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CONTENTS

CONTENTS

VOLUME 56

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ALMOST TACHIBANA RECURRENT MANIFOLDS

S. K. D. DUBEY* (Bhatpar Rani)

1. Introduction

We consider an n-dimensional differentiable manifold M of class C^∞ covered by a system of coordinate neighbourhoods $\{U, x^i\}$, where U denotes a neighbourhood and x^i denote the local coordinates in U with the indices $i, j, k, \ldots$ having the range $1, 2, 3, \ldots, n$. We choose the Jacobian determinant in M in such a way that

$$\Delta = \left| \frac{\partial x^{i'}}{\partial x^i} \right|$$

of the coordinate transformation ([1], [3])

$$x^{i'} = x^{i'}(x^1, x^2, \ldots, x^n)$$

occurring in every non-empty intersection of two coordinate neighbourhoods $\{U, x^h\}$ and $\{U', x^{h'}\}$ is always positive, then the manifold is said to be orientiable.

An almost Tachibana manifold is first of all an almost Hermite manifold, i.e., a $2n$-dimensional manifold with an almost complex structure F (Yano [8])

(1.1) $$F_j^i F_i^h = -A_j^h$$

and with Reimannian metric g_{ij} satisfying

(2.2) $$F_j^t F_j^s g_{ts} = g_{ji},$$

where

(1.3) $$F_{ji} = -F_{ij},, \quad F_{ji} = F_j^t g_{ti}.$$

The skew-symmetric tensor F_{ih} is a killing tensor if

(1.4) $$\nabla_j F_{ih} + \nabla_i F_{jh} = 0,$$

where

(1.5) $$\begin{cases} \nabla_j F_i^h + \nabla_i F_j^h = 0, \\ F_{jih} = \nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji}, \end{cases}$$

and

(1.6) $$F_i = -\nabla_j F_i^j = 0.$$

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In an almost Tachibana manifold, the Nijenhuis tensor $N_{ji}{}^{h}$ is given in the form

$$N_{ji}{}^{h}=-4(\nabla_j F_i^t)F_t^h+2G_{ij}{}^{t}F_t^h+F_j^tG_{ti}{}^{h}-F_i^tG_{tj}{}^{h}, \tag{1.7}$$

$$G_{ji}^{h}=\nabla_j F_i^h+\nabla_i F_j^h,$$

or

$$N_{ji}^{h}=-4(\nabla_j F_i^t)F_t^h \tag{1.8}$$

and consequently $\nabla_j F_h^i$ is pure in j and i. When the Nijenhuis tensor vanishes, the almost Tachibana manifold is called a Tachibana manifold, i.e.,

$$\nabla_j F_i^h=0, \tag{1.9}$$

which yields that a Tachibana manifold is a Kaehler manifold ([4], [6]). The Nijenhuis tensor N_{jih}, $N_{jit}F_h^t$ and $\nabla_j F_{ih}$ are skew-symmetric in all their corresponding indices and satisfy the following identities:

(1.10)

a) $3\nabla_j F_{ih}-F_{jih}=G_{jih}-G_{jhi}$,

b) $3\nabla_j F_{ih}=F_{jih}=-\dfrac{3}{4}N_{jit}F_h^t$,

c) $N_{jih}=\dfrac{4}{3}F_{jit}F_h^t$,

d) $\nabla_t\nabla_j F_i^t=(K_{jt}-K_{jt}^{*})F_i^t$,

e) $N_{ji}{}^{h}=-2O_{ji}^{ts}(\nabla_t F_t^r-\nabla_s F_t^r)F_n{}^{h}$.

The Ricci-tensors K_{ji} and K_{ji}^{*} are hybrid in an almost Tachibana manifold. A necessary and sufficient condition that an almost Tachibana manifold reduces to a Kaehler manifold is that

$$K_{jh}=K_{jh}^{*} \tag{1.11}$$

where

$$H_{kj}=\frac{1}{2}K_{kjih}F^{ih}.$$

For a conformally flat almost Tachibana manifold, the curvature tensor has the form

$$K_{kijh}=-[g_{kh}C_{ji}-g_{jh}C_{ki}+C_{kh}g_{ji}-C_{jh}g_{ki}] \tag{1.12}$$

where

$$C_{ji}=-\frac{K_{ji}}{2n-2}+\frac{Kg_{ji}}{2(2n-1)(2n-2)}, \tag{1.13}$$

$$H_{ji}=F_j^tC_{ti}-F_i^tC_{tj}, \tag{1.14}$$

$$K_{ji}^{*}=-H_{jt}F_i^t=-2{}^{*}O_{ji}^{ts}C_{ts}, \tag{1.15}$$

$${}^{*}O_{ji}^{ts}=C_{ts}=C_{ji},\ K_{ji}^{*}=-2C_{ji} \tag{1.16}$$

and

$$K_{ji}-K^*_{ji} = \frac{2n-4}{2n-2}K_{ji}+\frac{1}{(2n-1)(2n-2)}Kg_{ji}. \tag{1.17}$$

The Ricci identity having the tensor F_i^h, we can write (Yano [8])

$$\nabla_k\nabla_j F_i^h-\nabla_j\nabla_k F_i^h = K_{kjt}{}^h F_i^t-K_{kji}{}^t F_t^h, \tag{1.18}$$

from which

$$\nabla_t\nabla_j F_i^t = K_{jt}F_i^t-H_{ji}. \tag{1.19}$$

2. Almost Tachibana recurrent manifold of first order

DEFINITION (2.1). An almost complex structure F is recurrent of first order if it satisfies the relation

$$\nabla_j F_{ih} = v_j(\xi)F_{ih} \tag{2.1}$$

where $v_j(\xi)$ is a non-zero recurrence vector field in an almost Tachibana manifold.
Equations (1.4), (1.6) and (2.1) yield

$$v_jF_{ih}+v_iF_{jh} = 0 \tag{2.2}$$

which implies that

$$v_jF_i^h+v_iF_j^h = 0 \tag{2.3}$$

and

$$F_i+v_jF_i^j = 0. \tag{2.4}$$

THEOREM (2.1). *For a recurrent complex structure of first order, the Nijenhuis tensor satisfies the following identities*

$$N_{ji}{}^h+4v_jF_i^tF_t^h-2G_{ji}{}^tF_t^h-F_j^tG_{ti}{}^h+F_{ti}{}^hG_{tj}{}^h = 0, \tag{2.5}$$

$$N_{ji}{}^h+4v_jF_i^tF_t^h = 0;\ N_{ji}{}^h = 4v_jA_i^h, \tag{2.6}$$

and

$$N_{ji}{}^h+2O^{ts}_{ji}(v_tF_r^s-v_sF_t^r)F_r^h. \tag{2.7}$$

PROOF. In view of (2.1), equations (1.7), (1.8) and (1.10e) yield the theorem.
Using (1.10b), (2.1) for an almost complex recurrent structure, we can write

$$F_{jih} = 3v_jF_{ih}$$

for skew-symmetric in all its indices.

DEFINITION (2.2). An almost Tachibana manifold is called recurrent of first order if the Nijenhuis tensor satisfies the relation

$$\nabla_k N_{ji}{}^h = v_k(x)N_{ji}{}^h \tag{2.8}$$

where $v_k(x)$ is a non-zero recurrence vector field.

Let us take the coordinate system in almost Tachibana manifold

$$x^i = x^i(\xi), \tag{2.9}$$

the recurrence vector field $v_k(x)$ can be expressed as $v_k(\xi)$.

THEOREM (2.2). *In an almost Tachibana recurrent manifold of first order, the complex structure should be recurrent of first order but the converse is not true, in general.*

PROOF. The equations (1.7), (2.1), (2.8) and (2.9) prove the theorem.

THEOREM (2.3). *In an almost Tachibana recurrent manifold the following identity holds:*

$$v_k\left(N_{jih}-\frac{4}{3}F_{jit}F_h^{\ t}\right)-4(\nabla_k v_j+v_j v_k)F_{it}=0. \tag{2.10}$$

PROOF. The equations (1.10b), (1.10c) and (2.1) yield the theorem.

The hybrid tensors K_{jh} and K^*_{jh} in an almost Tachibana recurrent manifold, corresponding to recurrence vector fields w_i and w^*_i, can be expressed as

$$\nabla_i K_{jk} = w_i K_{jh} \tag{2.11}$$

and

$$K^*_{jh}\nabla_i K^*_{jh} = w^*_i \tag{2.12}$$

which yield the following by using (1.11).

THEOREM (2.4). *A necessary and sufficient condition for an almost Tachibana recurrent manifold to reduce to a Kaehler manifold is that the recurrence vector fields, corresponding to hybrid tensors, are equal, i.e.* $w_i = w^*_i$.

THEOREM (2.5). *If the differential form*

$$F_{ji}\, d\xi^j \Lambda d\xi^i$$

is closed in an almost Tachibana recurrent manifold, then an almost Kaehler manifold, in which the complex srtucture is recurrent, satisfies

$$v_j F_{ih}+v_i F_{hj}+v_h F_{ji}=0. \tag{2.13}$$

PROOF. Equations (1.5), (2.1) and the expression $F_{jih}=0$ will give the result.

3. Almost Tachibana bi-recurrent manifold

DEFINITION (3.1). An almost Tachibana manifold is recurrent of second order, i.e., almost Tachibana bi-recurrent manifold if it satisfies the relation

$$\nabla_k\nabla_j F_i^h = a_{kj}F_i^h \tag{3.1}$$

where a_{kj} is a non-zero recurrence tensor field.

Using equations (2.1) and (3.1), the recurrence tensor field can be expressed in terms of recurrence vector fields, i.e.,

$$a_{kj} = \nabla_k v_j + v_k v_j. \tag{3.2}$$

THEOREM (3.1). *Every almost Tachibana bi-recurrent manifold implies that there exists an almost Tachibana recurrent manifold of first order.*

PROOF. Equations (2.1), (3.2) give the above conclusion.

THEOREM (3.2). *The recurrence tensor field is symmetric, if in almost Tachibana recurrent manifold the recurrence vector field is gradient, i.e.,*

$$\nabla_k v_j - \nabla_j v_k = 0. \tag{3.3}$$

PROOF. Interchanging the indices k and j in (3.2), we get the result.

In an almost Tachibana bi-recurrent manifold, the Nijenhuis tensor yields the recurrence tensor field such that

$$\nabla_l \nabla_k N_{ji}{}^h = b_{kl} N_{ji}{}^h \tag{3.4}$$

in which

$$b_{kl} = a_{kl} \tag{3.5}$$

under the coordinate system $x^i = x^i(\xi)$.

Equations (2.1), (2.6), (2.8), (3.1) and (3.5) yield the following identity:

$$\begin{aligned} a_{kl} N_{ji}{}^h + 4(\nabla_l \nabla_k v_j) A_i^h + 8(\nabla_k v_j) v_l A_i^h + \\ + 8[(\nabla_l v_j) v_k + (\nabla_l v_k) v_j] A_i^h + 16 v_k v_j v_l A_i^h = 0. \end{aligned} \tag{3.6}$$

A killing vector is defined as a vector field $v_h(\xi)$ (Yano [8]) which satisfies

$$\nabla_j v_i + \nabla_i v_j = 0 \quad \text{and} \quad \nabla_i v^i = 0. \tag{3.7}$$

In a recurrence vector field, equations (3.2) and (3.7) yield

$$a_{kj} + a_{jk} = 2 v_j v_k, \tag{3.8}$$

from which it follows:

THEOREM (3.3). *A recurrence vector field as a killing vector generates a local one-parameter group of motions, if*

$$v_j v_k = \frac{1}{2}(a_{kj} + a_{jk}). \tag{3.9}$$

THEOREM (3.4). *If the recurrence tensor field is symmetric in an almost Tachibana bi-recurrent manifold, the recurrence vector field is a harmonic vector.*

PROOF. A harmonic vector is defined (Yano [8]) as a vector field $v^h(\xi)$ which satisfies

$$\nabla_j v_i - \nabla_i v_j = 0 \quad \text{and} \quad \nabla_i v^i = 0.$$

Thus, Theorem (3.2) gives that recurrence vector field is a harmonic vector.

If recurrence vector field $v^h(\xi)$ is treated as a killing vector field (Yano [7], [8])

$$\underset{v}{\mathcal{L}} g_{ij} = \nabla_j v_i + \nabla_i v_j = 0. \tag{3.10}$$

Equations (3.8) and (3.10) give

$$\underset{v}{\mathcal{L}} g_{ij} = a_{kj} + a_{jk} - 2v_j v_k = 0, \tag{3.11}$$

which yields the following:

THEOREM (3.5). *A necessary and sufficient condition that an almost Tachibana bi-recurrent manifold admits a transtitive group of motions if it satisfies the relation*

$$a_{kj} + a_{jk} - 2v_j v_k = 0. \tag{3.12}$$

THEOREM (3.6). *In an almost Tachibana bi-recurrent manifold, the Ricci identity relating to recurrence tensor field can be expressed as*

$$a_{kj} F_i^h + K_{kji}{}^t F_t^h = a_{jk} F_i^h + F_{kjt}{}^h F_{it}. \tag{3.13}$$

PROOF. Equations (1.18) and (3.1) yield (3.13).

Applying the conctraction with respect to k and h in equation (3.13), equations (1.19) and (1.10d) yield

$$a_{tj} F_i^t = K_{jt} F_i^t - H_{ji} \tag{3.14}$$

and

$$a_{tj} F_i^t = (K_{jt} - K_{jt}^*) F_i^t, \tag{3.15}$$

which gives the following:

THEOREM (3.7). *In order that* $K_{ji} = K_{ji}^*$ *in an almost Tachibana bi-recurrent manifold, it is necessary and sufficient that*

$$a_{tj} F_i^t = 0 \quad (a_{tj} \neq 0),$$

i.e., it reduces to an almost Kaehler manifold.

The Ricci-recurrence identity (3.13) can be expressed as

$$a_{kj} F_{ih} = a_{jk} F_{ih} - K_{kji}{}^t F_{th} - K_{kjh}{}^t F_{it}, \tag{3.16}$$

that is,

$$a_{kj} F_{ih} = a_{ji} F_{hk} + K_{kjit} F_h^t - K_{kjht} F_i^t \tag{3.17}$$

which yield the following:

THEOREM (3.8). *If the recurrence tensor field is an almost Tachibana bi-recurren manifold, the following identity holds*

$$K_{kjit} F_h^t = K_{kjht} F_i^t. \tag{3.18}$$

We take a contravariant almost analytic vector field w^h, in such a way that (Yano [7], [8])

$$\underset{v}{\mathcal{L}} F_i^h = w^t \nabla_t F_i^h - F_i^t \nabla_t w^h + F_t^h \nabla_i w^t = 0. \tag{3.19}$$

Using the recurrence vector field of equation (2.1) in (3.19), we have

$$\underset{v}{\mathscr{L}} F_i^h = w^t v_t F_i^h - F_i^t \nabla_t w^h + F_t^h \nabla_i w^t = 0 \tag{3.20}$$

or

$$w^t v_t F_{ih} - F_i^t \nabla_t w_h - F_h^t \nabla_i w_t = 0. \tag{3.21}$$

THEOREM (3.9). *If a recurrence vector field can be expressed as a contravariant almost analytic vector field in an almost Tachibana recurrent manifold, then it satisfies*

$$2n F_{ih} = F_i^t \nabla_t v_h + F_h^t \nabla_i v_t. \tag{3.22}$$

PROOF. We take $w^t = v^t$ in equation (3.21). Thus, in a $2n$-dimensional almost Tachibana recurrent manifold, equation (3.21) and $v^t v_t = 2n$ give the required expression.

THEOREM (3.10). *If the recurrence vector field can be expressed as a contravariant almost analytic vector field in an almost Tachibana bi-recurrent manifold, then*

$$2n a_{ks} F_{ih} = a_{ks} F_i^t \nabla_t v_h + F_i^t \nabla_k \nabla_s \nabla_t v_h + a_{ks} F_h^t \nabla_i v_t + F_h^t \nabla_k \nabla_s \nabla_i v_t. \tag{3.23}$$

PROOF. Equations (3.1) and (3.22) yield the identity.

4. Point-wise constant type almost Tachibana recurrent manifold

An almost Tachibana manifold has pointwise constant type (Yamaguchi, Chūman and Matsumoto [6], Gray [2]) if and only if there exists a scalar function α such that

$$\nabla_k F_j^r \nabla_i F_{hr} + \nabla_i F_j^r \nabla_k F_{hr} = \alpha(2g_{ki} g_{jh} - g_{kh} g_{ji} - g_{ih} g_{jk} + F_{kh} F_{ji} + F_{ih} F_{jk}). \tag{4.1}$$

Now, we have

THEOREM (4.1). *An almost Tachibana recurrent manifold has point-wise constant type if and only if*

$$v_k v_i F_j^r F_{hr} - \frac{\alpha}{2}(2g_{ki} g_{jh} - g_{kh} g_{ji} - g_{ih} g_{jk} + F_{kh} F_{ji} + F_{ih} F_{jk}) = 0. \tag{4.2}$$

PROOF. Equations (2.1) and (4.1) yield the required result.

The almost complex structure F in a special almost Tachibana manifold is a special killing 2-form with constant $\alpha(\neq 0)$ if

$$\nabla_k \nabla_j F_{ih} = -\alpha(g_{kj} F_{ih} - g_{ki} F_{jh} + g_{kh} F_{ji}). \tag{4.3}$$

Equations (3.1) and (4.3) give

$$a_{kj} F_{ih} + \alpha(g_{kj} F_{ih} - g_{ki} F_{jh} + g_{kh} F_{ji}) = 0. \tag{4.4}$$

Now, we have

THEOREM (4.2). *A special almost Tachibana bi-recurrent manifold is a special killing 2-form with a non-zero constant α if it satisfies*

$$(\nabla_k v_j + v_k v_j) F_{ih} + \alpha (g_{kj} F_{ih} - g_{ki} F_{jh} + g_{kh} F_{ji}) = 0, \tag{4.5}$$

i.e. it is of point-wise constant type.

PROOF. Equations (3.2), (4.1), (4.2) prove the theorem. The equations (Gray [2])

$$\nabla_j F_{sr} \nabla_i F^{sr} = R_{ji} - R^*_{ji} \tag{4.6}$$

and (2.1), give

$$v_i v_j F_{sr} F^{sr} = R_{ji} - R^*_{ji} = 2(n-1)\alpha g_{ji}, \tag{4.7}$$

which can be written as

$$v_j F_{ih} v^j F^{ih} = R - R^* = 4n(n-1)\alpha \tag{4.8}$$

where $R^* = R^*_{ji} g^{ji}$.

Takamatsu [5] has obtained the following identities

$$(R_{ji} - R^*_{ji})(R^{ji} - 5R^{*ji}) = 0, \tag{4.9}$$

and

$$(R_{ji} - R^*_{ji})(R^{ji} - R^{*ji}) = 4R_{kjih} O^{ih}_{sr} R^{kjsr}. \tag{4.10}$$

Thus, we have

THEOREM (4.3). *A special almost recurrent Tachibana manifold (non-Kaehlerian) is an almost Tachibana recurrent manifold ($n>1$) with point-wise constant type whose non-zero scalar function has the form*

$$\alpha = R/5n(n-1). \tag{4.11}$$

PROOF. Equations (4.8), (4.9) and (4.10) prove the theorem.

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LIMITING DISTRIBUTIONS OF ADDITIVE FUNCTIONS IN SHORT INTERVALS

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1. Introduction. In this paper we investigate the question of what choice of $y=y(x)$ insures that the real-valued functions $f_y(n)$ possess a limiting distribution in the interval $x-y<n\leqq x$ $(y\leqq x)$. Here we consider the two cases

$$f_y(n)=f(n), \tag{1}$$

where f is an additive function, and

$$f_y(n)=\frac{f(n)-A(y)}{B(y)}, \tag{2}$$

where f is strongly additive and

$$A(y):=\sum_{p\leqq y}\frac{f(p)}{p}, \quad B^2(y):=\sum_{p\leqq y}\frac{f^2(p)}{p}. \tag{3}$$

If we put

$$F_{x,y}(z):=y^{-1}\sum_{\substack{x-y<n\leqq x\\ f(n)<z}}1, \tag{4}$$

then the question about the (weak) convergence of the distribution functions $F_{x,y}$ to a limit distribution F is equivalent to the question concerning the convergence of the characteristic functions

$$\varphi_{x,y}(t):=y^{-1}\sum_{x-y<n\leqq x}e^{itf_y(n)} \tag{5}$$

to the characteristic function $\varphi(t)$ of F as y tends to infinity. Then $\varphi(t)$ is continuous in a neighbourhood of $t=0$ and $\varphi(0)=1$. Therefore the problem we have posed is equivalent to the investigation of the y's for which the sum in (5) converges to a non-zero limit for all sufficiently small t.

Let $f_y=f$ be an additive function. We show that one may choose $y=x^{h(x)}$, where $h(x)=1-\varepsilon(x)$ with $\varepsilon(x)>0$ and $\varepsilon(x)=o(1)$ as $x\to\infty$, so that the limiting distribution of an arbitrary real-valued additive function $f(n)$ in the interval $x-y<n\leqq x$ exists if and only if the limiting distribution of f in the usual sense exists, i.e. if the conditions of Erdös—Wintner hold (Corollary 1). An example given by Babu [1], p. 102, shows that this result is best possible. For, if $\varepsilon>0$ is given, one can construct an additive function f which fulfils the Erdös—Wintner conditions but possesses no limiting distribution in the interval $x-x^{1-\varepsilon}<n\leqq x$. Assuming that $f(p^m)\to 0$ as $p\to\infty$ $(m=1, 2, \dots)$ we choose $h(x)=o(1)$ where $o(1)$ depends on f (Corollary 2). Cases in which $h(x)$ lies between these extremes are treated in Corollary 3.

One of the few contributions in this area is due to Babu [2]. He showed that under the assumption

$$(6)\qquad y^{-1}\sum_{\substack{p^m\leqq x\\ |f(p^m)|\geqq\varepsilon}} 1 = o(1)\ (y\to\infty)\quad\text{for each}\quad \varepsilon>0,$$

where

$$(7)\qquad x^{\alpha}<y\leqq x\quad\text{for a fixed}\quad \alpha>0,$$

the limiting distribution in the interval $x-y<n\leqq x$ exists if and only if the conditions of Erdős—Wintner hold.

If y is chosen as in Corollary 1 then the assertion holds without condition (6). For smaller y an obvious modification of our proof gives a sharper result (Corollary 4).

In the case $f_y=(f-A(y))/B(y)$ we restrict ourselves to strongly additive functions belonging to the Kubilius class H (see Kubilius [8], Chapter IV) and characterize the cases in which the limit distribution has variance 1 (Theorem 4). Again one may choose $y(x)=x^{1-\varepsilon(x)}$ with $\varepsilon(x)>0$, and $\lim_{x\to\infty}\varepsilon(x)=0$. Under further restrictions on f (e.g. Lindeberg condition or $\max_{p\leqq x}|f(p)|=o(B(x))$) we obtain stronger results. As a corollary we prove a result of Babu [3] concerning the limiting distribution of $\omega(n)=\sum_{p|n}1$.

For the proof of the results we investigate the asymptotic behaviour of $\varphi_{x,y}(t)$. The methods used are elementary (sieve methods) and most of them were developed in the author's paper [6] to characterize uniformly summable multiplicative functions. The results and proofs of this paper also give a characterization of multiplicative functions g of modulus of at most 1 whose mean-value in the interval $x-y<n\leqq x$ differs from zero (Corollary 4).

NOTATION. We say that a strongly additive function $f\colon \mathbf{N}\to\mathbf{R}$ belongs to the Kubilius class H if $B(x)\to\infty$ as $x\to\infty$ and if there exists an unbounded, increasing function $r=r(x)$ such that

$$\frac{\log r(x)}{\log x}\to 0\quad\text{and}\quad\frac{B(r(x))}{B(x)}\to 1.$$

2. Results. THEOREM 1. (i) *Let $f_y=f\colon \mathbf{N}\to\mathbf{R}$ be an additive function. If the distribution functions $F_{x,y}$ possess a limit distribution as $y=y(x)\to\infty$, then the series*

$$(8)\qquad \sum_{\substack{p\\ |f(p)|>1}}\frac{1}{p},\qquad \sum_{\substack{p\\ |f(p)|\leqq 1}}\frac{f^2(p)}{p},\qquad \sum_{\substack{p\\ |f(p)|\leqq 1}}\frac{f(p)}{p}$$

converge.

(ii) *Let f_y be defined by* (2) *and* (3), *where $f\in H$. Further, let $y=y(x)\geqq x^{\varepsilon}$ where $\varepsilon>0$. If the distributions $F_{x,y}$ possess a limit distribution as $y\to\infty$, then the distribution functions*

$$F_x(z):=x^{-1}\left|\left\{n\leqq x\colon \frac{f(n)-A(x)}{B(x)}<z\right\}\right|$$

possess a limiting distribution.

In the following theorem we assume that the additive function f: $\mathbf{N}\to\mathbf{R}$ satisfies

$$f(p^m)\to 0 \quad \text{as} \quad p\to\infty, \quad m\geqq 1.$$

This means that

(9) $$\delta(x):=\sup_{x\leqq p}|f(p^m)|\searrow 0 \quad \text{as} \quad x\to\infty.$$

Now we determine a positive function $\lambda^*(x)$, which is monotonic and tends to zero as x tends to infinity, by

(10) $$\lambda^*(x)=\max\left(\delta(x^{1/\log\log x}),\ \frac{1}{\log\log x}\right).$$

Then obviously

(11) $$\delta(x^{\lambda^*(x)})\leqq\lambda^*(x).$$

Further we put

(12) $$\lambda(x):=\varrho(x)\lambda^*(x), \quad \text{where} \quad \lambda(x)\searrow 0 \quad \text{and} \quad \varrho(x)\to\infty \quad (x\to\infty),$$

and choose

(13) $$h(x)>2\varrho_1(x)\lambda(x) \quad \text{where} \quad h(x)\to 0 \quad \text{and} \quad \varrho_1(x)\to\infty \quad (x\to\infty).$$

THEOREM 2. *Let* f: $\mathbf{N}\to\mathbf{R}$ *be additive,* $f(p^m)\to 0$ *as* $p\to\infty$, $m\in\mathbf{N}$, *and let the series*

(14) $$\sum_p\frac{f^2(p)}{p} \quad \text{and} \quad \sum_p\frac{f(p)}{p}$$

be convergent. If $y\geqq x^{h(x)}$, *where* $h(x)\geqq 2\varrho_1(x)\lambda(x)$ *is defined in* (12) *and* (13), *then, as* $y\to\infty$,

(15) $$y^{-1}\sum_{x-y<n\leqq x}e^{itf(n)}=\prod_p\left(1-\frac{1}{p}\right)\left(1+\sum_{m=1}^{\infty}p^{-m}e^{itf(p^m)}\right)+o(1).$$

In the general case let f: $\mathbf{N}\to\mathbf{R}$ be additive, and let the series (8) be convergent. Then there exists a sequence $\varepsilon(p)\searrow 0$ $(p\to\infty)$, such that

$$\sum_{\substack{p\\|f(p)|\leqq 1}}\frac{f^2(p)}{p}\cdot\frac{1}{\varepsilon(p)}<\infty.$$

Put

$$\mathcal{T}:=\{p^m\colon |f(p^m)|^2>\varepsilon(p)\}.$$

Then

(16) $$\sum_{p^m\in\mathcal{T}}p^{-m}<\infty$$

and $|f(p^m)|^2<\varepsilon(p)$ for $p^m\notin\mathcal{T}$. Now define an additive function f^* by

$$f^*(p^m)=\begin{cases}f(p^m) & \text{if } p^m\notin\mathcal{T}\\ 0 & \text{otherwise}\end{cases}$$

and a multiplicative function g_t by the Dirichlet convolution

(17) $$e^{itf(\cdot)} = g_t(\,.\,)*e^{itf^*(\cdot)}.$$

Obviously, by (16),

$$\sum_{n=1}^{\infty}\frac{|g_t(n)|}{n}<\infty.$$

If we determine $\delta(x)$ as above (see (9)) by $\delta(x):=\sup_{x\leqq p}|f^*(p^m)|$ (i.e. $\delta(p)\leqq\sqrt{\varepsilon(p)}$), and if we put

$$G_t\left(\frac{y}{m}\right):=\sum_{\frac{x-y}{m}<n\leqq\frac{x}{m}}|g_t(n)|,$$

then we prove the following

THEOREM 3. *Let f: $\mathbf{N}\to\mathbf{R}$ be additive, and let the series* (8) *be convergent. With the same notations as above let $y\geqq x^{h(x)}$ such that, for every fixed $t\in\mathbf{R}$,*

(18) $$\sum_{m\leqq\frac{x}{y}+1}G_t\left(\frac{y}{m}\right)=o(y)\quad(y\to\infty).$$

Then

(19) $$y^{-1}\sum_{x-y<n\leqq x}e^{itf(n)}=\prod_p\left(1-\frac{1}{p}\right)\left(1+\sum_{m=1}^{\infty}p^{-m}e^{itf(p^m)}\right)+o(1).$$

The proof of Theorem 3 also shows that for $y/m\geqq x^{\varepsilon}$ $(\varepsilon>0)$ the estimation

$$G_t\left(\frac{y}{m}\right)\ll\frac{y}{m}\exp\left(\sum_{p\leqq y/m}\frac{|g_t(p)|-1}{p}\right)\ll\frac{y}{m}\Big/\log\frac{y}{m}$$

holds. Putting $y\geqq x^{1-\delta^*(x)}$ with $\delta^*(x)>0$, $\lim_{x\to\infty}\delta^*(x)=0$, we obtain

$$\sum_{m<\frac{x}{y}+1}G_t\left(\frac{y}{m}\right)\ll y\frac{\log x/y}{\log y^2/x}=y\frac{\delta^*(x)}{1-\delta^*(x)}=o(y).$$

This leads to

COROLLARY 1. *Let $f_y=f$: $\mathbf{N}\to\mathbf{R}$ be additive, and let $y\geqq x^{1-\delta^*(x)}$ with $\delta^*(x)>0$ and $\lim_{x\to\infty}\delta^*(x)=0$. Then the distribution functions $F_{x,y}$ possess a limiting distribution if and only if the series* (8) *converge.*

COROLLARY 2. *Let $f_y=f$ and $y\geqq x^{h(x)}$, $\lim_{x\to\infty}h(x)=0$ as in Theorem* 2. *Then the distribution functions $F_{x,y}$ converge to a limit law if and only if the series* (8) *converge.*

For a given sequence $\varepsilon(p)\searrow 0$ $(p\to\infty)$ let

(20) $$\sum_{\substack{p^m\\ |f(p^m)|>\varepsilon(p)}}\frac{1}{p^m}<\infty.$$

Further, let, if q_t is defined as in (17),

$$G_t\left(\frac{y}{m}\right) \ll \left(\frac{y}{m}\right)^{1-\delta} \tag{21}$$

for some $\delta>0$. Then we have

COROLLARY 3. *Let $f_y=f$: $\mathbf{N}\to\mathbf{R}$ be additive, and assume that* (20) *and* (21) *hold for some sequence $\{\varepsilon(p)\}$ and some $\delta>0$. Let $y>\varrho(x)x^{1/2}$ where $\varrho(x)\to\infty$ as $x\to\infty$. Then the distribution functions $F_{x,y}$ tend to a limit law if and only if the series* (8) *converge.*

The proof of Corollary 3 follows immediately from Theorem 3 and the estimate

$$\sum_{m<\frac{x}{y}+1} G_t\left(\frac{y}{m}\right) \ll y^{1-\delta}\left(\frac{x}{y}\right)^{\delta} = y(\varrho(x))^{-2} = o(y).$$

Corollary 1 and the following result sharpen the cited result of Babu [2].

COROLLARY 4. *Let $f_y=f$: $\mathbf{N}\to\mathbf{R}$ be additive, and assume that*

$$\sum_{\substack{p^m\le x\\ |f(p^m)|\ge\varepsilon}} 1 = o(z) \quad (z\to\infty) \quad \text{for each} \quad \varepsilon>0 \tag{22}$$

where z satisfies

$$x^{1/\log\log x} \le z \le x^{1-\varepsilon(x)} \quad (\varepsilon(x)\to 0 \ \text{as} \ x\to\infty). \tag{23}$$

Then there exists a function $h(x)\searrow 0$ $(x\to\infty)$, such that for all $y\ge\max(z, x^{h(x)})$ the distribution functions $F_{x,y}$ possess a limiting distribution if and only if the series (8) *converge.*

Next we consider strongly additive functions f of the Kubilius class H. We show

THEOREM 4. *Let $f\in H$ be strongly additive, and let $y=y(x)$ as in Corollary* 1. *Further, let f_y be defined by* (2) *and* (3). *Then in order that the distribution functions $F_{x,y}$ converge to a limit law with variance* 1, *it is necessary and sufficient that there exists a nondecreasing function $K(u)$ of unit variation such that at all points at which $K(u)$ is continuous*

$$\frac{1}{B^2(x)} \sum_{\substack{p\le x\\ f(p)<uB(x)}} \frac{f^2(p)}{p} \to K(u)$$

as $x\to\infty$.

It is not difficult to characterize the limit law in Theorem 4 under further asssumptions, for example, if the analogue of the Lindeberg-condition holds. We restrict ourselves to the following special case.

COROLLARY 5. *Let f: $\mathbf{N}\to\mathbf{R}$ be strongly additive with the property*

$$B(x)\to\infty, \ \left(\max_{p\le x}|f(p)|\right)(B(x))^{-1} = \varepsilon(x)$$

as $x\to\infty$, *where* $\lim_{x\to\infty}\varepsilon(x)=0$. *Put* $h(x)=\varepsilon^2(x)\,\varrho(x)$ *with* $\varrho(x)\to\infty\ (x\to\infty)$, *and let* f_y *be defined by* (2) *and* (3). *Then, if* $y\geqq x^{h(x)}$, *the distribution functions* $F_{x,y}$ *converge to the normal limit law*

$$F(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-u^2/2}\,du$$

as $y\to\infty$.

REMARK. In the case $f(n)=\omega(n)=\sum_{p|n}1$, we choose $h(x)=(\log\log x)^{-1}\varrho(x)$ $(\varrho(x)\to\infty)$ and obtain a result of Babu [3]. For multiplicative functions our proofs give

COROLLARY 6. *Let* $g\colon \mathbf{N}\to\mathbf{C}$ *be multiplicative with* $|g|\leqq 1$. *Further, let* $y\geqq x^{h(x)}$, *where* $h(x)$ *is defined as in Corollary* 1. *Then the following assertions hold:*

(i) *If the series*

$$\sum_p \frac{g(p)-1}{p} \tag{24}$$

converge, then

$$\lim_{y\to\infty} y^{-1}\sum_{x-y<n\leqq x} g(n)=\prod_p\left(1-\frac{1}{p}\right)\left(1+\sum_{m=1}^{\infty}p^{-m}g(p^m)\right). \tag{25}$$

(ii) *If the limit in* (25) *exists and is different from zero, then the series* (24) *converges.*

For the proof of Corollary 6 one observes that the convergence of (24) implies the convergence of

$$\sum_p \frac{|\mathrm{Re}\,g(p)-1|}{p},$$

i.e. for "almost all p" $g(p)\to 1$ (cf. (16)).

3. Proof of Theorem 1. By laying the intervals

$$(x-y(y),x],\quad (x-y(y),\quad (x-y(x)-y(x-y(x)),x-y(x)],$$

etc. end to end, one sees at once that the asymptotic relation

$$\sum_{x-y(x)<n\leqq x} e^{itf(n)}=(1+o(1))y(x)\varphi(t)$$

ensures that the relation

$$\sum_{n\leqq x} e^{itf(n)}=(1+o(1))x\varphi(t)$$

holds. By the well-known theorem of Erdös—Wintner (see, for example, Kubilius [8], Theorem 4.5) the series (8) converge. This proves (i).[1] For the proof of (ii) one only has to show that

$$\sum_{x-y<n\leqq x}\left(e^{itf_y(n)}-e^{it(f(n)-A(x))/B(x)}\right)=o(y)$$

[1] The proof which we had originally for (i) was an easy consequence of a large sieve inequality. The above proof was then suggested by Professor Kátai.

holds for $y \geqq x^{\varepsilon}$. But this is an easy consequence of the facts that $f \in H$ and that f_y possesses a limiting distribution.

REMARK. It is not difficult to prove a sharper result than (ii).

4. Proofs of Theorems 2 and 3. For the proof we use the methods from Indlekofer [6]. We put

$$\{a_n\} = \{[x-y]+n: n \leqq y\}$$

and $a_n = b_n d_n$ with

$$p(b_n) \leqq x^{\lambda(x)}, \quad q(d_n) > x^{\lambda(x)}.$$

Here $p(m)$ and $q(m)$ denote the largest and the smallest prime divisor of m, respectively. We formulate the main steps of the proof as lemmata.

LEMMA 1. *Let* $x^{\lambda(x)} \leqq p_1 < p_2 < \ldots < p_1 \leqq x$ *(p_i prime) and* $\prod_{i=1}^{l} p_i^{\alpha_i} \leqq x$. *Let* $t \in \mathbf{R}$. *Then, if f satisfies* (9),

$$\prod_{i=1}^{l} (1+\Theta_i t f(p_i^{\alpha_i})) = 1 + O\left(\frac{t}{\varrho(x)}\right),$$

where $\Theta_i \in \mathbf{C}$ *and* $|\Theta_i| \leqq 1$ $(i=1, \ldots 1)$.

PROOF. It is obvious that, if x is large enough,

$$\prod_{i=1}^{l} (1+\Theta_i t f(p_i^{\alpha_i})) = \exp\left(\sum_{i=1}^{l} \log\left(1+\Theta_i t f(p_i^{\alpha_i})\right)\right) =$$

$$= \exp\left(\left|\sum_{i=1}^{l} \Theta_i t f(p_i^{\alpha_i})\right|\right) + O\left(\sum_{i=1}^{l} |t f(p_i^{\alpha_i}|^2)\right).$$

The last two sums are

$$\ll |t| \delta(x^{\lambda(x)})\, l \ll |t| \frac{\lambda^*(x)}{\lambda(x)},$$

because of

$$(x^{\lambda(x)})^l \leqq p_1 \ldots p_l < x \quad \text{and} \quad l \leqq \frac{\log 2x}{\lambda(x) \log x}.$$

Now, $\exp\left(-(\log x)(\log p)^{-1}\right)/(\log p)$ increases monotonically. Therefore we have

LEMMA 2. *For all x the estimate*

$$\sum_{p \leqq x} p^{-1} \exp\left(-\frac{\log x}{\log p}\right) \ll 1$$

LEMMA 3. *Let* $z = x^{p_1(x)\lambda(x)}$. *Then there exists a constant c such that*

$$\sum_{\substack{z<m \\ p(m) \leqq w}} \frac{1}{m} \ll \exp\left(\sum_{p \leqq w} \frac{1}{p} - cu \log u\right)$$

holds with $u = \dfrac{\log z}{\log w}$ *for* $\log x < w < x^{\lambda(x)}$.

PROOF. By an idea of Rankin [9] one shows, if $\varepsilon\in(0,1/3)$ (cf. Indlekofer [6], p. 268)

$$\sum_{\substack{z<m\\ p(m)\leqq w}} \frac{1}{m} < \sum_{\substack{z<m\\ p(m)\leqq w}} \left(\frac{m}{z}\right)^{\varepsilon} \frac{1}{m} \ll \exp\left(\sum_{p\leqq w} \frac{1}{p}+c_1 w^{\varepsilon}-\varepsilon \log z\right).$$

With a suitably chosen c we put

$$\varepsilon = \frac{c}{\log w} \log \frac{\log z}{\log w}$$

such that $0<\varepsilon<1/3$, and the assertion of Lemma 3 holds.

LEMMA 4. *Let* $y=x^{h(x)}$, $z=x^{\varrho_1(x)\lambda(x)}$, $\mu<z^{3/2}$ *and* $p<x^{\lambda(x)}$. *Then*

$$\sum_{\substack{n\leqq y\\ \mu|b_n\\ q\left(\frac{b_n}{\mu}\right)\geqq p}} 1 = \frac{y}{\mu} \prod_{q\leqq p} \left(1-\frac{1}{q}\right)\{1+O(\exp(-u(\log-\log\log 3u-2))+$$
$$+O(\exp(-\sqrt{\log y/\mu})\}$$

where $u=\dfrac{\log y/\mu}{\log p}$.

PROOF. This "fundamental lemma" follows from Theorem 2.5 of Halberstam—Richert [5]. We observe that

$$p \leqq x^{\lambda(x)} \leqq x^{\varrho_1(x)\lambda(x)/2} \leqq y/\mu.$$

PROOF OF THEOREM 2. By Lemma 1 we obtain

$$\sum_{n\leqq y} e^{itf(a_n)} = \sum_{n\leqq y} e^{itf(b_n)}\{1+o(1)\} = \Sigma_1+o(y).$$

We divide Σ_1 into two parts, with $z=x^{\varrho_1(x)\lambda(x)}$. In Σ_{12} we sum over divisors μ of b_n, where $z<\mu<z^{3/2}$ and $q(b_n/\mu)>p(\mu)$. Then we have

$$\Sigma_{12} \leqq \sum_{z\leqq\mu\leqq z^{3/2}} \sum_{\substack{n\leqq y\\ \mu|b_n,\, q\left(\frac{b_n}{\mu}\right)\geqq p(\mu)}} 1.$$

If we put $\mu=p\mu_1$, where $p=p(\mu)$, we obtain

$$\Sigma_{12} \leqq \sum_{p\leqq x^{\lambda(x)}} \sum_{\substack{\frac{z}{p}\leqq\mu_1\leqq\frac{z^{3/2}}{p}\\ p(\mu_1)\leqq p}} \sum_{\substack{n\leqq y\\ \mu_1 p|b_n\\ q\left(\frac{b_n}{\mu_1 p}\right)\geqq p}} 1$$

and, by Lemma 4,

$$\Sigma_{12} \ll y\Big\{\sum_{p\leqq\log x} + \sum_{\log x<p\leqq x^{\lambda(x)}}\Big\} \sum_{\substack{\frac{z}{p}\leqq\mu_1\leqq\frac{z^{3/2}}{p}\\ p(\mu_1)\leqq p}} \frac{1}{\mu_1} = y(\Sigma_{12}^1+\Sigma_{12}^2).$$

Concerning the first sum we use a method of Rankin (cf. Lemma 3) and get

$$\Sigma_{12}^1 \ll z^{-1} \frac{\log x}{(\log\log x)^2} \sum_{\substack{\mu \leqq x \\ p(\mu) \leqq \log x}} 1 \ll$$

$$\ll \exp\left(-\varrho_1(x)\lambda(x)\log x + \log\log x - 2\log\log\log x + c\frac{\log x}{\log\log x}\right) \ll$$

$$\ll \exp\left(\frac{\log x}{\log\log x}(c' - \varrho_1(x))\right) = o(1)$$

as $x \to \infty$. For the second sum we use Lemma 2 and Lemma 3. Then

$$\Sigma_{12}^2 \ll \sum_{\log < x \leqq p \leqq x^{\lambda(x)}} p^{-1} \exp\left(-c' \frac{\log zp^{-1}}{\log p} \log \frac{\log zp^{-1}}{\log p}\right) \ll$$

$$\ll \sum_{\log x < p \leqq x^{\lambda(x)}} p^{-1} \exp\left(-c' \frac{(\varrho_1(x)-1)\log x^{\lambda(x)}}{\log p} \log(\varrho_1(x)-1)\right) \ll$$

$$\ll \sum_{\log x < p \leqq x^{\lambda(x)}} p^{-1} \exp\left(-\frac{\log x^{\lambda(x)}}{\log p}\right) \cdot \exp\left(-\frac{\log x^{\lambda(x)}}{\log p} \cdot \varrho_1(x)\right) \ll$$

$$\ll \exp(-\varrho_1(x)) = o(1) \quad \text{as} \quad y \to \infty.$$

Next we prove an upper estimate for Σ_{11}. We have

$$\Sigma_{11} = \sum_{\substack{b \leqq z \\ p(b) \leqq x^{\lambda(x)}}} e^{itf(b)} \sum_{\substack{n \leqq y \\ b | a_n,\, q\left(\frac{a_n}{b}\right) > x^{\lambda(x)}}} 1.$$

Because of $h(x) > 2\varrho_1(x)\lambda(x)$ Lemma 4 implies

$$\Sigma_{11} = y \prod_{p \leqq x^{\lambda(x)}} \left(1 - \frac{1}{p}\right) \sum_{\substack{b \leqq z \\ p(b) \leqq x^{\lambda(x)}}} \frac{e^{itf(b)}}{b} \{1 + o(1)\}.$$

For the innermost sum on the right side we obtain, using Lemma 3,

$$\sum_{\substack{b \leqq z \\ p(b) \leqq x^{\lambda(x)}}} \frac{e^{itf(b)}}{b} = \Big(\sum_{\substack{b \\ p(b) \leqq x^{\lambda(x)}}} - \sum_{\substack{z < b \\ p(b) \leqq x^{\lambda(x)}}} \Big) \frac{e^{itf(b)}}{b} =$$

$$= \prod_{p \leqq x^{\lambda(x)}} \Big(1 + \sum_{m=1}^{\infty} p^{-m} e^{itf(p^m)}\Big) + O\left(\exp\left\{\sum_{q \leqq x^{\lambda(x)}} \frac{1}{q} - c\varrho_1(x)\log\varrho_1(x)\right\}\right).$$

Hence, since $\Sigma_1 = \Sigma_{11} + \Sigma_{12} = \Sigma_{11} + O\big(y(\Sigma_{12}^1 + \Sigma_{12}^2)\big) = \Sigma_{11} + o(y)$, we have

$$\varphi_{x,y}(t) = \prod_{p \leqq x^{\lambda(x)}} \left(1 - \frac{1}{p}\right) \Big(1 + \sum_{m=1}^{\infty} p^{-m} e^{itf(p^m)}\Big) + o(1). \tag{26}$$

Now, the product in (26) converges because of the convergence of the series (14), and Theorem 2 is proved.

PROOF OF THEOREM 3. We use the decomposition (17). Then

$$\varphi_{x,y}(t) = y^{-1} \sum_{x-y<nm\leqq x} g_t(n)e^{itf^*(m)} = y^{-1}\Big(\sum_{n\leqq y} g_t(n) \sum_{\frac{x-y}{n}<m\leqq\frac{x}{n}} e^{itf^*(m)} +$$

$$+ \sum_{m<\frac{x}{y}+1} e^{itf^*(m)} \sum_{\frac{x-y}{m}<n\leqq\frac{x}{m}} g_t(n) = y^{-1}(\Sigma_1+\Sigma_2).$$

In view of the assumption of the theorem we have $\Sigma_2=o(y)$. Concerning Σ_1 we use for $n<K$ the result of Theorem 2 and, for $n>K$, the trivial esetimate

$$\sum_{\frac{x-y}{n}<m\leqq\frac{x}{n}} e^{itf^*(m)} \leqq \frac{y}{n}.$$

Then

$$y^{-1}\Sigma_1 = \prod_{p\leqq x^{\lambda(x)}}\left(1-\frac{1}{p}\right)\Big(1+\sum_{m=1}^{\infty} p^{-m}e^{itf^*(p^m)}\Big) \sum_{n\leqq K}\frac{g_t(n)}{n} + O\Big(\sum_{n\leqq K}\frac{|g_t(n)|}{n}\Big)+o(1).$$

Letting $K\to\infty$ we obtain

$$y^{-1}\Sigma_1 = \prod_{p\leqq x^{\lambda(x)}}\left(1-\frac{1}{p}\right)\Big(1+\sum_{m=1}^{\infty} p^{-m}e^{itf(p^m)}\Big)+o(1),$$

which implies the assertion of Theorem 3 because of the convergence of the series (8).

5. Proof of Corollary 4. We assume that $x^{(\log\log x)^{-1}}<y\leqq x^{1-\varepsilon(x)}$ holds with $\lim_{x\to\infty}\varepsilon(x)=0$. By the assumption (22) there exists a function $\delta(x)\searrow 0$ such that

$$\sum_{\substack{p^m\leqq x\\ |f(p^m)|\geqq\delta(x)}} 1 = o(y) \quad (y\to\infty). \tag{27}$$

Let $\mathscr{P}^*$ be the set of all prime powers which are counted in (27) $(y\to\infty)$. Then, by partial summation,

$$\sum_{p^m\in\mathscr{P}^*}\frac{1}{p^m}<\infty. \tag{28}$$

If we use the above mentioned function $\delta(x)$ in the proof of Theorem 2, and if we substitute the values $f(p^m)$ with $p^m\in\mathscr{P}^*$ and $x^{\lambda(x)}<p\leqq x$ by 0, we make an error of at most

$$< \sum_{\substack{x^{\lambda(x)}<p^m\leqq x\\ p^m\in\mathscr{P}^*}} \sum_{\substack{x-y<n\leqq x\\ n\equiv 0\,(p^m)}} 1 \leqq y \sum_{\substack{x^{\lambda(x)}<p^m\leqq y\\ p^m\in\mathscr{P}^*}} p^{-m} + \sum_{\substack{x^{\lambda(x)}<p^m\leqq x\\ p^m\in\mathscr{P}^*}} 1 = o(y).$$

This proves Corollary 5.

6. Proof of Theorem 4. The proof is about the same as in § 3. We only point out the necessary modifications, the rest of the proof is left to the reader. Further, we use Theorem 1, (ii), and Theorem 4.1 in Kubilius's book [8]. Put

$$g(n) = e^{itf(n)/B(y)}$$

and define a strongly multiplicative function g^* by

$$g^*(p) = \begin{cases} 1 & \text{if } x^{\varepsilon(x)} < p \leqq x \\ g(p) & \text{if } p \leqq x^{\varepsilon(x)}, \end{cases}$$

where $\varepsilon(x) = o(1)$ as $x \to \infty$ will be chosen later. Let $y_1 := x^{\varepsilon(x)}$. By the same methods as above we get

$$\text{(29)} \quad y^{-1} \sum_{x-y<n\leqq x} g^*(n) e^{-itA(y_1)/B(y_1)} = \prod_{p \leqq y_1} e^{-itf(p)/(pB(y_1))} \left(1 + \frac{g(p)-1}{p}\right) + o(1).$$

On the other hand

$$\text{(30)} \quad \Big| \sum_{x-y<n\leqq x} \left(e^{itf_y(n)} - g^*(n) e^{-itA(y_1/B(y_1))}\right) \Big| \leqq$$

$$\leqq \sum_{x-y<n\leqq x} \Big| \prod_{\substack{p|n \\ y_1<p\leqq x}} g(p) - e^{it(A(y)/B(y) - A(y_1)/B(y_1))} \Big|.$$

If $f \in H$ and if y and y_1 are larger than $r(x)$, we have $A(y)/B(y) - A(y_1)/B(y_1) = o(1)$ (cf. Kubilius [8], Chapter IV). Thus the right side of (30) is at most

$$\sum_{x-y<n\leqq x} \Big| \prod_{\substack{p|n \\ y_1<p\leqq x}} g(p) - 1 \Big| + o(y) <$$

$$< \sum_{\substack{x-y<n\leqq x \\ p(n)\leqq y}} \Big| \prod_{\substack{p|n \\ y_1<p}} g(p) - 1 \Big| + 2 \sum_{\substack{x-y<n\leqq x \\ p|n \text{ for some } p>y}} 1 + o(y) =: \Sigma_1 + \Sigma_2 + o(y).$$

For the first sum we use the estimate $|e^{i\alpha} - 1| \leqq |\alpha| \, (\alpha \in \mathbf{R})$ and obtain by the Cauchy—Schwarz-inequality

$$\Sigma_1 \leqq |t| \sum_{\substack{x-y<n\leqq x \\ p(n)\leqq y}} \sum_{\substack{p|n \\ y_1<p}} \frac{|f(p)|}{B(y)} \ll |t| y \sum_{y_1<p\leqq y} \frac{|f(p)|}{p} \frac{1}{B(y)} \leqq$$

$$\leqq |t| y \left(\frac{B^2(y) - B^2(y_1)}{B^2(y)}\right)^{1/2} \left(\log \frac{\log y}{\log y_1}\right)^{1/2}.$$

Now, $B^2(y_1) B^{-2}(y) = 1 + o(1)$, and therefore, by a suitable choice of $\varepsilon(x)$, we obtain

$$\text{(31)} \qquad \Sigma_1 = o(y).$$

For the estimate of Σ_2 we observe that, if $y > x^{2/3}$, each n, which is counted in Σ_2, has exactly one prime divisor $p > y$. Thus

$$\text{(32)} \quad \Sigma_2 \leqq \sum_{i\leqq x/y} \sum_{\frac{x-y}{i} < p \leqq \frac{x}{i}} 1 \ll \frac{y}{\log y} \sum_{i \leqq x/y} i^{-1} \ll \frac{y}{\log y} (\log x - \log y) = o(y)$$

if $\log x - \log y = o(\log x)$.

By the mentioned result of Kubilius and by Theorem 1, (ii), we only have to show that the condition in Theorem 4 is sufficient. Now, this condition implies (see [6], pp. 70—71), that the limit in (29) exists as $y \to \infty$ and determines the limiting distribution. Thus, by (31) and (32), the theorem is proved.

NOTE. The author gave a talk on the results of this paper during the Oberwolfach-conference on "Elementare und analytische Zahlentheorie" (September 21—September 27, 1986), and there he was informed by A. Hildebrand that a different proof of Corollary 1 is given in a forthcoming paper of Hildebrand on multiplicative functions.

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ON SOME ULTRAPARACOMPACT SPACES

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The following definitions are necessary in this paper:

a) A topological space X is called zero-dimensional if X has a base consisting of clopen sets ([6]).

b) A topological space X is called perfectly zero-dimensional if X is zero-dimensional and every open cover of X has a refinement consisting of open pairwise disjoint sets ([6]).

c) A topological space X is called ultraparacompact if X is T_2 and every open cover of X is refined by some locally finite clopen cover ([2]).

It is known that a T_1 or regular space is perfectly zero-dimensional if and only if every open cover of the space is refined by some discrete open cover ([6]), then a T_2 space is perfectly zero-dimensional if and only if it is ultraparacompact ([2]).

The Sorgenfrey line S (the set of real numbers topologized by the base $\mathcal{B}=\{[a, b)|a, b\in \mathbf{R}, a<b\}$) is ultraparacompact. Moreover every base for S has a discrete subfamily covering S (see 1.1 in [1]). Nevertheless, there exists bases for S without discrete subcovers, for example, the base $\mathcal{B}^*=\left\{[a, b)\setminus\left\{\frac{a+b}{2}\right\}\Big|[a, b)\in\mathcal{B}\right\}$. This motivates the study of some classes of ultraparacompact spaces.

DEFINITION 1. We will say that a topological space X verifies property (P_0) if X is T_2 and every base for X has a discrete subfamily covering X.

REMARKS. 1. A T_2 space X verifies property (P_0) if and only if given any base $\mathcal{B}$ for X, every open cover of X has a discrete subfamily of $\mathcal{B}$ refining it and covering X.

2. If X verifies property (P_0) then X is ultraparacompact. The Sorgenfrey line is ultraparacompact and does not verify property (P_0).

3. There exist spaces which are not discrete and verify property (P_0). In fact, let $A(\mathfrak{m})$ be the space (X, T) where X is a set of cardinal $\mathfrak{m}\geqq\aleph_0$, x_0 a point in X and T the family consisting of all subsets of X that do not contain x_0 and of all subsets of X that have finite complement.

4. If a space X verifies property (P_0) then X is totally paracompact. (A topological space is said to be totally paracompact [4] if every base contains a locally finite covering.)

* The results in this paper are contained in the author's Doctoral Dissertation, written under the direction of Professor E. Outerelo.

5. If a space X is T_2, totally paracompact and ultraparacompact, in general, X does not verify property (P_0). In fact let $\mathbf{Q}$ be the space of all rational numbers; the base $\mathscr{B}=\{(a, b)\cap\mathbf{Q}|a, b\in\mathbf{Q}, a<b\}$ of $\mathbf{Q}$ does not have discrete subcovers.

Definition 2. We will say that a topological space X verifies property (P_1) if X is T_2, zero-dimensional and every base $\mathscr{B}$ for X such that for each $B\in\mathscr{B}$ $\bar{B}$ is open, has a discrete subfamily covering X.

Remarks. 1. Clearly all spaces that verify property (P_0) also verify property (P_1), and all spaces that verify property (P_1) are ultraparacompact.

2. A T_2 space X verifies property (P_1) if and only if it is zero-dimensional and given any open cover $\mathscr{U}$ of X, every base $\mathscr{B}$ for X such that for each $B\in\mathscr{B}$ $\bar{B}$ is open, has a discrete subfamily refining $\mathscr{U}$ and covering X.

3. If a space X verifies property (P_1), in general, X is not extremally disconnected. In fact, in $A(\mathfrak{m})$ let U be an open set which does not contain x_0 and has infinite complement; then $\bar{U}=U\cup\{x_0\}$.

Proposition 1. *If a space X verifies property* (P_1), *then for every base $\mathscr{B}$ of X such that for each $B\in\mathscr{B}$, $\bar{B}$ is open, there exists a base $\mathscr{B}'$ of X, contained in $\mathscr{B}$, such that for each $B'\in\mathscr{B}'$ is $\bar{B}'=B'$.*

Proof. Let $\mathscr{B}$ be a base of X such that for each $B\in\mathscr{B}$ $\bar{B}$ is open. Then, since X is regular, $\mathscr{B}^*=\{\bar{B}|B\in\mathscr{B}\}$ is a base of X. Let $\mathscr{B}'=\mathscr{B}\cap\mathscr{B}^*=\{B\in\mathscr{B}|\bar{B}=B\}$. $\mathscr{B}'$ is a base of X, in fact: let $B\in\mathscr{B}$ and $x_0\in B$ then there is $B_1\in\mathscr{B}$ such that $x_0\in B_1\subset\bar{B}_1\subset B$. $\mathscr{U}=\{B\}\cup\{X\setminus\bar{B}_1\}$ covers X, then there exists $\mathscr{V}\subset\mathscr{B}$ which refines $\mathscr{U}$ and is discrete (then $\mathscr{V}\subset\mathscr{B}'$). Hence there exists $B'\in\mathscr{V}$ such that $x_0\in B'=\bar{B}'$. Since $\mathscr{V}$ refines $\mathscr{U}$, we have that $B'\subset B$ or $B'\subset X\setminus\bar{B}_1$, and since $x_0\in B'$, there is $x_0\in B'\subset B$ and $B'\in\mathscr{B}'$.

Theorem 1. *If a space X verifies property* (P_1) *then X is locally compact or X has infinitely many isolated points.*

Proof. Since X is zero-dimensional, there is a base $\mathscr{B}$ of X consisting of clopen sets.

Let $B\in\mathscr{B}$, then, since B is closed in X, we have that B is compact or is not countably compact (because X is paracompact).

Let $B\in\mathscr{B}$ not countably compact then there exists a sequence $S=(x_n)_{n\in\mathbf{N}}$ in B without cluster points in X (because B is closed). Let C_S be the set of all cluster points of S.

Let $\mathscr{B}_1^*=\{B_S^*=B\setminus S|B\in\mathscr{B}$ is not compact and there exists a sequence $S\subsetneqq B$ with $C_S=\emptyset\}$, $\mathscr{B}_0^*=\{B\in\mathscr{B}|B$ is compact$\}$ and $\mathscr{B}^*=\mathscr{B}_0^*\cup\mathscr{B}_1^*$.

We will prove that $\mathscr{B}^*$ is a base of X.

$\mathscr{B}^*$ is a family of open sets because, if S does not have cluster points, then S is closed in B, and B_S^* is open in X.

Let A be an open set of X and let $x\in A$. Then there exists $B\in\mathscr{B}$ such that B is compact and $x\in B\subset A$ (and so $B\in\mathscr{B}_0^*\subset\mathscr{B}^*$) or B is not compact for every $B\in\mathscr{B}$ such that $x\in B\subset A$. In this case, there exists $B\in\mathscr{B}$ and $S=(x_n)_{n\in\mathbf{N}}\subset B$ such that $C_S=\emptyset$, $\{x_n|n\in\mathbf{N}\}\neq B$ and $x\in B\subset A$. If $x\in\{x_n|n\in\mathbf{N}\}$ then there exists $n_x=\max\{n|x_n=x\}$ (because $x\notin C_S$). Let $y_m=x_{m+n_x}$, then $S'=(y_n)_{n\in\mathbf{N}}$ is a sequence in B, $C_{S'}=\emptyset$, $x\notin\{y_n|n\in\mathbf{N}\}$ and $x\in B_{S'}^*\subset B$ (where $B_{S'}^*\subset\mathscr{B}_1^*\subset\mathscr{B}^*$).

If $x\notin\{x_n|n\in\mathbf{N}\}$ then $x\in B_S^*\subset B$ (and $B_S^*\in\mathscr{B}_1^*\subset\mathscr{B}^*$).

Thus, $\mathscr{B}^*$ is a base of X.

For every $B^* \in \mathscr{B}^*$, $\overline{B^*}$ is open in X:

If $B^* \in \mathscr{B}_0^*$ then $B^* = B \in \mathscr{B}$ and B is compact. Then $\overline{B^*} = \bar{B} = B$ is open.

If $B^* \in \mathscr{B}_1^*$ then $B^* = B_S^*$, where $B \in \mathscr{B}$, B is not compact, $S \subset B$ and $C_S = \emptyset$. We have that S is closed in B (and also in X). Let $S_1 = \{x_k \in S | x_k$ is an isolated point of $B\}$ and $S_2 = \{x_k \in S | x_k$ is an accumulation point of $B\}$; clearly $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$. Then $\overline{B \setminus S} = B \setminus S_1$; in fact, if $x \in \overline{B \setminus S}$ then $x \in \bar{B} = B$; if x is in S_1, then $\{x\}$ is open in X and $x \notin \overline{B \setminus S}$, thus $x \in B \setminus S_1$. Conversely $B \setminus S_1 = (B \setminus S) \cup S_2$ and $B \setminus S \subset \overline{B \setminus S}$; let $x \in S_2$ then, for each open neighbourhood U^X of x, $(U^X \setminus \{x\}) \cap$ $\cap B \neq \emptyset$; since S is closed in X, $U^X \setminus (S \setminus \{x\})$ is an open neighbourhood of x in X and we have that $(U^X \setminus S) \cap B \neq \emptyset$ and $U^X \cap (B \setminus S) \neq \emptyset$; then $x \in \overline{B \setminus S}$ and $S_2 \subset \overline{B \setminus S}$. Thus $\overline{B \setminus S} = B \setminus S_1$. Since S_1 is closed in B then $\overline{B \setminus S} = \overline{B^*}$ is open in B (and also in X).

By the hypothesis X verifies property (P_1), then there exists a discrete subfamily $\mathscr{V}^*$ of $\mathscr{B}^*$ which covers X. If for each $B^* \in \mathscr{V}^*$, $B^* \in \mathscr{B}_0^*$, then X is locally compact. If there exists $B_S^* \in \mathscr{V}^*$ then for every $x_k \in S$, $x_k \in B$, hence there is $B^* \in \mathscr{V}^*$ such that $x_k \in B^*$. Since $\mathscr{V}^*$ is discrete, we have that $B^* \cap B_S^* = \emptyset$ and $x_k \in B^* \cap B \subset S$. Thus x_k is an isolated point of X (for every $k \in \mathbf{N}$).

Remark. The Sorgenfrey line, the space of rational numbers and the space of irrational numbers are ultraparacompact spaces and do not verify property (P_1).

Proposition 2. (a) *Every closed subspace of a space which verifies property* (P_0), *also verifies property* (P_0).

(b) *All spaces which verify property* (P_0) *are C-scattered.*

Proof. (a) Let X be a space which verifies property (P_0) and let F be a closed of X. Let $\mathscr{B}$ be a base for F. Then $\mathscr{B}^* = \{G | G$ is open in X and $G \cap F = \emptyset$ or $G \cap F \in \mathscr{B}\}$ is a base for X, and let $\mathscr{V}^*$ be a discrete subcovering of $\mathscr{B}^*$. Clearly, $\mathscr{V} = \{B \cap G | B \in \mathscr{V}^*$, $B \cap F \neq \emptyset\}$ is a discrete subcovering of $\mathscr{B}$. Thus F verifies property (P_0).

(b) Let X be a space which verifies property (P_0). From (a) it follows that all closed subspaces of X verify property (P_1) and, by Theorem 1, every closed subspace of X is locally compact or has infinite many isolated points. Then X is C-scattered.

Theorem 2. *For each cardinal number* $\mathfrak{m} \geqq \aleph_0$ *there exists a base* $\mathscr{B}_0$ *of the Cantor cube of weigth* $\mathfrak{m}$, $D^{\mathfrak{m}}$, *such that for every* $B \in \mathscr{B}_0$ *we have* $\bar{B} = B$; *for all* $B_1, B_2 \in \mathscr{B}_0$ *such that* $B_1 \setminus B_2 \neq \emptyset$ *we have that* $B_1 \setminus B_2$ *is the union of a discrete family of members of* $\mathscr{B}_0$, *and for every open cover* $\mathscr{U}$ *of* $D^{\mathfrak{m}}$ *there exists a discrete refinement consisting of members of* $\mathscr{B}_0$.

Proof. Let J be a set of cardinality $\mathfrak{m}$ and D the two-point discrete space.

We define $\mathscr{B}_0 = \{\prod_{j \in J} A_j | A_j \subset D$ for all $j \in J$, $A_j = D$ for each $j \in J \setminus F$, where $F \subset J$ is finite$\}$. $\mathscr{B}_0$ is a base of $D^{\mathfrak{m}}$.

For every $\prod_{j \in J} A_j$ of $\mathscr{B}_0$ we have $\overline{\prod_{j \in J} A_j} = \prod_{j \in J} A_j$.

1. For all $\prod_{j \in J} A_j$, $\prod_{j \in J} A'_j \in \mathscr{B}_0$ such that $\prod_{j \in J} A_j \setminus \prod_{j \in J} A'_j \neq \emptyset$, where $A_j = D$ for each $j \in J \setminus F$, F is a finite set, $A'_j = D$ for each $j \in J \setminus H$ and H is a finite set, we have

that $\prod_{j\in J} A_j \setminus \prod_{j\in J} A'_j$ is the union of a discrete family of members of $\mathscr{B}_0$. In fact, if $x=(x_j)_{j\in J}\in \prod_{j\in J} A_j\setminus \prod_{j\in J} A'_j$ then there exists $k\in H$ such that $x_k\notin A'_k$. Let for each $j\in J$,

$$C_j^k = \begin{cases} A_j, & \text{if } j\neq k \\ A_j\setminus A'_j, & \text{if } j=k \end{cases}$$

thus $\prod_{j\in J} A_j\setminus \prod_{j\in J} A'_j = \bigcup_{k\in H}(\prod_{j\in J} C_j^k)$.

Let $P=\bigcup_{k\in H}(\prod_{j\in J} C_j^k)$; clearly P is finite.

For every $p=(p_j)_{j\in H}\in P$, let

$$M_j^p = \begin{cases} A_j, & \text{if } j\notin H \\ \{p_j\}, & \text{if } j\in H. \end{cases}$$

We have

$$\bigcup_{k\in H}(\prod_{j\in J} C_j^k) = \bigcup_{p\in P}(\prod_{j\in J} M_j^p),$$

because, if $x=(x_j)_{j\in J}\in \prod_{j\in J} C_j^k$ for some $k\in H$, then $x_j\in C_j^k$ for all $j\in J$ and $x_j\in A_j$ for all $j\notin H$ (for some $k\in H$). Therefore $(x_j)_{j\in H}\in \prod_{j\in H} C_j^k\subset P$ and $x_j\in A_j$ for each $j\notin H$, thus there exists $p\in P$ such that $p=(p_j)_{j\in H}=(x_j)_{j\in H}$ and $x_j\in A_j$ for each $j\notin H$ and finally $x\in \prod_{j\in J} M_j^p$ for some $p\in P$.

Conversely, if $x=(x_j)_{j\in J}\in \prod_{j\in J} M_j^p$ for some $p=(p_j)_{j\in H}\in P$, then there exists $k\in H$ such that $p_j\in C_j^k$ for each $j\in H$, and so there exists $k\in H$ such that $p_j\in A_j$ for each $j\in H$ and $\{p_k\}=A_k\setminus A'_k$. Since $x_j\in M_j^p$ for each $j\in J$, we have that $x_j\in A_j$ for each $j\notin H$ and $\{x_j\}=\{p_j\}$ for each $j\in H$, thus there exists $k\in H$ such that $\{x_k\}=$ $=\{p_k\}=A_k\setminus A'_k$ and $x_j\in A_j$ for each $j\in J$, hence there exists $k\in H$ such that $x\in \prod_{j\in J} C_j^k$.

Therefore we have

$$\prod_{j\in J} A_j\setminus \prod_{j\in J} A'_j = \bigcup_{p\in P}(\prod_{j\in J} M_j^p.)$$

Now $\{\prod_{j\in J} M_j^p\}_{p\notin P}$ is a discrete subfamily of $\mathscr{B}_0$. In fact, if p, $p'\in P$ are distinct, there exists $j_0\in H$ such that $p_{j0}\neq p'_{j0}$, and so $M_{j0}^p\cap M_{j0}^{p'}=\emptyset$ and $(\prod_{j\in J} M_j^p)\cap(\prod_{j\in J} M_j^{p'})=$ $=\emptyset$; thus, the family $\{\prod_{j\in J} M_j^p\}_{p\in J}$ consists of clopen pairwise disjoint sets. Since $\prod_{j\in J} A_j\setminus \prod_{j\in J} A'_j$ is clopen, we have that $\{\prod_{j\in J} M_j^p\}_{p\in P}$ is discrete.

2. For any finite subfamily $B_1, \ldots, B_n\in\mathscr{B}_0$ (where $n\geqq 2$) such that $B_1\setminus\bigcup_{i=2}^n B_i\neq\emptyset$ we have that $B_1\setminus\bigcup_{i=2}^n B_i$ is the union of a discrete family of members of $\mathscr{B}_0$.

In fact, we proceed by induction: if $n=2$ the assertion is proved above. Supposing the assertion is true for $n-1$ members of $\mathcal{B}_0$, then

$$B_1 \setminus \bigcup_{i=2}^{n} B_i = (B_1 \setminus \bigcup_{i=2}^{n-1} B_i) \setminus B_n = (\bigcup_{d \in D} B_d) \setminus B_n = \bigcup_{d \in D} (B_d \setminus B_n) = \bigcup_{d \in D} (\bigcup_{a \in A_d} B_a)$$

where $\{B_d\}_{d \in D}$ and $\{B_a\}_{a \in A_d}$ (for each $d \in D\}$ are discrete subfamilies of $\mathcal{B}_0$, thus $\{B_a | a \in A_d,\ d \in D\}$ is a discrete subfamily of $\mathcal{B}_0$.

Finally, let $\mathcal{U}$ be an open covering of $D^{\mathfrak{m}}$. Since $\mathcal{B}_0$ is a base, there exists a finite refinement $\{B_1, \ldots, B_n\} \subset \mathcal{B}_0$ of $\mathcal{U}$. Then

$$D^{\mathfrak{m}} = \bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} (B_i \setminus \bigcup_{j>i} B_j)$$

and from paragraph 1 it follows that there exists a discrete refinement of $\mathcal{U}$ consisting of members of $\mathcal{B}_0$.

REMARK. For every cardinal number $\mathfrak{m} \geqq \aleph_0$, the Cantor cube of weight $\mathfrak{m}$ does not verify property (P_0):

Let J be a set of cardinality $\mathfrak{m}$, let $\mathcal{B}_0$ be the base of $D^{\mathfrak{m}}$ defined in Theorem 2 and let

$$\mathcal{B} = \{\prod_{j \in J} A_j \setminus \{x\} | x \in \prod_{j \in J} A_j,\ \prod_{j \in J} A_j \in \mathcal{B}_0\}.$$

Clearly, $\mathcal{B}$ is a base for $D^{\mathfrak{m}}$ and does not have discrete subcovers.

This motivates introduction of a new class of ultraparacompact spaces.

DEFINITION 3. If X is a zero-dimensional T_2 space, we will say that X verifies property (P_2) if for every open cover $\mathcal{U}$ of X and for every base $\mathcal{B}$ of X such that

(i) for each $B \in \mathcal{B}$ we have $\bar{B} = B$,

(ii) for all $B_1, B_2 \in \mathcal{B}$ such that $B_1 \setminus B_2 \neq \emptyset$ we have that $B_1 \setminus B_2$ is the union of a discrete family of members of $\mathcal{B}$;

there exists a discrete refinement $\mathcal{V}$ of $\mathcal{U}$, consisting Mf members of $\mathcal{B}$.

REMARK. If X is a zero-dimensional T_1 space, there exists a base verifying assertions (i) and (ii).

PROOF. 1. If $w(X)$ is finite there exists a finite base of X; since X is T_1 it follows that X is finite and discrete. The base consisting of all subsets of X verifies assertions i) and ii).

2. If $w(X) = \mathfrak{m} \geqq \aleph_0$ then X is embeddable in $D^{\mathfrak{m}}$. Let $\mathcal{B}_0$ be the base for $D^{\mathfrak{m}}$ defined in Theorem 2. $\mathcal{B}_0$ verifies assertions i) and ii). Let $\mathcal{B}_1 = \{B \cap X | B \in \mathcal{B}_0, B \cap X \neq \emptyset\}$, Clearly $\mathcal{B}_1$ verifies assertions i) and ii).

PROPOSITION 3. *Let X be a C-scattered and ultraparacompact space, then X verifies property* (P_2).

PROOF. The proof is completely analogous to the proof of Theorem 3.1 in [7].

PROPOSITION 4. *Let X be a zero-dimensional T_2 second-countable space, then X verifies property* (P_2).

PROOF. From the above remark it follows that there exists a countable base $\mathcal{B}_1$ verifing assertions (i) and (ii). Let $\mathcal{U}$ be an open cover of X. Let $\{B\in\mathcal{B}_1 | B\subset U$ for some $U\in\mathcal{U}\}=\{B_n\}_{n\in\mathbf{N}}$. Then $X=\bigcup_{n\in\mathbf{N}} B_n=\bigcup_{n\in\mathbf{N}}(B_n\setminus\bigcup_{i<n} B_i)$. For every $n\in\mathbf{N}$, $B_n\setminus\bigcup_{i<n} B_i$ is the union of a discrete family of members of $\mathcal{B}_1$. Thus $\bigcup_{n\in\mathbf{N}}(B_n\setminus\bigcup_{i<n} B_i)$ is the union of a discrete family $\mathcal{V}\subset\mathcal{B}_1$ (because B_n is clopen for each $n\in\mathbf{N}$). Finally, from the definition of $\{B_n\}_{n\in\mathbf{N}}$ it follows that $\mathcal{V}$ refines $\mathcal{U}$.

PROBLEMS. 1. If X is a space which verifies property (P_1) and F is a closed subspace of X, does F verify property (P_1)?

2. Is every space which verifies property (P_1) C-scattered?

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ON THE CLASSES OF ALMOST HERMITIAN STRUCTURES ON THE TANGENT BUNDLE OF AN ALMOST CONTACT METRIC MANIFOLD

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§ 1. Introduction

If (M, φ, ξ, η) is an almost contact manifold, its tangent bundle $T(M)$ is an almost Hermitian manifold with the Sasaki metric g^D and the almost complex structure $\tilde{J}$ defined by

$$\tilde{J} = \varphi^H + \eta^V \otimes \xi^V - \eta^H \otimes \xi^H.$$

In this paper following the classifications of Gray—Hervella [2] and Oubiña [3] for almost Hermitian an almost contact manifolds respectively, we study the type of almost contact structure M acquires when we consider a particular almost Hermitian structure on $T(M)$ and conversely.

In § 2 we give the results that will be needed in the sequel. In § 3 we define the $K_{i\varphi}$-curvature identities. Taking into account that if M is an almost contact (almost Hermitian) manifold, $M \times R$ is an almost Hermitian (almost contact) manifold, we prove that if M satisfies the $K_{i\varphi}$-curvature (K_i-curvature) identity, then $M \times R$ satisfies the K_i-curvature ($K_{i\varphi}$-curvature) identity. This fact allows us to obtain new subclasses of almost contact manifolds.

In § 4 we study the relationship among the different types of almost contact structures on (M, φ, ξ, η) and almost Hermitian structures on $\left(T(M), g^D, \tilde{J}\right)$.

§ 2. Almost contact structures

We recall here the necessary results from the theory of almost contact structures.

A $(2n+1)$-dimensional real differentiable manifold M of class C^∞ is said to have a (φ, ξ, η)-structure or an almost contact structure if it admits a field φ of endomorphisms of the tangent spaces, a vector field ξ and a 1-form η satisfying

$$\eta(\xi) = 1, \quad \varphi^2 = -I + \eta \otimes \xi$$

where I denotes the identity transformation. Then $\varphi\xi = 0$ and $\eta\varphi = 0$; moreover the endomorphism φ has rank $2n$.

Denote by $\mathscr{X}(M)$ the Lie algebra of C^∞-vector fields on M. If a manifold M with a (φ, ξ, η)-structure admits a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

where $X, Y \in \mathscr{X}(M)$, then M is said to have a (φ, ξ, η, g)-structure or an almost contact metric structure and g is called a compatible metric. A manifold with a (φ, ξ, η)-

structure admits a compatible metric g. The 2-form Φ on M defined by $\Phi(X, Y)=$ $=g(X, \varphi Y)$ is called the fundamental 2-form of the almost contact metric structure.

If ∇ is the Riemannian connection of g, the exterior derivatives of η and Φ are given by

$$2d\eta(X, Y) = (\nabla_X \eta)Y - (\nabla_Y \eta)X, \quad 3d\Phi(X, Y, Z) = \sigma(\nabla_X \Phi)(Y, Z$$

where σ denotes the cyclic sum over $X, Y, Z \in \mathscr{X}(M)$.

If $\{X_i, \varphi X_i, \xi; i=1, 2, \dots, n\}$ is a local orthonormal basis, defined on an open subset of M, the coderivative of η and Φ are computed to be

$$\delta\eta = -\sum_{i=1}^{n} \{(\nabla_{X_i}\eta)X_i + (\nabla_{\varphi X_i}\eta)\varphi X_i\},$$

$$\delta\Phi(X) = -\sum_{i=1}^{n} \{(\nabla_{X_i}\Phi)(X_i, X) + (\nabla_{\varphi X_i}\Phi)(\varphi X_i, X) - (\nabla_\xi \Phi)(\xi, X)\}.$$

On the other hand, being M an almost contact metric manifold, $M \times R$ is an almost Hermitian manifold with the almost complex structure J defined by

$$J\left(X, a\frac{d}{dt}\right) = \left(\varphi X - a\xi, \eta(X)\frac{d}{dt}\right)$$

and the metric

$$h\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right) = g(X, Y) + ab$$

where $X, Y \in \mathscr{X}(M)$ and a, b are C^∞-functions on $M \times R$. Bearing this fact in mind and the classification of A. Gray and L. M. Hervella [2] for almost Hermitian manifolds, J. A. Oubiña [3] gives a classification of almost contact manifolds. M is said to be

1) *Cosymplectic* (C) if $M \times R$ is *Kähler* (K),
2) *Nearly-K-cosymplectic* $(N-K-C)$ *if* $M \times R$ is nearly-Kähler $(NK = W_1)$,
3) *Almost-cosymplectic* $(A-C)$ if $M \times R$ is almost-Kähler $(AK = W_2)$,
4) *Semi-cosymplectic normal* $(S-C-N)$ if $M \times R$ is W_3,
5) *Trans-Sasakian* $(T-S)$ if $M \times R$ is W_4,
6) *Quasi-K-cosymplectic* $(Q-K-C)$ if $M \times R$ is quasi-Kähler $(QK = W_1 \oplus W_2)$,
7) *Normal* (N) if $M \times R$ is Hermitian $(H = W_3 \oplus W_4)$,
8) G_1*-semi-cosymplectic* (G_1-S-C) if $M \times R$ si $W_1 \oplus W_3$,
9) *Almost-trans-Sasakian* $(A-T-S)$ if $M \times R$ is $W_2 \oplus W_4$,
10) *Nearly-trans-Sasakian* $(N-T-S)$ if $M \times R$ is $W_1 \oplus W_4$,
11) G_2*-semi-cosymplectic* (G_2-S-C) if $M \times R$ is $W_2 \oplus W_3$,
12) *Semi-cosymplectic* $(S-C)$ if $M \times R$ is semi-Kähler $(SK = W_1 \oplus W_2 \oplus W_3)$,
13) *Quasi-trans-Sasakian* $(Q-T-S)$ if $M \times R$ is $W_1 \oplus W_2 \oplus W_4$,
14) G_1*-Sasakian* (G_1-S) if $M \times R$ is $G_1 = W_1 \oplus W_3 \oplus W_4$,
15) G_2*-Sasakian* (G_2-S) if $M \times R$ is $G_2 = W_2 \oplus W_3 \oplus W_4$.

On the other hand, if $(\bar{M}, \bar{h}, J)$ is an almost Hermitian manifold, $\bar{M}\times R$ can be endowed with an almost contact metric structure defining

$$\varphi\left(X, a\frac{d}{dt}\right) = (JX, 0); \quad \xi = \left(0, \frac{d}{dt}\right); \quad \eta\left(X, a\frac{d}{dt}\right) = a,$$

$$g\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right) = \bar{h}(X,Y) + ab$$

and we have the same relations between almost Hermitian and almost contact manifolds that we pointed out above.

§ 3. $K_{i\varphi}$-curvature identities

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold and (h, J) the almost Hermitian structure on $M\times R$ previously defined. We shall represent by ∇ and $\tilde{\nabla}$ the metric connections on M and $M\times R$ respectively and R and $\tilde{R}$ their curvature tensors.

Definition 3.1. We shall say that M satisfies the $K_{i\varphi}$-curvature identity, $i=1, 2, 3$ if for all $X, Y, Z, W\in\mathfrak{X}(M)$

$K_{1\varphi}$: $R(X, Y, Z, W) = R(\varphi X, \varphi Y, Z, W)$

$K_{2\varphi}$: $R(X, Y, Z, W) = R(\varphi X, \varphi Y, Z, W) + R(\varphi X, Y, \varphi Z, W) + R(\varphi X, Y, Z, \varphi W)$

$K_{3\varphi}$: $R(X, Y, Z, W) = R(\varphi X, \varphi Y, \varphi Z, \varphi W)$.

Theorem 3.1. *M satisfies the $K_{i\varphi}$-curvature identity, if and only if $M\times R$ satisfies the K_i-curvature identity* $(i=1, 2, 3)$.

Proof. Let $\{Z_1, Z_2, \ldots, Z_{2n+1}\}$ be a local basis in a coordinate open set U in M, then

$$\left\{(Z_1, 0), (Z_2, 0), \ldots, (Z_{2n+1}, 0), \left(0, \frac{d}{dt}\right)\right\}$$

is a local basis on $U\times R$ in $M\times R$.

Because of the tensorial character of $\tilde{R}$, to prove the case $i=1$, it will be sufficient to show the equivalence for the elements of the basis. Thus, to prove that $M\times R$ satisfies the K_1-curvature identity, it will suffice to prove:

a) $\tilde{R}(J\tilde{X}, J\tilde{Y})(Z_i, 0) = \tilde{R}(\tilde{X}, \tilde{Y})(Z_i, 0)$, $i=1, 2, \ldots, 2n+1$

b) $\tilde{R}(J\tilde{X}, J\tilde{Y})\left(0, \frac{d}{dt}\right) = \tilde{R}(\tilde{X}, \tilde{Y})\left(0, \frac{d}{dt}\right)$

where $\tilde{X} = \left(X, a\frac{d}{dt}\right)$, $\tilde{Y} = \left(Y, b\frac{d}{dt}\right)$ are arbitrary vector fields on $M\times R$.

Now, the curvatures R and $\tilde{R}$ are related as follows:

$$\tilde{R}\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right), \left(Z, c\frac{d}{dt}\right)\right) = \left(R(X,Y)Z, \left(bX\left(\frac{dc}{dt}\right) - aY\left(\frac{dc}{dt}\right) + \right.\right.$$

$$\left.\left. + a\frac{d(Y(c))}{dt} - b\frac{d(X(c))}{dt} + a\frac{db}{dt}\frac{dc}{dt} - b\frac{da}{dt}\frac{dc}{dt}\right)\frac{d}{dt}\right).$$

Taking into account this relation, we have

$$\tilde{R}(J\tilde{X}, J\tilde{Y})(Z_i, 0) = \big(R(\varphi X, \varphi Y)Z_i - aR(\xi, \varphi Y)Z_i - bR(\varphi X, \xi)Z_i, 0\big).$$

$$\tilde{R}(\tilde{X}, \tilde{Y})(Z_i, 0) = \big(R(X, Y)Z_i, 0\big).$$

But since M satisfies the K_1-curvature identity, $R(\xi, X)=0$ for all $X\in\mathscr{X}(M)$ and then a) follows.

Case b) is verified trivially because both members are zero as a consequence of the relation between R and $\tilde{R}$ mentioned above.

Conversely, let us suppose now that $M\times R$ satisfies the K_1-curvature identity, then setting $\tilde{X}=(X, 0)$, $\tilde{Y}=(Y, 0)$ for arbitrary vector fields X and Y on M, and using again the above mentioned relation between R and $\tilde{R}$, the result follows.

The remaining cases ($i=2, 3$) are obtained through a straightforward computation like in the case $i=1$.

Bearing in mind that $K_1\Rightarrow K_2\Rightarrow K_3$, this theorem leads us to the following

COROLLARY 3.1. *$K_{1\varphi}$-curvature$\Rightarrow K_{2\varphi}$-curvature$\Rightarrow K_{3\varphi}$-curvature.*

Now let $(\overline{M}, \bar{h}, J)$ be an almost Hermitian manifold and consider on $\overline{M}\times R$ the associated almost contact metric structure (φ, ξ, η, g).

The following theorem is obtained similarly to Theorem 3.1.

THEOREM 3.2. *M satisfies the K_i-curvature identity if and only if $M\times R$ satisfies the $K_{i\varphi}$-curvature identity.*

In the sequel we shall represent with the subindex $i\varphi$ ($i=1, 2, 3$) the class of almost contact structures that satisfies the $K_{i\varphi}$-curvature identity. Using this notation, subclasses of almost contact manifolds appear and their relation with some of the already known is given in the following

THEOREM 3.3. *In the lattice of almost contact structures the following relationships are true:*

(i) $C=C_{1\varphi}=C_{2\varphi}=C_{3\varphi}=(N-K-C)_{1\varphi}=(A-C)_{1\varphi}$,
(ii) $N-K-C=(N-K-C)_{3\varphi}=(N-K-C)_{2\varphi}\supset C$,
(iii) $N_{2\varphi}=N_{3\varphi}\subset N$,
(iv) $(S-C-N)_{2\varphi}=(S-C-N)_{3\varphi}$,
(v) $C\subset N_{1\varphi}$,
(vi) $N-K-C\subset(Q-K-C)_{3\varphi}$,
(vii) $C\subset(Q-K-C)_{1\varphi}$,
(viii) $(Q-K-C)_{3\varphi}\subseteqq Q-K-C$,
(ix) $(A-C)_{i\varphi}\subset(Q-K-C)_{i\varphi}$ ($i=1, 2, 3$).

PROOF. It is a consequence of Theorems 3.1 and 3.2 and the relation among the different kinds of almost Hermitian and almost contact manifolds, [2], [3], [4].

§ 4. The tangent bundle of an almost contact manifold

Let $X=X^i\frac{\partial}{\partial x^i}$, $\omega=\omega_i dx^i$ be the expressions in local coordinates of a vector field X and a 1-form ω defined in local charts (U, x^i) on M and let (x^i, y^i) be the local coordinates induced naturally by (U, x^i) on $T(M)$. Then the vertical lift of functions, vector fields and 1-forms is given by, [5]

$$f^V = f\circ\pi; \quad X^V = X^i\frac{\partial}{\partial x^i}; \quad \omega^V = \omega_i\, dx^i$$

and the vertical lift of a tensor field can be defined using the formula $(S\otimes T)^V = S^V\otimes T^V$.

The horizontal lift is defined with respect to a given connection ∇ on the base manifold by

$$f^H = 0; \quad X^H = X^i\frac{\partial}{\partial x^i} - \Gamma^i_{hs}X^hY^s\frac{\partial}{\partial y^i}; \quad \omega^H = \Gamma^i_{hs}y^s\omega_i\, dx^h + \omega_i\, dy^i$$

where Γ^i_{hs} are the components of ∇, and the horizontal lift of a tensor field can be given using the formula

$$(S\otimes T)^H = S^V\otimes T^H + S^H\otimes T^V.$$

Furthermore, if M is a Riemannian manifold with metric tensor g and metric connection ∇, $T(M)$ is also Riemannian and this with respect to the diagonal lift g^D (or Sasaki metric) which is defined by

$$g^D(X^H, Y^H) = g^D(X^V, Y^V) = g(X, Y)^V, \quad g^D(X^V, Y^H) = g^D(X^H, Y^V) = 0$$

for all $X, Y \varepsilon \mathscr{X}(M)$.

Now we consider the metric connection ∇^D of g^D and we have

THEOREM 4.1. *The connection ∇^D is determined by the following relations*

(i) $g^D(\nabla^D_{X^V}Y^V, Z^V) = g^D(\nabla^D_{X^V}Y^V, Z^H) = g^D(\nabla^D_{X^V}Y^H, Z^H) = 0$,
(ii) $g^D(\nabla^D_{X^H}Y^V, Z^V) = g^D(\nabla^D_{X^H}Y^H, Z^H) = \{g(\nabla_X Y, Z)\}^V$,
(iii) $g^D(\nabla^D_{X^V}Y^H, Z^H) = -1/2\, g^D(\gamma R(Z, Y), X^V)$,
(iv) $g^D(\nabla^D_{X^H}Y^V, Z^H) = -1/2 g^D(\gamma R(Z, X), Y^V)$,
(v) $g^D(\nabla^D_{X^H}Y^H, Z^V) = -1/2 g^D(\gamma R(X, Y), Z^V)$,

where $\gamma R(X, Y)$ is the vertical vector field given by

$$\gamma R(X, Y) = [X, Y]^H - [X^H, Y^H].$$

Let now $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold $(2n+1)$-dimensional with metric connection ∇. Then, if we consider the (1, 1) tensor field $\tilde{J}$ defined by

$$\tilde{J} = \varphi^H + \eta^V\otimes\xi^V - \eta^H\otimes\xi^H$$

the Sasaki metric g^D is a Hermitian metric with respect to $\tilde{J}$ and we may conclude that $(T(M), g^D, \tilde{J})$ is an almost Hermitian manifold.

Taking into account the relationship between the curvature tensors R^D and R of ∇^D and ∇ respectively, we may state the following

PROPOSITION 4.1. *If* $(T(M), g^D, \tilde{J})$ *satisfies the* K_i*-curvature identity, then* $(M, \varphi, \xi, \eta, g)$ *satisfies the* $K_{i\varphi}$*-curvature identity, for* $i=1, 2, 3$.

Next we study the relations among the different kinds of almost Hermitian structures on $T(M)$ and almost contact structures on M, considering on $T(M)$ and M the structures mentioned above.

THEOREM 4.2. *Let us suppose that the almost Hermitian manifold* $(T(M), g^D, \tilde{J})$ *satisfies the* K_i*-curvature identity* ($i=1, 2, 3$). *Then*

a) if $T(M)$ *is a* G_1*-manifold, then* M *is a* $N_{i\varphi}$*-manifold,*
b) if $T(M)$ *is a* G_2*-manifold, then* M *is a* $(G_2-S)_{i\varphi}$*-manifold,*
c) if $T(M)$ *is a* $W_1 \oplus W_4$*-manifold, then* M *is a cosymplectic manifold,*
d) if $T(M)$ *is a* $Q-K$*-manifold, then* M *is a* $(Q-K-C)_{i\varphi}$*-manifold.*

PROOF. It should be noted first that since M satisfies the K_i-curvature identity, we have

$$g^D(\gamma R(X, Y), \xi^V)=0, \quad \text{for all } X, Y \in \mathscr{X}(M).$$

a) Since $T(M)$ is a G_1-manifold

$$g^D(\nabla^D_{\tilde{X}}(\tilde{J})\tilde{Y}, \tilde{Z})+g^D(\nabla^D_{\tilde{Y}}(\tilde{J})\tilde{X}, \tilde{Z}) = g^D(\nabla^D_{\tilde{J}\tilde{X}}(\tilde{J})\tilde{J}\tilde{Y}, \tilde{Z})+g^D(\nabla^D_{\tilde{J}\tilde{Y}}(\tilde{J})\tilde{J}\tilde{X}, \tilde{Z})$$

for all $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathscr{X}(T(M))$; in particular for $\tilde{X}=\tilde{Y}=\xi^H$, $\tilde{Z}=Z^H$ we obtain $\nabla_\xi(\varphi)\xi=0$ and setting now $\tilde{X}=X^H$, $\tilde{Y}=Y^V$, $\tilde{Z}=Z^V$ the result follows.

b) By hypothesis

$$\sigma\{g^D(\nabla^D_{\tilde{X}}(\tilde{J})\tilde{Y}, \tilde{Z})-g^D(\nabla^D_{\tilde{J}\tilde{X}}(\tilde{J})\tilde{J}\tilde{Y}, \tilde{Z})\}=0$$

for all $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathscr{X}(T(M))$.

Then, setting $\tilde{X}=X^H$, $\tilde{Y}=Y^H$, $\tilde{Z}=Z^H$ in this expression we obtain that M is a (G_2-S)-manifold through a straightforward computation.

c) Using a) we may deduce that M is a normal manifold and therefore

$$\nabla_\xi(\varphi)Y=0, \quad \text{for } Y \in \mathscr{X}(M).$$

On the other hand, since $T(M)$ is a $W_1 \oplus W_4$-manifold we may write

$$g^D(\nabla^D_{(\varphi X)^H}(\tilde{J})Y^V, Z^V)+g^D(\nabla^D_{\tilde{J}(\varphi X)^H}(\tilde{J})\tilde{J}Y^V, Z^V)=0.$$

and this together with the fact of being $\nabla_\xi(\varphi)Y=0$ and Theorem 4.2 leads to

$$g(\nabla_{\varphi X}(\varphi)Y, Z)-g(\nabla_X(\varphi)\varphi Y, Z)-\eta(X)\eta(Y)g(\nabla_\xi\xi, Z)+\eta(Y)g(\nabla_X\xi, Z)=0.$$

Substituting X by φX in this expression we obtain that M is a $Q-K-C$-manifold, which together with being M a normal manifold leads to the result.

d) From the definition of $Q-K$-manifold we obtain

$$g^D(\nabla^D_{X^H}(\tilde{J})Y^V, Z^V)=-g^D(\nabla^D_{\tilde{J}(X^H)}(\tilde{J})\tilde{J}Y^V, Z^V)$$

and the result follows from theorem 4.1.

THEOREM 4.3. *If* $(T(M), g^D, \tilde{J})$ *is a* $(W_1 \oplus W_2 \oplus W_4)_i$*-manifold, then* M *is a* $(Q-K-C)_{i\varphi}$*-manifold if and only if* $\nabla_\xi \xi = 0$.

PROOF. The definition condition of $(W_1 \oplus W_2 \oplus W_4)_i$-manifold for vector fields $(\varphi X)^H$, Y^V, Z^V leads to the expression

$$g(\nabla_{\varphi X}(\varphi)Y, Z) - g(\nabla_X(\varphi)\varphi Y, Z) + \eta(X) g(\nabla_\xi(\varphi)\varphi Y, Z) + \\ + \eta(Y) g(\nabla_X \xi, Z) - \eta(X)\eta(Y) g(\nabla_\xi \xi, Z) = 0.$$

Substituting X by φX in this expression, we have

$$g((\nabla_X(\varphi)Y, Z) + g(\nabla_{\varphi X}(\varphi)\varphi Y, Z) - \eta(Y) g(\nabla_{\varphi X}\xi, Z) - \eta(X) g(\nabla_\xi(\varphi)Y, Z) = 0.$$

Now

$$\eta(X) g(\nabla_\xi(\varphi)Y, Z) = 0 \quad \text{if} \quad \text{and only if} \quad \nabla_\xi \xi = 0.$$

Since $T(M)$ is a $(W_1 \oplus W_2 \oplus W_4)_i$-manifold $g(\nabla_\xi(\varphi)\varphi Y, \varphi Z) = 0$ which is equivalent to

$$g(\nabla_\xi(\varphi)Y, Z) + \eta(Y) g(\varphi \nabla_\xi \xi, Z) + \eta(Z) g(\nabla_\xi \xi, Y) = 0$$

and the result follows inmediately.

Now, in order to study the case when M is an $S-K$-manifold, we need the following lemma.

LEMMA 4.1. *The coderivative of the Kähler 2-form* $\tilde{F}$ *of* $(T(M), g^D, \tilde{J})$ *is determined by*

(i) $\delta \tilde{F}(X^H) = \{\delta \Phi(X)\}^V$,

(ii) $\delta \tilde{F}(X^V) = \{\eta(X)\}^V (\delta \eta)^V + \sum_{i=1}^{n} g^D(\gamma R(E_i, \varphi E_i), X^V) - \{g(\nabla_\xi \xi, X)\}^V$,

where X *is a vector field on* M *and* $\{E_1, \ldots, E_n, \varphi E_1, \ldots, \varphi E_n, \xi\}$ *is a local orthonormal* φ*-basis on* M.

The proof follows directly from the definition of the coderivative and Theorem 4.1.

Using this lemma we may state

THEOREM 4.4. *If* $(T(M), g^D, \tilde{J})$ *is an* $(S-K)_i$*-manifold then* M *is an* $(S-C)_{i\varphi}$*-manifold.*

As a consequence of this theorem and Proposition 4.1 we have

COROLLARY 4.1. *Let* $(T(M), g^D, \tilde{J})$ *be an almost Hermitian manifold satisfying the* K_i*-curvature identity* $(i=1, 2, 3)$.

a) If $T(M)$ *is a* $W_1 \oplus W_3$*-manifold, then* M *is an* $(S-C-N)_{i\varphi}$*-manifold.*

b) If $T(M)$ *is a* $W_2 \oplus W_3$*-manifold, then* M *is a* $(G_2-S-C)_{i\varphi}$*-manifold.*

Finally, we point out some results about lifts of almost contact structures.

THEOREM 4.5. *If* $(M, \varphi, \xi, \eta, g)$ *is a normal manifold satisfyng the* $K_{1\varphi}$*-cuvature identity, then* $(T(M), g^D, \tilde{J})$ *is a Hermitian manifold.*

PROOF. $(T(M), g^D, \tilde{J})$ is a Hermitian manifold if and only if

$$g^D(\nabla^D_{\tilde{X}}(\tilde{J})\tilde{Y} - \nabla^D_{\tilde{J}\tilde{X}}(\tilde{J})\tilde{J}\tilde{Y}, \tilde{Z}) = 0$$

for all $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathscr{X}(T(M))$.

In order to prove this identity it will be sufficient to do it in the following cases:

(i) $\tilde{X}=X^V$, $\tilde{Y}=Y^V$, $\tilde{Z}=Z^V$,
(ii) $\tilde{X}=X^V$, $\tilde{Y}=Y^V$, $\tilde{Z}=Z^H$,
(iii) $\tilde{X}=X^V$, $\tilde{Y}=Y^H$, $\tilde{Z}=Z^H$,
(iv) $\tilde{X}=X^H$, $\tilde{Y}=Y^V$, $\tilde{Z}=Z^V$,
(v) $\tilde{X}=X^H$, $\tilde{Y}=Y^V$, $\tilde{Z}=Z^H$,
(vi) $\tilde{X}=X^H$, $\tilde{Y}=Y^H$, $\tilde{Z}=Z^H$.

(i) In this case $g^D(\nabla^D_{X^V}(\tilde{J})Y^V, Z^V)=0$ and the other term is also zero, because since M satisfies the $K_{1\varphi}$-curvature identity,

$$g^D(\nabla^D_{\tilde{J}X^V}(\tilde{J})\tilde{J}Y^V, Z^V) = -(\eta(X))^V\{g(\nabla_\xi(\varphi)\varphi Y, Z)\}^V +$$
$$+(\eta(X))^V\{g(\nabla_\xi\eta(Y)\xi, Z)-\eta(Z)g(\nabla_\xi\eta(Y)\xi, \xi)\}^V$$

and since M is normal $\nabla_\xi\varphi=0$ and $\nabla_\xi\xi=0$.

The proof in cases (ii) and (iii) is totally analogous to case (i).

(iv) Taking into account what we pointed out in case (i) we have

$$g^D(\nabla^D_{X^H}(\tilde{J})Y^V, Z^V) = \{g(\nabla_X(\varphi)Y, Z)\}^V,$$
$$g^D(\nabla^D_{\tilde{J}(X^H)}(\tilde{J})\tilde{J}Y^V, Z^V) = \{g(\nabla_{\varphi X}(\varphi)\varphi Y, Z)-\eta(Y)g(\nabla_{\varphi X}\xi, Z)\}^V$$

and the result follows from the fact that M is normal.

(v) Using Theorem 4.1 and since M satisfies the $K_{1\varphi}$-curvature identity, one can deduce that

$$g^D(\nabla^D_{X^H}(\tilde{J})Y^V, Z^H)-g^D(\nabla^D_{\tilde{J}X^H}(\tilde{J})\tilde{J}Y^V, Z^H) = \eta(Y)^V\{-g(\nabla_X\xi, Z)+g(\nabla_{\varphi X}\xi, \varphi Z)\}^V$$

and since M is normal the result follows.

(vi) In this case we obtain

$$g^D(\nabla^D_{X^H}(\tilde{J})Y^H, Z^H) = \{g(\nabla_X(\varphi)Y, Z)\}^V,$$
$$g^D(\nabla^D_{\tilde{J}X^H}(\tilde{J})\tilde{J}Y^H, Z^H) = \{g(\nabla_{\varphi X}(\varphi)\varphi Y, Z)-\eta(Y)g(\nabla_{\varphi X}\xi, Z)\}^V$$

and since M is normal the proof is finished.

Therefore, we have

COROLLARY 4.2. *If $(M, \varphi, \xi, \eta, g)$ is a cosymplectic manifold, then $(T(M), g^D, \tilde{J})$ is a Hermitian manifold.*

Furthermore,

THEOREM 4.6. *If $(M, \varphi, \xi, \eta, g)$ is an almost contact manifold, the tangent bundle $T(M)$ with the almost Hermitian structure $(g^D, \tilde{J})$ is an almost Kähler or a nearly Kähler manifold if and only if it is Kähler.*

PROOF. If $(T(M), g^D, \tilde{J})$ were a nearly Kähler manifold, then it would also be a $W_1\oplus W_4$-manifold and by Theorem 4.2, M would be cosymplectic.

On the other hand, if $(T(M), g^D, \tilde{J})$ were almost Kähler, we would have $\sigma g^D(\nabla^D_{\tilde{X}}(\tilde{J})\tilde{Y}, \tilde{Z})=0$ which for X^H, Y^V, Z^V, is reduced to $\{g(\nabla_X(\varphi)Y, Z)\}^V=0$ and M would be cosymplectic. Then the result follows using the last corollary.

THEOREM 4.7. *Let* $\{E_1, \ldots, E_n, \varphi E_1, \ldots, \varphi E_n, \xi\}$ *be a local orthonormal* φ*-basis on* M. *If* $(M, \varphi, \xi, \eta, g)$ *is a semi-cosymplectic manifold such that* $\nabla_\xi \xi = 0$ *and* $\sum_{i=1}^{n} R(E_i, \varphi E_i) = 0$, *then* $(T(M), g^D, \tilde{J})$ *is a semi-Kähler manifold.*

PROOF. Since M is a semi-cosymplectic manifold, $\delta\Phi = \delta\eta = 0$ and Lemma 4.1 leads to the result.

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QUASI-CONTINUITY OF MULTIVALUED MAPS WITH RESPECT TO THE QUALITATIVE TOPOLOGY

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A subset A of a topological space X is said to be:

— semi-open, if there exists an open set U such that $U \subset A \subset \overline{U}$ [8],

— semi-closed, if $X \setminus A$ is semi-open [2, 3].

The union of any family of semi-open sets is semi-open [8]. The union of all semi-open sets which are contained in A is called the semi-interior of A; we denote it by s-Int A [2, 3]. From the definition it immediately follows that each semi-open (semi-closed) set has the Baire property.

Let F be a multivalued map which assigns a non-empty subset $F(x)$ of a topological space Y to each point $x \in X$ (for simplicity we will write $F\colon X \to Y$). For any sets $A \subset X$, $B \subset Y$ we will denote [1]: $F(A) = \bigcup \{F(x)\colon x \in A\}$, $F^+(B) = \{x \in X\colon F(x) \subset B\}$, $F^-(B) = \{x \in X\colon F(x) \cap B \neq \emptyset\}$.

A multivalued map $F\colon X \to Y$ is said to be:

— upper (lower) quasi-continuous at a point $x_0 \in X$, if for each open set $V \subset Y$ such that $F(x_0) \subset V$ (resp. $F(x_0) \cap V \neq \emptyset$) and for each neighbourhood U of x_0 there exists an open non-empty set $U_1 \subset U$ such that $F(x) \subset V$ (resp. $F(x) \cap V \neq \emptyset$) for each $x \in U_1$ [10, 12[,

— quasi-continuous at $x_0 \in X$, if for any open sets $V_1, V_2 \subset Y$ such that $F(x_0) \subset V_1$ and $F(x_0) \cap V_2 \neq \emptyset$ and for each neighbourhood U of x_0 there exists an open non-empty set $U_1 \subset U$ such that $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \emptyset$ for $x \in U_1$ [13].

Any single valued map $f\colon X \to Y$ can be considered as a multivalued map with values $\{f(x)\}$. In this case each of the above three definitions gives the definition of quasi-continuity in the sense of Kempisty [6].

In the sequel the symbol $E_u(F)$, $E_l(F)$ and $E_*(F)$ will be used to denote the sets of all points at which a multivalued map F is upper quasi-continuous, lower quasi-continuous or quasi-continuous respectively. It follows from the definitions that $E_*(F) \subset E_u(F) \cap E_l(F)$; the equality does not hold in general [4].

A multivalued map F is called upper quasi-continuous (lower quasi-continuous, quasi-continuous) if it has this property at each point Equivalently F is upper (lower) quasi-continuous iff for each open set $V \subset Y$ the set $F^+(V)$ (resp. $F^-(V)$) is semi-open. Similarly F is quasi-continuous iff for any open sets $V_1, V_2 \subset Y$ the set $F^+(V_1) \cap F^-(V_2)$ is semi-open [13].

Let X be a topological space and let $T_q = \{U \setminus H\colon U$ is open, H is of the first category$\}$. Then T_q is a topology on X [5] which sometimes is called the qualitative topology.

LEMMA 1. *Let X be a Baire space.*

(1) *A set $A \subset X$ is of the first category if and only if it is nowhere dense in (X, T_q).*

(2) *A set* $A\subset X$ *is of the first category if and only if it is of the first category in* (X, T_q).

(3) *A set* $A\subset X$ *has the Baire property if and only if it has this property in* (X, T_q).

(4) (X, T_q) *is a Baire space.*

(5) *A set* $A\subset X$ *is* T_q*-semi-open* (T_q*-semi-closed*) *if and only if it is of the form* $A=A_1\setminus B$ $(A=A_1\cup B)$, *where* A_1 *is semi-open (semi-closed) and* B *is of the first category.*

PROOF. (1) follows from [5, Theorem 4]; (2) and (3) are consequences of (1). Property (4) follows from the assumptions and the definition of T_q. Thus we will prove (5). If A is T_q-semi-open, then there exists a set $V\in T_q$ such that $V\subset A\subset \overline{V}^{(q)}$ where $\overline{V}^{(q)}$ denotes the T_q-closure of V. Thus $V=U\setminus H$, U is an open set and H is of the first category. The set $H_1=H\cap U$ is T_q-closed, so we have $U\subset A\cup H_1\subset \subset \overline{U\setminus H}^{(q)}\cup H_1=\overline{(U\setminus H)\cup H_1}^{(q)}=\overline{U}^{(q)}\subset\overline{U}$ what means that $A\cup H_1$ is semi-open. Assuming $B=H_1\setminus A$, $A_1=A\cup H_1$ we have $A=A_1\setminus B$, A_1 is semi-open and B is of the first category. Conversely, let $A=A_1\setminus B$ where A_1 is semi-open and B of the first category. Then there exists an open set U such that $U\subset A_1\subset \overline{U}$. Since $\overline{U}= =\overline{U}^{(q)}=\overline{U\setminus B}^{(q)}$ and $U\setminus B\subset A_1\setminus B\subset \overline{U}$, the set $A_1\setminus B$ is T_q-semi-open. For T_q-semi-closed sets the statement follows from above and the definition of a semi-closed set.

For a multivalued map $F\colon X\to Y$ by $E_u(F, T_q)$, $E_1(F, T_q)$ and $E_*(F, T_q)$ will be denoted the sets of all points at which F is upper T_q-quasi-continuous, lower T_q-quasi-continuous or T_q-quasi-continuous respectively.

THEOREM 2. *Let* X *be a Baire space and* Y *a regular one. If* $F\colon X\to Y$ *is a multivalued map with compact values, then*

$$\operatorname{Int}\big(E_u(F, T_q)\cap E_1(F, T_q)\big)\subset E_u(F)\cap E_1(F).$$

PROOF. Assume that $x_0\in \operatorname{Int}\big(E_u(F, T_q)\cap E_1(F, T_q)\big)\setminus E_u(F)$. Then there exist open sets $U_0\subset X$ and $V\subset Y$ such that $F(x_0)\subset V$, $x_0\in U_0\subset \operatorname{Int}\big(E_u(F, T_q)\cap E_1(F, T_q)\big)$ and each open non-empty set $U\subset U_0$ contains a point x_u for which $F(x_u)\cap \cap (Y\setminus V)\neq\emptyset$ holds. Let us take an open set $W\subset Y$ satisfying $F(x_0)\subset W\subset \overline{W}\subset V$. Since $x_0\in E_u(F, T_q)$ we have $F(U\setminus H)\subset W$ for some non-empty open set $U\subset U_0$ and a set of the first category H. The set U contains a point x_u satisfying $F(x_u)\cap (Y\setminus V)\neq\emptyset$. Hence $x_u\in H$ and $F(x_u)\cap(Y\setminus\overline{W})\neq\emptyset$. The lower T_q-quasi-continuity of F at x_u implies the existence of a T_q-open set $\emptyset\neq A\subset U$ such that

$$(*)\qquad F(x')\cap(Y\setminus\overline{W})\neq\emptyset \quad\text{for}\quad x'\in A.$$

On the other hand because A is of the second category we have $A\cap(U\setminus H)\neq\emptyset$ and $F\big(A\cap(U\setminus H)\big)\subset W$, what is a contradiction to $(*)$. Thus we have shown

$$\operatorname{Int}\big(E_u(F, T_q)\cap E_1(F, T_q)\big)\subset E_u(F).$$

Now let us assume $x_0\in \operatorname{Int}\big(E_u(F, T_q)\cap E_1(F, T_q)\big)\setminus E_1(F)$. Then there exist open sets $V\subset Y$ and $U_0\subset X$ such that $F(x_0)\cap V\neq\emptyset$, $x_0\in U_0\subset \operatorname{Int}\big(E_u(F, T_q)\cap E_1(F, T_q)\big)$ and each open non-empty set $U\subset U_0$ contains a point x_u satisfying $F(x_u)\subset Y\setminus V$. Let $y_0\in F(x_0)\cap V$ and let $V_1\subset Y$ be an open set such that $y_0\in V_1\subset \overline{V}_1\subset V$. Since F

is lower T_q-quasi-continuous at x_0 there exist a non-empty open set $U_1 \subset U_0$ and a set M of the first category such that

$$(**) \qquad F(x) \cap V_1 \neq \emptyset \quad \text{for} \quad x \in U_1 \setminus M.$$

The set U_1 contains a point x_1 for which the inclusion $F(x_1) \subset Y \setminus \overline{V}_1$ holds. The condition $x_1 \in E_u(F, T_q)$ implies that there exists a T_q-open set $\emptyset \neq A \subset U_1$ such that $F(A) \subset Y \setminus \overline{V}_1$. The set A is of the second category so $A \cap (U_1 \setminus M) \neq \emptyset$ and $F(A \cap (U_1 \setminus M)) \subset Y \setminus \overline{V}_1$. But this is a contradiction to $(**)$. Hence we have $\operatorname{Int}(E_u(F, T_q) \cap E_1(F, T_q)) \subset E_1(F)$, what finishes the proof.

Corollary 3. *Let X be a Baire space and Y a regular one. A multivalued map $F\colon X \to Y$ with compact values is lower and upper quasi-continuous if and only if it is simultaneously lower and upper T_q-quasi-continuous.*

For single valued maps we obtain a known result:

Corollary 4. *Let X be a Baire space and Y a regular one. A single valued map $f\colon X \to Y$ is quasi-continuous if and only if it is T_q-quasi-continuous.*

Let us observe that the regularity of the space Y in Theorem 2 is essential.

Example 5. Let X be the set of real numbers with the natural topology, Q the set of rational numbers and let Y be the set of real numbers with the topology $\tau = \{(a, \infty)\colon a \in Y\} \cup \{\emptyset, Y\}$. A multivalued map $F\colon X \to Y$ is defined as

$$F(x) = \begin{cases} [0, 1] & \text{for} \quad x \in Q \\ [0, 2] & \text{for} \quad x \in X \setminus Q. \end{cases}$$

Then F has compact values and it is upper and lower T_q-semi-continuous. Bur $F^-((1, \infty)) = X \setminus Q$, $F^+((-2, 2)) = Q$ are not semi-open, so F is neither lower not upper quasi-continuous.

Theorem 6. *Let X be a Baire space and Y a regular one. If $F\colon X \to Y$ is a multivalued map with compact values, then* $\operatorname{Int} E_*(F, T_q) \subset E_*(F)$.

Proof. Suppose $x_0 \in \operatorname{Int} E_*(F, T_q) \setminus E_*(F)$. Then there exist open sets $U_0 \subset X$ and $V_1, V_2 \subset Y$ such that $F(x_0) \subset V_1$, $F(x_0) \cap V_2 \neq \emptyset$, $x_0 \in U_0 \subset E_*(F, T_q)$ and each open non-empty set $U \subset U_0$ contains a point x_u for which $F(x_u) \setminus V_1$ or $F(x_u) \cap \cap V_2 = \emptyset$ holds. Let $y_0 \in F(x_0) \cap V_2$; we can choose open sets $W_1, W_2 \subset Y$ satisfying the inclusions $F(x_0) \subset W_1 \subset \overline{W}_1 \subset V_1$ and $y_0 \in W_2 \subset \overline{W}_2 \subset V_2$. Since F is T_q-quasi-continuous at x_0 there exist an open non-empty set $U \subset U_0$ and a set H of the first category such that $F(x) \subset W_1$ and $F(x) \cap W_2 \neq \emptyset$ for $x \in U \setminus H$. For some $x_1 \in U$ we have $F(x_1) \cap (Y \setminus \overline{W}_1) \neq \emptyset$ or $F(x_1) \subset Y \setminus \overline{W}_2$. Assume that $F(x_1) \cap (Y \setminus \overline{W}_1) \neq \emptyset$. It follows from the condition $x_1 \in E_*(F, T_q)$ that there exists a non-empty T_q-open set $A \subset U$ such that $F(x) \cap (Y \setminus \overline{W}_1) \neq \emptyset$ for $x \in A$. The set A is of the second category, so $\emptyset \neq A \setminus H \subset U \setminus H$. Thus we have $F(x) \cap (Y \setminus \overline{W}_1) \neq \emptyset$ for $x \in A \setminus H$ and $F(A \setminus H) \subset W_1$, what is impossible. If $F(x_1) \subset Y \setminus \overline{W}_2$, then there exists a non-empty T_q-open set $A_1 \subset U$ such that $F(A_1) \subset Y \setminus \overline{W}_2$. The set A_1 is of the second category, so $\emptyset \neq A_1 \setminus H \subset U \setminus H$ and $F(A_1 \setminus H) \subset Y \setminus \overline{W}_2$. On the other hand we have $F(x) \cap W_2 \neq \emptyset$ for $x \in A_1 \setminus H$ and this is the contradiction finishing the proof.

As Example 5 shows, the assumption of regularity of Y in Theorem 6 cannot be omitted.

COROLLARY 7. *Let X be a Baire space and Y a regular one. A multivalued map $F: X \to Y$ with compact values is quasi-continuous if and only if it is T_q-quasi-continuous.*

If X is not a Baire space, then the above theorems are not true.

EXAMPLE 8. In the set R of real numbers we denote by T the natural topology, $\mathcal{N}$ — the σ-ideal of all subsets of Lebesgue measure zero and $T(\mathcal{N})=\{U\setminus B: U\in T, B\in\mathcal{N}\}$; then $T(\mathcal{N})$ is a topology on R, [5]. A set $A\subset R$ is of the first category in $(R, T(\mathcal{N}))$ if and only if it is the union of a set of the first category and a set belonging to $\mathcal{N}$ [5, Corollary 1, Lemmas 3 and 5]. Every subsets of R can be represented as the union of a set of Lebesgue measure zero and a set of the first category [11, p. 5], so $(R, T(\mathcal{N}))$ is not a Baire space and $(T(\mathcal{N}))_q$ contains all subsets of R. Therefore each multivalued map $F: R\to R$ is lower and upper $(T(\mathcal{N}))_q$-semicontinuous. Moreover for F given by

$$F(x)=\begin{cases}[0, 1] & \text{for} \quad x\in[0, 1]\cap Q\\ [-1, 0] & \text{for} \quad x\in R\setminus[0, 1]\cap Q\end{cases}$$

we have $F^+\left(\left(-\frac{1}{2}, 2\right)\right)=F^-((0, 1))=[0, 1]\cap Q\in\mathcal{N}$, so F is neither lower nor upper $T(\mathcal{N})$-quasi-continuous.

For lower or upper quasi-continuity only the theorem similar to Theorem 2 is not true. We will formulate some sufficient conditions under which lower (upper) T_q-quasi-continuity implies lower (upper) quasi-continuity. To begin with, we give some definitions.

Let $\mathcal{G}$ be the family of all non-empty open subsets of X, $\mathcal{B}_0$ the family of all sets of the second category with the Baire property and let $\mathcal{G}\subset\mathcal{B}\subset\mathcal{B}_0\cup\mathcal{G}$.

A multivalued map $F: X\to Y$ is said to be u-$\mathcal{B}$-continuous (l-$\mathcal{B}$-continuous) at $x_0\in X$ if for each open set $V\subset Y$ such that $F(x_0)\subset V$ (resp. $F(x_0)\cap V\neq\emptyset$) and for each neighbourhood U of x_0 there exists a set $B\in\mathcal{B}$ such that $B\subset U \subseteqq$and $F(x)\subset V$ (resp. $F(x)\cap V\neq\emptyset$) for $x\in B$ [9].

A multivalued map F is said to be u-$\mathcal{B}$-continuous (l-$\mathcal{B}$-continuous) if it has this property at each point.

We use the symbols $C_u(F)$ and $C_l(F)$ to denote the sets of all points at which F is upper or lower semicontinuous respectively.

THEOREM 9 [9, Theorems 1 and 2]. *Let F be a multivalued map defined on a topological spac X assuming compact values in a metric space Y. If F is u-$\mathcal{B}$-continuous (l-$\mathcal{B}$-continuous), then the set $X\setminus C_l(F)$ $(X\setminus C_u(F))$ is of the first category.*

For a set $A\subset X$ we denote by $D(A)$ the set of all points at which A is not of the first category [7, p. 88].

THEOREM 10. *Let X be a Baire space, (Y, d) a metric one and let $F: X\to Y$ be a multivalued map with compact values. If*

(a) *F is upper (lower) T_q-quasi-continous,*

(b) *$F^-(V)\subset D(F^-(V))$ (resp. $F^+(V)\subset D(F^+(V))$) for each open set $V\subset Y$, then F is upper quasi-continuous (lower quasi-continuous).*

PROOF. Let $\mathscr{B}$ be the family of all sets of the second category with the Baire property in X and let F be upper T_q-quasicontinuous. For each open set $V\subset Y$ the set $F^+(V)$ is T_q-semi-open. Thus for each open set $U\subset X$ if $F^+(V)\cap U\neq\emptyset$, then according to Lemma 1(5) it is of the second category and has the Baire property. From this it follows that F is u-$\mathscr{B}$-continuous and by Theorem 9 the set $X\setminus C_l(F)$ is of the first category. Let us take a point $x\in X$, a neighbourhood U of x and an open set $V\subset Y$ such that $F(x)\cap V\neq\emptyset$. Then $V=\bigcup_{n=1}^{\infty} M_n$, where the M_n are closed sets and $F^-(V)=$ $=\bigcup_{n=1}^{\infty} F^-(M_n)$. The sets $F^-(M_n)$ are T_q-semiclosed, so they have the Baire property. Thus $F^-(V)$ is a set with the Baire property; moreover the assumption (b) implies that $U\cap F^-(V)$ is of the second category. Hence F is l-$\mathscr{B}$-continuous and from Theorem 9 the set $X\setminus C_u(F)$ is of the first category. So we have shown

(1) $$X\setminus C_u(F)\cap C_l(F) \text{ is of the first category.}$$

For each open set $V\subset Y$ the set $F^+(V)$ is T_q-semi-open, so $F^+(V)=A\setminus H$, where A is semi-open and H is of the first category. Since X is a Baire space we have $\overline{A\setminus H}=$ $=\bar{A}$ is semi-open [2], thus s-Int $\overline{F^+(V)}=$s-Int $\bar{A}=\bar{A}\supset F^+(V)$. We have obtained

(2) $$F^+(V)\subset \text{s-Int}\,\overline{F^+(V)} \quad \text{for each open set } V\subset Y.$$

Now we are going to show that F is upper quasi-continuous. Let $x_0\in X$ be any point, U_0 a neighbourhood of x_0, $\varepsilon_0>0$ and $0<2\varepsilon<\varepsilon_0$. We denote $K(F(x_0),\varepsilon)=$ $=\bigcup\{K(y,\varepsilon)\colon y\in F(x_0)\}$, where $K(y,\varepsilon)$ is the open ball with center y and radius ε. Then it follows from (2) that $x_0\in A_1=U_0\cap$s-Int $\overline{F^+(K(F(x_0),\varepsilon))}$. Since A_1 is a semi-open set the condition (1) implies that there exists a non-empty open set $U\subset A_1$ such that

(3) $$F(x')\subset K(F(x''),\varepsilon) \quad \text{for} \quad x',x''\in U.$$

From the inclusion $U\subset\overline{F^+(K(F(x_0),\varepsilon))}$ we have $F(x_1)\subset K(F(x_0),\varepsilon)$ for some $x_1\in U$. This fact and (3) give $F(x)\subset K(F(x_1),\varepsilon)\subset K(F(x_0),\varepsilon_0)$ for $x\in U$, i.e. F is upper quasi-continuous at x_0. Now let F be lower T_q-quasi-continuous. Using similar arguments as in the first part of the proof we obtain that F is l-$\mathscr{B}$-continuous. So from Theorem 9 the set $X\setminus C_u(F)$ is of the first category. For an open set $V\subset Y$ such that $F^+(V)\neq\emptyset$ we denote:

$$W_n=\left\{y\in V\colon\ d(y,Y\setminus V)>\frac{1}{n}\right\}\quad M_n=\left\{y\in V\colon\ d(y,Y\setminus V)\geqq\frac{1}{n}\right\}$$

where $d(y,Y\setminus V)$ is the distance of y from $Y\setminus V$. Then $M_n\subset W_{n+1}\subset M_{n+1}\subset V$, for $n\geqq 1$ and $V=\bigcup_{n=1}^{\infty} W_n=\bigcup_{n=1}^{\infty} M_n$. Moreover there exists $n_0\geqq 1$ such that $W_n\neq\emptyset$ for $n\geqq n_0$. Let us take $x\in F^+(V)$. The increasing sequence $\{W_n : n\geqq 1\}$ is an open cover of the compact set $F(x)$, so $F(x)\subset W_n\subset M_n$ for some n. Hence we obtain $F^+(V)=\bigcup_{n=1}^{\infty} F^+(M_n)$. The sets $F^+(M_n)$ are T_q-semi-closed, hence $F^+(V)$ has the

Baire property. This fact and the assumption (b) imply that F is u-$\mathscr{B}$-continuous. According to Theorem 9 the set $X\setminus C_1(F)$ is of the first category and consequently

$$X\setminus C_u(F)\cap C_1(F) \quad \text{is of the first category.} \tag{4}$$

By the same way as (2) we can prove

$$F^-(V)\subset \text{s-Int}\,\overline{F^-(V)} \quad \text{for each open set} \quad V\subset Y. \tag{5}$$

The conditions (4) and (5) imply the lower quasi-continuity of F, what finishes the proof.

In Theorem 10 the assumption (b) cannot be omitted.

EXAMPLE 11. Let $F_1, F_2: R\to R$ be multivalued maps given by

$$F_1(x)=\begin{cases}[0,1] & \text{for} \quad x\in Q\\ [0,2] & \text{for} \quad x\in R\setminus Q,\end{cases} \qquad F_3(x)=\begin{cases}[0,1] & \text{for} \quad x\in R\setminus Q\\ [0,2] & \text{for} \quad x\in Q.\end{cases}$$

Then F_1 is lower T_q-semicontinuous and F_2 upper T_q-semicontinuous. Assuming $V=(-1,2)$ and $W=(1,2)$ we have $F_1^+(V)\not\subset D(F_1^+(V))$, $F_2^-(W)\not\subset D(F_2^-(W))$. The sets $F_1^-(W)$ and $F_2^+(V)$ are not semi-open, thus F_1 is not lower quasi-continuous and F_2 is not upper quasi-continuous.

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AN INVARIANCE PRINCIPLE FOR DEPENDENT RANDOM VARIABLES

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1. Introduction and notations

Let $\{X_n, n\geqq 1\}$ be a sequence of random variables on a probability space (Ω, F, P). Let $F_n^m=\sigma(X_i: n\leqq i\leqq m)$, $1\leqq n\leqq m\leqq\infty$. Define two measures of dependence by

$$\varphi(m)=\sup\{|P(B|A)-P(B)|: A\in F_1^n,\ P(A)\neq 0,\ B\in F_{n+m}^\infty,\ n\geqq 1\},$$

$$\psi(m)=\sup\left\{\left|\frac{P(A\cap B)}{P(A)P(B)}-1\right|: A\in F_1^n,\ B\in F_{n+m}^\infty,\ P(A)P(B)\neq 0,\ n\geqq 1\right\}.$$

The sequence $\{X_n, n\geqq 1\}$ is said to be φ-mixing or ψ-mixing according as $\varphi(n)\to 0$ or $\psi(n)\to 0$, respectively, as $n\to\infty$. Clearly a ψ-mixing sequence is φ-mixing.

Assume $EX_n=0$, $EX_n^2<\infty$ for every $n\geqq 1$ and let $S_n=\sum_{i=1}^n X_i$ and $s_n^2=ES_n^2$. We will always assume that $s_n^2\to\infty$ as $n\to\infty$. Consider a sequence $\{k_n, n\geqq 0\}$ of real numbers satisfying

$$(1)\qquad 0=k_0<k_1<k_2<\dots,\quad \lim_{n\to\infty}\max_{1\leqq i\leqq n}(k_i-k_{i-1})/k_n=0,$$

and for each $n\geqq 1$ define

$$W_n(t)=S_{m_n(t)}/s_n,\quad t\in[0,1],$$

where $m_n(t)=\max\{i\geqq 0: k_i\leqq tk_n\}$. The function $\omega\to W_n(t,\omega)$ is a measurable map from (Ω, F) into $(D[0,1], B)$, where $D[0,1]$ is the set of all functions, defined on the interval $[0,1]$, which have left hand limits and are continuous from the right at every point, and B is the Borel σ-field on $D[0,1]$ induced by the Skorohod topology (cf. [1]).

In this paper we present some sufficient and necessary conditions for the weak convergence, in $D[0,1]$, of the random elements $\{W_n, n\geqq 1\}$ to the standard Brownian process on $D[0,1]$, denoted by W in the sequel. We give the invariance principle for nonstationary, mixing-type sequences under Lindeberg's condition. This result improves the moment conditions used by McLeish and Peligrad in [6]—[11]. For example, we do not assume that $\lim s_n^2/n=\sigma^2>0$ and $\{X_n^2, n\geqq 1\}$ is uniformly integrable. Moreover, we consider the processes $\{S_{m_n(t)}/s_n, 0\leqq t\leqq 1\}$, while the authors of the above mentioned papers have investigated the processes $\{S_{[nt]}/\sigma n^{1/2}, 0\leqq t\leqq 1\}$. Thus we obtain a version for the dependent case of an invariance principle of Prohorov which treats the independent case under Lindeberg's condition (cf. [1, Problem 1, p. 77]).

2. Results

The main results of this paper are given in the following three theorems:

THEOREM 1. *Let* $\{X_i, i\geqq 1\}$ *be a centered* ψ*-mixing sequence of random variables having finite second moments, satisfying*

$$\lim_{n\to\infty} s_u^{-2} \sum_{i=1}^{n} EX_i^2 I(|X_i| > \varepsilon s_n) = 0 \quad \textit{for every} \quad \varepsilon > 0 \tag{2}$$

and

$$\lim_{n\to\infty} s_n^{-1}(\max_{1\leqq j\leqq n} E|X_j|) \sum_{i=1}^{n} \psi(i) = 0. \tag{3}$$

Then

$$W_n \to W \quad \textit{weakly in} \quad D[0, 1], \quad n\to\infty, \tag{4}$$

provided the sequence $\{k_n, n\geqq 0\}$ *satisfies* (1) *and*

$$s_n^2 = k_n h(k_n), \quad \textit{where} \quad h: R_+ \to R_+ \quad \textit{is a slowly varying function.} \tag{5}$$

THEOREM 2. *Let* $\{X_i, i\geqq 1\}$ *be a centered* ψ*-mixing sequence of random variables having finite* $(2+\delta)$*-th moments for some* $\delta>0$,

$$\lim_{n\to\infty} s_n^{-2-\delta} \sum_{i=1}^{n} E|X_i|^{2+\delta} = 0 \tag{2'}$$

and

$$\sum_{i=1}^{\infty} [\psi(i)]^{(2+\delta)/(1+\delta)} < \infty. \tag{3'}$$

Then, for every sequence $\{k_n, n\geqq 0\}$ *satisfying* (1) *and* (5), $W_n\to W$ *weakly in* $D[0, 1]$, *as* $n\to\infty$.

THEOREM 3. *Let* $\{X_i, i\geqq 1\}$ *be a centered* φ*-mixing sequence of random variables having finite second moments satisfying*

$$\lim_{n\to\infty} s_n^{-2}(E \sum_{i=1}^{n} |X_i| I(|X_i| > \varepsilon s_n))^2 = 0 \quad \textit{for every} \quad \varepsilon > 0, \tag{2''}$$

and

$$\sum_{i=1}^{\infty} \varphi(i) < \infty. \tag{3''}$$

Then, for every sequence $\{k_n, n\geqq 0\}$ *satisfying* (1) *and* (5), *the invariance principle* (4) *holds.*

REMARK 1. (i) If $\varphi(1)<1$, then condition (2) is necessary for the invariance principle. The proof of this fact is essentially the same as that given by Peligrad [11, Proposition 2.2].

(ii) Condition (5) (with $\{k_n, n\geqq 0\}$ satisfying (1)) is necessary for (4), too. In fact it is enough to apply the method presented by Herrndorf in [2, Remark 2.3] with some modifications (for a detailed proof see [12]). It is also worth noticing that condition (1) is equivalent (under $k_k\nearrow\infty$) to $\lim_{n\to\infty} k_{n+1}/k_n=1$.

REMARK 2. Let us observe that in our theorems we do not assume that $\lim s_n^2/n=\sigma^2>0$. This condition, which seems to be very restrictive even in the stationary case, was considered by many authors as basic one (cf. [9], [8]). We also show that the mixing-rate conditions presented in [1, Theorem 20.1], [6], [7] and [8] may be weakened. Out theorems extend all the above mentioned results ([2], [4], [6], [8]—[11]) to "essentially nonstationary" sequences of weakly dependent random variables.

3. Proofs

In what follows we need the following lemmas:

LEMMA 1 ([11, Lemmas 3.1 and 3.2]). *Let* $\{X_n, n\geqq 1\}$ *be a sequence of random variables,* $S_n=\sum_{i=1}^n X_i$. *Suppose that for some* $b>0$, $p\in N$, *and* $a_0>0$

$$\varphi(p)+\max_{1\leqq i\leqq m} P(|S_m-S_i|>2^{-1}ba_0)\leqq\eta<1. \tag{6}$$

Then for every $a\geqq a_0$ *and* $m>p$ *the following relations hold:*

$$P(\max_{1\leqq i\leqq m}|S_i|>(1+b)a)\leqq P(|S_m|>a)/(1-\eta)+P(\max_{1\leqq i\leqq m}|X_i|>ba/2(p-1))/(1-\eta) \tag{7}$$

and for every $A\geqq a_0^2$

$$\begin{aligned} ES_m^2I(S_m^2>(1+2b)^2A)&\leqq(1-\eta)^{-1}\eta(1+2b)^2ES_m^2I(S_m^2>A)+\\ &+4b^{-2}(1-\eta)^{-1}p^2(1+2b)^2E\max_{1\leqq i\leqq m}X_i^2I(\max_{1\leqq i\leqq m}X_i^2>(2p)^{-2}Ab^2). \end{aligned} \tag{8}$$

LEMMA 2 ([11, Proposition 2.1]). *Let* $\{X_n, n\geqq 1\}$ *be a centered* φ*-mixing sequence of random variables. Then,* $\{\max_{1\leqq i\leqq n} S_i^2/s_n^2, n\geqq 1\}$ *is uniformly integrable if and only if* $\{\max_{1\leqq i\leqq n} X_i^2/s_n^2, n\geqq 1\}$ *is uniformly integrable.*

LEMMA 3. *Let* $\{X_i, i\geqq 1\}$ *and* $\{\{k_n, n\geqq 0\}$ *be two sequences which satisfy the assumptions of Theorem* 1, 2 or 3. *Then there exists a sequence* $\{\varepsilon_n, n\geqq 1\}$ *of positive numbers tending to zero which satisfies each of the relations*

$$\lim_{n\to\infty} s_n^{-2}\varepsilon_n^{-1}\sum_{i=1}^n EX_i^2I(|X_i|>\varepsilon_n s_n)=0, \tag{9}$$

$$\lim_{n\to\infty} s_n^{-2}\max_{1\leqq j\leqq n} E(\sum_{i=1}^j X_iI(|X_i|>\varepsilon_n s_n))^2=0, \tag{10}$$

$$\max_{1\leqq k\leqq n-1} s_n^{-1}\Big|\sum_{i=k+1}^n E(X_iI(|X_i|\leqq\varepsilon_n s_n)|F_1^k)\leqq\delta_n, \quad \textit{for all}\quad n\geqq 1 \tag{11}$$

and for some sequence of positive constans $\delta_n\to 0$.

PROOF. It is easy to choose (using (2), (2′) or (2″), respectively) a sequence $\varepsilon_n \downarrow 0$, satisfying (9). Note that, by Lemma (1.2) [9]

$$s_n^{-2} \max_{1 \leqq j \leqq n} E\Big(\sum_{i=1}^{j} X_i I(|X_i| > \varepsilon_n s_n\Big)^2 \leqq s_n^{-2} E\Big(\sum_{i=1}^{n} |X_i| I(|X_i| > \varepsilon_n s_n)\Big)^2 \leqq$$

$$\leqq s_n^{-2} \sum_{i=1}^{n} EX_i^2 I(|X_i| > \varepsilon_n s_n) + \big(1+\psi(1)\big)\Big(s_n^{-1} \sum_{i=1}^{n} E|X_i| I(|X_i| > \varepsilon_n s_n)\Big)^2 \leqq$$

$$\leqq s_n^{-2} \sum_{i=1}^{n} EX_i^2 I(|X_i| > \varepsilon_n s_n) + \big(1+\psi(1)\big)\Big(s_n^{-2} \varepsilon_n^{-1} \sum_{i=1}^{n} EX_i^2 I(|X_i| > \varepsilon_n s_n)\Big)^2,$$

and

$$\max_{1 \leqq k \leqq n} s_n^{-1} \Big| \sum_{i=k+1}^{n} E\big(X_i I(|X_i| \leqq \varepsilon_n s_n) F_1^k\big) \leqq$$

$$\leqq \max_{1 \leqq k \leqq n} s_n^{-1} \sum_{i=k+1}^{n} E\big(X_i I(|X_i| \leqq \varepsilon_n s_n) | F_1^k\big) - EX_i I(|X_i| \leqq \varepsilon_n s_n)| +$$

$$+ s_n^{-1} \sum_{i=1}^{n} \big|EX_i\, I(|X_i| \leqq \varepsilon_n s_n)\big| \leqq \max_{1 \leqq k \leqq n} 2 s_n^{-1} \sum_{i=k+1}^{n} \psi(i-k) E|X_i| I(|X_i| \leqq \varepsilon_n s_n) +$$

$$+ s_n^{-2} \varepsilon_n^{-1} \sum_{i=1}^{n} EX_i^2 I(|X_i| > \varepsilon_n s_n).$$

Moreover, by Hölder's inequality and Lemma 1.1.8 [5]

$$\max_{1 \leqq k \leqq n} \sum_{i+1}^{n} \psi(i-k) E|X_i|/s_n \leqq$$

$$\leqq \Big(s_n^{-(2+\delta)} \sum_{i=1}^{n} E|X_i|^{2+\delta}\Big)^{1/(2+\delta)} \Big(\sum_{i=1}^{n} [\psi(i)]^{(2+\delta)/(1+\delta)}\Big)^{(1+\delta)/(2+\delta)}$$

and

$$\max_{1 \leqq k \leqq n} s_n^{-1} \sum_{i=k+1}^{n} \big|E\big(X_i I(|X_i| \leqq \varepsilon_n s_n)\big) - E\big(X_i I(|X_i| \leqq \varepsilon_n s_n) F_1^k\big| \leqq 2\varepsilon_n \sum_{i=1}^{n} \varphi(i).$$

Finally, taking into account the above inequalities, (9) and the assumptions of Theorem 1, 2, or 3, respectively, we obtain (10) and (11).

PROOF OF THE THEOREMS. We will apply Theorem 19.2 of Billingsley [1]. From the φ-mixing condition, by induction, it follows that $W_n(t)$ has asymptotically independent increments. (See the proof of Theorem 20.1 of [1].) Clearly $EW_n(t)=0$ and $EW_n^2(t)=s_{m_n}^2/s_n^2 \to t$ as $n \to \infty$, according to (1), (5) and the Karamata representation of slowly varying functions (cf. Theorem 1.2 [13]). To complete the proof, we have to verify that $\{W_n^2(t),\, n \geqq 1\}$ is uniformly integrable and W_n is tight. But by (2) it follows that $\big\{\max_{1 \leqq i \leqq n} X_i^2/s_n^2,\, n \geqq 1\big\}$ is uniformly integrable, whence by Lemma 2, we get the uniform integrability of $\{W_n^2(t),\, n \geqq 1\}$ for each t.

Let $\{\varepsilon_n, n \geqq 1\}$ be the sequence from Lemma 3. For every $1 \leqq i \leqq n$, $n \geqq 1$, let us put

$$X_{ni} = X_i I(|X_i| \leqq \varepsilon_n s_n)/s_n, \quad Y_{ni} = X_i/s_n - X_{ni},$$

$$X'_{ni} = X_{ni} - EX_{ni}, \quad Z_i(n) = \sum_{j=1}^{n} E(X'_{nj}|F_1^i),$$

$$U_i(n) = \sum_{j=i+1}^{n} E(X'_{nj}|F_1^i),$$

and define the random functions

$$Z'_n(t) = Z_{m_n(t)}(n), \quad U'_n(t) = U_{m_n(t)}(n),$$

$$W'_n(t) = \sum_{=1}^{m_n(t)} X_{ni}, \quad W''_n(t) = W_n(t) - W'_n(t), \quad t \in [0, 1].$$

Obviously, for every $t \in [0, 1]$

$$W_n(t) = Z'_n(t) - U'_n(t) + EW'_n(t) + W''_n(t)$$

and, for each n, $Z'_n(t)$ is a martingale. Let us observe that by Lemma 3

$$\sup_{0 \leqq t \leqq 1} |EW'_n(t)| \leqq s_n^{-1} \sum_{i=1}^{n} E|X_i| I(|X_i| > \varepsilon_n s_n) \to 0, \tag{12}$$

$$\sup_{0 \leqq s \leqq t \leqq 1} E\big(W''_n(t) - W''_n(s)\big)^2 \leqq 4 \sup_{0 \leqq t \leqq 1} E\big(W''_n(t)\big)^2 \leqq \tag{13}$$

$$4 s_n^{-2} \max_{1 \leqq j \leqq n} E\big(\sum_{i=1}^{j} X_i I(|X_i| > \varepsilon_n s_n)\big)^2 \to 0,$$

$$\sup_{0 \leqq s \leqq t \leqq 1} E\big(U'_n(t) - U'_n(s)\big)^2 \leqq 4 \sup_{0 \leqq t \leqq 1} E\big(U'_n(t)\big)^2 \leqq \tag{14}$$

$$\leqq 8\delta_n^2 + 8\big(s_n^{-1} \sum_{i=1}^{n} E|X_i| I(|X_i| > \varepsilon_n s_n)\big)^2 \to 0.$$

Therefore on account of (12), (13) and (14) we get

$$\sup_{0 \leqq t \leqq 1} E\big(W_n(t) - Z'_n(t)\big)^2 \to 0 \quad \text{as} \quad n \to \infty, \tag{15}$$

so that, for every $t \in [0, 1]$ by the already proved relation $EW_n^2(t) \to t$,

$$E\big(Z'_n(t)\big)^2 \to t \quad \text{as} \quad n \to \infty. \tag{16}$$

Now we show that the sequence $\{W_n, n \geqq 1\}$ is tight in $D[0, 1]$. As $P\big(W_n(0) = 0\big) = 1$, for the tightness of this sequence it is enough to prove (cf. Theorems 19.2, 8.2 and 8.3 [1]) that for every $\varepsilon > 0$

$$\lim_{\delta \downarrow 0; 1/\delta \in N} \limsup_{n \to \infty} \sum_{i=0}^{1/\delta - 1} P\big(\max_{i\delta \leqq s \leqq (i+1)\delta} |W_n(s) - W_n(i\delta)| > \varepsilon\big) = 0. \tag{17}$$

Using (15), (16), the martingale property of $Z_n'(t)$ and the Markov inequality we have for any $a>0$, $b>0$

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\max_{\substack{0\le i\le 1/\delta-1\\ i\delta\le s\le(i+1)\delta}} P\big((W_n((i+1)\delta)-W_n(s))^2>4^{-1}b^2a\big)=0. \tag{18}$$

Choose b, p, δ_0, and n_0 such that for every $\delta<\delta_0$ and $n>n_0$

$$\varphi(p)+\max_{\substack{0\le i\le 1/\delta-1\\ i\delta\le s\le(i+1)\delta}} P\big((W_n((i+1)\delta)-W_n(s))^2>4^{-1}b^2a\big)= \tag{19}$$

$$=\eta'(n,\delta,a)=\eta'<1 \quad\text{and}\quad (1+2b)^2\eta'(1-\eta')^{-1}<1.$$

From (19) and (8) we obtain for every $0\le i\le 1/\delta-1$

(20)

$$E_{(1+2b)^2a}\big(W_n((i+1)\delta)-W_n(i\delta)\big)^2\le(1+2b)^2\eta'(1-\eta')^{-1}E_a\big(W_n((i+1)\delta)-W_n(i\delta)\big)^2+$$

$$+\big(2p(1+2b)\big)^2b^{-2}(1-\eta')^{-1}\sum_{k=m_n(i\delta)+1}^{m_n((i+1)\delta)}E_{a(b/2p)^2}X_k^2/s_n^2,$$

where $E_AX=EXI(X>A)$. Moreover by (5), (15) and (16), for sufficiently small $\delta>0$ we get

$$\limsup_{n\to\infty}\sum_{i=0}^{1/\delta-1}E\big(W_n((i+1)\delta)-W_n(i\delta)\big)^2=O\Big(\sum_{i=0}^{1/\delta-1}\delta\Big)=O(1).$$

By (20), (5) and (2) it follows that

$$\limsup_{\delta\downarrow 0}\limsup_{n\to\infty}\sum_{i=0}^{1/\delta-1}E_{(1+2b)^2a}\big(W_n((i+1)\delta)-W_n(i\delta)\big)^2\le$$

$$\le\limsup_{\delta\downarrow 0}\limsup_{n\to\infty}(1+2b)^2\eta'(1-\eta')^{-1}\sum_{i=0}^{1/\delta-1}E_a\big(W_n((i+1)\delta)-W_n(i\delta)\big)^2.$$

Because both sides of the preceding inequality are decreasing functions in a, and $(1+2b)^2\eta'(1-\eta')^{-1}<1$ we obtain $(a\to 0)$, for every $\varepsilon>0$

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\sum_{i=1}^{1/\delta-1}P\big(|W_n((i+1)\delta)-W_n(i\delta)|>\varepsilon\big)=0. \tag{21}$$

Now (17) is a simple consequence of (7), (2), (5), (18) and (21). Thus the proof of the theorems is complete.

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ON SOME DENSE COMPACT OPERATOR RANGE SUBSPACES IN SEPARABLE FRÉCHET SPACES

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Introduction

Let E be a (Hausdorff) topological vector space (tvs) and X, Y two closed subspaces of E with $X \cap Y = 0$. If $X+Y$ is dense in E (and $X+Y \neq E$), then X, Y are called *(proper) quasi-complements* [4]. In [2] Drewnowski obtained some extensions to F-spaces, i.e. metrizable and complete tvs, of results concerning the existence and properties of quasi-complements in Banach spaces. Among other things he obtained the following results:

(a) *Every closed non-minimal infinite codimensional subspace in a separable F-space has a proper quasi-complement* (an analogue of the Murray—Mackey theorem).

b) *If X, Y are proper quasi-complements in a Fréchet space E, i.e. a locally convex F-space, then there exist quasicomplements $M \supset Y$ to X such that* $\dim(W/Y) = \infty$.

To obtain (b) Drewnowski first proved that whenever T is a continuous not relatively open linear operator from a Fréchet space F into a Fréchet space E, then E contains a closed infinite dimensional subspace N with $N \cap T(F) = 0$. Based on this result and inspired by Shevchik's Theorem 1 of [5] we prove the following: If $T\colon F \to E$ is a continuous not relatively open linear operator from a Fréchet space F into a separable Fréchet space E with dense range, then for every infinite dimensional separable Banach space Z there exists a compact injective linear operator $Q\colon Z \to E$ such that $Q(Z)$ is dense and $Q(Z) \cap T(F) = 0$ (a generalization of a result of Shevchik [5]). Making use of this we establish some property of quasi-complements in separable Fréchet spaces: If X, Y are proper quasi-complements in a separable Férchet space E, then E contains a dense subspace $G \supset Y$ with $G \cap X = 0$.

Results

The following result extends Theorem 1 of [5].

PROPOSITION. *Let T be a continuous not relatively open linear operator from a Fréchet space F into a separable Fréchet space E with dense range. Then for every infinite dimensional separable Banach space Z there exists a compact injective linear operator $Q\colon Z \to E$ such that $Q(Z) = E$ and $Q(Z) \cap T(F) = 0$.*

It is known (cf. e.g. [6], pp. 253—254) that whenever G is a dense non-barrelled subspace in a Fréchet space E, then E contains a dense non-barrelled subspace H (containing G) with a strictly finer metrizable and complete locally convex topology; hence the proposition has an equivalent formulation:

($*$) *Let Z be an infinite dimensional separable Banach space. Then for every dense non-barrelled subspace G in a separable Fréchet space E there exists a compact injective linear operator $Q\colon Z\to E$ such that $Q(Z)$ is dense and $G\cap Q(Z)=0$.*

We observe also (using a similar argument as in [1]) that the assumption "non-barrelled" cannot be replaced by "codim $G=\infty$".

PROOF. By $\|\,.\,\|$ we denote on E an F-norm defining the original topology of E. Using Drewnowski's result (mentioned above, [2], Theorem 5.6) we find in E a closed infinite dimensional subspace N such that $N\cap T(F)=0$. Choose in N a linearly independent sequence (y_n) with $\Sigma\|y_n\|<\infty$ which is m-independent, i.e. if $(a_n)\in l^\infty$ and $\Sigma a_n y_n=0$, then $(a_n)=0$ (this is possible by Theorem 1 of [3]). Let $(x_n : n\in\mathbf{N})$ be a dense linearly independent subset in $T(F)$. Since T is not relatively open, $T(F)$ admits a strictly finer metrizable and complete locally convex topology; by $|\,.\,|$ we denote an F-norm definining this topology. We find a sequence $0<c_n\leqq 1$ such that

$$\Sigma|c_n x_n|<\infty \quad\text{and}\quad \Sigma\|c_n(x_n+y_n)\|<\infty.$$

Take $a_n>0$ and $b_n>0$ such that $|a_n x_n|<2^{-n}$ and $\|b_n(x_n+y_n)\|<2^{-n}$, $n\in\mathbf{N}$. It is enough to put

$$c_n=a_n b_n(1+a_n)^{-1}(1+b_n)^{-1}.$$

Since $\|(x_n+y_n)-x_n\|\to 0$ and E is without isolated points, $\operatorname{lin}(z_n : n\in\mathbf{N})$ is dense in E, where $z_n=c_n x_n+c_n y_n$, $n\in\mathbf{N}$. By the assumption concerning Z there exists a biorthogonal system (u_n, f_n), $n\in\mathbf{N}$, where $(u_n)\subset Z$, $(f_n)\subset Z'$ (=the topological dual of Z), (f_n) is equicontinuous and total over Z. Define a compact linear operator $Q; Z\to E$ putting

$$Q(x)=\Sigma f_n(x)z_n.$$

Observe Q is injective: Since $N\cap T(F)=0$, then $\Sigma f_n(x)c_n y_n=0$ provided $Q(x)=0$. Hence $x=0$ (because (y_n) is m-independent). Clearly $Q(Z)\cap T(F)=0$ and $Q(Z)$ is dense in E; this completes the proof.

COROLLARY 1. *Let X, Y be proper quasi-complements in a separable Fréchet space E. Then E contains a dense subspace G such that $G\supset Y$ and $G\cap X=0$.*

PROOF. Let $T\colon E\to E/Y$ be the quotient map. Since the restricted map $T|X$ is not relatively open and $T(X)$ is dense in E/Y, we apply Proposition to find in E/Y a dense subspace S such that $S\cap T(X)=0$. Then $G=T^{-1}(S)$ is as required.

COROLLARY 2. *Let X be a closed infinite dimensional [non-minimal] subspace in a separable Banach [Fréchet] space E. If* $\dim(E/X)=\infty$, *then E contains a dense subspace G such that* $\dim D=2^{\aleph_0}$ *and* $G\cap X.\ =0$.

COROLLARY 3. *Let T be a continuous not relatively open linear operator from a separable Banach space E into a tvs W. If* $\dim\operatorname{Ker} T=\infty$, *then $T|G$ is injective for some dense subspace G of E.*

We shall say that a subspace V in a tvs E is a *compact operator range* if there exist an infinite dimensional separable Banach space Z and a compact linear operator defined on Z whose range is V.

Making use of Proposition (in its equivalent form) we note the following

COROLLARY 4. *Every infinite dimensional separable Fréchet space contains two dense non-barrelled compact operator range subspaces V and W such that* $V \cap W = 0$.

It is known that every Fréchet space E with $\dim E = 2^{\aleph_0}$ contains a dense barrelled subspace G of infinite codimension such that for every linearly independent sequence $(x_n) \subset E$ which is subseries summable there exists a subsequence (y_n) of (x_n) with $0 \neq \Sigma\, y_n \in G$, [1]. Hence E does not contain an infinite dimensional subspace V that is a continuous linear image of another Fréchet space and such that $G \cap V = 0$. Thus the assumption "G is non-barrelled" in Proposition (in its equivalent form) cannot be replaced by "codim $G = \infty$".

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SYMMETRY FACTORS FOR DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO ORTHOGONAL POLYNOMIALS

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§ 1. Introduction: Motivation and previous results

The classical orthonormal polynomials of Jacobi, Laguerre and Hermite are well known to share many common properties (e.g. three-term recurrence relations, generating functions, Rodrigues formulas, solutions of second order differential equations etc.). One rather intriguing fact about these three polynomial sets is that their orthogonalizing weight function is also a symmetry factor for the differential equation that the polynomials satisfy. More precisely, if $w(x)$ is the weight function and $L(y)=0$ is the second order differential equation, then $w(x)L(y)$ is symmetric.

Recently, there has been a renewed and increasing interest in finding nonclassical orthogonal polynomial solutions to higher order differential equations. In fact, at the present time, there are ten sets of orthogonal polynomials known to satisfy differential equations of the form:

$$\sum_{k=0}^{r} a_k(x) y^{(k)}(x) = \lambda_n y(x). \tag{1.1}$$

The search is continuing to find all differential equations of the form (1.1) that have a sequence of orthogonal polynomial solutions. The interested reader is encouraged to consult [3], [6], [7], [11], [13], [14] for indepth discussions of this work as well as applications of this work. Observing that all ten known differential equations can be made symmetric, Littlejohn set out to find conditions for when a differential expression of the form

$$L(y) = \sum_{k=0}^{r} a_k(x) y^{(k)}(x) \tag{1.2}$$

can be made symmetric. Here we are assuming that $a_k(x)$ is real valued, $x \in I$ where I is some interval of the real line and $a_r(x) \neq 0$ for all $x \in I$.

The Lagrange adjoint of (1.2) is

$$L^+(y) = \sum_{k=0}^{r} (-1)^{(k)} (a_k(x) y(x))^{(k)}.$$

The expression $L(y)$ is said to be symmetric (or formally self adjoint) if $L(y)=L^+(y)$. In [12], we defined a *symmetry factor* for $L(y)$ to be a function $f(x)$ such that $f(x)L(y)$ is symmetric. It is well known that *every* second order differential expression

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) \quad (a_i \in C^i)$$

can be made symmetric when multiplied by

$$(1.3)\qquad f(x)=\frac{\exp\left(\int^x a_1(t)/a_2(t)\,dt\right)}{a_2(x)}.$$

In fact, such an $f(x)$ can be found by solving the first order equation

$$(1.4)\qquad a_2(x)y'(x)+\left(a_2'(x)-a_1(x)\right)y(x)=0.$$

Until recently, however, very little was known on the existence and determination of symmetry factors for higher order differential expressions. It is apparent, though, that in order for a real differential expression to be symmetric, it must have even order. Henceforth, we shall assume $r=2n$. In [12], Littlejohn proved the following theorem:

THEOREM 1. *A function $f(x)$ is a symmetry factor for*

$$(1.2)\qquad L(y)=\sum_{k=0}^{2n} a_k(x)y^{(k)}(x)$$

if and only if $f(x)$ simultaneously satisfies the following system of n homogeneous differential equations:

$$(1.5)\qquad \sum_{s=k}^{n}\sum_{j=0}^{2s-2k+1}\binom{2s}{2k-1}\binom{2s-2k+1}{j}\frac{2^{2s-2k+2}-1}{s-k+1}B_{2s-2k+2}a_{2s}^{(2s-2k+1-j)}(x)y^{(j)}(x)$$
$$-a_{2k-1}(x)y(x)=0,\quad k=1,2,\dots,n,$$

where B_{2i} is the Bernoulli number defined by:

$$\frac{x}{e^x-1}=1-\frac{x}{2}+\sum_{i=1}^{\infty}\frac{B_{2i}x^{2i}}{(2i)!}.$$

Observe that the orders of the equations above are $1, 3, \dots, 2n-1$. When $k=n$, (1.5) becomes

$$(1.6)\qquad na_{2n}(x)y'(x)+\left(na_{2n}'(x)-a_{2n-1}(x)\right)y(x)=0.$$

If $n=1$, notice that equations (1.4) and (1.6) are identical. Solving (1.6), we find that (up to a constant multiple):

$$(1.7)\qquad f(x)=\frac{\exp\left(\frac{1}{n}\int^x a_{2n-1}(t)/a_{2n}(t)\,dt\right)}{a_{2n}(x)}.$$

Hence $f(x)$, given by (1.7), is a symmetry factor for (1.2) if and only if $f(x)$ simultaneously satisfies the system (1.5) for $k=1, 2, \dots, n-1$. We name the equations (1.5) the *symmetry equations* for $L(y)$.

EXAMPLES. 1. Let $L_1(y)=(x^2-x)^2y^{(4)}+2x(x-1)(5x-2)y^{(3)}+x(26x-20)y''+$ $+(18x-2)y'+6y$. In this case, the symmetry equations are:

(i) $x^2(x-1)^2y'+x^2(1-x)y=0$,

(ii) $x^2(x-1)^2y^{(3)}+(12x^3-18x^2+6x)y''+(10x^2-16x+6)y'+(-10x+6)y=0.$
It is easy to see that $y(x)=1-x$ satisfies (i) and (ii) so it is a symmetry factor for $L(y)$.

2. $L_2(y)=y^{(6)}-5y^{(5)}+2y^{(3)}-y''+6y'+3y$. From (1.5), the symmetry factor must necessarily be $e^{-5x/3}$. However, $e^{-5x/3}$ does not satisfy the other symmetry equations so $L_2(y)$ cannot be made symmetric.

§ 2. Orthogonal polynomials

In 1929, S. Bochner [1] solved the following classification problem: (up to a linear change of variable), find all orthogonal polynomials $\{\Phi_n(x)\}$ that satisfy the second order differential equation

$$a_2(x)y''(x)+a_1(x)y'(x) = \lambda_n y(x).$$

Bochner solved this problem showing that there are four such polynomial sets. Besides the three sequences of classical orthogonal polynomials, Bochner realized the existence of a fourth set: these polynomials were subsequently named the Bessel polynomials and studied by H. L. Krall and O. Frink [10]. Bochner's proof relied on his observation that if (1.1) has a polynomial solution of degree m, $m=0, 1, \dots r$, then $a_i(x)$ must be a polynomial of degree $\leqq i$. By considering the possible locations of the roots of $a_2(x)$, he was able to obtain his classification result. In his work, Bochner implicitly posed the following problem: classify all differential equations of the form (1.1) that have a sequence of orthogonal polynomial eigenfunctions. Some early success on this problem was obtained by Hahn [2] and especially by H. L. Krall [8], [9]. In 1938, Krall proved his "classification" theorem:

THEOREM 2. *Let $\Phi_m(x)$, $-\infty<x<\infty$, be a polynomial of degree m, $m=0, 1, 2, \dots$. Then $\{\Phi_m(x)\}$ is an orthogonal polynomial sequence and $\Phi_m(x)$ satisfies*

$$\sum_{i=1}^{r}\sum_{j=0}^{i} l_{ij}x^j y^{(i)}(x) = \lambda_m y(x)$$

if and only if the moments $\{\mu_m\}$ associated with $\{\Phi_m(x)\}$ satisfy:

$$\text{(i)}\qquad \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_m \\ \mu_1 & \mu_2 & & \mu_{m+1} \\ \vdots & \vdots & & \\ \mu_m & \mu_{m+1} & & \mu_{2m} \end{vmatrix} \neq 0, \quad m = 0, 1, \dots,$$

$$(2.1)\qquad \text{(ii)}\quad S_k(m) = \sum_{i=2n+1}^{r}\sum_{u=0}^{i}\binom{i-n-1}{n}P(m-2n-1, i-2n-1)l_{i,i-u}\mu_{m-u} = 0,$$

$$2n+1 \leqq r, \quad m = 2n+1, 2n+2, \dots$$

where $P(n, k)=n(n-1)\dots(n-k+1)$.

As a corollary to his theorem, Krall showed under the same hypotheses that r must be even. The proof of this theorem is very difficult and long. In the next section, we will show how easy the recurrence relations follow from our results. We call system (2.1) the *moment equations* for $\{\Phi_m(x)\}$.

In 1940, Krall [9] illustrated his theorem by classifying all differential equations of order four having orthogonal polynomial solutions. Besides rediscovering the four previously known orthogonal polynomial sets, Krall discovered three fourth order equations having nonclassical orthogonal polynomial solutions. Properties of these polynomials, including their weight functions, were not discussed until 1978 when A. M. Krall and R. D. Morton [4] found their weights and then in 1981, A. M. Krall studied the polynomials [3] and their appropriate boundary value problems.

From the recurrence relations (2.1), Krall and Morton found the moments $\{\mu_m\}$ associated with the polynomials $\{\Phi_m(x)\}$. They then showed that the formal series

$$w(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \mu_m \delta^{(m)}(x)}{m!} \tag{2.2}$$

acts as a formal weight function for $\{\Phi_m(x)\}$. By using the Fourier transform, they were able to obtain classical representations of (2.2).

Using Theorem 1, Littlejohn offered [13] an alternative for determining the weight function. We illustrate this alternative by considering one of the fourth order equations discovered by H. L. Krall:

EXAMPLE. The Laguerre type polynomials [3] satisfy:

$$L_4(y) = x^2y^{(4)} - (2x^2-4x)y^{(3)} + \big(x^2-(2R+6)x\big)y'' + \big((2R+2)x-2R\big)y' = \lambda_m y. \tag{2.3}$$

By Theorem 1, the symmetry equations are:

$$x^2y' + x^2y = 0 \tag{2.4}$$

and

$$x^2y^{(3)} + 6xy'' + \big(-x^2+(2R+6)x+6\big)y' + (2Rx+6)y = 0. \tag{2.5}$$

It is easy to check that $y(x)=e^{-x}$ simultaneously satisfies (2.4) and (2.5) and hence is a symmetry factor for $L_4(y)$. Unfortunately, e^{-x} is not a weight function for the Laguerre type polynomials. However, if we solve (2.4) and (2.5) *distributionally over polynomials* and require that the solution vanishes as $|x|\to\infty$, we do arrive at a weight function for the Laguerre type polynomials: Rewrite (2.4) as $x^2(e^xy)'=0$. Dividing by x^2 introduces the Dirac delta function and its derivative: $(e^xy)'=c_1\delta(x)+c_2\delta'(x)$. Integrating this latter equation yields: $e^xy(x)=c_1H(x)+c_2\delta(x)+c_3$, where $H(x)$ is Heaviside's function. Multiplication by e^{-x} yields $y(x)=c_1e^{-x}H(x)+c_2\delta(x)+x_3e^{-x}$. Since we require $y(x)\to 0$ as $|x|\to\infty$ we must have $c_3=0$. Substitution of $y(x)$ into (2.5) yileds $c_2R=c_1$. Choosing $c_1=1$, gives us $y(x)=e^{-x}H(x)+(1/R)\delta(x)$.

This example illustrates the following theorem of Littlejohn [13]:

THEOREM 3. *Suppose $\{\Phi_m(x)\}$ is a sequence of orthogonal polynomial solutions to the differential equation:*

$$\sum_{i=1}^{2n} \sum_{j=0}^{i} l_{ij} x^j y^{(i)}(x) = \lambda_m y(x).$$

Suppose Λ is a nontrivial solution to the system of distributional differential equations

(2.6)

$$\left\langle \sum_{s=k}^{n} \sum_{j=0}^{2s-2k+1} \binom{2s}{2k-1} \binom{2s-2k+1}{j} \frac{2^{2s-2k+2}-1}{s-k+1} B_{2s-2k+2}\, a_{2s}^{(2s-2k+1-j)}(x)\Lambda^{(j)}, \Phi \right\rangle =$$

$$= \langle a_{2k-1}(x)\Lambda, \Phi\rangle, \quad k = 1, 2 \ldots, n,$$

for all polynomials Φ, where

$$a_i(x) = \sum_{j=0}^{i} l_{ij} x^j.$$

Then Λ is an orthogonalizing weight distribution for $\{\Phi_m(x)\}$.

The equations (2.6), when solved in the above distributional sense, are called the *weight equations* for Λ.

§ 3. New results

Unfortunately, the symmetry equations given by (1.5) are quite complicated; the appearance of the Bernoulli numbers is intriguing on the one hand yet bothersome and cumbersome on the other hand. The following new result gives a very succinct characterization of an equivalent set of symmetry equations.

THEOREM 4. *Let*

$$L(y) = \sum_{k=0}^{2n} a_k(x) y^{(k)}(x),$$

where $a_k \in C^k(I)$ is real valued and $a_{2n}(x) \neq 0$ for all $x \in I$, I being some interval of the real line. Then $f(x)L(y)$ is symmetric if and only if $f(x)$ simultaneously satisfies the n homogeneous differential equations

$$\text{(3.1)} \qquad \sum_{i=2k+1}^{2n} (-1)^i \binom{i-k-1}{k} (a_i(x) f(x))^{(i-2k-1)} = 0, \quad k = 0, 1, \ldots, n-1.$$

PROOF (sketch). By definition, $f(x)L(y)$ is symmetric if and only if $f(x)$ simultaneously satisfies the following system of $2n$ equations:

$$A_{k+1} =: \sum_{j=0}^{2n-k} (-1)^{k+j} \binom{k+j}{j} (f(x) a_{k+j}(x))^{(j)} - f(x)\, a_k(x) = 0, \quad k = 0, 1, \ldots, 2n-1.$$

Define

$$C_{k+1} =: \sum_{i=2k+1}^{2n} (-1)^i \binom{i-k-1}{k} (a_i(x) f(x))^{(i-2k-1)}, \quad k = 0, 1, \ldots, n-1.$$

It is easy to see that $C_1'=A_1$, $C_2''+2C_1=A_2$ and it is not too difficult to establish for $3\leqq k\leqq 2n-1$:

$$C_k^{(k)}+kC_{k-1}^{(k-2)}+\sum_{j=3}^{\left[\frac{k+3}{2}\right]}\frac{k(k-j)(k-j-1)\dots(k-2j+3)}{(j-1)!}C_{k-j+1}^{(k-2j+2)}=A_k,$$

where $C_k=0$ if $k>n$ and $[\,.\,]$ is the greatest integer function. From these relations, it is clear that $A_{k+1}=0$, $k=0, 1, \dots, 2n-1$, if and only if $C_{k+1}=0$, $k=0, 1, \dots, \dots, n-1$. □

NOTE. As one can see, proving this theorem is not very difficult; *finding* the conditions (3.1) was much harder.

§ 4. An application of Theorem 4: A glance at a new proof of Krall's classification theorem

A. M. Krall and Littlejohn have recently [5] found a new proof of H. L. Krall's Theorem 2. We shall not reproduce it here; rather, we shall show how naturally the moment equations (2.1) follow from distributionally solving the sytem (3.1).

Assume then that $\{\Phi_m(x)\}$, $-\infty<x<\infty$, is an orthogonal polynomial sequence of solutions to

$$L_{2n}(y)=\sum_{i=1}^{2n}\sum_{j=0}^{i}l_{ij}x^j y^{(i)}(x)=\lambda_m y(x).$$

Assume a symmetry factor $f(x)$ exists for $L_{2n}(y)$. By Theorem 4, $f(x)$ satisfies

$$\sum_{i=2k+1}^{2n}(-1)^i\binom{i-k-1}{k}(a_i(x)f(x))^{(i-2k-1)}=0,\quad k=0, 1, \dots, n-1, \tag{4.1}$$

where $a_i(x)=\sum_{j=0}^{i}l_{ij}x^j$. Let $w(x)$ be the general distributional solution (found weakly on polynomials, as in Theorem 3) to the system (4.1). Then for any polynomial $\Psi(x)$, we have

$$\begin{aligned}0&=\left\langle\sum_{i=2k+1}^{2n}(-1)^i\binom{i-k-1}{k}(a_iw)^{(i-2k-1)},\ \Psi\right\rangle=\\&=\left\langle w,\ \sum_{i=2k+1}^{2n}\binom{i-k-1}{k}a_i\Psi^{(i-2k-1)}\right\rangle.\end{aligned}\tag{4.2}$$

Let $A_{k,\Psi}=\sum_{i=2k+1}^{2n}\binom{i-k-1}{k}a_i\Psi^{(i-2k-1)}$ so $\langle w, A_{k,\Psi}\rangle=0$. Then, it is easy to see that

$$\begin{aligned}0&=\sum_{k=0}^{n-1}(-1)^k\langle w, A_{k,\,\Phi_p^{(n-k)}\Phi_m^{(n-k-1)}-\Phi_m^{(n-k)}\Phi_p^{(n-k-1)}}\rangle=\\&=\langle w, \Phi_m L_{2n}(\Phi_p)-\Phi_p L_{2n}(\Phi_m)\rangle=(\lambda_p-\lambda_m)\langle w, \Phi_p\Phi_m\rangle\end{aligned}$$

so w is an orthogonalizing weight for $\{\Phi_m(x)\}$. Now let $\Psi(x)=x^{m-2k-1}$, $m\geqq 2k+1$, and substitute into

$$0=\left\langle w, \sum_{i=2k+1}^{2n}\binom{i-k-1}{k} a_i \Psi^{(i-2k-1)}\right\rangle=$$

$$=\left\langle w, \sum_{i=2k+1}^{2n}\sum_{j=0}^{i}\binom{i-k-1}{k} P(m-2k-1,\, i-2k-1)\, l_{ij}x^{m-i+j}\right\rangle.$$

Letting $u=i-j$, this latter equation becomes

$$=\left\langle w, \sum_{i=2k+1}^{2n}\sum_{u=0}^{i}\binom{i-k-1}{k} P(m-2k-1,\, i-2k-1)\, l_{i,i-u}x^{m-u}\right\rangle=$$

$$=\sum_{i=2k+1}^{2n}\sum_{u=0}^{i}\binom{i-k-1}{k} P(m-2k-1,\, i-2k-1)\, l_{i,i-u}\mu_{m-u}=S_k(m).$$

These calculations clearly show the explicit relationships that exist among the symmetry equations (1.5), (3.1), the weight equations (2.6), (4.2) and the moment equations (2.1).

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CIRCLES, HOROCYCLES AND HYPERCYCLES IN A FINITE HYPERBOLIC PLANE

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1. Introduction

Let **P** be a projective plane of order $n \equiv 3 \pmod 4$, and π a regular polarity whose absolute points form a conic (suitably defined in the finite case — see [5]). As in the real projective plane, there are two disjoint classes of nonabsolute points:

I $=\{$inner points, or points having 0 absolute lines$\}$;

O$=\{$outer points, or points having 2 absolute lines$\}$;

and their dual classes of non-absolute lines:

o $=\{$outer lines, or lines having 0 absolute points$\}$;

i $=\{$inner lines, or lines having 2 absolute points$\}$.

The incidence structure $HA(n)$ whose points are **I** and lines are **i**, with incidence as given in **P**, has been investigated as a finite analogue of the classical hyperbolic plane ([4]). One of its most interesting features is the existence of parallel points — points with no common line. In [6] we have studied the types of motions which exist in $HA(n)$ when defined over certain *desarguesian* planes, and in this paper we continue this study by investigating the finite analogues of circles, horocycles and hypercycles. Like those of the classical hyperbolic plane, these are shown to be conics.

In this paper the words "point" and "line" always refer to elements of $HA(n)$ unless the adjective "outer" or "absolute" is explicitly inserted. In the figures, lines of $HA(n)$ are represented by solid lines, absolute lines by dotted lines.

2. Finite analogue of a circle

If $HA(n)$ is defined over a desarguesian projective plane **P** of non-square order $n \equiv 3 \pmod 4$, then **P** is pappian ([3], p. 160) and also fanonian, i.e. the diagonal points of a quadrangle are not collinear ([7], pp. 190—194). Thus **P** satisfies the axioms of projective geometry as enunciated by Coxeter ([2], p. 25), and in particular we can exploit the properties of harmonic conjugates and involutory homologies.

DEFINITION 1. In **P**, $\sigma_{A,a}$ denotes the involutory homology with centre A, axis $a = A\pi$. Since only one of A, a is an inner element, $\sigma_{A,a}$ is called a *point reflection* σ_A or *line reflection* σ_a according as A or a belongs to $HA(n)$.

Many properties of point- and line-reflections are derived in [6], and a list of all possible products of reflections is given in Table 1 ([6], p. 493).

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DEFINITION 2. A *pencil of lines* in $HA(n)$ is the set of all lines incident with a common point, having a common absolute point (called an *end*) or having a common perpendicular. Such a pencil is called an *intersecting, parallel* or *ultraparallel* pencil respectively. See figure 1.

As usual, two lines are *perpendicular* if they are conjugate; that is, each is incident with the other's pole ([7], p. 222, [1], p. 157, [4], p. 137).

The three reflection theorem holds for these pencils, just as in the classical planes ([4], pp. 326—329 and [6], p. 292). Recalling a standard definition of a circle in the Euclidean plane as the locus of a point by reflection in all the lines of a pencil, we are led to the following three generalizations of circle, just as in the classical hyperbolic plane ([1], p. 213):

DEFINITION 3. In $HA(n)$, let $\mathbf{G}$ be a pencil of lines and X any point incident with a line of $\mathbf{G}$. Then $\mathbf{C}_X(\mathbf{G})=\{Y=\sigma_a(X)\colon a\in\mathbf{G}\}$ is called a *circle, horocycle* or *ultra-*

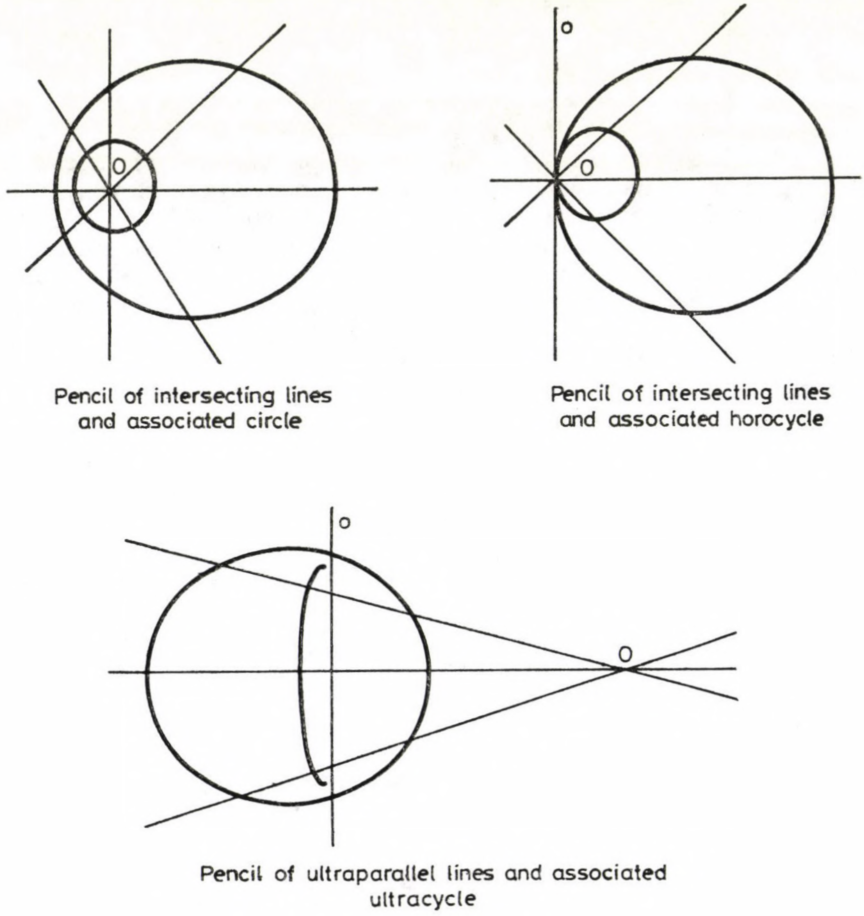

Fig. 1

cycle according as **G** is a pencil of intersecting, parallel or ultraparallel lines. See figure 1.

Note that $\mathbf{C}_X(\mathbf{G})$ is a subset of points of $HA(n)$ ([6], p. 487). Moreover, since X lies on a line of **G**, any point of $\mathbf{C}_X(\mathbf{G})$ must also line on a line of **G**.

DEFINITION 4. The unique line of **G** through any point of $\mathbf{C}_X(\mathbf{G})$ is called a *diameter* through that point.

To generate a horocycle, it is unnecessary to require that X lie on a line of **G**, because any inner point in **P** is joined to an absolute point by an inner line. Thus through any point of $HA(n)$ there passes a unique line of the parallel pencil **G**.

THEOREM 1. *If* $Y\in\mathbf{C}_X(\mathbf{G})$, *then* $\mathbf{C}_Y(\mathbf{G})=\mathbf{C}_X(\mathbf{G})$, *i.e. any circle, horocycle or ultracycle associated with a given pencil is determined by any of its points.*

PROOF. Let $\mathbf{C}_X(\mathbf{G})=\{\sigma_a(X): a\in\mathbf{G}\}$, $\mathbf{C}_Y(\mathbf{G})=\{\sigma_a(Y): a\in\mathbf{G}\}$ where $Y=\sigma_b(X)$ for some $b\in\mathbf{G}$, is a member of $\mathbf{C}_X(\mathbf{G})$. Let Z be any element of $\mathbf{C}_Y(\mathbf{G})$, so that $Z=\sigma_c(Y)$ for some $c\in\mathbf{G}$. Then $Z=\sigma_c(Y)=\sigma_c\sigma_b(X)$ and we wish to show $Z\in\mathbf{C}_X(\mathbf{G})$.

Denoting by d the line through X belonging to **G** (see figure 2), $Z=\sigma_c\sigma_b(X)=\sigma_c\sigma_b\sigma_d(X)=\sigma_f(X)$ where f, the fourth reflection line of $c, b, d,$ also belongs to **G** ([6], p. 492). Thus $Z\in\mathbf{C}_X(\mathbf{G})$ and so $\mathbf{C}_X(\mathbf{G})\supseteqq\mathbf{C}_Y(\mathbf{G})$. In the same way we can show that $\mathbf{C}_Y(\mathbf{G})\supseteqq\mathbf{C}_X(\mathbf{G})$, since $\sigma_b(d)$ is a line of **G** incident with Y. □

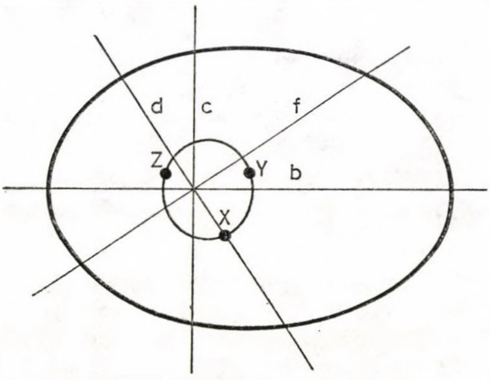

Fig. 2

Since $\mathbf{C}_X(\mathbf{G})$ is independent of the choice of point X, but depends upon the line pencil **G**, we shall refer to $\mathbf{C}_X(\mathbf{G})$ as $\mathbf{C}_\mathbf{G}$.

3. Finite hyperbolic circles

THEOREM 2. *In* $HA(n)$, *the number of points on a circle is* $n+1$.

PROOF. There are $(n+1)/2$ lines through O, the common point of **G** ([4], p. 319). Suppose $\sigma_a(X)=\sigma_b(X)$ for some $a, b\in G$. Then $\sigma_a\sigma_b(X)=X$ which implies $X=O$ (Table 1 of [6]). This contradiction shows there are at least $(n+1)/2$ points in $\mathbf{C}_\mathbf{G}$.

But each diameter intersects $\mathbf{C_G}$ in exactly two points. To see this, consider, without loss of generality, the diameter OX. Now $\sigma_O=\sigma_m\sigma_d$ where d is the line OX and m is the unique perpendicular to d through O ([6], Result 8, p. 488). Thus $\sigma_O(X)=$ $=\sigma_m\sigma_d(X)=\sigma_m(X)$ which is not X, since $\sigma_m(X)=X$ would imply both m and d incident with both O and X. Thus since $m\in G$, X and $\sigma_m(X)$ are two distinct points of $\mathbf{C_G}$ on the diameter OX. Hence every diameter contains two points of $\mathbf{C_G}$, giving precisely $n+1$ points on $\mathbf{C_G}$. □

We note as an interesting corollary that the $(n-3)/2$ points distinct from O on any diameter determine $(n-3)/4$ concentric circles about O.

THEOREM 3. *Any circle is a conic.*

PROOF. Since any oval in a desarguesian projective plane of odd order is a conic by Segre's theorem ([8]), we need only prove that $\mathbf{C_G}$ is an oval, a set of $n+1$ points, no three collinear.

Suppose there exist points $X, Y, Z\in \mathbf{C_G}$ which are incident with a common line c. Without loss of generality, we can assume $Y=\sigma_a(X)$, $Z=\sigma_b(X)$ for some $a, b\in G$. Then $Y=\sigma_a(X)$ implies A, X, Y collinear and similarly B, X, Z are collinear. Since X, Y, Z are incident with c, we must have $c=AB$. Thus c is the polar of the point $a\cdot b$, and so an outer line, which contradiction proves that X, Y, Z are not collinear.

4. Finite hyperbolic horocycles

THEOREM 4. *In $HA(n)$, the number of points on a horocycle is n.*

PROOF. Let

$$\mathbf{C_G}=\{\sigma_a(X)\colon a\in \mathbf{G},\ \text{a pencil of parallel lines with a common end } O\}$$

be a horocycle. Through O there are n lines (and one absolute line o, in $\mathbf{P}$), and no two images of X can be equal since $\sigma_a(X)=\sigma_b(X)$ implies $\sigma_b\sigma_a(X)=X$, a contradiction, for $\sigma_b\sigma_a$ is a parallel displacement ([6], p. 489). Moreover, no diameter of $\mathbf{C_G}$ has two points of the horocycle. For if X and $Y=\sigma_a(X)$ are on the diameter $OX=XY$, then A, X, Y are collinear so that $A\in XY$. But since $a\in\mathbf{G}$, $O\in a$ and so $A\in o$. Thus $A=XY\cdot o=O$, a contradiction, and so $\mathbf{C_G}$ has exactly n points, one on each diameter. □

Again, an interesting corollary is that the $(n-1)/2$ distinct points on any line of $\mathbf{G}$ yield $(n-1)/2$ "concentric" horocycles, i.e. horocycles with the same end.

THEOREM 5. *Any horocycle with its end is a conic.*

PROOF. As in Theorem 2, we need only prove that no three of these $n+1$ points are collinear. Clearly no three are incident with a diameter, so we need only consider the possibility that $X, Y=\sigma_a(X)$ and $Z=\sigma_b(X)$ (where $a, b\in\mathbf{G}$) have a common line c. But $Y=\sigma_a(X)$ and $Z=\sigma_b(X)$ imply $c=AB$ as in Theorem 3. Since $a, b\in\mathbf{G}$ a pencil of parallels with end O in $\mathbf{P}$, $a\cdot b=O$ and so $AB=o$, an absolute line. This is the required contradiction. □

5. Finite hyperbolic hypercycles

THEOREM 6. *In $HA(n)$, the number of points on an ultracycle is* $(n-1)/2$.

PROOF. Let $\mathbf{C_G}=\{\sigma_a(X): a\in G\}$ be an ultracycle generated by the point X belonging to the pencil $\mathbf{G}$ of ultraparallels. $\mathbf{G}$ has cardinality $(n-1)/2$ (dual of [4], Theorem 3) and arguments similar to those used in Theorem 4 show that there is precisely one point of $\mathbf{C_G}$ on each diameter through O. □

Now if X be chosen on the line o, the ultracycle is simply the line o. For if $Y=$ $=\sigma_a(X)$ for some $a\in\mathbf{G}$, then A, X, Y are collinear. But $a\in\mathbf{G}$ (i.e. $O\in a$ in $\mathbf{P}$) which implies $A\in o$, and so this common line of A, X, Y is o. Thus we have:

THEOREM 7. *For each pencil of ultraparallels G, there is one ultracycle which is also a line.* □

We wish to extend an ultracycle to a hypercycle (analogous to the classical equidistant curve) and show that it is also a conic.

LEMMA 1. *Let o be the common perpendicular of a pencil $\mathbf{G}$, and let o intersect the absolute $\mathbf{A}$ in absolute points M and N. Then M and N are not collinear with any point of an ultracycle $\mathbf{C_G}\neq o$, nor are any two points of $\mathbf{C_G}\neq o$ collinear with either M or N.*

PROOF. Since M and N are incident with o, the first part of the theorem is obvious. Now suppose that X and $Y=\sigma_a(X)\in\mathbf{C_G}$ are collinear with M: $X, Y, M\in$ line $\mathbf{q}$. Since $Y=\sigma_a(X)$ implies A, X, Y are collinear in $\mathbf{P}$, X, Y, M, A all $\in\mathbf{q}$. But $a\in\mathbf{G}$ implies $A\in o$, and so $A=q\cdot o=M$. Thus A is an absolute point, which is a contradiction. □

LEMMA 2. *Let $\mathbf{C_G}$ be an ultracycle belonging to the pencil $\mathbf{G}$ of ultraparallel lines with common perpendicular o. Then $\sigma_o(\mathbf{C_G})$ is also an ultracycle belonging to* $\mathbf{G}$

PROOF. See Figure 3. Let X and $Y=\sigma_a(X)$, for some $a\in\mathbf{G}$, be points of $\mathbf{C_G}$. Then $\sigma_o(Y)=\sigma_o\sigma_a(X)=\sigma_a\sigma_o(X)$ since a and o are perpendicular ([4], p. 321). Thus $Y'=\sigma_o(Y)=\sigma_a(X')$ where $X'=\sigma_o(X)$ is a point incident with the line of the pencil

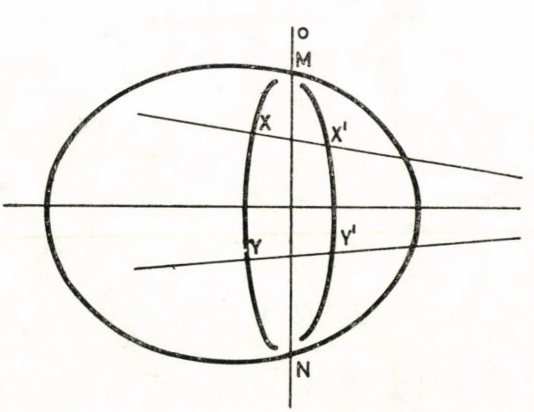

Fig. 3

$\mathbf{G}$ through Y. Since Y was arbitrary in $\mathbf{C_G}$, we have a new ultracycle $\mathbf{C'_G} = \{Y' = \sigma_a(X') \colon a \in \mathbf{G}\}$. If $X \notin \mathbf{o}$, X' also $\notin \mathbf{o}$. □

DEFINITION 5. A hypercycle belonging to the pencil $\mathbf{G}$ of ultraparallel lines with common perpendicular o is a set of points $\mathbf{C_G} \cup \mathbf{C'_G} \cup \{M, N\}$ where M, N are the points of intersection of o with the absolute $\mathbf{A}$, provided $\mathbf{C_G} \neq o$.

THEOREM 7. *Any hypercycle which is not a line is a conic.*

PROOF. Since a hypercycle has $2(n-1)/2+2=n+1$ points, we again need only prove that no three are collinear. Because of Lemma 1, we must show it is impossible for M (or N) to be collinear with two points of the hypercycle, one from each of the ultracycles. Suppose M is collinear with X and $Y' = \sigma_a(X') = \sigma_a \sigma_o(X)$ for some $a \in \mathbf{G}$. But as before, $Y' = \sigma_a(X')$ implies A, X', Y' are collinear. Since $A \in o$, A and M are both incident with the distinct lines o and $Y'X'$, so that $A = M$, a contradiction. Thus a hypercycle is an oval, and so a conic by Segre's theorem ([8]). □

The ultracycle described here is analogous to the two-branched equidistant curve of the classical hyperbolic plane ([1], pp. 216—217). Again we note that since any diameter of a hypercycle cuts the hypercycle twice, once in each branch, the $(n-1)/2$ distinct points on each line of $\mathbf{G}$ yield $(n-1)/4$ concentric hypercycles.

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SINGULARITIES OF THE STABILITY FRONTIER OF GENERIC FAMILIES OF LINEAR DIFFERENTIAL EQUATIONS

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I. Introduction

Let us consider the family of linear differential equations

$$a_n(\lambda)y^{(n)}+a_{n-1}(\lambda)y^{(n-1)}+\ldots+a_1(\lambda)y'+a_0(\lambda)y=0 \tag{1}$$

where $\lambda\in\Lambda$, Λ is a C^∞-manifold, $a_j\in C^\infty(\Lambda, \mathbf{R})$, $j=0, 1, \ldots, n$. The set of values of the parameter λ in which the greatest real part of the roots of the characteristic polynomial is annulled, is called stability frontier.

For arbitrary a_j, $j=0, 1, \ldots, n$ the stability frontier has a very complicated structure, even locally speaking [1], [2]. But under some conditions we are going to define below, its local structure can be classified. For the case $a_n(\lambda)=1$ L. V. Levantovsky [3] has shown that the stability frontier of the generic families has only a finite number of local models.

In this paper we shall give a list of local models to which the stability frontier of an arbitrary generic family is locally equivalent.

II. Stratification of the space of polynomials

Let M be the space of polynomials $a_nt^n+\ldots+a_1t+a_0$ where $a_n^2+a_{n-1}^2+\ldots+a_1^2\neq 0$, and let F be the stability frontier of this space, i.e. the set of polynomials whose greatest real parts of the roots is annulled.

DEFINITION 2.1. We denote by $L_l(k; k_1, \ldots, k_r)$ the subset of F in which the greatest real part is reached at one real root with multiplicity k, r pairs of complex roots with multiplicities $k_1, \ldots, k_r$, respectively and $a_n=a_{n-1}=\ldots=a_{n-l+1}=0$, $a_{n-l}\neq 0$ where a_j are coefficients of the polynomials in M, $j=0, 1, \ldots, n$.

THEOREM OF DECOMPOSITION (see [4]). *Each polynomial* $\sum_{j=0}^{n} a_jt_j$ *whose coefficients belong to the ring of germs of smooth functions at p can be decomposed into factors as follows:*

$$\sum_{j=0}^{n} a_jt^j=\prod_{j=1}^{r} P_j(t)\cdot \prod_{k=1}^{s} Q_k(t)\cdot \prod_{k=1}^{s} \bar{Q}_k(t)\cdot R(t)$$

where $a_j \in C_p^\infty$, *the ring of germs of smooth function at* p, $p \in \Lambda$, $j=0, 1, \dots n$,

$$P_j(t) = (t-\alpha_j)^{k_j} + \sum_{h=1}^{k_j} u_{jh}(t-\alpha_j)^{k_j-h},$$

$$Q_k(t) = (t-\beta_k-iw_k)^{l_k} + \sum_{h=1}^{l_k} v_{kh}(t-\beta_k-iw_k)^{l_k-h},$$

$$\bar{Q}_k(t) = (t-\beta_k+iw_k)^{l_k} + \sum_{h=1}^{l_k} \bar{v}_{kh}(t-\beta_k+iw_k),^{l_k-h},$$

$$R(t) = \sum_{h=0}^{L} q_h t^h, \quad L = n - \sum_{j=1}^{r} k_j - 2\sum_{k=1}^{s} l_k,$$

$u_{jh} \in M_p$, *the ideal of germs of smooth functions being annulled at the point* p, $h=1, 2, \dots, \dots, k_j$; $j=1, 2, \dots, r$, $v_{jh} \in \mathcal{M}_p$, *the ideal of germs of complex valued smooth functions being annulled at the point* p, $h=1, 2, \dots, l_j$, $j=1, 2, \dots, r$, $q_h \in M_p$, $h=1, 2, \dots L$, $q_0 \in C_p^\infty \setminus M_p$.

LEMMA 2.1. *$L_l(k, k_1, \dots, k_r)$ is a submanifold of M whose codimension equals* $l+k+2\sum_{i=1}^{r} k_i - r$.

PROOF. By using the Theorem of Decomposition we have

$$\sum_{i=0}^{n} x_i t^i = P(t) \cdot \prod_{k=1}^{r} Q_k(\tau_k) \cdot \prod_{k=1}^{r} \bar{Q}_k(\bar{\tau}_k) \cdot R(t) \cdot S(t)$$

where $x_i : (x_0, x_1, \dots x_n) \in \mathbf{R}^{n+1} \mapsto x_i$ $(i=0, 1, \dots, n)$,

$$\sum_{i=1}^{n} x_i(p) t^i \in L_l(k, k_1, \dots, k_r),$$

$$P(t) = (t-a_1)^k - \sum_{h=1}^{k} a_h(t-a_1)^{k-h},$$

$$Q_j(\tau_j) = (\tau_j - b_{j_1}) - \sum_{h=2}^{k_j} b_{jh}(\tau - b_{j1})^{k_j-h}, \quad \tau_j = t - iw_j,$$

$$S(t) = \sum_{h=0}^{l} c_h t^h.$$

This polynomial will become constant at the point p. $R(t) = R_1(t) \dots R_s(t)$ where the factors $R_j(t)$ $(j=1, \dots, s)$ have the form

$$R_j(t) = (t-\gamma_j)^m + \sum_{h=1}^{m} d_h(t-\gamma_j)^{m-h}$$

where the real parts of γ_j are negative.

It is clear that in a neighbourhood of p, the set $L_1(k; k_1, \dots k_r)$ is determined by the following equations:

$$(1) \qquad \begin{cases} a_h = 0, & h = 1, 2, \dots, k \\ \operatorname{Re} b_{jh} = 0, & h = 1, 2, \dots, k_j; \quad j = 1, 2, \dots, r \\ \operatorname{Im} b_j h = 0, & h = 2, 3, \dots, k_j; \quad j = 1, 2, \dots, r \\ c_h = 0, & h = 1, 2, \dots, l. \end{cases}$$

Besides, it is not difficult to show that the system (1) is composed of independent equations. So we have the conclusion of the lemma.

From this lemma and the definition of $L_l(k; k_1, \dots, k_r)$ it follows that

$F = \coprod L_l(k; k_1, \dots, k_r)$ (disjoint union)

where $l, k, k_1, \dots, k_r$ run throughout the set of natural numbers, each $L_l(k; k_1, \dots, \dots, k_r)$ is a submanifold. It is not difficult to prove that this decomposition of F is in fact a stratification of F. But in what follows we need only the decompositionin to disjoint submanifolds of F.

III. Local models of singularities

DEFINITION 3.1. Every mapping $\Lambda \to M$ transversal to every stratum $L_l(k; k_1, \dots, \dots, k_r)$ is called a generic family.

COROLLARY 3.1. *The generic families form an everywhere dense set in $C^\infty(\Lambda, M)$, more precisely a Baire set in $C^\infty(\Lambda, M)$.*

THEOREM 3.1. *For every generic family of differential equations*

$$a_n(\lambda) y^{(n)} + \dots + a_1(\lambda) y' + a_0(\lambda) y = 0$$

the stability frontier is locally equivalent to one of the following surfaces:

$$0 = \max_{1 \le j \le r} (x_1 + \mu(x_2, \dots x_k), y_{j1} + v_j(y_{j3}, y_{j4}, \dots, y_{j2k_j}), \xi(y_{01}, y_{02}, \dots y_{0l}))$$

where $x_1, x_2, \dots, x_k, \; y_{j1}, \; y_{j3}, \; y_{j4}, \dots, y_{j2k_j}, \; y_{01}, \; y_{02}, \dots, y_{0l}; j = 1, 2, \dots, r$ are the $k + l + 2 \sum_{j=1}^{r} k_j - r$ fist coordinates of $\mathbf{R}^{\dim \Lambda}$, $\mu(x_2, \dots, x_k)$ is the greatest real part of all the roots of the polynomial $t^k - \sum_{i=2}^{k} x_i t^{k-i}$, $v_j(y_{j3}, y_{j4}, \dots, y_{j2k_j})$ is the greatest real part of all the roots of the polynomial $t^{k_j} - \sum_{h=2}^{k_j} (y_{j2h-1} + i y_{j2h}) t^{kj-h}$, $\xi(y_{01}, y_{02}, \dots y_{0l})$ is the greatest real part of all the roots of the polynomial

$$y_{01} t^l + y_{02} t^{l-1} + \dots + y_{0l} t + 1 \quad \textit{if} \quad \sum_{j=1}^{l} y_{0j}^2 \neq 0,$$

$$\xi = -\infty \quad \textit{if} \quad \sum_{j=1}^{l} y_{0j}^2 = 0.$$

PROOF. Let us consider the generic family in a neighbourhood of $p\in L_l(k; k_1, \ldots, k_r)$. From the transversality of the family to $L_1(k; k_1, \ldots, k_r)$ and the equations (1) determining $L_l(k; k_1, \ldots, k_r)$ it follows that the stability frontier in a neighbourhood of λ_0 where the family belongs to the submanifold $L_l(k; k_1, \ldots, k_r)$ is equivalent to the surface mentioned above by a local diffeomorphism. Besides, we have the inequality

$$l+k+2\sum_{j=1}^{r} k_j - r \leqq \dim \Lambda.$$

REMARK. From Theorem 3.1 it is easy to see that there is only a finite number of local models for every fixed n and fixed $\dim \Lambda$ (see L. V. Levantovsky [3] for the case the generic family having the form

$$y^{(n)} + a_1(\lambda)y^{(n-1)} + \ldots + a_n(\lambda)y = 0$$

and the surface mentioned above can be modified as follows

$$0 = \max\left(x_1 + \mu(x_2, \ldots x_k), y_{j1} + \nu_j(y_{j3}, \ldots y_{j2k_j})\right)$$

and $k+2\sum_{j=1}^{r} k_j - r \leqq \dim \Lambda$).

IV. List of local models of the stability frontier

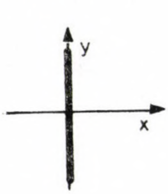
Codimension	Number of strata	Strata
0	0	$L_0(1);\ L_1(0\,1)$
1	2	
2	5	$L_0(2);\ L_0(0\,1,\,1),\ L_0(1\,1)$ $L_1(1);\ L_1(0\,1)$
3	10	...
4	17	...

PROPOSITION 4.1. *The stability frontier of generic families of* 1, 2, 3, 4 *parameters posseses exactly* 1, 4, 11, 23 *different local normal forms.*

PROOF. By calculating concretely we obtain the conclusion.

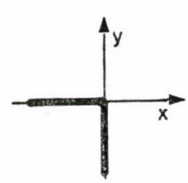
A. *The case* $\dim \Lambda = 1$. The stability frontier is composed of isolated points in the space of parameters.

B. *The case* $\dim \Lambda = 2$. The local models are germs at 0 of the following curves in $\mathbf{R}^2$:

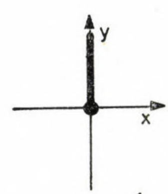


$\{0 = x\}$

$\{0 = x + \mu(y)\}$



$\{0 = \max.(x,y)\}$



$\left\{0 = \begin{cases} \max.(x, \frac{-1}{y}) \text{ if } y \neq 0 \\ \max.(x, -\infty) \text{ if } y = 0 \end{cases}\right\}$

C. *The case* dim $\Lambda=3$. The local models are germs at 0 of the following surfaces in $\mathbf{R}^3$

$$L_0(1),\quad L_0(0,1) \qquad \{0=x\}$$

$$L_0(1,1),\quad L_0(0,1,1) \qquad \{0=\max(x,y)\}$$

$$L_0(2) \qquad \left\{0=x+\begin{cases}\sqrt{y} & \text{if } y \geqq 0\\ 0 & \text{if } y<0\end{cases}\right\}$$

$$L_1(1),\quad L_1(0,1) \qquad \left\{0=\begin{cases}\max\left(x, -\dfrac{1}{y}\right) & \text{if } y \neq 0\\ \max(x, -\infty) & \text{if } y=0\end{cases}\right\}$$

$$L_0(3) \qquad \{0=x+\mu(y,z)\}$$

$$L_0(2,1) \qquad \left\{0=\max\left(x+\begin{cases}\sqrt{y} & \text{if } y \geqq 0\\ 0 & \text{if } y<0\end{cases}, z\right)\right\}$$

$$L_0(1,1,1),\quad L_0(0,1,1,1) \qquad \{0=\max(x,y,z)\}$$

$$L_1(1,1),\quad L_1(0,1,1) \qquad \left\{0=\begin{cases}\max\left(x, y, \dfrac{-1}{z}\right) & \text{if } z \neq 0\\ \max(x, y, -\infty) & \text{if } z=0\end{cases}\right\}$$

$$L_1(2) \qquad \left\{0=\begin{cases}\max\left(x+\begin{cases}\sqrt{y} & \text{if } y \geqq 0\\ 0 & \text{if } y<0\end{cases}, -\dfrac{1}{z}\right) & \text{if } z \neq 0\\ \max\left(x+\begin{cases}\sqrt{y} & \text{if } y \geqq 0\\ 0 & \text{if } y<0\end{cases}, -\infty\right) & \text{if } z=0\end{cases}\right.$$

$$L_2(1),\quad L_2(0,1) \qquad 0=\max\big(x, \xi(y,z)\big)\}.$$

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ON THE CONSTRUCTION OF A CLASS OF ABUNDANT SEMIGROUPS

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0. Introduction

On a semigroup S, the relation $\mathscr{L}^*$ ($\mathscr{R}^*$) is defined by the rule that $a\mathscr{L}^*b$ ($a\mathscr{R}^*b$) if and only if the elements a, b of S are related by Green's relation $\mathscr{L}$ ($\mathscr{R}$) in some oversemigroup of S. Following Fountain [7] we say that a semigroup in which each $\mathscr{L}^*$-class and each $\mathscr{R}^*$-class contains an idempotent is *abundant*. When the idempotents commute in an abundant semigruop, then it is called an *adequate* semigroup. An adequate semigroup S in which $eS\cap aS=eaS$ and $Se\cap Sa=Sae$ for any $e^2=e$, $a\in S$ is called a *type A* semigroup. Adequate semigroups as well as type A semigroups have been studied by Fountain [6]. Inverse semigroups are contained properly in the class of type A semigroups. Some properties of inverse semigroups were extended to type A semigroups (see Fountain [6]).

Recall that regular semigroups are abundant semigroups and in this case $\mathscr{L}^*=\mathscr{L}$ and $\mathscr{R}^*=\mathscr{R}$. Blyth and McFadden [2] described a contruction of all regular semigroups S that possess a normal medial idempotent u. It follows in that case that uSu is an inverse monoid. The "building bricks" in that construction are the idempotent-generated regular semigroups with a normal medial idempotent and the inverse monoids.

In this paper we extend that construction to a class of abundant semigroups S in which the idempotents generate a regular subsemigroup and possess a normal medial idempotent u such that uSu is a type A semigroup. Clearly, this class of abundant semigroups contains the class of regular semigroups considered by Blyth and McFadden [2]. The "building bricks" of this construction are the idempotent-generated regular semigroups with a normal medial idempotent and the type A monoids. The approach adopted closely follows that used by Blyth and McFadden [2].

In the first section we introduce the basic concepts. In Section 2 we give some properties of abundant semigrioups that possess a normal medial idempotent. Sections 3 and 4 are concerned with the general construction which includes a structure theorem for the class of semigroups under consideration. In the final section we indicate how this construction can be put to use.

1. Preliminaries

We begin by recalling some of the basic facts about the relations $\mathscr{L}^*$ and $\mathscr{R}^*$. As stated in the introduction, the relation $\mathscr{L}^*$ ($\mathscr{R}^*$) is defined on a semigroup S by the rule that $a\mathscr{L}^*b$ ($a\mathscr{R}^*b$) if and only if the elements, a, b of S are related by Green's relation $\mathscr{L}$ ($\mathscr{R}$) in some oversemigroup of S. Evidently $\mathscr{L}^*$ ($\mathscr{R}^*$) is a right (left) congruence on S. We put $\mathscr{H}^*=\mathscr{L}^*\cap\mathscr{R}^*$. An alternative description of $\mathscr{L}^*$ ($\mathscr{R}^*$) is provided by the following lemma.

LEMMA 1.1 [6]. *Let S be a semigroup and let $a, b\in S$. Then the following conditions are equivalent:*

1) $a\mathscr{L}^*b$ $(a\mathscr{R}^*b)$;
2) *for any* $x, y\in S^1$, $ax=ay$ $(xa=ya)$ *if and only if* $bx=by$ $(xb=yb)$.

As an easy consequence of Lemma 1.1 we have:

COROLLARY 1.2. *Let S be a semigroup, $a\in S$ and e be an idempotent of S. Then the following conditions are equivalent:*

1) aL^*e (aR^*e);
2) $ae=a(ea=a)$ *and, for* $x, y\in S^1$, $ax=ay$ $(xa=ya)$ *implies* $ex=ey$ $(xe=ye)$.

Obviously, on any semigroup S, we have $\mathscr{L}\subseteqq\mathscr{L}^*$, $\mathscr{R}\subseteqq\mathscr{R}^*$. It is well-known and easy to see for regular elements a, b in S, that $a\mathscr{L}^*b$ $(a\mathscr{R}^*b)$ if and only if $a\mathscr{L}b$ $(a\mathscr{R}b)$. In particular, if S is a regular semigroup, then $\mathscr{L}^*=\mathscr{L}$ and $\mathscr{R}^*=\mathscr{R}$ on S.

Let S be an abundant semigroup with the set E of idempotents and let U be an abundant subsemigroup of S. We say that U is a left (right) *-subsemigroup if for any $a\in U$, there exists $e\in U\cap E$ such that $a\mathscr{L}^{*S}e$ $(a\mathscr{R}^{*S}e)$. U is called a **-subsemigroup* if it is left and right *-subsemigroup. From El-Qallali [3] we have the following.

PROPOSITION 1.3. *Let S be an abundant semigroup and U be an abundant subsemigroup of S. U is a *-subsemigroup if and only if*

$$\mathscr{L}^{*U}=\mathscr{L}^{*S}\cap(U\times U) \quad \textit{and} \quad \mathscr{R}^{*U}=\mathscr{R}^{*S}\cap(U\times U).$$

COROLLARY 1.4. *Let S be an abundant semigroup and e an idempotent of S. Then eSe is a *-subsemigroup.*

If the set of idempotents of S forms a band B, then S is called *quasi-adequate*. For any element a in S, let

$$a^+\in R_a^*\cap B, \quad a^*\in L_a^*\cap B$$

and $E(e)$ as in [8] be the $\mathscr{D}$-class in B that contains the idempotent e. Define the relation δ on S by the rule $(x, y)\in\delta$ if and only if $x=eyf$, for some $e\in E(y^+)$, $f\in E(y^*)$.

A semigroup homomorphism and a congruence relation on S are *good* if they preserve the relations $\mathscr{L}^*$ and $\mathscr{R}^*$. The properties of good homomorphisms and the relation δ have been examined in [5]. One of the main properties of δ is included in the following proposition:

PROPOSITION 1.5 [5]. *If S is a quasi-adequate semigroup and δ is a congruence on S then δ is the minimum adequate good congruence on S.*

A quasi-adequate semigroup S is *idempotent-connected* when for each element a of S, there is a bijection $\alpha\colon\langle a^+\rangle\to\langle a^*\rangle$ satisfying $xa=a(x\alpha)$ for all $x\in\langle a^+\rangle$. A type A semigroup is an idempotent-connected adequate semigroup. This is equivalent to the definition given in the introduction (cf. [4]).

A quasi-adequate semigroup S with band of idempotents B is *split* when δ is a congruence on S and the natural homomorphism $\delta^\natural\colon S\to S/\delta$ is split in that there is a good homomorphism $\pi\colon S/\delta \to S$ such that $\pi \cdot S^{\natural} = 1_{S/\delta}$. In [3] we studied the split quasi-adequate semigroups which satisfy the following condition: There is a splitting homomorphism $\pi\colon S/\delta \to S$ such that for any $a \in \operatorname{Im} \pi$, there is an E-preserving bijection $\theta\colon a^+Ba^+ \to a^*Ba^*$ with $xa = a(x\theta)$ for any $x \in a^+Ba^+$, where E is the set of idempotents of $\operatorname{Im} \pi$.

It turns out that the semigroup which will be considered in Section 5 can be decomposed into two semigroups each one of them lies in the class of split quasi-adequate semigroups which was described in [3].

2. Normal medial idempotents

Let S be an abundant semigroup and E the set of its idempotents. Let $\langle E \rangle$ be the subsemigroup generated by E. Following Blyth and McFadden [2], we say that an idempotent u in S is *medial* if it is such that $x = xux$ for any x in $\langle E \rangle$. A medial idempotent u will be called *normal* if the subband $u\langle E \rangle u$ is a semilattice. Now assume throughout this section that S is an abundant semigroup with a normal medial idempotent u. As a generalization of Theorem 2.2 in [2] we have the following:

PROPOSITION 2.1. (i) *S is quasi-adequate if and only if u is a middle unit.*

(ii) *S is adequate if and only if u is an identity element.*

PROOF. i) If u is a middle unit then $x = xux = xx$ for any x in $\langle E \rangle$. Therefore $\langle E \rangle = E$ and S is quasi-adequate. Conversely, let S be quasi-adequate. Then E is a band, $\mathscr{D}$ is conguence on E and the $\mathscr{D}$-classes of E are the rectangular bands $E(e)$ where $e \in E$. Now, since

$$E(x) = E(xux) = E(uxu)$$

we have $x\mathscr{D}uxu$. Then $uxyu\mathscr{D}xy\mathscr{D}uxu \cdot uyu$ for any $x, y \in E$. Thus $uxyu = uxu \cdot uyu$, and

$$xy = xuxyuy = x \cdot uxu \cdot uyu \cdot y = xuy.$$

Hence u is a middle unit for E and hence also for S because for any $a, b \in S$, there exist $e, f \in E$ such that $a\mathscr{L}^*e$ and $b\mathscr{R}^*f$ whence

$$ab = aefb = aeufb = aub.$$

(ii) If S is adequate and $e \in E$ then

$$e = eue = e^2u = eu = ue.$$

Therefore, $au = aeu = ae = a$ and $ua = ufa = fa = a$ for any $a \in S$ where $e \in L_a^* \cap E$ and $f \in R_a^* \cap E$. Hence u is an identity element. Conversely, if u is the identity element of S then E is a band by (i) and

$$ef = ueu \cdot ufu = ufu \cdot ueu = fe$$

for any $e, f \in E$. Hence S is adequate.

PROPOSITION 2.2. *uSu is an adequate $*$-subsemigroup with u as an identity element.*

PROOF. uSu is an abundant *-subsemigroup by Corollary 1.4 and $u\langle E\rangle u$ is a semilattice. Hence uSu is adequate. Obviously u is the identity element of uSu.

S is said to be *locally adequate* if eSe is an adequate subsemigroup for any idempotent e in S. By a similar proof as in [2] we have the following.

PROPOSITION 2.3. *S is locally adequate.*

3. Construction

In this section we consider the abundant semigroups satisfying the following conditions:

A) The set of idempotents E of S generates a regular subsemigroup $\langle E\rangle$.

B) S contains a normal medial idempotent u.

C) uSu is a type A semigroup.

It is easy to see that this class of abundant semigroups properly contains the class of regular semigroups that possess a normal medial idempotent. Therefore, the construction of the class of semigroups considered in this section extends that considered in [2]. Indeed, our approach is inspired by that in [2]. In particular, the "building bricks" in this construction will be idempotent-generated regular semigroups with normal medial idempotent which are characterized in [2] and type A semigroups which are analogues of inverse semigroups.

Suppose then that $\langle E\rangle$ is an idempotent-generated regular semigroup with a normal medial idempotent u. Let E^0 be the semilattice $u\langle E\rangle u$ and suppose that S is an adequate semigroup whose semilattice of idempotents is isomorphic to E^0. For notational convenience, we shall identify this semilattice with E^0. In doing so, we shall see that u becomes the identity element of S as well as that of E^0; for, if $a\in S$ then we have $e\in L_a^*(S)\cap E^0$ so that $e=eu$ and it follows by Corollary 1.2 that $a=au$ and likewise $a=ua$.

Now let T_{E^0} be the set of all isomorphisms between the principal ideals of E^0. Given $\theta,\ \varphi\in T_{E^0}$ with $\theta\colon e_\theta E^0\to f_\theta E^0$ and $\varphi\colon e_\varphi E^0\to f_\varphi E^0$ we recall (see [8]) that $\theta\varphi\colon e_{\theta\varphi}E^0\to f_{\theta\varphi}E^0$ where $e_{\theta\varphi}=(f_\theta e_\varphi)\theta^{-1}$ and $f_{\theta\varphi}=(f_\theta e_\varphi)\varphi$.

Moreover, T_{E^0} is an inverse semigroup. Observe that if $x\in\langle E\rangle$ and $\theta\in T_{E^0}$ then

$$e_\theta x e_\theta=(e_\theta u)x(ue_\theta)=e_\theta(uxu)e_\theta\in e_\theta E^0 e_0=e_\theta E^0.$$

Consequently, if $\mathcal{T}(\langle E\rangle)$ denotes the full transformation semigroup on $\langle E\rangle$ then we can extend any $\theta\in T_{E^0}$ to a mapping $\bar\theta\in\mathcal{T}(\langle E\rangle)$ by defining

$$x\bar\theta=(e_\theta xe_\theta)\theta\quad\text{for any}\quad x\in\langle E\rangle.$$

Then we know from [2] that the mapping $T_{E^0}\to\mathcal{T}(\langle E\rangle)$ described by $\theta\to\bar\theta$ is a homomorphism.

For any element a in S there is a unique idempotent e in L_a^* and a unique idempotent f in R_a^* (cf. [6]), so let the idempotent in L_a^* be denoted by a^* and the idempotent in R_a^* be denoted by a^+. Suppose now that S is a type A semigroup. From [6] we have a homomorphism $\alpha\colon S\to T_{E^0}$ defined by $a\alpha=\alpha_a$ where $\alpha_a\colon a^+E^0\to a^*E^0$ such that $e\alpha_a=(ea)^*$, $f\alpha_a^{-1}=(af)^+$ for any $a\in S$, $e\in a^+E^0$, $f\in a^*E^0$.

Thus, for any $a, b \in S$, we have

$$\alpha_a \alpha_b = \alpha_{ab}\colon (ab)^+ E^0 \to (ab)^* E^0$$

where $(ab)^+ = (a^* b^+)\alpha_a^{-1}$ and $(ab)^* = (a^* b^+)\alpha_b$.

In what follows $\bar{\alpha}$ denotes the composite homomorphism $S \to T_{E^0} \to \mathscr{T}(\langle E \rangle)$ described by $a \to \alpha_a \to \overline{\alpha_a}$.

Continuing with the above hypotheses on $\langle E \rangle$, E^0 and S, let $\mathscr{L}$ and $\mathscr{R}$ denote Green's relations on $\langle E \rangle$. Then, as in Blyth and McFadden [2] and in El-Qallali [3] we have the following fundamental result.

PROPOSITION 3.1. *Given $a, b \in S$, let $g, h, v, w \in \langle E \rangle$ be such that*

$$g \mathscr{L} a^+, \quad h \mathscr{R} a^*, \quad v \mathscr{L} b^+, \quad w \mathscr{R} b^*.$$

Then $hv \in E^0$ and

$$g(hv)\overline{\alpha_a^{-1}} \mathscr{L} (ahvb)^+, \quad (hv)\bar{\alpha}_b \cdot w \mathscr{R} (ahvb)^*.$$

PROOF. Notice that

$$hv = a^* hv b^+ = u a^* hv b^+ u \in u \langle E \rangle u = E^0$$

and

$$g(hv)\overline{\alpha_a^{-1}}(ahvb)^+ = g(a^* hva^*)\alpha_a^{-1}(a^* hvb^+)\alpha_a^{-1} = g(a^* hva^* hvb^+)\alpha_a^{-1} =$$

$$= g(a^* hva^*)\alpha_a^{-1} = g(hv)\overline{\alpha_a^{-1}},$$

$$(ahvb)^+ g(hv)\overline{\alpha_a^{-1}} = (a^* hvb^+)\alpha_a^{-1} a^+ g(a^* hva^*)\alpha_a^{-1} = (a^* hvb^+ a^* hva^*)\alpha_a^{-1} =$$

$$= (a^* hvb^+)\alpha_a^{-1} = (ahvb^+)^+ = (ahvb)^+.$$

Consequently, we see that $g \cdot (hv)\overline{\alpha_a^{-1}} \mathscr{L} (ahvb)^+$. Similarly, we can show that $(hv)\overline{\alpha_b} \cdot w \mathscr{R} (ahvb)^*$.

Now, since $gu = ga^+u = ga^+ = g$ and $uh = ua^*h = a^*h = h$, we have $g \in \langle E \rangle u$ and $h \in u \langle E \rangle$. Also we notice that $\langle E \rangle u$ is a subband since, for any $y \in \langle E \rangle u$, $y = xu$ for some $x \in \langle E \rangle$ and $y \cdot y = xu \cdot xu = xu = y$. Similarly $u \langle E \rangle$ is a subband. Therefore Proposition 3.1 shows that if we define

$$W = W(E, S, \alpha) = \{(g, a, h) \in \langle E \rangle u \times S \times u \langle E \rangle;\ g \mathscr{L} a^+, h \mathscr{R} a^*\}$$

then the expression

$$(g, a, h)(v, b, w) = \big(g \cdot (hv)\overline{\alpha_a^{-1}}, ahvb, (hv)\overline{\alpha_b} \cdot w\big)$$

defines a binary operation on W. The following sequence of results provides considerably more information.

PROPOSITION 3.2. *W is a semigroup.*

PROOF. *Let (g, a, h), (v, b, w) and (x, c, y) be in W. Then*

$$[(g, a, h)(v, b, w)](x, c, y) = \big(g \cdot (hv)\overline{\alpha_a^{-1}}, ahvb, (hv)\overline{\alpha_b} \cdot w\big)(x, c, y) =$$

$$= \big(g(hv)\overline{\alpha_a^{-1}}((hv)\overline{\alpha_b} wx)\overline{\alpha_{ahvb}^{-1}}, ahvb(hv)\overline{\alpha_b} \cdot wxc, ((hv)\overline{\alpha_b} \cdot wx)\overline{\alpha_c} \cdot y\big)$$

and

$$(g, a, h)[(v, b, w)(x, c, y)] = (g, a, h)\big(v(wx)\overline{\alpha_b^{-1}}, bwxc, (wx)\overline{\alpha_c}y\big) =$$

$$= \big(g(hv(wx)\overline{\alpha_b^{-1}})\big)\overline{\alpha_a^{-1}}, ahv(wx)\overline{\alpha_b^{-1}}bwxc, \big(hv(wx)\overline{\alpha_b^{-1}}\big)\overline{\alpha_{bwxc}}\cdot(wx)\overline{\alpha_c}\cdot y\big).$$

Notice that

$$g(hv)\overline{\alpha_a^{-1}}\cdot((hv)\overline{\alpha_b}wx)\overline{\alpha_{ahvb}^{-1}} = g(hv)\overline{\alpha_a^{-1}}((hv)\overline{\alpha_b}wx)\overline{\alpha_b^{-1}}\overline{\alpha_{hv}^{-1}}\overline{\alpha_a^{-1}} =$$

$$= g\big[hv\big((hv)(wx)\overline{\alpha_b^{-1}}\big)\overline{\alpha_{hv}^{-1}}\big]\overline{\alpha_a^{-1}} = g[hv(wx)\overline{\alpha_b^{-1}}]\overline{\alpha_a^{-1}}.$$

Similarly

$$\big(hv(wx)\overline{\alpha_b^{-1}}\big)\overline{\alpha_{bwxc}}\cdot(wx)\bar{\alpha}_c y = [(hv)\overline{\alpha_b}wx]\overline{\alpha_c}\cdot y.$$

Since S is type A, then

$$b(hv)\overline{\alpha_b} = b(b^+hvb^+)\alpha_b = b^+hvb^+b = hvb$$

and

$$bwx = bb^*wxb^* = (b^*wxb^*)\alpha_b^{-1}bwx = (wx)\overline{\alpha_b^{-1}}bwx.$$

Therefore

$$ahvb(hv)\overline{\alpha_b}wxc = ahvbwxc = ahv(wx)\overline{\alpha_b^{-1}}bwxc$$

and the associativity holds.

PROPOSITION 3.3. *The set of idempotents of W is*

$$E(W) = \{(g, a, h)\colon a\cdot a = a,\ g = ghg,\ h = hgh\}.$$

PROOF. For any (g, a, h) in W we have

$$(g, a, h)\cdot(g, a, h) = \big(g(hg)\overline{\alpha_a^{-1}}, ahga, (hg)\overline{\alpha_a}h\big) =$$

$$= \big(g(a^*hga^*)\alpha_a^{-1}, ahga, (a^+hga^+)\alpha_a\cdot h\big) = \big(g(ahg)^+, ahga, (hga)^*h\big).$$

Thus, if (g, a, h) is an idempotent, then $a=ahga$ which by Corollary 1.2 implies $a^+=ahga^+=ahg$ and $a^*=a^*hga=hga$. Consequently, as $hg\in E^0$ is an idempotent, we have

$$a = ahga = ahg\cdot hga = a^+a^*\in E^0.$$

Thus a is an idempotent. It follows that $a^+=a=a^*$ and so

$$(g, a, h)(g, a, h) = (gahg, ahga, hgah) = (ghg, a, hgh).$$

Thus $g=ghg$ and $h=hgh$.

Suppose, conversely, that $(g, a, h)\in W$ is such that $a\cdot a=a$, $ghg=g$ and $hgh=h$. Then $a^+=a=a^*$. Since $g\mathscr{L}a$, therefore $ug=ua=a$ and $hu=au=a$ ([2], Theorem 2.1).

Now, since $ah=h$ and $a=ug$, we see that $ugh=h$ and $hu\cdot ugh=huh$ which imply $hugh=h$. Therefore

$$hg = hugh\cdot g = hughg = hug.$$

Thus

$$ahga = ahuga = (ahu)(uga) = (a \cdot a)(a \cdot a) = a \cdot a = a$$

and

$$(g, a, h)(g, a, h) = \big(g(ahg)^+, ahga, (hga)^*h\big) = (ga, a, ah) = (g, a, h).$$

Hence (g, a, h) is an idempotent.

PROPOSITION 3.4. *W is abundant.*

PROOF. Given $(g, a, h) \in W$. Consider (g, a^+, a^+). We conclude from Proposition 3.3 that (g, a^+, a^+) is an idempotent in W. Also

$$(g, a^+, a^+)(g, a, h) = \big(g(a^+g)\overline{\alpha_{a^+}^{-1}}, a^+a^+ga, (a^+g)\overline{\alpha_a}h\big) = (g, a, h).$$

Now let (v, b, w) and (x, c, y) be in W so that

$$(v, b, w)(g, a, h) = (x, c, y)(g, a, h)$$

i.e. $\big(v(wg)\overline{\alpha_b^{-1}}, bwga, (wg)\overline{\alpha_a}h\big) = \big(x(yg)\overline{\alpha_c^{-1}}, cyga, (yg)\overline{\alpha_a} \cdot h\big)$. Then we have

$$v(wg)\overline{\alpha_b^{-1}} = x(yg)\overline{\alpha_c^{-1}},$$

$bwga = cyga$ which by Corollary 1.2 implies $bwga^+ = cyga^+$. Similarly, $(wg)\overline{\alpha_a}h = (yg)\overline{\alpha_a}h$ implies $a^+wga^+ = a^+yga^+$, i.e. $(wg)\overline{\alpha_{a^+}} = (yg)\overline{\alpha_{a^+}}$.

However,

$$(v, b, w)(g, a^+, a^+) = \big(v(wg)\overline{\alpha_b^{-1}}, bwga^+, (wg)\bar{\alpha}_{a^+} \cdot a^+\big)$$

and

$$(x, c, y)(g, a^+, a^+) = \big(x(yg)\overline{\alpha_c^{-1}}, cyga^+, (yg)\overline{\alpha_{a^+}} \cdot a^+\big).$$

Thus we conlude

$$(v, b, w)(g, a^+, a^+) = (x, c, y)(g, a^+, a^+).$$

Now by Corollary 1.2, we get $(g, a, h)\mathscr{R}^*(g, a^+, a^+)$. Similarly, the idempotent (a^*, a^*, h) is $\mathscr{L}^*$-related to (g, a, h). Hence the result.

COROLLARY 3.5. *For any (g, a, h) and (v, b, w) in $E(W)$ we have*
1) $(g, a, h)\mathscr{L}(v, b, w)$ if and only if $a=b$ and $h=w$;
2) $(g, a, h)\mathscr{R}(v, b, w)$ if and only if $a=b$ and $g=v$.

PROOF. 1) From the proof of Proposition 3.4, we have

$$(g, a, h)L^*(a^*, a^*, h) \quad \text{and} \quad (v, b, w)\mathscr{L}^*(b^*, b^*, w).$$

If $a=b$ and $h=w$, then $a^*=b^*$ and we have

$$(a^*, a^*, h) = (b^*, b^*, w) \quad \text{and} \quad (g, a, h)\mathscr{L}^*(v, b, w).$$

Since they are regular elements, we obtain

$$(g, a, h)\mathscr{L}(v, b, w).$$

Suppose, conversely that $(g, a, h)\mathcal{L}(v, b, w)$. Then

$$(g, a, h)(v, b, w) = (g, a, h)$$

and

$$(v, b, w)(g, a, h) = (v, b, w),$$

that is

$$\left(g(hv)\overline{\alpha_a^{-1}},\ ahvb,\ (hv)\overline{\alpha_b}w\right) = (g, a, h)$$

and

$$\left(v(wg)\overline{\alpha_b^{-1}},\ bwga,\ (wg)\overline{\alpha_a}h\right) = (v, b, w).$$

Since a and b are idempotents, hence we get

$$(gahva,\ ahvb,\ bhvbw) = (g, a, h)$$

and

$$(vbwgb,\ bwga,\ awgah) = (v, b, w).$$

Thus

$$(ghv,\ hv,\ hvw) = (g, a, h) \quad \text{and} \quad (vwg,\ wg,\ whg) = (v, b, w).$$

It follows that $hv=a$, $hvw=h$ whence $aw=h$. Similarly, $wg=b$ and $bh=w$. Now $aw=h$, $bh=w$ imply $abh=aw=h$, whence $aba=a$. On the other hand, $baw=bh==w$ implies $bab=b$. Since $a, b\in E^0$, we obtain $a=b$ and $hRa=bRw$. Notice that $aw=h$ implies $hw=h$ and $bh=w$ implies $wh=w$ so that wLh. Thus hHw and $h=w$.

2) is proved similarly to (1).

Corollary 3.6. *The relations* $\mathcal{L}^*$, $\mathcal{R}^*$ *and* H^* *on* W *are given by*

1) $(g, a, h)\mathcal{L}^*(v, b, w)$ *if and only if* $a\mathcal{L}^*b$ *and* $h=w$;

2) $(g, a, h)\mathcal{R}^*(v, b, w)$ *if and only if* $a\mathcal{R}^*b$ *and* $g=v$;

3) $(g, a, h)\mathcal{H}^*(v, b, w)$ *if and only if* $a\mathcal{H}^*b$, $h=w$ *and* $g=v$.

Proof. Since (3) is an immediate consequence of (1) and (2), and (2) is the dual of (1), it suffices to prove (1).

From the proof of Proposition 3.4, we have

$$(g, a, h)\mathcal{L}^*(a^*, a^*, h), \quad (v, b, w)\mathcal{L}^*(b^*, b^*, w).$$

Then

$$(g, a, h)\mathcal{L}^*(v, b, w) \Leftrightarrow (a^*, a^*, h)\mathcal{L}(b^*, b^*, w) \Leftrightarrow$$

$$\Leftrightarrow a^* = b^* \quad \text{and} \quad h = w \text{ (Corollary 3.5)} \Leftrightarrow a\mathcal{L}^*b \quad \text{and} \quad h = w.$$

From [2] we have

PROPOSITION 3.7. *If $\langle E(W)\rangle$ denotes the subsemigroup of W generated by the set of idempotents $E(W)$, then*

$$\langle E(W)\rangle = \{(g, a, h)\in W\colon\ a\cdot a = a\}.$$

PROPOSITION 3.8. *(u, u, u) is a normal medial idempotent in W.*

PROOF. The same as in Blyth and McFadden [2].
Also from [2] we have

PROPOSITION 3.9. *$\langle E(W)\rangle$ is isomorphic to $\langle E\rangle$.*

Now consider the subsemigroup $(u, u, u)W(u, u, u)$ of W. Notice that, for any $(g, a, h)\in W$, we have $a^*=ha^*=hua^*=a^*hu=hu$ and $ug=a^+$. Then we obtain that

$$(u, u, u)(g, a, h)(u, u, u) = (u, u, u)\big(g(hu)\overline{\alpha_a^{-1}},\ ahu,\ (hu)\overline{\alpha_u}\cdot u\big) =$$

$$= (u, u, u)\big(g(a^*)\alpha_a^{-1},\ ahua^*,\ uhu\big) = (u, u, u)(ga^+,\ aa^*,\ a^*hu) =$$

$$= (u, u, u)(g, a, a^*) = \big(u(ug)\overline{\alpha_u^{-1}},\ uga,\ (ug)\overline{\alpha_a}\, a^*\big) =$$

$$=\big(u(a^+)\alpha_u^{-1},\ a,\ (a^+)\alpha_a a^*\big) = (a^+, a, a^*).$$

It is now clear that we can define a bijection

$$\theta\colon (u, u, u)W(u, u, u) \to S$$

by $(a^+, a, a^*)\theta=a$. Notice that

$$(a^+, a, a^*)(b^+, b, b^*) = \big(a^+(a^*b^+)\overline{\alpha_a^{-1}},\ aa^*b^+b,\ (a^*b^+)\overline{\alpha_b}b^*\big) =$$

$$= \big(a^+(ab)^+,\ ab,\ (ab)^*b^*\big) = \big((ab)^+,\ ab,\ (ab)^*\big).$$

Therefore

$$\big((a^+, a, a^*)(b^+, b, b^*)\big)\theta = \big((ab)^+, ab, (ab)^*\big)\theta = ab = (a^+, a, a^*)\theta(b^+, b, b^*)\theta.$$

Thus we see that θ is an isomorphism and the following result is established.

PROPOSITION 3.10. *$(u, u, u)W(u, u, u)$ is isomorphic to S.*

Summing up, we have the following theorem:

THEOREM 3.11. *Let $\langle E\rangle$ be an idempotent-generated regular semigroup with a normal medial idempotent u, and S be a type A semigroup whose semilattice of idempotents is $u\langle E\rangle u$. Then $W=W(E, S, \alpha)$ is an abundant semigroup which satisfies the following conditions:*

(i) *the set of idempotents of W generates a regular subsemigroup;*
(ii) *W contains a normal medial idempotent (u, u, u) and;*
(iii) *$(u, u, u)W(u, u, u)$ is a type A semigroup.*

In fact, the converse of this theorem is also true as the subsequent section (Theorem 4.5) will show.

4. Characterization

The aim of this section is to prove that every abundant semigroup which satisfies the conditions (A), (B) and (C) stated at the beginning of Section 3 is isomorphic to some semigroup $W(E, S, \alpha)$ constructed in that section. For this purpose, let S be an abundant semigroup with set of idempotents E that generates a regular subsemigroup $\langle E\rangle$ which possesses a normal medial idempotent u such that uSu is a type A semigroup. Denote the set of idempotents of uSu by E^0. Clearly, we have $E^0=u\langle E\rangle u$.

For any $x\in S$, $uxu\in uSu$. Since uSu is a *-subsemigroup by Corollary 1.4, then for any $e\in E$ such that eR^*xu, we get

$$ue\mathcal{R}^*uxu \quad \text{and} \quad (uxu)^+\mathcal{R}ue.$$

Then $ue=(uxu)^+ue$. Thus

$$ueu=(uxu)^+ueu=ueu(uxu)^+=ue(uxu)^+=(uxu)^+.$$

Observe that eu is an idempotent and $eu\mathcal{R}e$. Therefore $eu\mathcal{R}^*xu$. Also $ueu\mathcal{L}eu$.

On the other hand, for any $f\in E$ with $f\mathcal{L}^*ux$, $uf\mathcal{L}f$ and $uf\mathcal{L}^*ux$. Then $(uxu)^*=ufu$, also $uf\mathcal{R}ufu$.

However eu and uf are uniquely determined by x because if there exist $e'u$, uf' such that $(uxu)^+=ue'u$ and $(uxu)^*=uf'u$, then $eu\mathcal{R}^*xu\mathcal{R}^*e'u$ and $eu\mathcal{L}eue=(uxu)^+=ue'u\mathcal{L}e'u$. Thus $eu\mathcal{H}e'u$ and hence $eu=e'u$. Similarly $uf=uf'$.

In this case let eu be denoted by e_x and uf be denoted by f_x.

LEMMA 4.1. *For any* $x, y\in S$ *we have*
(i) $e_x\mathcal{R}^*x, f_x\mathcal{L}^*x$,
(ii) $e_{xy}=e_x(uxe_y)^+$,
(iii) $f_{xy}=(f_xyu)^*f_y$.

PROOF. (i) Since $e_xxu=xu$, then it follows that for any $g\in L_x^*\cap E$, $e_xxgug=xgug$. Therefore $e_xx=x$. Now, for any $s, t\in S$, we have

$$sx=tx\Rightarrow sxu=txu\Rightarrow se_x=te_x \quad (e_x\mathcal{R}^*xu).$$

Hence $e_x\mathcal{R}^*x$. Similarly $f_x\mathcal{L}^*x$.

(ii) $e_{xy}\mathcal{R}^*xyu\mathcal{R}^*xe_y=e_xuxe_y\mathcal{R}^*e_x(uxe_y)^+$ so that

$$e_x(uxe_y)^+\cdot e_{xy}=e_{xy}\Rightarrow ue_x(uxe_y)^+ue_{xy}=ue_{xy}\Rightarrow$$

$$\Rightarrow ue_{xy}ue_x(uxe_y)^+=ue_{xy}\Rightarrow e_{xy}ue_{xy}e_x(uxe_y)^+=e_{xy}ue_{xy}\Rightarrow e_{xy}e_x(uxe_y)^+=e_{xy}.$$

But $e_x(uxe_y)^+\mathcal{R}e_{xy}$. Hence $e_{xy}=e_x(uxe_y)^+$.

(iii) is similar to (ii).

Consider the semigroup $W=W(E, uSu, \alpha)$. Clearly $(e_x, uxu, f_x)\in W$ for any $x\in S$ and we have a mapping $\theta\colon S\to W$ defined by $x\theta=(e_x, uxu, f_x)$.

LEMMA 4.2. *θ is injective.*

PROOF. If $x\theta=y\theta$, then $e_x=e_y$, $uxu=uyu$ and $f_x=f_y$. Thus

$$uxuf_x = uyuf_y \Rightarrow uxf_x = uyf_y \Rightarrow ux = uy \Rightarrow$$
$$\Rightarrow e_xux = e_yuy \Rightarrow e_xx = e_yy \Rightarrow x = y \text{ (Lemma 4.1 (i))}.$$

Hence the result.

LEMMA 4.3. θ *is surjective.*

PROOF. Let $(g, x, h)\in W$. Then $x=uxu$, $g\mathcal{L}x^+$, $h\mathcal{R}x^*$. Notice that $gxh\in S$ and

$$(gxh)\theta = (e_{gxh}, ugxhu, f_{gxh}).$$

Also

$$ugxhu = ugu\cdot x^+\cdot xx^*uhu = x^+\cdot ugu\cdot x\cdot uhu\cdot x^* = x^+gxhx^* = x^+xx^* = x.$$

Consequently, $g\mathcal{L}x^+=(ugxhu)^+=ug(xhu)^+$ so that $gug(xhu)^+=g$, i.e. $g(xhu)^+= =g$. Therefore

$$e_{gxh}\mathcal{R}^*\ gxhu\ \mathcal{R}^*\ g(xhu)^+ = g.$$

Thus $e_{gxh}\cdot g=g$ so that

$$ue_{gxh}ugu = ug \Rightarrow ugu\cdot ue_{gxh} = ug \Rightarrow gug\,e_{gxh} = gug \Rightarrow ge_{gxh} = g.$$

But $e_{gxh}\mathcal{R}g$ and hence $g\cdot e_{gxh}=e_{gxh}$. Thus $e_{gxh}=g$. Likewise $f_{gxh}=h$. Hence $(gxh)\theta= =(g, x, h)$ and θ is surjective.

LEMMA 4.4. θ *is a homomorphism.*

PROOF. Let $x, y\in S$. Then

$$x\theta y\theta = (e_x, uxu, f_x)(e_y, uyu, f_y) =$$
$$= \left(e_x(f_xe_y)\overline{\alpha_{uxu}^{-1}},\ uxuf_xe_yuyu,\ (f_xe_y)\overline{\alpha_{uyu}}\cdot f_y\right).$$

Since $uf_x=f_x$, $e_yu=e_y$, $f_x\mathcal{L}^*ux$ and e_yR^*yu, we obtain

$$uxuf_xe_yuyu = uxyu.$$

Also

$$e_x(f_xe_y)\overline{\alpha_{uxu}^{-1}} = e_x\left((uxu)^*f_xe_y(uxu)^*\right)\alpha_{uxu}^{-1} = e_x(uxuf_xe_y)^+ =$$
$$= e_x(uxf_xe_y)^+ = e_x(uxe_y)^+ = e_{xy}\quad \text{(Lemma 4.1 (ii))}.$$

Similarly, $(f_xe_y)\overline{\alpha_{uyu}}\cdot f_y=f_{xy}$. Therefore

$$x\theta\cdot y\theta = (e_{xy}, uxyu, f_{xy}) = (xy)\theta$$

and θ is a homomorphism.

Thus we have proved the following.

THEOREM 4.5. *If S is an abundant semigroup with set of idempotents E such that*
(i) *E generates a regular subsemigroup $\langle E\rangle$;*
(ii) *S contains a normal medial idempotent u, and*
(iii) *uSu is a type A semigroup;*
then S is isomorphic to the semigroup $W(E, uSu, \alpha)$.

5. Decomposition

Let S be an abundant semigroup which satisfies the conditions A), B) and C) stated at the beginning of Section 3. Retain the notations of Section 4 about S, E, $\langle E\rangle$, u and e_x, f_x for any $x\in S$. Our objective in this section is to obtain a decomposition of S which is similar to that in Blyth and McFadden [1]. For this reason we need the following lemma:

LEMMA 5.1. *For any $x, y\in S$, if $uxu=uyu$, then $e_yx=yf_x$.*

PROOF. Since

$$y = e_yuyuf_y = e_yuxuf_y = e_yxf_y$$

therefore

$$y(uyu)^* = e_yxf_y(uyu)^* = e_yx(uxu)^* = e_yxu.$$

Thus

$$uy(uyu)^* = ue_yxu \Rightarrow uyu = ue_yxu \Rightarrow e_yuyu = e_yue_yxu \Rightarrow$$
$$\Rightarrow e_yyu = e_yxu \Rightarrow yu = e_yxu \Rightarrow yuf_x = e_yxuf_x \Rightarrow yf_x = e_yx.$$

Recall (see [2]) that F_u and ${}_uF$ are the equivalences associated with the translations

$$\lambda_u\colon x\mapsto ux \quad\text{and}\quad \varrho_u\colon x\mapsto xu$$

respectively. Equivalently, we have $(x, y)\in F_u$ if and only if $ux=uy$, and $(x, y)\in {}_uF$ if and only if $xu=yu$. If we denote by F the relation on S that is given by

$$xFy \Leftrightarrow uxu = uyu$$

then we have the following proposition which is similar to Theorem 6.1 [2].

PROPOSITION 5.2. $F_u\cdot {}_uF=F={}_uF\cdot F_u$.

PROOF. Suppose that $(x, y)\in F_u\cdot {}_uF$. Then there exists $z\in S$ such that $ux=uz$ and $zu=yu$. Then $uxu=uzu=uyu$ and so $(x, y)\in F$. Conversely, suppose that $(x, y)\in F$. Then $uxu=uyu$ and, by Lemma 5.1, $e_yx=yf_x=t$(say). Notice that

$$ut = uyf_x = uyuf_x = uxu\cdot f_x = uxf_x = ux,$$
$$tu = e_yuxu = e_yuyu = e_y\cdot yu = yu$$

and hence $(x, y)\in F_u\cdot {}_uF$. Thus $F_u\cdot {}_uF=F$. Similarly, we can show that ${}_uF\cdot F_u=F$.

If we now define $Su|\times|_F uS=\{(xu, uy); xFy\}$ then we have the following result.

PROPOSITION 5.3. *There is a one-to-one correspondence between S and $Su|\times|_F uS$.*

PROOF. It is clear that we have a map $\theta\colon S\to Su|\times|_F uS$ defined by $x\theta=(xu, ux)$ for any $x\in S$. Now, if $x, y\in S$ such that $x\theta=y\theta$, then $xu=yu$. It follows that $uxu=uyu$ and we get

$$e_x\mathcal{R}xu = yu\mathcal{R}^*e_y,\ e_x\,\mathcal{L}(uxu)^+ = (uyu)^+\,\mathcal{L}e_y$$

Then $e_x=e_y$. Similarly $f_x=f_y$. Therefore

$$x = e_x uxuf_x = e_y uyuf_y = y$$

and θ is injective. In order to see that θ is also surjective, let $(xu, uy)\in Su|\times|_F uS$. Then xFy, that is, $uxu=yuy$. By Lemma 5.1, $xf_y=e_x y=s$ (say), $s\in S$ and

$$us = uxf_y = uxuf_y = uyuf_y = uy,\quad su = e_x yu = e_x uyu = e_x uxu = xu.$$

Therefore

$$s\theta = (su, us) = (xu, uy).$$

Hence the result.

Now let S be quasi-adequate. By Proposition 2.1, u is a middle unit and for any (xu, uy), $(x'u, uy')$, (zu, ut), $(z'u, ut')$ in $Su|\times|_F uS$, the equalities $(xu, uy)=$ $=(x'u, uy')$, $(zu, ut)=(z'u, ut')$ imply

$$xzu = xuzu = x'uz'u = x'z'u$$

and

$$uyt = uyut = uy'ut' = uy't'.$$

Hence, we can define a binary operation on $Su|\times|_F uS$ by the following rule:

$$(xu, uy)\cdot(zu, ut) = (xzu, uyt)\quad \text{for any}\quad (xu, uy), (zu, ut)$$

in $Su|\times|_F uS$.

It is clear that this binary operation is associative and $\theta\colon S\to Su|\times|_F uS$ defined by $x\theta=(xu, ux)$ is a homomorphism.

It follows by Proposition 5.3 that θ is an isomorphism, and the following result is established.

THEOREM 5.4. *If S is a quasi-adequate semigroup which contains a normal medial idempotent u and uSu is a type A semigroup, then S is isomorphic to $Su|\times|_F uS$.*

The significance of this result relies heavily on the structure of each of the subsemigroups Su and uS. We now retain the hypotheses of Theorem 5.4 and proceed to investigate the main properties of Su and uS.

LEMMA. 5.5. *Su and uS are quasi-adequate.*

PROOF. Since S is quasi-adequate, then $\langle E\rangle=E$ and the set of idempotents of Su is Eu which is a band. Let x be an element in Su, say, $x=yu$ for some $y\in S$. Notice that

$$y^+ux = y^+uyu = y^+uy^+yu = y^+yu = yu = x$$

and for any $s, t \in Su$,

$$sx = tx \Rightarrow syu = tyu \Rightarrow syy^*uy^* = tyy^*uy^* \Rightarrow sy = ty \Rightarrow sy^+ = ty^+ \Rightarrow$$
$$\Rightarrow sy^+u = ty^+u.$$

Therefore, by Corollary 1.2, $x\mathcal{R}^*y^+u$ in Su. Since $y\mathcal{L}^*y^*$, then $yu\mathcal{L}^*y^*u$ and $x\mathcal{L}^*y^*u$ in Su. Hence Su is quasi-adequate. Similarly uS is quasi -adequate.

From the proof of Lemma 5.5, it is clear that each of the semigroups Su and uS is a *-subsemigroup of S. It is easily verified that Su and uS satisfy the hypotheses of Theorem 5.4. In what follows we shall prove some more properties for uS and the dual argument shows the same properties for Su.

LEMMA 5.6. *Let the relation δ on uS be as defined in Section* 1. *Then $(x, y)\in\delta$ if and only if $xu=yu$ for any $x, y\in uS$.*

PROOF. For any x and y in uS, we have $(x, y)\in\delta$ if and only if $x=eyf$ for some $e\in uS\cap E(y^+)$ and $f\in uS\cap E(y^*)$, which implies $xu=eyfu$; But

$$\begin{aligned}
xu &= eyfu && (\text{in } uSu)\\
&= yfu(eyfu)^* && (uSu \text{ is type } A)\\
=yy^*ufu(ey^+uyy^*ufu)^* &= yfuy^*u(y^+ueyfuy^*u)^* = yy^*fy^*(y^+ey^+yy^*fy^*u)^* = \\
&= y(yu)^* && (f\in E(y^*),\ e\in E(y^+))\\
&= yu(yu)^* && ((yu)^* = u(yu)^*)\\
&= yu.
\end{aligned}$$

On the other hand, let $xu=yu$. Then we have $x^*u=(xu)^*=(yu)^*=y^*u$ which implies $x^*ux^*=y^*ux^*$ i.e., $x^*=y^*x^*$, and $x^*uy^*=y^*uy^*$, i.e., $x^*y^*=y^*$. Therefore $x^*\mathcal{R}y^*$, in particular $x^*\in E(y^*)$. But also

$$xu = yu \Rightarrow xux^* = yux^* \Rightarrow x = y^+yx^*$$

where $y^+\in E(x^+)$, $x^*\in E(y^*)$. Hence $(x, y)\in\delta$.

COROLLARY 5.7. *δ is a congruence on uS.*

PROOF. It is an immediate consequence of Lemma 5.6 that δ is a left congruence. To show that δ is also right compatible, let $x, y, c\in uS$. Then we get

$$(x, y)\in\delta \Rightarrow xu = yu \Rightarrow xuc = yuc \Rightarrow xc = yc \quad (c\in uS) \Rightarrow$$
$$\Rightarrow xcu = ycu \Rightarrow (xc, yc)\in\delta.$$

Hence the result.

Now it follows by Proposition 1.5 that δ is the minimum adequate good congruence on uS. Difine $\pi\colon uS/\delta\to S$ by $(x\delta)\pi=xu$ for any $x\delta\in uS/\delta$. Since $x\delta=y\delta\Leftrightarrow$ $\Leftrightarrow(x, y)\in\delta\Leftrightarrow xu=yu$ and for any $x\delta$, $y\delta\in uS/\delta$,

$$(x\delta\cdot y\delta)\pi = (xy)\delta\pi = xyu = xuyu = (x\delta)\pi\cdot(y\delta)\pi,$$

therefore π is a monomorphism. To verify the goodness of π, let $x\delta$ and $y\delta$ be elements in uS/δ such that $x\delta\mathscr{L}^*y\delta$ in uS/δ. Since $x, y\in uS$, uS is a *-subsemigroup of S, $x\mathscr{L}^*x^*$, $y\mathscr{L}^*y^*$ and δ is good then $x\delta\mathscr{L}^*x^*\delta$ in uS/δ and $y\delta\mathscr{L}^*y^*\delta$ in uS/δ and we get $x^*\delta\mathscr{L}^*y^*\delta$ in uS/δ which is an adequate semigroup. Thus, $x^*\delta=y^*\delta$ which implies $x^*u=y^*u$. But $x^*u\mathscr{L}^*xu$, $y^*u\mathscr{R}^*yu$ in S. Therefore $xu\mathscr{L}^*yu$ in S i.e., $(x\delta)\pi\mathscr{L}^*(y\delta)\pi$ and π preserves the relation $\mathscr{L}^*$. To show that π preserves also the relation $\mathscr{R}^*$, let $x\delta$ and $y\delta$ be elements in uS/δ such that $x\delta\mathscr{R}^*y\delta$ in uS/δ. Then we get $x^+\delta\mathscr{R}^*x\delta$, $y^+\delta\mathscr{R}^*y\delta$ in the adequate semigroup uS/δ. Therefore $x^+\delta=y^+\delta$ which implies $x^+u=y^+u$. Since $x^+uxu=xu$ and

$$sxu = txu \Rightarrow sx = tx \Rightarrow sx^+u = tx^+u$$

for any $s, t\in S$, therefore $x^+u\mathscr{R}^*xu$. Similarly $y^+u\mathscr{R}yu$. Therefore $xu\mathscr{R}^*yu$ i.e., $(x\delta)\pi\mathscr{R}^*(y\delta)\pi$. Hence π is good. Moreover, we notice that $(x\delta)\pi\cdot\delta^\natural=(xu)\delta=x\delta$ which means $\pi\cdot\delta^\natural=1_{uS/\delta}$. Thus the following result is proved:

THEOREM 5.8. *uS is a split quasi-adequate semigroup.*

Notice that $\operatorname{Im}\pi=\{xu\colon x\in uS\}=uSu$ which is a type A semigroup whose semi lattice of idempotents is uEu. For any $a\in\operatorname{Im}\pi$, choose a^+ and a^* to be in uEu. Then

$$a^+uEa^+ = a^+uEua^+ = a^+uEu, \quad a^*uEa^* = a^*uEua^* = a^*uEu$$

and we have the bijection $\alpha_a\colon a^+uEa^+\to a^*uEa^*$ defined by $e\alpha_a=(ea)^*$. Hence we conclude that uS is a split quasi-adequate semigroup which satisfies the required condition for the class of semigroups considered in [3], and therefore uS can be described as in [3].

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EXTENSIONS OF DISCRETE AND EQUAL BAIRE FUNCTIONS

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0. Introduction In the paper [3], two stronger kinds of pointwise convergence of sequences of real-valued functions, discrete and equal convergence, have been introduced. [3], [4], [5] have investigated the Baire classes based on these convergence types; the respective definitions (in their final form) can be found in [5], Sections 1 and 3. Our present purpose is to study extension theorems corresponding to these classes, similar to those valid for Baire classes based on pointwise convergence.

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1. Perfect, normal δ-lattices. Let $\mathcal{M}$ be a system of subsets of a set X. We say that $\mathcal{M}$ is *normal* iff $M_1, M_2 \in \mathcal{M}$, $M_1 \cap M_2 = \emptyset$ imply the existence of $N_1, N_2 \in \mathcal{M}^c = \{X-M: M \in \mathcal{M}\}$ such that

$$M_i \subset N_i, \quad N_1 \cap N_2 = \emptyset.$$

LEMMA 1.1. *If Φ is a subtractive lattice* ([4], 1.3) *on X, then $\mathcal{P}(\Phi)$* ([4], 1.4) *is a perfect, normal lattice* ([4], 1.1) *on X such that $\emptyset$, $X \in \mathcal{P}(\Phi)$.*

PROOF. According to [4], 3.2, we only have to prove the normality of $\mathcal{P}(\Phi)$. If $P_1, P_2 \in \mathcal{P}(\Phi)$, $P_1 \cap P_2 = \emptyset$, let f_1, f_2 be taken from Φ such that $P_i = X(f_i = 0)$ and $f_i \geqq 0$ ([4], 3.1). Now

$$Q_1 = X(f_1 - f_2 > 0), \quad Q_2 = X(f_2 - f_1 > 0)$$

belong to $\mathcal{P}(\Phi)^c$ and satisfy $P_1 \subset Q_2$, $P_2 \subset Q_1$, $Q_1 \cap Q_2 = \emptyset$. $\square$

LEMMA 1.2. *Let $\mathcal{P}$ be a perfect, normal δ-lattice* ([4], 1.1) *on X such that $\emptyset$, $X \in \mathcal{P}$. Then $\Phi = \Phi(\mathcal{P})$* ([4], 1.4) *is a complete, ordinary class* ([4], 1.2, 1.3) *such that $\mathcal{P} = \mathcal{P}(\Phi)$.*

PROOF. By [4], 3.3, Φ is a complete, ordinary class. Clearly $\mathcal{P}(\Phi) \subset \mathcal{P}$, so we have to prove $\mathcal{P} \subset \mathcal{P}(\Phi)$.

For this purpose, we introduce a binary relation $<$ on the power set of X by putting $A < B$ iff there are $P \in \mathcal{P}$, $Q \in \mathcal{P}^c$ such that

$$A \subset P \subset Q \subset B.$$

It is easy to see that $<$ is a topogeneous order ([2], p. 12) on X, and $\{<\}$ is a topoge-

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nous structure ([2], (7.7)) because $A \subset P \subset Q \subset B$, $P \in \mathcal{P}$, $Q \in \mathcal{P}^c$ implies

$$A \subset P \subset Q' \subset P' \subset Q \subset B$$

for suitable sets $P' \in \mathcal{P}$, $Q' \in \mathcal{P}^c$ (by the normality of $\mathcal{P}$), hence $A < Q' < B$.

Now let P_1, $P_2 \in \mathcal{P}$, $P_1 \cap P_2 = \emptyset$ be given. By [2], (12.41), there is a function $f: X \to [0, 1]$ such that $f(x)=0$ for $x \in P_1$, $f(x)=1$ for $x \in P_2$, and $a, b \in \mathbf{R}$, $a<b$ imply

$$X(f \leqq a) < X(f < b).$$

This function belongs to Φ. In fact, for $c \in \mathbf{R}$, there are $P_n \in \mathcal{P}$, $Q_n \in \mathcal{P}^c$ such that

$$X(f \leqq c) \subset P_n \subset Q_n \subset X\left(f < c + \frac{1}{n}\right)$$

so that

$$X(f \leqq c) = \bigcap_1^\infty P_n \in \mathcal{P}$$

because $\mathcal{P}$ is a δ-lattice. $X(f \geqq c) \in \mathcal{P}$ is proved in a similar manner:

$$X\left(f \leqq c - \frac{1}{n}\right) \subset P_n \subset Q_n \subset X(f < c),$$

$P_n \in \mathcal{P}$, $Q_n \in \mathcal{P}^c$, hence $X(f < c) = \bigcup_1^\infty Q_n \in \mathcal{P}^c$.

For a given $P \in \mathcal{P}$, we write, by the perfectness of $\mathcal{P}$, $X - P = \bigcup_1^\infty P_n$, $P_n \in \mathcal{P}$, and choose functions $f_n \in \Phi$ such that $f_n(x)=0$ for $x \in P$, $f_n(x)=2^{-n}$ for $x \in P_n$, $0 \leqq f_n \leqq 2^{-n}$. Then $f = \sum_1^\infty f_n$ belongs to Φ, and $P = X(f \leqq 0) \in \mathcal{P}(\Phi)$. □

2. Restriction of Baire and Borel classes. Let Φ be a complete, ordinary class on a set X, $\mathcal{P} = \mathcal{P}(\Phi)$, and let $\mathcal{P}_\alpha$, $\mathcal{Q}_\alpha$ be defined as in [4], 2, Φ_α as in [4], 3.11.

For a given subset $A \subset X$, consider the trace

$$\mathcal{P}(A) = \mathcal{P}|A = \{P \cap A:\ P \in \mathcal{P}\}$$

of the system $\mathcal{P}$. Since $\mathcal{P}$ is a perfect δ-lattice ([4], 3.2), it is easily seen that $\mathcal{P}(A)$ is a perfect δ-lattice on A, $\emptyset$, $A \in \mathcal{P}(A)$, so that we can define

$$\mathcal{P}_0(A) = \mathcal{P}(A), \quad \mathcal{Q}_0(A) = \{A - P:\ P \in \mathcal{P}(A)\},$$

$$\mathcal{P}_\alpha(A) = \Big(\bigcup_{\beta<\alpha} \mathcal{Q}_\beta(A)\Big)^\delta, \quad \mathcal{Q}_\alpha(A) = \Big(\bigcup_{\beta<\alpha} \mathcal{P}_\beta(A)\Big)^\sigma,$$

$$\mathcal{A}_\alpha(A) = \mathcal{P}_\alpha(A) \cap \mathcal{Q}_\alpha(A).$$

Then all propositions in [4], Section 2 hold for these classes with A instead of X.

PROPOSITION 2.1. *For* $0 \leqq \alpha < \omega_1$, *we have*

$$\mathcal{P}_\alpha(A) = \mathcal{P}_\alpha|A, \quad \mathcal{Q}_\alpha(A) = \mathcal{Q}_\alpha|A.$$

PROOF. Transfinite induction. □

PROPOSITION 2.2. *We have*

$$\mathscr{A}_\alpha|A \subset \mathscr{A}_\alpha(A)$$

for $0 \leqq \alpha < \omega_1$. *If* $A \in \mathscr{P}_\alpha$ *and* $1 \leqq \alpha < \omega_1$, *then* $=$ *holds instead of* $\subset$.

PROOF. The first part follows from 2.1. If $B \in \mathscr{A}_\alpha(A)$, then $B = P \cap A = Q \cap A$ for suitable sets $P \in \mathscr{P}_\alpha$, $Q \in \mathscr{Q}_\alpha$. Since $B \in \mathscr{P}_\alpha$, provided $A \in \mathscr{P}_\alpha$, and $B \subset Q$, by [4], 2.8, there are sets Q', $Q'' \in \mathscr{A}_\alpha$ such that

$$Q' \subset X - B, \quad Q'' \subset Q, \quad Q' \cap Q'' = \emptyset, \quad Q' \cup Q'' = X.$$

Thus $B \subset Q'' \subset Q$, $B = Q'' \cap A \in \mathscr{A}_\alpha|A$. □

By [4], 3.3,

$$\Phi(A) = \Phi(\mathscr{P}(A))$$

is a complete, ordinary class, and $\mathscr{P}(A) = \mathscr{P}(\Phi(A))$. In fact, $\mathscr{P}(A) \supset \mathscr{P}(\Phi(A))$ by definition, and $\Phi|A \subset \Phi(A)$ implies

$$\mathscr{P}(A) \subset \mathscr{P}(\Phi|A) \subset \mathscr{P}(\Phi(A).)$$

Thus we can define, starting from $\Phi(A)$ and according to [4], 3.11, [5], Section 1, and [5], Section 3, respectively, the pointwise, discrete, and equal Baire classes

$$\Phi_\alpha(A), \quad \Phi_\alpha^{(d)}(A), \quad \Phi_\alpha^{(e)}(A)$$

with the underlying set A.

PROPOSITION 2.3. *We have, for* $0 \leqq \alpha < \omega_1$, $\Phi_\alpha|A \subset \Phi_\alpha(A)$, $\Phi_\alpha^{(d)}|A \subset \Phi_\alpha^{(d)}(A)$, $\Phi_\alpha^{(e)}|A \subset \Phi_\alpha^{(e)}(A)$.

PROOF. By [4], 3.14, $\Phi_\alpha = \Phi(\mathscr{P}_\alpha)$, hence $f \in \Phi_\alpha$ implies $f|A \in \Phi(\mathscr{P}_\alpha|A) = \Phi_\alpha(A)$. From this, the statement concerning $\Phi_\alpha^{(e)}$ follows by [5], 3.3 (or immediately for limit ordinals α). For the case of $\Phi_\alpha^{(d)}$, [5], 1.2 applies. □

For pointwise Baire classes, the following classical theorem holds:

THEOREM 2.4. *If* $0 \leqq \alpha < \omega_1$, $f \in \Phi_\alpha(A)$, *then there are a set* $A^* \in \mathscr{P}_{\alpha+1}$, $A^* \supset A$, *and a function* $g \in \Phi_\alpha(A^*)$ *such that* $f = g|A$; *if* $A \in \mathscr{P}_\alpha$, *then* $A^* = X$ *can be chosen.*

PROOF. Consider first a bounded function f, say $|f| \leqq c$, and the sets

$$A^+ = A\left(f \geqq \frac{c}{3}\right), \quad A^- = A\left(f \leqq -\frac{c}{3}\right).$$

Since $f \in \Phi_\alpha(A)$, we have $A^+ = A \cap P^+$, $A^- = A \cap P^-$ for suitable sets P^+, $P^- \in \mathscr{P}_\alpha$. Choose functions $g, h \in \Phi_\alpha$ such that $g, h \geqq 0$, $X(g=0) = P^+$, $X(h=0) = P^-$. Define $Q_1 = X(g+h > 0)$. Then $A \subset Q_1 \in \mathscr{Q}_\alpha$, and the function

$$g_1 = -\frac{c}{3} + \frac{2c}{3}\frac{g}{g+h}$$

satisfies

$$g_1 \in \Phi_\alpha(Q_1), \quad |g_1| \leqq \frac{c}{3}, \quad |f(x) - g_1(x)| \leqq \frac{2c}{3}$$

for $x \in A$.

We can repeat the construction for $f-(g_1|A)$ instead of f because $g_1|A\in\Phi_\alpha(A)$; then c is repleaced by $\frac{2c}{3}$. In this way we obtain sets $Q_n\in\mathcal{Q}_\alpha$ such that $A\subset Q_n$ and functions $g_n\in\Phi_\alpha(Q_n)$ satisfying

$$|g_n|\leqq\frac{2^{n-1}c}{3^n},\quad \left|f(x)-\sum_{i=1}^{n}g_i(x)\right|\leqq\frac{2^n c}{3^n}$$

for $x\in A$.

Now

$$A^*=\bigcap_1^\infty Q_n\in\mathcal{P}_{\alpha+1},\quad A\subset A^*,$$

and $g_n|A^*\in\Phi_\alpha(A^*)$ implies

$$g=\sum_1^\infty g_n\in\Phi_\alpha(A^*)$$

because the series converges uniformly. Clearly $f=g|A$.

For an arbitrary $f\in\Phi_\alpha(A)$, we consider $f'=\tanh\circ f$. Clearly $f'\in\Phi_\alpha(A)$. Choose a set $A'\supset A$, $A'\in\mathcal{P}_{\alpha+1}$ such that $f'=g'|A$ for a suitable function $g'\in\Phi_\alpha(A')$. Define

$$A^*=A'(|g'|<1)\in\mathcal{Q}_\alpha(A')\subset\mathcal{P}_{\alpha+1},$$

and

$$g=\operatorname{artanh}\circ(g'|A^*)\in\Phi_\alpha A^*),$$

then $A\subset A^*$, $f=g|A$.

In the case $A\in\mathcal{P}_\alpha$, we have A^+, $A^-\in\mathcal{P}_\alpha$, so that we can choose $P^+=A^+$, $P^-=A^-$, and we obtain $Q_0=X$. In the sequel $Q_n=X$, $A^*=X$, and the second part of the statement turns out to be valid for a bounded $f\in\Phi_\alpha(A)$. For an unbounded f, we define $f'\in\Phi_\alpha(A)$ as above and take $g'\in\Phi_\alpha$ such that $f'=g'|A$. Let $h'\in\Phi_\alpha$ be chosen such that $|h'|\leqq1$, $h'(x)=1$ for $x\in A$, $h'(x)=0$ for $x\in X(|g'|\geqq1)\in\mathcal{P}_\alpha$. Then $g'h'\in\Phi_\alpha$ and $|g'h'|<1$, $g'h'|A=f'$. Hence

$$g=\operatorname{artanh}\circ g'\in\Phi_\alpha$$

is the function looked for. □

The reader has certainly observed that the method of proof is taken from the classical theorem of Tietze—Urysohn. In [6], §35, VI, Theorem, a quite different method has been applied for the case $\alpha\geqq1$, while, for $\alpha=0$, the case $\Phi=C(A)$, X a metric space, has been treated by an elementary method that cannot be generalized for the case of an arbitrary complete ordinary class ([6], §35, I, Theorem 1).

COROLLARY 2.5 (see [1]; [6], §35, VI, Corollary). *If* $0\leqq\alpha<\omega_1$, $f\in\Phi_\alpha(A)$, *then there is a function* $g\in\Phi_{\alpha+1}$ *such that* $f=g|A$. □

3. Extension of discrete Baire functions. The second part of 2.4 says that the first inclusion in 2.3 can be replaced by equality provided $A\in\mathcal{P}_\alpha$. For the second inclusion, an essentially weaker statement can be proved:

THEOREM 3.1. *The equality*

$$\Phi_\alpha^{(d)}|A = \Phi_\alpha^{(d)}(A)$$

holds if $A\in\mathscr{P}_0$ *and* $0\leqq\alpha<\omega_1$ *or if* $A\in\mathscr{Q}_1$ *and* $\alpha\geqq 2$.

PROOF. We have to show that the sign $\supset$ holds. The case $A\in\mathscr{P}_0$, $\alpha=0$ is contained in 2.4. Assume $\alpha\geqq 1$.

Let $f\in\Phi_\alpha^{(d)}(A)$. Then, by [5], 1.2,

$$A=\bigcup_1^\infty A_i,\quad A_i\in\mathscr{Q}_\alpha(A),\quad f|A_i=g_i|A_i,\quad g_i\in\Phi(A).$$

If $A\in\mathscr{P}_0$, then $A_i\in\mathscr{Q}_\alpha(A)=\mathscr{Q}_\alpha|A\subset\mathscr{Q}_\alpha$, and $g_i=h_i|A$, $h_i\in\Phi$ by 2.4 (for $\alpha=0$). Hence we can define

$$g(x)=f(x)\quad\text{for}\quad x\in A,\quad g(x)=0\quad\text{for}\quad x\in X-A, \tag{3.1.1}$$

and $X-A\in\mathscr{Q}_0\subset\mathscr{Q}_\alpha$ implies $g\in\Phi_\alpha^{(d)}$ by [5], 1.2.

If $A\in\mathscr{Q}_1$, then $A=\bigcup_1^\infty B_j$, $B_j\in\mathscr{P}_0$. We introduce again the sets A_i and the functions g_i. By 2.4 $g_i|B_j=h_{ij}|B_j$ where $h_{ij}\in\Phi$. Thus

$$A=\bigcup_{i=1}^\infty\bigcup_{j=1}^\infty (A_i\cap B_j),\quad A_i\cap B_j\in\mathscr{Q}_\alpha,$$

further $g|A_i\cap B_j=h_{ij}|A_i\cap B_j$. If $\alpha\geqq 2$, then $X-A\in\mathscr{P}_1\subset\mathscr{Q}_\alpha$ and $g\in\Phi_\alpha^{(d)}$. □

Another very weak result is the following:

PROPOSITION 3.2. *If* $A\in\mathscr{A}_1$, *then*

$$\Phi_1^{(d)}|A=\Phi_1^{(d)}(A).$$

PROOF. By [5], 1.2 again,

$$A=\bigcup_1^\infty A_i,\quad A_i\in\mathscr{Q}_1(A),\quad f|A_i=g_i|A_i,\quad g_i\in\Phi(A).$$

Now $A_i\in\mathscr{Q}_1=\mathscr{P}^\sigma$ so that we can suppose $A_i\in\mathscr{P}$. Then by 2.4 $f|A_i=g_i|A_i=h_i|A_i$ for suitable functions $h_i\in\Phi$. By defining g according to (3.1.1), [5], 1.2 yields $g\in\Phi_1^{(d)}$ because $X-A\in\mathscr{Q}_1$. □

The hypothesis $A\in\mathscr{P}_0$ in 3.1 cannot be replaced by $A\in\mathscr{P}_1$:

EXAMPLE 3.3. Let $\Phi=C(\mathbf{R})$, and f be a monotone function that has a jump at every $x\in\mathbf{Q}$ and is continuous on $\mathbf{R}-\mathbf{Q}=\mathbf{Q}^c$.

If $A=\mathbf{Q}^c$, then $A\in\mathscr{P}_1$, $f|A\in\Phi_0(A)$ (because $\mathscr{P}(A)$ is composed of the sets relatively closed in A, hence $\Phi(A)$ consists of the functions continuous on A). However, if $g\colon\mathbf{R}\to\mathbf{R}$ satisfies $g|A=f|A$ then g cannot belong to any class $\Phi_\alpha^{(d)}$. In fact, if $\mathbf{R}=\bigcup_1^\infty A_i$, $g|A_i=g_i|A_i$ for $g_i\in C(\mathbf{R})$, then, by Baire's category theorem, $A_i\cap\mathbf{Q}^c$ is dense in $\mathbf{Q}^c\cap I$ for an open interval I and for at least one i; this leads to a contradiction at a point $r\in\mathbf{Q}\cap I$. □

Similarly, in 3.2, $A\in\mathscr{A}_1$ cannot be replaced by $A\in\mathscr{P}_1$ or $A\in\mathscr{Q}_1$; the first assertion follows from 3.3, the second one from 3.4:

EXAMPLE 3.4. Let Φ, f, $\mathbf{Q}^c$ be the same as in 3.3, and A be a countable dense subset of $\mathbf{Q}^c$. Now $A\in\mathscr{Q}_1$, $f|A\in\Phi_0(A)$, and $f|A=g|A$ is impossible for any $g\in\Phi_1^{(d)}$. In fact, if $\mathbf{R}=\bigcup\limits_1^\infty A_i$, $A_i\in\mathscr{Q}_1$, $g|A_i=g_i|A_i$, $g_i\in C(\mathbf{R})$, then we can suppose $A_i\in\mathscr{P}_0$ so that at least one A_i contains an open interval I. For $r\in\mathbf{Q}\cap I$, we obtain a contradiction again. □

We have better results concerning the analogue of 2.5:

THEOREM 3.5. *If* $f\in\Phi_\alpha^{(d)}(A)$, $\alpha\geqq 2$, *then there is a set* $A^*\supset A$ *such that* $A^*\in\mathscr{Q}_\alpha$, $f=g|A$, $g\in\Phi_{\alpha+1}^{(d)}(A^*)$.

PROOF. By [5] ,1.2,

$$A=\bigcup_1^\infty A_i,\quad A_i\in\mathscr{Q}_\alpha(A),\quad f|A_i=g_i|A_i,\quad g_i\in\Phi(A).$$

According to [4], 2.8, we can suppose that the sets A_i are pairwise disjoint. Set

$$A_i=Q_i\cap A,\quad Q_i\in\mathscr{Q}_\alpha,$$

further

$$B_i=Q_i-\bigcup_{j<i}Q_j\in\mathscr{A}_{\alpha+1}.$$

Clearly

$$A_i=B_i\cap A,\quad B=\bigcup_1^\infty Q_i=\bigcup_1^\infty B_i\in\mathscr{Q}_\alpha.$$

By 2.4, there exist sets $C_i\supset A$ satisfying

$$C_i\in\mathscr{P}_1,\quad g_i=h_i|A,\quad h_i\in\Phi(C_i)$$

for suitable functions h_i. Define

$$A^*=B\cap\bigcap_1^\infty C_i\supset A.$$

Then $A^*\in\mathscr{Q}_\alpha$ (because $\alpha\geqq 2$), $A^*\cap B_i\in\mathscr{A}_{\alpha+1}$. Since the sets B_i are pairwise disjoint, we can define a function g on A^* such that

$$g|A^*\cap B_i=h_i|A^*\cap B_i.$$

By $h_i|A^*\in\Phi(A^*)$, we get $g\in\Phi_{\alpha+1}^{(d)}(A^*)$, and clearly $f=g|A$. □

COROLLARY 3.6. *If* $\alpha=1$, *then the statement of* 3.5 *holds for a set* $A^*\in\mathscr{A}_2$.

PROOF. The above construction applies with the only change $A^*\in\mathscr{A}_2$ (because $B\in\mathscr{Q}_1$, $C_i\in\mathscr{P}_1$), $A^*\cap B_i\in\mathscr{A}_2$. □

In 3.6, $\mathscr{A}_2$ cannot be replaced by $\mathscr{Q}_1$. In fact, let Φ, f, A be the same as in 3.3. Now if $\mathbf{Q}^c=A\subset A^*\in\mathscr{Q}_1$ and $f=g|A$, then g cannot belong to any class $\Phi_\alpha^{(d)}(A^*)$. Assume $A^*=\bigcup\limits_1^\infty A_i$, $g|A_i=g_i|A_i$, $g_i\in\Phi(A^*)$. Then by the Baire category theorem, $A_i\cap\mathbf{Q}^c$

is dense in $\mathbf{Q}^c \cap I$ for an i and an open interval I. Since $A^* \cap I \in \mathcal{Q}_1$, $A^* \cap I \neq \mathbf{Q}^c \cap I$ because the right hand side is not an F_σ (by the category theorem again). At a point $x \in (A^* \cap I) - A$, we obtain a contradiction.

4. Extension of equal Baire functions. For the classes $\Phi_\alpha^{(e)}$, there exists a better analogue of the second part of 2.4:

THEOREM 4.1. *If $f \in \Phi_\alpha^{(e)}(A)$, $0 \leqq \alpha < \omega_1$, and $A \in \mathcal{A}_\alpha$, then $f = g|A$ for some $g \in \Phi_\alpha^{(e)}$, and $\Phi_\alpha^{(e)}|A = \Phi_\alpha^{(e)}(A)$.*

PROOF. If $\alpha = 0$ or α is a limit ordinal, then $\Phi_\alpha^{(e)}(A) = \Phi_\alpha(A)$, $\Phi_\alpha^{(e)} = \Phi_\alpha$, hence 2.4 can be applied. Assume $\alpha = \beta + 1$.

By [5], 3.6,

$$A = \bigcup_1^\infty A_i, \quad A_i \in \mathcal{Q}_\alpha(A), \quad f|A_i = g_i|A_i, \quad g_i \in \Phi_\beta(A).$$

By 2.1, $A_i \in Q_\alpha$ so that we may suppose $A_i \in \mathcal{P}_\beta$. Then, by 2.3, $g_i|A_i \in \Phi_\beta(A_i)$, and, by 2.4, $g_i|A_i = h_i|A_i$, $h_i \in \Phi_\beta$. By putting $g(x) = f(x)$ for $x \in A$, $g(x) = 0$ for $x \in X - A$, [5], 3.6 yields $g \in \Phi_\alpha^{(e)}$. $\square$

Observe that 3.2 is a particular case of 4.1 according to $\Phi_1^{(d)} = \Phi_1^{(e)}$ ([5], 3.3). For the same reason, 3.3 and 3.4 show that, in general, $A \in \mathcal{A}_\alpha$ cannot be replaced by $A \in \mathcal{P}_\alpha$ or $A \in \mathcal{Q}_\alpha$ in the hypothesis.

The following theorem is the analogue of 2.5:

THEOREM 4.2. *If $f \in \Phi_\alpha^{(e)}(A)$, $A \subset X$, $\alpha = \beta + 1 < \omega_1$, then $f = g|A$, $g \in \Phi_{\alpha+1}^{(e)}$.*

PROOF. As in the previous proof,

$$A = \bigcup_1^\infty A_i, \quad A_i \in \mathcal{Q}_\alpha(A), \quad f|A_i = g_i|A_i, \quad g_i \in \Phi_\beta(A).$$

Now we can proceed like in the proof of 3.5. We define Q_i, B_i, then $C_i \supset A$ satisfying $C_i \in \mathcal{P}_{\beta+1} = \mathcal{P}_\alpha$, $g_i = h_i|A$, $h_i \in \Phi_\beta(C_i)$. Now we construct A^* as above and conclude $A^* \in \mathcal{A}_{\alpha+1}$, $A^* \cap B_i \in \mathcal{A}_{\alpha+1}$; finally we define g on A^*, and $h_i|A^* \in \Phi_\beta(A^*) \subset \Phi_\alpha(A^*)$ implies $g \in \Phi_{\alpha+1}^{(e)}(A^*)$. By 4.1, we can extend g to be an element of $\Phi_{\alpha+1}^{(e)}$. $\square$

3.3 shows that the statement of 4.2 is not valid for $\alpha = 0$.

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ON THE DIVERGENCE OF SOME FUNCTION SERIES

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This paper is devoted to the study of the divergence of Fourier series. In the four sections below we deal with the a.e., resp. the norm divergence of $S_{\mu_n}f-S_{\nu_n}f$, where $\mu_n-\nu_n\neq O(1)$; the a.e. divergence for signed Toeplitz summations and another norm divergence problem. We also formulate two corresponding problems.

1. Investigate first the pointwise divergence of a sequence of type $S_{\mu_n}f-S_{\nu_n}f$.

LEMMA 1. *Suppose $\{\mu_n\}$ and $\{\nu_n\}$ are natural numbers such that $\mu_n\to+\infty$, $\nu_n\to+\infty$ $(n\to\infty)$ and define*

(1) $$T_n:=S_{\mu_n}-S_{\nu_n};\quad S_N=\sum_{|n|\leqq N}c_ne^{inx};\quad c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}\,dx.$$

Suppose that there exists $f_1\in L_1(0,2\pi)$ such that

$$\limsup_{n\to\infty}|T_nf_1(x)|>0$$

on a set of positive measure. Then there exists $\delta>0$ and for any $M>0$ there exists a polynomial $f=f_M$ satisfying $\|f\|_1\leqq K$ with a constant K independent of M and δ; further

(2) $$\sup_n|T_nf(x)|>M\quad(x\in E)$$

where $E\subset[0,2\pi]$ is a set of Lebesgue measure $|E|>\delta$.

PROOF. We can suppose that

$$\limsup_{n\to\infty}|T_nf_1(x)|\geqq c_0>0\quad(x\in E_1)$$

for a set E_1, $|E_1|>0$. Fix an arbitrary $M^*>0$. Let[1]

$$f(x):=M^*\cdot\vartheta_N\big(f_1-\sigma_m(f_1)\big)(x)$$

where N and m will be specified later and ϑ_N denotes the N-th de la Vallée-Poussin means. If m is large enough, $m>m_0(M^*,f_1)$, then

$$\|f\|_1\leqq cM^*\|f_1-\sigma_m(f_1)\|_1\leqq K.$$

[1] $\vartheta_N:=\dfrac{S_{N+1}+\ldots+S_{2N}}{N},\quad \sigma_N:=\dfrac{S_0+\ldots+S_N}{N+1}.$

Suppose now that m, M^* are fixed and vary the number N. Clearly

$$n > N_1(m) \Rightarrow \nu_n, \mu_n > m.$$

We know that

$$\limsup_{\substack{n\to\infty \\ n>N_1}} |T_n f_1(x)| \geqq c_0 > 0 \quad (x \in E_1)$$

so there exists a set E, $|E| > \frac{|E_1|}{2}$ and a number N_2 such that

$$\sup_{N_1<n<N_2} |T_n f_1(x)| \geqq c_0/2 \quad (x \in E).$$

Here N_2 depends only on f_1, μ_n, ν_n. Let N be so large that $N > \mu_n, \nu_n$ if $N_1 < n < N_2$. Then

$$T_n f = M^*\{[S_{\mu_n} f_1 - \sigma_m f_1] - [S_{\nu_n} f_1 - \sigma_m f_1]\} = M^* T_n f_1,$$

consequently

$$\sup_{N_1<n<N_2} |T_n f(x)| = M^* \sup |T_n f_1(x)| \geqq \frac{c_0}{2} M^* \quad (x \in E)$$

and then

$$M := \frac{c_0}{2} M^*, \quad \delta := \frac{|E_1|}{2}$$

satisfies the statement of Lemma 1.

Lemma 2. *Suppose that $\mu_n, \nu_n \to +\infty$ are natural numbers and $|\mu_n - \nu_n| \to +\infty$. Then there exists $\delta > 0$ and for any $M > 0$ there exists a polynomial g with $\|g\|_1 \leqq 3\pi$ and such that*

$$\sup_n |T_n g(x)| > M$$

in a set of x of measure $\geqq \delta$.

Proof. We can suppose $\mu_n > \nu_n$. Fix $n \in \mathbf{N}$ and define

$$a_i := \frac{4\pi i}{2n+1} \quad (i = 0, 1, \ldots, n)$$

and

$$g(x) := \frac{1}{n} \sum_{\substack{i=1 \\ (i \in I)}}^{n}{}' V_{\mu_{k_i}}(x - a_i)$$

where

$$V_\mu(x) = \frac{1}{2} + \sum_{j=1}^{\mu} \cos jx + \sum_{j=\mu+1}^{2\mu} \left(1 - \frac{j-1}{\mu+1}\right) \cos jx$$

is the de la Vallée-Poussin kernel and Σ' means that the summation is restricted to the indices $i \in I$, where the set I will be given later. Define a sequence k_i with the properties

$$n^4 < \nu_{k_0}, \quad (\mu_{k_i} >) \nu_{k_i} > 2\mu_{k_{i-1}} \quad (i = 1, 2, \ldots).$$

Let $1 \leqq j_0 \leqq n$, then

$$S_{\mu_{k_{j_0}}}(g)-S_{\nu_{k_{j_0}}}(g) = \frac{1}{n} \sum_{\substack{i\in I\\ i\geqq j_0}} \left(D_{\mu_{k_{j_0}}}(x-a_i)-D_{\nu_{k_{j_0}}}(x-a_i)\right) =$$

$$= \frac{1}{n} \sum_{\substack{i\in I\\ i\geqq j_0}} \frac{\sin\left(\mu_{k_{j_0}}+\frac{1}{2}\right)(x-a_i)-\sin\left(\nu_{k_{j_0}}+\frac{1}{2}\right)(x-a_i)}{2\sin\frac{x-a_i}{2}} =$$

$$= \frac{1}{n} \sum_{\substack{i\in I\\ i\geqq j_0}} \frac{\sin\frac{1}{2}(\mu_{k_{j_0}}-\nu_{k_{j_0}})(x-a_i)\cdot\cos\frac{1}{2}(\mu_{k_{j_0}}+\nu_{k_{j_0}}+1)(x-a_i)}{\sin\frac{(x-a_i)}{2}}.$$

We can suppose that the sequence k_i is chosen such that

$$\left.\begin{aligned} \mu_{k_i}-\nu_{k_i} &\equiv 2k \pmod{2n+1} \\ \mu_{k_i}+\nu_{k_i}+1 &\equiv 2l \pmod{2n+1} \end{aligned}\right\}, \ \forall i.$$

This means that

$$\frac{1}{2}(\mu_{k_{j_0}}-\nu_{k_{j_0}})\,a_i \equiv i\,\frac{4k\pi}{2n+1} \pmod{2\pi},$$

$$\frac{1}{2}(\mu_{k_{j_0}}+\nu_{k_{j_0}}+1)a_i \equiv i\,\frac{4l\pi}{2n+1} \pmod{2\pi}.$$

Let $j_0 < n-\sqrt{n}$, then

$$(*) \qquad \ln n \geqq \sum_{i=j_0+1}^{n} \frac{1}{i-j_0} \geqq \frac{\ln n}{3}.$$

Let $\varepsilon > 0$ be a fixed number and $\frac{\pi}{2}$ be some multiple of ε. Divide $[0, 2\pi]$ into disjoint segments of length ε. Then there exists a pair of segments $[r\varepsilon, (r+1)\varepsilon]$, $[s\varepsilon, (s+1)\varepsilon]$ with the following property. Let

$$I = I(k, l, \varepsilon, r, s) := \{1 \leqq i \leqq n\colon\ ka_i \in [r\varepsilon, (r+1)\varepsilon],\ \mathrm{mod}\ 2\pi;$$
$$la_i \in [s\varepsilon, (s+1)\varepsilon],\ \mathrm{mod}\ 2\pi\},$$

then

$$\sum_{\substack{n\geqq i>j_0\\ i\in I}} \frac{1}{i-j_0} \geqq c\varepsilon^2 \ln n.$$

This follows from $(*)$ and from the fact that there is $(2\pi/\varepsilon)^2$ such pair. Let

$$\hat{I} := [(r-1)\varepsilon, (r+2)\varepsilon] \cup [(r-1)\varepsilon+\pi, (r+2)\varepsilon+\pi] \cup$$

$$\cup [(s-1)\varepsilon, (s+2)\varepsilon] \cup \left[(s-1)\varepsilon \pm \frac{\pi}{2}, (s+2)\varepsilon \pm \frac{\pi}{2}\right],$$

then $|\hat{I}|\leqq 16\varepsilon$ and then $|[0, 2\pi]\setminus\hat{I}|\geqq 2(\pi-8\varepsilon)$. Denote

$$\Delta_{j_0} := [a_{j_0}, a_{j_0+1}],$$

$$A_{j_0} := \left\{x\in\Delta_{j_0}\colon \frac{1}{2}(\mu_{k_{j_0}}-\nu_{k_{j_0}})x\notin\hat{I}\right\}, \quad B_{j_0} := \left\{x\in\Delta_{j_0}\colon \frac{1}{2}(\mu_{k_{j_0}}+\nu_{k_{j_0}}+1)x\notin\hat{I}\right\},$$

then, for $\varepsilon<1/100$, say, we have

$$|A_{j_0}| \geqq \frac{3}{4}|\Delta_{j_0}|, \quad |B_{j_0}| \geqq \frac{3}{4}|\Delta_{j_0}|$$

and hence

$$|A_{j_0}\cap B_{j_0}| \geqq \frac{1}{4}|\Delta_{j_0}|.$$

Take the partition of $[0, 2\pi]\setminus\hat{I}$ into segments of length ε, then there exists r', s' such that the measure of the set

$$C_{j_0} := \left\{x\in\Delta_{j_0}\colon \frac{1}{2}(\mu_{k_{j_0}}-\nu_{k_{j_0}})x\in[r'\varepsilon, (r'+1)\varepsilon] \bmod 2\pi,\right.$$

$$\left.\frac{1}{2}(\mu_{k_{j_0}}+\nu_{k_{j_0}}+1)x\in[s'\varepsilon, (s'+1)\varepsilon] \bmod 2\pi\right\}$$

satisfies $|C_{j_0}|\geqq c\varepsilon^2|\Delta_{j_0}|$ (further the segments $[r'\varepsilon, (r'+1)\varepsilon]$, $[s'\varepsilon, (s'+1)\varepsilon]$ are disjoint from $\hat{I}$). Clearly $i\in I$ and $x\in C_{j_0}$ imply that

$$\left|\sin\frac{1}{2}\mu_{k_{j_0}}-\nu_{k_{j_0}})(x-a_i)\right| \geqq c\varepsilon, \quad \left|\cos\frac{1}{2}(\mu_{k_{j_0}}+\nu_{k_{j_0}}+1)(x-a_i)\right| \geqq c\varepsilon,$$

and the sign of the above sin and cos is independent if $i\in I$. Summarizing all our observations we can estimate from below the difference

$$|S_{\mu_{k_{j_0}}}(g)-S_{\nu_{k_{j_0}}}(g)| \geqq c\varepsilon^2\frac{1}{n}\sum_{\substack{i\in I\\ i\geqq j_0}}\frac{1}{x-a_i} \geqq c\varepsilon^2\frac{1}{n}\sum_{\substack{i\in I\\ i>j_0}}\frac{1}{a_i-a_{j_0}} =$$

$$= c\frac{\varepsilon^2}{n}\sum_{\substack{i\in I\\ i>j_0}}\frac{2n+1}{4\pi}\,\frac{1}{i-j_0} \geqq c\varepsilon^4\ln n \quad (j_0<n-\sqrt{n},\ x\in C_{j_0}).$$

So the measure of the set $C := \bigcup_{j_0=1}^{n-\sqrt{n}} C_{j_0}$ satisfies

$$|C| \geqq c\varepsilon^2\,\frac{n-\sqrt{n}}{n} \geqq c\varepsilon^2$$

and

$$\max_{1\leqq j_0\leqq n-\sqrt{n}} |S_{\mu_{k_{j_0}}}(g)-S_{\nu_{k_{j_0}}}(g)| \geqq c\varepsilon^4\ln n, \quad x\in C$$

which proves Lemma 2.

LEMMA 3. *Let* $T_n: L^1 \to L^0$ *be a transformation for which there exists* $\delta > 0$ *and for any* $M > 0$ *there exists a polynomial g such that*

$$\|g\|_1 \leqq 3\pi, \quad \sup_n |T_n g(x)| \geqq M \quad (x \in E)$$

with $|E| \geqq \delta$. *Then there exists* $f \in L^1$ *with*

$$\sup_n |T_n f(x)| = \infty \quad a.e.$$

PROOF. Suppose indirectly that for all $f \in L^1$, $\sup_n |T_n f| < \infty$ on a set of positive measure. Saks' theorem [3] states that there exists a measurable set $E \subset [0, 2\pi]$ such that

a) $\sup_n |T_n f(t)| < \infty$ for a.e. $t \in E$, $\forall f \in L^1$,

b) $\sup_n |T_n f(t)| = \infty$ for a.e. $t \notin E$, $\forall f \in L^1 \setminus B$

with some set $B \subset L^1$ of first category. In particular, if for all $f \in L^1$, $\sup |T_n f|$ is bounded on a set of positive measure then $|E| > 0$. Since T_n commutes with the translations, i.e.

$$T_n[f(\cdot + y)](t) = [T_n f](y + t)$$

where the addition is meant mod 2π, hence $E \subset [0, 2\pi]$ must be translation-invariant, so $E = [0, 2\pi]$. Consequently

$$\sup_n |T_n f| < \infty \quad \text{a.e. for all} \quad f \in L^1.$$

Now the Banach theorem on a.e. convergence ([4]) states that T_n is uniformly bounded with respect to the metric inducing the convergence in measure on $L^0(0, 1)$. In other words, for any $\varepsilon, \alpha > 0$ there exists $\eta > 0$ such that

$$\|f\|_1 \leqq \eta \Rightarrow \left|(\sup_n |T_n f| > \alpha)\right| \leqq \varepsilon. \tag{3}$$

Define $T^* f := \sup |T_n f|$. The assumptions of Lemma 3 yield the existence of a sequence $(f_k) \subset L^1(0, 2\pi)$, $\|f_k\|_1 \to 0$ and of a sequence $M_k \to +\infty$ such that $T^* f_k > M_k$ holds on a set of measure $\geqq \delta$. But this contradicts (3) and this contradiction proves Lemma 3.

LEMMA 4. *Let* $\varepsilon < 2\pi$, $0 < \eta < 1$, $M > 0$. *Suppose that there exists* $N \in \mathbf{N}$ *and a (trigonometric) polynomial g such that the level set*

$$A := \Big(\sup_{1 \leqq n \leqq N} |T_n g(x)| \leqq M \Big)$$

satisfies $|A| \leqq \varepsilon$. *Then there exists another polynomial* g' *with* $\|g'\|_1 \leqq 4M\varepsilon$ *and a number* $N' \in \mathbf{N}$ *such that*

$$\sup_{1 \leqq n \leqq N'} |T_n(g + \eta g')(x)| > \frac{\eta}{12} M, \quad \forall x \in [0, 2\pi].$$

PROOF. We shall prove that we can define the polynomial g' such that for some $N'\gg\deg g$ we have $S_{N'}(g')\equiv 0$. Then, of course,

$$\sup_n |T_n(g+\eta g')(x)| > M \quad (x\notin A).$$

Further we have

$$A=\bigcup_{j=1}^{l} I_j$$

where every set I_j is either a point or a (closed) segment. Let $N_2\gg\deg g$ be a number so large that $\nu_n>\deg g$ $(n>N_2)$. Introduce the functions

$$f_j(x):=\begin{cases} M & \text{if } x\in I_j \\ 0 & \text{if } x\notin 2I_j \pmod{2\pi} \\ f_j \text{ is linear if } x\in 2I_j\setminus I_j. \end{cases}$$

The partial sums of the Fourier series of f_j converge uniformly to f_j, consequently for sufficiently large $n_0>N_2$ we obtain

$$\left\|(S_{\mu_{n_0}}-S_{\nu_{n_0}})\left(e^{i[(\mu_{n_0}+\nu_{n_0})/2]x}\sigma_{[\mu_{n_0}-\nu_{n_0}/2]}(f)\right)-fe^{i[(\mu_{n_0}+\nu_{n_0})/2]x}\right\|_\infty<\varepsilon$$

where $f:=\sum_{j=1}^{l} f_j$. Denote

$$g':=e^{i[(\mu_{n_0}+\nu_{n_0})/2]x}\sigma_{[(\mu_{n_0}-\nu_{n_0})/2]}(f)$$

and $N':=n_0$. Then g' and N' fulfil the requirements of Lemma 4.

THEOREM 1. *Let μ_n and ν_n be given natural numbers satisfying*

$$\mu_n\to\infty,\ \nu_n\to\infty,\ |\mu_n-\nu_n|\to\infty \quad (n\to\infty).$$

Then there exists $f\in L^1(0,2\pi)$ such that

$$\sup_n |T_n f(x)|=+\infty \quad \textit{for every} \quad x\in[0,2\pi].$$

PROOF. We showed the existence of $f_2\in L^1(0,2\pi)$ with

$$\sup_n |T_n f_2(x)|=+\infty \quad \text{a.e.}$$

For any $k>0$ we shall give numbers $N_k'\in\mathbf{N}$, $\varepsilon_k>0$ and polynomials g_k such that

$$\|g_k\|_1=O(1),\quad \varepsilon_k=O(1)$$

and

$$\sup_{1\le n\le N_k'}\left|T_n\left(\sum_{j=1}^{k}\frac{\varepsilon_j}{2^j}g_j\right)(x)\right|>k,\quad \forall x.$$

Suppose that N_j',ε_j,g_j is given for $1\le j\le k$ and consider the index $k+1$. If $\varepsilon_{k+1}\ll\varepsilon_k$ is small enough and for some $h\in L^1$,

$$\left\|h-\sum_{j=1}^{k}\frac{\varepsilon_j}{2^j}g_j\right\|_1<\varepsilon_{k+1},$$

then

(4) $$\sup_{1\leqq n\leqq N'} |T_n h(x)| > k-1, \quad \forall x.$$

In what follows we shall make use of this property. We know that

$$\sup_n \left| T_n\left(\sum_{j=1}^{k} \frac{\varepsilon_j}{2^j} g_j + \frac{\varepsilon_{k+1}}{2^{k+1}} f_2, x\right)\right| = \infty \quad \text{a.e.,}$$

hence for any $\varepsilon>0$ there exists $N_k = N_k(\varepsilon)$ such that

$$\sup_{1\leqq n\leqq N_k} \left| T_n\left(\sum_{j=1}^{k} \frac{\varepsilon_j}{2^j} g_j + \frac{\varepsilon_{k+1}}{2^{k+1}} f_2\right)(x)\right| > \frac{1}{\varepsilon}$$

holds outside a set of measure $<\varepsilon$. If l_k is large enough, then

$$\sup_{1\leqq n\leqq N_k} \left| T_n\left(\sum_{j=1}^{k} \frac{\varepsilon_j}{2^j} g_j + \frac{\varepsilon_{k+1}}{2^{k+1}} \sigma_{l_k}(f_2)\right)(x)\right| > \frac{1}{\varepsilon}$$

outside a set of measure $<c\varepsilon$. Use Lemma 4 with $\eta := \frac{\varepsilon_{k+1}}{2^{k+1}}$, $N := \frac{1}{\varepsilon}$ to show the existence of a polynomial g'_{k+1} with

$$\|g'_{k+1}\|_1 \leqq 4\cdot\frac{1}{4}\cdot\varepsilon = 4,$$

and of a number N'_{k+1} satisfying

$$\sup_{1\leqq n\leqq N'_{k+1}} \left| T_n\left(\sum_{j=1}^{k} \frac{\varepsilon_j}{2^j} g_j + \frac{\varepsilon_{k+1}}{2^{k+1}} \left(\sigma_{l_k}(f_2) + g'_{k+1}\right)\right)(x)\right| > c\frac{1}{\varepsilon}\frac{\varepsilon_{k+1}}{2^{k+1}}.$$

Since $\varepsilon>0$ can be arbitraly small, hence we obtain

$$c\cdot\frac{1}{\varepsilon}\cdot\frac{\varepsilon_{k+1}}{2^{k+1}} > k+1.$$

Hence we can define indeed $N'_{k+1}, \varepsilon_{k+1}, g_{k+1}$, with the indicated properties. Finally define

$$f := \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2^k} g_k.$$

Now

$$\|f\|_1 \leqq c\sum \frac{\varepsilon_k}{2^k} \leqq c.$$

On the other hand

$$\left\| f - \sum_{j=1}^{k} \frac{\varepsilon_j}{2^j} g_j \right\|_1 \leqq c \sum_{j=k+1}^{\infty} \frac{\varepsilon_j}{2^j} < \varepsilon_{k+1},$$

whence (4) ensures that

$$\sup_{1\leq n\leq N_k'} |T_n f(x)| > k-1, \quad \forall x \forall k,$$

and this proves Theorem 1.

2. Next we investigate the phenomenon of the local norm divergence of Fourier series.

LEMMA 5. *Let* $1\leq p<\infty$ *and* $F\subset L^0(0,1)$ *be an arbitrary function set. Then there exists a partition* $(0,1)=E_1\cup^* E_2$ *such that*

a) *For any* $\varepsilon>0$ *there exists a set* E_ε *with* $|E_\varepsilon|<\varepsilon$ *and*

$$\{\|f\|_{L^p(E_1\setminus E)_\varepsilon}: f\in F\}$$

is bounded.

b) *For any* $E\subset E_2$, $|E|>0$ *there exists* $(f_n)\subset F$ *with*

$$\sup_n \|f_n\|_{L^p(E)}=\infty.$$

Moreover, if we know that F *is translation-invariant, i.e.* $f\in F$ *imples* $f(\cdot+t)\in F$ *(the summation meant periodically), then either* $E_1=\emptyset$ *or* $E_1=(0,1)$ *and in the latter case in part* a) *we can take* $|E_\varepsilon|=0$, *i.e.* F *is bounded in* $L^1(0,1)$.

PROOF. We call a set $E\subset(0,1)$ a good set if

$$\{\|f\|_{L^p(E)}: f\in F\}$$

is bounded. Let

$$\delta := \sup\{|E|: E \text{ is good}\}.$$

There are good sets E^n such that $|E^n|\to\delta$. Let

$$E_1 := \bigcup_{n=1}^{\infty} E^n, \quad E_2 := (0,1)\setminus E_1.$$

Since the union of finitely many good sets is also good, hence $|E_1|=\delta$, so a) holds. On the other hand E_1 contains any good sets (up to a set of measure 0) by the same arguments. So E_2 can not contain any good set, and then b) also holds.

Consider the case when F is invariant under translations. Suppose there exists a good set E, $|E|>0$. It is enough to prove that $E^c:=(0,1)\setminus E$ is also good. Suppose indirectly that for any $M>0$ there exists $f=f_M\in F$ such that

$$\|f\|^p_{L^p(E^c)} = \int_{E^c} |f|^p > M.$$

Applying Fubini's theorem we obtain

$$\int_0^1 \Big(\int_{E^c\cap(E+t)} |f(\tau)|^p\,d\tau\Big)\,dt = \int_0^1 \Big(\int_{E^c} \chi_{E+t}(\tau)|f(\tau)|^p\,d\tau\Big)\,dt =$$

$$= \int_{E^c} |f(\tau)|^p \Big(\int_0^1 \chi_{E+t}(\tau)\,dt\Big)\,d\tau = |E|\cdot \int_{E^c} |f(\tau)|^p\,d\tau > M|E|.$$

Hence there exists $t_0 \in [0, 1]$ with

$$M|E| < \int_{E^c \cap (E+t_0)} |f|^p \leqq \int_{E+t_0} |f|^p = \int_E |f(\tau+t_0)|^p \, d\tau.$$

Since $f(\cdot + t_0) \in F$ and M can be taken arbitrarily large independently of E, hence E can not be a good set. The contradiction proves Lemma 5.

As a corollary we get the following statement.

THEOREM 2. *Let $L_n : L^1(0, 1) \to L^0(0, 1)$ be a sequence of continuous linear and translation-invariant operators. Suppose that there exists $f \in L^1(0, 1)$ such that*

$$\sup_n \|L_n f\|_{L^1(0,1)} = \infty.$$

Then for every set E, $|E| > 0$ we have

$$\sup_n \|L_n f\|_{L^1(E)} = \infty$$

for every $f \in L^1(0, 1)$ except for a set of first category.

COROLLARY. *Let $\mu_n, \nu_n \in \mathbf{N}$ satisfy*

$$\mu_n \to \infty, \ \nu_n \to \infty, \ \sup_n |\mu_n - \nu_n| = \infty.$$

Then for an arbitrary set E, $|E| > 0$, there exists $f \in L^1(0, 2\pi)$ such that the sequence $(S_{\mu_n} f - S_{\nu_n} f)$ is not bounded in $L^1(E)$.

PROOF. The operators $T_n := S_{\mu_n} - S_{\nu_n}$ are translation invariant. By Theorem 2 it is enough to prove the existence of $f \in L^1(0, 1)$ with $\sup_n \|T_n f\|_{L^1(0,2\pi)} = \infty$. Suppose indirectly that there is no such f, then the Banach—Steinhaus theorem implies that

$$\|T_n f\|_{L^1(0,2\pi)} \leqq C \|f\|_{L^1(0,2\pi)} \quad (\forall f \in L^1(0, 2\pi)).$$

In particular, for every analytic trigonometric polynomial p we have

$$\|S_{|\mu_n - \nu_n|} p\|_{L^1(0,2\pi)} \leqq C \|p\|_{L^1(0,2\pi)},$$

with $C > 0$ independent of n and p. But this implies

$$(*) \qquad \|S_k p\|_{L^1(0,\pi)} \leqq C \|p\|_{L^1(0,\pi)} \quad (k = 0, 1, \ldots)$$

for arbitrary analytic polynomial. Indeed, if n is chosen so that $|\mu_n - \nu_n| > k$, then let

$$\hat{p} := e^{i[\max(\mu_n, \nu_n) - k]x} p,$$

whence

$$\|S_k p\|_{L^1(0,2\pi)} = \|S_{|\mu_n - \nu_n|} \hat{p}\|_{L^1(0,2\pi)} \leqq C \|\hat{p}\|_{L^1(0,2\pi)} = C \|p\|_{L^1(0,2\pi)}.$$

But $(*)$ is not true, see ([11], p. 599).

We raise here a problem concerning Theorem 2.

PROBLEM 1. Does there exist a set A of first category in $L^1(0, 1)$ such that for every set E, $|E| > 0$,

$$\sup_n \|L_n f\|_{L^1(E)} = \infty \quad (f \notin A)?$$

For the trigonometrical partial sums $L_n = S_n$ the answer is positive and we return to this in a next paper of the same journal.

Lemma 5 shows that the classical Saks theorem [3] has an analogy also for the norm convergence. Indeed, we can prove the

LEMMA 6. *Let X be an arbitrary set, $1 \leqq p < \infty$, and let $L_n : X \to L^0(0, 1)$ be an arbitrary sequence of mappings. Then there exits a decomposition $(0, 1) = E_1 \cup^* E_2$ such that*

a) $\forall x \in X \forall \varepsilon > 0 \ \exists E_\varepsilon \subset (0, 1), |E_\varepsilon| < \varepsilon$ *satisfying*

$$\sup_n \|L_n x\|_{L^p(E \setminus E_\varepsilon)} < \infty.$$

b) $\forall E \subset E_2, \ |E| > 0 \ \exists x = x_E \colon \sup_n \|L_n x\|_{L^p(E)} = \infty.$

Moreover if for every $h \in R$ there exists a mapping $x \to x_h$ on X such that $L_n x_h = \tau_h L_n x$ then either $E_1 = \emptyset$ or $E_1 = (0, 1)$ and in the latter case a) *holds with $|E_\varepsilon| = 0$, i.e.*

$$\sup_n \|L_n x\|_{L^p(0,1)} < \infty, \quad \forall x \in X.$$

PROOF. We say that the set E is good if

$$\sup_n \|L_n x\|_{L^p(E)} < \infty, \quad \forall x \in X$$

and apply the proof of Lemma 5.

3. Next we investigate the a.e. divergence for signed Toeplitz summations.

LEMMA 7. *Let $0 < \varepsilon < 2\pi$, $0 < \eta < 1$, $0 < M$, and denote by T an arbitrary Toeplitz summation whose coefficients satisfy*

a) $0 < c_0 \leqq \left|\sum_{k=1}^{\infty} t_{n,k}\right| \leqq \sum_{k=1}^{\infty} |t_{n,k}| < \infty \quad (\forall n),$

b) $\lim_{n \to \infty} t_{n,k} = 0 \quad (\forall k).$

Let

$$T_n(g, x) := \sum_{k=1}^{\infty} t_{n,k} S_k(g, x).$$

Suppose that there exists $N \in \mathbf{N}$ and an analytic trigonometric polynomial g satisfying

$$\left|\left(\sup_{1 \leqq n \leqq N} |T_n(g, x)| \leqq M\right)\right| < \varepsilon.$$

Then there exists another trigonometric analytic polynomial g' and $N' \in \mathbf{N}$ satisfying $\|g'\|_1 \leqq 4M\varepsilon$ and

$$\sup_{1 \leqq n \leqq N'} |T_n(g + \eta g')(x)| > \min\left\{\frac{1}{2}\eta \frac{M}{6} - 2, M\right\} \quad \forall x \in [0, 2\pi].$$

PROOF. Let $N \ll N_1 \ll N_2' \ll N_2$, $\deg g < N_1$, then by a) and b) we get

$$\Big|\sum_{k=N_1+1}^{N_2} t_{N_2'k}\Big| \geqq \frac{c_0}{2},$$

hence in case $|g(x)| > \eta\, M/3$ we obtain

$$|T_{N_2'}^{N_2}(g,x)| = \Big|\sum_{k=1}^{N_2} t_{N_2'k}\, S_k(g,x)\Big| \geqq \Big|\sum_{k=N_1+1}^{N_2} t_{N_2'k}\Big| \cdot |g(x)| -$$

$$-\Big|\sum_{k=1}^{N_1} t_{N_2'k}\Big|\, |S_k(g,x)| \geqq \frac{c_0}{2}\eta\frac{M}{3} - 1, \quad T_n^N(g,x) := \sum_{k=1}^{N} t_{nk} S_k(g,x)$$

(in the last estimate we used b) again). If the polynomial g' will be given such that it has zero coefficients of order $\leqq N_2$, then on the set

$$\left(|g| > \eta\frac{M}{3}\right) \cup \big(\sup_{1 \leqq n \leqq N} |T_n(g)| > M\big)$$

we have

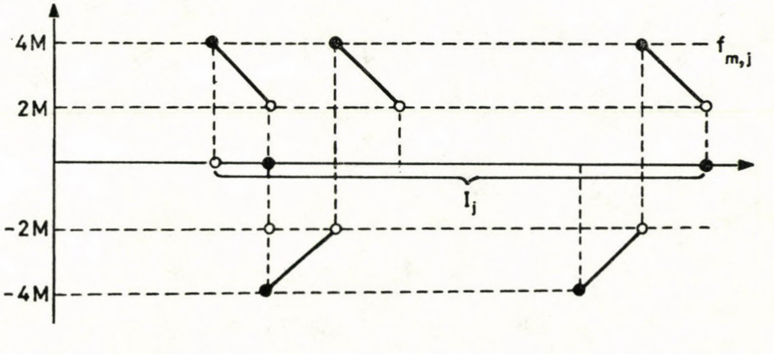
$$\sup_{1 \leqq n \leqq N_2} |T_n(h+\eta g', x)| \geqq \max\left(M, \frac{c_0}{2}\eta\frac{M}{3} - 1\right).$$

The set of x not yet considered has the form

$$\left(|g| \leqq \eta\frac{M}{3}\right) \cap \big(\sup_{1 \leqq n \leqq N} |T_n(g)| \leqq M\big) \subset$$

$$\subset \left(\left(|g| \leqq \eta\frac{M}{3}\right) \cap \big(\sup_{1 \leqq n \leqq N} |T_N^n(g)| \leqq M\right) = \bigcup_{j=1}^{l} I_j,$$

where I_j are closed segments and $\sum |I_j| < \varepsilon$. Take the function $f_{m,j}$ defined on I_j as the following figure shows (the interval I_j is divided into m equal parts):



This function was used also in Totik [2]. Take the sum

$$f_m := \sum_{j=1}^{l} f_{m,j}.$$

If x is a point of discontinuity of f_m then

$$|S_k(f_m, x)| \to f_m^*(x) := \frac{1}{2}|f_m(x-0)+f_m(x+0)| \geqq M \quad (k \to \infty),$$

hence for $N_3 \gg m$ we have

$$S_k(g+\eta f_m, x) = g(x)+\eta f_m^*(x)+o_k(1), \quad k \geqq N_3$$

and then

$$\Big|\sum_{k=N_3+1}^{N_4} t_{N_3' k} S_k(g+\eta f_m, x)\Big| \geqq \frac{c_0}{2}|g(x)+\eta f_m^*(x)| -$$

$$-o_{N_3}(1) \sum_{k=N_3+1}^{N_4} |t_{N_3' k}| \geqq c_0 \eta \frac{M}{3} - o_{N_3}(1)$$

holds if $N_3 \ll N_3' \ll N_4$. Let

$$h := \sigma_n(f_m) - S_{N_2}\sigma_n(f_m), \quad g' := P_+ h,$$

where P_+ denotes the Riesz projection i.e. the transform of taking the analytic part of the Fourier series. The constant n will be given later. We know that

$$\operatorname{Re} S_k g' = \frac{1}{2}\{S_0(h)+S_k(h)\} = \frac{1}{2} S_k(h),$$

since h is a real polynomial. Consequently

$$|T_{N_3'}^{N_4}(g+\eta g', x)| = \Big|\sum_{k=1}^{N_4} t_{N_3' k}\big(S_k g(x)+\eta S_k(g', x)\big)\Big| \geqq$$

$$\geqq \Big|\operatorname{Re}\big(\sum_{k=1}^{N_4} t_{N_3' k}\big(S_k g(x)+\eta S_k(g', x)\big) =$$

$$= \Big|\sum_{k=1}^{N_4} t_{N_3' k}\Big(\operatorname{Re} S_k g(x)+\frac{\eta}{2} S_k(h, x)\Big)\Big| \geqq$$

$$\geqq \Big|\sum_{k=N_3+1}^{N_4}\Big| t_{N_3' k} \max_{N_3 \leqq k \leqq N_4}\Big|\operatorname{Re} g(x)+\frac{\eta}{2} S_k(h, x)\Big| - 1/2.$$

Now if $N_2 \ll m$, then

$$\big\|S_{N_2}(\sigma_n(f_m))\big\|_\infty = o_m(1)$$

and if $N_4 \ll n$ then

$$\big\|S_k(\sigma_n(f_m)) - S_k(f_m)\big\|_\infty = o_{N_4, m}(1) \quad (k \leqq N_4)$$

so

$$S_k(h) = S_k(f_m)+o_{N_4, m}(1) \quad (k \leqq N_4).$$

Since $m \ll N_3$ hence

$$S_k(h, x) = f_m^*(x) + o_{N_3, N_4, m}(1) \quad (N_3 < k \leqq N_4)$$

and then

$$|T_{N_3'}^{N_4} T(g + \eta g', x)| > \frac{c_0}{2} \eta \frac{M}{6} - 1 \quad (x \in U)$$

holds for an open set $U = U(N_4)$ containing all discontinuity points of f_m. Outside U the sequence $S_k(f_m)$ converges uniformly to f_m hence for $N_5 \gg N_4$ we obtain

$$S_k(f_m, x) = f_m(x) + o_{N_5}(1) \quad (k \leqq N_5, x \notin U).$$

On the other hand, let $N_5 \ll N_5' \ll N_6 \ll n$, then

$$\|S_k(\sigma_n(f_m)) - S_k(f_m)\|_\infty = o_{N_6}(1) \quad (k \leqq N_6),$$

hence

$$\|S_k(h) - S_k(f_m)\|_\infty = o_{N_6}(1) \quad (k \leqq N_6)$$

and (taking real parts) we get

$$|T_{N_5'}^{N_6}(g + \eta g', x)| \geqq \left| \sum_{k=N_5+1}^{N_6} t_{N_5' k} \left(\operatorname{Re} g(x) + \frac{\eta}{2} S_k(h, x) \right) \right| - o(1) \geqq$$

$$\geqq \left| \sum_{k=N_5+1}^{N_6} t_{N_5' k} \right| \cdot \left| \operatorname{Re} g(x) + \frac{\eta}{2} f_m(x) \right| - 1 \geqq \frac{c_0}{2} \eta \frac{M}{6} - 1 \; (x \in \bigcup_{j=1}^{l} I_j \setminus U).$$

Lemma 7 is proved.

THEOREM 3. *Let T be a summation process satisfying* a) *and* b) *from Lemma* 7. *Suppose that there exists a power type $f \in L^1(0, 2\pi)$ satisfying*

$$\sup_n |T_n(f, x)| = \infty \quad a.e.$$

Then there exists a power type $f_1 \in L^1(0, 2\pi)$ such that

$$\sup_n |T_n(f, x)| = \infty \quad \forall x \in [0, 2\pi].$$

The proof repeats the ideas from Totik [2], Lemma 4, so we omit the details.

REMARK. We have proved in Theorem 1 the statement of Theorem 3 for the operator

$$T_n := S_{\mu_n} - S_{\nu_n}.$$

This operator, meant as a signed Toeplitz summation, does not satisfy condition a) of Lemma 7. Here the following question arises.

PROBLEM 2. Give a common generalization of Theorems 1 and 3.

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CHARACTERIZATION OF SUBPROJECTION SUBOPERATORS

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Introduction

After P. R. Halmos [2], a suboperator is a bounded linear transformation from a subspace of a Hilbert space into the whole space. A couple of problems initiated also by Halmos in the paper just mentioned arises when one asks for a characterization of subpositive, subprojection e.t.c. suboperators, that is for ones there are positive, projection e.t.c. operators that extend these suboperators. Of course, subselfadjoint suboperators are, in view of the now classical theorem of M. G. Krein, symmetric suboperators. A simple proof of this fact (together with extension not increasing the norm as usual) can be found in Z. Sebestyén [3]. Here an independent characterization of subpositive suboperators is proved as a starting point for the selfadjoint case. This turned out to be the natural approach.

As a matter of fact the so called Schwarz inequality is proved to be characteristic for subpositive suboperators in the author's paper [3]. In the present note we show that the Schwarz identity (with constant one) characterizes precisely the subprojection suboperators (Theorem 1).

As a corollary we get the characterization of Halmos [2, Proposition 3] and in a remark we prove the same result of Halmos for subpositive suboperators [2, Corollary 2] using our result.

Factorizations through projection are proved in Corollaries 2 and 3.

Characterization of subprojections

Given a (complex) Hilbert space H, a (closed) subspace H_0 in it and a suboperator $Q: H_0 \to H$, we are interested in searching for a (selfadjoint) projection P on H which restricted to H_0 is Q itself.

THEOREM 1. *Let $Q: H_0 \to H$ be a suboperator. Q is a subprojection if and only if the identity*

$$\|Qx\|^2 = (Qx, x) \quad (x \in H_0) \tag{1}$$

holds true.

PROOF. An operator P on a Hilbert space H is an orthogonal projection if and only if it is selfadjoint and idempotent:

$$P^* = P = P^2. \tag{2}$$

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In other words

$$\|Px\|^2 = (Px, Px) = (P^2x, x) = (Px, x)$$

holds true for any x in H. In the case when Q is the restriction of P to H_0 this reduces to (1).

On the other hand if we assume (1) to hold for Q, the approach of [3], [4] applies: define a semi-inner product $\langle , \rangle$ on H_0 by

$$\langle x, y\rangle := (Qx, y) \quad (x, y \in H_0). \tag{3}$$

Then another Hilbert space K arises by taking completion of the quotient space H_0/N with respect to the norm inherited from the inner product (denoted by the same symbol) on this space, where N is the nullspace of $\langle , \rangle$ in H_0. For x in H_0, $(x+N)$ is the corresponding vector in H_0/N so that

$$V(x+N) := Qx \quad (x \in H_0) \tag{4}$$

defines on the dense subset H_0/N of K a map $V: H_0/N \to H$ which is an isometry as well. Indeed,

$$\|V(x+N)\|^2 \overset{(4)}{=} \|Qx\|^2 \overset{(1)}{=} (Qx, x) \overset{(3)}{=} \langle x+N, x+N\rangle$$

holds true for all x in H_0. Here we use step by step (4), (1) and (3) respectively. We have thus a unique isometry, denoted also by V, of K into H as an extension of the former V. The desired projection of H will be $P := VV^*$. First, this is selfadjoint (moreover positive) and idempotent since V^*V is the identity operator on K by the isometry of V so that

$$P^2 = V(V^*V)V^* = VV^* = P.$$

That P restricted to H_0 is Q is a consequence of the characteristic identity

$$V^*x = x+N \quad (x \in H_0), \tag{5}$$

we have discovered in our previous works. It is implied by the identity (for any y in H_0):

$$\langle y+N, V^*x\rangle = (V(y+N), x) \overset{(4)}{=} (Qy, x) = \langle y+N, x+N\rangle \quad (y \in H_0).$$

Indeed (5) implies (using (4))

$$Px = V(V^*x) \overset{(5)}{=} V(x+N) \overset{(4)}{=} Qx$$

as desired. The proof is complete.

COROLLARY 1 (Halmos). *$Q: H_0 \to H$ is a subprojection if and only if $A^* = A$ and $A - A^2 = B^*B$ hold true for the operators $A: H_0 \to H_0$, $B: H_0 \to H \ominus H_0$ that represent Q as a "column matrix" $\begin{pmatrix} A \\ B \end{pmatrix}$.*

PROOF. In the representation just mentioned

$$Qx = Ax \oplus Bx, \quad Ax \in H_0, \quad Bx \in H \ominus H_0 \tag{6}$$

holds uniquely for all x in H_0. Therefore

$$\|Qx\|^2 \overset{(6)}{=} \|Ax\|^2+\|Bx\|^2 = ((A^*A+B^*B)x, x)$$

and

$$(Qx, x) \overset{(6)}{=} (Ax+Bx, x) \overset{(6)}{=} (Ax, x)$$

hold also true for any x from H_0. Thus (1) is (easily shown to be) equivalent to the requirements

$$A^* = A \quad \text{and} \quad A = A^*A+B^*B = A^2+B^*B.$$

This is nothing but the properties stated by Halmos.

Characterization of subpositive suboperators

REMARK 1 (Halmos). $Q\colon H_0\to H$ is subpositive if and only if $A\geqq 0$ and $\operatorname{ran} B^*\subset \subset \operatorname{ran}\sqrt{A}$ holds true for the operators A, B defined in Corollary 1.

PROOF. Theorem in [3] says that Q is subpositive if and only if there exists $M\geqq 0$ with the property

$$\|Qx\|^2 \leqq M(Qx, x) \quad (x\in H_0).$$

This implies (as before) $A^*=A$ and

$$\|Ax\|^2 \leqq \|Ax\|^2+\|Bx\|^2 = \|Qx\|^2 \leqq M(Qx, x) = M(Ax, x) \quad (x\in H_0).$$

That is $A\geqq 0$ and

$$(B^*Bx, x) = \|Bx\|^2 \leqq M(Ax, x) \quad (x\in H_0).$$

But this last inequality implies $\operatorname{ran} B^*\subset\operatorname{ran}\sqrt{A}$ [1, Theorem 1]. Moreover the last requirement is equivalent to the existence of $M\geqq 0$ with the property

$$\|Bx\|^2 = (B^*Bx, x) \leqq M(Ax, x) \quad (x\in H_0).$$

This means that for any x in H_0

$$\|Qx\|^2 = \|Ax\|^2+\|Bx\|^2 \leqq \|A\|(Ax, x)+M(Ax, x) = (\|A\|+M)(Qx, x)$$

holds, therefore Q is subpositive.

REMARK 2. For a subprojection suboperator $Q\colon H_0\to H$ and a projection R of H such that R extends Q we have $P\leqq R$, where P is the projection in the proof of Theorem 1.

PROOF. By an argument of [4] we have another Hilbert space L by taking a semi-inner product on H using R as follows:

$$\langle x, y\rangle_0 := (Rx, y) \quad (x, y\in H),$$

and the procedure as before (using Q on H_0). The identity map $H_0\to H$ induces an isometry $T\colon H_0/N\to H/N_0$, where N_0 is the nullspace of $\langle\,,\rangle_0$. Indeed

$$\|T(x+N)\|^2 = \|x+N_0\|^2 = (Rx, x) = (Qx, x) = \|x+N\|^2 \quad (x\in H_0)$$

holds true since R extends Q. Taking W as a counterpart of V by defining

$$W(x+N_0) := Rx \quad (x\in H)$$

we get the characteristic identity $V=WT$, where $T\colon K\to H$ is the unique isometric extension of the former isometry, since

$$WT(x+N) = W(x+N_0) = Rx = Qx \overset{(4)}{=} V(x+N)$$

holds for any x in H_0.

Finally we have thus the desired inequality as follows:

$$P = VV^* = WT(WT)^* = W(TT^*)W^* \leqq WW^* = R.$$

Factorization

COROLLARY 2. *Let A, B be bounded linear operators on the Hilbert space H. There exists a projection P such that*

$$A = PB \tag{7}$$

if and only if

$$A^*A = B^*A. \tag{8}$$

PROOF. Assuming (7) we get (8) easily:

$$A^*A = (B^*P)(PB) = B^*P^2B = B^*(PB) = B^*A.$$

Conversely, (8) implies (7) by Theorem 1 since the map $Q(Bx):=Ax$, $x\in H$ is well-defined and satisfies (1). Indeed, for any $x\in H$ we have

$$\|Q(Bx)\|^2 = \|Ax\|^2 = (A^*Ax, x) \overset{(8)}{=} (B^*Ax, x) = (Ax, Bx) = \big(Q(Bx), Bx\big),$$

where $Bx=0$ implies $Q(Bx)=Ax=0$. Q is defined on the range of B, a not necessarily closed subspace. Of course, this is not essential in Theorem 1.

COROLLARY 3. *Let A be a bounded linear operator on the Hilbert space H. There exists a projection P and a positive operator B on H such that*

$$A = PB \tag{9}$$

if and only if

$$(A^*A)^2 \leqq M\cdot A^*A^2 = M(A^*)^2A \quad \textit{for some } M \geqq 0. \tag{10}$$

PROOF. In view of (9) we arrive at (10) at once:

$$A^*A = (B^*P)(PB) = B^*P^2B = BPB,$$

$$(A^*A)^2 = (BPB)(BPB) = (BP)B^2(PB) \leqq BP(\|B\|B)PB = \|B\|\cdot A^*A\cdot A, \quad \|B\| = M.$$

Conversely, (10) implies by [3, Corollary 1] that there exists a positive operator B on H such that $A^*A=BA$. But Corollary 2 applies to get (9).

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THE SPECTRAL SINGULARITIES OF INJECTIVE WEIGHTED SHIFTS

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1. Introduction

The *set of the spectral singularities* of a (closed) operator in a Banach space was defined in Nagy [5; Definition 1, p. 319]. It is the intersection of all closed subsets S in the (extended) complex plane such that the operator T is S-spectral ([5; Theorem 5]). It will be denoted here by $S_1(T)$, and is, loosely speaking, the smallest subset of the spectrum sp (T), "outside which" the operator behaves like a spectral operator in the sense of Dunford and Bade (cf. [2]). The set of the spectral singularities *in the strict sense* $S_2(T)$ will be defined here as the smallest set in the class $\{K=\overline{K}\subset \mathrm{sp}\,(T):\ T$ is an $A(K)$-scalar operator$\}$ ([5; Theorem 6]), and may be thought of as the smallest set, "outside which" T behaves like a spectral operator of scalar type ([2]). Clearly, $S_1(T)\subset S_2(T)$.

Using the basic results of R. Gellar [3], [4] and the survey of A. Shields [6], in this note we determine these sets for the injective unilateral and bilateral weighted shifts on the spaces $l^2(N)$ and $l^2(Z)$, respectively (N, Z and C will denote the set of all positive integers, integers and complex numbers, respectively). As it is well-known (cf. e.g. [6; pp. 56—57] and [1; pp. 25—26]), general weighted shifts are direct sums of injective ones, and their spectral behaviour can be quite complicated. Since any weighted shift is unitarily equivalent to another one with weight sequence consisting of the moduli of the original one, we may and shall assume that all weights w_n are positive. Our basic reference and source of most notations will be [6]. We mention explicitly that if $\{\beta(m)\}$ is a two-sided sequence of positive numbers with $\beta(0)=1$, then the Hilbert space $L^2(\beta)$ and the commutative Banach algebra $L^\infty(\beta)$ are defined on [6; pp. 58 and 61], respectively. In this note operator will always mean a bounded linear operator in a Hilbert space, and $r(T)$ will stand for the spectral radius of the operator T.

2. The results

THEOREM 1. *Let T be an invertible bilateral weighted shift on $l^2(Z)$ with weight sequence $\{w_n: n\in Z\}$ with positive weights and such that $0<R=(r(T^{-1}))^{-1}=r(T)$. The following are equivalent:*

1° *T is a spectral operator.*

2° *T is spectral of scalar type.*

3° *T/R is similar to the (unitary) bilateral shift (for which all weights are 1).*

4° *There are positive numbers C_1, C_2 such that for all $m\leqq n$, $m, n\in Z$ we have $C_1R^{n-m+1}\leqq w_m\dots w_n\leqq C_2R^{n-m+1}$.*

5° *With the notation $\beta_R(n)=\beta(n)R^{-n}$ $(n\in Z)$ the spaces $L^2(\beta_R)$ and $L^2(1)$ are* *identical, and their (Hilbertian) norms (see* [6; *p.* 58]) *are equivalent. Here* 1 *denotes the particular sequence* $\beta(\,.\,)$ satisfying $\beta(n)=1$ *for every* $n\in Z$.

PROOF. Assume 1° and let N be a quasinilpotent operator commuting with T. By [6; Theorem 3(a)], then $N=UM_fU^{-1}$ with some unitary U, where M_f denotes the operator of multiplication by $f=\{\hat{f}(n)\colon n\in Z\}$ in $L^\infty(\beta)$. For any positive integer r then $f^r\in L^\infty(\beta)$ and $N^r=UM_{f^r}U^{-1}$, by [6; Proposition 9]. According to [4; Theorem 3], $f(z)=\sum_{n=-\infty}^{\infty}\hat{f}(n)z^n$ is the "Fourier series" of a bounded measurable function $\tilde{f}$ on the set $\{z\colon |z|=R\}$ (in the sense that for $z=Re^{i\varphi}$ the Fourier series of $\tilde{f}$ is $\sum_{n=-\infty}^{\infty}\hat{f}(n)R^{|n|}e^{in\varphi}$). Similarly, $\sum_{n=-\infty}^{\infty}\hat{f^r}(n)z^n$ is the "Fourier series" of the function $\tilde{f}^r$ for every positive integer r, and we have

$$|N^r| = |M_{f^r}| \geqq \operatorname*{ess\,sup}_{|z|=R} |\tilde{f}(z)|^r.$$

Hence $|N^r|^{1/r}\geqq \operatorname{ess\,sup}|\tilde{f}(z)|$. If the right-hand side is positive, then N is not quasinilpotent, a contradiction. Otherwise $N=0$, which shows that the quasinilpotent part of T vanishes, i.e. T is scalar. Therefore 1° implies 2°.

Assume now 2°. The spectrum of T is the circle $\{z\colon |z|=R\}$ (cf. [6; Theorem 5 (a)]). Hence there is $0<K<\infty$ such that for every function f continuous on this circle C_R we have $|f(T)|\leqq K\max\{|f(z)|\colon z\in C_R\}$, where the operator $f(T)$ is meant in the sense of the functional calculus for scalar operators. For every $n\in Z$ let $f_n(z)=z^n$ on C_R. Then $f_n(T)=T^n=UM_{f_n}U^{-1}$ with some unitary $U\colon L^2(\beta)\to l^2(Z)$. Applying [6; formulas (21) and (28)], we obtain that

$$R^nK \geqq |T^n| = \begin{cases}\sup\{[\beta(n+j)/\beta(j)]\colon j\in Z\} \\ \sup\{[\beta(j)/\beta(j-n)]\colon j\in Z\}\end{cases} \text{for} \quad \begin{matrix} n=0,1,\dots \\ n=-1,-2,\dots.\end{matrix}$$

From this and [6; Proposition 7] we have for $j\in Z$, $n=1,2,\dots$

$$0 < w_jw_{j+1}\dots w_{j+n-1} \leqq KR^n$$

and also $R^{-n}K\geqq\beta(j)/\beta(j+n)>0$. From this we obtain

$$0 < K^{-1}R^n \leqq \beta(j+n)/\beta(j) = w_jw_{j+1}\dots w_{j+n-1}.$$

Hence for every $m\leqq n$, $m,n\in Z$ we have $R^{n-m}K^{-1}\leqq w_m\dots w_n\leqq KR^{n-m}$, i.e. 4° holds.

Assume now 4°. By [6; Theorem 2 (a)], T/R is then similar to the ordinary bilateral shift, i.e. 3° follows. The implications 3°⇒2°⇒1° are clear. Finally, 5° is equivalent to the existence of $A,B>0$ such that for every $f\in L^2(\beta_R)=L^2(1)$

$$A^2\Sigma|\hat{f}(n)|^2 \leqq \Sigma|\hat{f}(n)|^2\beta_R(n)^2 \leqq B^2\Sigma|\hat{f}(n)|^2,$$

or, equivalently, $A\leqq\beta_R(n)\leqq B$ for every $n\in Z$. By [6; Proposition 7], this means that

$$A \leqq R^{-n}w_0\dots w_{n-1},\ R^n(w_{-1}\dots w_{-n})^{-1} \leqq B \quad (n>0)$$

or, equivalently, the existence of $a, b>0$ such that

$$aR^n \leqq w_0 \dots w_{n-1},\ w_{-1} \dots w_{-n} \leqq bR^n \quad (n>0).$$

5° is therefore equivalent to 4°, and the proof is complete. □

If T is an injective bilateral weighted shift, then $0 \leqq R_2 = (r(T^{-1}))^{-1} \leqq r(T) = R_1$ (in the sense that $R_2=0$ if and only if $0 \in \mathrm{sp}\,(T)$, cf. [3, Theorem 3]). With these notations we have

THEOREM 2. *Let T be an injective weighted shift on $l^2(Z)$ or on $l^2(N)$ with positive weight sequence $\{w_n\}$. If T is unilateral and $r(T)>0$, then the set of the spectral singularities (in both senses) is the spectrum $\{z \in C: 0 \leqq |z| \leqq r(T)\}$. If T is bilateral and $R_1<0$, then both sets of the spectral singularities are void exactly when the conditions of Theorem 1 hold, otherwise both sets are identical with the spectrum $\{z \in C: R_2 \leqq |z| \leqq \leqq R_1\}$. If in either case $r(T)=0$, then $S_1(T)=\emptyset$ and $S_2(T)=\{0\}$.*

PROOF. If T is unilateral, then [6; Corollary 2 to Theorem 3] shows that there is no nontrivial projection commuting with T. If T is bilateral and $R_2<R_1$, then [3; Corollary 2 to Theorem 4] yields the same conclusion. If T is bilateral and $0<R_2=R_1$, then the spectrum of T is the circle $\{z \in C: |z|=R_1\}$ (cf. [6; Theorem 5 (a)]). It is clear that (for any operator T) for any complex number $z \neq 0$ we have $S_i(zT) = zS_i(T)$ $(i=1, 2)$. On the other hand, if $|z|=1$, then the bilateral weighted shifts T and zT are unitarily equivalent (cf. [6; Corollary 2 to Proposition 1]), hence $zS_i(T)=S_i(zT)=S_i(T)$. Since $S_i(T)$ is a closed subset of the circle above, this implies either $S_i(T)=\emptyset$ or $S_i(T)=\mathrm{sp}\,(T)$ $(i=1, 2)$. By Theorem 1, $S_i(T)=\emptyset$ for $i=1$ or 2 implies that all the (equivalent) conditions there hold. Finally, $r(T)=0$ means that T is a (nonzero) quasinilpotent, hence the last assertion in Theorem 2 is valid. □

EXAMPLE. We give an example of a bilateral weighted shift T for which $R=R_1=R_2>0$, but T is not spectral. Let the corresponding weight sequence $\{w_n: n \in Z\}$ be $\dots 1\ 1\ 1\ 2\ 1^{3!}\ \frac{3}{2}\ 2\ 1^{4!}\ \frac{4}{3}\ \frac{3}{2}\ 2\ 1^{5!}\ \frac{5}{4}\ \frac{4}{3}\ \frac{3}{2}\ 2\ 1^{6!} \dots$ where the first 2 is on the 1st place $(n=1)$, and the notation $1^{n!}$ will mean that $n!$ copies of 1 follow after each other. It is clear that (cf. [3; Theorem 3])

$$R_2 = \lim_{n\to\infty} (\inf_m w_{m+1} \dots w_{m+n})^{1/n} = 1,$$

and we claim that

$$R_1 = \lim_{n\to\infty} (\sup_m w_{m+1} \dots w_{m+n})^{1/n} = 1. \tag{1}$$

For a fixed $n \in N$ consider an m such that $w_{m+1} = \dfrac{n+1}{n}$. Then

$$w_{m+1} \dots w_{m+n} = n+1.$$

By Theorem 1, this fact and $R_2=R_1=1$ show that T is not spectral. If j is greater than m_1, the first such m, then clearly

$$w_{j+1} \dots w_{j+n} \leqq n+1. \tag{2}$$

If $0 \leqq j < m_1$, then the first nonunit factor of $w_{j+1} \ldots w_{j+n}$ comes, say, from the kth "nonunitary subsequence" of the weight sequence, whereas the last nonunit one from the rth such subsequence $(k \leqq r)$. Then

$$w_{j+1} \ldots w_{j+n} \leqq (k+1)(k+2) \ldots (r+1) \leqq (r+1)!.$$

Now if $k<r$, then the construction of the weight sequence shows that $(r+1)!<n$, hence

$$w_{j+1} \ldots w_{j+n} < n.$$

On the other hand, if $k=r$, then the estimation (2) holds again. Hence in all cases for any $j \in Z$, $n \in N$

$$1 \leqq (w_{j+1} \ldots w_{j+n})^{1/n} \leqq (n+1)^{1/n}.$$

Therefore (1) holds, and the operator T has the stated properties. By Theorem 2, the sets of the spectral singularities $S_i(T)$ $(i=1, 2)$ coincide with $\mathrm{sp}\,(T)=C_R$, the circle with radius R. □

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ON Λ-STRONG CONVERGENCE OF DOUBLE NUMERICAL SEQUENCES AND FOURIER SERIES

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§ 1. Notation of Λ-strong convergence

Let $S=\{s_{mn}: m, n=0, 1, \ldots\}$ be a double sequence of complex numbers. In this paper we will use the notation

(1.1) $$\lim_{m,n\to\infty} s_{mn} = t$$

if s_{mn} converges to t as both m and n tend to ∞ independently of one another (that is, s_{mn} converges to t in Pringsheim's sense) and, in addition, s_{mn} is bounded:

(1.2) $$\|S\|_\infty = \sup_{m,n\geqq 0} |s_{mn}| < \infty.$$

We shall use the backward differences

$$\Delta_{10}s_{mn} = s_{mn}-s_{m-1,n}, \qquad \Delta_{01}s_{mn} = s_{mn}-s_{m,n-1},$$

$$\Delta_{11}s_{mn} = \Delta_{10}[\Delta_{01}s_{mn}] = \Delta_{01}[\Delta_{10}s_{mn}] = s_{mn}-s_{m-1,n}-s_{m,n-1}+s_{m-1,n-1}$$

defined for all $m, n=0, 1, \ldots,$ with the agreement that

(1.3) $$s_{-1,n} = s_{m,-1} = s_{-1,-1} = 0.$$

Let $\Lambda=\{\lambda_{mn}=\lambda_m^{(1)}\lambda_n^{(2)}: m, n=0, 1, \ldots\}$, where $\{\lambda_m^{(1)}\}$ and $\{\lambda_n^{(2)}\}$ are two single sequences of positive numbers, both nondecreasing and tending to ∞. Thus, we have

(1.4) $$\Delta_{10}\lambda_{mn} \geqq 0, \quad \Delta_{01}\lambda_{mn} \geqq 0, \quad \Delta_{11}\lambda_{mn} \geqq 0$$

and

(1.5) $$\lim_{m,n\to\infty} \frac{\lambda_{mq}}{\lambda_{mn}} = 0, \quad \lim_{m,n\to\infty} \frac{\lambda_{pn}}{\lambda_{mn}} = 0,$$

where p and q are fixed. We note that we actually need the product representation $\lambda_{mn}=\lambda_m^{(1)}\lambda_n^{(2)}$ only in the proof of Lemma 2 in Section 2.

We say that S *converges Λ-strongly* to t if

(1.6) $$\lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \sum_{p=0}^{m} \sum_{q=0}^{n} |\Delta_{11}[\lambda_{pq}(s_{pq}-t)]| = 0.$$

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* This research was completed while the author was a visiting professor at the Syracuse University, New York, U.S.A., in the academic year 1986/87.

The notion of Λ-strong convergence was introduced by Hyslop [1] and Tanović—Miller [6] for single sequences in the special case $\lambda_n = n+1$. (See also [3].)

The significance of conditions (1.4) and (1.5) is illustrated by the following.

LEMMA 1. *If conditions* (1.1), (1.4) *and* (1.5) *are satisfied, then*

$$\lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \sum_{p=0}^{m} [\Delta_{10}\lambda_{pn}]\,|s_{pn}-t| = 0,$$

$$\lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \sum_{q=0}^{n} [\Delta_{01}\lambda_{mq}]\,|s_{mq}-t| = 0,$$

$$\lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \sum_{p=0}^{m} \sum_{q=0}^{n} [\Delta_{11}\lambda_{pq}]\,|s_{pq}-t| = 0,$$

with the agreement that

$$\lambda_{-1,n} = \lambda_{m,-1} = \lambda_{-1,-1} = 0. \tag{1.7}$$

PROOF. It is routine.

We remind the reader that a sequence $S=\{s_{mn}\}$ is said to be *of bounded variation* if

$$\|S\|_{\mathrm{bv}} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\Delta_{11}s_{mn}| < \infty \tag{1.8}$$

(cf. agreement (1.3)). It will turn out in Section 2 that Λ-strong convergence is an intermediate notion between bounded variation and convergence in Pringsheim's sense.

We denote by bv and c the well-known Banach spaces of the double sequences of complex numbers that are of bounded variation or converge in Pringsheim's sense and bounded, respectively.

§ 2. Auxiliary results

First we characterize Λ-strong convergence as follows.

LEMMA 2. *A sequence* $S=\{s_{mn}\}$ *of complex numbers converges Λ-strongly to a number t if and only if*

(i) $\displaystyle\lim_{m,n\to\infty} s_{mn} = t,$

(ii) $\displaystyle\lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \sum_{p=1}^{m} \sum_{q=0}^{n} [\Delta_{01}\lambda_{p-1,q}]\,|\Delta_{10}s_{pq}| = 0,$

(iii) $\displaystyle\lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \sum_{p=0}^{m} \sum_{q=1}^{n} [\Delta_{10}\lambda_{p,q-1}]\,|\Delta_{01}s_{pq}| = 0,$

(iv) $\displaystyle\lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \sum_{p=1}^{m} \sum_{q=1}^{n} \lambda_{p-1,q-1}|\Delta_{11}s_{pq}| = 0.$

PROOF. *Necessity*. Dropping the absolute value bars in definition (1.6), we can write

$$\frac{1}{\lambda_{mn}} \sum_{p=0}^{m} \sum_{q=0}^{n} \Delta_{11}[\lambda_{pq}(s_{pq}-t)] = s_{mn}-t. \tag{2.1}$$

Thus, (1.6) implies (i) in an obvious way.

Next, a simple algebraic manipulation shows that

$$\sum_{q=0}^{n} \Delta_{11}[\lambda_{pq}(s_{pq}-t)] = \Delta_{10}[\lambda_{pn}(s_{pn}-t)] = [\Delta_{10}\lambda_{pn}](s_{pn}-t)+\lambda_{p-1,n}[\Delta_{10}s_{pn}]$$

(cf. (1.7)), whence

$$\frac{1}{\lambda_{mn}} \sum_{p=1}^{m} \lambda_{p-1,n}|\Delta_{10}s_{pn}| \leqq \frac{1}{\lambda_{mn}} \sum_{p=0}^{m} [\Delta_{10}\lambda_{pn}]|s_{pn}-t|+$$

$$+\frac{1}{\lambda_{mn}} \sum_{p=0}^{m} \sum_{q=0}^{n} |\Delta_{11}[\lambda_{pq}(s_{pq}-t)]|.$$

By Lemma 1 and (1.6), we have

$$\lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \sum_{p=1}^{m} \lambda_{p-1,n}|\Delta_{10}s_{pn}| = 0. \tag{2.2}$$

Using the product representation $\lambda_{mn}=\lambda_m^{(1)}\lambda_n^{(2)}$, we can write (2.2) as follows:

$$\lim_{m,n\to\infty} \frac{1}{\lambda_m^{(1)}} \sum_{p=1}^{m} \lambda_{p-1}^{(1)}|\Delta_{10}s_{pn}| = 0.$$

Forming the weighted means of the expression occurring here by means of the sequence $\{\lambda_n^{(2)}\}$, an elementary calculation yields

$$\lim_{m,n\to\infty} \frac{1}{\lambda_m^{(1)}\lambda_n^{(2)}} \sum_{p=1}^{m} \sum_{q=0}^{n} \lambda_{p-1}^{(1)}[\Delta_{01}\lambda_q^{(2)}]|\Delta_{10}s_{pq}| = 0,$$

which is identical with (ii).

Relation (iii) can be derived in an analogous way.

Finally, we apply the identity

$$\Delta_{11}[\lambda_{pq}(s_{pq}-t)] = [\Delta_{11}\lambda_{pq}](s_{pq}-t)+[\Delta_{01}\lambda_{p-1,q}]\Delta_{10}s_{pq}+ \tag{2.3}$$

$$+[\Delta_{10}\lambda_{p,q-1}]\Delta_{01}s_{pq}+\lambda_{p-1,q-1}\Delta_{11}s_{pq}$$

to get

$$\frac{1}{\lambda_{mn}} \sum_{p=1}^{m} \sum_{q=1}^{n} \lambda_{p-1,q-1}|\Delta_{11}s_{pq}| \leqq \frac{1}{\lambda_{mn}} \sum_{p=0}^{m} \sum_{q=0}^{n} |\Delta_{11}[\lambda_{pq}(s_{pq}-t)]|+$$

$$+\frac{1}{\lambda_{mn}} \sum_{p=0}^{m} \sum_{q=0}^{n} [\Delta_{11}\lambda_{pq}]|s_{pq}-t|+\frac{1}{\lambda_{mn}} \sum_{p=1}^{m} \sum_{q=0}^{n} [\Delta_{01}\lambda_{p-1,q}]|\Delta_{10}s_{pq}|+$$

$$+\frac{1}{\lambda_{mn}} \sum_{p=0}^{m} \sum_{q=1}^{n} [\Delta_{10}\lambda_{p,q-1}]|\Delta_{01}s_{pq}|.$$

Now, relation (iv) follows from (1.6), (i), (ii), (iii) and Lemma 1.

Sufficiency. It is an immediate consequence of (2.3) and Lemma 1.

Next, we reformulate conditions (ii) and (iii).

LEMMA 3. *Under* (iv), *conditions* (ii) *and* (iii) *are equivalent to the following two conditions:*

$$\text{(ii}'\text{)}\quad \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \sum_{p=1}^{m} \lambda_{p-1,n}|\Delta_{10}s_{pn}| = 0,$$

$$\text{(iii}'\text{)}\quad \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \sum_{q=1}^{n} \lambda_{m,q-1}|\Delta_{01}s_{mq}| = 0.$$

PROOF. By performing a summation by parts,

$$\sum_{q=0}^{n} [\Delta_{01}\lambda_{p-1,q}]|\Delta_{10}s_{pq}| = \lambda_{p-1,n}|\Delta_{10}s_{pn}| + \sum_{q=0}^{n-1} \lambda_{p-1,q}[|\Delta_{10}s_{pq}| - |\Delta_{10}s_{p,q+1}|] \leqq$$

$$\leqq \lambda_{p-1,n}|\Delta_{10}s_{pn}| + \sum_{q=1}^{n} \lambda_{p-1,q-1}|\Delta_{11}s_{pq}|$$

and similarly

$$\sum_{q=0}^{n} [\Delta_{01}\lambda_{p-1,q}]|\Delta_{10}s_{pq}| \geqq \lambda_{p-1,n}|\Delta_{10}s_{pn}| - \sum_{q=1}^{n} \lambda_{p-1,q-1}|\Delta_{11}s_{pq}|.$$

Hence it follows that

$$\frac{1}{\lambda_{mn}} \sum_{p=1}^{m} \sum_{q=0}^{n} [\Delta_{01}\lambda_{p-1,q}]|\Delta_{10}s_{pq}| \lesseqqgtr \tag{2.4}$$

$$\lesseqqgtr \frac{1}{\lambda_{mn}} \sum_{p=1}^{m} \lambda_{p-1,n}|\Delta_{10}s_{pn}| \pm \frac{1}{\lambda_{mn}} \sum_{p=1}^{m} \sum_{q=1}^{n} \lambda_{p-1,q-1}|\Delta_{11}s_{pq}|,$$

where "$\leqq$" corresponds to "$+$" and "$\geqq$" corresponds to "$-$", respectively. Now, the equivalence of (ii) and (ii′) follows from (2.4) and (iv).

The equivalence of (iii) and (iii′) under (iv) can be verified in a similar manner.

It is important to observe that condition (i) can be weakened in Lemma 2. To this effect, we introduce a Fejér type mean as follows

$$\sigma_{mn} = \frac{1}{\lambda_{mn}} \sum_{p=0}^{m} \sum_{q=0}^{n} [\Delta_{11}\lambda_{pq}]s_{pq}. \tag{2.5}$$

LEMMA 4. *The condition*

$$\text{(i}'\text{)}\quad \lim_{m,n\to\infty} \sigma_{mn} = t$$

together with (ii)—(iv) *are equivalent to* (i)–(iv).

PROOF. On the one hand, it is well-known that for a bounded sequence S condition (i) implies (i′), without any additional assumption.

On the other hand, due to (ii)—(iv) the converse implication is also true. In fact, it is easy to see that

$$s_{mn}-\sigma_{mn}=\frac{1}{\lambda_{mn}}\sum_{p=0}^{m}\sum_{q=0}^{n}[\Delta_{11}\lambda_{pq}](s_{mn}-s_{pq}).$$

Using the representation

$$s_{mn}-s_{pq}=[s_{mn}-s_{mq}-s_{pn}+s_{pq}]+[s_{mq}-s_{pq}]+[s_{pn}-s_{pq}]=$$

$$=\sum_{j=p+1}^{m}\sum_{k=q+1}^{n}\Delta_{11}s_{jk}+\sum_{j=p+1}^{m}\Delta_{10}s_{jq}+\sum_{k=q+1}^{n}\Delta_{01}s_{pk}$$

(the empty sums $\sum_{j=m+1}^{m}$ and $\sum_{k=n+1}^{n}$ are taken to be zero), we get

$$s_{mn}-\sigma_{mn}=\frac{1}{\lambda_{mn}}\sum_{p=0}^{m}\sum_{q=0}^{n}[\Delta_{11}\lambda_{pq}]\sum_{j=p+1}^{m}\sum_{k=q+1}^{n}\Delta_{11}s_{jk}+ \tag{2.6}$$

$$+\frac{1}{\lambda_{mn}}\sum_{p=0}^{m}\sum_{q=0}^{n}[\Delta_{11}\lambda_{pq}]\sum_{j=p+1}^{m}\Delta_{10}s_{jq}+\frac{1}{\lambda_{mn}}\sum_{p=0}^{m}\sum_{q=0}^{n}[\Delta_{11}\lambda_{pq}]\sum_{k=q+1}^{n}\Delta_{01}s_{pk}=$$

$$=\Sigma_1+\Sigma_2+\Sigma_3,\quad \text{say.}$$

Interchanging the summations yields

$$\Sigma_1=\frac{1}{\lambda_{mn}}\sum_{j=1}^{m}\sum_{k=1}^{n}[\Delta_{11}s_{jk}]\sum_{p=0}^{j-1}\sum_{q=0}^{k-1}\Delta_{11}\lambda_{pq}=\frac{1}{\lambda_{mn}}\sum_{j=1}^{m}\sum_{k=1}^{n}\lambda_{j-1,k-1}\Delta_{11}s_{jk}.$$

By (iv), Σ_1 tends to zero as $m,n\to\infty$.

Similarly,

$$\Sigma_2=\frac{1}{\lambda_{mn}}\sum_{j=1}^{m}\sum_{q=0}^{n}[\Delta_{10}s_{jq}]\sum_{p=0}^{j-1}\Delta_{11}\lambda_{pq}=\frac{1}{\lambda_{mn}}\sum_{j=1}^{m}\sum_{q=0}^{n}[\Delta_{01}\lambda_{j-1,q}]\Delta_{10}s_{jq}$$

also tends to zero as $m,n\to\infty$, due to (ii). The same is true for Σ_3, due to (iii).

Combining these with (2.6) results in the relation

$$\lim_{m,n\to\infty}[s_{mn}-\sigma_{mn}]=0$$

and (i′) obviously implies (i) in this case.

§ 3. Main results on numerical sequences

Denote by $c(\Lambda)$ the class of double sequences $S=\{s_{mn}\}$ of complex numbers that converge Λ-strongly. Clearly, $c(\Lambda)$ is a linear space. We endow $c(\Lambda)$ with the norm

$$\|S\|_{c(\Lambda)}=\sup_{m,n\geqq 0}\frac{1}{\lambda_{mn}}\sum_{p=0}^{m}\sum_{q=0}^{n}|\Delta_{11}[\lambda_{pq}s_{pq}]|, \tag{3.1}$$

which is obviously finite for every $S\in c(\Lambda)$.

REMARK 1. On the basis of Lemmas 2 and 3, the norm $|||\,.\,|||_{c(\Lambda)}$ defined by

$$|||S|||_{c(\Lambda)} = \|S\|_\infty + \sup_{m,n\geqq 0} \frac{1}{\lambda_{mn}} \sum_{p=1}^{m} \lambda_{p-1,n} |\Delta_{10} s_{pn}| +$$

$$+ \sup_{m,n\geqq 0} \frac{1}{\lambda_{mn}} \sum_{q=1}^{n} \lambda_{m,q-1} |\Delta_{01} s_{mq}| + \sup_{m,n\geqq 0} \frac{1}{\lambda_{mn}} \sum_{p=1}^{m} \sum_{q=1}^{n} \lambda_{p-1,q-1} |\Delta_{11} s_{pq}|$$

is equivalent to $\|\,.\,\|_{c(\Lambda)}$ in the sense that there exist two positive constants K_1 and K_2 such that for every $S\in c(\Lambda)$,

$$K_1 \|S\|_{c(\Lambda)} \leqq |||S|||_{c(\Lambda)} \leqq K_2 \|S\|_{c(\Lambda)}.$$

We remind the reader of definitions (1.2) and (1.8).

LEMMA 5. *For any sequence* S,

$$\|S\|_\infty \leqq \|S\|_{c(\Lambda)} \leqq 6 \|S\|_{\mathrm{bv}} \tag{3.2}$$

and, consequently,

$$\mathrm{bv} \subset c(\Lambda) \subset \mathrm{c}. \tag{3.3}$$

PROOF. The first half of (3.2) follows from (2.1) on putting $t=0$. The second half can be verified by using (2.3) also with $t=0$ and the identities

$$s_{pq} = \sum_{j=0}^{p} \sum_{k=0}^{q} \Delta_{11} s_{jk}, \quad \Delta_{10} s_{pn} = \sum_{q=0}^{n} \Delta_{11} s_{pq}, \quad \Delta_{01} s_{mq} = \sum_{p=0}^{m} \Delta_{11} s_{pq}.$$

Our main result reads as follows.

THEOREM 1. *The class* $c(\Lambda)$ *endowed with the norm* (3.1) *is a Banach space.*

PROOF. The only thing we have to prove is completeness. To this effect, let $\{S^{(r)}: r=1, 2, \ldots\}$ be a Cauchy sequence in the norm $\|\,.\,\|_{c(\Lambda)}$. Then by (3.2), $\{S^{(r)}\}$ is a Cauchy sequence in the norm $\|\,.\,\|_\infty$, as well. Thus, there exists a sequence $S\in \mathrm{c}$ such that

$$\lim_{r\to\infty} \|S^{(r)} - S\|_\infty = 0. \tag{3.4}$$

We will prove that $S\in c(\Lambda)$ and

$$\lim_{r\to\infty} \|S^{(r)} - S\|_{c(\Lambda)} = 0. \tag{3.5}$$

To see this, let an $\varepsilon>0$ be given. By assumption, there exists a $\nu=\nu(\varepsilon)$ such that

$$\|S^{(l)} - S^{(r)}\|_{c(\Lambda)} \leqq \varepsilon \quad \text{if} \quad l, r \geqq \nu. \tag{3.6}$$

Let $S^{(r)} = \{s_{jk}^{(r)}: j, k=0, 1, \ldots\}$ and $S=\{s_{jk}\}$. We fix (m, n) temporarily. Using (2.3) with $t=0$, it is not hard to see that

$$(3.7)\qquad \frac{1}{\lambda_{mn}}\sum_{p=0}^{m}\sum_{q=0}^{n}|\Delta_{11}[\lambda_{pq}(s_{pq}^{(r)}-s_{pq})]| \leqq \|S^{(r)}-S\|_\infty\Bigg[1+\frac{2}{\lambda_{mn}}\sum_{p=1}^{m}\lambda_{p-1,n}+$$

$$+\frac{2}{\lambda_{mn}}\sum_{q=1}^{n}\lambda_{m,q-1}+\frac{4}{\lambda_{mn}}\sum_{p=1}^{m}\sum_{q=1}^{n}\lambda_{p-1,q-1}\Bigg] \leqq \varepsilon,$$

provided r is large enough, due to (3.4). We apply the triangle inequality and take (3.6) and (3.7) into account to obtain

$$\frac{1}{\lambda_{mn}}\sum_{p=0}^{m}\sum_{q=0}^{n}|\Delta_{11}[\lambda_{pq}(s_{pq}^{(l)}-s_{pq})]| \leqq \frac{1}{\lambda_{mn}}\sum_{p=0}^{m}\sum_{q=0}^{n}|\Delta_{11}[\lambda_{pq}(s_{pq}^{(l)}-s_{pq}^{(r)})]|+$$

$$+\frac{1}{\lambda_{mn}}\sum_{p=0}^{m}\sum_{q=0}^{n}|\Delta_{11}[\lambda_{pq}(s_{pq}^{(r)}-s_{pq})]| \leqq \|S^{(l)}-S^{(r)}\|_{c(\Lambda)}+\varepsilon \leqq 2\varepsilon \quad \text{if} \quad l \geqq \nu.$$

Since this is valid for all (m, n), by definition

$$\|S^{(l)}-S\|_{c(\Lambda)} \leqq 2\varepsilon \quad \text{if} \quad l \leqq \nu,$$

which proves (3.5).

One can check the fulfillment of $S\in c(\Lambda)$ along the same lines. This completes the proof of Theorem 1.

PROBLEM 1. We conjecture that there is no Schauder basis in $c(\Lambda)$.

PROBLEM 2. What is the conjugate space to $c(\Lambda)$?

§ 4. Application to Fourier series

One can apply the notion of Λ-strong convergence to sequences of complex-valued functions, in particular to Fourier series, while using C-metric or L_p-metric. For the sake of concreteness, here we present in full details the results on the uniform Λ-strong convergence of double Fourier series of continuous functions on the two-dimensional torus $T^2=\{(x, y): -\pi\leqq x, y<\pi\}$.

Denote by C the Banach space of the complex-valued continuous functions $f(x, y)$, 2π-periodic in each variable and endowed with the norm

$$\|f\|_C = \max_{(x,y)\in T^2}|f(x, y)|.$$

Let

$$(4.1)\qquad \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\mu_{jk}A_{jk}(x, y)$$

be the double Fourier series of the function $f\in C$, where we systematically use the following notations:

$$A_{jk}(x, y) = a_{jk} \cos jx \cos ky + b_{jk} \sin jx \cos ky +$$
$$+ c_{jk} \cos jx \sin ky + d_{jk} \sin jx \sin ky,$$

$$a_{jk} = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x, y) \cos jx \cos ky \, dx \, dy,$$

and analogous representations for b_{jk}, c_{jk}, d_{jk}, and

$$\mu_{jk} = \begin{cases} \frac{1}{4} & \text{if} \quad j = k = 0, \\ \frac{1}{2} & \text{if} \quad j = 0 \quad \text{and} \quad k \geqq 1 \quad \text{or} \quad j \geqq 1 \quad \text{and} \quad k = 0, \\ 1 & \text{if} \quad j \geqq 1 \quad \text{and} \quad k \geqq 1. \end{cases}$$

We will consider the rectangular partial sums

$$s_{pq}(f) = s_{pq}(f; x, y) = \sum_{j=0}^{p} \sum_{k=0}^{q} \mu_{jk} A_{jk}(x, y)$$

and, following (2.5), the Fejér type means

$$\sigma_{mn}(f) = \sigma_{mn}(f; x, y) = \frac{1}{\lambda_{mn}} \sum_{p=0}^{m} \sum_{q=0}^{n} [\Delta_{11} \lambda_{pq}] s_{pq}(f) \tag{4.2}$$

of the series (4.1).

Denote by U, A and $S(\Lambda)$, respectively, the classes of functions f whose Fourier series converge uniformly with uniformly bounded rectangular partial sums, converge absolutely, and converge uniformly Λ-strongly on T^2. Clearly, in each case it follows that f must belong to C. To be more specific, we say that a function f belongs to $S(\Lambda)$ if (cf. (1.6))

$$\lim_{m, n \to \infty} \left\| \frac{1}{\lambda_{mn}} \sum_{p=0}^{m} \sum_{q=0}^{n} |\Delta_{11} [\lambda_{pq} (s_{pq}(f) - f)]| \right\|_C = 0.$$

It is a common place that U and A are Banach spaces with the norms

$$\|f\|_U = \sup_{m, n \geqq 0} \|s_{mn}(f)\|_C \quad \text{and} \quad \|f\|_A = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \mu_{jk} \varrho_{jk},$$

respectively, where

$$\varrho_{jk} = [a_{jk}^2 + b_{jk}^2 + c_{2k}^j + d_{jk}^2]^{1/2}.$$

Both statements can be proved along the same lines as the corresponding one-dimensional statements are proved (see, e.g. [2, pp. 6—8]).

According to (3.1), we introduce a norm in $S(\Lambda)$ as follows

$$\|f\|_{S(\Lambda)} = \sup_{m, n \geqq 0} \left\| \frac{1}{\lambda_{mn}} \sum_{p=0}^{m} \sum_{q=0}^{n} |\Delta_{11} [\lambda_{pq} s_{pq}(f)]| \right\|_C. \tag{4.3}$$

The norm inequalities corresponding to (3.2) are

$$\|f\|_U \leqq \|f\|_{S(\Lambda)} \leqq 6\|f\|_\Lambda,$$

which imply in turn (cf. (3.3)) $\Lambda \subset S(\Lambda) \subset U$.

The following results are the counterparts to Lemmas 2—4 and Theorem 1.

LEMMA 6. *A function f belongs to $S(\Lambda)$ if and only if the following four conditions are satisfied:*

(v) $\lim\limits_{m,n\to\infty} \|\sigma_{mn}(f)-f\|_C = 0,$

(vi) $\lim\limits_{m,n\to\infty} \left\| \frac{1}{\lambda_{mn}} \sum\limits_{p=1}^{m} \lambda_{p-1,n} |\Delta_{10} s_{pn}(f)| \right\|_C = 0,$

(vii) $\lim\limits_{m,n\to\infty} \left\| \frac{1}{\lambda_{mn}} \sum\limits_{q=1}^{n} \lambda_{m,q-1} |\Delta_{01} s_{mq}(f)| \right\|_C = 0,$

(viii) $\lim\limits_{m,n\to\infty} \left\| \frac{1}{\lambda_{mn}} \sum\limits_{p=1}^{m} \sum\limits_{q=1}^{n} \lambda_{p-1,q-1} |\Delta_{11} s_{pq}(f)| \right\|_C = 0.$

THEOREM 2. *The class $S(\Lambda)$ endowed with the norm* (4.3) *is a Banach space.*

We note that the Banach space $S(\Lambda)$ for one-dimensional Fourier series was introduced by Szalay [5] in the special case $\lambda_n = n+1$.

Lemma 7 below indicates that Λ-strong convergence exhibits some of the characteristic properties of absolute convergence. Namely, an analogue of the Denjoy—Luzin theorem holds true (cf. the original one-dimensional theorem in [8, pp. 232—233]).

LEMMA 7. *If the trigonometric series* (4.1) *converges Λ-strongly for all (x, y) belonging to a set of positive measure, then*

$$\lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \sum_{j=1}^{m} \sum_{k=1}^{n} \lambda_{j-1,k-1} \varrho_{jk} = 0. \tag{4.4}$$

Here and in the sequel, we mean the Lebesgue measure on the plane and denote it by $|\,.\,|$.

PROOF. We shall imitate the proofs of [4, Theorems 1 and 2], at least in the first part.

We may dismiss the $A_{jk}(x, y)$ for which $\varrho_{jk}=0$. Otherwise, let $B_{jk}(x, y)$ be defined by

$$A_{jk}(x, y) = \varrho_{jk} B_{jk}(x, y),$$

while let $u_{jk} \geqq 0$, $v_{jk} \geqq 0$, Φ_{jk} and Θ_{jk}, $-\pi \leqq \Phi_{jk}, \Theta_{jk} < \pi$, be defined by

$$a_{jk} = u_{jk} \cos \Phi_{jk}, \quad b_{jk} = u_{jk} \sin \Phi_{jk}, \tag{4.5}$$

$$c_{jk} = v_{jk} \cos \Theta_{jk}, \quad d_{jk} = v_{jk} \sin \Theta_{jk},$$

Then

$$A_{jk}(x, y) = u_{jk} \cos(jx - \Phi_{jk}) \cos ky + v_{jk} \cos(jx - \Theta_{jk}) \sin ky, \tag{4.6}$$

whence, applying the Schwarz inequality yields

$$|A_{jk}(x, y)| \leqq [u_{jk}^2+v_{jk}^2]^{1/2} = \varrho_{jk}.$$

Consequently,

$$|B_{jk}(x, y)| \leqq 1 \quad \text{for all} \quad (x, y).$$

According to the argument occurring in [4], in order to prove (4.4) it is enough to show that

$$\liminf_{j\to\infty \text{ or } k\to\infty} I_{jk} > 0, \tag{4.7}$$

where

$$I_{jk} = \iint_E B_{jk}^2(x, y)\, dx\, dy$$

and E is a set of positive measure.

Actually, the weaker relation

$$\liminf_{j\to\infty \text{ and } k\to\infty} I_{jk} > 0 \tag{4.8}$$

is proved in [4]. Now, if j is fixed and $k\to\infty$, then we make use of the representation

$$\begin{aligned}(4.9)\quad B_{jk}^2(x, y) = \frac{1}{4}+\frac{u_{jk}^2}{4\varrho_{jk}^2}&[\cos 2(jx-\Phi)+\cos 2ky+\cos 2(jx-\Phi)\cos 2ky]+\\ +\frac{v_{jk}^2}{4\varrho_{jk}^2}&[\cos 2(jx-\Theta)-\cos 2ky-\cos 2(jx-\Theta)\cos 2ky]+\\ +\frac{u_{jk}v_{jk}}{2\varrho_{jk}^2}&\sin 2ky[\cos(2kx-\Phi-\Theta)+\cos(\Theta-\Phi)],\end{aligned}$$

where $\Phi=\Phi_{jk}$ and $\Theta=\Theta_{jk}$. Taking into account the well-known fact that the Fourier coefficients of any integrable function converge to zero whenever at least one of the indices tends to ∞ (see, e.g. [8, p. 301]), it follows from (4.9) that for fixed j

$$\begin{aligned}I_{jk} = \frac{1}{4}|E|+\frac{u_{jk}^2}{4\varrho_{jk}^2}\iint_E \cos 2(jx-\Phi)\, dx\, dy+\\ +\frac{v_{jk}^2}{4\varrho_{jk}^2}\iint_E \cos 2(jx-\Theta)\, dx\, dy+o(1) \quad \text{as} \quad k\to\infty.\end{aligned}$$

It is easy to verify that for any fixed j, we have

$$\iint_E \cos 2(jx-\Phi)\, dx\, dy < |E|.$$

Consequently, (4.7) holds true as $k\to\infty$.

On the other hand, if k is fixed and $j\to\infty$, we can rely on the symmetric counterparts of (4.5), (4.6) and (4.9). To be more specific, let $\tilde{u}_{jk}$, $\tilde{v}_{jk}$, $\tilde{\Phi}_{jk}$ and $\tilde{\Theta}_{jk}$ be defined by

$$a_{jk} = \tilde{u}_{jk}\cos\tilde{\Phi}_{jk}, \quad c_{jk} = \tilde{u}_{jk}\sin\tilde{\Phi}_{jk},$$

$$b_{jk} = \tilde{v}_{jk}\cos\tilde{\Theta}_{jk}, \quad d_{jk} = \tilde{v}_{jk}\sin\tilde{\Theta}_{jk}.$$

Then

$$A_{jk}(x, y) = \tilde{u}_{jk} \cos jx \cos (ky - \tilde{\Phi}) + \tilde{v}_{jk} \sin jx \cos (ky - \tilde{\Theta})$$

and

$$B_{jk}^2(x, y) = \frac{1}{4} + \frac{\tilde{u}_{jk}^2}{4\varrho_{jk}^2} [\cos 2jx + \cos 2(ky - \tilde{\Phi}) + \cos 2jx \cos 2(ky - \tilde{\Phi})] +$$

$$+ \frac{\tilde{v}_{jk}^2}{4\varrho_{jk}^2} [-\cos 2jx + \cos 2(ky - \tilde{\Theta}) - \cos 2jx \cos 2(ky - \tilde{\Theta})] +$$

$$+ \frac{\tilde{u}_{jk}\tilde{v}_{jk}}{2\varrho_{jk}^2} \sin 2jx [\cos (2ky - \tilde{\Phi} - \tilde{\Theta}) + \cos (\tilde{\Theta} - \tilde{\Phi})],$$

where $\tilde{\Phi} = \tilde{\Phi}_{jk}$ and $\tilde{\Theta} = \tilde{\Theta}_{jk}$. Hence one can deduce that (4.7) holds true as $j \to \infty$ and this completes the proof of Lemma 7.

In the special case where $\lambda_{jk} = (j+1)(k+1)$, the mean $\sigma_{mn}(f)$ defined by (4.2) is the ordinary Fejér mean (in other words, the first arithmetic of the rectangular partial sums) of the Fourier series (4.1). By [8, p. 304], condition (v) is satisfied for every $f \in C$. Using the corresponding special case of Lemmas 6 and 7, we arrive at the following.

THEOREM 3. *Let* $\Lambda = \{(j+1)(k+1)\}$. *If* $f \in C$, *then the Fourier series* (4.1) *converges uniformly* Λ*-strongly to* $f(x, y)$ *on* T^2 *if and only if*

$$\text{(vi')} \quad \lim_{m,n\to\infty} \left\| \frac{1}{m+1} \sum_{j=1}^{m} j |\Delta_{10} s_{jn}(f)| \right\|_C = 0,$$

$$\text{(vii')} \quad \lim_{m,n\to\infty} \left\| \frac{1}{n+1} \sum_{k=1}^{n} k |\Delta_{01} s_{mk}(f)| \right\|_C = 0,$$

$$\text{(viii')} \quad \lim_{m,n\to\infty} \frac{1}{(m+1)(n+1)} \sum_{j=1}^{m} \sum_{k=1}^{n} jk\varrho_{jk} = 0.$$

We note that the conditions

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_{j=1}^{m} \sum_{k=0}^{\infty} j\varrho_{jk} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{\infty} \sum_{k=1}^{n} k\varrho_{jk} = 0$$

are sufficient for the fulfillment of (vi') and (vii'), respectively. But each of them is stronger than (viii').

PROBLEM 3. How to characterize conditions (vi') and (vii') in terms of ϱ_{jk}?

§ 5. Final observations

We finish our study with two more remarks.

REMARK 2. All results in Section 4 can be reformulated by substituting L_p-metric for C-metric where $1 \leq p < \infty$. It is well-known that L_p endowed with the norm

$$\|f\|_p = \left[\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x, y)|^p \, dx \, dy \right]^{1/p}$$

is a Banach space. We say that a function $f\in L_p$ belongs to the class $S_p(\Lambda)$ if

$$\lim_{m,n\to\infty}\left\|\frac{1}{\lambda_{mn}}\sum_{p=0}^{m}\sum_{q=0}^{n}\left|\Delta_{11}[\lambda_{pq}(s_{pq}(f)-f)]\right|\right\|_p=0$$

and introduce the norm

$$\|f\|_{S_p(\Lambda)}=\sup_{m,n\geqq 0}\left\|\frac{1}{\lambda_{mn}}\sum_{p=0}^{m}\sum_{q=0}^{n}|\Delta_{11}[\lambda_{pq}s_{pq}(f)]|\right\|_p.$$

The analogues of Lemma 6 and Theorems 2, 3 hold true in the case of L_p-metric, too. Their proofs are based on the results of Section 3 and the following two more auxiliary results. First, if $f\in L_p$ for some p, $1\leqq p<\infty$, then

$$\lim_{m,n\to\infty}\|\sigma_{mn}(f)-f\|_p=0$$

(see, e.g. [8, p. 304]). Second, if the trigonometric series (4.1) converges Λ-strongly in the L_p-metric restricted to a set of positive measure, then (4.4) holds true (which is the L_p-metric version of Lemma 7).

REMARK 3. The notion of Λ-strong convergence and the methods described in this paper clearly apply to multiple numerical sequences and higher dimensional Fourier series. The extension of these results to d-multiple and d-dimensional cases, where d is an integer greater than 2, is straightforward. As to the auxiliary results, in [8, Ch. 17] the whole presentation is done in this general setting, and concerning Lemma 7 we refer to [7] instead of [4].

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ON AFFINELY EMBEDDABLE SETS IN THE PROJECTIVE PLANE

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In this note we prove a conjecture of Bisztriczky and Schaer [1] about convex sets in the real projective plane P^2. It will be simpler to formulate the result for convex cones in R^3 and then show that it implies the conjecture. A cone $C \subset R^3$ is called pointed if it contains no line, i.e., when $x \in C$ and $-x \in C$ imply $x=0$. Here is the result:

THEOREM 1. *Assume $n \geqq 3$ and $C_1, \ldots, C_n \subset R^3$ are closed, pointed, convex cones with common apex the origin O. Assume that for $i \neq j$ $(i, j=1, 2, \ldots, n)$ there is an $e(i, j) \in \{-1, +1\}$ such that for all $k=1, \ldots, n$, $k \neq i, j$ and for both $e=1, -1$*

$$(i, j; k, e) \qquad (eC_k) \cap (C_1 + e(i, j) C_j) = \{O\}.$$

Then there is a plane P through O such that for all $i=1, \ldots, n$, $P \cap C_i = \{O\}$.

We will now translate this theorem from R^3 to P^2. For a convex pointed cone $C \subset R^3$ set $S(C) = S^2 \cap C$ where S^2 is the unit sphere of R^3. P^2 is obtained from S^2 by identifying antipodal points. With this identification the points of $S(C)$ and $-S(C) = S(-C)$ give rise to a set $P(C) \subset P^2$. Clearly, $P(C) = P(-C)$.

A set $A \subset P^2$ is called convex if there exists a line L in P^2 disjoint from A and A is convex in the affine plane $P^2 \setminus L$ (cf. [2] or [1]). A convex set A in P^2 gives rise to two connected subsets $S^+(A)$ and $S^-(A) = -S^+(A)$ of S^2, whose cone hulls are $C^+(A)$ and $C^-(A)$, respectively. Evidently, $C^+(A) = -C^-(A)$. In this way one can see that $A \subset P^2$ is convex if and only if $A = P(C)$ for some pointed convex cone $C \subset R^3$.

Now let $A_1, A_2 \subset P^2$ be convex. We want to define the convex hull of their union. Then $A_j = P(C_j)$ for some pointed convex cone $C_j \subset R^3$ and also $A_j = P(-C_j)$ $(j=1, 2)$. So the union of A_1 and A_2 will have, in general, two convex hulls: $H_1(A_1, A_2) = P(\text{conv}\,(C_1, C_2))$ and $H_2(A_1, A_2) = P(\text{conv}\,(C_1, -C_2))$. Of course, H_1 and H_2 will be convex only if $C_1 - C_2 = \text{conv}\,(C_1, -C_2)$ and $C_1 + C_2 = \text{conv}(C_1, C_2)$ are pointed cones.

We can now formulate Theorem 1 in P^2.

THEOREM 2. *Let $A_1, \ldots, A_n$ be closed convex sets in P^2 $(n \geqq 3)$. Assume that for $i \neq j$ $(i, j=1, \ldots, n)$ either $A_k \cap H_1(A_i, A_j) = \emptyset$ for all $k \neq i, j$ or $A_k \cap H_2(A_i, A_j) = \emptyset$ for all $k \neq i, j$. Then there is a line $L \subset P^2$ disjoint from each A_i.*

In [1], the collection of the sets $A_1, \ldots, A_n$ is called affinely embeddable when the conclusion of Theorem 2 holds.

In the proof of Theorem 1 we will use standard techniques from the theory of convex cones in finite dimensional spaces (cf. [3], [4] or [5]).

When proving Theorem 1 we will obtain its dual form which seems to be worth mentioning:

THEOREM 3. *Assume* $D_1, \dots, D_n \subset R^3$ $(n \geqq 3)$ *are closed, pointed, convex cones with common apex the origin. Suppose that for* $i \neq j$ $(i, j=1, \dots, n)$ *there is an* $e(i,j) \in \{-1, +1\}$ *such that for all* $k=1, \dots, n,$ $k \neq i, j$ *and for both* $e=1$ *and* -1 $(eD_k) \cap D_i \cap (e(i,j)D_j) \neq \{O\}$. *Then there are signs* $e_1, \dots, e_n$ $(e_i=+1$ or $-1)$ *and a vector* $p \in R^3 \setminus \{O\}$ *such that* $p \in e_i D_i$ *for all* $i=1, \dots, n$.

PROOF OF THEOREM 1. Assume the theorem is false and take a counterexample $C_1, \dots, C_n \subset R^3$ of closed, convex, pointed cones satisfying condition $(i, j; k, e)$ such that for all planes P through the origin there is an $i \in \{1, \dots, n\}$ with $P \cap C_i \neq \{O\}$.

We will modify this counterexample. We *claim* first that for $i \neq j$ both $C_i + C_j$ and $C_i - C_j$ are pointed and closed convex cones. We prove this for $C_i + C_j$, the proof for $C_i - C_j$ is identical. By condition $(i, k; j, -1)$

$$(-C_j) \cap C_i \subset (-C_j) \cap (C_i + e(i,k)C_k) = \{O\},$$

so C_i and $(-C_j)$ can be separated (strictly, because they are closed), i.e,. there exists $v \in R^3$ such that $v \cdot x < 0$ for all $x \in C_i \setminus \{O\}$ and $v \cdot y > 0$ for all $y \in (-C_j) \setminus \{O\}$. (Here $v \cdot x$ denotes the scalar product of $v, x \in R^3$.) Then $v \cdot z < 0$ for all $z \in (C_i + C_j) \setminus \{O\}$ proving that $(C_i + C_j)$ is pointed.

Now we prove that $C_i + C_j$ is closed. Assume it is not, then there are elements $x_m \in C_i$ and $y_m \in C_j$ with $x_m, y_m \in S^2$ and positive numbers α_m, β_m such that $z_m = \alpha_m x_m + \beta_m y_m$ is in $(C_i + C_j) \cap S^2$ but $z = \lim z_m$ is not. By the compactness of S^2 we may assume that $x = \lim x_m$ and $y = \lim y_m$ exists. Then α_m and β_m must tend to infinity and so $z_m \in S^2$ is possible only if $x + y = 0$. This implies that $C_i + C_j$ contains the line through x and $-x = y$ which is impossible because it is a pointed cone.

We define, for a closed pointed cone $C \subset R^3$ and for $\alpha > 0$ the set

$$C^\alpha = \{x \in R^3\colon \text{ there is } y \in C \text{ with } \sphericalangle xOy \leqq \alpha\},$$

where $\sphericalangle xOy$ denotes the angle of the triangle xOy at vertex O. C^α is clearly a convex, pointed cone with nonempty interior provided α is small enough.

Condition $(i, j; k, e)$ says that the two closed and pointed cones $C_i + e(i,j)C_j$ and eC_k are disjoint (except for the common apex). Then there is $\alpha(i, j; k, e) > 0$ such that for $0 < \alpha < \alpha(i, j; k, e)$

$$(eC_k^\alpha) \cap (C_i^\alpha + e(i,j)C_j^\alpha) = \{O\};$$

and $C_i^\alpha, C_j^\alpha, C_k^\alpha$, $C_i^\alpha + e(i,j)C_j^\alpha$ are all pointed, convex, closed cones. Set $\beta = \min \alpha(i, j; k, e)$ and take a closed polyhedral cone B_i with nonempty interior satisfying

$$C_i \subset B_i \subset C_i^\beta \quad \text{for} \quad i = 1, \dots, n.$$

We may choose the finitely many halflines generating the cones B_i to be in general position. We will clarify later what is meant by general position here.

This is what we have now: The cones B_i are convex, closed, pointed and polyhedral with nonempty interior, and they satisfy condition $(i, j; k, e)$. Moreover, for each plane P through the origin $P \cap \operatorname{int} B_i \neq \{O\}$ for some $i = 1, \dots, n$.

Consider now the polars $D_i=B_i^*$ of B_i defined as

$$D_i = \{x\in R^3:\ x\cdot y \leqq 0 \text{ for } y\in B_i\}.$$

The D_i's are convex, closed, pointed, polyhedral cones in R^3 with nonempty interior. We *claim* now that condition $(i,j;k,e)$ implies the following condition:

$$(i,j;\ k,e)^* \qquad (-eD_k)\cap D_i\cap(e(i,j)D_j) \neq \{O\},$$

and the last condition in the theorem implies this one: For each $p\in R^3\setminus\{O\}$ there is an $i\in\{1,\dots,n\}$ such that

$$(*) \qquad p\notin D_i \quad\text{and}\quad p\notin -D_i.$$

We prove this claim using standard techniques from the theory of convex polyhedral cones (cf. [4] or [5]). Condition $(i,j;k,e)$ for the cones B_i is of the form $B_k\cap(B_i+B_j)=\{O\}$ (here we dropped the signs) that has polar form $D_k+(D_i\cap D_j)=R^3$. Assume now that $(-D_k)\cap(D_i\cap D_j)=\{O\}$, then the cones $-D_k$ and $(D_i\cap D_j)$ can be separated, i.e., there is $v\in R^3\setminus\{O\}$ such that $v\cdot x\leqq 0$ for all $x\in -D_k$ and $v\cdot y\geqq 0$ for all $y\in D_i\cap D_j$. But then $v\cdot z\geqq 0$ for all $z\in D_k+(D_i\cap D_j)$, a contradiction. Let us see now the last condition:

$$P\cap \operatorname{int} B_i \neq \{O\},$$

and consider $q\in P\cap\operatorname{int} B_i$ with $q\neq O$. Write p for a normal of the plane P. Then $q\cdot p=0$ and $q\cdot x<0$ for all $x\in B_i^*\setminus\{O\}=D_i\setminus\{O\}$, so indeed, $\pm p\notin D_i$.

(As a matter of fact, from now on we will give the proof of Theorem 3 in the case when the sets D_i are polyhedral cones in R^3 with nonempty interior. The general case follows by a standard continuity argument.)

Choose a point $d_i\in\operatorname{int} D_i$ now for $i=1,\dots,n$ and shrink each set D_i to the point d_i linearly and simultaneously with a parameter $t\in[0,1]$, so that the shrinking set $D_i(t)$ equals D_i when $t=1$ and d_i when $t=0$. Write I for the set of indices i,j,k,e_i,e_j,e_k and set

$$D_I(t) = (e_iD_i(t))\cap(e_jD_j(t))\cap(e_kD_k(t))$$

when $t\in[0,1]$. We assume that the cones B_i and the points d_i are in general position to ensure that $D_I(1)\neq\{O\}$ implies that int $D_I(1)$ is nonempty. Moreover, as the cones $D_i(t)$ shrink, the cones $D_I(t)$ shrink as well and $D_I(t)=\{O\}$ for $t<t_0(I)$ where $t_0(I)$ is the smallest t for which $D_I(t)$ is different from $\{O\}$. (If, for some, $D_I(1)=\{O\}$ already, then $t_0(I)$ is not defined.) We assume that the cones B_i and the points d_i are in general position to ensure that $D_I(t)$ is a halfline when $t=t_0(I)$ and that $\operatorname{int} D_I(t)\neq\emptyset$ for $t>t_0(I)$.

As t decreases, condition $(*)$ remains true because the cones D_i get smaller and smaller. But conditions $(i,j;k,e)^*$ will fail for each $(i,j;k,e)$ for some t because $D_I(0)=\{O\}$ for all I. The condition $(i,j;k,e)^*$ holds for all $t>t(i,j;k,e)$ and fails for all $t\leqq t(i,j;k,e)$ where $t(i,j;k,e)$ is uniquely determined. Write t_0 for the largest $t(i,j;k,e)$, then $t_0=t(i,j;k,e)$ for some $(i,j;k,e)$. We may assume without loss of generality that $i=1$, $j=2$, $k=3$ and $e(1,2)=1$ and $e=-1$. So condition $(1,2;3,-1)^*$ fails, i.e.,

$$D_1(t_0)\cap D_2(t_0)\cap D_3(t_0) = K$$

where K is a halfline of the form $\{\alpha v: \alpha\geqq 0\}$ with $v\in R^3\setminus\{O\}$. We know that $D_1(t)\cap\cap D_2(t)\cap D_3(t)$ is $\{O\}$ for $t<t_0$ and has nonempty interior for $t>t_0$. We *claim* now that for each $j=1,2,\dots,n$, $v\in D_j(t_0)$ or $v\in -D_j(t_0)$. This will contradict condition $(*)$ and so prove the theorem.

The claim is evident when $j=1$, 2 and 3. We are going to prove it with notation $j=4$. There are two cases to consider.

1*st case*. When the intersection of two of the cones $D_j(t_0)$ $(j=1,2,3)$ is equal to K, $D_1(t_0)\cap D_2(t_0)=K$, say. From condition $(2,4;1,e=-1)$ we get for $t=t_0$ that

$$D_1(t_0)\cap D_2(t_0)\cap\big(e(2,4)D_4(t_0)\big)\neq\{O\}.$$

But $K=D_1(t_0)\cap D_2(t_0)$ and so $v\in K\subset e(2,4)D_4(t_0)$ indeed.

2*nd case*. When the intersection of any two cones $D_j(t_0)$ have nonempty interior $(j=1,2,3)$. Then, by a wellknown theorem (see [3], for instance), there are vectors $a_j\in R^3$ such that $a_j\cdot x\leqq 0$ for all $x\in D_j(t_0)$ $(j=1,2,3)$ and O is in the convex hull of a_1, a_2 and a_3. The case when some a_j is parallel with some other a_i has been dealt with in the first case. So we assume that every a_j is nonzero and $0=\alpha_1a_1+\alpha_2a_2+\alpha_3a_3$ and every $\alpha_j>0$. Then $a_j\cdot x\leqq 0$ $(j=1,2,3)$ implies that $x=\beta v$ for some real number β. Moreover, $a_j\cdot v=0$ for $j=1,2,3$.

Assume now that $\pm v\notin D_4(t_0)$. Then L, the line through v and $-v$ can be separated from $D_4(t_0)$, i.e., there exists a nonzero $a_4\in R^3$ such that $a_4\cdot x<0$ when $x\in D_4(t_0)\setminus\{O\}$ and $a_4\cdot x=0$ when $x\in L$. This shows that the vectors a_i $(i=1,2,3,4)$ are all orthogonal to v and so $a_4=\beta_1a_1+\beta_2a_2$ for some real numbers β_1 and β_2. We show now that β_1 and β_2 are both different from zero. Assume that $\beta_2=0$, say. Then a_1 and a_4 are parallel and, then $D_1(t_0)$ is separated either from $D_4(t_0)$ or from $-D_4(t_0)$, contradicting condition $(1,j;4,\pm1)^*$.

Consider now condition $(1,2;4,e)^*$: there exists an $x\in R^3\setminus L$ such that

$$x\in\big(-eD_4(t_0)\big)\cap D_1(t_0)\cap D_2(t_0).$$

Then $-ea_4\cdot x<0$, $a_1\cdot x\leqq 0$ and $a_2\cdot x\leqq 0$. This implies that β_1 and β_2 cannot be of the same sign. We may assume that $\beta_1>0$ and $\beta_2<0$.

Suppose now that $e(3,4)=1$ and consider condition $(3,4;2,-1)^*$. In the same way as above this implies the existence of an $x\in R^3\setminus L$ with $a_3\cdot x\leqq 0$, $a_4\cdot x<0$ and $a_2\cdot x\leqq 0$. Now a_1 is a positive linear combination of a_2 and a_4, so $a_1\cdot x<0$. But $a_1\cdot x<0$, $a_2\cdot x\leqq 0$, $a_3\cdot x\leqq 0$ is impossible. Assume now that $e(3,4)=-1$ and consider condition $(3,4;1,-1)^*$. Again, this implies the existence of an $x\in R^3\setminus L$ with $a_3\cdot x\leqq 0$, $a_4\cdot x>0$ and $a_1\cdot x\leqq 0$. Now a_2 is a positive linear combination of a_1 and $-a_4$, so $a_2\cdot x<0$. But $a_1\cdot x\leqq 0$, $a_2\cdot x<0$, $a_3\cdot x\leqq 0$ is impossible.

We mention finally that it is possible to extend these results to higher dimensional spaces but, unfortunately, the conditions in the theorems become rather unintelligible.

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DYADIC MARTINGALE HARDY AND VMO SPACES ON THE PLANE

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1. Introduction

It is well-known that in case $1<p<+\infty$ the classical Hardy space H^p corresponds to L^p, furthermore the dual space of H^1 is BMO and the dual space of VMO is H^1 ([2], [4]). The analogue of the first two parts were shown both in case of martingale Hardy ($\mathbf{H}^p$, $1\leqq p<+\infty$) and BMO spaces ([5]). The analogue of these third part of the theorem is also true for special type of Hardy and VMO spaces ([7]). In these last results the stochastic basis the definition of the spaces based on is linearly ordered.

In this paper we are going to investigate the dyadic Hardy and VMO space on the plane. We shall introduce an atomic Hardy space with respect to the two dimensional dyadic stochastic basis and it will be shown that its dual is the adequate BMO_1^+ space. The dual space of VMO belonging to this BMO_1^+ space will be investigated minutely and it will be proved that it contains strictly the above mentioned dyadic atomic Hardy space.

Moreover, an $\mathbf{H}^\infty$ space will be defined and its dual space will be found in case of one and two dimensional dyadic martingale. (Exact definitions are given later.)

2. Preliminaries and notations

i) In this paper some well-known definitions and theorems will be needed. The theorems will be stated without proof; the proofs can be found in [3] and [7].

Let $(X, \mathscr{A}, P)$ be an arbitrary probability measure space. Denote by $L^\infty(\mathscr{A}, l^2)$ the set of functions $f=(f_n, n\in\mathbf{N})$ such that $f_n: X\to\mathbf{R}$ are $\mathscr{A}$-measurable for all $n\in\mathbf{N}$ and

$$\sup_{x\in X} \mathrm{ess}\, \|f(x)\|_{l^2} < +\infty.$$

Endowe this space with the following norm:

$$\|f\|_{L^\infty(\mathscr{A}, l^2)} := \sup_{x\in X} \mathrm{ess}\, \|f(x)\|_{l^2} \quad (f\in L^\infty(\mathscr{A}, l^2)).$$

Now we introduce a subspace of $L^\infty(\mathscr{A}, l^2)$ whose dual has already been known. Denote by $L_0^\infty(\mathscr{A}, l^2)$ the set of functions $f=(f_n, n\in\mathbf{N})\in L^\infty(\mathscr{A}, l^2)$ such that

$$\lim_{N\to\infty} \|(f_N, f_{N+1}, \ldots)\|_{L^\infty(\mathscr{A}, l^2)} = 0.$$

To give the dual space of $L_0^\infty(\mathscr{A}, l^2)$, the space of totally continuous additive set functions of bounded variation mapping into l^2 is needed. This space will be denoted by $BA(X, \mathscr{A}, P, l^2) =: BA(\mathscr{A})$. Let $\mu: \mathscr{A} \to l^2$ be a set function. The total variation of μ will be defined as follows:

$$v(\mu) := \sup \left\{ \sum_{i=1}^{n} \|\mu(E_i)\|_{l^2} : \bigcup_{i=1}^{n} E_i = X,\ E_i \in \quad,\ E_i \cap E_j = \emptyset,\ i \neq j \right\}.$$

μ is a function of bounded variation if $v(\mu) < +\infty$ and it is totally continuous with respect to P if $\mu(A) = 0 \in l^2$ for all $A \in \mathscr{A}$ that satisfies $P(A) = 0$. Define the following norm on $BA(\mathscr{A})$:

$$\|\mu\|_{BA(\mathscr{A})} := v(\mu) \quad (\mu \in BA(\mathscr{A})).$$

It is easy to see that $BA(\mathscr{A})$ is complete. Denoting the coordinate functions of $\mu \in BA(\mathscr{A})$ by μ_n, i.e. $\mu = (\mu_n, n \in \mathbf{N})$, $\mu_n : \mathscr{A} \to \mathbf{R}$; it follows easily that all μ_n is additive, totally continuous relative to P and it is of bounded variation.

The function $f = (f_n, n \in \mathbf{N}) \in L_0^\infty(\mathscr{A}, l^2)$ is said to be a stationary 0-sequence if for all n large enough we have $f_n = 0$. It can be proved that the space of stationary 0-sequences is dense in $L_0^\infty(\mathscr{A}, l^2)$. On stationary 0-sequences an integral can be defined:

$$\int_X f\, d\mu := \sum_{n=0}^{\infty} \int_X f_n\, d\mu_n \quad (\mu = (\mu_n, n \in \mathbf{N}) \in BA(\mathscr{A})).$$

This integral is a bounded linear functional on a dense subspace of $L_0^\infty(\mathscr{A}, l^2)$ and its norm is $\|\mu\|_{BA(\mathscr{A})}$. Conversely, if I is a bounded linear functional on $L_0^\infty(\mathscr{A}, l^2)$ then there exists a $\mu = (\mu_n, n \in \mathbf{N}) \in BA(\mathscr{A})$ such that I has the following form on stationary 0-sequences:

$$I(f) = \int_X f\, d\mu.$$

Thus the following theorem holds:

THEOREM A [3]. *The dual space of* $L_0^\infty(\mathscr{A}, l^2)$ *is* $BA(\mathscr{A})$.

The following concept of orthogonality will be used ([6]). Let $\mathscr{C} \subset \mathscr{A}$ be an arbitrary σ-algebra. We say that $\Phi = \{\varphi_i : i \in I\} \subset L^2$ is a $\mathscr{C}$-orthogonal system ($\mathscr{C}$-o.s.) if for every $i, j \in I, i \neq j$, $E_{\mathscr{C}}(\varphi_i \varphi_j) = 0$ where $E_{\mathscr{C}}$ denotes the operator of conditional expectation with respect to the σ-algebra $\mathscr{C}$ and L^p denotes the space $L^p(X, \mathscr{A}, P)$ $(0 < p \leqq +\infty)$. Moreover, if for all $i \in I$, $E_{\mathscr{C}}(|\varphi_i|^2) = 1$ then Φ is said to be a $\mathscr{C}$-orthonormal system ($\mathscr{C}$-o. n. s). Suppose that $E_{\mathscr{G}}(f \cdot \varphi_i) = 0$ for all $i \in I$ where $f \in L^2$; if this implies $f = 0$ then the $\mathscr{C}$-o. n. s. Φ is complete in L^2. The following space is also needed. Denote by $L_{(\mathscr{C}, p, q)}$ the set of functions $f \in L^p$ such that

$$\|(E_{\mathscr{C}} |f|^p)^{1/p}\|_q < \infty.$$

Let us have a norm on this space like

$$\|f\|_{(\mathscr{C}, p, q)} := \|(E_{\mathscr{C}} |f|^p)^{1/p}\|_q \quad (1 \leqq p < \infty,\ 1 \leqq q \leqq \infty).$$

Then the following theorem holds:

THEOREM B ([3], [7]). *If the σ-algebra $\mathscr{C}$ is genereted by finite atoms then the dual space of* $L_{(\mathscr{C}, 2, \infty)}$ *is* $L_{(\mathscr{C}, 2, 1)}$.

Suppose that there exists a countable, $\mathscr{C}$-complete o. n. s. Φ in L^2. Then $L_{(\mathscr{C},2,\infty)}$ is isometrically isomorphic to $L^\infty(\mathscr{C}, l^2)$. The isometrical isomorphism is given by

$$\Psi\colon L_{(\mathscr{C},2,\infty} \to L^\infty(\mathscr{C}, l^2), \quad \Psi(f) = (\hat{f}_n, n\in\mathbf{N})$$

where $\hat{f}_n := E_{\mathscr{C}}(f\cdot\varphi_n)$ denotes the $\mathscr{C}$-Fourier coefficient (see [3]).

It is easy to give a space that is isomorphic to $L_0^\infty(\mathscr{C}, l^2)$ by the isometric isomorphism above. Let us denote by $L_{(\mathscr{C},2,\infty),0}$ the set of functions such that

$$\lim_{N\to\infty} \Big\|f - \sum_{n=0}^{N} \hat{f}_n\varphi_n\Big\|_{(\mathscr{C},2,\infty)} = 0.$$

Then $L_{(\mathscr{C},2,\infty),0}$ is isometrically isomorphic to $L_0^\infty(\mathscr{C}, l^2)$. Therefore the following theorem holds:

THEOREM C [3]. *The dual space of* $L_{(\mathscr{C},2,\infty),0}$ *is* $BA(\mathscr{C})$.

ii) Further on let $\Omega := [0,1)\times[0,1)$, let $\mathscr{B}$ be the class of Borel sets of the unit square and let λ be the Lebesgue measure on the unit square. We consider the probability measure space $(\Omega, \mathscr{B}, \lambda)$. We are going to use the following notations:

$$\mathscr{D} = (\mathscr{A}_{n,m};\ n, m\in\mathbf{N}),$$

$$\mathscr{A}_{n,m} := \sigma\big\{[k\cdot 2^{-n}, (k+1)\cdot 2^{-n})\times[l\cdot 2^{-m}, (l+1)\cdot 2^{-m}):$$
$$k, l\in\mathbf{N},\ 0\leqq k<2^n,\ 0\leqq l<2^m\big\},$$

$$\mathscr{A}_{n,\infty} := \sigma\big\{B\times[l\cdot 2^{-n}, (l+1)\cdot 2^{-n}):\ B\in\mathscr{B},\ l\in\mathbf{N},\ 0\leqq l<2^n\big\},$$

$$\mathscr{A}_{\infty,n} := \sigma\big\{[k\cdot 2^{-n}, (k+1)\cdot 2^{-n})\times B:\ B\in\mathscr{B},\ k\in\mathbf{N},\ 0\leqq k<2^n\big\}\quad (n, m\in\mathbf{N})$$

where $\sigma(\mathscr{A})$ denotes the σ-algebra generated by $\mathscr{A}$ for an arbitrary set system $\mathscr{A}$.

Moreover, let $L^p := L^p(\Omega, \mathscr{B}, \lambda)$,

$$L_0^p := \Big\{f\in L^p:\ \int_\Omega f\,d\lambda = 0\Big\}\quad (0<p\leqq\infty)$$

and let the operator of conditional expectation relative to $\mathscr{A}_{n,m}$ be denoted by $E_{n,m}$ $(n, m\in\mathbf{N}\cup\{\infty\})$.

To define the two dimensional dyadic Hardy space we need the concept of the atoms with respect to $\mathscr{D}$. The function $a\in L^\infty$ is said to be an atom if there exist $n, m\in\mathbf{N}$ and an atom H of $\mathscr{A}_{n,m}$ such that

(i) $\{a\neq 0\}\subseteqq H$

(ii) $\|a\|_\infty \leqq \dfrac{1}{\lambda(H)} = 2^n\cdot 2^m$

(iii) $\int_\Omega a\,d\lambda = 0.$

Let the set of $\mathscr{D}$-atoms in L^∞ be denoted by $\mathfrak{A}$.

Denote by $\mathscr{H}=\mathscr{H}(\mathscr{D})$ the set of functions $f\in L^1$ for which there exist a sequence of numbers $(\lambda_n, n\in \mathbf{N})$ and a sequence of atoms $(a_n, n\in\mathbf{N})$ satisfying

$$(1)\qquad \begin{cases} \text{(i)}\ \sum_{n=0}^{\infty}|\lambda_n|<\infty \\ \text{(ii)}\ f=\sum_{n=0}^{\infty}\lambda_a a_n. \end{cases}$$

Let the following norm be introduced:

$$\|f\|_{\mathscr{H}} := \inf\Big\{\sum_{n=0}^{\infty}|\lambda_n| : f=\sum_{n=0}^{\infty}\lambda_n a_n\Big\}$$

where the infimum is taken over all decompositions of f described in (1).

As for all atom $a\in\mathfrak{A}$, $\|a\|_1\leqq 1$, the series (1) (ii) is convergent in L^1-norm and also λ-a.e. and $\|f\|_1\leqq\|f\|_{\mathscr{H}}$ $(f\in\mathscr{H})$. It is easy to show that $\mathscr{H}$ is complete and $\mathscr{H}$. the series (1) (ii) is convergent in $\mathscr{H}$-norm, thus the linear envelop of $\mathfrak{A}$ is dense in

It can be proved (see [8]) that $L_0^p\subset\mathscr{H}$ $(1<p\leqq+\infty)$ and $\|f\|_{\mathscr{H}}\leqq C_p\|f\|_p$ $(f\in\mathscr{H},\ 1<p\leqq+\infty)$ where C_p depends only on p; moreover, an upper estimate of $\mathscr{H}$-norm can be given by C-fold L_1-norm of the maximal function $f^*:=\sup\limits_{n,m\in\mathbf{N}}|E_{n,m}f|$. The converse estimate does not hold (see [8]).

To give the dual space of $\mathscr{H}$, it is necessary to introduce the space BMO_i $(i=1,2)$, as the set of functions $\varphi\in L_0^i$ such that

$$\sup_{n,m\in\mathbf{N}}\|(E_{n,m}|\varphi-E_{n,m}\varphi|^i)^{1/i}\|_\infty<\infty$$

with the following norm:

$$\|\varphi\|_{\mathrm{BMO}_i} := \sup_{n,m\in\mathbf{N}}\|(E_{n,m}|\varphi-E_{n,m}\varphi|^i)^{1/i}\|_\infty .$$

It can be seen that $\mathrm{BMO}_1=\mathrm{BMO}_2$ and $\|\cdot\|_{\mathrm{BMO}_1}\sim\|\cdot\|_{\mathrm{BMO}_2}$ where $\sim$ denotes the equivalence of norms ([8]).

Note that spaces like this are usually called BMO_i^+ in the theory of martingales. It is easy to prove that $L_0^\infty\subset\mathrm{BMO}_2$.

Now the dual space of $\mathscr{H}$ can be given:

THEOREM D. *The dual of* $\mathscr{H}$ *is* BMO_1.

Under more general conditions the proof can be found in [8].

3. The dual of VMO

Define the concept of the space VMO in the following way: let the set of dyadic step functions be denoted by L and the dyadic step functions with 0-value integral be denoted by L_0. Let us close the space L_0 in BMO_2-norm and call it VMO. It is obvious that if $\varphi\in\mathrm{VMO}$ then

$$(2)\qquad \lim_{n,m\to\infty}\|(E_{n,m}|\varphi-E_{n,m}\varphi|^2)^{1/2}\|_\infty=0$$

where the meaning of "lim" is the following throughout this paper: for every $\varepsilon>0$ there exists a number K such that for all $n, m \geqq K$

$$\|(E_{n,m}|\varphi - E_{n,m}\varphi|^2)^{1/2}\|_\infty < \varepsilon.$$

To apply the theorems in 2. i, it is important to show that there exist for all $n\in \mathbf{N}$ a complete $\mathcal{A}_{\infty,n}$- and an $\mathcal{A}_{n,\infty}$-o.n.s. They are constructed from the one dimensional Haar-system in the following way:

$$\chi_{n,k}^{(1)}(x, y) := \chi_k(2^n \cdot x) \quad (k\in \mathbf{N})$$

where χ_k denotes the one dimensional, one periodic Haar-system. $\chi_{n,k}^{(2)}$ can be defined analogously. It is easy to show that $\{\chi_{n,k}^{(1)} : k\in \mathbf{N}\}$ is an $\mathcal{A}_{\infty,n}$-complete o.n.s. and $E_{\infty,n}(|\chi_{n,k}^{(1)}|^2)=1$. One can also easily see that the $\mathcal{A}_{\infty,n}$-Fourier series of every dyadic step function is formed by finite terms, thus $L_0 \subset L_{(\mathcal{A}_{\infty,n},2,\infty),0}$. Obviously, the analogues of these statements hold for the system $\{\chi_{n,k}^{(2)} : k\in \mathbf{N}\}$. Then Theorem C can be applied: the dual space of $L_{(\mathcal{A}_{\infty,n},2,\infty),0}$ is $BA\ (\mathcal{A}_{\infty,n})$ and the dual of $L_{(\mathcal{A}_{n,\infty},2,\infty),0}$ is $BA(\mathcal{A}_{n,\infty})$. The following notations will be useful:

$$\Psi_n^{(1)}(\varphi) := (E_{\infty,n}(\varphi\chi_n^{(1)});\ k\in \mathbf{N}).$$

$$\Psi_n^{(2)}(\varphi) := (E_{n,\infty}(\varphi\chi_{n,k}^{(2)});\ k\in \mathbf{N}) \quad (n\in \mathbf{N}).$$

A lemma is needed to give the dual of the space VMO.

LEMMA [8]. *Suppose that* $f=\sum_{n=0}^\infty f_n$ *λ-a.e.,* $f_n\in \mathcal{H}$ *and* $\sum_{n=0}^\infty \|f_n\|_{\mathcal{H}} < +\infty$. *Then* $f\in \mathcal{H}$ *and* $\|f\|_{\mathcal{H}} \leqq \sum_{n=0}^\infty \|f_n\|_{\mathcal{H}}$.

Now the dual of VMO can be presented:

THEOREM 1. *Suppose that* $f\in \mathcal{H}$, $\mu_n\in BA\,(\mathcal{A}_{\infty,n})$, $\nu_n\in BA\,(\mathcal{A}_{n,\infty})$, $\mu_{n,0}=\nu_{n,0}=0$ $(n\in \mathbf{N})$ *and*

$$\sum_{n=0}^\infty (\|\mu_n\|_{BA(\mathcal{A}_{\infty,n})} + \|\nu_n\|_{BA(\mathcal{A}_{n,\infty})}) < \infty \tag{3}$$

hold. Then

$$\Phi(\varphi) = \int_\Omega f\varphi\, d\lambda + \sum_{n=0}^\infty \Big(\int_\Omega \Psi_n^{(1)}(\varphi)\, d\mu_n + \int_\Omega \Psi_n^{(2)}(\varphi)\, d\nu_n\Big) \quad (\varphi\in L_0) \tag{4}$$

is a bounded linear functional on a dense subspace of VMO. *Conversely, if* $\Phi\in \mathrm{VMO}^*$ *then there exist* $f\in \mathcal{H}$, $\mu_n\in BA(\mathcal{A}_{\infty,n})$, $\nu_n\in BA\,(\mathcal{A}_{n,\infty})$, $\mu_{n,0}=\nu_{n,0}=0$ *such that* (3) *holds, moreover* Φ *takes the form* (4) *on the subspace* L_0 *and*

$$\frac{1}{6\cdot c_2}\Big[\|f\|_{\mathcal{H}} + \sum_{n=0}^\infty (\|\mu_n\|_{BA(\mathcal{A}_{\infty,n})} + \|\nu_n\|_{BA(\mathcal{A}_{n,\infty})}) \leqq \tag{5}$$

$$\leqq \|\Phi\| \leqq c_1\Big[\|f\|_{\mathcal{H}} + \sum_{n=0}^\infty (\|\mu_n\|_{BA(\mathcal{A}_{\infty,n})} + \|\nu_n\|_{BA(\mathcal{A}_{n,\infty})})\Big].$$

While proving Theorem 1 a duality theorem will be used which is very interesting in itself. Introduce some notations as follows: let $X_{n,m}:=L$,

$$\|\xi\|_{X_{n,m}} := \|(E_{n,m}|\xi|^2)^{1/2}\|_\infty \quad (n, m\in \mathbf{N}, \xi\in X_{n,m}).$$

By Theorem B

$$\left|\int_\Omega \xi\cdot\eta\, d\lambda\right| \leqq \|\xi\|_{X_{n,m}}\|(E_{n,m}|\eta|^2)^{1/2}\|_1 \quad (\xi, \eta\in X_{n,m}, n, m\in\mathbf{N}). \tag{6}$$

Moreover, if $A\in X^*_{n,m}$ then there uniquely eixsts an $f\in L^2$ such that

$$A(\xi) = \int_\Omega \xi f d\lambda \quad (\xi\in X_{n,m})$$

and

$$\|A\| = \|(E_{n,m}|f|^2)^{1/2}\|_1.$$

Now let $X:= \bigtimes_{n,m\in\mathbf{N}} X_{n,m}$. Let the space X be endowed with the following norm: for $\xi=(\xi_{n,m}\,;\, n, m\in\mathbf{N})\in X$

$$\|\xi\|_X := \sup_{n,m\in\mathbf{N}} \|\xi_{n,m}\|_{X_{n,m}}.$$

Denote by X_0 the set of elements $\xi\in X$ for which

$$\lim_{n,m\to\infty} \|\xi_{n,m}\|_{X_{n,m}} = 0$$

and for every fixed m there exists k such that if $n\geqq k$ then $\xi_{n,m}=\xi_{k,m}$ and for all fixed n there exists l satisfying $\xi_{n,m}=\xi_{n,l}$ if $m\geqq l$. The dual of X_0 is going to be given.

THEOREM 2. *Suppose that* $f_{k,l}\in L^2$ $(k, l\in\mathbf{N})$, $\mu'_n\in BA(\mathcal{A}_{\infty,n})$, $\nu'_n\in BA(\mathcal{A}_{n,\infty})$ *and*

$$C := \sum_{k=0}^\infty \sum_{l=0}^\infty \|(E_{l,k}|f_{l,k}|^2)^{1/2}\|_1 + \sum_{k=0}^\infty (\|\mu'_k\|_{BA(\mathcal{A}_{\infty,n})} + \|\nu'_k\|_{BA(\mathcal{A}_{n,\infty})}) < \infty \tag{7}$$

hold. Then

$$\Lambda(\xi) = \lim_{n\to\infty} \sum_{k=0}^n \Big(\sum_{l=k}^\infty \int_\Omega f_{l,k}\xi_{l,k}\, d\lambda + \int_\Omega \Psi_k^{(1)}(\xi_{\infty,k})\, d\mu'_k + \tag{8}$$

$$+ \sum_{m=k+1}^\infty \int_\Omega f_{k,m}\xi_{k,m}\, d\lambda + \int_\Omega \Psi_k^{(2)}(\xi_{k,\infty})\, d\nu'_k\Big)$$

$$(\xi\in X_0 \quad \text{and} \quad \xi_{\infty,k} := \lim_{n\to\infty} \xi_{n,k}, \quad \xi_{k,\infty} := \lim_{n\to\infty} \xi_{k,n})$$

is a bounded linear functional on X_0. *Conversely, if* $\Lambda\in X_0^*$ *then there exist* $f_{k,l}\in L^2$ $(k, l\in\mathbf{N})$, $\mu'_n\in BA(\mathcal{A}_{\infty,n})$, $\nu'_n\in BA(\mathcal{A}_{n,\infty})$ *such that* (7) *holds, furthermore,* Λ *can be written as in* (8) *and*

$$C \geqq \|\Lambda\| \geqq \frac{1}{2}C. \tag{9}$$

PROOF OF THEOREM 2. The first part of Theorem 2 and the left side of (9) follow from (6) and Theorem C.

In order to prove the converse statement some notations will be needed: for $\xi\in X_0$ let $\xi^N:=(\xi^N_{n,m}; n, m\in\mathbf{N})\in X_0$ where

$$\xi^N_{n,m} := \begin{cases} 0 & \text{if } n \geqq N \text{ and } m \geqq N \\ \xi_{n,m} & \text{otherwise.} \end{cases}$$

For $\xi\in X_0$ let $\xi^{N(i)}:=(\xi^{N(i)}_{n,m}; n, m\in\mathbf{N})\in X_0$ $(i=1, 2)$ where

$$\xi^{N(1)}_{n,m} := \begin{cases} \xi_{n,m} & \text{if } m = N \text{ and } n \geqq N \\ 0 & \text{otherwise} \end{cases}$$

and

$$\xi^{N(2)}_{n,m} := \begin{cases} \xi_{n,m} & \text{if } n = N \text{ and } m > N \\ 0 & \text{otherwise.} \end{cases}$$

In this case if N tends to ∞ then $\xi^N\to\xi$ in X-norm. Thus

$$\Lambda(\xi) = \lim_{n\to\infty} \Lambda(\xi^n) = \lim_{n\to\infty} \sum_{k=0}^{n} \left(\Lambda(\xi^{k(1)})+\Lambda(\xi^{k(2)})\right).$$

Therefore it is enough to give $\Lambda(\xi^{k(i)})$ $(i=1, 2)$. The restriction of Λ to the subset which has elements of the form $\xi^{k(i)}$ $(i=1, 2)$ will be denoted by $\Lambda^{(i)}_k$. Then

$$\|\Lambda\| = \sum_{k=0}^{\infty} (\|\Lambda^{(1)}_k\|+\|\Lambda^{(2)}_k\|) \tag{10}$$

and it follows that

$$\Lambda^{(1)}_n(0, 0, \dots, \overset{n}{\breve{\xi}}_{n,N}, 0, \dots) = \int_\Omega f_{n,N}\xi_{n,N}\,d\lambda \quad (n \geqq N,\ f_{n,N}\in L^2,\ \xi_{n,N}\in L).$$

Let $\xi_{k,N}$ be an $\mathcal{A}_{l,m}$-step function where $l\leqq k$ $(k\geqq N)$. Then

$$\|(0, 0, \dots, 0, \underbrace{\xi_{k,N}}^{k}, \underbrace{\xi_{k,N}}^{k+1}, \dots)\|_X = \|\xi_{k,N}\|_{X_{k,N}}$$

therefore there exists a function $g_{k,N}\in L^2$ which satisfies

$$\Lambda^{(1)}_N(0, \dots, 0, \underbrace{\xi_{k,N}}^{k}, \xi_{k,N}, \dots) = \int_\Omega g_{k,N}\xi_{k,N}\,d\lambda$$

and

$$\|\Lambda^{(1)}_N\| = \sup_{n\geqq N}\Big\{\sum_{k=N}^{n-1} \|(E_{k,N}|f_{k,N}|^2)^{1/2}\|_1+\|(E_{n,N}|g_{n,N}|^2)^{1/2}\|_1\Big\}. \tag{11}$$

If $\xi_{k,N}$ is an $\mathcal{A}_{l,m}$-step function $(l\leqq k)$ then

$$\Lambda^{(1)}_N(0, \dots, 0, \underbrace{\xi_{k,N}}^{k}, \xi_{k,N}, \dots) = \Lambda^{(1)}_N(0, \dots, 0, \underbrace{\xi_{k,N}}^{k}, 0, \dots, 0)+$$
$$+\Lambda^{(1)}_N(0, \dots, 0, \underbrace{\xi_{k,N}}^{k+1}, \xi_{k,N}, \dots).$$

Hence

$$\int_\Omega g_{k,N}\xi_{k,N}\,d\lambda = \int_\Omega f_{k,N}\xi_{k,N}\,d\lambda + \int_\Omega g_{k+1,N}\xi_{k,N} \quad (k \geqq N).$$

Applying induction we get

$$\int_\Omega g_{k,N}\xi_{k,N}\,d\lambda = \sum_{j=k}^{n}\int_\Omega f_{j,N}\xi_{k,N}\,d\lambda + \int_\Omega g_{n+1,N}\xi_{k,N}\,d\lambda. \tag{12}$$

But

$$\|f_{j,N}\|_1 \leqq \|(E_{j,N}|f_{j,N}|^2)^{1/2}\|_1$$

and (11) indicates that $\sum_{j=k}^{\infty} f_{j,N}$ converges in L_1-norm and λ-a. e. By (12) it follows that $\int_\Omega g_{n,N}\xi\,d\lambda$ converges for all $\xi\in L$ if n tends to $+\infty$. Let $B(\xi):=\lim_{n\to\infty}\int_\Omega g_{n,N}\xi\,d\lambda$ $(\xi\in L)$. Since

$$\|(E_{n,N}|\xi|^2)^{1/2}\|_\infty \leqq (E_{\infty,N}|\xi|^2)^{1/2}\|_\infty$$

thus

$$\left|\int_\Omega g_{n,N}\xi\,d\lambda\right| \leqq \|(E_{n,N}|g_{n,N}|^2)^{1/2}\|_1\cdot\|(E_{n,N}|\xi|^2)^{1/2}\|_\infty \leqq \|\Lambda_N^{(1)}\|\cdot\|(E_{\infty,N}|\xi|^2)^{1/2}\|_\infty.$$

Consequently, B is a bounded linear functional on a subspace of $L_{(\mathscr{A}_{\infty,n},2,\infty),0}$. Using Banach—Hahn's theorem, preserving its norm it can be extended onto the whole space i.e. there exists $\mu_N'\in BA\,(\mathscr{A}_{\infty,N})$ that satisfies

$$B(\xi) = \int_\Omega \Psi_N^{(1)}(\xi)\,d\mu_N' \quad \text{and} \quad \|\mu_N'\|_{BA(\mathscr{A}_{\infty,N})} \leqq \|\Lambda_N^{(1)}\|.$$

Similar statements can be said on $\Lambda_N^{(2)}$, too. Indeed, (10) and (11) imply (8) and (9). The proof Theorem 2 is completn.

Note that B can be written in the form $B(\xi)=\int_\Omega \xi\,d\nu$ where ν is a σ-additive measure but is not necessarily totally continuous with respect to λ.

Proof of Theorem 1. By Theorem D it follows that

$$\left|\int_\Omega f\cdot\varphi\,d\lambda\right| \leqq c_1\|f\|_{\mathscr{H}}\cdot\|\varphi\|_{\mathrm{BMO}_2} \quad (\varphi\in L_0,\ f\in\mathscr{H})$$

and by Theorem C we have

$$\left|\int_\Omega \Psi_n^{(1)}(\varphi)\,d\mu_n\right| \leqq \left|\int_\Omega \Psi_n^{(1)}(\varphi-E_{\infty,n}\varphi)\,d\mu_n\right| \leqq \|\mu_n\|_{BA(\mathscr{A}_{\infty,n})}\|\varphi-E_{\infty,n}\varphi\|_{(\mathscr{A}_{\infty,n},2,\infty)} \leqq$$

$$\leqq \|\mu_n\|_{BA(\mathscr{A}_{\infty,n})}\|\varphi\|_{\mathrm{BMO}_2}.$$

The same inequality holds for ν_n. Obviously, if Φ is of the form (4) then Φ is a bounded linear functional on a dense subspace of VMO and the right side of (5) is satisfied.

To show that if $\Phi\in\mathrm{VMO}^*$ then Φ is of the form (4) and the left side of (5) holds, we embed $(L_0,\|\cdot\|_{\mathrm{BMO}_2})$ into X_0, $R\colon L_0\to X_0$, $R\varphi:=(\varphi-E_{n,m}\varphi;\ m,n\in\mathbf{N})$.

From (2) we get that the range of R is in X_0. By the definition we have $\|\varphi\|_{\mathrm{BMO}_2}=$ $=\|R\varphi\|_X(\varphi\in L_0)$. Choose an arbitrary functional $\Phi\in\mathrm{VMO}^*$. It is easy to see that $\Phi\circ R^{-1}$ is a bounded linear functional on the range of R. Applying Banach—Hahn's theorem $\Phi\circ R^{-1}$ can be extended onto X_0 preserving its norm so by Theorem 2 it follows that there exist $f_{n,m}\in L^2$ $(n, m\in\mathbf{N})$, $\mu_n'\in BA(\mathscr{A}_{\infty,n})$, $\nu_n'\in BA(\mathscr{A}_{n,\infty})$ such that (7) and $2\|\Phi\circ R^{-1}\|=2\|\Phi\|\geqq C$ are satisfied and

$$(13)\qquad \Phi(\varphi)=\lim_{n\to\infty}\sum_{k=0}^{n}\Big(\sum_{l=k}^{\infty}\int_\Omega f_{l,k}(\varphi-E_{l,k}\varphi)\,d\lambda+\int_\Omega \Psi_k^{(1)}(\varphi-E_{\infty,k}\varphi)\,d\mu_k'+$$

$$+\sum_{m=k+1}^{\infty}\int_\Omega f_{k,m}(\varphi-E_{k,m}\varphi)\,d\lambda+\int_\Omega \Psi_k^{(2)}(\varphi-E_{k,\infty}\varphi)\,d\nu_k'\Big)\quad(\varphi\in L_0)$$

holds. Let $\mu_n=(\mu_{n,k}\,;\,k\in\mathbf{N})$, $\nu_n=(\nu_{n,k}\,;\,k\in\mathbf{N})$,

$$\mu_{n,k}:=\begin{cases}\mu_{n,k}' & \text{if } k>0\\ 0 & \text{if } k=0,\end{cases}\qquad \nu_{n,k}:=\begin{cases}\nu_{n,k}' & \text{if } k>0\\ 0 & \text{if } k=0.\end{cases}$$

Then $\mu_n\in BA(\mathscr{A}_{\infty,n})$, $\nu_n\in BA(\mathscr{A}_{n,\infty})$ and

$$(14)\qquad \|\mu_n\|_{BA(\mathscr{A}_{\infty,n})}\leqq\|\mu_n'\|_{BA(\mathscr{A}_{\infty,n})},\quad \|\nu_n\|_{BA(\mathscr{A}_{n,\infty})}\leqq\|\nu_n'\|_{BA(\mathscr{A}_{n,\infty})}.$$

Since

$$\int_\Omega f_{l,k}(\varphi-E_{l,k}\varphi)\,d\lambda=\int_\Omega (f_{l,k}-E_{l,k}f_{l,k})\varphi\,d\lambda,$$

(13) can be written as follows:

$$(15)\qquad \Phi(\varphi)=\lim_{n\to\infty}\sum_{k=0}^{n}\Big(\sum_{l=k}^{\infty}\int_\Omega (f_{l,k}-E_{l,k}f_{l,k})\varphi\,d\lambda+$$

$$+\int_\Omega \Psi_k^{(1)}(\varphi)\,d\mu_k+\sum_{m=k+1}^{\infty}\int_\Omega (f_{k,m}-E_{k,m}f_{k,m})\varphi\,d\lambda+\int_\Omega \Psi_k^{(2)}(\varphi)\,d\nu_k\Big)\quad(\varphi\in L_0).$$

We show that the series

$$(16)\qquad \sum_{k=0}^{n}\Big(\sum_{l=0}^{\infty}(f_{l,k}-E_{l,k}f_{l,k})+\sum_{m=k+1}^{\infty}(f_{k,m}-E_{k,m}f_{k,m})\Big)$$

converges to a function $\tilde f\in\mathscr{H}$ which satisfies

$$(17)\qquad \|\tilde f_{\mathscr{H}}\|\leqq 6c_2\|\Phi\|$$

in L_1-norm and λ-a. e. if $n\to+\infty$. Indeed, as

$$(18)\qquad \|E_{l,k}f_{l,k}\|_1\leqq\|f_{l,k}\|_1\leqq\|(E_{l,k}|f_{l,k}|^2)^{1/2}\|_1,$$

the series converges in L_1-norm and λ-a. e. Let the limit be denoted by $\tilde f$. To prove (17) let $\mathscr{L}_{l,k}\subset\mathscr{D}$ be a series of non-decreasing σ-algebras such that $\mathscr{A}_{0,0}$, $\mathscr{A}_{l,k}\in\mathscr{L}_{l,k}$ and for every fixed $N\in\mathbf{N}$, $\mathscr{L}_{l,k}$ contains at least one of $\mathscr{A}_{n,N}$ and one of $\mathscr{A}_{N,m}$. Then

$$\|f_{l,k}-E_{l,k}f_{l,k}\|_{\mathscr{H}}\leqq\|f_{l,k}-E_{l,k}f_{l,k}\|_{\mathscr{H}(\mathscr{L}_{l,k})}\leqq c_2\|f_{l,k}-E_{l,k}f_{l,k}\|_{\mathrm{H}(\mathscr{L}_{l,k})}$$

where $\mathscr{H}(\mathscr{L}_{l,k})$ is the atomic Hardy space, $\mathbf{H}(\mathscr{L}_{l,k})$ is the martingale Hardy space defined by the maximalfunction with respect to $\mathscr{L}_{l,k}$ and c_2 is an absolute constant. Using (18) and the well-known Doob's inequality

$$\|\sup_{n\in\mathbf{N}} |E_n f|\|_p \leqq \frac{p}{p-1} \|f\|_p \quad (1 < p \leqq \infty)$$

where $(E_n f, n\in\mathbf{N})$ is an arbitrary martingale, we have

$$\|f_{l,k} - E_{l,k} f_{l,k}\|_{\mathbf{H}(\mathscr{L}_{l,k})} = \|\sup_{n,m} \{|E(f_{l,k} - E_{l,k} f_{l,k} | \mathscr{A}_{n,m})| : \mathscr{A}_{n,m} \in \mathscr{L}_{l,k}\}\|_1 \leqq$$

$$\leqq \|\sup_{n,m} \{|E(f_{l,k}|\mathscr{A}_{n,m})| : \mathscr{A}_{n,m}\in\mathscr{L}_{l,k},\ \mathscr{A}_{n,m} \supset \mathscr{A}_{l,k}\}\|_1 + \|E_{l,k} f_{l,k}\|_1 \leqq 3\|(E_{l,k}|f_{l,k}|^2)^{1/2}\|_1.$$

Thus

$$\|f_{l,k} - E_{l,k} f_{l,k}\|_{\mathscr{H}} \leqq 3\cdot c_2 \|(E_{l,k}|f_{l,k}|^2)^{1/2}\|_1.$$

By the lemma it follows that $\tilde{f}\in\mathscr{H}$ and (17) hold. Taking (14) into consideration we get the left side of the inequality (5). Since the series (16) converges in L_1-norm, too, (15) implies (4). This completes the proof of Theorem 1,

Note that there is an exact but rather complicated way to give the subset of the space in Theorem 1 that maps all functions $\varphi\in L_0$ into zero i.e. the set of elements f, μ_n, ν_n $(n\in\mathbf{N})$ for which (4) takes zero for all $\varphi\in L_0$. The space of Theorem 1. factorized by this subspace is the dual space of VMO.

It is obvious that Theorem 1 implies $\mathrm{VMO}^*\supset\mathscr{H}$ and $\mathrm{VMO}^*\neq\mathscr{L}$.

4. The dual space of $\mathbf{H}^\infty$

(i) First we consider the case of one dimensional dyadic martingales. Further on denote by $\mathscr{B}$ the Borel sets of the interval $\Omega=[0, 1)$, by λ the Lebesgue measure on the unit interval and by L the set of one dimesional dyadic step functions. Let

$$\mathscr{A}_n := \sigma\{[k\cdot 2^{-n}, (k+1)\cdot 2^{-n}) : k\in\mathbf{N},\ 0\leqq k < 2^n\}$$

and E_n be the conditional expectation operator relative to $\mathscr{A}_n$. Interpreting $\mathbf{H}^\infty$-norm, the concept of quadratic variation is needed:

$$Qf := \Big(\sum_{n=0}^{\infty} |\Delta_n f|^2\Big)^{1/2} \quad (f\in L^1)$$

where $\Delta_n f = E_n f - E_{n-1} f$ $(n\geqq 0,\ E_{-1}f := 0)$. Let the $\mathbf{H}^\infty$-norm be introduced as follows

$$\|f\|_{\mathbf{H}^\infty} := \|Qf\|_\infty \quad (f\in L^1).$$

Obviously, if $f\in L$ then $\|f\|_{\mathbf{H}^\infty} < +\infty$. Let $\mathbf{H}^\infty$ be the closure of L in $\mathbf{H}^\infty$-norm. The other way to define $\mathbf{H}^\infty$ is the following: let $\mathbf{H}^\infty$ be the set of functions $f\in L^1$ for which $\lim_{N\to\infty} \|(\sum_{n=N}^{\infty} |\Delta_n f|^2)^{1/2}\|_\infty = 0$ holds. It is easy to see that these two definitions are equivalent. $\mathbf{H}^\infty$ can be embedded in the space $L_0^\infty(\mathcal{B}, l^2)$ isometrically:

$$R\colon \mathbf{H}^\infty \to L_0^\infty(\mathcal{B}, l^2), \quad Rf := (\Delta_n f, n\in\mathbf{N}).$$

Applying Theorem A if $\Phi\in(\mathbf{H}^\infty)^*$ is arbitrary then there exists $\mu=(\mu_n, n\in\mathbf{N})\in BA(\mathcal{B})$ such that $\|\Phi\| = \|\mu\|_{BA(\mathcal{B})}$ and

$$\Phi(f) = \int_\Omega R(f)\,d\mu = \sum_{n=0}^{\infty} \int_\Omega \Delta_n f\,d\mu_n \quad (f\in L).$$

Now we give the subspace M of $BA(\mathcal{B})$ in the following way: let $\mu\in M$ if the bounded linear functional Φ belonging to μ maps every function $f\in L$ into 0. It is enough to consider those functionals which map all Haar-functions into 0. But $\Delta_n f= = \sum_{k=2^{n-1}}^{2^n-1} \hat f(k)\chi_k$ where χ_k denotes the k^{th} Haar function and $\hat f(k)$ denotes the k^{th} Haar—Fourier coefficient. Consequently,

$$\Phi(\chi_k) = \int_\Omega \chi_k\,d\mu_n \quad (2^{n-1} \leqq k < 2^n).$$

Therefore, if $\Phi(\chi_k)=0$ for all $k\in\mathbf{N}$ then

$$\mu_n([2l\cdot 2^{-n}, (2l+1)\cdot 2^{-n})) = \mu_n([(2l+1)\cdot 2^{-n}, 2(l+1)\cdot 2^{-n}))$$

$$(n\in\mathbf{N},\ n>0,\ l\in\mathbf{N},\ 0 \leqq l \leqq 2^{n-1}-1)$$

and $\mu_0([0,1))=0$. Denote by M the space of set functions $\mu=(\mu_n, n\in\mathbf{N})\in BA(\mathcal{B})$ with the above mentioned property. A well-known theorem of functional analysis implies

THEOREM 3. *The dual space of* $\mathbf{H}^\infty$ *is the factor-space* $BA(\mathcal{B})/M$.

(ii) Let us see the two dimensional dyadic case and use the notations described in Sections 2 and 3 again. Quadratic variation is now defined as follows:

$$Qf = \Big(\sum_{n,m\in\mathbf{N}} |\Delta_{n,m} f|^2\Big)^{1/2} \quad (f\in L^1)$$

where

$$\Delta_{n,m} f := E_{n,m}f - E_{n-1,m}f - E_{n,m-1}f + E_{n-1,m-1}f, \quad E_{-1,n}f := E_{n,-1}f := 0 \quad (n, m\in\mathbf{N}).$$

We define $\mathcal{H}^\infty$-norm with the help of L^∞-norm of Qf. We call $\mathbf{H}^\infty$ the closure of the space L in $\mathbf{H}^\infty$ norm Similarly as in the one dimensional case $\mathbf{H}^\infty$ can also be defined here by limit. Denote by $\mathbf{H}^\infty$ the set of functions $f\in L^1$ for which

$$\lim_{N\to\infty} \Big\|\Big(\sum_{n\geqq N \text{ or } m\geqq N} |\Delta_{n,m} f|^2\Big)^{1/2}\Big\|_\infty = 0.$$

It is easy to show again that these two definitions are equivalent. Then the analogue of Theorem 3 can be proved in a similar way to the one in part i.

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OEPARTMENT OF NUMERICAL ANALYSIS
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INTEGER SETS CONTAINING NO ARITHMETIC PROGRESSIONS

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Introduction

K. F. Roth [1] proved 1953 using analytic methods that if a strictly increasing sequence of natural numbers $a_1 < a_2 < \ldots < a_k \leqq n$ contains no three term arithmetic progression then

$$k < \frac{c_1 n}{\log\log n}. \tag{1}$$

Very recently Heat—Brown [2] could improve considerably (1) by showing

$$k < \frac{c_2 n}{(\log n)^{c_3}} \quad (c_3 > 0). \tag{2}$$

The aim of the present work is to show that Roth's analytic method combined with some combinatorial ideas is are useful in the study of such type problems. Applying the method to the present problem the resulting inequality will be (2) whilst in [3] it was shown that if $a_1 < a_2 < \ldots < a_k \leqq n$ is a sequence $\mathscr{A}$ of natural numbers such that $\mathscr{A} - \mathscr{A}$ does not contain any positive square then

$$k < \frac{c_4 n}{(\log n)^{\log\log\log\log n/12}}.$$

It is possible that the present approach leads to new results in other problems of additive number theory too.

NOTATIONS. Let

$$\mathscr{N}_{(n)} = \{1, 2, \ldots, n\},$$
$$\mathscr{N}_{i,j,q,s} = \{iq+j,\ (i+1)q+j,\ (i+s-1)q+j\},$$
$$\mathscr{A}_{i,j,q,s} = \mathscr{N}_{i,j,q,s} \cap \mathscr{A}.$$

Let $|\mathscr{A}|$ be the number of elements in $\mathscr{A}$,

$$f_{\mathscr{A}}(\alpha) = \sum_{a \in \mathscr{A}} e(a\alpha),\ e(\alpha) = e^{2\pi i\alpha}, \quad \gamma = \frac{|\mathscr{A}|}{n}.$$

Let ε be a sufficiently small positive number and $n > n_0(\varepsilon)$. We suppose $\gamma > \dfrac{c_2}{\log^{c_3} n}$ otherwise the theorem is trivially true for $\mathscr{A}$.

Let us assume the assertion is proved for every $m \leqq \sqrt{n}$. Choosing c_2 sufficiently small it is clearly true for $n < c_4$. It is easy to see that either

a) $\mathscr{A}$ has a subset $\mathscr{A}' \subset \left[H, H+\frac{n}{\log n^3}\right]$ with

$$|\mathscr{A}'| > (1+\varepsilon)\gamma\frac{n}{(\log n)^3}, \quad 1 \leq H \leq n,$$

or

b) the total number of solutions of the equations $a_i+a_j=2a_k+n_0$ and $a_i+a_j=$ $=2a_k-n_0$ is at most

$$\frac{1}{2}(1+4\varepsilon)\gamma^3 n_0^2 \tag{3}$$

for some number $n_0\in\left[n, n+\frac{n}{\log^2 n}\right]$. The case a) can be settled easily (cf. the end of the proof). Now we are dealing with the case b). In the following we work with n_0 instead of n.

We write $\gamma_0=\gamma\cdot\frac{n}{n_0}$. Now

$$\frac{1}{n_0}\sum_{t=0}^{n_0-1} f_{\mathscr{A}}\left(\frac{t}{n_0}\right) f_{\mathscr{A}}\left(\frac{t}{n_0}\right) f_{\mathscr{A}}\left(\frac{-2t}{n_0}\right)$$

is the number of solutions of the equation $a_i+a_j\equiv 2a_k \pmod{n_0}$. In view of (3) this is less than $\left(\frac{1}{2}+3\varepsilon\right)\gamma_0^3 n_0^2$, since there is no 3-term arithmetic progression in $\mathscr{A}$.

Because the main term (corresponding to $t=0$) is $\gamma_0^3\cdot n_0^3$ it follows that

$$\frac{1}{n_0}\left|\sum_{t=1}^{n_0-1} f_{\mathscr{A}}\left(\frac{t}{n_0}\right) f_{\mathscr{A}}\left(\frac{t}{n_0}\right) f_{\mathscr{A}}\left(\frac{-2t}{n_0}\right)\right| > \left(\frac{1}{2}-3\varepsilon\right)\frac{\gamma_0^3 n_0^2}{2}.$$

Let us assume that for $t\neq 0$, $\left|f\left(\frac{t}{n_0}\right)\right| < \frac{|\mathscr{A}|}{2^{i_0}}$ with a fixed i_0, sufficiently large. There must be an $i=i_1$ with $2^{i_0}<2^{i_1}<(\log n)^{1/3}$ such that there exist $t_1, t_2, \ldots, t_q$ $\left(q=q(i_1)=\left[\frac{2^{3i_1}}{i_1^2}\right]+1\right)$, $t_\mu \not\equiv t_\nu \pmod{n_0}$ with $\left|f\left(\frac{t_\mu}{n_0}\right)\right| > \frac{|\mathscr{A}|}{2^{i_1}}$. Otherwise we would have

$$\frac{1}{n_0}\sum_{\substack{i\geq i_0 \\ 2^i\leq \log^{1/3} n}}\ \sum_{|\mathscr{A}|\cdot 2^{-i}\leq \max\left(\left|f_{\mathscr{A}}\left(\frac{t}{n_0}\right)\right|, \left|f_{\mathscr{A}}\left(-\frac{2t}{n_0}\right)\right|\right)\leq |\mathscr{A}|\cdot 2^{-i+1}} \left|f_{\mathscr{A}}\left(\frac{t}{n_0}\right)\right|^2 \left|f_{\mathscr{A}}\left(-\frac{2t}{n_0}\right)\right| < \frac{|\mathscr{A}|^3}{10 n_0}$$

while it follows from Parseval's identity that

$$\frac{1}{n_0}\sum_{\max\left(\left|f_{\mathscr{A}}\left(\frac{t}{n_0}\right)\right|, \left|f_{\mathscr{A}}\left(-\frac{2t}{n_0}\right)\right|\right)<|\mathscr{A}|\log^{-1/3} n} \left|f_{\mathscr{A}}\left(\frac{t}{n_0}\right)\right|^2 \left|f_{\mathscr{A}}\left(-\frac{2t}{n_0}\right)\right| \ll$$

$$\ll |\mathscr{A}|\cdot\frac{|\mathscr{A}|}{\log^{1/3} n} = o\left(\frac{|\mathscr{A}|^3}{n_0}\right).$$

We shall show the existence of a set

$$\mathcal{B} = \{b, 2b, \ldots, |\mathcal{B}|b\} = \{b_k\}_{k=1}^{|\mathcal{B}|}$$

such that

$$1 \leqq b \leqq n_0^{q/(q+1)}, \quad |\mathcal{B}| = \left[\frac{n^{1/(q+1)}}{(\log n)^2}\right],$$

$$jbt_\nu \equiv l_{i,j} \pmod{n_0}, \quad |l_{i,j}| < \frac{2n}{(\log n)^2}$$

for all $1 \leqq j \leqq |\mathcal{B}|, 1 \leqq \nu \leqq q = q(i_1)$. Dividing the set $\mathcal{N}_{(n_0)}$ into $n_0^{1/(q+1)}$ equal intervals $I_1, I_2, \ldots, I_{n_0^{1/(q+1)}}$, there must exist b' and b'' $(1 \leqq b' < b'' \leqq n_0^{q/(q+1)})$ such that $b't_\nu \pmod{n_0}$ lies in the same interval as $b''t_\nu$ for all ν (i.e. $|(b'-b'')t_\nu - k_\nu n_0| \leqq n_0^{q/(q+1)}$ with integer k_ν). The choise $b = |b''-b'|$ satisfies our requirements.

Now the number of solutions of $a_i - a_j \equiv b_k - b_l \pmod{n_0}$ is

$$\frac{1}{n_0} \sum_{t=0}^{n_0-1} f_{\mathcal{A}}\left(\frac{t}{n_0}\right) \overline{f_{\mathcal{A}}\left(\frac{t}{n_0}\right)} f_{\mathcal{B}}\left(\frac{t}{n_0}\right) \overline{f_{\mathcal{B}}\left(\frac{t}{n_0}\right)}$$

which is at least $(i = i_1, n > n_0(\varepsilon))$

$$(1-\varepsilon)\frac{1}{n_0}\frac{|\mathcal{A}|^2}{2^{2i}}|\mathcal{B}|^2 \cdot \frac{2^{3i}}{i^2} = (1-\varepsilon)\gamma_0^2 n_0 |\mathcal{B}|^2 \cdot \frac{2^i}{i^2}. \tag{4}$$

On the other hand the number of solutions of $a_i - a_j \equiv b_k - b_l \pmod{n_0}$ (with the notation $B' = \frac{|\mathcal{B}|}{T}$, T a large constant) is

$$(1+\delta(T)) \sum_{j=1}^{b} \sum_{i=0}^{n/B'} |\mathcal{A}_{B'i,j,b,B'}| \sum_{h=-T}^{T} |\mathcal{A}_{B'(i+h),j,b,B'}| \left(1 - \frac{|h|}{T}\right) |\mathcal{B}| \tag{5}$$

where $\delta(T) \to 0$ as $T \to \infty$.

There exists a set $\mathcal{A}_{B'\nu,j,b,B'}$ with

$$|\mathcal{A}_{B'\nu,j,b,B'}| \geqq (1-2\varepsilon)\frac{2^i}{i^2} B'\gamma_0 \tag{6}$$

since otherwise the sum in (4) would be with a fixed $T = T_0(\varepsilon)$

$$\leqq (1+\varepsilon)|\mathcal{A}|T \cdot \frac{2^i}{i^2}(1-2\varepsilon)B'\gamma_0|\mathcal{B}| < (1-\varepsilon)\frac{2^i}{i^2}\gamma_0^2 n_0 |\mathcal{B}|^2$$

in contradiction to (5).

Similarly if we have a t^* with

$$\left|f_{\mathcal{A}}\left(\frac{t^*}{n_0}\right)\right| > \frac{|\mathcal{A}|}{2^{i_0}}$$

then the same argument (with $q=1$) shows the existence of a

$\frac{\sqrt{n_0}}{\log^2 n} > B' > \sqrt{n_0}/\log^3 n$ with

$$|\mathcal{A}_{B'v,j,b,B'}| > (1-2\varepsilon)\left(1+\frac{1}{2^{2i_0}}\right)B'\gamma_0. \tag{6'}$$

In both cases (6) and (6′), using the set $\mathcal{A}_{B'v,j,b,B'}$ we obtain a new set $\mathcal{A}'\subset\subset\{1, \dots, B'\}$ with $B'\in\left[\frac{n^{1/(q+1)}}{\log^3 n}, n^{1/(q+1)}\right]$ such that $\mathcal{A}'$ contains no 3 terms arithmetic progressions and $|\mathcal{A}'|>c(q)B'\gamma_0$. Here either

i) $q=1,\ c(q)>1+\frac{1}{2^{2i_0+1}}$

or

ii) $q>\frac{2^{3i_0}}{i_0^2},\ c(q)>\frac{q^{1/3}}{\log^{4/3} q}$.

It is easy to see that we have in both cases

$$|\mathcal{A}'| > c_2\frac{c(q)}{\log^{c_3} n}B' > \frac{c_2 B'}{\log^{c_3} B'}$$

if c_3 was chosen sufficiently small. This contradicts our induction hypothesis and so proves the theorem,

REMARK. It is easy to see that if $\left|f_{\mathcal{A}}\left(\frac{t}{n}\right)\right|<\frac{|\mathcal{A}|}{2^{i_0}}$ for all $t=1, 2, \dots, n-1$ with a sufficiently large i_0, (i.e. we have case b) preceding (3)) then c_0 can be chosen near 1/3. ($c_0=1/3-\varepsilon$ is admissible if $i_0>c_0(\varepsilon)$). A more careful computation concerning case a) shows that $c_0>1/4$ can be chosen in the formulation of the Theorem.

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MULTIPLICATIVE FUNCTIONS WITH SMALL INCREMENTS. II

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1. Let $\mathcal{M}$ (resp. $\mathcal{M}^*$) denote the set of complex-valued multiplicative (completely multiplicative) functions. Let $\varrho: [1, \infty) \to [1, \infty)$ be a slowly varying function, i.e. such that

$$1-\varepsilon_x < \frac{\varrho(y)}{\varrho(x)} < 1+\varepsilon_x \quad \text{if} \quad \frac{x}{2} \leqq y \leqq 2x, \tag{1.1}$$

with a suitable function $\varepsilon_x(>0)$ tending to zero as $x\to\infty$. Let $\alpha \geqq 1$ be a constant, $\mathcal{L}_{\alpha,\varrho}$ the set of those $f\in\mathcal{M}$ for which

$$\sum_{n\leqq x} |f(n)|^\alpha \ll x\varrho^\alpha(x) \tag{1.2}$$

holds.

Let $K\in\mathbf{N}$, $f\in\mathcal{M}$ be such that

$$\sum_{n\leqq x} |\Delta_K f(n)|^\alpha \ll x\varrho^\alpha(x) \quad (x\to\infty), \quad \Delta_K f(n) = f(n+K)-f(n). \tag{1.3}$$

Our purpose in this paper is to characterize all these functions.

THEOREM. *If for $f\in\mathcal{M}$ (1.3) holds, then either $f(n)=n^s u(n)$, $\operatorname{Re} s\leqq 1$, $u(n+K)=u(n)$ $(\forall n\in\mathbf{N})$ or $f\in\mathcal{L}_{\alpha,\varrho}$.*

REMARK. The special case $\alpha=1$, $\varrho(x)\equiv 1$ has been considered in [2].

2. Lemmata. In this section we assume that $f\in\mathcal{M}$, $f\notin\mathcal{L}_{\alpha,\varrho}$ and that (1.3) is true.

LEMMA 1. *If H is an arbitrary positive constant, then*

$$\sum_{n\leqq x} \max_{1\leqq l_n\leqq H} |\Delta_{l_n K} f(n)|^\alpha \ll x\varrho^\alpha(x). \tag{2.1}$$

PROOF. It is clear; use only Hölder's inequality. □

LEMMA 2. *If there exists an integer D such that*

$$\sum_{\substack{n\leqq x\\ (n,D)=1}} |f(n)|^\alpha \ll x\varrho^\alpha(x) \quad (x\to\infty), \tag{2.2}$$

then $f\in\mathcal{L}_{\alpha,\varrho}$.

PROOF. For an arbitrary n, let $a(n)$ be the product of prime factors of n composed from the prime divisors of D and K, and let $b(n)$ be defined by $n=a(n)\cdot b(n)$. From the elementary theory of congruences we get that there are suitable constants H_1, H_2

such that $a(n+l_nK)\leqq H_1$ for a suitable l_n, $1\leqq l_n\leqq H_2$. Furthermore, $|f(n)|\leqq \leqq |f(n+l_nK)|+|\Delta_{l_nk}Kf(n)|$, and so, by Lemma 1,

$$\sum_{n\leqq x}|f(n)|^\alpha \ll \sum_{n\leqq x}|f(n+l_nK)|^\alpha + x\varrho^\alpha(x).$$

Since $f(a(n+l_nK))$ is bounded, the number of integers n with $b(n)=\nu$, $a(n)\leqq H_1$, is bounded as well, therefore the sum on the right hand side is

$$\ll \sum_{\substack{\nu\leqq x+H_2K\\(\nu,D)=1}}|f(\nu)|^\alpha.$$

This completes the proof. □

LEMMA 3. *For each prime p coprime to K, $f(p^a)=f(p)^a$.*

PROOF. From (2.1) we infer

$$\sum_{n\leqq x}|\Delta_{Kp}f(pn)-f(p)\Delta_K f(n)|^\alpha \ll x\varrho^\alpha(x).$$

Consequently, by summing over the integers n of the form $n=p^a\nu$, $(\nu,p)=1$, we have

$$\sum_{\substack{\nu\leqq x\\(\nu,p)=1}}|f(p)f(p^a\nu)-f(p^{a+1}\nu)|^\alpha \ll x\varrho^\alpha(x),$$

which leads to

$$|f(p)f(p^a)-f(p^{a+1})|^\alpha \sum_{\substack{\nu\leqq x\\(\nu,p)=1}}|f(\nu)|^\alpha \ll x\varrho^\alpha(x).$$

Then $f(p)f(p^a)=f(p^{a+1})$, since in the opposite case (2.2) would be satisfied with $p=D$, and by Lemma 2 this would imply that $f\in\mathcal{L}_{\alpha,\varrho}$. Hence the assertion immediately follows. □

Let $f_1(n)=\chi_0(n)|f(n)|$, where χ_0 is the principal character mod K. It is clear that (1.3) holds for f_1 instead of f as well, furthermore that $f_1\in\mathcal{M}^*$, $f_1\notin\mathcal{L}_{\alpha,\varrho}$ (see Lemma 2).

Let $g(n):=f_1^\alpha(n)$,

$$S(x):=\sum_{n\leqq x}g(n). \tag{2.3}$$

It is clear that

$$|g(n)-g(m)|=\left|\alpha\int_{f_1(n)}^{f_1(m)}u^{\alpha-1}\,du\right|\leqq \alpha|f_1(n)-f_1(m)|\big(f_1(n)^{\alpha-1}+f_1(m)^{\alpha-1}\big), \tag{2.4}$$

consequently we have

$$\sum_{n\leqq x}|g(n+K)-g(n)| \ll \sum_{n\leqq x}|\Delta_K f_1(n)|\cdot\big(f_1^{\alpha-1}(n+K)+f_1^{\alpha-1}(n)\big). \tag{2.5}$$

Hence, by Hölder's inequality,

$$\sum_{n\leqq x}|g(n+K)-g(n)| \ll \Big(\sum_{n\leqq x}|\Delta_K f_1(n)|^\alpha\Big)^{1/\alpha}\big(S(x+K)\big)^{(\alpha-1)/\alpha} \ll \tag{2.6}$$

$$\ll x^{1/\alpha}\varrho(x)\big(S(x+K)\big)^{(\alpha-1)/\alpha}.$$

Since

$$g(m) \leqq |g(m)-g(m-K)|+|g(m-K)-g(m-2K)|+\ldots,$$

therefore, by (2.6),

$$g(m) \ll m^{1/\alpha}\varrho(m)S(m)^{(\alpha-1)/\alpha}.$$

Hence we infer that

$$S(x+H)-S(x) \ll_H x^{1/\alpha}\varrho(x)S(x+H)^{(\alpha-1)/\alpha},$$

which implies easily that

$$S(x+H) \ll_H x\varrho^{\alpha}(x)+S(x) \tag{2.7}$$

for each fixed H.

LEMMA 4. *If H is an arbitrary positive constant, then*

$$\sum_{n\leqq x}\max_{1\leqq l_n\leqq H}|\Delta_{l_n K}g(n)| \ll x^{1/\alpha}\varrho(x)S(x)^{(\alpha-1)/\alpha}+x\varrho^{\alpha}(x). \tag{2.8}$$

PROOF. This is an obvious consequence of (2.6) and (2.7). □

Let q be an arbitrary natural number coprime to K. For each n, let l be the least nonnegative integer for which $q|n+lK$, and let $n_1=\left[\frac{n+lK}{q}\right]$. Then n determines l and n_1, furthermore a fixed $n_1(\geqq K)$ occurs exactly for q distinct n. From (2.8), we get

$$\Big|\sum_{n\leqq x}(g(n)-g(qn_1))\Big| \ll x\varrho^{1/\alpha}(x)S(x)^{(\alpha-1)/\alpha}+x\varrho^{\alpha}(x).$$

Observing that

$$\sum_{n\leqq x}g(qn_1) = qg(q)S\left(\frac{x}{q}\right)+O\big(x^{1/\alpha}\varrho(x)S(x)^{(\alpha-1)/\alpha}\big),$$

we get

$$\left|S(x)-qg(q)S\left(\frac{x}{q}\right)\right| \ll x^{1/\alpha}\varrho(x)S(x)^{(\alpha-1)/\alpha}+x\varrho^{\alpha}(x). \tag{2.9}$$

Let

$$\lambda(x):=\frac{S(x)}{x\varrho^{\alpha}(x)}.$$

Then, from (2.9),

$$\left|\lambda(x)-g(q)\left(\frac{\varrho(x/q)}{\varrho(x)}\right)^{\alpha}\lambda(x/q)\right| \ll \lambda^{(\alpha-1)/\alpha}(x)+1. \tag{2.10}$$

LEMMA 5. *If there exists some q, $(q,K)=1$, such that $g(q)<1$, then $\lambda(x)=O(1)$, consequently $f_1\in\mathscr{L}_{\alpha,\varrho}$.*

PROOF. Since $\varrho(x/q)\varrho^{-1}(x)\to 1$, as $x\to\infty$, therefore, from (1.10) we get:

$$\lambda(x) < (1-\varepsilon)\lambda(x/q)+A, \tag{2.11}$$

whenever x is larger than a constant y_0, with suitable positive constants ε, A. The deduction of $\lambda(x)=O(1)$, from (2.11), is immediate. Indeed, let $x_1<x_2<\ldots$ be such a sequence for which $\lambda(x_\nu)=\max_{x\le x_\nu}\lambda(x)$ is not bounded, then there exists such an infinite sequence x_ν, and $\lambda(x_\nu)\to\infty$. But from (2.11),

$$\lambda(x_\nu) < (1-\varepsilon)\lambda(x_\nu)+A,$$

which implies that $\varepsilon\lambda(x_\nu)<A$, a contradiction. □

LEMMA 6. *For* $q>1$ *and* $(q, K)=1$, *let* $\eta=\eta_q$ *be defined by* $g(q)=q^\eta$. *Then*

$$\lambda(x) \ll x^{\eta+\varepsilon}, \tag{2.12}$$

and for $\eta>0$,

$$\lambda(x) \gg x^{\eta-\varepsilon} \tag{2.13}$$

for every $\varepsilon>0$.

PROOF. First we prove (2.12). Assume that x is large enough, $x\ge y_0$, such that $\varrho(x/q)\cdot(\varrho(x))^{-1}<q^{\varepsilon_1/\alpha}$. Then, from (2.10), with a suitable constant c,

$$\lambda(x)-c\lambda^{(\alpha-1)/\alpha}(x) < q^{\eta+\varepsilon_1}\lambda\left(\frac{x}{q}\right)+c,$$

whence we have

$$\lambda(x)<q^{\eta+2\varepsilon_1}\lambda\left(\frac{x}{q}\right)+c_1 \tag{2.14}$$

with some constant c_1. Starting from some x_0, we define $x_\nu=q^\nu x_0$. Then, from (2.14),

$$\lambda(x_{\nu+1}) < q^{\eta+2\varepsilon_1}\lambda(x_\nu)+c_1,$$

which gives that

$$\lambda(x_\nu) \ll q^{\nu(\eta+2\varepsilon_1)} \quad (\nu\to\infty),$$

i.e. that $\lambda(x_\nu)\ll x_\nu^{\eta+2\varepsilon_1}$. Since for x, $x_\nu<x<x_{\nu+1}$, we have $\lambda(x)\ll x_{\nu+1}^{\eta+2\varepsilon_1}\ll x^{\eta+2\varepsilon_1}$, therefore (2.12) is true with $2\varepsilon_1=\varepsilon$. So (2.12) is true.

Let us prove now (2.13).

Let $\varepsilon_1>0$ be an arbitrary constant, y_1 so large that $\varrho(x/q)\varrho^{-1}(x)>q^{-\varepsilon_1/\alpha}$ whenever $x>y_1$. Then, from (2.10),

$$\lambda(x)+c\lambda^{(\alpha-1)/\alpha}(x)+c > q^{\eta-\varepsilon_1}\lambda\left(\frac{x}{q}\right) \quad (x>y_1). \tag{2.15}$$

Let $2\varepsilon_1<\eta$. There exists a constant A such that $\lambda(x/q)\ge A$ implies that $\lambda(x)\ge \lambda(x/q)$. From (2.15) we obtain

$$\lambda(x) > q^{\eta-2\varepsilon_1}\lambda\left(\frac{x}{q}\right) \tag{2.16}$$

assuming that $x>y_1$ and that $\lambda(x/q)\ge A_1$, A_1 is a suitable constant. Since $\lambda(x)$ is not bounded, therefore there exists x_0, $x_0>y_1$, such that $\lambda(x_0)>A_1$. Let $x_\nu=q^\nu x_0$. From (2.16) we obtain that $\lambda(x_\nu)>A_1$ $(\nu=1, 2, \ldots)$ holds as well, consequently

$$\lambda(x_\nu) > q^{\nu(\eta-2\varepsilon_1)}\lambda(x_0),$$

and so

$$\lambda(x_\nu) \gg x_\nu^{\eta-2\varepsilon_1} \quad (\nu \to \infty).$$

Since for x, $x_\nu \leqq x < x_{\nu+1}$, $\lambda(x) \gg \lambda(x_\nu)$, therefore (2.13) is true with $\varepsilon = 2\varepsilon_1$.

By this the proof is complete. □

3. Completion of the proof of the Theorem. If $g(q) \leqq 1$ for each q, then $\lambda(x) = O(1)$, i.e. $f \in \mathscr{L}_{\alpha,p}$ (see Lemma 2). If $g(q) < 1$ for at least one q coprime to K, then the same is true, see Lemma 5. Assume that $g(q) \geqq 1$ whenever $(q, K) = 1$.

Let q_1, q_2 be arbitrary integers, coprime to K, $q_1 > 1$, $q_2 > 1$, $g(q_1) = q_1^{\eta_1}$, $g(q_2) = q_2^{\eta_2}$. Assume that $\eta_1 > 0$. From Lemma 6 we obtain that

$$\frac{\log \lambda(x)}{\log x} \to \eta_1 \quad (x \to \infty).$$

Then $\eta_2 = 0$ cannot occur, since this would imply

$$\frac{\log \lambda(x)}{\log x} \to 0.$$

If $\eta_2 \neq 0$, then Lemma 6 gives

$$\frac{\log \lambda(x)}{\log x} \to \eta_2,$$

consequently $\eta_1 = \eta_2$.

We proved that $g(q) = q^\eta$ for each $(q, K) = 1$, with some positive constant η. Then $|f(n)| = n^\sigma$, $(n, K) = 1$, $\sigma = \eta/\alpha$. Since (1.3) holds for $|f|$ instead of f as well, furthermore $(n+K)^\sigma - n^\sigma \sim \sigma K n^{\sigma-1}$, therefore (1.3) implies $\sigma \leqq 1$.

Let $f_2(n) := \chi_0(n) f(n)$, and $t(n)$ be defined by $f_2(n) = n^\sigma t(n)$. It is clear that $|t(n)| = 1$ if $(n, K) = 1$, and $t(n) = 0$ if $(n, K) > 1$. Since

$$\Delta_K f_2(n) = (n+K)^\sigma \Delta_K t(n) + t(n)\big((n+K)^\sigma - n^\sigma\big),$$

therefore

$$\text{(3.1)} \quad \sum_{n \leqq x} (n+K)^\sigma |\Delta_K t(n)| \leqq \sum_{n \leqq x} |\Delta_K f_2(n)| + \sum_{n \leqq x} |t(n)| |(n+K)^\sigma - n^\sigma| = \Sigma_1 + \Sigma_2.$$

We have

$$\Sigma_2 \ll \sum_{n \leqq x} n^{\sigma-1} \ll x^\sigma \ll x.$$

Furthermore, $|\Delta_K f_2(n)| \leqq |\Delta_K f(n)|$, consequently by (1.3)

$$\Sigma_1 \leqq \Big(\sum_{n \leqq x} 1\Big)^{(\alpha-1)/\alpha} \Big(\sum_{n \leqq x} |\Delta_K f_2(n)|^\alpha\Big)^{1/\alpha} \ll x\varrho(x),$$

and so

$$\sum_{n \leqq x} (n+K)^\sigma |\Delta_K t(n)| \ll x\varrho(x).$$

This implies

$$\sum_{x/2 \leqq n \leqq x} |\Delta_K t(n)| \ll x^{1-\sigma} \varrho(x) \ll x^\delta$$

with some constant $\delta<1$. Hence we obtain

$$\sum_{n=1}^{\infty}\frac{|\varDelta_K t(n)|}{n}<\infty. \tag{3.2}$$

In [1] it was proved that such a function t has the form

$$t(n)=\chi_K(n)n^{i\tau},$$

where χ_K is a suitable character $\bmod K$. Consequently $f_2(n)=n^{\sigma+i\tau}\chi_K(n)$. Let us define $u(n)$ by $f(n)=n^{\sigma+i\tau}u(n)$. It is clear that $f_2(n)=f(n)$ if $(n,K)=1$. Then, from (1.3),

$$\sum_{n\leqq x}|(n+K)^{\sigma+i\tau}u(n+K)-n^{\sigma+i\tau}u(n)|^{\alpha}\ll x\varrho(x)^{\alpha}. \tag{3.3}$$

Repeating the argument used in [2] one can deduce that $u(n+K)=u(n)$ ($\forall n\in\mathbf{N}$).

The proof is finished. □

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INEQUALITIES FOR SUMS OF MULTIPOWERS

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1. Introduction

For $x=(x_1, \dots, x_n)\in \mathbf{R}_+^k$, $a=(a_1, \dots, a_k)\in \mathbf{R}^k$ the multipower x^a is defined by

$$x^a = (x_1, \dots, x_k)^{(a_1, \dots, a_k)} = x_1^{a_1} \dots x_k^{a_k}.$$

Using this notation, the Hölder inequality (see [5])

$$(1) \qquad \frac{1}{n}\sum_{i=1}^{n} x_i y_i \leqq \left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)^{1/p} \left(\frac{1}{n}\sum_{i=1}^{n} y_i^q\right)^{1/q}$$

(where $n\in\mathbf{N}$, $x_1, \dots, x_n, y_1, \dots, x_n>0$, $p, q>1$, $1/p+1/q=1$) can be rewritten as

$$(2) \qquad 1 \leqq \left(\frac{1}{n}\sum_{i=1}^{n} (x_i, y_i)^{(1,1)}\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} (x_i, y_i)^{(p,0)}\right)^{1/p} \left(\frac{1}{n}\sum_{i=1}^{n} (x_i, y_i)^{(q,0)}\right)^{1/q}.$$

The following inequality obviously generalizes (2):

$$(3) \qquad 1 \leqq \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{a_0}\right)^{\alpha_0} \dots \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{a_m}\right)^{\alpha_m}$$

where $\alpha_0, \dots, \alpha_m\in\mathbf{R}$, $a_0, \dots, a_m\in\mathbf{R}^k$ are fixed parameters and $n\in\mathbf{N}$, $x_1, \dots, x_n\in\mathbf{R}_+$ are arbitrary variables. (Indeed, letting $m=2$, $k=2$, $\alpha_0=-1$, $\alpha_1=1/p$, $\alpha_2=1/q$, $a_0=(1, 1)$, $a_1=(p, 0)$, $a_2=(0, q)$, we can see that (3) is equivalent to (2).) However (3) includes other important inequalities, for instance let $m=2$, $k=1$, $\alpha_0=c-b$, $\alpha_1=a-c$, $\alpha_2=b-a$, $a_0=a$, $a_1=b$, $a_2=c$, where $a\leqq b\leqq c$ are arbitrary real values. Then (3) reduces to

$$1 \leqq \left(\frac{1}{n}\sum_{i=1}^{n} x_i^a\right)^{c-b} \left(\frac{1}{n}\sum_{i=1}^{n} x_i^b\right)^{a-c} \left(\frac{1}{n}\sum_{i=1}^{n} x_i^c\right)^{b-a},$$

which is called Lyapunov's inequality (see [5, p. 27, Theorem 17]).

The key result of this paper is an approximation theorem for convex functions. Using this theorem, we obtain necessary and sufficient conditions for (3) to be valid for all $n\in\mathbf{N}$, $x_1, \dots, x_n\in\mathbf{R}_+^k$. (See the Corollary below.) Then we give new and simpler proofs for theorems obtained by Daróczy and Losonczi [3] and by the author [7], [8]. We note that our results also include those of Reznick [9] and Ursell [11].

2. Main results

LEMMA. *Let $n \in \mathbf{N}$, $x_1, \dots, x_n \in \mathbf{R}^k_+$ be fixed. Then the function*

$$\varphi(a) := \ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i^a\right), \quad a \in \mathbf{R}^k$$

is convex.

PROOF. Since f is continuous, it is enough to show that it is Jensen-convex, i.e.

$$\varphi\left(\frac{a+b}{2}\right) \leqq \frac{\varphi(a)+\varphi(b)}{2}, \quad a, b \in \mathbf{R}^k.$$

This inequality is equivalent to

$$\sum_{i=1}^{n} x_i^{a/2} \cdot x_i^{b/2} \leqq \left(\sum_{i=1}^{n} x_i^a\right)^{1/2}\left(\sum_{i=1}^{n} x_i^b\right)^{1/2},$$

which is an easy consequence of the Cauchy—Bunyakovsky—Schwartz inequality.

The following theorem is the key result of this paper.

THEOREM 1. (Approximation theorem for convex functions.) *Let $f\colon \mathbf{R}^k \to \mathbf{R}$ be an arbitrary convex function with $f(0)=0$. Then, for all compact sets $D \subset \mathbf{R}^k$ and for all $\varepsilon > 0$, there exist $0<\alpha<1$, $n \in \mathbf{N}$, $x_1, \dots, x_n \in \mathbf{R}^k_+$ such that*

$$\left| f(a) - \alpha \ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i^a\right)\right| < \varepsilon \quad \text{for} \quad a \in D. \tag{4}$$

PROOF. Let $\varepsilon > 0$ and $a^* \in \mathbf{R}^k$ be arbitrarily fixed. Then, using the convexity of f and the Hahn—Banach theorem, we can find a vector $u(a^*) \in \mathbf{R}^k$ and a constant $v(a^*) \in \mathbf{R}$ such that

$$f(a) \geqq \langle u(a^*), a\rangle + v(a^*) \quad \text{for} \quad a \in \mathbf{R}^k \tag{5}$$

and

$$f(a^*) = \langle u(a^*), a^*\rangle + v(a^*).$$

(Here $\langle , \rangle$) denotes the inner product on $\mathbf{R}^k$.) Write

$$G(a^*) := \{a \in \mathbf{R}^k | \langle u(a^*), a\rangle + v(a^*) > f(a) - \varepsilon/4\}.$$

Clearly, $G(a^*)$ is a neighbourhood of a^*. Since D is compact, there exist $a_1, \dots, a_m \in \mathbf{R}^k$ such that

$$D \subset \bigcup_{i=1}^{m} G(a_i). \tag{6}$$

Write $a_0 = 0$ and

$$u_i = u(a_i), \quad v_i = v(a_i), \quad i = 0, 1, \dots, m.$$

Since $f(0)=0$, we have $v_0 = 0$ and $v_1, \dots, v_m \leqq 0$. Using these notations, (6) and (5) yield

$$\max_{0 \leqq i \leqq m} (\langle u_i, a\rangle + v_i) > f(a) - \varepsilon/4 \quad \text{for} \quad a \in D. \tag{7}$$

Putting $a^* = a_i$ $(i=0, 1, \dots, m)$ into (5) we also find that

$$(8) \qquad f(a) \geqq \max_{0 \leqq i \leqq m} (\langle u_i, a\rangle + v_i) \quad \text{for all } a \in \mathbf{R}^k.$$

Now let $w_0 = 0$ and $w_i \in [v_i - \varepsilon/4, v_i]$ $(i=1, \dots, m)$ be arbitrary. Then by (7) and (8) we have

$$|f(a) \geqq \max_{0 \leqq i \leqq m} (\langle u_i, a\rangle + w_i)| > f(a) - \varepsilon/2$$

i.e.

$$(9) \qquad |f(a) - \max_{0 \leqq i \leqq m} (\langle u_i, a\rangle + w_i)| < \varepsilon/2 \quad \text{for} \quad a \in D.$$

On the other hand, for $t > (m+1)^{4/\varepsilon}$, we have the following estimate

$$\left| \frac{\ln\left(\sum_{i=0}^{m} t^{\langle u_i, a\rangle + w_i} / \sum_{i=0}^{m} t^{w_i} \right)}{\ln t} - \max_{0 \leqq i \leqq m} (\langle u_i, a\rangle + w_i) \right| =$$

$$= \frac{1}{\ln t} \left| \ln \sum_{i=0}^{m} t^{\langle u_i, a\rangle + w_i - \max_j (\langle u_j, a\rangle + w_j)} - \ln \sum_{i=0}^{m} t^{w_i} \right| \leqq$$

$$\leqq \frac{1}{\ln t} \left(\ln \sum_{i=0}^{m} 1 + \ln \sum_{i=0}^{m} 1 \right) = \frac{\ln (m+1)^2}{\ln t} < \varepsilon/2.$$

Let $t_0 > \max((m+1)^{4/\varepsilon}, e)$ be fixed and write $\alpha = 1/\ln t_0$. Then the above estimate and (9) yield

$$(10) \qquad \left| f(a) - \alpha \ln \left(\frac{\sum_{i=0}^{m} t_0^{\langle u_i, a\rangle + w_i}}{\sum_{i=0}^{m} t_0^{w_i}} \right) \right| < \varepsilon \quad \text{for} \quad a \in D.$$

This inequality is satisfied for all $w_i \in [v_i - \varepsilon/4, v_i]$, $i=1, \dots, k$. Therefore we can find values $w_i \in [v_i - \varepsilon/4, v_i]$ such that $t_0^{w_i}$ is rational for $i=1, \dots, k$. ($t_0^{w_0} = t_0^0 = 1$ is also rational.) Choose $q \in \mathbf{N}$ so that $p_i = q t_0^{w_i}$ is an integer for all $i=0, \dots, k$. If $u_i = (u_{i1}, \dots, u_{ik})$ then write $y_i = (t_0^{u_{i1}}, \dots, t_0^{u_{ik}})$. Clearly, $y_i^a = t_0^{\langle u_i, a\rangle}$. Therefore

$$\Delta := \frac{\sum_{i=0}^{m} t_0^{\langle u_i, a\rangle + w_i}}{\sum_{i=0}^{m} t_0^{w_i}} = \frac{\sum_{i=0}^{m} q t_0^{w_i} \cdot t_0^{\langle u_i, a\rangle}}{\sum_{i=0}^{m} q t_0^{w_i}} = \frac{\sum_{i=0}^{m} p_i y_i^a}{\sum_{i=0}^{m} p_i}.$$

Let $n := \sum_{i=0}^{m} p_i$ and define $x_1, \dots, x_n \in \mathbf{R}_+^k$ by

$$(x_1, \dots, x_n) := (\underbrace{y_0, \dots, y_0}_{p_0}, \underbrace{y_1, \dots, y_1}_{p_1}, \dots, \underbrace{y_m, \dots, y_m}_{p_m}).$$

Then Δ can be rewritten as $\frac{1}{n} \sum_{i=1}^{n} x_i^a$. Putting this expression of Δ into (10), we see that (4) is satisfied.

THEOREM 2. *Let* $F\colon \mathbf{R}_+^{m+1}\to\mathbf{R}$ *be a continuous function satisfying the following condition:*

(C) *If for some* $t_0, \dots, t_m > 0$, $F(t_0, \dots, t_m) \geqq 0$ *then* $F(t_0^\alpha, \dots, t_m^\alpha) \geqq 0$ *for all* $0<\alpha<1$.

Let further $a_0, \dots, a_m \subseteq a_m \in \mathbf{R}^k$ *be given vectors. Then*

$$F\left(\frac{1}{n}\sum_{i=1}^n x_i^{a_0}, \dots, \frac{1}{n}\sum_{i=1}^n x_i^{a_m}\right) \geqq 0 \tag{11}$$

holds for all $n\in\mathbf{N}$, $x_1, \dots, x_n \in \mathbf{R}_+^k$ *if and only if*

$$F(e^{f(a_0)}, \dots, e^{f(a_m)}) \geqq 0 \tag{12}$$

is satisfied for all convex functions $f\colon \mathbf{R}^k\to\mathbf{R}$ *with* $f(0)=0$.

PROOF. If (12) is satisfied for all convex functions f with $f(0)=0$, then it is satisfied by

$$f(a) = \ln\left(\frac{1}{n}\sum_{i=1}^n x_i^a\right),$$

which is convex by the Lemma and obviously satisfies $f(0)=0$. Thus we obtain (11) and have proved the "if" statement.

We prove the "only if" part of the theorem indirectly. Suppose that (11) holds for all $n\in\mathbf{N}$, $x_1, \dots, x_n\in\mathbf{R}^k$ but there exists a convex function $f^*\colon \mathbf{R}^k\to\mathbf{R}$ with $f^*(0)=0$ such that (12) is not satisfied. Then, by the continuity of $G(u_0, \dots, u_m) = F(e^{u_0}, \dots, e^{u_m})$, we can find an $\varepsilon>0$ such that $|u_i - f^*(a_i)|<\varepsilon$ $(i=0, \dots, m)$ implies $F(e^{u_0}, \dots, e^{u_m})<0$. By Theorem 1, we can find $0<\alpha<1$, $n\in\mathbf{N}$, $x_1, \dots, x_n\in\mathbf{R}_+^k$ such that

$$\left|f^*(a_i) - \alpha\ln\left(\frac{1}{n}\sum_{j=1}^n x_j^{a_i}\right)\right| < \varepsilon$$

for $i=0, \dots, m$. Therefore we have

$$F\left(\left(\frac{1}{n}\sum_{j=1}^n x_j^{a_0}\right)^\alpha, \dots, \left(\frac{1}{n}\sum_{j=1}^n x_j^{a_m}\right)^\alpha\right) < 0.$$

By the properties of F, this inequality implies

$$F\left(\frac{1}{n}\sum_{j=1}^n x_j^{a_0}, \dots, \frac{1}{n}\sum_{j=1}^n x_j^{a_m}\right) < 0,$$

which is a contradiction. $\square$

COROLLARY. *Let* $a_0, \dots, a_m\in\mathbf{R}_+^k$ *and* $\alpha_0, \dots, \alpha_m\in\mathbf{R}$ *be given parameters. Then* (3) *holds for all* $n\in\mathbf{N}$, $x_1, \dots, x_n\in\mathbf{R}_+^k$ *if and only if*

$$0 \leqq \alpha_0 f(a_0) + \dots + \alpha_m f(a_m) \tag{13}$$

is satisfied for all convex functions f *with* $f(0)=0$.

PROOF. The function $F(t_0, \ldots, t_m) = t_0^{\alpha_0} \ldots t_m^{\alpha_m} - 1$ obviously satisfies the conditions of Theorem 2. Thus (3) is equivalent to (12) which reduces to (13). $\square$

REMARKS. (1) There are important functions other than that used in the proof of Corollary, which satisfy the condition (C) of Theorem 2, for instance $F(t_0, \ldots, t_m) := t_1^{\alpha_1} + \ldots + t_n^{\alpha_n} - t_0^{\alpha_0}$. (Indeed, if $t_i^{\alpha_i} \geqq t_0^{\alpha_0}$ for some i then $F(t_0^\alpha, t_1^\alpha, \ldots, t_m^\alpha) \geqq \geqq 0$ holds for $\alpha > 0$; if $t_i^{\alpha_i} < t_0^{\alpha_0}$ for all i then $F(t_0, t_1, \ldots, t_m) \geqq 0$ yields

$$1 \leqq \frac{t_1^{\alpha_1}}{t_0^{\alpha_0}} + \ldots + \frac{t_m^{\alpha_m}}{t_0^{\alpha_0}} \leqq \left(\frac{t_1^{\alpha_1}}{t_0^{\alpha_0}}\right)^\alpha + \ldots + \left(\frac{t_m^{\alpha_m}}{t_0^{\alpha_0}}\right)^\alpha \quad \text{for} \quad 0 < \alpha < 1.$$

Thus we obtain $F(t_0^\alpha, \ldots, t_m^\alpha) \geqq 0$.)

(2) In the investigation of the inequality (13) the determination of the set

$$H := \{(u_0, \ldots, u_m) | \exists f\colon \mathbf{R}^k \to \mathbf{R} \quad \text{convex with} \quad f(0) = 0$$
$$\text{such that } u_i = f(a_i),\ i = 0, \ldots, m\}$$

is a crucial point. This set turns out to be polyhedral, i.e. it is the intersection of a finite number of closed half spaces. Any polyhedral set is determined by its extreme points and extremal rays. (See Rockafellar [10].) Thus (13) holds for all convex f with $f(0) = 0$ if and only if

$$0 \leqq \alpha_0 u_0 + \ldots + \alpha_m u_m$$

for all extreme points and extreme rays $(u_0, \ldots, u_m)$ of H.

In the next section we give some applications of the Corollary.

3. Applications

First we consider the one dimensional case, i.e. when $k = 1$.

THEOREM 3 (cf. Theorem 1 of [8]). *Let $a_0, \ldots, a_m, \alpha_0, \ldots, \alpha_m \in \mathbf{R}$ be given parameters with $\alpha_0 + \ldots + \alpha_m = 0$. Then*

$$1 \leqq \Big(\sum_{i=1}^{n} x_i^{a_0}\Big)^{\alpha_0} \ldots \Big(\sum_{i=1}^{n} x_i^{a_m}\Big)^{\alpha_m} \tag{14}$$

holds for all $n \in \mathbf{N}$, $x_1, \ldots, x_n > 0$ if and only if

$$0 \leqq \alpha_0 |a_0 - a_i| + \ldots + \alpha_m |a_m - a_i| \tag{15}$$

is valid for $i = 0, \ldots, m$.

PROOF. Since $\alpha_0 + \ldots + \alpha_m = 0$, (14) is equivalent to

$$1 \leqq \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{a_0}\right)^{\alpha_0} \ldots \left(\frac{1}{n} \sum_{i=1}^{n} x_i^{a_m}\right)^{\alpha_m}.$$

By Corollary 1 this holds for all $n, x_1, \ldots, x_n$ if and only if

$$0 \leqq \alpha_0 f(a_0) + \ldots + \alpha_m f(a_m) \tag{16}$$

is satisfied for all convex $f: \mathbf{R}\to\mathbf{R}$ with $f(0)=0$. However, using $\alpha_0+\ldots+\alpha_m=0$ again, we see that the restriction $f(0)=0$ can be omitted, i.e. (14) is valid for all $n, x_1, \ldots, x_n$ if and only if (16) holds for all convex $f: \mathbf{R}\to\mathbf{R}$. Taking $f(a)=|a-a_i|$ we find that (15) is a necessary condition.

To prove the sufficiency, assume that (15) holds for $i=0, \ldots, m$ and let $f: \mathbf{R}\to\mathbf{R}$ be an arbitrary convex function. Without loss of generality we can assume that $a_0<a_1<\ldots<a_m$ and that f is linear on the intervals $(-\infty, a_1]$, $[a_2, a_3]$, $\ldots$, $[a_{m-2}, a_{m-1}]$, $[a_{m-1}, \infty)$. Then f can be expressed in the form

$$f(a) = p_0+p_1|a-a_1|+\ldots+p_{m-1}|a-a_{m-1}|+p_m a. \tag{17}$$

By the convexity of f, we have that the left derivative $f_-(a)$ is not greater than the right derivative $f_+(a)$ at any point $a\in\mathbf{R}$, calculating these derivatives at $a=$ $=a_1, \ldots, a_{m-1}$, we find that $p_1, \ldots, p_{m-1}\geqq 0$. Putting $i=0$ into (15) we have

$$0 \leqq \alpha_0(a_0-a_0)+\ldots+\alpha_m(a_m-a_0).$$

Since $\alpha_0+\ldots+\alpha_m=0$, this reduces to

$$0 \leqq \alpha_0 a_0+\ldots+\alpha_m a_m.$$

Similarly, letting $i=m$ in (15) we find

$$0 \geqq \alpha_0 a_0+\ldots+\alpha_m a_m.$$

Thus

$$0 = \alpha_0 a_0+\ldots+\alpha_m a_m. \tag{18}$$

Now we can prove (16). Using (17), we have

$$\sum_{j=0}^{m} \alpha_j f(a_j) = p_0 \sum_{j=0}^{m} \alpha_j + \sum_{i=1}^{m-1} p_i \sum_{j=0}^{m} \alpha_j |a_j-a_i|+p_m \sum_{j=0}^{m} \alpha_j a_j.$$

Applying (15) for $i=1, \ldots, m-1$, (18), $\alpha_0+\ldots+\alpha_m=0$ and $p_1, \ldots, p_{m-1}\geqq 0$ we can see that (16) holds true. □

To formulate a corollary of Theorem 3, introduce the following notation: For $a, b\in\mathbf{R}$ let

$$M_{a,b}(x) = M_{a,b}(x_1, \ldots, x_n) = \begin{cases} \left(\sum_{i=1}^{n} x_i^a \Big/ \sum_{i=1}^{n} x_i^b\right)^{1/(a-b)}, & \text{if } a \neq b, \\ \exp\left(\sum_{i=1}^{n} x_i^a \ln x_i \Big/ \sum_{i=1}^{n} x_i^a\right), & \text{if } a = b, \end{cases}$$

where $n\in\mathbf{N}$, $x=(x_1, \ldots, x_n)\in\mathbf{R}_+^n$ are arbitrary. These means are sometimes called Gini means [4]. Putting $b=0$, $M_{a,b}$ reduces to a power means [5]. The means $M_{a+1,a}$ were investigated by Beckenbach [1]. Several authors (e.g. Brenner [2], Gini [4] found sufficient conditions for the comparison inequality

$$M_{a_0,b_0}(x) \leqq M_{a_1,b_1}(x). \tag{19}$$

The following necessary and sufficient condition is due to Daróczy and Losonczi [3].)

THEOREM 4. *Let* $a_0, b_0, a_1, b_1 \in \mathbf{R}$. *Then* (19) *holds for all* $n \in \mathbf{N}$, $x \in \mathbf{R}^n_+$ *if and only if*

(20) $$\min(a_0, b_0) \leqq \min(a_1, b_1) \quad \textit{and} \quad \max(a_0, b_0) \leqq \max(a_1, b_1).$$

PROOF. We may assume that $a_0 \leqq b_0$, $a_1 \leqq b_1$ because $M_{a,b} \equiv M_{b,a}$. First we deal with the case $a_0 < b_0$, $a_1 < b_1$. Then (19) can be rewritten as

$$1 \leqq \Big(\sum_{i=1}^{n} x_i^{a_1}\Big)^{1/(a_1-b_1)} \cdot \Big(\sum_{i=1}^{n} x_i^{b_1}\Big)^{1/(b_1-a_1)} \cdot \Big(\sum_{i=1}^{n} x_i^{b_0}\Big)^{1/(a_0-b_0)} \cdot \Big(\sum_{i=1}^{n} x_i^{a_0}\Big)^{1/(b_0-a_0)}.$$

By Theorem 3, this inequality is satisfied for all $n \in \mathbf{N}$, $x_1, \ldots, x_n > 0$ if and only if

(21) $$0 \leqq \frac{|t-a_1|}{a_1-b_1} + \frac{|t-b_1|}{b_1-a_1} + \frac{|t-b_0|}{a_0-b_0} + \frac{|t-a_0|}{b_0-a_0}$$

holds for $t \in \{a_0, a_1, b_0, b_1\}$.

Putting $t = b_1$ into (21) we have

$$0 \leqq -1 + \frac{|b_1-b_0|}{a_0-b_0} + \frac{|b_1-a_0|}{b_0-a_0}$$

whence we obtain

$$|b_1-b_0| + |b_0-a_0| \leqq |b_1-a_0|.$$

However this relation is the opposite of the triangle inequality. Therefore, this holds with equality and thus $a_0 < b_0$ implies $b_1 \geqq b_0$.

Similarly, the substitution $t = a_0$, yields $a_0 \leqq a_1$. Thus the necessity of (20) is proved. The proof of the sufficiency is a trivial computation, so we omit it.

To extend the theorem to the case $a_0 = b_0$ or $a_1 = b_1$, observe that $(a, b) \to M_{a,b}(x)$ is a continuous function on $\mathbf{R}^2$ for all fixed x. Furthermore it is an increasing function of both variables. (Indeed, if $a < a'$, then, for all $b \in \mathbf{R}$, $\min(a, b) \leqq \min(a', b)$, $\max(a, b) \leqq \max(a', b)$ and this implies $M_{a,b}(x) \leqq M_{a',b}(x)$ for all $b \notin \{a, a'\}$. However, using the continuity, this hold true for $b = a$ and $b = a'$.)

Let $b_0 \leqq a_0$, $b_1 \leqq a_1$ and assume that $M_{a_0,b_0}(x) \leqq M_{a_1,b_1}(x)$ for all x. Then, using the monotonicity of $M_{a,b}$,we have $M_{a_0,b}(x) \leqq M_{a,b_1}(x)$ for all x and $b < b_0$, $a_1 < a$. Thus necessarily $b \leqq b_1$ and $a_0 \leqq a$. Taking the limits $b \to b_0 - 0$, $a \to a_1 + 0$ we obtain that $b_0 \leqq b_1$ and $a_0 \leqq a_1$, i.e., (20) is necessary.

To prove the converse, assume that $a_0 \leqq a_1$ and $b_0 \leqq b_1$. Then for all $b < b_0$, $a_1 < a$ we have $a_0 \leqq a$ and $b \leqq b_1$, whence we obtain $M_{a_0,b}(x) \leqq M_{a,b_1}(x)$ for all x. Taking the limits $b \to b_0 - 0$, $a \to a_1 + 0$, we find that $M_{a_0,b_0}(x) \leqq M_{a_1,b_1}(x)$, i.e., (20) is also sufficient. □

The next result is a generalization of Hölder's inequality for the $M_{a,b}$ means. If $x_i = (x_{i1}, \ldots, x_{in}) \in \mathbf{R}^n_+$, $i = 1, \ldots, m$ then write

$$x_1 \ldots x_m = (x_{11} \ldots x_{m1}, \ldots, x_{1n} \ldots x_{mn}).$$

THEOREM 5 (cf. Theorem 3 of [8]). *Let* $k \geqq 2$ *and let* $b_0 \leqq a_0, \ldots, b_k \leqq a_k$ *be given parameters. Then*

(22) $$M_{-a_0,-b_0}(x_1 \ldots x_k) \leqq M_{a_1,b_1}(x_1) \ldots M_{a_k,b_k}(x_k)$$

holds for all $n\in\mathbf{N}$, $x_1, \ldots, x_k\in\mathbf{R}_+^n$ *if and only if*

$$a_i \geqq 0 \quad \textit{and} \quad c_i+b_i \geqq 0 \tag{23}$$

holds for all $i=0, \ldots, k$, *where*

$$c_i := \begin{cases} \Big(\sum\limits_{\substack{j=0\\ j\neq 0}}^{k} 1/a_j\Big)^{-1}, & \textit{if } \prod\limits_{\substack{j=0\\ j\neq 0}}^{k} a_j \neq 0,\\ 0, & \textit{if } \prod\limits_{\substack{j=0\\ j\neq 0}}^{k} a_j = 0. \end{cases}$$

PROOF. First we prove the theorem assuming that $b_0<a_0\neq 0, \ldots, b_k<a_k\neq 0$. Then (22) can be rewritten as

$$\Big(\frac{1}{n}\sum_{i=1}^{n}(x_{1i}, \ldots, x_{ki})^{(-a_0, \ldots, -a_0)}\Big)^{1/(b_0-a_0)}\Big(\frac{1}{n}\sum_{i=1}^{n}(x_{1i}, \ldots, x_{ki})^{(-b_0, \ldots, -b_0)}\Big)^{1/(a_0-b_0)} \leqq$$

$$\leqq \prod_{j=1}^{k}\Big[\Big(\frac{1}{n}\sum_{i=1}^{n}(x_{1i}, \ldots, x_{ki})^{(0, \ldots, a_j, \ldots, 0)}\Big)^{1/(a_j-b_j)}\Big(\frac{1}{n}\sum_{i=1}^{n}(x_{1i}, \ldots, x_{ki})^{(0, \ldots, b_j, \ldots, 0)}\Big)^{1/(b_j-a_j)}\Big].$$

Applying the Corollary, this holds for all $n\in\mathbf{N}$, $(x_{1i}, \ldots, x_{ki})\in\mathbf{R}_+^n$, $i=1, \ldots, n$ if and only if

(24)

$$\frac{f(-a_0, \ldots, -a_0)}{b_0-a_0}+\frac{f(-b_0, \ldots, -b_0)}{a_0-b_0} \leqq$$

$$\leqq \sum_{j=1}^{k}\Big[\frac{f(0, \ldots, a_j, \ldots, 0)}{a_j-b_j}+\frac{f(0, \ldots, b_j, \ldots, 0)}{b_j-a_j}\Big]$$

is satisfied for all convex functions $f\colon \mathbf{R}^k\to\mathbf{R}$ with $f(0)=0$. Since constant functions satisfy (24) with equality, therefore the condition $f(0)=0$ can be omitted.

For the sake of convenience, write

$$A_0 := (-a_0, \ldots, -a_0), \quad B_0 := (-b_0, \ldots, -b_0),$$

$$A_j := (0, \ldots, a_j, \ldots, 0), \quad B_j := (0, \ldots, b_j, \ldots, 0), \quad j=1, \ldots, k.$$

Thus (24) is equivalent to

$$\sum_{j=0}^{k}\frac{f(B_j)}{a_j-b_j} \leqq \sum_{j=0}^{k}\frac{f(A_j)}{a_j-b_j}. \tag{25}$$

Now we have to show that (25) is valid for all convex functions f if and only if (23) is satisfied for all $i=0, \ldots, k$.

Assume first that (25) holds. Then we prove that B_i is in the convex hull of $\{A_0, \ldots, A_k\}$ for all i. If this were not so then, by the Hahn—Banach separation theorem, there would exist a linear functional $\Phi\colon \mathbf{R}^k \to \mathbf{R}$ and a constant c such that

(26) $$\Phi(A_j) \leqq c, \quad j = 0, \dots, k \quad \text{and} \quad \Phi(B_i) > c.$$

Define f by

$$f(u) = \max\big(0,\ \Phi(u) - c\big),\quad u \in \mathbf{R}^k.$$

Then f is obviously a nonnegative convex function, furthermore, by (26),

$$f(A_j) = 0,\ j = 0, \dots, k \quad \text{and} \quad f(B_i) > 0.$$

Putting this f into (25) we find that the right hand side vanishes, however the ith term on the left hand side is positive and the others are nonnegative. This contradiction shows that the B_i $(i=0, \dots, k)$ belong to the convex hull of $\{A_0, \dots, A_k\}$. This means that there are nonnegative values $\lambda_{ij} \geqq 0$ $(i, j=0, \dots, k)$ with $\sum_{j=0}^{k} \lambda_{ij} = 1$ such that

$$B_i = \sum_{j=0}^{k} \lambda_{ij} A_j \quad (i = 0, \dots, k).$$

For $i=0$, we have the following equations

$$-b_0 = \lambda_{0j} a_j - \lambda_{00} a_0 \ (j = 1, \dots, k).$$

Thus $\lambda_{0j} = (\lambda_{00} a_0 - b_0)/a_j$. Substituting this into $\sum_{j=0}^{k} \lambda_{0j} = 1$, we get

$$\lambda_{00} a_0 \Big(\sum_{j=0}^{k} 1/a_j\Big) = 1 + b_0 \Big(\sum_{j=1}^{k} 1/a_j\Big).$$

If $c := \sum_{j=0}^{k} 1/a_j$ were zero then

$$0 = 1 + b_0 \Big(\sum_{j=1}^{k} 1/a_j\Big) = 1 + b_0(-1/a_0) = (a_0 - b_0)/a_0,$$

which is a contradiction. Therefore

$$\lambda_{00} = \Big(1 + b_0 \Big(\sum_{j=1}^{k} 1/a_j\Big)\Big) a_0^{-1} c^{-1}.$$

Now a simple computation yields

$$\lambda_{0j} = \frac{\lambda_{00} a_0 - b_0}{a_j} = \frac{a_0 - b_0}{a_0 a_j c}, \quad j = 1, \dots, k.$$

In the cases $i = 1, \dots, k$ we have the following equations:

$$0 = \lambda_{ij} a_j - \lambda_{i0} a_0 \ (j \neq i) \quad \text{and} \quad b_i = \lambda_{ii} a_i - \lambda_{i0} a_0.$$

Therefore

$$\lambda_{ij} = \frac{\lambda_{i0} a_0}{a_j} \ (j \neq i) \quad \text{and} \quad \lambda_{ii} = \frac{b_i + \lambda_{i0} a_0}{a_i}.$$

Substituting these values into $\sum_{j=0}^{k} \lambda_{ij}=1$, we find that

$$\lambda_{0i} = \frac{a_i-b_i}{a_0 a_i c}.$$

Thus we obtain

$$\lambda_{ij} = \begin{cases} \dfrac{a_i-b_i}{a_i a_j c}, & \text{if} \quad i \neq j, \\ \Big(1+b_i\big(\sum\limits_{\substack{j=0 \\ j\neq i}} 1/a_j\big)\Big) a_i^{-1}c^{-1}, & \text{if} \quad i = j, \end{cases}$$

and we can observe that these formulae are valid for all $i, j=0, \ldots, k$.

The conditions $\lambda_{ij} \geqq 0$ $(i \neq j)$ imply $a_i a_j c > 0$ for all $i \neq j$. If c were negative then $a_0 a_1$, $a_0 a_2$, $a_1 a_2$ would be also be negative, however their product is positive. This contradiction shows that $c>0$. Thus the numbers $a_0, \ldots, a_k$ have the same sign. Since $c=\sum_{j=0}^{k} 1/a_j>0$ therefore $a_0, \ldots, a_k$ are positive. The inequalities $\lambda_{ii} \geqq 0$ yield

$$1+b_i\big(\sum_{\substack{j=0 \\ j\neq i}} 1/a_j\big) \geqq 0, \quad i = 0, \ldots, k.$$

Rearranging this inequality we get $b_i+c_i \geqq 0$. Thus we have proved that (23) is a necessary condition.

To prove the converse, let $f\colon \mathbf{R}^k \to \mathbf{R}$ be an arbitrary convex function. Then

$$f(B_i) = f\big(\sum_{j=0}^{k} \lambda_{ij} A_j\big) \leqq \sum_{j=0}^{k} \lambda_{ij} f(A_j).$$

Therefore

$$\sum_{i=0}^{k} \frac{f(B_i)}{a_i-b_i} \leqq \sum_{i=0}^{k} \sum_{j=0}^{k} \frac{\lambda_{ij} f(A_j)}{a_i-b_i} = \sum_{j=0}^{k} \Big(\sum_{i=0}^{k} \frac{\lambda_{ij}}{a_i-b_i}\Big) f(A_j). \tag{27}$$

A simple calculation yields

$$\sum_{i=0}^{k} \frac{\lambda_{ij}}{a_i-b_i} = \frac{1}{a_j-b_j}, \quad j = 0, \ldots, k,$$

hence (27) is equivalent to (25), which was to be proved.

Thus we have shown that Theorem 5 is valid if $b_0<a_0 \neq 0, \ldots, b_k<a_k \neq 0$. Let $a_0 \geqq b_0, \ldots, a_k \geqq b_k$ be arbitrary values. Assume that (22) holds for all indicated values. If $a_i < a_i' \neq 0$, then, by Theorem 4,

$$M_{-a_0', -b_0}(x) \leqq M_{-a_0, -b_0}(x) \quad \text{and} \quad M_{a_i, b_i}(x) \geqq M_{a_i', b_i}$$

is valid for all x. Therefore it follows from (22) that

$$M_{-a_0', -b_0}(x_1 \ldots x_k) \leqq M_{a_1', b_1}(x_1) \ldots M_{a_k', b_k}(x_k) \tag{28}$$

for all $n\in\mathbf{N}$, $x_1, \ldots, x_k \in \mathbf{R}^n_+$. However, then it is necessary that

$$(29) \qquad a_i' \geqq 0, \quad b_i + \Big(\sum_{\substack{j=0\\ j\neq 0}}^{k} 1/a_j'\Big)^{-1} \geqq 0, \quad i = 0, \ldots, k.$$

Now, taking the limits $a_i' \to a_i + 0$ $(i=0, \ldots, k)$, we find that (23) is a necessary condition.

To prove the converse, assume that (23) is valid. Then for all values $a_i' > a_i$ $(i=0, \ldots, k)$ we have (29). Thus (28) is also satisfied. Letting $a_i' \to a_i + 0$ we obtain (22).

The proof of the theorem is complete. $\square$

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ON ORTHOGONALITY SPACES ADMITTING NONTRIVIAL EVEN ORTHOGONALLY ADDITIVE MAPPINGS

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Introduction

In this paper we consider a real orthogonality space $(X, \perp)$ in the Rätz sense (cf. [5]) and orthogonally additive mappings $F: X \to Y$ with values in an arbitrary abelian group $(Y, +)$. The general odd solution consists of all additive functions, while the even orthogonally additive mappings are quadratic (cf. [5]). The main problem is to select the even solutions from the class of quadratic functions.

The only known example of ortogonality spaces admitting nonzero even solutions is a real inner product space. For the Birkhoff—James orthogonality on a normed linear space (cf. [3], [4]) this example has proved to be unique (cf. under regularity conditions e.g. [2], [7] and in general e.g. [8]), while in the abstract case it is known only that such an orthogonality $\perp$ is necessarily symmetric and unique, as J. Rätz showed recently (cf. [6]).

Our purpose with this note is to continue this investigations. We obtain also the additivity of $\perp$, and show that any even orthogonally additive mapping is constant on concentric "spheres" if $\dim X \geqq 3$ (see Section 3 below). This makes it possible to define a real valued positive homogenous and positive definite "quasi norm" on X having the property that the values of an even orthogonally additive mapping in points of equal "quasi norm", are the same. This structure is now very close to the inner product space. Namely, our main result states that if $\dim X \geqq 3$ and there exists a not identically zero even orthogonally additive mapping, then X is an inner product space with the well known solutions (see Section 4 below).

Throughout the paper, $\mathbf{R}$ and $\mathbf{R}_+$ denote the set of real and nonnegative real numbers, respectively. Also, $\dim X$ and $\operatorname{lin} V$ stand for the linear dimension of X and the linear hull of a subset $V \subset X$, respectively. The constant mapping with values c is denoted by $\underset{\sim}{c}$. Finally, we use 0 for the zero vector, the number zero and for the identity element of the group Y.

1. Preliminaries

DEFINITION 1.1. Let $(X, +)$ and $(Y, +)$ be abelian groups. The mappings $A, Q: X \to Y$ are said to be *additive* and *quadratic*, respectively, if they satisfy the Cauchy- and the Jordan—von Neumann functional equations:

$$A(x+y) = A(x)+A(y) \quad \text{for all } x, y \in X, \tag{1.1}$$

$$Q(x+y)+Q(x-y) = 2Q(x)+2Q(y) \quad \text{for all } x, y \in X. \tag{1.2}$$

LEMMA 1.2 (Aczél [1], Theorem 2). *For any abelian groups* $(X, +)$ *and* $(Y, +)$ *and any quadratic function* $Q: X \to Y$ *there exists a mapping* $B: X \times X \to Y$ *with the properties:*

(i) *symmetric:* $B(x, y) = B(y, x)$ *for all* $x, y \in X$;
(ii) *biadditive:* $B(x, y+z) = B(x, y) + B(x, z)$ *for all* $x, y, z \in X$;
(iii) *representative:* $4Q(x) = B(x, x)$ *for all* $x \in X$.

The mapping B is uniquely determined by Q and we call it the *biadditive representation* of $4Q$. Notice that when X is a real space, then

$$Q(z) = Q\left(2\frac{x}{2}\right) = 4Q\left(\frac{x}{2}\right) = B\left(\frac{x}{2}, \frac{x}{2}\right), \quad x \in X,$$

i.e. in this case every quadratic mapping has a biadditive representation, namely $B': X \times X \to Y$ defined by $B'(z, y) = B(x/2, y/2, \ z, y \in X$.

DEFINITION 1.3 (Rätz [5], Definition 1). Let X be a real vector space of dimension $\geqq 2$ and $\perp$ a binary relation on X with the properties:

(O1) *total for zero:* $x \perp 0, 0 \perp x$ for every $x \in X$;
(O2) *independent:* if $x, y \in X \setminus \{0\}$, $x \perp y$, then x and y are linearly independent;
(O3) *homogeneous:* if $x, y \in X$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbf{R}$;
(O4′) *thalesian:* if P is a 2-dimensional linear subspace of X, $x \in P$ and $\lambda \in \mathbf{R}_+$, then there exists $y \in P$ such that $x \perp y$ and $(x+y) \perp (\lambda x - y)$.

Then $\perp$ is said to be an *orthogonality relation* on X and $(X, \perp)$ is called an *orthogonality space*.

EXAMPLES 1.4 (Rätz [5], Examples A, B, C). Each of the following relations satisfy the axioms (O1)—(O4′) of orthogonality:

(i) The "trivial" orthogonality on X, defined by (O1) and for $x, y \in X \setminus \{0\}$, $x \perp y \Leftrightarrow x, y$ are linearly independent.
(ii) The ordinary orthogonality on the inner product space $(X, \langle \cdot, \cdot \rangle)$ defined by $x \perp y \Leftrightarrow \langle x, y \rangle = 0$. In this case we refer to $(X, \perp)$ as an inner product space.
(iii) The Birkhoff—James orthogonality on the normed vector space $(X, \| \cdot \|)$ given by $x \perp y \Leftrightarrow \|x + \lambda y\| \geqq \|x\|$ for all $\lambda \in \mathbf{R}$.

DEFNITION 1.5 (Rätz [5], Definition 2). Let $(X, \perp)$ be an orthogonality space, $(Y, +)$ an abelian group. A mapping $F: X \to Y$ is called *orthogonally additive* if it satisfies the conditional Cauchy functional equation:

$$F(x+y) = F(x) + F(y) \quad \text{for all } x, y \in X \text{ with } x \perp y. \tag{1.3}$$

Further in this paper X and Y will denote an orthogonality space and an abelian group, respectively. Introduce the following notations:

$$\begin{aligned}
\mathrm{Hom}\,(X, Y) &= \{A\colon X \to Y \mid A \text{ is additive}\},\\
\mathrm{Quad}\,(X, Y) &= \{Q\colon X \to Y \mid Q \text{ is quadratic}\},\\
\mathrm{Hom}_{\perp}(X, Y) &= \{F\colon X \to Y \mid F \text{ is orthogonally additive}\},\\
\text{(o) } \mathrm{Hom}_{\perp}(X, Y) &= \{D\colon X \to Y \mid D \text{ is odd orthogonally additive}\},\\
\text{(e) } \mathrm{Hom}_{\perp}(X, Y) &= \{E\colon X \to Y \mid E \text{ is even orthogonally additive}\}.
\end{aligned}$$

THEOREM 1.6 (Rätz [5], Theorems 5, 6; Szabó [8], Theorem 1.8). *For any orthogonality space* $(X, \perp)$ *and any abelian group* $(Y, +)$, *we have*

(i) (o) $\mathrm{Hom}_{\perp}(X, Y) = \mathrm{Hom}\,(X, Y)$;
(ii) (e) $\mathrm{Hom}_{\perp}(X, Y) \subset \mathrm{Quad}\,(X, Y)$;
(iii) $\mathrm{Hom}_{\perp}(X, Y) = \mathrm{Hom}\,(X, Y) \Leftrightarrow$ (e) $\mathrm{Hom}_{\perp}(X, Y) = \{\underline{0}\}$.

EXAMPLE 1.7 (Rätz [5], Theorems 9, 16; Szabó [8], Corollary 1.7). If $(X, \|\,.\|)$ is a real normed linear space of dimension $\geqq 2$ with the Birkhoff—James orthogonality $\perp$, then (e) $\mathrm{Hom}_{\perp}(X, Y) \neq \{\underline{0}\} \Rightarrow X$ is an inner product space, and in this latter case we have

$$E \in \text{(e) } \mathrm{Hom}_{\perp}(X, Y) \Leftrightarrow E(x) = a(\|x\|^2) \quad \text{for all } x \in X$$

with some $a \in \mathrm{Hom}\,(\mathbf{R}, Y)$.

THEOREM 1.8 (Rätz [6], Theorem 2.3). *For any orthogonality space* $(X, \perp)$ *and any abelian group* $(Y, +)$ *such that* (e) $\mathrm{Hom}_{\perp}(X, Y) \neq \{\underline{0}\}$, *we have*

(i) $\perp$ *is symmetric;*
(ii) $\perp$ *is right unique.*

2. Basic properties of orthogonality and mappings

The following properties of orthogonality will play an important role in the rest of the paper.

DEFINITION 2.1. Let L be a real vector space of dimension $\geqq 2$. The binary relation $\vdash$ on L is said to be

(i) *symmetric,* if $x, y \in L$, $x \vdash y \Rightarrow y \vdash x$;
(ii) *right dense,* if for any 2-dimensional linear subspace $P \subset L$ and any $x \in P \setminus \{0\}$ there exists $y \in P \setminus \{0\}$ with $x \vdash y$;
(iii) *rigth additive,* if $x, y, z \in L$, $x \vdash y$, $x \vdash z \Rightarrow x \vdash (y+z)$;
(iv) *right homogeneous,* if $x, y \in L$, $x \vdash y \Rightarrow x \vdash \beta y$ for all $\beta \in \mathbf{R}$;
(v) *right projective,* if $x, y \in L \Rightarrow$ there is $\alpha \in \mathbf{R}$ with $x \vdash (y - \alpha x)$;
(vi) *right unique,* if $x, y \in L$, $x \neq 0 \Rightarrow$ there exists at most one $\alpha \in \mathbf{R}$ for which $x \vdash (y - \alpha x)$;

REMARK 2.2. In a similar way, one can define the corresponding "left sided" properties of the relation $\vdash$. In the "two sided" case the attributes "left" and "right" are omitted.

It will be fundamental in our considerations the following

LEMMA 2.3. *Suppose that L is a real vector space of dimension* $\geqq 2$ *and* $\vdash$ *is a right projective binary relation on L. If* $\vdash' \supset \vdash$ *is a right unique relation on L, then* $\vdash' = \vdash$.

PROOF. Let $x, y \in L$, $x \vdash' y = (y - 0x)$. Because of the right projectivity of $\vdash$, there exists $\alpha \in \mathbf{R}$ such that $x \vdash (y - \alpha x)$. Hence by the inclusion $\vdash' \supset \vdash$, we have also $x \vdash' (y - \alpha x)$. Now if $x = 0$, then $x = 0 \vdash (y - \alpha 0) = y$, else regarding the right uniqueness of $\vdash'$, $\alpha = 0$, i.e. also $x \vdash y$. Thus $\vdash' \subset \vdash$ holds, proving the lemma.

LEMMA 2.4. *For any orthogonality space* $(X, \perp)$, *we have*

(i) $\perp$ *is right dense and, what is more, for any 2-dimensional linear subspace* $P \subset X$ *and* $u \in P \setminus \{0\}$ *there exists* $v \in P \setminus \{0\}$ *such that* $u \perp v$ *and* $(u+v) \perp (u-v)$;

(ii) $\perp$ *is right projective.*

PROOF. (i) If P is a 2-dimensional linear subspace of X and $u \in P \setminus \{0\}$, then using axiom (O4′) for $x = u$ and $\lambda = 1$, we get $v \in P$ such that $u \perp v$ and $(u+v) \perp (u-v)$. Here $v = 0$ would imply $u \perp u$ that contradicts (O2).

(ii) Let $x, y \in X$. For $x = 0$ we have by (O1) that $x = 0 \perp y = (y - 1x)$. Otherwise, for $x \neq 0$, take a 2-dimensional linear subspace $x, y \in P \subset X$. The first part of the proof assures the existence of $v \in P \setminus \{0\}$ with $x \perp v$, whence by (O2) x and v are linearly independent. Since $y \in P = \operatorname{lin}\{x, v\}$, we have $y = \alpha x + \beta v$ for some, $\alpha, \beta \in \mathbf{R}$ and so by (O3) $x \perp \beta v = (y - \alpha x)$.

LEMMA 2.5. *Suppose the orthogonality* $\perp$ *on* X *is right unique. If* $u, v \in X \setminus \{0\}$ *such that* $u \perp v$, *then*

(i) *for any* $y \in \operatorname{lin}\{u, v\}$, $u \perp y$ *we have* $y = \beta v$ *with some* $\beta \in \mathbf{R}$;

(ii) *for any* $\lambda \in \mathbf{R}$ *there exists* $\mu \in \mathbf{R}$ *with* $(u + \mu v) \perp (\lambda u - \mu v)$.

PROOF. (i) Let $y \in \operatorname{lin}\{u, v\}$, $u \perp y = (y - 0u)$. Then $y = \alpha u + \beta v$, whence $u \perp \beta v = (y - \alpha u)$ and because of the right uniqueness of $\perp$, we have $\alpha = 0$, i.e. $y = \beta v$.

(ii) With respect to axiom (O2), we have $\dim(\operatorname{lin}\{u, v\}) = 2$, thus by (O4′) there exists $y \in \operatorname{lin}\{u, v\}$ such that $u \perp y$ and $(u+y) \perp (\lambda u - y)$. Finally part (i) completes the proof.

LEMMA 2.6. *Let* $Q \in \operatorname{Quad}(X, Y)$ *and let* B *be the biadditive representation of* $E = 4Q$. *Then* E *is even and*

(i) $E \in \operatorname{Hom}_\perp(X, Y) \Leftrightarrow 2B(x, y) = 0$ *for all* $x, y \in X$, $x \perp y$;

(ii) $E \in \operatorname{Hom}_\perp(X, Y) \Rightarrow E(\lambda u) = E(\lambda v)$ *for any* $u, v \in X$ *such that* $(u+v) \perp (u-v)$ *and every* $\lambda \in \mathbf{R}$;

(iii) $E(\lambda u) = E(\lambda v)$ *for some* $u, v \in X$ *and every* $\lambda \in \mathbf{R} \Rightarrow B(\lambda u, \mu u) = B(\lambda v, \mu v)$ *for all* $\lambda, \mu \in \mathbf{R}$.

PROOF. (i) For all $x, y \in X$ we have

$$E(x+y) = B(x+y, x+y) = B(x, x) + B(x, y) + B(y, x) + B(y, y) = \\ = E(x) + E(y) + 2B(x, y).$$

(ii) Since $E \in (\mathrm{e})\operatorname{Hom}_\perp(X, Y)$, thus for any $\lambda \in \mathbf{R}$ we get

$$E(\lambda u) = E\left(\frac{\lambda}{2}[u+v] + \frac{\lambda}{2}[u-v]\right) = E\left(\frac{\lambda}{2}[u+v]\right) + E\left(\frac{\lambda}{2}[u-v]\right) =$$

$$= E\left(\frac{\lambda}{2}[u+v]\right) + E\left(-\frac{\lambda}{2}[u-v]\right) = E\left(\frac{\lambda}{2}[u+v] - \frac{\lambda}{2}[u-v]\right) = E(\lambda v).$$

(iii) For any $\lambda, \mu \in \mathbf{R}$ we have

$$B(\lambda u, \mu u) = B\left(\frac{\lambda+\mu}{2}u, \frac{\lambda+\mu}{2}u\right) - B\left(\frac{\lambda-\mu}{2}u, \frac{\lambda-\mu}{2}u\right) =$$

$$= E\left(\frac{\lambda+\mu}{2}u\right) - E\left(\frac{\lambda-\mu}{2}u\right) = E\left(\frac{\lambda+\mu}{2}v\right) - E\left(\frac{\lambda-\mu}{2}v\right) =$$

$$= B\left(\frac{\lambda+\mu}{2}v, \frac{\lambda+\mu}{2}v\right) - B\left(\frac{\lambda-\mu}{2}v, \frac{\lambda-\mu}{2}v\right) = B(\lambda v, \mu v).$$

3. Orthogonality spaces with (e) $\mathrm{Hom}_\perp(X, Y) \neq \{\underline{0}\}$

LEMMA 3.1. *If* $E \in$ (e) $\mathrm{Hom}_\perp(X, Y) \setminus \{\underline{0}\}$, *then* $2E \in$ (e) $\mathrm{Hom}_\perp(X, Y) \setminus \underline{0}\}$ *holds, too.*

PROOF. Of course $2E$ is even and orthogonally additive. If it happened $2E = \underline{0}$, then for all $x \in X$ we have $E(x) + E(x) = 0$, i.e. $E(-x) = E(x) = -E(x)$. Thus E is odd and so by Theorem 1.6, part (i), E is additive. This means for any $x \in X$ that

$$E(x) = E(x/2 + x/2) = E(x/2) + E(x/2) = 2E(x/2) = 0$$

i.e. $E = \underline{0}$, which is a contradiction.

LEMMA 3.2. *Suppose B is the biadditive representation of a mapping $E \in$ (e)$\mathrm{Hom}_\perp(X, Y) \setminus \{\underline{0}\}$. If $x, y \in X$ are linearly independent and $2B(\lambda x, \mu y) = 0$ for all $\lambda, \mu \in \mathbf{R}$, then $x \perp y$.*

PROOF. By Lemma 2.4, part (ii), $\perp$ is right projective. Now let us define a new relation on X by

$$\perp' = \perp \cup \{(\lambda x, \mu y) | \lambda, \mu \in \mathbf{R}\}.$$

We have to show that $\perp' = \perp$. To do this, regarding the inclusion $\perp' \supset \perp$ and Lemma 2.3, we need only the right uniqueness of $\perp'$.

First observe that $\perp'$ is an orthogonality relation on X. Indeed, all of the axioms, but (O2), are satisfied trivially, while (O2) can be verified as follows: for $x', y' \in X \setminus \{0\}$, $x' \perp' y'$, we have

— either $x' \perp y'$ and so due to (O2) for $\perp$, x' and y' are linearly independent;

— or $x' = \lambda x$ and $y' = \mu y$ with some $\lambda, \mu \in \mathbf{R} \setminus \{0\}$, whence by our hypothesis, x' and y' are linearly independent, too.

Next we show that the mapping $E \neq 0$ is also $\perp'$-orthogonally additive. Indeed for $x', y' \in X$, $x' \perp y'$ we have

— either $x' \perp y'$, which implies $E(x' + y') = E(x') + E(y')$;

— or $x' = \lambda x$ and $y' = \mu y$ with some $\lambda, \mu \in \mathbf{R}$, whence $2B(x', y') = 2B(\lambda x, \mu y) = 0$ and by the proof of Lemma 2.6, part (i), $E(x' + y') = E(x') + E(y')$. Thus we have proved that (e) $\mathrm{Hom}_{\perp'}(X, Y) \neq \{0\}$.

Finally, we can refer to Theorem 1.8, part (ii), to obtain the right uniqueness of $\perp'$. This completes the proof.

COROLLARY 3.3. *If* (e) $\mathrm{Hom}_\perp(X, Y) \neq \{\underline{0}\}$ *and $x, y \in X$ are such that $x \perp y$, $(\alpha x + y) \perp (\beta x - y)$ with some $\alpha, \beta \in \mathbf{R}$, then $(\alpha x - y) \perp (\beta x + y)$ holds as well.*

PROOF. Let $E\in(\mathrm{e})\,\mathrm{Hom}_\perp(X, Y)\setminus\{\underline{0}\}$ with a biadditive representation B (see Lemma 3.1 above). Suppose $x, y\in X$, $\alpha, \beta\in\mathbf{R}$ with $x\perp y$ and $(\alpha x+y)\perp(\beta x-y)$. Of course we may assume that $y\neq 0$ and so by axiom (O2) also $x\neq 0$. Thus x and y are linearly independent. Hence $(\alpha x\pm y)$, $(\beta x\mp y)\neq 0$, therefore using again (O2), $(\alpha x-y)$ and $(\beta x+y)$ are linearly independent, too.

On the other hand, regarding Lemma 2.6, part (i), we have for all $\lambda, \mu\in\mathbf{R}$

$$\begin{aligned}
&2B(\lambda[\alpha x-y], \mu[\beta x+y]) = \\
&= 2B(\lambda\alpha x, \mu\beta x)+2B(\lambda\alpha x, \mu y)-2B(\lambda y, \mu\beta x)-2B(\lambda y, \mu y) = \\
&= 2B(\lambda\alpha x, \mu\beta x)-2B(\lambda\alpha x, \mu y)+2B(\lambda y, \mu\beta x)-2B(\lambda y, \mu y) = \\
&= 2B(\lambda[\alpha x+y], \mu[\beta x-y]) = 0.
\end{aligned}$$

Then Lemma 3.2 completes the proof.

COROLLARY 3.4. *If* $(e)\,\mathrm{Hom}_\perp(X, Y)\neq\{\underline{0}\}$, *then the orthogonality* $\perp$ *is (right) additive.*

PROOF. Let B be the biadditive representation of a mapping $E\in(e)\,\mathrm{Hom}_\perp(X, Y)\setminus\setminus\{0\}$. Now suppose $x, y, z\in X$ are such that $x\perp y$ and $x\perp z$. We may and do assume that $x, y, z, y+z\neq 0$. Then x and $y+z$ are linearly independent. Indeed, for if $y+z=\alpha x$, $\alpha\in\mathbf{R}$, we would have $y-\alpha x=-z$, i.e. $x\perp(-z)=(y-\alpha x)$ and $x\perp y=(y-0x)$ simultaneously. Thus the right uniqueness of $\perp$ gives $\alpha=0$ that contradicts $y+z\neq 0$.

On the other hand, for all $\lambda, \mu\in\mathbf{R}$, Lemma 2.6, part (i), implies

$$2B(\lambda x, \mu[y+z]) = 2B(\lambda x, \mu y)+2B(\lambda x, \mu z) = 0.$$

Finally Lemma 3.2 completes the proof.

THEOREM 3.5. *If* $\dim X\geqq 3$ *and* $(\mathrm{e})\,\mathrm{Hom}_\perp(X, Y)\neq\{\underline{0}\}$, *then there exists a functional* $\varrho\colon X\to\mathbf{R}$ *with the following properties:*

(i) $\varrho(0)=0$, $\varrho(x)>0$ *for all* $x\in X\setminus\{0\}$;
(ii) $\varrho(\lambda x)=|\lambda|\varrho(x)$ *for all* $x\in X$ *and* $\lambda\in\mathbf{R}$;
(iii) $E(x)=E(y)$ *for any* $E\in(\mathrm{e})\,\mathrm{Hom}_\perp(X, Y)$ *and every* $x, y\in X$ *such that* $\varrho(x)=\varrho(y)$.

PROOF. Remember that under our hypothesis, the orthogonality $\perp$ is symmetric, uniquely projective and additive, besides (O1)—(O4′).

Let $u\in X\setminus\{0\}$ be arbitrarily fixed. First we show that for any $s\in X\setminus\mathrm{lin}\,\{u\}$, there are $z\in X\setminus\{0\}$ and $\sigma\in\mathbf{R}\setminus\{0\}$ such that

$$u\perp z,\ (u+z)\perp(u-z) \quad\text{and}\quad z\perp s,\ (z+\sigma s)\perp(z-\sigma s) \tag{3.1}$$

whence by Lemma 2.6, part (ii), for any $E\in(\mathrm{e})\,\mathrm{Hom}_\perp(X, Y)$ and $\xi\in\mathbf{R}$ it follows that

$$E(\xi u) = E(\xi z) = E(\xi[\sigma s]). \tag{3.2}$$

By the assumption $\dim X\geqq 3$, we can choose $t\in X\setminus\mathrm{lin}\,\{u, s\}$. Using Lemma 2.4, part (i), we obtain

$$-v \in \operatorname{lin}\{u, s\} \setminus \{0\} \text{ such that } u \perp v \text{ and } (u+v) \perp (u-v);$$
$$-w \in \operatorname{lin}\{u, t\} \setminus \{0\} \text{ such that } u \perp w \text{ and } (u+w) \perp (u-w);$$
$$-z \in \operatorname{lin}\{v, w\} \setminus \{0\} \text{ such that } v \perp z \text{ and } (v+z) \perp (v-z).$$

Since $z=\alpha v+\beta w$ for some $\alpha, \beta \in \mathbf{R}$ and the orthogonality is linear, therefore $u \perp z$. Similarly, $s=\gamma u+\delta v$ implies $z \perp s$. Next we show that $(u+z) \perp (u-z)$. For this purpose choose B to be the biadditive representation of a mapping $E \in (\mathrm{e}) \operatorname{Hom}_{\perp}(X, Y) \setminus \{\underline{0}\}$. By axiom (O2), u and z are linearly independent and hence so are $u+z$ and $u-z$. Furthermore, for all $\lambda, \mu \in \mathbf{R}$, using Lemma 2.6, we gain

$$2B(\lambda[u+z], \mu[u-z]) = 2B(\lambda u, \mu u) - 2B(\lambda u, \mu z) +$$
$$+2B(\lambda z, \mu u) - 2B(\lambda z, \mu z) = 2B(\lambda v, \mu v) - 2B(\lambda v, \mu v) = 0.$$

Thus Lemma 3.2 assures $(u+z) \perp (u-z)$. Finally by Lemma 2.5, part (ii) with $\lambda=1$, there exists a scalar $\sigma \in \mathbf{R} \setminus \{0\}$ such that $(z+\sigma s) \perp (z-\sigma s)$, proving (3.1).

Now we are able to define the functional ϱ. The axiom of choice enables us to select a homogeneous basis S from X, i.e. $S \subset X$ such that for any $x \in X \setminus \{0\}$ there is a unique $s \in S$ and $\xi \in \mathbf{R} \setminus \{0\}$ with $x=\xi s$. By the first part of the proof, we can fix for each $s \in S$ a scalar $\sigma_s \in \mathbf{R} \setminus \{0\}$ satisfying (3.2) (if $s \in \operatorname{lin}\{u\}$, then let σ_s be such that $\sigma_s s=u$). Since $S'=\{\sigma_s s \mid s \in S\}$ is also a homogeneous basis, we may assume that $\sigma_s=1$ for all $s \in S$. Let then $\varrho\colon X \to \mathbf{R}$ be well defined by

$$\varrho(x) = \begin{cases} 0 & \text{if } x = 0 \\ |\xi| & \text{if } x = \xi s,\ s \in S,\ \xi \in \mathbf{R} \setminus \{0\}. \end{cases}$$

The desired properties of ϱ can be verified as follows:

(i) Obvious.

(ii) Let $x \in X$ and $\lambda \in \mathbf{R}$. The case $x=0$ is trivial, while for $x \neq 0$ there is a unique $s \in S$ and $\xi \in \mathbf{R} \setminus \{0\}$ such that $x=\xi s$ and so $\lambda x=(\lambda\xi)s$, whence by definition

$$\varrho(\lambda x) = |\lambda\xi| = |\lambda|\,|\xi| = |\lambda|\varrho(x).$$

(iii) Let $E \in (\mathrm{e}) \operatorname{Hom}_{\perp}(X, Y)$ and $x, y \in X$ with $\varrho(x)=\varrho(y)$. Then either $x=y=0$, when $E(x)=0=E(y)$, or $x, y \neq 0$, when $x=\xi s$ and $y=\eta t$ for some unique $s, t \in S$ and $\xi, \eta \in \mathbf{R} \setminus \{0\}$ with $|\xi|=|\eta|$. Hence

$$E(x) = E(\xi s) = E(\xi u) = E(|\xi| u) = E(|\eta| u) = E(\eta u) = E(\eta t) = E(y).$$

4. Inner product on orthogonality spaces

LEMMA 4.1. *Let L be a real vector space of dimension $\geqq 2$ and suppose that $\vdash$ is a binary relation on L having the properties:*

(P1) *right additive;*

(P2) *right homogeneous;*

(P3) *right uniquely projective;*

(P4) *symmetric.*

If $\varphi: L\to\mathbf{R}$ *is a functional with*
(N1) $\varphi(0)=0,\ \ \varphi(x)>0$ *for all* $x\in L\setminus\{0\}$;
(N2) $\varphi(\lambda x)=|\lambda|\varphi(x)$ *for all* $x\in L,\ \lambda\in\mathbf{R}$;
(PN) $u, v\in X,\ \ \varphi(u)=\varphi(v)=1$ *with* $u\vdash(v-\alpha u)\Rightarrow(u-\alpha v)\vdash v$,
then there exists an equivalent inner product $\langle\,.,\,.\,\rangle: L\times L\to\mathbf{R}$, *i.e.*
(i) $x\vdash y\Leftrightarrow\langle x, y\rangle=0$ *for all* $x, y\in L$;
(ii) $\varphi^2(x)=\langle x, x\rangle$ *for all* $x\in L$.

PROOF. Let $\langle\cdot,\ \cdot\rangle: L\times L\to\mathbf{R}$ be well defined by

$$\langle x, y\rangle=\alpha\varphi^2(x)\quad\text{for all } x, y\in L\quad\text{with}\quad x\vdash(y-\alpha x).$$

We show that $\langle\cdot,\ \cdot\rangle$ is an inner product on L:

Right additive: Let $x, y, z\in L$ and $\alpha,\beta\in\mathbf{R}$ be such that $x\vdash(y-\alpha x)$ and $x\vdash(z-\beta x)$. Then, regarding the right additivity of $\vdash$, we have $x\vdash([y+z]-[\alpha+\beta]x)$. This means that

$$\langle x, y+z\rangle=[\alpha+\beta]\varphi^2(x)=\alpha\varphi^2(x)+\beta\varphi^2(x)=\langle x, y\rangle+\langle x, z\rangle.$$

Right homogeneous: Let $x, y\in L$ and $\beta\in\mathbf{R}$. If $x\vdash(y-\alpha x)$ for some $\alpha\in\mathbf{R}$, then by the right homogeneity of $\vdash$, we have $x\vdash(\beta y-\beta\alpha x)$, i.e.

$$\langle x, \beta y\rangle=\beta\alpha\varphi^2(x)=\beta[\alpha\varphi^2(x)]=\beta\langle x, y\rangle.$$

Positive definite: If $x\in L\setminus\{0\}$, then there exists $\gamma\in\mathbf{R}$ such that $x\vdash(x-\gamma x)$ and by the right homogeneity of $\vdash$, it follows that $x\vdash 0(x-\gamma x)=(x-1x)$. Thus, regarding the right uniqueness of $\vdash$, we have $\gamma=1$ and so

$$\langle x, x\rangle=\gamma\varphi^2(x)=\varphi^2(x)>0,$$

proving also the assertion (ii).

Symmetric: Let $x, y\in L$ and $x\vdash(y-\alpha x)$. We may and do assume that $x, y\neq 0$. Then regarding the homogeneity of $\vdash$, we obtain

$$\frac{x}{\varphi(x)}\vdash\left(\frac{y}{\varphi(y)}-\frac{\alpha\varphi(x)}{\varphi(y)}\frac{x}{\varphi(x)}\right).$$

Hence, using property (PN), we have

$$\left(\frac{x}{\varphi(x)}-\frac{\alpha\varphi(x)}{\varphi(y)}\frac{y}{\varphi(y)}\right)\vdash\frac{y}{\varphi(y)}$$

and therefore, regarding once more the homogeneity and symmetry,

$$y\vdash\left(x-\frac{\alpha\varphi^2(x)}{\varphi^2(y)}y\right).$$

Thus

$$\langle x, y\rangle=\alpha\varphi^2(x)=\frac{\alpha\varphi^2(x)}{\varphi^2(y)}\varphi^2(y)=\langle y, x\rangle.$$

Finally, for $x=0$ or $y=0$ it holds clearly $\langle x, y\rangle=0$ and $x\vdash y$, while for $x, y\in\setminus\{0\}$ the definition implies

$$0=\langle x, y\rangle=\alpha\varphi^2(x)\Leftrightarrow\alpha=0\Leftrightarrow x\vdash y,$$

where $\alpha\in\mathbf{R}$ is the unique scalar for which $x\vdash(y-\alpha x)$.

THEOREM 4.2. (Main result.) *If* $\dim X\geqq 3$ *and* $(\mathrm{e})\,\mathrm{Hom}_\perp(X, Y)\neq\{\underline{0}\}$, *then there exists an equivalent inner product* $\langle\cdot,\cdot\rangle\colon X\times Y\to\mathbf{R}$, *i.e.* $x\perp y\Leftrightarrow\langle x, y\rangle=0$ *for all* $x, y\in X$, *and*

$$(\mathrm{e})\,\mathrm{Hom}_\perp(X, Y)=\{a\circ\|\cdot\|^2|a\in\mathrm{Hom}(\mathbf{R}, Y)\}.$$

PROOF. Let $E\in(\mathrm{e})\,\mathrm{Hom}_\perp(X, Y)\setminus\{\underline{0}\}$ be fixed with ist biadditive representation B. Under the hypothesis of our Theorem, the orthogonality $\perp$ is symmetric, additive, homogeneous and uniquely projective, i.e. properties (P1)—(P4) are satisfied. Also, by Theorem 3.5, we can define a functional $\varrho\colon X\to\mathbf{R}$ with properties (N1), (N2) and

$$E(x)=E(y)\quad\text{for all}\quad x, y\in X\text{ with }\quad\varrho(x)=\varrho(y).$$

Thus, with respect to the just proved lemma and Example 1.7, we have only to show that (PN) holds as well.

For this purpose let $u, v\in X$, $\varrho(u)=\varrho(v)=1$ and $u\perp(v-\alpha u)$ with some $\alpha\in\mathbf{R}$. We may assume that $\alpha\neq 0$ and $v-\alpha u\neq 0$, since otherwise (PN) would hold trivially (for instance $v=\alpha u\Rightarrow\varrho(v)=|\alpha|\varrho(u)\Rightarrow|\alpha|=1\Rightarrow\alpha v=u\Rightarrow u-\alpha v=0\perp v$). Hence by axiom (O2), u and v are linearly independent, i.e. $\dim P=2$ for $P=\mathrm{lin}\,\{u, v\}$.

Since $\varrho(\lambda u)=\varrho(\lambda v)$ and so $E(\lambda u)=E(\lambda v)$ for all $\lambda\in\mathbf{R}$, therefore Lemma 2.6 implies

$$0=2B(u, v-\alpha u)=2B(u, v)-2B(u, \alpha u)=2B(u, v)-2B(v, \alpha v)=$$
$$=2B(u, v)-2B(\alpha v, v)=2B(u-\alpha v, v).$$

Hence by the proof of Lemma 2.6, part (i), we obtain, using the notation $z=u-\alpha v+v\in P$, that

$$E(z)=E(z-v)+E(v). \tag{4.1}$$

Here $z\neq 0$ and, what is more, $\mathrm{lin}\,\{v, z\}=P$, since otherwise u and v would be linearly dependent.

Now choose a vector $w\in P$ with $\varrho(w)=1$ and $z\perp w$. Of course $\mathrm{lin}\,\{z, w\}=P$, thus $v=\zeta z+\omega w$ for some $\zeta, \omega\in\mathbf{R}$. We may assume that $\omega>0$, since $-\omega$ with $-w$ suit as well and $\omega=0$ would imply $v=\zeta z$ contradicting $\mathrm{lin}\,\{v, z\}=P$. Now the relation $z\perp w$ and formula (4.1) implies

$$[E(z-\zeta z)+E(-\omega w)]+[E(\zeta z)+E(\omega w)]= \tag{4.2}$$
$$=E(z-\zeta z-\omega w)+E(\zeta z+\omega w)=E(z-v)+E(v)=E(z).$$

Next we observe that due to axiom (O4′) and Corollary 3.3, there exists $y\in P$ such that $z\perp y$ and

$$(|\zeta|z\pm y)\perp(|1-\zeta|z\mp y)$$

(for $\zeta=0$ take $y=0$, otherwise apply (O4′) for $x=|\zeta|z$ and the scalar $\lambda=|1-\zeta|/|\zeta|$). There may be two cases:

Case $\zeta(1-\zeta)\geqq 0$. Then $(\zeta z\pm y)\perp([1-\zeta]z\mp y)$ follows. Because of the right uniqueness of $\perp$, Lemma 2.5, part (i), implies $y=\beta w$ for some $\beta\in\mathbf{R}$. Hence we obtain with $\mu=|\beta|\geqq 0$ the relation $(\zeta z+\mu w)\perp([1-\zeta]z-\mu w)$ and thus

$$(4.3)\qquad [E(z-\zeta z)+E(-\mu w)]+[E(\zeta z)+E(\mu w)] = \\ = E(z-\zeta z-\mu w)+E(\zeta z+\mu w) = E(z).$$

(4.2) and (4.3) give

$$(4.4)\qquad 2E(\omega w) = E(z)-E(z-\zeta z)-E(\zeta z) = 2E(\mu w).$$

Now we have only to prove that $\omega=\mu$, i.e.

$$u-\alpha v = z-v = ([1-\zeta]z-\omega w)\perp(\zeta z+\omega w) = v.$$

Namely, if e.g. $\omega>\mu\geqq 0$, then by the above argument we could choose $\gamma\in\mathbf{R}$ with $(\mu w+\gamma z)\perp([\omega-\mu]w-\gamma z)$. Hence

$$2E(\omega w) = 2E([\mu w+\gamma z]+[(\omega-\mu)w-\gamma z]) =$$

$$= 2E(\mu w+\gamma z)+2E([\omega-\mu]w-\gamma z) = 2E(\mu w)+2E(\gamma z)+2E([\omega-\mu]w-\gamma z)$$

and thus with respect to (4.4), we have

$$2E(\gamma z)+2E([\omega-\mu]w-\gamma z) = 0.$$

Finally, choosing a vector $s\in P$ with $s\perp([\omega-\mu]w+\gamma z)$, $\varrho(s)=\varrho(\gamma z)$, we obtain for $t=s+[\omega-\mu]w+\gamma z\neq 0$

$$2E(t) = 2E(s)+2E([\omega-\mu]w+\gamma z) = 0.$$

Using homogeneity properties, one can readily check that all of our considerations involved above are valid for vectors being a fixed scalar multiples of the original ones. Thus $2E\equiv 0$ on lin $\{t\}$, therefore Theorem 3.5 leads to the contradiction $2E=\underline{0}$.

In a similar way, one can exclude the possibility of $\mu>\omega>0$.

Case $\zeta(1-\zeta)<0$. We are going to derive a contradiction. Here $(\zeta z\pm y)\perp([\zeta-1]z\mp y)$ follows. By the above argument, we can take a scalar $\mu\geqq 0$ with $(\zeta z+\mu w)\perp([\zeta-1]z-\mu w)$. Then

$$(4.5)\qquad [E(z-\zeta z)+E(\mu w)]+[E(\zeta z)+E(\mu w)] = \\ = E(z-\zeta z+\mu w)+E(\zeta z+\mu w) = E(z+2\mu w) = \\ = E(z)+E(2\mu w) = E(z)+4E(\mu w)$$

whence with the aid of (4.2), we have

$$(4.6)\qquad 2E(\omega w) = E(z)-E(z-\zeta z)-E(\zeta z) = -2E(\mu w),$$

i.e. $2E(\omega w)+2E(\mu w)=0$. Let $v=\mu/\varrho(z)$. Then $\varrho(vz)=\mu=\varrho(\mu w)$ and therefore

$$2E(\omega w+vz) = 2E(\omega w)+2E(vz) = 0.$$

Thus, using the above mentioned homogeneity considerations, we have got $2E\equiv 0$ on lin $\{\omega w+vz\}$. Here, because of $\omega\neq 0$, also $\omega w+vz\neq 0$ holds, and so Theorem 3.5 implies that $2E=\underline{0}$, which is a contradiction.

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CONTENTS

1

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MULTIPLICATIVE SEMIGROUPS OF CONTINUOUS MAPPINGS

Á. CSÁSZÁR (Budapest), member of the Academy and E. THÜMMEL (Greifswald)

0. Introduction For a topological space X, we denote by $C(X)$ the set of all continuous real-valued functions on X, and by $C^*(X)$ the subset of $C(X)$ composed of bounded functions. Both $C(X)$ and $C^*(X)$ can be considered to be semigroups if the semigroup operation is the pointwise multiplication of functions.

A classical theorem [6] states that, if X_1, X_2 are compact Hausdorff spaces, then the isomorphy of the semigroups $C(X_1)$ and $C(X_2)$ implies the homeomorphy of X_1 and X_2. The paper [8] contains the statement that the same holds if X_1 and X_2 are realcompact spaces.

It is easy to deduce from these statements two slightly more general ones in which the conclusion is the homeomorphy of the Čech—Stone compactifications βX_1 and βX_2 and of the Hewitt realcompactifications υX_1 and υX_2, respectively (cf. [5]):

A. *Let X_1 and X_2 be Tikhonov spaces.*

(a) *If the semigroups $C^*(X_1)$ and $C^*(X_2)$ are isomorphic then βX_1 and βX_2 are homeomorphic;*

(b) *if $C(X_1)$ and $C(X_2)$ are isomorphic then υX_1 and υX_2 are homeomorphic.*

In some recent papers, generalizations of these statements can be found. The generalization goes in two directions: instead of $C(X)$ or $C^*(X)$, one considers semigroups composed of continuous mappings from X into suitable topological semigroups (not necessarily commutative), and semigroup isomorphy is replaced by weaker conditions.

In order to formulate the generalizations in question, let us recall the definitions of quasi-real and segment-like semigroups. In the following, $\mathbf{R}$, $[0, +\infty)$, $(0, +\infty)$, and $[0, 1]$ are considered to be topological semigroups equipped with the multiplication of real numbers and the Euclidean topology of $\mathbf{R}$ (or the subspace topology inherited from it, respectively).

(0.1) $\mathbf{S}$ is said to be a *quasi-real semigroup* [2] iff

(a) $\mathbf{S}$ is a topological semigroup containing $[0, +\infty)$ as a topological subsemigroup,

(b) 0 is a zero element and 1 is a unity element in $\mathbf{S}$,

(c) there is a continuous mapping $x \mapsto \frac{1}{x}$ from $\mathbf{S}-\{0\}$ into $\mathbf{S}$ such that $x\cdot\frac{1}{x}=\frac{1}{x}\cdot x=1$ for $x\in\mathbf{S}$, $x\neq 0$,

(d) there is a continuous homomorphism $x\mapsto|x|$ from $\mathbf{S}$ onto $[0, +\infty)$ such that $|x|=x$ for $x\in[0, +\infty)$,

(e) the sets $V_\varepsilon=\{x\in\mathbf{S}\colon |x|<\varepsilon\}$ $(\varepsilon>0)$ constitute a neighbourhood base of 0 in **S**.

(0.2) **S** is said to be a *segment-like semigroup* [3] iff

(a) **S** is a topological semigroup containing [0, 1] as a topological subsemigroup,

(b) 0 is a zero element and 1 is a unity element in **S**,

(c) there exists a continuous homomorphism $x\mapsto|x|$ from **S** into **R** such that $|x|=x$ for $x\in[0, 1]$,

(d) $|a|=0$ implies $a=0$ for $a\in\mathbf{S}$,

(e) $ab=a$ implies either $a=0$ or $b=1$ for $a, b\in\mathbf{S}$.

It is easily seen that every quasi-real semigroup is segment-like (cf. also 5.3).

We introduce the concept of a *weakly segment-like semigroup;* by this, we understand a semigroup fulfilling (0.2) (a)—(d). By [3], (2.9), a weakly segment-like semigroup need not be segment-like.

If X is a topological space, **S** is a weakly segment-like semigroup, let us denote by $S(X)$ the set of all continuous mappings $f\colon X\to\mathbf{S}$, equipped with the semigroup operation of pointwise multiplication. We denote by $S_0(X)$ and $S_c(X)$ the subsemigroups of $S(X)$ composed of the mappings into [0, 1] and of those with a compact support, respectively, where, for $f\in S(X)$,

$$Z(f)=\{x\in X\colon f(x)=0\},\quad Z^c(f)=X-Z(f),\quad \operatorname{supp} f=\overline{Z^c(f)}.$$

In a semigroup S, we introduce, for $f, g\in S$, the following relations according to [2] and [3]:

$f>_d g$ iff there is an $h\in S$ such that $f=hg$,

$f>_u g$ iff $f=fg$.

A bijection φ from a semigroup S_1 onto a semigroup S_2 is said to be a *d-isomorphism* or a *u-isomorphism* iff

$$f>_d g \Leftrightarrow \varphi(f)>_d\varphi(g)$$

or

$$f>_u g \Leftrightarrow \varphi(f)>_u\varphi(g)$$

for $f, g\in S_1$, where the relations on the left hand side are understood in S_1, those on the right hand side are taken in S_2. S_1 and S_2 are said to be *d-isomorphic* (*u-isomorphic*) iff there exists a d-isomorphism (u-isomorphism) from S_1 onto S_2. Semigroup isomorphy clearly implies d- isomorphy and u-isomorphy; the converses do not hold.

Now the generalizations mentioned above can be formulated as follows:

B ([3], Theorem 4.1). *Let X_1 and X_2 be Tikhonov spaces, $\mathbf{S}_1$ and $\mathbf{S}_2$ segment-like semigroups, S_i a subsemigroup of* $S_i(X_i)$ (i.e. of $S(X)$ for $X=X_i$, $\mathbf{S}=\mathbf{S}_i$) *satisfying*

$$S_{i0}(X_i)\subset S_i\subset S_i(X_i).$$

If S_1 and S_2 are u-isomorphic then βX_1 and βX_2 are homeomorphic.

C ([2], Theorem 3). *Let X_1 and X_2 be Tikhonov spaces, $\mathbf{S}_1$ and $\mathbf{S}_2$ quasi-real semigroups and let $S_i(X_i)$ be defined as in* B. *If $S_1(X_1)$ and $S_2(X_2)$ are d-isomorphic then υX_1 and υX_2 are homeomorphic.*

A further result, similar to B, is contained in [4]. In order to formulate it, let us denote, if X is a locally compact Hausdorff space, by $S_\infty(X)$ the subsemigroup of

$S(X)$ composed of the mappings vanishing at infinity, i.e. of those $f \in S(X)$ that possess a continuous extension f^* to the one-point compactification $X \cup \{\infty\}$ of X such that $f^*(\infty)=0$. Now we have:

D ([4], Theorem 9). *Let X_1 and X_2 be locally compact Hausdorff spaces,* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *segment-like semigroups, S_i a subsemigroup of $S_i(X_i)$ satisfying*

$$S_{i0}(X_i) \cap S_{ic}(X_i) \subset S_i \subset S_{i\infty}(X_i).$$

If S_1 and S_2 are u-isomorphic then X_1 and X_2 are homeomorphic.

Our main purpose is to prove theorems similar to B and D, in which u-isomorphy is replaced by another condition; the method of the proof furnishes also a result similar to C. Finally some remarks follow concerning semigroups of semicontinuous real-valued funtions.

1. t-isomorphic semigroups. Let S be a semigroup with zero element 0, and define, for $f, g \in S$,

$f >_t g$ iff there is $h \in S$ such that $gh=g$ and $fk=0$ implies $hk=0$ for $k \in S$.

LEMMA 1.1. *If $f >_t g$ then $fk=0$ implies $gk=0$ for $k \in S$.*

PROOF. If $h \in S$ is chosen according to the definition, and $fk=0$, then $gk=ghk=g \cdot 0=0$. □

LEMMA 1.2. *The relation $>_t$ is transitive in S.*

PROOF. Assume $f >_t g$ and $g >_t m$, and choose $h, p \in S$ such that

$$gh = g, \quad fk = 0 \text{ implies } hk = 0 \text{ for } k \in S,$$

$$mp = m, \quad gk = 0 \text{ implies } pk = 0 \text{ for } k \in S.$$

Then $mp=m$ and $k \in S$, $fk=0$ implies $gk=0$ by 1.1, hence $pk=0$ so that $f >_t m$. □

LEMMA 1.3. *$f >_t 0$ for any $f \in S$.*

PROOF. $0 \cdot 0=0$ and $fk=0$ implies $0 \cdot k=0$. □

If S_1 and S_2 are semigroups with zero elements, then a bijection φ from S_1 onto S_2 will be said to be a *t-isomorphism* iff

$$f >_t g \Leftrightarrow \varphi(f) >_t \varphi(g) \quad \text{for } f, g \in S_1$$

(where, as above, $>_t$ is to be taken in S_1 on the left hand side, in S_2 on the right hand side). S_1 and S_2 are said to be *t-isomorphic* iff there exists a t-isomorphism from S_1 onto S_2. Semigroup isomorphy implies t-isomorphy.

The converse of the last statement does not hold. Moreover, u-isomorphy and t-isomorphy are independent of each other.

LEMMA 1.4. *Let S be a totally ordered set, 0 its least element, and $fg=\min(f, g)$ for $f, g \in S$. Then, in the semigroup S,*

(a) $f >_u g$ *iff* $f \leqq g$,
(b) $f >_t g$ *iff* $f > 0$ *or* $g = 0$.

PROOF. (a) is obvious. $f \in S$, $g = 0$ implies $f >_t g$ by 1.3. If $g > 0$, then $gh = g$ implies $h > 0$, hence $fk = 0$ implies $hk = 0$ iff $fk = 0$ implies $k = 0$, i.e. iff $f > 0$. □

EXAMPLE 1.5. Let $S_1 = \{0, 1, 2, \ldots\}$ and $S_2 = \mathbf{Q} \cap [0, +\infty)$ be equipped with the order relation inherited from $\mathbf{R}$ and with the semigroup operation $fg = \min(f, g)$. By 1.4 S_1 and S_2 are not u-isomorphic, but a bijection $\varphi \colon S_1 \to S_2$ such that $\varphi(0) = 0$ is a t-isomorphism. □

EXAMPLE 1.6. Let $S_1 = \mathbf{R}$ be equipped with the multiplication of real numbers, and

$$S_2 = \{(x, y) \in \mathbf{R}^2 \colon 0 \leqq x < 1,\ 0 \leqq y < 1\} \cup \{(1, 1)\}$$

be equipped with the semigroup operation

$$(f_1, f_2) \cdot (g_1, g_2) = (f_1 g_1, f_2 g_2).$$

Clearly $f >_u g$ iff $f = 0$ or $g = 1$ both for $f, g \in S_1$ and $f, g \in S_2$ (in S_2, $0 = (0, 0)$, $1 = (1, 1)$). Hence $\varphi \colon S_1 \to S_2$ is a u-isomorphism provided it is bijective and $\varphi(0) = (0, 0)$, $\varphi(1) = (1, 1)$.

However, S_1 and S_2 are not t-isomorphic. In fact, $f \in S_1$, $f \neq 0$ implies $f >_t g$ for every $g \in S_1$: $g \cdot 1 = g$, and $fk = 0$ implies $k = 1 \cdot k = 0$ for $k \in S_1$.

On the other hand, in S_2, there is, for every $f = (a, 0) \in S_2$, $0 \leqq a < 1$, an element $g = \left(0, \frac{1}{2}\right) \in S_2$ such that $f >_t g$ does not hold: if $h \in S_2$, $gh = g$, then necessarily $h = (1, 1)$, and $fg = 0$, $hg \neq 0$. □

In a semigroup S with zero-element, a subset $I \subset S$ will be said to be a *t-ideal* iff

(1.7.1) $\emptyset \neq I \neq S$,

(1.7.2) $f \in S$, $g \in I$, $f >_t g$ imply $f \in I$,

(1.7.3) $f, g \in I$ implies the existence of $h \in I$ such that $f >_t h$, $g >_t h$.

If I is a t-ideal, then, in particular, $f \in I$ implies the existence of $h \in I$ such that $f >_t h$.

A t-isomorphism $\varphi \colon S_1 \to S_2$ carries the t-ideals of S_1 to the t-ideals in S_2.

2. t-ideals and t-filter bases. The following lemmas will be formulated in two variants; the first of them will be denoted by α) and the second one with β).

Let X be

α) a locally compact Hausdorff space,

β) a Tikhonov space,

$\mathbf{S}$ a weakly segment-like semigroup, S a subsemigroup of $S(X)$ satisfying

α) $S_0(X) \cap S_c(X) \subset S \subset S_c(X)$,

β) $S_0(X) \subset S \subset S(X)$,

finally $\mathfrak{T}$ the set of all subsets of X that are

α) compact closures of cozero-sets,

β) closures of cozero-sets.

A cozero-set in X is, of course, a set of the form $Z^c(f)$ for some $f\in C(X)$.

For $f\in S(X)$, we define

$$|f|(x) = |f(x)| \quad (x\in X).$$

Obviously $|f|\in C(X)$.

LEMMA 2.1. *$T\in\mathfrak{T}$ iff $T=\operatorname{supp} f$ for some $f\in S$.*

PROOF. If: $Z^c(f)=Z^c(|f|)$. Only if: $T=\overline{Z^c(g)}$, $g\in C(X)$ implies $T=\overline{Z^c(f)}$, where $f=\max(\min(g, 1), 0)\in S_0(X)$; in case α), we also have $f\in S_c(X)$. □

LEMMA 2.2. *Assume $f, h\in S$; $fk=0$ implies $hk=0$ for every $k\in S$ iff $\operatorname{supp} f\supset$ $\supset\operatorname{supp} h$.*

PROOF. Suppose $\operatorname{supp} f\supset\operatorname{supp} h$, $fk=0$, $k\in S$, and indirectly $h(x)k(x)\neq 0$ for some $x\in X$. Then $|h(x)|\cdot|k(x)|\neq 0$ by (0.2) (c) and (d), $x\in\operatorname{supp} h\subset\operatorname{supp} f$. Since $|k(x)|\neq 0$, there is a neighbourhood V of x such that $|k(y)|\neq 0$ for $y\in V$; by $x\in\operatorname{supp} f$ there is a $y\in V$ such that $f(y)\neq 0$. Then $|f(y)|\neq 0$, $|f(y)k(y)|\neq 0$, $f(y)k(y)\neq 0$: a contradiction.

Conversely suppose $x\in\operatorname{supp} h-\operatorname{supp} f$ for some $x\in X$. Then there is a neighbourhood V of x such that $V\cap\operatorname{supp} f=\emptyset$ and a $z\in\operatorname{int} V$ such that $h(z)\neq 0$; in case α) we can assume that V is compact. There is a continuous $k\colon X\to[0, 1]$ satisfying $k(z)=1$, $k(y)=0$ for $y\notin V$. Clearly $k\in S_0(X)$, in case α) $k\in S_c(X)$, hence $k\in S$, and $fk=0$, $h(z)k(z)\neq 0$. □

Observe that the statement remains valid if X is a locally compact Hausdorff space and

$$S_0(X)\cap S_c(X)\subset S\subset S(X).$$

The following lemma will motivate the somewhat strange definition of the relation $>_t$. In order to formulate it, let us denote by δ the Čech—Stone proximity of X, i.e., if we write $\bar\delta$ for non-δ, let us put $A\bar\delta B$ iff there is a continuous $s\colon X\to[0, 1]$ such that $s(x)=1$ for $x\in A$ and $s(x)=0$ for $x\in B$ $(A, B\subset X)$. Let us further write, for P, $Q\subset X$, $P<Q$ iff $P\bar\delta(X-Q)$. Then it is easy to check (cf. [1], (4.1.1)):

(2.3.1) $P<Q$ implies $X-Q<X-P$,

(2.3.2) $\emptyset<\emptyset$, $X<X$,

(2.3.3) $P'\subset P<Q\subset Q'$ implies $P'<Q'$,

(2.3.4) $P_i<Q_i$ for $i=1, \dots, n$ implies

$$\bigcap_1^n P_i<\bigcap_1^n Q_i,$$

(2.3.5) $P<Q$ implies the existence of $R\subset X$ such that

$$P<R<Q,$$

(2.3.6) $P<Q$ implies $\bar P\subset\operatorname{int} Q$.

LEMMA 2.4. *For* $f, g\in S$, *we have* $f>_t g$ *iff*

$$\operatorname{supp} g < \operatorname{supp} f.$$

PROOF. Suppose $\operatorname{supp} g<\operatorname{supp} f$, and let $h\colon X\to[0,1]$ be continuous, satisfying $h(x)=1$ for $x\in\operatorname{supp} g$, $h(x)=0$ for $x\in X-\operatorname{supp} f$. Then $h\in S_0(X)$, in case α) $h\in S_c(X)$ because $\operatorname{supp} h\subset\operatorname{supp} f$, $f\in S_c(X)$, hence $h\in S$. Clearly $gh=g$, and $fk=0$ implies $hk=0$ by 2.2 for $k\in S$.

Conversely suppose $gh=g$ for some $h\in S$ such that $fk=0$, $k\in S$ implies $hk=0$, i.e., by 2.2, such that $\operatorname{supp} h\subset\operatorname{supp} f$. For $s=\max(\min(|h|,1),0)$ we have $s\in S_0(X)$, and $g(x)\neq 0$ implies $|g(x)|\neq 0$ by (0.2)(d), hence $|h(x)|=1$, $s(x)=1$ for $x\in Z^c(g)$ and also for $x\in\operatorname{supp} g$. Finally $x\in X-\operatorname{supp} f$ implies $x\notin\operatorname{supp} h$, $h(x)=0$, $s(x)=0$. □

Let us now call a *t-filter base* a system $\mathfrak{t}$ of subsets of X such that

(2.5.1) $\mathfrak{t}\neq\emptyset$, $\emptyset\neq T\in\mathfrak{T}$ for $T\in\mathfrak{t}$,

(2.5.2) $T\in\mathfrak{t}$, $T\subset T'\in\mathfrak{T}$ imply $T'\in\mathfrak{t}$,

(2.5.3) $T_1, T_2\in\mathfrak{t}$ implies the existence of $T\in\mathfrak{t}$ such that $T<T_1\cap T_2$.

The terminology is justified by the fact that a t-filter base is a filter base by (2.5.1), (2.5.3), and (2.3.6).

LEMMA 2.6. *If I is a t-ideal in S then* $\operatorname{supp} I=\{\operatorname{supp} f\colon f\in I\}$ *is a t-filter base in X.*

PROOF. Denote $\mathfrak{t}=\operatorname{supp} I$. Then $I\neq\emptyset$ implies $\mathfrak{t}\neq\emptyset$, and by 2.1 $T\in\mathfrak{T}$ for $T\in\mathfrak{t}$. If $g\in I$ and $\operatorname{supp} g$ were empty, then $g=0$, and by 1.3 $f>_t g$ for $f\in S$ so that $I=S$ would follow by (1.7.2).

If $T\in\mathfrak{t}$, $T\subset T'\in\mathfrak{T}$, then $T=\operatorname{supp} g$ for some $g\in I$, $T'=\operatorname{supp} f$ for some $f\in S$ by 2.1, and there is $h\in I$ such that $g>_t h$ by (1.7.3), so that $\operatorname{supp} h<\operatorname{supp} g$ by 2.4. Hence $\operatorname{supp} h<\operatorname{supp} f$ by (2.3.3), $f>_t h$ by 2.4, and $f\in I$, $\operatorname{supp} f\in\mathfrak{t}$ by (1.7.2).

Suppose $T_1, T_2\in\mathfrak{t}$, $T_i=\operatorname{supp} f_i$, $f_i\in I$. By (1.7.3) there is $g\in I$ satisfying $f_i>_t g$ for $i=1,2$, hence $\operatorname{supp} g<\operatorname{supp} f_i=T_i$ by 2.4, and $T=\operatorname{supp} g\in\mathfrak{t}$, $T<T_1\cap T_2$ by (2.3.4). □

LEMMA 2.7. *If $\mathfrak{t}$ is a t-filter base in X then*

$$\operatorname{supp}^{-1}\mathfrak{t}=\{f\in S\colon \operatorname{supp} f\in\mathfrak{t}\}$$

is a t-ideal in S.

PROOF. $\mathfrak{t}\neq\emptyset$ implies $I=\operatorname{supp}^{-1}\mathfrak{t}\neq\emptyset$ by (2.5.1) and 2.1. Clearly $\operatorname{supp} 0=\emptyset\notin\mathfrak{t}$ implies $I\neq S$.

If $f\in S$, $g\in I$, $f>_t g$, then by 2.4 $\operatorname{supp} g<\operatorname{supp} f$, hence $\operatorname{supp} g\in\mathfrak{t}$ implies $\operatorname{supp} f\in\mathfrak{t}$, $f\in I$ by (2.3.6), 2.1 and (2.5.2).

If $f, g\in I$, $\operatorname{supp} f$, $\operatorname{supp} g\in\mathfrak{t}$, then by (2.5.3) there is $T\in\mathfrak{t}$ such that $T<\operatorname{supp} f\cap\operatorname{supp} g$. By 2.1 $T=\operatorname{supp} h$ for some $h\in S$, and clearly $h\in I$. By 2.4 $f>_t h$, $g>_t h$. □

COROLLARY 2.8. *The mappings*

$$\mathfrak{t}=\operatorname{supp} I \quad \textit{and} \quad I=\operatorname{supp}^{-1}\mathfrak{t}$$

are inverses of each other and establish a bijection from the set of all t-ideals in S onto the set of all t-filter bases in X.

Proof. For a t-ideal I, define $\mathfrak{t}=\operatorname{supp} I$, $I'=\operatorname{supp}^{-1}\mathfrak{t}$. Then clearly $I\subset I'$. If $f\in I'$, then $\operatorname{supp} f\in\mathfrak{t}=\operatorname{supp} I$, hence $\operatorname{supp} f=\operatorname{supp} g$ for some $g\in I$. By (1.7.3) there is $h\in I$ such that $g>_t h$, i.e. by 2.4 $\operatorname{supp} h<\operatorname{supp} g=\operatorname{supp} f$, $f>_t h$, and $f\in I$ by (1.7.2). Thus $I'=I$.

Conversely if $\mathfrak{t}$ is a t-filter base and $I=\operatorname{supp}^{-1}\mathfrak{t}$, then $\operatorname{supp} I=\mathfrak{t}$ because, by 2.1, each $T\in\mathfrak{t}$ has the form $T=\operatorname{supp} f$ for some $f\in S$, and f clearly belongs to I. □

Lemma 2.9. *For t-ideals I_1, I_2 in S, $I_1\subset I_2$ iff* supp $I_1\subset$supp I_2. *Hence I is a maximal t-ideal iff* supp *I is a maximal t-filter base.* □

Lemma 2.10. *For $x\in X$, the neighbourhoods of x belonging to $\mathfrak{T}$ constitute a neighbourhood base of x and a maximal t-filter base. Conversely, if $\mathfrak{t}'$ is a maximal t-filter base and $x\in\bigcap\mathfrak{t}'$, then $\mathfrak{t}'$ is composed of all neighbourhoods of x belonging to $\mathfrak{T}$.*

Proof. If V is a neighbourhood of x, let W be a closed neighbourhood of x such that $W\subset V$ and, in case α), let W be compact. Then there is $f\in S_0(X)$ such that $f(x)=1$, $f(y)=0$ for $y\in X-W$, and $T=\overline{U}$, $U=\left\{z\in X: f(z)>\frac{1}{2}\right\}$ yield a neighbourhood $T\in\mathfrak{T}$ of x such that $T\subset W\subset V$.

The collection of all neighbourhoods $T\in\mathfrak{T}$ of x is a t-filter base $\mathfrak{t}$; (2.5.1) and (2.5.2) are obvious, and if T_1, $T_2\in\mathfrak{t}$, then $\{x\}<T_i$ for $i=1,2$, hence $\{x\}<T_1\cap T_2$ by (2.3.4), $\{x\}<V<T_1\cap T_2$ for some V by (2.3.5), and V is a neighbourhood of x, hence there is $T\in\mathfrak{t}$ such that $T\subset V$, $T<T_1\cap T_2$ by (2.3.3).

If $\mathfrak{t}'\supset\mathfrak{t}$ is a t-filter base, let $T'\in\mathfrak{t}'$, and suppose $T'\notin\mathfrak{t}$. Choose by (2.5.3) $T''\in\mathfrak{t}'$ such that $T''<T'$. $x\notin T''$ would imply the existence of $T\in\mathfrak{t}$ such that $T\cap T''=\emptyset$ which is impossible since $T, T''\in\mathfrak{t}'$ and $\mathfrak{t}'$ is a filter base; hence $x\in T''$ and $T'\in\mathfrak{t}$ by (2.3.6). Thus $\mathfrak{t}$ is a maximal t-filter base.

If $\mathfrak{t}'$ is a maximal t-filter base and $x\in\bigcap\mathfrak{t}'$, then $T'\in\mathfrak{t}'$ implies $T''<T'$ for some $T''\in\mathfrak{t}'$, and $x\in T''$ yields $T'\in\mathfrak{t}$ by (2.3.6) again. Thus $\mathfrak{t}'\subset\mathfrak{t}$ and $\mathfrak{t}'=\mathfrak{t}$. □

3. Locally compact spaces. We continue by studying the consequences of the hypotheses α).

Lemma 3.1. *In X, every t-filter base is fixed (i.e. it has non-empty intersection), hence every maximal t-filter base coincides with the collection of all neighbourhoods belonging to $\mathfrak{T}$ of some point $x\in X$.*

Proof. A t-filter base is a filter base composed of compact sets hence it has a non-empty intersection. Then 2.10 applies. □

Corollary 3.2. *For $x\in X$, let $t(x)$ denote the maximal t-ideal* $\operatorname{supp}^{-1}\mathfrak{t}$ *in S, where* $\mathfrak{t}$ *is the collection of all neighbourhoods belonging to $\mathfrak{T}$ of x. Then t is a bijection from X onto the set of all maximal t-ideals in S.*

Proof. 2.10, 3.1 and 2.8 can be completed by the observation that, if $x, y\in X$, $x\neq y$, then $t(x)\neq t(y)$ because there are neighbourhoods T, $T'\in\mathfrak{T}$ of x and y, respectively, such that $T\cap T'=\emptyset$. □

LEMMA 3.3. *For $f\in S$, let $B(f)$ denote the set of all maximal t-ideals I in S such that $f\in I$. Then*

$$t^{-1}(B(f)) = \operatorname{int}\operatorname{supp} f.$$

PROOF. $t(x)\in B(f)$ iff $\operatorname{supp} f$ is a neighbourhood of x. □

COROLLARY 3.4. *Let the set Y of all maximal t-ideals in S be equipped with the topology for which the sets $B(f)$ $(f\in S)$ constitute a base. Then t is a homeomorphism from X onto Y.*

PROOF. By 2.1 and 2.10, the sets $\operatorname{int}\operatorname{supp} f$ $(f\in S)$ constitute a base in X. Hence 3.2 and 3.3 apply. □

THEOREM 3.5. *Let X_1 and X_2 be locally compact Hausdorff spaces, $\mathbf{S}_1$ and $\mathbf{S}_2$ weakly segment-like semigroups, $S_i(X_i)$ the semigroup of all continuous mappings from X_i into $\mathbf{S}_i$, $S_{i0}(X_i)$ and $S_{ic}(X_i)$ the subsemigroups of $S_i(X_i)$ composed of all mappings into* [0, 1] *and of those with compact support, respectively, and S_i a subsemigroup of $S_i(X_i)$ satisfying*

$$S_{i0}(X_i)\cap S_{ic}(X_i) \subset S_i \subset S_{ic}(X_i).$$

If S_1 and S_2 are t-isomorphic then X_1 and X_2 are homeomorphic. □

This theorem does not precisely correspond to Theorem D (because $S_i\subset S_{ic}(X_i)$ is supposed instead of $S_i\subset S_{i\infty}(X_i)$). However, we can prove a stronger result quite analogous to D.

LEMMA 3.6. *Let X be a locally compact Hausdorff space, $\mathbf{S}$ a weakly segment-like semigroup, and S a subsemigroup of $S(X)$ such that*

$$S_0(X)\cap S_c(X) \subset S \subset S_\infty(X).$$

Then $g\in S$ belongs to $S_c(X)$ iff there is an $f\in S$ satisfying $f>_t g$.

PROOF. If $f\in S$, $f>_t g$, then there exists $h\in S$ such that $gh=g$, Since $g(x)\neq 0$ implies $|h(x)|=1$, and $h\in S_\infty(X)$, $\operatorname{supp} g$ has to be compact. Conversely, if $\operatorname{supp} g$ is compact, we can construct an $h\in C(X)$ such that $\operatorname{supp} h$ is compact, $0\leqq h(x)\leqq 1$ for $x\in X$, $h(x)=1$ for $x\in\operatorname{supp} g$ (see, e.g., [4], Lemma 1). Clearly $h\in S_0(X)\cap S_c(X)$, $h\in S$, and $h>_t g$ because $gh=g$. □

LEMMA 3.7. *Under the hypotheses of 3.6, set $S'=S\cap S_c(X)$. Then S' is a subsemigroup of $S(X)$, and, for $f, g\in S'$, $f>_t g$ holds relative to S' iff it holds relative to S.*

PROOF. If $f>_t g$ relative to S', then there is $h\in S'\subset S$ such that $gh=g$, and, for $k\in S'$, $fk=0$ implies $hk=0$. However, according to the remark following 2.2, the latter condition means $\operatorname{supp} f\supset\operatorname{supp} h$, and then $fk=0$ implies $hk=0$ for every $k\in S$. Thus $f>_t g$ relative to S.

Assume, conversely, $f>_t g$ relative to S. Then there is $h\in S$ such that $gh=g$ and, if $fk=0$, $k\in S$, then $hk=0$. By 3.6 $\operatorname{supp} g$ is compact, hence there is $h_0\in S_0(X)\cap$ $\cap S_c(X)$ such that $h_0(x)=1$ for $x\in\operatorname{supp} g$. Clearly $hh_0\in S'$, $ghh_0=g$, and $fk=0$, $k\in S'$ implies $hh_0k=0$; therefore $f>_t g$ relative to S'. □

THEOREM 3.8. *Let X_1, X_2 be locally compact Hausdorff spaces, $\mathbf{S}_1$, $\mathbf{S}_2$ weakly* *segment-like semigroups, and S_i a subsemigroup of $S_i(X_i)$ satisfying*

$$S_{i0}(X_i)\cap S_{ic}(X_i)\subset S_i\subset S_{i\infty}(X_i)$$

($i=1, 2$). *If S_1 and S_2 are t-isomorphic then X_1 and X_2 are homeomorphic.*

Proof. By 3.6, a t-isomorphism $\varphi\colon S_1\to S_2$ carries $S_1'=S_1\cap S_{1c}(X_1)$ to $S_2'=S_2\cap S_{2c}(X_2)$. By 3.7, $\varphi|S_1'\colon S_1'\to S_2'$ is a t-isomorphism. Since $S_{i0}(X_i)\cap S_{ic}(X_i)\subset S_i'\subset S_{ic}(X_i)$, 3.5 applies. □

Thus, in Theorem D, segment-like semigroups can be replaced by weakly segment-like ones, and u-isomorphy by t-isomorphy.

4. Construction of βX. Now we adopt the hypotheses β), and our next purpose is to show that the knowledge of the relation $>_t$ in S enables us to construct the Čech—Stone compactification βX of X. For this purpose, let us recall the following construction of βX for a Tikhonov space X.

Let δ denote, as above, the Čech—Stone proximity in X and $<$ the corresponding order for the subsets of X. A filter $\mathfrak{s}$ in X is said to be *round* iff $S'\in\mathfrak{s}$ implies the existence of $S_1\in\mathfrak{s}$ such that $S_1<S'$, and $\mathfrak{s}$ is said to be *compressed* iff $S'\cap A\neq\emptyset\neq S'\cap B$ for every $S'\in\mathfrak{s}$ implies $A\delta B$ (see e.g. [1], pp. 250 and 186).

Now let $Z\supset X$ be a set such that there exists a bijection $\mathfrak{s}$ from $Z-X$ onto the set of all nonconvergent, round, compressed filters in X. For $x\in X$, let $\mathfrak{s}(x)$ denote the neighbourhood filter of x in X. Equip Z with the topology for which a base is composed of the sets $s(G)$, where G is open in X and

$$s(G)=\{z\in Z\colon\ G\in\mathfrak{s}(z)\}.$$

Then $Z=\beta X$ and, for $z\in Z$, $\mathfrak{s}(z)$ is the trace in X of the neighbourhood filter of z in Z.

It is well-known that (in general, in every proximity space) round, compressed filters coincide with maximal round filters, In fact, if $\mathfrak{s}$ is round and compressed, and $\mathfrak{s}'\supset\mathfrak{s}$ is a round filter, then, for $S'\in\mathfrak{s}'$, there is $S_1'\in\mathfrak{s}'$ such that $S_1'<S'$, $S_1'\bar{\delta}(X-S')$, and $S^*\cap S_1'\neq\emptyset$ for every $S^*\in\mathfrak{s}$ implies $S^*\subset S'$ for some $S^*\in\mathfrak{s}$, so that $S'\in\mathfrak{s}$. On the other hand, if $\mathfrak{s}$ is a maximal round filter, $S'\cap A\neq\emptyset\neq S'\cap B$ for every $S'\in\mathfrak{s}$, and we assume indirectly $A\bar{\delta}B$, $A<X-B=C$, then, by (2.3.5), there exist sets C_i for $i\in\mathbf{N}$ such that $A<C_{i+1}<C_i<C$ for every i. The filter $\mathfrak{s}'$ generated by the filter base composed of the intersections $S'\cap C_i$ for $S'\in\mathfrak{s}$, $i\in\mathbf{N}$, is easily seen to be round and $\mathfrak{s}'\supset\mathfrak{s}$, hence $\mathfrak{s}'=\mathfrak{s}$, and $S'\cap C_1\in\mathfrak{s}$ implies $C\in\mathfrak{s}$: a contradiction.

Lemma 4.1. *A t-filter base $\mathfrak{t}$ generates a round filter $\mathfrak{s}$ in X, and $\mathfrak{t}=\mathfrak{T}\cap\mathfrak{s}$.*

Proof. For $S'\in\mathfrak{s}$, choose $T\in\mathfrak{t}$, $T\subset S'$, and $T'\in\mathfrak{t}$, $T'<T$. Then $T'\in\mathfrak{s}$, $T'<S'$. Hence $\mathfrak{s}$ is a round filter. Clearly $\mathfrak{t}\subset\mathfrak{T}\cap\mathfrak{s}$. Conversely, if $T'\in\mathfrak{T}\cap\mathfrak{s}$, there is $T\in\mathfrak{t}$ such that $T\subset T'$, and $T'\in\mathfrak{t}$ by (2.5.2). □

Lemma 4.2. *If $\mathfrak{s}$ is a round filter in X, then $\mathfrak{T}\cap\mathfrak{s}$ is a t-filter base that generates $\mathfrak{s}$.*

Proof. If $S'\in\mathfrak{s}$, $S_1\in\mathfrak{s}$, $S_1<S'$, then choose $s\in S_0(X)$ such that $s(x)=1$ for $x\in S_1$, $s(x)=0$ for $x\in X-S'$. Then $\bar{G}\in\mathfrak{T}$ for $G=\left\{x\in X\colon s(x)>\frac{1}{2}\right\}$, and $S_1\subset\bar{G}$ implies $\bar{G}\in\mathfrak{T}\cap\mathfrak{s}$, $\bar{G}\subset S'$. Hence $\mathfrak{t}=\mathfrak{T}\cap\mathfrak{s}$ generates $\mathfrak{s}$.

$\mathfrak{t}$ is composed of non-empty elements of $\mathfrak{T}$, and (2.5.2) is clearly fulfilled. For $T_1, T_2 \in \mathfrak{t}$, we have $T_1 \cap T_2 \in \mathfrak{s}$. Choose $S' \in \mathfrak{s}$, $S' < T_1 \cap T_2$, and $T \in \mathfrak{t}$ such that $T \subset S'$. Then $T \in \mathfrak{t}$, $T < T_1 \cap T_2$. □

COROLLARY 4.3. *The constructions contained in* 4.1 *and* 4.2 *establish a bijection from the set of all t-filter bases in X to all round filters in X. In particular, maximal round filters are generated by maximal t-filter bases.* □

COROLLARY 4.4. *Let* $Z = \beta X$, $\mathfrak{s}(z)$ *be the trace in X of the neighbourhood filter of* $z \in Z$, $\mathfrak{t}(z) = \mathfrak{T} \cap \mathfrak{s}(z)$, $t(z) = \operatorname{supp}^{-1} \mathfrak{t}(z)$. *Then t is a bijection from Z onto the set Y of all maximal t-ideals in S.*

PROOF. 2.10, 4.3, 2.8, and the observation that $z_1 \neq z_2$ implies $\mathfrak{s}(z_1) \neq \mathfrak{s}(z_2)$. □

LEMMA 4.5. *The sets* $s(\operatorname{int} T)$, $T \in \mathfrak{T}$, *constitute a base in* $Z = \beta X$.

PROOF. For $z \in Z$ and a neighbourhood V of z, let G be open in X, $z \in s(G) \subset V$. Then $G \in \mathfrak{s}(z)$ and by 4.3 there is $T \in \mathfrak{t}(z)$ such that $T \subset G$. Choose $T' \in \mathfrak{t}(z)$, $T' < T$, then $T' \subset \operatorname{int} T$ by (2.3.6), and $\operatorname{int} T \in \mathfrak{s}(z)$, $z \in s(\operatorname{int} T) \subset s(G) \subset V$. □

LEMMA 4.6. *For* $f \in S$, $B(f) = \{I \in Y : f \in I\}$, *we have*

$$t^{-1}(B(f)) = s(\operatorname{int} \operatorname{supp} f).$$

PROOF. $t(z) \in B(f)$ iff $f \in t(z)$ iff $\operatorname{supp} f \in \mathfrak{t}(z)$. Now $\operatorname{int} \operatorname{supp} f \in \mathfrak{s}(z)$ clearly implies $\operatorname{supp} f \in \mathfrak{T} \cap \mathfrak{s}(z) = \mathfrak{t}(z)$, while $\operatorname{supp} f \in \mathfrak{t}(z)$ implies the existence of $T \in \mathfrak{t}(z)$, $T < \operatorname{supp} f$, hence $T \subset \operatorname{int} \operatorname{supp} f \in \mathfrak{s}(Z)$. □

COROLLARY 4.7. *If we equip Y with a topology for which a base is composed of the sets* $B(f)$, $f \in S$, *then* $t: Z \to Y$ *is a homeomorphism.* □

THEOREM 4.8. *Let* X_1, X_2 *be Tikhonov spaces,* $\mathbf{S}_1, \mathbf{S}_2$ *weakly segment-like semigroups,* $S_i(X_i)$ *and* $S_{i0}(X_i)$ *be defined as in* 3.5, S_i *a subsemigroup of* $S_i(X_i)$ *satisfying*

$$S_{i0}(X_i) \subset S_i \subset S_i(X_i).$$

If S_1 *and* S_2 *are t-isomorphic, then* βX_1 *and* βX_2 *are homeomorphic.* □

This theorem corresponds to Theorem B; segment-like semigroups are replaced by weakly segment-like ones, and u-isomorphy is replaced by t-isomorphy.

5. Pseudo-real semigroups. Now we look for a theorem analogous to Theorem C. It turns out that the method based on t-ideals can be applied if we replace quasi-real semigroups by a slightly more general concept.

For this purpose, let us say that $\mathbf{S}$ is a *pseudo-real semigroup* iff it fulfils (0.1) (a)–(d) (i.e. condition (e) is omitted). The argument used in the proof of [2], Theorem 1, easily furnishes:

LEMMA 5.1. *Let* $\mathbf{G}$ *be a topological group that contains* $(0, +\infty)$ *as a topological subgroup; suppose there is a continuous homomorphism* $\alpha: \mathbf{G} \to (0, +\infty)$ *such that* $\alpha(a) = a$ *for* $a \in (0, +\infty)$. *Let* $\mathbf{S} = \mathbf{G} \cup \{\omega\}$ *where* $\omega \notin \mathbf{G}$, *and define*

$$a \cdot \omega = \omega \cdot a = \omega \quad (a \in \mathbf{G}), \quad \omega \cdot \omega = \omega, \quad \alpha(\omega) = 0.$$

Equip $\mathbf{S}$ *with a topology such that* $\mathbf{G}$ *is a subspace of* $\mathbf{S}$ *and the neighbourhood filter of* ω *is composed of sets* $U\cup\{\omega\}$, *where* $U\in\mathfrak{s}$, *and* $\mathfrak{s}$ *is an open filter in* $\mathbf{G}$ *satisfying*
(a) *for* $U\in\mathfrak{s}$, *there is* $U'\in\mathfrak{s}$ *such that* $U'\cdot U'\subset U$,
(b) *for* $U\in\mathfrak{s}$, $a\in\mathbf{G}$, *there are* $U_a\in\mathfrak{s}$ *and a neighbourhood* V *of* a *in* $\mathbf{G}$ *such that* $U_aV\cup VU_a\subset U$,
(c) *for* $\varepsilon>0$, *there is* $U\in\mathfrak{s}$ *such that*

$$\alpha(x)<\varepsilon \quad \textit{for} \quad x\in U,$$

(d) *for* $U\in\mathfrak{s}$, *there is* $\varepsilon>0$ *such that*

$$0<x<\varepsilon \quad \textit{implies} \quad x\in U.$$

After having identified ω *with the real number* 0, $\mathbf{S}$ *will be a pseudo-real semigroup* (*with* $|x|=\alpha(x)$).

Conversely, every pseudo-real semigroup can be obtained from a topological group $\mathbf{G}$ *with the help of this construction.* □

Now we can show that the concept of a pseudo-real semigroup is strictly more general than that of a quasi-real semigroup:

EXAMPLE 5.2 (J. Gerlits). Let $\mathbf{G}=(0,+\infty)\times\mathbf{R}$ be equipped with the group operation

$$(x,y)\cdot(x',y')=(xx',y+y')$$

and with the product topology of the topology on $(0,+\infty)$ inherited from the Euclidean topology of $\mathbf{R}$ and of the discrete topology of $\mathbf{R}$. Denote by Φ the collection of all positive solutions of the functional equation

$$f(y+y')=f(y)f(y').$$

For a finite subset $\Phi'\subset\Phi$ and $\varepsilon>0$, define

$$U_{\Phi',\varepsilon}=\{(x,y):\ 0<x<\varepsilon f(y) \text{ for } f\in\Phi'\}.$$

The open subsets $U_{\Phi',\varepsilon}$ of $\mathbf{G}$ generate an open filter $\mathfrak{s}$ in $\mathbf{G}$. Define $\alpha(x,y)=x$ for $(x,y)\in\mathbf{G}$, and let us identify $(x,0)\in\mathbf{G}$ with $x\in(0,+\infty)$.

Now the hypotheses of 5.1 are fulfilled. In fact, $0<\varepsilon<1$ implies $U_{\Phi',\varepsilon}\cdot U_{\Phi',\varepsilon}\subset\subset U_{\Phi',\varepsilon}$, further, for $(a,b)\in\mathbf{G}$, $U_{\Phi',\delta}\cdot V\subset U_{\Phi',\varepsilon}$ provided $V=\left(\frac{a}{2},2a\right)\times\{b\}$ and $\delta<\frac{\varepsilon}{2a}\min\{f(b):f\in\Phi'\}$, finally, if $\varepsilon>0$, $f_1(y)=e^y$, $f_2(y)=e^{-y}$, we have $\alpha(x,y)==x<\varepsilon$ whenever $(x,y)\in U_{\{f_1,f_2\},\varepsilon}$, and, since $f(0)=1$ for any $f\in\Phi$, $0<x<\varepsilon$ implies $(x,0)\in U_{\Phi',\varepsilon}$ for any finite $\Phi'\subset\Phi$.

Hence 5.1 furnishes a pseudo-real semigroup $\mathbf{S}$. In $\mathbf{S}$, 0 does not possess any countable neighbourhood base. In fact, it is well-known that there is an infinite subset $B\subset\mathbf{R}$ (a Hamel base) such that the values of an $f\in\Phi$ taken on at the elements $b\in B$ can be quite arbitrarily prescribed positive numbers. Hence, if $\mathfrak{U}$ is a countable system composed of sets of the form $U_{\Phi',\varepsilon}$, and φ is an injection from $\mathfrak{U}$ into B, we can find a $g \in \Phi$ such that

$$g(b) < \varepsilon \min \{f(b) \colon f \in \Phi'\}$$

for $b = \varphi(U_{\Phi', \varepsilon})$, and then $U_{\Phi', \varepsilon} \subset U_{\{g\}, 1}$ does not hold for any $U_{\Phi', \varepsilon} \in \mathfrak{U}$.

Therefore $\mathbf{S}$ is not quasi-real, not only for the above embedding of $(0, +\infty)$ into $\mathbf{G}$ and the above definition of α, but for any possible embedding and absolute value. □

LEMMA 5.3. *Every quasi-real semigroup is pseudo-real and every pseudo-real semigroup is segment-like.* □

We know from [2], Theorem 2, that, for every topological group $\mathbf{T}$, $\mathbf{G} = \mathbf{T} \times \times (0, +\infty)$ satisfies the hypotheses of 5.1 provided (e, y) is identified with $y > 0$ (e is the unity element of $\mathbf{T}$), and $\alpha(x, y) = y$. It is worth while to observe that the converse is true if $\mathbf{G}$ is commutative:

LEMMA 5.4. *Let $\mathbf{G}$ be a commutative topological group satisfying the hypotheses in 5.1, and $\mathbf{T}$ the topological subgroup of $\mathbf{G}$ composed of the elements $a \in \mathbf{G}$ such that $\alpha(a) = 1$. Then $\mathbf{G}$ is isomorphic to $\mathbf{T} \times (0, +\infty)$.*

PROOF. Define $\varphi(t, x) = tx$ for $t \in \mathbf{T}$, $x \in (0, +\infty)$. Obviously $\varphi \colon \mathbf{T} \times (0, +\infty) \to \to \mathbf{G}$ is a continuous homomorphism. It is bijective because $a \in \mathbf{G}$ implies $a = = \left(\frac{1}{\alpha(a)} a\right) \cdot \alpha(a)$ where $\alpha\left(\frac{1}{\alpha(a)} a\right) = 1$, and $t_1 x_1 = t_2 x_2$, $\alpha(t_1) = \alpha(t_2) = 1$, $x_1, x_2 \in \in (0, +\infty)$ imply $x_1 = x_2$, $t_1 = t_2$. Since $a \mapsto \frac{1}{\alpha(a)} a$ and $a \mapsto \alpha(a)$ are continuous, φ^{-1} is continuous, as well. □

We know from [2], p. 135 that the condition of commutativity is essential in 5.4.

6. Construction of υX. We shall show that, under a suitable restriction of conditions β), the knowledge of the semigroup $S(X)$ determines the space υX. First we assume β) only.

LEMMA 6.1. *A t-ideal in S is a subsemigroup.*

PROOF. Let I be a t-ideal in S, $f, g \in I$. Choose $h \in I$ such that $f >_t h$, $g >_t h$, i.e. by 2.4

$$\operatorname{supp} h < \operatorname{supp} f, \quad \operatorname{supp} h < \operatorname{supp} g.$$

Put

$$G = Z^c(h), \quad G_1 = Z^c(f), \quad G_2 = Z^c(g).$$

Then G, G_1, G_2 are open sets, and $G \subset \bar{G}_1 \cap \bar{G}_2$. For every open set $\emptyset \neq H \subset G$, we have $H \cap G_1 \neq \emptyset$, then $H \cap G_1 \cap G_2 \neq \emptyset$, hence $G \subset \overline{G_1 \cap G_2}$ and $\bar{G} \subset \overline{G_1 \cap G_2}$. Since $G_1 \cap G_2 = Z^c(fg)$, we obtain $\operatorname{supp} h \subset \operatorname{supp} fg$ and $\operatorname{supp} fg \in \operatorname{supp} I$ by 2.6, $fg \in I$ by 2.8. □

From now on we suppose that $\mathbf{S}$ is a pseudo-real semigroup that is commutative or, more generally, *quasi-commutative* in the sense that $ab = ba$ for $a \in [0, +\infty)$, $b \in \mathbf{S}$. Let X be a Tikhonov space, $S = S(X)$; by 5.3, the conditions β) are fulfilled.

We recall that, if S is a semigroup,

$$S^* = \{h \in S\colon\ hf = fh \text{ for } f \in S\},$$

I is a subsemigroup of S such that $I \cap S^* \neq \emptyset$, and we define, for $f, g \in S$,

$$f \sim g \text{ iff there is } h \in I \cap S^* \text{ such that } fh = gh,$$

then $\sim$ is an equivalence relation, and the equivalence classes constitute a semigroup S/I provided $[f]_I[g]_I = [fg]_I$ where $[h]_I$ is the equivalence class containing $h \in S$.

If G is a semigroup with unity element e, an element $f \in G$ is said to be a unit iff there are $g,\ h \in G$ such that $fg = hf = e$.

LEMMA 6.2. *Let I be a t-ideal in S, $f \in S$. Then S/I exists, and $[f]_I$ is a unit in S/I iff there exists $k \in I$ such that* $\operatorname{supp} k \subset Z^c(f)$.

PROOF. By 2.8, $h \in I$ implies $|h| \in I$, hence $I \cap S^* \neq \emptyset$. Assume $[f]_I[g]_I = [1]_I$ for some $g \in S$ (1 denotes the constant 1; $[1]_I$ is a unity element in S/I). Then there is $k \in I$ such that $fgk = k$. If $k(x) \neq 0$, then $|k(x)| \neq 0$, hence $|f(x)| \cdot |g(x)| = 1$, $|f(x)| \neq 0$. By continuity, this is still valid for $x \in \operatorname{supp} k$. Thus $\operatorname{supp} k \subset Z^c(f)$.

Conversely, suppose $\operatorname{supp} k \subset Z^c(f)$, $k \in I$, and choose $h \in I$ such that $k >_t h$, i.e. by 2.4 $\operatorname{supp} h < \operatorname{supp} k$. By 2.8, we can assume $h(x) \in [0, +\infty)$ for $x \in X$ (replace h by $|h|$). There is $m \in S_0(X)$ satisfying $m(x) = 1$ for $x \in \operatorname{supp} h$, $m(x) = 0$ for $x \in X - \operatorname{supp} k$. Define

$$g(x) = m(x) \cdot \frac{1}{f(x)} \quad \text{for} \quad x \in Z^c(f),$$

$$g(x) = 0 \qquad\qquad \text{for} \quad x \in Z(f).$$

Then $g|Z^c(f)$ is continuous, and the same is true for $g|X - \operatorname{supp} k$. Finally g is continuous, $g \in S(X)$, and $fgh = h$, since $h(x) \neq 0$ implies $x \in Z^c(f)$, $m(x) = 1$. Similarly $gfh = h$, so that $[f]_I[g]_I = [1]_I$, $[g]_I[f]_I = [1]_I$. $\square$

Let us now introduce the spaces Y, Z, and the homeomorphism $t\colon Z \to Y$ just as in 4.

LEMMA 6.3. *Let I be a maximal t-ideal in S, and $f \in S$. Then $t^{-1}(I) \in \overline{Z(f)}$* (the closure is taken in Z) *iff $[f]_I$ is not a unit in S/I.*

PROOF. $z = t^{-1}(I)$ belongs to the closure of $Z(f)$ iff each element of $\mathfrak{s}(z)$ intersects $Z(f)$, i.e., by 4.4, iff each element of $\mathfrak{t}(z) = \operatorname{supp} I$ intersects $Z(f)$, i.e., by 6.2, iff $[f]_I$ is not a unit in S/I. $\square$

Let us now recall that υX is the subspace of $Z = \beta X$ composed of all points $z \in Z$ such that $\lim_{x \to z} h(x)$ exists and is finite for every $h \in C(X)$. For $f \in S$, denote by $Z'(f) \subset Y$ the set of all maximal t-ideals I such that $[f]_I$ is not a unit in S/I.

LEMMA 6.4. *For $z \in Z$, we have $z \in \upsilon X$ iff, for any $g \in C(Y)$ such that $g(t(z)) = 0$, there is $f \in S$ such that $\emptyset \neq Z'(f) \subset Z(g)$.*

PROOF. Suppose $z\in\upsilon X$ and $g\in C(Y)$, $g(t(z))=0$. Now $t(X)\subset Z^c(g)$ would imply, for $h(x)=\dfrac{1}{|g(t(x))|}$ $(x\in X)$, $h\in C(X)$ and $\lim_{x\to z} h(x)=+\infty$. Hence there is $x_0\in X$ such that $g(t(x_0))=0$, so that $f(x)=|g(t(x))|$ $(x\in X)$ defines a function $f\in S(X)$ such that $t(x_0)\in Z'(f)\subset Z(g)$, because $Z'(f)=t(\overline{Z(f)})$ by 6.3.

Conversely, suppose $z\in Z-\upsilon X$. Then there is $h\in C(X)$ such that $\lim_{x\to z}|h(x)|=+\infty$ (observe that any $h\in C(X)$ can be considered to be a continuous mapping from X into the one-point compactification $\mathbf{R}^*$ of $\mathbf{R}$, hence it has a continuous extension $h^*\colon Z\to\mathbf{R}^*$). By taking $\max(|h|, 1)$ instead of h, one can suppose $h\geqq 1$. Then there is a continuous extension $k^*\colon Z\to\mathbf{R}$ of $k=\dfrac{1}{h}\in C^*(X)$, and $g=k^*\circ t^{-1}\in C(Y)$ satisfies $g(t(z))=0$ and $t(X)\cap Z(g)=\emptyset$, so that $\emptyset\neq Z'(f)\subset Z(g)$ is impossible for $f\in S$ (use again $Z'(f)=t(\overline{Z(f)})$). □

If we know, for a quasi-commutative pseudo-real semigroup $\mathbf{S}$, the semigroup $S(X)$, then we can construct Y by 4.7, the set $Z'(f)\subset Y$ for any $f\in S(X)$, hence, according to 6.4, the subspace $t(\upsilon X)$ which is homeomorphic to υX.

THEOREM 6.5. *Let X_1 and X_2 be Tikhonov spaces, $\mathbf{S}_1$ and $\mathbf{S}_2$ quasi-commutative, pseudo-real semigroups, and $S_i(X_i)$ the set of all continuous mappings from X_i into $\mathbf{S}_i$, equipped with pointwise multiplication. If $S_1(X_1)$ and $S_2(X_2)$ are isomorphic, then υX_1 and υX_2 are homeomorphic.* □

This theorem corresponds to Theorem C; quasi-real semigroups are replaced by pseudo-real ones, but we assumed quasi-commutativity and isomorphy instead of d-isomorphy.

Let us remark that t-isomorphy of $C(X_1)$ and $C(X_2)$ does not imply homeomorphy of υX_1 and υX_2 in general. In fact, in [3], Theorem 1.5, it is shown that, if X is a non-compact, realcompact space, then there is a bijection $\varphi\colon C(X)\to C^*(X)$ such that

$$Z(\varphi(f))=Z(f),\quad Z(1-\varphi(f))=Z(1-f)$$

for $f\in C(X)$. From this, we easily deduce that $C(X)$ and $C^*(X)$ are t-isomorphic, i.e. the same holds for $C(X)$ and $C(\beta X)$.

7. Semigroups of semicontinuous functions. We add a remark on the multiplicative semigroup $U(X)$ (or $L(X)$, respectively) of nonnegative upper (lower) semicontinuous functions on a topological space X.

The following theorem strengthens the corresponding result on rings, which was established in [7].

THEOREM 7.1. *Let X_1 and X_2 be T_1-spaces. If $U(X_1)$ and $U(X_2)$ (or $L(X_1)$ and $L(X_2)$, respectively) are u-isomorphic, then X_1 and X_2 are homeomorphic.*

PROOF. We construct a space homeomorphic to X from the set $U(X)$ equipped with the relation $>_u$. For this purpose, let I denote the subset of $U(X)$ composed of all elements $f\in U(X)$ such that $f>_u f$ (i.e. $f=f^2$). Clearly I consists of all characteristic functions $\chi(F)$ of closed subsets of F, and, for closed sets $F_1, F_2\subset X$, $F_1\subset F_2$ iff $\chi(F_1)>_u\chi(F_2)$. Hence $f_0=\chi(\emptyset)$ is the unique element $f_0\in I$ such that $f_0>_u g$ for every $g \in I$. Let $Y \subset I$ be composed of all elements $g \in I$ such that $g \neq f_0$ and $f \in I$, $f >_u g$ imply $f = f_0$ or $f = g$. Then Y consists of the functions $\chi(\{x\})$ for $x \in X$. Therefore $\omega(x) = \chi(\{x\})$ defines a bijection $\omega: X \to Y$, and $\omega(F) = \{g \in Y: g >_u \chi(F)\}$ for any closed set $F \subset X$. Hence ω is a homeomorphism from X onto Y provided the latter is equipped with the topology for which the closed sets are those having the form $\{g \in Y: g >_u f\}$ $(f \in I)$.

A similar construction applies to $L(X)$. $\square$

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ÜBER EINE ZWEIPARAMETRIGE FAMILIE VON MITTELWERTEN

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1. Definition von $E(r, s;\ x, y)$

In einer Note aus dem Jahre 1975 hat K. B. Stolarsky [11] für positive reelle Zahlen x und y (mit $x \neq y$) und für reelle Parameter r und s folgende Familie von Mittelwerten eingeführt:

$$E(r, s;\ x, y) = \left[\frac{s}{r}\,\frac{x^r - y^r}{x^s - y^s}\right]^{1/(r-s)}, \quad r \neq s, \quad rs \neq 0,$$

$$E(r, 0;\ x, y) = \lim_{s \to 0} E(r, s;\ x, y) = \left[\frac{x^r - y^r}{r \ln (x/y)}\right]^{1/r}, \quad r \neq 0,$$

$$E(r, r;\ x, y) = \lim_{s \to r} E(r, s;\ x, y) = e^{-1/r}(x^{x^r}/y^{y^r})^{1/(x^r - y^r)}, \quad r \neq 0,$$

$$E(0, 0;\ x, y) = (xy)^{1/2}.$$

Die Mittelwertfamilie E enthält neben den drei klassischen Mittelwerten:

dem geometrischen Mittel:	$G(x, y) = E(r, -r; x, y) = (xy)^{1/2}$,
dem arithmetischen Mittel:	$A(x, y) = E(1, 2; x, y) = (x+y)/2$,
und dem harmonischen Mittel:	$H(x, y) = E(-2, -1; x, y) = 2xy/(x+y)$,

auch das logarithmische Mittel:

$$L(x, y) = E(1, 0;\ x, y) = \frac{x-y}{\ln x - \ln y}$$

als Spezialfall.

Dem Mittelwert L kommt bei praktischen Problemen aus den Gebieten Physik und Wirtschaftswissenschaft eine besondere Bedeutung zu (siehe [8—10]). Darüber hinaus ist er Gegenstand zahlreicher rein-mathematischer Untersuchungen. Insbesondere sind eine Reihe bemerkenswerter Ungleichungen für das logarithmische Mittel veröffentlicht worden; siehe [3; Chapter VI] und die dort angegebene Literatur.

Außer Stolarsky haben sich vor allem E. B. Leach und M. C. Sholander [5—7] mit E beschäftigt. Während sie sich sich in [5] vor allem mit Monotoniefragen befassen, behandeln sie in [6] das Problem, für welche Parameterpaare (r, s) und (r', s') die Ungleichung

$$E(r, s;\ x, y) \leqq E(r', s';\ x, y)$$

erfüllt ist; und in [7] untersuchen die beiden Autoren eine verallgemeinerte Mittelwertfamilie der Form $E(r, s; x_0, \ldots, x_n)$. Von Leach/Sholander ist für den Mittelwert

$$I(x, y) = E(1, 1; x, y) = \frac{1}{e}(x^x/y^y)^{1/(x-y)}$$

die Bezeichnung „identric mean“ gewählt worden. Auf Grund der (von Stolarsky [11] gefundenen) Integralformel

$$E(r, s; x, y) = \exp \frac{1}{s-r} \int_r^s \frac{1}{t} \ln I(x^t, y^t)\, dt$$

spielt I eine „central role“ [5, p. 209] innerhalb der Mittelwertfamilie E. Ungleichungen für I findet man außer in den erwähnten Arbeiten [5—7, 11] auch in [3, Chapter VI].

In dieser Note wollen wir die Untersuchungen über E fortsetzen. Unser Ziel ist es, den von Stolarsky bewiesenen Satz, daß $r \mapsto E(r, s; x, y)$ in $\mathbf{R}$ monoton steigt, zu verallgemeinern, indem wir nachweisen, daß es sich bei

$$r \mapsto E(r, s; x, y)/E(r, s; u, v)$$

um eine in $\mathbf{R}$ streng monoton steigende Funktion handelt, wenn die positiven reellen Zahlen x, y, u und v der Bedingung $x/y > u/v \geqq 1$ genügen.

Mit Hilfe dieses Resultats werden wir anschließend zeigen, wie sich die von Ky Fan stammende Ungleichung

$$\prod_{k=1}^{n} (x_k/(1-x_k))^{1/n} \leqq \sum_{k=1}^{n} x_k \Big/ \sum_{k=1}^{n} (1-x_k), \quad 0 < x_k \leqq 1/2, \quad k = 1, \dots, n,$$

für den Sonderfall $n=2$ verschärfen läßt.

2. Die Funktion $E(r, s; x, y)/E(r, s; u, v)$

Im ersten Teil dieses Abschnitts beweisen wir eine Ungleichung zwischen dem geometrischen und dem logarithmischen Mittel, die wir zum Beweis von Satz 2 benötigen.

SATZ 1. *Für alle positiven reellen Zahlen x, y, u und v mit $x/y > u/v > 1$ und für alle reellen Zahlen $r \neq 0$ gilt:*

$$G(x^r, y^r)/G(u^r, v^r) < L(x^r, y^r)/L(u^r, v^r). \tag{1}$$

BEWEIS. Wenn wir $a = x/y$ und $b = u/v$ setzen und mit f die Funktion

$$f(r) = f(r; a, b) = (b/a)^{r/2}(a^r - 1)/(b^r - 1), \quad r \neq 0,$$

$$f(0) = f(0; a, b) = \ln a/\ln b,$$

bezeichnen, dann sind die beiden Ungleichungen: (1) und $f(r) > f(0)$ für $r \neq 0$ einander äquivalent.

Differentiation von f ergibt für $r \neq 0$:

$$2r \frac{f'(r)}{f(r)} = g(a^r) - g(b^r) \quad \text{mit: } g(z) = \frac{z+1}{z-1} \ln(z).$$

Für $z>1$ gilt:

$$g'(z)(z-1)^2 = -2\ln(z)+z-1/z = \sum_{i=3}^{\infty}\left(\frac{z-1}{z}\right)^i(1-2/i) > 0;$$

somit ist g im Intervall $(1, \infty)$ streng monoton steigend und wir erhalten wegen $a^r > b^r > 1$ $(r>0)$ die Abschätzung $g(a^r)>g(b^r)$. Auf Grund von $f(r)>0$ gilt $f'(r)>0$ für $r>0$. Da f eine gerade Funktion ist, folgt: f ist in $\mathbf{R}^+$ streng monoton steigend und in $\mathbf{R}^-$ streng monoton fallend; insbesondere gilt für alle $r\neq 0$: $f(r)>f(0)$. □

BEMERKUNG. Die Ungleichung (1) ist auch für $u=v$ gültig. In diesem Fall ist (1) mit der Carlson Ungleichung

$$G(a, b) < L(a, b), \quad a, b > 0, \quad a \neq b,$$

identisch (siehe [4]).

Wir beweisen nun folgenden Monotoniesatz.

SATZ 2. *Es seine x, y, u und v positive reelle Zahlen mit $x/y>u/v\geqq 1$. Dann ist die Funktion*

$$E(r, s; x, y)/E(r, s; u, v)$$

bezüglich r in $\mathbf{R}$ streng monoton steigend,

BEWEIS. Für den Spezialfall $u=v$ ist Satz 2 (wie im ersten Abschnitt erwähnt) von Stolarsky bewiesen worden, so daß wir ohne Einschränkung $u/v>1$ voraussetzen können.

Wir bezeichnen mit h die Funktion

$$h(r) = h(r;\ x, y, u, v) = \ln\frac{x^r-y^r}{u^r-v^r}, \quad r\neq 0,$$

$$h(0) = h(0;\ x, y, u, v) = \ln\frac{\ln(x/y)}{\ln(u/v)}.$$

Eine kleine Rechnung ergibt für $r\neq 0$:

$$h''(r) = \left[\frac{1}{r}\,\frac{G(u^r, v^r)}{L(u^r, v^r)}\right]^2\left\{1-\left[\frac{L(u^r, v^r)}{L(x^r, y^r)}\,\frac{G(x^r, y^r)}{G(u^r, v^r)}\right]^2\right\}.$$

Nach Satz 1 erhalten wir $h''(r)>0$ für $r\neq 0$. Also ist h in $\mathbf{R}$ streng konvex und auf Grund von

$$\ln\frac{E(r, s;\ x, y)}{E(r, s;\ u, v)} = \begin{cases}\dfrac{h(r)-h(s)}{r-s}, & r\neq s,\\ h'(r), & r = s,\end{cases}$$

folgt, daß $\ln\dfrac{E(r, s; x, y)}{E(r, s; u, v)}$ und somit auch $\dfrac{E(r, s; x, y)}{E(r, s; u, v)}$ bezüglich r in $\mathbf{R}$ streng monoton steigt.

Da h' in $\mathbf{R}$ streng monoton steigt, haben wir insbesondere bewiesen, daß die Funktion

$$E(r, r;\ x, y)/E(r, r;\ u, v) = \exp h'(r)$$

dieselbe Monotonieeigenschaft besitzt. □

3. Verschärfung einer Ungleichung von Ky Fan

In ihrem Buch „Inequalities" haben E. F. Beckenbach und R. Bellman das folgende „unpublished result due to Ky Fan" [2, p. 5] angegeben:

Wenn mit $G(x_1, \ldots, x_n)$ das geometrische und mit $A(x_1, \ldots, x_n)$ das arithmetische Mittel von $x_1, \ldots, x_n$ bezeichnet wird, dann gilt für $0<x_k \leqq 1/2$, $k=1, \ldots, n$:

$$\frac{G(x_1, \ldots, x_n)}{G(1-x_1, \ldots, 1-x_n)} \leqq \frac{A(x_1, \ldots, x_n)}{A(1-x_1, \ldots, 1-x_n)}. \tag{2}$$

Das Gleichheitszeichen gilt in (2) genau dann, wenn $x_1 = \ldots = x_n$. Die Ungleichung (2) läst sich, wie Beckenbach/Bellman erwähnen, leicht durch „forward and backward induction" beweisen (vgl. [2]). Neue Beweise sowie Verallgemeinerungen und Verschärfungen der Fan Ungleichung sind in [1], [3] veröffentlicht worden.

Wir sind nun in der Lage, die Abschätzung (2) für den Sonderfall $n=2$ zu verschärfen, denn nach Satz 2 folgt:

$$\frac{G(x, y)}{G(1-x, 1-y)} < \frac{E(r, s;\ x, y)}{E(r, s;\ 1-x, 1-y)} < \frac{A(x, y)}{A(1-x, 1-y)},$$

$$0 < x,\ y \leqq \frac{1}{2}; \quad x \neq y,$$

für alle Parameterpaare (r, s) mit:

$$-1 \leqq -s < r \leqq 2, \quad (r, s) \neq (2, 1) \quad \text{oder} \quad -1 \leqq -r < s \leqq 2, \quad (r, s) \neq (1, 2).$$

Insbesondere erhalten wir als Gegenstück zu den in [5] und [11] bewiesenen Ungleichungen:

$$G(x, y) < L(x, y) < \frac{1}{2}\big(G(x, y)+A(x, y)\big) < I(x, y) < A(x, y), \quad x \neq y,$$

die folgenden Abschätzungen (man beachte: $E(1/2, 1; x, y) = \frac{1}{2}\big(G(x, y)+A(x, y)\big)$):

$$\frac{G(x, y)}{G(1-x, 1-y)} < \frac{L(x, y)}{L(1-x, 1-y)} < \frac{G(x, y)+A(x, y)}{G(1-x, 1-y)+A(1-x, 1-y)} <$$

$$< \frac{I(x, y)}{I(1-x, 1-y)} < \frac{A(x, y)}{A(1-x, 1-y)}, \quad 0 < x,\ y \leqq 1/2, \quad x \neq y.$$

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FONCTIONS ARITHMÉTIQUES TRONQUÉES

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1. Introduction

Soit $m \geqq 0$ un entier. Pour $n \geqq 1$ un entier, on dénote par $\omega(n)$ la fonction arithmétique qui désigne le nombre de nombres premiers distincts qui divisent n et par $\mu(n)$ la fonction de Möbius. Pour obtenir une approche générale de la méthode du crible combinatoire [3], on doit, par exemple, utiliser l'identité

$$\sum_{\substack{d \mid n \\ \omega(d) \leqq m}} \mu(d) = (-1)^m \binom{\omega(n)-1}{m}.$$

D'une façon générale toutefois, il n'est pas facile de trouver une formule fermée pour les sommes $\sum_{\substack{d \mid n \\ \omega(d) \leqq m}} g(d)$, où $g(n)$ est une fonction arithmétique arbitraire. Par exemple, pour $g(n)=1$, on a pour $n=p_1^{\alpha_1} \dots p_k^{\alpha_k}$

$$\sum_{\substack{d \mid n \\ \omega(d) \leqq m}} 1 = 1 + \sum_{1 \leqq i \leqq k} \alpha_i + \sum_{1 \leqq i < j \leqq k} \alpha_i \alpha_j + \sum_{1 \leqq i < j < r \leqq k} \alpha_i \alpha_j \alpha_r + \dots$$
$$\dots + \sum_{1 \leqq i_1 < \dots < i_m \leqq k} \alpha_{i_1} \dots \alpha_{i_m}.$$

En posant $\prod_{i=1}^{k} (1+\alpha_i t) = 1 + \sum_{i=1}^{k} a_i(\alpha_1, \dots, \alpha_k) t^i$ et en définissant le polynôme $q_m(\alpha_1, \dots, \alpha_k; t)$ de degré m $(m \leqq k)$ par

$$q_m(\alpha_1, \dots, \alpha_k;\ t) = 1 + \sum_{i=1}^{m} a_i(\alpha_1, \dots, \alpha_k) t^i,$$

on obtient

$$\sum_{\substack{d \mid n \\ \omega(d) \leqq m}} 1 = q_m(\alpha_1, \dots, \alpha_k;\ 1).$$

Dans le cas où n est libre de carrés, cette dernière identité devient

$$\sum_{\substack{d \mid n \\ \omega(d) \leqq m}} 1 = q_m(1, 1, \dots, 1;\ 1) = \binom{k}{0} + \binom{k}{1} + \dots + \binom{k}{m} =$$
$$= 2^m + \sum_{j=1}^{m} 2^{j-1} \binom{k-j}{m+1-j},$$

d'après ([5, p. 130]). Ceci nous amène à poser

$$g_m(n) = \sum_{\substack{d|n \\ \omega(d) \leqq m}} g(n/d) \tag{1}$$

où $g(n)$ est une fonction arithmétique connue. Les résultats que nous obtiendrons concernant $g_m(n)$ vont nous permettre de généraliser quelques résultats d'Alladi [1] lesquels sont utiles pour obtenir des informations sur le plus petit et le plus grand facteur premier de n. De plus, quelques généralisations de certains théorèmes de la théorie élémentaire des nombres seront obtenues.

2. Quelques lemmes

LEMME 1. *Soit m, r et k des entiers non-négatifs satisfaisant à $r \geqq k+m$. Alors on a*

$$\sum_{m+1 \leqq i_1 < i_2 < \ldots < i_k \leqq r} 1 = \binom{r-m}{k}.$$

DÉMONSTRATION. Puisque le nombre de «$r-m$» objets pris k à la fois est $\binom{r-m}{k}$, le résultat est immédiat.

LEMME 2. *Etant donnés m, r et k des entiers non-négatifs satisfaisant à $r \geqq k+m$, et soit (a_i) une suite d'entiers telle que $a_i \geqq 2$ pour chaque i. Alors*

$$\sum_{m+1 \leqq i_1 < i_2 < \ldots < i_k \leqq r} a_{i_k} = \sum_{j=k+m}^{r} \binom{j-m-1}{k-1} a_j.$$

DÉMONSTRATION. On utilise un raisonnement par induction sur k. Pour $k=1$, le résultat est trivial. Pour $k+1$, on a

$$\sum_{m+1 \leqq i_1 < \ldots < i_{k+1} \leqq r} a_{i_{k+1}} = \sum_{m+1 \leqq i_1 \leqq r-k} \Bigg(\sum_{i_1+1 \leqq i_2 < \ldots < i_{k+1} \leqq r} a_{i_{k+1}} \Bigg) \tag{2}$$

et en utilisant l'hypothèse d'induction, la somme de droite peut s'écrire sous la forme

$$\sum_{m+1 \leqq i_1 \leqq r-k} \left(\sum_{j=k+i_1}^{r} \binom{j-i_1-1}{k-1} a_j \right).$$

Ainsi (2) devient

$$\sum_{m+1 \leqq i_1 < \ldots < i_{k+1} \leqq r} a_{i_{k+1}} = \sum_{j=k+m+1}^{r} \binom{j-m-2}{k-1} a_j + \sum_{j=k+m+2}^{r} \binom{j-m-3}{k-1} a_j + \ldots \tag{3}$$

$$\ldots + \sum_{j=r}^{r} \binom{j-r+k-1}{k-1} a_j.$$

Soit $0 \leqq n \leqq r-k-m-1$, alors le coefficient qui multiplie $a_{k+n+m+1}$ est égal à

$$\sum_{j=1}^{n+1} \binom{k+n-j}{k-1} = \binom{k+n}{k}.$$

Dans ce cas, (3) devient

$$\sum_{m+1\leqq i_1<\dots<i_{k+1}\leqq r} a_{i_{k+1}} = \sum_{n=0}^{r-k-m-1} \binom{k+n}{k} a_{k+n+m+1} = \sum_{j=k+m+1}^{r} \binom{j-m-1}{k} a_j,$$

ce qui démontre le résultat.

LEMME 3. *Soit k et r des entiers positifs tels que $r>k$. Alors pour chaque entier $s\in[1,k]$, on a*

$$\sum_{1\leqq i_1<\dots<i_s<\dots<i_k\leqq r} a_{i_s} = \sum_{j=s}^{r-k+s} \binom{j-1}{s-1}\binom{r-j}{k-s} a_j.$$

DÉMONSTRATION. D'après le lemme 1, on peut écrire

$$\sum_{1\leqq i_1<\dots<i_s<\dots<i_k\leqq r} a_{i_s} = \sum_{1\leqq i_1<\dots<i_s\leqq r+s-k} \binom{r-i_s}{k-s} a_{i_s}$$

et en utilisant le lemme 2, on obtient le résultat.

LEMME 4. *Soit n et m des entiers non-négatifs. Alors pour tout entier positif k, on a*

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{k+j}{m} = (-1)^n \binom{k}{m-n}.$$

DÉMONSTRATION. Puisque

$$\sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j}\binom{k+j}{m} = \sum_{j=0}^{n+1} (-1)^j \left\{\binom{n}{j}+\binom{n}{j-1}\right\}\binom{k+j}{m} =$$

$$= \sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{k+j}{m} - \sum_{j=0}^{n} (-1)^j \binom{n}{j}\binom{k+j+1}{m}$$

alors en utilisant cette égalité ainsi que l'induction sur n, on obtient le résultat.

3. Séries de Dirichlet

Si dans (1), on remplace $g(n)$ par $\mu(n)$ alors il s'ensuit que

$$\mu_m(n) = \sum_{\substack{d|n\\ \omega(d)\leqq m}} \mu(n/d). \tag{4}$$

Il est immédiat que pour $m\geqq\omega(n)$, (4) devient $\mu_m(n)=\sum_{d|n}\mu(d)$, tandis que si $m=0$, $\mu_0(n)=\mu(n)$. Cependant, l'équation (4) peut s'écrire sous la forme

$$\mu_m(n) = \sum_{d|n} \mu(n/d)\beta(d)$$

qui est équivalent à

$$\sum_{d|n} \mu_m(d) = \beta(n) = \begin{cases} 1 & \text{si} \quad \omega(n)\leqq m \\ 0 & \text{si} \quad \omega(n)>m. \end{cases} \tag{5}$$

REMARQUE. Lorsque $m=0$, (5) se réduit à un résultat connu:

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{si} \quad n=1 \\ 0 & \text{si} \quad n>1. \end{cases}$$

Soit α_0 l'abscisse de convergence absolue des séries de Dirichlet, alors pour $\operatorname{Re} s>\alpha_0$ on a

$$\left(\sum_{n=1}^{\infty} \frac{\mu_m(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right) = \sum_{n=1}^{\infty} \left(\frac{\sum_{d|n} \mu_m(d)}{n^s}\right) = \sum_{\substack{n=1 \\ \omega(n)\leqq m}}^{\infty} \frac{1}{n^s}.$$

On définit maintenant formellement $\zeta_m(s)$ comme étant la série $\sum_{\substack{n=1 \\ \omega(n)\leqq m}}^{\infty} \frac{1}{n^s}$, ce qui nous permet d'écrire

$$\sum_{n=1}^{\infty} \frac{\mu_m(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \cdot \zeta_m(s). \tag{6}$$

Il est facile de voir que $\zeta_0(s)=1$ et que $\lim_{m\to\infty} \zeta_m(s)=\zeta(s)$ où $\zeta(s)$ désigne la fonction zeta de Riemann. Généralisons l'identité obtenue en (6).

THEORÈME 1. *Pour* $\operatorname{Re} s>\alpha_0$, *on a*

$$\sum_{n=1}^{\infty} \frac{g_m(n)}{n^s} = \sum_{n=1}^{\infty} \frac{g(n)}{n^s} \cdot \zeta_m(s),$$

où

$$g_m(n) = \sum_{\substack{d|n \\ \omega(d)\leqq m}} g(n/d).$$

DÉMONSTRATION. Ceci est immédiat d'après l'identité $g_m(n)=(g*\beta)(n)$, où $*$ désigne le produit de Dirichlet et $\beta(n)$ la fonction définie en (5).

COROLLAIRE. *Pour* $\operatorname{Re} s>1$, *on a*

$$\sum_{n=1}^{\infty} \frac{d_m(n)}{n^s} = \zeta^2(s)\zeta_m(s),$$

où $d_m(n) = \sum_{\substack{d|n \\ \omega(d)\leqq m}} d(n/d)$ et $d(n)=\sum_{d|n} 1$.

COROLLAIRE. *Soit* f *et* g *des fonctions arithmétiques arbitraires et soit* $h(n)=\sum_{d|n} g(d)f(n/d)$. *Alors pour* $\operatorname{Re} s>\alpha_0$, *on a*

$$\sum_{n=1}^{\infty} \left(\frac{\sum_{d|n} g_m(d)f(n/d)}{n^s}\right) = \zeta_m(s) \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$$

où $g_m(n) = \sum_{\substack{d|n \\ \omega(d)\leqq m}} g(n/d)$.

DÉMONSTRATION. En utilisant le théorème 1, on obtient

$$\sum_{n=1}^{\infty}\left(\frac{\sum_{d|n} g_m(d)f(n/d)}{n^s}\right)=\sum_{n=1}^{\infty}\frac{g_m(n)}{n^s}\cdot\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=\zeta_m(s)\cdot\sum_{n=1}^{\infty}\frac{g(n)}{n^s}\cdot\sum_{n=1}^{\infty}\frac{f(n)}{n^s},$$

ce qui démontre le résultat.

4. La fonction $\mu_m(n)$

THÉORÈME 2. *Soit* $n\geqq 1$ *un entier. Alors pour tout entier* $m\geqq 0$, *on a*

$$\mu_m(n)=(-1)^m\binom{\omega(n)-1}{m}\mu(n).$$

DÉMONSTRATION. D'après (5), il suffit de montrer que

$$\sum_{d|n}(-1)^m\binom{\omega(d)-1}{m}\mu(d)=\begin{cases}1 & \text{si}\quad \omega(n)\leqq m\\ 0 & \text{si}\quad \omega(n)>m.\end{cases}$$

Puisque $\binom{-1}{m}=(-1)^m$, alors pour $n=1$ on a le résultat. Soit $n>1$ et supposons que $1\leqq\omega(n)\leqq m$. Si $1<d|n$, alors $\binom{\omega(d)-1}{m}=0$ et ainsi $\sum_{\substack{d|n\\ d>1}}(-1)^m\binom{\omega(d)-1}{m}\mu(d)=$ $=0$. Cela prouve le résultat pour $0\leqq\omega(n)\leqq m$. Supposons maintenant que $\omega(n)>m$. En posant $n=p_1^{\alpha_1}\dots p_k^{\alpha_k}$, $k\geqq m+1$, alors pour $1<d|n$, il suffit de considérer les diviseurs « d » ayant la forme $d=p_1\dots p_j$, $m+1\leqq j\leqq k$. Ainsi, on peut écrire

$$\sum_{\substack{d|n\\ d>1}}(-1)^m\binom{\omega(d)-1}{m}\mu(d)=$$

$$=-\sum_{1\leqq i_1<\dots<i_{m+1}\leqq k}1+\binom{m+1}{m}\sum_{1\leqq i_1<\dots<i_{m+2}\leqq k}1+\dots+(-1)^k\binom{k-1}{m}$$

et d'après le lemme 1, on obtient

$$\sum_{\substack{d|n\\ d>1}}(-1)^m\binom{\omega(d)-1}{m}\mu(d)=-\left\{\binom{k}{m+1}-\binom{m+1}{m}\binom{k}{m+2}+\dots+(-1)^{k-1}\binom{k-1}{m}\right\}=$$

$$=\frac{-k!}{m!(k-m-1)!}\sum_{i=0}^{k-m-1}\frac{(-1)^i\binom{k-m-1}{i}}{m+1+i}.$$

Mais en utilisant les fractions partielles, il est facile de montrer que

$$\sum_{i=0}^{k-m-1}\frac{(-1)^i\binom{k-m-1}{i}}{m+1+i}=\frac{(k-m-1)!}{(m+1)(m+2)\dots k},$$

et ainsi on obtient le résultat pour $\omega(n)>m$. Ceci achève la démonstration du théorème 2.

THÉORÈME 3. *Pour chaque couple d'entiers* $r \geqq 0$, $m \geqq 0$ *on a*

$$\sum_{\substack{d|n \\ \omega(d) \leqq r}} \mu_m(d) = (-1)^m \sum_{j=0}^{r} (-1)^j \binom{j-1}{m} \binom{\omega(n)}{j}.$$

DÉMONSTRATION. Puisque

$$\sum_{\substack{d|n \\ \omega(d)=r}} \mu_m(d) = (-1)^{m+r} \binom{r-1}{m} \sum_{1 \leqq i_1 < \dots < i_r \leqq k} 1,$$

où $k=\omega(n)$, alors d'après le lemme 1 on obtient

$$\sum_{\substack{d|n \\ \omega(d)=r}} \mu_m(d) = (-1)^{m+r} \binom{r-1}{m} \binom{\omega(n)}{r}$$

et ceci démontre le résultat.

REMARQUES. 1) Lorsque $m=0$, on obtient l'identité déjà mentionnée dans l'introduction, à savoir

$$\sum_{\substack{d|n \\ \omega(d) \leqq r}} \mu(d) = (-1)^r \binom{\omega(n)-1}{r}.$$

2) En utilisant l'identité $\binom{\omega(n)}{j} = \binom{\omega(n)-1}{j} + \binom{\omega(n)-1}{j-1}$, l'expression du théorème précédent devient

$$\sum_{\substack{d|n \\ \omega(d) \leqq r}} \mu_m(d) = \begin{cases} 1 & \text{si} \quad r \leqq m \\ (-1)^{m+r}\, r \binom{r-1}{m} \displaystyle\sum_{j=1}^{k-m-1} \frac{\binom{m+j}{r}}{j} & \text{si} \quad r > m. \end{cases}$$

D'après le corollaire 2, on peut écrire

$$(7) \qquad \sum_{n=1}^{\infty} \left(\frac{\sum_{d|n} g_m(d) f(n/d)}{n^s} \right) = \sum_{n=1}^{\infty} \left(\frac{\sum_{\substack{d|n \\ \omega(d) \leqq m}} h(n/d)}{n^s} \right) = \sum_{n=1}^{\infty} \frac{h_m(n)}{n^s},$$

et ceci nous permet d'énoncer le résultat suivant.

THÉORÈME 4. *Soit* $n \geqq 1$ *et soit* f *une fonction airthmétique arbitraire. Alors pour tout entier* $m \geqq 0$, *on a*

$$\sum_{d|n} \mu_m(d) f(n/d) = \sum_{\substack{d|n \\ \omega(d) \leqq m}} g(n/d)$$

où $g(n) = \sum_{d|n} \mu(d) f(n/d)$.

THÉORÈME 5 (Inversion de Möbius). *Soit* $n \geqq 1$ *et soit* f *et* g *deux fonctions arithmétiques arbitraires. Alors pour tout entier* $m \geqq 0$, *on a*

$$\sum_{d|n} \mu_m(d) f(n/d) = g_m(n) \Leftrightarrow f_m(n) = \sum_{d|n} g_m(d),$$

où

$$f_m(n) = \sum_{\substack{d|n \\ \omega(d) \leqq m}} f(n/d) \quad et \quad g_m(n) = \sum_{\substack{d|n \\ \omega(d) \leqq m}} g(n/d).$$

DÉMONSTRATION. Soit $l(n)$ et $\beta(n)$ deux fonctions arithmétiques définies par $l(n)=1$, pour tout $n \geqq 1$ et $\beta(n) = \begin{cases} 1 \text{ si } \omega(n) \leqq m \\ 0 \text{ si } \omega(n) > m \end{cases}$. Alors on a $(\mu_m * f)(n) = g_m(n)$ ou encore

$$((1 * \mu_m) * f)(n) = (1 * g_m)(n).$$

En utilisant (5), cette dernière identité est équivalente à

$$(\beta * f)(n) = (1 * g_m)(n)$$

ou encore

$$\sum_{\substack{d|n \\ \omega(d) \leqq m}} f(n/d) = \sum_{d|n} g_m(d)$$

ce qui donne le résultat.

5. Identités contenant la fonction $\mu_m(n)$

L'objet de cette section est centré sur l'étude des sommes de la forme $\sum_{d|n} \mu_m(d) f(d)$ dans le cas où f est une fonction arithmétique choisie. Si $0 \leqq \omega(n) \leqq m$, il est immédiat que $\sum_{d|n} \mu_m(d) f(d) = f(l)$. Supposons maintenant que $\omega(n) > m$. En posant $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$, $k \geqq m+1$, alors pour $d|n$, $d>1$, il suffit de considérer les diviseurs « d » ayant la forme $d = p_1 \dots p_j$, $m+1 \leqq j \leqq k$. Donc

(8)
$$\sum_{\substack{d|n \\ d>1}} \mu_m(d) f(d) = - \sum_{1 \leqq i_1 < \dots < i_{m+1} \leqq k} f(p_{i_1} \dots p_{i_{m+1}}) +$$
$$+ \binom{m+1}{m} \sum_{1 \leqq i_1 < \dots < i_{m+2} \leqq k} f(p_{i_1} \dots p_{i_{m+2}}) + \dots + (-1)^{k+m} \binom{k-1}{m} f(p_1 \dots p_k).$$

Dans le cas où $f(n) = p_r(n)$, $(r \geqq 1)$, où $p_r(n)$ désigne le « r » ième plus petit facteur premier de n. L'équation (8) nous permet d'obtenir une relation intéressante qui lie $\sum_{d|n} \mu_m(d) p_r(d)$ avec une expression impliquant la fonction $P_r(n)$, qui désigne le « r » ième plus grand facteur premier de n. Plus précisément, on démontre le résultat suivant :

THÉORÈME 6. *Soit* $n\geqq 1$ *et* $k=\omega(n)$ *alors pour tout entier* $m\geqq 0$, *on a*

$$\sum_{d|n}\mu_m(d)p_r(d)=$$

$$=\begin{cases}\dfrac{(-1)^{m+r+k}}{\binom{m}{r-1}}\displaystyle\sum_{i=\max\{r,k-m\}}^{k+r-m-1}(-1)^i\binom{i-1}{r-1}\binom{k-i}{m-r+1}\binom{m}{m+i-k}P_{k-i+1}(n) & \\ & si\quad 1\leqq r\leqq m+1\\ (-1)^{m+r+k}\displaystyle\sum_{i=\max\{r,k-m\}}^{k}(-1)^i\binom{i-1}{r-1}\binom{r-1}{m-k+i}P_{k-i+1}(n) & si\quad r\geqq m+1.\end{cases}$$

DÉMONSTRATION. Soit $1\leqq r\leqq m+1$ et posons $f(n)=p_r(n)$. Puisque $p_r(1)=0$, alors pour $\omega(n)>m$, l'équation (8) devient

$$\sum_{d|n}\mu_m(d)p_r(d)=-\sum_{i\leqq i_1<\ldots<i_r<\ldots<i_{m+1}\leqq k}p_{i_r}+\binom{m+1}{m}\sum_{1\leqq i_1<\ldots<i_r<\ldots<i_{m+2}\leqq k}p_{i_r}-$$

$$-\binom{m+2}{m}\sum_{1\leqq i_1<\ldots<i_r<\ldots<i_{m+3}\leqq k}p_{i_r}+\ldots+(-1)^{m+k}\binom{k-1}{m}p_r,$$

et en utilisant le lemme 3, on peut écrire

$$\sum_{d|n}\mu_m(d)p_r(d)=$$

$$=-\sum_{j=r}^{k+r-m-1}\binom{j-1}{r-1}\binom{k-j}{m+1-r}p_j+\binom{m+1}{m}\sum_{j=r}^{k+r-m-2}\binom{j-1}{r-1}\binom{k-j}{m+2-r}p_j+\ldots$$

$$\ldots+(-1)^{k+r-m-i}\binom{k+r-i-1}{m}\sum_{j=r}^{i}\binom{j-1}{r-1}\binom{k-j}{k-i}p_j+\ldots+(-1)^{k+m}\binom{k-1}{m}p_r.$$

Or le coefficient de p_i, $1\leqq r\leqq i\leqq k+r-m-2$ est égal à

$$-\left\{\binom{i-1}{r-1}\binom{k-i}{m+1-r}-\binom{i-1}{r-1}\binom{m+1}{m}\binom{k-i}{m+2-r}+\ldots\right.$$

$$\left.\ldots+(-1)^{k+r-m-i-1}\binom{k+r-i-1}{m}\binom{i-1}{r-1}\right\}=$$

$$=-\binom{i-1}{r-1}\sum_{j=0}^{k+r-m-i-1}(-1)^j\binom{m+j}{m}\binom{k-i}{m+j+1-r}=$$

$$=-\frac{\binom{i-1}{r-1}\binom{k-i}{m+1-r}}{\binom{m}{r-1}}\sum_{j=0}^{k+r-m-i-1}(-1)^j\binom{m+j}{r-1}\binom{k+r-m-i-1}{j}$$

et d'après le lemme 4, ce dernier terme vaut

$$\frac{\binom{i-1}{r-1}\binom{k-i}{m+1-r}}{\binom{m}{r-1}}(-1)^{k+r-m-i}\binom{m}{m+i-k}.$$

De plus, le facteur de $p_{k+r-m-1}$ est $-\binom{k+r-m-2}{r-1}$, d'où pour $1\leqq r\leqq m+1$ on a

$$\sum_{d|n}\mu_m(d)p_r(d)=-\binom{k+r-m-2}{r-1}p_{k+r-m-1}+$$

$$+\frac{(-1)^{k+m+r}}{\binom{m}{r-1}}\sum_{i=\max\{r,k-m\}}^{k+r-m-2}(-1)^i\binom{i-1}{r-1}\binom{k-i}{m+1-r}\binom{m}{m+i-k}p_i=$$

$$=\frac{(-1)^{k+m+r}}{\binom{m}{r-1}}\sum_{i=\max\{r,k-m\}}^{k+r-m-1}(-1)^i\binom{i-1}{r-1}\binom{k-i}{m+1-r}\binom{m}{m+i-k}p_i$$

et ceci est le résultat de la première partie.

Supposons maintenant $m+1\leqq r\leqq\omega(n)=k$, alors (8) devient

$$\sum_{d|n}\mu_m(d)p_r(d)=$$

$$=(-1)^{m+r}\binom{r-1}{m}\sum_{1\leqq i_1<\ldots<i_r\leqq k}p_{i_r}+(-1)^{m+r+1}\binom{r}{m}\sum_{1\leqq i_1<\ldots<i_{r+1}\leqq k}p_{i_r}+\ldots$$

$$\ldots+(-1)^{m+k}\binom{k-1}{m}p_r.$$

En utilisant le lemme 3, cette dernière identité devient

$$\sum_{d|n}\mu_m(d)p_r(d)=(-1)^{m+r}\binom{r-1}{m}\sum_{j=r}^{k}\binom{j-1}{r-1}p_j+$$

$$+(-1)^{m+r+1}\binom{r}{m}\sum_{j=r}^{k-1}\binom{j-1}{r-1}\binom{k-j}{1}p_j+\ldots+(-1)^{m+k}\binom{k-1}{m}p_r.$$

Par conséquent le facteur qui multiplie p_i, $r\leqq i\leqq k-1$ est

$$(-1)^{m+r}\left\{\binom{r-1}{m}\binom{i-1}{r-1}-\binom{r}{m}\binom{i-1}{r-1}\binom{k-i}{1}+\ldots+(-1)^{k-i}\binom{r+k-i-1}{m}\binom{i-1}{r-1}\right\}=$$

$$=(-1)^{m+r}\binom{i-1}{r-1}\sum_{j=0}^{k-i}(-1)^j\binom{k-i}{j}\binom{r+j-1}{m}$$

et en utilisant le lemme 4, ce coefficient est égal à

$$(-1)^{m+r+k+i}\binom{i-1}{r-1}\binom{r-1}{m+i-k}.$$

Ainsi on obtient

$$\sum_{d|n} \mu_m(d) p_r(d) =$$

$$= (-1)^{m+r} \binom{r-1}{m} \binom{k-1}{r-1} p_k + \sum_{i=\max\{r,k-m\}}^{k-1} (-1)^{m+r+k+i} \binom{i-1}{r-1} \binom{r-1}{m+i-k} p_i$$

et ceci achève la démonstration de ce résultat.

Le prochain résultat est un cas particulier du théorème 6 et a été énoncé par Alladi [1].

COROLLAIRE. Soir $r \geqq 1$ un entier. Alors pour tout entier $m \geqq 0$, on a

$$\sum_{d|n} \mu(d) p_r(d) = (-1)^r \binom{\omega(n)-1}{r-1} P_1(n)$$

et

$$\sum_{d|n} \mu_m(d) p_1(d) = -P_{m+1}(n).$$

Inversant les rôles du plus grand facteur premier et du plus petit facteur premier, le théorème 6 devient

THÉORÈME 7. *Pour tout entier* $m \geqq 0$, *on a*

$$\sum_{d|n} \mu_m(d) F_r(d) =$$

$$= \begin{cases} \dfrac{(-1)^{m+r+k}}{\binom{m}{r-1}} \displaystyle\sum_{i=\max\{r,k-m\}}^{k+r-m-1} (-1)^i \binom{i-1}{r-1} \binom{k-i}{m+1-r} \binom{m}{m+i-k} p_{k+1-i}(n) & si \quad 1 \leqq r \leqq m+1 \\ (-1)^{m+r+k} \displaystyle\sum_{i=\max\{r,k=m\}}^{k} (-1)^i \binom{i-1}{r-1} \binom{r-1}{m+i-k} p_{k+1-i}(n) & si \quad r \geqq m+1 \end{cases}$$

où $k = \omega(n)$.

Encore une fois, le résultat suivant déjà obtenu par Alladi [1] découle immédiatement du théorème 7.

COROLLAIRE. Soit $r \geqq 1$ un entier. Alors pour tout entier $m \geqq 0$, on a

$$\sum_{d|n} \mu(d) P_r(d) = (-1)^r \binom{\omega(n)-1}{r-1} p_1(n)$$

et

$$\sum_{d|n} \mu_m(d) P_1(d) = -p_{m+1}(n).$$

THÉORÈME 8. *Pour tout entier* $m \geqq 0$, *on a*

$$\sum_{d|n} \mu_m(d)\mu(d) =$$

$$= \begin{cases} 1 & si \quad \omega(n) \leqq m \\ (-1)^m \left(1 + \dfrac{(-1)^{m+1}(\omega(n))!}{m!(\omega(n)-m-1)!} \displaystyle\sum_{j=1}^{m+1} (-1)^j \binom{m}{m+1-j} \dfrac{2^{\omega(n)+j-m-1}-1}{\omega(n)+j-m-1}\right) & \\ & si \quad \omega(n) > m. \end{cases}$$

DÉMONSTRATION. D'après (8), on a pour $k=\omega(n)>m$

$$\sum_{\substack{d|n \\ d>1}} \mu_m(d)\mu(d) = -\sum_{1 \leqq i_1 < \ldots < i_{m+1} \leqq k} (-1)^{m+1} + \binom{m+1}{m} \sum_{1 \leqq i_1 < \ldots < i_{m+2} \leqq k} (-1)^{m+2} -$$

$$-\binom{m+2}{m} \sum_{1 \leqq i_1 < \ldots < i_{m+3} \leqq k} (-1)^{m+3} + \ldots + (-1)^m \binom{k-1}{m},$$

et en utilisant le lemme 1, on a

$$\sum_{\substack{d|n \\ d>1}} \mu_m(d)\mu(d) = (-1)^m \sum_{j=0}^{k-m-1} \binom{m+j}{m} \binom{k}{m+j+1} =$$

$$= (-1)^m \frac{k!}{m!(k-m-1)!} \sum_{j=0}^{k-m-1} \frac{\binom{k-m-1}{j}}{m+1-j} =$$

$$= (-1)^m \frac{k!}{m!(k-m-1)!} \sum_{j=1}^{m+1} (-1)^{m+1-j} \binom{m}{m+1-j} \frac{2^{k-m-1+j}-1}{k-m-1+j}$$

où cette dernière égalité a été obtenue en utilisant l'identité (1.12) de [4].

REMARQUE. Il est intéressant de remarquer que dans le cas où $m=0$, on retrouve la relation bien connue $\sum_{d|n} \mu^2(d) = 2^{\omega(n)}$.

6. Autres fonctions arithmétiques

Considérons maintenant le cas où $g(n)=\varphi(n)$, la fonction φ d'Euler. Alors (1) devient

$$\varphi_m(n) = \sum_{\substack{d|n \\ \omega(d) \leqq m}} \varphi(n/d).$$

THÉORÈME 9. *Pour chaque entier* $m \geqq 0$, *on a*

$$\varphi_m(n) = \sum_{\substack{1 \leqq k \leqq n \\ \omega((k,n)) \leqq m}} 1.$$

DÉMONSTRATION. Soit $A = \{k \mid 1 \leqq k \leqq n,\ \omega((k,n)) \leqq m\}$. Séparons les entiers de A de la façon suivante : Si d est un diviseur de n tel que $\omega(d) \leqq m$, alors l'entier

« a » appartient à C_d si $(a, n)=d$, c'est-à-dire,

$$C_d = \{a|(a, n) = d,\ 1 \leqq a \leqq n,\ \omega(d) \leqq m\} =$$

$$= \left\{k \middle| \left(k, \frac{n}{d}\right) = 1,\ 1 \leqq k \leqq \frac{n}{d},\ \omega(d) \leqq m\right\}.$$

Ainsi, on obtient

$$\# C_d = \varphi(n/d) \quad \text{tel que} \quad \omega(d) \leqq m.$$

Puisque chaque entier de A appartient à exactement une classe C_d, alors

$$\# A = \sum_{\substack{d|n \\ \omega(d) \leqq m}} \varphi(n/d),$$

ce qui termine la preuve de ce résultat.

Définissons les fonctions suivantes

$$1(n) = 1 \quad \text{et} \quad I(n) = n \quad \text{pour chaque} \quad n \geqq 1.$$

Puisque

$$1(n) = \sum_{d|n} \mu(d) d(n/d) \quad \text{où} \quad d(n) = \sum_{d|n} 1$$

$$I(n) = \sum_{d|n} \mu(d) \sigma(n/d) \quad \text{où} \quad \sigma(n) = \sum_{d|n} d$$

$$J(k;\ n) = \sum_{d|n} \mu(d) \left(\frac{n}{d}\right)^k, \quad k \geqq 1,$$

elle est appelée la fonction de Jordan.

Pour $k=1$, on obtient

$$J(1;\ n) = \varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$$

a fonction d'Euler. En utilisant la notation de (1) et le théorème 4, on à le résultat suivant.

Théorème 10. *Soit* $n \geqq 1$, *alors pour chaque entier* $m \geqq 0$, *on a*

$$1_m(n) = \sum_{d|n} \mu_m(d)\, d(n/d), \quad I_m(n) = \sum_{d|n} \mu_m(d) \sigma(n/d),$$

$$J_m(k;\ n) = \sum_{d|n} \mu_m(d) \left(\frac{n}{d}\right)^k, \quad \varphi_m(n) = \sum_{d|n} \mu_m(d) \frac{n}{d}.$$

En utilisant ce théorème, on peut à l'aide du théorème 5 énoncer le résultat qui suit.

Théorème 11. *Soit* $n \geqq 1$, *alors pour chaque entier* $m \geqq 0$, *on a*

$$d_m(n) = \sum_{d|n} 1_m(d), \quad \sigma_m(n) = \sum_{d|n} I_m(d),$$

$$(I_m(n))^k = \sum_{d|n} J_m(k;\ d), \quad I_m(n) = \sum_{d|n} \varphi_m(d).$$

REMARQUE. Ces trois derniers résultats sont des identités connues lorsque $m=0$ (voir [2]). Nous terminerons cette section par une généralisation des sommes de Ramanujan. Soit k et n des entiers positifs. Il est bien connu que

$$\sum_{r=1}^{k} e^{2\pi i(nr/k)} = \begin{cases} 0 & \text{si } k \nmid n \\ n & \text{si } k \mid n. \end{cases} \tag{9}$$

Pour chaque entier $m \geqq 0$, posons

$$c_k(m;\, n) = \sum_{\substack{r=1 \\ \omega((r,k)) \leqq m}}^{k} e^{2\pi i(rn/k)}. \tag{10}$$

Lorsque $m=0$, on obtient les sommes de Ramanujan. De plus, on observe que lorsque $k|n$, (10) devient $c_k(m;\, n)=\varphi_m(k)$, d'après le théorème 9.

THÉORÈME 12. *Soit $k \geqq 1$ et $n \geqq 1$ des entiers. Alors pour chaque entier $m \geqq 0$ on a*

$$c_k(m;\, n) = \sum_{d|(k,n)} \mu_m(k/d)\, d.$$

DÉMONSTRATION. En utilisant (5), on peut écrire

$$c_k(m;\, n) = \sum_{\substack{r=1 \\ \omega((r,k)) \leqq m}}^{k} e^{2\pi i(rn/k)} = \sum_{r=1}^{k} e^{2\pi i(rn/k)} \sum_{d|(r,k)} \mu_m(d) =$$
$$= \sum_{d|k} \mu_m(d) \sum_{\substack{r \equiv 0 \pmod d \\ 1 \leqq r \leqq k}} e^{2\pi i(rn/k)}.$$

Soit $r=jd$, $j=1, 2, \ldots, k/d$, alors

$$c_k(m;\, n) = \sum_{d|k} \mu_m(d) \sum_{j=1}^{k/d} e^{2\pi i(jn/(k/d))} = \begin{cases} \sum_{d|k} \mu_m(d) \dfrac{k}{d} & \text{si } \dfrac{k}{d} \Big| n \\ 0 & \text{si } \dfrac{k}{d} \nmid n, \end{cases}$$

d'après (9). Soit $k/d=r$, alors $d=k/r$ et ainsi on obtient le résultat.

COROLLAIRE. *Pour chaque entier $m \geqq 0$, on a $c_k(m;\, 1)=\mu_m(k)$.*

THÉORÈME 13. *Pour chaque entier $m \geqq 0$, on a*

$$\sum_{d|n} c_k(m;\, d) = \begin{cases} I_m(n) & si \ \dfrac{n}{d} \Big| k \\ 0 & si \ \dfrac{n}{d} \nmid k. \end{cases}$$

DÉMONSTRATION. Posons

$$\delta_k(d) = \begin{cases} d & \text{si } d|k \\ 0 & \text{si } d \nmid k, \end{cases}$$

alors le théorème 12 peut s'écrire sous la forme

$$c_k(m;\, n) = (\mu_m * \delta_k)(n)$$

ou encore

$$(1 * c_k(m;\ n))(n) = ((1 * \mu_m) * \delta_k)(n) = (\beta * \delta_k)(n)$$

où $\beta(n)$ a été définie dans (5). Ainsi on obtient

$$\sum_{d|n} c_k(m;\ d) = \sum_{\substack{d|n \\ \omega(d) \leqq m}} \delta_k(n/d)$$

ce qui établit le résultat.

Lorsque $m=0$, on obtient le résultat suivant bien connu

COROLLAIRE. Soit $n \geqq 1$, alors

$$\sum_{d|n} c_k(0,\ d) = \begin{cases} n & si \quad n|k \\ 0 & si \quad n \nmid k. \end{cases}$$

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SUMMEN UNABHÄNGIGER ZUFALLSVARIABLEN, DIE DURCH DIE MAXIMALTERME DOMINIERT WERDEN

R. WITTMANN (Eichstätt)

Im folgenden sei (X_n) stets eine Folge von unabhängigen, symmetrisch verteilten Zufallsvariablen mit gemeinsamer Verteilungsfunktion F auf einem Wahrscheinlichkeitsraum $(\Omega, \mathscr{A}, P)$. Wir benützen die Bezeichnungen

$$S_n := \sum_{i=1}^{n} X_i, \quad S_n^* := \max_{1 \leqq i \leqq n} |S_i|, \quad S_0^* := 0.$$

Ziel dieser Arbeit ist der folgende

SATZ. *Es gilt*

$$\limsup_{n \to \infty} \frac{|X_n|}{S_{n-1}^*} = \infty \quad f.\,s.$$

genau dann wenn $E(X_1^2) = \infty$.

Für *positive* Zufallsvariablen wurde obiges Phänomen zuerst von Darling [2] beobachtet. Als Hauptresultat für positive Zufallsvariablen (insbesondere gilt $S_n^* = S_n$) kann die folgende unmittelbare Folgerung aus Kesten [5], Theorem 5 angesehen werden:

$$\limsup_{n \to \infty} \frac{X_n}{S_{n-1}} = \infty \quad \text{f. s.} \quad \Leftrightarrow \quad E(X_1) = \infty.$$

Dieser Fall unterscheidet sich also stark vom Fall symmetrisch verteilter Zufallsvariablen. Im symmetrischen Fall wurden die ersten Beispiele mit $E(|X_1|) < \infty$ in Feller [4] entdeckt. Unser Beweis ist dem in [1], S. 202 gegebenen Beweis des obigen Satzes von Kesten ähnlich.

Wir benötigen drei Lemmata.

LEMMA 1. *Für alle* $x \geqq 0$ *sei*

$$H(x) := \int_0^x 2tP\{|X_1| > t\}\, dt,$$

$$T_x(\omega) := \inf\{n \in \mathbf{N} : |S_n(\omega)| > x\} \quad (\omega \in \Omega).$$

Dann gilt für alle $x \geqq 0$

$$\frac{x^2}{H(2x)} \leqq E(T_x) \leqq \frac{9x^2}{H(2x)}.$$

BEWEIS. Sei $\varepsilon>0$ gegeben. Für alle $n\in\mathbf{N}$ sei

$$\tilde{X}_n := I_{\{|X_n|\leqq 2x+\varepsilon\}} X_n + (2x+\varepsilon)(I_{\{X_n>2x+\varepsilon\}} - I_{\{X_n<-2x-\varepsilon\}}),$$

$$\tilde{S}_n := \sum_{i=1}^{n} \tilde{X}_i.$$

Ist $|S_{n-1}(\omega)|\leqq x$, $S_{n-1}(\omega)=\tilde{S}_{n-1}(\omega)$ und $S_n(\omega)\neq\tilde{S}_n(\omega)$, dann gilt $|S_n(\omega)|$, $|\tilde{S}_n(\omega)|\geqq(2x+\varepsilon)-|S_{n-1}(\omega)|>x$. Daraus folgt

$$T_x(\omega) = \inf\{n\in\mathbf{N}\colon |\tilde{S}_n(\omega)| > x\} \quad (\omega\in\Omega).$$

Insbesondere gilt

$$x^2 \leqq \tilde{S}^2_{T_x} \leqq (3x+\varepsilon)^2, \quad \tilde{S}^2_{T_x\wedge n} \leqq (3x+\varepsilon)^2 \quad (n\in\mathbf{N}). \tag{1}$$

Wegen (1) und der 2. Identität von A. Wald (siehe [1], S. 139) gilt

$$E(T_x\wedge n)E(\tilde{X}_1^2) = E(\tilde{S}^2_{T_x\wedge n}) \leqq (3x+\varepsilon)^2$$

und folglich

$$E(T_x)E(\tilde{X}_1^2) = E(\tilde{S}^2_{T_x}) < \infty.$$

Zusammen mit (1) und $H(2x+\varepsilon)=E(\tilde{X}_1^2)$ folgt daraus schließlich

$$x^2 \leqq E(T_x)H(2x+\varepsilon) \leqq (3x+\varepsilon)^2.$$

Da $\varepsilon>0$ beliebig klein wählbar ist, folgt die Behauptung. □

Das nächste Lemma ist eine Variante des Borel—Cantelli—Lemmas. Einen Beweis hierfür kann man in [1], S. 95 finden.

LEMMA 2. *Seien (A_n) und (B_n) zwei Folgen in $\mathcal{A}$ mit*

$$P\Big(A_i \cap \bigcup_{j=1}^{\infty} A_{i+jk}\Big) \leqq P(A_i)P\Big(\bigcup_{j=k}^{\infty} B_j\Big) \quad (i, k\in\mathbf{N}).$$

Dann sind die folgenden beiden Aussagen äquivalent

(i) $\sum_{n=1}^{\infty} P(A_n) = \infty$,

(ii) $P\Big(\bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} B_j\Big) = 1.$

LEMMA 3. *Für alle $\varepsilon>0$ gilt*

$$\int_0^{\infty} \frac{x^2}{H(\varepsilon x)}\, dF(x) < \infty \Leftrightarrow E(X_1^2) < \infty.$$

BEWEIS. Wenden wir Erickson [3], S. 375 auf X_1^2 an, so folgt die Behauptung für $\varepsilon=1$. Daraus folgt der allgemeine Fall, denn

$$\frac{x^2}{H(x)} \leqq \frac{x^2}{H(\varepsilon x)} \leqq \varepsilon^{-2}\frac{x^2}{H(x)}$$

falls $0<\varepsilon\leqq 1$, und

$$\frac{x^2}{H(x)} \geqq \frac{x^2}{H(\varepsilon x)} \geqq \varepsilon^{-2}\frac{x^2}{H(x)}$$

falls $\varepsilon>1$. □

Beweis des Satzes. Sei $\varepsilon>0$ gegeben. Wir definieren

$$A_n := \{S_{n-1}^* \leqq \varepsilon |X_n|\}, \quad B_n := \{S_{n-1}^* \leqq 2\varepsilon |X_n|\}.$$

Indem wir die bedingte Verteilung bezüglich X_n bilden, folgt

$$P(A_n) = E\big(P(S_{n-1}^* \leqq \varepsilon|X_n|\,\big|\,|X_n|)\big) = \int_{-\infty}^{\infty} P\{S_{n-1}^* \leqq \varepsilon|x|\}\, dF(x). \tag{1}$$

Andererseits gilt

$$E(T_x) = \sum_{n=0}^{\infty} P\{S_n^* \leqq x\} \quad (x \geqq 0).$$

Zusammen mit (1) folgt daraus

$$\sum_{n=1}^{\infty} P(A_n) = 2\int_0^{\infty} \sum_{n-1}^{\infty} P\{S_{n-1}^* \leqq \varepsilon x\}\, dF(x) = 2\int_0^{\infty} E(T_{\varepsilon x})\, dF(x).$$

Wenden wir nun Lemma 1 an, so erhalten wir

$$2\int_0^{\infty} \frac{(\varepsilon x)^2}{H(2\varepsilon x)}\, dF(x) \leqq \sum_{n=1}^{\infty} P(A_n) \leqq 18\int_0^{\infty} \frac{(\varepsilon x)^2}{H(2\varepsilon x)}\, dF(x).$$

Zusammen mit Lemma 3 folgt daraus

$$\sum_{n=1}^{\infty} P(A_n) = \infty \Leftrightarrow E(X_1^2) = \infty. \tag{2}$$

Für alle $1\leqq i<j<\infty$ und $\omega\in A_j$ gilt

$$\Big|\sum_{m=i+1}^{p} X_m(\omega)\Big| \leqq \Big|\sum_{m=1}^{p} X_m(\omega)\Big| + \Big|\sum_{m=1}^{i} X_m(\omega)\Big| \leqq 2\varepsilon|X_j(\omega)| \quad (i<p<j)$$

und somit

$$A_i\cap A_j \subset A_i \cap \Big\{\max_{i<p<j}\Big|\sum_{m=i+1}^{p} X_m\Big| \leqq 2\varepsilon|X_j|\Big\}.$$

Da die Zufallsvariablen X_n außerdem unabhängig und identisch verteilt sind, gilt für alle $i, k\in\mathbf{N}$

$$P\Big(A_i \cap \bigcup_{j=i+k}^{\infty} A_j\Big) \leqq P\Big(A_i\cap \bigcup_{j=i+k}^{\infty}\Big\{\max_{i<p<j}\Big|\sum_{m=i+1}^{p} X_m\Big| \leqq 2\varepsilon|X_j|\Big\}\Big) =$$

$$= P(A_i)P\Big(\bigcup_{j=i+k}^{\infty}\Big\{\max_{i<p<j}\Big|\sum_{m=i+1}^{p} X_m\Big| \leqq 2\varepsilon|X_j|\Big\}\Big) = P(A_i)P\Big(\bigcup_{j=k}^{\infty} B_j\Big).$$

Also sind die Voraussetzungen von Lemma 2 erfüllt, und wir erhalten

$$\sum_{n=1}^{\infty} P(A_n) = \infty \Leftrightarrow P\Big(\bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} B_j\Big) = 1. \tag{3}$$

Fassen wir (2) und (3) zusammen, so ergibt sich

$$P\Big(\bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} B_j\Big) = 1 \Leftrightarrow E(X_1^2) = \infty.$$

Wegen

$$\left\{\limsup_{n\to\infty} \frac{|X_n|}{S_{n-1}^*} \geqq \frac{3}{\varepsilon}\right\} \subset \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} B_j \subset \left\{\limsup_{n\to\infty} \frac{|X_n|}{S_{n-1}^*} \geqq \frac{2}{\varepsilon}\right\}$$

und da wir $\varepsilon > 0$ beliebig klein wählen können, folgt die Behauptung. □

Danksagung. Dem Referenten möchte ich für das gründliche Durchlesen des Manuskriptes und insbesondere für die Entdeckung eines nicht-trivialen Fehlers im Beweis des Satzes danken.

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A GENERALIZATION OF BOUNDED VARIATION

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§1. Introduction. The concept of bounded p-variation was first introduced by N. Wiener [4] in 1924. Afterwards, Waterman [3] defined Λ-bounded variation and Chanturiya [2] introduced the concept of the modulus of variation.

In this paper, we shall define a new generalization of bounded variation and study some of its fundamental properties. We also consider the inclusion relations between Chanturiya's class and our class,

Let f be a function defined on $(-\infty, \infty)$ with period 1. Δ is said to be a partition with period 1, if

$$(1.1)\qquad \Delta:\ \ldots < t_{-1} < t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} < \ldots$$

satisfies $t_{k+m} = t_k + 1$ for $k = 0, \pm 1, \pm 2, \ldots$, where m is a positive integer.

We shall generalize the concept of bounded variation.

Definition 1.1. When $1 \leqq p_n \uparrow p$ as $n \to \infty$, where $1 \leqq p \leqq +\infty$, f is said to be a function of $BV(p_n \uparrow p)$ if and only if

$$V(f;\ p_n \uparrow p) = \sup_{n \geqq 1} \sup \left\{ \left(\sum_{k=1}^{m} |f(t_k) - f(t_{k-1})|^{p_n} \right)^{1/p_n} :\ \varrho(\Delta) \geqq 1/2^n \right\} < +\infty,$$

where $\varrho(\Delta) = \inf_k |t_k - t_{k-1}|$.

When $p_n = p$ for all n, it is easy to see that $BV(p_n \uparrow p)$ coincides with BV_p which is the Wiener's class of bounded p-variation. When $p = +\infty$, the space $BV(p_n \uparrow \infty)$ is especially useful and plays an important role in the uniform convergence of Fourier series.

§2. Preliminary results. Let $B[0, 1]$ denote the space of real valued functions f of period 1 such that

$$\|f\|_B = \sup \{|f(x)| :\ x \in [0, 1]\} < +\infty.$$

Lemma 2.1. *We have*

$$(2.1)\qquad \bigcup_{1 \leqq p < +\infty} BV_p \subseteqq BV(p_n \uparrow \infty) \subseteqq B[0, 1]$$

and

$$(2.2)\qquad \bigcup_{1 \leqq q < p} BV_q \subseteqq BV(p_n \uparrow p) \subseteqq BV_p \quad (1 < p < +\infty).$$

Proof. If $f \notin B[0, 1]$, then there exists a sequence of real numbers $\{x_j\,; j \geqq 1\}$ of the interval [0, 1] such that

$$(2.3)\qquad \begin{cases} \lim_{j\to\infty} |f(x_j)| = +\infty, & \lim_{j\to\infty} x_j = x_0 \in [0, 1], \\ |x_j - x_0| < 1/4 & \text{for} \quad j = 1, 2, 3, \ldots. \end{cases}$$

Put $t_0 = x_0 - 1/2$, $t_1 = x_j$ and $t_2 = x_0 + 1/2$. Then we get

$$\min\{x_j - t_0, t_2 - x_j \colon j \geqq 1\} \geqq 1/2^n \quad \text{for} \quad n = 2, 3, 4, \ldots .$$

From (2.3) we get

$$V(f; p_n\uparrow\infty) \geqq \{|f(x_j) - f(t_0)|^{p_n} + |f(t_2) - f(x_j)|^{p_n}\}^{1/p_n} \geqq$$
$$\geqq |f(x_j) - f(t_0)| \geqq |f(x_j)| - |f(t_0)| \to \infty \quad \text{as} \quad j \to \infty.$$

Hence $f \notin BV(p_n\uparrow\infty)$. Therefore $BV(p_n\uparrow\infty) \subseteqq B[0, 1]$ holds.

Next we prove $BV_p \subseteqq BV(p_n\uparrow\infty)$ for all $1 \leqq p < +\infty$. For any $1 \leqq p < \infty$, there exists an integer n_0 such that $p < p_n \leqq p_{n+1} \leqq \ldots$ for all $n \geqq n_0$. Let $f \in BV_p$ and Δ be an arbitrary partition defined by (1.1) such that $\varrho(\Delta) \geqq 1/2^n$. When $n > n_0$, we have

$$\Big\{\sum_{k=1}^{m} |f(t_k) - f(t_{k-1})|^{p_n}\Big\}^{1/p_n} \leqq \Big\{\sum_{k=1}^{m} |f(t_k) - f(t_{k-1})|^{p}\Big\}^{1/p} \leqq V_p(f) < +\infty.$$

When $1 \leqq n \leqq n_0$, we get

$$\Big\{\sum_{k=1}^{m} |f(t_k) - f(t_{k-1})|^{p_n}\Big\}^{1/p_n} \leqq 2\|f\|_B (m)^{1/p_n} \leqq 2\|f\|_B 2^{n/p_n} \leqq 2\|f\|_B 2^{n_0}.$$

Therefore $f \in BV(p_n\uparrow\infty)$ holds and (2.1) is proved.

Let $1 < p < +\infty$ and $1 \leqq p_n\uparrow p$. We prove (2.2). Since $1 \leqq p_n \leqq p$, it follows that

$$\Big\{\sum_{k=1}^{m} |f(t_k) - f(t_{k-1})|^{p}\Big\}^{1/p} \leqq \Big\{\sum_{k=1}^{m} |f(t_k) - f(t_{k-1})|^{p_n}\Big\}^{1/p_n}$$

for all Δ for which $\varrho(\Delta) \geqq 1/2^n$. Therefore we get $BV(p_n\uparrow p) \subseteqq BV_p$. Let $1 \leqq q < p$. Then there exists an integer n_0 such that $q < p_n \leqq p_{n+1} \leqq \ldots \leqq p$ for all $n \geqq n_0$. The proof of (2.2) can be finished similarly as that of (2.1).

Let $\bar\Delta$ be an arbitrary partition of the interval [0, 1], namely,

$$(2.4)\qquad \bar\Delta\colon\ 0 = t_0 < t_1 < \ldots < t_m = 1.$$

It is clear that $f \in BV(p_n\uparrow p)$ if and only if

$$(2.5)\qquad \sup_{n\geqq 1} \sup \Big\{\Big(\sum_{k=1}^{m} |f(a + t_k) - f(a + t_{k-1})|^{p_n}\Big)^{1/p_n} \colon \varrho(\bar\Delta) \geqq 1/2^n,\ a \in R\Big\} < +\infty.$$

LEMMA 2.2. *Let $\bar\Delta$ be an arbitrary partition defined by* (2.4). *If for some* $0 < p_n$ $(n = 1, 2, \ldots)$

$$(2.6)\qquad \sup_{n\geqq 1} \sup \Big\{\Big(\sum_{k=1}^{m} |f(t_k) - f(t_{k-1})|^{p_n}\Big)^{1/p_n} \colon \varrho(\bar\Delta) \geqq 1/2^n\Big\} < +\infty,$$

then $f \in B[0, 1]$ *holds.*

The proof of this lemma can be accomplished in the same way as that of (2.1).

LEMMA 2.3. *Let* $1 \leqq p_n \uparrow p$ *as* $n \to +\infty$, *where* $1 \leqq p \leqq +\infty$ *and let* f *be any function satisfying* (2.6). *Then* $f \in BV(p_n \uparrow p)$.

PROOF. Let $\bar{\Delta}$ be an arbitrary partition defined by (2.4). And suppose $\varrho(\bar{\Delta}) \geqq 1/2^n$. We consider any partition

$$a = a+t_0 < a+t_1 < \ldots < a+t_k < \ldots < a+t_m = a+1$$

of the interval $[a, a+1]$, where a is any real number and $a \not\equiv 0 \pmod 1$. Then there exist integers l_0 and k_0 such that $a+t_{k_0} < l_0 < a+t_{k_0+1}$. (Without loss of generality, we may suppose that $l_0 \neq a+t_{k_0}$ and $l_0 \neq a+t_{k_0+1}$.) Then it follows that

$$\Big\{\sum_{k=1}^{m} |f(a+t_k)-f(a+t_{k-1})|^{p_n}\Big\}^{1/p_n} =$$

$$= \{|f(a+t_{k_0})-f(a+t_{k_0-1})|^{p_n} + |f(a+t_{k_0+1})-f(a+t_{k_0})|^{p_n} +$$

$$+ |f(a+t_{k_0+2})-f(a+t_{k_0+1})|^{p_n} + \sum_{k=1}^{m}{}' |f(a+t_k)-f(a+t_{k-1})|^{p_n}\Big\}^{1/p_n},$$

where $\sum_{k=1}^{m}{}'$ denotes the sum for $k \neq k_0,\ k_0+1,\ k_0+2$. Since $f \in B[0, 1]$ by Lemma 2.2, we get

$$\Big\{\sum_{k=1}^{m} |f(a+t_k)-f(a+t_{k-1})|^{p_n}\Big\}^{1/p_n} \leqq$$

$$\leqq \Big\{3(2\|f\|_B)^{p_n} + \sum_{k=1}^{m}{}' |f(a+t_k)-f(a+t_{k-1})|^{p_n}\Big\}^{1/p_n} \leqq$$

$$\leqq 6\|f\|_B + \Big\{\sum_{k=1}^{m}{}' |f(a+t_k)-f(a+t_{k-1})|^{p_n}\Big\}^{1/p_n} \leqq$$

$$\leqq 6\|f\|_B + \Big\{|f(l_0)-f(a+t_{k_0-1})|^{p_n} + |f(a+t_{k_0+2})-f(l_0)|^{p_n} +$$

$$+ \sum_{k=1}^{m}{}' |f(a+t_k)-f(a+t_{k-1})|^{p_n}\Big\}^{1/p_n}.$$

We obtain a partition of the interval [0, 1]; namely,

$\bar{\Delta}_1$: $0 < a+t_{k_0+2}-l_0 < \ldots < a+1-l_0 < a+t_1-(l_0-1) < \ldots < a+t_{k_0-1}-(l_0-1) < 1$

which satisfies $\varrho(\bar{\Delta}_1) \geqq 1/2^n$. Hence and from (2.6) we get (2.5) and $f \in BV(p_n \uparrow p)$.

§3. Main results. THEOREM 3.1. *Let* $1 \leqq q \leqq p \leqq +\infty$. *When* $1 \leqq p_n \uparrow p$, $1 \leqq q_n \uparrow q$ *as* $n \to +\infty$, *and* $p_n/n \downarrow 0$, $q_n/n \downarrow 0$ *as* $n \to +\infty$, $BV(p_n \uparrow p) = BV(q_n \uparrow q)$ *if and only if*

(3.1) $$|n/q_n - n/p_n| = O(1) \quad as \quad n \to +\infty.$$

PROOF. We note that (3.1) implies $p=q$. First, we shall prove the sufficiency. Let Δ be a partition with period 1 such that $\varrho(\Delta) \geqq 1/2^n$ and

(3.2) $$\ldots < t_{-1} < 0 \leqq t_0 < t_1 < \ldots < t_{m-1} < 1 \leqq t_m < \ldots.$$

Let $f \in BV(p_n \uparrow p)$. When $1 \leqq q_n < p_n$, we have from (3.1)

$$\{\sum_{k=1}^{m} |f(t_k)-f(t_{k-1})|^{q_n}\}^{1/q_n} \leqq$$

$$\leqq [\{\sum_{k=1}^{m} |f(t_k)-f(t_{k-1})|^{p_n}\}^{q_n/p_n}\{\sum_{k=1}^{m} 1\}^{(1-q_n/p_n)}]^{1/q_n} \leqq$$

$$\leqq \{\sum_{k=1}^{m} |f(t_k)-f(t_{k-1})|^{p_n}\}^{1/p_n} \cdot m^{(1/q_n-1/p_n)} \leqq$$

$$\leqq V(f;\ p_n \uparrow p) \cdot 2^{n(1/q_n-1/p_n)} = V(f;\ p_n \uparrow p) O(1).$$

When $1 \leqq p_n \leqq q_n$, it is obvious that

$$\{\sum_{k=1}^{m} |f(t_k)-f(t_{k-1})|^{q_n}\}^{1/q_n} \leqq \{\sum_{k=1}^{m} |f(t_k)-f(t_{k-1})|^{p_n}\}^{1/p_n} \leqq V(f;\ p_n \uparrow p).$$

Therefore we proved that $BV(p_n \uparrow p) \subseteqq BV(q_n \uparrow q)$. Similarly we have $BV(q_n \uparrow q) \subseteqq \subseteqq BV(p_n \uparrow p)$ and consequently we get $BV(p_n \uparrow p) = BV(q_n \uparrow q)$.

Next, we shall prove the necessity. Assume that $\varlimsup_{n\to\infty} |n/q_n - n/p_n| = +\infty$. Without loss of generality we may assume that $\varlimsup_{n\to\infty} n\{1/q_n - 1/p_n\} = +\infty$. Then we construct $f \in BV(p_n \uparrow p)$ such that $f \notin BV(q_n \uparrow q)$ holds. We shall define a sequence of functions $\{f_n;\ n \geqq 1\}$ as follows. Set $d_n = (1/2)^{n/p_n}$. Since $n/p_n \uparrow \infty$ as $n \to +\infty$, $d_n \downarrow 0$ as $n \to +\infty$.

First, set

$$f_1(x) = \begin{cases} 2d_1 x, & \text{if } 0 \leqq x \leqq 1/2, \\ (-2d_1)(x-1), & \text{if } 1/2 \leqq x \leqq 1. \end{cases}$$

Next, we shall define $f_2(x)$ on $[0, 1/2]$ as follows:

$$f_2(x) = \begin{cases} 2^2(d_1-d_2)x, & \text{if } 0 \leqq x \leqq 1/2^2 \\ 2^2 d_2(x-1/2)+d_1, & \text{if } 1/2^2 \leqq x \leqq 1/2. \end{cases}$$

We extend $f_2(x)$ on $[1/2, 1]$ as follows: $f_2(x) = f_2(1-x)$ for $1/2 \leqq x \leqq 1$.

When $f_n(x)$ is defined, $f_{n+1}(x)$ will be defined as follows: When n is even, if $f_n(k/2^n) > f_n((k+1)/2^n)$, set

$$f_{n+1}(x) = \begin{cases} -2^{n+1}\{f_n(k/2^n) - f_n((k+1)/2^n) - d_{n+1}\}(x-k/2^n) + f_n(k/2^n), \\ \qquad\qquad\qquad \text{if } k/2^n \leqq x \leqq (2k+1)/2^{n+1}, \\ -2^{n+1} d_{n+1}\{x-(k+1)/2^n\} + f_n((k+1)/2^n), \\ \qquad\qquad\qquad \text{if } (2k+1)/2^{n+1} \leqq x \leqq (k+1)/2^n, \end{cases}$$

for $k = 0, 1, 2, \ldots, 2^n - 1$, and, if $f_n(k/2^n) < f_n((k+1)/2^n)$, set

$$f_{n+1}(x) = \begin{cases} 2^{n+1} d_{n+1}(x-k/2^n) + f_n(k/2^n), & \text{if } k/2^n \leqq x \leqq (2k+1)/2^{n+1}, \\ 2^{n+1}\{f_n((k+1)/2^n) - f_n(k/2^n) - d_{n+1}\}(x-(k+1)/2^n) + f_n((k+1)/2^n), \\ \qquad\qquad\qquad \text{if } (2k+1)/2^{n+1} \leqq x \leqq (k+1)/2^n, \end{cases}$$

for $k=0, 1, 2, \ldots, 2^n-1$. When n is odd, if $f_n(k/2^n)>f_n((k+1)/2^n)$, set

$$f_{n+1}(x) = \begin{cases} -2^{n+1} d_{n+1}(x-k/2^n)+f_n(k/2^n), & \text{if } k/2^n \leqq x \leqq (2k+1)/2^{n+1}, \\ -2^{n+1}\{f_n(k/2^n)-f_n((k+1)/2^n)-d_{n+1}\}(x-(k+1)/2^n)+f_n((k+1)/2^n), & \\ & \text{if } (2k+1)/2^{n+1} \leqq x \leqq (k+1)/2^n, \end{cases}$$

for $k=0, 1, 2, \ldots, 2^n-1$, and, if $f_n(k/2^n)<f_n((k+1)/2^n)$, set

$$f_{n+1}(x) = \begin{cases} 2^{n+1}\{f_n((k+1)/2^n)-f_n(k/2^n)-d_{n+1}\}(x-k/2^n)+f_n(k/2^n), & \\ & \text{if } k/2^n \leqq x \leqq (2k+1)/2^{n+1}, \\ 2^{n+1} d_{n+1}(x-(k+1)/2^n)+f_n((k+1)/2^n), & \\ & \text{if } (2k+1)/2^{n+1} \leqq x \leqq (k+1)/2^n, \end{cases}$$

for $k=0, 1, 2, \ldots, 2^n-1$. Continuing in this way, we get a sequence of functions $\{f_n; n\geqq 1\}$. Since $p_n\uparrow p$ and $d_n=(1/2)^{n/p_n}$,

$$d_n/d_{n+1} = 2^{\{(n+1)/p_{n+1}-n/p_n\}} \leqq 2^{\{(n+1)/p_n-n/p_n\}} = 2^{1/p_n} \leqq 2.$$

Therefore we get

(3.3) $$d_n \leqq 2d_{n+1} \quad \text{for} \quad n \geqq 1.$$

From (3.3) and $d_n\downarrow 0$, it follows that

(3.4) $$|f_n(x)-f_{n+N}(x)| \leqq d_n \quad \text{for all} \quad n, N \geqq 1, \quad x\in[0, 1].$$

Since $\{f_n; n\geqq 1\}$ is a Cauchy sequence by (3.4), there exists a function f such that $\lim_{n\to\infty} f_n(x)=f(x)$ and

(3.5) $$|f_n(x)-f(x)| \leqq d_n \quad \text{for all} \quad n \geqq 1 \quad \text{and} \quad x\in[0, 1].$$

We shall prove that $f\in BV(p_n\uparrow p)$. Let Δ be an arbitrary partition satisfying (3.2) such that $\varrho(\Delta)\geqq 1/2^n$ and set

$$\Gamma_k = \{1 \leqq j \leqq m:\ 1/2^{k+1} \leqq |t_j-t_{j-1}| < 1/2^k\} \quad \text{for} \quad k = 0, 1, 2, \ldots .$$

If $1/2^{n+1}\leqq|x-y|<1/2^n$, it follows from (3.5) that

$$|f(x)-f(y)| \leqq |f(x)-f_n(x)|+|f_n(x)-f_n(y)|+|f_n(y)-f(y)| \leqq$$
$$\leqq d_n+|f_n(x)-f_n(y)|+d_n \leqq 4d_n.$$

Then we get

$$\sum_{j=1}^{m} |f(t_j)-f(t_{j-1})|^{p_n} = \sum_{k=0}^{n-1} \sum_{j\in\Gamma_k} |f(t_j)-f(t_{j-1})|^{p_n} \leqq$$

$$\leqq \sum_{k=0}^{n-1} \sum_{j\in\Gamma_k} (4d_k)^{p_n} = 4^{p_n} \sum_{k=0}^{n-1} (1/2)^{kp_n/p_k} \sum_{j\in\Gamma_k} 1 \leqq$$

$$\leqq 4^{p_n} \sum_{k=0}^{n-1} (1/2)^k \sum_{j\in\Gamma_k} 1 \leqq 2\cdot 4^{p_n} \sum_{k=0}^{n-1} \sum_{j\in\Gamma_k} |t_j-t_{j-1}| \leqq 2\cdot 4^{p_n} \leqq 8^{p_n}.$$

Therefore it follows that

$$\{\sum_{j=1}^{m}|f(t_j)-f(t_{j-1})|^{p_n}\}^{1/p_n} \leqq 8 \quad \text{for all} \quad n \geqq 1.$$

Hence we get $f \in BV(p_n \uparrow p)$.

Last we prove $f \notin BV(q_n \uparrow q)$. From the definition of f,

$$\left|f((2k+1)/2^n)-f(2k/2^n)\right| = d_n$$

or

$$\left|f((2k+2)/2^n)-f((2k+1)/2^n)\right| = d_n \quad \text{for} \quad n \geqq 1.$$

Then we get

$$\{\sum_{k=1}^{2^n}|f(k/2^n)-f((k-1)/2^n)|^{q_n}\}^{1/q_n} \geqq \{2^{n-1}\cdot d_n^{q_n}\}^{1/q_n} = (1/2)^{1/q_n}\cdot 2^{n/q_n}\cdot d_n =$$
$$= (1/2)^{1/q_n}\cdot 2^{n\{1/q_n-1/p_n\}}.$$

We have already assumed that $\overline{\lim}_{n\to\infty} n\{1/q_n-1/p_n\}=+\infty$. Therefore it follows that $f \notin BV(q_n \uparrow q)$. We proved the theorem.

THEOREM 3.2. *Let* $1 \leqq p_n \uparrow \infty$ *as* $n \to +\infty$. *Then* $BV(p_n \uparrow \infty)=B[0, 1]$ *if and only if the sequence* $\{p_n; n \geqq 1\}$ *satisfies the following condition:*

(3.6) $$n/p_n \leqq C \quad \textit{for all} \quad n \geqq 1,$$

where $C>0$ *is a constant.*

PROOF. Suppose that $\{p_n; n\geqq 1\}$ satisfies (3.6). We prove $B[0, 1] \subseteq BV(p_n \uparrow \infty)$. Let $\bar{\Delta}$ be an arbitrary partition defined by (2.4) which satisfies $\varrho(\bar{\Delta}) \geqq 1/2^n$. From (3.6) and $m \leqq 2^n$, we get

$$\{\sum_{k=1}^{m}|f(t_k)-f(t_{k-1})|^{p_n}\}^{1/p_n} \leqq \{\sum_{k=1}^{m}(2\|f\|_B)^{p_n}\}^{1/p_n} =$$
$$= 2\|f\|_B\cdot m^{1/p_n} \leqq 2\|f\|_B\cdot 2^{n/p_n} \leqq 2^{C+1}\|f\|_B < +\infty.$$

Therefore $f \in BV(p_n \uparrow \infty)$ holds.

Next we suppose that $\{p_n; n \geqq 1\}$ does not satisfy (3.6). As an example we construct a bounded function which is not in $BV(p_n \uparrow \infty)$. Since $\{n/p_n; n\geqq 1\}$ is not bounded by hypothesis, there exists a sequence of integers $\{n_k; k \geqq 1\}$ such that

(3.7) $$\lim_{k\to\infty} n_k/p_{n_k} = +\infty.$$

We define two dense subsets E_1 and E_2 of the interval $[0, 1]$ as follows. The set E_1 contains all the points of the form $m/2^n$ where n and m are nonnegative integers; $E_2=[0, 1]\setminus E_1$. Put

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in E_1 \\ 1 & \text{if} \quad x \in E_2. \end{cases}$$

Let $\bar{\Delta}$ be a partition of the interval $[0, 1]$, namely,

$$\bar{\Delta}\colon 0 = t_0 < t_1 < t_2 < \ldots < t_{2^n} = 1,$$

where $t_{2k}=2k/2^n$, $t_{2k+1}\in E_2$ and $\varrho(\bar{\Delta})\geqq 1/2^{n+1}$. If we put $n=n_k-1$, we get from (3.7) that

$$\Big\{\sum_{j=1}^{2^n}|f(t_j)-f(t_{j-1})|^{p_{n+1}}\Big\}^{1/p_{n+1}}=2^{n/p_{n+1}}=2^{(n_k/p_{n_k})(1-1/n_k)}\to+\infty \quad\text{as}\quad k\to+\infty.$$

Therefore $f\notin BV(p_n\uparrow\infty)$ and proof is complete.

Wiener [4] showed that functions of the class BV_p could only have simple discontinuities. For functions of the class $BV(p_n\uparrow\infty)$ we have the following result.

THEOREM 3.3. *For the class $BV(p_n\uparrow\infty)$, there exists a function which has a discontinuity not of the first kind.*

PROOF. We consider the following example. Since $1\leqq p_n\uparrow\infty$ as $n\to+\infty$, we can choose a monotone increasing sequence of positive integers $\{n_k;\, k\geqq 1\}$ such that $n_1=1$ and

(3.8) $$p_{n_{k-1}}\geqq\log k \quad\text{for all}\quad k\geqq 2.$$

We construct a function f as follows. Set

(3.9)
$$f(x)=\begin{cases}1 & \text{if}\quad (1/2)\{(1/2^{n_k})+(1/2^{n_k-1})\}<x<(1/2^{n_k-1}) \quad\text{for}\quad k=1,2,3,\dots,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly, f is a bounded function defined on $[0, 1]$ and f has a discontinuity not of the first kind at $x=0$.

We prove that f is a function of the class $BV(p_n\uparrow\infty)$. Let $\bar{\Delta}$ be an arbitrary partition defined by (2.4) such that $\varrho(\bar{\Delta})\geqq 1/2^n$. Then, for every fixed index n, there exists a positive integer k such that

(3.10) $$2^{n_{k-1}}\leqq 2^n<2^{n_k}.$$

Hence we get $1/2^{n_k}<1/2^n\leqq 1/2^{n_{k-1}}$ and $p_{n_{k-1}}\leqq p_n\leqq p_{n_k}$. From (3.8), (3.9) and (3.10), it follows that

$$\Big\{\sum_{j=1}^{m}|f(t_j)-f(t_{j-1})|^{p_n}\Big\}^{1/p_n}\leqq(2k)^{1/p_n}\leqq 2\cdot k^{1/p_n}\leqq 2\cdot k^{1/p_{n_{k-1}}}\leqq 2\cdot k^{1/\log k}=2e<+\infty.$$

Therefore we get $f\in BV(p_n\uparrow\infty)$ and Theorem 3.3 is proved.

COROLLARY 3.1. *We have*

$$\bigcup_{1\leqq p<+\infty}BV_p\subsetneqq BV(p_n\uparrow\infty).$$

Let $\omega(f;h)$ be the modulus of continuity in the space $C(0, 1)$. We have the following theorem.

THEOREM 3.4. *Let $1\leqq p_n\uparrow\infty$ as $n\to+\infty$, $f\in C(0,1)$ and $\|f\|_B<1/2$. If $p_n\geqq$ $\geqq n/\log_4(1/\omega_n)$, where $\omega_n=\omega(f;1/2^n)$, then we get $f\in BV(p_n\uparrow\infty)$.*

PROOF. Let Δ be an arbitrary partition defined by (3.2) such that $\varrho(\Delta) \geqq 1/2^n$ holds. Then we have

$$\Big\{\sum_{j=1}^{m} |f(t_j)-f(t_{j-1})|^{p_n}\Big\}^{1/p_n} = \Big\{\sum_{k=0}^{n-1} \sum_{j\in\Gamma_k} |f(t_j)-f(t_{j-1})|^{p_n}\Big\}^{1/p_n} \leqq$$

$$\leqq \Big\{\sum_{k=0}^{n-1} 2^{k+1}\cdot \omega_k^{p_n}\Big\}^{1/p_n} \leqq 2\Big\{\sum_{k=0}^{n-1} 2^k \omega_k^{p_n}\Big\}^{1/p_n} \leqq 2\Big\{\sum_{k=0}^{n-1} 2^k \omega_k^{p_k}\Big\}^{1/p_n}.$$

Since $p_k \geqq k/\log_4(1/\omega_k)$, we get $2^k \omega_k^{p_k} \leqq 1/2^k$. Therefore it follows that

$$\Big\{\sum_{j=1}^{m} |f(t_j)-f(t_{j-1})|^{p_n}\Big\}^{1/p_n} \leqq 2\Big\{\sum_{k=0}^{n-1} (1/2^k)\Big\}^{1/p_n} \leqq 4.$$

Hence we have $f \in BV(p_n \uparrow \infty)$.

§4. Relations between $V[v]$ and $BV(p_n\uparrow\infty)$. In this section we consider the inclusion relations between Chanturiya's class $V[v]$ and our class $BV(p_n\uparrow\infty)$.

DEFINITION 4.1. The modulus of variation of a function f is the function

$$v(f\colon n) = \sup_{\Pi_n} \sum_{k=1}^{n} |f(I_k)|$$

where $f(I_k) = f(b_k) - f(a_k)$ and Π_n is an arbitrary system of n non-overlapping intervals $I_k = [a_k, b_k] \subseteqq [0, 1]$. Suppose v is a nondecreasing, convex upwards function on $[0, +\infty)$ and $v(0)=0$. If v is given, then $V[v]$ denotes the class of functions for which $v(f\colon n) = O\big(v(n)\big)$ as $n \to +\infty$.

We have the following theorem.

THEOREM 4.1. *Let* $1 < p_n \uparrow \infty$ *as* $n \to +\infty$. *When* $p_n \geqq \log v(2^n)$ *for* $n \geqq 1$, *we have* $V[v] \subseteqq BV(p_n\uparrow\infty)$.

PROOF. Let Δ be a partition defined by (1.1) such that $\varrho(\Delta) \geqq 1/2^n$. Since $v(n)\uparrow\infty$ as $n \to +\infty$, there exists an integer n_0 such that $p_n \geqq \log v(2^n) > 1$ for all $n \geqq n_0$. When $f \in V[v]$, there exists a constant $M > 1$ such that $|f(x)| \leqq M$ for all $x \in [0, 1]$. Therefore we get

$$\Big\{\sum_{k=1}^{m} |f(t_k)-f(t_{k-1})|^{p_n}\Big\}^{1/p_n} \leqq \Big\{\sum_{k=1}^{m} |f(t_k)-f(t_{k-1})|^{\log v(2^n)}\Big\}^{1/\log v(2^n)} \leqq$$

$$\leqq \Big\{(2M)^{\log v(2^n)-1} \sum_{k=1}^{m} |f(t_k)-f(t_{k-1})|\Big\}^{1/\log v(2^n)} \leqq$$

$$\leqq 2M\Big\{\sum_{k=1}^{m} |f(t_k)-f(t_{k-1})|\Big\}^{1/\log v(2^n)} \leqq 2M\{v(f\colon m)\}^{1/\log v(2^n)} \leqq$$

$$= O\big((v(2^n))^{1/\log v(2^n)}\big) = O(1) \quad \text{as} \quad n \to +\infty.$$

When $1 \leqq n \leqq n_0$, we get

$$\Big\{\sum_{k=1}^{m} |f(t_k)-f(t_{k-1})|^{p_n}\Big\}^{1/p_n} \leqq 2Mm \leqq 2M\cdot 2^n \leqq 2M\cdot 2^{n_0}.$$

Therefore we have $f\in BV(p_n\uparrow\infty)$.

THEOREM 4.2. $V[v]=B[0, 1]$ *if and only if*

$$\varlimsup_{n\to\infty} n/v(n) < +\infty.$$

PROOF. Suppose that $n\leqq Cv(n)$ $(n\geqq 1)$ for some constant $C>0$. When $f\in B[0,1]$, we get

$$\sum_{k=1}^{n} |f(I_k)| \leqq 2\|f\|_B n \leqq 2C\|f\|_B\cdot v(n).$$

Therefore we have $v(f\colon n)\leqq 2C\|f\|_B\cdot v(n)$ and $f\in V[v]$ holds.

Conversely, we assume that $V[v]=B[0, 1]$. We prove that $\varlimsup_{n\to\infty} n/v(n)<+\infty$. Let f be a function defined in Theorem 3.3. Set $\alpha_k=1/2^{n_k}$ and $\beta_k=1/2^{n_k}+2/(3\cdot 2^{n_k})$ for $k=1, 2, 3, \ldots$, and consider a system of intervals $I_k=[\alpha_k, \beta_k]$ for $k=1, 2, 3, \ldots$ $\ldots, n$. Then we get $n=\sum_{k=1}^{n} |f(I_k)|\leqq v(f\colon n)$. By our assumption, $f\in V[v]$. Therefore we have $v(f\colon n)\leqq Cv(n)$. Hence $n\leqq Cv(n)$ holds and the proof is complete.

THEOREM 4.3. *Let* $\varlimsup_{n\to\infty} n/v(n)=+\infty$ *and* $1<p_n\uparrow\infty$ *as* $n\to+\infty$. *Then there exists a function* $f\in BV(p_n\uparrow\infty)\cap C(0, 1)$ *such that* $f\notin V[v]$.

PROOF. Since $\varlimsup_{n\to\infty} n/v(n)=+\infty$, there exists a sequence $0<c_n<1$, $c_n\downarrow 0$ as $n\to+\infty$ such that $\varlimsup_{n\to\infty} nc_n/v(n)=+\infty$. We shall construct a function f as follows. We choose a monotone increasing sequence of positive integers $\{n_k\,;\, k\geqq 1\}$ such that $n_1=1$ and (3.8) hold. Set

$$f(x)=\begin{cases} 2^{n_k+1}c_k(x-1/2^{n_k}), & \text{if } 1/2^{n_k}\leqq x\leqq (1/2^{n_k}+1/2^{n_k-1})/2,\\ -2^{n_k+1}c_k(x-1/2^{n_k-1}), & \text{if } (1/2^{n_k}+1/2^{n_k-1})/2\leqq x\leqq 1/2^{n_k-1},\\ & \text{for } k=1, 2, 3, \ldots,\\ 0, & \text{otherwise.}\end{cases}$$

We get a function $f\in C(0, 1)$ and extend it periodically with period 1 on $(-\infty, \infty)$.

Next we show that $f\in BV(p_n\uparrow\infty)$. Let $\bar{\Delta}$ be an arbitrary partition of the interval $[0, 1]$ defined by (2.4) such that $\varrho(\bar{\Delta})\geqq 1/2^n$. For this fixed n, we can choose integers n_{k-1} and n_k for which $n_{k-1}\leqq n<n_k$ holds. Then it follows that $p_{n_{k-1}}\leqq p_n\leqq p_{n_k}$ and $1/2^{n_k}<1/2^n\leqq 1/2^{n_{k-1}}$. Hence we get from (3.8)

$$\Big\{\sum_{j=1}^{m} |f(t_j)-f(t_{j-1})|^{p_n}\Big\}^{1/p_n} \leqq \Big\{2\sum_{j=1}^{k} c_j^{p_n}\Big\}^{1/p_n} \leqq 2\cdot k^{1/p_{n_{k-1}}} \leqq 2\cdot k^{1/\log k} \leqq 2e<+\infty.$$

Therefore $f\in BV(p_n\uparrow\infty)$ holds.

Finally, we prove that $f \notin V[v]$. Set $a_k = 1/2^{n_k}$ and $b_k = (1/2^{n_k} + 1/2^{n_k - 1})/2$. We consider a system of intervals $I_k = [a_k, b_k]$ for $k = 1, 2, 3, \ldots$. Since $c_n \downarrow 0$, we get

$$\sum_{k=1}^{n} |f(I_k)| = \sum_{k=1}^{n} |f(b_k) - f(a_k)| = \sum_{k=1}^{n} c_k \geqq nc_n.$$

Assume that $f \in V[v]$. Then it follows that

$$\sum_{k=1}^{n} |f(I_k)| \leqq v(f\colon n) = O(v(n)) \quad \text{as} \quad n \to +\infty,$$

which means $\varlimsup_{n \to \infty} nc_n / v(n) \leqq C$. We arrive at a contradiction. Therefore we get $f \notin V[v]$ and the proof is complete.

Let $\Lambda = \{\lambda_n; n \geqq 1\}$ be an increasing sequence of positive numbers such that $\sum_{n=1}^{\infty} (1/\lambda_n) = +\infty$. A function f is said to be of Λ-bounded variation ($f \in \Lambda BV$), if for every choice of nonoverlapping intervals $\{I_n; n \geqq 1\}$ we have

$$\sum_{n=1}^{\infty} |f(I_n)|/\lambda_n < +\infty.$$

M. Avdispahić [1] showed the inclusion relation between Waterman's class ΛBV and Chanturiya's class $V(v)$. Our class $BV(p_n \uparrow \infty)$ plays an important role with respect to the problems of uniform convergence of Fourier series. We describe some results for the applications of Fourier series in another paper.

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GREEN'S THEOREM ON A FOLIATED RIEMANNIAN MANIFOLD AND ITS APPLICATIONS

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1. Introduction

Our main aim is to prove the following theorems that are well-known ([4]) in the cases of the foliations by points:

THEOREM A. *Let* $(M, g_M, \mathcal{F})$ *be a closed, oriented, connected Riemannian manifold with a transversally orientable foliation* $\mathcal{F}$ *of codimension* q *and a bundle-like metric* g_M *with respect to* $\mathcal{F}$. *Let* Q *be the normal bundle of* $\mathcal{F}$. *Then*

$$\int_M \operatorname{div}_D v \, d_M = \langle\langle \tau, v \rangle\rangle$$

for all $v \in \Gamma(Q)$, *where* $\operatorname{div}_D v$ *denotes the transverse divergence of* v *with respect to the transverse Riemannian connection* D *of* $\mathcal{F}$ *and* τ *denotes the tension field of* $\mathcal{F}$.

Let E be the integrable sub-bundle of the tangent bundle TM given by $\mathcal{F}$. We notice that $Q=TM/E$, and $\mathcal{F}$ is minimal if $\tau=0$. Let $\bar{V}(\mathcal{F})=\{v\in\Gamma(Q)|D_X v=0$ for all $X\in\Gamma(E)\}$.

THEOREM B. *Let* $(M, g_M, \mathcal{F})$ *be as in Theorem* A. *Suppose that* $\mathcal{F}$ *is minimal. If* $v\in \bar{V}(\mathcal{F})$ *satisfies*

$$\Delta v = \varrho_D(v) \quad \text{and} \quad \operatorname{div}_D v = 0,$$

then v *is a transverse Killing field of* $\mathcal{F}$. *Here* ϱ_D *is the Ricci operator with respect to* D.

Theorem A is analogous to the well-known Green's Theorem on a Riemannian manifold with or without boundary. Theorem B is proved by the integral formulas induced from Theorem A.

We shall be in C^∞-category and deal only with connected and oriented manifolds without boundary. We use the following convention on the range of indices: $1\leqq i, j\leqq p$ and $p+1\leqq a, b, c, d\leqq p+q$. The Einstein summation convention will be used.

2. Preliminaries

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional compact Riemannian manifold with a transversally orientable foliation $\mathcal{F}$ of codimension q and a bundlelike metric g_M with respect to $\mathcal{F}$ in the sense of Reinhart [7]. Let ∇ be the Levi-Civita connection with respect to g_M. The foliation $\mathcal{F}$ defines an integrable sub-bundle E of the tangent bundle TM over M. Let $Q=TM/E$ be the normal bundle of $\mathcal{F}$. The metric g_M

gives a splitting σ of the exact sequence

$$0 \to E \to TM \underset{\sigma}{\overset{\pi}{\rightleftarrows}} Q \to 0$$

with $\sigma(Q)=E^\perp$ (the orthogonal complement bundle of E). Then g_M induces a metric g_Q on Q:

$$g_Q(\nu, \mu) = g_M(\sigma(\nu), \sigma(\mu))$$

for all $\nu, \mu \in \Gamma(Q)$.

In a flat chart $U(x^i, x^a)$ with respect to $\mathscr{F}$ ([7]), a local frame $\{X_i, X_a\} = \{\partial/\partial x^i, \partial/\partial x^a - A_a^j \partial/\partial x^j\}$ is called the basic adapted frame to $\mathscr{F}$. Here A_a^j are functions on U with $g_M(X_i, X_a)=0$ ([7], [8]). We notice that $\{X_i\}$ and $\{X_a\}$ span $\Gamma(E|U)$ and $\Gamma(E^\perp|U)$ respectively. We set

$$g_{ij} = g_M(X_i, X_j), \quad g_{ab} = g_M(X_a, X_b),$$

$$(g^{ij}) = (g_{ij})^{-1}, \quad (g^{ab}) = (g_{ab})^{-1}.$$

A connection D in Q is defined by

$$D_X \nu = \pi([X, Y_\nu]) \quad \text{for all} \quad X \in \Gamma(E), \quad \nu \in \Gamma(Q) \quad \text{with} \quad Y_\nu = \sigma(\nu),$$

$$D_X \nu = \pi(\nabla_X Y_\nu) \quad \text{for all} \quad X \in \Gamma(E^\perp), \quad \nu \in \Gamma(Q) \quad \text{with} \quad Y_\nu = \sigma(\nu).$$

Then we have

PROPOSITION 1 ([2]). *The connection D in Q is torsion free and metric with respect to g_Q.*

Thus we have

DEFINITION 2 ([2], [3], [5], [6]). The connection D is called the transversal Riemannian connection of $\mathscr{F}$.

The curvature R_D of D is defined as follows: $R_D(X, Y)\nu = D_X D_Y \nu - D_Y D_X \nu - D_{[X,Y]}\nu$ for all $X, Y \in \Gamma(TM)$ and $\nu \in \Gamma(Q)$. We notice that $i(X) R_D = 0$ for all $X \in \Gamma(E)$, where i denotes the interior product ([2]). The Ricci operator $\varrho_D : \Gamma(Q) \to \Gamma(Q)$ of $\mathscr{F}$ is defined by

$$\varrho_D(\nu) = g^{ab} R_D(\sigma(\nu), X_a)\pi(X_b) \tag{1}$$

for all $\nu \in \Gamma(Q)$.

Let $V(\mathscr{F})$ be the space of all vector fields Y on M satisfying $[Y, Z] \in \Gamma(E)$ for all $Z \in \Gamma(E)$. An element of $V(\mathscr{F})$ is called an infinitesimal automorphism of $\mathscr{F}$ ([3], [6]). We set

$$\bar{V}(\mathscr{F}) = \{\nu \in \Gamma(Q) | \nu = \pi(Y), \ Y \in V(\mathscr{F})\}. \tag{2}$$

LEMMA 3. *An element ν of $\bar{V}(\mathscr{F})$ satisfies $D_X \nu = 0$ for all $X \in \Gamma(E)$.*

The transverse Lie derivation $\theta(Y)$ with respect to $Y \in V(\mathscr{F})$ is defined by

$$\theta(Y)\nu = \pi([Y, Y_\nu])$$

for all $\nu \in \Gamma(Q)$ with $Y_\nu = \sigma(\nu)$. Then we have

DEFINITION 4 ([3], [5], [6], [9]). If $Y \in V(\mathcal{F})$ is metrical, that is, $\theta(Y)g_Q=0$, then $v=\pi(Y)$ is called a transverse Killing field of $\mathcal{F}$.

For $Y \in V(\mathcal{F})$, we define an operator $A_D(Y)\colon \Gamma(Q)\to\Gamma(Q)$ by

$$A_D(Y)v = \theta(Y)v - D_Y v \tag{3}$$

for all $v \in \Gamma(Q)$. Then we have

$$A_D(Y)v = -D_{Y_v}\pi(Y) \tag{4}$$

where $Y_v=\sigma(v)$. This formula shows that (i) $A_D(Y)$ depends only on $v=\pi(Y)$, (ii) $A_D(Y)$ is a linear operator on $\Gamma(Q)$. Thus we may use $A_D(v)$ instead of $A_D(Y)$ ([3]).

DEFINITION 5 ([3]). If $Y \in V(\mathcal{F})$ preserves the connection D in Q, that is, $\theta(Y)D=0$, then $v=\pi(Y)$ is called a transverse affine field of $\mathcal{F}$.

Then we have

PROPOSITION 6 ([3]). *Let $v \in \bar{V}(\mathcal{F})$. Then the following conditions are equivalent:*

(i) *v is a transverse Killing field of $\mathcal{F}$.*
(ii) $A_D(v)g_Q=0$.
(iii) $g_Q(A_D(v)\mu_1, \mu_2)+g_Q(\mu_1, A_D(v)\mu_2)=0$ *for all* $\mu_1, \mu_2 \in \Gamma(Q)$.

PROPOSITION 7 ([3]). *Let $Y \in V(\mathcal{F})$ with $v=\pi(Y)$. Then the following conditions are equivalent:*

(i) *Y preserves D.*
(ii) $D_{\sigma(\mu)}A_D(v)=R_D(\sigma(\mu), Y)$ *for all* $\mu \in \Gamma(Q)$.

LEMMA 8 ([3]). *A transverse Killing field of $\mathcal{F}$ is a transverse affine field of $\mathcal{F}$.*

Let $\Omega^r(M, Q)$ be the space of all Q-valued r-forms on M. The inner product $\langle\,,\rangle$ on $\Omega^r(M, Q)$ is defined by

$$\langle\langle \eta, \eta' \rangle\rangle = \int_M g_Q(\eta \wedge * \eta') \tag{5}$$

([2]). For example, if $\eta=\xi\otimes v$, $\eta'=\xi'\otimes v' \in \Omega^1(M, Q)$, then $g_Q(\eta\wedge *\eta')= =g_Q(v, v')\xi\wedge *\xi'$. We notice that the bundle projection $\pi\colon TM\to Q$ is an element of $\Omega^1(M, Q)$.

The Q-valued bilinear form α on M is defined by

$$\alpha(X, Y) = -(D_X\pi)(Y) \tag{6}$$

for all $X, Y \in \Gamma(TM)$ ([2]). Since $\alpha(X, Y)=\pi(\nabla_X Y)$ for all $X, Y \in \Gamma(E)$, we call α the second fundamental form of $\mathcal{F}$ ([2]). The tension field τ of $\mathcal{F}$ is defined by

$$\tau = g^{ij}\alpha(X_i, X_j). \tag{7}$$

DEFINITION 9. The foliation $\mathcal{F}$ is minimal if all the leaves of $\mathcal{F}$ are minimal submanifolds.

Then we have

PROPOSITION 10 ([2]). *The foliation $\mathcal{F}$ is minimal if and only if $\tau=0$.*

Let $d_D: \Omega^r(M, Q) \to \Omega^{r+1}(M, Q)$ be the exterior differential operator, and let d_D^* be the adjoint operator of d_D with respect to $\langle , \rangle$ ([2]). The Laplacian Δ acting on $\Omega^r(M, Q)$ is defined by

$$\Delta = d_D d_D^* + d_D^* d_D. \tag{8}$$

An element ν of $\Gamma(Q)$ is regarded as an element of $\Omega^0(M, Q)$, that is, $\nu \in \Omega^0(M, Q)$. Then we have

PROPOSITION 11. *A transverse Killing field ν of $\mathcal{F}$ satisfies the equation:*

$$\Delta\nu = D_{\sigma(\tau)}\nu + \varrho_D(\nu).$$

PROOF. We first remark that $D_{X_i}\nu = 0$. Thus we have

$$\Delta\nu = d_D^* d_D \nu = -g^{ij}(D_{X_i}D_{X_j}\nu - D_{\nabla_{X_i}X_j}\nu) - g^{ab}(D_{X_a}D_{X_b}\nu - D_{\nabla_{X_a}X_b}\nu) =$$

$$= g^{ij}D_{\sigma(\pi(\nabla_{X_i}X_j))}\nu - g^{ab}(D_{X_a}D_{X_b}\nu - D_{\sigma(\pi(\nabla_{X_a}X_b))}\nu) =$$

$$= D_{\sigma(\tau)}\nu + g^{ab}\big(D_{X_a}A_D(\nu)\big)\big(\pi(X_b)\big).$$

By Proposition 7, we have

$$\big(D_{X_a}A_D(\nu)\big)\big(\pi(X_b)\big) = R_D\big(\sigma(\nu), X_a\big)\big(\pi(X_b)\big).$$

Thus we have, by (1),

$$\Delta\nu = D_{\sigma(\tau)}\nu + g^{ab}R_D\big(\sigma(\nu), X_a\big)\big(\pi(X_b)\big) = D_{\sigma(\tau)}\nu + \varrho_D(\nu).$$

Q.E.D.

DEFINITION 12 ([10]). An operator $\mathrm{div}_D: \Gamma(Q) \to R$ is defined by

$$\mathrm{div}_D\,\nu = g^{ab}g_Q\big(D_{X_a}\nu, \pi(X_b)\big).$$

We call $\mathrm{div}_D\,\nu$ the transverse divergence of ν with respect to D.

By Proposition 6, we have

PROPOSITION 13 ([10]). *If ν is a transverse Killing field of $\mathcal{F}$, then* $\mathrm{div}_D\,\nu = 0$.

3. Proof of Theorem A

Let $\mathrm{div}_\nabla X$ be the divergence of $X \in \Gamma(TM)$ with respect to ∇. For any $\nu \in \Gamma(Q)$, we have

$$\mathrm{div}_\nabla\,\sigma(\nu) = g^{ij}g_M\big(\nabla_{X_i}\sigma(\nu), X_j\big) + g^{ab}g_M\big(\nabla_{X_a}\sigma(\nu), X_b\big) =$$

$$= -g^{ij}g_M\big(\sigma(\nu), \sigma(\pi(\nabla_X X_j))\big) + g^{ab}g_M\big(\sigma(D_{X_a}\nu), \sigma(\pi(X_b))\big) =$$

$$= -g_Q(\nu, \tau) + g^{ab}g_Q\big(D_{X_a}\nu, \pi(X_b)\big) = -g_Q(\nu, \tau) + \mathrm{div}_D\,\nu.$$

By Green's Theorem, we have

$$0 = \int_M \mathrm{div}_\nabla\,\sigma(\nu)\,d_M = \int_M \mathrm{div}_D\,\nu\,d_M - \langle\langle \nu, \tau \rangle\rangle.$$

This completes the proof of Theorem A.

As Corollary of Theorem A, we have

COROLLARY 14 ([10]). *Let $(M, g_M, \mathscr{F})$ be as in Theorem* A. *If $\mathscr{F}$ is minimal, then*

$$\int_M \operatorname{div}_D \nu \, d_M = 0$$

for all $\nu \in \Gamma(Q)$.

4. Integral formulas and proof of Theorem B

We consider an element ξ of $\Gamma(Q)$ given by

(9) $$\xi = g^{cd} g_Q(D_{X_c}\nu, \mu)\pi(X_d),$$

where $\mu, \nu \in \bar{V}(\mathscr{F})$. Then we have

$$D_{X_a}\xi = X_a(g^{cd}) g_Q(D_{X_c}\nu, \mu)\pi(X_d) + g^{cd} g_Q(D_{X_a}D_{X_c}\nu, \mu)\pi(X_d) + $$
$$+ g^{cd} g_Q(D_{X_c}\nu, D_{X_a}\mu)\pi(X_d) + g^{cd} g_Q(D_{X_c}\nu, \mu) D_{X_a}\pi(X_d)$$

and

$$\operatorname{div}_D \xi = g^{ac} g_Q(D_{X_a}D_{X_c}\nu, \mu) + g^{ac} g_Q(D_{X_c}\nu, D_{X_a}\mu) -$$
$$- g^{ab} g^{cd} g_Q\big(\pi(X_d), D_{X_a}\pi(X_b)\big) g_Q(D_{X_c}\nu, \mu) =$$
$$= g^{ac} g_Q(D_{X_a}D_{X_c}\nu, \mu) + g^{ac} g_Q(D_{X_a}\nu, D_{X_c}\mu) - g_Q(D_{\sigma(\eta)}\nu, \mu),$$

where $\eta = g^{ab}\pi(\nabla_{X_a}X_b)$.

Since $\Delta\nu = D_{\sigma(\tau)}\nu - g^{ab} D_{X_a}D_{X_b}\nu + D_{\sigma(\eta)}\nu$, we have

$$-g_Q(\Delta\nu, \mu) = g^{ab} g_Q(D_{X_a}D_{X_b}\nu, \mu) - g_Q(D_{\sigma(\eta)}\nu, \mu) - g_Q(D_{\sigma(\tau)}\nu, \mu).$$

Thus we have

$$\operatorname{div}_D \xi = -g_Q(\Delta\nu, \mu) + g_Q(D_{\sigma(\tau)}\nu, \mu) + g^{ab} g_Q(D_{Xa}\nu, D_{X_b}\mu).$$

If we set $\tau = \tau^d\pi(X_d)$, then $\sigma(\tau) = \tau^d X_d$. Thus we have

$$g_Q(\xi, \tau) = g^{cd} g_Q(D_{X_c}\nu, \mu) g_Q\big(\pi(X_d), \tau\big) =$$
$$= \tau^d g_Q(D_{X_d}\nu, \mu) = g_Q(D_{\sigma(\tau)}\nu, \mu).$$

Therefore, by Theorem A, we have

THEOREM 15. *Let $(M, g_M, \mathscr{F})$ be as Theorem* A. *For* $\mu, \nu \in \bar{V}(\mathscr{F})$, *we have*

$$\langle\langle \Delta\nu, \mu \rangle\rangle = \langle\langle D\nu, D\mu \rangle\rangle,$$

where $\langle\langle D\nu, D\mu \rangle\rangle = \int_M g^{ab} g_Q(D_{X_a}\nu, D_{X_b}\mu) \, d_M$.

If ν is a transverse Killing field of $\mathscr{F}$, then, by Proposition 11 and Theorem 15, we have

$$\langle\langle \varrho_D(\nu), \nu \rangle\rangle + \langle\langle D_{\sigma(\tau)}\nu, \nu \rangle\rangle = \langle\langle D\nu, D\nu \rangle\rangle \geqq 0.$$

Thus we have

THEOREM 16 ([3]), [9]). *Let $(M, g_M, \mathcal{F})$ be as in Theorem* A. *Suppose that $\mathcal{F}$ is minimal and that the Ricci operator ϱ_D of $\mathcal{F}$ is non-positive everywhere and negative for at least one point of M. If v is a transverse Killing field of $\mathcal{F}$, then $v=0$.*

By direct calculation, we have, for all $v \in \Gamma(Q)$,

$$\operatorname{div}_D((\operatorname{div}_D v)v) = v(\operatorname{div}_D v) + (\operatorname{div}_D v)^2. \tag{10}$$

$$\operatorname{div}_D A_D(v)v = -g^{ab} g_Q(D_{X_a} D_{\sigma(v)} v, \pi(X_b)). \tag{11}$$

$$\operatorname{Tr} A_D(v) A_D(v) = g^{ab} g_Q(D_{\sigma(\nabla_{X_a} v)} v, \pi(X_b)), \tag{12}$$

where Tr denotes the trace operator.

$$v(\operatorname{div}_D v) = v(g^{ab}) g_Q(D_{X_a} v, \pi(X_b)) + \\ + g^{ab} g_Q(D_{\sigma(v)} D_{X_a} v, \pi(X_b)) + g^{ab} g_Q(D_{X_a} v, D_{\sigma(v)} \pi(X_b)). \tag{13}$$

Then we have

$$-\operatorname{div}_D A_D(v)v - \operatorname{div}_D((\operatorname{div}_D v)v) =$$

$$= g^{ab} g_Q(R_D(X_a, \sigma(v))v, \pi(X_b)) + \operatorname{Tr} A_D(v) A_D(v) - (\operatorname{div}_D v)^2 -$$

$$- g^{ab} g_Q(D_{\sigma(D_{\sigma(v)} \pi(X_a))} v, \pi(X_b)) + g^{ab} g_Q(D_{[X_a \sigma(v)]_E} v, \pi(X_b)) -$$

$$- v(g^{ab}) g_Q(D_{X_a} v, \pi(X_b)) - g^{ab} g_Q(D_{X_a} v, D_{\sigma(v)} \pi(X_b)),$$

where $[\ ,\ ]_E$ denotes the E-component of $[\ ,\ ]$. Since we have

$$-v(g^{ab}) g_Q(D_{X_a} v, \pi(X_b)) = g^{ac} g_Q(D_{X_a} v, D_{\sigma(v)} \pi(X_c)) + \\ + g^{db} g_Q(D_{\sigma(D_{\sigma(v)} \pi(X))} v, \pi(X_b)),$$

if $v \in \bar{V}(\mathcal{F})$, then

$$-\operatorname{div}_D A_D(v)v - \operatorname{div}_D((\operatorname{div}_D v)v) = \varrho_D(v) + \operatorname{Tr} A_D(v) A_D(v) - (\operatorname{div}_D v)^2.$$

Therefore, we have

THEOREM 17. *Let $(M, g_M, \mathcal{F})$ be as in Theorem* A. *Suppose that $\mathcal{F}$ is minimal. If $v \in \bar{V}(\mathcal{F})$, then*

$$\int_M \{\operatorname{Ric}_D(v) + \operatorname{Tr} A_D(v) A_D(v) - (\operatorname{div}_D v)^2\}\, d_M = 0,$$

where $\operatorname{Ric}_D(v) = g_Q(\varrho_D(v), v)$.

By Theorem 17 and the fact that

$$\operatorname{Tr} A_D(v) A_D(v) = -\operatorname{Tr} {}^t A_D(v) A_D(v) + (1/2) \operatorname{Tr} (A_D(v) + {}^t A_D(v))^2,$$

where ${}^t B$ denotes the transpose of B, we have

THEOREM 18. *Let $(M, g_M, \mathcal{F})$ be as Theorem* A. *Suppose that $\mathcal{F}$ is minimal. Then*

$$\int_M \{\operatorname{Ric}_D(v) - \operatorname{Tr} {}^t A_D(v) A_D(v) + (1/2) \operatorname{Tr} (A_D(v) + {}^t A_D(v))^2 - (\operatorname{div}_D v)^2\}\, d_M = 0$$

for all $v \in \bar{V}(\mathcal{F})$.

A proof of Theorem B is given as follows: We remark that

$$\mathrm{Tr}\,{}^t A_D(v) A_D(v) = g^{ab} g_Q(D_{X_a} v, D_{X_b} v),$$
$$\mathrm{Ric}_D(v) = g_Q(\Delta v, v).$$

Thus, by Theorem 18, we have

$$\langle\langle \Delta v, v\rangle\rangle - \langle\langle Dv, Dv\rangle\rangle + (1/2) \int_M \mathrm{Tr}\left(A_D(v) + {}^t A_D(v)\right)^2 d_M = 0.$$

By Theorem 15, the above equality implies

$$\int_M \mathrm{Tr}\left(A_D(v) + {}^t A_D(v)\right)^2 d_M = 0.$$

Since $A_D(v) + {}^t A_D(v)$ is symmetric, it follows that

$$A_D(v) + {}^t A_D(v) = 0$$

([4]). Thus Proposition 6 implies that v is a transverse Killing field of $\mathcal{F}$.

Finally, we have some coments: If M is complete and noncompact, then L^2-transverse Killing fields of $\mathcal{F}$ ([9]) on M are discussed by [1]. We have prepared the discussion of another geometric transverse fields of $\mathcal{F}$. In [10], the first author ought to have made reference to Molino's papers ([5], [6], etc.) that are concerned with the transversal Riemannian connection of $\mathcal{F}$ and transverse Killing fields of $\mathcal{F}$

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HARDY'S INEQUALITY FOR ORLICZ—LUXEMBURG NORMS

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1. Introduction

The original Hardy's Inequality [1 : Theorem 326] may be written

$$\|Ax\| \leqq C\|x\| \tag{1}$$

where $x=(x_n)$ is a sequence treated as a column matrix, $\|\,.\,\|$ is the l^p-norm with $p>1$, C is independent of x, and $A=(a_{mn})$ is the Cesaro matrix

$$a_{mn} = \begin{cases} 1/m & \text{if } \ 0<n\leqq m, \\ 0 & \text{otherwise.} \end{cases}$$

In this paper I propose to generalize this inequality, replacing A by a much more general matrix operator and $\|\,.\,\|$ by an Orlicz-type norm which is essentially due to Luxemburg [2]. This norm is defined below.

References [3] and [4] are related to this paper. In [3] the norms are weighted l^p-norms only. In [4] the norms are Orlicz—Luxemburg; but the dominance condition on the elements of A has a more complicated character and, except in one theorem, A is lower triangular.

Orlicz function is any continuous convex $\varphi\colon (0,\infty)\to(0,\infty)$ for which $\varphi(0+)=0$. Such a function is strictly increasing, and consequently has a continuous inverse $\varphi^{-1}\colon (0,\infty)\to(0,\infty)$. As usual in Orlicz theory the domain of φ is extended to the complex plane by $\varphi(z)=\varphi(|z|)$ and $\varphi(0)=0$.

Orlicz—Luxemburg norm of a complex-valued sequence $s=(s_n)$ is

$$\|s\| = \inf\left\{k>0\colon \sum_{n=1}^{\infty} \lambda_n \varphi\left(\frac{s_n}{k}\right) \leqq 1\right\} \tag{2}$$

for a fixed Orlicz function φ and fixed $\lambda_n>0$. Besides having the properties of a norm it has the monotonic property that if $|s_n|\leqq|t_n|$ for all n then $\|s\|\leqq\|t\|$.

EXAMPLE. If $\varphi(t)=t^p$ and $p\geqq 1$, the condition in (2) can be written

$$\sum_{n=1}^{\infty} \lambda_n |s_n|^p \leqq k^p,$$

and consequently

$$\|s\| = \Big(\sum_{n=1}^{\infty} \lambda_n |s_n|^p\Big)^{1/p},$$

a weighted l^p-norm.

2. Preliminaries

LEMMA 1. *If* $0<\|s\|<\infty$ *then* $\sum_{n=1}^{\infty}\lambda_n\varphi\left(\frac{s_n}{\|s\|}\right)\leqq 1$.

PROOF. Suppose that $k_m\downarrow\|s\|$ as $m\uparrow\infty$. Then for each m and N

$$\sum_{n=1}^{N}\lambda_n\varphi\left(\frac{s_n}{k_m}\right)\leqq 1.$$

Since φ is continuous,

$$\sum_{n=1}^{N}\lambda_n\varphi\left(\frac{s_n}{\|s\|}\right)\leqq 1.$$

The lemma follows from this by making $N\to\infty$.

LEMMA 2. *Let each function* $s_n(t)$ *be non-negative and measurable on* E, *a measurable subset of the real axis. If* $s(t)$ *denotes the sequence* $(s_n(t))$, *and* $\int_E s(t)\,dt$ *the sequence* $\left(\int_E s_n(t)\,dt\right)$, *then* $\|s(t)\|$ *is measurable on* E *and*

$$\left\|\int_E s(t)\,dt\right\|\leqq\int_E\|s(t)\|\,dt. \tag{3}$$

The same conclusions hold if each $s_n(t)$ *is complex-valued and integrable on* E.

PROOF. (i) To prove that $\|s(t)\|$ is measurable on E, consider the function

$$g(x,t)=\sum_{n=1}^{\infty}\lambda_n\varphi\left(\frac{s_n(t)}{x}\right).$$

Since φ is continuous, g is plane-measurable on $\{x\colon x>0\}\times\{t\colon t\in E\}$; thus

$$\{(x,t)\colon g(x,t)>1,\ x>0\ \&\ t\in E\}$$

is plane-measurable. So the measures of its sections t-constant form a measurable function of t on E. But g is a decreasing function of x for each fixed t, so these sections are line-segments $0<x<\|s(t)\|$ or $0<x\leqq\|s(t)\|$. Thus $\|s(t)\|$ is measurable on E.

(ii) Suppose that the right side of (3) is zero. Consider the null set $N=\{t\in E\colon \|s(t)\|>0\}$. For $t\in E-N$, $s_n(t)=0$ for all n, and so

$$0\leqq\int_E s_n(t)\,dt=\int_{E-N}s_n(t)\,dt=0.$$

Thus

$$\left\|\int_E s(t)\,dt\right\|=0\leqq\int_E\|s(t)\|\,dt=0,$$

and the required inequality is just an equality.

(iii) Suppose that the right side of (3) is not zero. Denote its value by $\varkappa$; we can suppose that $0<\varkappa<\infty$. Let

$$N=\{t\in E\colon \|s(t)\|=\infty\},\quad S=\{t\in E\colon 0<\|s(t)\|<\infty\}.$$

For $t \in (E-N)-S$, $\|s(t)\|=0$ and so $s_n(t)=0$ for all n. Thus

$$\sum_{n=1}^{\infty} \lambda_n \varphi\left(\frac{1}{\varkappa} \int_E s_n(t)\,dt\right) = \sum_{n=1}^{\infty} \lambda_n \varphi\left(\frac{1}{\varkappa} \int_{E-N} s_n(t)\,dt\right) = \tag{4}$$

$$= \sum_{n=1}^{\infty} \lambda_n \varphi\left(\frac{1}{\varkappa} \int_S s_n(t)\,dt\right) = \sum_{n=1}^{\infty} \lambda_n \varphi\left(\int_S \frac{\|s(t)\|}{\varkappa} \frac{s_n(t)}{\|s(t)\|}\,dt\right) \leqq$$

$$\leqq \sum_{n=1}^{\infty} \lambda_n \int_S \frac{\|s(t)\|}{\varkappa} \varphi\left(\frac{s_n(t)}{\|s(t)\|}\right) dt = \int_S \frac{\|s(t)\|}{\varkappa} \sum_{n=1}^{\infty} \lambda_n \varphi\left(\frac{s_n(t)}{\|s(t)\|}\right) dt \leqq \tag{5}$$

$$\leqq \int_S \frac{\|s(t)\|}{\varkappa}\,dt = 1. \tag{6}$$

In (4) we have used the fact that N is a null set; in (5) the convexity of φ and the fact that

$$\int_S \frac{\|s(t)\|}{\varkappa}\,dt = \int_{S \cup N} \frac{\|s(t)\|}{\varkappa}\,dt \leqq \frac{1}{\varkappa} \int_E \|s(t)\|\,dt = 1;$$

and the last has been used again in (6), as well as Lemma 1.

Now (3), whose right side is $\varkappa$, follows at once from (2) and (6).

(iv) For any complex-valued sequence $s=(s_n)$ write $|s|=(|s_n|)$. Then by (2) $\|s\|=\||s|\|$. So if each $s_n(t)$ is complex-valued and integrable on E we obtain, using the monotonic property and (3),

$$\left\|\int_E s(t)\,dt\right\| = \left\|\left|\int_E s(t)\,dt\right|\right\| \leqq \left\|\int_E |s(t)|\,dt\right\| \leqq \int_E \||s(t)|\|\,dt = \int_E \|s(t)\|\,dt.$$

This completes the proof of Lemma 2, which is of course an extension of Minkowski's Inequality.

LEMMA 3. *If $s_{\cdot r}$ denotes the sequence $s_{1,r}, s_{2,r}, s_{3,r}, \dots$ for $r=1, 2, \dots, \infty$, and for each fixed m and all positive integers r*

$$0 \leqq s_{m,r} \leqq s_{m,r+1} \to s_{m,\infty} \quad as \quad r \to \infty,$$

then $\|s_{\cdot r}\| \to \|s_{\cdot\infty}\|$ as $r \to \infty$.

PROOF. By the monotonic property of the norm,

$$\|s_{\cdot r}\| \leqq \|s_{\cdot r+1}\| \leqq \|s_{\cdot\infty}\|$$

for all r; and consequently the following limit exists and

$$\lim_{r\to\infty} \|s_{\cdot r}\| \leqq \|s_{\cdot\infty}\|.$$

There is nothing further to prove unless there is inequality here.

Suppose there is inequality here. Then there is $\varkappa$ such that

$$\lim_{r\to\infty} \|s_{\cdot r}\| < \varkappa < \|s_{\cdot\infty}\|. \tag{7}$$

It follows that

$$\sum_{m=1}^{n} \lambda_m \varphi\left(\frac{s_{m,r}}{\varkappa}\right) \leqq 1$$

holds for every positive integer n, since by (2) it holds with n replaced by ∞. Since it also holds for all r, and φ is continuous,

$$\sum_{m=1}^{n} \lambda_m \varphi\left(\frac{s_{m,\infty}}{\varkappa}\right) \leqq 1.$$

Making $n \to \infty$,

$$\sum_{m=1}^{\infty} \lambda_m \varphi\left(\frac{s_{m,\infty}}{\varkappa}\right) \leqq 1;$$

whence, by (2), $\|s_{\cdot\infty}\| \leqq \varkappa$. This contradicts (7), so proving Lemma 3.

3. Hardy's Inequality generalized

THEOREM. *Let $\varphi(t)$ be Orlicz and supermultiplicative. Let $\alpha(t)$ be non-negative and measurable and have a decreasing rearrangement $\bar{\alpha}(t)$, all on $(0, \infty)$. For all positive integers m and n let $\lambda_n > 0$ and $\Lambda_m = \sum_{n=1}^{m} \lambda_n$; also $\Lambda_0 = 0$. If $A = (a_{mn})$ and $x = (x_m)$ are complex matrices,*

$$|a_{mn}| \leqq \int_{\Lambda_{n-1}/\Lambda_m}^{\Lambda_n/\Lambda_m} \alpha(t)\,dt \quad \text{and} \quad C = \int_0^\infty \bar{\alpha}(t)\varphi^{-1}(t^{-1})\,dt,$$

then $\|Ax\| \leqq C\|x\|$.

If the sequence $(|x_n|)$ is decreasing, this inequality holds with C reduced in value to

$$\int_0^\infty \alpha(t)\varphi^{-1}(t^{-1})\,dt, \tag{8}$$

and α need not have a decreasing rearrangement.

PROOF. We may suppose that $\alpha(t)$ is not null, and consequently $C > 0$; for otherwise $a_{mn} = 0$ and the left side of the inequality would be zero. We may suppose also that $0 < \|x\| < \infty$; for if $\|x\| = 0$ the left side would again be zero, while if $\|x\| = \infty$ the right side would be infinite.

(i) Suppose that a_{mn} and x_n are non-negative, and that $x = (x_n)$ has a decreasing rearrangement. For all n let

$$g(u) = x_n \quad \text{for} \quad \Lambda_{n-1} < u \leqq \Lambda_n, \tag{9}$$

and let $g(u) = 0$ otherwise. Then $g(u)$ has a decreasing rearrangement $\bar{g}(u)$ on $(0, \infty)$, whether $\lim \Lambda_n$ is finite or infinite. We have

$$(10) \qquad 0 \leqq \sum_{n=1}^{\infty} a_{mn} x_n \leqq \sum_{n=1}^{\infty} \int_{\Lambda_{n-1}/\Lambda_m}^{\Lambda_n/\Lambda_m} \alpha(t) x_n \, dt =$$

$$= \sum_{n=1}^{\infty} \int_{\Lambda_{n-1}}^{\Lambda_n} \alpha\left(\frac{u}{\Lambda_m}\right) g(u) \frac{du}{\Lambda_m} \leqq \int_0^{\infty} \alpha\left(\frac{u}{\Lambda_m}\right) g(u) \frac{du}{\Lambda_m} =$$

$$= \int_0^{\infty} \alpha(t) g(\Lambda_m t) \, dt \leqq \int_0^{\infty} \bar{\alpha}(t) \bar{g}(\Lambda_m t) \, dt,$$

using [1 : Theorem 378] in the last step.

Now take norms of the sequences in (10). Using the monotonic property of the norm, and Lemma 2,

(11)

$$\left\| \sum_{n=1}^{\infty} a_{mn} x_n \right\| \leqq \left\| \int_0^{\infty} \bar{\alpha}(t) \bar{g}(\Lambda_m t) \, dt \right\| \leqq \int_0^{\infty} \|\bar{\alpha}(t) \bar{g}(\Lambda_m t)\| \, dt = \int_0^{\infty} \bar{\alpha}(t) \|\bar{g}(\Lambda_m t)\| \, dt;$$

here the mth member of each sequence concerned is written instead of the sequence itself.

(ii) By (2), for all $t > 0$,

$$(12) \qquad \|\bar{g}(\Lambda_m t)\| = \inf\left\{k > 0 \colon \sum_{n=1}^{\infty} \lambda_n \varphi\left(\frac{\bar{g}(\Lambda_n t)}{k}\right) \leqq 1\right\};$$

accordingly we estimate the sum in this for all $k > 0$ and $t > 0$.

$$(13) \qquad \sum_{n=1}^{\infty} \lambda_n \varphi\left(\frac{\bar{g}(\Lambda_n t)}{k}\right) = \sum_{n=1}^{\infty} \int_{\Lambda_{n-1}}^{\Lambda_n} \varphi\left(\frac{\bar{g}(\Lambda_n t)}{k}\right) ds \leqq \sum_{n=1}^{\infty} \int_{\Lambda_{n-1}}^{\Lambda_n} \varphi\left(\frac{\bar{g}(st)}{k}\right) ds \leqq$$

$$\leqq \int_0^{\infty} \varphi\left(\frac{\bar{g}(st)}{k}\right) ds = \int_0^{\infty} \varphi\left(\frac{\bar{g}(u)}{k}\right) t^{-1} \, du = \int_0^{\infty} \varphi\left(\frac{\bar{g}(u)}{k}\right) \varphi(\varphi^{-1}(t^{-1})) \, du \leqq$$

$$(14) \qquad \leqq \int_0^{\infty} \varphi\left(\frac{\bar{g}(u)}{k} \varphi^{-1}(t^{-1})\right) du;$$

in (13) we have used the (opposite) monotonies of φ and of $\bar{g}$, and at (14) the super-multiplicativity of φ. Writing $k' = k/\varphi^{-1}(t^{-1})$,

$$(15) \qquad \sum_{n=1}^{\infty} \lambda_n \varphi\left(\frac{\bar{g}(\Lambda_n t)}{k}\right) \leqq \int_0^{\infty} \varphi\left(\frac{\bar{g}(u)}{k'}\right) du = \int_0^{\infty} \varphi\left(\frac{g(u)}{k'}\right) du =$$

$$(16) \qquad = \sum_{n=1}^{\infty} \int_{\Lambda_{n-1}}^{\Lambda_n} \varphi\left(\frac{g(u)}{k'}\right) du = \sum_{n=1}^{\infty} \lambda_n \varphi\left(\frac{x_n}{k'}\right)$$

using rearrangement invariance in (15), and (9) in (16).

In the above, k and t are independent positive variables. Since $0<\|x\|<\infty$ we may impose the relation $k=k(t)=\|x\|\varphi^{-1}(t^{-1})$. This makes $k'=\|x\|$, so that (16) and Lemma 1 give

$$\sum_{n=1}^{\infty}\lambda_n\varphi\left(\frac{\bar g(\Lambda_n t)}{k(t)}\right) \leqq \sum_{n=1}^{\infty}\lambda_n\varphi\left(\frac{x_n}{\|x\|}\right) \leqq 1. \tag{17}$$

The required inequality follows from (11), (12) and (17), since they give

$$\|Ax\| \leqq \int_0^\infty \bar\alpha(t)\|\bar g(\Lambda_m t)\|\,dt \leqq \int_0^\infty \bar\alpha(t)k(t)\,dt = \tag{18}$$

$$= \int_0^\infty \bar\alpha(t)\|x\|\,\varphi^{-1}(t^{-1})\,dt = C\|x\|.$$

This proves the inequality in the situation laid down in (i).

(iii) Suppose that a_{mn} and x_n are still non-negative, but (cf. (i)) that (x_n) has no decreasing rearrangement; for instance, $x_n>0$ for all but a finite set of n. Let $x_{\cdot r}$ be the rth segment of x, that is,

$$x_{nr}=x_n \quad\text{for}\quad n\leqq r,\quad x_{nr}=0\quad\text{for}\quad n>r.$$

Since each sequence $x_{\cdot r}$ has a decreasing rearrangement, (18) gives, for each r,

$$\Big\|\sum_{n=1}^{r} a_{mn}x_n\Big\| = \Big\|\sum_{n=1}^{\infty} a_{mn}x_{nr}\Big\| \leqq C\|x_{mr}\|; \tag{19}$$

here again the mth member of each sequence concerned is written instead of the sequence itself. Making $r\to\infty$, Lemma 3 applied to the extreme members of (19) gives the required inequality.

(iv) Suppose that a_{mn} and x_n may be complex. For any complex-valued sequence $s=(s_n)$ we have $\|s\|=\||s|\|$, where $|s|$ is the sequence $(|s_n|)$. This, together with the monotonic property and the outcome of (i), (ii) and (iii), gives

$$\Big\|\sum_{n=1}^{\infty} a_{mn}x_n\Big\| = \Big\|\Big|\sum_{n=1}^{\infty} a_{mn}x_n\Big|\Big\| \leqq \Big\|\sum_{n=1}^{\infty} |a_{mn}|\,|x_n|\Big\| \leqq C\,\||x_m|\| = C\|x_m\|. \tag{20}$$

This completes the proof of the first paragraph, the main part, of the theorem.

(v) Suppose that a_{mn} and x_n are non-negative and that (x_n) is decreasing. Then $g(u)$ in (9) is decreasing, and the steps in (10), omitting the last, give

$$0\leqq \sum_{n=1}^{\infty} a_{mn}x_n \leqq \int_0^\infty \alpha(t)g(\Lambda_m t)\,dt.$$

The rest of the argument in (i) and (ii) now applies with $\bar\alpha$ and $\bar g$ replaced by α and g. At the end of this (18) becomes

$$\|Ax\| \leqq \int_0^\infty \alpha(t)\|x\|\,\varphi^{-1}(t^{-1})\,dt.$$

This proves the second paragraph of the theorem in the case where a_{mn} and x_n are non-negative. It will be noticed that no rearrangement of α is required. The claim that replacement of C by (8) *reduces* it is a consequence of [1: Theorem 378], because $\varphi^{-1}(t^{-1})$ is its own decreasing rearrangement.

(vi) Suppose that a_{mn} and x_n may be complex, and that $(|x_n|)$ is decreasing. Then (20) applies, except that use of (i), (ii) and (iii) is replaced by use of (v), and consequently C may be replaced by (8). This completes the proof.

REMARK. In the theorem it is supposed that φ is both Orlicz and supermultiplicative. That there are such functions is evident from the example $\varphi(t)=t^p$ with $p>1$. This example is actually multiplicative; but there are others which are not, for instance

$$\varphi(t)=\frac{t^{p+1}}{(t+1)^q} \quad \text{with} \quad p>q>0.$$

COROLLARY. *If* $p>1$, $\lambda_n>0$ *and* $X_m=\sum_{n=1}^{m}\lambda_n x_n/\sum_{n=1}^{m}\lambda_n$, *then*

$$\Big(\sum_{m=1}^{\infty}\lambda_m|X_m|^p\Big)^{1/p}\leqq\frac{p}{p-1}\Big(\sum_{m=1}^{\infty}\lambda_m|x_m|^p\Big)^{1/p}.$$

This is the case of the theorem in which $\varphi(t)=t^p$,

$$\alpha(t)=1 \quad \text{for} \quad 0<t\leqq 1, \quad \alpha(t)=0 \quad \text{otherwise},$$

$$a_{mn}=\lambda_n/\Lambda_m \quad \text{for} \quad 0<n\leqq m, \quad a_{mn}=0 \quad \text{otherwise}.$$

It is essentially [1: Theorem 332], and includes the original Hardy's Inequality as the case $\lambda_n=1$.

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STRONGLY REGULAR RINGS

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1. Introduction

Rings (all of which are assumed to be associative) with no nonzero nilpotent elements will be called reduced rings. In this note we prove that a ring R with identity, is Noetherian strongly regular if and only if R satisfies the following three conditions: (i) R is reduced, (ii) every non right unit in R is a zero divisor and (iii) 0 is the product of a finite number of prime elements.

2. Strongly regular rings

Throughout this paper R denotes an associative ring with identity. For any $a \in R$, the principal ideal (principal right ideal) generated by 'a' is denoted by $(a)aR$. An ideal P of R is prime if and only if $P \neq R$ and $aRb \subseteq P$ implies that $a \in P$ or $b \in P$, for all $a, b \in R$. An element $a \in R$ is said to be a prime element, if (a) is a prime ideal. All ideals are assumed to be proper. For undefined terms in this paper, the reader may refer to [1] and [4].

If R is a reduced ring, then $xy=0$ if and only if $yx=0$, for all $x, y \in R$ (see [3]).

LEMMA 1. *Let R be a reduced ring. For any $x, y, z \in R$, if $xyz=0$, then $xzy=0$.*

PROOF. Suppose $xyz=0$. Then $(yx)(yz)=0$, so that by hypothesis, $(yz)(yx)=0$ and hence $(yz)(yxz)=0$. Again since $y(zyxz)=0$, it follows that $0=(zyxz)y=$ $=(zy)(xzy)=(xzy)(xzy)=(xzy)^2$. As R is reduced, we have $xzy=0$. Hence proof of the lemma.

LEMMA 2. *Let R be a reduced ring. If $a_1 a_2 \ldots a_n = 0$ $(a_i \in R)$, then $a_{i_1} a_{i_2} \ldots a_{i_n} = 0$ for every permutation $i_1, i_2, \ldots, i_n$ of $1, 2, \ldots, n$.*

PROOF. The proof of the lemma follows from Lemma 1.

DEFINITION. A nonzero element $a \in R$ is said to be an atom if, for any $x \in R$, either $x^n a^m = 0$ for some $n, m \in Z^+$, or $xy=a$ for some $y \in R$.

LEMMA 3. *Suppose R is a reduced ring in which every non right unit is a zero divisor. Let x be a prime element of R. If $xy=0$ $(y \neq 0)$, then $xR+yR=R$. Moreover y is an atom.*

PROOF. Suppose $xR+yR \neq R$. Then $x+y$ is a non right unit, so that there is $d \in R$ such that $(x+y)d=0$ and $d \neq 0$. Observe that $xd=-yd$. We show that $yRd \subseteq (x)$. Let a be any element of R. Then by Lemma 1, $(x+y)da=0=(x+y)ad$; so $xad=-yad$ and hence $yad \in (x)$. Since (x) is a prime ideal, we have either $y \in (x)$

or $d\in(x)$. Suppose $y\in(x)$. Then $y=\sum_{i=1}^{n} r_i x s_i$ and so by Lemma 2, $y^2=y(\sum_{i=1}^{n} r_i x s_i)=$ $=\sum_{i=1}^{n} y r_i x s_i=0$. Again since R is reduced, it follows that $y=0$, a contradiction. Hence $y\notin(x)$. If $d\in(x)$, then $d=\sum_{i=1}^{n} a_i x b_i$; so that $yd=y(\sum_{i=1}^{n} a_i x b_i)=0$ and therefore $xd=0$. Since $xd=0$, by Lemma 2, $a_i x b_i d=0$ for $i=1, 2, \ldots, n$. Consequently $0=(\sum_{i=1}^{n} a_i x b_i)d=d^2$. As R is reduced, $d=0$, a contradiction. This shows that $xR+yR=R$.

Now we prove that y is an atom. Suppose $zy\neq 0$ for some $z\in R$. Using Lemma 2, it can be easily shown that, $z^m y^n\neq 0$ for all $n, m\in Z^+$. Since $zy\neq 0$, by above argument, $xR+zyR=R$; so that $1=xx_1+zyy_1$ for some $x_1, y_1\in R$. Therefore $y=1\cdot y=$ $=(xx_1+zyy_1)y=xx_1y+z(yy_1y)=z(yy_1y)$. Hence y is an atom. This completes the proof of the lemma.

A ring R is called regular if for each $a\in R$ there exists $x\in R$ such that $a=axa$. R is called strongly regular if for every $a\in R$, there exists $x\in R$ such that $a=a^2x$. It is well known that every strongly regular ring is a regular ring (see [1], Theorem 3.2) and in a strongly regular ring every one sided ideal is an ideal (see [1], Theorem 3.4) and every idempotent is a central idempotent (see [4], Proposition 4 and Corollary 5). Therefore by Theorem 9.5 of [2 page 186], we get that in a Noetherian strongly regular ring R, every maximal right ideal is a principal right ideal generated by some central idempotent. Now we prove the converse part.

LEMMA 4. *If every maximal right ideal of R is a principal right ideal generated by some central idempotent, then R is a Noetherian strongly regular ring.*

PROOF. First we prove that every right ideal is a principal right ideal generated by some central idempotent. Suppose not. Let $\mathscr{I}=\{I \mid I$ is a right ideal which is not a principal right ideal generated by some central idempotent$\}$. By Zorn's lemma, $\mathscr{I}$ has a maximal element say I. By hypothesis I is not a maximal right ideal. So $I\subset M$ (properly) for some maximal right ideal M of R. By hypothesis $M=eR$ for some central idempotent. Observe that $1-e\notin I$ and so $I+(1-e)R=fR$ for some central idempotent $f\in R$. Obviously $I\subseteq eR\cap fR=efR$. Since $f\in I+(1-e)R$, we have $f=$ $=i+(1-e)a$ for some $i\in I$ and $a\in R$; so that $ef=ei+e(1-e)a=ei=ie\in I$ and hence $efR\subseteq I$. This shows that $I=efR$, a contradiction. Therefore every right ideal is a principal right ideal generated by some central idempotent. Now we prove that R is a strongly regular ring. Let $a\in R$. By the above argument, $aR=eR$ for some central idempotent $e\in R$. So $a=ea=ae$ and $e=ax$ for some $x\in R$. Consequently, $a=ae=a^2x$ and hence R is a strongly regular ring. Again since R is a strongly regular ring, right ideals and ideals coincide (see [1], Theorem 3.4) and hence R is a Noetherian strongly regular ring.

LEMMA 5. *R is a Noetherian strongly regular ring if and only if R is reduced and for every maximal right ideal M of R, there is some idempotent atom $e\in R$ such that $e\notin M$.*

PROOF. The "only if" part is obvious. Now we prove the 'if' part. Since R is reduced, every idempotent is a central idempotent. Let M be any maximal right ideal of R. Choose an idempotent atom $e \in R$ such that $e \notin M$. We claim that $M=(1-e)R$. As $e \notin M$ and e is an idempotent atom, it follows that $em=0$ for every $m \in M$. So that, for any $m \in M$, $m=(1-e)m+em=(1-e)m$, and therefore $M \subseteqq (1-e)R$. Again since $(1-e)R \neq R$, we get $M=(1-e)R$, and therefore by Lemma 4, R is Noetherian strongly regular.

We characterize Noetherian strongly regular rings as follows.

THEOREM. *R is Noetherian strongly regular if and only if R satisfies the following conditions:*

(i) *R is reduced.*

(ii) *Every non right unit in R is a zero divisor.*

(iii) *0 is the product of a finite number of prime elements.*

PROOF. The "only if" part is obvious. Now we prove the "if" part. Suppose $0=a_1 \cdot a_2 \ldots a_n$ where a_i's are prime elements. As each a_i is a non right unit, there exist $b_i \in R (1 \leqq i \leqq n)$ such that $a_i b_i = 0$ and $b_i \neq 0$ for $1 \leqq i \leqq n$. As R is reduced, by Lemma 2, we have $(a_i) \cap (b_i) = (0)$ for $i=1, 2, \ldots, n$. Also by Lemma 3, $(a_i) + (b_i) = R$, so that $(a_i)=(e_i)$ and $(b_i)=(f_i)$ for some idempotents $\{e_i\}$, $\{f_i\} \subseteqq R$ $(1 \leqq i \leqq n)$. Observe that $e_i f_i = 0$ and $f_i \neq 0$ for $i=1, 2, \ldots, n$. Since each e_i is a prime element, $e_i f_i = 0$ and $f_i \neq 0$ for $i=1, 2, \ldots, n$; by Lemma 3, each f_i is an atom. As $\left(\sum_{i=1}^{n} (f_i)\right) + (e_i) = R$ for every $i \in \{1, 2, \ldots, n\}$, we have

$$R = \left(\sum_{i=1}^{n} (f_i)\right) + (e_1, \ldots, e_n) = \left(\sum_{i=1}^{n} (f_i)\right) + (0) = \sum_{i=1}^{n} (f_i).$$

Without loss of generality, we can assume that $f_i \neq f_j$ for $i \neq j$. Again since $\sum_{i=1}^{n} (f_i) = R$ and f_i's are pairwise disjoint idempotents, it follows that $\left(\sum_{i=1}^{n} f_i\right) = R$ and therefore $\sum_{i=1}^{n} f_i = 1$. Now the result follows from Lemma 5. This completes the proof of the theorem.

It is not hard to show that the conditions (i), (ii), and (iii) of the above theorem, are independent.

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UNIFORM BOUNDEDNESS FOR DELAY EQUATIONS

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1. Introduction. Let $D \subset R^n$ be an open set with $0 \in D$ and let $f: [0, \infty) \times D \to R^n$ be continuous. Then

$$x' = f(t, x) \tag{0}$$

is a system of differential equations. The classical theory of Liapunov's direct method concerns a Liapunov function $V: [0, \infty) \times D \to R$ which is continuous and locally Lipschitz in x, together with continuous strictly increasing functions $W_i: [0, \infty) \to [0, \infty)$ satisfying $W_i(0) = 0$. Two of the main theorems may be stated as follows.

THEOREM 0_1. *If*

(i) $W_1(|x|) \leqq V(t, x) \leqq W_2(|x|)$

and

(ii) $V'_{(0)}(t, x) \leqq -W_3(|x|)$

then $x = 0$ *is uniformly asymptotically stable.*

THEOREM 0_2. *If* $D = R^n$ *and*

(i) $W_1(|x|) \leqq V(t, x) \leqq W_2(|x|)$,

(ii) $V'_{(0)}(t, x) \leqq -W_3(|x|) + M, \quad M > 0$,

(iii) $W_1(r) \to \infty$ *as* $r \to \infty$,

(iv) $W_3(U) > M$ *for some* $U > 0$,

then solutions of (0) *are uniformly bounded (U. B.) and uniformly ultimately bounded for bound B (U. U. B.).*

When these theorems were extended to delay equations, serious problems were encountered which we briefly describe after we present the setting for delay equations.

Let $(C, \|\cdot\|)$ denote the Banach space of continuous functions $\varphi: [-h, 0] \to R^n$ with the supremum norm: $\|\varphi\| = \sup_{-h \leqq s \leqq 0} |\varphi(s)|$ and $|\cdot|$ is any convenient norm on R^n, while h is a positive constant. If $A > 0$ and $x: [-h, A) \to R^n$ is continuous, then $0 \leqq t < A$ implies that $x_t \in C$, where $x_t(s) = x(t+s)$.

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If $F: [0, \infty)\times C \to R^n$ is continuous and takes bounded sets into bounded sets, then

$$x' = F(t, x_t) \tag{1}$$

is a system of functional differential equations with finite delay which satisfies the following properties. For each $t_0 \geqq 0$ and each $\varphi \in C$, there is a solution $x(t_0, \varphi)$ of (i) with value $x(t, t_0, \varphi)$ which satisfies $x_{t_0}(t_0, \varphi)=\varphi$ and satisfies (1) on $[t_0, t_0+\alpha)$ for some $\alpha>0$. If $x(t, t_0, \varphi)$ remains bounded, then $\alpha=\infty$.

If $V: [0, \infty)\times C \to [0, \infty)$ is continuous, the derivative of V along a solution of (1) can be defined by

$$V'_{(1)}(t, \varphi) = \limsup_{\delta\to 0^+} [V(t+\delta, x_{t+\delta}(t, \varphi))]-V(t, \varphi)/\delta.$$

Details may be found in [9], for example. In the same reference one may see the classical extension of Liapunov theory from (0) to (1). In that presentation we see that in extending Theorem 0_1 to (1), investigators were forced to require that $|F(t, \varphi)|$ be bounded for $\|\varphi\|$ bounded when V satisfies the relation

$$W_1(|\varphi(0)|) \leqq V(t, \varphi) \leqq W_2(\|\varphi\|).$$

But it was noted in [2] that if we ask that

$$W_1(|\varphi(0)|) \leqq V(t, \varphi) \leqq W_2(|\varphi(0)|)+W_3(|||\varphi|||),$$

where $|||\varphi|||$ is the L^2-norm, then the boundedness of $|F(t, \varphi)|$ for $\|\varphi\|$ bounded may be dropped, making a clean generalization of Theorem 0_1 to (1).

But when it came to extending Theorem 0_2 to (1), Hale [9; p. 139] lists none at all. He complains that, though Yoshizawa [11; p. 206] gives such a generalization, it depends on the size of h. We point out the additional difficulty that it requires $F(t, \varphi)$ periodic in t.

In this paper we again use an L^1-norm and obtain a very clean generalization of Theorem 0_2 to (1). That is the content of our Theorem 3. But some progress has been made since the presentation in [9] and we now survey those results. To be definite here, we first give some definitions.

Definition. Solutions of (1) are uniformly bounded (U. B.) if for each $B_1>0$ there exists $B_2>0$ such that $[t_0\geqq 0, \|\varphi\|\leqq B_1]$ imply that $|x(t, t_0, \varphi)|<B_2$. Solutions of (1) are U. U. B. for bound B if for each $B_3>0$ here exists $T>0$ such that $[t_0\geqq 0, \|\varphi\|\leqq B_3, t\geqq t_0+T]$ imply that $|x(t, t_0, \varphi)|<B$.

An early result by Yoshizawa [11; p. 202] shows that if there is a V with

(i) $W_1(|\varphi(0)|) \leqq V(t, \varphi) \leqq W_2(|\varphi(0)|)+W_3(\|\varphi\|)$,

(ii) $V'_{(1)}(t, \varphi) \leqq 0$ for $|\varphi(0)|$ large,

and with

(iii) $W_1(r)-W_3(r) \to \infty$ as $r\to\infty$,

then solutions of (1) are U.B.

Burton—Zhang [8] show that if

(i) $W_1(|\varphi(0)|) \leqq V(t, \varphi) \leqq W_2(|\varphi(0)|)+W_3\Big(\int_{-h}^{0} W_4(|\varphi(s)|)\, ds\Big)$,

(ii) $V'_{(1)}(t, \varphi) \leqq -W_4(|\varphi(0)|)+M, \quad M>0$,

(iii) $W_1(r), \quad W_4(r) \to \infty$ as $r \to \infty$,

then solutions of (1) are U.B. and U.U.B.

This result has two main faults: W_4 must be the same in (i) and (ii), and W_4 in (ii) must be unbounded.

Burton [4] proves that if there are positive constants μ, U, β, and C with

(i) $0 \leqq V(t, \varphi) \leqq W_2(\|\varphi\|)$,

(ii) $V'_{(1)}(t, \varphi) \leqq -\mu|F(t, \varphi)|-c$ for $|\varphi(0)| \geqq U$,

(iii) $V'(t, \varphi) \leqq \beta$ if $|\varphi(0)| < U$,

then solutions of (1) are U.B. and U.U.B.

Burton—Hatvani [6] show that if $M>0$ and $U>0$ with

(i) $W_1(|\varphi(0)|) \leqq V(t, \varphi) \leqq W_2(|\varphi(0)|)+W_3\Big(\int_{-h}^{0} |\varphi(s)|\, ds\Big)$,

(ii) $V'_{(1)}(t, \varphi) \leqq -\Big[W_4(|\varphi(0)|)+W_5\Big(\int_{-h}^{0} |\varphi(s)|\, ds\Big)\Big]+M$,

(iii) $W_4(U)>M, \quad W_5(U)>M, \quad W_1(r) \to \infty$ as $r \to \infty$,

then solutions of (1) are U.B. and U.U.B.

Burton—Huang—Mahfoud [7] show that for general delay equations some type of fading memory is required in the functional V. The pair

$$V(t) \leqq \alpha(t)+\int_{0}^{t} \alpha(s)\, ds, \quad V'(t)=-\alpha(t)+1$$

has an unbounded solution $V(t)$. Fading memory can take the form

$$V(t) \leqq \alpha(t)+\int_{t-h}^{t} \alpha(s)\, ds,$$

for example.

While Theorem 3 is our most direct generalization of Theorem 0_2 to (1), the other results require much less on the growth of V'.

2. Boundedness. The result of Burton—Hatvani just quoted asks that

$$V' \leqq -\Big[W_5\Big(\int_{-h}^{0} |\varphi(s)|\, ds\Big)\Big]+W_4(|\varphi(0)|)+M$$

making the derivative, in fact, a function of V. We begin by separating this into two results which overcome that objection.

In most examples the integral of x_t appears in V (and frequently in V'). We express that in Theorem 1 by the L^2-norm, denoted by $|||x_t|||$, and in Theorems 2 and 3 by $\int_{t-h}^{t} |x(s)|\, ds$. These, and many others forms, are completely interchangeable.

THEOREM 1. *If there is a continuous functional* $V\colon [0, \infty)\times C \to [0, \infty)$ *with*

(i) $W_1(|\varphi(0)|) \leqq V(t, \varphi) \leqq W_2(|\varphi(0)|)+W_3(|||\varphi|||)$,

(ii) $V'(t, \varphi) \leqq -W_4(|||\varphi|||)+M, \quad M>0$,

(iii) $W_1(r)\to\infty$ *as* $r\to\infty$, $W_4(U/2)\geqq 12M$ *for some* $U>0$, *then solutions of* (1) *are U.B. and U.U.B.*

PROOF. Let $B_1 > U/\sqrt{h}$ be given and set

$$P = W_2(U/\sqrt{h})+W_3(U)$$

and

$$P^* = W_2(B_1)+W_3(B_1\sqrt{h}).$$

Then we will show that

$$B_2 = W_1^{-1}[P^*+4Mh+1]$$

and

$$B = W_1^{-1}[P+4Mh+1].$$

Let $t_0\geqq 0$, $\|\varphi\|\leqq B_1$, $I_j=[t_0+jh, t_0+(j+1)h]$, $x(t)=x(t, t_0, \varphi)$, and $V(t)= =V(t, x_t)$.

If $V(t)<P^*$ for all $t\geqq t_0$, there is nothing to be proved. Thus, we let t_1 be the first value of t past t_0 with $V(t_1)=P^*$. (This is the place referred to later as the beginning of our argument.) Now $t_1\in I_k$ for some k. Since $V'(t)\leqq M$, it follows that

$$V(t) \leqq P^*+4Mh \quad \text{on} \quad I_k^* = I_k\cup I_{k+1}\cup I_{k+2}\cup I_{k+3}.$$

I. Suppose there is a first $t_2\in I_{k+2}$ with $|||x_{t_2}|||\geqq U$. Then elementary properties of the integral show that $|||x_t|||\geqq U/2$ on an interval $\bar{I}_k$ of length $h/2$ lying in $I_k^{**}= =I_{k+1}\cup I_{k+2}\cup I_{k+3}$. Thus, $V'(t)\leqq -W_4(U/2)+M$ on $\bar{I}_k$ and so V decreases by at least $[W_4(U/2)-M]h/2\geqq 11Mh/2>5Mh$ units on $\bar{I}_k$. Thus, at the right end-point of I_{k+3} we have $V(t)<P^*-Mh$ and $V(t)<P^*$ on I_{k+4}. The argument begins all over again with the search for the first t_1 past I_{k+4} with $V(t_1)=P^*$.

II. Suppose $|||x_t|||<U$ on all of I_{k+2}. This means that there is a $t_0^*\in I_{k+2}$ with $|x(t_0^*)|<U/\sqrt{h}$; otherwise, $|x(t)|\geqq U/\sqrt{h}$ on all of I_{k+2} so that at the right end-point of I_{k+2} we have $|||x_t|||\geqq U$. Hence,

$$V(t_0^*) \leqq W_2(U/\sqrt{h})+W_3(U) = P < P^*,$$

and the argument begins again with t_0 replaced by t_0^*.

Because of (I) and (II) it is true that

$$W_1(|x(t)|) \leqq V(t) < P^*+4Mh+1 \quad \text{for} \quad t\geqq t_0$$

and this is the uniform boundedness.

For the U.U.B., let $B_3>0$ be given, $t_0\geqq 0$, $\|\varphi\|\leqq B_3$, $x(t)=x(t, t_0, \varphi)$, $V(t)=V(t, x_t)$, and $V(t_0)\leqq W_2(B_3)+W_3(B_3\sqrt{h})\overset{\text{def}}{=}P^{**}$. If $P^{**}\leqq P$, then the U.B. argument yields $|x(t)|<B$ for $t\geqq t_0$. Thus, we suppose $P^{**}>P$.

Arguing as before, either $V(t)<P$ at some point in $[t_0, t_0+3h]$ (so that $V(t)<P+4Mh+1$ for $t\geqq t_0+4h$) or $V(t)\geqq P$ on $[t_0, t_0+3h]$. If there is a t_2 in that interval with $|||x_{t_2}|||\geqq U$ then V decreases by at least $5Mh$ units and increases at most $4Mh$ units on $[t_0, t_0+4h]$, so that $V(t_0+4h)<V(t_0)-Mh$. If $P\leqq V(t)$ on $[t_0, t_0+3h]$ and there is no t_2 with $|||x_{t_2}|||\geqq U$, then $|x(t)|\geqq U/\sqrt{h}$ on all of $[t_0, t_0+3h]$, a contradiction. Hence, there is a $t\in[t_0, t_0+3h]$ with $V(t)<P$ or $V(t_0+4h)<V(t_0)-Mh$. If $N>P^{**}/Mh$, then there is a $t\in[t_0, t_0+4Nh]$ with $V(t)<P$ so that $V(t)<P+4Mh+1$ for $t\geqq t_0+4Nh$. This completes the proof.

The proof of the next result centers on three lemmas which go well beyond Theorem 2, which is itself simply an example illustrating how the three lemmas can be satisfied. When $V'\leqq -W_4(|x|)+M$ with W_4 convex downward, then the three lemmas are readily satisfied without the seemingly complicated form of (iii) in Theorem 2. That convexity is assumed in Theorem 3 and it results in a clean generalization of Theorem 0_2 to (1); but the price is moderately high since convexity requires that $W_4(r)\to\infty$ as $r\to\infty$ (among other things). Theorem 2 reduces the requirement through (iii) which allows us to construct a convex function under W_4 on short intervals.

Properties of convex functions and Jensen's inequality are found in Natanson [10; pp. 36—46]. Other applications of Jensen's inequality to stability theory are found in [1] and [5]. The three properties we use are:

(a) W is convex downward if

$$W([t_1+t_2]/2) \leqq [W(t_1)+W(t_2)]/2,$$

(b) If W is convex downward, then

$$\int_{t-h}^{t} W(|x(s)|)\,ds \geqq hW\Big(\int_{t-h}^{t} |x(s)|\,ds/h\Big),$$

and

(c) If W is increasing, then $\int_0^r W(s)\,ds$ is convex downward.

THEOREM 2. *Let $V:[0,\infty)\times C\to[0,\infty)$ be continuous and suppose there are positive constants $U<J_0$, $U<J_0h$, and M with*

(i) $W_1(|\varphi(0)|)\leqq V(t,\varphi)\leqq W_2(|\varphi(0)|)+W_3\Big(\int_{-h}^{0}|\varphi(s)|\,ds\Big)$,

$W_1(r)\to\infty$ *as* $r\to\infty$,

(ii) $V'_{(1)}(t,\varphi)\leqq -W_4(|\varphi(0)|)+M,\quad W_4(U)\geqq 5M,$

(iii) $J_0 \leqq J$ *implies that*

$$\int_0^J W_4(s)\,ds/W_1^{-1}[W_2(U)+5Mh+W_3(Jh)] > 5M.$$

Then solutions are U.B. and U.U.B.

The proof proceeds by way of three lemmas.

LEMMA 1. *Let* $\tilde{J}$ *be defined by* $W_1(\tilde{J})=W_2(U)+W_3(J_0 h)+5Mh$ *and suppose that* $x(t)$ *is a solution of* (1) *with* $|x(t_2)|=U$ *and* $|x(t_1)|=\|x_{t_2}\|\geqq\tilde{J}$ *with* $t_1\in[t_2-h, t_2]$. *If* $V(t)=V(t, x_t)$, *then* $V(t_2)\leqq V(t_2-h)-4Mh$.

PROOF. Suppose first that $V(t_1)\leqq V(t_2)+5Mh$. Then

$$W_1\big(|x(t_1)|\big) \leqq V(t_1) \leqq V(t_2)+5Mh \leqq W_2(U)+W_3\Big(\int_{t_2-h}^{t_2}|x(s)|\,ds\Big)+5Mh.$$

Since $|x(t_1)|\geqq\tilde{J}$, we have $X\overset{\text{def}}{=}\int_{t_2-h}^{t_2}|x(s)|\,ds\geqq J_0 h$. Now for $0\leqq r\leqq|x(t_1)|$ define W_5 by

$$W_5(r) = \int_0^r [W_4(s)/|x(t_1)|]\,ds.$$

Since W_4 is increasing we have

$$W_5(r) \leqq W_4(r)\,r/|x(t_1)| \leqq W_4(r).$$

Moreover, W_5 is convex downward. Hence,

$$V(t_2) \leqq V(t_2-h)-\int_{t_2-h}^{t_2} W_4\big(|x(s)|\big)\,ds+Mh \leqq$$

$$\leqq V(t_2-h)-\int_{t_2-h}^{t_2} W_5\big(|x(s)|\big)\,ds+Mh \leqq V(t_2-h)-hW_5\Big(\int_{t_2-h}^{t_2}|x(s)|\,ds/h\Big)+Mh =$$

(by Jensen's inequality)

$$= V(t_2-h)-hW_5(X/h)+Mh = V(t_2-h)-h\Big[\int_0^{X/h} W_4(s)\,ds/|x(t_1)|\Big]+Mh \leqq$$

$$\leqq Mh+V(t_2-h)-h\Big[\int_0^{X/h} W_4(s)\,ds/W_1^{-1}[W_2(U)+W_3(X)+5Mh\Big] \leqq$$

$$\leqq Mh+V(t_2-h)-5Mh = V(t_2-h)-4Mh.$$

The last conclusion is based on $V(t_1)\leqq V(t_2)+5Mh$. If $V(t_1)\geqq V(t_2)+5Mh$, then $V(t_1)\leqq V(t_2-h)+Mh$ since $V'(t)\leqq M$ and so

$$V(t_2-h)+Mh \geqq V(t_1) \geqq V(t_2)+5Mh$$

yielding

$$V(t_2-h) \geqq V(t_2)+4Mh,$$

as required.

LEMMA 2. *Let $\tilde{J}$ be defined in Lemma* 1 *and let $x(t)$ be a solution of* (1). *Denote $V(t)=V(t, x_t)$ and suppose there is a t with*

$$V(t) \geqq W_2(\tilde{J})+W_3(\tilde{J}h) \stackrel{\text{def}}{=\!=} P_0;$$

then there is a pair t_2, $t_2-h\in[t-2h, t+h]$ with

$$V(t_2) \leqq V(t_2-h)-4Mh.$$

PROOF. Either

(α) $|x(t)| \geqq \tilde{J}$

or

(β) $\int\limits_{t-h}^{t} |x(s)|\, ds \geqq \tilde{J}h.$

If (α) holds, then because of $\tilde{J}>U$, either

(αi) $|x(s)| > U$ on $[t, t+h]$ so that $V(t+h) \leqq V(t)-4Mh$,

or

(αii) there is a first $t_2\in[t, t+h]$ with $|x(t_2)| = U$

so that by Lemma 1 we have the desired conclusion.

If (β) holds, then there is an $\xi\in[t-h, t]$ with

$$\int\limits_{t-h}^{t} |x(s)|\, ds = |x(\xi)|\, h \geqq \tilde{J}h,$$

so that (α) holds for $t=\xi$ and, again, the desired conclusion is obtained. This proves Lemma 2.

LEMMA 3. *If $x(t)$ is a solution of* (1), *$V(t)=V(t, x_t)$, and $P\geqq P_0=W_2(\tilde{J})+ +W_3(\tilde{J}h)$, then the inequality $V(t)<P$ on $[t_0, t_0+h]$ implies that $V(t)<P$ for all $t\geqq t_0$.*

PROOF. If this lemma is false, then there is a first $t_1>t_0$ with $V(t_1)=P$. Thus, $|x(t_1)|<\tilde{J}$ because $V'(t_1)\geqq 0$. Hence,

$$\int\limits_{t_1-h}^{t_1} |x(s)|\, ds = |x(\xi)|\, h \geqq \tilde{J}h \quad \text{for some} \quad \xi\in(t_1-h, t_1);$$

this yields $|x(\xi)|\geqq\tilde{J}$. Since $V(\xi)<P$ and $V(t_1)=P$, there exists $\bar{t}_2\in[\xi, t_1]$ with $V'(\bar{t}_2)>0$ so that $|x(\bar{t}_2)|<U$ and there is a $t_2\in[\xi, t_1]$ with $|x(t_2)|=U$. By Lemma 1, V decreases by at least $4Mh$ units on $[t_2-h, t_2]$. But $V(t_2-h)<P$ so

$$V(t_2) \leqq V(t_2-h)-4Mh < P-4Mh.$$

This means that $V(t_1)<P$ because $V'(t)\leqq M$. Hence, $\int\limits_{t_1-h}^{t_1} |x(s)|\, ds<\tilde{J}h$ and $V(t)<P$ for all $t\geqq t_0$. This proves Lemma 3.

COROLLARY. *Given* $B_1>\tilde{J}$, $t_0\geqq 0$, *if* $\|\varphi\|\leqq B_1$, *then for* $x(t)=x(t,t_0,\varphi)$ *we have*

$$|x(t)| < W_1^{-1}[W_2(B_1)+W_3(B_1h)+1] \stackrel{\text{def}}{=} B_2,$$

so solutions are U.B.

To complete the proof of U.U.B., let $B_3>\tilde{J}$ be given and for $t_0\geqq 0$ and $\|\varphi\|\leqq \leqq B_3$ let $x(t)=x(t,t_0,\varphi)$ and $V(t)=V(t,x_t)$. By Lemma 2 if $V(t)\geqq P_0$ on an interval of length $3h$, then V decreases at least $4Mh$ units, while increasing at most $3Mh$ units. Hence, there is a $T=T(B_3)$ such that $V(t)<P_0$ at some point $t_1\in[t_0,t_0+T]$. Thus, $V(t)<P_0+Mh$ on $[t_1,t_1+h]$. By Lemma 3, $V(t)<P_0+Mh$ for all $t\geqq t_0+T+h$ so that

$$|x(t)| \leqq W_1^{-1}(P_0+Mh) \stackrel{\text{def}}{=} B.$$

This completes the proof.

In the following example, when a function is written without its argument, that argument is t.

EXAMPLE 2. Let $f(x)=x/(1+|x|)$ and consider the scalar equation

$$x'(t) = -(1+t\sin^2 t)f(x)-a(t)f\big(x(t-h)\big)+p\cos t \tag{3}$$

where $a(t)$ is continuous on $[0,\infty)$, $|a(t)|\leqq k$, $k<1-\alpha$, $k>0$, $p>0$, $\alpha/6>p$, and $k\alpha/3h>5p$. Then the conditions of Theorem 2 will be satisfied when we define

$$V(t,x_t) = |x|+k\int_{t-h}^{t}\big|f\big(x(s)\big)\big|\,ds$$

so that

$$V'(t,x_t) \leqq -|f(x)|+k\big|f\big(x(t-h)\big)\big|+k|f(x)|-k\big|f\big(x(t-h)\big)\big|+p$$
$$\leqq -\alpha|f(x)|+p = -W_4(|x|)+M.$$

Here, $W_1=W_2=(1/k)W_3$ and $W_4(r)=\alpha f(r)$. Thus, if $U=2$, then $W_4(U)=2\alpha/3> >4p$. For large J, $\int_0^J W_4(s)\,ds\geqq \alpha J/2$ so that for fixed U, M, and h, the left side of (iii) in Theorem 2 exceeds $k\alpha/2h$ for large J. And $k\alpha/3h>5p$ so (iii) is satisfied. Hence, solutions are U.B. and U.U.B.; yet, none of the cited results apply to this problem.

THEOREM 3. *Let* $V\colon[0,\infty)\times C\to[0,\infty)$ *be continuous with*

(i) $W_1(|\varphi(0)|)\leqq V(t,\varphi)\leqq W_2(|\varphi(0)|)+W_3\Big(\int_{-h}^{0}|\varphi(s)|\,ds\Big)$,

$W_1(r)\to\infty$ *as* $r\to\infty$, *and*

(ii) $V'_{(1)}(t,\varphi)\leqq -W_4(|\varphi(0)|)+M$

where W_4 *is convex downward and* $M>0$. *Then solutions of* (1) *are U.B. and U.U.B.*

PROOF. Following Lemma 1 we first note that W_4 convex implies that $W_4(r)\to\infty$ as $r\to\infty$; hence, there is a U with $W_4(U)=5M$. Next, we note that there is a $J_0>0$

such that $J \geqq J_0$ implies that $hW_4(J) > 5M$. Define $\tilde{J}$ by $W_1(\tilde{J}) = W_2(U) + W_3(J_0 h) + 5Mh$.

Suppose there is a solution $x(t)$ and a t_2 with $|x(t_2)| = U$, $\|x_{t_2}\| \geqq \tilde{J}$, $\|x_{t_2}\| = |x(t_1)|$ with $t_2 - h \leqq t_1 \leqq t_2$. If $V(t_1) \leqq V(t_2) + 5Mh$, then

$$W_1(\tilde{J}) \leqq W_1(|x(t_1)|) \leqq V(t_1) \leqq V(t_2) + 5Mh \leqq$$

$$\leqq W_2(U) + W_3\Big(\int_{t_2-h}^{t_2} |x(s)|\, ds\Big) + 5Mh$$

so that $\int_{t_2-h}^{t_2} |x(s)|\, ds \geqq J_0 h$. Thus,

$$V(t_2) \leqq V(t_2 - h) + Mh - \int_{t_2-h}^{t_2} W_4(|x(s)|)\, ds \leqq$$

$$\leqq V(t_2 - h) + Mh - hW_4\Big(\int_{t_2-h}^{t_2} |x(s)|\, ds/h\Big) \leqq$$

$$\leqq V(t_2 - h) + Mh - hW_4(J_0) \leqq V(t_2 - h) - 4Mh,$$

as claimed in Lemma 1. This depends on $V(t_1) \leqq V(t_2) + 5Mh$. If $V(t_1) \geqq V(t_2) + 5Mh$, then $V(t_1) \leqq V(t_2 - h) + Mh$ and so $V(t_2 - h) + Mh \geqq V(t_1) \geqq V(t_2) + 5Mh$ or $V(t_2) \leqq V(t_2 - h) - 4Mh$, as required. Hence, for these choices of U, J_0, and $\tilde{J}$, then Lemma 1 is satisfied.

We now consider Lemma 2 and suppose that $V(t) \geqq W_2(\tilde{J}) + W_3(\tilde{J}h) = P_0$. We must show that there is a pair t_2, $t_2 - h \in [t - 2h, t + h]$ with $V(t_2) \leqq V(t_2 - h) - 4Mh$. In fact, the proof is identical to that of Lemma 2.

Lemma 3 claims that if $V(t) < P$ on $[t_0, t_0 + h]$, then $V(t) < P$ for all $t \geqq t_0$. Again, the proof is precisely the same as the one already given.

Note added in proof (January 2, 1991). The final version of reference [6] did not contain the material on U. B. and U. U. B. mentioned this paper because of the length.

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SOME PARTITION RELATIONS FOR IDEALS ON $P_\varkappa\lambda$

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The problem of lifting theorems about a regular uncountable cardinal $\varkappa$ to an analogue about $P_\varkappa\lambda$ has always been a troublesome one, especially in the case where ideal theory is concerned (see [19]). In this paper we develop techniques whereby some partition relations for ideals on $P_\varkappa\lambda$ may be proved.

In §1 we introduce the class of seminormal ideals on $P_\varkappa\lambda$ and in §§1, 3 and 5 show that for ideals belonging to this class, the distributivity of the quotient algebra is strongly related to some ideal theoretic partition relations. In particular we prove (Theorem 5.3) that for a normal ideal I on $P_\varkappa\lambda$, the partition relation $I^+\to(I^+)^2_2$ may be characterised in terms of distributivity, thus extending a result of Menas [15].

We also consider some well known ineffability properties of $P_\varkappa\lambda$. For many of our results the cardinality of $P_\varkappa\lambda$ will play an important role, and hence in §2 we adapt a theorem of Solovay [17] to show that if $cf\lambda\geqq\varkappa$ and $\varkappa$ is λ-Shelah then $\lambda^{<\varkappa}=\lambda$.

In [11, 12] we introduced a property of ideals on uncountable cardinals, *WC*, which may be regarded as an ideal theoretic analogue of the "strong inaccessibility and tree propert y" equivalent of weak compactness. In § 3 we introduce an analogous concept for ideals on $P_\varkappa\lambda$, and show it to be related to $NSh_{\varkappa\lambda}$ (the ideal induced by the λ-Shelah property) and mild λ-ineffability. An interesting corollary of our results here is that if $\mathrm{cf}\,\lambda\geqq\varkappa$, $\varkappa$ is λ-Shelah, $A\in NSh_{\varkappa\lambda}$ and $\langle f_x|x\in A\rangle$ is a sequence such that $(\forall x\in A)(f_x\colon x\to x)$, then there is a function $f\colon\lambda\to\lambda$ such that $(\forall y\in P_\varkappa\lambda)(\{x\in A|x\supset y$ and $f_x\restriction y=f\restriction y\}\in NSh^+_{\varkappa\lambda})$. We also prove that the property *WC* is related to partition relations of the form $I^+\to(I^+,\varkappa)^2_2$.

In §4 we extend results of Carr [6a], Kunen [14] and Baumgartner [1] to show that ineffable and almost ineffable subsets of $P_\varkappa\lambda$ may be characterised in terms of regressive partition relations.

§0. Notation and terminology

Throughout $\varkappa$ will denote a regular uncountable cordinal and λ a cardinal $\geqq\varkappa$. $P_\varkappa\lambda=\{x\subset\lambda\,\|x|<\varkappa\}$ and $\lambda^{<\varkappa}$ is the cardinality of this set. For $x\in P_\varkappa\lambda$, $\hat{x}=\{y\in P_\varkappa\lambda|x\subset\subset y\}$, $\varkappa_x=\varkappa\cap x$ and $\bar{x}$ denotes the order type of x.

For $X\subseteq P_\varkappa\lambda$ and $n<\omega$, $[X]^n=\{(x_1,x_2,\dots,x_n)\in X^n|\emptyset\neq x_1\subset x_2\subset\dots\subset x_n\}$ and $[X]^n_<=\{(x_1,x_2,\dots,x_n)\in[X]^n|\forall 1\leqq i<n,\ |x_i|<|\varkappa\cap x_{i+1}|\}$. If $f\colon[X]^n\to\lambda$, then (i) f is said to be regressive iff for each $(x_1,x_2,\dots,x_n)\in[X]^n$, $f(x_1,x_2,\dots,x_n)\in x_1$ and (ii) a set $H\subseteq X$ is said to be homogeneous for f iff $|f''[H]^n|=1$. X is said to be unbounded iff $(\forall x\in P_\varkappa\lambda)(X\cap\hat{x}\neq\emptyset)$. Throughout $I_{\varkappa\lambda}$ will denote the ideal of not unbounded subsets of $P_\varkappa\lambda$ and I, a proper, non principal, $\varkappa$-complete ideal on $P_\varkappa\lambda$ extending $I_{\varkappa\lambda}$. Furthermore $I^+=\{X\subseteq P_\varkappa\lambda|X\notin I\}$ and I^* denotes the filter dual to I.

If $A\in I^+$ then $I|A=\{X\subseteq P_\varkappa\lambda|X\cap A\in I\}$ and an I-partition of A is a maximal collection $W\subseteq P(A)\cap I^+$ such that $X\cap Y\in I$ whenever $X, Y\in W$, $X\neq Y$. The I-partition W is said to be disjoint if distinct members of W are disjoint, and in this case for $x\in\bigcup W$, $W(x)$ denotes the unique member of W containing x. If W and T are I-partitions of A, we say W refines T (and write $W\leqq T$) iff for each $X\in W$ there is a $Y\in T$ such that $X\subseteq Y$.

I is said to be (μ, ν)-distributive iff whenever $A\in I^+$ and $\langle W_\gamma|\gamma<\mu\rangle$ is a sequence of I-partitions of A, each of cardinality $\leqq\nu$, there is a $B\in P(A)\cap I^+$ and a sequence $\langle X_\gamma|\gamma<\mu\rangle$ such that for each $\gamma<\mu$, $X_\gamma\in W_\gamma$ and $B-X_\gamma\in I$. (Hence I is (μ, ν)-distributive iff the quotient algebra $P(P_\varkappa\lambda)/I$ is (μ, ν)-distributive in the usual sense.)

I is said to be normal iff whenever $A\in I^+$ and $f\colon A\to\lambda$ is regressive, there is a $B\in P(A)\cap I^+$ such that $f\restriction B$ is constant. $NS_{\varkappa\lambda}$ denotes the non stationary ideal on $P_\varkappa\lambda$ and

$$SNS_{\varkappa\lambda}=\{X\subseteq P_\varkappa\lambda|(\exists f\colon X\to\lambda)\ (f \text{ is regressive and } \forall\alpha<\lambda,\ f^{-1}(\{\alpha\})\in I_{\varkappa\lambda})\}.$$

Clearly all these conceps could be similarly defined for $P_\varkappa A$ where A is any set of ordinals of cardinality $\geqq\varkappa$.

§1. Seminormality

A simple but important fact concerning I-partitions is that (by $\varkappa$-completeness) any I-partition of cardinality $\leqq\varkappa$ has a disjoint refinement (see for instance [10, Theorem 1.4.1.]). An interesting intermediate between $\varkappa$-completeness and normality, which enables us to obtain a similar results for I-partitions of cardinality $\leqq\lambda$ is the following.

Definition 1.1. I is said to be seminormal iff whenever $A\in I^+$, $\mu<\lambda$ and $f\colon A\to\mu$ is regressive, there is a $B\in P(A)\cap I^+$ such that $f\restriction B$ is constant.

Seminormal ideals may be regarded as λ-complete in that

Proposition 1.2. *If I is seminormal then $P(P_\varkappa\lambda)/I$ is λ-complete.*

Proof. Given $\{A_\alpha|\alpha<\mu<\lambda\}\subseteq P(P_\varkappa\lambda)$ then by seminormality, in $P(P_\varkappa\lambda)/I$, $[\{x\in P_\varkappa\lambda|\exists\alpha\varepsilon\mu\cap x,\quad x\in A_\alpha\}]=\bigvee\{[A_\alpha]|\alpha<\mu\}$. □

Lemma 1.3. *If I is seminormal, $A\in I^+$ and $W\subseteq P(A)\cap I^+$ is predense below A in I^+ and of cardinality $\leqq\lambda$, then there is a disjoint I-partition of A, T, of cardinality $\leqq\lambda$ such that $(\forall X\in T)(\exists Y\in W)(X\subseteq Y)$.*

Proof. Suppose $W=\{Y_\alpha|\alpha<\mu\leqq\lambda\}$ then by induction on $\alpha<\mu$ define $X_\alpha=\{x\in Y_\alpha|\alpha\in x$ and $\forall\beta<\alpha,\ x\notin X_\beta\}$ and let $T=\{X_\alpha|\alpha<\mu$ and $X_\alpha\in I^+\}$.

Clearly distinct members of T are disjoint and every element of T is contained in some element of W. Suppose $B\in P(A)\cap I^+$ and α is the least ordinal $<\mu$ such that $B\cap Y_\alpha\in I^+$. If $B\cap X_\alpha\in I$ then $\{x\in B\cap Y_\alpha|\exists\beta<\alpha,\ x\in X_\beta\}\in I^+$, and hence by seminormality $(\exists\beta<\alpha)(B\cap X_\beta\in I^+)$, contradicting our choice of α. □

Lemma 1.3 enables us to push through some well known arguments concerning ideals on $\varkappa$. For instance

THEOREM 1.4 (cf. [10, Theorem 1.4.1]). *If I is seminormal and λ^+-saturated then I is precipitous.*

We leave the proof to the reader.

In [3, p. 59] Baumgartner, Taylor and Wagon showed that (for an ideal J on $\varkappa$) the partition property $J^+\to(J^+,\omega+1)^2_2$ is related to weak selectivity and weak p-pointness. Using seminormality we may obtain a $P_\varkappa\lambda$ analogue of this result. First the $P_\varkappa\lambda$ analogues of the notions concerned.

DEFINITION 1.5. (a) I is a weak p-point (weak q-point) iff whenever $A\in I^+$ and $\{A_x|x\in A\}\subseteq P(A)\cap I$ $(P(A)\cap I_{\varkappa\lambda})$, there is a $B\in P(A)\cap I^+$ such that for each $x\in B$, $B\cap A_x\cap\hat{x}\in I_{\varkappa\lambda}(=\emptyset)$.

I is weakly selective iff I is a weak p-point and a weak q-point.

(b) If $P,Q,R\subseteq P(P_\varkappa\lambda)$ and $n<\omega$, then $P\to(Q,R)^n_2$ denotes the assertion "whenever $X\in P$ and $f\colon[X]^n\to 2$, either there is a $Y\in Q$ homogeneous for 0 or a $Z\in R$ homogeneous for 1". If $\alpha\leqq\varkappa$ and $R=\{Z\subseteq P_\varkappa\lambda|$ the order type (with respect to $\subset$) of Z is $\alpha\}$, we write $P\to(Q,\alpha)^n_2$, and if $Q=R$ we write $P\to(Q)^n_2$.

(c) I is weakly lean iff for each $A\in I^+$ there is a $B\in P(A)\cap I^+$ such that $|B|=\lambda$ (it is easy to show that if $B\in I^+_{\varkappa\lambda}$ then $|B|\geqq\lambda$).

THEOREM 1.6. *Suppose I is seminormal and weakly lean. Consider the following assertions:*

(a) *I is weakly selective.*

(b) $I^+\to(I^+,\omega+1)^2_2$.

(c) $I^+\to(I^+,\omega)^2_2$.

(d) *I is a weak p-point.*

Then (a)$\to$(b)$\to$(c)$\to$(d).

PROOF. (a)$\to$(b). Suppose $A\in I^+$ and $f\colon[A]^2\to 2$ is a function having no homogeneous set for 0 in I^+. Since I is weakly lean we may assume that $|A|=\lambda$, and for each $x\in A$ let $A_x=\{y\in A\cap\hat{x}|f(x,y)=1\}$.

Define a sequence of I-partitions of A, $\langle W_n|n<\omega\rangle$ such that $W_0\geqq W_1\geqq W_2\geqq\ldots$ by induction as follows. Let $W_0=\{A\}$. Given W_n and $X\in W_n$, $\{X\cap A_x|x\in X$ and $X\cap A_y\in I^+\}$ is predense below X in I^+ (for if $Y\in P(X)\cap I^+$ is such that $(\forall y\in Y)$ $(Y\cap A_y\in I)$, then by weak selectivity there is a $Z\in P(Y)\cap I^+$ such that for each $(y,z)\in[Z]^1$, $z\notin A_y$, and so Z is homogeneous for 0), and hence by Lemma 1.3 we may find a disjoint I-partition of A, W_{n+1} refining W_n such that whenever $Y\in W_{n+1}$, $X\in W_n$ and $Y\subseteq X$, there is an $x\in X$ such that $Y\subseteq A_x$.

Since each W_n is an I-partition of A, $A-\bigcup W_n\in I$, hence by $\varkappa$-completeness, $\bigcup\limits_{n<\omega}[A-\bigcup W_n]=A-\bigcap\limits_{n<\omega}\bigcup W_n\in I$ and in particular $\bigcap\limits_{n<\omega}\bigcup W_n\neq\emptyset$. Pick $x\in\bigcap\limits_{n<\omega}\bigcup W_n$, then since $W_0\geqq W_1\geqq W_2\geqq\ldots$, $W_0(x)\supseteq W_1(x)\supseteq W_2(x)\supseteq\ldots$ and for each $n<\omega$ there is an $x_n\in W_n(x)$ such that $W_{n+1}(x)\subseteq A_{x_n}$. Clearly $\{x_n|n<\omega\}\cup\{x\}$ is the required homogeneous set for 1.

(b)$\to$(c) is trivial.

(c)$\to$(d). Suppose $A\in I^+$ and $\{A_x|x\in A\}\subseteq P(A)\cap I$. Since I is weakly lean and extends $I_{\varkappa\lambda}$ we may assume that $|A|=\lambda$ and $A\subseteq\{\hat{0}\}$. Let $c\colon\lambda-\{0\}\to A$ be a bijection and for each $x\in A$ let $f(x)$ be the least ordinal $\alpha\in x-\{0\}$ such that $x\in A_{c(\alpha)}$; $f(x)=0$ if no such α exists. Define $g\colon[A]^2\to 2$ by $g(x,y)=0$ iff $f(x)\leqq f(y)$, then since $I^+ \to (I^+, \omega)^2_2$ there is a $B \in P(A) \cap I^+$ such that for each $(x, y) \in [B]^2$, $f(x) \leqq \leqq f(y)$. Suppose $\beta \in \lambda - \{0\}$, $c(\beta) = x$, say, and $B \cap A_x \in I^+_{\varkappa\lambda}$, then for each $y \in B$ $f(y) \leqq f(z) \leqq \beta$ whenever $z \in B \cap A_x \cap \hat{y} \cap \{\hat{\beta}\}$. But then by seminormality there is a, $C \in P(B) \cap I^+$ such that $f \restriction C$ is constant, hence $f''C = \{0\}$ and so for each $\alpha \in \lambda - \{0\}$, $C \cap A_{c(\alpha)} \cap \{\hat{\alpha}\} = \emptyset$. This proves the weak p-point property. $\square$

As in [11], distributivity is related to partition relations.

THEOREM 1.7. *Suppose I is seminormal, weakly lean and η is a cardinal, $\eta < \varkappa$. Consider the following assertions:*

(a) *I is weakly selective and (μ, λ)-distributive for each $\mu < \eta$.*
(b) $I^+ \to (I^+, \eta+1)^2_2$.
(c) $I^+ \to (I^+, \eta)^2_2$.
(d) *I is (μ, λ)-distributive for each $\mu < \eta$.*
Then (a)$\to$(b)$\to$(c)$\to$(d).

The proof of (a)$\to$(b) is similar to that of Theorem 1.6 (a)$\to$(b), using the distributivity of I to continue the construction of the I-partitions $\langle W_\alpha | \alpha < \eta \rangle$ at limit stages. The proof of (c)$\to$(d) is similar to the corresponding proof for ideals on $\varkappa$ ([11, Theorem 7]) using Lemma 1.3.

§2. The λ-Shelah property

The following ineffability property of $P_\varkappa \lambda$ was introduced by Carr [4] as the $P_\varkappa \lambda$ analogue of a combinatorial equivalent of weak compactness due to Shelah [16]. Let $NSh_{\varkappa\lambda}$ be the set of all $X \subseteq P_\varkappa \lambda$ which do not have the property "for any sequence $\langle f_x | x \in X \rangle$ such that $(\forall x \in X)(f_x \colon x \to x)$, there is a function $f \colon \lambda \to \lambda$ such that $(\forall y \in P_\varkappa \lambda)$ $(\{x \in X \cap \hat{y} | f_x \restriction y = f \restriction y\} \in I^+_{\varkappa\lambda})$". $\varkappa$ is said to be λ-Shelah iff $P_\varkappa \lambda \notin NSh_{\varkappa\lambda}$, and in this case Carr [4, Theorem 2.3] showed that $NSh_{\varkappa\lambda}$ is a normal ideal on $P_\varkappa \lambda$. In this section we wish to prove (Corollary 2.7 below) that if $cf\lambda \geqq \varkappa$ and $\varkappa$ is λ-Shelah, then $\lambda^{<\varkappa} = \lambda$.

If X is a set, a function $F \colon {}^\omega X \to X$ is ω-Jónsson for X if $(\forall Y \subseteq X)(|Y| = |X| \to \to F''{}^\omega Y = X)$. Erdös and Hajnal [8] showed that every infinite set has an ω-Jónsson function and Solovay (see [17]) used this concept to show that if U is a normal measure on $P_\varkappa \lambda$ and λ is regular, then the function $\langle \sup(x) | x \in P_\varkappa \lambda \rangle$ is injective on a set in U. We first show that Solovay's proof extends to yield the following.

THEOREM 2.1. *Suppose $\lambda = \mu^+$ and $\varkappa$ is λ-Shelah, then the function* $\langle \sup(x) | x \in P_\varkappa \lambda \rangle$ *is injective on a set in $NSh^*_{\varkappa\lambda}$.*

PROOF. Let $F \colon {}^\omega\lambda \to \lambda$ be an ω-Jónsson function for λ.

LEMMA 2.2. $\{x \in P_\varkappa \lambda | \forall g \in {}^\omega x,\ F(g) \in x\} \in NSh^*_{\varkappa\lambda}$.

PROOF OF LEMMA 2.2. Suppose not then $A = \{x \in P_\varkappa \lambda | x \supset \omega$ and $\exists g_x \in {}^\omega x$, $F(g_x) \notin x\} \in NSh^+_{\varkappa\lambda}$. For each $x \in A$ pick $g_x \in {}^\omega x$ such that $F(g_x) \notin x$, then by the λ-Shelah property there is a function $f \colon \omega \to \lambda$ such that $B = \{x \in A | g_x = f\} \in I^+_{\varkappa\lambda}$. But then for each $x \in B$, $F(f) = F(g_x) \notin x$, contradicting $B \in I^+_{\varkappa\lambda}$. $\square$

LEMMA 2.3. $\{x \in P_\varkappa \lambda | F | {}^\omega x$ *is an ω-Jónsson function for* $x\} \in NSh^*_{\varkappa\lambda}$.

PROOF OF LEMMA 2.3. Suppose not, then by Lemma 2.2, $\{x\in P_\varkappa\lambda|(\exists\alpha_x\in x)(\exists a_x\subseteq x)(|a_x|=|x|$ and $\alpha_x\notin F''{}^\omega a_x)\}\in NSh^+_{\varkappa\lambda}$. By normality of $NSh_{\varkappa\lambda}$, there is an $\alpha<\lambda$ such that $A=\{x\in P_\varkappa\lambda|\exists a_x\subseteq x,\ |a_x|=|x|$ and $\alpha\notin F''{}^\omega a_x\}\in NSh^+_{\varkappa\lambda}$. For each $x\in A$ let $a_x\subseteq x$ witness that $x\in A$ and $f_x\colon x\to a_x$ be a bijection. By the λ-Shelah property there exists a function $f\colon\lambda\to\lambda$ such that $(\forall y\in P_\varkappa\lambda)$ $(\{x\in A\cap\hat{y}|f_x\restriction y=f\restriction y\}\in I^+_{\varkappa\lambda})$. But then f is injective and so there exists an $s\in{}^\omega(\mathrm{im}\,(f))$ such that $F(s)=\alpha$. Let $y=\{\gamma<\lambda|f(\gamma)\in\mathrm{im}(s)\}$, then y is countable and so there is an $x\in A\cap\hat{y}$ such that $f_x\restriction y=f\restriction y$. Hence $s\in{}^\omega(\mathrm{im}\,(f_x))={}^\omega a_x$ contradicting $F(s)=\alpha\notin F''{}^\omega a_x$. □

LEMMA 2.4. $\{x\in P_\varkappa\lambda|\bar{x}=(\overline{x\cap\mu})^+\}\in NSh^*_{\varkappa\lambda}$.

PROOF OF LEMMA 2.4. Suppose $A=\{x\in P_\varkappa\lambda|\bar{x}<(\overline{x\cap\mu})^+\}\in NSh^+_{\varkappa\lambda}$. For each $x\in A$ let $f_x\colon x\to x\cap\mu$ be injective, then applying the λ-Shelah property yields an injective function $f\colon\lambda\to\mu$; contradiction.

Suppose $\{x\in P_\varkappa\lambda|\bar{x}>(\overline{x\cap\mu})^+\}\in NSh^+_{\varkappa\lambda}$, then by normality of $NSh_{\varkappa\lambda}$ there exists a γ such that $\mu<\gamma<\lambda$ and $\{x\in P_\varkappa\lambda|\overline{x\cap\gamma}=(\overline{x\cap\mu})^+\}\in NSh^+_{\varkappa\lambda}$. However if $g\colon\gamma\to\mu$ is bijective, then by normality, $\{x\in P_\varkappa\lambda|g''x\cap\gamma=x\cap\mu\}\in NSh^*_{\varkappa\lambda}$; contradiction. □

As in Lemma 2.2 we may also prove.

LEMMA 2.5. $\{x\in P_\varkappa\lambda|x$ *is closed under* ω*-limits*$\}\in NSh^*_{\varkappa\lambda}$.

The proof of Theorem 2.1 is now similar to that of [17, Theorem 3.4]: Let $X=\{x\in P_\varkappa\lambda|\bar{x}$ is regular, $\bar{x}>\omega$, x is closed under ω-limits and $F\restriction{}^\omega x$ is ω-Jónsson for $x\}$, then $X\in NSh^*_{\varkappa\lambda}$. Suppose $x,y\in X$ with $\sup(x)=\sup(y)=\delta$. As $\bar{x},\bar{y}$ are regular and x and y are closed under ω-limits $\sup(x\cap y)=\delta$ and hence $|x|=|x\cap y|=|y|$. But then by the ω-Jónsson property, $x=F''{}^\omega(x\cap y)=y$. □

COROLLARY 2.6. *If* $\lambda=\mu^+$ *and* $\varkappa$ *is* λ*-Shelah then* $\lambda^{<\varkappa}=\lambda$.

PROOF. By a result of Carr ([5, Proposition 1.2)], $\varkappa$ is strongly inaccessible and hence if $A\in NSh^*_{\varkappa\lambda}$ with $|A|=\lambda$, then $|P_\varkappa\lambda|=|\bigcup\{P(x)|x\in A\}|\leqq 2^{<\varkappa}\cdot|A|=\lambda$. □

COROLLARY 2.7. *If* $cf\,\lambda\geqq\varkappa$ *and* $\varkappa$ *is* λ*-Shelah then* $\lambda^{<\varkappa}=\lambda$.

PROOF. The case for successor cardinals is proved in Corollary 2.6, hence suppose λ is a limit cardinal. By a result of Carr ([5, Proposition 1.1]), $\varkappa$ is μ-Shelah whenever μ is a cardinal such that $\varkappa\leqq\mu\leqq\lambda$, and hence $cf\lambda\geqq\varkappa$ and Corollary 2.6 yield $|P_\varkappa\lambda|=|\bigcup\{P_\varkappa\mu^+|\varkappa\leqq\mu<\lambda\}|=\lambda$. □

§3. *WC* Ideals

In [11, 12] we introduced a property of ideals on $\varkappa$, *WC*, which may be regarded as an ideal theoretic analogue of the "strong inaccessibility and tree property" equivalent of weak compactness. The following seems to be the natural $P_\varkappa\lambda$ analogue of this notion.

DEFINITION 3.1. For $\mu\leqq\lambda$, I is $(\mu,\nu)-WC$ iff whenever $A\in I^+$ and $\langle W_\gamma|\gamma<\mu\rangle$ is a sequence of I-partitions of A, each of cardinality $\leqq\nu$, there is a sequence $\langle X_\gamma|\gamma<\mu\rangle$ such that for each $\gamma<\mu$, $X_\gamma\in W_\gamma$ and for each $y\in P_\varkappa\lambda$, $\bigcap\{X_\gamma|\gamma\in y\cap\mu\}\in I^+$.

In [12, §3] we mentioned that if $\varkappa$ is weakly compact and J is a normal ideal on $\varkappa$, then J is WC (in the sense of [12]) iff J extends the Π_1^1-indescribable ideal on $\varkappa$. Analogously we have.

THEOREM 3.2. *If $P_\varkappa\lambda$ carries a normal (λ, λ)-WC ideal I, then $\varkappa$ is λ-Shelah and $I \supseteq NSh_{\varkappa\lambda}$.*

PROOF. Suppose $A \in I^+$ and $\langle f_x | x \in A\rangle$ is a sequence such that $(\forall x \in A)$ $(f_x : x \to x)$. For each $\alpha, \varrho \in \lambda$ let $X_\alpha^\varrho = \{x \in A \cap \{\hat{\alpha}\} | f_x(\alpha) = \varrho\}$, then by normality, $W_\alpha = \{X_\alpha^\varrho | \varrho < \lambda$ and $X_\alpha^\varrho \in I^+\}$ is an I-partition of A. Since I is (λ, λ)-WC there is a function $f: \lambda \to \lambda$ such that for each $y \in P_\varkappa\lambda$, $\bigcap\{X_\alpha^{f(\alpha)} | \alpha \in y\} \in I^+$. In particular, $\{x \in A \cap \hat{y} | f_x \upharpoonright y = f \upharpoonright y\} \in I^+_{\varkappa\lambda}$. □

Before we can prove the converse we need the following preliminaries, the first of which may be regarded as the $P_\varkappa\lambda$ analogue of [11, Theorem 2].

THEOREM 3.3. *Suppose I is normal and $\mu < \varkappa$ then I is (μ, λ)-distributive iff whenever $A \in I^+$ and $f: A \to {}^\mu\lambda$ is such that $(\forall x \in A - \{\emptyset\})(f(x) \in {}^\mu x)$, there is a $B \in P(A) \cap \cap I^+$ such that $f \upharpoonright B$ is constant.*

PROOF. $(\to)$. Let A and f be as given. For each $\alpha < \mu$ and $\varrho < \lambda$ let $X_\alpha^\varrho = \{x \in A | f(x)(\alpha) = \varrho\}$, then by normality $W_\alpha = \{X_\alpha^\varrho | \varrho < \lambda$ and $X_\alpha^\varrho \in I^+\}$ is an I-partition of A. By (μ, λ)-distributivity there is a set $B \in P(A) \cap I^+$ and a function $g: \mu \to \lambda$ such that for each $\alpha < \mu$, $B - X_\alpha^{g(\alpha)} \in I$, and hence by $\varkappa$-completeness, $C = \bigcap\{X_\alpha^{g(\alpha)} | \alpha < \mu\} \in I^+$. Clearly for each $x \in C$, $f(x) = g$.

$(\leftarrow)$. Suppose $A \in I^+$ and $\langle W_\gamma | \gamma < \mu\rangle$ is a sequence of I-partitions of A, each of cardinality $\leq \lambda$. By Lemma 1.3 we may assume that each W_γ is disjoint. Let h: $\bigcup\{W_\gamma | \gamma < \mu\} \to \lambda$ be injective, then by $\varkappa$-completeness, $B = \{x \in A | \forall \gamma < \mu,\ x \in \bigcup W_\gamma$ and $h(W_\gamma(x)) \in x\} \in (I|A)^*$. By hypothesis there is a $g \in {}^\mu\lambda$ such that $\{x \in B | \forall \gamma < \mu,\ W_\gamma(x) = h^{-1}(g(\gamma))\} \in I^+$, and hence $\bigcap\{h^{-1}(g(\gamma)) | \gamma < \mu\} \in I^+$, thus proving (μ, λ)-distributivity. □

COROLLARY 3.4. (cf $\lambda \geq \varkappa$). *If $\varkappa$ is λ-Shelah and I is normal and extends $NSh_{\varkappa\lambda}$, then I is (μ, λ)-distributive for each $\mu < \varkappa$.*

PROOF. Using Theorem 3.3, suppose $\mu < \varkappa$, $A \in I^+$ and $f: A \to {}^\mu\lambda$ is such that $(\forall x \in A - \{\emptyset\})(f(x) \in {}^\mu x)$. By Corollary 2.7, $\lambda^\mu = \lambda$ and as in the proof of Lemma 2.2, if $c: {}^\mu\lambda \to \lambda$ is bijective then $\{x \in P_\varkappa\lambda | c''{}^\mu x \subseteq x\} \in NSh^*_{\varkappa\lambda} \subseteq I^*$. Hence

$$\{x \in A | c(f(x)) \in x\} \in I^+,$$

and by normality f is constant on a set in I^+. □

LEMMA 3.5 (Carr [6], $\lambda^{<\varkappa} = \lambda$). *If $\varkappa$ is λ-Shelah and $c: P_\varkappa\lambda \to \lambda$ and $b: \lambda^2 \to \lambda$ are bijective then $\{x \in P_\varkappa\lambda | \varkappa_x$ is an inaccessible cardinal, $c'' P_{\varkappa_x} x = x$ and $b'' x^2 = x\} \in NSh^*_{\varkappa\lambda}$.*

THEOREM 3.6. (cf $\lambda \geq \varkappa$). *If $\varkappa$ is λ-Shelah and I is normal and extends $NSh_{\varkappa\lambda}$, then I is $(\lambda, \lambda)-WC$.*

PROOF. Suppose $A \in I^+$ and $\langle W_\gamma | \gamma < \lambda\rangle$ is a sequence of I-partitions of A, each of cardinality $\leq \lambda$. By Lemma 1.3 we may assume that each W_γ is disjoint. By Corollary 2.7, $\lambda^{<\varkappa} = \lambda$ and by Lemma 3.5, if $c: \lambda \to P_\varkappa\lambda$ is bijective then

$$B = \{x\in P_\varkappa\lambda \mid \varkappa_x \text{ is an inaccessible cardinal and } c''x = P_{\varkappa_x}x\}\in NSh^*_{\mu\lambda} \subseteq I^*.$$

For each $\alpha<\lambda$ let $T_\alpha=\{\bigcap\{X_\gamma|\gamma\in c(\alpha)\}|\forall\gamma\in c(\alpha),\ X_\gamma\in W_\gamma\}\cap I^+$, then since I is $(|c(\alpha)|, \lambda)$-distributive, T_α is a disjoint I-partition of A. Since $\lambda^{<\varkappa}=\lambda$, $|T_\alpha|\leqq\lambda$, and by normality, if $h\colon \bigcup\{T_\alpha|\alpha<\lambda\}\to\lambda$ is injective then

$$C = \{x\in A\cap B \mid \forall\alpha\in x,\ x\in \bigcup T_\alpha \text{ and } h(T_\alpha(x))\in x\}\in (I/A)^*.$$

Hence by the λ-Shelah property there is a function $f\colon \lambda\to\lambda$ such that for each $y\in P_\varkappa\lambda$, $\{x\in C\cap\hat{y}|\forall\alpha\in y,\ f(\alpha)=h(T_\alpha(x))\}\in I^+_{\varkappa\lambda}$. In particular for each $\alpha<\lambda$, $h^{-1}(f(\alpha))\in T_\alpha$ and $\bigcap\{h^{-1}(f(\alpha))|\alpha\in y\}\in I^+_{\varkappa\lambda}$.

Suppose $y\in P_\varkappa\lambda$ and $\bigcap\{h^{-1}(f(\alpha))|\alpha\in y\}\in I$. Choose $\delta<\lambda$ such that $c(\delta)\in B\cap\hat{y}$, then $y\cup\{\delta\}\in P_\delta\lambda$ and hence $\bigcap\{h^{-1}(f(\alpha))|\alpha\in y\cup\{\delta\}\}\in I^+_{\varkappa\lambda}$. Now $h^{-1}(f(\delta))\in T_\varkappa$ and for each $\alpha\in y$, $c(\alpha)\subseteq c(\delta)$ (since $c(\delta)\in B$), hence $T_\delta\leqq T_\alpha$ and we may find a set $Y_\alpha\in T_\alpha$ such that $h^{-1}(f(\delta))\subseteq Y_\alpha$. But then there is a $\beta\in y$ such that $Y_\beta\neq h^{-1}(f(\beta))$ (for otherwise $\bigcap\{h^{-1}(f(\alpha))|\alpha\in y\}=\bigcap\{Y_\alpha|\alpha\in y\}\supseteq h^{-1}(f(\delta))\in I^+$), and hence since T_β is disjoint, $h^{-1}(f(\delta))\cap\bigcap\{h^{-1}(f(\alpha))|\alpha\in y\}\subseteq Y_\beta\cap h^{-1}(f(\beta))=\emptyset$, a contradiction.

Finally since each W_γ is refined by some T_α, this proves the $(\lambda,\lambda)-WC$ property. □

Immediately from (the proof of) Theorem 3.2 we now have.

COROLLARY 3.7. (cf $\lambda\geqq\varkappa$). *Suppose $\varkappa$ is λ-Shelah, $A\in NSh^+_{\varkappa\lambda}$ and $\langle f_x|x\in A\rangle$ is a sequence such that $(\forall x\in A)(f_x : x\to x)$. Then there is a function $f\colon \lambda\to\lambda$ such that for each $y\in P_\varkappa\lambda$, $\{x\in A\cap\hat{y}|\, f_x{\restriction}y=f{\restriction}y\}\in NSh^+_{\varkappa\lambda}$.*

The λ-Shelah property is ideal theoretically strong in that the associated ideal is normal. An ideal theoretically weak $P_\varkappa\lambda$ generalisation of weak compactness due to DiPrisco and Zwicker [7] is the following. Let $NMIn_{\varkappa\lambda}$ be the of all $X\subseteq P_\varkappa\lambda$ which do not have the property "for any sequence $\langle S_x|x\in X\rangle$ such that $(\forall x\in X)(S_x\subseteq x)$, there is an $S\subseteq\lambda$ such that $(\forall y\in P_\varkappa\lambda)(\{x\in X\cap\hat{y}|S_x\cap y=S\cap y\}\in I^+_{\varkappa\lambda})$". $\varkappa$ is said to be mildly λ-ineffable iff $P_\varkappa\lambda\notin NMIn_{\varkappa\lambda}$, and in this case Carr ([4, Proposition 1.4]) showed that $NMIn_{\varkappa\lambda}=I_{\varkappa\lambda}$. Mild λ-ineffability is related to the property WC by the following theorem whose (easy) proof we leave to the reader.

THEOREM 3.8. *The following are equivalent:*
(a) *$P_\varkappa\lambda$ carries a $(\lambda, 2)-WC$ ideal.*
(b) *$\varkappa$ is mildly λ-ineffable.*
(c) *$I_{\varkappa\lambda}$ is $(\lambda, 2)-WC$.*

In [11, Corollary 2] we showed that if $\varkappa$ is weakly compact then $I_\varkappa$ (the ideal on $\varkappa$ dual to the Fréchet filter) is $(\varkappa,\varkappa)$-distributive. This result suggests the following question which we have been unable to answer: if $\varkappa$ is midly λ-ineffable, is it the case that $I_{\varkappa\lambda}$ is $(\lambda, 2)$-distributive?

As in [11, Theorem 8], the property WC is related to partition relations.

THEOREM 3.9. *Suppose I is seminormal, weakly lean and weakly selective, then I is $(\varkappa,\lambda)-WC$ iff $I^+\to(I^+,\varkappa)^2_2$.*

The proof is similar to that of Theorem 1.6 and [11, Theorem 7], hence we leave the details to the reader.

In [1, Theorem 3.2], Baumgartner showed that if $\varkappa$ is weakly compact and J is the Π_1^1-indescribable ideal on $\varkappa$, then (in the terminology of [1]), $J^+ \to (J^+, \varkappa)_2^2$. Baumgartner and Carr [6] generalised this result to the $P_\varkappa \lambda$ context: if $\lambda^{<\varkappa}=\lambda$, $\varkappa$ is λ-Shelah, $A\in NSh_{\varkappa\lambda}^+$ and $f\colon [A]^2\to 2$, then either there is a set $B\in P(A)\cap NSh_{\varkappa\lambda}^+$ such that $f''[B]_<^2=\{0\}$ or a function $h\colon P_\varkappa\lambda\to A$ such that $(\forall (y,x)\in[P_\varkappa\lambda]_<^2)\big((y,h(y),h(x))\in\in[P_\varkappa\lambda]_<^3$ and $f(h(y),h(x))=1\big)$. The property $(\lambda,\lambda)-WC$ is related to a natural variant of this partition relation by the following.

THEOREM 3.10. *Suppose I is seminormal and weakly lean. Consider the following assertions:*

(a) $I^+\to(I^+,4)_2^3$.

(b) $I^+\to(I^+)_2^2$.

(c) *I is weakly selective and (λ,λ)-distributive.*

(d) *I is weakly selective and $(\lambda,\lambda)-WC$.*

(e) *whenever $A\in I^+$ and $f\colon[A]^2\to 2$, either there is a set $B\in P(A)\cap I^+$ such that $f''[B]^2=\{0\}$ or a set $C\in P(A)\cap I_{\varkappa\lambda}^+$ and a function $h\colon C\to A$ such that*

$$(\forall(y,x)\in[C]^2)\big(y\subset h(y)\subset h(x) \text{ and } f(h(y),h(x))=1\big).$$

Then (a)$\leftrightarrow$(b)$\leftrightarrow$(c)$\to$(d)$\to$(e).

PROOF. The implication (a)$\to$(b) is proved exactly as in [1, Theorem 6.2]. The proofs of (b)$\to$(c) and (c)$\to$(a) are similar to those of Theorems 5.3 (d)$\to$(a) and 6.2 (a)$\to$(b) respectively. (c)$\to$(d) is trivial.

(d)$\to$(e). Suppose $A\in I^+$ and $f\colon[A]^2\to 2$ is a function having no homogeneous set in I^+ for 0. Since I is weakly lean we may assume that $|A|=\lambda$, and since $\varkappa$ is strongly inaccessible (by Theorem 3.8 and [5, Proposition 1.2]) and I is weakly selective we may assume that $|y|<|x|$ whenever $(y,x)\in[A]^2$. For each $x\in A$ let $A_x==\{z\in A\cap\hat{x}\mid f(x,z)=1\}$, then as in the proof of Theorem 1.6, given $B\in P(A)\cap I^+$ and $r\in P_\varkappa\lambda$, $\{B\cap A_w\mid w\in B\cap\hat{r}\}\cap I^+$ is predense below B in I^+.

For each $x\in A$ we define, by induction on $|x|$, a disjoint I-partition of A, W_x, and a function $g_x\colon W_x\to A\cap\hat{x}$ such that for each $X\in W_x$, $X\subseteqq A_{g_x(X)}$ as follows. Suppose $x\in A$ and W_y and g_y have been defined for each $y\in A$ such that $|y|<|x|$ (and in particular for each $y\in A\cap P(x)-\{x\}$). As in the proof of Theorem 3.6, since I is $(|P(x)|,\lambda)$-distributive we may find a disjoint I-partition of A, T_x, such that $T_x\leqq W_y$ whenever $y\in A\cap P(x)-\{x\}$. Given $Z\in T_x$, $\{Z\cap A_w\mid w\in Z\cap\hat{x}\}\cap I^+$ is predense below Z in I^+, and hence by lemma 1.3 we may find a disjoint I-partition of A, W_x, and a function $g_x\colon W_x\to A\cap\hat{x}$ such that for each $X\in W_x$ there is a $Z\in T_x$ with $g_x(X)\in Z$ and $X\subseteqq Z\cap A_{g_x(X)}$.

Since I is $(\lambda,\lambda)-WC$ we may find a sequence $\langle X_x\mid x\in A\rangle$ such that for each $x\in A$, $X_x\in W_x$ and for each $s\in A$, $\bigcap\{X_x\mid x\in A\cap P(s)\}\in I^+$. Let $h\colon A\to A$ be given by $h(x)=g_x(X_x)$, then clearly $h(x)\in\hat{x}$. Suppose $(y,x)\in[A]^2$ and $Z\in T_x$ is such that $h(x)\in Z$ and $X_x\subseteqq Z\cap A_{h(x)}$. Since $X_x\cap X_y\in I^+$ and $T_x\leqq W_y$ we must have $Z\subseteqq X_y$, and since $X_y\subseteqq A_{h(y)}$, $h(x)\in A_{h(y)}$. Hence h is our required function. □

§4. Ineffability

In [9], Jech defined the following generalisations of ineffability and almost ineffability. A set $X\subseteq P_\varkappa\lambda$ is said to be λ-ineffable (almost λ-ineffable) iff for any sequence $\langle S_x|x\in X\rangle$ such that $(\forall x\in X)(S_x\subseteq x)$, there is an $S\subseteq\lambda$ such that $\{x\in X|S_x=S\cap x\}\in NS^+_{\varkappa\lambda}(I^+_{\varkappa\lambda})$.

$NIn_{\varkappa\lambda}=\{X\subseteq P_\varkappa\lambda|X$ is not λ-ineffable$\}$

$NAIn_{\varkappa\lambda}=\{X\subseteq P_\varkappa\lambda|X$ is not almost λ-ineffable$\}$

and by a result of Carr [4, Corollary 1.3], $NSh_{\varkappa\lambda}\subseteq NAIn_{\varkappa\lambda}\subseteq NIn_{\varkappa\lambda}$. $\varkappa$ is said to be λ-ineffable (almost λ-ineffable) iff $P_\varkappa\lambda\notin NIn_{\varkappa\lambda}(NAIn_{\varkappa\lambda})$, and in this case Carr [4, Theorem 1.2] showed that $NIn_{\varkappa\lambda}$ $(NAIn_{\varkappa\lambda})$ is a normal ideal on $P_\varkappa\lambda$.

In this section we generalise results of Baumgartner [1] and Kunen [14] to show that λ-ineffable and almost λ-ineffable subsets of $P_\varkappa\lambda$ may be characterised in terms of regressive partition relations.

THEOREM 4.1. (cf $\lambda\geqq\varkappa$). *Suppose $\varkappa$ is almost λ-ineffable and $A\in NAIn^+_{\varkappa\lambda}$, then for every regressive function $f\colon[A]^2\to\lambda$ there is a $B\in P(A)\cap I^+_{\varkappa\lambda}$ such that $|f''[B]^2_<|=1$.*

PROOF. By Corollary 2.7, $\lambda^{<\varkappa}=\lambda$ and by Lemma 3.5, if $c\colon P_\varkappa\lambda\to\lambda$ and $b\colon\lambda^2\to\lambda$ are bijective and

$$M=\{x\in P_\varkappa\lambda|\varkappa_x \text{ is an inaccessible cardinal, } c''P_{\varkappa_x}x=x \text{ and } b''x^2=x\},$$

then $M\in NAIn^*_{\varkappa\lambda}$.

Suppose now that $A\in NAIn^+_{\varkappa\lambda}$ and $f\colon[A]^2\to\lambda$ is regressive. Clearly we may assume that $A\subseteq M$. For each $x\in A$ let

$$S_x=\{b(c(y),f(y,x))|y\in P_{\varkappa_x}x\cap A\}(\subseteq x) \text{ and } B_x=\{y\in P_{\varkappa_x}x\cap A|S_y=S_x\cap y\}.$$

Let $E=\{x\in A|B_x\in SNS^+_{\varkappa_x x}\}$, then

LEMMA 4.2. $A-E\in NAIn_{\varkappa\lambda}$.

PROOF. Suppose $A-E\in NAIn^+_{\varkappa\lambda}$ and $x\in A-E$. Since $B_x\in SNS_{\varkappa_x x}$ there is a regressive function $f_x\colon B_x\to x$ such that $(\forall\alpha\in x)(f_x^{-1}(\{\alpha\})\in I_{\varkappa_x x})$, and hence for each $\alpha\in x$ there is a $g(x,\alpha)\in P_{\varkappa_x}x$ such that $f_x^{-1}(\{\alpha\})\subseteq P_{\varkappa_x}x-g(x,\hat{}\ \alpha)$. Let $G_x=\{b(\alpha,c(g(x,\alpha)))|\alpha\in x\}$ then $G_x\subseteq x$ and by coding $\langle G_x,S_x\rangle$ as a single subset of x (using b) and applying almost λ-ineffability, we may find subsets of λ, G and S such that $Z=\{x\in A-E|G_x=G\cap x$ and $S_x=S\cap x\}\in I^+_{\varkappa\lambda}$.

Pick $(y,x)\in[Z]^2_<$ then $S_y=S_x\cap y$ and hence $y\in B_x$. Suppose $f_x(y)=\alpha(\in y)$ then $y\in f_x^{-1}(\{\alpha\})\subseteq P_{\varkappa_x}x-g(x,\hat{}\ \alpha)$. Also $G_y=G_x\cap y$ hence (since G_x codes a function), $g(x,\alpha)=g(y,\alpha)$ and so $y\notin g(y,\hat{}\ \alpha)$, contradicting our definition of g. □

For each $x\in E$ we may (unambigously) define a regressive function $g_x\colon B_x\to x$ by $g_x(y)=\gamma$ iff there is a $z\in B_x\cap\hat{y}$ such that $|y|<\varkappa_z$ and $f(y,z)=\gamma$. Since $B_x\in SNS^+_{\varkappa_x x}$ there is a $\delta_x\in x$ such that $g_x^{-1}(\{\delta_x\})\in I^+_{\varkappa_x x}$, and hence by normality of $NAIn_{\varkappa\lambda}$ there is a $\delta<\lambda$ such that $F=\{x\in E|\delta_x=\delta\}\in NAIn^+_{\varkappa\lambda}$. For each $x\in F$ let $T_x=c''g_x^{-1}(\{\delta\})$, then $T_x\subseteq x$ and by almost λ-ineffability there is a set $T\subseteq\lambda$ such that $H=\{x\in F|T_x=T\cap x\}\in I^+_{\varkappa\lambda}$.

Let $B=c^{-1}{}''T$, then claim B is the required "homogeneous" set for f. Firstly suppose $v\in P_\varkappa\lambda$, then pick $x\in H\cap\hat{v}$ such that $|v|<\varkappa_x$. Since $g_x^{-1}(\{\delta\})\in I^+_{\varkappa_x x}$ we may find a $y\in g_x^{-1}(\{\delta\})\cap\hat{v}$ and hence $c(y)\in T_x\subseteq T$; thus $B\in I^+_{\varkappa\lambda}$. Suppose $(y,z)\in[B]^2_<$, then pick $x\in H\cap\hat{z}$ such that $|z|<\varkappa_x$. Since $x\in M$, $c(y), c(z)\in T\cap x=T_x$, hence $(y,z)\in[B_x]^2_<$ and $g_x(y)=\delta$. But then by our definition of g_x, $f(y,z)=\delta$. □

Conversely we have

THEOREM 4.3. *If $A\subseteq P_\varkappa\lambda$ and satisfies the conclusion of Theorem* 4.1, *then $A\notin NAIn_{\varkappa\lambda}$.*

PROOF. Suppose $\langle S_x|x\in A\rangle$ is a sequence such that $(\forall x\in A)(S_x\subseteq x)$. Define a function $f\colon[A]^2\to\lambda$ by $f(x,y)=\min\left(S_x\Delta(S_y\cap x)\right)$ if $S_x\Delta(S_y\cap x)\neq\emptyset$; $f(x,y)=$ $=\min(x)$ otherwise, then f is regressive and hence we may find an $\alpha<\lambda$ and a set $B\in P(A)\cap I^+_{\varkappa\lambda}$ such that $f''[B]^2_<=\{\alpha\}$. Pick $C\in P(B)\cap I^+_{\varkappa\lambda}$ such that either $(\forall x\in C)(\alpha\in S_x)$ or $(\forall x\in C)(\alpha\notin S_x)$, then $S_x=S_y\cap x$ whenever $(x,y)\in[C]^2_<$. Hence if $S=\bigcup\{S_x|x\in C\}$, then $S_x=S\cap x$ for each $x\in C$. □

For ineffability we have the following

THEOREM 4.4. (cf. $\lambda\geqq\varkappa$.) *Suppose $\varkappa$ is λ-ineffable and $A\in NIn^+_{\varkappa\lambda}$, then for every regressive function $f\colon[A]^2\to\lambda$ there is a $B\in P(A)\cap NS^+_{\varkappa\lambda}$ such that $|f''[B]^2_<|=1$.*

PROOF. Suppose $A\in NIn^+_{\varkappa\lambda}$ and $f\colon[A]^2\to\lambda$ is regressive. Let b, c and M be as in the proof of Theorem 4.1, then $M\in NIn^*_{\varkappa\lambda}$ and $A\cap M\in NIn^+_{\varkappa\lambda}$. For each $x\in A\cap M$ let $S_x=\{b(c(y),f(y,x))|y\in P_{\varkappa_x}x\cap A\}$ then $S_x\subseteq x$ and by λ-ineffability we may find an $S\subseteq\lambda$ such that $C=\{x\in A\cap M|S_x=S\cap x\}\in NS^+_{\varkappa\lambda}$.

We may (unambiguously) define a regressive function $g\colon C\to\lambda$ by $g(y)=\gamma$ iff there is an $x\in C\cap\hat{y}$ such that $|y|<\varkappa_x$ and $f(y,x)=\gamma$. Hence by normality of $NS_{\varkappa\lambda}$ there is a $\delta<\lambda$ such that $g^{-1}(\{\delta\})\in NS^+_{\varkappa\lambda}$, and clearly $f''\left[g^{-1}(\{\delta\})\right]^2_<=\{\delta\}$. □

THEOREM 4.5. *If $A\subseteq P_\varkappa\lambda$ and satisfies the conclusion of Theorem* 4.4, *then $A\notin NIn_{\varkappa\lambda}$.*

The proof is similar to that of Theorem 4.3 and we leave the details to the reader.

In order to obtain a version of Theorem 4.4 in which the set B is homogeneous for f, we may redefine λ-ineffability as in the following

THEOREM 4.6 (cf $\lambda\geqq\varkappa$). *Suppose $A\subseteq P_\varkappa\lambda$, then the following are equivalent:*

(a) *For every sequence $\langle S_x|x\in A\rangle$ such that $(\forall x\in A)(S_x\subseteq x)$, there is a set $H\in P(A)\cap NS^+_{\varkappa\lambda}$ such that for each $(x,y)\in[H]^2$, $|x|<|\varkappa\cap y|$ and $S_x=S_y\cap x$.*

(b) *For every regressive function $f\colon[A]^2\to\lambda$, there is a set $B\in P(A)\cap NS^+_{\varkappa\lambda}$ such hat $|f''[B]^2|=1$.*

The proof is similar to that of Theorem 4.4. We do not know if a version of Theorem 4.6 holds for "almost λ-ineffability".

Alternatively we may turn to Menas' combinatorial principle χ (see [15, p. 228]): Let $\chi(I)$ denote the statement "for every $A\in I^+$ there is a function $f\colon\varkappa\to\varkappa$ such that $\{x\in A|f(|\varkappa\cap x|)=|x|$ and $(\forall\alpha<|\varkappa\cap x|)(f(\alpha)<|\varkappa\cap x|)\}\in I^+$". As in [15, Lemma 5] we may show (using Lemma 2.3) that if $\varkappa$ is almost λ-ineffable and $\chi(NAIn_{\varkappa\lambda})$ holds then $\{A\in NAIn^+_{\varkappa\lambda}|\forall(x,y)\in[A]^2,|x|<|\varkappa\cap y|\}$ is dense in $NAIn^+_{\varkappa\lambda}$, and hence if $\operatorname{cf}\lambda\geqq\varkappa$, $A\in NAIn^+_{\varkappa\lambda}$ and $f\colon[A]^2\to\lambda$ is regressive, then there exists a $B\in P(A)\cap I^+_{\varkappa\lambda}$ homogeneous for f. Also as in the case of normal measures (see [15,

p. 228]), for certain cardinals $\lambda>\varkappa$, $\chi(NAIn_{\varkappa\lambda})$ will always hold. For instance (by Lemmas 2.4 and 3.5) if $\lambda=\varkappa^+$ and $\varkappa$ is almost λ-ineffable, then the function $f\colon \varkappa\to\varkappa$ given by $(\forall\alpha<\varkappa)(f(\alpha)=\alpha^+)$ witnesses that $\chi(NAIn_{\varkappa\lambda})$ holds.

Similar remarks may be made for λ-ineffability.

§5. Complete ineffability

In [11. Corollary 3], we showed that $\varkappa$ is completely ineffable iff $\varkappa$ carries a normal $(\varkappa, \varkappa)$-distributive ideal. Analogously let us define $\varkappa$ to be completely λ-ineffable iff $P_\varkappa\lambda$ carries a normal (λ, λ)-distributive ideal.

THEOREM 5.1. *The following are equivalent:*

(a) *I is normal and $(\lambda, 2)$-distributive.*

(b) *Whenever $A\in I^+$ and $\langle S_x|x\in A\rangle$ is a sequence such that $(\forall x\in A)(S_x\subseteq x)$, there is an $S\subseteq\lambda$ such that $\{x\in A|S_x=S\cap x\}\in I^+$.*

(c) *I is normal and (λ, λ)-distributive.*

PROOF. (a)$\to$(b). Let A and $\langle S_x|x\in A\rangle$ be as given. For each $\alpha<\lambda$ let $X_\alpha^0=\{x\in A|\alpha\notin S_x\}$ and $X_\alpha^1=\{x\in A|\alpha\in S_x\}$, then $W_\alpha=\{X_\alpha^i|i<2$ and $X_\alpha^i\in I^+\}$ is an I-partition of A. By $(\lambda, 2)$-distributivity we may find a $B\in P(A)\cap I^+$ and a function $f\colon\lambda\to 2$ such that for each $\alpha<\lambda$, $B-X_\alpha^{f(\alpha)}\in I$. By normality, $C=\{x\in B|\forall\alpha\in x,$ $x\in X_\alpha^{f(\alpha)}\}\in I^+$ and hence for each $x\in C$, $S_x=f^{-1}(\{1\})\cap x$.

(b)$\to$(c). Normality is clear. Suppose $A\in I^+$ and $\langle W_\alpha|\alpha<\lambda\rangle$ is a sequence of I-partitions of A, each of cardinality $\leqq\lambda$. By Lemma 1.3 we may assume that each W_α is disjoint. Let $h\colon\bigcup\{W_\alpha|\alpha<\lambda\}\to\lambda$ be injective, then by normality, $B=\{x\in A|\forall\alpha\in x, x\in\bigcup W_\alpha$ and $h(W_\alpha(x))\in x\}\in(I|A)^*$. For each $x\in B$ let $S_x=\{h(W_\alpha(x))|\alpha\in x\}$, then $S_x\subseteq x$, and by our hypothesis there is an $S\subseteq\lambda$ such that $C=\{x\in B|S_x=S\cap x\}\in I^+$. Suppose $\alpha<\lambda$, $X, Y\in W_\alpha$ with $C\cap X$, $C\cap Y\in I^+$. Pick $x, y, z\in C\cap\{\hat{\alpha}\}$ such that $x\in X$, $y\in Y$ and $z\supset\{h(X), h(Y)\}$. Then $h(X)=h(W_\alpha(x))\in S_x\subseteq S$, $h(Y)=h(W_\alpha(y))\in S_y\subseteq S$, hence $\{h(X), h(Y)\}\subseteq S\cap z=S_z$ and so $z\in X\cap Y$. But then since W_α is disjoint $X=Y$, thus proving (λ, λ)-distributivity.

(c)$\to$(a) is trivial. $\square$

THEOREM 5.2. *If $\varkappa$ is completely λ-ineffable and μ is a cardinal such that $\varkappa\leqq\mu<\lambda$, hen $\varkappa$ is completely μ-ineffable.*

PROOF. Suppose I is a normal (λ, λ)-distributive ideal on $P_\varkappa\lambda$ and for each $x\in P_\varkappa\lambda$, $h(x)=x\cap\mu$. Then $h\colon P_\varkappa\lambda\to P_\varkappa\mu$ and it is easy to check that $h_*(I)=\{X\subseteq P_\varkappa\mu|h^{-1}(X)\in I\}$ is an ideal on $P_\varkappa\mu$. Using Theorem 5.1 we show that $h_*(I)$ is normal and (μ, μ)-distributive. Suppose $X\in h_*(I)^+$ and $\langle T_r|r\in X\rangle$ is a sequence such that $(\forall r\in X)(T_r\subseteq r)$. Then $h^{-1}(X)\in I^+$ and for each $x\in h^{-1}(X)$ let $S_x=T_{h(x)}$. By Theorem 5.1 we may find an $S\subseteq\lambda$ (indeed $S\subseteq\mu$) such that $B=\{x\in h^{-1}(X)|S_x=S\cap x\}\in I^+$. But then $h''B\in h_*(I)^+$ and for each $x\in B$, $T_{h(x)}=S_x=S\cap x=S\cap\mu\cap x=S\cap h(x)$. $\square$

In [15, Proposition 1], Menas showed that if I is normal and prime, then I is a weak q-point iff there exists a set $A\in I^*$ such that $(\forall(x, y)\in[A]^2)(|x|<|\varkappa\cap y|)$ iff $I^+\to(I^+)^2_2$. As in [11]] this latter partition relation is related to distributivity.

THEOREM 5.3. (cf $\lambda \geqq \varkappa$). *The following are equivalent:*

(a) *I is normal, (λ, λ)-distributive and a weak q-point.*

(b) *I is normal, (λ, λ)-distributive and $\{B \in I^+ | \forall (x, y) \in [B]^2,\ |x| < |\varkappa \cap y|\}$ is dense in I^+.*

(c) *Whenever $A \in I^+$, $n < \omega$ and $f \colon [A]^n \to \lambda$ is regressive, there is a $B \in P(A) \cap I^+$ such that $|f''[B]^n| = 1$.*

(d) *I is normal and $I^+ \to (I^+)^2_2$.*

PROOF. (a)$\to$(b). As in [15, Proposition 1], given $A \in I^+$ let $q \colon A \to P_\varkappa \lambda$ be any function such that $(\forall x \in A)(x \subseteq q(x))$ and $|\varkappa \cap q(x)| > |x|)$. By the weak q-point property we may find a $B \in P(A) \cap I^+$ such that for each $(x, y) \in [B]^2$, $q(x) \subset y$ and hence $|x| < |\varkappa \cap y|$.

The proof of (b)$\to$(c) is similar to that of Theorem 4.4.

(c)$\to$(d) is trivial.

(d)$\to$(a). Firstly the weak q-point property. As in [15, Proposition 1], if $A \in I^+$ and $\{A_x | x \in A\} \subseteq P(A) \cap I_{\varkappa\lambda}$, define $f \colon [A]^2 \to 2$ by $f(x, y) = 0$ iff $y \in A_x$. Clearly if $B \in P(A) \cap I^+$ is homogeneous for f then $f''[B]^2 = \{1\}$, and hence for each $x \in B$, $B \cap A_x \cap \hat{x} = \emptyset$.

Now suppose $A \in I^+$ and $\langle W_\alpha | \alpha < \lambda \rangle$ is a sequence of I-partitions of A, each of cardinality $\leqq \lambda$. By Lemma 1.3 we may assume that each W_α is disjoint. Let $h \colon \bigcup \{W_\alpha | \alpha < \lambda\} \to \lambda$ be injective, then by normality, $B = \{x \in A | \forall \alpha \in x,\ x \in \bigcup W_\alpha\} \in (I|A)^*$. Define $g \colon [B]^2 \to 2$ by $g(x, y) = 1$ iff $(\exists \alpha \in x)\ (W_\alpha(x) \neq W_\alpha(y))$ and if α is the least such ordinal then $h(W_\alpha(x)) > h(W_\alpha(y))$.

Suppose $C \in P(B) \cap I^+$ is homogeneous for g, say $g''[C]^2 = \{1\}$, $\beta < \lambda$ and for each $\alpha < \beta$ there is a set $X_\alpha \in W_\alpha$ such that $C - X_\alpha \in I$. By normality, $D = \{x \in C | \forall \alpha \in \beta \cap x,\ W_\alpha(x) = X_\alpha\} \in (I|C)^*$, and hence if X and Y are distinct members of W_β, say $h(X) < h(Y)$, such that $C \cap X \in I^+$ and $C \cap Y \in I^+$, then $f(x, y) = 0$ whenever $x \in D \cap X \cap \{\hat{\beta}\}$ and $y \in D \cap Y \cap \hat{x}$, a contradiction. A similar argument carries through when $g''[C]^2 = \{0\}$, and hence this proves (λ, λ)-distributivity. □

§6. Non seminormal ideals

To date we have restricted ourselves to considering mainly seminormal weakly lean ideals. In this section we briefly mention two results relating distributivity to partition relations which hold for ideals in general. Since I may not be weakly lean we first make the following definitions. Let $L = \{A \in I^+ | \forall B \in P(A) \cap I^+,\ |B| = |A|\}$, then clearly L is dense in I^+ and so I may be broken down into ideals which are lean in the following sense.

DEFINITION 6.1. For a cardinal $\eta \geqq \lambda$, I is said to be η-lean iff there is an $A \in I^*$ such that $|A| = \eta$ and $(\forall B \in I^+)(|B| \geqq \eta)$.

THEOREM 6.2. *Suppose I is η-lean, then the following are equivalent:*

(a) *I is weakly selective and $(\eta, 2)$-distributive.*

(b) *For each $n < \omega$ and $\mu < \varkappa$, $I^+ \to (I^+)^n_\mu$.*

PROOF. (a)$\to$(b). Firstly note that (by Theorem 3.8 and [5, Proposition 1.2]) since $\eta \geqq \lambda$ and I is $(\eta, 2)$-distributive, $\varkappa$ is weakly compact (and in particular strongly inaccessible).

By induction on n now; the case $n=1$ is immediate from $\varkappa$-completeness, hence suppose $\mu<\varkappa$, $A\in I^+$ and $f\colon [A]^{n+1}\to\mu$. Since I is η-lean we may assume that $|A|=\eta$ and for each $a=(x_1, x_2, \dots, x_n)\in[A]^n$ and $\varrho<\mu$ let $X_a^\varrho=\{y\in A\cap\hat{x}_n|f(x_1, x_2, \dots, x_n, y)=\varrho\}$. By $(\eta, 2)$-distributivity there is a $B\in P(A)\cap I^+$ such that for each $a\in[A]^n$ and $\varrho<\mu$, $B-X_a^\varrho\in I$ or $B\cap X_a^\varrho\in I$, and hence by $\varkappa$-completeness there is a unique function $g\colon [A]^n\to\mu$ such that for each $a\in[A]^n$, $B-X_a^{g(a)}\in I$. By inductive hypothesis there is a $\gamma<\mu$ and a $C\in P(B)\cap I^+$ such that $g''[C]^n=\{\gamma\}$. For each $x\in C$ let

$$A_x = \{y\in C\cap\hat{x}|\exists(x_1, x_2, \dots, x_{n-1})\in[C\cap P(x)-\{x\}]^{n-1}, f(x_1, x_2, \dots, x_{n-1}, x, y)\neq\gamma\}$$

then $A_x\in I$ (since $|P(x)|<\varkappa$ and I is $\varkappa$-complete), and hence by weak selectivity we may find an $E\in P(C)\cap I^+$ such that for each $(x, y)\in[E]^2$, $y\notin A_x$. Clearly $f''[E]^{n+1}=\{\gamma\}$.

(b)$\to$(a). Weak selectivity is proved as in Theorem 5.3 (d)$\to$(a).

Suppose $A\in I^+$ and $\langle W_\gamma|\gamma<\eta\rangle$ is a sequence of I-partitions of A, each of cardinality $\leqq 2$. By $\varkappa$-completeness we may assume that each W_γ is disjoint and since I is η-lean we may suppose that $|A|=\eta$. Hence reindex $\langle W_\gamma|\gamma<\eta\rangle$ as $\langle W_x|x\in A\rangle$, and for each $x\in A$ let $T_x=\{\bigcap\{X_y|y\in P(x)\cap A\}|\forall y\in P(x)\cap A,\ X_y\in W_y\}\cap I^+$. Straightforward arguments show that $\varkappa$ is weakly compact (and in particular strongly inaccessible), and hence by $\varkappa$-completeness, I is $(|P(x)|, 2)$-distributive. Thus T_x is a disjoint I-partition of A.

Define $h\colon [A]^3\to 2$ by $h(x, y, z)=0$ iff $y, z\in\bigcup T_x$ and $T_x(y)=T_x(z)$. If $B\in P(A)\cap I^+$ is homogeneous for h, then clearly we must have $h''[B]^3=\{0\}$, and hence for each $x\in B$ there is a $Y_x\in T_x$ such that $B-Y_x\in I$.

Finally, given $y\in A$, $W_y\geqq T_x$ whenever $x\in B\cap\hat{y}$, and so this proves $(\eta, 2)$-distributivity. □

Note that in proving the implication (b)$\to$(a) we have used $I^+\to(I^+)_2^3$ rather that $I^+\to(I^+)_2^2$. To obtain $(\eta, 2)$-distributivity from this latter partition relation we (seem to) need η many such partitions.

THEOREM 6.3. *Suppose I is η-lean, then the following are equivalent:*

(a) *I is weakly selective and $(\eta, 2)$-distributive.*

(b) *whenever $A\in I^+$ and $f_\gamma\colon [A]^2\to 2$ $(\gamma<\eta)$ is a family of functions, there is a $B\in P(A)\cap I^+$ and a function $t\colon\gamma\to I_{\varkappa\lambda}$ such that for each $\gamma<\eta$, $|f_\gamma''[B-t(\gamma)]^2|=1$.*

The proof is similar to that of [12, Theorem 4.7] and we leave the details to the reader.

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ON ALMOST NILPOTENT RINGS AND IDEALS

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1. Introduction

An associative ring A is said to be almost nilpotent if every nonzero ideal of A strictly contains some power of A (cf. [2]). This concept is a generalization of 'nilpotent' and it was shown in [2] that an almost nilpotent ring is either a prime ring or a nilpotent ring.

The purpose of this note is in the first instance, to characterize prime almost nilpotent rings. Some properties analogous to those of nilpotent ideals of a ring are being investigated in the rest of the paper.

All rings to be considered will be associative and $I \lhd A$ will mean 'I is an ideal of A'.

2. Prime almost nilpotent rings

For our first theorem we will need the following lemma:

LEMMA 1. *If A is an almost nilpotent ring then* $\bigcap_{m=k}^{\infty} A^m = 0$ *for all* $k \geqq 1$.

PROOF. *If* $\bigcap_{m=k}^{\infty} A^m \neq 0$ there is a positive integer n such that $A^n \subsetneqq \bigcap_{m=k}^{\infty} A^m$ and we may obviously choose $n \geqq k$. But then $A^n \subsetneqq A^n$, which is absurd.

THEOREM 1. *The following are equivalent for the ring R:*

(1) *R is a prime almost nilpotent ring.*

(2) *R is a semiprime ring with a proper essential almost nilpotent ideal P and R/P is nilpotent.*

(3) *R is a semiprime ring such that every proper homomorphic image of R is nilpotent.*

PROOF. (1)⇒(2). Let R be a prime almost nilpotent ring. By definition R cannot be a simple ring, so it must have a proper nonzero ideal P, and the rest is obvious.

(2)⇒(3). Let $0 \neq A \lhd R$. We want to show that R/A is nilpotent. If $A = P$ there is nothing to prove. So assume $A \neq P$. We consider three cases:

(a) $P \subseteqq A$. Then $\dfrac{R/P}{A/P} \simeq R/A$ is nilpotent since R/P is.

(b) $A \subseteqq P$. Then $\dfrac{R/A}{P/A} \cong R/P$ so that R/A is nilpotent since both R/P and P/A is.

(c) $X = P \cap A$ with $A \subsetneqq P$ and $P \subsetneqq A$. Now P/X is nilpotent since P is almost nilpotent, and $R/X/P/X \cong R/P$ is nilpotent so that R/X is nilpotent. Then $R/X/A/X \cong \cong R/A$ is nilpotent.

(3)⇒(1) For any two nonzero ideals I and J of R we have $R^k \subseteq I$ and $R^l \subseteq J$. So if $IJ=0$ then $R^{k+l}=0$, a contradiction. Hence R is prime. Furthermore, if X is any nonzero ideal of R then $R^p \subseteq X$, $p \in \mathbf{N}$ and if no power of R is strictly contained in X then $R^p = X$. But $R^n \subseteq P$ for certain $n \in \mathbf{N}$, so that by Lemma 1,

$$\bigcap_{m=k}^{\infty} (R^n)^m \subseteq \bigcap_{m=k}^{\infty} P^m = 0 \quad \text{for} \quad k \geqq 1.$$

This means that $X = \bigcap_{m=k}^{\infty} (R^p)^m = 0$, a contradiction which implies that R is almost nilpotent.

3. Almost nilpotent ideals of a ring

It is well known that the sum of two nilpotent ideals in a ring is a nilpotent ideal but that the sum of all nilpotent ideals need not be nilpotent (see Example 3 in [1]). We now investigate the almost nilpotent analogue of this property.

First of all, by the very example mentioned above, we already know that the sum of all almost nilpotent ideals need not be almost nilpotent. Furthermore it is easy to show that the sum of a prime almost nilpotent ideal and a nilpotent ideal of a ring R need not be almost nilpotent.

Let for instance

$$W = \left\{ \frac{2x}{2y-1} \middle| (2x, xy+1) = 1 \text{ and } x, y \in Z \right\}$$

(cf [1], p. 103), and consider the ring $R = W \oplus N$, where N is any nilpotent ring. It is well known that W is a prmie almost nilpotent ring and that the only nonzero ideals of W are of type $(2)^n$, $n = 1, 2, \ldots$, and $W = (2)$. So obviously R is neither prime nor nilpotent, so cannot be almost nilpotent.

For our next result let us assume that R is a subdirectly indecomposible ring, that is a ring in which the intersection of any two nonzero ideals always is nonzero (cf [3]). Clearly every prime ring is subdirectly indecomposible.

THEOREM 2. *If P and Q are two prime almost nilpotent ideals of a subdirectly indecomposable ring R, then $P+Q$ is almost nilpotent.*

PROOF. Let $0 \neq X \lhd P+Q$. Then $X_R^3 \neq 0$ for $X_R^3 = 0$ would imply that P has a nonzero nilpotent ideal $X_R \cap P$. Hence $0 \neq X_R^3 \cap P \subseteq X \cap P \lhd P$ so that $P^r \subset X$, $r \in N$. Analogously $Q^s \subset X$, $s \in N$, and so $(P+Q)^{r+s} \subseteq X$. The inclusion is strict for if no power of $(P+Q)^{r+s}$ is strictly contained in X then all powers of $(P+Q)^{r+s}$ equals X, so that $\bigcap_{n=k}^{\infty} [(P+Q)^{r+s}]^n \subseteq \bigcap_{n=k}^{\infty} (P+Q)^n$ for $k \geqq 1$. But $(P+Q)^p \subseteq P$ so that $\bigcap_{m=k}^{\infty} [(P+Q)^p]^m \subseteq \bigcap_{m=k}^{\infty} P^m = 0$ for any $k \geqq 1$. This obiously implies that $X=0$, a contradiction which proves the theorem.

Corollary. *Let P and Q be two either nilpotent or prime almost nilpotent ideals of a subdirectly indecomposable ring R. Then $P+Q$ is almost nilpotent if and only if $P+Q$ is either a nilpotent or a prime ring.*

To close our discussion, one last observation. A class M of rings is said to satisfy the extension property if $R/A \in M$ and $A \in M$ imply $R \in M$.

While we know that nilpotent rings satisfy this property we offer the following.

Theorem 3. *If R is a semiprime ring, R/A a nilpotent ring and A a prime ring which is a nonzero almost nilpotent ideal of R, then R is a prime almost nilpotent ring.*

Proof. $R^n \subseteqq A$, $n \in N$, so R^n is almost nilpotent. Let then $0 \neq X \lhd R$. Then $0 \neq X^n \lhd R^n$ and since R^n is almost nilpotent we have $R^{nt} \subset X^n \subseteqq X$, which proves that R is almost nilpotent.

Remark. One cannot expect R/A and A prime almost nilpotent rings to imply R prime almost nilpotent. In fact R almost nilpotent implies R/A nilpotent which immediately gives a contradiction.

Example. Consider the ring $R = W \oplus N$ mentioned earlier. Then $R/W \cong N$, so R/W and also W is almost nilpotent but R is not.

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THE INFLUENCE OF π-QUASINORMALITY OF SOME SUBGROUPS ON THE STRUCTURE OF A FINITE GROUP

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1. Introduction

Throughout this paper, only finite groups are considered and our notation is standard.

Ito, Buckley, Van der Waall and Asaad have proved the following theorems for a finite group G.

THEOREM (Ito [6]). *Suppose that G is of odd order and that every subgroup of G' of prime order is normal in G. Then G' is nilpotent.*

THEOREM (Buckley [2]). *Suppose that G is of odd order and that every subgroup of G of prime order is normal in G. Then G is supersolvable.*

THEOREM (Van der Waall [9]). *Suppose that every subgroup of G of prime order is normal in G. Then the following two assertions are equivalent:*
(1) *G is supersolvable.*
(2) *G is 2-nilpotent.*

THEOREM (Asaad [1]). *Suppose that every subgroup of G of prime order is quasinormal in G, and that every cyclic subgroup of G of order 4 is quasinormal in G. Then G is supersolvable.*

The object here is to improve these results.

2. Definitions and assumed results

DEFINITIONS. Subgroups H and K of G permute if $\langle H, K\rangle = HK = KH$. A subgroup of G is called π-quasinormal in G if it permutes with all Sylow subgroups of G.

The group G is said to have an ordered Sylow Tower property, that is to say there is a series $1 = G_0 < G_1 < G_2 < \ldots < G_n = G$ of normal subgroups of G such that for each $i = 1, 2, \ldots, n$, $G_i/G_{i-1} \cong$ to a Sylow p_i-subgroup of G where $p_1, p_2, \ldots, p_n$ are the distinct prime divisors of G and $p_1 > p_2 > \ldots > p_n$.

We now list for an easy reference some known results which are frequently used later:

(2.1) [7]. *A π-quasinormal subgroup of G is subnormal in G.*
(2.2) [7]. *If $H \subseteq K \subseteq G$ and H is π-quasinormal in G, then H is π-quasinormal in K.*
(2.3) [7]. *If H is π-quasinormal Hall subgroups of G, then $H \lhd G$.*

PROOF. By (2.1), H is subnormal subgroup of G. Hence H is subnormal Hall subgroup of G. This implies that $H \lhd G$.

(2.4) [4]. *If A is a p'-group of automorphisms of the abelian p-group P which acts trivially on $\Omega_1(P)$, then $A=1$.*

(2.5) [5]. *If H is a normal Hall subgroup of G, then there exists a subgroup K of G such that $G/H \cong K$.*

(2.6) [8, exercise, 7.2.22, p. 159]. *If G/H and G/K are supersolvable, then $G/H \cap K$ is supersolvable.*

Proof. $G/H \cap K \cong G/H \times G/K$.

(2.7) [8]. *If $H \lhd G$, then $\Phi(H) \subseteqq \Phi(G)$.*

(2.8) [5]. *G is supersolvable iff $G/\Phi(G)$ is supersolvable.*

(2.9) [4]. *If $G=\Phi(G)H$ for some subgroup H of G, then $G=H$.*

(2.10) (Burnside Basis Theorem [8]). *If G is a finite p-group, where p is a prime, $|G/\Phi(G)|=p^n$,*

$$G/\Phi(G) = \langle x_1\Phi(G), x_2\Phi(G), \dots, x_n\Phi(G)\rangle,$$

then $G=\langle x_1, x_2, \dots, x_n\rangle$.

(2.11) [5]. *Suppose that G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent. Then G is a group which is not nilpotent but whose proper subgroups are all nilpotent.*

Proof. See [5, IV, 5.4, p. 434]

(2.12) [5]. *Suppose that G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then*

(i) *G has a normal Sylow p-subgroup P for some prime p and $G/P \cong Q$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.*

(ii) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*

(iii) *If P is non-abelian and $p \neq 2$, then the exponent of P is p.*

(iv) *If P is non-abelian and $p=2$, then the exponent if P is 4.*

(v) *If P is abelian, then P is of exponent p.*

Proof. (i), (ii), (iii) and (iv): see [5, III, 5.2, p. 281].

(v) We argue that $\Omega_1(P)=P$. If not, $\Omega_1(P)Q$ is a proper subgroup of G. Clearly, $\Omega_1(P)Q=\Omega_1(P)XQ$. Now applying (2.4), it follows that $G=P\times Q$, a contradiction. Hence $\Omega_1(P)=P$.

(2.13) [8]. *If G is supersolvable, then.*

(i) *G possesses an ordered Sylow tower.*

(ii) *G is p_n-nilpotent, where p_n is the smallest prime dividing $|G|$.*

(iii) *G has a normal Sylow p_1-subgroup, where p_1 is the largest prime dividing $|G|$.*

Proof. (i) See [8, 7.2.19, p. 158].

(ii) and (iii) are immediate consequences of (i).

(2.14) [7]. *If $N \subseteqq H \subseteqq G$ and N is normal in G, then H is π-quasinormal in G if and only if H/N is π-quasinormal in G/N.*

(2.15) [3]. *Suppose that G is a group which is not supersolvable but whose proper subgroups are all supersolvable. Then*

(i) *G has a normal Sylow p-subgroup P for some prime p.*

(ii) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*

(iii) *If* $p\neq 2$, *then the exponent of* P *is* p.
(iv) *If* P *is non-abelian and* $p=2$, *then* P *is of exponent* 4.
(v) *If* P *is abelian, then the exponent of* P *is* p.

PROOF. (i), (ii), (iii) and (iv): See [3].

(v) If $p\neq 2$, then the exponent of P is p by (iii). We may, therefore, assume that $p=2$.

By Hilfssatz C [3 (b), p. 198—199], G is 2-nilpotent or G is a group which is not nilpotent but all of whose proper subgroups are nilpotent. If G is 2-nilpotent, then there exists a normal 2′-Hall subgroup K of G such that $G=PK$ and $P\cap K=1$. Since $P\lhd G$ and $K\lhd G$, it follows that $G=P\times K$. Therefore, G is supersolvable. This is impossible as G is not supersolvable. Thus G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then $\Omega_1(P)=P$ by (2.12, v). Therefore, the exponent of P is 2.

3. Generalized results

In this section, we prove the following results:

THEOREM 3.1. *Suppose that* H *is a normal* p*-subgroup of* G, *and that* G/H *is supersolvable. Suppose further that every subgroup of* H *of order* p *is* π*-quasinormal in* G *and that one of the following conditions holds:*
(i) H *is abelian.*
(ii) $p\neq 2$.
(iii) $p=2$, *and every cyclic subgroup of* H *of order* 4 *is* π*-quasinormal in* G.
Then G *is supersolvable.*

PROOF. Suppose that the theorem is false and let G be a counter-example of smallest order. Clearly, the hypotheses are inherited by all proper subgroups of G. Thus G is a group which is not supersolvable but whose proper subgroups are all supersolvable. Then by (2.15, i), there exists $P\lhd G$, where P is a Sylow subgroup of G. By (2.5), there exists a subgroup K of G such that $G/P\cong K$. Clearly, K is supersolvable. If $(|P|, |H|)=1$, then by (2.6), $G=G/P\cap H\underset{\sim}{\subseteq} G/P\times G/H$. Since $G/P\times G/H$ is supersolvable, it follows that G is supersolvable, a contradiction. Thus $H\subseteq P$. By (2.15, ii), $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. Now it follows that either $H\Phi(P)=\Phi(P)$ or $H\Phi(P)=P$. If $H\Phi(P)=\Phi(P)$, then $H\subseteq\Phi(P)$ and so $G/\Phi(P)$ is supersolvable. By (2.7), $\Phi(P)\subseteq\Phi(G)$. Hence $G/\Phi(G)$ is supersolvable and so G is supersolvable by (2.8), a contradiction. If $H\Phi(P)=P$, then $H=P$ by (2.9) ,and so H is a Sylow p-subgroup of G.

Suppose first that H is abelian. Let A be a subgroup of H of order p. Let Q be a Sylow q-subgroup of G, where $q\neq p$. By hypothesis, AQ is a subgroup of G. By (2.2), A is π-quasinormal in AQ. Hence by (2.3), $A\lhd AQ$ and so $Q\subseteq N_G(A)$. Thus $O^p(G)\subseteq N_G(A)$, where $O^p(G)$ is the subgroup of G generated by all p'-elements of G. Since $A\lhd H$ and $O^p(G)\subseteq N_G(A)$, it follows that $A\lhd G$. By (1.15), iii) and (2.15, v), $\Phi(H)=1$. By (1.15, ii), $A=H$. Since G/H is supersolvable and $|H|=$prime, it follows that G is supersolvable, a contradiction.

Suppose next that H is nonabelian. Set $|H/\Phi(H)|=p^n$ and $H/\Phi(H)=\langle x_1\Phi(H), x_2\Phi(H), \ldots, x_n\Phi(H)\rangle$. Then by (2.10), $H=\langle x_1, x_2, \ldots, x_n\rangle$. By (2.15, iv) and (2.15, v), the exponent of H is p or 4. Then $|\langle x_i\rangle|=p$ or 4 for all $i=1, 2, \ldots, n$. We argue that $\langle x_i\rangle$ are nonnormal subgroups of G for all $i=1, 2, \ldots, n$. If not, there exists $\langle x_J\rangle \lhd G$ for some $1\leqq J\leqq n$. Now it follows from (2.15, ii) that $\langle x_J\rangle \Phi(H)=H$ and so $\langle x_J\rangle=H$. This is impossible as H is nonabelian. Thus $\langle x_i\rangle$ are all nonnormal subgroups of G for all $i=1, 2, \ldots, n$. Let Q be a Sylow q-subgroup of G, where $q\neq p$. By hypothesis, $\langle x_i\rangle Q$ is a subgroup of G. Now it follows from (2.2) and (2.3), that $\langle x_i\rangle \lhd \langle x_i\rangle Q$ and so $Q\subseteqq N_G(\langle x_i\rangle)$. Thus $O^p(G)\subseteqq N_G(\langle x_i\rangle)$. Since $N_G(\langle x_i\rangle)\subset G$, it follows that $O^p(G)\subset G$. Since G/H is supersolvable and $G/O^p(G)$ is a p-group, it follows that $G/H\cap O^p(G)$ is supersolvable. Clearly, $H\cap O^p(G)\subset H$. Then by (2.15, ii), $H\cap O^p(G)\subseteqq \Phi(H)$. Thus $G/\Phi(H)$ is supersolvable and so G is supersolvable, a final contradiction.

THEOREM 3.2. *Let p be the smallest prime dividing $|G|$. Suppose that every subgroup of order p is π-quasinormal in G and that one of the following conditions holds:*

(i) *The Sylow p-subgroups of G are abelian.*

(ii) $p\neq 2$.

(iii) $p=2$, *and every cyclic subgroup of G of order 4 is π-quasinormal in G.*

Then G is p-nilpotent.

PROOF. Suppose that the theorem is false and let G be a counter-example of smallest order. The hypotheses are inherited by all proper subgroups of G. Thus G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent. Now (2.11), implies that G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then by (2.12), G has a normal Sylow p-subgroup P and $G/P\cong Q$, where Q is a nonnormal cyclic Sylow q-subgroup of G. We consider the following three cases:

Case 1. P is abelian. Let T be a subgroup of P of order p. By hypothesis, T is π-quasinormal in G. Then TQ is a subgroup of G. We argue that $T\lhd G$. If not, TQ is a proper subgroup of G. Then TQ is nilpotent subgroup of G, and so $T\lhd TQ$. Also $T\lhd P$ as P is abelian. Thus $T\lhd G$, But by (2.12, iii), P is a minimal normal subgroup of G, and so $T=P$, Since $|\mathrm{Aut}\,(T)|=p-1$, it follows that $q/p-1$. This is a contradiction as $q>p$.

Case 2: $p\neq 2$ and P is nonabelian. Set $|P/\Phi(P)|=p^n$ and $P/\Phi(P)= =\langle x_1\Phi(P), \ldots, x_n\Phi(P)\rangle$. Then by (2.10), $P=\langle x_1, x_2, \ldots, x_n\rangle$. By (2.15, iii), P has exponent p, so $|\langle x_i\rangle|=p$ for all $i=1, 2, \ldots, n$. We argue that $\langle x_i\rangle$ are nonnormal subgroups of G for all $i=1, 2, \ldots, n$. If not, there exists $\langle x_J\rangle$, for some $1\leqq J\leqq n$, such that $\langle x_J\rangle$ is normal subgroup of G. By (2.12, ii), $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, and so $\langle x_J\rangle\Phi(P)=P$. By (2.15, ii), $P=\langle x_J\rangle]$ and so $|P|= =|\langle x_J\rangle|$. By (2.12, iii), P has exponent p, and so $|P|=|\langle x_J\rangle|=p$. Since $|\mathrm{Aut}\,(P)|= =p-1$, it follows that $q/p-1$. This is a contradiction as $q>p$. Thus $\langle x_i\rangle$ are nonnormal subgroups of G for all $i=1, 2, \ldots, n$. By hypothesis, $\langle x_i\rangle$ are π-quasinormal subgroups of G for all $i=1, 2, \ldots, n$. Then $\langle x_i\rangle Q$ are subgroups of G for all $i= =1, 2, \ldots, n$. In fact $\langle x_i\rangle Q$ are proper nilpotent subgroups of G, and so $\langle x_i\rangle Q= =\langle x_i\rangle\times Q$. Therefore, $P\subseteqq C_G(Q)$ and that is a contradiction as Q is nonnormal Sylow q-subgroup of G.

Case 3: $p=2$ *and P is nonabelian.* Set $|P/\Phi(P)|=2^n$ and $P/\Phi(P)=\langle x_1\Phi(P), \ldots, \ldots, x_n\Phi(P)\rangle$ By (2.10), $P=\langle x_1, x_2, \ldots, x_n\rangle$. By (2.15, iv), P has exponent 4, and so

$|\langle x_i\rangle|=2$ or 4 for all $i=1, 2, \ldots, n$. By hypothesis, $\langle x_i\rangle$ are π-quasinormal in G, and so $\langle x_i\rangle Q$ are subgroups of G for all $i=1, 2, ;\ldots, n$. Since P is nonabelian, it follows that $\langle x_i\rangle Q$ are proper subgroups of G for all $i=1, 2, \ldots, n$. Then $\langle x_i\rangle Q$ are nilpotent subgroups of G, and so $\langle x_i\rangle Q=\langle x\rangle\times Q$, for all $i=1, 2, \ldots, n$. Now it follows that $P\subseteqq C_G(Q)$, a contradiction as Q is not normal Sylow q-subgroup of G. This proves case 3, and the Theorem.

THEOREM 3.3. *Set $\pi(G)=\{p_1, p_2, \ldots, p_n\}$, where $p_1>p_2>\ldots>p_n$. Suppose that every subgroup of G of prime order p_i, where $i=2, 3, \ldots, n$, is π-quasinormal in G. Suppose further that one of the following conditions holds:*

(i) $p_n\neq 2$.

(ii) $p_n=2$, *and the Sylow* 2-*subgroups of G are abelian.*

(iii) $p_n=2$, *and every cyclic subgroup of G of order* 4 *is π-quasinormal in G.*

Then G possesses an ordered Sylow tower.

PROOF. By Theorem 3.2, G is p_n-nilpotent. Then $G=P_nK$, where P_n is a Sylow p_n-subgroup of G, and K is a normal p_n'-Hall subgroup of G. By induction on $|G|$, K possesses an ordered Sylow tower. Therefore, G possesses an ordered Sylow tower. This completes the proof.

THEOREM 3.4. *Let H be a proper normal subgroup of G. Suppose that G/H is supersolvable and that every subgroup of H of prime order is π-quasinormal in G. Suppose further that one of the following conditions holds:*

(i) $2\nmid |H|$.

(ii) $2|\ |\mathrm{H}|$ *and the Sylow* 2-*subgroups of are abelian.*

(iii) $2|\ |H|$ *and every cyclic subgroup of H of order* 4 *is π-quasinormal in G.*

Then G is supersolvable.

PROOF. If H is a group of prime power order, then G is supersolvable by Theorem 3.1. Now we consider the case where $|H|$ is divisible by at least two distinct primes.

By Theorem 3.3, H possesses an ordered Sylow tower- Then by (2.13), $P\lhd H$, where P is a Sylow p-subgroup of H and p is the largest prime dividing $|H|$. Since P char $H\lhd G$, we have $P\lhd G$. By (2.14), every subgroup of H/P of of prime order is π-quasinormal in G/P. Also, $G/P/H/P\cong G/H$ is supersolvable. Now by induction on $|G|$, G/P, is supersolvable. Thus by Theorem 3.1, G is supersolvable. This completes the proof.

THEOREM 3.5. *Under the assumptions of Theorem* 3.3, *G possesses an ordered Sylow tower and G/P_1 is supersolvable, where P_1 is a Sylow p_1-subgroup of G, and p_1 is the largest prime dividing $|G|$.*

PROOF. By Theorem 3.3, G possessses an ordered Sylow tower. Then by (2.13, iii), $P_1\lhd G$, where P_1 is a Sylow p_1-subgroup of G, and p_1 is the largest prime dividing $|G|$. By (2.5), $G=P_1K$, where K is a p_1'-Hall subgroup of G. By Theorem 3.3, K possesses an ordered Sylow tower. Then by (1.13, ii), $K=QL$, where Q is a Sylow q-subgroup of K, q is the smallest prime dividing $|K|$, and L is a normal q'-Hall subgroup of K. Clearly, $K/L\cong Q$. It follows now from Theorem 3.4, that K is supersolvable. Therefore, $G/P_1\cong K$ is supersolvable. This completes the proof.

As an immediate consequence of Theorem 3.1 and 3.5, we have:

COROLLARY 3.6. *If G is a finite group of odd order and all subgroups of G of prime order are π-quasinormal in G, then G is supersolvable.*

COROLLARY 3.7 (Asaad [1]). *Suppose that every subgroup of prime order is quasinormal in G, and that every cyclic subgroup of order 4, is quasinormal in G. Then G is supersolvable.*

COROLLARY 3.8. *Let p_n be the smallest prime dividing $|G|$. Suppose that every subgroup of G of prime order is π-quasinormal in G, then the following two conditions are equivalent:*

(i) *G is supersolvable.*

(ii) *G is p_n-nilpotent.*

PROOF. (1)$\to$(2). Suppose that G is supersolvable. Then by (2.13, i), G possesses an ordered Sylow tower, and by (2.13, ii), G is p_n-nilpotent.

(2)$\to$(1). Let $G=P_nK$, where P_n is a Sylow p_n-subgroup of G, p_n is the smallest prime dividing $|G|$, and K is a normal p_n'-Hall subgroup of G. Clearly, $G/K\cong P_n$. Now applying Theorem 3.4, it follows that G is supersolvable.

COROLLARY 3.9. *Suppose that $G'\subseteqq H$, where G' is the commutator subgroup of G and H is a subgroup of G, and that every subgroup of H of prime order is π-quasinormal in G. Suppose further that one of the following conditions holds:*

(i) $2\nmid |H|$.

(ii) $2|\ |H|$ *and the Sylow 2-subgroups of H are abelian.*

(iii) $2|\ |H|$ *and every cyclic subgroup of H of order 4, is π-quasinormal in G.*

Then G is supersolvable.

PROOF. If $H=G$, then G is supersolvable by Theorems 3.5 and 3.1. Now we consider the case where $H\subset G$. Clearly, G/H is abelian. Applying Theorem 3.4, it follows that G is supersolvable.

As an immediate consequence of Corollary 3.8, we have:

ITO'S THEOREM [5, 177, 5.3 (b), p. 282; see also 6]. *Suppose that G is a group of odd order and that every subgroup of G' of prime order is normal in G. Then G' is nilpotent.*

THEOREM 3.10. *Suppose that $p\geqq q$ for every prime q dividing $|G|$, $O_p(G)=1$, and every subgroup of prime order $q\neq p$ is π-quasinormal in G. Suppose further that one of the following conditions holds:*

(i) $2\nmid |G|$.

(ii) $2|\ |G|$ *and the Sylow 2-subgroups of G are abelian.*

(iii) $2|\ |G|$ *and every cyclic subgroup of G of order 4 is π-quasinormal in G.*

Then G is supersolvable p'-group.

PROOF. We argue that G is a p'-group. If not, p dividing $|G|$. Now Theorem 3.3, implies that G possesses an ordered Sylow tower, and so $P\lhd G$, where P is a Sylow p-subgroup of G and p is the largest prime dividing $|G|$. Since $O_p(G)=1$, we have $P=1$. Thus G is a p'-group. Theorem 3.5, implies that G/P_1 is supersolvable, where P_1 is a Sylow P_1-subgroup of G and p_1 is the largest prime dividing $|G|$. Thus by Theorem 3.1, G is supersolvable. This completes the proof.

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REPRESENTING COMPLETELY BOUNDED MULTILINEAR OPERATORS

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In an important recent article [1] Christensen and Sinclair introduced the notion of a completely bounded multilinear operator $\Phi: A^k \to L(H)$, where A is a C^*-algebra and $L(H)$ the space of bounded linear operators on a Hilbert space H. They showed that Φ is completely bounded if, and only if, Φ is representable in the sense that there are Hilbert spaces $H_1, \ldots, H_k$, *-representations $\theta_j: A \to L(H_j)$ for $j=1, \ldots, k$, and bounded linear operators $V_j: H_{j+1} \to H_j$ (notation: $V_j \in L(H_{j+1}, H_j)$) for $j=0, \ldots, k$, where $H_0 = H_{k+1} = H$, such that

$$\Phi(a_1, \ldots, a_k) = V_0 \theta_1(a_1) V_1 \theta_2(a_2) V_2 \ldots \theta_k(a_k) V_k \tag{1}$$

for all $a_1, \ldots, a_k \in A$. Moreover, the so-called completely bounded norm $\|\Phi\|_{cb}$ equals the representation norm $\|\Phi\|_{rep}$ defined as the infinum of $\|V_0\| \cdot \|V_1\| \ldots \|V_k\|$ over all representations of Φ in the form (1) (see [1, Theorem 5.2]).

This note contains an elementary proof of the fact, arrived at in [3] via a different route, that a representable Φ always has also a representation of a simpler type, namely,

$$\Phi(a_i, \ldots, a_k) = V_0' \theta_1' (a_1) \theta_2'(a_2) \ldots \theta_{k-1}'(a_{k-1}) \theta_k'(a_k) V_k' \tag{2}$$

where all the *-representations θ_j' act on the same Hilbert space K, and $V_0' \in L(K, H)$, $V_k' \in L(H, K)$. Moreover, if in (1) the operators $V_0, \ldots, V_k$ are so normalized (by multiplying by suitable constants) that $\|V_j\| \leqq 1$ for $j=1, \ldots, k-1$, then we may take $\|V_0'\| = \|V_0\|$ and $\|V_k'\| = \|V_k\|$ in (2). This shows that $\|\Phi\|_{rep}$ equals the infimum of $\|V_0'\| \cdot \|V_k'\|$ over the representations of Φ in the form (2).

The following lemma is the key ingredient of the proof.

LEMMA. *Let H_j for $j=0, 1, 2, 3$ be Hilbert spaces and $V_j \in L(H_{j+1}, H_j)$ for $j=0, 1, 2$, with $\|V_1\| \leqq 1$. Let A_j for $i=0, \ldots, n$ be C^*-algebras, and let $\theta_0: A_0 \to L(H_1)$ and $\vartheta_i: A_i \to L(H_2)$ for $i=1, \ldots, n$ be *-representations. Then there exist a Hilbert space K, bounded linear operators $V_0': K \to H_0$, $V_2': H_3 \to K$, and *-representations $\theta_i': A_i \to L(K)$ for $i=0, \ldots, n$ such that $\|V_0'\| = \|V_0\|$, $\|V_2'\| = \|V_2\|$, and*

$$V_0 \theta_0(a_0) V_1 \theta_1(a_1) \ldots \theta_n(a_n) V_2 = V_0' \theta_0'(a_0) \theta_1'(a_1) \ldots \theta_n'(a_n) V_2'$$

for all $a_i \in A_i$, $i=0, \ldots, n$.

PROOF. Denote $H = H_1 \oplus H_2$ for short and define $\tilde{V}_1 \in L(H)$ by the formula $\tilde{V}_1(\xi_1, \xi_2) = (V_1 \xi_2, 0)$. Borrowing a trick from the proof of [2, Theorem 2.4] (see also [7, p. 638]) we note that since $\|\tilde{V}_1\| \leqq 1$, $\tilde{V}_1$ has a unitary dilation, i.e., there is a Hilbert space $K \supset H$ such that for some unitary operator $U: K \to K$ $\tilde{V}_1 = P_H U | H$

where P_H is the orthogonal projection of K onto H (see [6, Problem 177] or [5]). Write $K=H_1\oplus H_2\oplus H^\perp$. Define $V_2' : H_3\to K$ by the formula $V_2'(\xi)=(0, V_2\xi, 0)$, write $\theta_i'(a_i)=0\oplus\theta_i(a_i)\oplus 0$ for $i=1, \dots, n$, $\theta_0'(a_0)=U^*(\theta_0(a_0)\oplus 0\oplus 0)U$, and finally denote $V_0'=V_0P_{H_1}U\colon K\to H_0$. Then

$$V_0'\theta_0'(a_0)\theta_1'(a_1)\dots\theta_n'(a_n)V_2'\xi =$$

$$= [V_0P_{H_1}UU^*(\theta_0(a_0)\oplus 0\oplus 0)P_{H_1}U(0\oplus \prod_{i=1}^{n}\theta_i(a_i)\oplus 0)](0, V_2\xi, 0) =$$

$$= V_0\theta_0(a_0)P_{H_1}U(0, (\prod_{i=1}^{n}\theta_i(a_i))V_2\xi, 0) = V_0\theta_0(a_0)P_{H_1}\tilde{V}_1(0, (\prod_{i=1}^{n}\theta_i(a_i))V_2\xi) =$$

$$= V_0\theta_0(a_0)V_1(\prod_{i=1}^{n}\theta_i(a_i))V_2\xi$$

for all $\xi\in H_3$. □

We formulate the main result in slightly greater generality than the situation described at the beginning would require.

THEOREM. *Let $H_0, H_1, \dots, H_{k+1}$ be Hilbert spaces and $A_1, \dots, A_k$ C^*-algebras. Suppose that $V_j\in L(H_{j+1}, H_j)$ for $j=0, \dots, k$ and $\|V_j\|\leq 1$ for $j=1, \dots, k-1$. Let $\theta_j : A_j\to L(H_j)$ be *-representations for $j=1, \dots, k$. Then there exist a Hilbert space K, two operators $V_0'\in L(K, H_0)$ and $V_k'\in L(H_{k+1}, K)$ with $\|V_j'\|=\|V_j\|$ for $j=0$, k, and for each $j=1, \dots, k$ a *-representation $\theta_j' : A_j\to L(K)$ such that*

$$V_0\theta_1(a_1)V_1\theta_2(a_2)V_2\dots\theta_k(a_k)V_k = V_0'(\prod_{j=1}^{k}\theta_j'(a_j))V_k' \tag{3}$$

for all $a_i\in A_i$, $i=1, \dots, k$.

PROOF. The proof is by induction on k. The claim is trivial for $k=1$. Suppose the theorem is true for $k-1$. Then the left-hand side of (3) equals $V_0\theta_1(a_1)V_1''\theta_2''(a_2)\dots$ $\dots\theta_k''(a_k)V_k''$, where for some Hilbert space K', $V_1''\in L(K', H_1)$, $\|V_1''\|=\|V_1\|(\leq 1)$, $V_k''\in L(H_k, K')$, $\|V_k''\|=\|V_k\|$, and $\theta_j : A_j\to L(K')$ is a *-representation for each $j=2, \dots, k$. Applying the Lemma we see that this has the type of form indicated by the right-hand side. □

REMARK. (a) After completing the above arguments the author found out from E. G. Effros that [2] also contains an inductive dilation proof which shows that in the Christensen—Sinclair representation the *-representations θ_j can be replaced by ones acting on the same Hilbert space, and the bridging maps V_j, $j=1, \dots, k-1$ can be removed. Our proof is more elementary in the sense that it does not depend on the existence of the square root of a positive operator. Indeed, in [9, Proposition 3.1] a square root free proof of the type of dilation theorem we used above is given.

(b) Paulsen and Smith [8] have proved a representation theorem for completely contractive multilinear operators on the product of subspaces of C^*-algebras, analogous to (1) with $\|V_j\|\leq 1$ for $j=1, \dots, k-1$, V_k an isometry, and V_0 the adjoint of an isometry in $L(H_0, H_1)$ [8, Theorem 3.2]. The above proofs (and the argument in [2] referred to above) show that there (and in [8, Theorem 2.9 and Corollary 2.10]) the appropriate bridging maps can also be removed.

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A DECOMPOSITION OF CONTINUITY

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In 1922 Blumberg [1] introduced the notion of a real valued function on Euclidean space being densely approached at a point in its domain. Continuous functions satisfy this condition at each point of their domains. This concept was generalized by Ptàk [7] in 1958 who used the term nearly continuous, and by Husain [3] in 1966 under the name of almost continu ty. More recently, Mashhour et al. [5] have called this property of functions between arbitrary topological spaces precontinuity.

In this paper we define a new property of functions between topological spaces which is the dual of Blumberg's original notion, in the sense that together they are equivalent to continuity. Thus we provide a new decomposition of continuity in Theorem 4 (iv) which is of some historical interest.

In a recent paper [10], Tong introduced the notion of an $\mathcal{A}$-set in a topological space and the concept of $\mathcal{A}$-continuity of functions between topological spaces. This enabled him to produce a new decomposition of continuity. In this paper we improve Tong's decomposition result and provide a decomposition of $\mathcal{A}$ -continuity.

Let S be a subset of a topological space $(X, \mathcal{T})$. We denote the closure of S and the interior of S with respect to $\mathcal{T}$ by $\mathcal{T}$ cl S and $\mathcal{T}$ int S respectively, often suppressing the $\mathcal{T}$ when there is no possibility of confusion.

DEFINITION 1. A subset S of $(X, \mathcal{T})$ is called
(i) an α-set if $S \subset \mathcal{T}\text{ int}\,(\mathcal{T}\text{ cl}\,(\mathcal{T}\text{ int } S))$,
(ii) a semiopen set if $S \subset \mathcal{T}\text{ cl}\,(\mathcal{T}\text{ int } S)$,
(iii) a preopen set if $S \subset \mathcal{T}\text{ int}\,(\mathcal{T}\text{ cl } S)$,
(iv) an $\mathcal{A}$-set if $S = U \cap F$ where U is open and F is regular closed,
(v) locally closed if $S = U \cap F$ where U is open and F is closed.

Recall that S is regular closed in $(X, \mathcal{T})$ if $S = \mathcal{T}\text{ cl}\,(\mathcal{T}\text{ int } S)$. We shall denote the collections of regular closed, locally closed, preopen and semiopen subsets of $(X, \mathcal{T})$ by $RC(X, \mathcal{T})$, $LC(X, \mathcal{T})$, $PO(X, \mathcal{T})$ and $SO(X, \mathcal{T})$ respectively. The collection of $\mathcal{A}$-sets in $(X, \mathcal{T})$ will be denoted $\mathcal{A}(X, \mathcal{T})$. Following the notation of Njåstad [6], $\mathcal{T}^\alpha$ will denote the collection of all α-sets in $(X, \mathcal{T})$.

The notions in Definition 1 were introduced by Njåstad [6], Levine [4], Mashhour et al. [5], Tong [10] and Bourbaki [2] respectively. Stone [9] used the term FG for a locally closed subset. We note that a subset S of $(X, \mathcal{T})$ is locally closed iff $S = U \cap \text{cl } S$ for some open set U ([2], I. 3.3, Proposition 5).

Corresponding to the five concepts of generalized open set in Definition 1, we have five variations of continuity.

DEFINITION 2. A function $f\colon X \to Y$ is called α-continuous (semicontinuous, precontinuous, $\mathcal{A}$-continuous, LC-continuous respectively) if the inverse image under f

of each open set in Y is an α-set (semiopen, preopen, $\mathscr{A}$-set, locally closed respectively) in X.

Njåstad [6] introduced α-continuity, Levine [4] semicontinuity and Tong [10] $\mathscr{A}$-continuity, while LC-continuity seems to be a new notion. It is clear that $\mathscr{A}$-continuity implies LC-continuity. We now provide an example to distinguish these concepts.

EXAMPLE 1. Let $(X, \mathscr{T})$ be the set N of positive integers with the cofinite topology. Define the function $f: X \to X$ by $f(1)=1$ and $f(x)=2$ for all $x \neq 1$. Then $V = X - \{2\}$ is open and $f^{-1}(V) = \{1\}$ which is (locally) closed but not an $\mathscr{A}$-set. Note that the only regular closed subsets of $(X, \mathscr{T})$ are $\emptyset$ and X. For any subset V of X, $f^{-1}(V)$ is $\{1\}$, $X - \{1\}$, $\emptyset$ or X, and these are all locally closed subsets of X. Hence f is LC-continuous but not $\mathscr{A}$-continuous.

Our first two results improve Theorems 3.1 and 3.2 of Tong [10].

THEOREM 1. *Let S be a subset of a topological space $(X, \mathscr{T})$. Then S is an $\mathscr{A}$-set if and only if S is semiopen and locally closed.*

PROOF. Let $S \in \mathscr{A}(X, \mathscr{T})$, so $S = U \cap F$ where $U \in \mathscr{T}$ and $F \in RC(X, \mathscr{T})$. Clearly S is locally closed. Now $\operatorname{int} S = U \cap \operatorname{int} F$, so that $S = U \cap \operatorname{cl}(\operatorname{int} F) \subset \operatorname{cl}(U \cap \operatorname{int} F) = \operatorname{cl}(\operatorname{int} S)$, and hence S is semiopen.

Conversely, let S be semiopen and locally closed, so that $S \subset \operatorname{cl}(\operatorname{int} S)$ and $S = U \cap \operatorname{cl} S$ where U is open. Then $\operatorname{cl} S = \operatorname{cl}(\operatorname{int} S)$ and so is regular closed. Hence S is an $\mathscr{A}$-set.

THEOREM 2. *For a subset S of a topological space $(X, \mathscr{T})$ the following are equivalent.*

(1) *S is open.*
(2) *S is an α-set and locally closed.*
(3) *S is preopen and locally closed.*

PROOF. (1) implies (2) and (2) implies (3) are obvious. (3) implies (1): Let S be preopen and locally closed, so that $S \subset \operatorname{int}(\operatorname{cl} S)$ and $S = U \cap \operatorname{cl} S$. Then $S \subset U \cap \operatorname{int}(\operatorname{cl} S) = \operatorname{int}(U \cap \operatorname{cl} S) = \operatorname{int} S$, hence S is open.

THEOREM 3. *For a topological space $(X, \mathscr{T})$ the following are equivalent.*

(1) $\mathscr{A}(X, \mathscr{T}) = \mathscr{T}$.
(2) *$\mathscr{A}(X, \mathscr{T})$ is a topology on X.*
(3) *The intersection of any two $\mathscr{A}$-sets in X is an $\mathscr{A}$-set.*
(4) *$SO(X, \mathscr{T})$ is a topology on X.*
(4) *$(X, \mathscr{T})$ is extremally disconnected.*

PROOF. (1) implies (2) and (2) implies (3) are clear.

(3) implies (4): Let $S_1, S_2 \in SO(X, \mathscr{T})$. We wish to show $S_1 \cap S_2 \in SO(X, \mathscr{T})$. Suppose there is a point $x \in S_1 \cap S_2$ such that $x \notin \operatorname{cl}(\operatorname{int}(S_1 \cap S_2))$. So there is an open neighbourhood U of x such that $U \cap \operatorname{int} S_1 \cap \operatorname{int} S_2 = \emptyset$. Thus $U \cap \operatorname{cl} S_1 \cap \operatorname{int} S_2 = \emptyset$, and hence we have $U \cap \operatorname{int}(\operatorname{cl} S_1) \cap \operatorname{cl} S_2 = \emptyset$. Therefore $U \cap \operatorname{int}(\operatorname{cl} S_1 \cap \operatorname{cl} S_2) = \emptyset$, so that $x \notin \operatorname{cl}(\operatorname{int}(\operatorname{cl} S_1 \cap \operatorname{cl} S_2))$. But, on the other hand we have $\operatorname{cl} S_1, \operatorname{cl} S_2 \in RC(X, \mathscr{T})$, so that $\operatorname{cl} S_1, \operatorname{cl} S_2 \in \mathscr{A}(X, \mathscr{T}) \subset SO(X, \mathscr{T})$. Then $x \in \operatorname{cl} S_1 \cap \operatorname{cl} S_2$ implies $x \in \operatorname{cl}(\operatorname{int}(\operatorname{cl} S_1 \cap \operatorname{cl} S_2))$, which is a contradiction. Thus no such point x exists, and so $S_1 \cap S_2 \in SO(X, \mathscr{T})$.

(4) implies (5) is due to Njåstad [6].

(5) implies (1): If A is an $\mathcal{A}$-set then $A=U\cap F$ where $U\in\mathcal{T}$ and $F\in RC(X,\mathcal{T})$. Since $(X,\mathcal{T})$ is extremally disconnected $F\in\mathcal{T}$. Hence $A\in\mathcal{T}$.

Theorem 1 and 2 show that in any topological space $(X,\mathcal{T})$ we have the following fundamental relationships between the classes of subsets of X which we are considering, namely

(i) $\mathcal{A}(X,\mathcal{T})=SO(X,\mathcal{T})\cap LC(X,\mathcal{T})$.

(ii) $\mathcal{T}=\mathcal{T}^{\alpha}\cap LC(X,\mathcal{T})$.

(iii) $\mathcal{T}=PO(X,\mathcal{T})\cap LC(X,\mathcal{T})$.

(i) and (iii) immediately imply.

(iv) $\mathcal{T}=PO(X,\mathcal{T})\cap\mathcal{A}(X,\mathcal{T})$.

(v) $\mathcal{T}^{\alpha}=PO(X,\mathcal{T})\cap SO(X,\mathcal{T})$ is due to Reilly and Vamanamurthy [8].

These relationships provide immediate proofs for the following decompositions. We note that (ii) of Theorem 4 is an improvement of Tong's decomposition of continuity [10, Theorem 4.1], and that (iii) of Theorem 4 is due to Reilly and Vamanamurthy [8]. Theorem 4 (i), (iv) and (v) seem to be new results and provide new decompositions of continuity.

THEOREM 4. *Let $f\colon X\to Y$ be a function. Then*

(i) *f is $\mathcal{A}$-continuous if and only if f is semicontinuous and LC-continuous.*

(ii) *f is continuous if and only if f is α-continuous and LC-continuous.*

(iii) *f is α-continuous if and only if f is precontinuous and semicontinuous.*

(iv) *f is continuous if and only if f is precontinuous and LC-continuous.*

(v) *f is continuous if and only if f is precontinuous and $\mathcal{A}$-continuous.*

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A GENERAL SADDLE POINT THEOREM AND ITS APPLICATIONS

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Let X and Y be nonempty sets, f and g be real-valued functions on the Cartesian product $X\times Y$ of these sets. A point (x, y) in $X\times Y$ is said to be a *saddle point* of the functions f, g if

$$\text{(SP)} \qquad g(u, y) \leqq f(x, v) \quad \text{for every} \quad (u, v) \quad \text{in} \quad X\times Y$$

holds true. For a single function f the well-known notion of saddle point follows here by letting $g\equiv f$ in (SP). It should also be noted that the existence of a saddle point implies the following minimax inequality

$$\text{(MMI)} \qquad \inf_{y\in Y}\sup_{x\in X} g(x, y) \leqq \sup_{x\in X}\inf_{y\in Y} f(x, y).$$

In the case when $f\leqq g$, especially when g equals f, the latter property is known as the statement of the two variable generalized version of the celebrated von Neumann's minimax theorem, namely

$$\text{(MME)} \qquad \inf_{y\in Y}\sup_{x\in X} g(x, y) = \sup_{x\in X}\inf_{y\in Y} f(x, y).$$

Our aim is to prove a general but rather elementary theorem first on the existence of saddle points (Theorem 1), secondly, as a consequence, on the existence of minimax inequality and equality respectively — giving necessary and sufficient conditions for them. Our condition is general enough and not only of convexity type. The results so obtained are a common generalization of our previous ones and many other known theorems of concave-convex type. Our approach is essentially the same as our earlier one. We use the finite dimensional separation argument for disjoint convex sets in a similar but essentially simpler way as in [1, Theorem 2.5.1] and Riesz's well-known theorem concerning a common point of compact sets with finite intersection property. The compactness here follows by Alexander's subbase theorem [6].

Concerning minimax type inequalities see S. Simmons [10], J. Kindler [5] and Z. Sebestyen [8, 9]. Minimax theorems are e.g. in Belakrishnan [1], Z. Sebestyen [7, 8, 9], I. Joó [3] and I. Joó—L. L. Stachó [4].

Let now f, g be two real-valued functions defined on the Cartesian product $X\times Y$ of two nonempty sets X, Y. As a notation, for a nonemty set $K\subset X\times Y$, for a point (u, v) in $X\times Y$ and for a positive real number c let

$$K^c_{u,v} = \{(x, y)\in K\colon\ 0 \leqq f(x, v) - g(u, y) + c\}.$$

This is why for a point (x, y) in $X\times Y$ to be a saddle point is nothing else but each $K^c_{u,v}$ being nonempty for the one point set $K=\{(x, y)\}$.

THEOREM 1. *Let f, g be real-valued functions on $X\times Y$. There exists a saddle point for f, g if and only if there exists a nonempty set $K\subset Y\times Y$ such that :*

$$\text{(1)} \qquad \min_{(u,v)\in G} \sum_{(x,y)\in F} \lambda(x, y)[f(x, v)-g(u, y)] \leqq \sup_{(x,y)\in K} \min_{(u,v)\in G} [f(x, v)-g(u, y)]$$

for all finite sets $F\subset K$, $G\subset X\times Y$ and a probability measure λ on F;

$$\text{(2)} \qquad 0 \leqq \inf_{(u,v)\in X\times Y} \sup_{(x,y)\in K} [f(x, v)-g(u, v)] \leqq \sup_{(x,y)\in K} \sum_{(u,v)\in G} \mu(u, v)[f(x, v)-g(u, y)]$$

for every finite set $G\subset X\times Y$ and a probability measure μ on G;

(3) if $D\subset(0, +\infty)\times X\times Y$ has the property that for any (x, y) in K there exists (c, u, v) in D with $f(x, v)-g(u, y)+c<0$, then a finite subset of D exists with the same property.

PROOF. Assume first that a point (x, y) in $X\times Y$ is a saddle point for the functions f, g on $X\times Y$. The one point subset $K=\{(x, y)\}$ of $X\times Y$ clearly satisfies conditions (1), (2) and (3)

To prove the sufficiently let K be as in the assumption. Let further $U^c_{u,v}=K\setminus K^c_{u,v}$ be the complements in K of the subsets $K^c_{u,v}$ introduced before.

Topologize K by taking $\{U^c_{u,v} : (c, u, v)\in(0, +\infty)\times X\times Y\}$ as a family of open subbase for this topology. Condition (3) says that if K is covered by a subfamily $\{U^c_{u,v} : (c, u, v)\in D\}$ then K is also covered by a finite subcollection of the family indexed by D. By Alexander's well-known subbase lemma K is thus compact in the topology so introduced. But the subsets $K^c_{u,v}$ of K are thus closed hence compact with respect to this topology on K. Now a point (x, y) in $X\times Y$ satisfies (SP) if and only if

$$0 \leqq f(x, v)-g(u, y)+c \quad \text{holds for all} \quad (c, u, v)\in(0, +\infty)\times X\times Y,$$

in other words (x, y) belongs to each of $K^c_{u,v}$. To prove that a saddle point exists is therefore nothing else but to prove that the sets $K^c_{u,v}$ have a common point. But the compactness of $K^c_{u,v}$'s allows us, refering to Riesz, to prove the finite intersection property of the family $K^c_{u,v}$. Let $0<c_i$, $(u_i, v_i)\in X\times Y$ for $i=1, 2, \dots, n$ have a finite family of subsets $K^{c_i}_{u_i,v_i}$ in K indexed by $i=1, 2, \dots, n$. Since with $c=\{\min c_i : 1\leqq i\leqq n\}$

$$K^c_{u_i,v_i} \subset K^{c_i}_{u_i,v_i} \quad \text{for} \quad i = 1, 2, \dots, n,$$

$\bigcap_{i=1}^n K^c_{u_i,v_i}\neq\emptyset$ will imply the desired nonvoid intersection property for the chosen finite family $\{K^{c_i}_{u_i,v_i} : i=1, 2, \dots n\}$. Assume the contrary: $\bigcap_{i=1}^n K^c_{u_i,v_i}=\emptyset$. Then we conclude that for any (x, y) in K there exists a natural number i, $1\leqq i\leqq n$ such that $(x, y)\notin K^c_{u_i,v_i}$, i.e. $f(x, v_i)-g(u_i, y)+c<0$.

This implies the following property:

$$\text{(4)} \qquad \min_{1\leqq i\leqq n} [f(x, V_i)-g(u_i, y)] < -c \quad \text{for any} \quad (x, y) \quad \text{in} \quad K.$$

Let now Φ_c be the R^n-valued function on K defined as follows:

$$\Phi_c(x, y) := \big(f(x, v_1)-g(u_1, y)+c, \dots, f(x, v_n)-g(u_1, y)+c\big).$$

We have thus that $\Phi_c(X, Y)$, the range of Φ_c, does not meet

$$\mathbf{R}^n_+ := \{t = (t_1, \dots, t_n) \in \mathbf{R}^n\colon\ 0 \leqq t_i \text{ for } i = 1, 2, \dots, n\},$$

the positive cone in $\mathbf{R}^n$. But we state more that this, namely that the convex hull of $\Phi_c(X, Y)$ also does not meet but the interior of $\mathbf{R}^n_+$. Otherwise there would be a finite set $F \subset X \times Y$, probability measure λ on F such that

$$0 < \sum_{(x,y)\in F} \lambda(x, y)[f(x, v_i) - g(u_i, y) + c] \quad \text{for} \quad i = 1, 2, \dots, n.$$

But then, in view of (1) and (4), we have

$$-c < \min_{1 \leqq i \leqq n} \sum_{(x,y)\in F} \lambda(x, y)[f(x, v_i) - g(u_i, y) \leqq \sup_{(x,y)\in K} \min_{1\leqq i \leqq n} [f(x, v_i) - g(u_i, y)] \leqq -c,$$

a contradiction. The separation argument thus applies: there exists a nonzero vector $\mu = (\mu_1, \dots, \mu_n) \in \mathbf{R}^n$ separating the mentioned two convex sets in $\mathbf{R}^n$. This can be expressed by the following property

$$\sum_{i=1}^{n} \mu_i[f(x, v_i) - g(u_i, y) + c] \leqq \sum_{i=1}^{n} \mu_i t_i \quad \text{for} \quad (x, y) \quad \text{in} \quad K, 0 \leqq t_i, \quad i = 1, 2, \dots, n.$$

As an easy consequence we have $0 \leqq \mu_i$ for $i = 1, 2, \dots, n$. We can thus assume $\sum_{i=1}^{n} \mu_i = 1$, i.e. that μ is a probability measure on the finite set $G = \{(u_i, v_i)\colon i = 1, 2, \dots, n\}$. But (2) thus gives us the following contradiction

$$0 \leqq \inf_{(u,v)\in X\times Y} \sup_{(x,y)\in K} [f(x, v) - g(u, y)] \leqq \sup_{(x,y)\in K} \sum_{i=1}^{n} \mu_i[(f(x, v_i) - g(u_i, y))] \leqq -c.$$

The proof of the theorem is now complete.

COROLLARY 1. *Let X, Y be convex subsets of real linear spaces, and let f, g be real-valued functions on $X \times Y$ such that* (5) *$f(-g)$ is concave in its first (second), and convex in its second (first) variable.*

Then there exists a saddle point in $X \times Y$ for f, g if and only if there exists a nonempty subset K in $X \times Y$ with (3) *and such that*

$$0 \leqq \sup_{(x,y)\in K} [f(x, v) - g(u, y)] \quad \textit{for every} \quad (u, v) \in X \times Y. \tag{6}$$

PROOF. For concave-convex functions $f, -g$, as (5) assumes, we have for every finite sets $F, G \subset X \times Y$ and probability measures λ, μ on them, respectively, such that

$$\sum_{(x,y)\in F} \lambda(x, y)[f(x, v) - g(u, y)] \leqq f\Big(\sum_{(x,y)\in F} \lambda(x, y)x, v\Big) - g\Big(u, \sum_{(x,y)\in K} \lambda(x, y)y\Big),$$

$$f\Big(x, \sum_{(u,v)\in G} \mu(u, v)v\Big) - g\Big(\sum_{(u,v)\in G} \mu(u, v)u, y\Big) \leqq \sum_{(u,v)\in G} \mu(u, v)[f(x, v) - g(u, v)].$$

Properties (1), (2) are easy consequences of these and (6). Therefore Theorem 1 applies.

REMARK 1. A known result is a consequence of Corollary 1 in the case when X, Y are compact convex subsets of real topological linear spaces and $f=g$ is continuous (or at least $f(x, v)-g(u, y)$ is upper semicontinuous in (x, y) for every (u, v)) concave-convex real-valued function on $X\times Y$.

THEOREM 2. *Let g, f be real-valued functions on the Cartesian product $X\times Y$ of the nonempty sets X, Y. The minimax inequality* (MMI) *holds true for f, g if and only if for each positive real number ε there exists a nonempty subset K_ε of $X\times Y$ such that conditions* (1), (2), (3) *of Theoerm* 1 *hold true with K_ε, $f+\varepsilon$ instead of K and f, respectively.*

PROOF. The minimax inequality (MMI) clearly holds if and only if for any $\varepsilon>0$ the following inequality is satisfied

$$\inf_{y\in Y}\sup_{x\in X} g(x, y) < \sup_{x\in X}\inf_{y\in Y}\big(f(x, y)+\varepsilon\big).$$

Equivalently, when there exists y_ε in Y such that

$$\sup_{x\in X} g(x, y_\varepsilon) < \sup_{x\in X}\inf_{y\in Y}\big(f(x, y)+\varepsilon\big),$$

then there exists x_ε in X such that

$$\sup_{x\in X} g(x, y_\varepsilon) < \inf_{y\in Y}\big(f(x_\varepsilon, y)+\varepsilon\big).$$

But this is (SP) for $f+\varepsilon$, g with $(x_\varepsilon, y_\varepsilon)$ in $X\times Y$ indeed. Theorem 1 is therefore applies.

As a further consequence we have [3, Theorem] in an improved form instead of its minimax formulation in [3]:

COROLLARY 1a. *Let f be a real-valued function on $X\times Y$ such that $\inf_{y\in Y}\sup_{u\in X} f(u, y)\in\mathbf{R}$. There exists $x_0\in X$ such that*

$$\inf_{y\in Y}\sup_{x\in X} f(u, y) \leqq f(x_0, v) \quad \textit{for every} \quad v\in Y \tag{7}$$

holds if and only if there exists a nonempty set $X_0\subset X$ such that the following properties hold:

$$\min_i \sum_j \lambda_j f(x_j, v_i) \leqq \sup_{x\in X_0}\min_i f(x, v_i) \tag{8}$$

for any finite sets $(x_j, v_i)\in X_0\times Y$ and $\lambda_j\geqq 0$, $\sum_j \lambda_j=1$;

$$\inf_{v\in Y}\sup_{x\in X_0} f(x, v) \leqq \sup_{x\in X_0}\sum_i \mu_i f(x, v_i) \tag{9}$$

for any finite sets $v_i\in Y$ and $\mu_i\geqq 0$, $\sum_i \mu_i=1$;

(10) *if $C\subset(0, +\infty)\times X$ has the property that for any $x\in X_0$ there exists (c, v)* in C *with*

$$f(x, v)+c < \inf_{y\in Y}\sup_{u\in X_0} f(u, y)$$

then a finite subset of C exists with the same property.

PROOF. The validity of the conditions (8), (9), (10) for the one point set $X_0=\{x_0\}$ where x_0 satisfies (7), is evident. Thus the necessity part of the theorem is clear. To prove the sufficiency let g be the constant

$$g:=\inf_{u\in Y}\sup_{u\in X_0} f(u,y)$$

as a function on $X\times Y$ to apply Theorem 1 with $X_0\times\{y_0\}$ as K in the hope that (x_0, y_0) is saddle point for f, g with any $y_0\in Y$, as (7) requires. With this K, f and g conditions (1)—(3) reduce clearly to conditions (8)—(10), respectively. Theorem 1 hence applies, thus the proof is complete.

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ON A PROBLEM OF R. PÖSCHEL ON LOCALLY INVERTIBLE MONOIDS

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To the memory of my Parents

0. Introduction and statement of results

Automorphisms of algebraic structures have been widely investigated. In connection with the characterzation of properties of endomorphism monoids, in [*P*1] it was formulated a problem which we are going to investigate with a set-theoretic approach.

Before formulating the set theoretic version of the problem we need a few definitions:

Definition 0.0 (i) For sets A and B denote by ${}^{B}A$ the set of functions mapping from B to A and the set of permutations on A by S_A.

(ii) $\circ$ and $\restriction$ denote the operations of composition and restriction respectively. (That is for $f, g \in {}^{A}A$, $D \subset A$ and $b \in A$ $(f \circ g)(b) = f(g(b))$, $f \restriction D \in {}^{D}A$ and for every $d \in D$ $(f \restriction D)(d) = f(d)$.) Let further $f''D = \mathrm{Range}\,(f \restriction D)$ for $f \in {}^{B}A$ and $D \subset \mathrm{Range}\,(f)$

(iii) a monoid $M \subset {}^{A}A$ is called *locally invertible* iff for every $f \in M$ and finite subset D of A there is a $g \in M$ such that $(g \circ f) \restriction D = \mathrm{id}$.

(iv) for $F \subset {}^{A}A$ let $\mathrm{Loc}\,(F) = \{f \in {}^{A}A \colon \forall D \subset A,\ D$ finite, $\exists g \in F\ f \restriction D = g \restriction D\}$ the *local closure* of F.

Then the problem is whether the following statement is true:

$P(A) =$ "$M \subset \mathrm{Loc}\,(S_A \cap \mathrm{Loc}\,(M))$ holds for every locally invertible monoid $M \subset {}^{A}A$."

We denote the negation of $P(A)$ by $\neg P(A)$.

Remarks. (a) R. Pöschel raised the problem in Szeged, 1983 (see [C, p. 653]). The problem first appeared in [P3], the original problem is whether "a clone of relations closed with respect to complementation" is an equivalent definition of Krasner clones of 2^{nd} kind. (For more algebraic background and intuition see [P1, p. 161] or [P2].)

(b) Stone's definition in [St, p. 41] is, in fact, equivalent to (iii).

The validity of $P(A)$ depends on the cardinality of A. It is easy to see that for cardinal numbers $\lambda < \varkappa$ $P(\varkappa)$ implies $P(\lambda)$. (If A is finite then $P(A)$ is trivially true.) The affirmative answer for countable A was first given by J. Kollár (see eg. [P1, p. 164]) and the reader can make up a proof himself also for this case.

Our results are the following (for the definitions of CH or MA see [K]):

Theorem 2.1. CH *implies* $\neg P(2^{\aleph_0})$.

Theorem 3.1. (a) MA *(Martin's axiom) implies* $\neg P(2^{\aleph_0})$.
(b) MA (λ) *implies* $P(\lambda)$ *for* $\lambda < 2^{\aleph_0}$ *for countable monoids* M.

Theorem 3.2. $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 + \neg\mathrm{MA} + \neg P(2^{\aleph_0})$ *is relative consistent with* ZFC.

THEOREM 3.4. $2^\lambda=\lambda^+$ *implies* $\neg P(2^\lambda)$ *for any cardinal* λ.

(The same argument shows $\neg P(\varkappa)$ for any $\varkappa$ if $2^\lambda \leqq \varkappa$ for every $\lambda<\varkappa$.

It remains open whether $\neg P(2^{\aleph_0})$ follows from ZFC or $P(2^{\aleph_0})$ is consistent. Further, very little is known about $P(\lambda)$ for $\aleph_0<\lambda<2^{\aleph_0}$.

Though Theorem 3.4 generalises all the other theorems, we prove it at the end because of the following reason. Theorem 3.1 (b) shows that the cardinality of the monoid plays an important role and in the proof of Theorem 3.4 we construct a monoid of size λ while the other proofs (using slightly different arguments) give countable monoids. Theorem 3.1.b shows that countable counterexamples can not be given in ZFC alone.

In Section 1 we prove Lemma 1.2 which is the key to our results. In Section 2 we prove our main result: Theorem 2.1. Using the same ideas (but forcing techniques) we prove generalizations of this theorem (Theorems 3.1, 3.2 and 3.4). The author thanks R. Pöschel, P. Komjáth and P. Prőhle for helpful discussions.

1. The Lemma

In this Section we prove a lemma which is the starting point of our proofs and introduce some useful definitions which throw some light to the behaviour of our monoids.

We start with the notations and definitions we need. ω_0 is the set $\{0, 1, 2, \dots\}$ and $k<\omega_0$ means $k\in\omega_0$ and $i<k$ means $i\in\{0, 1, \dots, k-1\}$ for $k\in\omega_0$.

DEFINITION 1.0. (i) $M\subset {}^AA$ is a *free monoid* iff it has no nontrivial $\circ$-equations. (That is for every $f_1, f_2, f_3, f_4\in M$ if $f_1\circ f_2=f_3\circ f_4$ then there are $g_j\in M$ $(j=1, 2, \dots, n$ for some $n<\omega_0)$ such that $f_1=g_1\circ\dots\circ g_k$, $f_2=g_{k+1}\circ\dots\circ g_p$ and $f_3=g_1\circ\dots\circ g_l$, $f_4=g_{l+1}\circ\dots\circ g_n$ for some $k, l<n$.)

(ii) For a one-to-one function $f\in {}^AA$ we denote by f^{-1} the partial inverse of f: $\mathrm{Dom}\,(f^{-1})=\mathrm{Range}\,(f)$ and $f^{-1}(a)=b$ iff $f(b)=a$ for $a\in\mathrm{Dom}\,(f^{-1})$.

(iii) For a set $F\in {}^AA$ we denote by $\langle F, \circ\rangle$ $(\langle F, \circ, -1\rangle)$ the set of functions generated from F with the help of operation $\circ$, the composition (with the help of $\circ$ and -1, the partial inverse, resp.). (To be more precise, $g\in\langle F, \circ\rangle$ and $h\in\langle F, \circ, -1)$ iff there are $k\in\omega_0$, $f_1, \dots, f_k\in F$ and $\varepsilon_1, \dots, \varepsilon_k\in\{+1, -1\}$ such that $g=f_1\circ\dots\circ f_k$ and $h=f_1^{\varepsilon_1}\circ\dots f_k^{\varepsilon_k}$ where we write f^{+1} for f and f^{-1} for the partial inverse of f.) (Sometimes we write f^0 for id.)

(iv) A monoid $M\subset {}^AA$ is finitely generated iff there is $F\subset M$ finite such that $M=\langle F, \circ\rangle$.

In the next section we extend the elements of the monoid $M\subset {}^AA$ (constructed in this section) to $N\subset {}^BB$, $B\supset A$ $M=\{f\upharpoonright A: f\in N\}$. In the meantime we want to "kill" (every) permutation $\pi\in\mathrm{Loc}\,(M)\cap S_A-\{\mathrm{id}_A\}$, that is $\pi\neq\varrho\upharpoonright A$ for all $\varrho\in\mathrm{Loc}\,(N)\cap S_B$. To achieve this, while extending the elements of M to B, we have to extend their local inverses in such a way that the partially killed π will not rise again. This is ensured by requiring the existence of local inverses for all $f\in M$ with good properties (and Lemma 1.2 (iv); for further details see the sets E_i and Lemma 2.2). These good properties are declared in the following definition. (Any of the finite sets may be empty.)

DEFINITION 1.1 (weaker version). A set of functions $F \subset {}^A A$ (A is an arbitrary set) is called *fairly complete* iff:

FOR EVERY $f \in F$, $D \subset A$ finite, $v < \omega$, $D_m \subset A$ finite and one-to-one function g_m mapping from D_m to A, $g_m \neq \mathrm{id} \upharpoonright D_m$ and $\mathscr{H} \subset F$ finite such that $\varphi \upharpoonright D_m \neq g_m$ for all $\varphi \in \langle \mathscr{H}_m, \circ, -1 \rangle$ and $m < V$, THERE ARE infinitely many $t \in F$ such that $t \circ f \upharpoonright D = \mathrm{id} \upharpoonright D$ and for every $m < v$ and every $\psi \in \langle \mathscr{H}_m \cup \{t\}, \circ, -1 \rangle$ we have $\psi \upharpoonright D_m \neq g_m$ for $m < v$.

Roughly speaking t is a local inverse for $f \upharpoonright D$ and moreover makes no forbidden functions in $\langle \mathscr{H}_m \cup \{t\}, \circ, -1 \rangle$ with respect to g_m simultaneously for $m < v$.

(In the construction of the next section, one of the g's is will be the permutation π to be killed, see also Lemma 2.2.)

Observe that if F is fairly complete then it is locally invertible. Further if F is locally invertible (fairly complete) then so is $\langle F, \circ \rangle$ too.

However, in proving Theorem 2.1 (see Case 3 in Claim 2.3) we need a stronger property:

DEFINITION 1.1 (stronger version). *A* set of functions $F \subset {}^A A$ (A is an arbitrary set) is caled *strongly fairly complete* iff:

FOR EVERY $f \in F$, $D \subset A$ finite, $v < \omega_0$, $D_m \subset A$ finite and one-to-one function g_m, mapping from D_m to A, $g_m \neq \mathrm{id} \upharpoonright D_m$ and $\mathscr{H}_m \subset F$ finite for $m < v$, THERE ARE infinitely many $t \in F$ such that $t \circ f \upharpoonright D = \mathrm{id} \upharpoonright D$ and for every $m < v$ and every $\psi \in \langle \mathscr{H}_m \cup \{t\}, \circ, -1 \rangle$ $\psi \upharpoonright D = g_m$ implies $D \subset \mathrm{Dom}(\psi')$ and $\psi' \upharpoonright D_m = \psi \upharpoonright D_m$ where ψ' results if we replace t^{-1} by $(f \upharpoonright D)^{-1}$ (and t^{-1} by $f \upharpoonright D$) in ψ everywhere.

We need this stronger version because in the main construction (see the proof of Claim 2.3) we can not ensure that $\psi \upharpoonright D_m \neq g_m$ for all $\psi \in \langle \mathscr{H}_m, \circ, -1 \rangle$ but for ψ' only if ψ' is defined as above.

In what follows we always use the stronger version of Definition 1.1.

The following lemma is the key to our results:

LEMMA 1.2. *There exists a countable monoid $M \subset {}^A A$ on a countable set A with the following properties:*

(i) *M is not finitely generated and has independent $\circ$ — generators $F = \{f_i: i < \omega_0\}$,*

(ii) *F is strongly fairly complete,*

(iii) $\mathrm{Loc}(\langle \{f_i: i < j\}, \circ, -1 \rangle) \cap S_A \subseteqq \{\mathrm{id}_A\}$ *for every* $j < \omega_0$.

REMARKS M is free by (i) and locally invertible by (ii). We will use (iii) in the next section to construct some sets E_j for $j < \omega_0$. Using their properties and the fairly completeness of M we will be able to kill $\pi \in \mathrm{Loc}(M) \cap S_A$.

PROOF. We will construct an increasing sequence of countable sets $\langle A_n: n < \omega_0 \rangle$, $A_n \subset A_{n+1}$ for $n < \omega_0$ and we will take $A = \bigcup \{A_n: n < \omega_0\}$. In order to construct M, in each step $n < \omega_0$ we will build monoids $M_{n+1} \subset {}^{A_{n+1}}(A_{n+1})$ by extending the elements of M_n to A_{n+1} and adding some (countable many) elements from ${}^{A_{n+1}}(A_{n+1})$. More precisely we construct the free $\circ$—generators of M_{n+1}. Finally every element of M_n will be extended to A for every $n<\omega_0$ and at the end we will take the set of generators of M to be the set of these extended functions.

(In terms of formulas, the set of generators of M is $F=\{\psi_{i,j}: i, j<\omega_0\}\subset {}^A A$ which we intend to define, $\psi_{i,n}$ "appears first" when constructing M_n (see below for definition). In step $n<\omega_0$ we will define the elements of the set $F_n=\{\psi_{i,j}\restriction A_n: i<\omega_0,\ j\leqq n\}\subset {}^{A_n}(A_n)$ only. This is the set of the generators of $M_n\subset {}^{A_n}(A_n)$. Since in the n^{th} step we have not constructed $\psi_{i,j}$ but $\psi_{i,j}\restriction A_n$ only, we write $\psi_{i,j}^{(n)}$ instead of $\psi_{i,j}\restriction A_n$ and define after the construction $\psi_{i,j}:=\bigcup\{\psi_{i,j}^{(n)}: n\geqq j\}$. For convenience we enumerate F_n as $\{\varphi_u^{(n)}: u<\omega_0\}$.)

From algebraic point of view, if M^* and M_n^* are the abstract monoids represented by M and M_n $(n<\omega_0)$, then M_n^* is a homomorph image of a submonoid of M_{n+1}^* for all $n<\omega_0$ and so M^* is the inverse limit of the system $\{\langle M_n^*, \vartheta_n\rangle: n<\omega_0\}$ where ϑ_n is the homomorphism mentioned above. Since every element of M map hierarchically (that is Range $(f\restriction(A_{n+1}\setminus A_n))\subset A_{n+1}\setminus A_n$ for every $f\in M$ and $n<\omega_0$ if f appeared before n) and is one-to-one; further the sets D, D_m for $m<v$, and v are finite in the definition of the local invertibility and the fairly completeness, these things are handled in A_m, and so in M_m for some $m<\omega_0$ large enough. Further, we construct the free generators of M_n and M only, so we can manage (i) through (iii) easily.

Now, let us get down into the details. Let A_0 be an arbitrary countable set, $F_0\subset {}^{A_0}(A_0)$ an arbitrary countable set of $\circ$—independent injective functions on it, and put $M_0:=\langle F_0, \circ\rangle$.

Suppose that we have already constructed A_n and M_n and now we want to construct A_{n+1} and M_{n+1}. We have $M_m=\langle F_m, \circ\rangle$ by construction, where $F_m=\{\psi_{i,j}^{(m)}: i<\omega_0,\ j\leqq m\}$ for all $m\leqq n$, and $\psi_{i,j}^{(k)}=\psi_{i,j}^{(m)}\restriction A_k$ for $i<\omega_0$, $j\leqq k\leqq m\leqq n$. We want to extend the elements of F_n to A_{n+1} and find infinitely many local inverses for them on A_{n+1} as independent from each other as possible. To this end choose countable sets $B_{\psi,D,u}^{(n)}$ and $B_\psi^{(n)}$ disjoint from each other and from A_n for $\psi\in F_n$, $D\subset A_n$ finite, $u<\omega_0$ and, let P_{n+1} be not element of any of these sets and put

$$A_{n+1}:=A_n\cup\bigcup\{B_\psi^{(n)}\cup B_{\psi,D,u}^{(n)}: \psi\in F_n,\quad D\subset A_n\quad \text{finite},\quad u<\omega_0\}\cup\{P_{n+1}\}.$$

P_{n+1} ensures (iii); for further details see Lemma 1.3. $A_n\cup B_\psi^{(n)}$ will be the Range os $\psi\in F_n$ after extending it to A_{n+1} and $A_n\cup B_{\psi,D,u}^{(n)}$ will be the Range of a new element of F_{n+1}, the uth local inverse of $\psi\restriction D$ where $\psi\in F_n$ and $D\subset A_n$ is finite, $u<\omega_0$. The disjointness of the sets $B_\psi^{(n)}$ and $B_{\psi,D,u}^{(n)}$ is the main trick in the construction which ensures (i) through (iii). To be more precise, first extend all $\psi\in F_n$ to A_{n+1} to be one-to-one arbitrarily such that $\psi''(A_{n+1}\setminus A_n)\subset B_\psi^{(n)}$ and let $\{\psi_{i,j}^{(n+1)}: i<\omega_0, j\leqq n\}$ enumerate the set of these extended functions so that $\psi_{i,j}^{(n+1)}\restriction A_n=\psi_{i,j}^{(n)}$. (Recall that every $\psi\in F_n$ has the form $\psi_{i,j}^{(n)}$ for some $i<\omega_0$, $j\leqq n$.) Next let $l_{\psi,D,u}$ be the following injective function from A_{n+1} to $A_n\cup B_{\psi,D,u}^{(n)}$ for $\psi\in F_n$, $D\subset A_n$ finite for $u<\omega_0$: $l_{\psi,D,u}\restriction X=(\psi\restriction D)^{-1}$ and $l_{\psi,D,u}''(A_{n+1}\setminus X)\subset B_{\psi,D,u}^{(n)}$ where $X=\psi''D$. Finally put

$$\{\psi_{i,n+1}^{(n+1)}: i<\omega_0\}=\{l_{\psi,D,u}: \psi\in F_n,\ D\subset A_n \text{ is finite},\ u<\omega_0\}.$$

It is easy to see that all the functions $\psi_{i,j}^{(n+1)}$ for $i<\omega_0$, $j\leqq n+1$ are $\circ$— and -1—independent by the disjointness of the sets $B_\psi^{(n)}$, $B_{\psi,D,u}^{(n)}$, A_n for $\psi\in F_n$, $D\subset A_n$ finite, $u<\omega_0$. So we can define F_{n+1}, the set of generators of M_{n+1} as $F_{n+1}=$

$=\{\psi_{i,j}^{(n+1)}:\ i<\omega_0,\ \ j\leqq n+1\}$. Finally put $M=\langle F, \circ\rangle$ and $F=\bigcup\{\psi_{i,j}:\ i, j<\omega_0\}$ where $\psi_{i,j}=\bigcup\{\psi_{i,j}^{(n)}:\ n\geqq j\}$ for $i, j<\omega_0$. So we can say that the function $\psi=\psi_{i,j}\in F$ (or $\psi_{i,j}^{(j)}=\psi_{i,j}\restriction A_j$) *appeared first* in M_j, or shortly, at j, for any $i,\ j<\omega_0$.

So we have constructed A and M. Now we have to show that they have properties (i) through (iii).

It is easy to see that the elements of F are independent, so (i) holds.

Now we prove (iii). (Recall that F is enumerated as $\{f_i:\ i<\omega_0\cup\}$.)

LEMMA 1.3. *There is no permutation except* id_A *in* $\mathrm{Loc}\,(\langle H, \circ, -1\rangle)$ *for any* $H\subset F$ *finite.*

PROOF. We are given an $H\subset F$ finite and we must show that $\mathrm{Loc}\,(\langle H, \circ, -1\rangle)$ contains no permutation except id_A.

First choose an $n<\omega_0$ large enough such that all the members of H appeared first far below n (e.g. if $h\in H$ appeared first in M_{n_h}, $n_h<\omega$, then $n>n_h+1$ for $h\in H$.)

Suppose now $\pi\in\mathrm{Loc}\,(\langle H, \circ, -1\rangle)\cap S_A\setminus\{\mathrm{id}_A\}$. Let $a\in A$ be such that $\pi(a)\neq a$ and $D=\{P_{n+1}, \pi^{-1}(P_{n+1}), a\}$. Then we have

$$\pi\restriction D = (\varphi_k^{\varepsilon_k}\circ\varphi_{k-1}^{\varepsilon_{k-1}}\circ\ldots\circ\varphi_1^{\varepsilon_1}\circ\varphi_0^{\varepsilon_0})\restriction D$$

for some $k<\omega_0$, $\varphi_i\in H$ and $\varepsilon_i\in\{+1, -1\}$ for $i\leqq k$.

Using the facts that $\mathrm{Range}\,(\psi\restriction(A_{n+1}\setminus A_n))\subset B_\psi^{(n)}$ and $P_{n+1}\notin B_\psi^{(n)}$ for every $\psi\in F$ appeared first before n, clearly $\varepsilon_0=+1$ and since the sets $B_\psi^{(n)}$ for $\psi\in F$ are pairwise disjoint we can see (by induction on i) that $\varepsilon_i\neq\varepsilon_{i+1}$ implies $\varphi_i=\varphi_{i+1}$. Since $\pi(a)\neq a$ we can suppose that $\varepsilon_i=1-\varepsilon_{i+1}$ and $\varphi_i=\varphi_{i+1}$ holds for no $i\leqq k$. This means that $\varepsilon_i=+1$ for all $i\leqq k$. Finally $P_{n+1}\notin\mathrm{Range}\,(\varphi)$ for all $\varphi\in H$ but $P_{n+1}\in\in\pi''D$ shows a contradiction. $\square$

So Lemma 1.2 (iii) holds.

Now we prove (ii).

LEMMA 1.4. *F is strongly fairly complete.*

Let us be given $\nu<\omega_0$, D_m, $\mathscr{H}_m$, g_m for $m<\nu$ and $f\in F$, $D\subset A$ finite as in Definition 1.1. We have to find some $t\in F$ with good properties.

Choose an $n<\omega_0$ large enough so that all these things appeared below n. That is we require that $\hat{D}\subset A_n$ where

$$\hat{D}:=D\cup f''D\cup\bigcup\{D_m\cup\mathrm{Range}\,(g_m):\ m<\nu\}$$

and every element ψ of $\hat{H}:=\{f\}\cup\bigcup\{\mathscr{H}_m:\ m<\nu\}$ appeared first below n. Write $\bar{f}$ for $f\restriction A_n$, so $\bar{f}\in F_n$.

When we built F_{n+1} we defined some local inverses $\bar{l}=l_{\bar{f},D,u}\in F_{n+1}$ $(u<\omega_0)$ for the present f and D and this $\bar{l}$ appears in the sequence $\{f_i\restriction A_{n+1}:\ i<\omega_0\}=F_{n+1}$ infinitely many times.

We now show that the functions $t\in F$ for which $t\restriction A_{n+1}=\bar{l}$ works. So, fix such a $t\in F$ and let $u<\omega_0$ its index.

We may work in M_{n+1} and A_{n+1} since $\psi''(A_{m+1}\setminus A_m)\subset A_{m+1}\setminus A_m$ for every. $\psi\in\hat{H}$ and $m>n$ since the elements of $\hat{H}$ appeared first at last $n+1$ and $\hat{H}\subset A_n$. (That is, all the functions we use from now on, we can suppose are elements of M_{n+1}, their Dom is A_{n+1}.) Let $m<\nu$ be fixed. Roughly speaking our construction works because we defined the values of our functions as independently as it was possible, that is ($\bar{f}$ stands for $f \upharpoonright A_n$, $\bar{f} \in F$):

(!) $\operatorname{Range}(t \upharpoonright (A_n \setminus f''D)) \subset B^{(n)}_{\bar{f}, D, u}$ but $\operatorname{Range}(f \upharpoonright A_n) \subset A_n$

(!!) $\operatorname{Range}(\varphi \upharpoonright B^{(n)}_{\bar{f}, D, u}) \subset A_{n+1} \setminus A_n$ for $\varphi \in M$ appeared before n

(!!!) $B^{(n)}_{\bar{f}, D, u} \cap A_n = \emptyset$ and $D \subset A_n$.

Now we verify in details. We have to verify: if $g_m \neq \mathrm{id} \upharpoonright D_m$ and $g_m \neq \varphi \upharpoonright D_m$ for all $\varphi \in \langle \mathscr{H}_m, \circ, -1 \rangle$ then for all $\varphi \in \langle \mathscr{H}_m \cup \{t\}, \circ, -1 \rangle$ we have $\varphi \upharpoonright D_m \neq g_m$. Namely we prove the following:

STATEMENT 1.5. *For arbitrary* $\psi \in \langle \mathscr{H}_m \cup \{t\}, \circ, -1 \rangle$
(a) *EITHER* t *can be replaced by* $(f \upharpoonright D)^{-1}$ *and* t^{-1} *by* $f \upharpoonright D$ *in* ψ *everywhere and for the resulted* ψ' *we have* $D_m \subset \operatorname{Dom}(\psi')$ *and* $\psi' \upharpoonright D_m = \psi \upharpoonright D_m$,
(b) *OR* $\operatorname{Range}(\psi) \cap (A_{n+1} \setminus A_n) = \emptyset$.

This statement clearly implies that F is strongly fairly complete.

PROOF. Let $\psi \in \langle \mathscr{H}_m \cup \{t\}, \circ, -1 \rangle$ be fixed. We can write ψ in the form

$$\psi = y_h^{\varepsilon_h} \circ y_{h-1}^{\varepsilon_{h-1}} \circ \ldots \circ y_2^{\varepsilon_2} \circ y_1^{\varepsilon_1} \circ y_0^{\varepsilon_0} \tag{1}$$

where $y_i \in \mathscr{H}_m \cup \{t\}$ and $\varepsilon_i \in \{+1, -1\}$ for $i \leqq h$ for some $h < \omega_0$.

Our goal is to replace t by $(f \upharpoonright D)^{-1}$ in ψ as required in (a) whenever it is possible. We try to replace t in each of its occurrence in ψ separately step by step. (We are allowed to make a replacement if for the resulted $\tilde{\psi}$ we have $D_m \subset \operatorname{Dom}(\tilde{\psi})$.) If we succeed to replace all the occurrences of t by $(f \upharpoonright D)^{-1}$ (and t^{-1} by $f \upharpoonright D$) then we reach case (a). If not, we get a breakdown somewhere, we reach case (b).

Now we examine not only the structure of ψ but the "route" of D_m. That is, if $\psi_{i_0}''D_m$ once pops into $A_{n+1} \setminus A_n$ (ψ_{i_0} is an initial segment of ψ) then, by our construction, it does for all $i > i_0$, so finally ψ satisfies case (b). In the remainder part of the proof we verify the above in details. Now let the sequence $\langle i_r \colon r < w \rangle$ enumerate the indices $i \leqq h$ in increasing order for which $y_i = t$. We can clearly suppose that $w \neq 0$.

Case I: $\varepsilon_i = +1$ for all $i < h$. (In this case $\psi \in \langle \mathscr{H}_m \cup \{t\}, \circ \rangle$.) Now define $\psi_0 = \psi_0' = \psi_0^{(0)} = \mathrm{id}$ and for $r < w$, $r > 0$ put

$$\psi_r^{(0)} = y_{i_r - 1} \circ y_{i_r - 2} \circ \ldots \circ y_j \circ \psi_{r-1}$$

$$\psi_r = t \circ \psi_r^{(0)}$$

$$\psi_r' = (f \upharpoonright D)^{-1} \circ y_{i_r - 1} \circ \ldots \circ y_j \circ \psi_{r-1}'$$

where $j = i_{r-1} + 1$.

(ψ_r and ψ_r' are the initial parts of ψ and ψ' resp., showing the procedure of replacing each occurrence of t by $(f \upharpoonright D)^{-1}$ in ψ.)

Now our task is to prove by induction on $r < w$ that

$$\begin{cases} \text{(a)} \quad \text{EITHER} \quad D_m \subset \operatorname{Dom}(\psi_r') \ \text{and} \ \psi_r' D_m \upharpoonright = \psi_r \upharpoonright D_m \\ \text{(b)} \quad \text{OR} \qquad \operatorname{Range}(\psi_r \upharpoonright D_m) \cap (A_{n+1} \setminus A_n) \neq \emptyset. \end{cases} \tag{2}$$

Obviously (2) for all $r<w$ implies that we are done. To see this observe that $\psi=y_h\circ\ldots\circ y_j\circ\psi_{w-1}$ and $\psi'=y_h\circ\ldots\circ y_j\circ\psi'_{w-1}$ where $j=i_{w-1}+1$.

Then (2) (a) for $r=w-1$ implies $D_m\subset\mathrm{Dom}(\psi')$ and $\psi'\restriction D_m=\psi\restriction D_m$ while (2) (b) for $r=w-1$ implies Range $(\psi_r\restriction D_m)\cap(A_{n+1}\setminus A_n)\neq\emptyset$ as required for Statement 1.5.

So, the induction step for (2): If case (b) holds in (2) for some $r_0<w$ then it is easy to see that for every $r>r_0$ case (b) holds in (2). So w.l.o.g. case (a) holds for every $r<w$. In this case, if we denote $(\psi_r^{(0)})''D_m$ by x, we have two subcases depending on the position of x:

Subacase (i): $x\subset f''D$. Then t can obviously be replaced by $(f\restriction D^{-1})$ in ψ_r and $\psi'_r\restriction D_m=\psi_r\restriction D_m$.

Subcase (ii): $x\nsubseteq f''D$. Then it is easy to see that t can not be replaced by $(f\restriction D)^{-1}$ since $t''(x\setminus f''D)\subset B^n_{f,D,u}\subset A_{n+1}\setminus A_n$ and so $D_m\nsubseteq\mathrm{Dom}(\psi'_r)$. So Range $(\psi'_r\restriction D_m)\cap(A_{n+1}\setminus A_n)\neq\emptyset$ which proves the induction step for r and so we proved Case I, too.

Case II: $\varepsilon_i=-1$ for some $i<h$. The method for this case is similar to the previous one but we have to be more careful.

Obviously we may suppose that there is no part like $y\circ y^{-1}$ in (1), i.e.

(3) for no $i<h$ we have $y_{i+1}=y_i$ and $\varepsilon_{i+1}=1-\varepsilon_i$

(since $g_m\neq\mathrm{id}\restriction D_m$). Again we examine the route of D_m. Put now $Y_{-1}=D_m$ and $Y_i=(y_i^{\varepsilon_i})''Y_{i-1}$ for $i\leqq h$. Let further e_0 be the smallest $e\leqq h$ such that $Y_l\cap(A_{n+1}\setminus A_n)\neq\emptyset$ if such an e does exist. Again we have two subcases:

Subcase (i): e_0 does exist. Then we know that for every $\varphi\in M_n$ we have Range $(\varphi^{+1})\subset A_n$ and Range $(\varphi^{-1})\subset A_n$. But e_0 was minimal and $D_m\subset A_n$ and $\hat{H}\subset M_n$, so we must have $e_0=i_{r_0}$ for some $r_0<w$. (The sequence $\langle i_r\colon r<w\rangle$ was defined before Case I.) In other words $y_{e_0}=t$. Further, by the construction of t and by the minimality of e_0 we have $\varepsilon_{e_0}=+1$ and $Y_{e_0}\cap(A_{n+1}\setminus A_n)\subset B^{(n)}_{f,D,u}$.

We know that for every $\varphi\in F_n$, $D\subset A_n$ finite and $i<\omega_0$ all the sets $B^{(n)}_\varphi$ and $B^{(n)}_{\varphi,D,u}$ are all pairwise disjoint, and for every $\varphi\in M_n$ we have Range $\big(\varphi\restriction(A_{n+1}\setminus A_n)\big)\subset B^{(n)}_\varphi$.

So, by (3) we can prove by induction on i, $e_0\leqq i\leqq h$ the following fact (as in Lemma 1.3), using $\hat{H}\subset M_n$: $e_i=+1$ and there is a $\varphi=\varphi(i)\in F_n$ such that $Y_i\cap\cap(A_{n+1}\setminus A_n)\subset B^{(n)}_\varphi$ or $Y_i\cap(A_{n+1}\setminus A_n)\subset B^n_{\varphi,D,u}$. (This means that $\psi''D_m\cap\cap(A_{m+1}\setminus A_m)\neq\emptyset$.) This proves Subcase (i).

Subcase (ii): e_0 does not exist. Then for $i\leqq h$ we have $Y_i\subset A_n$. Now define ψ_r, $\psi_r^{(0)}$ and ψ'_r and prove (2) by induction on $r<w$ exactly on the same way as in Case I.

The induction step in case ("t^{ε_i} can be replaced by $(f\restriction D)^{1-\varepsilon_i}$ in ψ_r for every $r<w$") is as follows:

Let $x=(\psi_r^{(0)})''D_m$ and $\varepsilon=\varepsilon_{i_r}$. If $x\subset D$ and $\varepsilon=+1$ or $x\subset f''D$ and $\varepsilon=-1$ then clearly we can replace t^ε by $(f\restriction D)^{1-\varepsilon}$ in ψ_r and $\psi'_r\restriction D_m=\psi_r\restriction D_m$. In any other case we would have $Y_{i_n}=(t^\varepsilon)''x\subset A_n$ by the definition of t, which is impossible.

So we proved Lemma 1.4. □

So (ii) also holds in Lemma 1.2 and we concluded the proof of Lemma 1.2. □

2. Proof of the main theorem

In this section we prove:

THEOREM 2.1. CH *implies* $\neg P(2^{\aleph_0})$.

PROOF. Our task is to define a monoid $N\subset {}^B B$ on some set B such that $\mathrm{Loc}(N)\cap S_B=\{\mathrm{id}_B\}$. We start with the monoid $M\subset {}^A A$ constructed in the previous section. Then, using the main ideas of the previous section to extend the generators to larger and larger sets as independently as possible, step by step we extend M to B, killing the elements of $\mathrm{Loc}(M)\cap S_A\setminus\{\mathrm{id}_B\}$. We do not add any new generator, we only extend the elements of F (= the set of free generators of $M\subset {}^A A$) to B. Finally we will get N as the generatum of these extended generators.

So, let A, F and M be guaranteed by Lemma 1.2 and let $\{\pi_i\colon i<\omega_1\}$ enumerate $\mathrm{Loc}(M)\cap S_A\setminus\{\mathrm{id}_A\}$. In each step $j<\omega_1$ we extend A and the elements of F to a larger set B_{j+1} $(B_j\supset\bigcup\{B_u\colon u<j\}$ for all $j<\omega_1$, $B_1=A$, $B_0=\emptyset)$ in such a way that π_i does not extend to B_{j+1} for some $j\geqq i$. (This means that for no $\varrho\in\mathrm{Loc}(M_{j+1})$ $\varrho\restriction A=\pi_i$ where $M_{j+1}\subset {}^{B_{j+1}}(B_{j+1})$ is the extended monoid.) In this case we say that we "killed π_i".

To be somewhat more precise, let A_i be arbitrary countable infinite sets disjoint from each other for $i<\omega_1$ and $A_0=A$. Put $B_j=\bigcup\{A_i\colon i<j\}$ for $j\leqq\omega_1$ and $B=B_{\omega_1}$. (So $B_0=\emptyset$ and $B_1=A$.) Fix further a booking function δ mapping $\omega_1\setminus\{0,1\}$ onto $\omega_1\times\omega_1$ with the property: $h\leqq j$ if $\delta(j)=(h,k)$ for some $j<\omega_1$ and for all $j,h<\omega_1$. (In the j^{th} step we will kill the $\delta(i)=(h,k)^{\text{th}}$ permutation, that is the k^{th} permutation of $\mathrm{Loc}(M_h)\subset^{(B_h)}B_h$ (the h^{th} level). We are forced to use such a booking function since $\varrho\restriction A=\mathrm{id}_A$ for many $\varrho\in\mathrm{Loc}(M_h)\setminus\{\mathrm{id}_{B_h}\}$, $h<\omega_1$ and finally we want to kill every elements of $\mathrm{Loc}(N)\setminus\{\mathrm{id}_B\}$ and $A\subset B_h\subset B$.)

Step by step we will extend (the generators of) M to B as follows. Let $M_1=M$. Denote M_j the monoid already extended to B_j (so $M_j\subset {}^{B_j}B_j$ and $M_0=B_0=\emptyset$, $M_{\omega_1}=N\subseteqq {}^B B$) for $j\leqq\omega_1$. The set of generators of M_j for $j\leqq\omega_1$, $j\neq\emptyset$ is $F_j=\{f_k^{(j)}\colon k<\omega_0\}\subset^{(B_j)}B_j$ and they have the property $f_k^{(t)}=f_k^{(j)}\restriction B_t$ for $k<\omega_0$ and $0<t<j\leqq\omega_1$ by the construction.

Let further $\{\pi_{j,k}\colon k<\omega_1\}$ enumerate $\mathrm{Loc}(M_j)\cap S_{B_j}\setminus\{\mathrm{id}_{B_j}\}$ for $j<\omega_1$. Now let i be given, $2\leqq i<\omega_1$ and suppose that we have already constructed M_j for all $j<i$. Now we want to construct M_i. (Recall that $M_j=\langle F_{j,0}\rangle$ for $j<i$ and the elements of F_{j_1} extend the elements of F_{j_2} for $j_1<j_2<i$.)

In case i is limit we clearly take

$$f_k^{(i)}=\bigcup\{f_k^{(j)}\colon j<i\}\quad\text{for}\quad k<\omega_0;$$

and

$$F_i=\{f_k^{(i)}\colon k<\omega_0\},\quad M_i=\langle F_{i,0}\rangle\subset(B_j)B_i.$$

If $i=j+1$ then we extend the $\circ$—generators of M_j to B_i $(=B_{j+1}=B_j\cup A)$ in such a way that the resulted M_i will have the properties (*i*) through (iii) of Lemma 1.2 and $\pi_{\delta(j)}$ will have been killed. The latter means that there will be no permutation $\varrho\in S_{B_i}$ in $\mathrm{Loc}(M_i)$ such that $\varrho\restriction B_h=\pi_{\delta(j)}$, where $\delta(j)=(h,k)$ for some $k<\omega_1$ (since $\pi_{\delta(j)}$ is the k^{th} element of $\mathrm{Loc}(M_h)\cap S_{B_h}\setminus\{\mathrm{id}\restriction B_h\}$, $k<\omega_1$, $h\leqq j$).

This construction ensures that finally we will have a locally invertible (and, moreover, a still strongly fairly complete) monoid N $(=M_{\omega_1})$ on B $(=B_{\omega_1})$ such that $\mathrm{Loc}(N)\cap S_B=\{\mathrm{id}\}$. (To see this use the fact that every element of N maps $A_n=B_{n+1}\setminus B_n$ into A_n for every $n<\omega_1$ and so does every element of $\mathrm{Loc}(N)\cap S_B$. If then $\pi\in\mathrm{Loc}(N)\cap S_B$, $\pi\neq\mathrm{id}_B$, then $\pi\restriction B_h\neq\mathrm{id}\restriction B_h$ for some $h\in\omega_1$, and so, by the construction, $\pi\restriction B_h\in\mathrm{Loc}(M_h)\cap S_{B_h}\setminus\{\mathrm{id}_{B_h}\}$ say $\pi\restriction B_h=\pi_{h,k}$ for some $k<\omega_1$. Then $\delta(j)=(h,k)$ for some $j<\omega_1$, $j\geqq h$. In the j^{th} step, defining the elements of N on $A_j=B_{j+1}\setminus B_j$ we killed $\pi_{h,k}$, so $\varrho\restriction B_h\neq\pi_{h,k}=\pi\restriction B_h$ for each $\varrho\in\mathrm{Loc}(N_j)\cap\cap S_{B_j}$ which so holds for each $\varrho\in\mathrm{Loc}(N)\cap S_B$, a contradiction.)

Now we present a construction for $M_2=\langle F_2,\circ\rangle$, the other successor steps $i=j+1$ are the same. Write for convenience π and A_π instead of $\pi_{\delta(1)}$ and A_1. (Recall that $B_1=A_0=A$, $B_2=A\cup A_\pi$ and $M_1=M\subset{}^{B_1}B_1$, $M=\langle F,\circ\rangle$.) Step by step we extend the elements of F to A_π in ω_0 steps (A_π and F are countable) and we take these extended functions into $F_2=\{f_k^{(2)}:\ k<\omega_0\}\subset{}^{B_2}B_2$. We intend to define the values of $f_k^{(2)}$, $k<\omega_0$ on A_π as independent as possible.

After the n^{th} step we will have extended the first $k^{(n)}$ many elements of F to a finite set $W^{(n)}\subset A_\pi$. (The only important thing is that we extended only finitely many elements of F. We choose the first $k^{(n)}$ elements of F for convenience only.) Further we will have fixed finite sets $E_i\subset A$ for every $i\leqq k^{(n)}$. These $E_i=E_i^\pi$ sets are the most important objects in our construction. We require that $E_i\supset E_j$ for $i>j$ and $\varphi\restriction E_i\neq\pi\restriction E_i$ for every $\varphi\in\langle\{f_j:\ j\leqq i\},\circ,-1\rangle$. This can be done by Lemma 1.2 (iii). The sets E_i play an important role in choosing locally inverses for the extended functions and taking care of the fairly completeness of F_2 (see Case 3). Furthermore, in Lemma 2.2 we prove that if $a\in A_\pi$ is fixed, $\hat\pi\restriction A=\pi$ for some $\hat\pi\in\mathrm{Loc}(M_2)\cap\cap S_{A\cup A_\pi}$ and $m<\omega_0$ is large enough then for all $\varphi\in\langle H,\circ,-1\rangle$ either $\varphi(a)\neq\hat\pi(a)$ or $\varphi\restriction E_m\neq\pi\restriction E_m$ where $H=\{f_k^{(2)}:\ i\leqq m\}$; moreover this property is preserved in all further steps, that is H can be any finite subset of $F^{(2)}$. This clearly justifies that π will be killed.

Denote the extended functions by $\hat f_i$, that is $\mathrm{Dom}(\hat f_i)=A\cup W^{(n)}$ and $\hat f_i\restriction A=f_i$ for $i\leqq k^{(n)}$. To summarize: after the n^{th} step $(n<\omega_0)$ we will have $W^{(n)}\subset A_\pi$ finite, $k^{(n)}<\omega_0$ and $\{\hat f_i:\ i\leqq k^{(n)}\}$ where $\hat f_i$ extends f_i to $A\cup W^{(n)}$. ($\hat f_i$ depends upon n but we do not indicate this.) Finally let $A_\pi=\{a_j:\ j<\omega_0\}$ and let γ be a function from ω_0 onto the set

$$\omega_0\times[A_0]^{<\omega}\times[A_\pi]^{<\omega}\times\omega_0\times[[A]^{<\omega}]^{\omega}\times[[A_\pi]^{<\omega}]^{<\omega}\times[[F]^{<\omega}]^{<\omega}\times[A^*]^{<\omega}$$

and γ takes every value infinitely many times, where $A^*=\{g\restriction D:\ g\in{}^AA,\ D\in[A]^{<\omega}\}$ and $[X]^{<\omega}=\{Y\subset X:\ Y\text{ is finite}\}$ for any set X.

The role of γ is similar to that of δ: enumerates the requirements for M_2 to be locally invertible and fairly complete. The requirements listed by γ will be satisfied during the construction, in Case 3, $n=3l$.

Now let us see the construction itself.

In the 0^{th} step we do nothing: $W^{(0)}=\emptyset$ and no element of F is extended, $k^{(0)}=0$.

The $(n+1)^{\text{th}}$ step: let $W=W^{(n)}\subset A_\pi$ be the set constructed in the previous step and the function $f_0,f_1,\dots,f_k$ already extended be $\hat f_0,\hat f_1,\dots,\hat f_k$ with fixed sets $E_0,E_1,\dots,E_k$ where $k=k^{(n)}$. In ω_0 steps we have to define $\hat f(a)$ for all $f\in F$, $a\in A_\pi''$, and infinitely many locally inverses of $\hat f\restriction D$ for all $f\in F$, $D\subset A\cup A_\pi$ finite. In each step $n<\omega_0$ we either define $\hat f(a)$ for a new $a\in A_\pi$ or for a new $f\in F$ or we define some local inverse of an $\hat f\restriction D$, and we have to make each type of steps cofinally many times. Enumerate first A_π and F as $A_\pi=\{a_j:\ j<\omega_0\}$ and $F=\{f_k:\ k<\omega_0\}$.

Since the order of the steps is unimportant, for easier understanding we work modulo 3 and distinguish three cases:

Case 1: $n=3l+1$ for some $l<\omega_0$. If $a_l \in W^{(n)}$ then we have nothing to do i. e. $W^{(n+1)}=W^{(n)}$, $k^{(n+1)}=k^{(n)}$. Otherwise extend $f_0, f_1, \ldots, f_k$ $(k=k^{(n+1)}=k^{(n)})$ to $W^{(n+1)}=W^{(n)}\cup\{a_l\}$ totally independently from each other and the points used before. That is, for $i\leqq k$ let $\hat{f}_i(a_l)$ be an arbitrary element of the set

$$A_\pi - W^{(n)} - \{a_l\} - \cup\{\text{Range}(\hat{f}_j)\colon j\leqq k\} - \{\hat{f}_j(a_l)\colon j<i\}.$$

Then we put $W^{(n+1)}=W^{(n)}\cup\{a_l\}$ and $k^{(n+1)}=k^{(n)}$.

Case 2: $n=3l+2$ for some $l<\omega_0$. If $k=k^{(n)}\geqq l$ then we have nothing to do. (I.e. $W^{(n+1)}=W^{(n)}$, $k^{(n+1)}=k^{(n)}$.) If not, then extend all the functions $f_{k+1}, f_{k+2}, \ldots, f_l$ step by step to $W=W^{(n)}$ independently from each other and the points used before. That is, if $W=\{a_u\colon u<w\}$ for some $w<\omega_0$ then let for $u<w$ and i, $k<i\leqq l$ $\hat{f}_i(a_u)$ be an arbitrary element of the set

$$A_\pi - W - \cup\{\text{Range}(\hat{f}_j)\colon j<i\} - \{\hat{f}_i(a_t)\colon t<u\}.$$

Further, for every i, $k<i\leqq l$ by Lemma 1.2 (iii) (and by the induction hypothesis, that is M_m satisfies Lemma (i) through (iii) for every $m\leqq\omega_1$) we know that there is no bijection in $\text{Loc}(\langle f_j\colon j\leqq i\}, \circ, -1\rangle)$ except id_A. So we can choose an $E_i\subset A$ for $k<i\leqq l$ be finite such that $E_i\supset E_j$ and $\pi\restriction E_i\neq\text{id}\restriction E_i$ for $j<i$ and $\varphi\restriction E_i\neq\pi\restriction E_i$ for every $\varphi\in\text{Loc}(\langle\{f_t\colon t\leqq i\}, \circ, -1\rangle)$ and $k\leqq j<i\leqq l$. So in this case we construct $W^{(n+1)}=W^{(n)}$, $k^{(n+1)}=l$, $\hat{f}_{k+1}, \hat{f}_{k+2}, \ldots, \hat{f}_l$ and $E_{k+1}, E_{k+2}, \ldots, E_l$, too, $(k=k^{(n)})$ while $\hat{f}_i$ and E_i for $i\leqq k$ remain unchanged.

Case 3: $n=3l$ for some $l<\omega_0$. Now we have to do something only if $\gamma(l)$ codes a requirement for F_2 to be fairly complete.

First we clarify when $\gamma(l)$ codes such a requirement. We have

$$\gamma(l) = (l_1, X, Y, m_l, S_1, S_2, \zeta, G\rangle$$

where l_1, $m_l<\omega_0$ and $S_i=\{T_m^{(i)}\colon m<\nu^{(i)}\}\subset[A_i]^{<\omega_0}$ for some $\nu^{(i)}<\omega_0$, $i=1,2$ recall that $A_1=A$, $A_2=A_\pi$) and

$$\zeta = \{\mathscr{H}_m\colon m \nleqq \nu^{(3)}\} \subset [F]^{<\omega_0} \quad \text{for some} \quad \nu^{(3)}<\omega_0$$

and

$$G = \{g_m\colon m \nleqq \nu^{(4)}\} \subset A^* \quad \text{for some} \quad \nu^{(4)}<\omega_0.$$

Then $\gamma(l)$ codes such a requirement iff $|G|=|S_1|=|S_2|=|\zeta|=\nu$ and for every $m<\nu$ $\text{Dom}(g_m)=T_m^{(1)}$.

If the above statement does not hold then we have nothing to do. If it does, then we will construct m_l many locally inverses of $\hat{f}_{l_1}(X\cup Y)$ with taking care of the fairly completeness of F_2 with respect to $\mathscr{H}_m$ and

$$\hat{D}_m := T_m^{(1)}\cup T_m^{(2)} \quad (m<\nu).$$

Now do the following construction m_l times, repeatedly. (Repeatedly here means true physically repetitions: after one construction ends we start the whole procedure once more again from the very beginning, repeatedly increasing $k^{(n+1)}$ and $W^{(n+1)}$.)

First we have to suppose that $k=k^{(n)}$ and $W=W^{(n)}$ are large enough, that is $k\geqq l_1$, $k>\max\{t<\omega_0\colon f_t\in\mathscr{H}_m,\ m<\nu\}$ and $W\supset Y\cup\bigcup S_2\cup(\hat f_{l_1})''Y$. (If not, use the constructions described in Cases 1 and 2.)

In the construction we use the fairly completeness of F. We have already g_m, $\mathscr{H}_m$ for $m<\nu$. Now link the sequence $\pi\restriction E_i$ and $\{f_j\colon j\leqq i\}$ for $i\leqq k^{(n)}$ to the above sequence, that is put

$$g_{\nu+i}:=\pi\restriction E_i \quad\text{and}\quad \mathscr{H}_{\nu+i}:=\{f_j\colon j\leqq {}_i\}\quad\text{for}\quad i\leqq k^{(n)},\quad\text{so}\quad v=\nu+k^{(n)}+1.$$

Further, write f_{l_1} and X instead of f and D in Definition 1.1. Since F is fairly complete we have a function $t\in F$ with good properties; moreover such that $t\notin\{f_i\colon i\leqq k^{(n)}\}$. t is good in A. We will extend it to $W^{(n)}$ taking care of $\hat f$, Y and $T_m^{(2)}$ and arbitrary functions $\hat g_m$ on $T_m^{(2)}$ for $m<\nu$. The sets $T_m^{(2)}$ for $m<\nu$ are settled since $\bigcup\{T_m^{(2)}\colon m<\nu\}\subset W^{(n)}$. The sets $T_m^{(2)}$ for $\nu\leqq m<v$ and the functions $\hat g_m$ on $T_m^{(2)}$ for $m<v$ are unimportant since we will define $\hat t$ on $W^{(n)}$ (and later on further sets) totally independently from the other functions.

Now use the construction described in Case 2 to extend the functions f_i for $k<i\leqq k(t)$ to $W^{(n)}$ and determine the sets E_i with the method described in Case 2 with the restriction $\hat t\restriction(\hat f''Y)=(\hat f\restriction Y)^{-1}$ where $k(t)$ is defined as $t=f_{k(t)}\in F$. Though $R(\hat t)$ is not disjoint from $W^{(n+1)}=W^{(n)}$, we will see in the proof (see Lemmas 2.2 and 2.4) it does not make any trouble. Finally we put $W^{(n+1)}=W^{(n)}$ and $k^{(n+1)}=k(t)$ (and possibly repeat the construction m_l-1 times again). (To be somewhat more precise: let $W^{(n)}=\{a_u\colon u<w\}$ and for i, $k<i\leqq k(t)$ and $u<w$ if $i\neq k(t)$ or $a_u\notin f''Y$ then let $f_i(a_u)$ be an arbitrary element of the set

$$A_\pi\setminus W^{(n)}\cup\{\hat f_r W^{(n)}\colon r\leqq k\}\setminus\{\hat f_r(a_s)\colon s\leqq u,\ r<i\}).$$

This ends the construction. □

So we have extended all the generators of M to A_π. Let the $\circ$—generators of M_2 be F_2, the set of these extended $\hat f_i$ functions, $i<\omega_0$. We have to show that M_2 satisfies the requirements (i) through (iii) in Lemma 1.2 and that π does not extend to A_π. (i) and (iii) can be easily verified.

We only have to check that F_2 is strongly fairly complete and that π does not extend to A_π. (The other requirements are clearly satisfied.)

LEMMA 2.2. *π is not extended to A_π.*

PROOF. We prove a bit more: π can not be extended to an element of $\mathrm{Loc}(\langle F_2,\circ,-1\rangle)\cap S_{A\cup A_\pi}$.

Suppose it does. Let $a\in A_\pi$ be arbitrary fixed. If there is a $\hat\pi\in\mathrm{Loc}(\langle F_2,\circ,-1\rangle)\cap\cap S_{A\cup A_\pi}$ such that $\hat\pi\restriction A=\pi$ then $b=\hat\pi(a)=(\hat f_{i_1}\circ\hat f_{i_2}\circ\ldots\circ\hat f_{i_s})(a)$ for some $i_0, i_1, \ldots, \ldots, i_s<\omega_0$ and $s<\omega_0$. By the construction there is an $n<\omega_0$ large enough such that we have already extended all the functions f_{i_j} $(j\leqq s)$ till the n^{th} step so that $(\hat f_{i_0}\circ\hat f_{i_1}\circ\ldots\circ\hat f_{i_s})(a)$ is meaningful and equals $b\in W^{(n)}$. (That is, $k^{(n)}\geqq i_j$ for $j\leqq s$ and $(\hat f_{i_t}\circ\ldots\circ\hat f_{i_s})(a)$ for $0<t\leqq s$ and b are elements of $W^{(n)}$.)

Now fix such an arbitrary $n<\omega_0$. Recall that till the n^{th} step we have extended f_i $(i\leqq k\leqq k^{(n)})$ to $W=W^{(n)}$ and fixed the sets $E_i\subset A$ $(i\leqq k)$. By the definition of the set E_k we have $\varphi\restriction E_k\neq\pi\restriction E_k=\hat\pi\restriction E_k$ for every $\varphi\in\langle\{f_i\colon i\leqq k\},\circ,-1\rangle$ and $\pi\restriction E_k\neq\mathrm{id}\restriction E_k$. But by our indirect assumption there is a $k'<\omega_0$ such that $\psi\restriction E_k=$

$=\hat{\pi}\restriction E_k=\pi\restriction E_k$ for some $\psi\in\mathrm{Loc}\,\langle\{f_j\colon j\leqq k'\},\circ\rangle$ since $\hat{\pi}\in\mathrm{Loc}\,(M_2)$. Clearly we have $k'>k$ and we must have extended the functions $f_k, \ldots, f_{k'}$ till the n^{th} step, $n'>n$ and $k'=k^{(n')}$.

However the following result can be proved by induction on m, $m>n$, using that F_1 is fairly complete:

CLAIM 2.3. *For arbitrary* $m\geqq n$ *if we have extended the functions* f_i $(i\leqq k^{(n)})$ *to* $W^{(m)}$ *in steps* 0, 1, ..., m *then for every* $\varphi\in\langle\{\hat{f}_i\colon i\leqq k^{(m)}\},\circ,-1\rangle$ *either* $\varphi(a)\neq b$ *or* $\varphi\restriction E_k\neq\pi\restriction E_k$ *(here* $k=k^{(n)}$ *and* n *is fixed.)*

Obviously this claim proves Lemma 2.2.

PROOF. The proof is an easy induction on m, examining the effect of the construction in all three cases. The heart of our construction is that we always extended the functions totally independently from everything (the other functions and the points used before with a small restriction in Case 3).

The proof is rather easy but technical. The claim for $m=n$ is valid. Let $m\geqq n$ and $k^{(m)}$, $W^{(m)}$ as usual. We prove for $m+1$.

Fix any $\varphi\in\langle\{\hat{f}_i\colon i\leqq k^{(m+1)}\},\circ,-1\rangle$, say

$$\varphi=\hat{y}_0^{\varepsilon_0}\circ\hat{y}_1^{\varepsilon_1}\circ\ldots\circ\hat{y}_s^{\varepsilon_s}\quad\left(s<\omega_0,\ \varepsilon_n\in\{+1,-1\}\ \text{for}\ u\leqq s\right)\tag{4}$$

where $\hat{y}_u=\hat{f}_{i_u}$, $i_u\leqq k^{(m)}$. We have to show that either $\varphi(a)\neq b$ of $\varphi\restriction E_k\neq\pi\restriction E_k$ $(k=k^{(n)}$ is fixed), using that this statement holds for m, that is for all $\psi\in\langle\{\hat{f}_i\colon i\leqq k^{(m)}\},\circ,-1\rangle$. Suppose that $\varphi\restriction E_k=\pi\restriction E_k$ and $\varphi(a)=a$. Then we have $\varphi\restriction E_k\neq\mathrm{id}\restriction E_k$ since $\pi\restriction E_k\neq\mathrm{id}\restriction E_k$. So we may suppose that there is no part like $f\circ f^{-1}$ or $f^{-1}\circ f$ in φ. (That is in (4) there is no $u<s$ such that $y_u=y_{u+1}$ and $\varepsilon_u=1-\varepsilon_{u+1}$.)

According to the construction we have to distinguish three cases—which one was carried out to construct $k^{(m+1)}$, $W^{(m+1)}$, etc.

Case 1: $m=3l+1$ for some $l<\omega_0$. Then we extended the functions (among other functions) $\hat{y}_u$ $(u\leqq s)$ to $W^{(m+1)}=W^{(m)}\cup(a_l)$ totally independently from each other and the points used before. Since the induction hypothesis holds for m, by the construction it also holds for $m+1$.

Case 2: $m=3l+2$ for some $l<\omega_0$. Then $k^{(m+1)}\geqq l$ and we extended the functions $f_{k^{(m)}+1}, \ldots, f_l$ to $W^{(m+1)}=W^{(m)}$ totally independently from each other and the points used before. Since the induction hypothesis holds for m, it also holds for $m+1$, as well. (If $i_u>k^{(m)}$, that is there is a new function in (4), not constructed till the m^{th} step, we must have $\varphi(a)\neq b$. If not, then φ was constructed in the m^{th} step, and so we can use the induction hypothesis.)

Case 3: $m=3l$ for some $l<\omega_0$. This is the most crucial part of our proof.

In this case we constructed for some $l_1\leqq k^{(m)}$ (several) locally inverses $\hat{f}_t$ of the function $\hat{f}_{l_1}$ with respect to a set $X\cup Y\subset A\cup A_\pi$ $(l_1\leqq k^{(m)}<t\leqq k^{(m+1)},\ Y\subset W^{(m)})$. We took an $f_t\in F_1$, using the strongly fairly completeness of F_1, with respect to (among others) $\hat{\pi}\restriction E_k$ $(k=k^{(n)}$ is fixed) and extended f_t to $W^{(m+1)}=W^{(m)}$ totally independently from the functions and points used before (with the only restriction that $\hat{f}_t\circ\hat{f}_{l_1}\restriction Y=\mathrm{id}\restriction Y$ but this causes no trouble since $Y\cup(\hat{f}_{l_1})''Y\subset W^{(m)}$).

If $i_u\neq t$ for all $u<s$ (that is $\hat{f}_t$ does not appear in φ in (4)) then by (4) we know

that φ has already been constructed before the $(m+1)^{\text{th}}$ step and using the induction hypothesis we are done.

So $\hat{f}_t$ appears in (4).

Write φ' for the function we get by replacing $\hat{f}_t$ by $(\hat{f}_{l_1}\restriction(X\cup Y))^{-1}$ and $(\hat{f}^t)^{-1}$ by $\hat{f}_{l_1}\restriction(X\cup Y)$ in φ. Using the good properties of f_t by the strongly fairly completeness of F_1 and our indirect assumption $\varphi\restriction E_k=\pi\restriction E_k$ we may replace f_t by $(f_{l_1}\restriction X)^{-1}$ in $\varphi\restriction A$ everywhere and so we have $\varphi'\restriction E_n=\varphi\restriction E_k=\pi\restriction E_k$ $(k=k^{(n)}$ is fixed).

Next we show that we can derive $\varphi'(a)=b$ using the indirect assumption $\varphi(a)=b$. We defined $\hat{f}_t$ totally independently on $W^{(m)}\setminus\text{Range}(\hat{f}_l\restriction Y)$ from the points used before and we defined $\hat{f}_t$ on $\text{Range}(\hat{f}_{l_1}\restriction Y)$ to be the inverse of $\hat{f}_{l_1}\restriction Y$. It follows that supposing $\varphi(a)$ is meaningful and equals to b we have that $\varphi'(a)$ is meaningful and equals $\varphi(a)=b$ (since b was an old point, too, that is $a, b\in W^{(m)}$ and $Y\cup(\hat{f}_{l_1})''Y\subset W^{(m)}$).

So we have $\varphi'\restriction E_k=\pi\restriction E_k$ and $\varphi'(a)=\varphi(a)=b=\hat{\pi}(a)$. But φ' only consists of functions constructed before the $(m+1)^{\text{th}}$ step and by the induction hypothesis this is a contradiction.

So we proved Claim 2.3 and so Lemma 2.2. too. $\square$

In order to carry out our construction in further steps (for $M_3, M_4, \ldots$ and for any M_{i+1} $(i<\omega_1)$) we must also to preserve the strongly fairly completeness of F.

LEMMA 2.4. *F_2 is strongly fairly complete.*

PROOF. The proof is mainly included in the construction: in Case 3 we manage the fairly completeness of F_2, and do not destroy it in further steps.

Observe first that the following fact is true: for every $n_1<n_2<\omega_0$ if untill the n_i-th steps $(i=1,2)$ we have extended the functions $\{f_j\colon j\leqq k^{(n_i)}\}$ to the functions $\{\hat{f}_j^{(i)}\colon j\leqq k^{(n_i)}\}$, $\text{Dom}(\hat{f}_j^{(i)})=A\cup W^{(n_i)}$ for $j\leqq k^{(n_i)}$ and $i=1,2$ then we have

$$W^{(n_1)}\subseteqq W^{(n_2)} \quad\text{and}\quad \hat{f}_j^{(1)}\subseteqq\hat{f}_j^{(2)} \quad\text{for}\quad j\leqq k^{(1)} \tag{5}$$

$$\text{Range}(\hat{f}_i^{(1)}\restriction W^{(n_1)})\cap\text{Range}(\hat{f}_j^{(2)}\restriction(W^{(n_2)}\setminus W^{(n_1)}))=\emptyset \quad\text{for}\quad i,j<k^{(n_1)}. \tag{6}$$

(That is: (5) says that we keep extending our functions, and (6) says that we define all functions independently from each other and the points used before.)

This fact can be proved by a simple induction on n_2, $n_1\leqq n_2<\omega_0$.

Now, recall that $F_2=\{\hat{f}_i\colon i<\omega_0\}$, $\hat{f}_i\restriction A=f_i\in F_1$ for $i<\omega_0$. Let us be given $v<\omega_0$, $D\subset A\cup A_\pi$, $\hat{f}_j\in F_2$, $\mathscr{H}_m\subset F_2$, $D_m\subset A\cup A_\pi$ finite and $g_m\colon D_m\to A\cup A_\pi$ for $m<v$ as in the definition of strongly fairly completeness. We have to find some $t=t(j)<\omega_0$ such that $\hat{f}_t$ has certain good preperties.

Clearly we may suppose that

$$\text{Range}(g_m\restriction(A\cap D_m))\subset A \quad\text{and}\quad \text{Range}(g_m\restriction(A_\pi\cap D_m))\subset A_\pi$$

for $m<v$. Choose an $n_0<\omega_0$ large enough such that in the n_0-th step we can talk about the above functions and sets, that is we have already extended all the elements of the set

$$\hat{H}=\{f_j\}\cup\{f_u\colon \hat{f}_u\in\mathscr{H}_m,\ m<v\}$$

to the set $W^{(n_0)}\subset A_\pi$, and $\hat{D}\cap A_\pi\subset W^{(n_0)}$ where

$$\hat{D}=D\cup\bigcup\{D_m\cup\text{Range}(g_m)\colon m<v\}\cup f_j''D.$$

We know that there are infinitely many $l_0>n_0$ large enough such that in the $n=3l_0$-th step we found a local inverse f_t of $\hat{f}_j\upharpoonright D$ with respect to g_m and $\mathscr{H}_m$ $(m<v)$ taking care of the fairly completeness of F_2. (See Case 3 of the construction.) By the construction we have exactly one $\hat{f}_t\in F_2$ such that $\hat{f}_t\upharpoonright A=f_t$. We claim that $\hat{f}_t$ works.

Roughly speaking, $\hat{f}_t$ was extended as independently from $W^{(n)}$, the points and the functions used before as possible and this causes $\hat{f}_t$ to work.

Obviously we have $\hat{f}_t\circ(\hat{f}_j\upharpoonright D)=\mathrm{id}$. We have no trouble with the sets $D\cap A$, $D_m\cap A$ and $g_m\upharpoonright(D_m\cap A)$ $(m<v)$ since all members of F_2 map A into A and A_π into A_π and F_1 was strongly fairly complete. We also do not have trouble with the sets $D\cap A_\pi$, $D_m\cap A_\pi$ and $g_m\upharpoonright(D_m\cap A_m)$ $(m<v)$ using the construction (that is $\hat{f}_t$ was defined totally independently) and (5) and (6) for induction for $m>n$. By the construction the set $\{\hat{f}_i^{(n)}\colon i\leqq k^{(n)}\}$ is strongly fairly complete for the full sets D, D_m, g_m and $\mathscr{H}_m$ $(m<v)$. Further (5) and (6) ensure that we can not damage these good properties of $\hat{f}_t$ in any further step $m<\omega_0$ for $m>n$.

Finally, since this holds for all $m<\omega_0$ (m is large enough), it must hold for F_2 also (better to say, for $\hat{f}_t\in F_2$).

This proves Lemma 2.4 and so Theorem 2.1. □

3. Further results

In this section we use the ideas of Sections 1 and 2 to prove further theorems.

THEOREM 3.1. (a) MA *implies* $\neg P(2^{\aleph_0})$.
(b) MA(λ) *implies* $P(\lambda)$ *for* $\lambda<2^{\aleph_0}$ *and for countable monoids.*

PROOF. (a) The method is rather similar to the one presented in the proof of Theorem 2.1. Let $\{\pi_i\colon i<2^{\aleph_0}\}$ enumerate $\mathrm{Loc}\,(M)\cap S_A-\{\mathrm{id}_A\}$, let A_j be pairwise disjoint countable sets for $j<2^{\aleph_0}$, $A_0=A$ and let $B_i=\bigcup\{A_j\colon j<i\}$ for $i\leqq 2^{\aleph_0}$. Extend the elements of M succesively to B_i by killing π_i (and of course use the coding function $\delta\colon 2^{\aleph_0}\to 2^{\aleph_0}\times 2^{\aleph_0}$ as in Theorem 2.1 and use the fact that MA implies $2^\tau=2^{\aleph_0}$ for $\tau<2^{\aleph_0}$). The only difference is the succesive step: killing a permutation $\pi\in\mathrm{Loc}\,(M)\cap S_A$.

First we briefly sketch how to find a suitable forcing notion $\langle P,\leqq\rangle$ in the proof of Theorem 2.1. We know that the set of generators of M is $F=\{f_i\ i<\omega_0\}$ and there is no permutation in $\mathrm{Loc}\,(\langle\{f_j\colon j<i\},\circ,-1\rangle)$ for every $i<\omega_0$. So for every $i<\omega_0$ we can fix a finite subset $E_i\subset A$ such that $\varphi\upharpoonright E_i\neq\pi\upharpoonright E_i$ for every $\varphi\in\mathrm{Loc}\,(\langle\{f_j\colon j<i\},0,-1\rangle)$ and $E_i\subset E_j$ for $i<j<\omega_0$, Let $\langle P^{(0)},\leqq^{(0)}\rangle$ be the following forcing notion: $P^{(0)}$ consists of the forcing conditions of the form

$$p=\langle D^{(p)},\langle\hat{f}_1^{(p)},\ldots,\hat{f}_{k^{(p)}}^{(p)}\rangle\rangle$$

such that $k^{(p)}<\omega_0$, $D^{(p)}$ is a finite subset of A_π and $\hat{f}_i^{(p)}$ is a one-to-one extension of f_i to $A\cup D$ for $i\leqq k^{(p)}$.

Define the partial order $\leqq^{(0)}$ on $P^{(0)}$ as $p_1\leqq^{(0)}p_2$ iff $k^{(2)}\leqq k^{(1)}$ and for every $i\leqq k^{(p_2)}$ $\hat{f}_i^{(p_2)}\subseteqq\hat{f}_i^{(p_1)}$. Now define the subordering $\leqq$ of $\leqq^{(0)}$ as $p_1\leqq p_2$ iff we obtained p_1 from p_2 using some (but finite) steps described in the proof of theorem 2.1. Clearly the largest element of P^0 is $1_P=\langle 0,0\rangle$. Then we define $\langle P,\leqq\rangle$ as $P=\{p\in P^{(0)}\colon p\leqq 1_P\}$ and we have already defined $\leqq$ above.

P is countable so it satisfies the ccc.

The following subsets of P are dense:

$$D_a = \{p \in P\colon a \in D^{(p)}\} \quad \text{for} \quad a \in A_\pi,$$

$$D_j = \{p \in P\colon j \leqq k^{(p)}\} \quad \text{for} \quad j \leqq \omega_0,$$

and

$$D_{j,m,D} = \{p \in P\colon j \leqq k^{(p)} \text{ and } D \cup \hat{f}_j'' D \subset D^{(p)}$$

and $\hat{f}_j \restriction D$ has at least m locally inverse among the functions $\{\hat{f}_j\colon j \leqq k^{(p)}\}\}$

for $j, m < \omega_0$ and $D \subset A \cup A_\pi$ finite .

Applying Martin's axiom we get the desired extension of our monoid M to $A \cup A_\pi$ as in Theorem 2.1. $\square$

(b) Let $|A| = \lambda$. The forcing notions

$$P_{f,D} = \{g \restriction H;\ g \in M,\ H \in [A]^{<\omega},\ f \restriction (H \cap D) = g \restriction (H \cap D)\} \quad (f \in M,\ D \in [A]^{<\omega})$$

ordered by reversed inclusion satisfy the ccc since M is countable. By MA we get a generic subset $G \subset P$ intersecting all the dense sets $D_a = \{g \restriction H \in P_{f,D}\colon a \in H\ \&\ a \in \mathrm{Range}\,(g \restriction H)\}$ for $a \in A$. This proves Theorem 3.1. $\square$

THEOREM 3.2. $2^{\aleph_0} = \aleph_2 + \neg\mathrm{MA}$ *with* $\neg P(2^{\aleph_0})$ *is consistent.*

PROOF. The forcing notion P defined in the proof of Theorem 3.1 is countable so we can apply a weak form of Martin's axiom which is consistent with $2^{\aleph_0} = \aleph_2 + {} + \neg\mathrm{MA}$:

THEOREM 3.3 (C. Hernik, [W, Theorem 5.7, p. 848]). *If there is a model of set theory then there is one in which we have*

(i) $2^{\aleph_0} = \aleph_2$,
(ii) SH,
(iii) MA($\aleph_0$-*linked*)
(iv) $\neg$ MA.

(For the definitions see e.g. [*K*] or [*W*].)

We only have to know that every countable poset is $\aleph_0$-linked. Then we proceed as in the proof of Theorem 3.1 (a) and apply Herink's theorem. Use the fact that MA($\aleph_0$-linked) also implies $2^\tau = 2^{\aleph_0}$ for $\tau < 2^{\aleph_0}$. This proves Theorem 3.2.

REMARK. We could get a suitable model for Theorem 3.2 simply adding $\aleph_2$ Cohen reals to an arbitrary model of ZFC (well-known or see e.g. [*W*]).

THEOREM 3.4. $2^\lambda = \lambda^+$ *implies* $\neg P(2^\lambda)$ *for any cardinal* λ.

PROOF. First construct a set C and a monoid $M_\lambda \subset {}^C C$ both of power λ taking λ disjoint copies of M constructed in Lemma 1.2. (In other words let $C = \bigcup \{C_i\colon i < \lambda\}$ where C_i are pairwise disjoint sets of power $\aleph_0$ and let $M_i \subset {}^{C_i} C_i$ be a monoid isomorphic to M of Lemma 1.2 with generator set F_i.) Put $\hat{F}_i = \{f \in {}^C C\colon f \restriction C_i = f'$ and $f \restriction (C - C_i) = \mathrm{id}$ for some $f' \in F_i\}$ and let $F_\lambda = \bigcup \{\hat{F}_i\colon i < \lambda\}$ and $M_\lambda = \langle F_\lambda, 0 \rangle$. Clearly F_λ satisfies the properties described in Lemma 1.2. Now extend M_λ step by step to a set of power of λ^+ by killing every permutation in Loc (M_λ) using a coding function δ: $\lambda \to \lambda \times \lambda$. When we kill a single permutation π we extend the elements of F_λ to $C \cup C_\pi$ where $|C_\pi| = \lambda$, in λ setps (where the sets C_π are pairwise disjoint). I do not think the details are worth writing down. □

The same argument proves $\neg P(\varkappa)$ for $\varkappa$ strong limit.

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OSCILLATION AND NON-OSCILLATION THEOREMS FOR A CLASS OF SECOND ORDER QUASILINEAR DIFFERENTIAL EQUATIONS

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1. Introduction

We consider the quasilinear ordinary differential equation

$$(p(t)\varphi(y'))'+f(t, y) = 0 \tag{1}$$

subject to the hypotheses:

$$\text{(2)}\quad\begin{cases}\text{(a) } p\colon [0,\infty)\to(0,\infty) \text{ is continuous;}\\ \text{(b) } \varphi\colon \mathbf{R}\to\mathbf{R} \text{ is continuous, strictly increasing and such that } \operatorname{sgn}\varphi(u)=\operatorname{sgn} u\\ \quad\text{and } \varphi(\mathbf{R})=\mathbf{R};\\ \text{(c) } f\colon [0,\infty)\times\mathbf{R}\to\mathbf{R} \text{ is continuous, nondecreasing with respect to the second}\\ \quad\text{variable and such that } \operatorname{sgn} f(t, v)=\operatorname{sgn} v \text{ for each fixed } t\geqq 0.\end{cases}$$

A prototype of (1) satisfying (2) is

$$((y')^{m*})'+q(t)y^{n*} = 0, \tag{3}$$

where m and n are positive constants, $q\colon [0,\infty)\to(0,\infty)$ is a continuous function, and use is made of the notation

$$u^{\lambda *} = |u|^{\lambda}\operatorname{sgn} u, \quad \lambda > 0. \tag{4}$$

Our main objective is to investigate in detail the oscillatory and nonoscillatory behavior of proper solutions of (1). By a proper solution of (1) we mean a function $y\colon [T_y,\infty)\to\mathbf{R}$ which satisfies equation (1) (so that $p(t)\varphi(y'(t))$ is continuously differentiable) for all sufficiently large t and $\sup\{|y(t)|\colon t\geqq T\}>0$ for any $T\geqq T_y$. A proper solution is called oscillatory if it has arbitrarily large zeros, and nonoscillatory otherwise. Under additional hypotheses on p, φ and f, first we study the structure of the set of nonoscillatory solutions of (1), and then establish criteria for all proper solutions of (1) to be oscillatory. Thus we are able to indicate a wide class of equations of the form (1), including (3) with $m\neq n$, for which the oscillation of all proper solutions can be completely characterized.

The qualitative behavior of equation (3) has been studied by several authors including Elbert [2—5], Izjumova and Mirzov [7], Kitamura and Kusano [8], Mirzov [9—11] and Piros [12]; however, equation (1) in its general form does not seem to have been the object of systematic investigation.

2. Existence of nonoscillatory solutions

Throughout the paper we make the following assumption without further mentioning:

$$(5) \qquad \int_0^\infty \left|\varphi^{-1}\left(\frac{k}{p(t)}\right)\right| dt = \infty \quad \text{for every constant} \quad k \neq 0,$$

where φ^{-1}: $\mathbf{R}\to\mathbf{R}$ denotes the inverse function of φ, and employ the notation:

$$(6) \qquad \begin{cases} \Phi_{k,T}(p;\ t) = \displaystyle\int_T^t \varphi^{-1}\left(\frac{k}{p(s)}\right) ds, & t \geqq T, \\ \Phi_k(p;\ t) = \Phi_{k,0}(p;\ t), \quad t \geqq 0. \end{cases}$$

From (5) and (6) it is obvious that $\Phi_{k,T}(p; T)=0$,

$$\lim_{t\to\infty} |\Phi_{k,T}(p;\ t)| = \infty \quad \text{for every} \quad k \neq 0,$$

$$|\Phi_{k,T}(p; t)| > |\Phi_{l,T}(p; t)|, \quad t > T, \quad \text{for} \quad |k| > |l| \quad \text{with} \quad kl > 0,$$

and

$$\lim_{k\to 0} \Phi_{k,T}(p;\ t) = 0 \quad \text{for each} \quad t \geqq T.$$

We begin by classifying all possible nonoscillatory solutions of equation (1) according to their asymptotic behavior as $t\to\infty$.

LEMMA 1. *Any nonoscillatory solution $y(t)$ of (1) is of one of the following three types:*

I. $\lim\limits_{t\to\infty} p(t)\varphi(y'(t)) = \text{const} \neq 0;$

II. $\lim\limits_{t\to\infty} p(t)\varphi(y'(t)) = 0, \quad \lim\limits_{t\to\infty} |y(t)| = \infty;$

III. $\lim\limits_{t\to\infty} p(t)\varphi(y'(t)) = 0, \quad \lim\limits_{t\to\infty} y(t) = \text{const} \neq 0.$

PROOF. Let $y(t)$ be a nonoscillatory solution of (1). Without loss of generality we may suppose by (b), (c) that $y(t)>0$ for $t\geqq t_0>0$. From (1), $(p(t)\varphi(y'(t)))'= = -f(t, y(t))<0$, $t\geqq t_0$, and so $p(t)\varphi(y'(t))$ is decreasing for $t\geqq t_0$. We claim that $p(t)\varphi(y'(t))>0$, $t\geqq t_0$, so that $\lim\limits_{t\to\infty} p(t)\varphi(y'(t))\geqq 0$. In fact, if $p(t_1)\varphi(y'(t_1))= =-k<0$ for some $t_1\geqq t_0$ and $k>0$, then $p(t)\varphi(y'(t))\leqq -k$ for $t\geqq t_1$, which is equivalent to $y'(t)\leqq\varphi^{-1}(-k/p(t))$, $t\geqq t_1$. Integrating the last inequality from t_1 to t and letting $t\to\infty$, we see in view of (5) that $y(t)\to-\infty$ as $t\to\infty$. But this contradicts the assumed positivity of $y(t)$. Therefore, $p(t)\varphi(y'(t))>0$ for $t\geqq t_0$, as claimed. A consequence of this observation is that $y'(t)>0$ for $t\geqq t_0$, i.e., the function $y(t)$ is strictly increasing.

The limit $\lim\limits_{t\to\infty} p(t)\varphi(y'(t))$ is either positive or zero. In the first case, $y(t)$ is unbounded, since there are positive constants k_1, k_2, and t_0 ($k_1<k_2$) such that

$$\Phi_{k_1,t_0}(p;\ t) \leqq y(t)-y(t_0) \leqq \Phi_{k_2,t_0}(p;\ t), \quad t \geqq t_0.$$

In the second case, since $y(t)$ is increasing, $y(t)$ tends to a positive limit, finite or infinite, as $t\to\infty$. This completes the proof.

Now we give criteria for the existence of nonoscillatory solutions of (1) of types I, II and III.

THEOREM 1. *Suppose that for each fixed* $k\neq 0$ *and* $T\geqq 0$,

$$\lim_{l\to 0,\, lk>0} \frac{\Phi_{l,T}(p;\, t)}{\Phi_{k,T}(p;\, t)} = 0 \tag{7}$$

uniformly on any interval of the form $[T', \infty)$, $T'>T$. *Then equation* (1) *has a nonoscillatory solution of type* I *if and only if*

$$\int_0^\infty |f(t, c\Phi_k(p;\, t))|\, dt < \infty \tag{8}$$

for some constants $k\neq 0$ *and* $c>0$.

THEOREM 2. *Equation* (1) *has a nonoscillatory solution of type* III *if and only if*

$$\int_0^\infty \left|\varphi^{-1}\left(\frac{1}{p(t)}\int_t^\infty f(s, c)\, ds\right)\right| dt < \infty \tag{9}$$

for some constant $c\neq 0$.

PROOF OF THEOREM 1. (The "only if" part.) Let $y(t)$ be a nonoscillatory solution of type I of (1). Without loss of generality $y(t)$ may be assumed to be eventually positive. There exist positive constants c_1, k_1 and t_0 such that $c_1\Phi_{k_1}(p;\, t)\leqq y(t)$ for $t\geqq t_0$. An integration of (1) yields $\int_t^\infty f(t, y(t))dt<\infty$, which combined with the above inequality leads to

$$\int_{t_0}^\infty f(t, c_1\Phi_{k_1}(p;\, t))\, dt < \infty.$$

(The "if" part.) Suppose (8) holds for some $c>0$ and $k>0$. Because of (7) we can choose $l>0$ and $T>0$ so that $l<k/2$ and

$$\int_T^\infty f(t, \Phi_{2l}(p;\, t))\, dt \leqq l. \tag{10}$$

Define the subset Y of $C[T, \infty)$ and the mapping $\mathcal{F}\colon Y\to C[T, \infty)$ by

$$Y = \{y\in C[T, \infty)\colon \Phi_{l,T}(p;\, t) \leqq y(t) \leqq \Phi_{2l,T}(p;\, t),\ t\geqq T\} \tag{11}$$

and

$$\mathcal{F}y(t) = \int_T^t \varphi^{-1}\left(\frac{1}{p(s)}\left[l+\int_s^\infty f(r, y(r))\, dr\right]\right) ds, \quad t\geqq T. \tag{12}$$

(i) *$\mathscr{F}$ maps Y into itself.* If $y \in Y$, then since

$$0 \leqq \int_s^\infty f(r, y(r))\, dr \leqq \int_T^\infty f(r, \Phi_{2l}(p;\, r))\, dr \leqq l, \quad s \geqq T,$$

we obtain from (12)

$$\int_T^t \varphi^{-1}\left(\frac{l}{p(s)}\right) ds \leqq \mathscr{F} y(t) \leqq \int_T^t \varphi^{-1}\left(\frac{2l}{p(s)}\right) ds, \quad t \geqq T,$$

implying that $\mathscr{F} y \in Y$.

(ii) *$\mathscr{F}$ is continuous.* Let $\{y_\nu\}$ be a sequence of elements of Y converging to $y \in Y$ as $\nu \to \infty$ in the topology of $C[T, \infty)$. The Lebesgue dominated convergence theorem shows that

$$\int_T^\infty f(t, y_\nu(t))\, dt \to \int_T^\infty f(t, y(t))\, dt \quad \text{as} \quad \nu \to \infty,$$

and so

$$\int_t^\infty f(s, y_\nu(s))\, ds \to \int_t^\infty f(s, y(s))\, ds \quad \text{as} \quad \nu \to \infty$$

uniformly on $[T, \infty)$. It follows that $\mathscr{F} y_\nu(t) \to \mathscr{F} y(t)$ as $\nu \to \infty$ uniformly on compact subintervals of $[T, \infty)$, which implies the convergence $\mathscr{F} y_\nu \to \mathscr{F} y$ in $C[T, \infty)$.

(iii) *$\mathscr{F}(Y)$ is relatively compact.* This follows from the relation

$$0 \leqq (\mathscr{F} y)'(t) = \varphi^{-1}\left(\frac{1}{p(t)}\left(l + \int_t^\infty f(s, y(s))\, ds\right)\right) \leqq$$

$$\leqq \varphi^{-1}\left(\frac{1}{p(t)}\left(l + \int_t^\infty f(s, \Phi_{2l}(p;\, s))\, ds\right)\right), \quad t \geqq T,$$

holding for all $y \in Y$.

Therefore, applying the Schauder—Tychonoff fixed point theorem [1, 6], we see that there exists an element $y \in Y$ such that $y = \mathscr{F} y$. Differentiation of the integral equation $y(t) = \mathscr{F} y(t)$, $t \geqq T$, shows that $y(t)$ is a positive solution of equation (1) for $t \geqq T$. It is obvious that $y(t)$ is of type I.

If (8) holds for some $k < 0$ and $c > 0$, a similar argument is used to construct a negative solution of type I of equation (1).

Proof of Theorem 2. (The "only if" part.) Let $y(t)$ be a positive solution of type III of (1). There are positive constants c_1 and t_0 such that $y(t) \geqq c_1$ for $t \geqq t_0$. Integrating (1) from t to ∞, we have

$$p(t)\varphi(y'(t)) = \int_t^\infty f(s, y(s))\, ds, \quad t \geqq t_0,$$

which implies

$$y'(t) = \varphi^{-1}\left(\frac{1}{p(t)} \int_t^\infty f(s, y(s))\, ds\right), \quad t \geqq t_0.$$

Integrating this equation again and using $y(t)\geqq c_1$, we conclude that

$$\int_{t_0}^{\infty} \varphi^{-1}\left(\frac{1}{p(t)}\int_t^{\infty} f(s, c_1)\, ds\right) dt < \infty.$$

A parallel argument holds if $y(t)$ is supposed to be a negative solution of (1).

(The "if" part.) Suppose that (9) holds for some $c>0$. Let $T>0$ be such that

$$\int_T^{\infty} \varphi^{-1}\left(\frac{1}{p(t)} f(s, c)\, ds\right) dt \leqq \frac{c}{2} \tag{13}$$

and define

$$Y = \left\{y\in C[T, \infty)\colon \frac{c}{2} \leqq y(t) \leqq c,\ t \geqq T\right\} \tag{14}$$

and

$$\mathcal{F} y(t) = c - \int_t^{\infty} \varphi^{-1}\left(\frac{1}{p(s)}\int_s^{\infty} f(r, y(r))\, dr\right) ds, \quad t \geqq T. \tag{15}$$

It is easy to verify that $\mathcal{F}$ is continuous and maps Y into a compact subset of Y, and hence $\mathcal{F}$ has a fixed element y in Y, which gives the desired solution of type III of equation (1). Similarly, (1) is shown to possess a negative solution of type III if (9) holds for some $c<0$. This completes the proof.

Nonoscillatory solutions of types I and III have thus been characterized. A characterization of type II solutions is difficult to obtain, and so we content ourselves with sufficient conditions for the existence of such solutions of (1).

THEOREM 3. *Suppose* (7) *holds. Equation* (1) *has a nonoscillatory solution of type* II *if* (8) *holds for some* $k\neq 0$ *and* $c>0$ *and*

$$\int_0^{\infty} \left|\varphi^{-1}\left(\frac{1}{p(t)}\int_t^{\infty} f(s, d)\, ds\right)\right| dt = \infty \tag{16}$$

for every nonzero constant d such that $kd>0$.

PROOF. It suffices to consider the case where $k>0$ and $d>0$. Let $a>0$ be an arbitrary fixed constant, and choose $l>0$ small enough and $T>0$ large enough so that $a+\Phi_l(p; t)\leqq c\Phi_k(p; t)$ for $t\geqq T$ and

$$\int_T^{\infty} f(t, a+\Phi_l(p;\ t))\, dt \leqq l. \tag{17}$$

This is possible because of (7) and the fact that $\lim\limits_{t\to\infty} \Phi_k(p; t)=\infty$. Now consider the set Y and the mapping $\mathcal{F}$ defined by

$$Y = \{y\in C[T, \infty)\colon a \leqq y(t) \leqq a+\Phi_l(p; t),\ t \geqq T\} \tag{18}$$

and

$$\mathcal{F} y(t) = a + \int_T^{t} \varphi^{-1}\left(\frac{1}{p(s)}\int_s^{\infty} f(r, y(r))\, dr\right) ds, \quad t \geqq T. \tag{19}$$

As is easily verified, $\mathscr{F}$ has a fixed element $y \in Y$ by the Schauder—Tychonoff theorem. That $y=y(t)$ is a solution of equation (1) follows from differentiation of the integral equation $y(t)=\mathscr{F}y(t)$, $t \geqq T$. From this equation we also see that

$$p(t)\varphi(y'(t)) = \int_t^\infty f(s, y(s))\, ds \to 0 \quad \text{as} \quad t \to \infty$$

and by (16)

$$y(t) \geqq a + \int_T^\infty \varphi^{-1}\left(\frac{1}{p(s)} \int_s^\infty f(r, a)\, dr\right) ds \to \infty \quad \text{as} \quad t \to \infty.$$

It follows therefore that $y(t)$ is a solution of type II. This finishes the proof.

EXAMPLE 1. Consider the equation

$$((y')^{m*})' + q(t) y^{n*} = 0, \tag{3}$$

where $m>0$, $n>0$ and $q\colon [0, \infty) \to (0, \infty)$ is continuous. This is a special case of (1) in which

$$p(t) = 1, \quad \varphi(u) = u^{m*} \quad \text{and} \quad f(t, v) = q(t) v^{n*}$$

and we have

$$\varphi^{-1}(u) = u^{1/m*} \quad \text{and} \quad \Phi_{k,T}(p; t) = k^{1/m*}(t-T),$$

so that conditions (5) and (7) are satisfied for equation (3).

The possible types of asymptotic behaviour at infinity of nonoscillatory solutions of (3) are as follows:

I. $\lim\limits_{t\to\infty} [y(t)/t] = \text{const} \neq 0;$

II. $\lim\limits_{t\to\infty} [y(t)/t] = 0, \quad \lim\limits_{t\to\infty} |y(t)| = \infty;$

III. $\lim\limits_{t\to\infty} y(t) = \text{const} \neq 0.$

From Theorems 1 and 2 it follows that (3) has a solution of type I if and only if

$$\int_0^\infty t^n q(t)\, dt < \infty, \tag{20}$$

and that (3) has a solution of type III if and only if

$$\int_0^\infty \Big(\int_t^\infty q(s)\, ds\Big)^{1/m} dt < \infty. \tag{21}$$

Theorem 3 implies that the conditions (20) and

$$\int_0^\infty \Big(\int_t^\infty q(s)\, ds\Big)^{1/m} dt = \infty \tag{22}$$

are sufficient for the existence of type II solution of (3).

Conditions (20) and (22) are not always consistent. In fact, let $q(t)=(t+1)^\lambda$, where λ is a constant. Then, (20) holds if and only if $\lambda < -1-n$, and (22) holds if and only if $\lambda\geqq -1-m$, and hence these two conditions are inconsistent if $m\leqq n$. Thus, if $m>n$ and $-1-m\leqq\lambda<-1-n$, then there exists a solution of type II for the equation

$$((y')^{m*})'+(t+1)^{\lambda}y^{n*}=0.$$

3. Oscillation of all solutions

We are interested in the situation in which equation (1) has no nonoscillatory solution, or equivalently, all proper solutions of (1) are oscillatory, and show that a characterization for this situation can be obtained provided additional hypotheses are placed on the nonlinearity of (1).

We say that equation (1) is *strongly superlinear* if there exists a constant $\gamma>0$ such that $|v|^{-\gamma}|f(t,v)|$ is nondecreasing in $|v|$ for each fixed t and

$$(23)\qquad \int_M^{\infty}\frac{dv}{\varphi^{-1}(v^{\gamma})}<\infty \quad\text{and}\quad \int_{-M}^{-\infty}\frac{dv}{\varphi^{-1}(v^{\gamma *})}<\infty \quad\text{for any}\quad M>0;$$

equation (1) is *strongly sublinear* if there exists a constant $\delta>0$ such that $|v|^{-\delta}|f(t,v)|$ is nonincreasing in $|v|$ for each fixed t and

$$(24)\qquad \int_0^{N}\frac{dv}{[\varphi^{-1}(v)]^{\delta}}<\infty \quad\text{and}\quad \int_{-N}^{0}\frac{dv}{[\varphi^{-1}(v)]^{\delta *}}<\infty \quad\text{for any}\quad N>0.$$

According to the above definition, equation (3) is strongly superlinear if $m<n$ and strongly sublinear if $m>n$.

THEOREM 4. *Suppose equation* (1) *is strongly superlinear. Suppose moreover that*

$$(25)\qquad \varphi^{-1}(uv)\geqq\varphi^{-1}(u)\varphi^{-1}(v)\quad \textit{for all } u,v \textit{ with } uv>0.$$

Then all proper solutions of (1) *are oscillatory if and only if*

$$(26)\qquad \int_0^{\infty}\left|\varphi^{-1}\left(\frac{1}{p(t)}\int_t^{\infty}f(s,c)\,ds\right)\right|dt=\infty$$

for every constant $c\neq 0$.

THEOREM 5. *Let equation* (1) *be strongly sublinear and suppose* (7) *and* (25) *hold. Then all proper solutions of* (1) *are oscillatory if and only if*

$$(27)\qquad \int_0^{\infty}\left|f(t,c\Phi_k(p;\ t))\right|dt=\infty$$

for all constants $k\neq 0$ *and* $c>0$.

PROOF OF THEOREM 4. The "only if" part follows from Theorem 2.

To prove the "if" part, assume for contradiction that (1) has a nonoscillatory solution $y(t)$. We may assume without loss of generality that $y(t)>0$ for $t\geqq t_0$.

Integrating (1) from t to ∞ and noting that $\lim_{t\to\infty} p(t)\varphi(y'(t)) \geqq 0$, we have

$$p(t)\varphi(y'(t)) \geqq \int_t^\infty f(s, y(s))\, ds, \quad t \geqq t_0,$$

which implies

$$y'(t) \geqq \varphi^{-1}\left(\frac{1}{p(t)} \int_t^\infty f(s, y(s))\, ds\right), \quad t \geqq t_0. \tag{28}$$

We divide the above by $\varphi^{-1}((y(t))^\gamma)$, where $\gamma > 0$ is the constant of strong superlinearity of (1), and use (25) to obtain

$$\frac{y'(t)}{\varphi^{-1}((y(t))^\gamma)} \geqq \varphi^{-1}\left(\frac{1}{p(t)} \int_t^\infty \frac{f(s, y(s))}{(y(t)^\gamma)}\, ds\right) \geqq \varphi^{-1}\left(\frac{1}{p(t)} \int_t^\infty \frac{f(s, y(s))}{(y(s))^\gamma}\, ds\right),$$
$$t \geqq t_0.$$

Since $y(t) \geqq c_0$, $t \geqq t_0$, for some constant $c_0 > 0$ we have, in view of the strong superlinearity of (1),

$$(y(t))^{-\gamma} f(t, y(t)) \geqq c_0^{-\gamma} f(t, c_0), \quad t \geqq t_0,$$

so that

$$\frac{y'(t)}{\varphi^{-1}((y(t))^\gamma)} \geqq \varphi^{-1}\left(c_0^{-\gamma} \frac{1}{p(t)} \int_t^\infty f(s, c_0)\, ds\right) \geqq$$

$$\geqq \varphi^{-1}(c_0^{-\gamma})\varphi^{-1}\left(\frac{1}{p(t)} \int_t^\infty f(s, c_0)\, ds\right), \quad t \geqq t_0.$$

Integrating the last inequality over $[t_0, t_1]$, we have

$$\varphi^{-1}(c^{-\gamma}) \int_{t_0}^{t_1} \varphi^{-1}\left(\frac{1}{p(t)} \int_t^\infty f(s, c_0)\, ds\right) dt \leqq \int_{y(t_0)}^{y(t_1)} \frac{dv}{\varphi^{-1}(v^\gamma)}$$

which, because of (23), implies

$$\int_{t_0}^\infty \varphi^{-1}\left(\frac{1}{p(t)} \int_t^\infty f(s, c_0)\, ds\right) dt < \infty.$$

But this contradicts (26) and completes the proof of the "if" part of Theorem 4.

Proof of Theorem 5. The "only if" part is a consequence of Theorem 1. The "if" part is proved as follows. Let $y(t)$ be a positive solution of (1) for $t \geqq t_0$. First we note that $y'(t) > 0$, $t \geqq t_0$, and

$$y'(t) \geqq \varphi^{-1}\left(\frac{1}{p(t)}\right)\varphi^{-1}(p(t)\varphi(y'(t))), \quad t \geqq t_0.$$

Integrating over $[t_0, t]$ and using the decreasing nature of $p(t)\varphi(y'(t))$ we have

$$y(t) - y(t_0) \geqq \varphi^{-1}(p(t)\varphi(y'(t)))\Phi_{1,t_0}(p;\, t), \quad t \geqq t_0. \tag{29}$$

Now define

$$F(u)=\int_{u}^{u_0}\frac{dv}{[\varphi^{-1}(v)]^{\delta}},\tag{30}$$

where $\delta>0$ is the strong sublinearity constant and $u_0=p(t_0)\varphi(y'(t_0))>0$. Then from (30) and (29) we find

$$[F(p(t)\varphi(y'(t)))]'=\frac{f(t,y(t))}{[\varphi^{-1}(p(t)\varphi(y'(t)))]^{\delta}}\geqq \tag{31}$$

$$\geqq[\Phi_{1,t_0}(p;\ t)]^{\delta}(y(t))^{-\delta}f(t,y(t)),\quad t\geqq t_0.$$

From the strong sublinearity and the inequality $y(t)\leqq c_0\Phi_{k,t_0}(p;t)$, $t\geqq t_0$, where $c_0>0$ is a constant, it follows that

$$(y(t))^{-\delta}f(t,y(t))\geqq c_0^{-\delta}[\Phi_{k,t_0}(p;\ t)]^{-\delta}f(t,c_0\Phi_{k,t_0}(p;\ t))\tag{32}$$

for $t>t_0$. Substituting (32) into (31) and using the inequality

$$\Phi_{1,t_0}(p;\ t)/\Phi_{k,t_0}(p;\ t)\geqq\varphi^{-1}(1/k),\quad t>t_0.$$

which follows from (25), we have by integration over $[t_1,t_2]$, $t_1>t_0$,

$$c_0^{-\delta}\varphi^{-1}(1/k)\int_{t_1}^{t_2}f(t,c_0\Phi_{k,t_0}(p;\ t))\,dt\leqq\int_{v_2}^{v_1}\frac{dv}{[\varphi^{-1}(v)]^{\delta}},$$

where $v_i=p(t_i)\varphi(y'(t_i))$, $i=1,2$. Letting $t_2\to\infty$ and using (24) we obtain

$$\int_{t_1}^{\infty}f(t,c_0\Phi_{k,t_0}(p;\ t))\,dt<\infty,$$

which contradicts (27). This completes the proof of the "if" part of Theorem 5.

The following are variants of Theorems 4 and 5.

THEOREM 6. *Suppose* (25) *holds and there exist continuous functions* $q\colon[0,\infty)\to\to(0,\infty)$ *and* $g\colon(0,\infty)\to(0,\infty)$ *such that* g *is increasing,*

$$|f(t,v)|\geqq q(t)g(|v|)\quad for\quad (t,v)\in[0,\infty)\times\mathbf{R},\tag{33}$$

and

$$\int_{M}^{\infty}\frac{du}{\varphi^{-1}\circ g(u)}<\infty\quad for\ any\quad M>0.\tag{34}$$

Then, all proper solutions of equation (1) *are oscillatory if and only if*

$$\int_{0}^{\infty}\varphi^{-1}\left(\frac{1}{p(t)}\int_{t}^{\infty}q(s)\,ds\right)dt=\infty.\tag{35}$$

THEOREM 7. *Let* $q(t)$ *and* $g(v)$ *be as in* (33). *Suppose* (7) *and* (25) *hold and*

$$g(uv)\geqq g(u)g(v)\quad for\ any\quad u,v>0\tag{36}$$

and

$$(37)\qquad \int_0^N \frac{du}{g\circ|\varphi^{-1}(u)|}<\infty \quad \text{and} \quad \int_{-N}^0 \frac{du}{g\circ|\varphi^{-1}(u)|}<\infty \quad \textit{for any} \quad N>0.$$

Then all proper solutions of equation (1) *are oscillatory if and only if*

$$(38)\qquad \int_0^\infty q(t)g(\Phi_k(p;\, t))\,dt=\infty$$

for every constant $k\neq 0$.

Proof of Theorem 6. Since the "only if" part follows from Theorem 2, it suffices to prove the "if" part. Assume to the contrary that (1) has a nonoscillatory solution $y(t)>0$ for $t\geqq t_0>0$. As in the proof of Theorem 4 we have (28), which implies

$$y'(t)\geqq \varphi^{-1}\left(\frac{1}{p(t)}\int_t^\infty q(s)g(y(s))\,ds\right),\quad t\geqq t_0.$$

Dividing the above by $\varphi^{-1}(g(y(t)))$ and using (25) we obtain

$$(39)\qquad \frac{y'(t)}{\varphi^{-1}(g(y(t)))}\geqq \varphi^{-1}\left(\frac{1}{p(t)}\int_t^\infty q(s)\frac{g(y(s))}{g(y(t))}\,ds\right)\geqq$$

$$\geqq \varphi^{-1}\left(\frac{1}{p(t)}\int_t^\infty q(s)\,ds\right),\quad t\geqq t_0,$$

where we have used the fact that $g(y(s))/g(y(t))\geqq 1$ for $s\geqq t$ since $g(y(s))$ is increasing. Integration of (39) over $[t_0, t_1]$ yields

$$\int_{t_0}^{t_1} \varphi^{-1}\left(\frac{1}{p(t)}\int_t^\infty q(s)\,ds\right)dt\leqq \int_{t_0}^{t_1}\frac{y'(t)}{\varphi^{-1}(g(y(t)))}\,dt=\int_{y(t_0)}^{y(t_1)}\frac{du}{\varphi^{-1}\circ g(u)}$$

which, in view of (34), implies

$$\int_{t_0}^\infty \varphi^{-1}\left(\frac{1}{p(t)}\int_t^\infty q(s)\,ds\right)dt<\infty.$$

But this contradicts (35) and the proof of Theorem 6 is complete.

Proof of Theorem 7. The "only if" part is a consequence of Theorem 1.
To prove the "if" part let $y(t)$ be a positive solution of (1) for $t\geqq t_0$. As in the proof of Theorem 5, (29) holds, and so (36) implies

$$(40)\qquad g(y(t))\geqq g(\varphi^{-1}(p(t)\varphi(y'(t))))g(\Phi_{1,t_0}(p;\, t)),\quad t\geqq t_0.$$

Define

$$(41)\qquad F(u)=\int_u^{u_0}\frac{dv}{g\circ\varphi^{-1}(v)},\quad u>0,$$

where $u_0=p(t_0)\varphi\big(y'(t_0)\big)>0$. Then by (41) and (40) we have

$$[F(p(t)\varphi(y'(t)))]' = \frac{f\big(t, y(t)\big)}{g\big(\varphi^{-1}(p(t)\varphi(y'(t))\big)} \geqq \tag{42}$$

$$\geqq \frac{q(t)g\big(y(t)\big)}{q\big(\varphi^{-1}(p(t)\varphi(y'(t))\big)} \geqq q(t)g\big(\Phi_{1,t_0}(p;\ t)\big), \quad t>t_0.$$

Integration of (42) over $[t_1, t_2]$, $t_1>t_0$, shows that

$$\int_{t_1}^{t_2} q(t)g\big(\Phi_{1,t_0}(p;\ t)\big)\,dt \leqq \int_{v_2}^{v_1} \frac{dv}{g\circ\varphi^{-1}(v)},$$

where $v_i=p(t_i)\varphi\big(y'(t_i)\big)$, which because of (37), implies

$$\int_{t_1}^{\infty} q(t)g\big(\Phi_{1,t_0}(p;\ t)\big)\,dt<\infty.$$

This contradiction proves the truth of the "if" part of Theorem 7, and the proof is complete.

EXAMPLE 2. Consider equation (3) again. Since $\varphi^{-1}(u)=u^{1/m*}$ and $f(t, v)= =q(t)v^{n*}$, equation (3) is strongly superlinear or strongly sublinear according to whether $m<n$ or $m>n$. Therefore from Theorems 4 and 5 it follows that a necessary and sufficient condition for oscillation of all proper solutions of (3) is

$$\int_0^{\infty}\Big(\int_t^{\infty} q(s)\,ds\Big)^{1/m}dt=\infty \quad \text{if} \quad m<n,$$

$$\int_0^{\infty} t^n q(t)\,dt=\infty \quad \text{if} \quad m>n.$$

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SOME GOOD POINT SYSTEMS FOR DERIVATIVES OF LAGRANGE INTERPOLATORY OPERATORS*

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1. Introduction

1.1. Let us consider an arbitrary projection operator $L_n\colon C\to\mathcal{P}_{n-1}$ (i.e. L_n is a linear bounded operator, $L_n(f,x)\in\mathcal{P}_{n-1}$ if $f\in C$ and $L_n(f,x)\equiv f(x)$ iff $f\in\mathcal{P}_{n-1}$; here C is the set of continuous functions on $[-1,1]$, $\mathcal{P}_{n-1}$ denotes the set of algebraic polynomials of degree $\leqq n-1$).

We investigate the expression $\|L_n^{(r)}(f,x)\|$. By the usual definition $\left(|||L_n^{(r)}|||:=\sup\limits_{f\in C}\|L_n^{(r)}(f,x)\|\right.$ where $\|g(x)\|$ is the supremum norm$\left.\right)$ $|||L_n^{(r)}|||\geqq cn^{2r}$ (cf. D. L. Berman [10]). However, using a new norm introduced and investigated for projection operators by J. Szabados [3] we can have much better results. Namely, let

$$\Lambda_{n\mu}^{[s]}:=\sup_{\substack{f\in C\\ |f(x)|\leqq(\sqrt{1-x^2})^{\mu}}}\|L_n^{(s)}(f,x)\|,\quad s=0,1,2,\ldots,\quad \mu\geqq 0 \tag{1.1}$$

(cf. [3, (2)]).

If L_n is the Lagrange interpolation, J. Szabados [3] constructed a "good" point system for each fixed s such that

$$\Lambda_{ns}^{[s]}\leqq c(r)n^s\log n,\quad s=1,2,\ldots$$

(when $s=1$ or 2, see also N. S. Baiguzov [2] and P. O. Runck [3]).. This estimation, considering that *for arbitrary* $L_n\colon C\to\mathcal{P}_{n-1}$

$$\Lambda_{n\mu}^{[s]}\geqq c(s,\mu)n^s\log n,\quad s=0,1,\ldots;\quad \mu\geqq 0 \tag{1.2}$$

(proved for Lagrange interpolation in J. Szabados [3] and in general by P. Vértesi [4]), is the best possible in order. This is why our paper is devoted to find other good systems (in the previous sense) for Lagrange interpolation.

1.2. So *hence let L_n be the Lagrange interpolatory operator based on the node system*

$$X=\{x_{kn}\},\quad 1\leqq k\leqq n,\quad n=1,2,\ldots.$$

Concerning $\Lambda_{n\mu}^{[s]}$, we are interested in two problems. Let $r\geqq 0$ be given.

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A. If $f^{(r)}\in C$, ω is the modulus of continuity, then, using the Gopengauz theorem [5, p. 168] stating that there exist polynomials $p_n\in\mathcal{P}_{n-1}$ such that

$$|f^{(i)}(x)-p_n^{(i)}(x)| \leq c\left(\frac{\sqrt{1-x^2}}{n}\right)^{r-i}\omega\left(f^{(r)},\frac{1}{n}\right),\quad |x|\leq 1,\quad i=0,1,\dots,r,$$

we get, if l_{kn} denote fundamental polynomials,

$$|f^{(i)}(x)-L_n^{(i)}(f,x)| \leq |f^{(i)}(x)-p_n^{(i)}(x)|+|L_n^{(i)}(f-p_n,x)| \leq$$

$$\leq c\left[\frac{\omega\left(f^{(r)},\frac{1}{n}\right)}{n^r}n^i+\sum_{k=1}^{n}\left(\sqrt{1-x_{kn}^2}\right)^r|l_{kn}^{(i)}(x)|\right)\right],\quad i=0,1,\dots,r.$$

So using (1.2) and the relation

$$\Lambda_{nr}^{[i]} = \sum_{k=1}^{n}\left(\sqrt{1-x_{kn}^2}\right)^r|l_{kn}^{(i)}(x)|$$

holding for Lagrange interpolations (see [3, (6)] (or (1.1))), we have

$$\|f^{(i)}(x)-L_n^{(i)}(f,x)\| \leq \frac{c}{n^r}\Lambda_{nr}^{[i]}\omega\left(f^{(r)},\frac{1}{n}\right),\quad 0\leq i\leq r. \tag{1.3}$$

B. If we know only $f^{(s)}\in C$, $s\geq 0$, then for this derivative (again by the same Gopengauz theorem)

$$\|f^{(s)}(x)-L_n^{(s)}(f,x)\| \leq \frac{c}{n^s}\Lambda_{ns}^{[s]}\omega\left(f^{(s)},\frac{1}{n}\right) \tag{1.4}$$

(c. f. [3, Theorem 2]).

Then in the light of (1.2)—(1.4), it would be interesting

A) to find a node-system $X(r)$ for any fixed r such that for the corresponding norms

$$\Lambda_{nr}^{[i]} \leq cn^i\log n,\quad 0\leq i\leq r, \tag{1.5}$$

B) to find a node-system X such that for the corresponding norms

$$\Lambda_{ns}^{[s]} \leq cn\log n,\quad \text{for certain values of } s. \tag{1.6}$$

As it turned out the first problem (or (1.5)) can be solved for *any given* $r\geq 0$. (Actually, the system in [3] does the job.) On the other hand if for a given matrix X, $\Lambda_{ns}^{[s]}\leq cn^s\log n$ and $\Lambda_{nq}^{[q]}\leq cn^q\log n$ *then* $|s-q|\leq 2$ (cf. J. Szabados, P. Vértesi [6, Theorem 2] where we gave a corresponding matrix, too).

The aim of this paper is to give infinitely many matrices solving the above problems.

2. Results

2.1. To get our matrices we apply the method of [3] for other points systems. Namely, consider the Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$, $\alpha>-1$, orthogonal on $[-1,1]$ with the weight $(1-x^2)^\alpha$, with the roots $x_{kn}^{(\alpha,\alpha)}=\cos\theta_{kn}^{(\alpha,\alpha)}$ ($x_k=\cos\theta_k$, shortly).

If $t \geqq 1$ is a fixed integer, then let

$$y_i = y_{in} = \cos \eta_{in} \quad \text{where} \quad \eta_{in} = \frac{(i-1)\theta_{1n}}{t}, \quad y_{-i} = -y_i, \quad 1 \leqq i \leqq t. \tag{2.1}$$

Consider the polynomial

$$\Omega_{n+2t}(x) = c \prod_{i=1}^{t} (x^2 - y_i^2) P_n^{(\alpha,\alpha)}(x), \quad n > 2t \geqq 0. \tag{2.2}$$

If $t=0$, let $\Omega_n(x) = P_n^{(\alpha,\alpha)}(x)$.

Now we take the Lagrange interpolation $L_{nt\alpha}(f, x)$ based on the roots of $\Omega_{n+2t}(x)$ (if $\alpha = -1/2$, see [3]).

THEOREM 2.1. *If $t \geqq 0$ and $r \geqq 0$ are fixed then for* $L_{nt\alpha}\colon C \to \mathcal{P}_{n+2t-1}$

$$\Lambda^{[i]}_{n+2t,r} \leqq cn^i \log n, \quad 0 \leqq i \leqq r, \tag{2.3}$$

whenever

$$\max\left(-1, -\frac{5}{2} + 2t - r\right) \leqq \alpha \leqq -\frac{1}{2} + 2t - r, \quad \alpha > -1. \tag{2.4}$$

2.2. First let us give solutions for (1.5). If $r = 2r_1$, the possible values of t are, by (2.4), $t = r_1, r_1+1, r_1+2, \ldots$, whence α is from $\left(-1, -\frac{1}{2}\right], \left[-\frac{1}{2}, \frac{3}{2}\right], \left[\frac{3}{2}, \frac{7}{2}\right], \ldots$, respectively. If $r = 2r_1+1$, the possible values of t are, by (2.4), $t = r_1+1, r_1+2, r_1+3, \ldots$, whence α is from $\left(-1, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{5}{2}\right], \left[\frac{5}{2}, \frac{9}{2}\right], \ldots$, respectively.

I.e. *for any fixed r, arbitrary α can be good choosing the proper value of t.*

2.3. To solve (1.6) take three consecutive values $s, s+1$ and $s+2$. By the previous consideration, if $s = 2s_1$, whence $s+1 = 2s_1+1$ and $s+2 = 2(s_1+1)$, we get the same node-systems for $s, s+1$ and $s+2$ whenever $t = s_1+1, s_1+2, \ldots$. If $t = s_1+1$, say, then the corresponding intervals of α for $s, s+1$ and $s+2$ are $\left[-\frac{1}{2}, \frac{3}{2}\right], \left(-1, \frac{1}{2}\right]$ and $\left(-1, -\frac{1}{2}\right]$, respectively. The solution is their only common value, $\alpha = -\frac{1}{2}$. Similarly, if $t = s_1+1+k$, $k \geqq 0$, we get $\alpha = -\frac{1}{2} + 2k$, respectively.

Now let $s = 2s_1+1$. Then, by similar considerations, the proper values of t and α are $t = s_1+2+k$ and $\alpha = \frac{1}{2} + 2k$, $k \geqq 0$, respectively (when $k=0$, see [6, Theorem 1]).

2.4. Our statements are valid if, instead of $P_n^{(\alpha,\alpha)}(x)$, we condsier generalized Jacobi polynomials, i.e. when the weight function w, instead of $(1-x^2)^\alpha$ $\alpha > -1$, is $g(x)(1-x^2)^\alpha$, $a > -1$, with $g \in C$, $g(x) > 0$ and $\int_0^1 \frac{\omega(g, t)}{t} dt < \infty$ (shortly $w \in GJ$).

The only thing we have to consider is relations (3.6) and (3.7) which are valid for $w \in GJ$, too (c.f. P. Nevai, P. Vértesi [8, §2 and Lemma 2], say). Results concerning the weight $g(x)(1-x^2)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$ can be obtained, too. We omit the details.

3. Proof

3.1. If $l_k(x)$ denote the fundamental functions based on the roots of $P_n^{(\alpha,\alpha)}(x)$ ($=P_n(x)$, shortly), the fundamental functions of $L_{nt\alpha}$ are

$$(3.1)\qquad \begin{cases} \varphi_k(x) = \left(\prod_{i=1}^{t} \dfrac{x^2-y_i^2}{x_k^2-y_i^2}\right) l_k(x), & 1 \leqq k \leqq n, \\ \psi_u(x) = \dfrac{P_n(x)}{P_n(y_u)} \left(\prod_{\substack{i\neq u\\ i=1}}^{t} \dfrac{x^2-y_i^2}{y_u^2-y_i^2}\right) \dfrac{x+y_u}{2y_u}, & 1 \leqq |u| \leqq t. \end{cases}$$

If

$$(3.2)\qquad \lambda_{n+2t}^{[i]}(x, r, t) := \sum_{k=1}^{n} (\sqrt{1-x_k^2})^r |\varphi_k^{(i)}(x)| + \sum_{|u|=1}^{t} (\sqrt{1-y_n^2})^r |\psi_u^{(i)}(x)| := S_1 + S_2$$

then, as it comes from (1.1), for $L_{nt\alpha}$

$$(3.3)\qquad \Lambda_{n+2t,r}^{[i]} = \|\lambda_{n+2t}^{[i]}(x, r, t)\|, \quad 0 \leqq i \leqq r.$$

To get (2.3), by (3.3) it is enough to prove

$$(3.4)\qquad \lambda_{n+2t}^{[i]}(x, r, t) \leqq cn^i \left(\sqrt{1-x^2}+\frac{1}{n}\right)^{r-i} \log n, \quad 0 \leqq i \leqq r.$$

3.2. First we prove that with $\lambda_n^{[0]} = \lambda_n$

$$(3.5)\qquad \lambda_{n+2t}(x, r, t) \leqq c\left(\sqrt{1-x^2}+\frac{1}{n}\right)^r \log n.[1]$$

Indeed, let $|x-x_j| = \min_{1\leqq k\leqq n} |x-x_k|$. Then with $x=\cos\theta$ $\left(\text{whence } \sqrt{1-x^2}+\frac{1}{n} \sim \sin\theta_j\right)$, $\theta_{0n}=0$ and $\theta_{n+1,n}=\pi$

$$(3.6)\qquad \theta_{k+1,n}-\theta_{kn} \sim \frac{1}{n}, \quad 0 \leqq k \leqq n,$$

$$(3.7)\qquad |l_k(x)| \sim (n|\theta-\theta_j|)\left(\frac{\sin\theta_k}{\sin\theta_j}\right)^{\alpha+1/2} \frac{\sin\theta_k}{n|x-x_k|}, \quad 1 \leqq k \leqq n,$$

uniformly in k (cf. P. Nevai, P. Vértesi [8, §2 and Lemma 2], say) we can write if $x \geqq 0$, say,

$$(3.8)\qquad \sum_{k=1}^{n} (\sqrt{1-x_k^2})^r |\varphi_k(x)| = \sum_{k=1}^{3n/4} + \sum_{k=3n/4+1}^{n} := I_1 + I_2.$$

[1] The use of (3.5) and Lemma 3.1 below (instead of long calculations for derivatives) are due to J. Szabados and A. K. Varma [9].

Here, by $|x^2-y_i^2|\leqq c\cdot\sin^2\theta_j$, $x_k^2-y_i^2\sim\sin^2\theta_k$ and (2.4)

$$I_1\leqq\frac{c}{n^r j^{\alpha-2t+1/2}}\Big(\sum_{k<j/2}+\sum_{j/2\leqq k<2j}+\sum_{k\geqq 2j}\Big)\frac{k^{\alpha+r-2t+3/2}}{(k+j)(|k-j|+1)}\sim$$

$$\sim\left(\frac{j}{n}\right)^r\left[\log 2j+\log 2j+\left(1+\left(\frac{n}{j}\right)^{\alpha+r-2t+1/2}\right)\right]\sim\left(\frac{j}{n}\right)^r\log 2j\leqq c\left(\sqrt{1-x^2}+\frac{1}{n}\right)^r\log n.$$

Using similar arguments for I_2 and S_2 (where the relations $P_n(x)P_n^{-1}(y_u)\leqq cj^{-\alpha-1/2}$ and $y_n^2-y_i^2\sim n^{-2}$ $(i\neq u)$ (cf. [8, (20), (21), (17)]) can be used) we obtain (3.5) which is (3.4) when $i=0$.

3.2. To go further first we quote the next

LEMMA 3.1. *If $q_n\in\mathcal{P}_n$ and with fixed ϱ and $A>0$*

$$|q_n(x)|\leqq\left(\sqrt{1-x^2}+\frac{A}{n}\right)^\varrho,\quad |x|\leqq 1,$$

then with an absolute constant $c>0$

$$|q_n'(x)|\leqq cn\left(\sqrt{1-x^2}+\frac{A}{n}\right)^{\varrho-1},\quad |x|\leqq 1$$

(cf. A. F. Timan [7; 4.8.72.]).

For $x_0\in[-1,1]$ arbitrary fixed, consider

$$q_{n+2t}(x,x_0):=\sum_{k=1}^{n}\left(\sqrt{1-x_k^2}\right)^r\varphi_k(x)\cdot\operatorname{sign}\varphi_k'(x_0)+\sum_{|u|=1}^{t}\left(\sqrt{1-y_u^2}\right)^r\psi_u(x)\cdot\operatorname{sign}\psi_u'(x_0).$$

By (3.3), (3.5) and Lemma 3.1

$$\lambda_{n+2}^{[1]}(x_0,r,t)=q_{n+2t}'(x_0,x_0)\leqq cn\left(\sqrt{1-x_0^2}+\frac{1}{n}\right)^{r-1}\log n,$$

i.e., using that x_0 was arbitrary, we get (3.4) if $i=1$. Next, we use analogous argument for

$$q_{n+2t-1}(x,x_0):=\sum_{k=1}^{n}\left(\sqrt{1-x_k^2}\right)^r\varphi_k'(x)\cdot\operatorname{sign}\varphi_k''(x_0)+$$

$$+\sum_{|u|=1}^{t}\left(\sqrt{1-y_u^2}\right)^r\psi_u'(x)\cdot\operatorname{sign}\psi_u''(x_0)$$

to get (3.4) when $i=2$, etc. We omit the details.

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NONLINEAR ELLIPTIC DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

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In this paper it will be proved the existence of weak solutions of second order differential equations

$$\sum_{|\alpha|\leqq 1}(-1)^{|\alpha|}D^\alpha[f_\alpha(x,u,Du)]+g(x,u,Du)=F \quad \text{in} \quad \Omega \tag{0.1}$$

with nonlocal and nonlinear boundary conditions

$$\sum_{|\alpha|=1} f_\alpha(x,u,Du)\nu_\alpha+h_1(x,u(x))+h_2^*(x,u(\Phi(x)))=0, \quad x\in\partial\Omega \tag{0.2}$$

where $\alpha=(\alpha_1,\dots,\alpha_n)$ is a multiindex, $D=(D_1,\dots,D_n)$, $D_j=\dfrac{\partial}{\partial x_j}$, $D^\alpha=D_1^{\alpha_1}\dots D_n^{\alpha_n}$, by ν_α are denoted the coordinates of the exterior normal unit vector on the boundary $\partial\Omega$ ($|\alpha|=1$) and Φ is a given C^1-diffeomorphism in a neighbourhood of $\partial\Omega$ which maps $\partial\Omega$ onto $\Gamma\subset\bar\Omega$. The domain $\Omega\subset R^n$ may be unbounded but its boundary $\partial\Omega$ is supposed to be bounded. The functions f_α, g, h_1, h_2^* satisfy certain polynomial growth conditions. Similar equations with usual boundary conditions have been considered in [1] and [2].

If u is a classical solution of (0.1), (0.2) then by the formula

$$-\sum_{|\alpha|=1}\int_\Omega D^\alpha[f_\alpha(x,u,Du)]v\,dx=$$

$$=\sum_{|\alpha|=1}\int_\Omega f_\alpha(x,u,Du)D^\alpha v\,dx-\sum_{|\alpha|=1}\int_{\partial\Omega} f_\alpha(x,u,Du)\nu_\alpha v\,d\sigma$$

we obtain that for all $v\in C_0^\infty(\mathbf{R}^n)$

$$\sum_{|\alpha|\leqq 1}\int_\Omega f_\alpha(x,u,Du)D^\alpha v\,dx+\int_\Omega g(x,u,Du)v\,dx+$$

$$+\int_{\partial\Omega}[h_1(x,u)+h_2^*(x,u\circ\Phi)]v\,d\sigma=\int_\Omega Fv\,dx,$$

i.e.

$$\sum_{|\alpha|\leqq 1}\int_\Omega f_\alpha(x,u,Du)D^\alpha v\,dx+\int_\Omega g(x,u,Du)v\,dx+ \tag{0.3}$$

$$+\int_{\partial\Omega}h_1(x,u)v\,d\sigma+\int_\Gamma h_2(x,u)(v\circ\Phi^{-1})\,d\sigma=\int_\Omega Fv\,dx.$$

Weak solutions of (0.1), (0.2) will be defined by (0.3).

Elliptic equations with nonlocal boundary conditions have been considered first by T. Carleman [3] and the importance of such problems has been emphasized by A. A. Samarskij in [4]. A. L. Skubachevskij has developed the theory of linear elliptic equations of order $2m$ with rather general linear nonlocal boundary conditions (see [5]).

§1. The formulation of the results

Let $\Omega\subset\mathbf{R}^n$ be a (possibly) unbounded domain with a bounded, continuously differentiable boundary $\partial\Omega$, $p>1$ a fixed number. The usual Sobolev space with the norm

$$\|u\|_{W_p^1(\Omega)}=\Big\{\sum_{|\alpha|\leqq 1}\int_\Omega |D^\alpha u|^p\,dx\Big\}^{1/p}$$

will be denoted by $W_p^1(\Omega)$. The points of $\mathbf{R}^{n+1}$ will be written in the form $\xi=(\eta,\zeta)$ where $\eta\in\mathbf{R}^n$, $\zeta\in\mathbf{R}$.

Assume that

I. The functions f_α: $\Omega\times\mathbf{R}^{n+1}\to\mathbf{R}$ satisfy the Carathéodory conditions, i.e. $f_\alpha(x,\xi)$ are measurable in x for each fixed $\xi\in\mathbf{R}^{n+1}$ and they are continuous in ξ for a.e. $x\in\Omega$.

II. There exist a constant $c_1>0$ and a function $k_1\in L^q(\Omega)$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$ such that

$$|f_\alpha(x,\xi)|\leqq c_1|\xi|^{p-1}+k_1(x)$$

for a.e. $x\in\Omega$, all $\xi\in\mathbf{R}^{n+1}$ if $|\alpha|\leqq 1$.

III. For a.e. $x\in\Omega$, each $\eta\in\mathbf{R}$, $\zeta,\zeta'\in\mathbf{R}^n$ with $\zeta\neq\zeta'$

$$\sum_{|\alpha|=1}[f_\alpha(x,\eta,\zeta)-f_\alpha(x,\eta,\zeta')](\zeta_\alpha-\zeta'_\alpha)>0.$$

IV. There exist a constant $c_2>0$ and a nonnegative $k_2\in L^1(\Omega)\cap L^\infty(\Omega)$ such that

$$\sum_{|\alpha|\leqq 1}f_\alpha(x,\xi)\xi_\alpha\geqq c_2\sum_{|\alpha|\leqq 1}|\xi_\alpha|^p-k_2(x)$$

for all $\xi\in\mathbf{R}^{n+1}$, a.e. $x\in\Omega$.

V. The function g: $\Omega\times\mathbf{R}^{n+1}\to\mathbf{R}$ satisfies the Carathéodory conditions.

VI. If $n<p$ then for each $s>0$ there exist $k_s\in L^q(\Omega)$ and a constant $c_s>0$ such that for a.e. $x\in\Omega$, all $\zeta\in\mathbf{R}^n$

$$|g(x,\xi)|\leqq c_s|\zeta|^\varrho+k_s(x)\quad\text{if}\quad|\eta|\leqq s$$

with some positive number ϱ, satisfying $p-1<\varrho\leqq p$ and if $p\leqq n$ then there exist $k_3\in L^q(\Omega)$ and a constant $c_3>0$ such that for a.e. $x\in\Omega$, all $\xi\in\mathbf{R}^{n+1}$

$$|g(x,\xi)|\leqq c_3|\xi|^\varrho+k_3(x)$$

with some number ϱ, satisfying $p-1<\varrho<p-1+p/n$.

VII. There exist a nonnegative constant $c_4<c_2$ and $k_4\in L^1(\Omega)\cap L^\infty(\Omega)$ such that $k_4\geqq 0$ and for a.e. $x\in\Omega$, all $\xi\in\mathbf{R}^{n+1}$

$$g(x,\xi)\eta\leqq -c_4|\eta|^p-k_4(x).$$

VIII. The functions h_1: $\partial\Omega\times\mathbf{R}\to\mathbf{R}$, h_2: $\Gamma\times\mathbf{R}\to\mathbf{R}$ satisfy the Carathéodory conditions.

IX. If $n\geqq p$ then for a.e. $x\in\partial\Omega$ resp. $x\in\Gamma$, each $\eta\in\mathbf{R}$, $j=1, 2$

$$|h_j(x,\eta)|\leqq c_5|\eta|^{\varrho_j}+k_5^j(x)$$

with some constants $c_5>0$, $0<\varrho_j<\dfrac{n(p-1)}{n-p}$ and $k_5^1\in L^{1+1/\varrho_1}(\partial\Omega)$, $k_5^2\in L^{1+1/\varrho_2}(\Gamma)$ (in the case $n=p$ for ϱ_j an arbitrary positive number can be chosen); if $n<p$ then for each $s>0$ there exist $h_s^1\in L^1(\partial\Omega)$, $h_s^2\in L^1(\Gamma)$ such that for a.e. $x\in\partial\Omega$ resp. $x\in\Gamma$, $|\eta|\leqq s$, $j=1, 2$

$$|h_j(x,\eta)|\leqq h_s^j(x).$$

X. In the case $p>2$ there are functions $h_j^*\in C^1(U_j\times\mathbf{R})$ (where U_1, U_2 denote some open and bounded neighbourhood of $\partial\Omega$ resp Γ) such that for a.e. $x\in\partial\Omega$ resp. $x\in\Gamma$, each $\eta\in\mathbf{R}$

$$h_1(x,\eta)\eta\geqq-|h_1^*(x,\eta)\eta|,\quad |h_2(x,\eta)|\leqq|h_2^*(x,\eta)|$$

where for $j=1, 2$ the functions h_j^* satisfy

$$\lim_{|\eta|\to\infty}\frac{|h_j^*(x,\eta)|^q+\sum_{k=1}^n|D_kh_j^*(x,\eta)|^q+|D_{n+1}h_j^*(x,\eta)|^{p/(p-2)}}{\sum_{|\alpha|\leqq1}f_\alpha(x,\xi)\xi_\alpha+g(x,\xi)\eta}=0$$

uniformly in $x\in U_j'$ and $\zeta\in\mathbf{R}^n$ where $U_j'=U_j\cap\bar{\Omega}$.

In the case $p\leqq2$ there exist numbers c_j^*, ϱ_j^* and $k_1^*\in L^{1+1/\varrho_1^*}(\partial\Omega)$, $k_2^*\in L^{1+1/\varrho_2^*}(\Gamma)$ such that $0<\varrho_j^*<p-1$ and for a.e. $x\in\partial\Omega$ resp. $x\in\Gamma$, each $\eta\in\mathbf{R}$

$$h_1(x,\eta)\eta\geqq-c_1^*[|\eta|^{\varrho_1^*}+k_1^*(x)]|\eta|,\quad |h_2(x,\eta)|\leqq c_2^*|\eta|^{\varrho_2^*}+k_2^*(x).$$

XI. V is a closed linear subspace of $W_p^1(\Omega)$ with the following property: there exists a positive number R such that $\varphi\in C_0^\infty(\mathbf{R}^n)$, $\varphi(x)=1$ for $|x|<R$, $v\in V$ imply $\varphi v\in V$.

THEOREM. *Assume that conditions* I—XI *are fulfilled. Then for any $G\in V^*$ (i.e. any linear continuous functional on V) there exists $u\in V$ such that for all $v\in V$*

$$\sum_{|\alpha|\leqq1}\int_\Omega f_\alpha(x,u,Du)D^\alpha v\,dx+\int_\Omega g(x,u,Du)v\,dx+ \tag{1.1}$$

$$+\int_{\partial\Omega}h_1(x,u)\,v\,d\sigma+\int_\Gamma h_2(x,u)(v\circ\Phi^{-1})\,d\sigma=\langle G,v\rangle.$$

REMARK 1. In the case $p\leqq2$ the estimations on h_2 in IX follow from assumption X.

REMARK 2. The existence theorem can be easily extended to the case of "sufficiently good" unbounded $\partial\Omega$.

REMARK 3. The following conditions are sufficient for the uniqueness of the solution of (1.1):

$$\sum_{|\alpha|\leq 1} [f_\alpha(x, \xi)-f_\alpha(x, \tilde{\xi})](\xi_\alpha-\tilde{\xi}_\alpha)+[g(x, \xi)-g(x, \tilde{\xi})](\xi_0-\tilde{\xi}_0) \geqq c_2|\xi-\tilde{\xi}|^2$$

with some positive constant $c_2>0$, $\eta\mapsto h_1(x, \eta)$ is monotone increasing (for a.e. $x\in\partial\Omega$), $D_{n+1}h_2$ exists and $\sup_{\Gamma\times\mathbf{R}} |D_{n+1}h_2|$ is sufficiently small.

REMARK 4. The following example shows that in the case $\varrho_2^*=p-1$ (or $\varrho_1^*==p-1$, see X) problem (1.1) may have no solution. Consider in $B_{1,2}=\{x\in\mathbf{R}^2: 1<|x|<2\}$ the problem

$$\Delta u-u=0 \quad \text{in } B_{1,2} \tag{1.2}$$

$$\partial_\nu u(x)=\beta_j u(\gamma_j x)+\delta_j, \quad x\in S_j, \quad j=1,2 \tag{1.3}$$

where $S_j=\{x\in\mathbf{R}^2: |x|=j\}$, $\partial_\nu u$ is the normal derivative of u, β_j, γ_j are given numbers, $\gamma_1>1$, $\frac{1}{2}<\gamma_2<1$.

Then all conditions of the existence theorem are fulfilled but in X $\varrho_2^*=1=p-1$. If u is a weak solution of (1.2), (1.3) then by well known results on regularity of weak solutions of the Neumann problem for (1.2) one obtains that u is a classical solution of (1.2), (1.3). Set

$$U(r, \varphi)=u(x_1, x_2) \quad \text{where} \quad x_1=r\cos\varphi, \quad x_2=r\sin\varphi$$

and

$$v(r)=\int_0^{2\pi} U(r, \varphi)\, d\varphi$$

then (1.2), (1.3) imply

$$v''(r)+\frac{1}{r}v'(r)-v(r)=0, \quad 1<r<2 \tag{1.4}$$

$$v'(j)=\beta_j v(j\gamma_j)+\delta_1 \quad (j=1,2). \tag{1.5}$$

It is easy to show that numbers β_j, γ_j, δ_j can be chosen such that problem (1.4), (1.5) has no solution.

Further, numbers β_j, γ_j, δ_j can be chosen such that problem (1.2), (1.3) has more than one solution.

§2. The proof of the existence theorem

First we formulate a lemma. For arbitrary natural number k define functions g_n by

$$g_k(x, \xi)=\begin{cases} g(x, \xi) & \text{if } |x|\leqq k \\ 0 & \text{if } |x|>k. \end{cases}$$

Then the formulas

$$\langle T(u), v\rangle = \sum_{|\alpha|\leqq 1} \int_\Omega f_\alpha(x, u, Du) D^\alpha v\, dx,$$

$$\langle S_k(u), v\rangle = \int_\Omega g_k(x, u, Du) v\, dx,$$

$$\langle Q(u), v\rangle = \int_{\partial\Omega} h_1(x, u) v\, d\sigma + \int_\Gamma h_2(x, u)(v\circ\Phi^{-1})\, d\sigma; \quad u, v\in V$$

define bounded (nonlinear) operators T, S_k, Q: $V\to V^*$.

LEMMA. *The operators* $T+S_k+Q$ *are pseudomonotone and coercive.*

PROOF. Assumptions I—IV imply that the operator T is pseudomonotone (see [6]). Further, if $(u_l)\to u$ weakly in V then by VI, IX and known compact imbedding theorems (see e.g. [7]) there is a subsequence (u_l') of (u_l) such that $(u_l')\to u$ in $L^{p^*}(\Omega_k)$ and a.e. in Ω_k where $\Omega_k=\Omega\cap B_k$, $B_k=\{x\in\mathbf{R}^n, |x|<k\}$ and p^* is defined by $\frac{1}{p/\varrho}+\frac{1}{p^*}=1$. Further, if $n\geqq p$ then $(u_l')\to u$ in $L^{\varrho_1+1}(\partial\Omega)$, $L^{\varrho_1+1}(\Gamma)$ and a.e. on $\partial\Omega$, Γ $\left(\text{since for } p<n \text{ we have } 1<\varrho_1+1<\frac{p(n-1)}{n-p}\right)$; if $p>n$ then $(u_l')\to u$ in $C(\partial\Omega)$ and in $C(\Gamma)$. Consequently, V VI, IX and Hölder's inequality imply

$$\lim_{l\to\infty} \langle S_k(u_l'), u_l'-u\rangle = 0, \quad \lim_{l\to\infty} \langle Q(u_l'), u_l'-u\rangle = 0$$

and by using Vitali's theorem, Hölder's inequality, assumptions V, VI, VIII, IX it follows

$$\lim_{l\to\infty} \langle S_k(u_l'), v\rangle = \langle S_k(u), v\rangle, \quad \lim_{l\to\infty} \langle Q(u_l'), v\rangle = \langle Q(u), v\rangle$$

for any fixed $v\in V$. Thus it is easy to show that also $T+S_k+Q$ is pseudomonotone.

Now we prove coercivity of $T+S_k+Q$. Firstly, consider the case $p>2$. Since the trace operator compactly maps $W_p^1(U_j')$ into $L^{p_1}(\partial\Omega)$ resp. $L^{p_1}(\Gamma)$ for $p_1<\frac{p(n-1)}{n-p}$ if $p<n$ and for any $p_1>1$ if $p\geqq n$ thus it is easy to see that numbers $p_1, q_1>1$ can be chosen such that $1/p_1+1/q_1=1$ and the trace operator compactly maps $W_p^1(U_j')$ into $L^{p_1}(\partial\Omega)$ resp. $L^{p_1}(\Gamma)$ and $W_q^1(U_j')$ into $L^{q_1}(\partial\Omega)$ resp. $L^{q_1}(\Gamma)$.

By Hölder's inequality and X for any $u\in C^\infty(\mathbf{R}^n)$

$$\text{(2.1)} \qquad \langle Q(u), u\rangle \geqq -\int_{\partial\Omega} |h_1^*(x, u) u|\, d\sigma - \int_\Gamma |h_1^*(x, u)(u\circ\Phi^{-1})|\, d\sigma \geqq$$

$$\geqq -\|h_1^*(x, u)\|_{L^{q_1}(\partial\Omega)} \|u\|_{L^{p_1}(\partial\Omega)} - c_1' \|h_2^*(x, u)\|_{L^{q_1}(\Gamma)} \|u\|_{L^{p_1}(\partial\Omega)} \geqq$$

$$\geqq -c_2' \|h_1^*(x, u)\|_{W_q^1(U_1')} \|u\|_{W_p^1(U_1')} - c_3' \|h_2^*(x, u)\|_{W_q^1(U_2')} \|u\|_{W_p^1(U_1')}$$

with some positive constants c'_j. Further, Young's inequality implies

(2.2)

$$\|h_1^*(x,u)\|_{W_q^1(U_1')} = \Big\{\int_{U_1'} [|h_1^*(x,u)|^q + \sum_{k=1}^n |D_k h_1^*(x,u) + D_{n+1} h_1^*(x,u) D_k u|^q]\, dx\Big\}^{1/q} \leq$$

$$\leq \Big\{\int_{U_1'} [|h_1^*(x,u)|^q + c_1(q) \sum_{k=1}^n [|D_k h_1^*(x,u)|^q + |D_{n+1} h_1^*(x,u)|^q |D_k u|^q]]\, dx\Big\}^{1/q} \leq$$

$$\leq \Big\{\int_{U_1'} \Big(|h_1^*(x,u)|^q + c_1(q) \sum_{k=1}^n \Big[|D_k h_1^*(x,u)|^q +$$

$$+ \frac{1}{\varepsilon^{q_2 q^2/p}} \frac{|D_{n+1} h_1^*(x,u)|^{qq_2}}{q_2} + \varepsilon^q \frac{(|D_k u|^q)^{p/q}}{p/q}\Big]\Big)\, dx\Big\}^{1/q} \leq$$

$$\leq c_2(q,\varepsilon) \Big\{\int_{U_1'} [|h_1^*(x,u)|^q + \sum_{k=1}^n |D_k h_1^*(x,u)|^q + |D_{n+1} h_1^*(x,u)|^{qq_2}]\, dx\Big\}^{1/q} +$$

$$+ c_3(q)\varepsilon \|u\|_{W_p^1(U_1')}^{p/q} =$$

$$= c_2(q,\varepsilon) \Big\{\int_{U_1'} [|h_1^*(x,u)|^q + \sum_{k=1}^n |D_k h_1^*(x,u)|^q + |D_{n+1} h_1^*(x,u)|^{p/(p-2)}]\, dx\Big\}^{1/q} +$$

$$+ c_3(q)\varepsilon \|u\|_{W_p^1(U_1')}^{p-1}$$

where q_2 is defined by $\dfrac{1}{p/q} + \dfrac{2}{q_2}$, i.e. $q_2 = \dfrac{p-1}{p-2}$ and so $qq_2 = \dfrac{p}{p-2}$.

In virtue of X for each $m>0$ there exists $a>0$ such that $|\eta|>a$ implies

$$|h_1^*(x,\eta)|^q + \sum_{k=1}^n |D_k h_1^*(x,\eta)|^q + |D_{n+1} h_1^*(x,\eta)|^{p/(p-2)} \leq$$

$$\leq \frac{1}{m^q} \Big[\sum_{|\alpha| \leq 1} f_\alpha(x,\xi)\xi_\alpha + g(x,\xi)\eta\Big]$$

whence by using notation $U_{1,a} = \{x \in U_1' : |u(x)| \leq a\}$, IV, VII, IX

$$\int_{U_1'} [|h_1^*(x,u)|^q + \sum_{k=1}^n |D_k h_1^*(x,u)|^q + |D_{n+1} h_1^*(x,u)|^{p/(p-2)}]\, dx = \tag{2.3}$$

$$= \int_{U_{1,a}} [|h_1^*(x,u)|^q + \sum_{k=1}^n |D_k h_1^*(x,u)|^q + |D_{n+1} h_1^*(x,u)|^{p/(p-2)}]\, dx +$$

$$+ \int_{U_1' \setminus U_{1,a}} [|h_1^*(x,u)|^q + \sum_{k=1}^n |D_k h_1^*(x,u)|^q + |D_{n+1} h_1^*(x,u)|^{p/(p-2)}]\, dx \leq$$

$$\leq c_4(q,m) + \frac{1}{m^q} \int_{U_1'} \Big[\sum_{|\alpha| \leq 1} f_\alpha(x,u,Du) D^\alpha u + g(x,u,Du)u + k_2 + k_4\Big]\, dx.$$

(VI, Sobolev's imbedding theorem and Hölder's inequality imply that $g(x, u, Du)u$ is integrable over Ω.) Combining (2.2) and (2.3) we obtain

$$\|h_1^*(x, u)\|_{W_q^1(U_1')} \leqq c_5(q, m, \varepsilon)+ \tag{2.4}$$
$$+\frac{c_6(q, \varepsilon)}{m}\Big\{\int_{U_1'}\Big[\sum_{|\alpha|\leqq 1} f_\alpha(x, u, Du)D^\alpha u+g(x, u, Du)u+k_2+k_4\Big]\,dx\Big\}^{1/q}+$$
$$+c_3(q)\varepsilon\|u\|_V^{p-1}.$$

Similarly can be shown that $h_2^*(x, u)$ satisfies the same inequality as $h_1^*(x, u)$ in (2.4).

By IV, VII for sufficiently large k

$$\langle T(u), u\rangle+\langle S_k(u), u\rangle \geqq \tag{2.5}$$
$$\geqq \frac{1}{2}\int_{U_1'}\Big[\sum_{|\alpha|\leqq 1} f_\alpha(x, u, Du)D^\alpha u+g(x, u, Du)u+k_2+k_4\Big]\,dx+$$
$$+\frac{c_2-c_4}{2}\|u\|_V^p-\int_\Omega (k_2+k_4)\,dx \geqq \frac{c_2-c_4}{2}\|u\|_V^p-\int_\Omega (k_2+k_4)\,dx$$

and (2.1), (2.4) imply

$$\langle Q(u), u\rangle \geqq -c_7(q, m, \varepsilon)\|u\|_V- \tag{2.6}$$
$$-\frac{c_8(q, \varepsilon)}{m}\int_{U_1'}\Big[\sum_{|\alpha|\leqq 1} f_\alpha(x, u, Du)D^\alpha u+g(x, u, Du)u+k_2+k_4\Big]dx-c_9(q)\varepsilon\|u\|_V^p.$$

Choosing sufficiently small ε and sufficiently large m one obtains from (2.5), (2.6) the inequality

$$\langle T(u), u\rangle+\langle S_k(u), u\rangle+\langle Q(u), u\rangle \geqq \frac{c_2-c_4}{4}\|u\|_V^p+ \tag{2.7}$$
$$+\frac{1}{4}\int_{U_1'}\Big[\sum_{|\alpha|\leqq 1} f_\alpha(x, u, Du)D^\alpha u+g(x, u, Du)u+k_2+k_4\Big]\,dx-$$
$$-c_7\|u\|_V-c_{10} \geqq \frac{c_2-c_4}{4}\|u\|_V^p-c_7\|u\|_V-c_{10}$$

for any $u\in C^\infty(\mathbf{R}^n)$ and, consequently, for any $u\in V$ since restrictions of functions of $C^\infty(\mathbf{R}^n)$ to Ω are dense in $W_p^1(\Omega)$ ($\partial\Omega$ is supposed to be continuously differentiable). (2.7) implies coercivity of $(T+S_k+Q)\colon V\to V^*$ in the case $p>2$.

In the case $p\leqq 2$ by X and Hölder's inequality

$$\langle Q(u), u\rangle \geqq \int_{\partial\Omega} h_1(x, u)u\,d\sigma-\Big|\int_\Gamma h_2(x, u)(u\circ\Phi^{-1})\,d\sigma\Big| \geqq$$
$$\geqq -\int_{\partial\Omega}[c_1^*|u|^{\varrho_1^*}+k_1^*]|u|\,d\sigma-\int_\Gamma [c_2^*|u|^{\varrho_2^*}+k_2^*]|u\circ\Phi^{-1}|\,d\sigma \geqq$$
$$\geqq -\|c_1^*|u|^{\varrho_1^*}+k_1^*\|_{L^{p_1/\varrho_1^*}(\partial\Omega)}\|u\|_{L^{p_1}(\partial\Omega)}-\|c_2^*|u|^{\varrho_2^*}+k_2^*\|_{L^{p/\varrho_2^*}(\Gamma)}\|u\circ\Phi^{-1}\|_{L^{p_2}(\Gamma)}$$

where $p_j=\varrho_j^*+1<p<\dfrac{p(n-1)}{n-p}$ (if $n=p=2$ then the right hand side is considered to be $+\infty$). Consequently,

$$\langle Q(u), u\rangle \geqq -[c_1^*\|u\|_{L^{p_1}(\partial\Omega)}^{\varrho_1^*}+\|k_1^*\|_{L^{p_1/\varrho_1^*}(\partial\Omega)}]\|u\|_{L^{p_1}(\partial\Omega)}-$$

$$-[c_2^*\|u\|_{L^{p_2}(\Gamma)}^{\varrho_2^*}+\|k_2^*\|_{L^{p_2/\varrho_2^*}(\Gamma)}]\|u\circ\Phi^{-1}\|_{L^{p_2}(\Gamma)} \geqq -c_1'\|u\|_V^{\varrho_1^*+1}-c_2'\|u\|_V^{\varrho_2^*+1}-c_3'\|u\|_V.$$

$$\varrho_1^*+1,\ \varrho_2^*+1<p$$

and by (2.5)

$$\langle T(u), u\rangle+\langle S_k(u), u\rangle+Q(u), u\rangle \geqq \frac{c_2-c_4}{2}\|u\|_V^p- \tag{2.8}$$

$$-c_1'\|u\|_V^{\varrho_1^*+1}-c_2'\|u\|_V^{\varrho_2^*+1}-c_3'\|u\|_V-c_4'$$

which implies coercivity of $T+S_k+Q$.

Proof of Theorem. By the lemma for each positive integer k there exists $u_k\in V$ such that for all $v\in V$

$$\langle (T+S_k+Q)(u_k), v\rangle = \langle G, v\rangle. \tag{2.9}$$

Applying (2.9) to $v=u_k$, by (2.7), (2.8) (where constants do not depend on k) and the inequality $|\langle G, u_k\rangle|\leqq\|G\|\|u_k\|_V$ we obtain that $\|u_k\|_V$ is bounded. Thus there exist a subsequence (u_{k_l}) of (u_k) and $u\in V$ such that

$$(u_{k_l})\to u \quad \text{weakly in } V, \tag{2.10}$$

$$(u_{k_l})\to u \quad \text{a.e. in } \Omega. \tag{2.11}$$

Now consider an arbitrary fixed bounded domain ω such that $\omega\subset\Omega$ and a function $\Theta\in C_0^\infty(\mathbf{R}^n)$ satisfying

$$\theta\geqq 0 \quad\text{and}\quad \theta(x)=1 \quad\text{for}\quad x\in\omega.$$

By VI, IX and theorems on compact imbeddings (see [7]), the subsequence (u_{k_l}) may be supposed to be chosen such that

$$(u_{k_l})\to u \quad\text{in}\quad L^p(\Omega\cap\operatorname{supp}\theta), \tag{2.12}$$

$$(u_{k_l})\to u \quad\text{in}\quad L^{p^*}(\Omega\cap\operatorname{supp}\theta) \tag{2.13}$$

where p^* is defined by $\dfrac{1}{p/\varrho}+\dfrac{1}{p^*}=1$;

$$(u_{k_l})\to u \quad\text{in}\quad L^{\varrho_1+1}(\partial\Omega) \quad\text{and a.e. on } \partial\Omega \text{ for } n\geqq p \tag{2.14}$$

and in $C(\partial\Omega)$ for $n<p$;

$$(u_{k_l})\to u \quad\text{in}\quad L^{\varrho_2+2}(\Gamma),\quad\text{a.e. on } \Gamma \text{ for } n\geqq p \tag{2.15}$$

and in $C(\Gamma)$ for $n<p$.

In virtue of assumption XI $\theta(u_{k_l}-u)\in V$ and so by (2.9) we have

$$\sum_{|\alpha|\le 1}\int_\Omega f_\alpha(x, u_{k_l}, Du_{k_l})D^\alpha[\theta(u_{k_l}-u)]\,dx+\int_\Omega g_{k_l}(x, u_{k_l}, Du_{k_l})[\theta(u_{k_l}-u)]\,dx+$$

$$+\int_{\partial\Omega} h_1(x, u_{k_l})[\theta(u_{k_l}-u)]\,d\sigma+\int_\Gamma h_2(x, u_{k_l})[\theta(u_{k_l}-u)]\circ\Phi^{-1}\,d\sigma=\langle G, \theta(u_{k_l}-u)\rangle.$$

This equality can be written in the form

$$(2.16)\qquad \sum_{|\alpha|\le 1}\int_\Omega [f_\alpha(x, u_{k_l}, Du_{k_l})-f_\alpha(x, u_{k_l}, Du)]\theta D^\alpha(u_{k_l}-u)\,dx=$$

$$=\sum_{|\alpha|\le 1}\int_\Omega f_\alpha(x, u_{k_l}, Du)\theta D^\alpha(u-u_{k_l})\,dx+\sum_{|\alpha|\le 1}\int_\Omega f_\alpha(x, u_{k_l}, Du_{k_l})(D^\alpha\theta)(u-u_{k_l})\,dx+$$

$$+\int_\Omega f_0(x, u_{k_l}, Du_{k_l})\theta(u-u_{k_l})\,dx+\int_\Omega g_{k_l}(x, u_{k_l}, Du_{k_l})\theta(u-u_{k_l})\,dx+$$

$$+\int_{\partial\Omega} h_1(x, u_{k_l})\theta(u-u_{k_l})\,d\sigma+\int_\Gamma h_2(x, u_{k_l})[\theta(u-u_{k_l})]\circ\Phi^{-1}\,d\sigma+\langle G, \theta(u_{k_l}-u)\rangle.$$

Now we show that all the terms in the right of (2.16) tend to 0 as $l\to\infty$. Since by (2.11) and I

$$(2.17)\qquad f_\alpha(\cdot, u_{k_l}, Du)\theta\to f_\alpha(\cdot, u, Du)\theta\quad \text{a.e. in}\quad \Omega$$

thus II, (2.12) and Vitali's theorem imply that (2.17) holds also in the norm of $L^q(\Omega)$. Consequently, the first term in the right of (2.16) tends to 0 as by (2.10) $D^\alpha(u-u_{k_l})\to 0$ weakly in $L^p(\Omega)$. Further, from I, II and (2.10) it follows that $f_\alpha(\cdot, u_{k_l}, Du_{k_l})$ is bounded in $L^q(\Omega)$ and so (2.12) implies that the second and third terms in the right of (2.16) converge to 0.

Assumptions V, VI imply the boundedness of $g_{k_l}(\cdot, u_{k_l}, Du_{k_l})$ in $L^{p/\varrho}(\Omega\cap\operatorname{supp}\theta)$ whence we obtain by (2.13) that also the fourth term in the right of (2.16) tends to 0 as $l\to\infty$. Assumption IX implies that $h_j(\cdot, u_{k_l})$ is bounded in $L^{1+1/\varrho_1}(\partial\Omega)$ resp. in $L^{1+1/\varrho_2}(\Gamma)$ for $n\ge p$ and in $L^1(\partial\Omega)$ resp. $L^1(\Gamma)$ for $n<p$ thus by (2.14), (2.15) also the fifth and sixth terms in the right of (2.16) converge to 0. Finally, (2.10) implies that the last term in the right of (2.16) tends to 0.

We have proved that the right of (2.16) converges to 0 as $l\to\infty$ for arbitrary bounded ω with the property $\omega\subset\Omega$ which implies that there is a subsequence (u'_{k_l}) of (u_{k_l}) such that for each $j=1, 2, \dots, n$

$$(2.18)\qquad (D_j u'_{k_l})\to D_j u\quad \text{a.e. in}\quad \Omega.$$

(See e.g. Lemma 2. of [1] or [8].) Thus from I it follows

$$f_\alpha(\cdot, u'_{k_l}, Du'_{k_l})\to f_\alpha(\cdot, u, Du)\quad \text{a.e. in}\quad \Omega$$

and so II, Hölder's inequality and Vitali's theorem imply

$$(2.19)\qquad \lim_{l\to\infty}\langle T(u'_{k_l}), v\rangle=\langle T(u), v\rangle$$

for each fixed $v\in V$. Similarly, by (2.18), V and the definition of g_k we have

$$g'_{k_l}(\cdot, u'_{k_l}, Du'_{k_l})\to g(\cdot, u, Du)\quad \text{a.e. in}\quad \Omega$$

and so VI, Hölder's inequality and Vitali's theorem imply

$$\lim_{l\to\infty} \langle S'_{k_l}(u'_{k_l}), v\rangle = \int_{\Omega} g(x, u, Du)v\,dx \tag{2.20}$$

for each $v\in V$. Finally, from (2.14), (2.15), VIII, IX, Hölder's inequality and Vitali's theorem we obtain

$$\lim_{l\to\infty} \langle Q(u'_{k_l}), v\rangle = \int_{\partial\Omega} h_1(x, u)v\,d\sigma + \int_{\Gamma} h_2(x, u)(v\circ\Phi^{-1})\,d\sigma \tag{2.21}$$

for each $v\in V$. Combining (2.19)—(2.21), (2.10) with (2.9) we obtain the statement of our theorem.

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CROFTIAN SEQUENCES

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1. H. T. Croft proved in [2] that if f: $\mathbf{R}\to\mathbf{R}$ is continuous and $\lim_{n\to\infty} f(nx)=0$ for every $x>0$ then $\lim_{x\to\infty} f(x)=0$. This is an easy consequence of the following statement: for every open, unbounded set $G\subset\mathbf{R}^+$ there is a positive number x such that $nx\in G$ for infinitely many n. (This result was found earlier by D. J. Newman and W. E. Weissblum [4].) Actually, Croft proves slightly more; he shows that, under the same hypothesis on G, every subinterval of the positive reals contains a point x such that $nx\in G$ infinitely often.

These results were generalized by J. F. C. Kingman in [3]. Kingman considers the additive variant of Croft's theorems. He proves that for every increasing sequence $c_n\to\infty$ the following are equivalent.

(1) $\lim_{n\to\infty}(c_{n+1}-c_n)=0$.

(2) If G is an open subset of $\mathbf{R}$, unbounded from above then, in any open interval of $\mathbf{R}$, there is an x such that $x+c_n\in G$ for infinitely many n.

(3) If f: $\mathbf{R}\to\mathbf{R}$ is continuous and $\lim_{n\to\infty} f(x+c_n)=0$ for every x in some open interval, then $\lim_{x\to\infty} f(x)=0$.

The statement (3) can be replaced by the following: If f: $\mathbf{R}\to\mathbf{R}$ is continuous and $\lim_{n\to\infty} f(x+c_n)$ exists for every x in some open interval, then $\lim_{x\to\infty} f(x)$ exists.

Now Croft's theorem follows from the observation that the sequence $c_n=\log n$ satisfies (1). (These results proved to be useful in the theory of regularly varying functions; see [1, p. 50].)

As Kingman remarks, the theorem above leaves open the possibility that a sequence $\{c_n\}$ does not satisfy (1), but the existence of $\lim_{n\to\infty} f(x+c_n)$ *for all* x implies the existence of $\lim_{x\to\infty} f(x)$. This happens, for example, when $\{c_n\}$ consists of the numbers of the form $2m+k2^{-m}$ $(m=0, 1, \ldots;\ k=0, 1, \ldots, 2^m)$ arranged in ascending order. (We shall discuss this example in Remark 1.)

This observation raises the following problem. What is the exact characterization of those sequences $\{c_n\}$ for which

(4) if f: $\mathbf{R}\to\mathbf{R}$ is continuous and $\lim_{n\to\infty} f(x+c_n)$ exists for every x, then $\lim_{x\to\infty} f(x)$ exists; or

(5) if f: $\mathbf{R}\to\mathbf{R}$ is continuous and $\lim_{n\to\infty} f(x+c_n)=0$ for every $x\in\mathbf{R}$, then $\lim_{x\to\infty} f(x)=0$.

In this paper we shall consider the second of these questions. We shall say that the sequence $\{c_n\}$ is *thick*, if (5) holds. A sequence will be called *thin*, if it is not thick. We shall give several characterizations of thin sequences. The characterizations given by our Theorem 1 may not seem completely satisfactory, since they are difficult to check and, in fact, they are not very far from the definition. However, most of our other results will be based on these conditions.

One would expect that a "real" characterization involves the speed of the sequence: slow sequences are thick and fast sequences are thin. We show, however, that it is not the case: there are arbitrarily slow sequences which are thin (Theorem 2). Yet, there are conditions in terms of the speed of the sequence which are sufficient and necessary, respectively. Thus, as Kingman showed, (1) is sufficient for $\{c_n\}$ to be thick. In Theorem 3 we shall prove that $\lim\limits_{n\to\infty} c_n/n=0$ is a necessary condition. On the other hand, as we shall see, these conditions are far from being characterizations.

We also consider the following generalization of the problem. We call a set $H\subset\mathbf{R}$ thick if the following statement is true: whenever $f\colon \mathbf{R}\to\mathbf{R}$ is continuous and $\lim\limits_{y\to\infty, y\in H} f(x+y)=0$ for every $x\in\mathbf{R}$, then $\lim\limits_{x\to\infty} f(x)=0$.

The exact characterization of thick sets seems to be difficult. We give sufficient conditions for a set to be thick and prove that there is a set H such that H is thick and whenever $c_n\in H$ and $c_n\to\infty$ then $\{c_n\}$ is thin (Theorems 4 and 5.)

Acknowledgement. We are grateful to Paul Erdős for several stimulating discussions on this subject. He proved first that $\limsup\limits_{n\to\infty}(c_{n+1}-c_n)<\infty$ holds for thick sequences (cf. (iii) of Theorem 1). He also constructed a thick sequence with $\limsup\limits_{n\to\infty}(c_{n+1}-c_n)>0$ and conjectured that $\lim\limits_{n\to\infty} c_n/n=0$ is a necessary condition.

2. We shall use the following notation. For every $H\subset\mathbf{R}$ and $a\in\mathbf{R}$ we denote $H+a=\{x+a\colon x\in H\}$ and $-H=\{-x\colon x\in H\}$. The cardinality, closure, derived set and outer Lebesgue measure of H will be denoted by card H, $\bar{H}$, H' and $|H|$, respectively. If the range of indices is not indicated, it will mean that this range is the set $\mathbf{N}$ of positive integers. Thus $\{c_n\}$ denotes $\{c_n\colon n\in\mathbf{N}\}$ and $\{a_k-c_n\}$ denotes $\{a_k-c_n\colon k, n\in\mathbf{N}\}$.

We begin with some preliminary remarks. If $f\colon\mathbf{R}\to\mathbf{R}$ is continuous then $\lim\limits_{x\to\infty} f(x)=0$ if and only if the open set $\{x\colon |f(x)|>\varepsilon\}$ is bounded from above for every $\varepsilon>0$. This easily implies that a sequence $\{c_n\}$ is thick if and only if the following conditions holds:

(6) For every open set $G\subset\mathbf{R}$, unbounded from above, there is a point x such that $x+c_n\in G$ for infinitely many n.

If a sequence $\{c_n\}$ has a convergent subsequence then $\{c_n\}$ is thick. Indeed, in this case for every non-empty open G there is a point x such that $x+c_n\in G$ for infinitely many n. Therefore we shall only consider sequences with $|c_n|\to\infty$. In this case, deleting the negative terms from $\{c_n\}$ does not effect the property of thickness. We may also rearrange the remaining terms and hence we may confine ourselves to increasing sequences tending to infinity.

THEOREM 1. *Suppose that $\{c_n\}$ is increasing and tends to infinity. Then the following statements are equivalent.*

(i) $\{c_n\}$ *is thin.*

(ii) *There is a sequence* $\{a_k\}$ *such that* $a_k \to \infty$ *and* $\{a_k - c_n\}$ *is nowhere dense.*

(iii) *Either* $\limsup_{n\to\infty} (c_{n+1} - c_n) = \infty$ *or there is a subsequence* $\{c_{n_k}\}$ *such that* $\{c_{n_k} - c_n\}$ *is nowhere dense.*

(iv) *There is a sequence of open intervals* $\{I_k\}$ *and a sequence of real numbers* $\{a_k\}$ *such that* $\bigcup_{k=1}^{\infty} I_k$ *is everywhere dense in* $\mathbf{R}$, $a_k \to \infty$ *and* $\{c_n\} \cap \left(\bigcup_{j=1}^{k} I_j + a_k\right) = \emptyset$ *for every* $k \in \mathbf{N}$.

PROOF. (i)⇒(ii): Suppose (i). Then (6) is false and hence there is an open set $G = \bigcup_{k=1}^{\infty} (a_k, b_k)$ such that $a_k \to \infty$ and, for every x, $x + c_n \notin G$ if n is large enough.

Let $X_k = \bigcup_{j>k} (G - c_j)$ $(k \in \mathbf{N})$, then $\bigcap_{k=1}^{\infty} X_k = \emptyset$.

We prove that $\{a_k - c_n\}$ is nowhere dense. Suppose this is not true and let $\{a_k - c_n\}$ be dense in an open interval I. Since $a_k \to \infty$ and $c_n \to \infty$, this implies that $\{a_i - c_j \colon i, j > k\}$ is also dense in I for every $k \in \mathbf{N}$. Therefore $X_k \cap I$ is a dense open subset of I for every k and hence $\bigcap_{k=1}^{\infty} X_k \neq \emptyset$, a contradiction.

(ii)⇒(i): Suppose that $\{a_k\}$ satisfies the requirements of (ii). In order to prove (i) it is enough to show that there are numbers $0 < \delta_k < \varepsilon_k$ such that for every x, $x + c_n \notin \bigcup_{k=1}^{\infty} (a_k + \delta_k, a_k + \varepsilon_k)$ if n is large enough.

Let $P = \{a_k - c_n\}$; then P is nowhere dense. For each k we choose an index n_k such that $c_n > k + a_k$ for every $n > n_k$. The set $\bigcup_{j=1}^{n_k} (\bar{P} + c_j - a_k)$ is nowhere dense and hence we can select $0 < \delta_k < \varepsilon_k < 1/k$ such that

$$(\delta_k, \varepsilon_k) \cap \bigcup_{j=1}^{n_k} (\bar{P} + c_j - a_k) = \emptyset.$$

Let x be arbitrary and suppose that $x + c_n \in \bigcup_{k=1}^{\infty} (a_k + \delta_k, a_k + \varepsilon_k)$ holds for infinitely many n. Then there are sequences $n_i \to \infty$, $k_i \to \infty$ such that

$$a_{k_i} - c_{n_i} + \delta_{k_i} < x < a_{k_i} - c_{n_i} + \varepsilon_{k_i}$$

for every i, and hence $x \in \bar{P}$. If $x + c_n \in (a_k + \delta_k, a_k + \varepsilon_k)$ then $(\bar{P} + c_n - a_k) \cap (\delta_k, \varepsilon_k) \neq \emptyset$ and then, by the definition of δ_k and ε_k, we have $n > n_k$. Thus $c_n > k + a_k$ and, as $x < a_k - c_n + 1 < -k + 1$, we obtain $k < -x + 1$. Therefore $k_i < -x + 1$ for every i, which is impossible.

(ii)⇒(iii): Suppose (ii) and $\limsup_{n\to\infty} (c_{n+1} - c_n) < K < \infty$. Let $\{a_k\}$ be a sequence satisfying the conditions of (ii). Then for every k large enough we can choose an index n_k with $|c_{n_k} - a_k| < K$. By selecting a subsequence we may assume that $\{c_{n_k} - a_k\}$ is convergent. This easily implies that $\{c_{n_k} - c_n\}$ is nowhere dense.

(iii)⇒(iv): Suppose first that $\limsup_{n\to\infty} (c_{n+1} - c_n) = \infty$ and let $\{n_n\}$ be a sequence of indices with $\lim_{k\to\infty} d_k = \infty$, where $d_k = c_{n_k+1} - c_{n_k}$.

We may assume that $\{d_k\}$ is increasing. Then the sequences of intervals $I_k = =(-d_k/2, d_k/2)$ and numbers $a_k = c_{n_k} + \frac{1}{2} d_k$ satisfy the requirements of (iv).

If there is a subsequence $\{c_{n_k}\}$ such that $P = \{c_n - c_{n_k}\}$ is nowhere dense then let $\{I_k\}$ be an enumeration of the components of the exterior of P and let $a_k = c_{n_k}$.

(iv)⇒(ii): Suppose that $\{I_k\}$ and $\{a_k\}$ satisfy the requirements formulated in (iv). We prove that $P = \{a_k - c_n\}$ is nowhere dense. Let x be a point of accumulation of P. Then there are sequences of indices k_i, n_i such that $x = \lim_{i \to \infty} (a_{k_i} - c_{n_i})$. Since $a_k \to \infty$, $c_k \to \infty$, we may assume that $k_i \to \infty$ and $n_i \to \infty$. For every fixed k, $c_{n_i} \notin I_k + a_{k_i}$ if $k_i \geqq k$ and hence $a_{k_i} - c_{n_i} \notin -I_k$ for i large enough. This implies $x \notin -I_k$ for every k and hence

$$x \in \mathbf{R} - \bigcup_{k=1}^{\infty} (-I_k) \stackrel{\text{def}}{=} S.$$

By assumption, S is nowhere dense. This shows that $P' \subset S$ is nowhere dense and then so is P. □

REMARK 1. By Kingman's theorem, every sequence satisfying condition (1) is thick. Making use of Theorem 1 we can prove this as follows. Suppose that the sequence $\{c_n\}$ tends to infinity and satisfies (1). It is easy to see that if $a_k \to \infty$ then $\{a_k - c_n\}$ is everywhere dense and thus, by (ii) of Theorem 1, $\{c_n\}$ is thick.

On the other hand, the following example given by Kingman shows that (1) is not necessary for a sequence to be thick. Let the sequence (d_n) consists of the numbers $2m + k2^{-m}$ $(m=0, 1, \dots; k=0, 1, \dots, 2^m)$.

Then the sequence consisting of the elements d_n, d_n+1 satisfies conditions (1) and hence, by Kingman's theorem, it is thick. This implies that $\{d_n\}$ itself is thick, too. Indeed, it is easy to see that if there is a finite set $\{h_1, \dots, h_k\}$ such that the sequence of numbers $\{c_n + h_i: n=1, 2, \dots; i=1, \dots, k\}$ is thick that $\{c_n\}$ is also thick. (This is an easy consequence of (6).)

We may ask whether the following condition is equivalent to the property of thickness. There is a finite set $\{h_1, \dots, h_k\}$ such that the sequence of numbers $\{c_n + h_i: n=1, 2, \dots; i=1, \dots, k\}$ satisfies (1).

The answer is negative. In fact, one can construct a thick sequence $\{c_n\}$ such that the sequence of numbers $\{c_n + h_i: n=1, 2, \dots; i=1, \dots, k\}$ does not satisfy (1) for any finite set $\{h_1, \dots, h_k\}$; moreover, for every bounded and nowhere dense set $H \subset \mathbf{R}$, the complement of $\{c_n\} + H$ contains infinitely many disjoint intervals of length $\geqq d = d(H) > 0$. Since we shall not use this fact in the sequel, we omit the proof.

LEMMA 1. *Let $c_n \to \infty$ be thin, and let $\varepsilon_n \to 0$. If $d_n \to \infty$ and $\{d_n\} \subset \bigcup_{k=1}^{\infty} (c_k - \varepsilon_k, c_k + \varepsilon)$, then $\{d_n\}$ is thin.*

PROOF. By (ii) of Theorem 1, there is a sequence $\{a_k\}$ such that $a_k \to \infty$ and $\{a_k - c_n\}$ is nowhere dense. The condition on $\{d_n\}$ easily implies that $\{a_k - d_n\}' \subset \subset \{a_k - c_n\}'$ and hence $\{a_k - d_n\}$ is also nowhere dense. Therefore, by Theorem 1, $\{d_n\}$ is thin. □

THEOREM 2. *There are arbitrarily slow thin sequences. More precisely, for every* $c_n\to\infty$ *there is a thin sequence* $\{d_n\}$ *such that* $d_n<c_n$ *for every* n.

PROOF. Let m be an integer with $m<\inf\{c_n\}$. It is easy to construct a sequence $\{d_n\}$ such that $d_n\to\infty$ and for every n, $d_n<c_n$ and $d_n\in\bigcup_{i=m}^{\infty}(i, i+(|i|+1)^{-1})$. Since $\{i\colon i=m, m+1, \dots\}$ is thin by (ii) of Theorem 1, Lemma 1 implies that $\{d_n\}$ is thin as well. □

THEOREM 3. *Let* $\{c_n\}$ *be an increasing sequence tending to infinity. If* $\{c_n\}$ *is thick then* $c_n/n\to 0$.

PROOF. Suppose that $c_n/n\not\to 0$ and let $c>0$ be such that $c_n\geqq cn$ for infinitely many n. Let $\{I_k\}$ be a sequence of open intervals such that $\bigcup_{k=1}^{\infty} I_k$ is everywhere dense in $\mathbf{R}$ and $\sum_{k=1}^{\infty}|I_k|<c/4$.

We shall prove that for every k there is a number $a_k\geqq ck/4$ such that $\{c_n\}\cap\bigcup_{j=1}^{k}(I_j+a_k)=\emptyset$. By (iv) of Theorem 1, this implies that $\{c_n\}$ is thin and this will provide the desired contradiction.

Let k be fixed, let $H=\bigcup_{j=1}^{k} I_j$ and choose $n>k$ such that $H\subset(-\infty, cn/2)$ and $c_n\geqq cn$. Let A denote the set of points $x\in[cn/4, cn/2]$ for which $\{c_j\}\cap(H+x)=\emptyset$. It is enough to show that $A\neq\emptyset$. If $x\in[cn/4, cn/2]$ and $c_j\in H+x$, then $c_j<cn$ and hence $j<n$ and, consequently, $x\in\bigcup_{j=1}^{n-1}(-H+c_j)$.

It follows that

$$[cn/4, cn/2]-A\subset\bigcup_{j=1}^{n-1}(-H+c_j)\quad\text{and hence}\quad |[cn/4, cn/2]-A|\leqq(n-1)|H|<cn/4.$$

Therefore $A\neq\emptyset$, and the proof is complete. □

REMARK 2. If $s_n/n\to 0$ then there are thick sequences with $c_n/s_n\not\to 0$. Indeed, if $s_n/n\to 0$ then there is an increasing sequence $\{c_n\}$ such that $c_{n+1}-c_n\to 0$ and $\limsup_{n\to\infty} c_n/s_n>0$. By Kingman's theorem, $\{c_n\}$ is thick (but $c_n/s_n\not\to 0$).

REMARK 3. The condition $c_n/n\to 0$ is not sufficient for $\{c_n\}$ to be thick (this is an immediate consequence of Theorem 2). In fact, this condition is very far from being sufficient, as the following argument shows. Let $\{c_n\}$ be an increasing sequence tending to infinity. We say that $\{c_n\}$ is *very thin* if there is a sequence $a_k\to\infty$ such that $\{a_k-c_n\}'$ is discrete, that is, if $\{a_k-c_n\}'\cap[-K, K]$ is finite for every $K>0$. Since every discrete set is nowhere dense, it follows from Theorem 1 that very thin sequences are thin. By Theorem 3, if $c_n/n\not\to 0$ then $\{c_n\}$ is thin. Now we claim that if $c_n/n\not\to 0$ then $\{c_n\}$ is, in fact, very thin.

We briefly sketch the proof of this statement. The proof is based on the following combinatorial lemma.

Let I be an interval and let H be a finite subset of I with $\operatorname{card}(H) \leqq c|I|$. Then there is a point a in the middle third of I such that $\operatorname{card}(H \cap [a-m, a+m]) \leqq 9cm^3$ holds for every $m \in \mathbf{N}$, $m \geqq 2$.

(Proof: for every fixed $m \geqq 2$, the number of integers a with $\operatorname{card}(H \cap [a-m, a+m]) > 9cm^3$ is at most

$$\frac{c|I|(2m+1)}{9cm^3} = \frac{|I|}{3} \cdot \frac{2m+1}{3m^3}.$$

Since $\sum_{m=2}^{\infty} (2m+1)/3m^3 < 1$, there is an integer with the required properties.)

Now suppose that $\{c_n\}$ is increasing and $c_n/n \not\to 0$. Let $c_{n_k} > cn_k$ $(k=1, 2, \ldots)$. Applying the lemma with $I=[0, cn_k]$ and $H=\{c_n:\ n<n_k\}$ we obtain a number $a_k \in [cn_k/3, 2cn_k/3]$ such that

$$\operatorname{card}\left(\{c_n:\ a_k - m \leqq c_n \leqq a_k + m,\ n < n_k\}\right) \leqq \frac{9}{c} m^3$$

for every $m \geqq 2$. If $m < cn_k/3$ then $a_k + m < cn_k$ and hence $c_n < a_k + m$ implies $n < n_k$. Consequently, for $m < cn_k/3$ we have

$$\operatorname{card}\left(\{a_k - c_n:\ n \in N\} \cap [-m, m]\right) \leqq \frac{9}{c} m^3.$$

For every fixed m we can select a positive integer p and a subsequence $\{a_{k_i}\}$ of $\{a_k\}$ such that $\operatorname{card}\left(\{a_{k_i} - c_n:\ n \in \mathbf{N}\} \cap [-m, m]\right) = p$ for every i. Let

$$\left(\{a_{k_i} - c_n:\ n \in N\} \cap [-m, m]\right) = \{t_{i,1}, \ldots, t_{i,p}\},$$

where $p \leqq \frac{9}{c} m^3$ is fixed. By selecting another subsequence, we may assume that $\{t_{i,j}:\ i \in \mathbf{N}\}$ is convergent for every $j \leqq p$, and hence $\left(\{a_{k_i} - c_n\} \cap [-m, m]\right)'$ is finite. Now we select subsequences of $\{a_k\}$ successively for $m=2, 3, \ldots$ and then, using a diagonal argument, select a final subsequence $\{a_{k_i}\}$ such that $\left(\{a_{k_i} - c_n\} \cap \cap [-m, m]\right)'$ is finite for every m. □

3. Now we turn to the investigation of thick sets. As we mentioned in the introduction, the exact characterization of thickness proves to be difficult. There is a stronger property, however, which is easier to handle. A set $H \subset \mathbf{R}$ will be called *very thick* if the following statement is valid. Whenever a function: f: $\mathbf{R} \to \mathbf{R}$ satisfies $\lim_{y \to \infty, y \in H} f(x+y) = 0$ for every $x \in \mathbf{R}$, then $\lim_{x \to \infty} f(x) = 0$.

This is easily seen to be equivalent to the following property: for every sequence $a_k \to \infty$ there is a point x such that $x + a_k \in H$ for infinitely many k. That is, H is very thick if and only if, for every $a_k \to \infty$,

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} (H - a_k) \neq \emptyset.$$

From this condition it is obvious that every residual or full measure set is very thick. Another sufficient condition is given by the next theorem.

THEOREM 4. *Suppose that H is measurable and that there is an $a>0$ such that*

$$\liminf_{n\to\infty} |H\cap[(n-1)a, na]| \stackrel{\text{def}}{=} A > 0.$$

Then H is very thick.

PROOF. Let $\{a_k\}$ be a sequence tending to infinity and put

$$H_n = \bigcup_{k=n}^{\infty} (H-a_k)\cap[0, 2a] \quad (n\in N).$$

Then H_n is measurable, $|H_n|\geqq A$, and $H_n\supset H_{n+1}$ holds for every n. This implies $|\bigcap_{n=1}^{\infty} H_n|\geqq A$. In particular, $\bigcap_{n=1}^{\infty} H_n\neq\emptyset$ and hence H is very thick. □

THEOREM 5. *There exists a very thick set H such that whenever $c_n\in H$ and $c_n\to\infty$ then $\{c_n\}$ is thin.*

PROOF. Let $\{I_k\}$ be a sequence of open intervals such that $\bigcup_{k=1}^{\infty} I_k$ is everywhere dense in **R** and $\sum_{k=1}^{\infty} |I_k|<1/2$. Let N_k be chosen such that $\bigcup_{j=1}^{k} I_j\subset[-N_k, N_k]$ $(k\in\mathbf{N})$, and let $\{a_k\}$ be an increasing sequence of natural numbers such that the intervals $[-N_k+a_k, N_k+a_k]$ are pairwise disjoint. We define

$$H = \mathbf{R}\setminus\bigcup_{k=1}^{\infty} \left(\bigcup_{j=1}^{k} I_j+a_k\right).$$

Then H is measurable and $|H\cap[n, n+1]|>1/2$ holds for every $n\in\mathbf{N}$. By Theorem 4, this implies that H is very thick. On the other hand, if $c_n\in H$ and $c_n\to\infty$ then $\{c_n\}$ is thin. Indeed, we have

$$\{c_n\}\cap\bigcup_{j=1}^{k} (I_j+a_k) = \emptyset,$$

and (iv) of Theorem 1 applies. □

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ON BOUNDEDLY DIVERGENT WALSH—FOURIER SERIES

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Dedicated to Professor Z. Daróczy on his 50th birthday

1. Introduction

The first example of everywhere divergent trigonometric Fourier series was given by Kolmogorov [5], [6]. Later, Marcinkiewicz [9] constructed a function whose trigonometric Fourier series diverges a.e., but the partial sums are bounded. For other counterexamples concerning the trigonometric system and for the history of this field see Ul'janov [15].

In this paper we will investigate the bounded divergence of Walsh—Fourier series. The Walsh system $(w_n, n\in\mathbf{N})$ is taken in Paley's enumeration, i.e. for $n\in\mathbf{N}=\{0, 1, 2, \ldots\}$ with dyadic expansion

$$n = \sum_{k=0}^{\infty} n_k 2^k \quad (n_k = 0, 1) \tag{1}$$

we set

$$w_n = \prod_{k=0}^{\infty} r_k^{n_k}, \tag{2}$$

where $(r_n, n\in\mathbf{N})$ is the Rademacher system. The partial sums of the Walsh—Fourier series Sf of $f\in L=L^1[0, 1)$ are denoted by $S_n f$ $(n\in\mathbf{N})$ and the dyadic maximal function is defined by

$$f^*(x) = \sup_n |(S_{2^n} f)(x)| = \sup_I \frac{1}{|I|} \Big| \int_I f(t)\,dt \Big|$$

where in the last expression the supremum is taken over all dyadic subintervals I of $[0, 1)$ containing $x\in[0, 1)$. The function $f\in L$ belongs to the dyadic Hardy space $\mathbf{H}$ if

$$\|f\|_{\mathbf{H}} = \int_0^1 f^*(x)\,dx < \infty.$$

More generally, for every continuous, increasing function $\Phi\colon [0, \infty)\to[0, \infty)$ let $\mathbf{H}\Phi(\mathbf{H})$ be the set of functions $f\in L$ satisfying

$$\int_0^1 f^*(x)\Phi(f^*(x))\,dx < \infty. \tag{3}$$

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The Walsh—Fourier series Sf is said to diverge boundedly at $x\in[0, 1)$ if Sf diverges at x but

$$(S^*f)(x) = \sup_n |(S_nf)(x)| < \infty,$$

i.e. Sf diverges at x because of oscillation.

The existence of divergent Walsh—Fourier series was first proved by Stein [13].

We show that there exist a.e. boundedly divergent Walsh—Fourier series. Moreover the following claim is true.

THEOREM 1. *Let* $\Phi\colon [0, \infty)\to[0, \infty)$ *be a continuous, monotone increasing function which satisfies*

$$\Phi(t) = o(\log\log t) \quad as \quad t\to\infty. \tag{4}$$

Then there is a function $F\in\mathbf{H}\Phi(\mathbf{H})$ *such that* SF *diverges boundedly a.e. and diverges everywhere.*

This is a generalization of an earlier result of the author [11]. The existence of functions in $\mathbf{H}$ with everywhere unboundedly divergent Walsh—Fourier series was proved by Ladhawala and Pankratz [8]. Functions belonging to $\mathbf{H}\Phi(\mathbf{H})$ with everywhere unboundedly divergent Walsh—Fourier series were constructed in [12]. For analogous result concerning the trigonometric system see Körner [7] andTandori [14].

The a.e. bounded divergence cannot be replaced by everywhere bounded divergence, since the following theorem is true (see Chen [3]).

THEOREM 2. *If* $f\in L$ *and* Sf *diverges a.e., then* Sf *diverges unboundedly on a dense subset of* $[0, 1)$.

2. Auxiliary results

In our construction we will use the term of strongly multiplicative system. Let $(\varphi_n,\ n\in\mathbf{N})$ be a system of real functions defined on $[0, 1)$. The product system of the system in question is defined by

$$\psi_n = \prod_{k=0}^{\infty} \varphi_k^{n_k},$$

where the n_k's are defined in (1). The system $(\varphi_n,\ n\in\mathbf{N})$ is called strongly multiplicative, if $(\psi_n,\ n\in\mathbf{N})$ is orthogonal (see Alexits [1]).

Waterman [16] showed that if the strongly multiplicative system satisfies

$$\int_0^1 \psi_n^2(x)\,dx = K \quad (n \geqq n_0) \tag{5}$$

with a constant K, then $(\psi_n,\ n\in\mathbf{N})$ is the Walsh system in disguise.

THEOREM A. *Let* $(\varphi_n,\ n\in\mathbf{N})$ *be a strongly multiplicative system satisfying* (5) *and* $|\varphi_n|\leqq 1$ *a.e.* $(n\in\mathbf{N})$. *Then there exists a measure preserving mapping* π *of* $[0, 1)$ *into itself such that*

$$\psi_n = w_n\circ\pi \quad a.e. \quad (n\in\mathbf{N}). \tag{6}$$

Denote

(7) $$D_n = \sum_{k=0}^{n-1} w_k \quad (n \in \mathbf{P} = \mathbf{N} \setminus \{0\})$$

the n-th Walsh—Dirichlet kernel and set $D_0=0$. It is well-known (see e.g. [4]) that

(8) $$D_{2^k}(x) = \begin{cases} 2^k, & \text{if } 0 \leqq x < 2^{-k} \\ 0, & \text{if } 2^{-k} \leqq x < 1 \end{cases}$$

and

(9) $$|D_n(x)| < \frac{2}{x} \quad (n \in \mathbf{N},\ x \in (0, 1)).$$

It is easy to see that for $n \leqq 2^m$ the kernels D_n are constant on every dyadic interval with length 2^{-m}. Furthermore, it follows from the explicit form of the Lebesgue constants for the Walsh system that

(10) $$\begin{cases} \text{i)} \quad L_n = \int_0^1 |D_n(t)|\, dt \leqq m, & \text{for } 0 \leqq n \leqq 2^m \\ \text{ii)} \quad L_{N_m} \geqq cm, & \text{for all } m \in \mathbf{P} \end{cases}$$

where

$$N_m = \sum_{2k<m} 2^{2k} \quad (m \in \mathbf{P})$$

and $c>0$ is an absolute constant (see Fine [4]).

Our construction is based on the Walsh—Kolmogorov polynomials introduced in [11]. For every $m \in \bar{P}$ set

$$q_m = \begin{cases} 1, & \text{on } \{D_{N_m} > 0\} \\ -1, & \text{on } \{D_{N_m} \leqq 0\}. \end{cases}$$

Then it follows from (10) ii) that for every $m \in \mathbf{P}$ and $x \in [0, 2^{-m})$ we have

(11) $$(S_{N_m} g)(x) = (S_{N_m} g)(0) = \int_0^1 g_m(t) D_{N_m}(t)\, dt \geqq cm.$$

For every $m \in \mathbf{P}$ we introduce a sequence of Walsh polynomials $(\varphi_k^m,\ k \in \mathbf{N})$ by setting

(12) $$\varphi_k^m(x) = \begin{cases} r_{m+k}(x) g_m(x \dotplus k2^{-m}), & \text{if } 0 \leqq k < 2^m \\ r_{m+k}(x), & \text{if } k \geqq 2^m, \end{cases}$$

where $x \in [0, 1)$ and $\dotplus$ is Fine's operation (see [4]).

It is clear that the Walsh spectrum of φ_k^m lies in $[2^{m+k}, 2^{m+k+1})$ and

$$|\varphi_k^m| = 1 \quad (k \in \mathbf{N},\ m \in \mathbf{P}).$$

This implies that $(\varphi_k^m,\ k \in \mathbf{N})$ is a strongly multiplicative system and its product system $(\psi_k^m,\ k \in \mathbf{N})$ satisfies

$$\int_0^1 |\psi_k^m(t)|^2\, dt = 1 \quad (k \in \mathbf{N},\ m \in \mathbf{P}).$$

Thus, by Waterman's theorem for every $m\in\mathbf{P}$ there exists a measure preserving transformation π_m of $[0, 1)$ such that

(13) $$\psi_k^m = w_k\circ\pi_m \quad (k\in\mathbf{N},\ m\in\mathbf{P}).$$

The Walsh—Kolmogorov polynomials mentioned above are defined by

(14) $$Q_m = \prod_{k=0}^{2^m-1} (1+\varphi_k^m) \quad (m\in\mathbf{P}).$$

Denote $\|f\|_p$ $(1\leqq p)$ the $L^p[0, 1)$-norm of f and $|E|$ the Lebesgue measure of the set $E\subseteqq[0, 1)$.

We need the following properties of the Q_m's.

Lemma. *For every $m\in\mathbf{P}$ the function Q_m is a Walsh polynomial with properties*

(15) $$\begin{cases} \text{i)} & \|Q_m\|_1 = 1, \\ \text{ii)} & \sup\limits_{k,l} |S_kQ_m - S_lQ_m| \geqq cm, \\ \text{iii)} & \left|\{S^*Q_m > 2m\}\right| \leqq 2/m, \end{cases}$$

where c is the same constant as in (11).

Proof. From the definition of the product system and from (8) and (13) it follows that

$$Q_m = \sum_{n=0}^{2^{2^m}-1} \psi_n^m = \sum_{n=0}^{2^{2^m}-1} w_n\circ\pi_m = D_{2^{2^m}}\circ\pi_m \geqq 0.$$

Thus from (8) we get

$$\|Q_m\|_1 = \|D_{2^{2^m}}\|_1 = 1.$$

Observe that the definition of ψ_n^m implies that

(16) $$\psi_n^m = w_{n2^m} g_m^n$$

for some Walsh polynomial g_m^n which only takes the values 1 and -1 and has a spectrum in $[0, 2^m)$. Moreover,

$$\psi_{2^k}^m(x) = w_{2^{m+k}}(x) g_m(x\dotplus k2^{-m}) \quad \big(x\in[0, 1)\big).$$

Hence and from (11) we get that for all $0\leqq k<2^m$ and $t\in\lfloor k2^{-m}, (k+1)2^{-m})$

$$|(S_{2^{m+k}+N_m}Q_m)(t) - (S_{2^{m+k}}Q_m)(t)| = |(S_{N_m}g_m)(t\dotplus k2^{-m})| \geqq cm.$$

Consequently (15) ii) is satisfied.

To prove (15) iii) we use the fact that by (16) the spectrum of each ψ_n^m is contained in $\lfloor n2^m, (n+1)2^m)$ and, consequently,

$$S^*Q_m \leqq \max_{0\leqq l<2^{2^m}} \left|\sum_{n=0}^{l} \psi_n^m\right| + \max_{0\leqq n<2^{2^m}} |S^*\psi_n^m| =$$

$$= \max_{0\leqq l<2^{2^m}} |D_l\circ\pi_m| + \max_{0\leqq n<2^{2^m}} S^*g_m^n = A_1 + A_2.$$

Since the spectrum of g_m^n's lies in $[0, 2^m)$, by (10) for A_2 we get

$$A_2 \leqq \max_{0 \leqq l < 2^n} \|D_l\|_1 \max_{0 \leqq n < 2^{2^m}} \|g_m^n\|_\infty \leqq m.$$

Thus

$$|\{S^* Q_m > 2m\}| \leqq |\{A_1 > m\}| = |\{D^* \circ \pi_m > m\}| = |\{D^* > m\}|,$$

where

$$D^*(x) = \sup_n |D_n(x)| \leqq \frac{2}{x} \quad (0 < x < 1)$$

(compare (9)). Hence (15) iii) follows and the Lemma is proved.

3. Proof of the theorem

Using the Walsh—Kolmogorov polynomials we can construct a function F with the required properties.

PROOF OF THEOREM 1. Let $\varepsilon_n \to 0$ $(n \to \infty)$ be a decreasing sequence of positive numbers such that

$$\Phi(n) = \varepsilon_n \log_2 \log_2 n \quad (n \geqq 4) \tag{17}$$

and set

$$\beta_n = \frac{n\varepsilon_n}{\Phi(n)} \quad (n \in \mathbf{N}).$$

Then $\beta_n \to \infty$ as $n \to \infty$, and we can choose an increasing sequence $(\nu_n, n \in \mathbf{N})$ of indices with $\nu_0 \geqq 4$ such that

$$\sum_{n=0}^{\infty} \frac{1}{\nu_n} < \infty \tag{18}$$

and for $l_n = 2^{2^{\nu_n}}$ $(n \in \mathbf{N})$ the following properties hold:

$$\begin{cases} \text{i)} & \beta_{l_n} \leqq \dfrac{1}{2} \beta_{l_{n+1}} \\ \text{ii)} & \dfrac{\varepsilon_{l_n}}{\Phi(l_n)} < \dfrac{1}{2} \quad (n \in \mathbf{N}). \\ \text{iii)} & \displaystyle\sum_{n=0}^{\infty} \varepsilon_{l_n} < \infty \end{cases} \tag{19}$$

The function F is defined by

$$F = \sum_{n=0}^{\infty} \frac{1}{\nu_n} r_{2^{\nu_n}+1} Q_{\nu_n}.$$

Conditions (18) and (15) i) imply that this series converges in L-norm. By definition, the Walsh spectrum of $r_{2^{\nu_n}+1} Q_{\nu_n}$ satisfies

$$\operatorname{spec}(r_{2^{\nu_n}+1} Q_{\nu_n}) \subset [2^{2^{\nu_n}+1}, 2^{2^{\nu_n}+1} + 2^{\nu_n} + \nu_n) \subset [2^{2^{\nu_n}+1}, 2^{2^{\nu_{n+1}}+1}] \tag{20}$$

therefore the series in question is the Walsh—Fourier series of F. Furthermore, from (15) ii) it follows that

$$\sup_{k,l} |S_k F - S_l F| \geqq c > 0$$

and, consequently, the Walsh—Fourier series of F diverges everywhere on $[0, 1)$.

To verify $F \in \mathbf{H}\Phi(\mathbf{H})$ we use $2^{\nu_n + 2^{\nu_n}} < 2^{2^{\nu_n}+1}$ and $Q_{\nu_n} \geqq 0$ to see that

$$F^* \leqq \sum_{n=0}^{\infty} \frac{Q_{\nu_n}}{\nu_n}.$$

Decompose $[0, 1) = E^* \cup \left(\sum_{n=0}^{\infty} E_n\right)$, where

$$E^* = \limsup_{n\to\infty} \{Q_{\nu_n} > 0\}$$

and

$$E_n = \{Q_{\nu_n} > 0 \text{ and } Q_{\nu_k} = 0 \text{ for all } k > n\}$$

for $n \in \mathbf{N}$. Recall that Q_{ν_n} only takes the values 0 and $2^{2^{\nu_n}}$. Consequently, the definitions E_n and β_n imply that

$$\chi_{E_n} F^* \leqq \sum_{k=0}^{n} 2^{2^{\nu_n}}/\nu_n = \sum_{k=0}^{n} \beta_{l_k},$$

where χ_E is the characteristic function of the set $E \subseteqq [0, 1)$. Since by (19) $\beta_{l_k} \leqq \frac{1}{2} \beta_{l_{k+1}}$ and $\beta_{l_n} < \frac{1}{2} l_n$, we have

$$\chi_{E_n} F^* \leqq 2\beta_{l_n} < l_n$$

for each $n \in \mathbf{N}$. But (15) i) and the fact that Q_{ν_n} only takes the values 0 and $2^{2^{\nu_n}}$ implies

$$\left|\{Q_{\nu_n} \neq 0\}\right| = 2^{-2^{\nu_n}} = \frac{1}{l_n}.$$

Consequently, $|E_n| \leqq 1/l_n$ and $|E^*| = 0$. Therefore, it follows (see (19)) that

$$\int_0^1 F^*(t)\Phi\big(F^*(t)\big)\,dt = \sum_{n=0}^{\infty} \int_{E_n} F^*(t)\Phi\big(F^*(t)\big)\,dt \leqq$$

$$\leqq 2 \sum_{n=0}^{\infty} \beta_{l_n} \Phi(l_n)|E_n| = 2 \sum_{n=0}^{\infty} \varepsilon_{l_n} < \infty.$$

Thus, it remains to prove that S^*F is finite a.e. To this end observe that by (20)

$$S^* F \leqq F^* + \sup_n \frac{S^* Q_{\nu_n}}{\nu_n} \tag{21}$$

and since $F \in \mathbf{H}\Phi(\mathbf{H})$, we have that F^* is finite a.e. To estimate the second term set

$$H_n = \left\{\frac{S^* Q_{\nu_n}}{\nu_n} > 2\right\} = \{S^* Q_{\nu_n} > 2\nu_n\}$$

for $n\in\mathbf{N}$ and let $H^*=\bigcap_{n=0}^{\infty}\bigcup_{k=n}^{\infty}H_k$. By (15) iii) $\sum_{n=0}^{\infty}|H_n|<\infty$. Consequently, $|H^*|=0$ and the second term is finite off H^*.

Thus, Theorem 1 is proved.

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HERMITE AND HERMITE—FEJÉR INTERPOLATIONS OF HIGHER ORDER. II (MEAN CONVERGENCE)

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To Professor A. Sharma on his 70th birthday

Weighted L^p convergence of Hermite and Hermite—Fejér interpolations of higher order on the zeros of Jacobi polynomials is investigated. The results which cover the classical Hermite—Fejér case give necessary and in many cases sufficient conditions for such convergence for all continuous functions. Uniform convergence is considered, too.

1. Notations. Preliminary results

General notations. The symbols "const", "c", "c_1", etc. denote some positive constant being independent of the variables and indices. In each formula they may take different value. The symbol "$\sim$" means as follows. If A and B are two expressions depending on some variables and indices then

$$A \sim B \quad \text{iff} \quad |A/B| \leqq c_1 \quad \text{and} \quad |B/A| \leqq c_2.$$

C and L^p spaces. Let C denote the set of continuous functions on $[-1, 1]$ with the usual norm

$$\|f\| := \max_{-1 \leqq x \leqq 1} |f(x)|.$$

Sometimes we use the notation $\|f\|_{[a,b]} := \max\limits_{a \leqq x \leqq b} |f(x)|$, $f \in C$, $[a, b] \subset [-1, 1]$. Further $f \in L^p$, $0 < p < \infty$ iff

$$\|f\|_p := \Big(\int_{-1}^{1} |f(x)|^p \, dx \Big)^{1/R} < \infty$$

where $R = \max(1, p)$. If $0 < p < 1$, $\|\cdot\|_p$ is a distance, not a norm, but we keep this notation for convenience. It satisfies

$$\|cf\|_p = c^{p/R} \|f\|_p, \quad \|f+g\|_p \leqq \|f\|_p + \|g\|_p, \quad 0 < p < \infty,$$

whenever $f, g \in L^p$. As above, $\|f\|_{p,[a,b]} := \Big(\int_a^b |f|^p \Big)^{1/R}$.

Hermite—Fejér interpolation of higher order. Let $X = \{x_{kn} = \cos \vartheta_{kn}\} \subset [-1, 1]$,

$$-1 \leqq x_{nn} < x_{n-1,n} < \ldots < x_{1n} \leqq 1, \quad n = 1, 2, \ldots, \tag{1.1}$$

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be an infinite triangular interpolatory matrix. We consider the *unique interpolatory polynomials* $H_{nm}(f, X, x)$ *of degree* $\leqq mn-1$ for $f \in C$ defined as follows:

$$(1.2)\quad \begin{cases} H_{nm}(f, X, x_{kn}) = f(x_{kn}), & 1 \leqq k \leqq n, \\ H_{nm}^{(t)}(f, X, x_{kn}) = 0, & 1 \leqq t \leqq m-1, \quad 1 \leqq k \leqq n, \\ m = 1, 2, \ldots \text{ is fixed; if } m = 1, & \text{we prescribe the function values only.} \end{cases}$$

By definition, H_{n1} are the Lagrange-, H_{n2} the Hermite—Fejér, and H_{n4} the so called Krylov—Stayermann interpolatory polynomials. By (1.2), H_{nm} can be written as follows:

$$(1.3)\qquad H_{nm}(f, x) = \sum_{k=1}^{n} f(x_{kn}) h_{knm}(x), \quad n = 1, 2, \ldots,$$

where the polynomials h_k of degree exactly $nm-1$ are

$$(1.4)\qquad h_{knm}(x) = l_{kn}^{m}(x) \sum_{i=0}^{m-1} e_{iknm}(x-x_{kn})^{i}, \quad 1 \leqq k \leqq n.$$

Here the coefficients e_{ik} can be obtained from

$$(1.5)\quad \begin{cases} h_{knm}(x_l) = \delta_{lk}, & 1 \leqq k, \ l \leqq n, \\ h_{knm}^{(t)}(x_l) = 0, & 1 \leqq k, \ l \leqq n, \quad 1 \leqq t \leqq m-1; \\ \text{if } m = 1, & \text{we omit the second row;} \end{cases}$$

further, $l_{kn}(x)$ are the Lagrange fundamental polynomials of degree exactly $n-1$, i.e. with $\omega_n(x) := c_n \prod_{k=1}^{n} (x-x_{kn})$,

$$(1.6)\qquad l_{kn}(x) = \frac{\omega_n(x)}{\omega_n'(x_{kn})(x-x_{kn})}.$$

Here and later we use some obvious short notations: e.g. we write $H_{nm}(f, x)$, $H_{nm}(x)$ or H_{nm} for $H_{nm}(f, X, x)$; $h_{knm}(x)$, $h_{kn}(x)$, $h_k(x)$ or h_k for $h_{knm}(X, x)$, etc.

Hermite interpolation. If $f^{(m-1)} \in C$, the Hermite interpolatory polynomial $\mathcal{H}_{nm}(f, X, x)$ of degree $\leqq mn-1$ based on the nodes (1.1) is defined by

$$\mathcal{H}_{nm}^{(t)}(f, X, x_{kn}) = f^{(t)}(x_{kn}), \quad 1 \leqq k \leqq n, \quad 0 \leqq t \leqq m-1.$$

They can be written as

$$\mathcal{H}_{nm}(f, X, x) = \sum_{t=0}^{m-1} \sum_{k=1}^{n} f^{(t)}(x_{kn}) h_{tknm}(X, x), \quad m = 1, 2, \ldots,$$

where, by $h_{tknm}^{(i)}(x_{jn}) = \delta_{ti}\delta_{kj}$, we write

$$(1.7)\quad h_{tknm}(X, x) = l_{kn}^{m}(X, x) \frac{(x-x_{kn})^{t}}{t!} \sum_{i=0}^{m-1-t} e_{tiknm}(x-x_{kn})^{i}, \quad 0 \leqq t \leqq m-1.$$

Here $h_{0k} = h_k$ and $e_{0ik} = e_{ik}$ (cf. (1.4)), further

$$(1.8)\qquad e_{trknm} = e_{0rknm}, \quad r = 0, 1, \ldots, m-1-t, \quad 0 \leqq t \leqq m-1.$$

Relation (1.7) can be proved by induction with respect to t using the conditions for h_{tk} (cf. [2, 3.1]).

By definition, for $P\in\mathcal{P}_{mn-1}$ (= the set of polynomials of degree $\leqq mn-1$)

$$P(x)\equiv\mathcal{H}_{nm}(P,X,x)=H_{nm}(P,X,x)+\sum_{t=1}^{m-1}\sum_{k=1}^{n}P^{(t)}(x_k)h_{tk}(x). \tag{1.9}$$

Jacobi polynomials. $w=w^{(\alpha,\beta)}$ is a Jacobi weight function with parameters α,β (shortly $w\in J$ or $w\in J(\alpha,\beta)$) iff

$$w(x)=(1-x)^\alpha(1+x)^\beta,\quad \alpha,\beta>-1,\quad |x|\leqq 1. \tag{1.10}$$

The corresponding system of orthogonal Jacobi polynomials is denoted by $\{p_n(w)\}_0^\infty$, i.e. $p_n(w)=\gamma_n(w)x^n+$lower degree terms, $\gamma_n(w)>0$ and

$$\int_{-1}^{1}p_n(w)p_N(w)w=\delta_{nN},\quad n,N\geqq 0.$$

The zeros of $p_n(w)$ are $x_{kn}(w)$ and they are ordered so that

$$-1<x_{nn}(w)<\ldots<x_{2n}(w)<x_{1n}(w)<1.$$

2. Results

Let $A_m:=-\dfrac{1}{2}-\dfrac{2}{m}$, $B_m:=-\dfrac{1}{2}+\dfrac{1}{m}$ and $C_m:=-\dfrac{1}{2}-\dfrac{1}{m}$, $m=1,2,\ldots$. Statements concerning the uniform convergence of $H_{nm}(f)$ to f show that $\alpha,\beta\geqq A_m$ are necessary conditions (cf. P. Vértesi [7]). This is why we suppose them from now on.

Let $\gamma=\min(\alpha,\beta)$ and $\Gamma=\max(\alpha,\beta)$.

The process H_{nm}

THEOREM 2.1. *Let $m=2,4,\ldots$, be fixed, even, $p>0$, $u^p\in J$, $w\in J(\alpha,\beta)$ and*

$$(1)\ \gamma\geqq C_m\quad or\quad (2)\ A_m\leqq\gamma<C_m\quad and\quad \Gamma-\gamma\leqq 2/m. \tag{2.1}$$

Then

$$\lim_{n\to\infty}\|(H_{nm}(f,w)-f)u\|_p=0\quad\forall f\in C \tag{2.2}$$

if

$$v(x):=\frac{u(x)\sqrt{1-x^2}}{\left(w(x)\sqrt{1-x^2}\right)^{m/2}}\in L^p. \tag{2.3}$$

Further using

$$0<e_{1,m-2,knm}\sim\left(\frac{n}{\sin\vartheta_k}\right)^{m-2}\quad if\quad K\geqq M, \tag{2.4}$$

(2.2) *implies* (2.3). *Here* $K=\min(k,n-k+1)$, M *is fixed.*

REMARKS 1. Conditions (2.4) say that the order of the corresponding coefficients is as big as it could be (cf. (1.8), (3.5) and (3.6)). They were verified very recently in R. Sakai, P. Vértesi [8].

2. If $m=2$ (when, as it is well known, $e_{10kn2}\equiv 1$) *the result was proved* using another approach in P. Nevai, P. Vértesi [3, Theorem 5] even for $w\in GJC$ (i.e. $w=w_1 g$, $w_1\in J$ and $g>0$, $g'\in\mathrm{Lip}\,1$ on $[-1,1]$).

However *the method applied here has some advantages.* (a) It does not use results on mean convergence of Lagrange interpolation demanding a rather fine technique (cf. P. Nevai [2; Lemma and Theorem 6] and [3, Theorem 5]). (b) The proof directly works for each fixed $p>0$ (cf. [3, p. 57]).

Our direct approach is based on the precise pointwise estimation of $\sum |h_{tk}(x)|$ taking shape in Lemma 3.2 (cf. (3.2), too). Using Lemma 3.2, one can prove mean convergence on subintervals, too. The efficiency of Lemma 3.2 shows *that at the estimation* of $\sum a_{tk}(f)h_{tk}(x)$ *we do not have to consider the possibly different signs of* $h_{tk}(x)$ — a phenomenon which certainly does not occur taking odd values of m (cf. [2], when $m=1$, i.e. Lagrange interpolation).

3. Let $w\in GJC$. While our method works in this case, too, if $m=2$ (cf. [3, Lemma 2]), the relations (3.5) seem to be difficult to verify whenever $m>2$.

The process $\mathcal{H}_{nm}$

THEOREM 2.2. *Let* $m=2, 4, \ldots,$ *be fixed, even,* $p>0$, $u^p\in J$, $w\in J(\alpha,\beta)$ *and suppose* (2.1). *Then*

$$\lim_{n\to\infty} \left\|\left(\mathcal{H}_{nm}(f,w)-f\right)u\right\|_p = 0 \quad \forall f \quad \text{with} \quad f^{(m-1)}\in C, \tag{2.5}$$

if (2.3) *holds true.*

Using Lemma 3.2 we can obtain results on *uniform approximation,* too (cf. [7]). Namely let $-1<a<1$. Then

STATEMENT 2.3. *Let* $m=2, 4, \ldots,$ *be fixed, even, and suppose* (3.7). *Then*

$$\lim_{n\to\infty} \|\mathcal{H}_{nm}(f,w)-f\|_{[a,1]} = 0 \quad \forall f \quad \text{with} \quad f^{(m-1)}\in C \tag{2.6}$$

if $\alpha<B_m$.

Using $p_n^{(\alpha,\beta)}(x)=(-1)^n p_n^{(\beta,\alpha)}(-x)$ ([4, (4.1.3)]), we obtain analogous results for $[-1,a]$, and finally

COROLLARY 2.4. *Let* $m=2, 4, \ldots$ *be fixed even numbers, and suppose* (2.1). *Then*

$$\lim_{n\to\infty} \|\mathcal{H}_{nm}(f,w)-f\| = 0 \quad \forall f \text{ with } \quad f^{(m+1)}\in C \tag{2.7}$$

if $\Gamma<B_m$.

REMARKS 1. *Uniform convergence* of $H_{nm}(f)$ to f can be obtained, too (cf. Propositions 2.3 and 2.4). The results slightly improve [7, Theorem 2.1] namely, [7, (2.1) and (2.4)] can be replaced by $\alpha\in[A_m, B_m)$ and $\alpha-\beta\leqq 2/m$, respectively. The proof is analogous to that of Statement 2.3.

Again we get *nearly the same necessary and sufficient conditions* for H_{nm} and $\mathcal{H}_{nm}$.

2. The previous Remark 3 holds true for $\mathcal{H}_{nm}$, too.

3. For theorems of "iff" type see P. Vértesi, Y. Xu [9].

3. Proofs

3.1. From now on, unless otherwise stated, we suppose that $X=X(w^{(\alpha,\beta)})=$ $=\{x_{kn}(w)=\cos\vartheta_{kn}(w)\}$, $k=1,2,\ldots,n$; $n=1,2,\ldots$, $\alpha,\beta>-1$.
Here with $x=\cos\vartheta$, $0\leqq\vartheta\leqq\pi$,

$$w^{(\alpha,\beta)}(x)=(1-x)^\alpha(1+x)^\beta=\varrho^{(\alpha,\beta)}(\vartheta)=2^{\alpha+\beta}\sin^{2\alpha}\frac{\vartheta}{2}\cos^{2\beta}\frac{\vartheta}{2},\quad \alpha,\beta>-1.$$

Using obvious short notations, we have

LEMMA 3.1. *With* $\vartheta_{0n}=0$, $\vartheta_{n+1,n}=\pi$ *and* $|x_j-x|=\min_{1\leqq k\leqq n}|x-x_k|$,

$$\vartheta_{k+1,n}-\vartheta_{kn}\sim\frac{1}{n} \tag{3.1}$$

uniformly for $0\leqq k\leqq n$, $n\in\mathbf{N}$,

$$|p_n(x)|=|p_n(\cos\vartheta)|\sim\frac{n|\vartheta-\vartheta_j|}{\left(w(x_j)\sqrt{1-x_j^2}\right)^{1/2}}, \tag{3.2}$$

uniformly for $-1\leqq x=\cos\vartheta\leqq 1$ *and* $n\in\mathbf{N}$,

$$|p_n'(x_k)|=|p_n'(\cos\vartheta_k)|\sim\frac{n}{\sqrt{1-x_k^2}}\cdot\frac{1}{\left(w(x_k)\sqrt{1-x_k^2}\right)^{1/2}}, \tag{3.3}$$

uniformly for $1\leqq k\leqq n$ $n\in\mathbf{N}$.

Here (3.1) comes from [1], (3.2) from [6, Lemma 4.3], finally (3.3) is essentially [4, (8.9.2)].

If $K:=\min(k,n-k+1)$ and $J:=\min(j,n-j+1)$, $1\leqq k,j\leqq n$, by (3.1) it is easy to get the following relations, frequently used later:

$$\begin{cases} |x-x_k|\sim\dfrac{(K+J)|K-J|}{n^2}, & 1\leqq k,j\leqq n,\ k\neq j,\\ |x-x_j|\sim\dfrac{|\vartheta-\vartheta_j|J}{n}\leqq c\dfrac{J}{n^2}, & 1\leqq j\leqq n,\\ \sin\vartheta_k\sim\dfrac{K}{n}, & 1\leqq k\leqq n,\\ \text{uniformly for } k,j \text{ and } n\in\mathbf{N}. \end{cases} \tag{3.4}$$

Using Lemma 3.1 and the differential equation for the Jacobi polynomials *for* $X^{(\alpha,\beta)}$ *and any fixed* $m\geqq 1$, *we have the relations*

$$|e_{0sknm}|\leqq cI(n,k,s),\quad 0\leqq s\leqq m, \tag{3.5}$$

where

$$I(n,k,s):=\begin{cases}\left(\dfrac{n}{\sin\vartheta_{kn}}\right)^s & \textit{if}\quad s=0,2,4,\ldots,\\ \dfrac{n^{s-1}}{\sin^{s+1}\vartheta_{kn}} & \textit{if}\quad s=1,3,5,\ldots\end{cases} \tag{3.6}$$

(see [7, Lemma 3.11]).

Now we prove our basic statement.

LEMMA 3.2. *Let* $m=2, 4, \dots,$ *fixed, further let*

$$\text{(3.7)}\qquad \begin{cases} (1)\quad \alpha, \beta \geqq C_m \quad \text{or} \quad (2)\quad \alpha \geqq C_m > \beta \geqq A_m \quad \text{and} \quad \alpha-\beta \leqq 2/m \quad \text{or} \\ (3)\quad \beta \geqq C_m > \alpha \geqq A_m \quad \text{or} \quad (4)\quad A_m \leqq \alpha, \beta \leqq C_m. \end{cases}$$

If $-1<a<1$ *then uniformly for* n *and* x

$$\text{(3.8)}\qquad \sum_{k=1}^{n} |h_{tk}(x)| \leqq \frac{c}{n^t}\left[(\sqrt{1-x_j^2})^t + \frac{\log n}{n}\frac{1}{(w(x_j)\sqrt{1-x_j^2})^{m/2}}\right],$$

$$x\in[a, 1],\ t=0, 2, \dots, m-2,$$

$$\text{(3.9)}\qquad \sum_{k=1}^{n} |h_{tk}(x)| \leqq c\frac{\log n}{n^t}\left[\frac{1}{(w(x_j)\sqrt{1-x_j^2})^{m/2}}+1\right], \quad x\in[a, 1],\ t=1, 3, \dots, m-1.$$

Further, supposing (2.1), *relations* (3.8) *and* (3.9) *hold for* $|x|\leqq 1$.

PROOF OF LEMMA 3.2. First *let* $m-t$ *be even,* $m=2, 3, \dots$. By (1.7), (1.8) and (3.6), for arbitrary $|x|\leqq 1$

$$|h_{tk}(x)| \leqq c|l_k^m(x)|\,|x-x_k|^t\left[1+\frac{|x-x_k|}{\sin^2\vartheta_k}+\left(\frac{n|x-x_k|}{\sin\vartheta_k}\right)^2+\dots\right.$$

$$\left.+\left(\frac{n|x-x_k|}{\sin\vartheta_k}\right)^{m-t-2}+\left(\frac{n|x-x_k|}{\sin\vartheta_k}\right)^{m-t-2}\frac{|x-x_k|}{\sin^2\vartheta_k}\right].$$

Here by (3.4) $n|x-x_k|/\sin\vartheta_k \sim (K+J)|K-J|K^{-1}\geqq c,\quad k\neq j$ whence, writing $[\dots]=\sum_{i=0}^{m-1-t} q(i)$, we have $q(0)\leqq cq(2)\leqq\dots\leqq cq(m-t-2)$ and by $q(2r+1)=$ $=(|x-x_k|/\sin^2\vartheta_k)$

$$q(2r),\ q(1) \leqq cq(3) \leqq \dots \leqq cq(m-t-1).$$

So

$$|h_{tk}(x)| \leqq c\,|l_k^m(x)|\,|x-x_k|^t\left(\frac{n|x-x_k|}{\sin\vartheta_k}\right)^{m-t-2}\left[1+\frac{|x-x_k|}{\sin^2\vartheta_k}\right], \quad k\neq j.$$

On the other hand, if $k=j$ we get $q(i)\leqq c$ for every i, whence

$$|h_{tj}(x)| \leqq c\,|l_j^m(x)|\,|x-x_j|^t.$$

First we estimate $h_{tj}(x)$. If $x\in[x_{j+1}, x_j]$, $1\leqq j\leqq n-1$, say, and $4n|\vartheta-\vartheta_j|\geqq$ $\geqq|\vartheta_{j+1}-\vartheta_j|$, $h_{tj}(x)$ can be handled as $h_{t,j+1}(x)$. If $4n|\vartheta-\vartheta_j|<|\vartheta_{j+1}-\vartheta_j|$,

$$|l_j^m(x)|\,|x-x_j|^t \leqq c(J/n^2)^t \sim n^{-t}\sin^t\vartheta_j$$

(cf. (3.4), the first term of (3.8)). The other cases can be treated analogously.

To go further, by (3.3) we write

$$\sum_{k=1}^{n}{}' |h_{tk}(x)| \leqq c|p_n^m(x)| \left[\sum_k{}' \frac{(w(x_k))^{m/2}(\sin\vartheta_k)^{t+2+m/2}}{n^{t+2}|x-x_k|^2}+\right.\tag{3.10}$$

$$\left.+\sum_k{}' \frac{(w(x_k))^{m/2}(\sin\vartheta_k)^{t+m/2}}{n^{t+2}|x-x_k|}\right] := c|p_n^m(x)|(I_1+I_2), \quad m-t \text{ is even}, \quad m=2,3,\ldots.$$

Here and later $\sum\limits_k{}'$ means that we omit the term $k=j$.

First remark that by (3.7) $\alpha, \beta \geqq A_m$. Now let $x \geqq a$. Then by (3.10)

$$I_1 \leqq \sum_{k=1}^{[3n/4]}{}' + \sum_{k=[3n/4]}^{n} \sim \frac{1}{n^{t-2}} \sum_{k=1}^{n}{}' \frac{(k/n)^{m(\alpha+1/2)+t+2}}{(k+j)^2(k-j)^2}+$$

$$+\frac{1}{n^{t+2}}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{m(\beta+1/2)+t+2} := S_1+S_2.$$

Considering S_1, we write

$$\sum_{k=1}^{n}{}' \frac{k^{m(\alpha+1/2)+t+2}}{(k+j)^2(k-j)^2} \leqq \frac{1}{j^4}\sum_{k=1}^{[j/2]} k^{m(\alpha+1/2)+t+2}+$$

$$+\sum_{k=[j/2]}^{2j}{}' \frac{j^{m(\alpha+1/2)+t}}{(k-j)^2}+\sum_{k=2j}^{n} k^{m(\alpha+1/2)+t-2} \leqq c\left(j^{m(\alpha+1/2)+t}+n^{m(\alpha+1/2)+t-1}\log n\right)$$

using $m(\alpha+1/2)+t+2 \geqq m(\alpha+1/2)+2 \geqq 0$ (by $\alpha \geqq A_m$). By $\beta \geqq A_m$, $m(\beta+1/2)+ +t+2 \geqq m(\beta+1/2)+2 \geqq 0$, i.e. S_2 can be estimated by cn^{-t-1}. So

$$I_1 \leqq \frac{c}{n^t}\left[\left(\frac{j}{n}\right)^{m(\alpha+1/2)+t}+\frac{\log n}{n}\right], \quad x \geqq 0,\ t \geqq 0.\tag{3.11}$$

To estimate I_2, again by (3.4)

$$I_2 \leqq \frac{c}{n^t}\sum_{k=1}^{n}{}' \frac{(k/n)^{m(\alpha+1/2)+t}}{(k+j)|k-j|}+\frac{c}{n^{t+2}}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{m(\beta+1/2)+t} := S_3+S_4.$$

By arguments similar to the above ones, further using that $t=0, 2, \ldots$ ($m-t$ is even, m is even)

$$S_3 \begin{cases} \leqq \dfrac{c}{n^{t+1}}\left[\left(\dfrac{j}{n}\right)^{m(\alpha+1/2)+t-1}\log 2j+\log n\right] & \text{if } t \geqq 2 \text{ or } t=0 \text{ and } m(\alpha+1/2) \geqq -1,\\[2ex] \sim \dfrac{1}{j^2 n^{m(\alpha+1/2)}} & \text{if } t=0 \text{ and } m(\alpha+1/2) < -1,\end{cases}$$

$$S_4 \begin{cases} \sim \dfrac{1}{n^{t+1}} & \text{if } t \geqq 2,\\[2ex] \leqq c\dfrac{\log n}{n} & \text{if } t=0 \text{ and } m(\beta+1/2) \geqq -1,\\[2ex] \sim n^{-m(\beta+1/2)-2} & \text{if } t=0 \text{ and } m(\beta+1/2) < -1.\end{cases}$$

I.e.,

$$I_2 \leqq \frac{c}{n^t}\left[\frac{\log 2j}{n}\left(\frac{j}{n}\right)^{m(\alpha+1/2)+t-1}+\frac{\log n}{n}\right] \quad \text{if } t \geqq 2 \text{ or } t=0 \text{ and } \alpha,\beta \geqq C_m,$$

whence by (3.10), (3.11), $w(x_j)\sim(j/n)^{2\alpha}$ and $\sqrt{1-x_j^2}=\sin\vartheta_j\sim j/n$, we obtain (3.8) when $\alpha,\beta\geqq C_m$.

If $t=0$, $m(\alpha+1/2)\geqq-1$ but $m(\beta+1/2)<-1$, for our new term coming from S_4 we have, using (3.2) and (3.7),

$$\frac{p_n^m(x)}{n^{m(\beta+1/2)+2}} \leqq \frac{c}{j^{m(\beta+1/2)+2}}\left(\frac{j}{n}\right)^{m(\beta-\alpha)+2} \leqq c,$$

i.e. we obtain (3.8) for this case, too.

If $t=0$, $m(\beta+1/2)>-1$ but $m(\alpha+1/2)<-1$, the new term coming from S_3 is

$$\frac{p_n^m(x)}{j^2 n^{m(\alpha+1/2)}} \leqq \frac{c}{j^{m(\alpha+1/2)+2}} \leqq c$$

(by $\alpha\geqq A_m$), i.e. (3.8) holds true.

Finally, if $t=0$, $m(\alpha+1/2)<-1$ and $m(\beta+1/2)<-1$, we combine the last two cases to finish the proof of (3.8).

Now *let* $m-t$ *be odd*, $m=1, 2, \dots$. By (1.7), (1.8), (3.5) and (3.6) for arbitrary $|x|\leqq 1$

$$|h_{tk}(x)| \leqq c|l_k^m(x)|\,|x-x_k|^t\left[1+\frac{|x-x_k|}{\sin^2\vartheta_k}+\left(\frac{n|x-x_k|}{\sin\vartheta_k}\right)^2+\dots\right.$$

$$\left.+\left(\frac{n|x-x_k|}{\sin\vartheta_k}\right)^{m-t-3}\frac{|x-x_k|}{\sin^2\vartheta_k}+\left(\frac{n|x-x_k|}{\sin\vartheta_k}\right)^{m-t-1}\right].$$

If $[\dots]:=\sum_{i=0}^{m-1-k} q(i)$, then by

$$q(2r) = n^2|x-x_k|\,q(2r-1)\sim|K-J|\,|K+J|\,q(2r-1) \geqq cq(2r-1),$$

$$k\neq j,\quad r=1,2,\dots,(m-t-1)/2$$

and $n|x-x_k|\sin^{-1}\vartheta_k\geqq c$, $k\neq j$, we get that $[\dots]\leqq cq(m-t-1)$ which means

$$|h_{tk}(x)| \leqq c|l_k^m(x)|\,|x-x_k|^t\left(\frac{n|x-x_k|}{\sin\vartheta_k}\right)^{m-t-1}, \quad k\neq j. \tag{3.12}$$

If $k=j$, then $q(i)\leqq c$ for all i, whence $|h_{tj}(x)| \leqq c|l_j^m(x)|\,|x-x_j|^t$. This term can be estimated as before. If $k\neq j$,

$$\sum_{k=1}^{n}{}' |h_{tk}(x)| \leqq c|p_n^m(x)|\sum_k{}' \frac{(w(x_k))^{m/2}(\sin\vartheta_k)^{t+2+m/2}}{n^{t+1}|x-x_k|}, \tag{3.13}$$

$$m-t \text{ is odd},\ m=1,2,\dots.$$

Let $x \geqq a$. Denoting the sum by I_3, we write

$$I_3 \leqq \sum_{k=1}^{[3n/4]}{}' + \sum_{k=[3n/4]}^{n} \sim \frac{1}{n^{t-1}} \sum_{k=1}^{n}{}' \frac{(k/n)^{m(\alpha+1/2)+t+1}}{(k+j)|k-j|} +$$

$$+\frac{1}{n^{t+1}} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^{m(\beta+1/2)+t+1} := S_5+S_6.$$

Using considerations as before further using that $t \geqq 1$ ($m-t$ is odd, m is even), we get

$$S_5 \leqq \frac{c}{n^t}\left[\log 2j\left(\frac{j}{n}\right)^{m(\alpha+1/2)+t}+\log n\right], \quad S_6 \leqq c\frac{\log n}{n^t}$$

whence, by $\log 2j(\sqrt{1-x_j^2})^t \leqq c \log n$, we get (3.9).

Changing the roles of α and β we get four other conditions and the corresponding statements for $[-1, a]$. It is easy to check that (2.1) includes all the eight conditions. This completes the proof for the whole interval $[-1, 1]$. □

3.2. Proof of Theorem 2.1. Let $\varepsilon>0$ be fixed and take a polynomial Q such that $\|f-Q\| \leqq \varepsilon$. Then

$$||H_{nm}(f, x)-f(x)| \leqq |H_{nm}(f, x)-Q(x)|+|f(x)-Q(x)| \leqq$$

$$\leqq |H_{nm}(f-Q, x)|+\sum_{t=1}^{m-1}\sum_{k=1}^{n} |Q^{(t)}(x_k)h_{tk}(x)|+\varepsilon \leqq$$

$$\leqq \varepsilon \sum_{k=1}^{n} |h_{0k}(x)|+M\sum_{t=1}^{m-1}\sum_{k=1}^{n} |h_{tk}(x)|+\varepsilon, \quad n \geqq n_0$$

where $M:=\max\limits_{1 \leqq t \leqq m-1} \|Q^{(t)}\|$. *So for any fixed X and $\varepsilon>0$ there is a constant M such that for $n \geqq n_0(\varepsilon)$*

$$|H_{nm}(f, X, x)-f(x)| \leqq \varepsilon\Big[\sum_{k=1}^{n} |h_{0k}(x)|+1\Big]+M\sum_{t=1}^{m-1}\sum_{k=1}^{n} |h_{tk}(x)|. \tag{3.14}$$

3.3. "(2.3)⇒(2.2)". For $x \in [a, 1]$ by (3.14), (3.8) and (3.9)

$$|H_{nm}(f, x)-f(x)| \leqq \varepsilon\Big[\sum_{k=1}^{n} |h_{0k}(x)|+1\Big]+M\sum_{t=1,3,\dots}\sum_{k=1}^{n} |h_{tk}(x)|+ \tag{3.15}$$

$$+M\sum_{t=2,4,\dots}\sum_{k=1}^{n} |h_{tk}(x)| \leqq c\left\{\varepsilon\left[1+\frac{\log n}{n}\frac{1}{(w(x_j)\sqrt{1-x_j^2})^{m/2}}\right]+\right.$$

$$\left.+\frac{\log n}{n}\left[\frac{1}{(w(x_j)\sqrt{1-x_j^2})^{m/2}}+1\right]+\frac{1}{n^2}\left(1+\frac{\log n}{n}\frac{1}{(w(x_j)\sqrt{1-x_j^2})^{m/2}}\right)\right\} \leqq$$

$$\leqq c\left[\varepsilon+\frac{\log n}{n}\frac{1}{(w(x_j)\sqrt{1-x_j^2})^{m/2}}\right] \sim \varepsilon+\frac{\log n}{n}\frac{1}{(\sqrt{1-x_j^2})^{m(\alpha-1/2)}}.$$

First let $m(\alpha+1/2)-1\geqq 0$. By $v\in L^p$ we can choose a ϱ, $0<\varrho\leqq 1$, such that $v_1:=v/(\sqrt{1-x^2})^\varrho\in L^p$. By the triangular inequality, (3.15) and $1/(n\sqrt{1-x_j^2})^{1-\varrho}\leqq c$ we have

$$\|(H_{nm}(f)-f)u\|_{p,[0,1]}\leqq c\|\varepsilon u\|_p+$$

$$+c\left\{\left(\frac{\log n}{n^\varrho}\right)^p\int_0^1\left[\frac{u(x)}{(\sqrt{1-x_j^2})^{m(\alpha+1/2)-1}(\sqrt{1-x_j^2})^\varrho}\,\frac{1}{(n\sqrt{1-x_j^2})^{1-\varrho}}\right]^p dx\right\}^{1/R}\leqq$$

$$\leqq c\varepsilon^{p/R}+c\left\{\left(\frac{\log n}{n^\varrho}\right)^p\int_0^1\left[\frac{v(x)}{(\sqrt{1-x^2})^\varrho}\right]^p dx\right\}^{1/R}$$

$$\leqq c\varepsilon^{p/R}+c\left[\frac{\log n}{n^\varrho}\right]^{p/R}\|v_1\|_p=c\varepsilon^{p/R}+o(1).$$

If $1-m\left(\alpha+\dfrac{1}{2}\right)=A>0$ with $0<\zeta\leqq A$, $\zeta\leqq 1$, we write

$$\|(H_{nm}(f)-f)u\|_{p,[0,1]}\leqq c\|\varepsilon u\|_p+$$

$$+c\left\{\left(\frac{\log n}{n^\zeta}\right)^p\int_0^1\left[\frac{u(x)(\sqrt{1-x_j^2})^{1-m(\alpha+1/2)-\zeta}}{(n\sqrt{1-x_j^2})^{1-\zeta}}\right]^p dx\right\}^{1/R}\leqq$$

$$\leqq c\varepsilon^{p/R}+c\left(\frac{\log n}{n^\zeta}\right)^{p/R}\Big(\int_0^1 u^p(x)\,dx\Big)^{1/R}=c\varepsilon^{p/R}+o(1).$$

Using similar arguments for the interval $[-1,0]$, we get (2.2).

3.4. "(2.2)$\Rightarrow$(2.3)". First let $\alpha+\dfrac{1}{2}=\varrho>0$. By (1.7), (1.8), (3.5), (3.6), (2.4) and Lemma 3.1

$$h_{1k}(x)\geqq l_k^m(x)(x-x_k)\left\{c\left[\frac{n}{\sin\vartheta_k}(x-x_k)\right]^{m-2}-1-\right.\tag{3.18}$$

$$\left.-\frac{(x-x_k)}{\sin^2\vartheta_k}-\ldots-\left[\frac{n(x-x_k)}{\sin\vartheta_k}\right]^{m-4}\cdot\frac{x-x_k}{\sin^2\vartheta_k}\right\}\geqq cl_k^m(x)(x-x_k)\left[\frac{n}{\sin\vartheta_k}(x-x_k)\right]^{m-2},$$

uniformly for $K\geqq M$, $(x_1+1)/2:=a_n\leqq x\leqq b_n:=(x_1+2)/3$.

By (3.18), (3.5), (2.4), Lemma 3.1 and (3.12), we get as before

$$\begin{cases}\displaystyle\sum_{(1)}h_{1k}(x)\geqq\frac{c}{n}\,\frac{1}{(w(x_1)\sqrt{1-x_1^2})^{m/2}}\sim\frac{1}{n^{1-m(\alpha+1/2)}}=\frac{1}{n^{1-m\varrho}},\\ \displaystyle\sum_{(2)}|h_{1k}(x)|\leqq\frac{c}{n^2},\\ \displaystyle\sum_{(3)}h_{1k}(x)>0\end{cases}\tag{3.19}$$

uniformly for $a_n\leqq x\leqq b_n$ and $n\in\mathbf{N}$. Here $\sum\limits_{(1)}$, $\sum\limits_{(2)}$ and $\sum\limits_{(3)}$ mean summation for

k with (1) $k>M$, $0\leqq x_k$, (2) $1\leqq k\leqq M$, (3) $x_k<0$, respectively. If $u\in J(\mu, v)$, by $f_1(x)=x$ and (3.19)

$$\text{(3.20)}\qquad \|(H_{nm}(f_1)-f_1)u\|_p^R = \int_{-1}^{1} \Big|\sum_{k=1}^{n} h_{1k}\Big|^p u^p \geqq \int_{a_n}^{b_n} \Big|\sum_{k=1}^{n} h_{1k}\Big|^p u^p \geqq$$

$$\geqq c \int_{a_n}^{b_n} \Big|\sum_{(1)} h_{1k}\Big|^p u^p \geqq \frac{c}{n^2}\left(\frac{1}{n^{1-m(\alpha+1/2)+2\mu}}\right)^p := g(n).$$

By (2.2), $g(n)\to 0$, whence $(2\mu+1-m(\alpha+1/2))p+2>0$ i.e.

$$\left((\mu+1/2)-\frac{m}{2}(\alpha+1/2)\right)p>-1.$$

So v^p is integrable in $[0, 1]$. If $\alpha+1/2<0$, we use $v(x)\leqq u(x)$ $(x\geqq 0)$ and $u\in L^p$. The argument for $[-1, 0]$ is similar. □

3.5. Proof of Theorem 2.2. If Q is as that in 3.2, further $M=\max_{1\leqq t\leqq m-1}\|Q^{(t)}-f^{(t)}\|$ we get, using Lemma 3.2 as at 3.2—3.3

$$\text{(3.21)}\qquad |\mathscr{H}_{nm}(f, x)-f(x)| \leqq c\left[\varepsilon+\frac{\log n}{n}\frac{1}{(\sqrt{1-x_j^2})^{m(\alpha+1/2)}}\right],\quad x\in[a, 1].$$

The remaining part is as that for H_{nm}. □

3.6. Proof of Statement 2.3. If we suppose (3.7), we get (3.21), whence if $\alpha+1/2\leqq 0$, $|\mathscr{H}_{nm}(f, x)-f(x)|\leqq c(\varepsilon+\log n/n)$, $x\in[a, 1]$, from where we obtain (2.6). When $\alpha+1/2>0$, by (3.21) $|\mathscr{H}_{nm}(f, x)-f(x)|\leqq c(\varepsilon+n^{m(\alpha+1/2)-1}\log n)=c\varepsilon+o(1)$ using $\alpha<\beta_m$. □

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