, c_2 are explicitly calculable positive absolute constants.

In the course of proof we shall use the notations

$$\Pi(x) \overset{\text{def}}{=} \sum_{p^m \leqq x} \frac{1}{m} = \sum_{m \leqq x} \frac{\Lambda(n)}{\log n}, \tag{2.2}$$

$$\lg x \overset{\text{def}}{=} \sum_{2 \leqq n \leqq x} \frac{1}{\log n} \quad (\lg x = 0 \quad \text{for} \quad x < 2), \tag{2.3}$$

$$\Delta_2(x) = \Pi(x) - \lg x, \tag{2.4}$$

$$\Delta_1(x) = \pi(x) - \operatorname{Li}(x), \tag{2.5}$$

$$y = 4u \geqq 4. \tag{2.6}$$

In the proof c_i will denote explicitly calculable absolute constants with $c_i > 0$ except perhaps c_5.

By partial integration we get for $\sigma > 1$

$$-\int_1^\infty \Pi(x) \frac{d}{dx}(x^{-s})\, dx = \sum_{n=1}^\infty \frac{\Lambda(n)}{\log n \cdot n^s} = \int_2^s \frac{\zeta'}{\zeta}(z)\, dz + c_3, \tag{2.7}$$

$$\int_1^\infty \lg x \frac{d}{dx}(x^{-s})\, dx = -\sum_{n=2}^\infty \frac{1}{\log n \cdot n^s} = \int_2^s (\zeta(z) - 1)\, dz - c_4. \tag{2.8}$$

Adding the above two inequalities we have

$$\int_1^\infty \frac{\Delta_2(x)}{x^{s+1}}\, dx = \frac{1}{s} \left\{ \int_2^s \left(\frac{\zeta'}{\zeta}(z) + \zeta(z) - 1 \right) dz + c_5 \right\} \tag{2.9}$$

i.e.

$$\int_1^\infty \frac{\Delta_2(x)}{x^s}\, dx = \frac{1}{s-1} \left\{ \int_2^{s-1} \left(\frac{\zeta'}{\zeta}(z) + \zeta(z) - 1 \right) dz + c_5 \right\} \overset{\text{def}}{=} \varphi(s) \tag{2.10}$$

being valid for $\sigma > 2$.

Further we shall use the formula ($A > 0$, B arbitrary complex)

$$\frac{1}{2\pi i} \int_{(3)} e^{As^2 + Bs} ds = \exp\left(-\frac{B^2}{4A}\right) \cdot \frac{1}{2\pi i} \int_{(3)} e^{\left(\sqrt{A}s + \frac{B}{2\sqrt{A}}\right)^2} ds = \tag{2.11}$$

$$= \exp\left(-\frac{B^2}{4A}\right) \cdot \frac{1}{\sqrt{A}} \cdot \frac{1}{2\pi i} \int_{(0)} e^{z^2} dz = \frac{1}{2\sqrt{\pi A}} \exp\left(-\frac{B^2}{4A}\right).$$

(2.10) and (2.11) together give

$$U \stackrel{\text{def}}{=} \frac{1}{2\sqrt{\pi u}} \int_1^\infty \Delta_2(x) \exp\left(-\frac{\log^2 x}{4u}\right) dx = \tag{2.12}$$

$$= \int_1^\infty \Delta_2(x) \cdot \frac{1}{2\pi i} \int_{(3)} e^{us^2} \cdot x^{-s} ds\, dx = \frac{1}{2\pi i} \int_{(3)} e^{us^2} \varphi(s)\, ds.$$

Instead of $\sigma = 3$ we can integrate on the broken line ℓ defined for $t \geqq 0$ by

$$\begin{cases} I_1 : \sigma = 3 & \text{for} \quad t \geqq 3 \\ I_2 : 1.1 \leqq \sigma \leqq 3 & \text{for} \quad t = 3 \\ I_3 : \sigma = 1.1 & \text{for} \quad 0 \leqq t \leqq 3 \end{cases} \tag{2.13}$$

and for $t \leqq 0$ by reflection on the real axis since $\varphi(s)$ is obviously regular right of ℓ and on ℓ. Further we have

$$|\varphi(s)| \leqq c_6 \quad \text{for} \quad s \in \ell \tag{2.14}$$

and

$$|e^{us^2}| \leqq e^{\frac{9}{8}u} \quad \text{for} \quad s \in \ell. \tag{2.15}$$

Thus we have

$$|U| = \left| \frac{1}{2\pi i} \int_{(\ell)} \varphi(s) e^{us^2} ds \right| \leqq \frac{c_6}{2\pi} \int_{(\ell)} |e^{us^2}|\, |ds| \leqq \tag{2.16}$$

$$\leqq \frac{c_6}{2\pi} \left(10 e^{\frac{9}{8}u} + 2 \int_3^\infty e^{(9-t^2)u} dt \right) \leqq 2 c_6 e^{\frac{9}{8}u}.$$

3. On the other hand by Chebyshev's theorem

$$\Pi(x) - \pi(x) \geqq \frac{1}{2}\pi(\sqrt{x}) > c_7 \frac{\sqrt{x}}{\log x} \tag{3.1}$$

and by the trivial remark

$$\lg x = \mathrm{Li}\,(x) + O(1) \tag{3.2}$$

we have

$$\Delta_2(x) - \Delta_1(x) > c_8 \frac{\sqrt{x}}{\log x} \quad \text{for} \quad x > c_9. \tag{3.3}$$

From this we get

$$(3.4)\qquad \int_1^\infty (\Delta_2(x) - \Delta_1(x)) \exp\left(-\frac{\log^2 x}{4u}\right) dx >$$

$$> \int_{c_9}^{e^{3u}} c_8 \frac{\sqrt{x}}{\log x} \exp\left(-\frac{\log^2 x}{4u}\right) dx + O(1) >$$

$$> \frac{c_8}{3u} \int_{c_9}^{c^{3u}} \sqrt{x} \exp\left(-\frac{3\log x}{4}\right) dx + O(1) = \frac{4}{3} \cdot \frac{c_8}{3u} \cdot e^{\frac{9}{4}u} + O(1).$$

Now (2.16) and (3.4) together prove our Theorem.

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
H–1053 BUDAPEST, REÁLTANODA U. 13–15
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[1] G. Szegő, *Orthogonal polynomials*, AMS Coll. Publ. Vol. XXXIII (Providence, 1939).
[2] A. Zygmund, Smooth functions, *Duke Math. J.*, (1945), 47–76.

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