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PARALLELIZABILITY IN BANACH SPACES: APPLICATIONS OF NEGLIGIBILITY THEORY

B. M. GARAY (Budapest)

1. Introduction

1. Definitions. Let (X, d) be a metric space and let π be a dynamical system on X. Our terminology and notation follows [3]: part will be recapitulated for the convenience of readers.

The relation $\sim$ of being on the same trajectory is an equivalence relation on X. The set of equivalence classes modulo $\sim$ is endowed with the quotient topology, denoted by X/C, and called the orbit space of X. Given a point $x\in X$, we define $J^+(x)$, $J^-(x)$ (first positive resp. negative prolongational limit set) as follows: $y\in J^+(x)$ if and only if $x_n\pi t_n\to y$ for some $x_n\to x$, $t_n\to\infty$; $y\in J^-(x)$ if and only if $x_n\pi t_n\to y$, for some $x_n\to x$, $t_n\to-\infty$.

A point $x\in X$ is called dispersive, if $J^+(x)\cup J^-(x)=\emptyset$. If this holds for all points $x\in X$, then the entire dynamical system is termed dispersive. π is called completely unstable, if $x\notin J^+(x)\cup J^-(x)$ for all $x\in X$.

π is called parallelizable if there is a global section S for π, i.e. a subset $S\subset X$ such that for every $x\in X$, there is a unique $\tau(x)\in R$ for which $x\pi\tau(x)\in S$ and the mapping $\tau\colon X\to R$, $x\to\tau(x)$ is continuous.

If π is parallelizable, then the mapping $X/C\to S$, $[x]\to x\pi\tau(x)$ is a homeomorphism. Here, of course, $[x]$ denotes the equivalence class under $\sim$ that contains x.

If π is parallelizable, then $h\colon R\times S\to X$ defined by $h(t, s)=s\pi t$ is a homeomorphism and for each $x\in X$, $h^{-1}(x)=(-\tau(x), x\pi\tau(x))$. On the other hand, if $S\subset X$ and $h\colon R\times S\to X$ is a homeomorphism with the property that $h(0, s)=s$ for all $s\in S$, then there exists a unique dynamical system, say π_h, on X for which S is a global section (thus, π_h is parallelizable) and $s\pi_h t=h(t, s)$ for all $t\in R$, $s\in S$. In fact, $x\pi_h t$ is to be defined as $h(t+\alpha(x), \beta(x))$ where $h^{-1}(x)=(\alpha(x), \beta(x))\in R\times S$ for each $x\in X$, $t\in R$.

Up to orbit-conjugacy, parallelizable dynamical systems are determined by their sections. In fact, if (X_1, d_1) and (X_2, d_2) are metric spaces, π_1 and π_2 are parallelizable dynamical systems on X_1 and X_2, and S_1 and S_2 are global sections for π_1 and π_2 respectively, then the following conditions are equivalent [26]:

(*) S_1 is homeomorphic to S_2

(**) there exists a homeomorphism of X_1 onto X_2 which carries the trajectories of π_1 onto the trajectories of π_2.

Consequently [26], the classification problem of parallelizable dynamical systems on a fixed space X can be reduced to a purely topological one: how can the space be represented in the form $X=W\times R$?

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2. A brief survey of parallelizability theory in metric spaces. The following theorem which is a slightly weakened version of [14, Theorem 7] (Hajek's original result is valid in a Tichonov space setting as well) is fundamental:

HAJEK'S THEOREM [14]. *Let (X, d) be a metric space and let π be a dynamical system on X. Then π is parallelizable if and only if it is completely unstable and the orbit space X/C is paracompact.*

The "only if" part is clear. In fact, if π is a parallelizable dynamical system, then π is obviously dispersive (and hence, it is completely unstable) and the orbit space X/C is homeomorphic to S (in particular, X/C is metrizable and hence, it is paracompact). Therefore, results guaranteeing that the orbit space of a dispersive dynamical system is metrizable are of extreme importance. Unfortunately, to our best knowledge, there are few results [14], [20] into this direction. The underlying topological problem, the metrizability problem of quotient spaces, is far from being simple; the difficulties are discussed at great length e.g. in [10]. On further topological and differentiability properties of the orbit space X/C, see the references cited in the beautiful paper of Goel and Neumann [13]. Cf. also [19].

Now we recall two sufficient conditions for the metrizability of the orbit space X/C. We need two definitions first.

DEFINITION A [14]. A dynamical system is called to have the *regularity property* if every invariant neighborhood of any point contains a closed invariant neighborhood of the point, i.e. if the orbit space X/C is a regular topological space.

THEOREM A [14]. *Let (X, d) be a separable metric space and let π be a dynamical system on X. Assume that π is dispersive and has the regularity property. Then X/C is metrizable.*

In what follows M is assumed to be a compact invariant subset of X.

DEFINITION B [20]. M is called to be *globally locally uniformly asymptotically stable* if it is stable and, for any $x \in X \setminus M$, there is a neighborhood U of x such that for any neighborhood V of M there exists a $T \in R$ such that $ut \in V$ for all $u \in U$, $t \geqq T$.

THEOREM B [20]. *Let (X, d) be a metric space and let π be a dynamical system on X. If a compact invariant subset M of X is globally locally uniformly asymptotically stable, then $(X \setminus M)/C$ is metrizable.*

Since global asymptotic stability of M implies that π restricted to $X \setminus M$ is dispersive, a direct application [20] of Hajek's theorem yields

MCCANN'S THEOREM [20]. *Let (X, d) be a metric space and let π be a dynamical system on X. If a compact invariant subset M of X is globally locally uniformly asymptotically stable, then π restricted to $X \setminus M$ is parallelizable.*

In case of M being globally uniformly asymptotically stable, the global section S for π restricted to $X \setminus M$ can be constructed as a level surface of a Liapunov function suitably chosen [3, p. 210]. (Throughout this paper, a continuous function $V: X \to R^+$ is called to be a Liapunov function if $V^{-1}(0) = M$ and V is strictly

decreasing along the trajectories in $X \setminus M$.) This remains true [3, p. 211] when global uniform asymptotic stability is weakened to global asymptotic stability provided M is an isolated invariant set, i.e. if M has a neighborhood N such that M is the largest (under inclusion) invariant set in N.

3. A brief survey of parallelizability theory in finite-dimensional normed spaces. The problem of parallelizability in finite-dimensional normed spaces is solved by the famous Nemyckii—Stepanov theorem [3, p. 190] stating that a dynamical system on a locally compact separable metric space is parallelizable if and only if it is dispersive. Since the regularity property is implied [14] by dispersiveness provided that (X, d) is locally compact, Theorem A shows that Hajek's theorem is a generalization of the Nemyckii—Stepanov theorem. Similarily, in case of (X, d) being a locally compact separable metric space, McCann's theorem is also a restatement of classical results.

As it was indicated in Section 1.1, the classification problem of parallelizable dynamical systems can be reduced to a purely topological one. The results of [30], [4], [16], [17], [6] concerning the representation of R^n in form of $R^n = W \times R$ imply the following corollary which summarizes several remarks and observations from [18], [26], [5], [21], [13], [11].

COROLLARY. a) *Assume that π_1 and π_2 are parallelizable dynamical systems on R^n, $n=1, 2, 3$. Then π_1 and π_2 are orbit-conjugate. On the other hand, for $n \geqq 4$, there exist uncountably many parallelizable dynamical systems on R^n that are not mutually orbit-conjugate.*

b) *Let π_1 and π_2 be dynamical systems on R^n, $n=1, 2, 3$. Assume that $P_1 \in R^n$ and $P_2 \in R^n$ are globally asymptotically stable equilibrium points of π_1 and π_2, respectively. Then π_1 and π_2 are orbit-conjugate. On the other hand, for $n \geqq 4$, there exist dynamical systems π and ϱ on R^n that are not orbit-conjugate to each other but both π and ϱ have the property that the origin is a globally asymptotically stable equilibrium point.*

We have stated this corollary here, because, in most cases, the authors of [18], [26], [5], [21], [13], [11] seem to be unaware of each other's work as well as of some of the underlying topological results. (For differentiability aspects of this corollary see [31], [18], [21], [13].) To our best knowledge, the infinite-dimensional Banach space aspects of this corollary are not cleared. (Cf. the Problem in Section 3.1.)

It is worth to mention that Z, the linear span of the usual orthonormal base of the separable Hilbert space l_2, can be represented in the form $Z = W \times R$ where W is not homeomorphic to Z [15]. Observe that Z can be expressed as $Z = Z \times R$. Therefore, two parallelizable dynamical systems on Z need not be orbit-conjugate.

4. The results of the present paper. The content of this paper manifests itself in the headings.

The results of this paper are applications of negligibility theory. Negligibility theory is a part of infinite-dimensional topology: a brief outline is given in Section 2.2. Applying Cutler's theorem, a fundamental homotopy result from negligibility theory, we modify linear flows (translation flow, radial flow, standard hyperbolic flow) in infinite-dimensional Banach spaces. The resulting new dynamical systems have

some unusual properties. It is pointed out that none of these properties can be shared by dynamical systems in locally compact metric spaces.

The main result of Section 2 is Bebutov's example (version D) which we state here as

THEOREM 1. *Let* $(E, \|\cdot\|)$ *be an infinite-dimensional Banach space. Then there exists a dynamical system* ϱ *on* E *with the following properties:*

(a) 0_E, *the origin of* E, *is a globally asymptotically stable equilibrium point of* ϱ;

(b) *the function* $V: E \to R^+$ *defined by* $V(e)=\|e\|$ *is a Liapunov function on* E;

(c) ϱ *restricted to* $E \setminus \{0_E\}$ *is dispersive but not parallelizable.*

The Bebutov's examples are followed by a more or less elementary study on McCann's theorem.

Section 3 is devoted to topologically hyperbolic equilibria. Loosely speaking, it turns out that all aspects of the infinite-dimensional counterpart of Coleman's conjecture are false. The main result is the example constructed in Section 2.4 which we state here as

THEOREM 2. *Let* E, F *be Banach spaces. Assume that* E *is infinite-dimensional. Then there exists a dynamical system* η *on* $E \times F$ *satisfying the following conditions:*

(d) *for all positive* c, $D_{E\times F}(c)$ *is a normal block of type* (E, F);

(e) *there exists a sequence* $\{e_n\}_{n=1}^{\infty} \subset E$ *converging to* 0_E *such that* $\gamma((e_n, 0_F))$, $n=1, 2, \ldots$, *the trajectory of* η *through* $(e_n, 0_F) \in E \times F$ *is not attracted by* $(0_E, 0_F)$.

Finally, in Section 3.5, the link between parallelizability and asymptotic stability is investigated. The main result is Example 3.4 which is restated here as

THEOREM 3. *Let* E *be an infinite-dimensional Banach space. Then there exists a dynamical system* π *on* E *with the following properties:*

(f) π *restricted to* $E \setminus \{0_E\}$ *is parallelizable;*

(g) *for all* $e \neq 0_E$, *there holds* $J^+(e) \cup J^-(e) = \emptyset$.

2. Bebutov's example in infinite-dimensional Banach spaces

1. The question whether dispersiveness implies parallelizability has been solved in the negative by Bebutov. In what follows we describe two versions of his counterexample. (A detailed bibliography of Bebutov's papers on dynamical systems is given and discussed in [3].)

BEBUTOV'S EXAMPLE (Version A) [3, p. 185]. Let π_1 be the planar dynamical system determined by the system of differentiable equations in cartesian coordinates $\dot{x}=f(x, y)$, $\dot{y}=0$, where f is a continuously differentiable function such that $f(n, n^{-1})=0$ for every positive integer n and $f(x, y)>0$ otherwise. Let X denote the plane with the half lines $\{(x, y) | y=n^{-1}, x \leqq n\}$ deleted and let π denote π_1 restricted to X. It is clear that π is dispersive but does not have the regularity property. In fact, $U=\{(x, y) | y \neq n^{-1}\}$ is an invariant neighborhood of $(0, 0)$ in X that contains no closed invariant neighborhoods. Hence, π is dispersive but not parallel-

izable. The Nemyckii—Stepanov theorem [3, p. 190] traces this to absence of local compactness of X: the point $(0, 0)$ has no compact neighborhoods in X.

BEBUTOV'S EXAMPLE (Version B) [20]. Let π_2 be the planar dynamical system determined by the system of differential equations in polar coordinates $\dot{r}=g(r, \varphi)$ $\dot{\varphi}=0$, where g is a continuously differentiable function such that $g(n^{-1}, n^{-1})=0$ for every positive integer n, $g(0, 0)=0$ and $g(r, \varphi)<0$ otherwise. Let X denote the plane with the half lines $\{(r, \varphi)|\varphi=n^{-1}, r\geqq n^{-1}\}$ deleted and let π denote π_2 restricted to X. Then π restricted to $X\setminus\{(0, 0)\}$ is dispersive but not parallelizable. Note that the trajectories of π are rays or portions of rays emanating from the origin. The motion along these trajectories is toward the origin. Hence, the origin is globally asymptotically stable and $V(r, \varphi)=r$ defines a Liapunov function on X for the origin.

2. The construction of Bebutov's example is based on deleting some points from the plane. The resulting phase space is a proper subset of the plane. In infinite-dimensional Banach spaces, the procedure of deleting some points from the plane can be replaced by the procedure of pushing some points off the Banach space so that the resulting phase space be the original Banach space itself.

We need some definitions first. Let X be a topological space and K be a subset of X. An invertible isotopy pushing K off X is a homeomorphism $\mathscr{F}: (X\times I)\setminus(K\times\{1\})\to X\times I$ which preserves the second coordinate (i.e. if $(y, t)=\mathscr{F}(x, s)$, then $s=t$) and which is the identity on $X\times\{0\}$. Such an isotopy is said to be limited by $\mathscr{G}$, a collection of open subsets of X, if for each $x\in X$, either $\mathscr{F}(x, s)=(x, s)$ for all $s\in I$, or $\{\mathscr{F}(x, s), \mathscr{F}^{-1}(x, s)\}_{s\in I}$ is contained in $G\times I$ for some $G\in\mathscr{G}$. K is extractible from X if for every collection $\mathscr{G}$ of open subsets of X containing K in their union, there is an invertible isotopy pushing K off X which is limited by $\mathscr{G}$. Now we are in a position to recall one of the fundamental results of negligibility theory (a part of infinite-dimensional topology established among others by V. L. Klee, R. D. Anderson, R. H. Bing and T. A. Chapman: for references, see [1], [29]):

CUTLER'S THEOREM [8]. *Let X be a Fréchet manifold (i.e. a manifold modelled on an infinite-dimensional Fréchet space) and let $K\subset X$ be a countable union of locally compact sets. Then K is extractible from X. In other words, K is negligible in X (i.e. $X\setminus K$ is homeomorphic to X) and the homeomorphism of $X\setminus K$ onto X may be chosen so as to be isotopic to the identity by an isotopy which moves points as little as desired.*

3. The basic construction. Let $(E, \|\cdot\|)$ be an infinite-dimensional Banach space. Choose an $e_0\in E$ so that $\|e_0\|=1$. Consider a continuous linear functional $L: E\to R$ for which $L(e_0)=1$. Set $X=\{e\in E|L(e)=0\}$. Since $e-L(e)e_0\in X$ and

$$e=(e-L(e)e_0)+L(e)e_0$$

for all $e\in E$, E can be represented as $E=X\times R$ and $e\in E$ can be written as $e=(x, \lambda)\in X\times R$ where $x=e-L(e)e_0$ and $\lambda=L(e)$. Note that $X\subset E$ is an infinite-dimensional Banach space. Its origin will be denoted by 0_x. The origin of E will be denoted by $(0_x, 0)$.

For $A>0$, we define a dynamical system π_A on E satisfying the following conditions:

(1) if $\|x\| \geqq A$, then $(x, 0)\pi_A t=(x, t)$ for all $t\in R$;

(2) $X\times\{0\}\subset E$ is a global section for π_A and for each $x\in X$, the function $L_x: R\to R$ defined by $L_x(t)=L((x, 0)\pi_A t)$ is strictly increasing.

(3) $$L_{0_x}(t)\to-\infty \quad \text{as} \quad t\to-\infty \quad \text{and} \quad L_{0_x}(t)\to 1 \quad \text{as} \quad t\to\infty;$$

and

(4) for each $x\in X\setminus\{0_x\}$, we have

$$L_x(t)\to-\infty \quad \text{as} \quad t\to-\infty \quad \text{and} \quad L_x(t)\to\infty \quad \text{as} \quad t\to\infty.$$

For $a>0$, set $M_A(a)=\max\{A/a, 1\}$ and let

$$Q_A = \{(0, t)\in R\times R | t \geqq 0\}\cup\{(a, t)\in R\times R | a>0, \quad 0\leqq t\leqq M_A(a)\}.$$

Finally, let $q_A: Q_A\to R$ be a continuous function with the following properties (q_A can easily be constructed explicitly): the function $q_A^0: [0, \infty)\to R$ defined by $q_A^0(t)=q_A(0, t)$ is strictly increasing, $q_A^0(0)=0$ and $q_A^0(t)\to 1$ as $t\to\infty$; for each $a\in(0, A)$, the function $q_A: [0, A/a]\to R$ defined by $q_A^a(t)=q_A(a, t)$ is strictly increasing, $q_A^a(0)=0$, $q_A^a(A/a)=1$; and for each $a\in[A, \infty)$, $t\in[0, 1]$, there holds $q_A(a, t)=t$.

By elementary real analysis, the map

$$Q_A \to ([0, \infty)\times I)\setminus\{(0, 1)\}, \quad (a, t)\to (a, q_A(a, t))$$

is a homeomorphism. Consequently, the map

$$\{(t, (x, 0))\in R\times(X\times\{0\}) | (\|x\|, t)\in Q_A\} \to (X\times I)\setminus(\{0_x\}\times\{1\})$$
$$(t, (x, 0)) \to (x, q_A(\|x\|, t))$$

is a homeomorphism as well.

π_A will be defined as π_{h_A} for a suitable homeomorphism $h_A: R\times(X\times\{0\})\to E$. In defining h_A, we make use of Cutler's theorem in the special case when X is an infinite-dimensional Banach space, $K=\{0_x\}$,

$$G = \{x\in X | \|x\| < A\} \quad \text{and} \quad \mathcal{G} = \{G\}.$$

For $(t, (x, 0))\in R\times(X\times\{0\})$, let

$$h_A(t, (x, 0)) = \begin{cases} (x, t) & \text{if} \quad x\in X, \quad t\leqq 0 \\ \mathcal{F}(x, q_A(\|x\|, t)) & \text{if} \quad (\|x\|, t)\in Q_A \\ \mathcal{F}(x, 1)+(0_x, t-M_A(\|x\|)) & \text{if} \quad x\neq 0_x, \quad t\geqq M_A(\|x\|). \end{cases}$$

First observe that $\mathcal{F}(x, q_A(\|x\|, 0))=\mathcal{F}(x, 0)=(x, 0)$ for all $x\in X$ and $\mathcal{F}(x, q_A(\|x\|, M_A(\|x\|)))=\mathcal{F}(x, 1)$ for all $x\in X\setminus\{0_x\}$. Thus, h_A is well-defined. It

can be readily verified that h_A maps

$$\{(t,(x,0))\in R\times(X\times\{0\})\,|\,x\in X,\quad t\leqq 0\},$$

$$\{(t,(x,0))\in R\times(X\times\{0\})\,|\,(\|x\|,t)\quad\in Q_A\},$$

$$\{(t,(x,0))\in R\times(X\times\{0\})\,|\,x\neq 0_x,\quad t\geqq M_A(\|x\|))\}$$

homeomorphically onto $X\times(-\infty,0]$, $X\times I$, $X\times[1,\infty)$, respectively. It follows that h_A is a homeomorphism.

Set $\pi_A=\pi_{h_A}\cdot\pi_A$. is a dynamical system on E. (1), (2), (3) and (4) follow immediately from the properties of h_A. In fact, since $\mathscr{F}$ is limited by $G=\{x\in X\,|\,\|x\|<A\}$ and $M_A(\|x\|)=1$, $q_A(\|x\|,t)=t$ whenever $\|x\|\geqq A$, $t\in[0,1]$; we have $\mathscr{F}(x,q_A(\|x\|,t))=(x,t)$ for all $x\in X\setminus G$, $t\in R$. Thus, if $\|x\|\geqq A$, then $(x,\lambda)\pi_A t=(x,\lambda+t)$ for all $\lambda\in R$, $t\in R$. By the definition of π_h for $h=h_A$, π_A is parallelizable and $X\times\{0\}\subset E$ is a global section for π_A. The rest follows from the simple observation that

$$L_x(t)=\begin{cases} t & \text{if}\quad x\in X, \quad t\leqq 0\\ q_A(0_x,t) & \text{if}\quad x=0_x,\quad t\geqq 0\\ q_A(\|x\|,t) & \text{if}\quad x\neq 0_x,\quad 0\leqq t\leqq M_A(\|x\|)\\ 1+t-M_A(\|x\|) & \text{if}\quad x\neq 0_x,\quad t\geqq M_A(\|x\|)\end{cases}$$

and from the properties of q_A. Note that

$(5)_1$ for each $x\in X$, if $t\leqq 0$, then $L_x(t)=t$,

and

$(5)_2$ for each $x\in X$, if $L_x(t_1)\geqq 1$ and $L_x(t_2)\geqq 1$, then $L_x(t_1)-L_x(t_2)=t_1-t_2$.

In the special case when X is a separable Hilbert space,

$$K=\{0_x\},\quad G=\{x\in X\,|\,\|x\|<A\}\quad\text{and}\quad\mathscr{G}=\{G\}$$

the invertible isotopy pushing K off X which is limited by $\mathscr{G}$ can be constructed explicitly [1]. Moreover, a careful step by step reconsideration of the construction in [1] as well as that of our method for defining π_A shows that, in this particular case, both

$$\mathscr{F}\colon (X\times I)\setminus(K\times\{1\})\to X\times I\quad\text{(cf. [29])}$$

and $h_A\colon R\times(X\times\{0\})\to E$ can be chosen for C^∞-diffeomorphism. This implies that in case of E being a separable Hilbert space, the dynamical system π_A can be constructed within the framework of the theory of autonomous ordinary differential equations, i.e. if π_A is suitably chosen, then it can be induced by a C^∞ vector field. This remark holds for the dynamical systems we derive from π_A as well. (Differentiability at $(0_x,0)$ follows by changing the time parametrization of the trajectories.)

4. Set $Z=S(0_x,1)\times R\subset E$ where $S(0_x,1)=\{x\in X\,|\,\|x\|\leqq 1\}$ and set

$$W=\{e\in E\,|\,e=\mu(y,1)\text{ for some }y\in S(0_x,1),\ \mu>0\}.$$

Define a map $H: Z \to W$ by

$$H((x, \lambda)) = 2^{\lambda}(x, 1)/\|(x, 1)\| \in W$$

for all $(x, \lambda) \in Z$. Since H is a continuous bijection and

$$H^{-1}(\mu(y, 1)) = (y, (\ln\|\mu(y, 1)\|/\ln 2) \in Z$$

for all $\mu(y, 1) \in W$, H is a homeomorphism. For each $\lambda \in R$, H maps $S(0_x, 1) \times \times \{\lambda\} \subset Z$ homeomorphically onto $\{e \in W \mid \|e\| = 2^{\lambda}\}$. For each $x \in S(0_x, 1)$, H maps $\{x\} \times R \subset Z$ homeomorphically onto $\{\mu(x, 1) \in W \mid \mu > 0\}$, in particular, $\{0_x\} \times R$ is mapped onto $\{\mu e_0 \mid \mu > 0\}$ homeomorphically and $H((0_x, 0)) = (0_x, 1) = e_0$.

For $x \in X$, $A > 0$, set

$$S(x, A) = \{y \in X \mid \|y - x\| \leqq A\}.$$

Let the sequences $\{x_n\}_{n \in N} \subset S(0_x, 1/2)$, $\{A_n\}_{n \in N} \subset (0, 1/2)$ be chosen so that $x_n \to 0_x$ as $n \to \infty$ and $\|x_n\| > A_n$, $n \in N$ and

$$S(x_n, A_n) \cap S(x_m, A_m) = \emptyset, \; n, m \in N, \quad n \neq m.$$

Observe that

$$\cup \{S(x_n, A_n) \mid n \in N\} \subset S(0_x, 1) \setminus \{0_x\}.$$

In what follows we define four dynamical systems, π, ϱ, σ, τ, on E.

BEBUTOV'S EXAMPLE (Version C). In virtue of (1) and (5) by letting

$$(x, \lambda)\pi t = \begin{cases} ((x, \lambda) + (-x_n, n))\pi_{A_n} t - (-x_n, n) & \text{if } x \in S(x_n, A_n) \text{ for some } n \in N \\ (x, \lambda + t) & \text{otherwise} \end{cases}$$

for $(x, \lambda) \in X \times R$, $t \in R$, we obtain a dynamical system π on E. It follows easily from (2) that π is dispersive. By (3) and (4), every closed invariant neighborhood of $(0_x, 0) \in X \times R$ in E contains $\gamma((x_n, -n))$, the trajectory of π through $(x_n, -n) \in X \times R$, for sufficiently large n, but $E \setminus \cup \{\gamma((x_n, -n)) \mid n \in N\}$ is an invariant neighborhood of $(0_x, 0) \in X \times R$ in E. Thus, π does not have the regularity property and consequently, π is not parallelizable.

BEBUTOV'S EXAMPLE (Version D). Since π is a dynamical system on E and for $x \in X \setminus S(0_x, 1)$, $\lambda \in R$, $t \in R$ there holds $(x, \lambda)\pi t = (x, \lambda + t)$; by letting

$$e\varrho t = \begin{cases} H(H^{-1}(e)\pi(-t)) & \text{if } e \in W \\ 2^{-t} e & \text{if } e \in E \setminus (W \cup \{(0_x, 0)\}) \end{cases}$$

for all $e \in E \setminus \{(0_x, 0)\}$, $t \in R$, we obtain a dynamical system ϱ on $E \setminus \{(0_x, 0)\}$. In virtue of (5), ϱ can be extended continuously onto E by setting $(0_x, 0)\varrho t = (0_x, 0)$ for all $t \in R$. It follows immediately that $(0_x, 0)$ is a globally asymptotically stable equilibrium point and the function $V: E \to R^+$ defined by $V(e) = \|e\|$ is a Liapunov function for ϱ on E. Since $\gamma(H((x_n, -n)))$, the trajectory of ϱ through $H((x_n, -n))$, is contained in $\{e \in E \mid \|e\| < 2^{-n+1}\}$, we conclude that $\{(0_x, 0)\}$ is not an isolated invariant set. Dispersiveness and nonparallelizability of ϱ restricted to $E \setminus \{(0_x, 0)\}$ follow from those of π. By McCann's theorem, nonparallelizability of ϱ restricted

to $E \setminus \{(0_x, 0)\}$ is traced to the absence of globally local uniform asymptotic stability: ϱ is not locally uniformly asymptotically stable at e whenever $e \in \{\mu e_0 | \mu > 0\}$. On the other hand, the conditions of McCann's theorem are fulfilled if ϱ is restricted to $E \setminus \{\mu e_0 | \mu \geqq 0\}$.

For $(x, \lambda) \in X \times R$, $t \in R$, set

$$(x, \lambda)\sigma t = \begin{cases} ((x, \lambda) + (-x_n, -n))\pi_{A_n} t - (-x_n, -n) & \text{if} \quad x \in S(x_n, A_n) \\ (x, \lambda + t) & \text{otherwise} \end{cases}$$

for some $n \in N$.

It follows easily from (1), (2), and (5) that σ is a dynamical system on E and $X \times \{0\}$ is a global section for σ.

For $e \in E$, $t \in R$, set

$$e\tau t = \begin{cases} H(-(-H^{-1}(e)\sigma t)) & \text{if} \quad e \in W \\ 2^{-t} e & \text{if} \quad e \in E \setminus W. \end{cases}$$

By the same arguments used in constructing Bebutov's example (Version D), we obtain that τ is a dynamical system on E, $\{(0_x, 0)\}$ is an isolated invariant set as well as a stable equilibrium point of τ, the function $V: E \to R^+$ defined by $V(e) = \|e\|$ is a Liapunov function for τ on E and that τ restricted to $E \setminus \{(0_x, 0)\}$ is parallelizable. On the other hand, $(0_x, 0)$ is not asymptotically stable. In fact, $\gamma(H((-x_n, -n)))$, the trajectory of τ through $H((-x_n, -n))$, is contained in

$$\{e \in E | \|e\| > 2^{-n-1}\}.$$

5. The braked radial flow. Next we point out by an example that, in general, global asymptotic stability of the origin in E does not imply globally local uniform asymptotic stability even if the dynamical system restricted to $E \setminus \{(0_x, 0)\}$ is parallelizable. The example we give below is a simplification of Bebutov's example (Version D).

Select a sequence $\{y_n\}_{n \in N} \subset S(0_x, 1/2)$ that contains no convergent subsequences. This can be done since $S(0_x, 1/2)$ is not totally bounded. For each $x \in \mathrm{cl}\,(X \setminus S(0_x, 1))$, set $r(x) = 1$ and set $r(y_n) = n + 1$, $n \in N$. An obvious application of Tietze's extension theorem yields the existence of a continuous function $r: X \to [1, \infty)$ for which $\sup\{r(x) | x \in X\} = \infty$ but $r(x) = 1$ whenever $\|x\| \geqq 1$. Let the sequences

$$\{x_n\}_{n \in N} \subset S(0_x, 1/2), \quad \{B_n\} \subset (0, 1/2)$$

be chosen so that $x_n \to 0_x$ as $n \to \infty$ and that $\|x_n\| > B_n$, $n \in N$ and $S(x_n, B_n) \cap \cap S(x_m, B_m) = \emptyset$, $n, m \in N$, $n \neq m$. If $x \in S(x_n, B_n)$ for some $n \in N$ and $t \in R$, then let

$$h_B(t, (x, 0)) = \begin{cases} (x, t) & \text{if} \quad -n \leqq t \\ (x, -n + (t + n)/r_n(x)) & \text{if} \quad -n - r_n(x) \leqq t \leqq -n \\ (x, t + r_n(x) - 1) & \text{if} \quad t \leqq -n - r_n(x) \end{cases}$$

where $r_n(x) = r((x - x_n)/B_n)$, $n \in N$. If $x \in X \setminus \cup \{S(x_n, B_n) | n \in N\}$ and $t \in R$, then let $h_B(t, (x, 0)) = (x, t)$.

It can be readily verified that h_B: $R\times(X\times\{0\})\to E$ is a homeomorphism and that the dynamical system defined by

$$e\tilde{\omega}t=\begin{cases}H(H^{-1}(e)\pi_{h_B}(-t)) & \text{if } e\in W,\ t\in R\\ 2^{-t}e & \text{if } e\in E\setminus W,\ t\in R\end{cases}$$

has all the desired properties. Here, of course, H: $Z\to W$ is the homeomorphism defined in constructing Bebutov's example (Version D). The origin is globally asymptotically stable and $V(e)=\|e\|$ defines a Liapunov function on E whose level surfaces are global sections for $\tilde{\omega}$ restricted to $E\setminus\{(0_x, 0)\}$. On the other hand, $\tilde{\omega}$ is not locally uniformly asymptotically stable at e whenever $e\in\{\mu e_0|\mu>0\}$. The details are left to the reader. A similar example (an accelerated radial flow) has been given in [12].

Geometrically, $\tilde{\omega}$ is a braked radial flow. The trajectories of $\tilde{\omega}$ are the same as those of ω, the radial flow, which is defined by $e\omega t=2^{-t}e$, $e\in E$, $t\in R$.

6. A simple remark. Select a sequence

$$\{e_n\}_{n\in N}\subset\{e\in E\,|\,\|e\|=1\}$$

that contains no convergent subsequences. Set

$$Y=\cup\big\{\{e\in E|e=\lambda e_n \text{ for some } \lambda\in[0, 1/(n+1))\}|n\in N\big\}$$

and consider a dynamical system, say $\varkappa$, on Y for which the origin is globally asymptotically stable. It is clear that $\varkappa$ restricted to Y save the origin is parallelizable but no global section can be given as a level surface of a Liapunov function.

3. Topologically hyperbolic equilibria in infinite-dimensional Banach spaces

1. Topologically hyperbolic equilibria of type $(E, \emptyset)$. Let $(E, \|\cdot\|)$ be a Banach space and let π be a dynamical system on E. We say that the origin is a topologically hyperbolic equilibrium of type $(E, \emptyset)$, if $\{e\in E|\|e\|=1\}$ consists of strict ingress points and the origin is the largest invariant set in $\{e\in E|\|e\|\leqq 1\}$.

It is obvious that topologically hyperbolic equilibria of type $(R^m, \emptyset)$ are asymptotically stable.

PROPOSITION 3.1. *Let π be a dynamical system on E. Assume that the origin is a globally asymptotically stable topologically hyperbolic equilibrium of type $(E, \emptyset)$. Then there is a homeomorphism J: $E\to E$ which carries the trajectories of π onto the trajectories of the radial flow ω defined in Section 2.5. In other words, π and ω are orbit-conjugate.*

PROOF. The origin of E is denoted by 0_E. The conditions imposed on π imply that for each $e\in E\setminus\{0_E\}$, there is a unique $\tau(e)\in R$ such that $\|e\pi\tau(e)\|=1$. It is easy to show that the function τ: $E\setminus\{0_E\}\to R$ is continuous. Thus, π restricted to $E\setminus\{0_E\}$ is parallelizable and $\{e\in E|\|e\|=1\}$ is a global section for π in $E\setminus\{0_E\}$.

Applying the well-known Yoshizawa—Zubov method [3, p. 144] we define a Liapunov function V: $E\to R^+$ such that $V(e\pi t)\to 0$ resp. $V(e\pi t)\to\infty$ as $t\to\infty$

resp. $t \to -\infty$ for each $e \in E \setminus \{0_E\}$, $V(e)=1$ whenever $\|e\|=1$ and that $V(e) \geqq \geqq \|e\|/3$ for each $e \in E$. In fact, for each $e \in E$, let

$$V_2(e) = \sup\{(1+\arctan(t))\| e\pi t\| \mid t \geqq 0\},$$

$$V_1(e) = \begin{cases} V_2(e)/V_2(e\pi\tau(e)) & \text{if} \quad e \neq 0_E \\ 0 & \text{if} \quad e = 0_E, \end{cases}$$

$$V(e) = \begin{cases} V_1(e) & \text{if} \quad \|e\| \leqq 1 \\ 1+\tau(e) & \text{if} \quad \|e\| \geqq 1. \end{cases}$$

The desired homeomorphism $J: E \to E$ can be defined by

$$J(e) = \begin{cases} V(e) \cdot (e\pi\tau(e)) & \text{if} \quad e \neq 0_E \\ 0_E & \text{if} \quad e = 0_E. \end{cases}$$

The (lengthy but straightforward) details are left to the reader. It is worth mentioning that the inverse mapping of the "obvious" candidate $\tilde{J}$ for J,

$$\tilde{J}(e) = \exp(\tau(e)) \cdot (e\pi\tau(e)) \quad \text{if} \quad e \neq 0_E,$$

$\tilde{J}(0_E)=0_E$ is not necessarily (e.g. when $\pi=\tilde{\omega}$, the braked radial flow defined in Section 2.5) continuous at the origin. Q.E.D.

In what follows we point out by examples that none of the conditions of Proposition 3.1 can be dropped.

EXAMPLE 3.1. Assume that E is infinite-dimensional. Since $\{e \in E \mid \|e\|=1\}$ is not compact, it is easy to construct a dynamical system on E with the following properties: the origin is a topologically hyperbolic equilibrium of type $(E, \emptyset)$; the origin is globally attractive but the origin is not stable.

The details are left to the reader. It is obvious that the conclusion of Proposition 3.1 fails to hold.

EXAMPLE 3.2. Assume that E is infinite-dimensional. Consider dynamical system τ constructed in Section 2.4. The origin is a topologically hyperbolic equilibrium of type (E, Φ). More precisely, for all $c>0$, $\{e \in E \mid \|e\|=c\}$ consists of strict ingress points and the origin is the largest invariant set in $\{e \in E \mid \|e\| \leqq c\}$. On the other hand, for arbitrary $c>0$, there does not exist any homeomorphism of $\{e \in E \mid \|e\| \leqq c\}$ onto itself which carries the trajectory segments of τ onto the trajectory segments of ω. Essentially, this is a consequence of the failure of attraction and can be seen as follows. With the notation used in Section 2.4, $\gamma(H((-x_n, -n)))$, the trajectory of τ through $H((-x_n, -n)) \in \{e \in E \mid \|e\|=2^{-n}\}$ is contained in $\{e \in E \mid \|e\| > 2^{-n-1}\}$, $n \in N$, contrary to the fact that the origin is contained in the closure of any trajectory segment of ω in $\{e \in E \mid \|e\| \leqq c\}$, $c>0$.

EXAMPLE 3.3. Assume that E is infinite-dimensional. Consider the dynamical system ϱ constructed in Section 2.4 as Bebutov's example (Version D). The origin is a globally asymptotically stable equilibrium point. For all $c>0$, $\{e \in E \mid \|e\|=c\}$ consists of strict ingress points. On the other hand, with the notation used in Section 2.4, $\gamma(H((x_n, -n)))$, the trajectory of ϱ through $H((x_n, -n))$, is contained

in $\{e\in E\|\|e\|<2^{-n+1}\}$, $n\in N$. Thus, the conditions of Proposition 3.1 are satisfied except the one requiring the isolatedness of the origin as an invariant set. Since $\{0_E\}$ is an isolated invariant set of the radial flow ω, the conclusion of Proposition 3.1 fails to hold.

PROBLEM. Suppose we are given an infinite-dimensional Banach space E and a dynamical system π on E. Assume that the origin is a globally asymptotically stable equilibrium point as well as an isolated invariant subset of E. Does there exist a homeomorphism $J: E\to E$ which carries the trajectories of π onto the trajectories of the radial flow ω? In other words: is it true that π is orbit-conjugate to ω? We conjecture that the answer is, in general, negative. In a weaker sense, the answer is positive: the proof of Theorem 2.1 of [11] shows that there exists a Banach space F and a homeomorphism $H: E\to H(E)\subset F$ such that $H(E)$ is invariant with respect to the radial flow of F and H carries the trajectories of π onto the trajectories of the radial flow of F restricted to $H(E)$.

The answer to the similar problem concerning conjugacy is negative. In fact, consider the braked radial flow $\tilde{\omega}$ constructed in Section 2.5. Suppose that there exists a homeomorphism $J: E\to E$ such that $J(e\omega t)=J(e)\tilde{\omega}t$ for all $e\in E$, $t\in R$. It is evident that for the radial flow ω, the origin is a globally locally uniformly asymptotically stable equilibrium point. On the other hand, for the braked radial flow $\tilde{\omega}$, recall that the origin is not globally locally uniformly asymptotically stable. The contradiction follows immediately.

2. Normal blocks and topologically hyperbolic equilibria of type (E, F). Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces. In order to avoid unnecessary overlaps with Section 3.1, it is assumed that $\dim(E)\geqq 1$, $\dim(F)\geqq 1$. For $(e,f)\in E\times F$, define $\|(e,f)\|_{E\times F}=\max\{\|e\|_E, \|f\|_F\}$. Then $(E\times F, \|(\cdot,\cdot)\|_{E\times F})$ is a Banach space. The origin of E, F, $E\times F$ is denoted by 0_E, 0_F, $(0_E, 0_F)$, respectively. For $c>0$, let

$$D_{E\times F}(c) = \{(e, f)\in E\times F \mid \|(e, f)\|_{E\times F} \leqq c\}.$$

$D_E(c)$ and $D_F(c)$ are defined analogously.

Let π be a dynamical system on $E\times F$. We say that $D_{E\times F}(c)$ is a normal block of type (E, F) if the positively invariant set in $D_{E\times F}(c)$ is $D_E(c)\times\{0_F\}$, the negatively invariant set in $D_{E\times F}(c)$ is $\{0_E\}\times D_F(c)$ and in addition, $\partial D_E(c)\times(D_F(c)\setminus\partial D_F(c))$ consists of strict ingress points, $(D_E(c)\setminus\partial D_E(c))\times\partial D_F(c)$ consists of strict egress points and $\partial D_E(c)\times\partial D_F(c)$ is the tangency set. (Observe that the definition implicitly includes that the origin is an equilibrium point and it is the largest invariant set in $D_{E\times F}(c)$.) Finally, we say that $(0_E, 0_F)$ is a topologically hyperbolic equilibrium point of type (E, F) if $D_{E\times F}(1)$ is a normal block of type (E, F).

If $E=R^m$, $F=R^n$, then, essentially, the definition of topologically hyperbolic equilibria of type (E, F) turns into one of Wilson's topologically hyperbolic equilibria of type (m, n). By Wilson's definition [32], a dynamical system in $R^m\times R^n$ is topologically hyperbolic of type (m, n) at an equilibrium point p if there is a C^1 system of coordinates near p with $p\sim(0_E, 0_F)$ and so that in these coordinates $D_{E\times F}(1)$ is a normal block of type (E, F), where $E=R^m$, $F=R^n$. Therefore, if $D_{E\times F}(c)$ is a normal block of type (E, F) for some $c>0$ and $E=R^m$, $F=R^n$, then the origin is a topologically hyperbolic equilibrium of type (m, n). These remarks hold for topologically hyperbolic equilibria of $(R^m, \emptyset)$ resp. $(m, 0)$ as well.

THE STANDARD EXAMPLE. Consider the linear hyperbolic vector field $\dot{e}=-e$, $\dot{f}=f$ on $E\times F$. The induced dynamical system ξ is usually referred to as the standard example. Explicitly, for $(e,f)\in E\times F$, $t\in R$,

$$(e,\ f)\,\xi t = \big((\exp(-t))\,e,\ (\exp(t))f\big).$$

It is clear that $D_{E\times F}(c)$ is a normal block of type (E, F) for $c>0$, arbitrary.

3. The Poincaré map of normal blocks. Let π be a dynamical system on $E\times F$. If $D_{E\times F}(c)$ is a normal block of type (E, F) then the trajectory segments of π induce a Poincaré map

$$P(c,\pi):\ \partial D_E(c)\times\big(D_F(c)\setminus\partial D_F(c)\big)\to\big(D_E(c)\setminus\partial D_E(c)\big)\times\partial D_F(c)$$

that maps each entry point in $\partial D_E(c)\times\big(D_F(c)\setminus\partial D_F(c)\big)$ to the exit point of the same trajectory.

Let $\tilde{\pi}$ be a dynamical system on $E\times F$. For both π and $\tilde{\pi}$, assume that the origin is a topologically hyperbolic equilibrium of type (E, F) and $P(1,\pi)=$ $=P(1,\tilde{\pi})$. In case of $E=R^m$, $F=R^n$, there exists [33] a homeomorphism of $D_{E\times F}(1)$ onto itself which carries the trajectory segments of π onto the trajectory segments of $\tilde{\pi}$ (and preserves the orientation of the trajectories). This is no longer true if infinite dimensional spaces are allowed.

In fact, if $\dim(F)=\infty$, it is possible to modify the standard example ξ so that the new dynamical system $\tilde{\xi}$ has the following properties: for each $c>0$, $D_{E\times F}(c)$ is a normal block of type (E, F); $P(1,\tilde{\xi})=P(1,\xi)$; for each

$$(e,f)\in\big(E\times\{0_F\}\big)\cup\big(\{0_E\}\times F\big),\quad t\in R,$$

there holds $(e,f)\tilde{\xi}t=(e,f)\xi t$ and, last but not least there exists a sequence $\{f_n\}_{n\in N}\subset$ $\subset D_F(1)$ that contains no convergent subsequences and

$$(0_E,\ 0_F)\in\mathrm{cl}\big(\cup\{\tilde{\gamma}((e,f_n))\,|\,n\in N\}\big)$$

whenever $e\in\partial D_E(1)$. Here, of course, $\tilde{\gamma}((e,f_n))$ denotes the trajectory segment of $\tilde{\xi}$ (inside $D_{E\times F}(1)$) emanating from (e,f_n), $n\in N$.

Assume that E is finite-dimensional. The last property of $\tilde{\xi}$ implies immediately that there does not exist a homeomorphism of $D_{E\times F}(1)$ onto itself which carries the trajectory segments of $\tilde{\xi}$ onto the trajectory segments of ξ. In the contrary, suppose that there exists a homeomorphism

$$\Phi: D_{E\times F}(1)\to D_{E\times F}(1),\ (e,f)\to\big(\Phi_1(e,f),\ \Phi_2(e,f)\big)\in E\times F$$

that carries the trajectory segments of $\tilde{\xi}$ onto the trajectory segments of ξ. Fix an $e\in\partial D_E(1)$. Since $\Phi(e,f_n)\xi t_n\to\Phi(0_E,0_F)=(0_E,0_F)$ for some sequence $\{t_n\}_{n\in N}\subset R^+$, it follows that $\Phi_2(e,f_n)\to 0_F$. On the other hand, observe that the sequence $\{\Phi(e,f_n)\}_{n\in N}$, a fortiori the sequence $\{\Phi_1(e,f_n)\}_{n\in N}$ contains no convergent subsequences, contradicting the assumption that E is finite-dimensional. Thus $\tilde{\xi}$ is not conjugate to ξ near $(0_E,0_F)$ (or more generally, to a dynamical system induced by a linear hyperbolic vector field) although the information one derives from the isolating block analysis of $\tilde{\xi}$ is the same as one derives from the isolating block analysis of ξ. Consequently, the infinite-dimensional case [23] of the famous Hartman—Grobman Lemma implies that $\tilde{\xi}$ can not be induced by a C^1 vector field.

Conceptually, the construction of $\tilde{\xi}$ is rather simple. On the other hand, the technical details are not trivial and of considerable length. Besides, our method of constructing $\tilde{\xi}$ has nothing in common with negligibility theory: therefore, the details — together with some further results concerning infinite-dimensional dynamical systems — will be published elsewhere.

The dynamical system $\tilde{\xi}$ can be interpreted as a refutation to Coleman's conjecture [7] in infinite dimensions. Coleman's conjecture is that a suitable topological notion of hyperbolicity for an equilibrium point x_0 of a dynamical system π in R^{n+m} should guarantee that near x_0, π is conjugate to a dynamical system induced by a linear hyperbolic vector field. Wilson [32] clarified this conjecture by defining topologically hyperbolic equilibria of type (m, n) and showed that the reformulated conjecture is true if $\min\{m, n\}=0$ (this is trivial) or $\min\{m, n\}=1$ [33]. The conjecture is false for all types of m, n with $\min\{m, n\}\geqq 2$ [22], [24]. For a survey of these developments, see [28].

4. The method of Section 2 in product spaces. In what follows we apply the method of Section 2 in the vicinity of a saddle point in the infinite-dimensional Banach space $E\times F$. Here E is assumed to be infinite-dimensional. We construct a dynamical system η on $E\times F$ satisfying the following conditions:

(d) for all positive c, $D_{E\times F}(c)$ is a normal block of type (E, F),

(e) there exists a sequence $\{e_n\}_{n=1}\subset E$ converging to 0_E such that $\gamma((e_n, 0_F))$, $n=1, 2, \ldots$, the trajectory of η through $(e_n, 0_F)\in E\times F$ is not attracted by $(0_E, 0_F)$.

The construction of η is modelled after the one of τ given in Section 2.4. (The dynamical system τ satisfies both (d) and (e) provided $\dim(F)=0$.) Let us remark an immediate consequence of (e) and (d): the dynamical system η is not orbit-conjugate (in fact, near $(0_E, 0_F)$, not locally orbit-conjugate) to the Standard Example ξ.

Let $(E, \|\cdot\|_E)$ be an infinite-dimensional Banach space. Choose an $e_0\in E$, $e_0\neq 0_E$ and consider a continuous linear functional $L: E\to R$ so that $L(e_0)=1$. As in Section 2.3, E can be represented as $E=X\times R$ and $e\in E$ can be written as $e=(x, \lambda)\in X\times R$ where

$$X=\{e\in E\,|\,L(e)=0\},\quad x=e-L(e)\,e_0\quad\text{and}\quad\lambda=L(e).$$

In particular, the origin of E can be written as $0_E=(0_x, 0)$.

Let F be a Banach space. To avoid unnecessary overlaps with Section 2.4, it is assumed that $\dim(F)\geqq 1$. The projections of the product $E\times F$ onto its first and second factor are denoted by P_1 and P_2, respectively.

For $C>0$, we define a dynamical system π_C on $E\times F$ satisfying the following conditions:

(1)′ if $\|((x, 0), f)\|_{E\times F}\geqq C$, then $((x, 0), f)\pi_C t=((x, t), f)$

(2′) $(X\times\{0\})\times F\subset E\times F$ is a global section for π_C and for each $(x, f)\in X\times F$, the function $L_{x,f}: R\to R$ defined by

$$L_{x,f}(t)=L(P_1(((x, 0), f)\pi_C t))$$

is strictly increasing.

(3)′ $\quad L_{0_x, 0_F}(t)\to-\infty$ as $t\to-\infty$ and $L_{0_x, 0_F}(t)\to 1$ as $t\to\infty$.

(4)′ for each $(x,f)\in(X\times F)\setminus\{(0_x, 0_F)\}$, we have

$$L_{x,f}(t)\to-\infty \quad \text{as} \quad t\to-\infty \quad \text{and} \quad L_{x,f}(t)\to\infty \quad \text{as} \quad t\to\infty.$$

(5)′$_1$ for each $x\in X, f\in F$; if $t\leqq 0$, then $L_{x,f}(t)=t$.
(5)′$_2$ for each $x\in X, f\in F$; if $L_{x,f}(t_1)\geqq 1$ and $L_{x,f}(t_2)\geqq 1$, then $L_{x,f}(t_1)-L_{x,f}(t_2)=t_1-t_2$.

(6) for each $f\in F$, the set $E\times\{f\}$ is invariant under π_C. In particular, $E\times\{0_F\}$ is invariant.

The core of the construction of π_C is a version of Cutler's theorem in product spaces:

LEMMA IN NEGLIGIBILITY THEORY. *There exists an invertible isotopy Φ pushing $\{(0_x, 0_F)\}$ off $X\times F$ which is limited by $\mathcal{G}=\{G\}$, $G=\{(x,f)\in X\times F \,|\, \|(x,0),f)\|_{E\times F}<C\}$ and takes $X\times(F\setminus\{0_F\})\times I$ onto $X\times(F\setminus\{0_F\})\times I$.*

PROOF. The proof is a simple application of a usual method of negligibility theory (cf. the proof of Lemma 3 of Cutler's original paper [8]).

By Cutler's theorem, there exists a homeomorphism $\mathcal{F}\colon (X\times I)\setminus\{(0_x, 1)\}\to X\times I$ which preserves the second coordinate (i.e. $\mathcal{F}(x,t)=(\varphi(x,t),t)$) and which is the identity on $(X\times\{0\})\cup(\{x\in X \,|\, \|(x,0)\|_E\geqq C\}\times I)$.

For $f\in F$, $\|f\|_F<C$, define $k(f)=1-\|f\|_F/C$. Observe that $k(0_F)=1$ and $k(f)$ approaches 0 as $\|f\|_F$ approaches C. Finally, for $(x,f,t)\in(X\times F\times I)\setminus\{(0_x, 0_F, 0)\}$, let

$$\Phi(x,f,t)=\begin{cases}(k(f)\mathcal{F}(x/k(f), t\cdot k(f)), f, t) & \text{if} \quad \|f\|_F<C\\ (x,f,t) & \text{if} \quad \|f\|_F\geqq C.\end{cases}$$

It is not hard to show that $\Phi\colon (X\times F\times I)\setminus\{(0_x, 0_F, 1)\}\to X\times F\times I$ is a homeomorphism with all the desired properties. Q.E.D.

The projections of the product $X\times F\times I$ onto its first, second and third factor are denoted by p_1, p_2 and p_3, respectively. We define a homeomorphism

$$\Psi\colon(X\times F\times I)\setminus\{(0_x, 0_F, 1)\}\to(X\times I)\times F\subset E\times F$$

by letting

$$\Psi(x,f,t)=\big(p_1\Phi(x,f,t),\ p_3\Phi(x,f,t),\ p_2\Phi(x,f,t)\big).$$

The desired dynamical system π_C will be defined as $\pi_C=\pi_{h_C}$, where $h_C\colon R\times(X\times\{0\})\times F\to E\times F$ is a homeomorphism suitably chosen. The definition of h_C follows a similar pattern to the one of h_A given in Section 2.3. With A replaced by C, the set Q_C and the functions $q_C\colon Q_C\to R$ and $M_C\colon(0,\infty)\to R$ are to be defined as Q_A, q_A and M_A, respectively. Finally, for $(t,(x,0),f)\in R\times(X\times\{0\})\times F$, $h_C(t,(x,0),f)$ is to be defined by

$$h_C(t,(x,0),f)=\begin{cases}((x,t),f) & \text{if} \quad x\in X, \quad t\leqq 0, \quad f\in F\\ \Psi(x,f,q_C(\|x,f\|,t)) & \text{if} \quad (\|x,f\|,t)\in Q_C\\ \Psi(x,f,1)+((0_x, t-M_C(\|x,f\|)), 0_F) & \text{if} \quad x\neq 0_x, \quad f\in F,\\ & \qquad t\geqq M_C(\|x,f\|).\end{cases}$$

For sake of brevity, we have written $\|x, f\|$ instead of $\|((x, 0), f)\|_{E\times F}$. By the same arguments used in Section 2.3, it is not difficult to prove that h_C is a homeomorphism and the dynamical system $\pi_C = \pi_{h_C}$ satisfies conditions (1)′—(6)′.

Set

$$Y = \{((y, \mu), g) \in (X\times R)\times F = E\times F \mid \mu > 0\}.$$

Define a map $K\colon E\times F = (X\times R)\times F \to Y$ by

$$K((x, \lambda), f) = (((\exp(-\lambda))x, \exp(-\lambda))/\|(x, 1)\|_E, (\exp(\lambda))f)$$

for all $((x, \lambda), f) \in (X\times R)\times F$. Since K is a continuous bijection and

$$K^{-1}((y, \mu), g) = (y/\mu, -\ln(\|(y, \mu)\|_E), \|(y, \mu)\|_E g)$$

for all $((y, \mu), g) \in Y$, K is a homeomorphism. For each $(\lambda, f) \in R\times F$, K maps $(X\times\{\lambda\})\times\{f\}$ homeomorphically onto

$$\{(e, (\exp(\lambda))f) \in Y \mid \|e\|_E = \exp(-\lambda)\}.$$

For $x \in X$, $f \in F$, $C > 0$, set

$$S(x, f, C) = \{((y, 0), g) \in (X\times R)\times F \mid \|((y-x, 0), g-f)\|_{E\times F} \leqq C\}.$$

Let the sequences $\{((x_n, 0), 0_F)\}_{n\in N} \subset S(0_x, 0_F, 1/2)$, $\{C_n\}_{n\in N} \subset (0, 1/2)$ be chosen so that $x_n \to 0_x$ as $n \to \infty$ and that $\|(x_n, 0)\|_E > C_n$, $n \in N$ and $S(x_n, 0_F, C_n) \cap \cap S(x_m, 0_F, C_m) = \emptyset$, $n, m \in N$, $n \neq m$. Observe that

$$\cup\{S(x_n, 0_F, C_n) \mid n \in N\} \subset S(0_x, 0_F, 1) \setminus \{((0_x, 0), 0_F)\}.$$

In virtue of (1)′ and (5)′, by letting

$$((x, \lambda), f)\zeta t = \begin{cases} ((x, \lambda), f) + ((-x_n, -n), 0_F)\pi_{C_n} t - ((-x_n, -n), 0_F) & \\ \qquad \text{if } ((x, 0), f) \in S(x_n, 0_F, C_n) \text{ for some } n \in N & \\ ((x, \lambda + t), f) \qquad \text{otherwise} & \end{cases}$$

for $((x, \lambda), f) \in (X\times R)\times F = E\times F$, $t \in R$, we obtain a dynamical system on $E\times F$. By (2)′, $(X\times\{0\})\times F$ is a global section for ζ.

Assume that, in addition,

$$C_n \leqq \exp(-2n-2), \quad \text{for each } n \in N \tag{7}$$

and finally, for $((y, \mu), g) \in (X\times R)\times F$, $t \in R$, let

$$((y, \mu), g)\eta t = \begin{cases} K((K^{-1}((y, \mu), g))\zeta t) & \text{if } ((y, \mu), g) \in Y \\ ((y, \mu), g)\xi t & \text{otherwise.} \end{cases}$$

(Recall that ξ is the dynamical system defined as the Standard Example of Section 3.2.)

In virtue of (5)′, we have

$$((x, \lambda), f)\zeta t = ((x, \lambda + t), f) \tag{8}$$

whenever

$$((x, 0), f) \in S(x_n, 0_F, C_n), \quad \lambda \geqq n+1, \quad t \geqq 0.$$

Consequently, applying (7) and (8), a direct computation shows that

$$((y, \mu), g)\eta t = ((y, \mu), g)\xi t \tag{9}$$

whenever

$$((y, \mu), g) \in Y, \ \|P_1((y, \mu), g)\|_E \leqq \|P_2((y, \mu), g)\|_F, \quad t \geqq 0.$$

In fact, if $((y, \mu), g) \in Y$, $((y/\mu, 0), \|(y, \mu)\|_E \cdot g) \in S(x_n, 0_F, C_n)$ for some $n \in N$ and $\|P_1((y, \mu), g)\|_E \leqq \|P_2((y, \mu), g)\|_F$, then

$$\|(y, \mu)\|_E^2 \leqq \|g\|_F \cdot \|(y, \mu)\|_E \leqq C_n \leqq \exp(-2n-2)$$

or equivalently, $-\ln(\|(y, \mu)\|_E) \geqq n+1$ and, therefore

$$((y, \mu), g)\eta t = K((y/\mu, -\ln(\|(y, \mu)\|_E)+t), \|(y, \mu)\|_E \cdot g) =$$
$$= ((y, \mu)\exp(-t), (\exp(t))g) = ((y, \mu), g)\xi t$$

for $t \geqq 0$. On the other hand, for any fixed $((y, \mu), g) \in Y$, in virtue of (2)′, $L(P_1(K^{-1}((y, \mu), g)\zeta t))$ is a strictly increasing function of t. It follows immediately that

(10) for any fixed $((y, \mu), g) \in Y$, $\|P_1(((y, t), g)\eta t)\|_E$ is a strictly decreasing function of t.

Now we are in a position to show that η is a dynamical system with the desired properties.

If $((y, \mu), g) \in (X \times R) \times F = E \times F$ and $\mu \neq 0$, $t \in R$, then η is clearly continuous at $(((y, \mu), g), t)$. In case of $((y, 0), g) \neq ((0_x, 0), 0_F)$, $t \in R$, the continuity of η at $(((y, 0), g), t)$ is a simple consequence of (9). Finally, for all $t \in R$, (5)′ implies that η is continuous at $(((0_x, 0), 0_F), t)$.

Let c be any fixed positive number and consider the set $D_{E \times F}(c)$. By the construction of η, especially in virtue of (6), the positively invariant set in $D_{E \times F}(c)$ is $D_E(c) \times \{0_F\}$. It is evident that the negatively invariant set in $D_{E \times F}(c)$ is $\{0_E\} \times \times D_F(c)$. Finally, (9) and (10) imply immediately that $\partial D_E(c) \times (D_F(c) \setminus \partial D_F(c))$ consists of strict ingress points, $(D_E(c) \setminus \partial D_E(c)) \times (D_F(c))$ consists of strict egress points and $\partial D_E(c) \times \partial D_F(c)$ is the tangency set. Thus, $D_{E \times F}(c)$ is a normal block of type (E, F). In other words, η satisfies condition (d).

Finally, for each fixed $n \in N$, by the construction of η especially in virtue of (3)′ and (4)′, it follows that $\gamma(K((x_n, n), 0_F))$, the trajectory through $K((x_n, n), 0_F)$, does not tend to $((0_x, 0), 0_F)$. On the other hand, for each $c > 0$,

$$K((x_n, n), 0_F) \in D_E(c) \times \{0_F\}$$

for n sufficiently great. Thus, η satisfies condition (e) as well: e_n can be taken for $P_1(K((x_n, n), 0_F))$, $n = 1, 2, \ldots$.

5. A further application of Cutler's theorem. In the locally compact case, the behaviour of dynamical systems near compact isolated invariant sets is well understood. For references, see [25]. A typical result into this direction is the following Lemma of Ura and Kimura, which will be applied to prove a simple result con-

cerning the link between parallelizability and asymptotic stability in locally compact metric spaces.

LEMMA [27]. *Let (X, d) be locally compact and $M \subset X$ a non-open compact isolated invariant set. Then one and only one of the following alternatives holds:* (A) *M is positively asymptotically stable,* (B) *M is negatively asymptotically stable,* (C) *there exist points $x \in X \setminus M$ and $y \in X \setminus M$ such that the trajectory through x resp. y is attracted resp. repelled by M.*

PROPOSITION 3.2. *Let (X, d) be a non-compact, locally compact separable metric space and let π be a dynamical system on X. Let M be a compact invariant set. Assume that $X \setminus M$ as well as X are connected. Finally, assume that π restricted to $X \setminus M$ is parallelizable. Then M is (positively or negatively) globally asymptotically stable.*

PROOF. Let $\hat{X}$ denote the one-point-compactification of X. Following [2], we extend the dynamical system π to a dynamical system $\hat{\pi}$ on $\hat{X}$ by defining $x_0 \hat{\pi} t = x_0$, for all $t \in R$, where x_0 denotes the point at infinity. The first positive and negative prolongational limit sets pertaining to $\hat{\pi}$ are denoted by $\hat{J}^+(x)$ and $\hat{J}^-(x)$, respectively.

Since parallelizability implies dispersiveness, it follows easily that $J^+(x) \cup \cup J^-(x) \subset M$ and $\hat{J}^+(x) \cup \hat{J}^-(x) \subset M \cup \{x_0\}$ for all $x \in X \setminus M$. Since $\hat{X}$ is a compact metrizable space, $\hat{J}^+(x)$ is nonvoid and connected [3, p. 123]. Therefore, $X \setminus M = P^+ \cup N^+$ where

$$P^+ = \{x \in X \setminus M \mid \emptyset \neq \hat{J}^+(x) \subset M\} \quad \text{and} \quad N^+ = \{x \in X \setminus M \mid \hat{J}^+)(x) = x_0\}.$$

Next we show that P^+ is closed in $X \setminus M$. In fact, suppose that $\{x_n\}_1^\infty \subset P^+$, $x_n \to x \in X \setminus M$. For $n = 1, 2, 3, \ldots$, choose $m_n \in \hat{J}^+(x_n) \subset M$. Since M is compact, there is no loss of generality in assuming that $m_n \to m$ for some $m \in M$. It can be readily verified that $m \in \hat{J}^+(x)$ and consequently, $x \in P^+$. Similarly, N^+ is closed in $X \setminus M$, too. By the same arguments, the sets

$$P^- = \{x \in X \setminus M \mid \emptyset \neq \hat{J}^-(x) \subset M\}$$

and

$$N^- = \{x \in X \setminus M \mid \hat{J}^-(x) = x_0\}$$

are closed in $X \setminus M$ and $X \setminus M = P^- \cup N^-$. Observe that $P^+ \cap N^+ = P^- \cap N^- = \emptyset$.

In virtue of the connectedness of $X \setminus M$, we have to distinguish four cases according as $P^+ = N^- = X \setminus M$ (Case 1) or $P^- = N^+ = X \setminus M$ (Case 2) or $P^+ = P^- = X \setminus M$ (Case 3) or $N^+ = N^- = X \setminus M$ (Case 4).

The connectedness assumptions imposed on $X \setminus M$ and X imply that M is not open. Applying the Lemma, the fourth case is impossible and the first two cases correspond to (positive or negative) global asymptotic stability. Finally, the third case is excluded by the Lemma when applied to $\{x_0\}$ instead of M. Q.E.D.

It is pointed out by Egawa [9] that the usual description of dynamical systems near compact isolated invariant sets does not hold when local compactness is dropped. More precisely, modifying the so-called Bebutov shift system, Egawa has constructed [9] an arcwise connected and complete separable metric space X and a dynamical system π on X with a non-open equilibrium point x^* isolated from closed invariant sets such that no trajectory of π (excepting x^* itself) is attracted or repelled by x^*.

Next we show that similar examples can be given in every infinite-dimensional Banach space E. Moreover, π can be chosen so that

(f) π restricted to $E\setminus\{0_E\}$ is parallelizable and

(g) for all $e\neq 0_E$, there holds $J^+(e)\cup J^-(e)=\emptyset$.

EXAMPLE 3.4. Let E be an infinite-dimensional Banach space and let $e_0\in E$, $e_0\neq 0_E$. As in Sections 2.3 and 3.4, E can be represented as $E=X\times R$. Consider a dynamical system, say v, on E with the following properties: there are no equilibrium points but 0_E, the line $\{\lambda e_0|\lambda\in R\}$ consists of three trajectories, and any other trajectory is parallel to the line $\{\lambda e_0|\lambda\in R\}$.

It is trivial that v restricted to $E\setminus\{\lambda e_0|\lambda\in R\}$ is parallelizable and $X\setminus\{0_x\}$ is a global section. By Cutler's theorem, the set $\{\lambda e_0|\lambda\in R\setminus\{0\}\}$ is extractible from E. In other words, there exists a homeomorphism

$$H: E\setminus\{\lambda e_0|\lambda\in R\setminus\{0\}\}\to E.$$

Without loss of generality, we may assume that $H(0_E)=0_E$. It is clear that

$$e\vartheta t = H\big(H^{-1}(e)vt\big),\quad e\in E,\quad t\in R$$

defines a dynamical system ϑ on E whose restriction onto $E\setminus\{0_E\}$ is parallelizable and $H(X\setminus\{0_x\})$ is a global section. Similarly, it is evident that ϑ satisfies (g) as well.

The dynamical system ϑ constructed above as well as trivial examples show that none of the conditions of Proposition 3.2 can be dropped.

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ÜBER DIE MITTEL VON ORTHOGONALEN FUNKTIONEN. III

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Herrn Professor P.L. Ul'yanov zum 60. Geburtstag gewidmet

1. In dieser Arbeit werden wir die vorhergehenden Untersuchungen [3], [4], [5] weiterführen.

Es sei $\mathfrak{B}$ die Klasse der Matrizen $B=\|b_{n,k}\|_{n,k=1}^{\infty}$ mit $\{b_{n,k}\}_{k=1}^{\infty}\in l^2$ $(n=1,2,\ldots)$ und

$$\sum_{k=1}^{\infty} b_{n,k}^2 \to 0 \quad (n\to\infty). \tag{1}$$

Für $1\leqq K\leqq\infty$ bezeichnen wir mit $\Omega(K)$ die Klasse der in $(0,1)$ orthonormierten Funktionensysteme $\varphi=\{\varphi_k(x)\}_1^{\infty}$, für die

$$|\varphi_k(x)| \leqq K \quad (x\in(0,1);\ k=1,2,\ldots)$$

besteht.

Ist $B\in\mathfrak{B}$, so konvergiert die Reihe

$$\sum_{k=1}^{\infty} b_{n,k}\varphi_k(x) \quad (n=1,2,\ldots)$$

für jedes $\varphi\in\Omega(\infty)$ in der Metrik von $L^2(0,1)$; die Summe dieser Reihe bezeichnen wir mit $s_n(B,\varphi;x)$.

Wir werden die folgenden Sätze beweisen.

SATZ I. *Für jedes* K $(1\leqq K\leqq\infty)$ *gibt es eine Klasse von Matrizen* $B\in\mathfrak{B}$ *mit folgenden Eigenschaften. Ist* $B\in M(K)$, *so gilt*

$$s_n(B,\varphi;x)\to 0 \quad (n\to\infty) \tag{2}$$

für jedes $\varphi\in\Omega(K)$ *fast überall in* $(0,1)$. *Ist aber* $B\notin M(K)$, *so gibt es ein* $\Phi\in\Omega(K)$ *derart, daß*

$$s_n(B,\Phi;x)\not\to 0 \quad (n\to\infty) \tag{3}$$

fast überall in $(0,1)$ *gilt.*

SATZ II. *Für jedes* $1<K<\infty$ *gilt* $M(K)=M(1)$.

BEMERKUNGEN. a) Bekanntlich [1] ist die Bedingung (1) notwendig dafür, daß das Mittel $s_n(B,r;x)$ der Rademacherschen Funktionen $r_k(x)=\operatorname{sign}\sin 2^k\pi x$ fast überall in $(0,1)$ gegen 0 konvergiert.

b) Mit der in [6] angewandten Methode kann man ähnliche Sätze im Falle beweisen, daß $\Omega(K)$ die Klasse der Systeme $\varphi=\{\varphi_k(x)\}_1^{\infty}$ der stochastisch un-

abhängigen Funktionen ist, für die

$$|\varphi_n(x)| \leqq K \quad (x \in (0, 1),\ k = 1, 2, \ldots),$$

$$\int_0^1 \varphi_k(x)\, dx = 0, \quad \int_0^1 \varphi_k^2(x)\, dx = 1 \quad (k = 1, 2, \ldots)$$

bestehen.

Es sei $1 \leqq K \leqq \infty$. Für eine Matrix $B = \|b_{n,k}\|_{n,k=1}^{\infty}$ und für eine positive ganze Zahl N setzen wir

$$\|B, K, N\| = \sup_{\varphi \in \Omega(K)} \left\{ \int_0^1 \sup_{n \geqq N} s_n^2(B, \varphi, x)\, dx \right\}^{1/2}.$$

Die Klasse $M(K)$ kann man folgenderweise charakterisieren.

SATZ III. *Es seien* $1 \leqq K \leqq \infty$ *und* $B \in \mathfrak{B}$. $B \in M(K)$ *gilt dann und nur dann, wenn*

$$\lim_{N \to \infty} \|B, K, N\| = 0.$$

Im Falle $K = \infty$ gilt noch die folgende Behauptung.

SATZ IV. *Es sei* $B \in \mathfrak{B}$. $B \in M(\infty)$ *gilt dann und nur dann, wenn*

$$\|B, \infty, 1\| < \infty.$$

2. Zum Beweis unserer Sätze müssen wir gewisse Bezeichnungen und Hilfssätze vorausschicken.

Für eine in $(0, 1)$ definierte Funktion $f(x)$ und für ein Intervall $I = (a, b)$ $(a < b)$ setzen wir

$$f(I;\ x) = \begin{cases} f\left(\dfrac{x-a}{b-a}\right), & x \in I, \\ 0 & \text{sonst}, \end{cases}$$

und für eine Menge $H (\subseteqq (0, 1))$ sei $H(I)$ jene Menge, die aus H unter der Transformation $y = (b-a)x + a$ hervorgeht.

Die Menge H nennen wir *einfach,* wenn sie die Vereinigung endlichvieler Intervalle ist.

HILFSSATZ I. *Es sei* $1 \leqq K \leqq \infty$. *Gilt für eine Matrix* $B \in \mathfrak{B}$ $\lim_{N \to \infty} \|B, K, N\| = 0$, *so besteht* (2) *für jedes* $\varphi \in \Omega(K)$ *in* $(0, 1)$ *fast überall.*

BEWEIS DES HILFSSATZES I. Es sei $\varphi \in \Omega(K)$. Wir setzen

$$F_N(x) = \sup_{n \geqq N} s_n^2(B, \varphi, x) \quad (N = 1, 2, \ldots).$$

Dann gilt

$$F_N(x) \geqq F_{N+1}(x) \geqq 0 \quad (x \in (0, 1),\ N = 1, 2, \ldots). \tag{4}$$

Auf Grund der Definition von $\|\cdot, K, N\|$ haben wir

$$\left\{\int_0^1 F_N(x)\,dx\right\}^{1/2} \leqq \|B, K, N\|.$$

Daraus und aus (4), folgt durch Anwendung des Fatouschen Lemmas $\lim_{N\to\infty} F_N(x)=0$ fast überall in (0, 1), also gilt (2) in (0, 1) fast überall.

Es seien $B=\|b_{n,k}\|_{n,k=1}^{\infty}$ eine Matrix und N_1, N_2, M_1, M_2 $(N_1\leqq N_2,\ M_1\leqq M_2)$ positive ganze Zahlen. Für ein K $(1\leqq K\leqq\infty)$ sei

$$\|B, K, N_1, N_2, M_1, M_2\| = \sup_{\varphi\in\Omega(K)}\left\{\int_0^1 \max_{N_1\leqq n\leqq N_2}\Big(\sum_{k=M_1}^{M_2} b_{n,k}\varphi_k(x)\Big)^2 dx\right\}^{1/2}.$$

HILFSSATZ II. *Es seien B eine Matrix und N_1, N_2, M_1, M_2 $(N_1\leqq N_2,\ M_1\leqq M_2)$ positive ganze Zahlen. Dann gibt es ein orthonormiertes System von Treppenfunktionen $\psi_k(x)$ $(k=M_1, \ldots, M_2)$ in $(0, 1)$ und eine einfache Menge $E(\subseteq(0, 1))$ derart, daß* mes $E\geqq 1/2$ *und*

$$\max_{N_1\leqq n\leqq N_2}\left|\sum_{k=M_1}^{M_2} b_{n,k}\psi_k(x)\right| \geqq \frac{1}{2}\|B, \infty, N_1, N_2, M_1, M_2\| \quad (x\in E)$$

erfüllt sind.

BEWEIS DES HILFSSATZES II. Ohne Beschränkung der Allgemeinheit können wir $\|B, \infty, N_1, N_2, M_1, M_2\|>0$ voraussetzen. Nach der Definition von

$$\|\cdot, \infty; N_1, N_2, M_1, M_2\|$$

gibt es ein orthonormiertes System $\varphi_k(x)$ $(k=M_1, \ldots, M_2)$ in $(0, 1)$ mit

$$\int_0^1 \max_{N_1\leqq n\leqq N_2}\Big(\sum_{k=M_1}^{M_2} b_{n,k}\varphi_k(x)\Big)^2 dx > \frac{1}{4}\|B, \infty, N_1, N_2, M_1, M_2\|^2. \tag{5}$$

Für ein $\eta>0$ seien $\chi_k(x)$ $(k=M_1, \ldots, M_2)$ Treppenfunktionen in $(0, 1)$ mit

$$\int_0^1 (\varphi_k(x)-\chi_k(x))^2\,dx < \eta \quad (k = M_1, \ldots, M_2).$$

Aus (5) folgt

$$\int_0^1 \max_{N_1\leqq n\leqq N_2}\Big(\sum_{k=M_1}^{M_2} b_{n,k}\chi_k(x)\Big)^2 dx > \frac{1}{4}\|B, \infty, N_1, N_2, M_1, M_2\|^2, \tag{6}$$

wenn die Zahl η genügend klein ist. Wir setzen

$$\alpha_{i,j} = \int_0^1 \chi_i(x)\chi_j(x)\,dx \quad (i, j = M_1, \ldots, M_2).$$

Es sei $I_{i,j}$ $(i,j=M_1,\dots,M_2,\ i\neq j)$ eine Unterteilung des Intervalls $(1/2,1)$ in paarweise disjunkte Intervalle gleicher Länge. Dann setzen wir

$$\bar{\chi}_k(x) =$$

$$= \begin{cases} \sqrt{2}\,\chi_k(2x), & x\in(0,1/2), \\ \sqrt{\dfrac{|\alpha_{k,l}|}{2\operatorname{mes} I_{k,l}}}, & x\in I_{k,l},\quad l=M_1,\dots,M_2,\quad l\neq k, \\ -\sqrt{\dfrac{|\alpha_{k,l}|}{2\operatorname{mes} I_{k,l}}}\operatorname{sign}\alpha_{k,l}, & x\in I_{l,k},\quad l=M_1,\dots,M_2,\quad l\neq k, \\ 0 & \text{sonst.} \end{cases} \qquad (k=M_1,\dots,M_2),$$

Offensichtlich sind diese Funktionen Treppenfunktionen, die ein orthogonales System in $(0,1)$ bilden; weiterhin folgt aus (6)

$$(7)\qquad \int_0^1 \max_{N_1\leqq n\leqq N_2}\Big(\sum_{k=M_1}^{M_2} b_{n,k}\bar{\chi}_k(x)\Big)^2 dx \geqq \int_0^{1/2} \max_{N_1\leqq n\leqq N_2}\Big(\sum_{k=M_1}^{M_2} b_{n,k}\bar{\chi}_k(x)\Big)^2 dx =$$

$$= 2\int_0^{1/2} \max_{N_1\leqq n\leqq N_2}\Big(\sum_{k=M_1}^{M_2} b_{n,k}\chi_k(2x)\Big)^2 dx = \int_0^1 \max_{N_1\leqq n\leqq N_2}\Big(\sum_{k=M_1}^{M_2} b_{n,k}\chi_k(x)\Big)^2 dx >$$

$$> \frac{1}{4}\|B,\infty,N_1,N_2,M_1,M_2\|^2.$$

Wir setzen

$$\bar{\psi}_k(x) = \bar{\chi}_k(x)\Big/\Big\{\int_0^1 \bar{\chi}_k^2(x)\,dx\Big\}^{1/2} \quad (k=M_1,\dots,M_2).$$

Diese Funktionen sind Treppenfunktionen und bilden ein orthonormiertes System in $(0,1)$; weiterhin folgt aus (7)

$$(8)\qquad b^2 = \int_0^1 \max_{N_1\leqq n\leqq N_2}\Big(\sum_{k=M_1}^{M_2} b_{n,k}\bar{\psi}_k(x)\Big)^2 dx \geqq \frac{1}{4}\|B,\infty,N_1,N_2,M_1,M_2\|^2$$

für genügend kleines η.

Wir setzen

$$b'_{n,k} = \sqrt{2}\,b_{n,k}/b \quad (n=N_1,\dots,N_2,\ k=M_1,\dots,M_2)$$

und

$$F(x) = \max_{N_1\leqq n\leqq N_2}\Big|\sum_{k=M_1}^{M_2} b'_{n,k}\bar{\psi}_k(x)\Big|.$$

Aus (8) ergibt sich

$$(9)\qquad \int_0^1 F^2(x)\,dx = 2.$$

Da $F(x)$ eine Treppenfunktion ist, gibt es eine Zerlegung des Intervalls $(0,1)$ in paarweise disjunkte Intervalle $I_r=(a_r,b_r)$ $(b_r>a_r)$ $(r=1,\dots,\varrho)$ derart, daß $F(x)$

in jedem I_r konstant ist; den Wert von $F(x)$ in I_r bezeichnen wir mit w_r $(r=1, \ldots, \varrho)$. Es seien $1 \leqq r_1 < \ldots < r_\sigma \leqq \varrho$ diejenigen Indizes r, für welche $w_r \geqq 1$ gilt; die übrigen Indizes r $(1 \leqq r \leqq \varrho)$ bezeichnen wir mit $1 \leqq s_1 < \ldots < s_{\varrho-\sigma} \leqq \varrho$. Aus (9) folgt

(10)
$$c = \sum_{p=1}^{\sigma} w_{r_p}^2 \operatorname{mes} I_{r_p} = \sum_{r=1}^{\varrho} w_r^2 \operatorname{mes} I_r - \sum_{q=1}^{\varrho-\sigma} w_{s_q}^2 \operatorname{mes} I_{s_q} \geqq 2 - \sum_{q=1}^{\varrho-\sigma} \operatorname{mes} I_{s_q} \geqq 1.$$

Es seien $J_p' = (a_p', b_p')$ $(p=1, \ldots, \sigma)$ paarweise disjunkte Intervalle in $(0, c)$ mit $\operatorname{mes} J_p' = w_{r_p}^2 \operatorname{mes} I_{r_p}$ $(p=1, \ldots, \sigma)$ und $J_q'' = (a_q'', b_q'')$ $(q=1, \ldots, \varrho-\sigma)$ paarweise disjunkte Intervalle in $(c, 2)$ mit $\operatorname{mes} J_q'' = \operatorname{mes} I_{s_q}$ $(q=1, \ldots, \varrho-\sigma)$. Wir setzen

$$\psi_k^*(x) =$$

$$= \begin{cases} \dfrac{1}{w_{r_p}} \bar{\psi}_k\left(\dfrac{x-a_p'}{b_p'-a_p'}(b_{r_p}-a_{r_p})+a_{r_p}\right), & x \in J_p', \quad p = 1, \ldots, \sigma, \\ \bar{\psi}_k(x-a_{s_q}), & x \in J_p'', \quad q = 1, \ldots, \varrho-\sigma \end{cases} \qquad (k = M_1, \ldots, M_2)$$

und

$$E^* = \bigcup_{p=1}^{\sigma} J_p'.$$

Endlich sei

$$\psi_k(x) = \sqrt{2}\,\psi_k^*(2x) \quad (x \in (0, 1),\ k = M_1, \ldots, M_2),$$

und E bezeichne die Menge, die aus E^* unter der Transformation $y = x/2$ entsteht.

Es ist klar, daß die Funktionen $\psi_k(x)$ Treppenfunktionen sind, in $(0, 1)$ ein orthonormiertes System bilden, E einfach ist und $\operatorname{mes} E \geqq 1/2$ besteht, weiterhin

$$\max_{N_1 \leqq n \leqq N_2} \left| \sum_{k=M_1}^{M_2} b_{n,k}' \psi_k(x) \right| = \sqrt{2} \quad (x \in E)$$

gilt. Daraus und aus (8) folgt

$$\max_{N_1 \leqq n \leqq N_2} \left| \sum_{k=M_1}^{M_2} b_{n,k} \psi_k(x) \right| = b \geqq \frac{1}{2} \|B, \infty, N_1, N_2, M_1, M_2\|$$

für $x \in E$.

Damit haben wir Hilfssatz II bewiesen.

HILFSSATZ III. *Es seien* $1 < K < \infty$, B *eine Matrix und* N_1, N_2, M_1, M_2 $(N_1 \leqq N_2,\ M_1 \leqq M_2)$ *positive ganze Zahlen mit* $\|B, K, N_1, N_2, M_1, M_2\| > 0$. *Dann gibt es ein System* $\psi \in \Omega(K)$ *von Treppenfunktionen* $\psi_{M_1}(x), \ldots, \psi_{M_2}(x)$ *mit*

$$\int_0^1 \max_{N_1 \leqq n \leqq N_2} \Big(\sum_{k=M_1}^{M_2} b_{n,k} \psi_k(x) \Big)^2 dx > \frac{1}{2} \|B, K, N_1, N_2, M_1, M_2\|^2.$$

BEWEIS DES HILFSSATZES III. Auf Grund der Definition von

$$\|\cdot, K, N_1, N_2, M_1, M_2\|$$

gibt es ein System $\bar{\varphi}\in\Omega(K)$ von Funktionen $\bar{\varphi}_{M_1}(x), \dots, \bar{\varphi}_{M_2}(x)$ mit

$$\int_0^1 \max_{N_1\leqq n\leqq N_2} \Big(\sum_{k=M_1}^{M_2} b_{n,k}\bar{\varphi}_k(x)\Big)^2 dx > \|B, K, N_1, N_2, M_1, M_2\|^2 - \frac{\varepsilon}{3} \tag{11}$$

$$(\varepsilon = \|B, K, N_1, N_2, M_1, M_2\|^2/4).$$

Es werde a $(0<a<1)$ so gewählt, daß die Ungleichungen

$$\big(1-(1-a^2)^2(1-a)\big)/a \leqq K^2, \tag{12}$$

(13)

$$(1-a)(1-a^2)\int_0^1 \max_{N_1\leqq n\leqq N_2} \Big(\sum_{k=M_1}^{M_2} b_{n,k}\bar{\varphi}_k(x)\Big)^2 dx \geqq \int_0^1 \max_{N_1\leqq n\leqq N_2} \Big(\sum_{k=M_1}^{M_2} b_{n,k}\bar{\varphi}_k(x)\Big)^2 dx - \frac{\varepsilon}{3}$$

erfüllt werden. Es seien $\varphi_k(x)$ $(k=M_1, \dots, M_2)$ Treppenfunktionen mit

$$\int_0^1 (\bar{\varphi}_k(x)-\varphi_k(x))^2 dx < \eta \quad (k = M_1, \dots, M_2)$$

und

$$|\varphi_k(x)| \leqq |\bar{\varphi}_k(x)| \quad (x\in(0, 1),\ k = M_1, \dots, M_2). \tag{14}$$

Wir setzen

$$\alpha_{i,j} = \int_0^1 \varphi_i(x)\varphi_j(x)\,dx \quad (i, j = M_1, \dots, M_2).$$

Wir teilen das Intervall $(1-a^2, 1)$ in $(M_2-M_1)(M_2-M_1-1)$ paarweise disjunkte Intervalle $I_{i,j}$ $(i,j=M_1, \dots, M_2,\ i\neq j)$ gleicher Länge. Ist η genügend klein $(\eta\leqq\eta_1)$, so gelten die Abschätzungen

$$\int_0^1 \max_{N_1\leqq n\leqq N_2} \Big(\sum_{k=M_1}^{M_2} b_{n,k}\varphi_k(x)\Big)^2 dx > \int_0^1 \max_{N_1\leqq n\leqq N_2} \Big(\sum_{k=M_1}^{M_2} b_{n,k}\bar{\varphi}_k(x)\Big)^2 dx - \frac{\varepsilon}{3}, \tag{15}$$

$$\frac{\sqrt{1-a^2}}{a}\,\frac{(M_2-M_1)(M_2-M_1-1)}{2}\max_{\substack{M_1\leqq i,\ j\leqq M_2,\\ i\neq j}} |\alpha_{i,j}| \leqq K^2, \tag{16}$$

$$\alpha_{k,k} \geqq 1-a^2 \quad (k = M_1, \dots, M_2). \tag{17}$$

Wir setzen

$$\bar{\psi}_k(x) = \begin{cases} \varphi_k\left(\dfrac{x}{1-a^2}\right), & x\in(0, 1-a^2), \\ \sqrt{\dfrac{(M_2-M_1)(M_2-M_1-1)}{2a^2}|\alpha_{k,i}|}\cdot\sqrt{1-a^2}, & x\in I_{k,i}, \\ & i = M_1, \dots, M_2,\ i\neq k, \\ -\sqrt{\dfrac{(M_2-M_1)(M_2-M_1-1)}{2a^2}|\alpha_{k,i}|}\cdot \operatorname{sign}\alpha_{k,i}\sqrt{1-a^2}, & x\in I_{i,k}, \\ & i = M_1, \dots, M_2,\ i\neq k, \\ 0 & \text{sonst.} \end{cases}$$

Offensichtlich bilden die Treppenfunktionen $\overline{\psi}_k(x)$ $(k=M_1, \ldots, M_2)$ ein orthogonales System in $(0, 1)$, weiterhin gelten auf Grund von (14), (16), (17) und der Definition der Funktionen $\overline{\psi}_k(x)$ die folgenden Ungleichungen:

$$|\overline{\psi}_k(x)| \leqq K \quad (x\in(0, 1),\ k = M_1, \ldots, M_2), \tag{18}$$

$$\int_0^1 \max_{N_1\leqq n\leqq N_2} \Big(\sum_{k=M_1}^{M_2} b_{n,k}\overline{\psi}_k(x)\Big)^2 dx \geqq \int_0^{1-a^2} \max_{N_1\leqq n\leqq N_2} \Big(\sum_{k=M_1}^{M_2} b_{n,k}\overline{\psi}_k(x)\Big)^2 dx = \tag{19}$$

$$= \int_0^{1-a^2} \max_{N_1\leqq n\leqq N_2} \left(\sum_{k=M_1}^{M_2} b_{n,k}\varphi_k\left(\frac{x}{1-a^2}\right)\right)^2 dx = (1-a^2)\int_0^1 \max_{N_1\leqq n\leqq N_2} \Big(\sum_{k=M_1}^{M_2} b_{n,k}\varphi_k(x)\Big)^2 dx,$$

$$\int_0^1 \overline{\psi}_k^2(x)\,dx \geqq \int_0^{1-a^2} \overline{\psi}_k^2(x)\,dx = \int_0^{1-a^2} \varphi_k^2\left(\frac{x}{1-a^2}\right) dx = (1-a^2)\alpha_{k,k} \geqq (1-a^2)^2 \tag{20}$$

$$(k = M_1, \ldots, M_2).$$

Es sei endlich

$$\psi_k(x) = \begin{cases} \overline{\psi}_k\left(\dfrac{x}{1-a}\right), & x\in(0, 1-a), \\ m_k r_k\left(\dfrac{x-1+a}{a}\right), & x\in(1-a, 1) \end{cases} \qquad (k = M_1, \ldots, M_2),$$

wobei m_k derart bestimmt ist, daß die Funktionen $\psi_k(x)$ normiert sind; d. h. es gilt

$$(1-a)\int_0^1 \overline{\psi}_k^2(x)\,dx + m_k^2 a = 1 \quad (k = M_1, \ldots, M_2). \tag{21}$$

Auf Grund von (12), (20) und (21) folgt:

$$m_k \leqq \sqrt{\frac{1-(1-a^2)^2(1-a)}{a}} \leqq K \quad (k = M_1, \ldots, M_2).$$

Die Funktionen $\psi_{M_1}(x), \ldots, \psi_{M_2}(x)$ sind offensichtlich Treppenfunktionen, und aus (18) erhalten wir $\psi = \{\psi_k(x)\}_{M_1}^{M_2} \in \Omega(K)$, weiterhin gilt

$$\int_0^1 \max_{N_1\leqq n\leqq N_2} \Big(\sum_{k=M_1}^{M_2} b_{n,k}\psi_k(x)\Big)^2 dx \geqq \int_0^{1-a} \max_{N_1\leqq n\leqq N_2} \Big(\sum_{k=M_1}^{M_2} b_{n,k}\psi_k(x)\Big)^2 dx =$$

$$= (1-a)\int_0^1 \max_{N_1\leqq n\leqq N_2} \Big(\sum_{k=M_1}^{M_2} b_{n,k}\overline{\psi}_k(x)\Big)^2 dx.$$

Daraus erhalten wir

$$\int_0^1 \max_{N_1\leqq n\leqq N_2} \Big(\sum_{k=M_1}^{M_2} b_{n,k}\psi_k(x)\Big)^2 dx > \|B, K, N_1, N_2, M_1, M_2\|^2 - \varepsilon >$$

$$> \frac{1}{2}\|B, K, N_1, N_2, M_1, M_2\|^2$$

auf Grund von (11), (13), (15) und (19).

Damit haben wir Hilfssatz III bewiesen.

HILFSSATZ IV. *Es seien* $1 \leqq K \leqq \infty$, *B eine Matrix und* N_1, N_2, M_1, M_2 ($N_1 \leqq N_2$, $M_1 \leqq M_2$) *positive ganze Zahlen mit*

$$\|B, K, N_1, N_2, M_1, M_2\|^2 \geqq 128K^2 \max_{N_1 \leqq n \leqq N_2} \sum_{k=M_1}^{M_2} b_{n,k}^2.$$

Dann gibt es ein System $\psi = \{\psi_k(x)\}_{M_1}^{M_2} \in \Omega(1)$ *von Treppenfunktionen mit folgender Eigenschaft. Es gilt*

$$\max_{N_1 \leqq n \leqq N_2} \left| \sum_{k=M_1}^{M_2} b_{n,k} \psi_k(x) \right| \geqq \frac{1}{4K} \|B, K, N_1, N_2, M_1, M_2\| \quad (x \in E),$$

wobei $E (\subseteq (0, 1))$ *eine einfache Menge ist, für die* $\operatorname{mes} E \geqq 1/10$ *besteht.*

BEWEIS DES HILFSSATZES IV. Wir gebrauchen eine Idee von B. S. Kašin [2]. Der Satz ist nur im Falle $\|B, K, N_1, N_2, M_1, M_2\| > 0$ zu bewiesen; ohne Beschränkung der Allgemeinheit können wir

$$\|B, K, N_1, N_2, M_1, M_2\| = 4$$

voraussetzen. Durch Anwendung des Hilfssatzes III erhalten wir ein System $\overline{\psi} = \{\overline{\psi}_k(x)\}_{M_1}^{M_2} \in \Omega(K)$ von Treppenfunktionen mit

$$\int_0^1 \max_{N_1 \leqq n \leqq N_2} \left(\sum_{k=M_1}^{M_2} b_{n,k} \overline{\psi}_k(x) \right)^2 dx > \frac{1}{2} \|B, K, N_1, N_2, M_1, M_2\|^2 \quad (=2). \tag{22}$$

Es sei $I_r = (a_r, b_r)$ $(b_r > a_r)$ $(r = 1, \ldots, \varrho)$ eine Einteilung des Intervalls $(0, 1)$ in paarweise disjunkte Teilintervalle derart, daß jede Funktion $\overline{\psi}_k(x)$ in jedem I_r konstant ist. Den Wert der Funktion

$$\max_{N_1 \leqq n \leqq N_2} \left| \sum_{k=M_1}^{M_2} b_{n,k} \overline{\psi}_k(x) \right|$$

im Intervall I_r bezeichnen wir mit w_r. Nach (22) gilt

(23)

$$4 = \|B, K, N_1, N_2, M_1, M_2\|^2 \geqq \sum_{r=1}^{\varrho} w_r^2 \operatorname{mes} I_r > \frac{1}{2} \|B, K, N_1, N_2, M_1, M_2\|^2 = 2.$$

Es seien $1 \leqq r_1 < \ldots < r_\lambda \leqq \varrho$ diejenigen Indizes r, für die $w_r \geqq 1$ ist: jene Indizes r $(1 \leqq r \leqq \varrho)$, die von $r_1, \ldots, r_\lambda$ verschieden sind, bezeichnen wir der Reihe nach mit $s_1, \ldots, s_{\varrho - \lambda}$. Aus (23) folgt

$$4 \geqq \sum_{l=1}^{\lambda} w_{r_l}^2 \operatorname{mes} I_{r_l} > 1.$$

Wir setzen

$$a = \sum_{l=1}^{\lambda} w_{r_l}^2 \operatorname{mes} I_{r_l}, \quad b = \sum_{l=1}^{\varrho - \lambda} \operatorname{mes} I_{s_l},$$

wobei offensichtlich

$$1 < a \leqq 4, \quad 0 \leqq b \leqq 1 \tag{24}$$

gelten. Es seien $J_l' = (a_l', b_l')$ $(l=1, ..., \lambda)$ disjunkte Intervalle in $(0, a)$ mit mes $J_l' = w_{r_l}^2 \operatorname{mes} I_{r_l}$ und $J_l'' = (a_l'', b_l'')$ $(l=1, ..., \varrho-\lambda)$ disjunkte Intervalle in $(a, a+b)$ mit mes $J_l'' = \operatorname{mes} I_{s_l}$. Wir setzen

$$\tilde{\psi}_k(x) =$$

$$= \begin{cases} \bar{\psi}_k\left(\dfrac{x-a_l''}{b_l''-a_l''}(b_{s_l}-a_{s_l})+a_{s_l}\right), & x \in J_l'', \quad l=1, ..., \varrho-\lambda, \\ \dfrac{1}{w_{r_l}}\bar{\psi}_k\left(\dfrac{x-a_l'}{b_l'-a_l'}(b_{r_l}-a_{r_l})+a_{r_l}\right), & x \in J_l', \quad l=1, ..., \lambda \end{cases} \qquad (k = M_1, ..., M_2).$$

Es sei

$$\psi_k^*(x) = \tilde{\psi}_k((a+b)x)/K \quad (x \in (0, 1),\ k = M_1, ..., M_2).$$

Offensichtlich bilden die Treppenfunktionen $\psi_k^*(x)$ $(k=M_1, ..., M_2)$ in $(0, 1)$ ein orthogonales System, und es gilt

$$|\psi_k^*(x)| \leqq 1 \quad (x \in (0, 1),\ k = M_1, ..., M_2). \tag{25}$$

Es sei $\bar{E}$ die Bildmenge des Intervalls $(0, a)$ unter der linearen Transformation $y=x/(a+b)$. Aus (24) folgt:

$$\operatorname{mes} \bar{E} \geqq 1/5. \tag{26}$$

Weiterhin gilt auf Grund der Definition der Funktionen $\psi_k^*(x)$

$$\max_{N_1 \leqq n \leqq N_2} \left| \sum_{k=M_1}^{M_2} b_{n,k}\psi_k^*(x) \right| = \frac{1}{K} = \frac{1}{2K} \|B, K, N_1, N_2, M_1, M_2\| \quad (x \in \bar{E}). \tag{27}$$

Es sei J_s $(s=1, ..., \sigma)$ eine Einteilung des Intervalls $(0, 1)$ in paarweise disjunkte Intervalle derart, daß jede Funktion $\psi_k^*(x)$ in jedem J_s konstant ist, und $\bar{E}$ die Vereinigung einiger J_s ist. Den Wert von $\psi_k^*(x)$ im Intervall J_s bezeichnen wir mit $\varrho_s^{(k)}$. Für jeden Index s $(1 \leqq s \leqq \sigma)$ sei $\{\chi_s^{(k)}(x)\}_{M_1}^{M_2}$ ein orthogonales System von Treppenfunktionen derart, daß

$$\int_0^1 \chi_s^{(k)}(x)\,dx = 0 \quad (k = M_1, ..., M_2)$$

gilt und jede Funktion $\chi_s^{(k)}(x)$ den Wertbereich $\{1-\varrho_s^{(k)}, -1-\varrho_s^{(k)}\}$ besitzt. (Im Falle $\varrho_s^{(k)}=1$ setzt man $\chi_s^{(k)}(x) \equiv 0$.) Aus (25) folgt

$$|\chi_s^{(k)}(x)| \leqq 2 \quad (x \in (0, 1),\ k = M_1, ..., M_2,\ s = 1, ..., \sigma). \tag{28}$$

Es sei

$$\psi_k(x) = \psi_k^*(x) + \sum_{s=1}^{\sigma} \chi_s^{(k)}(J_s;\ x) \quad (k = M_1, ..., M_2).$$

Offensichtlich sind $\psi_k(x)$ $(k=M_1, \ldots, M_2)$ Treppenfunktionen, und man sieht leicht ein, daß $\psi=\{\psi_k(x)\}_{M_1}^{M_2}\in\Omega(1)$ ist. Für jeden Index s $(1\leqq s\leqq\sigma)$ sei n_s eine positive ganze Zahl mit $N_1\leqq n_s\leqq N_2$ und

$$\Big|\sum_{k=M_1}^{M_2} b_{n_s,k}\psi_k^*(x)\Big| = \max_{N_1\leqq n\leqq N_2}\Big|\sum_{k=M_1}^{M_2} b_{n,k}\psi_k^*(x)\Big| \quad (x\in J_s).$$

Dann gilt

$$(29)\quad \operatorname{mes}\Big\{x\in J_s\colon \max_{N_1\leqq n\leqq N_2}\Big|\sum_{k=M_1}^{M_2} b_{n,k}\psi_k(x)\Big| \geqq \frac{1}{4K}\|B, K, N_1, N_2, M_1, M_2\|\Big\} \geqq$$

$$\geqq \operatorname{mes}\Big\{x\in J_s\colon \Big|\sum_{k=M_1}^{M_2} b_{n_s,k}\psi_k(x)\Big| \geqq \frac{1}{4K}\|B, K, N_1, N_2, M_1, M_2\|\Big\} \geqq$$

$$\geqq \operatorname{mes}\Big\{x\in J_s\colon \Big|\sum_{k=M_1}^{M_2} b_{n_s,k}\psi_k^*(x)\Big| \geqq \frac{1}{2K}\|B, K, N_1, N_2, M_1, M_2\|\Big\} -$$

$$-\operatorname{mes}\Big\{x\in J_s\colon \Big|\sum_{k=M_1}^{M_2} b_{n_s,k}\chi_s^{(k)}(J_s;\, x)\Big| \geqq \frac{1}{4K}\|B, K, N_1, N_2, M_1, M_2\|\Big\} =$$

$$= \operatorname{mes}\Big\{x\in J_s\colon \max_{N_1\leqq n\leqq N_2}\Big|\sum_{k=M_1}^{M_2} b_{n,k}\psi_k^*(x)\Big| \geqq \frac{1}{2K}\|B, K, N_1, N_2, M_1, M_2\|\Big\} -$$

$$-\operatorname{mes}\Big\{x\in J_s\colon \Big|\sum_{k=M_1}^{M_2} b_{n_s,k}\chi_s^{(k)}(J_s;\, x)\Big| \geqq \frac{1}{4K}\|B, K, N_1, N_2, M_1, M_2\|\Big\}.$$

Nach (28) ergibt sich durch Anwendung der Tschebyscheffschen Ungleichung

$$\operatorname{mes}\Big\{x\in J_s\colon \Big|\sum_{k=M_1}^{M_2} b_{n_s,k}\chi_s^{(k)}(J_s;\, x)\Big| \geqq \frac{1}{4K}\|B, K, N_1, N_2, M_1, M_2\|\Big\} \leqq$$

$$\leqq \operatorname{mes} J_s\cdot 16K^2\sum_{k=M_1}^{M_2} b_{n_s,k}^2\int_0^1 (\chi_s^{(k)}(x))^2\,dx/\|B, K, N_1, N_2, M_1, M_2\|^2 \leqq$$

$$\leqq \operatorname{mes} J_s\cdot 64K^2\sum_{k=M_1}^{M_2} b_{n_s,k}^2/\|B, K, N_1, N_2, M_1, M_2\|^2 \leqq$$

$$\leqq \operatorname{mes} J_s\cdot 64K^2\cdot\max_{N_1\leqq n\leqq N_2}\sum_{k=M_1}^{M_2} b_{n,k}^2/\|B, K, N_1, N_2, M_1, M_2\|^2 \leqq \frac{1}{2}\operatorname{mes} J_s$$

auf Grund der Voraussetzung des Hilfssatzes IV. Daraus und aus (27), (29) folgert man

(30)

$$\operatorname{mes}\Big\{x\in J_s\colon \max_{N_1\leqq n\leqq N_2}\Big|\sum_{k=M_1}^{M_2} b_{n,k}\psi_k(x)\Big| \geqq \frac{1}{4K}\|B, K, N_1, N_2, M_1, M_2\|\Big\} \geqq \operatorname{mes} J_s/2.$$

Es sei

$$E = \bigcup_{J_s \subseteq E} \left\{ x \in J_s : \max_{N_1 \leqq n \leqq N_2} \left| \sum_{k=M_1}^{M_2} b_{n,k} \psi_k(x) \right| \geqq \frac{1}{4K} \| B, K, N_1, N_2, M_1, M_2 \| \right\}.$$

E ist offensichtlich einfach, und aus (26), (30) erhalten wir mes $E \geqq 1/10$.

Damit haben wir Hilfssatz IV bewiesen.

HILFSSATZ V. *Es seien* $1 \leqq K \leqq \infty$ *und* $B \in \mathfrak{B}$. *Dann gilt*

$$\lim_{N_2 \to \infty} \| B, K, N_1, N_2 \| = \| B, K, N_1 \| \tag{31}$$

für jede positive ganze Zahl N_1.

BEWEIS DES HILFSSATZES V. a) Fall $\| B, K, N_1 \| < \infty$. Es sei $\varepsilon > 0$ beliebig. Auf Grund der Definition von $\| \cdot, K, N_1 \|$ gibt es ein System $\varphi \in \Omega(K)$ mit

$$\int_0^1 \sup_{n \geqq N_1} \left(\sum_{k=1}^{\infty} b_{n,k} \varphi_k(x) \right)^2 dx > \| B, K, N_1 \|^2 - \varepsilon. \tag{32}$$

Da

$$\max_{N_1 \leqq n \leqq N_2} \left(\sum_{k=1}^{\infty} b_{n,k} \varphi_k(x) \right)^2 \nearrow \sup_{n \geqq N_1} \left(\sum_{k=1}^{\infty} b_{n,k} \varphi_k(x) \right)^2 \quad (x \in (0, 1),\ N_2 \nearrow \infty) \tag{33}$$

ist, erhalten wir

$$\int_0^1 \max_{N_1 \leqq n \leqq N_2} \left(\sum_{k=1}^{\infty} b_{n,k} \varphi_k(x) \right)^2 dx \nearrow \int_0^1 \sup_{n \geqq N_1} \left(\sum_{k=1}^{\infty} b_{n,k} \varphi_k(x) \right)^2 dx \quad (N_2 \nearrow \infty). \tag{34}$$

Daraus und aus (32) folgt;

$$\| B, K, N_1, N_2 \|^2 \geqq \int_0^1 \max_{N_1 \leqq n \leqq N_2} \left(\sum_{k=1}^{\infty} b_{n,k} \varphi_k(x) \right)^2 dx \geqq \| B, K, N_1 \|^2 - \varepsilon$$

für genügend großes N_2; d. h. es gilt (31).

b) Fall $\| B, K, N_1 \| = \infty$. Es sei $\omega > 0$ beliebig. Auf Grund der Definition von $\| \cdot, K, N_1 \|$ gibt es ein System $\varphi \in \Omega(K)$ mit

$$\int_0^1 \sup_{n \geqq N_1} \left(\sum_{k=1}^{\infty} b_{n,k} \varphi_k(x) \right)^2 dx > \omega. \tag{35}$$

Da (33) besteht, erhalten wir (34). Aus (34) und (35) folgt

$$\| B, K, N_1, N_2 \|^2 \geqq \int_0^1 \max_{N_1 \leqq n \leqq N_2} \left(\sum_{k=1}^{\infty} b_{n,k} \varphi_k(x) \right)^2 dx \geqq \omega$$

für genügend großes N_2; d. h. es gilt (31).

HILFSSATZ VI. *Es seien* $1 \leqq K \leqq \infty$ *und* $B \in \mathfrak{B}$. *Dann gilt*

$$\| B, K, N \| \geqq \| B, K, N+1 \| \quad (N = 1, 2, \ldots).$$

BEWEIS DES HILFSSATZES VI. Es sei $\varphi \in \Omega(K)$. Dann gilt

$$\sup_{n \geqq N+1} \left(\sum_{k=1}^{\infty} b_{n,k} \varphi_k(x) \right)^2 \leqq \sup_{n \geqq N} \left(\sum_{k=1}^{\infty} b_{n,k} \varphi_k(x) \right)^2 \quad (x \in (0, 1),\ N = 1, 2, \ldots),$$

woraus sich

$$(36) \quad \int_0^1 \sup_{n \geqq N+1} \left(\sum_{k=1}^{\infty} b_{n,k} \varphi_k(x) \right)^2 dx \leqq \int_0^1 \sup_{n \geqq N} \left(\sum_{k=1}^{\infty} b_{n,k} \varphi_k(x) \right)^2 dx \quad (N = 1, 2, \ldots)$$

ergibt. Da auf Grund der Definition von $\| \cdot, K, N \|$

$$\int_0^1 \sup_{n \geqq N} \left(\sum_{k=1}^{\infty} b_{n,k} \varphi_k(x) \right)^2 dx \leqq \| B, K, N \|^2$$

gilt, folgt aus (36)

$$\int_0^1 \sup_{n \geqq N+1} \left(\sum_{k=1}^{\infty} b_{n,k} \varphi_k(x) \right)^2 dx \leqq \| B, K, N \|^2 \quad (\varphi \in \Omega(K)),$$

und so gilt die Behauptung des Hilfssatzes VI.

HILFSSATZ VII. *Es seien* $1 \leqq K \leqq \infty$ *und* $B \in \mathfrak{B}$ *mit*

$$(37) \qquad \| B, K, N \| \not\to 0 \quad (N \to \infty).$$

Dann gibt es eine positive Zahl ϱ_1 *derart, daß für jede positive ganze Zahl* N_0 *positive ganze Zahlen* N_1, N_2 $(N_0 \leqq N_1 \leqq N_2)$ *mit*

$$(38) \qquad \| B, K, N_1, N_2 \| \geqq \varrho_1$$

existieren.

BEWEIS DES HILFSSATZES VII. Die Behauptung ergibt sich unmittelbar auf Grund der Hilfssätze V, VI.

HILFSSATZ VIII. *Es seien* $1 \leqq K \leqq \infty$ *und* $B \in \mathfrak{B}$ *mit* (37). *Dann gibt es eine positive Zahl* ϱ *mit folgender Eigenschaft. Für beliebige positive ganze Zahlen* N_0, M_1 *existieren positive ganze Zahlen* N_1, N_2, M_2 $(N_0 \leqq N_1 \leqq N_2,\ M_1 \leqq M_2)$, *mit*

$$(39) \qquad \| B, K, N_1, N_2, M_1, M_2 \| \geqq \varrho.$$

BEWEIS DES HILFSSATZES VIII. Wegen (1) gibt es eine positive ganze Zahl $\overline{N}_0 (\geqq N_0)$ mit

$$(40) \qquad \sum_{k=1}^{M_1 - 1} \sup_{n \geqq \overline{N}_0} |b_{n,k}| \leqq \frac{\varrho_1}{6}.$$

Wegen des Hilfssatzes VII gibt es positive ganze Zahlen N_1, N_2 $(\overline{N}_0 \leqq N_1 \leqq N_2)$ mit (38). Endlich gibt es wegen (1) eine positive ganze Zahl $M_2 (\geqq M_1)$ mit

$$(41) \qquad \sum_{k=N_1}^{N_2} \sqrt{\sum_{k=M_2+1}^{\infty} b_{n,k}^2} \leqq \frac{\varrho_1}{6}.$$

Auf Grund der Definition von $\|\cdot, K, N_1, N_2\|$ gibt es ein System $\varphi\in\Omega(K)$ mit

$$\int_0^1 \max_{N_1\leqq n\leqq N_2} \Big(\sum_{k=1}^{\infty} b_{n,k}\varphi_k(x)\Big)^2 dx \geqq \varrho_1/2. \tag{42}$$

Es gilt

$$\Big\{\int_0^1 \max_{N_1\leqq n\leqq N_2} \Big(\sum_{k=M_1}^{M_2} b_{n,k}\varphi_k(x)\Big)^2 dx\Big\}^{1/2} \geqq \Big\{\int_0^1 \max_{N_1\leqq n\leqq N_2} \Big(\sum_{k=1}^{\infty} b_{n,k}\varphi_k(x)\Big)^2 dx\Big\}^{1/2} - \tag{43}$$

$$-\Big\{\int_0^1 \max_{N_1\leqq n\leqq N_2} \Big(\sum_{k=1}^{M_1-1} b_{n,k}\varphi_k(x)\Big)^2\Big\}^{1/2} - \Big\{\int_0^1 \max_{N_1\leqq n\leqq N_2} \Big(\sum_{k=M_2+1}^{\infty} b_{n,k}\varphi_k(x)\Big)^2 dx\Big\}^{1/2}.$$

Da

$$\Big|\sum_{k=1}^{M_1-1} b_{m,k}\varphi_k(x)\Big| \leqq \sum_{k=1}^{M_1-1} |b_{m,k}||\varphi_k(x)| \leqq \sum_{k=1}^{M_1-1} \max_{N_1\leqq n\leqq N_2} |b_{n,k}||\varphi_k(x)| \leqq$$

$$\leqq \sum_{k=1}^{M_1-1} \sup_{n\geqq N_0} |b_{n,k}||\varphi_k(x)| \quad (x\in(0,1),\ m = N_1, \ldots, N_2)$$

ist, gilt

$$\max_{N_1\leqq n\leqq N_2} \Big|\sum_{k=1}^{M_1-1} b_{n,k}\varphi_k(x)\Big| \leqq \sum_{k=1}^{M_1-1} \sup_{n\geqq N_0} |b_{n,k}||\varphi_k(x)| \quad (x\in(0,1)),$$

und so folgt

$$\Big\{\int_0^1 \max_{N_1\leqq n\leqq N_2} \Big(\sum_{k=1}^{M_1-1} b_{n,k}\varphi_k(x)\Big)^2 dx\Big\}^{1/2} \leqq \Big\{\int_0^1 \Big(\sum_{k=1}^{M_1-1} \sup_{n\geqq N_0} |b_{n,k}||\varphi_k(x)|\Big)^2 dx\Big\}^{1/2} \leqq \tag{44}$$

$$\leqq \sum_{k=1}^{M_1-1} \Big\{\int_0^1 \Big(\sup_{n\geqq N_0} |b_{n,k}|\Big)^2 \varphi_k^2(x)\, dx\Big\}^{1/2} = \sum_{k=1}^{M_1-1} \sup_{n\geqq N_0} |b_{n,k}| \leqq \frac{\varrho_1}{6}$$

auf Grund von (40). Wegen

$$\max_{N_1\leqq n\leqq N_2} \Big|\sum_{k=M_2+1}^{\infty} b_{n,k}\varphi_k(x)\Big| \leqq \sum_{n=N_1}^{N_2} \Big|\sum_{k=M_2+1}^{\infty} b_{n,k}\varphi_k(x)\Big| \quad (x\in(0,1))$$

folgt

$$\Big\{\int_0^1 \max_{N_1\leqq n\leqq N_2} \Big(\sum_{k=M_2+1}^{\infty} b_{n,k}\varphi_k(x)\Big)^2 dx\Big\}^{1/2} \leqq \Big\{\int_0^1 \Big(\sum_{n=N_1}^{N_2} \Big|\sum_{k=M_2+1}^{\infty} b_{n,k}\varphi_k(x)\Big|\Big)^2 dx\Big\}^{1/2} \leqq \tag{45}$$

$$\leqq \sum_{n=N_1}^{N_2} \Big\{\int_0^1 \Big(\sum_{k=M_2+1}^{\infty} b_{n,k}\varphi_k(x)\Big)^2 dx\Big\}^{1/2} = \sum_{n=N_1}^{N_2} \sqrt{\sum_{k=M_2+1}^{\infty} b_{n,k}^2} \leqq \frac{\varrho_1}{6}$$

auf Grund von (41). Aus (42), (43), (44), (45) erhalten wir

$$\|B, K, N_1, N_2, M_1, M_2\| \geqq \Big\{\int_0^1 \max_{N_1\leqq n\leqq N_2} \Big(\sum_{k=M_1}^{M_2} b_{n,k}\varphi_k(x)\Big)^2 dx\Big\}^{1/2} \geqq \frac{\varrho_1}{2} - 2\frac{\varrho_1}{6} = \frac{\varrho_1}{6}.$$

Damit haben wir Hilfssatz VIII bewiesen.

3. BEWEIS DES SATZES III. Es sei $1 \leqq K \leqq \infty$. Es sei $M(K)$ die Klasse der Matrizen $B \in \mathfrak{B}$, für die (2) bei jedem System $\varphi \in \Omega(K)$ fast überall in (0, 1) besteht. Offensichtlich ist die Klasse $\Omega(K)$ nicht leer.

Aus dem Hilfssatz I folgt, daß im Falle $B \in \mathfrak{B}$ und $\lim_{N \to \infty} \|B, K, N\| = 0$ (2) bei jedem $\varphi \in \Omega(K)$ fast überall in (0, 1) besteht; d. h. es gilt $B \in M(K)$.

Wir werden zeigen, daß es im Falle $B \in \mathfrak{B}$ und (37) ein System $\Phi \in \Omega(K)$ der Treppenfunktionen gibt derart, daß (3) fast überall in (0, 1) besteht.

(a) *Fall* $1 < K < \infty$. Auf Grund von (1) und des Hilfssatzes VIII kann man Folgen von positiven ganzen Zahlen $\{\nu_m\}_1^\infty$, $\{\nu'_n\}_1^\infty$ $\{\mu_m\}_1^\infty$ derart angeben, daß die folgenden Ungleichungen für jedes $m (= 1, 2, \ldots)$ erfüllt werden:

(i) $$\nu_m < \nu'_m < \nu_{m+1}, \quad \mu_m < \mu_{m+1} \quad (\mu_1 = 1),$$

(ii) $$\|B, K, \nu_m, \nu'_m, \mu_m, \mu_{m+1} - 1\| \geqq \varrho \quad (> 0),$$

(iii) $$\|B, K, \nu_m, \nu'_m, \mu_m, \mu_{m+1} - 1\|^2 \geqq 128 K^2 \max_{\nu_m \leqq n < \nu_{m+1}} \sum_{k=\mu_m}^{\mu_{m+1}-1} b_{n,k}^2,$$

(iv) $$\sum_{k=1}^{\mu_m - 1} \sup_{n \geqq \nu_m} |b_{n,k}| \leqq \frac{\varrho}{12K},$$

(v) $$\sum_{n=\nu_m}^{\nu'_m} \sqrt{\sum_{k=\mu_{m+1}}^{\infty} b_{n,k}^2} \leqq \frac{\varrho}{12 \cdot 2^m \cdot K}.$$

Es sei m eine positive ganze Zahl. Wegen (iii) können wir den Hilfssatz IV im Falle $N_1 = \nu_m$, $N_2 = \nu'_m$, $M_1 = \mu_m$, $M_2 = \mu_{m+1} - 1$ anwenden; die so erhaltenen Treppenfunktionen, bzw. die so erhaltene einfache Menge bezeichnen wir mit $\varphi_k^{(m)}(x)$ $(k = \mu_m, \ldots, \mu_{m+1} - 1)$ bzw. mit $E^{(m)}$. Dann gelten für jedes $m (= 1, 2. \ldots)$

(I) $$\{\varphi_k^{(m)}(x)\}_{\mu_m}^{\mu_{m+1}-1} \in \Omega(1),$$

(II) $$\operatorname{mes} E^{(m)} \geqq 1/10,$$

(III) $$\max_{\nu_m \leqq n < \nu_{m+1}} \left| \sum_{k=\mu_m}^{\mu_{m+1}-1} b_{n,k} \varphi_k(x) \right| \geqq \varrho/4K \quad (x \in E^{(m)})$$

auf Grund von (ii).

Dann wählen wir ein System $\Phi = \{\Phi_k(x)\}_1^\infty \in \Omega(1)$ von Treppenfunktionen und eine stochastisch unabhängige Folge $\{F_m\}_1^\infty$ von einfachen Mengen $F_m (\subseteq (0, 1))$ derart, daß für jedes $m (= 1, 2, \ldots)$ die folgenden Ungleichungen erfüllt werden:

(α) $$\operatorname{mes} F_m \geqq 1/10,$$

(β) $$\max_{\nu_m \leqq n < \nu_{m+1}} \left| \sum_{k=\mu_m}^{\mu_{m+1}-1} b_{n,k} \Phi_k(x) \right| \geqq \varrho/4K \quad (x \in F_m).$$

Wir setzen

$$\Phi_k(x) = \varphi_k^{(1)}(x) \quad (k = \mu_1, \ldots, \mu_2 - 1), \quad F_1 = E^{(1)}.$$

Auf Grund von (I)—(III) erhalten wir, daß $\{\Phi_k(x)\}_1^{\mu_2-1}\in\Omega(1)$ ist, und (α), (β) im Falle $m=1$ erfüllt sind. Weiterhin sind $\Phi_k(x)$ $(k=1, \ldots, \mu_2-1)$ Treppenfunktionen, und F_1 ist einfach.

Es sei m_0 eine positive ganze Zahl. Wir nehmen an, daß die einfachen Mengen $F_m(\subseteq(0,1))$ $(m=1, \ldots, m_0)$ und die Treppenfunktionen $\Phi_k(x)$ $(k=1, \ldots, \mu_{m_0+1}-1)$ schon derart definiert sind, daß $\{\Phi_k(x)\}_1^{\mu_{m_0+1}-1}\in\Omega(1)$ ist, die Mengen F_m $(m=1, \ldots, m_0)$ stochastisch unabhängig sind, weiterhin (α), (β) in den Fällen $m=1, \ldots, m_0$ erfüllt werden.

Dann gibt es eine Einteilung des Intervalls $(0, 1)$ in paarweise disjunkte Intervalle I_r $(\text{mes } I_r>0)$ $(r=1, \ldots, \varrho)$ derart, daß jede Funktion $\Phi_k(x)$ $(k=1, \ldots, \mu_{m_0+1}-1)$ in jedem I_r konstant ist, und jede Menge F_m $(m=1, \ldots, m_0)$ die Vereinigung gewisser I_r ist. Die zwei Hälfte von I_r bezeichnen wir mit I_r' bzw. mit I_r'' $(r=1, \ldots, \varrho)$. Dann setzen wir

$$\Phi_k(x) = \sum_{r=1}^{\varrho} \varphi_k^{(m_0+1)}(I_r'; x) - \sum_{r=1}^{\varrho} \varphi_k^{(m_0+1)}(I_r''; x) \quad (k = \mu_{m_0+1}, \ldots, \mu_{m_0+2}-1),$$

$$F_{m_0+1} = \bigcup_{r=1}^{\varrho} \left(E^{(m_0+1)}(I_r')\cup E^{(m_0+1)}(I_r'')\right).$$

Offensichtlich sind die Funktionen $\Phi_k(x)$ $(k=\mu_{m_0+1}, \ldots, \mu_{m_0+2}-1)$ Treppenfunktionen, die Menge F_{m_0+1} ist einfach, die Mengen F_m $(m=1, \ldots, m_0+1)$ sind stochastisch unabhängig, weiterhin ist auf Grund von (I)—(III) $\{\Phi_k(x)\}_1^{\mu_{m_0+2}-1}\in\Omega(1)$, und (α), (β) sind in den Fällen $m=1, \ldots, m_0+1$ erfüllt.

Die Mengenfolge $\{F_m\}_1^{\infty}$ und das System Φ mit den geforderten Eigenschaften erhalten wir also durch Induktion.

Es sei $F=\overline{\lim_{m\to\infty}} F_m$. Da die Mengen F_m stochastisch unabhängig sind und (α) für jedes m besteht, erhalten wir durch Anwendung des zweiten Borel—Cantellischen Lemmas $\text{mes } F=1$. Aus (β) ergibt sich also, daß für fast jeden Punkt $x(\in(0,1))$ und für unendlich viele m

$$\max_{\nu_m\leqq n<\nu_{m+1}} \left|\sum_{n=\mu_m}^{\mu_{m+1}-1} b_{n,k}\Phi_k(x)\right| \geqq \frac{1}{4K}\varrho \tag{46}$$

besteht. Aus (iv) bekommen wir:

$$\max_{\nu_m\leqq n<\nu_{m+1}} \left|\sum_{k=1}^{\mu_m-1} b_{n,k}\Phi_k(x)\right| \leqq \max_{\nu_m\leqq n<\nu_{m+1}} \sum_{k=1}^{\mu_m-1} |b_{n,k}||\Phi_k(x)| \leqq$$

$$\leqq \sum_{k=1}^{\mu_m-1} \sup_{n\geqq\nu_m} |b_{n,k}| \leqq \frac{\varrho}{12K} \quad (x\in(0,1),\ m=1,2,\ldots).$$

Daraus und aus (46) erhalten wir, daß für fast jeden Punkt $x(\in(0,1))$ und für unendlich viele m

$$\max_{\nu_m\leqq n<\nu_{m+1}} \left|\sum_{k=1}^{\mu_{m+1}-1} b_{n,k}\varphi_k(x)\right| \geqq \frac{2}{12K}\varrho \tag{47}$$

besteht.

Es sei

$$f_m(x) = \max_{\nu_m \leqq n < \nu_{m+1}} \left| \sum_{k=\mu_{m+1}}^{\infty} b_{n,k} \Phi_k(x) \right| \quad (m = 1, 2, \ldots).$$

Durch einfache Rechnung bekommen wir

$$\left\{ \int_0^1 f_m^2(x)\, dx \right\}^{1/2} \leqq \sum_{n=\nu_m}^{\nu_{m+1}-1} \sqrt{\sum_{k=\mu_m+1}^{\infty} b_{n,k}^2} \leqq \frac{\varrho}{12 \cdot 2^m \cdot K} \quad (m = 1, 2, \ldots)$$

auf Grund von (v). Daraus erhalten wir

$$\operatorname{mes}\left\{ x \in (0, 1):\ f_m(x) \geqq \frac{\varrho}{12K} \right\} \frac{\varrho}{12K} \leqq \int_0^1 f_m(x)\, dx \leqq$$

$$\leqq \left\{ \int_0^1 f_m^2(x)\, dx \right\}^{1/2} \leqq \frac{\varrho}{12 \cdot 2^m \cdot K},$$

und so ist

$$\operatorname{mes}\left\{ x \in (0, 1):\ f_m(x) \geqq \frac{\varrho}{12K} \right\} \leqq \frac{1}{2^m} \quad (m = 1, 2, \ldots).$$

Durch Anwendung des ersten Borel—Cantellischen Lemmas erhalten wir daraus, daß für fast jeden Punkt $x (\in (0, 1))$ eine positive ganze Zahl $m_0 = m_0(x)$ derart existiert, daß

$$f_m(x) < \frac{\varrho}{12K} \quad (m = m_0, m_0 + 1, \ldots).$$

Daraus und aus (47) folgt, daß für fast jeden Punkt $x (\in (0, 1))$

$$\max_{\nu_m \leqq n < \nu_{m+1}} \left| \sum_{k=1}^{\infty} b_{n,k} \Phi_k(x) \right| \geqq \frac{1}{12K} \varrho$$

für unendlich viele m besteht, d. h. es gilt $s_n(B, \Phi, x) \not\to 0$ $(n \to \infty)$ fast überall in $(0, 1)$. Da offensichtlich $\Phi \in \Omega(1) \subseteqq \Omega(K)$ ist, bekommen wir die Behauptung im Falle a).

b) *Fall* $K = 1$. Da für jedes $B \in \mathfrak{B}$ und für jede positive ganze Zahl N

$$\|B, K_0, N\| \geqq \|B, 1, N\|$$

für eine beliebige Zahl $1 < K_0 < \infty$ ist, gilt im Falle $\|B, 1, N\| \not\to 0$ $(N \to \infty)$ auch $\|B, K_0, N\| \not\to 0$ $(N \to \infty)$. Unter a) haben wir bewiesen, daß in diesem Falle ein System $\Phi \in \Omega(1)$ von Treppenfunktionen derart existiert, daß fast überall in $(0, 1)$ $s_n(B, \Phi, x) \not\to 0$ $(n \to \infty)$ gilt. Also ist die Behauptung im Falle b) richtig.

c) *Fall* $K = \infty$. Wir werden zeigen: gilt $\|B, \infty, N\| \not\to 0$ $(N \to \infty)$, so gibt es ein System $\Phi = \{\Phi_k(x)\}_1^\infty \in \Omega(\infty)$ von Treppenfunktionen derart, daß

$$\overline{\lim_{n \to \infty}} |s_n(B, \Phi, x)| = \infty$$

fast überall in $(0, 1)$ besteht.

Die Beweisführung ist ähnlich zu jener im Fall a).

Auf Grund von (1) und des Hilfssatzes VIII kann man Folgen von positiven ganzen Zahlen $\{\nu_m\}_1^\infty$, $\{\nu'_m\}_1^\infty$, $\{\mu_m\}_1^\infty$ derart angeben, daß die folgenden Ungleichungen für jedes $m(=1, 2, \ldots)$ erfüllt werden:

(a) $\nu_m < \nu'_m < \nu_{m+1}$, $\mu_m < \mu_{m+1}$ $(M_1 = 1)$,

(b) $\|B, \infty, \nu_m, \nu'_m, \mu_m, \mu_{m+1}-1\| \leqq \varrho$ (> 0),

(c) $$\sum_{k=1}^{\mu_m-1} \sup_{n \geqq \nu_m} |b_{n,k}| \leqq \frac{\varrho}{6 \cdot 2^m},$$

(d) $$\sum_{n=\nu_m}^{\nu'_m} \sqrt{\sum_{k=\mu_{m+1}}^{\infty} b_{n,k}^2} \leqq \frac{\varrho}{6 \cdot 2^m}.$$

Es sei m eine positive ganze Zahl. Wir können den Hilfssatz II im Falle $N_1 = \nu_m$, $N_2 = \nu'_m$, $M_1 = \mu_m$, $M_2 = \mu_{m+1}-1$ anwenden, die so erhaltenen Treppenfunktionen, bzw. die so erhaltene einfache Menge bezeichnen wir mit $\varphi_k^{(m)}(x)$ $(k = \mu_m, \ldots, \mu_{m+1}-1)$, bzw. mit $E^{(m)}$. Dann gelten für jedes $m(=1, 2, \ldots)$

(a_1) $\{\varphi_k^{(m)}(x)\}_{\mu_m}^{\mu_{m+1}-1} \in \Omega(\infty)$,

(b_1) $\operatorname{mes} E^{(m)} \geqq 1/2$,

(c_1) $$\max_{\nu_m \leqq n < \nu_{m+1}} \left| \sum_{k=\mu_m}^{\mu_{m+1}-1} b_{n,k} \varphi^{(m)}(x) \right| \geqq \frac{1}{2} \varrho \quad (x \in E^{(m)})$$

auf Grund von (b).

Wir setzen

$$\bar{\varphi}_k^{(m)}(x) = \sqrt{m}\, \varphi_k^{(m)}\left(\left(0, \frac{1}{m}\right); \; x\right) \quad (k = \mu_m, \ldots, \mu_{m+1}-1), \quad \bar{E}^{(m)} = E^{(m)}((0, 1/m))$$

für jedes $m(=1, 2, \ldots)$. Dann sind die Funktionen $\bar{\varphi}_k^{(m)}(x)$ $(k = \mu_m, \ldots, \mu_{m+1}-1$, $m = 1, 2, \ldots)$ Treppenfunktionen und die Mengen $\bar{E}^{(m)}$ einfach, weiterhin folgen aus (a_1), (b_1), (c_1) für jedes $m(=1, 2, \ldots)$

(a_2) $\{\bar{\varphi}_k^{(m)}(x)\}_{\mu_m}^{\mu_{m+1}-1} \in \Omega(\infty)$,

(b_2) $\operatorname{mes} \bar{E}^{(m)} \geqq 1/2m$,

(c_2) $$\max_{\nu_m \leqq n < \nu_{m+1}} \left| \sum_{k=\mu_m}^{\mu_{m+1}-1} b_{n,k} \bar{\varphi}_k^{(m)}(x) \right| \geqq \frac{\sqrt{m}}{2} \varrho \quad (x \in \bar{E}^{(m)}).$$

Auf Grund von (a_2), (b_2), (c_2), können wir mit der im Falle a) angewandte Methode ein System $\Phi = \{\Phi_k(x)\}_1^\infty \in \Omega(\infty)$ von Treppenfunktionen und eine stochastisch unabhängige Folge $\{F_m\}_1^\infty$ von einfachen Mengen $F_m (\subseteq (0, 1))$ derart definieren, daß für jedes $m=(1, 2, \ldots)$ die folgenden Ungleichungen erfüllt werden:

(a_3) $\operatorname{mes} F_m \geqq 1/2m$,

(b_3) $$\max_{\nu_n \leqq n < \nu_{m+1}} \left| \sum_{k=\mu_m}^{\mu_{m+1}-1} b_{n,k} \Phi_k(x) \right| \geqq \frac{\sqrt{m}}{2} \varrho \quad (x \in F_m).$$

Es sei $F=\overline{\lim_{m\to\infty}} F_m$. Da die Mengen F_m stochastisch unabhängig sind und (a_3) für jedes $m(=1, 2, \ldots)$ besteht, erhalten wir mes $F=1$ auf Grund des zweiten Borel—Cantellischen Lemmas. Aus (b_3) bekommen wir, daß für fast jeden Punkt $x(\in(0, 1))$

$$\overline{\lim_{n\to\infty}} \max_{\nu_m\leqq n<\nu_{m+1}} \Big|\sum_{k=\mu_m}^{\mu_{m+1}-1} b_{n,k}\Phi_k(x)\Big| = \infty \tag{48}$$

ist.

Es sei

$$g_m(x) = \max_{\nu_m\leqq n<\nu_{m+1}} \Big|\sum_{k=1}^{\mu_m-1} b_{n,k}\Phi_k(x)\Big| \quad (m = 1, 2, \ldots).$$

Durch einfache Rechnung bekommen wir auf Grund von (c)

$$\Big\{\int_0^1 g_m^2(x)\,dx\Big\}^{1/2} \leqq \sum_{k=1}^{\mu_m-1} \sup_{n\geqq\nu_m} |b_{n,k}| \leqq \frac{\varrho}{6\cdot 2^m} \quad (m = 1, 2, \ldots).$$

Daraus erhalten wir

$$\operatorname{mes}\Big\{x\in(0, 1):\ g_m(x) \geqq \frac{\varrho}{6}\Big\}\frac{\varrho}{6} \leqq \int_0^1 g_m(x)\,dx \leqq \Big\{\int_0^1 g_m^2(x)\,dx\Big\}^{1/2} \leqq \frac{\varrho}{6\cdot 2^m},$$

und so ist

$$\operatorname{mes}\Big\{x\in(0, 1):\ g_m(x) \geqq \frac{\varrho}{6}\Big\} \leqq 1/2^m \quad (m = 1, 2, \ldots).$$

Durch Anwendung des ersten Borel—Cantellischen Lemmas erhalten wir daraus, daß für fast jeden Punkt $x(\in(0, 1))$ eine positive ganze Zahl $m_1=m_1(x)$ derart existiert, daß

$$g_m(x) \leqq \varrho/6 \quad (m = m_1, m_1+1, \ldots) \tag{49}$$

gilt.

Es sei

$$f_m(x) = \max_{\nu_m\leqq n<\nu_{m+1}} \Big|\sum_{k=\mu_{m+1}}^{\infty} b_{n,k}\Phi_k(x)\Big| \quad (m = 1, 2, \ldots).$$

Durch einfache Rechnung bekommen wir auf Grund von (d)

$$\Big\{\int_0^1 f_m^2(x)\,dx\Big\}^{1/2} \leqq \sum_{n=\nu_m}^{\nu_{m+1}} \sqrt{\sum_{k=\mu_{m+1}}^{\infty} b_{n,k}^2} \leqq \frac{\varrho}{6\cdot 2^m} \quad (m = 1, 2, \ldots).$$

Daraus folgt:

$$\operatorname{mes}\Big\{x\in(0, 1):\ f_m(x) \geqq \frac{\varrho}{6}\Big\}\frac{\varrho}{6} \leqq \int_0^1 f_m(x)\,dx \leqq \Big\{\int_0^1 f_m^2(x)\,dx\Big\}^{1/2} \leqq \varrho/6\cdot 2^m,$$

und so ist

$$\operatorname{mes}\left\{x\in(0,1):\ f_m(x)\geqq\frac{\varrho}{6}\right\}\leqq 1/2^m\quad(m=1,2,\ldots).$$

Durch Anwendung des ersten Borel—Cantellischen Lemmas folgt daraus, daß für fast jeden Punkt $x(\in(0,1))$ eine positive ganze Zahl $m_2=m_2(x)$ derart existiert, daß

$$f_m(x)\leqq\varrho/6\quad(m=m_2,m_2+1,\ldots)\tag{50}$$

gilt. Aus (48), (49), (50) ergibt sich endlich, daß

$$\overline{\lim_{m\to\infty}}\max_{\nu_m\leqq n<\nu_{m+1}}\left|\sum_{k=1}^{\infty}b_{n,k}\Phi_k(x)\right|=\infty$$

fast überall in (0, 1) besteht, und so gilt die Behauptung im Falle c).

Damit haben wir Satz III bewiesen.

BEWEIS DES SATZES I. Auf Grund der Definition von $M(K)$ und des Satzes III ist die Behauptung des Satzes I offensichtlich.

BEWEIS DES SATZES II. Es sei $1<K<\infty$. Da $M(K)\subseteq M(1)$ offensichtlich gilt, haben wir nur zu beweisen, daß im Falle $B\in\mathfrak{B}$, $B\notin M(K)$ auch $B\notin M(1)$ gilt.

Es sei B eine Matrix mit $B\in\mathfrak{B}$, $B\notin M(K)$. Auf Grund des Satzes III gilt aber $\|B,K,N\|\not\to 0$ $(N\to\infty)$. Auf Grund der Ergebnisse des Falles a) im Beweis des Satzes III gibt es ein System $\Phi\in\Omega(1)$ von Treppenfunktionen derart, daß $s_n(B,\Phi,x)\not\to 0$ $(n\to\infty)$ fast überall in (0, 1), d. h. es gilt $B\notin M(1)$.

Damit haben wir Satz II bewiesen.

BEWEIS DES SATZES IV. Auf Grund der Punkt c) des Beweises von Satz III gibt es im Falle $B\in\mathfrak{B}$, $B\notin M(\infty)$ ein System $\Phi\in\Omega(\infty)$ von Treppenfunktionen derart, daß

$$\overline{\lim_{n\to\infty}}\,|s_n(B,\Phi,x)|=\infty$$

fast überall in (0, 1), und so ist

$$\int_0^1\sup_{n\geqq 1}\left(s_n(B,\Phi,x)\right)^2dx=\infty.$$

Daraus folgt $\|B,\infty,1\|=\infty$. Andererseits aus dem Satz III folgt leicht, daß im Falle $B\in M(\infty)$ $\|B,\infty,1\|<\infty$ gilt.

Damit haben wir Satz IV bewiesen.

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FOLIATIONS AND APPLICATIONS TO STRUCTURE PROBLEMS

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§ 0. Introduction

A representation of a monoid M and a lattice L by functions and subsets on a set A is *algebraic* if there is an algebra over A with M and L as its endomorphisms and subalgebras. We show that M and L in fact have an algebraic representation iff M and L have an apparently weaker type of set representation which need not, itself, be algebraic.

By way of application we describe completely the lattices L and singly generated monoids M which have an algebraic representation. A monoid generated by g is denoted by $[g]$. Each singly generated monoid is uniquely determined by a pair of numbers. The *period* and *preperiod* of $[g]$ are, respectively, the cycle length and precycle length of the graph of the Cayley representation for $[g]$. For a lattice L and a map $g: L \to L$, we say g is *join continuous* provided g preserves joins $(g(p \vee q) = g(p) \vee g(q))$ and g maps principal ideals of L to principal ideals of L with compact elements of the domain mapped to compact elements of the range. If L is algebraic it follows that g also preserves infinite joins, and each compact element of the range is the image of a compact element.

We will prove the following:

THEOREM. *An algebraic lattice L and a singly generated monoid $[f]$ are algebraic iff $[f]$ is infinite or there exists a join continuous map $g: L \to L$ for which the monoid $[g]$ is either isomorphic to $[f]$ or else satisfies: the preperiod of $[g]$ equals the preperiod of $[f]$, and the period of $[g]$ divides the period of $[f]$, and $g(p) = p$ for some compact $p \neq 0$.*

Every endomorphism of an algebra induces a join continuous map on the subalgebra lattice. The result above shows that the converse is also true: each join continuous map of an algebraic lattice L is induced by some endomorphism on a suitable algebra A with $\mathrm{Su}\, A \cong L$.

§ 1. Foliations

Let M be a monoid and L a complete compactly generated lattice. $\mathscr{A} = \langle A; M, L \rangle$ is a *representation structure of type* M, L if $\mathscr{A}$ is a relational structure with unary operations $f \in M$ and unary relations $p \in L$ (subsets of $\mathscr{A}$) which satisfy:

(i) The relations indexed by L form an algebraic closure system which is order isomorphic to L;

(ii) The functions indexed by M form a transformation monoid isomorphic to M.

For $D \subseteq A$, the rank of D is $R(D) = \bigwedge_{\substack{D \subset q \\ q \in L}} q$. So the rank of D is the closure of D in the algebraic closure system L, and hence for $S \subseteq A$ we have $R(S) = \bigvee_{\substack{D \subseteq S \\ D \text{ finite}}} R(D)$. For $n \in \omega$ and $x \in A^n$ the rank of x, $R(x)$, is the rank of the range of x.

We say $p \in L$ is *singly generated* in $\mathscr{A}$ if there is an element $a \in A$ with $R(a) = p$. The monoid M is *locally closed* in $\mathscr{A}$ if every function from A to A which agrees on each finite subset of A with some element of M is already an element of M.

A map $\alpha: \mathscr{A} \to \mathscr{B}$ for two relational structures of the same type is a homomorphism if for each relation S, and each n-tuple $a_1, a_2, \ldots, a_n$ for which $S(a_1, \ldots, a_n)$ holds, $S(\alpha a_1, \ldots, \alpha a_n)$ also holds. The *substitutive* operations on $\mathscr{A}$ are the homomorphisms from $\mathscr{A}^n$ to $\mathscr{A}$. The *dual algebra* $\mathscr{A}^* = \langle A; \mathscr{T} \rangle$ is the algebra of all substitutive operations on $\mathscr{A}$. We say that $\mathscr{A} = \langle A; M, L \rangle$ is *pre-algebraic* iff $L = \mathrm{Su}\, \mathscr{A}^*$. This is equivalent to the following 'mapping condition': For each $r \in \omega$ and $\underset{\sim}{x} \in A^r$ and $a \in A$ with $R(\underset{\sim}{x}) \cong R(a)$ there exists a relational homomorphism $\alpha: \mathscr{A}^r \to \mathscr{A}$ with $\alpha(\underset{\sim}{x}) = a$. Observe for $r = 0$ this simply requires: for all $a \in A$ with $R(a) = 0$, $f(a) = a$ for all $f \in M$. (Here $\mathscr{A}^r$ is the r-fold cartesian product, hence a relational structure of the same type.)

We say the representation structure $\mathscr{A} = \langle A; M, L \rangle$ is *algebraic* if $\langle A; M, L \rangle = \langle A; \mathrm{End}\, \mathscr{A}^*, \mathrm{Su}\, \mathscr{A}^* \rangle$. Observe $\mathrm{End}\, \mathscr{A}^* \supseteq M$ and $\mathrm{Su}\, \mathscr{A}^* \supseteq L$. Of course M and L are *algebraic* if and only if there exists an algebraic representation structure $\langle A; M, L \rangle$. In fact M and L are *algebraic* (as we shall see) if and only if there exists a representation structure $\langle A; M, L \rangle$ which is pre-algebraic, even though this representation itself need not be algebraic.

In order to prove this assertion, let $\mathscr{A} = \langle A; M, L \rangle$ be a representation structure of a monoid M and an algebraic lattice L. For each $x \in A$ index the elements of A by x to get an isomorphic copy, $\mathscr{A}_x$ of $\mathscr{A}$. That is let $\mathscr{A}_x = \langle A_x; M, L \rangle$ with $A_x = \{a_x;\ a \in A\}$ and $f(a_x) = (f(a))_x$ for $f \in M$. For $p \in L$, $a_x \in p$ on $\mathscr{A}_x$ iff $a \in p$ on $\mathscr{A}$. Clearly then $\mathscr{A}^0 = \overset{0}{\bigcup_{x \in A}} \mathscr{A}_x$ is a representation structure of M and L on the disjoint union $A^0 = \overset{0}{\bigcup_{x \in A}} \mathscr{A}x$.

Next we identify certain elements of $\mathscr{A}^0$, using an appropriate congruence relation. Define a binary relation $\sim$ on A^0 by:

$$a_x \sim a_y \Leftrightarrow \begin{cases} a_x = a_y & \text{or} \\ (f(x) = a \quad \text{and} \quad g(y) = a) & \text{for some } f, g \in M. \end{cases}$$

We let $\mathscr{F}(A) = A^0/\sim$ and introduce the term *foliation* of $\mathscr{A}$ to denote $\mathscr{F}(\mathscr{A}) = \mathscr{A}^0/\sim$. Observe that no two different elements a_x, b_x in the same copy $\mathscr{A}_x$ of $\mathscr{A}$ are ever $\sim$ related. Furthermore for $a_x \sim a_y$ and $f \in M$, $f(a_x) \sim f(a_y)$ and for $p \in L$, $a_x \in p$ in A_x iff $a_y \in p$ in A_y. It is obvious now (cf. [2]) that the foliation $\mathscr{F}(\mathscr{A}) = \langle \mathscr{F}(A); M, L \rangle$ is a representation structure of M and L, in which $f(a_x/\sim) = (f(a))_x/\sim$, and $a_x/\sim \in p$ iff $a \in p$.

THEOREM 1.1. *If $\mathscr{A}=\langle A; M, L\rangle$ is a pre-algebraic representation in which M is locally closed and each compact element of L is singly generated, then $\mathscr{F}(\mathscr{A})$ is algebraic and moreover each compact element of L is singly generated and the induced action of M on L in $\mathscr{F}(\mathscr{A})$ is the same as the induced action of M on L in $\mathscr{A}$.*

PROOF. Observe whenever $a\in A$ generates $p\in L$ in $\mathscr{A}$ $(R(a)=p)$, then $a_x/\sim$ generates p in $\mathscr{F}(\mathscr{A})$. Hence each compact element of L is singly generated in $\mathscr{F}(\mathscr{A})$. Moreover the action of M on L in $\mathscr{F}(\mathscr{A})$ is the same as that of M on L in $\mathscr{A}$.

Next we prove that $\mathscr{F}(\mathscr{A})$ is pre-algebraic by verifying the mapping condition. Fix $\underline{a}=(a'_{x_1}/\sim, \ldots, a^n_{x_n}/\sim)\in(\mathscr{F}(A))^n$ and $c=b_y/\sim\in\mathscr{F}(A)$ with $R(\underline{a})\geqq R(b)$. Then $R(a^1, \ldots, a^n)\geqq R(b)$ and hence there exists a morphism $\alpha: A^n\to A$ with $\alpha(a^1, \ldots, a^n)=b$. It is easy to check that $\beta: (A^0)^n\to(A^0)^n$ by $\beta(e^1_{z_1}, \ldots, e^n_{z_n})=(e^1_y, \ldots, e^n_y)$ is a morphism and that $\alpha_y: A^n_y\to A_y$ by $\alpha_y(e^1_y, \ldots, e^n_y)=(\alpha(e^1, \ldots, e^n))_y$ is a morphism. Thus $\alpha_y\beta$ is a morphism from $(A^0)^n\to A_y$ with $\alpha_y\beta(a^1_{x_1}, \ldots, a^n_{x_n})=b_y$. In fact $\ker(\alpha_y\beta)\supseteqq\sim$ and hence $\alpha_y\beta$ is a morphism from $(\mathscr{F}(\mathscr{A}))^n\to\mathscr{F}(\mathscr{A})$ with $\alpha_y\beta(\underline{a})=c$.

To see that M is locally closed in $\mathscr{F}(\mathscr{A})$, we first show that M is locally closed in $\mathscr{A}^0$. Let $h: A^0\to A^0$ be in the local closure of M on $\mathscr{A}^0$. Observe for each $x\in A$ that $h: A_x\to A_x$ since M does so on A^0 (i.e., for $a_x\in A_x\exists g\in M$ with $h(a_x)=g(a_x)=(g(a))_x\in A_x$). Since M is locally closed on A_x there is $h_x\in M$ with $h_x|A_x=h|A_x$. We claim $h_x=h_y$ for all $x, y\in A$. For if $h_x\neq h_y$ we have $h_x(a)\neq h_y(a)$ for some $a\in A$. Since $\{a_x, a_y\}$ is finite there is some $g\in M$ with $g|\{a_x, a_y\}=h|\{a_x, a_y\}$. But then $(h_x(a))_x=h_x(a_x)=h(a_x)=g(a_x)=(g(a))_x$ hence $h_x(a)=g(a)$. Similarly $h_y(a)=g(a)$, contrary to $h_x(a)\neq h_y(a)$. Thus M is locally closed in $\mathscr{A}^0$.

We show now that the local closure is inherited by $\mathscr{F}(\mathscr{A})=\mathscr{A}^0/\sim$. Let h be in the local closure of M on $\mathscr{F}(\mathscr{A})$. Define $h^0: A^0\to A^0$ as follows. For $a_x\in A_x$, if $h(a_x/\sim)=b_y/\sim$ let $h^0(a_x)=b_x$. Note that h^0 is well defined, for if $c_z\sim d_u$ then $c=d$. Further h^0 is in the local closure of M on $\mathscr{A}^0$: for finite $B\subseteqq\mathscr{A}^0$, clearly $B/\sim=\{b/\sim | b\in B\}$ is also finite hence there is some $g\in M$ with $g|B/\sim=h|B/\sim$. Now for $a_x\in B$ suppose $h(a_x/\sim)=b_y/\sim$; then $g(a_x/\sim)=b_y/\sim$ hence $g(a)=b$ and $x=y$ and $h^0(a_x)=b_x=(g(a))_x=g(a_x)$. But M is locally closed on $\mathscr{A}^0$, so for some $f\in M$, f and h^0 agree on $\mathscr{A}^0$. In fact f and h agree on $\mathscr{F}(\mathscr{A})$ as follows:

$$f(a_x/\sim)=f(a_x)/\sim=h^0(a_x)/\sim=h(a_x/\sim).$$

To verify the last equality fix $a_x/\sim\in\mathscr{F}(A)$ with say $h(a_x/\sim)=b_y/\sim$. Then for some $g\in M$ $h(a_x/\sim)=g(a_x/\sim)=(g(a))_x/\sim$ and hence $g(a)=b$. But now $h^0(a_x)=b_x$, hence $h^0(a_x)/\sim=b_x/\sim=(g(a))_x/\sim=h(a_x/\sim)$.

To see that $\mathscr{F}(\mathscr{A})$ is algebraic it suffices to see that $\mathrm{End}\,\mathscr{F}(\mathscr{A})^*=M$. Let $h: \mathscr{F}(A)\to\mathscr{F}(A)$ be a homomorphism of $F(\mathscr{A})^*$. We show that $h\in M$ by showing that h is in the local closure of M in $\mathscr{F}(\mathscr{A})$. To that end let D be finite, $D\subseteqq F(\mathscr{A})$. $R(D)=p$ is compact. Because each compact element is singly generated, there is a $d\in\mathscr{F}(A)$ with $R(d)=p$, say $d=a_x/\sim$. Note $R(a_a/\sim)=R(a_x/\sim)$. Hence we may assume that $d=a_a/\sim$. Now $h(a_a/\sim)=b_x/\sim$ for some $b_x\in A_x$ and some $x\in A$.

For any homomorphism $\delta: \mathscr{F}(\mathscr{A})\to\mathscr{F}(\mathscr{A})$ observe that $h\delta(t)=\delta h(t)$ for all $t\in\mathscr{F}(\mathscr{A})$ since δ is an operation of $\mathscr{F}(\mathscr{A})^*$.

Observe that $\varepsilon: \mathscr{F}(A)\to\mathscr{F}(A)$ given by $\varepsilon(c_z/\sim)=c_c/\sim$ is an endomorphism of $\mathscr{F}(A)$. Similarly for each $y\in A$. $\varepsilon_y: \mathscr{F}(A)\to\mathscr{F}(A)$ given by $\varepsilon_y(c_z/\sim)=c_y/\sim$ is an endomorphism of $\mathscr{F}(A)$. Now $h(a_a/\sim)=h\varepsilon(a_a/\sim)=\varepsilon h(a_a/\sim)=\varepsilon(b_x/\sim)=b_b/\sim$ and $h(a_a/\sim)=h\varepsilon_a(a_a/\sim)=\varepsilon_a h(a_a/\sim)=\varepsilon_a(b_x/\sim)=b_a/\sim$.

Thus $b_b \sim b_a$, and for some $f \in M$ we have $f(a)=b$. Also $f(a_a/\sim)=f(a_a)/\sim = f(a)_a/\sim = b_a/\sim = b_b/\sim = h(a_a/\sim)$. Hence h agrees with f on $a_a/\sim$. In fact h agrees with f on all of D because $a_a/\sim$ generates D in $\mathscr{F}(\mathscr{A})^*$ and both h and f are homomorphisms of $\mathscr{F}(\mathscr{A})^*$. □

It is well known that 'finitely generated' can often be replaced by 'singly generated' by a procedure which replaces finite sets by sequences (cf. [1]). For our purposes we need:

LEMMA 1.2. *If $\mathscr{A}=\langle A; M, L\rangle$ is a pre-algebraic representation structure with M locally closed, then there is a pre-algebraic representation structure $\mathscr{B}=\langle B; M, L\rangle$ in which additionally each compact $t \in L$ is singly generated. In fact $\mathscr{B}$ can be taken to be A^ω, the set of all finite sequences from A.*

PROOF. Let $B=A^\omega$, and let each $f \in M$ act coordinatewise on elements of B. The rank of $\underline{a} \in B$ is $R(a_1, \ldots, a_n)=\bigvee_{i=1}^{n} R(a_i)$; elements $p \in L$ are represented by the set of all elements of B whose rank is less than or equal to p. The rank of the empty sequence is zero in L, and $f(\emptyset)=\emptyset$. It is straightforward to verify that $\langle B, M, L\rangle$ is a representation structure. Moreover each compact $t \in L$ is clearly singly generated in $\langle B, M, L\rangle$ because it is finitely generated in $\langle A, M, L\rangle$. To see M is locally closed in the new representation, fix $h: B \to B$ and assume h agrees on each finite subset of B with some $m \in M$. Now since M is locally closed in $\langle A; M, L\rangle$ there is some $f \in M$ which agrees with h on all sequences of length one. In fact h agrees with f everywhere in B as follows: fix $\underline{a}=(a_1, \ldots, a_n) \in B$. Let $D=\{\underline{a}, (a_1), (a_2), \ldots, (a_n)\}$, thus D is a finite subset of B, and there is some $g \in M$ and $g|D=h|D$. But $h(\underline{a})=g(\underline{a})=(ga_1, ga_2, \ldots, ga_n)=(ha_1, \ldots, ha_n)=(fa_1, \ldots, fa_n)=f(\underline{a})$. Thus M is locally closed. It remains to show the mapping condition holds. For $n \in \omega$ and $\underline{x} \in B^n$, $y \in B$ with $R(\underline{x}) \cong R(y)$ we show there is a homomorphism $\alpha: B^n \to B$ with $\alpha(\underline{x})=y$.

If $n=0$ it suffices to see $f(a)=a$ for each $a \in B$ with $R(a)=0$. Clearly $R(a)=0$ implies a is the empty sequence from A or a is a non-empty sequence of elements of A each of rank zero. For $n>0$ by concatenation we may view $\underline{x} \in B^n$ as a sequence $\hat{x}$ in A^m. Thus if y is a sequence of length one, the mapping condition on $\langle A, M, L\rangle$ provides the desired homomorphism. If $y \in A^r$, then each component $y_i \in A$ also has rank $y_i \leqq$ rank $\underline{x}$ so there is a homomorphism $\varepsilon_i: B^n \to B$ with $\varepsilon_i(\underline{x})=y_i$. (Here $\varepsilon_i(\underline{z})=\underline{z}$ if $\hat{\underline{z}} \notin A^m$.) Then the map $\varepsilon: B^n \to B$ given by $\varepsilon(\underline{z})= =(\varepsilon_1 \underline{z}, \ldots, \varepsilon_n \underline{z})$ for $\underline{z} \in B^n$ is the desired map. Finally observe that the action of M on L in $\mathscr{A}^\omega$ is the same as the action of M on L in $\mathscr{A}$. □

COROLLARY 1.3. *If $\mathscr{A}=\langle A; M, L\rangle$ is a pre-algebraic representation structure with M locally closed, then there exists a universal algebra $\underline{A}$ with* End $\underline{A} \cong M$, Su $\underline{A} \cong L$ *such that the action of M on L induced by $\langle A; M, L\rangle$ is the same as the action of M on L induced by $\underline{A}$.*

PROOF. $(\mathscr{F}(\mathscr{A}^\omega))^*$ is the required algebra. □

§ 2. M-spaces over lattices

Given a representation of M (not necessarily faithful) by functions from L into L so that for each $f\in M$ the action of f on L is a join continuous map and the identity element of M acts like the identity transformation on L, we call the algebra $\mathscr{R}=\langle L; 0, 1, \vee, f\rangle_{f\in M}$ an *M-space over L*. If $\underline{A}$ is a universal algebra with $\mathrm{End}\,\underline{A}\cong M$ and $\mathrm{Su}\,\underline{A}\cong L$ then because each endomorphism in $\mathrm{End}\,\underline{A}$ maps subalgebras of $\underline{A}$ onto subalgebras of $\underline{A}$, M acts on L. It is not difficult to see that with this action of M on L, $\mathrm{Sp}\,(\underline{A})=\langle L, 0, 1, \vee, M\rangle$ is an M-space over L, the space of $\underline{A}$. An M-space $\mathscr{R}=\langle L; 0, 1, \vee, M\rangle$ is *algebraic* if there exists a universal algebra $\underline{A}$ with $\mathrm{Sp}\,(\underline{A})$ isomorphic to $\mathscr{R}$. For M singly generated we characterize the algebraic M-spaces. Thus we find not only the lattices algebraic with M, but the permissible induced actions of M on such lattices. (See Theorem 4.2.)

For integers k, t we write $k|t$ for "k divides t". If M is not the free monoid generated by f, then M has finite "preperiod" $\varrho(M)$ and finite "period" $\pi(M)$: If $n<m$ and $f^n=f^m$ and n, m are minimal in this respect, then $\varrho(M)=n$ and $\pi(M)=m-n$. If M is free then we put $\varrho(M)=\pi(M)=\infty$ and will agree that every integer n divides ∞, $n|\infty$. Furthermore $\infty|\infty$ and $\infty\pm\infty=\infty$, and for n and m integers $n\equiv m$ modulo (∞) iff $n=m$. Observe now that with these definitions we get:

$$(*)\qquad f^n=f^m \text{ iff } \{n=m, \text{ or } m, n\geqq\varrho(M) \text{ and } n\equiv m \bmod(\pi(M))\}.$$

Note here that the symbol f^n will be used in two ways: f^n denotes the n-fold product $f\cdot f\cdot\ldots\cdot f$ and also (for n suitably small) names an element of M. Clearly if $n<\varrho(M)+\pi(M)$ then we may identify these interpretations of f^n without confusion.

For a given representation of M (not necessarily faithful) by functions on some set A we define *preperiod* and *period* for elements, $a\in A$ and for A itself: If for $a\in A$ there exist numbers $n<m$ with $f^n(a)=f^m(a)$ and n and m are minimal under these conditions then let $\varrho(a)=n$ and $\pi(a)=m-n$. Otherwise (if there are no such numbers m and n) let $\varrho(a)=\pi(a)=\infty$. Observe that: $f^n(a)=f^m(a)$ iff $\{n=m,$ or $m, n\geqq\varrho(a)$ and $n\equiv m \bmod(\pi(a))\}$. Note also $\varrho(a)\leqq\varrho(M)$ and $\pi(a)|\pi(M)$. If there is an $a\in A$ such that for all $b\in A(\varrho(a)\geqq\varrho(b))$, we let $\varrho(A)=\varrho(a)$ and put $\varrho(A)=\infty$ otherwise. Clearly $\varrho(A)\leqq\varrho(M)$. If there is a smallest number n such that for all $b\in A(\pi(b)|n)$, this number n is called the period of A, $\pi(A)$. Otherwise we let $\pi(A)=\infty$. Clearly $\pi(A)|\pi(M)$.

Let M be a monoid generated by the single element f and $\mathscr{R}=\langle L; 0, 1, \vee, M\rangle$ an M-space over L. Because f maps elements of L_k, the compact elements of L, to elements of L_k, M acts on L_k. M also acts on L_k^n $(n\geqq 0)$ by $f(p_1, \ldots, p_n)=(f(p_1), \ldots, f(p_n))$. (For $n=0$, $f(\Phi)=\Phi$.) Observe that $\varrho(p_1, \ldots, p_n)=\max\{\varrho(p_1), \ldots, \varrho(p_n)\}$ and $\pi(p_1, \ldots, p_n)=$least common multiple of $\{\pi(p_1, \ldots, \pi(p_n))\}$.

LEMMA 2.1. *If $\varrho(M)$ is finite and $(p_1, \ldots, p_n)\in L_k^n$ and $q\in L_k$ with $\bigvee_{i=1}^{n} p_i\geqq q$, then $\varrho(p_1, \ldots, p_n)\geqq\varrho(q)$.*

PROOF. We first prove that the lemma holds for $\varrho(p_1, \ldots, p_n)=0$. In this case there exists for every $m\geqq 0$ some $(p_1', \ldots, p_n')$ such that $f^m(p_1')=p_1, \ldots, f^m(p_n')=p_n$.

But then $f^m \bigvee_{i=1}^{n} p_i' = \bigvee_{i=1}^{n} f^m(p_i') = \bigvee_{i=1}^{n} p_i \geqq q$. Because f^m maps principal ideals onto principal ideals by mapping the compact elements onto the compact elements, there exists $q' \leqq \bigvee_{i=1}^{n} p_i'$ with $f^m(q') = q$. This means that q is in the range of f^m for every $m \geqq 0$. Because $\varrho(M)$ is finite, this is only possible if $\varrho(q) = 0$.

If in general $\varrho(p_1, \ldots, p_n) = m$, then $\varrho(f^m(p_1), \ldots, f^m(p_n)) = 0$. Because $f^m(q) \leqq \leqq f^m \bigvee_{i=1}^{n} p_i = \bigvee_{i=1}^{n} f_m(p_i)$ we get from the first part of the proof that $\varrho(f^m(q)) = 0$ and hence $\varrho(q) \leqq m$. □

LEMMA 2.2. *If* $(p_1, \ldots, p_n) \in L_k^n$ *with* $t = \varrho(p_1, \ldots, p_n)$ *and* $f^t(p_1, \ldots, p_n) = (0, 0, \ldots, 0)$ *and* $q \in L_k$ *with* $\bigvee_{i=1}^{n} p_i \geqq q$, *then* $t \geqq \varrho(q)$ *and* $f^t(q) = 0$.

PROOF. Note

$$f^t(q) \leqq f^t\left(\bigvee_{i=1}^{n} p_i\right) = \bigvee_{i=1}^{n} f^t(p_i) = 0. \quad \square$$

§ 3. The word-algebra of a singly generated monoid M over a lattice L

Let M be a monoid generated by the single element f and $\mathscr{R} = \langle L; 0, 1, \vee, M \rangle$ an M-space over L. Let $N = N(\mathscr{R}) = \{p \in L_k;\ 0 < \varrho(p) < \infty$ and $f^n(p) \neq 0$ for any $n \in \omega\}$. Let $C = C(\mathscr{R}) = \{p \in L_k;\ p \neq 0,\ \varrho(p) = 0\}$. Let $E = E(\mathscr{R}) = \{p \in L_k;\ \varrho(p) = \infty$ or there exists $n \in \omega$ $f^n(p) = 0\}$. Let $E^* = \{p^*; p \in E\}$, a disjoint copy of E.

Then $W(\mathscr{R})$, the word-algebra of $\mathscr{R}$, is defined on words W (strings of symbols) where:

$$W = E^* \cup \{f^n q;\ q \in C,\ 0 \leqq n < \pi(M)\} \cup$$

$$\cup \{pf^n q;\ p \in N,\ q \in C,\ 0 \leqq n < \pi(M),\ f^{\varrho(p)}(p) = f^n(q)\}.$$

We intend to construct a representation structure $\mathscr{B} = \langle W; M, L \rangle$ of M and L on W. Towards this end we first define a rank function $R\colon W \to L_k$ by:

$R(p^*) = p$ and $R(f^n q) = f^n(q)$ and $R(pf^n q) = p$. Construct a representation of L on W induced by R, representing each $p \in L$ by the set of words $w \in W$ with $R(w) \leqq p$.

Observe that this representation of L constitutes an algebraic closure system in which the only word with rank 0 is 0^*.

Next we define an action of M on W by stipulating that:

$$f(p^*) = (f(p))^*,$$

$$f(pf^n q) = \begin{cases} f(p) f^n q & \text{if } f(p) \neq f^n(q), \\ f^n q & \text{otherwise,} \end{cases}$$

$$f(f^n q) = \begin{cases} f^{n+1}(q) & \text{if } n+1 < \pi(M), \\ f^0 q & \text{otherwise.} \end{cases}$$

If $w \in W$ then $f^0(w) = w$ and $f^{n+1}(w)$ is defined inductively by $f^{n+1}(w) = f^n(f(w))$.

LEMMA 3.1. (i) *If* $w\in E^*$ *then* $f^n(w)=f^m(w)\Leftrightarrow n=m$, *or* $n, m\geqq\varrho(R(w))$.

(ii) *If* $\varrho(M)=\infty$ *and* $w\in W-E^*$ *then* $f^n(w)=f^m(w)\Leftrightarrow n=m$.

(iii) *If* $\varrho(M)<\infty$ *and* $w\in W-E^*$ *then* $f^n(w)=f^m(w)\Leftrightarrow n=m$ *or* $n, m\geqq\varrho(R(w))$ *and* $n\equiv m \bmod(\pi(M))$.

PROOF. (i) If $p\in E$ with $\varrho(p)=\infty$ and $f^n(p^*)=f^m(p^*)$, then $(f^n(p))^*=(f^m(p))^*$ and $f^n(p)=f^m(p)$; so $n=m$. If $p\in E$ with $\varrho(p)$ finite, then again $f^n(p^*)=f^m(p^*)$ implies $f^n(p)=f^m(p)$ and so if $n\neq m$, then $n, m\geqq\varrho(p)$.

(ii) This is immediate from the action of f on $W-E^*$ by observing that $n<\infty$ for any number n.

(iii) If $w=f^sp$, then $\varrho(R(w))=\varrho(f^s(p))=0$ and the statement follows. If not, then $w=pf^sq$ and the proof follows readily from the observation that $f^{\varrho(p)}(pf^sq)=f^sq$. □

To prove that the action of M on W is properly defined we have to show that whenever $f^n, f^m, f^r\in M$ and $f^n\cdot f^m=f^r$, then for all $w\in W$, $f^n(f^m(w))=f^r(w)$. But because $f^n(f^m(w))=f^{n+m}(w)$ we have to show that $f^{n+m}(w)=f^r(w)$. But now if $n+m=r$, this equality trivially holds. Otherwise $n+m\geqq\varrho(M)$ and $r\geqq\varrho(M)\geqq\varrho(R(w))$ and $n+m\equiv r \bmod(\pi(M))$, because in M, $f^{n+m}=f^r$ and $f^n=f^m$ iff $\{n=m$, or $m, n\geqq\varrho(M)$ and $n\equiv m \bmod(\pi(M))\}$. (See (iii) above.)

So we deduce the desired equality from Lemma 3.1. Because M acts on W, M also acts on $W^r(r=0, 1, 2, 3, \ldots)$ by acting coordinatewise on the product.

We deduce then from Lemma 3.1 the following corollaries for the r-fold cartesian product W^r. For $w\in W^r$ the rank of w is $R(w)=\bigvee_{i\in r} R(w_i)$.

COROLLARY 3.2. *If* $\varrho(M)=\infty$ *and* $r\geqq 1$ *and* $w\in W^r$, *then* $f^n(w)=f^m(w)$ *iff either* $n=m$ *or for each* $i\in r$:

(i) $(w_i\in E^*$ *with* $(R(w_i)))<\infty$, *and*

(ii) $(\varrho(R(w_1), \ldots, R(w_r))=\max_{1\leqq i\leqq r}\varrho(R(w_i)))\geqq n, m$.

PROOF. Observe that $w_i\in E^*$ with $\varrho(R(w_i))<\infty$ means that $f^{\varrho(R(w_i))}(R(w_i))=0$. Thus $f^n(w)=f^m(w)\Leftrightarrow n=m$ or for each i $(w_i\in E^*$ with $f^{\varrho(R(w_i))}(R(w_i)))=0$ and $n, m\geqq\varrho(R(w_1), \ldots, R(w_r))$. (For $w\in W^0$, namely $w=\emptyset$, $f^n(\emptyset)=f^m(\emptyset)$ for all pairs of numbers n, m.) □

COROLLARY 3.3. *Let* $\varrho(M)<\infty$ *and* $r\geqq 1$ *and* $w\in W^r$.

(i) *If all of the* w_i *belong to* E^* *then* $f^n(w)=f^m(w)\Leftrightarrow n=m$ *or* $n, m\geqq\varrho(R(w_1), \ldots, R(w_r))$,

(ii) *otherwise* $f^n(w)=f^m(w)\Leftrightarrow n=m$ *or* $n, m\geqq\varrho(R(w_1), \ldots, R(w_r))$ *and* $n\equiv m \bmod(\pi(M))$. □

LEMMA 3.4. *For each* $w\in W$, $(f(R(w))=R(f(w)))$.

PROOF. If $p\in E$, then $f(R(p^*))=f(p)=R((f(p))^*)$. If $m=f^nq$, then $fR(f^nq)=f(f^n(q))=f^{n+1}(q)$. Now if $n+1<\pi(M)$ we get $R(f(f^nq))=R(f^{n+1}q)=f^{n+1}(q)$ and if $n+1=\pi(M)$ then $R(f(f^nq))=R(f^0q)=q=f^{\pi(M)}(q)=f^{n+1}(q)$. $(q=f^{\pi(M)}(q)$ because $\pi(q)$ divides $\pi(M)$.) If $w=pf^nq$, then $f(R(pf^nq))=f(p)$. If $f(p)\neq f^n(q)$, then $R(f(pf^nq))=R(f(p)f^nq)=f(p)$. If $f(p)=f^n(q)$, then $R(f(pf^nq))=R(f^nq)=f^n(q)=f(p)$. □

LEMMA 3.5. *If* $v\in W$ *and* $w\in W^r$ *with* $R(w)\geqq R(v)$, *then* $f^n(w)=f^m(w)$ *implies* $f^n(v)=f^m(v)$.

PROOF. We can assume that $n\neq m$ otherwise the implication is trivially true.

(i) If $r=0$ then $f^n(\emptyset)=f^m(\emptyset)$ for all $n, m\in\omega$. But clearly if $R(\emptyset)\geqq R(v)$, then $v=0^*$ and $f^n(0^*)=(f^n(0))^*=(f^m(0))^*=f(0^*)$. ($\emptyset$ is the empty function, and 0 is the smallest element in L.)

(ii) If $r\geqq 1$ and $\varrho(M)=\infty$, then we get from Corollary 3.2 for all i with $1\leqq i\leqq r$, $f^{\varrho(R(w_i))}(R(w_i))=0$. Hence we have $f^{\varrho(R(w_1),\ldots,R(w_r))}(R(w_1),\ldots,R(w_r))=$ $=(0,0,\ldots,0)$. Furthermore $n, m\geqq\varrho(R(w_1),\ldots,R(w_r))$, so applying Lemma 2.2 we deduce that $v\in E^*$ with $\varrho(R(w_1),\ldots,R(w_r))\geqq\varrho((Rv))$ and hence $n, m\geqq\varrho(R(v))$. Lemma 3.1 (i) states that then $f^n(v)=f^m(v)$.

(iii) If $r\geqq 1$ and $\varrho(M)<\infty$ and all of the coordinates of w are in E^* then the same proof as for (ii) applies.

(iv) If $r\geqq 1$ and $\varrho(M)<\infty$ and at least one of the coordinates of w is not in E^*, then we get from Corollary 3.3 that $n, m\geqq\varrho(R(w_1),\ldots,R(w_r))$ and from Lemma 3.1 that $\varrho(R(w_1),\ldots,R(w_r))\geqq\varrho(R(v))$. Hence we see from Lemma 3.1 (iii) that $f^n(v)=$ $=f^m(v)$. □

LEMMA 3.6. *If for* $r\geqq 1$, $w\in W^r$, $v\in W$ *with* $R(f(w))\geqq R(v)$, *then there is some word* $b\in W$ *with* $R(b)\leqq R(w)$ *and* $f(b)=v$.

PROOF. Observe first that $f(R(w))=R(f(w))\geqq R(v)$, hence there exists $l\in L_k$ with $l\leqq R(w)$ and $f(l)=R(v)$, since f is join continuous.

Now if $v=p^*\in E^*$ then let $b=l^*$ and we get $R(b)=R(l^*)=l\leqq R(w)$ and $f(l^*)=(f(l))^*=(R(v))^*=p^*=v$. Obviously then $l\in E$. If $v=qf^np$ put $b=lf^np$. Then $R(b)=R(lf^np)=l\leqq R(w)$ and $f(b)=f(lf^np)=f(l)f^np=R(v)f^np=qf^np$. Clearly $q\neq f^n(p)$ because $\varrho(q)>0$ and $\varrho(f^n(p))=0$. Furthermore $\varrho(l)=\varrho(q)+1>0$. If $v=f^nq$ and $l\in N$ then put $b=lf^nq$. If $v=f^nq$ and $l\in C$ put $b=f^mq$ with $m\equiv n-1 \mod(\pi(M))$ and $0\leqq m<\pi(M)$. Of course l then is not in E and to check in these last two cases that b satisfies the requirements is straightforward. □

LEMMA 3.7 *The representation of* M *on* $W=W(\mathscr{R})$ *is faithful if any one of the following conditions is satisfied:*

(i) *There is some* $n\in\omega$ *with* $f^n(1)=0, \pi(M)=1$ *and moreover for each* $m\leqq\varrho(M)$ *there exists* $p\in L_k$ *with* $\varrho(p)\geqq m$.

(ii) *There is no* $n\in\omega$ *with* $f^n(1)=0$, $\varrho(M)$ *is finite, some* $p\in L_k$ *has* $\varrho(p)=$ $=\varrho(M)$ *and for all* $q\in L_k$ *we have* $\pi(q)|\pi(M)$.

(iii) *There is no* $n\in\omega$ *with* $f^n(1)=0$, *and* $\pi(M)$ *is infinite.*

PROOF. (i) Note in this case $W=E^*$ and $E=L_k$. Now if f^n and f^m are distinct elements of M, say $n<m$, choose $p\in L_k$ with $\varrho(p)=m$. Clearly $f^n(p^*)\neq f^m(p^*)$.

(ii) There is $p\in L_k$ with $\varrho(p)=\varrho(M)$, and so there is $w\in W$ (e.g. $w=pf^0q$ with $q=f^{\varrho(p)}(p)$) with $\varrho(w)=\varrho(p)=\varrho(M)$. Furthermore there is $v\in W$ with $\pi(v)=$ $=\pi(M)$; (for $v=f^0q$ with $q\in C(\mathscr{R})$).

(iii) There is $w\in W$ such that $w\notin E^*$; hence by (ii) of Lemma 3.1 $f^n(w)=$ $=f^m(w)\Leftrightarrow n=m$. Thus M is faithfully represented on W. □

COROLLARY 3.8 *If any of the conditions of Lemma* 3.7 *hold, then* $\langle W; M, L\rangle$ *is a representation structure of type* M, L.

LEMMA 3.9. *If any of the conditions of Lemma* 3.7 *hold, then the representation of* M *on* W *is locally closed.*

PROOF. If $\varrho(M)<\infty$ then M is finite. Every representation of a finite monoid is locally closed. So assume that $\varrho(M)=\infty$ and that $h\in$l.c.M. Then there exists $w\in W$ with $f^n(w)=f^m(w)\Leftrightarrow n=m$. Now for some r, $h(w)=f^r(w)$. Also for all $v\in W$, h agrees with some element of M on the set $\{w, v\}$. Because $f^n(w)=f^m(w)\Leftrightarrow$ $\Leftrightarrow n=m$, h agrees with f^r on $\{w, v\}$. This means that h agrees with f^r on all of W. □

§ 4. Characterising algebraic M-spaces for monoids with a single generator

LEMMA 4.1. *If* $M=[f]$ *is a singly generated monoid and* L *an algebraic lattice, and the* M*-space* $\mathscr{R}=\langle L; 0, 1, \vee, M\rangle$ *satisfies one of the following:*

(i) *There is some* $n\in\omega$ *with* $f^n(1)=0$, $\pi(M)=1$ *and moreover there is some* $p\in L_k$ *with* $\varrho(p)=\varrho(M)$.

(ii) *There is no* $n\in\omega$ *with* $f^n(1)=0$, $\varrho(M)$ *is finite, some* $p\in L_k$ *has* $\varrho(p)=$ $=\varrho(M)$.

(iii) *There is no* $n\in\omega$ *with* $f^n(1)=0$, *and* $\pi(M)$ *is infinite.*

Then $\mathscr{B}=\langle W(\mathscr{R}); M, L\rangle$ *is a representation structure in which* M *is locally closed and the mapping condition holds. Moreover compact elements of* L *are singly generated in* $\mathscr{B}$.

PROOF. Assume that $\mathscr{R}=\langle L; 0, 1, \vee, M\rangle$ is an M-space over L satisfying either (i), (ii), or (iii) and that M is generated by the element f. In view of § 3, we have only to prove that the 'mapping condition' holds. Let $r=0$, observe that $f(0^*)=0^*$. We assume now that $r\geqq 1$, $\underline{x}\in W^r$, $y\in W$ with $R(\underline{x})\geqq R(y)$ and recursively construct a morphism ε from $\mathscr{B}^r\to\mathscr{B}$ (i.e. an operation of $\mathscr{B}^*$) with $\varepsilon(\underline{x})=y$. Let $X_0=\{f^n(x); n\in\omega\}$; then ε_0 is defined on X_0 as follows:

$$\varepsilon_0(f^n(\underline{x}))=f^n(y).$$

Because of Lemma 3.5 ε_0 is a well defined function from X_0 into W. Furthermore ε_0 maps 'down' in rank, e.g., $R(\varepsilon_0 f^n(\underline{x}))=R(f^n(y))=f^nR(y)\leqq f^nR(\underline{x})=$ $=Rf^n(\underline{x})$. Assume now that ε_i: $X_i\to W$ is already defined and that $X_k\cap X_j=\emptyset$ for $k\neq j$ and $0\leqq k, j\leqq i$, and that ε_i maps down in rank. Then for each $w\in W^r$ such that $w\notin X_i$ and $f(w)\in X_i$ choose (using Lemma 3.6) an element $h_w\in W$ with $R(h_w)\leqq R(w)$ and $f(h_w)=\varepsilon_i f(w)$. Let $X_{i+1}=\{w\in W^r;$ with $w\notin X_i$ and $f(w)\in X_i\}$ and put $\varepsilon_{i+1}(w)=h_w$. Obviously for all $0\leqq j\leqq i$ $(X_{i+1}\cap X_j=\emptyset)$ and ε_{i+1} maps down in rank.

Let $Y=\bigcup_{i=0}^{\infty} X_i$ and $\varepsilon|Y=\bigcup_{i=0}^{\infty}\varepsilon_i$, and let $\varepsilon(w)=w_0$ (the first coordinate) if $w\notin Y$. We claim now that ε is a morphism from $\mathscr{B}^r\to\mathscr{B}$. First observe that ε is a morphism as far as the lattice representation is concerned because ε maps down in rank. So we have to check only that for all $w\in W^r$, $\varepsilon f(w)=f\varepsilon(w)$. If $w\notin Y$, then $f(w)\notin Y$ and $\varepsilon f(w)=(f(w))_0=f(w_0)=f\varepsilon(w)$. If $w\in X_i$ with $i\geqq 1$, then $f(w)\in X_{i-1}$ and $\varepsilon(w)=\varepsilon_i(w)=h_w$ with $f(h_w)=\varepsilon f(w)$ and so $\varepsilon f(w)=f(h_w)=f\varepsilon(w)$. If $w\in X_0$,

then $w=f^n(x)$ for some $n\in\omega$ and then $\varepsilon f(w)=\varepsilon f(f^n(x))=\varepsilon f^{n+1}(x)=f^{n+1}(y)=$ $=f(f^n(y))=f\varepsilon f^n(x)=f\varepsilon(w)$. Thus ε is a morphism from $\mathscr{B}'$ into $\mathscr{B}$. □

Note $f\in M$ maps subsets of W to subsets of W in a natural way. In fact by Lemma 3.6 we see that f maps subsets which represent lattice elements to other such subsets. Thus we have an action of M on L induced by $\langle W(\mathscr{R}); M, L\rangle$. Observe with the aid of Lemma 3.4 that this induced action of M on L is just the same as the action of M on L in the M-space $\mathscr{R}$.

THEOREM 4.2. *An algebraic lattice L and a singly generated monoid M are isomorphic to* Su $\underline{A}$ *and* End $\underline{A}$ *for some algebra $\underline{A}$ iff there exists a join continuous map $g\colon L\to L$ satisfying either:*

(A) *M is free and $g^n(1)\neq 0$ for any $n\in\omega$, or*

(B) *$M\cong[g]$, or*

(C) *M has the same preperiod as $[g]$, the period of $[g]$ divides the period of M, and for some $p\in L_k$, $g(p)=p\neq 0$.*

Moreover, for any such g, the algebra $\underline{A}$ can be chosen so that the induced action of $f\in$ End $\underline{A}=[f]$ on $L=$ Su $\underline{A}$ is the same as that of g on L.

PROOF. Let $\underline{A}=\langle A, P\rangle$ be an algebra with $M=[f]=$ End $\underline{A}$ and $L=$ Su $\underline{A}$. If M is free, then clearly (A) holds with g the identity map.

If M is not free then M is finite and we have to show that either (B) or (C) holds. We first argue that $\varrho(a)=\varrho([a])$ for all $a\in A$. Here $[a]\in$ Su $\underline{A}$ is the subalgebra generated by a. In order to do this we observe that for $a\in A$, $\varrho[a]=0$ implies that $\varrho(a)=0$. Let to the contrary $\varrho(a)=n$ with $n\geqq 1$; then $\varrho(f^n(a))=0$ and therefore for all x in $[f^n(a)]$, $\varrho(x)=0$. Because $\varrho[a]=0$, there exists $m\in\omega$ with $f^m(f^n[a])=[a]$, and therefore there is $x\in[f^n(a)]$ with $f^m(x)=a$. But then $\varrho(a)\leqq$ $\leqq\varrho(x)=0$, so $\varrho(a)=0$. If $\varrho([a])=n$ for some $n\geqq 1$, then $\varrho(f^n([a]))=0$ and hence $\varrho(f^n(a))=0$, so $\varrho(a)\leqq\varrho[a]$. On the other hand, if $\varrho(a)=0$, then there is $n\in\omega$ with $f^n(a)=a$ and so $f^n[a]=[f^n(a)]=[a]$, which means that $\varrho([a])=0$. If $\varrho[a]=n$, then $\varrho(f^n[a])=0$ and $0=\varrho[f^n(a)]=\varrho(f^n([a]))$, which means that $\varrho(a)\leqq\varrho([a])$. Hence $\varrho(a)=\varrho([a])$ for all $a\in A$.

Now let $g\colon L\to L$ be the map induced by f on $L=$ Su $\underline{A}$ and observe that $[g]$ is a homomorphic image of $[f]$, hence $\pi[g]|\pi[f]$ and $\varrho[g]\leqq\varrho[f]$. Because $\varrho(a)=$ $=\varrho[a]$ for all $a\in A$ we have in fact $\varrho[g]=\varrho[f]$. Recall that f is fixed on the minimal subalgebra. If $p=f^{\varrho(f)}(A)$ is the minimal subalgebra, then $\pi[f]=1$, hence $\pi[g]=1$ and $[g]\cong[f]$. If p is not the minimal subalgebra, then $g(p)=p\neq 0$, and (C) holds.

Conversely assume that $g\colon L\to L$ is a join continuous map, and that (A), (B) or (C) hold. To show the existence of an appropriate algebra $\underline{A}$, it will suffice to see that there is an M-space $\mathscr{R}$ which satisfies (i), (ii) or (iii) of Lemma 4.1, and apply Theorem 1.1. Let the action of M on L be given by $f(p)=g(p)$ for $p\in L$. Clearly $\mathscr{R}=\langle L; 0, 1, \vee, M\rangle$ is an M-space because in each of the three cases (A), (B), and (C) we have that $[g]$ is a homomorphic image of M. If (A) holds, then M is free and $f^n(1)\neq 0$ for any $n\in\omega$ (since $g^n(1)\neq 0$); hence 4.1 (iii) holds. If (B) holds but (A) does not hold, then either: $g^n(1)=0$ and $f^n(1)=0$ and 4.1 (i) holds, or $g^n(1)\neq 0$ for any $n\in\omega$ and M is not free and 4.1 (ii) holds. If (C) holds but (A) does not hold then $g^n(1)\neq 0$ (since $g(p)=p\neq 0$) so $f^n(1)\neq 0$ for any $n\in\omega$. Since (A) does not hold we have $\varrho(M)$ finite. For some $q\in L_k$, $\varrho(q)=\varrho([g])=\varrho(M)$.

Moreover $[g]$ is a homomorphic image of M hence for all $q \in L_k$ we have $\pi(q)|\pi[g]$ and $\pi[g]|\pi(M)$, so 4.1 (ii) holds.

Clearly the induced action of f is the same as that for g by virtue of Corollary 1.3. □

REMARK. Theorem 4.2 is a stronger version of the Theorem stated in the Introduction.

§ 5. Applications and examples

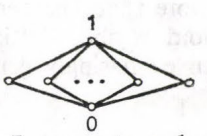
COROLLARY 5.1. *If L is an algebraic lattice and $f\colon L \to L$ is a join continuous map, then the monoid M generated by f is algebraic with L. Furthermore there exists a universal algebra $\underline{A} = \langle A, \mathscr{P} \rangle$ with $L \cong \mathrm{Su}\,\underline{A}$ and $M \cong \mathrm{End}\,\underline{A}$ such that f acts on $\mathrm{Su}\,\underline{A}$ in just the same way as f acts on L, and each finitely generated subalgebra of $\underline{A}$ is singly generated.*

PROOF. An immediate application of Theorem 4.2 with $M = [f]$. □

COROLLARY 5.2. *If L is an algebraic lattice with at least one atom p then L is algebraic with all singly generated monoids M with preperiod* 1.

PROOF. The transformation $f\colon L \to L$ with $f(x) = p$ for all $x \neq 0$ and $f(0) = 0$ is a join continuous map of L. Apply Theorem 4.2. □

We show how to apply Theorem 4.2 to obtain information about monoids algebraic with M_n's.

EXAMPLE 5.3. Let $L = M_n$: 

a_1 a_n, for some $n \geqq 2$.

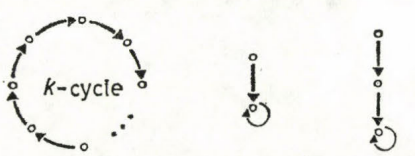
Note the join continuous maps $g\colon L \to L$ generate only the following monoids $[g]$ where g^0 denotes the identity map: $[g] = G$, a cyclic subgroup of S_n; or;

$[g]$	g^0	g
g^0	g^0	g
g	g	g

or

$[g]$	g^0	g	g^2
g^0	g^0	g	g^2
g	g	g^2	g^2
g^2	g^2	g^2	g^2

The Cayley "diagrams" below display the possible preperiods and periods for $[g]$.

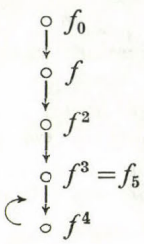


$\pi[g] = k, \varrho[g] = 0$ $\pi[g] = 1, \varrho[g] = 1$ $\pi[g] = 1, \varrho[g] = 2$

Using Theorem 4.2 and consolidating the results for all three cases: a singly generated monoid $[f]$ is algebraic with some M_n iff $[f]$ is free or $[f]$ has $\varrho \leqq 1$, or $\pi = 1$ and $\varrho = 2$.

Theorem 4.2 is equally useful for discovering algebraic incompatibility of singly generated monoids with particular lattices:

EXAMPLE 5.4. A monoid $M=[f]$ with $\varrho(M)=3$ and $\pi(M)=2$ has Cayley diagram:



This monoid M is not algebraic with the non-modular lattice $L=N_5$:

Note for $g: L \to L$, join continuous with $\varrho(g)=3$, we necessarily have $g^3(1)=0$. Since g is order preserving, $g^3(x)=0$ for any x. Hence neither (A) nor (C) of Theorem 4.2 hold. Moreover $\pi(g)=1 \neq \pi(M)$, hence (B) does not hold, and we conclude M is not algebraic with N_5 since there is no appropriate join continuous map.

As a final remark we note that Theorem 4.2 provides a strong necessary condition for an arbitrary monoid M and a lattice L to be algebraic since such a lattice must also be algebraic with each singly generated submonoid $[f] \subset M$ (cf. Corollary 4 of [2]).

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ОВ ОДНОМ КЛАССЕ КОЛЕЦ С СУЩЕСТВЕННЫМ ПРАВЫМ ЦОКОЛЕМ

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1. Введение

Пусть M — правый модуль над кольцом A, N — неприводимый правый A-модуль. Обозначим через $\mathrm{Soc}(M)$ сумму всех неприводимых подмодулей модуля M, через $\mathrm{Soc}(M; N)$ — сумму всех неприводимых подмодулей, изоморфных модулю N. Напомним, что $\mathrm{Soc}(M; N)$ называется однородной компонентой цоколя $\mathrm{Soc}(M)$ модуля M. Модуль M_A называется конечно копорожденным, если для любого семейства подмодулей $\{M_i\}_{i\in I}$ модуля M такого, что $\bigcap\{M_i | i\in I\}=0$, существует конечное подмножество $F\subseteq I$, такое, что $\bigcap\{M_i | i\in F\}=0$. (Каш [6], определение 2.3.14.) Конечно копорожденные модули (возможно под другим названием) изучались в Ahsan [1], Beachy [2], Kamil [5], Onodera [8, 9, 10, 11], Smith [12], Vámos [14], Yahya [15, 16]. Отметим следующее утверждение.

Предложение 1. (Фейс [3], предложение 26.1B). *Следующие свойства правого A-модуля M эквивалентны:*

1. *M — конечно копорожденный модуль;*
2. *$\mathrm{Soc}(M)$ — существенный артинов подмодуль модуля M.*

Напомним, что абелева группа M называется конечно копорожденной, если существует такое конечное подмножество $0\notin L\subseteq M$, что $L\cap N\neq\emptyset$ для любой ненулевой подгруппы $N\subseteq M$. Конечно копорожденные абелевы группы характеризуются следующей теоремой.

Теорема 1. (Курош [7], Yahya [15]). *Для любой абелевой группы M эквивалентны следующие условия:*

1. *M — конечно копорожденная группа;*
2. *M — существенное расширение конечной группы;*
3. *M — прямая сумма конечного числа коциклических групп;*
4. *подгруппы группы M удовлетворяют условию минимальности.*

Так как неприводимыми Z-модулями являются только циклические группы простого порядка, то абелева группа конечно копорождена тогда и только тогда когда она конечно копорождена как Z-модуль (где Z — кольцо целых чисел). По аналогии с соответствующим понятием для абелевых групп, подмножество $0\notin L\subseteq M$ правого A-модуля M называется системой кообразующих, если $L\cap N\neq\emptyset$ для любого ненулевого подмодуля $N\subseteq M$. Из теоремы 1 вытекает, что Z-модуль M конечно копорожден тогда и только тогда, когда он обладает конечной системой кообразующих. В книге F. Szász [13] поставил вопрос о характеризации колец A таких, что правый A-модуль A обладает ко-

нечной системой кообразующих (Szász [13], Problem 94). В данной работе охарактеризованы правые модули, обладающие конечной системой кообразующих и показано, что в общем случае конечно копорожденный модуль может не иметь конечной системы кообразующих.

2. Модули с конечной системой кообразующих

Обозначим через $|S|$ мощность множества S, через $M_n(A)$ кольцо $n\times n$-матриц над кольцом A.

Лемма 1. *Пусть M — правый модуль над кольцом A, обладающий конечной системой кообразующих. Тогда M — конечно копорожденный A-модуль.*

Доказательство. Пусть L — конечная система кообразующих модуля M, $\{M_i\}_{i\in I}$ — такое семейство подмодулей модуля M, что $\bigcap\{M_i|i\in I\}=0$. Тогда для элемента $l\in L$ существует индекс $i(l)\in I$ такой, что $l\notin M_{i(l)}$. Положим $N=\bigcap\{M_{i(l)}|l\in L\}$. Тогда $N\cap L=\emptyset$. Так как L — система кообразующих, то $N=0$ и лемма доказана.

Теорема 2. *Следующие свойства правого модуля M над кольцом A эквивалентны;*

1. *модуль M обладает конечной системой кообразующих;*
2. *выполнены следующие условия:*
   а) *цоколь* $\mathrm{Soc}(M)$ *является существенным артиновым подмодулем модуля M;*
   б) *множество неприводимых подмодулей модуля M конечно;*
3. *выполнены следующие условия:*
   а) *цоколь* $\mathrm{Soc}(M)$ *является существенным артиновым подмодулем модуля M;*
   б) *если N — неприводимый подмодуль модуля M и* $|\mathrm{End}(N_A)|=\infty$ *то* $\mathrm{Soc}(M,N)=N$.

Доказательство. Пусть выполнено условие 1. Тогда из леммы 1 и предложения 1 следует, что $\mathrm{Soc}(M)$ — существенный артинов подмодуль модуля M. Пусть L — конечная система кообразующих модуля M, N_1 и N_2 — неприводимые подмодули модуля M. Тогда, очевидно, равенство $L\cap N_1=L\cap N_2$ влечет равенство $N_1=N_2$. Поэтому множество неприводимых подмодулей модуля M конечно. Следовательно выполнено условие 2.

Пусть выполнено условие 2, N — неприводимый подмодуль A-модуля M и $|\mathrm{End}(N_A)|=\infty$. Предположим, что $\mathrm{Soc}(M;N)\neq N$. Тогда существует неприводимый подмодуль K модуля M, изоморфный подмодулю N такой, что $K\neq N$. Ясно, что $K\cap N=0$ и $K+N$ — прямая сумма подмодулей. Пусть $f\colon N\to K$ — изоморфизм правых A-модулей и $D=\mathrm{End}(N_A)$. Зафиксируем ненулевой элемент $a\in N$. Для каждого $u\in D$ обозначим через N_u подмодуль модуля M, порожденный элементом $a+uf(a)$. Ясно, что $N_u\subseteq K+N\subseteq\mathrm{Soc}(M)$. Покажем, что для $v\in D$, $v\neq u$ выполнено соотношение $N_u\cap N_v=0$. Пред-

положим противное. Тогда существуют такие целые числа m, n и элементы $x, y \in A$, что выполнены соотношения:

$$(*) \quad (a+uf(a))m+(a+uf(a))x = (a+vf(a))n+(a+vf(a))y$$

$$(**) \quad (a+uf(a))m+(a+uf(a))x \neq 0.$$

Так как $K+N$ — прямая сумма подмодулей, то из соотношения $(*)$ следует, что

$$(***) \quad am+ax = an+ay$$

$$(****) \quad uf(a)m+uf(a)x = vf(a)n+vf(a)y.$$

Из соотношения $(***)$ вытекает равенство $f(am+ax)=f(an+ay)$. Так как f — изоморфизм правых A-модулей, то $f(a)m+f(a)x=f(a)n+f(a)y$. Из соотношения $(****)$ теперь следует, что $(u-v)(f(a)m+f(a)x)=0$. Так как $u \neq v$, то $u-v$ ненулевой элемент кольца D. Далее, N — неприводимый A-модуль. Значит по лемме Шура, D — тело. Поэтому $f(a)m+f(a)x=0$. Но $f(a)m+f(a)x=f(am+ax)$ и f — изоморизм A-модулей. Значит, $am+ax=0$ и $f(a)m+f(a)x=0$. Получили противоречие с соотношением $(**)$. Итак, $N_u \cap N_v=0$ для всех $u \neq v \in D$. Далее $N_u \subseteqq \mathrm{Soc}(M)$ для всех $u \in D$. Следовательно модуль M обладает бесконечным числом неприводимых подмодулей. Получили противоречие с условием 2.

Пусть выполнено условие 3. Покажем, что тогда выполнено условие 2. Так как каждый неприводимый подмодуль модуля M содержится в какой либо однородной компоненте цоколя модуля M и цоколь $\mathrm{Soc}(M)$ является прямой суммой своих однородных компонент (Каш [6] лемма 8.1.8), то достаточно показать, что модуль $\mathrm{Soc}(M; N)$ содержит конечное число подмодулей для каждого неприводимого подмодуля $N \subseteqq M$. Из условия следует, что подмодуль $\mathrm{Soc}(M; N)$ является прямой суммой конечного числа неприводимых подмодулей $L_1, L_2, \ldots, L_n$ изоморфных подмодулю N. Если $|\mathrm{End}(N_A)|=\infty$, то $n=1$ и $\mathrm{Soc}(M; N)=N$. Поэтому достаточно рассмотреть тот случай, когда $|\mathrm{End}(N_A)|<\infty$. Пусть $D=\mathrm{End}(N_A)$. Тогда кольцо $\mathrm{End}(\mathrm{Soc}\, M; N))=\mathrm{End}((L_1+L_2+L_3+\ldots+L_n)_A)$ изоморфно кольцу $M_n(D)$. Значит, $|\mathrm{End}(\mathrm{Soc}(M; N)_A)|<\infty$. Пусть K — подмодуль модуля $\mathrm{Soc}(M; N)$. Тогда подмодуль K выделяется прямым слагаемым в $\mathrm{Soc}(M; N)$, (Каш [6], теорема 8.1.3). Обозначим через e_k проекцию на прямое слагаемое K модуля $\mathrm{Soc}(M; N)$. Тогда $e_k \in \mathrm{End}(\mathrm{Soc}(M; N))$ и $K=e_k \mathrm{Soc}(M; N)$. Наше утверждение теперь следует из соотношения $|\mathrm{End}(\mathrm{Soc}(M; N)_A)|<\infty$. Итак, из условия 3 вытекает условие 2.

Пусть выполнено условие 2 и $\{N_i | i=1, 2, 3, \ldots, m\}$ — множество всех неприводимых подмодулей модуля M. Выберем из каждого модуля N_i ненулевой элемент n_i и положим $L=\{n_1, n_2, \ldots, n_m\}$. Далее, пусть N — ненулевой подмодуль модуля M. Так как $\mathrm{Soc}(M)$ — существенный подмодуль модуля M, то $N_i \subseteqq N$ для некоторого $1 \leqq i \leqq m$. Значит $N \cap L \ni n_i$ и модуль M конечно копорожден.

Отметим, что теорема 2 дает решение упомянутой проблемы F. Szász. Далее, Ginn [4] построил пример примитивной слева и справа области главных левых идеалов A и неартиного левого A-модуля с существенным точным неприводимым цоколем. Тем самым показано, что конечно копорожденный модуль

с конечной системой кообразующих может не быть артиновым (сравнить с условием 4) из теорема 1. В связи с условием 3 из теоремы 1 ниже приводится пример кольца A и конечно копорожденного модуля M с конечной системой кообразующих не являщегося прямой суммой модулей с существенным неприводимым цоколем.

Пример 1. Пусть $F[X, Y]$ — кольцо многочленов над полем F, I — идеал кольца $F[X, Y]$ порожденный X^2, XY, Y^2, $A=F[X, Y]/I$, $x=X+I$, $y=Y+I$. Очевидно, что xF и yF — минимальный правые идеалы кольца A, $\mathrm{Soc}(A)=xF+yF$ — существенный подмодуль правого A-модуля A, A — локальное кольцо с нильпотентным радикалом Джекобсона $J(A)=xF+yF$. Далее, $xF\neq yF$. Поэтому $\mathrm{Soc}(A)$ не является неприводимым правым A-модулем. С другой стороны, $\mathrm{Soc}(A)$ — существенный артинов подмодуль модуля A. Следодовательно A-модуль A конечно копорожден. Предположим, что $A=L_1+L_2+\ldots+L_n$ — прямая сумма ненулевых правых A-модулей с неприводимым существенным цоколем. Так как $xF\neq yF$, то $n\geqq 2$. Ясно, что $1=a_1+a_2+\ldots+a_n$ для некоторых $a_i\in L_i$, $1\leqq i\leqq n$. Далее, так как A — коммутативное кольцо и $L_i\cap L_j=0$ то $L_iL_j=0$ для всех $i\neq j$. Значит, $1a_i=a_1a_i+a_2a_i+\ldots+a_na_i=a_i^2$ и a_i идемпотент кольца A. Но A — локальное кольцо. Поэтому либо $a_i=0$, либо $a_i=1$. Значит, найдется номер j такой, что $a_j=1$, $a_i=0$ для всех $i\neq j$. Но тогда $L_j=A$ и $L_i=0$ для всех $i\neq j$. Получили противоречие с соотношениями $n\geqq 2$, $L_i=0$ для всех $1\leqq i\leqq n$. Итак, правый A-модуль A не является прямой суммой подмодулей с существенным неприводимым цоколем.

Ниже будет приведен пример конечно копорожденного кольца не имеющего конечной системы кообразующих.

Пример 2. Пусть F — бесконечное поле, $A=M_2(F)$ — кольцо 2×2-матриц над полем F и e_{11} — матрица у которой 1 на $(1, 1)$-месте и нули на всех остальных. Ясно, что A — конечно копорожденное кольцо. Далее, очевидно, что $e_{11}A$ — неприводимый правый A-модуль и $\mathrm{End}(e_{11}A_A)=F$. Так как $|F|=\infty$ и $\mathrm{Soc}(A; e_{11}A)=A$, то из теоремы 2 следует, что правый A-модуль A не имеет конечной системы кообразующих.

Следствие 1. *Следующие свойства кольца A эквивалентны:*

1. *каждый конечно копорожденный правый A-модуль обладает конечной системой кообразующих;*

2. *для каждого неприводимого правого A модуля M выполнено соотношение* $|\mathrm{End}(M_A)|<\infty$.

Доказательство. Ввиду теоремы 2 достаточно доказать, что из условия 1 вытекает условие 2. Пусть M — неприводимый правый A-модуль и $N=M\oplus M$. Ясно, что N — конечно копорожденный правый A-модуль. Поэтому модуль N обладает конечной системой кообразующих. Так как $\mathrm{Soc}(N; M)=N\neq M$, то из теоремы 2 следует, что $|\mathrm{End}(M_A)|<\infty$.

Следствие 2. *Следующие свойства коммутативного кольца A с единицей эквивалентны:*

1. *Каждый конечно копорожденный правый A-модуль обладает конечной системой кообразующих;*

2. *для каждого максимального идеала M кольца A выполнено соотношение* $|A/M|<\infty$.

Доказательство. Пусть выполнено условие 1 и M — максимальный идеал кольца A. Положим $\bar{A}=A/M$. Тогда $\bar{A}$ — поле и $\bar{A}$ — неприводимый правый A-модуль и $\mathrm{End}\,(\bar{A}_A)=\bar{A}$. Значит, $|\bar{A}|<\infty$ (см. следствие 2). Итак, из 1 следует 2.

Пусть выполнено условие 2 и N — неприводимый правый A-модуль. Положим $M=\{a\in A|Na=0\}$. Тогда M — максимальный идеал кольца A, правые A-модули A/M и N изоморфны и кольцо $\mathrm{End}\,(N_A)$ изоморфно кольцу $\mathrm{End}\,((A/M)_A)\cong A/M$. Так как $|A/M|<\infty$, то $|\mathrm{End}\,(N_A)|<\infty$. Из следствия 1 теперь вытекает, что выполнено условие 1.

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CHARACTERIZATIONS OF GENERALIZED NOETHERIAN RINGS

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1. Introduction. In Proposition 4 of [4], several characterizations were given for a left $\aleph_\beta$-noetherian ring with several objects. However, a number of characterizations of the left noetherian condition from the familiar ring with identity setting were not extended to this more general situation. For example, a ring is left noetherian if and only if it satisfies either the ascending chain condition or the maximum condition on left ideals. In this paper we look at versions of such conditions in the several objects setting for arbitrary β. In particular, $\aleph_\beta$-chain and $\aleph_\beta$-maximum conditions are introduced to generalize the usual ascending chain and maximum conditions (see [5]). Also, the notion of $\aleph_\beta$-coproduct is introduced to provide the setting for the generalization of the condition that a ring is left noetherian if and only if every direct sum of injective modules is injective ([1], [2], [5]). This generalized coproduct is also used to generalize results of C. Faith and E. Walker [3] and Z. Papp [7] concerning the decomposition of injective modules over left noetherian rings. Finally, examples are presented which investigate the commutativity of the injective hull and $\aleph_\beta$-coproduct functors in the setting of a left $\aleph_\beta$-noetherian ring.

2. $\aleph_\beta$-chain and maximum conditions in $\mathscr{A}\ell^{\mathscr{C}}$. In this section we shall relate the conditions given in Proposition 4 of [4] for a ring $\mathscr{C}$ to be left $\aleph_\beta$-noetherian to generalized notions of chain and maximum conditions on "left ideals" of $\mathscr{C}$. Recall that $M\in|\mathscr{A}\ell^{\mathscr{C}}|$ is $\aleph_\beta$-noetherian if and only if every subobject of M is $\aleph_\beta$-generated. We shall begin by generalizing to the several objects setting some results of A. Simis [10].

LEMMA 2.1. *If $M\in|\mathscr{A}\ell^{\mathscr{C}}|$ is strictly $\aleph_\beta$-generated, then for every $\alpha<\beta$ there exists a well-ordered strictly increasing chain*

$$M_0 \overset{o_0^1}{\subset\!\!\longrightarrow} M_1 \overset{o_1^2}{\subset\!\!\longrightarrow} \ldots \subset\!\!\longrightarrow M_\lambda \overset{o_\lambda^\mu}{\subset\!\!\longrightarrow} M_\mu \subset\!\!\longrightarrow$$

of subobjects of M indexed by Ω_α such that each M_λ is $\aleph_\alpha$-generated.

PROOF. We may clearly assume that $\beta\geqq 1$ and $\alpha\geqq 0$. If $\bigoplus_\Lambda \mathscr{C}(p_\lambda, -)\xrightarrow{\eta} M$ is an epimorphism with $|\Lambda|=\aleph_\beta$, take $\lambda_0\in\Lambda$ and define $M_0=\operatorname{Im}(\mathscr{C}(p_{\lambda_0}, -)\rightarrowtail \bigoplus_\Lambda \mathscr{C}(p_{\lambda_0}, -)\xrightarrow{\eta} M)$. Now let $\nu\in\Omega_\alpha$ and suppose that for every $\mu<\nu$, $M_\mu=\operatorname{Im}\left(\bigoplus_{\gamma\leqq\mu} \mathscr{C}(p_{\lambda_\gamma}, -)\longrightarrow M\right)$ has been defined such that whenever $\mu<\mu'$, then $\{\lambda_\gamma|\gamma\leqq\mu\}\subseteq\{\lambda_\gamma\,|\gamma\leqq\mu'\}\subseteq\Lambda$ and $M_\mu\subset\!\!\longrightarrow M_{\mu'}$ properly.

If $\{\sigma_\lambda|\lambda\in\Lambda\}$ is the generating set for M, then there is $\lambda_\nu\in\Lambda$ such that $\sigma_{\lambda_\nu}\notin\bigcup_{\mu<\nu}M_\mu(p_{\lambda_\nu})$. Putting $M_\nu=\mathrm{Im}\left(\bigoplus_{\gamma\leq\nu}\mathscr{C}(p_{\lambda_\gamma},-)\to M\right)$, we see that M_ν is $\aleph_\alpha$-generated, and for every $\mu<\nu$ we have $M_\mu\subset\to M_\nu$ properly.

REMARK. If Ω_α is a regular ordinal, then since for every $\nu\in\Omega_\alpha$ we have $|\nu|<\aleph_\alpha$, the subobjects M_ν of M of Lemma 2.1 are generated by less than $\aleph_\alpha$ elements.

The following example illustrates the construction of the previous lemma.

EXAMPLE. Let S be a ring and let $R=S[\{x_\lambda\}_{\lambda\in\Omega_\beta}]$. If I is the left ideal of R generated by $\{x_\lambda\}_{\lambda\in\Omega_\beta}$, then I is strictly $\aleph_\beta$-generated. If $\alpha<\beta$ then for each $\nu\in\Omega_\alpha$ the construction of Lemma 2.1 yields M_ν equal to the left ideal of R generated by $\{x_\lambda|\lambda\leqq\nu\}$.

If M is a left R-module with no strictly $\aleph_\beta$-generated submodules and if $\aleph_\beta$ is regular, then Simis [10] has shown that, in fact, every submodule of M can be generated by less than $\aleph_\beta$ elements. We now generalize this result to the ring with several ovjects setting.

LEMMA 2.2. *If $M\in|\mathscr{A}\ell^{\mathscr{C}}|$ and if $\aleph_\beta$ is a regular cardinal such that M has no strictly $\aleph_\beta$-generated subobjects, then every subobject of M can be generated by fewer than $\aleph_\beta$ elements.*

PROOF. Suppose that N is a strictly $\aleph_\alpha$-generated subobject of M for some $\alpha>\beta$. By Lemma 2.1 there exists a well-ordered strictly increasing chain $\{N_\lambda,\varrho^\mu_\lambda\}_{\lambda\in\Omega_\beta}$ of subobjects of N such that each N_λ is $\aleph_\beta$-generated. But then $\bigcup_{\lambda\in\Omega_\beta}N_\lambda$ is generated by fewer than $\aleph_\beta$ elements, say by $\{\tau_\mu\in\bigcup_{\lambda\in\Omega_\beta}N_\lambda(p_\mu)|\mu\in\Omega\}$, where $|\Omega|<\aleph_\beta$. We let ν_μ be the least index for which $\tau_\mu\in N_{\nu_\mu}(p_\mu)$.

Now since Ω_β is regular and $|\Omega|<\aleph_\beta$, we have $\varrho=\sup\{\nu_\mu|\mu\in\Omega\}<\Omega_\beta$ and hence $\varrho+1<\Omega_\beta$. Since $\nu_\mu<\varrho+1$ for every $\mu\in\Omega$, $N_{\nu_\mu}\subset\to N_{\varrho+1}$ and hence $\bigcup_{\mu\in\Omega}N_{\nu_\mu}\subset\to N_{\varrho+1}$. Now $\varrho+2<\Omega_\beta$ and $N_{\varrho+2}\subset\to\bigcup_{\lambda\in\Omega_\beta}N_\lambda=\bigcup_{\mu\in\Omega}N_{\nu_\mu}\subset\to N_{\varrho+1}$ so that $N_{\varrho+1}=N_{\varrho+2}$, contrary to the fact that $\{N_\lambda,\varrho^\mu_\lambda\}_{\lambda\in\Omega_\beta}$ is strictly increasing.

The following corollaries generalize results of Simis [10].

COROLLARY 1. *If every countably generated subobject of $M\in|\mathscr{A}\ell^{\mathscr{C}}|$ is finitely generated, then M is noetherian.*

COROLLARY 2. *If every countably generated subobject of $M\in|\mathscr{A}\ell^{\mathscr{C}}|$ is cyclic, then every subobject of M is cyclic.*

We recall that a left ideal of a ring R is just a subobject of the functor $R(1,-)\in|\mathscr{A}\ell^R|$. This notion is generalized to the several objects setting in the following definition.

DEFINITION. A *left ideal* of a ring $\mathscr{C}$ is a subobject of $\mathscr{C}(p,-)$ for some $p\in|\mathscr{C}|$.

COROLLARY 3. *If every countably generated left ideal of $\mathscr{C}$ is cyclic, then every left ideal of $\mathscr{C}$ is cyclic; that is, $\mathscr{C}$ is a principal ideal ring.*

Using the last two lemmas, we can now characterize a chain condition for $\mathscr{A}\ell^{\mathscr{C}}$ (see [5], [8]).

LEMMA 2.3. *For* $M\in|\mathscr{A}\mathscr{b}^{\mathscr{C}}|$, *the following are equivalent.*

1. *For every well-ordered increasing chain* $(M_\lambda)_{\lambda\in\Omega_{\beta+1}}$ *of subobjects of* M *there exists* $\varrho\in\Omega_{\beta+1}$ *such that* $M_\lambda=M_\varrho$ *for* $\lambda\geqq\varrho$.

2. *For every regular cardinal* $\aleph_\alpha$ *with* $\alpha>\beta$ *and for every well-ordered increasing chain* $(M_\lambda)_{\lambda\in\Omega_\alpha}$ *of subobjects of* M *there exists* $\varrho\in\Omega_\alpha$ *such that* $M_\lambda=M_\varrho$ *for* $\lambda\geqq\varrho$.

PROOF. That (2) implies (1) is immediate since $\beta+1>\beta$ and $\aleph_{\beta+1}$ is regular. To show that (1) implies (2), it suffices to consider the case $\alpha>\beta+1$. First we notice that by Lemma 2.1 M can have no strictly $\aleph_{\beta+2}$-generated subobjects. But then if $(M_\lambda)_{\lambda\in\Omega_\alpha}$ is a well-ordered increasing chain of subobjects of M, $\bigcup_{\lambda\in\Omega_\alpha} M_\lambda$ is $\aleph_{\beta+1}$-generated, say by $\{\tau_\mu|\mu\in\boldsymbol{\Omega}\}$. Each $\tau_\mu\in M_{\nu_\mu}(p_\mu)$ for some $\nu_\mu\in\boldsymbol{\Omega}_\alpha$, so $\varrho=\sup\{\nu_\mu|\mu\in\boldsymbol{\Omega}\}\in\boldsymbol{\Omega}_\alpha$. It follows that $M_\lambda=M_p$ whenever $\lambda\geqq\varrho$.

Notice that in the case $\beta=-1$, condition (1) asserts that M satisfies the usual ascending chain condition on subobjects. This observation leads us to the following generalization for arbitrary β.

DEFINITION. $M\in|\mathscr{A}\mathscr{b}^{\mathscr{C}}|$ satisfies the *ascending* $\aleph_\beta$*-chain condition on subobjects* if either of the conditions of Lemma 2.3 holds.

We recall that a left R-module M satisfies the maximum condition on submodules if and only if every nonempty set of submodules of M has a maximal element. It is well known that the conditions M is noetherian, M satisfies the ascending chain condition on submodules, and M satisfies the maximum condition on submodules are equivalent. It is natural to try to obtain a corresponding result for arbitrary β in $\mathscr{A}\mathscr{b}^{\mathscr{C}}$. First, however, we need to generalize the notion of maximum condition (see [5]).

DEFINITION. $M\in|\mathscr{A}\mathscr{b}^{\mathscr{C}}|$ satisfies the $\aleph_\beta$*-maximum condition on subobjects* if every nonempty set $\mathscr{S}$ of subobjects of M with the property that every well-ordered increasing chain $(M_\mu)_{\mu\in\boldsymbol{\Omega}}$, $|\boldsymbol{\Omega}|\leqq\aleph_\beta$, of elements of $\mathscr{S}$ has an upper bound in $\mathscr{S}$ has a maximal element.

We see that the $\aleph_{-1}$-maximum condition is just the usual maximum condition, since finite chains of elements of $\mathscr{S}$ automatically have upper bounds in $\mathscr{S}$. We now show that the $\aleph_\beta$-chain and maximum conditions are equivalent to the $\aleph_\beta$-noetherian condition.

LEMMA 2.4. *For* $M\in|\mathscr{A}\mathscr{b}^{\mathscr{C}}|$, *the following are equivalent.*

1. M *is* $\aleph_\beta$*-noetherian.*
2. *Every* $\aleph_{\beta+1}$*-generated subobject of* M *is* $\aleph_\beta$*-generated.*
3. M *satisfies the ascending* $\aleph_\beta$*-chain condition on subobjects.*
4. M *satisfies the* $\aleph_\beta$*-maximum condition on subobjects.*

PROOF. It is clear that (1) and (2) are equivalent. The remaining implications will be proved according to the following diagram:

$$1\to 3\to 4\to 1.$$

If M is $\aleph_\beta$-noetherian and if $(M_\lambda)_{\lambda\in\Omega_{\beta+1}}$ is a well-ordered increasing chain of subobjects of M, then $\bigcup_{\lambda\in\Omega_{\beta+1}} M_\lambda$ is $\aleph_\beta$-generated, say by $\{\tau_\mu|\mu\in\boldsymbol{\Omega}\}$, and each

$\tau_\mu \in M_{\nu_\mu}(p_\mu)$ for some $\nu_\mu \in \Omega_{\beta+1}$. Since $\varrho = \bigcup_{\mu\in\Omega} \nu_\mu \in \Omega_{\beta+1}$ and since $\bigcup_{\lambda\in\Omega_{\beta+1}} M_\lambda = \bigcup_{\mu\in\Omega} M_{\nu_\mu}$, we see that $M_\lambda = M_\varrho$ whenever $\lambda \cong \varrho$, so (3) holds.

Now assume (3) and suppose $\mathscr{S}$ is a nonempty set of subobjects of M with the property that every well-ordered increasing chain $(M_\mu)_{\mu\in\Omega}$, $|\Omega| \leqq \aleph_\beta$, of elements of $\mathscr{S}$ has an upper bound in $\mathscr{S}$. If $\mathscr{S}$ has no maximal element, then a transfinite induction argument yields a well-ordered strictly increasing chain $(N_\mu)_{\mu\in\Omega_{\beta+1}}$ of subobjects of M, contrary to (3). Thus (4) holds.

Finally, assume that (4) is true and suppose $N \rightarrowtail M$. If $\mathscr{S}$ consists of all $\aleph_\beta$-generated subobjects of N, then by (4) $\mathscr{S}$ contains a maximal element N_0. It is easy to see that $N_0 = N$, from which (1) follows.

The following definitions for a ring with several objects generalize two familiar ring concepts.

DEFINITION. 1. $\mathscr{C}$ satisfies the *ascending $\aleph_\beta$-chain condition on left ideals* if $\mathscr{C}(p, -)$ satisfies the ascending $\aleph_\beta$-chain condition on subobjects for every $p \in |\mathscr{C}|$.

2. $\mathscr{C}$ satisfies the *$\aleph_\beta$-maximum condition on left ideals* if $\mathscr{C}(p, -)$ satisfies the $\aleph_\beta$-maximum condition on subobjects for every $p \in |\mathscr{C}|$.

The above definitions, Proposition 4 of [4], and Lemma 2.4 are now combined to give the following expanded list of equivalent conditions under which $\mathscr{C}$ is left $\aleph_\beta$-noetherian. Note that in the case $\beta = -1$, we have the usual characterizations of a left noetherian ring.

PROPOSITION 2.5. *For a ring $\mathscr{C}$, the following are equivalent.*

1. *$\mathscr{C}(p, -)$ is $\aleph_\beta$-noetherian for every $p \in |\mathscr{C}|$.*
2. *Every $\aleph_\beta$-generated $M \in |\mathscr{A}\ell^{\mathscr{C}}|$ is $\aleph_\beta$-noetherian.*
3. *Every cyclic $M \in |\mathscr{A}\ell^{\mathscr{C}}|$ is $\aleph_\beta$-noetherian.*
4. *Every $\aleph_\beta$-generated $M \in |\mathscr{A}\ell^{\mathscr{C}}|$ is $\aleph_\beta$-presented.*
5. *Every $\aleph_\beta$-generated $M \in |\mathscr{A}\ell^{\mathscr{C}}|$ has an $\aleph_\beta$-generated free resolution.*
6. *$\mathscr{C}$ satisfies the ascending $\aleph_\beta$-chain condition on left ideals.*
7. *$\mathscr{C}$ satisfies the $\aleph_\beta$-maximum condition on left ideals.*

We conclude this section with some examples of genuinely left $\aleph_\beta$-noetherian rings.

EXAMPLES. 1. For $\beta \geqq 0$, we construct a genuinely left $\aleph_\beta$-noetherian ring. Let S be a ring with at most $\aleph_\beta$ elements and define R to be the polynomial ring $S[\{x_\lambda\}_{\lambda\in\Omega_\beta}]$. Since $|R| = \aleph_\beta$, R is left $\aleph_\beta$-noetherian. However, the left ideal of R generated by $\{x_\lambda\}_{\lambda\in\Omega_\beta}$ is not $\aleph_\beta$-generated for $\alpha < \beta$, so R is genuinely left $\aleph_\beta$-noetherian.

2. We exhibit a genuinely left $\aleph_1$-noetherian ring. Let R be the ring of continuous, real valued functions on $[0, 1]$. Using the continuum hypothesis, it is easy to see that $|R| = \aleph_1$ so that R is left $\aleph_1$-noetherian. Now consider the left ideal of R consisting of all $f \in R$ which vanish at some $x_0 \in [0, 1]$. K. Yale [11] has shown by an argument similar to that used in the proof of the Stone—Weierstrass theorem that I is not $\aleph_0$-generated. Therefore R is genuinely left $\aleph_1$-noetherian.

3. $\aleph_\beta$-coproducts in $\mathscr{A}\ell^{\mathscr{C}}$. It is well known [1], [2] that a ring is left noetherian if and only if every direct sum of injective left R-modules is injective, which is in turn equivalent to requiring that every direct limit of injective left R-modules is

injective. We generalize this result to the arbitrary β setting by replacing the notion of coproduct with a new structure, the $\aleph_\beta$-coproduct (see [5]).

DEFINITION. In $\mathscr{A}\ell^{\mathscr{C}}$, the $\aleph_\beta$-*coproduct* of $\{M_\lambda|\lambda\in\Lambda\}$, denoted by $\aleph_\beta-\bigoplus_\Lambda M_\lambda$, is the subobject of $\prod_\Lambda M_\lambda$ defined by

$$(\aleph_\beta-\bigoplus_\Lambda M_\lambda)(p)=\Big\{f\in\prod_\Lambda M_\lambda(p)\Big|\big|\{\lambda|f(\Lambda)\neq 0\}\big|\leqq\aleph_\beta\Big\}.$$

It is clear that $\aleph_{-1}-\bigoplus_\Lambda M_\lambda=\bigoplus_\Lambda M_\lambda$. Further, $\aleph_\beta-\bigoplus_\Lambda M_\lambda=\prod_\Lambda M_\lambda$ if and only if $|\Lambda|\leqq\aleph_\beta$. We recall that $\bigoplus_\Lambda M_\lambda=\varinjlim_{\mathscr{F}(\Lambda)}\bigoplus_S M_\lambda=\varinjlim_{\mathscr{F}(\Lambda)}\prod_S M_\lambda$, where $\mathscr{F}(\Lambda)$ is the collection of finite subsets of Λ. We now generalize this result to give a universal mapping property characterization for the $\aleph_\beta$-coproduct. In the following, $P_\beta(\Lambda)$ will denote the collection of subsets of Λ of cardinality at most $\aleph_\beta$. In particular, $P_{-1}(\Lambda)=\mathscr{F}(\Lambda)$.

LEMMA 3.1. *In* $\mathscr{A}\ell^{\mathscr{C}}$, $\aleph_\beta-\bigoplus_\Lambda M_\lambda=\varinjlim_{P_\beta(\Lambda)}\prod_S M_\lambda$.

PROOF. Take $X\in|\mathscr{A}\ell^{\mathscr{C}}|$ and suppose that for every $S\in P_\beta(\Lambda)$ there is $\prod_S M_\lambda\xrightarrow{\mu_S}X$ such that $\mu_T\pi_{ST}=\mu_S$ whenever $S\subseteq T$, where $\prod_S M_\lambda\xrightarrow{\pi_{ST}}\prod_T M_\lambda$ is the natural map. Define $\aleph_\beta-\bigoplus_\Lambda M_\lambda\xrightarrow{\mu}X$ by $(\mu_p)(f)=(\mu_W)_p(f|_W)$, where $W=\{\lambda\in\Omega|f(\lambda)\neq 0\}$. $\mu\in\mathscr{A}\ell^{\mathscr{C}}$ is the unique map satisfying $\mu\pi_S=\mu_S$ for all $S\in P_\beta(\Lambda)$, where $\prod_S M_\lambda\xrightarrow{\pi_S}\aleph_\beta-\bigoplus_\Lambda M_\lambda$ is the natural map.

Recall that a partially ordered set Λ is directed if and only if every finite subset of Λ has an upper bound in Λ. More generally, we shall say that a partially ordered set Λ is $\aleph_\beta$-*filtered* if and only if every subset of Λ of cardinality at most $\aleph_\beta$ has an upper bound in Λ.

We are now ready to characterize left $\aleph_\beta$-noetherian rings with several objects in terms of $\aleph_\beta$-coproducts of injectives, a generalization of the usual $\beta=-1$ result.

PROPOSITION. 3.2. *For a ring* $\mathscr{C}$, *the following are equivalent.*

1. $\mathscr{C}(p,-)$ *is* $\aleph_\beta$-*noetherian for every* $p\in|\mathscr{C}|$.

2. *If* Λ *is* $\aleph_\beta$-*filtered and if* $\{E_\lambda\to E_\mu|\lambda,\mu\in\Lambda,\ \lambda\leqq\mu\}$ *is a direct system of injectives in* $\mathscr{A}\ell^{\mathscr{C}}$, *then* $\varinjlim_\Lambda E_\lambda$ *is injective.*

3. *If* $\{E_\lambda|\lambda\in\Lambda\}$ *is a collection of injective objects of* $\mathscr{A}\ell^{\mathscr{C}}$, *then* $\aleph_\beta-\bigoplus_\Lambda E_\lambda$ *is injective.*

REMARK. It is not enough to use the usual coproduct in (3), for Bass [1] has shown that if R is left noetherian (hence left $\aleph_0$-noetherian), then there exist countably many injective left R-modules whose coproduct is not injective.

PROOF OF PROPOSITION. We establish the implications $1\to 2\to 3\to 1$. If (3) holds and if $\mathscr{C}(p,-)$ is not $\aleph_\beta$-noetherian for some $p\in|\mathscr{C}|$, then there is a strictly increasing chain $(I_\mu)_{\mu\in\Omega_{\beta+1}}$ of subobjects of $\mathscr{C}(p,-)$. Put $I=\bigcup_{\Omega_{\beta+1}} I_\mu$ and for each

μ let $E_\mu = E(\mathrm{Cok}\,(I_\mu \rightarrowtail I))$, where E is the injective hull functor. Also, let $I \overset{\pi_\mu}{\twoheadrightarrow} \mathrm{Cok}\,(I_\mu \rightarrowtail I)$ and $\mathrm{Cok}\,(I_\mu \rightarrowtail I) \overset{1_\mu}{\rightarrowtail} E$ be the natural maps. These maps determine a map $I \overset{f}{\longrightarrow} \aleph_\beta - \bigoplus_{\Omega_{\beta+1}} E_\mu \in \mathscr{A}\ell^{\mathscr{C}}$, so using the injectivity of $\aleph_\beta - \bigoplus_{\Omega_{\beta+1}} E_\mu$ we have a commutative diagram

$$\begin{array}{ccc} I & \rightarrowtail & \mathscr{C}(p, -) \\ & \searrow^{f} & \downarrow \bar{f} \\ & & \aleph_\beta - \bigoplus_{\Omega_{\beta+1}} E_\mu. \end{array}$$

We define $g = \bar{f}_p(1_p)$. $|\{\mu | g(\mu) \neq 0\}| \leqq \aleph_\beta$, so $\varrho = \sup\{\mu | g(\mu) \neq 0\} < \Omega_{\beta+1}$ and hence $\varrho+1, \varrho+2 < \Omega_{\beta+1}$. Thus there is $a \in I_{\varrho+2}(q) - I_{\varrho+1}(q)$ for some $q \in |\mathscr{C}|$. Notice that $(\pi_{\varrho+1})_q(a) \neq 0$ whereas $g(\varrho+1) = 0$. On the other hand, $a \cdot g = ((\iota_\mu \pi_\mu)_{\mu \in \Omega_{\beta+1}})_q(a)$, so at $\mu = \varrho+1$ we have $0 = a \cdot 0 = a \cdot g(\varrho+1) = (a \cdot g)(\varrho+1) = (\iota_{\varrho+1})_q((\pi_{\varrho+1})_q(a)) \neq 0$, an impossibility. Thus (1) holds.

That (2) implies (3) follows at once from Lemma 3.1 and the fact that $P_\beta(\Lambda)$ is $\aleph_\beta$-filtered.

The proof that (1) implies (2) is a straightforward generalization of the usual $\beta = -1$ result (see [9]) and is omitted

If we combine Propositions 2.5 and 3.2, we get the following characterization of a left $\aleph_\beta$-noetherian ring.

THEOREM 3.3. *For a ring $\mathscr{C}$, the following are equivalent.*

1. *$\mathscr{C}(p, -)$ is $\aleph_\beta$-noetherian for every $p \in |\mathscr{C}|$.*
2. *Every $\aleph_\beta$-generated $M \in |\mathscr{A}\ell^{\mathscr{C}}|$ is $\aleph_\beta$-noetherian.*
3. *Every cyclic $M \in |\mathscr{A}\ell^{\mathscr{C}}|$ is $\aleph_\beta$-noetherian.*
4. *Every $\aleph_\beta$-generated $M \in |\mathscr{A}\ell^{\mathscr{C}}|$ is $\aleph_\beta$-presented.*
5. *Every $\aleph_\beta$-generated $M \in |\mathscr{A}\ell^{\mathscr{C}}|$ has an $\aleph_\beta$-generated free resolution.*
6. *$\mathscr{C}$ satisfies the ascending $\aleph_\beta$-chain condition on left ideals.*
7. *$\mathscr{C}$ satisfies the $\aleph_\beta$-maximum condition on left ideals.*
8. *If Λ is $\aleph_\beta$-filtered and if $\{E_\lambda \to E_\mu | \lambda, \mu \in \Lambda, \lambda \leqq \mu\}$ is a direct system of injectives in $\mathscr{A}\ell^{\mathscr{C}}$, then $\varinjlim_{\Lambda} E_\lambda$ is injective.*
9. *If $\{E_\lambda | \lambda \in \Lambda\}$ is a collection of injective objects of $\mathscr{A}\ell^{\mathscr{C}}$, then $\aleph_\beta - \bigoplus_{\Lambda} E_\lambda$ is injective.*

4. Some results for left $\aleph_\beta$-noetherian rings. In [3], C. Faith and E. Walker characterize a left noetherian ring R by the condition that every injective left R-module may be decomposed as a coproduct of c-generated modules for some predetermined cardinal c. In this section we shall generalize this result to the setting of left $\aleph_\beta$-noetherian rings with several objects. First we prove several lemmas.

LEMMA 4.1. *In $\mathscr{A}\ell^{\mathscr{C}}$, $\aleph_\beta - \bigoplus_{\Omega} (\aleph_\beta - \bigoplus_{\Lambda_\omega} M_{\lambda_\omega}) = \aleph_\beta - \bigoplus_{\Omega^{\Lambda_\omega}} M_\lambda$, where ${}_\Omega\Lambda_\omega = {}_\Omega(\Lambda_\omega \times \{\omega\})$ and where $M_\lambda = M_{\lambda_\omega}$ whenever $\lambda = (\lambda_\omega, \omega) \in {}_\Omega\Lambda_\omega$.*

PROOF. Define $\aleph_\beta - \bigoplus_{\Omega} (\aleph_\beta - \bigoplus_{\Lambda_\omega} M_{\lambda_\omega}) \xrightarrow{\varphi} \aleph_\beta - \bigoplus_{\Omega^{\Lambda_\omega}} M_\lambda$ by $\varphi_p((f_\omega)_{\omega\in\Omega}) = h_{(f_\omega)_{\omega\in\Omega}}$ where $h_{(f_\omega)_{\omega\in\Omega}}(\lambda_\omega, \omega) = f_\omega(\lambda_\omega) \in M_{\lambda_\omega}(p)$. Also, define $\aleph_\beta - \bigoplus_{\Omega^{\Lambda_\omega}} M_\lambda \to$ $\to \aleph_\beta - \bigoplus_{\Omega} (\aleph_\beta - \bigoplus_{\Lambda_\omega} M_{\lambda_\omega})$ by $\psi_p(h) = (f_\omega)_{\omega\in\Omega}$, where $f_\omega(\lambda_\omega) = h(\lambda_\omega, \omega) \in M_{\lambda_\omega}(p)$. It is routine to check that φ and ψ have the correct codomains, are natural, and that $\psi\varphi$ and $\varphi\psi$ are the appropriate identities.

LEMMA 4.2. *In* $\mathscr{A}\ell^{\mathscr{C}}$, $\aleph_\beta - \bigoplus_{\Lambda} (M_\lambda \oplus N_\lambda) = (\aleph_\beta - \bigoplus_{\Lambda} M_\lambda) \oplus (\aleph_\beta - \bigoplus_{\Lambda} N_\lambda)$.

PROOF. The inverse isomorphisms are the maps

$$\aleph_\beta - \bigoplus_{\Lambda} (M_\lambda \oplus N_\lambda) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} (\aleph_\beta - \bigoplus_{\Lambda} M_\lambda) \oplus (\aleph_\beta - \bigoplus_{\Lambda} N_\lambda),$$

where $\varphi_p((m_\lambda, n_\lambda)_{\lambda\in\Lambda}) = ((m_\lambda)_{\lambda\in\Lambda}, (n_\lambda)_{\lambda\in\Lambda})$ and $\psi_p(f, g) = (f(\lambda), g(\lambda))_{\lambda\in\Lambda}$.

We recall that left $\aleph_\beta$-noetherian rings with several objects are characterized by the property that $\aleph_\beta$-coproducts of injective objects of the corresponding functor category are injective. On the other hand, if an $\aleph_\beta$-coproduct of objects of $\mathscr{A}\ell^{\mathscr{C}}$ is injective, it is natural to ask under what conditions each of the summands is injective. The next result shows this to be the case for all rings.

LEMMA 4.3. *In* $\mathscr{A}\ell^{\mathscr{C}}$, *if* $\aleph_\beta - \bigoplus_{\Lambda} M_\lambda$ *is injective, then* M_λ *is injective for every* $\lambda \in \Lambda$.

PROOF. Choose $\lambda \in \Lambda$ and suppose

$$\begin{array}{ccccc} 0 & \to & A & \xrightarrow{g} & B \\ & & & \searrow^{f} & \\ & & & & M_\lambda \end{array}$$

is in $\mathscr{A}\ell^{\mathscr{C}}$ with top row exact. Let $M_\lambda \xrightarrow{\iota_\lambda} \aleph_\beta - \bigoplus_{\Lambda} M_\lambda$ and $\aleph_\beta - \bigoplus_{\Lambda} M_\lambda \xrightarrow{\pi_\lambda} M_\lambda$ be the canonical maps. Since $\aleph_\beta - \bigoplus_{\Lambda} M_\lambda$ is injective, there is $B \xrightarrow{h} \aleph_\beta - \bigoplus_{\Lambda} M_\lambda$ with $hg = \iota_\lambda f$. But then $B \xrightarrow{\pi_\lambda h} M_\lambda \in \mathscr{A}\ell^{\mathscr{C}}$ and $\pi_\lambda hg = \pi_\lambda \iota_\lambda f = f$.

If M is a subobject of an $\aleph_\beta$-coproduct of objects of $\mathscr{A}\ell^{\mathscr{C}}$, say $M \rightarrowtail \aleph_\beta - \bigoplus_{\Lambda} N_\lambda$, it can be useful to know whether in fact $M \rightarrowtail \aleph_\beta - \bigoplus_{\Omega} N_\lambda$ for some subset Ω of Λ of given cardinality. The next lemma gives a condition on M under which such containment holds.

LEMMA 4.4. *In* $\mathscr{A}\ell^{\mathscr{C}}$, *if* M *is a* c*-generated subobject of* $\aleph_\beta - \bigoplus_{\Lambda} N_\lambda$ *and if* $c \geqq \aleph_\beta$, *then* M *is a subobject of an* $\aleph_\beta$*-coproduct of at most* c *of the* N_λ.

PROOF. If M is generated by $\mathscr{S} = \{x_\omega \in M(p_\omega) | \omega \in \Omega\}$, $|\Omega| \leqq c$, then each x belongs to an $\aleph_\beta$-coproduct of $\aleph_\beta$ many of the $N_\lambda(p_\lambda)$, say $x_\omega \in \aleph_\beta - \bigoplus_{\Lambda_{x_\omega}} N_\lambda(p_\omega)$.

If for each ω, $\hat{x}_\omega$ is the corresponding element in $\aleph_\beta - \bigoplus_{\bigcup_{\mathscr{S}} \Lambda_{x_\omega}} N_\lambda(p_\omega)$, then for each $p \in |\mathscr{C}|$ we have a monomorphism

$$M(p) \xrightarrow{\varphi_p} \aleph_\beta - \bigoplus_{\bigcup_{\mathscr{S}} \Lambda_{x_\omega}} N_\lambda(p)$$

induced by the $\hat{x}_\omega$. But then the φ_p determine a monomorphism

$$M \to \aleph_\beta - \bigoplus_{\bigcup_{\mathscr{S}} \Lambda_{x_\omega}} N_\lambda$$

in $\mathscr{A}\ell^{\mathscr{C}}$, and $\left|\bigcup_{\mathscr{S}} \Lambda_{x_\omega}\right| \leqq |\mathscr{S}| \cdot \aleph_\beta \leqq c \cdot \aleph_\beta = c$.

Our final lemma is concerned with the intersection of subobjects of a product of objects of $\mathscr{A}\ell^{\mathscr{C}}$.

LEMMA 4.5. *In $\mathscr{A}\ell^{\mathscr{C}}$, if $\prod_\Lambda M_\lambda = N_\omega \to \prod_\Omega N_\omega$, if $N \rightarrowtail \prod_\Omega N_\omega$, and if $M_\lambda \cap N = M_{\lambda'} \cap N$ for some $\lambda \neq \lambda'$, then $M_\lambda \cap N = 0$.*

PROOF. For each $\lambda \in \Lambda$, identify $M_\lambda \cap N$ with its image in $\prod_\Omega N_\omega$. The result is then immediate.

We now give a generalized version of a theorem of Faith and Walker [3]. Our result provides a sufficient condition for a ring with several objects to be left $\aleph_\beta$-noetherian.

THEOREM 4.6. *If $\mathscr{C}$ is a ring with several objects and if there exists a cardinal number c such that every injective object of $\mathscr{A}\ell^{\mathscr{C}}$ is an $\aleph_\beta$-coproduct of c-generated objects of $\mathscr{A}\ell^{\mathscr{C}}$, then $\mathscr{C}$ is left $\aleph_\beta$-noetherian.*

PROOF. Suppose such a cardinal c exists. To show that $\mathscr{C}$ is left $\aleph_\beta$-noetherian, it suffices by hypothesis and by Lemmas 4.1 and 4.3 to show that if $M = \aleph_\beta - \bigoplus_\Lambda M_\lambda$, where each $M_\lambda \in |\mathscr{A}\ell^{\mathscr{C}}|$ is c-generated, injective, then M is injective. Clearly we may assume that Λ is infinite.

We choose a cardinal $\bar{c}$ satisfying $\bar{c} \geqq \max(\aleph_0, \aleph_\beta, c, |\{\text{morphims of } \mathscr{C}\}|)$. Also, we select a set B such that $|B| > 2^{\bar{c}|\Lambda|}$. For each $\lambda \in \Lambda$ define

$$N_\lambda = \prod_{b \in B} M_{\lambda, b} \text{ (product of } |B| \text{ copies of } M_\lambda)$$

and define $P = \prod_{\lambda \in \Lambda} N_\lambda$. By assumption we may write $P = \aleph_\beta - \bigoplus_G Q_g$, where each Q_g is c-generated and hence $\bar{c}$-generated.

We well-order Λ and take one of the factors of N_1, say M_{1,b_1}. By Lemma 4.4,

$$M_{1.b_1} \rightarrowtail \aleph_\beta - \bigoplus_{G_1} Q_g = P_1$$

for some $G_1 \subseteqq G$ with $|G_1| \leqq \bar{c}$. If $\hat{P}_1$ denotes the representative class of subobjects of P_1, notice that $|\hat{P}_1| \leqq \|\mathscr{C}\| \cdot 2^{\bar{c}} \leqq 2^{\bar{c}}$.

Now consider the subobjects $\{M_{2,b} \cap P_1 | b \in B\}$ of P_1. Since $|B| > 2^{\bar{c}}$, $M_{2,b} \cap P_1 = M_{2,b'} \cap P_1$ for some $b, b' \in B$, $b \neq b'$. Applying Lemma 4.5, we get $M_{2,b_2} \cap P_1 = 0$ for some $b_2 \in B$. By Lemma 4.4

$$M_{2,b_2} \rightarrowtail \aleph_\beta - \bigoplus_{G_2} Q_g = P_2$$

for some $G_2 \subseteq G$ with $|G_2| \leqq \bar{c}$. Since $M_{2,b_2} \cap P_1 = 0$, we may clearly take G_2 so that $G_1 \cap G_2 = \emptyset$.

Now take $\alpha \in \Lambda$ and suppose that there exist mutually disjoint subsets $\{G_\gamma\}_{\gamma<\alpha}$ of G with $|G_\gamma| \leqq \bar{c}$ for each γ such that $P_\gamma = \aleph_\beta - \bigoplus_{G_\gamma} Q_g$ contains a copy of M_{γ,b_γ} for some $b_\gamma \in B$. Define $H_\alpha = \bigcup_{\gamma<\alpha} G_\gamma$ and put

$$S = \aleph_\beta - \bigoplus_{H_\alpha} Q_g = \aleph_\beta - \bigoplus_{\gamma<\alpha} P_\gamma .$$

Then if $\hat{S}_\alpha$ denotes the representative class of subobjects of S_α, it is easy to see that $|\hat{S}_\alpha| = 2^{\bar{c}|\Lambda|}$.

We now consider the subobjects $\{M_{\alpha,b} \cap S_\alpha | b \in B\}$ of S_α. Since $|B| > 2^{\bar{c}|\Lambda|}$, we must as above have $M_{\alpha,b_\alpha} \cap S_\alpha = 0$ for some $b_\alpha \in B$. Also

$$M_{\alpha,b_\alpha} \rightarrowtail \aleph_\beta - \bigoplus_{G_\alpha} Q_g = P_\alpha$$

for some $G_\alpha \subseteq G$ with $|G_\alpha| \leqq \bar{c}$, where G_α may be chosen so that $\emptyset = G_\alpha \cap H_\alpha = \bigcup_{\gamma<\alpha} (G_\alpha \cap G_\gamma)$. Thus $G_\alpha \cap G_\gamma = \emptyset$ for all $\gamma < \alpha$, so by transfinite induction we have mutually disjoint subsets $\{G_\alpha\}_{\alpha \in \Lambda}$ of G and elements $b_\alpha \in B$ such that

$$M_{a,b_\alpha} \rightarrowtail \aleph_\beta - \bigoplus_{G_\alpha} Q_g = P_\alpha$$

with $|G_\alpha| \leqq \bar{c}$ for every $\alpha \in \Lambda$.

Now define $H = \bigcup_{\alpha \in \Lambda} G_\alpha$. Then

$$P = \left(\aleph_\beta - \bigoplus_{H} Q_g\right) \oplus \left(\aleph_\beta - \bigoplus_{g \notin H} Q_g\right) = \left(\aleph_\beta - \bigoplus_{\Lambda} P_\alpha\right) \oplus \left(\aleph_\beta - \bigoplus_{g \notin H} Q_g\right).$$

Now each M_{α,b_α} is injective, so M_{α,b_α} is a direct summand of P_α for every α, say $P_\alpha = M_{\alpha,b_\alpha} \oplus T_\alpha$. But then

$$\aleph_\beta - \bigoplus_{\Lambda} P_\alpha = \aleph_\beta - \bigoplus_{\Lambda} (M_{\alpha,b_\alpha} \oplus T_\alpha) = \left(\aleph_\beta - \bigoplus_{\Lambda} M_{\alpha,b_\alpha}\right) \oplus \left(\aleph_\beta - \bigoplus_{\Lambda} T_\alpha\right).$$

Since P is injective, we have $\aleph_\beta - \bigoplus_{\Lambda} P_\alpha$ is injective and therefore $M = \aleph_\beta - \bigoplus_{\Lambda} M_\lambda = \aleph_\lambda - \bigoplus_{\Lambda} M_{\alpha,b_\alpha}$ is injective.

Next we shall apply the previous theorem to generalize a result of Z. Papp [7]. In so doing, we shall obtain yet another sufficient condition for a ring $\mathscr{C}$ to be left $\aleph_\beta$-noetherian. We shall need the following definition for our work.

DEFINITION. $M \in |\mathscr{A}\mathscr{b}^{\mathscr{C}}|$ is *indecomposable* if its only direct summands are 0 and M.

We recall that the class of objects of $\mathscr{AL}^{\mathscr{C}}$ is a proper class. In what follows, we shall need to consider the class of isomorphism classes of injective, indecomposable objects of $\mathscr{AL}^{\mathscr{C}}$. The next lemma assures us that this class is in fact a set.

LEMMA 4.7. *In $\mathscr{AL}^{\mathscr{C}}$, the class of isomorphism classes of injective, indecomposable objects is a set.*

PROOF. Let $\mathscr{M}$ (respectively $\mathscr{S}$) denote the class of isomorphism classes of injective, indecomposable (respectively cyclic) objects of $\mathscr{AL}^{\mathscr{C}}$. There is a natural injection $\mathscr{M}\to\mathscr{S}$. Now the class of left ideals of $\mathscr{C}$ is a set, so $\mathscr{S}$ is a set and hence so too is $\mathscr{M}$.

In [7], Papp has proved that if every injective left R-module is a direct sum of injective, indecomposable modules, then R is left noetherian. Using the last lemma and the generalized theorem of Faith and Walker, it is an easy matter to generalize Papp's result.

THEOREM 4.8. *In $\mathscr{AL}^{\mathscr{C}}$, if every injective object is an $\aleph_\beta$-coproduct of injective, indecomposable objects, then $\mathscr{C}$ is left $\aleph_\beta$-noetherian.*

PROOF. Let $\mathscr{M}$ denote the set of isomorphism classes of injective, indecomposable objects of $\mathscr{AL}^{\mathscr{C}}$. For each $N\in\mathscr{M}$, let c_N be the cardinality of a generating set for N and define $c=\sum_{N\in\mathscr{M}} c_N$. Then every injective $M\in|\mathscr{AL}^{\mathscr{C}}|$ is an $\aleph_\beta$-coproduct of c-generated objects of $\mathscr{AL}^{\mathscr{C}}$, so $\mathscr{C}$ is left $\aleph_\beta$-noetherian.

E. Matlis [6] has proved that if R is left noetherian, then every injective module may be expressed as a coproduct of injective, indecomposable modules. As we have seen, there exists a cardinal c such that every injective, indecomposable left R-module is c-generated. Thus we see that the converse of Theorem 4.6 holds in the left noetherian setting. It is not known if the converse generalizes to the left $\aleph_\beta$-noetherian case.

5. Examples. It is well known [6] that if R is a ring with identity, then $E\left(\bigoplus_{i=1}^{n} M_i\right)=\bigoplus_{i=1}^{n} E(M_i)$ for a family $\{M_i|i=1,\ldots,n\}$ of left R-modules, and that if R is left noetherian then the finiteness condition can be removed. A natural question to ask is whether a corresponding result can be obtained for $E(\aleph_\beta-\bigoplus_{\Lambda} M_\lambda)$ when R is a left $\aleph_\beta$-noetherian ring and $\{M_\lambda|\lambda\in\Lambda\}$ is a family of left R-modules. We shall see, however, that the obvious generalization does not carry over to the $\aleph_\beta$ case. First we need some lemmas.

LEMMA *If R is an integral domain and if M is a torsion-free divisible R-module, then M is injective.*

PROOF. Suppose I is an ideal of R, $I\xrightarrow{f}M\in{}_R\mathrm{Mod}$. If $f\neq 0$ there is $i_0\in I$ with $f(i_0)\neq 0$. Now M is divisible so there is $m_0\in M$ such that $f(i_0)=i_0m_0$. We define $R\xrightarrow{g}M$ by $r\mapsto rm_0$. Then for every $i\in I$, $i_0g(i)=i_0f(i)$, so since M is torsion-free we have $g(i)=f(i)$. But then $g|_I=f$ so M is injective.

Notice that if R is an integral domain with quotient field Q, then Q is a torsion-free divisible R-module and hence injective. Since $R\subset' Q$, where $\subset'$ denotes essential extension, we then have that $E(R)=Q$.

Our next lemma computes the injective hull of an $\aleph_\beta$-coproduct of a family of ideals of an integral domain.

LEMMA 5.2. *Let R be an integral domain and let $\{I_\lambda|\lambda\in\Lambda\}$ be a family of non-zero ideals of R. If $0\neq p\in R$, then*

$$E(\aleph_\beta-\bigoplus_\Lambda I_\lambda)=\{f\in\aleph_\beta-\bigoplus_\Lambda R_{\{0\}}|\, pf(\lambda)\in I_\lambda \text{ for every } \lambda\in\Lambda\},$$

where $R_{\{0\}}$ is the quotient field of R.

PROOF. For $0\neq p\in R$, let $J_p=\{f\in\aleph_\beta-\bigoplus_\Lambda R_{\{0\}}|\, pf(\lambda)\in I_\lambda$ for every $\lambda\in\Lambda\}$. If P is a nonzero submodule of J_p and if $0\neq f\in P$, then $0\neq pf\in(\aleph_\beta-\bigoplus_\Lambda I_\lambda)\cap P$ and hence $\aleph_\beta-\bigoplus_\Lambda I_\lambda\subset' J$. It is clear that J_p is divisible and torsion-free, so J is injective by Lemma 5.1.

We now look at several examples of rings R where $E(\aleph_\beta-\bigoplus_\Lambda M_\lambda)$ and $\aleph_\beta-\bigoplus_\Lambda E(M_\lambda)$ are not necessarily the same for families $\{M_\lambda|\lambda\in\Lambda\}$ of left R-modules. We consider only situations where $\beta=0$, but it will be seen that these examples can readily be modified to hold for arbitrary β.

EXAMPLE 5.3. In $_{\mathbf{Z}}$Mod, we show $E(\aleph_0-\bigoplus_{\mathbf{N}}\langle n\rangle)\neq\aleph_0-\bigoplus_{\mathbf{N}}E(\langle n\rangle)$. We have $\langle n\rangle\subset'\mathbf{Z}$ for every $n\in\mathbf{N}$, so since $E(\mathbf{Z})=\mathbf{Q}$ and since $\subset'$ is transitive, $E(\langle n\rangle)=Q$ for every $n\in N$. Therefore $\aleph_0-\bigoplus_{\mathbf{N}}E(\langle n\rangle)=\prod_{\mathbf{N}}\mathbf{Q}$, which is injective. Now take $0\neq P$ to be the submodule of $\prod_{\mathbf{N}}\mathbf{Q}$ generated by $(2, 2, 2, \ldots)$. For $0\neq k\in\mathbf{Z}$, $2k\in\langle n\rangle$ for at most finitely many $n\in\mathbf{N}$, so $P\cap\prod_{\mathbf{N}}\langle n\rangle=0$ and hence $E(\aleph_0-\bigoplus_{\mathbf{N}}\langle n\rangle)\neq$ $\neq\aleph_0\bigoplus_{\mathbf{N}}E(\langle n\rangle)$. Indeed, by Lemma 5.2 $E(\aleph_0-\bigoplus_{\mathbf{N}}\langle n\rangle)=\{(a_1/a, a_2/a, a_3/a, \ldots)|a_i\in\langle i\rangle\}$ for $0\neq a$ fixed in $\mathbf{Z}$.

We notice that in Example 5.3, our ring $R=\mathbf{Z}$ is left noetherian and hence left $\aleph_0$-noetherian. Next we consider the situation where the ring R is genuinely left $\aleph_0$-noetherian.

EXAMPLE 5.4. For the genuinely left $\aleph_0$-noetherian ring $R=\mathbf{Z}[x_1, x_2, \ldots]$ with quotient field $R_{\{0\}}$, we show that $E(\aleph_0-\bigoplus_N\langle x_i\rangle)\neq\aleph_0-\bigoplus_N E(\langle x_i\rangle)$. For each i, $\langle x_i\rangle\subset' R$, so it follows that $\aleph_0-\bigoplus_N E(\langle x_i\rangle)=\prod_N R_{\{0\}}$, which is injective. Now if $0\neq P$ is the R-submodule of $\prod_N R_{\{0\}}$ generated by $(x_1, x_1, \ldots)$, then for every $0\neq p\in R$, $p(x_1, x_1, \ldots)\notin\prod_N\langle x_i\rangle$ so that $P\cap\prod_N\langle x_i\rangle=0$. Therefore $E(\aleph_0-\bigoplus_N\langle x_i\rangle)\neq$ $\neq\aleph_0-\bigoplus_N E(\langle x_i\rangle)$. Again one can compute $E(\aleph_0-\bigoplus_N\langle x_i\rangle)$ by Lemma 5.2.

In both of the above examples, we notice that the $\aleph_0$-coproduct of our family of R-modules is just the product. We now examine the relationship between $E(\aleph_0-\bigoplus_\Lambda M_\lambda)$ and $\aleph_0-\bigoplus_\Lambda E(M_\lambda)$ in the setting of a genuinely left $\aleph_0$-noetherian ring with $\aleph_0-\bigoplus_\Lambda M_\lambda\subsetneq\prod_\Lambda M_\lambda$.

EXAMPLE 5.5. We consider the genuinely left $\aleph_0$-noetherian ring $R=Z[x_1, x_2, \ldots]$ and exhibit left R-modules $\{M_\lambda|\lambda\in\Lambda\}$ such that $\aleph_0-\bigoplus_\Lambda M_\lambda\subseteq\neq\prod_\Lambda M_\lambda$ and $E(\aleph_0-\bigoplus_\Lambda M_\lambda)\neq\aleph_0-\bigoplus_\Lambda E(M_\lambda)$. Put $I=\{x_1, x_2, \ldots\}$ and for each nonempty subset S of I define M_S to be the left ideal of R generated by S. Assuming the Continuum Hypothesis, we notice that $|\{M_S|S\subseteq I\}|=\aleph_1$, so that $\aleph_0-\bigoplus_{S\subseteq I} M_S\subseteq\neq\prod_{S\subseteq I} M_S$. We now compute $E(\aleph_0-\bigoplus_{S\subseteq I} M_S)$ and $\aleph_0-\bigoplus_{S\subseteq I} E(M_S)$. Since R is left $\aleph_0$-noetherian, note that the latter module is injective.

Suppose now that $\emptyset\neq S\subseteq I$ and that P is a nonzero left ideal of R. If $x_i\in M_S$ and if $0\neq p\in P$, then $0\neq px_i\in P\cap M_S$ so we have $M_S\subset' R$. But $E(R)=R_{\{0\}}$, so $E(M_S)=R_{\{0\}}$ for every nonempty subset S of I and therefore $\aleph_0-\bigoplus_{S\subseteq I} E(M_S)=\aleph_0-\bigoplus_{S\subseteq I} R_{\{0\}}$.

Now define $f\in\prod_{S\subseteq I} R_{\{0\}}$ by

$$f(S)=\begin{cases}x_1 & \text{if } |S|<\aleph_0\\ 0 & \text{if } |S|=\aleph_0\end{cases}$$

and define Q to be the submodule of $R_{\{0\}}$ generated by f. Now for every $0\neq p\in R$ there is i_0 such that $px_1\notin\langle x_{i_0}\rangle=M_{\{x_{i_0}\}}$. However, $(pf)(\{x_{i_0}\})=px_1$, so $pf\notin\aleph_0-\bigoplus_{S\subseteq I} M_S$ and hence $Q\cap(\aleph_0-\bigoplus_{S\subseteq I} M_S)=0$. But then $\aleph_0-\bigoplus_{S\subseteq I} M_S\not\subset'\aleph_0-\bigoplus_{S\subseteq I} E(M_S)$, so $E(\aleph_0-\bigoplus_{S\subseteq I} M_S)\subseteq\neq\aleph_0-\bigoplus_{S\subseteq I} E(M_S)$. Again, Lemma 5.2 can be used to actually compute $E(\aleph_0-\bigoplus_{S\subseteq I} M_S)$.

We conclude this series of examples by generalizing the conclusion of Example 5.5. If $R=\mathbf{Z}[\{x_\lambda|\lambda\in\Omega_\beta\}]$, then R is genuinely left $\aleph_\beta$-noetherian. If, for every nonempty subset S of $I=\{x_\lambda|\lambda\in\Omega_\beta\}$, M_S is the left ideal of R generated by S, then assuming the Generalized Continuum Hypothesis we have $|\{M|S\subseteq I\}|=\aleph_{\beta+1}$ and hence $\aleph_\beta-\bigoplus_{S\subseteq I} M_S\subseteq\neq\prod_{S\subseteq I} M_S$. We have that $\aleph_\beta-\bigoplus_{S\subseteq I} E(M_S)$ is injective. However, the same argument as given in Example 5.5 shows $\aleph_\beta-\bigoplus_{S\subseteq I} M_S\not\subset'\aleph_\beta-\bigoplus_{S\subseteq I} E(M_S)$, so that $E(\aleph_\beta-\bigoplus_{S\subseteq I} M_S)\subseteq\neq\aleph_\beta-\bigoplus_{S\subseteq I} E(M_S)$.

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LAGRANGE INTERPOLATION OF FUNCTIONS OF GENERALIZED BOUNDED VARIATION

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1. Introduction

Let $f\in C[-1, 1]$ and let

$$-1 \leqq x_{nn} < \ldots < x_{2n} < x_{1n} \leqq 1$$

be the roots of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ defined by

$$P_n^{(\alpha,\beta)}(x) = (1-x)^{-\alpha}(1+x)^{+\beta}\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}] \quad (\alpha, \beta > -1).$$

When $\alpha = -1$, $P_n^{(-1,\beta)}(x)$ is defined as follows

$$nP_n^{(-1,\beta)}(x) = (n+\beta)(x-1)\frac{1}{2}P_{n-1}^{(1,\beta)}(x)$$

(see [3]). Define the Lagrange interpolation polynomial by

$$L_n^{(\alpha,\beta)}(f, x) = \sum_{k=1}^{n} f(x_{kn}^{(\alpha,\beta)}) l_{kn}^{(\alpha,\beta)}(x), \tag{1.1}$$

where

$$l_{kn}^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)/[P_n'^{(\alpha,\beta)}(x_{kn}^{(\alpha,\beta)})(x-x_{kn}^{(\alpha,\beta)})] \quad (k = 1, \ldots, n).$$

(For convenience, we shall omit some superfluous notation; for example, $l_k(x) = l_{kn}^{(\alpha,\beta)}(x)$.)

D. L. Berman [1] proved that if $-1<\alpha, \beta<0$, then the Lagrange interpolation (1.1) tends to $f(x)$ for arbitrary $x\in[-1, 1]$ whenever $f\in C[-1, 1]$ is of bounded variation.

Later, P. Vértesi [2], [12] proved that if $-1<\alpha, \beta<\frac{1}{2}$, then

$$\lim_{n\to\infty} \|L_n^{(\alpha,\beta)}(f, x) - f(x)\| = 0 \tag{1.2}$$

for a continuous function f with bounded variation, and (1.2) generally does not hold for $\alpha = \frac{1}{2}$.

Recently, A. A. Kelzon [3] considered the convergence of (1.1) for continuous f with bounded variation of order $p\geqq 1$ (BV_p, shortly) and proved that if $-1\leqq\alpha, \beta<p^{-1}-2^{-1}$, then (1.2) holds for $f\in C[-1, 1]\cap \mathrm{BV}_p(p\geqq 1)$, but he did not give an estimate of the rate of convergence.

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In this paper the first purpose is to estimate the rate of convergence for $f\in C\Lambda BV$. It is well-known that if $f'\in C[-1, 1]$, then $L_n(f, x)-f(x)=O(\omega(f', 1/n)\|L_n\|/n)$, where $\|L_n\|$ is the Lebesgue constant of Lagrange interpolation and $\|L_n\|\cong c\cdot\log n$. Therefore, generally, we can not obtain $L_n(f, x)-f(x)=O(1/n)$ if $f'\in C[-1, 1]$. What kind of condition implies that $L_n(f, x)-f(x)=O(1/n)$? The second purpose of the paper is to answer this question.

DEFINITION. A function f is said to be of Λ-bounded variation (ΛBV), if the numbers

$$\sum_{k=1}^{n}|f(b_k)-f(a_k)|/\lambda_k \tag{1.3}$$

are bounded uniformly over the sequences of nonoverlapping intervals $I_k=[a_k, b_k]$ in $I=[a, b]$, where $\Lambda=\{\lambda_n\}$ is a nondecreasing sequence of positive numbers such that $\sum_{n=1}^{\infty}\lambda_n^{-1}=\infty$. If $f\in\Lambda BV$, the Λ-variation of f is defined to be the supremum of (1.3), denoted by $V_\Lambda(f, [a, b])$ (see [4]). When $\Lambda=\{n\}$, the class is called the class of functions of harmonic bounded variation (HBV). For $\lambda_n\equiv 1$, it is clear that the class is the usual class of functions of bounded variation (BV). If $f\in C[-1, 1]\cap\Lambda BV$, then we say $f\in C\Lambda BV$.

2. Main results

Now we state our main results.

THEOREM 2.1. *Let* $f\in C\Lambda BV$ *with* $\Lambda=\{k^t\}$ $(0\leqq t<1)$ *and* $t<\gamma=\min\left\{\frac{1}{2}-\alpha, \frac{1}{2}-\beta\right\}$. *Then we have for some* $c>0$

$$L_n^{(\alpha,\beta)}(f, x)-f(x)=$$

$$=O(1)\sum_{k=1}^{n}V_\Lambda\left(f, \left[x-c\left(\frac{k}{n}\sqrt{1-x^2}+\frac{k^2}{n^2}\right), x+c\left(\frac{k}{n}\sqrt{1-x^2}+\frac{k^2}{n^2}\right)\right]\right)\times$$

$$\times\left(\frac{1}{k}+\frac{1}{k^\gamma}\right)\left(\sum_{i=1}^{k}\frac{1}{\lambda_i}\right)^{-1},$$

where the sign "O" is independent of f, n *and* x *and depends only on* α *and* β, *and* $V_\Lambda(f, [a, b])=V_\Lambda(f, [a, b]\cap[-1, 1])$.

THEOREM 2.2. *Let* $f\in CHBV$ *and* $-1\leqq\alpha, \beta\leqq-\frac{1}{2}$, *then for some* $c>0$

$$L_n^{(\alpha,\beta)}(f, x)-f(x)=O(1)\Big\{V_H\Big(f, \left[x-c(\varepsilon_n\sqrt{1-x^2}+\varepsilon_n^2), x+c(\varepsilon_n\sqrt{1-x^2}+\varepsilon_n^2)\right]\Big)+$$

$$+V_H(f, [-1, 1])\Big(\sum_{i=1}^{[n\varepsilon_n]}1/\lambda_i\Big)^{-1}+$$

$$+\sum_{k=[n\varepsilon_n]+1}^{n}V_H\left(f, \left[x-c\left(\frac{k}{n}\sqrt{1-x^2}+\frac{k^2}{n^2}\right), x+c\left(\frac{k}{n}\sqrt{1-x^2}+\frac{k^2}{n^2}\right)\right]\right)\Big(k\sum_{i=1}^{k}1/\lambda_i\Big)^{-1}\Big\},$$

where $\varepsilon_n\downarrow 0$ *and* $\log\varepsilon_n=o(\log n)$.

Set $\varepsilon_n = 1/\log n$; then we have the following

COROLLARY. *For* $f \in \mathrm{CHBV}$ *and* $-1 \leqq \alpha, \beta \leqq -\frac{1}{2}$,

$$L_n^{(\alpha,\beta)}(f, x) - f(x) = O(1)\left\{V_H(f, [-1, 1]) \log\log n/\log n + \right.$$

$$\left. + V_H\left(f, \left[x - c\left(\frac{\sqrt{1-x^2}}{\log n} + \frac{1}{\log^2 n}\right), x + c\left(\frac{\sqrt{1-x^2}}{\log n} + \frac{1}{\log^2 n}\right)\right]\right)\right\}.$$

THEOREM 2.3. *Suppose that* $f' \in \mathrm{CBV}$ *and* $-1 \leqq \alpha \leqq -\frac{1}{2}$. *Then we have*

$$L_n^{(\alpha,\alpha)}(f, x) - f(x) = O(1/n) V(f', [-1, 1]) \quad (n \geqq 2).$$

3. Lemmas and preliminaries

In order to prove the theorems we need some known results and the following lemmas.

Denote the zeros of $P_n^{(\alpha,\beta)}(x)$ and $P_n'^{(\alpha,\beta)}(x)$ by x_k and z_k, respectively. Let $x_k = \cos\theta_k$ and $z_k = \cos\varphi_k$ $(0 \leqq \theta_k, \varphi_k \leqq \pi)$. We have

(3.1) $$|P_n'^{(\alpha,\beta)}(x_k)| \sim k^{-\alpha-3/2} n^{\alpha+2}, \quad (0 < \theta_k \leqq \pi/2) \quad [5, (8.9.2)]$$

(3.2) $$\theta_k \sim n^{-1}(k\pi + O(1)), \quad (0 < \theta_k \leqq \pi/2) \quad [5, (8.9.1)]$$

(3.3) $$P_n^{(\alpha,\beta)}(z_k) \sim k^{-\alpha-1/2} n^{\alpha}, \quad (0 < \varphi_k \leqq \pi/2) \quad [6, \text{Lemma } 3.4]$$

(3.4) $$P_n'^{(\alpha,\beta)}(x) = \frac{1}{2}(n+\alpha+\beta+1) P_{n-1}^{(\alpha+1,\beta+1)}(x), \quad [5, (4.21.7)]$$

(3.5) $$P_n^{(\alpha,\beta)}(\cos\theta) = \begin{cases} O(n^{\alpha}), & 0 \leqq \theta \leqq c/n \quad [7] \\ (\sin\theta)^{-\alpha-1/2} O(n^{-1/2}), & c/n \leqq \theta \leqq \pi/2, \end{cases}$$

(3.6) $$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x). \quad [5, (4.1.3)].$$

Let m be a positive integer such that $x_{m+1} < x \leqq x_m$, then (see [3])

(3.7) $$\left|\sum_{i=1}^{k} l_i(x)\right| \leqq |l_k(x)| \quad (k = 1, \ldots, m),$$

$$\left|\sum_{i=k}^{n} l_i(x)\right| \leqq |l_k(x)| \quad (k = m+1, \ldots, n).$$

LEMMA 3.1. *Let* x_j *be the zero of* $P_n^{(\alpha,\beta)}(x)$ *which is closest to* x, *set* $i = |k-j|$. *Then we have*

(3.8) $$l_{kn}^{(\alpha,\beta)}(x) = O(i^{-1} + i^{\alpha-1/2}) \quad (0 \leqq x \leqq 1, \ k \neq j)$$

(3.9) $$l_{kn}^{(\alpha,\beta)}(x) = O(i^{-1} + i^{\beta-1/2}) \quad (-1 \leqq x \leqq 0, \ k \neq j)$$

(3.10) $$l_{jn}^{(\alpha,\beta)}(x) = O(1).$$

PROOF. Because of (3.6), without loss generality, we can suppose $0 \leqq x \leqq 1$. We shall distinguish three different cases.

Case (i): $-1 \leqq \alpha \leqq -\frac{1}{2}$. If $0 \leqq \theta = \arccos x \leqq c/n$, then from (3.1)—(3.2) and (3.5)

$$|l_{kn}^{(\alpha,\beta)}(x)| = O(1)|n^{-1/2}P_n^{(\alpha,\beta)}(x)\sin^{\alpha+3/2}\theta_k| \Big/ \left|\sin\frac{1}{2}(\theta-\theta_k)\sin\frac{1}{2}(\theta+\theta_k)\right| =$$

$$= O(1)\, n^{\alpha-1/2}\left|\sin^{-1}\frac{1}{2}(\theta-\theta_k)\,\sin^{\alpha+1/2}\frac{1}{2}(\theta+\theta_k)\right| = O(i^{\alpha-1/2}).$$

If $c/n \leqq \theta \leqq \pi/2$, from (3.5) it follows that

$$|l_{kn}^{(\alpha,\beta)}(x)| = O(1)\left|n^{-1/2}P_n^{(\alpha,\beta)}(x)\sin^{\alpha+3/2}\theta_k\right| \Big/ \left[\sin\frac{1}{2}(\theta-\theta_k)\sin\frac{1}{2}(\theta+\theta_k)\right] = O(i^{-1}).$$

Hence (3.8) holds for $-1 \leqq \alpha \leqq -\frac{1}{2}$.

Case (ii): $-\frac{1}{2} < \alpha$. We have

$$|l_{kn}^{(\alpha,\beta)}(x)| = O(1)n^{-1/2}\left|P_n^{(\alpha,\beta)}(x)\,\sin^{\alpha+1/2}\theta_k / \sin\frac{1}{2}(\theta-\theta_k)\right| =$$

$$= O(1)\left\{n^{-1/2}\left|P_n^{(\alpha,\beta)}(x)\sin^{\alpha+1/2}\theta / \sin\frac{1}{2}(\theta-\theta_k)\right| + \right.$$

$$\left. + n^{-1/2}|P_n^{(\alpha,\beta)}(x)| \Big/ \left|\sin\frac{1}{2}(\theta-\theta_k)\right|^{1/2-\alpha}\right\} = O(i^{\alpha-1/2}).$$

Now we turn to (3.10). It is easy to see that

$$|l_{jn}^{(\alpha,\beta)}(x)| = |P_n^{(\alpha,\beta)}(x) - P_n^{(\alpha,\beta)}(x_j)| / |P_n'^{(\alpha,\beta)}(x_j)(x-x_j)| = |P_n'^{(\alpha,\beta)}(u_j)/P_n'^{(\alpha,\beta)}(x_j)|,$$

where u_j is between x and x_j. Noticing that $|P_n'^{(\alpha,\beta)}(x)|$ takes its maximum at the zeros of $P_n''^{(\alpha,\beta)}(x)$ and applying (3.3), we obtain (3.10).

LEMMA 3.2. *The following estimates are valid:*

$$x - x_k = O\left(\frac{i}{n}\sqrt{1-x^2} + \frac{i^2}{n^2}\right), \quad (k \neq j) \tag{3.11}$$

$$x - x_j = O\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right), \tag{3.12}$$

where the sign "O" does not depend on n and x, only on α and β.

Lemma 3.2 is a consequence of (3.2).

4. Proofs

PROOF OF THEOREM 2.1. Let m be a positive integer such that $x_{m+1}<x\leqq x_m$. Write

$$L_n(f,x)-f(x)=\sum_{k=1}^{m}[f(x_k)-f(x)]\,l_k(x)+\sum_{k=m+1}^{k}[f(x_k)-f(x)]\,l_k(x):=I_1+I_2. \tag{4.1}$$

Applying Abel Transformation to I_1, we have

$$I_1=[f(x_m)-f(x)]\sum_{k=1}^{m}l_k(x)+\sum_{k=1}^{m-1}[f(x_k)-f(x_{k+1})]\sum_{i=1}^{k}l_i(x):=I_{11}+I_{12}. \tag{4.2}$$

From (3.7), (3.10) and (3.12)

$$I_{11}=O\left(V_A(f,[x,x_m])\right)=O\left(V_A\left(f,\left[x,x+c\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)\right]\right)\right). \tag{4.3}$$

Using (3.7), (3.8), (3.9) and Lemma 3.2, we get

$$I_{12}=O(1)\sum_{i=1}^{m-1}|f(x_{m+1-i})-f(x_{m-i})|\,(i^{-1}+i^{-\gamma}), \tag{4.4}$$

wher $\gamma=\min\left\{\frac{1}{2}-\alpha,\frac{1}{2}-\beta\right\}$.

Applying Abel transformation again, we obtain

$$I_{12}=O(1)\Big\{\sum_{i=1}^{m-1}|f(x_{m+1-i})-f(x_{m-i})|\left((m-1)^{-1}+(m-1)^{-\gamma}\right)+ \tag{4.5}$$

$$+\sum_{k=1}^{m-2}(k^{-2}+k^{-1-\gamma})\sum_{i=1}^{k}|f(x_{m+1-i})-f(x_{m-i})|\Big\}.$$

Arrange $|f(x_{m+1-i})-f(x_{m-i})|$ $(i=1,\ldots,m-1)$ in decreasing order:

$$|f(x_{m+1-k_1})-f(x_{m-k_1})|\geqq|f(x_{m+1-k_2})-f(x_{m-k_2})|\geqq\ldots\geqq|f(x_{m+1-k_{m-1}})-f(x_{m-k_{m-1}})|.$$

By Chebyshev inequality (see [11]) we get

$$k^{-1}\sum_{i=1}^{k}|f(x_{m+1-i})-f(x_{m-i})|=\left[k^{-1}\sum_{i=1}^{k}\lambda_i^{-1}\sum_{i=1}^{k}|f(x_{m+1-k_i})-f(x_{m-k_i})|\right]\Big/\left(\sum_{i=1}^{k}\lambda_i^{-1}\right)\leqq$$

$$\leqq\left[\sum_{i=1}^{k}\lambda_i^{-1}|f(x_{m+1-k_i})-f(x_{m-k_i})|\right]\Big/\left(\sum_{i=1}^{k}\lambda_i^{-1}\right)\leqq V_A(f,[x,x_{m-k}])\Big/\left(\sum_{i=1}^{k}\lambda_i^{-1}\right)\leqq$$

$$\leqq V_A\left(f,\left[x,x+c\left(\frac{k}{n}\sqrt{1-x^2}+\frac{k^2}{n^2}\right)\right]\right)\left(\sum_{i=1}^{k}\lambda_i^{-1}\right)^{-1}.$$

Hence and from (4.5)

$$I_{12}=O(1)\left\{V_A\left(f,\left[x,\,x+c\left(\frac{m-1}{n}\sqrt{1-x^2}+\frac{(m-1)^2}{n^2}\right)\right]\right)\right\}\times$$

$$\times\Big(\Big(\sum_{i=1}^{m-1}\lambda_i^{-1}\Big)^{-1}+(m-1)^{1-\gamma}\Big(\sum_{i=1}^{m-1}\lambda_i^{-1}\Big)^{-1}\Big)+$$

$$+\sum_{k=1}^{m-2}V_A\left(f,\left[x,\,x+c\left(\frac{k}{n}\sqrt{1-x^2}+\frac{k^2}{n^2}\right)\right]\right)(k^{-1}+k^{-\gamma})\Big(\sum_{i=1}^{k}\lambda_i^{-1}\Big)^{-1}.$$

Observing that for $\lambda_k=k^t$ $(0\leqq t<1)$

$$V_A\left(f,\left[x,\,x+c\left(\frac{m-1}{n}\sqrt{1-x^2}+\frac{(m-1)^2}{n^2}\right)\right]\right)(1+(m-1)^{1-\gamma})\Big(\sum_{i=1}^{m-1}\lambda_i^{-1}\Big)^{-1}=$$

$$=O(1)\sum_{k=m-1}^{n}V_A\left(f,\left[x,\,x+c\left(\frac{k}{n}\sqrt{1-x^2}+\frac{k^2}{n^2}\right)\right]\right)(k^{-1}+k^{-\gamma})\Big(\sum_{i=1}^{k}\lambda_i^{-1}\Big)^{-1},$$

we obtain that

$$I_{12}=O(1)\sum_{k=1}^{n}V_A\left(f,\left[x,\,x+c\left(\frac{k}{n}\sqrt{1-x^2}+\frac{k^2}{n^2}\right)\right]\right)(k^{-1}+k^{-\gamma})\Big(\sum_{i=1}^{k}\lambda_i^{-1}\Big)^{-1}. \tag{4.6}$$

Combining (4.2)—(4.3) with (4.6), we have

$$I_1=O(1)\sum_{k=1}^{n}V_A\left(f,\left[x,\,x+c\left(\frac{k}{n}\sqrt{1-x^2}+\frac{k^2}{n^2}\right)\right]\right)(k^{-1}+k^{-\gamma})\Big(\sum_{i=1}^{k}\lambda_i^{-1}\Big)^{-1}. \tag{4.7}$$

Similarly,

$$I_2=O(1)\sum_{k=1}^{n}V_A\left(f,\left[x-c\left(\frac{k}{n}\sqrt{1-x^2}+\frac{k^2}{n^2}\right),\,x\right]\right)(k^{-1}+k^{-\gamma})\Big(\sum_{i=1}^{k}\lambda_i^{-1}\Big)^{-1}. \tag{4.8}$$

Theorem 2.1 follows from (4.7) and (4.8).

PROOF OF THEOREM 2.2. From (4.4)

$$I_{12}=O(1)\Big(\sum_{i=1}^{[n\varepsilon_n]}+\sum_{i=[n\varepsilon_n]+1}^{m-1}\Big)\,|f(x_{m+1-i})-f(_{m-i})|\cdot i^{-1}:=r'_{12}+r''_{12}. \tag{4.9}$$

If $m-1=O(n\varepsilon_n)$, then r''_{12} vanishes. It is evident that

$$r'_{12}=O(1)V_H\big(f,\,[x,\,x+c(\varepsilon_n\sqrt{1-x^2}+\varepsilon_n^2)]\big) \tag{4.10}$$

and

$$r''_{12}=O(1)\Big\{(m-1)^{-1}\sum_{i=[n\varepsilon_n]+1}^{m-1}|f(x_{m+1-i})-f(x_{m-i})|+ \tag{4.11}$$

$$+\sum_{k=[n\varepsilon_n]+1}^{m-2}k^{-2}\sum_{i=[n\varepsilon_n]+1}^{k}|f(x_{m+1-i})+f(x_{m-i})|\Big\}.$$

Hence from (4.2)—(4.3) and (4.10)—(4.11) we have

$$(4.12)\qquad I_1 = O(1)\left\{V_H\left(f, [x, x+c(\varepsilon_n\sqrt{1-x^2}+\varepsilon_n^2)]\right)+V_H\left(f, [x, 1]\right)\Big(\sum_{i=1}^{[n\varepsilon_n]}\lambda_i^{-1}\Big)^{-1}+\right.$$

$$\left.+\sum_{k=[n\varepsilon_n]+1}^{n} V_H\left(f, \left[x, x+c\left(\frac{k}{n}\sqrt{1-x^2}+\frac{k^2}{n^2}\right)\right]\right)\Big(k\sum_{i=1}^{k}\lambda_i^{-1}\Big)^{-1}\right\}.$$

Similarly,

$$(4.13)\qquad I_2 = O(1)\left\{V_H\left(f, [x-c(\varepsilon_n\sqrt{1-x^2}+\varepsilon_n^2), x]\right)+V_H\left(f, [-1, x]\right)\Big(\sum_{i=1}^{[n\varepsilon_n]}\lambda_i^{-1}\Big)^{-1}+\right.$$

$$\left.+\sum_{k=[n\varepsilon_n]+1}^{n} V_H\left(f, \left[x-c\left(\frac{k}{n}\sqrt{1-x^2}+\frac{k^2}{n^2}\right), x\right]\right)\Big(k\sum_{i=1}^{k}\lambda_i^{-1}\Big)^{-1}\right\}.$$

Now, Theorem 2.2 follows from (4.12) and (4.13).

PROOF OF THEOREM 2.3. For $n\geqq 2$, we have

$$(4.14)\quad L_n^{(\alpha,\alpha)}(f, x)-f(x) = \Big(\sum_{k=1}^{m}+\sum_{k=m+1}^{n}\Big)[f(x_k)-f(x)-f'(x)(x_k-x)]\,l_k(x) := R_1+R_2.$$

On using Abel transformation,

$$(4.15)\qquad R_1 = [f(x_m)-f(x)-f'(x)(x_m-x)]\sum_{k=1}^{m} l_k(x)+$$

$$+\sum_{k=1}^{m-1}[f(x_k)-f(x_{k+1})-f'(x)(x_k-x_{k+1})]\sum_{i=1}^{k} l_i(x) =$$

$$= [f'(\zeta_m)-f'(x)](x_m-x)\sum_{k=1}^{m} l_k(x)+$$

$$+\sum_{k=1}^{m-1}[f'(\xi_k)-f'(x)](x_k-x_{k+1})\sum_{i=1}^{k} l_i(x) := R_{11}+R_{12}.$$

where $x<\zeta_m<x_m$ and $x_{k+1}<\xi_k<x_k$.

It is easy to see that

$$(4.16)\qquad R_{11} = O\left(n^{-1}V\left(f', \left[x, x+c\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)\right]\right)\right).$$

For R_{12}, using Abel transformation again, we have

$$(4.17)\qquad R_{12} = \sum_{k=1}^{m-1}[f'(\xi_k)-f'(x)](x_k-x_{k+1})\sum_{i=1}^{k} l_i(x) =$$

$$= [f'(\xi_{m-1})-f'(x)]\sum_{k=1}^{m-1}(x_k-x_{k+1})\sum_{i=1}^{k} l_i(x)+$$

$$+\sum_{k=1}^{m-2}[f'(\xi_k)-f'(\xi_{k+1})]\sum_{l=1}^{k}(x_l-x_{l+1})\sum_{i=1}^{l} l_i(x) := R'_{12}+R''_{12}.$$

Since

$$\sum_{l=1}^{k}(x_l-x_{l+1})\sum_{i=1}^{l}l_i(x)=\sum_{l=1}^{k}x_l\sum_{i=1}^{l}l_i(x)-\sum_{l=2}^{k+1}x_l\sum_{l=1}^{l-1}l_i(x)=$$

$$=\sum_{i=1}^{k}(x_i-x)l_i(x)+(x-x_{k+1})\sum_{i=1}^{k}l_i(x),$$

in view of (3.7) and the fact that sign $P_n'^{(\alpha,\alpha)}(x_k)=(-1)^{k+1}$ ([8]) and $P_n'^{(\alpha,\alpha)}(x_k)$ decreases for $x_k\in[-1,0]$ and increases for $x_k\in[0,1]$ ([9]), we obtain

$$\sum_{l=1}^{k}(x_l-x_{l+1})\sum_{i=1}^{l}l_i(x)=O(1)\Big\{\Big|\sum_{i=1}^{k}P_n(x)\big(P_n'(x_i)\big)^{-1}\Big|+ \tag{4.18}$$

$$+|(x-x_{k+1})l_k(x)|\Big\}=O(1)\Big(\big|P_n(x)\big(P_n'(x_k)\big)^{-1}\big|+n^{-1}|l_k(x)|\Big)=$$

$$=(|x-x_m|+|x_m-x_k|)|l_k(x)|+O(n^{-1})=O(n^{-1}).$$

From (4.17)—(4.18)

$$R_{12}=O(n^{-1})\left\{V\left(f',\left[x,x+c\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)\right]\right)+V(f',[x,1])\right\}. \tag{4.19}$$

Combining (4.15)—(4.16) with (4.19),

$$R_1=O\big(n^{-1}V(f',[x,1])\big).$$

Similarly

$$R_2=O\big(n^{-1}V(f',[-1,x])\big).$$

Finally, combining (4.14) with (4.20)—(4.21), we complete the proof.

5. Some remarks

5.1. It is well-known that for $f\in C\Lambda BV$, the function $V_\Lambda(f,[x,y])$ is continuous in the variables $x,y\in[-1,1]$ ([10]). Therefore the function $V_\Lambda(f,[x,y])$ is uniformly continuous in $-1\leqq x\leqq y\leqq 1$. We introduce the so-called modulus of uniform Λ-variation:

$$V_\Lambda(f,\delta)=\sup_{|x-y|\leqq\delta}V_\Lambda(f,[x,y]).$$

It is obvious that $V_\Lambda(f,\delta)$ has the following properties which are similar to that of the modulus of continuity:

(i) $V_\Lambda(f,\delta)\downarrow 0$ if $\delta\downarrow 0$;

(ii) $V_\Lambda(f,\delta_1+\delta_2)\leqq V_\Lambda(f,\delta_1)+V_\Lambda(f,\delta_2)$;

(iii) $V_\Lambda(f,\lambda\delta)\leqq(1+\lambda)V_\Lambda(f,\delta)$;

(iv) $\omega(f,\delta)\leqq c\cdot V_\Lambda(f,\delta)$.

In view of the proofs of the theorems and the properties of $V_\Lambda(f,\delta)$, we have the following

THEOREM 2.1′. *Under the conditions of Theorem* 2.1,

$$\|L_n^{(\alpha,\beta)}(f,x)-f(x)\|_{[-1,1]}=O(1)\sum_{k=1}^{n}V_\Lambda\left(f,\frac{k}{n}\right)\Big(\sum_{i=1}^{k}1/\lambda_i\Big)^{-1}(k^{-1}+k^{-\gamma}). \tag{5.1}$$

THEOREM 2.2′. *Under the conditions of Theorem* 2.2

$$\|L_n^{(\alpha,\beta)}(f,x)-f(x)\|_{[-1,1]}= \tag{5.2}$$

$$=O(1)\Big\{V_H(f,\varepsilon_n)+V_H(f,[-1,1])\Big(\sum_{i=1}^{[n\varepsilon_n]}1/\lambda_i\Big)^{-1}+\sum_{k=[n\varepsilon_n]+1}^{n}V_H\left(f,\frac{k}{n}\right)\Big(k\sum_{i=1}^{k}1/\lambda_i\Big)^{-1}\Big\}.$$

where $\varepsilon_n\downarrow 0$ *and* $\log\varepsilon_n=o(\log n)$.

THEOREM 2.3′. *Under the conditions of Theorem* 2.3,

$$\|L_n^{(\alpha,\alpha)}(f,x)-f(x)\|_{[-1,1]}=O\big(n^{-1}V(f',[-1,1])\big).$$

5.2. Under the conditions of Theorem 2.1′, the right side of (5.1) is $o(1)$. In fact,

$$\sum_{k=1}^{n}V_\Lambda\left(f,\frac{k}{n}\right)\Big(\sum_{i=1}^{k}1/\lambda_i\Big)^{-1}(k^{-1}+k^{-\gamma})=$$

$$=\Big(\sum_{k=1}^{[\sqrt{n}]}+\sum_{k=[\sqrt{n}]+1}^{n}\Big)V_\Lambda\left(f,\frac{k}{n}\right)\Big(\sum_{i=1}^{k}1/\lambda_i\Big)^{-1}(k^{-1}+k^{-\gamma})=$$

$$=V_\Lambda(f,n^{-1/2})\sum_{k=1}^{[\sqrt{n}]}k^{-1-(\gamma-t)}+V_\Lambda(f,1)\sum_{k=[\sqrt{n}]+1}^{n}k^{-1-(\gamma-t)}=$$

$$=O\big(V_\Lambda(f,n^{-1/2})+V_\Lambda(f,1)/n^{1/2(\gamma-t)}\big)=o(1).$$

5.3. For $f\in\mathrm{CHBV}$, i.e., $\Lambda=\{k\}$, the estimate (5.1) is not adequate in some cases, for example, if $V_\Lambda(f,\delta)\sim 1/\log|\log\delta|$, we have

$$\sum_{k=1}^{n}V_\Lambda(f,k/n)/(k\log k)\geqq\sum_{k=1}^{[\sqrt{n}]}V_\Lambda(f,k/n)/(k\log k)\geqq$$

$$\geqq c\sum_{k=1}^{[\sqrt{n}]}1/\big(k\log k\log\log(n/k)\big)\geqq c\big(\log\log\sqrt{n}\big)^{-1}\sum_{k=1}^{[\sqrt{n}]}(k\log k)^{-1}\geqq c>0.$$

But, under the condition of Theorem 2.2, the right of (5.2) is $o(1)$. It is easy to see that

$$\Big(\sum_{k=1}^{[n\varepsilon_n]}1/k\Big)^{-1}=O\big((\log n\varepsilon_n)^{-1}\big)=O(\log^{-1}n),$$

$$\sum_{k=[n\varepsilon_n]+1}^{n}V_\Lambda(f,k/n)/(k\log k)=O\big(V_\Lambda(f,[-1,1])\sum_{k=[n\varepsilon_n]+1}^{n}(k\log k)^{-1}=$$

$$=O\big(V_\Lambda(f,[-1,1])(\log\log n-\log\log(n\varepsilon_n)\big)=O\big(|\log\varepsilon_n|(\log n)^{-1}=o(1).$$

Take $\varepsilon_n=\log^{-1}n$, then Corollary follows.

5.4. Let $f\in \mathrm{BV}_p$ $(p\geqq 1)$ and $-1\leqq\alpha, \beta<p^{-1}-2^{-1}$. Choose $(p-1)p^{-1}<t<$ $<\min\left\{\frac{1}{2}-\alpha, \frac{1}{2}-\beta\right\}$. Since for $q=p(p-1)^{-1}$

$$\sum_{k=1}^{n} k^{-1}|f(x_k)-f(x_{k-1})| \leqq \Big(\sum_{k=1}^{n} |f(x_k)-f(x_{k-1})|^p\Big)^{1/p}\Big(\sum_{k=1}^{n} k^{-tq}\Big)^{1/q} =$$

$$= O\Big(\Big(\sum_{k=1}^{n} |f(x_k)-f(x_{k-1})|^p\Big)^{1/p}\Big) = O\big(V_p(f,[a,b])\big).$$

Hence $\mathrm{BV}_p\subset \Lambda\mathrm{BV}$ $(\Lambda=\{k^t\},\ t>(p-1)p^{-1})$. Therefore, our results include the convergence results of P. Vértesi and A. A. Kelzon.

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О СУЩЕСТВОВАНИИ ЛИНЕЙНОГО ПОЛИНОМИАЛЬНОГО ОПЕРАТОРА НА ТОПОЛОГИЧЕСКИХ ГРУППАХ

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1. Мы будем рассматривать функции, определенные на бикомпактной коммутативной группе G. Поскольку функции $\{e^{ikx}\}_{k=-\infty}^{\infty}$ являются характерами группы вращения окружности, то тригонометрический случай теории полиномиальных операций является частным случаем наших рассмотрений. Естественным обобщением понятия обыкновенного тригонометрического полинома являются конечные линейные комбинации характеров группы G. Будем предполагать, что в G установлено инвариантное интегрирование и что мера $\mu(G)=1$. Тогда характеры $\{\chi_k\}$ группы G образуют ортонормированную систему функций относительно меры μ[1]

$$\int \chi_k(t)\,\bar{\chi}_j(t)\,d\mu(t) = \delta_{kj}.$$

Обобщенным тригонометрическим полиномом на группе G называется функция вида

$$p(x) = \sum_{k=1}^{m} \lambda_k \chi_k(x) \quad x \in G, \tag{1}$$

где $\{\lambda_k\}_{k+1}^{m}$ — комплексные числа. Совокупность характеров $\{\chi_j(x)\}_{j=1}^{m}$ будем обозначать через $C(P)$ и называть спектром полинома P. Пусть P — фиксированный обобщенный тригонометрический полином и Q — произвольный обобщенный тригонометрический полином. Мы говорим, что Q спектра P, если $C(Q)\subset C(P)$. Множество всех обобщенных тригонометрических полиномов спектра $C(P)$ обозначим через $\Pi(P)$. Через $L'=L'(G)$ обозначим совокупность всех μ-измеримых функций f, для которых $\int |f|\,d\mu(t)<\infty$. В дальнейшем мы пользуемся аддитивной записью операции в G. Через $f_t(x)$ обозначим функцию $f(x+t)$, где $t\in G$.

Рассмотрим функциональное пространство $E=E(G)$, которое определяется с помощью следующих аксиом: 1. Элементы E суть функции из L'. 2. E является линейным нормированным пространством. При этом сложение функций и умножение функции на число определяются обычным образом. 3. Если $f\in E$, то $f_t\in E$ при любом $t\in G$, причем $\|f_t\|\leqq\|f\|$. 4. E содержит множество всех обобщенных тригонометрических полиномов.

Определение. Пусть E_1 и E_2 — пространства типа E. Будем говорить, что $U=U(f)$ — обобщенный линейный тригонометрический полиномиальный

[1] В дальнейшем предполагается, что интеграл берется относительно всей группы G.

оператор спектра $C(P)$ на группе G, если: 1) $U(f)$ есть линейный оператор из E_1 и E_2; 2) для любой $f\in E_1$, $U(f)\in\Pi(P)$, где P — заданный фиксированный обобщенный тригонометрический полином на группе G вида (1). Множество всех таких U обозначим через $\mathfrak{M}(P)$. Через $\mathfrak{N}(P)$ обозначим подмножество $\mathfrak{M}(P)$, состоящее из операторов, коммутативных со сдвигом. Таким образом, если $U\in\mathfrak{N}(P)$, то для любой $f\in E_1$ и любого $t\in G$ имеет место равенство $U(f_t)=(U(f))_t$. Оператор $U\in\mathfrak{M}(P)$ называется положительным, если из $f(x)\geqq 0$, $x\in G$, следует, что $U(f,x)\geqq 0$, $x\in G$.

2. Множество всех линейных положительных операторов из $\mathfrak{M}(P)$ обозначим через $\Delta(P)$. Пусть $U\in\mathfrak{M}(P)$. Введем оператор[2]

$$\tilde{U}(f,x)=\int U(f_t,x-t)\,d\mu(t). \tag{2}$$

Относительно $\tilde{U}(f,x)$ справедливы следующие утверждения [1]: 1) Для любого $U\in\mathfrak{M}(P)$, $\tilde{U}\in\mathfrak{N}(P)$. 2) Если $U\in\mathfrak{N}(P)$, то $\tilde{U}=U$. 3) $\|\tilde{U}\|\leqq\|U\|$.

Теорема 1. *Если в $\mathfrak{M}(P)$ имеется линейный положительный полиномиальный оператор, то и в $\mathfrak{N}(P)$ имеется линейный положительный полиномиальный оператор.*

Доказательство. Пусть $U_0\in\Delta(P)$. Из (2) очевидно, что и $\tilde{U}_0\in\Delta(P)$. Воспользуемся свойством 1) оператора $\tilde{U}$, тогда получим требуемое. Из теоремы 1, в частности, следует, что линейные положительные операторы следует искать среди линейных операторов, коммутативных со сдвигом.

Теорема 2. *Пусть операция $U\in\mathfrak{M}(P)$ такова, что существует свертка вида*

$$\sigma(f,x)=\int f(x+t)\overline{K}(t)\,dt, \tag{3}$$

где $K\in\Pi(P)$, совпадающая с оператором U на $\Pi(P)$. Тогда для того чтобы в $\mathfrak{M}(P)$ существовал линейный положительный оператор необходимо и достаточно, чтобы $\sigma(f)$ был линейный положительный оператор.

Доказательство. Достаточность очевидна, ибо из определения оператора (3) видно, что он из $\mathfrak{M}(P)$. Необходимость следует из того, что если $U\in\Delta(P)$, то и оператор $\tilde{U}\in\Delta(P)$. В [1] доказано, что $\tilde{U}=\sigma$. Поэтому $\sigma\in\Delta(P)$.

Теорема 3. *Пусть оператор $U\in\mathfrak{M}(P)$ коммутирует со сдвигом. Тогда для любой $f\in E_1$ и любого $x\in G$ имеет место равенство*

$$U(f,x)=\int f(x+t)U\Big(\sum_{\chi\in C(p)}\chi,-t\Big)\,d\mu(t). \tag{4}$$

То есть $U(f,x)$ может быть представлен в виде свертки (3).

Доказательство. Уже отмечалось, что если $U\in\mathfrak{N}(P)$, то $\tilde{U}=U$. Пусть $S(f,P)$ — частная сумма ряда Фурье функции f спектра $C(P)$:

$$S(f,p)=\sum_{\chi\in C(p)}C_\chi\chi,\quad C_\chi=C_\chi(f)=\int f\bar{\chi}\,d\mu(t). \tag{5}$$

[2] Тривиальный нулевой оператор исключается из наших рассмотрений.

В [1] доказано, что если $U\in\mathfrak{M}(P)$, то для любой $f\in E_1$ имеет место тождество

$$\tilde{U}(f-S(f,p))=0. \tag{6}$$

Так как $\tilde{U}=U$, то (6) принимает вид:

$$U(f-S(f,p))=0. \tag{7}$$

Из (5) ясно, что

$$S(f,p)(x)=\int f(t)\sum_{\chi\in C(p)}\chi(x)\bar{\chi}(t)\,d\mu(t).$$

Отсюда, в силу свойства характеров группы G, получаем, что

$$S(f,p)(x)=\int f(t)\sum_{\chi\in C(p)}\chi(x-t)\,d\mu(t). \tag{8}$$

В силу инвариантности интегрирования и перестановки оператора и интеграла из (7), (8) выводим (4). См. [2], [3].

Теорема 4. *Пусть $U\in\mathfrak{M}(P)$ и коммутирует со сдвигом. Для того чтобы U был положительным оператором необходимо и достаточно, чтобы выполнялось условие*

$$K(t)\overset{\text{def}}{=}U\Big(\sum_{\chi\in C(p)}\chi,-t\Big)\geqq 0,\quad t\in G. \tag{9}$$

Доказательство. Достаточность непосредственно следует из (4) и (9). Докажем необходимость условия (9) для того, чтобы оператор $U\in\mathfrak{M}(P)$ был положительным. Положим в (4) $x=0$, тогда получим, что

$$U(f,0)=\int f(t)\overline{K}(-t)\,d\mu(t).$$

Допустим, что для $t_0\in G$ выполняется неравенство

$$\overline{K}(-t_0)<0. \tag{10}$$

Тогда в силу свойства группы G, (10) выполняется в некоторой окрестности положительной μ-меры. Стало быть, легко построить такую функцию $f_0(t)\in E_1$, что $f_0(t)\geqq 0$, $t\in G$ и $U(f_0,0)<0$.

Обозначим через $\Lambda(x)$ функцию, обладающую тем свойством, что для произвольного $x\in G$, $\Lambda(x)=1$. Среди различных операторов $\mathfrak{N}(P)$ выделим подкласс для которых $U(\Lambda,x)=\Lambda(x)$, $x\in G$. Будем считать, что $\|\Lambda\|=1$. Имеет место

Следствие 1. *Пусть $U\in\mathfrak{M}(P)$ положительный оператор, коммутирующий со сдвигом и пусть $U(\Lambda,x)=\Lambda(x)$. Тогда $\|U\|=1$.*

Доказательство. Из $U(\Lambda,x)=\Lambda(x)$ и $\|\Lambda\|=1$ следует, что

$$\|U\|\geqq 1. \tag{11}$$

С другой стороны, так как оператор и интеграл перестановочны, то в силу

теоремы 3, ([2], [3]), имеем, что[3]

$$\|U(f)\| \leqq \|f\| \int |\overline{K}(-t)|\, d\mu(t). \tag{12}$$

Воспользуемся теперь теоремой 4, тогда из (12) получим, что

$$\|U(f)\| \leqq \|f\| \int \overline{K}(-t)\, d\mu(t). \tag{13}$$

По условию $U(\Lambda, x)=\Lambda(x)$. Поэтому из (4) выводим, что

$$\int \overline{K}(-t)\, d\mu(t) = 1. \tag{14}$$

Из (13) и (14) вытекает, что

$$\|U\| \leqq 1. \tag{15}$$

В силу (11) и (15) получаем, что $\|U\|=1$.

3. Рассмотрим теперь тригонометрический случай, которой, как уже отмечалось, является частным случаем наших рассмотрений. Естественно, что в элементарном случае можно получить более сильные результаты. Пусть $\tilde{C}$ — пространство всех 2π-периодических функций с нормой $\|f\| = \operatorname{Max}_{0 \leqq x < 2\pi} |f(x)|$. Обозначим через $\Omega_{k,n}$ множество всех линейных операторов $U(f)$ из $\tilde{C}$ в $\tilde{C}$, облающих свойствами: 1) для любой $f \in \tilde{C}$, $U(f)$ есть полином порядка $\leqq n$; 2) если f есть тригонометрический полином порядка $\leqq k$, то $U(f)=f$. Очевидно, что $\Omega_{n,n} C \Omega_{n-1,n} C \ldots C \Omega_{1,n}$. Настоящий пункт посвящен вопросу о существовании линейного положительного оператора в $\Omega_{k,n}$. Ответ на этот вопрос дает

Следствие 2. *Пусть* $1 \leqq k \leqq n$, *тогда в классе операторов* $\Omega_{k,n}$ *нет линейного положительного полиномиального оператора. В классе* $\Omega_{0,n}$ *существуют линейные положительные полиномиальные операторы.*

Доказательство. Очевидно, что в первой части следствия 2 нужно доказать, что нет линейного положительного полиномиального оператора в классе $\Omega_{1,k}$. Из предыдущего пункта следует, что искомый оператор должен быть вида

$$U(f, x) = \frac{1}{\pi} \int_0^{2\pi} f(x+t) K(t)\, dt, \tag{16}$$

где $K(t)$ — тригонометрический полином порядка n. Поскольку по условию $U \in \Omega_{1,n}$, то $U(f)=f$, если f — тригонометрический полином порядка $\leqq 1$. Стало быть, $K(t)$ можно записать в виде

$$K(t) = \frac{1}{2} + \cos t + \ldots + \alpha_n \cos nt. \tag{17}$$

В силу теоремы 4 получаем, что оператор (16) положительный тогда и только тогда, когда $K(t) \geqq 0$, $-\infty < t < \infty$. Воспользуемся теперь теоремой Л. Фе-

[3] См. например [2], стр. 93—95.

йера [4], согласно которой полином $p(t)=\frac{1}{2}+\beta_1\cos t+\beta_2\cos 2t+\ldots+\beta_n\cos nt$ удовлетворяет неравенству $p(t)\geqq 0$, $-\infty<t<\infty$, тогда, когда $|\beta_1|\leqq\cos\frac{\pi}{n+2}$. Очевидно, что полином (17) этому неравенству не удовлетворяет. Итак, первая часть следствия 2 доказана.

Доказательство второй части следствия 2 следует, например, из того факта, что интеграл Л. Фейера

$$F_n(f,x)=\frac{1}{(n+1)\pi}\int_0^{2\pi} f(x+t)\left[\sin^2\frac{n+1}{2}\,t/2\sin^2\frac{t}{2}\right]dt$$

задает линейный положительный полиномиальный оператор из класса $\Omega_{0,n}$[4]. Любопытно, что в случае класса операторов $\Omega_{n,n}$ можно доказать теорему 6 без использования упомянутой теоремы Л. Фейера [4]. Действительно, известно [5], что если $U\in\Omega_{n,n}$, то

$$\frac{1}{2\pi}\int_0^{2\pi} U(f_t,x-t)\,dt=S_n(f,x),$$

где $S_n(f,x)$ — частная сумма порядка n ряда Фурье функции f. Поэтому если бы существовал линейный положительный полиномиальный оператор в $\Omega_{n,n}$, то $S_n(f,x)$ был бы положительным оператором. Напишем формулу для $S_n(f,x)$

$$S_n(f,x)=\frac{1}{\pi}\int_0^{2\pi} f(x+t)D_n(t)\,dt,$$

где

$$D_n(t)=\sin\left(n+\frac{1}{2}\right)t/2\sin\frac{t}{2}. \tag{18}$$

Из (18) очевидно, что $D_n(t)$ не удовлетворяет всюду неравенству $D_n(t)\geqq 0$. Поэтому $S_n(f,x)$ не является положительным оператором.

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[4] Рецензент заметил, что следствие 2 также может быть доказано с помощью теоремы П. П. Коровкина о положительных операторах.

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ON A GENERAL OPEN MAPPING THEOREM BY M. VALDIVIA

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1. A general open mapping theorem in locally convex spaces

In this first part, we take locally convex space for Hausdorff, locally convex topological vector space.

In [2], M. Valdivia obtained general theorems of isomorphism and homomorphism by introducing the W_r, W, V_r and V spaces, which are defined as follows:

DEFINITION 1. A locally convex space $\mathbf{E}[\tau]$ is a W_r space if for every locally convex topology τ' on $\mathbf{E}$, strictly finer than τ, there is a locally convex and barrelled topology on $\mathbf{E}$, strictly coarser than τ'.

Obviously, every barrelled space is a W_r space (Theorem 1 of [2]).

DEFINITION 2. A locally convex space $\mathbf{E}[\tau]$ is a W space if for every locally convex topology τ' on $\mathbf{E}$, strictly finer than τ, there is a closed linear subspace G of $\mathbf{E}[\tau']$ and a locally convex and barrelled topology T on $\mathbf{E}/G$, strictly coarser than the topology induced by τ' on $\mathbf{E}/G$.

Every W_r space is a W space (see Theorem 2 of [2]).

DEFINITION 3. A locally convex space $\mathbf{E}[\tau]$ is a V_r space if every dense linear subspace of $\mathbf{E}'[\sigma(\mathbf{E}', \mathbf{E})]$ in which every closed and bounded subset is equicontinuous and compact, coincides with $\mathbf{E}'$.

DEFINITION 4. A locally convex space $\mathbf{E}[\tau]$ is a V space if every linear subspace of $\mathbf{E}'[\sigma(\mathbf{E}', \mathbf{E})]$ in which every closed and bounded subset is equicontinuous and compact, is closed.

According to Theorems 3 and 5 of [2], every V space and every B_r-complete space is a V_r space. According to Theorems 4 and 6 of [2], every B-complete-space (or Ptàk-space) and every space with a countable Hamel base, are V spaces.

It is very easy to characterize the V_r and V spaces without taking part the dual. We give these characterizations, with abbreviated demonstrations because of their simplicity. The theorems of the isomorphism and the homomorphism of M. Valdivia can be immediately deduced from them. Also we can extend these theorems for other classes of spaces.

PROPOSITION 1. *A locally convex space* $\mathbf{E}[\tau]$ *is a* V_r *space if and only if there is no locally convex and barrelled topology on* $\mathbf{E}$, *strictly coarser than* τ.

PROOF. Let $\mathbf{E}'$ be the dual space of $\mathbf{E}[\tau]$. The barrelled locally convex topologies strictly coarser than τ correspond to the quasi-complete, proper linear subspaces of $\mathbf{E}'$ whose bounded subsets are τ-equicontinuous.

PROPOSITION 2. *A locally convex space* $\mathbf{E}[\tau]$ *is a V space if and only if every separated quotient of it is a* V_r *space.*

PROOF. Let $\mathbf{F}$ be a closed linear subspace of $\mathbf{E}[\tau]$. The barrelled locally convex topologies on $\mathbf{E}/\mathbf{F}$ that are strictly coarser than the ones induced by τ on this space, are in correspondence to the proper, closed, quasi-complete linear subspaces of $\mathbf{F}$ which bounded subsets are equicontinuous.

This proposition is contained in Theorems 3, 10 and 11 of [2].

From Definition 1 and Proposition 1 we can obtain directly the next result.

PROPOSITION 3. *A locally convex space* $\mathbf{E}[\tau]$ *is a* W_r *space if and only if there is no locally convex topology* τ' *on* $\mathbf{E}$ *strictly finer than* τ *such that* $\mathbf{E}[\tau']$ *is a* V_r *space.*

The isomorphism theorem by M. Valdivia is a direct consequence of Propositions 1 and 3. The following result contains Theorems 7, 8 and 9 of [2].

THE ISOMORPHISM THEOREM OF M. VALDIVIA. *Let* $\mathbf{E}[\tau]$ *be a locally convex space which is a* V_r *space, let* $\mathbf{F}[v]$ *be a locally convex space which is a* W_r *space, and let u be a continuous, one-one, linear mapping from* $\mathbf{E}[\tau]$ *onto* $\mathbf{F}[v]$. *Then u is an isomorphism. If* $\mathbf{E}[\tau]$ *is a locally convex space which is not a* V_r *space, there is a locally convex space* $\mathbf{F}[v]$, *which is* W_r *space, and there is a continuous, one-one, linear mapping u from* $\mathbf{E}[\tau]$ *onto* $\mathbf{F}[v]$, *which is not an isomorphism. If* $\mathbf{F}[v]$ *is a locally convex space which is not a* W_r *space, there is a locally convex space* $\mathbf{E}[\tau]$, *which is* V_r *space, and there is a continuous, one-one, linear mapping u from* $\mathbf{E}[\tau]$ *onto* $\mathbf{F}[v]$ *which is not an isomorphism.*

PROPOSITION 4. *A locally convex space* $\mathbf{E}[\tau]$ *is a* W_r *space if and only if there is no locally convex topology* τ' *on* $\mathbf{E}$ *strictly finer than* τ *such that* $\mathbf{E}[\tau']$ *is a V space.*

PROOF. It is an easy consequence of Definition 2 and Proposition 2.

The homomorphism theorem by M. Valdivia, (Theorems 13 and 14 of [2]) are direct consequences of our Propositions 3 and 4. The following theorem contains these results.

THE HOMOMORPHISM THEOREM OF M. VALDIVIA. *Let* $\mathbf{E}[\tau]$ *be a locally convex space which is a V space, let* $\mathbf{F}[v]$ *be a locally convex space which is a W space, and let u be a continuous linear mapping from* $\mathbf{E}[\tau]$ *onto* $\mathbf{F}[v]$. *Then u is a homomorphism. If* $\mathbf{E}[\tau]$ *is a locally convex space which is not a V space, it is possible to find a locally convex space* $\mathbf{F}[v]$ *which is a W space and a continuous linear mapping from* $\mathbf{E}[\tau]$ *onto* $\mathbf{F}[v]$ *which is not an homomorphism. If* $\mathbf{F}[v]$ *is a locally convex space which is not a W space, it is possible to find a locally convex space* $\mathbf{E}[\tau]$ *which is V space and a continuous linear mapping from* $\mathbf{E}[\tau]$ *onto* $\mathbf{F}[v]$ *which is not an homomorphism.*

2. Generalization of classes of topological spaces

Let ζ be a class of topological spaces. Let v_r be the family of all topological spaces (X, τ) such that there is no topology μ on X strictly coarser than τ, such that $(X, \mu) \in \zeta$. If ζ does not contain any antidiscrete spaces, the family v_r is not empty. Let ω_r be the family of all topological spaces (X, τ) such that there is no topology

μ on X, strictly finer than τ and such that $(X, \mu) \in \nu_r$. Obviously $\zeta \subset \omega_r$, and if $(X, \tau) \in \omega_r$ and μ is a topology on X finer than τ, $(X, \mu) \in \omega_r$, and if $(X, \tau) \in \nu_r$ and μ is a topology on X coarser than τ, then $(X, \mu) \in \nu_r$. The following theorem is a very easy consequence of previous definitions.

THE HOMEOMORPHISM THEOREM. *Every continuous, one-one mapping from a space of ν_r class onto a space of class ω_r is a homeomorphism. If (X, τ) is a topological space which does not belong to the class ν_r, there is a topological space (Y, μ), which belongs to the class ω_r and there is a continuous, one-one mapping from (X, τ) onto (Y, μ) which is not a homeomorphism. If (Y, μ) is a topological space which does not belong to the class ω_r, there is a topological space (X, τ), which belongs to the class ν_r, and there is a continuous, one-one mapping from (X, τ) onto (Y, μ) which is not a homeomorphism.*

For example if ζ is the class of all Hausdorff topological spaces, the corresponding class ν_r is the class of all topological spaces (X, τ) such that if τ' is a finer topology on X than τ, (X, τ') is Hausdorff. The corresponding class ω_r is the class of all Hausdorff topological spaces. Restricting us to the class of Hausdorff topological spaces, if ζ is the class of all compact spaces, considering that every continuous, one-to-one mapping from a compact Hausdorff space on a Hausdorff topological space is open (see [1] § 3.2. (6)), we obtain that $\zeta = \nu_r$ and ω_r is the class of all Hausdorff topological spaces.

Let ν be the subclass of ν_r stable for the formation of final topologies, ν is not empty if ζ has this property. Let ω be the class of those topological spaces (X, τ) for which there is no topology μ on X, strictly finer than τ such that $(X, \mu) \in \nu$. Obviously we can obtain the following result.

THE OPEN MAPPING THEOREM. *Every continuous mapping from a space of class ν onto a space of class ω is open. If (X, τ) is a topological space which does not belong to the class ν, there is a topological space (Y, μ) which belongs to the class ω and there is a continuous mapping from (X, τ) onto (Y, μ) which is not open. If (Y, μ) is a topological space which does not belong to the class ω, there is a topological space (X, τ) which belongs to the class ν, and there is a continuous mapping from (X, τ) onto (Y, μ) which is not open.*

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ON STRONG GENERALIZED HAUSDORFF SUMMABILITY

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Introduction

For a series $\sum_0^\infty a_k$, let $s_n = \sum_{k=0}^n a_k$. Let $Q = \{q_{n,k}\}$ $(n, k = 0, 1, \ldots)$ be a matrix and let

$$\sigma_n = Q(s_n) = \sum_{k=0}^\infty q_{n,k} s_k .$$

The series $\sum_0^\infty a_k$ is said to be summable Q to s if σ_n exists for $n = 0, 1, \ldots$ and tends to s as n tends to infinity. In this case we write $s_n \to s(Q)$. The symbol P is reserved for matrices $\{p_{n,k}\}$ with $p_{n,k} \geqq 0$, and I denotes the identity matrix. We now recall the definition of strong summability introduced by Borwein [1].

Strong summability. A series $\sum_0^\infty a_k$ is said to be summable $[P, Q]_\beta$ $(\beta > 0)$ to s if $\sum_{k=0}^\infty p_{n,k} |\sigma_k - s|^\beta$ exists for $n = 0, 1, \ldots$ and tends to zero as n tends to infinity. In this case we write $s_n \to s[P, Q]_\beta$.

For summability methods V and W, the notation $V \subseteq W$ means that any series summable V to s is also summable W to s. The notation $V \simeq W$ means that both $V \subseteq W$ and $W \subseteq V$.

Generalized Hausdorff matrices. Suppose throughout that $\lambda = \{\lambda_n\}$ is a sequence of real numbers with

$$\lambda_0 \geqq 0, \quad \inf_{n \geqq 1} \lambda_n > 0 \quad \text{and} \quad \sum_{n=0}^\infty 1/\lambda_n = \infty .$$

Let Ω be a simply connected region that contains every positive λ_n, and suppose, for $n = 0, 1, \ldots$, that Γ_n is a positively sensed Jordan contour lying in Ω and enclosing every $\lambda_k \in Q$ with $0 \leqq k \leqq n$. Suppose that f is holomorphic in Ω and that $f(\lambda_0)$ is defined even when $\lambda_0 \notin \Omega$. Define

$$(1) \qquad \lambda_{n,k} = \begin{cases} -\lambda_{k+1} \ldots \lambda_n \dfrac{1}{2\pi i} \displaystyle\int_{\Gamma_n} \frac{f(z)\, dz}{(\lambda_k - z) \ldots (\lambda_n - z)} + \delta_k & \text{for} \quad 0 \leqq k \leqq n \\ 0 \quad \text{for} \quad k > n \end{cases}$$

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where $\delta_k=f(\lambda_0)$ if $k=0$ and $\lambda_0\notin\Omega$, and $\delta_k=0$ otherwise. Here and elsewhere we observe the convention that products like $\lambda_{k+1}\dots\lambda_n=1$ when $k=n$. Denote the triangular matrix $\{\lambda_{n,k}\}$ by $(\lambda;f)$. This is called a generalized Hausdorff matrix. The set of all such matrices is denoted by $\mathscr{H}_\lambda$.

For α any real number, the generalized Hausdorff matrix H_α is defined to be the matrix $(\lambda;f)$ with $f(z)=(z+1)^{-\alpha}$. For $\alpha>-1$, the generalized Cesàro matrix C_α is defined to be the matrix $(\lambda;f)$ with

$$f(z)=\frac{\Gamma(\alpha+1)\Gamma(z+1)}{\Gamma(\alpha+z+1)}.$$

These reduce to the standard Hölder and Cesàro matrices when $\lambda_n=n$. (See [1].)

Preliminary results

For $0<t\leqq 1$, let $\lambda_{n,k}(t)$ denote the value of $\lambda_{n,k}$ obtained from (1) with $f(z)=t^z$, and let $\lambda_{n,k}(0)=\lambda_{n,k}(0+)$. Let

$$D_0=(1+\lambda_0)\,d_0=1;$$

$$D_n=\left(1+\frac{1}{\lambda_1}\right)\dots\left(1+\frac{1}{\lambda_n}\right)=(1+\lambda_n)\,d_n \quad \text{for} \quad n\geqq 1.$$

Then, (see [3]),

$$\int_0^1 \lambda_{n,k}(t)\,dt=\frac{d_k}{D_n} \quad \text{for} \quad 0\leqq k\leqq n.$$

If

$$f(z)=\int_0^1 t^z\,d\chi(t) \quad \text{with} \quad \chi\in\mathrm{BV} \tag{2}$$

where BV is the space of functions of bounded variation on the closed interval [0, 1], then

$$\lambda_{n,k}=\int_0^1 \lambda_{n,k}(t)\,d\chi(t).$$

It follows that

$$C_1(s_n)=\frac{1}{D_n}\sum_{k=0}^{n} d_k s_k$$

so that

$$s_n-C_1(s_n)=C_1(\lambda_n a_n). \tag{3}$$

If f satisfies (2), $\chi(1)-\chi(0)=1$ and $\chi(0+)=\chi(0)$, then $X=(\lambda;f)$ is regular, i.e. $s_n\to s(X)$ whenever $s_n\to s$. (See [2; Theorem 1].)

Lemma 2 of [2] shows that if g and h are holomorphic in Ω and defined at λ_0, $X=(\lambda;g)$ and $Y=(\lambda;h)$, then

$$XY=YX=(\lambda;gh). \tag{4}$$

Lemma 1 of [3] shows that if $X=(\lambda;f)$ with f satisfying (2), $\tilde{X}=(\lambda;\tilde{f})$ with $\tilde{f}(z)=\int_0^1 t^z\,|d\chi(t)|$, and $\beta\geqq 1$, then, for any sequence $\{w_n\}$,

$$|X(w_n)|^\beta \leqq M^{\beta-1}\tilde{X}(|w_n|^\beta) \tag{5}$$

where $M=\int_0^1 |d\chi(t)|$.

From (4) it can be seen that $H_\alpha H_\delta = H_{\alpha+\delta}$ for all real α, δ. Theorem 2 of [2] shows that

$$C_\alpha \simeq H_\alpha \quad \text{for} \quad \alpha > -1. \tag{6}$$

(See also [5] and [6].) Thus

$$C_\alpha C_\delta \simeq C_{\alpha+\delta} \quad \text{for} \quad \alpha > -1, \quad \delta > -1, \quad \alpha+\delta > -1. \tag{7}$$

Some theorems on strong summability

The first theorem generalizes Theorem 5 in [1].

THEOREM 1. *Suppose Q is a matrix, P is a regular matrix in $\mathcal{H}_\lambda$, and $X=(\lambda;f)$ where $f(z)=\int_0^1 t^z\,d\chi(t)$ with $\chi\in$BV, $\chi(1)-\chi(0)=1$ and $\chi(0+)=\chi(0)$. Then, for $\beta\geqq 1$, $[P,Q]_\beta\subseteqq[P,XQ]_\beta$.*

PROOF. Let $\tilde{X}=\{\tilde{x}_{n,k}\}=(\lambda;\tilde{f})$ where $\tilde{f}(z)=\int_0^1 t^z|d\chi(t)|$. Since $\chi\in$BV and $\chi(0+)=\chi(0)$, it follows that $\lim_{n\to\infty}\tilde{x}_{n,k}=0$ for $k=0,1,\ldots$ and $\sup_{n\geqq 0}\sum_{k=0}^{n}|\tilde{x}_{n,k}|<\infty$. (See [2, Theorem 1].) Hence $\tilde{X}(u_n)\to 0$ whenever $u_n\to 0$. (See [4, Theorem 4].)

Let $\{s_n\}$ be a sequence, $\sigma_n=X(s_n)$ and $w_n=s_n-s$. In view of the regularity of X we have $\sigma_n-s=X(w_n)+\varepsilon_n$ where $\varepsilon_n\to 0$. From (4) and (5) it follows that

$$P(|X(w_n)|^\beta) \leqq M^{\beta-1}P\tilde{X}(|w_n|^\beta) = M^{\beta-1}\tilde{X}P(|s_n-s|^\beta) \tag{8}$$

where $M=\int_0^1 |d\chi(t)|$. Next, by Minkowski's inequality,

$$(P(|X(w_n)+\varepsilon_n|^\beta))^{1/\beta} \leqq (P(|X(w_n)|^\beta))^{1/\beta}+(P(|\varepsilon_n|^\beta))^{1/\beta}. \tag{9}$$

Suppose now that $P(|s_n-s|^\beta)\to 0$. Then, by (8), $P(|X(w_n)|^\beta)\to 0$ so that, by (9), $P(|\sigma_n-s|^\beta)\to 0$. Hence $[P,I]_\beta\subseteqq[P,X]_\beta$, from which it follows that $[P,Q]_\beta\subseteqq\subseteqq[P,XQ]_\beta$. □

The next two theorems generalize corollaries to Theorem 7 in [1].

THEOREM 2. *If* $X \in \mathscr{H}_\lambda$ *and* $\beta \geqq 1$, *then necessary and sufficient conditions for a series* $\sum_0^\infty a_n$ *to be summable* $[C_1, X]_\beta$ *to s are that it be summable* C_1X *to s and that* $\lambda_n a_n \to 0[C_1, C_1X]_\beta$.

PROOF. It follows from Theorem 1 in [1] that $\sum_0^\infty a_n$ is summable $[C_1, X]_\beta$ to s if and only if it is summable C_1X to s and summable $[C_1, (I-C_1)X]_\beta$ to 0. Further, by (3) and (4),

$$(I-C_1)X(s_n) = X\big(s_n - C_1(s_n)\big) = C_1X(\lambda_n a_n).$$

The result follows. □

In conformity with notation introduced earlier (see [1; p. 123]), the generalized strong Cesàro method $[C_1, C_{\alpha-1}]_\beta$ will be denoted by $[C, \alpha]_\beta$ and the generalized strong Hölder method $[H_1, H_{\alpha-1}]_\beta$ by $[H, \alpha]_\beta$. We require the following known result (see [8]).

LEMMA 1. *Let*

$$g(z) = \frac{\Gamma(\delta+1)\Gamma(z+1)(z+1)^\delta}{\Gamma(\delta+z+1)}, \quad \delta > -1.$$

Then both $g(z)$ *and* $1/g(z)$ *can be expressed as Mellin transforms of the form* $\int_0^1 t^z \, d\chi(t)$ *with* $\chi \in \mathrm{BV}$, $\chi(1)-\chi(0)=1$ *and* $\chi(0+)=\chi(0)$.

THEOREM 3. *If* $\alpha \geqq 0$ *and* $\beta \geqq 1$, *then necessary and sufficient conditions for a series* $\sum_0^\infty a_n$ *to be summable* $[C, \alpha]_\beta$ *to s are that it be summable* C_α *to s and that* $\lambda_n a_n \to 0[C, \alpha+1]_\beta$.

PROOF. It follows from Theorem 2 that $\sum_0^\infty a_n$ is summable $[C, \alpha]_\beta$ to s if and only if it is summable $C_1C_{\alpha-1}$ to s and $\lambda_n a_n \to 0[C_1, C_1C_{\alpha-1}]_\beta$. Next, it follows from Lemma 1 with $\delta = \alpha-1$ and Theorem 1 that $\lambda_n a_n \to 0[C_1, C_1C_{\alpha-1}]_\beta$ if and only if $\lambda_n a_n \to 0[C_1, H_\alpha]_\beta$. Applying Lemma 1 and Theorem 1 again, we see that $\lambda_n a_n \to 0[C_1, C_1C_{\alpha-1}]_\beta$ if and only if $\lambda_n a_n \to 0[C_1, C_\alpha]_\beta$. This together with (7) yields the result. □

The above theorem suggests the following extension of the definition of $[C, \alpha]_\beta$ to the case $\alpha = 0$: $\sum_0^\infty a_n$ is summable $[C, 0]_\beta$ to s if the series is convergent with sum s and $\sum_{k=0}^n d_k |\lambda_k a_k|^\beta = o(D_n)$. When $\lambda_n = n$, this definition reduces to the one given by Hyslop [7].

The next theorem is an analogue of the equivalence relation (6) for strong summability.

THEOREM 4. *For* $\alpha \geqq 0$, $\beta \geqq 1$, $[C, \alpha]_\beta \simeq [H, \alpha]_\beta$.

PROOF. The case $\alpha=0$ follows from Theorem 2 and the definition of $[C, 0]_\beta$. Suppose therefore that $\alpha>0$. By Theorem 3, $\sum_0^\infty a_n=s[C, \alpha]$ if and only if $\sum_0^\infty a_n=s(C_\alpha)$ and $\lambda_n a_n\to 0[C_1, C_\alpha]_\beta$. Further, by Theorem 2, $\sum_0^\infty a_n=s[H, \alpha]_\beta$ if and only if $\sum_0^\infty a_n=s(H_\alpha)$ and $\lambda_n a_n\to 0[C_1, H_\alpha]_\beta$. The result now follows from (6), Lemma 1 and Theorem 1. □

Generalized Hausdorff matrices associated with L^p functions

Let L^p denote the function space $L^p(0, 1)$. In this section we deal with Hausdorff matrices $(\lambda; f)$ with $f(z)=\int_0^1 t^z\varphi(t)\,dt$ where $\varphi\in L^p$ for some $p>1$. An ordinary Hausdorff matrix $\{x_{n,k}\}$ satisfies these conditions if and only if $\sum_{k=0}^n |x_{n,k}|^p<$ $<M(n+1)^{1-p}$ for $n=0, 1, \ldots$ where M is independent of n. (See [4, Theorem 215].)

The following lemma is needed for the proof of Theorem 5.

LEMMA 2. *Let $\varphi\in L^p$ with $p>1$. Let $X=(\lambda; f)$ and $X^{(p)}=(\lambda; f^{(p)})$ where $f(z)=\int_0^1 t^z\varphi(t)\,dt$ and $f^{(p)}(z)=\int_0^1 t^z|\varphi(t)|^p\,dt$. If $\mu>\beta\geqq 1$ and $1/p=1/\mu-1/\beta$, then for any sequence $\{w_n\}$,*

$$|X(w_n)|^\mu \leqq M^{\mu(1-1/\beta)}\big(C_1(|w_n|^\beta)\big)^{\mu/\beta-1}X^{(p)}(|w_n|^\beta)$$

where $M=\int_0^1 |\varphi(t)|^p\,dt$.

PROOF. Let $f_n(t)=\sum_{k=0}^n \lambda_{n,k}(t)w_k$ where $0\leqq t\leqq 1$. Then, by Hölder's inequality,

$$|f_n(t)|^\beta \leqq \sum_{k=0}^n \lambda_{n,k}(t)\,|w_k|^\beta.$$

(See [3, (8)].) Hence

$$\int_0^1 |f_n(t)|^\beta\,dt \leqq \sum_{k=0}^n |w_k|^\beta \int_0^1 \lambda_{n,k}(t)\,dt = \frac{1}{D_n}\sum_{k=0}^n d_k|w_k|^\beta = C_1(|w_n|^\beta) \tag{10}$$

and

$$\int_0^1 |\varphi(t)|^p\,|f_n(t)|^\beta\,dt \leqq \sum_{k=0}^n |w_k|^\beta \int_0^1 \lambda_{n,k}(t)\,|\varphi(t)|^p\,dt = X^{(p)}(|w_n|^\beta).$$

It follows, by Hölder's inequality, that

$$|X(w_n)| = \left|\int_0^1 \varphi(t) f_n(t)\, dt\right| \leqq$$

$$\leqq \left(\int_0^1 |\varphi(t)|^p\, dt\right)^{1-1/\beta} \left(\int_0^1 |f_n(t)|^\beta\, dt\right)^{1/\beta-1/\mu} \left(\int_0^1 |\varphi(t)|^p |f_n(t)|\, dt\right)^{1/\mu} \leqq$$

$$\leqq M^{1-1/\beta}\left(C_1(|w_n|^\beta)^{1/\beta-1/\mu} X^{(p)}(|w_n|^\beta)\right)^{1/\mu}. \quad \square$$

The following theorem generalizes Theorem 10 in [1].

THEOREM 5. *Let* $\mu>\beta\geqq 1$, $1/p=1+1/\mu-1/\beta$. *Let* $X=(\lambda;f)$ *where* $f(z)=\int_0^1 t^z\varphi(t)\, dt$ *with* $\varphi\in L^p$ *and* $\int_0^1 \varphi(t)\, dt=1$. *Then, for any matrix* Q, $[C_1, Q]_\beta \subseteqq [C_1, XQ]_\mu$.

The theorem remains valid when $\mu=\infty$ (with $1/p=1-1/\beta$ if $\lambda>1$ and $p=\infty$ if $\beta=1$) provided $[C_1, XQ]_\infty$ is interpreted to mean XQ.

PROOF. We use the notation introduced in Lemma 2, and note that X is regular and $X^{(p)}(v_n)\to 0$ whenever $v_n\to 0$. Suppose that $s_n\to s[C_1, Q]_\beta$, and let $\sigma_n=Q(s_n)$, $w_n=\sigma_n-s$, and $v_n=C_1(|w_n|^p)$.

(i) Suppose μ is finite. By hypothesis, $v_n\to 0$ and hence, by Lemma 2,

$$C_1(|X(w_n)|^\mu) \leqq KC_1 X^{(p)}(|w_n|^\beta) = KX^{(p)}(v_n) \to 0$$

where $K=M_n^{\mu(1-1/\beta)} \sup v_n^{\mu/\beta-1}$. Also, by the regularity of X, we have $X(\sigma_n)-s=X(w_n)+\varepsilon_n$ where $\varepsilon_n\to 0$. Thus, by Minkowski's inequality,

$$\left(C_1(|X(\sigma_n)-s|^\mu)\right)^{1/\mu} \leqq \left(C_1(|X(w_n)|^\mu)\right)^{1/\mu} + \left(C_1(|\varepsilon_n|^\mu)\right)^{1/\mu} \to 0,$$

i.e. $s_n\to s[C_1, XQ]_\mu$.

(ii) Suppose now that $\mu=\infty$. By Hölder's inequality,

$$|X(w_n)|^\beta = \left|\int_0^1 f_n(t)\varphi(t)\, dt\right|^\beta \leqq m\int_0^1 |f_n(t)|^\beta\, dt$$

where $m=M^{\beta-1}$ if $\beta>1$ and $m=\operatorname{ess\,sup}_{0<t<1} |\varphi(t)|$ if $\beta=1$. Since (10) holds under the operative hypotheses, it follows that

$$|X(w_n)|^\beta \leqq mC_1(|w_n|^\beta) = mv_n \to 0,$$

and hence that $s_n\to s(XQ)$. $\square$

THEOREM 6. *Let* $\varrho>1/\beta-1/\mu$ *where* $\mu\geqq\beta\geqq 1$. *Then, for any matrix* Q, $[C_1, Q]_\beta \subseteqq [C_1, C_\varrho Q]_\mu$.

PROOF. When $\mu=\beta$, the result follows from Theorem 1. Suppose that $\mu>\beta$ and let $1/p=1+1/\mu-1/\beta$. Then $C_\varrho=(\lambda;f)$, where $f(z)=\int_0^1 t^z\varphi(t)\,dt$ with $\varphi(t)=$ $=\varrho(1-t)^{\varrho-1}$. Since $\varphi\in L^p$, Theorem 5 now yields the result. □

THEOREM 7. *Let* $\gamma>\alpha+1/\beta-1/\mu$ *where* $\mu\geqq\beta\geqq 1$ *and* α *is any real number. Then* $[H,\alpha]_\beta\subseteq[H,\gamma]_\mu$.

PROOF. Applying first Theorem 6 and then Theorem 1 together with Lemma 1, we get

$$[H,\alpha]_\beta=[H_1,H_{\alpha-1}]_\beta\subseteqq[H_1,C_{\gamma-\alpha}H_{\alpha-1}]_\mu\simeq[H_1,H_{\gamma-\alpha}H_{\alpha-1}]_\mu=$$
$$=[H_1,H_{\gamma-1}]_\mu=[H,\gamma]_\mu.\quad\square$$

REMARK. It is known that, in the special case $\lambda_n=n$, Theorem 6 also holds when $\varrho=1/\beta-1/\mu$ and Theorem 7 when $\gamma=\alpha+1/\beta-1/\mu$. (See [1] and the references there given.) Whether the same is true for more general λ_n is an open question.

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SEMIDISCRETE HILBERT SPACES

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In 1938 Sz. Nagy [14] characterized L^2-spaces of commutative W^*-algebras in the following way:

A Hilbert space H, which is ordered by a selfdual cone H^+ having the Riesz interpolation property, is isomorphic to $L^2(X, \mu)$ for some measure space (X, μ).

Grothendieck [9, Ch. I §, 5.3], [16, Ch. IV, Theorem 2.4] has shown that the identity on $L^2(X, \mu)$ can be approximated in the strong operator topology by factorisation through $\mathbf{C}^m$ with positive linear maps. As it can be seen from a result of Wittstock [23, Theorem 3.1] Grothendieck's property implies the Riesz interpolation property for any Hilbert space H with selfdual cone H^+. Hence it characterizes as well the spaces $L^2(X, \mu)$. In this paper an analogous theory is formulated for L^2-spaces of semidiscrete W^*-algebras.

A W^*-algebra $\mathcal{M}$ is called semidiscrete, if the identity on $\mathcal{M}$ can be approximated in the point-$\sigma(\mathcal{M}, \mathcal{M}_*)$ topology by factorisation through the $m \times m$ matrices M_m with completely positive, normal, unital maps. Defining semidiscreteness this way is due to Choi and Effros [4]. It is equivalent [4, pp. 75—76] to the usual notion of semidiscreteness introduced by Effros and Lance [8]. Effros and Lance [8] proved that a semidiscrete W^*-algebra is injective, while the converse statement was obtained by Wassermann [22]. Wittstock [24] characterized injective W^*-algebras by a non-commutative analogue of the Riesz interpolation property. Haagerup [11], Araki and Masuda [2] and others developed L^p-spaces of W^*-algebras. The L^2-spaces are the Hilbert spaces arising in standard forms of W^*-algebras introduced by Araki [1], Connes [5] and Haagerup [10]. Wittstock and the author [18] characterized L^2-spaces of arbitrary W^*-algebras among matrix ordered Hilbert spaces. Based on this, L^2-spaces of injective W^*-algebras were characterized in [17] by Wittstock's matricial analogue of the Riesz interpolation property. A. M. Torpe [21] has shown that a W^*-algebra is semidiscrete if and only if the identity on the corresponding L^p-space can be approximated in the strong operator topology with completely positive operators of finite rank and norm $\leqq 1$. We shall make use of her techniques and [17] to obtain our result. Our notation is the same as in [17].

DEFINITION 1. Let H be a complex Hilbert space, which is matrix ordered by a family of selfdual cones $H_n^+ \subseteqq H \otimes M_n$. H is called semidiscrete, if the identity of H can be approximated in the strong operator topology by factorisation through M_m with completely positive, linear maps.

So for $\varepsilon > 0$ and $\xi_1, \ldots, \xi_n \in H$ there exist $m \in \mathbf{N}$ and completely positive, linear maps $\sigma: H \to M_m$, $\tau: M_m \to H$ such that

$$\|\tau \circ \sigma(\xi_\nu) - \xi_\nu\| < \varepsilon \quad \text{for} \quad 1 \leqq \nu \leqq n.$$

THEOREM 2. *For a complex Hilbert space H which is matrix ordered by a family of selfdual cones $H_n^+ \subseteq H \otimes M_n$, the following statements are equivalent:*

(a) *There exists a semidiscrete W^*-algebra $\mathcal{M}$ such that $(H, H_n^+, n \in \mathbf{N})$ is completely isomorphic to the space $(L^2(\mathcal{M}), L^2(\mathcal{M})_n^+, n \in \mathbf{N})$.*

(b) *H is semidiscrete.*

If H is semidiscrete, then the approximation of the identity by maps $\tau \circ \sigma$ can be arranged in such a way that $\|\tau \circ \sigma\| \leqq 1$.

For the proof of Theorem 2 we shall make use of the following two lemmata due to A. M. Torpe [21].

LEMMA 3. *Suppose $\mathcal{M}$ is a semidiscrete, σ-finite W^*-algebra. Let $\varphi \in \mathcal{M}_*^+$ be a faithful state on $\mathcal{M}$. Then there exists a net $\{T_\alpha, \alpha \in A\}$ of normal, completely positive, linear maps $T_\alpha \colon \mathcal{M} \to \mathcal{M}$ with finite rank such that*

(a) $\|T_\alpha\| \leqq 1$

(b) $T_\alpha \circ \varphi = \lambda_\alpha \varphi$ *with* $0 \leqq \lambda_\alpha \leqq 1$ *and* $\lambda_\alpha \to 1$.

(c) T_α *converges to the identity on $\mathcal{M}$ in the point $\sigma(\mathcal{M}, \mathcal{M}_*)$ topology.*

PROOF [21]. Since $\mathcal{M}$ is semidiscrete, we can find a net $\{S_\alpha, \alpha \in A\}$ of completely positive, linear maps $S_\alpha \colon \mathcal{M}_* \to \mathcal{M}_*$ with finite rank such that

$$\|S_\alpha \psi - \psi\| \to 0 \quad \text{for} \quad \alpha \to \infty, \quad \psi \in \mathcal{M}_*,$$

S_α maps states into states. Let $S'_\alpha \colon \mathcal{M}_* \to \mathcal{M}_*$ be defined by

$$S'_\alpha(\psi) = S_\alpha(\psi) + \psi(1)\big(S_\alpha(\varphi) - \varphi\big)_-, \quad \psi \in \mathcal{M}_*.$$

S'_α is a completely positive map of finite rank such that

$$\|S'_\alpha - S_\alpha\| \leqq \|S_\alpha(\varphi) - \varphi\|, \quad S'_\alpha(\varphi) = S_\alpha(\varphi) + \big(S_\alpha(\varphi) - \varphi\big)_- \geqq \varphi.$$

From Sakai's Radon—Nikodym theorem [20, Theorem 5.21] one gets the existence of $x_\alpha \in \mathcal{M}$, $0 \leqq x_\alpha \leqq 1$ such that

$$\varphi(x) = S'_\alpha(\varphi)(x_\alpha x x_\alpha), \quad x \in \mathcal{M}.$$

Since

$$\varphi(1 - x_\alpha^2) \leqq S'_\alpha(\varphi)(1 - x_\alpha^2) = \big(S'_\alpha(\varphi) - \varphi\big)(1),$$

one obtains by [6, Ch. I. 4.6 Proposition 4] that

$$(1 - x_\alpha^2)^{1/2} \to 0$$

in the strong operator topology. Hence $x_\alpha \to 1$ strongly since

$$\|(1 - x_\alpha)\xi\|^2 \leqq \|(1 - x_\alpha^2)^{1/2}\xi\|^2.$$

Consequently $x_\alpha \times x_\alpha \to x$ for all $x \in \mathcal{M}$ in the $s(\mathcal{M}, \mathcal{M}_*)$ topology. Now $S''_\alpha \colon \mathcal{M}_* \to \mathcal{M}_*$ is defined by

$$S''_\alpha(\psi)(x) = S'_\alpha(\psi)(x_\alpha \times x_\alpha), \quad \psi \in \mathcal{M}_*, \quad x \in \mathcal{M}.$$

Clearly S_α is a completely positive, linear map of finite rank which has the following properties:

$$\langle S_\alpha''(\psi)-\psi, x\rangle = \langle S_\alpha'(\psi)-\psi, x_\alpha \times x_\alpha\rangle + \langle \psi, x_\alpha \times x_\alpha - x\rangle, \quad \psi\in\mathcal{M}_*,$$

$$S_\alpha''(\psi)(1) \leqq S_\alpha'(\psi)(1) \leqq 1+\|S_\alpha(\varphi)-\varphi\|, \quad \psi \geqq 0, \quad \psi(1)=1,$$

$$S_\alpha''(\varphi) = \varphi.$$

Now the net $T_\alpha=(1+\|S_\alpha(\varphi)-\varphi\|)^{-1}(S_\alpha'')^*$ has the desired properties. □

In [21, Lemma 2.4] the existence of a net T_α with $T_\alpha(1)=1$ and $T_\alpha\circ\varphi=\varphi$ is claimed. But there are some faults in the proof that had to be removed.

LEMMA 4 *Let* $(\mathcal{M}, H, H_n^+)$ *be the matrix ordered standard form* [17], [18] *of a semidiscrete,* σ*-finite* W^**-algebra* $\mathcal{M}$*. Then there exists a net* $\{F_\alpha, \alpha\in A\}\subseteq B(H)$ *of completely positive, linear maps of finite rank such that*

(a) $\|F_\alpha\| \leqq 1$

(b) $\|F_\alpha(\xi)-\xi\| \to 0$ *for* $\xi\in H$.

PROOF. [21] By convexity it suffices to show that F_α approximates the identity in the weak operator topology. Let $\varphi\in\mathcal{M}_*^+$ be a faithful state, $\xi\in H_1^+$ the corresponding separating vector and Δ the associated modular operator [20]. Let $\{T_\alpha, \alpha\in A\}$ be a net as constructed in Lemma 3.

One defines

$$F_\alpha^0\colon \Delta^{1/4}\mathcal{M}\xi \to \Delta^{1/4}\mathcal{M}\xi \quad \text{by}$$

$$F_\alpha^0\colon (\Delta^{1/4}\times\xi) \mapsto \Delta^{1/4}T_\alpha(x)\xi \quad \text{for} \quad x\in\mathcal{M}$$

and a sesquilinear form on $\mathcal{M}\times\mathcal{M}$ by

$$s_\alpha(x, y) = \langle \Delta^{1/4}T_\alpha(x)\xi, \quad \Delta^{1/4}T_\alpha(y)\xi\rangle \quad \text{for} \quad x, y\in\mathcal{M}.$$

Using that T_α is positive and [20, Sect. 10.23, Lemma 3] we see that $s_\alpha(x, y)\geqq 0$ for $x, y\in\mathcal{M}^+$. In addition

$$s_\alpha(x, 1) = \langle \Delta^{1/4}T_\alpha(x)\xi, \ \Delta^{1/4}T_\alpha(1)\xi\rangle \leqq \langle \Delta^{1/4}T_\alpha(x)\xi, \ \Delta^{1/4}\xi\rangle =$$

$$= \langle T_\alpha(x)\xi, \xi\rangle = \varphi\circ T_\alpha(x) \leqq \varphi(x), \quad x\geqq 0.$$

Hence $s_\alpha(x, x)=\|F_\alpha^0(\Delta^{1/4}\times\xi)\|^2\leqq\|\Delta^{1/4}\times\xi\|^2$ by [25, Theorem 1.1].

So F_α^0 has a continuous linear extension $F_\alpha\colon H\to H$ with $\|F_\alpha\|\leqq 1$. F_α has finite rank and is completely positive since T_α and by [17, Lemma 1.4] F_α^0 are completely positive. Using the remarks following [17, Definition 1.2] we obtain from $T_\alpha(x)\to x$ in the $\sigma(\mathcal{M}, \mathcal{M}_*)$-topology that $\Delta^{1/4}T_\alpha(x)\xi\to\Delta^{1/4}\times\xi$ weakly in H. Therefore $F_\alpha\to\mathrm{id}_H$ in the weak operator topology since $\|F_\alpha\|\leqq 1$. □

LEMMA 5. *Suppose* $(\mathcal{M}^{(k)}, H^{(k)}, H_n^{(k)+}, n\in\mathbf{N})$, $k=1, 2$, *are matrix ordered standard forms* [17], [18]. *Then every completely positive Hilbert—Schmidt operator* $T\colon H^{(1)}\to H^{(2)}$ *can be approximated in the Hilbert—Schmidt norm by operators* $\tau\circ\sigma$, *where* $\sigma\colon H^{(1)}\to M_m$ *and* $\tau\colon M_m\to H^{(2)}$, $m\in\mathbf{N}$, *are completely positive, linear maps.*

PROOF. Let

$$C = \left\{ \sum_{ij} \xi_{ji} \otimes \eta_{ij} \mid (\xi_{ij}) \in H_n^{(1)+}, (\eta_{ij}) \in H_n^{(2)+} \right\}.$$

By [3, Lemmas 4.1 and 4.3], C is the convex cone of completely positive Hilbert—Schmidt operators of the form $\tau \circ \sigma$ as above. The dual cone of C is the set of completely positive Hilbert—Schmidt operators [7, pp. 19—20]. Since the completely positive Hilbert—Schmidt operators form a selfdual cone [19] the proof is complete. □

COROLLARY 6. *Suppose* $(\mathcal{M}, H, H_n^+, n \in \mathbf{N})$ *is a matrix ordered standard form. If the identity on* H *can be approximated in the strong operator topology by completely positive Hilbert—Schmidt operators, then* H *is semidiscrete.* □

PROOF OF THEOREM 2. (a)⇒(b). With the aid of [10, Lemma 2.6] the proof is reduced to the σ-finite case. Now (b) is a direct consequence of Lemma 4 and Corollary 6. Lemmas 4 and 5 prove the last statement of Theorem 2.

(b)⇒(a). By the argument indicated in [7, p. 29] and the remarks following [17, Definition 2.3] H has the approximative factorisation property of [17]. So (a) follows from [17, Theorem 3.2].

By a matrix ordered Banach space we shall unterstand a Banach space E with continuous involution J, which is matrix ordered by a family of cones $E_n^+ \subseteqq E \otimes M_n$ such that E_1^+ is an α-normal and α-generating cone [13] in $E_h = \{x \in E \mid x = Jx\}$, $\alpha \geqq 1$. Using the matrix order it is not hard to show that each of the cones E_n^+ is a normal cone in $(E_n)_h$. By [13, Theorem 3.2], E_n^+ is also α_n-generating for some $\alpha_n \geqq 1$. The following corollary stands in contrast to Lemma 5:

COROLLARY 7. *Suppose* H *is a Hilbert space which is matrix ordered by a family of selfdual cones* $H_n^+ \subseteqq H \otimes M_n$. *Then the following statements are equivalent:*

(a) H *is semidiscrete,*

(b) *The identity on* H *can be approximated in the topology of uniform convergence on compact sets by factorisation through* M_m *with completely positive, linear maps (of norm* $\leqq 1$*).*

(c) *For every matrix ordered Banach space* E *every completely positive, compact operator* $E \to H$ *can be approximated uniformly by factorisation through* M_m *with completely positive, linear maps.*

(d) *For every matrix ordered Banach space* E *every completely positive, compact operator* $H \to E$ *can be approximated uniformly by factorisation through* M_m *with completely positive, linear maps.*

PROOF. (a)⇒(b) This follows directly from the last statement in Theorem 2.

(b)⇒(d) is obtained in a similar way as [15, Theorem 9 5: (c)⇒(d)].

(d)⇒(c) is obtained by considering adjoint maps.

(c)⇒(a). Let $\xi_1, \ldots, \xi_k \in H_1^+$ and consider the matrix ordered space. $(E := \operatorname{span}_{\mathbf{C}} \{\xi_1, \ldots, \xi_k\}, E_n^+)$, where the cones E_n^+ are defined as $E_n^+ = E_n \cap H_n^+$ Since by hypothesis the natural injection $E \to H$ can be approximated uniformly by factorisation through M_m with completely positive, linear maps one obtains that H is semidiscrete. □

REMARK 8. (a) If (E, E^+) is a regular ordered Banach space, then the Riesz interpolation property is equivalent to Grothendieck's property. ($\Rightarrow$: pass to the bidual and use [23, Lemma 3.2]. $\Leftarrow$: [23, Theorem 3.1]).

(b) Using [12, Lemma I. 1.2] one can show that a Hilbert space H is regular ordered if and only if the cone H^+ is selfdual. This shows that for a Hilbert lattice H the canonical isomorphism onto the dual H^* is an isomorphism of lattices. So [16, Theorem 6.7] essentially follows from [14] in the separable case.

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A UNIFIED APPROACH TO METRIZATION PROBLEMS

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In literature on general topology, three kinds of metrization problems are discussed. These pertain to (i) topological spaces as in [4], [5], [11], [12], [13], [14], [15], [16], [17] (ii) proximity spaces as in [7], [10] and (iii) uniform spaces as in [2], [9]. There is a general belief that these metrization problems are quite different from each other. Kelley [9, Page 186] even states that the solution of (iii) "is a rather unsatisfactory solution to the topological metrization problem". It is the main purpose of this paper to show that, contrary to this belief and Kelley's statement, there does exist *one* metrization theorem — namely, the Alexandroff—Urysohn uniform metrization theorem [2] — from which many other metrization theorems can be easily deduced. This confirms the well known dictum of E. H. Moore, that whenever there are parallel theories, there is one which subsumes them all.

This paper is of an expository nature and so proofs, that are easily accessible in the literature, are omitted. Further, in all the proofs that we do give, only the sufficiency part is given because the necessity follows easily.

A proof of the following main theorem can be found in Kelley [9, Pages 185—186].

1. THEOREM (Alexandroff—Urysohn). *A T_1 uniform space is metrizable if and only if its uniformity has a countable base.*

A semi-metric provides a bridge between general topological spaces and metric spaces.

2. DEFINITIONS. (a) A *semi-metric* d on a nonempty set X is a real valued non-negative function on $X\times X$ such that

(i) for all x, y in X, $d(x, y)=d(y, x)$,

(ii) $d(x, y)=0$ iff $x=y$.

(b) A semi-metric d is said to be *compatible* with a topological space X iff for each $x\in X$ and each subset M of X, $x\in M^-$ iff $D(x, M)=0$, where for subsets A, B of X,

$$D(A, B) = \operatorname{Inf}\{d(a, b): a\in A,\ b\in B\}.$$

The pair (X, d) is called a semi-metric space.

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3. THEOREM (Chittenden [5]). *A T_1 space X is metrizable if and only if there exists a compatible semi-metric d on X such that for sequences (x_n), (y_n), (z_n) of X,*

4. $d(x_n, y_n)\to 0$, $d(y_n, z_n)\to 0$ *implies* $d(x_n, z_n)\to 0$.

PROOF. Suppose X is T_1 and d a compatible semi-metric on X satisfying (4). In view of Theorem 1, it is enough to prove that the family

$$\mathbf{B} = \{V_n : n\in\mathbf{N}\} \quad \text{where} \quad V_n = \{(x, y)\in X\times Y : d(x, y) < n^{-1}\}$$

is a base for a compatible uniformity on X. To prove this, it is sufficient to verify the triangle inequality, since the other axioms follow easily. If $\mathbf{B}$ does not satisfy the triangle inequality, then there exists an n in $\mathbf{N}$ such that $V_m^2 \not\subset V_n$ for all m in $\mathbf{N}$. Hence there exist sequences (x_m), (y_m), (z_m) in X such that (x_m, y_m), (y_m, z_m) are in V_m but (x_m, z_m) is not in V_n. Clearly, (4) is violated, and this proves the triangle inequality for $\mathbf{B}$. Finally the compatibility of $\mathbf{B}$ with the topology of X follows from the compatibility of d.

5. THEOREM (Niemytzki 12, Wilson [18]). *A T_1 space X is metrizable if and only if it has a compatible semi-metric d such that for x in X and sequences (x_n), (y_n) in X.*

6. $d(x_n, x)\to 0$ *and* $d(x_n, y_n)\to 0$ *implies* $d(y_n, x)\to 0$.

PROOF. We give the main steps of the proof due to Wilson [18] which consists in showing that (6) implies the existence of an equivalent semi-metric d' which satisfies (4). Hence from Theorem 3, it follows that X is metrizable.

It is easy to show that (6) is equivalent to

7. For each $x\in X$ and $k>0$, there is an $r>0$ such that $r\leqq k$ and $d(x, y)\geqq k$ implies that for all $z\in X$, $d(x, z)+d(y, z)\geqq r$.

The set of all r's satisfying (7), for given $x\in X$ and $k>0$, has a supremum $f(x, k)\leqq k$. For each $x\in X$ we define inductively: $s_0(x)=1$, $2r_{2i-1}(x)=f[x, s_{i-1}(x)]$, $2r_{2i}(x)=f[x, r_{2i-1}(x)]$, $2s_i(x)=r_{2i}(x)$, for $i\geqq 1$. It follows that for each $x\in X$ and $i\geqq 0$, $s_i(x)\leqq 2^{-3i}$ Set $U(z, k)=\{p\in X : d(p, z)<k\}$ and $U_i(x)=U(x, s_i(x))$. We now show that

8. $U_i(x)\cap U_i(y)\neq\emptyset$ implies $U_i(x)\cup U_i(y)\subset U_{i-1}(p)$ for some $p\in\{x, y\}$. To prove (8), we may assume that $s_i(x)\geqq s_i(y)$. If $u\in U_i(x)\cap U_i(y)$, then $d(x, u)+$ $+d(u, y)<2s_i(x)=r_{2i}(x)$. So $d(x, y)<r_{2i-1}(x)$. If $z\in U_i(y)$, then $d(y, z)<s_i(y)\leqq$ $\leqq s_i(x)<r_{2i-1}(x)$. Hence $d(x, y)+d(y, z)<2r_{2i-1}(x)=f[x, s_{i-1}(x)]$ and so $d(x, z)<$ $<s_{i-1}(x)$. If $z\in U_i(x)$, then $d(x, z)<s_i(x)<s_{i-1}(x)$.

We now define a new semi-metric d' on X by

9.

$$d'(x,y) = \begin{cases} 2^{-i} \text{ where } i \text{ is the greatest integer such that } \{x, y\}\subset U_i(p) \text{ for some } p\in X, \\ 1 \text{ if no such } i \text{ exists.} \end{cases}$$

We set $U'(x, k)=\{z\in X : d'(x, z)<k\}$, and show that

10. d' is equivalent to d.

To prove (10) suppose $x\in X$ and $\varepsilon>0$. Choose i, k such that $2^{-i}<\varepsilon$ and $2^{1-3k}<f(x, \varepsilon)$. The result then follows from

$$U_{i+1}(x)\subset U'(x, 2^{-i})\subset U'(x, \varepsilon), \quad \text{and} \quad U'(x, 2^{-k})\subset U(x, \varepsilon).$$

Finally, we show that d' satisfies (4). Suppose $d'(x, y)=2^{-i}$, $d'(y, z)=2^{-j}$ where $j \geqq i$. Then $\{x, y\} \subset U_i(p)$ and $\{y, z\} \subset U_i(q)$ for some $p, q \in X$. So by (8), $U_i(p) \cup U_i(q) \subset U_{i-1}(w)$ for some $w=p$ or q. So $d'(x, z) \leqq 2^{-i+1}$. Thus if $d'(x_n, y_n) \to 0$, $d'(y_n, z_n) \to 0$, then $d'(x_n, z_n) \to 0$.

11. DEFINITION. A binary relation $\delta=\delta(d)$ (a generalized proximity) may be defined on the power set of X, where (X, d) is a semi-metric space, as follows:

12. $A \delta B$ iff $D(A, B)=0$.

13. THEOREM (Arhangel'skii [3]). *A T_1 space X is metrizable if and only if it has a compatible semi-metric d such that $\delta(d)$ is an Efremovič or* EF*-proximity on X (see Efremovič* [6]*).*

PROOF. Suppose d is a semi-metric on X such that $\delta(d)$ is an EF-proximity on X. We now prove that d satisfies (6) Suppose not. Then there exists an x in X and sequences (x_n), (y_n) in X such that $d(x_n, x) \to 0$, $d(x_n, y_n) \to 0$ but $d(y_n, x) \not\to 0$. Without any loss of generality, we may suppose that x is not a cluster point of $\{y_n: n \in \mathbf{N}\}=B$, i.e. $x \bar{\delta} B$. Since δ is EF, there is a subset E of X such that $x \bar{\delta} E$ and $(X-E) \bar{\delta} B$. But $d(x_n, x) \to 0$ and $x \bar{\delta} E$ implies that (x_n) is eventually in $X-E$. This contradicts the facts $d(x_n, y_n) \to 0$ and $(X-E) \bar{\delta} B$.

14. REMARK. The above proof shows that the result is still true if δ is an R-proximity as defined by Harris [8], thereby strengthening the sufficiency part of Theorem 13.

15. THEOREM (Arhangel'skii [3]). *A Hausdorff space X is metrizable if and only if there is a compatible semi-metric d satisfying.*

16. *For every pair of disjoint closed subsets A, B of X, one of which is countably compact, $D(A, B)>0$.*

PROOF. We show that if d satisfies (16) then it also satisfies (6). For if not, then there exists an x in X and sequences (x_n), (y_n) in X such that $d(x_n, x) \to 0$, $d(x_n, y_n) \to 0$ but $d(x, y_n) \not\to 0$. Without any loss of generality, we may assume that $A \cap B=\emptyset$, where $A=\{x\} \cup \{x_n: n \in \mathbf{N}\}$, and $B=\{y_n: n \in \mathbf{N}\}^-$. But A is compact, B is closed and $D(A, B)=0$, a contradition.

17. THEOREM (A. H. Stone [15]). *A T_0-space X is metrizable if and only if there exists a countable collection (G_n) of open covers of X satisfying:*

18. *for each x in X and ecery neighbourhood U of x, there exists a neighbourhood V of x and an n in* N *such that* $\mathrm{St}(V, G_n) \subset U$.

PROOF. We first note that (18) makes (G_n) a neighborhood-star development of X (see Rolfsen [13]) and that a T_0-space which admits a neighbourhood-star development is Hausdorff. We may also assume that G_{n+1} is a refinement of G_n. Then the real-valued function d on $X \times X$ defined by

$$d(x, y) = \mathrm{Inf}\{n^{-1}: y \in St(x, G_n)\}$$

is a compatible semi-metric on X. Our proof is completed if we show that d satisfies (6). Suppose not. Then there exists an x in X and sequences (x_n), (y_n) in X such that $d(x_n, x) \to 0$, $d(x_n, y_n) \to 0$ but $d(y_n, x) \not\to 0$. Without any loss of generality, we assume that $x \notin B^-$ where $B=\{y_n: n \in \mathbf{N}\}$. By (18) there is a neighbourhood N

of x and an $m \in \mathbf{N}$ such that $\mathrm{St}(N, G_n) \subset X - B^-$ for each $n > m$. But this is impossible since $d(x_n, x) \to 0$ and $d(x_n, y_n) \to 0$.

19. THEOREM. *A topological space is metrizable if it satisfies one of the following conditions:*

20. *X is T_0 and has a regular development* (Alexandroff—Urysohn [2]).

21. *X is T_3 and has a countable base* (Urysohn [17], Tychonoff [16]).

22. *X is T_3 and has a σ-locally discrete base* (Bing [4]).

23. *X is T_3 and has a σ-locally finite base* (Nagata [11], Smirnov [14]).

PROOF. Rolfsen [13] has shown that each of (20) to (23) implies that X admits a neighbourhood-star development and hence, by Theorem 17, the result follows.

Finally, we consider "proximity" metrization theorems.

24. DEFINITION. An *EF*-proximity space (X, δ) is said to be metrizable iff there exists a metric d on X such that

$$A \delta B \quad \text{iff} \quad D(A, B) = 0.$$

25. DEFINITION. For sequences $(x_n)(y_n)$ in an *EF*-proximity space (X, δ), we say that $(x_n) \sim (y_n)$ iff $\{x_n : n \in M\} \delta \{y_n : n \in M\}$ for every infinite subset M of $\mathbf{N}$. It is easy to show that $\sim$ is an equivalence relation.

26. DEFINITION. If (X, δ) is an *EF*-proximity space, then a subset U of $X \times X$ is called an *admissible entourage* iff whenever $(x_n) \sim (y_n)$, (x_n, y_n) is in U for some $n \in \mathbf{N}$ (and hence eventually).

27. THEOREM (Efremovič and Švarc [7]). *An EF-proximity space (X, δ) is metrizable if and only if*

28. (i) *$A \delta B$ implies there exist sequences (x_n) in A, (y_n) in B such that $(x_n) \sim (y_n)$,*
(ii) *the system $\cup$ of admissible entourages has a countable base* $\mathbf{B}$.

PROOF. It is sufficient to show that $\cup$ is a uniformity for by (ii) it is metrizable and by (i) it is compatible with δ. It also follows that the metric which induces $\cup$ also induces δ. We need only verify that $\mathbf{B}$ satisfies the triangle inequality, since the other axioms follow easily. If $\mathbf{B}$ does not satisfy the triangle inequality, then without any loss of generality, we may assume that $\mathbf{B} = \{V_n : n \in \mathbf{N}\}$ such that $V_{n+1} \subset V_n$ but $\overset{2}{V}_n \not\subset V_1$ for each $n \in \mathbf{N}$. Hence there exist sequences (x_n), (y_n), (z_n) in X such that (x_n, y_n), (y_n, z_n) are in V_n but (x_n, z_n) is not in V_1. Clearly, $(x_n) \sim (y_n)$ and $(y_n) \sim (z_n)$ but $(x_n) \not\sim (z_n)$, a contradiction since $\sim$ is an equivalence relation.

The following result can be proved similarly:

29. THEOREM (Leader [10]). *An EF-proximity space (X, δ) is metrizable if and only if there is a sequence (U_n) of symmetrical admissible entourages with $U_{n+1} \subset U_n$ and such that $A \delta B$ iff $(A \times B) \cap U_n \neq \emptyset$ for each $n \in \mathbf{N}$.*

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AN EXTENSION OF THE q-SAALSCHÜTZIAN THEOREM

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1. Introduction

For real or complex q, $q \neq 1$, let

$$(1) \qquad (\lambda;\, q)_0 = 1 \quad \text{and} \quad (\lambda;\, q)_l = \prod_{j=1}^{l} (1-\lambda q^{j-1}), \quad l = 1, 2, 3, \ldots,$$

and denote by ${}_r\Phi_s$ a q-hypergeometric series with r numerator and s denominator parameters (see Slater [1, Chapter 3] for details). Also let the Gaussian (or q-binomial) coefficient be defined, for all non-negative integers n and k, by

$$(2) \qquad \begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} 1, & \text{if} \quad k = 0, \\ \prod_{j=1}^{k} \left(\dfrac{1-q^{n-j+1}}{1-q^j} \right), & \text{if} \quad 1 \leqq k \leqq n, \\ 0, & \text{if} \quad k > n. \end{cases}$$

We shall prove the following

THEOREM. *Let n and N be arbitrary non-negative integers. Then*

$$(3) \qquad {}_3\Phi_2 \left[\begin{matrix} a, b, q^{-N}; \\ cq^n, abq^{1-N}/c; \end{matrix} \; q, q \right] =$$

$$= \frac{(cq^n/a;\, q)_{N-n}(cq^n/b;\, q)_{N-n}}{(cq^n;\, q)_{N-n}(c/ab;\, q)_N} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a;\, q)_k (b;\, q)_k (c/ab;\, q)_{n-k}}{(cq^N;\, q)_k} \left(\frac{c}{ab} \right)^k,$$

provided that each side exists, exceptional values of the complex parameters a, b, c, and q being tacitly excluded.

REMARK 1. In its special case when $n=0$, the summation formula (3) reduces immediately to the familiar q-Saalschützian theorem [1, p. 97, Equation (3.3.2.2)]:

$$(4) \qquad {}_3\Phi_2 \left[\begin{matrix} a, b, q^{-N}; \\ c, abq^{1-N}/c; \end{matrix} \; q, q \right] = \frac{(c/a;\, q)_N (c/b;\, q)_N}{(c;\, q)_N (c/ab;\, q)_N},$$

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which holds true for every non-negative integer N. Furthermore, upon replacing a, b, and c by q^a, q^b, and q^c, respectively, and letting $q \to 1$, Formula (4) yields the well-known Saalschütz's theorem [1, p. 49, Equation (2.3.1.3)].

2. Proof of the Theorem

Our proof of the summation formula (3) is based upon the following

LEMMA. *Let $\{\Omega_n\}_{n=0}^{\infty}$ denote a bounded sequence of complex numbers. Then, for arbitrary complex parameters λ and μ,*

$$\sum_{l,m=0}^{\infty} \Omega_{l+m}(\lambda;\, q)_l(\mu;\, q)_m \frac{(\mu z)^l}{(q;\, q)_l} \frac{z^m}{(q;\, q)_m} = \sum_{n=0}^{\infty} \Omega_n(\lambda\mu;\, q)_n \frac{z^n}{(q;\, q)_n}, \tag{5}$$

provided that each of the series involved converges absolutely.

REMARK 2. Formula (5) provides a q-analogue of a well-known series identity [2, p. 55, Equation (12)].

PROOF. Setting $l+m=n$ and $m=n-l$ $(n \geqq 0;\ 0 \leqq l \leqq n)$, we readily have

$$\sum_{l,m=0}^{\infty} \Omega_{l+m}(\lambda;\, q)_l(\mu;\, q)_m \frac{(\mu z)^l}{(q;\, q)_l} \frac{z^m}{(q;\, q)_m} =$$

$$= \sum_{n=0}^{\infty} \Omega_n(\mu;\, q)_n \frac{z^n}{(q;\, q)_n} {}_2\Phi_1\left[\begin{matrix} \lambda,\ q^{-n}; \\ q^{1-n}/\mu; \end{matrix}\ q,\ q\right] = \sum_{n=0}^{\infty} \Omega_n(\mu;\, q)_n \frac{(q^{1-n}/\lambda\mu;\, q)_n}{(q^{1-n}/\mu;\, q)_n} \frac{(\lambda z)^n}{(q;\, q)_n},$$

by q-Vandermonde's theorem [1, p. 247, Equation (IV.1)], and the lemma follows upon appropriately simplifying this last series.

Now we turn to the proof of the theorem. Denote, for convenience, the left-hand side of the summation formula (3) by S. Applying the series identity (5) asserted by the lemma with, of course,

$$z = q, \quad \lambda = q^{-n}, \quad \mu = q^{-N+n}, \quad \text{and} \quad \Omega_l = \frac{(a;\, q)_l(b;\, q)_l}{(cq^n;\, q)_l(abq^{1-N}/c;\, q)_l}, \quad l \geqq 0, \tag{6}$$

we thus find that

$$S = \sum_{l,m \geqq 0} \frac{(a;\, q)_{l+m}(b;\, q)_{l+m}}{(cq^n;\, q)_{l+m}(abq^{1-N}/c;\, q)_{l+m}} (q^{-n};\, q)_l(q^{-N+n};\, q)_m \frac{q^{(1-N+n)l}}{(q;\, q)_l} \frac{q^m}{(q;\, q)_m} =$$

$$= \sum_{l=0}^{n} \frac{(a;\, q)_l(b;\, q)_l(q^{-n};\, q)_l}{(cq^n;\, q)_l(abq^{1-N}/c;\, q)_l} \frac{q^{(1-N+n)l}}{(q;\, q)_l} {}_3\Phi_2\left[\begin{matrix} aq^l,\ bq^l,\ q^{-N+n}; \\ cq^{n+l},\ abq^{1-N+l}/c; \end{matrix}\ q,\ q\right].$$

Summing this ${}_3\Phi_2$ series by the q-Saalschützian theorem (4) with a, b, c, and N replaced by

$$aq^l, \quad bq^l, \quad cq^{n+l}, \quad \text{and} \quad N-n,$$

respectively, we have

$$S =$$

$$= \frac{(cq^n/a;\, q)_{N-n}(cq^n/b;\, q)_{N-n}}{(cq^n;\, q)_{N-n}} \sum_{l=0}^{n} \frac{(a;\, q)_l(b;\, q)_l(q^{-n};\, q)_l}{(cq^N;\, q)_l(abq^{1-N}/c;\, q)_l(cq^{n-l}/ab;\, q)_{N-n}} \frac{q^{(1-N+n)l}}{(q;\, q)_l} =$$

$$= \frac{(cq^n/a;\, q)_{N-n}(cq^n/b;\, q)_{N-n}}{(cq^n;\, q)_{N-n}(c/ab;\, q)_N} \sum_{l=0}^{n} (-1)^l q^{(1/2)l(1+2n-l)} \cdot$$

$$\cdot \frac{(a;\, q)_l(b;\, q)_l(q^{-n};\, q)_l}{(cq^N;\, q)_l} \frac{(c/ab)^l}{(q;\, q)_l} (c/ab;\, q)_{n-l}.$$

From the definition (2) it is easily verified that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;\, q)_n}{(q;\, q)_k(q;\, q)_{n-k}} = (-1)^k q^{(1/2)k(1+2n-k)} \frac{(q^{-n};\, q)_k}{(q;\, q)_k}, \quad 0 \leqq k \leqq n, \tag{8}$$

which, when used in the last expression in (7), yields

$$S = \frac{(cq^n/a;\, q)_{N-n}(cq^n/b;\, q)_{N-n}}{(cq^n;\, q)_{N-n}(c/ab;\, q)_N} \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix} \frac{(a;\, q)_l(b;\, q)_l(c/ab;\, q)_{n-l}}{(cq^N;\, q)_l} \left(\frac{c}{ab}\right)^l,$$

thus completing the proof of the theorem.

3. Applications

Just as Saalschütz's theorem and its q-analogue (4), the general summation formula (3) has great potential for applications in various areas of combinatorial analysis (see, for details, Takács [3] and the references cited there). We choose to record here the following consequence of our theorem, which can indeed be proved by comparing coefficients of like powers of z on both sides, using the summation formula (3):

COROLLARY. *For every bounded sequence* $\{\Omega_n\}_{n=0}^{\infty}$ *of complex numbers, and for complex parameters* α, β, γ, *and* q, $|q|<1$,

$$\sum_{l,m=0}^{\infty} \Omega_{l+m} \frac{(\alpha;\, q)_l(\beta;\, q)_l(\gamma/\alpha\beta;\, q)_m}{(\gamma q^n;\, q)_l} \frac{(\gamma z/\alpha\beta)^l}{(q;\, q)_l} \frac{z^m}{(q;\, q)_m} = \tag{9}$$

$$= \frac{(\gamma;\, q)_n}{(\gamma/\alpha;\, q)_n(\gamma/\beta;\, q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (\alpha;\, q)_k(\beta;\, q)_k(\gamma/\alpha\beta;\, q)_{n-k} \left(\frac{\gamma}{\alpha\beta}\right)^k \cdot$$

$$\cdot \sum_{N=0}^{\infty} \Omega_N \frac{(\gamma/\alpha;\, q)_N(\gamma/\beta;\, q)_N}{(\gamma;\, q)_{N+k}} \frac{z^N}{(q;\, q)_N},$$

provided that each series involved converges absolutely.

REMARK 3. For $n=0$, this last formula (9) yields a q-analogue of a known series identity [2, p. 313, Equation (143) *with* $y=0$]. More interestingly, (9) with $n=0$ and $\Omega_l=1$ $(l\geqq 0)$ gives us the familiar Euler transformation for ${}_2\Phi_1$ [1, p. 97, Equation (3.3.2.3)].

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THE KUROSH—ORE EXCHANGE PROPERTY

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1. Introduction

In 1935 Oystein Ore published a paper with the title "Foundations of Abstract Algebra" which turned out to be one of the fundamental papers for a new and independent branch of mathematics, lattice theory. Ore was led to his abstractions from his studies of representation and decomposition properties which arise in algebraic number and ring theory. He worked out a general and fundamental exchange property which is known today as the Kurosh—Ore Property. Kurosh had investigated this concept independently from Ore. A detailed historical analysis can be found in [11].

In this paper we will convey this classical exchange property with respect to some new developments in lattice theory. Firstly we characterize lattices which have the Kurosh—Ore Property as so-called consistent lattices. This allows us to apply some recent results of J. P. S. Kung about combinatorial properties of join and meet irreducible elements of consistent lattices. Then we discuss the Kurosh—Ore property with respect to closure structures. Matroids and convex geometries turn out to be two contrary cases of closure structures having the Kurosh—Ore property. We show that the concept of minors, as known from matroid theory, can be generalized. At the end of this paper the Kurosh—Ore property in semimodular lattices is investigated. This is connected to results of U. Faigle on "Geometries on Partially Ordered Sets" [5]. We will characterize semimodular lattices with the Kurosh—Ore property as glued geometric lattices. For this purpose some useful gluing techniques for modular lattices are generalized to so-called balanced lattices.

2. Preliminaries

Terminology and notations of lattices not explained in the following can be found in [2] and [6].

The elements of $J(L):=\{x\in L|x\neq 0_L$ and $x=y\vee z$ implies $x=y$ or $x=z\}$ are called the *join irreducibles* of L. The *meet irreducibles*, denoted by $M(L)$, are defined dually. We set $J_0(L):=J(L)\cup\{0_L\}$ and $M^1(L):=M(L)\cup\{1_L\}$. In a lattice L of finite length the unique lower (upper) cover of a join irreducible u (resp. meet irreducible s) is denoted by u_* (resp. s^*). We write $x\prec y$ if y covers x and define $a_+:=\wedge\{b\in L|b\prec a\}$ and $a^+:=\vee\{b\in L|b\succ a\}$ for $a\in L$. A lattice is said to be *atomistic* if each element of L is the join of upper covers of 0_L (atoms). If the dual of L is atomistic then L is said to be *coatomistic*. If $a=x_1\vee\ldots\vee x_n$ for $a\in L$ and $x_1,\ldots,x_n\in J_0(L)$ then we say that $x_1\vee\ldots\vee x_n$ is a $\vee$-*representation* of a. We will only be concerned with lattices in which each element has a $\vee$-representation.

(This is, for example, the case if L fulfills the descending chain condition; i.e., each subset of L has a minimal element.) In fact, with the exception of Theorem 1 we only consider lattices of finite length. A lattice L has the *Kurosh—Ore property* (KOP) if for all $a \in L$ with two $\vee$-representations $x_1 \vee \ldots \vee x_n$ and $y_1 \vee \ldots \vee y_m$ the following exchange property holds: For each x_i there is a y_j such that $a = x_1 \vee \ldots \vee x_{i-1} \vee y_j \vee x_{i+1} \vee \ldots \vee x_n$.

A $\vee$-representation of a is called *irredundant* if no join irreducible of the $\vee$-representation can be omitted to obtain still a $\vee$-representation of a. It can be easily seen that by the above finiteness assumption it suffices to look at irredundant $\vee$-representations in order to see whether a lattice has the KOP or not.

We finally recall some facts about tolerance relations and gluing which will be needed in Section 5 (cf. [7] and [13]). A binary relation Θ on a lattice L is called a *tolerance relation* if Θ is reflexive, symmetric, and compatible with the operations $\wedge$ and $\vee$; i.e., $x_1 \Theta y_1$ and $x_2 \Theta y_2$ imply $x_1 \wedge x_2 \Theta y_1 \wedge y_2$ and $x_1 \vee x_2 \Theta y_1 \vee y_2$. We define $a_\Theta := \wedge \{x \in L | a \Theta x\}$ and $a^\Theta := \vee \{x \in L | a \Theta x\}$ for $a \in L$ and call the intervals $[a]_\Theta := [a_\Theta, (a_\Theta)^\Theta]$ *blocks* of Θ. The compatibility property of Θ makes it possible to build a *factor lattice* L/Θ whose elements are the blocks of Θ. A tolerance relation Θ is called a *glued tolerance relation* if A covers B in L/Θ implies $A \cap B \neq \emptyset$ for $A, B \in L/\Theta$. A glued tolerance relation Θ of L contains all covering pairs of L; this property characterizes glued tolerance relations (Proposition 9 in [12]). From this characterization it follows that there exists a smallest glued tolerance relation of L, denoted by $\Sigma(L)$ and called the *skeleton relation* of L.

3. Consistent lattices

In a recent paper [8] Kung has proved deep theorems about "Matchings in modular lattices" using Radon transform methods and the concept of consistent $\vee$-irreducible elements and consistent lattices. We will show that the concept of consistency, which at the first glance seems to be purely technical, relates to the KOP. This is also connected to a characterization of the KOP given by Crawley in [1].

Following [8], we say that an element u of $J_0(L)$ is *consistent* if $x \vee u \in J_0([x, 1_L])$ for all $x \in L$. In case that all join irreducible elements of a lattice L are consistent then L itself is said to be consistent.

THEOREM 1. *In a lattice L, in which each element has a $\vee$-representation, the* KOP *holds if and only if L is consistent.*

PROOF. Suppose L does not have the KOP. Then there exists an x with two representations $x_1 \vee \ldots \vee x_n$ and $y_1 \vee \ldots \vee y_m$ not fulfilling the exchange condition. This means that there exists an x_i such that $x_1 \vee \ldots \vee x_{i-1} \vee y_j \vee x_{i+1} \vee \ldots \vee x_n < x$ for all y_j. Set $a := x_1 \vee \ldots \vee x_{i-1} \vee x_{i+1} \vee \ldots \vee x_n$. We have $a \vee x_i = x = (a \vee y_1) \vee \ldots \vee (a \vee y_m)$ and, as $a \vee y_j < a \vee x_i$ by the foregoing, it follows that $a \vee x_i \notin J_0(a, 1_L])$. Thus, x_i is not a consistent join irreducible in L. Now suppose that L is not consistent. This means that there exist $a \in L$ and $u \in J_0(L)$ with $a \vee u \notin J_0([a, 1])$. Thus there exist elements $a_1, \ldots, a_l$ of $[a, 1]$ with $a_i < a \vee u$ and $a_1 \vee \ldots \vee a_l = a \vee u$. From representations of each a_i we obtain a representation of $a \vee u$, say $y_1 \vee \ldots \vee y_m$, with

the property that $a \vee y_j < a \vee u$. A further representation of $a \vee u$, say $x_1 \vee \ldots \vee x_n \vee u$, can be obtained from a representation of a. The join irreducible u in this second representation cannot be replaced by any y_j to get still $a \vee u$. Thus, L does not fulfill the KOP.

Theorem 1 is helpful, because it is often easier to check the consistence condition than the KOP in a lattice. It is, for example, easy to see that a finite modular lattice is consistent, since it has the interval isomorphism property. It can also easily be seen that the consistence condition is hereditary to intervals. Thus it follows:

COROLLARY 1. *If a lattice has the* KOP *then all its intervals have this property too.*

We are now in a postion to reinterpret Kungs results ([8], and for survey [9]), which have been expressed with the consistence property, in term of the more meaningful KOP. Let $I(M^1|J_0)$ denote the incidence matrix where the rows are indexed by M^1 and the columns are indexed by J_0 and in which the m, j-entry is 1 if $m \geqq j$ and 0 otherwise. For subsets X and Y of a lattice L, a function f of X to Y is called a *matching from X to Y* if f is one-to-one and $x \leqq f(x)$ for all $x \in X$. Here are the results:

— Let L be a finite lattice having the KOP. Then $I(M^1(L)|J_0(L))$ has rank $|J_0(L)|$.

— For a finite lattice L having the KOP there is a matching from $J_0(L)$ to $M^1(L)$.

— Let L be a finite lattice having the KOP. Then $|J(L)| \leqq |M(L)|$.

In [12] the author has studied matchings for linearly indecomposable modular lattices. A finite lattice L is called *linearly indecomposable* if the diagram of L cannot be disconnected by removing one edge. D. Race observed [10] that our results of [12] are more generally valid for consistent lattices. Thus the following holds:

— Let L be a finite linearly indecomposable lattice having the KOP and let F be an order filter of L. Then $|J(L) \cap F| \leqq |M(L) \cap F|$.

— For a finite linearly indecomposable lattice having the KOP there is a matching of $J(L)$ to $M(L)$.

4. Closure structures and exchange properties

We shall study in this section the KOP from the view of closure structures. Given a finite set X, a function Cl which maps the set $\mathcal{P}(S)$ of all subsets of S to $\mathcal{P}(S)$ is called a closure structure, denoted by (X, Cl), if it satisfies the properties:
(1) $\mathrm{Cl}(A) = \mathrm{Cl}(\mathrm{Cl}(A))$,
(2) $A \subseteqq \mathrm{Cl}(A)$,
(3) if $A \subseteqq B$ then $\mathrm{Cl}(A) \subseteqq \mathrm{Cl}(B)$.
We will be concerned with the following exchange properties:

Kurosh—Ore Property (KOP): Let $\mathrm{Cl}(A) = \mathrm{Cl}(B)$. For $a \in A$ there exists $b \in B$ such that $\mathrm{Cl}(A) = \mathrm{Cl}((A \setminus \{a\}) \cup \{b\})$.

Steinitz—MacLane Property (SMP): Let $p, q \notin \mathrm{Cl}(A)$ and $p \neq q$. Then $p \in \mathrm{Cl}(A \cup \{q\})$ implies $q \in \mathrm{Cl}(A \cup \{p\})$.

Antiexchange Property (AP): Let $p, q \notin \mathrm{Cl}(A)$ and $p \neq q$. Then $p \in \mathrm{Cl}(A \cup \{q\})$ implies $q \notin \mathrm{Cl}(A \cup \{p\})$.

The connection to Section 3 can be clarified as follows: A closure structure has the KOP if and only if the lattice of its closed subsets has this property. We will point out that the KOP implies the other exchange properties. Closure structures with the SMP, known as matroids (pregeometries), are extensively studied. The corresponding lattices are called geometric lattices and are characterized as atomistic and semimodular lattices. It is easy to see that matroids fulfill the KOP or, alternatively, that geometric lattices are consistent.

Closure structures with the AP have been studied more recently by several authors (see [4] for survey). Following [4] we call such a closure structure, as well as its lattice, convex geometry. A convex geometry, considered as lattice, is characterized in [3] as meet-distributive. A finite lattice L is called meet-distributive if $[a_+, a]$ is a boolean interval of L for all $a \in L$.

LEMMA 1. *A finite meet-distributive lattice is consistent.*

PROOF. Let L be meet-distributive but not consistent. Then there exists $a \in L$ and $u \in J(L)$ with $a \vee u \notin J_0([a, 1])$. If there are other elements $a \in L$ and $u \in J(L)$ with this property, take a and u such that $a \vee u$ is minimal. It follows that there exist three different lower covers c_1, c_2, c_3 of $a \vee u$ with $a \leqq c_1, c_2$ and $u \leqq c_3$. Define $a' := c_1 \wedge c_2 \wedge c_3$. Since c_1, c_2, c_3 span a boolean lattice it follows that $a' \vee u = c_3$. In addition, we conclude that $a' \vee u = c_3 \notin J_0([a', 1])$, which contradicts our assumption of minimality.

LEMMA 2. *Let y cover x in a meet-distributive lattice L and let $u \in J(L)$ with $u \leqq y$, $u \not\leqq x$. If x' denotes the complement of x in $[y_+, y]$, then x' covers y_+ and $u \leqq x'$, $u \not\leqq y_+$.*

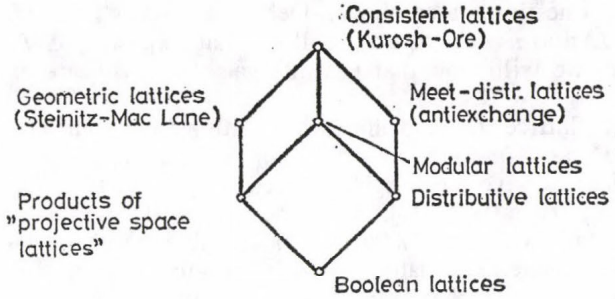
PROOF. $u \not\leqq x$ implies $u \not\leqq y_+$. Thus $y_+ \vee u \in (y_+, y]$. Since L is consistent, it follows that $y_+ \vee u$ has to cover y_+. But x' is the only element of $(y_+, y]$ which covers y_+ and which is not below x.

By a repeated application of this Lemma it follows that in a meet-distributive lattice a covering pair is distinguished by a unique join irreducible. Let us denote this unique join irreducible by $\alpha(x, y)$ if $x \prec y$.

LEMMA 3. *Let a be an element of a meet-distributive lattice L, then $\vee \{\alpha(b, a) | b \prec a\}$ is the unique irredundant $\vee$-representation of a.*

PROOF. Let $x_1 \vee \ldots \vee x_n$ be an arbitrary $\vee$-representation of a. For each lower cover b of a there must exist a x_i with $x_i \not\leqq b$. By the foregoing remark x_i must be $\alpha(b, a)$ and it follows that $\{\alpha(b, a) | b \prec a\} \subseteq \{x_1, \ldots, x_n\}$. On the other hand $\vee \{\alpha(b, a) | b \prec a\} = a$, since $\vee \{\alpha(b, a) | b \prec a\} \prec a$ would imply $\vee \{\alpha(b, a) | b \prec a\} \leqq c$ for some lower cover c of a and hence $\alpha(c, a) \leqq c$, a contradiction.

Looking at these lemmas, we can say that convex geometries or meet-distributive lattices behave in respect to the KOP and join representations as usual distributive lattices. The following figure illustrates the hierarchy of several classes of lattices:



At the end of this section we show that the constructions of minors, as done in matroid theory, also works in the case of closure structures with the KOP. Let (X, Cl) be a closure structure and $Y \subseteq X$. The set Y together with the operator Cl_Y: $A \mapsto \mathrm{Cl}(A) \cap Y$ for $A \subseteq Y$ is called the *restriction of* (X, Cl) *to* Y, denoted by $(X, \mathrm{Cl})|Y$.

The set $X \setminus Y$ together with the operator $\mathrm{Cl}_{X/Y}$: $A \mapsto \mathrm{Cl}(A \cup Y) \setminus Y$ for $A \subseteq X \setminus Y$ is called the *contraction of* (X, Cl) *through* Y, denoted by $(X, \mathrm{Cl})/Y$.

THEOREM 2. *Let* (X, Cl) *be a Kurosh—Ore closure structure and* $Y \subseteq X$. *Then*
a) $(X, \mathrm{Cl})|Y$ *is a Kurosh—Ore closure structure*
b) $(X, \mathrm{Cl})/Y$ *is a Kurosh—Ore closure structure.*

PROOF. a) Let $A, B \subseteq Y$ and $\mathrm{Cl}_Y(A) = \mathrm{Cl}_Y(B)$. Observe that this implies $\mathrm{Cl}(A) = \mathrm{Cl}(B)$, because $A \subseteq \mathrm{Cl}(A) \cap Y = \mathrm{Cl}(B) \cap Y$ yields $A \subseteq \mathrm{Cl}(B)$ and thus $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(B)$. Now, for $a \in A$ there exists $b \in B$ with $\mathrm{Cl}(A) = \mathrm{Cl}(A \setminus \{a\}) \cup \{b\})$. Thus, for $a \in A$ we can choose this b to obtain $\mathrm{Cl}_Y(A) = \mathrm{Cl}_Y((A \setminus \{a\}) \cup \{b\})$.

b) Let $A, B \subseteq X \setminus Y$ and $\mathrm{Cl}_{X/Y}(A) = \mathrm{Cl}_{X/Y}(B)$; i.e., $\mathrm{Cl}(A \cup Y) \setminus Y = \mathrm{Cl}(B \cup Y) \setminus Y$. This implies $\mathrm{Cl}(A \cup Y) = \mathrm{Cl}(B \cup Y)$, since $Y \subseteq \mathrm{Cl}(A \cup Y)$ and $Y \subseteq \mathrm{Cl}(B \cup Y)$. Let $a \in A$, then there exists $b \in B \cup Y$ with $\mathrm{Cl}(A \cup Y) = \mathrm{Cl}(((A \cup Y) \setminus \{a\}) \cup \{b\})$. We will show that we find even a $b' \in B$ with $\mathrm{Cl}(A \cup Y) = \mathrm{Cl}(((A \cup Y) \setminus \{a\}) \cup \{b'\})$. If $b \in B$, clearly take $b' := b$. If $b \in Y$ then it follows that $\mathrm{Cl}(A \cup Y) = \mathrm{Cl}((A \cup Y) \setminus \{a\})$ and we can choose b' arbitrarily from B. Observe that $((A \cup Y) \setminus \{a\}) \cup \{b'\} = ((A \setminus \{a\}) \cup \{b'\}) \cup Y$ because $A, B \subseteq X \setminus Y$. Thus, we have found $b' \in B$ with $\mathrm{Cl}_{X/Y}(A) = \mathrm{Cl}(A \cup Y) \setminus Y = \mathrm{Cl}(((A \setminus \{a\}) \cup \{b'\}) \cup Y) \setminus Y = \mathrm{Cl}_{X/Y}(A \setminus \{a\} \cup \{b'\})$.

5. Balanced lattices and gluing

This section can be considered as a preparation for the last one. How can we find the skeleton relation of a given lattice L? If L is modular then there is an answer. The maximal intervals of L which are atomistic (this is equivalent to complemented) are the blocks of $\Sigma(L)$. The approach is the following: We know that $(x, x^+) \in \Sigma(L)$ and $(x_+, x) \in \Sigma(L)$ for each x of an arbitrary finite lattice L, because $\Sigma(L)$ contains all covering pairs of L and is compatible with $\vee$ and $\wedge$. Thus, the intervals $[x_+, (x_+)^+]$ are contained in blocks of $\Sigma(L)$. Now, in case that L is modular, it turns out that the intervals $[x_+, (x_+)^+]$ are exactly the blocks of $\Sigma(L)$. This is implicitly shown in Lemma 6.1 of [7]. We now consider a more general

approach to the question stated above. Define $x^{\uparrow} := \wedge\{s^* | s \in M(L \text{ and } x \leqq s\}$ and $x_{\downarrow} := \vee\{u_* | u \in J(L) \text{ and } u \leqq x\}$. Again it follows that $(x_{\downarrow}, (x_{\downarrow})^{\uparrow}) \in \Sigma(L)$. For so called balanced lattices we will show that the intervals $[x_{\downarrow}, (x_{\downarrow})^{\uparrow}]$ are exactly the blocks of $\Sigma(L)$.

For a finite lattice L we define two relations between $J(L)$ and $M(L)$ by $u \nearrow s: \Leftrightarrow u \vee s = s^*$ and $u \swarrow s: \Leftrightarrow u \wedge s = u_*$ for $u \in J(L)$ and $s \in M(L)$ with $u \not\leqq s$. These arrow relations have been originally used by R. Wille to investigate congruence relations of complete lattices [14]. A lattice L is said to be a *balanced* lattice if $u \nearrow s \Leftrightarrow u \swarrow s$ for all $u \in J(L)$, $s \in M(L)$. For example, modular lattices, geometric lattices, and more general all lattices which are atomistic as well as coatomistic, are balanced lattices. A further example is shown in Figure 1:

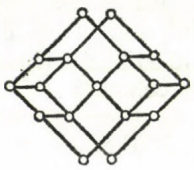


Fig. 1

LEMMA 4. *L is a balanced lattice if and only if $a_{\downarrow} \leqq b \Leftrightarrow a \leqq b^{\uparrow}$ for all $a, b \in L$; i.e., if $(\uparrow, \downarrow)$ is a mixed Galois connection.*

PROOF. First assume that $a_{\downarrow} \leqq b \Leftrightarrow a \leqq b^{\uparrow}$ for all $a, b \in L$. For $u \in J(L)$, $s \in M(L)$ with $u \not\leqq s$ it follows that $u \vee s = s^* \Leftrightarrow u \leqq s^{\uparrow} \Leftrightarrow u_{\downarrow} \leqq s \Leftrightarrow u \wedge s = u_*$. Now suppose that L is a balanced lattice. By duality it is enough to show that $a_{\downarrow} \leqq b \Rightarrow a \leqq b^{\uparrow}$. Let $a_{\downarrow} \leqq b$ but $a \not\leqq b^{\uparrow}$. Then there exists $s \in M(L)$ with $s \geqq b$ and $s^* \not\geqq a$. Now in turn there exists $u \in J(L)$ with $u \leqq a$ and $u \not\leqq s^*$. We conclude that $u_* \leqq a_{\downarrow} \leqq b \leqq s$ and thus $u \wedge s = u_*$, but $u \vee s \neq s^*$, a contradiction.

COROLLARY 2. *For a balanced lattice L with $a, b \in L$ the following statements and the duals hold:*
$a_{\downarrow} \leqq a$, $a \leqq (a_{\downarrow})^{\uparrow}$, $a \leqq b \Rightarrow a_{\downarrow} \leqq b_{\downarrow}$, $a_{\downarrow} \vee b_{\downarrow} = (a \vee b)_{\downarrow}$.

PROOF. The first statement is a direct consequence of the definition of $a_{\downarrow}$. All the other statements hold by standard conclusions of Galois connections.

THEOREM 3. *The intervals $[a_{\downarrow}, (a_{\downarrow})^{\uparrow}]$ for $a \in L$ are the blocks of the skeleton relation of a balanced lattice L.*

PROOF. Let $a_{\downarrow} \leqq x$, $x' \leqq (a_{\downarrow})^{\uparrow}$ and $b_{\downarrow} \leqq y$, $y' \leqq (b_{\downarrow})^{\uparrow}$. Then, $(a \vee b)_{\downarrow} = a_{\downarrow} \vee b_{\downarrow} \leqq$ $\leqq x \vee y$ (resp. $x' \vee y') \leqq (a_{\downarrow})^{\uparrow} \vee (b_{\downarrow})^{\uparrow} \leqq (a_{\downarrow} \vee b_{\downarrow})^{\uparrow} = ((a \vee b)_{\downarrow})^{\uparrow}$ and $((a_{\downarrow} \wedge b_{\downarrow})^{\uparrow})_{\downarrow} \leqq a_{\downarrow} \wedge b_{\downarrow} \leqq x \wedge y$ (resp. $x' \wedge y') \leqq (a_{\downarrow})^{\uparrow} \wedge (b_{\downarrow})^{\uparrow} = (a_{\downarrow} \wedge b_{\downarrow})^{\uparrow}$. Thus the intervals $[a_{\downarrow}, (a_{\downarrow})^{\uparrow}]$ are the blocks of a tolerance relation, say Θ. We will show that Θ is a glued tolerance relation. For $u \in J(L)$ it holds $(u_*, u) \in \Theta$ because of $u_{\downarrow} \leqq u_* \leqq u$. For a covering pair $x \prec y$ of L let $u \in L$ be a minimal element with the property to be less than or equal to y but not less than or equal to x. It follows that $u \in J(L)$, $u_* \leqq x$ and $u \vee x = y$. By the compatibility property of tolerance relations, $(u, u_*) \in \Theta$ implies $(u \vee x, u_* \vee x) =$ $= (x, y) \in \Theta$. The fact that Θ contains all covering pairs implies that Θ is a glued

tolerance relation. Moreover, as remarked before, every glued tolerance relation of L must contain Θ. Thus Θ is proved to be a skeleton relation.

LEMMA 5. *A lattice L, in which all blocks of $\Sigma(L)$ are atomistic and coatomistic, is a balanced lattice.*

PROOF. We show that $u \nearrow s$ implies $u \swarrow s$ for $u \in J(L)$ and $s \in M(L)$. Let $u \in J(L)$, $s \in M(L)$ with $u \nearrow s$, that is $u \vee s = s^*$, $u \not\leqq s$. From $(s, s^*) \in \Sigma(L)$ it follows that $(s \wedge u, u) \in \Sigma(L)$. This means that there exists an atomistic block containing $s \wedge u$ and u. Since u is join irreducible, it has to be an atom of this block. Therefore $s \wedge u = u_*$.

The converse of Lemma 5 does not hold as can be seen from the example represented in Fig. 1. We finish this section studying the relation between the operations $\downarrow$ and $+$.

LEMMA 6. *$a_\downarrow \leqq a_+$ holds in a balanced lattice L with $a \in L$.*

PROOF. Suppose $a_\downarrow \not\leqq a_+$. There exists a $c \in L$ with $c \prec a$ and $c \not\leqq a_\downarrow$. There exists, in turn, a $s \in M(L)$ with $s \geqq c$ and $s \not\geqq a$. Once more, there exists a $u \in J(L)$ with $u \leqq a$ and $u_* \not\leqq c$. It follows $u_* \not\leqq s$, since $u_* \leqq s$ and $u_* \leqq a$ imply $u_* \leqq s \wedge a = c$. Thus $u \vee s = s^*$, but $u \wedge s \neq u_*$, a contradiction.

LEMMA 7. *For a modular lattice L, $a_\downarrow = a_+$ holds for $a \in L$.*

PROOF. $a_\downarrow \leqq a_+$ holds by Lemma 6. Now,

$$a = \vee \{u | u \in J(L),\ u \leqq a\} = \vee \{u \vee a_\downarrow | u \in J(L),\ u \leqq a\}.$$

By modularity, $u \vee a_\downarrow$ covers $a_\downarrow$. Thus, a is the join of some upper covers of $a_\downarrow$, which means that the interval $[a_\downarrow, a]$ is atomistic. By modularity $[a_\downarrow, a]$ is also coatomistic and hence $a_\downarrow \geqq a_+$ holds.

6. Glued geometric lattices

In this section we are concerned with semimodular lattices which have the KOP. We give a description of these lattices as glued geometric lattices. This gives a possibility to construct such lattices.

A finite lattice L is said to be *glued by geometric lattices* if all blocks of $\Sigma(L)$ are geometric lattices. In [5], U. Faigle has shown (beside others) that so-called strongly semimodular lattices are exactly the semimodular lattices which have the KOP; a finite semimodular lattice is said to be *strongly semimodular* if for $u_1, u_2 \in J(L)$ and $x \in L$, $u_1 \prec u_2$ and $u_2 \leqq x \vee u_1$ implies $u_2 \leqq x$. He has also found the equivalence of the KOP and the consistence condition for semimodular lattices, which has us inspired to think of Theorem 1. The results are collected in

THEOREM 4. *For a finite lattice L the following conditions are equivalent:*
(1) *L is semimodular and has the* KOP.
(2) *L is glued by geometric lattices.*
(3) *L is a semimodular balanced lattice.*
(4) *L is semimodular and consistent.*
(5) *L is strongly semimodular.*

PROOF. The equivalence of (1), (4) and (5) is shown in [5]. Nevertheless, a complete proof is given.

(3)→(4). Suppose L is not consistent. There are elements $u\in J(L)$ and $x\in L$ such that $u\vee x\notin J_0([x, 1])$. By semimodularity $x\vee u$ covers $x\vee u_*$. Because of $x\vee u\notin J_0([x, 1])$, there exists besides $x\vee u_*$ another lower cover, say c, of $x\vee u$ with $c\geqq x$. We conclude $u^*\ngeq c$, because $x\vee u_*\ngeq c$. Choose a greatest element m with $m\geqq c$ and $m\ngeq x\vee u$. It follows that m is meet irreducible. In addition $m\ngeq u_*$, since otherwise m and $x\vee u$ do not have a unique greatest lower bound. Thus $u\vee m=m^*$, but $u\wedge m\neq u_*$.

(4)→(5). (This is the proof of Lemma 4 in [5].) Suppose L is semimodular and consistent but not strongly semimodular. Let $x\in L$ be of minimal height such that there are $u_1, u_2\in J(L)$ with $u_1<u_2$ and $u_1, u_2\ngeq x$ and $x\vee u_1=x\vee u_2$. It is clear that $x\neq 0_L$. If $y\in L$ is covered by x, then $y\vee u_1<y\vee u_2$. Now $y\vee u_1\neq x\vee u_1$, since otherwise $y\vee u_1=y\vee u_2$. By semimodularity $y\vee u_1$ is covered by $x\vee u_1$. It follows that $x\vee u_2=x\vee u_1=y\vee u_2$. Now $y\vee u_2\notin J_0([y, 1_L])$ because of $y\vee u_2=$ $=x\vee(y\vee u_1)$, a contradiction.

(5)→(3). By semimodularity $u\wedge m=u_*$ implies $u\vee m=m^*$ for $u\in J(L)$, $m\in M(L)$. Suppose L is strongly semimodular but $u\vee m=m^*$ and $u_*\ngeq m$ for some $u\in J(L)$ and $m\in M(L)$. There exists a $u'\in J(L)$ with $u'\leqq u_*$ and $u'\ngeq m$. Now $u'\vee m\geqq u$ and $m\ngeq u$ yield a contradiction.

(3)→(2). The blocks of $\Sigma(L)$ are of the form $[a_\downarrow, (a_\downarrow)^\dagger]$. Equivalently they can be described as $[(a^\dagger)_\downarrow, a^\dagger]$. They are semimodular, since they are intervals of the semimodular lattice L. It remains to show that they are atomistic. Set $b:=a^\dagger$ and let $x\in[b_\downarrow, b]$. Then, $x=\vee\{u\in J(L)|u\leqq x\}=\vee\{u\vee b_\downarrow|u\in J(L), u\leqq x\}$. If $u\ngeq b_\downarrow$, then $u\wedge b_\downarrow=u$ by the definition of $b_\downarrow$. Hence the elements $u\vee b_\downarrow$ which are not $b_\downarrow$ itself cover $b_\downarrow$ by the semimodularity of L. Thus x is $b_\downarrow$, or x is the join of atoms of $[b_\downarrow, b]$.

(2)→(3). By Lemma 5, L is a balanced lattice. It remains to show that L is semimodular. Let $x\wedge y\prec x, y$ for $x, y\in L$. Since $\Sigma(L)$ contains all covering pairs there exists a block which contains $x\wedge y$, x, y and $x\vee y$. Thus $x\vee y\succ x, y$ follows from the semimodularity of this block.

(1)↔(4). By Theorem 1.

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GENERATING AN INJECTIVE CLASS OF TORSION-FREE ABELIAN GROUPS

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§ 0. Introduction

In this paper we consider, for a set of torsion-free abelian groups of finite rank (henceforth, plainly referred to as groups), Λ, those groups (t.f.f.r) C such that each $A\in\Lambda$ is injective relative to any pure exact sequence $0\to B\to C\to K\to 0$. Let $\mathscr{C}^0(\Lambda)$ be the class of those groups C. In Section 2, we consider $\mathscr{C}^0(\Lambda)$ and give a characterization of this class when Λ is a set of rank-1 groups.

Associated with $\mathscr{C}^0(\Lambda)$ is a class $\mathscr{E}$ of exact sequences $E\colon 0\to B\to C\to K\to 0$ with $C\in\mathscr{C}^0(\Lambda)$. In Section 2, we consider the class $\mathscr{I}\mathscr{C}^0(\Lambda)$ which is the class of all groups injective with respect to each $E\in\mathscr{E}$. We will characterize $\mathscr{I}\mathscr{C}^0(\Lambda)$ when Λ is a set of rank-1 groups. We say that Λ generates the injective class (i.e., the class of all groups injective with respect to a given class of exact sequences), $\mathscr{I}\mathscr{C}^0(\Lambda)$.

In the sections to follows, Z, Q and Π are the sets (or rings) of integers, rationals and primes respectively. The rings Z_p and Z_{p-1} denote the integers localized at p and $\Pi\setminus\{p\}$ respectively. For a group A, let $\operatorname{supp} A=\{p\,|\,pA\neq A\}$.

§ 1. The lattice of types

In this section we will discuss certain lattice properties of the set of types. They will be implemented in the next section.

A type is an equivalence class of height vectors (see [2]). Recall that the set of types forms a distributive lattice if we define $t\wedge t'=[m]$ and $t\vee t'=[M]$ where $m_p=\min\{h_p, h'_p\}$ and $M_p=\max\{h_p, h'_p\}$ for $h\in t$ and $h'\in t'$.

Let $\bar{n}$ be the type of the height sequence which assigns to each prime the value n ($n=0, 1, \ldots, \infty$). If S is a set of types, S may not have a least upper bound or a greatest lower bound. For instance, if $\Pi=\{p_1, p_2, p_3, \ldots\}$ is the set of primes, let $h_j\colon \Pi\to\mathbf{N}\cup\{\infty\}$ be defined by $h_j(p_i)=0$ if $i\leqq j$ and ∞ if $i>j$. For $t_j=[h_j]$, it is not hard to see that $\bar{n}<t_j$ for each j and every finite n. In this case, $\{t_j\,|\,j=1, 2, \ldots\}$ doesn't have a g.l.b.

For height vectors h and h', we'll write $h>h'$ if $h(p)\geqq h'(p)$ fo all p and $[h]>[h']$.

The explanation for the above example is found in

LEMMA 1.1. *Let S be a countable set of types. If $\wedge S$ exists then there are $t_1\ldots t_n\in S$ such that $t_1\wedge\ldots\wedge t_n=\wedge S$.*

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PROOF. Put $s=\wedge S$ and enumerate $S=\{t_1, t_2, \ldots\}$. By considering $s_n=\bigwedge_{i=1} t_{n_i}$, and then relabeling, we may assume that S is a monotonically decreasing set. Moreover, if $t_n=t_m$ for all $m\geqq n$ then $s=t_1\wedge\ldots\wedge t_n$. Otherwise, by discarding redundant types, we will assume that $t_i<t_j$ if $i>j$.

Assume to the contrary that s cannot be written as a finite inf. Choose $h_1\in t_1$ and $h\in S$ with $h_1>h$. Inductively, assume $h_i\in t_i$ for $i=1, \ldots, n$, have been chosen, with $h_j>h_i$ if $i>j$. Since $t_{n+1}>t_n$, there is an $h_{n+1}\in t_{n+1}$ with $h_{n+1}>h_n$.

Set $S_n=\{p|h_n(p)>h(p)\}$ so that $S_n\supseteq S_{n+1}$, and $\bigcap_n S_n$ is a finite set ($s=\bigwedge_{n=1}^{\infty} t_n$). Since $h_n(p)>h(p)$ for every n and every $p\in\bigcap_m S_m$, for n large enough, h_n assumes only finite values on $\bigcap_m S_m$. If we redefine $h_1, h_2, \ldots$ and h, to be zero on $\bigcap_m S_m$, then we have changed only a finite number of the types. Thus, w.l.o.g., we will assume that $\bigcap_m S_m=\emptyset$.

For each n, there is an $m_n>n$ with $S_{m_n}\subset S_n$. If not then $S_n=S_m$ for all $m>n$ which implies that $S_n=\emptyset$, and, in turn, that $h_m=h$ for all $m>n$. This contradicts our assumption. In this way we get a sequence $m_0=1<m_1<m_2<\ldots$ with $S_{m_j}\subset S_{m_{j-1}}$ for $j=1, 2, \ldots$. Choose $p_j\in S_{m_{j-1}}\setminus S_{m_j}$ and let h_0 be defined by $h_0(p)=h(p)$ if $p\neq p_j$ (some j) and $h_0(p_j)=h(p_j)+1$, for $j=1, 2, \ldots$.

Since $\{p_j|\, j=1, 2, \ldots\}$ is an infinite set, $h_0>h$. For any n, choose j so that $m_j>n$. Thus $\{p_i\}_{i=j+1}^{\infty}\subseteq S_{m_j}\subseteq S_n$ and consequently $h_n(p)\geqq h_0(p)$ for all p, except possibly for $p=p_1\ldots p_j$. But $\{p_1\ldots p_j\}\subseteq S_1$ which implies that $h(p_i)<h_1(p_i)\leqq\infty$, i.e., $h(p)$ and hence $h_0(p_i)$ are finite for $i=1, \ldots, j$. Therefore, $t_n=[h_n]\geqq[h_0]>s$, a contradiction. □

The dual statement for reducing countable sup's does not hold. Also, the infimum of an uncountable set, in general, cannot be reduced to the infimum of a finite subcollection.

EXAMPLE 2.1. Let $(-n, n)$ denote the interval of rationals between $-n$ and n, and let f: $\mathbf{Q}\to\Pi$ be a bijection. Consider $\mathscr{S}=\{S\subseteq\Pi|S$ is infinite and either $S=f(-n, n)$ for some n, or $S\cap f(-n, n)$ is finite for all $n>0\}$.

For each $S\in\mathscr{S}$, let t_s be the type of the subring of $\mathbf{Q}$ generated by $1/p$ for $p\notin S$.

Claim: $\bar{0}=\wedge\{t_s|S\in\mathscr{S}\}$.

To show this we will show that if $t>\bar{0}$, then $t_s\ngeqq t$ for some $S\in\mathscr{S}$. Let $h\in t$ and $A=\{p\in\Pi|h(p)>0\}$. If A is infinite then clearly there is an $S\in\mathscr{S}$ with $S\cap A$ infinite. For this S, $t_s\ngeqq t$.

If A is finite, then $h(p)=\infty$ for some $p\in A$. Since $\cup\{S|S\in\mathscr{S}\}=\Pi$, $p\in S$ for some $S\in\mathscr{S}$. For this S, $t_s\ngeqq t$.

If $\bar{0}=\wedge\{t_s|S\in\mathscr{S}'\}$ for some $\mathscr{S}'\subseteq\mathscr{S}$, then $\operatorname{supp}\bar{0}=\Pi=\cup\{\operatorname{supp} t_s|S\in\mathscr{S}'\}=$ $=\cup\{S|S\in\mathscr{S}'\}$ (where $\operatorname{supp} t=\{p|h(p)\neq\infty\}$, for some $h\in t$). But Π is not the finite union of sets from $\mathscr{S}$, so that $\bar{0}$ cannot be written as a finite infimum from $\{t_s|S\in\mathscr{S}\}$. □

A result similar to Lemma 1.1 can be achieved for the infimum of an arbitrary set. This will be useful for what follows.

LEMMA 1.3. *Let $T=\{t_\alpha|\alpha\in I\}$ be a set of types, none of which is $\overline{\infty}$ and $S_\alpha=\operatorname{supp} t_\alpha$. Suppose s is the g.l.b. for T. Then there is a partition $I_1, \ldots, I_k$ of I, into disjoint nonempty sets, such that:*

1. *$\{t_\alpha|\alpha\in I_j\}$ has a g.l.b. for $j=1, 2, \ldots, k$;*
2. *If $s_j=\wedge\{t_\alpha|\alpha\in I_j\}$ for $j=1, \ldots, k$ then $\operatorname{supp} s_i\cap\operatorname{supp} s_j=\emptyset$ for $i\neq j$;*
3. *$s=s_1\wedge\ldots\wedge s_k$ (or equivalently, $I=\bigcup_{j=1}^{k} I_j$);*
4. *For each j and any $\alpha, \beta\in I_j$, there are $\alpha_0, \ldots, \alpha_m\in I_j$ with $\alpha=\alpha_0$, $\beta=\alpha_m$ and $S_{\alpha_i}\cap S_{\alpha_{i+1}}\neq\emptyset$ for $i=0, \ldots, m-1$; and*
5. *If $\alpha\in I$, then $t_\alpha\geqq s_j$ for some j.*

PROOF. Define an equivalence relation on I as follows: for $\alpha, \beta\in I$, $\alpha\sim\beta$ iff there is a sequence $\alpha_0, \ldots, \alpha_m\in I$ with $\alpha=\alpha_0$, $\beta=\alpha_m$ and $S_{\alpha_i}\cap S_{\alpha_{i+1}}\neq\emptyset$ for $i=0, 1, \ldots, m-1$. After only minimal reflection, one sees that this is an equivalence relation on I.

For $\alpha\in I$, let I_α denote the equivalence class of α. For $K_\alpha=\bigcup_{\beta\in I_\alpha} S_\beta$ and J a set formed by choosing one member from each equivalence class, we have that $K_\alpha\cap K_\beta=\emptyset$ if $\beta\notin I_\alpha$.

Since $s=\wedge\{t_\alpha|\alpha\in I\}$, $\operatorname{supp} S=\bigcup_{\alpha\in I} S_\alpha=\bigcup_{\beta\in J} K_\beta$, a set which is at most countable. Thus J is at most countable since $\{K_\beta|\beta\in J\}$ are mutually disjoint.

We will show that $\{t_\beta|\beta\in I_\alpha\}$ has a g.l.b. s_α for each $\alpha\in J$. When this has been done, we will have shown that $s=\bigwedge_{\alpha\in J} S_\alpha$, so by Lemma 1.1, s is a finite inf.

Let $\alpha\in J$ and $h\in s$. Define $h_\alpha(p)=\infty$ for $p\notin K_\alpha$ and $h(p)$ for $p\in K_\alpha$. Take $s_\alpha=[h_\alpha]$. Since for each $\beta\in I_\alpha$, $p\notin\operatorname{supp} t_\beta$ if $p\notin K_\alpha$, then for $h'\in t_\beta$ with $h'\geqq h$, $h'\geqq h_\alpha$. Hence $t_\beta\geqq s_\alpha$. We will show that $s_\alpha=\wedge\{t_\beta|\beta\in I_\alpha\}$.

If $t\leqq t_\beta$ for every $\beta\in I_\alpha$ but $t>s_\alpha$ then choose $h_1\in t$ with $h_1>h$ (since $s_\alpha\geqq s$ this can be done). Let $h_0(p)=h_1(p)$ if $p\in K_\alpha$ and $h(p)$ if $p\notin K_\alpha$, and set $s_0=[h_0]$. Note that if $s_\alpha=s$ then $I_\alpha=I$ since if $\beta\in I$, $t_\beta\geqq s_\alpha$ implies that $\alpha\sim\beta$. In this case no such t can exist.

We assume that $s_\alpha>s$. Because $t>s_\alpha$, $\sum_{p\in K_\alpha}(h_1(p)-h_\alpha(p))=\infty$ so that $\sum_{p\in K_\alpha}(h_0(p)-h(p))=\infty$, and consequently $h_0>h$ (i.e., $s_0>s$ and $h_0\geqq h$).

Claim: $s_0\leqq t_\beta$ for every $\beta\in I$.

Since $h_1\geqq h_0$, $s_0\leqq t_\beta$ for all $\beta\in I_\alpha$. If $\gamma\in I\setminus I_\alpha$ then $\operatorname{supp} t_\gamma=S_\gamma\subseteq\setminus\Pi K_\alpha$. If we choose $h'\in t_\gamma$ with $h'\geqq h$ then $h'(p)=\infty$ for $p\in K_\alpha$ and $h'(p)\geqq h(p)$ otherwise. Hence $h'\geqq h_0$ and therefore $t_\gamma\geqq s$.

This contradicts the hypothesis, $s=\wedge\{t_\beta|\beta\in I\}$. Therefore, no such t can exist, and consequently, $s_\alpha=\wedge\{t_\beta|\beta\in I_\alpha\}$.

If $t'\leqq s_\alpha$ for every $\alpha\in J$ then $t'\leqq t_\beta$ for every $\beta\in I$. Therefore $t'\leqq s$, which shows that $s=\wedge\{s_\alpha|\alpha\in J\}$.

But J is at most countable, and therefore by Lemma 3.6, there are $\alpha_1\ldots\alpha_k\in J$, with $s=s_{\alpha_1}\wedge\ldots\wedge s_{\alpha_k}$. But, for $\alpha\in J$, $\operatorname{supp} s_\alpha=K_\alpha$ and the K_α's are mutually disjoint. Hence $J=\{\alpha_1\ldots\alpha_k\}$.

Write s_j for s_{α_j} and I_j for I_{α_j}. It is clear that Properties 1 through 5 hold. □

For the set $\{t_S|S\in\mathscr{S}\}$ in Example 2.1, $k=1$ and $\mathscr{S}$ is the desired partition of $\mathscr{S}$.

We will end this section by recording an elementary result for later use. If G is a group with $x\in G$ let $h^G(x)$ denote the height vector of x in G.

LEMMA 1.4. *Let s, r and t be types with $s=r\wedge t$ and $\operatorname{supp} t\cap\operatorname{supp} r=\emptyset$. If Y has rank-1 subgroups X, A and B, of types s, r and t respectively, and if $X\lhd A$ with $X\cong A\oplus B$, then $A\oplus B\lhd Y$.*

PROOF. We will first show that $A, B\lhd Y$, by showing that $A\lhd Y$.

Let $A_*=\langle A\rangle_*$. We have $X\cong A\oplus B\cong A_*\oplus B\cong Y$. Let $0\neq x\in X$ and $x=a+b\in A\oplus B$. We will first show that $h^A(a)=h^A{}*(a)$ (note that $a, b\neq 0$ since $s<t, r$).

By the purity of X in Y, $h^X(x)=\min\{h^A(a), h^B(b)\}=\min\{h^{A*}(a), h^B(b)\}$. If $p\in\operatorname{supp} t$, $p\in\operatorname{supp} r$ so that $h_p^A(a)=\infty=h_p^{A*}(a)$ (since $h^{A*}(a)\cong h^A(a)$). Otherwise $p\in\operatorname{supp} t$, and therefore $h_p^B(b)=\infty$, and $h_p^X(x)=h_p^A(a)=h_p^{A*}(a)$.

If $y\in A_*$ then there are nonzero relatively prime integers m and n with $ny=ma$. But $h^{A*}(ma)=h^A(ma)$ so ma is divisible by n in A. Thus $y=m/n\, a\in A$, and $A\lhd Y$. Similarly, $B\lhd Y$.

If $my=a+b\in A\oplus B$ for some $y\in Y$, then write $m=n_1n_2$ where $n_1A=A$ and $n_2B=B$ and $(n_1, n_2)=1$ ($\operatorname{supp} A\cap\operatorname{supp} B=\emptyset$). Write $a=n_1a_1$ and $b=n_2a_2$. Then $n_1n_2y=n_1a_1+n_2b_1$, and since $B\not\lhd Y$, $n_1(n_2y-a)=n_2b$ implies that $n_2y-a_1=n_3b'$ for some $b'\in B$. By the purity of A in Y, there is an $a'\in A$ with $y-b'=a'$. Thus $A\oplus B\lhd Y$. □

§ 2. The classes $\mathscr{IC}^0(\Lambda)$ and $\mathscr{C}^0(\Lambda)$

In the paper [1], the authors studied the groups A such that $A\in\mathscr{C}^0(A)$. In that paper, a complete characterization of such groups were given. In this section, we will study $\mathscr{C}^0(\Lambda)$ where Λ is a set of groups, and as a consequence, we will give characterizations of the classes $\mathscr{IC}^0(\Lambda)$ and $\mathscr{C}^0(\Lambda)$ when Λ is a set of rank-1 groups.

Several basic facts about $\mathscr{C}^0(\Lambda)$ should be pointed out before we proceed. To simplify the notation, given an exact sequence $0\to A\xrightarrow{i}B\xrightarrow{g}C\to 0$, we will identity A with iA and C with B/A whenever appropriate.

PROPOSITION 2.1. *If $0\to A\to B\to C\to 0$ is an exact sequence of groups (in particular, the sequence is pure exact), and if $B\in\mathscr{C}^0(\Lambda)$, then $A, C\in\mathscr{C}^0(\Lambda)$.*

PROOF. Let K be a pure subgroup of A. Then K is a pure subgroup of B, and since $B\in\mathscr{C}^0(\Lambda)$, and $f\in\operatorname{Hom}(K, G)$ can be lifted through B, where $G\in\Lambda$. If $f\in\operatorname{Hom}(K, G)$, for some $G\in\Lambda$, then let $\bar f$ be such that $\bar f|_K=f$. Then $g=\bar f|_A$ is a lifting of f through A.

Let H/A be a pure subgroup of $C=B/A$. If $G\in\Lambda$ and $f\in\operatorname{Hom}(H/A, G)$, let $\pi: B\to C$ be the natural map. Then H is pure in B by the correspondence theorem, and $f\pi=g|_H$, then $A\subset\ker g$, so that $g\in\operatorname{Hom}(C, G)$. Also $g|_{H/A}=f$. □

By default, every rank-1 group is in $\mathscr{C}^0(\Lambda)$. Hence the converse to the proposition almost never holds, for if it did then one would conclude that every group is in $\mathscr{C}^0(\Lambda)$. This implies that Λ contains only divisible groups, and the characterization of $\mathscr{IC}^0(\Lambda)$ is trivial in this case. In fact, we will assume that Λ contains no divisible groups. Moreover, we may write a group A as $D\oplus R$, where R is reduced

and D is divisible. Since A is injective with respect to a particular exact sequence E iff R is injective with respect to E, we will assume that Λ contains only reduced groups.

LEMMA 2.2. *The following are equivalent:*
1. $Y\in\mathscr{I}\mathscr{C}^0(\Lambda)$.
2. *The diagram*

$$\begin{array}{ccc} 0 \to B & \xrightarrow{j} & C \\ {\scriptstyle f}\downarrow & & \\ Y & & \end{array}$$

can be completed whenever j is a pure embedding, $C\in\mathscr{C}^0(\Lambda)$, and $\operatorname{rank} f(B)=1$.

PROOF. (1→2) Clear.

(2→1) We will show that any $f\in\operatorname{Hom}(B, Y)$ can be lifted to an $\bar{f}\in\operatorname{Hom}(C, Y)$, where B is a pure subgroup of C and $C\in\mathscr{C}^0(\Lambda)$, by induction on $\operatorname{rank} f(B)=n$. The case when $n=1$ is covered by 2. Inductively, assume that whenever $C\in\mathscr{C}^0(\Lambda)$ and B is pure in C, any $f\in\operatorname{Hom}(B, Y)$ with $\operatorname{rank} f(B)\leqq n-1$ has a lifting in $\operatorname{Hom}(C, Y)$. Let $C\in\mathscr{C}^0(\Lambda)$, B be pure in C and $f\in\operatorname{Hom}(B, Y)$ have $\operatorname{rank} f(B)=n$.

Clearly

(∗)
$$\begin{array}{ccc} 0 \to B & \to C \\ {\scriptstyle f}\downarrow & \\ Y & \end{array}$$

can be completed iff

$$\begin{array}{ccc} 0 \to B/\ker f & \to C/\ker f \\ {\scriptstyle f}\downarrow & \\ Y & \end{array}$$

can be completed. By Proposition 2.1, $C/\ker f\in\mathscr{C}^0(\Lambda)$, so we may as well assume that f is injective, in the diagram (∗).

Let $0\neq x\in B$. By the hypothesis, there is $g\in\operatorname{Hom}(C, Y)$ making the diagram

$$\begin{array}{ccc} 0 \to \langle x\rangle_* & \to C \\ {\scriptstyle f}\downarrow & \\ Y & \end{array}$$

commutative. Consider the map $f-g\in\operatorname{Hom}(B, Y)$. By induction, there is a map $g'\in\operatorname{Hom}(C, Y)$ with $g'|_B=f-g$. It is not hard to see that $g'+g$ provides a lifting of through C. □

From this it is easy to see that if G is a group such that typeset $G\subset\{\text{type } X|X\in\Lambda\}$, where Λ is a set of rank-1 groups, then $G\in\mathscr{I}\mathscr{C}^0(\Lambda)$. However, $G\in\mathscr{I}\mathscr{C}^0(\Lambda)$ does not necessarily imply that typeset $G\subset\{\text{type } X|X\in\Lambda\}$. For example, let X_1 and X_2 be two rank-1 groups with incomparable types. Then $X_1\oplus X_2\in\mathscr{I}\mathscr{C}^0(\{X_1, X_2\})$, since $\mathscr{I}\mathscr{C}^0$ is always closed under finite direct sums. But type $X_1\wedge\text{type } X_2<\text{type } X_i$.

LEMMA 2.3. *The following are equvialent:*
1. $Y\in\mathscr{I}\mathscr{C}^0(\Lambda)$.

2. *Any diagram of the form*

$$\begin{array}{ccc} 0 \to B & \to & C \\ & {\scriptstyle f}\downarrow & \\ & Y & \end{array}$$

can be completed, when $C \in \mathscr{C}^0(A)$ *and* B *is a pure corank-*1 *subgroup of* C.

PROOF. (1→2) Clear.

(2→1) We will show that we can complete any diagram:

$$\begin{array}{ccc} 0 \to B & \to & C \\ & {\scriptstyle f}\downarrow & \\ & Y & \end{array}$$

with $C \in \mathscr{C}^0(A)$ and B pure in C. As was done in the proof of Lemma 2.2, reduce to the case that f is injective.

Since C is of finite rank, there are pure subgroups, $B_0 = B, B_1, \ldots, B_n = C$, of C such that $B_j \subset B_{j+1}$ and $\operatorname{rank} B_{j+1}/B_j = 1$, for $j = 0, \ldots, n-1$. By Proposition 2.1, each $B_j \in \mathscr{C}^0(A)$. By the hypothesis, for each $j \geqq 1$, there is an $f_j \in \operatorname{Hom}(B_j, Y)$ such that $f_j|_{B_j} = f_{j-1}$ $(f_0 = f)$. Thus f_n is a lifting of f. □

In the paper [4], it is shown that if $OT(C) \leqq IT(A)$ then $C \in \mathscr{C}^0(A)$, for any groups C and A. Call a subgroup B of C an S-pure subgroup, for S a set of primes, provided that C/B is p-torsion-free for every $p \in S$. For a group A, let $R(A)$ be the subring generated by 1 and $1/p$ for every p such that $pA = A$. Note that since A is torsion-free, A is naturally an $R(A)$-module and therefore, $R(A) \otimes A = A$.

LEMMA 2.4. *Assume that* $OT(C) \leqq IT(A)$ *and that* B *is a* supp A*-pure subgroup of* C. *Then* A *is injective with respect to the embedding* $0 \to B \to C$.

PROOF. Let X be a rank-1 group having type $X = IT(A)$. The hypothesis implies that there is an embedding $0 \to C \to X^n$ (where $n = \operatorname{rank} C$). Note that $R(A) = R(X)$ since $pX = X$ iff $pA = A$. Then $0 \to C \otimes R(X) \to X^n \otimes R(X)$ and $C \otimes R(X)$ is isomorphic to a subgroup of $X^n \cong X^n \otimes R(X)$. Therefore $OT(C \otimes R(A)) \leqq$ $\leqq$ type $X = IT(A)$.

By the remarks preceding the statement of the lemma, A is injective with respect to the pure embedding $0 \to B \otimes R(A) \to C \otimes R(A)$. But the rectangle is commutative

$$\begin{array}{ccc} B & \longrightarrow & C \\ \downarrow & & \downarrow \\ B \otimes R(A) & \to & C \otimes R(A) \end{array}$$

with monic columns. If $f\colon B \to A$, then $f \otimes 1\colon B \otimes R(A) \to A$ has a lifting $\bar{f}$ through $C \otimes R(A)$. The restriction of $\bar{f}$ to C is a lifting of f. □

Before attempting to characterize $\mathscr{C}^0(A)$, we will need a few definitions. The first term is standard, the other two are not.

DEFINITION. For groups C and A, the A-radical of C, written $C[A]$, is defined as $C[A] = \cap \{\ker f \mid f \in \operatorname{Hom}(C, A)\}$. The notation $C'[A]$ will be used to in-

dicate the *set*, $C'[A]=\cup\{X\lhd C \mid \mathrm{Hom}(X, A)=0 \text{ and rank } X=1\}$, and $C^1[A]=\langle X\lhd C \mid \text{rank } X=1 \text{ and } \mathrm{Hom}(X, A)=0\rangle$.

Generally $C'[A]$ need not be a subgroup of C, while $C[A]$ is always a pure subgroup of C. A characterization of $C[A]$ (in C) is that it is the unique smallest subgroup K of C such that C/K is isomorphic to a subgroup of A^n (for some n).

PROPOSITION 2.5. *For any group A, if $C\in\mathscr{C}^0(A)$ then $C[A]=C'[A]=C^1[A]$.*

PROOF. Clearly $C[A]\supseteq C'[A]$. Let $x\in C[A]$ and $X=\langle x\rangle_*$. If $\mathrm{Hom}(X, A)\neq 0$, then the diagram:

$$\begin{array}{ccc} 0 \to X & \to & C \\ {\scriptstyle \neq 0}\downarrow & & \\ A & & \end{array}$$

can be completed, contrary to the assumption that $X\leqq C[A]$. Thus $X\subseteq C'[A]$. □

For groups A and C, the C-socle of A is the subgroup of A generated by the set $\mathrm{Hom}(C, A)C$. We denote this subgroup by $S_C(A)$. Clearly, $C\in\mathscr{C}^0(A)$ iff $C\in\mathscr{C}^0(S_C(A))$. For a set of primes S, let $S^\omega C$ represent the pure subgroup of C, $\bigcap_{p\in S} p^\omega C$ (where as usual, $p^\omega C=\cap\, {}_{n}p^n C$).

THEOREM 2.6. *Let $S=\mathrm{supp}\, S_C(A)$, where A is any reduced group. The following are equivalent:*

1. $C\in\mathscr{C}^0(A)$.
2. $C'[A]=C$ *or* $C[A]=S^\omega C$ *and* $C/C[A]\in\mathscr{C}^0(A)$.
3. $C/C[A]\in\mathscr{C}^0(A)$, $C'[A]=C[A]$ *and for any pure subgroup B of C, $B+C[A]$ is S-pure in C.*

PROOF. (1→2) Assume that $C'[A]\neq C$ (so that $C[A]\neq C$). Since A is reduced and $\mathrm{Hom}(S^\omega C, A)$ is divisible, $S^\omega C\subset C[A]$. Suppose $C[A]\setminus S^\omega C\neq\emptyset$. Choose $x\in C[A]\setminus S^\omega C$ such that the p-height of x is 0, for some $p\in S$. For this p there is a $y\in S_C(A)$ with zero p-height. Represent y as $\Sigma_j f_j(c_j)$. For some i, $f_i(c_i)$ has zero p-height in $S_C(A)$. Call $y'=f_i(c_i)$. Then $f_i(c_i+x)=y'$, so that f_i provides an embedding of $\langle c_i+x\rangle_*$ into $\langle y'\rangle_*$. Also $f_i(pc_i+x)=py'$ so that $t^C(pc_i+x)\geqq t$ where $t=$type of y' in $S_C(A)$. If h is the height vector of y' in $S_p^C(A)$, then $h_p^C(pc_i+x)=0=h_p^C(x)=h_p$. Therefore, there is an embedding, $f\colon \langle pc_i+x\rangle_*\to A$ with $f(pc+x)=y'$. But if $\bar{f}$ lifts f through C, then $\bar{f}(pc_i+x)=\bar{f}(pc_i)=p\bar{f}(c_i)=f(pc_i+x)=y'$, contrary to the fact that y' has 0 p-height in $S_C(A)$. Thus $C[A]=S^\omega C$.

By Proposition 2.1, $C/C[A]\in\mathscr{C}^0(A)$.

(2→3) If $C'[A]=C$, then $C[A]=C$, since in general, $C'[A]\subset C[A]$. Condition 3 holds trivially in this case. If $C[A]=S^\omega C$, then for any pure subgroup X of $C[A]$, $\mathrm{Hom}(X, A)=0$, since A is reduced. Thus $C'[A]=C[A]$. In either case, $C/C[A]\in\mathscr{C}^0(A)$. We must show that if B is a pure subgroup of C, then $B+C[A]$ is an S-pure subgroup of C.

Suppose $pc=x\in B+C[A]$. Write $x=b+v$, for some $b\in B$, and $v\in C[A]$. But v is p-divisible, by condition 2, so that $v=pv'$ for some $v'\in C[A]$. Thus $b=pc-pv'$, and by the purity of B, $b=pb'$ for some $b'\in B$. This implies that $c=b'+v'$, and that $B+C[A]$ is S-pure in C.

(3→1) First suppose $K \in \mathscr{C}^0(A)$. Since A is naturally an $R(A)$-module, $K \otimes R(A) \in \mathscr{C}^0(A \otimes R(A)) = \mathscr{C}^0(A)$. We must show that any diagram of the form:

$$\begin{array}{ccccc} 0 & \to & B & \to & C \\ & & \downarrow f & & \\ & & A & & \end{array}$$

can be completed, under the assumption that B is a pure subgroup. Since we are assuming that $C'[A]=C[A]$, if $x \in C[A] \cap B$, then $\mathrm{Hom}(\langle x \rangle_*, A)=0$.

In particular, $f(x)=0$. Therefore, it suffices to show that the diagram:

$$(*) \qquad \begin{array}{ccccc} 0 & \to & B/(B \cap C[A]) & \to & C/C[A] \\ & & \downarrow f & & \\ & & A & & \end{array}$$

can be completed, where the top row is an S-pure embedding as guaranteed by statement 3. Tensoring the top row by $R(A)$, yields a pure exact row, and a diagram which can be completed by the first remark.

Let $\bar{f} \in \mathrm{Hom}(R(A) \otimes C/C[A], A)$ be a lifting to $1 \otimes f \in \mathrm{Hom}(R(A) \otimes B/(BC[A]), A)$. Then since the rectangle

$$\begin{array}{ccc} B/(B \cap C[A]) & \longrightarrow & C/C[A] \\ \downarrow & & \downarrow \\ R(A) \otimes B/(B \cap C[A]) & \to & R(A) \otimes C/C[A] \end{array}$$

is commutative, the restriction of $\bar{f}$ to $C/C[A]$ yields a lifting of f in $(*)$. □

Generally, this theorem does not provide satisfying conditions for determining which groups are in $\mathscr{C}^0(A)$. However, if A is rank-1 then $C/C[A] \in \mathscr{C}^0(A)$ for any C, by Lemma 2.4. Hence a characterization of $\mathscr{C}^0(\Lambda)$ is now at hand, for Λ a set of rank-1 groups. Call an embedding $0 \to B \xrightarrow{f} C$ an S-pure embedding provided that $f(B)$ is an S-pure subgroup of C.

THEOREM 2.7. *Let Λ be a set of rank-1 groups. The following are equivalent:*

1. $C \in \mathscr{C}^0(\Lambda)$.

2. *For each $X \in \Lambda$, either there is no nonzero $c \in C$ with an embedding of $\langle c \rangle_*$ into X, or, $C[X]=p^\omega C$ for every $p \in \mathrm{supp}\, X$.*

3. *If $X \in \Lambda$, then for any pure subgroup B of C, $B[X]=C[X] \cap B$, and the natural map from $B/B[X] \to C/C[X]$ is an embedding which is S-pure, for $S=\mathrm{supp}\, X$.*

PROOF. (1→2) Statement 2 in this theorem, is the same as Statement 2 in Theorem 2.6, taking into account the special properties of rank-1 groups. For instance, it is always true that $OT(C/C[X]) \leqq \mathrm{type}\, X$, so by Lemma 2.4, $C/C[X] \in \mathscr{C}^0(X)$. The condition that $C'[X]=C$ is equivalent to the condition that there is no $0 \neq c \in C$ with an embedding of $\langle c \rangle_*$ into X. Since X is rank-1, $p^\omega X=0$ for every $p \in \mathrm{supp}\, X$. This implies that $p^\omega C \subset C[X]$, and therefore by Theorem 2.6, $C[X]=p^\omega C$ for all such p.

(2→3) If B is a pure subgroup of C, then $p^\omega B=p^\omega C \cap B$ for every p. If $C'[X]=C$, then $B \subseteqq C'[X]$, so that $B[X]=B'[X]=B$. In this case, $B[X]=C[X] \cap B$ and the embedding is one of two zero groups, hence is pure. In the case that $C[X]=$

$=p^\omega C$ for all $p\in\operatorname{supp} X$, we must have that $B[X]=p^\omega B$ for those p. To see this, suppose $b\in Bp^\omega B$ for $p\in\operatorname{supp} X$. Then $b\notin p^\omega C=C[X]$, so there is an $f\in\operatorname{Hom}(C, X)$ with $f(b)\neq 0$. Then $f|_B(b)\neq 0$, so that $b\notin B[X]$. The containment $p^\omega B\subseteq B[X]$ always holds, so that $B[X]=C[X]\cap B$, and as in 2→3 in Theorem 2.6, the embedding is supp X-pure.

(3→1) Let $B\lhd C$ and $f\colon B\to X$. Since $B[X]\subseteq\ker f$, to complete the diagram:

$$\begin{array}{ccc} 0 \to B & \to & C \\ & {\scriptstyle f}\downarrow & \\ & X & \end{array}$$

it suffices to complete the diagram

$$\begin{array}{ccccc} 0 \to B/B[X] & \to & C/C[X]. & & \\ & {\scriptstyle f}\downarrow & & & \\ & X & & & \end{array}$$

The latter diagram can be completed by Lemma 2.4. □

For now, let Λ be a fixed set of rank-1 groups and $T(\Lambda)=\{t\,|\,t=\text{type } X \text{ for some } X\in\Lambda\}$.

The following result is a contribution of W. Wickless, the proof of which appears in [5].

THEOREM. *If Y is a reduced group, then Y contains at most countably many rank-2 subgroups G such that if A and B are pure independent rank-1 subgroups of G then A and B are almost pure in Y (i.e., $\langle A\rangle_*/A$ and $\langle B\rangle_*/B$ are finite).*

DEFINITION. If A is a rank-1 group, let $\tau(A)=\{t\in T(\Lambda)\,|\,t\cong\text{type } A\}$.

The next result will be useful in determining $\mathscr{IC}^0(\Lambda)$.

THEOREM 2.8. *If $Y\in\mathscr{IC}^0(\Lambda)$ then for any $X\lhd Y$, X reduced and rank-1, type $X=\wedge\tau(X)$.*

PROOF. If $Y=R\oplus D$ with R reduced and D divisible, and if $X\lhd Y$ with X reduced then X is isomorphic to a pure subgroup of R, with $R\in\mathscr{IC}^0(\Lambda)$. Hence, we may as well assume that Y is reduced.

Assume, to the contrary, that for some $X\lhd Y$, with rank $X=1$, there is a type $t>\text{type } X=s$ such that $t\geqq r$ for every $r\in\tau(X)$. Furthermore, assume that s is a maximal type in typeset Y, with this occurence.

We will first show that $\tau(X)\neq\emptyset$. If $\tau(X)=\emptyset$, then $s\not<r$ for any $r\in T(\Lambda)$. Consequently, if G is a group which is homogeneous of type $=s$ and cohomogeneous of type $=\overline{\infty}$, then $\operatorname{Hom}(\langle y\rangle_*, A)=0$, for any $0\neq y\in G$ and $A\in\Lambda$. Thus $G\in C^0(\Lambda)$, by Theorem 2.6.

For such $a\in G$, consider

$$\begin{array}{ccc} 0 \to X & \xrightarrow{\alpha} & G \\ & {\scriptstyle f}\downarrow & \\ & Y & \end{array}$$

with a pure row. Any completion of f must be an embedding since every factor of G is isomorphic to Q, and $X \lhd Y$. Moreover, if $y \in G \setminus \alpha X$ then $Y = \langle f(y) \rangle_*$ must have type $Y = s$, partially due to the fact that type $Y \geqq s$. But if type $Y > s$ then type $Y = \Lambda\tau(\text{type } Y)$ and in particular $\tau(Y) \subseteq \tau(X)$ is nonempty, contrary to our current assumption. Hence if $B = \langle y \rangle_* \subseteq G$ then $f(B)$ is almost pure in Y. Since Y contains at most countably many subgroups $f(G)$ and there are uncountably many such groups G, there is a group G' and a diagram:

$$\begin{array}{ccc} 0 \to X & \xrightarrow{\alpha'} & G' \\ & \downarrow & \\ & Y & \end{array}$$

which cannot be completed. Moreover, $G' \in \mathscr{C}^0(\Lambda)$, which is a contradiction. Therefore $\tau(X) \neq \emptyset$.

Let G be a group which is homogeneous of type s and cohomogeneous of type t (such groups are easy to construct). Then $G \in \mathscr{C}^0(\Lambda)$. To see this, let $r \in T(\Lambda)$, and $A \in \Lambda$ with type $A = r$. If $r \in \tau(X)$ then $r \geqq t$ so that $OT(G) \leqq r$. By Lemma 2.4, $G \in \mathscr{C}^0(A)$. If $r \notin \tau(X)$ then $\operatorname{Hom}(X, A) = 0$. Consequently, $\operatorname{Hom}(\langle y \rangle_*, A) = 0$ for all $y \in G$. By Theorem 2.7, $G \in \mathscr{C}^0(A)$. Thus $G \in \bigcap \mathscr{C}^0(A) = \mathscr{C}^0(\Lambda)$.

For such groups G, any completion f of the diagram

$$\begin{array}{ccc} 0 \to X & \to & G \\ {\scriptstyle f}\downarrow & & \\ Y & & \end{array}$$

must be an embedding. By counting the number of subgroups $f(G)$ of Y, we see that there is a rank-2 group G', homogeneous of type s, cohomogeneous of type t and $X \lhd G'$ with an embedding $h: G' \to Y$ (extending $X \subseteq Y$) such that type $\langle h(y) \rangle_* = s' > s$ for some $y \in G' X$.

Let $Y = \langle h(y) \rangle_*$. By the maximality of s, $s' = \wedge\tau(Y)$, and since $\tau(Y) \subseteq \tau(X)$, $s' \geqq t$. In particular, $s' \geqq OT(G')$. Consider the diagram:

$$\begin{array}{ccc} 0 \to \langle y \rangle_* & \to & G. \\ {\scriptstyle f}\downarrow & & \\ Y & & \end{array}$$

Embed G'/X into Y. Call the embedding $g: G'/X \to Y$, and let $\pi: G' \to G'/X$ be the natural map. Since Y is rank-1, there are nonzero integers n and m with $mh(y) = ng\pi(y)$. Therefore the diagram:

$$\begin{array}{ccc} 0 \to \langle y \rangle_* & \to & G' \\ {\scriptstyle nh}\downarrow & & \\ y & & \end{array}$$

is commutative.

But the map $\mu = mh - ngu \in \operatorname{Hom}(G', Y)$ satisfies $\mu(y) = 0$ while for $0 \neq x \in X$, $\mu(x) = mh(x) \in X$ and $\mu(x) \neq 0$. This contradicts the fact that G' is cohomogeneous of type t and completes the proof. □

As a consequence, we have

COROLLARY 2.9. *Let A be a rank-1 group, and Y a reduced group. Then $Y=\mathscr{C}^0(\{A\})$ iff* typeset $Y=\{\text{type } A\}$.

PROOF. The sufficiency was noted earlier. The necessity follows from theorem 2.8. □

We will use Lemma 1.2 to give a characterization of the rank-1 groups in $\mathscr{C}^0(\Lambda)$. Write Λ as an indexed set $\{X_\alpha|\alpha\in I\}$. Let $T=T(\Lambda)=\{t_\alpha|t_\alpha=\text{type } X_\alpha \text{ and } \alpha\in I\}$, for a type s let $\tau(s)=\tau(X)$ where X is any rank-1 group of type s, and $l(T)=$ $=\{s|s \text{ is a type and } s=\wedge\tau(s)\}$. Finally, let $\mu(s)=\{\alpha\in I|t_\alpha\in\tau(s)\}$.

DEFINITION. We can define a map $\eta: l(T)\to Z^+$ as follows: For $s\in l(T)$, $s=\tau(s)$. Partition $\mu(s)$ into $I_1\dots I_k$ as in Lemma 1.2. Define $\eta(s)=k$.

If $s=s_1\wedge\dots\wedge s_k$ as in Lemma 1.2, then $\eta(s_i)=1$ so that $\eta(s)=\Sigma\eta(s_i)$. We can now generalize Corollary 2.9.

THEOREM 2.10. *Let X be a rank-1 group and $s=$ type X. The following are equivalent:*

1. $X\in\mathscr{I}\mathscr{C}^0(\Lambda)$.
2. $s=\wedge\{t_\alpha|t_\alpha\geqq s\}$ *and* $\eta(s)=1$.

PROOF. (1→2) By Theorem 2.8, $s=\wedge\tau(s)$. We will assume that $\eta(s)>1$ and reach a contradiction by verifying the claim below.

If $\eta(s)=k>1$ and write $s=s_1\wedge\dots\wedge s_k$ as in Lemma 1.2. Call $t=s_2\wedge\dots\wedge s_k$. Let C and B be subgroups of $\mathbf{Q}$ containing Z, of types s_1 and t respectively, and let $A=C\oplus B$. For $(1, 1)\in A$, type $\langle(1, 1)\rangle_*=s_1$ $t=s$, so there is a pure embedding $i: X\cong\langle(1, 1)\rangle_*\subseteq A$. Clearly X is not a summand of A.

Claim: $A\in\mathscr{C}^0(\Lambda)$.

We will show that $A\in\mathscr{C}^0(\Lambda)$ by showing that $A\in\mathscr{C}^0(\{X_\alpha\})$ for each $X_\alpha\in\Lambda$.

Case 1: $\alpha\notin\mu(s)$. Since $IT(A)=s$ and $t_\alpha\not\geqq s$, Hom $(\langle a\rangle_*, X_\alpha)=0$ for any $a\in A$. By Theorem 2.7, $A\in\mathscr{C}^0(\{X_\alpha\})$.

Case 2: $\alpha\in\mu(S)$. In this case, we need to show that $p^\omega A=A[t_\alpha]$ for every $p\in$ supp t_α.

By Lemma 1.2, Property 5, either $t_\alpha\geqq s_j\geqq t$ for some $j=2, 3, \dots, k$ or $t_\alpha\geqq s_1$. The cases are handled similarly, so will assume that $t_\alpha\geqq t$.

If $p\in$supp t_α then $p\in$supp $t\subseteq\Pi\setminus$supp s_1, so that $p^\omega A=C$. On the other hand, $A/C=B$ which implies that $C=A[t_\alpha]$. By Theorem 2.7, $A\in\mathscr{C}^0(\{X_\alpha\})$, which proves the claim.

The claim contradicts the assumption that $\mathscr{C}^0(\Lambda)\subseteqq\mathscr{C}^0(\{X\})$ since $A\in\mathscr{C}^0(\{X\})$. Thus $\eta(s)=1$.

(2→1) We will show that $\mathscr{C}^0(\Lambda)\subseteqq\mathscr{C}^0(\{X\})$. Let $A\in\mathscr{C}^0(\Lambda)$. If $A'[X]=A$ then $A\in\mathscr{C}^0(\{X\})$. Otherwise, if $0\neq x\in A$ with type $\langle x\rangle_*=t^A(x)\geqq$ type X, then $t^A(x)\geqq t_\alpha$ for every $\alpha\in\mu(S)$. By Theorem 2.7, $A[t_\alpha]=p^\omega A$ for every $p\in$supp t_α.

Claim: for any $\alpha, \beta\in\mu(S)$, $p^\omega A=A[t_\alpha]=A[t_\beta]$ for all $p\in$supp t_α.

To show this, it is enough to show that $p^\omega A=q^\omega A$ for every $p, q\in$supp S. Let $p\in$supp X_α and $q\in$supp X_β. Since $\eta(s)=1$, there is a sequence $\alpha_0, \dots, \alpha_m\in\mu(s)$

with $\alpha_0=\alpha$, $\alpha_m=\beta$ and $\operatorname{supp} X_{\alpha_i}\cap\operatorname{supp} X_{\alpha_{i+1}}\neq\emptyset$ for $i=0,\ldots,m-1$. By Theorem 2.7, $p_iA=p_{i+1}A$ since both $p_i, p_{i+1}\in\operatorname{supp} X_{\alpha_{i+1}}$. Thus $p_0^\omega A=p_1^\omega=\ldots=p_m^\omega A$ and $p^\omega A=p_0^\omega A=p_m^\omega A=q^\omega A$.

Hence $p^\omega A=A[t_\alpha]$ for every $\alpha\in\mu(s)$ and $p\in\operatorname{supp} s$. For $p\in\operatorname{supp} s$, $OT(A/p^\omega A)=OT(A/A[t_\alpha])\geqq t_\alpha$ for every $\alpha\in\mu(s)$. Since $s=\tau(s)$, $OT(A/p^\omega A)\geqq s$. Thus $p^\omega A\subseteqq A[s]$ for every $p\in\operatorname{supp} s$. But $p^\omega X=0$ for $p\in\operatorname{supp} s$, so that $A[s]=p^\omega A$ for those primes p. Thus $A\in\mathscr{C}^0(\{X\})$. □

We are now ready to give a characterization of $\mathscr{IC}^0(\Lambda)$, which will depend on the characterizations of the rank-1 groups in $\mathscr{IC}^0(\Lambda)$.

THEOREM 2.11. *Let* $\Lambda=\{X_\alpha|\alpha\in I\}$ *be a set of rank-1 groups and* $T(\Lambda)=\{t_\alpha|t_\alpha=\text{type } X_\alpha, \alpha\in I\}$. *For any group* G, $G\in\mathscr{IC}^0(\Lambda)$ *iff for any pure rank-1 subgroup* X *of* G, *there are pure rank-1 subgroups* $G_1\ldots G_k\leqq G$ *such that* $X\leqq G_1\oplus\ldots\oplus G_k\leqq G$ *and* $G_i\in IC^0(\Lambda)$ *for* $i=1, 2, \ldots, k$.

In this case, if k *is minimal,* $k=\eta(\text{type } X)$.

PROOF. ($\rightarrow$) If $X\lhd G$ and $X\cong Q$ then $X\in IC^0(\Lambda)$ and the condition holds. Therefore, we will assume without loss of generality, that G is reduced. Since typeset $G\subseteqq I(T)$ by Theorem 2.8, we will show by induction on $\eta(s)=k$, for $s\in$typeset G, that the condition holds.

If $s\in$typeset G with $\eta(s)=1$, and $X\lhd G$ of rank-1 with type $X=s$, then $X\in IC^0(\Lambda)$ by Theorem 2.10.

Assume for every $t\in$typeset G with $1\leqq\eta(t)<k$, the condition holds for any rank-1 pure subgroup A of G with type $A=t$.

Let $s\in$typeset G, $X\lhd G$ of rank-1 with type $X=s$ and $\eta(s)=k$. Write $s=s_1\wedge\ldots\wedge s_k$ as in Lemma 2.1, and let $t=s_2\wedge\ldots\wedge s_k$. Let C and B be subgroups of $\mathbf{Q}$, containing Z, of types s_1 and t respectively. Then, as in the proof of Theorem 2.10, $A=C\oplus B\in\mathscr{C}^0(\Lambda)$.

Let $i\colon X\rightarrowtail A$ be a pure embedding. Let $g\in\operatorname{Hom}(A, G)$ making the diagram:

$$\begin{array}{ccccc} 0 & \to & X & \xrightarrow{i} & A \\ & & \downarrow & \swarrow_{g} & \\ & & G & & \end{array}$$

commutative.

Clearly g is an embedding, so that $X\leqq g(C)\oplus g(B)\leqq G$. By Lemma 1.4 both $g(C)$ and $g(B)$ are pure in G. By the remark preceding Theorem 2.10, $\eta(s_1)=1$ and $\eta(t)=k-1$. Hence, by the induction hypothesis, there are $G_1, \ldots, G_k\in IC^0(\Lambda)$ such that each G_i is a rank-1 pure subgroup of G and $g(B)\leqq G_1\oplus\ldots\oplus G_{k-1}\leqq G$. Call $g(C)=G_k$.

Let $p\in\operatorname{supp} G_k$. Any nonzero element of $G_k\cap(G_1\oplus\ldots\oplus G_{k-1})$ has finite p-height as well as being p-divisible ($pG_i=G_i$ for $i=1, \ldots, k-1$). Thus $X\leqq G_1\oplus\ldots\oplus G_{k-1}\oplus\oplus G_k\leqq Y$ as desired.

($\leftarrow$) Let $A\in\mathscr{C}^0(\Lambda)$, $B\lhd A$ and $f\colon B\rightarrow G$. By Lemma 2.2, we need only consider maps f with rank $f(B)=1$. Let $X=\langle f(B)\rangle$. By the hypothesis, there are $G_1\ldots G_k\in\mathscr{IC}^0(\Lambda)$ with $X\leqq G_1\oplus\ldots\oplus G_k\leqq G$.

For each $j=1, 2, \ldots, k$, let $\pi_j: \bigoplus_{i=1}^{k} G_i \to G_j$ be the coordinate projection, and consider $\pi_j f: B \to G_j$. Let $\bar{f}_j: A \to G_j$ lift $\pi_j f$ through A. Consider $\bar{f}=\bar{f}_1+\ldots+\bar{f}_k: A \to G$. For $b \in B$, $f(b)=\pi_1 f(b)+\ldots+\pi_k f(b)=\bar{f}_1(b)+\ldots+\bar{f}_k(b)=\bar{f}(b)$. □

§ 3. A remark on the dual, projective class

The dual class to $\mathscr{C}^0(\Lambda)$ defined as the class of those groups C such that each $A \in \Lambda$ is projective with respect to any pure exact sequence $0 \to B \to C \to C/B \to 0$, was studied in [5]. In this section we will compare notes and point out the lack of duality.

The authors, C. Vinsonhaler and W. Wickless determined $\mathscr{C}_0(\Lambda)$ and $\mathscr{PC}_0(\Lambda)$ (defined analogously to $\mathscr{IC}^0(\Lambda)$) when Λ is a set of rank-1 groups. The following appear in that paper:

THEOREM 3.1. *Let Λ be a set of rank-1 groups. The following are equivalent:*
1. $C \in \mathscr{C}^0(\Lambda)$.
2. *For each $A \in \Lambda$, either C has no factors of type $\cong$ type A, or $IT(C) \cong$ $\cong$ type X.*

And

THEOREM 3.2. *Let $\Lambda=\{A_\alpha | \alpha \in I\}$ be a set of rank-1 groups and $T(\Lambda)=$ $=\{t_\alpha | t_\alpha=\text{type } A_\alpha, \alpha \in I\}$. Then $Y \in PC_0(\Lambda)$ iff for any rank-1 factor X of Y, if $X \ncong Z$ then* type $X=\vee\{t_\alpha | t_\alpha \cong \text{type } X, \alpha \in I\}$.

Theorem 3.2 states that the dual to Theorem 2.8, gives a characterization of $\mathscr{PC}_0(\Lambda)$. However, if we take p and q to the distinct primes, $\Lambda=\{Z_q, Z_p\}$ and $X=\langle (1, 1)\rangle_* \subseteq Z_p \oplus Z_q$, then type $X=$type $Z_p \wedge$ type Z_q. But $\eta(\text{type } X)=2$ since Z_p and Z_q have disjoint supports. Therefore, $X \in \mathscr{IC}^0(\Lambda)$, by Theorem 2.10. Let DS be the dual statement.

DS: For any group Y, $Y \in \mathscr{IC}^0(\Lambda)$ iff for any $X \lhd Y$, X rank-1 and reduced, type $X=\tau(X)$.

Recalling notation from Section 2,

THEOREM 3.3. *Let Λ be a set of rank-1 groups. Then the statement* DS *is true iff $\eta(s)=1$ for any $s \in I(T(\Lambda))$.*

PROOF. Assuming $\eta(s)=1$ for every $s \in I(T(\Lambda))$, we must verify that the converse to Theorem 2.8 holds in this case. Let $C \in \mathscr{C}^0(\Lambda)$, $B \lhd C$ and $f: B \to Y$ with $X=\langle f(B)\rangle_*$ rank-1. Assuming type $X=\wedge\tau(X)$ and therefore $\eta(\text{type } X)=1$, by Theorem 2.10, $X \in \mathscr{IC}^0(\Lambda)$. Hence f can be lifted to $f \in \text{Hom}(C, X)$.

Conversely, assume DS is true we must show that $\eta(s)=1$ for every $s \in I(T(\Lambda))$. If $\eta(s)>1$ for some $s \in I(T(\Lambda))$, then we may proceed as in Theorem 2.10, to construct a group $A \in \mathscr{C}^0(\Lambda)$ while $A \in \mathscr{C}^0(\{X\})$ where X is a rank-1 group of type s. But DS guarantees that $X \in \mathscr{IC}^0(\Lambda)$ since type $X=\wedge\tau(X)$ (i.e., $s \in I(T(\Lambda))$), a contradiction. □

COROLLARY 3.4. DS *characterizes $\mathscr{IC}^0(\Lambda)$ iff $\mathscr{IC}^0(\Lambda)$ is closed under pure subgroup taking.*

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CONVERGENCE OF CERTAIN POLYNOMIAL INTERPOLANTS TO A FUNCTION DEFINED ON THE UNIT CIRCLE

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1. Introduction. Let $f(z)$ be continuous or more generally, R-integrable on the unit circle Γ. Let $L_{n-1}(z,f)$ be the Lagrange interpolant to f on the n^{th} roots of unity. In an interesting extension of a result of Meray, Walsh [4] proved that

$$\lim_{n\to\infty} L_{n-1}(z,f) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)}{t-z}\,dt \tag{1.1}$$

for $|z|<1$ and uniformly for $|z|\leqq\delta<1$.

In this note we extend this result of Walsh to three kinds of polynomials. In Section 2 we suppose that $f^{(r-1)}(z)$ exists and is R-integrable along the circle. We show that a result of type (1.1) persists for polynomials $h_{rn-1}(z,f)$ which interpolate f in the Hermite sense at the zeros of $(z^n-1)^r$. In Section 3, we replace the polynomial $h_{rn-1}(z,f)$ by the polynomial of l_2-approximation, i.e. by $Q_{n-1}(z,f)$ which minimizes

$$\sum_{k=0}^{m-1}|f(w^k)-p(w^k)|^2 \tag{1.2}$$

over all $p(z)\in\pi_{n-1}$, $m\geqq n$. Here w is a primitive m^{th} root of unity. In Section 4, we show that the result of type (1.1) holds for the average of partial sums of the Hermite interpolant $h_{rn-1}(z,f)$. It may be mentioned that the convergence of the average of partial sums of a real polynomial interpolant has been discussed in Zygmund ([5], p. 20).

2. Hermite interpolant. Suppose $f^{(j)}(z)$ exists along Γ for $j=0, 1, \ldots, r-1$ and $h_{rn-1}(z,f)$ is the polynomial of degree $nr-1$ which satisfies the conditions:

$$h^{(j)}_{rn-1}(w^k,f) = f^{(j)}(w^k), \quad k=1,\ldots,n;\ j=0,1,\ldots,r-1, \tag{2.1}$$

where $w^n=1$. Then we shall prove the following

THEOREM 1. *Let $f^{(r-1)}(z)$ exist and be Riemann integrable along Γ. If $h_{rn-1}(z,f)$ is the Hermite interpolant to $f(z)$ satisfying* (2.1) *then*

$$\lim_{n\to\infty} h_{rn-1}(z,f) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)}{t-z}\,dt \tag{2.2}$$

for $|z|<1$ and uniformly for $|z|\leqq\delta<1$.

In order to prove Theorem 1 we need the following lemma.

LEMMA 1. *Let $f(z)$ be Riemann integrable on Γ and let $L_{n-1}(z,f)$ be the Lagrange interpolant to $f(z)$ at the n^{th} roots of unity. Then*

$$\lim_{n\to\infty} L_{n-1}^{(p)}(z,f) = \frac{p!}{2\pi i}\int_{\Gamma}\frac{f(t)}{(t-z)^{p+1}}\,dt, \tag{2.3}$$

for any given nonnegative integer p and for $|z|<1$. The convergence is uniform for $|z|\leqq\delta<1$.

PROOF. We first notice that (2.3) with $p=0$ is the same as (1.1). We only sketch the proof of (2.3) as it is similar to that of (1.1). We have

$$L_{n-1}(z,f) = \frac{1}{n}\sum_{k=0}^{n-1} f(w^k)\sum_{j=0}^{n-1} w^{-kj}z^j.$$

On differentiating p times, we get

$$L_{n-1}^{(p)}(z,f) = \frac{p!}{n}\sum_{k=0}^{n-1}\frac{f(w^k)w^k}{(w^k-z)^{p+1}} - \frac{z^n}{n}\sum_{l=0}^{p}\binom{p}{l}(n)_l z^{-l}(p-l)!\sum_{k=0}^{n-1}\frac{f(w^k)w^k}{(w^k-z)^{p-l+1}},$$

where $(n)_l = n(n-1)\ldots(n-l+1)$. (2.3) follows from this when we notice that (cf. [4])

$$\lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n-1}\frac{f(w^k)w^k}{(w^k-z)^{p-l+1}}\right] = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)}{(t-z)^{p-l+1}}\,dt$$

and that for any $\tau>0$, $|z|^n n^\tau\to 0$ as $n\to\infty$ for $|z|<1$.

PROOF OF THEOREM 1. We first observe that the special case $r=1$ of our Theorem 1 is the same as (1.1) and therefore it is enough to consider the case when $r>1$. Let us write

$$h_{rn-1}(z,f) = L_{n-1}(z,f) + \sum_{j=1}^{r-1}(1-z^n)^j P_{n,j}(z,f) \tag{2.4}$$

where each $P_{n,j}(z,f)$ is a polynomial of degree $n-1$. Thus for the proof of Theorem 1 it is enough to show that $\lim_{n\to\infty} P_{n,j}(z,f)=0$, $j=1,2,\ldots,r-1$, for then we have

$$\lim_{n\to\infty}\{h_{rn-1}(z,f)-L_{n-1}(z,f)\} = 0, \quad |z|<1. \tag{2.5}$$

and (2.2) follows from (1.1). However, we shall prove the stronger result that

$$\lim_{n\to\infty} P_{n,j}^{(l)}(z,f) = 0, \quad j=1,2,\ldots,r-1, \quad l=0,1,\ldots,|z|<1. \tag{2.6}$$

On taking $r=2$ in (2.4) and differentiating for $z=w^k$, in view of (2.1) we obtain

$$wh'_{2n-1}(w^k,f) = wf'(w^k) = wL'_{n-1}(w^k,f) - nP_{n,1}(w^k,f), \quad k=0,1,\ldots,n-1,$$

which yields

$$P_{n,1}(z,f) = \frac{1}{n}[zL'_{n-1}(z,f) - L_{n-1}(z,zf')]. \tag{2.7}$$

Differentiating (2.7) l times, we obtain

$$P_{n,1}^{(l)}(z,f)=\frac{1}{n}[zL_{n-1}^{(l+1)}(z,f)+lL_{n-1}^{(l)}(z,f)-L_{n-1}^{(l)}(z,zf')]. \tag{2.8}$$

Using Lemma 1 in (2.8) we see that (2.6) is true for $j=1$. We can now complete the proof of (2.6) by induction. Suppose (2.6) is valid for $j=1, 2, \ldots, r-2$. Then from (2.4) we have

$$w^{(r-1)k}h_{rn-1}^{(r-1)}(w^k,f)=w^{(r-1)k}L_{n-1}^{(r-1)}(w^k,f)+(-1)^{r-1}(r-1)!n^{r-1}P_{n,r-1}(w^k,f)+$$

$$+w^{(r-1)k}\sum_{j=1}^{r-2}\sum_{s=j}^{r-1}\binom{r-1}{s}\left(\frac{d^s}{dz^s}(1-z^n)^j\right)_{z=w^k}P_{n,j}^{(r-1-s)}(w^k,f).$$

Then (2.1) yields

$$(-1)^{r-1}(r-1)!P_{n,r-1}(z,f)=\frac{1}{n^{r-1}}L_{n-1}(z,z^{r-1}f^{(r-1)})-\left(\frac{z}{n}\right)^{r-1}L_{n-1}^{(r-1)}(z,f)-$$

$$-\sum_{j=1}^{r-2}\sum_{s=j}^{r-1}\binom{r-1}{s}\frac{z^{r-1-s}P_{n,j}^{(r-1-s)}(z,f)}{n^{r-1}}\sum_{t=1}^{j}\binom{j}{t}(-1)^t(nt)_s.$$

Differentiating l times, and using the induction hypothesis and Lemma 1 we see that (2.6) is also true for $j=r-1$ and the proof is complete. □

3. Least square minimization. Suppose $m\geqq n$. We shall prove

THEOREM 2. *Let $f(z)$ be Riemann integrable on Γ and let $Q_{n-1}(z,f)$ be the unique polynomial which minimizes* (1.2). *Then*

$$\lim_{n\to\infty}Q_{n-1}(z,f)=\frac{1}{2\pi i}\int_\Gamma\frac{f(t)}{t-z}\,dt \tag{3.1}$$

for $|z|<1$ and uniformly for $|z|\leqq\delta<1$.

It may be observed that for the special case $m=n$, $Q_{n-1}(z,f)$ is the Lagrange interpolant to f at the n^{th} roots of unity and (3.1) reduces to (1.1).

We need the following result for the proof of Theorem 2.

LEMMA 2. *The polynomial $Q_{n-1}(z,f)$ which minimizes* (1.2) *is given by* $\sum_{j=1}^{n-1}c_jz^j$ *where*

$$c_j=\frac{1}{m}\sum_{k=0}^{m-1}f(w^k)w^{-kj},\quad j=0,1,\ldots,n-1, \tag{3.2}$$

and $w^m=1$.

Rivlin [3] has shown (as is also easy to see) that the polynomial $Q_{n-1}(z,f)$ is the Taylor section of degree $n-1$ of the polynomial $L_{m-1}(z,f)$. Lemma 2 follows immediately from this.

PROOF OF THEOREM 2. In view of Lemma 2, we see that

$$Q_{n-1}(z,f)=\frac{1}{m(w-1)}\sum_{k=0}^{m-1}\frac{f(w^k)(w^{k+1}-w^k)}{w^k-z}-\frac{z^n}{m}\sum_{k=0}^{m-1}\frac{f(w^k)w^{-(n-1)k}}{w^k-z}. \tag{3.3}$$

We also notice that as $n \to \infty$,

$$\left| \frac{z^n}{m} \sum_{k=0}^{m-1} \frac{f(w^k) w^{-(n-1)k}}{w^k - z} \right| = O(|z|^n) = o(1) \tag{3.4}$$

for $|z|<1$ and

$$\lim_{n \to \infty} m(w-1) = 2\pi i. \tag{3.5}$$

Thus, combining (3.3)—(3.5) we prove (3.1) which completes the proof of Theorem 2. □

Let $\mathscr{P}(f)$ denote the class of all polynomials $P(z) \in \pi_{m+n-1}$ which have the representation

$$P(z) = L_{m-1}(z, f) - (z^m - 1) \sum_{j=0}^{m-1} d_j z^j.$$

Let f be differentiable along Γ and let $Q_{m+n-1}(z, f)$ be the unique polynomial in $\mathscr{P}(f)$ which minimizes

$$\sum_{k=0}^{m-1} |f'(w^k) - P'(w^k)|^2, \quad w^m = 1$$

over all $P(z) \in \mathscr{P}(f)$. Following [1], we can see that

$$Q_{m+n-1}(z, f) = L_{m-1}(z, f) - (z^m - 1) \sum_{j=0}^{n-1} c_j z^j \tag{3.6}$$

where

$$c_j = \frac{1}{m^2} \sum_{k=0}^{m-1} [jf(w^k) - w^k f'(w^k)] w^{-kj}, \quad j = 0, 1, \ldots, n-1.$$

An easy calculation shows that

$$\sum_{j=0}^{n-1} c_j z^j = \frac{z}{m^2} \sum_{k=0}^{m-1} \frac{f(w^k) w^k}{(z - w^k)^2} [1 - w^{-kn} z^n) +$$

$$+ \frac{n}{m^2} z^n \sum_{k=0}^{m-1} \frac{f(w^k)}{z - w^k} w^{-k(n-1)} + \frac{1}{m^2} \sum_{k=0}^{m-1} \frac{f'(w^k) w^{2k}}{z - w^k} [1 - w^{-kn} z^n].$$

It follows that $\sum_{j=0}^{n-1} c_j z^j \to 0$ as $n \to \infty$. This, combined with (1.1) and (3.6) shows that

$$\lim_{n \to \infty} Q_{m+n-1}(z, f) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t - z} dt$$

for $|z|<1$ and uniformly for $|z| \leqq \delta < 1$.

It may be observed that the analogue of Theorem 2 with $Q_{n-1}(z)$ replaced by a certain average of Lagrange interpolants in (3.1) is also valid (cf. [2]).

4. Partial sums of interpolants. Following Zygmund ([5], p. 20) we shall study in this section the convergence of the average of the partial sums of the Hermite interpolant $h_{rn-1}(z, f)$. For this purpose let us write

$$h_{rn-1}(z, f) = \sum_{k=0}^{nr-1} \gamma_k z^k.$$

The average of the partial sums $\sum_{k=0}^{j} \gamma_k z^k$, $j=0, 1, \ldots, rn-1$, is given by

$$A_{rn-1}(z,f) = \frac{1}{rn} \sum_{j=0}^{rn-1} \sum_{k=0}^{j} \gamma_k z^k.$$

We shall now prove

THEOREM 3. *Let $f^{(r-1)}$ exist along Γ and be Riemann integrable on Γ. Let $A_{rn-1}(z,f)$ be the average of partial sums of the Hermite interpolant of $f(z)$ satisfying* (2.1). *Then*

$$\lim_{n\to\infty} A_{rn-1}(z,f) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t-z}\, dt$$

for $|z|<1$ and uniformly for $|z|\leqq\delta<1$.

PROOF OF THEOREM 3. By a change in the order of summation we see that

$$A_{rn-1}(z,f) = \frac{1}{rn} \sum_{k=0}^{rn-1} (rn-k)\gamma_k z^k = h_{rn-1}(z,f) - \frac{1}{nr}\, z h'_{rn-1}(z,f).$$

Now, we see from (2.4) that

$$z h'_{rn-1}(z,f) = z L'_{n-1}(z,f) + z \sum_{k=0}^{r-1} (1-z^n)^j P'_{n,j}(z,f) - nz^n \sum_{j=1}^{r-1} j(1-z^n)^{j-1} P_{n,j}(z,f).$$

Thus, using Lemma 1 and (2.5) we see that

$$\lim_{n\to\infty} \frac{z}{n}\, h'_{rn-1}(z,f) = 0.$$

This, combined with Theorem 1 proves Theorem 3. □

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SOME REMARKS ON A PROBLEM OF ERDŐS AND GRAHAM

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To Professor P. Erdős on his 75th birthday

1. Introduction

For a sequence $A=\{a_0<a_1<\ldots\}$ of natural numbers let $P(A)$ be defined by

$$P(A)=\Big\{\sum_{k=0}^{\infty}\varepsilon_k a_k:\ \varepsilon_k\in\{0,1\},\ \sum_{k=0}^{\infty}\varepsilon_k<\infty\Big\}.$$

Let us call an integer m representable if $m\in P(A)$. We shall say that a sequence $A=\{a_0<a_1\ldots\}$ of integers is complete if $P(A)$ contains all sufficiently large integers.

Erdős and Graham asked in [1]: Suppose α and β are positive reals with α/β irrational and let S denote the sequence

$$S=\{[\alpha],[\beta],\ldots,[2^n\alpha],[2^n\beta],\ldots\}.$$

Is S complete?

See also [2] for related problems.

Let α be a positive real number and write α in base two i.e. let

$$\alpha=\varepsilon_{-k}\varepsilon_{-k+1}\ldots\varepsilon_0\cdot\varepsilon_1\ldots\varepsilon_n\ldots$$

or

$$\alpha=\sum_{i=-k}^{\infty}\varepsilon_i 2^{-i},\quad \varepsilon_i\in\{0,1\}. \tag{1.1}$$

It is well-known that α is uniquely representable in the form $\sum_{i=-k}^{\infty}\varepsilon_i 2^{-i}$ if we also assume that $\varepsilon_i=0$ infinitely many times. Let us call a real number α an infinite diadical fraction (briefly i.d.f.) if $\varepsilon_i=1$ infinitely many times, otherwise let us call α a finite diadical fraction (briefly f.d.f.).

Let A_α (resp. A_β, A_γ, ...) be defined as

$$A_\alpha=\{[\alpha],\ldots,[2^n\alpha],\ldots\}=\{a_0<a_1<\ldots\}. \tag{1.2}$$

Obviously we can assume without loss of generality that $a_0\geqq 1$ (resp. $b_0\geqq 1$, $c_0\geqq 1$, ...).

Finally denote

$$A_{\alpha_1\alpha_2\ldots\alpha_s}:=\bigcup_{i=1}^{s}A_{\alpha_i}.$$

Then the Erdős—Graham conjecture can be formulated as follows: If α/β is irrational, then $A_{\alpha\beta}$ is complete.

2. Some preliminary remarks

REMARK 1.

$$a_{n+1} = 2a_n + \varepsilon_{n+1}. \tag{1.3}$$

Indeed

$$2^n\alpha = {}_2\underset{\sim}{\varepsilon}_{-k}\varepsilon_{-k+1}\dots\varepsilon_0\varepsilon_1\dots\varepsilon_n\cdot\varepsilon_{n+1}\dots$$

and

$$2^{n+1}\alpha = {}_2\underset{\sim}{\varepsilon}_{-k}\dots\varepsilon_0\varepsilon_1\dots\varepsilon_{n+1}\cdot\varepsilon_{n+2}\dots$$

and so

$$a_{n+1} = [2^{n+1}\alpha] = {}_2\underset{\sim}{\varepsilon}_{-k}\dots\varepsilon_{n+1} = 2\cdot[2^n\alpha]+\varepsilon_{n+1} = 2\cdot a_n+\varepsilon_{n+1}.$$

REMARK 2.

$$a_j > a_{j-1}+\dots+a_0 \quad (j>0). \tag{1.4}$$

In fact $a_n \geqq 2a_{n-1}$ follows from (1.3) and since $a_1 > a_0 \geqq 1$ we have (1.4)

REMARK 3. It is clear that

$$P(A_\alpha) = \bigcup_{i=0}^{\infty} P_i$$

where

$$P_0 := \{a_0\}\cup\{0\}, \quad P_{n+1} := P_n\cup\{P_n+a_{n+1}\} \tag{1.5}$$

and

$$P(A_\alpha)\cap[1, a_{k+1}) = P_k.$$

REMARK 4. We obtain from (1.3), (1.4) and (1.5) that if $m\in P(A_\alpha)$ then m is uniquely representable in the form $m=\sum_i \delta_i a_i$, $\delta_i\in\{0, 1\}$.

3. On the sequence $A_{\alpha\beta}$

We shall distinguish three cases:

1. α, β are f.d.f.
2. α is f.d.f., β is i.d.f.
3. α, β are i.d.f.

As for Case 1, we merely mention the following example which shows that $A_{\alpha\beta}$ may be incomplete.

EXAMPLE. Let $\alpha = {}_2\underset{\sim}{1}11.1$ and $\beta = {}_2\underset{\sim}{1}0001.1$. Then we have

$$P(A_{\alpha\beta}) = \{x\colon x \ 0,2 \text{ or } 4 \pmod 5\}$$

apart from finitely many terms.

As for Case 2, we shall prove that $A_{\alpha\beta}$ is always complete, which is a partial result of the Erdős—Graham conjecture.

THEOREM 1. *If α is i.d.f. and β is f.d.f. then $A_{\alpha\beta}$ is complete.*

I think that the Erdős—Graham conjecture is also true in the following stronger form:

CONJECTURE. If α or β is i.d.f. and $\alpha/\beta \neq 2^n$ ($n \in \mathbf{Z}$) then $A_{\alpha\beta}$ is complete.

If this conjecture is true this is best possible by

THEOREM 2. *Let $\alpha \geqq 2$ and $\beta = 2^n\alpha$ ($n \in \mathbf{N}$). Then $A_{\alpha\beta}$ is not complete.*

On the other hand we have the following result:

THEOREM 3. *If α or β is i.d.f. then $P(A_{\alpha\beta})$ contains arbitrarily large intervals.*

In fact we shall prove the following stronger

THEOREM 3′. *Let α be an i.d.f. and $B = \{b_1 < b_2 < \ldots\}$ an arbitrary increasing sequence. Then $P(A_\alpha) + P(B)$ contains arbitrarily large intervals.*

Since $P(A_{\alpha\beta}) = P(A_\alpha) + P(A_\beta)$ we only have to prove Theorem 3′.

4. On the sequence $A_{\alpha\beta\gamma}$

It is easy to construct an example of an incomplete $A_{\alpha\beta\gamma}$ with α, β and γ finite diadic fractions.

But these are the only exceptions:

THEOREM 4. *If α is i.d.f. then $A_{\alpha\beta\gamma}$ is complete.*

Finally we state two more results which can be proved by similar methods.

1. A_α is complete iff $\alpha = 1/2^n$ for some $n \in \mathbf{N}$.

2. If α is an i.d.f. then arbitrarily large gaps occur in A_α. Moreover, if $f(x)$ denotes the largest gap till x, then

$$\limsup f(x)/\log_2 x \leqq 1 \quad \text{and} \quad \lim f(x)/\log_2 x = 1$$

can be attained for a set of continuous of values α.

5. Proofs

5.1. PROOF OF THEOREM 2. It is easy to see by induction that for every $n \geqq 1$

$$a_n > a_{n-1} + a_{n-2} + \ldots + a_0 + 1 \tag{5.1.1}$$

if $\alpha \geqq 2$. Next we will prove that for every $m \geqq 1$

$$x_m = a_m + a_{m-1} + \ldots + a_0 + 1 \tag{5.1.2}$$

is not representable. Because of $\beta = 2^n\alpha$ (for some n) there exists a k such that $b_s = a_{s+k}$. In other words, a_r occurs in $A_{\alpha\beta}$ exactly twice if $r \geqq k$. It is clear that x_0 is not representable. Now indirectly let us suppose that there exists an m such that if $n < m$ then $x_n \notin P(A_{\alpha\beta})$ but $x_m \in P(A_{\alpha\beta})$. As observed before, in the representation of x_m and a_i occurs twice at most.

We shall treat three cases, according to the multiplicity of a_m in the representation of x_m.

First case: The multiplicity of a_m is two. Then $x_m \geqq 2a_m$ contradicting (5.11).

Second case: The multiplicity of a_m is zero (i.e. a_m does not occur in the representation of x_m). Then again by (5.1.1) $x_m \leqq 2(a_0+\ldots+a_{m-1}) < a_0+\ldots+a_m+1=x_m$; another contradiction.

Third case: The multiplicity of a_m is one. Then (1.5) implies $x_{m-1}= =x_m-a_m \in P(A_{\alpha\beta})$ contradicting the choice of m. With this the theorem is proved.

5.2. PROOF OF THEOREM 1. We shall need a lemma.

LEMMA 1. *If α is i.d.f. then to every s and m there is an $x_{m,s} \in P(A_\alpha)$ such that*

$$x_{m,s} \equiv s \pmod m.$$

PROOF OF LEMMA 1. Let m be a natural number. Select those indices $k=k_1, k_2, \ldots$ for which $\varepsilon_{k+1}=1$. By the pigeon-hole principle there exists a subsequence $\{k_{i_j}\}_{j=1}^{\infty}$ of $\{k_i\}_{i=1}^{\infty}$ such that

$$a_{k_{i_j}} \equiv c \pmod m \quad 0 \leqq c \leqq m-1$$

for some c and so

$$a_{k_{i_j}+1} = 2a_{k_{i_j}}+1 \equiv 2c+1 \equiv u \pmod m, \quad 0 \leqq u \leqq m-1. \tag{5.2.1}$$

Let $(c, u)=v$. Since $(a_{k_{i_j}}, a_{k_{i_j}+1})=(a_{k_{i_j}}, 2a_{k_{i_j}}+1)=1$. therefore $(m, v)=1$. Let $c=c'v$, $u=u'v$. Thus $(c', u')=1$. It is well-known that if $N>(c'-1)(u'-1)$ then there are nonnegative integers x, y such that

$$N = c'x+u'y. \tag{5.2.2}$$

Thus, for every r, $0 \leqq r \leqq m-1$ there exist $x, y \geqq 0$ such that

$$r \equiv v(c'x+u'y) = cx+uy \pmod m.$$

In conclusion: for every m and r $0 \leqq r \leqq m-1$ there exist $x, y \geqq 0$ such that

$$\sum_{j=1}^{x} a_{k_{i_j}} + \sum_{j=1}^{y} a_{k_{i_j}+1} \equiv cx+uy \equiv r \pmod m$$

which proves Lemma 1.

Since β is f.d.f., there exists an m_0 such that for $m \geqq m_0$ $\varepsilon_m=0$, and so we have from (1.3)

$$b_{k+m_0} = 2^k b_{m_0}. \tag{5.2.4}$$

Next we claim that for every $t \in \mathbf{N}$, $t \cdot b_{m_0} \in P(A_\beta)$. Indeed if $t=\sum_{i=0}^{r} \delta_i 2^i$, $\delta_i \in \{0, 1\}$ then

$$t \cdot b_{m_0} = b_{m_0} \sum_{i=0}^{r} \delta_i 2^i = \sum_{i=0}^{r} \delta_i b_{i+m_0} \in P(A_\beta).$$

Denote b_{m_0} by M. In view of Lemma 1 we can select elements $A_1, A_2, \ldots, A_M$ of $P(A_\alpha)$ which form a complete residue system mod M. We assert that if $n \geqq \max_j \{A_j\}$ then n is representable. Indeed if $n=M \cdot j+s$, $0 \leqq s \leqq M-1$ and $A_s \equiv s \pmod M$ then also $n=t \cdot M+A_s$, where $t \cdot M \in P(A_\beta)$, $A_s \in P(A_\alpha)$ hence $n \in P(A_{\alpha\beta})$. Thus $A_{\alpha\beta}$ is complete.

5.3. PROOF OF THEOREM 3'. We assume that for some x and t the interval $[x, x+t]$ lies in $P(A_\alpha)+P(B)$. We shall prove the existence of an x' such that $[x', x'+t+1] \subset P(A_\alpha)+P(B)$ thus establishing our theorem.

Note that if $[x, x+t] \subset P(A_\alpha)+P(B)$ and $x+t \leqq a_k$ then also $[x+a_m, x+a_m+t]$ lies in $P(A_\alpha)+P(B)$ for every $m>k$.

Let now b_r be an element of B which we have not yet used and for which

$$b_r \geqq a_{k+1} - \sum_{i=0}^{k} a_i + x + t + 1. \tag{5.3.1}$$

If in (5.3.1) equality holds then the theorem follows because $b_r + \sum_{i=0}^{k} a_i = a_{k+1} + x + t + 1$ and so $[x+a_{k+1}, x+a_{k+1}+t+1]$ lies in $P(A_\alpha)+P(B)$. If in (5.3.1) the inequality is strict, then let $k_1 > k+1$ be the first index for which $\varepsilon_{k_1} = 1$. But in this case

$$\Delta_{k+1} := a_{k+1} - \sum_{i=0}^{k} a_i = 2a_{k+1} - \sum_{i=0}^{k+1} a_i = a_{k+2} - \sum_{i=0}^{k+1} a_i = \Delta_{k+2} = \ldots = \Delta_{k_1-1}$$

and

$$\Delta_{k_1} = 2a_{k_1-1} + 1 - \sum_{i=0}^{k_1-1} a_i = a_{k_1} - \sum_{i=0}^{k_1-1} a_i = \Delta_{k+1} + 1$$

and therefore

$$b_r \geqq a_{k_1} - \sum_{i=0}^{k_1-1} a_i + x + t + 1 \tag{5.3.2}$$

also holds.

If in (5.3.2) the inequality is strict then repeat the previous procedure with k_1 substituted for $k+1$, thus we obtain indices k_2, k_3 etc. This is possible because α is i.d.f.

But the numbers $a_{k_j} - \sum_{i=0}^{k_j-1} a_i + x + t + 1$ increase by one as j increases by one.

Therefore there is a j such that $b_r = a_{k_j} - \sum_{i=0}^{k_j-1} a_i + x + t + 1$, and we are done.

5.4. PROOF OF THEOREM 4. If β or γ is f.d.f. then we can use Theorem 1, i.e. $A_{\alpha\beta}$, hence a fortiori $A_{\alpha\beta\gamma}$, is complete. Therefore we only have to investigate the case when α, β, γ are i.d.f. We shall need two lemmas.

LEMMA 2. *Suppose X, Y are sequences of integers such that X contains arbitrarily large intervals, and the differences of the adjacent elements in Y are bounded. Then $X+Y$ contains all sufficiently large integers.*

We omit the easy proof.

Let $A_\beta = \{b_0 < b_1 < \ldots\}$ and $A_\gamma = \{c_0 < c_1 < \ldots\}$. It is clear that if $b_0 \leqq c_{i_0}$ then for every $i \geqq i_0$ there exists a t such that $b_t \in [c_i, c_{i+1})$ (this follows from (1.3))

LEMMA 3. *Let $A_\beta = \{b_0 < b_1 < \ldots\}$, $A_\gamma = \{c_0 < c_1 < \ldots\}$ and let*

$$A_{\beta 2} = \{b \in A_\beta : b \in [c_m, c_{m+1}) \, m \geqq i_0 \text{ iff } \varepsilon^\gamma_{m+1} = 1\}$$

where $\varepsilon_{m+1}^{\gamma}$ denotes the $m+1$-st diadic digit of γ. Then the set $P(A_{\beta 2})+P(A_{\gamma})$ has bounded gaps.

Now we deduce the theorem from Lemmas 2 and 3.
It is clear that

$$P(A_{\alpha\beta\gamma}) = P(A_{\alpha})+P(A_{\beta})+P(A_{\gamma}) = P(A_{\alpha})+P(A_{\beta 1})+P(A_{\beta 2})+P(A_{\gamma})$$

where $A_{\beta 1}=A_{\beta}\setminus A_{\beta 2}$ is an infinite increasing sequence (indeed $\varepsilon_s^{\beta}=0$ holds infinitely many times).

Now Theorem 3′ implies that $P(A_{\alpha})+P(A_{\beta 1})$ contains arbitrarily large intervals. Furthermore Lemma 3 implies that the set $P(A_{\beta 2})+P(A_{\gamma})$ has bounded gaps. Finally Lemma 2 applied with $X:=P(A_{\alpha})+P(A_{\beta 1})$ and $Y:=P(A_{\beta 2})+P(A_{\gamma})$ yields that $X+Y=P(A_{\alpha})+P(A_{\beta 1})+P(A_{\beta 2})+P(A_{\gamma})$, and hence a fortiori $P(A_{\alpha\beta\gamma})$, contains all sufficiently large integers which proves Theorem 4.

PROOF OF LEMMA 3. Let $U=P(A_{\beta 2})+P(A_{\gamma})=\{u_1<u_2<\ldots\}$. We claim that for every $j>0$, $u_{j+1}-u_j\leqq c_{i_0}$. Assume that this is not true. Thus there exists a p such that $u_{p+1}-u_p>c_{i_0}$ and for every $0<j\leqq p-1$, $u_{j+1}-u_j\leqq c_{i_0}$.

Let t be defined by

$$c_t \leqq u_p < u_{p+1} \leqq c_{t+1}.$$

Here $\varepsilon_{t+1}^{\gamma}=1$. Indeed let $[1, c_t]\cap U=\{u_1<u_2<\ldots<u_k\}$ $(u_k=c_t)$. But

$$\{c_t+\{u_j\}_{j=1}^{k-1}\}\subset L := U\cap[c_t, c_{t+1}]$$

and if $\varepsilon_{t+1}^{\gamma}=0$ then $c_{t+1}=2c_t$ thus $c_t+u_k\in L$ is also true. This means that the biggest gap in $[c_t, c_{t+1}]$ is not bigger than c_{i_0}.

Thus $\varepsilon_{t+1}^{\gamma}=1$ and so there exists a $b\in A_{\beta 2}$ for which $b\in[c_t, c_{t+1})$. Now $\{b+\{u_j\}_{j=1}^{k}\}\subset U$ and $b+c_t\geqq 2c_t=c_{t+1}-1$ i.e. the biggest gap in $[b, c_{t+1}]\cap U$ is $\leqq c_{i_0}$, so we have $u_p, u_{p+1}\in[c_t, b]$. If $c_t+u_{k-1}>b$ then (because of $\{c_t+\{u_j\}_{j=1}^{k-1}\}\subset U$) the biggest gap in $\{c_t+\{u_j\}_{j=1}^{k-1}\}\cap[c_t, b]$ is not bigger than c_{i_0} which is impossible.

Thus $c_t+u_{k-1}\leqq b$ and so $u_p, u_{p+1}\in[c_t+u_{k-1}, b]$. Hence

$$b-(c_t+u_{k-1}) \leqq c_{t+1}-1-(c_t+u_{k-1}) = 2c_t-c_t-u_{k-1} =$$

$$= c_t-u_{k-1} = u_k-u_{k-1} \leqq c_{i_0},$$

a contradiction, which proves the Lemma.

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ON THE LOWER ESTIMATION OF NON-AVERAGING SETS

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A set S of positive integers is called non-averaging if the arithmetic mean of two or more members of S never belongs to S. Denote by $f(n)$ the cardinality of a largest non-averaging subset of $\{1, 2, ..., n\}$.

Straus [4], raising the problem of estimation of $f(n)$, proved that

$$f(n) > \exp(c_1\sqrt{\log n}).$$

(Here and in the sequel, $c_1, c_2, \ldots$ are positive absolute constants.)

Erdős and Straus [3] have the result

$$f(n) < c_2 n^{2/3}.$$

Abbott [1] proved that

$$f(n) > c_3 n^{1/10};$$

later, in [2], be obtained

$$f(n) > c_5 n^{1/5}$$

for all n and

$$f(n) > c_5 n^{1/5} (\log\log n)^{2/5}$$

for infinitely many n.

In this paper we show that

$$f(n) > c_6 n^{1/4}$$

improving a method of Abbott.

THEOREM. *For some* $c_6>0$ *and all sufficiently large* n *we have*

$$f(n) > c_6 n^{1/4}. \tag{1}$$

PROOF. Let $n=2q^4$, q an integer. Without loss of generality, it is enough to show (1) only for such n's.

First, we introduce the following points in the plane:

$$(x_i, y_i) = \left(iq, \frac{i(i+1)}{2}\right) \quad (i = 1, 2, ..., q-1).$$

We remark here that $x_i<q^2$, $y_i<q^2$. Clearly, $x_1<x_2<\ldots<x_{q-1}$, and the points (x_i, y_i) lie on a convex curve (a parabole), thus they are non-averaging in a stronger

sense, namely for any $\lambda_1, \ldots, \lambda_k > 0$, $i_1, i_2, \ldots, i_k$ and j

$$\frac{\sum_{l=1}^{k} \lambda_l (x_{i_l}, y_{i_l})}{\sum_{l=1}^{k} \lambda_l} \neq (x_j, y_j), \tag{2}$$

assuming that there are different numbers among $i_1, \ldots, i_k$.

Let us introduce the numbers

$$n_i = x_i q^2 + y_i \quad (i = 1, \ldots, q-1). \tag{3}$$

Clearly, these numbers are different integers, and

$$n_i < q^4 + q^2 \leqq 2q^4 = n.$$

We prove that the set $\{n_1, n_2, \ldots, n_{q-1}\}$ is non-averaging.

Let us proceed indirectly. This yields that

$$n_j = \frac{n_{i_1} + \ldots + n_{i_k}}{k}$$

for some j, and $i_1, \ldots, i_k$ are different. Hence

$$n_j = \frac{n_{i_1} + \ldots + n_{i_k} + n_{i_{k+1}} + \ldots + n_{i_q}}{q},$$

where we choose $i_{k+1} = i_{k+2} = \ldots = i_q = j$.

Using (3),

$$x_j q^2 + y_j = \frac{x_{i_1} + \ldots + x_{i_q}}{q} \cdot q^2 + \frac{y_{i_1} + \ldots + y_{i_q}}{q}. \tag{4}$$

The definition of $x_{i_1}, \ldots, x_{i_q}$ implies that

$$\frac{x_{i_1} + \ldots + x_{i_q}}{q}$$

is an integer. By (4)

$$\frac{y_{i_1} + \ldots + y_{i_q}}{q}$$

is also an integer.

Applying $x_{i_1}, x_{i_q} < q^2$, $y_{i_1}, \ldots, y_{i_q} < q^2$ we have

$$\frac{x_{i_1} + \ldots + x_{i_q}}{q} < q^2, \quad \frac{y_{i_1} + \ldots + y_{i_q}}{q} < q^2.$$

This and (4) imply

$$y_j = \frac{y_{i_1} + \ldots + y_{i_q}}{q}, \quad x_j = \frac{x_{i_1} + \ldots + x_{i_q}}{q}$$

and these equations contradict (2). The theorem is proved.

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ON THE EXTERIOR LINEAR MEASURE

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THEOREM 1. *The exterior linear measure of a connected set is the same as that of its closure.*

Based on a theorem of Besicovitch ([1], Theorem 2, p. 424) M. Laczkovich has proved the theorem for plane sets.

We now prove the theorem for sets lying in an arbitrary metric space. In effect, this Theorem 1 will be considered as a corollary of the following:

THEOREM 2. *If the exterior linear measures of a sequence of connected sets have an upper bound B then the exterior linear measure of the limit inferior of this sequence cannot exceed B.*

Preparing the proof of this second theorem we recall some basic notions. Also, we shall formulate a third theorem, the background of the two others. This third one will be considered as the main theorem of this paper.

1. We first recall the definition of the exterior linear measure introduced by C. Carathéodory [2].

Let R be a metric space with the metric d. For any nonempty subset M of R let $\delta(M)$ be the diameter of M. The case $\delta(M)=\infty$ is not excluded. If $M=\emptyset$ then let $\delta(M)=0$.

Let $\mathbf{R}^+$ be the set of positive reals and let $\mathbf{N}$ be the set of positive integers. We shall adhere to these notations throughout.

A *system* $\mathfrak{M}$ *of subsets of* R consists of a nonempty subset N^* of $\mathbf{N}$ and a function which assigns to each $r\in N^*$ a subset M_r of R, i.e.

$$\mathfrak{M}=(N^*, \{M_r;\ r\in N^*\}).$$

The cardinality of N^* is said to be *the cardinality of the system* $\mathfrak{M}$. The system $\mathfrak{M}$ is said to be *open* if for each $r\in N^*$ M_r is an open set in R.

Now let $\mathfrak{M}=(N^*, \{M_r;\ r\in N^*\})$ be a system of subsets of R and let $\Sigma_{\mathfrak{M}}=\infty$ if for any $r\in N^*$, $\delta(M_r)=\infty$. On the other hand let

$$\Sigma_{\mathfrak{M}}=\sum_{r\in N^*}\delta(M_r)$$

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if $\delta(M_r)<\infty$ for each $r\in N^*$. It may happen that $\Sigma_{\mathfrak{M}}=\infty$ in this latter case too. Generally we have

$$0 \leqq \Sigma_{\mathfrak{M}} \leqq \infty.$$

Let $\varepsilon\in\mathbf{R}^+$ and $X\subset R$. The system $\mathfrak{M}=(N^*, \{M_r; r\in N^*\})$ of subsets of R is said to be an *ε-system* if $\delta(M_r)<\varepsilon$ for each $r\in N^*$ and it is said to be an *ε^+-system* if $\delta(M_r)\leqq\varepsilon$ for each $r\in N^*$. $\mathfrak{M}$ is called an *ε covering* (*ε^+ covering* respectively) of X if it is an ε-system (ε^+-system respectively) and $X\subset\bigcup\{M_r; r\in N^*\}$. The set of all ε coverings of X is denoted by $U(X, \varepsilon.)$ Now let

$$L_\varepsilon(X) = \infty \quad \text{if} \quad U(X,\varepsilon)=\emptyset$$

and

$$L_\varepsilon(X) = \inf\{\Sigma_{\mathfrak{M}};\ \mathfrak{M}\in U(X,\varepsilon)\} \quad \text{if} \quad U(X,\varepsilon)\neq\emptyset.$$

It may happen that $L_\varepsilon(X)=\infty$ in this latter case, too.

If ε decreases, $L_\varepsilon(X)$ cannot decrease, consequently

$$\lim_{\varepsilon\to 0} L_\varepsilon(X) = L^*(X)$$

always exists.

We call this $L^*(X)$ the *exterior linear measure* of the set X.

Observe that for each $\varepsilon\in\mathbf{R}^+$ we have

$$L_\varepsilon(X) \leqq L^*(X). \tag{1}$$

2. Now we formulate the main theorem of this paper.

THEOREM 3. *Let Q be a connected subset of R such that $L^*(Q)<\infty$. Let $\eta, \zeta\in\mathbf{R}^+$ and let C be an integer higher than $\dfrac{4L^*(Q)}{\eta}$. Then there is an η^+ covering $\mathfrak{M}$ of Q of cardinality $\leqq C$ such that*

$$\Sigma_{\mathfrak{M}} < L^*(Q)+\zeta.$$

3. First we formulate and prove a Lemma.

LEMMA. *Let P be a nonempty subset of R with $L^*(P)<\infty$. Let $\varepsilon, \xi\in\mathbf{R}^+$. Then there is an open ε^+ covering $\mathfrak{M}'=(N'^*, \{M'_k; k\in N'^*\})$ of P satisfying the following conditions.*

(2) *For all $k\in N'^*$ we have $M'_k\cap P\neq\emptyset$.*

(3) $\Sigma_{\mathfrak{M}'}\leqq L^*(P)+\xi$.

(4) *$k, k'\in N'^*$, $k\neq k'$, $\delta(M'_k)\leqq\dfrac{\varepsilon}{2}$ and $\delta(M'_{k'})\leqq\dfrac{\varepsilon}{2}$ imply $M'_k\cap M'_{k'}=\emptyset$.*

PROOF. Let $\mathfrak{M}=(N^*, \{M_r; r\in N^*\})$ be an ε covering of P such that

$$\Sigma_{\mathfrak{M}} \leqq L^*(P)+\xi. \tag{5}$$

Since by (1) $L\varepsilon(P)\leqq L^*(P)$, such a covering $\mathfrak{M}$ exists. Without loss of generality we may suppose that all the M_r-s are open subsets of R and for $r\in N^*$ we have

$$M_r\cap P\neq\emptyset \tag{6}$$

and of course

$$\delta(M_r) < \varepsilon. \tag{7}$$

We now define the subsets $N_0, N_1, \ldots, N_k, \ldots$ of N^* in a recursive way as follows.

Let $N_0=\emptyset$. Suppose that $k\geqq 1$ and that $N_0, \ldots, N_{k-1}$ have already been selected. If

$$N^k = N^* \setminus \bigcup_{i=0}^{k-1} N_i = \emptyset \tag{8}$$

then let $N_k=\emptyset$.

Suppose that $N^k\neq\emptyset$ and let x_k be the least member of N^k. Let $\mathfrak{F}^k$ be the family of all subsets S of N^k satisfying the conditions:

$$x_k\in S, \tag{9}$$

$$\delta\Big(\bigcup_{r\in S} M_r\Big) \leqq \varepsilon, \tag{10}$$

$$\delta\Big(\bigcup_{r\in S} M_r\Big) \leqq \sum_{r\in S} \delta(M_r). \tag{11}$$

We then have $\{x_k\}\in\mathfrak{F}^k$ and thus $\mathfrak{F}^k\neq\emptyset$.

Consider now a chain $\mathfrak{F}'$ in $\mathfrak{F}^k$ and let $S'=\cup\mathfrak{F}'$. We show that $S'\in\mathfrak{F}^k$.

In fact, by $\mathfrak{F}'\neq\emptyset$ (9) is fulfilled for S'.

Now let $q_1, q_2\in \bigcup_{r\in S'} M_r$. Let $q_1\in M_{r_1}$ and $q_2\in M_{r_2}$ with $r_1\in S_1$ and $r_2\in S_2$ ($S_1, S_2\in\mathfrak{F}'$). Since $\mathfrak{F}'$ is a chain we can clearly suppose $S_1\subset S_2$ and thus $r_1, r_2\in S_2$. By $S_2\in\mathfrak{F}^k$ we then have

$$d(q_1, q_2) \leqq \delta\Big(\bigcup_{r\in S_2} M_r\Big) \leqq \varepsilon$$

and thus $\delta\big(\bigcup_{r\in S'} M_r\big)\leqq\varepsilon$.

(10) is fulfilled for S' as well.

Finally, to show (11) for S' we argue by contradiction. Suppose that

$$\delta\Big(\bigcup_{r\in S'} M_r\Big) > \sum_{r\in S'} \delta(M_r). \tag{12}$$

Then there are $q_1, q_2\in \bigcup_{r\in S'} M_r$ with

$$d(q_1, q_2) > \sum_{r\in S'} \delta(M_r). \tag{13}$$

Let $q_1\in M_{r_1}$, $q_2\in M_{r_2}$ with $r_1\in S_1$, $r_2\in S_2$ ($S_1, S_2\in\mathfrak{F}'$). We may suppose again $S_1\subset S_2$. Then $r_1, r_2\in S_2$ and since $S_2\subset S'$ (13) implies

$$\delta\Big(\bigcup_{r\in S_2} M_r\Big) \geqq d(q_1, q_2) > \sum_{r\in S'} \delta(M_r) \geqq \sum_{r\in S_2} \delta(M_r).$$

Taking also $S_2\in\mathfrak{F}^k$ into account, this last relation is in contradiction with (11). Hence the assumption (12) was false, (11) is fulfilled for S' indeed. Consequently $S'\in\mathfrak{F}^k$ as required.

Now we can apply the Kuratowski—Zorn Lemma and thus there are maximal subsets S of N^k satisfying (9), (10) and (11). Let N_k be such a maximal subset. We then have

$$N_k \in \mathfrak{F}^k \tag{14}$$

and thus

$$x_k \in N_k. \tag{15}$$

Now we can state that $N_0, N_1, \ldots, N_k, \ldots$ are pairwise disjoint subsets of N^*, i.e.

$$k \neq k' \Rightarrow N_k \cap N_{k'} = \emptyset \quad (k, k' \in \mathbf{N} \cup \{0\}) \tag{16}$$

moreover

$$N_1 \neq \emptyset \tag{17}$$

and taking also (15) into account we have

$$N^* = \bigcup_{k=1}^{\infty} N_k. \tag{18}$$

Now let

$$N'^* = \{k \in \mathbf{N};\ N_k \neq \emptyset\} \tag{19}$$

and for $k \in N'^*$ let

$$M'_k = \bigcup_{r \in N_k} M_r. \tag{20}$$

Then by (17) we have $N'^* \neq \emptyset$ and thus $\mathfrak{M}' = (N'^*, \{M'_k;\ k \in N'^*\})$ is a system of subsets of R. Moreover (20), (19), (14) and (11) imply that for $k \in N'^*$

$$\delta(M'_k) \leqq \sum_{r \in N_k} \delta(M_r). \tag{21}$$

We are going to show that

$$\mathfrak{M}' = (N'^*, \{M'_k;\ k \in N'^*\})$$

satisfies all the requirements of the Lemma.

Since $\mathfrak{M}$ is an open covering of P it follows by (18), (19) and (20) that $\mathfrak{M}'$ is an open covering of P as well. Moreover (20), (19), (14) and (10) show that $\mathfrak{M}'$ is an open ε^+ covering of P. (19), (6) and (20) also imply that $\mathfrak{M}'$ satisfies relation (2). Moreover by (21), (18), (16) and (5) we have

$$\Sigma_{\mathfrak{M}'} = \sum_{k \in N^*} \delta(M'_k) \leqq \sum_{k \in N'^*} \Big(\sum_{r \in N_k} \delta(M_r) \Big) = \sum_{r \in N^*} \delta(M_r) = \Sigma_{\mathfrak{M}} \leqq L^*(P) + \xi$$

and thus $\mathfrak{M}'$ also satisfies relation (3).

To prove the Lemma we only need to show that $\mathfrak{M}'$ satisfies relation (4) too.

We argue by contradiction. Suppose that $k, k' \in N'^*$, $k \neq k'$, $\delta(M'_k) \leqq \frac{\varepsilon}{2}$, $\delta(M'_{k'}) \leqq$

$\leqq \frac{\varepsilon}{2}$ and $M'_k \cap M'_{k'} \neq \emptyset$. Without loss of generality we can suppose $k<k'$. We then clearly have

$$\delta(M'_k \cup M'_{k'}) \leqq \delta(M'_k)+\delta(M'_{k'}) \leqq \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon. \tag{22}$$

Moreover denoting $N_k \cup N_{k'}$ by S we have obviously

$$N_k \neq S = N_k \cup N_{k'} \subset N^k \tag{23}$$

(see also (8)).

$x_k \in S$ (see also (15)),

$$\delta\Big(\bigcup_{r\in S} M_r\Big) = \delta\Big(\Big(\bigcup_{r\in N_k} M_r\Big) \cup \Big(\bigcup_{r\in N_{k'}} M_r\Big)\Big) = \delta(M'_k \cup M'_{k'}) \leqq \varepsilon \tag{24}$$

(see also (20) and (22)) and taking also (24), (22), (21) and (16) into account one has

$$\delta\Big(\bigcup_{r\in S} M_r\Big) = \delta(M'_k \cup M'_{k'}) \leqq \delta(M'_k)+\delta(M'_{k'}) \leqq$$

$$\leqq \sum_{r\in N_k} \delta(M_r) + \sum_{r\in N_{k'}} \delta(M_r) = \sum_{r\in S} \delta(M_r).$$

Hence $S=N_k \cup N_{k'}$ satisfies all the conditions (9), (10) and (11) and thus $S \in \mathfrak{F}^k$. Consequently by (23) N_k fails to be a maximal subset of $\mathfrak{F}^k$ and this is a contradiction. Our assumption was false and thus $\mathfrak{M}'$ satisfies condition (4) as well.

The proof of the Lemma is complete.

4. We are going to prove Theorem 3.

If $Q=\emptyset$ then the theorem holds obviously. Thus we can suppose $Q \neq \emptyset$.

Select $\xi \in \mathbf{R}^+$ such that

$$\xi < \zeta \tag{25}$$

and

$$\frac{4(L^*(Q)+\xi)}{\eta} < C. \tag{26}$$

Let $\mathfrak{M}'=(N'^*, \{M'_k; k\in N'^*\})$ be an open $\left(\frac{\eta}{2}\right)^+$ covering of Q satisfying the conditions:

$$\text{For } k\in N'^* \quad M'_k \cap Q \neq \emptyset, \tag{27}$$

$$\Sigma_{\mathfrak{M}'} \leqq L^*(Q)+\xi, \tag{28}$$

$$k, k' \in N'^*, \quad k \neq k', \quad \delta(M'_k) \leqq \frac{\eta}{4} \quad \text{and} \tag{29}$$

$$\delta(M'_{k'}) \leqq \frac{\eta}{4} \quad \text{imply} \quad M_k \cap M'_{k'} = \emptyset.$$

By the Lemma of Section 3 there is such a covering $\mathfrak{M}$.

Now let N_a be the subset of the $k \in N'^*$-s with

$$\frac{\eta}{4} < \delta(M_k') \leqq \left(\frac{\eta}{2}\right) \tag{30}$$

and let

$$Q_a = \bigcup_{k \in N_a} M_k'.$$

Obviously N_a is a finite subset of N'^* and its cardinality K does not exceed

$$\frac{\Sigma_{\mathfrak{M}'}}{\frac{\eta}{4}} \leqq \frac{4(L^*(Q)+\xi)}{\eta} < C$$

(see also (28) and (26)) and thus

$$K \leqq C. \tag{31}$$

Let N_b be the subset of N'^* consisting of the $k \in N'^*$-s with $\delta(M_k') \leqq \frac{\eta}{4}$ and $M_k' \cap Q_a \neq \emptyset$ and let

$$N_c = N'^* \setminus (N_a \cup N_b).$$

There are two possibilities.

Case 1: $N_a = \emptyset$. We then clearly have $N_b = \emptyset$ and thus

$$N'^* = N_c. \tag{32}$$

In this case let $\mathfrak{M} = \mathfrak{M}'$. By (29), (27) and (32) the members of the covering $\mathfrak{M}$ of Q are pairwise disjoint open sets and each of them meets Q. However Q is a nonempty connected set and thus $N'^* = N_c$ consists of a single element. Consequently the cardinality of $\mathfrak{M}$ is $1 \leqq C$. $\mathfrak{M}$ is an $\frac{\eta^+}{4}$ — and thus an η^+ covering of Q and by (25) and (28) we have

$$\Sigma_{\mathfrak{M}} < L^*(Q) + \zeta$$

as required.

Case 2: $N_a \neq \emptyset$. Then $Q_a \neq \emptyset$.

We first show that $N_c = \emptyset$.

In fact, let $Q_b = \bigcup_{k \in N_b} M_k'$. Then $Q_a \cup Q_b$ is clearly disjoint to all the M_k-s with $k \in N_c$ and thus

$$\{Q_a \cup Q_b\} \cup \{M_k';\ k \in N_c\}$$

is an open covering of the connected set Q and the members of the covering are pairwise disjoint (see also (29)). Moreover for $k \in N_c$ we have $M_k' \cap Q \neq \emptyset$ (see (27)) and for the same reasons $N_a \neq \emptyset$ implies

$$(Q_a \cup Q_b) \cap Q \neq \emptyset.$$

Consequently $N_c = \emptyset$ indeed. Hence

$$N'^* = N_a \cup N_b. \tag{33}$$

Now for each $k' \in N_b$ select an $i(k') \in N_a$ such that

$$M'_k \cap M'_{i(k')} \neq \emptyset \tag{34}$$

and for each $k \in N_a$ let

$$M_k = M'_k \cup \left(\bigcup \{M'_{k'};\ k' \in N_b \text{ and } i(k') = k\} \right).$$

Then

$$\mathfrak{M} = (N_a, \{M_k; k \in N_a\})$$

is an open covering of Q of cardinality $\leqq C$ (see (31)).

$\mathfrak{M}$ is an η^+-system. For if not then there would exist $k \in N_a$ and $p, q \in M_k$ with

$$d(p, q) > \eta. \tag{35}$$

By (30) $p, q \in M'_k$ cannot occur.

Consider the case $p \in M'_k$ and $q \in M'_{k'}$ with $k' \in N_b$ and $i(k') = k$. Let $z \in M'_{k'} \cap \cap M'_{i(k')}$. (34) shows the existence of such a z. However in this case we would have

$$d(p, q) \leqq d(p, z) + d(q, z) \leqq \frac{\eta}{2} + \frac{\eta}{4} < \eta$$

in contradiction with (35).

Finally, if $p \in M'_{k_1}$ and $q \in M'_{k_2}$ with $k_1, k_2 \in N_b$ and $i(k_1) = i(k_2) = k$ then select $z_1 \in M'_k \cap M'_{k_1}$ and $z_2 \in M'_k \cap M'_{k_2}$. Hence

$$d(p, q) \leqq d(p, z_1) + d(z_1, z_2) + d(z_2, q) \leqq \frac{\eta}{4} + \frac{\eta}{2} + \frac{\eta}{4} = \eta$$

in contradiction with (35). Thus (35) cannot occur. $\mathfrak{M}$ is an η^+ covering of Q indeed.

To prove Theorem 3 we only need to show that

$$\Sigma_{\mathfrak{M}} < L^*(Q) + \zeta.$$

First for $k \in N_a$ let

$$\delta(k) = \delta(M'_k) + \sum_{i(k')=k} \delta(M'_{k'}). \tag{36}$$

Then

$$\delta(M_k) \leqq \delta(k). \tag{37}$$

In fact let $p, q \in M_k$. If $p, q \in M'_k$ then

$$d(p, q) \leqq \delta(M'_k) \leqq \delta(k).$$

If $p, q \in M'_{k'}$ for some $k' \in N_b$ with $i(k') = k$ then

$$d(p, q) \leqq \delta(M'_{k'}) < \delta(k).$$

If $p \in M'_k$ and $q \in M'_{k'}$ with $k' \in N_b$ and $i(k') = k$ then select $z \in M'_k \cap M'_{k'}$. We then obtain

$$d(p, q) \leqq d(p, z) + d(z, q) \leqq \delta(M'_k) + \delta(M'_{k'}) \leqq \delta(k).$$

Finally if $p \in M'_{k_1}$ and $q \in M'_{k_2}$ with $k_1 \neq k_2$, $k_1, k_2 \in N_b$ and $i(k_1) = i(k_2) = k$ then

select $z_1 \in M'_k \cap M'_{k_1}$ and $z_2 \in M'_k \cap M'_{k_2}$. Hence we get

$$d(p, q) \leqq d(p, z_1) + d(z_1, z_2) + d(z_2, q) \leqq$$
$$\leqq \delta(M'_{k_1}) + \delta(M'_k) + \delta(M'_{k_2}) \leqq \delta(k).$$

Thus for each $p, q \in M_k$ we have $d(p, q) \leqq \delta(k)$. Consequently relation (37) holds true indeed.

Now (37), (33), (28) and (25) imply

$$\Sigma_{\mathfrak{M}} = \sum_{k \in N_a} \delta(M_k) \leqq \sum_{k \in N_a} \delta(k) = \sum_{k \in N_a} \left(\delta(M'_k) + \sum_{i(k')=k} \delta(M'_{k'})\right) =$$
$$= \sum_{k'' \in N'^*} \delta(M'_{k''}) = \Sigma_{\mathfrak{M}'} \leqq L^*(Q) + \xi < L^*(Q) + \zeta$$

as required.

The proof of Theorem 3 is complete.

5. For $q \in R$ and $\eta \in \mathbf{R}^+$ let $S(q, \eta)$ be the open η-ball with center q. On the other hand, for $X \subset R$ let $S(X, \eta)$ be the η-neighbourhood of X, i.e. $S(X, \eta) = \bigcup_{q \in X} S(q, \eta)$.

Let $P_1, \ldots, P_k, \ldots$ be an infinite sequence of subsets of R. The set of all pints $q \in R$ such that each $S(q, \eta)$ intersects all but a finite number of the sets P_k is called the *limit inferior* of the sequence. This limit inferior is a closed subset of R.

Observe that if P is the limit inferior of the sequence $P_1, \ldots, P_k, \ldots$ and $S(q_1, \eta_1), \ldots, S(q_m, \eta_m)$ are open balls in R such that $\{q_1, \ldots, q_m\} \subset P$ then all but a finite number of the sets P_k intersect every ball $S(q_i, \eta_i)$ and thus there exists a $k \in \mathbf{N}$ such that P_k intersects every ball $S(q_i, \eta_i)$.

6. We are going to prove Theorem 2. Let $P_1, \ldots, P_k, \ldots$ be an infinite sequence of connected sets lying in the metric space R. Let B be a nonnegative real number and suppose that for $k = 1, 2, \ldots$

$$L^*(P_k) \leqq B. \tag{38}$$

Let P be the limit inferior of the sequence $P_1, \ldots, P_k, \ldots$.

If $P = \emptyset$ then we have $L^*(P) \leqq B$. Thus we can suppose $P \neq \emptyset$.

We first show that P is totally bounded. We argue by contradiction. Suppose the existence of an $\varepsilon \in \mathbf{R}^+$ and of an infinite sequence $q_1, \ldots, q_m, \ldots$ in P such that $i < j$ implies

$$d(q_i, q_j) \geqq \varepsilon. \tag{39}$$

Let C be an integer larger than $\frac{8B}{\varepsilon}$, i.e.

$$\frac{8B}{\varepsilon} < C \tag{40}$$

and let $n = C + 1$. Consider the $\frac{\varepsilon}{4}$-balls $S\left(q_1, \frac{\varepsilon}{4}\right), \ldots, S\left(q_n, \frac{\varepsilon}{4}\right)$ and select $k \in \mathbf{N}$ such that P_k intersects all these balls. Since P is the limit inferior of the sequence $P_1, \ldots, P_k, \ldots$ it follows the existence of such a k. We then clearly have $P_k \neq \emptyset$.

Let $\mathfrak{M}=(N^*, \{M_r; r\in N^*\})$ be an $\left(\frac{\varepsilon}{2}\right)^+$ covering of P_k of the cardinality $\leqq C$ such that

$$\Sigma_{\mathfrak{M}} \leqq L^*(P_k)+1.$$

(40), (38) and Theorem 3 show the existence of such a covering $\mathfrak{M}$.

However, for $r\in N^*$, $\delta(M_r)\leqq\frac{\varepsilon}{2}$ and (39) show that M_r meets at most one of the balls $S\left(q_i, \frac{\varepsilon}{4}\right)$ $(i=1, \ldots, n=C+1)$ and thus the subset P_k of $\bigcup_{r\in N^*} M_r$ cannot meet each $S\left(q_i, \frac{\varepsilon}{4}\right)$ $(i=1, \ldots, n)$. This is a contradiction.

Hence for each $\varepsilon\in\mathbf{R}^+$ there is a finite covering of P by ε-balls and thus P is totally bounded indeed.

Next observe that for each $\eta\in\mathbf{R}^+$ there is a $k\in\mathbf{N}$ such that P is confined to the η-neighbourhood of P_k.

In fact, since P is totally bounded, P is confined to a finite number $S\left(q_1, \frac{\eta}{2}\right), \ldots, S\left(q_n, \frac{\eta}{2}\right)$ of $\frac{\eta}{2}$ -balls where $q_1, \ldots, q_n\in P$. Select now $k\in\mathbf{N}$ such that P_k meets all these balls. Then P is confined to the η-neighbourhood of P_k.

To prove Theorem 2 we only need to show that for arbitrary $\varepsilon, \xi\in\mathbf{R}^+$, $L_\varepsilon(P)< <B+\xi$.

Now let $\varepsilon, \xi\in\mathbf{R}^+$. Let C be an integer larger than $\frac{8B}{\varepsilon}$ and let $\eta=\min\left(\frac{\varepsilon}{5}, \frac{\xi}{3C}\right)$. Select a member P_k of the sequence $P_1, P_2, \ldots$ such that P is confined to the η-neighbourhood of P_k. Then $P_k\neq\emptyset$.

Let $\mathfrak{M}=(N^*, \{M_r; r\in N^*\})$ be an $\left(\frac{\varepsilon}{2}\right)^+$ covering of P_k of the cardinality $\leqq C$ such that $\Sigma_{\mathfrak{M}}<B+\frac{\xi}{3}$. By (38) and Theorem 3 we have such a covering $\mathfrak{M}$.

For each $r\in N^*$ let M_r' be the η-neighbourhood of M_r. Then

$$M' = (N^*, \{M_r'; r\in N^*\})$$

is a covering of P. Moreover for $r\in N^*$ we have

$$\delta(M_r') \leqq \frac{\varepsilon}{2}+\frac{2\varepsilon}{5} < \varepsilon$$

and thus $\mathfrak{M}'\in U(P, \varepsilon)$ (see Section 1). Finally we have

$$\Sigma_{\mathfrak{M}'} = \sum_{r\in N^*} \delta(M_r') \leqq \sum_{r\in N^*} \delta(M_r)+2C\eta \leqq$$

$$\leqq \Sigma_{\mathfrak{M}}+2C\frac{\xi}{3C} < B+\frac{\xi}{3}+\frac{2\xi}{3} = B+\xi.$$

Consequently

$$L_s(P) < B+\xi$$

as required.

The proof of Theorem 2 is complete.

7. Finally we prove Theorem 1. Let Q be a connected subset of R. If $L^*(Q)=\infty$ then we have $L^*(\bar{Q})=\infty$ and thus $L^*(\bar{Q})=L^*(Q)$.

Now suppose that $L^*(Q)<\infty$. For $k\in\mathbf{N}$ let $P_k=Q$. We then have $L^*(P_k)=L^*(Q)$. The limit inferior of the sequence $P_1, \ldots, P_k, \ldots$ is $\bar{Q}$. Hence by Theorem 2 we have $L^*(\bar{Q})\leqq L^*(Q)$, and since $Q\subset\bar{Q}$ we also have $L^*(Q)\leqq L^*(\bar{Q})$. Consequently $L^*(\bar{Q})=L^*(Q)$. The proof of Theorem 1 is complete.

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TWO NOTES ON INDEPENDENT SUBSETS IN LATTICES

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The present paper deals with two kinds of independent subsets in lattices. A subset H of a lattice L is called *weakly independent* (cf. [1]) iff for all $h, h_1, h_2, \ldots, h_n \in H$ satisfying $h \leqq h_1 \vee h_2 \vee \ldots \vee h_n$ there is an $i \in \{1, 2, \ldots, n\}$ such that $h \leqq h_i$. Analogously, let us call a subset H of L *$*$-independent* iff for all $h, h_1, h_2, \ldots, h_n \in H$ satisfying $h = h_1 \vee h_2 \vee \ldots \vee h_n$ there is an $i \in \{1, 2, \ldots, n\}$ such that $h = h_i$. A maximal weakly independent subset, resp. maximal $*$-independent subset, is called a *weak basis*, resp. *$*$-basis*, of L.

Denoting the set of join-irreducible elements of a lattice L by $J_0(L)$, it is well-known that

THEOREM A (cf., e.g., Grätzer [2]). *For every maximal chain C in a finite distributive lattice L, $|J_0(L)| = |C|$.*

As $J_0(L)$, provided L is distributive, and every maximal chain are always weak bases of finite lattices (cf. [1]), the following assertion is a generalization of Theorem A.

THEOREM B ([1]). *Any two weak bases of a finite distributive lattice have the same number of elements.*

Our first goal is to present another generalization of Theorem A. First we observe that, for every maximal chain $C = \{0 = c_0 < c_1 < c_2 < \ldots < c_{n-1} < c_n = 1\}$ in an arbitrary finite lattice L, $|J_0(L)| \geqq |C|$. Indeed, let $H_0 = \{0\}$ and, for $i \in \{1, 2, \ldots, n\}$, let H_i be the set of minimal elements of $(c_i] \setminus (c_{i-1}]$. Then the H_i are pairwise distinct and nonempty, and $H_0 \cup H_1 \cup \ldots \cup H_n \subseteq J_0(L)$. Therefore the equality in Theorem A is equivalent (modulo lattice theory) to the inequality $|J_0(L)| \leqq |C|$. As $J_0(L)$ and all maximal chains are $*$-bases of a finite lattice L, the following statement generalizes Theorem A, indeed.

THEOREM 1. *Every $*$-basis of a finite distributive lattice L has at least $|J_0(L)|$ elements.*

It was observed in [1] that Theorem B fails to hold for all finite modular lattices. However, modularity is still relevant as we have

THEOREM 2. *If any two weak bases of a finite lattice L have the same cardinality then L is modular.*

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PROOF OF THEOREM 1. Let H be a $*$-basis of L. It suffices to give a map φ from $J_0(L)$ into $P(H)$, the power set of H, such that $x\varphi \neq \emptyset$ and $x\varphi \cap y\varphi = \emptyset$ for any two distinct $x, y \in J_0(L)$. Define φ as follows. For $x \in J_0(L)$ put $x\varphi = \{x\}$ if $x \in H$ and put

$$x\varphi = \{h \in H: \text{ there is a positive integer } n \text{ and there are } h_1, h_2, \ldots, h_n \in H \text{ such that } h \notin \{x, h_1, h_2, \ldots, h_n\} \text{ and } h = x \vee h_1 \vee h_2 \vee \ldots \vee h_n\}$$

if $x \notin H$. As H is a *maximal* $*$-independent subset, $x\varphi$ is never empty. To check $x\varphi \cap y\varphi = \emptyset$ for $x \neq y \in J_0(L)$, the following three cases have to be considered. If $x, y \in H$ then $x\varphi \cap y\varphi = \emptyset$ is evident. If $x \notin H$ and $y \in H$ but $x\varphi \cap y\varphi \neq \emptyset$ then $y \in x\varphi$. Thus $y = x \vee h_1 \vee h_2 \vee \ldots \vee h_n$ and $y \notin \{x, h_1, h_2, \ldots, h_n\}$ for some $h_1, h_2, \ldots, h_n \in H$, which contradicts $y \in J_0(L)$. If $x \notin H$ and $y \notin H$ but $x\varphi \cap y\varphi \neq \emptyset$, say $h \in x\varphi \cap y\varphi$, then there are $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n \in H$ such that $x \vee a_1 \vee a_2 \vee \ldots \vee a_m = h = y \vee b_1 \vee b_2 \vee \ldots \vee b_n$ and $h \notin \{x, a_1, a_2, \ldots, a_m\} \cup \{y, b_1, b_2, \ldots, b_n\}$. We may assume that $x \not\leqq y$. Then, by distributivity, $x = x \wedge h = x \wedge (y \vee b_1 \vee b_2 \vee \ldots \vee b_n) = (x \wedge y) \vee (x \wedge b_1) \vee (x \wedge b_2) \vee \ldots \vee (x \wedge b_n)$. As $x \in J_0(L)$ and $x \neq x \wedge y$, we infer $x = x \wedge b_i$, i.e. $x \leqq b_i$, for some $i \in \{1, 2, \ldots, n\}$. Hence $h = b_i \vee h = b_i \vee x \vee a_1 \vee a_2 \vee \ldots \vee a_m = b_i \vee a_1 \vee a_2 \vee \ldots \vee a_m$. The $*$-independence of H yields $h \in \{b_i, a_1, a_2, \ldots, a_m\}$, a contradiction. Q.e.d.

PROOF OF THEOREM 2. In order to recall a result from Jakubík [3], let us call a sublattice S of a lattice L a *c-sublattice* of L iff for any $u, v \in S$ whenever v covers u in S then v covers u in L. What we need here is only the following weakened version of Jakubík [3, Theorem 1; note the misprint, "sublattice" should be "*c*-sublattice"]: If a finite lattice L is not modular then L includes a c-sublattice S such that S has a four element chain and either 1_S, the greatest element of S, is the join of two atoms of S or 0_S is the meet of two coatoms of S.

Now suppose L is a finite non-modular lattice and consider an above-mentioned c-sublattice S. Let

$$\{0_L = c_0 \prec c_1 \prec c_2 \prec \ldots \prec c_{m-1} \prec c_m = 0_S\}$$

and

$$\{1_S = d_0 \prec d_1 \prec d_2 \prec \ldots \prec d_{n-1} \prec d_n = 1_L\}$$

be maximal chains in the intervals $[0_L, 0_S]$ and $[1_S, 1_L]$, respectively $(0 \leqq m, n)$, and let D be an at least four element maximal chain in S. Then $C = \{c_0, c_1, \ldots, c_{m-1}\} \cup D \cup \{d_1, d_2, \ldots, d_n\}$ is a maximal chain in L, whence it is a weak basis of L. Now we distinguish two cases. (If both conditions hold, we can choose the first one.)

Case A. If $1_S = a_0 \vee a_1$ for some atoms a_0 and a_1 of S then put $G = \{0_S, a_0, a_1\}$.

Case B. If $0_S = b_0 \wedge b_1$ for some coatoms b_0 and b_1 of S then put $G = \{0_S, b_0, b_1\}$.

We claim that $H = \{c_0, c_1, \ldots, c_{m-1}\} \cup G \cup \{d_1, d_2, \ldots, d_n\}$ is a weak basis of L. It is easy to see that H is weakly independent, and we have to show that for every $x \in L \setminus H$ the subset $H \cup \{x\}$ is not weakly independent. Suppose the contrary, i.e., let $H \cup \{x\}$ be weakly independent for some $x \in L \setminus H$. We have $x \leqq d_0 = 1_S$ as otherwise $i = \min\{j: x \leqq d_j\}$ would be positive, and all the three possibilities $i = 1$ in Case A, $i = 1$ in Case B, $i > 1$ would contradict the weak independence of $H \cup \{x\}$ via $d_1 = a_0 \vee a_1 \vee x$, $d_1 = b_0 \vee b_1 \vee x$, $d_i = d_{i-1} \vee x$, respectively. Similarly, $x \not\leqq c_m = 0_S$ as otherwise $c_k = c_{k-1} \vee x$, where $k = \min\{j: x \leqq c_j\}$, would be a con-

tradiction. In Case A, $x \leqq 1_S = a_0 \vee a_1$ implies $x \leqq a_i$ for some $i \in \{0, 1\}$. Since $x \not\leqq 0_S$ and a_i covers 0_S in L, $0_S < 0_S \vee x \leqq a_i$ yields $a_i \leqq 0_S \vee x$, which contradicts the weak independence of H. In Case B, $x \not\leqq 0_S = b_0 \wedge b_1$ yields $x \not\leqq b_i$ for some $i \in \{0, 1\}$. As 1_S covers b_i and $x \leqq 1_S$, $1_S = b_i \vee x$. But then $b_{1-i} \leqq b_i \vee x$ implies $b_{1-i} \leqq x$. Hence $x \leqq 1_S = b_i \vee b_{1-i}$ contradicts the weak independence of $H \cup \{x\}$. Now we have seen that H is a weak basis, and the proof is complete by $|H| < |C|$.

Concluding remarks. Theorem 1 is sharp in the sense that the distributivity of L cannot be omitted and a $*$-basis may have more element than $J_0(L)$. The five element non-distributive modular lattice and the eight element Boolean lattice are appropriate counterexamples. However, we have the following open

PROBLEM. Does every $*$-basis of an arbitrary finite modular lattice L have at least $|C|$ elements where C is a maximal chain in L?

An affirmative answer would be a generalization of Theorem 1. Modularity or some stronger assumption seems to be essential as we have the following example. Let $C = \{0 < a < 1\}$ be a three element chain, and insert a new element into each of the intervals $[(0, 0), (0, a)]$ and $[(0, 0), (a, 0)]$ of C^2. Then $\{(0, 0), (0, 1), (a, 1), (1, 0), (1, a)\}$ is a five element $*$-basis of the eleven element lattice L we obtained, albeit all maximal chains in L have exactly six elements.

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FURTHER PROBABILISTIC RESULTS ON THE SYMMETRIC p-GROUPS

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1. Introduction. In this paper we pursue our investigations of the symmetric p-groups P_n from the point of view of statistical (or probabilistic) group theory (cf. [5], [6]). Let p be a fixed prime; the symmetric p-group P_n $(n=1, 2, \ldots)$ is the Sylow p-subgroup of the symmetric group S_{p^n} of degree p^n.

In [6] we proved that the logarithms of the degrees of the complex irreducible characters of P_n show an asymptotically normal distribution with respect to the uniform measure. In his review of that paper S. V. Kerov [3] pointed out that the Plancherel measure seems to be more natural than the uniform measure. In the present paper we prove that with respect to the Plancherel measure the distribution of the logarithms of the degrees of the complex irreducible characters of P_n is also asymptotically normal. In order to make the paper self-contained we give the details of the proof in the first part (Sections 2—5), although it is quite similar to the one we gave in the case of a uniform measure in [6].

In the second part of the paper (Sections 6—7) we compare the obtained parameters and formulae for the two measures.

In Sections 8—11 we investigate the random variable $\log_p |C_{P_n}(g)|$, $g \in P_n$, where $C_{P_n}(g)$ denotes the centralizer of the element $g \in P_n$ and we choose g with probability $1/|P_n|$ for all $g \in P_n$. We prove that there exists a continuous distribution function $F(x)$ such that, for a randomly chosen element g of P_n, we have

$$\lim_{n\to\infty} \mathrm{Prob}\left(\log_p |C_{P_n}(g)| < x\left(2-\frac{1}{p}\right)^n\right) = F(x).$$

We remind the reader of the related result of P. Erdős and P. Turán [1, Theorem III, p. 414] according to which, choosing the conjugacy classes of S_n with equal probabilities, for the elements g of a randomly chosen conjugacy class of S_n, the relation

$$\log |C_{S_n}(g)| = (1+o(1))\frac{\sqrt{6}}{4\pi}\sqrt{n}\log^2 n$$

holds with probability $1-o(1)$ as $n\to\infty$.

In Section 12 we investigate the analogous problem for P_n and obtain an asymptotically normal distribution (with respect to the uniform measure for the conjugacy classes of P_n).

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The distribution of the character degrees with respect to the Plancherel measure

2. The recurrence formula. Let ξ_n denote a random variable which assigns the logarithm of the degree of χ (to the base p) to a randomly chosen complex irreducible character χ of P_n, i.e., $\xi_n(\chi)=\log_p \chi(1)$, ξ_n is integer-valued. For the probability measure we take the Plancherel measure for which the probability of a character χ is $\chi^2(1)/|P_n|$. If a_{nk} is the number of complex irreducible characters of P_n with $\chi(1)=p^k$ then the distribution of ξ_n is reflected by the generating function

$$B_n(z) = \sum_{k=0}^{\infty} a_{nk}\, p^{2k} z^k. \tag{2.1}$$

This sum is clearly finite, so $B_n(z)$ is a polynomial. We shall use the notation

$$g_n = |P_n| = B_n(1) = p^{(p^n-1)/(p-1)}. \tag{2.2}$$

We have

$$g_{n+1} = p^{(p^{n+1})/(p-1)} = p g_n^p. \tag{2.3}$$

In [6] we investigated the distribution of the character degrees of P_n with respect to the uniform probability measure. There the polynomials

$$A_n(z) = \sum_{k=0}^{\infty} a_{nk} z^k \tag{2.4}$$

played the central role. We established the recurrence formula (see [6, p. 140])

$$A_{n+1}(z) = \frac{z}{p} A_n^p(z) + \left(p - \frac{z}{p}\right) A_n(z). \tag{2.5}$$

Now $B_n(z)=A_n(p^2 z)$, hence we obtain

$$\begin{aligned} B_{n+1}(z) = A_{n+1}(p^2 z) &= pz A_n^p(p^2 z) + (p-pz) A_n(p^{2p} z^p) = \\ &= pz B_n^p(z) + (p-pz) B_n(p^{2p-2} z^p). \end{aligned} \tag{2.6}$$

We have $B_0(z)=1$, $B_1(z)=p$, $B_2(z)=(p^{p+1}-p^2)z+p^2$, etc. For notational convenience we shall sometimes write

$$r = p^{2p-2} \tag{2.7}$$

for the factor appearing in the recurrence formula (2.6).

Obviously, $B_n(z)$ has the same degree as $A_n(z)$, so we can borrow from [6, Proposition 1]:

$$\deg B_n(z) = \begin{cases} \dfrac{p^{n-1}-1}{p-1} & \text{for } p>2,\ n\geqq 1 \text{ and } p=2,\ n=1,2; \\ 3\cdot 2^{n-3}-1 & \text{for } p=2,\ n\geqq 3. \end{cases} \tag{2.8}$$

3. The expectation and the variance. As usual, the expectation $M(\xi_n)$ and the variance $D^2(\xi_n)$ can be expressed in terms of the derivatives of the generating function, namely,

$$M(\xi_n) = \frac{B_n'(1)}{g_n}, \tag{3.1}$$

$$D^2(\xi_n) = \frac{B_n''(1)}{g_n} + \frac{B_n'(1)}{g_n} - \left(\frac{B_n'(1)}{g_n}\right)^2. \tag{3.2}$$

Now we are going to derive recurrence formulae for the expectations and the variances. Differentiating (2.6) and substituting $z=1$ (note that $B_n(1)=g_n$, $g_{n+1}=pg_n^p$), we get

$$B_{n+1}'(1) = g_{n+1} + \frac{pg_{n+1}}{g_n} B_n'(1) - pB_n(p^{2p-2})$$

and

$$B_{n+1}''(1) = \frac{2pg_{n+1}}{g_n} B_n'(1) + \frac{p(p-1)g_{n+1}}{g_n^2} B_n'(1)^2 +$$

$$+ \frac{pg_{n+1}}{g_n} B_n''(1) - 2p^{2p} B_n'(p^{2p-2}).$$

Now, making use of the notation $r=p^{2p-2}$, we have

$$M(\xi_{n+1}) = \frac{B_{n+1}'(1)}{g_{n+1}} = 1 + p\frac{B_n'(1)}{g_n} - \frac{pB_n(r)}{g_{n+1}} = pM(\xi_n) + \left(1 - \frac{B_n(r)}{g_n^p}\right)$$

and, by (3.2),

$$D^2(\xi_{n+1}) = \left\{2p\frac{B_n'(1)}{g_n} + p(p-1)\left(\frac{B_n'(1)}{g_n}\right)^2 + p\frac{B_n''(1)}{g_n} - \frac{2p^{2p}B_n'(r)}{g_{n+1}}\right\} +$$

$$+ \left\{p\frac{B_n'(1)}{g_n} + 1 - \frac{B_n(r)}{g_n^p}\right\} - \left\{p\frac{B_n'(1)}{g_n} + 1 - \frac{B_n(r)}{g_n^p}\right\}^2 = pD^2(\xi_n) +$$

$$+ \frac{2pB_n(r)}{g_n^p} \cdot \frac{B_n'(1)}{g_n} - \frac{2prB_n'(r)}{g_n^p} + \frac{B_n(r)}{g_n^p} - \left(\frac{B_n(r)}{g_n^p}\right)^2.$$

Thus we have the recurrence formulae:

$$M(\xi_{n+1}) = pM(\xi_n) + 1 - \frac{B_n(r)}{g_n^p} \tag{3.3}$$

and

$$D^2(\xi_{n+1}) = pD^2(\xi_n) + \frac{B_n(r)}{g_n^p}\left\{1 - \frac{B_n(r)}{g_n^p} - 2p\left(r\frac{B_n'(r)}{B_n(r)} - \frac{B_n'(1)}{B_n(1)}\right)\right\} \tag{3.4}$$

for $n \geqq 1$. Notice that $M(\xi_1)=0$ and $D^2(\xi_1)=0$.

PROPOSITION 1. *For* $n \geqq 1$, *we have*

$$\alpha p^n - \frac{1}{p-1} < M(\xi_n) < \alpha p^n - \frac{1}{p-1} + \frac{2}{p-1} p^{-cp^n-1}, \tag{3.5}$$

where $\alpha = \alpha(p)$ *is a constant defined by*

$$\alpha = \frac{1}{p(p-1)} - \sum_{k=1}^{\infty} \frac{B_k(p^{2p-2})}{p^{k+1} g_k^p}, \tag{3.6}$$

and

$$c = \begin{cases} 1 - \dfrac{2}{p} & \text{for } p > 2, \\ \dfrac{1}{4} & \text{for } p = 2. \end{cases} \tag{3.7}$$

PROOF. Making use of (2.8) we have for $p > 2$, $k \geqq 1$

$$0 < \frac{B_k(p^{2p-2})}{g_k^p} \leqq \frac{g_k (p^{2p-2})^{(p^{k-1}-1)/(p-1)}}{g_k^p} = p^{-p^k+2p^{k-1}-1} = p^{-cp^k-1}$$

and for $p = 2$, $k \geqq 3$

$$0 < \frac{B_k(p^{2p-2})}{g_k^p} \leqq \frac{g_k 2^{2(3 \cdot 2^{k-3}-1)}}{g_k^2} = 2^{-2^{k-2}-1} = 2^{-c2^k-1}.$$

So for $p > 2$, $k \geqq 1$ or $p = 2$, $k \geqq 3$

$$0 < \frac{B_k(p^{2p-2})}{g_k^p} \leqq p^{-cp^k-1} \tag{3.8}$$

holds, where c is defined by (3.7). In order to include the cases $p = 2$, $k = 1, 2$ as well we shall use also the estimation

$$0 < \frac{B_k(p^{2p-2})}{g_k^p} \leqq 2p^{-cp^k-1}. \tag{3.9}$$

Now we see that the series defining α is convergent and

$$\frac{1}{p(p-1)} - \frac{1}{p^{p+1}} > \alpha > \frac{1}{p(p-1)} - \frac{1}{p^{p+1}} - \frac{2}{p^2(p-1)} p^{-cp^2-1} >$$

$$> \frac{1}{p(p-1)} \left\{ 1 - \frac{1}{p^{p-1}} + \frac{1}{p^p} - \frac{2}{p^2} \right\} > 0.$$

Rewriting (3.3) we get

$$\frac{M(\xi_{k+1})}{p^{k+1}} = \frac{M(\xi_k)}{p^k} + \frac{1}{p^{k+1}} - \frac{B_k(r)}{p^{k+1} g_k^p}$$

for $k \geqq 1$. Since $M(\xi_1)=0$, we have

$$\frac{M(\xi_n)}{p^n} = \sum_{k=1}^{n-1}\left(\frac{1}{p^{k+1}} - \frac{B_k(r)}{p^{k+1}g_k^p}\right) = \frac{1}{p-1}\left(\frac{1}{p} - \frac{1}{p^n}\right) -$$

$$-\sum_{k=1}^{\infty}\frac{B_k(r)}{p^{k+1}g_k^p} + \sum_{k=n}^{\infty}\frac{B_k(r)}{p^{k+1}g_k^p}$$

for $n \geqq 1$. Hence,

$$M(\xi_n) = \alpha p^n - \frac{1}{p-1} + \sum_{k=n}^{\infty}\frac{B_k(r)}{p^{k+1-n}g_k^p}$$

and here by (3.9)

$$0 < \sum_{k=n}^{\infty}\frac{B_k(r)}{p^{k+1-n}g_n^p} \leqq \sum_{k=n}^{\infty} 2p^{-cp^k-1-k-1+n} < \frac{2}{p-1}p^{-cp^n-1}$$

for $n \geqq 1$.

COROLLARY 2. *For $n \to \infty$, we have*

$$M(\xi_n) = \alpha p^n - \frac{1}{p-1} + o(1), \tag{3.10}$$

where $\alpha=\alpha(p)$ is given by (3.6).

PROPOSITION 3. *For $n \geqq 1$, we have*

$$|D^2(\xi_n) - \beta^2 p^n| \leqq \frac{4}{(p-1)^2}p^{-cp^n+n-1}, \tag{3.11}$$

where $\beta^2=\beta^2(p)$ is a constant defined by

$$\beta^2 = \sum_{k=1}^{\infty}\frac{1}{p^{k+1}}\cdot\frac{B_k(p^{2p-2})}{g_k^p}\left\{\left(1-\frac{B_k(p^{2p-2})}{g_k^p}\right) - \right. \tag{3.12}$$

$$\left. -2p\left(p^{2p-2}\frac{B_k'(p^{2p-2})}{B_k(p^{2p-2})} - \frac{B_k'(1)}{g_k}\right)\right\}.$$

PROOF. We shall make use of the following facts. Let $B(x)$ be a non-zero polynomial with nonnegative coefficients. Then for all $x>0$ we have

$$\left(\frac{xB'(x)}{B(x)}\right)' \geqq 0 \tag{3.13}$$

and

$$0 \leqq \frac{xB'(x)}{B(x)} \leqq \deg B(x). \tag{3.14}$$

Indeed, if $B(x)=\sum_{j=0}^{d} b_j x^j$ then

$$\left(\frac{xB'(x)}{B(x)}\right)' = \frac{(xB''(x)+B'(x))B(x)-x(B'(x))^2}{B(x)^2} =$$

$$= \frac{1}{xB(x)^2}\left\{\left(\sum_{j=0}^{d} j^2 b_j x^j\right)\left(\sum_{j=0}^{d} b_j x^j\right)-\left(\sum_{j=0}^{d} j b_j x^j\right)^2\right\} \geqq 0$$

by the Schwarz inequality. (3.14) is obvious. Hence in our case

$$r\frac{B_k'(r)}{B_k(r)} \geqq 1\,\frac{B_k'(1)}{B_k(1)} = \frac{B_k'(1)}{g_k}.$$

Thus, for $k\geqq 1$,

$$\frac{B_k(r)}{g_k^p}\left\{\left(1-\frac{B_k(r)}{g_k^p}\right)-2p\left(r\frac{B_k'(r)}{B_k(r)}-\frac{B_k'(1)}{g_k}\right)\right\} < \frac{B_k(r)}{g_k^p} \leqq 2p^{-cp^k-1}(<1)$$

by (3.9). On the other hand, applying (3.14) and (2.8) we get

$$\frac{B_k(r)}{g_k^p}\left\{\left(1-\frac{B_k(r)}{g_k^p}\right)-2p\left(r\frac{B_k'(r)}{B_k(r)}-\frac{B_k'(1)}{g_k}\right)\right\} > -\frac{B_k(r)}{g_k^p}2p\left(r\frac{B_k'(r)}{B_k(r)}\right) \geqq$$

$$\geqq -\frac{B_k(r)}{g_k^p}2p\deg B_k(z) \geqq -2p^{-cp^k-1}2p\frac{p^{k-1}-1}{p-1} > -\frac{4}{p-1}p^{-cp^k+k-1}.$$

Hence the series defining β^2 is convergent and

$$\beta^2 < \frac{1}{p^{p+1}}-\frac{1}{p^{2p}}+\sum_{k=2}^{\infty}\frac{1}{p^{k+1}}2p^{-cp^k-1} \leqq \frac{1}{p^{p+1}}-\frac{1}{p^{2p}}+\frac{2}{p^2(p-1)}p^{-cp^2-1}.$$

Now we are going to show that β^2 is positive. If $p>2$ then instead of (3.9) we may refer to (3.8), hence we get

$$\frac{B_k(r)}{g_k^p}\left\{\left(1-\frac{B_k(r)}{g_k^p}\right)-2p\left(r\frac{B_k'(r)}{B_k(r)}-\frac{B_k'(1)}{g_k}\right)\right\} > -\frac{2}{p-1}p^{-cp^k+k-1}.$$

The exponent $-cp^k+k-1$ is a decreasing function of k so we obtain

$$\beta^2 > \frac{1}{p^{p+1}}-\frac{1}{p^{2p}}-\sum_{k=2}^{\infty}\frac{1}{p^{k+1}}\cdot\frac{2}{p-1}p^{-cp^k+k-1} >$$

$$> \frac{1}{p^{p+1}}-\frac{1}{p^{2p}}-\frac{2}{p^2(p-1)^2}p^{-(1-2/p)p^2+1} =$$

$$= \frac{1}{p^{p+1}}\left(1-\frac{1}{p^{p-1}}-\frac{2\left(\frac{p}{p-1}\right)^2}{p^{p(p-3)+2}}\right) \geqq \frac{1}{p^{p+1}}\left(1-\frac{1}{3^2}-\frac{2\left(\frac{3}{2}\right)^2}{3^2}\right) > 0.$$

If $p=2$ then we can use the estimation (3.8) only for $k\geqq 3$, then

$$\frac{B_k(r)}{g_k^p}\left\{\left(1-\frac{B_k(r)}{g_k^p}\right)-2p\left(r\frac{B_k'(r)}{B_k(r)}-\frac{B_k'(1)}{g_k}\right)\right\}\geqq-\frac{B_k(r)}{g_k^p}2p\deg B_k(z)\geqq$$

$$\geqq-2^{-(1/4)2^k-1}4(3\cdot 2^{k-3}-1)>-3\cdot 2^{-(1/4)2^k+k-2}.$$

Evaluating the first four terms of the series defining β^2 we get

$$0.0625,\quad -0.0200195,\quad -0.0004813,\quad -0.0006371,$$

so

$$\beta^2>0.041362-\sum_{k=5}^{\infty}\frac{1}{2^{k+1}}3\cdot 2^{-(1/4)2^k+k-2}>0.038>0.$$

Similarly as for the expectation we obtain

$$D^2(\xi_n)=\beta^2 p^n-\sum_{k=n}^{\infty}\frac{1}{p^{k+1-n}}\cdot\frac{B_k(r)}{g_k^p}\left\{\left(1-\frac{B_k(r)}{g_k^p}\right)-2p\left(r\frac{B_k'(r)}{B_k(r)}-\frac{B_k'(1)}{g_k}\right)\right\},$$

hence

$$|D^2(\xi_n)-\beta^2 p^n|\leqq\frac{4}{(p-1)^2}p^{-cp^n+n-1}.$$

COROLLARY 4. *For $n\to\infty$, we have*

$$D^2(\xi_n)=\beta^2 p^n+o(1), \tag{3.15}$$

where $\beta^2=\beta^2(p)$ is given by (3.12).

4. The higher moments. Now we investigate the random variables

$$\eta_n=\frac{\xi_n-\alpha p^n+\dfrac{1}{p-1}}{\beta p^{n/2}}. \tag{4.1}$$

(η_n is, within a small error, the standardized form of ξ_n.) Observe that η_n takes on only finitely many values, hence every moment $M(\eta_n^k)$ exists and is finite.

PROPOSITION 5. *For every $k=1, 2, \ldots$ we have*

$$\lim_{n\to\infty}M(\eta_n^k)=\begin{cases}1\cdot 3\cdot\ldots\cdot(k-1) & \text{for } k \text{ even},\\ 0 & \text{for } k \text{ odd};\end{cases} \tag{4.2}$$

i.e., the k-th moment of the standard Gaussian distribution.

PROOF. For $k=1, 2$ the assertion follows from Corollaries 2 and 4. For $k\geqq 3$ we proceed by induction on k. The moments of η_n appear as coefficients in the characteristic function $\psi_n(t)$ of η_n, namely, for real t,

$$\psi_n(t)=M(e^{\eta_n it})=\sum_{k=0}^{\infty}\frac{M(\eta_n^k)}{k!}(it)^k.$$

The definition (4.1) of η_n yields that

$$\psi_n(t) = \exp\left(\frac{-\alpha p^n + \frac{1}{p-1}}{\beta p^{n/2}} it\right) \frac{1}{g_n} B_n(e^{it/\beta p^{n/2}}).$$

Now the recurrence formula (2.6) implies that

$$\psi_{n+1}(t) = \exp\left(\frac{-\alpha p^{n+1} + \frac{1}{p-1}}{\beta p^{(n+1)/2}} it\right) \frac{1}{g_{n+1}} \{p e^{it/\beta p^{(n+1)/2}} B_n^p(e^{it/\beta p^{(n+1)/2}}) +$$

$$+ p(1 - e^{it/\beta p^{(n+1)/2}}) B_n(p^{2p-2} e^{pit/\beta p^{(n+1)/2}})\} =$$

$$= \psi_n^p\left(\frac{t}{\sqrt{p}}\right) + \frac{1}{g_n^p} \exp\left(\frac{-\alpha p^{n+1} + \frac{1}{p-1}}{\beta p^{(n+1)/2}} it\right)(1 - e^{it/\beta p^{(n+1)/2}}) \times$$

$$\times \sum_{j=0}^{\infty} a_{nj} p^{2j} (p^{2p-2} e^{pit/\beta p^{(n+1)/2}})^j. \tag{4.3}$$

This formula involves a recurrence for the moments of η_{n+1}. The coefficient of $(it)^k/k!$ in $\psi_n^p(t/\sqrt{p})$ is a polynomial in $M(\eta_n^0), \ldots, M(\eta_n^k)$, more exactly, it is

$$F_k(M(\eta_n^0), \ldots, M(\eta_n^{k-1})) + p \frac{M(\eta_n^k)}{\sqrt{p^k}},$$

where F_k is a polynomial. Denoting the moments of the standard Gaussian distribution by m_j $(j=0, 1, \ldots)$, the equation

$$(\exp(-(t/\sqrt{p})^2/2))^p = \exp(-t^2/2)$$

(the characteristic function of the standard Gaussian distribution) implies that F_k satisfies the equation

$$F_k(m_0, \ldots, m_{k-1}) + p \frac{m_k}{\sqrt{p^k}} = m_k. \tag{4.4}$$

Now we infer from (4.3) that

$$M(\eta_{n+1}^k) = F_k(M(\eta_n^0), \ldots, M(\eta_n^{k-1})) + p^{1-k/2} M(\eta_n^k) +$$

$$+ \frac{1}{g_n^p} \sum_{j=0}^{\infty} a_{nj} p^{2pj} \left\{\left(\frac{-\alpha p^{n+1} + \frac{1}{p-1} + pj}{\beta p^{(n+1)/2}}\right)^k - \left(\frac{-\alpha p^{n+1} + \frac{1}{p-1} + 1 + pj}{\beta p^{(n+1)/2}}\right)^k\right\}.$$

Consequently,

$$M(\eta_{n+1}^k) - m_k = p^{1-k/2}(M(\eta_n^k) - m_k) + (F_k(M(\eta_n^0), \ldots, M(\eta_n^{k-1})) - F_k(m_0, \ldots, m_{k-1})) +$$

$$+ \frac{1}{g_n^p} \sum_{j=0}^{\infty} a_{nj} p^{2j} p^{2(p-1)j} \left\{\left(\frac{-\alpha p^{n+1} + \frac{1}{p-1} + pj}{\beta p^{(n+1)/2}}\right)^k - \left(\frac{-\alpha p^{n+1} + \frac{1}{p-1} + 1 + pj}{\beta p^{(n+1)/2}}\right)^k\right\}.$$

By the induction hypothesis $\lim_{n\to\infty} M(\eta_n^j)=m_j$ for $j=0, \ldots, k-1$. Thus

$$\lim_{n\to\infty} F_k(M(\eta_n^0), \ldots, M(\eta_n^{k-1})) = F_k(m_0, \ldots, m_{k-1}).$$

Moreover, since $\deg B_n(z)\leqq(p^{n-1}-1)/(p-1)$ and $\sum_{j=0}^{\infty} a_{nj}p^{2j}=g_n$, we have

$$\left|\frac{1}{g_n^p}\sum_{j=0}^{\infty} a_{nj}p^{2j}p^{2(p-1)j}\left\{\left(\frac{-\alpha p^{n+1}+\frac{1}{p-1}+pj}{\beta p^{(n+1)/2}}\right)^k-\left(\frac{-\alpha p^{n+1}+\frac{1}{p-1}+1+pj}{\beta p^{(n+1)/2}}\right)^k\right\}\right| <$$

$$< \frac{1}{g_n^{p-1}}p^{2(p^{n-1}-1)}2\left\{\left(\frac{\alpha}{\beta}p^{(n+1)/2}\right)^k+\left(\frac{\frac{p^{n-1}-1}{p-1}p+1+\frac{1}{p-1}}{\beta p^{(n+1)/2}}\right)^k\right\} =$$

$$= 2\left(\left(\frac{\alpha}{\beta}\right)^k+\left(\frac{1}{p(p-1)\beta}\right)^k\right)p^{2p^{n-1}-2-p^n+1+k(n+1)/2}.$$

For fixed $k\geqq 3$ and $p>2$ it tends to 0 if $n\to\infty$. For $p=2$, $n\geqq 3$, $\deg B_n(z)=$ $=3\cdot 2^{n-3}-1$ so

$$\left|\frac{1}{g_n^p}\sum_{j=0}^{\infty} a_{nj}p^{2j}p^{2(p-1)j}\left\{\left(\frac{-\alpha p^{n+1}+\frac{1}{p-1}+pj}{\beta p^{(n+1)/2}}\right)^k-\left(\frac{-\alpha p^{n+1}+\frac{1}{p-1}+1+pj}{\beta p^{(n+1)/2}}\right)^k\right\}\right| <$$

$$< \frac{1}{g_n}2^{2(3\cdot 2^{n-3}-1)}2\left\{\left(\frac{\alpha}{\beta}2^{(n+1)/2}\right)^k+\left(\frac{3}{8\beta}2^{(n+1)/2}\right)^k\right\} =$$

$$= \left(\left(\frac{\alpha}{\beta}\right)^k+\left(\frac{3}{8\beta}\right)^k\right)2^{-2^n+1+3\cdot 2^{n-2}-2+1+k(n+1)/2},$$

so again it converges to 0 as $n\to\infty$. Hence

$$M(\eta_{n+1}^k)-m_k = p^{1-k/2}(M(\eta_n^k)-m_k)+o(1).$$

So for any $\varepsilon>0$, there exists an $n_1=n_1(\varepsilon, k)$ such that

$$|M(\eta_{n+1}^k)-m_k| \leqq p^{1-k/2}|M(\eta_n^k)-m_k|+\varepsilon$$

for all $n\geqq n_1$. Consequently, since $p^{1-k/2}<1$,

$$\limsup_{n\to\infty}|M(\eta_n^k)-m_k| \leqq \frac{\varepsilon}{1-p^{1-k/2}}$$

and this holds for every $\varepsilon>0$, therefore $\lim_{n\to\infty} M(\eta_n^k)=m_k$.

5. The limit distribution. By the Moment Convergence Theorem ([4, p. 185]) we infer that the distribution functions of η_n converge to the distribution function

of the standard Gaussian distribution. Obviously, the term $(p-1)^{-1}$ does not play a significant role hence we obtain

THEOREM 6. *With the constants* $\alpha=\alpha(p)$, $\beta=\beta(p)$ *defined by* (3.6) *and* (3.12) *we have*

$$\lim_{n\to\infty} \mathrm{Prob}\left(\log_p \chi(1) < \alpha p^n + x\beta p^{n/2}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt.$$

Comparison of the distributions with respect to the uniform and the Plancherel measure

6. The parameters. By Corollary 2 the expectation of $\log_p \chi(1)$ with respect to the Plancherel measure is

$$\alpha_{\mathrm{Pl}} p^n - \frac{1}{p-1} + o(1),$$

where α_{Pl} is defined by (3.6). Since the Plancherel measure assigns greater probabilities to large dimensional representations, this is greater than the expectation with respect to the uniform measure (see [6, Corollary 2 and (4.6)]) which is

$$\alpha_{\mathrm{U}} p^n - \frac{1}{p-1} + o(1),$$

where

$$\alpha_{\mathrm{U}} = \frac{1}{p(p-1)} - \sum_{k=2}^{\infty} \frac{h_{k-1}}{p^{k-1} h_k}$$

and h_k is the class number of P_k (see [6, Proposition 2]). In order to see the difference, we calculate the limit of the expectation divided by the logarithm of the maximal dimension (i.e. the degree of $B_n(z)$), and denote it by $1-\alpha^*_{\mathrm{Pl}}$ and $1-\alpha^*_{\mathrm{U}}$, respectively. For $p>2$, $n\geqq 1$, the logarithm of the maximal dimension (to the base p) is $\frac{1}{p(p-1)} p^n - \frac{1}{p-1}$ (see (2.8)), hence

$$\alpha^*_{\mathrm{Pl}} = 1 - \lim_{n\to\infty} \frac{\alpha_{\mathrm{Pl}} p^n - \frac{1}{p-1} + o(1)}{\frac{1}{p(p-1)} p^n - \frac{1}{p-1}} = p(p-1)\left(\frac{1}{p(p-1)} - \alpha_{\mathrm{Pl}}\right) =$$

$$= p(p-1) \sum_{k=1}^{\infty} \frac{B_k(p^{2p-2})}{p^{k+1} g_k^p},$$

and similarly we obtain

$$\alpha^*_{\mathrm{U}} = p(p-1) \sum_{k=2}^{\infty} \frac{h_{k-1}}{p^{k-1} h_k}.$$

From the proof of Proposition 1, we see that

$$\frac{p-1}{p^p} < \alpha_{\mathrm{Pl}}^* < \frac{p-1}{p^p} + \frac{2}{p^{p^2-2p+2}}.$$

Also, the proof of [6, Proposition 5] yields that

$$\frac{p(p-1)}{p^{p-1}+p^2-1} < \alpha_{\mathrm{U}}^* < \frac{p(p-1)}{p^{p-1}+p^2-1}\left(1+\frac{1}{p^{p^2-3p+2}}\right).$$

For $p=2$, $n\geqq 3$, the degree of $B_n(z)$ is $3\cdot 2^{n-3}-1$ (see (2.8)), so the formulae differ slightly. By direct calculation we obtain:

p	α_{Pl}^*	α_{U}^*
2	0.12236	0.39709
3	0.07652	0.35967
5	0.00128	0.03082
7	0.00001	0.00036

(rounded to 10^{-5}). For $p\geqq 11$, the approximate values are

$$\alpha_{\mathrm{Pl}}^* \approx \frac{p-1}{p^p}, \quad \alpha_{\mathrm{U}}^* \approx \frac{p-1}{p^{p-2}}$$

within relative accuracy $p^{-p+3}(<10^{-8})$.

It turns out that the variance is smaller for the Plancherel measure. By Corollary 4 it is $\beta_{\mathrm{Pl}}^2 p^n + o(1)$ where β_{Pl}^2 is defined by (3.12), while for the uniform measure it is $\beta_{\mathrm{U}}^2 p^n - o(1)$ with

$$\beta_{\mathrm{U}}^2 = \sum_{k=2}^{\infty} \frac{h_{k-1}}{p^{k-1}h_k}\left(1-\frac{ph_{k-1}}{h_k}\right)\prod_{l=k+1}^{\infty}\left(1+\frac{(p^2-1)(p-1)h_{l-1}}{ph_l}\right)$$

(see [6, Corollary 3 and (4.9)]). In the proof of Proposition 3, for $p>2$, we have shown

$$\frac{1}{p^{p+1}}-\frac{1}{p^{2p}}-\frac{2}{(p-1)^2 p^{p^2-2p+1}} < \beta_{\mathrm{Pl}}^2 < \frac{1}{p^{p+1}}-\frac{1}{p^{2p}}+\frac{2}{(p-1)\,p^{p^2-2p+3}}.$$

Also, by [6, Proposition 5] we have

$$\frac{1}{p^{(p-1)/2}}\left(1-\frac{1}{p^{p-3}}\right) < \beta_{\mathrm{U}} < \frac{1}{p^{(p-1)/2}}\left(1-\frac{1}{p^{p-3}}+\frac{2}{p^{p-2}}\right).$$

For small primes we determined these constants by direct calculation:

p	β_{Pl}	β_{U}
2	0.20278	0.32923
3	0.10540	0.17396
5	0.00799	0.03849
7	0.00042	0.00291

(rounded to 10^{-5}). For $p\geqq 11$, the approximate values are

$$\beta_{\mathrm{Pl}} \approx \frac{1}{p^{(p+1)/2}}, \quad \beta_{\mathrm{U}} \approx \frac{1}{p^{(p-1)/2}}$$

within relative accuracy $p^{-p+3}(<10^{-8})$.

7. The formulae. We have found that the distribution of the logarithm of the character degrees of P_n is asymptotically normal both for the uniform and the Plancherel measure. The proofs ran very close to each other. In case of the Plancherel measure the recurrence formulae for the derivatives of the generating polynomial $B_n(z)$ are somewhat more complicated than those of $A_n(z)$, and this causes some technical difficulties. However, the main terms in the recurrence formulae for the variances (3.4) and the characteristic functions (4.3) become simpler for the Plancherel measure (cf. [6, (4.4) and (5.3)]). This can be regarded as an evidence supporting Kerov's suggestion that the Plancherel measure is more natural (cf. [7]).

Centralizers in P_n

8. Recurrence formula. Now we are going to investigate the random variable ζ_n which assigns to an element $g\in P_n$ the logarithm of the order of its centralizer, i.e., $\zeta_n(g)=\log_p |C_{P_n}(g)|$. We choose g with probability $1/|P_n|$ for all $g\in P_n$. Our random variable is integer-valued, we take its generating function

$$C_n(z) = \frac{1}{|P_n|} \sum_{g\in P_n} z^{\log_p |C_{P_n}(g)|}. \tag{8.1}$$

Obviously $C_1(z)=z$. Further, for all $n\geqq 1$ we have $C_n(1)=1$ and

$$C_n(p) = \frac{1}{|P_n|} \sum_{g\in P_n} |C_{P_n}(g)| = h_n, \tag{8.2}$$

the number of conjugacy classes in P_n.

PROPOSITION 7. *For $n\geqq 1$ we have*

$$C_{n+1}(z) = \frac{1}{p} C_n^p(z) + \frac{z-1}{p} C_n(z^p/p^{p-1}) + \left(1-\frac{1}{p}\right) z C_n(z). \tag{8.3}$$

PROOF. As it is well known, P_n is an iterated wreath product of n copies of the cyclic group Z_p of order p. Therefore we have

$$P_{n+1} = P_n \mathbf{wr} Z_p.$$

According to the definition of the wreath product, the elements of P_{n+1} are expressed in the form $x=(f_x, x')$ where $f_x: Z_p \to P_n$ and $x' \in Z_p$. The elements x for which $x'=e$, the unit element of Z_p, form a normal subgroup $D \cong P_n^p$ of index p in P_{n+1}. Let $x=(f_x, x')$ and $y=(f_y, y')$ be elements of P_{n+1}. Then $xy=(f_{xy}, x'y')$ where $f_{xy}(z)=f_x(z)f_y(x'z)$ for $z \in Z_p$. Hence x and (f_y, e) commute if and only if $f_x(z)f_y(x'z)=f_{xy}(z)=f_{yx}(z)=f_y(z)f_x(z)$, i.e. $f_y(x'z)=f_x(z)^{-1}f_y(z)f_x(z)$ holds for all $z \in Z_p$.

Now take $g=(f_g, g') \in P_{n+1}$ and determine $|C_{P_{n+1}}(g)|$. We distinguish three cases.

(i) $g'=e$ and $\{f_g(z) | z \in Z_p\}$ lies in a single conjugacy class of P_n. Then we can find an $x \in C_{P_{n+1}}(g)$ for which $x' \neq e$. Indeed, fixing $x' \neq e$, we can choose $f_x(x'^j)$ for $j=0, 1, \ldots, p-1$ so as to satisfy $f_g(x'^{j+1})=f_x(x'^j)^{-1}f_g(x'^j)f_x(x'^j)$. Thus $|C_{P_{n+1}}(g): C_D(g)|=p$. Furthermore, $(f_x, e) \in C_D(g)$ iff $f_g(z)=f_x(z)^{-1}f_g(z)f_x(z)$ for all $z \in Z_p$, hence $|C_D(g)| = \prod_{z \in Z_p} |C_{P_n}(f_g(z))| = |C_{P_n}(f_g(e))|^p$, as the elements $f_g(z)$ are conjugate in P_n. Therefore $|C_{P_{n+1}}(g)| = p|C_{P_n}(f_g(e))|^p$.

(ii) $g'=e$ and $\{f_g(z) | z \in Z_p\}$ does not lie in a single conjugacy class of P_n. Then for any $x \in C_{P_{n+1}}(g)$ we have $x'=e$ and $f_g(z)=f_x(z)^{-1}f_g(z)f_x(z)$ for all $z \in Z_p$. Hence $|C_{P_{n+1}}(g)| = \prod_{z \in Z_p} |C_{P_n}(f_g(z))|$.

(iii) $g' \neq e$. Then $C_{P_{n+1}}(g)$ contains elements outside D, for example g itself. So $|C_{P_{n+1}}(g): C_D(g)|=p$. Now $y=(f_y, e) \in C_D(g)$ iff $f_y(g'z)=f_g(z)^{-1}f_y(z)f_g(z)$ for all $z \in Z_p$, hence

$$f_y(e) \in C_{P_n}\Big(\prod_{j=0}^{p-1} f_g(g'^j)\Big) \quad \text{and}$$

$$f_y(g'^k) = \Big(\prod_{j=0}^{k-1} f_g(g'^j)\Big)^{-1} f_y(e) \Big(\prod_{j=0}^{k-1} f_g(g'^j)\Big) \quad \text{for} \quad k = 1, 2, \ldots, p-1.$$

Thus $|C_{P_{n+1}}(g)| = p \cdot \Big|C_{P_n}\Big(\prod_{j=0}^{p-1} f_g(g'^j)\Big)\Big|$.

Now we break

$$C_{n+1}(z) = \frac{1}{|P_{n+1}|} \sum_{g \in P_{n+1}} z^{\log_p |C_{P_{n+1}}(g)|}$$

into three sums according to the three cases we considered:

$$\sum_{(i)} z^{\log_p |C_{P_{n+1}}(g)|} = \sum_{a \in P_n} \left(\frac{|P_n|}{|C_{P_n}(a)|}\right)^{p-1} z^{1+\log_p |C_{P_n}(a)|^p} =$$

$$= |P_n|^p z \frac{1}{|P_n|} \sum_{a \in P_n} \left(\frac{z^p}{p^{p-1}}\right)^{\log_p |C_{P_n}(a)|} = \frac{|P_{n+1}|}{p} z C_n(z^p/p^{p-1}),$$

$$\sum_{(ii)} z^{\log_p |C_{P_{n+1}}(g)|} = \sum_{\substack{a_1, \ldots, a_p \in P_n \\ \text{not all conjugate}}} z^{\log_p \left(\prod_{j=1}^{p} |C_{P_n}(a_j)|\right)} = \frac{|P_{n+1}|}{p} (C_n^p(z) - C_n(z^p/p^{p-1})),$$

and

$$\sum_{(iii)} z^{\log_p |C_{P_{n+1}}(g)|} = \sum_{g' \neq e} \sum_{a_1, \ldots, a_p \in P_n} z^{1+\log_p |C_{P_n}(a_1 \cdot \ldots \cdot a_p)|} = \frac{(p-1)|P_{n+1}|}{p} z C_n(z).$$

Thus we get

$$C_{n+1}(z) = \frac{z}{p} C_n(z^p/p^{p-1}) + \frac{1}{p}\left(C_n^p(z) - C_n(z^p/p^{p-1})\right) + \left(1-\frac{1}{p}\right) z C_n(z).$$

COROLLARY 8. *For the class number h_n of P_n we have the recurrence*

$$h_{n+1} = \frac{1}{p} h_n^p + \left(p - \frac{1}{p}\right) h_n. \tag{8.4}$$

PROOF. Substitute $z=p$ in (8.3) and use (8.2).

In this way we avoided the application of representation theory for proving the recurrence formula (8.4) (cf. [6, (3.1)]). In our earlier paper [6, Proposition 2] we derived that

$$h_n = [p^{\gamma p^n + 1/(p-1)}] - \begin{cases} 0 & \text{if } p > 2 \\ 1 & \text{if } p = 2 \end{cases} \tag{8.5}$$

where

$$\gamma = \frac{p-2}{p(p-1)} + \frac{1}{\log p} \sum_{k=2}^{\infty} \frac{1}{p^k} \log\left(1 + \frac{p^2-1}{h_{k-1}^{p-1}}\right).$$

COROLLARY 9. (a) *The center of P_n has order p for all $n \geqq 1$.*

(b) *For any $g \in P_n$, $|C_{P_n}(g)| \geqq p^n$. Equality holds for $\left(1-\frac{1}{p}\right)^{n-1} |P_n|$ elements if $p > 2$ and for $\frac{3}{2^n} |P_n|$ elements if $p=2$, $n \geqq 2$.*

PROOF. (a) is equivalent to saying that the highest degree term in $C_n(z)$ is $\frac{p}{|P_n|} z^{\log_p |P_n|}$, (b) is equivalent to the statement that the lowest degree term of $C_n(z)$ is $\left(1-\frac{1}{p}\right)^{n-1} z^n$ for $p>2$ and $\frac{3}{2^n} z^n$ for $p=2$, $n \geqq 2$. We prove these assertions by induction on n. Obviously P_1 is cyclic of order p, so $C_1(z)=z$, hence (a) and (b) hold. Owing to (8.3) the highest degree term of $C_{n+1}(z)$ becomes

$$\frac{z}{p} \frac{p}{|P_n|} \left(\frac{z^p}{p^{p-1}}\right)^{\log_p |P_n|} = \frac{p}{p\,|P_n|^p} z^{1+p\log_p |P_n|} = \frac{p}{|P_{n+1}|} z^{\log_p |P_{n+1}|}.$$

For $p>2$, the lowest degree term of $C_{n+1}(z)$ turns out to be

$$\left(1-\frac{1}{p}\right) z \left(1-\frac{1}{p}\right)^{n-1} z^n = \left(1-\frac{1}{p}\right)^n z^{n+1}.$$

Finally, let $p=2$. Then $C_2(z)=\frac{1}{4}z^3+\frac{3}{4}z^2$, and, by induction, the lowest degree term of $C_{n+1}(z)$ is

$$\left(1-\frac{1}{2}\right)z\frac{3}{2^n}z^n=\frac{3}{2^{n+1}}z^{n+1}.$$

9. The expectation of ζ_n is $C_n'(1)$. We have $C_n'(1)=1$ and making use of the recurrence formula (8.3) and the fact that $C_n(1)=1$ we obtain for $n\geqq 1$

$$C_{n+1}'(1)=C_n'(1)+\frac{1}{p}C_n(1/p^{p-1})+\left(1-\frac{1}{p}\right)+\left(1-\frac{1}{p}\right)C_n'(1)= \tag{9.1}$$

$$=\left(2-\frac{1}{p}\right)C_n'(1)+1-\frac{1}{p}+\frac{1}{p}C_n(1/p^{p-1}).$$

If we take $C_0(z)=1$, (9.1) remains valid for $n=0$ as well. We shall denote

$$K=2-\frac{1}{p}. \tag{9.2}$$

PROPOSITION 10. *For $n\geqq 1$, we have*

$$LK^n-1-\frac{1}{p}\left(\frac{1}{p^{p-1}}\right)^n<C_n'(1)<LK^n-1 \tag{9.3}$$

where $L=L(p)$ is a constant defined by

$$L=1+\frac{1}{p}\sum_{k=0}^{\infty}\frac{1}{K^{k+1}}C_k\left(\frac{1}{p^{p-1}}\right). \tag{9.4}$$

PROOF. By Corollary 9(b), it follows that

$$C_k\left(\frac{1}{p^{p-1}}\right)\leqq\left(\frac{1}{p^{p-1}}\right)^k, \tag{9.5}$$

hence the series defining L is obviously convergent. Rewriting (9.1) we get

$$\frac{C_{n+1}'(1)+1}{K^{n+1}}=\frac{C_n'(1)+1}{K^n}+\frac{1}{pK^{n+1}}C_n(1/p^{p-1}).$$

Since $C_0'(1)=0$, we obtain

$$C_n'(1)=K^n\left(1+\frac{1}{p}\sum_{k=0}^{\infty}\frac{1}{K^{k+1}}C_k(1/p^{p-1})-\frac{1}{p}\sum_{k=n}^{\infty}\frac{1}{K^{k+1}}C_k(1/p^{p-1})\right)-1=$$

$$=LK^n-1-\frac{1}{p}\sum_{k=n}^{\infty}\frac{1}{K^{k+1-n}}C_k(1/p^{p-1}).$$

Here, applying (9.5) we get

$$0 < \frac{1}{p}\sum_{k=n}^{\infty}\frac{1}{K^{k+1-n}}C_k(1/p^{p-1}) \leqq \frac{K^{n-1}}{p}\sum_{k=n}^{\infty}\left(\frac{1}{Kp^{p-1}}\right)^k =$$

$$= \frac{1}{(Kp^{p-1}-1)\,p^{1+(p-1)(n-1)}} \leqq \frac{1}{p}\left(\frac{1}{p^{p-1}}\right)^n.$$

10. The characteristic function. We shall consider the characteristic functions of the random variables ζ_n/K^n, so let

$$\varphi_n(t) = C_n(e^{it/K^n}). \tag{10.1}$$

By (8.3) we have

$$\varphi_{n+1}(Kt) = \left(1-\frac{1}{p}\right)e^{it/K^n}\varphi_n(t)+\frac{1}{p}\varphi_n^p(t)+ \tag{10.2}$$

$$+\frac{1}{p}(e^{it/K^n}-1)C_n(e^{pit/K^n}/p^{p-1}).$$

Now we introduce the function

$$\Delta_n(t) = \varphi_n(t)-\varphi_{n-1}(t)-\frac{it}{K^n}\left(1-\frac{1}{p}+\frac{1}{p}C_{n-1}(1/p^{p-1})\right). \tag{10.3}$$

Then, making use of (10.2) we obtain

$$\Delta_{n+1}(Kt) = \varphi_{n+1}(Kt)-\varphi_n(Kt)-\frac{iKt}{K^{n+1}}\left(1-\frac{1}{p}+\frac{1}{p}C_n(1/p^{p-1})\right) = \tag{10.4}$$

$$= \left(1-\frac{1}{p}\right)\varphi_n(t)+\left(1-\frac{1}{p}\right)(e^{it/K^n}-1)\,\varphi_n(t)+\frac{1}{p}\varphi_n^p(t)+$$

$$+\frac{1}{p}(e^{it/K^n}-1)C_n(e^{pit/K^n}/p^{p-1})-\left(1-\frac{1}{p}\right)\varphi_{n-1}(t)-$$

$$-\left(1-\frac{1}{p}\right)(e^{it/K^{n-1}}-1)\,\varphi_{n-1}(t)-\frac{1}{p}\varphi_{n-1}^p(t)-$$

$$-\frac{1}{p}(e^{it/K^{n-1}}-1)C_{n-1}(e^{pit/K^{n-1}}/p^{p-1})-\frac{it}{K^n}\left(1-\frac{1}{p}+\frac{1}{p}C_n(1/p^{p-1})\right) =$$

$$= \left\{\varphi_n(t)-\varphi_{n-1}(t)-\frac{it}{K^n}\left(1-\frac{1}{p}+\frac{1}{p}\,C_{n-1}(1/p^{p-1})\right)\right\}\times$$

$$\times\left\{1-\frac{1}{p}+\frac{1}{p}\sum_{j=0}^{p-1}\varphi_n^j(t)\,\varphi_{n-1}^{p-1-j}(t)\right\}+$$

$$+\left(e^{it/K^n}-1-\frac{it}{K^n}\right)\left(\left(1-\frac{1}{p}\right)\varphi_n(t)+\frac{1}{p}C_n(e^{pit/K^n}/p^{p-1})\right)-$$

$$-\left(e^{it/K^{n-1}}-1-\frac{it}{K^{n-1}}\right)\left(\left(1-\frac{1}{p}\right)\varphi_{n-1}(t)+\frac{1}{p}C_{n-1}(e^{pit/K^{n-1}}/p^{p-1})\right)+$$

$$+\frac{it}{K^n}\left\{\left(1-\frac{1}{p}+\frac{1}{p}C_{n-1}(1/p^{p-1})\right)\left(1-\frac{1}{p}+\frac{1}{p}\sum_{j=0}^{p-1}\varphi_n^j(t)\,\varphi_{n-1}^{p-1-j}(t)\right)+\right.$$

$$+\left(1-\frac{1}{p}\right)\varphi_n(t)+\frac{1}{p}C_n(e^{pit/K^n}/p^{p-1})-\left(1-\frac{1}{p}\right)K\varphi_{n-1}(t)-$$

$$\left.-\frac{1}{p}KC_{n-1}(e^{pit/K^{n-1}}/p^{p-1})-\left(1-\frac{1}{p}+\frac{1}{p}C_n(1/p^{p-1})\right)\right\}=$$

(10.4.1) $$=\left(1-\frac{1}{p}+\frac{1}{p}\sum_{j=0}^{p-1}\varphi_n^j(t)\varphi_{n-1}^{p-1-j}(t)\right)\Delta_n(t)+$$

(10.4.2) $$+\left(e^{it/K^n}-1-\frac{it}{K^n}\right)\left(\left(1-\frac{1}{p}\right)\varphi_n(t)+\frac{1}{p}C_n(e^{pit/K^n}/p^{p-1})\right)-$$

(10.4.3) $$-\left(e^{it/K^{n-1}}-1-\frac{it}{K^{n-1}}\right)\left(\left(1-\frac{1}{p}\right)\varphi_{n-1}(t)+\frac{1}{p}C_{n-1}(e^{pit/K^{n-1}}/p^{p-1})\right)+$$

(10.4.4) $$+\frac{it}{K^n}\cdot\frac{1}{p}(C_n(e^{pit/K^n}/p^{p-1})-C_n(1/p^{p-1}))-$$

(10.4.5) $$-\frac{it}{K^n}\cdot\frac{K}{p}(C_{n-1}(e^{pit/K^{n-1}}/p^{p-1})-C_{n-1}(1/p^{p-1}))+$$

(10.4.6) $$+\frac{it}{K^n}\left(1-\frac{1}{p}\right)(\varphi_n(t)-1)-$$

(10.4.7) $$-\frac{it}{K^n}\left(1-\frac{1}{p}\right)K(\varphi_{n-1}(t)-1)+$$

(10.4.8) $$+\frac{it}{K^n}\cdot\frac{1}{p}\left(\sum_{j=0}^{p-1}(\varphi_n^j(t)\,\varphi_{n-1}^{p-1-j}(t)-1)\right)\left(1-\frac{1}{p}+\frac{1}{p}C_{n-1}(1/p^{p-1})\right).$$

Let us denote the terms in lines (10.4.1)–(10.4.8) by $T_1, \dots,$ and T_8, respectively. Now we estimate them separately. Since for real t, $|\varphi_n(t)|\leqq 1$, it follows that

$$|T_1|\leqq K|\Delta_n(t)|.$$

Since C_n is a polynomial with nonnegative coefficients, $|C_n(z)|\leqq C_n(1)=1$ whenever $|z|\leqq 1$. Hence

$$|T_2|\leqq\left|e^{it/K^n}-1-\frac{it}{K^n}\right|\left(\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)\leqq\frac{1}{2}\left(\frac{t}{K^n}\right)^2.$$

Similarly, we have

$$|T_3| \leqq \frac{1}{2}\left(\frac{t}{K^{n-1}}\right)^2.$$

Now let $0<|z|\leqq 1/2$ and $C_n(z)=\sum c_k z^k$. Then

$$\left|C_n(z)-C_n(|z|)\right| = \left|\sum c_k(z^k-|z|^k)\right| \leqq \sum c_k|z-|z||k|z|^{k-1} \leqq$$

$$\leqq \frac{1}{2}\left|\frac{z}{|z|}-1\right|\sum c_k(2|z|)^k = \frac{1}{2}\left|\frac{z}{|z|}-1\right| C_n(2|z|) \leqq \frac{1}{2}\left|\frac{z}{|z|}-1\right|.$$

Hence we obtain

$$|T_4| \leqq \frac{|t|}{K^n}\cdot\frac{1}{p}|e^{pit/K^n}-1| \leqq \frac{t^2}{K^{2n}},$$

and similarly

$$|T_5| \leqq \frac{t^2}{K^{2n-2}}.$$

Furthermore, owing to Proposition 10 we have

$$|\varphi_n(t)-1| = \left|\sum c_k(e^{kit/K^n}-1)\right| \leqq \sum c_k\frac{k|t|}{K^n} = \frac{|t|}{K^n}C_n'(1) < L|t|.$$

Thus

$$|T_6| \leqq \frac{|t|}{K^n}\left(1-\frac{1}{p}\right)L|t| = L\left(1-\frac{1}{p}\right)\frac{t^2}{K^n}$$

and

$$|T_7| \leqq L\left(1-\frac{1}{p}\right)\frac{t^2}{K^{n-1}}.$$

Finally,

$$|T_8| \leqq \frac{|t|}{K^n}\cdot\frac{1}{p}\sum_{j=0}^{p-1}\left(|\varphi_n^j(t)-1|\cdot|\varphi_{n-1}^{p-1-j}(t)|+|\varphi_{n-1}^{p-1-j}(t)-1|\right) \leqq$$

$$\leqq \frac{|t|}{pK^n}\sum_{j=0}^{p-1}\left(j|\varphi_n(t)-1|+(p-1-j)|\varphi_{n-1}(t)-1|\right) =$$

$$= \frac{|t|}{K^n}\cdot\frac{p-1}{2}\left(|\varphi_n(t)-1|+|\varphi_{n-1}(t)-1|\right) \leqq \frac{(p-1)Lt^2}{K^n}.$$

Summarizing, we obtain

$$|\Delta_{n+1}(Kt)| \leqq K|\Delta_n(t)|+R\frac{t^2}{K^n}$$

for some constant R. By induction this yields that

$$|\Delta_n(t)| \leqq \sum_{j=1}^{n-1} K^{j-1} R \frac{(t/K^j)^2}{K^{n-j}} + K^{n-1}|\Delta_1(t/K^{n-1})| = \tag{10.5}$$

$$= (n-1)\,R\frac{t^2}{K^{n+1}} + K^{n-1}|\Delta_1(t/K^{n-1})| \leqq \frac{t^2}{K^n}\left(\frac{(n-1)R}{K} + \frac{1}{2K}\right),$$

since

$$|\Delta_1(t)| = \left|e^{it/K} - 1 - \frac{it}{K}\right| \leqq \frac{1}{2}\left(\frac{t}{K}\right)^2.$$

Referring to the definition (10.3) of $\Delta_n(t)$ we see that

$$|\varphi_n(t) - \varphi_{n-1}(t)| \leqq \frac{|t|}{K^n} + |\Delta_n(t)|. \tag{10.6}$$

(10.5) and (10.6) yield that the (continuous) characteristic functions $\varphi_n(t)$ converge to a function $\varphi(t)$ and the convergence is uniform in every finite interval $[-t_0, t_0]$. Consequently, $\varphi(t)$ is a continuous limit. From (10.2) we infer that $\varphi(t)$ satisfies the functional equation

$$\varphi(Kt) = \left(1 - \frac{1}{p}\right)\varphi(t) + \frac{1}{p}\varphi^p(t). \tag{10.7}$$

Furthermore, notice that $\varphi'(0) = iL \neq 0$, since by (10.5) and (10.6)

$$|\varphi(t) - \varphi_n(t)| \leqq \sum_{j=n+1}^{\infty}\left(\frac{|t|}{K^j} + \frac{t^2}{K^j}\left(\frac{(j-1)R}{K} + \frac{1}{2K}\right)\right) =$$

$$= \frac{|t|}{K^n(K-1)} + \frac{t^2}{K^{n+1}(K-1)}\left(\left(n + \frac{1}{K-1}\right)R + \frac{1}{2}\right),$$

and so $\varphi'(0)$ exists and is equal to $\lim_{n\to\infty} \varphi_n'(0) = \lim_{n\to\infty} iC_n'(1)/K^n = iL$, by (9.3). In particular, $\varphi(t)$ is not identically equal to 1.

11. The limit distribution. Owing to the previous calculations we are able to show the existence of the limit distribution, however, we could not find a formula for the distribution function.

THEOREM 11. *There exists a continuous monotonic real function $F(x)$ with $F(x) = 0$ for $x \leqq 0$ and $\lim_{x\to+\infty} F(x) = 1$ such that, for a randomly chosen element $g \in P_n$, we have*

$$\lim_{n\to\infty} \operatorname{Prob}\left(\log_p |C_{P_n}(g)| < x\left(2 - \frac{1}{p}\right)^n\right) = F(x). \tag{11.1}$$

PROOF. Let $\varphi(t) = \lim_{n\to\infty} \varphi_n(t)$. We have seen that $\varphi(t)$ is continuous. By the well-known continuity theorem (see, e.g., [4, p. 191]), the distribution functions of the random variables ζ_n/K^n converge to a *distribution function* $F(x)$ at each point

of continuity of $F(x)$, $F(-\infty)=0$, $F(+\infty)=1$ and $\varphi(t)$ is the characteristic function of $F(x)$. Since each ζ_n/K^n is positive, we have $F(x)=0$ for $x\leqq 0$.

In order to establish the continuity of $F(x)$, we have to show that for any x

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\varphi(t)e^{-itx}\,dt=0, \tag{11.2}$$

see, e.g., [2, Theorem 3.8.1, p. 102].

Obviously, the limit distribution is not a lattice distribution, hence $|\varphi(t)|<1$ for any $t\neq 0$. Since $\varphi(t)$ is continuous

$$M=\sup_{1\leqq|t|\leqq K}|\varphi(t)|<1. \tag{11.3}$$

Furthermore, put

$$\sigma=\frac{M^{p-1}+p-1}{p}<1.$$

Then we show by induction that

$$\sup_{K^n\leqq|t|\leqq K^{n+1}}|\varphi(t)|\leqq\sigma^n M$$

for $n=0, 1, 2, \ldots$. By (11.3) the assertion is valid for $n=0$. Let $n>0$. Making use of (10.7) we obtain

$$\sup_{K^n\leqq|t|\leqq K^{n+1}}|\varphi(t)|=\sup_{K^{n-1}\leqq|t|\leqq K^n}|\varphi(Kt)|=$$

$$=\sup_{K^{n-1}\leqq|t|\leqq K^n}\left|\frac{\varphi^p(t)+(p-1)\varphi(t)}{p}\right|\leqq$$

$$\leqq\frac{1}{p}\left((\sigma^{n-1}M)^p+(p-1)\sigma^{n-1}M\right)=\frac{1}{p}\left((\sigma^{n-1}M)^{p-1}+p-1\right)\sigma^{n-1}M\leqq\sigma^n M.$$

Now let $K^n\leqq T\leqq K^{n+1}$. Then

$$\left|\frac{1}{2T}\int_{-T}^{T}\varphi(t)e^{-itx}\,dt\right|\leqq\frac{1}{2T}\int_{-T}^{T}|\varphi(t)|\,dt\leqq$$

$$\leqq\frac{2}{2K^n}\left(1+(K-1)M+(K^2-K)\sigma M+\ldots+(K^{n+1}-K^n)\sigma^n M\right)=$$

$$=\frac{1}{K^n}\left(1+(K-1)M\sum_{j=0}^{n}(K\sigma)^j\right)\to 0,$$

if $n\to\infty$, since $\sigma<1$, $K>1$. In virtue of (11.2) it proves the continuity of $F(x)$.

12. Cardinalities of conjugacy classes of P_n. For an arbitrary conjugacy class H of P_n, $g\in H$ implies that

$$|C_{P_n}(g)|=\frac{|P_n|}{|H|}.$$

In what follows we investigate the random variable ξ_n which assigns to a conjugacy class H of P_n the logarithm of the common order of the centralizers of its elements, i.e.,

$$\xi_n(H) = \log_p \frac{|P_n|}{|H|}. \tag{12.1}$$

We choose H with probability $1/h_n$ for all H's, where h_n is the number of conjugacy classes in P_n, see (8.5). ξ_n is integer-valued, we take its generating function

$$\frac{1}{h_n} H_n(z) = \frac{1}{h_n} \sum_H z^{\log_p(|P_n|/|H|)}. \tag{12.2}$$

Since, by (8.1),

$$H_n(z) = \sum_H \frac{1}{|H|} \sum_{g \in H} z^{\log_p |C_{P_n}(g)|} = \frac{1}{|P_n|} \sum_{g \in P_n} |C_{P_n}(g)| z^{\log_p |C_{P_n}(g)|} = C_n(pz),$$

we have $H_1(z)=pz$ and (8.3) implies that, for $n \geqq 1$,

$$H_{n+1}(z) = \frac{1}{p} H_n^p(z) + \left(z - \frac{1}{p}\right) H_n(z^p) + (p-1) z H_n(z). \tag{12.3}$$

For the expectation $M(\xi_n)$ and the variance $D^2(\xi_n)$, we have

$$M(\xi_n) = \frac{H_n'(1)}{h_n} \tag{12.4}$$

and

$$D^2(\xi_n) = \frac{H_n''(1)}{h_n} + \frac{H_n'(1)}{h_n} - \left(\frac{H_n'(1)}{h_n}\right)^2. \tag{12.5}$$

Taking into account the relations $H_n(1)=h_n$ and

$$h_{n+1} = \frac{1}{p} h_n^p + \left(p - \frac{1}{p}\right) h_n,$$

the recurrence formula (12.3) yields that

$$\frac{H_{n+1}'(1)}{h_{n+1}} = p\left(1 - (p-1)^2 \frac{h_n}{p h_{n+1}}\right) \frac{H_n'(1)}{h_n} + \frac{p h_n}{h_{n+1}}$$

and

$$\frac{H_{n+1}''(1)}{h_{n+1}} = p \frac{H_n''(1)}{h_n} + (p^2-p)\left(1 - \left(p - \frac{1}{p}\right) \frac{h_n}{h_{n+1}}\right) \left(\frac{H_n'(1)}{h_n}\right)^2 +$$

$$+ (p^2+2p-1) \frac{h_n}{h_{n+1}} \cdot \frac{H_n'(1)}{h_n}.$$

Thus we obtain the recurrence formulae

$$M(\xi_{n+1}) = p\left(1 - (p-1)^2 \frac{h_n}{p h_{n+1}}\right) M(\xi_n) + \frac{p h_n}{h_{n+1}} \tag{12.6}$$

and

$$D^2(\xi_{n+1}) = pD^2(\xi_n) + \left(\frac{H_n'(1)}{h_n}\right)^2 (p-1)^3 \frac{h_n}{h_{n+1}} \left(1 - \frac{(p-1)h_n}{h_{n+1}}\right) - \tag{12.7}$$

$$- \frac{H_n'(1)}{h_n} 2(p-1)^2 \frac{h_n}{h_{n+1}} \left(1 - \frac{ph_n}{h_{n+1}}\right) + \frac{ph_n}{h_{n+1}} \left(1 - \frac{ph_n}{h_{n+1}}\right)$$

for $n \geqq 1$. We notice that $M(\xi_1)=1$, $D^2(\xi_1)=0$ and

$$(p-1)^2 \frac{h_n}{ph_{n+1}} \leqq \frac{(p-1)^2}{h_n^{p-1}} \leqq \frac{(p-1)^2}{p^{p-1}} \leqq \frac{1}{2}.$$

Let $h_0=1$. Similarly to the proof of Proposition 4 of [6], we obtain that, for $n \to \infty$,

$$M(\xi_n) = \lambda p^n + o(1), \tag{12.8}$$

where $\lambda=\lambda(p)$ is a constant defined by

$$\lambda = \sum_{k=1}^{\infty} \frac{1}{p^{k-1}} \cdot \frac{h_{k-1}}{h_k} \sum_{l=k+1}^{\infty} \left(1 - (p-1)^2 \frac{h_{l-1}}{ph_l}\right). \tag{12.9}$$

Next, (12.7) and (12.8) yield that, for $n \to \infty$,

$$D^2(\xi_n) = \mu^2 p^n - o(1), \tag{12.10}$$

where $\mu^2=\mu^2(p)$ is a constant defined by

$$\mu^2 = \sum_{k=2}^{\infty} \frac{1}{p^k} \cdot \frac{h_{k-1}}{h_k} \left\{ \left(\frac{H_{k-1}'(1)}{h_{k-1}}\right)^2 (p-1)^3 \left(1 - \frac{(p-1)h_{k-1}}{h_k}\right) - \right. \tag{12.11}$$

$$\left. - \frac{H_{k-1}'(1)}{h_{k-1}} 2(p-1)^2 \left(1 - \frac{ph_{k-1}}{h_k}\right) + p\left(1 - \frac{ph_{k-1}}{h_k}\right) \right\}.$$

Taking into account (8.5) and the estimate

$$H_n'(1) \leqq h_n \log_p |P_n| = h_n \frac{p^n - 1}{p-1},$$

we see that the series (12.9) and (12.11) are convergent and

$$\mu^2 > \sum_{k=2}^{\infty} \frac{1}{p^k} \cdot \frac{h_{k-1}}{h_k} \left(1 - \frac{ph_{k-1}}{h_k}\right) \left\{ (p-1)\left((p-1)\frac{H_{k-1}'(1)}{h_{k-1}} - 1\right)^2 + 1 \right\} > 0.$$

For the characteristic functions

$$\psi_n(t) = \frac{1}{h_n} e^{-\lambda p^{n/2} it/\mu} H_n(e^{it/\mu p^{n/2}})$$

of the random variables

$$\eta_n = \frac{\xi_n - \lambda p^n}{\mu p^{n/2}},$$

from (12.3) we get the recurrence formula

$$\psi_{n+1}(t)=\left(1-\frac{(p^2-1)h_n}{ph_{n+1}}\right)\psi_n^p(t/\sqrt{p})+\frac{h_n}{h_{n+1}}\left(e^{it/\mu p^{(n+1)/2}}-\frac{1}{p}\right)\psi_n(\sqrt{p}t)+$$

$$+\frac{h_n}{h_{n+1}}(p-1)\exp\left(-\frac{it}{\mu}\left(\lambda p^{(n-1)/2}(p-1)-p^{-(n+1)/2}\right)\right)\psi_n(t/\sqrt{p}).$$

By means of the arguments in § 5 of [6], we obtain

THEOREM 12. *With the positive constants* $\lambda=\lambda(p)$, $\mu=\mu(p)$ *defined by* (12.9) *and* (12.11), *for a randomly chosen conjugacy class* H *of* P_n, *we have*

$$\lim_{n\to\infty}\operatorname{Prob}\left(\log_p\frac{|P_n|}{|H|}<\lambda p^n+x\mu p^{n/2}\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-t^2/2}\,dt, \tag{12.12}$$

or equivalently,

$$\lim_{n\to\infty}\operatorname{Prob}\left(\log_p|H|<\left(\frac{1}{p-1}-\lambda\right)p^n+x\mu p^{n/2}\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-t^2/2}\,dt. \tag{12.13}$$

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ON CERTAIN SATURATION PROBLEMS

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1. Introduction

1.1. In his paper [1] G. Somorjai proved a saturation theorem for the positive interpolatory linear operators

$$L_n(f,x) := \frac{\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\left|x-\frac{k}{n}\right|^{-r}}{\sum_{k=0}^{n}\left|x-\frac{k}{n}\right|^{-r}}, \quad 0 \leqq x \leqq 1, \quad n = 1, 2, \ldots, \tag{1.1}$$

where $r>2$ is a fixed real number, $f\in C[0, 1]$. He considered the expressions

$$M_f(x) := \overline{\lim_{y\to x}} \frac{|f(y)-f(x)|}{|y-x|}, \quad M_f := \sup_{0\leqq x\leqq 1} M_f(x) \tag{1.2}$$

for arbitrary $f\in C[0, 1]$ and proved that

$$\begin{cases} M_f(x) = 0 \quad \textit{for each} \quad x\in[0, 1] \quad \textit{iff} \quad f(x) = \textit{const}, \\ M_f < +\infty \quad \textit{iff} \quad f\in \mathrm{Lip}\, 1. \end{cases} \tag{1.3}$$

(Here and later "const", $c_1, c_2, \ldots$ denote absolute positive real numbers.)

By (1.2) and using that every number $x_0\in[0, 1]$ can be approximated with certain positive fractions $\{k_t/n_t\}$ (k_t, n_t are integers) such that

$$\left|x_0-\frac{k_t}{n_t}\right| \leqq \frac{c}{n_t^2}, \quad t = 1, 2, \ldots, \tag{1.4}$$

(Dirichlet-theorem), and finally, exploiting that the fractions $\{k_t/n_t\}$ are interpolatory nodes for the operator (1.1), *he succeeded in proving that*

$$\begin{cases} \|L_n(f,x)-f(x)\| = o(1/n) \quad \textit{iff} \quad f = \mathrm{const}, \\ \|L_n(f,x)-f(x)\| = O(1/n) \quad \textit{iff} \quad f\in \mathrm{Lip}\, 1 \end{cases} \tag{1.5}$$

where $\|\cdot\|$ stands for the supremum norm.

Later in T. Hermann, P. Vértesi [2] we tried to extend his argument for other interpolatory operators, but as F. Pintér very recently remarked, [2; Lemma 3.2] was incorrect.

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1.2. The aim of this paper is to give some useful generalizations of (1.3) and (1.4) which help to get some other saturation theorems. Especially, we can use them to correct the proof of [2; Theorem 2.4] (cf. [8]).

2. Results

2.1. Using the previous notations first we verify the following

STATEMENT 2.1. *Let* $f\in C[0,1]$ *and* $E\subset[0,1]$ *a set such that* $CE:=[0,1]\setminus E$ *is countable. Then, by* $M_f(E):=\sup_{x\in E} M_f(x)$, *we get*

$$M_f(E)=M_f. \tag{2.1}$$

2.2. A simple consequence of the previous theorem is that

$$\begin{cases} M_f(x)=0 \quad \textit{for each} \quad x\in E \quad \textit{iff} \quad f(x)=\textit{const}, \\ M_f(E)<\infty \quad \textit{iff} \quad f\in \mathrm{Lip}\,1 \end{cases} \tag{2.2}$$

(see (1.3)).

2.3. Now we state a Dirichlet-type approximation relation. Namely

THEOREM 2.2. *Let* $x_0\in(0,1)$ *be a fixed irrational number,* $\{y_r\}_{r=1}^{\infty}$ *an arbitrary sequence with* $y_r\neq x_0$, $r=1,2,\ldots$ *and* $\lim_{r\to\infty} y_r=x_0$, $0<\varrho\leqq 1/3$ *and* $0\leqq\gamma$, $\delta<1$ *fixed real numbers. Then there exists a sequence* $\{x_k\}\subset\{y_r\}$ *and positive integers* $\{l_k\}_{k=1}^{\infty}$ *and* $\{n_k\}_{k=1}^{\infty}$ *with*

$$1<n_1<n_2<\ldots, \quad \text{i.e.} \quad \lim_{k\to\infty} n_k=\infty \tag{2.3}$$

such that for the fractions $\dfrac{l_k+\gamma}{n_k+\delta}$ *the relations*

$$\left|x_0-\frac{l_k+\gamma}{n_k+\delta}\right|=o\left(\frac{1}{n_k}\right), \quad k=1,2,\ldots, \tag{2.4}$$

$$\frac{\varrho}{2n_k}\leqq\left|x_k-\frac{l_k+\gamma}{n_k+\delta}\right|\leqq\frac{4\varrho}{n_k} \tag{2.5}$$

hold true.

2.4. For our purposes Theorem 2.2 is quite satisfactory. On the other hand, using its proof the following slight generalization can be proved (see Section 3.4).

THEOREM 2.3. *Let* $x_0\in(0,1)$ *be a fixed irrational number,* $\{y_r\}_{r=1}^{\infty}$ *be an arbitrary sequence with* $y_r\neq x_0$, $r=1,2,\ldots,$ $\lim_{r\to\infty} y_r=x_0$. *Further, let* $0<\varrho\leqq 1/3$ *(real),* $p,q>0$, $(p,q)=1$ *(integers),* $0\leqq\gamma<p$, $0\leqq\delta<q$ *(reals), be fixed numbers. Then there exist a sequence* $\{x_k\}\subset\{y_r\}$ *and positive integers* $\{l_k\}_{k=1}^{\infty}$ *and* $\{n_k\}_{k=1}^{\infty}$ *with*

$$1<n_1<n_2<\ldots, \quad \textit{i.e.} \quad \lim_{k\to\infty} n_k=\infty$$

such that relations

$$\left|x_0-\frac{pl_k+\gamma}{qn_k+\delta}\right| = o\left(\frac{1}{n_k}\right), \quad k=1, 2, \ldots, \tag{2.6}$$

$$\frac{\varrho}{2n_k} \leqq \left|x_k-\frac{pl_k+\gamma}{qn_k+\delta}\right| \leqq \frac{(2p+2)\varrho}{n_k}, \quad k=1, 2, \ldots \tag{2.7}$$

hold true.

2.5. As it was mentioned earlier, Theorems 2.1—2.2 are very useful in some saturation problems. They can be applied, e.g., to get the following statement.

Let

$$t_{kn} = \frac{2k\pi}{2n+1}, \quad k=0, \pm 1, \pm 2, \ldots, \tag{2.8}$$

and consider the trigonometric polynomials

$$p_n(f, t) = \sum_{k=-n}^{n} f(t_{kn}) u_{kn}(t), \quad n=1, 2, \ldots,$$

for the continuous 2π-periodic f (shortly $f \in \tilde{C}$). Here

$$u_{kn}(t) = 4l_{kn}^3(t) - 3l_{kn}^4(t), \quad k=0, \pm 1, \pm 2, \ldots,$$

where $l_{kn}(t)$ are the fundamental functions of trigonometric interpolation based on (2.8). One can prove that

$$\deg p_n \leqq 4n,$$

$$p_n(f, t_{kn}) = f(t_{kn}), \quad k=0, \pm 1, \pm 2, \ldots,$$

$$p_n(g, t) \equiv 1 \quad \text{if} \quad g(t) \equiv 1,$$

$$|p_n(f, t) - f(t)| \leqq 7\omega\left(f, \frac{1}{n}\right)$$

($\omega(f, \delta)$ is the modulus of continuity of f; cf. [6]). *Then* (see A. K. Varma, P. Vértesi [3])

$$\begin{cases} \|p_n(f, t) - f(t)\| = o\left(\frac{1}{n}\right) & \textit{iff} \quad f = \textit{const}, \\ \|p_n(f, t) - f(t)\| = O\left(\frac{1}{n}\right) & \textit{iff} \quad f \in \text{Lip } 1. \end{cases} \tag{2.9}$$

2.6. Other applications can be find in [8] and [4].

3. Proofs

3.1. PROOF OF STATEMENT 2.1.[1] As a simple application of [7; Ch. 6, § 7, Theorem 7.2.] we get that *if* $h \in C[0, 1]$ *further*

$$\overline{\lim_{y \to x+0}} \frac{h(y)-h(x)}{y-x} > 0 \quad \textit{for each} \quad x \in E \tag{3.1}$$

then h is strictly monotone increasing on [0, 1].

Now let us suppose that

$$M_f(E) < c < M_f. \tag{3.2}$$

Then, for

$$g(x) := cx - f(x), \tag{3.3}$$

$g \in C[0, 1]$, further for any fixed $x \in E$

$$\frac{g(x)-g(y)}{x-y} = c - \frac{f(x)-f(y)}{x-y} > 0 \tag{3.4}$$

if y is close enough to x (see (3.2)). Especially,

$$\overline{\lim_{y \to x+0}} \frac{g(x)-g(y)}{x-y} > 0, \quad x \in E. \tag{3.5}$$

By (3.5) and the quoted theorem, g is strictly monotone increasing i.e. for *arbitrary* $x, y \in [0, 1]$, $x \neq y$ we have

$$\frac{f(x)-f(y)}{x-y} < c, \quad x, y \in [0, 1], \quad x \neq y. \tag{3.6}$$

Applying the same argument for the function

$$t(x) := cx + f(x) \tag{3.7}$$

we get

$$\frac{f(x)-f(y)}{x-y} > -c, \quad x, y \in [0, 1], \quad x \neq y. \tag{3.8}$$

By (3.6) and (3.8)

$$\left| \frac{f(x)-f(y)}{x-y} \right| < c, \quad x, y \in [0, 1], \quad x \neq y, \tag{3.9}$$

from where

$$M_f = \sup_{0 \leq x \leq 1} \overline{\lim_{y \to x}} \left| \frac{f(x)-f(y)}{x-y} \right| \leq c, \tag{3.10}$$

a contradiction.

[1] This argument, which is much simpler than our previous one, is due to G. Petruska.

3.2. PROOF OF THEOREM 2.2. By [5; Theorem 7.11, p. 196], for any fixed irrational number $x_0 \in (0, 1)$ there exist integer numbers $\{u_s\}$, $\{v_s\}$ such that

$$(3.11)\qquad \left|x_0 - \frac{u_s}{v_s}\right| < \frac{1}{v_s v_{s+1}}, \quad (u_s, v_s) = 1, \quad 1 < v_1 < v_2 < \ldots, \quad \text{i.e.} \quad \lim_{s\to\infty} v_s = \infty.$$

Then, by $y_l \to x_0$ and $\lim_{s\to\infty} v_s = \infty$, one can find infinitely many different $x_k \in \{y_l\}$ for which

$$(3.12)\qquad \frac{\varrho}{v_{k+1}} < |x_k - x_0| \leqq \frac{\varrho}{v_k}, \quad k = k_1, k_2, \ldots.$$

If

$$(3.13)\qquad M_k := \left[\frac{\varrho}{|x_k - x_0|}\right],$$

then, if k_1 is big enough (which will be supposed from now on), by (3.12) and (3.13)

$$(3.14)\qquad \frac{1}{v_{k+1}} < \frac{1}{M_k} < \frac{2}{v_k}, \quad k = k_1, k_2, \ldots.$$

Now let $A_k := v_k\gamma - u_k\delta$. For fixed u_k and v_k consider the expression

$$\left|\frac{u_k}{v_k} - \frac{l+\gamma}{n+\delta}\right| = \left|\frac{nu_k - lv_k - A_k}{v_k(n+\delta)}\right|.$$

3.3. Here, by $(u_k, v_k) = 1$, the Diophantine equation $nu_k - lv_k = 1$ is solvable; let (n^*, l^*) be a solution. Take $n_0 := n^*[A_k]$ and $l_0 := l^*[A_k]$. Then, by $n^*u_k - l^*v_k = 1$, we get as follows.

$$(3.15)\qquad \left|\frac{u_k}{v_k} - \frac{l_0+\gamma}{n_0+\delta}\right| = \left|\frac{n_0u_k - l_0v_k - A_k}{v_k(n_0+\delta)}\right| = \left|\frac{[A_k](n^*u_k - l^*v_k) - A_k}{v_k(n_0+\delta)}\right| \leqq \frac{1}{|v_k(n_0+\delta)|}.$$

Now we state that by a proper shifting of n_0 and l_0 we get the solution (n_k, l_k) with the relations

$$(3.16)\qquad M_k \leqq n_k < 3M_k,$$

$$(3.17)\qquad \left|\frac{u_k}{v_k} - \frac{l_k+\gamma}{n_k+\delta}\right| \leqq \frac{1}{v_k(n_k+\delta)}.$$

a) Indeed, if $n_0 < M_k$, consider $n' := n_0 + v_k$ and $l' := l_0 + u_k$. By definition $n'u_k - l'v_k = n_0u_k - l_0v_k = [A_k]\ (n^*u_k - l^*v_k) = [A_k]$, i.e. we have (3.15), if we replace (n_0, l_0) by (n', l'). If $n' \geqq M_k$, then by (3.14)

$$n_0 < M_k \leqq n_0 + v_k(=n') < M_k + v_k \leqq M_k + 2M_k = 3M_k,$$

i.e. for $n_k := n'$ we have both (3.16) and (3.17). On the other hand if $n' < M_k$, with a proper integer t, $n_0 + (t-1)v_k < M_k \leqq n_0 + tv_k$. Then $n_k := n_0 + tv_k$ and $l_k := l_0 + tu_k$ will satisfy (3.16) and (3.17).

b) If $n_0 \in [M_k, 3M_k)$, by $n_k := n_0$ and $l_k := l_0$ we get (3.16) and (3.17). If $n_0 \geqq 3M_k$, with a proper integer $\tilde{q}$ let $n_k := n_0 - \tilde{q}v_k$, $l_k := l_0 - \tilde{q}u_k$ be chosen such that $n_k - v_k < M_k \leqq n_k$. Then, as above we can verify (3.16) and (3.17).

Now, by (3.11), (3.17), (3.14), (3.16) and $\lim_{k\to\infty} v_k = \infty$

$$(3.18) \qquad \left|x_0 - \frac{l_k+\gamma}{n_k+\delta}\right| \leqq \left|x_0 - \frac{u_k}{v_k}\right| + \left|\frac{u_k}{v_k} - \frac{l_k+\gamma}{n_k+\delta}\right| < \frac{1}{v_k v_{k+1}} +$$

$$+ \frac{2}{v_k n_k} < \frac{1}{v_k M_k} + \frac{2}{v_k n_k} \leqq \frac{3}{v_k n_k} + \frac{2}{v_k n_k} = o\left(\frac{1}{n_k}\right), \quad k = k_1, k_2, \ldots,$$

which actually gives (2.4). To get (2.5) we write for any fixed $k = k_1, k_2, \ldots, k_1 \geqq k_0$,

$$(3.19) \qquad \left|x_k - \frac{l_k+\gamma}{n_k+\delta}\right| \geqq \left||x_k - x_0| - \left|x_0 - \frac{l_k+\gamma}{n_k+\delta}\right|\right| < \frac{0.9\varrho}{M_k} - \frac{\varrho^2}{n_k} > \frac{\varrho}{2n_k}$$

(see (3.13) and (3.18)), further

$$\left|x_k - \frac{l_k+\gamma}{n_k+\delta}\right| \leqq |x_k - x_0| + \left|x_0 - \frac{l_k+\gamma}{n_k+\delta}\right| \leqq \frac{1.1\varrho}{M_k} + \frac{\varrho^2}{n_k} < \frac{4\varrho}{n_k}$$

(see (3.13) and (3.18)) which complete the proof.

3.4. PROOF OF THEOREM 2.3. Being very similar to the previous one, we only sketch it. The first difference is that instead of $nu_k - lv_k = 1$, we solve $qu_k n - pv_k l = B_k$, where $B_k := (qu_k, pv_k)$ (cf. 3.3.). By $(u_k, v_k) = (p, q) = 1$, it is easy to see that $B_k \leqq p \cdot q$. Then, if (n^*, l^*) are solutions, with

$$n_0 = n^*\left[\frac{A_k}{B_k}\right] \quad \text{and} \quad l_0 := l^*\left[\frac{A_k}{B_k}\right]$$

we get that

$$\left|\frac{u_k}{v_k} - \frac{pl_0+\gamma}{qn_0+\delta}\right| \leqq \frac{B_k}{v_k(qn_0+\delta)} \leqq \frac{pq}{v_k(qn_0+\delta)}$$

(cf. (3.15)). Now if $n_0 < M_k$, say, then we take $n' := n_0 + pv_k$ and $l' := l_0 + q\mu_k$ finally we get $M_k \leqq n_k < (2p+1)M_k$ (cf. (3.16)). We omit the further details.

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A DECIDABILITY RESULT FOR HOMOMORPHIC REPRESENTATION OF AUTOMATA BY α_0-PRODUCTS

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It is an open problem for a long time if it is decidable for a finite automaton **A** and a finite class $\mathscr{K}$ of finite automata whether or not **A** can be homomorphically represented by an α_0-product of automata from $\mathscr{K}$. Here we give a partial solution for this question. Namely, it will be shown that the above representation problem is decidable whenever **A** is not metrically complete. Since none of the quasi-nilpotent and nilpotent automata is metrically complete, we obtain decidability for two important types of automata.

1. Preliminaries

In this paper an automaton is a system $\mathbf{A}=(A, X, \delta)$ where A is the nonempty set of states, X is the finite nonempty set of input signs and $\delta: A\times X\to A$ is the transition function. Thus, the input set is always finite but the state set may be infinite. If A is finite as well, we call **A** a finite automaton. The transition function is also used in the extended sense, i.e. as a mapping $\delta: A\times X^*\to A$. If no confusion arises we write ap for $\delta(a, p)$, $a\in A$, $p\in X^*$. For the definition of the α_0-product see [5]. If $\mathscr{K}$ is a class of automata we put

$\mathbf{P}_{\alpha_0}(\mathscr{K})$:= all α_0-products of automata from $\mathscr{K}$,
$\mathbf{P}_{f\alpha_0}(\mathscr{K})$:= all finite α_0-products of automata from $\mathscr{K}$,
$\mathbf{P}(\mathscr{K})$:= all direct products of automata from $\mathscr{K}$,
$\mathbf{P}_f(\mathscr{K})$:= all finite direct products of automata from $\mathscr{K}$,
$\mathbf{S}(\mathscr{K})$:= all subautomata of automata from $\mathscr{K}$,
$\mathbf{H}(\mathscr{K})$:= all homomorphic images of automata from $\mathscr{K}$.

Let $\mathscr{K}$ be a class of automata. We call $\mathscr{K}$ an α_0-variety if $\mathscr{K}$ is closed under **H**, **S** and $\mathbf{P}_{\alpha_0}$. Similarly, a class $\mathscr{K}$ of finite automata is referred to an $f\alpha_0$-variety if it is closed under **H**, **S** and $\mathbf{P}_{f\alpha_0}$. Since the direct product is a special case of the α_0-product, $\mathbf{P}(\mathscr{K})\subseteq\mathscr{K}$ for any α_0-variety $\mathscr{K}$, and $\mathbf{P}_f(\mathscr{K})\subseteq\mathscr{K}$ for $f\alpha_0$-varieties. Given a class $\mathscr{K}$, $\mathbf{HSP}_{\alpha_0}(\mathscr{K})$ is the smallest α_0-variety containing $\mathscr{K}$. Similarly, if $\mathscr{K}$ is a class of finite automata, $\mathbf{HSP}_{f\alpha_0}(\mathscr{K})$ is exactly the $f\alpha_0$-variety generated by $\mathscr{K}$.

To study α_0-varieties, we introduced p-identities in [1]. By definition, a p-identity has the form

(1) $(m, k)=(m, l)$, or
(2) $m=n$,

where m, n, k and l are nonnegative integers. It is said that an automaton satisfies p-identity (1), or (1) holds in $\mathbf{A}$, if for arbitrary $u, v, w \in X^*$ with length m, k and l, respectively, in notation $|u|=m$, $|v|=k$, $|w|=l$, the usual identity $zuv=zuw$ holds in $\mathbf{A}$ where z is any variable. Similarly, (2) holds in $\mathbf{A}$ if for every $u, v \in X^*$ with $|u|=m$, $|v|=n$, the identity $z_1u=z_2v$ is satisfised by $\mathbf{A}$, where z_1 and z_2 are distinct variables. Let $\mathcal{K}$ be a class of automata. We denote by $\mathcal{K}^*$ the set of all p-identities that hold in any $\mathbf{A} \in \mathcal{K}$. Conversely, if Σ is a set of p-identities, we put Σ^* to denote the class of those automata $\mathbf{A}$ satisfying all p-identities contained in Σ. In [1] we proved the following result:

THEOREM 1.1 [1]. $\mathbf{HSP}_{\alpha_0}(\mathcal{K})=\mathbf{HSPP}_{f\alpha_0}(\mathcal{K})=\mathcal{K}^{**}$ *for any class* $\mathcal{K}$. □

COROLLARY 1.2. *The following are equivalent for every class* $\mathcal{K}$*:*
(i) $\mathcal{K}$ *is an* α_0*-variety,*
(ii) $\mathcal{K}$ *is closed under* $\mathbf{H}$, $\mathbf{S}$, $\mathbf{P}$ *and* $\mathbf{P}_{f\alpha_0}$,
(iii) $\mathcal{K}=\Sigma^*$ *for a set* Σ *of p-identities,*
(iv) $\mathcal{K}=\mathcal{K}^{**}$. □

Or even, we pointed out that every α_0-variety has a finite base of p-identities. For reasons explained elsewhere (see [4], [5]), we call a class $\mathcal{K}$ metrically complete if $\mathbf{HSP}_{\alpha_0}(\mathcal{K})=\emptyset^*$. Equivalently, this means that only trivial p-identities are satisfied by $\mathcal{K}$. (A p-identity is called trivial if it is of the form $(n, 0)=(n, 0)$.)

Fact 1.3 [1]. For every class $\mathcal{K}$, either $\mathcal{K}$ is metrically complete or $\mathbf{HSP}_{\alpha_0}(\mathcal{K})=\Sigma^*$ for a finite set Σ of p-identities of one of the following 3 forms:

(1) $$\Sigma\colon\ (n, 1)=(n, 1),\quad (n_1, k_1)=(n_1, k_1), \ldots, (n_t, k_t)=(n_t, k_t)$$

with $n_i+k_i \leqq n$, $i=1, \ldots, t$, $t \geqq 0$;

(2) $$\Sigma\colon\ (n, m)=(n, 0),\quad (n_1, k_1)=(n_1, k_1), \ldots, (n_t, k_t)=(n_t, k_t)$$

with $m>0$, $n_i+k_i \leqq n$, $i=1, \ldots, t$, $t \geqq 0$;

(3) $$\Sigma\colon\ n=n,\ (n_1, k_1)=(n_1, k_1), \ldots, (n_t, k_t)=(n_t, k_t),\ n_i+k_i \leqq n,$$

$i=1, \ldots, t$, $t \geqq 0$.

If in addition $\mathcal{K}$ is finite and consists of finite automata then either $\mathcal{K}$ is metrically complete or $\mathbf{HSP}_{\alpha_0}(\mathcal{K})=\Sigma^*$ with Σ of the form (2) or (3) above, and Σ is effectively constructible. Further, it is decidable if $\mathcal{K}$ is metrically complete. □

Let $\mathcal{K}$ be a finite class of finite automata. Suppose that $\mathcal{K}$ is not metrically complete. Given a finite nonvoid set X, denote by $\mathcal{K}_X$ the class of all automata having X as input set. The class $\mathcal{K}'=\mathbf{HSP}_{\alpha_0}(\mathcal{K}) \cap \mathcal{K}_X=\mathbf{HSPP}_{f\alpha_0}(\mathcal{K}) \cap \mathcal{K}_X=\mathbf{HSP}(\mathbf{P}_{f\alpha_0}(\mathcal{K}) \cap \mathcal{K}_X)$ is a variety of automata in the ordinary sense. Since $\mathbf{HSP}_{\alpha_0}(\mathcal{K})=\Sigma^*$ with Σ of type (2) or (3) above, it is easily seen that every finitely generated free automaton is finite in $\mathcal{K}'$. By making use of this fact and Theorem 1.1, the following was shown as a part of the proof of the Theorem in [3]:

Fact 1.4 [3]. Let $\mathcal{K}$ be a finite class of finite automata that is not metrically complete. Then a finite automaton $\mathbf{A}$ is in $\mathbf{HSP}_{f\alpha_0}(\mathcal{K})$ if and only if $\mathbf{A}$ is in $\mathbf{HSP}_{\alpha_0}(\mathcal{K})$. □

In the sequel we shall make use of the following simple finite automata. Let $n, r \geqq 0$ and $m \geqq 1$ be integers.

$$\mathbf{C}_{n,m} = (\{0, \ldots, n+m-1\}, \{x\}, \delta)$$

with $\delta(i, x) = i+1$, $i = 0, \ldots, n+m-2$, $\delta(n+m-1, x) = n$.

$$\mathbf{F}_r = (\{0, \ldots, r, r+1, (r+1)'\}, \{x, y\}, \delta)$$

with

$$\delta(i,x) = i+1,\ i = 0, \ldots, r,\ \delta(r+1, x) = \delta(r+1, y) = r+1,\ \delta(i, y) = i+1,\ i = 0, \ldots, r-1,$$

$$\delta(r, y) = (r+1)',\ \delta((r+1)', x) = \delta((r+1)', y) = (r+1)'.$$

$$\mathbf{D}_0 = (\{0, 1\}, \{x\}, \delta),\quad \delta(i, x) = i,\quad i = 1, 2.$$

2. The result

Recall that a class $\mathscr{K}$ is called metrically complete if $\mathscr{K}^*$ contains only trivial p-identities. Accordingly, an automaton $\mathbf{A}$ is said to be metrically complete if only trivial p-identities hold in $\mathbf{A}$.

THEOREM 2.1. *Let $\mathscr{K}$ be a finite class of finite automata and $\mathbf{A}$ a finite automaton that is not metrically complete. It is then decidable if* $\mathbf{A} \in \mathbf{HSP}_{f\alpha_0}(\mathscr{K})$.

PROOF. Obviously, $\mathbf{A} \in \mathbf{HSP}_{f\alpha_0}(\mathscr{K})$ if and only if $\mathbf{HSP}_{f\alpha_0}(\{\mathbf{A}\}) \subseteq \mathbf{HSP}_{f\alpha_0}(\mathscr{K})$. By Fact 1.4, $\mathbf{HSP}_{f\alpha_0}(\{\mathbf{A}\})$ consists of all finite members of the α_0-variety $\mathbf{HSP}_{\alpha_0}(\{\mathbf{A}\})$. By Fact 1.3, $\mathbf{HSP}_{\alpha_0}(\{\mathbf{A}\}) = \Sigma^*$ for an effectively constructible set of p-identities. We distinguish two cases.

Case 1. Σ: $(n, m) = (n, 0)$, $(n_1, k_1) = (n_1, k_1)$, ..., $(n_t, k_t) = (n_t, k_t)$ with $m > 0$, $k_i > 0$, $n_i + k_i \leqq n$, $i = 1, \ldots, t$, $t \geqq 0$.

We claim that $\mathbf{A} \in \mathbf{HSP}_{f\alpha_0}(\mathscr{K})$ if and only if the following hold:

(1) For every p-identity $(n', k') = (n', k')$ with $n' + k' < n$, if $(n', k') = (n', k')$ is not in $\{\mathbf{A}\}^*$ then it is not in $\mathscr{K}^*$ either,

(2) $\mathbf{C}_{n,m} \in \mathbf{HSP}_{f\alpha_0}(\mathscr{K})$,

(3) $\mathbf{D}_0 \in \mathbf{HSP}_{f\alpha_0}(\mathscr{K})$,

(4) Let $n_0 = \min\{n_i, n | i = 1, \ldots, t\}$. If $n_0 > 0$ then $\mathbf{F}_r \in \mathbf{HSP}_{f\alpha_0}(\mathscr{K})$ with $r = n_0 - 1$.

The necessity of (1) is obvious by Theorem 1.1. Since all the p-identities in Σ are satisfied by both automata $\mathbf{C}_{n,m}$ and $\mathbf{D}_0$, $\mathbf{C}_{n,m}, \mathbf{D}_0 \in \mathbf{HSP}_{\alpha_0}(\{\mathbf{A}\})$. Similarly, $\mathbf{F}_r \in \mathbf{HSP}_{\alpha_0}(\{\mathbf{A}\})$ if $n_0 > 0$. Apply Fact 1.4.

For the converse we are going to construct a finite class $\mathscr{K}_0 \subseteq \mathbf{HSP}_{f\alpha_0}(\mathscr{K})$ that is not metrically complete and such that $\mathscr{K}_0^* \subseteq \{\mathbf{A}\}^*$. It follows then that $\mathbf{A} \in \mathbf{HSP}_{\alpha_0}(\mathscr{K}_0)$, and by Fact 1.4, $\mathbf{A} \in \mathbf{HSP}_{f\alpha_0}(\mathscr{K}_0)$. Since $\mathscr{K}_0 \subseteq \mathbf{HSP}_{f\alpha_0}(\mathscr{K})$, also $\mathbf{A} \in \mathbf{HSP}_{f\alpha_0}(\mathscr{K})$.

Let $(n', k') = (n', k')$ be a p-identity not in $\{\mathbf{A}\}^*$ and such that $n' + k' < n$. Let $\mathbf{B} \in \mathscr{K}$ be an automaton in which this p-identity does not hold. It is easy to con-

struct an α_0-product of $\mathbf{C}_{n,1}$ with $\mathbf{B}$ (in fact a quasi-direct product suffices) such that a homomorphic image $\mathbf{B}'$ of the product

(a) satisfies the p-identity $(n, 1)=(n, 0)$,
(b) does not satisfy $(n', k')=(n', k')$.

Let $\mathscr{K}_0$ be the collection of all these automata $\mathbf{B}'$ and the automata $\mathbf{D}_0$, $\mathbf{C}_{n,m}$, $\mathbf{F}_r$, the latter only if $n_0>0$. Since $\mathbf{C}_{n,1}$ is a homomorphic image of $\mathbf{C}_{n,m}$ we have $\mathscr{K}_0 \subseteq \mathbf{HSP}_{f\alpha_0}(\mathscr{K})$. Further, $\mathscr{K}_0$ is not metrically complete because $(n, 1)=(n, 0)\in\mathscr{K}_0^*$.

Take a nontrivial p-identity in $\mathscr{K}_0^*$. Since $\mathbf{D}_0\in\mathscr{K}_0$, it can only be one of the p-identities $(n', k')=(n', l')$. Since $\mathbf{F}_r\in\mathscr{K}_0$ if $n_0>0$, it follows that $n'\geqq n_0$ regardless whether $n_0>0$ or $n_0=0$. Since $\mathbf{C}_{n,m}\in\mathscr{K}_0$ either $n'+k'$, $n'+l'\geqq n$ and m divides $k'-l'$ or $n'+k'=n'+l'<n$. In the first case $(n', k')=(n', l')$ is a consequence of the p-identities $(n, m)=(n, 0)$ and $(n_0, n-n_0)=(n_0, n-n_0)$, and is therefore satisfied by $\mathbf{A}$. (Note that $(n_0, n-n_0)=(n_0, n-n_0)$ is a consequence of the p-identities in Σ.) Assume that the p-identity is $(n', k')=(n', k')$ with $n'+k'<n$. Since it is satisfied by all the automata $\mathbf{B}'$ constructed above, it is in $\{\mathbf{A}\}^*$.

Case 2. Σ: $n=n$, $(n_1, k_1)=(n_1, k_1)$, ..., $(n_t, k_t)=(n_t, k_t)$ with $n_i+k_i\leqq n$, $k_i>0$, $i=1, ..., t$, $t\geqq 0$.

We now have the following equivalence: $\mathbf{A}\in\mathbf{HSP}_{f\alpha_0}(\mathscr{K})$ if and only if

(1) None of the p-identities $(n', k')=(n', k')$ with $n'+k'<n$ which does not hold in $\mathbf{A}$ is satisfied by $\mathscr{K}$,
(2) $\mathbf{C}_{n,1}\in\mathbf{HSP}_{f\alpha_0}(\mathscr{K})$.

The necessity of these conditions can be seen as in the previous case. For the converse we can again assign an automaton $\mathbf{B}'\in\mathbf{HSP}_{f\alpha_0}(\mathscr{K})$ to every p-identity $(n', k')=(n', k')\notin\{\mathbf{A}\}^*$ with $n'+k'<n$ such that

(a) $\mathbf{B}'$ satisfies the p-identity $(n, 1)=(n, 0)$,
(b) $\mathbf{B}'$ does not satisfy $(n', k')=(n', k')$.

Let $\mathbf{C}'_{n,1}$ be the automaton $(\{0, ..., n, 0', ..., (n-1)'\}, \{x\}, \delta)$, $\delta(i, x)=i+1$, $i=0, ..., n-1$, $\delta(i', x)=(i+1)'$, $i=0, ..., n-2$, $\delta((n-1)', x)=n$, $\delta(n, x)=n$. Obviously, $\mathbf{C}'_{n,1}$ is isomorphic to a subautomaton of the second direct power of $\mathbf{C}_{n,1}$. Let $\mathscr{K}_0$ be the collection of all the automata $\mathbf{B}'$ and $\mathbf{C}'_{n,1}$. $\mathscr{K}_0$ is finite, not metrically complete and $\mathscr{K}_0\subseteq\mathbf{HSP}_{f\alpha_0}(\mathscr{K})$. It suffices to show that $\mathscr{K}_0^*\subseteq\{\mathbf{A}\}^*$.

Suppose a p-identity $n'=m'$ is in $\mathscr{K}_0^*$. Since $\mathbf{C}'_{n,1}\in\mathscr{K}_0$, $n', m'\geqq n$. Consequently, $n'=m'\in\{\mathbf{A}_0\}^*$. Next take a nontrivial p-identity $(n', k')=(n', l')$ in $\mathscr{K}_0^*$. Since $\mathbf{C}'_{n,1}\in\mathscr{K}_0$, either $n'+k'$, $n'+l'\geqq n$ or $n'+k'=n+l'<n$. Both cases yield $(n', k')=(n', l')\in\{\mathbf{A}_0\}^*$.

Condition (1) is obviously decidable in both cases. To complete the proof we must establish decidability for any of the automata $\mathbf{C}_{n,m}$, $\mathbf{D}_0$ and $\mathbf{F}_r$. As regards $\mathbf{C}_{n,m}$ and $\mathbf{D}_0$, it has been done in [2]. The decidability of $\mathbf{F}_r\in\mathbf{HSP}_{f\alpha_0}(\mathscr{K})$ is proved later, see Lemma 2.4. □

For arbitrary integers $n\geqq 0$, $k_1, m_1, k_2, m_2>0$ let $\mathbf{C}_{n,k_1,m_1,k_2,m_2}$ denote the automaton with state set

$$\{0, ..., n, n+1, ..., n+k_1, n+k_1+1, ..., n+k_1+m_1-1,$$

$$(n+1)', ..., (n+k_2)', (n+k_2+1)', ..., (n+k_2+m_2-1)'\},$$

input set $\{x, y\}$, and transition δ defined as follows:

(i) $$\delta(i, x) = \begin{cases} i+1 & \text{if } 0 \leqq i < n+k_1+m_1-1, \\ n+k_1 & \text{if } i = n+k_1+m_1-1, \end{cases}$$

(ii) $$\delta(i', x) = \begin{cases} (i+1)' & \text{if } n+1 \leqq i < n+k_2+m_2-1, \\ (n+k_2)' & \text{if } i = n+k_2+m_2-1, \end{cases}$$

(iii) $$\delta(i, y) = \begin{cases} \delta(i, x) & \text{if } 0 \leqq i \leqq n+k_1+m_1-1, \quad i \neq n, \\ (n+1)' & \text{if } i = n, \end{cases}$$

(iv) $\delta(i', y) = \delta(i', x)$, all i with $n+1 \leqq i \leqq n+k_2+m_2-1$.

We shall write $\mathbf{C}_{n,k,m}$ for $\mathbf{C}_{n,k,m,k,m}$. Notice that $\mathbf{F}_n = \mathbf{C}_{n,1,1}$.

LEMMA 2.2. *Let n be a nonnegative integer and $\mathcal{K}$ a class of finite automata. If $\mathbf{F}_n \in \mathbf{HSP}_{f\alpha_0}(\mathcal{K})$ then the following hold for some positive integers m and k:*

(i) $(n, k) = (n, k)$ *is not in* $\mathcal{K}^*$,

(ii) $\mathbf{C}_{n+k,m} \in \mathbf{HSP}_{f\alpha_0}(\mathcal{K})$,

(iii) *There exist* $\mathbf{A} = (A, X, \delta) \in \mathcal{K}$, $a_0, a_1, a_2 \in A$, $u, v_1, v_2, w \in X^*$ *with* $|u| = n$, $|v_1| = |v_2| = k$ *and* $|w| = m$ *for which*

(a) $a_1 \neq a_2$,

(b) $a_0 u v_1 = a_1$, $a_0 u v_2 = a_2$,

(c) $a_1 w = a_1$, $a_2 w = a_2$.

PROOF. Consider all α_0-products

$$\mathbf{B} = \mathbf{C}_{n+i,j} \times \mathbf{A}_1 \times \ldots \times \mathbf{A}_l[\{x, y\}, \varphi]$$

$(i, j \geqq 1)$ such that $\mathbf{B}$ has an initially connected subautomaton with initial state b_0 that can be mapped homomorphically onto $\mathbf{F}_n$, and such that $pr_1(b_0) = 0$, $\mathbf{C}_{n+i,j} \in \mathbf{HSP}_{f\alpha_0}(\mathcal{K})$, $\mathbf{A}_t = (A_t, X_t, \delta_t) \in \mathcal{K}$, $t = 1, \ldots, l$. Since $\mathbf{F}_n \in \mathbf{HSP}_{f\alpha_0}(\mathcal{K})$ implies $\mathbf{C}_{n+1,1} \in \mathbf{HSP}_{f\alpha_0}(\mathcal{K})$, there is at least one such $\mathbf{B}$. Especially, choose $\mathbf{B}$ with minimal number of factors. Thus, $\mathbf{F}_n \notin \mathbf{HS}(\{\overline{\mathbf{B}}\})$ for any α_0-product

$$\overline{\mathbf{B}} = \mathbf{C}_{n+i,j} \times \mathbf{A}'_1 \times \ldots \times \mathbf{A}'_r[\{x, y\}, \overline{\varphi}]$$

with $\mathbf{C}_{n+i,j} \in \mathbf{HSP}_{f\alpha_0}(\mathcal{K})$, $\mathbf{A}'_1, \ldots, \mathbf{A}'_r \in \mathcal{K}$, $r < l$.

Define the α_0-product

$$\mathbf{B}' = \mathbf{C}_{n+i,j} \times \mathbf{A}_1 \times \ldots \times \mathbf{A}_l[\{x, y\}, \varphi']$$

by $\varphi'_1 = \varphi_1$ and

$$\varphi'_{1+r}(s, b_1, \ldots, b_{r-1}, x) = \varphi_{1+r}(s, b_1, \ldots, b_{r-1}, x),$$

$$\varphi'_{1+r}(s, b_1, \ldots, b_{r-1}, y) = \begin{cases} \varphi_{1+r}(s, b_1, \ldots, b_{r-1}, x) & \text{if } s \neq n, \\ \varphi_{1+r}(x, b_1, \ldots, b_{r-1}, y) & \text{if } s = n, \end{cases}$$

where $1 \leqq r \leqq l$, $0 \leqq s \leqq n+i+j-1$, $b_1 \in A_1, \ldots, b_l \in A_l$. It should be obvious that the subautomaton generated by b_0 in $\mathbf{B}'$ is isomorphic to an automaton $\mathbf{C}_{n,i_1,j_1,i_2,j_2}$ with $i \geqq i_1, i_2$, $j \mid j_1, j_2$. Put $k = \max\{i_1, i_2\}$, $m = \text{l.c.m.}\{j_1, j_2\}$. Since

$$\mathbf{C}_{n,i_1,j_1,i_2,j_2} \in \mathbf{HSP}_{f\alpha_0}(\mathcal{K}),$$

both automata $\mathbf{C}_{n+i_1,j_1}$ and $\mathbf{C}_{n+i_2,j_2}$ are in $\mathbf{HSP}_{f\alpha_0}(\mathscr{K})$. From this follows that $\mathbf{C}_{n+k,m}\in\mathbf{HSP}_{f\alpha_0}(\mathscr{K})$, i.e. condition (ii) is fulfilled. Since the p-identity $(n,k)=(n,k)$ does not hold in $\mathbf{C}_{n,i_1,j_1,i_2,j_2}$, it does not hold in $\mathscr{K}$ either. This proves (i). For (iii), let h be the unique homomorphism $\mathbf{C}_{n+k,m}$ onto $\mathbf{C}_{n+i,j}$. Form the α_0-product

$$\mathbf{B}''=\mathbf{C}_{n+k,m}\times\mathbf{A}_1\times\ldots\times\mathbf{A}_1[\{x,y\},\varphi'']$$

by defining $\varphi''_1=\varphi'_1$ and

$$\varphi''_{1+r}(s,b_1,\ldots,b_{r-1},z)=\varphi'_{1+r}(h(s),b_1,\ldots,b_{r-1},z)$$

for all $1\leqq r\leqq l$, $0\leqq s\leqq n+k+m-1$, $b_1\in A_1,\ldots,b_l\in A_l$ and $z=x,y$. The subautomaton $\mathbf{B}_0=(B_0,\{x,y\},\delta_0)$ generated by b_0 in $\mathbf{B}''$ is obviously isomorphic to $\mathbf{C}_{n,k,m}$. Now define the congruence Θ on B_0 by $b\Theta c$ if and only if $pr_1(b)=pr_1(c),\ldots,pr_l(b)=pr_l(c)$, i.e. the first l coordinates pairwise agree. The special choice of $\mathbf{B}$ yields that $\mathbf{B}_0/\Theta$ is isomorphic to a factor automaton of $\mathbf{C}_{n,k,m}$ that collapses $n+k$ and $(n+k)'$. It follows that $\mathbf{A}_l$ satisfies (iii).

The converse of Lemma 2.2 holds, too.

LEMMA 2.3. *Let $k,m\geqq 1$ and $n\geqq 0$ be integers, and $\mathscr{K}$ a class of finite automata. Assume that the conclusions of Lemma* 4.2 *are satisfied. Then* $\mathbf{F}_n\in\mathbf{HSP}_{f\alpha_0}(\mathscr{K})$.

PROOF. Since $\mathscr{K}$ does not satisfy the p-identity $(n,k)=(n,k)$, it does not satisfy any p-identity $(n,l)=(n,l)$ with $1\leqq l\leqq k$. Moreover, by our assumptions, $\mathbf{C}_{n+k,m}\in\mathbf{HSP}_{f\alpha_0}(\mathscr{K})$. Let $\mathbf{B}=\mathbf{C}_{n,k,m}/\Theta$, where Θ is the congruence generated by the pair $((n+k),(n+k)')$. It is easy to see that $\mathbf{B}\in\mathbf{HSP}_{f\alpha_0}(\mathscr{K})$. Further, it is routine work to show that a subautomaton of an α_0-product of $\mathbf{B}$ by $\mathbf{A}$ (in condition (iii) of Lemma 2.2) can be mapped homomorphically onto $\mathbf{F}_n$. □

Based on Lemmas 2.2 and 2.3, we show that for a finite class $\mathscr{K}$ of finite automata and a nonnegative integer n, the inclusion $\mathbf{F}_n\in\mathbf{HSP}_{f\alpha_0}(\mathscr{K})$ is decidable.

LEMMA 2.4. *Let $\mathscr{K}$ be a finite class of finite automata and $n\geqq 0$ an integer. There exists an algorithm to decide if* $\mathbf{F}_n\in\mathbf{HSP}_{f\alpha_0}(\mathscr{K})$.

PROOF. Let $p_1,\ldots,p_l$ be all the prime numbers less than or equal to the maximum of the state numbers of automata in $\mathscr{K}$. Set

$$L_1(\mathscr{K})=\{p_{i_1}\ldots p_{i_t}|\mathbf{C}_{0,m}\in\mathbf{HSP}_{f\alpha_0}(\mathscr{K}),\ m=p_{i_1}\cdot\ldots\cdot p_{i_t}\}.$$

As shown in [2], $L_1(\mathscr{K})$ is a regular language and can be given effectively. Moreover, let

$$L_2(\mathscr{K})=\{x^k|\mathbf{C}_{n+k,1}\in\mathbf{HSP}_{f\alpha_0}(\mathscr{K})\}.$$

By the Remark following Corollary 11 in [2], $L_2(\mathscr{K})$ is also an effectively regular language. Further, let

$$L_3(\mathscr{K})=\{x^k|(n,k)=(n,k)\not\in\mathscr{K}^*\}.$$

From Fact 1.3, it follows that $L_3(\mathscr{K})$ is an effectively regular language. (If $\mathscr{K}$ is metrically complete, then none of the non-trivial p-identities hold in $\mathscr{K}$. Otherwise construct the base Σ as stated in Fact 1.3. If Σ contains a p-identity $r=r$ or

$(r, s)=(r, 0)$ with $s>0$ then for every k with $n+k\geqq r$, $(n, k)=(n, k)\in\mathscr{K}^*$ if and only if $n\geqq r$ or $n<r$ and $(n, r-n)=(n, r-n)\in\mathscr{K}^*$.)

Finally, for fixed $\mathbf{A}=(A, X, \delta)\in\mathscr{K}$ and $a_1, a_2\in A$ with $a_1\neq a_2$ consider the language

$$L_{a_1,a_2} = \{x^k p_{i_1}\dots p_{i_t} | \exists a_0', a_0\in A,\ u, v_1, v_2, w\in X^*,$$
$$|u| = n, |v_1| = |v_2| = k,\ |w| = m, m = p_{i_1}\cdot\dots\cdot p_{i_t},$$
$$a_0' u = a_0,\ a_0 v_1 = a_1,\ a_0 v_2 = a_2,\ a_1 w = a_1,\ a_2 w = a_2\}.$$

We give a regular gramar $G=(N, T, S, P)$ generating L_{a_1,a_2}. Let $N=\{S\}\cup$ $\cup P(\{\delta_u | u\in X^*\})\cup(A\times A)$, where P is the power-set operator, and $T=\{x\}\cup$ $\cup\{p_1, \dots, p_l\}$. The productions of G are given by the following five rules:

(1) If for $b, c\in A$ there are $x', y'\in X$ with $\delta(b, x')=a_1$ and $\delta(c, y')=a_2$, then the production $S\to x(b, c)$ is in P.

(2) For all $b_1, b_2, c_1, c_2\in A$ and $x', y'\in X$ with $\delta(b_2, x')=b_1$ and $\delta(c_2, y')=c_1$ the production $(b_1, c_1)\to x(b_2, c_2)$ is in P.

(3) If there are $a_0, a_0'\in A$ and $u\in X^n$ such that $a_0' u=a_0$, then $(a_0, a_0)\to$ $\to\{\delta_y | y\in X\}$ is in P.

(4) For all $U\in P(\{\delta_u | u\in X^*\})$ and p_i $(1\leqq i\leqq l)$, the production

$$U\to p_i\{\delta_{u_1}\dots\delta_{u_{p_i}} | \delta_{u_1}, \dots, \delta_{u_{p_i}}\in U\}$$

is in P.

(5) If for a $U\in P(\{\delta_u | u\in X^*\})$ there is a $\delta_u\in U$ with $\delta_u(a_1)=a_1$ and $\delta_u(a_2)=a_2$, then $U\to\lambda$ is in P (λ is the empty word).

It is immediate to see that $\mathbf{F}_n\in\mathbf{HSP}_{f\alpha_0}(\mathscr{K})$ if and only if

$$((L_2(\mathscr{K})\cap L_3(\mathscr{K}))L_1(\mathscr{K})\cap L_{a_1,a_2}\neq\emptyset$$

for an automaton $\mathbf{A}\in\mathscr{K}$ and two distinct states a_1, a_2 of A. □

Let us call automaton $\mathbf{A}$ quasi-nilpotent [nilpotent] if the p-identity $(n, 1)=$ $=(n, 0)$ $[n=n]$ holds in $\mathbf{A}$ for some $n\geqq 0$. Note that every nilpotent automaton is quasi-nilpotent, and our definition of a nilpotent automaton is equivalent to the usual one, see e.g. [6]. Since none of the quasi-nilpotent automata is metrically complete, from Theorem 2.1 we directly obtain:

COROLLARY 2.5. *Let* $\mathbf{A}$ *be a finite quasi-nilpotent automaton and* $\mathscr{K}$ *a finite class of finite automata. It is then decidable if* $\mathbf{A}\in\mathbf{HSP}_{f\alpha_0}(\mathscr{K})$. □

We end the paper with characterizations of classes of finite automata complete for the class of all finite quasi-nilpotent and nilpotent automata with respect to the finite α_0-product.

THEOREM 2.6. *An* $f\alpha_0$*-variety* $\mathscr{K}$ *contains the class of all finite quasi-nilpotent automata if and only if is metrically complete and* $\mathbf{F}_n\in\mathscr{K}$ *for every* $n\geqq 0$. □

THEOREM 2.7. *An* $f\alpha$*-variety* $\mathscr{K}$ *contains the class of all finite nilpotent automata if and only if* $\mathscr{K}$ *is metrically complete and* $\mathbf{C}_{n,1}\in\mathscr{K}$ *for every* $n\geqq 0$. □

REMARKS. Let $\mathcal{K}$ be a finite class of finite automata. It is decidable if $\mathcal{K}$ is metrically complete. Define the language $L_2'(\mathcal{K})$ by

$$L_2'(\mathcal{K}) = \{w | w \in \{x, y\}^*,\ |w| = n,\ \mathbf{C}_{n,1} \in \mathbf{HSP}_{f\alpha_0}(\mathcal{K})\}.$$

Moreover, for fixed $\mathbf{A}=(A, X, \delta) \in K$ and $a_1, a_2 \in A$ with $a_1 \neq a_2$, let

$$L'_{a_1,a_2} = \{x^k y^n p_{i_1} \dots p_{i_t} | \exists a_0', a_0 \in A, u, v_1, v_2, w \in X^*,$$
$$|u| = n, |v_1| = |v_2| = k, |w| = m, m = p_{i_1} \cdot \dots \cdot p_{i_t},$$
$$a_0' u = a_0, a_0 v_1 = a_1, a_0 v_2 = a_2, a_1 w = a_1, a_2 w_2 = a_2)\}.$$

Set $L'(\mathcal{K}) = \bigcup (L'_{\alpha_1,a_2} | \exists \mathbf{A}=(A, X, \delta) \in \mathcal{K},\ a_1, a_2 \in A,\ a_1 \neq a_2)$, and let $L(\mathcal{K}) = (L_2'(\mathcal{K}) L_1(\ ^c)) \cap L'(\mathcal{K})$, where $L_1(\mathcal{K})$ is defined in the proof of Lemma 2.4. Let τ: $\{x, y, p_1, \dots, p_l\} \to \{y, \lambda\}$ be the substitution given $\tau(y)=y$ and $\tau(z)=\lambda$ in all other cases. Then $\tau(L(\mathcal{K}))$ is regular and can be given effectively. Moreover, $\mathbf{F}_n \in \mathbf{HSP}_{f\alpha_0}(\mathcal{K})$ for every $n \geqq 0$ if and only if, $\tau(L(\mathcal{K})) = \{y\}^*$. The validity of this equality is decidable. Therefore it is decidable whether $\mathbf{HSP}_{f\alpha_0}(\mathcal{K})$ contains all finite quasi-nilpotent automata. Based on the remark following Corollary 11 in [2], one can give a procedure to decide if $\mathbf{HSP}_{f\alpha_0}(\mathcal{K})$ contains all finite nilpotent automata.

Let us note that Imreh [5] gives necessary and sufficient conditions for a class to be complete with respect to nilpotent automata. Finally, we remark that, by Corollary 8 in [2], there is no finite class $\mathcal{K}$ of finite automata such that $\mathbf{HSP}_{f\alpha_0}(\mathcal{K})$ would contain every finite, not metrically complete automaton.

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THE DISTRIBUTION OF r-FREE NUMBERS

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§ 1. Introduction

A positive integer is said to be r-free if it is divisible by no r^{th} power of a prime. An elementary argument shows that

$$\sum_{\substack{n\leqq x\\ n\ r\text{-free}}} 1-\frac{x}{\zeta(r)} \ll x^{1/r}.$$

Let $R_r(x)$ denote the left hand side of the above. It is easy to show that $R_r(x)=\Omega(x^{1/2r})$ (see [4] or [8]). It seems plausible that $R_r(x)\ll x^{1/2r+\varepsilon}$, but this is not known, even assuming the Riemann Hypothesis (RH). Montgomery and Vaughan [7] assumed RH and obtained

$$R_r(x) \ll x^{1/(r+1)+\varepsilon}. \tag{1.1}$$

The object of this paper is to improve this result.

The case $r=2$ has received special attention. Montgomery and Vaughan (loc. cit.) obtained (on RH)

$$R_2(x) \ll x^{9/28+\varepsilon}.$$

Graham [5] improved 9/28 to 8/25, and Baker and Pintz [1] and Jia Chao-hua (written communication) have improved 8/25 to 7/22.

We shall prove

THEOREM 1. *Assume* RH. *For every* $\varepsilon>0$,

$$R_r(x) \ll_{\varepsilon,r} x^{D(r)+\varepsilon}, \tag{1.2}$$

where

$$D(r)=\begin{cases} 7/(8r+6) & \text{if } 2\leqq r\leqq 5,\\ 67/514 & \text{if } r=6,\\ 11(r-4)/(12r^2-37r-41) & \text{if } 7\leqq r\leqq 12,\\ 23(r-1)/(24r^2+13r-37) & \text{if } 13\leqq r\leqq 20.\end{cases}$$

For $r\geqq 21$, (1.2) *is true with* $D(r)$ *defined by the following procedure. Let* $q\geqq 2$ *and* $Q=2^q$. *For every* $r\geqq 21$, *there is a unique* q *such that*

$$(12Q-q-5)/2 < r \leqq (24Q-q-6)/2.$$

With this value of q, define

$$D(r) = \frac{(12Q-1)r-12Q+1}{12Qr^2+(6Qq+1)r-(6Qq+12Q+1)}.$$

Note that $D(2)=7/22$, the same exponent obtained by Baker, Pintz, and Jia.

The proof of Theorem 1 uses a two-dimensional version of van der Corput's method for estimating exponential sums. It can be shown that as r tends to infinity,

$$(D(r))^{-1} = r+\log r/(2\log 2)+O(1).$$

For large values of r, we can improve this by appealing to Vinogradov's method. Our result is

THEOREM 2. *Assume* RH. *Then*

$$R_r(x) \ll_r x^{E(r)},$$

where $E(r)=1/(r+cr^{1/3})$ *for some constant* c.

We begin with the following result of Montgomery and Vaughan. Let $\psi(w)=w-[w]-1/2$, and let $N>0$. Define

$$S_r(x, N) = \sum_{n\leqq N} \mu(n)\psi(xn^{-r}).$$

If RH is assumed, then

$$R_r(x) = -S_r(x, N)+O(x^{1/2+\varepsilon}N^{(1-r)/2}+N^{1/2+\varepsilon}) \tag{1.3}$$

([7], Theorem 1). The result (1.1) follows immediately from the trivial estimate $S_r(x, N)\ll N$. Here we shall obtain non-trivial estimates for S_r. If we let $N_1=x^{1/2}N^{(1-r)/2}$, then we can describe our task as showing that $S_r(x, N)\ll N^{\varepsilon}N_1$ for N as large as possible. A trivial argument suffices for those $n\leqq N_1$, so we need only show that

$$\sum_{N_1<n\leqq N} \mu(n)\psi(xn^{-r}) \ll N^{\varepsilon}N_1. \tag{1.4}$$

NOTATION. Throughout this paper, we will use $e(w)$ to denote $e^{2\pi iw}$, and we will use $\mathscr{L}$ to denote $\log x$. To simplify writing logarithms, we will assume that all parameters are bounded by a power of x.

§ 2. Vaughan's identity

We use Vaughan's identity [13] to transform the sum in (1.4). There is apparently no advantage in using the more complicated identity of Heath—Brown [6].

Let u and v be parameters to be chosen later; for now assume only that $u\leqq N_1$ and $v\leqq N_1$. Define

$$F(s) = \sum_{n\leqq v} \mu(n)n^{-s}, \quad M(s) = \sum_{m\leqq u} \mu(m)m^{-s}.$$

Then

$$\frac{1}{\zeta} = (1-\zeta M)\left(\frac{1}{\zeta}-F\right)-FM\zeta+M+F.$$

Equating coefficients on both sides gives an identity for $\mu(n)$. We multiply this identity by $\psi(xn^{-r})$ and sum from N_1 to N to get

$$\sum_{N_1<n\leqq N}\mu(n)\psi(xn^{-r})=\sum\nolimits_1-\sum\nolimits_2-\sum\nolimits_3, \tag{2.1}$$

where

$$\sum\nolimits_1=\sum_{u<m\leqq N/v}\ \sum_{\substack{N_1/m<n\leqq N/m\\ v<n}} A(m)\mu(n)\psi(xm^{-r}n^{-r}),$$

$$\sum\nolimits_2=\sum_{u<m\leqq uv}\ \sum_{N_1/m<n\leqq N/m} B(m)\psi(xm^{-r}n^{-r}),$$

$$\sum\nolimits_3=\sum_{m\leqq u}\ \sum_{N_1/m<n\leqq N/m} B(m)\psi(xm^{-r}n^{-r}),$$

$$A(m)=-\sum_{\substack{d\mid m\\ d\leqq u}}\mu(d),\quad B(m)=\sum_{\substack{de=m\\ d\leqq u,\ e\leqq v}}\mu(d)\mu(e).$$

Note that $A(m)\ll N^{\varepsilon}$ and $B(m)\ll N^{\varepsilon}$.

In the nomenclature of Vaughan [13], $\sum_1$ and $\sum_2$ are type II sums, and $\sum_3$ is a type I sum. It will transpire that our type II estimates have the form

$$\sum_{X<m\leqq 2X}\ \sum_{\substack{Y<n\leqq 2Y\\ N_1<mn\leqq N}} a(m)b(n)\psi(xm^{-r}n^{-r})\ll N^{1/2+\varepsilon}X^{1/2}+\text{other terms}.$$

Motivated by this, we choose u and v so that $uv=N/v$ and $N^{1/2}u^{1/2}v^{1/2}=N_1$. Moreover, it is notationally convenient to define δ by the relation $x=N^{r+1-2\delta}$. It will also be convenient to define $w=xN_1^{-1}$. For ease of future reference, we note that

$$v=N^{2\delta},\quad u=N^{1-4\delta},\quad N_1=N^{1-\delta},\quad w=N^{r-\delta},\quad\text{and}\quad x=N^{r+1-2\delta}.$$

Note that our requirements that $u\leqq N_1$ and $v\leqq N_1$ are satisfied if $0\leqq\delta\leqq 1/3$. Note also that once (1.4) is proved, (1.2) follows with

$$D(r)=(1-\delta)/(r+1-2\delta). \tag{2.2}$$

§ 3. Lemmas

As usual in work of this nature, we convert sums involving ψ into finite exponential sums. The following lemma provides an elegant way of doing this.

LEMMA 1. *Suppose* $J>0$. *There is a function* $\psi^*(x)$ *such that*

(i) $$\psi^*(x)=\sum_{1\leqq|j|\leqq J}\gamma(j)e(jx),$$

(ii) $$\gamma(j)\ll 1/|j|,$$

(iii) $$|\psi^*(x)-\psi(x)|\leqq(2J+2)^{-1}\sum_{|j|\leqq J}(1-|j|/J)e(jx).$$

REMARKS. This is contained in Theorem 18 of Vaaler [12]. Vaaler gives $\gamma(j)$ explicitly; if $t=j/(J+1)$ then

$$\gamma(j) = -\big(\pi t(1-|t|)\cot \pi t + |t|\big)/2\pi ij.$$

Suppose we have an estimate of the form $L \leqq AZ^a + BZ^{-b}$ valid for any $Z>0$. If we choose Z so that $AZ^a = BZ^{-b}$, we get $L \ll (A^b B^a)^{1/(a+b)}$, and this is best possible, apart from the implied constant. Our second lemma generalizes this principle.

LEMMA 2. *Suppose A_i, B_j, a_i, and b_j are all positive numbers. Define*

$$L(Z) = \sum_{i=1}^{m} A_i Z^{a_i} + \sum_{j=1}^{n} B_j Z^{-b_j}.$$

If Z_1 and Z_2 are real numbers with $0 \leqq Z_1 \leqq Z_2$, then there is some Z such that $Z_1 \leqq Z \leqq Z_2$ and

$$L(Z) \ll \sum_{i=1}^{m} \sum_{j=1}^{n} (A_i^{b_j} B_j^{a_i})^{1/(a_i+b_j)} + \sum_{i=1}^{m} A_i Z_1^{a_i} + \sum_{j=1}^{n} B_j Z_2^{-b_j}.$$

This is Lemma 3 of Srinivasan [10].

Finally, we mention a simple but very useful convexity principle. If $X_1, \ldots, X_k$ are positive and $a_1, \ldots, a_k$ are non-negative real numbers such that $a_1 + \ldots + a_k = 1$, then

$$\min(X_1, \ldots, X_k) \leqq X_1^{a_1} \ldots X_k^{a_k} \leqq \max(X_1, \ldots, X_k).$$

The proof is obvious.

§ 4. Type II estimates

In this section, we obtain upper bounds for $\sum_1$ and $\sum_2$. Our argument is similar to the corresponding arguments in Heath—Brown [6] and Baker and Pintz [1]; however, we use the lemmas of the previous section to simplify the results.

We also use the exponent pair theory of Phillips [9]. Here is a brief summary of the relevant results of this theory. If (k, l) is an exponent pair, then

$$\sum_{X < m \leqq 2X} e(\lambda m^{-r}) \ll |\lambda|^k X^{l-(r+1)k} + |\lambda|^{-1} X^{1+r}. \tag{4.1}$$

If (k, l) is an exponent pair then so are

$$A(k, l) = \left(\frac{k}{2k+2}, \frac{k+l+1}{2k+2}\right)$$

and $B(k, l) = (l - 1/2, k + 1/2)$. Finally, $(0, 1)$ is an exponent pair, and for any integer $q \geqq 0$,

$$A^q B(0, 1) = \left(\frac{1}{4Q-2}, 1 - \frac{q+1}{4Q-2}\right)$$

is an exponent pair $(Q = 2^q)$.

LEMMA 3. Let $a(m)$, $b(n)$, *and* $c(k)$ *be sequences supported on* $(X, 2X]$, $(Y, 2Y]$, *and* $(H, 2H]$ *respectively. Assume further that all of these sequences are bounded above by* 1. *Define*

$$S = H^{-1} \sum_{h,m,n} a(m)b(n)c(h)e(hxm^{-r}n^{-r}),$$

and let $F = xX^{-r}Y^{-r}$. *Assume that*

$$X^r Y^{r+1} \leqq x, \tag{4.2}$$

that (k, l) *is an exponent pair, and that* $(\varkappa, \lambda) = A(k, l)$. *Then*

$$S \ll x^{\varepsilon}\{F^{\varkappa}X^{\lambda}Y^{1-\varkappa} + H^{-1/2}XY^{1/2}\}.$$

PROOF. By Cauchy's inequality,

$$|S|^2 \leqq XH^{-2} \sum_{X<m\leqq 2X} \Big|\sum_{h,n} b(n)c(h)e(hxm^{-r}n^{-r})\Big|^2 =$$

$$= XH^{-2} \sum_{X<m\leqq 2X} \Big|\sum_{d\in\mathscr{D}} g(d)e(xdm^{-r})\Big|^2,$$

where $\mathscr{D} = \{hn^{-r}\colon H < h \leqq 2H,\ Y < n \leqq 2Y\}$ and

$$g(d) = \sum_{hn^{-r}=d} b(n)c(h).$$

Let Z be a parameter to be chosen later; for now we assume only that $Z \leqq HY^{-r}$. Let $\mathscr{I}_q$ denote the interval

$$\big(HY^{-r} + qZ,\ HY^{-r} + (q+1)Z\big]$$

for $0 \leqq q \leqq 3HY^{-r}Z^{-1}$. Then

$$\Big|\sum_{d\in\mathscr{D}} g(d)e(xdm^{-r})\Big|^2 \leqq \sum_q \Big|\sum_{d\in\mathscr{I}_q}\Big|^2 \ll HY^{-r}Z^{-1} \sum_q \sum_{s,t\in\mathscr{I}_q} g(s)\overline{g(t)}e\big(x(s-t)m^{-r}\big).$$

We obtain

$$|S|^2 \ll \frac{X}{HY^rZ} \sum_q \sum_{s,t\in\mathscr{I}_q} |g(s)\overline{g(t)}| \sum_{\substack{t\\ |s-t|\leqq Z}} \Big|\sum_{X<m\leqq 2X} e\big(x(s-t)m^{-r}\big)\Big| \ll$$

$$\ll H^{-1}XY^{-r}Z^{-1} \sum_{(4.3)} \Big|\sum_{X<m\leqq 2X} e(axm^{-r})\Big|.$$

Here, (4.3) denotes that the summation is over h_1, h_2, n_1, n_2 satisfying

$$H < h_1, h_2 \leqq 2H, \quad Y < n_1, n_2 \leqq 2Y, \quad |a| \leqq Z, \tag{4.3}$$

and a denotes $h_1n_1^{-r} - h_2n_2^{-r}$. If (k, l) is an exponent pair then

$$\sum_{X<m\leqq 2X} e(axm^{-r}) \ll (ax)^k X^{l-(r+1)k} + (ax)^{-1}X^{r+1}. \tag{4.4}$$

We also have the trivial upper bound X for this sum.

It is now convenient to define

$$M(\lambda) = \big|\{(h_1, h_2, n_1, n_2)\colon H < h_1,\ h_2 \leqq 2H,\ Y < n_1,\ n_2 \leqq 2Y,\ |a| \leqq \lambda\}\big|.$$

Using (4.4) when $a>x^{-1}X^r$ and the trivial upper bound X otherwise, we find that

$$|S|^2 \ll \mathscr{L}H^{-1}XY^{-r}Z^{-1}\{M(Z)x^kX^{l-(r+1)k}Z^k+x^{-1}X^{r+1}\max_{\lambda\leqq x^{-1}X^r}M(\lambda)\lambda^{-1}\}.$$

Now $M(\lambda)\ll\lambda HY^{2+r}+HY\mathscr{L}$ by the argument of Heath—Brown [6] after his equation (16). Using this, we find that

$$|S|^2 \ll x^\varepsilon\{x^kX^{1+l-(r+1)k}Y^2Z^k+x^kX^{1+l-(r+1)k}Y^{1-r}Z^{k-1}+ \tag{4.5}$$
$$+x^{-1}X^{r+2}Y^2Z^{-1}+X^2Y^{1-r}Z^{-1}\}.$$

Up to now, we have assumed only that $Z\leqq HY^{-r}$. We arbitrarily impose the condition $Z\geqq Y^{-1-r}$. This condition is useful because it implies that the first term on the right hand side of (4.5) dominates the second term. From Lemma 2, we get

$$|S| \ll x^\varepsilon\{F^\varkappa X^\lambda Y^{1-\varkappa}+X^\lambda Y+H^{-1/2}XY^{1/2}+H^{-1/2}F^{-1/2}XY+$$
$$+F^{k/2}X^{(1+l-k)/2}Y^{1-k/2}\} = x^\varepsilon\{E_1+E_2+E_3+E_4+E_5\},$$

say. We also have the trivial estimate $S\ll XY$. Since

$$\min(XY, E_5) \leqq (XY)^{k/(k+1)}E_5^{1/(k+1)} = E_1,$$

it follows that

$$S \ll x^\varepsilon\{E_1+E_2+E_3+E_4\}. \tag{4.6}$$

Up to this point, we have not used our assumption that $X^rY^{r+1}\leqq x$. This assumption is equivalent to $FY^{-1}\geqq 1$. Since $E_1=E_2(FY^{-1})^\varkappa$ and $E_3=E_4(FY^{-1})^{1/2}$, we get $S\ll x^\varepsilon\{E_1+E_3\}$, which is the desired result.

It should be noted that the conditions $|a(m)|\leqq 1$, $|b(n)|\leqq 1$, and $|c(k)|\leqq 1$ are normalizations. If we assume instead that $|a(m)|\leqq A$, $|b(n)|\leqq B$, and $|c(k)|\leqq C$, we obtain the same result except that the left hand side is multiplied by ABC.

LEMMA 4. *Assume all of the hypotheses of the previous lemma. Suppose that* $N_1\leqq M\leqq N$ *and that*

$$T(M) = \sum_{u<m\leqq uv}\ \sum_{M/m<n\leqq 2M/m} a(m)\,b(n)\,\psi(xm^{-r}n^{-r}).$$

If $\delta\leqq 1/6$ *then*

$$T(M) \ll x^\varepsilon\{N_1+x^\varkappa M^{1-(r+1)\varkappa}u^{\lambda+k-1}\}.$$

PROOF. By Lemma 1 with $J=MN_1^{-1}$,

$$T(M) \leqq \mathscr{L}N_1/2+\sum_{1\leqq|j|\leqq J}\gamma(j)\sum_{u<m\leqq uv}\ \sum_{M/m<n\leqq 2M/m} a(m)b(n)e(xjm^{-r}n^{-r})+$$
$$+\sum_{1\leqq|j|\leqq J}\gamma_1(j)\sum_{u<m\leqq uv}\ \sum_{M/m<n\leqq 2M/m}|a(m)|\,|b(n)|e(xjm^{-r}n^{-r}),$$

where $\gamma_1(j)=(J-|j|)/(J(2J+2))$ and $\gamma(j)$ is as in Lemma 1. These sums may be broken into $\ll\mathscr{L}^3$ sums of the form

$$H^{-1}\sum_{H<h\leqq 2H}\ \sum_{X<m\leqq 2X}\ \sum_{\substack{Y<n\leqq 2Y\\ M<mn\leqq 2M}} a'(m)\,b'(n)\,c'(h)e(hxm^{-r}n^{-r}), \tag{4.7}$$

where $1 \leqq H \leqq J$, $u \leqq X \leqq uv$, $M/2 < XY \leqq 2M$, $|a'(m)| \leqq 1$, $|b'(n)| \leqq 1$, and $|c'(h)| \leqq 1$. To apply Lemma (4.7), we need to remove the condition $M < mn \leqq 2M$.

It is no loss of generality to assume that $M - 1/4$ is an integer. Let $\sigma = 1/\log M$ and $U = X^2 Y^2$. We use the relation

$$\frac{1}{2\pi i} \int_{\sigma - iU}^{\sigma + iU} \frac{y^s}{s}\, ds = \begin{cases} 1 + O(y^\sigma U^{-1} |\log y|^{-1}) & \text{if } y > 1 \\ O(y^\sigma U^{-1} |\log y|^{-1}) & \text{if } y < 1 \end{cases}$$

(see [2], p. 105). We then see that the sum in (4.7) is

$$= \frac{1}{2\pi i} \int_{\sigma - iU}^{\sigma + iU} H^{-1} \sum_{h,m,n} a'(m) m^s b'(n) n^s c'(h) e(hxm^{-r} n^r) \cdot$$

$$\cdot \left(M^{-s} - (2M)^{-s}\right) \cdot s^{-1}\, ds + O(M^2 U^{-1}).$$

We now apply Lemma 3 pointwise. Note that $X^r Y^{r+1} \ll N^{r+1} u^{-1} = N^{r+4\delta} \leqq x$ since $\delta \leqq 1/6$. We get the upper bound

$$x^\varepsilon \{x^\varkappa M^{1-(r+1)\varkappa} X^{\lambda+\varkappa-1} + H^{-1/2} M^{1/2} X^{1/2}\}$$

for the right hand side of (4.6). We then sum over values of H, X, and Y. Since $(k.\, l)$ is an exponent pair, $l \leqq 1$ by definition; consequently

$$\lambda + \varkappa - 1 = \frac{l-1}{2\varkappa + 2} \leqq 0.$$

Therefore

$$T(M) \ll x^\varepsilon \{N_1 + x^\varkappa M^{1-(r+1)\varkappa} u^{\lambda+\varkappa-1} + M^{1/2} u^{1/2} v^{1/2}\}.$$

By our definitions of u and v, we have $M^{1/2} u^{1/2} v^{1/2} \leqq N_1$.

LEMMA 5. *Let $\sum_1$ and $\sum_2$ be as defined in Section 2. The estimate $\sum_1 + \sum_2 \ll N^\varepsilon N_1$ is true provided $\delta \leqq \beta(r)$, where $\beta(r)$ is defined as follows. If $2 \leqq r \leqq 5$, then $\beta(r) = 1/8$. If $r \geqq 6$, then there is a unique integer $q \geqq 2$ such that*

$$2Q - 3 < r \leqq 4Q - 3 \quad (Q = 2^q).$$

Then

$$\beta(r) = \frac{2Qq - 4Q + r + 3}{2Qr + 8Qq - 18Q + 4r + 12}.$$

PROOF. For $2 \leqq r \leqq 5$, the result follows immediately from Lemma 4 with $(\varkappa, \lambda) = AB(0, 1) = (1/6, 2/3)$.

For $r \geqq 6$, we first apply Lemma 4 with

$$(\varkappa, \lambda) = A^q B(0, 1) = \left(\frac{1}{4Q-2}, 1 - \frac{q+1}{4Q-2}\right)$$

to get

$$T(M) \ll N^\varepsilon N_1 + N(xM^{4Q-3-r} u^{-q})^{1/(4Q-2)} = N^\varepsilon (N_1 + E_q),$$

say. We then apply the same lemmas with $(\varkappa, \lambda)=A^{q-1}B(0, 1)$ to get

$$T(M) \ll N^{\varepsilon}N_1+N^{\varepsilon}(xM^{2Q-3-r}u^{-q+1})^{1/(2Q-2)} = N^{\varepsilon}(N_1+E_{q-1}),$$

say. By convexity,

$$T(M) \ll N^{\varepsilon}(N_1+E_{q-1}^{c}E_q^{d}),$$

provided $c\geqq 0$, $d\geqq 0$ and $c+d=1$. We choose c and d so that the M exponent is zero. Thus we take

$$c=\frac{(2Q-2)(4Q-3-r)}{2(r+1)Q}, \quad d=\frac{(4Q-2)(r+3+2Q)}{2(r+1)Q}.$$

Note that our assumptions on r assure that $c\geqq 0$ and $d\geqq 0$. Thus $\sum_1+\sum_2 \ll$ $\ll N^{\varepsilon}(N_1+N^{a})$, where

$$a=\frac{A+B}{2Q(r+1)},$$

$$A=2Qr-2Qq+6Q-r-3, \quad \text{and} \quad B=8Qq-20Q+4r+12.$$

We therefore require that $a\leqq 1-\delta$; i.e. $\delta\leqq\beta(r)$.

§ 5. Poisson summation formula

This section and the next will be concerned in the type I estimates. We begin by quoting two lemmas. The first is a well-known result of van der Corput ([11], Theorem 5.9); the second is due to Heath—Brown ([16], Lemma 6).

LEMMA 6. *Suppose g is real valued and has two continuous derivatives on $[a, b]$. If $\lambda\leqq|g''(x)|\leqq h\lambda$ on this interval, then*

$$\sum_{a<n\leqq b} e(g(n)) \ll h(b-a)\lambda^{1/2}+\lambda^{-1/2}.$$

LEMMA 7. *Let $0<a<b\leqq 2a$. Let $g(z/a)$ be holomorphic on an open convex set R containing the real line segment $[1, b/a]$. Assume also that (1) $|g''(z/a)|\leqq\Lambda$ on R, (2) $g(x)$ is real when x is real, (3) $g''(x)\leqq -c\Lambda$ for some $c>0$. Let $g'(b)=\alpha$, $g'(a)=\beta$. For each integer v in the range $\alpha\leqq v\leqq\beta$, define x_v by $g'(x_v)=v$. Then*

$$\sum_{a<n\leqq b} e(g(n)) = \sum_{\alpha<v\leqq\beta} |g''(x_v)|^{-1/2}e(g(x_v)-vx_v-1/8)+$$

$$+O(\Lambda^{-1/2}+\log(2+\Lambda(b-a))).$$

We shall be applying these lemmas to a function of two variables. Assume that $f(u, v)$ is defined and real-valued on the rectangle $(X, 2X]\times(Y, 2Y]$ and that $\mathscr{D}$ is a subdomain of this rectangle. We wish to estimate

$$S=\sum_{(m, n)\in\mathscr{D}} e(f(m, n)),$$

and we shall need several assumptions in $\mathscr{D}$ and f. For convenience, we list these assumptions as (Ω_1), (Ω_2), ..., (Ω_6). The first three conditions are as follows

(Ω_1) f has partial derivatives of all orders, and there is some constant C_1 such that

$$\left|\frac{\partial^{a+b} f}{\partial u^a \partial v^b}\right| \leqq C_1 F X^{-a} Y^{-b}$$

for all a and b with $0 \leqq a \leqq 2$ and $0 \leqq b \leqq 2$.

(Ω_2) Three is some constant C_2 such that

$$|f_{uu}| \geqq C_2 F X^{-2} \quad \text{and} \quad |f_{vv}| \geqq C_2 F X^{-2}.$$

(Ω_3) There is some constant C_3 such that for each fixed n, $Y < n \leqq 2Y$, the set $\mathscr{I}(n) = \{m\colon (m, n) \in \mathscr{D}\}$ consists of at most C_3 intervals. Similarly, for each fixed m, $X < m \leqq 2X$, the set $\mathscr{J}(m) = \{n\colon (m, n) \in \mathscr{D}\}$ consists of at most C_3 intervals.

When using the Ω-conditions, we will allow the implied constants to depend on the C_i's. All estimates will be uniform in F, X, and Y.

LEMMA 8. *Assume* (Ω_1), (Ω_2), *and* (Ω_3). *Then*

$$S \ll \min(F^{1/2}X, F^{1/2}Y) + F^{-1/2}XY.$$

PROOF. We write S as

$$\sum_{Y < n \leqq 2Y} \sum_{m \in \mathscr{I}(n)} e(f(m, n)).$$

Now apply Lemma 6 to the inner sum to get

$$S \ll \sum_{Y < n \leqq 2Y} (F^{1/2} + F^{-1/2}X) \ll F^{1/2}Y + F^{-1/2}XY.$$

Our hypotheses are symmetric in X and Y, so we also have

$$S \ll F^{1/2}X + F^{-1/2}XY,$$

and the result follows.

To fully exploit Lemma 7 we must assume

(Ω_4) There is a constant C_4 such that for each fixed v, $Y < v \leqq 2Y$, the function $g(z) = f(z, v)$ is analytic in the region $R = \{z\colon |z - u| < C_4 X$ for some u, $X < u \leqq 2X\}$. Also, there is a constant C_4' such that $|g''(z)| \leqq C_4' F X^{-2}$ on R.

We define $\psi(w, v)$ by the relation

$$f_u(\psi(w, v), v) = w, \tag{5.1}$$

and we will assume

(Ω_5) There is a constant C_5 such that for each fixed l, the set

$$\mathscr{I}^*(l) = \{n\colon (\psi(l, n), n) \in \mathscr{D}\}.$$

consists of at most C_5 intervals.

Let

$$g(w, v) = f(\psi(w, v), v) - w\psi(w, v), \tag{5.2}$$

and let $\mathscr{H}f$ denote the Hessian of f; i.e.

$$\mathscr{H}f = f_{uu}f_{vv} - f_{uv}^2.$$

Our next lemma expresses the first and second derivatives of g in terms of f.

LEMMA 9. *We have*

$$g_v = f_v(\psi, v), \quad \text{and} \quad g_{vv} = \frac{\mathscr{H}f(\psi, v)}{f_{uu}(\psi, v)}.$$

PROOF. We differentiate (5.1) with respect to v to get

$$f_{uu}(\psi, v)\psi_v + f_{uv}(\psi, v) = 0,$$

whence

$$\psi_v = \frac{-f_{uv}(\psi, v)}{f_{uu}(\psi, v)}. \tag{5.3}$$

Differentiation of g with respect to v yields

$$g_v = f_u\psi_v + f_v - w\psi_v = f_v.$$

This proves the first assertion. We differentiate again to get $g_{vv} = f_{uv}\psi_v + f_{vv}$, and the second assertion by combining this with (5.3).

This lemma motivates our final condition.

(Ω_6) There is some constant C_6 such that $|\mathscr{H}f| \geqq C_6 F^2 X^{-2} Y^{-2}$.

LEMMA 10. *Assume* $(\Omega_1), \ldots, (\Omega_6)$. *Then*

$$S \ll F + F^{-1/2}XY\mathscr{L}.$$

PROOF. Without loss of generality, we may assume that $f_{uu} < 0$. By Lemma 7,

$$S = \sum_{Y < n \leqq 2Y} \sum_{m \in \mathscr{I}(n)} e(f(m, n)) =$$

$$= \sum_{Y < n \leqq 2Y} \sum_{l \in \mathscr{K}(n)} e(-1/8 + g(l, n)) |f_{uu}(\psi, n)|^{-1/2} + O(F^{-1/2}XY + Y\mathscr{L}),$$

where $\mathscr{K}(n) = \{l: (\psi(l, n), n) \in \mathscr{D}\}$. Changing the order of summation yields

$$S = \sum_{L_1 \leqq l \leqq L_1 + L} \sum_{n \in \mathscr{I}^*(l)} e(-1/8 + g(l, n)) |f_{uu}|^{-1/2} + O(F^{-1/2}XY + Y\mathscr{L}).$$

From (Ω_1) with $a = 0$ and $b = 1$, we see that $L \ll FX^{-1}$ and $L_1 \ll FX^{-1}$. Using partial summation and Lemma 6 for the inner sum, we get

$$S \ll \sum_{L_1 \leqq l \leqq L_1 + L} F^{-1/2}X\{|\mathscr{I}^*(l)| F^{1/2}Y^{-1} + F^{-1/2}Y\} + F^{-1/2}XY + Y\mathscr{L}.$$

Furthermore,

$$\sum_l |\mathscr{I}^*(l)| = \sum_n \sum_{(l, n) \in \mathscr{D}} 1 \ll \sum_{Y < n \leqq 2Y} \left\{ \int_{\mathscr{I}(n)} f_{uu}(t, y)\, dt + 1 \right\} \ll FX^{-1}Y + Y.$$

It follows that

$$S \ll XY^{-1}(FX^{-1}Y+Y)+F^{-1}XY(FX^{-1}+1)+F^{-1/2}XY+Y\mathscr{L} \ll$$

$$\ll F+X\mathscr{L}+Y\mathscr{L}+F^{-1/2}XY+F^{-1}XY.$$

We may assume that $F>1$, since our theorem is trivial otherwise. Therefore

$$S \ll F+X\mathscr{L}+Y\mathscr{L}+F^{-1/2}XY.$$

We combine this with Lemma 8 to get

$$S \ll F+\min(F^{1/2}Y, X\mathscr{L})+\min(F^{1/2}X, Y\mathscr{L})+F^{-1/2}XY\mathscr{L}. \tag{5.4}$$

By the convexity principle,

$$\min(F^{1/2}Y, X\mathscr{L}) \leqq F^{1/4}X^{1/2}Y^{1/2}\mathscr{L}^{1/2} \leqq \max(F,\ F^{-1/2}XY\mathscr{L}),$$

so the second term on the right hand side of (5.4) may be removed. Similarly, the third term may also be removed.

§ 6. Type I sums

In this section, we first derive bounds for sums of the form

$$S = \sum_{(m,n)\in\mathscr{D}} a(m)e(f(m, n)), \tag{6.1}$$

where $|a(m)|\leqq 1$, $f(m, n)=ym^{-r}n^{-r}$, and $\mathscr{D}$ is an appropriate subset of $(X, 2X]\times$ $\times(Y, 2Y]$. Our primary tools are Lemma 10 and the following two lemmas.

LEMMA 11. *Let $\mathscr{I}$ be a subinterval of $(Y, 2Y]$ and let H be a positive integer. For any complex $z(n)$,*

$$\Big|\sum_{n\in\mathscr{I}} z(n)\Big|^2 \leqq (1+YH^{-1})\sum_{|h|\leqq H}(1-|h|H^{-1})\sum_{n,n+h\in\mathscr{I}} z(n)\overline{z(n+h)}.$$

REMARK. This is Lemma 5 of Heath—Brown [6].

LEMMA 12. *Let $\mathscr{D}$ be a subset of $(X, 2X]\times(Y, 2Y]$. For any complex $z(m, n)$ with $|z(m, n)|\leqq 1$ and $a(m)$, define*

$$U = \sum_{(m,n)\in\mathscr{D}} a(m)z(m, n).$$

Assume that $\mathscr{D}$ satisfies (Ω_3).
(a) *If $0<H\leqq Y$ and $|a(m)|\leqq 1$ then*

$$|U|^2 \ll \frac{X^2Y^2}{H}+\frac{XY}{H}\sum_{1\leqq|h|\leqq H}\Big|\sum_{\substack{(m,n)\in\mathscr{D}\\(m,n+h)\in\mathscr{D}}} z(m, n)\overline{z(m,n+h)}\Big|$$

(b) *If* $0<H\leqq X$ *and* $a(m)=1$ *then*

$$|U|^2 \ll \frac{X^2Y^2}{H}+\frac{XY}{H}\sum_{1\leqq|h|\leqq H}\Big|\sum_{\substack{(m,n)\in\mathscr{D}\\(m+h,n)\in\mathscr{D}}} z(m,n)\overline{z(m+h,n)}\Big|.$$

PROOF. Assume momentarily that H is a positive integer. By Cauchy's inequality and the previous lemma,

$$|U|^2 \leqq X\sum_{X<m\leqq 2X}\Big|\sum_{\substack{n\\(m,n)\in\mathscr{D}}} z(m,n)\Big|^2 \leqq$$

$$\leqq X(YH^{-1}+1)\sum_{|h|<H}(1-|h|H^{-1})\sum_{\substack{(m,n)\in\mathscr{D}\\(m,n+h)\in\mathscr{D}}} z(m,n)\overline{z(m,n+h)}.$$

Since $H\leqq Y$, we obtain

$$|U|^2 \ll \frac{X^2Y^2}{H}+\frac{XY}{H}\sum_{1\leqq|h|\leqq H}\Big|\sum_{\substack{(m,n)\in\mathscr{D}\\(m,n+h)\in\mathscr{D}}} z(m,n)\overline{z(m,n+h)}\Big|. \tag{6.2}$$

To complete the proof of (a), we need to remove the assumption that H is an integer. If $H\geqq 1$, we can remove this assumption by observing that $H^{-1}\leqq 2[H]^{-1}$. If $H<1$, then (6.2) is trivial. This proves (a). Statement (b) follows by interchanging the roles of m and n.

In the theory of exponent pairs, Lemma 11 is known as the "A process", and the Poisson summation formula is the "B process". Here, we refer to Lemma 12b as A_1, Lemma 12a as A_2, and Lemma 10 as B. Our next lemma is A_2^qB. In other words, we use Lemma 12a q times and Lemma 10 once. Similarly, Lemma 15 is $A_2A_1^{q-1}B$.

LEMMA 13. *Let* S *be as in* (6.1) *with*

$$\mathscr{D}=\{(m,n):\ X<m\leqq 2X,\ Y<n\leqq 2Y,\ N_1<mn\leqq N\}.$$

Let q *be a positive integer and* $Q=2^q$. *Define* $F=|y|X^{-r}Y^{-r}$. *Then*

$$S \ll \mathscr{L}\{F^{1/(3Q-2)}X^{1-1/(3Q-2)}Y^{1-(q+1)/(3Q-2)}+$$

$$+(XY)^{1-1/3Q}+XY^{1-1/Q}+F^{-1/2Q}XY^{1+(q-2)/2Q+1/Q^2}\}.$$

PROOF. From Lemma 12a, we see that if $H_1\leqq Y$ then

$$|S|^2 \ll \frac{X^2Y^2}{H_1}+\frac{XY}{H_1}\sum_{1\leqq|h_1|\leqq H_1}|S_1(h_1)|, \tag{6.3}$$

where

$$S_1(h_1)=\sum_{(m,n)\in\mathscr{D}_1(h_1)} e\big(f_1(m,n;h_1)\big),$$

$$f_1(m,n;h_1)=f(m,n+h)-f(m,n)=\int_0^1\frac{\partial}{\partial t}f(m,n+ht)\,dt,$$

$$\mathscr{D}_1(h_1)=\{(m,n):\ (m,n)\in\mathscr{D}\ \text{and}\ (m,n+h)\in\mathscr{D}\}.$$

We apply Lemma 12a $q-1$ more times, and we find that if $H_1 \leqq Y, H_2 \leqq Y, \dots,$ and $H_q \leqq Y$ then

$$|S|^Q \ll \frac{(XY)^Q}{H_q} + \frac{(XY)^Q}{H_{q-1}^2} + \dots + \frac{(XY)^Q}{H_1^{Q/2}} + \frac{(XY)^{Q-1}}{H_1 \dots H_q} \sum_{\mathbf{h} \in \mathscr{H}} |S_q(\mathbf{h})|, \tag{6.4}$$

where

$$S_q(\mathbf{h}) = \sum_{(m,n) \in \mathscr{D}_q(\mathbf{h})} e\big(f_q(m, n; \mathbf{h})\big),$$

$$f_q(m, n; \mathbf{h}) = \int_0^1 \dots \int_0^1 \frac{\partial^q}{\partial t_1 \dots \partial t_q} f(m, n + h_1 t_1 + \dots + h_q t_q)\, dt_1 \dots dt_q,$$

$$\mathscr{D}_q(\mathbf{h}) = \{(m, n): (m, n + t_1 h_1 + \dots + t_q h_q) \in \mathscr{D} \text{ for all } \mathbf{t} = (t_1, \dots, t_q) \text{ with } t_i = 1 \text{ or } 0\},$$

$$\mathscr{H} = \{\mathbf{h} = (h_1, \dots, h_q): 1 \leqq |h_i| \leqq H_i \text{ for all } i\}.$$

To simplify (6.4), we assume there is some H such that $H \leqq Y$, $H_q = H$, $H_{q-1} = H^{1/2}, \dots, H_1 = H^{2/Q}$. Then

$$|S|^Q \ll \frac{(XY)^Q}{H} + \frac{(XY)^{Q-1}}{H_1 \dots H_q} \sum_{\mathbf{h} \in \mathscr{H}} |S_q(\mathbf{h})|.$$

We now apply Lemma 10 to $S_q(\mathbf{h})$. We will show later that $S_q(\mathbf{h})$ satisfies (Ω_1) through (Ω_6) with f replaced by $f_q(u, v; \mathbf{h})$, F replaced by $|h_1 \dots h_q y| X^{-r} Y^{-r-q}$, and $\mathscr{D}$ replaced by $\mathscr{D}(\mathbf{h})$. Assume this momentarily and let $G = FY^{-q}$. We see that

$$|S|^2 \ll \frac{(XY)^Q}{H} + \frac{(XY)^{Q-1}}{H_1 \dots H_q} \sum_{\mathbf{h}} \{|h_1 \dots h_q|\, G + \mathscr{L}\, |h_1 \dots h_q|^{-1/2} G^{-1/2} XY\} \ll$$

$$\ll H^{-1}(XY)^Q + H^{2-2/Q} G (XY)^{Q-1} + \mathscr{L} H^{-1+1/Q} G^{-1/2} (XY)^Q.$$

This lemma follows by applying Lemma 2.

To complete the proof, we need to justify the claims made for $S_q(\mathbf{h})$. From the formula for f_q, we see that

$$\frac{\partial^{a+b} f_q}{\partial^a u\, \partial^b v} = (-1)^{q+a+b} (r)_{q+b} (r)_a y h_1 \dots h_q u^{-r-a} v^{-r-q-b} \{1 + O(HY^{-1})\}. \tag{6.5}$$

Here, $(r)_k$ is defined by $(r)_0 = 1$ and $(r)_{k+1} = (r+k)(r)_k$. If $H \leqq cY$ and c is sufficiently small, the error term does not dominate, and (Ω_1) and (Ω_2) are satisfied. From (6.5), we find that

$$\mathscr{H} f_q = (2r+q+1)\, r(r+q) \big(y h_1 \dots h_q u^{-r-1} v^{-r-q-1} (r)_q\big)^2 \{1 + O(HY^{-1})\},$$

so (Ω_6) is satisfied. (Ω_4) is clear from the definition of f_q. For (Ω_3), observe that

$$\mathscr{I}(n) = \{m: X < m \leqq 2X, \text{ and } N_1 < m(n + \mathbf{t} \cdot \mathbf{h}) \leqq N \text{ for all } \mathbf{t} \in \mathscr{V}\},$$

where $\mathscr{V} = \{\mathbf{t} = (t_1, \dots, t_q): t_i = 0 \text{ or } 1\}$. Since $\mathscr{I}(n)$ is an intersection of intervals, it is an interval itself. Similarly, $\mathscr{J}(m)$ is an interval for every m.

To show (Ω_5), observe that in the present context, $\psi(v, w)$ is defined by

$$\frac{\partial f_q}{\partial u}(\psi(v, w), v; \mathbf{h}) = w.$$

For fixed l and $\mathbf{h}$, define $G(n)=\psi(l, n)$. Then

$$\mathscr{J}^*(l) = \{n: X < G(n) \leqq 2X \text{ and } N_1 < (n+\mathbf{t}\cdot\mathbf{h})G(n) \leqq N \text{ for all } \mathbf{t}\in\mathscr{V}\}.$$

From (6.5) and (5.3), we see that

$$G'(n) = \frac{-(r+q)}{(r+1)}\frac{\psi(l, n)}{n}\{1+O(HY^{-1})\}.$$

Therefore G is monotonic, and the set $\{n: X<G(n)\leqq 2X\}$ is an interval. For each fixed $\mathbf{t}\in\mathscr{V}$, the function $(n+\mathbf{t}\cdot\mathbf{h})G(n)$ has derivative

$$(n+\mathbf{t}\cdot\mathbf{h})G'(n)+G(n) = -(q-1)(r+1)^{-1}\psi(l, n)\{1+O(HY^{-1})\}.$$

This shows that if $q>1$, then the set

$$\{n: N_1 < (n+\mathbf{t}\cdot\mathbf{h})G(n) \leqq N\}$$

is an interval, and $\mathscr{J}^*(l)$ is an interval since it is an intersection of intervals.

The case $q=1$ must be argued more carefully. From (5.3) and the definition of f_1 we see that

$$f_{1uu}(\psi, n)\{(n+th)G'(n)+G(n)\} = f_{1uu}(\psi, n)\cdot\psi-(n+th)\, f_{1uv}(\psi, n) =$$

$$= -r^2(r+1)\, hy\psi^{-r-1}\int_0^1 (n+\tau h)^{-r-2}(\tau-t)\, d\tau,$$

where $t=0$ or 1. Again, we see that $(n+th)G(n)$ is monotonic, and the desired result follows.

LEMMA 14. *In the notation of Lemma* 13,

$$S \ll \mathscr{L}\{F^{1/(3Q-2)}X^{1-1/(3Q-2)}Y^{1-(q+1)/(3Q-2)}+(XY)^{1-1/3Q}+XY^{1/2}+F^{-1/4}XY\}.$$

PROOF. For $q=1$, this follows from Lemma 13. For larger q, we supplement Lemma 13 with the following simple estimate. Note that

$$|S| \leqq \sum_{X<m\leqq 2X}\Big|\sum_{\substack{n\\(m,n)\in\mathscr{D}}} e(f(m, n))\Big|,$$

and apply the exponent pair $\mathrm{A}^{q-1}\mathrm{B}$ to the inner sum to get

$$S \ll \max(F^{1/(2Q-2)}XY^{1-(q+1)/(2Q-2)}, F^{-1}XY\}. \tag{6.6}$$

Our proof is by induction on q. We already have the result for $q=1$, so assume that it is true with q replaced by $q-1$. From the induction hypothesis, we get

$$S \ll \mathscr{L}\max(F^{2/(3Q-4)}X^{1-2/(3Q-4)}Y^{1-2q/(3Q-4)}, (XY)^{1-2/3Q}, XY^{1/2}, F^{-1/4}XY). \tag{6.7}$$

From Lemma 13, we get

$$S \ll \mathscr{L} \max (F^{1/(3Q-2)} X^{1-1/(3Q-2)} Y^{1-(q+1)/(3Q-2)}, \tag{6.8}$$

$$(XY)^{1-1/3Q}, XY^{1-1/Q}, F^{-1/2Q} XY^{1+(q-2)/2Q+1/Q^2}) = \max (E_1, E_2, E_3, E_4),$$

say. Let M_1, M_2, M_3 denote the right hand sides of (6.6), (6.7), and (6.8) respectively. The desired result is immediate unless

$$M_1 = F^{1/(2Q-2)} XY^{1-(q+1)/(2Q-2)},$$

$$M_2 = F^{2/(3Q-4)} X^{1-2/(3Q-4)} Y^{1-2q/(3Q-4)},$$

and $M_3 = E_3$ or E_4. If $M_3 = E_3$, then

$$S \ll \mathscr{L} (M_2^{3Q-4} M_3^{3Q})^{1/(6Q-4)} = \mathscr{L} F^{1/(3Q-2)} X^{1-1/(3Q-2)} Y^{1-(2q+3)/(6Q-4)},$$

so the lemma follows. If $M_3 = E_4$ then

$$S \ll \mathscr{L} M_1^a M_2^b M_3^c \leqq \mathscr{L} F^{1/(3Q-2)} X^{1-1/(3Q-2)} Y^{1-(q+1)/(3Q-2)-d}$$

where

$$a = \frac{3Q(2Q-2)}{(6Q-4)(4Q-2)}, \quad b = \frac{3Q-4}{6Q-4},$$

$$c = \frac{6Q^2}{(6Q-4)(4Q-2)}, \quad d = \frac{(Q-2)}{(3Q-2)(8Q-4)}.$$

Since $d > 0$, the lemma follows in this case as well.

LEMMA 15. *In the notation of Lemma* 13,

$$S \ll \{F^{1/(3Q-2)} X^{1-q/(3Q-2)} Y^{1-2/(3Q-2)} + (XY)^{1-1/3Q} + XY^{1/2} + F^{-1/4} XY\} N^{\varepsilon}.$$

PROOF. The proof is similar to the previous two lemmas, so we shall only sketch it here. Note that we may assume that $q \geqq 2$; the case $q = 1$ follows from Lemma 14.

We apply Lemma 12(b) to (6.3) $q-1$ times to get

$$|S|^Q \ll \frac{(XY)^Q}{H} + \frac{(XY)^{Q-1}}{H_1 \dots H_q} \sum_{\mathbf{h} \in \mathscr{H}} |S_q(\mathbf{h})|,$$

where

$$H_q = H, \quad H_{q-1} = H^{1/2}, \dots, H_1 = H^{2/Q}, \quad H \leqq \min (X, Y^{Q/2}),$$

$$S_q(\mathbf{h}) = \sum_{(m,n) \in \mathscr{D}(\mathbf{h})} e(f_q(m, n; \mathbf{h})),$$

$$f_q(m, n; \mathbf{h}) = \int_0^1 \dots \int_0^1 \frac{\partial^q f(m + t_2 h_2 + \dots + t_q h_q, n + t_1 h_1)}{\partial t_1 \dots \partial t_q} \, dt_1 \dots dt_q,$$

and

$$\mathscr{D}(\mathbf{h}) = \{(m, n) : (m + t_2 h_2 + \dots + t_q h_q, n + t_1 h_1) \in \mathscr{D} \text{ for all}$$

$$(t_1, \dots, t_q) \text{ with } t_i = 0 \text{ or } 1\}.$$

We apply Lemma 10 to $S_q(\mathbf{h})$ and use Lemma 2 to get

$$(6.9)\quad S \ll \mathscr{L} \max (F^{1/(3Q-2)}X^{1-q/(3Q-2)}Y^{1-2/(3Q-2)}, (XY)^{1-1/3Q}, X^{1-1/Q}Y, XY^{1/2},$$
$$F^{-1/2Q}X^{1+(q-3)/2Q+1/Q^2}Y^{1+1/2Q}, F^{-1/2Q}X^{1+(q-1)/2Q}Y^{1/2+1/Q}).$$

We can supplement this with an estimate along the lines of Lemma 3. Let (k, l) be an exponent pair, and let $(\varkappa, \lambda)=A(k, l)$. From equation (4.6), we see that

$$\sum_{X<m\leqq 2X}\ \sum_{Y<n\leqq 2Y} a(m)e(ym^{-r}n^{-r}) \ll N^{\varepsilon}\{F^{\varkappa}X^{\lambda}Y^{1-\varkappa}+X^{\lambda}Y+XY^{1/2}+F^{-1/2}XY\}.$$

By using the integration argument of Lemma 4, we can show the above estimate also holds when the extra condition $(m, n)\in\mathscr{D}$ is imposed on the sum. Taking $(\varkappa, \lambda)=A^{q-1}B(0, 1)=(1/(2Q-2),\ 1-q/(2Q-2))$ yields

$$(6.10)\quad S \ll N^{\varepsilon} \max \{F^{1/(2Q-2)}X^{1-q/(2Q-2)}Y^{1-1/(2Q-2)}, X^{1-q/(2Q-2)}Y, XY^{1/2}, F^{-1/2}XY\}.$$

The proof can then be completed by using induction and convexity.

LEMMA 16. *Assume that* $q\geqq 2$. *In the notation of Lemma* 13,

$$S \ll N^{\varepsilon}\{F^{1/(3Q-2)}(XY)^{1-(q+2)/(6Q-4)}+(XY)^{11/12}+XY^{1/2}+F^{-1/4}XY\}.$$

PROOF. From Lemma 14, Lemma 15, and convexity, we obtain

$$(6.11)\quad S \ll \max \left(F^{1/(3Q-2)}(XY)^{1-(q+2)/(6Q-4)}, (XY)^{1-1/3Q}, XY^{1/2}, F^{-1/4}XY\right)N^{\varepsilon}.$$

(Note that the convexity argument requires $q\geqq 2$.) When $q=2$, (6.11) is the desired result. We complete the proof by induction.

Assume the lemma holds with $q-1$ in place of q. Then

$$(6.12)\quad S \ll \max \left(F^{2/(3Q-4)}(XY)^{1-(q+1)/(3Q-4)}, (XY)^{11/12}, XY^{1/2}, F^{-1/4}XY\right)N^{\varepsilon}.$$

Let E_1 and E_2 denote the right hand sides of (6.11) and (6.12) respectively. Our lemma is immediate unless

$$E_1 = (XY)^{1-1/3Q}, \quad E_2 = F^{2/(3Q-4)}(XY)^{1-(q+1)/(3Q-4)}.$$

In this case, we have

$$S \ll N^{\varepsilon}(E_1^{3Q}E_2^{3Q-4})^{1/(6Q-4)} = N^{\varepsilon}F^{1/(3Q-2)}(XY)^{1-(q+2)/(6Q-4)},$$

which is the desired result.

Now we consider $\sum_3$. Recall that

$$\sum\nolimits_3 = \sum_{(m,n)\in\mathscr{D}} B(m)\psi(xm^{-r}n^{-r})$$

where $\mathscr{D}=\{(m, n)\colon m\leqq u,\ N_1<mn\leqq N\}$. We can divide $\mathscr{D}$ into the appropriate rectangles and obtain

$$\sum\nolimits_3 \ll \mathscr{L}^2 \max_{X,Y} \Big| \sum_{X<m\leqq 2X}\ \sum_{\substack{Y<n\leqq 2Y\\(m,n)\in\mathscr{D}}} B(m)\psi(xm^{-r}n^{-r})\Big|,$$

the maximum being taken over X and Y with $1/2 \leqq X \leqq u$ and $N_1/4 < XY \leqq N$. From Lemma 1 we see that

$$\sum_{X<m\leqq 2X} \sum_{\substack{Y<n\leqq 2Y \\ (m,n)\in\mathscr{D}}} B(m)\psi(xm^{-r}n^{-r}) \ll N^{\varepsilon}(N_1+U(X,Y)),$$

where

$$U(X,Y) = \sum_{j\leqq XYN_1^{-1}} j^{-1}\Big|\sum_{X<m\leqq 2X} \sum_{\substack{Y<n\leqq 2Y \\ (m,n)\in\mathscr{D}}} a(m)e(xjm^{-r}n^{-r})\Big|,$$

and the maximum is taken over all sequences $a(m)$ with $|a(m)|\leqq 1$. To prove $\sum_3 \ll N^{1-\delta+\varepsilon}$, it therefore suffices to prove that $U(X,Y)\ll N^{\varepsilon}N_1$ for X and Y with $1/2\leqq X\leqq u$ and $N_1/4\leqq XY\leqq N$. We do this by means of Lemmas 15 and 16. The final result is

LEMMA 17. *For r satisfying $2\leqq r\leqq 20$, let $\alpha(r)$ be defined by*

$$\alpha(r) = \begin{cases} \dfrac{1}{6} & \text{if } 2 \leqq r \leqq 4, \\ \dfrac{1}{7} & \text{if } r = 5 \\ \dfrac{r-3}{12r-47} & \text{if } 6 \leqq r \leqq 12, \\ \dfrac{r+14}{24r-9} & \text{if } 13 \leqq r \leqq 20. \end{cases}$$

For $r>20$, let $\alpha(r)$ be defined as follows. Let $q=q(r)$ be the unique integer satisfying $(Q=2^q)$

$$12Q-q-5 < 2r \leqq 24Q-q-6, \tag{6.13}$$

and let

$$\alpha(r) = \frac{60q+r+2}{12Qr+6Qq-12Q+3}.$$

If $\delta\leqq\alpha(r)$ then $\sum_3 \ll N^{\varepsilon}N_1$.

PROOF. We write $M=XY$ and $w=xN_1^{-1}=N^{r-\delta}$. From Lemma 15 with $q=1$, we get

$$U(X,Y) \ll u^{1/4}w^{1/4}M^{(3-r)/4}+N_2, \tag{6.14}$$

where

$$N_2 = N^{5/6}+u^{1/2}N^{1/2}+x^{-1/4}N^{1+r/4}.$$

Observe that $N_2\ll N_1$ if $\delta\leqq 1/6$.

Assume that $r=2$ or 3. From (6.14) we see that

$$U(X,Y) \ll N^{(4-5\delta)/4}+N_2 \ll N_1$$

provided $\delta\leqq 1/6$.

From Lemma 16 with $q=2$ we obtain

$$U(X,Y) \ll N^{\varepsilon}\{w^{1/10}M^{(9-r)/10}+M^{11/12}+N_2\}. \tag{6.15}$$

Assume that $4\leqq r\leqq 8$. By convexity, we see that

$$\min\left((wu)^{1/4}M^{(3-r)/4},\ w^{1/10}M^{(9-r)/10}\right) \leqq (w^6u^{9-r})^{1/(6r+6)}$$

and

$$\min\left((wu)^{1/4}M^{(3-r)/4},\ M^{11/12}\right) \leqq (wu)^{11/(12r+8)}.$$

Consequently, statements (6.14) and (6.15) imply that

$$U(X,Y) \ll N^{\varepsilon}\{(w^6u^{9-r})^{1/(6r+6)}+(wu)^{11/(12r+8)}+N_2\} \ll N^{\varepsilon}N_1$$

provided

$$\delta \leqq \min\left(\frac{r-3}{10r-36},\ \frac{r-3}{12r-47},\ \frac{1}{6}\right).$$

This proves the result for $4\leqq r\leqq 9$.

From Lemma 16 with $q=3$ we get

$$U(X,Y) \ll N^{\varepsilon}\{w^{1/22}M^{(41-2r)/44}+M^{11/12}+N_2\}. \tag{6.16}$$

Now assume that $10\leqq r\leqq 20$. From convexity, (6.14), (6.15), and (6.16), we obtain

$$U(X,Y) \ll N^{\varepsilon}\{w^{23/(24r+14)}+(wu)^{11/(12r+8)}+N_2\} \ll N_1^{\varepsilon}N_1$$

provided

$$\delta \leqq \min\left(\frac{r+14}{24r-9},\ \frac{r-3}{12r-47},\ \frac{1}{6}\right).$$

This proves the result for $10\leqq r\leqq 20$.

Now suppose $r\geqq 21$, and let q be defined by (6.13). From Lemma 16 with q replaced by $q+1$, we see that

$$U(X,Y) \ll N^{\varepsilon}\{w^{1/(6Q-2)}M^{1-(q+1+2r)/(12Q-4)}+N^{11/12}+N_2\}.$$

Similarly, we can replace q by $q+2$ in Lemma 16 and obtain

$$U(X,Y) \ll N^{\varepsilon}\{w^{1/(12Q-2)}M^{1-(q+2+2r)/(24Q-4)}+N^{11/12}+N_2\}.$$

We apply convexity and obtain

$$U(X,Y) \ll N^{\varepsilon}\{w^{(A+B)/C}+N^{11/12}+N_2\},$$

where $A=2r-12Q+q+5$, $B=-2r+24Q-q-6$, and $C=(12Q-2)A+(6Q-2)\mathrm{B}$. Consequently, $U(X,Y)\ll N^{\varepsilon}N_1$ whenever

$$\delta \leqq \min\left(\frac{C-Ar-Br}{C-A-B},\ \frac{1}{12}\right) = \min\left(\frac{6Qq+r+2}{12Qr+6Qq-12Q+3},\ \frac{1}{12}\right).$$

To complete the proof, it suffices to show that

$$\frac{6Qq+r+2}{12Qr+6Qq-12Q+3} \leqq \frac{1}{12}, \tag{6.17}$$

when $21 \leqq r < 44$, this may be shown by direct computation. Assume that $r \geqq 44$, i.e. $q \geqq 3$ and $Q \geqq 8$. From the definition of q, we see that the left hand side of (6.17) is

$$\leqq \frac{6Qq+12Q}{12Qr+6Qq-12Q} = \frac{q+2}{2r+q-2}.$$

To prove (6.17) it therefore suffices to show that

$$2r \geqq 11q+26. \tag{6.18}$$

From the inequalities $q \geqq 3$ and $Q \geqq 8q/3$, we see that $2r \geqq 12Q-q-4 \geqq 31q-4 \geqq$ $\geqq 11q+56$. This proves (6.18), so (6.17) follows.

§ 7. Proof of Theorem 1

From Lemmas 5 and 17, we see that

$$\sum_{N_1 < n \leqq N} \mu(n)\psi(xn^{-r}) \ll N^{\varepsilon}N_1$$

provided $\delta \leqq \min(\alpha(r), \beta(r))$. After we decide which of these two quantities is the minimum, we can appeal to (2.2) to finish the proof.

A direct computation shows that $\beta(r) \leqq \alpha(r)$ when $2 \leqq r \leqq 6$ and $\alpha(r) < \beta(r)$ when $7 \leqq r \leqq 61$. We now complete the proof by showing that $\alpha(r) \leqq \beta(r)$ for $r \geqq 62$.

For fixed r, let q be the unique integer such that

$$2Q-3 < r \leqq 4Q-3$$

(as usual, $Q=2^q$) and let q' be the unique integer such that

$$12Q'-q'-5 < 2r \leqq 24Q'-q'-6$$

($Q'=2^{q'}$). By definition,

$$\alpha(r) = \frac{6Q'q'+r+2}{12Q'r+6Q'q'-12Q'+3}$$

and

$$\beta(r) = \frac{2Qq-4Q+r+3}{2Qr+8Qq-18Q+4r+12}.$$

From the definitions of q and q', we see that

$$\beta(r) \geqq \frac{2Qq-4Q+2Q}{2Qr+8Qq-2Q} = \frac{q-1}{r+4q-1}$$

and

$$\alpha(r) \leqq \frac{6Q'q'+12Q'}{12Q'r+6Q'q'-12Q'} = \frac{q'+2}{2r+q'-2}. \tag{7.1}$$

On the other hand, $4Q-3 \geqq r \geqq 6Q'-q'/2-5/2$, so $4Q > 6Q'-q'-2 \geqq 5Q'$. Since q and q' are integers, we must have $q \geqq q'+1$. From (7.1,) we see that

$$\alpha(r) \leqq \frac{q+1}{2r+q-3}.$$

We can therefore complete the proof if we can show that

$$\frac{q+1}{2r+q-3} \leqq \frac{q-1}{r+4q-1}.$$

This statement is equivalent to

$$3q^2+7q-4 \leqq r(q-3).$$

By $r \geqq 62$, we have $q \geqq 5$, and so

$$3q^2+7q-4 \leqq 3q^2+7q+8q(q-5) = 11q(q-3) \leqq r(q-3)$$

since $r \geqq 2Q-3 = 2^{q+1}-3 \geqq 11q$ for $q \geqq 5$.

§ 8. Proof of Theorem 2

As in the proof of Theorem 2, we need an estimate of the form

$$\sum_{N_1 < n \leqq N} \mu(n)\psi(xn^{-r}) \ll N^{\varepsilon}N_1.$$

As before, we write $x = N^{r+1-2\delta}$, $N_1 = N^{1-\delta}$, and

$$\sum_{N_1 < n \leqq N} \mu(n)\psi(xn^{-r}) = \sum\nolimits_1 - \sum\nolimits_2 - \sum\nolimits_3,$$

where $\sum_1$, $\sum_2$, and $\sum_3$ are defined after equation (2.1). We deviate from the proof of Theorem 1 in our choice of u and v; here we take $u = v = N^{2\delta}$. Our primary tool is the following consequence of Vinogradov's mean value theorem.

LEMMA 18. *Let M, N be natural numbers with $1 \leqq M \leqq N$. Suppose $f: [N, 2N] \to \mathbf{R}$ is L times differentiable and that there is some $\lambda \geqq 1/2$ such that*

$$\frac{cN^{\lambda-l}}{4^l} \leqq \left|\frac{f^{(l)}(x)}{l!}\right| \leqq N^{\lambda-l}$$

for $l = 1, \ldots, L$. Assume moreover that $L \geqq 6\lambda$. Then

$$\left|\sum_{n=N+1}^{N+M} e(f(n))\right| \leqq 3c^{-1}N^{1-1/49152\lambda^2}.$$

REMARK. This can be proved by following the argument of Théorème 10.3 of Ellison and Mendès-France [3] and making the following changes. They hypothesize bounds of $f^{(l)}$ for all l, but in fact they use these bounds only for $l \leqq k+1$. In their notation, $k = 2h = 2[2\lambda]$ (p. 408, line —9) so $k+1 \leqq 4\lambda+1 \leqq 6\lambda$. Further-

more, they assume $c=1$. To allow for arbitrary c, we need only observe that on p. 409, lines 6—8, we have

$$q^{2r}|\alpha_r| \geqq c2^{-4r}N^{\lambda-2r\eta}$$

and therefore in lines 10—11 we have

$$\theta_r(\alpha_r) \prec c^{-1}2^{4r}N^{2r\eta-\lambda}\gamma^{-1}q^{r\gamma}.$$

(Note that in line 11, $N^{2r\eta-\lambda}$ is misprinted as $N^{2r\gamma-\lambda}$.) In view of this, we have, in line 15,

$$\sum_{r=1}^{k}\theta_r(\alpha_r) \leqq c^{-k}(q^{\gamma k^2}\gamma^{-k}2^{2h(3h+1)}N^{\eta h(3h+1)-\lambda h}).$$

Since $b=lk^2$ (p. 406, line -1) and $l>1$ (p. 410, line -1) we have $k/4b^2<1$, so we do not lose more than c^{-1}.

In our applications, we need an estimate of the form

$$\sum_{N<m\leqq N+M} e(Am^{-r}) \ll_r N\exp(-c\Lambda^{-2}\log N), \tag{8.1}$$

where $A=N^{r+\Lambda}$ and $\Lambda\ll r^{1/3}$. Let $f(m)=Am^{-r}$. On the interval $[N, 2N]$,

$$\frac{(r+l-1)!\,A}{l!(r-1)!(2N)^{r+l}} \leqq \frac{|f^{(l)}(m)|}{l!} \leqq \frac{(r+l-1)!\,A}{l!(r-1)!\,N^{r+l}}$$

for all l. Thus the hypotheses of the previous lemma are satisfied with

$$N^{\lambda} = \frac{N^{\Lambda}(r+L-1)!}{L!(r-1)!}, \quad c = \frac{L!(r-1)!}{(r+L-1)!\,2^r}.$$

We choose $L=r$, and get

$$\lambda = \Lambda + (\log D_r)/\log N,$$

where D_r depends only on r. Since we are not striving for uniformity in r, we may assume that $N\geqq N_0(r)$. By making an appropriate choice of N_0, we can make $\Lambda\geqq\lambda(1-\varepsilon)$ for any $\varepsilon>0$. Estimate (8.1) then follows with some $c<1/49152$ ($c=1/50000$, say).

We now consider Type II sums.

LEMMA 19. *There is some constant c such that if $\delta\leqq cr^{-2/3}$ then $\sum_1+\sum_2\ll$ $\ll N^{1-\delta+\varepsilon}$.*

PROOF. By a splitting-up argument, we see that

$$\sum_1+\sum_2 \ll N^{\varepsilon}\max_{X,Y}|U(X,Y)| \tag{8.2}$$

where the maximum is taken over X and Y with $u\leqq X\leqq N/v$, $v\leqq Y\leqq N/u$, $N_1/4<$ $<XY\leqq N$, and

$$U(X,Y) \leqq \sup\Big|\sum_{X<m\leqq 2X}\sum_{\substack{Y<n\leqq 2Y\\ N_1<mn\leqq N}} a(m)\,b(n)\,\psi(xm^{-r}n^{-r})\Big|;$$

the supremum being taken over those sequences $a(m)$ and $b(n)$ with $|a(m)| \leqq 1$ and $|b(n)| \leqq 1$. We write $M = XY$ and apply Lemma 1 to get

$$(8.3)\quad U(X,Y) \ll N_1 + \sup_{j \leqq MN_1^{-1}} j^{-1} \Big| \sum_{X < m \leqq 2X} \sum_{\substack{Y < n \leqq 2Y \\ N_1 < mn \leqq N}} a(m)\, b(n)\, e(xjm^{-r} n^{-r}) \Big|.$$

Because of the symmetry between m and n, we may assume that $X \geqq Y$. It follows that $M^{1/2} \leqq X \leqq MN^{-2\delta}$. We also write $M = N^{1-\eta}$, so $0 \leqq \eta \leqq \delta$. Assume also that $\delta \leqq 1/8$.

Let

$$(8.4)\qquad V = \sum_{X < m \leqq 2X} \sum_{\substack{Y < n \leqq 2Y \\ N_1 < mn \leqq N}} a(m)\, b(n)\, e(xjm^{-r} n^{-r}).$$

By Cauchy's inequality,

$$|V|^2 \leqq \sum_m |a(m)|^2 \sum_{n,q} b(n)\, \overline{b(q)} \sum_m e\left(\frac{xj}{m^r} \left(\frac{1}{n^r} - \frac{1}{q^r} \right) \right) =$$

$$= \sum_m |a(m)|^2 \sum_{n,q} b(n)\, \overline{b(q)}\, V(q,n).$$

Removing the diagonal terms, we see that

$$(8.5)\qquad |V|^2 \ll X^2 Y + XY^2 \max_{q \neq r} |V(q,n)|.$$

It thus suffices to bound $V(q,n)$ when $q \neq n$.

Let $H = xj|q^{-r} - n^{-r}|$, and write $H = X^{r+\lambda}$. Before applying Lemma 18, we need to show that

$$(8.6)\qquad 1/2 \leqq \lambda \leqq 16(1+r\eta)/7.$$

To prove the left-hand side of the above, note that $H \gg xY^{-r-1}$. Since $X \geqq Y$, we get

$$X^{r+1/2} Y^{r+1} \leqq (XY)^{r+3/4} \leqq N^{r+3/4}.$$

Since $r + 1 - 2\delta \geqq r + 3/4$, the right hand side is $\leqq x$. In other words, $H \gg xY^{-r-1} \gg X^{r+1/2}$, so $\lambda \geqq 1/2$.

To prove the right hand side of (7.6), we note that

$$X^{r+\lambda} = H \ll xMN_1^{-1} Y^{-r} = N^{r-\delta} MY^{-r},$$

so

$$X^{\lambda} \ll (NX^{-1}Y^{-1}) MN^{-\delta} \ll (NM^{-1})^r N = N^{1+r\eta}.$$

Since $X \gg M^{1/2} = N^{(1-\eta)/2} \geqq N^{7/16}$, the desired result follows. From Lemma 18, we see that

$$V(q,r) \ll X \exp(-c_1 \lambda^{-2} \log X) \ll X \exp(-c_2 (1+r\eta)^{-2} \log N).$$

Therefore

$$|V|^2 \ll X^2 Y + X^2 Y^2 \exp(-c_2 (1+r\eta)^{-2} \log N) \ll N^{2-2\delta} + N^2 \exp(-f(\eta) \log N),$$

where $f(\eta)=2\eta+c_2(1+r\eta)^{-2}$. To complete the proof, we need to show that $f(\eta)\gg \gg r^{-2/3}$ for all η. If $\eta\leqq 1/r$ then $f(\eta)\gg 1$. If $\eta>1/r$, then $f(\eta)\gg\min(\eta, r^{-2}\eta^{-2})\geqq \geqq\eta^{2/3}(r^{-2}\eta^{-2})^{1/3}=r^{-2/3}$. Thus there is some c such that $V\ll N^{1-\delta}$ when $\delta\leqq cr^{1/3}$. Combining this with (7.3) and (7.2) gives the result stated in the lemma.

LEMMA 20. *There is some constant c such that if $\delta\leqq cr^{-2/3}$ then $\sum_3\ll N^{1-\delta+\varepsilon}$.*

PROOF. By a splitting-up argument, we see that

$$\sum_3 \ll N^{\varepsilon}\max_{X,Y}\left|\left(U(X,Y)\right)\right|, \tag{8.7}$$

where

$$U(X,Y)=\sum_{Y<m\leqq 2Y}\Big|\sum_{\substack{X<n\leqq 2X\\ N_1<mn\leqq N}}\psi(xm^{-r}n^{-r})\Big|,$$

and the maximum is taken over X and Y with $N^{1-2\delta}\leqq X\leqq N$ and $N_1/4\leqq XY\leqq N$. From Lemma 1, we see that if $XY=M$ then

$$U(X,Y)\ll N_1+\sum_{j\leqq MN_1^{-1}}j^{-1}\sum_{Y<m\leqq 2Y}\Big|\sum_{\substack{X<n\leqq 2X\\ N_1<mn\leqq 2N}}e(xjm^{-r}n^{-r})\Big|.$$

Now let $H=xjm^{-r}=X^{r+\lambda}$ and $M=N^{1-\eta}$. Note that $0\leqq\eta\leqq 2\delta$. We may also assume that $\delta\leqq 1/8$, so $\eta\leqq 1/4$.

To apply Lemma 18, we need upper and lower bounds for λ. Now $X^{r+\lambda}\geqq xY^{-r}$, so $X^{\lambda}\geqq xM^{-r}\geqq xN^{-r}=N^{1-2\delta}\geqq X^{3/4}$. Therefore $\lambda\geqq 3/4$. On the other hand,

$$X^{r+\lambda}\leqq\frac{xM}{N_1Y^r}=\frac{N^{r-\delta}M}{Y^r},\quad\text{so}\quad X^{\lambda}\leqq N^{r-\delta}M^{1-r}\leqq N^{1+\eta r}.$$

Since $X\geqq N^{1-2\delta}\geqq N^{3/4}$, we have $\lambda\leqq 4(1+r\eta)/3$.

From Lemma 18, we get

$$U(X,Y)\ll N_1+\mathscr{L}XY\exp\left(-c_1\lambda^{-2}\log X\right)\ll$$

$$\ll N_1+\mathscr{L}XY\exp\left(-c_2(1+r\eta)^{-2}\log N\right)\ll N_1+\mathscr{L}\exp\left(-g(\eta)\log N\right),$$

where $g(\eta)=\eta+c_2(1+\eta r)^{-2}$. The argument given at the end of the proof of Lemma 19 shows that $g(\eta)\gg r^{-2/3}$, so the result follows.

Finally, we note that Theorem 2 follows immediately from Lemma 18, Lemma 19, and (2.2).

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BERNSTEIN TYPE INEQUALITIES FOR A CLASS OF POLYNOMIALS

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Let n be a nonnegative integer, $0<\omega\leqq\pi$, and denote by T_n the set of all real trigonometric polynomials of order at most n. In [1] we introduced the following class of trigonometric polynomials:

$$\mathscr{T}_n(\omega)=\left\{p(t)=\sum_{l=0}^{2n}a_l\sin^l\frac{\omega-t}{2}\sin^{2n-l}\frac{t+\omega}{2}\;\middle|\;\text{all } a_l\geqq 0 \text{ or all } a_l\leqq 0\right\}.$$

This proved to be a wide subset of T_n (e.g. it contains all the trigonometric polynomials $p(t)$ of order at most n for which $p(x+iy)\neq 0$ if $\operatorname{ch} y\cos\omega\geqq\cos x$, $-\pi\leqq x<\pi$, see [1]), which possessed a number of interesting properties. In this paper we will show that the analogue of the Bernstein type inequality shows an essential improvement in $\mathscr{T}_n(\omega)$ (in fact, even in a slightly wider class, see below).

For an arbitrary $p(t)\in T_n$ we have

$$|p'(t)|\leqq\frac{c_1 n}{\sqrt{\omega^2-t^2}}\|p\|\quad(|t|<\omega)\tag{1}$$

where c_1 (and in what follows $c_2, c_3, \ldots$) is an absolute positive constant and $\|\cdot\|$ denotes the sup norm over the interval $[-\omega,\omega]$ (see Videnskiĭ [2]). In [3], the second named author improved this to

$$|p'(t)|\leqq\frac{c_2\sqrt{n}}{\omega(\omega^2-t^2)}\|p\|\quad(|t|<\omega)$$

for trigonometric polynomials having only real roots outside $(-\omega,\omega)$ (mod 2π), the latter class $T_n^0(\omega)$ being a subset of $\mathscr{T}_n(\omega)$. Now we show

THEOREM 1. *If* $q(t)=p(t)r(t)$, $p(t)\in\mathscr{T}_n(\omega)$, $r(t)\in T_k$ *then*

$$|q'(t)|\leqq c_3(k+1)^2\sqrt{\frac{n+1}{\omega^2-t^2}}\|q\|\quad(|t|<\omega\leqq\pi,\ n,k\geqq 0).\tag{2}$$

Of course, Theorem 1 gives a reasonable estimate only if $k=o(\sqrt[4]{n})$, otherwise the Bernstein type inequality (1) is better. It would be important to improve $(k+1)^2\sqrt{n+1}$ to $\sqrt{(n+k)(k+1)}$ in (2) but at present this is an open problem.

For the proof we need a series of lemmas.

LEMMA 1. *Let* $k\geqq 1$, $r(x)\in T_k$, $\varphi>0$, $\tilde{\varphi}:=\varphi\left(1+\frac{1}{8k^2}\right)\leqq\frac{\pi}{2}$. *Then*

(i) $$\max_{|x|\leqq\tilde{\varphi}}|r(x)|\leqq 2\max_{|x|\leqq\varphi}|r(x)|,$$

(ii) $$\max_{|x|\leqq\tilde{\varphi}}|r'(x)|\leqq\frac{8k^2}{\varphi}\max_{|x|\leqq\varphi}|r(x)|.$$

PROOF. Using a Markov type estimate of Videnskiĭ [1], we have

(3) $$\max_{|x|\leqq\tilde{\varphi}}|r'(x)|\leqq 2k^2\cot\frac{\tilde{\varphi}}{2}\max_{|x|\leqq\tilde{\varphi}}|r(x)|\leqq\frac{4k^2}{\tilde{\varphi}}\max_{|x|\leqq\tilde{\varphi}}|r(x)|.$$

By the mean value theorem and (3), if $|x_1|, |x_2|\leqq\tilde{\varphi}$, $|x_1-x_2|\leqq\frac{\varphi}{8k^2}$ then

$$|r(x_1)-r(x_2)|\leqq|x_1-x_2|\max_{|x|\leqq\tilde{\varphi}}|r'(x)|\leqq\frac{1}{2}\max_{|x|\leqq\tilde{\varphi}}|r(x)|.$$

Therefore

$$\max_{|x|\leqq\tilde{\varphi}}|r(x)|\leqq\max_{|x|\leqq\varphi}|r(x)|+\frac{1}{2}\max_{|x|\leqq\tilde{\varphi}}|r(x)|,$$

hence (i) follows. By (3) and the just proved inequality (i), we get (ii). □

LEMMA 2. *Let* $0<\omega\leqq\frac{\pi}{2}$ *and* $p_{ln}(t)=\sin^l\frac{\omega-t}{2}\sin^{2n-l}\frac{t+\omega}{2}$ $(l=0, 1, \ldots, 2n)$. *Then*

$$|p'_{ln}(t)|\leqq\frac{18}{\delta}\max\{p_{ln}(t), p_{ln}(t\pm\delta)\}$$

$$\left(|t|\leqq\frac{n-1}{n+1}\omega,\ 0<\delta\leqq\sqrt{\frac{\omega^2-t^2}{n}},\ l=0, 1, \ldots, 2n\right).$$

PROOF. Let

(4) $$\omega_1=\omega_1(t)=t-\delta\sqrt{n\frac{\omega+t}{\omega-t}},\quad \omega_2=\omega_2(t)=t+\delta\sqrt{n\frac{\omega-t}{\omega+t}}.$$

Then, by $|t|\leqq\frac{n-1}{n+1}\omega$ and $0<\delta\leqq\sqrt{\frac{\omega^2-t^2}{n}}$, obviously $(\omega_1, \omega_2)\subset(-\omega, \omega)$, whence a representation

$$p_{ln}(\tau)=\sum_{j=0}^{2n}b_{jlnt}\sin^j\frac{\omega_2-\tau}{2}\sin^{2n-j}\frac{\tau-\omega_1}{2}\quad(b_{jlnt}\geqq 0,\ j, l=0, 1, \ldots, 2n)$$

holds (cf. [1], Theorem 1). Thus, with the notation

$$q_{jnt}(\tau)=\sin^j\frac{\omega_2-\tau}{2}\sin^{2n-j}\frac{\tau-\omega_1}{2}\quad(j=0, 1, \ldots, 2n),$$

it suffices to prove the inequalities

$$|q'_{jnt}(t)| \leqq \frac{18}{\delta} \max \{q_{jnt}(t), q_{jnt}(t\pm\delta)\} \quad (j = 0, 1, \ldots, 2n) \tag{5}$$

under the conditions of the lemma. $\left(\text{Here } q'_{jnt}(t)=\left.\frac{\partial q_{jnt}}{\partial \tau}\right|_{\tau=t}, \text{ to be precise.}\right)$ In proving (5) we distinguish two cases.

Case 1: t is such that $q''_{jnt}(\tau)=\frac{\partial^2 q_{jnt}(\tau)}{\partial \tau^2}>0$ when $|t-\tau|<\delta$. Hence $q'_{jnt}(\tau)$ is monotone increasing in $(t-\delta, t+\delta)\subset(\omega_1, \omega_2)$ (see (4)), thus if e.g. $q'_{jnt}(t)\geqq 0$, then by the mean value theorem

$$0 \leqq q'_{jnt}(t) \leqq q'_{jnt}(\xi) = \frac{q_{jnt}(t+\delta)-q_{jnt}(t)}{\delta} \leqq \frac{q_{jnt}(t+\delta)}{\delta} \quad (t < \xi < t+\delta).$$

Similarly, if $q'_{jnt}(t)<0$ then $q'_{jnt}(t)\geqq -\frac{q_{jnt}(t-\delta)}{\delta}$. Hence in this case $|q'_{jnt}(t)|\leqq$ $\leqq \frac{1}{\delta}\max q_{jnt}(t\pm\delta)$ which proves (5).

Case 2: t is such that $q''_{jnt}(\tau^*)<0$ for some $|\tau^*-t|\leqq\delta$. Since

$$4\sin^{2-j}\frac{\omega_2-\tau^*}{2}\sin^{2+j-2n}\frac{\tau^*-\omega_1}{2}q''_{jnt}(\tau^*) =$$

$$= \left[(n-j)\sin\frac{\omega_2-\omega_1}{2}-(n-1)\sin\left(\tau^*-\frac{\omega_1+\omega_2}{2}\right)\right]\cdot$$

$$\cdot\left[(n-j)\sin\frac{\omega_2-\omega_1}{2}-n\sin\left(\tau^*-\frac{\omega_1+\omega_2}{2}\right)\right]-$$

$$-2n\sin\frac{\omega_2-\tau^*}{2}\sin\frac{\tau^*-\omega_1}{2}\cos\left(\tau^*-\frac{\omega_1+\omega_2}{2}\right)\leqq 0,$$

with the notation

$$x = (n-j)\sin\frac{\omega_2-\omega_1}{2}-n\sin\left(\tau^*-\frac{\omega_1+\omega_2}{2}\right),$$

using also $\frac{\omega+|t|}{\omega-|t|}\leqq n$ and (4), we get

$$x^2+x\sin\left(\tau^*-\frac{\omega_1+\omega_2}{2}\right)\leqq\frac{n}{2}(\omega_2-\tau^*)(\tau^*-\omega_1)\leqq$$

$$\leqq\frac{n}{2}(\omega_2-t+|t-\tau^*|)\cdot(|\tau^*-t|+t-\omega_1)\leqq$$

$$\leqq\frac{n}{2}\left(\delta\sqrt{n\frac{\omega-t}{\omega+t}}+\delta\right)\left(\delta+\delta\sqrt{n\frac{\omega+t}{\omega-t}}\right)=$$

$$=\frac{n\delta^2}{2}\left(\sqrt{n\frac{\omega-t}{\omega+t}}+\sqrt{n\frac{\omega+t}{\omega-t}}+n+1\right)<2n^2\delta^2.$$

Hence

$$|x| \leqq \frac{\left|\sin\left(\tau^* - \frac{\omega_1+\omega_2}{2}\right)\right| + \sqrt{\sin^2\left(\tau^* - \frac{\omega_1+\omega_2}{2}\right) + 8n^2\delta^2}}{2} \leqq$$

$$\leqq \sin\frac{\omega_2-\omega_1}{2} + \sqrt{2}n\delta < \omega\delta\sqrt{\frac{n}{\omega_2-t^2}} + \sqrt{2}n\delta < \frac{5}{2}n\delta.$$

Thus we obtain

$$\frac{|q'_{jnt}(t)|}{q_{jnt}(t)} = \frac{\left|(n-j)\sin\frac{\omega_2-\omega_1}{2} - n\sin\left(t-\frac{\omega_1+\omega_2}{2}\right)\right|}{2\sin\frac{\omega_2-t}{2}\sin\frac{t-\omega_1}{2}} \leqq$$

$$\leqq \frac{|x|+n|\tau^*-t|}{\frac{2}{\pi^2}(\omega_2-t)(t-\omega_1)} \leqq \frac{7\pi^2}{4}\frac{n\delta}{n\delta^2} < \frac{18}{\delta}$$

and Lemma 2 is completely proved. □

Let

$$\delta_n(t) := \sqrt{\frac{\omega^2-t^2}{n}}.$$

LEMMA 3. *Let* $n \geqq 1$, $k \geqq 1$, $r \in T_k$, $\mu \geqq 300$ *and consider the intervals*

$$I_1(t, \mu) := \left[t - \frac{\delta_n(t)}{\mu}\left(1+\frac{1}{16k^2}\right),\ t - \frac{\delta_n(t)}{16\mu k^2}\right]$$

and

$$I_2(t, \mu) := \left[t - \frac{\delta_n(t)}{16\mu k^2},\ t + \frac{\delta_n(t)}{\mu}\left(1+\frac{1}{16k^2}\right)\right]$$

where $|t| < \omega \leqq \frac{\pi}{2}$ *is fixed. Choose a* $\xi_j \in I_j(t, \mu)$ *for which*

$$|r(\xi_j)| = \max_{x \in I_j(t,\mu)} |r(x)| \quad (j = 1, 2).$$

Then for $q_l := p_{ln} r$ $(l = 0, 1, \ldots, 2n)$ *we have*

$$|q'_l(t)| \leqq \frac{c_4 \mu k^2}{\delta_n(t)} \max\{|q_l(\xi_1)|, |q_l(\xi_2)|\}$$

if $|t| \leqq \left(1 - \frac{c_5}{n}\right)\omega$ *with a suitable* c_5.

PROOF. Choose a $c_5 \geqq 2$ such that

$$|t| + \delta_n(t) \leqq \frac{n-1}{n+1}\omega \tag{6}$$

and

$$\delta_n(t) \leqq 2\delta_n\left(t \pm \delta_n(t)\right) \tag{7}$$

hold when $|t| \leqq \left(1-\frac{c_5}{n}\right)\omega$. (A simple calculation shows that such a c_5 exists.) From now on let t be fixed such that $|t| \leqq \left(1-\frac{c_5}{n}\right)\omega$.

Now if $p_{ln}(\tau)$ is monotone in the interval

$$I_3(t, \mu) := \left[t-\frac{\delta_n(t)}{\mu}\left(1+\frac{1}{16k^2}\right),\ t+\frac{\delta_n(t)}{\mu}\left(1+\frac{1}{16k^2}\right)\right]$$

(e.g. monotone decreasing in $I_3(t, \mu)$, the other case is similar), then

$$0 < p_{ln}(t) \leqq p_{ln}(\xi_1). \tag{8}$$

On the other hand, if there exists an $\eta_l \in \operatorname{int} I_3(t, \mu)$ such that $p'_{ln}(\eta_l)=0$, then $p_{ln}(\eta_l)=\|p_{ln}\|$. Using the mean value theorem, Lemma 2, (6) and (7), further recalling the definition of $I_3(t, \eta)$, for all $\xi \in I_3(t, \eta)$ we get

$$|p_{ln}(\eta_l)-p_{ln}(\xi)| \leqq |\eta_l-\xi_1| \sup_{z \in I_3(t, \mu)} |p'_{ln}(z)| \leqq$$

$$\leqq \frac{3\delta_n(t)}{\mu} \sup_{z \in I_3(t, \mu)} \left\{\frac{18}{\delta_n(z)}\|p_{ln}\|\right\} \leqq \frac{54\delta_n(t)\,p_{ln}(\eta_l)}{\mu\delta_n(|t|+\delta_n(t))} < \frac{108 \cdot p_{ln}(\eta_l)}{\mu} < \frac{1}{2}p_{ln}(\eta_l).$$

Therefore

$$p_{ln}(t) \leqq \|p_{ln}\| = p_{ln}(\eta_l) < 2p_{ln}(\xi) \quad (\xi \in I_3(t, \mu)); \tag{9}$$

in particular (9) holds for $\xi=\xi_1$.

If $p_{ln}(\tau)$ is monotone decreasing in $I_3(t, \mu)$, then using Lemma 2 with $\delta = t-\xi_1 \geqq \frac{\delta_n(t)}{16\mu k^2}$, we get

$$|p'_{ln}(t)| \leqq 18\frac{16\mu k^2}{\delta_n(t)}p_{ln}(\xi_1). \tag{10}$$

On the other hand, if there exists an $\eta_l \in \operatorname{int} I_3(t, \mu)$ such that $p'_{ln}(\eta_l)=0$, then by Lemma 2 and (9) we get

$$|p'_{ln}(t)| \leqq \frac{18}{\delta_n(t)}\|p_{ln}\| < \frac{36}{\delta_n(t)}p_{ln}(\xi_1). \tag{11}$$

Now from Lemma 1, by definition of $I_1(t, \mu)$ and ξ_1 we easily deduce

$$|r(t)| \leqq 2|r(\xi_1)| \tag{12}$$

and

$$|r'(t)| \leqq \frac{16\mu k^2}{\delta_n(t)}|r(\xi_1)|. \tag{13}$$

Thus (8)—(13) yield

$$|q_l'(t)| \leqq |p_{ln}'(t)r(t)| + |p_{ln}(t)r'(t)| \leqq$$

$$\leqq \frac{c_6 \mu k^2}{\delta_n(t)} p_{ln}(\xi_1)|r(\xi_1)| = \frac{c_6 \mu k^2}{\delta_n(t)} |q_l(\xi_1)|$$

and Lemma 3 is proved. □

PROOF OF THEOREM 1. In case of $n=0$ the statement is contained in (1), so it may be supposed that $n \geqq 1$. First let $|t| \leqq \left(1-\frac{c_5}{n}\right)\omega$ $\left(0<\omega \leqq \frac{\pi}{2}\right)$. Then $c_5 \geqq 2$ implies that $\xi_1, \xi_2 \in I_1(t, \mu) \cup I_2(t, \mu) \subset [-\omega, \omega]$, and using the representation $q(t) = p(t)r(t) = \sum_{l=0}^{2n} a_l p_{ln}(t)r(t)$ with all $a_l \geqq 0$ or all $a_l \leqq 0$, we get the theorem by Lemma 3. For $\left(1-\frac{c_5}{n}\right)\omega < |t| < \omega$ the theorem follows from the Markov type inequality

$$\|q'\| \leqq \frac{c_7(n+k)(k+1)}{\omega}\|q\| \leqq c_8(k+1)^2\sqrt{\frac{n+1}{\omega^2-t^2}}\|q\|$$

(cf. Erdélyi [4], Theorem 1). By this Theorem 1 is proved when $0<\omega \leqq \frac{\pi}{2}$. For $\frac{\pi}{2} < \omega \leqq \pi$, recalling the fact that $0<\omega_1 \leqq \omega_2 \leqq \pi$ implies $\mathcal{T}_n(\omega_2) \supset \mathcal{T}_n(\omega_1)$ (see [1], Theorem 1), we can apply the just proved part of the theorem to the (overlapping) intervals $[-\omega, 0], \left[-\frac{\omega}{2}, \frac{\omega}{2}\right], [0, \omega]$ of length $\omega \leqq \pi$. (Evidently a statement similar to (2) holds for intervals not symmetric to 0.) □

Denote by $S_n^k(\omega, \varepsilon)$ $(0 \leqq k \leqq 2n, 0<\omega \leqq \pi, 0<\varepsilon \leqq 1)$ the set of those trigonometric polynomials from T_n which have all but at most k roots outside the ellipse

$$\left\{z = x+iy \in \mathbf{C} \,\middle|\, x^2 + \frac{y^2}{\varepsilon^2} < \omega^2\right\}.$$

According to Theorem 2 of [1], $p \in S_n^0(\omega, \varepsilon)\left(0<\omega \leqq \frac{\pi}{4}, 0<\varepsilon \leqq 1\right)$ implies $p \in \mathcal{T}_l(-\omega, \omega)$ if $l \geqq c_9 \frac{n}{\varepsilon^2}$. From this and Theorem 1 we get

COROLLARY 1. *Let* $0<\omega \leqq \pi$, $0<\omega \leqq 1$ *and* $q \in S_n^k(\omega, \varepsilon)$. *Then*

$$|q'(t)| \leqq c_{10}\frac{(k+1)^2}{\varepsilon^2}\sqrt{\frac{2n-k+1}{\omega^2-t^2}}\|q\|. \tag{14}$$

(First we get (14) for $0<\omega \leqq \frac{\pi}{4}$ and from this the general case $0<\omega \leqq \pi$ is straightforward similarly to the proof of Theorem 1.)

The following example shows how close (or how far) is the estimate of Theorem 1 from being sharp when $2 \leqq k \leqq 2n$ (the case $k=0, 1$ will be handled separately, see Example 2 below). Denote by $T_n^k(\omega)$ $(0 \leqq k \leqq 2n,\ 0 < \omega \leqq \pi)$ the set of those polynomials from T_n which have all but at most k roots in $[-\pi, \pi) \setminus (-\omega, \omega)$ (mod 2π).

EXAMPLE 1. There exists a $p_1(t) \in T_n^{2k}(\omega)$ such that

(a) $$\left\|p_1'(t)\sqrt{\omega^2 - t^2}\right\| \geqq |p_1'(0)|\,\omega \geqq \sqrt{\frac{nk}{3}}\,\|p_1\| \quad (1 \leqq k \leqq n,\ 0 < \omega \leqq \pi),$$

(b) $$\max_{|t| \leqq \omega/2} |p_1^{(m)}(t)| \geqq c_1(m)(nk)^{m/2}\omega^{-m}\|p_1\| \quad (n \geqq 900m^2,\ k \geqq m+1,\ 0 < \omega \leqq \pi)$$

where $c_1(m)$ (and in what follows $c_2(m), c_3(m), \ldots$) is a positive constant depending only on m. Namely let

$$p_1(t) = \left(\frac{\cos t - \cos\omega}{1 - \cos\omega}\right)^{n-k} C_{2k-1}\left(\sqrt{\frac{n}{3k}}\,\frac{\sin\frac{t}{2}}{\sin\frac{\omega}{2}}\right)\cos\frac{t}{2}$$

where $C_{2k-1}(x) = \cos\left((2k-1)\arccos x\right)$ is the Chebyshev polynomial of degree $2k-1$ (compare also Erdélyi [5] where the algebraic version of this example is considered). Obviously $p_1(t) \in T_n^{2k}(\omega)$. First we prove that $\|p_1\| \leqq 1$. For $n/3 \leqq k \leqq n$ this is evident since $|C_{2k-1}(x)| \leqq 1$ if $|x| \leqq 1$. Now for $1 \leqq k \leqq n/3$ again it is sufficient to estimate $p_1(t)$ for $\sqrt{\frac{3k}{n}}\sin\frac{\omega}{2} \leqq \left|\sin\frac{t}{2}\right| \leqq \sin\frac{\omega}{2}$. In this case using the inequality $|C_{2k-1}(x)| \leqq (2|x|)^{2k-1}$ $(|x| > 1)$, we obtain

$$|p_1(t)| \leqq \left(1 - \frac{\sin^2\frac{t}{2}}{\sin^2\frac{\omega}{2}}\right)^{n-k}\left(2\sqrt{\frac{n}{3k}}\,\frac{\sin\frac{t}{2}}{\sin\frac{\omega}{2}}\right)^{2k-1}.$$

Since the function $(1-u^2)^{n-k}u^{2k-1}$ attains its absolute maximum in $[-1, 1]$ at $|u| = \sqrt{\frac{2k-1}{2n-1}} < \sqrt{\frac{3k}{n}}$, we get

$$|p_1(t)| \leqq \left(1 - \frac{3k}{n}\right)^{n-k} 2^{2k-1} < \exp\left(-\frac{3k}{n}\cdot\frac{2n}{3}\right)2^{2k-1} = \frac{1}{2}\left(\frac{2}{e}\right)^{2k} < 1.$$

Thus

$$\frac{|p_1'(0)|\,\omega}{\|p_1\|} \leqq |p_1'(0)|\,\omega = \frac{\omega\sqrt{\frac{n}{3k}}}{\sin\frac{\omega}{2}}\,\frac{2k-1}{2} \geqq \sqrt{\frac{nk}{3}}$$

so (a) is proved.

To prove (b) for the sake of brevity let

$$s(t) = C_{2k-1}\left(\sqrt{\frac{n}{3k}}\,\frac{\sin\frac{t}{2}}{\sin\frac{\omega}{2}}\right)$$

and define $t_j \in [0, \omega]$ by

$$\sin\frac{t_j}{2} = \sqrt{\frac{3k}{n}}\,\sin\left(\frac{2j+1}{2k-1}\,\frac{\pi}{2}\right)\sin\frac{\omega}{2} \quad (j = 0, 1, \ldots, m \leqq k-1).$$

Then

$$s(t_j) = -s(t_{j+1}) = (-1)^{k-j-1} \quad (0 \leqq j \leqq m-1). \tag{15}$$

Further a simple calculation shows

(16)

$$0 < t_j \leqq \frac{\pi}{2}\sqrt{\frac{3k}{n}}\,\frac{2m+1}{2k-1}\,\frac{\pi}{2}\,\omega \leqq 15m\frac{\omega}{\sqrt{kn}} \leqq \frac{\omega}{2} \quad (0 \leqq j \leqq m \leqq k-1,\ n \geqq 900m^2).$$

From (16) it is clear that

$$\left(\frac{\cos t - \cos\omega}{1-\cos\omega}\right)^{n-k} = \left(1 - \frac{\sin^2\frac{t}{2}}{\sin^2\frac{\omega}{2}}\right)^{n-k} \geqq \left(1-\frac{t^2}{\omega^2}\right)^{n-k} \geqq \tag{17}$$

$$\geqq \left(1-\frac{225m^2}{kn}\right)^{n} \geqq 2c_2(m) \quad (0 \leqq t \leqq t_m).$$

By (15) and (17)

$$|p_1(t_j)| \geqq c_2(m) \quad (0 \leqq j \leqq m) \tag{18}$$

and

$$\operatorname{sgn} p_1(t_j) = -\operatorname{sgn} p_1(t_{j+1}) \quad (0 \leqq j \leqq m-1). \tag{19}$$

Thus if $\Omega(t) = \prod_{j=0}^{m}(t-t_j)$, then by (16)

$$|\Omega'(t_j)| \leqq \left(\frac{15m\omega}{\sqrt{nk}}\right)^m \quad (0 \leqq j \leqq m-1) \tag{20}$$

and evidently

$$\operatorname{sgn}\Omega'(t_j) = -\operatorname{sgn}\Omega'(t_{j+1}) \quad (0 \leqq j \leqq m-1). \tag{21}$$

Using (18)—(21), and recalling a well-known relation for the m^{th} order divided differences, we obtain

$$|p_1^{(m)}(\xi)| = m! \left| \sum_{j=0}^{m} \frac{p_1(t_j)}{\Omega'(t_j)} \right| = m! \sum_{j=0}^{m} \left| \frac{p_1(t_j)}{\Omega'(t_j)} \right| \geqq$$

$$\geqq (m+1)! c_2(m) \left(\frac{\sqrt{nk}}{15m\omega} \right)^m = c_1(m)(nk)^{m/2} \omega^{-m}$$

for a suitable $\xi \in [t_0, t_m] \subset \left[0, \frac{\omega}{2}\right]$. □

The properties possessed by the trigonometric polynomials in Theorem 1 are, in general, not inherited by the derivatives of these polynomials. Thus a straightforward generalization of the statement for higher derivatives seems difficult. We could prove inequalities only for the classes $T_n^k(\omega)$ and $\mathcal{T}_n(\omega)$.

THEOREM 2. *If* $p(t) \in T_n^k(\omega)$ *then*

$$|p^{(m)}(t)| \leqq c_3(m)(k+1)^{2m} \left(\frac{n}{\omega^2 - t^2} \right)^{m/2} \|p\| \quad (|t| < \omega \leqq \pi,\ n, m \geqq 1,\ 0 \leqq k \leqq 2n).$$

PROOF. Let e.g. $t \geqq 0$, $t_i = t + (m-i)\frac{\omega - t}{m}$ $(i = 0, 1, \ldots, m-1)$, and apply Theorem 1 successively to $p^{(i)}(t) = q_i(t) r_i(t)$, where $q_i(t) \in T_{n-k-i}^0(\omega) \subset \mathcal{T}_{n-k-i}(\omega) \cup T_0$ (let $T_l^0(\omega) = \mathcal{T}_l(\omega) := T_0$ for a negative l) $r_i(t) \in T_{k+i}$, on the interval $[-t_i, t_i]$ at $t = t_{i+1}$ $(i = 0, 1, \ldots, m-1)$. We obtain

$$|p^{(m)}(t)| \leqq c_3^m (n+1)^{m/2} \prod_{i=0}^{m-1} \frac{(k+i+1)^2}{\sqrt{t_i^2 - t_{i+1}^2}} \|p\| \leqq$$

$$\leqq (c_3 m)^m (k+m)^{2m} \left(\frac{n+1}{\omega^2 - t^2} \right)^{m/2} \|p\|,$$

whence the theorem follows. □

THEOREM 3. *If* $p(t) \in \mathcal{T}_n(\omega)$, *then*

$$|p^m(t)| \leqq c_4(m) \left(\frac{n}{\omega^2 - t^2} \right)^{m/2} \|p\| \quad (n, m \geqq 1,\ |t| < \omega \leqq \pi).$$

PROOF. It may be supposed that $0 < \omega \leqq \frac{\pi}{2}$, otherwise, similarly to the proof of Theorem 1, the interval $[-\omega, \omega]$ can be divided into overlapping subintervals of length not greater than π and repeated application of the theorem to the subintervals gives the desired result. We need the following

LEMMA 4. *Let* $m, n, k \geqq 1$, $k \leqq 2n$, $0 < \omega \leqq \frac{\pi}{2}$, $s \in T_n^k(\omega)$ *and consider the intervals*

$$K_i(t) = \left[t - \frac{i\delta_n(t)}{300m},\ t + \frac{i\delta_n(t)}{300m} \right] \quad (1 \leqq i \leqq m)$$

where $\delta_n(t)=\sqrt{\dfrac{\omega^2-t^2}{n}}$ *again. Then we have*

$$|s^{(i)}(t)| \leqq c(i,m)\left(\frac{k^2}{\delta_n(t)}\right)^i \max_{\xi\in K_i(t)}|s(\xi)| \quad (1\leqq i\leqq m) \tag{22}$$

if $|t|\leqq\left(1-\dfrac{c_{11}}{n}\right)\omega$ *with an appropriate* c_{11}.

PROOF. A simple calculation shows that there exists a $c_{11}\geqq 2$ such that in case of $|t|\leqq\left(1-\dfrac{c_{11}}{n}\right)\omega$ we have

$$K_m(t)\subset(-\omega,\omega), \tag{23}$$

$$|\xi|+\delta_n(\xi)\leqq\frac{n-1}{n+1}\omega \quad \text{for all} \quad \xi\in K_m(t) \tag{24}$$

and

$$\frac{1}{2}\delta_n(t)\leqq\delta_n(\xi)\leqq 2\delta_n(t) \quad \text{for all} \quad \xi\in K_m(t). \tag{25}$$

Let $|t|\leqq\left(1-\dfrac{c_{11}}{n}\right)\omega$ be fixed. We prove the statement by induction on i $(1\leqq i\leqq m)$. We have $s(\tau)=q(\tau)r(\tau)$ where $q(\tau)\in T^0_{n-k}(\omega)\subset\mathcal{T}_{n-k}(\omega)\subset\mathcal{T}_n(\omega)$ $\left(0<\omega\leqq\dfrac{\pi}{2}\right)$ and $r(\tau)\in T_k$, so from Lemma 3 with $\mu=400m$ we get Lemma 4 for $i=1$ with $c(1,m)=400c_4m$. Suppose that (22) holds for an $1\leqq i\leqq m-1$ and we will show that it is true for $i+1$ as well. Applying (22) to $s'\in T_n^{k+1}(\omega)$ (let $T_n^{k+1}(\omega):=T_n$ if $k+1\geqq 2n$), then using Lemma 3 with $\mu=700m$, finally by (25) we get

$$|s^{(i+1)}(t)| \leqq c(i,m)\left(\frac{(k+1)}{\delta_n(t)}\right)^i \max_{\xi\in K_i(t)}|s'(\xi)| \leqq \tag{26}$$

$$\leqq c(i,m)4^i\left(\frac{k^2}{\delta_n(t)}\right)^i c_4 700m \max_{\xi\in K_i(t)}\frac{k^2}{\delta_n(\xi)}\max_{\tau\in I_3(\xi,\,700m)}|s(\tau)| \leqq$$

$$\leqq c(i,m)4^i\left(\frac{k^2}{\delta_n(t)}\right)c_4 700m\frac{2k^2}{\delta_n(t)}\max_{\tau\in K_{i+1}}|s(\tau)| \leqq$$

$$\leqq c(i+1,m)\left(\frac{k^2}{\delta_n(t)}\right)^{i+1}\max_{\tau\in K_{i+1}(t)}|s(\tau)|$$

with $c(i+1,m)=c(i,m)2\cdot 4^i\cdot c_4 700m$. (Because of $|t|\leqq\left(1-\dfrac{c_{11}}{n}\right)\omega$, (23), (24) and (25) each step is correct in (26).) □

Applying Lemma 4 to $p_{ln}(t)\in T_n^0(\omega)$, we get

COROLLARY 1. *We have*

$$|p_{ln}^{(m)}(t)| \leqq c_5(m)\left(\frac{n}{\omega^2-t^2}\right)^{m/2} \max_{\tau\in K_m(t)} |p(\tau)| \quad \left(|t| \leqq \left(1-\frac{c_{11}}{n}\right)\omega,\ 0<\omega\leqq\frac{\pi}{2}\right)$$

where c_{11} *is defined in Lemma* 4 *and* $c_5(m)=c(m, m)$.

Observe that $K_m(t)\subset I_3(t, 300)$ $\left(|t|\leqq\left(1-\frac{c_{11}}{n}\right)\omega\right)$. Denote the two endpoints of $K_m(t)$ by $\tau_1<\tau_2$. We show that

$$\max_{\tau\in K_m(t)} p_{ln}(\tau) \leqq 2\max\{p_{ln}(\tau_1), p_{ln}(\tau_2)\} \quad \left(|t|\leqq\left(1-\frac{c_{11}}{n}\right)\omega\right). \tag{27}$$

This estimate is trivial when p_{ln} is monotone in $K_m(t)$ and it follows from $K_m(t)\subset \subset I_3(t, 300)$ and (9) otherwise. By Corollary 1 and (27) we get

$$|p_{ln}^{(m)}(t)| \leqq 2c_5(m)\left(\frac{n}{\omega^2-t^2}\right)^{m/2} \max\{p_{ln}(\tau_1), p_{ln}(\tau_2)\} \quad \left(|t|\leqq\left(1-\frac{c_{11}}{n}\right)\omega\right). \tag{28}$$

Now let $p\in\mathcal{T}_n(\omega)$. Then using the representation

$$p(t) = \sum_{l=0}^{2n} a_l \sin^l\frac{\omega-t}{2}\sin^{2n-l}\frac{t+\omega}{2}$$

with all $a_l\geqq 0$ or all $a_l\leqq 0$, from (28) we get the desired result. In case of $\left(1-\frac{c_{11}}{n}\right)\omega\leqq|t|<\omega$ the theorem follows from the Markov type inequality

$$\|p^{(m)}\| \leqq c_6(m)\, n^m\|p\| \quad (p\in\mathcal{T}_n(\omega))$$

(cf. Erdélyi [4], Theorem 1). Thus Theorem 3 is completely proved. □

We now prove that, in a sense, Theorem 2 is sharp for $k=0$.

EXAMPLE 2. For the polynomial

$$p_2(t) = \left(\frac{\cos t-\cos\omega}{2}\right)^n \in T_n^0(\omega) \quad (0<\omega\leqq\pi)$$

we have

$$\max_{|t|\leqq\omega/2} |p_2^{(m)}(t)| \geqq c_7(m)\left(\frac{n}{\omega^2}\right)^{m/2}\|p_2\| \quad (n\geqq c_8(m)). \tag{29}$$

To show this, at first observe that

$$p_2^{(m)}(t) = \left(\frac{\cos t-\cos\omega}{2}\right)^{n-m} s_m(t) \tag{30}$$

where $s_m(t)\in T_m$ and

$$|s_m(t)| \leqq (a\omega\sqrt{n})^m \quad \text{if} \quad |t|\leqq\frac{a\omega}{\sqrt{n}}, \quad n\geqq 4a^2\geqq 32m^2. \tag{31}$$

This follows at once from the recurrence relation

$$s_{m+1}(t) = (m-n)\frac{\sin t}{2} s_m(t) + \frac{\cos t - \cos \omega}{2} s'_m(t)$$

and the Markov type inequality

$$|s'_m(t)| \leqq 2m^2 \cot \frac{a\omega}{2\sqrt{n}} \max_{|x| \leqq \frac{a\omega}{\sqrt{n}}} s_m(x) \quad \left(n \geqq 4a^2, \ |t| \leqq \frac{a\omega}{\sqrt{n}}\right)$$

(see Videnskiĭ [2]). Thus from (30) and (31) we get

$$|p_2^{(m)}(\tilde{t})| \leqq \sin^{2n-2m}\frac{\omega}{2}\left(1 - \frac{\sin^2(\tilde{t}/2)}{\sin^2(\omega/2)}\right)^{n-m} (a\omega\sqrt{n})^m \leqq \tag{32}$$

$$\leqq \|p_2\| (a\omega\sqrt{n})^m \exp\left(-(n-m)\frac{\tilde{t}^2}{\omega^2}\right) \sin^{-2m}\frac{\omega}{2} \leqq \pi^{2m} a^m \exp(1-a^2)\omega^{-m} n^{m/2} \|p_2\|$$

$$\left(n \geqq 4ma^2, \ a \geqq \sqrt{8}m, \ \tilde{t} = \frac{a\omega}{\sqrt{n}}\right).$$

We now prove that for each $m \geqq 0$ and $0 < \omega \leqq \pi$ there exist $a_m, b_m > 0$ and $t_{m,\omega}$ such that

$$|p_2^{(m)}(t_{m,\omega})| \geqq b_m \omega^{-m} n^{m/2} \|p_2\| \tag{33}$$

$$\left(n \geqq 4ma_m^2, \ a_m \geqq \sqrt{8}m, \ |t_{m,\omega}| \leqq \frac{a_m \omega}{\sqrt{n}}\right).$$

This will prove (29) with $c_7(m) = b_m$ and $c_8(m) = 4ma_m^2$, since $|t_{m,\omega}| \leqq \frac{\omega}{2}$. To show (33) we proceed by induction. For $m = 0$ it is true with $a_0 = t_{0,\omega} = 0$ and $b_0 = c_7(0) = 1$. Assume that it holds for m, and let us choose $a = a_{m+1} \geqq \sqrt{8}m$ so large that

$$\pi^{2m} a_{m+1}^m \exp(1 - a_{m+1}^2) \leqq \frac{b_m}{2}.$$

Then by (32)

$$|p_2^{(m)}(\tilde{t})| \leqq \frac{b_m}{2} \omega^{-m} n^{m/2} \|p_2\| \quad \left(n \geqq 4ma_{m+1}^2, \ \tilde{t} = \frac{a_{m+1}\omega}{\sqrt{n}}\right).$$

Hence and from (33), by the mean value theorem there exists a $|t_{m+1,\omega}| \leqq \frac{a_{m+1}\omega}{\sqrt{n}}$

such that

$$|p_2^{(m+1)}(t_{m+1,\omega})| = \frac{p_2^{(m)}\left(\frac{a_{m+1}\omega}{\sqrt{n}}\right) - p_2^{(m)}(t_{m,\omega})}{\frac{a_{m+1}\omega}{\sqrt{n}} - t_{m,\omega}} \geqq$$

$$\geqq \frac{\frac{b_m}{2}\omega^{-m}n^{m/2}\|p_2\|}{\frac{a_{m+1}+a_m}{\sqrt{n}}} = \frac{b_m}{2(a_{m+1}+a_m)}\omega^{-m-1}n^{(m+1)/2}\|p_2\|,$$

i.e. (30) holds with $m+1$ instead of m if we choose $b_{m+1}=\frac{b_m}{2(a_{m+1}+a_m)}$ and $n\geqq 4(m+1)a_{m+1}^2$. □

Examples 1 and 2 give the following

COROLLARY 1. *Let* $0\leqq k\leqq 2n$, $0<\omega\leqq\pi$, $m\geqq 1$, $n\geqq c_9(m)$. *Then*

$$\sup_{p\in T_n^k(\omega)} \frac{\max_{|t|\leqq\omega/2}|p^{(m)}(t)|}{\|p\|} \geqq c_{10}(m)\left(\frac{n(k+1)}{\omega^2}\right)^{m/2}.$$

From Theorems 1 and 2 we can get Bernstein type estimates for algebraic polynomials of special type. First we need some definitions. Denote by Π_n the set of all real algebraic polynomials of degree at most n, by P_n^k $(0\leqq k\leqq n)$ the set of those polynomials from Π_n which have all but at most k roots in $\mathbf{R}\setminus(-1,1)$, by $S_n^k(\varepsilon)$ $(0\leqq k\leqq n,\ 0<\varepsilon\leqq 1)$ the set of those polynomials from Π_n which have all but at most k roots outside the ellipse

$$\left\{z=x+iy\in\mathbf{C}\,\middle|\,x^2+\frac{y^2}{\varepsilon^2}<1\right\}.$$

Obviously $P_n^k\subset S_n^k(\varepsilon)$. Further let $\mathscr{P}_n$ be the set of polynomials of the form

$$p(x)=\sum_{i=0}^{n} a_i(1-x)^i(x+1)^{n-i}$$

with each $a_i\geqq 0$ or each $a_i\leqq 0$. By an observation of G. G. Lorentz, $S_n^0(1)\subset\mathscr{P}_n$. Finally, denote by H_{n+k}^k the set of polynomials of the form $q(x)=p_n(x)r_k(x)$ where $p_n(x)\in\mathscr{P}_n$ and $r_k(x)\in\Pi_k$. From $S_n^0(1)\subset\mathscr{P}_n$ it is obvious that

$$P_n^k\subset S_n^k(1)\subset H_n^k \quad (0\leqq k\leqq n).$$

In [6] we proved the following relation:

$$S_n^0(\varepsilon)\subset\mathscr{P}_l \quad \left(l\geqq\frac{3n}{\varepsilon^2}\right),$$

and from this we get

$$S_n^k(\varepsilon)\subset H_l^k \quad \left(l\geqq\frac{3(n-k)}{\varepsilon^2}\right). \tag{34}$$

THEOREM 4. *Let* $q(x)\in H_{n+k}^k$ $(n, k\geqq 0)$. *Then*

$$|q'(x)| \leqq c_{12}(k+1)^2\sqrt{\frac{n+1}{1-x^2}}\,\|q\| \quad (|x|<1),$$

where (and in what follows) $\|\cdot\|$ *denotes the sup norm over the interval* $[-1, 1]$.

This is a direct consequence of Theorem 1 if we apply the change of variable $x=\dfrac{\sin t}{\sin\omega}$ with $\omega=\dfrac{\pi}{4}$.

Recalling (34), from Theorem 4 we get

COROLLARY 2. *If* $q(x)\in S_n^k(\varepsilon)$ *then*

$$|q'(x)| \leqq c_{13}\frac{(k+1)^2}{\varepsilon}\sqrt{\frac{n-k+1}{1-x^2}}\,\|q\| \quad (|x|<1).$$

Similarly to the trigonometric case the properties possessed by polynomials from H_{n+k}^k are, in general, not inherited by the derivatives of these polynomials. That is why we were able to prove an inequality only for the class P_n^k. Namely, observe that $q\in P_n^k$ implies $q'\in P_n^{k+1}$ (let $P_n^{k+1}:=\Pi_n$ if $k+1\geqq n$). Then from Corollary 2 by induction on m we get

THEOREM 5. *Let* $q(x)\in P_n^k$, $m\geqq 1$. *Then*

$$|q^{(m)}(x)| \leqq c_{11}(m)\left((k+1)^2\sqrt{\frac{n+1}{1-x^2}}\right)^m\|q\| \quad (|x|<1).$$

The proof of this statement is almost word for word the same as that of Theorem 2.

NOTE. In [7] G. G. Lorentz proved the inequality of Theorem 4 when $k=0$, moreover he proved for higher derivatives that

$$|q^{(m)}(x)| \leqq c_{12}(m)\left(\frac{n}{1-x^2}\right)^{m/2}\|q\| \quad (q\in\mathscr{P}_n,\ |x|<1).$$

We conjecture that in Theorems 4 and 5 $(k+1)^2\sqrt{n+1}$ can be replaced by $\sqrt{(n+k)(k+1)}$, but like in the trigonometric case, at present this is an open problem.

Finally we mention a result to show how sharp the estimates in Theorems 4 and 5 are. Similarly to the trigonometric case (see Examples 1 and 2) we have

THEOREM 6. *Let* $0\leqq k\leqq n$, $n\geqq m\geqq 1$. *Then*

$$\sup_{p\in P_n^k}\frac{\max\limits_{|x|\leqq 1/2}|p^{(m)}(x)|}{\max\limits_{|x|\leqq 1}|p(x)|} \geqq c_{13}(m)\big(n(k+1)\big)^{m/2}.$$

(This inequality was proved in [5] for $m=1$ and in [8] for $k=0$.)

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ON A CLASS OF PROBLEMS ON COVERING OF A BOUNDED SET

S. V. YAKOVLEV (Harkov)

1. Introduction

At present there is a great number of publications on problems of packing and covering in space. One may find an extensive literature on this problem, e.g. in [1]. At the same time studies on coverings and packings in bounded domains are not rather numerous (in particular, [2—4]).

The present paper describes a constructive approach to the search of admissible coverings of a polyhedron by translates of polyhedra. Such a location of the covering objects when each point of the domain of the covering belongs to at least one object will be referred to as an admissible covering. On the one hand, a problem is to find at least one admissible covering and, on the other hand, to determine the set of all admissible coverings.

The main aim of the paper is to show the major stages of solution to a class of problems on covering. The author describes briefly algorithms of solution of auxiliary problems, occuring at each stage, by referring to the respective literature. A numerical example, given at the end of the paper, shows acceptability of the approach proposed, and, on the whole, shows difficulties of its realization.

2. The problem

Let S_0 be the domain of a covering (a point set of the space E^k), and S_i, $i=1, \ldots, n$ be the covering objects (point sets of the space E^k). A Cartesian system of coordinates $Ox_1^i x_2^i \ldots x_k^i$ is connected with each set S_i, $i=0, \ldots, n$ and its center is called the pole of the object S_i. The position of the object S_i in the space E^k with the Cartesian system of coordinates $Ox_1 x_2 \ldots x_k$ is determined by the coordinates of its pole and the angles of rotation of $Ox_1^i x_2^i \ldots x_k^i$ w.r.t. $Ox_1 x_2 \ldots x_k$. The parameters describing the position of the system of coordinates $Ox_1^i x_2^i \ldots x_k^i$ by $Ox_1 x_2 \ldots x_k$ are called the parameters of the location ([3]) of the object S_i. One fixes all the angle parameters and the parameters of location of the domain S_0, putting them to be equal to zero.

We denote the result of the translation of the object S_i by the vector $x^i = (x_1^i, x_2^i, \ldots, x_k^i)$ by $S_i(x^i)$. Let $S_0(0)$ be domain of the covering. We have to determine such translations of the objects S_i, $i=1, \ldots, n$ for which we have an admissible covering of the domain $S_0(0)$ by the set of objects $S_i(x^i)$, $i=1, \ldots, n$.

So the problem is to describe analytically the domain

$$\Omega = \{(x^1, x^2, \ldots, x^n)/S_0(0) \subset \bigcup_{i=1}^{n} S_i(x^i)\}. \tag{1}$$

The main attention is paid to the case when the objects S_i, $i=0, \ldots, n$ are polyhedra of the space E^k.

3. Main concepts

We present the concept of a structure of inequalities according to [5].

DEFINITION 1. An ordered set $F(x)$ of inequalities of the type $\{f_i(x) \geqq 0, i=1, \ldots, m, x \in E^k\}$ with conjunction and disjunction operations defined for each pair of inequalities, presented as a symmetric matrix $\Delta = \|\delta_{ji}\|_{m \times m}$ is called a structure of inequalities $\mathscr{S}(F(x), \Delta, m)$.

Suppose that $\delta_{ij}=1$ corresponds to the conjunction between the i-th and the j-th inequalities, and $\delta_{ij}=0$; $(i=1, \ldots, m, j=1, \ldots, m)$ corresponds to the disjunction; $\delta_{ii}=1$, $i=1, \ldots, m$.

DEFINITION 2. The set $\tilde{F}(x) \subset F(x)$ of inequalities of the structure $\mathscr{S}(F(x), \Delta, m)$ is called a complete system of inequalities of this structure, if all the inequalities of the set $\tilde{F}(x)$ are connected by the conjunction operation with each other, and among the inequalities of the set $F(x)$ none is connected simultaneously by conjunctions with all inequalities from $\tilde{F}(x)$.

It is clear that the number of complete systems of inequalities of the structure $\mathscr{S}(F(x), \Delta, m)$ is finite. Algorithms for finding all complete systems of inequalities of the structure are described in [5].

DEFINITION 3. The set of values $x \in E^k$ for which at least one complete system of inequalities of the structure $\mathscr{S}(F(x), \Delta, m)$ is satisfied is called the set determined by this structure.

Intersection and union operations are defined on the set of structures of inequalities. Let

$$F^0(x) = \{f_i^0(x) \geqq 0,\ i = 1, \ldots, m_0\}, \quad \Delta_0 = \|\delta_{ij}^0\|_{m_0 \times m_0};$$

$$F^1(x) = \{f_i^1(x) \geqq 0,\ i = 1, \ldots, m_1\}, \quad \Delta_1 = \|\delta_{ij}^1\|_{m_1 \times m_1};$$

$$F^2(x) = \{f_i^2(x) \geqq 0,\ i = 1, \ldots, m_2\}, \quad \Delta_2 = \|\delta_{ij}^2\|_{m_2 \times m_2}.$$

DEFINITION 4. The structure $\mathscr{S}(F^0(x), \Delta_0, m_0)$ is called the intersection of the structures $\mathscr{S}(F^1(x), \Delta_1, m_1)$ and $\mathscr{S}(F^2(x), \Delta_2, m_2)$, and is denoted as

$$\mathscr{S}(F^0(x), \Delta_0, m_0) = \mathscr{S}(F^1(x), \Delta_1, m_1) \cap \mathscr{S}(F^2(x), \Delta_2, m_2),$$

if

$$m_0 = m_1 + m_2; \tag{2}$$

$$f_i^0(x) = \begin{cases} f_i^1(x), & \text{if } i \leqq m_1, \\ f_{i-m_1}^2(x), & \text{if } i > m_1; \end{cases} \tag{3}$$

$$\delta_{ij}^0 = \begin{cases} \delta_{ij}^1, & \text{if } i \leqq m_1,\ j \leqq m_1, \\ \delta_{i-m_1, j-m_1}^2, & \text{if } i > m_1,\ j > m_1, \\ 1, & \text{in the other cases;} \end{cases}$$

$i=1, \ldots, m_0$; $j=1, \ldots, m_0$.

DEFINTION 5. The structure $\mathcal{S}(F^0(x), \Delta_0, m_0)$ is called the union of the structure $\mathcal{S}(F^1(x), \Delta_1, m_1)$ and $\mathcal{S}(F^2(x)\Delta_2, m_2)$ and is denoted as

$$\mathcal{S}(F^0(x), \Delta_0, m_0) = \mathcal{S}(F^1(x), \Delta_1, m_1) \cup \mathcal{S}(F^2(x), \Delta_2, m_2),$$

if conditions (2) and (3) are satisfied, and

$$\delta^0_{ij} = \begin{cases} \delta^1_{ij}, & \text{if } i \leqq m_1,\ j \leqq m_1, \\ \delta^2_{i-m_1, j-m_1}, & \text{if } i > m_1,\ j > m_1, \\ 0, & \text{in the other cases;} \end{cases}$$

$i=1, \ldots, m_0$, $j=1, \ldots, m_0$.

It is evident that the intersection (union) of structures of inequalities determines the intersection (union) of the sets, determined by these structures. Both open and closed sets may be considered respectively, using signs of strict and non-strict inequality (provided $f_i(x)$ is continuous).

DEFINITION 6. A structure $\mathcal{S}(F(x), \Delta, m)$, all inequalities of the set $F(x)$ of which are linear, is referred to as a structure of linear inequalities.

Any polyhedral set of the space E^k is evidently determined by a structure of linear inequalities.

4. Solution of the problem

We represent the set Ω given by the expression (1) in the following way.

ASSERTION. *The set Ω is equivalently presented as follows:*

$$\Omega = c \operatorname{Pr}_{E^{kn}} \Big\{ \bigcup_{(x^1, \ldots, x^n) \in E^{kn}} \Big[(x^1, \ldots, x^n) \times \Big(\bigcup_{i=1}^{n} cS_i(x^i) \cap S_0(0) \Big) \Big] \Big\} = \tag{4}$$

$$= \operatorname{Pr}_{E^{kn}} \Big\{ \bigcup_{(x^1, \ldots, x^n) \in E^{kn}} \Big[(x^1, \ldots, x^n) \times \Big(\bigcup_{i=1}^{n} S_i(x^i) \ominus S_0 \Big) \Big] \cap E^{kn} \times \{0\} \Big\} =$$

$$= \operatorname{Pr}_{E^{kn}} \Big\{ \bigcup_{(x^1, \ldots, x^n) \in E^{kn}} \Big[(x^1, \ldots, x^n) \times c \Big(\bigcap_{i=1}^{n} cS_i(x^i) \oplus \bar{S}_0 \cap E^{kn} \times \{0\} \Big\},$$

where c is the symbol of the operation of complementation; $\operatorname{Pr}_{E^{kn}}$ is the operator of the orthogonal projection on the subspace E^{kn} of the variables $(x^1, x^2, \ldots, x^n)$; $\ominus$ and $\oplus$ are the symbols of Minkowski's operations of difference and sum, respectively; $\{0\}$ is the zero of the space E^k; $\bar{S}_0$ is the object reflected about $\{0\}$.

PROOF. It is evident that the covering holds if and only if

$$\bigcap_{i=1}^{n} cS_i(x^i) \cap S_0(0) = \emptyset. \tag{5}$$

The collection of points $(x^1, \ldots, x^n) \in E^{kn}$ for which the condition (5) is not satisfied determines the set

$$W = \bigcup_{(x^1, \ldots, x^n) \in E^{kn}} \Big[(x^1, \ldots, x^n) \times \Big(\bigcap_{i=1}^{n} cS_i(x^i) \cap S_0(0) \Big) \Big] \tag{6}$$

in the space $E^{kn+k}=E^k\times E^{kn}$, where E^k is the subspace of vectors $(x_1, \ldots, x_k)$ and E^{kn} is the subspace of vectors $(x^1, \ldots, x^n)$. The orthogonal projection of the set W on the subspace E^{kn} of vectors $(x^1, \ldots, x^n)$ describes the set of parameters for which the covering does not hold. Subsequently we have to find the complement of the set obtained, in the whole space E^{kn}.

The other representations of Ω in (4) can be obtained by equivalent transformations, connecting the operations of intersection, union and Minkowski's sum and difference.

Let $\mathcal{S}(F^i(x), \Delta_{i,m_i})$, $i=0, \ldots, n$ be the structures of inequalities determining the sets $S_0(0)$, $cS_i(0)$, $i=1, \ldots, n$ respectively in the space E^k of variables $x=(x_1, \ldots, x_k)$. Then the set $W\subset E^{kn+k}$ in (6) is described by the structure

$$\mathcal{S}(\Psi(x, x^1, \ldots, x^n), \Delta, m) = \mathcal{S}(F^0(x), \Delta_0, m_0)\cap\Big[\bigcap_{i=1}^{n}\mathcal{S}(F^i(x-x^i), \Delta_i, m_i)\Big]. \tag{7}$$

The formation of the structure $\mathcal{S}(\Psi(x, x^1, \ldots, x^n), \Delta, m)$ is realized according to the rules, mentioned in § 3.

We denote by

$$W_0 = \mathrm{Pr}_{E^{kn}}W \tag{8}$$

the orthogonal projection of the set W on the subspace with the system of coordinates $Ox^1x^2\ldots x^n$. The problem of the construction of the orthogonal projection has not yet been solved for the general case. However if the set W is determined by a structure of linear inequalities then there are effective algorithms for the construction of its orthogonal projection. It is clear that the structure $\mathcal{S}(\Psi(x, x^1, \ldots, x^n), \Delta, m)$ will be linear if the structures of inequalities $\mathcal{S}(F^i(x), \Delta_i, m_i)$, $i=0, \ldots, n$ are linear. It is evident that in this case one has polyhedral sets.

One of the algorithms of the orthogonal projection of the set determined in the space E^s by a structure of linear inequalities is described in [5] and is as follows. Let the inequalities of the set $F(x)$ of the structure $\mathcal{S}(F(x), \Delta, m)$ have the form

$$\sum_{j=1}^{s} a_{ij}x_j+b_i \geqq 0$$

where a_{ij}, b_i, $i=1, \ldots, m$ are given values.

Let $N_s=\{1, 2, \ldots, s\}$. Fix a certain number $r\in N_s$ and, if it is possible, choose inequalities, indexed by numbers p and q, from the set $F(x)$, such that

$$a_{pr}a_{qr} < 0, \quad \delta_{pq} = 1, \quad p, q\in N_m. \tag{9}$$

Consider the linear combination

$$\sum_{j=1}^{s}(\alpha a_{pj}+\beta a_{qj})x_j+\alpha b_p+\beta b_q \geqq 0 \tag{10}$$

where

$$\alpha = |a_{pr}|^{-1}, \quad \beta = |a_{qr}|^{-1}.$$

We index the inequality (10) with the number $m+1$ and include it in the set $F(x)$. At the same time we add to the matrix Δ a row and a column with the

elements

$$\delta_{m+1,i} = \delta_{pi}\delta_{qi}, \quad \delta_{i,m+1} = \delta_{ip}\delta_{iq}, \quad \delta_{m+1,m+1} = 1, \quad i = 1, \ldots, m.$$

We perform the above operations with all pairs of inequalities of the initial set $F(x)$ satisfying condition (9). It is evident that the total number of inequalities and the dimension of the matrix Δ will increase step-by-step. Denote the structure thus obtained by $\mathcal{S}(F^*(x), \Delta^*, m^*)$, $x \in E^s$.

We select the structure $\mathcal{S}(\bar{F}(x), \bar{\Delta}, \bar{m})$ $x \in E^{s-1}$ from the structure

$$\mathcal{S}(F^*(x), \Delta^*, m^*),$$

the set $\bar{F}(x)$ consisting of those inequalities of the set $F^*(x)$ where the coefficients of x_r are equal to zero, and the matrix $\bar{\Delta}$ being constructed by deleting the rows and columns of the matrix Δ^*, corresponding to the inequalities excluded from $F^*(x)$. As proved in [5], the structure $\mathcal{S}(\bar{F}(x), \bar{\Delta}, \bar{m})$ describes the orthogonal projection of the polyhedral set, determined in E^s by the structure $\mathcal{S}(F(x), \Delta, m)$, on the subspace E^{s-1} with the system of coordinates $Ox_1 \ldots x_{r-1}x_{r+1} \ldots x_s$. It is not difficult to see that the above described algorithm of projection is polynomial with complexity $O(m^2)$ (except the calculation of $\bar{\Delta}$ which has $O(m^4)$ entries thus needs $O(m^4)$ steps).

Subsequent projections on subspaces of codimension one make it possible to obtain the orthogonal projection of the set on the subspace with the required system of coordinates. As in the paper given above, if the problem of covering a planar domain by the system of polyhedra is considered, then it is necessary to use the projection procedure twice in a step-by-step manner. The complexity of the algorithm will be $O(m^4)$ (for $\bar{\Delta}$, $O(m^8)$). Note that the orthogonal projection of the set given by the structure will coincide with the whole subspace if there are no pairs of inequalities satisfying (9) and also there are no inequalities in which x_r has coefficient 0.

The last step of forming the set Ω is to construct the complement of the set W_0 given by the structure of inequalities $\mathcal{S}(\bar{F}(x), \bar{\Delta}, \bar{m})$. Choose all complete systems of inequalities of the structure and enumerate them. Let λ be the number of the complete systems of inequalities.

Consider the collection of logical variables $z_1, z_2, \ldots, z_{\bar{m}}$, by establishing a correspondence between the index of the inequality from $\bar{F}(x)$ and the index of the logical variable. Then a conjunction of variables $\bigwedge_{i \in I_j} z_i$ corresponds to each complete system of inequalities of the structure $\mathcal{S}(\bar{F}(x), \bar{\Delta}, \bar{m})$, where $I_j \subset N_{\bar{m}}$ is the set of the indices of the inequalities forming the j-th complete system of inequalities. The set W_0 corresponds to the disjunctive normal form (d.n.f.)

$$\bigvee_{j=1}^{\lambda} \left(\bigwedge_{i \in I_j} z_i \right)$$

and the complement of the set W_0 in E^{kn} will correspond to the conjunctive normal form (c.n.f.)

$$D = \bigwedge_{j=1}^{\lambda} \left(\bigvee_{i \in I_j} \bar{z}_i \right) \tag{11}$$

where the logical variable $\bar{z}_i$ corresponds to the i-th inequality of the structure $\mathscr{S}(\bar{F}(x), \bar{A}, \bar{m})$, in which the sign $\geqq$ is replaced by the sign $<$ and the sign $>$ is replaced by the sign $\leqq$.

We transform the c.n.f. (11) into a d.n.f. and present D as follows:

$$D = D_1 \vee D_2 \tag{12}$$

where D_1 and D_2 are disjunctions of such conjunctions of the variables to which compatible, and/or incompatible systems of inequalities correspond, respectively. It is important to know how D_1 is described.

The disjunctions in (11) are multiplied subsequently. At the t-th step, after transforming the c.n.f. $\bigwedge_{j=1} (\bigvee_{i \in I_j} \bar{z}_i)$, $t \leqq \lambda$ into a d.n.f. and after omitting the conjunctions corresponding to the incompatible systems, we obtain a d.n.f. as follows:

$$Q = \bigvee_{j=1}^{h_t} (\bigwedge_{i \in L_j^t} \bar{z}_i) = \bigvee_{j=1}^{h_t} K_j$$

where the value h_t and the sets $L_j^t \subset N_{\bar{m}}$, $j=1, \ldots, h_t$ are determined by the sets I_j, $j=1, \ldots, t$.

Every conjunction K_j corresponds to the $\bar{m}$-dimensional row vector $(g_{j1}, g_{j2}, \ldots, g_{j\bar{m}})$ which has 1 at the i-th place if $i \in L_j^t$ and 0 otherwise. Then the d.n.f. Q will correspond to the Boolean matrix $G = \| g_{ji} \|_{h_t \times \bar{m}}$.

Consider the $t+1$-st step and transforms the logical expression

$$(\bigvee_{j=1}^{h_t} K_j) \wedge (\bigvee_{i \in I_{t+1}} \bar{z}_i) \tag{13}$$

into a d.n.f. Let the conjunction K_j (i.e. the j-th row of the matrix G) correspond to the characteristic number $\mu(K_j)$ according to the rule:

$$\mu(K_j) = \begin{cases} -1, & \text{if } g_{ji} = 1 \text{ for at least two numbers } i \in I_{t+1}, \\ 0, & \text{if } g_{ji} = 0 \text{ for any } i \in I_{t+1}, \\ r, & \text{if } g_{jr} = 1, g_{ji} = 0 \text{ for any } i \in I_{t+1} \setminus \{r\}, r \in I_{t+1}. \end{cases}$$

While transforming the expression (13), one uses the following rules:

I. If $\mu(K_j) \neq 0$, then K_j is not multiplied by any variable $\bar{z}_i$, $i \in I_{t+1}$.

II. If $\mu(K_j) = 0$ and $\mu(K_p) = i$ for any $p \in N_{h_t}$, then K_j is multiplied by $\bar{z}_i$.

III. If $\mu(K_j) = 0$ and if for some set $L^* \subset N_{h_t}$ of indices $\mu(K_p) = i$, $p \in L^*$ then K_j is multiplied by $\bar{z}_i$ if and only if for any $p \in L^*$ there exists $q \in N_{\bar{m}} \setminus \{i\}$ such that $g_{jq} < g_{pq}$.

IV. When multiplying K_j by $\bar{z}_i$, a new row of the matrix G is formed (possibly) differing from the j-th row by the number 1 at the i-th place.

V. After the end of the t-th stage those rows to which there correspond incompatible systems of inequalities or which were not used to form new rows of the matrix G, are deleted from the matrix G.

ASSERTION. *After the end of the λ steps of the proposed algorithm the unique, minimal and shortest representation of the d.n.f. D_1 satisfying condition* (12) *is constructed.*

The proof of the above assertion directly follows from the following lemma.

LEMMA [5]. *If a c.n.f. whose all variables have equal indices of negation, is to be transformed into a d.n.f. according to the laws of distributivity and the operations of absorption then the obtained d.n.f. will be unique, minimal and shortest.*

So the set Ω is presented as the union of sets, determined by the compatible systems of inequalities, the indices of which correspond to the indices of the logical variables in D_1. Note that if the problem is to determine at least one admissible covering then it is not necessary to construct the d.n.f. D_1 completely. It suffices to find at least one conjunction in D_1 and to choose an arbitrary point from the set determined by the system of inequalities corresponding to this conjunction. The search of such a conjunction is easily realized by the above proposed algorithm.

The greatest interest have optimization problems on coverings in which given function $\varphi(x^1, x^2, \ldots, x^n)$ should attain its extremum on the set Ω. A simple approach to solve such problems is the complete search of all sets, consisting Ω and the optimization of $\varphi(x^1, x^2, \ldots, x^n)$ on each of these sets. However, effective procedures of directed exhaustive search of such sets can be proposed.

5. Numerical realization

We consider the following numerical example to illustrate the above approach to determine the set of admissible coverings.

We have a domain of covering S_0 and two covering objects S_1 and S_2 (Fig. 1). The initial structure $\mathcal{S}^* = \mathcal{S}(\Psi(x, x^1, x^2), \Delta, m)$ of the type (7) will be determined

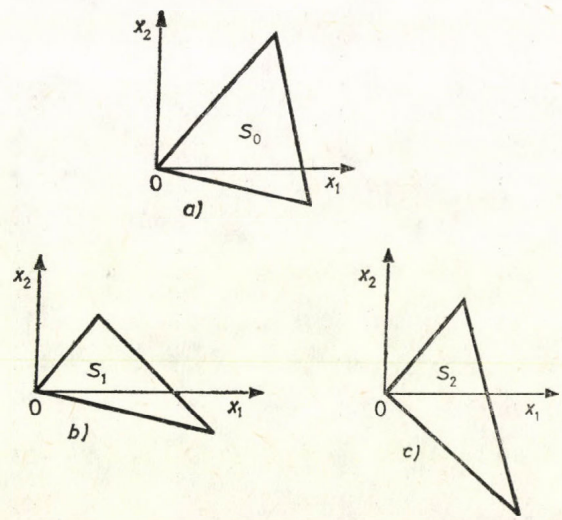

Fig. 1. The domain of covering and the covering objects

as follows: $m=9$;

$$\Psi(x, x^1, x^2)=\begin{cases} 7x_1-6x_2 & \geqq 0 \\ x_1+4x_2 & \geqq 0 \\ -9x_1-2x_2+68 & \geqq 0 \\ -4x_1+3x_2+4x_1^1-3x_2^1 & > 0 \\ -2x_1-9x_2+2x_1^1+9x_2^1 & > 0 \\ x_1+x_2-x_1^1-x_2^1-7 & > 0 \\ -5x_1+4x_2+5x_1^2-4x_2^2 & > 0 \\ -6x_1-7x_2+6x_1^2+7x_2^2 & > 0 \\ 11x_1+3x_2-11x_1^2-3x_2^2-59 & > 0, \end{cases} \quad \Delta=\begin{Vmatrix} 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&0&0&1&1&1 \\ 1&1&1&0&1&0&1&1&1 \\ 1&1&1&0&0&1&1&1&1 \\ 1&1&1&1&1&1&1&0&0 \\ 1&1&1&1&1&1&0&1&0 \\ 1&1&1&1&1&1&0&0&1 \end{Vmatrix}.$$

Table 1

The coefficients in the inequalities of the system $\bar{F}(x_1^1, x_2^1, x_1^2, x_2^2)$

i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	b_i
1	−68	−51	0	0	2 261
2	0	0	85	−68	2 856
3	0	0	102	119	2 023
4	−1	−1	0	0	6
5	0	0	−187	−51	476
6	748	−561	−748	−204	1 343
7	−5	−5	5	−4	−28
8	28	−21	0	0	6
9	68	−51	0	0	0
10	224	−168	5	−4	0
11	476	−357	18	21	0
12	140	−105	−297	−81	651
13	116	−87	−11	−3	−59
14	238	1071	0	0	2 550
15	112	504	125	−100	0
16	238	1071	450	525	0
17	14	63	−1485	−405	3 255
18	0	0	595	−476	68
19	8	−6	665	−532	0
20	0	0	85	−68	0
21	−18	−18	245	−196	10
22	−2	−2	65	−52	−14
23	0	0	714	833	2 890
24	340	−255	798	931	0
25	0	0	−6	7	0
26	−765	−765	294	343	425
27	2	9	0	0	2
28	22	99	−22	−6	68
29	−6	−6	−6	7	−40
30	48	216	5	−4	0
31	34	153	6	7	0
32	10	45	−99	−27	217
33	−17	−17	18	21	−119
34	−51	−51	42	49	−289
35	110	495	−9317	−2541	19 595

Table 2

The entries of the matrix $\bar{\Delta}$

11101101111110000111001110000000000
11010011110001100111110000100100000
10110001101001010000001111101010110
01111010000000000101111011001000110
10011101100111001000000000110001001
10001101100110000000000000000000000
01010010000000000101110000000000000
11101101111110000111001110000000000
11101101111110000111001110000000000
11000001110000000111000000000000000
10100001101000000000001110000000000
10001101100110000000000000000000000
10001101100110000000000000000000000
01101000000001111101001010110111001
01000000000001100101000000100100000
00100000000000101000000101010010000
00001000000000100100000000110001001
11010001111000000111110000100100000
11000001110000000111000000000000000
11010011110001100111110000100100000
01010010000000000101110000000000000
01010010000000000101110000000000000
10110001101001010000001111101010110
10100001101000000000001110000000000
10110001101001010000001111101010110
00110000000000000000000101100100011 0
01101000000001111101001010110111001
00001000000000100100000000110001001
00110000000000000000000101100100011 0
01000000000001100101000000100100000
00100000000000101000000101010010000
00001000000000100100000000110001001
00110000000000000000000101100100011 0
00110000000000000000000101100100011 0
00001000000000100100000000110001001

The orthogonal projection of the set W determined by the structure $\mathscr{S}^*$ on the subspace E^4 with the system of coordinates $Ox_1^1x_2^1x_1^2x_2^2$ will be determined by the structure $\mathscr{S}(\bar{F}(x_1^1, x_2^1, x_1^2, x_2^2), \bar{\Delta}, \bar{m})$, where $\bar{m}=35$; the set $\bar{F}(x_1^1, x_2^1, x_1^2, x_2^2)$ consists of inequalities of the type $\bar{f}_i(x_1^1, x_2^1, x_1^2, x_2^2)=a_{i1}x_1^1+a_{i2}x_2^1+a_{i3}x_1^2+a_{i4}x_2^2+$ $+b_i>0$, coefficients of which are given in Table 1; the matrix $\bar{\Delta}$ is presented in Table 2.

We choose all complete systems of inequalities of the structure

$$\mathscr{S}(\bar{F}(x_1^1, x_2^1, x_1^2, x_2^2), \bar{\Delta}, \bar{m}).$$

Then the set W_0 will correspond to the d.n.f.

$$z_1z_2z_8z_9z_{10}z_{18}z_{19}z_{20} \vee z_1z_3z_8z_9z_{11}z_{23}z_{24}z_{25} \vee z_1z_5z_6z_8z_9z_{12}z_{13} \vee$$
$$z_2z_4z_7z_{18}z_{20}z_{21}z_{22} \vee z_2z_{14}z_{15}z_{18}z_{20}z_{27}z_{30} \vee$$
$$z_3z_4z_{23}z_{25}z_{26}z_{29}z_{33}z_{34} \vee z_3z_{14}z_{16}z_{23}z_{25}z_{27}z_{31} \vee z_4z_5 \vee$$
$$z_5z_{14}z_{17}z_{27}z_{28}z_{32}z_{35}.$$

The complement of the set W_0 in the whole space E^4 will correspond to a d.n.f. consisting of 311 simple conjunctions (so D_1 is not given completely):

$$D = \bar{z}_5\bar{z}_{11}\bar{z}_{19}\bar{z}_{22}\bar{z}_{30}\bar{z}_{31}\bar{z}_{34} \vee \bar{z}_5\bar{z}_{11}\bar{z}_{18}\bar{z}_{27}\bar{z}_{34} \vee \bar{z}_8\bar{z}_{22}\bar{z}_{30}\bar{z}_{31}\bar{z}_{34}\bar{z}_{35} \vee$$

$$\bar{z}_9\bar{z}_{17}\bar{z}_{22}\bar{z}_{30}\bar{z}_{31}\bar{z}_{34} \vee \bar{z}_{11}\bar{z}_{13}\bar{z}_{18}\bar{z}_{31}\bar{z}_{34}\bar{z}_{35} \vee \dots .$$

The variable $\bar{z}_i$ will correspond to inequalities of the type $a_{i1}x_1^1 + a_{i2}x_2^1 + a_{i3}x_1^2 + a_{i4}x_2^2 + b_i \leqq 0$, whose coefficients are the same as in Table 1.

It is clear that any point $(x_1^1, x_2^1, x_1^2, x_2^2) \in E^4$, which satisfies at least one system of inequalities corresponding to a simple conjunctions in the presentation of D_1 belongs to the domain Ω. In particular, the vector $(-1, 0, 2, 3)$ is such a point and the corresponding covering is given in Fig. 2.

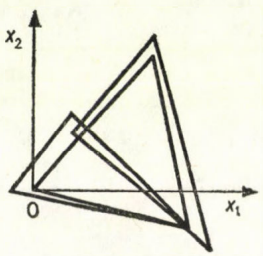

Fig. 2. An admissible covering

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POTENZREIHEN MIT FASTGERADEN KOEFFIZIENTEN

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1. Einleitung und Definitionen

L. Rubel und K. Stolarsky [3] bestimmten alle Teilmengen A von $\mathbf{N}$, so daß

$$g(x) = \sum_{n \in A} \frac{x^n}{n!}$$

wie die Exponentialfunktion auf der negativen reellen Achse beschränkt ist. Es ergaben sich genau vier echte Teilreihen von $e^x - 1$, nämlich die Potenzreihen für

$$g_1(x) = \frac{1}{2}(e^x + \cos x + \sin x) - 1$$

$$g_2(x) = \frac{1}{2}(e^x - \cos x - \sin x)$$

$$g_3(x) = \frac{1}{2}(e^x + \cos x - \sin x) - 1$$

$$g_4(x) = \frac{1}{2}(e^x - \cos x + \sin x).$$

Dabei wird zum Beweis u. a. eine Aussage über die Fortsetzbarkeit der Potenzreihen $\sum \chi_A(n) z^n$ über den Rand des Konvergenzkreises hinaus verwendet. Die charakteristischen Funktionen $\chi_A(\,.\,)$ ersetzte W. Schwarz durch allgemeinere zahlentheoretische Funktionen und zeigte analoge Ergebnisse für die Potenzreihen

$$E_f(z) = \sum_{n=1}^{\infty} f(n) \frac{z^n}{n!}.$$

Bezeichne

$$c_r(n) = \sum_{\substack{1 \leqq a \leqq r \\ (a,r)=1}} e^{2\pi i (a/r) n} = \sum_{d \mid (r,n)} d \cdot \mu\left(\frac{r}{d}\right) \tag{1}$$

die r-te Ramanujan-Summe, $\mathscr{B}$ den von den Funktionen $c_r(\,.\,)$ aufgespannten $\mathbf{C}$-Vektorraum (Vektorraum der „geraden Funktionen" [2], [6]) und $\mathscr{B}^2$ den Abschluß von $\mathscr{B}$ bezüglich der (Halb-)Norm

$$\|f\|_2 := \left\{ \limsup_{x \to \infty} \frac{1}{x} \sum_{n \leqq x} |f(n)|^2 \right\}^{1/2}.$$

Für die Ramanujan-Summen gelten die Orthogonalitätsrelationen

$$M(c_r \cdot c_s) = \begin{cases} 0 & r \neq s \\ \varphi(r) & r = s, \end{cases}$$

wobei

$$M(f) = \lim_{x \to \infty} \frac{1}{x} \sum_{n \leqq x} f(n)$$

(im Falle der Existenz dieses Grenzwertes) den Mittelwert der Funktion f bezeichnet. Diese Relationen erlauben es, Ramanujan-Entwicklungen für zahlentheoretische Funktionen

$$f(.) \sim \sum_{r=1}^{\infty} a_r c_r(.)$$

zu betrachten, wobei für die Entwicklungskoeffizienten

$$a_r = a_r(f) = \frac{1}{\varphi(r)} M(f \cdot c_r)$$

gilt, falls diese Mittelwerte existieren (Dies ist für $f \in \mathscr{B}^2$ gesichert; vgl. [5]). W. Schwarz zeigte nun unter Verwendung der Methode von Rubel und Stolarsky [5]:

SATZ A. *Sei $f \in \mathscr{B}^2$ und f werde durch seine Ramanujan-Entwicklung dargestellt. Ferner sei $E_f(z) = \sum_{n=1}^{\infty} f(n) \frac{z^n}{n!}$ auf der negativen reellen Achse beschränkt.*

Dann gilt:

(i) *$f \in \mathscr{B}$;*

(ii) *Der Entwicklungskoeffizient a_r ist Null, wenn eine primitive r-te Einheitswurzel ω existiert mit* $\operatorname{Re} \omega \leqq 0$.

Die o. g. Aussage über die Fortsetzbarkeit der Potenzreihe $\sum_{n=1}^{\infty} f(n) z^n$ über den Rand des Konvergenzkreises hinaus liefert dabei der folgende Satz (vgl. [5]):

SATZ B. *Sei $f \in \mathscr{B}^2$. Sind unendlich viele Ramanujan-Koeffizienten $a_r(f)$ von Null verschieden, so ist $\sum_{n=1}^{\infty} f(n) z^n$ nicht über den Konvergenzkreis hinaus fortsetzbar. Wird f durch seine Ramanujan-Entwicklung dargestellt und ist $a_r(f) \neq 0$ nur für endlich viele r, so ist $\sum_{n=1}^{\infty} f(n) z^n$ eine rationale Funktion.*

Multiplikative Funktionen $f \in \mathscr{B}^2$ mit von Null verschiedenem Mittelwert $M(f)$ werden durch ihre Ramanujan-Entwicklung dargestellt (vgl. [5]). Für diese Funktionen zeigte W. Schwarz [5]

SATZ C. *Sei* $f\in\mathscr{B}^2$, *multiplikativ und* $M(f)\neq 0$. $E_f(z)$ *sei auf der negativeu reellen Achse beschränkt. Dann ist* $f(n)=1$ *für alle* $n\in\mathbf{N}$ *oder*

$$f(n)=\begin{cases}1 & n\equiv 1(2)\\ 1-c & n\equiv 2(4)\\ 1+c & n\equiv 0(4)\end{cases}$$

mit einem beliebigen komplexen Parameter c.

In der vorliegenden Arbeit soll nun die Beschränktheit von $E_f(z)$ auf der negativen reellen Achse ersetzt werden durch eine Wachstumsbedingung, d. h. wir suchen Funktionen $f\in\mathscr{B}^2$ mit (*) $E_f(z)\leqq c\cdot e^{\vartheta|z|}$ für negative, reelle z und $\vartheta>0$. Das ist jedoch nur für $\vartheta<1$ interessant, da jede Funktion $f\in\mathscr{B}^2$ einer solchen Wachstumsbedingung mit $\vartheta>1$ genügt und für beschränkte Funktionen $f\in\mathscr{B}^2$ (*) mit $\vartheta=1$ erfüllt ist. Wir zeigen

SATZ 1. *Die Funktion* $f\in\mathscr{B}^2$ *werde durch ihre Ramanujan-Entwicklung dargestellt,* ϑ *sei gegeben mit* $0\leqq\vartheta<1$. *Aus*

$$(*)\qquad |E_f(z)|\leqq c\cdot e^{\vartheta|z|}\quad \textit{für}\quad z\in\mathbf{R},\quad z<0$$

folgt:

(i) f *hat eine endliche Ramanujan-Entwicklung* $f(n)=\sum_{r=1}^{R}a_r c_r(n)$ *d. h.* f *ist sogar aus* $\mathscr{B}$;

(ii) *Der Entwicklungskoeffizient* a_r *ist Null, falls eine primitive r-te Einheitswurzel* $\omega_{m,r}=e^{2\pi i(m/r)}$, $(m,r)=1$ *existiert mit* $\operatorname{Re}\omega_{m,r}<-\vartheta$.

Nach Aussage (ii) von Satz 1 ist es erforderlich, zu gegebenem ϑ, $0\leqq\vartheta<1$, eine primitive r-te Einheitswurzel $\omega_{m,r}$ mit $\operatorname{Re}\omega_{m,r}<-\vartheta$ zu suchen. Dazu dient das folgende Lemma:

LEMMA 1. *Ist* $\omega_{m,r}=e^{2\pi i(m/r)}$, $(m,r)=1$, *so ist* $\operatorname{Re}\omega_{m,r}$ *minimal, wenn*

(i) $m=\dfrac{r-1}{2}$ *für* $r\equiv 1(2)$,

(ii) $m=\dfrac{r}{2}-1$ *für* $r\equiv 0(4)$,

(iii) $m=\dfrac{r}{2}-2$ *für* $r\equiv 2(4)$, $r\neq 2$,

(iv) $m=1$ *für* $r=2$

ist. $\left(\textit{Gleichen Realteil erhält man für } m=\dfrac{r+1}{2},\ \dfrac{r}{2}+1 \textit{ bzw. } \dfrac{r}{2}+2\right)$.

Diese Ergebnisse gestatten es, spezielle fastgerade Funktionen zu untersuchen und für multiplikative Funktionen schärfere Aussagen zu formulieren.

LEMMA 2. *Sei* $f\in\mathscr{B}^2$, *multiplikativ und* $M(f)\neq 0$. *Aus* (*) *folgt* $a_r=0$ *für* $r\equiv 2(4)$.

Um in Abhängigkeit von ϑ diejenigen natürlichen Zahlen r zu bestimmen, für die $a_r\neq 0$ sein kann, zeigen wir

LEMMA 3. *Sei* $0\leqq\vartheta<1$ *gegeben. Ist* $r>\dfrac{\pi}{\arccos\vartheta}$ *für ungerades* r *bzw.* $r>\dfrac{2\pi}{\arccos\vartheta}$ *für* $r\equiv 0(4)$, *so ist* $a_r=0$.

Bezeichnen wir mit $U(\vartheta)$ die Menge der natürlichen Zahlen r, für die der Entwicklungskoeffizient a_r von Null verschieden sein kann, so ist für multiplikative Funktionen $f\in\mathscr{B}^2$ mit $M(f)\neq 0$

$$U(\vartheta)=\left\{r\in\mathbf{N},\, r\equiv 1(2),\, r\leqq\frac{\pi}{\arccos\vartheta}\right\}\cup\left\{r\in\mathbf{N},\, r\equiv 0(4),\, r\leqq\frac{2\pi}{\arccos\vartheta}\right\}.$$

Damit erhält man

BEISPIELE:

$$U(0)=\{1,4\};\quad U\left(\frac{1}{2}\right)=\{1,3,4\};\quad U\left(\frac{\sqrt{2}}{2}\right)=\{1,3,4,8\}.$$

Schließlich zeigen wir

SATZ 2. *Gegeben sei ein* ϑ *mit* $0\leqq\vartheta<1$.
(i) *Sei* $f\in\mathscr{B}^2$, *multiplikativ mit* $M(f)\neq 0$, *und es gelte Bedingung* (*). *Dann existiert ein* $r\in U(\vartheta)$, *so daß*

$$f(n)=f_r(n)=\prod_{p\mid r}\left(1+\sum_{\substack{k\geqq 1\\ p^k\mid(r,n)}}p^k\cdot\frac{a_{p^k}-a_{p^{k+1}}}{a_1-a_p}\right)$$

ist.

(ii) *Sei umgekehrt* $r\in U(\vartheta)$, $a_1\neq 0$, $a_p\neq a_1$ *für alle Primteiler* p *von* r, $a_{p^k}=0$ *falls* $p^k\nmid r$ *und* $a_2=0$. *Dann ist die Funktion* $f_r(\,.\,)$ *multiplikativ, gerade, besitzt einen von Null verschiedenen Mittelwert und genügt der Wachstumsbedingung* (*).

BEISPIELE. Für $\vartheta=0$ ist neben der Funktion $f_1\equiv 1$ auch

$$f_4(n)=\begin{cases}1 & 2\nmid n\\ 1-c & 2\parallel n\\ 1+c & 4\mid n\end{cases}$$

mit beliebigem komplexen Parameter c möglich.

Für $\vartheta=\dfrac{1}{2}$ kommt

$$f_2(n)=\begin{cases}1+c & 3\mid n\\ 1 & 3\nmid n\end{cases}$$

mit $c \neq -3$ hinzu. Zusätzlich ergibt sich für $\vartheta = \frac{\sqrt{2}}{2}$ noch die Funktion

$$f_8(n) = \begin{cases} 1 & 2 \nmid n \\ 1-c_1 & 2 \| n \\ 1+c_1-c_2 & 2^2 \| n \\ 1+c_1+c_2 & 8 | n \end{cases}$$

mit beliebigen $c_1, c_2 \in \mathbf{C}$.

2. Beweis von Satz 1

Analog zu Rubel und Stolarsky [3] betrachten wir die Laplace-Transformierte von E_f

$$F(z) = \int_0^\infty E_f(-t) e^{-tz}\, dt \tag{2}$$

$$= \sum_{n=1}^{\infty} (-1)^n f(n) \left(\frac{1}{z}\right)^{n+1}. \tag{3}$$

Nach Boas [1] Theorem 5.3.1 folgt aus (*) die Holomorphie von $F(z)$ für $\operatorname{Re} z > \vartheta$. Außerdem ergibt sich aus (3), daß $F(z)$ analytisch fortsetzbar ist auf $|z| > 1$, da aus der Endlichkeit von $\| f \|_2$ die Beziehung $|f(n)| \leqq c' \sqrt{n}$ folgt. Daher ist $-F\left(-\frac{1}{z}\right)$ im Innern des Einheitskreises holomorph und über den Einheitskreis hinaus analytisch fortsetzbar. Andererseits rechnet man leicht nach, daß

$$-F\left(-\frac{1}{z}\right) = z \sum_{n=1}^{\infty} f(n) z^n$$

gilt. Daraus ergibt sich unter Verwendung von Satz B, daß

$$f(n) = \sum_{r=1}^{R} a_r c_r(n)$$

ist. Das beweist also (i) von Satz 1. Mit diesem Ergebnis wird

$$E_f(z) = \sum_{n=1}^{\infty} f(n) \frac{z^n}{n!} = \sum_{r=1}^{R} a_r \sum_{\substack{1 \leqq m \leqq r \\ (m,r)=1}} \sum_{n=1}^{\infty} \frac{1}{n!} (\omega_{m,r} z)^n =$$

$$= \sum_{r=1}^{R} a_r \sum_{\substack{1 \leqq m \leqq r \\ (m,r)=1}} \{e^{\omega_{m,r} z} - 1\}$$

ein Exponentialpolynom. Dieses läßt sich in der Form $\sum_{j=1}^{k} b_j e^{\lambda_j z}$ schreiben, wobei

eines der λ_j (o. B. d. A. λ_k)$=0$ ist mit dem Koeffizienten

$$b_k = -\sum_{r=1}^{R} \sum_{\substack{1 \leqq m \leqq r \\ (m,r)=1}} a_r .$$

Die übrigen λ_j sind primitive r-te Einheitswurzeln, $1 \leqq r \leqq R$, mit den Koeffizienten $b_j = a_r$, falls $\lambda_j = e^{2\pi i (m/r)}$, $(m, r)=1$, ist. Wegen der Bedingung (*) ist für $x<0$

$$\sum_{j=1}^{k} b_j e^{\lambda_j x - \vartheta|x|} = \sum_{j=1}^{k} b_j e^{(\lambda_j + \vartheta)x}$$

beschränkt. Nach Rubel und Stolarsky [3], Lemma 2, erfüllt ein für $x<0$ beschränktes Exponentialpolynom $\sum_{j=1}^{k} d_j e^{\delta_j z}$ für jedes j, für das der Realteil von δ_j negativ ist, $d_j=0$. Daraus folgt $a_r=0$ für jedes r, zu dem eine primitive r-te Einheitswurzel $\omega_{m,r}$ existiert mit $\operatorname{Re} \omega_{m,r} < -\vartheta$.

3. Beweis der Lemmata

a) Lemma 1: Es läßt sich unmittelbar nachrechnen, daß $(m, r)=1$ ist. Ebenso leicht sieht man, daß der ggT von r und $(m+1)$ und zusätzlich von r und $(m+2)$ für $r \equiv 2(4)$ größer als 1 ist.

b) Lemma 2: Nach Satz 1 ist $f \in \mathscr{B}$ und $a_2=0$ für alle ϑ, $0 \leqq \vartheta < 1$ da der Realteil der primitiven zweiten Einheitswurzel $e^{i\pi} = -1$ ist. Aus der Multiplikativität der Abbildung $r \to \frac{a_r}{a_1}$ für multiplikative Funktionen $f \in \mathscr{B}$ mit $M(f) = a_1 \neq 0$ (vgl. [4]) folgt die Behauptung.

c) Lemma 3: Sei im folgenden ω_r die primitive r-te Einheitswurzel mit kleinstem Realteil und positivem Imaginärteil, also $\omega_r = e^{2\pi i (m/r)}$ mit m wie in Lemma 1. Ist r ungerade und $r > \frac{\pi}{\arccos \vartheta}$, so ist für $\omega_r \frac{m}{r} = \frac{1}{2} - \frac{1}{2r} > \frac{1}{2} - \frac{\arccos \vartheta}{2\pi}$. Dies gilt auch für $r \equiv 0(4)$ und $r > \frac{2\pi}{\arccos \vartheta}$. Aus der Existenz einer primitiven r-ten Einheitswurzel mit $\frac{m}{r} > \frac{1}{2} - \frac{\arccos \vartheta}{2\pi}$ ergibt sich $\cos 2\pi \frac{m}{r} = \operatorname{Re} \omega_{m,r} < -\vartheta$, woraus nach Satz 1 $a_r=0$ folgt.

4. Beweis von Satz 2

(i) a) Sei $r_0 := \max \{r \in U(\vartheta)\}$, $a_{r_0} \neq 0$, $p^\ell \| r_0$. Wegen der Multiplikativität der Abbildung $r \to \frac{a_r}{a_1}$ ist $a_{p^\ell} \neq 0$ und $a_{p^k}=0$ für $k > \ell$. Speziell folgt $a_d=0$, falls d kein Teiler von r_0 ist.

b) Nach Satz 1 ist

$$f(n) = \sum_{d|r_0} a_d c_d(n) = a_1 \sum_{d|r_0} \frac{a_d}{a_1} c_d(n) =$$

$$= a_1 \prod_{p|r_0} \left(1 + \sum_{\substack{k\geqq 1 \\ p^k|r_0}} \frac{a_{p^k}}{a_1} c_{p^k}(n)\right).$$

c) Sei $a_{r_0}=0$. Setze $r_1 := \max\{r\in U(\vartheta),\ r\neq r_0\}$, $a_{r_1}\neq 0$. Man verfährt wie in b) und erhält schließlich zu jedem $r\in U(\vartheta)$ eine Funktion

$$f(n) = f_r(n) = a_1 \prod_{p|r} \left(1 + \sum_{\substack{k\geqq 1 \\ p^k|r}} \frac{a_{p^k}}{a_1} c_{p^k}(n)\right). \tag{4}$$

d) Aus (1) ergibt sich

$$c_{p^k}(n) = \begin{cases} 0 & p^{k-1}\nmid n \\ -p^{k-1} & p^{k-1}\| n \\ p^k - p^{k-1} & p^k | n. \end{cases} \tag{5}$$

Daraus folgt

$$1 = f(1) = a_1 \prod_{p|r} \left(1 - \frac{a_p}{a_1}\right)$$

und speziell $a_p \neq a_1$ für alle Primteiler p von r. Weiter erhält man aus (4) unter Verwendung von (5)

$$f(n) = a_1 \prod_{p|r} \left(\sum_{\substack{k\geqq 0 \\ p^k|(r,n)}} p^k \left(\frac{a_{p^k}}{a_1} - \frac{a_{p^{k+1}}}{a_1} \right) \right).$$

Aus diesen beiden Gleichungen folgt sofort die Behauptung.

(ii) a) Man rechnet leicht nach, daß die Funktion $f_r(\,.\,)$ multiplikativ und gerade mod r ist. Weiter folgt unmittelbar die Multiplikativität der Abbildung $r \mapsto f_r(\,.\,)$. Daher ist $f_r(n) = \prod_{p^\ell \| r} f_{p^\ell}(n)$.

b) Aus (1) folgt

$$f_{p^\ell}(n) = \frac{1}{a_1 - a_p} \sum_{j=0}^{\ell} a_{p^j} c_{p^j}(n)$$

und daher

$$f_r(n) = \sum_{d|r} b_d c_d(n)$$

mit

$$b_1 = M(f_r) = \prod_{p|r} \left(\frac{a_1}{a_1 - a_p}\right) \neq 0$$

und

$$\frac{b_d}{b_1} = \prod_{p^m \| d} \frac{a_{p^m}}{a_1}.$$

c) Für $x\in\mathbf{R}$, $x<0$ ist

$$E_{f_r}(x)=\sum_{n=1}^{\infty}f_r(n)\frac{x^n}{n!}=\sum_{d|r}b_d\sum_{\substack{1\leqq m\leqq d\\(m,d)=1}}e^{-\operatorname{Re}\omega_{m,d}|x|+ix\operatorname{Im}\omega_{m,d}}-\sum_{d|r}b_d\varphi(d).$$

Wegen $r\in U(\vartheta)$ existiert keine primitive r-te Einheitswurzel, deren Realteil $<-\vartheta$ ist. Aus Lemma 1 folgt für alle primitiven d-ten Einheitswurzeln $\omega_{m,d}$, $d|r$, $d\neq 2$

$$\operatorname{Re}\omega_{m,d}>-\vartheta.$$

Da

$$b_2=a_2\frac{b_1}{a_1}=0$$

ist, ergibt sich

$$|E_{f_r}(x)|\leqq 2\sum_{d|r}b_d\varphi(d)e^{\vartheta|x|}.$$

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BMO-SEQUENCES AND AMARTS

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1. Introduction

Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $(\mathscr{F}_n)_{n\geq 0}$ be an increasing sequence of sub-σ-fields of $\mathscr{F}$. The notion of the BMO (bounded mean oscillation) for regular martingales has been studied by several authors (see for instance [5], [6]). The purpose of this paper is to deal with the remarkable connection between the generalized BMO-sequences (i.e. adapted sequences of real random variables $(X_n, \mathscr{F}_n)_{n\geq 0}$ having the property that $\sup_{m\geq n\geq 1} \|E(|X_m - X_{n-1}|\,|\mathscr{F}_n)\|_\infty < +\infty$) and the theory of amarts (semiamarts, uniform semiamarts, amarts of finite order, amarts). We show that every BMO-sequence is a uniform semiamart (this notion has been given by G. A. Edgar in [3]) having the optional sampling property. Moreover, if the BMO-sequence $(X_n)_{n\geq 0}$ converges in probability to some random variable X, then $(X_n, \mathscr{F}_n)$ is an amart of finite order. If we assume that the net $(X_\tau)_{\tau\in T}$ converges in probability to X as τ runs over the set of all bounded stopping times, then $(X_n, \mathscr{F}_n)$ is an amart. Finally, we give a necessary and sufficient condition for an amart to be a BMO-sequence.

2. Notation and terminology

Throughout this paper, let $(\Omega, \mathscr{F}, P)$ be a probability space and let $(\mathscr{F}_n)_{n\geq 0}$ be an increasing sequence of sub-σ-fields of $\mathscr{F}$. For the sake of commodity we suppose that $\mathscr{F} = \sigma\left(\bigcup_{n=0}^{\infty} \mathscr{F}_n\right)$. We shall denote by T the set of all bounded stopping times (with respect to $(\mathscr{F}_n)_{n\geq 0}$), i.e. the set of all mappings $\tau\colon \Omega \to \mathbf{N} = \{0, 1, 2, \ldots\}$ such that $\{\tau = n\} \in \mathscr{F}_n$ for each $n \in \mathbf{N}$ and τ assumes only finitely many values. If $\tau, \sigma \in T$ we define $(\tau \vee \sigma)(\omega) = \max[\tau(\omega), \sigma(\omega)]$ (resp. $(\tau \wedge \sigma)(\omega) = \min[\tau(\omega), \sigma(\omega)]$). It is clear that $\tau \vee \sigma, \tau \wedge \sigma \in T$. If $(X_n, \mathscr{F}_n)_{n\geq 0}$ is an adapted sequence and $\tau \in T$ then the random variable X_τ is defined for all $\omega \in \Omega$ by $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$ and is measurable with respect to the σ-field

$$\mathscr{F}_\tau = \{A \in \mathscr{F}\colon A \cap \{\tau = n\} \in \mathscr{F}_n \text{ for all } n \in \mathbf{N}\}.$$

If $\sigma \in T$, we write

$$T(\sigma) = \{\tau \in T\colon \tau(\omega) \geq \sigma(\omega) \text{ for almost all } \omega \in \Omega\}.$$

Given a sub-σ-field $\mathscr{G} \subset \mathscr{F}$ and $X \in L_1(\Omega, \mathscr{F}, P)$ we denote by $E(X|\mathscr{G})$ the conditional expectation of X with respect to $\mathscr{G}$.

We introduce the following definitions and notations.

DEFINITION 1. Let $(X_n)_{n\in\mathbf{N}}$ be an adapted sequence of integrable random variables (i.e. $X_n\in L_1(\Omega,\mathscr{F}_n, P)$ for each $n\in\mathbf{N}$).

(i) $(X_n)_{n\in\mathbf{N}}$ is called a BMO (resp. BMO$^+$)-sequence if

$$\sup_{m\geqq n\geqq 1}\|E(|X_m-X_{n-1}|\,|\mathscr{F}_n)\|_\infty<+\infty \tag{2.1}$$

(resp.

$$\sup_{m\geqq n\geqq 0}\|E(|X_m-X_n|\,|\mathscr{F}_n)\|_\infty<+\infty). \tag{2.2}$$

(ii) $(X_n)_{n\in\mathbf{N}}$ is called a $\mathscr{H}_p$ (resp. $\mathscr{H}_p^+$)-sequence, where $1\leqq p\leqq+\infty$, if there exists a random variable $\gamma\in L_p$ such that

$$\sup_{m\geqq n}E(|X_m-X_{n-1}|\,|\mathscr{F}_n)\leqq E(\gamma|\mathscr{F}_n)\quad\text{a.s. for each } n\geqq 1, \tag{2.3}$$

(resp.

$$\sup_{m\geqq n}E(|X_m-X_n|\,|\mathscr{F}_n)\leqq E(\gamma|\mathscr{F}_n)\quad\text{a.s. for each } n\geqq 0). \tag{2.4}$$

We put

$$\Gamma^p_{(X_n)}=\{\gamma\in L_p:\ \sup_{m\geqq n}E(|X_m-X_{n-1}|\,|\mathscr{F}_n)\leqq E(\gamma|\mathscr{F}_n)\ \text{a.s. for each } n\geqq 1\},$$

$$\Gamma^{+p}_{(X_n)}=\{\gamma\in L_p:\ \sup_{m\geqq n}E(|X_m-X_n|\,|\mathscr{F}_n)\leqq E(\gamma|\mathscr{F}_n)\ \text{a.s. for each } n\geqq 0\}.$$

REMARKS. 1. If the sequence $(X_n,\mathscr{F}_n)_{n\geqq 0}$ is a regular martingale (i.e. there is $X\in L_1(\Omega,\mathscr{F},P)$ such that $X_n=E(X|\mathscr{F}_n)$, $n\geqq 0$, $X_0=0$ a.s.) then our definition is equivalent to the classical definition of the BMO (resp. BMO$^+$, $\mathscr{H}_p$, $\mathscr{H}_p^+$)-sequence in the following sense:

$$E(|X_m-X_{n-1}|\,|\mathscr{F}_n)\uparrow E(|X-X_{n-1}|\mathscr{F}_n)\quad\text{a.s. and in } L_1 \text{ as } \quad m\to+\infty$$

(resp. for BMO$^+$, $\mathscr{H}_p$ and $\mathscr{H}_p^+$).

2. It is easy to see that every BMO (resp. $\mathscr{H}_p$)-sequence is also a BMO$^+$ (resp. $\mathscr{H}_p^+$)-sequence.

DEFINITION 2. Let $(X_n,\mathscr{F}_n)_{n\in\mathbf{N}}$ be an adapted sequence of integrable random variables. We say that

(i) $(X_n,\mathscr{F}_n)_{n\in\mathbf{N}}$ is a semiamart iff $\sup_{\tau\in T}\left|\int_\Omega X_\tau\,dP\right|<+\infty$ (see [7]),

(ii) $(X_n,\mathscr{F}_n)_{n\in\mathbf{N}}$ is a uniform semiamart iff

$$\lim_{\sigma\in T}\sup_{\tau\in T(\sigma)}\int_\Omega|E(X_\tau-X_\sigma|\mathscr{F}_\sigma)\,dP<+\infty$$

(see [3]),

(iii) $(X_n,\mathscr{F}_n)_{n\in\mathbf{N}}$ is an amart of finite order iff for every positive integer d, the net

$$\left(\int_\Omega X_\tau\,dP\right)_{\tau\in T^d}$$

converges (to a finite number) as τ runs over the set T^d, where T^d denotes the set of all stopping times each of which takes essentially at most d values (see [2]).

(iv) $(X_n, \mathscr{F}_n)_{n\in\mathbf{N}}$ is an amart iff the net $\left(\int\limits_\Omega X_\tau\, dP\right)_{\tau\in T}$ converges (to a finite number) as τ runs over the set T of all bounded stopping times (see [4]).

REMARK. It is clear that every amart is also an amart of finite order (resp., every amart is a uniform semiamart and every amart is a semiamart; see [3], [7]).

3. Results

We first prove the following

THEOREM. 1. *Let* $(X_n, \mathscr{F}_n)_{n\in\mathbf{N}}$ *be a* $\mathscr{H}_p^+$*-sequence* $(1 \leqq p \leqq +\infty)$. *Then for every* $\tau, \sigma \in T$ *and for arbitrary* $\gamma \in \Gamma^{+p}_{(X_n)}$ *we have*

$$E(|X_\tau - X_\sigma| \,|\, \mathscr{F}_{\tau\wedge\sigma}) \leqq E(2\gamma|\mathscr{F}_{\tau\wedge\sigma}) \quad a.s. \tag{3.1}$$

PROOF. Choose a positive integer N_0 such that $N_0 \geqq \tau \vee \sigma$ holds. Then we can write

$$E(|X_{N_0} - X_\tau| \,|\, \mathscr{F}_\tau) = E\Big[\Big(\sum_{i=1}^{k} \mathbf{1}_{\{\tau=n_i\}} |X_{N_0} - X_{n_i}|\Big) \Big| \mathscr{F}_\tau\Big] =$$

$$= \sum_{i=1}^{k} E[\mathbf{1}_{\{\tau=n_i\}} |X_{N_0} - X_{n_i}| \,|\, \mathscr{F}_\tau].$$

Let $A \in \mathscr{F}_\tau$ be arbitrary. From the above relation we get

$$\int\limits_A E(|X_{N_0} - X_\tau| \,|\, \mathscr{F}_\tau)\, dP = \int\limits_A \Big[\sum_{i=1}^{k} E(\mathbf{1}_{\{\tau=n_i\}} |X_{N_0} - X_{n_i}| \,|\, \mathscr{F}_\tau)\Big] dP = \tag{3.2}$$

$$= \sum_{i=1}^{k} \Big[\int\limits_A \mathbf{1}_{\{\tau=n_i\}} |X_{N_0} - X_{n_i}|\, dP\Big] = \sum_{i=1}^{k} \Big[\int\limits_{A\cap\{\tau=n_i\}} |X_{N_0} - X_{n_i}|\, dP\Big] =$$

$$= \sum_{i=1}^{k} \Big[\int\limits_{A\cap\{\tau=n_i\}} E(|X_{N_0} - X_{n_i}| \,|\, \mathscr{F}_{n_i})\, dP\Big] \leqq \sum_{i=1}^{k} \Big[\int\limits_{A\cap\{\tau=n_i\}} E(\gamma|\mathscr{F}_{n_i})\, dP\Big] =$$

$$= \sum_{i=1}^{k} \Big[\int\limits_{A\cap\{\tau=n_i\}} \gamma\, dP\Big] = \sum_{i=1}^{k} \int\limits_A \mathbf{1}_{\{\tau=n_i\}} \gamma\, dP = \int\limits_A \gamma\, dP = \int\limits_A E(\gamma|\mathscr{F}_\tau)\, dP.$$

Since (3.2) is true for arbitrary $A \in \mathscr{F}_\tau$ we deduce that

$$E(|X_{N_0} - X_\tau| \,|\, \mathscr{F}_\tau) \leqq E(\gamma|\mathscr{F}_\tau) \quad \text{a.s.} \tag{3.3}$$

holds. Note that $\mathscr{F}_{\tau\wedge\sigma} \subseteq \mathscr{F}_\tau$. Taking the conditional expectation of both sides of (3.3) with respect to $\mathscr{F}_{\tau\wedge\sigma}$ we get

$$E(|X_{N_0} - X_\tau| \,|\, \mathscr{F}_{\tau\wedge\sigma}) \leqq E(\gamma|\mathscr{F}_{\tau\wedge\sigma}) \quad \text{a.s.} \tag{3.4}$$

Now for arbitrary $\tau, \sigma \in T$ (3.4) implies

$$E(|X_\tau - X_\sigma| \,|\, \mathscr{F}_{\tau\wedge\sigma}) \leqq E(|X_{N_0} - X_\tau| \,|\, \mathscr{F}_{\tau\wedge\sigma}) + E(|X_{N_0} - X_\sigma| \,|\, \mathscr{F}_{\tau\wedge\sigma}) \leqq$$
$$\leqq 2E(\gamma|\mathscr{F}_{\tau\wedge\sigma}) = E(2\gamma|\mathscr{F}_{\tau\wedge\sigma}) \quad \text{a.s.}$$

which completes the proof of our theorem.

Theorem 1 has the following two corollaries.

COROLLARY 2. *Every $\mathscr{H}_p^+$-sequence, where $1 \leqq p \leqq +\infty$ is some "power", is a uniform semiamart (resp., semiamart).*

PROOF. Let $\tau, \sigma \in T$ such that $\tau \in T(\sigma)$ and let $\gamma \in \Gamma^{+p}_{(X_n)}$ be arbitrary. Then on the basis of (3.1) we have

$$E(|X_\tau - X_\sigma| \mid \mathscr{F}_\sigma) \leqq E(2\gamma \mid \mathscr{F}_\sigma) \quad \text{a.s.}$$

This implies

$$\limsup_{\sigma \in T} \sup_{\tau \in T(\sigma)} \int_\Omega \left| E[(X_\tau - X_\sigma) \mid \mathscr{F}_\sigma] \right| dP \leqq \int_\Omega 2\gamma \, dP < +\infty.$$

Hence $(X_n, \mathscr{F}_n)_{n \in \mathbf{N}}$ is a uniform semiamart. Also, on the basis of (3.1) for every $\tau \in T$ we can write

$$E(|X_\tau - X_0| \mid \mathscr{F}_0) \leqq E(2\gamma \mid \mathscr{F}_0) \quad \text{a.s.}$$

This implies

$$\sup_{\tau \in T} \left| \int_\Omega X_\tau \, dP \right| \leqq \sup_{\tau \in T} \int_\Omega |X_\tau| \, dP \leqq 2 \int_\Omega \gamma \, dP + \int_\Omega |X_0| \, dP < +\infty.$$

Consequently, $(X_n, \mathscr{F}_n)_{n \in \mathbf{N}}$ is a semiamart.

COROLLARY 3 (an "optional sampling theorem" for $\mathscr{H}_p^+$-sequences). *Let $(X_n, \mathscr{F}_n)_{n \in \mathbf{N}}$ be a $\mathscr{H}_p^+$-sequence, where $1 \leqq p \leqq +\infty$ is some "power" and let $(\tau_n)_{n \in \mathbf{N}}$ be a nondecreasing sequence of bounded stopping times corresponding to $(\mathscr{F}_n)$. Then $(X_{\tau_n}, \mathscr{F}_{\tau_n})_{n \in \mathbf{N}}$ is also a $\mathscr{H}_p^+$-sequence. In particular, if σ is a (not necessarily bounded) stopping time, then $(X_{n \wedge \sigma}, \mathscr{F}_{n \wedge \sigma})_{n \in \mathbf{N}}$ is also a $\mathscr{H}_p^+$-sequence.*

PROOF. Let $\tau_0 \leqq \tau_1 \leqq \tau_2 \leqq \ldots$ be an increasing sequence of bounded stopping times, i.e. belonging to T. Then from the inequality (3.1) we have that

$$E(|X_{\tau_m} - X_{\tau_n}| \mid \mathscr{F}_{\tau_n}) \leqq E(2\gamma \mid \mathscr{F}_{\tau_n}) \quad \text{a.s.}$$

for all $n \in \mathbf{N}$ and for arbitrary $\gamma \in \Gamma^{+p}_{(X_n)}$.

Now we recall a result whose proof can be found in [5, Lemma III. 5.1., p. 116 and Theorem III. 2.1., p. 75].

LEMMA 4. *Let $(X_n, \mathscr{F}_n)_{n \in \mathbf{N}}$ be a $\mathscr{H}_p$-sequence, where $1 < p < +\infty$ is some power. Then setting*

$$X_n^* = \max_{1 \leqq k \leqq n} |X_k - X_0|$$

we have for all $\gamma \in \Gamma^p_{(X_n)}$

$$E(X_n^{*p} \mid \mathscr{F}_1) \leqq (3pqe)^p E(\gamma^p \mid \mathscr{F}_1) \quad \text{a.s.} \quad (p^{-1} + q^{-1} = 1). \tag{3.5}$$

Moreover, if $p = +\infty$, then

$$E(X_n^* \mid \mathscr{F}_1) \leqq 8 \sup_{m \geqq n \geqq 1} \| E(|X_m - X_{n-1}| \mid \mathscr{F}_n) \|_\infty \quad \text{a.s.}$$

From this result it follows that for $X_0 = 0$, $1 < p < +\infty$

$$E\left[(\sup_{n \geqq 0} |X_n|)^p \right] \leqq (3pqe)^p E(\gamma^p) < +\infty$$

and for $p=+\infty$

$$E\left(\sup_{n\geqq 0}|X_n|\right)\leqq 8\sup_{m\geqq n\geqq 1}\|E(|X_m-X_{n-1}|\,|\mathscr{F}_n)\|_\infty<+\infty.$$

This means that for $1<p<+\infty$ the $\mathscr{H}_p$-sequence $(X_n,\mathscr{F}_n)$ is uniformly integrable. It also follows that for $1<p<+\infty$ the family $(|X_\tau|^p)_{\tau\in T}$ and for $p=+\infty$ the family $(|X_\tau|)_{\tau\in T}$ is uniformly integrable. This follows from the fact that

$$\sup_{\tau\in T}|X_\tau|^p\leqq\left[\sup_{n\geqq 0}|X_n|\right]^p\quad(1\leqq p<+\infty).$$

Now we formulate a sufficient condition for a $\mathscr{H}_p$-sequence to be an amart of finite order.

THEOREM 5. *Let $(X_n,\mathscr{F}_n)_{n\in\mathbf{N}}$ be a $\mathscr{H}_p$-sequence, where $1<p\leqq+\infty$ is some "power". If X_n converges in probability to some random variable X as $n\to+\infty$, then for every positive integer d we have*

$$\lim_{\tau\in T^d}\int_\Omega|X_\tau-X|\,dP=0.$$

In particular, $(X_n,\mathscr{F}_n)_{n\in\mathbf{N}}$ is an amart of finite order.

PROOF. Without loss of the generality we can suppose that $X_0=0$. (Indeed, denote $X_n'=X_n-X_0$; then $X_0'=0$ and $(X_n,\mathscr{F}_n)_{n\in\mathbf{N}}$ is an amart of finite order if and only if $(X_n',\mathscr{F}_n)_{n\in\mathbf{N}}$ is an amart of finite order).

Since $X_n\to X$ in probability as $n\to+\infty$ and $(X_n,\mathscr{F}_n)_{n\in\mathbf{N}}$ is uniformly integrable we deduce that $X\in L_1(\Omega,\mathscr{F},P)$ and

$$\lim_{n\to+\infty}\int_\Omega|X_n-X|\,dP=0.$$

This implies that given any positive ε and an arbitrary fixed positive integer d, there exists a positive integer $n(\varepsilon,d)$ such that for $n\geqq n(\varepsilon,d)$ we have

$$\int_\Omega|X_n-X|\,dP\leqq\frac{\varepsilon}{d}.$$

Consequently, for $\tau\in T^d$ such that $\tau\geqq n(\varepsilon,d)$ we see that

$$\int_\Omega|X_\tau-X|\,dP=\sum_{i=1}^k\left[\int_{\{\tau=n_i\}}|X_{n_i}-X|\,dP\right]\leqq d\,\frac{\varepsilon}{d}=\varepsilon.$$

Since $\varepsilon>0$ is arbitrary from this it follows that

$$\lim_{\tau\in T^d}\int_\Omega|X_\tau-X|\,dP=0.$$

This proves the assertion.

In [4] (Proposition 2.2) the authors proved that every adapted sequence of random variables satisfying

$$\int_\Omega\left(\sup_{n\geqq 0}|X_n|\right)dP<+\infty$$

is an amart if and only if (X_n) converges a.s. Our following assertion gives a necessary and sufficient condition for a $\mathscr{H}_p$-sequence to be an amart, where $1<p\leqq+\infty$ is some "power".

THEOREM 6. *Let $(X_n, \mathscr{F}_n)_{n\in\mathbf{N}}$ be a $\mathscr{H}_p$-sequence, where $1<p\leqq+\infty$ is some "power". Then the following assertions are equivalent:*

(i) $(X_n, \mathscr{F}_n)_{n\in\mathbf{N}}$ *is an amart,*
(ii) $(X_n)_{n\in\mathbf{N}}$ *converges a.s.,*
(iii) *the net* $(X_\tau)_{\tau\in T}$ *converges in probability,*
(iv) *the net* $(X_\tau)_{\tau\in T}$ *converges in* L_p, *if* $1<p<+\infty$ *and in* L_1, *if* $p=+\infty$.

PROOF. We can assume again, as in the proof of Theorem 5 that $X_0=0$. We prove the following chain of implications: (i)$\Leftrightarrow$(ii)$\Rightarrow$(iv)$\Rightarrow$(iii)$\Rightarrow$(i).

(i)$\Leftrightarrow$(ii) is a consequence of Proposition 2.2 of [4], since in the case $1<p<+\infty$ we have

$$\sup_{n\geqq 0}|X_n| \leqq \sup_{n\geqq 0}[\max(1, |X_n|)] \leqq 1+\sup_{n\geqq 0}|X_n|^p,$$

and for

$$p=+\infty, \quad \sup_{n\geqq 0}|X_n|\in L_1.$$

Consequently, for $1<p\leqq+\infty$ we have $\int_\Omega (\sup_{n\geqq 0}|X_n|)\,dP<+\infty$.

(ii)$\Rightarrow$(iv). By Proposition 2.2 of [4] $(\mathbf{X}_n, \mathscr{F}_n)_{n\in\mathbf{N}}$ is an amart since by the $\mathscr{H}_p$-property

$$\int_\Omega \sup_{n\geqq 0}|X_n|\,dP<+\infty.$$

Suppose that (iv) does not hold, i.e. $(X_\tau)_{\tau\in T}$ does not converge in L_p, where $1<p<+\infty$. Then there exists an increasing sequence of stopping times $\tau_0\leqq\tau_1\leqq\leqq\tau_2\leqq\ldots$ such that the sequence $(X_{\tau_k})_{k\geqq 0}$ does not converge in L_p. However since $(X_n, \mathscr{F}_n)_{n\geqq 0}$ is an amart by Proposition 1.6 of [4] we deduce that $(X_{\tau_k}, \mathscr{F}_{\tau_k})_{k\in\mathbf{N}}$ is an amart. By Theorem of [1] it then follows that $X_{\tau_k}\to X$ in L_p, since $(|X_{\tau_k}|^p)_{k\geqq 0}$ is uniformly integrable. This is a contradiction. For $p=+\infty$ we argue similarly.

This proves that (ii)$\Rightarrow$(iv).

The implication (iv)$\Rightarrow$(iii) is trivial.

Now we prove that (iii)$\Rightarrow$(i). The fact that $(X_\tau)_{\tau\in T}$ converges in probability and that the family $(X_\tau)_{\tau\in T}$ is uniformly integrable implies that for some random variable X we have

$$\lim_{\tau\in T}\int_\Omega |X_\tau - X|\,dP = 0.$$

Hence it follows that $(X_n, \mathscr{F}_n)_{n\in\mathbf{N}}$ is an amart.

This proves the assertion.

THEOREM 7. *Let $(X_n, \mathscr{F}_n)_{n\in\mathbf{N}}$ be an amart. Then the following two assertions are equivalent.*

(i) $(X_n, \mathscr{F}_n)_{n\in\mathbf{N}}$ *is a BMO-sequence*

(ii) *X_n can be written in the form*

$$X_n = Y_n + Z_n,$$

where $(Y_n, \mathscr{F}_n)_{n\in\mathbf{N}}$ *is a regular martingale satisfying*

$$\sup_{m\geqq n\geqq 1} \|E(|Y_m - Y_{n-1}|\mathscr{F}_n)\|_\infty < +\infty$$

and $(Z_n, \mathscr{F}_n)_{n\in\mathbf{N}}$ *is an amart with the property* $\left\|\sup_{n\geqq 0} |Z_n|\right\|_\infty < +\infty$.

PROOF. It is obvious that (ii) implies (i). Now we prove that (i)⇒(ii). Using the Riesz decomposition for amarts (see [4], Theorem 3.2, p. 209) we can write $X_n = Y_n + Z_n$ where $(Z_n, \mathscr{F}_n)_{n\geqq 0}$ is an amart and $(Y_n, \mathscr{F}_n)_{n\geqq 0}$ is a martingale such that

$$Y_n = \lim_{m\to\infty} E(X_m|\mathscr{F}_n) \quad \text{a.s.}$$

Since $(X_n, \mathscr{F}_n)_{n\geqq 0}$ is a BMO-sequence we have (by denoting

$$B = \sup_{m\geqq n\geqq 1} \|E(|X_m - X_{n-1}|\mathscr{F}_n)\|_\infty)$$

$$\left|E(X_m|\mathscr{F}_n) - X_n\right| \leqq E(|X_m - X_n|\mathscr{F}_n) \leqq B \quad \text{a.s.}$$

When $m\to+\infty$ this implies

$$|Z_n| = |Y_n - X_n| \leqq B \quad \text{a.s.}$$

Consequently,

$$\left\|\sup_{n\geqq 0} |Z_n|\right\|_\infty \leqq B < +\infty.$$

It is clear that $(Z_n, \mathscr{F}_n)_{n\geqq 0}$ is a BMO-sequence since

$$\sup_{m\geqq n\geqq 1} \|E(|Z_m - Z_{n-1}|\mathscr{F}_n)\|_\infty \leqq E\left(2\sup_{k\geqq 0} |Z_k|\mathscr{F}_n\right) \leqq 2B.$$

Hence $(Y_n, \mathscr{F}_n)_{n\geqq 0}$ is also a BMO-sequence. This implies by Lemma 4 that $(Y_n, \mathscr{F}_n)_{n\geqq 0}$ is uniformly integrable and so $(Y_n, \mathscr{F}_n)_{n\geqq 0}$ is a regular martingale.

This proves the assertion.

Now to finish with we give two examples. The first one shows that there is a BMO-sequence which is not an amart of finite order. In the second we construct a BMO-sequence which converges in probability (consequently it is an amart of finite order), but it is not an amart.

EXAMPLE 1. Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $X_{2k}(\omega) \equiv 1$, $X_{2k+1}(\omega) \equiv -1$, $k = 0, 1, 2, \ldots$. Let further $\mathscr{F}_n = \sigma(X_0, X_1, \ldots, X_n)$. It is easily seen that

$$\sup_{m\geqq n\geqq 1} \|E(|X_m - X_{n-1}|\mathscr{F}_n)\|_\infty \leqq 2.$$

Thus $(X_n, \mathscr{F}_n)_{n\in\mathbf{N}}$ is a uniform semiamart. But $(X_n, \mathscr{F}_n)_{n\in\mathbf{N}}$ is not an amart of finite order since

$$\int_\Omega X_{2k}\, dP = 1 \quad \text{and} \quad \int_\Omega X_{2k+1}\, dP = -1.$$

EXAMPLE 2. Let $(\Omega, \mathscr{F}, P)=([0, 1), \mathscr{B}, \lambda)$, where $\mathscr{B}$ is the σ-field of the Borel-sets of $[0, 1)$ and λ denotes the Lebesgue measure. For $m=1, 2, \ldots$ and for $k=1, 2, \ldots, 2^m$ define the random variables $Y_{m,k}$ as follows:

$$Y_{m,k} = \begin{cases} (-1)^{k-1} & \text{if } \omega \in [(k-1)2^{-m}, k2^{-m}) \\ 0 & \text{if } \omega \notin [(k-1)2^{-m}, k2^{m}). \end{cases}$$

For example

$$Y_{1,1} = \begin{cases} 1 & \text{if } \omega \in [0, 2^{-1}) \\ 0 & \text{if } \omega \notin [0, 2^{-1}); \end{cases} \quad Y_{1,2} = -\begin{cases} -1 & \text{if } \omega \in [2^{-1}, 1) \\ 0 & \text{if } \omega \notin [2^{-1}, 1). \end{cases}$$

Then the sequence $(X_n)_{n \geqq 0}$ is defined from the sequence $(Y_{m,k})$ in the following way:

$$(X_0, X_1, X_2, \ldots) = (Y_{1,1}, Y_{1,2}, Y_{2,1}, Y_{2,2}, Y_{2,3}, Y_{2,4}, \ldots, Y_{m,1}, \ldots, Y_{m,k}, \ldots).$$

Let further $\mathscr{F}_n=\sigma(X_0, X_1, \ldots, X_n)$. It is easy to see that

$$\sup_{m \geqq n \geqq 1} \| E(|X_m - X_{n-1}| \mid \mathscr{F}_n) \|_\infty \leqq 2$$

and that

$$\lim_{n \to +\infty} P(|X_n| > \varepsilon) = 0 \quad \text{for all} \quad \varepsilon > 0.$$

Consequently, $(X_n, \mathscr{F}_n)_{n \in \mathbf{N}}$ is a BMO-sequence and its limit in probability is equal to 0.

Further for each fixed positive integer d we trivially have

$$\lim_{\tau \in T^d} \int_\Omega |X_\tau| \, dP = 0.$$

Now we define two increasing sequences of stopping times with respect to $(\mathscr{F}_n)$ as follows: for $m=0, 1, 2, \ldots$ let $\tau_0(\omega) \equiv 0$,

$$\sigma_0(\omega) \equiv 0; \text{ for } m \geqq 1 \text{ let } \tau_m(\omega) = k + \sum_{i=1}^{m} 2^i \quad \text{if} \quad \omega \in \left[k \cdot 2^{-m-1}, (k+2)2^{-m-1}\right)$$

where $k=0, 2, 4, \ldots, Z^{m+1}-2$. Also let

$$\sigma_m(\omega) = (k+1) + \sum_{i=1}^{m} 2^i \quad \text{if} \quad \omega \in \left[k \cdot 2^{-m-1}, (k+2)2^{-m-1}\right)$$

where $k=0, 2, 4, \ldots, 2^{m+1}-2$.

It is not difficult to show that $(\tau_m)_{m \in \mathbf{N}}$ and $(\sigma_m)_{m \in \mathbf{N}}$ are stopping times with respect to $(\mathscr{F}_n)_{n \in \mathbf{N}}$. Also, trivially $\tau_m, \sigma_m \in T^{2^m}$ and

$$\int_\Omega X_{\tau_m} dP \equiv \frac{1}{2}, \quad \int_\Omega X_{\sigma_m} dP \equiv -\frac{1}{2}.$$

Consequently, $(X_n, \mathscr{F}_n)_{n \in \mathbf{N}}$ is not an amart.

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CONVERGENCE ORDER OF INTERPOLATION PROCESS BY BERNSTEIN POLYNOMIALS

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1. Introduction. Let $f(x)\in C[-1,1]$ and $T_n(x)=\cos n\vartheta$ ($x=\cos\vartheta, 0\leqq\vartheta\leqq\pi$) the first kind of Chebyshev polynomial of degree n. $T_n(x)$ has all its zeros in $(-1,1)$ at the following points

$$x_k=\cos\vartheta_k,\ \vartheta_k=\frac{(2k-1)\pi}{2n},\quad k=1,2,\ldots,n. \tag{1.1}$$

The Bernstein interpolation process $F_n(f;x)$ with zeros of the $T_n(x)$ is given by

$$F_n(f;x)=\sum_{k=1}^{n} f(x_k)\,m_k(x) \tag{1.2}$$

where

$$\begin{cases} m_1(x)=\dfrac{1}{4}\left(3\mu_1(x)+\mu_2(x)\right) \\ m_k(x)=\dfrac{1}{4}\left(\mu_{k-1}(x)+2\mu_k(x)+\mu_{k+1}(x)\right),\quad k=2,3,\ldots,n-1 \\ m_n(x)=\dfrac{1}{4}\left(\mu_{n-1}(x)+3\mu_n(x)\right). \end{cases} \tag{1.3}$$

$\mu_k(x)$ are called the fundamental polynomials of Lagrange interpolation based on the nodes x_k:

$$\mu_k(x)=(-1)^{k+1}\frac{\sqrt{1-x_k^2}\,T_n(x)}{n(x-x_k)},\quad k=1,2,\ldots,n. \tag{1.4}$$

For $F_n(f;x)$, A. K. Varma [1] has proved the following theorem

THEOREM (Varma). *Let $f(x)\in C[-1,1]$. Then*

$$|F_n(f;x)-f(x)|\leqq c\left(\omega\left(f,\frac{\sqrt{1-x^2}}{n}\right)+\omega\left(f,\frac{1}{n^2}\right)\right),\quad -1\leqq x\leqq 1, \tag{1.5}$$

where c is a constant, and $\omega(f,\delta)$ is the modulus of continuity of $f(x)$.

From (1.5) we know that the convergence order attains the order of best approximation when using the interpolation process $F_n(f;x)$ to approximate $f(x)\in C[-1,1]$ on $[-1,1]$. It is natural to raise the very interesting question: If the function $f(x)$ has continuous derivative function $f'(x)$ on $[-1,1]$, then using the interpolation process $F_n(f;x)$ to approximate $f(x)\in C^1[-1,1]$ on

$[-1, 1]$, can the convergence order reach the order of best approximation? In this paper we give an answer to this question.

THEOREM. *Let $f(x)$ be an arbitrary continuous function and $f(x)\in C^1[-1, 1]$. Then*

$$F_n(f; x)-f(x)=O\left(\frac{1}{n}\omega\left(f', \frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)+\frac{\|f'\|}{n^2}\right), \quad -1\leqq x\leqq 1, \tag{1.6}$$

where O is independent of n, f and f', $\|f'\|=\max\limits_{-1\leqq x\leqq 1}|f'(x)|$.

2. Some lemmas. To prove the theorem we need some lemmas.

LEMMA 1 (cf. [1, 4]). *Let x_j be the nearest node to x. Then*

$$f'(y)-f'(x)=O\left(\omega\left(f', \frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)\right), \quad k=j-1, j, j+1. \tag{2.1}$$

If $1<k=j-i<j-1$ or $j+1<k=j+i<n$, then

$$f'(y)-f'(x)=O\left(\omega\left(f', \frac{i\sqrt{1-x^2}}{n}+\frac{i^2}{n^2}\right)\right), \tag{2.2}$$

$$f'(t_k)-f'(t_{k+1})=O\left(\omega\left(f', \frac{\sqrt{1-x^2}}{n}+\frac{i}{n^2}\right)\right) \tag{2.3}$$

where $x_k\leqq t_k\leqq x_{k-1}$, $y=x+a_k(x_k-x)$ $(0<a_k<1)$ or $y=t_k$.

LEMMA 2. *The following estimates are valid in $-1\leqq x\leqq 1$:*

$$m_k(x)=O\left(\frac{|\cos n\vartheta|}{n^3\cdot\left|\prod\limits_{j=k-1}^{k+1}\sin\frac{\vartheta-\vartheta_j}{2}\right|}\right), \quad k=2, 3, \ldots, n-1, \tag{2.4}$$

$$p_k(x)=O\left(\frac{|\cos n\vartheta|}{n^5\cdot\left|\prod\limits_{j=k-2}^{k+2}\sin\frac{\vartheta-\vartheta_j}{2}\right|}\right), \quad k=3, 4, \ldots, n-2, \tag{2.5}$$

where $p_k(x)=m_{k-1}(x)+2m_k(x)+m_{k+1}(x)$.

PROOF. (Cf. [1].) By (2.4) we have

$$m_k(x)+m_{k+1}(x)=(-1)^k\frac{\cos n\vartheta}{n}\sin^3\frac{\pi}{2n}[I_k(\vartheta)+I_k(-\vartheta)]$$

where

$$I_k=\frac{2\cos\frac{\pi}{2n}+\cos\left(\vartheta_k+\vartheta+\frac{\pi}{2n}\right)}{\prod\limits_{j=k-1}^{k+2}\sin\frac{\vartheta+\vartheta_j}{2}}. \tag{2.6}$$

Hence, from (2.6) and

$$\left|\sin\frac{\vartheta-\vartheta_k}{2}\right| \leqq \left|\sin\frac{\vartheta+\vartheta_k}{2}\right|$$

we get

$$p_k(x) = (-1)^k \frac{T_n(x)}{n} \sin^3\frac{\pi}{2n}\left[(I_k(\vartheta)-I_{k-1}(\vartheta))+(I_k(-\vartheta)-I_{k-1}(-\vartheta))\right] =$$

$$= (-1)^k \frac{T_n(x)}{n} \sin^3\frac{\pi}{2n}\left\{\frac{2\sin\frac{\pi}{n}\cos\frac{\vartheta+\vartheta_k}{2}\left(2\cos\frac{\pi}{2n}+\cos\left(\vartheta_k+\vartheta+\frac{\pi}{2n}\right)\right)}{\prod_{j=k-2}^{k+2}\sin\frac{\vartheta+\vartheta_j}{2}}+\right.$$

$$+\frac{2\sin\frac{\pi}{2n}\sin\frac{\vartheta+\vartheta_k}{2}\sin\frac{\vartheta+\vartheta_{k-2}}{2}}{\prod_{j=k-2}^{k+2}\sin\frac{\vartheta+\vartheta_j}{2}}+\frac{2\sin\frac{\pi}{2n}\sin\frac{\vartheta_k-\vartheta}{2}\sin\frac{\vartheta_{k-2}-\vartheta}{2}}{\prod_{j=k-2}^{k+2}\sin\frac{\vartheta_j-\vartheta}{2}}+$$

$$\left.+\frac{2\sin\frac{\pi}{n}\cos\frac{\vartheta_k-\vartheta}{2}\left(2\cos\frac{\pi}{2n}+\cos\left(\vartheta_k-\vartheta+\frac{\pi}{2n}\right)\right)}{\prod_{j=k-2}^{k+2}\sin\frac{\vartheta_j-\vartheta}{2}}\right\} =$$

$$= O\left(\frac{|\cos n\vartheta|}{n^5\cdot\left|\prod_{j=k-2}^{k+2}\sin\frac{\vartheta-\vartheta_j}{2}\right|}\right),$$

thus (2.5) is valid.

LEMMA 3. *The following estimation is valid in* $[-1, 1]$:

$$\sum_{k=3}^{n-2}\left[(x_{k-1}-x)m_{k-1}(x)+2(x_k-x)m_k(x)+(x_{k+1}-x)m_{k+1}(x)\right] = O\left(\frac{1}{n^2}\right).$$

PROOF. We have

$$(2.7)\qquad 4\sum_{k=3}^{n-2}\left[(x_{k-1}-x)m_{k-1}(x)+2(x_k-x)m_k(x)+(x_{k+1}-x)m_{k+1}(x)\right] =$$

$$= 4\sum_{k=4}^{n-3}\left[(x_{k-1}-x)\mu_{k-1}(x)+2(x_k-x)\mu_k(x)+(x_{k+1}-x)\mu_{k+1}(x)\right]+$$

$$+\left\{\sum_{k=2}^{3}+\sum_{k=n-2}^{n-1}\right\}\beta_k\left[(x_{k-1}-x)\mu_{k-1}(x)+2(x_k-x)\mu_k(x)+(x_{k+1}-x)\mu_{k+1}(x)\right]+$$

$$+\left\{\sum_{k=2}^{3}\left[(x_k-x_{k-1})\mu_{k-1}(x)+2(x_{k+1}-x_k)\mu_k(x)+(x_{k+2}-x_{k+1})\mu_{k+1}(x)\right]+\right.$$

$$+\sum_{k=n-2}^{n-1}[(x_{k-2}-x_{k-1})\mu_{k-1}(x)+2(x_{k-1}-x_k)\mu_k(x)+(x_k-x_{k+1})\mu_{k+1}(x)]\}+$$

$$+\sum_{k=4}^{n-3} W_k[\mu_{k-1}(x)+2\mu_k(x)+\mu_{k+1}(x)]+$$

$$+\sum_{k=4}^{n-3}[(W_{k+1}-W_k)\mu_{k+1}(x)+(W_{k-1}-W_k)\mu_{k-1}(x)]=$$

$$=\sum_{j=1}^{5} B_j,$$

where $w_k=x_{k-1}-2x_k+x_{k+1}$, $\beta_2=\beta_{n-1}=1$, $\beta_3=\beta_{n-2}=3$.

Using the relations

$$(2.8)\quad (x_j-x)\mu_j(x)=(-1)^{j+1}\cdot\frac{1}{n}\cdot T_n(x)\sqrt{1-x_j^2}=O\left(\frac{1}{n^2}\right),\quad j\leqq 4 \text{ or } j\geqq n-3,$$

$$(2.9)\qquad x_j-x_{j+1}=O\left(\frac{1}{n^2}\right),\quad j\leqq 4 \text{ or } j\geqq n-3,$$

$$(2.10)\qquad \mu_k(x)=O(1),\quad k=1,2,\ldots,n.$$

we obtain

$$B_2=O\left(\frac{1}{n^2}\right),\quad B_3=O\left(\frac{1}{n^2}\right).$$

By $x_k=\cos\dfrac{(2k-1)\pi}{2n}$ we have

$$(2.11)\qquad (x_{k-1}-x)\mu_{k-1}(x)+2(x_k-x)\mu_k(x)+(x_{k+1}-x)\mu_{k+1}(x)=$$

$$=(-1)^k\frac{T_n(x)}{n}\left((1-x_{k-1}^2)^{1/2}-2(1-x_k^2)^{1/2}+(1-x_{k+1}^2)^{1/2}\right)=$$

$$=(-1)^k\frac{T_n(x)}{n}\left(4\sin^2\frac{\pi}{2n}\sin\frac{(2k-1)\pi}{2n}\right)=O\left(\frac{1}{n^3}\right).$$

$$W_k=-4\sin^2\frac{\pi}{2n}\cos\vartheta_k=O\left(\frac{1}{n^2}\right).$$

$$W_{k+1}-W_k=-4\sin^2\frac{\pi}{2n}(\cos\vartheta_{k+1}-\cos\vartheta_k)=O\left(\frac{1}{n^3}\right).$$

Thus, on using (2.10) and $\sum_{k=1}^{n}|m_k(x)|=O(1)$ (cf. [2]) we obtain

$$B_1=O\left(\frac{1}{n^2}\right)\cdot B_4=O\left(\frac{1}{n^2}\right)\cdot B_5=O\left(\frac{1}{n^2}\right).$$

Combining B_j $(j=1,2,\ldots,5)$ proves Lemma 3.

3. Proof of the theorem. By $\sum_{k=1}^{n} m_k(x)=\sum_{k=1}^{n} \mu_k(x)=1$ we have

$$F_n(f;\, x)-f(x)=\sum_{k=1}^{n}\big(f(x_k)-f(x)\big)m_k(x).$$

On using Lagrange mean-value theorem we obtain

$$f(x_k)-f(x)=f'\big(x+a_k(x_k-x)\big)(x_k-x),\quad 0<a_k<1,\quad k=1,2,\ldots,n$$

$$f(x_k)-f(x_{k-1})=f'(t_k)(x_k-x_{k-1}),\quad x_k<t_k<x_{k-1},\quad k=2,3,\ldots,n.$$

Let

$$A_1=\Big\{\sum_{k=1}^{2}+\sum_{k=n-1}^{n}\Big\}e_k\big(f(x_k)-f(x)\big)m_k(x),\quad e_1=e_n=4,\quad e_2=e_{n-1}=2,$$

$$A_2=\sum_{k=3,n-1} f'(t_k)(x_{k-1}-x_k)m_{k-1}(x)+\sum_{k=2}^{3}\big(f'(t_k)-f'(x)\big)(x_k-x_{k+1})m_k(x)+$$

$$+\sum_{k=n-2}^{n-1}\big(f'(t_k)-f'(x)\big)(x_k-x_{k-1})m_k(x),$$

$$A_3=\sum_{k=3,n-1}\big(f(x_k)-f(x)\big)\big(m_k(x)+m_{k-1}(x)\big),$$

$$A_4=f'(x)\sum_{k=3}^{n-2}\big((x_{k-1}-x)m_{k-1}(x)+2(x_k-x)m_k(x)+(x_{k+1}-x)m_{k+1}(x)\big),$$

$$A_5=\sum_{k=3}^{n-2}\big[f'\big(x+a_k(x_k-x)\big)-f'(x)\big](x_k-x)\,p_k(x),$$

$$A_6=\sum_{k=3}^{n-2}\big(f'(t_k)-f'(t_{k-1})\big)(x_{k-1}-x_k)m_{k-1}(x),$$

$$A_7=\sum_{k=4}^{n-3}\big(f'(x)-f'(t_k)\big)(x_{k-1}-2x_k+x_{k+1})m_k(x).$$

Then

$$4\big(F_n(f;\, x)-f(x)\big)=\sum_{j=1}^{7}A_j. \tag{3.1}$$

Underneath, we estimate the convergence order of A_j respectively. On using (2.9) and (2.10) we have

$$A_2=O\left(\frac{\|f'\|}{n^2}\right). \tag{3.2}$$

By (2.8), (2.9), (2.10) and Lagrange mean-value theorem we have

$$A_3 = f'(\eta_1)(x_3-x)\big(m_2(x)+m_3(x)\big)+f'(\eta_2)(x_{n-1}-x)\big(m_{n-1}(x)+m_{n-2}(x)\big) = \tag{3.3}$$

$$= \frac{1}{4}f'(\eta_1)[(x_4-x)\mu_4(x)+3(x_3-x)\mu_3(x)+3(x_2-x)\mu_2(x)+(x_1-x)\mu_1(x)+$$

$$+(x_3-x_4)\mu_4(x)+3(x_3-x_2)\mu_2(x)+(x_3-x_1)\mu_1(x)]+$$

$$+\frac{1}{4}f'(\eta_2)[(x_n-x)\mu_n(x)+3(x_{n-1}-x)\mu_{n-1}(x)+3(x_{n-2}-x)\mu_{n-2}(x)+$$

$$+(x_{n-3}-x)\mu_{n-3}(x)+(x_{n-1}-x_n)\mu_n(x)+3(x_{n-1}-x_{n-2})\mu_{n-2}(x)+$$

$$+(x_{n-1}-x_{n-3})\mu_{n-3}(x)] = O\left(\frac{\|f'\|}{n^2}\right).$$

Similarly we have

$$A_1 = O\left(\frac{\|f'\|}{n^2}\right). \tag{3.4}$$

On using the Lemma 3 we get

$$A_4 = O\left(\frac{\|f'\|}{n^2}\right). \tag{3.5}$$

To estimate A_5, A_6 and A_7 let x_j be the nearest node to x. Then

$$A_5 = \Big\{\sum_{k=3}^{j-3} + \sum_{k=j+3}^{n-2} + \sum_{k=j-2}^{j+2}\Big\}\big[f'(x+a_k(x_k-x))-f'(x)\big](x_k-x)p_k(x) = V_1+V_2+V_3.$$

For V_3, on using (2.1), (2.10) and $x-x_k=O\left(\frac{1}{n}\right)$ we have

$$V_3 = O\left(\frac{1}{n}\,\omega\left(f', \frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)\right).$$

For V_1 and V_2 when using (2.2), (2.5) and

$$x_k-x = O(|\vartheta-\vartheta_k|), \quad \omega(f', k\delta) \leqq (k+1)\omega(f', \delta)$$

we obtain

$$V_1 = O\left(\sum_{k=3}^{j-3}\omega\left(f', \frac{i\sqrt{1-x^2}}{n}+\frac{i^2}{n^2}\right)\cdot\frac{i}{n}\cdot\frac{1}{i^5}\right) = O\left(\frac{1}{n}\,\omega\left(f', \frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)\right),$$

$$V_2 = O\left(\frac{1}{n}\,\omega\left(f', \frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)\right)$$

therefore

$$A_5 = O\left(\frac{1}{n}\,\omega\left(f', \frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)\right). \tag{3.6}$$

By the same way, on using (2.1)—(2.4), (2.10), (2.11) we have

$$A_6 = O\left(\frac{1}{n}\omega\left(f', \frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)\right), \tag{3.7}$$

$$A_7 = O\left(\frac{1+\ln n}{n^2}\omega\left(f', \frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)\right). \tag{3.8}$$

Combining (3.2)—(3.8) from (3.1) we have

$$F_n(f; x)-f(x) = O\left(\frac{1}{n}\omega\left(f', \frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)+\frac{\|f'\|}{n^2}\right).$$

The theorem is proved.

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THE RESULT DERIVATIVE, DISTRIBUTIVE RESULTS

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In this paper certain properties of the Heaviside operator are described and the notion of distributive result in the Bittner operational calculus is introduced. The distributive result and the result derivative are analogons of the distribution and distributive derivative in the operational calculus.

1. Preliminaries

The notion of the Bittner operational calculus and definitions of the results and of the operators [2, 3], which will be used further in the paper, will be presented in this part.

The operational calculus is referred to as the system

$$\mathrm{CO}\,(L^0, L^1, S, T_q, s_q, q, Q),$$

where L^0 and L^1 are linear spaces (over the same field Γ of real or complex numbers) such that $L^1 \subset L^0$; the linear operation $S\colon L^1 \to L^0$ (denoted by the symbol $S \in L(L^1, L^0)$), called derivative, is a surjection, i.e. $S(L^1) = L^0$. The elements of the kernel of S, i.e. of the set

$$\operatorname{Ker} S := \{c \in L^1 \colon Sc = 0\}$$

are called the constants of S.

Q is an arbitrary set of indices q for the operations $T_q \in L(L^0, L^1)$ (called integrals) such that $ST_q w = w$, $w \in L^0$ and for the operations $s_q \in L(L^1, L^1)$ (called limit conditions) such that $T_q S x = x - s_q x$, $x \in L^1$.

By induction we define a sequence of spaces L^k, $k \in N$ such that

$$L^k := \{x \in L^{k-1} \colon Sx \in L^{k-1}\}.$$

Then

$$\ldots \subset L^k \subset L^{k-1} \subset \ldots \subset L^1 \subset L^0$$

and

$$w = \underbrace{S(S(\ldots(Sx)\ldots))}_{k \text{ times}} =: S^k x,$$

where

$$w \in L^0, \quad x \in L^k, \quad S^k \in L(L^k, L^0), \quad S^k(L^k) = L^0, \quad k \in N.$$

We also determine

$$L^{\infty} := \bigcap_{k=0}^{\infty} L^k.$$

For $x \in L^k$, $k \in N$ the following Taylor formula holds:

$$x = s_q x + T_q s_q S x + \ldots + T_q^{k-1} s_q S^{k-1} x + T_q^k S^k x. \tag{1}$$

The operations t_q^k defined by the formula

$$t_q^k c := k! T_q^k c, \tag{2}$$

where $c \in \operatorname{Ker} S$, $k \in N$ will be called the abstract variables corresponding to T_q.

The endomorphism $A \in L(L^0, L^0)$ such that

$$A|_{L^1} \in L(L^1, L^1), \quad SA = AS, \quad s_q A = A s_q, \quad (Sx = Ax \wedge s_q x = 0) \Rightarrow (x = 0), \quad x \in L^1$$

will be called the logarithm (with a fixed $q \in Q$).

Let $\pi(X)$ be a commutative semigroup of injective endomorphisms of the linear space X (over the field Γ). When a space X and the semigroup $\pi(X)$ are given, we can introduce ordered couples

$$\xi := [x, A], \quad x \in X, \quad A \in \pi(X)$$

and the relation

$$([x, A] = [y, B]) \overset{\text{def}}{\Leftrightarrow} (Bx = Ay), \quad x, y \in X, \ A, B \in \pi(X) \tag{3}$$

which is an equivalence. By this relation the entire set of the considered couples is divided into equivalence classes. These classes will be called results. An arbitrary representative $\xi = [x, A]$ of such a class will also be called a result. For such a representative the fraction symbol $\xi = \frac{x}{A}$ will be applied. If we introduce in the set of results $\Xi(X, \pi(X))$ (denoted briefly by $\Xi(X)$) the operations

$$\frac{x}{A} + \frac{y}{B} := \frac{Bx + Ay}{AB}, \quad \gamma\left(\frac{x}{A}\right) := \frac{\gamma x}{A},$$

where $x, y \in X$, $\gamma \in \Gamma$, $A, B \in \pi(X)$, then $\Xi(X)$ is a linear space over the field Γ. The elements of X can be referred to as results, as the map

$$x \mapsto \frac{x}{I}, \quad x \in X, \quad I := \mathrm{id}_X \in \pi(X)$$

is an isomorphism.

In $\Xi(X)$ we can also define the operation

$$B\left(\frac{x}{A}\right) := \frac{Bx}{A},$$

where $x \in X$, $A \in \pi(X)$, $B \in L(X, X)$, $AB = BA$.

Given the endomorphism $R\in L(X, X)$ commutative with the operations $A\in\pi(X)$, the linear operation given by the formula

$$\mu\frac{x}{B}:=\frac{Rx}{AB}$$

will be called an operator on the space of results $\Xi(X)$ and denoted by $\mu=\frac{R}{A}$.

The operator $\mu_0=\frac{R}{I}$ such that

$$\mu_0\frac{x}{I}=\frac{Rx}{I}\mapsto Rx$$

will be identified with the endomorphism R.

The sum of commutative operators, the product of an operator by an element of Γ and the superposition of commutative operators is also a commutative operator. The division by operators, the numerators of which are injections, defined by the formula

$$\frac{\frac{x}{B}}{\frac{A_1}{A_2}}:=\frac{A_2}{A_1}\frac{x}{B}=\frac{A_2x}{A_1B} \tag{4}$$

renders also an operator.

Given a fixed derivative S and a fixed integral T_q, select a fixed commutative semigroup of endomorphisms being simultaneously injections of L^0 containing T_q (and hence also the iterations of T_q). A semigroup of this kind will be called a semigroup with an integral and denoted by $\pi_{T_q}(L^0)$. As

$$T_{q_1}T_{q_2}\neq T_{q_2}T_{q_1},\quad q_1, q_2\in Q$$

therefore only one integral T_q belongs to $\pi_{T_q}(L^0)$.

The operator

$$P_q:=\frac{I}{T_q},$$

where $I:=\mathrm{id}_{L^0}$, $T_q\in\pi_{T_q}(L^0)$ will be called the Heaviside operator with the integral T_q. Hence we get

$$T_q=\frac{I}{P_q}.$$

For any two results

$$\xi=F(P_q)c,\quad \eta=G(P_q)c \tag{5}$$

being operational functions of the same constant $c\in\mathrm{Ker}\,S$ we denote by $\xi*\eta$ the result

$$\xi*\eta:=\frac{F(P_q)G(P_q)}{P_q}c, \tag{6}$$

where $F(P_q)G(P_q)$ means the usual multiplication of functions of the variable P_q

(this multiplication corresponds to the composition of operators). The result $\xi * \eta$ will be called the convolution of the elements (5).

It is easy to verify that the operation of convolution is associative, commutative and distributive with respect to addition. It can be shown that the convolution (6) of continuous functions $\xi = \{\xi(t)\}$, $\eta = \{\eta(t)\}$, $t \in [\alpha, \beta]$ and sequences $\xi = (\xi_k)$, $\eta = (\eta_k)$, $k \in N_0 := N \cup \{0\}$ can be represented by the formulas

$$\xi * \eta = \left\{ \int_0^t \xi(\tau)\eta(t-\tau)\, d\tau \right\}, \quad \xi * \eta = \left(\sum_{i=0}^{k-1} \xi_i \eta_{k-1-i} \right), \quad \sum_{i=0}^{-1} := 0.$$

2. Result derivative

In this section the Dirac result will be defined, the notion of k-th order result will be introduced and multiplication of such result by an element of the space L^∞ will be determined. In this connexion some algebraic properties of Heaviside operator and Dirac result will be shown.

In the sequel the Heaviside operator P_q corresponding to T_q will also be called a result derivative and denoted by D. Notice that the space of results $\Xi(L^0) = \Xi\left(L^0, \pi_{T_q}(L^0)\right)$ contains more elements than L^0, as

$$\xi = \frac{c}{T_q}, \quad c \in \operatorname{Ker} S \tag{7}$$

does not belong to L^0 if $c \neq 0$.

It follows from (7) that $T_q \xi = c$. If $\xi \in L^0$ then $T_q \xi \in L^1$ and $s_q T_q \xi = s_q c$. Hence $0 = c$ what contradicts the assumption [3].

DEFINITION 1. The element $\xi \in L^0$ is said to be a regular result, the element $\xi \in \Xi(L^0) - L^0$ is said to be a singular result.

It follows from definition of the Heaviside operator that for the result $\xi = \frac{x}{A} \in \Xi(L^0)$ the formula

$$D\xi = \frac{x}{T_q A} \tag{8}$$

holds. Using definition (4) we can also write

$$D\xi = \frac{\xi}{T_q} = P_q \xi. \tag{8'}$$

It is easy to verify that the result derivative (8) does not depend on the form of representing the result ξ by a fraction. Moreover, D is an injection and

$$D: \Xi(L^0) \xrightarrow{\text{onto}} \Xi(L^0).$$

This follows from the fact that for any result $\eta \in \Xi(L^0)$ there exists a result $\xi = T_q \eta \in \Xi(L^0)$ such that $D\xi = \eta$.

COROLLARY 1. *For any result there exists a result derivative of arbitrarily high order.*

Let L^0 be a commutative algebra with the unit l belonging to Ker S.

DEFINITION 2. The singular result

$$\delta := \frac{l}{T_q} = P_q l \tag{9}$$

will be called the Dirac delta-result.

Hencefrom

$$T_q \delta = l \tag{10}$$

and

$$ST_q \delta = 0. \tag{11}$$

By formulas (2) and (9) we get

$$D^k \delta = P_q^{k+1} l, \quad D^k t_q^{k-1} l = (k-1)! \delta, \quad k \in N. \tag{12}$$

DEFINITION 3.

$$\frac{c}{A} \odot \frac{l}{T_q^k} := \frac{c}{AT_q^k}, \quad c \in \operatorname{Ker} S, \ A \in \pi_{T_q}(L^0), \ k \in N_0. \tag{13}$$

Particularily

$$l \odot \delta = \delta \odot l = \delta.$$

THEOREM 1. *For any regular result* $x \in L^k$, $k \in N$ *the following formula holds:*

$$D^k x = S^k x + s_q x \odot D^{k-1} \delta + s_q S x \odot D^{k-2} \delta + \ldots + s_q S^{k-1} x \odot \delta. \tag{14}$$

In particular,

$$D^k x = S^k x \tag{15}$$

if

$$s_q S^i x = 0, \quad i = 0, 1, \ldots, k-1. \tag{16}$$

PROOF. For $x \in L^k$, $k \in N$ from Taylor formula (1) we get

$$\frac{x}{T_q^k} = \frac{s_q x}{T_q^k} + \frac{s_q S x}{T_q^{k-1}} + \ldots + \frac{s_q S^{k-1} x}{T_q} + S^k x.$$

As $s_q S^i x \in \operatorname{Ker} S$, $i = 0, 1, \ldots, k-1$ and

$$\frac{s_q S^i x}{T_q^{k-i}} = \frac{s_q S^i x}{I} \odot \frac{l}{T_q^{k-i}} = s_q S^i x \odot D^{k-1-i} \delta, \quad i = 0, 1, \ldots, k-1$$

therefore from the last relation we obtain (14). (15) is a particular case of (14).

COROLLARY 2. *For the singular result* $\xi = \frac{x}{A}$, *where* $x \in L^k$, $k \in N$, $A \in \pi_{T_q}(L^0)$ *the following formula holds:*

$$D^k \xi = \frac{S^k x}{A} + \frac{s_q x}{A} \odot D^{k-1} \delta + \frac{s_q S x}{A} \odot D^{k-2} \delta + \ldots + \frac{s_q S^{k-1} x}{A} \odot \delta. \tag{17}$$

In particular,

$$D^k\xi = \frac{S^k x}{A} \tag{18}$$

for

$$s_q S^i x = 0, \quad i = 0, 1, \ldots, k-1. \tag{19}$$

Theorem 9 [3] follows from Theorem 1.

THEOREM 2.

$$D^k(\xi * \eta) = D^k\xi * \eta = \xi * D^k\eta, \quad k \in N_0. \tag{20}$$

PROOF. Let ξ and η have the form (5). From definition (8′) of the result derivative and formula (6) we get

$$D^k\xi * \eta = P_q^k F(P_q)c * G(P_q)c = \frac{P_q^k F(P_q)G(P_q)}{P_q}\, c = P_q^k\left[\frac{F(P_q)G(P_q)}{P_q}\, c\right] = D^k(\xi * \eta)$$

and

$$\xi * D^k\eta = F(P_q)c * P_q^k G(P_q)c = \frac{F(P_q)P_q^k G(P_q)}{P_q}\, c = P_q^k\left[\frac{F(P_q)G(P_q)}{P_q}\, c\right] =$$

$$= D^k(\xi * \eta), k \in N_0.$$

THEOREM 3. *The Dirac result δ is a modulus of the convolution multiplication in the domain of results of the form $\xi = F(P_q)l$.*

PROOF. From (6) we obtain

$$\xi * \delta = F(P_q)l * P_q l = \frac{F(P_q)P_q}{P_q}\, l = F(P_q)l = \xi.$$

From this last theorem and from (20) we obtain

$$\delta * \delta = \delta, \quad \delta * l = l, \tag{21}$$

$$D^k(\xi * \delta) = \xi * D^k\delta = D^k\xi * \delta = D^k\xi, \quad k \in N_0. \tag{22}$$

Hence

$$D^k\delta * D^m\delta = D^{k+m}\delta, \quad k, m \in N_0. \tag{23}$$

EXAMPLE 1. Given a semigroup $\pi_{T_q}(L^0)$ such that $I - T_q \in \pi_{T_q}(L^0)$, in the space $\Xi(L^0, \pi_{T_q}(L^0))$ of results we shall solve the convolution equation

$$(D\delta - \delta) * \xi = \delta. \tag{24}$$

Let $\xi = F(P_q)l$. As $\delta = P_q l$ and $D\delta = P_q^2 l$, therefore equation (24) can be rewritten in the form

$$(P_q^2 - P_q)l * F(P_q)l = P_q l.$$

Therefrom and from definition (6) we obtain

$$\frac{(P_q^2 - P_q)F(P_q)}{P_q}\, l = P_q l,$$

i.e.

$$\xi = F(P_q)l = \frac{P_q}{P_q - I}\, l = \frac{l}{I - T_q}.$$

Notice that if the result ξ is regular (it is, in this case, denoted by $e^{t_q} l$ (see [3])) then this result is also the unique solution of the abstract differential equation

(25) $$Sx = x$$

with the limit condition

(26) $$s_q x = l$$

because problem (25), (26) is equivalent to the abstract integral equation

$$(I - T_q)x = l.$$

DEFINITION 4. The set of endomorphisms

$$\{I, T_q, T_q^2, \ldots, T_q^k, \ldots\}$$

of L^0 will be referred to as the cyclic semigroup generated by the integral T_q.

COROLLARY 3. *If $\pi_{T_q}(L^0)$ is the cyclic semigroup generated by T_q, then for any result $\xi \in \Xi(L^0)$ there exists an element $y \in L^0$ and a number $k \in N_0$ such that $\xi = D^k y$.*

DEFINITION 5. The element $\xi = \frac{x}{A}$, where $x \in L^0$, $A \in \pi_{T_q}(L^0)$ will be called a result of order k if there exists an element $y \in L^0$ and a number $k \in N_0$ such that

(27) $$Ay = T_q^k x.$$

The smallest non-negative integer k such that (27) holds will be referred to as the fundamental order of ξ.

COROLLARY 4. *If $\pi_{T_q}(L^0)$ is the cyclic semigroup generated by T_q, then any element $\xi \in \Xi(L^0)$ is a result of some order.*

It can be easily seen, that (27) can be rewritten in the form

(28) $$y = T_q^k \frac{x}{A} = T_q^k \xi.$$

COROLLARY 5. *The element $\xi \in \Xi(L^0)$ is a result of order k if $T_q^k \xi$ is regular. The fundamental order of a regular result is equal to zero.*

From the definition of the result derivative and formula (28) we get

$$D^k y = \xi.$$

COROLLARY 6. *If $\xi \in \Xi(L^0)$ is a result of order k, then there exists an element $y \in L^0$ such that $\xi = D^k y$.*

EXAMPLE 2. A) The fundamental order of the singular result

$$\xi = \frac{c}{T_q^n}, \quad c \in \operatorname{Ker} S - \{0\}, \quad n \in N$$

is equal to n.

B) If A is a logarithm (with a fixed $q \in Q$) then we can consider a semigroup $\pi_{T_q}(L^0)$ such that

$$I - T_q A \in \pi_{T_q}(L^0).$$

If, moreover, for any regular result $y \neq 0$ the elements

$$\xi = \frac{y}{I - T_q A}$$

are singular, then the result

$$\eta = \frac{x}{I - T_q A}, \quad x \in L^0 - \{0\}$$

has no defined order.

Let L^k, $k \in N$ be a subalgebra of the commutative algebra L^0.

DEFINITION 6 (cf. [8]). We say that the derivative S satisfies the Leibniz condition if

$$S(x \cdot y) = Sx \cdot y + x \cdot Sy, \quad x, y \in L^1 \tag{29}$$

holds. We say that the limit condition s_q is multiplicative if

$$s_q(x \cdot y) = s_q x \cdot s_q y, \quad x, y \in L^1. \tag{30}$$

THEOREM 4. *If the derivative S satisfies the Leibniz condition* (29), *then*

$$x \cdot S^k y = \sum_{i=0}^{k} (-1)^i \binom{k}{i} S^{k-i}(S^i x \cdot y), \quad x, y \in L^k, \ k \in N_0. \tag{31}$$

PROOF. For $k=0$, (31) is obviously true. Assume that (31) holds for a fixed k. Then

$$x \cdot S^{k+1} y = S(x \cdot S^k y) - Sx \cdot S^k y = \sum_{i=0}^{k} (-1)^i \binom{k}{i} S^{k+1-i}(S^i x \cdot y) +$$

$$+ \sum_{i=0}^{k} (-1)^{i+1} \binom{k}{i} S^{k-1}(S^{i+1} x \cdot y) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} S^{k+1-i}(S^i x \cdot y) +$$

$$+ \sum_{i=1}^{k+1} (-1)^i \binom{k}{i-1} S^{k+1-i}(S^i x \cdot y) = S^{k+1}(x \cdot y) +$$

$$+ \sum_{i=1}^{k} (-1)^i \left[\binom{k}{i} + \binom{k}{i-1} \right] S^{k+1-i}(S^i x \cdot y) + (-1)^{k+1} S^{k+1} x \cdot y =$$

$$= \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} S^{k+1-i}(S^i x \cdot y).$$

Thus (31) has been proved by induction for $k \geqq 0$.

If the derivative S satisfies the Leibniz condition and if the limit condition s_q is multiplicative, then (31) suggests the following definition of the product of a regular result $x\in L^\infty$ by the result ξ of order k, $k\in N_0$:

DEFINITION 7.

$$x\cdot\xi := \sum_{i=0}^{k}(-1)^i\binom{k}{i}D^{k-i}(S^i x\cdot y),\quad x\in L^\infty,\ \xi = D^k y,\ y\in L^0,\ k\in N_0. \tag{32}$$

It is obvious that $x\cdot\xi=\xi\cdot x$ and

$$(x_1+x_2)\xi = x_1\cdot\xi+x_2\cdot\xi,\quad x(\xi_1+\xi_2) = x\cdot\xi_1+x\cdot\xi_2,\quad (x_1\cdot x_2)\xi = x_1(x_2\cdot\xi),$$

where $x, x_1, x_2\in L^\infty$ and ξ, ξ_1, ξ_2 are results of order k, $k\in N_0$.

THEOREM 5. *The result* (32) *does not depend on the way of representing the result* ξ *in the form* $\xi=D^k y$.

PROOF. It is sufficient to demonstrate that if $\xi=D^{k+1}(T_q y)$, then

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}D^{k-i}(S^i x\cdot y) = \sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}D^{k+1-i}(S^i x\cdot T_q y). \tag{33}$$

As the limit condition s_q is multiplicative and $s_q T_q=0$ therefore

$$s_q(S^i x\cdot T_q y) = 0,\quad i\in N_0.$$

Hence we obtain from Theorem 1

$$S(S^i x\cdot T_q y) = D(S^i x\cdot T_q y),\quad i\in N_0.$$

Hence and from the Leibniz formula (29) we get

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}D^{k-i}(S^i x\cdot y) = \sum_{i=0}^{k}(-1)^i\binom{k}{i}D^{k-i}[S(S^i x\cdot T_q y)-S^{i+1}x\cdot T_q y] =$$

$$= \sum_{i=0}^{k}(-1)^i\binom{k}{i}D^{k+1-i}(S^i x\cdot T_q y)+\sum_{i=0}^{k}(-1)^{i+1}\binom{k}{i}D^{k-i}(S^{i+1}x\cdot T_q y) =$$

$$= \sum_{i=0}^{k}(-1)^i\binom{k}{i}D^{k+1-i}(S^i x\cdot T_q y)+\sum_{i=1}^{k+1}(-1)^i\binom{k}{i-1}D^{k+1-i}(S^i x\cdot T_q y) =$$

$$= D^{k+1}(x\cdot T_q y)+\sum_{i=1}^{k}(-1)^i\left[\binom{k}{i}+\binom{k}{i-1}\right]D^{k+1-i}(S^i x\cdot T_q y)+$$

$$+(-1)^{k+1}S^{k+1}x\cdot T_q y = \sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}D^{k+1-i}(S^i x\cdot T_q y),$$

which is (33).

If $k=0$ then (31) and (32) are identical and consistent with the multiplication in the algebra L^0.

Notice that if $y\in L^k$, $k\in N$ and

$$s_q S^i y = 0, \quad i = 0, 1, \ldots, k-1$$

then

$$D^k y = S^k y \quad \text{and} \quad D^{k-i}(S^i x \cdot y) = S^{k-i}(S^i x \cdot y), \quad i = 0, 1, \ldots, k,$$

which follows from Theorem 1 and (29), (30).

Also in this case the multiplication defined by formula (32) is identical with the multiplication (31) of regular results x and $S^k y$, $k\in N$.

THEOREM 6. *If $s_q x=0$ then the result derivative D satisfies the Leibniz condition*

$$D(x\cdot\xi) = Dx\cdot\xi + x\cdot D\xi, \tag{34}$$

where $x\in L^\infty$ and ξ is a result of order k, $k\in N_0$.

PROOF. From the assumption $s_q x=0$ we obtain by Theorem 1 $Dx=Sx$. Now by (32) it is sufficient to prove that

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}D^{k+1-i}(S^i x\cdot y) = \sum_{i=0}^{k}(-1)^i\binom{k}{i}D^{k-i}(S^{i+1}x\cdot y) +$$

$$+\sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}D^{k+1-i}(S^i x\cdot y).$$

The proof of this relation proceeds as follows:

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}D^{k-i}(S^{i+1}x\cdot y)+\sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}D^{k+1-i}(S^i x\cdot y) =$$

$$= \sum_{i=1}^{k}(-1)^i\binom{k}{i-1}D^{k+1-i}(S^i x\cdot y)+(-1)^k S^{k+1}x\cdot y + D^{k+1}(x\cdot y) +$$

$$-\sum_{i=1}^{k}(-1)^{i-1}\binom{k+1}{i}D^{k+1-i}(S^i x\cdot y)-(-1)^k S^{k+1}x\cdot y = D^{k+1}(x\cdot y)+$$

$$+\sum_{i=1}^{k}(-1)^{i-1}\left[\binom{k}{i-1}-\binom{k+1}{i}\right]D^{k+1-i}(S^i x\cdot y) = \sum_{i=0}^{k}(-1)^i\binom{k}{i}D^{k+1-i}(S^i x\cdot y).$$

THEOREM 7.

$$x\cdot D^k\delta = \sum_{i=0}^{k}(-1)^i\binom{k}{i}s_q S^i x\odot D^{k-i}\delta, \quad x\in L^\infty,\ k\in N_0. \tag{35}$$

PROOF. From (32), the axiom $x-T_q Sx=s_q x$ and Definition 3 we obtain

$$x\cdot\delta = x\cdot Dl = Dx-Sx = \frac{x-T_q Sx}{T_q} = \frac{s_q x}{T_q} = s_q x\odot\delta,$$

i.e. (35) for $k=0$.

Let $k \in N$. From (14) we get

$$(36)\quad D^{k+1}x - D^k Sx + (-1)^k DS^k x + (-1)^{k+1} S^{k+1}x = S^{k+1}x + s_q x \odot D^k\delta +$$
$$+ s_q Sx \odot D^{k-1}\delta + \ldots + s_q S^k x \odot \delta - (S^{k+1}x + s_q Sx \odot D^{k-1}\delta + \ldots + s_q S^k x \odot \delta) +$$
$$+ (-1)^k (S^{k+1}x + s_q S^k x \odot \delta) - (-1)^k S^{k+1}x = s_q x \odot D^k\delta + (-1)^k s_q S^k x \odot \delta$$

as well as

$$(37)\quad D^{k+1-i}S^i x - D^{k-i}S^{i+1}x = S^{k+1}x + s_q S^i x \odot D^{k-i}\delta + s_q S^{i+1}x \odot D^{k-1-i}\delta + \ldots$$
$$\ldots + s_q S^k x \odot \delta - (S^{k+1}x + s_q S^{i+1}x \odot D^{k-1-i}\delta + \ldots + s_q S^k x \odot \delta) =$$
$$= s_q S^i x \odot D^{k-i}\delta, \quad i = 1, 2, \ldots, k-1.$$

From Definition 7 and formulas (36) and (37) we obtain

$$x \cdot D^k\delta = x \cdot D^{k+1}l = \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} D^{k+1-i}(S^i x \cdot l) =$$

$$= D^{k+1}x + \sum_{i=1}^{k} (-1)^i \binom{k+1}{i} D^{k+1-i}S^i x + (-1)^{k+1}S^{k+1}x =$$

$$= D^{k+1}x + \sum_{i=1}^{k} (-1)^i \left[\binom{k}{i-1} + \binom{k}{i}\right] D^{k+1-i}S^i x + (-1)^{k+1}S^{k+1}x =$$

$$= D^{k+1}x + \sum_{i=1}^{k} (-1)^i \binom{k}{i-1} D^{k+1-i}S^i x + \sum_{i=1}^{k} (-1)^i \binom{k}{i} D^{k+1-i}S^i x + (-1)^{k+1}S^{k+1}x =$$

$$= D^{k+1}x + \sum_{i=0}^{k-1} (-1)^{i+1} \binom{k}{i} D^{k-i}S^{i+1}x + \sum_{i=1}^{k-1} (-1)^i \binom{k}{i} D^{k+1-i}S^i x +$$
$$+ (-1)^k DS^k x + (-1)^{k+1}S^{k+1}x = D^{k+1}x - D^k Sx + (-1)^k DS^k x + (-1)^{k+1}S^{k+1}x +$$
$$+ \sum_{i=1}^{k-1} (-1)^i \binom{k}{i} [D^{k+1-i}S^i x - D^{k-i}S^{i+1}x] = s_q x \odot D^k\delta +$$
$$+ \sum_{i=1}^{k-1} (-1)^i \binom{k}{i} s_q S^i x \odot D^{k-i}\delta + (-1)^k s_q S^k x \odot \delta = \sum_{i=0}^{k} (-1)^i \binom{k}{i} s_q S^i x \odot D^{k-i}\delta,$$

which is (35).

THEOREM 8.

$$(38)\qquad t_q^k l \cdot D^m\delta = \begin{cases} 0 & \text{for} \quad m < k \\ (-1)^k k!\,\delta & \text{for} \quad m = k,\ k, m \in N_0. \\ (-1)^k \binom{m}{k} k!\, D^{m-k}\delta & \text{for} \quad m > k \end{cases}$$

PROOF. From (35) and definition (2) of the abstract variables we get

$$t_q^k l \cdot D^m \delta = \sum_{i=0}^{m} (-1)^i \binom{m}{i} s_q S^i t_q^k l \odot D^{m-i} \delta = \tag{39}$$

$$= \sum_{i=0}^{m} (-1)^i \binom{m}{i} k! s_q S^i T_q^k l \odot D^{m-i} \delta.$$

If $m<k$ then for $i=0, 1, \ldots, m$ we have

$$(-1)^i \binom{m}{i} k! s_q S^i T_q^k l \odot D^{m-i} \delta = (-1)^i \binom{m}{i} k! s_q T_q^{k-i} l \odot D^{m-i} \delta = 0, \tag{40}$$

because $s_q T_q = 0$. Hence and from (39) we obtain the assertion of the Theorem for $m<k$. If $m=k$ then for $i=0, 1, \ldots, k-1$ we obtain (40), whereas for $i=k$ we have

$$(-1)^k k! s_q S^k T_q^k l \odot \delta = (-1)^k k! s_q l \odot \delta = (-1)^k k! \delta,$$

as $s_q l = l$ and $l \odot \delta = \delta$. Hence and from (39) we obtain formula (38) for $m=k$. If $m>k$ then for $i=0, 1, \ldots, k-1$ we obtain (40), for $i=k+1, k+2, \ldots, m$ we get

$$(-1)^i \binom{m}{i} k! s_q S^i T_q^k l \odot D^{m-i} \delta = (-1)^i \binom{m}{i} k! s_q S^{i-k} l \odot D^{m-i} \delta = 0,$$

as $Sl = 0$, whereas for $i=k$ we have

$$(-1)^k \binom{m}{k} k! s_q S^k T_q^k l \odot D^{m-k} \delta = (-1)^k \binom{m}{k} k! s_q l \odot D^{m-k} \delta = (-1)^k \binom{m}{k} k! D^{m-k} \delta,$$

as $s_q l = l$ and $l \odot D^{m-k} \delta = D^{m-k} \delta$.

Herefrom and from (39) we get formula (38) for $m>k$.

3. Operational convergence. Distributive results

In [3] Bittner considers partially ordered linear spaces, which are referred to as Mikusiński spaces. In these spaces he introduces a convergence, by means of which he defines the convergence in the space L^0 and the operational convergence of results, which will be referred to as the convergence in narrow sense.

In this section a more general definition of operational convergence will be introduced and distributive results will be defined. Some analytic properties of result derivative will also be presented.

DEFINITION 8. A real linear space Z is said to be a Mikusiński space if in Z a cone K is distinguished with the following properties:

1. $0 \in K$,
2. if $z_1, z_2 \in K$ then $z_1 + z_2 \in K$,
3. if $z \in K$, $\alpha \geqq 0$ then $\alpha z \in K$,
4. if $z_1 \in K$ and there exists $z_2 \in K$ that for any $n \in N$ $z_2 - n z_1 \in K$, then $z_1 = 0$,
5. for any $z \in Z$ there exist $z_1, z_2 \in K$ such that $z = z_1 - z_2$.

If $z_1, z_2 \in Z$ then we assume by definition

$$z_1 \leqq z_2 \quad \text{if and only if} \quad z_2 - z_1 \in K.$$

The notation $z_2 \geqq z_1$ is also assumed to be correct.

Moreover, for any element $z \in Z$ a modulus $|z| \in K$ is defined with the following properties:

6. $|z| = z$ for $z \geqq 0$,
7. if $|z| = 0$ then $z = 0$,
8. $|z_1 + z_2| \leqq |z_1| + |z_2|$,
9. $|\alpha z| = |\alpha| |z|$, $z, z_1, z_2 \in Z$, $\alpha \in R^1$.

In a Mikusiński space the notion of regular convergence (convergence with a regulator f) can be introduced (cf. [4]).

DEFINITION 9. We say that the sequence of elements $z_n \in Z$, $n \in N$ converges to some $z \in Z$, denoting this by the symbol $(z_n) \Rightarrow z$, if and only if

$$\bigvee_{f \in K} \bigwedge_{\varepsilon > 0} \bigvee_{m \in N} \bigwedge_{n \geqq m} |z_n - z| \leqq \varepsilon \cdot f. \tag{41}$$

In [3] the convergence (41) is referred to as the Mikusiński convergence (cf. [5]).

The convergence in the linear space L^0 can be introduced as the convergence in a space normed by elements belonging to a cone K in the Mikusiński space Z.

DEFINITION 10. We say that in L^0 a norm

$$\|x\| \in Z, \quad x \in L^0$$

is defined, i.e. a norm with values belonging to the Mikusiński space Z if

1. $\|x\| \geqq 0$; if $\|x\| = 0$ then $x = 0$,
2. $\|x_1 + x_2\| \leqq \|x_1\| + \|x_2\|$,
3. $\|\gamma x\| = |\gamma| \|x\|$, $x, x_1, x_2 \in L^0$, $\gamma \in \Gamma$.

We say that the sequence $(x_n)_{n \in N}$ of elements of L^0 is convergent to some $x \in L^0$, denoting this by $(x_n) \Rightarrow x$, if and only if

$$(\|x_n - x\|) \Rightarrow 0$$

in Z. L^0 together with the norm $\|\cdot\|$ defined in this space, i.e. the couple $(L^0, \|\cdot\|)$ is referred to as a normed space.

Let L^0 be a normed space. By $\pi^c_{T_q}(L^0)$ we denote the semigroup (with the integral T_q) of continuous endomorphisms of L^0.

DEFINITION 11. We say that the sequence $(\xi_n)_{n \in N}$ of results of the space $\Xi^c(L^0) = \Xi(L^0, \pi^c_{T_q}(L^0))$ is operationally convergent in the broad sense to the result ξ (this being denoted by the symbol $(\xi_n) \to \xi$) if and only if there exists an operation A and a sequence $(A_n)_{n \in N}$ of elements of the semigroup $\pi^c_{T_q}(L^0)$ such that $A_n \xi_n$, $A\xi \in L^0$, $n \in N$ and

$$(A_n) \Rightarrow A, \quad (A_n \xi_n) \Rightarrow A\xi,$$

i.e.

$$\lim_{n \to \infty} \|A_n - A\| = 0, \quad \lim_{n \to \infty} \|A_n \xi_n - A\xi\| = 0.$$

If $A_n := A$, $n \in N$ and $(A\xi_n) \Rightarrow A\xi$ then we say that the sequence $(\xi_n)_{n\in N}$ is operationally convergent in the narrow sense to ξ, denoting this by the symbol $(\xi_n) \dashrightarrow \xi$.

Any sequence $(x_n)_{n\in N}$ of regular results convergent according to the norm $\|\cdot\|$ is operationally convergent, because

$$[(x_n) \Rightarrow x] \equiv [(\mathrm{id}_{L^0} x_n) \to \mathrm{id}_{L^0} x] \equiv [(\mathrm{id}_{L^0} x_n) \dashrightarrow \mathrm{id}_{L^0} x].$$

THEOREM 9. 1. *The convergence in the space of results $\Xi^c(L^0)$ is defined uniquely.*

2. *The operations of addition of results and multiplication by a number are operationally continuous.*

3. *The endomorphism $R \in L(L^0, L^0)$ continuous and commutative with the operations of the semigroup $\pi^c_{T_q}(L^0)$ is operationally continuous.*

PROOF. 1. Assume that $(\xi_n) \to \xi$ and $(\xi_n) \to \eta$. There exist operations A, B and sequences (A_n), (B_n) of elements of $\pi^c_{T_q}(L^0)$ such that

$$(A_n) \Rightarrow A \qquad (B_n) \Rightarrow B,$$

$$(A_n \xi_n) \Rightarrow A\xi, \quad (B_n \xi_n) \Rightarrow B\eta.$$

Hence $(A_n B_n) \Rightarrow AB$ and

$$(A_n B_n \xi_n) \Rightarrow AB\xi, \quad (A_n B_n \xi_n) \Rightarrow AB\eta. \tag{42}$$

In the space L^0 the convergence "$\Rightarrow$" (according to the norm $\|\cdot\|$) is unique. Hence and from (42) we have

$$[AB\xi = AB\eta] \equiv [AB(\xi - \eta) = 0] \equiv [\xi - \eta = 0],$$

as $AB \in \pi^c_{T_q}(L^0)$.

2. Let $(\xi_n) \to \xi$ and $(\eta_n) \to \eta$. Then there exist operations A, B and sequences (A_n), (B_n) of elements of $\pi^c_{T_q}(L^0)$ such that

$$(A_n) \Rightarrow A, \qquad (B_n) \Rightarrow B,$$

$$(A_n \xi_n) \Rightarrow A\xi, \quad (B_n \eta_n) \Rightarrow B\eta.$$

Hence $(A_n B_n) \Rightarrow AB$ and

$$(A_n B_n [\xi_n + \eta_n]) = (A_n B_n \xi_n) + (A_n B_n \eta_n) \Rightarrow AB\xi + AB\eta = AB[\xi + \eta],$$

i.e. $(\xi_n) + (\eta_n) \to \xi + \eta$.

Moreover, if

$$\lim_{n\to\infty} \gamma_n = \gamma, \quad \gamma_n, \gamma \in \Gamma, \ n \in N$$

then $(\gamma_n A_n) \Rightarrow \gamma A$ and

$$(A_n \gamma_n \xi_n) = (\gamma_n A_n \xi_n) \Rightarrow \gamma A\xi = A\gamma\xi,$$

i.e. $(\gamma_n \xi_n) \to \gamma\xi$.

3. Let $(\xi_n) \to \xi$. There exists an operation A and a sequence (A_n) of elements of $\pi^c_{T_q}(L^0)$ such that

$$(A_n) \Rightarrow A, \quad (A_n \xi_n) \Rightarrow A\xi.$$

Therefore for any operation $R \in L(L^0, L^0)$ continuous and commutative with the operations of the semigroup $\pi^c_{T_q}(L^0)$ we have

$$(A_n R) \Rightarrow AR, \quad (A_n R\xi_n) \Rightarrow AR\xi,$$

i.e. $(R\xi_n) \to R\xi$.

Theorem 43 of [3] is a particular case of Theorem 9.

THEOREM 10. *The result derivative D is operationally continuous.*

PROOF. Let $(\xi_n)\to\xi$. Hence there exists an operation A and a sequence (A_n) of elements of $\pi^c_{T_q}(L^0)$ such that

$$(A_n)\Rightarrow A, \quad (A_n\xi_n)\Rightarrow A\xi.$$

Therefore $(T_qA_n)\Rightarrow T_qA$ and

$$(T_qA_nD\xi_n)\Rightarrow T_qAD\xi,$$

i.e. $(D\xi_n)\to D\xi$.

THEOREM 11. *If* $s_qS^ix_n=0$, $x_n\in L^k$, $k, n\in N$, $i=0, 1, \ldots, k-1$ *and* $(x_n)\to\xi$, $\xi\in\Xi^c(L^0)$, *then*

$$(S^kx_n)\to D^k\xi.$$

PROOF. Let $(x_n)\to\xi$, where $x_n\in L^k$, $k, n\in N$, $\xi\in\Xi^c(L^0)$. Hence there exists an operation A and a sequence (A_n) of elements of $\pi^c_{T_q}(L^0)$ such that

$$(A_n)\Rightarrow A, \quad (A_nx_n)\Rightarrow A\xi.$$

From Taylor's formula (1) for $x_n\in L^k$ we obtain $T^k_qS^kx_n=x_n$, because $s_qS^ix_n=0$, $i=0, 1, \ldots, k-1$. Obviously $(A_nT^k_q)\Rightarrow AT^k_q$. Moreover,

$$(A_nT^k_qS^kx_n)=(A_nx_n)\Rightarrow A\xi=AT^k_qD^k\xi.$$

Hence $(S^kx_n)\to D^k\xi$.

Given two sequences (ξ_n), (η_n) of results from $\Xi^c(L^0)$. If there exist operations A, B and sequences (A_n), (B_n) of elements of $\pi^c_{T_q}(L^0)$ such that $A_n\xi_n$, $B_n\eta_n$, $A\xi$, $B\xi\in L^0$, $n\in N$ and

$$(A_n)\Rightarrow A, \qquad (B_n)\Rightarrow B,$$

$$(A_n\xi_n)\Rightarrow A\xi, \quad (B_n\eta_n)\Rightarrow B\xi,$$

then we shall denote this by $(\xi_n)\to\leftarrow(\eta_n)$.

If, however, $A\xi_n$, $B\eta_n$, $A\xi$, $B\xi\in L^0$, $n\in N$ and

$$(A\xi_n)\Rightarrow A\xi, \quad (B\eta_n)\Rightarrow B\xi,$$

then the symbol $(\xi_n)\dot{\to}\dot{\leftarrow}(\eta_n)$ will be applied.

It is easy to verify that the relation

$$[(\xi_n)\sim(\eta_n)]\overset{\text{def}}{\Leftrightarrow}[(\xi_n)\to\leftarrow(\eta_n)] \tag{43}$$

is an equivalence relation. It divides the set of all sequences of results operationally convergent in the broad sense into equivalence classes, which will be referred to as distributive results in the broad sense. A distributive result in the broad sense determined by the class of sequences being in relation (43) with (ξ_n) will be denoted by $\xi=\langle\xi_n\rangle$, where $(\xi_n)\to\xi$. Similarly, the relation

$$[(\xi_n)\dot{\sim}(\eta_n)]\overset{\text{def}}{\Leftrightarrow}[(\xi_n)\dot{\to}\dot{\leftarrow}(\eta_n)]$$

determines distributive results in the narrow sense $\xi\doteq\langle\xi_n\rangle$, where $(\xi_n)\dot{\to}\xi$.

EXAMPLE 3. Given a semigroup with the integral $\pi^c_{T_q}(L^0)$ such that

$$\gamma_n(I+n^2T_q^2),\quad \gamma_n(I+nT_q)\in\pi^c_{T_q}(L^0),\quad \gamma_n\in\Gamma-\{0\},\quad n\in N,$$

take two sequences

$$\delta_{n,1} := \frac{n^2T_q l}{I+n^2T_q^2} = \frac{T_q l}{\frac{1}{n^2}I+T_q^2}$$

and

$$\delta_{n,2} := \frac{nl}{I+nT_q} = \frac{l}{\frac{1}{n}I+T_q},\quad n\in N$$

of results from $\Xi^c(L^0)$. These sequences are operationally convergent in the broad sense to the delta-result of Dirac, because for

$$A_n := \frac{1}{n^2}I+T_q^2,\quad A := T_q^2,\quad B_n := \frac{1}{n}I+T_q,\quad B := T_q,\quad n\in N$$

we have

$$(A_n)\Rightarrow A,\qquad (B_n)\Rightarrow B,$$

$$(A_n\delta_{n,1})\Rightarrow A\delta,\quad (B_n\delta_{n,2})\Rightarrow B\delta.$$

Therefore $(\delta_{n,1})\sim(\delta_{n,2})$. If the result $\delta_{n,1}$, $n\in N$ is regular, then we denote it by $n\sin nt_q l$, $n\in N$ (cf. [3]). It is then the solution of the problem

$$\begin{cases} S^2x+n^2x=0,\\ s_qx=0,\ s_qSx=n^2l,\end{cases}\quad n\in N.$$

The problem is equivalent to the integral equation

$$(I+n^2T_q^2)x = n^2T_q l,\quad n\in N.$$

If the result $\delta_{n,2}$, $n\in N$ is regular, then we denote it by $ne^{-nt_q}l$, $n\in N$ (cf. [3]). It is then the solution of the problem

$$\begin{cases} Sx+nx=0,\\ s_qx=nl,\end{cases}\quad n\in N.$$

The problem is equivalent to the integral equation

$$(I+nT_q)x = nl,\quad n\in N.$$

Notice that

$$(\delta'_{n,1}) := \left(\frac{n^2 l}{I+n^2T_q^2}\right)\to D\delta.$$

If the result $\delta'_{n,1}$, $n\in N$ is regular, then we shall denote it by $n^2\cos nt_q l$, $n\in N$ (cf. [3]). It is then the solution of the problem

$$\begin{cases} S^2x+n^2x=0,\\ s_qx=n^2l,\ s_qSx=0,\end{cases}\quad n\in N.$$

The problem is equivalent to the integral equation

$$(I+n^2T_q^2)x = n^2 l, \quad n\in N.$$

For regular results $\delta_{n,1}$, $\delta'_{n,1}$, $n\in N$ we have obviously

$$S\delta_{n,1} = Sn\sin nt_q l = ST_q \frac{n^2 l}{I+n^2T_q^2} = \frac{n^2 l}{I+n^2T_q^2} = \delta'_{n,1} = n^2\cos nt_q l, \quad n\in N$$

and

$$(S\delta_{n,1}) \to D\delta$$

which is consistent with the last Theorem, as $s_q\delta_{n,1}=0$. We can also write

$$\delta = \langle\delta_{n,1}\rangle = \langle n\sin nt_q l\rangle$$

and

$$D\delta = \langle\delta'_{n,1}\rangle = \langle n^2\cos nt_q l\rangle.$$

EXAMPLE 4. The set $Z:=C(N)$ of real sequences $z=(z_n)_{n\in N}$ is a real linear space with the usual addition of sequences and the multiplication of a sequence by a real number. It is also a Mikusiński space with the ordering

$$z \geqq 0 \quad \text{if and only if} \quad z_n \geqq 0 \quad \text{for any} \quad n\in N$$

and the modulus

$$|z| := (|z_n|)_{n\in N}.$$

If with every element $x\in L^0$ we associate a sequence of real quasi-norms $\|x\|_n$, $n\in N$ such that for any $n\in N$ the condition $\|x\|_n=0$ implies $x=0$, then

$$\|x\| := (\|x\|_n)\in C(N) \tag{44}$$

is a norm in L^0.

Given the operational calculus in which $L^k:=C^k([0,+\infty), R^1)$, $k\in N_0$, the space $L^0=C^0([0,+\infty), R^1)$ is a normed space if

$$\|x\|_n := \sup\{|x(t)|: t\in[\alpha_n, \beta_n]\}, \quad n\in N, \quad x=\{x(t)\}\in L^0,$$

where (α_n) and (β_n) are fixed real sequences such that

$$\lim_{n\to\infty}\alpha_n = 0, \quad \lim_{n\to\infty}\beta_n = +\infty$$

and $[\alpha_n, \beta_n]\subset[0,+\infty)$, $n\in N$. Then the norm (44) induces an almost uniform convergence in L^0 on the interval $[0,+\infty)$.

Furthermore, let

$$S := \frac{d}{dt}, \quad T_{t_0} := \int_{t_0}^{t}, \quad s_{t_0} := |_{t=t_0}, \quad q = t_0\in Q := [0,+\infty).$$

Put $t_0=0$. The set L^0 is a commutative algebra with the usual multiplication of functions. The modulus of this multiplication is the function

$$l := \{1\}\in \operatorname{Ker} S.$$

Take the following sequence of elements of L^0:

$$\{\delta_{n,1}(t)\} := \left\{\frac{2n}{\pi(1+n^2t^2)}\right\}, \quad n \in N.$$

Because of

$$\left(\left\{\int_0^t \delta_{n,1}(\tau)\,d\tau\right\}\right) = \left(\left\{\frac{2}{\pi}\operatorname{arc\,tg} nt\right\}\right) \Rightarrow \{1\}$$

we have

$$\left(\left\{\int_0^t \frac{2}{\pi}\operatorname{arc\,tg} n\tau\,d\tau\right\}\right) \Rightarrow \{t\}$$

and therefore

$$(T_0^2\{\delta_{n,1}(t)\}) \Rightarrow T_0^2\frac{\{1\}}{T_0},$$

i.e.

$$(\{\delta_{n,1}(t)\}) \dashrightarrow \{\delta(t)\},$$

where $\{\delta(t)\}$ is the delta-result of Dirac $\frac{l}{T_0}$. Therefore we can write

$$\{\delta(t)\} \doteq \langle\{\delta_{n,1}(t)\}\rangle.$$

Notice that the sequence of continuous functions on the interval $[0, +\infty)$ of the form

$$\{\delta_{n,2}(t)\} := \begin{cases} 2(n-n^2t) & \text{for } 0 \leqq t \leqq \frac{1}{n}, \\ 0 & \text{for } t > \frac{1}{n}, \end{cases} \qquad n \in N$$

is an element of the class $\langle\{\delta_{n,1}(t)\}\rangle$ determining the distributive delta-result of Dirac. Namely we have

$$\left(\left\{\int_0^t \delta_{n,2}(\tau)\,d\tau\right\}\right) = \begin{cases} 2nt-n^2t^2 & \text{for } 0 \leqq t \leqq \frac{1}{n} \\ 1 & \text{for } t > \frac{1}{n} \end{cases} \Rightarrow \{1\},$$

i.e.

$$(T_0\{\delta_{n,2}(t)\}) \Rightarrow T_0\frac{\{1\}}{T_0}$$

and hence

$$(\{\delta_{n,2}(t)\}) \dashrightarrow \{\delta(t)\}.$$

If $x=\{x(t)\}\in L^1$ then from (14) we get

$$D\{x(t)\} = \{x'(t)\}+\{x(0)\odot\delta(t)\},$$

i.e. the formula of the result derivative of the regular result x, where

$$\left(\left\{\frac{2nx(0)}{\pi(1+n^2t^2)}\right\}\right) \dashrightarrow \{x(0)\odot\delta(t)\}.$$

As

$$\left\{\int_0^t n\cos n^2\tau\, d\tau\right\} = \left\{\frac{\sin n^2 t}{n}\right\}, \quad n\in N$$

and

$$\left(\left\{\frac{\sin n^2 t}{n}\right\}\right) \Rightarrow \{0\}$$

therefore the sequences

$$\left(\left\{\frac{\sin n^2 t}{n}\right\}\right) \dot{\sim} (\{0\}) \dot{\sim} (\{n\cos n^2 t\})$$

in the considered model of the operational calculus determine a zero distributive result in the narrow sense.

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MODULES OVER ENDOMORPHISM RINGS. II

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Introduction

This paper could be regarded as continuation of [7] by A. K. Gupta and K. Varadarajan. Throughout this paper we will be following the standard convention in ring theory. The rings we consider will be associative rings with an identity element $1 \neq 0$. All the modules we consider will be unitary. The category of left (resp. right) R-modules will be denoted by R-mod (resp. mod-R). Morphisms will be written on the side opposite to that of scalars. Composition of maps will thus depend on which side of the argument the maps are written. We write ${}_RM$ (resp. M_R) to denote that M is a left (resp. right) R-module. If $S = \mathrm{End}\ {}_RM$, the endomorphism ring of ${}_RM$, then in a natural way M is a left R, right S bimodule. For any ${}_RN$, the module $H = \mathrm{Hom}_R(M, N)$ is in a natural way a left S-module, which we denote by ${}_SH$. For any $A \in R$-mod, let $\tau_M(A) = \sum_{f \in \mathrm{Hom}(M, A)} (M)f$. Then τ_M is an idempotent pre-radical in R-mod. A will be said to be an M-cotorsionless (resp. an M-cotorsion) module if $\tau_M(A) = A$ (resp. $\tau_M(A) = 0$). Let M be a finitely generated (in future abbreviated as f.g.) quasi-projective module. In [7] A. K. Gupta and K. Varadarajan using the $(1-1)$ correspondence between the submodules of ${}_SH$ and M-cotorsionless submodules of ${}_RN$ have successfully tackled the problem of determining necessary and sufficient conditions for ${}_SH$ to be respectively noetherian, artinian, co-noetherian or co-artinian. Under the additional assumption that M is N-projective (in the sense of Azumaya) and $\tau_M(N) = N$ they prove that $\mathrm{corank}\ {}_SH = \mathrm{corank}\ {}_RN$ where corank is the invariant dual to Goldie dimension introduced in [16]. The pre-radical τ_M is said to be N-faithful if $0 \neq N' \subset {}_RN \Rightarrow \tau_M(N') \neq 0$. Also when $\tau_M(N) = N$ and τ_M is N-faithful it is shown that $\dim {}_SH = \dim {}_RN$ where dim stands for Goldie dimension.

In the present paper we continue the study of the module ${}_SH = \mathrm{Hom}_R({}_RM, {}_RN)$. In § 1 we record some preliminary results about ${}_SH$ that we need for our later study. In § 2 we introduce respectively the concepts of τ_M-simple submodule, τ_M maximal submodule, τ_M-small submodule of a given R-module ${}_RN$. We define the τ_M-Jacobson radical of N to be τ_M $(\cap T | T$ a τ_M-maximal submodule of $N)$ and denote it by $\tau_M - J(N)$. The τ_M-socle of N, denoted by $\tau_M - \mathrm{Soc}(N)$ is defined to be the sum of the τ_M-simple submodules of N. When M is finitely generated and N-projective we give alternative descriptions for $\tau_M - J(N)$ (Theorem 2.12). When M is f.g. quasi-projective and N-projective we prove that

$$\mathrm{Soc}({}_SH) = \mathrm{Hom}_R(M, \tau_M - \mathrm{Soc}(N))$$

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and

$$J({}_SH) = \mathrm{Hom}_R(M, \tau_M - J(N))$$

(Theorem 2.18). Using these we also prove that

$$\mathrm{Soc}({}_SH) = \sum \{Sf \mid f \in {}_SH,\ (M)f \text{ is } \tau_M\text{-simple in } N\},$$

$$J({}_SH) = \{f \mid f \in {}_SH,\ (M)f \text{ is } \tau_M\text{-small in } N\}.$$

In § 3 we deal with the problem of determining the singular submodule of ${}_SH = \mathrm{Hom}_R(M, N)$ under suitable assumptions on M and N. For this purpose we introduce the concept of a τ_M-essential submodule in any R-module N. When M is f.g. quasi-projective we prove that $Z({}_SH) = \{f \mid f \in {}_SH, \mathrm{Ker} f \text{ is } \tau_M\text{-essential in } M\}$ (Theorem 3.11). When M is R-projective in the sense of Azumaya, for any $N \in R$-mod we introduce the concept of the τ_M-singular submodule $Z_{\tau_M}(N)$ of N. If ${}_RP$ is f.g. projective, for any ${}_RN$ we show that $Z({}_SH) = \mathrm{Hom}_R(P, Z_{\tau_P}(N))$ where $S = \mathrm{End}\ {}_RP$ and ${}_SH = \mathrm{Hom}_R(P, N)$ (Theorem 3.18). Using some results of A. V. Jategaonkar [8] we show that if ${}_RM$ is a generator for R-mod and $N \in R$-mod, then the singularity of ${}_SH$ implies the singularity of ${}_RN$ and the non-singularity of ${}_RN$ implies the non-singularity of ${}_SH$ (Corollary 3.7). Using results of J. M. Zelmanowitz [19] we show that if M is a torsionless module over a semiprime ring R and $N \in R$-mod is arbitrary, then $Z({}_SH) = \{f \in {}_SH \mid \mathrm{Ker} f \text{ is essential in } M\}$ (Theorem 3.36).

In § 4 we introduce the concepts of τ_M-localness, τ_M-hollowness and τ_M-semi-hollowness. In terms of these concepts it turns out that if ${}_RM$ is f.g. quasi-projective, then ${}_SH = \mathrm{Hom}_R(M, N)$ is local, hollow, resp. semi-hollow if and only N is τ_M-local, τ_M-hollow, resp. τ_M-semi-hollow (Proposition 4.4). We show that our results imply that the endomorphism ring of a f.g. quasi-projective hollow module is local. We also show that our results in § 4 yield quicker proofs of two theorems due to P. Fleury (Theorems 4.10 and 4.11).

In § 5 we introduce the concept of τ_M-Krull dimension for any R-module N. When ${}_RM$ is f.g. quasi-projective, for any R-module N we prove that $K - \dim {}_SH = \tau_M - K - \dim {}_RN$ (Theorem 5.3).

When ${}_RR$ has Krull dimension, ${}_RM$ is f.g. quasi-projective and ${}_R\tau_M(N)$ has Krull dimension, we prove that ${}_SH$ has Krull dimension. Moreover

$$K - \dim {}_SH \leqq K - \dim {}_SS \leqq K - \dim {}_RM \leqq K - \dim {}_RR$$

and $K - \dim {}_SH \leqq K - \dim {}_R\tau_M(N) \leqq K - \dim {}_RR$ (Corollary 5.12).

While characterising rings with the property that a module has Krull dimension if and only if the module is noetherian, B. Sarath [15] introduced the concept of a tall module. We introduce the concept of a τ_M-tall module and show that if M is f.g. quasi-projective and N-projective, then ${}_SH$ is tall if and only if ${}_RN$ is τ_M-tall. If in addition, M is a generator for R-mod, this implies that ${}_SH$ (resp. ${}_SS$) is tall $\Leftrightarrow {}_RN$ (resp. ${}_RM$) is tall (Proposition 5.15 and Corollary 5.19). A ring R is said to be (left) tall if every non-noetherian left R-module is tall. Using our results in § 5, we also show that tallness is a Morita invariant property (Theorem 5.25).

In [11] Erika Mares has defined the concept of semiperfectness for projective modules. It turns out that for a projective module, this property is equivalent to requiring that every submodule have a supplement. We can define a module N,

not necessarily projective, to be semiperfect if every submodule of N has a supplement. We then introduce the concepts of τ_M-supplements and τ_M semiperfectness. When M is f.g. quasi-projective and N-projective we prove that ${}_SH$ is semiperfect $\Leftrightarrow {}_RN$ is τ_M-semiperfect (Theorem 6.15). In addition if $\tau_M(N)=N$, we show that ${}_SH$ is semiperfect $\Leftrightarrow {}_RN$ is semiperfect. In particular if ${}_RM$ is a f.g., quasi-projective, then $S=\mathrm{End}\,{}_RM$ is a semiperfect ring $\Leftrightarrow {}_RM$ is semiperfect. In case ${}_RM$ is a progenerator, this is a result of Erika Mares [11]. When ${}_RM$ is a progenerator, ${}_SH=\mathrm{Hom}_R(M, N)$ is projective $\Leftrightarrow {}_RN$ is projective. In this case, our result asserts that ${}_SH$ is semiperfect in the sense of Erika Mares $\Leftrightarrow {}_RN$ is semiperfect in the sense of Erika Mares.

We conclude by showing that factor modules of semiperfect modules in our sense are semiperfect. Modules with finite spanning dimension in the sense of P. Fleury [4] are semiperfect in our sense. In particular all hollow modules, all artinian modules are semiperfect in our sense. All semisimple modules are semiperfect in our sense.

§ 1. Preliminary results on ${}_SH$

Let $S=\mathrm{End}\,{}_RM$ and ${}_SH=\mathrm{Hom}_R({}_RM, {}_RN)$. When there is no possibility of confusion we write $\mathrm{Hom}(M, N)$ for $\mathrm{Hom}_R({}_RM, {}_RN)$. For any ${}_RN'\subset {}_RN$ we identify $\mathrm{Hom}_R(M, N')$ with an S-submodule of H. We denote this S-submodule of H by $\beta(N')$. Also for any ${}_SL\subset {}_SH$ let $\gamma(L)=\sum_{f\in L}(M)f$. Then $\gamma(L)$ is an R-submodule of N. Also we have

$$\gamma\beta(N')=\sum_{f\in \mathrm{Hom}(M,N')}(M)f=\tau_M(N')\subset N' \tag{1.1}$$

and

$$\beta\gamma(L)=\mathrm{Hom}\big(M, \gamma(L)\big)\supset L. \tag{1.2}$$

The inclusion $\mathrm{Hom}\big(M, \gamma(L)\big)\supset L$ is a consequence of the fact that for any $f\in L$ we have $(M)f\subset\gamma(L)$.

From $(M)f\subset\tau_M(N')$ for any $f\in\mathrm{Hom}(M, N')$ we immediately get $\mathrm{Hom}(M, N')\subset$ $\subset\mathrm{Hom}\big(M, \tau_M(N')\big)$. From $\tau_M(N')\subset N'$ we get $\mathrm{Hom}\big(M, \tau_M(N')\big)\subset\mathrm{Hom}(M, N')$. Thus

$$\beta\tau_M(N')=\beta\gamma\beta(N')=\mathrm{Hom}\big(M, \tau_M(N')\big)=\mathrm{Hom}(M, N')=\beta(N'). \tag{1.3}$$

From (1.2) we get $\gamma\beta\gamma(L)\supset\gamma(L)$. Also

$$\gamma\beta\gamma(L)=\sum_{f\in\mathrm{Hom}(M,\gamma(L))}(M)f\subset\gamma(L).$$

Hence

$$\tau_M\big(\gamma(L)\big)=\gamma\beta\gamma(L)=\gamma(L). \tag{1.4}$$

From (1.1) we see that if N' is an M-cotorsionless submodule of N, then $N'=$ $=\gamma\big(\beta(N')\big)$. From (1.4) we see that for any $L\subset{}_SH$, $\gamma(L)$ is an M-cotorsionless submodule of N. For any family $\{N_\alpha\}_{\alpha\in I}$ of submodules of ${}_RN$, we have

$$\beta\Big(\bigcap_{\alpha\in I}N_\alpha\Big)=\mathrm{Hom}\Big(M, \bigcap_{\alpha\in I}N_\alpha\Big)=\bigcap_{\alpha\in I}\mathrm{Hom}(M, N_\alpha)=\bigcap_{\alpha\in I}\beta(N_\alpha). \tag{1.5}$$

Let $\{L_\alpha\}_{\alpha\in I}$ be any family of submodules of ${}_SH$. Then $f\in\sum_{\alpha\in I}L_\alpha\Rightarrow f=f_{\alpha_1}+\ldots+f_{\alpha_r}$ with $f_{\alpha_i}\in L_{\alpha_i}$ for a finite number of α_i in I. This yields $(M)f\subset(M)f_{\alpha_1}+\ldots+(M)f_{\alpha_r}$ implying $\gamma(\sum_{\alpha\in I}L_\alpha)\subset\sum_{\alpha\in I}\gamma(L_\alpha)$. Always we have $\gamma(\sum_{\alpha\in I}L_\alpha)\supset\gamma(L_\alpha)$ hence $\gamma(\sum_{\alpha\in I}L_\alpha)\supset$ $\supset\sum_{\alpha\in I}\gamma(L_\alpha)$. Combining these two inclusions, we get

$$\gamma\Big(\sum_{\alpha\in I}L_\alpha\Big)=\sum_{\alpha\in I}\gamma(L_\alpha). \tag{1.6}$$

Since the family of M-cotorsionless submodules of N agrees with the family $\{\gamma(L)|L\subset_S H\}$ it follows from (1.6) that the sum of any family of M-cotorsionless submodules of N is again M-cotorsionless.

In general $\bigcap_{\alpha\in I}\gamma(L_\alpha)\supset\gamma(\bigcap_{\alpha\in I}L_\alpha)$. But $\gamma(\bigcap_{\alpha\in I}L_\alpha)$ is M-cotorsionless. Hence

$$\tau_M\Big(\bigcap_{\alpha\in I}\gamma(L_\alpha)\Big)\supset\gamma\Big(\bigcap_{\alpha\in I}L_\alpha\Big), \tag{1.7}$$

Also,

$$\beta\Big(\sum_{\alpha\in I}N_\alpha\Big)\supset\sum_{\alpha\in I}\beta(N_\alpha)\,. \tag{1.8}$$

$$\gamma\Big(\bigcap_{\alpha\in I}\beta(N_\alpha)\Big)=\gamma\Big(\beta\Big(\bigcap_{\alpha\in I}N_\alpha\Big)\Big)=\gamma\beta\Big(\bigcap_{\alpha\in I}N_\alpha\Big)=\tau_M\Big(\bigcap_{\alpha\in I}N_\alpha\Big). \tag{1.9}$$

LEMMA 1.10. *Let M be N-projective and $\{N_i\}_{i=1}^k$ any finite collection of submodules of N. Then*

$$\sum_{i=1}^{K}\tau_M(N_i)=\tau_M\Big(\sum_{i=1}^{k}N_i\Big).$$

PROOF. Follows from Proposition 3.18 in [7] and induction on k.

LEMMA 1.11. *Let M be f.g. and N-projective. Then for any family $\{N_\alpha\}_{\alpha\in J}$ of submodules of N we have*

$$\sum_{\alpha\in J}\tau_M(N_\alpha)=\tau_M\Big(\sum_{\alpha\in J}N_\alpha\Big).$$

PROOF. We know that $\tau_M(\sum_{\alpha\in J}N_\alpha)\supset\tau_M(N_\alpha)$ for each $\alpha\in J$. Hence

$$\tau_M\Big(\sum_{\alpha\in J}N_\alpha\Big)\supset\sum_{\alpha\in J}\tau_M(N_\alpha). \tag{1.12}$$

Let $a\in\tau_M(\sum_{\alpha\in J}N_\alpha)$. Then there exist finitely many elements $f_1,\ldots,f_k$ in $\mathrm{Hom}(M,\sum_{\alpha\in J}N_\alpha)$ such that $a\in\sum_{i=1}^{k}(M)f_i$. Since M is f.g. we get a finite subset K of J with $(M)f_i\subset\sum_{\alpha\in K}N_\alpha$ for each i. Thus $a\in\tau_M(\sum_{\alpha\in J}N_\alpha)$. From Lemma 1.10 we have $a\in\sum_{\alpha\in K}\tau_M(N_\alpha)\subset\sum_{\alpha\in J}\tau_M(N_\alpha)$. Hence

$$\tau_M\Big(\sum_{\alpha\in J}N_\alpha\Big)\subset\sum_{\alpha\in J}\tau_M(N_\alpha). \tag{1.13}$$

Combining (1.12) and (1.13) we get the desired equality. □

The following result played a crucial role in [7] and will again play a crucial role in the present paper.

PROPOSITION 1.14. *Let ${}_RM$ be f.g. quasi-projective. Then the correspondence $L \mapsto \gamma(L)$ is an inclusion preserving $(1-1)$ correspondence between submodules of ${}_SH = \mathrm{Hom}_R({}_RM, {}_RN)$ and M-cotorsionless submodules of ${}_RN$. Moreover $\beta\gamma(L) = L$ for every $L \subset {}_SH$.*

This is Proposition 4.2 in [7].

LEMMA 1.15. *Let M be f.g. quasi-projective and N-projective. Then $\beta(\sum_{\alpha\in J} N_\alpha) = \sum_{\alpha\in J} \beta(N_\alpha)$ for any family $\{N_\alpha\}_{\alpha\in J}$ of submodules of N.*

PROOF. From Lemma 1.11 we have $\tau_M(\sum_{\alpha\in J} N_\alpha) = \sum_{\alpha\in J} \tau_M(N_\alpha)$. This means

$$\gamma\beta(\sum_{\alpha\in J} N_\alpha) = \sum_{\alpha\in J} \gamma\beta(N_\alpha) = \gamma(\sum_{\alpha\in J} \beta(N_\alpha))$$

by (1.6). But γ is $(1-1)$ by Proposition 1.14. Hence $\beta(\sum_{\alpha\in J} N_\alpha) = \sum_{\alpha\in J} \beta(N_\alpha)$. □

REMARKS 1.16. (1) If M is f.g. quasi-projective, but not necessarily N-projective, we have

$$\beta(\sum_{\alpha\in J} \gamma(L_\alpha)) = \sum_{\alpha\in J} L_\alpha = \sum_{\alpha\in J} \beta\gamma(L_\alpha) \tag{1.17}$$

for any family $\{L_\alpha\}_{\alpha\in J}$ of submodules of ${}_SH$.

This is Lemma 4.16 in [7].

(2) If M is f.g. quasi-projective, for any $L \subset {}_SH$ we have $L = \beta(\gamma(L))$. Let $\{L_\alpha\}_{\alpha\in J}$ be any family of submodules of ${}_SH$. Setting $N_\alpha = \gamma(L_\alpha) \subset N$ from (1.9) we get

$$\gamma(\bigcap_{\alpha\in J} L_\alpha) = \tau_M(\bigcap_{\alpha\in J} \gamma(L_\alpha)). \tag{1.18}$$

This is Lemma 4.7 in [7].

LEMMA 1.19. *Let M be N-projective and N' any M-cotorsionless submodule of N. Let $N' \subseteq P \subseteq N$. Then P is M-cotorsionless $\Leftrightarrow P/N'$ is M-cotorsionless.*

PROOF. Let $\eta\colon P \to P/N'$ be the canonical quotient map. Since M is N-projective, from Proposition 3.3 (3) of [7] we see that M is P-projective. Now, Lemma 3.16 of [7] implies $(\tau_M(P))\eta = \tau_M(P/N')$. Also $P \supset N' \Rightarrow \tau_M(P) \supset \tau_M(N') = N'$. Hence $\tau_M(P/N') = (\tau_M(P))\eta = \tau_M(P)/N'$, i.e. $\tau_M(P/N') = P/N' \Leftrightarrow \tau_M(P) = P$. □

DEFINITION 1.20 [7]. N is said to be τ_M-artinian (resp. τ_M-noetherian) if the family of M-cotorsionless submodules of N has d.c.c. (resp. has a.c.c.).

As observed already in [7], as an immediate consequence of Proposition 1.14 we see that ${}_SH$ is noetherian (resp. artinian) $\Leftrightarrow {}_RN$ is τ_M-noetherian (resp. τ_M-artinian) whenever M is f.g. quasi-projective.

REMARK 1.21. Let M be N-projective and $0 \to N' \to N \to N'' \to 0$ be exact. Then $0 \to \mathrm{Hom}(M, N') \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, N'') \to 0$ is exact.

This is immediate from the definition of N-projectivity.

LEMMA 1.22. *Let M be f.g., quasi-projective and N-projective. Let $N' \subset N$. Then ${}_S(H/\beta(N'))$ is noetherian (resp. artinian) $\Leftrightarrow {}_R(N/N')$ is τ_M-noetherian (resp. τ_M-artinian).*

PROOF. From Remark 1.21, we see that ${}_S(H/\beta(N')) \cong \mathrm{Hom}_R(M, N/N')$. Now, Lemma 1.22 is an immediate consequence of Proposition 1.14. □

LEMMA 1.23. *Let M be f.g., quasi-projective and N-projective. Let $H' \subset {}_SH$. Then ${}_S(H/H')$ is noetherian (resp. artinian) $\Leftrightarrow {}_R(N/\gamma(H'))$ is τ_M-noetherian (resp. τ_M-artinian).*

PROOF. We have $\beta\gamma(H') = H'$. Hence Lemma 1.23 follows immediately from Lemma 1.22 by taking $N' = \gamma(H')$. □

This is actually Lemma 4.6 in [7]. But the proof presented here is considerably more elegant than the proof presented in [7].

§ 2. Socle and the Jacobson radical of ${}_SH$

Throughout this paper, we will be studying the module ${}_SH = \mathrm{Hom}_R({}_RM, {}_RN)$. We will just write H for ${}_SH$.

DEFINITION 2.1. A submodule K of N will be called a τ_M-simple submodule of N if $\tau_M(K) = K \neq 0$ and there is no proper, non-zero M-cotorsionless submodule of K.

DEFINITION 2.2. A submodule T of N will be called a τ_M-maximal submodule of N if it is a maximal M-cotorsionless submodule of N in the sense that $\tau_M(T) = T \neq \tau_M(N)$ and the only M-cotorsionless submodule of N strictly containing T is $\tau_M(N)$.

Recall that the idempotent pre-radical τ_M was termed N-faithful in [7] if $0 \neq N' \subset N \Rightarrow \tau_M(N') \neq 0$.

REMARKS 2.3. (i) Any M-cotorsionless, simple (resp. M-cotorsionless, maximal) submodule of N is clearly a τ_M-simple (resp. τ_M-maximal) submodule of N.

(ii) If τ_M is N-faithful, then any simple submodule of N is automatically M-cotorsionless and the class of τ_M-simple submodules of N is the same as the class of simple submodules of N.

LEMMA 2.4. *Let M be f.g. quasi-projective and N-projective. Then an M-cotorsionless submodule K of N is τ_M-maximal in N if and only if $\tau_M(N)/K$ is τ_M-simple.*

PROOF. Let K be τ_M-maximal in N. If $\tau_M(N)/K$ were not τ_M-simple, then there exists a K' with $K \subset K' \subset \tau_M(N)$, $0 \neq K'/K \neq \tau_M(N)/K$ and $\tau_M(K'/K) = K'/K$. From Lemma 1.19 we get $K' = \tau_M(K')$. Clearly $K \underset{+}{\subset} K' \underset{+}{\subset} \tau_M(N)$, a contradiction to the τ_M-maximality of K in N.

The converse is proved similarly using Lemma 1.19. □

LEMMA 2.5. *Let ${}_RM$ be f.g. quasi-projective and $N \in R$-mod.*

(1) *$L \subset_S H$ is simple (resp. maximal in H) $\Leftrightarrow \gamma(L)$ is τ_M-simple (resp. τ_M-maximal in N).*

(2) *An M-cotorsionless submodule N' of N is τ_M-simple (resp. τ_M-maximal in N) $\Leftrightarrow \beta(N')$ is simple (resp. maximal in ${}_SH$).*

PROOF. Immediate consequence of Proposition 1.14. □

DEFINITION 2.6. Define an M-cotorsionless submodule K of N to be τ_M-small if $K+L=\tau_M(N)$, $\tau_M(L)=L \Rightarrow L=\tau_M(N)$.

It is clear that if A is an M-cotorsionless submodule of the M-cotorsionless submodule K, and if K is τ_M-small in N then A is τ_M-small in N. In general a submodule B (not necessarily M-cotorsionless) of N will be called τ_M-small in N if B is contained in a τ_M-small M-cotorsionless submodule K of N.

We write $A \ll N$ to denote that A is small in N and write $K \overset{\tau_M}{\ll} N$ to denote that K is τ_M-small in N.

PROPOSITION 2.7. *Let M be f.g. quasi-projective. Then*

(i) $L \ll H \Leftrightarrow \gamma(L) \overset{\tau_M}{\ll} N$.

(ii) $N' \overset{\tau_M}{\ll} N \Rightarrow \beta(N') \ll H$. *Conversely, if N' is M-cotorsionless, $\beta(N') \ll H \Rightarrow N' \overset{\tau_M}{\ll} N$.*

PROOF. Let $L \ll H$. Suppose $\gamma(L)+C=\tau_M(N)$ with $\tau_M(C)=C$. By Proposition 1.14, there is a unique $V \subset H$ with $C=\gamma(V)$. Then $\gamma(L+V)=\gamma(L)+\gamma(V)$ (by (1.6)) $=\gamma(L)+C=\tau_M(N)=\gamma(H)$. Since γ is $(1-1)$ we get $L+V=H$. Since $L \ll H$, this yields $V=H$, hence $C=\gamma(H)=\tau_M(N)$. Thus $L \ll H \Rightarrow \gamma(L) \overset{\tau_M}{\ll} N$.

Conversely, assume $\gamma(L) \overset{\tau_M}{\ll} N$. Suppose $L+V=H$. Then

$$\gamma(L)+\gamma(V) = \gamma(L+V) = \gamma(H) = \tau_M(N) \quad \text{by (1.6).}$$

Since $\gamma(L) \overset{\tau_M}{\ll} N$, it follows that $\gamma(V)=\tau_M(N)=\gamma(H)$. The $(1-1)$ nature of γ yields $V=H$. Hence $L \ll H$. This completes the proof of (i).

Let $N' \overset{\tau_M}{\ll} N$. Then $N' \subset K$ with $\tau_M(K)=K$ and $K \overset{\tau_M}{\ll} N$. Let $\beta(N')+V=H$. Since $N' \subset K$ we get $\beta(K)+V=H$. Hence $\gamma(\beta(K)+V)=\gamma(H)=\tau_M(N)$. But $\gamma(\beta(K)+V)=\gamma\beta(K)+\gamma(V)$ by (1.6). Thus we get $K+\gamma(V)=\tau_M(N)$. Since $K \overset{\tau_M}{\ll} N$, we get $\gamma(V)=\tau_M(N)=\gamma(H)$. Hence $V=H$. This proves that $\beta(N') \ll H$.

Now assume that N' is M-cotorsionless and that $\beta(N') \ll H$. Let $N'+C=\tau_M(N)$ with $\tau_M(C)=C$. Since N' and C are M-cotorsionless, we have $N'=\gamma\beta(N')$, $C=\gamma\beta(C)$. Hence $\gamma\beta(N')+\gamma\beta(C)=\tau_M(N)=\gamma(H)$. This yields $\gamma(\beta(N')+\beta(C))=\gamma(H)$, hence $\beta(N')+\beta(C)=H$. Since $\beta(N') \ll H$ we get $\beta(C)=H$. Hence $C=\gamma\beta(C)=\gamma(H)=\tau_M(N)$. This proves that $N' \overset{\tau_M}{\ll} N$. This completes the proof of (ii). □

LEMMA 2.8. *Let M be f.g. quasi-projective and N-projective. Further let $\tau_M(N)=N$. Let $N' \subset N$. Then $N' \ll N \Leftrightarrow \tau_M(N') \overset{\tau_M}{\ll} N$.*

PROOF. Assume that $N' \ll N$. Let $\tau_M(N')+C=\tau_M(N)=N$. Since $\tau_M(N') \subset N'$ and $N' \ll N$ we get $C=N=\tau_M(N)$. This proves that $\tau_M(N') \overset{\tau_M}{\ll} N$.

Conversely, assume that $\tau_M(N') \overset{\tau_M}{\ll} N$. Let $N'+A=N$. Since M is N-projective, from Lemma 1.10 we get $\tau_M(N')+\tau_M(A)=\tau_M(N'+A)=\tau_M(N)=N$. Since $\tau_M(A) \overset{\tau_M}{\ll} N$ we get $\tau_M(A)=\tau_M(N)=N$. Clearly $\tau_M(A) \subset A \subset N$. Hence $A=N$. This proves that $N' \ll N$. □

COROLLARY 2.9. *Let M be f.g. quasi-projective, N-projective and $\tau_M(N)=N$. Then on the class of M-cotorsionless submodules of N, the concepts of τ_M-smallness and usual smallness agree.*

COROLLARY 2.10. *Let M be f.g. quasi-projective, N-projective and $\tau_M(N)=N$. Then*

(i) $H' \ll H \Leftrightarrow \gamma(H') \ll N$

(ii) $N' \ll N \Leftrightarrow \beta(N') \ll H$.

PROOF. (i) is an immediate consequence of Proposition 2.7 (i) and Corollary 2.9.

From Lemma 2.8, $N' \ll N \Leftrightarrow \tau_M(N') \overset{\tau_M}{\ll} N$. Now, Proposition 2.7 (ii) yields $\tau_M(N') \overset{\tau_M}{\ll} N \Leftrightarrow \beta\tau_M(N') \ll H$. But $\beta\tau_M(N')=\beta(N')$ by (1.3). Hence $N' \ll N \Leftrightarrow \beta(N') \ll H$. This proves (ii). □

LEMMA 2.11. *Let M be N-projective and $\{N_i\}_{i=1}^k$ a finite number of τ_M-small submodules of N. Then $\sum_{i=1}^{k} N_i$ is τ_M-small in N.*

PROOF. By definition $N_i \subset C_i \subset N$ with $\tau_M(C_i)=C_i$ and C_i τ_M-small in N. By Lemma 1.10, $\sum_{i=1}^{k} \tau_M(C_i)=\tau_M\left(\sum_{i=1}^{k} C_i\right)$. Hence $\sum_{i=1}^{k} C_i=\sum_{i=1}^{k} \tau_M(C_i)=\tau_M\left(\sum_{i=1}^{k} C_i\right)$. Since $\sum_{i=1}^{k} N_i \subset \sum_{i=1}^{k} C_i$, it suffices to prove that $\sum_{i=1}^{k} C_i$ is τ_M-small. It suffices to prove this for $k=2$. Let $C_1+C_2+A=\tau_M(N)$ with $\tau_M(A)=A$. By Lemma 1.10, $C_2+A=\tau_M(C_2+A)$. Since $C_1 \overset{\tau_M}{\ll} N$ we get $C_2+A=\tau_M(N)$. Since $C_2 \overset{\tau_M}{\ll} N$ we get $A=\tau_M(N)$. This proves that $C_1+C_2 \overset{\tau_M}{\ll} N$. □

For any M, N in R-mod, $K_3=\cap\{T|T$ a τ_M-maximal submodule of $N\}$ is clearly a submodule of N. Let M be N-projective. Then we claim that $K_1=\{x|x \in N,\ \tau_M(Rx) \overset{\tau_M}{\ll} N\}$ is a submodule of N. In fact if x, y are in K_1 we have $\tau_M(R(x+y)) \subset \tau_M(Rx+Ry)=\tau_M(Rx)+\tau_M(Ry)$ by Lemma 1.10. Also $\tau_M(Rx)+\tau_M(Ry) \overset{\tau_M}{\ll} N$ by Lemma 2.11. Hence $x+y \in K_1$. Also for any $\lambda \in R$, $\tau_M(R\lambda x) \subset \tau_M(Rx) \overset{\tau_M}{\ll} N$. Hence $\tau_M(R\lambda x) \overset{\tau_M}{\ll} N$ yielding $\lambda x \in K_1$. Now, assume M to be f.g. and N-projective. Then $K_2=\sum\{K \subset N|K \overset{\tau_M}{\ll} N\}$ is an M-cotorsionless submodule of N. In fact any such K is contained in an M-cotorsionless τ_M-small submodule of N. Thus $K_2=\sum\{K \subset N|\tau_M(K)=K,\ K \overset{\tau_M}{\ll} N\}$. By Lemma 1.11, since M is f.g. and N-projective, it follows that $\tau_M(K_2)=K_2$.

THEOREM 2.12. *Let M be f.g. and N-projective. Then* $\tau_M(K_1)=K_2=\tau_M(K_3)$.

PROOF. We have $K_1=\sum_{x\in K_1} Rx$. From Lemma 1.11 we get $\tau_M(K_1)=\sum_{x\in K_1}\tau_M(Rx)$. By the very definition of K_1 we have $\tau_M(Rx)\overset{\tau_M}{\ll}N$. Hence

$$\tau_M(K_1)\subset K_2. \tag{2.13}$$

Next, we show that $K_2\subset K_3$. For this it suffices to show that every M-cotorsionless, τ_M-small submodule A of N is contained in every τ_M-maximal submodule T of N. If $A\not\subset T$, then $T+A$ is an M-cotorsionless submodule (by Lemma 1.10) of N with $T\underset{+}{\subset}T+A$. Hence $T+A=\tau_M(N)$. Since $A\overset{\tau_M}{\ll}N$ we get $T=\tau_M(N)$ contradicting the fact that T is a τ_M-maximal submodule of N. Thus $K_2\subset K_3$. Hence

$$K_2=\tau_M(K_2)\subset\tau_M(K_3). \tag{2.14}$$

Next we show that $K_3\subset K_1$. Let $x\in K_3$. We have to show that $\tau_M(Rx)\overset{\tau_M}{\ll}N$. If this is not the case, there exists a $K\subset\tau_M(N)$ with $\tau_M(K)=K\underset{+}{\subset}\tau_M(N)$ and $\tau_M(Rx)+K=\tau_M(N)$ It is clear that $x\notin K$, for if $x\in K$ then $\tau_M(Rx)\subset Rx\subset K$. Hence $\tau_M(Rx)+K=K=\tau_M(N)$, a contradiction. Let $\Gamma=\{C\subset N\mid\tau_M(C)=C,\ K\subset C$ and $x\notin C\}$. Clearly $K\in\Gamma$. Partially order Γ by inclusion. If $\{C_\alpha\}_{\alpha\in I}$ is any totally ordered subset of Γ then $\bigcup_{\alpha\in I}C_\alpha=\sum_{\alpha\in I}C_\alpha$ is a submodule of N with $x\notin\bigcup_{\in\alpha I}C_\alpha\supset K$.

Moreover, by Lemma 1.11, $\sum_{\alpha\in I}C_\alpha$ is M-cotorsionless. Hence any totally ordered subset of Γ has an upper bound in Γ. By Zorn's Lemma, Γ has a maximal element, say A. We claim that A is a τ_M-maximal submodule of N. Since $A\in\Gamma$, we have $\tau_M(A)=A$. If A is not τ_M-maximal in N, there exists a $B\subset N$ with $\tau_M(B)=B$ and $A\underset{+}{\subset}B\underset{+}{\subset}\tau_M(N)$. Since A is a maximal element in Γ we see that $B\notin\Gamma$. Hence $x\in B$. Then $B\supset Rx+K\subset\tau_M(Rx)+K=\tau_M(N)$, a contradiction. This contradiction shows that A is τ_M-maximal in N. Hence $K_3\subset A$. But then $x\notin A\Rightarrow$ $\Rightarrow x\notin K_3$ a contradiction. Hence $K_3\subset K_1$ yielding

$$\tau_M(K_3)\subset\tau_M(K_1). \tag{2.15}$$

From (2.13), (2.14) and (2.15) we get $\tau_M(K_1)=K_2=\tau_M(K_3)$. □

DEFINITION 2.16. For any $N\in R$-mod, the τ_M-socle and the τ_M-Jacobson radical of N, denoted respectively by $\tau_M-\mathrm{Soc}\,(N)$ and $\tau_M-J(N)$ are defined as follows:

$\tau_M-\mathrm{Soc}\,(N)$=the sum of τ_M-simple submodules of N,

$\tau_M-J(N)=\tau_M\big(\cap\{T\mid T a\tau_M$-maximal submodule of $N\}\big)$.

REMARK 2.17. When M is f.g. and N-projective, then the τ_M-Jacobson radical $\tau_M-J(N)$ can be characterised by

$$\tau_M-J(N)=\begin{cases}\tau_M\{x\mid x\in N,\ \tau_M(Rx)\overset{\tau_M}{\ll}N\}\\ \sum\{K\mid K\subset N,\ K\overset{\tau_M}{\ll}N\}=\sum\{K\mid K\subset N,\ \tau_M(K)=K,\ K\overset{\tau_M}{\ll}N\}\\ \tau_M\big(\cap\{T\mid T\text{ a }\tau_M\text{-maximal submodule of }N\}\big).\end{cases}$$

This follows from Theorem 2.12.

THEOREM 2.18. *Let M be f.g. quasi-projective and N-projective. Then*

$$\mathrm{Soc}({}_SH) = \mathrm{Hom}_R(M, \tau_M - \mathrm{Soc}(N)), \tag{2.19}$$

$$J({}_SH) = \mathrm{Hom}_R(M, \tau_M - J(N)). \tag{2.20}$$

PROOF. Let $\mathscr{S}$ be the class of simple submodules of ${}_SH$ and $\mathscr{K}$ the class of τ_M-simple submodules of N. Then $\mathrm{Soc}({}_SH) = \sum_{L\in\mathscr{S}} L$. Hence

$$\mathrm{Soc}({}_SH) = \beta\gamma(\mathrm{Soc}({}_SH)) = \beta\gamma\Big(\sum_{L\in\mathscr{S}} L\Big) = \beta\Big(\sum_{L\in\mathscr{S}} \gamma(L)\Big) \subset \tag{2.21}$$

$$\subset \beta\Big(\sum_{K\in\mathscr{K}} K\Big) \subset \beta(\tau_M - \mathrm{Soc}(N)) \quad \text{by (1.6)}$$

by Lemma 2.5 (1). On the other hand

$$\beta(\tau_M - \mathrm{Soc}(N)) = \beta\Big(\sum_{K\in\mathscr{K}} K\Big) = \sum_{K\in\mathscr{K}} \beta(K) \subset \bigcap_{L\in\mathscr{S}} L \subset \mathrm{Soc}({}_SH) \quad \text{by (1.15)} \tag{2.22}$$

by (1.18) and by Lemma 2.5 (2). (2.21) and (2.22) together yield

$$\mathrm{Soc}({}_SH) = \mathrm{Hom}_R(M, \tau_M - \mathrm{Soc}(N)).$$

Let $\mathscr{M}$ denote the class of maximal submodules of ${}_SH$ and $\mathscr{T}$ the class of τ_M-maximal submodules of N. Then

$$J({}_SH) = \beta\gamma(J({}_SH)) = \beta\gamma\Big(\bigcap_{L\in\mathscr{M}} L\Big) = \beta\tau_M\Big(\sum_{L\in\mathscr{M}} \gamma(L)\Big) \supset \beta(\tau_M - J(N)) \tag{2.23}$$

Lemma 2.5 (1). On the other hand,

$$\beta(\tau_M - J(N)) = \beta\Big(\tau_M\Big(\bigcap_{T\in\mathscr{T}} T\Big)\Big) = \beta\Big(\bigcap_{T\in\mathscr{T}} T\Big) = \bigcap_{T\in\mathscr{T}} \beta(T) \supset J({}_SH) \tag{2.24}$$

by (1.3), (1.5) and Lemma 2.5 (2). Combining (2.23) and (2.24) we get $J({}_SH) = \mathrm{Hom}_R(M, \tau_M - J(N))$. □

In all the remarks in (2.25) below the basic assumption is that M is f.g. quasi-projective and N-projective. We adopt the notations in Theorem 2.18.

REMARKS 2.25. (1) Let $D = \bigcap_{T\in\mathscr{T}} T$ where $\mathscr{T}$ is the class of τ_M-maximal submodules of N. Then $\tau_M - J(N) = \tau_M(D)$ by the very definition of $\tau_M - J(N)$.

Also,

$$J({}_SH) = \beta(\tau_M - J(N)) = \beta(\tau_M(D)) = \beta(D) = \mathrm{Hom}_R(M, D) \tag{2.26}$$

by (2.20), and (1.3).

(2) $$\mathrm{Soc}({}_SH) = \mathrm{Hom}_R(M, \tau_M - \mathrm{Soc}(N)) = \beta\Big(\sum_{K\in\mathscr{K}} K\Big) =$$

$$= \sum_{K\in\mathscr{K}} \beta(K) = \sum_{K\in\mathscr{K}} \mathrm{Hom}_R(M, K)$$

by Lemma (1.15). Thus one can write

$$\mathrm{Soc}({}_SH) = \sum \{Sf \mid f \in {}_SH,\ (M)f \text{ is } \tau_M\text{-simple in } N\}. \tag{2.27}$$

(3) From 2.17, we see that $\tau_M - J(N) = \sum \{K | K \overset{\tau_M}{\ll} N\}$. Using this expression for $\tau_M - J(N)$, from (2.20) we get

$$J({}_SH) = \sum \beta\{K | K \overset{\tau_M}{\ll} N\} = \sum \{\beta(K) | K \overset{\tau_M}{\ll} N\} = \sum_{K \overset{\tau_M}{\ll} N} \mathrm{Hom}_R(M, K)$$

by Lemma 1.15. Any element $f \in \sum_{K \overset{\tau_M}{\ll} M} \mathrm{Hom}_R(M, K)$ is of the form $\varphi_1 + \ldots + \varphi_k$ with $\varphi_i \in \mathrm{Hom}_R(M, K_i)$, $K_i \overset{\tau_M}{\ll} N$. Then $f \in \mathrm{Hom}_R(M, K_1 + \ldots + K_k)$. From Lemma 2.11, $\sum_{i=1}^{k} K_i \overset{\tau_M}{\ll} N$. Thus $(M)f \overset{\tau_M}{\ll} N$. Conversely, if $(M)f \overset{\tau_M}{\ll} N$, from $f \in \mathrm{Hom}_R(M, (M)f)$ we see that $f \in J({}_SH)$. Thus we obtain

$$J({}_SH) = \{f \in {}_SH | (M)f \overset{\tau_M}{\ll} N\}. \tag{2.28}$$

(4) Let us assume further that $\tau_M(N) = N$. From Corollary 2.9 we see that an M-cotorsionless submodule of N is τ_M-small $\leftrightarrow$ it is small. Moreover for any $f \in {}_SH$, clearly $(M)f$ is an M-cotorsionless submodule of N. Hence (2.28) in this case immediately yields

$$J({}_SH) = \{f \in {}_SH | (M)f \ll N\}. \tag{2.29}$$

When M is quasi-projective, since $\tau_M(M) = M$ we get the following as a Corollary of (2.29).

COROLLARY 2.30. *Let M be a f.g. quasi-projective module and $S = \mathrm{End}_R M$. Then $J({}_SS) = \{f \in S | (M)f \ll M\}$.*

This is the well-known characterization of the Jacobson radical of the endomorphism ring of a f.g. quasi-projective module.

COROLLARY 2.31. *Let M be f.g. quasi-projective and N-projective, assume further that τ_M is N-faithful. Then*

$$\mathrm{Soc}({}_SH) = \mathrm{Hom}_R(M, \mathrm{Soc}\, N) =$$

$$= \sum \{Sf | f \in {}_SH, M(f) \text{ a simple submodule of } N\}.$$

PROOF. When τ_M is N-faithful, from Remark 2.3 (ii) we see that $\tau_M - \mathrm{Soc}(N) = \mathrm{Soc}\, N$. Hence $\mathrm{Soc}({}_SH) = \mathrm{Hom}_R(M, \mathrm{Soc}\, N)$ from (2.19).

Also (2.27) and Remark 2.3 (ii) yield

$$\mathrm{Soc}({}_SH) = \sum \{Sf | f \in {}_SH, (M)f \text{ a simple submodule of } N\}. \quad \square$$

COROLLARY 2.32. *Let M be f.g. quasi-projective and τ_M be M-faithful. Then*

$$\mathrm{Soc}({}_SS) = \mathrm{Hom}_R(M, \mathrm{Soc}\, M) =$$

$$= \sum \{Sf | f \in S, (M)f \text{ is a simple submodule of } M\}.$$

In particular S is a semi-simple ring $\leftrightarrow M$ is a semi-simple module.

REMARK. Compare this with Theorem 4.8 in [20].

§ 3. Singular submodule of ${}_SH$

DEFINITION 3.1. A submodule N' of N will be said to be τ_M-essential (or τ_M-large) in N if for every non-zero M-cotorsionless submodule A of N we have $\tau_M(A \cap N') \neq 0$.

We write $N' \underset{e}{\subset} N$ to denote that N' is essential in N in the usual sense and $N' \underset{\tau_M - e}{\subset} M$ to denote that N' is τ_M-essential in N. Clearly $N' \subset N'' \subset N$ and $N' \underset{\tau_M - e}{\subset} N \Rightarrow N'' \underset{\tau_M - e}{\subset} N$.

PROPOSITION 3.2. *Let M be f.g. quasi-projective. Then*

(1) $H' \underset{e}{\subset} H \Leftrightarrow \gamma(H') \underset{\tau_M - e}{\subset} N$.

(2) $N' \underset{\tau_M - e}{\subset} N \Leftrightarrow \beta(N') \underset{e}{\subset} H$.

PROOF. Assume $H' \underset{e}{\subset} H$. Let $0 \neq A \subset N$ with $\tau_M(A) = A$. Then, from Proposition 1.14 we get $\beta(A) \neq 0$ in H. Hence $H' \cap \beta(A) \neq 0$. Again, by Proposition 1.14 we get $\gamma(H' \cap \beta(A)) \neq 0$. But

$$\gamma(H' \cap \beta(A)) = \tau_M(\gamma(H') \cap \gamma\beta(A)) = \tau_M(\gamma(H') \cap A)$$

by 1.18. Hence $\tau_M(\gamma(H') \cap A) \neq 0$. This proves that $\gamma(H') \underset{\tau_M - e}{\subset} N$.

Conversely, assume $\gamma(H') \underset{\tau_M - e}{\subset} N$. Let $0 \neq L \subset H$. Then $\gamma(L)$ is a non-zero M-cotorsionless submodule of N. Hence $\tau_M(\gamma(H') \cap \gamma(L)) \neq 0$. This yields $\gamma(H' \cap L) \neq 0$ by (1.18). Using Proposition 1.14 we get $H' \cap L \neq 0$. Hence $H' \underset{e}{\subset} H$. This completes the proof of (1):

Let $N' \underset{\tau_M - e}{\subset} N$. Let $0 \neq L \subset H$. Then $\gamma(L)$ is a non-zero M-cotorsionless submodule of N. Hence $\tau_M(N' \cap \gamma(L)) \neq 0$. This means $\gamma\beta(N' \cap \gamma(L)) \neq 0$. Hence $\gamma(\beta(N') \cap \beta\gamma(L)) \neq 0$ by (1.5). Since $\beta\gamma(L) = L$ we get $\beta(N') \cap L \neq 0$. Thus $\beta(N') \underset{e}{\subset} H$.

Conversely, assume $\beta(N') \underset{e}{\subset} H$. Let $0 \neq A \subset N$ satisfy $\tau_M(A) = A$. Then $A = \gamma(L)$ for some $0 \neq L \subset H$. Then $\beta(N') \cap L \neq 0$. Hence $\gamma(\beta(N') \cap L) \neq 0$. This yields $\gamma(\beta(N') \cap \beta\gamma(L)) \neq 0$. Hence $\gamma\beta(N' \cap \gamma(L)) \neq 0$ by (1.5). This means $\tau_M(N' \cap \gamma(L)) \neq 0$ or $\tau_M(N' \cap A) \neq 0$. Thus $N' \underset{\tau_M - e}{\subset} N$. This completes the proof of (2). □

COROLLARY 3.3. *Let M be f.g. quasi-projective and $S = \mathrm{End}_R M$.*

(i) $I \underset{e}{\subset} {}_SS \Leftrightarrow MI \underset{\tau_M - e}{\subset} M$.

(ii) $M' \underset{\tau_M - e}{\subset} M \Leftrightarrow \mathrm{Hom}_R(M, M') \underset{e}{\subset} {}_SS$.

PROOF. Take $N = M$ in Proposition 3.2. □

REMARK 3.4. When τ_M is N-faithful the concept of τ_M-essentiality in N agrees with the notion of usual essentiality in N.

PROOF. Let $N' \subset_e N$. Let $0 \neq A \subset N$ satisfy $\tau_M(A)=A$. From $N' \subset_e N$ we get $N' \cap A \neq 0$. Since τ_M is N-faithful, we get $\tau_M(N' \cap A) \neq 0$. Hence $N' \subset_{\tau_M - e} N$.

Conversely, assume $N' \subset_{\tau_M - e} N$. Let $0 \neq C \subset N$. Then $\tau_M(C) \neq 0$, because τ_M is N-faithful. From $N' \subset_{\tau_M - e} N$ we get $\tau_M(N' \cap \tau_M(C)) \neq 0$. Hence $N' \cap \tau_M(C) \neq 0$. But $N' \cap C \supset N' \cap \tau_M(C)$. Thus $N' \cap C \neq 0$.

PROPOSITION 3.5. *Let M be f.g. quasi-projective and τ_M be N-faithful. Then*

(1) $H' \subset_e H \Leftrightarrow \gamma(H') \subset_e N$.

(2) $N' \subset_e N \Leftrightarrow \beta(N') \subset_e H$.

PROOF. Immediate consequence of Proposition 3.2 and Remark 3.4. □

COROLLARY 3.6. *Let M be f.g. quasi-projective and τ_M be M-faithful. Then*

(i) $I \subset_e {}_SS \Leftrightarrow MI \subset_e M$.

(ii) $M' \subset_e M \Leftrightarrow \mathrm{Hom}_R(M, M') \subset_e {}_SS$.

PROOF. Take $N=M$ in Proposition 3.5. □

Let us now recall the well-known Johnson's lemma.

LEMMA 3.7 (Johnson). *Let $f: A \to B$ be a map in R-mod and $C \subset_e B$. Let $Cf^{-1}=\{x \mid x \in A,\ (x)f \in C\}$. Then $Cf^{-1} \subset_e A$.*

COROLLARY 3.8. *Let $C \subset_e B$ and $b \in B$. Let $(b:C)=\{\lambda \in R \mid \lambda b \in C\}$. Then $(b:C) \subset_e R_R$.*

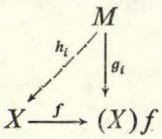
LEMMA 3.9 (τ_M-Johnson's Lemma). *Let $f: A \to B$ be an R-homomorphism and let M be A-projective. Let $K \subset_{\tau_M - e} B$. Then $Kf^{-1} \subset_{\tau_M - e} A$.*

PROOF. Let $0 \neq X$ be an M-cotorsionless submodule of A. We want to show that $\tau_M(Kf^{-1} \cap X) \neq 0$.

Case (i): $X \subset Kf^{-1}$. Then $\tau_M(Kf^{-1} \cap X)=\tau_M(X)=X \neq 0$.

Case (ii): $X \not\subset Kf^{-1}$. Then $(X)f \neq 0$. Also $(X)f$ is M-cotorsionless. In fact, if $y \in (X)f$, then there exists an $x \in X$ with $y=(x)f$. Since $\tau_M(X)=X$, there exist $u_1, \ldots, u_k$ in M and $\varphi_1, \ldots, \varphi_k$ in $\mathrm{Hom}_R(M, X)$ with $\sum_{i=1}^{k} (u_i)\varphi_i = x$. Then $\sum_{i=1}^{k} (u_i)\varphi_i f = (x)f$ and $\varphi_i \circ f \in \mathrm{Hom}_R(M, (X)f)$. Thus $\tau_M((X)f)=(X)f$. From $K \subset_{\tau_M - e} B$ we get $\tau_M(K \cap (X)f) \neq 0$. This means, there exists an element $a \in X$ with $0 \neq f(a) \in K$, elements $w_1, \ldots, w_r$ in M and homomorphisms $g_i: M \to K \cap (X)f$ $(1 \leq i \leq r)$ with $(a)f = \sum_{i=1}^{r} (w_i)g_i$. Since M is A-projective and $X \subset A$ it follows that M is X-projec-

tive. Hence there exist maps h_i: $M \to X$ such that



is commutative. From $(M)g_i \subset K$ we get $(M)h_i \subset Kf^{-1}$. Thus $h_i \in \mathrm{Hom}_R(M, Kf^{-1} \cap X)$. Now, $(\sum_{i=1}^{r}(w_i)h_i)f = \sum_{i=1}^{r}(w_i)g_i = (a)f \neq 0$. Hence $0 \neq \sum_{i=1}^{r}(w_i)h_i \in \tau_M(Kf^{-1} \cap X)$. □

COROLLARY 3.10. *Let M be R-projective in the sense of Azumaya. Let $N' \underset{\tau_M - e}{\subset} N$ and $x \in N$. Then $(x:N') \underset{\tau_M - e}{\subset} {}_RR$.*

PROOF. Let f: $R \to N$ be the map $(\lambda)f = \lambda x$. Then $N'f^{-1} = (x:N')$. Now, apply Lemma 3.9 to the map f.

THEOREM 3.11. *Let M be f.g. quasi-projective. Then*

$$Z({}_SH) = \{f \mid f \in {}_SH = \mathrm{Hom}_R(M, N), \ker f \underset{\tau_M - e}{\subset} M\}.$$

PROOF. Let $h \in Z({}_SH)$. Then there exists a left ideal I of ${}_SS$ with $Ih = 0$ and $I \underset{e}{\subset} {}_SS$. From Corollary 3.3 (i) we see that $MI \underset{\tau_M - e}{\subset} M$. Also $(MI)h = 0$. Thus $\ker h \supset MI$, hence $\ker h \underset{\tau_M - e}{\subset} M$.

Conversely, let $f \in {}_SH$ be such that $\ker f \underset{\tau_M - e}{\subset} M$. Let $M' = \ker f$. From Corollary 3.3 (ii) we see that $\beta(M') = \mathrm{Hom}_R(M, M') \underset{e}{\subset} {}_SS$. We claim that $\beta(M')f = 0$ in ${}_SH$.

If on the contrary $\beta(M')f \neq 0$ in ${}_SH$, then $(M)\beta(M')f \neq 0$ in N. But $(M)\beta(M') = \tau_M(M') \subset M'$ since $(M')f = 0$ we arrive at a contradiction. Thus $\beta(M')f = 0$ in ${}_SH$. Since $\beta(M') \underset{e}{\subset} {}_SS$ we get $f \in Z({}_SH)$. Hence

$$Z({}_SH) = \{f \mid f \in {}_SH = \mathrm{Hom}_R(M, N), \ker f \underset{\tau_M - e}{\subset} M\}. \quad \square$$

LEMMA 3.12. *Let N_1, N_2 be any two τ_M-essential submodules of N. Then $N_1 \cap N_2 \underset{\tau_M - e}{\subset} N$.*

PROOF. Let $0 \neq A \subset N$ satisfy $\tau_M(A) = A$. Since $N_1 \underset{\tau_M - e}{\subset} N$ we get $\tau_M(N_1 \cap A) \neq 0$. Clearly $\tau_M(N_1 \cap A)$ is M-cotorsionless. Hence $N_2 \underset{\tau_M - e}{\subset} N \Rightarrow \tau_M(N_2 \subset \tau_M(N_1 \cap A)) \neq 0$. Since $\tau_M(N_2 \cap N_1 \cap A) \supset \tau_M(N_2 \cap \tau_M(N_1 \cap A))$ we get $\tau_M(N_2 \cap N_1 \cap A) \neq 0$. Hence $N_1 \cap N_2 \underset{\tau_M - e}{\subset} N$. □

DEFINITION 3.13. Let M be R-projective. For any $N \in R$-mod, let $U_M(N) = \{x \in N \mid l_R(x) \underset{\tau_M - e}{\subset} R\}$.

LEMMA 3.14. *When M is R-projective, $U_M(N)$ is a submodule of N.*

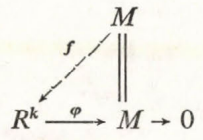
PROOF. Let x, y be any two elements of $U_M(N)$. Then $l(x)$ and $l(y)$ are τ_M-essential in R. From $l(x\pm y)\supset l(x)\cap l(y)$ and Lemma 3.12 we see that $l(x\pm y) \underset{\tau_M-e}{\subset} R$. Hence $x\pm y\in U_M(N)$. Since M is R-projective, for any $\lambda\in R$, $(\lambda\colon l(x))=l(\lambda x)$ is τ_M essential in R by Corollary 3.10. Hence $\lambda x\in U_M(N)$. □

DEFINITION 3.15. Let M be R-projective. The τ_M-singular submodule of N, denoted by $Z_{\tau_M}(N)$, is defined by $Z_{\tau_M}(N)=\tau_M(U_M(N))$ where

$$U_M(N)=\{x\in N\,|\,l(x)\underset{\tau_M-e}{\subset} R\}.$$

LEMMA 3.16. *Let ${}_RM$ be any f.g. quasi-projective. Then M is R-projective in the sense of Azumaya$\Leftrightarrow M$ is a projective R-module.*

PROOF. Any projective R-module is clearly R-projective. Let ${}_RM$ be f.g. quasi-projective and R-projective. Since M is f.g., there exists an epimorphism $\varphi\colon R^k\to M$. Since M is R-projective, from Proposition 3.3 (3) in [7] we see that M is R^k-projective. Hence, there exists a map $f\colon M\to R^k$ with



commutative. Thus M is isomorphic to a direct summand of R^k, hence projective.

LEMMA 3.17. *Let k be an integer $\geqq 1$, $\{N_i\}_{i=1}^k$ be R-modules. Suppose ${}_RM$ is N_i projective for $1\leqq i\leqq k$. Suppose $A_i \underset{\tau_M-e}{\subset} N_i$. Then*

$$\bigoplus_{i=1}^{k} A_i \underset{\tau_M-e}{\subset} \bigoplus_{i=1}^{k} N_i.$$

PROOF. It suffices to prove this result for $k=2$. Let $p_i\colon N_1\oplus N_2\to N_i$ $(i=1,2)$ be the canonical projections. From Proposition 3.3 (3) in [7] we see that M is $N_1\oplus N_2$ projective. From Lemma 3.9, we see that $A_ip_i^{-1} \underset{\tau_M-e}{\subset} N_1\oplus N_2$ $(i=1,2)$. Thus from Lemma 3.12 we get $A_1p_2^{-1}\cap A_2p_2^{-1} \underset{\tau_M-e}{\subset} N_1\oplus N_2$. But $A_1\oplus A_2=A_1p_1^{-1}\cap A_2p_2^{-1}$. Thus $A_1\oplus A_2 \underset{\tau_M-e}{\subset} N_1\oplus N_2$. □

THEOREM 3.18. *Let ${}_RP$ be a f.g. projective module, $S=\mathrm{End}\,{}_RP$ and $N\in R$-mod. Let ${}_SH=\mathrm{Hom}_R({}_RP, {}_RN)$. Then*

$$Z({}_SH)=\mathrm{Hom}_R(P, Z_{\tau_P}(N)).$$

PROOF. Let $U_P(N)=\{x\in N\,|\,l(x)\underset{\tau_M-e}{\subset} R\}$. Then $Z_{\tau_P}(N)=\tau_P(U_P(N))$. Hence $\beta(Z_{\tau_P}(N))=\beta\tau_P(U_P(N))=\beta(U_P(N))$ (by 1.3). We will first show that

$$\beta(U_P(N))\subset Z(H). \tag{3.19}$$

Let $h\in\beta(U_P(N))=\mathrm{Hom}_R(P, U_P(N))$. Let $a_1, \ldots, a_k$ generate P. Since $a_ih\in U_P(N)$

there exists an $I_i \underset{\tau_P - e}{\subset} R$ with $I_i a_i h = 0$ $(i=1, \ldots, k)$. Write $x_i = a_i h$. Let $\varphi: R_i \to \sum_{i-1}^{k} Rx_i$ be the obvious epimorphism, namely $(\lambda_1, \ldots, \lambda_k)\varphi = \sum_{i=1}^{k} \lambda_i x_i$. Clearly $(P)h \subset \sum_{i=1}^{k} Rx_i$. Since P is projective, we get a map $f: P \to R^k$ with

$$\begin{array}{ccc} & & P \\ & \overset{f}{\swarrow} & \downarrow h \\ R^k & \xrightarrow{\varphi} & \sum_{i=1}^{k} Rx_i \to 0 \end{array}$$

commutative. Lemma 3.17 implies that $I_1 \oplus \ldots \oplus I_k \underset{\tau_P - e}{\subset} R^k$. From Lemma 3.9 we see that $(I_1 \oplus \ldots \oplus I_k) f^{-1} \underset{\tau_P - e}{\subset} P$. Let $P' = (I_1 \oplus \ldots \oplus I_k) f^{-1}$. From $I_i x_i = 0$ we see that $(P')h = 0$. Thus $\ker h \underset{\tau_P - e}{\subset} P$. Theorem 3.11 now yields $h \in Z(H)$. This proves (3.19).

To complete the proof of Theorem 3.18 we have only to show that $Z(H) \subset \beta(Z_{\tau_P}(N))$. Using Proposition 1.14, this is equivalent to proving $\gamma(Z(H)) \subset Z_{\tau_P}(N)$. We will show that $\gamma(Z(H)) \subset U_P(N)$. Since $\gamma(Z(H))$ is P-cotorsionless, this will imply $\gamma(Z(H)) \subset \tau_P(U_P(N)) = Z_{\tau_P}(N)$. To prove that $\gamma(Z(H)) \subset U_P(N)$ we have only to show that for any $h \in Z(H)$ and $a \in P$, the element ah is in $U_P(N)$. Since $h \in Z(H)$, there exists an $I \underset{e}{\subset} {}_S S$ with $Ih = 0$ in H. This means $PIh = 0$. From Corollary 3.3 (i) we get $PI \underset{\tau_P - e}{\subset} P$. From Corollary 3.10 we see that $(a:PI) = \{r \in R \mid ra \in PI\}$ is τ_P-essential in R. Now, $(a:PI)ah \subset PIh = 0$. Thus $l_R(ah) \supset (a:PI)$. Since $(a:PI) \underset{\tau_P - e}{\subset} R$, we get $ah \in U_P(N)$. This proves (3.20).

(3.19) and (3.20) together yield $Z(H) = \mathrm{Hom}_R(P, Z_{\tau_P}(N))$. □

REMARK 3.21. With the same hypotheses as in Theorem 3.18, we have

$$\gamma(Z({}_S H)) = Z_{\tau_P}(N) \tag{3.22}$$

and

$$Z({}_S H) = \mathrm{Hom}_R(P, Z_{\tau_P}(N)) = \mathrm{Hom}_R(P, U_P(N)). \tag{3.23}$$

Defining N to be τ_P-singular if $Z_{\tau_P}(N) = \tau_P(N)$ and to τ_P-nonsingular if $Z_{\tau_P}(N) = 0$ we immediately see that ${}_S H$ is singular or non-singular according as N is τ_P-singular or τ_P-nonsingular.

From Remark 3.14 we see that whenever τ_M is R-faithful, the concept of τ_M-essentiality in R agrees with the usual concept of essentiality in R. Hence if M is R-projective and τ_M is R-faithful, then for any $N \in R$-mod, $Z_{\tau_M}(N) = \tau_M(Z(N))$ where $Z(N)$ is the usual singular submodule of N. If M is a generator for R-mod, then $\tau_M(A) = A$ for all $A \in R$-mod and then τ_M is A-faithful for all $A \in R$-mod. Using these observations from Theorem 3.18 we get the following:

COROLLARY 3.24. *Let ${}_R P$ be a progenerator, $S = \mathrm{End}\, {}_R P$ and $N \in R$-mod. Let ${}_S H = \mathrm{Hom}_R(P, N)$. Then $Z({}_S H) = \mathrm{Hom}_R(P, Z({}_R N))$.*

Thus in this case ${}_S H$ is singular or non-singular according as ${}_R N$ is singular or non-singular.

A part of the later half of Corollary 3.24 can be extended when ${}_RP$ is just a generator (not necessarily a progenerator). For this purpose we need the following result of A. V. Jategaonkar [8].

PROPOSITION 3.25. *Let* ${}_RM$ *be a generator for* R-mod *and* $S=\mathrm{End}\,{}_RM$. *Then a left ideal* K *of* S *is essential in* $S \Leftrightarrow MK$ *is an essential submodule of* ${}_RM$.

This is (3) of Theorem 1.3 in [8].

PROPOSITION 3.26. *Let* ${}_RM$ *be a generator for* R-mod. *Then* $\gamma(Z({}_SH)) \subset Z({}_RN)$.

PROOF. The proof is similar to the proof of the later half of Theorem 3.18. Still we will spell out the proof. We have only to show that for any $m \in M$ and $h \in Z({}_SH)$, the element mh is in $Z({}_RN)$. Since $h \in Z({}_SH)$, there exists a $K \underset{e}{\subset} {}_SS$ with $Kh=0$ in H. Since $K \underset{e}{\subset} {}_SS$, from Proposition 3.25 we get $MK \underset{e}{\subset} {}_RM$. By the usual Johnson's Lemma (Lemma 3.7) we see that $(m:MK) \underset{e}{\subset} {}_RR$. From $(m:MK)mh \subset MKh=0$ since $Kh=0$ in H we see that $l_R(mh) \supset (M:MK)$. Hence $l_R(mh) \underset{e}{\subset} {}_RR$. This yields $mh \in Z({}_RN)$. □

COROLLARY 3.27. *Let* M *be a generator for* R-mod. *Then*

(i) ${}_SH$ *singular* $\Rightarrow {}_RN$ *singular and*
(ii) ${}_RN$ *non-singular* $\Rightarrow {}_SH$ *non-singular.*

PROOF. (i) Suppose ${}_SH$ is singular. Then $H=Z(H)$. Hence $\gamma(Z(H))=\gamma(H)=$ $= \sum_{f \in \mathrm{Hom}(M,N)} (M)f = \tau_M(N)=N$ since M is a generator. From Proposition 3.26, $\gamma(Z(H)) \subset Z(N)$. Hence $N=\gamma(Z(H)) \subset Z(N) \subset N$. This yields $N=Z(N)$. Thus ${}_RN$ is singular.

(ii) If ${}_RN$ is non-singular, then $Z(N)=0$. Let $h \in Z(H)$. Then in the proof of Proposition 3.26 we have shown that $mh \in Z(N)$ for all $m \in M$. Thus $mh=0$ for all $m \in M$. This means $h=0$. Hence $Z(H)=0$.

COROLLARY 3.28. *Let* M *be a generator for* R-mod *and* $S=\mathrm{End}\,{}_RM$.
(i) *If* ${}_SS$ *is singular so is* ${}_RM$.
(ii) *If* ${}_RM$ *is non-singular so is* ${}_SS$.

(ii) of Corollary 3.28 is (4) of Theorem 1.3 in [8].

We will show that Corollary 3.27 (i) is also valid under a different set of conditions on ${}_RM$. For this purpose we need some results from [19].

DEFINITION 3.29. A module ${}_RM$ is said to be torsionless if it is contained in a direct product of copies of R or equivalently

$$\cap \{\ker f \mid f \in \mathrm{Hom}_R(M, R)\} = 0.$$

PROPOSITION 3.30. *Let* M *be a torsionless module over a semiprime ring* R. *Then*

(1) *For each essential submodule* K *of* M, $\mathrm{Hom}_R(M, K)$ *is an essential left ideal of* $S=\mathrm{End}\,{}_RM$.
(2) *For each essential left ideal* I *of* S, MI *is an essential submodule of* ${}_RM$.

These are respectively (iii) and (iv) of Proposition 1.2 in [19].

LEMMA 3.31. *Let ${}_RN$ be a non-singular module, $M \in R$-mod, $S = \mathrm{End}\,{}_RM$ and ${}_SH = \mathrm{Hom}_R(M, N)$. Then the only element of ${}_SH$ whose kernel is an essential submodule of M is the zero element.*

PROOF. Let $h \in {}_SH$ be such that $\ker h \underset{e}{\subset} M$. Let $K = \ker h$. Let $x \in M$ be arbitrary. By Corollary 3.8, $(x:K) = \{r \in R \mid rx \in K\}$ is an essential left ideal of R. Now, $(x:K)xh \subset Kh = 0$. Since $(x:K) \underset{e}{\subset} R$, we get $xh \in Z({}_RN) = 0$. Thus $xh = 0$. Since $x \in M$ is arbitrary, this implies $h = 0$ in H. □

COROLLARY 3.32. *Let f: $N \to N$ be an endomorphism of a non-singular module N. Suppose $\ker f \underset{e}{\subset} N$. Then $f = 0$.*

PROOF. Take $M = N$ in Lemma 3.31.

Actually Corollary 3.32 is Lemma 1.3 in [19].

COROLLARY 3.33. *If in addition to the hypothesis of Lemma 3.31, we assume that M is uniform then for any $h \in {}_SH = \mathrm{Hom}_R(M, N)$ either $h = 0$ or h is monomorphism.*

PROOF. If h is not a monomorphism, then $\ker h$ is an essential submodule of M, since M is assumed uniform. By Lemma 3.31 we get $h = 0$.

PROPOSITION 3.34. *Let M be a torsionless module over a semiprime ring R and N a non-singular R-module. Then ${}_SH = \mathrm{Hom}_R(M, N)$ is non-singular.*

PROOF. Let $h \in Z({}_SH)$. Then $Ih = 0$ in H for some $I \underset{e}{\subset} {}_SS$. From (2) of Proposition 3.30, we get $MI \underset{e}{\subset} M$. Also $Ih = 0$ in $H \Rightarrow MIh = 0 \Rightarrow MI \subset \ker h$. Since $MI \underset{e}{\subset} M$, from Lemma 3.31 we get $h = 0$. Thus $Z({}_SH) = 0$. □

COROLLARY 3.35. *Let M be a torsionless module over a semi-prime ring R. Then $Z({}_RM) = 0 \Rightarrow Z({}_SS) = 0$ where $S = \mathrm{End}\,{}_RM$.*

PROOF. Take $N = M$ in Proposition 3.34.

Actually Corollary 3.35 is part of Theorem 2.2 in [19]. □

THEOREM 3.36. *Let M be a torsionless module over a semiprime ring R and N any R-module. Then*

$$Z({}_SH) = \{f \in {}_SH = \mathrm{Hom}_R(M, N) \mid \ker f \underset{e}{\subset} M\}.$$

PROOF. The proof of this theorem is similar to that of Theorem 3.11. Still we spell it out. Let $f \in Z({}_SH)$. Then $If = 0$ in H for some $I \underset{e}{\subset} {}_SS$. From (2) Proposition 3.30 we conclude that $MI \underset{e}{\subset} M$. Also $If = 0$ in $H \Rightarrow MI \subset \ker f$. Thus $\ker f \underset{e}{\subset} M$.

Conversely, let $f\in H$ satisfy $\ker f\underset{e}{\subset} M$. Let $K=\ker f$. Then by (1) of Proposition 3.30 we see that $\mathrm{Hom}_R(M, K)\underset{e}{\subset}{}_SS$. Let $I=\mathrm{Hom}_R(M, K)$. We claim that $If=0$ in H. If $If\neq 0$, then $MIf\neq 0$. But $MI\subset K$. Hence $MIf\subset Kf=0$, a contradiction. Since $I\underset{e}{\subset}{}_SS$, it follows that $f\in Z({}_SH)$.

Thus $Z({}_SH)=\{f\in{}_SH|\ker f\underset{e}{\subset} M\}$. □

REMARK 3.37. Observe that if M is f.g., quasi-projective and τ_M is M-faithful, from Theorem 3.11 and Proposition 3.5 we get

$$Z({}_SH)=\{f\in{}_SH\}\ \ker f\underset{e}{\subset} M.$$

§ 4. Conditions for ${}_SH$ to be local, hollow or semi-hollow

DEFINITION 4.1 [12]. Let A be an R-module.

(i) A is said to be local if it has a unique maximal submodule J which contains every proper submodule of A.

(ii) A is called a hollow module if every proper submodule of A is small in A.

(iii) A is said to be semi-hollow if every proper f.g. submodule of A is small in A.

In case A is local, the unique maximal submodule J of A is exactly the Jacobson radical of A. For any $x\in A$ with $x\notin J$, we should have $Rx=A$, because every proper submodule of A is contained in J. Thus A is cyclic. Hence $J\ll A$. It follows that every proper submodule of A is small in A. Thus A is hollow. Clearly every hollow module is semi-hollow. Any noetherian semi-hollow module is hollow.

DEFINITION 4.2. (i) $N\in R$-mod will be called τ_M-local if there exists a unique τ_M-maximal submodule K of N which contains every proper, M-cotorsionless submodule of $\tau_M(N)$. In this case $\tau_M-J(N)=K$.

(ii) N will be referred to as τ_M-hollow if every proper, M-cotorsionless submodule of $\tau_M(N)$ is τ_M-small in N.

(iii) N will be referred to as τ_M-semi-hollow if every f.g., proper, M-cotorsionless submodule of $\tau_M(N)$ is τ_M-small in N.

Let N be τ_M-local. Suppose $A\subset N$ satisfies $\tau_M(A)=A$. Then if $A\neq\tau_M(N)$, by the very definition of a τ_M-local module, we have $A\subset K$, hence $K+A=K\neq$ $\neq\tau_M(N)$. It follows that $K\overset{\tau_M}{\ll}N$. Since every, proper M-cotorsionless submodule of $\tau_M(N)$ is in K we see that N is τ_M-hollow. Any τ_M-hollow module is clearly τ_M-semi-hollow.

REMARK 4.3. For any $N\in R$-mod, clearly $\tau_M(N)$ is the largest M-cotorsionless submodule of N. We will refer to any M-cotorsionless, proper submodule of $\tau_M(N)$ shortly as a τ_M-proper submodule of N.

PROPOSITION 4.4. *Let ${}_RM$ be f.g. quasi-projective and $N\in R$-mod. Then*

(1) *${}_SH$ is local $\Leftrightarrow{}_RN$ is τ_M-local.*

(2) *${}_SH$ is hollow $\Leftrightarrow{}_RN$ is τ_M-hollow.*

(3) *${}_SH$ is semi-hollow $\Leftrightarrow{}_RN$ is τ_M-semi-hollow.*

PROOF. (1) is an immediate consequence of the two facts that $L \subset_S H$ is maximal $\Leftrightarrow \gamma(L)$ is τ_M-maximal in N and that an M-cotorsionless submodule A of N is τ_M-maximal in $N \Leftrightarrow \beta(A)$ is maximal in ${}_SH$ (Lemma 2.5).

(2) Is an immediate consequence of Proposition 2.7.

(3) From Remark 4.19 in [7] we infer that $L \subset_S H$ is f.g. $\Leftrightarrow \gamma(H)$ is a f.g. submodule of ${}_RN$ and that if A is an M-cotorsionless submodule of ${}_RN$, then A is f.g. $\Leftrightarrow \beta(A)$ is a f.g. submodule of ${}_SH$.

Now (3) is an immediate consequence of the above observations and Proposition 2.7. □

PROPOSITION 4.5. *An R-module A is semi-hollow $\Leftrightarrow$ either A is local or A has no maximal submodules.*

This is Proposition 2.2 of [12].

Since every f.g. module always has a maximal submodule, as an immediate consequence of Proposition 4.5, one sees that if A is f.g. then

$$A \text{ is a semi-hollow } \Leftrightarrow A \text{ is hollow } \Leftrightarrow A \text{ is local.}$$

PROPOSITION 4.6. *Let M be f.g. quasi-projective and $\tau_M(N)$ be f.g. Then the following are equivalent.*

(1) *N is τ_M-local.*
(2) *N is τ_M-hollow.*
(3) *N is τ_M-semi-hollow.*
(4) ${}_SH = \mathrm{Hom}_R(M, N)$ *is semi-hollow.*
(5) ${}_SH$ *is hollow.*
(6) ${}_SH$ *is local.*

PROOF. (1)$\Rightarrow$(2)$\Rightarrow$(3) are clear. (3)$\Leftrightarrow$(4) by Proposition 4.4 (3). (4)$\Leftrightarrow$(5)$\Leftrightarrow$(6) follow, since finite generation of $\tau_M(N)$ implies the finite generation of ${}_SH$ (Remark 4.19 in [7]). (6)$\Leftrightarrow$(1) by Proposition 4.4 (1).

If M is f.g. quasi-projective, N-projective and $\tau_M(N) = N$ it was shown in [7] that $\mathrm{corank}\, {}_SH = \mathrm{corank}\, {}_RN$. Hollow modules are precisely the modules with corank 1. In the case of hollowness the proof of this result is quite easy as we will presently show.

PROPOSITION 4.7. *Let M be f.g. quasi-projective, N-projective and $\tau_M(N) = N$. Then ${}_SH$ is hollow $\Leftrightarrow {}_RN$ is hollow.*

PROOF. Let ${}_SH$ be hollow. Let $N' \underset{+}{\subset} N$. Then we claim $\beta(N') \neq H$. If on the contrary $\beta(N') = H$, we would have $\tau_M(N') = \gamma\beta(N') = \gamma(H) = \tau_M(N) = N$. But $\tau_M(N') \subset N' \underset{+}{\subset} N$, a contradiction. Since $\beta(N') \neq H$, from the hollowness of H we get $\beta(N') \ll H$. Corollary 2.10 (ii) yields $N' \ll N$. Hence ${}_RN$ is hollow.

Conversely, assume ${}_RN$ hollow. Let $L \underset{+}{\subset}_S H$. Then $\gamma(L) \underset{+}{\subset} \gamma(H) = \tau_M(N) = N$. Thus $\gamma(L) \ll N$. From Lemma 2.10 (i) we get $L \ll H$. Hence ${}_SH$ is hollow. □

Observe that in this proof we had to check that $\beta(N') \neq H$, because N' need not be M-cotorsionless.

PROPOSITION 4.8. *Let* ${}_RM$ *be f.g. quasi-projective and* $S=\mathrm{End}\,{}_RM$. *Then the following are equivalent.*

(1) *M is local.*
(2) *M is hollow.*
(3) *M is sem-hollow.*
(4) ${}_SS$ *is semi-hollow.*
(5) ${}_SS$ *is hollow.*
(6) *S is a local ring.*

PROOF. (1)⇔(2)⇔(3) because M is f.g.
(4)⇔(5)⇔(6) because ${}_SS$ is cyclic.
(2)⇔(5) by Proposition 4.7.

REMARK 4.9. Equivalence of (2) and (5) is proved for an arbitrary quasi-projective module ${}_RM$ in [12].

The following result is proved in [5] and [12].

THEOREM 4.10. *Let I be a two-sided ideal in R. If* ${}_R(R/I)$ *is hollow, then* $\mathrm{End}_R(R/I)$ *is a local ring.*

This is Theorem 4.1 in [5] or Corollary 2.8 in [12].

When I is a two-sided ideal, R/I is a quasi-projective R-module [Corollary to Lemma 4.1 in [13]]. Since R/I is cyclic as an R-module, Proposition 4.8 immediately yields Theorem 4.10.

Also the following is proved in [5].

THEOREM 4.11. *Let I be a left ideal of R and R/I be hollow in* R-mod. *If R/I is projective or injective in* R-mod, *then* $\mathrm{End}_R(R/I)$ *is local.*

This is Theorem 4.4 in [5].

REMARK 4.12. In Theorem 4.11 we can replace projectivity or injectivity by quasi-projectivity or quasi-injectivity. When ${}_R(R/I)$ is quasi-projective the result is a consequence of Proposition 4.8. When ${}_R(R/I)$ is hollow, it is indecomposable. It is well-known that indecomposable quasi-injective modules have local endomorphism rings. This is already observed by K. M. Rangaswamy [12].

§ 5. Krull dimension and tallness of ${}_SH$

We first recall the definition of deviation [10] of a partially ordered set (poset) $(E, \leqq)$. Let $\Delta(E)=\{(e, e')|(e, e')\in E\times E,\ e\leqq e'\}$. An infinite sequence $\{e_n\}_{n\geqq 1}$ of elements will be said to be stationary if there exists some integer $k\geqq 1$ with $e_n=e_k$ for all $n\geqq k$. E is said to be noetherian (resp. artinian) if every increasing (resp. decreasing) infinite sequence $\{e_n\}_{n\geqq 1}$ is stationary. For any $(e, e')\in\Delta(E)$, let $[e, e']=\{x\in E,\ e\leqq x\leqq e'\}$. By transitive induction, for every ordinal α, define $\Delta_\alpha(E)$ as follows:

$$\Delta_0(E)=\{(e, e')|(e, e')\in\Delta(E),\ [e, e'] \text{ artinian}\}.$$

Assume that $\Delta_\beta(E)$ is defined for all $\beta<\alpha$. Then define $\Delta_\alpha(E)=\{(e, e')\}|e, e')\in\Delta(E)$;

for every infinite decreasing sequence $e_1 \geqq e_2 \geqq e_3 \geqq e_4 \geqq \ldots \ldots$ contained in $[e, e']$ we have $(e_{n+1}, e_n) \in \bigcup_{\beta<\alpha} \Delta_\beta(E)$ for all but a finite number of $n\}$. Clearly if $\Delta_\alpha(E)=\Delta(E)$, then $\Delta_\gamma(E)=\Delta_\alpha(E)$ for all $\gamma \geqq \alpha$.

DEFINITION 5.1 [10]. We say that E has a deviation if there exists an ordinal α with $\Delta_\alpha(E)=\Delta(E)$. Then the deviation of E is defined to be the least ordinal α with $\Delta_\alpha(E)=\Delta(E)$.

Let A be an R-module. The Krull dimension of A denoted by $K-\dim A$ is defined as follows. If $A=0$, then $K-\dim A=-1$. If $A \neq 0$, let $T(A)$ be the partially ordered set of submodules of A. Then $K-\dim A=\operatorname{dev} T(A)$, where $\operatorname{dev} T(A)$ is the deviation of $T(A)$.

We now introduce the concept of τ_M-Krull dimension of N, denoted by $\tau_M K-\dim N$.

DEFINITION 5.2. If $\tau_M(N)=0$ we set $\tau_M-K-\dim N=-1$. If $\tau_M(N) \neq 0$, let $T_{\tau_M}(N)$ be the poset of M-cotorsionless submodules of N. We set $\tau_M-K-\dim N=\operatorname{dev} T_{\tau_M}(N)$.

From Proposition 2 of [10] we get $\tau_M-K-\dim N \leqq K-\dim N$. From Proposition 1.14, we immediately get the following.

THEOREM 5.3. *Let ${}_RM$ be f.g. quasi-projective and $N \in R$-mod arbitrary. Then*

$$K-\dim {}_SH = \tau_M-K-\dim {}_RN \leqq K-\dim {}_RN.$$

COROLLARY 5.4. *Let ${}_RM$ be f.g. quasi-projective and $S=\operatorname{End} {}_RM$. Then*

$$K-\dim {}_SS \leqq \tau_M-K-\dim {}_RM \leqq K-\dim {}_RM.$$

DEFINITION 5.5. Define N to be hereditarily M-cotorsionless if $N'=\tau_M(N')$ for all $N' \subset N$.

When N is hereditarily M-cotorsionless, then $T_{\tau_M}(N)=T(N)$. When M is a generator for R-mod, then every $N \in R$-mod is hereditarily M-cotorsionless.

COROLLARY 5.6. *Let M be f.g. quasi-projective and N hereditarily M-cotorsionless. Then*

$$K-\dim {}_SH = K-\dim {}_RN.$$

M will be called self-generator if M is hereditarily M-cotorsionless.

COROLLARY 5.7. *Let ${}_RM$ be a f.g. quasi-projective self-generator and $S=\operatorname{End} {}_RM$. Then $K-\dim {}_SS=K-\dim {}_RM$.*

We now record some results from [6] which we need.

LEMMA 5.8. (1) *If $B \subset A$ then $K-\dim A=\operatorname{Sup}\{K-\dim B, K-\dim A/B\}$ when either exists.*

(2) *If R is a ring, then $K-\dim R=\operatorname{Sup}\{K-\dim M|_R M$ f.g.$\}$ if either side exists.*

This is Lemma 1.1 in [6].

LEMMA 5.9. *Let R be a ring with Krull dimension. If ${}_RM$ is any module with Krull dimension, then* $K-\dim {}_RM \leqq K-\dim {}_RR$.

This is Corollary 4.4 in [6].

PROPOSITION 5.10. *Let R be a ring such that ${}_RR$ has Krull dimension. Let ${}_RM$ be f.g. quasi-projective and ${}_RN$ have Krull dimension. Then ${}_SH$ and ${}_SS$ have Krull dimension; moreover*

$$K-\dim {}_SH \leqq K-\dim {}_SS \leqq K-\dim {}_RM \leqq K-\dim {}_RR$$

and

$$K-\dim {}_SH \leqq K-\dim {}_RN \leqq K-\dim {}_RR.$$

PROOF. When ${}_RN$ has Krull dimension, by Theorem 5.3, ${}_SH$ has Krull dimension and $K-\dim {}_SH \leqq K-\dim {}_RN$. By Lemma 5.8, ${}_RM$ has Krull dimension. By Corollary 5.4, ${}_SS$ has Krull dimension and $K-\dim {}_SS \leqq K-\dim {}_RM$. From Lemma 5.9, we now get $K-\dim {}_SH \leqq K-\dim {}_SS$, $K-\dim {}_RM \leqq K-\dim {}_RR$ and $K-\dim {}_RN \leqq K-\dim {}_RR$. Combining all these we get

$$K-\dim {}_SH \leqq K-\dim {}_SS \leqq K-\dim {}_RM \leqq K-\dim {}_RR$$

and

$$K-\dim {}_SH \leqq K-\dim {}_RN \leqq K-\dim {}_RR.$$

REMARK 5.11. When ${}_RR$ has Krull dimension, by Lemma 5.8 any f.g. ${}_RN$ has Krull dimension. Thus Proposition 5.10 is valid whenever ${}_RN$ is f.g.

When R is a ring with Krull dimension and ${}_RM$ is f.g. projective, it is proved in [6] that

$$K-\dim {}_SS \leqq K-\dim {}_RR.$$

COROLLARY 5.12. *In Proposition* 5.10 *replace the assumption that ${}_RN$ has Krull dimension by the assumption that ${}_R\tau_M(N)$ has Krull dimension and keep the other assumptions unaltered. Then*

$$K-\dim {}_SH \leqq K-\dim {}_SS \leqq K-\dim {}_RM \leqq K-\dim {}_RR$$

and

$$K-\dim {}_SH \leqq K-\dim {}_R\tau_M(N) \leqq K-\dim {}_RR.$$

PROOF. This follows immediately from Proposition 5.10 and the observation ${}_SH = \mathrm{Hom}_R(M, \tau_M(N))$.

We recall the definition of a tall module as introduced in [15].

DEFINITION 5.13. An R-module N is called a tall module if there exists a submodule $A \subset N$ with A and N/A both non-noetherian.

DEFINITION 5.14. We define a module N to be τ_M-tall if there exists a submodule $A \subset N$ with A and N/A both τ_M-non-noetherian.

PROPOSITION 5.15. *Let M be f.g., quasi-projective and N-projective. Then ${}_SH$ is tall $\Leftrightarrow$ ${}_RN$ is τ_M-tall.*

PROOF. Let ${}_SH$ be tall. Then there exists a submodule $L \subset {}_SH$ with ${}_SL$ and ${}_S(H/L)$ both non-noetherian. From Proposition 1.14, it follows that ${}_R\gamma(L)$ is τ_M-non-noetherian. From Lemma 1.23, we see that ${}_R(N/\gamma(L))$ is τ_M-non-noetherian.

Hence ${}_RN$ is τ_M-tall. The converse is proved similarly using Lemma 1.22 and Proposition 1.14. □

COROLLARY 5.16. *Let M be f.g., quasi-projective. Then $S=\mathrm{End}\,{}_RM$ is tall as a left S-module $\Leftrightarrow {}_RM$ is τ_M-tall.*

REMARKS 5.17. (1) If N is hereditarily M-cotorsionless, then it is clear that N is τ_M-noetherian (resp. τ_M-artinian) $\Leftrightarrow N$ is noetherian (resp. artinian).

(2) If A is M-cotorsionless and $A\xrightarrow{f}A''\to 0$ is exact, then A'' is M-cotorsionless. The proof of this is straightforward. This fact is actually indicated in the proof of Lemma 3.9.

(3) If A is hereditarily M-cotorsionless and $A\xrightarrow{f}A''\to 0$ is exact, then A'' is hereditarily M-cotorsionless. In fact if $C''\subset A''$ and $C=C''f^{-1}$, then $C\xrightarrow{f}C''\to 0$ is exact. Since $\tau_M(C)=C$ we get $\tau_M(C'')=C''$ from (2).

COROLLARY 5.18. *Let M be f.g. quasi-projective and N-projective. Suppose N is hereditarily M-cotorsionless. Then ${}_SH$ is tall $\Leftrightarrow {}_RN$ is tall.*

COROLLARY 5.19. *Let M be f.g. quasi-projective and ${}_RM$ a generator for R-mod. Then for any $N\in R$-mod, ${}_SH=\mathrm{Hom}_R(M,N)$ is tall $\Leftrightarrow {}_RN$ is tall. In particular the left S-module ${}_SS$ is tall $\Leftrightarrow {}_RM$ is tall.*

At this juncture a *word of caution* may not be out of place. In [15] B. Sarath defines a ring R to be left tall if every non-noetherian left R-module is tall. This is *not* the same thing as saying that the left R-module ${}_RR$ is tall.

In [15] B. Sarath associates to any module M, two submodules $G(M)$ and $H(M)$ as follows.

Let

$$\mathscr{A}_M=\{A\subset M\mid M/A \text{ is noetherian}\}$$

and

$$\mathscr{B}_M=\{B\subset M\mid B \text{ is a non noetherian submodule of } M\}.$$

If M is noetherian, set $G(M)=H(M)=0$. If M is *not* noetherian define $G(M)=\bigcap_{A\in\mathscr{A}_M} A$ and $H(M)=\bigcap_{B\in\mathscr{B}_M} B$. When M is not noetherian, clearly $\mathscr{A}_M\subset\mathscr{B}_M$. Hence $G(M)\supset H(M)$. When M is noetherian $G(M)=0=H(M)$. Hence $G(M)\supset H(M)$ always. The following is proved in [15].

LEMMA 5.20. *Suppose $M/G(M)$ is noetherian and $H(G(M))\neq G(M)$ then M is tall.*

This is Lemma 2.5 in [15]. Later, in this section we will show that the assumption of noetherianness of $M/G(M)$ can be deleted for the validity of this lemma. But before this, we prove two related propositions.

PROPOSITION 5.21. *If $H(M)\neq G(M)$ then M is tall.*

PROOF. If $H(M)\neq G(M)$ then $\mathscr{A}_M\subset\mathscr{B}_M$. Hence there exists a $B\in\mathscr{B}_M$ with $B\notin\mathscr{A}_M$. Then B is a non-noetherian submodule of M, with M/B non-noetherian. Hence M is tall. □

When M is tall, it is true that $\mathscr{A}_M \subset_+ \mathscr{B}_M$. But from this we can not conclude that $G(M) \supset_+ H(M)$. The following is a partial converse to Proposition 5.21.

PROPOSITION 5.22. *If M is tall and $M/G(M)$ is noetherian, then $H(M) \subset G(M)$.*

PROOF. Since M is tall, we get $\mathscr{A}_M \subset_+ \mathscr{B}_M$. Let $B \in \mathscr{B}_M$ be such that $B \notin \mathscr{A}_M$. We wish to show that $H(M) \neq G(M)$. If on the contrary, $G(M) = H(M)$, we would have $G(M) = \bigcap_{C \in \mathscr{B}_M} C$. Since $B \in \mathscr{B}_M$, we get $G(M) \subset B$. This implies that M/B is a quotient of $M/G(M)$. Since $M/G(M)$ is noetherian by assumption, it follows that M/B is noetherian — contradicting the fact that $B \in \mathscr{B}_M$. □

PROPOSITION 5.23. *Let $M \in R$-mod be such that $M/G(M)$ is noetherian. Then M is tall if and only if $H(M) \neq G(M)$.*

Immediate consequence of Propositions 5.21 and 5.22.

PROPOSITION 5.24. *If $H(G(M)) \neq G(M)$ then M is tall.*

PROOF. $H(G(M)) \neq G(M) \Rightarrow G(M) \neq 0 \Rightarrow M$ is non-noetherian. Again $H(G(M)) \neq G(M) \Rightarrow \exists$ a non-noetherian module $B \subset_+ G(M)$. We claim that M/B is non-noetherian. Suppose on the contrary M/B is noetherian. Then $B \in \mathscr{A}_M$. Hence $G(M) \subset B$, a contradiction. Thus M is tall. □

As commented earlier this is an improvement upon Lemma 2.5 of [15]. As remarked earlier, a ring R is called a left tall ring if every non-noetherian left R-module is tall. Our next result shows that the concept of tallness for a ring is a Morita invariant property.

THEOREM 5.25. *Let ${}_RP$ be a progenerator and $S = \mathrm{End}\,{}_RP$. Then S is a left tall ring if and only if R is a left tall ring.*

PROOF. It is know, from Morita's theorems, that in the above situation, any left S-module ${}_SH$ is isomorphic to $\mathrm{Hom}_R(P, N)$ for some $N \in R$-mod. From our results we see that ${}_SH$ is noetherian $\Leftrightarrow {}_RN$ is noetherian (because τ_P-noetherianness is the same as noetherianness when P is a generator). Hence ${}_SH$ is non-noetherian $\Leftrightarrow {}_RN$ is non-noetherian. From Corollary 5.19 we see that ${}_SH$ is tall $\Leftrightarrow {}_RN$ is tall. Hence S is a tall ring $\Leftrightarrow R$ is as tall ring. □

§ 6. Semi-perfectness of ${}_SH$

In [11] Erika Mares has defined the concept of semiperfectness for projective modules and proved structure theorems for these modules generalising the structure theorems of H. Bass [3] for semiperfect rings. In this section we will extend the notion of semiperfectness to modules which need not be projective and then obtain conditions for ${}_SH$ to be semi-perfect.

For this purpose we need to deal with the concept of supplement of a submodule N of M in M.

DEFINITION 6.1 [4]. Let $N \subset M$. A submodule I of M is called a supplement of N in M if $N+K=M$ and $N+K' \neq M$ for any $K' \underset{+}{\subset} K$.

The following are well-known.

LEMMA 6.2. *Let $M=N+K$. Then K is a supplement of $N \Leftrightarrow K \cap N \ll K$.*

LEMMA 6.3. *Let $K \subset M$ be a supplement of some submodule in M. Suppose K itself has a supplement L in M. Then K is also a supplement of L in M.*

Proofs of these two lemmas could be found in [16].

DEFINITION 6.4 [11]. A projective module P is said to be semiperfect if every factor module of P has a projective cover. The following is a result due to F. Kasch and E. Mares [9].

PROPOSITION 6.5. *A projective module P is semi-perfect if and only if every submodule of P has a supplement in P.*

Motivated by this we introduce the following.

DEFINITION 6.6. A module M will be called semi-perfect if every submodule of M has a supplement in M.

In an earlier paper [16] one of the authors referred to this property as property (P_1).

DEFINITION 6.7. Let A be an M-cotorsionless submodule of N. An M-cotorsionless submodule C of N will be called a τ_M-supplement of A in N if $A+C=\tau_M(N)$ and $A+C' \neq \tau_M(N)$ for any M-cotorsionless $C' \subset C$.

PROPOSITION 6.8. *Let M be N-projective and $\tau_M(N)=A+K$ where both A and K are M-cotorsionless. Then K is a τ_M-supplement of A in $N \Leftrightarrow \tau_M(A \cap K) \overset{\tau_M}{\ll} K$.*

PROOF. Let K be a τ_M-supplement of A in N. We want to show that $\tau_M(A \cap K)$ is τ_M-small in K. Let $\tau_M(A \cap K)+B=K$ with $\tau_M(B)=B$. Then

$$A+B \supset \tau_M(A \cap K)+B \supset K \Rightarrow A+B \supset A+K = \tau_M(N).$$

Hence $A+B=\tau_M(N)$. Since K is a τ_M-supplement of A in N and B is an M-cotorsionless submodule of K, from the definition of a τ_M-supplement, we get $B=K$. This proves $\tau_M(A \cap K) \overset{\tau_M}{\ll} K$. Conversely, assume that $\tau_M(A \cap K) \overset{\tau_M}{\ll} K$. Let $B \subset K$ with $\tau_M(B)=B$ satisfy $A+B=\tau_M(N)$. Then $K=K \cap \tau_M(N)=K \cap (A+B)=K \cap A+B$ since $K \supset B$. Since M is N-projective, by Lemma 1.10, we get $K=\tau_M(K)=$ $=\tau_M(K \cap A)+\tau_M(B)=\tau_M(K \cap A)+B$. Since $\tau_M(K \cap A) \overset{\tau_M}{\ll} K$, we get $B=K$. This proves that K is a τ_M-supplement of A.

LEMMA 6.9. *Let $A \subset B \subset N$, $\tau_M(A)=A$ and $A \overset{\tau_M}{\ll} B$. Assume that M is N-projective. Then $A \overset{\tau_M}{\ll} N$.*

PROOF. Let $A+C=\tau_M(N)$ with $\tau_M(C)=C$. Then $A+C \cap B=(A+C) \cap B=$ $=\tau_M(N) \cap B \supset \tau_M(B)$. Hence $\tau_M(A+C \cap B) \supset \tau_M(\tau_M(B))=\tau_M(B)$. Since M is N-pro-

jective, by Lemma 1.10, we get $\tau_M(A)+\tau_M(C\cap B)=\tau_M(A+C\cap B)$. Since $B\supset A$, we have $\tau_M(B)\supset\tau_M(A)+\tau_M(C\cap B)=\tau_M(A+C\cap B)\supset\tau_M(B)$. Hence $A+\tau_M(C\cap B)=$ $=\tau_M(B)$. Since $A\overset{\tau_M}{\ll}B$, we conclude that $\tau_M(C\cap B)=\tau_M(B)$. In particular $\tau_M(C)\supset$ $\supset\tau_M(B)\supset\tau_M(A)$. From $A+C=\tau_M(N)$ and N-projectivity of M, we get $\tau_M(A)+$ $+\tau_M(C)=\tau_M(N)$. Since $\tau_M(C)\supset\tau_M(A)$, this yields $\tau_M(C)=\tau_M(N)$. Hence $A\overset{\tau_M}{\ll}N$. □

PROPOSITION 6.10. *Let M be N-projective. Let K be a τ_M-supplement in N of some M-cotorsionless submodule of N. Let $L=\tau_M(L)\subset N$ be any τ_M-supplement of K in N. Then K is a τ_M-supplement of L in N.*

PROOF. Let K be a τ_M-supplement of $B=\tau_M(B)$ in N. We are given that L is a τ_M-supplement of K in N. Hence $L+K=\tau_M(N)$ and $\tau_M(K\cap L)\overset{\tau_M}{\ll}L$ by Proposition 6.8. We want to prove that K is a τ_M-supplement of L in N. Since, $L+K=$ $=\tau_M(N)$, by Proposition 6.8, it suffices to prove that $\tau_M(K\cap L)\overset{\tau_M}{\ll}K$.

Let $\tau_M(K\cap L)+C=K=\tau_M(K)$ with $\tau_M(C)=C$. We know that $K+B=\tau_M(N)$. Hence $C+\tau_M(K\cap L)+B=\tau_M(N)$. Hence $C+\tau_M(K\cap L)+B=\tau_M(N)$. From $\tau_M(K\cap L)\overset{\tau_M}{\ll}L$ and Lemma 6.9, we get $\tau_M(K\cap L)\overset{\tau_M}{\ll}N$. Hence $C+B=\tau_M(N)$. Now $C\subset K$ and K is a τ_M-supplement of B. Hence $C=K$. This proves that $\tau_M(K\cap L)\overset{\tau_M}{\ll}K$. □

PROPOSITION 6.11. *Let M be f.g. quasi-projective and N-projective. Then*

(i) *$L\subset_S H$ has a supplement in $_SH\Leftrightarrow\gamma(L)$ has a τ_M-supplement in N.*

(ii) *Let N' be an M-cotorsionless submodule of N. Then N' has a τ_M-supplement in $N\leftrightarrow\beta(N')$ has a supplement in $_SH$.*

PROOF. (i) Let U be a supplement of L in $_SH$. Then $H=U+L$ and $U\cap L\ll U$. Then $\tau_M(N)=\gamma(H)=\gamma(U+L)=\gamma(U)+\gamma(L)$ by (1.6). Also $\mathrm{Hom}_R(M,\gamma(U))=$ $=\beta\gamma(U)=U$. Thus $U\cap L\ll U\Rightarrow\gamma(U\cap L)\overset{\tau_M}{\ll}\gamma(U)$ by Proposition 2.7. From (1.18) we have $\gamma(U\cap L)=\tau_M(\gamma(U)\cap\gamma(L))$. Thus $\gamma(U),\gamma(L)$ are M-cotorsionless submodules of N satisfying $\gamma(U)+\gamma(L)=\tau_M(N)$ and $\tau_M(\gamma(U)\cap\gamma(L))\overset{\tau_M}{\ll}\gamma(U)$. From Proposition 6.8, we see that $\gamma(U)$ is a τ_M-supplement of $\gamma(L)$.

Conversely, suppose $\gamma(L)$ has a τ_M-supplement C in N. Then $\tau_M(C)=C$ and $\gamma(L)+C=\tau_M(N)$. Now, $\gamma(L+\beta(C))=\gamma(L)+\gamma\beta(C)=\gamma(L)+\tau_M(C)=\tau_M(N)=\gamma\beta(N)$ by (1.6). Since γ is (1−1) we get $L+\beta(C)=\beta(N)=_SH$. Also $\tau_M(\gamma(L)\cap C)\overset{\tau_M}{\ll}C$. Hence $\beta\tau_M(\gamma(L)\cap C)\ll\beta(C)$ by Proposition 2.7, (ii). However $\beta\tau_M=\beta$ by (1.3). Hence $\beta(\gamma(L)\cap C)\ll\beta(C)$. But $\beta(\gamma(L)\cap C)=\beta\gamma(L)\cap\beta(C)=L\cap\beta(C)$ by (1.5). Thus $L+\beta(C)=_SH$ and $L\cap\beta(C)\ll\beta(C)$, showing that $\beta(C)$ is a supplement of L in $_SH$.

(ii) Since N' is M-cotorsionless we have $\gamma\beta(N')=N'$. Hence (ii) is an immediate consequence of (i), if we set $L=\beta(N')$.

REMARK 6.12. If N is hereditarily M-cotorsionless, then any $N'\subset N$ is M-cotorsionless and $K\supset N$ will be a τ_M-supplement of $N'\leftrightarrow K$ is a supplement of N. Hence the following is an immediate corollary of Proposition 6.11.

COROLLARY 6.13. *Let M be f.g. quasi-projective and N-projective. Further let N be hereditarily M-cotorsionless. Then*

(i) *$L \subset {}_SH$ has a supplement in ${}_SH \Leftrightarrow \gamma(L)$ has a supplement in ${}_RN$.*
(ii) *$N' \subset {}_RN$ has a supplement in ${}_RN \Leftrightarrow \beta(N')$ has a supplement in ${}_SH$.*

DEFINITION 6.14. A module N will be called τ_M-semiperfect, if every M-cotorsionless submodule of N has a τ_M-supplement in N.

LEMMA 6.15. *Let M be N-projective and $\tau_M(N)=N$. Then N is τ_M-semiperfect $\Leftrightarrow$ it is semiperfect.*

PROOF. Assume N to be τ_M-semiperfect. Let N' be an arbitrary submodule of N. Then $\tau_M(N')$ has a τ_M-supplement say K in N. Then $\tau_M(K)=K$ and $\tau_M(N')+K=\tau_M(N)=N$. From $N' \supset \tau_M(N')$ we get $N'+K=N$. We claim that K is actually a supplement of N' in N. If not, there exists a $K' \underset{+}{\subset} K$ with $N'+K'=N$. From 1.10, we get $\tau_M(N')+\tau_M(K')=\tau_M(N)=N$. But $\tau_M(K') \subset K' \underset{+}{\subset} K=\tau_M(K)$. This contradicts the fact that K is a τ_M-supplement of $\tau_M(N')$ in N.

Conversely, assume that N is semiperfect, Let A be any M-cotorsionless submodule of N. Since N is semiperfect, there exists a supplement K of A in N. From 1.10 and $A+K=N$ we get $\tau_M(A)+\tau_M(K)=\tau_M(N)$ or $A+\tau_M(K)=N$. Since $\tau_M(K) \subset K$ and K is a supplement of A in N we get $\tau_M(K)=K$. It is now clear that K is a τ_M-supplement of A in N. □

THEOREM 6.16. *Let M be f.g. quasi-projective and N-projective. Then ${}_SH$ is semiperfect $\Leftrightarrow {}_RN$ is τ_M-semiperfect. If in addition $\tau_M(N)=N$, then ${}_SH$ is semiperfect $\Leftrightarrow {}_RN$ is semiperfect.*

PROOF. Immediate consequence of Propositions 1.14, 6.11 and Lemma 6.15. □

COROLLARY 6.17. *Let ${}_RM$ be f.g. quasi-projective and $S=\mathrm{End}\ {}_RM$. Then S is a semiperfect ring $\Leftrightarrow M$ is semiperfect.*

REMARKS 6.18. (1) Corollary 6.17, where ${}_RM$ is a progenerator has been proved by Erika Mares [11].

(2) When ${}_RM$ is a progenerator, ${}_SH$ is projective $\Leftrightarrow {}_RN$ is projective. From Theorem 6.16 and Proposition 6.5, then see that ${}_SH$ is semiperfect in the sense of Erika Mares $\Leftrightarrow {}_RN$ is semiperfect in the sense of Erika Mares.

(3) All modules having finite spanning dimension in the sense of Patrick Fleury [4] have property (P_2) described in Section 2 of [16], hence are semiperfect in the sense of Definition 6.6 This class includes the class of hollow modules and the class of artinian modules. Also every semisimple module is semiperfect in this sense (Lemma 3.1 in [16]).

We conclude this section by showing that homomorphic images of semiperfect modules are semiperfect.

LEMMA 6.19. *Let $M \xrightarrow{\varphi} M'' \to 0$ be exact with $K=\mathrm{Ker}\ \varphi$. Let $K \subset N \subset M$. Suppose N has a supplement H in M. Then $H\varphi$ is a supplement of $N\varphi$ in M''.*

This is Lemma 2.22 in [16].

PROPOSITION 6.20. *Every homomorphic image of a semiperfect module is semiperfect.*

PROOF. Let $M \xrightarrow{\varphi} M'' \to 0$ be exact with M semiperfect. Let $A \subset M''$. Then $A\varphi^{-1}$ is a submodule of M containing $\operatorname{Ker}\varphi$. Since M is semiperfect, $A\varphi^{-1}$ has a supplement B in M. Then by Lemma 6.18, $B\varphi$ is a supplement of $(A\varphi^{-1})\varphi = A$ in M''. Hence M'' is semiperfect. □

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RINGS IN WHICH DERIVATIONS SATISFY CERTAIN ALGEBRAIC CONDITIONS

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The primary purpose of this paper is to investigate some commutator conditions for rings, which were suggested by group-theoretic results of F. W. Levi, I. D. Macdonald and N. D. Gupta. Most of these conditions can be simply interpreted in terms of inner derivations, and they suggest further questions about arbitrary derivations.

In [8] and [9] Levi essentially established the equivalence of the following properties for a group G:

(i) The commutator operation in G is associative.
(ii) Commutators are in the center of G.
(iii) The commutator operation is right distributive with the group composition.
(iv) The commutator operation is left distributive with the group composition.

(Cf. Kurosh [7], pp. 99—101.) Denoting by $[a, b]$ the group commutator $a^{-1}b^{-1}ab$, we can reformulate conditions (i)—(iv) in terms of identities. The added condition (v) is obviously equivalent to (ii).

(i) $[x, [y, z]] = [[x, y], z]$ for all $x, y, z \in G$;

(ii) $[[x, y], z] = 1$ for all $x, y, z \in G$;

(iii) $[xy, z] = [x, z][y, z]$ for all $x, y, z \in G$;

(iv) $[z, xy] = [z, x][z, y]$ for all $x, y, z \in G$;

(v) $G = Z_2(G)$, the second center of G.

For rings, if we use the usual notion of commutator, namely $[a, b] = ab - ba$, conditions (i)—(iv) all have obvious analogues; and in Section 1 we study equivalences among them. A less obvious analogue for (v) appears in the process.

For a group G and $a \in G$ we define the map $d_a: G \to G$ by $d_a(x) = [x, a]$ for all $x \in G$. In [10] I. D. Macdonald shows that d_a is a homomorphism of G if and only if $a \in Z_2(G)$. For a ring R and $a \in R$ the analogue maps $d_a(x) = [x, a]$ for all $x \in R$ are examples of derivations — that is, additive endomorphisms d satisfying the "product rule" $d(xy) = xd(y) + d(x)y$ for all $x, y \in R$; and the d_a are known as inner derivations. The ring analogue of condition (iii) is simply the statement that all inner derivations d_z are ring endomorphisms; moreover, in the group

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case, (iv) is equivalent to the identity $[xy, z]=[y, z][x, z]$, for which the ring analogue is just the assertion that all d_z are ring anti-endomorphisms.

Lemma 1 in Section 1, which can be viewed as the ring analogue of Macdonald's result in [10], provides information as to when an inner derivation is a ring endomorphism or anti-endomorphism; and it suggests investigating when arbitrary derivations, not necessarily inner, are endomorphisms or anti-endomorphisms. Section 2 contains some results of this investigation.

For either groups or rings, define $[x, y]_1$ to be the usual commutator, and continue inductively by $[x, y]_n=[[x, y]_{n-1}, y]$. Gupta [4] showed that for groups G, the identity $[x, y]=[x, y]_n$, $n\geqq 2$, implies that any solvable or finite G is abelian. For $n=2$ or 3 this result holds without any restrictions on the structure of G. In rings, for any integer $n\geqq 2$, the identity $[x, y]=[x, y]_n$ implies that R is commutative. This result is an immediate corollary of a recent and powerful commutativity theorem of W. Streb [11]. In our final section we extend this result to the case where n is allowed to vary with y; in other words, we prove that R is commutative if for each inner derivation d there exists an integer $n=n(d)>1$ for which $d^n=d$.

1. Some commutator conditions

Let R denote an arbitrary ring with center $Z=Z(R)$, and let $C_R(R^2)=$ $=\{a\in R|[xy, a]=0,\ x, y\in R\}$. Consider the following conditions:

(I) $[x, [y, z]]=[[x, y], z]$ for all $x, y, z\in R$;

(II) $[[x, y], z]=0$ for all $x, y, z\in R$;

(III) $[xy, z]=[x, z][y, z]$ for all $x, y, z\in R$;

(IV) $[xy, z]=[y, z][x, z]$ for all $x, y, z\in R$;

(V) $R^2\subseteqq Z(R)$.

THEOREM 1. *Conditions* (I) *and* (II) *are equivalent. Conditions* (III), (IV), *and* (V) *are equivalent. Condition* (V) *implies* (II).

We begin the proof with a local version of the equivalence of conditions (III)—(V).

LEMMA 1. *Define* $H(R)=\{a\in R|d_a$ *is an endomorphism of* $R\}$ *and* $H^*(R)=$ $=\{a\in R|d_a$ *is an anti-endomorphism of* $R\}$. *Then* $H(R)=H^*(R)=C_R(R^2)$.

PROOF. (i) $H^*(R)=H(R)\subseteqq C_R(R^2)$. Let $a\in H(R)$. Then for all $x\in R$, $[xa, a]=$ $=[x, a][a, a]=0$ and $[ax, a]=[a, a][x, a]=0$, hence $a^2x=xa^2=axa$. It follows by straightforward computation that $[x, a][y, a]=0=[y, a][x, a]$ for all $x, y\in R$, hence $H(R)\subseteqq H^*(R)$ and $H(R)\subseteqq C_R(R^2)$. Essentially the same argument shows that $a\in H^*(R)$ implies $a\in H(R)$.

(ii) $C_R(R^2)\subseteqq H(R)$. Let $a\in C_R(R^2)$. Then, by definition, $[xy, a]=0$ for all $x, y\in R$. Moreover, $[x, a][y, a]=(xa-ax)(ya-ay)=xaya-axya-xa^2y+axay=$ $=xa(ya)-a(xya)-xa^2y+a(xa)y=x(ya)a-(xya)a-xa^2y+(xa)ay=0$; therefore $a\in H(R)$.

PROOF OF THEOREM 1. The equivalence of (III), (IV), and (V) is immediate from the lemma. That (II) implies (I) is obvious. To show that (I) implies (II), suppose that $[[x, y], z]=[x, [y, z]]=-[[y, z], x]$. Since $[[x, y], z]+[[y, z], x]+$ $+[[z, x], y]=0$ by the Jacobi identity, we conclude that $[[z, x], y]=0$. Finally, the obvious identity $[[x, y], z]=[xy, z]-[yx, z]$ shows that (V) implies (II).

COROLLARY 1. *Let R be a ring with 1 or, more generally, a ring for which $R^2=R$. If every inner derivation of R is a ring endomorphism, then R is commutative.*

COROLLARY 2. *Let R be semiprime. If d_a is a ring endomorphism, then $d_a=0$. Thus, if every inner derivation is an endomorphism, then R is commutative.*

PROOF. Let d_a be an endomorphism. Then by Lemma 1, $a\in C_R(R^2)$ and $[x, a][y, a]=[xy, a]=0$ for all $x, y\in R$. Consequently,

$$[x, a]y[x, a] = [x, a]([yx, a]-[y, a]x) = 0$$

for all $x, y\in R$, and semiprimeness of R yields $[x, a]=0$ for all $x\in R$.

The hypothesis that all inner derivations are endomorphisms is clearly inherited by homomorphic images. This observation, together with the fact that R is semiprime if and only if the Baer radical $B(R)$ is zero, shows that the final sentence of Corollary 2 is equivalent to the following corollary.

COROLLARY 3. *Let R be any ring in which every inner derivation is a ring endomorphism. Then $R/B(R)$ is commutative.*

We note that neither (II) nor (V) implies commutativity in general, since a non-commutative R with $R^3=\{0\}$ satisfies both (II) and (V). We observe also that (II) does not imply (V); we obtain a counterexample by letting S be any non-commutative ring with 1 and taking

$$R = \left\{ \begin{bmatrix} c & a & b \\ 0 & c & a \\ 0 & 0 & c \end{bmatrix} \middle| \; a, b\in S \text{ and } c\in Z(S) \right\},$$

with the operations on R being the usual matrix operations.

2. Derivations as homomorphisms and anti-homomorphisms

Corollary 2 and Lemma 1 suggest the following theorem.

THEOREM 2. *Let R be a semiprime ring. If d is a derivation of R which is either an endomorphism or an anti-endomorphism, then $d=0$.*

Of course, derivations which are not endomorphisms or anti-endomorphisms may behave as such on certain subsets of R; for example, any derivation d behaves as the zero endomorphism on the subring C consisting of all constants (i.e. elements x for which $d(x)=0$). In fact, in a semiprime ring, d may behave as an endomorphism on a proper ideal of R. As an example of such R and d, let S be any semiprime ring with a nonzero derivation δ, take $R=S\oplus S$, and define d by $d((r_1, r_2))=$

$=(\delta(r_1), 0)$. Our next theorem shows that in prime rings, the behavior of d is somewhat more restricted.

If $\emptyset \neq T \subseteq R$, we shall say that d acts as a homomorphism or an anti-homomorphism on T if $d(xy)=d(x)d(y)$ or $d(xy)=d(y)d(x)$ respectively for all $x, y \in T$. (Note that we do not require that $d(T) \subseteq T$.)

THEOREM 3. *Let R be a prime ring and U a nonzero right ideal of R. If d is a derivation of R which acts as a homomorphism or an anti-homomorphism on U, then $d=0$ on R.*

We begin the proofs of these theorems with the following lemma.

LEMMA 2. (a) *Let U be a subring of a ring R, and let d be a derivation of R which acts as a homomorphism on U. Then*

$$d(x)x(y-d(y)) = 0 \quad \text{for all} \quad x, y \in U.$$

(b) *Let V be a right ideal of R, and d a derivation of R acting as an anti-homomorphism on V. Then*

$$d(x)y[r, d(x)] = 0 \quad \text{for all} \quad x, y \in V \quad \text{and all} \quad r \in R.$$

PROOF. (a) Since d acts as a homomorphism on U, we have

$$d(xy) = xd(y)+d(x)y = d(x)d(y) \quad \text{for all} \quad x, y \in U. \tag{1}$$

Substituting x^2 for x, we get

$$x^2 d(y)+xd(x)y+d(x)xy = xd(x)d(y)+d(x)xd(y),$$

or equivalently,

$$x(xd(y)+d(x)y)+d(x)xy = xd(x)d(y)+d(x)xd(y).$$

Recalling (1), we conclude that

$$d(x)x(y-d(y)) = 0 \quad \text{for all} \quad x, y \in U. \tag{2}$$

(b) Since d acts as an anti-homomorphism on V, we have

$$d(xy) = xd(y)+d(x)y = d(y)d(x) \quad \text{for all} \quad x, y \in V; \tag{3}$$

and replacing y by xy, we see that

$$xd(xy)+d(x)xy = xd(y)d(x)+d(x)yd(x).$$

Since the first terms on the two sides of this equation are equal, we conclude that

$$d(x)xy = d(x)yd(x) \quad \text{for all} \quad x, y \in V. \tag{4}$$

Replacing y by yr in (4) gives $d(x)xyr=d(x)yrd(x)$; on the other hand, right-multiplying (4) by r gives $d(x)xyr=d(x)yd(x)r$. Thus,

$$d(x)y[r, d(x)] = 0 \quad \text{for all} \quad x, y \in V \quad \text{and all} \quad r \in R.$$

PROOF OF THEOREM 2. First suppose that d is an endomorphism. Applying Lemma 2(a) with $U=R$ gives

$$d(x)x(y-d(y))=0 \quad \text{for all} \quad x,y\in R. \tag{5}$$

Replacing y by yr for arbitrary $r\in R$ yields

$$d(x)x(y-d(y))r-d(x)xyd(r)=0,$$

which together with (5) gives $d(x)xRd(R)=\{0\}$. In particular, $d(x)xRd(x)x=\{0\}$; and semiprimeness of R gives $d(x)x=0$ for all $x\in R$. Similarly, $xd(x)=0$, so taking $x=y$ in (1) shows that $(d(x))^2=0$ for all $x\in R$. But Giambruno and Herstein [3] have recently shown that semiprime rings admit no nonzero derivations for which $(d(x))^n=0$ for a fixed positive integer n and all $x\in R$; consequently, our derivation d must be trivial.

Now consider the case of d an anti-endomorphism. Then by Lemma 2(b) with $R=V$, we conclude that

$$d(x)y[r,d(x)]=0 \quad \text{for all} \quad x,y,r\in R.$$

Since y can be replaced by ry in this equation, we get $[r,d(x)]R[r,d(x)]=\{0\}$, from which we conclude that $d(R)\subseteq Z$ and hence d is an endomorphism.

PROOF OF THEOREM 3. We begin by assuming that d acts as a homomorphism on U. By Lemma 2 (a), we have

$$d(x)x(y-d(y))=0 \quad \text{for all} \quad x,y\in U. \tag{6}$$

Replacing y by yr for arbitrary $r\in R$ gives $d(x)x(y-d(y))r-d(x)xyd(r)=0$, which implies that $d(x)xyd(r)=0$; and replacing r by rs yields

$$d(x)xyRd(R)=\{0\}. \tag{7}$$

Assume that $d(R)\neq\{0\}$. Then (7) gives

$$d(x)xy=0 \quad \text{for all} \quad x,y\in U, \tag{8}$$

and it follows from (6) that

$$d(x)xd(y)=0 \quad \text{for all} \quad x,y\in U. \tag{9}$$

Note now that $d(x(xy))=d(x)d(xy)$ for all $x,y\in U$ — that is, $xd(xy)+d(x)xy=d(x)d(xy)$, or equivalently,

$$x^2d(y)+xd(x)y+d(x)xy=d(x)xd(y)+d(x)d(x)y.$$

In view of (8) and (9), this reduces to

$$x^2d(y)=(d(x)d(x)-xd(x))y \quad \text{for all } x,y\in U. \tag{10}$$

Since $d(x^2)=(d(x))^2$ for all $x\in U$, we have $(d(x))^2-xd(x)=d(x)x$, hence (8) and (10) yield

$$x^2d(y)=0 \quad \text{for all} \quad x,y\in U. \tag{11}$$

Letting r be an arbitrary element of R and replacing y by yr in (11) gives

$x^2yd(R)=\{0\}$, from which we conclude that $x^2yRd(R)=\{0\}$ and hence $x^2y=0$ for all $x, y\in U$. In particular, $x^3=0$ for all $x\in U$. But by a result of Levitzki [6, Lemma 1.1] a prime ring has no nil right ideals of bounded index, so we have a contradiction. Thus, $d(R)=\{0\}$.

Now assume that d acts as an anti-homomorphism on U, in which case 2(b) yields

$$d(x)y[r, d(x)] = 0 \quad \text{for all} \quad x, y\in U \quad \text{and} \quad r\in R.$$

Replacing y by ys for arbitrary $s\in R$ then gives

$$d(x)yR[r, d(x)] = \{0\} \quad \text{for all} \quad x, y\in U \quad \text{and} \quad r\in R.$$

Thus, for fixed $x\in U$, either $d(x)U=\{0\}$ or $d(x)\in Z$. Since the subsets of U for which these two alternatives hold are additive subgroups of U whose union is U, at least one must be equal to U; thus, either $d(U)U=\{0\}$ or $d(U)\subseteq Z$. In the latter case, d acts as a homomorphism on U, and there is nothing further to be done; thus, we assume that $d(U)U=\{0\}$.

Since $xd(y)+d(x)y=d(y)d(x)$ for all $x, y\in U$, we have

$$xd(y) = d(y)d(x) \quad \text{for all} \quad x, y\in U. \tag{12}$$

Replacing x by xr, we get $xrd(y)=d(y)(xd(r)+d(x)r)$, hence

$$xrd(y) = d(y)d(x)r \quad \text{for all} \quad x, y\in U \quad \text{and} \quad r\in R.$$

From (12) we see that $d(y)d(x)r$ is also equal to $xd(y)r$, hence $x[r, d(y)]=0$. Replacing x by xs yields $xR[r, d(y)]=\{0\}$ for all $x, y\in U$ and $r\in R$; and therefore $d(U)\subseteq Z$.

3. On potent derivations

Motivated by Gupta's results in [4], we now investigate derivations d with the property that there exists an integer $n>1$ for which $d^n=d$; such derivations will be called *potent*. The major result deals with rings in which all inner derivations are potent, which we shall refer to as PD-*rings* (potent derivation rings). Note that a ring R is a PD-ring if and only if for each $y\in R$, there exists $n>1$ such that $[x, y]=$ $=[x, y]_n$ for all $x\in R$.

We shall employ some results of Streb from [11]. Let $\mathbf{Z}\langle X, Y\rangle$ denote the ring of polynomials with integer coefficients in two noncommuting indeterminates, and $\mathbf{Z}[X, Y]$ the ring of polynomials in two commuting indeterminates. Consider polynomials $f(X, Y)\in\mathbf{Z}\langle X, Y\rangle$ which have trivial image under the natural homomorphism from $\mathbf{Z}\langle X, Y\rangle$ to $\mathbf{Z}[X, Y]$, and call such a polynomial *admissible* if each monomial has total degree at least 3 and Y-degree at least 2. A ring R is called a $(\mathbf{Z}, \beta)$-ring if for each $x, y\in R$ there exists an admissible polynomial $f(X, Y)$ such that $[x, y]=f(x, y)$; if the $f(X, Y)$ may be chosen independent of x and y, R is called a $(\mathbf{Z}, \bar{\beta})$-ring. We are interested in these notions because every PD-ring is a $(\mathbf{Z}, \beta)$-ring; and if there exists a fixed $n>1$ such that $d^n=d$ for all inner derivations d, then R is a $(\mathbf{Z}, \bar{\beta})$-ring.

We shall also capitalize on the following easy and well-known result, the proof of which we omit.

LEMMA 3. *Let R be a ring of prime characteristic p, and let m denote an arbitrary positive integer. If $x\in R$ and $u=x^{p^m}$, then $(d_x)^{p^m}=d_u$.*

The d_x notation is a cumbersome one for dealing with this result; hence, we shall use instead of d_x the notation $\operatorname{ad} x$, which is in common use among ring theorists. The conclusion of the lemma then takes the form $(\operatorname{ad} x)^{p^m}=\operatorname{ad} x^{p^m}$.

Derivations d with $d^2=d$ are somewhat scarce, at least in the absence of 2-torsion. Indeed, it is an easy exercise to show that a semiprime 2-torsion-free ring admits no nontrivial derivation d with $d^2=d$. On the other hand, if R_0 is the ring of 2×2 matrices over $GF(2)$, Lemma 3 shows that $(d_x)^2=d_x$ if and only if $x^2-x\in Z$ — a property readily verified in the case of the noncentral element $x=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$. Obviously, this d_x also satisfies $(d_x)^n=d_x$ for all $n>2$.

For $n>2$, it is easy to exhibit inner derivations d with $d^n=d$, with no restrictions whatever on the additive orders of elements of R. For example, if R is any ring containing a noncentral idempotent e, then d_e is a nontrivial inner derivation satisfying $(d_e)^n=d_e$ for every odd n. However, as our final theorem shows, a noncommutative ring cannot have too many potent derivations.

THEOREM 4. *Every PD-ring is commutative.*

Two lemmas will facilitate the proof.

LEMMA 4. *If R is a PD-ring in which the nilpotent elements form an ideal containing the commutator ideal, then R is commutative.*

PROOF. As we have previously observed, R is a $(\mathbf{Z}, \beta)$-ring. Streb has proved that all $(\mathbf{Z}, \beta)$-rings have the following properties.

(a) [11, Lemma 7(2)] If B is any nil subset of the Jacobson radical, the ideal generated by B is commutative.

(b) [11, Lemma 9(1)] If A is any commutative ideal, then $A\subseteq Z$.

Under our hypothesis, applying these results to the ideal N of nilpotent elements shows that $N\subseteq Z$; hence all commutators are central. Thus $[x, y]_n=0$ for all $x, y\in R$ and all $n\geq 2$, so R is commutative.

Our second lemma disposes of our problem for periodic rings R, which are defined as rings such that for each $x\in R$, there exist distinct positive integers m and n for which $x^m=x^n$.

LEMMA 5. *Every periodic PD-ring is commutative.*

PROOF. Let e be any idempotent. Then for $x\in R$, there exists $n\geq 2$ such that $[e, [e, xe]]=[e, [e, xe]]_n$, which reduces to $xe-exe=0$. Similarly, $ex-exe=0$, so idempotents are central. Since some power of each element of R is idempotent [1], R is radical over its center; thus, by a theorem of Herstein [5], the nilpotent elements form an ideal containing the commutator ideal. By Lemma 4, therefore, R is commutative.

PROOF OF THEOREM 4. In view of Lemma 4, we need only prove that if R has no nonzero nil ideals, then R is commutative. But such a ring R is isomorphic to a subdirect product of prime rings with no nonzero nil ideals, so we assume henceforth that R is a prime ring with no nonzero nil ideals.

Note that if d is an inner derivation with $d^n=d$, then $d^j=d^k$ for all positive integers j, k with $j\equiv k \bmod (n-1)$. In particular, if d_1 and d_2 are two inner derivations with $d_1^{n_1}=d_1$ and $d_2^{n_2}=d_2$, then $d_1^N=d_1$ and $d_2^N=d_2$ for $N=(n_1-1)(n_2-1)+1$.

Assume that R is noncommutative and $[x, y]\neq 0$. Choose $N>1$ such that $(\operatorname{ad} y)^N=\operatorname{ad} y$ and $(\operatorname{ad} 2y)^N=\operatorname{ad} 2y$. Then $[x, y]=[x, y]_N$ and $[x, 2y]=[x, 2y]_N$, hence $(2^N-2)[x, y]=0$. Thus, R has prime characteristic p.

Now consider any fixed $y\in R$, and take $n>1$ such that $(\operatorname{ad} y)^n=\operatorname{ad} y$. Since there are only finitely many congruence classes mod $n-1$, the pigeonhole principle gives us distinct powers p^r and p^s, each greater than $n-1$, for which $p^r\equiv p^s \bmod (n-1)$. Thus, $(\operatorname{ad} y)^{p^r}=(\operatorname{ad} y)^{p^s}$, so that by Lemma 3 $[x, y^{p^r}]=[x, y^{p^s}]$ for all $x\in R$, which is to say that $y^{p^r}-y^{p^s}\in Z$.

Suppose now that each central element is periodic. Then for each $y\in R$, there exist distinct positive integers j and k and distinct positive integers r and s for which $(y^{p^r}-y^{p^s})^j=(y^{p^r}-y^{p^s})^k$; and it follows by [2, Theorem 1], a theorem originally proved by Chacron, that R is periodic. Thus, R is commutative by Lemma 5.

The remaining case is that of R having a nonperiodic central element z_0. For fixed $y\in R$, choose a single n such that $(\operatorname{ad} y)^n=\operatorname{ad} y$ and $(\operatorname{ad} z_0y)^n=\operatorname{ad} z_0y$; and repeat the previous argument, obtaining r, s such that $y^{p^r}-y^{p^s}\in Z$ and $(z_0y)^{p^r}-(z_0y)^{p^s}=z_0^{p^r}y^{p^r}-z_0^{p^s}y^{p^s}\in Z$. It follows at once that $(z_0^{p^r}-z_0^{p^s})y^{p^s}\in Z$. Noting that $z_0^{p^r}-z_0^{p^s}\neq 0$ and that the center of a prime ring contains no nonzero divisors of zero, we show easily that $y^{p^s}\in Z$. Thus R is commutative, again by the theorem of [5].

We remark in closing that Streb has proved the amazing result that every $(\mathbf{Z}, \bar{\beta})$-ring is commutative. Therefore, as we mentioned in the introduction, the special case of Theorem 4 in which the same n works for all inner derivations d is immediate from Streb's work.

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THE EXCEPTIONAL SET FOR THE SUM OF A PRIME AND A SQUARE

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1. Introduction

In this paper we investigate the following well known conjecture of Hardy and Littlewood [6]:

CONJECTURE H. Every large integer n is either a square or the sum of a prime and a square. The number $N(n)$ of representations is given asymptotically by

$$N(n)\sim\frac{\sqrt{n}}{\log n}\prod_{p>2}\left(1-\frac{\left(\frac{n}{p}\right)}{p-1}\right).$$

$\left(\left(\frac{n}{p}\right)\right.$ denotes, as usual, the Legendre symbol$\left.\right)$.

Davenport and Heilbronn [3] proved in 1937 that the conjecture is true for almost all natural numbers. More precisely they showed that Conjecture H holds for all but $O(X\log^{-c}X)$ positive integers $n\leqq X$, for some $c>0$. This was shown to be true for arbitrary $c>0$ by Miech [7] in 1968. Let

$$E(X)=\left|\{n\leqq X:\ n \text{ is not the sum of a prime and a square}\}\right|.$$

The above result implies that

$$E(X)\ll X\log^{-c}X$$

for every $c>0$. Although the problem of estimating $E(X)$ appears to be analogous to that of estimating the exceptional set in Goldbach's problem, various new difficulties appear and, in fact, Miech needed to use density theorems for the zeros of L-functions to have adequate control over the singular series which arises.

After the important work of Vaughan [14] and of Montgomery and Vaughan [10], where they show that the cardinality of the exceptional set in Goldbach's problem is $O(X^{\theta})$ for some $\theta<1$, one can perhaps expect a similar estimate to hold also for our problem. Polyakov [12] (see also [11]) used the method of Montgomery—Vaughan to obtain

$$N(n)=\frac{\sqrt{n}}{\log n}\prod_{p>2}\left(1-\frac{\left(\frac{n}{p}\right)}{p-1}\right)\left\{1+O\left(\exp\left(-\frac{\log^{1/2}n}{\log^{3}\log n}\right)\right)\right\}$$

for all but $O(X\exp(-c\sqrt{\log X}))$ positive integers $n\leqq X$. However, due to the possible existence of the Siegel-zero, such a result is unlikely to be provable in the

present state of knowledge and, indeed, a serious mistake occurs in the crucial estimate (14.5) of [12].

The main ideas of Montgomery—Vaughan's method can be briefly described as follows.

i) The effect of the possible Siegel-zero has to be calculated exactly. In this way one obtains a second singular series and a "second main term" attached to the Siegel-zero.

ii) Instead of the Siegel—Walfisz theorem one has to use the much stronger Gallagher's prime number theorem (see Theorem 7 of [5]).

The problem of the sum of a prime and a square seems, however, to be genuinely more difficult than Goldbach's problem and similar ideas are not apparently sufficiently powerful to yield $E(X) \ll X^{\theta}$, $\theta < 1$.

We proceed by modifying the method of Montgomery—Vaughan as follows.

iii) We calculate exactly the effect of the possibly existing zeros of the L-functions in the domain

$$\sigma > 1 - \frac{c' \log\log X}{\log X}, \quad |t| \leqq X^{c''}$$

thus obtaining a number of singular series and "main terms" attached to these zeros.

iv) Finally we estimate the effect of these "main terms" using the deep density theorem of Bombieri (see Théorème 14 of [1]).

Working in this way we also need a modification of Gallagher's prime number theorem (see (18) and (19)), and the appearance of the new "main terms" leads to various trigonometric sums, which are estimated by van der Corput's method (see Lemmas 12—15).

The above method yields the following

THEOREM. *There exist effectively computable constants* $\theta < 1$ *and* $C > 0$ *such that*

$$E(X) \leqq CX^{\theta}.$$

REMARK. After the first version of manuscript was completed, the work of A. I. Vinogradov [15] appeared in which he proves the same result, using another method. Nevertheless we think our work is still of interest, due to the following reasons:

a) Vinogradov uses the original approach of Hardy—Littlewood via theta-functions in contrast to us or [8] and [10].

b) He avoids the crucial difficulties mentioned in iii) and iv) in another way, estimating the sum $S_3(\alpha)$ of (3.10) of [15] in the way given in (3.15), although this method is only very briefly sketched.

c) Vinogradov's proof is in many parts somewhat sketchy. Our aim is here to give a proof as complete as possible, apart that we just indicate the proofs for some standard lemmas (some of them appear in earlier works too, perhaps in slightly modified form).

2. Notations

Throughout the paper we will use the following notations.

$\| \|$=distance from the nearest integer, $\log_k$=k times iterated logarithm, $e(x)=e^{2\pi ix}$, p=prime number, cond χ=conductor of χ, $|\mathscr{A}|$=cardinality of the set $\mathscr{A}$, $s=\sigma+it$.

X, T and P will denote sufficiently large positive numbers. All the constants involved in the $\ll$ and O-notations will be absolute and effectively computable. The value of an empty sum will be 0, the value of an empty product 1. We will denote by $\varrho=\beta+i\gamma$ the zeros of the L-functions. We set

$$S(\alpha)=\sum_{X/2\leqq p<X}\log p e(\alpha p),\quad F(\alpha)=\sum_{\sqrt{X}/2\leqq k\leqq\sqrt{X}} e(\alpha k^2),$$

$$S(\chi,\eta)=\sum_{X/2\leqq p\leqq X}\chi(p)\log p e(p\eta),\quad r(X,n)=\sum_{\substack{k^2+p=n\\ X/2\leqq p<X\\ \sqrt{X}/2\leqq k<\sqrt{X}}}\log p,$$

$$V(a,q)=\sum_{k=1}^{q} e\left(k^2\frac{a}{q}\right),$$

$$T_\varrho(\eta)=\sum_{X/2\leqq m<X} m^{\varrho-1}e(m\eta),\quad T(\eta)=T_1(\eta)$$

$\chi_{0,q}$=principal character $(\mathrm{mod}\, q)$, $\tau(\chi)=\sum_{a=1}^{q}\chi(a)e\left(\frac{a}{q}\right)$,

$$\sum_{a=1}^{q}{}^{*}=\sum_{\substack{a=1\\(a,q)=1}}^{q},\quad \sum_{\chi(\mathrm{mod}\, q)}^{*}=\sum_{\substack{\chi(\mathrm{mod}\, q)\\ \chi\ \text{primitive}}},$$

$$H(\chi,q,n)=\sum_{a=1}^{q}\chi(a)V(a,q)e\left(-\frac{an}{q}\right),\quad H(q,n)=H(\chi_{0,q},q,n),$$

$$L_\varrho(X,n)=\sum_{\substack{m+k^2=n\\ X/2\leqq m<X\\ \sqrt{X}/2\leqq k<\sqrt{X}}} m^{\varrho-1},\quad L(X,n)=L_1(X,n),$$

$$\sigma(n,R,l)=\sum_{\substack{q\leqq R\\(q,l)=1}}\frac{\mu(q)}{q\varphi(q)}H(q,n),\quad \sigma(n,R)=\sigma(n,R,1),$$

$$T(\chi,r,n)=\frac{\tau(\bar{\chi})H(\chi,r,n)}{r\varphi(r)},$$

$$N(\sigma,T,\chi)=\left|\{\varrho\colon L(\varrho,\chi)=0,\ \beta\geqq\sigma,\ |\gamma|\leqq T\}\right|,$$

$\mu(n)$=Möbius' function, $\varphi(n)$=Euler's function, $\omega(n)$=number of prime factors of n, $\varrho_n(d)$=number of solutions k of $k^2\equiv n\ (\mathrm{mod}\, d)$, $r(n)$=number of representations of n as sum of two squares. $c_1, c_2, \ldots$ will denote positive, absolute and effectively computable constants.

3. Dissection of the unit interval

We need the following

LEMMA 1. *There exists a constant* $c_1>0$ *such that* $L(s,\chi)\neq 0$ *whenever*

$$\sigma \geqq 1-\frac{3\cdot 10^{-3}}{\log T}, \quad |t| \leqq T^{15}$$

for all primitive characters $\chi \pmod q$, $q\leqq T$, *with the possible exception of at most one real primitive character* $\tilde\chi \pmod{\tilde r}$. *If it exists, the unique exceptional zero of* $L(s,\tilde\chi)$ *is a real simple zero* $\tilde\beta$ *satisfying*

$$\frac{c_1}{\tilde r^{1/2}\log^2 \tilde r} \leqq 1-\tilde\beta \leqq \frac{3\cdot 10^{-2}}{\log T}.$$

This result is contained in Davenport [2], Ch. 14. The choice of the numerical constant $3\cdot 10^{-3}$ is justified by the work of Miech [8]. The exceptional zero, if it exists, is called Siegel zero (relative to T).

Let $P_1=X^{b_1}$, where b_1 is a sufficiently small positive constant for which an upper bound will be given at the end of the proof. Let us choose $T=P_1$ in Lemma 1. Let

$$P_2=\begin{cases} P_1 & \text{if } \tilde r < P_1^{1/200},\\ P_1^{1/200} & \text{otherwise.}\end{cases}$$

Define b_2 by

$$P_2 = X^{b_2}. \tag{1}$$

Then it is clear that Lemma 1 remains true with $T=P_2$, the value $15\cdot 10^{-6}$ in place of $3\cdot 10^{-3}$ and $\tilde r \leqq P_2^{1/200}$ (if it exists).

We define the P_2-*excluded zeros* as those zeros of the functions $L(s,\chi)$, where χ is any primitive character $\pmod q$, $q\leqq P_2$, lying in the region

$$\sigma \geqq 1-\frac{10\log_2 X}{\log X}, \quad |t| \leqq P_2^{15},$$

the possible Siegel zero being excluded. Then we define the P_2-*excluded characters* as the primitive characters $\chi \pmod r$, $r\leqq P_2$, for which $L(\varrho,\chi)=0$, ϱ being a P_2-excluded zero. The P_2-*excluded moduli* are the moduli of the P_2-excluded characters.

Let $N^-(\alpha,T,T^{15})$ denote the total number of zeros of the functions $L(s,\chi)$, where χ is any primitive character $\pmod q$, $q\leqq T$, in the region $\sigma\geqq\alpha$, $|t|\leqq T^{15}$, the Siegel zero (relative to T) being again excluded. An estimate for the number of P_2-excluded zeros may be obtained by using the following

LEMMA 2. *There exist positive constants* c_2 *and* c_3 *such that*

$$N^-(\alpha,T,T^{15}) \leqq c_2 G(T) T^{c_3(1-\alpha)}$$

where

$$G(T)=\begin{cases} 1 & \text{if } \tilde\beta \text{ does not exist}\\ (1-\tilde\beta)\log T & \text{if } \tilde\beta \text{ exists.}\end{cases}$$

Lemma 2 is a reformulation of Théorème 14 of Bombieri [1]. Using Lemma 2 we obtain that

$$N^{-}\left(1-\frac{10\log_2 X}{\log X}, P_2, P_2^{15}\right) \leqq c_2 G(P_2)\exp(10b_2c_3\log_2 X) \leqq G(P_2)\log^{1/3} X,$$

provided b_2 is sufficiently small. Hence

$$|\{P_2\text{-excluded zeros}\}| \leqq G(P_2)\log^{1/3} X. \tag{2}$$

By (2) there are at most $G(P_2)\log^{1/3} X$ P_2-excluded moduli. Hence by (1) we can choose $P\in[P_2^{9/10}, P_2]$ such that for any P_2-excluded modulus $r\leqq P$ we have

$$\frac{P}{r} \geqq \exp(\sqrt{\log X}). \tag{3}$$

From now on we will use the following notations:

$$P = X^b, \quad G = G(P), \tag{4}$$

$$\mathscr{E} = \{P\text{-excluded characters}\}, \quad \mathscr{E}' = \{P\text{-excluded zeros}\},$$

$$\mathscr{S} = \{\text{Siegel character (relative to } P)\}, \quad \mathscr{S}' = \{\text{Siegel zero (relative to } P)\}.$$

It is clear that Lemma 1 remains true with $T=P$, the value 10^{-5} in place of $3\cdot 10^{-3}$ and

$$\tilde{r} < P^{1/180} \tag{5}$$

(if it exists).

Let

$$Q = XP^{-11}. \tag{6}$$

For $q\leqq P$ and $(a, q)=1$, $a<q$, we let $M(a/q)$ be the major arc

$$\left[\frac{a}{q}-\frac{1}{Qq}, \frac{a}{q}+\frac{1}{Qq}\right];$$

The major arcs are non-overlapping, and let

$$\mathscr{M} = \bigcup_{q\leqq P} \bigcup_{\substack{a=1\\(a,q)=1}}^{q} \mathscr{M}\left(\frac{a}{q}\right).$$

Moreover, let $\mathscr{m}$ denote the minor arcs, i.e.

$$\mathscr{m} = \left[\frac{1}{Q}, 1+\frac{1}{Q}\right]\setminus\mathscr{M}.$$

Then we have

$$r(X,n) = \int_{1/Q}^{1+(1/Q)} F(\alpha)S(\alpha)e(-n\alpha)\,d\alpha =$$

$$= \int_{\mathfrak{M}} F(\alpha)S(\alpha)e(-n\alpha)\,d\alpha + \int_{\mathfrak{m}} F(\alpha)S(\alpha)e(-n\alpha)\,d\alpha =$$

$$= r_1(X,n)+r_2(X,n), \quad \text{say.}$$

Since the sets $\mathfrak{M}$ and $\mathfrak{m}$ are even (mod 1) it is clear that $r_1(X,n)$ and $r_2(X,n)$ are real.

4. Some lemmas

In this section we prove several lemmas which will be needed in the course of the proof of the Theorem. First we assert without proof the standard

LEMMA 3. *Let* $(q_1,q_2)=1$ *and* χ_{q_i} *be a character* (mod q_i), $i=1,2$. *Then*

$$H(\chi_{q_1}\chi_{q_2}, q_1q_2, n) = \chi_{q_1}(q_2)\chi_{q_2}(q_1)H(\chi_{q_1}, q_1, n)H(\chi_{q_2}, q_2, n).$$

LEMMA 4. *Let* $\left(\frac{n}{p}\right)$ *denote the Legendre symbol. Then*

$$H(p,n) = \begin{cases} 0 & \text{if } p|2n \\ p\left(\frac{n}{p}\right) & \text{if } p\nmid 2n. \end{cases}$$

PROOF. Indeed,

$$H(p,n) = \sum_{a=1}^{p-1} e\left(-\frac{an}{p}\right) \sum_{k=1}^{p} e\left(k^2\frac{a}{p}\right) =$$

$$= -p + p \sum_{\substack{k=1 \\ k^2\equiv n \,(\mathrm{mod}\, p)}}^{p} 1 = p(e_n(p)-1)$$

and Lemma 4 follows from the relation

$$\varrho_n(p) = \begin{cases} 1 & \text{if } p|2n \\ \left(\frac{n}{p}\right)+1 & \text{if } p\nmid 2n. \end{cases} \tag{7}$$

LEMMA 5. *Let* χ (mod r) *be a primitive character. Then*

$$|H(\chi,r,n)| \leqq r^{3/2} \prod_{\substack{p|r \\ p|2n}} \left(1-\frac{1}{p}\right) \prod_{\substack{p|r \\ p\nmid 2n}} \left(1-\frac{\left(\frac{n}{p}\right)+1}{p}\right) \leqq r^{3/2}.$$

PROOF. We have

$$|H(\chi, r, n)| = \left|\sum_{k=1}^{r}\sum_{a=1}^{r}\chi(a)e\left(a\frac{k^2-n}{r}\right)\right| =$$

$$= |\tau(\chi)|\left|\sum_{k=1}^{r}\bar{\chi}(k^2-n)\right| \leqq r^{3/2}\sum_{d|r}\mu(d)\frac{\varrho_n(d)}{d}$$

and Lemma 5 follows from (7) and the multiplicativity of $\varrho_n(d)$.

LEMMA 6. *Let* $\chi \pmod q$ *be induced by* $\chi^* \pmod r$. *Then*

$$\tau(\chi) = \mu\left(\frac{q}{r}\right)\chi^*\left(\frac{q}{r}\right)\tau(\chi^*).$$

This is Lemma 5.2 of [10].

LEMMA 7. *Let* $\chi \pmod r$ *be a primitive character. Then*

$$\sum_{\substack{q\leqq P\\ q\equiv 0 \pmod r}} \frac{|\tau(\chi_{0,q}\chi)|}{q\varphi(q)}|H(\chi_{0,q}\chi, q, n)| \ll \frac{r}{\varphi(r)}\log\frac{2P}{r}. \tag{8}$$

PROOF. Denoting by $\sum$ the left hand side of (8) and setting $q=lr$, we obtain from Lemmas 3, 4, 5 and 6

$$\sum \leqq \sum_{\substack{l\leqq P/r\\ (l,r)=1}} \frac{r^{1/2}}{rl\varphi(rl)} r^{3/2} l \ll \frac{r}{\varphi(r)}\log\frac{P}{r}.$$

LEMMA 8. *Let* $(a, q)=1$. *Then*

$$F\left(\frac{a}{q}+\eta\right) = \frac{V(a,q)}{q}F(\eta)+O\left((q\log q)^{1/2}(1+|\eta|X)\right),$$

$$V(a,q) = O\left((q\log q)^{1/2}\right),$$

$$F\left(\frac{a}{q}+\eta\right) = O\left(\left(\frac{X\log q}{q}\right)^{1/2}+(q\log q)^{1/2}(1+|\eta|X)\right).$$

Lemma 8 follows by partial summation from Lemma 3 of Davenport—Heilbronn [4].

LEMMA 9. *We have*

$$\int_0^1 |F(\eta)|^4\,d\eta = O(X\log X).$$

PROOF. We have by Parseval's identity

$$\int_0^1 |F(\eta)^2|^2\,d\eta \leqq \sum_{n\leqq 2X} r(n)^2 \ll X\log X.$$

LEMMA 10. *Let* $u_1, \dots, u_N$ *be real numbers. Then for any* $\delta>0$

$$\int_{-\delta}^{\delta} \Big|\sum_{n\leqq N} u_n e(n\eta)\Big|^2 d\eta \ll \int_{-\infty}^{+\infty} \Big|\delta \sum_{t}^{t+(2\delta)^{-1}} u_n\Big|^2 dt.$$

This is Lemma 1 of Gallagher [5].

LEMMA 11. *Let* $3X^{-1}<\delta<X^{-1/2}$. *Then*

$$\int_{-\delta}^{\delta} |F(\eta)|^2 d\eta = O(1).$$

PROOF. Let us choose $N=X$ and

$$u_n = \begin{cases} 1 & \text{if } \ n = m^2 \in [X/4, X) \\ 0 & \text{otherwise} \end{cases}$$

in Lemma 10. Then

$$\int_{-\infty}^{+\infty} \Big|\delta \sum_{t}^{t+(2\delta)^{-1}} u_n\Big|^2 dt \ll \int_{X/20}^{X} \left(\delta \frac{\delta^{-1}}{\sqrt{t}}\right)^2 dt \ll 1.$$

LEMMA 12. *Let* $|\gamma|<X/Qq$ *and* $(Qq)^{-1}\leqq|\eta|\leqq 1/2$. *Then*

$$T_Q(\eta) \ll \frac{X^{\beta-1}}{|\eta|}.$$

PROOF. By Abel's inequality we have

$$T_Q(\eta) = \sum_{X/2\leqq m<X} m^{\beta-1} e\left(m\eta + \frac{\gamma}{2\pi}\log m\right) \ll$$

$$\ll X^{\beta-1} \max_{X/2\leqq Y<X} \left|\sum_{X/2\leqq m<Y} e\left(m\eta + \frac{\gamma}{2\pi}\log m\right)\right|.$$

Setting $f(u)=u\eta+\frac{\gamma}{2\pi}\log u$ we have

$$|f'(u)| = \left|\eta + \frac{\gamma}{2\pi u}\right| \gg |\eta|,$$

f' is monotonic and $\operatorname{sgn} f'(u)=\operatorname{sgn}\eta$, $|f'(u)|\leqq 3/5$. Lemma 12 follows then from Lemmas 4.2 and 4.8 of Titchmarsh [13].

LEMMA 13. *Let* $(3/4)X\leqq n\leqq 2X$ *and* $|\gamma|\leqq X^{1/2}$. *Then there exists a positive constant* c_4 *such that*

$$|L_Q(X, n)| \leqq c_4 \frac{X^{1/2}}{1+|\gamma|} X^{\beta-1}.$$

PROOF. Clearly

$$|L_Q(X, n)| = \Big| \sum_{\substack{\sqrt{X}/2\leqq k<\sqrt{X} \\ n-X<k^2\leqq n-(X/2)}} (n-k^2)^{Q-1}\Big| \leqq c_5 X^{1/2} X^{\beta-1}$$

if $|\gamma| \leqq 1$ and, by Abel's inequality,

$$|L_q(X, n)| \leqq c_6 X^{\beta-1} \max_{\sqrt{X}/2 \leqq Y < \sqrt{X}} \left| \sum_{\substack{\sqrt{X}/2 \leqq m < Y \\ n-X < k^2 \leqq n-(X/2)}} e\left(\frac{\gamma}{2\pi} \log(n-k^2)\right) \right|.$$

Setting $f(u) = \frac{\gamma}{2\pi} \log(n-u^2)$ we have

$$|f'(u)| = \left| \frac{-2u\gamma}{2\pi(n-u^2)} \right| \geqq c_7 \frac{\gamma}{\sqrt{X}},$$

f' is monotonic and $\operatorname{sgn} f'(u) = \operatorname{sgn}(-\gamma)$, $|f'(u)| \leqq \frac{2}{\pi}$. Lemma 13 follows then from Lemmas 4.2 and 4.8 of [13].

LEMMA 14. *Let* $|\gamma| < X^{1/2}$. *Then*

$$\int_0^1 |F(\eta) T_q(\eta)|^2 \, d\eta \ll \frac{X^2}{1+|\gamma|^2} X^{2(\beta-1)}.$$

PROOF. From Lemma 13 we get

$$\int_0^1 |F(\eta) T_q(\eta)|^2 \, d\eta = \int_0^1 \Big| \sum_{(3/4)X \leqq n < 2X} L_q(X, n) e(n\eta) \Big|^2 \, d\eta =$$

$$= \sum_{(3/4)X \leqq n < 2X} |L_q(X, n)|^2 \ll \frac{X^2}{1+|\gamma|^2} X^{2(\beta-1)}.$$

LEMMA 15. *Let* $|\gamma| \leqq X/Qq$. *Then*

$$\int_{1/Qq \leqq |\eta| \leqq 1/2} |F(\eta) T_q(\eta)| \, d\eta \ll (XQq \log X)^{1/4} X^{\beta-1}.$$

PROOF. By Lemmas 9 and 12 we have

$$\int_{1/Qq \leqq |\eta| \leqq 1/2} |F(\eta) T_q(\eta)| \, d\eta \leqq \Big(\int_0^1 |F(\eta)|^4 \, d\eta \Big)^{1/4} \Big(\int_{1/Qq \leqq |\eta| \leqq 1/2} |T_q(\eta)|^{4/3} \, d\eta \Big)^{3/4} \ll$$

$$\ll (X \log X)^{1/4} (Qq)^{1/4} X^{\beta-1}.$$

LEMMA 16. *Let* $\chi \pmod q$, $\chi \neq \chi_{0,q}$. *Let* $T^{-2} \leqq \eta \leqq \frac{\delta}{2} \leqq \frac{1}{4}$ *and* $N(1-\delta, 2T, \chi) = 0$. *Let* $D = \{s: |\sigma - 1| \leqq \eta, |t| \leqq T\}$. *Then there exists a constant* $c_8 > 0$ *such that*

$$\max_{s \in D} \left| \frac{1}{L(s, \chi)} \right| \leqq \exp\left\{ \frac{c_8}{\eta} (\log qT)^{4\eta/\delta} \right\}.$$

This is essentially Lemma 19 of Miech [7].

LEMMA 17. *Let* $n \in [(9/10)X, X]$ *and* $\tilde{\chi} \in S$. *Then there exists a constant* $c_9 > 0$ *such that*

$$L(X, n) - L_\beta(X, n) \geqq c_9 G \sqrt{X}.$$

PROOF. In fact,

$$\sum_{\substack{m+k^2=n \\ X/2\leqq m<X \\ \sqrt{X}/2\leqq k<\sqrt{X}}} (1-m^{\beta-1}) \ll c_{10}\sqrt{X}(1-P^{\beta-1}) \ll c_9\sqrt{X}(\beta-1)\log P = c_9 G\sqrt{X}.$$

5. The minor arcs

The estimate of the contribution of the minor arcs proceeds as usual. From Bessel's inequality and the prime number theorem we get

$$\sum_{(9/10)X\leqq n<X} r_2(X,n)^2 \ll \int_{\mathfrak{m}} |F(\alpha)S(\alpha)|^2\,d\alpha \ll \sup_{\alpha\in\mathfrak{m}}|F(\alpha)|^2 \int_0^1 |S(\alpha)|^2\,d\alpha \ll$$

$$\ll X\log X \sup_{\alpha\in\mathfrak{m}}|F(\alpha)|^2.$$

If $\alpha\in\mathfrak{m}$ there exist a and q, $(a,q)=1$ and $P<q\leqq Q$ such that

$$\left|\alpha-\frac{a}{q}\right| \leqq \frac{1}{Qq}.$$

Since $QP<X$, from Lemma 8 we have

$$\sup_{\alpha\in\mathfrak{m}}|F(\alpha)|^2 \ll \frac{X\log X}{P}.$$

Hence

$$\sum_{(9/10)X\leqq n<X} r_2(X,n)^2 \ll \frac{X^2\log^2 X}{P}. \tag{9}$$

6. Approximation of $F(\alpha)$ and $S(\alpha)$ on $M(a/q)$

If $\alpha\in\mathfrak{M}(a/q)$ we write $\alpha=a/q+\eta$. Since $q\leqq P$, we have that if $p>P$ then $(q,p)=1$ and hence

$$e(\alpha p) = \frac{1}{\varphi(q)}\sum_{\chi}\tau(\bar{\chi})\chi(ap)e(p\eta).$$

Thus we have

$$S(\alpha) = \frac{1}{\varphi(q)}\sum_{\chi}\tau(\bar{\chi})\chi(a)S(\chi,\eta).$$

We define $W(\chi,\eta)$ in the following way.

i) If $\chi=\chi_{0,q}$ then

$$W(\chi_{0,q},\eta) = S(\chi_{0,q},\eta)-T(\eta).$$

ii) If $\chi \pmod q$ is induced by $\chi^*\in\mathscr{E}\cup\mathscr{S}$ then $\chi=\chi_{0,q}\chi^*$ and

$$W(\chi_{0,q}\chi^*,\eta) = S(\chi_{0,q}\chi^*,\eta)+\sum_{\substack{\varrho\in\mathscr{E}'\cup\mathscr{S}' \\ L(\varrho,\chi^*)=0}} T_\varrho(\eta).$$

iii) In all other cases $W(\chi,\eta)=S(\chi,\eta)$.

Hence we have

$$S\left(\frac{a}{q}+\eta\right)=\frac{\mu(q)}{\varphi(q)}T(\eta)+D(a,q,\eta)+E(a,q,\eta), \tag{10}$$

where

$$D(a,q,\eta)=\frac{1}{\varphi(q)}\sum_{\chi}\tau(\bar{\chi})\chi(a)W(\chi,\eta)$$

and

$$E(a,q,\eta)=-\sum_{\substack{\chi\in\mathcal{E}\cup\mathcal{S}\\ \operatorname{cond}\chi\mid q}}\sum_{\substack{\varrho\in\mathcal{E}'\cup\mathcal{S}'\\ L(\varrho,\chi)=0}}\frac{\chi_{0,q}\chi(a)\tau(\overline{\chi_{0,q}\chi})}{\varphi(q)}T_{\varrho}(\eta).$$

We use Lemma 8 for the approximation of $F(\eta)$. We have

$$F\left(\frac{a}{q}+\eta\right)=\frac{V(a,q)}{q}F(\eta)+\Delta(a,q,\eta), \tag{11}$$

where

$$\Delta(a,q,\eta)=O\big((q\log q)^{1/2}(1+|\eta|X)\big).$$

From (10) and (11) we get

$$r_1(X,n)=\sum_{q\leqq P}\sideset{}{^*}\sum_{a=1}^{q}e\left(-\frac{an}{q}\right)\int_{-(1/Qq)}^{1/Qq}F\left(\frac{a}{q}+\eta\right)S\left(\frac{a}{q}+\eta\right)e(-n\eta)\,d\eta= \tag{12}$$

$$=\sum_{q\leqq P}\frac{\mu(q)}{q\varphi(q)}H(q,n)\int_{-(1/Qq)}^{1/Qq}F(\eta)T(\eta)e(-n\eta)\,d\eta+$$

$$+\sum_{q\leqq P}\frac{1}{q}\sideset{}{^*}\sum_{a=1}^{q}V(a,q)e\left(-\frac{na}{q}\right)\int_{-(1/Qq)}^{1/Qq}F(\eta)D(a,q,\eta)e(-n\eta)\,d\eta+$$

$$+\sum_{q\leqq P}\frac{1}{q}\sideset{}{^*}\sum_{a=1}^{q}V(a,q)e\left(-\frac{na}{q}\right)\int_{-(1/Qq)}^{1/Qq}F(\eta)E(a,q,\eta)e(-n\eta)\,d\eta+$$

$$+\sum_{q\leqq P}\sideset{}{^*}\sum_{a=1}^{q}e\left(-\frac{na}{q}\right)\int_{-(1/Qq)}^{1/Qq}\Delta(a,q,\eta)S\left(\frac{a}{q}+\eta\right)e(-n\eta)\,d\eta=$$

$$=S_1+S_2+S_3+S_4,\quad\text{say.}$$

7. Estimate of S_4

By (6) and the prime number theorem we have

$$S_4\ll\sum_{q\leqq P}\sideset{}{^*}\sum_{a=1}^{q}\int_{-(1/Qq)}^{1/Qq}(q\log q)^{1/2}(1+|\eta|X)\left|S\left(\frac{a}{q}+\eta\right)\right|d\eta\ll \tag{13}$$

$$\ll\frac{X}{Q}\sum_{q\leqq P}(q^{-1}\log q)^{1/2}\Big(\int_{-(1/Qq)}^{1/Qq}d\eta\Big)^{1/2}\Big(\int_0^1|S(\eta)|^2\,d\eta\Big)^{1/2}\ll P^{18}.$$

8. Evaluation of S_1

Clearly

$$S_1 = \sum_{q \leqq P} \frac{\mu(q)}{q\varphi(q)} H(q, n) \int_0 F(\eta) T(\eta) e(-n\eta)\, d\eta + \tag{14}$$

$$+ O\Big(\sum_{q \leqq P} \frac{|H(q, n)\mu(q)|}{q\varphi(q)} \int_{1/Qq \leqq |\eta| \leqq 1/2} |F(\eta) T(\eta)| d\eta \Big).$$

By Lemmas 3, 4, 9 and 12 the error term in (14) is

$$\ll \sum_{q \leqq P} \frac{1}{\varphi(q)} \Big(\int_0^1 |F(\eta)|^4 d\eta \Big)^{1/4} \Big(\int_{1/Qq \leqq |\eta| \leqq 1/2} |\eta|^{-4/3} d\eta \Big)^{3/4} \ll \tag{15}$$

$$\ll \sum_{q \leqq P} \frac{1}{\varphi(q)} (X \log X)^{1/4} (Qq)^{1/4} \ll \sqrt{X} P^{-2}.$$

Since

$$\int_0^1 F(\eta) T(\eta) e(-n\eta)\, d\eta = L(X, n),$$

(14) and (15) give

$$S_1 = \sigma(n, P) L(X, n) + O(\sqrt{X} P^{-2}). \tag{16}$$

9. Estimate of S_2

Using Lemma 11 we have that

$$S_2 \ll \sum_{q \leqq P} \frac{1}{q\varphi(q)} \sum_{\chi} |\tau(\bar{\chi}) H(\chi, q, n)| \Big(\int_{-(1/Qq)}^{1/Qq} |F(\eta)|^2 d\eta \Big)^{1/2} \cdot \Big(\int_{-(1/Qq)}^{1/Qq} |W(\chi, \eta)|^2 d\eta \Big)^{1/2} \ll$$

$$\ll \sum_{r \leqq P} \sum_{\substack{q \leqq P \\ q \equiv 0 \,(\mathrm{mod}\, r)}} \frac{1}{q\varphi(q)} \sum_{\chi \,(\mathrm{mod}\, r)}^{*} |\tau(\overline{\chi_{0,q}\chi}) H(\chi_{0,q}\chi, q, n)| \cdot \Big(\int_{-(1/Qq)}^{1/Qq} |W(\chi_{0,q}\chi, \eta)|^2 d\eta \Big)^{1/2}.$$

Since $q \leqq P$ and $p > P$ we clearly have

$$W(\chi_{0,q}\chi, \eta) = W(\chi, \eta).$$

Hence by Lemma 7 we get

$$S_2 \ll \sum_{r \leqq P} \sum_{\chi \,(\mathrm{mod}\, r)}^{*} \Big(\int_{-(1/Qr)}^{1/Qr} |W(\chi, \eta)|^2 d\eta \Big)^{1/2} \cdot \tag{17}$$

$$\cdot \sum_{\substack{q \leqq P \\ q \equiv 0 \,(\mathrm{mod}\, r)}} \frac{|\tau(\overline{\chi_{0,q}\chi}) H(\chi_{0,q}\chi, q, n)|}{q\varphi(q)} \ll$$

$$\ll \log^2 X \sum_{r \leqq P} \sum_{\chi \,(\mathrm{mod}\, r)}^{*} \Big(\int_{-(1/Qr)}^{1/Qr} |W(\chi, \eta)|^2 d\eta \Big)^{1/2}.$$

Let

$$\sum_t^{t+h}{}^{\#} \chi(p)\log p = \begin{cases} \sum_t^{t+h} \log p - \sum_t^{t+h} 1 & \text{if } r=1 \\ \sum_t^{t+h} \chi(p)\log p + \sum\limits_{\substack{\varrho\in\mathscr{E}'\cup\mathscr{S}' \\ L(\varrho,\chi)=0}} \sum_t^{t+h} m^{\varrho-1} & \text{if } r>1. \end{cases}$$

By Lemma 10 with $\delta=\dfrac{1}{Qr}$ we obtain that

$$\int_{-(1/Qr)}^{1/Qr} |W(\chi,\eta)|^2\,d\eta \ll X \max_{X/5\le t\le X} \max_{h\le X} (h+XP^{-11})^{-2} \Big|\sum_t^{t+h}{}^{\#} \chi(p)\log p\Big|^2,$$

hence by (17)

$$S_2 \ll \sqrt{X}\log^2 X \sum_{r\le P} \sum_{\chi (\mathrm{mod}\, r)}^{*} \max_{X/5\le t\le X} \max_{h\le X} (h+XP^{-11})^{-1} \Big|\sum_t^{t+h}{}^{\#} \chi(p)\log p\Big| = \tag{18}$$

$$= \sqrt{X}\log^2 X \cdot W, \quad \text{say.}$$

Now, following the ideas of the proof of Theorem 7 of [5] and using Lemma 4.8 of [13], we get

$$W \ll \sum_{r\le P} \sum_{\chi (\mathrm{mod}\, r)}^{*} \Big(\sum_{\substack{\varrho\notin\mathscr{E}'\cup\mathscr{S}' \\ L(\varrho,\chi)=0 \\ |\gamma|\le P^{15}}} X^{\beta-1} + \frac{P^{11}}{X}\frac{X}{P^{15}}\log^2 X\Big).$$

Using Lemma 2 we obtain

$$W \ll G \int_0^{1-(10\log_2 X/\log X)} X^{1/2(\alpha-1)}\log X\,d\alpha + X^{-1/2} + P^{-1} \ll G\log^{-5} X + P^{-1}. \tag{19}$$

From (18) and (19) we have

$$S_2 \ll \sqrt{X}G\log^{-3} X + \frac{\sqrt{X}\log^2 X}{P}; \tag{20}$$

10. Evaluation of S_3

Let

$$\mathscr{E}_1' = \{\varrho\in\mathscr{E}'\cup\mathscr{S}' : |\gamma| \le P^{10}\}, \quad \mathscr{E}_2' = \mathscr{E}'\setminus\mathscr{E}_1'.$$

As in the last section, we have

(21)

$$S_3 = -\sum_{\chi\in\mathscr{E}\cup\mathscr{S}} \sum_{\substack{q\le P \\ \mathrm{cond}\,\chi\mid q}} \frac{\tau(\overline{\chi_{0,q}\chi})}{q\varphi(q)} H(\chi_{0,q}\chi, q, n) \sum_{\substack{\varrho\in\mathscr{E}_1' \\ L(\varrho,\chi)=0}} \int_{-(1/Qq)}^{1/Qq} F(\eta)T_\varrho(\eta)\,e(-n\eta)\,d\eta -$$

$$- \sum_{\chi\in\mathscr{E}} \sum_{\substack{q\le P \\ \mathrm{cond}\,\chi\mid q}} \frac{\tau(\overline{\chi_{0,q}\chi})}{q\varphi(q)} H(\chi_{0,q}\chi, q, n) \sum_{\substack{\varrho\in\mathscr{E}_2' \\ L(\varrho,\chi)=0}} \int_{-(1/Qq)}^{1/Qq} F(\eta)T_\varrho(\eta)\,e(-n\eta)\,d\eta =$$

$$= S_{3,1} + S_{3,2}, \quad \text{say.}$$

By Lemmas 7 and 14 we have

(22)
$$S_{3,2} \ll \sum_{\chi \in \mathscr{E}} \sum_{\substack{q \leqq P \\ \mathrm{cond}\,\chi \mid q}} \frac{|\tau(\overline{\chi_{0,q}\chi})|}{q\varphi(q)} |H(\chi_{0,q}\chi, q, n)| \sum_{\substack{\varrho \in \mathscr{E}_2' \\ L(\varrho,\chi)=0}} \left(\frac{1}{Qq} \int_0^1 |F(\eta) T_\varrho(\eta)|^2 \, d\eta \right)^{1/2} \ll$$

$$\ll \log^3 XQ^{-1/2} XP^{-1\varrho} \ll \sqrt{X} P^{-4},$$

since $|\mathscr{E}'| \ll \log^{1/3} X$.

We have

$$S_{3,1} = - \sum_{\chi \in \mathscr{E} \cup \mathscr{S}} \sum_{\substack{q \leqq P \\ \mathrm{cond}\,\chi \mid q}} \frac{\tau(\overline{\chi_{0,q}\chi})}{q\varphi(q)} H(\chi_{0,q}\chi, q, n) \sum_{\substack{\varrho \in \mathscr{E}_1' \\ L(\varrho,\chi)=0}} \int_0^1 F(\eta) T_\varrho(\eta) e(-n\eta) \, d\eta +$$

$$+ O\left(\sum_{\chi \in \mathscr{E} \cup \mathscr{S}} \sum_{\substack{q \leqq P \\ \mathrm{cond}\,\chi \mid q}} \frac{|\tau(\overline{\chi_{0,q}\chi})|}{q\varphi(q)} |H(\chi_{0,q}\chi, q, n)| \sum_{\substack{\varrho \in \mathscr{E}_1' \\ L(\varrho,\chi)=0}} \int_{1/Qq \leqq |\eta| \leqq 1/2} |F(\eta) T_\varrho(\eta)| \, d\eta \right).$$

By Lemmas 3, 6, 7 and 15 we obtain

(23)
$$S_{3,1} = - \sum_{r \leqq P} \sum_{\substack{\chi \in \mathscr{E} \cup \mathscr{S} \\ \chi \,(\mathrm{mod}\, r)}} \frac{\tau(\bar{\chi}) H(\chi, r, n)}{r\varphi(r)} \sum_{l \leqq P/r} \frac{\mu(l) \bar{\chi}(l)}{l\varphi(l)} \chi(l) \chi_{0,l}(r) \cdot$$

$$\cdot H(\chi_{0,l}, l, n) \sum_{\substack{\varrho \in \mathscr{E}_1' \\ L(\varrho,\chi)=0}} L_\varrho(X, n) + O\left(\log^3 X (XQP \log X)^{1/4}\right) =$$

$$= - \sum_{r \leqq P} \sum_{\substack{\chi \in \mathscr{E} \cup \mathscr{S} \\ \chi \,(\mathrm{mod}\, r)}} \frac{\tau(\bar{\chi}) H(\chi, r, n)}{r\varphi(r)} \sum_{\substack{l \leqq P/r \\ (l,r)=1}} \frac{\mu(l)}{l\varphi(l)} H(l, n) \sum_{\substack{\varrho \in \mathscr{E}_1' \\ L(\varrho,\chi)=0}} L_\varrho(X, n) + O\left(\sqrt{X} P^{-2}\right).$$

Hence from (21)—(23) we get

(24)
$$S_3 = - \sum_{r \leqq P} \sum_{\substack{\chi \in \mathscr{E} \cup \mathscr{S} \\ \chi \,(\mathrm{mod}\, r)}} T(\chi, r, n) \sigma\left(n, \frac{P}{r}, r\right) \sum_{\substack{\varrho \in \mathscr{E}_1' \\ L(\varrho,\chi)=0}} L_\varrho(X, n) + O\left(\sqrt{X} P^{-2}\right).$$

From (12), (13), (16), (20) and (24) we finally obtain that

(25)
$$r_1(X, n) = \sigma(n, P) L(X, n) - \sum_{r \leqq P} \sum_{\substack{\chi \in \mathscr{E} \cup \mathscr{S} \\ \chi \,(\mathrm{mod}\, r)}} T(\chi, r, n) \sigma\left(n, \frac{P}{r}, r\right) \cdot$$

$$\cdot \sum_{\substack{\varrho \in \mathscr{E}_1' \\ L(\varrho,\chi)=0}} L_\varrho(X, n) + O\left(\frac{G\sqrt{X}}{\log^3 X} + \frac{\sqrt{X} \log^2 X}{P} \right).$$

11. The singular series

Let n be any natural number not a perfect square. Then n can be uniquely written as $n=n^*\tilde{n}^2$, where $n^*>1$ is square-free. Let

$$N = N(n) = \begin{cases} n^* & \text{if} \quad n^* \equiv 1 \pmod 4 \\ 4n^* & \text{if} \quad n^* \not\equiv 1 \pmod 4. \end{cases}$$

Then it is well known (see e.g. [2], Ch. 5) that the Kronecker symbol $\left(\frac{N}{\cdot}\right)$ is a real primitive character (mod N), which we denote by $\chi^{(n)}$. Moreover, it is easy to see that

$$\left(\frac{n}{p}\right) = \chi^{(n)}(p) \tag{26}$$

if $p\nmid 2n$, where the symbol on the left hand side of (26) is the Legendre symbol.

Let

$$Z(s, n, l) = \sum_{\substack{k=1 \\ (k,l)=1}}^{\infty} \frac{\mu(k)}{\varphi(k)} H(k, n) k^{-s}.$$

Now Lemma 4 implies that $|H(k, n)| \leqq k$ for square-free k and hence it is clear that $Z(s, n, l)$ is convergent for $\sigma>1$.

LEMMA 18. *For* $\sigma>1/2$ *we have that*

$$Z(s, n, l) = \frac{I(s, n, l)}{L(s, \chi^{(n)})},$$

where

$$I(s, n, l) = \prod_{p\mid 2nl}\left(1-\frac{\chi^{(n)}(p)}{p^s}\right)^{-1} \prod_{p\nmid 2nl}\left(1-\frac{\chi^{(n)}(p)}{\varphi(p)p^s}-\frac{p}{\varphi(p)p^{2s}}\right)\left(1-\frac{1}{p^{2s}}\right).$$

PROOF. We have

$$Z(s, n, l) = \prod_{p\nmid 2nl}\left(1-\frac{p}{\varphi(p)}\left(\frac{n}{p}\right)\frac{1}{p^s}\right) =$$

$$= \prod_{p\nmid 2nl}\left(1-\frac{\chi^{(n)}(p)}{p^s}\right) \prod_{p\nmid 2nl}\left(1-\frac{p\chi^{(n)}(p)}{\varphi(p)p^s}\right)\left(1+\frac{\chi^{(n)}(p)}{p^s}\right) \prod_{p\nmid 2nl}\left(1-\frac{\chi^{(n)}(p)}{p^s}\right)^{-1} \times$$

$$\times\left(1+\frac{\chi^{(n)}(p)}{p^s}\right)^{-1} = \frac{I(s, n, l)}{L(s, \chi^{(n)})}.$$

LEMMA 19. *We have*

$$I(1, n, l) \gg \frac{\varphi(nl)}{nl}$$

and, for $\sigma\geqq 1-\delta$, $0<\delta\leqq\frac{1}{4}$,

$$I(s, n, l) \ll \exp\left(\frac{1}{\delta}\omega(nl)^{\delta}\right).$$

PROOF. The first assertion is straight-forward. Moreover, it is easy to see that, for $\sigma \geqq 1-\delta$,

$$I(s, n, l) \ll \prod_{p|2nl} (1-p^{\delta-1})^{-1} \ll \prod_{p|2nl} (1+p^{\delta-1}) \ll$$

$$\ll \exp\Big(\sum_{m=1}^{\omega(nl)} m^{\delta-1}\Big) \ll \exp\left(\frac{1}{\delta}\omega(nl)^{\delta}\right).$$

Let

$$A(X) = \left\{n \in [(9/10)X, X]: n \neq m^2 \text{ and } N\left(\frac{19}{20}, 2X, \chi^{(n)}\right) > 0, \text{ or } n = m^2\right\}$$

and

$$B(X) = [(9/10)X, X] \setminus A(X).$$

LEMMA 20. *We have* $|A(X)| \ll X^{9/10}$.

PROOF. Theorem 12.2 of [9] implies that

$$\sum_{\substack{q \leqq 4X \\ N(19/20, 2X, \chi) > 0}} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} 1 \ll X^{3/8} \log^{c_{11}} X.$$

Since for any $n_0 \in [(9/10)X, X]$ there are at most $O(\sqrt{X})$ $n \in [(9/10)X, X]$ such that $\chi^{(n)} = \chi^{(n_0)}$, it is clear that

$$|A(X)| \ll X^{1/2} X^{3/8} \log^{c_{11}} X \ll X^{9/10}.$$

LEMMA 21. *Let* $R, l \leqq X$ *and* $n \in B(X)$. *Then there exists a constant* $c_{12} > 0$ *such that*

$$\sigma(n, R, l) = Z(1, n, l) + O\left(\exp\left(c_{12} \log^{2/5} X - \frac{\log R}{200}\right)\right).$$

PROOF. Indeed, letting $\eta = \dfrac{1}{200}$ we have

$$\sigma(n, R, l) = \frac{1}{2\pi i} \int_{1+\eta-iX}^{1+\eta+iX} Z(s, n, l) \frac{R^{s-1}}{s-1}\, ds + O\left(\frac{R^{\eta}}{\eta X} + \frac{\log^2 R}{X}\right) = \tag{27}$$

$$= Z(1, n, l) + \frac{1}{2\pi i} \int_{\Gamma} Z(s, n, l) \frac{R^{s-1}}{s-1}\, ds + O\left(\frac{R^{\eta}}{\eta X} + \frac{\log^2 R}{X}\right),$$

where

$$\Gamma = [1+\eta-iX, 1-\eta-iX] \cup [1-\eta-iX, 1-\eta+iX] \cup [1-\eta+iX, 1+\eta+iX].$$

The integral in (27) is

$$\ll \max_{s \in \Gamma} |Z(s, n, l)| \left(R^{\eta} \int_{1-\eta}^{1+\eta} \frac{d\sigma}{|1-\sigma \pm iX|} + R^{-\eta} \int_{-X}^{X} \frac{dt}{|\eta - it|}\right) \ll \tag{28}$$

$$\ll \max_{s \in \Gamma} |Z(s, n, l)|\, R^{-\eta} \log X.$$

From Lemmas 16 and 19 we get

$$\max_{s\in I}|Z(s, n, I)| \ll \exp\left(\frac{c_{12}}{2}\log^{2/5} X\right), \tag{29}$$

and Lemma 21 follows from (27)—(29).

12. Proof of the Theorem

We will need two more lemmas.

LEMMA 22. *Let* $n\neq m^2$ *and* χ *be a primitive character* (mod r). *Then*

$$|T(\chi, r, n)Z(1, n, r)| \leqq Z(1, n, 1).$$

PROOF. By Lemma 5 we have

$$|T(\chi, r, n)| \leqq \frac{r}{\varphi(r)} \prod_{\substack{p|r\\ p|2n}}\left(1-\frac{1}{p}\right) \prod_{\substack{p|r\\ p\nmid 2n}}\left(1-\frac{1+\left(\frac{n}{p}\right)}{p}\right) = \prod_{\substack{p|r\\ p\nmid 2n}}\left(1-\frac{\left(\frac{n}{p}\right)}{\varphi(p)}\right).$$

Hence

$$|T(\chi, r, n)Z(1, n, r)| \leqq \prod_{\substack{p_1 r\\ p\nmid 2n}}\left(1-\frac{\left(\frac{n}{p}\right)}{\varphi(p)}\right) \prod_{\substack{p\nmid r\\ p\nmid 2n}}\left(1-\frac{\left(\frac{n}{p}\right)}{\varphi(p)}\right) = Z(1, n, 1).$$

LEMMA 23. *Let* $n\neq m^2$. *Then*

$$Z(1, n, 1) \gg \log^{-2} X.$$

PROOF. It is well known that

$$L(1, \chi^{(n)}) \ll \log X, \tag{30}$$

and Lemma 23 follows from (30) and Lemma 19.

Now we are ready to finish the proof.

From (9) we obtain that

$$|r_2(X, n)| \ll \sqrt{X}P^{-1/4} \tag{31}$$

for every $n\in[(9/10)X, X]\setminus C(X)$, where

$$|C(X)| \ll \frac{X\log^2 X}{\sqrt{P}}.$$

On the other hand, from (3), (5), (25) and Lemma 21 we have for every $n\in B(X)$ that (here the second term has to be deleted if there is no Siegel zero)

$$r_1(X,n)=Z(1,n,1)L(X,n)-T(\bar\chi,\bar r,n)Z(1,n,\bar r)L_{\tilde\beta}(X,n)- \tag{32}$$
$$-\sum_{r\leqq P}\sum_{\substack{\chi\in\mathscr{E}\\ \chi(\mathrm{mod}\,r)}}T(\chi,r,n)Z(1,n,r)\sum_{\substack{\varrho\in\mathscr{E}_1'\setminus\{\tilde\beta\}\\ L(\varrho,\chi)=0}}L_\varrho(X,n)+$$
$$+O\big(G\sqrt{X}\log^{-3}X+\sqrt{X}P^{-1}\log^2X\big)+$$
$$+O\big(\sqrt{X}P^{-1/250}+\sqrt{X}\exp\big(-c_{13}\sqrt{\log X}\big)\big|\mathscr{E}_1'\setminus\{\tilde\beta\}\big|\big),$$

since $|L_\rho(X,n)|\ll\sqrt{X}$ for every $\varrho\in\mathscr{E}'$. By (2) the error terms in (32) are

$$O\big(G\sqrt{X}\log^{-3}X+\sqrt{X}P^{-1/250}\big). \tag{33}$$

Hence by Lemmas 17 and 22, (32) and (33) we get

$$r_1(X,n)\geqq Z(1,n,1)\Big(c_9G\sqrt{X}-\sum_{\varrho\in\mathscr{E}_1'\setminus\{\tilde\beta\}}|L_\varrho(X,n)|\Big)+ \tag{34}$$
$$+O\big(G\sqrt{X}\log^{-3}X+\sqrt{X}P^{-1/250}\big).$$

From Lemmas 2 and 13 we obtain that

$$\sum_{\varrho\in\mathscr{E}_1'\setminus\{\tilde\beta\}}|L_\varrho(X,n)|\leqq c_4\sqrt{X}\sum_{\varrho\in\mathscr{E}_1'\setminus\{\tilde\beta\}}X^{\beta-1}\leqq \tag{35}$$
$$\leqq c_2c_4G\sqrt{X}\int_{1-(10\log_2X/\log X)}^{1-(10^{-5}/\log P)}X^{1/2(\alpha-1)}\log X\,d\alpha+O(1)\leqq$$
$$\leqq 2c_2c_4G\sqrt{X}\exp\Big(-\frac{10^{-5}}{2b}\Big)+O(1)\leqq\frac{1}{2}c_9G\sqrt{X}+O(1)$$

if b is sufficiently small. But by (5) and Lemma 1 the Siegel zero $\tilde\beta$, if it exists, satisfies

$$\frac{G}{\log P}=1-\tilde\beta\geqq\frac{c_1}{P^{1/360}\log^2X}. \tag{36}$$

Hence from (34)—(36) and Lemma 23 we finally get

$$r_1(X,n)\gg G\sqrt{X}\log^{-2}X \tag{37}$$

for every $n\in B(X)$. From (5), (31), (36) and (37) we have that

$$r(X,n)\gg G\sqrt{X}\log^{-2}X$$

for every $n\in[(9/10)X,X]$, with at most $O(X^\theta)$ exceptions, where $\theta<1$.

Let

$$X_j=\Big(\frac{10}{9}\Big)^j.$$

Then there are at most $O(X_j^{\theta})$ integers $n \in [(9/10)X_j, X_j]$ for which $r(X_j, n) = 0$. The Theorem follows summing $O(X_j^{\theta})$ over $j = 1, 2, \ldots, \left[\frac{\log X}{\log \frac{10}{9}}\right] + 1$.

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A STRING EQUATION WITH SPECIAL BOUNDARY CONDITIONS

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Consider the system

$$(1)\qquad \begin{cases} w(x)\dfrac{\partial^2 y(x,t)}{\partial t^2} = \dfrac{\partial}{\partial x}\left(p(x)\dfrac{\partial y(x,t)}{\partial x}\right)+\delta(x-a)u(t), \quad 0<x<1,\ 0<t<T, \\ y(\cdot,0)=y_0, \quad \dfrac{\partial}{\partial t}y(\cdot,0)=\hat{y}_0,\ U_1(y(\cdot,t))=U_2(y(\cdot,t))=0. \end{cases}$$

Here and throughout this paper let $p, w>0$, $p\in C^1[0,1]$, $w\in C[0,1]$. The initial conditions y_0, $\hat{y}_0$ and boundary conditions U_1, U_2 will be given later. The system (1) has been investigated by many authors. We mention the works [1], [2] of I. Joó who considered the simplest boundary conditions $U_1 z:=z(0)=0$; $U_2 z:=z(1)=0$, and the paper [3] of M. Horváth where the case of strongly regular boundary conditions is studied. In [4] the authors considered the analogous problem for the infinite string. Further references on this topic can be found in [1]—[4].

The aim of this paper is to extend the investigations of I. Joó and M. Horváth to some family of boundary conditions. As a theorem of Mihajlov and Kesselman states the eigenfunctions of a boundary value problem with strongly regular boundary conditions form a Riesz basis in $L^2(0,1)$, and M. Horváth in [3] proved an analogous result for the derivatives of the eigenfunctions. The boundary conditions (i)—(v), which will be defined and studied in this work ensures not only Riesz basis property but orthogonality of the eigenfunctions and its derivatives with respect to some weight and the boundary condition (i) is not necessarily strongly regular. So in some sense the mentioned orthogonality relations extend the Mihajlov—Kesselman theorem.

In the study of the system (1) using the Fourier method as it was used e.g.] in [1]—[4], we have to introduce the following system of eigenfunctions:

$$(2)\qquad (pu_k')'+\lambda_k w u_k = 0, \quad U_1(u_k)=U_2(u_k)=0.$$

If the boundary conditions U_1, U_2 are given such that for any function $u, v\in C^2[0,1]$

$$(3)\qquad U_i(u)=U_i(v)=0, \quad i=1,2 \quad \text{implies} \quad [pu'v]_0^1=0$$

then we can obtain the following observation

LEMMA 1. *If* (3) *holds, there is a system* (u_k) *of eigenfunctions of* (2) *forming a complete orthonormal sequence in* $L_w^2(0,1)$ *such that the system* $(u_k'\lambda_k^{-1/2},\ \lambda_k\neq 0)$ *is an orthonormal sequence in* $L_p^2(0,1)$. *The eigenvalues* λ_k *are nonnegative and to each eigenvalue there corresponds one or two eigenvectors* u_k.

PROOF. The operator $Lu=(pu')'$ is (formally) selfadjoint and the boundary conditions are also selfadjoint with respect to L since for any u, v in the domain of the boundary value problem (2) we have

$$\int_0^1 (Lu)v = [pu'v-puv']_0^1+\int_0^1 u(Lv) = \int_0^1 u(Lv).$$

Now the existence of a complete orthonormal sequence (u_n) of eigenfunctions of (2) is the special case of [5, II. 5.2] or of [6, II. Kap. I, 2.6] (the crucial point is the completeness). Since for $\lambda_n\neq 0$ we have

$$\delta_{n,k} = \int_0^1 u_n u_k w = -\frac{1}{\lambda_n}\int_0^1 (pu_n')' u_k = \frac{1}{\lambda_n}\int_0^1 pu_n' u_k,$$

the orthonormality of $(u_k'\lambda_k^{-1/2})$ is also proved. The remaining statements of Lemma 1 are trivial.

REMARK 1. The following boundary conditions ensures the property (3):

(i) $$a_1 u'(0)+b_1 u'(1) = 0$$
$$b_1 p(0)u(0)+a_1 p(1)u(1) = 0$$

(ii) $$u'(0) = u'(1) = 0$$

(iii) $$u(0) = u'(1) = 0$$

(iv) $$u'(0) = u(1) = 0$$

(v) $$u(0) = u(1) = 0.$$

These boundary conditions are strongly regular (cf. [5]) except for the most important case (i) if $a_1=\pm b_1$ or $a_1 p(1)=\pm b_1 p(0)$ is also assumed.

Define

(4) $$H := \left\{h\in H^1(0,1): \sum |\lambda_k||c_k|^2<\infty, \quad c_k := \int_0^1 hu_k w\right\}.$$

By Lemma 1 we see that H consists precisely of those functions $h\in H^1(0,1)$ for which the expansion $\sum c_k u_k$ converges to h and $\sum c_k u_k'$ converges to h', both in L^2-norm. (Here we use the fact $L^2=L_w^2=L_p^2$.)

LEMMA 2. *In case of the boundary conditions* (i), (ii), (iii), (iv) *or* (v), *H is the set of $h\in H^1(0,1)$ satisfying those of the corresponding boundary conditions which do not contain derivatives.*

PROOF. Since for $\lambda_n\neq 0$ we have

$$c_n\sqrt{\lambda_n} = \sqrt{\lambda_n}\int_0^1 hu_n w = -\frac{1}{\sqrt{\lambda_n}}\int_0^1 h(pu_n')' = -\frac{1}{\sqrt{\lambda_n}}[pu_n'h]_0^1+\frac{1}{\sqrt{\lambda_n}}\int_0^1 h'pu_n',$$

which implies by Lemma 1 that $(c_n\sqrt{\lambda_n})\in l_2$ if and only if

$$\left(\frac{1}{\sqrt{\lambda_n}}[pu_n'h]_0^1\right)\in l_2. \tag{5}$$

Now if $h\in H^1(0,1)$ satisfies the mentioned boundary conditions then $(c_n\sqrt{\lambda_n})\in l_2$ since $[pu_n'h]_0^1=0$ for all n. This assertion is obvious in cases (ii)—(v); in case (i) we argue as follows. Since the vectors $(u_k(0), u_k(1))$ and $(h(0), h(1))$ are orthogonal to $(b_1p(0), a_1p(1))\neq 0$ hence

$$[pu_n'h]_0^1 = \text{const}\,[pu_n'u_k]_0^1 = 0$$

unless $u_k(0)=u_k(1)$ for all k; but this is impossible. Conversely, we show that any $h\in H$ satisfies the boundary conditions without those involving derivatives. It will be enough to prove that $\sum c_nv_n$ converges uniformly. Indeed, for any $a, b\in[0,1]$

$$\left|\sum_N^{N+M} c_nv_n(b)-\sum_N^{N+M} c_nv_n(a)\right| \leqq \varepsilon_N \to 0 \tag{6}$$

with $\varepsilon_N := \left\|\sum_N^{\infty} c_nv_n'\right\|_1$, not depending on a, b and M. If $\sum c_nv_n(a)$ is convergent then (6) proves the uniform convergence of the series $\sum c_nv_n$. If $\sum c_nv_n(a)$ is divergent i.e. if there are N_k, M_k and $\delta>0$ such that

$$\left|\sum c_nv_n(a)\right| \geqq \delta$$

then for any $b\in[0,1]$ we get

$$\left|\sum c_nv_n(b)\right| \geqq \delta-\varepsilon_{N_k}$$

which contradicts the L^2-convergence of $\sum c_nv_n$. Lemma 2 is proved.

Next we investigate the system (1) by the method used in [1], [2], [3]. Introduce the following

DEFINITION. Let

$$(y_0, \hat{y}_0)\in Z := H\oplus L^2(0,1). \tag{7}$$

The solution of the system (1) is a function

$$y(x,t)\in L^2((0,1)X(0,T))$$

for which

$$\int_0^T z(a,t)u(t)\,dt = \int_0^1\int_0^T y(x,t)\{z_{tt}(x,t)w(x)-[p(x)z_x(x,t)]_x\}\,dt\,dx + \tag{8}$$

$$+\int_0^1 y_0(x)z_t(x,0)w(x)\,dx-\int_0^1 \hat{y}_0(x)z(x,0)w(x)\,dx$$

holds for all functions

$$z(x,t)\in G^2([0,1]\times[0,T])$$

satisfying

$$z(x,T)=z_t(x,T)=0,\quad U_1(z(x,T))=U_2(z(x,T))=0. \tag{9}$$

THEOREM 1. *Suppose that*

$$\left|\{n: k \leqq \sqrt{\lambda_n} < k+1\}\right| \leqq c, \quad u_n(a) = O(1). \tag{10}$$

Then for any given $(y_0, \hat{y}_0) \in Z$, $u \in L^2(0, T)$ *the system* (1) *has a unique solution* $y(x, t)$, *it belongs to* $H^1((0, 1)\times(0, T))$ *and has the expansion*

$$y(x, t) = \sum c_n(t) u_n(x)$$

convergent in $L^2((0, 1)\times(0, T))$. *Further the coefficients* $c_n(t)$ *have the form* (13), (13′) *below and*

$$y_x(x, t) = \sum c_n(t) u_n'(x), \quad y_t(x, t) = \sum c_n'(t) u_n(x),$$

both series being convergent in $L^2((0, 1)\times(0, T))$. *For each* $0 \leqq t \leqq T$ *we have*

$$A_t := (y(\cdot, t), y_t(\cdot, t)) \in Z$$

and the trajectory A_t $t \leqq T$ *is a continuous in* Z.

PROOF. We sketch it only, because it is similar to that of the corresponding results of [1], [2], [3]. Define the function

$$z(x, t) = u_n(x) b(t) \tag{11}$$

where $b \in C^2[0, T]$, $b(T) = b'(T) = 0$. If we set

$$y_0 =: \sum c_n^0 u_n, \quad \hat{y}_0 =: \sum \hat{c}_n^0 u_n \tag{12}$$

then putting (11) in (8) we get for the function $c_n(t)$ the second order boundary value problem

$$u_n(a) \int_0^T b(t) u(t)\, dt = \int_0^T c_n(t)[b''(t) + \lambda_n b(t)]\, dt - b'(0) c_n^0 + b(0) \hat{c}_n^0.$$

Its solution is

$$c_n(t) = c_n^0 \cos \sqrt{\lambda_n} t + \hat{c}_n^0 \frac{\sin \sqrt{\lambda_n} t}{\sqrt{\lambda_n}} + u_n(a) \int_0^t u(\tau) \frac{\sin \sqrt{\lambda_n}(t-\tau)}{\sqrt{\lambda_n}}\, d\tau \tag{13}$$

or

$$c_n(t) = c_n^0 + \hat{c}_n^0 t + u_n(a) \int_0^t u(\tau)(t-\tau)\, d\tau \quad \text{if} \quad \lambda_n = 0.$$

It can be written in the more compact form

$$c_n'(t) \pm i\sqrt{\lambda_n} c_n(t) = e^{\pm i \sqrt{\lambda_n} t} \left[\hat{c}_n^0 \pm i \sqrt{\lambda_n} c_n^0 + u_n(a) \int_0^t u(\tau) e^{\mp i \sqrt{\lambda_n} \tau}\, d\tau\right]. \tag{13′}$$

Using a theorem of Avdonin [7] on the stability of exponential bases, we can easily verify that if $\left|\{n: k \leqq \sqrt{\lambda_n} < k+1\}\right| \leqq c$, $c \in \mathbf{N}$, then the set $\{\pm\sqrt{\lambda_n}\}$ of exponents can be extended to a set $\{\mu_k\} \supset \{\pm\sqrt{\lambda_n}\}$ such that the system $\{e^{i\mu_k t}\}$ forms a Riesz basis in $L^2(0, 2\pi c)$. With respect to (13′) this means that

$$\sum \left| \int_0^t u(\tau) e^{Fi\sqrt{\lambda_n}\tau}\, d\tau \right|^2 \leqq C_T \|u\|_2 \quad (t < T)$$

and hence

$$\int_0^T \sum \left(|c_n'(t)|^2+\lambda_n |c_n(t)|^2\right) <\infty. \tag{14}$$

If we define the function $y(x, t)$ by the expansion

$$y(x, t) = \sum c_n(t) u_n(x)$$

where $c_n(t)$ is the function (13), then by (14) we see that this expansion converges in $L^2((0, 1)\times(0, T))$ and for fixed t it converges uniformly; so $y(x, t)$ is everywhere defined. Further the series $\sum c_n(t)u_n'(x)$ and $\sum c_n'(t)u_n(x)$ converge in $L^2((0, 1)\times(0, T))$ hence $y\in H^1((0, 1)\times(0, T))$ and $y_t(x, t)=\sum c_n'(t)u_n(x)$, $y_x(x, t)=$ $=\sum c_n(t)u_n'(x)$. The assertion $A_t\in Z$ and the continuity of the trajectory $t\mapsto A_t$ again follows from (14). Hence it remains to verify that the function $y(x, t)$ so defined is a solution of (1) in our sense, i.e. that (8) is satisfied for every $z(x, t)$ of type (9). By the construction of $y(x, t)$ (8) fulfils for $z(x, t)$ in (11). Now expanding any $z(x, t)$ in (9) with respect to (u_n) we can show using limit arguments that (8) fulfils also in this case. The details are the same as in [1] or [3], so we omit them. Theorem 1 is proved.

Next we prove a controllability result. Define the reachability set as follows:

$$\mathscr{D}_a(T) = \left\{(y(\cdot, T), y_t(\cdot, T)):\ u\in L^2(0, T) \quad \text{for any fixed} \quad (y_0, \hat{y}_0)\in Z\right\}.$$

THEOREM 2. *Suppose* (10) *and that* $\lambda_n\neq 0$ *for all* n. *Then any point* $a\in(0, 1)$, *except for a denumerable set, has the following property. For any fixed initial condition* $(y_0, \hat{y}_0)\in Z$ *and for each* $T\geqq 2\pi c$ *the sets* $\mathscr{D}_a(T)$ *are dense in* Z *and*

$$\mathscr{D}_a(T) = \mathscr{D}_a(2\pi c).$$

PROOF. The mapping

$$I:\ H\oplus L^2(0, 1) \to l_2,$$

$$\left(\sum c_n u_n, \sum \hat{c}_n u_n\right) \mapsto \left(\hat{c}_n + i\sqrt{\lambda_n} c_n\right)$$

is an isomorphism (onto l_2), so we can investigate the properties of

$$\mathscr{B}_a(T) := I\mathscr{D}_a(T) = \left\{\left(c_n'(T)+i\sqrt{\lambda_n} c_n(T)\right):\ u\in L^2(0, T)\right\}$$

instead of $\mathscr{D}_a(T)$.

It is well-known that for any $p, w\in C[0, 1]$, $p>0$, $w>0$ any function $u(x)$ satisfying

$$(pu')'+\lambda wu = 0 \quad \text{on} \quad [0, 1]$$

has only finitely many zeros; it follows e.g. from the Sturm oscillation theorem of the form given in [8, Ch. XI, Corollary 3.1]. We exclude from $[0, 1]$ the denumerable set of the zeros of every u_n; the other points $a\in[0, 1]$ satisfy $u_n(a)\neq 0$ for all n. Since the set $(e^{\pm i\sqrt{\lambda_n} x})$ is a Riesz basis in its closed linear hull in $L^2(0, 2\pi c)$, we have obviously for $T\geqq 2\pi c$

$$\mathscr{B}_a(T) = \left\{\left(u_n(a) b_n\right):\ (b_n)\in l_2\right\}$$

and this formula implies the density of $\mathscr{B}_a(T)$ in l_2. Theorem 2 is proved.

In Theorems 1 and 2 we observed that the controllability investigations of the system (1) depend on (10) i.e. on the uniform estimations of the eigenfunctions and on the distribution of the eigenvalues. In the case when p and w are sufficiently smooth $(p, w \in C^2[0, 1])$ then (2) can be reduced to a boundary value problem with some Schrödinger equation $v''+qv+\lambda v=0$ by the substitution

$$v := (u\circ\varphi)((p\circ\varphi)(w\circ\varphi))^{1/4} \tag{15}$$

where $\varphi := f^{-1}$, $f(x) := \int_0^x \left(\frac{w}{p}\right)^{1/2}$; and we know the asymptotic distribution of the eigenvalues λ_n [5, Kap. II, § 4. Theorem 9.2] and hence that of the eigenfunctions.

Now we shall prove (10) for not necessarily smooth functions p, w. Our result is of spectral theoretic character, we continue the investigations of Joó [9], [10]. We have to assume $p=w$.

LEMMA 3. *Let $p\in C^1[0, 1]$, $p>0$ and consider any solution $u(x)$ of the problem*

$$(pu')'+\lambda pu = 0.$$

Then in case $\lambda>0$ we have

a) $\|u'\|_\infty \leqq c\|u'\|_2$

b) $\|u'\|_\infty \leqq c(1+\sqrt{\lambda})\|u\|_\infty$

c) $\|u\|_\infty \leqq c\|u\|_2$.

PROOF. We have

$$\lambda \int_{x-t}^{x+t} u(\xi)\frac{\sin\sqrt{\lambda}(t-|x-\xi|)}{\sqrt{\lambda}}\,d\xi =$$

$$= -\int_{x-t}^{x+t} [p(\xi)u'(\xi)]'\frac{1}{p(\xi)}\frac{\sin\sqrt{\lambda}(t-|x-\xi|)}{\sqrt{\lambda}}\,d\xi =$$

$$= -\int_{x-t}^{x+t} \frac{p'(\xi)}{p(\xi)}u'(\xi)\frac{\sin\sqrt{\lambda}(t-|x-\xi|)}{\sqrt{\lambda}}\,d\xi +$$

$$+\int_{x-t}^{x+t} u'(\xi)\cos\sqrt{\lambda}(t-|x-\xi|)\operatorname{sgn}(\xi-x)\,d\xi.$$

Another integration by parts in the last term (separately on $\int_{x-t}^{x}$ and $\int_{x}^{x+t}$) gives a Titchmarsh type formula

$$u(x+t)+u(x-t) = 2u(x)\cos\sqrt{\lambda}t + \tag{16}$$

$$+\int_{x-t}^{x+t} \frac{\sin\sqrt{\lambda}(t-|x-\xi|)}{\sqrt{\lambda}}\frac{p'(\xi)}{p(\xi)}u'(\xi)\,d\xi.$$

Differentiating in x we get

$$u'(x+t)+u'(x-t) = 2u'(x)\cos\sqrt{\lambda}t+ \tag{16'}$$

$$+\int_{x-t}^{x+t} \operatorname{sgn}(\xi-x)\cos\sqrt{\lambda}(t-|x-\xi|)\frac{p'(\xi)}{p(\xi)}u'(\xi)\,d\xi.$$

a) Substitute $x+t$ instead of x in (16') to get

$$|u'(x)| \leqq |u'(x+2t)|+2|u'(x+t)|+\left\|\frac{p'}{p}\right\|_{L^1(x,x+2t)}\cdot\|u'\|_\infty.$$

Integrate this in t from 0 to R; then

$$R|u'(x)| \leqq \sqrt{R}\|u'\|_2+2\sqrt{R}\|u'\|_2+R\left\|\frac{p'}{p}\right\|_{L^1(x,x+2R)}\cdot\|u'\|_\infty.$$

$$|u'(x)| \leqq \frac{3}{\sqrt{R}}\|u'\|_2+\left\|\frac{p'}{p}\right\|_{L^1(x,x+2R)}\cdot\|u'\|_\infty.$$

Now if R is so small that

$$\max_x\left\|\frac{p'}{p}\right\|_{L^1(x,x+2R)} \leqq \frac{1}{2}$$

then

$$\|u'\|_\infty \leqq \frac{6}{\sqrt{R}}\|u'\|_2.$$

b) Integrate (16) in t from 0 to R. Then

$$2|u'(x)|\left|\frac{\sin\sqrt{\lambda}R}{\sqrt{\lambda}}\right| \leqq 4\|u\|_\infty+R\left\|\frac{p'}{p}\right\|_{L^1(x-R,x+R)}\cdot\|u'\|_\infty.$$

Fix R such that

$$\max_x\left\|\frac{p'}{p}\right\|_{L^1(x-R,x+R)} \leqq \frac{1}{10}.$$

Now if $\sqrt{\lambda}R \leqq 2$ then $\left|\frac{\sin\sqrt{\lambda}R}{\sqrt{\lambda}R}\right| > \frac{1}{2}$, consequently

$$|u'(x)|\,R \leqq 4\|u\|_\infty+R\left\|\frac{p'}{p}\right\|_{L^1(x-R,x+R)}\cdot\|u'\|_\infty$$

which gives

$$\|u'\|_\infty \leqq \frac{8}{R}\|u\|_\infty \leqq c(1+\sqrt{\lambda})\|u\|_\infty.$$

If $\sqrt{\lambda}R > 2$ then from the identity

$$|\sin z|^2 = \operatorname{sh}^2(\operatorname{Im} z)+\sin^2(\operatorname{Re} z)$$

we see that there exists some $t_0<\pi|\lambda|^{-1/2}$ such that $|\sin\sqrt{\lambda}t_0|>3^{-1}$. Hence

$$\frac{2\|u'\|_\infty}{3\sqrt{\lambda}} \leqq 4\|u\|_\infty + t_0 \left\|\frac{p'}{p}\right\|_{L^1(x-t_0,\,x+t_0)} \cdot \|u'\|_\infty$$

implies

$$\|u'\|_\infty \leqq 8(1+\sqrt{\lambda})\|u\|_\infty$$

also in this case.

c) From (16) we see

$$|u(x)| \leqq |u(x+2t)|+2|u(x+t)|+$$

$$+\left|\int_x^{x+2t} \frac{\sin\sqrt{\lambda}(t-|x+t-\xi|)}{\sqrt{\lambda}} \frac{p'(\xi)}{p(\xi)} u'(\xi)\,d\xi\right| \leqq$$

$$\leqq |u(x+2t)|+2|u(x+t)|+\frac{1}{\sqrt{\lambda}}\left\|\frac{p'}{p}\right\|_{L^1(x,\,x+2t)} \cdot \|u'\|_\infty \leqq$$

$$\leqq |u(x+2t)|+2|u(x+t)|+C\left\|\frac{p'}{p}\right\|_{L^1(x,\,x+2t)} \cdot \|u\|_\infty .$$

Integrate in t from 0 to R; then

$$R\|u\|_\infty \leqq 3\sqrt{R}\|u\|_2 + RC\left\|\frac{p'}{p}\right\|_{L^1(x,\,x+2R)} \cdot \|u\|_\infty .$$

If we choose $R>0$ such that

$$C\max_x \left\|\frac{p'}{p}\right\|_{L^1(x,\,x+2R)} < \frac{1}{2}$$

then we get

$$\|u\|_\infty \leqq 6R^{-1/2}\|u\|_2 .$$

Lemma 3 is proved

THEOREM 3. *Let* $p\in C^1[0,1]$, $p>0$ *and consider the system of eigenfunctions*

$$(pu_n')'+\lambda_n pu_n = 0$$

with one of the boundary conditions (i) *to* (v). *Then*

a) $\sup\limits_{\mu>0} \sum\limits_{|\sqrt{\lambda_n}-\mu|\leqq 1} \|u_n\|_\infty^2 < \infty$,

b) $\sup\limits_{k\in\mathbf{N}} \left|\{n\colon k^2 \leqq \lambda_n < (k+1)^2\}\right| < \infty$.

PROOF. Let $0\leqq x\leqq 2^{-1}$ be fixed (the case $2^{-1}\leqq x\leqq 1$ is analogous). Define the function

$$w(y) := \begin{cases} 2\cos\mu(y-x)-2^{-1} & \text{if} \quad x\leqq y\leqq x+R \\ -2^{-1} & \text{if} \quad x+R<y\leqq x+2R \\ 0 & \text{otherwise,} \end{cases}$$

where $R:=R(p)>0$ is to be specified later. Suppose that $|\sqrt{\lambda_n}-\mu|\leqq 1$ then

$$|\cos\sqrt{\lambda_n}t-\cos\mu t| \leqq Ct \quad (0<t<R).$$

From (16) and (16′) we get, taking into consideration the relation

$$\int_0^R [2u_n(x+t)\cos \mu t - u_n(x+2t)]\,dt = \langle u_n, w\rangle$$

that

$$R|u_n(x)| \leqq |\langle u_n, w\rangle| + CR^2\|u_n\|_\infty + \frac{R}{\sqrt{\lambda_n}}\left\|\frac{p'}{p}\right\|_{L^1(x,x+2R)} \cdot \|u_n'\|_\infty$$

and

$$\frac{R|u_n'(x)|}{\sqrt{\lambda_n}} \leqq \frac{|\langle u_n', w\rangle|}{\sqrt{\lambda_n}} + CR^2\frac{\|u_n'\|_\infty}{\sqrt{\lambda_n}} + R\left\|\frac{p'}{p}\right\|_{L^1(x,x+2R)} \cdot \frac{\|u_n'\|_\infty}{\sqrt{\lambda_n}},$$

$$R\left\{|u_n(x)| + \frac{|u_n'(x)|}{\sqrt{\lambda_n}}\right\} \leqq |\langle u_n, w\rangle| + \frac{|\langle u_n', w\rangle|}{\sqrt{\lambda_n}} +$$

$$+ CR^2\left(\|u_n\|_2 + \frac{\|u_n'\|_2}{\sqrt{\lambda_n}}\right) + R\left\|\frac{p'}{p}\right\|_{L^1(x,x+2R)} \cdot \left(\|u_n\|_2 + \frac{\|u_n'\|_2}{\sqrt{\lambda_n}}\right).$$

Using repeatedly the inequalities

$$a^2 + b^2 \leqq (a+b)^2 \leqq 2(a^2+b^2) \quad (a, b) \geqq 0)$$

we get that

$$C_1R^2\left(|u_n(x)|^2 + \frac{|u_n'(x)|^2}{\lambda_n}\right) \leqq |\langle u_n, w\rangle|^2 + \frac{|\langle u_n', w\rangle|^2}{\lambda_n} +$$

$$+ C_2R^4\left(\|u_n\|_2^2 + \frac{\|u_n'\|_2^2}{\lambda_n}\right) + C_3R^2\left\|\frac{p'}{p}\right\|_{L^1(x,x+2R)}^2 \cdot \left(\|u_n\|_2^2 + \frac{\|u_n'\|_2^2}{\lambda_n}\right).$$

Take any finite set

$$I \subset I_\mu := \{u\colon |\mu - \sqrt{\lambda_n}| \leqq 1\}.$$

Since (u_n) and $(u_n'\lambda_n^{-1/2})$ are orthonormal systems in L_p^2 and the L_p^2-norm (so the L^2-norm) of w have an upper bound independent of x, we have

$$\sum_{n\in I}\left(|\langle u_n, w\rangle|^2 + \frac{|\langle u_n', w\rangle|^2}{\lambda_n}\right) \leqq C_4$$

consequently

$$C_1R^2\sum_{n\in I}\left(|u_n(x)|^2 + \frac{|u_n'(x)|^2}{\lambda_n}\right) \leqq C_4 + \frac{C_1}{2}R^2\sum_{n\in I}\left(\|u_n\|_2^2 + \frac{\|u_n'\|_2^2}{\lambda_n}\right)$$

for sufficiently small $R = R(C_1, C_3, p)$. Integration in x from 0 to 1 gives

$$\sum_{n\in I}\left(\|u_n\|_2^2 + \frac{\|u_n'\|_2^2}{\lambda_n}\right) \leqq \frac{2C_4}{C_1R^2}.$$

Consequently by Lemma 3

$$\sum_{|\sqrt{\lambda_n}-\mu|\leqq 1}\|u_n\|_\infty^2 \leqq C_5\sum_{n\in I_\mu}\|u_n\|_2^2 = O(1).$$

b) From a) we know that

$$\sum_{|\sqrt{\lambda_n}-\mu|\leq 1} p(x)\,|u_n(x)|^2 \leq C$$

independently of μ and x. Integrating in x we get

$$\sum_{|\sqrt{\lambda_n}-\mu|\leq 1} 1 \leq C$$

which completes the proof of Theorem 3.

COROLLARY. *If* $p=w\in C^1[0,1]$, $p>0$ *and* (u_n) *is the system of eigenfunctions of*

$$(pu_n')'+\lambda_n p u_n = 0$$

subject to one of the boundary conditions (i) *to* (v), *then* (10) *is satisfied, so the assertion of Theorem* 2 *is valid.*

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ON THE EQUATION $f(x)=y^m$ OVER FINITELY GENERATED DOMAINS

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1. Introduction

The purpose of this paper is to give effective results for the solutions of the hyperelliptic equation in integral domains of finite type over **Z**. Equations of this type play an important role in the theory of diophantine equations and its applications. C. L. Siegel [10], [11] and W. J. LeVeque [7] obtained general ineffective finiteness theorems for the integer solutions of this equation over algebraic number fields. For an ineffective extension to the case when the ground ring is an arbitrary integral domain of finite type over **Z** see S. Lang [6]. In the number field case A. Baker [1] proved the first general effective result for hyperelliptic equations. Baker's theorem was later improved and generalized by V. G. Sprindzuk [12], L. A. Trelina [14] and B. Brindza [2].

W. M. Schmidt [13], R. C. Mason [8] and R. C. Mason and B. Brindza [9] gave effective bounds for the heights of the S-integral solutions of hyperelliptic equations considered over function fields.

In [3], [4] and [5] K. Győry proved several general effective results for Thue's equation, norm form, discriminant form and index form equations when the ground ring is an arbitrary finitely generated integral domain over **Z**. In this general case the hyperelliptic equation can not be reduced to the Thue's equation directly because there are no appropriate effective results on units and unit equations. By using Győry's method we shall reduce the hyperelliptic equation considered to hyperelliptic equations over algebraic number fields.

In order to be able to state our result we introduce some concepts and notation. Let G be a finitely generated extension field of the rational number field **Q**. Then G can be written in the form

$$G = \mathbf{Q}(z_1, \ldots, z_q, u)$$

where $\{z_1, \ldots, z_q\}$ is a transcendence basis of G over **Q** and u is integral over the polynomial ring $\mathbf{Z}[z_1, \ldots, z_q]$.

Let

$$F(X) = X^\delta + F_1 X^{\delta-1} + \ldots + F_\delta$$

be the minimal polynomial of u over $\mathbf{Q}(z_1, \ldots, z_q)$. For brevity let us write

$$D = \max_i \deg F_i, \quad \mathscr{F} = \max_i L(F_i)$$

where $L(P)$ denotes the length of the polynomial $P \in \mathbf{Z}[z_1, \ldots, z_q]$ (i.e. the sum of the absolute values of the coefficients of P). Since $\mathbf{Z}[z_1, \ldots, z_q]$ is integrally closed, all F_i lie in $\mathbf{Z}[z_1, \ldots, z_q]$.

Any element α of G can be written in the form

$$\alpha = \frac{P_0 + P_1 u + \ldots + P_{\delta-1} u^{\delta-1}}{Q} \tag{1}$$

where $P_0, \ldots, P_{\delta-1}$ and Q are uniquely determined (up to the factors ± 1) relatively prime polynomials from $\mathbf{Z}[z_1, \ldots, z_q]$. Let $\operatorname{Deg} P$ denote the total degree in $z_1, \ldots, z_q$ of an element P of $\mathbf{Z}[z_1, \ldots, z_q]$ and define the *degree* of α as

$$\operatorname{Deg} \alpha = \max \{\operatorname{Deg} P_0, \ldots, \operatorname{Deg} P_{\delta-1}, \operatorname{Deg} Q\}.$$

The *size* of $\alpha \neq 0$ (with respect to the generating set $\{z_1, \ldots, z_q, u\}$ of G) will be defined as

$$s(\alpha) = \max \{s(P_0), \ldots, s(P_{\delta-1}), s(Q)\}$$

where

$$s(P) = \max \left\{\log H(P), \ 1 + \max_i \deg_{z_i} P\right\}$$

and $H(P)$ denotes the height of $P \in \mathbf{Z}[z_1, \ldots, z_q]$ (i.e. the maximum of the absolute values of the coefficients of P), (cf. [4]). It is clear that there are only finitely many elements in G with bounded size. Let $R = \mathbf{Z}[u_1, \ldots, u_t]$ be a finitely generated subring of G. Further, let $f \in G[X]$ be a polynomial having all its roots in G. We assume throughout that

$$f(X) = a \prod_{i=1}^{n} (X - \alpha_i)^{r_i}$$

with $a \neq 0$, $n > 0$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. Let $m > 1$ be an integer and consider the equation

$$f(x) = y^m \quad \text{in} \quad x, y \in R. \tag{2}$$

To avoid some techniqual difficulties, we shall consider the solutions of (2) in the larger subring

$$R_1 = \mathbf{Z}\left[z_1, \ldots, z_q, \frac{1}{b}, u\right]$$

of G where b denotes the product of the denominators of $a, \alpha_1, \ldots, \alpha_n, u_1, \ldots, u_t$. It is easy to see that $b \in \mathbf{Z}[z_1, \ldots, z_q]$ and $R \subseteq R_1$. In the special case when $q = 0$ and $\{u_1, \ldots, u_t\}$ is an integral basis of G is just the ring of integers of G.

We shall give an effectively computable bound for the *degree* and *size* of the solutions x, y in R_1 which depend only on G, R_1, f and m. This will imply the finiteness of the number of solutions. Further, if $q, \delta, F_1, \ldots, F_\delta$ and the elements $u_1, \ldots, u_t$ are effectively given (cf. [3] or [4]) then our result make it possible to determine all the solutions of the equation (2).

THEOREM. *Put* $t_i = m/(m, r_i)$, $i = 1, \ldots, n$, *and suppose that* $\{t_1, \ldots, t_n\}$ *is not a permutation of the n-tuples*

$$\text{(a)} \quad \{t, 1, \ldots, 1\} \quad \text{and} \quad \text{(b)} \quad \{2, 2, 1, \ldots, 1\}.$$

Then all solutions $x, y \in R_1$ *of the equation* (2) *satisfy*

$$\max\{s(x), s(y)\} < C_1$$

where C_1 *is an effectively computable constant depending only on* G, R_1, $f(X)$ *and* m.

We note that the assumption concerning $\{t_1, \ldots, t_n\}$ is in general necessary (cf. [7]).

2. Auxiliary results

Let k be an algebraically closed field of characteristic 0, and let $\mathbf{K}$ be an algebraic extension of the rational function field $k(z)$ with genus $g(\mathbf{K})$. Let $\mathcal{S}$ be a finite set of (additive) valuations of $\mathbf{K}/k$ with value group $\mathbf{Z}$. An element $\alpha \in \mathbf{K}$ is called S-integral if $v(\alpha) \geqq 0$ for every valuation $v \notin \mathcal{S}$. For $x \neq 0$ in $\mathbf{K}$, the height $H_{\mathbf{K}}(x)$ of x is defined by

$$H_{\mathbf{K}}(x) = -\sum_v \min\{0, v(x)\}$$

where v runs through the valuations of $\mathbf{K}/k$ with value group $\mathbf{Z}$. Let $P(X) \in \mathbf{K}[X]$ and suppose that $P(X)$ has the representations

$$P(X) = P\prod_{i=1}^{n}(X-\beta_i)^{r_i}$$

(in the splitting field of $P(X)$), where $n>0$ and $\beta_i \neq \beta_j$ for $i \neq j$. Further, let $m>1$ be a fixed natural number and we put again $t_i = m/(m, r_i)$, $i=1, \ldots, n$.

LEMMA 1. (R. C. Mason, B. Brindza [9].) *Suppose that* $\{t_1, \ldots, t_n\}$ *is not a permutation of the n-tuples* (a) *and* (b). *Then all S-integral solutions* x, y *of the equation* $P(x)=y^m$ *satisfy*

$$\max\{H_{\mathbf{K}}(x), H_{\mathbf{K}}(y)\} \leqq 78H_{\mathbf{K}}(P)+12g(\mathbf{K})+6|\mathcal{S}|$$

where $H_{\mathbf{K}}(P)$ *is the maximum of the heights of the coefficients of* P *and* $|\mathcal{S}|$ *denotes the cardinality of* $\mathcal{S}$.

We note that the above result have been proved by using Mason's powerful inequality [8]. In case $m=2$ a similar result was obtained by W. M. Schmidt [13] and he gave upper bound for the heights of solutions of Thue's equation even in the more general case when the unknowns are arbitrary elements of $\mathbf{K}$.

Now let $\mathbf{L}$ be an algebraic number field and let

$$g_0\prod_{i=1}^{n}(X-\gamma_i)^{r_i} = g(X) \in \mathbf{L}[X]$$

be a given polynomial where $n>0$ and $\gamma_i \neq \gamma_j$ for $i \neq j$. Further, let $\mathfrak{P}_1, \ldots, \mathfrak{P}_s$ ($s \geqq 0$) be distinct prime ideals in $\mathbf{L}$. Write $\mathcal{P}=\max_i N\mathfrak{P}_i$ (with $\mathcal{P}=1$ if $s=0$) and let $\mathcal{S}$ denote the set of all (additive) valuations of $\mathbf{L}$ corresponding to $\mathfrak{P}_1, \ldots, \mathfrak{P}_s$. We recall that an element α of $\mathbf{L}$ is said to be S-integral if $v(\alpha) \geqq 0$ for all valuations $v \notin \mathcal{S}$. The following Lemma is an effective version of LeVeque's theorem [7]. Using the above notation we have

LEMMA 2. (B. Brindza [2].) *Suppose that* $\{t_1, \ldots, t_n\}$ *is not a permutation of the n-tuples* (a) *and* (b). *Then all S-integral solutions* x, y *of the equation* $g(x)=y^m$ *satisfy*

$$\max\{H(x), H(y)\} < \exp\exp\{C_2 P^2 (s+1)^2\}$$

where C_2 *is an effectively computable constant depending only on* $g(X)$, $\mathbf{L}$ *and* m.*

By using Győry's method we shall reduce the proof of our Theorem to Lemma 2. We note here that Lemmas 3 and 5 below of Győry are implicitly proved also in Győry [3] and [4].

Now we suppose that $q>0$ and let

$$k_i = \mathbf{Q}(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_q)$$

for an arbitrary but fixed index i with $1 \leqq i \leqq q$. Further, let $\bar{k}_i$ denote its algebraic closure (in an fixed algebraic closure of G). Moreover, we write z for z_i. If ω is an element of the polynomial ring $\mathbf{Q}[z_1, \ldots, z_q]$ then denote by ω^* the element of $k_i[z]$ obtained from ω by substituting $z_i \mapsto z$. Let $u^{(1)}=u, \ldots, u^{(\delta)}$ be the conjugates of u over $\mathbf{Q}(z_1, \ldots, z_q)$ and let

$$M_i = \bar{k}_i(z)(u^{(1)}, \ldots, u^{(\delta)}).$$

Set $\Delta_i=[M_i: \bar{k}_i(z)]$ and denote by $g(M_i)$ the genus of $M_i/\bar{k}_i$. Then $\Delta_i \leqq \delta!$. Since $u^{(1)}, \ldots, u^{(\delta)}$ are integral over $\mathbf{Z}[z_1, \ldots, z_q]$, hence they are also integral over $\bar{k}_i[z]$; i.e. $v(u^{(j)}) \geqq 0$ $(1 \leqq j \leqq \delta)$ for every finite valuation v of $M_i/\bar{k}_i$.

We have seen that any element α of G can be written in the form

$$\alpha = \frac{P_0 + P_1 u + \ldots + P_{\delta-1} u^{\delta-1}}{Q}$$

where $P_0, \ldots, P_{\delta-1}$ and Q are relatively prime polynomials from $\mathbf{Z}[z_1, \ldots, z_q]$. Let

$$\alpha^{(j)} = \frac{P_0^* + P_1^* u^{(j)} + \ldots + P_{\delta-1}^* (u^{(j)})^{\delta-1}}{Q^*}$$

for $j=1, \ldots, \delta$. Then $\alpha^{(1)}, \ldots, \alpha^{(\delta)} \in M_i$ and the Degree of α with respect to the generating set $\{z_1, \ldots, z_q, u\}$ is equal to the Degree of $\alpha^{(j)}$ with respect to the generating set $\{z_1, \ldots, z_q, u^{(j)}\}$ of G.

LEMMA 3. (K. Győry [5].) *If* $\alpha \in R_1$ *then*

$$\operatorname{Deg} \alpha < C_3 + C_4 \max_j H_{M_i}(\alpha^{(j)})$$

where C_3 *and* C_4 *are effectively computable constants depending only on* G *and* R_1.

We assume again that $q>0$ and let $P \in \mathbf{Z}[z_1, \ldots, z_q]$ be an arbitrary polynomial of degree less then d in each z_i for some positive integer d. Then the non-zero coefficients of the polynomial

$$\Phi_P(t) = P(t, t^d, \ldots, t^{d^{q-1}})$$

* By the height $H(\alpha)$ of an algebraic number α we mean the height of the minimal defining polynomial of α with relatively prime rational integer coefficients.

are equal to the coefficients of P and $\deg \Phi_P(t)<d^q$. Further, if $Q\in\mathbf{Z}[z_1, \ldots, z_q]$ and

$$d > \max_i \{\deg_{z_i} Q, \deg_{z_i}(PQ)\}$$

then $\Phi_{P+Q}=\Phi_P+\Phi_Q$ and $\Phi_{PQ}=\Phi_P\cdot\Phi_Q$ (cf. [3] or [4]).

Let D_F denote the discriminant of $F(X)$. It is easy to see that $\operatorname{Deg} D_F\leqq 2(\delta-1)D$. Supposing

$$d > \max\{\operatorname{Deg} D_F, \operatorname{Deg} b\},$$

$\Phi_{D_F}(t)$ and $\Phi_b(t)$ are non-vanishing polynomials and

$$\max\{\deg \Phi_{D_F}(t), \deg \Phi_b(t)\} < d^q.$$

Let t_0 be a rational integer such that $\Phi_{D_F}(t_0)\Phi_b(t_0)\neq 0$ and let $U_1, \ldots, U_\delta(\in\mathbf{C})$ be the roots of the polynomial

$$F_{t_0}(X) = X^\delta+\Phi_{F_1}(t_0)X^{\delta-1}+\ldots+\Phi_{F_\delta}(t_0)\in\mathbf{Z}[X].$$

Since $\Phi_{D_F}(t_0)\neq 0$ hence $U_i\neq U_j$ for $i\neq j$. Put $M_{t_0}=\mathbf{Q}(U_1, \ldots, U_\delta)$. The substitution

$$z_1 \mapsto t_0, \ldots, z_q \mapsto t_0^{d^{q-1}}, \quad u \mapsto U_k$$

define a homomorphic mapping of the ring R_1 into the algebraic number field M_{t_0}, and

$$\Phi_\alpha(t_0, k) = \frac{\Phi_{P_0}(t_0)+\Phi_{P_1}(t_0)U_k+\ldots+\Phi_{P_{\delta-1}}(t_0)U_k^{\delta-1}}{\Phi_Q(t_0)} \tag{3}$$

will be the image of $\alpha\in R_1$ under this mapping ($\Phi_Q(t_0)$ is different from zero because of $\Phi_b(t_0)\neq 0$).

If

$$P(X) = a_nX^n+\ldots+a_0 = a_n\prod_{i=1}^{n}(X-\alpha_i)$$

is the defining polynomial of an algebraic number α over $\mathbf{Z}$ then we recall that

$$h(\alpha) = \frac{1}{n}\log\left(a_n\prod_{i=1}^{n}\max\{1, |\alpha_i|\}\right)$$

is the logarithmic height of α (cf. [15]) and

$$h(\alpha)\leqq \log H(\alpha)+1, \quad H(\alpha) < \exp\{n(h(\alpha)+\log 2)\}.$$

LEMMA 4. (K. Győry [3] or [4].) *Let $\gamma_1, \ldots, \gamma_n$ be non-zero elements of R_1, d a rational integer such that*

$$d > (n+z)\delta D+\delta\sum_{i=1}^{n}\operatorname{Deg}\gamma_i+\operatorname{Deg} b$$

and $\mathcal{M}$ a finite subset of $\mathbf{Z}\setminus\{0\}$ with cardinality $|\mathcal{M}|>d^q$, Then there are at least $|\mathcal{M}|-(d^q-1)$ elements t_0 in $\mathcal{M}$ such that the numbers $\Phi_{D_F}(t_0)$, $\Phi_b(t_0)$, $\Phi_{\gamma_i}(t_0, k)$

$(1 \leqq i \leqq n)$ *are all different from zero and*

$$\max_k h(\Phi_{\gamma_i}(t_0, k')) < C_5(d^q \log|t_0| + (q+1)s(\gamma_i) + \log \mathscr{F} + 1)$$

for all i *where* C_5 *is an effectively computable constant depending only on* G *and* R_1.

LEMMA 5. (K. Győry [5].) *Let* α *be a non-zero element of* R_1 *and let* $d > \max\{\operatorname{Deg} D_F, \operatorname{Deg} \alpha\}$ *be an integer. Suppose that there are at least* (d^q+1) *non-zero rational integers* t_0 *such that* $|t_0| \leqq d^q$, $\Phi_{D_F}(t_0) \neq 0$ *and*

$$\max_k h(\Phi_\alpha(t_0, k)) \leqq H$$

for some positive H. *Then*

$$s(\alpha) \leqq C_6(2d)^{4q}(H + \log(C_7 \mathscr{F}|t_0|))$$

where C_6 *and* C_7 *are effectively computable constants depending only on* G *and* R_1.

3. Proof of the Theorem

Let $x, y \in R_1$ be an arbitrary but fixed solution of the equation (2). First, we shall give an upper bound for $\operatorname{Deg} x$ and $\operatorname{Deg} y$. In this proof $c_1, \ldots, c_8$ will denote positive computable constants which depend only on G, R_1, $f(X)$ and m. We may assume that $q > 0$. Supposing the contrary, from Lemma 2 our Theorem immediately follows. Let $\mathscr{S}$ denote the set of all (additive) valuations v of $M_i/\overline{k}_i$ such that $v(b^*) \neq 0$. Then the cardinality of $\mathscr{S}$ is bounded by c_1 and $x^{(j)}$, $y^{(j)}$ $\alpha_1^{(j)}, \ldots, \alpha_n^{(j)}$ $(j = 1, \ldots, \delta)$ are S-integers in M_i. From (2) we get

$$(a^{(j)}) \prod_{i=1}^{n} (x^{(j)} - \alpha_i^{(j)})^{r_i} = (y^{(j)})^m, \quad 1 \leqq j \leqq \delta$$

where $\alpha_{i_1}^{(j)} \neq \alpha_{i_2}^{(j)}$ for $i_1 \neq i_2$. But $\{t_1, \ldots, t_n\}$ is not a permutation of the n-tuples (a) and (b), hence by Lemma 1 we obtain

$$\max_j \{H_{M_i}(x^{(j)}), H_{M_i}(y^{(j)})\} < c_2.$$

We can now apply Lemma 3 to obtain

$$\max\{\operatorname{Deg} x, \operatorname{Deg} y\} < c_3.$$

Let

$$d > \max\Big\{c_3, (n^2+2)\delta D + \delta\Big(\sum_{i=1}^{n} \operatorname{Deg} \alpha_i + \operatorname{Deg} a + \sum_{1 \leqq i < j \leqq n} \operatorname{Deg}(\alpha_i - \alpha_j)\Big) + \operatorname{Deg} b\Big\} = c_4$$

be an integer and assume that $\mathscr{M}$ consist of all non-zero rational integers having absolute value $\leqq d^q$. By virtue of Lemma 4 there are at least (d^q+1) elements $t_0 \in \mathscr{M}$ such that the numbers

$$\Phi_{D_F}(t_0),\ \Phi_b(t_0),\ \Phi_a(t_0, k),\ \Phi_{\alpha_i}(t_0, k), \quad (1 \leqq i \leqq n,\ 1 \leqq k \leqq \delta)$$

are all different from zero and $\Phi_{\alpha_i}(t_0, k) \neq \Phi_{\alpha_j}(t_0, k)$ for $i \neq j$. Further,

$$\max_i \{h(\Phi_a(t_0, k), h(\Phi_{\alpha_i}(t_0, k))\} < c_5.$$

From (2) we get

$$\Phi_a(t_0, k) \prod_{i=1}^{n} (\Phi_x(t_0, k)-\Phi_{\alpha_i}(t_0, k))^{r_i} = (\Phi_y(t_0, k))^m.$$

Let $\mathscr{S}_k$ be the set of all (additive) valuation v of the algebraic number field M_{t_0} such that $v(\Phi_b(t_0, k))\neq 0$. Then $|\mathscr{S}_k|<c_6$ and $\Phi_x(t_0, k)$, $\Phi_y(t_0, k)$ are $\mathscr{S}_k$-integers in M_{t_0}. From Lemma 2 we have

$$\max_k \{H(\Phi_x(t_0, k), H(\Phi_y(t_0, k))\} < c_7,$$

$$\max_k \{h(\Phi_x(t_0, k)), h(\Phi_y(t_0, k))\} < c_8.$$

We note that the constant C_2 in Lemma 2 depends only on $g(x)$, m and the parameters (degree, regulator, discriminant) of the algebraic number field $\mathbf{L}$ (cf. [14]) and the elements $U_1, \ldots, U_\delta$ depend only on the polynomial $F_{t_0}(X)$. Hence the parameters of M_{t_0} can be estimated in terms of G, R_1 and F.

Finally, we can apply Lemma 5 to complete our proof.

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A BROUWER TYPE THEOREM

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The well known and very plausible conjecture, that any contractible compact set of $\mathbf{R}^n$ has the fixed point property was disproved by S. Kinoshita in 1953. He gave a two dimensional contractible continuum in [1] without the fixed property. The aim of the present note is to prove the fixed point theorem to the "star-like" compact sets of $\mathbf{R}^2$.

In [1] it is proved that in $\mathbf{R}^n$, $n \geqq 4$ there exists a compact cone without the fixed point property. It is open whether any compact cone or more generally any star-like compact subset of $\mathbf{R}^3$ has the fixed point property. For special star-like compact sets in $\mathbf{R}^n$ the fixed point property was proved in [2].

We shall prove the following

THEOREM. *Let $\emptyset \neq \mathscr{K} \subset \mathbf{R}^2$ be a compact set such that $w \in \mathscr{K}$ implies $[0, w] \subset \mathscr{K}$ (i.e. $\mathscr{K}$ is star-like). Then every continuous function $f\colon \mathscr{K} \to \mathscr{K}$ on $\mathscr{K}$ has a fixed point.*

PROOF. We shall use the following known and simple

LEMMA. *Let (W, ϱ) be a compact metric space and suppose that for every $\varepsilon > 0$ there exists a continuous function $g_\varepsilon\colon W \to W$ on W such that*

1. $\varrho(w, g_\varepsilon(w)) < \varepsilon$ *for all* $w \in W$,
2. $g_\varepsilon(W)$ *has the fixed point property, i.e. every continuous function* $f\colon g_\varepsilon(W) \to g_\varepsilon(W)$ *has a fixed point.*

Then W has also the fixed point property.

(In what follows we consider $\mathbf{R}^2$ as the complex plane.)

Hence we fix an $0 < \varepsilon < \frac{\pi}{2}$ and give a map $f\colon \mathscr{K} \to \mathscr{K}$ such that $|f(w) - w| < 5\varepsilon$ for all w and $f(\mathscr{K})$ has the fixed point property. Denote $K(t) := \max\{l \geqq 0\colon [0, le^{it}] \subset \mathscr{K}\}$, $0 \leqq t \leqq 2\pi$. We can obviously suppose $\max\limits_{[0, 2\pi]} K \leqq 1$. Let us divide $\mathbf{R}^2$ into closed angular domains whose vertex is 0 and whose angle is $< \varepsilon$. Take the intersection of these domains with $\mathscr{K}$. On each of these intersections we define a map, taking it into itself, keeping fixed the points lying on the boundary of the angular domain and moving each point at a distance $< 5\varepsilon$ such that the range of this map is a polyhedron, star-like with respect to the origin. This will be sufficient since these 5ε-mappings define a 5ε-mapping $f\colon \mathscr{K} \to \mathscr{K}$ such that $f(\mathscr{K})$ is a star-like polyhedron which has the fixed point property by the Lefschetz fixed point theorem ([3], p. 151).

Now take any domain of the partition mentioned above, bounded by the rays of angles x and y, $0 \leqq x < y \leqq 2\pi$, $y - x < \varepsilon$. Take a point $x < z < y$ such that $K(z) \leqq \leqq \inf\limits_{[x, y]} K + \varepsilon$. We only give the construction of the mapping of the intersection of

$\mathscr{K}$ with the angular domain bounded by the rays of angles x and z (and such that the points of $\mathscr{K}$ along the ray of angle z remain also fixed); the case of the other angular domain, bounded by the rays of angles z and y can be dealt with symmetrically.

Now we shall construct a sequence $x=x_0\leqq\ldots\leqq x_{2n}=z$ such that

$$K(x_{2j})-\varepsilon\leqq\inf_{[x_{2j},x_{2j+1}]}K,\quad K(x_{2j+1})+\varepsilon\geqq\max_{[x_{2j},x_{2j+1}]}K, \tag{1}$$

$$K(x_{2j+2})-\varepsilon\leqq\inf_{[x_{2j+1},x_{2j+2}]}K,\quad K(x_{2j+1})+\varepsilon\geqq\max_{[x_{2j+1},x_{2j+2}]}K, \tag{2}$$

$$K(x_{2j})\leqq K(x_{2j+1}),\quad K(x_{2j+2})\leqq K(x_{2j+1}). \tag{3}$$

In fact, if $x_{2j}<z$ and $K(x_{2j})-\varepsilon\leqq\inf_{[x_{2j},z]}K$, define $x_{2j+2}=z$ and $x_{2j+1}\in[x_{2j},x_{2j+2}]$ such that

$$K(x_{2j+1})=\max_{[x_{2j},x_{2j+2}]}K.$$

Otherwise, if $x_{2j}<z$ define

$$\tau:=\inf\{t:\ z\geqq t>x_{2j},\ K(t)<K(x_{2j})-\varepsilon\}$$

and let x_{2j+1} be a point in $[x_{2j},\tau]$ such that $K(x_{2j+1})=\max_{[x_{2j},\tau]}K$; further let $x_{2j+2}=\tau+\eta$ (where $\tau<x_{2j+2}<z$) be given such that $K(x_{2j+2})-\frac{\varepsilon}{2}\leqq\inf_{[\tau,x_{2j+2}]}K$. If $\eta>0$ is small enough, then the upper semicontinuity of K implies $\max_{[\tau,x_{2j+2}]}K\leqq K(\tau)+\varepsilon\leqq K(x_{2j+1})+\varepsilon$. So we can define inductively a sequence (x_s) satisfying (1), (2) and (3). Since $x_{2j+2}<z$ implies $0<K(x_{2j+2})+\frac{\varepsilon}{2}\leqq K(x_{2j})$ we must obtain $x_{2n}=z$ in finitely many steps. Now let $s\in\{0,1,\ldots,2n-1\}$ and

$$\mathscr{K}_s:=\{le^{it}\in\mathscr{K}:\ l\geqq 0,\ x_s\leqq t\leqq x_{s+1}\}.$$

Hence to prove the Theorem, we only need to construct a 5ε-mapping $f_s\colon\mathscr{K}_s\to\mathscr{K}_s$ keeping fixed the points le^{it} where $l\geqq 0$ and $t=x_s$ or $t=x_{s+1}$ (the bounding segments) such that $f_s(\mathscr{K}_s)=P_{\varepsilon,s}$ is a star-like polyhedron, i.e. $w\in P_{\varepsilon,s}$ implies $[0,w]\subset P_{\varepsilon,s}$. We consider only the case $s=2j$, the case $s=2j+1$ is similar. If $x_{2j}=x_{2j+1}$ then we have nothing to do. If $x_{2j}<x_{2j+1}$ then by (1) and (3) we can define a continuous 2ε-mapping $r_{2\varepsilon}\colon\mathscr{K}_{2j}\to\mathscr{K}_{2j}$ which is a retraction of any segment $[0,w]$, $w\in\mathscr{K}_{2j}$ onto a subsegment $[0,w']$, $w'\in[0,w]$ such that $r_{2\varepsilon}$ keeps fixed the points of segments bounding $\mathscr{K}_{2j}$, the function $\tilde{K}(t):=\max\{l\geqq 0\colon[0,le^{it}]\subset r_{2\varepsilon}(\mathscr{K}_{2j})\}$ satisfies $K(x_{2j+1})=\tilde{K}(x_{2j+1})=\max_{[x_{2j},x_{2j+1}]}\tilde{K}$ and there exists $x_{2j}<t_1<t_2<t_3<x_{2j+1}$ such that for some $0\leqq\delta<\varepsilon$ and $\delta>0$ if $\tilde{K}(t_2)>0$

(a) $t_2-t_1=t_3-t_2$, $\tilde{K}(t_2)+\delta=\tilde{K}(t_1)=\tilde{K}(t_3)$,
(b) $\tilde{K}$ is linear in $[t_1,t_2]$ and in $[t_2,t_3]$,
(c) for $t\in[x_{2j},t_1]$ and $t\in[t_3,x_{2j+1}]$ we have $\tilde{K}(t)\geqq\tilde{K}(t_2)+\delta$,
(d) $\tilde{K}(t)\leqq\tilde{K}(x_{2j})=K(x_{2j})$ for $t\in[x_{2j},t_1]$.

(The property (d) can be obtained using the upper semicontinuity of K at x_{2j}.)

Now define an ε-mapping $h_\varepsilon\colon r_{2\varepsilon}(\mathscr{K}_{2j})\to r_{2\varepsilon}(\mathscr{K}_{2j})$ keeping fixed the points of the segments bounding $\mathscr{K}_{2j}$ as follows. Let $le^{it}\in r_{2\varepsilon}(\mathscr{K}_{2j})$. If $l\leqq\mathscr{K}(t_2)$ then let $h_\varepsilon(le^{it})=$

$=le^{it}$. If $l \geqq \tilde{K}(t_2)+\delta$ then in case $t \leqq t_1$ let $h_\varepsilon(le^{it})=le^{ix_{2j}}$, in case $t \geqq t_3$ let $h_\varepsilon(le^{it})=$ $=le^{ix_{2j+1}}$. Finally if $\tilde{K}(t_2)+\alpha\delta=l$, $0<\alpha<1$, then in case $t<t_2$ define $h_\varepsilon(le^{it})=$ $=le^{i(\alpha x_{2j}+(1-\alpha)t)}$, in case $t>t_2$ let $h_\varepsilon(le^{it})=le^{i(\alpha x_{2j+1}+(1-\alpha)t)}$. The set $h_\varepsilon(r_{2\varepsilon}(\mathcal{K}_{2j}))$ being constructed, we can define a 2ε-mapping $i_{2\varepsilon}$: $h_\varepsilon(r_{2\varepsilon}(\mathcal{K}_{2j}))\to h_\varepsilon(r_{2\varepsilon}(\mathcal{K}_{2j}))$ such that $P_{\varepsilon,2j}:=i_{2\varepsilon}(h_\varepsilon(r_{2\varepsilon}(\mathcal{K}_{2j})))$ is a star-like polyhedron and $i_{2\varepsilon}$ is the identity on the bounding segments. Indeed, define the points $Q:=(\tilde{K}(t_2)+\delta+1)e^{i(x_{2j}+x_{2j+1})/2}$, $A:=\tilde{K}(t_2)e^{ix_{2j}}$, $B:=\tilde{K}(t_2)e^{ix_{2j+1}}$. If $le^{it}\in h_\varepsilon(r_{2\varepsilon}(\mathcal{K}_{2j}))$, in case $[0, le^{it}]\cap[A, B]=\emptyset$ let $i_{2\varepsilon}(le^{it})=le^{it}$ and in case $[0, le^{it}]\cap[A, B]\neq\emptyset$ let $i_{2\varepsilon}(le^{it})$ be the central projection of le^{it} from the center Q into the boundary of the domain

$$\{re^{iu}\in\mathbf{R}^2:\ x_{2j} \leqq u \leqq x_{2j+1},\ [0, re^{iu}]\cap[A, B]\neq 0\}.$$

Finally we get the desired 5ε-mapping $f_{2j}:=i_{2\varepsilon}\circ h_\varepsilon\circ r_{2\varepsilon}$: $\mathcal{K}_{2j}\to\mathcal{K}_{2j}$ such that $f_{2j}(\mathcal{K}_{2j})=$ $=P_{\varepsilon,2j}$ is a star-like polyhedron. We can analogously define f_s in case $s=2j+1$ and this completes the proof of the Theorem.

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ON OSCULATION OF FINSLER-TYPE CONNECTIONS

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The concept of osculating Riemannian spaces of Finsler spaces was introduced by A. Nazim [6], later O. Varga [10] studied it in details and used in the determination of Cartan's connection in Finsler spaces. The method of osculation was extended to the connections of Finsler spaces and those of the line element manifold by A. Moór [4], [5] and to areal spaces by L. Tamássy [9]. Roughly speaking, the osculation means that to a rather complicated structure (Finsler structure, Finsler connection), a relatively simple structure (a Riemannian metric, an affine or linear connection) is associated, which approximates the former in some sense. Based on this link there is a hope to obtain results on the more complicated structure. Our aim in this paper is to perform and investigate osculations for Finsler-type connections in a global setting.

In Section 1 the necessary preliminaries on connections are given. The Finsler-type connections most often used in this paper are Finsler pair connections (see [7], [1]), which are analogous to Matsumoto's Finsler connections [3]. Using the Cartan vector field C, in Section 2 the notion of C-osculating horizontal map is given, and the role of the C_1 condition is investigated. In Section 3 the osculation of a connection of the Finsler vector bundle $V\xi$ along a section of ξ is given and its properties are studied. This osculation means the construction of a connection in the base vector bundle ξ, whose parallel transport is closely related to the original connection in $V\xi$. In the last section the h-osculation, related to the connection is considered, and the link with the classical results [4] is given.

§ 1. Preliminaries

In the sequel we use the standard differential geometric notations and conditions: All manifolds and maps are of class C^∞.

Given a real vector bundle $\xi=(E, \pi, B, F)$ with finite rank, consider the canonical short exact sequence:

$$0 \to V\xi \xrightarrow{\iota} \tau_E \xrightarrow{\tilde{d\pi}} \pi^*(\tau_B) \to 0 \tag{1}$$

where $V\xi$ is the vertical subbundle of the tangent bundle $\tau_E=(TE, \pi_E, E, \mathbf{R}^{n+r})$, $\pi^*(\tau_B)$ denotes the pull-back of the tangent bundle of B by π, ι is the inclusion map, and $\tilde{d\pi}(\mathbf{a}):=(\pi_E(\mathbf{a}), d\pi(\mathbf{a}))$ for $\mathbf{a}\in TE$. A splitting $\mathscr{H}: \pi^*(\tau_B)\to\tau_E$ of (1) for which $\tilde{d\pi}\circ\mathscr{H}=\mathrm{id}$ holds is called a horizontal map for ξ. It is well known that all fundamental data of the connection theory can be directly derived from a horizontal map. Therefore, the terms "connection in ξ" or "horizontal map for ξ" will be equivalently used. For our purposes the following are needed:

$h := \mathscr{H} \circ \widetilde{d\pi}$ — the horizontal projection
$v := \mathrm{id} - h$ — the vertical projection
$K := \alpha \circ v$ — the Dombrowski map induced by $\mathscr{H}$,
where α: $V\xi \to \xi$ is the canonical morphism.

Given a standard local coordinate system $(x^i, y^\alpha, i = 1, \ldots, \dim B; \alpha = 1, \ldots, \operatorname{rank} \xi)$ for ξ over $U \subset B$, every horizontal map $\mathscr{H}$ can be characterized by differentiable functions Γ_i^α: $\pi^{-1}(U) \to \mathbf{R}$ such that the system

$$\left\{\left(\frac{\partial}{\partial x^i}\right)_z - \Gamma_i^\alpha(z)\left(\frac{\partial}{\partial y^\alpha}\right)_z,\ i = 1, \ldots, \dim B\right\}$$

determines a base of the horizontal subspace $\operatorname{Im} \mathscr{H}_z$ at $z \in \pi^{-1}(U) \subset E$. These functions are called the connections parameters.

Now the *pre-covariant derivation* ∇: $\mathscr{X}(B) \times \operatorname{Sec} \xi \to \operatorname{Sec} \xi$ is given as follows: $(X, \sigma) \mapsto \nabla_X \sigma := K \circ d\sigma(X)$; $X \in \mathscr{X}(B)$, $\sigma \in \operatorname{Sec} \xi$ (see [7]). This map does not behave as a derivation yet. It satisfies the usual Koszul-axioms if (and only if) $\mathscr{H}$ satisfies the homogeneity conditions [7]:

(HC) $\qquad \forall t \in \mathbf{R}$: $h \circ d\mu_t = d\mu_t \circ h$ where μ_t: $z \in E \mapsto tz \in E$.

In this case $\Gamma_i^\alpha(x, y) = \Gamma_{\beta i}^\alpha(x) y^\beta$, and we speak of a *linear connection* and call the map ∇ a covariant derivation.

A section $\sigma \in \operatorname{Sec} \xi$ is called parallel along a curve φ in B if $\mathscr{H}(\sigma \circ \varphi, \dot{\varphi}) = d\sigma(\dot{\varphi})$, where $\dot{\varphi}$ is the tangent curve in TB belonging to φ. It is easy to see that this requirement is equivalent to $\nabla_{\dot{\varphi}} \sigma = 0$. As it is known the parallelism induces a map between the fibers of ξ along φ, called the parallel transport along φ. In the case of linear connection the parallel transport is a linear map between the fibers.

Now we sketch the notion of certain Finsler-type connection for a vector bundle. This is analogous to Matsumoto's notion of Finsler connection, based on principal bundles.

There are several equivalent definitions of Finsler-type connections of a vector bundle ξ; here only two of them are listed ([7], [1]):

Let ξ be a vector bundle. A pair $(\mathscr{H}^F, \mathscr{H})$ is called a *Finsler pair connection* for ξ in $\mathscr{H}^F$ is a horizontal map for the vertical bundle $V\xi = (VE, \pi_V, E, F)$, i.e. a splitting of the exact sequence

$$0 \to V(V\xi) \xrightarrow{\iota} \tau_{VE} \xrightarrow{\widetilde{d\pi_V}} \pi_V^*(\tau_E) \to 0,$$

and $\mathscr{H}$ is a horizontal map for ξ.

The *quasi-vertical subspace* $V_{\mathbf{a}}^q(VE)$ at $\mathbf{a} \in VE$ is defined as the kernel of $d(\pi \circ \pi_V)$ at $\mathbf{a} \in VE$. This subspace is of dimension $2r$ and contains the vertical subspace $V_{\mathbf{a}}(VE) = \operatorname{Ker}(d\pi_V)_{\mathbf{a}}$. Consider now the short exact sequence

$$0 \to V^q(V\xi) \xrightarrow{\iota} \tau_{VE} \xrightarrow{\widetilde{d(\pi \circ \pi_V)}} (\pi \circ \pi_V)^*(\tau_B) \to 0. \tag{2}$$

A splitting $\mathscr{H}^h$: $(\pi \circ \pi_V)^*(\tau_B) \to \tau_{VE}$ is called an *h-horizontal map* for ξ. On the other hand, a splitting $\mathscr{H}^v$ of the exact short sequence

$$0 \to V(V\xi) \xrightarrow{\iota} V^q(V\xi) \xrightarrow{\widetilde{d\pi_V}} \pi_V^*(V\xi) \to 0 \tag{3}$$

is called a *v-horizontal map* for ξ. A pair $(\mathscr{H}^h, \mathscr{H}^v)$ consisting of an h-horizontal and a v-horizontal map is called an (h, v)-*Finsler-connection* when, in addition, the next condition hold for $\mathscr{H}^h$:

$$d\pi_V(\mathscr{H}^h(\tilde{\mu}_t(\mathbf{a}), \mathbf{v})) = d\pi_V(\mathscr{H}^h(\mathbf{a}, \mathbf{v})) \quad \text{for all} \quad \mathbf{a}\in VE, \quad \mathbf{v}\in TB \tag{4}$$

where $\tilde{\mu}_t$: $VE\to VE$ is the extension of μ_t defined by the relations $\pi_V\circ\tilde{\mu}_t=\mu_t\circ\pi_V$, and $\alpha\circ\tilde{\mu}_t=\alpha$.

As it was mentioned, there is a one-to-one correspondence between the Finsler pair-connections and the (h, v)-Finsler-connections.

For the proof we refer to [7], [1], only the explicit relation is written here:

$$\mathscr{H}^h(\mathbf{a}, \mathbf{v}) = \mathscr{H}^F(\mathbf{a}, \mathscr{H}(\pi_V(\mathbf{a}), \mathbf{v})); \quad \mathbf{a}\in VE, \quad \mathbf{v}\in TB,$$

$$\mathscr{H}^v(\mathbf{a}, \mathbf{b}) = \mathscr{H}^F(\mathbf{a}, \mathbf{b}); \quad \mathbf{a}, \mathbf{b}\in VE.$$

For geometrical reasons $\mathscr{H}^F$ is usually supposed to satisfy the homogeneity condition. Then we speak of linear Finsler connection, and in this case its parallel transport is a linear map between the fibres.

Considering standard local coordinate systems for $V\xi$ and ξ, the connection parameters of $\mathscr{H}^F$ will be denoted by $F_i^\alpha(\mathbf{a})$, $C_\beta^\alpha(\mathbf{a})$ and the parameters of $\mathscr{H}$ by $N_i^\beta(z)$.

§ 2. The C-osculating horizontal map

Starting with a Finsler pair-connection $(\mathscr{H}^F, \mathscr{H})$ for ξ, in general there is no dependence between $\mathscr{H}^F$ and $\mathscr{H}$. However, in some cases, which are closely related to the classical local Finsler connection, the horizontal map $\mathscr{H}^F$ for $V\xi$ uniquely determines a horizontal map $\mathscr{H}$ for ξ. In this determination the Cartan intrinsic vector field C: $E\to VE$ occurs, which is characterized by the requirements $\pi_V\circ C=\mathrm{id}$ and $\alpha\circ C=\mathrm{id}$.

First a special type of osculation is introduced.

PROPOSITION 1. *Let $(\mathscr{H}^F, \mathscr{H})$ be a Finsler pair-connection. Then the bundle map* $\overset{C}{\mathscr{H}}$: $\pi^*(\tau_B)\to\tau_E$ *given by*

$$\overset{C}{\mathscr{H}}(z, \mathbf{v}) = (d\alpha)_{C(z)}(\mathscr{H}^F(C(z), \mathscr{H}(z, \mathbf{v})))$$

defines a horizontal map for ξ. It will be called the C-osculating horizontal map of $(\mathscr{H}^F, \mathscr{H})$. Its connection parameters are given as

$$\overset{C}{\Gamma}{}_i^\alpha(z) = F_i^\alpha(C(z)) - N_i^\beta(z)C_\beta^\alpha(C(z)).$$

PROOF. We have to show that $d\pi\circ\overset{C}{\mathscr{H}}=pr_2$, where pr_2: $\pi^*(\tau_B)\to\tau_B$. Using the fact $\pi\circ\pi_V=\pi\circ\alpha$ we get

$$d\pi(\overset{C}{\mathscr{H}}(z, \mathbf{v})) = d\pi\circ d\alpha\circ\mathscr{H}^F(C(z), \mathscr{H}(z, \mathbf{v})) =$$

$$= d\pi\circ d\pi_V\circ\mathscr{H}^F(C(z), \mathscr{H}(z, \mathbf{v})) = d\pi\circ\mathscr{H}(z, \mathbf{v}) = \mathbf{v},$$

since $\mathscr{H}$ is also a horizontal map for ξ. Thus the first statement is proved. The local relation can be verified by an easy computation.

It should be remarked that $\overset{C}{\mathscr{H}}$ does not satisfy the homogeneity condition in general, even when the homogeneity condition (HC) is supposed for $\mathscr{H}$.

DEFINITION. A horizontal map $\mathscr{H}^F$ for V_ξ satisfies the *C_1-condition* if $d\alpha(\mathscr{H}^F(C(z), \mathbf{b}))=0$ for all $\mathbf{b}\in V_z E$.

A short computation shows that this requirement locally means that $C^\gamma_\alpha(C(z))=0$. When the linearity of $\mathscr{H}^F$ is supposed, then this can be written as $C^\gamma_{\beta\alpha}(x, y)y^\beta=0$.

Then C_1-condition is a basic and natural requirement in the classical connection theory of Finsler spaces, namely, the C_1-condition is one of the characterizing properties of the Cartan connection (see [3]). The next theorem shows the importance of the C_1-condition also in our general situation.

THEOREM 1 [2]. *If a horizontal map $\mathscr{H}^F$ for $V\xi$ satisfies the C_1-condition, then the C-osculating horizontal map $\overset{C}{\mathscr{H}}$ of $(\mathscr{H}^F, \mathscr{H})$ is independent from the choice of $\mathscr{H}$. So, it is induced by $\mathscr{H}^F$ alone.*

This Theorem show us how a horizontal map $\mathscr{H}^F$ for $V\xi$ determines a horizontal map for ξ under C_1 condition. The role of the C_1 and C_2 conditions in the theory of Finsler-type connections of vector bundles are investigated in more details in [2].

§ 3. Osculation of connections on $V\xi$

Now we describe an osculating connection for a connection in the vertical bundle $V\xi$ and investigate its fundamental properties and especially, the relations between the original and the osculating connections. Finally the metrical compatibility is pointed out.

PROPOSITION 2. *Let $\mathscr{H}^F$ be a horizontal map for the vertical bundle $V\xi$, and $\sigma\in\operatorname{Sec}\xi$. Then the map*

$$\overset{\sigma}{\mathscr{H}}:\ \pi^*(\tau_B)\to\tau_E,\quad \overset{\sigma}{\mathscr{H}}(z, \mathbf{v}) := d\alpha\big(\mathscr{H}^F(\beta_\sigma(z), d\sigma(\mathbf{v}))\big)$$

is a horizontal map for the vector bundle ξ where $\beta_\sigma: E\to VE$ denotes the map defined by the relations $\alpha\circ\beta_\sigma=\mathrm{id}$ and $\pi_V\circ\beta_\sigma=\sigma\circ\pi$. If $\mathscr{H}^F$ satisfies the homogeneity condition, then so does also $\overset{\sigma}{\mathscr{H}}$.

PROOF. We are to show that $d\pi\circ\overset{\sigma}{\mathscr{H}}=pr_2$. For all $\mathbf{v}\circ T_x B$, $x\in B$

$$d\pi\big(\overset{\sigma}{\mathscr{H}}(z, \mathbf{v})\big) = d\pi\circ d\alpha\circ\mathscr{H}^F\big(\beta_\sigma(z), d\sigma(\mathbf{v})\big) =$$
$$= d\pi\circ d\pi_V\circ\mathscr{H}^F\big(\beta_\sigma(z), d\sigma(\mathbf{v})\big) = d\pi\big(d\sigma(\mathbf{v})\big) = \mathbf{v},$$

since $\pi\circ\alpha=\pi\circ\pi_V$ and $\pi\circ\sigma=\mathrm{id}_B$. Thus $\overset{\sigma}{\mathscr{H}}$ is a horizontal map for ξ. The homo-

geneity condition for $\mathscr{H}^F$ means that $\mathscr{H}^F \circ \tilde{v}_t = dv_t \circ \mathscr{H}^F$, where

$$v_t: VE \to VE, \quad \mathbf{a} \mapsto t\mathbf{a}; \quad \tilde{v}_t: VEx_E TE \to VEx_E TE, \quad (\mathbf{a}, \mathbf{b}) \mapsto (v_t(\mathbf{a}), \mathbf{b}).$$

Using the relation $\alpha \circ v_t = \mu_t \circ \alpha$ we obtain

$$d\mu_t \circ \overset{\sigma}{\mathscr{H}}(z, \mathbf{v}) = d\mu_t \circ d\alpha \circ \mathscr{H}^F(\beta_\sigma(z), d\sigma(\mathbf{v})) = d\alpha \circ dv_t \circ \mathscr{H}^F(\beta_\sigma(z), d\sigma(\mathbf{v})) =$$

$$= d\alpha \circ \mathscr{H}^F \circ \tilde{v}_t(\beta_\sigma(z), d\sigma(\mathbf{v})) = d\alpha \circ \mathscr{H}^F(v_t(\beta_\sigma(z), d\sigma(\mathbf{v}))) =$$

$$= d\alpha \circ \mathscr{H}^F(\beta_\sigma(\mu_t(z)), d\sigma(\mathbf{v})) = \overset{\sigma}{\mathscr{H}}(\mu_t(z), \mathbf{v}) = \overset{\sigma}{\mathscr{H}} \circ \tilde{\mu}_t(z, \mathbf{v}),$$

thus $\overset{\sigma}{\mathscr{H}}$ satisfies also the homogeneity condition.

DEFINITION. The horizontal map $\overset{\sigma}{\mathscr{H}}$ constructed in Proposition 2 is called *the osculating horizontal map* of $\mathscr{H}^F$ *along* $\sigma \in \operatorname{Sec} \xi$.

REMARK. If $(F_i^\alpha, C_\gamma^\alpha)$, $(\overset{\sigma}{\Gamma}{}_i^\alpha)$ denote the local connection parameters of $\mathscr{H}^F$ and $\overset{\sigma}{\mathscr{H}}$, resp., then a simple but lengthy computation shows that

$$\overset{\sigma}{\Gamma}{}_i^\alpha(z) = F_i^\alpha(\beta_\sigma(z)) - \frac{\partial \sigma^r}{\partial x^i}(\pi(z)) C_\gamma^\alpha(\beta_\sigma(z)), \quad z \in E,$$

holds in general, while

$$\overset{\sigma}{\Gamma}{}_{i\beta}^\alpha(x) = F_{i\beta}^\alpha(\sigma(x)) - \frac{\partial \sigma^\gamma}{\partial x^i}(x) C_{\gamma\beta}^\alpha(\sigma(x)), \quad x \in B,$$

holds in the linear case.

In the sequel the next lemma will be helpful.

LEMMA. *If* $\overset{\sigma}{\mathscr{H}}$ *is the osculating horizontal map of* $\mathscr{H}^F$ *along* $\sigma \in \operatorname{Sec} \xi$, *then its horizontal and vertical projections* h^σ *and* v^σ *satisfy the relations*

$$\overset{\sigma}{h} = d\alpha \circ h^F \circ d\beta_\sigma, \tag{5}$$

$$\overset{\sigma}{v} = \alpha^V \circ v^F \circ d\beta_\sigma, \tag{6}$$

where $\alpha^V: V(V\xi) \to V\xi$ *is the canonical epimorphism belonging to* $V\xi$.

PROOF. As to the relation (5), for all $\mathbf{a} \in T_z E$

$$\overset{\sigma}{h}(\mathbf{a}) = \overset{\sigma}{\mathscr{H}} \circ \widetilde{d\pi}(\mathbf{a}) = \overset{\sigma}{\mathscr{H}}(z, d\pi(\mathbf{a})) =$$

$$= d\alpha \circ \mathscr{H}^F(\beta_\sigma(z), d\sigma(d\pi(\mathbf{a}))) = d\alpha \circ \mathscr{H}^F(\beta_\sigma(z), d\pi_V(d\beta_\sigma(\mathbf{a}))) =$$

$$= d\alpha \circ \mathscr{H}^F \circ \widetilde{d\pi}_V(d\beta_\sigma(\mathbf{a})) = d\alpha \circ h^F \circ d\beta_\sigma(\mathbf{a})$$

holds, since obviously $\sigma \circ \pi = \pi_V \circ \beta_\sigma$. The relation (6) can be verified similarly.

DEFINITION. For $\eta \in \operatorname{Sec} \xi$ the lift $\eta^{\uparrow} \in \operatorname{Sec} V\xi$ of η is given as

$$\eta^{\uparrow}(z) := \alpha_z^{-1}(\eta(\pi(z))).$$

Conversely, for $\Sigma \in \operatorname{Sec} V\xi$ the sunk $\Sigma^{\downarrow} \in \operatorname{Sec} \xi$ of Σ along σ is defined by $\Sigma^{\downarrow} = \alpha \circ \Sigma \circ \sigma$. ($\sigma \in \operatorname{Sec} \xi$ is thought from here as a fixed section.)

The following relations will often used in the proofs:

$$(\eta^{\uparrow})^{\downarrow} = \eta \quad \text{and} \quad \eta^{\uparrow} \circ \sigma = \beta_{\sigma} \circ \eta \quad \text{for} \quad \eta \in \operatorname{Sec} \xi,$$

$$(\Sigma^{\downarrow})^{\uparrow} = \Sigma \circ \sigma \quad \text{and} \quad \Sigma \circ \sigma = \beta_{\sigma} \circ \Sigma^{\downarrow} \quad \text{for} \quad \Sigma \in \operatorname{Sec} V\xi.$$

They can be verified directly from the definition.

THEOREM 2. *Let $\varphi: I \to B$ be a smooth curve in B. If $\Sigma \in \operatorname{Sec} V\xi$ is parallel along the curve $\sigma \circ \varphi$ with respect to $\mathscr{H}^F$, then the sunk $\Sigma^{\downarrow} \in \operatorname{Sec} \xi$ is parallel along φ with respect to the osculating connection $\overset{\sigma}{\mathscr{H}}$.*

PROOF. Calculate first $d\Sigma^{\downarrow}(\dot{\varphi})$. Using the relation $\Sigma \circ \sigma = \beta_{\sigma} \circ \Sigma^{\downarrow}$ we obtain

$$d\Sigma^{\downarrow}(\dot{\varphi}) = d\alpha \circ d\Sigma \circ d\sigma(\dot{\varphi}) = d\alpha \circ \mathscr{H}^F(\Sigma \circ \sigma \circ \varphi, d\sigma(\dot{\varphi})) =$$

$$= d\alpha \circ \mathscr{H}^F(\beta_{\sigma} \circ \Sigma^{\downarrow} \circ \varphi, d\sigma(\dot{\varphi})) = \overset{\sigma}{\mathscr{H}}(\Sigma^{\downarrow} \circ \varphi, \dot{\varphi}),$$

since the tangent curve of $\sigma \circ \varphi$ is $d\sigma(\dot{\varphi})$. This means that $\Sigma^{\downarrow}$ is parallel along φ.

COROLLARY. *If $\eta \in \operatorname{Sec} \xi$ is a parallel section along φ with respect to $\overset{\sigma}{\mathscr{H}}$ and $\Sigma \in \operatorname{Sec} V\xi$ is a parallel section along $\sigma \circ \varphi$ with respect to $\mathscr{H}^F$, furthermore if there is a $t_0 \in I$ such that $\Sigma^{\downarrow}(\varphi(t_0)) = \eta(\varphi(t_0))$ then $\Sigma^{\downarrow} \circ \varphi = \eta \circ \varphi$ in a neighborhood of t_0.*

PROOF. It follows immediately from the local uniqueness of parallel sections along curves.

THEOREM 3. *If $\eta \in \operatorname{Sec} \xi$ is a parallel section along a curve φ in B with respect to $\overset{\sigma}{\mathscr{H}}$, then the lift $\eta^{\uparrow} \in \operatorname{Sec} V\xi$ is parallel along $\sigma \circ \varphi$ with respect to $\mathscr{H}^F$.*

PROOF. Let $\Sigma \in \operatorname{Sec} V\xi$ be a section of $V\xi$ such that is parallel along $\sigma \circ \varphi$ with respect to $\mathscr{H}^F$ and starts from $\beta_{\sigma}(\eta(\varphi(0)))$. We state $\eta^{\uparrow} \circ \sigma \circ \varphi = \Sigma \circ \sigma \circ \varphi$. In fact, according to the Corollary $\Sigma^{\downarrow} \circ \varphi = \eta \circ \varphi$ holds, so using the relations $\eta^{\uparrow} \circ \sigma = \beta_{\sigma} \circ \eta$ and $\beta_{\sigma} \circ \Sigma^{\downarrow} = \Sigma \circ \sigma$ we have

$$\eta^{\uparrow} \circ \sigma \circ \varphi = \beta_{\sigma} \circ \eta \circ \varphi = \beta_{\sigma} \circ \Sigma^{\downarrow} \circ \varphi = \Sigma \circ \sigma \circ \varphi.$$

From these theorems we can see that the notion of osculation could be defined by means of the parallelism structure arising from the connection as well. The next theorem will point out the fact that the covariant derivation may also be a base of osculating.

THEOREM 4. *Denoting the covariant derivations belonging to $\mathscr{H}^F$, $\overset{\sigma}{\mathscr{H}}$ by ∇^F, ∇^{σ} resp., for all $\eta \in \operatorname{Sec} \xi$ and $X \in \mathfrak{X}(B)$*

$$\alpha \circ \nabla^F_{d\sigma(X)} \eta^{\uparrow} \circ \sigma = \nabla^{\sigma}_X \eta$$

holds, and for all $\Sigma\in\operatorname{Sec} V\xi$ *and* $X\in\mathfrak{X}(B)$

$$(\nabla^F_{d\sigma(X)}\Sigma)^\downarrow = \nabla^{\overset{\sigma}{v}}_X\Sigma^\downarrow$$

holds.

PROOF. By $\eta^\uparrow\circ\sigma=\beta_\sigma\circ\eta$ and the representation of $\overset{\sigma}{v}$ in the Lemma, we obtain

$$\alpha\circ\nabla^F_{d\sigma(X)}\eta^\uparrow\circ\sigma = \alpha\cdot\alpha^V\circ v^F\circ d\eta^\uparrow\circ d\sigma(X) =$$

$$= \alpha\circ\alpha^V\circ v^F\cdot d\beta_\sigma\circ d\eta(X) = \alpha\circ\overset{\sigma}{v}\circ d\eta(X) = \nabla^{\overset{\sigma}{v}}_X\eta.$$

Similarly, by $\Sigma\circ\sigma=\beta_\sigma\circ\Sigma^\downarrow$, we get the second relation.

Now we show that the process of osculation is in accordance with the metrical structures. Namely, let g be a Riemannian metric in the vertical bundle $V\xi$, regarded as Finsler metric (possibly constructed from a Finsler fundamental function). Fixed a section $\sigma\in\operatorname{Sec}\xi$, the notion of *osculating Riemannian metric in* ξ can be defined as follows:

$$\overset{\sigma}{g}_x(z_1, z_2) = g(\beta_\sigma(z_1), \beta_\sigma(z_2)) \quad \text{where} \quad z_1, z_2\in E, \quad \pi(z_1)=\pi(z_2)=x.$$

$\overset{\sigma}{g}$ is called the osculating Riemannian metric along σ. So we have two similar constructions: one for connections and an other for metrics. We are to show they are in accordance with each other.

THEOREM 5. *If the horizontal map* $\mathscr{H}^F$ *is metrical with respect to a Riemannian metric* g *in* $V\xi$, *then the osculating horizontal map* $\overset{\sigma}{\mathscr{H}}$ *along* σ *is metrical with respect to the osculating Riemannian metric* $\overset{\sigma}{g}$ *along* σ *as well.*

PROOF. Take a parallel section $\eta\in\operatorname{Sec}\xi$ with respect to $\overset{\sigma}{\mathscr{H}}$ along a curve φ in B. It is to be shown that the square of the length $\overset{\sigma}{g}(\eta\circ\varphi, \eta\circ\varphi)$ is a constant function. By Theorem 3 $\eta^\uparrow\in\operatorname{Sec} V\xi$ is parallel along $\sigma\circ\varphi$ with respect to $\mathscr{H}^F$, thus $g(\eta^\uparrow\circ\sigma\circ\varphi, \eta^\uparrow\circ\sigma\circ\varphi)$ is constant. By the definition of $\overset{\sigma}{g}$, and using the relation $\beta_\sigma\circ\eta=\eta^\uparrow\circ\sigma$ we have

$$\overset{\sigma}{g}(\eta\circ\varphi, \eta\circ\varphi) = g(\beta_\sigma\circ\eta\circ\varphi, \beta_\sigma\circ\eta\circ\varphi) = g(\eta^\uparrow\circ\sigma\circ\varphi, \eta^\uparrow\circ\sigma\circ\varphi),$$

which was to be proved.

§ 4. The *h*-osculation of Finsler connections

In this section we intend to sketch briefly how the concept of osculation applied here does fit to the classical works of A. Moór ([4], [5]).

An analogous definition to that of [4] can be given using the *h*-horizontal map $\mathscr{H}^h$ based upon the following statement.

PROPOSITION 3. *Let $\mathcal{H}^h$ be an h-horizontal map of the vertical bundle $V\xi$, and $\sigma \in \operatorname{Sec} \xi$. Then the map*

$$\overset{\sigma!}{\mathcal{H}}: \pi^*(\tau_B) \to \tau_E, \quad \overset{\sigma!}{\mathcal{H}}(z, v) := d\alpha\left(\mathcal{H}^h(\beta_\sigma(z), \mathbf{v})\right)$$

gives a horizontal map for ξ.

The proof is very similar to those of Propositions 2 and 3.

REMARK 1. When an h-horizontal map $\mathcal{H}^h$ is derived from a Finsler connection $(\mathcal{H}^F, \mathcal{H})$, the horizontal map $\overset{\sigma!}{\mathcal{H}}$ constructed in Proposition 4 is called *the h-osculating connection* (or horizontal map) of the Finsler connection $(\mathcal{H}^F, \mathcal{H})$ along σ. Then its connection parameters $\overset{\sigma!}{\Gamma}{}_i^\alpha$ can be calculated as follows:

$$\overset{\sigma!}{\Gamma}{}_i^\alpha(z) = F_i^\alpha(\beta_\sigma(z)) - N_i^\gamma(\sigma(x)) C_\gamma^\alpha(\beta_\sigma(z)).$$

Now we obtain the osculating connection described by A. Moór in [4] when $\mathcal{H}^F$ is supposed to satisfy the C_1-condition and, in the Finsler pair-connection $(\mathcal{H}^F, \mathcal{H})$ the C-osculating horizontal map $\overset{C}{\mathcal{H}}$ is chosen for $\mathcal{H}$ (and, of course, $\mathcal{H}^F$ is homogeneous in the support element).

REMARK 2. It is easy to see that under special circumstances the concepts of osculating and h-osculating connections coincide. Namely, if $\sigma \in \operatorname{Sec} \xi$ is absolute parallel with respect to $\mathcal{H}$, then the osculating horizontal map $\overset{\sigma}{\mathcal{H}}$ of $\mathcal{H}^F$ and the h-osculating connection $\overset{\sigma!}{\mathcal{H}}$ of $(\mathcal{H}^F, \mathcal{H})$ are equal. In fact, $\sigma \in \operatorname{Sec} \xi$ is absolute parallel with respect to $\mathcal{H}$ if and only if $d\sigma(\mathbf{v}) = \mathcal{H}(\sigma(x), \mathbf{v})$ for all $\mathbf{v} \in T_x B$, $x \in B$. Thus our statement follows from the explicit formulas given in Propositions 3 and 4. Note that this condition is also assumed in [4], p. 400; more exactly, σ is constructed from a line element sequence in such a way that σ is absolute parallel with respect to $\mathcal{H}$.

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REMARKS ON CONVERGENCE OF GAUSSIAN QUADRATURE FOR SINGULAR INTEGRALS

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1. Introduction and preliminary results

1.1. Consider a distribution function γ: $(-1, 1)\to\mathbf{R}$, the corresponding orthogonal polynomials $p_n(x)$ $(n=0, 1, \ldots)$ of degree exactly n with the roots

$$-1 < x_{nn} < x_{n-1,n} < \ldots < x_{1n} < 1, \quad n = 0, 1, \ldots, \tag{1.1}$$

and the Christoffel numbers

$$\lambda_{kn} := \frac{1}{\sum_{l=0}^{n-1} p_l^2(x_{kn})}, \quad n = 0, 1, \ldots. \tag{1.2}$$

Then for the *Gaussian quadrature* of f: $(-1, 1)\to\mathbf{R}$

$$Q_n(f) := \sum_{k=1}^{n} \lambda_{kn} f(x_{kn}), \quad n = 1, 2, \ldots, \tag{1.3}$$

we have $Q_n(f)=I(f):=\int_{-1}^{1} f(x)\,d\gamma(x)$ whenever f is a polynomial of degree at most $2n-1$ (cf. [1; 15.1—15.3]). For other f, we can consider

$$R_n(f) := I(f)-Q_n(f), \quad n = 1, 2, \ldots. \tag{1.4}$$

1.2. We shall frequently take some fixed point $y\in(-1, 1)$ where $f(x)$ may, or may not, have a singularity. Throughout $x_{j(n),n}$, $x_{l(n),n}$ and $x_{r(n),n}$ denote the abscissas from (1.1) which are as follows:

$$\begin{cases} |y-x_{jn}| := \min\limits_i \{|y-x_{in}|\}, \\ y-x_{ln} := \min\limits_i \{y-x_{in};\ y \geqq x_{in}\}, \\ x_{rn}-y := \min\limits_i \{x_{in}-y;\ y < x_{in}\}. \end{cases} \tag{1.5}$$

If $2y=x_l+x_r$ we take $x_j=x_l$. *We let*

$$Q_n^*(f) := \sum_{\substack{k=1 \\ k\neq j}}^{n} \lambda_{kn} f(x_{kn}), \quad Q_n^{**}(f) := \sum_{\substack{k=1 \\ k\neq l,r}}^{n} \lambda_{kn} f(x_{kn}) \tag{1.6}$$

so that Q_n^* (or Q_n^{**}) avoids the singularity by omitting the closest abscissa (the closest

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abscissas from the left and right) to y. Further we define

$$R_n^*:=I(f)-Q_n^*(f),\quad R_n^{**}(f):=I(f)-Q_n^{**}(f). \tag{1.7}$$

To formulate some results proved by D. S. Lubinsky and P. Rabinovitz [2] we say that $d\gamma(x)$ *is bounded above and below near* y if there exist positive constants m and M such that

$$m\leqq\frac{\gamma(x_2)-\gamma(x_1)}{x_2-x_1}\leqq M$$

for all x_1 and x_2 in a neighbourhood of y. Then if

$$a(x):=\begin{cases}|x-y|^{-\delta}, & x\neq y\\ 0, & x=y,\end{cases}\quad 0<\delta<1,$$

$$l(x):=\begin{cases}-\log|x-y|, & x\neq y\\ 0, & x=y,\end{cases}$$

we have the following

STATEMENT 1.1. *Let* $y\in(-1,1)$. *Assume* $d\gamma(x)$ *is bounded above and below near* y. *Then*

(i) $R_n^{**}(a)\sim n^{-1+\delta}$, $R_n^{**}(l)\sim\dfrac{\log n}{n}$.

(ii) $R_n^*(a)=O(n^{-1+\delta})$, $R_n^*(l)=O\left(\dfrac{\log n}{n}\right)$.

(iii) *For those* n *for which* $y\neq x_{j(n),n}$,

$$R_n(a)=-\lambda_{jn}|x_{jn}-y|^{-\delta}+O(n^{-1+\delta})$$

$$R_n(l)=-\lambda_{jn}\log|x_{jn}-y|+O(n^{-1}\log n)\quad \textit{where}\quad \lambda_{jn}\sim n^{-1}.$$

(iv) *For any given* $\varphi>1$ *there is a set* $\mathscr{E}_\varphi\subset(-1,1)$ *with* $|\mathscr{E}_\varphi|=0$ *such that*

$$R_n(a)=O(n^{-1+2\delta}(\log n)^{\delta}(\log\log n)^{\varphi\delta})\quad \textit{whenever}\quad y\notin\mathscr{E}_\varphi$$

and there is a set $\mathscr{E}\subset(-1,1)$ *with* $|\mathscr{E}|=0$ *such that*

$$R_n(l)=O\left(\frac{\log n}{n}\right)\quad \textit{if}\quad y\notin\mathscr{E}.$$

(v) *Let* $d\gamma(x)$ *be a Jacobi distribution given by*

$$\gamma(x)=(1-x)^{\alpha}(1+x)^{\beta}\quad \textit{where}\quad \alpha,\beta=\pm\frac{1}{2}.$$

Then there is a set $\mathscr{E}\subset(-1,1)$ *with* $|\mathscr{E}|=0$ *for which*

$$\varlimsup_{n\to\infty}\frac{|R_n(a)|}{n^{-1+2\delta}(\log n)^{\delta}(\log\log n)^{\delta}}>0,\quad \varlimsup_{n\to\infty}\frac{|R_n(l)|}{n^{-1}\log n}>0\quad \textit{if}\quad y\notin\mathscr{E}. \tag{1.8}$$

On the other hand if $y=\cos\frac{p}{q}\pi$, $\frac{p}{q}\in(0,1)$ *rational, then*

$$R_n(a)=O(n^{-1+\delta}),\quad R_n(l)=O\left(\frac{\log n}{n}\right). \tag{1.9}$$

(cf. [2; Corollaries 4.4, 4.6 and Theorem 5.1]; for "$\sim$", see [1; 1.1].)

2. Results

2.1. The proof of (1.8) and (1.9) is based on the fact that for the given Jacobi weigths the zeros x_{kn} are known explicitly. So to get (1.8) and (1.9) for other values of α, β it would be desirable to develop a Dirichlet or Duffin—Schaffer type theory ([5; I. § 2]) for the corresponding Jacobi-roots $x_{kn}^{(\alpha,\beta)}$ (cf. Lubinsky's problem in [3]). Perhaps a first approach can be found in P. Erdős and P. Vértesi [4; Theorems 2.2 and 2.3]. By similar reasonings we are going to prove the next

STATEMENT 2.1. *Let* $x_0\in(0,1)$ *be a fixed irrational and* $0\leqq\varepsilon,\ \eta<1$ *be fixed real numbers. Then there exist positive integer numbers* $\{l_k, n_k\}_{k=1}^{\infty}$ *with* $1<n_1<n_2<\dots$ *such that*

$$\left|x_0-\frac{l_k+\varepsilon}{n_k+\eta}\right|<\frac{6}{n_k^2},\quad k=1,2,\dots. \tag{2.1}$$

Using (2.1) we prove

THEOREM 2.2. *For any sequence* $\varrho=\{\varrho_n\}_{n=1}^{\infty}$ *with* $\varrho_n>0$ *and* $\sum_{n=1}^{\infty} n\varrho_n<\infty$ *there exists a set* $\mathscr{A}=\mathscr{A}(\varrho,\alpha,\beta)\subset(-1,1)$, $|\mathscr{A}|=2$, *such that if* $y\in\mathscr{A}$ *is fixed, then*

$$0<\varrho_n\leqq|y-x_{jn}^{(\alpha,\beta)}|=O\left(\frac{1}{n^2}\right),\quad n=n_1,n_2,\dots, \tag{2.2}$$

with a proper sequence n_k, $1<n_1<n_2<\dots$ *whatever are the Jacobi parameters* $\alpha,\beta>-1$.

By Theorem 2.2 we can gain the next generalization of (1.8).

THEOREM 2.3. *For any fixed* $\alpha,\beta>-1$ *there is a set* $\mathscr{A}\subset(-1,1)$ *with* $|\mathscr{A}|=2$ *for which*

$$\overline{\lim_{n\to\infty}}\frac{|R_n(a)|}{n^{-1+2\delta}}>0,\quad \overline{\lim_{n\to\infty}}\frac{|R_n(l)|}{n^{-1}\log n}>0\quad \text{if}\quad y\in\mathscr{A}. \tag{2.3}$$

To generalize (1.9) first we prove

STATEMENT 2.4. *Let* $y=\cos\frac{p}{q}\pi\in(-1,1)$, $(p,q)=1$, *and* $\alpha,\beta>-1$ *be arbitrary fixed. Then we can give a sequence* $\{n_k\}$, $1<n_1<n_2<n_3<\dots$, *such that*

$$|x_{jn}^{(\alpha,\beta)}-y|\geqq\frac{c}{n}\quad \text{if}\quad n=n_1,n_2,\dots. \tag{2.4}$$

By the previous statement and Statement 1.1, (iii) we immediately gain the next

THEOREM 2.5. *Let* $y=\cos\frac{p}{q}\pi\in(-1,1)$, $(p,q)=1$, *and* $\alpha,\beta>-1$ *be arbitrary fixed. Then we can give a sequence* $\{n_k\}$, $1<n_1<n_2<\dots$, *such that*

$$R_n(a)=O(n^{-1+\delta}),\quad R_n(l)=O\left(\frac{\log n}{n}\right). \tag{2.5}$$

2.2. As a final remark let me mention that [2; Theorem 7.6, (ii)] can analogously be extended for all parameters $\alpha,\beta>-1$, too. We omit the further details.

3. Proofs

3.1. PROOF OF STATEMENT 2.1. The argument is based on our paper [4; 3.2]. By the Dirichlet theorem, for any fixed irrational number $x_0\in(0,1)$ there exist positive integers $\{u_k\}$, $\{v_k\}$ such that

$$\left|x_0-\frac{u_k}{v_k}\right|<\frac{1}{v_k^2},\quad (u_k,v_k)=1,\ 1<v_1<v_2<\dots. \tag{3.1}$$

Let $A_k:=v_k\varepsilon-u_k\eta$. For fixed u_k and v_k consider the Diophantine equation $nu_k-lv_k=1$. By $(u_k,v_k)=1$, it is solvable; let $\{n^*,l^*\}$ be a solution. Consider $n_0:=n^*[A_k]$ and $l_0:=l^*[A_k]$.

a) If $n_0<v_k$ we fix the integer $t>0$ such that

$$n_0+(t-1)v_k<v_k\leqq n_0+tv_k. \tag{3.2}$$

Then let

$$n_k:=n_0+tv_k\quad\text{and}\quad l_k:=l_0+tu_k. \tag{3.3}$$

By $n_ku_k-l_kv_k=[A_k](n^*u_k-l^*v_k)=[A_k]$ we have

$$\left|\frac{u_k}{v_k}-\frac{l_k+\varepsilon}{n_k+\eta}\right|=\left|\frac{n_ku_k-l_kv_k-A_k}{v_k(n_k+\eta)}\right|\leqq\frac{1}{v_k(n_k+\eta)}:=I. \tag{3.4}$$

Using (3.2), $n_k-v_k<v_k\leqq n_k$, from where

$$v_k\leqq n_k<2v_k. \tag{3.5}$$

By (3.4) and (3.5)

$$I\leqq\frac{1}{v_kn_k}<\frac{2}{n_k^2}. \tag{3.6}$$

b) If $n_0\geqq 2v_k$, with a proper p, $\tilde{n}_0:=n_0-pv_k$, $\tilde{l}_0:=l_0-pu_k$ will be a solution for which $\tilde{n}_0<v_k$, i.e. we have case a). So in both cases, using (3.1), (3.4), (3.6) and (3.5)

$$\left|x_0-\frac{l_k+\varepsilon}{n_k+\eta}\right|\leqq\left|x_0-\frac{u_k}{v_k}\right|+\left|\frac{u_k}{v_k}-\frac{l_k+\varepsilon}{n_k+\eta}\right|<\frac{6}{n_k^2}.$$

3.2. PROOF OF THEOREM 2.2. First we consider the left side of (2.2). To this end let

$$\mathscr{B}_n := \bigcup_{k=1}^{n} (x_{kn}-\varrho_n, x_{kn}+\varrho_n), \quad n = 1, 2, \dots,$$

$$\mathscr{B} := \{x;\ x\in(-1, 1),\ x\in\mathscr{B}_n \text{ for infinitly many } n\}.$$

Obviously $|\mathscr{B}_n|\leqq 2n\varrho_n$, i.e. by the Borell—Cantelli lemma (see e.g. [5; I., Lemma 1]) $|\mathscr{B}|=0$. By definition, if $y\in\mathscr{A}_1:=[-1, 1]\setminus\mathscr{B}$, then $|y-x_{j(n)}|\geqq\varrho_n$ for all large n. Obviously $|\mathscr{A}_1|=2$.

Now we deal with the right-hand side. Let $\frac{\vartheta_0}{\pi}\in(0, 1)$ be irrational. Then by (2.1) one can choose the double sequence $\{t_k, n_k\}$ such that with $\varepsilon=\frac{\alpha-\frac{1}{2}}{2}$ and $\eta=\frac{\alpha+\beta+1}{2}$ for the fractions $\psi_{tn}=\frac{\pi(t+\varepsilon)}{n+\eta}$

$$|\vartheta_0-\psi_{t_k,n_k}| < \frac{6\pi}{n_k^2}, \quad k = 1, 2, \dots. \tag{3.7}$$

On the other hand, by Vértesi [6, 4.8] the interval

$$\left[\psi_{t_k,n_k}-\frac{c_1}{n_k^2}, \psi_{t_k,n_k}+\frac{c_1}{n_k^2}\right], \quad c_1 = c_1(\alpha, \beta, \vartheta_0) \tag{3.8}$$

contains exactly one zero of $P_n(\cos\vartheta)$ (denoted by ϑ_{j_k,n_k}) whenever n_k is big enough which we suppose from now on. By this reason we call $\cos\psi_{t_n}$ quasi roots (or quasi nodes). By (3.7) and (3.8)

$$|\vartheta_0-\vartheta_{j_k,n_k}^{(\alpha,\beta)}| = O\left(\frac{1}{n_k^2}\right), \quad k = 1, 2, \dots.$$

By $\vartheta_{i+1}-\vartheta_i\sim n^{-1}$ (see e.g. [6; (4.10)]) ϑ_{j_k,n_k} is the nearest root to ϑ_0, indeed. Then

$$|x_0-x_{j_k,n_k}^{(\alpha,\beta)}| = \left|2\sin\frac{\vartheta_0+\vartheta_{j_k}}{2}\sin\frac{\vartheta_0-\vartheta_{j_k}}{2}\right| = O\left(\frac{1}{n_k^2}\right), \quad k = 1, 2, \dots$$

whenever $x_0\in\mathscr{A}_2:=\left\{x_0=\cos\vartheta_0;\ \frac{\vartheta_0}{\pi}\in(0, 1), \text{ irrational}\right\}$. Again $|\mathscr{A}_2|=2$. Now, if $\mathscr{C}:=[-1, 1]\setminus\mathscr{A}_2$, by $\mathscr{A}:=\mathscr{A}_1\cap\mathscr{A}_2=[-1, 1]\setminus(\mathscr{B}\cup\mathscr{C})$ we get the theorem.

3.3. PROOF OF THEOREM 2.3. Let $y\in\mathscr{A}$ defined in Section 2.2. Now (2.3) immediately follows from Statement 1.1, (iii) (taking into account that all Jacobi weights $(1-x)^\alpha(1+x)^\beta$ are bounded above and below near arbitrary fixed $y\in(-1, 1)$) using (2.2).

3.4. PROOF OF STATEMENT 2.4. Let us consider the nearest quasi root (shortly qroot) $\cos\psi_{Jn}$ to $y=\cos\vartheta_0=\cos\frac{p}{q}\pi$, $n=1, 2, \dots$. Then with $r=r(\alpha, \beta, p, q)=$

$=\left(\alpha+\frac{1}{2}\right)q-(\alpha+\beta+1)p$ we have for arbitrary fixed $n\geqq n_0$

$$\left|\psi_{Jn}-\frac{p}{q}\pi\right| = \frac{|(2J-1)q-2np+r|}{(2n+\alpha+\beta+1)q}\pi := \frac{|D_n|}{N_n} := F_n = O\left(\frac{1}{n}\right). \tag{3.9}$$

a) If *r is an integer and* $|D_n|>0$ then we obviously have

$$F_n \geqq \frac{\pi}{(2n+\alpha+\beta+1)q} \geqq \frac{c}{n}, \tag{3.10}$$

i.e. by $|\vartheta_{jn}-\psi_{Jn}|=O(n^{-2})$, we get with $n\geqq n_0$ by (3.9) and (3.10)

$$|y-x_{jn}| = 2\sin\frac{\vartheta_0+\vartheta_j}{2}\sin\frac{|\vartheta_0-\vartheta_j|}{2} \geqq$$

$$\geqq 2\sin\left(\vartheta_0\pm\frac{c}{n}\right)\sin\frac{|\vartheta_0-\psi_J+\psi_J-\vartheta_j|}{2} \geqq \frac{c}{n},$$

i.e. n may be member of $\{n_k\}$.

b) If *r is an integer and* $D_n=0$, then we state that $(n+1)\in\{n_k\}$. Indeed, if $|D_{n+1}|\geqq 1$, we are ready (see case a)). On the other hand if $D_{n+1}=0$, too, we can write

$$\left.\begin{aligned} 2np-(2J-1)\,q &= r \\ 2(n+1)p-(2t-1)q &= r \end{aligned}\right\},$$

from where $p=(t-J)q$, a contradiction.

c) Finally *if r is not an integer, let* $r_1:=\min(r-[r],[r+1]-r)$. Then obviously

$$F_n \geqq \frac{r_1\pi}{(2n+\alpha+\beta+1)q} \geqq \frac{c}{n}, \quad n\geqq n_0,$$

i.e. we can apply the argument of a) to get $n\in\{n_k\}$.

By a), b) and c) we can define the sequence $\{n_k\}$.

Addendum. The interested reader may find some other connected results and/or improvements in papers [7] and [8], appeared recently.

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PROJECTIVE AND INDUCTIVE GENERATIONS OF RELATOR SPACES

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Introduction

A nonvoid family $\mathscr{R}$ of reflexive relations on a set X is called a relator on X, and the ordered pair $X(\mathscr{R})=(X, \mathscr{R})$ is called a relator space. Relators are straightforward generalizations of the various generalized uniformities [16, p. 3]. They were proposed in our former paper [21] as the most suitable basic terms that topology and analysis should be based on.

A relation f from a relator space $X(\mathscr{R})$ into another $Y(\mathscr{S})$ is called mildly continuous, or more precisely mildly $(\mathscr{R}, \mathscr{S})$-continuous, if $f^{-1}\circ\mathscr{S}\circ f\subset\mathscr{R}$, i.e., $f^{-1}\circ S\circ f\in\mathscr{R}$ for all $S\in\mathscr{S}$. This concept has also been introduced in our former paper [21], where we showed that the most important continuity properties of a function can be obtained as particular cases of the above definition.

In the present paper, projective and inductive generations of relator spaces by relations are defined and studied in accordance with the above concepts of mild continuity. As the most important applications subspaces, products, quotients and free sums of relator spaces are mentioned. Most of the results obtained extend standard theorems on uniform or generalized uniform spaces.

For instance, in connection with quotients we prove that if ϱ is a nonmingled-valued relation from $X(\mathscr{R})$ onto $Y(\mathscr{S})$ such that both ϱ and ϱ^{-1} are mildly continuous, then the inverse φ_ϱ^{-1} of the projection φ_ϱ of $X(\mathscr{R})$ onto the quotient space $X|\varrho$ $(\mathscr{R}|\varrho)$ is also mildly continuous, and $\psi_\varrho=\varphi_\varrho\circ\varrho^{-1}\circ\varphi_{\varrho^{-1}}^{-1}$ is an isomorphism of $Y|\varrho^{-1}(\mathscr{S}|\varrho^{-1})$ onto $X|\varrho(\mathscr{R}|\varrho)$, moreover $Y|\varrho^{-1}(\mathscr{S}|\varrho^{-1})=X|\varrho^{-1}\circ\varrho(\mathscr{R}|\varrho^{-1}\circ\varrho)$.

The only prerequisites for reading this paper are a basic knowledge of relations and a familiarity with a few things from our former paper [21]. But, for the sake of brevity, these will not be laid out here. Moreover, we shall also omit several proofs, which are either straightforward enough, or parallel to the usual ones. However, for the reader's convenience, several illustrating examples will be included.

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1. Projective generation

DEFINITION 1.1. If f_α is a relation from a set X into a relator space $Y_\alpha(\mathscr{S}_\alpha)$ for each α in a nonvoid set Γ and

$$\mathscr{R}=\bigcup_{\alpha\in\Gamma} f_\alpha^{-1}\circ\mathscr{S}_\alpha\circ f_\alpha,$$

then we say that the relator space $X(\mathscr{R})$ is projectively generated from the spaces

$Y_\alpha(\mathscr{S}_\alpha)$ by the relations f_α and write

$$X(\mathscr{R}) = \operatorname{proj}\operatorname{gen}_{\alpha\in\Gamma} f_\alpha Y_\alpha(\mathscr{S}_\alpha).$$

The appropriateness of the above definition is apparent from the following two obvious characterization theorems.

THEOREM 1.2. *If*

$$X(\mathscr{R}) = \operatorname{proj}\operatorname{gen}_{\alpha\in\Gamma} f_\alpha Y_\alpha(\mathscr{S}_\alpha),$$

then $\mathscr{R}$ is the smallest relator on X such that f_α is mildly continuous for all $\alpha\in\Gamma$.

THEOREM 1.3. *If f_α is a relation from a relator space $X(\mathscr{R})$ into another $Y_\alpha(\mathscr{S}_\alpha)$ for each α in a nonvoid set Γ, then the following assertions are equivalent:*

(i) $X(\mathscr{R}) = \operatorname{proj}\operatorname{gen}_{\alpha\in\Gamma} f_\alpha Y_\alpha(\mathscr{S}_\alpha)$;

(ii) *a relation g from a relator space $Z(\mathscr{T})$ into $X(\mathscr{R})$ is mildly continuous if and only if $f_\alpha\circ g$ is mildly continuous for all $\alpha\in\Gamma$.*

To prove the implication (ii)⇒(i), note that even the very particular cases of (ii) when either $Z(\mathscr{T})=X(\mathscr{R})$ or $Z(\mathscr{T})=\operatorname{proj}\operatorname{gen}_{\alpha\in\Gamma} f_\alpha Y_\alpha(\mathscr{S}_\alpha)$ and g is the identity function of X do already imply (i).

The following reduction theorem is an immediate consequence of the above characterization theorems.

THEOREM 1.4. *If f_α is a relation from a relator space $X(\mathscr{R})$ into another $Y_\alpha(\mathscr{S}_\alpha)$ for each α in a nonvoid set Γ and*

$$Y_\alpha(\mathscr{S}_\alpha) = \operatorname{proj}\operatorname{gen}_{\beta\in\Gamma_\alpha} g_{\alpha\beta} Z_{\alpha\beta}(\mathscr{T}_{\alpha\beta})$$

for all $\alpha\in\Gamma$, then the following assertions are equivalent:

(i) $X(\mathscr{R}) = \operatorname{proj}\operatorname{gen}_{\alpha\in\Gamma} f_\alpha Y_\alpha(\mathscr{S}_\alpha)$;

(ii) $X(\mathscr{R}) = \operatorname{proj}\operatorname{gen}_{(\alpha,\beta)\in\bigcup_{\alpha\in\Gamma}\Gamma\times\Gamma_\alpha} g_{\alpha\beta}\circ f_\alpha Z_{\alpha\beta}(\mathscr{T}_{\alpha\beta})$.

From [21, Proposition 4.12], one can easily derive the following theorem which shows the full compatibility of projective generations with mild hyperspaces.

THEOREM 1.5. *If f_α is a relation from a relator space $X(\mathscr{R})$ into another $Y_\alpha(\mathscr{S}_\alpha)$ for each α in a nonvoid set Γ, then the following assertions are equivalent:*

(i) $X(\mathscr{R}) = \operatorname{proj}\operatorname{gen}_{\alpha\in\Gamma} f_\alpha Y_\alpha(\mathscr{S}_\alpha)$;

(ii) $X(\mathscr{R}) = \operatorname{proj}\operatorname{gen}_{\alpha\in\Gamma} \varphi_{f_\alpha} \tilde{Y}_\alpha(\tilde{\mathscr{S}}_\alpha)$;

(iii) $\tilde{X}(\tilde{\mathscr{R}}) = \operatorname{proj}\operatorname{gen}_{\alpha\in\Gamma} \Phi_{f_\alpha} \tilde{Y}_\alpha(\tilde{\mathscr{S}}_\alpha)$.

Using [21, Theorem 6.4] and the corresponding definitions, one can easily prove the next two theorems which describe limits and closures in projectively generated relator spaces.

THEOREM 1.6. *If*

$$X(\mathscr{R}) = \operatorname{proj\,gen}_{\substack{f_\alpha \\ \alpha\in\Gamma}} Y_\alpha(\mathscr{S}_\alpha)$$

and (x_ν) *and* (y_ν) *are nets in* X, *then the following assertions are equivalent:*

(i) $y_\nu \in \operatorname{Lim}_{\nu,\mathscr{R}} x_\nu$;

(ii) $f_\alpha(y_\nu) \in \operatorname{Lim}_{\nu,\widetilde{\mathscr{S}}_\alpha} f_\alpha(x_\nu)$ *for all* $\alpha\in\Gamma$.

COROLLARY 1.7. *If*

$$X(\mathscr{R}) = \operatorname{proj\,gen}_{\substack{f_\alpha \\ \alpha\in\Gamma}} Y_\alpha(\mathscr{S}_\alpha)$$

and f_α *is a function for all* $\alpha\in\Gamma$, *then*

$$\lim_{\nu,\mathscr{R}} x_\nu = \bigcap_{\alpha\in\Gamma} f_\alpha^{-1}\left(\lim_{\nu,\mathscr{S}_\alpha} f(x_\nu)\right)$$

for any net (x_ν) *in* X.

THEOREM 1.8. *If*

$$X(\mathscr{R}) = \operatorname{proj\,gen}_{f_\alpha} Y_\alpha(\mathscr{S}_\alpha)$$

and $A, B\subset X$, *then the following assertions are equivalent:*

(i) $B\in \operatorname{Cl}_{\mathscr{R}}(A)$;

(ii) $f_\alpha(B)\in \operatorname{Cl}_{\mathscr{S}_\alpha}(f_\alpha(A))$ *for all* $\alpha\in\Gamma$.

COROLLARY 1.9. *If*

$$X(\mathscr{R}) = \operatorname{proj\,gen}_{\substack{f_\alpha \\ \alpha\in\Gamma}} Y_\alpha(\mathscr{S}_\alpha)$$

and f_α *is a function for all* $\alpha\in\Gamma$, *then*

$$\operatorname{cl}_{\mathscr{R}}(A) = \bigcap_{\alpha\in\Gamma} f_\alpha^{-1}(\operatorname{cl}_{\mathscr{S}_\alpha}(f_\alpha(A)))$$

for any $A\subset X$.

The following theorem shows that projective generations are, to some extent, also compatible with the basic refinement operations on relators.

THEOREM 1.10. *If*

$$X(\mathscr{R}) = \operatorname{proj\,gen}_{\substack{f_\alpha \\ \alpha\in\Gamma}} Y_\alpha(\mathscr{S}_\alpha) \quad \text{and} \quad X(\mathscr{U}) = \operatorname{proj\,gen}_{\substack{f_\alpha \\ \alpha\in\Gamma}} Y_\alpha(\mathscr{V}_\alpha),$$

then the following assertions hold:

(i) *If* $\mathscr{S}_\alpha\subset\mathscr{V}_\alpha^*$ *for all* $\alpha\in\Gamma$, *then* $\mathscr{R}\subset\mathscr{U}^*$.

(ii) *If* $\mathscr{S}_\alpha\subset\mathscr{V}_\alpha^\#$ *for all* $\alpha\in\Gamma$, *then* $\mathscr{R}\subset\mathscr{U}^\#$.

(iii) *If* $\mathscr{S}_\alpha\subset\widehat{\mathscr{V}_\alpha}$ *and* f_α *is a function for all* $\alpha\in\Gamma$, *then* $\mathscr{R}\subset\widehat{\mathscr{U}}$.

The proof of this theorem is also rather straightforward. For instance, we prove (ii). For this, assume that $\mathscr{S}_\alpha\subset\mathscr{V}_\alpha^\#$ for all $\alpha\in\Gamma$, and let $R\in\mathscr{R}$ and $A\subset X$.

Then, by the definition of $X(\mathscr{R})$, there exist $\alpha\in\Gamma$ and $S\in\mathscr{S}_\alpha$ such that $R=f_\alpha^{-1}\circ S\circ f_\alpha$. Moreover, by the assumption $\mathscr{S}_\alpha\subset\mathscr{V}_\alpha^\#$, there exists $V\in\mathscr{V}_\alpha$ such that $V(f_\alpha(A))\subset S(f_\alpha(A))$. This implies that $(f_\alpha^{-1}\circ V\circ f_\alpha)(A)\subset(f_\alpha^{-1}\circ S\circ f_\alpha)(A)=R(A)$. And hence, since $f_\alpha^{-1}\circ V\circ f_\alpha\in\mathscr{U}$, it is clear that $R\in\mathscr{U}^\#$.

The next example shows that the extra condition that each f_α is a function cannot be omitted from the assertion (iii) of Theorem 1.10.

EXAMPLE 1.11. Let $X=Y=\{0,1,2\}$, and define the relation f from X into Y such that $f(0)=\{0,1\}$ and $f(x)=\{x\}$ if $x\in\{1,2\}$. Moreover, let $\mathscr{S}=\{\Delta_Y\}$, where Δ_Y is the diagonal of $Y\times Y$, and $\mathscr{V}=\{V_y\}_{y\in Y}$, where V_y is the relation on Y such that $V_y(z)=\{y\}$ if $z=y$ and $V_y(z)=Y$ if $z\in Y\setminus\{y\}$. Furthermore, define $\mathscr{R}=f^{-1}\circ\mathscr{S}\circ f$ and $\mathscr{U}=f^{-1}\circ\mathscr{V}\circ f$. Then, it is clear that

$$X(\mathscr{R})=\operatorname{proj}\operatorname{gen}_f Y(\mathscr{S}) \quad\text{and}\quad X(\mathscr{U})=\operatorname{proj}\operatorname{gen}_f Y(\mathscr{V}),$$

and moreover $\mathscr{S}\subset\hat{\mathscr{V}}$, but $\mathscr{R}\not\subset\hat{\mathscr{U}}$.

To check this latter assertion, note that

$$(f^{-1}\circ V_y\circ f)(0)=X\not\subset\{0,1\}=(f^{-1}\circ\Delta_Y\circ f)(0)$$

for all $y\in Y$.

REMARK 1.12. In addition to Theorem 1.10, it is also worth mentioning that if

$$X(\mathscr{R})=\operatorname*{proj\,gen}_{\alpha\in\Gamma,\,f_\alpha} Y_\alpha(\mathscr{S}_\alpha)$$

and f_α is an injection of X onto Y_α and $\mathscr{S}_\alpha^*=\mathscr{S}_\alpha$ for all $\alpha\in\Gamma$, then $\mathscr{R}^*=\mathscr{R}$.

To check this, note that if $T\in\mathscr{R}^*$, then there exists $R\in\mathscr{R}$ such that $R\subset T$. Moreover, there exist $\alpha\in\Gamma$ and $S\in\mathscr{S}_\alpha$ such that $R=f_\alpha^{-1}\circ S\circ f_\alpha$. Thus, $f_\alpha^{-1}\circ S\circ f_\alpha\subset T$, and hence $S=f_\alpha\circ(f_\alpha^{-1}\circ S\circ f_\alpha)\circ f_\alpha^{-1}\subset f_\alpha\circ T\circ f_\alpha$. Consequently, $f_\alpha\circ T\circ f_\alpha^{-1}\in\mathscr{S}_\alpha^*=\mathscr{S}_\alpha$, and hence $T=f_\alpha^{-1}\circ(f_\alpha\circ T\circ f_\alpha^{-1})\circ f_\alpha\in\mathscr{R}$.

Finally, we prove some unusual statements which do really require relations.

THEOREM 1.13. *Let f be a relation from a set X onto a relator space $Y(\mathscr{S})$ and $(\psi_\alpha)_{\alpha\in\Gamma}$ be a nonvoid family of selection relations for f^{-1}. Moreover, define*

$$X(\mathscr{R})=\operatorname{proj}\operatorname{gen}_f Y(\mathscr{S}) \quad\textit{and}\quad Y(\mathscr{V})=\operatorname*{proj\,gen}_{\alpha\in\Gamma,\,\psi_\alpha} X(\mathscr{R}).$$

Then $\mathscr{V}\subset\mathscr{S}^*$, and moreover $\mathscr{V}=\mathscr{S}$ if f is a function.

PROOF. If $V\in\mathscr{V}$, then there exist $\alpha\in\Gamma$ and $R\in\mathscr{R}$ such that $V=\psi_\alpha^{-1}\circ R\circ\psi_\alpha$. Moreover, there exists $S\in\mathscr{S}$ such that $R=f^{-1}\circ S\circ f$. Thus, we have

$$V=\psi_\alpha^{-1}\circ(f^{-1}\circ S\circ f)\circ\psi_\alpha=(f\circ\psi_\alpha)^{-1}\circ S\circ(f\circ\psi_\alpha)\supset\Delta_Y^{-1}\circ S\circ\Delta_Y=S,$$

whence $V\in\mathscr{S}^*$ follows.

To prove the remaining part, note that if f is a function, then we have

$$\psi_\alpha^{-1}\circ\mathscr{R}\circ\psi_\alpha=\psi_\alpha^{-1}\circ(f^{-1}\circ\mathscr{S}\circ f)\circ\psi_\alpha=(f\circ\psi_\alpha)^{-1}\circ\mathscr{S}\circ(f\circ\psi_\alpha)=\Delta_Y^{-1}\circ\mathscr{S}\circ\Delta_Y=\mathscr{S}$$

for all $\alpha\in\Gamma$, whence it is clear that $\mathscr{V}=\mathscr{S}$.

To briefly formulate an essential improvement of a useful consequence of Theorems 1.2 and 1.13, it seems appropriate to introduce a substantial strengthening of mild continuity.

DEFINITION 1.14. A relation f from a relator space $X(\mathscr{R})$ into another $Y(\mathscr{S})$ will be called perfectly mildly continuous, or more precisely perfectly mildly $(\mathscr{R}, \mathscr{S})$-continuous, if $\varphi_2^{-1} \circ \mathscr{S} \circ \varphi_1 \subset \mathscr{R}$ for any two selection relations φ_1 and φ_2 for f.

REMARK 1.15. Note that the perfect mild continuity of f always implies the selectional mild continuity of f which means only that each selection relation φ for f is mildly continuous.

The appropriateness of the above concept of perfect mild continuity is apparent from the next simple but important

THEOREM 1.16. *If f is a relation from a set X onto a relator space $Y(\mathscr{S})$ and*

$$X(\mathscr{R}) = \operatorname{proj\,gen}_f Y(\mathscr{S}),$$

then f is mildly $(\mathscr{R}, \mathscr{S})$-continuous and f^{-1} is perfectly mildly $(\mathscr{S}^*, \mathscr{R})$-continuous. Moreover, if f is a function, then f^{-1} is perfectly mildly $(\mathscr{S}, \mathscr{R})$-continuous.

PROOF. The mild $(\mathscr{R}, \mathscr{S})$-continuity of f is immediate from the definition of $\mathscr{R}$ by Theorem 1.2.

To prove the perfect mild $(\mathscr{S}^*, \mathscr{R})$-continuity of f^{-1}, note that if $R \in \mathscr{R}$, then there exists $S \in \mathscr{S}$ such that $R = f^{-1} \circ S \circ f$. Thus, if ψ_1 and ψ_2 are selection relations for f^{-1}, then similarly as in the proof of Theorem 1.13, we have

$$\psi_2^{-1} \circ R \circ \psi_1 = (f \circ \psi_2)^{-1} \circ S \circ (f \circ \psi_1) \supset \Delta_Y \circ S \circ \Delta_Y = S,$$

whence $\psi_2^{-1} \circ R \circ \psi_1 \in \mathscr{S}^*$ follows.

Finally, if f is a function, then again in a similar way as in the proof of Theorem 1.13, we also have

$$\psi_2^{-1} \circ \mathscr{R} \circ \psi_1 = (f \circ \psi_2)^{-1} \circ \mathscr{S} \circ (f \circ \psi_1) = \Delta_Y \circ \mathscr{S} \circ \Delta_Y = \mathscr{S},$$

which proves that f^{-1} is perfectly mildly $(\mathscr{S}, \mathscr{R})$-continuous.

The next simple example shows that the assertions of Theorems 1.13 and 1.16 cannot be sharpened in general.

EXAMPLE 1.17. Let $X = \{0\}$ and $Y = \{0, 1\}$, and denote by f the unique relation from X onto Y. Moreover, let $\mathscr{R} = \{\Delta_X\}$, $\mathscr{S} = \{\Delta_Y\}$ and $\mathscr{V} = \{Y \times Y\}$. Then, it is clear that $X(\mathscr{R}) = \operatorname{proj\,gen}_f Y(\mathscr{S})$ and $Y(\mathscr{V}) = \operatorname{proj\,gen}_{f^{-1}} X(\mathscr{R})$, but $\mathscr{V} \not\subset \mathscr{S}$, and thus f^{-1} is not mildly $(\mathscr{S}, \mathscr{R})$-continuous.

Moreover, it is also worth noticing that f^{-1} is now a function and we also have $\mathscr{S} \not\subset \hat{\mathscr{V}}$.

Theorem 1.16 can also be supplemented by the following useful

THEOREM 1.18. *If f is a continuous function from a relator space $X(\mathscr{R})$ onto another $Y(\mathscr{S})$ such that f^{-1} is mildly continuous, then*

$$Y(\mathscr{S}) = \operatorname{proj\,gen}_{f^{-1}} X(\mathscr{R}).$$

PROOF. Since f is continuous, $f^{-1} \circ \mathscr{S} \circ f \subset \mathscr{R}$, whence

$$\mathscr{S} = f \circ f^{-1} \circ \mathscr{S} \circ f \circ f^{-1} \subset f \circ \mathscr{R} \circ f^{-1}.$$

On the other hand, since f^{-1} is mildly continuous, $f \circ \mathscr{R} \circ f^{-1} \subset \mathscr{S}$. Consequently, $\mathscr{S} = f \circ \mathscr{R} \circ f^{-1}$.

Combining Theorems 1.16 and 1.18, we can at once state

COROLLARY 1.19. *If f is a function from a relator space $X(\mathscr{R})$ onto another $Y(\mathscr{S})$, then any of the following assertions implies the subsequent one:*

(i) $X(\mathscr{R}) = \operatorname{proj} \operatorname{gen}_f Y(\mathscr{S})$:
(ii) *f is continuous and f^{-1} is perfectly mildly continuous;*
(iii) *f is continuous and f^{-1} is mildly continuous;*
(iv) $Y(\mathscr{S}) = \operatorname{proj} \operatorname{gen}_{f^{-1}} X(\mathscr{R})$.

REMARK 1.20. From Example 1.17, it is clear that the assertions (ii) and (iv) do not, in general, imply the former ones. Moreover, our forthcoming Example 2.12 will show that (iii) does not, in general, imply (ii). Thus, the assertion of Corollary 1.19 cannot be improved.

But, of course, in particular, we also have

COROLLARY 1.21. *If f is an injection from a relator space $X(\mathscr{R})$ onto another $Y(\mathscr{S})$, then the following assertions are equivalent:*

(i) $X(\mathscr{R}) = \operatorname{proj} \operatorname{gen}_f Y(\mathscr{S})$;
(ii) *f and f^{-1} are continuous;*
(iii) $Y(\mathscr{S}) = \operatorname{proj} \operatorname{gen}_{f^{-1}} X(\mathscr{R})$.

For some further purposes, it seems also convenient to introduce the following terminology.

DEFINITION 1.22. An injection f from a relator space $X(\mathscr{R})$ onto another $Y(\mathscr{S})$ will be called an isomorphism, or more precisely a relator space isomorphism of $X(\mathscr{R})$ onto $Y(\mathscr{S})$ if both f and f^{-1} are continuous.

REMARK 1.23. Uniform, proximal, resp. topological isomorphisms of $X(\mathscr{R})$ onto $Y(\mathscr{S})$, due to [21, Theorem 6.6], may be defined simply as relator space isomorphisms of $X(\mathscr{R}^*)$, $X(\mathscr{R}^\#)$ and $X(\hat{\mathscr{R}})$ onto $Y(\mathscr{S}^*)$, $Y(\mathscr{S}^\#)$ and $Y(\hat{\mathscr{S}})$, respectively.

2. Applications of projective generation

DEFINITION 2.1. If $Y(\mathscr{S})$ is a relator space and $X \subset Y$, then the relator space

$$X(\mathscr{R}) = \operatorname{proj} \operatorname{gen}_{\Delta_X} Y(\mathscr{S}),$$

where Δ_X is the identity function of X will be called the subspace of $Y(\mathscr{S})$ generated by X.

As an immediate consequence of this definition, we can at once state

THEOREM 2.2. *If $X(\mathscr{R})$ is a subspace of a relator space $Y(\mathscr{S})$, then*

$$\mathscr{R} = \mathscr{S} \wedge \{X \times X\}.$$

Several further properties of subspaces can be easily derived from the results of Section 1. For instance, Theorem 1.4 yields

THEOREM 2.3. *If f_α is a relation from a relator space $X(\mathscr{R})$ into a subspace $Y_\alpha(\mathscr{S}_\alpha)$ of a relator space $Z_\alpha(\mathscr{T}_\alpha)$ for each α in a nonvoid set Γ, then the following assertions are equivalent:*

(i) $X(\mathscr{R}) = \operatorname{proj\,gen}_{\alpha\in\Gamma} Y_\alpha(\mathscr{S}_\alpha)$;

(ii) $X(\mathscr{R}) = \operatorname{proj\,gen}_{\alpha\in\Gamma} Z_\alpha(\mathscr{T}_\alpha)$.

Another important application of projective generation is offered by the next trivial

THEOREM 2.4. *If $\mathscr{R}_\alpha$ is a relator on a set X for each α in a nonvoid set Γ and $\mathscr{R} = \bigcup_{\alpha\in\Gamma} \mathscr{R}_\alpha$, then*

$$X(\mathscr{R}) = \operatorname{proj\,gen}_{\alpha\in\Gamma} X(\mathscr{R}_\alpha).$$

This theorem allows us to derive several properties of unions of relators from the results of Section 1. For instance, using Theorem 2.4, from Theorems 1.5, 1.6, 1.8 and 1.10, we can easily get

COROLLARY 2.5. *If $\mathscr{R}_\alpha$ is a relator on a set X for each α in a nonvoid set Γ and $\mathscr{R} = \bigcup_{\alpha\in\Gamma} \mathscr{R}_\alpha$, then*

(i) $\tilde{\mathscr{R}} = \bigcup_{\alpha\in\Gamma} \tilde{\mathscr{R}}_\alpha$;

(ii) $\operatorname{Lim}_{\mathscr{R}} = \bigcap_{\alpha\in\Gamma} \operatorname{Lim}_{\mathscr{R}_\alpha}$ *and* $\operatorname{Cl}_{\mathscr{R}} = \bigcap_{\alpha\in\Gamma} \operatorname{Cl}_{\mathscr{R}_\alpha}$;

(iii) $\mathscr{R}^\square = \left(\bigcup_{\alpha\in\Gamma} \mathscr{R}_\alpha^\square\right)^\square$, *where* $\square \in \{*, \#, \wedge\}$.

REMARK 2.6. Note that in particular, by Remark 1.12, we also have $\left(\bigcup_{\alpha\in\Gamma} \mathscr{R}_\alpha^*\right)^* = \bigcup_{\alpha\in\Gamma} \mathscr{R}_\alpha^*$.

Now, we shall turn to product spaces, the most important applications of projective generation.

DEFINITION 2.7. If $Y_\alpha(\mathscr{S}_\alpha)$ is a relator space for each α in a nonvoid set Γ and $X = \bigtimes_{\alpha\in\Gamma} Y_\alpha$, then the relator space

$$X(\mathscr{R}) = \operatorname{proj\,gen}_{\alpha\in\Gamma} Y_\alpha(\mathscr{S}_\alpha)$$

where π_α is the projection of X onto Y_α, will be called the projectively generated product of the spaces $Y_\alpha(\mathscr{S}_\alpha)$ and the notation

$$X(\mathscr{R}) = \bigtimes_{\alpha\in\Gamma} Y_\alpha(\mathscr{S}_\alpha)$$

will be used.

As an immediate consequence of this definition, we can at once state

THEOREM 2.8. *If*

$$X(\mathscr{R}) = \bigtimes_{\alpha\in\Gamma} Y_\alpha(\mathscr{S}_\alpha),$$

then $\mathscr{R}$ consists of all relations R on X which can be written in the form

$$R(x) = \bigtimes_{\alpha\in\Gamma} R_\alpha(x_\alpha) \qquad (x = (x_\alpha)_{\alpha\in\Gamma}\in X),$$

where $R_\alpha\in\mathscr{S}_\alpha$ for some $\alpha\in\Gamma$ and $R_\beta = Y_\beta\times Y_\beta$ for all $\beta\in\Gamma\setminus\{\alpha\}$.

To briefly formulate an important addition to the corresponding particular case of Theorem 1.2, it seems convenient to introduce another strengthening of mild continuity which is, in general, much weaker than selectional mild continuity.

DEFINITION 2.9. A relation f from a relator space $X(\mathscr{R})$ into another $Y(\mathscr{S})$ will be called lower semiperfectly mildly continuous, or more precisely lower semiperfectly mildly $(\mathscr{R}, \mathscr{S})$-continuous, if $f^{-1}\circ\mathscr{S}\circ\varphi\subset\mathscr{R}$ for any selection relation φ for f.

REMARK 2.10. Note that even the lower semiperfect mild topological continuity of f implies the usual topological lower semicontinuity of f which means simply that f^{-1} takes open sets onto open sets.

Linear relations of preseminormed spaces provide an important particular case when lower semiperfect mild uniform continuity coincides with the usual topological lower semicontinuity. (See [20] and [22].)

Now, we are ready to formulate the promised supplement to the corresponding particular case of Theorem 1.2 which is an analogue of the classical statement that the projections of product spaces are open [11, p. 90].

THEOREM 2.11. *If*

$$X(\mathscr{R}) = \bigtimes_{\alpha\in\Gamma} Y_\alpha(\mathscr{S}_\alpha),$$

then the inverse π_α^{-1} of the projection π_α of X onto Y_α is lower semiperfectly mildly $(\mathscr{S}_\alpha\cup\{Y_\alpha\times Y_\alpha\}, \mathscr{R})$-continuous for all $\alpha\in\Gamma$.

PROOF. Fix $\alpha\in\Gamma$, and let ψ be a selection relation for π_α^{-1}. Define

$$X(\mathscr{R}_\alpha) = \operatorname{proj}\operatorname{gen}_{\pi_\alpha} Y_\alpha(\mathscr{S}_\alpha).$$

Then, by Theorem 1.16, π_α^{-1} is perfectly mildly $(\mathscr{S}_\alpha, \mathscr{R}_\alpha)$-continuous. Therefore, if $R\in\mathscr{R}_\alpha$, then we have $(\pi_\alpha^{-1})^{-1}\circ R\circ\psi\in\mathscr{S}_\alpha$.

On the other hand, if $R\in\mathscr{R}\setminus\mathscr{R}_\alpha$, then there exist $\beta\in\Gamma\setminus\{\alpha\}$ and $S\in\mathscr{S}_\beta$ such that $R = \pi_\beta^{-1}\circ S\circ\pi_\beta$. Thus, we have

$$(\pi_\alpha^{-1})^{-1}\circ R\circ\psi = (\pi_\alpha\circ\pi_\beta^{-1})\circ S\circ(\pi_\beta\circ\psi) = Y_\alpha\times Y_\alpha$$

since $\pi_\alpha\circ\pi_\beta^{-1} = Y_\beta\times Y_\alpha$ and $S\circ(\pi_\beta\circ\psi)$ is a relation from Y_α into Y_β.

Consequently, π_α^{-1} is lower semiperfectly mildly $(\mathscr{S}_\alpha\cup\{Y_\alpha\times Y_\alpha\}, \mathscr{R})$-continuous.

The next example shows that π_α^{-1} is not even, in general, selectionally mildly $(\hat{\mathscr{S}}_\alpha, \mathscr{R})$-continuous, and thus Theorem 2.11 cannot be improved.

EXAMPLE 2.12. Let $Y_1=Y_2=\{0, 1\}$ and $\mathscr{S}_1=\mathscr{S}_2=\{S\}$ where S is a relation on $\{0, 1\}$ such that $S(0)=\{0\}$ and $S(1)=\{0, 1\}$. Moreover, define

$$X(\mathscr{R}) = Y_1(\mathscr{S}_1)\times Y_2(\mathscr{S}_2).$$

Then, the inverse π_1^{-1} of the projection π_1 of X onto Y_1 is not selectionally mildly $(\hat{\mathscr{S}}_1, \mathscr{R})$-continuous.

To check this, note that the function ψ defined on Y_1 by $\psi(0)=(0, 1)$ and $\psi(1)=$ $=(1, 0)$ is a selection function for π_1^{-1} such that ψ is not $(\hat{\mathscr{S}}_1, \mathscr{R})$-continuous. Namely, the relation

$$T = \psi^{-1}\circ(\pi_2^{-1}\circ S\circ\pi_2)\circ\psi$$

cannot belong to $\hat{\mathscr{S}}_1$ since $T(1)=\{1\}$, and hence $S(1)\not\subset T(1)$.

From Corollary 1.9, one can easily derive the following unusual statement which apparently contradicts the corresponding particular case of Theorem 1.6.

THEOREM 2.13. *If*

$$X(\mathscr{R}) = \underset{\alpha\in\Gamma}{\times} Y_\alpha(\mathscr{S}_\alpha)$$

and $A\subset X$, *then*

$$\mathrm{cl}_{\mathscr{R}}(A) = \underset{\alpha\in\Gamma}{\times} \mathrm{cl}_{\mathscr{S}_\alpha}(A_\alpha),$$

where A_α *is the projection of* A *into* Y_α.

In this respect, it is also worth mentioning that in contrast to [13, Theorem 1.37] now we also have

THEOREM 2.14. *If*

$$X(\mathscr{R}) = Y_1(\mathscr{S}_1)\times Y_2(\mathscr{S}_2)$$

and $A\subset X$, *then*

$$\mathrm{cl}_{\mathscr{R}}(A) = \cap\mathscr{S}_2^{-1}\circ\{A\circ X^{-1}\circ A\}\circ\mathscr{S}_1.$$

PROOF. If $(x, y)\in\mathrm{cl}_{\mathscr{R}}(A)$, then by the definition of $\mathrm{cl}_{\mathscr{R}}$, we have $R(x,y)\cap A\neq\emptyset$ for all $R\in\mathscr{R}$. Hence, by Theorem 2.8, it is clear that then we also have $(S(x)\times Y_2)\cap$ $\cap A\neq\emptyset$ and $(Y_1\times T(y))\cap A\neq\emptyset$ for all $S\in\mathscr{S}_1$ and $T\in\mathscr{S}_2$. This means that, for each $S\in\mathscr{S}_1$ and $T\in\mathscr{S}_2$, there exist $z\in S(x)$, $w\in Y_2$ and $\omega\in Y_1$, $\tau\in T(y)$ such that $(z, w)\in A$ and $(\omega, \tau)\in A$. Thus, we have

$$(x, z)\in S,\quad (z, w)\in A,\quad (w, \omega)\in X^{-1},\quad (\omega, \tau)\in A,\quad (\tau, y)\in T^{-1}$$

whence $(x, y)\in T^{-1}\circ A\circ X^{-1}\circ A\circ S$ follows. Since $S\in\mathscr{S}_1$ and $T\in\mathscr{S}_2$ were arbitrary, hence it is clear that we also have $(x, y)\in\cap\mathscr{S}_2^{-1}\circ\{A\circ X^{-1}\circ A\}\circ\mathscr{S}_1$.

This proves one half of the theorem. The other half can be proved by reversing the above argument.

REMARK 2.15. Note that

$$A\circ X^{-1}\circ A = \mathrm{dom}(A)\times\mathrm{rng}(A)$$

and hence $A\subset A\circ X^{-1}\circ A$.

REMARK 2.16. Theorems 2.13 and 2.14 show that the projectively generated product relator $\mathscr{R}$ given in Definition 2.7 is too small for several purposes.

Therefore, we have also to consider some larger relators on the product set X. An immediate candidate for this is

$$\mathscr{R}' = \{\cap\mathscr{A}\colon \emptyset \neq \mathscr{A} \subset \mathscr{R} \text{ is finite}\}.$$

However, note that by taking finite intersections some useful properties of $\mathscr{R}$ may be lost. For instance, the assertion of Theorem 1.3 fails to hold for $\mathscr{R}'$ instead of $\mathscr{R}$.

Moreover, there are cases when a "direct product" of relators, which corresponds to the "box product" of topologies, proves to be more suitable than $\mathscr{R}$ or $\mathscr{R}'$.

The importance of projectively generated products as particular projective generations lies mainly in the following reduction.

THEOREM 2.17. *If*

$$X(\mathscr{R}) = \operatorname*{proj\,gen}_{\alpha\in\Gamma} f_\alpha Y_\alpha(\mathscr{S}_\alpha) \quad \text{and} \quad Y(\mathscr{S}) = \bigtimes_{\alpha\in\Gamma} Y_\alpha(\mathscr{S}_\alpha),$$

and f is the relation from X into Y defined by

$$f(x) = \bigtimes_{\alpha\in\Gamma} f_\alpha(x)$$

then

$$X(\mathscr{R}) = \operatorname{proj\,gen}_f Y(\mathscr{S}).$$

PROOF. Note that, for $\alpha\in\Gamma$, we have $f_\alpha = \pi_\alpha \circ f$, where π_α is the projection of Y onto Y_α. Thus the implication (ii)⇒(i) in Theorem 1.4 can be applied to get the assertion.

From this theorem, using Theorem 2.3 and Corollary 1.21, one can easily derive

COROLLARY 2.18. *If*

$$X(\mathscr{R}) = \operatorname*{proj\,gen}_{\alpha\in\Gamma} f_\alpha Y_\alpha(\mathscr{S}_\alpha)$$

and $(f_\alpha)_{\alpha\in\Gamma}$ is a separating family of functions on X, then the function f defined on X by $f(x) = (f_\alpha(x))_{\alpha\in\Gamma}$ is an isomorphism of $X(\mathscr{R})$ onto a subspace of $\bigtimes_{\alpha\in\Gamma} Y_\alpha(\mathscr{S}_\alpha)$.

Moreover, applying Corollary 2.18 to the particular case of Theorem 2.4 when each $\mathscr{R}_\alpha$ is a singleton, we get

COROLLARY 2.19. *If $X(\mathscr{R})$ is a relator space, then the function f defined on X by $f(x) = (x)_{R\in\mathscr{R}}$ is an isomorphism of $X(\mathscr{R})$ onto a subspace of $\bigtimes_{R\in\mathscr{R}} X(\{R\})$.*

3. Inductive generation

DEFINITION 3.1. If f_α is a relation from a relator space $X_\alpha(\mathscr{R}_\alpha)$ into a set Y for each α in a nonvoid set Γ and the family

$$\mathscr{S} = \{\Delta_Y \subset S \subset Y\times Y\colon f_\alpha^{-1}\circ S\circ f_\alpha \in \mathscr{R}_\alpha \text{ for all } \alpha\in\Gamma\}$$

is nonvoid, then we say that the relator space $Y(\mathscr{S})$ is inductively generated from the spaces $X_\alpha(\mathscr{R}_\alpha)$ by the relations f_α and write

$$Y(\mathscr{S}) = \operatorname{ind}\operatorname{gen}^Y_{\substack{f_\alpha \\ \alpha\in\Gamma}} X_\alpha(\mathscr{R}_\alpha).$$

REMARK 3.2. Note that if $X_\alpha\times X_\alpha\in\mathscr{R}_\alpha$, or more specially $\mathscr{R}^*_\alpha=\mathscr{R}_\alpha$ for all $\alpha\in\Gamma$, then $Y\times Y\in\mathscr{S}$.

In the sequel, for the sake of simplicity, we shall usually omit the superscript Y from the above notation.

The appropriateness of the above definition is apparent from the next two obvious characterization theorems.

THEOREM 3.3. *If*

$$Y(\mathscr{S}) = \operatorname{ind}\operatorname{gen}_{\substack{f_\alpha \\ \alpha\in\Gamma}} X_\alpha(\mathscr{R}_\alpha),$$

then $\mathscr{S}$ is the largest relator on Y such that f_α is mildly continuous for all $\alpha\in\Gamma$.

THEOREM 3.4. *If f_α is a relation from a relator space $X_\alpha(\mathscr{R}_\alpha)$ into another $Y(\mathscr{S})$ for each α in a nonvoid set Γ. Then the following assertions are equivalent:*

(i) $Y(\mathscr{S}) = \operatorname{ind}\operatorname{gen}_{\substack{f_\alpha \\ \alpha\in\Gamma}} X_\alpha(\mathscr{R}_\alpha)$;

(ii) *a relation g from $Y(\mathscr{S})$ into a relator space $Z(\mathscr{T})$ is mildly continuous if and only if $g\circ f_\alpha$ is mildly continuous for all $\alpha\in\Gamma$.*

To prove the implication (ii)⇒(i), note that even the very particular cases of (ii) when either $Z(\mathscr{T})=Y(\mathscr{S})$ or $Z(\mathscr{T})=\operatorname{ind}\operatorname{gen}_{\substack{f_\alpha \\ \alpha\in\Gamma}} X_\alpha(\mathscr{R}_\alpha)$ and g is the identity function of Y do already imply (i).

The following reduction theorem is an immediate consequence of the above characterization theorems.

THEOREM 3.5. *If f_α is a relation from a relator space $X_\alpha(\mathscr{R}_\alpha)$ into another $Y(\mathscr{S})$ for each α in a nonvoid set Γ and*

$$X_\alpha(\mathscr{R}_\alpha) = \operatorname{ind}\operatorname{gen}_{\substack{g_{\alpha\beta} \\ \alpha\in\Gamma}} Z_{\alpha\beta}(\mathscr{T}_{\alpha\beta})$$

for all $\alpha\in\Gamma$, then the following assertions are equivalent:

(i) $Y(\mathscr{S}) = \operatorname{ind}\operatorname{gen}_{\substack{f_\alpha \\ \alpha\in\Gamma_\alpha}} X_\alpha(\mathscr{R}_\alpha)$;

(ii) $Y(\mathscr{S}) = \operatorname{ind}\operatorname{gen}_{\substack{f_\alpha\circ g_{\alpha\beta} \\ (\alpha,\beta)\in \bigcup_{\alpha\in\Gamma}\Gamma\times\Gamma_\alpha}} Z_{\alpha\beta}(\mathscr{T}_{\alpha\beta})$.

The above straightforward reduction theorem can be supplemented by the next mixed type reduction.

THEOREM 3.6. *Let f_α be a relation from a relator space $X_\alpha(\mathscr{R}_\alpha)$ into a set Y for each α in a nonvoid set Γ and g be a relation from Y into a set Z such that g^{-1} is a*

function. Moreover, let

$$Z(\mathscr{T}) = \operatorname{ind} \operatorname*{gen}_{\alpha\in\Gamma} {}_{g\circ f_\alpha} X_\alpha(\mathscr{R}_\alpha) \quad \textit{and} \quad Y(\mathscr{S}) = \operatorname{proj} \operatorname{gen}_g Z(\mathscr{T}).$$

Then

$$Y(\mathscr{S}) = \operatorname{ind} \operatorname*{gen}_{\alpha\in\Gamma} {}_{f_\alpha} X_\alpha(\mathscr{R}_\alpha) \quad \textit{and} \quad Z(\mathscr{T}) = \operatorname{ind} \operatorname{gen}_g Y(\mathscr{S}).$$

PROOF. By Theorems 3.3 and 1.3, it is clear that each f_α is mildly continuous. Thus, we may define

$$Y(\mathscr{V}) = \operatorname{ind} \operatorname*{gen}_{\alpha\in\Gamma} {}_{f_\alpha} X_\alpha(\mathscr{R}_\alpha)$$

and we clearly have $\mathscr{S}\subset\mathscr{V}$.

To prove the converse inclusion, assume that $V\in\mathscr{V}$, and define a reflexive relation T on Z such that

$$T(z) = (g\circ V\circ g^{-1})(z) \quad \text{if} \quad z\in g(Y).$$

Then, it is clear that

$$g^{-1}\circ T\circ g = g^{-1}\circ(g\circ V\circ g^{-1})\circ g = (g^{-1}\circ g)\circ V\circ(g^{-1}\circ g) = \Delta_Y\circ V\circ\Delta_Y = V,$$

and hence

$$(g\circ f_\alpha)^{-1}\circ T\circ(g\circ f_\alpha) = f_\alpha^{-1}\circ(g^{-1}\circ T\circ g)\circ f_\alpha = f_\alpha^{-1}\circ V\circ f_\alpha\in\mathscr{R}_\alpha$$

for all $\alpha\in\Gamma$. Consequently, $T\in\mathscr{T}$, and hence $V=g^{-1}\circ T\circ g\in\mathscr{S}$.

This proves the first assertion of the theorem. The second assertion follows immediately from the first one by the implication (ii)⇒(i) in Theorem 3.5.

The next simple example shows that the extra condition that g^{-1} is a function cannot be omitted from Theorem 3.6.

EXAMPLE 3.7. Let $X=Y=\{0, 1\}$ and $Z=\{0\}$. Define $f=\Delta_X$, and let g be the unique function from Y into Z. Moreover, let $\mathscr{R}=\{\Delta_X\}^*$, $\mathscr{S}=\{Y\times Y\}$ and $\mathscr{T}=\{\Delta_Z\}$. Then, it is clear that

$$Z(\mathscr{T}) = \operatorname{ind} \operatorname{gen}_{g\circ f} X(\mathscr{R}) \quad \text{and} \quad Y(\mathscr{S}) = \operatorname{proj} \operatorname{gen}_g Z(\mathscr{T}),$$

but

$$Y(\mathscr{S}) \neq \operatorname{ind} \operatorname{gen}_f X(\mathscr{R}).$$

To feel the force of the second assertion of Theorem 3.6, the reader may also consider our forthcoming Example 3.20.

As a useful consequence of Theorem 3.6, we can now state that inductive generations are also compatible with subspaces.

COROLLARY 3.8. *If*

$$Y(\mathscr{S}) = \operatorname{ind} \operatorname{gen}^Y_{f_\alpha} X_\alpha(\mathscr{R}_\alpha)$$

and $Z(\mathscr{T})$ is a subspace of $Y(\mathscr{S})$ such that $f_\alpha(X_\alpha)\subset Z$ for all $\alpha\in\Gamma$, then

$$Z(\mathscr{T}) = \operatorname{ind} \operatorname*{gen}^Z_{\alpha\in\Gamma} {}_{f_\alpha} X_\alpha(\mathscr{R}_\alpha).$$

From [21, Proposition 4.12], one can at once derive much of the following theorem which states only a partial compatibility of inductive generations and mild hyperspaces.

THEOREM 3.9. *If*

$$Y(\mathscr{S}) = \operatorname{ind}\operatorname{gen}_{f_\alpha} X_\alpha(\mathscr{R}_\alpha)$$

and

$$\tilde{Y}(\mathscr{H}) = \operatorname{ind}\operatorname{gen}_{\Phi_{f_\alpha}} X_\alpha(\mathscr{R}_\alpha) \quad \text{and} \quad \tilde{Y}(\mathscr{K}) = \operatorname{ind}\operatorname{gen}_{\Phi_{f_\alpha}} \tilde{X}_\alpha(\tilde{\mathscr{R}}_\alpha),$$

then

$$\tilde{\mathscr{S}} \subset \mathscr{K} \subset \mathscr{H} \quad \text{and} \quad \tilde{\mathscr{S}} = \mathscr{H} \cap (\{\Delta_Y\}^*)^\sim.$$

To prove the inclusion $\mathscr{K} \subset \mathscr{H}$, note that if $K \in \mathscr{K}$ and $\alpha \in \Gamma$, then there exists $R \in \mathscr{R}_\alpha$ such that $\Phi_{f_\alpha}^{-1} \circ K \circ \Phi_{f_\alpha} = \tilde{R}$, whence by noticing that $(x, z) \in \varphi_{f_\alpha}^{-1} \circ K \circ \varphi_{f_\alpha}$ if and only if $(x, z) \in R$, it follows that $\varphi_{f_\alpha}^{-1} \circ K \circ \varphi_{f_\alpha} = R$, which proves $K \in \mathscr{H}$.

Note also that as an immediate consequence of this theorem, we also have

$$\tilde{\mathscr{S}} = \mathscr{K} \cap (\{\Delta_Y\}^*)^\sim.$$

The fact that the assertion of Theorem 3.9 cannot be sharpened follows at once from the next

EXAMPLE 3.10. Let $X = \{0, 1\}$, $Y = \{0, 1, 2\}$, $\mathscr{R} = \{\Delta_X\}$, $f = \Delta_X$. Moreover, define

$$Y(\mathscr{S}) = \operatorname{ind}\operatorname{gen}_f X(\mathscr{R})$$

and

$$\tilde{Y}(\mathscr{H}) = \operatorname{ind}\operatorname{gen}_{\varphi_f} X(\mathscr{R}) \quad \text{and} \quad \tilde{Y}(\mathscr{K}) = \operatorname{ind}\operatorname{gen}_{\Phi_f} \tilde{X}(\tilde{\mathscr{R}}).$$

Then $\mathscr{H} \not\subset \mathscr{K} \not\subset \tilde{\mathscr{S}}$.

To check this, define

$$H = \tilde{\Delta}_Y \setminus \{(\{0\}, \{0, 1\})\} \quad \text{and} \quad K = \tilde{\Delta}_Y \setminus \{(\{0\}, \{0, 1, 2\})\},$$

and note that $H \in \mathscr{H}$ and $K \in \mathscr{K}$, but $H \notin \mathscr{K}$ and $K \notin \tilde{\mathscr{S}}$.

Note also that by making the little change $Y = \{0, 1\}$ we still have a reasonable example to Theorem 3.9, but in this case the relation $\mathscr{K} \not\subset \tilde{\mathscr{S}}$ is no longer true.

In connection with the relationship of inductive generations to the basic refinement operations on relators, we can only prove the following

THEOREM 3.11. *If*

$$Y(\mathscr{S}) = \operatorname{ind}\operatorname{gen}_{f_\alpha} X_\alpha(\mathscr{R}_\alpha),$$

then the following assertions hold:

(i) *If* $\mathscr{R}_\alpha^* = \mathscr{R}_\alpha$ *for all* $\alpha \in \Gamma$, *then* $\mathscr{S}^* = \mathscr{S}$.
(ii) *If* $\mathscr{R}_\alpha^\# = \mathscr{R}_\alpha$ *for all* $\alpha \in \Gamma$, *then* $\mathscr{S}^\# = \mathscr{S}$.
(iii) *If* $\hat{\mathscr{R}}_\alpha = \mathscr{R}_\alpha$ *and* f_α *is a function for all* $\alpha \in \Gamma$, *then* $\hat{\mathscr{S}} = \mathscr{S}$.

The proof of this theorem is also rather straightforward. For instance, we prove (ii). For this, assume that $\mathscr{R}_\alpha^\# = \mathscr{R}_\alpha$ for all $\alpha \in \Gamma$, and pick $T \in \mathscr{S}^\#$. Then, by the definition of $\mathscr{S}^\#$, for any fixed $\alpha \in \Gamma$ and $A \subset X_\alpha$, there exists $S \in \mathscr{S}$ such that $S(f_\alpha(A)) \subset T(f_\alpha(A))$. Hence, it follows that $(f_\alpha^{-1} \circ S \circ f_\alpha)(A) \subset (f_\alpha^{-1} \circ T \circ f_\alpha)(A)$. This, since $f_\alpha^{-1} \circ S \circ f_\alpha \in \mathscr{R}_\alpha$, implies that $f_\alpha^{-1} \circ T \circ f_\alpha \in \mathscr{R}_\alpha^\#$. Hence, since $\mathscr{R}_\alpha^\# = \mathscr{R}_\alpha$, it is clear that $T \in \mathscr{S}$.

The next simple example shows that the extra condition that each f_α is a function cannot be omitted from the assertion (iii) of Theorem 3.11.

EXAMPLE 3.12. Let $X=\{0, 1\}$ and $Y=\{0, 1, 2\}$, and define the relation f from X into Y such that $f(0)=\{0, 1\}$ and $f(1)=\{2\}$. Moreover, let R be a relation on X such that $R(0)=\{0, 1\}$ and $R(1)=\{1\}$, and define $\mathscr{R}=\{R, X\times X\}$. Then, it is clear that $\hat{\mathscr{R}}=\mathscr{R}$, and

$$Y(\mathscr{S}) = \operatorname{ind} \operatorname{gen}_f X(\mathscr{R})$$

exists. But, despite this, now we still have $\hat{\mathscr{S}}\neq\mathscr{S}$.

Namely, if $\mathscr{V}$ is as in Example 1.11, then one can easily check that $\mathscr{V}\subset\mathscr{S}$, whence it is clear that $\Delta_Y\in\hat{\mathscr{S}}$.

On the other hand, now we clearly have

$$f^{-1}\circ\Delta_Y\circ f=f^{-1}\circ f=\Delta_X\notin\mathscr{R},$$

whence $\Delta_Y\notin\mathscr{S}$ follows.

The inconvenience that the definition of inductive generation is not, in contrast to that of the projective one, a constructive one can be partly eliminated by proving

THEOREM 3.13. *Let f_α be a relation from a relator space $X_\alpha(\mathscr{R}_\alpha)$ into a set Y for each α in a nonvoid set Γ, and $(\varphi_{\alpha\beta})_{\beta\in\Gamma_\alpha}$ be a nonvoid family of selection relations for f_α for each $\alpha\in\Gamma$. Moreover, let*

$$Y(\mathscr{S}) = \operatorname*{ind\,gen}_{\substack{\varphi_{\alpha\beta}\\(\alpha,\beta)\in\bigcup_{\alpha\in\Gamma}\Gamma\times\Gamma_\alpha}} X_\alpha(\mathscr{R}_\alpha)$$

and

$$\mathscr{V}=\Big\{\bigcup_{\alpha\in\Gamma} f_\alpha\circ R_\alpha\circ f_\alpha^{-1}\colon R_\alpha\in\mathscr{R}_\alpha \text{ for all } \alpha\in\Gamma\Big\}.$$

Then

(i) *$\mathscr{V}\subset\mathscr{S}$ if $\mathscr{R}_\alpha^*=\mathscr{R}_\alpha$ for all $\alpha\in\Gamma$ and $Y=\bigcup_{\alpha\in\Gamma} f_\alpha(X_\alpha)$;*

(ii) *$\mathscr{S}\subset\mathscr{V}^*$ if f_α is a function for all $\alpha\in\Gamma$ and $Y=\bigcup_{\alpha\in\Gamma} f_\alpha(X_\alpha)$;*

(iii) *$\mathscr{S}\subset\mathscr{V}$ if f_α is a function for all $\alpha\in\Gamma$ and $Y\times Y=\bigcup_{\alpha\in\Gamma} f_\alpha(X_\alpha)\times f_\alpha(X_\alpha)$.*

PROOF. If $V\in\mathscr{V}$, then for each $\alpha\in\Gamma$, there exists $R_\alpha\in\mathscr{R}_\alpha$ such that $V=\bigcup_{\alpha\in\Gamma} f_\alpha\circ R_\alpha\circ f_\alpha^{-1}$. Hence, it is clear that

$$\varphi_{\alpha\beta}^{-1}\circ V\circ\varphi_{\alpha\beta}\supset\varphi_{\alpha\beta}^{-1}\circ f_\alpha\circ R_\alpha\circ f_\alpha^{-1}\circ\varphi_{\alpha\beta}\supset\Delta_{X_\alpha}^{-1}\circ R_\alpha\circ\Delta_{X_\alpha}=R_\alpha,$$

and hence $\varphi_{\alpha\beta}^{-1}\circ V\circ\varphi_{\alpha\beta}\in\mathscr{R}_\alpha^*$ for all $\alpha\in\Gamma$ and $\beta\in\Gamma_\alpha$. Thus, we have $V\in\mathscr{S}$ if the conditions of (i) hold. (Note that the second condition of (i) is necessary in order that Y be the domain of V.)

On the other hand, if $S\in\mathscr{S}$, then $\varphi_{\alpha\beta_\alpha}^{-1}\circ S\circ\varphi_{\alpha\beta_\alpha}\in\mathscr{R}_\alpha$ for any $\alpha\in\Gamma$ and $\beta_\alpha\in\Gamma_\alpha$. Thus

$$W=\bigcup_{\alpha\in\Gamma} f_\alpha\circ\varphi_{\alpha\beta_\alpha}^{-1}\circ S\circ\varphi_{\alpha\beta_\alpha}\circ f^{-1}\in\mathscr{V}.$$

Moreover, if f_α is a function for all $\alpha\in\Gamma$, then $\varphi_{\alpha\beta_\alpha}=f_\alpha$, and hence $\Delta_\alpha=f_\alpha\circ\varphi_{\alpha\beta_\alpha}^{-1}=$

$=f_\alpha \circ f_\alpha^{-1}$ is the identity function of $Y_\alpha = f_\alpha(X_\alpha)$ for all $\alpha \in \Gamma$. Thus, we also have

$$W = \bigcup_{\alpha\in\Gamma} \Delta_\alpha \circ S \circ \Delta_\alpha^{-1} = \bigcup_{\alpha\in\Gamma} S \cap (Y_\alpha \times Y_\alpha) = S \cap \bigcup_{\alpha\in\Gamma} Y_\alpha \times Y_\alpha .$$

Hence, it is clear that $S \in \mathscr{V}^*$ if $Y = \bigcup_{\alpha\in\Gamma} Y_\alpha$, and $S \in \mathscr{V}$ if $Y \times Y = \bigcup_{\alpha\in\Gamma} Y_\alpha \times Y_\alpha$. (Note that this latter condition implies the former one, but not conversely.)

Important particular cases are when Γ_α is a singleton and $\varphi_{\alpha\beta} = f_\alpha$ for all $\alpha \in \Gamma$, or Γ is a singleton.

In this latter particular case, Theorem 3.13 yields the following analogue of Theorem 1.13.

COROLLARY 3.14. *Let f be a relation from a relator space $X(\mathscr{R})$ onto a set Y and $(\varphi_\alpha)_{\alpha\in\Gamma}$ be a nonvoid family of selection relations for f. Moreover, let*

$$Y(\mathscr{S}) = \operatorname{ind} \operatorname*{gen}_{\substack{\varphi_\alpha \\ \alpha\in\Gamma}} X(\mathscr{R}) \quad \text{and} \quad Y(\mathscr{V}) = \operatorname{proj} \operatorname{gen}_{f^{-1}} X(\mathscr{R}).$$

Then $\mathscr{V} \subset \mathscr{S}$ if $\mathscr{R}^* = \mathscr{R}$, and $\mathscr{S} \subset \mathscr{V}$ if f is a function.

From the first assertion of Corollary 3.14, using Theorems 1.2 and 3.3, one can easily get a useful particular case of Theorem 1.16.

However, in the present context, it is more interesting to prove

THEOREM 3.15. *If f is a relation from a uniformly fine relator space $X(\mathscr{R})$ into a set Y and*

$$Y(\mathscr{S}) = \operatorname{ind} \operatorname{gen}_f X(\mathscr{R}),$$

then both f and f^{-1} are mildly continuous.

PROOF. The mild continuity of f is apparent from Theorem 3.3. To prove that of f^{-1}, note that if $Z(\mathscr{T})$ is the subspace of $Y(\mathscr{S})$ generated by $f(X)$, then by Corollary 3.8, we also have

$$Z(\mathscr{T}) = \operatorname{ind} \operatorname{gen}_f^{\mathrm{Z}} X(\mathscr{R}).$$

Therefore, if

$$Z(\mathscr{U}) = \operatorname{proj} \operatorname{gen}_{f^{-1}} X(\mathscr{R}).$$

Then by the first assertion of Corollary 3.14, we clearly have $\mathscr{U} \subset \mathscr{T}$, whence by Theorem 1.2, the mild $(\mathscr{T}, \mathscr{R})$-continuity of f^{-1} is also obvious.

Note that the mild continuity of f^{-1} can also be easily proved with the help of Theorem 3.4 and [21, Corollary 4.5].

The next simple example shows that the condition $\mathscr{R}^* = \mathscr{R}$ cannot be omitted from the above assertions even if f is a function.

EXAMPLE 3.16. Let $X = \{0, 1, 2\}$ and $Y = \{0, 1\}$. Define the function f from X onto Y such that $f(x) = x$ if $x \in \{0, 1\}$ and $f(2) = 0$. Moreover, let $\mathscr{R} = \{\Delta_X, X \times X\}$, $\mathscr{S} = \{Y \times Y\}$ and $\mathscr{V} = \{\Delta_Y, Y \times Y\}$. Then, it is clear that

$$Y(\mathscr{S}) = \operatorname{ind} \operatorname{gen}_f X(\mathscr{R}) \quad \text{and} \quad Y(\mathscr{V}) = \operatorname{proj} \operatorname{gen}_{f^{-1}} X(\mathscr{R}),$$

but $\mathscr{V} \not\subset \mathscr{S}$, and thus f is not $(\mathscr{R}, \mathscr{V})$-continuous and f^{-1} is not mildly $(\mathscr{S}, \mathscr{R})$-continuous.

Using the second assertion of Corollary 3.14, we can also prove a partial converse to Theorem 3.15 which is an analogue of the statement (7.4.12) in Császár [5].

THEOREM 3.17. *If f is a continuous function from a relator space $X(\mathscr{R})$ onto another $Y(\mathscr{S})$ such that f^{-1} is mildly continuous, then*

$$Y(\mathscr{S}) = \operatorname{ind} \operatorname{gen}_f X(\mathscr{R}).$$

PROOF. Because of the continuity of f, we may define

$$Y(\mathscr{V}) = \operatorname{ind} \operatorname{gen}_f X(\mathscr{R}).$$

Moreover, by Theorem 3.3, we clearly have $\mathscr{S} \subset \mathscr{V}$.

On the other hand, by Theorem 1.18, now we also have

$$Y(\mathscr{S}) = \operatorname{proj} \operatorname{gen}_{f^{-1}} X(\mathscr{R}).$$

Hence, by the second assertion of Corollary 3.14, it is clear that $\mathscr{V} \subset \mathscr{S}$ is also true.

Applying this theorem to Theorem 2.11, one can at once state an analogue of the statement (7.4.13) in Császár [5].

COROLLARY 3.18. *If*

$$X(\mathscr{R}) = \bigtimes_{\alpha \in \Gamma} Y_\alpha(\mathscr{S}_\alpha)$$

and π_α is the projection of X onto Y_α, then

$$Y_\alpha(\mathscr{S}_\alpha) = \operatorname{ind} \operatorname{gen}_{\pi_\alpha} X(\mathscr{R})$$

whenever $Y_\alpha \times Y_\alpha \in \mathscr{S}_\alpha$.

From Theorem 3.17, by using the implication (i)⇒(iii) in Corollary 1.19, one can also immediately get

COROLLARY 3.19. *If f is a function from a set X onto a relator space $Y(\mathscr{S})$ and*

$$X(\mathscr{R}) = \operatorname{proj} \operatorname{gen}_f Y(\mathscr{S}),$$

then

$$Y(\mathscr{S}) = \operatorname{ind} \operatorname{gen}_f X(\mathscr{R}).$$

The next simple example shows that Theorem 3.17 and Corollary 3.19 do not, in general, hold for relations instead of functions.

EXAMPLE 3.20. Let $X = \{0\}$ and $Y = \{0, 1\}$. Denote by f the unique relation from X onto Y. Moreover, let $\mathscr{R} = \{\Delta_X\}$ and $\mathscr{S} = \{Y \times Y\}$. Then, it is clear that

$$X(\mathscr{R}) = \operatorname{proj} \operatorname{gen}_f Y(\mathscr{S}),$$

but

$$Y(\mathscr{S}) \neq \operatorname{ind} \operatorname{gen}_f X(\mathscr{R})$$

despite that f^{-1} is now an $(\mathscr{S}, \mathscr{R})$-continuous function.

In this respect, it is also worth mentioning that we can also prove a certain converse to Corollary 3.19.

THEOREM 3.21. *If f is a relation from a relator space $X(\mathscr{R})$ onto a set Y such that f^{-1} is a function, and*

$$Y(\mathscr{S}) = \operatorname{ind} \operatorname{gen}_f X(\mathscr{R}),$$

then

$$X(\mathscr{R}) = \operatorname{proj} \operatorname{gen}_f Y(\mathscr{S}).$$

PROOF. Define

$$X(\mathscr{U}) = \operatorname{proj\,gen}_f Y(\mathscr{S}).$$

Then, by Theorems 3.3 and 1.2, it is clear that $\mathscr{U} \subset \mathscr{R}$.

On the other hand, if $R \in \mathscr{R}$, then $S = f \circ R \circ f^{-1}$ is a reflexive relation on Y such that

$$f^{-1} \circ S \circ f = f^{-1} \circ f \circ R \circ f^{-1} \circ f = \Delta_X \circ R \circ \Delta_X = R,$$

whence it is clear that $S \in \mathscr{S}$ and $R = f^{-1} \circ S \circ f \in \mathscr{U}$. Consequently, we also have $\mathscr{R} \subset \mathscr{U}$.

Note that the particular case of this theorem when $\mathscr{R}^* = \mathscr{R}$ is already an immediate consequence of Theorem 3.15 and Theorem 1.18, while the theorem itself is certainly not.

The fact that the extra condition that f^{-1} is a function cannot, even in the best cases, be omitted from Theorem 3.21 follows at once from the next simple.

EXAMPLE 3.22. Let $X = \{0, 1\}$ and $Y = \{0\}$. Denote by f the unique function from X onto Y. Moreover, let $\mathscr{R} = \{\Delta_X\}^*$ and $\mathscr{S} = \{\Delta_Y\}$. Then, it is clear that

$$Y(\mathscr{S}) = \operatorname{ind\,gen}_f X(\mathscr{R}),$$

but

$$X(\mathscr{R}) \neq \operatorname{proj\,gen}_f Y(\mathscr{S})$$

despite that f is now a function and $\mathscr{R}^* = \mathscr{R}$.

Finally, we remark that combining Theorem 3.21 with Corollary 3.19, we can supplement Corollary 1.21 by stating

COROLLARY 3.23. *If f is an injection from a relator space $X(\mathscr{R})$ onto another $Y(\mathscr{S})$, then the following assertions are equivalent:*

(i) $X(\mathscr{R}) = \operatorname{proj\,gen}_f Y(\mathscr{S})$;

(ii) $Y(\mathscr{S}) = \operatorname{ind\,gen}_f X(\mathscr{R})$.

4. Application of inductive generation

DEFINITION 4.1. If ϱ is a relation defined on a relator space $X(\mathscr{R})$ and $X|\varrho = \varphi_\varrho(X)$, then the relator space

$$X|\varrho(\mathscr{R}|\varrho) = \operatorname{ind\,gen}_{\varphi_\varrho} X(\mathscr{R}),$$

whenever it exists, will be called the quotient space of $X(\mathscr{R})$ generated by ϱ.

REMARK 4.2. The set-valued mapping φ_ϱ induced by ϱ will now be called the projection, or the canonical mapping of X onto $X|\varrho$.

Note that under the identification of singletons with their elements, because of Definition 4.1, we always have $X|\varrho(\mathscr{R}|\varrho) = X|\varphi_\varrho(\mathscr{R}|\varphi_\varrho)$.

Several properties of quotient spaces can be easily derived from the results of Section 3. For instance, Corollary 3.8 and Theorem 3.9 yield

THEOREM 4.3. *If ϱ is a relation from a relator space $X(\mathscr{R})$ into a set Y such that the relator space*

$$Y(\mathscr{S}) = \operatorname{ind\,gen}_\varrho X(\mathscr{R})$$

exists, then the quotient space $X|\varrho(\mathscr{R}|\varrho)$ *also exists, moreover*

$$\tilde{\mathscr{S}}\wedge\{(X|\varrho)\times(X|\varrho)\} = (\mathscr{R}|\varrho)\cap((\{\Delta_Y\}^*)^\sim\wedge\{(X|\varrho)\times(X|\varrho)\}).$$

As a useful consequence of this theorem, we have

COROLLARY 4.4. *If* ϱ *is a mildly continuous relation from a relator space* $X(\mathscr{R})$ *into another* $Y(\mathscr{S})$, *then the quotient space* $X|\varrho(\mathscr{R}|\varrho)$ *exists, moreover*

$$\tilde{\mathscr{S}}\wedge\{(X|\varrho)\times(X|\varrho)\}\subset\mathscr{R}|\varrho.$$

Furthermore, for instance, Remark 3.2 and Theorems 3.11 and 3.15 yield

THEOREM 4.5. *If* ϱ *is a relation defined on a uniformly fine relator space* $X(\mathscr{R})$, *then the quotient space* $X|\varrho(\mathscr{R}|\varrho)$ *exists and is also uniformly fine. Moreover, the inverse* φ_ϱ^{-1} *of the projection* φ_ϱ *is also mildly continuous.*

REMARK 4.6. By Example 3.16, it is clear that the extra condition $\mathscr{R}^*=\mathscr{R}$ cannot be omitted from this theorem even if ϱ is a function.

However, for nonmingled-valued relations, we can still establish an essential improvement of the second part of Theorem 4.5.

DEFINITION 4.7. A relation ϱ will be called nonmingled-valued if $\varrho(x)\cap\varrho(y)\neq\emptyset$ implies $\varrho(x)=\varrho(y)$ for any two elements x and y of the domain of ϱ.

The above terminology and the first half of the next proposition are due to Whyburn [23, p. 1495].

PROPOSITION 4.8. *If* ϱ *is a relation, then the following properties are equivalent:*

(i) ϱ *is nonmingled-valued;*

(ii) $\varrho\circ\varrho^{-1}\circ\varrho=\varrho$;

(iii) $\varrho^{-1}\circ\varrho=\varrho^{-1}\circ\varphi$ *for any selection* φ *for* ϱ.

PROOF. We shall only prove the equivalence of (i) and (iii) since that of (i) and (ii) has been proved in [12].

Suppose first that (i) holds, and let φ be a selection for ϱ. Then, since $\varphi\subset\varrho$, it is clear that $\varrho^{-1}\circ\varphi\subset\varrho^{-1}\circ\varrho$. On the other hand, if $y\in\varrho^{-1}(\varrho(x))$, then $\varrho(y)\cap\varrho(x)\neq\emptyset$, and hence by (i) we have $\varrho(y)=\varrho(x)$. Thus, since $\varphi(x)\in\varrho(x)$, we also have $\varphi(x)\in\varrho(y)$, whence $y\in\varrho^{-1}(\varphi(x))$. Consequently, $\varrho^{-1}\circ\varrho\subset\varrho^{-1}\circ\varphi$ is also true.

Suppose now that (iii) holds and $\varrho(x)\cap\varrho(y)\neq\emptyset$. Then $y\in\varrho^{-1}(\varrho(x))$, and hence by (iii), we have $y\in\varrho^{-1}(\varphi(x))$, i.e., $\varphi(x)\in\varrho(y)$ for any selection φ for ϱ. Hence, by the axiom of choice, it is clear that $\varrho(x)\subset\varrho(y)$.

Using this proposition, one can easily verify the next simple but useful assertions.

COROLLARY 4.9. *A relation is nonmingled-valued if and only if its inverse is nonmingled-valued.*

COROLLARY 4.10. *If* ϱ *is a nonmingled-valued relation, then* $\varrho^{-1}\circ\varrho$ *is an equivalence relation.*

REMARK 4.11. By defining a relation ϱ on $\{0, 1\}$ such that $\varrho(0)=\{0, 1\}$ and $\varrho(1)=\{1, 2\}$, one can at once see that the converse of the above assertion is not, in general, true.

COROLLARY 4.12. *A relation is nonmingled-valued if and only if it is the composition of an equivalence relation and a function.*

COROLLARY 4.13. *A reflexive relation is nonmingled-valued if and only if it is an equivalence relation.*

The next proposition offers some more relevant characterizations of nonmingled-valued relations.

PROPOSITION 4.14. *If ϱ is a relation, then the following properties are equivalent:*

(i) *ϱ is nonmingled-valued;*
(ii) $\varphi_\varrho^{-1}\circ\varphi_\varrho=\varrho^{-1}\circ\varrho$;
(iii) $\psi_\varrho=\varphi_\varrho\circ\varrho^{-1}\circ\varphi_{\varrho^{-1}}^{-1}$ *is a function.*

PROOF. The equivalence of (i) and (ii) is immediate from the fact that $y\in\varphi_\varrho^{-1}(\varphi_\varrho(x))$ iff $\varrho(y)=\varrho(x)$, and $y\in\varrho^{-1}(\varrho(x))$ iff $\varrho(y)\cap\varrho(x)\neq\emptyset$.

To prove the equivalence of (i) and (iii), it seems convenient to use that by [13, Lemma 1.35], we have

$$\psi_\varrho=\{(\varrho^{-1}(y),\ \varrho(x))\colon (x, y)\in\varrho\}.$$

Namely, if $(x, y)\in\varrho$ and $(z, w)\in\varrho$ such that $\varrho^{-1}(y)=\varrho^{-1}(w)$, then $y\in\varrho(x)$ and $z\in\varrho^{-1}(w)=\varrho^{-1}(y)$, whence $y\in\varrho(z)$. Thus, $\varrho(x)\cap\varrho(z)\neq\emptyset$, whence $\varrho(x)=\varrho(z)$ follows if (i) holds.

On the other hand, if $\varrho(x)\cap\varrho(y)\neq\emptyset$, then there exists $z\in\varrho(x)$ such that $z\in\varrho(y)$. Thus $(\varrho^{-1}(z),\ \varrho(x))\in\psi_\varrho$ and $(\varrho^{-1}(z),\ \varrho(y))\in\psi_\varrho$, whence $\varrho(x)=\varrho(y)$ follows if (iii) holds.

REMARK 4.15. Another important property of the relation ψ_ϱ associated with ϱ is that

$$\psi_\varrho^{-1}=\varphi_{\varrho^{-1}}\circ\varrho\circ\varphi_\varrho^{-1}=\psi_{\varrho^{-1}}.$$

Hence, by Corollary 4.9 and Proposition 4.14, it is clear that we can also state

COROLLARY 4.16. *A relation ϱ is nonmingled-valued if and only if the associated relation ψ_ϱ is an injection.*

Using Proposition 4.14, we can also easily establish a useful criterion for the mild continuity of φ_ϱ^{-1} whenever ϱ is nonmingled-valued.

THEOREM 4.17. *If ϱ is a nonmingled-valued relation defined on a relator space $X(\mathscr{R})$ such that the quotient space $X|\varrho(\mathscr{R}|\varrho)$ exists, then φ_ϱ^{-1} is mildly continuous if and only if $\varrho^{-1}\circ\varrho$ is mildly continuous.*

PROOF. By Theorem 3.4, it is clear that φ_ϱ^{-1} is mildly continuous if and only if $\varphi_\varrho^{-1}\circ\varphi_\varrho$ is mildly continuous. On the other hand, by Proposition 4.14, now we also have $\varphi_\varrho^{-1}\circ\varphi_\varrho=\varrho^{-1}\circ\varrho$. Thus the proof is complete.

The important particular case of this theorem, when ϱ is an equivalence relation yields and analogue of the first part of Theorem 3.10 in Kelley [11, p. 97].

COROLLARY 4.18. *If ϱ is an equivalence relation on a relator space $X(\mathscr{R})$ such that the quotient space $X|\varrho(\mathscr{R}|\varrho)$ exists, then φ_ϱ^{-1} is mildly continuous if and only if ϱ is mildly continuous.*

Using Theorem 4.17, we can also easily establish an extension of a standard isomorphism theorem on classical quotient spaces of uniform spaces.

THEOREM 4.19. *If ϱ is a nonmingled-valued relation from a relator space $X(\mathscr{R})$ onto another $Y(\mathscr{S})$ such that both ϱ and ϱ^{-1} are mildly continuous, then ψ_ϱ is an isomorphism of $Y|\varrho^{-1}(\mathscr{S}|\varrho^{-1})$ onto $X|\varrho(\mathscr{R}|\varrho)$.*

PROOF. From Corollary 4.16, we know that ψ_ϱ is an injection. Moreover, Corollary 4.4 guarantees that the corresponding quotient space exists. Thus, it remains only to show that ψ_ϱ and ψ_ϱ^{-1} are continuous.

By Corollary 4.9, ϱ^{-1} is also nonmingled-valued. Moreover, by [21, Theorem 4.6], $\varrho\circ\varrho^{-1}$ is mildly continuous. Hence, using Theorem 4.17 with ϱ^{-1} instead of ϱ, we can infer that $\varphi_{\varrho^{-1}}^{-1}$ is mildly continuous. On the other hand, by Theorem 3.3, φ_ϱ is continuous. Hence, again by [21, Theorem 4.6], it is clear that $\psi_\varrho = \varphi_\varrho\circ\varrho^{-1}\circ\varphi_{\varrho^{-1}}^{-1}$ is continuous.

The continuity of ψ_ϱ^{-1} follows immediately from that of ψ_ϱ by Corollary 4.7 and Remark 4.15.

From this theorem, using Theorems 1.16 and 3.17, one can at once derive

COROLLARY 4.20. *If ϱ is a function from a set X onto a relator space $Y(\mathscr{S})$ then $\varphi_{\varrho^{-1}}^{-1}$ is an isomorphism of $Y|\varrho^{-1}(\mathscr{S}|\varrho^{-1})$ onto $Y(\mathscr{S})$.*

To make the above result even more familiar to the reader we also prove

THEOREM 4.21. *If ϱ is a nonmingled-valued relation from a relator space $X(\mathscr{R})$ onto another $Y(\mathscr{S})$ such that both ϱ and ϱ^{-1} are mildly continuous, then*

$$Y|\varrho^{-1}(\mathscr{S}|\varrho^{-1}) = X|\varrho^{-1}\circ\varrho(\mathscr{R}|\varrho^{-1}\circ\varrho).$$

PROOF. By Proposition 4.8, we have $\varrho^{-1}(\varrho(x)) = \varrho^{-1}(y)$, and hence $\varphi_{\varrho^{-1}\circ\varrho}(x) = \varphi_{\varrho^{-1}}(y)$ for any $y\in\varrho(x)$. This shows that

$$X|\varrho^{-1}\circ\varrho = Y|\varrho^{-1} \quad \text{and} \quad \varphi_{\varrho^{-1}\circ\varrho} = \varphi_{\varrho^{-1}}\circ\varrho.$$

If $T\in\mathscr{S}|\varrho^{-1}$, then by the definition of $\mathscr{S}|\varrho^{-1}$, we have $\varphi_{\varrho^{-1}}^{-1}\circ T\circ\varphi_{\varrho^{-1}}\in\mathscr{S}$. Hence, since ϱ is mildly continuous, it is clear that

$$\varphi_{\varrho^{-1}\circ\varrho}^{-1}\circ T\circ\varphi_{\varrho^{-1}\circ\varrho} = \varrho^{-1}\circ(\varphi_{\varrho^{-1}}^{-1}\circ T\circ\varphi_{\varrho^{-1}})\circ\varrho\in\mathscr{R},$$

whence $T\in\mathscr{R}|\varrho^{-1}\circ\varrho$ follows.

Conversely, if $T\in\mathscr{R}|\varrho^{-1}\circ\varrho$, then a similar argument as above shows that $\varrho^{-1}\circ\varphi_{\varrho^{-1}}^{-1}\circ T\circ\varphi_{\varrho^{-1}}\circ\varrho\in\mathscr{R}$. Hence, by the mild continuity of ϱ^{-1}, it follows that

$$\varrho\circ\varrho^{-1}\circ\varphi_{\varrho^{-1}}^{-1}\circ T\circ\varphi_{\varrho^{-1}}\circ\varrho\circ\varrho^{-1}\in\mathscr{S}.$$

On the other hand, by Corollary 4.9 and Proposition 4.14 and 4.8, we clearly have

$$\varphi_{\varrho^{-1}} \circ \varrho \circ \varrho^{-1} = \varphi_{\varrho}^{-1} \circ \varphi_{\varrho^{-1}}^{-1} \circ \varphi_{\varrho^{-1}} = \varphi_{\varrho^{-1}}.$$

Thus, $\varphi_{\varrho^{-1}}^{-1} \circ T \circ \varphi_{\varrho^{-1}} \in \mathscr{S}$, whence $T \in \mathscr{S}|\varrho^{-1}$ follows.

From this theorem, using Theorem 1.16 and Corollary 4.9, one can at once derive

COROLLARY 4.22. *If ϱ is a nonmingled-valued relation from a relator space $X(\mathscr{R})$ onto a set Y such that either $\mathscr{R}^* = \mathscr{R}$ or ϱ^{-1} is a function, and*

$$Y(\mathscr{S}) = \operatorname{proj} \operatorname{gen}_{\varrho^{-1}} X(\mathscr{R}),$$

then

$$X|\varrho(\mathscr{R}|\varrho) = Y|\varrho \circ \varrho^{-1}(\mathscr{S}|\varrho \circ \varrho^{-1}).$$

REMARK 4.23. Note that by Corollaries 4.9 and 4.10, $\varrho \circ \varrho^{-1}$ is now an equivalence relation.

And thus quotient spaces defined by nonmingled-valued relations can usually be obtained by equivalences.

Another important application of inductive generation is offered by the next trivial

THEOREM 4.24. *If $\mathscr{R}_\alpha$ is a relator on a set X for each α in a nonvoid set X such that $\mathscr{R} = \bigcap_{\alpha \in \Gamma} \mathscr{R}_\alpha$ is nonvoid, then*

$$X(\mathscr{R}) = \operatorname{ind} \operatorname{gen}_{\Delta_X} X(\mathscr{R}_\alpha).$$

This theorem allows us to derive several properties of intersections of relators from the results of Section 3. For instance, using Theorem 4.24, from Theorems 3.9, 3.3 and 3.11, one can easily get an analogue of Corollary 2.5.

COROLLARY 4.25. *If $\mathscr{R}_\alpha$ is a relator on a set X for each α in a nonvoid set X such that $\mathscr{R} = \bigcap_{\alpha \in \Gamma} \mathscr{R}_\alpha$ is nonvoid, then*

(i) $\tilde{\mathscr{R}} = \bigcap_{\alpha \in \Gamma} \tilde{\mathscr{R}}_\alpha$;

(ii) $\bigcup_{\alpha \in \Gamma} \operatorname{Lim}_{\mathscr{R}_\alpha} \subset \operatorname{Lim}_{\mathscr{R}}$ *and* $\bigcup_{\alpha \in \Gamma} \operatorname{Cl}_{\mathscr{R}_\alpha} \subset \operatorname{Cl}_{\mathscr{R}}$;

(iii) $\mathscr{R}^\square = \mathscr{R}$ *if* $\mathscr{R}_\alpha^\square = \mathscr{R}_\alpha$ *for all* $\alpha \in \Gamma$, *where* $\square \in \{*, \#, \wedge\}$.

The fact that the equalities do not, in general, hold in (ii) can be seen at once from the next simple

EXAMPLE 4.26. If $X = \{0, 1\}$, $\mathscr{R}_1 = \{\Delta_X, X \times X\}$ and $\mathscr{R}_2 = \{\Delta_X \cup \{(0, 1)\}, X \times X\}$ then the relator $\mathscr{R} = \mathscr{R}_1 \cap \mathscr{R}_2$ has the properties

$$\operatorname{Lim}_{\mathscr{R}} \neq \operatorname{Lim}_{\mathscr{R}_1} \cup \operatorname{Lim}_{\mathscr{R}_2} \quad \text{and} \quad \operatorname{Cl}_{\mathscr{R}} \neq \operatorname{Cl}_{\mathscr{R}_1} \cup \operatorname{Cl}_{\mathscr{R}_2}.$$

To check this, consider for instance the sequences $x = (0)$ and $y = (1)$, and the sets $A = \{0\}$ and $B = \{1\}$.

Finally, we shall briefly discuss free sums which seem to play the same role in the theory of inductive generations as the cartesian products do in that of projective generations.

DEFINITION 4.27. If $X_\alpha(\mathscr{R}_\alpha)$ is a relator space for each α in a nonvoid set Γ,

$$Y = \sum_{\alpha\in\Gamma} X_\alpha = \bigcup_{\alpha\in\Gamma} X_\alpha\times\{\alpha\},$$

and j_α is the canonical injection of X_α into Y defined by $j_\alpha(x)=(x, \alpha)$, then the relator space

$$Y(\mathscr{S}) = \operatorname*{ind\,gen}_{j_\alpha\ \alpha\in\Gamma} X_\alpha(\mathscr{R}_\alpha),$$

provided it exists, will be called the free sum of the spaces $X_\alpha(\mathscr{R}_\alpha)$, and the notation

$$Y(\mathscr{S}) = \sum_{\alpha\in\Gamma} X_\alpha(\mathscr{R}_\alpha)$$

will be used.

Several properties of free sums can be easily derived from the results of Section 3. Moreover, in addition to Theorems 3.13 and 3.3, we can now prove

THEOREM 4.28. *If $X_\alpha(\mathscr{R}_\alpha)$ is a relator space for each α in a nonvoid set Γ, then the free sum*

$$Y(\mathscr{S}) = \sum_{\alpha\in\Gamma} X_\alpha(\mathscr{R}_\alpha)$$

exists, moreover if

$$\mathscr{V} = \Big\{\bigcup_{\alpha\in\Gamma} j_\alpha\circ R_\alpha\circ j_\alpha^{-1}\colon R_\alpha\in\mathscr{R}_\alpha \text{ for all } \alpha\in\Gamma\Big\},$$

then

(i) $\mathscr{V}\subset\mathscr{S}\subset\mathscr{V}^*$;

(ii) *the subspaces of $Y(\mathscr{V})$ and $Y(\mathscr{S})$ generated by $j_\alpha(X_\alpha)$ coincide for all $\alpha\in\Gamma$;*

(iii) *the canonical injection j_α is an isomorphism of $X_\alpha(\mathscr{R}_\alpha)$ onto a subspace of $Y(\mathscr{S})$ for all $\alpha\in\Gamma$.*

PROOF. If $V\in\mathscr{V}$, then for each $\alpha\in\Gamma$, there exists $R_\alpha\in\mathscr{R}_\alpha$ such that $V=\bigcup_{\alpha\in\Gamma} j_\alpha\circ R_\alpha\circ j_\alpha^{-1}$. Hence, we can now infer that

$$j_\beta^{-1}\circ V\circ j_\beta = \bigcup_{\alpha\in\Gamma} j_\beta^{-1}\circ j_\alpha\circ R_\alpha\circ j_\alpha^{-1}\circ j_\beta = R_\beta\in\mathscr{R}_\beta$$

for any $\beta\in\Gamma$. This shows that $V\in\mathscr{S}$. Consequently, we have $\mathscr{V}\subset\mathscr{S}$. The other inclusion in (i) follows directly from Theorem 3.13.

If $\alpha\in\Gamma$, then the $(\mathscr{R}_\alpha, \mathscr{S})$-continuity of j_α is immediate from Theorem 3.3. On the other hand, if $R_\alpha\in\mathscr{R}_\alpha$ then by choosing $R_\beta\in\mathscr{R}_\beta$ for each $\beta\in\Gamma\setminus\{\alpha\}$ and defining $V=\bigcup_{\beta\in\Gamma} j_\beta\circ R_\beta\circ j_\beta^{-1}$, we clearly have

$$j_\alpha\circ R_\alpha\circ j_\alpha^{-1} = j_\alpha\circ j_\alpha^{-1}\circ V\circ j_\alpha\circ j_\alpha^{-1} = V\cap\big(j_\alpha(X_\alpha)\times j_\alpha(X_\alpha)\big).$$

Whence, since $V\in\mathscr{V}$, it is clear that j_α^{-1} is a continuous function of a subspace of $Y(\mathscr{V})$ onto $X_\alpha(\mathscr{R}_\alpha)$.

Thus, since $\mathscr{V}\subset\mathscr{S}$, we can state that j_α is an isomorphism of $X_\alpha(\mathscr{R}_\alpha)$ onto a subspace of both $Y(\mathscr{V})$ and $Y(\mathscr{S})$. Consequently, the assertions (ii) and (iii) are also true.

The next simple example shows that the first assertion of the above theorem cannot be improved by stating that $\mathscr{S}=\mathscr{V}$.

EXAMPLE 4.29. Let $\Gamma=\{0, 1\}$, and for $\alpha\in\Gamma$, define $X_\alpha=\{\alpha\}$ and $\mathscr{R}_\alpha=\{\Delta_{X_\alpha}\}$. Moreover, let $Y(\mathscr{S})$ and $\mathscr{V}$ be as in Theorem 4.28. Then, it is clear that $Y=\Delta_\Gamma$, $\mathscr{S}=\{\Delta_Y\}^*$ and $\mathscr{V}=\{\Delta_Y\}$, and thus $\mathscr{S}\not\subset\mathscr{V}$.

The importance of free sums as particular inductive generations lies mainly in the following reduction

THEOREM 4.30. *If*

$$Y(\mathscr{S}) = \operatorname{ind}\operatorname{gen}_{f_\alpha} X_\alpha(\mathscr{R}_\alpha) \quad \textit{and} \quad X(\mathscr{R}) = \sum_{\alpha\in\Gamma} X_\alpha(\mathscr{R}_\alpha),$$

and f is the relation from X into Y defined by $f((x, \alpha))=f_\alpha(x)$, then

$$Y(\mathscr{S}) = \operatorname{ind}\operatorname{gen}_f X(\mathscr{R}).$$

PROOF. Note that, for $\alpha\in\Gamma$, we have $f_\alpha=f\circ j_\alpha$, where j_α is the canonical injection of X_α into X. Thus the implication (ii)⇒(i) in Theorem 3.5 can be applied to get the assertion.

From this theorem, using Theorems 3.11, 3.15 and 4.21 and Corollary 4.20, one can easily derive

COROLLARY 4.31. *If f_α is a function from a uniformly fine relator space $X_\alpha(\mathscr{R}_\alpha)$ into a set Y for each α in a nonvoid set Γ such that $Y=\bigcup_{\alpha\in\Gamma} f_\alpha(X_\alpha)$, moreover*

$$X(\mathscr{R}) = \sum_{\alpha\in\Gamma} X_\alpha(\mathscr{R}_\alpha) \quad \textit{and} \quad Y(\mathscr{S}) = \operatorname{ind}\operatorname{gen}_{f_\alpha} X_\alpha(\mathscr{R}_\alpha),$$

and f is the function defined on X by $f((x, \alpha))=f_\alpha(x)$, then $\varphi_{f^{-1}}^{-1}$ is an isomorphism of $X|f^{-1}\circ f(\mathscr{R}|f^{-1}\circ f)$ onto $Y(\mathscr{S})$.

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