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WIENER'S TAUBERIAN THEOREMS FOR GENERALIZED MEASURES DEFINED ON THE DYADIC FIELD

K. YONEDA (Sakai)

The following Wiener's tauberian theorems are very important in analysis (see [6]).

THEOREM A. *If h is an L^∞-integrable function and* $\lim_{x\to\infty} h(x)=A$, *then we have*

$$\lim_{x\to\infty} (k*h)(x) = A \int_{-\infty}^{\infty} k(y)\,dy$$

for each $k\in L^1(-\infty,\infty)$.

THEOREM B. *When k_1 is an integrable function, and*

$$\lim_{x\to\infty} (k_1*h)(x) = A \int_{-\infty}^{\infty} k_1(y)\,dy$$

for each $h\in L^\infty(-\infty,\infty)$, *then*

$$\lim_{x\to\infty} (k*h)(x) = A \int_{-\infty}^{\infty} k(y)\,dy$$

for each $k\in L^1(-\infty,\infty)$ *if and only if* $\hat{k}_1(x)\neq 0$ *for each x (see* [2], [5]*).*

In this paper, we shall generalize Theorems A and B for generalized measures, so-called dyadic measures or quasi-measures, defined on the dyadic field.

For details of properties of the dyadic field we refer to [7]. The dyadic field is the set of all sequences $(\dots, x_i, \dots)$ such that $x_i=0$ or 1 and $\lim_{i\to-\infty} x_i=0$. In [3] and [4], N. J. Fine investigated many properties of the dyadic group and the dyadic field. In these two papers Fine defined the group operations $\dotplus$ and $\cdot$. The additive group of the dyadic field is a locally compact abelian group and its characters are called Walsh functions. A Walsh function is defined by the following equation:

$$w_y(x) = (-1)^{\sum_{i+j=1} x_i y_j}$$

where $x=(\dots, x_i, \dots)$ and $y=(\dots, y_j, \dots)$. When m is a dyadic measure which is a generalized measure defined on the dyadic field, we can define the Walsh—Fourier transform of m by the following dyadic measure $\hat{m}$:

$$\hat{m}(I_n(x)) = 1/2^n \int_0^{2^{n-}} w_y(x)\,m(dx),$$

where $I_n(x)$ is the dyadic interval of rank n containing x. $\hat{m}$ has the following property: $\hat{\hat{m}}=m$. This definition contains the original definition of Walsh—Fourier transforms.

If m and h are dyadic measures satisfying the conditions

$$(1)\qquad \sum_{p=0}^{\infty} |m[p/2^n, (p+1)^-/2^n]| < \infty$$

and

$$(2)\qquad \sup_x \left|h\big(I_n(x)\big)\right| < \infty$$

for all n, then we can define the convolution $m*h$ by

$$(m*h)[p/2^n, (p+1)^-/2^n] = \sum_{k=0}^{\infty} m\big[(p\dotplus k)/2^n, ((p\dotplus k)+1)^-/2^n\big] h[k/2^n, (k+1)^-/2^n].$$

From condition (1), it follows that there exists a continuous function $\hat{f}$ such that for each dyadic interval I

$$\hat{m}(I) = \int_I \hat{f}(x)\,dx,$$

and

$$(m*h)\hat{}\,(I) = \int_I \hat{f}(x)\,\hat{h}(dx)$$

(see Theorem 2 of [7]).

THEOREM 1. *Let h be a dyadic measure satisfying*

$$(3)\qquad \lim_{x\to\infty} h\big(I_n(x)\big) = A/2^n$$

for all n where A is a constant and m is a dyadic measure satisfying (1). *Then we have*

$$\lim_{x\to\infty} (m*h)\big(I_n(x)\big) = A/2^n m[0, \infty)$$

for all n where $m[0, \infty)=\lim\limits_{n\to\infty} m[0, 2^{-n}]$.

PROOF. From (3), we have

$$|(m*h)[p/2^n, (p+1)^-/2^n]-A/2^n m[0, \infty)| =$$

$$= \left|\sum_{k=0}^{\infty} m[(p\dotplus k)/2^n, ((p\dotplus k)+1)^-/2^n] h[k/2^n, (k+1)^-/2^n] - \right.$$

$$\left. -A/2^n \sum_{k=0}^{\infty} m\big[(p\dotplus k)/2^n, ((p\dotplus k)+1)^-/2^n\big]\right| = \sum_{k=0}^{\infty} m\big[(p\dotplus k)/2^n, ((p\dotplus k)+1)^-/2^n\big]\cdot$$

$$\cdot \{h[k/2^n, (k+1)^-/2^n]-A/2^n\}\Big| \leqq \left|\sum_{k=0}^{N}\right| + \left|\sum_{k=N+1}^{\infty}\right| \leqq \left|\Big\{\sum_{k=p-N}^{\infty} |m[k/2^n, (k+1)^-/2^n]|\Big\}\cdot\right.$$

$$\cdot \big\{\max_k |h[k/2^n, (k+1)^-/2^n]| + |A|/2^n\big\} + \Big\{\sum_{k=0}^{\infty} |m[k/2^n, (k+1)^-/2^n]|\Big\}\cdot$$

$$\cdot \Big\{\sup_{k\geqq N+1} |h[k/2^n, (k+1)^-/2^n, (k+1)^-/2^n]-A/2^n|\Big\} \equiv I_1+I_2.$$

By hypothesis, for each $\varepsilon>0$ we may take N so large that $I_2<\varepsilon/2$. Further, for fixed N take p so large that $I_1<\varepsilon/2$. Therefore we have

$$|(m*h)[p/2^n, (p+1)^-/2^n]-A/2^n m[0, \infty)| < \varepsilon.$$

COROLLARY 1. *If f is an L^∞-integrable function and satisfies $\lim_{x\to\infty} f(x)=A$, then*

$$\lim_{x\to\infty} (g*f)(x) = A\int_0^\infty g(y)\,dy$$

for each $g\in L^1[0, \infty)$.

PROOF. This follows immediately from the fact that $f*g$ is uniformly continuous on $[0, \infty)$.

THEOREM 2. *When m_1 is a dyadic measure satisfying* (1), *and*

$$\lim_{x\to\infty} (m_1*h)(I_n(x)) = A/2^n m_1[0, \infty) \tag{4}$$

for a dyadic measure h satisfying (2) *for each n, then*

$$\lim_{x\to\infty} (m*h)(I_n(x)) = A/2^n m[0, \infty) \tag{5}$$

for a dyadic measure m satisfying (1) *if and only if $\hat{f}(x)\neq 0$ for all x where $m_{\hat{f}}=\hat{m}_1$.*

PROOF. At first we shall prove the necessity. Assume that there exists x_0 such that $\hat{f}(x_0)=0$. There exists a dyadic measure m satisfying $\hat{g}(x_0)\neq 0$ where $m_{\hat{g}}=\hat{m}_1$. Set

$$h(I_n(x)) = \int_{I_n(x)} w_{x_0}(y)\,dy.$$

h is a dyadic measure satisfying $|h(I_n(x))|\leq 1/2^n$ for all x. $\hat{h}$ is the dyadic measure which has mass 1 at x_0. Then we have

$$(m_1*h)[p/2^n, (p+1)^-/2^n] = 1/2^n \int_0^{2^{n-}} w_y(p/2^n)\hat{f}(y)\,\hat{h}(dy) =$$

$$= 1/2^n \hat{f}(x_0) w_{x_0}(p/2^n) = 0$$

if $x_0\in[0, 2^{-n}]$. Moreover we have

$$1/2^n \int_0^{2^{n-}} w_y(p/2^n)\hat{f}(y)\,\hat{h}(dy) = 0$$

if $x_0\notin[0, 2^{-n}]$. Therefore we proved that

$$\lim_{x\to\infty} (m_1*h)(I_n(x)) = 0/2^n m_1[0, \infty) = 0. \tag{6}$$

On the other hand, for sufficiently large n the following equality holds:

$$(m*h)[p/2^n, (p+1)^-/2^n] = 1/2^n \int_0^{2^{n-}} w_y(p/2^n)\hat{g}(y)\hat{h}(dy) = 1/2^n \hat{g}(x_0) w_{x_0}(p/2^n).$$

Hence we obtain

(7) $$\lim_{x\to\infty}(m*h)(I_n(x)) = \lim_{x\to\infty} 1/2^n\hat{g}(x_0)w_{x_0}(x) \neq 0.$$

(6) and (7) contradict the hypothesis. We proved that $\hat{f}(x)\neq 0$ for all x.

Next we shall prove the sufficiency. Since m_1 satisfies (2), we can write

$$\hat{f}(2^n x) = \sum_{p=0}^{\infty} m_1[p/2^n, (p+1)^-/2^n]w_p(x);$$

the last series converges absolutely for each n and $x\in[0, 1^-]$ and $\hat{m}_1=m_{\hat{f}}$. Set $\hat{g}(x)=1/\hat{f}(x)$. Since $\hat{f}(x)\neq 0$ for all x, the Walsh—Fourier series of $\hat{g}(x)$ converges absolutely (see [1]). Set $\overline{m}=m_{\hat{g}}$. m satisfies (2). Then we can write

$$\hat{g}(2^n x) = \sum_{p=0}^{\infty}\overline{m}[p/2^n, (p+1)^-/2^n]w_p(x) \quad \text{(absolutely convergent)}$$

for each n and $x\in[0, 1^-]$. Since $(m_1*\overline{m})\hat{}=m_{\hat{f}\cdot\hat{g}}$ and $\hat{f}(x)\hat{g}(x)\equiv 1$, $m_1*\hat{m}$ has mass 1 at 0. Hence we have

$$(m_1*h*\overline{m}*m)(I) = (m_1*\overline{m}*m*h)(I) = (m*h)(I).$$

$m*\overline{m}$ satisfies

$$\sum_{p=0}^{\infty}|(m*\overline{m})[p/2^n, (p+1)^-/2^n]| \leq \Big(\sum_{k=0}^{\infty}|m[k/2^n, (k+1)^-/2^n]|\Big)\cdot$$

$$\cdot\Big(\sum_{j=0}^{\infty}|\overline{m}[j/2^n, (j+1)^-/2^n]|\Big) < \infty.$$

m_1*h satisfies (5). Therefore from Theorem 1, we obtain

$$\lim_{x\to\infty}(m*h)(I_n(x)) = \lim_{x\to\infty}(m*\overline{m}*m_1*h)(I_n(x)) = A/2^n m_1[0,\infty)\cdot(m*\overline{m})[0,\infty).$$

We easily get

$$\overline{m}[0,\infty) = \sum_{p=0}^{\infty}\overline{m}[p/2^n, (p+1)^-/2^n]w_p(0/2^n) = \hat{g}(0) = 1/\hat{f}(0) = 1/m_1[0,\infty).$$

Set

$$|\overline{m}[0,\infty)\cdot m[0,\infty)-(m*\overline{m})[0, 2^{-N}]| = \Big|\sum_{k=0}^{\infty} m[k2^N, (k+1)2^{N-}]\cdot$$

$$\cdot\sum_{j=0}^{\infty}\overline{m}[j2^N, (j+1)2^{N-}] - \sum_{p=0}^{\infty} m[p2^N, (p+1)2^{N-}]\overline{m}[p2^N, (p+1)2^{N-}]\Big| =$$

$$= \Big|\sum_{p\neq p'} m[p2^N, (p+1)2^{N-}]\overline{m}[p'2^N, (p'+1)2^{N-}] \leq |m[0, 2^{N-}]|\cdot$$

$$\cdot\Big\{\sum_{k=1}^{\infty}|\overline{m}[k2^N, (k+1)2^{N-}]|\Big\} + \Big\{\sum_{k=1}^{\infty}|m[k2^N, (k+1)2^{N-}]|\Big\}\Big\{\sum_{j=0}^{\infty}|\overline{m}[j2^N, (j+1)2^{N-}]|\Big\} =$$

$$\equiv A_N + B_N.$$

Since $\left\{\sum_{k=1}^{\infty} m[k2^N, (k+1)2^{N}-]\right\}_N$ is decreasing with N, we have

$$\sup_{N\geqq 0}\left\{\sum_{k=0}^{\infty} |m[k2^N, (k+1)2^{N}-]|\right\} \equiv K < \infty.$$

$\overline{m}$ satisfies (2) for $n=0$ and

$$\sum_{k=1}^{\infty} |\overline{m}[k2^N, (k+1)2^{N}-]| \leqq \sum_{k=2^N}^{\infty} |\overline{m}[k, (k+1)^{-}]|,$$

then we get

$$\lim_{N\to\infty} \sum_{k=1}^{\infty} |\overline{m}[k2^N, (k+1)2^{N}-]| = 0.$$

Hence we can prove $\lim_{N\to\infty} A_N=0$. In the same way we shall get $\lim_{N\to\infty} B_N=0$. We proved that

$$(m*\overline{m})[0, \infty) = m[0, \infty)\cdot \overline{m}[0, \infty).$$

Consequently we get

$$\lim_{x\to\infty} (m*h)\big(I_n(x)\big) = A/2^n m_1[0, \infty)\cdot \overline{m}[0, \infty)\cdot m[0, \infty) = A/2^n m[0, \infty).$$

COROLLARY 2. *When* k_1 *is an integrable function, and*

$$\lim_{x\to\infty} (k_1*h)(x) = A\int_0^{\infty} k_1(y)\,dy$$

for $h\in L^{\infty}[0, \infty)$, *then*

$$\lim_{x\to\infty} (k_1*h)(x) = A\int_0^{\infty} k(y)\,dy$$

for each $k\in L^1[0, \infty)$ *if and only if* $\hat{k}_1(x)\neq 0$ *for all* x.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OSAKA PREFECTURE
SAKAI, OSAKA
JAPAN

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INVERSE LIMITS AND SMOOTHNESS OF CONTINUA

J. J. CHARATONIK and W. J. CHARATONIK (Wrocław)

Relations between the inverse limit operation and smoothness of continua are studied in this paper. Inverse systems with monotone bonding mappings are mainly discussed. It is shown that smoothness of continua is preserved under the inverse limits of inverse systems with monotone bonding mappings provided there exists a thread composed of points at which the factor spaces are smooth; in particular it follows that the inverse limit of an inverse sequence of smooth dendroids with monotone bonding mappings is a smooth dendroid if the corresponding thread does exist.

Topological spaces considered throughout this paper are assumed to be compact (thus Hausdorff, see [7], p. 165) and the mappings are assumed to be continuous. By a *continuum* we mean a compact connected space.

The following notation will be used. $\{X^\lambda, f^{\lambda\mu}, \Lambda\}$ denotes an inverse system of the topological spaces X^λ with continuous bonding mappings $f^{\lambda\mu}\colon X^\mu \to X^\lambda$ for any $\lambda \leqq \mu$, where $\lambda, \mu \in \Lambda$ and Λ is a set directed by the relation $\leqq$. We assume that $f^{\lambda\lambda}$ is the identity, and we denote by $X = \varprojlim \{X^\lambda, f^{\lambda\mu}, \Lambda\}$ the inverse limit space. Further, $\pi^\lambda\colon X \to X^\lambda$ denotes the projection from the inverse limit space into the λ-th factor space. In a particular case when Λ is the set N of natural numbers with the natural ordering $\leqq$ we write $\{X^i, f^i\}_{i=1}^\infty$ and π^i instead of $\{X^\lambda, f^{\lambda\mu}, \Lambda\}$ and π^λ respectively, where $f^i\colon X^{i+1} \to X^i$ are bonding maps, and then $\{X^i, f^i\}_{i=1}^\infty$ is called an inverse sequence.

Given a point $p \in X = \varprojlim \{X^\lambda, f^{\lambda\mu}, \Lambda\}$, we put $p^\lambda = \pi^\lambda(p) \in X^\lambda$ and we write $p = \{p^\lambda\}$. If $\Lambda = N$, we write $p = \{p^i\}$. Obviously we have

$$f^{\lambda\mu}(p^\mu) = p^\lambda \quad \text{for any} \quad \lambda, \mu \in \Lambda \quad \text{with} \quad \lambda \leqq \mu. \tag{1}$$

A point $p \in X$, i.e., a system of points $p^\lambda \in X^\lambda$ for $\lambda \in \Lambda$ satisfying (1) is called a *thread*.

Let a continuum X be given. Consider an arbitrary decomposition of X into two of its subcontinua A and B, i.e., $X = A \cup B$, and let $r(A, B)$ denote the number of components of $A \cap B$ less one. The *multicoherence degree* $r(X)$ is then defined by $r(X) = \sup \{r(A, B)\colon A \text{ and } B \text{ are subcontinua of } X \text{ and } A \cup B = X\}$ (see [6], p. 159: cf. [14], p. 83).

Let a point $p \in X$ be fixed. The *hereditarily multicoherence degree* $r(X, p)$ *of* X at the point p is defined by $r(X, p) = \sup \{r(C)\colon C \text{ is a subcontinuum of } X \text{ and } p \in C\}$.

In other words we have $r(X, p) = \sup \{r(A, B)\colon A \text{ and } B \text{ are subcontinua of } X \text{ and } p \in A \cap B\}$.

A continuum X is said to be *hereditary unicoherent at a point* $p \in X$ provided that the intersection of any two subcontinua of X, each of which contains p, is connected (see [8], p. 52). Thus a continuum X is hereditarily unicoherent at $p \in X$ if and only if $r(X, p)=0$.

We have the following

PROPOSITION 1. *Let X denote the inverse limit of an inverse system $\{X^\lambda, f^{\lambda\mu}, \Lambda\}$ of the continua X^λ. Let $k \geqq 0$ be a fixed integer. If there exists a thread $p=\{p^\lambda\}$ such that $r(X^\lambda, p^\lambda) \leqq k$ for each $\lambda \in \Lambda$, then $r(X, p) \leqq k$.*

PROOF. Note that X is a continuum ([3], 2.10, p. 236; cf. [7], 6.1.18, p. 436). Let A and B be subcontinua of X such that $p \in A \cap B$. Put $Y=A \cup B$, and $g^{\lambda\mu}=f^{\lambda\mu}|Y^\mu$, where $Y^\lambda=\pi^\lambda(Y) \subset X^\lambda$ for each $\lambda \in \Lambda$. Then $\{Y^\lambda, g^{\lambda\mu}, \Lambda\}$ is an inverse system with surjective mappings $g^{\lambda\mu}$, and Y is the inverse limit of this system (see [3], (2.8), p. 235; cf. [7], 2.5.7, p. 138). Further, we have $p^\lambda \in Y^\lambda$ for each $\lambda \in \Lambda$. Let $C^\lambda=\pi^\lambda(A) \cap \pi^\lambda(B) \subset Y^\lambda$ and put $h^{\lambda\mu}=g^{\lambda\mu}|C^\mu$. It follows from [3], (2.9), p. 235 that $\{C^\lambda, h^{\lambda\mu}, \Lambda\}$ is an inverse system having $A \cap B$ as its inverse limit. Since $\pi^\lambda(A)$ and $\pi^\lambda(B)$ are subcontinua of X^λ both containing p^λ, their intersection C^λ has no more than $k+1$ components by assumption. So by Lemma 1 of [11], p. 227, the intersection $A \cap B$ has no more than $k+1$ components. Thus the proof is complete.

COROLLARY 1. *Let X denote the inverse limit of an inverse system $\{X^\lambda, f^{\lambda\mu}, \Lambda\}$ of the continua X^λ. If there exists a thread $p=\{p^\lambda\} \in X$ such that X^λ is hereditarily unicoherent at p^λ for each $\lambda \in \Lambda$, then X is a continuum which is hereditarily unicoherent at the point p.*

The authors do not know whether the assumption that the points at which the continua X^λ are hereditarily unicoherent form a thread of the inverse system is essential in Corollary 1. Thus one can raise the following

PROBLEM 1. *Let X denote the inverse limit of an inverse system $\{X^\lambda, f^{\lambda\mu}, \Lambda\}$ of the continua X^λ each of which is hereditarily unicoherent at a point and such that all bonding mappings are onto. Does it follow that X is hereditarily unicoherent at some point?*

Recall that a continuum X is hereditarily unicoherent at a point $p \in X$ if and only if for every point $x \in X$ there exists in X a unique subcontinuum px which is irreducible between p and x (see [8], Theorem 1.3, p. 52). A continuum X is said to be *smooth at a point* $p \in X$ (in the sense of Gordh, [8], p. 52) if X is hereditarily unicoherent at p and for each convergent net $\{a_n; n \in D\}$ (where D is a directed set) of points of X the condition $\lim a_n=a$ implies that the net $\{pa_n; n \in D\}$ of subcontinua of X converges to the limit continuum pa. The point p is then called an *initial point* of X, and the set of all points at which a continuum X is smooth is called an *initial set* of X and is denoted by $I(X)$. If $I(X) \neq \emptyset$, then the continuum X is said to be *smooth*.

A *quasi-order* on a set X is a reflexive and transitive relation. A quasi-order on a topological space X is said to be *closed* if its graph is closed in $X \times X$. If a continuum X is hereditarily unicoherent at a point p, then the quasi-order $\leqq$ on X defined by the condition $x \leqq y$ if and only if $px \subset py$ is said to be a *weak cut-point order with respect to* p ([9], p. 63). It is known that a continuum X which is hereditarily unicoherent at a point $p \in X$ is smooth at this point if and only if the weak cutpoint order with respect to p is closed ([9], Theorem 3.1, p. 65).

We discuss now some relations between smoothness of continua and the inverse limit operation. We start with an example.

Let X be the simplest Knaster indecomposable continuum with one end point (see [10], § 48, V, Example 1, p. 204 and Fig. 4, p. 205). It is well known that X is the inverse limit of an inverse sequence $\{X^i, f^i\}_{i=1}^{\infty}$, where X^i is the closed unit segment $[0, 1]$ and $f^i\colon [0, 1] \to [0, 1]$ is a fixed mapping defined by $f^i(t) = 1 - |2t-1|$, $t \in [0, 1]$, for each $i = 1, 2, \ldots$. Note that each continuum X^i is hereditarily unicoherent at each point and it is smooth at each point, each bonding mapping is open, and yet the limit continuum X is hereditarily unicoherent at each point while it is smooth at none. So we see that smoothness of continua is not preserved by the inverse limit operation, even if the inverse limit space is hereditarily unicoherent at each point and if bonding mappings are very simple ones (in particular open). Therefore a natural question arises concerning conditions (on factor spaces and/or on bonding mappings) under which the inverse limit continuum is smooth at a point provided that the factor spaces are so. The following theorem gives a partial answer to this question.

THEOREM 1. *Let X be the inverse limit of an inverse system $\{X^\lambda, f^{\lambda\mu}, \Lambda\}$, where X^λ are continua and $f^{\lambda\mu}$ are monotone mappings. If there exists a thread $p = \{p^\lambda\}$ such that X^λ is smooth at p^λ for each $\lambda \in \Lambda$, then X is a continuum which is smooth at the point p.*

PROOF. By Corollary 1 the inverse limit space X is a continuum which is hereditarily unicoherent at p. By Theorem 3.1 of [9], p. 65 for each $\lambda \in \Lambda$ there exists a closed weak cutpoint order $\leqq_\lambda$ with respect to the point p^λ. Define a relation $\leqq$ on X by $x \leqq y$ if and only if $x^\lambda \leqq_\lambda y^\lambda$ for each $\lambda \in \Lambda$. Note that $\leqq$ is transitive and reflexive, i.e., it is a quasi-order. We claim that $\leqq$ is closed. To see this, consider two convergent nets $\{x_n; n \in D\}$ and $\{y_n; n \in D\}$ of points of X having points x and y of X as their limits respectively. Assume that $x_n \leqq y_n$ for each $n \in D$. Thus $x_n^\lambda = \pi^\lambda(x_n) \leqq_\lambda y_n^\lambda = \pi^\lambda(y_n)$ for each $\lambda \in \Lambda$ by the definition of the quasi-order $\leqq$ on X. Since a net $\{x_n; n \in D\}$ of points in the inverse limit space X converges to a limit point x if and only if the nets $\{x_n^\lambda; n \in D\}$ converge to x^λ for each $\lambda \in \Lambda$ (see [7], 2.3.34, p. 119), and since each quasi-order $\leqq_\lambda$ is closed, we have $x^\lambda \leqq_\lambda y^\lambda$ for each $\lambda \in \Lambda$, and thus the claim is proved.

To complete the proof we need only to show that the quasi-order defined above is just the weak cutpoint order with respect to p, i.e., that $x \leqq y$ holds if and only if $px \subset py$. To this end let us take for $\lambda \leqq \mu$ the partial mapping $g^{\lambda\mu} = f^{\lambda\mu}|p^\mu y^\mu$ from the unique irreducible continuum $p^\mu y^\mu$ into the continuum X^λ.

It follows from Theorem 4.1 (ii) of [8], p. 56 that $g^{\lambda\mu}(p^\mu y^\mu) = p^\lambda y^\lambda$, so we can consider the mapping $g^{\lambda\mu}\colon p^\mu y^\mu \to p^\lambda y^\lambda$ as a surjection. Obviously we have $g^{\lambda\mu}(p^\mu) = p^\lambda$ and $g^{\lambda\mu}(y^\mu) = y^\lambda$. Since $\{\pi^\lambda(X), f^{\lambda\mu}|\pi^\mu(X), \Lambda\}$ is an inverse system ([3], 2.8, p. 235), it can be easily verified that $\{p^\lambda y^\lambda, g^{\lambda\mu}, \Lambda\}$ is also an inverse system. Let L denote its inverse limit. Obviously $p, y \in L$. Further, L is a continuum which is irreducible from p to y. Indeed, if M is a subcontinuum of L containing the points p and y, then for each $\lambda \in \Lambda$ the set $\pi^\lambda(M)$ is a continuum containing p^λ and y^λ and contained in $p^\lambda y^\lambda$. Therefore $\pi^\lambda(M) = p^\lambda y^\lambda$ by irreducibility of $p^\lambda y^\lambda$, whence by [3], 2.8, p. 235 it follows that $M = \varprojlim \{p^\lambda y^\lambda, g^{\lambda\mu}, \Lambda\} = L$. Thus $L = py$, since by hereditary unicoherence of X at p there is only one continuum irreducible from p to y ([8], Theorem 1.3, p. 52).

Furthermore, since $\{p^\lambda y^\lambda, g^{\lambda\mu}, \Lambda\}$ is an inverse system, we conclude that the condition $x \leqq y$, i.e., $x^\lambda \leqq_\lambda y^\lambda$ for each $\lambda \in \Lambda$ (which means that $p^\lambda x^\lambda \subset p^\lambda y^\lambda$ for each $\lambda \in \Lambda$), is equivalent to $px \subset py$. So the proof is finished.

The remainder of the paper deals with metrizable spaces only, therefore we restrict the inverse systems to be considered as inverse sequences (cf. [7], Corollary 4.2.4, p. 324 for justification). We recall that the inverse limit of an inverse sequence of metrizable spaces is metrizable ([7], 4.2.5, p. 325) and that the inverse limit of an inverse system of continua (i.e. of compact connected topological spaces) is a continuum ([7], 6.1.18, p. 436), whence it follows that

(2) the inverse limit of an inverse sequence of metric continua is a metric continuum.

By a *dendroid* we mean a metric continuum which is arcwise connected and hereditarily unicoherent (i.e. an arcwise connected metric continuum such that the intersection of each two of its subcontinua is connected). Nadler ([11], Theorem 4, p. 229) and Bellamy ([2], Lemma 1, p. 192) have proved the following result about inverse limits of dendroids:

THEOREM A (Nadler, Bellamy). *Let X denote the inverse limit of an inverse sequence $\{X^i, f^i\}_{i=1}^\infty$ where X^i is a dendroid for each $i=1, 2, \ldots$.*

1. *If X is arcwise connected, then X is a dendroid.*
2. *If X is locally connected, then X is a dendrite.*
3. *If X^i is a dendrite and $f^i\colon X^{i+1} \to X^i$ is monotone for each $i=1, 2, \ldots$, then X is a dendrite.*
4. *If the mapping $f^i\colon X^{i+1} \to X^i$ is monotone for each $i=1, 2, \ldots$, then X is a dendroid.*

Recall that for dendroids the notion of smoothness presented above and due to G. R. Gordh, Jr. ([8], p. 52) coincides with a previous one due to the first author and C. A. Eberhart ([5], p. 298). Thus Theorems 1 and A (Part 4) imply the following

COROLLARY 2. *Let X be the inverse limit of an inverse sequence $\{X^i, f^i\}_{i=1}^\infty$ of the dendroids X^i with monotone mappings $f^i\colon X^{i+1} \to X^i$. If there exists a thread $p=\{p^i\}$ such that X^i is smooth at p^i for each $i=1, 2, \ldots$, then X is a dendroid which is smooth at p.*

Similarly to a previous question (see Problem 1) the authors do not know whether the existence of a thread composed of initial points of X^i (i.e. of points at which each X^i is smooth) is an essential assumption in Corollary 2. Thus we have

PROBLEM 2. *Let a dendroid X be the inverse limit of an inverse sequence of smooth dendroids with monotone bonding mappings. Is then X smooth?*

Monotonicity of bonding mappings is an essential assumption in Corollary 2 even in the case when the limit continuum is a dendroid. This can be seen by the following

EXAMPLE. There exists an inverse sequence $\{X^i, f^i\}_{i=1}^\infty$ such that X^i is a finite dendrite, $X^i \subset X^{i+1}$, and f^i is a retraction for each $i=1, 2, \ldots$ whose inverse limit space X is a non-smooth (and even non-contractible and non-selectible) dendroid.

Indeed, let A_n be the line segment joining $(-1, 0)$ and $(1, 2^{-n})$ in the plane, for $n=0, 1, 2, \ldots$ and let T be the line segment joining $(-1, 0)$ and $(1, 0)$. Then $D_1 = T \cup \bigcup_{n=0}^{\infty} A_n$ is a dendroid (the so called harmonic fan). Let D_2 be the reflection of D_1 about the origin, and put $X = D_1 \cup D_2$ (see [12], Fig. 1, p. 372). It is evident that X is a non-smooth dendroid, and it is known that X is non-contractible ([13], Theorem 2.1, p. 838) and that the hyperspace $C(X)$ of its subcontinua admits no continuous selection ([12], Theorem 2, p. 372). For each $i = 1, 2, \ldots$ put $D_1(i) = T \cup \bigcup_{n=1}^{i} A_n$, let $D_2(i)$ denote the reflection of $D_1(i)$ about the origin, and define $X^i = D_1(i) \cup D_2(i)$. Thus $X^i \subset X$ is a finite dendrite for each $i = 1, 2, \ldots$ and we have

$$T \subset X^1 \subset X^2 \subset \ldots \subset X^i \subset X^{i+1} \subset \ldots \subset X = \overline{\bigcup_{i=1}^{\infty} X^i}.$$

We define the mapping $f^i \colon X^{i+1} \to X^i$ as follows:

$$f^i(x, y) = \begin{cases} (x, y) & \text{if } (x, y) \in X^i, \\ (x, 0) & \text{if } (x, y) \in X^{i+1} \setminus X^i. \end{cases}$$

Thus f^i projects the $(i+1)$-th line segment A_{i+1} in $D_1(i+1)$ and its reflection about the origin in $D_2(i+1)$ perpendicularly onto T, and f^i is the identity on the rest. So $f^i \colon X^{i+1} \to X^i$ is a retraction. The equality $X = \varprojlim \{X^i, f^i\}_{i=1}^{\infty}$ can be seen by Theorem I of [1], p. 348.

In light of the above example one can ask questions about conditions (regarding the bonding mappings f^i) under which some other properties such as contractibility or the existence of a continuous selection on the hyperspace of subcontinua are transferred from X^i to $X = \varprojlim (X^i, f^i)$. Studying these and similar problems is left for the future.

Let us keep our attention for a while on a very particular kind of dendroids, nemely on fans. By a *fan* we understand a dendroid having exactly one ramification point (called the *top* of the fan).

Observe the following easy characterization of fans.

PROPOSITION 2. *A dendroid F is a fan with top p if and only if for every two points $x, y \in F$ the condition $px \cap py \setminus \{p\} \neq \emptyset$ implies either $px \subset py$ or $py \subset px$ (we consider here an arc as a degenerate fan whose top is an arbitrary point).*

THEOREM 2. *Let X denote the inverse limit of an inverse sequence $\{X^i, f^i\}_{i=1}^{\infty}$, where X^i is a fan and f^i is a monotone mapping of X^{i+1} into X for each $i = 1, 2, \ldots$. Then X is a fan (an arc or a singleton).*

PROOF. Let p_i be the top of the fan X^i for each $i = 1, 2, \ldots$. Applying Theorem 12 of [4], p. 32 we have $f^i(p_{i+1}) = p_i$ so that there exists a thread $p \in X$ whose coordinates $\pi^i(p) = p^i = p_i$ are just the tops of the fans X^i. It follows from Part 4 of Theorem A that X is a dendroid. We will show that X is a fan with top p. To this end, let x and y be two distinct points of X such that $px \cap py \setminus \{p\} \neq \emptyset$. Let $q \in px \cap$

$\cap py \setminus \{p\}$. Consider the partial mappings $g^i = f^i | p^{i+1}x^{i+1}$ and $h^i = f^i | p^{i+1}y^{i+1}$ of $p^{i+1}x^{i+1}$ and $p^{i+1}y^{i+1}$ into X^i for each $i = 1, 2, \ldots$. They are monotone (see e.g. [5], Proposition 1, p. 307), whence it follows that $g^i(p^{i+1}x^{i+1})$ is the arc p^ix^i and, similarly, $h^i(p^{i+1}y^{i+1})$ is the arc p^iy^i (cf. [14], Chapter IX, (1.1), p. 165). Thus we can consider the mappings $g^i\colon p^{i+1}x^{i+1} \to p^ix^i$ and $h^i\colon p^{i+1}y^{i+1} \to p^iy^i$ as monotone and onto for each $i = 1, 2, \ldots$. They preserve end points of the arcs, whence we conclude that $\{g^i, p^ix^i\}_{i=1}^{\infty}$ and $\{h^i, p^iy^i\}_{i=1}^{\infty}$ are inverse sequences, the inverse limits of which are the arcs px and py, respectively (see [3], Theorem 4.8, p. 244). Thus we have $q^i \in p^ix^i \cap p^iy^i \setminus \{p^i\}$ for each $i = 1, 2, \ldots$. Since the X^i are fans, Proposition 2 implies that for every $i = 1, 2, \ldots$ we have either $p^ix^i \subset p^iy^i$ or $p^iy^i \subset p^ix^i$. Note that since g^i and h^i are monotone and onto, either the former inclusion holds for all $i = 1, 2, \ldots$ or the latter one is satisfied for all $i = 1, 2, \ldots$. In the first case we have $px \subset py$, in the second one the inclusion $py \subset px$ is true; therefore X is a fan with top p by Proposition 2.

The next corollary summarises some results on inverse sequences of dendroids with monotone bonding mappings.

COROLLARY 3. *The following properties are preserved under the inverse limits of inverse sequences with monotone bonding mappings:* (a) *to be an arc;* (b) *to be a dendrite;* (c) *to be a dendroid;* (d) *to be a smooth dendroid, provided that there is a thread of initial points;* (e) *to be a fan;* (f) *to be a smooth fan.*

PROOF. (a) see [3], Theorem 4.8, p. 244; (b) [11], Part 3 of Theorem 4, p. 229; (c) [2], Lemma 1, p. 192; (d) Corollary 2; (e) Theorem 2; (f) see (d) and (e) above and note that the existence of a proper thread is shown in the proof of (e).

Results of the present paper will be applied by the first author in a forthcoming paper to the construction of a dendroid composed of endpoints and of ramification points only.

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INSTITUTE OF MATHEMATICS
UNIVERSITY OF WROCŁAW
Pl. GRUNWALDZKI 2/4
50-384 WROCŁAW
POLAND

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ON THE JACOBSON RADICAL OF ASSOCIATIVE 2-GRADED RINGS

A. SULIŃSKI (Warsaw) and J. F. WATTERS (Leicester)

In this note definitions of the Jacobson radical of 2-graded rings are given which are analogous to the usual definitions for associative rings in terms of maximal modular one-sided ideals and of primitive ideals. We show that these definitions lead to the same ideal and are left-right symmetric. The relationship between the Jacobson radical of the 2-graded ring and that of the associated associative ring is also investigated.

Preliminaries

Recall from [1] that an abelian group G which is the direct sum of fixed subgroups G_0 and G_1 is said to be 2-graded and we write $G=(G_0, G_1)$. The elements of G_i, $i=0, 1$, are called homogeneous and if $0\neq a\in G_i$ then i is called the degree of a. By convention 0 has degree 0 and 1. A graded subgroup $H=(H_0, H_1)$ has $H_0\subseteq G_0$ and $H_1\subseteq G_1$. If $\{H^\lambda\}$ is a family of graded subgroups, then $\cap\{H^\lambda\}$ is the graded subgroup $K=(K_0, K_1)$ with $K_0=\bigcap_\lambda H_0^\lambda$ and $K_1=\bigcap_\lambda H_1^\lambda$.

A 2-graded abelian group $R=(R_0, R_1)$ is a 2-graded ring if an associative multiplication is defined for all homogeneous elements $a_i\in R_i$, $b_j\in R_j$ such that $a_ib_j\in R_{i+j}$, $i, j\in\{0, 1\}$ and $i+j$ is addition modulo 2; we also require this multiplication to be right and left distributive with respect to the addition of elements of the same degree. For brevity we shall refer to 2-graded rings as graded rings.

If $A=(A_0, A_1)$ and $B=(B_0, B_1)$ are additive graded subgroups of the graded ring R we define the homogeneous subgroups A_iB_j, $(A_i\,.\!\cdot B_j)$ and $(A_i\cdot\!.\, B_j)$ as follows:

$$A_iB_j=\{\sum_k a_{ik}b_{jk}\colon\ a_{ik}\in A_i,\ b_{jk}\in B_j\}\subseteq R_{i+j},$$

where the summation is finite;

$$(A_i\,.\!\cdot B_j)=\{r\in R\colon\ rB_j\subseteq A_i\}\subseteq R_{i+j};$$

and

$$(A_i\cdot\!.\, B_j)=\{r\in R\colon\ B_jr\subseteq A_i\}\subseteq R_{i+j};$$

in all cases with $i, j\in\{0, 1\}$ and $i+j$ being addition modulo 2.

An additive graded subgroup $I=(I_0, I_1)$ will be called a left ideal of R if, for all i, j, $R_iI_j\subseteq I_{i+j}$. Right ideals and ideals are similarly defined. Note that if I is a left ideal of R then I_0 is a left ideal of R_0 and $I_1\subseteq(I_0\cdot\!.\, R_1)$, so $I\subseteq(I_0, (I_0\cdot\!.\, R_1))$ which is a left ideal of R whenever I_0 is a left ideal of R_0.

We will write

$$\hat{I}_0 = (I_0 .\!\cdot R_0) \cap (I_1 .\!\cdot R_1) = \{a_0 \in R_0 \colon a_0 R_i \subseteqq I_i \quad \text{for} \quad i = 0, 1\},$$

$$\hat{I}_1 = (I_0 .\!\cdot R_1) \cap (I_1 .\!\cdot R_0) = \{a_1 \in R_1 \colon a_1 R_i \subseteqq I_{i+1} \quad \text{for} \quad i = 0, 1\},$$

and $\hat{I} = (\hat{I}_0, \hat{I}_1)$.

If I is a left ideal of R then it is easily verified that $\hat{I}$ is an ideal of R. Furthermore, if K is an ideal of R and $K \subseteqq I$, then $K \subseteqq \hat{I}$ for $K_i R_j \subseteqq K_{i+j} \subseteqq I_{i+j}$.

Modular left ideals

We define a left ideal I of the graded ring $R=(R_0, R_1)$ to be modular if there is an element $e \in R_0$ such that $x - xe \in I$ for every homogeneous element x of R. Of necessity we have $e \in R_0$ so that $x - xe$ is homogeneous.

PROPOSITION 1. *If I is a modular left ideal of R, then $(I_j .\!\cdot R_0) \subseteqq I_j$ for $j=0, 1$ and $\hat{I}$ is the largest ideal of R contained in I.*

PROOF. If $a \in (I_j .\!\cdot R_0) \subseteqq R_j$ then $a - ae \in I_j$ whilst $ae \in (I_j .\!\cdot R_0) R_0 \subseteqq I_j$. Hence $(I_j .\!\cdot R_0) \subseteqq I_j$ and so $\hat{I} \subseteqq I$. We have already noted that $\hat{I}$ contains every ideal of R within I and this proves the proposition.

The left Jacobson radical $J_l(R)$ is defined to be $\cap \{\hat{M} \colon M$ is a maximal modular left ideal of $R\}$. The right Jacobson radical $J_r(R)$ is defined similarly. From the definitions it is clear that $J_l(R)$ and $J_r(R)$ are ideals of R. These definitions are analogoues of the definition of the Jacobson radical in rings in terms of primitive ideals. Again by analogy we define $T_l(R) = \cap \{M \colon M$ is a maximal modular left ideal of $R\}$. Since $\hat{M} \subseteqq M$ it is clear that $J_l(R) \subseteqq T_l(R)$. Our aim is to establish the opposite inclusion by characterizing the maximal modular left ideals of R.

PROPOSITION 2. *If $M = (M_0, M_1)$ is a maximal modular left ideal of R, then M_0 is a maximal modular left ideal of R_0.*

PROOF. From the definitions it is clear that M_0 is a modular left ideal of R_0. If $M_0 = R_0$ then $R_1 M_0 = R_1 R_0 \subseteqq M_1$ and $R_1 \subseteqq (M_1 .\!\cdot R_0) \subseteqq M_1$ by Proposition 1 so that $M = R$. Thus $M_0 \neq R_0$. If K_0 is a left ideal of R_0 properly containing M_0, then $K = (K_0, M_1 + R_1 K_0)$ is a left ideal of R properly containing M. Therefore $K = R$, $K_0 = R_0$, and M_0 is maximal.

PROPOSITION 3. *Let I_0 be a modular left ideal of R_0 where $R = (R_0, R_1)$ is a graded ring. Then $I = (I_0, (I_0 \cdot\!. R_1))$ is a modular left ideal of R and if $K = (I_0, K_1)$ is a modular left ideal of R, then $K \subseteqq I$.*

PROOF. With the usual notation put $L_1 = R_1(1-e) + R_1 I_0$. Then $L = (I_0, L_1)$ is a left ideal of R (observe that $R_1 L_1 \subseteqq R_0(1-e) + I_0 \subseteqq I_0$ since $R_0(1-e) \subseteqq I_0$ by modularity). By construction L is modular.

For any left ideal $K = (I_0, K_1)$ we have $R_1 K_1 \subseteqq I_0$. Thus $K_1 \subseteqq (I_0 \cdot\!. R_1)$ and, in particular, $L_1 \subseteqq (I_0 \cdot\!. R_1)$.

From $R_1(R_1 I_0) \subseteqq I_0$ we deduce that $R_1 I_0 \subseteqq (I_0 \cdot\!. R_1)$. Also $R_1[R_0(I_0 \cdot\!. R_1)] \subseteqq I_0$ implies that $R_0(I_0 \cdot\!. R_1) \subseteqq (I_0 \cdot\!. R_1)$. The inclusions $R_0 I_0 \subseteqq I_0$ and $R_1(I_0 \cdot\!. R_1) \subseteqq I_0$ are clear. Therefore I is a left ideal of R and I is modular since $L \subseteqq I$.

THEOREM 1. *$M=(M_0, M_1)$ is a maximal modular left ideal of the graded ring $R=(R_0, R_1)$ if and only if M_0 is a maximal modular left ideal of R_0 and $M_1=(M_0 \dot{.} R_1)$.*

PROOF. If M is a maximal modular left ideal of R, then M_0 is a maximal modular left ideal of R_0 from Proposition 2 and $M_1=(M_0 \dot{.} R_1)$ from Proposition 3.

Conversely, if M_0 is a maximal modular left ideal of R_0, then $M=(M_0, (M_0 \dot{.} R_1))$ is a modular left ideal of R by Proposition 3. If M is not maximal then there is a proper modular left ideal $N=(N_0, N_1)$ with $M_0 \subseteq N_0$ and $(M_0 \dot{.} R_1) \subseteq N_1$ but $N \supset M$. If $N_0=M_0$ then $N=M$ from Proposition 3. Hence $N_0=R_0$. But then, arguing as in the proof of Proposition 2, we have $N=R$. Thus M is maximal.

The radical

It is clear from the definition of $T_l(R)$ and Theorem 1 that $T_l(R)=(J(R_0), T_1)$ where

$$T_1=\cap\{(M_0 \dot{.} R_1)\colon M_0 \text{ is a maximal modular left ideal of } R_0\}$$
$$=\{a_1 \in R_1\colon R_1 a_1 \subseteq \cap M_0=J(R_0)\}=(J(R_0) \dot{.} R_1).$$

Likewise $T_r(R)=(J(R_0), (J(R_0) . \dot{} R_1))$. To show that $T_l(R)=T_r(R)$ we prove first that $T_l(R)=J_l(R)$.

THEOREM 2. *$T_l(R)=J_l(R)$.*

PROOF. The inclusion $J_l(R) \subseteq T_l(R)$ is immediate from the definitions and Proposition 1.

Now suppose that $M=(M_0, M_1)$ is a maximal modular left ideal of R and $T_l(R) \subseteqq\!\!\!\!/\; \hat{M}$. If $T_0 R_j \subseteq M_j$ and $T_1 R_j \subseteq M_{1+j}$ for all j, then $T_0 \subseteq \hat{M}_0$ and $T_1 \subseteq \hat{M}_1$. Hence, for some i and j, we have an element $x \in R_j$ with $T_i x \subseteqq\!\!\!\!/\; M_{i+j}$, in particular $x \notin M_j$.

Consider the left ideal $N=(M_0+T_j x, M_1+T_{j+1} x)$ of R. Since $T_i x \subseteqq\!\!\!\!/\; M_{i+j}$ we have $N \supset M$. Hence $N=R$ by the maximality of M. From $M_0+T_j x=R_0$ and $M_1+T_{j+1} x=R_1$ we deduce that $M_j+T_0 x=R_j$. Therefore $x-ax \in M_j$ for some $a \in T_0$. But $T_0=J(R_0)$ so a has a quasi-inverse $a' \in T_0$ and $(1-a')(x-ax)=x \in M_j$, a contradiction. Thus $T_l(R) \subseteq \hat{M}$ and $T_l(R)=J_l(R)$.

THEOREM 3. *$T_l(R)=T_r(R)$.*

PROOF. From Theorem 2 we have that $T_l(R)=(T_0, T_1)$ is an ideal of R. Hence $T_1 R_1 \subseteq T_0=J(R_0)$. Therefore $T_1 \subseteq (J(R_0) . \dot{} R_1)$ so that $T_l(R) \subseteq T_r(R)$ and, by symmetry, $T_l(R)=T_r(R)$.

We now consider the class of graded ringgs $\mathcal{J}=\{R\colon J(R)=R\}$ where $J(R)=J_l(R)=J_r(R)$. If $R \in \mathcal{J}$ then $J(R_0)=R_0$. On the other hand if $J(R_0)=R_0$, then $R_1^2 \subseteq J(R_0)$ so $R_1 \subseteq T_1$ and $J(R)=R$. Therefore $\mathcal{J}=\{R\colon J(R_0)=R_0\}$. It now follows from the known properties of the Jacobson radical in associative rings that $\mathcal{J}$ is homomorphically closed, $J(R) \in \mathcal{J}$ for any graded ring R, and if I is an ideal of R with $I \in \mathcal{J}$ then $I \subseteq J(R)$, and finally $\mathcal{J}(R/\mathcal{J}(R))=0$ for any graded ring R. Thus $\mathcal{J}$ determines a radical property for graded rings in accordance with the definition of such properties given in [1].

The associated associative ring

Whenever we have a graded ring R we can define an associative ring $S(R)$ on the additive group $R_0 \oplus R_1$ by defining a product

$$(a_0, a_1)(b_0, b_1) = (a_0 b_0 + a_1 b_1, a_0 b_1 + a_1 b_0).$$

We now prove that $S(J(R)) \subseteqq J(S(R))$ and give an example to show that this inclusion can be strict.

THEOREM 4. $S(J(R)) \subseteqq J(S(R))$.

PROOF. Note first that $S(J(R))$ is an ideal of $S(R)$. Let $(a_0, a_1) \in S(J(R))$. Then $a_0 \in J(R_0)$ and $\bar{a}_0 = a_0 + a_1^2 - a_1 a_0' a_1 \in J(R_0)$ where a_0' is the quasi-inverse of a_0, that is $a_0 + a_0' - a_0 a_0' = 0$. Let $\bar{a}_0'$ be the quasi-inverse in R_0 of $\bar{a}_0$. Now it can be verified that (b_0, b_1) is a right quasi-inverse of (a_0, a_1) where $b_0 = a_0' \circ (a_1 b_1)$ and $b_1 = (1 - \bar{a}_0')(a_1 a_0' - a_1)$. Hence $S(J(R))$ is a right quasi-regular ideal of $S(R)$ and so $S(J(R)) \subseteqq J(S(R))$.

EXAMPLE. Let F be a field and $G = \{a : a^2 = e\}$, the cyclic group of order 2. Set $S = FG$, the group algebra. Then $S = R_0 \oplus R_1$ with $R_0 = Fe$ and $R_1 = Fa$. Thus we have a graded ring $R = (R_0, R_1)$ with $S(R) = S$. For this graded ring $J(R) = 0$ since $J(R_0) = 0$ and $(J(R_0)^{\cdot} . R_1) = \{r_1 \in R_1 : R_1 r_1 = 0\} = 0$. However if char $F = 2$ then $J(S) = F(e+a) \neq 0$ and the inclusion $S(J(R)) \subset J(S(R))$ is strict.

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INSTITUTE OF MATHEMATICS,
UNIVERSITY OF WARSAW,
PKIN, 00-901,
WARSAW, POLAND

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY,
LEICESTER, LE1 7RH
ENGLAND

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ON COHOMOLOGICAL DIMENSION AND THE SUM THEOREMS

S. DEO (Jammu) and R. A. SHUKLA (Allahabad)

1. Introduction

Let X be a locally paracompact space. Then it always admits a paracompactifying family Φ of supports with its extent as X, viz. the family of all those closed subsets of X which have a paracompact neighbourhood in X. In this paper all spaces are assumed to be Hausdorff and a "module" does not necessarily mean "unitary module". Suppose L is a ring, and for any sheaf $\mathcal{A}$ of L-modules on X, let $H^i_\Phi(X, \mathcal{A})$ denote the sheaf cohomology of X with supports in Φ. Then the smallest integer n (or ∞) such that $H^i_\Phi(X, \mathcal{A})=0$ for each sheaf $\mathcal{A}$ on X and $i>n$, is called the *cohomological dimension* of X and is denoted by $\dim_L(X)$. It is well-known that $\dim_L(X)$ is independent of Φ [3, p. 74]. The above definition of cohomological dimension is essentially due to H. Cartan [5] and it is this cohomological dimension of a paracompact space X which has been of much use in the cohomological theory of topological transformation groups [2, 4, 9]. The main objective of this paper is to obtain all forms of sum theorems for the above dimension for locally paracompact spaces. Some results of Quillen [13] turn out to be straightforward corollaries of these sum theorems or of the method of the proof used in proving them. The basic tool for proving these results is a well-known theorem which characterises $\dim_L(X)=n$ in terms of existence of a soft resolution of length n for each sheaf on X. This characterization is analogous to the characterization of left global dimension (l. gl. dim $L=n$) of a ring L in terms of existence of an injective resolution of length n for each left L-module C [10, p. 202].

In 1954 Cohen [6] defined a cohomological dimension of a locally compact space X with respect to a non-zero coefficient group G as follows: $\mathrm{Cd}(X, G)\leqq n$ iff $H^m_c(U, G)=0$ for each $m>n$ and each open set U of X. Here H^*_c denotes the Alexander—Spanier cohomology with compact supports. If L is a P.I.D. then it is a theorem of Floyd—Grothendieck that for a locally compact space X, $\mathrm{Cd}(X, L)=\dim_L(X)$. Motivated by Cohen's definition, Okuyama [12] gave the following definition of cohomological dimension: Let $H^*(X, G)$ denote the Čech cohomology of X with coefficient in a non-zero abelian group G. Then the smallest integer n such that for each $m\geqq n$ and each closed set A of X the map $i^*\colon H^m(X, G)\to H^m(A, G)$ induced by the inclusion map $i\colon A\to X$ is onto, is called the cohomological dimension of X and is denoted by $D(X, G)$. He also shows that if X is paracompact and locally compact then $\mathrm{Cd}(X, G)=D(X, G)$. One form of sum theorem for $\mathrm{Cd}(X, G)$ was proved in [6] for a locally compact space X and if X is paracompact then other forms of sum theorems were proved for $D(X, G)$ in [12]. Now the sum theorems for $\dim_L(X)$ cannot be deduced from those for $D(X, L)$ even if X is assumed to be paracompact. As a matter of fact for paracompact X and for any abelian group G we have $D(X, G)\leqq\dim_L(X)$ and whether or not there will be an equality when $G=L$ is an open problem. We discuss these matters in Section 3. In Section 4, first of all we prove some results about the softness of sheaves which directly yield the sum theorems. These results are stated more generally in terms of Φ-dimension for any paracompactifying family Φ of supports. For locally paracompact spaces the sum theorems are simple corollaries. When X is locally compact, Cohen's sum theorem for $\mathrm{Cd}(X, L)=\dim_L(X)$ becomes a special case of one of our sum theorems.

The authors are thankful to the referee for suggesting an improvement in the proof of a Proposition.

2. Preliminaries

First of all we recall Cartan's definition of cohomological Φ-dimension of a space X where Φ is a paracompactifying family of supports on X. For any ring L, let $\mathscr{A}$ denote a sheaf of L-modules on X and $H_\Phi^*(X, \mathscr{A})$ be the sheaf cohomology of X with supports in Φ. Then the smallest integer n (or ∞) such that $H_\Phi^i(X, \mathscr{A})=0$ for each $i>n$ and each sheaf $\mathscr{A}$ of L-modules on X is called the cohomological Φ-dimension of X over L and is denoted by $\dim_{\Phi,L}(X)$. It turns out that if Φ, Ψ are two paracompactifying families of supports on X having the same extents then $\dim_{\Phi,L}(X)=\dim_{\Psi,L}(X)$ [3, p. 74]. Thus if X admits a paracompactifying family Φ of supports, such that $E(\varphi)=X$, then we can define the cohomological dimension of X over L, denoted by $\dim_L(X)$, to be $\dim_{\Phi,L}(X)$. Locally paracompact spaces, which include all locally compact spaces and all paracompact spaces, form such a class for which $\dim_L(X)$ is always defined. However, if Φ is not paracompactifying or its extent does not equal to X then $\dim_{\Phi,L}(X)$ may turn out to be different from the desirable one [7, 8]. Let us recall that a sheaf $\mathscr{A}$ on X is said to be Φ-soft if each section of $\mathscr{A}$ defined on any member of Φ has an extension to the whole X. If Φ is the family cld of all closed subsets of X then a Φ-soft sheaf is said to be soft. We follow [3] for various sheaf theoretic standard definitions, notations and results.

Now let us recall [10, p. 202] that for any ring L the *left global dimension of* $L\leqq n$ if and only if for each left L-module A whenever

$$0 \to A \to X_0 \to X_1 \to \ldots \to X_{n-1} \to X_n \to 0$$

is a resolution of A of length n by left L-modules and each X_i is injective, $i=0, 1, \ldots, n-1$, then X_n is also injective. A characterization of $\dim_{\Phi,L}(X)$ for paracompactifying family similar to the above one is [3, p. 73] as follows:

THEOREM 2.1. *Let Φ be a paracompactifying family of supports on a space X. Then the following statements are equivalent:*

(a) $\dim_{\Phi,L}(X)\leqq n$.

(b) *For any sheaf $\mathscr{A}$ of L-modules on X if $0\to\mathscr{A}\to\mathscr{L}^0\to\mathscr{L}^1\to\ldots\to\mathscr{L}^{n-1}\to\to\mathscr{L}^n\to 0$ is a resolution of $\mathscr{A}$ in which $\mathscr{L}^i$ is Φ-soft for each $i<n$ then $\mathscr{L}^n$ is also Φ-soft.*

We shall make repeated use of the above theorem. We shall also need the following two results. The first is elementary and the second follows easily from the above theorem.

LEMMA 2.2. *Let Φ be a paracompactifying family of supports on a space X. Then any sheaf $\mathcal{A}$ on X is Φ-soft if and only if $\mathcal{A}|K$ is soft for each $K\in\Phi$.*

PROPOSITION 2.3 (subset theorem). *Let X be any space and Φ be a paracompactifying family of supports on X. Suppose A is locally closed in X. Then*

$$\dim_{\Phi\cap A,L}(A) \leqq \dim_{\Phi,L}(X).$$

3. Various cohomological dimensions and the covering dimension

For the results on covering dimension ($\dim X$) we refer to [11]. In the Appendix of that book $D(X, G)$ (notation changed to $d(X, G)$) has been studied in great detail. For paracompact spaces it is a theorem of Sklyarenko [14] that $D(X, G)\leqq n$ if and only if for each closed set A of X, $H^m(X, A, G)=0$ for each $m>n$. We use this in the following

PROPOSITION 3.1. *Let X be a paracompact space, G be any abelian group and L be a ring. Then*

$$D(X, G) \leqq \dim_L(X) \leqq \dim X.$$

Further, if $\dim X<\infty$ *and* $L=Z$, *then*

$$D(X, Z) = \dim_Z(X) = \dim X.$$

PROOF. Note that for a paracompact space X Čech cohomology and the sheaf cohomology with supports in the family cld of all closed sets of X are naturally isomorphic. Let G be an abelian group and suppose $D(X, G)=n$. Then by Sklyarenko's theorem there exists a closed subset A of X such that $H_n(X, A, G)\neq 0$. Since A is closed

$$H^n(X, A, G) \approx H^n_{\mathrm{cld}|(X-A)}(X-A, G)$$

where $\mathrm{cld}|(X-A)$ denotes the family of all closed subsets of X which are contained in $X-A$. Now because $\mathrm{cld}|(X-A)$ is paracompactifying on the locally paracompact space $X-A$ and its extent is $X-A$, we find by regarding G as an L-module that $\dim_L(X-A)\geqq n$. Since the subset theorem for $\dim_L$ is true for any locally closed subset of X we find that $\dim_L(X)\geqq n$. Next using Čech's definition of sheaf cohomology one can easily see that $\dim_L(X)\leqq \dim X$. Finally, if $\dim X$ is finite for a paracompact space X then it has been proved in [11, p. 206, 210] that $D(X, Z)=\dim X$. This completes the proof.

REMARK 3.2. If X is paracompact and $\dim X=\infty$, then by the above proposition

(a) $D(X,L) \leqq \dim_L(X)$ and,

(b) $\dim_L(X) \leqq \dim X = \infty$.

Whether or not there is equality in each of (a) and (b) is not known. In fact when X is also locally compact and L is a P.I.D. then we give an elementary proof below, that there must be equality in (a). However (b) has been a long standing problem in cohomological dimension theory. In fact, even if we assume that the space is compact then whether or not there will be equality for $L=Z$ in (b) is a famous problem of Aleksandrov [1].

REMARK 3.3. Various forms of sum theorems for $D(X, G)$ when X is paracompact have also been obtained by Kodama [11, Appendix]. However, his proofs appear to be inadequate. In fact he has obtained a characterization of $D(X, G)$ in terms of extension of maps from closed sets of X into Eilenberg—MacLane spaces under the assumption that $\dim X$ is finite. It is this characterization which has been used to prove the sum theorems. By the above remark it is possible (unless proved otherwise) that there is a paracompact space X of infinite covering dimension which has finite cohomological dimension with respect to some non-zero abelian group G. For such a space X if there is a countable closed covering $\{F_p\}$ such that $D(F_p, G) \leqq n$, for each p then Kodama's proof cannot be applied to conclude that $D(X, G) \leqq n$.

PROPOSITION 3.4. *If X is paracompact and locally compact then for any principal ideal domain L*

$$D(X,L) = \dim_L (X).$$

PROOF. Suppose $\dim_L (X)=n$. Then by [3, p. 75, Theorem 5.11] there exists an open set U of X such that $H^n_c(U, L)\neq 0$. This means $H^n_c(X, X-U, L)\neq 0$. Now if we put $X-U=B$ then by [15, p. 322]

$$H^n_c(X, B;\ L) = \varinjlim H^n(X, V;\ L)$$

where V runs over all cobounded neighbourhoods of B in X. Now it is easy to verify that in a locally compact space X any cobounded neighbourhood of B contains a closed cobounded neighbourhood of B. For, U cobounded neighbourhood of B implies $\overline{X-U}$ is compact and therefore $\overline{X-U}$ and B can be separated by disjoint open sets say V_1 and V_2. Also compactness of $\overline{X-U}$ in V_1 means there is an open neighbourhood W of $\overline{X-U}$ contained in V_1 such that $\overline{W}$ is compact. Then $N=X-W$ is a closed cobounded neighbourhood of B contained in U. Therefore closed cobounded neighbourhoods of B form a cofinal set in the set of all neighbourhoods of B and so $H^n(X, N, L)\neq 0$ for some closed subset N of X. By Sklyarenko's theorem, this implies that, $D(X, L)\geqq n$.

We have seen that if X is a paracompact space and L is any ring then $\dim_L (X)\leqq \dim X$. Also we pointed out that the converse of this is a classical open problem. However, one special case of this converse when $\dim_L (X)=0$ is quite elementary.

PROPOSITION 3.5. *Let X be a paracompact space and suppose* $\dim_L (X)=0$ *for some ring L. Then* $\dim X=0$.

PROOF. The abelian group $Z_2=\{0, 1\}$ can always be regarded as an L-module (possibily trivial). We can identify Z_2 with the 0-sphere S^0. Thus to prove the proposition it suffices to show [11, p. 51] that any continuous map f from any closed subset A of X to S^0 can be extended to the whole X. Let us regard S^0 as the constant sheaf of L-modules on X. Then f can obviously be regarded as a section of S^0 defined on A. Since $\dim_L(X)=0=\dim_{\mathrm{cld}, L}(X)$ we find by [3, p. 73] the that each sheaf on X must be soft and therefore f can be extended to a continuous map defined on the whole X.

By a similar argument one can prove the following

PROPOSITION 3.6. *If X is locally paracompact and $\dim_L(X)=0$ for a ring L then X must be totally disconnected. If X is locally compact then the converse is also true.*

4. Softness of sheaves and sum theorems

First of all we prove the following results about the softness of sheaves which are basic to the proof of various forms of sum theorems and other results.

PROPOSITION 4.1. *Let $\{F_p|\, p=1, 2, \dots, n, \dots\}$ be a countable closed cover of a space X and Φ be a paracompactifying family of supports on X. If $\mathcal{A}$ is a sheaf of L-modules on X such that $\mathcal{A}|F_p$ is $\Phi|F_p$-soft for each $p=1, 2, \dots, n, \dots,$ then $\mathcal{A}$ itself is Φ-soft.*

PROOF. Since Φ is paracompactifying it suffices, by Lemma 2.2, to prove that $\mathcal{A}|K$ is soft for each member K of Φ. Again because $\mathcal{A}|F_p$ is $\Phi|F_p$-soft for each p and each closed subset of $K\cap F_p$ is a member of Φ we find that $\mathcal{A}|K\cap F_p$ is soft for each p. Thus it suffices to show that if X is paracompact, $\{F_p\}$ is a countable closed cover of X and $\mathcal{A}|F_p$ is soft for each p then $\mathcal{A}$ itself is soft. To prove this let K be a closed subset of X and $s_0\in\mathcal{A}(K)$. Since $K\cap F_1$ is closed in F_1 and $\mathcal{A}|F_1$ is soft there is an extension $s'\in\mathcal{A}(K\cup F_1)$ of s_0. Now $K\cup F_1$ being closed in X and X being paracompact there exists an open set U containing $K\cup F_1$ and a section $s''\in\mathcal{A}(U)$ which extends s'. Let V_1 be an open set of X such that $K\cup F_1\subset V_1\subset\overline{V}_1\subset U_1$. Then $s_1=s''|\overline{V}_1$ is a section on $\overline{V}_1$ and extends s_0. Substituting $\overline{V}_1$ for K we can find another open set V_2 containing $\overline{V}_1\cup F_2$ and a section $s_2\in\mathcal{A}(V_2)$ which extends s_1. Proceeding inductively we can find, for each n, and open set $V_n\supset F_n\supset\overline{V}_{n-1}$ and a section $s_n\in\mathcal{A}(V_n)$ which extends s_{n-1}. Now if we define $s\colon X\to\mathcal{A}$ so that $s|\overline{V}_n=s_n$ for each n, then s is easily seen to be a section of $\mathcal{A}$ and is a required extension of s_0.

Before we come to the next Proposition, let us recall that if $\{F_\alpha\}$ is a closed covering of a topological space X, then X is said to have the *weak topology with respect to* $\{F_\alpha\}$ if (i) for any sub family $\{F_\beta\}$ of $\{F_\alpha\}$, $\cup\{F_\beta\}$ is closed in X, and (ii) a subset F of $\cup\{F_\beta\}$ is closed in $\cup\{F_\beta\}$ iff the intersection of F with each member of $\{F_\beta\}$ is closed in $\cup\{F_\beta\}$. This notion is due to K. Morita. With this definition we have

PROPOSITION 4.2. *Let X be a space which has the weak topology defined by a closed covering $\{F_\alpha|\alpha\in I\}$. Suppose $\mathcal{A}$ is a sheaf of L-modules on X and Φ is a paracompactifying family of supports on X. If $\mathcal{A}|F_\alpha$ is Φ/F_α-soft for each α, then $\mathcal{A}$ itself is Φ-soft.*

PROOF. Just as in Proposition 4.1 we can assume that X is paracompact and $\Phi=\text{cld}$. Let K be any closed subset of X and $t\in\mathcal{A}(K)$. Well-order the index set I and for each $\alpha\in I$ put $E_\alpha=(\cup\{F_\beta|\beta<\alpha\})\cup K$. By transfinite induction we shall define $s_\alpha\in\mathcal{A}(E_\alpha)$ such that for each $\beta<\alpha$, $s_\alpha|E_\beta=s_\beta$. Suppose s_β has been defined for each $\beta<\alpha$ satisfying the given condition. If α is a limit ordinal then $E_\alpha=\bigcap\limits_{\beta<\alpha} E_\beta$ and we can define $s_\alpha\in\mathcal{A}(E_\alpha)$ by setting $s_\alpha(x)=s_\beta(x)$ if $x\in E_\beta$. Notice that since X has the weak topology defined by the covering $\{F_\alpha\}$, each E_α is a closed set and the topology of E_α is also the weak topology defined by $\{F_\beta | \beta < \alpha\}$. Therefore s_α is continuous. On the other hand if α is a successor of α' then because $\mathscr{A}|F_\alpha$ is soft we can extend $s_{\alpha'}$ to $s_\alpha \in \mathscr{A}(E_\alpha)$. Finally, we define $s: X \to \mathscr{A}$ so that for each α, $s|E_\alpha = s_\alpha$. Then s is a section of $\mathscr{A}$ on X which extends t. Thus $\mathscr{A}$ is soft and this completes the proof of the proposition.

PROPOSITION 4.3. *Let Φ be a paracompactifying family of supports on a space X and F be a closed subset of X. Suppose $\mathscr{A}$ is a sheaf of L-modules on X such that $\mathscr{A}|F$ is $\Phi|F$-soft and $\mathscr{A}|(X-F)$ is $\Phi|(X-F)$-soft. Then $\mathscr{A}$ itself is Φ-soft.*

PROOF. Once again we assume that X is paracompact and $\Phi = \text{cld}$. Let K be a closed set of X and $s \in \mathscr{A}(K)$. Since $K \cap F$ is closed in F there is an $s' \in \in \mathscr{A}(K \cup F)$ which extends s. Now $K \cup F$ being a closed subset of a paracompact space X implies that there is an open set U containing $K \cup F$ and a section $s'' \in \mathscr{A}(U)$ which extends s'. Let V be another open set such that $K \cup F \subset V \subset \bar{V} \subset U$. Since $\mathscr{A}|(X-F)$ is $\text{cld}\,|(X-F)$-soft we find that $\mathscr{A}|X-U$ is $(\text{cld}\,|\,X-V) \cap \cap (X-U) = \text{cld} \cap (X-U)$-soft. Now $s''|(\bar{V}-V) \in \mathscr{A}(\bar{V}-V)$ and $\bar{V}-V$ being closed in X, $\bar{V}-V \in \text{cld}\,|\,(X-V)$. Now because $\mathscr{A}|(X-F)$ is $\text{cld}\,|\,(X-F)$-soft we find that $\mathscr{A}|X-F$ is $\text{cld}\,|\,(X-V)$-soft and hence there is an extension $s''' \in \in \mathscr{A}(X-V)$ extending $s''|(\bar{V}-V)$. Since s''' and s'' agree on the common part $\bar{V}-V$ there is a section $s^{(\text{iv})} \in \mathscr{A}(X)$ which extends s.

The following proposition now immediately follows from Lemma 2.2 and the above Proposition.

PROPOSITION 4.4. *Let Φ be a paracompactifying family of supports on a space X and $\mathscr{A}$ be a sheaf of L-modules on X. Suppose F is a closed subset of X. If $\mathscr{A}|F$ is $\Phi|F$-soft and $\mathscr{A}|A$ is $\Phi|A$-soft for each closed subset A of X disjoint from F then $\mathscr{A}$ itself is Φ-soft.*

Now we can prove all forms of sum theorems for the cohomological dimension of locally paracompact spaces over any ring L. We state the theorems in most general forms for Φ-dimensions and the usual forms of sum theorems are their immediate corollaries. We have

THEOREM 4.5. *Let $\{F_p | p = 1, 2, \dots, n, \dots\}$ be a countable closed covering of a space X and Φ be a paracompactifying family of supports on X. Then for any ring L*

$$\dim_{\Phi, L}(X) = \text{Sup}\,\{\dim_{\Phi|F_p, L}(F_p) | p = 1, 2, \dots, n, \dots\}.$$

PROOF. By the subset theorem for Φ-dimension (Proposition 2.3) $\dim_{\Phi|F_p, L}(F_p) \leqq \leqq \dim_{\Phi, L}(X)$ for each p. Conversely suppose $\dim_{\Phi|F_p, L}(F_p) \leqq n$ for each p. Let $\mathscr{A}$ be any sheaf of L-modules on X and let

$$0 \to \mathscr{A} \to \mathscr{L}^0 \to \mathscr{L}^1 \dots \to \mathscr{L}^{n-1} \to \mathscr{L}^n \to 0$$

be a resolution of $\mathscr{A}$ in which $\mathscr{L}^i$ is Φ-soft for each $i = 0, 1, \dots, n-1$. This means, for each p,

$$0 \to \mathscr{A}|F_p \to \mathscr{L}^0|F_p \to \mathscr{L}^1|F_p \to \dots \to \mathscr{L}^{n-1}|F_p \to \mathscr{L}^n|F_p \to 0$$

is a resolution of $\mathscr{A}|F_p$ in which $\mathscr{L}^i|F_p$ is $\Phi|F_p$-soft for each $i = 0, 1, \dots, n-1$. Since $\dim_{\Phi|F_p, L}(F_p) \leqq n$, Proposition 2.1 implies that $\mathscr{L}^n|F_p$ is $\Phi|F_p$-soft for each p.

Then we can apply 4.1 to conclude that $\mathscr{L}^n$ itself is Φ-soft. By Proposition 2.1 again this means $\dim_{\Phi,L}(X)\leqq n$.

If X is locally paracompact, let Φ be a paracompactifying family of supports on X such that $E(\Phi)=X$. Then for each closed subset F of X, $\Phi|F$ is paracompactifying and $E(\Phi|F)=F$. This gives the following usual *countable closed covering sum theorem.*

COROLLARY 4.6. *Let X be a locally paracompact space and $\{F_p|\, p=1,2,\dots,n,\dots\}$ be a countable closed covering of X. Then for any ring L*

$$\dim_L(X)=\operatorname{Sup}\{\dim_L(F_p)|p=1,2,\dots,n,\dots\}.$$

By using the same arguments as in the proof of Theorem 4.5 and using Proposition 4.2 or 4.3 or 4.4 appropriately one can prove all of the following results:

THEOREM 4.7. *Suppose a space X has the weak topology defined by a closed covering $\{F_\alpha|\alpha\in I\}$ of X and let Φ be a paracompactifying family of supports on X. Then for any ring L*

$$\dim_{\Phi,L}(X)=\operatorname{Sup}\{\dim_{\Phi|F_\alpha,L}(F_\alpha)|\alpha\in I\}.$$

Note that if $\{F_\alpha\}$ is a locally finite closed covering of a space X then the topology of X is always the weak topology defined by $\{F_\alpha\}$. Hence the following includes the usual *second form of sum theorem* for $\dim_L$.

COROLLARY 4.8. *Suppose a locally paracompact space X has the weak topology defined by a closed covering $\{F_\alpha\mid\alpha\in I\}$. Then for any ring L*

$$\dim_L(X)=\operatorname{Sup}\{\dim_L(F_\alpha)|\alpha\in I\}.$$

The following result, sometimes considered as a sum theorem, is known to be not generally valid for covering dimension or $D(X,G)$ and therefore shows a better behaviour of $\dim_L$.

THEOREM 4.9. *Let F be any closed subset of a space X and Φ be a paracompactifying family of supports on X. Then for any ring L*

$$\dim_{\Phi,L}(X)=\operatorname{Sup}\{\dim_{\Phi|F,L}(F),\ \dim_{\Phi|(X-F),L}(X-F)\}.$$

COROLLARY 4.10. *If X is locally paracompact and F is a closed subset of X then for any ring L*

$$\dim_L(X)=\operatorname{Sup}\{\dim_L(F),\ \dim_L(X-F)\}.$$

The following was first proved (for paracompact spaces) and usefully exploited by Quillen [13] in proving that if a compact Lie group G acts continuously on a paracompact space X then $\dim_L(X\mid G)\leqq\dim_L(X)$.

COROLLARY 4.11. *Let Φ be a paracompactifying family of supports on a space X and F be a closed subset of X. For any ring L, if $\dim_{\Phi|F,L}(F)\leqq n$ and $\dim_{\Phi|A,L}(A)\leqq n$ for each closed subset A disjoint from F then $\dim_{\Phi,L}(X)\leqq n$.*

COROLLARY 4.12. *Let X be locally paracompact and F be a closed subset of X. For any ring L, if $\dim_L(F)\leqq n$ and $\dim_L(A)\leqq n$ for each closed subset A of X disjoint from F then $\dim_L(X)\leqq n$.*

Quillen used the above corollary to prove the following result. However, by applying our second form of sum theorem it can be proved directly. The only thing to be remembered is that for any n-cell σ, $\dim_L(\sigma) \leqq n$ for any L.

COROLLARY 4.13. *Let X be a CW complex of dimension n. Then for any ring L, $\dim_L(X) \leqq n$.*

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF JAMMU
JAMMU — 180 001
INDIA

DEPARTMENT OF MATHEMATICS
V.S. MEHTA COLLEGE OF SCIENCE
BHARWARI, ALLAHABAD
INDIA

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DERIVATIVES AND CLOSED SETS

J. MAŘÍK (East Lansing)

In their article [1] G. Petruska and M. Laczkovich proved (among other things) that a function defined on a perfect set S and differentiable relative to S can be extended to a function differentiable on the whole real line R. This note contains an elementary proof of a more general theorem where the set S is supposed only to be closed in R.

NOTATION. The word function means a mapping to $R=(-\infty, \infty)$. Let $a\in S\subset R$ and let F be a function. If $S\cap(a, b)\neq\emptyset$ for each $b>a$, we define

$$F_S'^{+}(a)=\lim\,(F(x)-F(a))/(x-a)\ \ (x\in S,\ x\searrow a)$$

provided that this limit exists. We define analogously the meaning of $F_S'^{-}(a)$ and $F_S'(a)$. (Note that $F_S'(a)$ may exist even if $F_S'^{+}(a)$ is undefined.) The symbols $F'^{+}(a)$, $F'^{-}(a)$ and $F'(a)$ will have the usual meaning (i.e. $F'^{+}(a)=F_R'^{+}(a)$ etc.). Points in $R\times R$ will be denoted by $\langle\cdot,\cdot\rangle$.

1. *Let* $a, b\in R$, $a<b$ *and let* $J=[a, b]$. *Let* φ *and* ψ *be functions continuous on* J. *Let* φ *be convex,* ψ *concave,* $\varphi=\psi$ *on* $\{a, b\}$. *Set* $s=(\varphi(b)-\varphi(a))/(b-a)$. *Let* $\alpha, \beta, M, N\in R$, $\varphi'^{+}(a)\leqq\alpha\leqq\psi'^{+}(a)$, $\psi'^{-}(b)\leqq\beta\leqq\varphi'^{-}(b)$, $M<\min(\alpha, \beta, s)$, $\max(\alpha, \beta, s)<N$. *Then there is a function* G *continuously differentiable on* J *such that* $G'^{+}(a)=\alpha$, $G'^{-}(b)=\beta$, $M<G'<N$ *on* (a, b) *and that, for each* $x\in(a, b)$, $G(x)=\varphi(a)+s(x-a)$ *or* $\varphi(x)<G(x)<\psi(x)$.

PROOF. We may assume that $\varphi=\psi=0$ on $\{a, b\}$. Then $s=0$. Let $c=(a+b)/2$. We construct a function H continuously differentiable on J such that $H'^{+}(a)=\alpha$, $H=0$ on (c, b), $M<H'<N$ on (a, b) and that, for each $x\in(a, b)$, $H(x)=0$ or $\varphi(x)<H(x)<\psi(x)$. If $\alpha=0$, we choose $H=0$ on J. Now let, e.g., $\alpha>0$. Choose an $\varepsilon\in(0, -M)$ and set $\mu(x)=\psi'^{+}(x)$ $(x\in[a, b))$. We have $\alpha\leqq\mu(a)=\mu(a^{+})$. There is an $a_1\in(a, c)$ such that ψ increases on (a, a_1). There is an $a_2\in(a, a_1)$ and a function p continuous and decreasing on $[a, a_2]$ such that $\alpha(a_2-a)<\varepsilon(a_1-a_2)$, $p(a)=\alpha$, $p<\mu$ on (a, a_2) and $p(a_2)=0$. Since $\int_a^{a_2} p<\alpha(a_2-a)<\varepsilon(a_1-a_2)$, there is a function q continuous on $[a_2, a_1)$ such that $0\leqq q\leqq\varepsilon$, $\int_{a_2}^{a_1} q=\int_a^{a_2} p$ and that $q=0$ on $\{a_2, a_1\}$. Set $h=p$ on $[a, a_2)$, $h=-q$ on $[a_2, a_1]$, $h=0$ on $(a_1, b]$ and $H(x)=\int_a^x h$ for each $x\in J$. It is easy to see that $-\varepsilon\leqq H'(x)<\alpha$ and $0\leqq H(x)<\psi(x)$ for each $x\in(a, b)$.

In an analogous way we construct a function K continuously differentiable on J such that $K=0$ on (a, c), $K'^-(b)=\beta$, $M<K'<N$ on (a, b) and that, for each $x\in(a, b)$, $K(x)=0$ or $\varphi(x)<K(x)<\psi(x)$. Now it suffices to take $G=H+K$.

2. *Let a, b and J be as in* 1. *Let P be a function on J such that the derivatives $\alpha=P'^+(a)$, $\beta=P'^-(b)$ exist. Set $s=(P(b)-P(a))/(b-a)$. Let $M, N\in R$, $M<\min(\alpha, \beta, s)$, $\max(\alpha, \beta, s)<N$. Then there is a function G continuously differentiable on J such that the graph of G is contained in the convex hull of the graph of P and that $G'^+(a)=\alpha$, $G'^-(b)=\beta$, $G=P$ on $\{a, b\}$ and $M<G'<N$ on (a, b).*

PROOF. Let Φ and Ψ be functions continuous on J such that $\Phi=\Psi=P$ on $\{a, b\}$, Φ is convex, Ψ is concave, $\Phi'^+(a)=\Psi'^-(b)=-\infty$, $\Psi'^+(a)=\Phi'^-(b)=\infty$. Set $P_0=(P\vee\Phi)\wedge\Psi$. Obviously $\alpha=P_0'^+(a)$, $\beta=P_0'^-(b)$. Let C and C_0 be the convex hulls of the graphs of P and P_0 respectively. It is easy to see that $C_0\subset C$. Let φ be the greatest convex function on J such that $\varphi\leqq P_0$ and let ψ be the smallest concave function on J such that $P_0\leqq\psi$. Let C_1 be the set of all points $\langle x, y\rangle$ such that $x\in(a, b)$ and that $y=P(a)+s(x-a)$ or $\varphi(x)<y<\psi(x)$. Then $C_1\subset C_0$. Now we apply 1.

3. *Let S be a nonempty set closed in R. Let $A, B\in R\cup\{-\infty, \infty\}$. Let P be a function on R such that $A<P'(x)<B$ for each $x\in S$ and that*

$$A<(P(y)-P(x))/(y-x)<B,$$

whenever $x, y\in S$, $x\neq y$. Then there is a function G differentiable on R such that $G=P$, $G'=P'$ on S and $A<G'<B$ on R.

PROOF. We may suppose that $\inf S=-\infty$, $\sup S=\infty$. Let (a, b) be a component of $R\setminus S$ and let α, β, s be as in 2. There are $M, N\in R$ such that $A<M<\min(\alpha, \beta, s)$, $\max(\alpha, \beta, s)<N<B$. Construct a function G according to 2. In this way we define G on $R\setminus S$; further we set $G=P$ on S. It is easy to see that G has the required properties.

4. *Let $x_0, y_0, s\in R$. For each $\gamma\in(0, \infty)$ define*

$$W_\gamma=\{\langle x, y\rangle\in R\times R;\ |y-y_0-s(x-x_0)|<\gamma(x-x_0)\}. \tag{1}$$

Let $\varepsilon\in(0, \infty)$ and let $\langle x_1, y_1\rangle$, $\langle b, c\rangle\in W_\varepsilon$, $3x_1\leqq 4b-x_0$. Then $\langle 2b-x_1, 2c-y_1\rangle\in W_{5\varepsilon}$.

PROOF. We may suppose that $x_0=y_0=0$. Then $6x_1\leqq 8b$ and hence $|2c-y_1-s(2b-x_1)|\leqq 2|c-sb|+|y_1-sx_1|<\varepsilon(2b+x_1)\leqq\varepsilon(10b-5x_1)=5\varepsilon(2b-x_1)$.

REMARK. The geometric meaning of W_γ is obvious. To see the geometric meaning of assertion 4 the reader should realize that $3x_1\leqq 4b-x_0$ means the same as $x_1-x_0\leqq\frac{4}{3}(b-x_0)$ and that $\langle b, c\rangle$ is the center of the segment with end points $\langle x_1, y_1\rangle$ and $\langle 2b-x_1, 2c-y_1\rangle$.

5. *Let $x_0, y_0, s\in R$. For each $\gamma\in(0, \infty)$ define W_γ by* (1). *Let $\varepsilon\in(0, \infty)$ and let $\langle x_1, y_1\rangle$, $\langle b, c\rangle$, $\langle x_2, y_2\rangle\in W_\varepsilon$, $x_1<b<x_2$, $x\in R$, $3|x-b|\leqq b-x_1$. Let $q=(y_2-y_1)/(x_2-x_1)$. Then $\langle x, c+q(x-b)\rangle\in W_{3\varepsilon}$.*

PROOF. We may suppose that $x_0=y_0=0$. Set $y=c+q(x-b)$, $Z=|x-b|(x_1+x_2)/(x_2-x_1)$. As $3|x-b|<\min(x_2-x_1, b)$, we have $3Z<\min(x_1+x_2, b(x_1+x_2)/(x_2-x_1))$. If $x_2\leqq 2b$, then $x_1+x_2<3b$; if $x_2>2b$, then

$$(x_1+x_2)/(x_2-x_1)<(b+x_2)/(x_2-b)<3.$$

Thus in either case $Z<b$.

Obviously $|q-s|=|y_2-sx_2-(y_1-sx_1)|/(x_2-x_1)\leqq\varepsilon(x_1+x_2)/(x_2-x_1)$; therefore $|y-xs|=|c-sb+(x-b)(q-s)|\leqq\varepsilon b+\varepsilon Z<2\varepsilon b$. Since $x=b-(b-x)>2b/3$, we have $|y-sx|<3\varepsilon x$.

6. *Let S be a set closed in R. Let F be a function on S such that $F'_S(x)$ is finite for each accumulation point x of S. Then there is a function H on R differentiable at each point of S such that $H=F$ on S.*

PROOF. We may suppose that $\inf S=-\infty$, $\sup S=\infty$. Set

$$A^+=\{x\in S;\ S\cap(x, y)\neq\emptyset \quad \text{for each } y>x\},$$

$$A^-=\{x\in S;\ S\cap(y, x)\neq\emptyset \quad \text{for each } y<x\},$$

$I^+=A^-\setminus A^+$, $I^-=A^+\setminus A^-$, $I=S\setminus(A^+\cup A^-)$. Define a function f on S as follows: If $b\in A^+\cup A^-\ (=S\setminus I)$, set $f(b)=F'_S(b)$. If $b\in I$, find $x_1, x_2\in S$ such that $S\cap(x_1, x_2)=\{b\}$ and set

$$f(b)=\big(F(x_2)-F(x_1)\big)/(x_2-x_1).$$

For each $b\in S$ define a set M_b as follows:

If $b\in A^+\cap A^-$, let $M_b=\{b\}$.

If $b\in I^+\cup I^-$, choose a $d_b>0$ such that either $S\cap(b, b+3d_b)=\emptyset$ or $S\cap\cap(b-3d_b, b)=\emptyset$ and set

$$M_b=\{x;\ 2b-x\in S\cap[b-d_b, b+d_b]\}.$$

If $b\in I$, choose a $d_b>0$ such that $S\cap(b-3d_b, b+3d_b)=\{b\}$ and set $M_b==[b-d_b, b+d_b]$.

Let $M=\bigcup M_b$ $(b\in S)$. Obviously $b\in M_b$ for each $b\in S$ and $M_a\cap M_b=\emptyset$, whenever $a, b\in S$, $a\neq b$. If (a, b) is a component of $R\setminus S$, then $M_c\cap(a, b)=\emptyset$ for each $c\in S\setminus\{a, b\}$. Thus $(a, b)\setminus M=(a, b)\setminus(M_a\cup M_b)$ which is open. Therefore $R\setminus M=(R\setminus S)\setminus M$ is open, M is closed.

There is a unique function G on M with the following properties: $G=F$ on S; if $x\in M_b$, $b\in I^+\cup I^-$, then $G(x)=2F(b)-F(2b-x)$; if $x\in M_b$, $b\in I$, then $G(x)=F(b)+(x-b)f(b)$.

Let $x_0\in S$. We shall prove that

$$G'^{+}_{M}(x_0)=f(x_0). \tag{2}$$

The case $x_0\notin A^+$ is left to the reader. Now let $x_0\in A^+$ and let $\varepsilon\in(0, \infty)$. Set $s=f(x_0)$ $(=F'_S(x_0))$. For each $\gamma\in(0, \infty)$ define W_γ by (1). There is a $z>x_0$ such that $\langle x, F(x)\rangle\in W_\varepsilon$ for each $x\in S\cap(x_0, z)$. There are $z_1, z_2\in S$ such that $x_0<z_2<z$ and that $0<z_1-x_0<\frac{3}{4}(z_2-x_0)$ (so that $x_0<z_1<z_2$). Let $x\in M\cap(x_0, z_1)$. If $x\in S$,

then, obviously, $\langle x, G(x)\rangle \in W_\varepsilon$. Thus, let $x \notin S$ and let (a, b) be the component of $R \setminus S$ containing x. We have $x_0 < a < x < b \leqq z_1$. There are the following four possibilities:

1. $x \in M_b$, $b \in I^-$. Set $x_1 = 2b - x$. Then $x_1 \in S$, $0 < x_1 - b \leqq d_b \leqq (b-a)/3 < (b-x_0)/3$, therefore $3x_1 < 4b - x_0$, and $x_1 < z_1 + (z_1 - x_0)/3 = x_0 + \frac{4}{3}(z_1 - x_0) < z_2$. Set $c = F(b)$, $y_1 = F(x_1)$. We have $\langle b, c\rangle, \langle x_1, y_1\rangle \in W_\varepsilon$, $x = 2b - x_1$, $G(x) = 2c - y_1$ so that, by 4, $\langle x, G(x)\rangle \in W_{5\varepsilon}$.

2. $x \in M_b$, $b \in I$. There is an $x_2 \in S \cap (b, \infty)$ such that $S \cap (b, x_2) = \emptyset$. Obviously $x_2 \leqq z_2$. Then $G(x) = F(b) + (x-b)f(b)$, $0 < b - x \leqq d_b \leqq (b-a)/3$ so that by 5 with $x_1 = a$, $q = f(b)$ etc. we have $\langle x, G(x)\rangle \in W_{3\varepsilon}$.

3. $x \in M_a$, $a \in I^+$. Proceeding as in 1 we get $\langle x, G(x)\rangle \in W_{5\varepsilon}$.

4. $x \in M_a$, $a \in I$. Proceeding as in 2 we get $\langle x, G(x)\rangle \in W_{3\varepsilon}$.

This proves (2). Similarly, it can be shown that $G_M'^-(x_0) = f(x_0)$ for each $x_0 \in S$. Now it suffices to choose for H the function that equals G on M and is linear on the closure of each component of $R \setminus M$.

7. *Let T be a closed set in R, $V = R \setminus T$, $Q \subset V$ and let Q be isolated in V. Let g be a function on Q. Then there is a function K differentiable on R such that $K = 0$ on $T \cup Q$, $K' = 0$ on T and $K' = g$ on Q.*

PROOF. Let φ be a function differentiable on R such that $\varphi = 0$ on $\{0\} \cup \cup (R \setminus (-1, 1))$, $\varphi'(0) = 1$, $|\varphi| < 1$ on R. There is a function ω continuous on R such that $\omega = \omega' = 0$ on T and that $\omega > 0$ on V. There are positive numbers ε_q $(q \in Q)$ such that the intervals $J_q = [q - \varepsilon_q, q + \varepsilon_q]$ are pairwise disjoint and that $J_q \subset V$ for each q. Now let $\eta_q = \min\{\omega(x);\, x \in J_q\}$, $c_q = \max(1/\varepsilon_q, |g(q)|/\eta_q)$ and, for each $x \in R$, let

$$K(x) = \sum_{q \in Q} \frac{g(q)}{c_q} \varphi(c_q(x-q)).$$

Obviously $|K| \leqq \omega$ on R. It is easy to see that K satisfies our requirements.

REMARK. The following assertion is a generalization of Theorem 5.5.3 in [1].

8. *Let S be a nonempty set closed in R. Let F and f be functions on S such that $F_S'(x) = f(x)$ for each accumulation point x of S. Let $A, B \in R \cup \{-\infty, \infty\}$. Suppose that $A < f(x) < B$ for each $x \in S$ and that $A < (F(y) - F(x))/(y - x) < B$, whenever $x, y \in S$ and $x \neq y$. Then there is a function G differentiable on R such that $G = F$, $G' = f$ on S and $A < G' < B$ on R.*

PROOF. Let T be the set of all accumulation points of S. Let H be as in 6. By 7 there is a function K differentiable on R such that $K = 0$ on S, $K' = 0$ on T and that $K' = f - H'$ on $S \setminus T$. Set $P = H + K$. Obviously $P = F$ and $P' = f$ on S. Now we apply 3.

REMARK. It has been mentioned in [1] that there is a perfect set S and a function F on S such that $|F_S'(x)| \leqq 1$ for each $x \in S$ and that G' is unbounded for each function G differentiable on R such that $G = F$ on S. The following example shows a little more.

Let $1=x_0>x_1>\dots$, $x_n\to 0$, $y_n=x_{n-1}-x_n^2(x_{n-1}-x_n)$ $(n=1,2,\dots)$. It is easy to see that $x_n<y_n<x_{n-1}$. Set $S=\left(\bigcup_{n=1}^{\infty}[x_n,y_n]\right)\cup\{0\}$. Define a function F on S setting $F(0)=0$ and $F(x)=x_n^2$ for each $x\in[x_n,y_n]$. Then S is perfect and $F_S'=0$ on S. Now let G be a function differentiable on R such that $G=F$ on S. Then $G(x_{n-1})-G(y_n)=x_{n-1}^2-x_n^2>2x_n(x_{n-1}-x_n)=2(x_{n-1}-y_n)/x_n$ so that $(G(x_{n-1})-$ $-G(y_n))/(x_{n-1}-y_n)\to\infty$ $(n\to\infty)$. We see that G' is unbounded on $(0,1)$.

Thus, we have constructed a perfect set S and a function on S twice (actually, infinitely many times) differentiable relative to S that cannot be extended to a function twice differentiable on R.

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DEPARTMENT OF MATHEMATICS
MICHIGAN STATE UNIVERSITY
EAST LANSING, MICHIGAN 48824
USA

Let $1 = x_0 > y_1 > x_1 > \cdots$, $y_0 = 0$, $y_n = x_{n-1} - \frac{1}{2}(x_{n-1} - x_n)$ $(n = 1, 2, \ldots)$. It is easy to see that [illegible]. Let $S = \left\{ \bigcup [x_n, y_n] \right\} \cup \{0\}$. Define a function f on S by $f(0) = 0$ [illegible] for $x \in [x_n, y_n]$. Then S is perfect and f [illegible] S. Now let G be a function differentiable on R such that $G = f$ on S. [illegible] so that [illegible]. We see that G' is unbounded on $(0, 1)$.

Thus we have constructed a perfect set S and a function on S twice (actually infinitely many times) differentiable relative to S that cannot be extended to a function twice differentiable on R.

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SELECTIVE, BI-SELECTIVE, AND COMPOSITE DIFFERENTIATION

R. J. O'MALLEY* (Milwaukee) and C. E. WEIL (East Lansing)

Only the third notion of differentiation is new here. The results show when it implies each of the first two and when the second implies the third.

1. Introduction

The real line is denoted by R and all functions will be realvalued and defined on R. The closure of a set $A\subset R$ will be denoted by $\mathrm{Cl}\,A$. The selective and bi-selective derivatives have been studied by the first author in [2] and [6] respectively. The composite derivative is introduced here and we show that a composite derivative is a bi-selective derivative, determine when a composite derivative is a selective derivative, and find conditions under which a bi-selective derivative is a composite derivative. The paper is concluded with an example showing that sometimes selective and composite derivatives must be different.

2. Preliminaries

In this section we give the necessary definitions and state some known facts.

2.1. Definition. By a bi-selection, b, we mean an ordered pair, s and v, of interval functions defined on the family of all nondegenerate, closed subintervals of R satisfying $x<s([x, y])<y$ for all $x<y$. The interval function s is called a selection.

2.2. Definition. Let b be a bi-selection and let f and g be functions. Then g is said to be the bi-selective derivative of f relative to b if for each x

$$\lim_{y\to x}\big(v([x, y])-f(x)\big)/\big(s([x, y])-x\big) = g(x).$$

Here $[x, y]$ denotes the interval $[y, x]$ if $y<x$.

We note that if the limit defining $g(x)$ exists, it is unique. So it is permissible to denote g by bf'.

2.3. Definition. Let s be a selection and let f be a function. If f has a bi-selective derivative relative to the bi-selection b where $v([a, b])=f\big(s([a, b])\big)$, then we say f has a selective derivative relative to s and denote the derivative by sf'.

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2.4. DEFINITION. By a decomposition of R we mean closed sets E_n for $n=1, 2, 3, \ldots$ such that $\bigcup_{n=1}^{\infty} E_n = R$. A function f is said to have the function g as a composite derivative relative to the decomposition E_n if for each n and each $x \in E_n$

$$\lim_{y \to x,\, y \in E_n} (f(y)-f(x))/(y-x) = g(x).$$

In this case the above limit need not be unique unless x is a limit point of E_n. On the other hand if x lies in two different sets E_n, the two limits must be the same. So it may always be assumed that $E_n \subset E_{n+1}$.

The following facts are obvious or can be found in the references listed.

2.5. All selective derivatives are bi-selective derivatives.

2.6. Bi-selective derivatives are equivalent to l—r derivatives. (See [1] and [6].)

2.7. All selective derivatives have the Darboux property. (See [2].)

2.8. All bi-selective derivatives are of honorary Baire class 2 (see [6]) but even selective derivatives are not necessarily of Baire class 1. (See [2].)

2.9. All approximate derivatives are selective derivatives (see [2]) and composite derivatives. (See [5].)

We close this section by showing that the situation for the composite derivative relative to Baire 1 and Darboux is exactly opposite to that for the selective derivative; namely every composite derivative is of Baire class 1, but need not have the Darboux property.

2.10. THEOREM. *All composite derivatives are of Baire class* 1.

PROOF. Let the sets E_n, $n=1, 2, \ldots$, be a decomposition and let f and g be functions such that g is a composite derivative of f relative to the sets E_n. To shows that g is of Baire class 1 let E be any perfect set. Then $E = \bigcup_{n=1}^{\infty} (E \cap E_n)$ and each $E \cap E_n$ is closed. So there is a closed interval I such that $\emptyset \neq I \cap E \subset E \cap E_n$ for some n. Assume as we may that $I \cap E$ is perfect. Then g is a derivative relative to $I \cap E$. Thus by Theorem 5.5.2 on page 209 of [7], g is the restriction of a function $\bar{g}$ which is a derivative on R. The function $\bar{g}$ has a point of continuity, $x \in I \cap E$ relative to $I \cap E$, and since $g = \bar{g}$ on $I \cap E$, x is a point of continuity of g relative to $I \cap E$.

2.11. EXAMPLE. *A composite derivative need not have the Darboux property.*

PROOF. Let $f(x) = |x|$ and for $n=1, 2, \ldots$, let $E_n = R \setminus (0, 1/n)$. Then $g(x) = |x|/x$ for $x \neq 0$ and $g(0) = -1$ is a composite derivative of f (in fact *the* composite derivative of f) relative to the sets E_n, but g does not have the Darboux property.

Observe that g is also a bi-selective derivative of f. Thus a bi-selective derivative need not have the Darboux property.

3. Main theorems

3.1. THEOREM. *Let the sets* E_n, $n=1,2,\dots$ *be a decomposition of* R, *and let* f *and* g *be functions. Suppose* g *is a composite derivative of* f *relative to the sets* E_n. *Then there is a bi-selection* b *such that* $bf'(x)=g(x)$.

PROOF. Assume, as we may, that for each n, $E_n\subset E_{n+1}$. For each $x\in R$ let $n(x)=\min\{n\colon x\in E_n\}$. Thus $x\in E_{n(x)}\setminus E_{n(x)-1}$ where $E_0=\emptyset$. By the distinguished endpoint of an interval $[a,b]$ we mean that endpoint, d, satisfying $n(d)=\min\{n(a),n(b)\}$ with the honor going to the right endpoint in case $n(a)=n(b)$. To define the bi-selection we let $[a,b]$ be an interval and let d be the distinguished endpoint. If $E_{n(d)}\cap(a,b)\neq\emptyset$, then let $s([a,b])\in E_{n(d)}\cap(a,b)$ and let $v([a,b])=f(s([a,b]))$. If $E_{n(d)}\cap(a,b)=\emptyset$, then let $s([a,b])=\dfrac{a+b}{2}$ and $v([a,b])=f(d)+g(d)(b-a)/2$.

To show that the bi-selective derivative, bf', of f exists and is g, let $x\in R$ and let $\varepsilon>0$. There is a $\delta_1>0$ such that if $|y-x|<\delta_1$, then $n(y)\geqq n(x)$ for if not, then x would be the limit of a sequence from $E_{n(x)-1}$ and since each E_n is closed, x would belong to $E_{n(x)-1}$ contrary to the earlier observation that $x\in E_{n(x)}\setminus E_{n(x)-1}$. Next there is a $\delta_2>0$ such that if $y\in E_{n(x)}$ and if $|y-x|<\delta_2$, then $|g(x)-(f(y)-f(x))/(y-x)|<\varepsilon$. If x is a right hand limit point of $E_{n(x)}$, then let $\delta_3=+\infty$. If not, then there is a $\delta_3>0$ such that $E_{n(x)}\cap(x,x+\delta_3)=\emptyset$. Let $\delta=\min\{\delta_1,\delta_2,\delta_3\}$. If $0<|y-x|<\delta$, then since $\delta_1\geqq\delta$, $n(y)\geqq n(x)$. If $E_{n(x)}\cap\cap(x,y)\neq\emptyset$, then by definition of s, $s([x,y])\in E_{n(x)}\cap(x,y)$ and hence by definition of v,

$$|g(x)-(v([x,y])-f(x))/(s([x,y])-x)| = |g(x)-(f(s([x,y]))-f(x))/(s([x,y])-x)| < \varepsilon$$

since $\delta\leqq\delta_2$. Now suppose $E_{n(x)}\cap(x,y)=\emptyset$. If $x-\delta<y<x$, then $n(y)\geqq n(x)$ implies x is the distinguished endpoint of $[x,y]$. If on the other hand $x<y<x+\delta$, then the supposition $E_{n(x)}\cap(x,y)=\emptyset$ implies that x is not a right hand limit point of $E_{n(x)}$. Thus $\delta_3<+\infty$ and since $\delta\leqq\delta_3$, $y\notin E_{n(x)}$. Consequently $n(y)>n(x)$ and again x is the distinguished endpoint. Therefore by definition of s and v,

$$|g(x)-(v([x,y])-f(x))/(s([x,y])-x)| = |g(x)-(g(x)(y-x)/2+ +f(x)-f(x))/((y+x)/2-x)| = 0 < \varepsilon.$$

3.2. THEOREM. *Let* E_n, f, *and* g *be as in* 3.1. *Suppose in addition that for each* x, $g(x)$ *is a bilateral derived number of* f *at* x. *Then there is a selection,* s, *such that* $g=sf'$ *on* R.

PROOF. As in the proof of 3.1 assume that $E_n\subset E_{n+1}$, define the positive integer $n(x)$ for each $x\in R$ and the distinguished endpoint of a closed interval. Let $[a,b]$ be closed interval and let d denote its distinguished endpoint. As in the previous proof if $E_{n(d)}\cap(a,b)\neq\emptyset$, then let $s([a,b])\in E_{n(d)}\cap(a,b)$. However if $E_{n(d)}\cap(a,b)=\emptyset$, then since $g(d)$ is a bilateral derived number of f at d, we may select $s([a,b])\in(a,b)$ such that $|g(d)-(f(s([a,b]))-f(d))/(s([a,b])-d)|<|b-a|$.

Let $x \in R$ and let $\varepsilon > 0$. Let $\delta = \min\{\delta_1, \delta_2, \delta_3, \varepsilon\}$ where δ_1, δ_2, and δ_3 are defined as in the proof of 3.1. Let $0 < |x-y| < \delta$. If $E_{n(x)} \cap (x, y) \neq \emptyset$, then as before $|g(x) - (f(s([x, y])) - f(x))/(s([x, y]) - x)| < \varepsilon$. Suppose $E_{n(x)} \cap (x, y) = \emptyset$. Again as before x is the distinguished endpoint of $[x, y]$ and consequently by definition of s

$$|g(x) - (f(s([x, y])) - f(x))/(s([x, y]) - x)| < |y - x| < \delta \leqq \varepsilon.$$

3.3. Corollary. *Let E_n, f, and g be as in* 3.1. *If for each x there is an n such that x is a bilateral limit point of E_n, then there is a selection, s, such that $sf' = g$.*

Proof. The hypotheses of 3.3 imply those of 3.2.

The condition on a decomposition given in 3.3 besides yielding that the composite derivative is Darboux also gives that it will have Zahorski's M_2 property as well as the Denjoy—Clarkson property. (See [2].)

To complete the circle we now present a condition by which a bi-selective derivative becomes a composite derivative. It is clear that some condition is needed since a bi-selective derivative need not be Baire 1 and hence need not be a composite derivative. The idea comes from [4] and requires that we first introduce the following notion.

3.4. Definition. By a family of tangential paths we mean a function t on $\{(x, h): x \in R, |h| > 0\}$ such that for all $x \in R$ and $|h| > 0$, $t(x, h) \neq h$ and for each $x \in R$, $\lim_{h \to 0} t(x, h)/h = 1$.

3.5. Theorem. *Let b be a bi-selection and let f and g be functions such that $bf' = g$ on R. Suppose there is a family of tangential paths, t, such that for each $x \in R$*

$$\lim_{h \to 0} (v([x+h, x+t(x, h)]) - f(x))/(s([x+h, x+t(x, h)]) - x) = g(x).$$

Then there is a decomposition of R into sets E_n, $n = 1, 2, \ldots$, such that g is a composite derivative of f relative to the sets E_n.

Proof. For each n let

$$A_n = \{x \colon |(v([x, x+\delta]) - f(x))/(s([x, x+\delta]) - x)| < n \text{ if } 0 < |\delta| < 1/n\},$$

and let $E_n = \operatorname{Cl} A_n$. By Lemma 2 of [6] if $x_1, x_2 \in E_n$ and if $|x_1 - x_2| < 1/n$, then $|f(x_1) - f(x_2)| \leqq n|x_1 - x_2|$. Since $bf' = g$ on R, $\bigcup_{n=1}^{\infty} A_n = R$. So the sets E_n are a decomposition of R.

Let n be a positive integer, let $x \in E_n$ and let $\{x_k\}$ be a sequence in E_n converging to x. Without loss of generality we may assume that $x = 0$, $f(0) = 0$, and $0 < |x_k| < 1/n$ for each k. For each k there is an $h_k \in A_n$ such that $|h_k - x_k| < |x_k|/k < 1/n$ and $0 < |h_k| < 1/n$. Let $t_k = t(0, h_k)$, $v_k = v([h_k, t_k])$ and $s_k = s([h_k, t_k])$. Then

$$f(x_k)/x_k = \{((f(x_k) - f(h_k))/(x_k - h_k))((x_k - h_k)/x_k)\} +$$
$$+ \{((f(h_k) - v_k)/(h_k - s_k))((h_k - s_k)/h_k)(h_k/x_k)\} + \{(v_k/s_k)(s_k/h_k)(h_k/x_k)\}.$$

The first factor of the first term is bounded by n. (If we were fortunate enough to be able to choose $h_k=x_k$, then we just let the first term be zero.) The second factor tends to 0 as $k\to\infty$. So the first term has limit 0 as $k\to\infty$. Since $h_k\in A_n$, the first factor of the second term is bounded by n provided $|t_k-h_k|<1/n$ which is true for k sufficiently large since t_k/h_k has limit 1 as $k\to\infty$, which also implies that s_k/h_k has limit 1 as $k\to\infty$ and hence the second factor has limit 0 as $k\to\infty$. Finally the first factor of the third term has limit $g(0)$ as $k\to\infty$ by assumption; the second and third factors both have limit 1 when $k\to\infty$ as was just mentioned. Thus the third factor has limit $g(0)$ as $k\to\infty$ which completes the proof.

To see how close a general bi-selective derivative comes to being a composite derivative, the reader is referred to Proposition 4 of [6].

We close this paper with an example of a function, f, which has both a selective derivative and a composite derivative. In general it is possible for a given function to have several selective derivatives depending on the selection and likewise many composite derivatives. The important aspect of this example is that any selective derivative of f must differ from any composite derivative of f.

3.6. Example. Let $C\subset[0,1]$ be the Cantor set. Let $f=0$ on $(R\setminus[0,1])\cup C$. Let (a,b) be a component interval of $[0,1]\setminus C$. We let f be a continuously differentiable function on $[a,b]$ whose derivative from the right at a is 0, whose derivative from the left at b is 1 and $f\big((a+b)/2\big)=3(b-a)/2$. Clearly the function $g=0$ on $(R\setminus[0,1])\cup C$ and $g=f'$ on the component intervals of $[0,1]\setminus C$ is a composite derivative of f.

Let K be the set of all right hand endpoints of component intervals of $[0,1]\setminus C$. We assert that there is a selection, s, such that $sf'=0$ on $C\setminus K$, $sf'=1$ on K, and of course $sf'=f'$ elsewhere. To define s let $x<y$. First suppose $x\notin K$. If $C\cap(x,y)\neq\emptyset$, then let $s([x,y])\in C\cap(x,y)$. If $C\cap(x,y)=\emptyset$, then let $s([x,y])=$ $=(x+y)/2$. Now suppose $x\in K$. If z is the midpoint of a component interval of $[0,1]\setminus C$ to the right of x, then $\big(f(z)-f(x)\big)/(z-x)=1$. If $y\notin C$, then let $s([x,y])$ be any such midpoint. If $y\in C$, then we choose $s([x,y])$ to be the midpoint of one of these intervals that is so close to x that $\big(f(y)-f(s([x,y]))\big)/\big(y-s([x,y])\big)>$ $>x-y$. By a straightforward but tedious argument it can be shown that sf' is as claimed.

To show that no composite derivative of f can be a selective derivative of f we recall that a selective derivative has the Darboux property and we now show that no composite derivative of f can have that property. Let the sets E_n, $n=1,2,\ldots$ be a decomposition of R and let g be a composite derivative of f relative to the sets E_n. Then $C=\bigcup_{n=1}^{\infty}(E_n\cap C)$. Thus there is an interval I with $\emptyset\neq I\cap C\subset E_n\cap C$ for some n. Since $f=0$ on $E_n\cap C$, $g=0$ on $E_n\cap C$. Since K is dense in C, $g=0$ at some points of K. Since $g=f'$ on the component intervals of $[0,1]\setminus C$, since f' is continuous on these intervals and has value 1 at the right hand endpoints, we see that g is not Darboux.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN
MILWAUKEE, WISCONSIN 53201
USA

DEPARTMENT OF MATHEMATICS
MICHIGAN STATE UNIVEFSITY
EAST LANSING, MICHIGAN 48824
USA

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DILATABLE OPERATOR VALUED FUNCTIONS ON C^*-ALGEBRAS

Z. SEBESTYÉN (Budapest)

Introduction

Our recent papers [6], [7] on moment theorems with respect to C^*-algebras offer a way to treat these questions in a setting of the dilation theory due to Halmos, Naimark and Sz.-Nagy [5]. We shall do this investigations here.

Let A be a (complex) C^*-algebra, not necessarily with unit, let G be a multiplicative semigroup in A, closed with respect to the involution of A (briefly a *-semigroup) and such that its linear span is a norm dense *-subalgebra in A. Given an operator valued function f on G, $f: G \to B(H)$, where $B(H)$ is the C^*-algebra of all bounded linear operators on the (complex) Hilbert space H, we say that f is dilatable with respect to A if there is a Hilbert space K, a continuous linear operator V of K into H and a *-representation S of A on K such that

$$f(g) = VS_gV^* \tag{1}$$

holds for each g in G.

In the scalar valued case, when $H=\mathbf{C}$ (i.e. $\dim H=1$), V is a continuous linear functional on K, hence by the Riesz Representation Theorem there is a vector x in K such that (1) is of the form [6]

$$f(g) = (S_gx, x) \quad (g \in G) \tag{1$'$}$$

where in addition $V^*1=x$. In this case

$$\varphi(a) = (S_ax, x) \quad (a \in A) \tag{2$'$}$$

is a (unique) positive linear extension of f giving a solution of a moment theorem with respect to the C^*-algebra A.

In the previous case, similarly,

$$\varphi(a) = VS_aV^* \quad (a \in A) \tag{2}$$

defines not only a positive linear, but also a dilatable extension of f. These two notions coincide in the case when A is commutative (Theorem 1), which is an easy consequence of [7, Theorem 4], due to the author, and generalizes a theorem of Stinespring (and Naimark too). As a corollary, we give a new characterization of subnormal operators on Hilbert space differing from that of Halmos—Bram, MacNerny and Embry (Theorem 2).

In the case when A is noncommutative, our result (Theorem 3) subsumes the previous ones and gives a common generalization of theorems of Sz.-Nagy [5] and Stinespring [8].

Dilatable functions with respect to commutative C^*-algebras

The first result is a simple consequence of our previous one proved in [7, Theorem 4].

THEOREM 1. *Let G be a multiplicative *-semigroup in a C^*-algebra A such that G generates a norm dense *-subalgebra in A. Given an operator valued function f of G with values in $B(H)$, the C^*-algebra of all bounded linear operators on the Hilbert space H, it is dilatable with respect to A if and only if there is a positive constant M such that if $0\neq x\in H$, then*

$$\frac{1}{M\|x\|^2}\left|\sum_g c_g(f(g)x,x)\right|^2 \leqq \sum_{g,h} c_g\bar{c}_h(f(h^*g)x,x) \leqq M\|x\|^2\left\|\sum_g c_g g\right\|^2 \tag{3}$$

holds for each finite sequence $\{c_g\}$ of complex numbers indexed by elements of G.

PROOF. The necessity is an easy consequence of (1) which holds by assumption:

$$\left|\sum_g c_g(f(g)x,x)\right|^2 = \left|\left(\left(\sum_g c_gS_g\right)V^*x, V^*x\right)\right|^2 \leqq \|V^*\|^2\|x\|^2\left\|\left(\sum_g c_gS_g\right)V^*x\right\|^2 =$$

$$= \|V^*\|^2\|x\|^2\sum_{g,h} c_g\bar{c}_h(VS_{h^*g}V^*x,x) = \|V^*\|^2\|x\|^2\sum_{g,h} c_g\bar{c}_h(f(h^*g)x,x) \leqq$$

$$\leqq \|V^*\|^4\|x\|^4\left\|S\sum_g c_g g\right\|^2 \leqq \|V^*\|^4\|x\|^4\|S\|^2\left\|\sum_g c_g g\right\|^2$$

where $\|S\|\leqq 1$, as S is a *-representation of the C^*-algebra A.

To prove the sufficiency of (1) assume (3) and conclude by [7, Theorem 4], (an operator valued moment theorem, if A is considered via the commutative Gelfand—Naimark Theorem as $C_0(\Omega)$, the complex valued continuous functions vanishing at infinity over the locally compact Hausdorff space Ω), that there is a positive operator valued measure $F(\cdot)$ on Ω, dilatable by our Naimark-type result [7, Theorem 2] to a spectral measure $E(\cdot)$ on a Hilbert space K which has a suitable continuous linear operator V into H such that

$$F(.) = VE(.)V^* \tag{4}$$

holds for these two operator measures $F(\cdot)$ and $E(\cdot)$ on Ω. Define now a *-representation S of A by

$$S_a = \int_\Omega a(t)E(dt) \quad (a\in A). \tag{5}$$

We have the desired property of S given in (1):

$$(f(g)x,x) = \int_\Omega g(t)(F(dt)x,x) = \int_\Omega g(t)(VE(dt)V^*x,x) =$$

$$= \int_\Omega g(t)E(dt)(V^*x,V^*x) = (S_gV^*x,V^*x) = (VS_gV^*x,x).$$

The proof is complete.

As a consequence we give a new characterization of subnormal operators, showing that an operator B on a Hilbert space is subnormal if and only if the operator double-sequence $\{B^{*m}B^n\}_{m,n=0}^{\infty}$ is a moment sequence on the compact subset Ω of the complex plane (the spectrum of B in the sense of MacNerny [4]). Recall that B is subnormal if there is a Hilbert space K containing H, and a normal operator N on K with H as invariant subspace and extending B. In other words for the orthogonal projection P of K onto H

$$PN^{*m}N^n x = B^{*m}B^n x \tag{5}$$

holds for any natural numbers $m, n \geqq 0$ [5].

THEOREM 2. *B is a subnormal operator on the Hilbert space H if and only if*

$$0 \leqq \sum_{\substack{m,n\\k,l}} c_{m,n}\bar{c}_{k,l}(B^{*(l+m)}B^{k+n}x, x) \leqq \|x\|^2 \max_{\lambda\in\Omega} \Big|\sum_{m,n} c_{m,n}(\bar{\lambda})^m \lambda^n\Big|^2 \quad (x\in H) \tag{6}$$

holds for any double-sequence $\{c_{m,n}\}_{m,n=0}^{\infty}$ of complex numbers, where Ω denotes the spectrum of B.

PROOF. The necessity of (6) is a simple consequence of the spectral theorem with respect to normal operators and, of course, (5):

$$\sum_{k,l,m,n} c_{m,n}\bar{c}_{k,l}(B^{*(l+m)}B^{k+n}x, x) = \sum_{k,l,m,n} c_{m,n}\bar{c}_{k,l}(PN^{*(l+m)}N^{k+n}x, x) =$$

$$= \sum_{k,l,m,n} c_{m,n}\bar{c}_{k,l}(N^{*(l+m)}N^{k+n}x, x) =$$

$$= \Big\|\sum_{m,n} c_{m,n}N^{*m}N^n x\Big\|^2 \leqq \|x\|^2 \Big\|\sum_{m,n} c_{m,n}N^{*m}N^n\Big\|^2 \leqq \|x\|^2 \max_{\lambda\in\Omega} \Big|\sum_{m,n} c_{m,n}(\bar{\lambda})\lambda^n\Big|^2.$$

To prove the sufficiency of (6), we shall prove that the function $f((\bar{\lambda})^m\lambda^n) = B^{*m}B^n$ $(m, n = 0, 1, 2, \ldots)$ is dilatable with respect to the C^*-algebra $C(\Omega)$ of the complex continuous functions on the spectrum Ω of B. Here by the Stone—Weierstrass Theorem the *-semigroup $\{(\bar{\lambda})^m\lambda^n\}_{m,n=0}^{\infty}$ $(\lambda\in\Omega)$ of polynomials generates a norm dense *-subalgebra in $C(\Omega)$. But our assumption (6) proves (3) with $M=1$, only the left hand side of (3) is not seen at once. In this special case the *-semigroup (and the C^*-algebra) in question has a unit element (the constant 1 function on Ω) so that this difficulty vanishes if we take the left hand side of (3) as a Schwarz inequality for the numerical function $(f(\cdot)x, x)$ $(x\in H)$, which is positive definite by assumption, indeed.

The general case

Our next result is a common generalization of the Naimark, Sz.-Nagy and Stinespring dilation theorems.

THEOREM 3. *Let A be a C^*-algebra, G a multiplicative *-semigroup in A, generating a norm dense *-subalgebra in A. Given an operator valued function $f: G \to B(H)$ on G with values in $B(H)$, the C^*-algebra of all bounded linear operators on the Hilbert space H, f is dilatable with respect to A if and only if there is* *a positive constant M such that*

$$\frac{1}{M}\|\sum_g f(g)x_g\|^2 \leqq \sum_{g,h}\big(f(h^*g)x_g, x_h\big), \tag{7}$$

$$\sum_{g,h} c_g\bar{c}_h\big(f(h^*g)x, x\big) \leqq M\|x\|^2\,\|\sum_g c_g g\|^2 \quad (x\in H) \tag{8}$$

holds for any finite sequences $\{x_g\}$ and $\{c_g\}$ in H and $\mathbf{C}$, respectively.

PROOF. The necessity of (7) and (8) are easy consequences of the dilatability with respect to A, since (1) implies

$$\|\sum_g f(g)x_g\|^2 = \|V(\sum_g S_g V^* x_g)\|^2 \leqq \|V\|^2 \sum_{g,h}(VS_{h^*g}V^*x_g, x_h) =$$

$$= \|V\|^2 \sum_{g,h}\big(f(h^*g)x_g, x_h\big),$$

$$\sum_{g,h} c_g\bar{c}_h\big(f(h^*g)x, x\big) = \big((\sum_{g,h} c_g\bar{c}_h S_{h^*g})V^*x, V^*x\big) \leqq$$

$$\leqq \|V^*\|^2\,\|x\|^2\,\|\sum_g c_g S_g\|^2 = \|V\|^2\|x\|^2\|S(\sum_g c_g g)\|^2 \leqq \|V\|^2\|x\|^2\|S\|^2\|\sum_g c_g g\|^2,$$

where $\|S\|\leqq 1$ as in the proof of Theorem 1.

To prove the sufficiency, assume (7) and (8) and consider the linear space F of H-valued functions with finite support on G with semi-inner product (via (7)) defined by

$$\langle \sum_h h\otimes x_h,\ \sum_k k\otimes y_k\rangle = \sum_{h,k}\big(f(k^*h)x_h, y_k\big), \tag{9}$$

where the generating element $h\otimes x_h$ is the function with value $x_h\in H$ in $g\in G$ and 0 otherwise on G, and Σ denotes (as always in this paper) a finite sum (over the *-semigroup G). We thus get a Hilbert space K by first factoring F with respect to the nullspace of $\langle\cdot,\cdot\rangle$ and then completing this quotient space with respect to the norm obtained from the arising inner product. For simplicity we take F as a norm dense subspace of K, denoting elements of it and the inner product of K also by the symbols introduced before, respectively.

We have a shift operation S_g on F (for a g in G) given by

$$S_g(\sum_h h\otimes x_h) = \sum_h gh\otimes x_h \quad (g\in G). \tag{10}$$

Our first aim is to prove that S_g generates a bounded linear operator, denoted by the same symbol, on K. For $u=\sum_h h\otimes x_h$ in K we have by (9)

$$\|S_g u\|^2 = \langle S_g u, u\rangle = \langle S_{g^*g}u, u\rangle \leqq \|S_{g^*g}u\|\,\|u\|,$$

$$\|S_g u\|^{2^{n+1}} = \|S_{(g^*g)^{2^n-1}}u\|^2\|u\|^{2^{n+1}-2} = \|\sum_h (g^*g)^{2^n-1}h\otimes x_h\|^2\|u\|^{2^{n+1}-2} =$$

$$= \|u\|^{2^{n+1}-2}\sum_{h,k}\langle (g^*g)^{2^n-1}h\otimes x_h,\ (g^*g)^{2^n-1}k\otimes x_k\rangle \leqq$$

$$\leqq \|u\|^{2^{n+1}-2}\sum_{h,k}\|(g^*g)^{2^n-1}h\otimes x_h\|\,\|(g^*g)^{2^n-1}k\otimes x_k\| = \|u\|^{2^{n+1}-2}\big(\sum_h\|(g^*g)^{2^n-1}h\otimes x_h\|\big)^2$$

for $n=1, 2, \ldots,$ by induction. Using (8) this implies

$$\|S_g u\|^{2^{n+1}} \leqq \|u\|^{2^{n+1}-2}\big(\sum_h (f(h^*(g^*g)^{2^n}h)x_h, x_h)^{\frac{1}{2}}\big)^2 \leqq$$

$$\leqq \|u\|^{2^{n+1}-2}\big(\sum_h \sqrt{M}\,\|x_h\|\,\|(g^*g)^{2^{n-1}}h\|\big)^2 \leqq \|u\|^{2^{n+1}-2}M\|(g^*g)^{2^{n-1}}\|^2\big(\sum_h \|x_h\|\,\|h\|\big)^2 =$$

$$= \|u\|^{2^{n+1}-2}M\|g\|^{2^{n+1}}\big(\sum_h \|x_h\|\,\|h\|\big)^2,$$

giving as $n\to\infty$

$$\|S_g\| \leqq \|g\| \tag{11}$$

for any g in G. It is easy to prove that $(S_g)^*=S_{g^*}$ and $S_{gg'}=S_gS_{g'}$, for any g, g' in G. In other words $S\colon G\to B(K)$ is a *-representation of G on K. To prove that it has a (unique) extension to a *-representation of A on K we have only to show instead of (11)

$$\Big\|\sum_g \lambda_g S_g\Big\| \leqq \Big\|\sum_g \lambda_g g\Big\| \tag{12}$$

for any finite sequence $\{\lambda_g\}$ of complex numbers indexed by elements of G. For simplicity let $S_a=\sum_g \lambda_g S_g$ for an $a=\sum_g \lambda_g g$ in A. Then we have, similarly as before, for any $u=\sum_h h\otimes x_h$

$$\|S_a u\|^2 = \langle S_{a^*a}u, u\rangle \leqq \|S_{a^*a}u\|\,\|u\|,$$

$$\|S_a u\|^{2^{n+1}} \leqq \big\|S_{(a^*a)^{2^{n-1}}}u\big\|^2\,\|u\|^{2^{n+1}-2} = \|u\|^{2^{n+1}-2}\Big\|\sum_{h,s} g_s\, h\otimes\lambda_s x_h\Big\|^2 =$$

$$= \|u\|^{2^{n-1}-2}\sum_{h,k}\Big\langle \sum_s g_s h\otimes\lambda_s x_h,\ \sum_t g_t k\otimes\lambda_t x_k\Big\rangle \leqq$$

$$\leqq \|u\|^{2^{n+1}-2}\sum_{k,h}\Big\|\sum_s g_s h\otimes\lambda_s x_h\Big\|\,\Big\|\sum_t g_t k\otimes\lambda_t x_k\Big\| = \|u\|^{2^{n+1}-2}\Big(\sum_h\Big\|\sum_s g_s h\otimes\lambda_s x_h\Big\|\Big)^2$$

where $(a^*a)^{2^{n-1}}=\sum_s \lambda_s g_s$ stands for the sake of simplicity, so that

$$\|S_a u\|^{2^{n+1}} \leqq \|u\|^{2^{n+1}-2}\Big(\sum_h\big(\sum_{s,t}\lambda_s\bar\lambda_t(f(h^*g_t^*g_sh)x_h, x_h)\big)^{\frac{1}{2}}\Big)^2 \leqq$$

$$\leqq \|u\|^{2^{n+1}-2}\Big(\sum_h \sqrt{M}\|x_h\|\,\Big\|\sum_h \lambda_s g_s h\Big\|\Big)^2 \leqq$$

$$\leqq \|u\|^{2^{n+1}-2}M\|(a^*a)^{2^{n-1}}\|^2\big(\sum_h \|x_h\|\,\|h\|\big)^2 = M\big(\sum_h \|x_h\|\,\|h\|\big)^2\|a\|^{2^{n+1}}\|u\|^{2^{n+1}-2}$$

thus giving (12) as $n\to\infty$, indeed.

Finally, we have a bounded linear operator V of K into H, if we define it on F by

$$V\big(\sum_h h\otimes x_h\big) = \sum_h f(h)x_h. \tag{13}$$

It is densely defined and bounded by (7) so that V has a (unique) bounded linear extension to K, denoted also by V. To prove (1), it suffices to show

$$S_gV^*x = g\otimes x \quad (g\in G, x\in H) \tag{14}$$

since then $VS_gV^*x=V(g\otimes x)=f(g)x$ holds for any x in H, proving (1). To prove (14), we need only to show that

$$\langle\sum_h h\otimes x_h, S_gV^*x\rangle = (VS_{g^*}(\sum_h h\otimes x_h), x) = (V\sum_h g^*h\otimes x_h, x) =$$
$$= \sum_h (f(g^*h)x_h, x) = \langle\sum_h h\otimes x_h, g\otimes x\rangle$$

for any element $\sum_h h\otimes x_h$ in F, since they are norm dense in K. The proof is complete.

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DEPARTMENT OF MATH. ANALYSIS II
EÖTVÖS LORÁND UNIVERSITY
MÚZEUM KÖRÚT 6—8
1088 BUDAPEST, HUNGARY

Acta Math. Hung.
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ON IRREDUCIBLE OPERATOR *-ALGEBRAS ON BANACH SPACES

J. VUKMAN (Maribor)

This work is in close relationship with our earlier research (see [9], [11], and [12]). Throughout this paper we denote by $L(X)$ the algebra of all bounded linear operators on the Banach space X. A subalgebra $\mathscr{B}\subset L(X)$ is irreducible on X (or acts irreducibly on X), if for each pair $x, y\in X$, $x\neq 0$ there exists $A\in\mathscr{B}$ such that $Ax=y$. We follow C. E. Rickart [8], and call a subalgebra $\mathscr{B}\subset L(X)$ strongly irreducible if for each $y\in X$ there exists a constant α_y with the following property: If $x\in X$, $\|x\|=1$, then there exists $A\in\mathscr{B}$ such that $Ax=y$, and $\|A\|\leqq\alpha_y$. Note that a subalgebra $\mathscr{B}\subset L(X)$ is strongly irreducible if it contains all linear operators with finite dimensional range. Let $\mathscr{A}$ be a real or complex Banach $*$-algebra with the identity element e. We say that $\mathscr{A}$ is symmetric if $(e+a^*a)^{-1}$ exists for each $a\in\mathscr{A}$. As usual, we call a linear functional on the complex Banach $*$-algebra $\mathscr{A}$ positive, if $f(a^*a)\geqq 0$ for all $a\in\mathscr{A}$. In the real case we call a linear functional f positive if $f(a^*a)\geqq 0$ and $f(a^*)=f(a)$ are fulfilled for all $a\in\mathscr{A}$. We denote by K_f the left kernel which corresponds to the positive functional f acting on the real or complex Banach $*$-algebra $\mathscr{A}$ ($K_f=\{a; f(a^*a)=0\}$). We shall write $r(a)$ for the spectral radius of $a\in\mathscr{A}$ and $p(a)$ for $r(a^*a)^{1/2}$. It should be mentioned that the spectral radius of an element a in a real Banach $*$-algebra $\mathscr{A}$ is defined to be equal to the spectral radius of a as an element of the complexification of $\mathscr{A}$ (see [8, p. 5], and [10] for details).

Let X be such a real or complex Banach space that there exists an involution $A\to A^*$ on $L(X)$ satisfying the condition $A^*A\neq 0$ for each nonzero $A\in L(X)$. According to the classical result of S. Kakutani and G. W. Mackey (see [3] and [4]) there exists an inner product on X such that the corresponding norm is equivalent to the given norm on X and that A^* is the adjoint of A relative to the inner product. It should be mentioned that J. Bognár obtained a simple and elementary proof of this result (see [1] and [2]). Some results in the sense of the Kakutani—Mackey theorem can be found in [9], [11] and [12]. The main purpose of this paper is to prove the following result which also characterizes a Banach space with an equivalent Hilbert norm among all Banach spaces. The proof is based on N. Namsraj's result concerning the existence of positive functionals on complex symmetric Banach $*$-algebras (see [5]), and recent extension of this result to the real case which can be found in our earlier paper [10].

THEOREM 1. *Let X be a real or complex Banach space. Suppose there exists a strongly irreducible symmetric Banach $*$-algebra $\mathscr{B}\subset L(X)$ which contains the identity operator I. In this case there exists an inner product on X such that the corresponding norm is equivalent to the given norm, and that for each $A\in\mathscr{B}$, A^* is the adjoint of A relative to the inner product.*

REMARKS. The algebra $\mathscr{B}$ in the theorem above is semisimple since it is irreducible. Therefore by [6, (5,5)] and [10, Theorem 4] $A^*A\neq 0$ is fulfilled for each nonzero $A\in\mathscr{B}$. On the other hand we did not require the existence of operators with finite dimensional range, minimal idempotents or minimal ideals in $\mathscr{B}$. Therefore we cannot introduce an inner product into X using the dual space of X (see [11]) or a minimal hermitian idempotent (see [7], [8, Theorem (4.10.7)] and [9]). In our case an inner product will be introduced into X via a positive functional.

For the proof of Theorem 1 we need the following lemmas.

LEMMA 1. *Let $\mathscr{A}$ be a real or complex symmetric Banach $*$-algebra with the identity element e. Then the following statements are fulfilled.*

1° To each proper left ideal $\mathscr{L}\subset\mathscr{A}$ there corresponds a positive functional f such that $\mathscr{L}\subset K_f, f(e)=1$.

2° There exists a constant M such that the relation $r(a^*a)^{1/2}\leqq M\|a\|$ is fulfilled for each $a\in\mathscr{A}$.

PROOF. For the proof of the complex version of 1° see the proof of Theorem 1 in [5]. The real version of 1° is contained in [10, Theorem 11]. The proof of the complex version of 2° can be found in [6] (see (8,1) and (8,2)). The proof is based on the subadditivity of $p(\cdot)$, and the fact that the radical of a complex symmetric Banach $*$-algebra contains exactly those elements a for which the relation $p(a)=0$ is fulfilled. Since both of those results are proved also for real symmetric Banach $*$-algebras (see [10, Theorem 4]), the real version of 2° can be proved in the same way.

LEMMA 2. *Let f be a positive functional acting on a real or complex Banach $*$-algebra $\mathscr{A}$ with the identity element e. The following statements are fulfilled.*

1° $f(a^*a)\leqq f(e)r(a^*a)$ for all $a\in\mathscr{A}$.

2° $f(b^*a^*ab)\leqq r(a^*a)f(b^*b)$ for all pairs $a, b\in\mathscr{A}$.

PROOF. The complex version is well known (see [6, (2,4)]). For the real version see the proof of Lemma 8 in [10].

PROOF OF THEOREM 1. Let $u\in X$ be a fixed nonzero vector. First observe that each $x\in X$ can be written in the form $x=Au$ for some $A\in\mathscr{B}$ since $\mathscr{B}$ is by assumption strongly irreducible. It is easy to prove that the left ideal $\mathscr{L}=\{A;\, A\in\mathscr{B}, Au=0\}$ is maximal. Since $\mathscr{L}$ is closed, the quotient space $\mathscr{B}/\mathscr{L}$ is complete in the norm $\|A+\mathscr{L}\|_0=\inf\limits_{B\in\mathscr{L}}\|A+B\|$. By the open mapping theorem the isomorphism $A+\mathscr{L}\mapsto Au$, which maps $\mathscr{B}/\mathscr{L}$ onto X, is bicontinuous if we equip $\mathscr{B}/\mathscr{L}$ with the norm $\|\cdot\|_0$. By Lemma 1 there exists a positive functional $f, f(I)=1$ such that $\mathscr{L}\subset K_f$. Since $\mathscr{L}$ is maximal, we have $\mathscr{L}=K_f$. Therefore $\mathscr{B}/\mathscr{L}$ is a pre-Hilbert space with the inner product $(A+\mathscr{L}, B+\mathscr{L})=f(B^*A)$, $A, B\in\mathscr{B}$. Denote the norm corresponding to the inner product by $\|\cdot\|_1$. Let us prove that there exists a constant α such that for all $A+\mathscr{L}\in\mathscr{B}/\mathscr{L}$ the inequality

$$\|A+\mathscr{L}\|_1\leqq\alpha\|A+\mathscr{L}\|_0 \tag{1}$$

is fulfilled. From the first statement of Lemma 2 it follows $\|A+\mathscr{L}\|_1^2=f(A^*A)=$ $=f\big((A+B)^*(A+B)\big)\leqq r\big((A+B)^*(A+B)\big)$. Since by the second statement of Lemma 1 there exists a constant M such that $r\big((A+B)^*(A+B)\big)^{1/2}\leqq$ $\leqq M\|A+B\|$, we obtain $\|A+\mathscr{L}\|_1\leqq M\|A+B\|$ where B is any operator from $\mathscr{L}$.

Therefore $\|A+\mathscr{L}\|_1 \leqq M \inf_{B\in\mathscr{L}} \|A+B\| = M\|A+\mathscr{L}\|_0$ which completes the proof of (1). The isomorphism $A+\mathscr{L} \to Au$ allows us to introduce an inner product into X as follows

$$(x, y) = (A+\mathscr{L}, B+\mathscr{L}) = f(B^*A), \quad x = Au, \quad y = Bu.$$

Denote the norm induced by this inner product by $\|\cdot\|_2$. Then combining (1) with the fact that there exists a bicontinuous isomorphism between $\mathscr{B}/\mathscr{L}$ (equipped with the norm $\|\cdot\|_0$), and X (equipped with the original norm $\|\cdot\|$), we obtain that the inequality

$$\|x\|_2 \leqq \beta\|x\| \tag{2}$$

is fulfilled for some constant β, and each $x\in X$. Let us prove that

$$\|x\| \leqq \gamma\|x\|_2 \tag{3}$$

for some γ and all $x\in X$. For this purpose let us first prove the relation

$$\|Ax\|_2 \leqq r(A^*A)^{1/2}\|x\|_2\,, \tag{4}$$

where $A\in\mathscr{B}$ and $x\in X$ are arbitrary. Let $A\in\mathscr{B}$, and $x=Bu, B\in\mathscr{B}$ be given. Then $\|Ax\|_2^2=(Ax, Ax)=((AB)u, (AB)u)=f((AB)^*(AB))=f(B^*A^*AB)$. Using the second statement of Lemma 2 we obtain $\|Ax\|_2^2=f(B^*A^*AB)\leqq r(A^*A)f(B^*B)= =r(A^*A)\|x\|_2^2$, which completes the proof of (4). From (4) and the second statement of Lemma 1 it follows that

$$\|Ax\|_2 \leqq M\|A\|\|x\|_2 \tag{5}$$

for some constant M, all $A\in\mathscr{B}$ and all $x\in X$. Let a fixed vector $e\in X$, $\|e\|_2=1$ be given. Then by strong irreducibility there exists for each $x\in X$ an operator $A\in\mathscr{B}$ such that $Ax=\|x\|e$, and $\|A\|\leqq C$, where C is some constant. Therefore $\|x\|=\|Ax\|_2$, and by (5) $\|x\|=\|Ax\|_2\leqq M\|A\|\,\|x\|_2\leqq MC\|x\|_2$, which completes the proof of (3). From (2) and (3) it follows that the norm induced by the inner product is equivalent with the given norm on X. It remains to prove that $(Ax, y)= =(x, A^*y)$ for $A\in\mathscr{B}$, and all pairs $x, y\in X$. Let $A\in\mathscr{B}$ and $x, y\in X$ be given. There exist $A_1, A_2\in\mathscr{B}$ such that $x=A_1u, y=A_2u$. Since $Ax=(AA_1)u, A^*y= =(A^*A_2)u$, we obtain $(Ax, y)=f(A_2^*AA_1)$, and $(x, A^*y)=f((A^*A_2)^*A_1)=f(A_2^*AA_1)$. The proof of the theorem is complete.

We conclude with the result below which can be considered as a consequence of Kadison's remarkable result concerning representations of B^*-algebras (see [8, Theorem (4.9.10)]).

THEOREM 2. *Let X be a complex Banach space, and suppose that there exists an irreducible B^*-algebra $\mathscr{B}\subset L(X)$ which contains the identity operator. In this case there exists an inner product on X such that the corresponding norm is equivalent to the given norm, and that for each $A\in\mathscr{B}$, A^* is the adjoint of A relative to the inner product.*

PROOF. The proof will be similar to the proof of Theorem 1. Let therefore $u\in X$ be a fixed nonzero vector, and denote by $\mathscr{L}$ the left ideal $\{A;\, A\in\mathscr{B},\, Au=0\}$, which is by irreducibility of $\mathscr{B}$ maximal. Since a complex B^*-algebra is symmetric, there exists by Theorem (4.7.14) in [8] a pure state f such that $K_f=\mathscr{L}$. Therefore, as in the proof of Theorem 1, an inner product can be introduced into $\mathscr{B}/\mathscr{L}$ as follows: $(A+\mathscr{L}, B+\mathscr{L})=f(B^*A)$, $A, B\in\mathscr{B}$. By Lemma (4.9.11) in [8] the norm induced by this inner product is equivalent to the norm $\|A+\mathscr{L}\|_0=\inf_{B\in\mathscr{L}}\|A+B\|$. Combining this with the fact that there exists a bicontinuous isomorphism between X and $\mathscr{B}/\mathscr{L}$ equipped with the norm $\|\cdot\|_0$, it follows that an inner product can be introduced into X such that the corresponding norm is equivalent to the given norm on X. The rest of the proof goes through as in the proof of Theorem 1.

REMARK. It seems that the proof of Theorem 2 cannot be used for the real case, since real B^*-algebras are not necessarily symmetric, and since the proof of Theorem 2 depends heavily on Lemma (4.9.11) in [8], which is an immediate consequence of Kadison's result mentioned above, and which is by our knowledge proved only for complex B^*-algebras.

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UNIVERSITY OF MARIBOR
VEKS
62000 MARIBOR
RAZLAGOVA 14
YUGOSLAVIA

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ON THE THREE-DIMENSIONAL FINSLER SPACES WITH T-TENSOR OF A SPECIAL FORM

H. D. PANDE and J. P. PANDEY (Gorakhpur)

In this paper a special form of the T-tensor T_{hijk} has been proposed and studied in three-dimensional Finsler spaces.

Introduction

Let $C_{ijk}(x, y)$ (cf. Matsumoto [3]) be the $(h)hv$-torsion tensor of an n-dimensional Finsler space F_n with the metric function $L(x, y)$ where x is a point and y is an element of support.

The v-curvature tensor S_{hijk} (cf. Matsumoto [3]) is defined by

$$S_{hijk} = C_{hkr}C^r_{i\,j} - C_{hjr}C^r_{i\,k}. \tag{1.1}$$

The v-covariant derivative of a tensor T^i_j is given by

$$T^i_j|_k = \partial T^i_j/\partial y^k + T^r_j C^i_{rk} - T^i_r C^r_{jk}. \tag{1.2}$$

The T-tensor T_{hijk} (cf. Matsumoto [4]) is completely symmetric and is defined by the equation

$$T_{hijk} = LC_{hij}|_k + C_{hij}l_k + C_{hik}l_j + C_{hkj}l_i + C_{kij}l_h \tag{1.3}$$

where $l_i = L^{-1}y_i$.

A Finsler space in which the $h(hv)$-torsion tensor C_{ijk} is of the form

$$(n+1)C_{ijk} = h_{ij}C_k + h_{jk}C_i + h_{ki}C_j, \tag{1.4}$$

is called a C-reducible Finsler space (cf. Matsumoto [5]).

A Finsler space F_n $(n \geqq 3)$ with the non-zero length C of the torsion vector C^i is called semi-C-reducible (cf. Matsumoto and Shibata [7]), if the $(h)hv$-torsion tensor C_{ijk} is of the form

$$C_{ijk} = \frac{p}{n+1}(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j) + \frac{q}{C^2}C_iC_jC_k \tag{1.5}$$

where p and $q\,(=1-p)$ do not vanish. p is called the characteristic scalar of the F_n.

The purpose of the present paper is to study a special form of the T-tensor T_{hijk} in special Finsler spaces.

Special form of the T-tensor T_{hijk}

We assume that that the T-tensor T_{hijk} is written in the form

$$T_{hijk} = h_{hi}Q_{jk}+h_{hj}Q_{ik}+h_{hk}Q_{ij}+h_{ij}Q_{hk}+h_{ik}Q_{hj}+h_{jk}Q_{hi}+\lambda C_h C_i C_j C_k \tag{2.1}$$

where Q_{ij} are components of a certain tensor field and λ is a scalar.

Since T_{hijk} is symmetric in all the indices, hence we get

$$h_{jk}(Q_{hi}-Q_{ih}) = 0 \tag{2.2}$$

which implies that Q_{ij} is a symmetric tensor.

Contracting (2.1) with respect to y^k and using $T_{hijk}y^k=h_{jk}y^k=0$ we obtain

$$h_{ji}Q_{jo}+h_{hj}Q_{io}+h_{ij}Q_{ho} = o \tag{2.3}$$

where "o" means contraction with respect to the element of support.

Further contraction of (2.3) with respect to g^{hi} yields

$$Q_{io}=o \quad (n>2) \tag{2.4}$$

which means that Q_{ij} is an indicatory tensor.

Therefore, we have

LEMMA (2.1). *If the T-tensor of a Finsler space $F_n(n>2)$ is of the form* (2.1), *then Q_{ij} is a symmetric and indicatory tensor.*

In a two-dimensional Finsler space (cf. Berwald [1]), the angular metric tensor h_{ij} and the $(h)hv$-torsion tensor C_{ijk} are written in the form

$$\text{a) } h_{ij} = m_i m_j \quad \text{b) } LC_{ijk} = Im_i m_j m_k \tag{2.5}$$

where the function I is called the principal scalar by L. Berwald.

From (1.3) and (2.5), the T-tensor of F_2 is given by

$$LT_{hijk} = I_{;2}m_h m_i m_j m_k \tag{2.6}$$

where

$$I_{;2}= L\frac{\partial I}{\partial y^k}m^k. \tag{2.7}$$

Hence the T-tensor T_{hijk} of an F_2 is of the form (2.1) since Q_{ij} and λ are found from (2.5) and (2.6) in the form

$$Q_{ij} = \frac{L^{-1}I_{;2}m_i m_j}{7}, \quad \lambda = \frac{L^{-1}I_{;2}}{7C^4}. \tag{2.8}$$

The T-tensor of a semi-C-reducible Finsler space (Matsumoto and Shibata [7]) of the first kind is given by

$$T_{hijk} = L(T_1^{(1)}H_{hijk}+T_2^{(1)}H_{hijk}^{(c)}+T_3^{(1)}C_{hijk}^{(4)}) \tag{2.9}$$

where

$$T_1^{(1)} = (p\alpha + p_c)pC^2/(n+1)(n+1-2p), \quad T_2^{(1)} = q(p\alpha + p_c)/(n+1-2p),$$

$$T_3^{(1)} = [\{n+1-(n+3)p\}q\alpha + \{(n+3)p - 2(n+1)\}p_c]/(n+1-2p)C^2,$$

$$H_{hijk} = h_{hi}h_{jk} + h_{hj}h_{ki} + h_{hk}h_{ij},$$

$$H^{(c)}_{hijk} = h_{hi}C_jC_k + h_{ij}C_kC_h + h_{hj}C_iC_k + h_{jk}C_iC_h + h_{hk}C_iC_j + h_{ki}C_jC_h,$$

$$C^{(4)}_{hijk} = C_hC_iC_jC_k, \quad p_i = \frac{\partial p}{\partial y^i} \quad \text{and} \quad p_c = p_iC^i/C^2, \quad \alpha = pC^i|_i/C^2.$$

We see that the T-tensor T_{hijk} of a semi-C-reducible Finsler space of the first kind is of the form (2.1) since Q_{ij} and λ are found from (2.9) as

$$Q_{ij} = L(\tfrac{1}{2}T_1^{(1)}h_{ij} + T_2^{(1)}C_iC_j), \quad \lambda = LT_3^{(1)}. \tag{2.10}$$

The indicatorized tensor T_{hijkl} (Fukui and Yamada [2] and Yamada [9]) of $S_{hijk}|_l$ is given by

$$T_{hijkl} = S_{hijk}|_l + L^{-1}(21_lS_{hijk} + l_hS_{lijk} + l_iS_{hljk} + l_jS_{hilk} + l_kS_{hijl}). \tag{2.11}$$

Differentiating (1.1) with respect to y^k, we get

$$S_{hijk}|_l = C_{hkr}|_lC_i^r{}_j + C_{hkr}C_i^r{}_j|_l - C_{hjr}|_lC^r_{ik} - C_{hjr}C^r_{ik}|_l. \tag{2.12}$$

Indicatorizing (2.12) and using the indicatory properties of C_{ijk}, Q_{ij} and h_{ij}, we obtain with the help of (2.1), (2.12),

$$\begin{aligned} LT_{hijkl} = {} & h_{kj}a_{lhi} + h_{kl}a_{jhi} + h_{jl}a_{khi} + h_{hi}a_{lkj} + \\ & + h_{hl}a_{ijk} + h_{il}a_{hkj} - h_{ki}a_{lhj} - h_{kl}a_{ihj} - h_{il}a_{khj} - h_{hj}a_{lki} - h_{hl}a_{jik} - h_{jl}a_{hki} + \\ & + C_{hil}Q_{kj} + C_{kjl}Q_{hi} - C_{hjl}Q_{ki} - C_{kil}Q_{hj} + \lambda(C_kC_lC_jE_{hi} + C_hC_iC_lE_{kj} - C_kC_iC_lE_{hj} - \\ & - C_hC_lE_{ki}) \end{aligned} \tag{2.13}$$

where

$$a_{lij} = Q_{ml}C^m_{ij} \quad \text{and} \quad E_{ij} = C_mC^m_{ij}. \tag{2.14}$$

Therefore we have

THEOREM (2.1). *If the indicatorized tensor of $L\,C_{hij}|_k$ is of the form (2.1), then the indicatorized tensor of $S_{hijk}|_l$ is given by (2.13).*

Three dimensional Finsler spaces

In this section we give the fundamental formulae of three-dimensional Finsler spaces. Matsumoto [6] developed the theory of three-dimensional Finsler spaces with respect to the orthogonal frame $e_{(\alpha)i}$, $\alpha=0, 1, 2$ where $e_{(0)i}=L^{-1}y^i$, $e_{(1)i}=C^{-1}C_i$ where C is the length of the torsion vector C_i. The third vector $e_{(2)i}$ is given by

$$e_{(2)i} = e_{ijk}e_{(0)}{}^je_{(1)}{}^k. \tag{3.1}$$

The $(h)hv$-torsion tensor C_{ijk} are written as

(3.2) $$LC_{ijk} = C_{\alpha\beta\gamma} e_{(\alpha)i} e_{(\beta)j} e_{(\gamma)k}$$

where $L^{-1}C_{\alpha\beta\gamma}$ are called the scalar components of C_{ijk}. We also have

(3.3) $$C_{\alpha\beta 0} = 0, \quad C_{111} = H, \quad C_{122} = I, \quad C_{222} = -C_{112} = J.$$

H, I and J are called the main scalars and satisfy

(3.4) $$H+I = LC.$$

The v-covariant derivative of the vector $e_{(\alpha)}{}^i$ are given by

(3.5) $$e_{(\alpha)}{}^i|_j \cdot L = \delta_{0\alpha}\delta^i_j - e_{(0)}{}^i e_{(\alpha)j} + \delta^{012}_{0\alpha\beta} e_{(\beta)}{}^i v_j$$

where v_j are the components of the v-connection vector. The scalar components $T_{\alpha\beta\gamma}$ of a tensor $T^i_{.jk}$ are defined by

(3.6) $$T_{\alpha\beta\gamma} = T^i_{.jk} e_{(\alpha)i} e_{(\beta)}{}^j e_{(\gamma)}{}^k.$$

The scalar components $T_{\alpha\beta;\gamma}$ of $T^i{}_j|_k$ L are written in the form

(3.7) $$T_{\alpha\beta;\gamma} = L\frac{\partial T_{\alpha\beta}}{\partial y^k} e_{(\gamma)}{}^k + T_{\mu\beta}\Gamma^{(v)}_{(\mu)\alpha\gamma} + T_{\alpha\mu}\Gamma^{(v)}_{(\mu)\beta\gamma}$$

where the quantities $\Gamma^{(v)}_{(\alpha)\beta\gamma}$ are such that

(3.8) $$\Gamma^{(v)}_{(\alpha)\beta\gamma} = -\Gamma^{(v)}_{(\beta)\alpha\gamma}, \quad \Gamma^{(v)}_{(0)\beta\gamma} = \delta_{\beta\gamma} - \delta_{0\beta}\delta_{0\gamma}$$

and $\Gamma^{(v)}_{(1)2\gamma} = v_\gamma$. The v-covariant derivatives of $T^i_{.j}$ are given by

(3.9) $$T^i_{.j}|_k \cdot L = T_{\alpha\beta;\gamma} e_{(\alpha)}{}^i e_{(\beta)j} e_{(\gamma)k}.$$

Since $y_i = L\dfrac{\partial L}{\partial y^i}$, from (3.2) and (3.9), we obtain

(3.10) $$L^2 \cdot C_{hij}|_k + L \cdot C_{hij} e_{(0)k} = C_{\alpha\beta\gamma;\delta} e_{(\alpha)h} e_{(\beta)i} e_{(\gamma)j} e_{(\delta)k}.$$

From (3.3), (3.4), (3.7) and (3.8), we obtain

(3.11) $$\begin{cases} C_{0\beta\gamma;\delta} = -C_{\beta\gamma\delta}, \\ C_{111;\delta} = H_{;\delta} + 3Jv_\delta, \\ C_{112;\delta} = -J_{;\delta} + (H-2I)v_\delta, \\ C_{122;\delta} = I_{;\delta} - 3Jv_\delta, \\ C_{222;\delta} = J_{;\delta} + 3Iv_\delta. \end{cases}$$

where $H_{;\delta} = L(\partial H/\partial y^i)e_{(\delta)}{}^i$.

The v-connection vector also satisfy the following relations:

(3.12) $$\begin{cases} \text{(i)} & (H-2I)v_1 - 3Jv_2 = J_{;1} + H_{;2}, \\ \text{(ii)} & 3Jv_1 + (H-2I)v_2 = I_{;1} + J_{;2}, \\ \text{(iii)} & 3Iv_1 + 3Jv_2 = -J_{;1} + I_{;2}. \end{cases}$$

From (1.3) and (3.2) we get

(3.13) $$LT_{hijk} = \{C_{\alpha\beta\gamma;\delta} + C_{\beta\gamma\delta}\delta_{0\alpha} + C_{\alpha\gamma\delta}\delta_{0\beta} + C_{\alpha\beta\delta}\delta_{0\gamma}\} e_{(\alpha)h} e_{(\beta)i} e_{(\gamma)j} e_{(\delta)k}.$$

Three dimensional Finsler space with T-tensor of the form (2.1)

Let F_3 be the three-dimensional Finsler space whose T-tensor T_{hijk} is of the form (2.1). Let $Q_{\alpha\beta}$ be the scalar components of LQ_{ij} i.e.

$$LQ_{ij} = Q_{\alpha\beta} e_{(\alpha)i} e_{(\beta)j}. \tag{4.1}$$

Since Q_{ij} is a symmetric and indicatory tensor, hence we have

$$Q_{\alpha\beta} = Q_{\beta\alpha} \quad \text{and} \quad Q_{0\alpha} = o. \tag{4.2}$$

The scalar component of the angular metric tensor h_{ij} are given by

$$h_{ij} = (\delta_{\alpha\beta} - \delta_{0\alpha}\delta_{0\beta}) e_{(\alpha)i} e_{(\beta)j}. \tag{4.3}$$

From (2.1), (3.13), (4.1), and (4.3), we obtain

$$\begin{aligned} C_{\alpha\beta\gamma;\delta} + C_{\beta\gamma\delta}\delta_{0\alpha} + C_{\alpha\gamma\delta}\delta_{0\beta} + C_{\alpha\beta\delta}\delta_{0\gamma} &= (\delta_{\alpha\beta} - \delta_{0\alpha}\delta_{0\beta}) Q_{\gamma\delta} + (\delta_{\alpha\gamma} - \delta_{0\alpha}\delta_{0\gamma}) Q_{\beta\delta} + \\ &+ (\delta_{\alpha\delta} - \delta_{0\alpha}\delta_{0\delta}) Q_{\beta\gamma} + (\delta_{\beta\gamma} - \delta_{0\beta}\delta_{0\gamma}) Q_{\alpha\delta} + (\delta_{\beta\delta} - \delta_{0\beta}\delta_{0\delta}) Q_{\alpha\gamma} + \\ &+ (\delta_{\gamma\delta} - \delta_{0\gamma}\delta_{0\delta}) Q_{\alpha\beta} + L_1 \lambda C^4 \delta_{1\alpha}\delta_{1\beta}\delta_{1\gamma}\delta_{1\delta}. \end{aligned} \tag{4.4}$$

From (3.3), (3.11), (4.2) and (4.4), we get

$$\left\{ \begin{array}{ll} \text{(i)} & H_{;1} + 3Jv_1 = 6Q_{11} + L\lambda C^4, \\ \text{(ii)} & H_{;2} + 3Jv_2 = 3Q_{12}, \\ \text{(iii)} & -J_{;1} + (H-2I)v_1 = 3Q_{12}, \\ \text{(iv)} & -J_{;2} + (H-2I)v_2 = Q_{22} + Q_{11}, \\ \text{(v)} & I_{;1} - 3Jv_1 = Q_{11} + Q_{22}, \\ \text{(vi)} & I_{;2} - 3Jv_2 = 3Q_{12}, \\ \text{(vii)} & J_{;1} + 3Iv_1 = 3Q_{12}, \\ \text{(viii)} & J_{;2} + 3Iv_2 = 6Q_{22}. \end{array} \right. \tag{4.5}$$

By virtue of the equations (3.12), equations (ii) and (iii), equations (iv) and (v) and equations (vi) and (vii) are identical. From equations (ii) and (vi) of (4.9) and (3.4) we obtain

$$LC_{;2} = 6Q_{12} \quad \text{i.e.} \quad Q_{12} = \frac{1}{6} LC_{;2}. \tag{4.6}$$

Also adding (iii) and (vii) and using (3.4), we get

$$Q_{12} = \frac{1}{6} LCv_1. \tag{4.7}$$

Therefore $Q_{12} = \frac{1}{6} LC_{;2} = \frac{1}{6} LCv_1$. Hence $v_1 = C^{-1}C_{;2}$. Adding (iv) and (viii) of (4.5) and using (3.4), we get

$$Q_{11} + 7Q_{22} = (H+I)v_2 = LCv_2. \tag{4.8}$$

From (v) (4.5) and (4.8) we obtain

$$(4.9)\qquad Q_{22} = \frac{1}{6}(LCv_2 - I_{;1} + 3Jv_1) \quad\text{and}\quad Q_{11} = \frac{1}{6}(7I_{;1} - 21Jv_1 - LCv_2).$$

With the help of (i) (4.5) and (4.9), we have

$$(4.10)\qquad \lambda = C^{-4}\{C_{;1} + Cv_2 - 8L^{-1}(I_{;1} - 3Jv_1)\}.$$

Therefore we have

THEOREM (4.1). *If the T-tensor of a three-dimensional Finsler space is of the form* (2.1), *then the scalar components* $Q_{\alpha\beta}$ *of* LQ_{ij} *and the scalar* λ *are given by* (4.2), (4.7), (4.9) *and* (4.10).

We also have

COROLLARY (4.1). *If the T-tensor of a three-dimensional Finsler space is of the form* (2.1), *then the v-connection vector* v_i *vanishes if and only if* $Q_{12}=o$ *and* $Q_{11}+7Q_{22}=o$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF GORAKHPUR
GORAKHPUR
INDIA

DEPARTMENT OF MATHEMATICS
S. H. KISAN P. G. COLLEGE
BASTI (U. P.) — 272001
INDIA

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ON A PROBLEM OF F. A. SZÁSZ

S. FEIGELSTOCK (Ramat-Gan)

1. The study of rings satisfying the minimum condition for principal (right) two-sided ideals, (MHR) MHI-rings, was initiated by F. A. SzÁsz [2, 3, and 4]. These rings provoke interesting questions in the radical theory of rings, see [5, pp. 128—129, problems 29—40]. Szász has asked for necessary and sufficient conditions under which an MHR-ring is embeddable into a unital MHR-ring. In this note, the structure of the additive groups of unital MHR, and MHI-rings will be given. Using this structure theory, it will be shown that an (MHR) MHI-ring may be embedded into a unital (MHR) MHI-ring if and only if its torsion part is bounded.

The author is indebted to Professor Szász for posing the above problem, and for his suggestion that it might be solved by studying the additive groups of MHR-rings.

2. *Notation:*

R	a ring
R^+	the additive group of R
R_t	the torsion part of R^+
R_p	the p-primary component of R_t, p a prime
$I \lhd R$	I is an ideal in R
$\langle x \rangle$	the ideal in R generated by $x \in R$
Q	the field of rational numbers.

THEOREM 1. *Let G be a torsion free group. The following are equivalent:*
1) *G is the additive group of an* MHR*-ring.*
2) *G is the additive group of an* MHI*-ring.*
3) *G is divisible.*

PROOF. Clearly 1)⇒2).

2)⇒3). Let R be an MHI-ring with $R^+=G$, $x \in G$, and let n be a positive integer. There exists a positive integer k such that $\langle n^k x \rangle = \langle n^{k+1} x \rangle$. Therefore there exists $y \in R$ such that $n^k x = n^{k+1} y$. Hence $n^k(x-ny)=0$. Since G is torsion free, $x=ny$, and so G is divisible.

3)⇒1). If G is a divisible torsion free group, then G is the additive group of a field.

LEMMA 2. *Let R be a torsion free* (MHR) MHI*-ring, and let $I \lhd R$. Then I^+ is divisible.*

PROOF. Similar to the proof of the implication 2)⇒3) in Theorem 1.

COROLLARY 3. *Let R be an* (MHR) MHI-*ring, and let $I \lhd R/R_t$. Then I^+ is divisible.*

COROLLARY 4. *Let R be a torsion free* (MHR) MHI-*ring, and let S be an extension of R by Q. Then S is an* (MHR) MHI-*ring.*

PROOF. It follows easily from Lemma 2, that every (right) ideal in R is a (right) ideal in S. The statement of Corollary 4 is a direct consequence of this fact.

A ring satisfying the minimum condition for principal ideals generated by a torsion free element will be called an MHTFI-ring. It is easily observed that if R is an MHTFI-ring, then R/R_t is an MHI-ring.

THEOREM 5. *Let G be a group. The following are equivalent:*

1) *G is the additive group of a unital* MHR-*ring.*

2) *G is the additive group of a unital* MHI-*ring.*

3) *G is the additive group of a unital* MHTFI-*ring.*

4) $G \cong \bigoplus_{\alpha} Q^+ \oplus \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{k_i} \bigoplus_{\alpha_{ij}} Z(p_i^j)$, *$n, k_i$ positive integers, p_i a prime, and α, α_{ij} arbitrary cardinals, $i=1, \dots, n; j=1, \dots, k_i$.*

PROOF. Clearly 1)$\Rightarrow$2)$\Rightarrow$3).

3)$\Rightarrow$4). Let R be a unital MHTFI-ring, with unity e. If R^+ is a torsion group, then $|e|=n<\infty$, and $nx=(ne)x=0$ for all $x \in R$. Hence R^+ is bounded, and is of form 4) with $\alpha=0$, [1, Theorem 17.2]. It may therefore be assumed that R^+ is not a torsion group, in which case e is torsion free. Let p be a prime. There exists a positive integer k such that $\langle p^k e\rangle = \langle p^{k+l} e\rangle$ for every non-negative integer l. Hence $p^k e = p^{k+l} y_l$, for some $y_l \in R$, or $p^k(e - p^l y_l) = 0$. Therefore $e = p^l y_l + z_l$, with $p^k z_l = 0$. Let $x \in R_p$, $x \neq 0$, and choose l such that $|x| = p^l$. Then $x = ex = z_l x$. Hence $p^k x = (p^k z_l)x = 0$, i.e., $p^k R_p = 0$. This implies that $R_p^+ = \bigoplus_{j=1}^{k} \bigoplus_{\alpha_j} Z(p^j)$, α_j a cardinal, $j=1, \dots, k$, [1, Theorem 17.2].

Suppose that $R_{p_i} \neq 0$ for infinitely many primes p_i, $i=1, 2, 3, \dots$. It was shown above that there exists a positive integer k_i such that $p_i^{k_i} R_{p_i} = 0$, $i=1, 2, 3, \dots$. There exists a positive integer n, such that $\langle p_1^{k_1} p_2^{k_2} \dots p_n^{k_n} e\rangle = \langle p_1^{k_1} p_2^{k_2} \dots p_n^{k_n} p_{n+1}^{k_{n+1}} e\rangle$. Put $s = p_1^{k_1} p_2^{k_2} \dots p_n^{k_n}$. There exists $f \in R$ such that $se = p_{n+1}^{k_{n+1}} sf$. Let $x \in R_{p_{n+1}}$, $x \neq 0$. Since $(s, |x|) = 1$, there exist integers u, v such that $us + v|x| = 1$. Hence $x = ex = usex = usfp_{n+1}^{k_{n+1}} x = 0$, a contradiction. Therefore $R_t = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{k_i} \bigoplus_{\alpha_{ij}} Z(p_i^j)$, α_{ij} a cardinal, $i=1, \dots, n$; $j=1, \dots, k_i$. Since R_t is bounded, $R^+ = R_t \oplus H$, H a torsion free group, [1, Theorem 100.1]. By Lemma 2, H is divisible, and so $H \cong \bigoplus_{\alpha} Q^+$, [1, Theorem 23.1].

4)$\Rightarrow$1). Let G be a group satisfying 3). Let F be a field with $F^+ \cong \bigoplus_{\alpha} Q^+$, and let $Z_{p_i^j}$ be the ring of integers modulo p_i^j, $i=1, \dots, n$; $j=1, \dots, k_i$. Then $R = F \oplus \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{k_i} \bigoplus_{\alpha_{ij}} Z_{p_i^j}$ is a unital MHR-ring with $R^+ \cong G$.

For an alternate proof of the equivalence of 1) and 4) in Theorem 5, see [3, Satz 3.2].

COROLLARY 6. *Let* R *be a unital* (MHR) MHI-*ring. Then* R *is a ring direct sum* $R=R_t\oplus R_0$, *with* R_t *a bounded* (MHR) MHI-*ring, and* R_0 *a torsion free, divisible* (MHR) MHI-*ring.*

COROLLARY 7. *Let* R *be an* (MHR) MHI-*ring.* R *may be embedded into a unital* (MHR) MHI-*ring if and only if* R_t *is bounded.*

PROOF. If R is a subring of a unital (MHR) MHI-ring then R_t is bounded by Corollary 6.

Conversely, let R be an (MHR) MHI-ring with R_t a bounded group. Then $R^+=R_t\oplus R_0$, [1, Theorem 100.1]. By Corollary 3, R_0 is divisible. Let n be a positive integer such that $nR_t=0$. Now $R_t\cdot R_0=R_t\cdot(nR_0)=(nR_t)\cdot R_0=0$, and similarly $R_0\cdot R_t=0$. Hence $R=R_t\oplus R_0$ is a ring direct sum, with R_t a bounded (MHR) MHI-ring, and R_0 a torsion free, divisible (MHR) MHI-ring. It clearly suffices to embed R_t and R_0 separately into unital (MHR) MHI-rings. The following procedure is essentially that in [1, Lemma 123.2]. Let Z_n be the ring of integers modulo n. Clearly, R_t is a Z_n-algebra. Put $G_t=Z_n^+\oplus R_t$. For $a_i\in Z_n^+$, $b_i\in R_t$, $i=1,2$, define $(a_1+b_1)(a_2+b_2)=a_1a_2+b_1b_2+a_1b_2+a_2b_1$, where the products a_1a_2, b_1b_2 are the products in Z_n and R_t respectively, and a_1b_2, a_2b_1 are defined by the action of Z_n on R_t. This multiplication induces a unital (MHR) MHI-ring structure S_t, with $R_t\lhd S_t$. Replacing Z_n with Q, and R_t with R_0; the same procedure yields a unital ring S_0, with $R_0\lhd S_0$, and $S_0/R_0\cong Q$. By Corollary 4, S_0 is an (MHR) MHI-ring.

OBSERVATION. The proof of Corollary 7 shows that if an MHR (MHI)-ring R is embeddable into a unital (MHR) MHI-ring as a subring, then R may be embedded into a unital (MHR) MHI-ring as an ideal.

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BAR-ILAN UNIVERSITY
RAMAT-GAN
ISRAEL

COROLLARY 6. Let R be a unital (MHR) MHI-ring. Then R is a ring direct sum [illegible] with R_t a [illegible] (MHI) MHI-ring, and R_t a torsion [illegible] (MHR) MHI-ring.

COROLLARY 7. Let R be [illegible] MHI-ring. R may [illegible]. Then a unital (MHR) MHI-ring [illegible]

PROOF. If R is a subring of a unital (MHR) MHI-ring [illegible] by Corollary 6.

Conversely, let R be an [illegible] MHI-ring with [illegible] group. Then [illegible]

[illegible] following process [illegible]

[illegible] integers [illegible] the products [illegible] are defined [illegible] (MHR) MHI-ring structure [illegible] R with [illegible] the same process [illegible] By Corollary 6 [illegible] MHI-ring.

OBSERVATION. [illegible] (MHR) MHI-ring [illegible] embedded [illegible] unital (MHR) MHI-ring [illegible]

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A GENERALIZATION OF STRONGLY REGULAR RINGS

V. GUPTA (Tripoli)

In this paper we introduce the notion of s-weakly regular rings. We give some characterizations of s-weakly regular rings with unit element. Finally we show that every ring A has a unique maximal two sided s-weakly regular ideal $S(A)$ with some radical like properties.

All rings are assumed to be associative. A ring A is called s-weakly regular if for all $a\in A$ we have $a\in aAa^2A$. The class of s-weakly regular rings lies strictly between the class of right (or left) weakly regular ring and strongly regular rings. Following are some examples.

EXAMPLE 1 (Fisher [3], Example 2). $K[y, D]$, the ring of differential polynomials in the indeterminate y with cofficients in K, where K is a universal differential field with derivation D, is s-weakly regular but not strongly regular.

EXAMPLE 2. A_n, the ring of $n\times n$ matrices over a divison ring is a right (left) weakly regular but not s-weakly regular.

We use the following notations.

$\langle x\rangle$ = the two sided ideal generated by $x\in A$.
$X^T = \{a\in A|Xa=aX=0\}$, the annihilator of a non empty set $X\subseteq A$.
$X^r = \{a\in A|Xa=0\}$, the right annihilator of a non empty set $X\subseteq A$.
X^1 = the left annihilator of a non empty set $X\subseteq A$.

Concerning our terminology we refer to the papers by V. A. Andrunakievic and Ju. M. Rjabuhin [1], V. S. Ramamurthy [7] and E. T. Wong [9].

A ring A is called reduced if it is without non zero nilpotent elements. Now we give the following two lemmas which will be used frequently in the subsequent study.

LEMMA 3 (K. Chiba and H. Tominaga [2]). *Let A be a reduced ring and let $a, b\in A$.*

(i) *If $ab=0$ then $ba=0$ and $(a)^r=(a)^1$.*

(ii) *If $a\neq 0$ then $A/(a)^r$ is reduced and the residue class $\bar{a}$ of a* mod $(a)^r$ *is a non zero divisor.*

(iii) *If A is a prime ring then A contains no non zero divisor (i.e. $(a)^r=(a)^1=0$ for $a\neq 0$).*

PROOF. It is obvious.

LEMMA 4. *If A is a prime s-weakly regular ring then A is a simple ring with unit element.*

PROOF. It is clear that A is reduced. Let $a\in A$, then $a=ax$ where $x\in\langle a^2\rangle$. Now x acts as a unit element of A, since for all $\alpha\in A$, $x\alpha-\alpha\in(a)^r=0$. Thus $x\alpha=\alpha$. It is obvious that $\alpha x=\alpha$. Moreover $A=Ax\subseteq AAa^2A\subseteq AaA$. Thus A is a simple ring with unit element.

Now we formulate the following characterizations of s-weakly regular rings.

THEOREM 5. *The following are equivalent for a ring with unit element.*

(i) *A is s-weakly regular.*

(ii) *A is a right (or left) weakly regular ring and is a subdirect sum of simple reduced rings.*

(iii) *A is reduced and right (or left) weakly regular.*

(iv) *A is reduced and A/P is right (or left) weakly regular for every proper prime ideal P of A.*

(v) *A is reduced and A/P is s-weakly regular for every proper prime ideal P of A.*

(vi) *A is reduced and every proper prime ideal is maximal.*

(vii) *A is reduced and every proper completely prime ideal is maximal.*

PROOF. (i)$\Rightarrow$(ii). It is clear that A is right (or left) weakly regular and hence semisimple in the sense of Jacobson (Ramamurthy [7]). Now A is a subdirect sum of primitive rings A_α each of which is a homomorphic image of A and hence s-weakly regular. Now by Lemma 4 each of A_α is a simple reduced ring.

(ii)$\Rightarrow$(iii)$\Rightarrow$(iv). These are obvious.

(iv)$\Rightarrow$(vii). Let P be a proper completely prime ideal of A. Since A/P is right (or left) weakly regular, it can be verified that A/P is a simple ring. Thus P is a maximal ideal.

(vii)$\Rightarrow$(i). Let $0\neq a\in A$. Then $\bar{A}=A/(a)^r$ is reduced and $\bar{a}$ is a non zero divisor of $\bar{A}$. Every proper completely prime ideal of $\bar{A}$ is a maximal ideal of $\bar{A}$. Let M be the multiplicative semigroup generated by all elements $\bar{a}-\bar{a}\bar{x}$ where $x\in\langle a^2\rangle$. We claim that $0\in M$. Suppose $0\notin M$ then there exists a completely prime ideal $\bar{P}$ with $\bar{P}\cap M=\varnothing$ (Andrunakievic and Rjabuhin [1]). Let $\bar{a}\in\bar{A}$. Then $\langle\bar{a}^2\rangle\subseteq\bar{P}$ or there exists $\alpha\in\langle\bar{a}^2\rangle$ such that $\alpha\notin\bar{P}$. If $\langle\bar{a}^2\rangle\subseteq P$ then $\bar{a}^2\in\bar{P}$. Since $\bar{P}$ is completely prime, we have $\bar{a}\in\bar{P}$. Now $\bar{a}-\bar{a}\bar{x}\in\bar{P}\cap M=\varnothing$ for $\bar{x}\in\langle\bar{a}^2\rangle$ which gives a contradiction. If there exists $\alpha\in\langle\bar{a}^2\rangle$ such that $\alpha\notin\bar{P}$ then we have $\langle\alpha+\bar{P}\rangle=\bar{A}/\bar{P}$, since $\bar{A}/\bar{P}$ is simple. In particular $\bar{1}-\sum\bar{u}_i\alpha\bar{u}'_j\in\bar{P}$. Thus $\bar{a}-\bar{a}\Sigma\bar{u}_i\alpha\bar{u}'_j\in\bar{P}\cap\cap M=\varnothing$ which is a contradiction. Hence $0\in M$. Now

$$0=(\bar{a}-\bar{a}\bar{x}_1)(\bar{a}-\bar{a}\bar{x}_2)\dots(\bar{a}-\bar{a}\bar{x}_n)$$

where $x_i\in\langle a^2\rangle$. Since $\bar{A}$ is reduced and $\bar{a}$ is a non zero divisor, by using Lemma 3 it can be verified that $\bar{1}=\bar{x}$ for some $x\in\langle a^2\rangle$. Thus $1-x\in(a)^r$. Now we have $a=ax$.

(i)$\Rightarrow$(v)$\Rightarrow$(vi)$\Rightarrow$(vii). These are obvious.

Let A' be a ring and A a subring of A' containing the identity of A'. A' is called an integral extension of A if for every $x\in A'$, there exists a positive integer n and elements $a_{n-1},\dots,a_0$ in A such that $x^n+a_{n-1}x^{n-1}+\dots+a_0=0$.

LEMMA 6. *Let A' be an integral extension of A and let A' be an integral domain. If A is simple then A' is a simple ring.*

PROOF. It is sufficient to prove that for every non zero $x\in A'$, we have $A'xA'==A'$. Let $0\neq x\in A'$ then x is integral over A, thus $x^n+a_{n-1}x^{n-1}+\ldots+a_0=0$. Since A' is an integral domain, we can assume $a_0\neq 0$. Now consider the ideal generated by a_0 in A. Since A is a simple ring we have $Aa_0A=A$. Now it can be shown easily that $A'xA'=A'$. Hence A' is a simple ring.

Analogously to the corollary of Wong [9] we have the following.

THEOREM 7. *A reduced integral extension A' of a right (or left) weakly regular ring A is an s-weakly regular ring.*

PROOF. Let P' be a proper completely prime ideal of A'. Then $P=P'\cap A$ is a completely prime ideal of A. A/P is a simple ring. A'/P' is an integral domain and an integral extension of A/P. Now by Lemma 6 A'/P' is a simple ring. Thus P' is a maximal ideal of A'. By Theorem 5, A' is s-weakly regular.

An element $a\in A$ is called s-weakly regular if there exists $x\in\langle a^2\rangle$ such that $a=ax$. A two sided ideal I is called s-weakly regular ideal if each of its elements is s-weakly regular.

Let $S(A)=\{a\in A|\langle a\rangle$ is an s-weakly regular$\}$. Now we give the following lemma which will be used in proving our next theorem.

LEMMA 8. *Let I be an ideal of A. I is s-weakly regular ideal of A if and only if $a\in aAa^2A$ for any $a\in I$.*

PROOF. Let $a\in I$ and $a=ax$ where $x\in\langle a^2\rangle$. From this $x\in\langle axa\rangle\subseteqq Ia^2I$. Thus I is an s-weakly regular ideal. The converse is trivial.

LEMMA 9. *If $x\in aAa^2A$ and $a-x\in(a-x)A(a-x)^2A$ then $a\in aAa^2A$.*

PROOF. Since $x\in aAa^2A$, we have $(a-x)A\subseteqq aA$. Moreover

$$(a-x)\in(a-x)A(a-x)^2A=(a-x)A(a^2+x^2-ax-xa)A\subseteqq aAa^2A.$$

Since $x\in aAa^2A$, we have $a\in aAa^2A$.

THEOREM 10. (i) *$S(A)$ is the unique maximal s-weakly regular ideal of A.*

(ii) $S(A/S(A))=0$.

(iii) *If I is an ideal of A then $S(I)=S(A)\cap I$.*

(iv) *$S(A_n)=0$ where A_n denotes the full matrix ring of order n over A.*

(v) *If $A/U(A)$ is an s-weakly regular ring then $S(A)=0$ if and only if $U(A)^T\subseteqq\subseteqq U(A)$ where $U(A)$ denotes the upper nil radical of A.*

PROOF. (i) Let $\alpha\in A$ and $u\in S(A)$, then clearly αu and $u\alpha\in S(A)$. Now suppose that u_1 and $u_2\in S(A)$ then we show that $u_1-u_2\in S(A)$. Let $a\in\langle u_1-u_2\rangle$ then $a=z_1-z_2$ where $z_i\in\langle u_i\rangle$ for $i=1,2$. $z_1=z_1\Sigma r_iz_1^2r_i'$ for some $r_i, r_i'\in A$. Now consider

$$a-a\sum r_ia^2r_i'=(z_1-z_2)-(z_1-z_2)\sum r_i(z_1^2+z_2^2-z_1z_2-z_2z_1)r_i'=$$

$$=-z_2-z_1\sum r_i(z_2^2-z_1z_2-z_2z_1)r_i'+z_2\sum r_i(z_1^2+z_2^2-z_1z_2-z_2z_1)r_i'\in\langle z_2\rangle.$$

Now by Lemma 9 $a\in aAa^2A$. Thus $S(A)$ is an ideal. Now the result is immediate.

(ii) Let $\bar{a}$ denote the residue class modulo $S(A)$ which contains the element a of A. If $\bar{b} \in S(A/S(A))$ and $a \in \langle b \rangle$ then $\bar{a} \in \langle \bar{b} \rangle$. Thus $\bar{a} = \bar{a}\bar{x}$ where $x \in \langle a^2 \rangle$. Thus $a - ax \in S(A)$. By Lemma 9 $a \in aAa^2A$. Hence $b \in S(A)$ and $\bar{b} = 0$.

(iii) By (i) and Lemma 8, $S(A) \cap I \subseteqq S(I)$. Conversely, if $a \in S(I)$ then $ax \in aIa^2Ix \subseteqq IaI \subseteqq S(I)$. Similarly $xa \in S(I)$. Thus $S(I)$ is an ideal of A. We have $S(I) \subseteqq S(A) \cap I$ by Lemma 8.

(iv) By using (iii), it suffices to prove the result for a ring with unit element. Let $0 \neq X \in S(A_n)$ and let $0 \neq a$ be the (i, j)th entry in X. Then

$$Y = E_{1i} X E_{j2} = \begin{bmatrix} 0 & a & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in S(A_n).$$

This implies that $Y = Y \Sigma Z_i Y^2 Z_i' = 0$ which is a contradiction. Hence $S(A_n) = 0$.

(v) It is easy to verify that $U(A) \cap S(A) = 0$. Since $U(A) \cdot S(A)$ and $S(A) \cdot U(A) \subseteqq U(A) \cap S(A) = 0$ we have $S(A) \subseteqq U(A)^T \subseteqq U(A)$. Hence $S(A) = 0$.

Conversely, let $S(A) = 0$. First we will show that $U(A) \cap (U(A)^T)^2 = 0$. Let $x \in U(A) \cap (U(A)^T)^2$ then $x \in U(A)$ and $x = \sum_{i=1}^{n} a_i b_i$ where $a_i, b_i \in U(A)^T$. Since $A/U(A)$ is s-weakly regular, we have $a_i - \sum_{j=1}^{m} x_j a_i^2 x_j' a_i = u_i$ for some $u_i \in U(A)$. Now

$$x = \sum_{i=1}^{n} \left(u_i + \sum_{j=1}^{m} x_j a_i^2 x_j' a_i \right) b_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} x_j a_i^2 x_j' a_i \right) b_i.$$

We use induction on n.

If $n = 1$ then

$$x = \sum_{j=1}^{m} x_j a_1^2 x_j' a_1 b_1 = \sum_{j=1}^{m} x_j a_1^2 x_j' x = 0.$$

If $n > 1$ then we have

$$a_n b_n = x - \sum_{i=1}^{n-1} a_i b_i.$$

Thus

$$x = \sum_{i=1}^{n-1} \left(\sum_{j=1}^{m} x_j a_i^2 x_j' a_i \right) b_i + \sum_{j=1}^{m} x_j a_n^2 x_j' \left(x - \sum_{i=1}^{n-1} a_i b_i \right) =$$

$$= \sum_{i=1}^{n-1} \left(\sum_{j=1}^{m} x_j a_i^2 x_j' - \sum_{j=1}^{m} x_j a_n^2 x_j' \right) a_i b_i.$$

$U(A)^T$ is a two sided ideal. Since

$$\left(\sum_{j=1}^{m} x_j a_i^2 x_j' - \sum_{j=1}^{m} x_j a_n x_j'\right) a_i, \quad b_i \in U(A)^T,$$

we have $x=0$ by induction hypothesis. Now let $a \in (U(A)^T)^2$ then $\langle a \rangle \subseteqq (U(A)^T)^2$. Moreover, if $\alpha \in \langle a \rangle$ then $\alpha - \sum x_j \alpha^2 x_j' \alpha \in U(A) \cap (U(A)^T)^2 = 0$. Now $a \in S(A) = 0$. Thus $U(A)^T \subseteqq U(A)$.

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P. O. BOX 13261
DEPARTMENT OF MATHEMATICS
AL-FATEH UNIVERSITY
TRIPOLI
LIBYA

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THE SPECTRUM OF A CLASS OF SINGULAR INTEGRAL OPERATORS

N. S. FAOUR (Kuwait)

Introduction

Let E be a bounded measurable subset of the real line R, and let $L_n^2(R)$ be the usual Lebesgue space of C^n valued square integrable functions on R $(n>1)$. The space $L_{M_n}^\infty(R)$ is the set of $n\times n$ matrices (Φ_{ij}) $(1\leqq i,j\leqq n)$, where each of the functions $\Phi_{ij}\in L^\infty(R)$.

The operators of interest are the singular integral operators S defined on $L_n^2(E)$ by

$$Sf(s)=sf(s)+\frac{B^*(s)}{\pi}\int_E \frac{B(t)f(t)}{s-t}\,dt,$$

where $B\in L_{M_n}^\infty(E)$, and B^* is the adjoint of B.

The singular integral operator S is hyponormal, that is the selfcommutator $[S^*, S]=S^*S-SS^*$ is a non-negative operator. Moreover, $[S^*, S]$ is n-dimensional.

In this paper it is proved that the spectrum of S is the set of all complex numbers $z=x+iy$ such that x is in the essential closure of E and $|y|\leqq =\operatorname{ess}\lim_{t=x}\sup\|B(t)\|^2$. It should be remarked that a complete description of the spectrum of S for the case $n=1$ was given by Clancey and Putnam [1].

The spectrum

In this section the spectrum of the singular integral operator S defined on $L_n^2(E)$ by

$$Sf(s)=sf(s)+\frac{B^*(s)}{\pi}\int_E \frac{B(t)f(t)}{s-t}\,dt \tag{1}$$

is studied. To do that some definition and lemmas are needed.

If g is a non-negative essentially bounded measurable function defined on a subset of F of the real line, then g will automatically be extended to be zero off F, and $g^\#(x)$ will be defined by

$$g^\#(x)=\operatorname{ess}\lim_{t=x}\sup g(t)=\lim_{|\Delta|\to 0}\operatorname{ess}\sup_{\Delta\cap F} g(t)$$

where the Lebesgue measure of $\Delta\cap F$ is greater than zero for every neighborhood Δ of x.

If F is a measurable subset of the real line R, then the set of all real numbers x such that every neighborhood of x intersects F in a set of positive measure is the essential closure of F and will be denoted by F^e. The set $C(g)$ is $C(g)=$

$=\{x+iy: x\subset F^e, |y|\leqq g^{\#}(x)\}$. It is clear that $C(g)$ is a closed subspace of the complex plane.

Let $B\in L^{\infty}_{M_n}(E)$. In this paper the norm of B is defined by $|||B|||=\operatorname{ess\,sup}_{t\in E}\|B(t)\|$, where $\|B(t)\|$ denotes the norm of the matrix $B(t)$ as an operator on C^n with the Euclidean norm, also, the spectrum of the operator S is denoted by $\sigma(S)$.

The main theorem of the paper is the following

THEOREM 1. *Let S be the singular integral operator defined on $L^2_n(E)$ by* (1)*. Then the spectrum of S is equal to*

$$\{x+iy:\ x\in E^e,\ |y|\leqq \operatorname{ess}\limsup_{t=x}\|B(t)\|^2\}.$$

The proof of Theorem 1.1 requires some preliminary lemmas.

LEMMA 1. *Let T be the singular integral operator defined on $L^2(E)$ by*

$$Tf(s)=sf(s)+\frac{b^*(s)}{\pi}\int_E \frac{b(t)f(t)}{s-t}\,dt,$$

where $b\in L^{\infty}(E)$. Then the spectrum of T is equal to $C(|b|^2)$, where b^* is the complex conjugate of b.

PROOF. See Clancey and Putnam [1].

LEMMA 2. *Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of non-negative essentially bounded measurable functions defined on a measurable set F contained in the real line R. Suppose $g_1\leqq g_2\leqq g_3\leqq\ldots\leqq g$, where g is a non-negative essentially bounded function on F, and $g_n\to g$ uniformly on F. Then $\bigcap_{n=1}^{\infty}\left(\bigcup_{m\geqq n}^{\infty}C(g_m)\right)^-=C(g)$, where $\left(\bigcup_{m\geqq n}^{\infty}C(g_m)\right)^-$ denotes the closure of $\left(\bigcup_{m\geqq n}^{\infty}C(g_m)\right)$.*

PROOF. Since $g_1\leqq g_2\leqq g_3\leqq\ldots\leqq g$, then it follows that $C(g_1)\subseteqq C(g_2)\subseteqq$ $\subseteqq C(g_3)\subseteqq\ldots C(g)$. Since $C(g)$ is closed, it follows that $\bigcap_{n=1}^{\infty}\left(\bigcup_{m\geqq n}^{\infty}C(g_m)\right)^-\subseteqq C(g)$. To prove the other inclusion, let $z=x+iy\in C(g)$. Choose a sequence $\{z_n\}_{n=1}^{\infty}$, where $z_n=x_n+iy_n$, and $|y_n|=g_n^{\#}(x)=\lim_{|\Delta|\to 0}\operatorname{ess\,sup}_{\Delta\cap F} g_n(t)$. Note that $\{y_n\}_{n=1}^{\infty}$ is bounded, and hence $\lim_{n\to\infty} y_n$ exists. If $|y|<|y_n|$ for some n, then the result follows. Suppose $|y|\geqq|y_n|$ for any n. Since $g_n\to g$ uniformly on F, then it follows that $(g-g_n)(t)<\varepsilon$ for all $n\geqq N$. From this it follows that $g^{\#}(x)\leqq g_n^{\#}(x)+\varepsilon$ for all $n\geqq N$. Therefore $|y|\leqq g_n^{\#}(x)+\varepsilon$ for all $n\geqq N$. Hence, $|y|-|y_n|\leqq g_n^{\#}(x)-|y_n|+\varepsilon$. From this it follows that $|y|-|y_n|<\varepsilon$ for all $n\geqq N$. Hence, $|y|=\lim_{n\to\infty}|y_n|=|\lim_{n\to\infty}y_n|$.

From this it follows that $z\in\bigcap_{n=1}^{\infty}\left(\bigcup_{m\geqq n}^{\infty}C(g_m)\right)^-$, and that ends the proof of the lemma.

Before we state the next lemma, the following definition is needed. A sequence of operators $\{T_n\}_{n=1}^{\infty}$ is said to converge to T in the strong star sense in case $T_n\to T$ and $T_n^*\to T^*$ strongly. We will write $T_n\to T(s-*)$ for this convergence.

The following lemma is a result which appears in Howe [2], p. 643.

LEMMA 3. *Let* $\{T_n\}_{n=1}^{\infty}$ *be a sequence of hyponormal operators on a Hilbert space* H *such that* $T_n \to T(s-*)$, *where* T *is hyponormal, then*

$$\bigcap_{n=1}^{\infty}\Big(\bigcap_{m\geqq n}^{\infty}\sigma(T_m)\Big)^{-}=\sigma(T).$$

PROOF OF THEOREM 1. It should be remarked that B can be approximated by a sequence $\{\Phi_n\}_{n=1}^{\infty}$ of simple functions such that $\Phi_n\to B$ uniformly and $|||\Phi_1|||\leqq\leqq|||\Phi_2|||\leqq\ldots\leqq|||B|||$. In view of Lemmas 2, 3 it suffices to prove Theorem 1 for the operator S defined on $L_n^2(E)$ by

$$Sf(s)=sf(s)+\frac{\Phi^*(s)}{\pi}\int_E\frac{\Phi(t)f(t)}{s-t}\,dt$$

where Φ is a simple function. Since Φ is simple, then it can be written as $\Phi==\sum_{j=1}^{m}\Phi_j\chi_{F_j}$, where Φ_j $(1\leqq j\leqq m)$ is a constant $n\times n$ matrix, and the sets F_j $(1\leqq j\leqq m)$ are disjoint Borel sets whose union is E. It is easily seen that $\sigma(S)==\bigcup_{i=1}^{m}\sigma(S_j)$, where the singular integral operator S_j is defined by

$$S_jf(s)=sf(s)+\frac{\Phi_j^*}{\pi}\int_{F_j}\frac{\Phi_jf(t)}{s-t}\,dt.$$

The operator S_j can be written as

$$S_jf(s)=sf(s)+(\Phi_j^*\Phi_j)^{\frac12}\int_{F_j}\frac{(\Phi_j^*\Phi_j)^{\frac12}f(t)}{s-t}\,dt.$$

The matrix $(\Phi_j^*\Phi_j)^{\frac12}$ is unitarily equivalent to the $n\times n$ diagonal matrix with diagonal entries $\varphi_1^{(j)},\ldots,\varphi_n^{(j)}$, $1\leqq j\leqq m$. It follows that S_j is unitarily equivalent to the orthogonal direct sum of the n-operators defined on $L^2(F_j)$ by

$$S_{ij}f(s)=sf(s)+\frac{\varphi_i^{(j)}}{\pi}\int_{F_j}\frac{\varphi_i^{(j)}f(t)}{s-t}\,dt,\quad 1\leqq i\leqq n;\quad 1\leqq j\leqq m.$$

From Lemma 1, it follows that $\sigma(S_{ij})=\{x+iy\colon x\in F_j^e,\ |y|\leqq|\varphi_i^{(j)}|^2\}$. From this it follows that $\sigma(S_j)=\{x+iy\colon x\in F_j^e,\ |y|\leqq|||\Phi_j|||^2\}$. Since $\sigma(S)=\bigcup_{j=1}^{m}\sigma(S_j)$, it follows that $\sigma(S)=\{x+iy\colon x\in E,\ |y|\leqq|||\Phi|||^2\}$, and that ends the proof.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KUWAIT
KUWAIT

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$\wedge$-RIGHT CONGRUENCES AND A CLASSIFICATION OF ORTHOGROUPS

F. CATINO (Siena)

Introduction

The importance of Green's relations $\mathcal{H}, \mathcal{L}, \mathcal{R}, \mathcal{D}, \mathcal{T}$ in the theory of semigroups is well-known. In fact, investigations carried out by means of Green's relations have often led to the characterization of the structure of several types of semigroups and have allowed to give meaningful and useful classifications.

In the present work we characterize, by means of Green's relations, the orthogroups with band of idempotents of type $\mathcal{P}$, where $\mathcal{P}$ is anyone of the types of bands classified by Petrich in [3].

To this aim it is useful to introduce the concept of $\wedge$-right [$\wedge$-left] congruence as well as the band S/ϱ where S is an orthogroup and ϱ is a $\wedge$-right [$\wedge$-left] congruence.

We omit the duals of all the theorems, namely all the theorems obtained by interchanging $\mathcal{R}$ with $\mathcal{L}$, "right" with "left" and by changing, in a suitable way, the equalities in the last point of each theorem.

For the terminology and material used here, the reader is referred to [1] and [2].

1. If a is a completely regular (c.r.) element of a semigroup S, we denote the unit element of H_a by $\hat{a}$ and the inverse of a in H_a by a^{-1}.

Definition 1. Let S be a c.r. semigroup and let ϱ be a relation of equivalence on S; ϱ is said to be a *$\wedge$-relation* if $a\,\varrho\,\hat{a}$ for every $a \in S$.

Green's relations on a c.r. semigroup S provide examples of $\wedge$-relations.

Definition 2. Let S be a c.r. semigroup and let ϱ be a $\wedge$-relation on S; ϱ is said to be a *$\wedge$-right* [respectively *$\wedge$-left*] congruence if it is a left [resp. right] congruence and

$$a\varrho b \Rightarrow \hat{a}c\varrho\hat{b}c \quad (a, b\in S) \quad [a\varrho b \Rightarrow c\hat{a}\varrho c\hat{b} \quad (a, b\in S)]$$

for every $c\in S$.

Notice that a $\wedge$-relation ϱ is a congruence iff ϱ is both a $\wedge$-right and a $\wedge$-left congruence.

Let S be a c.r. semigroup; if $\{\varrho_i\}_{i\in I}$ is a family of $\wedge$-right congruences of S that contain a relation $\mathcal{K}$, then

$$\varrho = \bigcap_{i\in I} \varrho_i$$

is a $\wedge$-right congruence and contains $\mathcal{K}$. Let us denote by $\mathcal{K}'$ the intersection of all the $\wedge$-right congruences σ of S such that $\mathcal{K} \subseteq \sigma$.

THEOREM 1. *Let S be a completely regular semigroup. Then $\mathcal{H}'$ is a congruence iff any $\wedge$-right congruence is a congruence.*

PROOF. Let $\mathcal{H}'$ be a congruence, it is clear that $\mathcal{H}'=\mathcal{H}^{\#}$, where $\mathcal{H}^{\#}$ is the congruence on S generated by $\mathcal{H}$. Then if ϱ is a $\wedge$-right congruence $\hat{a}ac\varrho\hat{a}\hat{a}c$ for every $a, c\in S$, it follows that ϱ is a congruence.

Let S be a c.r. semigroup and let ϱ be a $\wedge$-right congruence; then $a\varrho$ denotes the ϱ-class of $a\,(a\in S)$ and S/ϱ denotes the set of equivalence classes. On these the product $a\varrho\cdot b\varrho=(\hat{a}b)\varrho$ is defined for every pair $a\varrho, b\varrho$ of ϱ-classes of S.

If a $\wedge$-relation ϱ is a congruence, then S/ϱ is the quotient semigroup.

We recall that an orthogroup is an orthodox c.r. semigroup.

THEOREM 2. *If S is an orthogroup and ϱ is a $\wedge$-right congruence then S/ϱ is a band.*

We recall that a band E is said *left* [respectively *right*] *regular* iff $ax=axa$ [resp. $xa=axa$] for every $a, x\in E$.

THEOREM 3. *In every orthogroup S, $\mathcal{R}'$ is the smallest $\wedge$-right congruence ϱ for which S/ϱ is a left regular band.*

PROOF. It follows from Theorem 2 that $S/\mathcal{R}'$ is a band; moreover $\hat{a}\hat{x}S=\hat{a}\hat{x}\hat{a}S$, for every $a, x\in S$; hence $\hat{a}\hat{x}\mathcal{R}'\hat{a}\hat{x}\hat{a}$. Therefore $S/\mathcal{R}'$ is a left regular band.

Let ϱ be a $\wedge$-right congruence such that S/ϱ is a left regular band and let $a, b\in S$ be such $a\mathcal{R}b$. Then $a\varrho\mathcal{R}b\varrho$ in S/ϱ; hence, since S/ϱ is a left regular band, $a\varrho=b\varrho$, so that $\mathcal{R}\subseteq\varrho$. Therefore $\mathcal{R}'\subseteq\varrho$.

2. If B is a subsemigroup of a semigroup S, we shall denote by $\mathcal{H}^B$, $\mathcal{L}^B$, $\mathcal{R}^B$, $\mathcal{D}^B$, $\mathcal{T}^B$, respectively, Green's relations $\mathcal{H}, \mathcal{L}, \mathcal{R}, \mathcal{D}, \mathcal{T}$ on the semigroup B.

Recall now that a band E is said to be a *right* [resp. *left*] *semiregular* band iff $yxa=yxyayxa$ [resp. $axy=axyayxy$] for every $a, x, y\in E$. A band E is said to be a *regular* band iff $axya=axaya$ for every $a, x, y\in E$.

THEOREM 4. *In an orthogroup S with set of idempotents E the following statements are equivalent:*

(i) *E is a right semiregular band;*

(ii) *$\mathcal{R}$ is a $\wedge$-right congruence;*

(iii) *$axyS=axayS$ for every $a, x\in E$, $y\in S$.*

PROOF. (i)$\Rightarrow$(ii). Let $a, b\in S$ be such that $a\mathcal{R}b$; then from Theorem 3 of [3] $\hat{a}\hat{c}\mathcal{R}^E\hat{b}\hat{c}$ for every $c\in S$. Since $ES=S$ and $s\hat{c}S=scS$ for every $s\in S$, we have $\hat{a}cS=\hat{b}cS$.

(ii)$\Rightarrow$(iii). Let $a, x\in E$, then $ax\mathcal{R}axa$ with ax and axa idempotents; therefore, from the assumption, $axy\mathcal{R}axay$ for every $y\in S$.

(iii)$\Rightarrow$(i). Let $a, x, y\in E$; by assumption, $yxa\mathcal{R}yxya$, hence, $yxa=yxyayxa$ (on the other hand, (iii)$\Rightarrow$(i) for Theorem 7 of [4] and Theorem 3 of [3]).

THEOREM 5. *In an orthogroup S with set of idempotents E the following statements are equivalent:*

(i) *E is a regular band;*

(ii) *$\mathcal{R}$ is a $\wedge$-right congruence and $\mathcal{L}$ is a $\wedge$-left congruence;*

(iii) *$axySxya=axaySxaya$ for every $a, x, y\in E$.*

PROOF. The implications (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) follow immediately from Theorem 4 and its dual. The implication (iii)$\Rightarrow$(i) follows immediately from Theorem 4 of [3] and Corollary 2 of [4].

Recall that a band E is said to be *right* [resp. *left*] *seminormal* iff $yxa=yayxa$ [resp. $axy=axyay$] for every $a, x, y\in E$; a band E is said to be *left* [resp. *right*] *normal* iff $axy=ayx$ [resp. $yxa=xya$] for every $a, x, y\in E$.

THEOREM 6. *In an orthogroup S with set of idempotents E the following statements are equivalent:*

(i) *E is a right seminormal band;*

(ii) *$\mathcal{R}$ is a $\wedge$-right congruence and $S/\mathcal{R}$ is a left normal band;*

(iii) *$\mathcal{R}$ is a congruence and $S/\mathcal{R}$ is a left normal band;*

(iv) *$axyS=ayxS$ for every $a, x, y\in S$.*

PROOF. It follows from Theorem 8 of [4] and Theorem 5 of [3] that the statements (i), (iii), (iv) are equivalent. The implication (iii)$\Rightarrow$(ii) is trivial.

(ii)$\Rightarrow$(i). Let $a, x, y\in E$; as $S/\mathcal{R}$ is a left normal band $yxa\mathcal{R}yax$. Therefore $z\in S$ exists such that $yxa=yaxz$. Thus $yayxa=ya(yaxz)=yaxz=yxa$.

Recall that a band E is said to be *right* [resp. *left*] *quasinormal* iff $yxa=yaxa$ [resp. $axy=axay$] for every $a, x, y\in E$; a band E is said to be *normal* iff $axya=ayxa$ for every $a, x, y\in E$.

THEOREM 7. *In an orthogroup S with set of idempotents E the following statements are equivalent:*

(i) *E is a right quasinormal band;*

(ii) *$\mathcal{L}$ is a $\wedge$-left congruence, $\mathcal{R}$ is a $\wedge$-right congruence and $S/\mathcal{R}$ is a left normal band;*

(iii) *$\mathcal{L}$ is a $\wedge$-left congruence, $\mathcal{R}$ is a congruence and $S/\mathcal{R}$ is a left normal band;*

(iv) *$axySxya=ayxSxaya$ for every $a, y\in E$, $x\in S$.*

PROOF. The implications (i)$\Rightarrow$(ii), (ii)$\Rightarrow$(iii), (iii)$\Rightarrow$(iv) follow immediately from Theorem 6 and from the dual of Theorem 4. The implication (iv)$\Rightarrow$(i) follows from Corollary 2 of [4].

THEOREM 8. *In an orthogroup S with set of idempotents E the following statements are equivalent:*

(i) *E is a normal band;*

(ii) *$\mathcal{R}$ is a $\wedge$-right congruence, $S/\mathcal{R}$ is a left normal band, $\mathcal{L}$ is a $\wedge$-left congruence and $S/\mathcal{L}$ is a right normal band;*

(iii) *$\mathcal{R}, \mathcal{L}$ are congruences, $S/\mathcal{R}$ is a left normal band and $S/\mathcal{L}$ is a right normal band;*

(iv) *$axySxya=ayxSyxa$ for every $a, x, y\in S$.*

PROOF. The implications (i)$\Rightarrow$(ii), (ii)$\Rightarrow$(iii), (iii)$\Rightarrow$(iv) follow immediately from Theorem 6 and its dual.

The implication (iv)$\Rightarrow$(i) follows from Corollary 3 of [4].

THEOREM 9. *In an orthogroup S with set of idempotents E the following statements are equivalent:*

(i) *E is a right regular band;*

(ii) $\mathcal{R}=\mathcal{D}$;
(iii) $xaS=axaS$ *for every* $a, x\in S$.

PROOF. It follows from Theorem 8 of [3] and Theorem 9 of [4] that the statements (i) and (ii) are equivalent.

(ii)⇒(iii). For every $a, x\in S$, $SxaS=Sxa\hat{a}S=S\hat{a}xaS=Sa^{-1}axaS\subseteq SaxaS$, hence $SxaS=SaxaS$ and, from the assumption, $xaS=axaS$.

(iii)⇒(i). It follows by assumption that, for every $a, x\in E$, $xa=axaxa=axa$.

THEOREM 10. *In an orthogroup S with set of idempotents E the following statements are equivalent:*

(i) *E is a right normal band;*
(ii) *$\mathcal{R}=\mathcal{D}$, $\mathcal{L}$ is a $\wedge$-left congruence and $S/\mathcal{L}$ is a right normal band;*
(iii) *$\mathcal{R}=\mathcal{D}$, $\mathcal{L}$ is a congruence and $S/\mathcal{L}$ is a right normal band;*
(iv) *$xaSxya=axaSyxa$ for every $a, x, y\in S$.*

PROOF. The implications (i)⇒(ii), (ii)⇒(iii), (iii)⇒(iv) follow immediately from Theorem 9 and from the dual of Theorem 6.

(iv)⇒(i). For every $a, x\in E$, $xa=(xa)^2=(xa)a(xaa)\in xaSxaa=axaSaxa\subseteq axaS$; analogously $axa\in xaS$, hence

$$xaS = axaS. \tag{1}$$

For every $a, x, y\in E$, because of (1)

$$xyaS = x(ya)S = (ya)x(ya)S = y(a(xy)aS) =$$
$$= y(xy)aS = yx(ya)S = yxayaS \subseteq yxaS;$$

analogously $yxaS\subseteq xyaS$, hence

$$xyaS = yxaS. \tag{2}$$

Besides,

$$xya = (xya)^2 = (xya)a(xya)\in xyaSxya = yxaSxya \text{ (from (2))} \subseteq SxaSxya =$$
$$= SaxaSyxa \subseteq Syxa;$$

$$yxa = (yxa)^2 = (yxa)a(yxa)\in yaxaSyxa \text{ (from (1))} \subseteq SaxaSyxa =$$
$$= SaxaSyxa \subseteq Sxya$$

i.e. $Syxa=Sxya$. Then the idempotents xya and yxa belong to the same $\mathcal{H}$-class; therefore $xya=yxa$.

THEOREM 11. *In an orthogroup S with set of idempotents E the following statements are equivalent:*

(i) *E is a semilattice;*
(ii) *$\mathcal{R}=\mathcal{L}=\mathcal{D}$;*
(iii) *$aS=Sa$ for every $a\in S$.*

PROOF. It follows from Theorem 10 of [3] and Corollary 3 of [4] that the statements (i), (ii), (iii) are equivalent.

I wish to thank Professor F. Migliorini with whom I had several discussions while the paper was being prepared.

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ISTITUTO MATEMATICO UNIVERSITÀ
VIA DEL CAPITANO, 15
53100 SIENA
ITALIA

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ON THE SUMMABILITY OF EIGENFUNCTION EXPANSIONS. I

I. JOÓ (Budapest)

Dedicated to Professor A. Kósa on his 50th birthday

The aim of the present paper is to give a necessary and sufficient condition of the Lebesgue summability of eigenfunction expansions associated with the Schrödinger operator in any bounded three dimensional domain. The results obtained generalize those of the paper [2] for more general potential. The conditions for the potential in the present paper are close to the necessary ones.

Let Ω be any bounded domain in $\mathbf{R}^3$, $x_0 \in \Omega$, q be a function of the form

$$q(x) = \frac{a(|x-x_0|)}{|x-x_0|} + q_1(x) \quad (= q_0(x) + q_1(x)),$$

where a is a non-negative function with the property

$$(*) \qquad \int_{+0} a(t)\,dt < \infty,$$

and $q_1 \in L_\infty(\Omega)$ is a non-negative function. Consider the Schrödinger operator

$$L = -\Delta + q(x)\cdot$$

with domain $C_0^\infty(\Omega)$. Denote by $\hat{L}$ an arbitrary non-negative selfadjoint extension of L with discrete spectrum. According to a well-known theorem of K. O. Friedrichs (cf. [7]) there exists such an extension, if $q \in L_2(\Omega)$. Denote by $0 \leqq \lambda_1 \leqq \lambda_2 \leqq \dots$ the sequence of eigenvalues and let $\{u_n\}_1^\infty$ be the complete orthonormal system in $L_2(\Omega)$ of the corresponding eigenfunctions of the operator $\hat{L}$. The expansion of $f \in L_2(\Omega)$ with respect to the system $\{u_n\}$ is said to be Lebesgue summable at $x \in \Omega$ if the limit

$$\lim_{h \to +0} \sum_{n=1}^{\infty} \frac{\sin \sqrt{\lambda_n}\, h}{\sqrt{\lambda_n}\, h} (f, u_n)\, u_n(x)$$

exists. We shall prove the following

THEOREM. *Let $f \in \mathring{W}_2^1(\Omega)$ be arbitrary and suppose $a(t) = O(1/\sqrt{t})$. Then the expansion of f with respect to the system $\{u_n\}$ is Lebesgue summable in x_0 if and only if the limit $\lim_{r \to +0} \int_\theta f(x_0 + r\theta)\,d\theta$ exists. Here $\int_\theta f(x_0 + r\theta)\,d\theta$ denotes the integral of f over the sphere of radius r with centre in x_0, with respect to the normalized Lebesgue measure.*

For the proof of the theorem we need some lemmas, which are given in Sections 1 and 2 below.

1. The mean value formula and its applications

Set

$$h(t)=\min(1, 1/t) \quad (t>0), \quad b(r)=\int_0^r a(t)\,dt \quad (r>0).$$

LEMMA 1.1. *We have*

$$\frac{1}{r\mu}\int_0^r h(t\mu)|\sin\mu(t-r)|a(t)\,dt \leqq c_1 b(r)h(r\mu) \quad (r\geqq 0,\ \mu\geqq 0). \tag{1.1}$$

PROOF. If $r\mu\leqq 1$, then — using the notation I for the left hand side of (1.1) — we obtain

$$I\leqq \int_0^r a(t)\,dt = 1\cdot b(r)h(r\mu).$$

If $r\mu>1$, then $I=\int_0^r=\int_0^{1/\mu}+\int_{1/\mu}^r=I_1+I_2$ and

$$I_1\leqq \int_0^{1/\mu} a(t)dt\leqq 1\cdot b(r)h(r\mu).$$

Further

$$I_2\leqq \frac{1}{r\mu}\int_{1/\mu}^r \frac{1}{\mu t}a(t)\,dt\leqq h(r\mu)\int_{1/\mu}^r a(t)\,dt\leqq 1\cdot b(r)h(r\mu).$$

Lemma 1.1 is proved.

Set

$$v_0(r,\mu)=\frac{\sin r\mu}{r\mu}, \quad v_k(r,\mu)=\frac{1}{r\mu}\int_0^r v_{k-1}(t,\mu)\sin\mu(t-r)a(t)\,dt,$$

$$w_0(r,\mu)=\frac{1}{r\mu}\int_0^r \big(q_1(x_0+t\theta)u(x_0+t\theta,\mu)d\theta\big)t\cdot\sin\mu(t-r)\,dt,$$

$$w_k(r,\mu)=\frac{1}{r\mu}\int_0^r w_{k-1}(t,\mu)\sin\mu(t-r)a(t)\,dt,$$

where $u(x)=u(x,\mu)$ is an arbitrary eigenfunction of the operator $-\Delta+q(x)$, with eigenvalue μ^2 and $\|u\|_{L_2(\Omega)}=1$.

LEMMA 1.2. *We have*

$$|v_k(r,\mu)|\leqq c_2 h(r\mu)[c_1 b(r)]^k, \tag{1.2}$$

$$|w_k(r,\mu)|\leqq c_2\Big(\int_0^r|\varphi(t,\mu)|t^{-2}dt\Big)[c_1 b(r)]^k \quad (0\leqq r\leqq 1,\ \mu\geqq 0;\ k=0,1,\dots), \tag{1.3}$$

where

$$\varphi(x, t) = \begin{cases} q_1(x) & \text{if } |x-x_0| \leqq t \\ 0 & \text{if } |x-x_0| > t, \end{cases} \tag{1.4}$$

$$\varphi(t, \mu) = \int_{x_0+tB} q_1(y)\, u(y, \mu)\, dy, \tag{1.5}$$

B denotes the unit ball of $\mathbf{R}^3$ *with centre* 0.

PROOF. Use induction in k. First prove (1.2). The case $k=0$ is trivial. Using (1.1) and the induction hypothesis it follows

$$|v_k(r, \mu)| \leqq \frac{1}{r\mu} \int_0^r |v_{k-1}(t, \mu)|\, |\sin \mu(t-r)|\, a(t)\, dt \leqq$$

$$\leqq c_2[c_1 b(r)]^{k-1} \frac{1}{r\mu} \int_0^r h(t\mu) |\sin \mu(t-r)|\, a(t)\, dt \leqq c_2 h(r\mu)\, [c_1 b(r)]^k.$$

Now we prove (1.3) for the case $k=0$. Integrating by parts we obtain

$$w_0(r, \mu) = \frac{1}{r\mu} \int_0^r \Big(\int_\theta q_1(x_0+t\theta)\, u(x_0+t\theta, \mu)\, d\theta \Big)\, t \sin \mu(t-r)\, dt =$$

$$= \frac{1}{4\pi^2} \cdot \frac{1}{r\mu} \int_0^r \frac{1}{t^2} \Big(\frac{d}{dt} \int_{x_0+tB} q_1(y)\, u(y, \mu)\, dy \Big)\, t \sin \mu(t-r)\, dt =$$

$$= \frac{-1}{4\pi^2\mu} \int_0^r \varphi(t, \mu) \frac{d}{dt} \Big(\frac{1}{tr} \sin \mu(t-r) \Big)\, dt.$$

On the other hand

$$\Big| \frac{1}{\mu} \frac{d}{dt} \Big(\frac{1}{tr} \sin \mu(t-r) \Big) \Big| \leqq 1 + \frac{1}{t^2} \leqq \frac{2}{t^2} \quad (0 < t \leqq 1),$$

consequently

$$|w_0(r, \mu)| \leqq \frac{1}{2\pi^2} \int_0^r |\varphi(t, \mu)|\, t^{-2} dt.$$

Using the induction hypothesis it follows

$$|w_k(r, \mu)| \leqq \frac{1}{r\mu} \int_0^r |w_{k-1}(t, \mu)|\, |\sin \mu(t-r)|\, a(t)\, dt \leqq$$

$$\leqq c_2[c_1 b(r)]^{k-1} \Big(\int_0^r |\varphi(t, \mu)|\, t^{-2} dt \Big) \frac{1}{r\mu} \int_0^r |\sin \mu(t-r)|\, a(t)\, dt \leqq$$

$$\leqq c_2[c_1 b(r)]^k \Big(\int_0^r |\varphi(t, \mu)|\, t^{-2} dt \Big).$$

Lemma 1.2 is proved.

COROLLARY. *For the functions*

$$\alpha(r,\mu)\overset{\text{def}}{=}\sum_{k=1}^{\infty} v_k(r,\mu),\quad \beta(r,\mu)\overset{\text{def}}{=}\sum_{k=0}^{\infty} w_k(r,\mu)$$

the estimates

$$|\alpha(r,\mu)|\leqq c_2 h(r\mu)[c_1 b(r)], \tag{1.6}$$

$$|\beta(r,\mu)|\leqq c_2\int_0^r |\varphi(t,\mu)|t^{-2}dt, \tag{1.7}$$

hold for $0\leqq r\leqq r_0$, *where* r_0 *satisfies* $c_1 b(r_0)<1$.

LEMMA 1.3. *For every* $r\in[0,r_0]$ *the following equation holds*

$$\int_\theta u(x_0+r\theta,\mu)\,d\theta = u(x_0,\mu)\left[\frac{\sin r\mu}{r\mu}+\alpha(r,\mu)\right]+\beta(r,\mu). \tag{1.8}$$

PROOF. By the mean value formula of E. C. Titchmarsh [19]

$$\int_\theta u(x_0+r\theta,\mu)\,d\theta = u(x_0,\mu)\frac{\sin r\mu}{r\mu}+$$

$$+\frac{1}{r\mu}\int_0^r\Big(\int_\theta q(x_0+t\theta)\,u(x_0-t\theta,\mu)\,d\theta\Big)\sin\mu(t-r)\,dt.$$

For our special q we obtain

$$\int_\theta u(x_0+r\theta,\mu)\,d\theta = u(x_0,\mu)v_0(r,\mu)+\frac{1}{r\mu}\int_0^r\Big(\int_\theta u(x_0+t\theta,\mu)\,d\theta\Big)\cdot$$

$$\cdot\sin\mu(t-r)a(t)\,dt+w_0(r,\mu),$$

i.e. the function

$$v(r,\mu)\overset{\text{def}}{=}\int_\theta u(x_0+r\theta,\mu)\,d\theta$$

is the (unique) solution of the integral equation

$$f(r)=u(x_0,\mu)v_0(r,\mu)+\frac{1}{r\mu}\int_0^r f(t)\sin\mu(t-r)(t)\,dt+w_0(r,\mu).$$

Solving this equation by successive approximation, beginning with

$$f_0(r)=u(x_0,\mu)v_0(r,\mu)+w_0(r,\mu)$$

we obtain for the solution v the Neumann series

$$v(r,\mu)=u(x_0,\mu)\sum_{k=0}^{\infty} v_k(r,\mu)+\sum_{k=0}^{\infty} w_k(r,\mu),$$

and hence (1.8) follows. Lemma 1.3 is proved.

LEMMA 1.4. *The estimate*

$$\sum_{|\mu_n-\mu|\le 1}^{n} |u_n(x_0)|^2 \leqq c_3\mu^2 \quad (\mu \geqq 1, \mu_n \overset{\text{def}}{=} \sqrt{\lambda_n}) \tag{1.9}$$

holds. The constant c_3 *does not depend on* μ.

PROOF. We use the method of V. A. Il'in [3]. A different method for such estimation was given by B. M. Levitan [5].

Consider the function

$$d(r,\mu) = \begin{cases} \mu\dfrac{\sin r\mu}{r} & \text{if} \quad R < r < 2R \\ 0 & \text{if} \quad r \notin (R, 2R), \end{cases}$$

where $R\in(0, r_0/2)$, $r=|x-x_0|$, $\mu>0$. Calculate the Fourier coefficients of d with respect to the system $\{u_n\}$:

$$d_n = d_n(\mu) = \int_\Omega d(|x-x_0|, \mu)\, u_n(x)\, dx =$$

$$= \left[\frac{\mu}{\mu_n}\int_R^{2R} \sin\mu r \sin\mu_n r dr + \mu\int_R^{2R} r\sin\mu r\alpha(r,\mu_n)\, dr\right] u_n(x_0) +$$

$$+\mu\int_R^{2R} r\sin\mu r\beta(r,\mu_n)\, dr.$$

Obviously,

$$\int_R^{2R} \sin\mu r\sin\mu_n r dr \geqq \frac{R}{2}\cos 2R$$

and, using (1.6),

$$\left|\mu\int_R^{2R} r\sin\mu r\alpha(r,\mu_n)\,dr\right| \leqq c_2\mu\int_R^{2R} r\bar{h}(r\mu)\,[c_1 b(r)]\, dr \leqq c_1 c_2 R b(2R),$$

if $\mu \geqq \mu_0$ and $|\mu-\mu_n|\leqq 1$.

On the other hand, by an easy computation we obtain

$$\sum_{|\mu_n-\mu|\le 1}^{n} \left|\mu\int_R^{2R} r\sin\mu r\beta(r,\mu_n)\,dr\right|^2 \leqq \mu^2 R \sum_{|\mu_n-\mu|\le 1}^{n} \Big(\int_R^{2R} |\beta(r,\mu_n)|\, dr\Big)^2 \leqq$$

$$\leqq O(\mu^2R^2) \sum_{|\mu_n-\mu|\le 1}^{n} \int_R^{2R} |\beta(r,\mu_n)|^2 dr \leqq O(\mu^2R^2)\int_R^{2R}\Big(\sum_{|\mu_n-\mu|\le 1}^{n}\Big[\int_0^r |\varphi(t,\mu_n)|t^{-2}dt\Big]^2\Big)\, dr \leqq$$

$$\leqq O(\mu^2R^2)\int_R^{2R}\Big(\sum_{|\mu_n-\mu|\le 1}^{n}\int_0^r |\varphi(t,\mu_n)|^2 t^{-4+1-\varepsilon}dt\Big)\, dr \leqq$$

$$\leqq O(\mu^2R^2)\int_R^{2R}\int_0^r\Big(\int_{x_0+tB}|q_1(y)|^2 dy\Big)t^{-3+\varepsilon}dt dr \leqq O(\mu^2R^{4-\varepsilon}).$$

We have used the Cauchy and Bessel inequalities and (1.7). Summarizing our estimates, using the Bessel inequality for the function d, we obtain (1.9). Lemma 1.4 is proved.

LEMMA 1.5. *We have*

$$\mu\left|\int_R^{2R} r\sin\mu r v_k(r,\mu_n)\,dr\right| \leqq c_2c_4[c_1b(2R)]^k\frac{\mu}{\mu_n}\cdot\frac{1}{|\mu_n-\mu|} \tag{1.10}$$

$$(0<R<r_0/2,\ \mu\geqq 0,\ \mu_n\geqq 0;\quad k=0,1,\ldots).$$

The constant c_4 *does not depend on* μ, μ_n, R *and* k.

PROOF. Use induction on k. For $k=0$ the estimateion (1.10) follows by an easy computation. Now, using (1.2), the Fubini theorem and the induction hypothesis, we get

$$\mu\int_R^{2R} r\sin\mu r v_k(r,\mu_n)\,dr = \mu\int_R^{2R} r\sin\mu r\frac{1}{\mu_n r}\int_0^r v_{k-1}(t,\mu_n)\sin\mu_n(t-r)a(t)\,dt\,dr =$$

$$=\frac{\mu}{\mu_n}\int_0^{2R}\Big(\int_{\max(t,R)}^{2R}\sin\mu r\cos\mu_n r\,dr\Big)v_{k-1}(t,\mu_n)\sin\mu_n t a(t)\,dt-$$

$$-\frac{\mu}{\mu_n}\int_0^{2R}\Big(\int_{\max(t,R)}^{2R}\sin\mu r\sin\mu_n r\,dr\Big)v_{k-1}(t,\mu_n)\cos\mu_n t a(t)\,dt,$$

$$\mu\left|\int_R^{2R} r\sin\mu r v_k(r,\mu_n)\,dr\right| \leqq c_4\frac{\mu}{\mu_n}\frac{1}{|\mu_n-\mu|}\int_0^{2R}|v_{k-1}(t,\mu_n)|a(t)\,dt\leqq$$

$$\leqq c_4c_2[c_1b(2R)]^{k-1}\frac{\mu}{\mu_n}\cdot\frac{1}{|\mu_n-\mu|}\int_0^{2R}h(t\mu_n)a(t)\,dt\leqq$$

$$\leqq c_4c_2[c_1b(2R)]^k\frac{\mu}{\mu_n}\cdot\frac{1}{|\mu_n-\mu|}.$$

Lemma 1.5 is proved.

COROLLARY. *For any* $R\in(0, r_0/2)$ *the estimate*

$$\mu\left|\int_R^{2R} r\sin\mu r\alpha(r,\mu_n)\,dr\right| \leqq c_5b(2R)\frac{\mu}{\mu_n}\cdot\frac{1}{|\mu_n-\mu|}\quad(\mu\geqq 0,\ \mu_n\geqq 0) \tag{1.11}$$

holds. The constant c_5 *does not depend on* μ, μ_n, R.

LEMMA 1.6. *There exist* $M>0$ *and* $c_6>0$ *such that*

$$\sum_{|\mu_n-\mu|\leqq M}{}_n\ |u_n(x_0)|^2\geqq c_6\mu^2\quad(\mu\geqq 0). \tag{1.12}$$

PROOF. We adapt the method of V. A. Il'in [4]; we use (1.11) and also other estimates of the present work. We only sketch the proof.

By the Parseval equality

(1.13)
$$\mu^2\left(\frac{R}{2}+O\left(\frac{1}{\mu}\right)\right) = \int_\Omega d^2(|x-x_0|,\mu)\,dx = \Big(\sum_{|\mu_n-\mu|\le M}{}_n + \sum_{|\mu_n-\mu|>M}{}_n\Big)d_n^2 = \sum{}_1+\sum{}_2.$$

First we prove for any fixed $M>0$

(1.14)
$$\sum{}_1 = O(R^2)\sum_{|\mu_n-\mu|\le M}{}_n |u_n(x_0)|^2+O(\mu^2R^{4-\varepsilon})\quad (\varepsilon>0,\ \mu\ge 2M).$$

Use an estimate given in the proof of Lemma 1.4:

$$|d_n| \le |u_n(x_0)|\left\{\frac{\mu}{\mu_n}\left|\int_R^{2R}\sin r\mu \sin\mu_n r\,dr\right|+c_1c_2b(2R)R\right\}+\mu\left|\int_R^{2R} r\sin\mu r\beta(r,\mu_n)\,dr\right|.$$

If $\mu\ge 2M$ and $|\mu_n-\mu|\le M$, then $\mu_n\in\left(\frac{\mu}{2},\ \frac{3}{2}\mu\right)$ and hence $|\mu/\mu_n|\le 2$. We get

$$\sum{}_1 \le (2R+c_1c_2b(2R)R)^2\sum_{|\mu_n-\mu|\le M}{}_n|u_n(x_0)|^2+\mu^2\sum_{n=1}^{\infty}\left|\int_R^{2R} r\sin\mu r\beta(r,\mu_n)\,dr\right|^2.$$

Using the estimate for the last term given in the proof of Lemma 1.4 the desired estimate (1.14) follows. Now consider Σ_2:

(1.15)
$$\sum{}_2 = \sum_{\mu_n\le 1}{}_n+\Big(\sum_{1\le\mu_n\le\frac{\mu}{2}}{}_n+\sum_{\mu_n>\frac{3}{2}\mu}{}_n\Big)+\sum_{M\le|\mu_n-\mu|\le\frac{\mu}{2}}{}_n = S_1+S_2+S_3.$$

An easy calculation shows

$$S_1 = O(\mu^2R^2),\quad S_2 = O\left(\mu^2\left[\frac{1}{\sqrt{\mu}}+R^{4-\varepsilon}\right]\right),\quad S_3 = O(\mu^2[2^{-m}+R^{4-\varepsilon}]),$$

where $\varepsilon>0$ and $1/2^m=M$. Indeed, using (1.9) and (1.15) we obtain

$$S_1 \le \sum_{\mu_n\le 1}{}_n|u_n(x_0)|^2\left\{\frac{\mu}{\mu_n}\int_R^{2R}\mu_n r\,dr+c_1c_2b(2R)R\right\}^2+O(\mu^2R^{4-\varepsilon}) = O(\mu^2R^2).$$

Applying (1.11) and (1.15) it follows

$$S_2 \le \Big(\sum_{1<\mu_n\le\frac{\mu}{2}}{}_n+\sum_{\mu_n>\frac{3}{2}u}{}_n\Big)|u_n(x_0)|^2O\left([1+b(2R)]\frac{\mu}{\mu_n}\frac{1}{|\mu_n-\mu|}\right)^2+O(\mu^2R^{4-\varepsilon}).$$

In this case $|\mu_n-\mu|\ge\frac{\mu}{2}$ and $|\mu_n-\mu|\ge\frac{1}{2}\mu_n$, consequently

$$\frac{1}{|\mu_n-\mu|} = \frac{O(1)}{\mu_n^{3/4}\mu^{1/4}}.$$

We get

$$S_2 \le \Big(\sum_{1<\mu_n\le\frac{\mu}{2}}+\sum_{\mu_n>\frac{3}{2}\mu}\Big)\frac{\mu^2}{\sqrt{\mu}}\frac{|u_n(x_0)|^2}{\mu_4^{3/2+2}}+O(\mu^2R^{4-\varepsilon}) = O\left(\mu^2\left[\frac{1}{\sqrt{\mu}}+R^{4-\varepsilon}\right]\right).$$

At last consider S_3. Suppose $M=2^m$ and set

$$p = \min\left\{k\colon\ 2^k \geqq \frac{\mu}{2}\right\}.$$

In this case $|\mu_n-\mu|\leqq\frac{\mu}{2}$ and hence $\mu/\mu_n\leqq 2$. Taking into consideration also (1.9) and (1.15) we obtain

$$S_3 = O(1)\sum_{k=m}^{p-1}\left[\sum_{2^{k-1}\leqq|\mu_n-\mu|\leqq 2^k}\frac{|u_n(x_0)|^2}{|\mu_n-\mu|^2}\right]+O(\mu^2R^{4-\varepsilon}) =$$

$$= O(1)\sum_{k=m}^{p-1}\frac{1}{4^{k-1}}\left[\sum_{|\mu_n-\mu|\leqq 2^k}^{n}|u_n(x_0)|^2\right]+O(\mu^2R^{4-\varepsilon}) =$$

$$= O(\mu^2)\sum_{k=m}^{p-1}2^{-k}+O(\mu^2R^{2-\varepsilon}) = O(\mu^2[2^{-m}+R^{4-\varepsilon}]).$$

The desired estimate (1.12) follows from (1.14) and (1.15) choosing m i.e. M large enough.

Lemma 1.6 is proved.

COROLLARY. *There exist infinitely many n with $u_n(x_0)\neq 0$.*

REMARK. Up to this point we used only the assumption ($*$). This raises the question, whether this condition is necessary or not. Next we show that this condition is close to the necessary, namely, if the singularity of q at x_0 is of order $1/|x-x_0|^2$, then for any eigenfunction $u(x,\mu)$ of the operator $-\Delta+q(x)$, $u(x_0,\mu)=0$ holds. This statement is true in any dimension. The reason of this fact is that in this case the operator $q\cdot$ is not only perturbation in $-\Delta+q\cdot$.

We prove our statement for radially symmetrical eigenfunctions (then the general case follows by expanding the eigenfunction $u(x,\mu)$ in hyperspherical functions. It is enough to remark that the coefficients $R_{nk}(r)$ in the expansion $u(r,\theta)\sim\sum_{n,k}R_{n,k}(r)Y_{n,k}(\theta)$, $r=|x-x_0|$ are spherically symmetrical eigenfunctions of a Schrödinger operator with spherically symmetrical potential; we left the details to the reader).

Indeed, let $\Omega\subset\mathbf{R}^N$ ($N\geqq 3$) be an arbitrary domain, $x_0\in\Omega$, $q(x)=a/|x-x_0|^2$ ($a>0$). Then for any spherically symmetrical eigenfunction $u(x,\mu)$ of the operator $-\Delta+q\cdot$ the equality $u(x_0,\mu)=0$ is fulfilled. To prove this we use spherical coordinates

$$-\frac{d^2u}{dr^2}+\frac{N-1}{r}\frac{du}{dr}+q(r)u = \mu^2u,$$

i.e. the function $v(r,\mu)\overset{\text{def}}{=}r^{-\nu}u(r,\mu)$ satisfies the Bessel equation

$$-v''+\frac{1}{r}v'+[\mu^2r^2-(a+\nu^2)]\,v = 0,\quad \nu = \frac{N-2}{2}.$$

The general solution of the last equation on $(0, \infty)$ is

$$v = c_1 J_{\nu+\delta} + c_2 Y_{\nu+\delta} \quad (\nu+\delta = \sqrt{a+\nu^2}).$$

$u(r)=r^\nu v(r)$ is bounded as $r \to +0$ and hence $c_2=0$. On the other hand

$$u(r) = c_1 \frac{J_{\nu+\delta}(r)}{r^\nu} \to 0 \quad (r \to +0)$$

and our statement is proved for spherically symmetrical eigenfunctions. It is easy to see from the proof, that if $a<0$, then $|u(r)| \to +\infty$ as $r \to +0$, that is in this case there is no any eigenfunction of the operator $-\Delta+q\cdot$ in the classical sense.

At last we remark, that developing the ideas of the present paper, it is possible to generalize Lemma 1.4 for arbitrary dimension for potentials of the form

$$q(x) = \frac{a(|x-x_0|)}{|x-x_0|} + q_1(x),$$

where $a \geqq 0$, $\int\limits_{+0} a(t)\,dt < \infty$, $q_1 \in C^l(\Omega)$ and $l > (N-4)/2$. In this case we have

$$\sum_{|\mu_n-\mu| \leqq 1} |u_n(x_0)|^2 \leqq c\mu^{N-1} \quad (\mu \geqq 0,\ \mu_n = \sqrt{\lambda_n}).$$

2. Proof of the Theorem

We need some lemmas.

LEMMA 2.1. *For any* $f \in \mathring{W}_2^1(\Omega)$

$$\sum_{n=1}^{\infty} |(f, u_n)|^2 \mu_n^2 \leqq c_7 \|f\|_{W_1^2}. \tag{2.1}$$

PROOF. It is wellknown that ([6])

$$\mathring{W}_2^1(\Omega) = \overline{C_0^\infty(\Omega)}\big|_{W_2^1(\Omega)}$$

(the closure of $C_0^\infty(\Omega)$ in the metric $W_2^1(\Omega)$), hence it is enough to prove (2.1) for $f \in C_0^\infty(\Omega)$. By the Parseval equation we get

$$\sum_{n=1}^{\infty} |f_n|^2 \mu_n^2 = \sum_{n=1}^{\infty} |f_n|^2 \lambda_n = \int\limits_\Omega f(-\Delta f + qf) =$$

$$= \int\limits_\Omega |\nabla f|^2 + \int\limits_\Omega q|f|^2 \leqq \text{const} \|f\|_{W_2^1} + \int\limits_\Omega (q_0+q_1)|f|^2 \leqq$$

$$\leqq \text{const} \|f\|_{W_2^1} + \|q_1\|_{L_2} \|f\|_{L_4}^2 + \int\limits_\Omega |f|^2 q_0 \leqq c\|f\|_{W_2^1} + \int\limits_\Omega q_0 |f|^2,$$

taking into account the imbedding $W_2^1 \to L_4$. At least apply the Hölder inequality with $p=3/2$, $q=3$ and use $W_2^1 \to L_6$. It follows

$$\int\limits_\Omega q_0|f|^2 \leqq \|q_0\|_{L_p} \|f^2\|_{L_q} = \|q_0\|_{L_p} \|f\|_{L_{2q}}^2 \leqq c\|f\|_{W_2^1}.$$

Lemma 2.1 is proved.

LEMMA 2.2. *For any* $f\in\mathring{W}_2^1(\Omega)$ *the estimate*

$$\sum_{k=1}^{\infty}|\alpha(r,\mu_n)||f_n u_n(x_0)|\leqq c_8\|f\|_{W_2^1} \tag{2.2}$$

holds, uniformly in $r\in(0,1)$, *i.e. the constant* c_8 *does not depend on* r.

PROOF. By (1.6), (2.1) and the Schwarz inequality we have

$$\left(\sum_{n=1}^{\infty}|\alpha(r,\mu_n)||f_n u_n(x_0)|\right)^2\leqq c_7\|f\|_{W_2^1}^2\sum_{n=1}^{\infty}\left|\frac{\alpha(r,\mu_n)u_n(x_0)}{\mu_n}\right|^2\leqq$$

$$\leqq cb^2(r)\sum_{n=1}^{\infty}h^2(r\mu_n)\frac{|u_n(x_0)|^2}{\mu_n^2}=\sum_{r\mu_n\leqq 1}{}_n+\sum_{r\mu_n>1}{}_n=A_1+A_2.$$

Taking into account (1.9) we get

$$A_1\leqq cb^2(r)\sum_{\mu_n\leqq\frac{1}{r}}{}_n\frac{|u_n(x_0)|^2}{\mu_n^2}\leqq cb^2(r)\sum_{k=1}^{\left[\frac{1}{r}\right]}\frac{1}{k^2}\sum_{k\leqq\mu_n\leqq k+1}|u_n(x_0)|^2\leqq$$

$$\leqq c\frac{b^2(r)}{r}=O(1)\frac{1}{r}\left(\int_0^r O\left(\frac{1}{\sqrt{t}}\right)dt\right)^2=O(1),$$

and

$$A_2\leqq c\frac{b^2(r)}{r}\sum_{\mu_n\leqq\frac{1}{r}}\frac{|u_n(x_0)|^2}{\mu_n^4}=O\left(\frac{b^2(r)}{r}\right)\sum_{k=\left[\frac{1}{r}\right]+1}^{\infty}\frac{1}{k^2}=O\left(\frac{b^2(r)}{r}\right)=O(1).$$

LEMMA 2.3. *For any* $f\in\mathring{W}_2^1(\Omega)$

$$\sum_{n=1}^{\infty}|\beta(r,\mu_n)||f_n|\leqq c_9\|f\|_{W_2^1} \tag{2.3}$$

holds uniformly in $r\in(0,1)$.

PROOF. It is easy to see that

$$\left(\sum_{n=1}^{\infty}|\beta(r,\mu_n)||f_n|\right)^2\leqq \text{const}\,\|f\|_{W_2^1}\sum_{n=1}^{\infty}|\beta(r,\mu_n)|^2$$

and, taking into account (1.7)

$$\sum_{n=1}^{\infty}|\beta(r,\mu_n)|^2\leqq\text{const}\left(\int_0^r|\varphi(t,\mu_n)|t^{-2}dt\right)^2\leqq\text{const}\sum_{n=1}^{\infty}\int_0^r|\varphi(t,\mu_n)|^2t^{-3-\varepsilon}dt=$$

$$=\text{const}\int_0^r\left(\sum_{n=1}^{\infty}|\varphi(t,\mu_n)|^2\right)t^{-3-\varepsilon}dt=O(1)\int_0^r\Big(\int_{x_0+tB}q_1^2(y)dy\Big)^{-3-\varepsilon}t\,dt=O(1).$$

Lemma 2.3 is proved.

Using our estimates the proof of the Theorem goes along the same line as in [2] for the special case $(q_1 \equiv 0)$.

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT II OF ANALYSIS
1088 BUDAPEST
MÚZEUM KRT 6—8
HUNGARY

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ON QUASI-IDEALS IN RINGS

H. J. WEINERT (Clausthal)

Dedicated to Professor O. Steinfeld on the occasion of the thirtieth anniversary of quasi-ideals

§ 1. Introduction

Let A be a ring, which always means an associative one throughout this paper. A subgroup Q of $(A, +)$ is called a *quasi-ideal* of A, iff $AQ \cap QA \subseteq Q$ holds, where AQ denotes all finite sums $\Sigma a_i q_i$ with $a_i \in A$, $q_i \in Q$. This concept, generalizing the notion of one-sided ideals of rings, and the corresponding one for semigroups are due to O. Steinfeld (cf. [5], [6]). We refer to his monography [7] for the far-reaching theory on quasi-ideals which has developed in the meantime.

Clearly, the intersection $L \cap R$ of a left ideal L and a right ideal R of a ring A is always a quasi-ideal of A, and the same statement holds for semigroups. But whereas each quasi-ideal of a semigroup can be obtained in this way, it was unknown for about 20 years whether or not analogously each quasi-ideal Q of a ring A is such an intersection

(1) $\qquad Q = L \cap R$ for suitable left and right ideals L and R of A.

The answer was in the negative and given by A. H. Clifford, who constructed an algebra A of dimension 3 over the field $\mathscr{K} = \{0, 1\}$, containing a quasi-ideal Q such that (1) does not hold (published in [7], Expl. 2.1, p. 8). We shall deal with this example in § 2 and show that A may be obtained as the contracted semigroup algebra $A = \mathscr{K}_0[S]$ of a certain semigroup $S = S^0$ over $\mathscr{K} = \{0, 1\}$. In this interpretation, Clifford's counter-example does no longer depend on the special choice of the field $\mathscr{K}$. More generally: For any commutative ring $\mathscr{R}$ with an identity the contracted semigroup algebra $\mathscr{R}_0[S]$ of this semigroup S contains a quasi-ideal Q violating (1), a statement which remains true if one defines $\mathscr{R}_0[S]$ for non-commutative rings $\mathscr{R}$ in a suitable way (cf. Proposition 2.1 and Remark 2.3). Further, there are various semigroups $T = T^0$ obtained from S such that each of the rings $A = \mathscr{R}_0[T]$ also has at least one quasi-ideal not satisfying (1) (cf. Remark 2.4).

In this situation, we say that a quasi-ideal Q of a ring A has the *intersection property* iff (1) holds and formulate the following questions:

PROBLEM a) (cf. [7], p. 9). Give sufficient and/or necessary conditions for a ring A such that each quasi-ideal Q of A has the intersection property.

PROBLEM b). If Q is a quasi-ideal of a ring A, give sufficient and/or necessary conditions such that Q has the intersection property.

PROBLEM c) (A. H. Clifford, correspondence communication). Give sufficient and/or necessary conditions for a semigroup $S = S^0$ such that for any field $\mathscr{K}$ each quasi-ideal Q of $\mathscr{K}_0[S]$ has the intersection property.

In the main part of this paper (§§ 3—5) we give some contributions to these problems. Concerning the first and the last one, we are far away from complete solutions, and some of our results disprove hopeful conjectures. Only for Problem b) we obtain general necessary and sufficient conditions in Proposition 3.1. They are repeatedly used in what follows and imply known sufficient conditions concerning the Problems b) and a) immediately (cf. Corollary 3.2).

In order to include minimal quasi-ideals [1] in our considerations, we give a description of the quasi-ideals $(X)_q$ and $(x)_q$ of a ring A generated by a subset $X\subseteq A$ or by an element $x\in A$,[2] and a characterization of minimal quasi-ideals (cf. Lemma 3.3). From these results we obtain that a minimal quasi-ideal Q of a ring A satisfying $Q^2\neq\{0\}$ has the intersection property (together with a short proof of known basic statements for those quasi-ideals, cf. Theorem 3.4), whereas for $Q^2=\{0\}$ both cases are possible (cf. Remark 3.7). By the first statement, each minimal quasi-ideal of a ring A without nilpotent elements has the intersection property (cf. Corollary 3.5). Unfortunately, one can not go on in this direction (for instance, with respect to Problem a)), since there are rings A containing quasi-ideals Q which do not satisfy the intersection property, whereas each minimal quasi-ideal of A has the intersection property (cf. Theorem 3.8 and also Theorem 5.1 and Remark 5.5).

All examples of rings A which contain quasi-ideals without the intersection property presented in §§ 2 and 3 are obtained as contracted semigroup algebras $A=\mathscr{R}_0[S]$ of some semigroups $S=S^0$. In order to combine these examples and to give a partial answer to Problem c), we characterize in Theorem 4.1 a class of semigroups S by some conditions such that each contracted semigroup algebra $\mathscr{R}_0[S^0]$ of S^0 contains at least one quasi-ideal Q without the intersection property. In the following remarks we show that all semigroups we have used to obtain those rings (and also other ones) satisfy these conditions. But we do not believe that all semigroups S satisfying the above statement for all $\mathscr{R}_0[S^0]$ belong to the class described in Theorem 4.1, and we formulate a corresponding problem at the end of § 4.

On the other hand, all these examples of rings containing quasi-ideals without the intersection property have a lot of zero divisors. This fact and Corollary 3.5 lead to the question, whether all quasi-ideals of a ring without zero divisors satisfy the intersection property. The answer is in the negative, and § 5 is devoted to construct a class of rings A without zero divisors and to prove that they have a quasi-ideal Q which does not satisfy the intersection property. The question, whether these rings A can be obtained as contracted semigroup algebras, remains open, and we refer to Problem 5.6 in this context.

As a by-product, we use these rings of § 5 to deal with two problems posed by L. Márki (cf. § 6 up to Proposition 6.1), recently solved by P. N. Stewart ([9], Example 1). The rings considered there as well as our rings mentioned above provide an answer to both problems in the positive, and even a stronger one than asked for since all these rings have no zero divisors (cf. Proposition 6.1). Note further that

[1] A quasi-ideal Q of A is called minimal if $\{0\}$ and $Q\neq\{0\}$ are all quasi-ideals of A contained in Q.

[2] Clearly, $(X)_q$ is defined as the intersection of all quasi-ideals of A containing X. Similarly, we denote by $(X)_l$, $(X)_r$ and (X) the left, the right and the (two-sided) ideal of A, resp., and by $\langle X\rangle$ the subgroup of $(A, +)$ generated by $X\subseteq A$.

the rings of [9], Example 1 have an identity, whereas our rings do not satisfy the intersection property. Moreover, in Proposition 6.2 we give an answer to a strengthened version of one of these problems: There are rings A (even without zero divisors) with a quasi-ideal Q such that Q^2 is not a qasi-ideal of A.

§ 2. Clifford's example and its generalization

A. H. Clifford considers the algebra A of dimension 3 over the two-element field $\mathscr{K}=\{0, 1\}$, defined by the basis $\{e, a, b\}$ and the left hand multiplication table $(2l)$ (cf. [7], p. 8):

$$(2)\qquad \begin{array}{c|ccc} & e & a & b \\ \hline e & e & a+b & 0 \\ a & b & 0 & 0 \\ b & b & 0 & 0 \end{array} \qquad\qquad \begin{array}{c|ccc} & e & b & c \\ \hline e & e & 0 & c \\ b & b & 0 & 0 \\ c & 0 & 0 & 0 \end{array}$$

Straightforward calculations, depending on the characteristic 2 of $\mathscr{K}$, show that A is associative and that $Q=\{0, a\}$ is a quasi-ideal of A, not satisfying the intersection property. The proof of these statements is easier if we use $\{e, b, c\}$ instead of $\{e, a, b\}$ as a basis of A, where $c=a+b$ and hence $a=b+c$. Obviously, the right hand table $(2r)$ above describes the same multiplication on A. Moreover, $(2r)$ defines a semigroup $S=\{e, b, c, 0\}$ with zero 0, which is easily checked by Light's associativity test (cf. [1], § 1.2) or by considering the non-trivial cases

$$e(ec) = ec = (ee)c, \qquad e(be) = eb = 0 = (eb)e,$$

$$e(ce) = 0 = ce = (ec)e, \quad b(ee) = be = (be)e.$$

Thus A is the contracted semigroup algebra $\mathscr{K}_0[S]$ (cf. [1], § 5.2). In this interpretation, Clifford's example has the following generalization:

Proposition 2.1. *Let $S=\{e, b, c, 0\}$ be the above semigroup, $\mathscr{R}$ a commutative ring with zero 0 and identity 1, and let $A=\mathscr{R}_0[S]$ be the contracted semigroup algebra of S over $\mathscr{R}$, i.e. the algebra defined by the base $\{e, b, c\}$ and the multiplication table $(2r)$. Then the subgroup*

$$Q = \langle b+c\rangle = \{\zeta(b+c) | \zeta \in \langle 1\rangle \subseteq \mathscr{R}\}$$

of $(A, +)$ generated by $b+c$ is a quasi-ideal of A which does not satisfy the intersection property. Moreover, one has $Q^2=\{0\}$, and Q is a minimal quasi-ideal of A iff the characteristic of $\mathscr{R}$ is a prime number.

Proof. From $(\varepsilon e+\beta b+\gamma c)\zeta(b+c)=\varepsilon\zeta c$ for all $\varepsilon, \beta, \gamma \in \mathscr{R}$ and $\zeta\in\langle 1\rangle$ we obtain $AQ=\{\varrho c|\varrho\in\mathscr{R}\}$ and similarly $QA=\{\varrho' b|\varrho'\in\mathscr{R}\}$. Hence Q is a quasi-ideal of A by $AQ\cap QA=\{0\}\subseteq Q$. To disprove the intersection property, we consider any left ideal L and any right ideal R of A such that $Q\subseteq L\cap R$. Then $c\in AQ\subseteq L$ and $b+c\in L$ yield $b\in L$, and we also have $b\in QA\subseteq R$, hence $b\in L\cap R$ for $b\notin Q$. Thus $Q\subseteq L\cap R$ implies $Q\subset L\cap R$. Finally, $Q^2=\{0\}$ is trivial, and Q is a minimal quasi-ideal of A iff $(Q, +)$ is simple, i.e. iff $\langle 1\rangle$ has prime order.

REMARK 2.2. Clearly, each (simple) subgroup $\mathcal{U}$ of $(\mathcal{R}, +)$ determines a (minimal) quasi-ideal $Q_{\mathcal{U}}=\{\mu(b+c)|\mu\in\mathcal{U}\}$ of A which has the same properties as $Q=Q_{\langle 1\rangle}$ above. The same holds, for instance, for $Q'=\langle b+c+c\rangle\neq Q$ if the characteristic of $\mathcal{R}$ is not 2, and so on. On the other hand, one easily checks that (cf. (10) of Lemma 3.3a))

$$(e)_q=\{\alpha e\},\quad (e+b+c)_q=\{\alpha(e+b+c)\},$$

$$(e+b)_q=\{\alpha(e+b)\},\quad (e+c)_q=\{\alpha(b+c)\}\quad \text{for all } \alpha\in\mathcal{R}$$

are quasi-ideals of A which are not one- or two-sided ideals, but all of them have the intersection property.

REMARK 2.3. The statements of Proposition 2.1 and also the following ones dealing with contracted semigroup algebras (also in §§ 3 and 4) do not depend on the commutativity of $\mathcal{R}$: Let $H=H^0$ be a semigroup, $\mathcal{R}$ a ring with identity, and let A be the left vector space over $\mathcal{R}$ with $H\setminus\{0\}$ as a basis. We identify the zeros of $\mathcal{R}$ and H, and write $\Sigma\alpha_i h_i$ with $\alpha_i\in\mathcal{R}, h_i\in H\setminus\{0\}$ for the elements of A, where $\alpha_i=0$ holds for almost all α_i. Then A is a ring with respect to the multiplication

$$\Big(\sum_i \alpha_i h_i\Big)\Big(\sum_j \beta_j h_j\Big)=\sum_{i,j}\alpha_i\beta_j h_i h_j,$$

where $h_i h_j$ denotes the product in H. We call A a contracted semigroup algebra and write $A=\mathcal{R}_0[H]$ also in this case. In fact, if $\mathcal{R}$ is not commutative, $A=\mathcal{R}_0[H]$ is a special case of a "generalized algebra" introduced by G. Pickert [3] or of a "monomial ring" as considered by L. Rédei [4], § 66, but in general not an algebra over $\mathcal{R}$ in the usual meaning. We refer to [10], §§ 2 and 4, for more details in this context. A reader not interested in this generalization may assume $\mathcal{R}$ to be commutative for all contracted semigroup algebras $A=\mathcal{R}_0[H]$ considered in the following.

REMARK 2.4. Starting with the above semigroup $S=\{e, b, c, 0\}$, there are different ways to obtain semigroups $T=T^0$ which contain S as a subsemigroup such that the quasi-ideal $Q=(b+c)_q$ of $A=\mathcal{R}_0[T]$ has the same properties as stated in Proposition 2.1. For instance, let T be any inflation of S (cf. [1], § 3.2), or any semigroup $(T, \cdot)\supseteqq(S, \cdot)$ such that

$$(3)\qquad (T\setminus S)S\subseteqq\{c, 0\}\quad\text{and}\quad S(T\setminus S)\not\ni c$$

hold. Another way is to consider a semigroup $(T, \cdot)\supseteqq(S, \cdot)$ such that for all $t_1, t_2\in T$ one has

$$(4)\qquad t_1t_2=b\Rightarrow t_1=b,\ t_2=e,\quad t_1t_2=c\Rightarrow t_1=e,\ t_2=c;$$

in this case the quasi-ideal $Q=(b+c)_q$ of A may contain $\langle b+c\rangle$ properly.

In this context we note that all these semigroups T satisfy the conditions (16) and (17) of Theorem 4.1, which imply that $A=\mathcal{R}_0[T^0]$ has a quasi-ideal Q without the intersection property (cf. the corresponding remarks in § 4).

§ 3. Some general statements on quasi-ideals

Let Q be a quasi-ideal of a ring A. If Q satisfies the intersection property $Q=L\cap R$, one clearly may choose L and R minimal, i.e. $L=(Q)_l=AQ+Q$ and $R=(Q)_r=QA+Q$. Thus the general situation is described by

$$AQ+QA\subseteqq Q\subseteqq(AQ+Q)\cap(QA+Q), \tag{5}$$

and we state the following solution of Problem b) in § 1:

PROPOSITION 3.1. *Let Q be a quasi-ideal of a ring A. Then each of the following statements is equivalent to the intersection property of Q:*

$$Q=(AQ+Q)\cap(QA+Q), \tag{6}$$

$$AQ\cap(QA+Q)\subseteqq Q, \tag{7}$$

$$QA\cap(AQ+Q)\subseteqq Q. \tag{7'}$$

PROOF. By the above considerations, Q has the intersection property iff (6) holds, which in turn implies (7) and (7′). Conversely, for any subgroups U, V and Q of the group $(A, +)$, one obviously has

$$U\cap(V+Q)\subseteqq Q\Rightarrow(U+Q)\cap(V+Q)\subseteqq Q.$$

Hence (7) as well as (7′) imply $(AQ+Q)\cap(QA+Q)\subseteqq Q$, which is equivalent to (6) by the right hand inclusion in (5).

As a consequence of (7), a quasi-ideal Q of A does not have the intersection property iff there is a finite number of elements $q_i, p_j, p\in Q$ and $a_i, b_j\in A$ such that

$$\sum_{i=1}^{n}a_iq_i=\left(\sum_{j=1}^{m}p_jb_j\right)+p\notin Q \tag{8}$$

holds. Clearly, both sums as well as p are not 0 in (7), and also the quasi-ideal $(\{q_i, p_j, p\})_q\subseteqq Q$ does not have the intersection property. The simplest case of such a formula (8) with $n=m=1$ was in fact the key point of the examples given in § 2, namely

$$e(b+c)=(b+c)(-e)+(b+c)\notin Q.$$

Subsequently, we shall use similar formulas to disprove the intersection property for a quasi-ideal.

The following consequences of Proposition 3.1 are already known (cf. [7], Proposition 2.8 and Corollary 2.9). The second one seems to be the only known sufficient condition with respect to Problem a) — apart from the trivial assumptions that A is a commutative ring or a division ring:

COROLLARY 3.2. a) *If a quasi-ideal Q of a ring A satisfies $Q\subseteqq AQ$ or $Q\subseteqq QA$, then Q has the intersection property.*

b) *If a ring A contains a left or a right identity, then each quasi-ideal Q of A has the intersection property.*

PROOF. Since b) clearly follows from a), we only state that $Q\subseteqq AQ$ implies (7′) by $QA\cap(AQ+Q)=QA\cap AQ\subseteqq Q$.

LEMMA 3.3. a) *The quasi-ideals* $(X)_q$ *and* $(x)_q$ *of a ring* A *generated by a subset* $X \subseteqq A$ *or by an element* $x \in A$, *respectively, are given by*

(9) $$(X)_q = \langle X \rangle + (AX \cap XA) = \mathbf{Z}X + (AX \cap XA),$$

(9′) $$(x)_q = \langle x \rangle + (Ax \cap xA) = \mathbf{Z}x + (Ax \cap xA).$$

b) *A quasi-ideal* Q *of* A *is minimal iff* $Q \neq \{0\}$ *and*

(10) $$Q = \mathbf{Z}x + (Ax \cap xA) \text{ holds for each } x \in Q \setminus \{0\}.$$

PROOF. a) It is enough to show (9). Clearly, $(X)_q$ contains $\langle X \rangle = \mathbf{Z}X$ as well as $AX \cap XA$. Both are subgroups of $(A, +)$, the latter since $AX \cap XA$ is a quasi-ideal of A. Thus $\langle X \rangle + (AX \cap XA) \subseteqq (X)_q$ is a subgroup, and even a quasi-ideal of A by

$$A(\langle X \rangle + (AX \cap XA)) \cap (\langle X \rangle + (AX \cap XA))A \subseteqq$$

$$\subseteqq A(\langle X \rangle + AX) \cap (\langle X \rangle + XA)A \subseteqq AX \cap XA.$$

b) Since $(x)_q = Q$ for each $x \in Q \setminus \{0\}$ holds iff a quasi-ideal $Q \neq \{0\}$ of A is minimal, (10) follows from (9′).

The next theorem is essentially Theorem 6.5 of [7]; due to Lemma 3.3 b), the following proof will be more convenient.

THEOREM 3.4. *Let* Q *be a minimal quasi-ideal of a ring* A. *Then the following statements are equivalent:*

a) $Q^2 \neq \{0\}$,

b) Q *has no zero divisors,*

c) Q *is a division ring.*

If this is the case, Q *has the intersection property, moreover,*

(11) $$Ay \cap xA = Q \text{ holds for all } x, y \in Q \setminus \{0\}.$$

In particular, we have $Q = Ae \cap eA = eAe$ *for the identity* e *of* Q.

PROOF. Assume $xy = 0$ for $x, y \in Q \setminus \{0\}$. Then we obtain from (10)

$$Q^2 = (x)_q (y)_q \subseteqq (\mathbf{Z}x + Ax)(\mathbf{Z}y + yA) = \{0\},$$

hence a) implies b). Now we suppose b). Then (11) holds since each quasi-ideal $Ay \cap xA$ satisfies $\{0\} \neq Ay \cap xA \subseteqq AQ \cap QA \subseteqq Q$ and Q is minimal. To obtain c), it is enough to show that for $x, y \in Q \setminus \{0\}$ there exists an element $z \in Q$ satisfying $x = zy$. Applying (11) to $xy \in Q \setminus \{0\}$, there are $a, b \in A$ such that

$$x = axy = xyb, \quad \text{hence} \quad z = ax = axyb = xb \in Q$$

is such an element. Since $c) \Rightarrow a)$ is trivial, all statements are proved.

COROLLARY 3.5. *If a minimal quasi-ideal* Q *of a ring* A *does not satisfy the intersection property,* $Q^2 = \{0\}$ *holds. Thus for a ring* A *without non-zero nilpotent elements, in particular for one without zero divisors, each minimal quasi-ideal of* A *has the intersection property.*

REMARK 3.6. A quasi-ideal Q of a ring A satisfying (11) or merely $Ax \cap xA = Q$ for all $x \in Q \setminus \{0\}$ is a minimal one by Lemma 3.3 b). Thus (11) implies that a quasi-ideal Q is minimal and has the intersection property, but there are examples satisfying also $Q^2 = \{0\}$ (cf. iii) below). Hence (11) is not equivalent to a), b) and c) in Theorem 3.4.

REMARK 3.7. For a minimal quasi-ideal Q of a ring A such that $Q^2 = \{0\}$ the following (extreme) cases are possible:

i) Q does not satisfy the intersection property, hence

(12) $$Ay \cap xA = \{0\} \text{ holds for all } x, y \in Q.$$

ii) Q satisfies the intersection property, but (12) and even

(13) $$Ay \cap (xA + \mathbf{Z}x) = (Ay + \mathbf{Z}x) \cap xA = \{0\} \text{ hold for all } x, y \in Q.$$

iii) Q satisfies the intersection property and (11).

PROOF. Examples for i) are given in Proposition 2.1. For ii), let $S = \{x, a, b, c, 0\}$ be the semigroup with zero 0 defined by

(14) $$ax = b, \quad xa = c, \quad \text{all other products } 0.$$

Let $\mathcal{R}$ be a ring with zero 0 and identity 1 and consider the contracted semigroup algebra $A = \mathcal{R}_0[S]$ (cf. Remark 2.3). For each simple subgroup $\mathcal{U}$ of $(\mathcal{R}, +)$, the subgroup $Q = \{\mu x | \mu \in \mathcal{U}\}$ is the intersection of

$$(x)_l = \{\beta b + \mu x | \beta \in \mathcal{R}, \mu \in \mathcal{U}\} \quad \text{and} \quad (x)_r = \{\gamma c + \mu x | \gamma \in \mathcal{R}, \mu \in \mathcal{U}\}.$$

Hence $Q = (x)_q$ is a minimal quasi-ideal of A with intersection property and $Q^2 = \{0\}$, and (13) holds by

$$A\eta x \cap (\xi x)_r = (\eta x)_l \cap \xi x A = \{0\} \quad \text{for all} \quad \xi, \eta \in \mathcal{U}.$$

To obtain examples for iii), we consider $A = \mathcal{K}_0[S^1]$, where $\mathcal{K}$ is a finite prime field and S^1 is the semigroup defined by (14) together with an adjoined identity e (which clearly differs from the identity 1 of $\mathcal{K}$). Then $Q = \{\mu x | \mu \in \mathcal{K}\}$ is the intersection of $A\eta x = (\eta x)_l$ and $\xi x A = (\xi x)_r$ for all $\xi, \eta \in \mathcal{K} \setminus \{0\}$, hence a minimal quasi-ideal $Q = (x)_q$ of A as stated in iii).

At this stage of our considerations, one clearly would like to reduce questions concerning quasi-ideals without the intersection property to minimal ones of this kind. Unfortunately, this is not possible in general:

THEOREM 3.8. a) *There are (finite as well as infinite) rings A such that A has a quasi-ideal Q which does not satisfy the intersection property, whereas each minimal quasi-ideal of A has the intersection property.*

b) *There are semigroups $S = S^0$ such that for each ring $\mathcal{R}$ with an identity, but without zero divisors, the contracted semigroup algebra $A = \mathcal{R}_0[S]$ is a ring as described at* a).

PROOF. It is enough to present a finite semigroup satisfying b). We use a semigroup $S=S^0=\{e, b, c, d, 0\}$ introduced in [11] by [3]

	e	b	c	d
e	e	0	c	d
b	b	0	0	0
c	0	d	0	0
d	d	0	0	0

From (9′) and the table it follows that the quasi-ideal Q of $A=\mathscr{R}_0[S]$ generated by $b+c$ or by $b+c+d$ is

$$Q=(b+c)_q=(b+c+d)_q=\{\zeta(b+c)+\varrho d|\zeta\in\langle 1\rangle,\ \varrho\in\mathscr{R}\}.$$

Since $e(b+c)=(b+c)(-e)+(b+c)=c\notin Q$ holds, Q does not satisfy the intersection property by (8). Moreover, Q contains the ideal $(d)=\{\varrho d|\ \varrho\in\mathscr{R}\}$ of A properly. To show that each minimal quasi-ideal of A has the intersection property, we consider all quasi-ideals $(x)_q$ of A generated by one element. Depending on the choice of

$$x=\varepsilon e+\beta b+\gamma c+\delta d\in A \quad (\varepsilon, \beta, \gamma, \delta\in\mathscr{R}), \tag{15}$$

we shall see that either $(x)_q$ is not a minimal quasi-ideal of A, or that $(x)_q$ has the intersection property, regardless whether or not it is minimal. If $\varepsilon\neq 0$ holds in (15), we obtain $xd=dx=\varepsilon d\in(x)_q$, hence $(\varepsilon d)_q\neq\{0\}$ is a quasi-ideal of A properly contained in $(x)_q$. The same follows for $\varepsilon=0$ and $\beta\neq 0\neq\gamma$ from $x^2=\gamma\beta d\in(x)_q$ and $\gamma\beta\neq 0$ [4] (the quasi-ideal Q above is one of these cases). Finally, if $\varepsilon=\beta=0$ or $\varepsilon=\gamma=0$, then $(x)_q$ is checked to be the intersection of $(x)_l$ and $(x)_r$.

§ 4. A criterion for contracted semigroup algebras

All semigroups $S=S^0$ used so far to obtain rings $A=\mathscr{R}_0[S]$ which contain at least one quasi-ideal without the intersection property will turn out to be semigroups satisfying the conditions of the following theorem. It is also applicable to semigroups $S\neq S^0$; in this case, clearly, the contracted semigroup algebra $A=\mathscr{R}_0[S^0]$ of S^0 is just the semigroup algebra $A=\mathscr{R}[S]$ of S.

THEOREM 4.1. *Let S be a semigroup containing a left ideal L_S, a right ideal R_S and elements $b\neq c$ such that the following conditions hold:*

$$b\notin L_S,\quad c\in Sb\cup Sc\subseteq(L_S\cap R_S)\cup\{c\}, \tag{16}$$

$$c\notin R_S,\quad b\in bS\cup cS\subseteq(L_S\cap R_S)\cup\{b\}. \tag{17}$$

[3] For an easy comparison with the considerations in § 2, we changed the notation used in [11] according to $\alpha=e$, $\beta=d$, $\gamma=b$, $\delta=c$. In [11] we proved that a ring-theoretical example constructed in [8] as an algebra A over the field $\mathscr{K}=\{0, 1\}$ can also be obtained as $A=\mathscr{K}_0[S]$.

[4] Only here we use that $\mathscr{R}$ has no zero divisors. In this context we note that there are rings $\mathscr{R}$ with zero divisors such that, for suitable $\beta, \gamma\in\mathscr{R}$, the quasi-ideal $Q=(\beta b+\gamma c)$ of $\mathscr{R}_0[S]$ (not satisfying the intersection property as above) is minimal.

Then, for each ring $\mathscr{R}$ with an identity, the contracted semigroup algebra $\mathscr{R}_0[S^0]$ of S^0 has a quasi-ideal Q which does not satisfy the intersection property.

PROOF. If the semigroup S has no zero, (16) and (17) imply the corresponding formulas with S^0, $L_{S^0}=L_S\cup\{0\}$ and $R_{S^0}=R_S\cup\{0\}$ instead of S, L_S and R_S, respectively. Thus, to simplify our notation, we may assume that S already has a zero, hence $S=S^0$ in the following.

Further, $D_S=L_S\cap R_S$ is a quasi-ideal of S, and we have $b, c\notin D_S$ by (16) and (17). In $A=\mathscr{R}_0[S]$ we define

$$Q=\{\zeta(b+c)+\sum_i \varrho_i d_i | \zeta\in\langle 1\rangle, \varrho_i\in\mathscr{R}, d_i\in D_S\},$$

clearly a subgroup of $(A, +)$ such that $b, c\notin Q$. From (16) it follows that each element of AQ is a linear combination of $L_S\cup\{c\}$ (with coefficients in $\mathscr{R}$), and each element of QA is a linear combination of $R_S\cup\{b\}$ by (17). Since $b\notin L_S$ and $c\notin R_S$ we obtain

$$(L_S\cup\{c\})\cap(R_S\cup\{b\}) = L_S\cap R_S = D_S,$$

and therefore $AQ\cap QA\subseteq\{\sum_i \varrho_i d_i\}\subseteq Q$, which proves Q to be a quasi-ideal of A.

We are going to disprove the intersection property for Q. Again by (16) and (17), there are elements $s_1, s_2\in S$ such that

$$s_1 b = c \quad\text{or}\quad s_1 c = c \tag{18}$$

and

$$bs_2 = b \quad\text{or}\quad cs_2 = b \tag{19}$$

hold.

But $s_1b=c$ and $bs_2=b$ would imply $c=s_1b=s_1bs_2=cs_2$, hence by (17) $c\in cS\subseteq L_S\cap R_S\subseteq R_S$ contradicting $c\notin R_S$. Similarly, $s_1c=c$ and $cs_2=b$ would yield $b=s_1b$ and $b\in L_S$ by (16), whereas $b\notin L_S$ was assumed. Consequently, exactly one of the two equations (18) and exactly one of the two equations (19) is valid, and the other product is contained in $L_S\cap R_S=D_S$ by (16) and (17). Therefore we obtain in A

$$s_1(b+c) = c+d_1 \quad\text{for some}\quad s_1\in S,\quad d_1\in D_S, \tag{18'}$$

$$(b+c)s_2 = b+d_2 \quad\text{for some}\quad s_2\in S,\quad d_2\in D_S. \tag{19'}$$

Since $c\notin Q$ and $d_1\in Q$, we have

$$s_1(b+c) = (b+c)(-s_2)+(b+c+d_1+d_2) = c+d_1\notin Q$$

for elements $b+c\in Q$, $b+c+d_1+d_2\in Q$, $s_1\in A$ and $-s_2\in A$. This is a formula (8), which proves that Q does not have the intersection property.

REMARK 4.2. Obviously, the semigroup $S=\{e, b, c, 0\}$ of Proposition 2.1 satisfies the conditions (16) and (17) for the elements b, c and $L_S=R_S=D_S=\{0\}$. The same holds for the semigroup $S=\{e, b, c, d, 0\}$, used in the proof of Theorem 3.8, with respect to $L_S=R_S=D_S=\{d, 0\}$.

REMARK 4.3. Let S be a semigroup satisfying (16) and (17) and let T be an inflation of S. Then (16) and (17) are also valid for T, since $L_S=L_T$ is a left and $R_S=R_T$ a right ideal of T, too.

For more examples, we show that also the other semigroups $T=T^0$ mentioned in Remark 2.4 (in fact, more general ones) satisfy the conditions (16) and (17) of Theorem 4.1 for $b, c\in S\subseteqq T$ and suitable left and right ideals L_T and R_T of T:

REMARK 4.4. Let T be a semigroup containing the semigroup $S=\{e, b, c, 0\}$ of Proposition 2.1 as a subsemigroup such that (4) holds. Note that T need not have a zero or may have a zero which is not that of S. Then (4) implies

$$t_1t_2 = e \Rightarrow t_1 = t_2 = e \quad \text{for all} \quad t_1, t_2\in T,$$

since $b=be=(bt_1)t_2$ and $c=ec=t_1(t_2e)$ yield $t_2=e$ and $t_1=e$ by (4). Hence $T\setminus\{e, b, c\}$ is a two-sided ideal of T, and we have $b, c\notin T$ and

$$c\in Tb\cup Tc\subseteqq\{c, 0, T\setminus S\} = (T\setminus\{e, b, c\})\cup\{c\},$$

$$b\in bT\cup cT\subseteqq\{b, 0, T\setminus S\} = (T\setminus\{e, b, c\})\cup\{b\}.$$

Thus $b, c\in T$ and $L_T=R_T=T\setminus\{e, b, c\}$ satisfy the conditions (16) and (17) for T.

REMARK 4.5. Let T be a semigroup containing the semigroup $S=\{e, b, c, 0\}$ of Proposition 2.1 as a subsemigroup; instead of (3) we merely suppose

$$(20) \qquad (T\setminus S)\{b, c\}\subseteqq\{c, 0\} \quad \text{and} \quad \{b, c\}(T\setminus S)\not\ni c.$$

Again we have no assumptions concerning a zero of T. We shall prove that T satisfies (16) and (17) with respect to $b, c\in T$, the right ideal $R_T=bT\cup cT$ and the left ideal

$$L_T = Tb\cup Tc\cup(R_T\setminus\{b\})\cup T(R_T\setminus\{b\}).$$

Using the multiplication of S without comment, from (20l) we obtain $Tb\cup Tc= =\{c, 0\}$. Thus for $b\notin L_T$ it remains to show that $T(R_T\setminus\{b\})$ does not contain b. By way of contradiction, $b\in T(R_T\setminus\{b\})$ would yield

$$b\in TR_T = T(bT\cup cT)\subseteqq TbT\cup TcT\subseteqq\{c, 0\}T,$$

which is impossible according to

$$b = ct \Rightarrow b = (ec)t = e(ct) = eb = 0,$$

$$b = 0t \Rightarrow b = (e0)t = e(0t) = eb = 0.$$

From (20r) it follows that $c\notin R_T$, and we note $0\in cS\subseteqq R_T$ as well as $b\in bS\subseteqq R_T$. Now one easily checks

$$c\in Tb\cup Tc = \{0, c\} \subseteqq (L_T\cap R_T)\cup\{c\} \quad \text{and} \quad b\in bT\cup cT = R_T \subseteqq (L_T\cap R_T)\cup\{b\},$$

which are the remaining statements we were to show.

We conclude this section with the following version of Problem c):

PROBLEM 4.6. Characterize semigroups S for which the conditions of Theorem 4.1 are necessary and sufficient in order that for each ring $\mathcal{R}$ with identity the contracted semigroup algebra $\mathcal{R}_0[S^0]$ has a quasi-ideal Q which does not satisfy the intersection property.

§ 5. Rings without zero divisors

The purpose of this section is to prove the following

THEOREM 5.1. *There are rings A without zero divisors such that A has a quasi-ideal Q which does not satisfy the intersection property.*

To construct those rings, we use a certain generalization of polynominal rings, essentially due to Ore [2].

LEMMA 5.2. a) *Let $\mathcal{S}$ be a ring with identity* 1 *and η an endomorphism of $\mathcal{S}$. Using the elements $\sum a_i y^i$ and the addition of the usual polynomial ring $\mathcal{S}[y]$ in one indeterminate y over $\mathcal{S}$, we define another multiplication by*

$$(21) \qquad \Big(\sum_{i=0}^{n} a_i y^i\Big)\Big(\sum_{j=0}^{m} b_j y^j\Big) = \sum_{k=0}^{n+m}\Big(\sum_{i+j=k} a_i b_j^{\eta^i}\Big)y^k$$

(the key point is $yb=b^\eta y$ instead of $yb=by$). In this way one obtains a ring which we denote by $\mathcal{S}[y]_\eta$.

b) *The ring $\mathcal{S}[y]_\eta$ has no zero divisors iff $\mathcal{S}$ has none and $b\neq 0$ implies $b^\eta\neq 0$ for all $b\in\mathcal{S}$.*

PROOF. a) This statement is easily checked by direct computation. We refer to [10], § 1, in particular for a more general statement and the interrelation to the concept considered in [2].

b) Let $\mathcal{S}$ and η satisfy our conditions and suppose $a_n\neq 0\neq b_m$ in (21). Then $a_n b_m^{\eta^n} y^{n+m}$ is the only term of highest degree on the right side, and $a_n b_m^{\eta^n}\neq 0$ holds by our assumptions. The converse statement is clear.

COROLLARY 5.3. *Let $\mathcal{R}$ be a ring without zero divisors and with identity* 1, *and consider the endomorphism η of the usual polynomial ring $\mathcal{S}=\mathcal{R}[x]$ defined by $f(x)^\eta=f(1-x)$. Then the ring $R=\mathcal{S}[y]_\eta=\mathcal{R}[x][y]_\eta$ consists of all elements*

$$(22) \qquad \sum_{i=0}^{n} f_i(x)\,y^i \quad \text{with} \quad f_i(x)\in\mathcal{S}=\mathcal{R}[x],$$

and the multiplication is given by

$$(21') \qquad \Big(\sum_i f_i(x)y^i\Big)\Big(\sum_j g_j(x)y^j\Big) = \sum_{i,j} f_i(x)g_j(x)^{\eta^i}y^{i+j},$$

where $g_j(x)^{\eta^i}=g_j(x)$ if $2|i$ and $g_j(x)^{\eta^i}=g_j(1-x)$ if $2\nmid i$.

Moreover, $R=\mathcal{R}[x][y]_\eta$ has no zero divisors, and all elements (22) *such that $f_0(x)$ is contained in the ideal $(x)=x\mathcal{R}[x]$ of $\mathcal{R}[x]$ form a subring A of R.*

PROOF. With respect to Lemma 5.2, we only have to state that $f(x)\neq 0$ implies $f(x)^{\eta}=f(1-x)\neq 0$ and that $f(x)^{\eta^2}=f(x)$ holds for all $f(x)\in\mathcal{S}=\mathcal{R}[x]$.

Clearly, these subrings A will turn out to be those rings satisfying Theorem 5.1. Another way to obtain them (may be a more direct one, but more tedious to prove) is as follows: Let $\mathcal{R}[x, y]$ be the "polynomial ring" in the non-commutative indeterminates x and y, i.e. the semigroup ring $\mathcal{R}[F^1]$, where F denotes the free semigroup generated by $\{x, y\}$. Then the ideal $(y-xy-yx)$ of $\mathcal{R}[x, y]$ corresponds to the relation

$$yx = (1-x)\,y = y-xy \tag{23}$$

(cf. (21′)), and one has to check that the ring $\mathcal{R}[x, y]/(y-xy-yx)$ coincides with the ring $R=\mathcal{R}[x][y]_\eta$ of Corollary 5.3. Finally, we have

$$A = (x, y)/(y-xy-yx)\subset\mathcal{R}[x, y]/(y-xy-yx).$$

In particular, if we choose $\mathcal{R}=\mathbf{Z}$, then A is the ring generated by the elements x, y subject to the relation (23). Now, Theorem 5.1 will be proved by the following

LEMMA 5.4. *Let A be the subring of a ring $R=\mathcal{R}[x][y]_\eta$ as introduced in Corollary 5.3. Let Q be the set of all elements*

$$\sum_{\nu=1}^{n} h_\nu(x)\,y^\nu \tag{24}$$

such that $h_1(x)\in\mathcal{R}[x]$ satisfies

$$h_1(x) = \alpha_1+\bar{h}_1(x),\quad \alpha_1\in\langle 1\rangle\subseteqq\mathcal{R},\quad \bar{h}_1(x)\in\mathcal{R}[x]\ \text{ and }\ \bar{h}_1(0) = \bar{h}_1(1) = 0. \tag{25}$$

(Note that each of α_1 and $\bar{h}_1(x)$, may equal 0.) Then Q is a quasi-ideal of the ring A without zero divisors, in fact the quasi-ideal $Q=(y)_q$ generated by y, and Q does not satisfy the intersection property.

PROOF. Obviously, Q is contained in A. Since the difference of the polynomials

$$h_1(x) = \alpha_1+\bar{h}_1(x) \quad\text{and}\quad k_1(x) = \beta_1+\bar{k}_1(x) \tag{26}$$

occurring in (25) is again such a polynomial, Q is a subgroup of $(A, +)$. In order to check $AQ\cap QA\subseteqq Q$, we consider any element

$$\Big(\sum_{i=0}^{s} f_i(x)\,y^i\Big)\Big(\sum_{\nu=1}^{n} h_\nu(x)\,y^\nu\Big) = \Big(\sum_{\mu=1}^{m} k_\mu(x)\,y^\mu\Big)\Big(\sum_{j=0}^{t} g_j(x)\,y^j\Big)$$

of $AQ\cap QA$. Such an element, say $c_0(x)+c_1(x)y+c_2(x)y^2+\dots$ is contained in Q by (24) if $c_0(x)=0$ holds (which is clear) and if

$$c_1(x) = f_0(x)\,h_1(x) = k_1(x)\,g_0(1-x)\in\mathcal{R}[x]$$

satisfies (25). To show the latter, we use $h_1(x)$ and $k_1(x)$ as denoted in (26) and recall that $f_0(0)=g_0(0)=0$ holds by the definition of A. Then we obtain

$$c_1(0) = f_0(0)\alpha_1 = 0 \quad \text{and} \quad c_1(1) = \beta_1 g_0(0) = 0,$$

hence $c_1(x)=0+\bar{c}_1(x)$ satisfies (25), and Q is proved to be a quasi-ideal of A. Similar considerations show that $Ay\cap yA$ contains all elements (24) of Q such that $\alpha_1=0$ holds in (25), i.e. $Q=(y)_q$.

Finally, we have $y\in Q$ and $xy\notin Q$ by (24) and (25), and our relation (23) provides

$$xy = y(-x)+y\notin Q,$$

i.e. a formula (8) proving that Q does not satisfy the intersection property.

Remark 5.5. For all rings denoted by A in this section, each minimal quasi-ideal has the intersection property by Corollary 3.5. Hence the quasi-ideal Q considered in Lemma 5.4 can not be a minimal one, and Theorem 5.1 provides a statement parallel to Theorem 3.8 a). On the one hand, this new statement is stronger, since it presents those rings without zero divisors; on the other hand, there are no finite rings of this kind (which would be division rings and hence only contain the trivial quasi-ideals).

We further note that we could not clear whether or not the rings A of this section can be obtained as contracted semigroup algebras $\mathscr{R}_0[S^0]$, i.e. as semigroup algebras $\mathscr{R}[S]$ for a suitable semigroup $S \neq S^0$ since A has no zero divisors. A possibility to decide this question would be to answer the following problem in the positive:

Problem 5.6. If a semigroup algebra $A=\mathscr{K}[S]$ of a semigroup $S\neq S^0$ over a division ring $\mathscr{K}$ has no zero divisors, is it true that each quasi-ideal Q of A satisfies the intersection property?

§ 6. On two problems posed by L. Márki

The problems under consideration read as follows (cf. [7], § 3, p. 16):

Problem d). Does there exist a ring A with a left ideal L and a right ideal R such that RL is not a quasi-ideal of A?

Problem e). Does there exist a ring A with quasi-ideals Q_1 and Q_2 such that $Q_1 Q_2$ is not a quasi-ideal of A?

Proposition 6.1. There are even rings A without zero divisors containing a left ideal $L=(y)_l$ and a right ideal $R=(y)_r$ generated by the same element $y\in A$ such that RL is not a quasi-ideal of A.

Proof. As already mentioned in the introduction, the rings A considered in [9], Example 1 satisfy this statement for a suitable element $y\in A$. We shall show that the same holds for the subring A of each ring $\mathscr{R}[x][y]_\eta$ as introduced in

Corollary 5.3 with respect to the generating element y. Firstly, we have by (21')

$$L = (y)_l = \{\sum_i f_i(x)y^{i+1} + \zeta_1 y | f_0(0) = 0, \zeta_1 \in \langle 1 \rangle\},$$

$$R = (y)_r = \{\sum_j g_j(1-x)y^{i+1} + \zeta_2 y | g_0(0) = 0, \zeta_2 \in \langle 1 \rangle\}.$$

We want to show that $\alpha xy^2 \in A$ for each $\alpha \neq 0$ of $\mathcal{R}$ is not contained in RL. By way of contradiction, assume $\alpha xy^2 \in RL$. Since $\alpha xy^2 \neq \zeta_2 y \cdot \zeta_1 y$, this yields that αxy^2 should be a product

$$g_0(1-x)y \cdot \zeta_1 y \quad \text{or} \quad \zeta_2 y \cdot f_0(x)y = \zeta_2 f_0(1-x)y^2 \quad \text{or} \quad g_0(1-x)y \cdot f_0(x)y =$$
$$= g_0(1-x)f_0(1-x)y^2,$$

which is impossible by $g_0(1-1) = f_0(1-1) = 0$ and $\alpha 1 = \alpha \neq 0$. But then RL is not a quasi-ideal of A, since for a quasi-ideal Q of A which contains y^2 one has

$$\alpha xy^2 = y^2 \alpha x \in AQ \cap QA \subseteq Q.$$

It is clear that this solution of Problem d) is also an affirmative solution of Problem e). But now one may pose a stronger version of Problem e), namely the same question with $Q_1 = Q_2$. We present a solution by

PROPOSITION 6.2. *There are rings A without zero divisors containing a quasi-ideal Q such that Q^2 is not a quasi-ideal of A.*

PROOF. We shall show that the quasi-ideal Q of A considered in Lemma 5.4 is one of this kind. By (24), Q^2 consists of the elements

$$\left(\sum_{\nu=1}^{n} h_\nu(x)y^\nu\right)\left(\sum_{\mu=1}^{m} k^\mu(x)y^\mu\right) = \sum_{i=2}^{n+m} c_i(x)y^i$$

with $h_1(x) = \alpha_1 + \bar{h}_1(x)$ and $k_1(x) = \beta_1 + \bar{k}_1(x)$ according to (25). Hence we have

$$c_2(x) = \alpha_1\beta_1 + \alpha_1\bar{k}_1(1-x) + \bar{h}_1(x)\beta_1 + \bar{h}_1(x)\bar{k}_1(1-x),$$

which is clearly again a polynomial satisfying (25). On the other hand, αx for some $\alpha \neq 0$ of $\mathcal{R}$ is not such a polynomial. This proves that $Q^2 = Q'$ is not a quasi-ideal of A, since otherwise $y^2 \in Q'$ would imply

$$\alpha xy^2 = y^2 \alpha x \in AQ' \cap Q'A \subseteq Q'.$$

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TECHNISCHE UNIVERSITÄT CLAUSTHAL
INSTITUT FÜR MATHEMATIK
D-3392 CLAUSTHAL—ZELLERFELD
ERZSTRASSE 1
BUNDESREPUBLIK DEUTSCHLAND

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ON THE FISSILITY OF SEMIPRIMARY RINGS

DINH VAN HUYNH (Halle)[1]

We only consider associative rings. A ring A is called *fissile,* if the maximal torsion ideal of A is a ring-direct summand of A. The fissility of rings satisfying some chain conditions was investigated by several authors (cf. [1], [2], [4], [5], [6]). Artinian rings and rings with minimal condition for principal right ideals are examples of fissile rings. A ring A is called *semiprimary* if the Jacobson radical $J(A)$ of A is nilpotent and $A/J(A)$ is (right) artinian. By a famous result of H. Bass, every semiprimary ring with an identity is a ring with minimal condition for principal rigt ideals, therefore it is fissile. But obviously this result is no longer true for semiprimary rings without identity. Even there are semiprimary rings which are not fissile as one can easily see.

Let A be a semiprimary ring. Then $A/J(A)$ has an identity $\bar{e}$. By the nilpotency of $J(A)$, there is an idempotent e in A with $e\in\bar{e}$. Such an idempotent is called a *principal idempotent of* A. By $\operatorname{Ann}(e)$ we denote the set of all $x\in A$ with $xe=ex=0$. The symbols $\oplus$ and $(+)$ stand for the group-direct and ring-direct sum, respectively.

Now we begin with

LEMMA 1. *Let A be a semiprimary ring and e be a principal idempotent of A. If $A/J(A)$ is torsion free and* $\operatorname{Ann}(e)=(0)$, *then A is divisible.*

PROOF. In general, a semiprimary ring A has a decomposition

$$A = eAe\oplus(1-e)Ae\oplus(1-e)Ae\oplus(1-e)A(1-e), \tag{1}$$

where

$$eA(1-e)\overset{\text{def}}{=}\{ea-eae|a\in A\},\ (1-e)A(1-e)\overset{\text{def}}{=}\{a-ea-ae+eae|a\in A\}.$$

By assumption we get $(1-e)A(1-e)=\operatorname{Ann}(e)=(0)$, therefore

$$A = eAe\oplus eA(1-e)\oplus(1-e)Ae. \tag{2}$$

Clearly $eA(1-e)\oplus(1-e)Ae\subseteq J(A)$. Then $A/J(A)\cong eAe/J(eAe)$, so $eAe/J(eAe)$ is torsion free. Hence it is divisible. For each natural integer m there is an $f\in eAe$ and a $g\in J(eAe)=eJ(A)e$ with

$$mf = e+g. \tag{3}$$

[1] The results of this note are presented in a paper of the author "Über die Spaltbarkeit der halbprimären Ringe" (Hanoi, January 1980).

We prove first the divisibility of eAe. Let n be the nilpotent degree of $eJ(A)e$. For $n=1$, $eJ(A)e=(0)$, so eAe is divisible. Let us assume that the assertion is already proved for all natural integers k with $k<n$. Since $eAe/(eJ(A)e)^{n-1}$ satisfies the conditions of Lemma 1, $eAe/(eJ(A)e)^{n-1}$ is divisible by the induction hypothesis. For each $a\in(eJ(A)e)^{n-1}$ we get by (3) $m(fa)=(mf)a=(e+g)a=ea+$ $+ga=ea=a$, since $ga\in eJ(A)e(eJ(A)e)^{n-1}=(0)$. Hence each equation $mx=a$ with $a\in(eJ(A)e)^{n-1}$ has a solution $x=fa$ in $(eJ(A)e)^{n-1}$, i.e. $(eJ(A)e)^{n-1}$ is divisible. Thus eAe is divisible. Hence for each natural integer m there is an $f'\in eAe$ with $mf'=e$. Consequently $m(bf')=b(mf')=be=b$ for each $b\in(1-e)Ae$. This means that $x=bf'$ is a solution of the equation $mx=b$ in $(1-e)Ae$, proving the divisibility of $(1-e)Ae$. The same holds also for $eA(1-e)$. By (2), A is divisible.

LEMMA 2. *Let A be a semiprimary ring with a principal idempotent e. If* $\mathrm{Ann}(e)=(0)$, *A is fissile.*

PROOF. As is well-known, $\bar{A}\overset{\mathrm{def}}{=}A/J(A)$ is fissile, i.e. $\bar{A}=\bar{F}(+)\bar{T}$, where $\bar{T}$ is the maximal torsion ideal and $\bar{F}$ is a torsion free ideal of $\bar{A}$. The ideals $\bar{F}$ and $\bar{T}$ have identities $\bar{e}_1$ and $\bar{e}_2$, respectively. Then $\bar{e}=\bar{e}_1+\bar{e}_2$, where $\bar{e}$ is the image of e in $\bar{A}$. Since $J(A)$ is nilpotent, there exist orthogonal idempotents e_1, e_2 in A with $e=e_1+e_2$ and $e_i\in\bar{e}_i$ $(i=1,2)$. For e_2 there is a natural integer n with $ne_2\in J(A)$. Hence there exists a natural integer k with $(ne_2)^k=0$, i.e. $n^ke_2=0$.

Since by assumption $\mathrm{Ann}(e)=(0)$, A has a group-direct decomposition (2). By $e_1Ae_1/J(e_1Ae_1)\cong\bar{F}$ and by Lemma 1, e_1Ae_1 is divisible and torsionfree, since otherwise, e_1Ae_1 contained an additive quasicyclic subgroup U with $e_1Ae_1U=(0)$, in particular $e_1U=(0)$, a contradiction. Let $a\in(1-e)Ae$. Then there is a $b\in A$ with $a=be-ebe=b(e_1+e_2)+(e_1+e_2)b(e_1+e_2)=be_1+be_2-e_1be_1-e_2be_2-e_1be_2-e_2be_1$. By Lemma 1 there is an f in e_1Ae_1 with $n^kf=e_1$. Hence $e_1be_2=(n^kf)be_2=fb(n^ke_2)=$ $=0$. Similarly $e_2be_1=0$. From this and (2) we get

$$A=e_1Ae_1\oplus e_1A(1-e_1)\oplus(1-e_1)Ae_1\oplus e_2\oplus e_2Ae_2\oplus e_2A(1-e_2)\oplus(1-e_2)Ae_2. \tag{4}$$

By Lemma 1, $F\overset{\mathrm{def}}{=}e_1Ae_1\oplus e_1A(1-e_1)\oplus(1-e_1)Ae_1$ is a torsionfree and divisible subring of A and obviously $T\overset{\mathrm{def}}{=}e_2Ae_2\oplus e_2A(1-e_2)\oplus(1-e_2)Ae_2$ is the maximal torsion ideal of A. Since clearly $TF=FT=(0)$, we get $A=F(+)T$.

COROLLARY 3. *Every semiprimary ring with a right (or left) identity is fissile.*

Now we can formulate the main result of this paper.

THEOREM. *Let A be a semiprimary ring with a principal idempotent e. Then A contains a fissile subring B with $A=B\oplus\mathrm{Ann}(e)$, where $\mathrm{Ann}(e)$ is contained in $J(A)$.*

PROOF. A has a direct decomposition (1), hence $A=B\oplus\mathrm{Ann}(e)$ with $\mathrm{Ann}(e)\subseteq J(A)$. By Lemma 2, B is fissile.

COROLLARY 4. *Let A be a semiprimary ring with a principal idempotent e. Then*
(a) *If $\mathrm{Ann}(e)$ is torsion, then A is fissile.*

(b) *If* $J(A)$ *is an artinian A-right (or left) module, i.e.* A *is right (or left) artinian, then* A *is fissile* (cf. [6]).

PROOF. By Theorem, (a) is trivial.

(b) By $\mathrm{Ann}\,(e)\subseteqq A(1-e)\subseteqq J(A)$ and by [3], $\mathrm{Ann}\,(e)$ is contained in a right artinian nilpotent ring $A(1-e)$, which is (as well-known) a ring with minimal condition on additive subgroups. Such rings are torsion. Hence (b) follows from (a).

REMARK. In a discussion Dr. Widiger told me that he has obtained also the same results as in my paper but with other methods.

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SEKTION MATHEMATIK DER
MARTIN-LUTHER-UNIVERSITÄT HALLE—WITTENBERG
4020 HALLE
UNIVERSITÄTSPLATZ $^{\circ}$
GDR

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MULTIPLICATIVE FUNCTIONS WITH REGULARITY PROPERTIES. II

I. KÁTAI (Budapest), corresponding member of the Academy

1.

Let $\mathcal{M}$ and $\mathcal{M}^*$ be the set of complex valued multiplicative and completely multiplicative functions, respectively. Let $\mathcal{L} \subseteqq \mathcal{M}$ denote the set of those functions $f(n)$ for which

$$\sum_{n=1}^{\infty} \frac{|f(n)|}{n} < \infty \tag{1.1}$$

holds.

The letters p, q, π denote prime numbers, P, Q prime-powers.

Let

$$R(f, p) = \sum_{\alpha=1}^{\infty} \frac{|f(p^\alpha)|}{p^\alpha}. \tag{1.2}$$

It is obvious that (1.1) is equivalent with

$$\sum_p R(f, p) < \infty. \tag{1.3}$$

Our main purpose in this paper is to prove the following

THEOREM 1. *Let $f, g \in \mathcal{M}$. Assume that*

$$\sum_{n=1}^{\infty} \frac{|g(n+1) - f(n)|}{n} < \infty. \tag{1.4}$$

Then f and $g \in \mathcal{L}$, or $f(n) = g(n) = n^{\sigma+i\tau}$, $0 \leqq \sigma < 1$.

Without any important change in the proof we could prove the following

THEOREM 2. *Let $a(n), b(n) \in \mathcal{M}$. Assume that*

$$\sum_{n=1}^{\infty} |b(n+1) - a(n)| < \infty.$$

There are two possibilities: either

$$\sum_{n=1}^{\infty} |a(n)| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |b(n)| < \infty \tag{1}$$

or

$$a(n) = b(n) = n^{\sigma+i\tau}, \ -1 < \sigma < 0. \tag{2}$$

The special case $f = g$ in Theorem 1 seems to be nontrivial too.

THEOREM 3. *Let* $f\in\mathcal{M}$, *and*

$$\sum_{n=1}^{\infty}\frac{|f(n+1)-f(n)|}{n}<\infty. \tag{1.5}$$

Then $f\in\mathcal{L}$, *or* $f(n)=n^{\sigma+i\tau}$, $0<\sigma<1$.

We shall deduce Theorem 1 from Theorem 3 and from the following

THEOREM 4. *Assume that* $f, g\in\mathcal{M}$, *and* (1.4) *holds. If* $f, g\notin\mathcal{L}$, *then* $f(n)=g(n)$ *for every* n.

To prove Theorem 3 it is convenient to assume that $f\in\mathcal{M}^*$. Therefore we prove

THEOREM 5. *Let* k *be a positive integer,* $f\in\mathcal{M}$ *and*

$$\sum_{n=1}^{\infty}\frac{1}{n}|f(n+K)-f(n)|<\infty. \tag{1.6}$$

Assume that $f\in L$. *Then for each prime* p *coprime to* K *we have* $f(p^{\nu})=f(p)^{\nu}$ ($\nu=1, 2, \ldots$).

Especially for $K=1$ we have $f\in\mathcal{M}^*$.

We remark that J. Mauclaire and Leo Murata [2] proved the following assertion earlier: if $|f(n)|\equiv 1$, $f\in\mathcal{M}$, and

$$\frac{1}{x}\sum_{n<x}|f(n+1)-f(n)|\to 0,$$

then $f\in\mathcal{M}^*$.

Now we do not try to generalize this result since Theorem 5 has only an auxiliary character.

2. Proof of Theorem 5

We shall use the notation E, I, introduced in [1]. So (1.6) can be written in the form

$$\sum_{n}\frac{1}{n}|(E^k-I)f(n)|<\infty. \tag{2.1}$$

Let $m\geqq 1$. Since z^K-1 is a divisor of $z^{Km}-1$, therefore

$$\sum_{n}\frac{1}{n}|(E^{Km}-I)f(n)|<\infty,$$

consequently

$$\sum_{n}\frac{1}{mn}|(E^{Km}-I)f(mn)|<\infty. \tag{2.2}$$

Let

$$\begin{aligned}\Delta(n, m) &= (E^{Km}-I)f(mn)-f(m)(E^K-I)f(n)= \\ &= f(m(n+K))-f(m)f(n+K)-(f(mn)-f(m)f(n)).\end{aligned} \tag{2.3}$$

Then, by (2.1), (2.2)

$$\sum_{n=1} \frac{|\Delta(n;\, m)|}{n} < \infty. \tag{2.4}$$

Let $m=p$, $(p,K)=1$, $n=p^\nu u$, $(u,Kp)=1$. Then $(m,n+K)=1$, and so

$$\Delta(n,m) = -|f(p^{\nu+1})-f(p)f(p^\nu)|f(u). \tag{2.5}$$

Assume that $f(p^{\nu+1})\neq f(p)f(p^\nu)$. Then, from (2.4)

$$\sum_{(u,Kp)=1} \frac{|f(u)|}{u} < \infty. \tag{2.6}$$

Hence we deduce that $f\in\mathcal{L}$, and by this the proof will be completed.

Since $(p^\alpha+K, Kp)=1$ for $\alpha\geqq 1$, therefore by (1.6) and (1.10)

$$\sum_{\alpha=1}^{\infty} \frac{|f(p^\alpha)|}{p^\alpha} \leqq \sum_{\alpha=1}^{\infty} \frac{1}{p^\alpha}|f(p^\alpha+K)-f(p^\alpha)| + \sum_{\alpha=1}^{\infty} \frac{1}{p^\alpha}|f(p^\alpha+K)| < \infty.$$

Consequently

$$\sum_{(u,K)=1} \frac{|f(u)|}{u} < \infty. \tag{2.7}$$

Let q be a prime factor of K, $q^\beta \| K$. Let $K=q^\beta K_1$. We have

$$\sum_{\gamma=\beta+1}^{\infty} \frac{|f(q^\gamma)|}{q^\gamma} \leqq \sum_{\gamma=\beta+1}^{\infty} \frac{1}{q^\gamma}|f(q^\gamma+K)-f(q^\gamma)| + \frac{|f(q^\beta)|}{q^\beta} \sum_{\gamma=\beta+1}^{\infty} \frac{|f(q^{\gamma-\beta}+K_1)|}{q^\gamma}.$$

The last sum is finite, since $(q^{\gamma-\beta}+K_1, K)=1$. Consequently

$$\sum_{q|K} R(f,q) < \infty.$$

This, by (2.7) gives the desired result.

3. Proof of Theorem 3

We may assume that $f\notin\mathcal{L}$ and so that $f\in\mathcal{M}^*$. Let

$$\varepsilon(n+1) = |f(n+1)-f(n)|,$$

$q>2$ be an arbitrary integer, $\mathcal{A}_M=\mathcal{A}_M^{(q)}$ $(M=1,2,\dots)$ be the set of integers in $[q^{M-1}, q^M)$.

After q being fixed, we define for every $N>1$ the sequence

$$N_j = \left[\frac{N}{q^j}\right] \quad (i=0,1,2,\dots).$$

By this we have

$$(N=)\,N_0 = qN_1+a_0, \quad N_1 = qN_2+a_1, \dots$$

where

$$a_j\in\{0,1,\dots,q-1\} = \mathcal{A}_1^{(q)}.$$

If $N_0 \in \mathscr{A}_M^{(q)}$, then $N_j \in \mathscr{A}_{M-j}^{(q)}$ $(j=1, \ldots, M-1)$. Fixing any integer $k \in \mathscr{A}_{M-j}^{(q)}$, $N_j = k$ occurs q^j times when N_0 runs over $\mathscr{A}_M^{(q)}$.

Let $K = \max_{j<q-1} |f(j)|$. It is obvious that

$$|f(N_j)-f(q)f(N_{j+1})| \leqq \varepsilon(qN_{j+1}+1)+\ldots+\varepsilon(N_j), \tag{3.1}$$

and so

$$\sum_{N_j \in \mathscr{A}_{M-j}^{(q)}} |f(N_j)-f(q)f(N_{j+1})| \leqq q \sum_{n \in \mathscr{A}_{M-j}^{(q)}} \varepsilon(n). \tag{3.2}$$

Furthermore

$$|f(N_0)-f(q)^{M-1}f(N_{M-1})| \leqq \sum_{j=0}^{M-2} |f(N_j)-f(q)f(N_{j+1})|\,|f(q)|^j. \tag{3.3}$$

First we prove that $|f(n)| \geqq 1$ holds for every n. Assume in the contrary that $f(q) = \Lambda$, $|\Lambda| < 1$. Then, from (3.3)

$$\sum_{N_0 \in \mathscr{A}_M} |f(N_0)-\Lambda^{M-1}f(N_{M-1})| < \sum_{j=0}^{M-2} \sum_{N_j \in A_{M-j}} q^j |\Lambda|^j |f(N_j)-\Lambda f(N_{j+1})|.$$

Since $|f(N_{M-1})| \leqq K$, we have

$$q^{-M} \sum_{N_0 \in \mathscr{A}_M} |f(N_0)| \leqq K|\Lambda|^{M-1} + \sum_{j=0}^{M-2} |\Lambda|^j \sum_{n \in \mathscr{A}_{M-j}} \frac{\varepsilon(n)}{n}.$$

Summing up for $M = 1, 2, \ldots$, we get

$$\sum_{M \geqq 1} q^{-M} \sum_{n \in \mathscr{A}_M} |f(n)| \leqq K \sum_{M \geqq 1} |\Lambda|^M + \sum_{t \geqq 1} \sum_{n \in A_t} \frac{\varepsilon(n)}{n} \left\{ \sum_{M=t}^{\infty} |\Lambda|^{M-t} \right\} \ll$$

$$\ll \frac{K}{1-|\Lambda|} + \frac{1}{1-|\Lambda|} \sum_{n \geqq 1} \frac{\varepsilon(n)}{n} < \infty,$$

that is $f \in \mathscr{L}$.

Let $|f(n)| = e^{u(n)}$, where $u(n)$ is a completely additive function. It is non-negative, since $|f(n)| \geqq 1$. Since $u(n) \geqq 0$, $u(n+1) \geqq 0$, therefore

$$|\Delta u(n)| = |u(n+1)-u(n)| \leqq \big| |f(n+1)| - |f(n)| \big|.$$

Furthermore $\big| |f(n+1)| - |f(n)| \big| \leqq |f(n+1)-f(n)|$, and so by (1.5) we have

$$\sum_{n=1}^{\infty} \frac{|\Delta u(n)|}{n} < \infty,$$

and consequently

$$\frac{1}{x} \sum_{n \leqq x} |\Delta u(n)| \to 0. \tag{3.4}$$

In [3] it was proved that (3.4) involves that $u(n)$ is a constant multiple of $\log n$, $u(n) = \sigma \log n$. So $|f(n)| = n^{\sigma}$. Since

$$\sum_n \frac{1}{n} |(n+1)^{\sigma} - n^{\sigma}| < \infty$$

holds only for $\sigma < 1$, we may assume that $0 < \sigma < 1$.

Let now $t(n)$ be defined by $t(n)=f(n)n^{-\sigma}$. Then $|t(n)|=1$, $t\in\mathcal{M}^*$. Furthermore, from (1.5) it follows immediately that

$$\sum_n \frac{1}{n}|t(n+1)-t(n)|<\infty.$$

Now it has remained to prove the theorem under the condition $|f(n)|=1$. We assume that $|f(n)|=1$ $(n=1, 2, \dots)$. Let

$$\varrho(m)=\sum_{n\geqq m}\frac{\varepsilon(n)}{n}.$$

Let $q>1$ be given, and $N_0, N_1, \dots$ be defined as earlier. Let $\nu<M$. We start from the inequality

$$\text{(3.5)}\qquad \sum_{N_0\in\mathcal{A}_M^{(q)}}|f(N_0)-f(q)^\nu f(N_\nu)|\leqq\sum_{l=0}^{\nu-1}q^l\sum_{N_l\in\mathcal{A}_{M-l}}|f(N_l)-f(q)f(N_{l+1})|\leqq$$

$$\leqq\sum_{l=0}^{\nu-1}q^{l+1}\sum_{n\in\mathcal{A}_{M-l}}\varepsilon(n),$$

an obvious consequence of (3.1), (3.2).

Hence we get immediately that

$$\text{(3.6)}\qquad q^{-M}\sum_{N_0\in\mathcal{A}_M^{(q)}}|f(N_0)-f(q)^\nu f(N_\nu)|<q\varrho(q^{M-\nu}).$$

Let now assume that q_1, q_2 are positive integers, $q_2^2<q_1$. Let H_1 be a large fixed integer, $M_1>H_1$. It is clear that $\mathcal{A}_{M_2}^{(q_2)}\subseteq\mathcal{A}_{M_1}^{(q_1)}$ for a suitable M_2. After fixing H_1, let ν_1, ν_2 be a pair of large integers such that

$$\text{(3.7)}\qquad \left|\frac{q_2^{\nu_2}}{q_1^{\nu_1}}-1\right|<q_1^{-H_1}$$

holds. Now we define $M_1=\nu_1+H_1$ and M_2 by the property $\mathcal{A}_{M_2}^{(q_2)}\subseteq\mathcal{A}_{M_1}^{(q_1)}$. Let $H_2=M_2-\nu_2$. It is clear that $H_2\to\infty$ whenever $H_1\to\infty$.

Now we rewrite (3.6) with $q=q_1$ and $q=q_2$. We get

$$\text{(3.8)}\qquad \sum_{N_0\in\mathcal{A}_{M_1}^{(q_1)}}|f(N_0)-f(q_1)^{\nu_1}f(N_{\nu_1})|\ll q_1^{M_1}\varrho(q_1^{H_1-1}),$$

$$\text{(3.9)}\qquad N_{\nu_1}=\left[\frac{N_0}{q^{\nu_1}}\right],$$

$$\text{(3.10)}\qquad \sum_{N_0\in\mathcal{A}_{M_2}^{(q_2)}}|f(N_0)-f(q_2)^{\nu_2}f(N_{\nu_2})|\ll q_2^{M_2}\varrho(q_2^{N_2-1}),$$

$$\text{(3.11)}\qquad n_{\nu_2}=\left[\frac{N_0}{q_2^{\nu_2}}\right].$$

Each $N_0\in\mathcal{A}_{M_2}^{(q_2)}$ can be written in the form

$$N_0=hq_2^{\nu_2}+b,\quad 0\leqq b<q_2^{\nu_2},\quad h\in\mathcal{A}_{q_2}^{(H_2)}.$$

It is obvious that $n_{\nu_2}=h$. Furthermore,

$$N_{\nu_1}=\left[\frac{hq^{\nu_2}+b}{q_1^{\nu_1}}\right],$$

and so $|N_{\nu_1}-n_{\nu_2}|\leqq 3$, if H_1, H_2, ν_1, ν_2 are greater than a positive constant. Assuming this, we have

$$(3.12)\qquad \sum_{N_0\in\mathcal{A}_{M_2}^{(q_2)}}|f(N_\nu)-f(n_\nu)|\leqq\sum_b\sum_{h\in\mathcal{A}_{M_2}^{(q_2)}}(\varepsilon(h-3)+\ldots+\varepsilon(h+3))\ll$$

$$\ll q_2^{M_2}\sum_{h>q_2^{H_2-1}}\frac{\varepsilon(h)}{h}\ll q_2^{M_2}(q_2^{H_2-1}).$$

From (3.8), (3.10) we get

$$(3.13)\qquad \sum_{N_0\in\mathcal{A}_{M_2}^{(q_2)}}|f(q_1)^{\nu_1}f(N_\nu)-f(q_2)^{\nu_2}f(n_\nu)|\ll q_1^{M_1}\varrho(q_1^{H_1-1})+q_2^{M_2}\varrho(q_2^{H_2-1}).$$

Hence, by (3.12),

$$(3.14)\qquad |f(q_1)^{\nu_1}-f(q_2)^{\nu_2}|\cdot\Big(\sum_{N_0\in\mathcal{A}_{M_2}^{(q_2)}}1\Big)\ll q_1^{M_1}\varrho(q_1^{H_1-1})+q_2^{M_2}\varrho(q_2^{H_2-1}).$$

Since for fixed H_1, $q_2^{M_2}\ll q_1^{M_1}$, $q_1^{M_1}\ll q_2^{M_2}$, we have

$$(3.15)\qquad |1-\overline{f(q_1)}^{\nu_1}f(q_2)^{\nu_2}|\ll\varrho(q_1^{N_1-1})+\varrho(q_2^{H_2-1}).$$

We can see that the right hand side tends to zero as $H_1\to\infty$.

For a real x let $\|x\|$ denote its distance from the nearest integer. Let $f(q_l)=$ $=e^{2\pi i A_l\log q_l}$ $(l=1,2)$.

We have proved the fulfilment of the following assertion: If $q_2^2<q_1$ and $(\nu_1^{(j)},\nu_2^{(j)})$ $(j=1,2,\ldots)$ is an arbitrary sequence of pairs of positive integers tending to infinity such that $\nu_1^{(j)}\log q_1-\nu_2^{(j)}\log q_2\to 0$, then

$$\|\nu_1^{(j)}A_1\log q_1-\nu_2^{(j)}A_2\log q_2\|\to 0.$$

Let us assume that $f(2)=1$.

Now we choose $q_2=2$. Let $4<q$. Then we may put $A_2=0$. Let $\alpha=\dfrac{\log q_1}{\log 2}$. Assume that q_1 is not a power of 2. Then α is an irrational number. Furthermore $\|\nu_1^{(j)}\alpha\|\to 0$ involves that $\|\nu_1^{(j)}A_1(\log 2)\alpha\|\to 0$.

In [1] we proved the following assertion.

(A): Let α be an irrational number, β be an arbitrary real number. Assume that for every sequence $m_1<m_2<\ldots$ of integers satisfying $\|m_j\alpha\|\to 0$ the relation $\|m_j\beta\|\to 0$ holds. Then β is an integer, or $\beta=k\alpha$, with a suitable integer k.

That is $A_1\log q_1=$integer, or $A_1\log 2=$integer.

In the second case

$$(3.16)\qquad A_1=\frac{k(q_1)}{\log 2},\quad k(q_1)=\text{integer}.$$

Since $A_1=0$ can be stated in the first case, therefore (3.16) holds if $q_1<4, q_1\neq 2^t$. This representation is good for $q_1=2^t$, if we put $k(2^t)=0$.

By using that $f(mn)=f(m)f(n)$ holds for every m, n, we deduce that

$$k(mn)\frac{\log mn}{\log 2} \equiv k(m)\frac{\log m}{\log 2}+k(n)\frac{\log n}{\log 2} \pmod 1$$

for odd integers m, n. Hence

$$(mn)^{k(mn)} = m^{k(m)}\cdot n^{k(n)}\cdot 2^{\mathcal{I}_{m,n}}$$

with a suitable integer $\mathcal{I}_{m,n}$. By using the unicity of prime-decomposition, we get $\mathcal{I}_{m,n}=0$, and so $k(mn)=k(m)=k(n)$ if m and n are coprime odd integers, $m,n>4$. Consequently $k(n)=L=$constant for odd $n>4$. So we have $f(n)=e^{2\pi i L\frac{\log n}{\log 2}}$, if $(n,2)=1$, $n>4$. Since $f(3)=f(15)\,\overline{f(5)}=e^{2\pi i L\frac{\log 3}{\log 2} i}$, therefore $f(n)=e^{2\pi i\tau\log n}$ with $\tau=L/\log 2$. But this holds for $f(2^t)$, since $\tau\log 2^t\equiv 0 \pmod 1$, and $f(2^t)=0$.

The assumption $f(2^t)=1$ is not a restriction. If this condition does not hold, $f(2)=e^{2\pi i\lambda}$, then we consider the function $\tilde f(n)=f(n)e^{-2\pi\frac{\lambda\log n}{\log 2}i}$. I twill suffice the conditions of the theorem, and $\tilde f(2)=1$.

This completes the proof of the theorem.

4. Proof of Theorem 4. Preliminary lemmas

Given a subset $\mathcal{S}$ of natural numbers, we shall write

$$(4.1)\qquad \mathcal{F}(n|\mathcal{S})=\sum_{n\in\mathcal{S}}\frac{f(n)}{n};\quad \mathcal{G}(n|\mathcal{S})=\sum_{n\in\mathcal{S}}\frac{g(n)}{n}$$

permitting that the series do not converge. We shall say that $\mathcal{F}$ and $\mathcal{G}$ are finite, if the series are absolutely convergent.

The following assertions are obvious consequences of (1.4).

LEMMA 1. (1) *If $\mathcal{F}(n|\mathcal{S})$ is finite, then so is $\mathcal{G}(n|\mathcal{S}')$, where $\mathcal{S}'=\{n|n-1\in\mathcal{S}\}$. If $\mathcal{G}(n|\mathcal{S})$ is finite, then so is $\mathcal{F}(n|\mathcal{S}')$, where $\mathcal{S}'=\{n|n+1\in\mathcal{S}\}$.*

(2) *If $\mathcal{F}(n|\mathcal{S})$ is finite, $\mathcal{S}_Q=\{m|mQ\in\mathcal{S},\ (m,Q)=1\}$ and $f(Q)\neq 0$, then so is $\mathcal{F}(n|\mathcal{S}_Q)$. The same assertion is true for $\mathcal{G}(n|\mathcal{S})$, when $g(Q)\neq 0$.*

LEMMA 2. *Assume that $f,g\in\mathcal{M}$, and (1.4) holds, furthermore that $f(2)\neq 0$, $g(2)\neq 0$.*

If for a suitable odd integer N

$$(4.2)\qquad \mathcal{F}(r|(r,2N)=1)\ \text{ is finite,}$$

or

$$(4.3)\qquad \mathcal{G}(r|(r,2N)=1)\ \text{ is finite,}$$

then $f,g\in\mathcal{L}$.

PROOF. Let $N=\pi_1^{\alpha_1}\dots\pi_r^{\alpha_r}$, $\pi_1,\dots,\pi_r$ be distinct primes. Assume that (4.2) holds. We may assume that for every π_j there exists a suitable exponent $\gamma_j>0$, for which $f(\pi_j^{\gamma_j})\neq 0$. In the opposite case, i.e. if $f(\pi_j^{\alpha})=0$ $(\alpha=1, \ldots)$ holds for a suitable j, then we may change N by $N_1=\pi_j^{-\alpha_j}N$ in (4.2). After carrying out all the possible reductions we come to such an N satisfying the requirement. Let $\gamma_1, \ldots, \gamma_r$ be exponents such that $f(\pi_j^{\gamma_j})\neq 0$ $(j=1, \ldots, r)$. Let $\{i_1, \ldots, i_s\}$ be an arbitrary subset of $\{1, \ldots, r\}$. Let

$$\mathcal{K}=\{m|m=\pi_{i_1}^{\beta_{i_1}}\ldots\pi_{i_s}^{\beta_{i_s}};\ \beta_{i_1}\geqq 1, \ldots, \beta_{i_s}\geqq 1\}, \quad \mathcal{K}^*=\{m^*|m^*=m\prod_{j\neq i_l}\pi_j^{\gamma}\}.$$

First we observe that $(4m^*+1, 2N)=1$. Then $\mathcal{F}(4m^*+1|m^*\in\mathcal{K}^*)$ is finite. By using Lemma 1 and $f(2)\cdot g(2)\neq 0$ repeatedly, we get that the series $\mathcal{G}(4m^*+2|m^*\in\mathcal{K}^*), \mathcal{G}(2m^*+1|m^*\in\mathcal{K}^*), \mathcal{F}(2m^*|m^*\in\mathcal{K}^*), \mathcal{F}(m^*|m^*\in\mathcal{K}^*), \mathcal{F}(m|m\in\mathcal{K})$ are finite. Consequently $\mathcal{F}\big(r|(r, 2)=1\big)$ is finite, and so

$$\sum_{\alpha=1}^{\infty} 2^{-\alpha}|g(2^\alpha)|<\infty.$$

From Lemma 1 we get that the series $\mathcal{F}(2r-1|(r, 2)=1)$, $\mathcal{G}(2r|(r, 2)=1)$, and finally that $\mathcal{G}\big(r|(r, 2)=1\big)$ are finite, consequently $g\in\mathcal{L}$.

In the proof of the second assertion only a slight change is needed. First we observe that $\mathcal{G}(4m^*-1|m^*\in\mathcal{K}^*)$ is finite, and consider the chain of the finite series:

$$\mathcal{G}(4m^*-1|m^*\in\mathcal{K}^*), \quad \mathcal{F}(4m^*-2|m^*\in\mathcal{K}^*), \quad \mathcal{F}(2m^*-1|m^*\in\mathcal{K}^*),$$
$$\mathcal{G}(2m^*|m^*\in\mathcal{K}^*), \quad \mathcal{G}(m^*|m^*\in\mathcal{K}^*), \quad \mathcal{G}(m|m\in\mathcal{K}).$$

We can continue the proof in the same way that was used earlier. □

LEMMA 3. *Assume that* $f, g\in\mathcal{M}$, *and* (1.4) *holds, furthermore that* $f(2)\neq 0$, $g(2)\neq 0$. *Let* a *and* N *be nonzero integers,* $(a, N)=1$, N *odd. Assume that there exists a suitable integer* N^* *that contains all the prime divisors of* N *at least on the first power and does not contain any others, and that* $f(N^*)=0$. *If*

$$\mathcal{F}\big(r|r\equiv 1 \pmod 2),\ (ar+1, N)=1\big)$$

is finite, then $f, g\in\mathcal{L}$.

The same assertion holds if $g(N^*)\neq 0$ *and*

$$\mathcal{G}\big(r|r\equiv 1 \pmod 2), (ar+1, N)=1\big)$$

is finite.

PROOF. Let s be coprime to $2N^*$. Then sN^* is odd, and $(a\cdot sN^*+1, N)=1$. Then $\mathcal{F}\big(sN^*|(s, 2N^*)=1\big)$ and so $\mathcal{F}\big(s|(s, 2N^*)=1\big)$ is finite. From Lemma 2 we get the assertion. The proof of the second assertion is the same. □

LEMMA 4. *Assume that* $f, g\in\mathcal{L}$, (1.4) *holds, and* $f(2)\neq 0, g(2)\neq 0$. *Then*

$$\sum_{n\equiv 3 \pmod 8}\frac{1}{n}|Cg(n)-f(n)|<\infty;\quad C=\frac{g(4)}{g(2)f(2)}. \tag{4.4}$$

$$\sum_{n\equiv 5 \pmod 8}\frac{1}{n}|g(n)-C_1f(n)|<\infty;\quad C_1=\frac{f(4)}{g(2)f(2)}. \tag{4.5}$$

PROOF. If $g(4)=0$, then (4.4) is obvious from (1.4), similarly if $f(4)=0$, then (4.5) follows directly from (1.4).

Let $\Delta(n)=g(n+1)-f(n)$. We have

$$f(4k+1)=\frac{1}{f(2)}f(8k+2)=\frac{1}{f(2)}\big(g(8k+3)-\Delta(8k+2)\big),$$

and for $g(4)\neq 0$

$$g(4k+2)=g(2)\,g(2k+1)=\frac{g(2)}{g(4)}\big(f(8k+3)+\Delta(8k+3)\big).$$

Hence

(4.6)

$$\Delta(4k+1)=\frac{g(2)}{g(4)}f(8k+3)-\frac{1}{f(2)}g(8k+3)+\frac{g(2)}{g(4)}\Delta(8k+3)+\frac{1}{f(2)}\Delta(8k+2).$$

Similarly, if $f(4)\neq 0$, then

$$f(2k+1)=\frac{1}{f(4)}f(8k+4)=\frac{1}{f(4)}g(8k+5)-\frac{1}{f(4)}\Delta(8k+4).$$

$$f(2k+1)=\frac{1}{f(2)}f(4k+2)=\frac{1}{f(2)}\big(g(4k+3)-\Delta(4k+2)\big)=$$

$$=\frac{1}{f(2)g(2)}g(8k+6)-\frac{1}{f(2)}\Delta(4k+2)=$$

$$=\frac{1}{f(2)g(2)}f(8k+5)+\frac{1}{f(2)g(2)}\Delta(8k+5)-\frac{1}{f(2)}\Delta(4k+2),$$

and so

$$0=\frac{1}{f(4)}g(8k+5)-\frac{1}{f(2)g(2)}f(8k+5)-\frac{1}{f(4)}\Delta(8k+4)+ \tag{4.7}$$

$$+\frac{1}{f(2)}\Delta(4k+2)-\frac{1}{f(2)g(2)}\Delta(8k+5).$$

(4.4), (4.5) immediately follow from (4.6) and (4.7), we have to take into account only (1.4). □

LEMMA 5. *Assume that* $f,g\in\mathcal{M}, f(2)\neq 0,\ g(2)\neq 0,$ *and that* (1.4) *holds. Let* Q *be an odd integer.*

(1) *If* $\mathcal{G}(n|n\equiv 3 \pmod 8), (n,Q)=1)$ *is finite, then so is* $\mathcal{G}(r|r\equiv 1 \pmod 2), (4r+3,Q)=1)$. *If additionally* $g(4)\neq 0$, *then* $\mathcal{G}(r|r\equiv 1 \pmod 2), (4r-1,Q)=1)$ *is finite.*

(2) *If* $\mathcal{F}(n|n\equiv 5 \pmod 8),\ (n,Q)=1)$ *is finite, then so is* $\mathcal{F}(r|r\equiv 1 \pmod 2), (4r+1,Q)=1)$. *If additionally* $(Q,3)=1$ *and* $g(Q)\neq 0$, *then* $\mathcal{G}(t|t\equiv 1 \pmod 2), (Q,t)=1)$ *is finite.*

PROOF. All the following series are finite if so is the first one: $\mathcal{G}(n|n\equiv 3 \pmod 8), (n,Q)=1)$, $\mathcal{F}(n|n\equiv 2 \pmod 8),\ (n+1,Q)=1)$, $\mathcal{F}(n|n\equiv 1 \pmod 4), (2n+1,Q)=1)$,

$\mathcal{G}(n|n\equiv 2 \pmod 4), (2n+3, Q)=1)$, $\mathcal{G}(r|r\equiv 1 \pmod 2), (4r+3, Q)=1)$. If $g(4)\neq 0$, then $C\neq 0$ in Lemma 4, and so the series $\mathcal{F}(n|n\equiv 3 \pmod 8), (n, Q)=1)$, $\mathcal{G}(n|n\equiv 4 \pmod 8), (n-1, Q)=1)$, $\mathcal{G}(r|r\equiv 1 \pmod 2), (4r-1, Q)=1)$ are finite. By this (1) is proved.

Now we prove (2). If $\mathcal{F}(n|n\equiv 5 \pmod 8), (n, Q)=1)$ is finite, then so are the following series: $\mathcal{G}(n|n\equiv 6 \pmod 8), (n-1, Q)=1)$, $\mathcal{G}(n|n\equiv 3 \pmod 4), (2n-1, Q)=1)$, $\mathcal{F}(n|n\equiv 2 \pmod 4), (2n+1, Q)=1)$, $\mathcal{F}(r|r\equiv 1 \pmod 2), (4r+1, Q)=1)$. Consequently $\mathcal{G}(s|s\equiv 0 \pmod 2), (4s-3, Q)=1)$ is finite. Let us consider only the subset $s=2Qt$, $(t, 2Q)=1$. We get $(4s-3, Q)=1$, if $(Q, 3)=1$. Since $g(Q)\neq 0$, we get that $\mathcal{G}(t|t\equiv 1 \pmod 2), (Q, t)=1)$ is finite. Recalling Lemma 2, the proof is complete. □

5. Proof of Theorem 4 under the condition $f(2)\neq 0$, $g(2)\neq 0$, $f(4)\neq 0$, $g(4)\neq 0$

Let us assume in this section that the conditions of Theorem 4 as well as $f(2)\neq 0$, $g(2)\neq 0$, $f(4)\neq 0$, $g(4)\neq 0$ hold, furthermore that $f, g\notin \mathcal{L}$.

We define the function $H(n)$ by $f(n)/g(n)$ if the ratio has meaning. We shall say that a prime power Q is irregular, if $f(Q)\neq g(Q)$.

In our case $C\neq 0$, $C_1\neq 0$.

We shall reach our aim by proving a sequence of auxiliary assertions that we denote by (a), (b), etc.

(a) *If* $N\equiv 1 \pmod 8$ *and* $f(N)=0$, *then* $g(N)=0$, *and vice versa.*

PROOF. Assume that $g(N)=0$ and $f(N)\neq 0$. Then $\mathcal{F}(n|n\equiv 5 \pmod 8), (n, N)=1)$ is finite, and so from (2) in Lemma 5 we get that so is $\mathcal{F}(r|r\equiv 1 \pmod 2), (4r+1, N)=1)$. The conditions of Lemma 3 are valid, since $f(N)\neq 0$, so $f, g\in \mathcal{L}$.

Assume now that $f(N)=0$ and $g(N)\neq 0$. Then by (1) in Lemma 5, $\mathcal{G}(r|r\equiv 1 \pmod 2), (4r-1, Q)=1)$ is finite, whence by Lemma 3 $(g(N)\neq 0)$ we get immediately that $f, g\in \mathcal{L}$. □

(b) *If* $N\equiv 1 \pmod 8$, *then* $f(N)=g(N)$.

PROOF. We may assume that $g(N)\neq 0$. Let us assume that $H(N)\neq 1$. Starting from (4.4),

$$\sum_{\substack{Nn\equiv 3 \pmod 8 \\ (n, N)=1}} \frac{1}{n} |Cg(N)g(n)-f(N)f(n)| < \infty,$$

and so

$$\sum_{\substack{n\equiv 3 \pmod 8 \\ (n, N)=1}} \frac{1}{n} \left| \frac{C}{H(N)} g(n)-f(n) \right| < \infty.$$

Comparing this with (4.4), we get that $\mathcal{G}(n|n\equiv 3 \pmod 8), (n, N)=1)$ is finite. Hence, by Lemma 5 $\mathcal{G}(r|r\equiv 1 \pmod 2), (4r-1, N)=1)$ is finite, and so by Lemma 3 $g, f\in \mathcal{L}$ $(g(N)=0!)$. □

(c̃) *If* P *is a prime-power,* $P\equiv -1 \pmod 8$, *then* $f(P)=0$ *involves that* $g(P)=0$, *and vice versa.*

PROOF. Assume that $g(P)=0, f(P)\neq 0$. If $Pn\equiv 5 \pmod 8$, then $n\equiv 3 \pmod 8$. Let us substitute n by Pn in (4.5), and sum up for $n\equiv 3 \pmod 8$, $(n, P)=1$. Since

$g(Pn)=0$, we get that $\mathcal{F}(n|n\equiv 3 \pmod 8), (n,P)=1)$ is finite, and by Lemma 5 so is $\mathcal{G}(r|r\equiv 1 \pmod 2), (4r-1,P)=1)$. Consequently $\mathcal{F}(s|s\equiv 0 \pmod 2), (4s-3,P)=1)$ is finite. Extending the summation only for $2\|s$, we get that $\mathcal{F}(t|t\equiv 1 \pmod 2), (8t-3,P)=1)$ is finite. Since P is a prime-power, $\equiv -1 \pmod 8$, therefore $(P,3)=1$. Now we put tP into the place of t in the last series. The condition $(8(tP)-3,P)=1$ holds automatically, and so by $f(P)\neq 0$ we get that $\mathcal{F}(t|(t,2P)=1)$ is finite. From Lemma 2 we get that $f, g\in\mathcal{L}$.

Assume now that $f(P)=0$ and $g(P)\neq 0$. If $n\equiv 3 \pmod 8$, $(n,P)=1$, then $Pn\equiv 5 \pmod 8$ and $f(Pn)=0$. From (4.5) we get that $\mathcal{G}(n|n\equiv 3 \pmod 8), (n,P)=1)$ is finite. From Lemma 5 (see (1)) and Lemma 3, $g, f\in\mathcal{L}$. □

(c) *If* $N\equiv -1 \pmod 8$ *and* $f(N)g(N)\neq 0$, *then* $f(N)=g(N)$.

Proof. Assume that $f(N)g(N)\neq 0$. Replacing n by nN in (4.5), we get

$$\sum_{\substack{n\equiv 3 \pmod 8\\ (n,N)=1}} \frac{1}{n}\,|g(N)g(n)-C_1 f(N)f(n)| < \infty, \tag{5.1}$$

which by (4.4), involves

$$\sum_{\substack{n\equiv 3 \pmod 8\\ (n,N)=1}} \left|\frac{1}{C_1 H(N)}-C\right|\cdot\frac{1}{n}\,|g(n)| < \infty.$$

Hence, by Lemmas 5 and 3 we deduce immediately that

$$H(N)=\frac{1}{CC_1}. \tag{5.2}$$

Now we consider the inequality

$$\sum_{\substack{n\equiv 5 \pmod 8\\ (n,N)=1}} \frac{1}{n}\,|Cg(N)g(n)-f(N)f(n)| < \infty, \tag{5.3}$$

whence, after comparing it with (4.5),

$$\sum_{\substack{n\equiv 5 \pmod 8\\ (n,N)=1}} \left|C_1-\frac{H(N)}{C}\right|\cdot\frac{1}{n}\,|f(n)| < \infty.$$

Hence, by Lemmas 5 and 3,

$$H(N)=CC_1. \tag{5.4}$$

(5.2) and (5.4) imply $CC_1=\pm 1$. The case $H(N)=CC_1=-1$ is impossible. Indeed, assuming that $f(N)=-g(N)$, from (5.1), by repeating the above argument we deduce that $1+C_1=0$, and similarly, from (5.3) that $C+1=0$. But this leads to $CC_1=1$, a contradiction. □

(d) *Let* $P\equiv l \pmod 8$ *be a prime-power*, $l\equiv 3$ *or* $5 \pmod 8$. *If* $g(P)=0$, *then* $f(P)=0$.

PROOF. Assume in the contrary, that $f(P)\neq 0, g(P)=0$. If $P\equiv 3 \pmod 8$, then by (4.4), putting there Pn into the place of n, we deduce that

$$\mathscr{F}(n|n \equiv 1 \pmod 8), \ (n, P) \equiv 1) \text{ is finite.} \tag{5.4}$$

For $P\equiv 5 \pmod 8$ we consider (4.5) instead of (4.4) and get (5.4) too. From (5.4) it follows that the series $\mathscr{G}(n|n\equiv 2 \pmod 8), (n-1, P)=1)$, $\mathscr{G}(n|n\equiv 1 \bmod 4)$, $(2n-1, P)=1)$, $\mathscr{F}(m|m\equiv 0 \pmod 4), (2m+1, P)=1)$ are finite. Summing in the last sum only for $2^2\|m$, we get that $\mathscr{F}(r|r\equiv 1 \pmod 2), (8r+1, P)=1)$ is finite. Since $f(P)\neq 0$, Lemma 3 gives that $f\in\mathscr{L}$. □

(e) *If* $f(2^\alpha)=0$ *for every* $\alpha\geqq 3$, *then* $f\in\mathscr{L}$.

PROOF. By using (a) we get that all the series $\mathscr{G}(n|n\equiv 1 \pmod 8))= =\mathscr{F}(n|n\equiv 1 \bmod 8))$, $\mathscr{G}(n|n\equiv 2 \pmod 8))$, $\mathscr{G}(n|n\equiv 1 \pmod 4))$, $\mathscr{F}(n|n\equiv 0 \pmod 4))$, $\mathscr{F}(r|(r, 2)=1)$ are absolutely convergent. So $f\in\mathscr{L}$. □

(f) *Let* $P\equiv 3$ *or* $5 \pmod 8$ *be a prime-power*, $f(P)\neq 0$. *Then* $g(P)=C_1 f(P)$ *for* $P\equiv 5 \pmod 8$, *and* $Cg(P)=f(P)$ *for* $P\equiv 3 \pmod 8$.

PROOF. We change n by Pn in (4.4) and use (a). We get

$$\sum_{\substack{n\equiv 1 \pmod 8\\ (n,P)=1}} |Cg(P)-f(P)|\cdot\frac{1}{n}|f(n)| < \infty \quad (P\equiv 3 \pmod 8)),$$

$$\sum_{\substack{n\equiv 1 \pmod 8\\ (n,P)=1}} |g(P)-C_1 f(P)|\cdot\frac{1}{n}|f(n)| < \infty \quad (P\equiv 5 \pmod 8)).$$

If one of the assertions does not hold, then $\mathscr{F}(n|n\equiv 1 \pmod 8), (n, P)=1)$ is finite. Repeating the argument that was used by the proof of (e), we get that $f, g\in\mathscr{L}$. □

Let $\mathscr{I}_l$ be the set of irregular prime powers in the residue class $\equiv l \pmod 8$. As we have seen before, $\mathscr{I}_1$ and $\mathscr{I}_{-1}$ are empty.

Let P_1 be an irregular odd prime power in $\mathscr{I}_l$ such that

$$g(P_1)\neq 0, \quad f(P_1)=0. \tag{5.5}$$

Let $n\equiv l \pmod 8), (n, P_1)=1$. Then $Pn\equiv 1 \pmod 8)$, $g(Pn)=f(Pn)$, so $f(n)= =g(n)=0$ for every $n\equiv l \pmod 8)$, $(n, P)=1$.

If $l=3$, then $\mathscr{G}(n|n\equiv 3 \pmod 8), (n, P_1)=1)$ is finite, and by Lemmas 5 and 3, $g\in\mathscr{L}$.

Let $l=5$. Then $P_1\neq 3^\gamma$, $(P, 3)=1$, furthermore $f(n)=0$ if $n\equiv 5 \pmod 8)$, $(n, P_1)=1$, consequently $\mathscr{F}(n|n\equiv 5 \pmod 8), (n, P_1)=1)$ is finite. So, by Lemma 6, $\mathscr{G}(t|(2P_1, t)=1)$ is finite, which by Lemma 2 leads to $g, f\in\mathscr{L}$.

So we proved that (5.5) cannot occur, if P_1 is an odd prime power. □

(g) *If* $f(n)=g(n)=0$ *holds for every* $n\equiv -1 \pmod 8)$, *then* $f, g\in\mathscr{L}$.

PROOF. Since $g(n)=0$ for every $n\equiv 7 \pmod 8)$, therefore the series $\mathscr{F}(n|n\equiv 6 \pmod 8))$, $\mathscr{F}(n|n\equiv 3 \pmod 4))$, $\mathscr{G}(n|2^2\|n)$, $\mathscr{G}(r|(r, 2)=1)$ are finite. □

(h) *We have* $CC_1=1$.

We may assume that there exists an $N\equiv -1 \pmod 8$, for which $f(N)\neq 0$, (c) and (5.4) give (h). □

(i) *Assume that* f *and* g *are nonnegative functions, and that there exists an irregular odd prime power* P. *Then for every prime power* $Q\equiv 3$ *or* $5 \pmod 8$, *if* $(Q, P)=1$, *then* $g(Q)=f(Q)=0$.

PROOF. Assume that $P\equiv Q \pmod 8$ and that $g(Q)\neq 0$. First we observe that $g(PQ)=f(PQ)$, since $PQ\equiv 1 \pmod 8$. From (f) we get that $C^2=1$, or $C_1^2\equiv 1$, according to $P\equiv 3$ or $5 \pmod 8$. Since f and g are nonnegative, therefore $C=1$ or $C_1=1$, and by (h), $C=C_1=1$.

Assume that $P\equiv -Q \pmod 8$. Then $PQ\equiv -1 \pmod 8$, and so by (c), $g(PQ)=$ $=f(PQ)$ that by (f) involves $C=C_1$. From (h) we get $C=C_1=1$. □

(j) *If* f *and* g *are nonnegative functions, then there do not exist irregular odd prime powers.*

PROOF. Let P be irregular, $(P, 2)=1$. Then $f(P)\neq 0$, $g(P)\neq 0$. Let $N\equiv 3$ or $5 \pmod 8$, $(N, P)=1$. Then N contains a prime-power $Q\equiv 3$ or $5 \pmod 8$, and so by (i), $f(Q)=g(Q)=0$, $f(N)=g(N)=0$. So $f(N)=g(N)=0$ whenever $N\equiv 3$ or $5 \pmod 8$, $(N, P)=1$. This involves that $\mathscr{G}(n|n\equiv 3 \pmod 8, (n, P)=1)$ is finite that by Lemma 5 gives $f, g\in\mathscr{L}$. □

So we have proved the following assertion.

(k) *If* f *and* g *are nonnegative, then* $f(n)=g(n)$ *holds for every odd* n.

Now we consider $f(2^\alpha)$ and $g(2^\alpha)$. In the next step we do not assume the nonnegativity of f and g.

(l) *If* $f(n)=g(n)$ *for every odd* n, *then*

$$\begin{cases} f(2^{\alpha+2}) = g(2)f(2^{\alpha+1}) \\ g(2^{\alpha+2}) = f(2)g(2^{\alpha+1}) \end{cases} \quad (\alpha = 0, 1, 2, \dots). \tag{5.6}$$

PROOF. We start from the relation

$$\Delta(4k)+f(4k) = g(4k+1) = f(4k+1) = g(4k+2)-\Delta(4k+1) =$$

$$= g(2)g(2k+1)-\Delta(4k+1) = g(2)[f(2k)+\Delta(2k)]-\Delta(4k+1),$$

whence by (1.4),

$$\sum_{k=1}^{\infty}\frac{1}{k}|f(4k)-g(2)f(2k)| < \infty.$$

Extending the summation only for $k=2^\alpha m$, $(m, 2)=1$, and observing that $\mathscr{F}(m|(m, 2)=1)$ cannot be finite, hence we deduce immediately the first relation in (5.6).

To prove the second, we consider the relation

$$g(4k)=f(4k-1)+\Delta(4k-1)=g(4k-1)+\Delta(4k-1)=f(4k-2)+$$
$$+\Delta(4k-1)+\Delta(4k-2)=f(2)f(2k-1)+\Delta(4k-1)+\Delta(4k-2)=$$
$$=f(2)\big(g(2k)-\Delta(2k-1)\big)+\Delta(4k-1)+\Delta(4k-2)$$

whence by (1.4)

$$\sum_{k=1}^{\infty}\frac{1}{k}\,|g(4k)-f(2)g(2k)|<\infty.$$

Repeating the argument that was used earlier, we get the second equation in (5.6). □

(m) *If f and g are nonnegative, then* $f(2)=g(2)$.

Proof. Under the assumption stated we have $f(3)=g(3)$. Let n be running over the residue class $7 \pmod{12}$. For such an n we have $(n(n+1), 3)=1$. We start from the equation

$$g(3)\Delta(n)=g(3n+3)-f(3n)=\big(g(3n+3)-f(3n+2)\big)+$$
$$+\big(f(3n+2)-f(3n+1)\big)+\big(f(3n+1)-f(3n)\big)=$$
$$=\Delta(3n+2)+\Delta(3n+1)+\big(f(3n+1)-f(3n)\big).$$

Hence, by (1.4) we have

$$\sum_{n\equiv 7 \pmod{12}}\frac{1}{n}\,|f(3n+1)-f(3n)|<\infty,$$

and so

$$\sum_{n\equiv 7 \pmod{12}}\frac{1}{n}\,|g(3n+1)-f(3n+1)|<\infty.$$

Assume that $g(2)\neq f(2)$. Since $3n+1$ runs over the elements of $22 \pmod{36}$ and $2\|3n+1$, therefore

$$g(3n+1)-f(3n+1)=g(2)g\left(\frac{3n+1}{2}\right)-f(2)f\left(\frac{3n+1}{2}\right)=\big(g(2)-f(2)\big)g\left(\frac{3n+1}{2}\right),$$

and so from the last inequality we get

$$\sum_{t=1}^{\infty}\frac{1}{t}\,g(11+18t)<\infty. \tag{5.7}$$

So, the series $\mathcal{F}(n|n\equiv 10 \pmod{18})$, $\mathcal{F}(n|n\equiv 5 \pmod 9)$, $\mathcal{F}(n|n\equiv 5 \pmod{18})=\mathcal{G}=(n|n\equiv 5 \pmod{18})$, $\mathcal{F}(n|n\equiv 4 \pmod{18})$, $\mathcal{F}(n|n\equiv 4 \pmod{4\cdot 18})$, $\mathcal{F}(n|n\equiv 1 \pmod{18})=\mathcal{G}(n|n\equiv 1 \pmod{18})$, $\mathcal{F}(n|n\equiv 0 \pmod{18})$ are convergent. If there exists a suitable $\beta>0$ for which $f(3^\beta)=g(3^\beta)\neq 0$, then from $\mathcal{F}(n|n\equiv 0 \pmod{18})<\infty$ we get $\mathcal{F}(m|m\equiv 2\cdot 3^\beta v, (v,6)=1)<\infty$, consequently $\mathcal{G}(v|(v,6)=1)=\mathcal{F}(v|(v,6)=1)<\infty$ which leads to $f, g\in\mathcal{L}$ (see Lemma 2).

Assume now that $f(3^\beta)=g(3^\beta)=0$ for $\beta=1,2,\dots$. Then, by (1.4), $\mathcal{G}(n|n\equiv \equiv 1 \pmod 3)<\infty$, $\mathcal{F}(n|n\equiv -1 \pmod 3)<\infty$. Since $g(n)=f(n)$ for odd n,

$$\mathcal{G}(n|(n,6)=1)=\mathcal{F}(n|(n,6)=1)<\infty,$$

whence by Lemma 2 we deduce that $f,g\in\mathcal{L}$. □

(n) *If f and g are nonnegative, then $f(n)=g(n)$ for every n.*

PROOF. This is an immediate consequence of (l), (m) and (k). □

(o) *We have* $|f(n)|=|g(n)|=n^\sigma$ $(0<\sigma<1)$ *for every* n.

PROOF. If (1.4) holds for f and g, then it holds after substituting them by $|f(n)|$, $|g(n)|$, respectively. For them by (n) we get $|f(n)|=|g(n)|$ and so by Theorem 3 $|f(n)|=|g(n)|=n^\sigma$ $(0\le\sigma<1)$.

(p) *There do not exist irregular odd prime powers.*

PROOF. Assume that the assertion does not hold. From (h) we know that $CC_1=1$. Let $P_1\equiv P_2\equiv l \pmod 8$, $l\equiv 3$ or $5 \pmod 8$, $(P_1,P_2)=1$. Since by (o) $g(P_j)\ne 0$, and $P_1P_2\equiv 1 \pmod 8$, therefore $g(P_1P_2)=f(P_1P_2)=f(P_1P_2), g(P_1)g(P_2)= =f(P_1)f(P_2)$, and so by (f), $C_1^2=1, C^2=1$. Hence $C_1=C_2=\ne 1$. If there exists any irregular prime power P, then it has to be $\equiv 3$ or $5 \pmod 8$, and by (f), all the prime powers $P\equiv 3$ or $5 \pmod 8$ are irregular, furthermore $f(P)=-g(P)$ holds for all of them. This involves

$$f(n)=-g(n) \quad \text{if} \quad n\equiv 3 \quad \text{or} \quad 5 \pmod 8. \tag{5.8}$$

As we know, $f(n)=g(n)$ if $n\equiv\pm 1 \pmod 8$, and $|f(n)|=|g(n)|=n^\sigma\ge 1$.

Let $n\equiv 1 \pmod{24}$ Since $g(3)=-f(3)$, therefore $g(3(n+1))+f(3n)=g(3)\Delta(n)$. Since

$$g(3n+3)=f(3n+2)+\Delta(3n+2), \quad f(3n)=g(3n+1)-\Delta(3n),$$

we have

$$f(3n+2)+g(3n+1)=g(3)\Delta(n)+\Delta(3n)-\Delta(3n+2).$$

Since $3n+2\equiv 5 \pmod 8$, we have

$$f(3n+2)=-g(3n+2)=-f(3n+1)-\Delta(3n+2),$$

and so by (1.4)

$$\sum_{n\equiv 1 \pmod{24}} \frac{1}{n}|g(3n+1)-f(3n+1)|<\infty.$$

By putting there $n=1+24t$, $3n+1=4(1+18t)$, we get

$$\sum_{t=1}^{\infty}\frac{1}{t}|g(1+18t)-H(4)f(1+18t)|<\infty.$$

First we consider the sum for $t\equiv 0 \pmod 4$. Let $t=4\tau$. Since $g(1+18\cdot 4\tau)= =f(1+18\cdot 4\tau)$ we have

$$\sum\frac{1}{\tau}|H(4)-1||f(1+18\cdot 4\tau)|<\infty,$$

and so $H(4)=1$.

By putting $t=4\tau+1$, we have $1+18t=19+18\cdot 4\tau\equiv 3 \pmod 8$, that gives $g(19+18\tau)=-f(19+18\tau)$, whence

$$\sum_{\tau\geqq 1}\frac{1}{\tau}\,|H(4)+1|\,|f(19+18\tau)|<\infty,$$

i.e. $H(4)=-1$. This is a contradiction. □

(q) *We have* $f(2)=g(2)$.

Proof. We have to repeat the argument used in the proof of (m). Since $g(3)=f(3)$, therefore from the assumption $f(2)\neq g(2)$ we are led to the consequence

$$\sum_{t=1}^{\infty}\frac{1}{t}\,|g(11+18t)|<\infty,$$

which by $g\geqq 1$ is impossible. □

(r) *Under the conditions* $f(2)\neq 0$, $g(2)\neq 0$, $f(2^2)\neq 0$, $g(2^2)\neq 0$, *Theorem* 4 *is true.*

Proof. From (q) and (l) we have $f(2^\alpha)=g(2^\alpha)$ $(\alpha=1,2,\ldots)$ that together with (p) give $f(n)=g(n)$ $(n=1,2,\ldots)$. The conditions stated in Theorem 3 hold. □

6. Proof of Theorem 4 under the condition $f(2)g(2)f(4)g(4)=0$

We shall prove that in this case (1.4) involves that $f, g\in\mathscr{L}$.

Since (1.4) implies

$$\sum\frac{\big||g(n+1)|-|f(n)|\big|}{n}<\infty,$$

and $f\in\mathscr{L}$ if and only if $|f|\in\mathscr{L}$, we may assume that f and g are nonnegative.

Throughout this section, we shall assume that $f, g\geqq 0, f, g\in\mathscr{M}$ and that (1.4) holds.

Lemma 6. *If* $\mathscr{G}\big(n|(n,2)=1\big)<\infty$, *then* $f, g\in\mathscr{L}$.

Proof. If $\mathscr{G}\big(n|(n,2)=1\big)<\infty$, then $\mathscr{G}\big(2n|n\equiv 1 \pmod 2\big)<\infty$, and so $\mathscr{F}\big(n|n\equiv 1 \pmod 4\big)<\infty$. If $f(n)=0$ for every $n\equiv -1 \pmod 4$, then we are ready, since then $\mathscr{F}\big(n|(n,2)=1\big)<\infty$, and $\mathscr{G}(2^\beta|\beta=1,2,\ldots)\ll\mathscr{F}\big(n|(n,2)=1\big)+1$.

Let us assume that $f(n)\neq 0$ for a suitable $n\equiv -1 \pmod 4$, i.e. that $f(Q)\neq 0$, for $Q=q^\delta\equiv -1 \pmod 4$. Since $\mathscr{F}\big(Qm|m\equiv -1 \pmod 4\big)<\infty$, therefore $\mathscr{F}\big(m|m\equiv -1 \pmod 4, (m,Q)=1\big)<\infty$. Consequently

$$\sum_{\substack{p\,\mathrm{odd}\\ p\neq q}} R(f,p)<\infty. \tag{6.1}$$

If there exist coprime pairs $Q_1, Q_2\equiv -1 \pmod 4$ with the property $f(Q_1)\neq 0$, $f(Q_2)\neq 0$, then (6.1) holds extending the summation for every odd p, that is $\mathscr{F}\big(n|(n,2)=1\big)<\infty$, whence we get $f, g\in\mathscr{L}$.

It has remained the case when $f(n)=0$ for every $n \equiv -1 \pmod 4$ coprime to Q.

It may occur that $f(n)=0$ for every odd n coprime to Q. But then $f(2^\beta-1)=0$ except when $2^\beta-1$ being a power of q. Since as it is well known the equation $2^\beta-1=q^\gamma$ has only finitely many solutions, therefore $f(2^\beta-1)=0$ for every large β, consequently $R(g,2)<\infty$, and so $g\in\mathscr{L}$.

Finally we assume that there exists an odd prime power P, $(P,Q)=1, f(P)\neq 0$. Let s be so large that $2^s \nmid P-1$. We observe that for $q^\gamma\equiv -1 \pmod{2^s}$, $P_q^\gamma \equiv \equiv -P (\not\equiv -1) \pmod{2^s}$. Since the exponent of 2 in $P_q^\gamma+1$ is bounded by s if $q^\gamma \equiv -1 \pmod{2^s}$, therefore

$$\mathscr{G}(P_q{}^\gamma+1 | q^\gamma \equiv 1 \pmod{2^s}) \leqq \mathscr{G}(n|(n,2)=1) < \infty,$$

and so

$$\frac{f(P)}{P}\mathscr{F}(q^\gamma | q^\gamma \equiv -1 \pmod{2^s}) < \infty.$$

For the subset $q^\gamma \not\equiv -1 \pmod{2^s}$ we use the inequality

$$\mathscr{F}(q^\gamma | q^\gamma \not\equiv 1 \pmod{2^s}) \ll \mathscr{G}(q^\gamma+1 | q^\gamma+1 \not\equiv 0 \pmod{2^s}) + O(1).$$

But the sum in the right hand side is convergent, since every $q^\gamma+1$ contains 2 at most on the power $s-1$. So $\mathscr{F}(q^\gamma|\gamma=1,2,\ldots)<\infty$, and so $\mathscr{F}(n|(n,2)=1)<\infty$ that leads to the aim immediately. □

LEMMA 7. *If* $\mathscr{F}(n \nmid (n,2)-1)<\infty$, *then* $f, g\in\mathscr{L}$.

PROOF. If there exists a $\beta\geqq 1$ with $g(2^\beta)\neq 0$, then we are ready, since in this case

$$\mathscr{G}(m|(m,2)=1) \ll \mathscr{G}(2^\beta m|(m,2)=1) \ll \mathscr{F}(n|n\equiv 1 \pmod 2)+O(1) \ll 1,$$

and this case has been considered in Lemma 7.

We assume that $g(2^\beta)=0$ for $\beta=1,2,\ldots$.

From $\mathscr{F}(n|(n,2)=1)<\infty$ we get that $\mathscr{F}(n|2\|n)<\infty$ and so $\mathscr{G}(n|n\equiv -1 \pmod 4)<\infty$.

First we consider the case when there exists a $Q\equiv -1 \pmod 4$, $Q=q^\delta$, $g(Q)\neq 0$. Then, as above, $\mathscr{G}(n|n\equiv 1 \pmod 4,\ (n,Q)=1)<\infty$, that leads to $\sum_{p\nmid Q} R(g,p)<\infty$. If there exist $Q_1, Q_2, (Q_1,Q_2)=1$ with this property, then we are ready. Let Q be unique, i.e. $g(n)=0$ for $n\equiv 1 \pmod 4$, $(n,Q)=1$. We have to see that

$$R(g,q)=\sum \frac{g(q^\gamma)}{q^\gamma} < \infty.$$

The contribution of the terms $q^\gamma\equiv -1 \pmod 4$ is finite since $\mathscr{G}(n|n\equiv -1 \pmod 4)<\infty$.

Let P be an odd prime power coprime to Q such that $g(P)\neq 0$. Let s be so large that $P\not\equiv 1 \pmod{2^s}$. Since for $2^s|q^\gamma-1, 2^s|Pq^\gamma-1$, therefore

$$\frac{g(P)}{P}\mathscr{G}(q^\gamma|q^\gamma\equiv 1 \pmod{2^s}) \ll \mathscr{F}(n|2^s\nmid n)+O(1),$$

and here the right hand side is finite.

Furthermore

$$\mathscr{G}\big(q^\gamma | q^\gamma \not\equiv 1 (\mathrm{mod}\ 2^s)\big) < \mathscr{F}(q^\gamma - 1 | 2^s \nmid q^\gamma - 1) + O(1) \ll \mathscr{F}(n | 2^s \nmid n) + O(1).$$

Consequently $\mathscr{G}\big(n|(n,2)=1\big) < \infty$ and by Lemma 6 we are ready. If there does not exist such a P then $g(n)=0$ for $(n, 2Q)=1$, $g(2^\beta+1)=0$ for all but finitely many β, and so $R(f,2)<\infty$, and we are ready.

It has remained the case when $g(n)=0$ for every $n \equiv -1 \ (\mathrm{mod}\ 4)$.

If $g(n)=0$ for every odd n, then we are ready. We assume that this condition does not hold.

Let α be the least integer for which there exists an odd n_0, $n_0 \not\equiv 1 \ (\mathrm{mod}\ 2^{\alpha+1})$, $g(n_0) \neq 0$.

The case $\alpha=1$ has been considered. Assume that $\alpha \geqq 2$. We may assume that $n_0=Q$ is a prime-power, $Q=q^\eta$, $Q \equiv l_\alpha (\mathrm{mod}\ 2^{\alpha+1})$, $l_\alpha = 2^\alpha+1$.

If $n \equiv l_\alpha \ (\mathrm{mod}\ 2^{\alpha+1})$, then $2^\alpha \| n-1$, and so

(6.3)
$$\mathscr{G}\big(n | n \equiv l_\alpha (\mathrm{mod}\ 2^{\alpha+1})\big) \ll \mathscr{F}\big(2^\alpha m | (m,2)=1\big) + O(1) \ll \mathscr{F}\big(m|(m,2)=1\big) + O(1) \ll 1.$$

Since $uQ \equiv l_\alpha \ (\mathrm{mod}\ 2^{\alpha+1})$ when $u \equiv 1 \ (\mathrm{mod}\ 2^{\alpha+1})$, and $g(u) \ll g(uQ)$, when $(u,Q)=1$, we get

$$\mathscr{G}\big(u | u \equiv 1 (\mathrm{mod}\ 2^{\alpha+1}), (u,Q)=1\big) < \infty. \tag{6.4}$$

From (6.3) and (6.4), $\mathscr{G}\big(n|(n,2Q)=1\big)<\infty$, and so by the assumption $g(2^\beta)=0$, $\mathscr{G}\big(n|(n,q)=1\big)<\infty$. We distinguish two cases according to whether there exists or not another P with $(P,Q)=1$ such that $g(P)\neq 0$.

We can continue the proof in both cases on the way that was used in the proof of Lemma 6. We omit the details. □

LEMMA 8. *If* $f(2^\alpha)=0$ *for every* $\alpha \geqq 1$, *or* $g(2^\beta)=0$ *for every* $\beta \geqq 1$, *then* $f, g \in \mathscr{L}$.

PROOF. It is clear, since the conditions of Lemma 6 or 7 are satisfied obviously. □

LEMMA 9. *Assume that* $f(2^\alpha)\neq 0$, $g(2^\beta) \neq 0$. *Let* $\mathscr{S}$ *be an arbitrary subset of odd integers. Then*

$$\mathscr{G}(m | m \in \mathscr{S}) \ll \mathscr{G}\big(2^{\alpha+\beta} m - (2^\alpha - 1) | m \in \mathscr{S}\big) + 1 \tag{6.5}$$

and

$$\mathscr{F}(m | m \in \mathscr{S}) \ll \mathscr{F}\big(2^{\alpha+\beta} m + (2^\beta - 1) | m \in \mathscr{S}\big) + 1. \tag{6.6}$$

PROOF. Since $f(2^\alpha)\neq 0$, $g(2^\beta)\neq 0$, therefore

$$\mathscr{G}(m|m\in\mathscr{S}) \ll f(2^\alpha) g(2^\beta) \mathscr{G}(m|m\in\mathscr{S}) = f(2^\alpha)\mathscr{G}(2^\beta m | m \in \mathscr{S}) \ll$$

$$\ll f(2^\alpha)\mathscr{F}(2^\beta m - 1 | m \in \mathscr{S}) + 1 \ll \mathscr{F}(2^{\alpha+\beta} m - 2^\alpha | m \in \mathscr{S}) + 1 \ll$$

$$\ll \mathscr{G}\big(2^{\alpha+\beta} m - (2^\alpha - 1) | m \in \mathscr{S}\big) + 1.$$

The proof of (6.6) is similar and so we omit it.

LEMMA 10. *If* $g(2)=0$, *then* $f, g \in \mathscr{L}$.

PROOF. $g(2m)=0$ for odd m, so $\mathscr{F}(n|n\equiv 1 \pmod 4)<\infty$. Let $Q=q^\gamma\equiv -1 \pmod 4$ be a prime power, $f(Q)\neq 0$. Then $\mathscr{F}(n|n\equiv -1 \pmod 4, (n, q)=1)<\infty$. Hence we have $\sum_{(p, 2q)=1} R(f, p)<\infty$.

Let $\mathscr{A}_s$ be the set of those odd integers n for which $q^s\nmid n$. Since $\max_{j\leqq s-1} |f(q^j)|\ll 1$, therefore, from $\mathscr{F}(n|(n, 2q)=1)<\infty$ we have $\mathscr{F}(n|n\in\mathscr{A}_s)<\infty$ for every s.

By Lemma 8 we may assume that $f(2^\alpha)\neq 0$, $g(2^\beta)\neq 0$ for suitably chosen α and $\beta(\geqq 2)$. Let s be so large that $q^s\nmid 2^\beta-1$. Then by putting $\mathscr{S}=\{q^\gamma|\gamma=1, 2, ...\}$, and observing that $2^{\alpha+\beta}q^\gamma+(2^\beta-1)\in\mathscr{A}_s$ for every large γ, from (6.6) we deduce that $R(f, q)<\infty$, and so that $\mathscr{F}(n|(n, 2)=1)<\infty$. This, by Lemma 7 completes the proof. □

LEMMA 11. *Assume that* $\mathscr{F}(n|(2N, n)=1)<\infty$ *or* $\mathscr{G}(n|(2N, n)=1)<\infty$ *holds for a suitable* N. *Then* $f, g\in\mathscr{L}$.

PROOF. We may assume that N is odd, $N=\pi_1...\pi_r$, where π_j are distinct primes, furthermore that for suitable α and β, $f(2^\alpha)\neq 0$, $g(2^\beta)\neq 0$. The assertion is true for $N=1$. We shall treat only the case $\mathscr{F}(n|(2N, n)=1)<\infty$. If $R(f, \pi_j)<\infty$, then we can reduce it to $N=\frac{1}{\pi_j}N$. So we may assume that $R(f, \pi_j)=\infty$ $(j=1, ..., r)$. Let s be an integer such that $\pi_j^s\nmid 2^\beta-1$. Since $R(f, \pi_j)=\infty$, therefore $f(\pi_j)\neq 0$. for infinitely many γ, so there exists $\delta_j\geqq s$ for which $f(\pi_j^{\delta_j})\neq 0$. Let $A=\pi_1^{\delta_1}...\pi_{r-1}^{\delta_{r-1}}$, and consider the set $\mathscr{S}=\{\pi_r^\gamma A|\gamma=1, 2, ...\}$. Since $f(\pi_r^\gamma)\ll f(\pi_r^\gamma A)$, by (6.6) we get

$$\mathscr{F}(\pi_r^\gamma|\gamma=1, 2, ...)\ll\mathscr{F}(\pi_r^\gamma A|\gamma=1, 2 ...)\ll\mathscr{F}(2^{\alpha+\beta}\pi_r^\gamma A+(2^\beta-1)|\gamma=1, 2, ...)+1.$$

But the integers $2^{\alpha+\beta}\pi_r^\gamma A+(2^\beta-1)$ contain the primes π_j at most on the power $s-1$, if γ is large, consequently $R(f, \pi_\gamma)<\infty$, a contradiction. The proof of the second assertion is the same, and so we omit it. □

LEMMA 12. *Let* $f(2)=0$, $g(2)\neq 0$. *Then* $f, g\in\mathscr{L}$.

PROOF. We shall give an indirect proof. Assume that $f, g\notin\mathscr{L}$.

Then $f(2^\alpha)\neq 0$ for a suitable α (see Lemma 8). Let α be the smallest exponent for which $f(2^\alpha)\neq 0$.

(A) $g(n)=0$ *if* n *is odd and* $n\not\equiv 1 \pmod{2^\alpha}$.

PROOF. If n is odd, $n\equiv 1 \pmod{2^\alpha}$, then the exponent of 2 in $n-1$ is smaller than α, so $f(n-1)=0$. Then $\mathscr{G}(n|n\not\equiv 1 \pmod{2^\alpha}, (n, 2)=1)<\infty$.

If (A) is not true then there exists an odd prime power P, $\not\equiv 1 \pmod{2^\alpha}$, $g(P)\neq 0$. Then $\mathscr{G}(m|m\equiv 1 \pmod{2^\alpha}, (m, P)=1)<\infty$, and so $\mathscr{G}(m|(m, 2P)=1)<\infty$, which by Lemma 11 finishes the proof. □

(B) $f(n)=0$ *if* $n\equiv 5 \pmod 8$.

PROOF. Since $\alpha\geqq 2$, therefore from (A) it follows that $g(n)=0$ if $n\equiv 3 \pmod 4$, and so if $n\equiv 6 \pmod 8$, consequently $\mathscr{F}(n|n\equiv 5 \pmod 8)<\infty$.

Assume in the contrary that there exists an $N_0\equiv 5 \pmod 8$ for which $f(N_0)\neq 0$. We may assume that N_0 is "primitive" in the following sense: either $N_0=P=$

$=$prime power$\equiv 5 \pmod 8$, or $N_0=P_1 \cdot P_2$ where $P_1\equiv 3 \pmod 8$, $P_2\equiv 7 \pmod 8$, P_1, P_2 being coprime prime powers. We have

$$\mathscr{F}(n|n \equiv 1 \pmod 8,\ (n, N_0) = 1) < \infty,$$

and so by $g(2)\neq 0$ and (1.4)

$$\mathscr{G}(\nu|\nu \equiv 1 \pmod 4,\ (2\nu-1, N_0) = 1) < \infty.$$

Letting the values ν run only over $\nu=1+2^\alpha\mu, (\mu, 2)=1$, we get

$$\mathscr{F}(\mu|\mu\equiv 1 \pmod 2,\ (2^{\alpha+1}\mu+1, N_0) = 1) < \infty,$$

whence by putting $\mu=\sigma N_0$,

$$\mathscr{F}(\sigma|\sigma \equiv 1 \pmod 2,\ (\sigma, N_0) = 1) < \infty,$$

that by Lemma 11 gives the desired result. □

Now we prove that $f(n)=0$ for all elements of one of the residue classes $3 \pmod 8$, $7 \pmod 8$. Assume that this is not true. Then there exist $P_1\equiv 3 \pmod 8$, $P_2\equiv 7 \pmod 8$ such that $f(P_1)\neq 0, f(P_2)\neq 0$. (We take into account that $f(m)=0$ for $m\equiv 5 \pmod 8$!) If $(P_1, P_2)=1$, then $f(P_1P_2)=f(P_1)f(P_2)\neq 0$, but $P_1P_2\equiv$ $\equiv 5 \pmod 8$, and this is impossible. So $P_1=p^{\lambda_1}$, $P_2=p^{\lambda_2}$. Let $m\equiv 3 \pmod 4$, $(m, p)=1$. Since P_1m or $P_2m\equiv 5 \pmod 8$, therefore $f(m)=0$.

We distinguish two cases according to $g(4)=0$ or $g(4)\neq 0$.

If $g(4)=0$, then $\mathscr{F}(n|n\equiv 3 \pmod 8)<\infty$, and from $f(P_1)\neq 0$ we have $\mathscr{F}(n|n\equiv 1 \pmod 8, (n, p)=1)<\infty$. This involves that $\mathscr{G}(\nu|\nu\equiv 1 \pmod 4, (2\nu-1, p)=$ $=1)<\infty$. Summing up only for $\nu=1+2^\alpha\mu, (\mu, 2)=1$, we get

$$\mathscr{F}(\mu|\mu \equiv 1 \pmod 2,\ (2^{\alpha+1}\mu+1, p) = 1) < \infty.$$

By putting $\mu=\sigma P_1$, we have $\mathscr{F}(\sigma|(\sigma, 2p)=1)<\infty$, and this leads to the assertion.

Assume that $g(4)\neq 0$. Since $f(n)=0$ for $(n, p)=1$, $n\equiv 3 \pmod 8$, therefore $\mathscr{G}(4|\nu(\nu, 2)=1, (4\nu-1, p)=1)<\infty$, and so $\mathscr{G}(\nu|(\nu, 2)=1, (4\nu-1, p)=1)<\infty$. Let $\nu=1+2^\alpha\mu, (\mu, 2)=1$. Since $f(2^\alpha)\neq 0$, we get $\mathscr{F}(\mu|(2^{\alpha+2}\mu+3, p)=1, (\mu, 2)=1)<\infty$. If $p\neq 3$, then we put $\mu=\sigma P_1$ and deduce that $\mathscr{F}(\sigma|(\sigma, 2p)=1)<\infty$. If $p=3$, then the condition $(2^{\alpha+2}\mu+3, p)\equiv 1$ is equivalent with $\mu\not\equiv 0 \pmod 3$, and so $\mathscr{F}(\sigma|(\sigma, 2p)=1)<\infty$ holds in this case too. □

So we have proved the following assertion.

(C) *Either*

(a1) $f(n)=0$ *for every* $n\equiv 3 \pmod 8$

or

(a2) $f(n)=0$ *for every* $n\equiv 7 \pmod 8$.

Now we shall prove that (a1) is equivalent with $g(4)=0$. Assume that $g(4)=0$. Then $\mathscr{F}(n|n\equiv 3 \pmod 8)<\infty$. If (a1) does not hold, then $f(P)\neq 0$ for a suitable $P\equiv p^\gamma\equiv 3 \pmod 8$, so $\mathscr{F}(n|n\equiv 1 \pmod 8, (n, p)=1)<\infty$. In this case (a2) is satisfied. So we have $\mathscr{F}(n|(n, 2p)=1)<\infty$ that leads to $f, g\in\mathscr{L}$.

Assume that (a1) holds and $g(4)\neq 0$. Then $\mathscr{G}(n|2^2\|n)<\infty$, and so $\mathscr{G}(\nu|(\nu, 2)=$ $=1)<\infty$ and this gives that $f, g\in\mathscr{L}$.

We shall prove that (a2) is equivalent with $g(2^\beta)=0$ $(\beta=3, 4, \ldots)$. Assume first that $g(2^\beta)=0$ for every $\beta\geqq 3$. Then $\mathcal{F}(n|n\equiv 7 \pmod 8)<\infty$. If (a2) does not hold, then (a1) is satisfied, and there exists a $P\equiv p^\gamma\equiv 7 \pmod 8$ such that $f(P)\neq 0$. This involves that $\mathcal{F}(n|n\equiv 1 \pmod 8), (n, p)=1)<\infty$, and so by (B), (a2), we get $\mathcal{F}(n|(n, 2p)=1)<\infty$.

Assume now that (a2) holds and there exists a $\beta\geqq 3$ for which $g(2^\beta)\neq 0$. Since (a2) involves $\mathcal{G}(n|2^3\|n)<\infty$, we have

$$\mathcal{G}(\nu|(\nu, 2) = 1) \ll \mathcal{G}(2^\beta \nu|(\nu, 2) = 1) < \infty,$$

and so, we are ready. □

So we have proved the following assertion.

(D) *Either*

(b1) $g(4)=0$ *and* (a1) *holds*,

or

(b2) $g(2\ )=0$ *for every* $\beta\geqq 3$ *and* (a2) *holds.*

First we prove that

(E) *The case* $f(3)=0$ *is impossible.*

Proof. Let $f(3)=0$. From (A) we know that $g(3)=0$. Hence, by (1.4) we get that $\mathcal{G}(n|n\equiv 4, 7 \pmod 9)<\infty$. Since $g(2)\neq 0$, putting $n=2m$, $(m, 2)=1$, we get immediately that $\mathcal{G}(m|m\equiv 2, 8 \pmod 9), (m, 2)=1)<\infty$. By putting $m=1+2^\alpha\nu$, $(\nu, 2)=1$, from $f(2^\alpha)\neq 0$ we deduce that

$$\mathcal{F}(\nu|\nu \equiv 1 \pmod 2,\quad 2^\alpha \nu \equiv 1, 7 \pmod 9) < \infty. \tag{6.7}$$

From $g(3)=0$ it follows immediately that $\mathcal{F}(n|n\equiv 2, 5 \pmod 9)<\infty$, and so

$$\mathcal{F}(m|m \equiv 1 \pmod 2,\quad 2^\alpha m \equiv 2, 5 \pmod 9) < \infty. \tag{6.8}$$

Let $A_\alpha=2^\alpha+1$. If $(m, 2)=1$, $m\equiv 1 \pmod 9$, then $2^{\alpha+1}m-(2^\alpha-1)\equiv A_\alpha \pmod 9$. So, by Lemma 9

$$\mathcal{G}(m|m \equiv 1 \pmod 9),\quad (m, 2) = 1) < \infty \tag{6.9}$$

if

$$A_\alpha \pmod 9 \in \{4, 7, 2, 8, 3, 6\}. \tag{6.10}$$

Furthermore, if $m\equiv 5 \pmod 9$, then $2^{\alpha+1}m-(2^\alpha-1)\equiv 1 \pmod 9$, and so

$$\mathcal{G}(m|m \equiv 5 \pmod 9) < \infty \tag{6.11}$$

under the condition (6.10). Consequently $\mathcal{G}(m|(m, 6)=1)<\infty$, and so by Lemma 11, $f, g\in\mathcal{L}$.

(6.10) holds if $\alpha\equiv 1, 4, 5, 0 \pmod 6$. Consider now the cases $\alpha\equiv 2, 3 \pmod 6$. If $\alpha\equiv 2 \pmod 6$, then $2^\alpha\equiv 4 \pmod 9$, and so, from (6.7), (6.8) we deduce that

$$\mathcal{F}(\nu|\nu \equiv 1 \pmod 2,\quad \nu \equiv 2, 5, 8, 7, 4 \pmod 9) < \infty. \tag{6.12}$$

Similarly, if $\alpha\equiv 3 \pmod 6$, then $2^\alpha\equiv 8 \pmod 9$, and we get (6.12) as well .

If $g(3^\gamma)\neq 0$ for a suitable $\gamma\geqq 2$, then we get $\mathcal{G}(m|(m, 6)=1)<\infty$, since $\nu\equiv 8 \pmod 9$ is included in (6.12), and so $\mathcal{G}(2\cdot 3^\gamma\nu|(\nu, 6)=1)<\infty$.

Assume now that $g(3^\gamma)=0$ $(\gamma=1,2,\dots)$. Then

$$\mathscr{F}(n|n\equiv 2 \pmod 3)<\infty. \tag{6.13}$$

If $\alpha\equiv 3 \pmod 6$, then $2^\alpha\equiv 2 \pmod 3$, and, by putting $n=2^\alpha m$, $m\equiv 1 \pmod 3$, we get $\mathscr{F}(m|m\equiv 1 \pmod 3, (m,2)=1)<\infty$, which gives $\mathscr{F}(m|(m,6)=1)<\infty$.

It has remained to prove the case when $\alpha\equiv 2 \pmod 6$, $g(3^\gamma)=0$ $(\gamma=1,2,\dots)$. We know that $\mathscr{G}(n|n\equiv 2,4,7,8 \pmod 9, (n,2)=1)<\infty$.

If we could prove that

$$\mathscr{G}(n|n\equiv 5 \pmod 9,\ (n,2)=1)<\infty, \tag{6.14}$$

then by Lemma 9 $(\beta=1)$ it would follow that $\mathscr{G}(n|n\equiv 1 \pmod 9, (n,2)=1)<\infty$. Indeed, in this case $\Lambda_\alpha\equiv 5 \pmod 9$. But then $g\in\mathscr{L}$.

Let us assume that (6.14) does not hold. Then there exists at least one $n\equiv 5 \pmod 9$, $(n,2)=1$ for which $g(n)\neq 0$. So there exists an odd prime power P such that $P\not\equiv 1 \pmod 9$, $g(P)\neq 0$. We shall deduce that

$$\mathscr{G}(n|(n,6P)=1)<\infty, \tag{6.15}$$

which involves $g,f\in\mathscr{L}$.

(6.15) is obvious if $P\equiv 4,7,8 \pmod 9$, since in these cases $P^{-1} \pmod 9\equiv$ $\equiv 7,8,4 \pmod 9$ and so $P^{-1}\{2,4,7,8\} \pmod 9\supseteq\{1,5\}$, furthermore $\mathscr{G}(m|(m,6P)=1, Pm\equiv\{2,4,7,8\} \pmod 9)<\infty$.

First we consider the case $P\equiv 5 \pmod 9$. Then $P^{-1}\equiv 2 \pmod 9$, and so

$$\mathscr{G}(n|n\equiv 5 \pmod 9,\ (n,6P)=1)<\infty. \tag{6.16}$$

We consider now $\mathscr{G}(n|n\equiv 1 \pmod 9, (n,2P)=1)$.

Let $m=m_1$ be odd, $m_1\equiv 1 \pmod 9$, m_2, m_3, m_4 be defined by the relation

$$m_{i+1}=2^{\alpha+1}m_i-(2^\alpha-1) \quad (i=1,2,3). \tag{6.17}$$

Let $P=p^\theta$.

Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be the set of those m for which additionally $p\nmid m_2$ and $p\nmid m_4$ holds, respectively. From (6.16) and Lemma 9 we get $\mathscr{G}(m|m\in\mathscr{S}_1)<\infty$ and $\mathscr{G}(m|m\in\mathscr{S}_2)<\infty$. First we prove that every odd m in $1 \pmod 9$ belongs to $\mathscr{S}_1\cup\mathscr{S}_2$ if $p\nmid(2^{\alpha+1}+1)(2^\alpha-1)$. Indeed, assume that $p|m_2$, $p|m_4$. Then, from (6.17) we deduce that

$$0\equiv m_4\equiv 2^{\alpha+1}(2^{\alpha+1}m_2-(2^\alpha-1))-(2^\alpha-1)\equiv -(2^{\alpha+1}+1)(2^\alpha-1) \pmod p,$$

i.e. $p|(2^{\alpha+1}+1)(2^\alpha-1)$. So (6.15) holds if $p\nmid(2^{\alpha+1}+1)(2^\alpha-1)$.

If $p|2^\alpha-1$, then $(p,m_2)=1$ holds if and only if $(p,m_1)=1$, and so by Lemma 9 we get (6.15) immediately.

Let now $2^{\alpha+1}+1\equiv 0 \pmod p$. $m_2\equiv 0 \pmod p$ is equivalent with $m_1\equiv 1-2^\alpha$ $\pmod p$, and so by Lemma 9,

$$\mathscr{G}(m|m\not\equiv 1-2^\alpha \pmod p)\ll\mathscr{G}(m_2|m_2\not\equiv 0 \pmod p)<\infty,$$

consequently

$$\mathscr{G}(n|n\equiv 1 \pmod 9,\quad (n,2)=1,\ n\not\equiv 1-2^\alpha \pmod p)<\infty. \tag{6.18}$$

Let us assume now that $f(3^\lambda)\neq 0$ for a suitable λ. If $n=1+2^\alpha 3^\lambda\mu$, then $n\equiv 1 \pmod 9$, and the condition $n\equiv 1-2^\alpha \pmod p$ is equivalent with $2^\alpha 3^\lambda\mu\equiv\equiv -2^\alpha \pmod p$, i.e. with $3^\lambda\mu+1\equiv 9 \pmod p$. So, from (6.18) we get

$$\text{(6.19)}\qquad \mathscr{F}(\mu|(\mu, 6) = 1,\ \ 3^\lambda\mu+1 \not\equiv 0 (\text{mod}\, p)) < \infty.$$

By putting $\mu=2\Lambda-1$, $\Lambda\equiv 1 \pmod 9$, hence we deduce that

$$\text{(6.20)}\qquad \mathscr{G}(\Lambda|\Lambda \equiv 1 (\text{mod}\, 9),\ (\Lambda, 2) = 1,\ 3^\lambda(2\Lambda-1)+1 \not\equiv 0\ (\text{mod}\, p)) < \infty.$$

If the congruences $n\equiv 1-2^\alpha \pmod p$, $2\cdot 3^\lambda n\equiv(3^\lambda-1) \pmod p$ are not equivalent, then (6.19), (6.20) together involve that $\mathscr{G}(n|n\equiv 1 \ (\text{mod}\ 9), (n, 2) = 1) < \infty$ that is enough. If they are equivalent, then $2\cdot 3^\lambda(1-2^\alpha)\equiv 3^\lambda-1 \pmod p$, whence by $p|2^{\alpha+1}+1$ we get

$$\text{(6.21)}\qquad 2\cdot 3^\lambda+1 \equiv 0\ (\text{mod}\, p).$$

Assume that (6.21) holds and consider (6.19). If $f(p^\beta)\neq 0$ holds for a suitable power p^β of p, then by substituting $\mu=p^\beta\tau$, $(\tau, 6p)=1$ into (6.19), we deduce that $\mathscr{F}(\tau|(\tau, 6p)=1)<\infty$.

Let Q be a prime power, $(Q, 6)=1, f(Q)\neq 0$. Let $(\mu, Q)=1$. If $Q\not\equiv 1 \pmod p$, then the relations $3^\lambda\mu+1\equiv 0 \pmod p$, $3^\lambda Q\mu+1\equiv 0 \pmod p$ cannot hold simultaneously. This gives $\mathscr{F}(\tau|(\tau, 6Q)=1)<\infty$ and leads to aim immediately.

Assume that there does not exists such a Q. Then every integer n, $(n, 6)=1$, such that $f(n)\neq 0$ has to be composed from prime powers $Q_j\equiv 1 \pmod p$, i.e. $n = Q_1 \dots Q_s$, consequently $n\equiv 1 \pmod p$. To prove that $\mathscr{F}(\mu|(\mu, 6p) = 1) <\infty$, it is enough to show that $3^\lambda+1\not\equiv 0 \pmod p$ (see (6.19)), but this is obvious from (6.21).

Now we treat the case $f(3^\beta)=0$ $(\beta=1, 2, \dots)$. Then $\mathscr{G}(n|n\equiv 1 \pmod 3)<\infty$. If there exists an $N_0\equiv -1 \pmod 3$, such that $g(N_0)\neq 0$, then $\mathscr{G}(n|(n, N_0)=1,$ $n\equiv -1 \pmod 3)<\infty$, and we are ready. In the opposite case $g(n)=0$ for every $n\equiv 2 \pmod 3$, and so $\mathscr{G}(n|(n, 3)=1)<\infty$.

Now we may assume that $g(P)=0$ for every odd prime power P in the residue classes 4, 7, 8, 5 (mod 9). Consequently, if $(n, 2)=1$, $n\equiv 5 \pmod 9$ and $g(n)\neq 0$, then n contains at least four mutually coprime prime powers, $P_1, P_2, P_3, P_4\equiv\equiv 2 \pmod 9$, such that $g(P_j)\neq 0$. Let $P_j=p_j^{\beta_j} (j=1, 2, 3, 4)$. Since $P_1^{-1} \pmod 9 =5$, we get $\mathscr{G}(m|(m, 2p_1)=1, m\equiv 1 \pmod 9)<\infty$. Since $N_0=P_1P_2\equiv 4 \pmod 9$, $g(P_1P_2)\neq\neq 0$, we have $\mathscr{G}(m|(m, 2p_1p_2)=1, m\equiv 5 \pmod 9)<\infty$, and so $\mathscr{G}(m|(m, 6p_1p_2)==1)<\infty$. We are ready. □

(F) *The case* $f(3)\neq 0, f(4)\neq 0$ *is impossible.*

PROOF. Since $f(3)\neq 0$, therefore (a1) does not hold, so (a2) holds. Since $g(4)=0$ involves (a1), therefore $g(4)\neq 0$. We know that $g(3)=0$.

Consequently $\mathscr{F}(m|m\equiv 2, 5 \pmod 9)<\infty$. Since for $m\equiv 8 \pmod 9$, $3\|4m+1$, we get

$$\mathscr{F}(m|m \equiv 8 (\text{mod}\, 9),\ (m, 2) = 1) \ll \mathscr{F}(4m|4m \equiv 5 (\text{mod}\, 9))+1 \ll$$
$$\ll \mathscr{G}(n|n \equiv 6 (\text{mod}\, 9))+1 \ll 1.$$

Thus $\mathscr{F}(n|(n, 2)=1, n\equiv 2 \pmod 3)<\infty$. Hence we have $\mathscr{G}(n|n\equiv 0 \pmod 6)<\infty$ and this gives $\mathscr{G}(n|(n, 6)=1)<\infty$ if $g(3^\gamma)\neq 0$ holds for a suitable γ.

Assume that $g(3^\gamma)\equiv 0$. Then $\mathscr{F}(n|n\equiv 2 \pmod 3)<\infty$, and so, by $f(3)\neq 0$, $\mathscr{F}(n|n\equiv 6 \pmod 9)<\infty$. So $\mathscr{G}(n|n\equiv 7 \pmod 9)<\infty$. By taking $n=2m$ and $n=4m$, $(m,2)=1$ into the bracket after $\mathscr{G}$, we get

$$\mathscr{G}(m|(m,2)=1,\ m\equiv 4, 8 \pmod 9)<\infty.$$

Let $m=1+3\cdot 4\cdot \nu$, $(\nu,6)=1$, $m\equiv 4 \pmod 9$, i.e. $\nu\equiv 1 \pmod 3$. We have $\mathscr{F}(\nu|\nu\equiv \equiv 1 \pmod 3), (\nu,2)=1)<\infty$, which leads to $\mathscr{F}(n|(n,6)=1)<\infty$. Hence the assertion follows by Lemma 11 immediately. □

(G) *The case* $f(3)\neq 0, f(4)=0$ *is impossible.*

Proof. Now $g(4)\neq 0$. Since the minimal α satisfying $f(2^\alpha)\neq 0$ is greater than 2, therefore $g(5)=0$ (see (A)). $f(5)=0$ is true from (B).

Since $g(5)=0$, therefore $\mathscr{F}(n|5\|n+1)\ll\mathscr{G}(n+1|5\|n+1)+1\ll 1$, and so

$$\mathscr{F}(n|n\equiv 4, 9, 14, 19 \pmod{25})<\infty. \tag{6.22}$$

By putting $n=3m$, $(m,3)=1$, from $f(3)\neq 0$ we get $\mathscr{F}(m|m\equiv 5^{-1}\{4,9,14,19\} \pmod{25}, (m,3)=1)<\infty$. We have $5^{-1}\equiv 12 \pmod{25}$, and so

$$\mathscr{F}(m|(m,3)=1,\ m\equiv -2, 8, 1, 18, 3 \pmod{25})<\infty.$$

Consequently

$$\mathscr{G}(n|(n-1,3)=1,\ n\equiv -1, 9, 19, 4 \pmod{25})<\infty, \tag{6.23}$$

and so for $n=4m$, $m\equiv 1 \pmod{25}$, $(4m-1,3)=1$ the condition stated in (6.23) holds, whence by $g(4)\neq 0$

$$\mathscr{G}(m|(4m-1,3)=1,\ (m,2)=1,\ m\equiv 1 \pmod{25})<\infty.$$

Let $m=\theta+1$. $(4m-1,3)=1$ is equivalent with $(\theta,3)=1$, while $(m,2)=1$ with $\theta=$even. Therefore

$$\mathscr{F}(\theta|2|\theta,\ 25|\theta,\ (3,\theta)=1)<\infty. \tag{6.24}$$

If there exists a suitable β such that $f(5^\beta)\neq 0$, then by choosing $\theta=2^\alpha\cdot 5^\beta \nu$, $(\nu,30)=1$ in (6.24), we have $\mathscr{F}(\nu|(\nu,30)=1)<\infty$, that is sufficient.

It has remained to consider the case when $f(5^\beta)=0$ $(\beta=1,2,\dots)$. Then $\mathscr{G}(n|n\equiv 1 \pmod 5)<\infty$. Since $g(2)\neq 0$, $g(4)\neq 0$, by putting there $n=2m$, $n=4m$, $(m,2)=1$, we deduce that $\mathscr{G}' m|(m,2)=1$, $m\equiv 3, 4 \pmod 5)<\infty$. By choosing $m=1+2^\alpha \nu$, $(\nu,2)=1$, we get

$$\mathscr{F}(\nu|(\nu,2)=1,\ \nu\equiv 2^{-\alpha}\{3,4\} \pmod 5)<\infty. \tag{6.25}$$

If $\alpha\equiv 2 \pmod 4$, then $2^\alpha\equiv 4 \pmod 5$, $2^{-\alpha}\equiv 4 \pmod 5$, while for $\alpha\equiv 4 \pmod 4$ we have $2^\alpha\equiv 1 \pmod 5$, $2^{-\alpha}\equiv 1 \pmod 5$. Consequently $\mathscr{F}(\nu|(\nu,2)=1, \nu\equiv\{2,3\} \pmod 5)<\infty$ holds in these cases. If we take $\nu+1=2\mu$, $\mu\equiv 2 \pmod 5$, $(\mu,2)=1$, then $\nu\equiv 3 \pmod 5$, $(\nu,2)=1$, and so

$$\mathscr{G}(\nu|\nu\equiv 3 \pmod 5,\ (\nu,2)=1)<\infty.$$

Hence $\mathscr{G}(n|(n,10)=1)<\infty$, and thus we can deduce easily the desired result.

If $\alpha\equiv 1 \pmod 4$ then $2^\alpha\equiv 2 \pmod 5$, $2^{-\alpha}\equiv 3 \pmod 5$; if $\alpha\equiv 3 \pmod 4$, then $2^\alpha\equiv 3 \pmod 5$, $2^{-\alpha}\equiv 2 \pmod 5$, and so $2^{-\alpha}\{3,4\}\equiv\{1,4\} \pmod 5$.

In these cases $\mathscr{G}(n|n\equiv 0 \pmod{10})<\infty$, and this gives the desired result, except when $g(5^\beta)=0$ $(\beta=1, 2 \dots)$.

Assume that $g(5^\beta)=0$ $(\beta=1, 2, \dots)$. Then $\mathscr{F}(n|n\equiv 4 \pmod 5)<\infty$. Put $n=2^\alpha m\equiv 4 \pmod 5$, $(m, 2)=1$. We have

$$\mathscr{F}(m|m \equiv 2 \pmod 5),\ (m, 2) = 1) < \infty \quad \text{if} \quad \alpha \equiv 1 \pmod 4, \tag{6.26}$$

$$\mathscr{F}(m|m \equiv 3 \pmod 5),\ (m, 2) = 1) < \infty \quad \text{if} \quad \alpha \equiv 3 \pmod 4. \tag{6.27}$$

In the case $\alpha\equiv 1 \pmod 4$ we put $4n-1\equiv 2 \pmod 5$, i.e. $n\equiv 2 \pmod 5$, $(n,2)=1$, and by $g(4)\neq 0$, from (6.26) we deduce that

$$\mathscr{G}(n|(n, 2) = 1,\ n \equiv 2 \pmod 5)) < \infty. \tag{6.28}$$

In the case $\alpha\equiv 3 \pmod 4$ we put $2n-1\equiv 3 \pmod 5$, i.e. $n\equiv 2 \pmod 5$, and by $g(2)\neq 0$, from (6.26) we deduce (6.28). Collecting our results we have $\mathscr{G}(n|(n, 10)=1)<\infty$, that by Lemma 11 leads to $f, g\in\mathscr{L}$.

The proof of Lemma 12 is finished. □

LEMMA 13. *Let* $f(2)\neq 0$, $g(2)\neq 0$, $f(3)=0$. *Then* $f, g\in\mathscr{L}$.

PROOF. Assume in the contrary that $f, g\notin\mathscr{L}$. Since $f(3)=0$, therefore $\mathscr{G}(n|n\equiv 4, 7 \pmod 9))<\infty$. By putting $n=2m$, $(m, 2)=1$, from $g(2)\neq 0$ we deduce that $\mathscr{G}(m|(m, 2)=1, m\equiv 2, 8 \pmod 9))<\infty$. Hence $\mathscr{F}(n|2\|n, n\equiv 1, 7 \pmod 9))<\infty$, i.e.

$$\mathscr{F}(m|(m, 2) = 1,\ m \equiv 5, 8 \pmod 9)) < \infty. \tag{6.29}$$

(6.29) involves that $\mathscr{G}(n|2|n, n\equiv 6 \pmod 9))<\infty$, $\mathscr{G}(n|n\equiv 0 \pmod{18}))<\infty$.

If there exists a $\gamma\geqq 2$ such that $g(3^\gamma)\neq 0$, then $\mathscr{G}(n|(n, 6)=1)<\infty$, that is sufficient. If $g(3)\neq 0$, then from the first inequality, by putting there $n=6\mu$, $\mu\equiv 1 \pmod 3$, we have

$$\mathscr{G}(\mu|(\mu, 2) = 1,\ \mu \equiv 1 \pmod 3)) < \infty. \tag{6.30}$$

If there does not exist any prime power $P\equiv -1 \pmod 3$ such that $g(P)\neq 0$, then $g(n)=0$ for every odd n, $\equiv -1 \pmod 3$, and we are ready. If there exists such a P, then

$$\mathscr{G}(\mu|(\mu, 2P) = 1,\ \mu \equiv -1 \pmod 3)) < \infty$$

and this is enough to guarantee that $g, f\in\mathscr{L}$.

If $g(3^\beta)=0$ for $\beta=1, 2, \dots$, then $\mathscr{F}(n|n\equiv 2 \pmod 3))<\infty$, and by putting $n=2m$, $(m, 2)=1$, we deduce that $\mathscr{F}(m|m\equiv 1 \pmod 3))<\infty$. This involves that $\mathscr{F}(n|(n, 6)=1)<\infty$ which by Lemma 11 leads to $f, g\in\mathscr{L}$. □

LEMMA 14. *Assume that* $f(2)\neq 0$, $g(2)\neq 0$, $g(4)=0$. *Then* $f, g\in\mathscr{L}$.

PROOF. Assume that $f, g\notin\mathscr{L}$. From $g(4)=0$ we get $\mathscr{F}(n|n\equiv 3 \pmod 8))<\infty$. Since $g(3)\neq 0$, therefore $\mathscr{F}(n|n\equiv 1 \pmod 8), (n, 3)=1)<\infty$. Hence $\mathscr{G}(n|(n-1, 3)=1$, $n\equiv 2 \pmod 8))<\infty$, whence by $g(2)\neq 0$ we deduce that $\mathscr{G}(m|2m-1, 3)=1$, $m\equiv 1 \pmod 4))<\infty$. By putting $m=1+2^\gamma t$ $(\gamma\geqq 2)$, we deduce that

$$\sum_{\gamma\geqq 2} \mathscr{F}(2^\gamma t|(2^{\gamma+1}t+1, 3) = 1,\ (t, 2) = 1) < \infty.$$

Assume first that $f(2^{\gamma})\neq 0$ for a suitable $\gamma\geqq 2$. Then, from the last inequality we get

$$\mathcal{F}\left(t|(2^{\gamma+1}t+1, 3) = 1,\ (t, 2) = 1\right) < \infty,$$

whence by $f(3)\neq 0$, after substituting $t=3\tau$, $(\tau, 6)=1$, we get $\mathcal{F}(t|(t, 6)=1)<\infty$ which leads to $f, g\in\mathcal{L}$.

Let us assume finally that $f(2^{\alpha})=0$ ($\alpha=2, 3, \dots$). Then $\mathcal{G}(n|n\equiv 1 \pmod 4)<\infty$. If $g(N_0)\neq 0$ for a suitable $N_0\equiv -1 \pmod 4$, then $\mathcal{G}(n|n\equiv -1 \pmod 4), (n, N_0)=1)<<\infty$, and so $\mathcal{G}(n|(n, 2N_0)=1)<\infty$, while in the case $g(n)=0$ for all elements of $n\equiv -1 \pmod 4$, we get $\mathcal{G}(n|(n, 2)=1)<\infty$ immediately.

By this the proof has been finished. □

LEMMA 15. *If* $f(2)\neq 0, g(2)\neq 0, g(4)\neq 0, f(4)=0$, *then* $f, g\in\mathcal{L}$.

PROOF. We assume that $f, g\notin\mathcal{L}$. Now (4.4) holds, i.e.

$$\sum_{n\equiv 3 \pmod 8} \frac{1}{n}|Cg(n)-f(n)| < \infty, \quad C = \frac{g(4)}{g(2)f(2)}. \tag{6.31}$$

Furthermore, from $f(4)=0$ we have

$$\mathcal{G}\left(n|n \equiv 5 \pmod 8\right) < \infty. \tag{6.32}$$

Let us assume first that $g(n_0)\neq 0$ holds for a suitable $n_0\equiv 7 \pmod 8$. Then, from (6.32) we get

$$\mathcal{G}\left(n|n \equiv 3 \pmod 8),\ (n, n_0) = 1\right) < \infty, \tag{6.33}$$

whence

$$\mathcal{F}\left(t|t \equiv 1 \pmod 4),\ (2t+1, n_0) = 1\right) < \infty.$$

After substituting $t+1=2\xi$, $(\xi, 2)=1$, this leads to

$$\mathcal{G}\left(\xi|(\xi, 2) = 1,\ (4\xi-1, n_0) = 1\right) < \infty.$$

We finish the proof by applying Lemma 3.

Assume now that $g(n)=0$ holds for every $n\equiv 7 \pmod 8$. Consequently $\mathcal{F}(n|n\equiv 6 \pmod 8))<\infty$, and by $f(2)\neq 0$ we get $\mathcal{F}(n|n\equiv 3 \pmod 4))<\infty$. By Lemma 13 we may assume that $f(3)\neq 0$, and so from the last inequality we get $\mathcal{F}(n|n\equiv 1 \pmod 4), (n, 3)=1)<\infty$. So $\mathcal{F}(n|(n, 6)=1)<\infty$, we are ready.

By this all the possible cases have been discussed. Theorem 4 is proved.

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE
BUDAPEST, MÚZEUM KRT. 6—8.
H-1088

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SIMPLE RINGS WHOSE LOWER RADICALS ARE ATOMS*

B. J. GARDNER (Hobart)

Introduction

A 1972 paper of Snider [14] initiated the study in detail of lattices of radicals of associative rings. The subject of radical lattices has subsequently attracted a fair amount of attention in various contexts: much work has been done with rings and algebras (see, e.g. [7], [13], and their bibliographies) and, independently, with modules (e.g. [5], [6], [15]). Almost all of this work has, however, been concerned with hereditary radicals, [6] being a notable exception.

Snider [14] proved that in the lattice of hereditary radicals of associative rings, the lower radical $L(S)$ defined by every simple ring S is an atom. Not much is known about atoms in the lattice of all radicals (of associative rings), not even about those simple rings S for which $L(S)$ is an atom. Problem 7 of the recent book of Andrunakievich and Ryabukhin [3] asks for a description of such simple rings. Certainly not every simple ring has such a lower radical: the zeroring $Z(p)^0$ on a cyclic group of prime order p defines a lower radical class which contains the zeroring $Z(p^\infty)^0$, while the latter, having no maximal ideals, defines a properly smaller lower radical class.

Our approach to the problem is to seek "$Z(p^\infty)$-like" rings which will similarly disqualify other simple rings. The existence of such rings is closely connected with the existence of non-trivial ring extensions of a simple ring by itself.

It has been noted by Puczyłowski [12] that $L(S)$ is an atom whenever S is simple with identity. The latter property is equivalent to S being a direct summand whenever it is an ideal. We show that $L(S)$ is an atom when S (simple) satisfies the following, weaker condition:

$$(*) \qquad S \triangleleft R \,\&\, R/S \cong S \Rightarrow \exists I \triangleleft R \quad \text{such that} \quad R = S \oplus I.$$

We show also that $L(S)$ being an atom is equivalent to S satisfying a condition which appears to be (and possibly is) a good deal weaker that (*). It remains unknown whether S has to satisfy (*) if $L(S)$ is an atom. This being so, there is some interest in the status of this implication in other settings. We show that the implication is vacuously true — $L(S)$ is never an atom; S never satisfies (*) — in the class of all (not necessarily associative) rings and that it is sometimes false in the class of modules over a ring.

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1. Associative rings

All rings considered in this section are associative.

THEOREM 1.1. *Let S be a simple ring. Consider the following conditions:*
(i) *S satisfies* (*).
(ii) *$L(S)$ is an atom in the lattice of all radicals.*
(iii) *There is a ring R with a series*

$$0 = I_0 \lhd I_1 \lhd I_2 \lhd \ldots \lhd I_n \lhd I_{n+1} \lhd \ldots$$

such that $I_{n+1}/I_n \cong S$ for each n, $R = \bigcup_{n \geqq 0} I_n$ and R has no ideals but the I_n.
We have the following implications: (i)$\Rightarrow$(ii)$\Rightarrow\sim$(iii).

PROOF. (i)$\Rightarrow$(ii). Simple zerorings do not satisfy (*), so it can be assumed that $S^2 = S$. Let $A \neq 0$ be a ring in $L(S)$. Since the lower radical construction over $\{S\}$ terminates in two steps, we may assume that $S \lhd A$. We consider first the case where S is an essential ideal. Suppose $S \neq A$. Then A/S is a non-zero ring in $L(S)$ so as above, A/S has an ideal $I/S \cong S$. Since S satisfies (*), we have $I = S \oplus J$ for some $J \lhd I$ with $J \cong S$. But then, as $J^2 = J$, we have $J \lhd A$ and $J \cap S = 0$, which violates the assumption that S is essential. Thus $A = S$ if S is essential. If S is not essential, let $M \lhd A$ be maximal with respect to having zero intersection with S. Then $(S+M)/M$ is essential in A/M (see [1] or [4]) and $(S+M)/M \cong S/S \cap M \cong S$. Thus, as above, $A/M \cong S$. In any case, A has S as a homomorphic image, so $L(S) \subseteqq L(A) \subseteqq L(S)$. This proves that $L(S)$ is an atom.
(iii)$\Rightarrow\sim$(ii). Let R be as described. Clearly $R \in L(S)$; clearly also, R has no maximal ideals, so $S \notin L(R)$ and thus $L(R) \subsetneqq L(S)$. □

Two comments should be made about condition (iii) of Theorem 1.1.

Firstly, if R is as described, then the exact sequences

$$0 \to I_1 \cong S \to I_2 \to I_2/I_1 \cong S \to 0, \quad 0 \to I_2 \to I_3 \to I_3/I_2 \cong S \to 0,$$

etc. are non-split. Thus (iii) is (ostensibly) a stronger version of $\sim$(*). Moreover, the existence of non-split exact sequences

$$0 \to S \to T_1 \to S \to 0, \quad 0 \to T_1 \to T_2 \to S \to 0, \ldots$$

subject to the requirement that there be suitably compatible isomorphisms $T_{n+1}/S \cong$ $\cong T_n$ (cf. [10], Lemma 7) implies the existence of a ring R satisfying (iii), viz. $R = \bigcup T_n$ (or $\varinjlim T_n$).

The second point to note is that, when S is simple and idempotent, $L(S)$ consists of all rings A having a series

$$0 = I_0 \subseteqq I_1 \subseteqq I_2 \subseteqq \ldots \subseteqq I_\alpha \subseteqq I_{\alpha+1} \subseteqq \ldots \subseteqq I_\mu = A$$

where $I_\alpha \lhd A$ for each α, $I_{\alpha+1}/I_\alpha \cong S$ for each α and $I_\beta = \bigcup_{\alpha<\beta} I_\alpha$ when β is a limit ordinal. (See [2], Proposition 2.2 or [16].) Thus simple rings whose lower radicals are atoms can be described in terms of the possible order types of composition series of members of their lower radical classes.

It remains unclear whether (*) is equivalent to $\sim$(iii). In fact not too much is known about (*) for simple rings.

Let K be a field, $F(K)$ the ring of linear transformations of finite rank of an $\aleph_0$-dimensional K-vector space. Leavitt [8] has shown that $F(K)$ does not satisfy (*). By an extension of the argument used in the proof of this, Leavitt and van Leeuwen [10] have shown that, in fact, $F(K)$ satisfies (iii). Thus $L(F(K))$ is not an atom.

Of course the zerorings $Z(p)^0$ also fail to satisfy (*) and correspondingly the zerorings $Z(p^\infty)^0$ satisfy (iii).

Non-unital examples of simple rings satisfying (*) have been obtained by Leavitt [9] and Leavitt and van Leeuwen [11].

Some time after the submission of this paper the author was informed that the results of this section had, in effect, been obtained also by K. I. Beidar and by Halina Korolczuk.

2. Non-associative rings

In this section we shall work in the class of all (not necessarily associative) rings. Radical classes here also form a complete lattice, the only complication being that we must define

$$\bigvee_{\lambda\in\Lambda}\mathcal{R}_\lambda = L\Big(\bigcup_{\lambda\in\Lambda}\mathcal{R}_\lambda\Big)$$

(where $L(\ \ )$ is the lower radical) as semi-simple classes are not well-behaved with respect to intersections ([2], Proposition 1.2 a⁰). It turns out that we can always construct a "$Z(p^\infty)$-like" ring for a simple ring.

THEOREM 2.2. *Let S be a simple ring. Then $L(S)$ is not an atom in the lattice of all radicals.*

PROOF. If $S\cong Z(p)^0$, we can argue as in the associative case; thus we may assume that $S^2=S$.

For $n=1, 2, \ldots$, let R_n be the ring which is additively the direct sum of n copies of S and whose multiplication is given by

$$(a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n) =$$
$$=\Big(\sum_{1\le j\le n}\sum_{1\le i\le j} a_ib_j,\ \sum_{2\le j\le n}\sum_{2\le i\le j} a_ib_j,\ \ldots,\ a_{n-1}b_{n-1}+a_{n-1}b_n+a_nb_n,\ a_nb_n\Big).$$

Then for $m<n$, R_n contains a copy of R_m in its first m components — we shall call this copy R_m — and $R_m\lhd R_n$ for all $m\le n$. Also $R_1\cong S$.

Let I be a non-zero ideal of R_n. Suppose I contains an element $(a_1, a_2, \ldots, a_n)$ with $a_1\ne 0$. Then for every $b\in S$, we have

$$(a_1b, 0, 0, \ldots, 0) = (a_1, a_2, \ldots, a_n)(b, 0, 0, \ldots, 0)\in I$$

and

$$(ba_1, 0, 0, \ldots, 0) = (b, 0, 0, \ldots, 0)(a_1, a_2, \ldots, a_n)\in I.$$

Since S has zero two-sided annihilator, this means that $I\cap R_1\ne 0$, so that $I\supseteq R_1$. If, on the other hand, I contains an element $(a_1, a_2, \ldots, a_n)$ with $a_1=\ldots=a_{m-1}=$ $=0\ne a_m$, then I contains any element

$$(\underbrace{0, \ldots, 0, b}_{m}, 0, \ldots, 0)(a_1, \ldots, a_n) = (ba_m, \ldots),$$

as well as any

$$(a_1, a_2, \ldots, a_n)(0, \ldots, \underset{\longleftarrow m \longrightarrow}{-b, b}, 0, \ldots, 0) =$$

$$= (a_1 0 + (a_1 + a_2)0 + \ldots + (a_1 + \ldots + a_{m-1})(-b) + (a_1 + \ldots + a_m)b + 0 + \ldots + 0, \ldots) =$$

$$= (a_m b, \ldots).$$

Since S has zero annihilator, we see that I must contain an element with non-zero first component, so as above, $I \supseteqq R_1$.

The natural projection of R_n onto the last k components, for any k, produces an exact sequence.

$$0 \to R_{n-k} \to R_n \to R_k \to 0.$$

Moreover, if $i \leqq m \leqq n$, the induced diagram

$$\begin{array}{ccc} R_m & \subseteqq & R_n \\ \downarrow & & \downarrow \\ R_{m-i} & \subseteqq & R_{n-i} \end{array}$$

commutes. Using this, we can show by induction that the only ideals of R_n are $0, R_1, R_2, \ldots, R_n$. Let $R = \varinjlim R_n =$ the ring on the direct sum of $\aleph_0$ copies of S constructed using the obvious generalization of our multiplication. Then R has a series

$$0 \triangleleft R_1 \triangleleft R_2 \triangleleft \ldots \triangleleft R_n \triangleleft R_{n+1} \triangleleft \ldots \triangleleft R_\omega = R$$

where $R_n \triangleleft R$ and $R_{n+1}/R_n \cong S$ for each n and $R_\omega = \bigcup_{n<\omega} R_n$. It follows that R is in $L(S)$.

Let $0 \neq J \triangleleft R$. Then $J \not\supseteq R_n$ for some n, so $J \cap R_n = R_m$ for some $m < n$. Now for $k > n$, we have $J \cap R_k \supseteqq J \cap R_n = R_m$. If this inclusion were ever proper, we should have $J \cap R_k \supseteqq R_{m+1}$, whence $R_m = J \cap R_n \supseteqq R_{m+1}$ — a contradiction. It follows that $J = R_m$. In particular, J is not maximal, so $L(R)$ contains no simple ring. Thus $L(R) \subsetneqq L(S)$ and $L(S)$ is not an atom. $\square$

Thus (in a sense trivially) (*) is equivalent to $L(S)$ being an atom in the lattice of radicals for the universal class of all rings.

3. A remark on modules

By the same argument (actually a slightly simpler one) as was used to prove Theorem 1.1, we can show that if S is a simple (unital) module over a ring R, and if S satisfies (*), then $L(S)$ is an atom in the lattice of all radicals of unital R-modules. The following example shows that the converse need not be true.

Example 3.1. Let p be a prime. Then the only simple module over the ring Z_{p^2} of integers $\bmod p^2$ is $Z(p)$, and this module does not satisfy (*). However, $L(Z(p)) = \mathrm{Mod}(Z_{p^2})$ is the only non-zero radical class and hence an atom. (Anybody who is unhappy with this example can modify it by replacing Z_{p^2} by $Z_{p^2} \oplus R$ for any unital ring R.) The attempt to build a "$Z(p^\infty)$-like" module here (cf. Theorem 1.1) fails: any non-split element of $\mathrm{Ext}_{Z_{p^2}}(Z(p), Z(p))$ has the form

$$0 \to Z(p) \to Z(p^2) \to Z(p) \to 0$$

while any exact sequence

$$0 \to Z(p^2) \to M \to Z(p) \to 0$$

with $M \in \operatorname{Mod}(Z_{p^2})$ splits over Z, and hence over Z_{p^2}.

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MATHEMATICS DEPARTMENT
UNIVERSITY OF TASMANIA
GPO BOX 252C
HOBART. TAS. 7001
AUSTRALIA

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DIVERGENCE OF LAGRANGE INTERPOLATION ON A SET OF SECOND CATEGORY

P. VÉRTESI (Budapest)

1. Introduction

1.1. Let X be a triangular matrix of interpolation in $[-1, 1]$, i.e. $X=\{x_{kn}\}$, $k=1, 2, ..., n$; $n\in\mathbf{N}$ (=the set of natural numbers) with

$$(1.1)\qquad -1 \equiv x_{n+1,n} \leqq x_{nn} < x_{n-1,n} < \ldots < x_{2n} < x_{1n} \leqq 1 \equiv x_{0n}, \quad n\in\mathbf{N}.$$

Further let for $f\in C$ (=f is continuous on $[-1, 1]$)

$$(1.2)\qquad L_n(f, x) = L_n(f, X, x) = \sum_{k=1}^{n} f(x_{kn})\, l_{kn}(X, x),$$

$$l_{kn}(x) = l_{kn}(X, x) = \frac{\Omega_n(X, x)}{\Omega_n'(X, x_{kn})\,(x-x_{kn})}, \quad \Omega_n(x) = \Omega_n(X, x) = \prod_{k=1}^{n} (x-x_{kn}).$$

Let $\omega(t)\not\equiv 0$ be a modulus of continuity on $[0, 2]$ (see [1], 3.2). Finally, let

$$C(\omega_m) = \{f;\ \omega_m(f, t) = O_f(\omega_m(t))\}, \quad C^*(\omega_m) = \{f;\ \omega_m(f, t) = o_f(\omega_m(t))\},$$

where $\omega_m(f, t)$ is the m-th modulus of smoothness of f, $\omega_m(t)=\omega(t^m)$.

As it was proved in [2] by P. Erdős and me (see further [12], 3.3), *for any X there exists an $F\in C$ such that for the Lagrange interpolatory polynomials L_n*

$$(1.3)\qquad \varlimsup_{n\to\infty} |L_n(F, X, x)| = \infty \quad on \quad S\subset[-1, 1]$$

where $|S|=2$ and S is of second category. That means supposing merely continuity we obtain a rather strong divergence theorem. But if we tried to characterize $\omega(F, t)$ or to give lower estimations for $|L_n(F, X, x)-F(x)|$ using the Lebesgue function

$$\lambda_n(x) = \lambda_n(X, x) = \sum_{k=1}^{n} |l_{kn}(X, x)|$$

and $\omega(F, t)$, we should encounter practically insolvable difficulties.

With another word if we want to involve the modulus of continuity and the Lebesgue function, generally, we are not able to prove results involving measure, at least not at present (compare 2.2).

1.2 To be more precise we quote some recent results of this type. In his paper [10] O. Kis proved as follows.

If

$$(1.4)\qquad \lim_{t=+0} \frac{\omega_m(t)}{t^m} = \infty$$

then for any fixed X *there exists an* $f\in C(\omega_m)$ *such that*

$$\overline{\lim_{n\to\infty}}\frac{\|L_n(f,X,x)-f(x)\|}{\lambda_n\omega_m(d_n)}\geqq 1.$$

Here $\|.\|$ is the maximum norm on $[-1,1]$, $\lambda_n=\lambda_n(X)=\|\lambda_n(x)\|$ (Lebesgue constant) and

(1.5) $$d_n=\min_{1\leqq k\leqq n-1}(x_k-x_{k+1}).$$

Recently W. Dickmeis and P. Nessel [4] proved that *for any* $\omega_2(t)$ *there exists an* $f_1\in C(\omega_2)$ *such that*

(1.6) $$\overline{\lim_{n\to\infty}}\frac{|L_n(f,X,x)-f(x)|}{\omega_2(d_n)}=\infty$$

on a dense set of second category in $[-1,1]$. *(X is given.)* Questions of different type can be considered by investigating the expression $\omega(t)|\ln t|$.

As it is wellknown, if $\lim\limits_{t=0}\omega(t)|\ln t|=0$, then $\lim\limits_{n\to\infty}\|L_n(f,T,x)-f(x)\|=0$ for *any* $f\in C$ (where, as usual $T=\{\cos\frac{2k-1}{2n}\pi\}$, $k=1,2,\dots,n$; $n\in\mathbf{N}$, is the Chebyshev matrix), i.e. to obtain divergence type results for an *arbitrary matrix* X we have to suppose, say, that

(1.7) $$\underline{\lim_{t=0}}\,\omega(t)|\ln t|>0.$$

The case

(1.8) $$\lim_{t=0}\omega(t)|\ln t|=\infty$$

was investigated by A. A. Privalov [3], [6]. He stated that if X is given and $\lim\limits_{t=0}\omega(t)|\ln t|=\infty$[1] then there exists an $f\in C^*(\omega)$ such that $\overline{\lim\limits_{n\to\infty}}|L_n(f,X,x)|=\infty$ on a dense set of second category in $[-1,1]$. Unfortunately, there is a mistake in his proof.[2]

[1] Actually, he supposed the weaker condition

(1.8*) $$\overline{\lim_{t=0}}\,\omega(t)|\ln t|=\infty,$$

but he used the condition (1.8). I was not able to carry out the proof with (1.8*).

[2] Namely, in [6], in the proof of Lemma 6, he states that if $X\subset[-1,1]$ and

$$\max_{-1\leqq x\leqq 1}\min_{1\leqq k\leqq n}|x-x_{kn}|>2\ln\ln n/n \quad\text{supposing } n\geqq n_1,$$

then there exists an $x_0\in[-1,1]$ and a subsequence $\{n_i\}$ such that

$$\min_{1\leqq k\leqq n}|x_0-x_{kn}|\geqq 2\ln\ln n/n \quad\text{if}\quad n=n_1,n_2,\dots.$$

But this is not the case. Indeed, consider the matrix $x_{1n}=1$, $x_{2n}=1-6\ln\ln n/n$, $x_{kn}=x_{2n}-(k-2)(2-6\ln\ln n/n)/(n-2)$, $k=3,4,\dots,n$; $n=3,4,\dots$. (Obviously $x_0\neq 1$ because 1 is a node for each n. On the other hand, if $x_0\in[-1,1)$, then $x_{2n}>x_0$ if $n\geqq n_1$ (where n_1 is the first integer for which $6\ln\ln n_1/n_1<1-x_0$). But then $\min\limits_{1\leqq k\leqq n}|x_0-x_{kn}|\leqq(1-3\ln\ln n/n)/(n-2)<1/(n-2)<2\ln\ln n/n$ if $n\geqq n_2$.)

The above incorrect statement was used, for example, in his paper [16], too.

1.3 The aim of this paper is threefold. First, we want to answer a question raised by S. B. Steckin at a conference in Gdansk, 1979 and later at a conference in Varna, 1981: If $\omega(t)|\ln t|=1$, can we prove or not for a suitable function $f\in C(\omega)$ a corresponding divergence theorem on a set of second category? Secondly I shall give a correct proof of the above statement of A. A. Privalov, a proof, which, because of the mentioned mistake, essentially differs from the original one. Finally I prove and quote some further theorems.

2. Results

2.2 First we state the next

THEOREM 2.1. *Let X be given.*

a) *If we suppose* (1.7) *then there exists an $f_1\in C(\omega)$ such that*

$$\overline{\lim_{n\to\infty}}\,|L_n(f_1, X, x)-f_1(x)| \geqq 1 \tag{2.2}$$

on a dense set of second category in $[-1, 1]$.

b) *If we suppose* (1.8) *then for an $f_2\in C^*(\omega)$*

$$\overline{\lim_{n\to\infty}}\,|L_n(f_2, X, x)| = \infty \tag{2.2}$$

on a dense set of second category in $[-1, 1]$.

Sometimes the next result is sharper.

THEOREM 2.2. *If for a given X we have* (1.7), *then for a certain $f_3\in C(\omega)$*

$$\overline{\lim_{n\to\infty}}\,\frac{|L_n(f_3, X, x)-f_3(x)|}{\omega\left(\frac{1}{\mu_n}\right)\ln \mu_n} \geqq 1 \tag{2.3}$$

on a dense set of second category in $[-1, 1]$. *Here $\mu_n=n^{3+\varepsilon}$ where $\varepsilon>0$ is arbitrarily small.*

In many cases we can apply the following result.

COROLLARY 2.3. *If for a given X, $c>0$ and T_0 we have*

$$\omega(T)|\ln T| \leqq c\omega(t)|\ln t| \quad 0 < t \leqq T \leqq T_0 \tag{2.4}$$

then with a certain $f_4\in C(\omega)$

$$\overline{\lim_{n\to\infty}}\,\frac{|L_n(f_4, X, x)-f_4(x)|}{\omega\left(\frac{1}{n}\right)\ln n} \geqq 1 \tag{2.5}$$

on a dense set of second category in $[-1, 1]$.

(From now on $c, c_0, c_1, \ldots$ are arbitrary absolute positive constants.)
Indeed, by $\mu_n>n$ and (2.4) we obtain Corollary 2.3 by the previous theorem.

2.2. It is worthwhile remarking that for the matrix T we can prove a stronger result:

If we have (2.4) *then with a certain* $f_5 \in C(\omega)$

$$\overline{\lim_{n\to\infty}} \frac{|L_n(f_5, T, x) - f_5(x)|}{\omega\left(\frac{1}{n}\right)\ln n} \geqq 1 \quad \text{for every} \quad x \in [-1, 1] \tag{2.6}$$

(see P. Vértesi [15], Theorem 1.4).

3. Proofs

3.1. The main ideas are as follows. We use a general divergence theorem (Part 3.2) which requires a rather deep analysis of the Lebesgue function restricted for a subinterval (Part 3.4). By this divergence theorem, in Theorems 1 and 2 first we construct a *countable* dense set. (Namely, for every subinterval $[a, b]$ we choose a proper point $\tilde{x}_0$ of the interval considered.) To obtain $\tilde{x}_0$ we consider two types of intervals (Parts 3.5 and 3.6). To obtain the set of second category we use a generalization of a nice idea of Orlicz (Part 3.8).

3.2. Proof of Theorem 2.2. We intend to use the following statement which is a special case of a recent work of W. Dickmeis and R. J. Nessel [4], Theorems 1 and 2.

Let B be a Banach space, Y a normed linear space (with norms $\|\cdot\|_B$, $\|\cdot\|_Y$ respectively), and let $[B, Y]$ be the space of bounded linear operators from B into Y.

Theorem 3.1 ([4]). *Let* $\{T_{nj}\}_{n, j\in\mathbf{N}} \subset [C, Y]$ *and let* $\{\delta_{nj} > 0\}_{n, j\in\mathbf{N}}$ *satisfy* $\lim\limits_{n\to\infty} \delta_{nj} = 0$ *for each* $j \in \mathbf{N}$. *Suppose that for each* $n, j \in \mathbf{N}$ *there exists a function* g_{nj} *such that for a fixed* m

$$\begin{cases} g_{nj}^{(m)} \in C, \\ \|g_{nj}\|_C \leqq c_3, \\ \|g_{nj}^{(m)}\|_C \leqq c_4 \delta_{nj}^{-m}. \end{cases} \tag{3.1}$$

If

$$\overline{\lim_{n\to\infty}} \|T_{nj} g_{nj}\|_Y \geqq c_5 > 0 \tag{3.2}$$

then for $\omega_m(t)$ *satisfying* (1.4) *one can find an* $f \in C(\omega_m)$, *independent of* $j \in \mathbf{N}$, *for which*

$$\overline{\lim_{n\to\infty}} \frac{\|T_{nj} f\|_Y}{\omega_m(\delta_{nj})} \geqq 1 \quad \text{for each} \quad j \in \mathbf{N}. \tag{3.3}$$

We allow $c_5 = \infty$ *when one can find an* $\tilde{f} \in C^*(\omega_m)$ *for which*

$$\overline{\lim_{n\to\infty}} \frac{\|T_{nj} \tilde{f}\|_y}{\omega_m(\delta_{nj})} = \infty \quad \text{for each} \quad j \in \mathbf{N}. \tag{3.4}$$

Remark. Compare the conditions and the theorem with P. Vértesi [5].

3.3. To apply this theorem first we prove — roughly speaking — that in an arbitrary interval $[a, b]\subset[-1, 1]$ there exists a point $\tilde{x}_0\in[a, b]$ and a "good" function $g_n(x)$ such that $\sum_{\substack{k \\ x_{kn}\in[a,b]}} g_n(x_{kn})l_{kn}(\tilde{x}_0)\geqq c\ln n$ for infinitely many n.

First we need some definitions. Let $J_{kn}=[x_{k+1,n}, x_{kn}]$, $1\leqq k\leqq n-1$. For the interval $[a, b]\subset[-1, 1]$ let $s_n=s_n(a, b)=\bigcup J_{kn}$ where $J_{kn}\subset[a, b]$ and $|J_{kn}|\leqq \leqq \varrho_n\overset{\text{def}}{=}n^{-1/6}$, $S_n=S_n(a, b)=(a, b)\setminus s_n$.

3.4. Now we prove an interesting statement for the "restricted Lebesgue function" $\chi_n(X, s_n, x)=\chi_n(s_n, x)$ (3.4—3.4.3).

LEMMA 3.2. *Let $[a, b]\subset[-1, 1]$ be a fixed interval. Then for $\varepsilon>0$ there exist a constant $\eta=\eta(\varepsilon)>0$ and sets $H_n\subset s_n(a, b)$, $|H_n|\leqq(b-a)\varepsilon$, such that*

$$(3.5)\qquad \chi_n(s_n, x)\overset{\text{def}}{=}\sum_{\substack{k \\ x_k\in s_n}}|l_{kn}(x)|>\eta(\varepsilon)\ln n \quad \textit{if} \quad x\in s_n\setminus H_n \quad \textit{and} \quad n\geqq n_0(\varepsilon).^{3}$$

Here $\eta=c_0\varepsilon^3$ [where $c_0=(6\cdot 8\cdot 144\cdot 56)^{-1}$]; on the other hand $n_0(\varepsilon)$ depends on the length of $[a, b]$, too.

The next proof is analogous to P. Erdős, P. Vértesi [13] Part 3.3—3.6.1. We introduce the following notations.

$$J_k(q)=J_{kn}(q)=[x_{k+1}+q|J_k|,\quad x_k-q|J_k|]\quad (1\leqq k\leqq n-1),$$

where $0\leqq q\leqq 1/2$. Let $z_k=z_{kn}(q)$ be defined by

$$(3.6)\qquad |\omega_n(z_k)|=\min_{x\in J_k(q)}|\omega_n(x)|,\quad k=0, 1, \dots, n,$$

finally let

$$|J_i, J_k|=\max(|x_{i+1}-x_k|,\ |x_{k+1}-x_i|)\quad (0\leqq i,\ k\leqq n).$$

In [2], Lemma 4.2 we proved

LEMMA 3.3. *If $1\leqq k, r<n$ then for arbitrary $0<q\leqq 1/2$*

$$(3.7)\qquad |l_k(x)|+|l_{k+1}(x)|\geqq q^2\frac{|\omega_n(z_r)|}{|\omega_n(z_k)|}\frac{|J_k|}{|J_r, J_k|}\quad \textit{if}\quad x\in J_r(q).$$

We shall also use Lemma 3.3 from [13] which can be stated as follows (see further Vértesi [14], Lemma 3.3).

LEMMA 3.4. *Let $I_k=[a_k, b_k]$, $1\leqq k\leqq t$, $t\geqq 2$, be any t intervals in $[a, b]$ with $|I_k\cap I_j|=0$ $(k\neq j)$, $|I_k|\leqq\varrho$ $(1\leqq k\leqq t)$, $\sum_{k=1}^{t}|I_k|=\mu$. Supposing that for a certain integer $R\geqq 2$ we have $\mu\geqq 2^R\varrho$, there exists an index s, $1\leqq s\leqq t$, such that*

$$(3.8)\qquad S=\sum_{k=1}^{t}\frac{|I_k|}{|I_s, I_k|}\geqq\frac{R\mu}{4(b-a)}.$$

I_s will be called accumulation interval of $\{I_k\}_{k=1}^{t}$.

---

[3] The meaning of $\chi_n(X, A, x)$ where $A\subset[-1,1]$ is analogous to (3.5).

(Here and later mutatis mutandis we apply the notations of 3.4 for arbitrary intervals.)

Note that we do not require $b_k \leqq a_{k+1}$.

From now on we suppose $s_n \neq \emptyset$, $n=1, 2, \dots$. (If $s_n = \emptyset$, Lemma 3.2 is trivial.)

3.4.1. Suppose $x \in J_{kn}(q) \subset s_n$ $(1 \leqq k \leqq n-1)$. Whenever $\chi_n(s_n, x) \leqq \eta(\varepsilon) \ln n$ (η will be determined later), the point x, the intervals J_{kn} and $J_{kn}(q)$, finally the index k will be called *exceptional*. Let $q=\varepsilon/12$.

We shall prove

$$\sum_k{}' |J_{kn}| \overset{\text{def}}{=} \mu_n \leqq \frac{\varepsilon}{6}(b-a) \quad (n \geqq n_0 = n_0(\varepsilon)). \tag{3.9}$$

Here and later the dash indicates that the summation is extended only over the exceptional indices k.

To prove (3.9) it is enough to consider those indices $\{n_i\}_{i=1}^{\infty} \overset{\text{def}}{=} N_1$ for which $\mu_{n_i} \geqq \varepsilon(b-a)/10$, say.

We can apply Lemma 3.4 for the exceptional J_{kn}'s with $\mu=\mu_n$, $\varrho=\varrho_n$ and $R=[\overset{2}{\log}\, n^{1/7}]+1$ if $n \in N_1$ and $n \geqq n_0(\varepsilon)$ (shortly $n \in N_2$).

Denote by $M_1 = M_{1n}$ the accumulation interval. Dropping M_1, we apply Lemma 3.4 again for the remaining exceptional intervals with $\mu = \mu_n - |M_1| > \mu_n/2$ and the above ϱ and R, supposing $\mu_n \geqq \varrho 2^{R+1}$ whenever $n \in N_2$. We denote the accumulation interval by M_2. At the i-th step $(2 \leqq i \leqq \psi_n)$ we drop $M_1, M_2, \dots, \dots, M_{i-1}$ and apply Lemma 3.4 for the remaining exceptional intervals with $\mu = \mu_n - \sum_{j=1}^{i-1} |M_i|$ using the same ϱ and R.

Here ψ_n is the first index for which

$$\sum_{i=1}^{\psi_n - 1} |M_i| \leqq \frac{\mu_n}{2} \quad \text{but} \quad \sum_{i=1}^{\psi_n} |M_i| > \frac{\mu_n}{2}, \quad n \in N_2.$$

If we denote by $M_{\psi_n+1}, M_{\psi_n+2} \dots M_{\varphi_n}$ the remaining (i.e. not accumulation) exceptional intervals (by $|M_i| \leqq \varrho_n$, $(\varepsilon(b-a)/20)n^{1/6} < \psi_n < \varphi_n$), by (3.8) we can write

$$\sum_{k=r}^{\varphi_n} \frac{|M_k|}{|M_r, M_k|} \geqq \frac{\mu_n \ln n}{56(b-a)} \quad \text{if} \quad 1 \leqq r \leqq \psi_n \quad (n \in N_2). \tag{3.10}$$

3.4.2. To go further proving (3.9) let $\eta = c_1 \varepsilon^3/6$, $u_{in} \in M_{in}(q)$ $(1 \leqq i \leqq \varphi_n,\ n \in N_2)$ be exceptional points, where c_1 will be determined later.

If for a fixed $n \in N_2$ there exists t, $1 \leqq t \leqq \varphi_n$, such that

$$\chi_n(s_n, u_{tn}) \geqq \frac{c_1 \varepsilon^2 \mu_n \ln n}{b-a}, \tag{3.11}$$

by $\eta \ln n \geqq \chi_n(s_n, u_{tn})$ we obtain (3.9) for this n. *We shall prove* (3.11) *for arbitrary* $n \in N_2$. Indeed, let us suppose that for a certain $m \in N_2$

$$\chi_m(s_m, u_{rm}) < \frac{c_1 \varepsilon^2 \mu_m \ln m}{b-a} \quad \text{when} \quad u_{rm} \in M_{rm}(q), \quad 1 \leqq r \leqq \varphi_m. \tag{3.12}$$

By (3.12) we obtain

$$\sum_{r=1}^{\varphi_m} |M_{rm}|\chi_m(s_m, u_{rm}) < \frac{c_1\varepsilon^2\mu_m^2 \ln m}{b-a} \quad \text{where} \quad m\in N_2. \tag{3.13}$$

On the other hand, by (3.7), for *arbitrary* $n\in N_2$

$$|M_r| \sum_{\substack{k \\ x_k\in s_n}} |l_k(u_r)| \geqq \frac{1}{2}|M_r| \sum_k{}' [|l_k(u_r)|+|l_{k+1}(u_r)|] \geqq$$

$$\geqq \frac{q^2}{2}\sum_{k=1}^{\varphi_n}\left|\frac{\omega(\bar z_r)}{\omega(\bar z_k)}\right| \frac{|M_r|\,|M_k|}{|M_r, M_k|}, \quad (1\leqq r\leqq \varphi_n),$$

so, by (3.10) and (3.11) we have

$$\sum_{r=1}^{\varphi_n} |M_r|\chi_n(s_n, u_r) = \sum_{r=1}^{\varphi_n}|M_r| \sum_{\substack{k \\ x_k\in s_n}} |l_k(u_r)| \geqq \frac{q^2}{2}\sum_{r=1}^{\varphi_n}\sum_{k=1}^{\varphi_n}\left|\frac{\omega(\bar z_r)}{\omega(\bar z_k)}\right| \frac{|M_r|\,|M_k|}{|M_r, M_k|} \geqq$$

$$\geqq \frac{1}{2}\frac{q^2}{2}\sum_{r=1}^{\varphi_n}\sum_{k=r}^{\varphi_n}\left[\left|\frac{\omega(\bar z_r)}{\omega(\bar z_k)}\right| + \left|\frac{\omega(\bar z_k)}{\omega(\bar z_r)}\right|\right]\frac{|M_r|\,|M_k|}{|M_r, M_k|} \geqq$$

$$\geqq \frac{q^2}{4}\sum_{r=1}^{\psi_n}|M_r|\sum_{k=r}^{\varphi_n}\frac{|M_k|}{|M_r, M_k|} > \frac{q^2}{4}\frac{\mu_n}{2}\frac{\mu_n \ln n}{56(b-a)} = \frac{c_1\varepsilon^2\mu_n^2 \ln n}{b-a}$$

if $c_1=(8\cdot 144\cdot 56)^{-1}$. This contradicts (3.13), i.e. (3.11) is valid for arbitrary $n\in N_2$, which proves (3.9).

3.4.3. The exceptional intervals should belong to H_n. Moreover, by definition if $J_{kn}\subset s_n$ and if it is not exceptional, then for any $x\in J_{kn}(q)$ (3.5) holds.

The sets $J_{kn}\setminus J_{kn}(q)$ of aggregate measure c_2 should also belong to H_n. Obviously $c_2\leqq 2q \sum_{\substack{k \\ J_k\subset[a,b]}} |J_{kn}|\leqq 2q(b-a)=\frac{\varepsilon(b-a)}{6}$. So using this, and (3.9), we can write

$$|H_n| \leqq \mu_n + c_2 = \frac{\varepsilon}{6}(b-a)+\frac{\varepsilon}{6}(b-a) < \varepsilon(b-a)$$

which completes the proof of Lemma 3.2.

3.5. *Let us consider an arbitrary interval* $\Delta_1=[a,b]\subset[-1,1]$ *for which*

$$b-a = |s_n(a,b)|+o(1), \quad n\in\mathbf{N}. \tag{3.14}$$

By (3.14) and Lemma 3.2, if $\varepsilon=1/10$, say, $\chi_{n_1}(\Delta_1)\overset{\text{def}}{=}\max_{x\in\Delta_1}\chi_{n_1}(\Delta_1, x)\geqq 2c_7\ln n_1$ if n_1 is large enough.[4]

[4] The meaning of $\chi_n(\Delta)$ for an arbitrary interval $\Delta\subset[-1,1]$ is analogous.

If $\chi_{n_1}(\Delta_1) \geqq n_1^\gamma$ (where $\gamma > 1$ is any fixed number) let $\Delta_2 \subset \Delta_1$ be a closed interval for which $2\lambda_{n_1}(x) \geqq \max_{x \in \Delta_1} \lambda_{n_1}(x) \geqq n_1^\gamma$ if $x \in \Delta_2$.

If $\chi_{n_1}(\Delta_1) < n_1^\gamma$, consider the interval

$$(3.15) \qquad \tilde{\Delta}_1 = [\tilde{a}_1, \tilde{b}_1] = \left[\min s_{n_1}(\Delta_1) + \frac{|s_{n_1}(\Delta_1)|}{10}, \quad \max s_{n_1}(\Delta_1) - \frac{|s_{n_1}(\Delta_1)|}{10}\right].$$

If n_1 was large enough then $2|\tilde{\Delta}_1| \geqq |\Delta_1|$, say, i.e. again by (3.14) and Lemma 3.2, $\chi_{n_1}(\tilde{\Delta}_1) \geqq 2c_7 \ln n_1$; further there are intervals $J_{kn_1} \subset s_{n1}(\Delta_1)$ in $[a, \tilde{a}_1]$ and in $[\tilde{b}_1, b]$.

Let $\Delta_2 \subset \tilde{\Delta}_1$ be a closed interval for which

$$(3.16) \qquad \chi_{n_1}(\tilde{\Delta}_1, x) \geqq c_7 \ln n_1 \quad \text{if} \quad x \in \Delta_2.$$

Continuing this process we find a sequence $\{\Delta_i\}$ of embedded closed intervals and a sequence of indices $\{n_i\}$ such that if

$$(3.17) \qquad \chi_{n_i}(\Delta_i) \geqq n_i^\gamma$$

then

$$(3.18) \qquad 2\lambda_{n_i}(x) \geqq \max_{x \in \Delta_i} \lambda_{n_i}(x) \geqq n_i^\gamma \quad \text{if} \quad x \in \Delta_{i+1}.$$

Otherwise, i.e. if

$$(3.19) \qquad n_i^\gamma > \chi_{n_i}(\Delta_i)$$

we have for $\tilde{\Delta}_1 = [\tilde{a}_i, \tilde{b}_i]$ (see (3.15))

$$(3.20) \qquad \chi_{n_i}(\tilde{\Delta}_i, x) \geqq c_7 \ln n_i \quad \text{if} \quad x \in \Delta_{i+1}.$$

Moreover there are intervals J_{kn_i} from $s_{n_i}(\Delta_i)$ in $[a_i, \tilde{a}_i]$ and $[\tilde{b}_i, b_i]$ ($i \in \mathbf{N}$; $a_1 = a$, $b_1 = b$).

For further purposes we can suppose

$$(3.21) \qquad |\Delta_i| n_i^\varphi \geqq 1, \quad i \in \mathbf{N}, \quad \varphi > 0 \quad \text{is fixed.}$$

3.5.1. Let $\tilde{x}_0 \in \bigcap_{i \in \mathbf{N}} \Delta_i \neq \varnothing$. *If we have* (3.17) *and* (3.18) for certain $\{n_i\} \overset{\text{def}}{=} \mathbf{M}_1$ and $\{\Delta_i\}$ then let for any fixed $n \in M_1$

$$g_n(x) = \begin{cases} \operatorname{sign} l_{kn}(\tilde{x}_0) & \text{if} \quad x = x_{kn}, \quad 1 \leqq k \leqq n, \\ g_n(x_{1n}) & \text{if} \quad x_{1n} \leqq x \leqq 1, \\ g_n(x_{nn}) & \text{if} \quad -1 \leqq x \leqq x_{nn}; \end{cases}$$

between interpolation points $[x_{k+1,n}, x_{kn}]$ let $g_n(x)$ be the Hermite interpolatory polynomials of degree $\leqq 2m+1$ satisfying $g_n^{(j)}(x_{kn}) = g_n^{(j)}(x_{k+1,n}) = 0$ ($1 \leqq j \leqq m$, $1 \leqq k \leqq n-1$). Let

$$(3.22) \qquad \delta_n = \frac{1}{n^2 \lambda_n}.$$

Obviously $|g_n(x)| \leqq 1$ and $g_n^{(m)} \in C$. To obtain $|g_n^{(m)}(x)| \leqq c_8 \delta_n^{-m}$, we need as follows (see P. Erdős and P. Turán, [8], § 6).

If $J_{kn}\subset\Delta\overset{\text{def}}{=}[\alpha,\beta]\subset[-1,1]$ *then*

$$d_n(\Delta)\overset{\text{def}}{=}\min_{\substack{k\\ J_k\subset\Delta}}|J_k|\geqq\frac{|\Delta|}{2n^2\lambda_n(\Delta)}. \tag{3.23}$$

By (3.23) $d_n=d_n([-1,1])\geqq(n^2\lambda_n)^{-1}=\delta_n$ from where it is easy to get $|g_n^{(m)}(x)|\leqq c_8\delta_n^{-m}$ (see e.g. [10]). Moreover, by definition and (3.18)

$$L_n(g_n,\tilde{x}_0)=\sum_{k=1}^{n}g_n(\tilde{x}_0)l_{kn}(\tilde{x}_0)=\sum_{k=1}^{n}|l_{kn}(\tilde{x}_0)|=\lambda_n(\tilde{x}_0)\geqq 0.5n^{\gamma}. \tag{3.24}$$

3.5.2. *Now let us suppose* (3.19) *and* (3.20) for certain $n_i\in\mathbf{M}_2$ and Δ_i. Let for any fixed i

$$g_{n_i}(x)=\begin{cases}\operatorname{sign} l_{kn_i}(\tilde{x}_0) & \text{if } x=x_{kn_i}\in\tilde{\Delta}_i,\\ 0 \quad \text{if} \quad x=x_{kn_i}\notin\tilde{\Delta}_i & \text{or} \quad x\geqq x_{1n_i} \quad \text{or} \quad x\leqq x_{n_i,n_i}.\end{cases} \tag{3.25}$$

Between the nodes let g_{n_i} be as above.

Let

$$\delta_{n_i}=\frac{1}{2n_i^{2+\gamma+\varphi}}. \tag{3.26}$$

Then using that there are certain nodes x_{kn_i} in the intervals $[a_i,\tilde{a}_i)$ and $(\tilde{b}_i,b_i]$, by (3.19), (3.21) and (3.23)

$$d_{n_i}(\tilde{\Delta}_i)\geqq d_{n_i}(\Delta_i)>\frac{1}{2n_i^{2+\gamma+\varphi}}=\delta_{n_i}. \tag{3.27}$$

By (3.25) and (3.20)

$$L_{n_i}(g_{n_i},\tilde{x}_0)=\sum_{\substack{k\\ x_{kn_i}\in\tilde{\Delta}_i}}|l_{kn_i}(\tilde{x}_0)|\geqq c_7\ln n_i, \tag{3.28}$$

moreover, by definition it is easy to see that $|g_{n_i}|\leqq 1$, $g_{n_i}^{(m)}\in C$ and $|g_{n_i}^{(m)}(x)|\leqq c_8\delta_{n_i}^{-m}$.

3.6. *Now we are going to settle the case when instead of* (3.14) *we have that for infinitely many* n

$$|S_n(a,b)|\geqq 4c_9>0,\ n=n_1,n_2,\ldots. \tag{3.29}$$

If for a certain $n=n_i$, $[a,b]$ is free of the nodes $\{x_{kn}\}_{k=0}^{n+1}$, let $\tilde{A}_n=(a,b)$. Let x_{sn} and x_{tn} be the smallest and biggest nodes in $[a,b]$, respectively $(n=n_1,n_2,\ldots)$. By definition $(a,x_{sn})\subset S_n$ and $(x_{tn},b)\subset S_n$. If $0\leqq x_{sn}-a\leqq\varrho_n$, we omit (a,x_{sn}) from S_n. The same should be done with the interval (x_{tn},b). Let us denote by $\tilde{A}_n$ the remaining part of S_n. Then $\tilde{A}_n=\bigcup_{j=1}^{r}I_{jn}$ where for any fixed j

a) there exists a k such that $I_{jn}=(x_{k+1,n},x_{kn})$; or
b) $I_{jn}=(a,x_{sn})$; or $I_{jn}=(x_{tn},b)$; or
c) $\tilde{A}_n=(a,b)$;

moreover one can suppose that $|\tilde{A}_n|\geqq 3c_9$ $(n=n_1, n_2, \ldots)$. By construction if $I_{jn}\subset\tilde{A}_n$, then $|I_{jn}|>\varrho_n$ so if $I_{jn}=(\alpha_{jn}, \beta_{jn})$ then for the sets

$$(3.30)\qquad A_n = \bigcup_{j=1}^{r}\left[\alpha_{jn}+\frac{\beta_{jn}-\alpha_{jn}}{3},\quad \beta_{jn}-\frac{\beta_{jn}-\alpha_{jn}}{3}\right]$$

we have $|A_n|\geqq c_9$, $A_n\subset[a, b]$, moreover if $x\in A_n$

$$(3.31)\qquad \min_{0\leqq k\leqq n-1}|x-x_{kn}| > \varrho_n/3 \quad (n = n_1, n_2, \ldots).$$

First we remark that *there exists a set* $B\subset[a, b]$ *of measure* $\geqq c_9$ *such that for any* $x\in B$ *one can find a subsequence* $\{p_i\}\subset\{n_i\}$ *for which* $x\in A_n$ *whenever* $n=p_1, p_2, \ldots$. Indeed, let $B_i=\bigcup_{k=i}^{\infty}A_{n_k}$ and $B=\bigcap_{k=i}^{\infty}B_i$. Obviously $B_1\supset B_2\ldots$ and $|B_i|\geqq c_9$, $i\in\mathbf{N}$, from where $|B|\geqq c_9$. On the other hand if $x\in B$, then $x\in B_i$ $(i\in\mathbf{N})$ from where $x\in A_{n_k}$ for infinitely many k as it was stated.

3.1.6. So let $\tilde{x}_0\in B$ be a fixed point. Then $\tilde{x}_0\in A_n$ $(n=p_1, p_2, \ldots)$, i.e. by (3.30) and (3.31)

$$(3.32)\qquad \min_{0\leqq k\leqq n+1}|\tilde{x}_0-x_{kn}| > \frac{\varrho_n}{3} \quad \text{if} \quad n = p_1, p_2, \ldots.$$

In [7], pp. 116—117, S. Bernstein proved that for the polynomial

$$P_{4s}(y) = \cos 2s \arccos\frac{2y^2-(\alpha^2+\beta^2)}{\beta^2-\alpha^2} \qquad (\beta>\alpha)$$

of degree $4s$, $|P_{4s}(y)|\leqq 1$ if $y\in[-\beta, -\alpha]\cup[\alpha, \beta]$ and

$$(3.33)\qquad P_{4s}(0) = \frac{1}{2}\left[\left(1+\frac{2\alpha}{\beta-\alpha}\right)^{2s}+\left(1-\frac{2\alpha}{\beta-\alpha}\right)^{2s}\right].$$

Moreover, it is easy to see that for the roots of $P_{4s}(y)$ we have

$$(3.34)\qquad \begin{cases} -\beta<y_{-1}<y_{-2}<\ldots<y_{-2s}<-\alpha<\alpha<y_{2s}<y_{2s-1}<\ldots<y_1<\beta, \\ y_{\pm k} = \pm\left(\dfrac{\beta^2-\alpha^2}{2}\cos\dfrac{2k-1}{4s}\pi+\dfrac{\alpha^2+\beta^2}{2}\right)^{1/2}, \quad k = 1, 2, \ldots, 2s. \end{cases}$$

Let $\alpha=\varrho_n/6$, $\beta=2$ and $s=\left[\dfrac{n-1}{4(m+1)}\right]$. We have with $y=x-\tilde{x}_0$

$$P_{4s}^{m+1}(y) = P_{4s}^{m+1}(x-\tilde{x}_0) \stackrel{\text{def}}{=} G(x) = \sum_{k=1}^{n} G(x_k)l_{kn}(X, x).$$

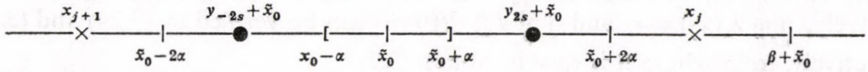
The polynomial $G(x)$ of degree $4s(m+1)$ has the following properties:

a) $|G(x)|\equiv|P_{4s}(x-\tilde{x}_0)|^{m+1}\leqq 1$ whenever $x\in[-\beta+\tilde{x}_0, -\alpha+\tilde{x}_0]\cup[\alpha+\tilde{x}_0, \beta+\tilde{x}_0]$, especially if $x=x_k$, $k=1, 2, \ldots, n$ $(n=p_1, p_2, \ldots$; see (3.32));

b) $G^{(t)}(y_{\pm i}+\tilde{x}_0)=0$ if $t=0, 1, \ldots, m$ and $i=1, 2, \ldots, 2s$.

Now let

$$g_n(x) = \begin{cases} G(x) & \text{if} \quad x\in[-\beta+\tilde{x}_0,\ y_{-2s}+\tilde{x}_0]\cup[y_{2s}+\tilde{x}_0,\ \beta+\tilde{x}_0] \\ 0 & \text{if} \quad x\in[y_{-2s}+\tilde{x}_0,\ y_{2s}+\tilde{x}_0]. \end{cases}$$



Evidently, one can find a $j=j(n)$, $0\leqq j\leqq n$, such that $\tilde{x}_0\in(x_{j+1,n}, x_{jn})$.[5] First we state that

$$\tilde{x}_0+\alpha < y_{2s}+\tilde{x}_0 < \tilde{x}_0+2\alpha < x_j \leqq 1 \quad \text{if} \quad n = p_1, p_2, \ldots,$$

and p_1 is large enough which we suppose. Indeed, by (3.34) and (3.32)

$$\text{(3.35)} \qquad y_{2s} = \left[\alpha^2+\frac{\beta^2-\alpha^2}{2}\left(1-\cos\frac{\pi}{4s}\right)\right]^{1/2} < 2\alpha < x_j-\tilde{x}_0.$$

Similarly $-1\leqq x_{j+1}<\tilde{x}_0-2\alpha<y_{-2s}+\tilde{x}_0<\tilde{x}_0-\alpha$. By these, a) and b) we get $|g_n(x)|\leqq 1$ and $g_n(x_k)=G(x_k)$, $1\leqq k\leqq n$, i.e.

$$L_n(g_n, \tilde{x}_0) = \sum_{k=1}^{n} g_n(x_k)\, l_k(\tilde{x}_0) = \sum_{k=1}^{n} G(x_k)\, l_k(\tilde{x}_0) = G(\tilde{x}_0) = P_{4s}^{m+1}(0).$$

By definition it is easy to see that $|g_n^{(m)}(x)|=0$ if $x\in[y_{-2s}+\tilde{x}_0,\ y_{2s}+\tilde{x}_0]$, moreover $|g_n^{(m)}(x)|=|G^{(m)}(x)|\leqq c_{10}n^{2m}$ whenever $x\in[-\beta+\tilde{x}_0, y_{-2s}+\tilde{x}_0]\cup[y_{2s}+\tilde{x}_0, \beta+\tilde{x}_0]$ (Markov theorem) i.e. we have (3.1) for g_n $(n=p_1, p_2, \ldots)$ with $\delta_n=n^{-2}$.

3.7. Using the above considerations for every interval $[A_j, B_j]\subset[-1, 1]$ with rational A_j and B_j $(j=1, 2, \ldots)$ we can state by Theorem 3.1 as follows.

If $\omega_m(t)$ satisfies (1.4) then there exists an $f\in C(\omega_m)$ for which

$$\text{(3.36)} \qquad \frac{|L_n(f, X, \tilde{x}_j)-f(\tilde{x}_j)|}{\omega_m(\delta_{nj})\,\lambda_{nj}} \geqq 1 \quad \text{if} \quad n\in P_j = \{n_j\}_{i=1}^{\infty}; \quad j\in\mathbf{N}.$$

Here $\{\tilde{x}_j\}_{j\in\mathbf{N}}$ is a dense set in $[-1, 1]$, moreover for $n\in P_j$ $(j\in\mathbf{N})$

$$\delta_{nj} = \begin{cases} \dfrac{1}{n^2\lambda_n} & \text{if we have (3.14), (3.17) and (3.18),} \\ \dfrac{1}{2n^{2+\gamma+\varphi}} & \text{if we have (3.14), (3.19) and (3.20),} \\ \dfrac{1}{n^2} & \text{if we have (3.29),} \end{cases}$$

$$\lambda_{nj} = \begin{cases} \lambda_n(\tilde{x}_j) & \text{if we have (3.14), (3.17) and (3.18),} \\ c_7 \ln n & \text{if we have (3.14), (3.19) and (3.20),} \\ P_{4s}^{m+1}(0) & \text{if we have (3.29)} \end{cases}$$

for the corresponding interval $[A_j, B_j]$ $(j=1, 2, \ldots)$.

[5] This interval is not the empty set.

Indeed, we can consider the functionals $T_{nj}f=[L_n(f,\tilde{x}_j)-f(\tilde{x}_j)]/\lambda_{nj}$ for $f\in C$. (By the previous notations, if $[a,b]=[A_j,B_j]$, then $n_{ji}=n_i$, which is actually $n_i(A_j,B_j)$; or $\delta_{n_{ji},j}=\delta_{n_i}(=\delta_{n_i(A_j,B_j)})$ (if we have (3.14), say).)

Clearly $T_{nj}\in[C,Y]$ where $Y=(-\infty,\infty)$. The requirements (3.1) and (3.2) for the corresponding functions g_{nj} can be found at 3.5.1, 3.5.2 and 3.6.1 respectively, $\lim\limits_{n\in P_j}\lambda_n(\tilde{x}_j)=\infty$ and $\lim\limits_{n\in P_j}P_{4s}^{m+1}(0)=\infty$ can be verified by (3.24) and (3.33), respectively, from where it is easy to obtain (3.2).

3.7.1. Let us remark that until now we have used neither condition (1.7) nor condition (1.8).

3.8. To go further we quote the following

LEMMA 3.5. *Let A denote a topological space of second category and $D\subset A$ a dense subset.* Let $\{h_n\}_{n\in\mathbf{N}}$ *be a sequence of continuous functions on A such that for each $t\in D$*

$$\overline{\lim_{n\to\infty}}\, h_n(t)\geqq c_{11}>0. \tag{3.37}$$

Then the set

$$S=\{t\in A;\ \overline{\lim_{n\to\infty}}\, h_n(t)\geqq c_{11}\}, \tag{3.38}$$

$D\subset S\subset A$, is dense and of second category in A. We allow the case $c_{11}=\infty$.

For the proof if $c_{11}=\infty$ or $0<c_{11}<\infty$ see W. Orlicz [11] and W. Dickmeis, R. J. Nessel [4], respectively.

3.8.1. If we want to use this lemma for the left hand side of (3.36) (as h_n) we have to choose another denominator to ensure the continuity. For this aim by the notations of 3.7 we prove:

If $m=1$ and we have (1.7) *then for any fixed j $(j=1,2,\ldots)$*

$$\omega(\delta_{nj})\lambda_{nj}\geqq c_{12}\,\omega\left(\frac{1}{\mu_n}\right)\ln\mu_n \quad \textit{if} \quad n=n_{j1},n_{j2},\ldots;\quad j=1,2,\ldots \tag{3.39}$$

where $\mu_n=n^{3+\varepsilon}$, $\varepsilon>0$ is arbitrarily small, n_{j1} is large enough.

1. Indeed, if we have (3.14), (3.17) and (3.18), with the corresponding indices $n=n_{ji}$ $(i=1,2,\ldots)$, by (1.7) and 3.7

$$\omega(\delta_{nj})\lambda_{nj}=\omega(\delta_{nj})\ln\frac{1}{\delta_{n_j}}\,\frac{\lambda_n(\tilde{x}_j)}{\ln\dfrac{1}{\delta_{n_j}}}\geqq c_{13}\frac{\lambda_n(\tilde{x}_j)}{2\ln n+\ln\lambda_n}=P.$$

To estimate λ_n we use the next estimation which was essentially proved by I. P. Natanson [9] (see 1/IX/§3).

Let $[\alpha,\beta]\subset[-1,1]$. If $P_n(x)$ is a polynomial of degree $\leqq n$ and $\max\limits_{\alpha\leqq x\leqq\beta}|P_n(x)|\leqq 1$, *then*

$$\|P_n\|<\left(\frac{8}{\beta-\alpha}\right)^n. \tag{3.40}$$

Indeed, if we consider the polynomial

$$Q_n(u) = P_n\left[\frac{(\beta-\alpha)u+\alpha+\beta}{2}\right]$$

then $|Q_n(u)|\leqq 1$ if $u\in[-1, 1]$, i.e. by [9], 1/(63) $|Q_n(u)|\leqq[|u|+\sqrt{u^2-1}]^n$ whenever $|u|>1$. Let $\beta<x\leqq 1$, say. Then $u=(2x-\alpha-\beta)(\beta-\alpha)^{-1}>1$, i.e. $|P_n(x)|=$ $=|Q_n(u)|\leqq[u+\sqrt{u^2-1}]^n<(2u)^n=\left(\frac{4x-2\alpha-2\beta}{\beta-\alpha}\right)^n<\left(\frac{8}{\beta-\alpha}\right)^n$ as it was stated.

By (3.40) and (3.21) we can write with a certain s, $1\leqq s\leqq n$, as follows.

$$\lambda_n \leqq n\|l_{sn}\| < n(8n^{\varphi})^n \max_{x\in\Delta_i} |l_{sn}(x)| \leqq n(8n^{\varphi})^n \max_{x\in\Delta_i} \lambda_n(x).$$

So by (3.18), for the denominator of P

$$2\ln n+\ln\lambda_n \leqq 3\ln n+2\varphi n\ln n+\ln\lambda_n(\tilde{x}_j).$$

If $2\varphi n\ln n\geqq\ln\lambda_n(\tilde{x}_j)$ we have

$$P > \frac{c_{13}n^{\gamma}}{5\varphi n\ln n} > \ln\mu_n > \omega\left(\frac{1}{\mu_n}\right)\ln\mu_n$$

whenever n_{j1} is large enough which we can suppose. On the other hand when $2\varphi n\ln n<\ln\lambda_n(x_j)$,

$$P \geqq c_{13}\frac{\lambda_n(\tilde{x}_j)}{3\ln\lambda_n(\tilde{x}_j)} > \sqrt{\lambda_n(\tilde{x}_j)} > \omega\left(\frac{1}{\mu_n}\right)\ln\mu_n$$

(whenever n_{j1} is large enough).

2. If we have (3.14), (3.19) and (3.20), we can write

$$\omega(\delta_{nj})\lambda_{nj} = \omega\left(\frac{1}{2n^{2+\gamma+\varphi}}\right)c_7\ln n \geqq c_{12}\omega\left(\frac{1}{\mu_n}\right)\ln\mu_n.$$

if n_{j1} is large enough and $\gamma+\varphi=1+\varepsilon$, which can be attained.

3. If we have (3.29) then by $\alpha=1/(6n^{1/6})$ and $\beta=2$, using (3.33) we have

$$\omega(\delta_{nj})\lambda_{nj} = \omega\left(\frac{1}{n^2}\right)P_{4s}^{m+1}(0) > \frac{c_{14}}{n^2}n^3 > \omega\left(\frac{1}{\mu_n}\right)\ln\mu_n,$$

as we stated.

3.8.2. So by (3.3) and (3.39) we have:

If $\omega(t)$ satisfies (1.7) then there exists an $f\in C(\omega)$ for which

$$\varlimsup_{n=\infty}\frac{|L_n(f, X, \tilde{x}_j)-f(\tilde{x}_j)|}{\omega\left(\frac{1}{\mu_n}\right)\ln\mu_n} \geqq 1 \quad\text{for}\quad j=1, 2, \ldots.$$

Here $\{\tilde{x}_j\}_{j=1}^{\infty}$ is a dense set in $[-1, 1]$.

3.8.3. If we apply Lemma 3.5 with $D=\{\tilde{x}_j\}$, $A=[-1,1]$ and $h_n(t)=$ $=[L_n(f,t)-f(t)][\omega\left(\frac{1}{\mu_n}\right)\ln\mu_n]^{-1}$ we obtain Theorem 2.2.

3.9. Proof of Theorem 2.1. Statement a) immediately follows from Theorem 2.2. To prove b) first we consider the notations and ideas of Parts 3.2 and 3.7.

According these for each fixed interval $[A_j, B_j]$ we have the functions $\{g_{nj}\}$ $(n\in P_j, j=1,2,\dots)$ such that with $\alpha_n=\left[\omega\left(\frac{1}{\mu_n}\right)\ln\mu_n\right]^{-1/2}$ we can write $\lim\limits_{n\in P_j}\|T_{nj}g_{nj}\|=\infty$ where $T_{nj}f=[L_n(f,\tilde{x}_j)-f(\tilde{x}_j)]$ $(\alpha_n\lambda_{nj})^{-1}$ if $f\in C$. Then by (3.4) with a proper $\tilde{f}\in C^*(\omega)$

$$\lim_{n\in P_j}\frac{|L_n(\tilde{f},\tilde{x}_j)-\tilde{f}(\tilde{x}_j)|}{\alpha_n\lambda_{nj}\omega(\delta_{nj})}=\infty,\quad j=1,2,\dots.$$

But for the denominator using (3.39) and (1.8)

$$\alpha_n\lambda_{nj}\omega(\delta_{nj})\geqq c_{12}\alpha_n\omega\left(\frac{1}{\mu_n}\right)\ln\mu_n\geqq c_{12}\left[\omega\left(\frac{1}{\mu_n}\right)\ln\mu_n\right]^{1/2},$$

i.e.

$$\lim_{n\in P_j}\frac{|L_n(\tilde{f},\tilde{x}_j)-\tilde{f}(\tilde{x}_j)|}{c_{12}\left[\omega\left(\frac{1}{\mu_n}\right)\ln\mu_n\right]^{1/2}}=\infty$$

from where by Lemma 3.5 we get b).

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
H-1053 BUDAPEST
REÁLTANODA U. 13—15
HUNGARY

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GAUSSIAN APPROXIMATION OF MIXING RANDOM FIELDS

I. BERKES (Budapest)

1. Introduction

Let Z^2 and N^2 denote, respectively, the set of all two-dimensional vectors, with integral, resp. positive integral coordinates. Let $\{\xi_\nu, \nu \in N^2\}$ be a random field with index set N^2. We assume that

$$(1.1) \qquad E\xi_\nu = 0, \quad E|\xi_\nu|^{2+\delta} \leqq C_1 < \infty, \quad \nu \in N^2$$

for some constants $C_1 > 0$, $0 < \delta \leqq 1$ and that $\{\xi_\nu, \nu \in N^2\}$ satisfies the following strong mixing condition:

$$(1.2) \qquad \varrho(H_1, H_2) \stackrel{\text{def}}{=} \sup_{\substack{A \in \sigma\{\xi_\nu, \nu \in H_1\} \\ B \in \sigma\{\xi_\nu, \nu \in H_2\}}} |P(AB) - P(A)P(B)| \leqq C_2 \big(\inf_{\mu \in H_1, \nu \in H_2} |\mu - \nu| \big)^{-\gamma}$$

for some $C_2 > 0$, a large enough $\gamma > 0$ and for any disjoint nonempty sets $H_1, H_2 \subset N^2$. (Here $\sigma\{\cdot\}$ denotes the σ-field generated by the r.v.'s in the brackets.) Set $S_n = \sum_{\nu \leqq n} \xi_\nu$ where, for any $m, n \in Z^2$, the relation "$m \leqq n$" means inequality coordinatewise. Put finally, for any $n = (n_1, n_2) \in Z^2$, $[n] = |n_1 n_2|$.

In [1] approximation of partial sums of stationary random fields satisfying (1.1), (1.2) with two-parameter Wiener process was studied. It was shown that under (1.1), (1.2) there exists a two-parameter Wiener process $\{W(t), t \in [0, \infty)^2\}$ such that

$$(1.3) \qquad S_n - W(n) \ll [n]^{1/2 - \lambda} \quad \text{a.s.}$$

holds with the exception of lattice points $n \in N^2$ lying "near" the coordinate axes; here λ is a positive constant and $\ll$ is an alternative symbol for the big O notation. In the same paper it was shown by a simple example that for all $n \in N^2$ not only (1.3) but even

$$(1.4) \qquad S_n - W(n) = o([n] \log\log [n])^{1/2} \quad \text{a.s.} \quad \text{as} \quad [n] \to \infty$$

is generally impossible.[1] In the mentioned example ES_n^2 behaves irregularly in the sense that the limit

$$(1.5) \qquad \lim_{[n] \to \infty} ES_n^2 / EW(n)^2$$

does not exist; more exactly, $ES_n^2/EW(n)^2$ converges to different limits on different lines parallel to the x-axis. Obviously then, (1.4) is impossible for any Wiener

[1] Relation (1.4) is meant in the sense $\lim_{t \to \infty} \sup_{[n] \geqq t} (S_n - W(n)/a_n = 0$ a. s. where $a_n - ([n] \log\log [n])^{1/2}$ The same convention applies for relations of the type (1.4) appearing later.

process W by the law of the iterated logarithm. (Far away from the axes ES_n^2 behaves nicely i.e. $ES_n^2 \sim [n]$.) This example shows that the natural candidate for approximating S_n in the whole first quadrant is not $W(n)$ but a Gaussian field $\{H(n), n \in N^2\}$ such that

$$\lim_{[n]\to\infty} ES_n^2/EH(n)^2 = 1. \tag{1.6}$$

The simplest such field is $H(n) = \sum_{\nu \leqq n} \zeta_\nu$ where $\{\zeta_\nu, \nu \in N^2\}$ is a stationary Gaussian field with mean zero and the same covariance structure as $\{\xi_\nu, \nu \in N^2\}$. We shall show that for this $H(n)$ the approximation

$$S_n - H(n) \ll [n]^{1/2} (\log\log [n]^{-\lambda} \quad \text{a.s.} \tag{1.7}$$

holds for all $n \in N^2$ where λ is a positive constant. ((1.7) is meant, as usual, in the sense that the fields $\{\xi_\nu, \nu \in N^2\}$ and $\{H(n), n \in N^2\}$ can be jointly defined on a suitable probability space, without changing their distribution, such that (1.7) holds.) More generally, we shall see that (1.7) holds for any Gaussian field $H(n)$ with stationary increments satisfying (1.6) and a simple regularity condition. This result, while being a natural two-parameter analogue of standard a.s. invariance principles for mixing sequences (see [13]) has the novel feature that the approximating field $H(n)$ has dependent increments. This fact causes substantial difficulties in the proof and though we will eventually be able to reduce the problem to the "standard" situation, we have to follows a rather indirect way.

It is worth noticing that while the Wiener process is, in general, not suitable for approximating the partial sum field $\{S_n, n \in N^2\}$ in the "uniform" sense (1.3), (1.4) i.e. when a nontrivial remainder term depending only on $[n]$ is required, allowing non-uniform remainder terms one can find a Wiener process W providing a satisfactory approximation. Theorem 2 below gives an example for such a "non-uniform" approximation theorem. Here the remainder term depends individually on both coordinates of n which has the consequence that $S_n - W(n)$ has, in terms of $[n]$, different order of magnitude in different domains of N^2. Along the axes this order is $O([n] \log\log [n])^{1/2}$ (in accordance with the fact that (1.4) is generally impossible) and it becomes gradually better as we move away from the axes. E.g., $O([n] \log\log [n])^{1/2}$ becomes $o([n] \log\log [n])^{1/2}$ as soon as $[n] \to \infty$ in such a way that both coordinates of n tend to infinity; moving even deeper into the first quadrant this order of magnitude improves continually until it reaches $O([n]^{1/2} \log^{-\lambda} [n])$ with a positive λ on the line $y = x$. This phenomenon is in accordance with the results of [1] where it was already observed that uniform bounds on $S_n - W(n)$ are necessarily of different form in different domains of N^2. While in [1], however, only very special domains were considered, here we get uniform bounds for a much larger class of domains.

We now formulate our results in detail. Define, for any weakly stationary field $\{\xi_\nu, \nu \in N^2\}$,

$$\begin{cases} c_k = \sum\limits_{\{\nu = (\nu_1, \nu_2) \in Z^2 :\, \nu_1 = k\}} r(\nu) & k = 0, \pm 1, \pm 2, \ldots, \\ d_l = \sum\limits_{\{\nu = (\nu_1, \nu_2) \in Z^2 :\, \nu_2 = l\}} r(\nu) & l = 0, \pm 1, \pm 2, \ldots \end{cases} \tag{1.8}$$

where r is the covariance function of the field; the series converge if the field satisfies (1.1) and (1.2) with a sufficiently large γ (see Lemma 4 below). Evidently $r(v)=$ $=r(-v)$ and thus $c_k=c_{-k}$, $d_l=d_{-l}$. Now we have

THEOREM 1. *Let* $\{\xi_\nu, \nu\in N^2\}$ *be a weakly stationary random field satisfying* (1.1) *and* (1.2) *with* $\gamma\geqq K_0/\delta$ *where* K_0 *is a large absolute constant; let* $r(v)$ *be the covariance function of the field. Let further* $\{\zeta_\nu, \nu\in N^2\}$ *be a stationary Gaussian field with mean zero and covariance function* $r^*(v)$ *satisfying*

$$|r^*(v)| \ll |v|^{-(2+\varepsilon)} \quad (v \neq 0) \tag{1.9}$$

$$r^*(0)-\sum_{v\neq 0}|r^*(v)| > 0 \tag{1.10}$$

for some $\varepsilon>0$. *Let* c_k, d_l *be defined by* (1.8) *and denote by* c_k^*, d_l^* *the analogous quantities for the field* $\{\zeta_\nu, \nu\in N^2\}$. *Then the following statements are equivalent:*

(A) *The fields* $\{\xi_\nu, \nu\in N^2\}$ *and* $\{\zeta_\nu, \nu\in N^2\}$ *can be defined jointly on a suitable probability space such that*

$$\sum_{\nu\leqq n}\xi_\nu-\sum_{\nu\leqq n}\zeta_\nu = o([n]\log\log[n])^{1/2} \quad a.s. \quad as \quad [n]\to\infty. \tag{1.11}$$

(B) *The fields* $\{\xi_\nu, \nu\in N^2\}$ *and* $\{\zeta_\nu, \nu\in N^2\}$ *can be defined jointly on a suitable probability space such that*

$$\sum_{\nu\leqq n}\xi_\nu-\sum_{\nu\leqq n}\zeta_\nu \ll [n]^{1/2} \quad (\log\log[n])^{-\lambda} \quad a.s. \tag{1.12}$$

holds for some positive constant λ.

(C) $c_k = c_k^* \quad (k = 0, 1, \ldots)$ *and* $d_l = d_l^* \quad (l = 0, 1, \ldots)$.

(D) $\lim_{[n]\to\infty} E(\sum_{\nu\leqq n}\xi_\nu)^2/E(\sum_{\nu\leqq n}\zeta_\nu)^2 = 1.$

Condition (1.9) implies that the field $\{\zeta_\nu, \nu\in N^2\}$ is of weakly dependent type (as contrasted to the strongly dependent Gaussian fields studied, e.g. in [4]). For strongly dependent $\{\zeta_\nu, \nu\in N^2]$ relations (1.11), (1.12) are impossible for variance reasons. E.g. if $r(v)\sim\text{const}\,|v|^{-\alpha}$ for some $0<\alpha<2$ then, as a simple calculation shows, for vectors $n\in N^2$ of the form $n=(k, k)$ we have

$$E(\sum_{\nu\neq n}\zeta_\nu)^2 \gg [n]^{2-\alpha/2}.$$

Hence (1.1) cannot hold even for these special n since the first sum on the left side is $o([n]^{1-\alpha/4})$ by $\alpha<2$ and the law of the iterated logarithm and the second sum, divided by $[n]^{1-\alpha/4}$, is a normal r.v. with mean zero and variance $\gg 1$ and thus it does not tend to 0 even in probability. Condition (1.10), on the other hand, is a technical condition needed in the proof to establish a certain mixing condition for the field $\{\zeta_\nu, \nu\in N^2\}$ (see the proof of Lemma 10(*)). Although this condition seems to be irrelevant for Theorem 1, we were not able to prove the result without it.

As an invariance principle, Theorem 1 implies various limit theorems connected with the law of the iterated logarithm for the field $\{\xi_\nu, \nu\in N^2\}$. However, the remainder term $O([n]^{1/2}(\log\log[n])^{-\lambda})$ is not strong enough to get upper and lower class tests; for such results at least a remainder term $O([n]^{1/2}\log^{-\lambda}[n])$ would be necessary with $\lambda>0$. If such an improvement of Theorem 1 holds remains open.

The proof of Theorem 1 yields an explicit value for the constant K_0; for example, $K_0=4098$ will do. $K_0\geqq 2$ is necessary even for the simplest estimates in the proof; the large numerical value 4098 is needed for the proof of Lemma 7 (except this lemma, $K_0=82$ would do). We shall not make any attempt to minimize this constant.

We finally mention that the constant λ in (1.12) can be chosen as large as desired i.e. if any of statements (A), (C), (D) in Theorem 1 holds then statement (B) will be valid with any prescribed positive λ (the construction of the sequences ξ_ν, ζ_ν will, however, depend on λ).

For the rest of the paper let $K_0=4098$.

COROLLARY (1.1). *Let $\{\xi_\nu, \nu\in N^2\}$ be a weakly stationary random field satisfying* (1.1) *and* (1.2) *with $\gamma\geqq K_0/\delta$; assume that the covariance function $r^*(\nu)$ of the field satisfies* (1.10). *Let $\{\zeta_\nu, \nu\in N^2\}$ be the stationary Gaussian field with mean zero and the same covariance structure as $\{\xi_\nu, \nu\in N^2\}$. Then the fields $\{\xi_\nu, \nu\in N^2\}$ and $\{\zeta_\nu, \nu\in N^2\}$ can be defined jointly on a suitable probability space such that* (1.12) *holds with a positive constant λ.*

COROLLARY (1.2). *Let $\{\xi_\nu, \nu\in N^2\}$ be a weakly stationary random field satisfying* (1.1) *and* (1.2) *with $\gamma\geqq K_0/\delta$; let $r(\nu)$ be the covariance function of the field. Then the following statements are equivalent:*

(A_1) *There exists a Wiener process $\{W(t), t\in[0,\infty)^2\}$ such that* (1.4) *holds.*

(B_1) *There exists a Wiener process $\{W(t), t\in[0,\infty)^2\}$ such that*

$$\sum_{\nu\leqq n}\xi_\nu - W(n) \ll [n]^{1/2}(\log\log[n])^{-\lambda} \quad a.s. \tag{1.13}$$

for some positive constant λ.

(C_1) $c_k=0$ $(k=1,2,\ldots)$ *and* $d_l=0$ $(l=1,2,\ldots)$ *where* c_k, d_l *are defined by* (1.8).

(D_1) $\lim\limits_{[n]\to\infty} E\big(\sum\limits_{\nu\leqq n}\xi_\nu\big)^2/[n]$ *exists.*

In the last corollary (and everywhere in our paper) "Wiener process" means any constant multiple of a standard Wiener process i.e. $c=EW(1,1)^2$ is allowed to be any nonnegative number. We shall see that the actual value of c in Corollary (1.2) is equal to the limit in (D_1) and also to $\sum\limits_{\nu\in Z^2} r(\nu)$. Hence the case $\sum\limits_{\nu\in Z^2} r(\nu)=0$ is degenerate with $W=0$. (Note that because of condition (1.10), Corollary (1.2) follows from Theorem 1 only if $\sum\limits_{\nu\in Z^2} r(\nu)\neq 0$. The case $\sum\limits_{\nu\in Z^2} r(\nu)=0$ will follow from Theorem 4 and the argument proving (A)⇒(C), (C)⇔(D) in the proof of Theorem 1.) We also note that replacing $[n]\to\infty$ by $n_1\wedge n_2\to\infty$ (where $a\wedge b$ and $a\vee b$ denote $\min(a,b)$ and $\max(a,b)$, respectively) the limit in (D_1) always exists under the mere conditions (1.1), (1.2) (see the corollary of Lemma 4). Hence condition (D_1) is a restriction on the behaviour of $E\big(\sum\limits_{\nu\leqq n}\xi_\nu\big)^2$ for lattice points n lying near the coordinate axes. Obviously, conditions (C_1) and (D_1) are satisfied if the r.v.'s ξ_ν are orthogonal.

THEOREM 2. *Let* $\{\xi_\nu, \nu\in N^2\}$ *be a weakly stationary random field satisfying* (1.1) *and* (1.2) *with* $\gamma\geqq K_0/\delta$. *Then the field* $\{\xi_\nu, \nu\in N^2\}$ *can be redefined on a new probability space together with a Wiener process* $\{W(t), t\in[0, \infty)^2\}$ *such that*

$$\sum_{\nu\leqq n} \xi_\nu - W(n) \ll ([n]\log\log[n])^{1/2}\big((\log n_1)^{-\lambda}+(\log n_2)^{-\lambda}\big) \quad a.s.^2 \tag{1.14}$$

for some positive constant λ *where* $n=(n_1, n_2)$. *(Actually,* λ *can be chosen as large as desired.)*

COROLLARY (2.1). *Let* $\{\xi_\nu, \nu\in N^2\}$ *be a weakly stationary random field satisfying* (1.1) *and* (1.2) *with* $\gamma\geqq K_0/\delta$. *Then there exists a Wiener process* $\{W(t), t\in[0, \infty)^2\}$ *such that*

$$\sum_{\nu\leqq n} \xi_\nu - W(n) = o([n]\log\log[n])^{1/2} \quad a.s. \quad as \quad n_1\wedge n_2 \to \infty \tag{1.15}$$

where $n=(n_1, n_2)$.

Thus, while (1.4) is generally impossible under (1.1) and (1.2), the slightly weaker approximation (1.15) can always be attained. It follows also that the trouble in (1.4) is caused by lattice points $n\in N^2$ lying along the coordinate axes. It is worth noticing that Corollary (2.1) is best possible in the sense that replacing $o([n]\log\log[n])^{1/2}$ in (1.15) by $o\big(f([n])\big)$ where $f(t)=o(t\log\log t)^{1/2}$ $(t\to\infty)$ is any prescribed function, the statement of Corollary (2.1) becomes false. This follows immediately from Theorem 3 of [1] or the theorem formulated after Corollary (2.2).

To discuss further consequences of Theorem 2 let us define, for any function $0\leqq f(t)\leqq t$ $(t\geqq 0)$

$$G_f = \{n = (n_1, n_2)\in N^2\colon\ n_1 \geqq f(n_2), n_2 \geqq f(n_1)\}. \tag{1.16}$$

The set G_f can be used to measure how far a point $n\in N^2$ lies from the coordinate axes: the larger the f is, the deeper the points of G_f lie inside N^2. Now Theorem 2 implies

COROLLARY (2.2). *Let* $\{\xi_\nu, \nu\in N^2\}$ *be a weakly stationary random field satisfying* (1.1) *and* (1.2) *with* $\gamma\geqq K_0/\delta$. *Then there exists a Wiener process* $\{W(t), t\in[0, \infty)^2\}$ *such that for any function* $f(t)$ *satisfying* $0\leqq f(t)\leqq t$ $(t\geqq 0)$ *and* $\sup_{k\geqq 1} f(k^2)/f^2(k)<\infty$ *we have*

$$\sum_{\nu\leqq n} \xi_\nu - W(n) \ll ([n]\log\log[n])^{1/2}\log^{-\lambda} f([n]) \quad a.s. \quad in \quad G_f \tag{1.17}$$

with a positive constant λ. *(Actually,* λ *can be chosen as large as desired.)*

Corollary (2.2) shows that the approximation of S_n by $W(n)$ gets gradually better (in terms of $[n]$) as we move more and more deeply into the first quadrant.

It is worth comparing Corollary (2.2) with the following special case of Theorem 3 of [1]:

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[2] Throughout this paper, $\log t$ and $\log\log t$ are meant as $\log(t\vee e)$ and $\log\log(t\vee e)$, respectively.

THEOREM. *There exists a stationary 2-dependent Gaussian random field* $\{\xi_\nu, \nu\in N^2\}$ *such that* $E\xi_\nu=0$ *and, for any Wiener process* $\{W(t), t\in[0,\infty)^2\}$ *and any positive nondecreasing function* $f(t)$, $t\geqq 0$ *satisfying the conditions*

$$f(t)\leqq c_1(\log\log t)^{1/2}\ (t\geqq t_0)\ \textit{for a sufficiently small}\ c_1>0,$$

$$\lim_{t\to\infty} f(t)=\infty,$$

$$\sup_{t\geqq 1}|f(2t)-f(t)|<\infty,$$

the approximation

$$\sum_{\nu\leqq n}\xi_\nu-W(n)\ll([n]\log\log[n])^{1/2}f([n])^{-\gamma}\quad \textit{a.s. in } G_f$$

cannot hold for any $\gamma>1$.

In other words, $\log^{-\lambda}f([n])$ in (1.17) cannot be replaced by $f([n])^{-\gamma}$ for $\gamma>1$ provided that f grows sufficiently slowly. If $\log^{-\lambda}f([n])$ can be replaced by $f([n])^{-\gamma}$ for some $0<\gamma\leqq 1$ remains open.

For $f(x)=x^\alpha$ $(0<\alpha<1)$ and $f(x)=(\log x)^\alpha$ $(\alpha>0)$ (1.17) yields

$$\sum_{\nu\leqq n}\xi_\nu-W(n)\ll[n]^{1/2}\log^{-\lambda/2}[n]\quad \text{a.s. in}\quad G_f \tag{1.18}$$

$$\sum_{\nu\leqq n}\xi_\nu-W(n)\ll[n]^{1/2}(\log\log[n])^{-\lambda/2}\quad \text{a.s. in}\quad G_f \tag{1.19}$$

respectively. These two special cases were treated also in [1] using a different method. (1.19) is the same as Theorem 2 of [1] except that in [1] only the case of large α was considered (where α is the constant in the definition of f). On the other hand, (1.18) is weaker than Theorem 1 of [1] where on the right side one had $[n]^{1/2-\lambda}$ instead of $[n]^{1/2}\log^{-\gamma/2}[n]$.

At this point we would like to point out an error in [1]: the proof of Lemma 4 is not correct. To get a correct proof, see the proof of Lemma 3 of the present paper. Here, however, $\gamma\geqq 82/\delta$ is assumed and to have the q-dimensional proof all right, the exponent $q(1+\varepsilon)(1+2/\delta)$ in (1.2) of [1] should be replaced by A_q/δ where $A_q=5\cdot 2^{5q-6}+2$.

It is worth noticing that a non-uniform approximation can be obtained in the context of Theorem 1 as well: the theorem remains valid if relation (1.12) is sharpened to

$$\sum_{\nu\leqq n}\xi_\nu-\sum_{\nu\leqq n}\zeta_\nu\ll[n]^{1/2}(\log\log[n])^{-\lambda}\big((\log n_1)^{-\lambda}+(\log n_2)^{-\lambda}\big)\quad \text{a.s.} \tag{1.20}$$

where $n=(n_1,n_2)$. Along the coordinate axes (1.20) is identical with (1.12) and like (1.14), it gets gradually stronger (in terms of $[n]$) as we move away from the axes. Contrary to Theorem 2, however, (1.20) adds very little new information to (1.12) since in domains $G\subset N^2$ where (1.20) is substantially sharper than (1.12), the partial sum process $\sum_{\nu\leqq n}\xi_\nu$ can actually be approximated by Wiener process in the same order. For example, in the domain G_f defined by (1.16) relation (1.20) yields (under the assumptions made on f in Corollary (2.2))

$$\sum_{\nu\leqq n}\xi_\nu-\sum_{\nu\leqq n}\zeta_\nu\ll[n]^{1/2}(\log\log[n])^{-\lambda}\log^{-\lambda}f([n])\quad \text{a.s. in}\quad G_f. \tag{1.21}$$

The presence of $\log^{-\lambda} f([n])$ on the right side of (1.21) means improvement to (1.12) only if $\log f(t)$ grows faster than any power of $\log\log t$; in this case, however, the right side of (1.21) and (1.17) are basically of the same order. Hence, for such domains G_f it is clearly more preferable to apply (1.17) than (1.21).

As far as practical applicability is concerned, a useful improvement of Theorem 1 would be, as we have already pointed out, if relation (1.12) could be replaced by

$$\sum_{\nu\leqq n}\xi_\nu-\sum_{\nu\leqq n}\zeta_\nu \ll [n]^{1/2}\log^{-\lambda}[n] \quad \text{a.s.} \tag{1.22}$$

for some $\lambda>0$. If (1.22) is possible remains open.

Finally we formulate one more approximation theorem of the type (1.7) where $H(n)$ is a slightly perturbed Wiener process. Given any random field $\{\zeta_\nu, \nu\in N^2\}$, a nondecreasing function $0\leqq f(t)\leqq t$ $(t\geqq 0)$ and a constant $\sigma^2\geqq 0$, define the field $\{\zeta_\nu^{(f,\sigma^2)}, \nu\in N^2\}$ by

$$\zeta_\nu^{(f,\sigma^2)}=\begin{cases}\omega_\nu & \text{if} \quad \nu\in G_f\\ \zeta_\nu & \text{if} \quad \nu\notin G_f\end{cases}$$

where G_f is defined by (1.16) and ω_ν are independent $N(0,\sigma^2)$ r.v.'s which are also independent of the field $\{\zeta_\nu, \nu\in N^2\}$. Now we have

THEOREM 3. *Let* $\{\xi_\nu, \nu\in N^2\}$ *be a weakly stationary random field satisfying* (1.1) *and* (1.2) *with* $\gamma\geqq K_0/\delta$; *set* $\sigma^2=\sum_{\nu\in Z^2} r(\nu)$ *where* $r(\nu)$ *is the covariance function of the field. Let* $\{\zeta_\nu, \nu\in N^2\}$ *be the stationary Gaussian field with mean zero and the same covariance function* $r(\nu)$. *Then there exists a nondecreasing function* $f(t)$ *satisfying* $f(t)\ll\exp((\log t)^\alpha)$ *for some* $0<\alpha<1$ *such that the fields* $\{\xi_\nu, \nu\in N^2\}$, $\{\zeta_\nu^{(f,\sigma^2)}, \nu\in N^2\}$ *can be redefined on a rich enough probability space so that*

$$\sum_{\nu\leqq n}\xi_\nu-\sum_{\nu\leqq n}\zeta_\nu^{(f,\sigma^2)} \ll [n]^{1/2}(\log\log[n])^{-\lambda} \quad \textit{a.s.}$$

holds with a positive constant λ. *(Actually* λ *can be chosen as large as desired and* α *as small as desired.)*

The r.v.'s $\zeta_\nu^{(f,\sigma^2)}$ being i.i.d. normal in G_f i.e. in the largest part of N^2, the field $H(n)=\sum_{\nu\leqq n}\zeta_\nu^{(f,\sigma^2)}$ is "almost Wiener"; on the other hand, it preserves the covariance structure of $\{\xi_\nu, F\in N^2\}$ in a narrow strip along the coordinate axes. Despite the nonstationarity of $\{\zeta_\sigma^{(f,\sigma^2)}, \nu\in N^2\}$, Theorem 3 is more suitable for applications than, for example, Corollary (1.1). As a matter of fact, Theorem 2 follows easily from Theorem 3 and the proof of Theorem 1 will also depend crucially on Theorem 3. (For technical reasons we shall actually work with a slightly modified version of $\zeta_\nu^{(f,\sigma^2)}$, namely the field $\{\zeta_\nu^{(H)}, \nu\in N^2\}$ in Theorem 4.) It would be interesting to minimize f in Theorem 3 but we shall not deal with this question here.

In conclusion we note that while in our paper we consider only two-parameter fields, all results and their proofs can be extended, without any difficulty, for q-parameter fields, $q\geqq 3$.

2. Preparatory lemmas

LEMMA 1. (Dvoretzky [5].) *Let* ξ *be a (possibly complex valued) random variable with* $|\xi|\leqq 1$ *and let* $\mathcal{F}$ *be the* σ*-field generated by* ξ. *Then for any* σ*-field* $\mathcal{G}$

$$E|E(\xi|\mathcal{G})-E\xi| \leqq 2\pi \sup_{A\in\mathcal{F}, B\in\mathcal{G}} |P(AB)-P(A)P(B)|.$$

LEMMA 2. (Davidov [3].) *Let* ξ *and* η *be (possibly complex valued) random variables measurable* $\mathcal{F}$ *and* $\mathcal{G}$, *respectively. Let* $p_1, p_2, p_3\geqq 1$ *with* $p_1^{-1}+p_2^{-1}+p_3^{-1}=1$. *If* $\|\xi\|_{p_1}<\infty$ *and* $\|\eta\|_{p_2}<\infty$ *(where* $\|\cdot\|_p$ *denotes the* L_p *norm) then*

$$|E\xi\eta-E\xi\cdot E\eta| \leqq 10\Big(\sup_{A\in\mathcal{F}, B\in\mathcal{G}} |P(AB)-P(A)P(B)|\Big)^{1/p_3}\|\xi\|_{p_1}\|\eta\|_{p_2}.$$

From Lemma 2 it follows (setting $p_1=p_2=2+\delta$, $p_3=(2+\delta)/\delta$) that if $\{\xi_\nu, \nu\in N^2\}$ satisfies (1.1) and (1.2) and it is weakly stationary with covariance function $r(\nu)$ then

$$|r(\nu)| \ll |\nu|^{-\gamma\delta/3} \ll ([\nu]\vee 1)^{-\gamma\delta/6} \tag{2.1}$$

for any $\nu\in Z^2$, $\nu\neq 0$.

LEMMA 3. *Let* $\{\xi_\nu, \nu\in N^2\}$ *be a (not necessarily stationary) random field satisfying* (1.1) *and* (1.2) *with* $\gamma\geqq 82/\delta$. *Put* $\alpha=\delta/1024$. *Then we have*

$$E\Big|\sum_{\mu+e\leqq\nu\leqq\mu+n} \xi_\nu\Big|^{2+\alpha} \leqq B[n]^{1+\alpha/2}$$

for any $\mu=(\mu_1, \mu_2)\geqq 0$ *and* $n\in N^2$; *here* B *is a positive constant and* $e=(1, 1)$.

PROOF. Set

$$S_a(y) = n_1^{-1/2} \sum_{a+1\leqq\nu_1\leqq a+n_1,\, \nu_2=y} \xi_\nu$$

for each $y=1, 2, \ldots$ and $a\geqq 0$ where $n=(n_1, n_2)$. Applying Lemma (2.5) of [8] with $\varepsilon=1/4$ and noting that $82/\delta\geqq(1+2/\delta)\cdot 5/4$ we get $E|S_\alpha(y)|^{2+\delta_1}\leqq B_1$ uniformly in a and y where $\delta_1=\delta/32$ and B_1 is a positive constant. Then, because $82/\delta\geqq(1+2/\delta_1)\cdot 5/4$ and since the random variables $\{S_\alpha(y),\ y=1, 2, \ldots\}$ are strong mixing in y with zero means and uniformly bounded $(2+\delta_1)$-th moments, we conclude by the same reasoning that

$$T_{a,b} = n_2^{-1/2} \sum_{b+1\leqq y\leqq b+n_2} S_a(y)$$

satisfies $E|T_{a,b}|^{2+\alpha}\leqq B$ uniformly in a, b where $\alpha=\delta_1/32$ and B is a positive constant.

For the following lemma we need some notation. Given any bounded sets I, J on the real line and any integer k, let $\varphi_{I,J}(k)$ denote the number of pairs (i, j) such that i, j are integers, $i\in I, j\in J$ and $j-i=k$. Further, for any bounded set I in a Euclidean space let $|I|$ denote the number of lattice points (i.e. points with integral coordinates) contained in I.

LEMMA 4. *Let* $\{\xi_\nu, \nu\in N^2\}$ *be a weakly stationary random field with mean zero and covariance function* $r(\nu)$ *satisfying*

$$|r(\nu)| \ll |\nu|^{-(2+\varepsilon)} \quad (\nu\neq 0) \tag{2.2}$$

for some $\varepsilon>0$. *(By* (2.1), *this is the case if* $\{\xi_\nu, \nu\in N^2\}$ *satisfies* (1.1) *and* (1.2) *with* $\gamma>6/\delta$.*) Then* $\sum_{\nu\in Z^2}|r(\nu)|<\infty$ *and thus the series defining* c_k *and* d_l *in* (1.8) *are convergent. Let* I, J *and* G *be closed intervals on the positive line and set* $S_1=\sum_{\nu\in I\times G}\xi_\nu$, $S_2=\sum_{\nu\in J\times G}\xi_\nu$. *Then*

$$ES_1S_2=|G|\left(\sum_{k=-\infty}^{+\infty}\varphi_{I,J}(k)c_k+\mathrm{const}\cdot\theta\cdot(|I|\wedge|J|)|G|^{-\varepsilon/4}\right). \tag{2.3}$$

(The series on the right side is convergent since $\varphi_{I,J}(k)=0$ for all but finitely many k.) If in the definition of S_1 and S_2 we replace $I\times G$ and $J\times G$ by $G\times I$ and $G\times J$, respectively, then (2.3) remains valid with the modification that c_k is to be replaced by d_k (defined by (1.8)).

(Here, and in the sequel, θ denotes various numbers satisfying $|\theta|\leqq 1$ and all the constants (including those implied by relations $\ll$) will depend only on the field $\{\xi_\nu, \nu\in N^2\}$.)

COROLLARY. *Under the conditions of Lemma* 4 *we have*

$$E\Big(\sum_{\nu\leqq n}\xi_\nu\Big)^2=[n]\,(a_{n_1}+\mathrm{const}\cdot\theta\cdot n_2^{-\varepsilon/4})=[n]\,(b_{n_2}+\mathrm{const}\cdot\theta\cdot n)_1^{-\varepsilon/4})= \tag{2.4}$$
$$=[n]\,(\sigma^2+\mathrm{const}\cdot\theta\cdot(n_1\wedge n_2)^{-\varepsilon/4})$$

where $n=(n_1, n_2)$ *and*

(2.5)
$$a_k=\sum_{i=-k}^{k}(1-|i|/k)c_i,\quad b_k=\sum_{i=-k}^{k}(1-|i|/k)d_i,\quad \sigma^2=\sum_{\nu\in Z^2}r(\nu)=\sum_{k=-\infty}^{+\infty}c_k=\sum_{l=-\infty}^{+\infty}d_l$$

where c_k, d_l *are defined by* (1.8). *Moreover, the sequences* $\{c_k, k\geqq 0\}$ *and* $\{d_l, l\geqq 0\}$ *are nonnegative definite. (The number* σ^2 *is nonnegative by the third equality of* (2.4).*)*

PROOF OF LEMMA 4. Let $m=|G|$. Obviously, for any given $\nu=(\nu_1, \nu_2)\in Z^2$ the number of those pairs $\mu^{(1)}, \mu^{(2)}$ such that $\mu^{(1)}\in(I\times G)\cap N^2$, $\mu^{(2)}\in(J\times G)\cap N^2$ and $\mu^{(2)}-\mu^{(1)}=\nu$ is $\varphi_{I,J}(\nu_1)(m-|\nu_2|)^+$. Hence, by the weak stationarity of the field $\{\xi_\nu, \nu\in N^2\}$ we have

(2.6)
$$ES_1S_2=\sum_{\{\nu=(\nu_1,\nu_2)\in Z^2:\,|\nu_2|\leqq m\}}\varphi_{I,J}(\nu_1)(m-|\nu_2|)r(\nu)=m\Bigg(\sum_{\{\nu\in Z^2:\,|\nu_2|\leqq m\}}\varphi_{I,J}(\nu_1)r(\nu)-$$
$$-\sum_{\{\nu\in Z^2:\,|\nu_2|\leqq m\}}\varphi_{I,J}(\nu_1)r(\nu)\frac{|\nu_2|}{m}\Bigg).$$

Let Σ_1 and Σ_2 denote the two sums appearing in the brackets in the last expression. Observing that $\varphi_{I,J}(k)\leqq|I|\wedge|J|$ for any k and

$$\sum_{\{\nu\in Z^2:\,|\nu_2|\geqq L\}}|r(\nu)|\ll L^{-\varepsilon/2}$$

for any $L>0$ by (2.2), it follows that

$$\Sigma_1=\sum_{\nu\in Z^2}\varphi_{I,J}(\nu_1)r(\nu)+\mathrm{const}\cdot\theta\cdot(|I|\wedge|J|)m^{-\varepsilon/2}.$$

Breaking Σ_2 into two parts according as $|v_2|\leqq m^{1/2}$ or $|v_2|>m^{1/2}$ and using the same inequalities as above we get $|\Sigma_2|\ll(|I|\wedge|J|)m^{-1/2}+(|I|\wedge|J|)m^{-\varepsilon/4}$. Hence

$$ES_1S_2 = m\Big(\sum_{v\in Z^2}\varphi_{I,J}(v_1)r(v)+\text{const}\cdot\theta\cdot(|I|\wedge|J|)m^{-\varepsilon/4}\Big) =$$

$$= m\left(\sum_{k=-\infty}^{+\infty}\varphi_{I,J}(k)c_k+\text{const}\cdot\theta\cdot(|I|\wedge|J|)m^{-\varepsilon/4}\right)$$

which proves (2.3).

Specializing (2.3) (and the analogous equation involving d_k) to the case $I=J=[1,n_1]$, $G=[1,n_2]$ and observing that $\varphi_{I,I}(k)=(l-|k|)^+$ for any interval I where $l=|I|$ we get the first two equalities of the corollary. To get the third equality note that $|c_k|\ll k^{-(1+\varepsilon/2)}$ by (2.2) and thus

$$|a_k-\sigma^2| = \left|\sum_{|i|>k}c_i+\sum_{|i|\leqq k}{}' \frac{|i|}{k}c_i\right| \ll k^{-\varepsilon/4}$$

as it follows again by breaking the second sum into two parts according as $|i|\leqq k^{1/2}$ or $|i|>k^{1/2}$. Hence the third equality of (2.4) follows from the first one. Finally, to show that $\{c_k,\ k\geqq 0\}$ is nonnegative definite, put

$$S_i^{(l)} = \sum_{\substack{v_1=i\\ 1\leqq v_2\leqq l}}\xi_v \quad (i\geqq 1,\ l\geqq 1).$$

By (2.3) we have

$$ES_i^{(l)}S_j^{(l)} = l(c_{j-i}+\text{const}\cdot\theta\cdot l^{-\varepsilon/4})$$

and thus

$$E\left|\sum_{i=1}^{r}\lambda_iS_i^{(l)}\right|^2 = l\left(\sum_{i,j=1}^{r}c_{j-i}\lambda_i\bar\lambda_j+\text{const}\cdot\theta\cdot l^{-\varepsilon/4}\sum_{i,j=1}^{r}|\lambda_i\lambda_j|\right)$$

for any $r\geqq 1$ and any complex numbers $\lambda_1,\ldots,\lambda_r$. Dividing by l and letting $l\to\infty$ we get that $\sum_{i\cdot j=1}^{r}c_{j-1}\lambda_i\bar\lambda_j\geqq 0$ what was to be proved.

We note that the sequences $\{c_k,\ k\geqq 0\}$, $\{d_l,\ l\geqq 0\}$ will actually be positive definite if, in addition to (2.2), the covariance function $r(v)$ satisfies the condition

$$\sigma_0 = r(0)-\sum_{v\neq 0}|r(v)| > 0 \tag{2.7}$$

(see Lemma 14). This fact, however, will not be needed in the sequel.

3. The associated Gaussian fields

In this section we introduce three Gaussian fields which will play a crucial role in the proof of our theorems and prove a central limit theorem related to them.

Assume that $\{\xi_v,\ v\in N^2\}$ is weakly stationary with mean zero and covariance function $r(v)$ satisfying (2.2). By the Corollary to Lemma 4 the sequences $\{c_k,\ k\geqq 0\}$ and $\{d_k,\ k\geqq 0\}$ (defined by (1.8)) are nonnegative definite. Let $\{\zeta_v^{(1)},\ v\in N^2\}$ be a Gaussian field such that for every fixed $l=1,2,\ldots$ the one-parameter process $\{\zeta_{(k,l)}^{(1)},\ k=1,2,\ldots\}$ is Gaussian with mean zero and covariance sequence $\{c_k,\ k\geqq 0\}$

and moreover, the just mentioned one-parameter processes are independent. Analogously, let $\{\zeta_\nu^{(2)}, \nu\in N^2\}$ be a Gaussian field such that the one-parameter processes $\{\zeta_{(l,k)}^{(2)}, k=1, 2, \ldots\}$ $(l=1, 2, \ldots)$ are independent Gaussian processes with mean zero and covariance sequence $\{d_k, k\geqq 0\}$. Finally, given any partition $H=(H_1, H_2, H_3)$ of N^2 into three disjoint sets H_1, H_2, H_3 we define the Gaussian field $\{\zeta_\nu^{(H)}, \nu\in N^2\}$ as follows:

$$\zeta_\nu^{(H)} = \begin{cases} \zeta_\nu^{(1)} & \text{if } \nu\in H_1 \\ \zeta_\nu^{(2)} & \text{if } \nu\in H_2 \\ \omega_\nu & \text{if } \nu\in H_3 \end{cases} \tag{3.1}$$

where $\{\zeta_\nu^{(1)}, \nu\in N^2\}$, $\{\zeta_1^{(2)}, \nu\in N^2\}$ are the fields defined above which, in addition, are assumed to be independent of each other and $\{\omega_\nu, \nu\in H_3\}$ are independent normal r.v.'s with mean zero and variance σ^2 which are independent of both the $\zeta_\nu^{(1)}$'s and the $\zeta_\nu^{(2)}$'s. Here σ^2 is the number defined by (2.5).

LEMMA 5. *Let $\{\xi\ , \nu\in N^2\}$ be a weakly stationary random field satisfying* (1.1) *and* (1.2) *with $\gamma\geqq 82/\delta$. Let $r\geqq 1$ be an arbitrary integer and let $I_1, I_2, \ldots, I_r$, G be closed intervals on the positive line. Let $u_k=|I_k\times G|$ $(k=1, \ldots, r)$, $m=|G|$ and set*

$$S_k = \sum_{\nu\in I_k\times G} \xi_\nu,\quad S_k^{(1)} = \sum_{\nu\in I_k\times G} \zeta_\nu^{(1)}\quad (k = 1, \ldots, r)$$

where $\{\zeta_\nu^{(1)}, \nu\in N^2\}$ is the field defined above. Then for any vector $\lambda=(\lambda_1, \ldots, \lambda_r)\in R^2$ we have

$$\left| E\left\{\exp\left(\frac{i\lambda_1}{\sqrt{u_1}} S_1+\ldots+\frac{i\lambda_r}{\sqrt{u_r}} S_r\right)\right\} - E\left\{\exp\left(\frac{i\lambda_1}{\sqrt{u_1}} S_1^{(1)}+\ldots+\frac{i\lambda_r}{\sqrt{u_r}} S_r^{(1)}\right)\right\}\right| \ll m^{-\tau} \tag{3.2}$$

provided that $|\lambda| r\leqq m^\tau$; here $\tau>0$ is a suitable constant. The same inequality holds if $I_k\times G$ is replaced everywhere by $G\times I_k$ and $\{\zeta_\nu^{(1)}, \nu\in N^2\}$ is replaced by $\{\zeta_\nu^{(2)}, \nu\in N^2\}$. If we drop the condition that $\{\xi_\nu, \nu\in N^2\}$ is weakly stationary then still there exist numbers $g_{i,j}$, $1\leqq i, j\leqq r$ (depending on the field $\{\xi_\nu, \nu\in N^2\}$ and on the intervals $I_1, \ldots, I_r, G$) such that $|g_{i,j}|\ll 1$ and

$$\left| E\left\{\exp\left(\frac{i\lambda_1}{\sqrt{u_1}} S_1+\ldots+\frac{i\lambda_r}{\sqrt{u_r}} S_r\right)\right\} - \exp\left(-\frac{1}{2}\sum_{i,j=1}^{r} g_{ij}\lambda_i\lambda_j\right)\right| \ll m^{-\tau}$$

for $|\lambda| r\leqq m^\tau$.

For the proof of Lemma 5 we need a trivial property of the field $\{\zeta_\nu^{(1)}, \nu\in N^2\}$, quite analogous to Lemma 4 which we formulate here as a separate lemma.

LEMMA 6. *Assume the conditions of Lemma 5, let I, J, G be closed intervals on the positive line and set*

$$\bar S_1 = \sum_{\nu\in I\times G} \zeta_\nu^{(1)},\quad \bar S_2 = \sum_{\nu\in J\times G} \zeta_\nu^{(1)}.$$

Then

$$E\bar S_1 \bar S_2 = |G|\left(\sum_{k=-\infty}^{+\infty} \varphi_{I,J}(k) c_k\right). \tag{3.3}$$

In other words, (2.3) is valid for the field $\{\zeta_\nu^{(1)}, \nu\in N^2\}$ as well, even without the error term on the right side. Relation (3.3) is obvious since the covariance function $r_1(\nu)$ of the field $\xi_\nu^{(1)}, \nu\in N^2\}$ vanishes except for those $\nu=(\nu_1, \nu_2)$ such that $\nu_2=0$ and thus the argument leading to the first equality of (2.6) yields directly (3.3).

Lemmas 4 and 6 imply the covariance matrices of the vectors $(S_1/\sqrt{u_1}, \ldots, S_r/\sqrt{u_r})$ and $(S_1^{(1)}/\sqrt{u_1}, \ldots, S_r^{(1)}/\sqrt{u_r})$ are close to each other if $|G|$ is large. Starting out from this fact, the proof of Lemma 5 follows the standard pattern of proving central limit theorems for mixing processes (see [6]). However, the calculations are somewhat tedious and thus we postpone the proof to section 6 in order not to digress from the main line of the proof of our theorems.

COROLLARY OF LEMMA 5. *Let $\{\xi_\nu, \nu\in N^2\}$ satisfy the conditions of Lemma 5 and let $S_n=\sum_{\nu\leqq n}\xi_\nu$. Then we have*

$$(3.4)\qquad |E\{\exp(i\lambda S_n/[n]^{1/2})\}-\exp(-\sigma^2\lambda^2/2)| \ll (n_1\wedge n_2)^{-\varrho} \quad \text{for} \quad |\lambda|\leqq(n_2\wedge n_2)^{\varrho}$$

and

$$(3.5)\qquad P\{|S_n|\geqq t[n]^{1/2}\}\ll \exp(-Bt^2)+[n]^{-\varrho} \quad \text{for} \quad t\geqq 0$$

where $n=(n_1, n_2)$, σ^2 is defined by (2.5) *and B, ϱ are positive constants.* (3.5) *is valid even without the assumption of the weak stationarity of $\{\xi_\nu, \nu\in N^2\}$.*

LEMMA 7. *Let $\{\xi_\nu, \nu\in N^2\}$ be a (not necessarily stationary) random field satisfying* (1.1) *and* (1.2) *with $\gamma\geqq 4098/\delta$ and let $S_n=\sum_{\nu\leqq n}\xi_\nu$. Then we have for all $n\in N^2$*

$$(3.6)\qquad P\left\{\max_{\nu\leqq n}|S_\nu|\geqq t[n]^{1/2}\right\}\ll\exp(-Dt^2) \quad \text{for} \quad 0\leqq t\leqq D\log^{1/2}[n]$$

with a positive constant D.

The proof of Lemma 7, like that of Lemma 5, is basically routine but tedious and thus it will be postponed to section 6.

LEMMA 8. *The conclusion of Lemma 5 remains valid if instead of the conditions made on $\{\xi_\nu, \nu\in N^2\}$ we assume that $\{\xi_\nu, \nu\in N^2\}$ is a stationary Gaussian field with mean zero and covariance function $r(\nu)$ satisfying* (2.2). *Lemmas 3 and 7 and the statement of Lemma 5 concerning the non-stationary case remain valid if instead of* (1.1), (1.2) *we assume that $\{\xi_\nu, \nu\in N^2\}$ is a Gaussian field satisfying*

$$(3.7)\qquad E\xi_\nu=0,\quad E\xi_\nu^2\ll 1 \quad (\nu\in N^2)$$

and

$$(3.8)\qquad |E\xi_\mu\xi_\nu|\ll|\mu-\nu|^{-(2+\varepsilon)} \quad \mu\neq\nu$$

for some $\varepsilon>0$.

PROOF. Assume first that $\{\xi_\nu, \nu\in N^2\}$ is a Gaussian field satisfying (3.7) and (3.8). Then

$$E\Big(\sum_{\nu\in G}\xi_\nu\Big)^2\leqq\sum_{\nu\in G}E\xi_\nu^2+\sum_{\{\nu\in Z^2,\,\nu\neq 0\}}\;\sum_{\{\mu:\,\mu\in G,\,\mu+\nu\in G\}}|E\xi_\mu\xi_{\mu+\nu}|\ll$$

$$\ll|G|+|G|\Big(\sum_{\{\nu\in Z^2,\,\nu\neq 0\}}|\nu|^{-(2+\varepsilon)}\Big)\ll|G|$$

uniformly for any finite set $G\subset N^2$. Since the sum $\sum_{v\in G}\xi_v$ is normally distributed with mean zero, the last estimate implies

$$E\big(\sum_{v\in G}\xi_v\big)^4 \ll |G|^2 \tag{3.9}$$

and

$$P\big\{\big|\sum_{v\in G}\xi_v\big| \geqq t|G|^{1/2}\big\} \ll \exp(-Bt^2) \quad (t\geqq 0) \tag{3.10}$$

uniformly in G with a positive constant B. (3.9) shows that the conclusion of Lemma 3 holds with $\alpha=2$, on the other hand, (3.10) implies (3.6) by a maximal inequality of Móricz ([11], Theorem 1).

Assume now that $\{\xi_v, v\in N^2\}$ is stationary Gaussian with mean zero and covariance function $r(v)$ satisfying (2.2). In this case, both vectors $(S_1, \dots, S_r)$ and $(S_1^{(1)}, \dots, S_r^{(1)})$ in Lemma 5 are Gaussian with mean zero and thus the left side of (3.2) equals

$$\left|\exp\left(-\frac{1}{2}\sum_{i,j=1}^{r} a_{i,j}\lambda_i\lambda_j\right) - \exp\left(-\frac{1}{2}\sum_{i,j=1}^{r} a_{i,j}^{(1)}\lambda_i\lambda_j\right)\right| \tag{3.11}$$

where

$$a_{ij} = (u_iu_j)^{-1/2}E(S_iS_j), \quad a_{i,j}^{(1)} = (u_iu_j)^{-1/2}E(S_i^{(1)}S_j^{(1)}).$$

By Lemmas 4 and 6 we have $|a_{i,j}-a_{i,j}^{(1)}|\ll m^{-\varepsilon/4}$ $(1\leqq i, j\leqq r)$. Since both matrices $(a_{i,j})^{r\times r}$ and $(a_{i,j}^{(1)})^{r\times r}$ are nonnegative definite, the expression (3.11) is

$$\ll m^{-\varepsilon/4}\left(\sum_{i=1}^{r}|\lambda_i|\right)^2 \leqq m^{-\varepsilon/4}|\lambda|^2 r$$

and thus (3.2) holds for $|\lambda|r\leqq m^\tau$ where $\tau=\varepsilon/16$. If now $\{\xi_v, v\in N^2\}$ is Gaussian satisfying (3.7) and (3.8) then the last inequality of Lemma 5 holds (with the left side equal to zero) with $g_{i,j}=(u_iu_j)^{-1/2}E(S_iS_j)$. It remains now to notice that $|g_{i,j}|\ll 1$ by the Cauchy—Schwarz inequality and the analogue of Lemma 3 proved above.

LEMMA 9. *Let* $\{\xi_k, k\geqq 1\}$ *be a stationary Gaussian process with mean zero and covariance function* $c_n\ll n^{-(1+\varepsilon)}$ *for some* $\varepsilon>0$. *Let* $\sigma^2=c_0+2\sum_{k=1}^{\infty}c_k$ *and set* $t_k=\sum_{i=1}^{k}[\exp(i^\alpha)]$ *for some* $0<\alpha<1$ *where* [ ] *denotes integral part.*[3] *Then there exists (after possibly redefining the sequence* ξ_k *on a new probability space) independent normal r.v's* $\{\zeta_k, k\geqq 1\}$ *with mean zero and variance* σ^2 *such that*

$$\Big\|\sum_{i\leqq t_k}(\xi_i-\zeta_i)\Big\|_3 \ll t_k^{1/2}(\log t_k)^{-(1-\alpha)/2\alpha} \quad (k=1,2,\dots) \tag{3.12}$$

where $\|\cdot\|_3$ *is the* L_3 *norm.*

Lemma 9 is implicit in [12] but we shall give a simple direct proof in section 6. (Our argument will also yield an alternative proof of the results of [12]).

[3] This notation is not to be confused with $[n]$ for $n\in N^2$, introduced in section 1.

4. Sketch of the proof of the theorems

Because of the rather technical nature of the proof of Theorems 1—3 it is worth outlining the basic idea behind the proofs. The real problem is to prove implication (C)⇒(B) in Theorem 1, the rest is quite easy. Assume that $\{\xi_\nu, \nu\in N^2\}$ satisfies (1.1) and (1.2) with a large γ. Divide the set of positive integers into consecutive intervals $I_1, J_1, \dots, I_k, J_k, \dots$ in such a way that $|I_k|\to\infty$, $|J_k|\to\infty$, $|J_k|/|I_k|\to 0$ in a suitable way. Set

$$X_{m,n} = |I_m\times I_n|^{-1/2} \sum_{\nu\in I_m\times I_n} \xi_\nu.$$

In analogy with the usual method for proving a.s. invariance principles for mixing sequences of r.v.'s (see [13], [2]) one might attempt to show that the sequence $\{X_{m,n}, m\geqq 1, n\geqq 1\}$ is "asymptotically independent" in the sense that

$$\varrho^*_{m,n} = \varrho^*\big\{\sigma\{X_{mn}\},\quad \sigma\{X_{i,j}\colon (i,j)\neq(m,n)\}\big\} \to 0 \quad\text{as}\quad m\vee n\to\infty$$

with a proper speed where $\sigma\{\cdot\}$ denotes the σ-field generated by the r.v.'s in the brackets and

$$\varrho^*\{\mathcal{F},\mathcal{G}\} = \sup_{A\in\mathcal{F}, B\in\mathcal{G}} |P(AB)-P(A)P(B)|.$$

However, $\varrho^*_{m,n}$ will not be small if one of m and n is small; for example, the separation between the index sets belonging to $X_{m,1}$ and $X_{m,2}$ is independent of m and thus if we assume nothing more than (1.1), (1.2) then generally $\varrho^*_{m,1}\not\to 0$. We remedy this trouble by introducing the vectors $X^*_m=(X_{1,m}, X_{2,m}, \dots, X_{r_m,m})$ and $X^{**}_m=(X_{m,1}, X_{m,2}, \dots, X_{m,r_m})$ where $r_m=o(m)$, $r_m\uparrow\infty$ is a suitable sequence of positive integers and by replacing the sequence $\{X_{m,n}, m\geqq 1, n\geqq 1\}$ by a new sequence formed by the r.v.'s $\{X_{m,n}, m>r_n, n>r_m\}$ and the vectors X^*_m, X^{**}_m ($m=2, 3, \dots$). As one can easily see, this new sequence is then asymptotically independent in the above sense. Moreover, if $\{\zeta^{(1)}_\nu, \nu\in N^2\}$, $\{\zeta^{(2)}_\nu, \nu\in N^2\}$ are the Gaussian fields introduced in Section 3 then by Lemma 5 the distribution of the vectors X^*_m, X^{**}_m are asymptotically equal, as $m\to\infty$, with those of the vectors Y^*_m, Y^{**}_m where Y^*_m, Y^{**}_m are defined in the same way as X^*_m, X^{**}_m just with the underlying field $\{\zeta_\nu, \nu\in N^2\}$ replaced by $\{\zeta^{(1)}_\nu, \nu\in N^2\}$, $\{\zeta^{(2)}_\nu, \nu\in N^2\}$, respectively. The distribution of the r.v.'s $\{X_{m,n}, m>r_n, n>r_m\}$ being asymptotically $N(0,\sigma^2)$ by the Corollary to Lemma 5, the above facts show that the (one and higher dimensional) r.v.'s $\{X_{m,n}, m>r_n, n>r_m\}$, X^*_m, X^{**}_m ($m=2, 3, \dots$) are asymptotically independent and are close in distribution to the r.v.'s $\{Z_{m,n}, m>r_n, n>r_m\}$, Z^*_m, Z^{**}_m ($m=2, 3, \dots$) defined analogously just with the field $\{\xi_\nu, \nu\in N^2\}$ replaced by the Gaussian field $\{\zeta^{(H)}_\nu, \nu\in N^2\}$ introduced in Section 3 where $H=(H_1, H_2, H_3)$ is the partition of N^2 defined by

$$H_1 = \bigcup_{m=2}^{\infty}\bigcup_{i=1}^{r_m} (K_i\times K_m),\quad H_2 = \bigcup_{m=2}^{\infty}\bigcup_{i=1}^{r_m} (K_m\times K_i),\quad H_3 = N^2\setminus(H_1\cup H_2)$$

where $K_m=I_m\cup J_m$. Also, the quantities Z are (strictly) independent. Hence, by the approximation method of [2] the r.v.'s $\{X_{m,n}, m>r_n, n>r_m\}$, X^*_m, X^{**}_m ($m=2,3,\dots$)

can be redefined on a suitable new probability space together with the r.v.'s $\{Z_{m,n}, m>r_n, n>r_m\}$, Z_m^*, Z_m^{**} $(m=2, 3, \dots)$ such that the joint distributions of both sequences remain unchanged and the X's are close to the corresponding Z's. From here we get easily a joint construction of the fields $\{\xi_\nu, \nu\in N^2\}$, $\{\zeta_\nu^{(H)}, \nu\in N^2\}$ on the same probability space such that their partial sums are close to each other.

As a matter of fact, in the proof we shall actually work with more complicated vectors than X_m^*, X_m^{**} defined above in order to avoid certain technical difficulties. However, the basic idea remains unchanged.

The above argument applies also in the case when instead of (1.1), (1.2) we assume that $\{\xi_\nu, \nu\in N^2\}$ is a stationary Gaussian field with mean zero and covariance function $r(\nu)$ satisfying (2.2), (2.7). The only difference is that in this case we need the Kolmogorov—Rozanov theorem (instead of (1.2)) to show the "asymptotically independent" character of the r.v.'s $\{X_{m,n}, m>r_n, n>r_m\}$, X_m^*, X_m^{**} $(m=2, 3, \dots)$. Hence in this case we can also approximate the partial sums of $\{\xi_\nu, \nu\in N^2\}$ by those of $\{\zeta_\nu^{(H)}, \nu\in N^2\}$ where H is the same partition. Observe now that $\{\zeta_\nu^{(H)}, \nu\in N^2\}$ is defined in terms of the quantities c_k, d_l in (1.8) and the partition H; hence if we have two stationary fields $\{\xi_\nu, \nu\in N^2\}$ one of which is mixing satisfying (1.1), (1.2) and the other is centered Gaussian satisfying (2.2), (2.7) and, moreover, the quantities c_k, d_l are the same for the two fields then the approximating field $\{\zeta_\nu^{(H)}, \nu\in N^2\}$ will also be the same in the two cases and thus the partial sums of the two fields can be approximated also by each other in a good order. This is exactly implication (C)⇒(B) of Theorem 1.

5. Proof of the theorems

Let $\alpha>0$, $c_0>0$ be sufficiently small constants such that $c_0<\alpha/3$; set $p_m=[\exp(m^\alpha)]-[\exp(m^{\alpha/2})]$, $p'_m=[\exp(m^{\alpha/2})]$, $t_m=\sum_{i=1}^{m}[\exp(i^\alpha)]$ $(t_0=0)$, $t'_m=t_{m-1}+p_m$, $q_m=t_{[(c_0\log m)^{1/\alpha}]}$, $r_m=[(c_0\log m)^{1/\alpha}]$; here [ ] denotes integral part.[4] It is easy to see that, as $m\to\infty$

$$t_m \sim c_1 m^{1-\alpha}\exp(m^\alpha), \tag{5.1}$$

$$p_m \sim c_2\frac{t_m}{(\log t_m)^{(1-\alpha)/\alpha}}, \tag{5.2}$$

$$q_m \sim c_3(\log m)^{1-\alpha/\alpha} m^{c_0} \tag{5.3}$$

with suitable positive constants c_1, c_2, c_3; the symbol $\sim$ means asymptotic equality. Let

$$\begin{cases} H_1 = \bigcup_{m=2}^{\infty}\{(i, j\in N^2\colon\ t_{m-1}<j\leqq t_m,\quad 1\leqq i\leqq t_{[m^{\alpha/3}]}\},\\ H_2 = \bigcup_{m=2}^{\infty}\{(i, j)\in N^2\colon\ t_{m-1}<i\leqq t_m,\quad 1\leqq j\leqq t_{[m^{\alpha/3}]},\\ H_3 = N^2\setminus\{H_1\cup H_2\}. \end{cases} \tag{5.4}$$

[4] Cf. footnote 3.

Evidently $H_1 \cap H_2 = \emptyset$. We are going to prove the following theorem from which each of Theorems 1—3 follows easily:

THEOREM 4. *Let $\{\xi_\nu, \nu \in N^2\}$ be a weakly stationary random field which either satisfies (1.1) and (1.2) with $\gamma \geqq 4098/\delta$ or is Gaussian with mean zero and covariance function r satisfying (2.2), (2.7). Let $H=(H_1, H_2, H_3)$ be the partition defined by (5.4) and let $\{\zeta_\nu^{(H)}, \nu \in N^2\}$ be the Gaussian field defined by (3.1). Then the fields $\{\xi_\nu, \nu \in N^2\}$ and $\{\zeta_\nu^{(H)}, \nu \in N^2\}$ can be jointly defined on a suitable probability space such that*

$$(5.5)\qquad \sum_{\nu \leqq n} \xi_\nu - \sum_{\nu \leqq n} \zeta_\nu^{(H)} \ll [n]^{1/2} (\log\log [n])^{-\tau} \big((\log n_1)^{-\tau} + (\log n_2)^{-\tau} \big) \quad a.s.$$

where $n=(n_1, n_2)$ and $\tau = (1-3\alpha)/8\alpha$.

PROOF. Assume first that $\{\xi_\nu, \nu \in N^2\}$ is weakly stationary and satisfies (1.1) and (1.2) with $\gamma \geqq 4098/\delta$. Let

$$(5.6)\qquad \begin{cases} X_m^{(i)} = p_m^{-1/2} \displaystyle\sum_{\nu = t_{m-1}+1,\, i}^{t'_m,\, i} \xi_\nu \quad (m \geqq 1,\ i \geqq 1), \\ Y_n^{(i)} = p_n^{-1/2} \displaystyle\sum_{\nu = i,\, t_{n-1}+1}^{i,\, t'_n} \xi_\nu \quad (n \geqq 1,\ i \geqq 1), \\ Z_{m,n} = (p_m p_n)^{-1/2} \displaystyle\sum_{\nu = t_{m-1}+1,\, t_{n-1}+1}^{t'_m,\, t'_n} \xi_\nu \quad (m \geqq 1,\ n \geqq 1), \\ \Delta_{m,n} = \displaystyle\sum_{\nu = t_{m-1}+1,\, t_{n-1}+1}^{t_m,\, t_n} \xi_\nu - \sum_{\nu = t_{m-1}+1,\, t_{n-1}+1}^{t'_m,\, t'_n} \xi_\nu \end{cases}$$

where, for any positive integers a, b, c, d the symbol $\sum\limits_{\nu=a,b}^{c,d} \xi_\nu$ means the (possibly empty) sum $\sum\limits_{(a,b) \leqq \nu \leqq (c,d)} \xi_\nu$. Define now a sequence $U_{m,n}$ $(m \geqq 1,\ n \geqq 1)$ of random vectors as follows:

$$(5.7)\quad U_{m,n} = \begin{cases} (X_m^{(1)}, X_m^{(2)}, \dots, X_m^{(q_m)},\ Z_{m, r_m+1}, \dots, Z_{m, [m^{\alpha/3}]}) & \text{if } m>1,\ n=1, \\ (Y_n^{(1)}, Y_n^{(2)}, \dots, Y_n^{(q_n)}, Z_{r_n+1, n}, \dots, Z_{[n^{\alpha/3}], n}) & \text{if } m=1,\ n>1, \\ Z_{m,n} \quad \text{if } m > [n^{\alpha/3}],\ n > [m^{\alpha/3}], \\ 0 \quad \text{otherwise.} \end{cases}$$

Define the sequences $\tilde{X}_m^{(i)}, \tilde{Y}_n^{(i)}, \tilde{Z}_{m,n}, \tilde{\Delta}_{m,n}, \tilde{U}_{m,n}$ analogously as in (5.6), (5.7), just replacing ξ_ν everywhere by $\zeta_\nu^{(H)}$ where H is the partition (5.4). The field $\{\zeta_\nu^{(H)}, \nu \in N^2\}$ may be defined on a probability space different from the space supporting the ξ_ν's. It follows immediately from definiton (5.4) of the partition $H = (H_1, H_2, H_3)$ that the variables $\tilde{U}_{m,n}$ $(m \geqq 1, n \geqq 1)$ are independent. Our purpose is to apply Theorem 1 of [2] to the sequence $U_{m,n}$; to this end we arrange the terms of this sequence linearly as follows. Let

$$\mathcal{G}_k = \{(k,1), (k,2), \dots, (k, k-1), (1,k),\ (2,k), \dots, (k-1,k), (k,k)\}$$

and define an ordering $\prec$ of all vectors $\nu\in N^2$ as follows: $\mu\prec\nu$ iff $\mu\in\mathcal{G}_k$, $\nu\in\mathcal{G}_l$ for some $k<l$ or $\mu,\nu\in\mathcal{G}_k$ for some $k\geqq 1$ and μ precedes ν in the ordering of $\mathcal{G}_k$. Let $d_{m,n}$ denote the dimension of the vector $U_{m,n}$ and set

$$\lambda_{m,n}(u) = E|E\{\exp i\langle u, U_{m,n}\rangle|\mathcal{F}^0_{m,n}\}-E\{\exp i\langle u, \tilde{U}_{m,n}\rangle\}|$$

for $u\in R^{d_{m,n}}$ where $\langle\cdot,\cdot\rangle$ is the inner product and $\mathcal{F}^0_{m,n}$ denotes the σ-field generated by those $U_{i,j}$ whose index (i,j) precedes (m,n) in the above ordering. Then we have

LEMMA 10. *For* $|u|\ll(mn)^6$ *we have*

$$\lambda_{m,n}(u)\ll\begin{cases}(p'_m p'_n)^{-1} & \text{in the first and second case of (5.7)},\\ (p'_{[m^{\alpha/3}]}p'_{[n^{\alpha/3}]})^{-1} & \text{in the third and fourth case of (5.7)}.\end{cases}\tag{5.8}$$

PROOF. We show (5.8) separetely in the four cases of (5.7). The estimation of $\lambda_{m,n}(u)$ in the first and second case of (5.7) is almost the same, on the other hand, $\lambda_{m,n}(u)=0$ in the fourth case. It suffices, therefore, to treat the first and third case of (5.7).

Let $H_{m,n}$ denote the set of those $\nu\in N^2$ such that the variable ξ_ν appears in the definition of $U_{m,n}$ and write

$$\lambda_{m,n}(u)\leqq E|E\{\exp i\langle u, U_{m,n}\rangle|\mathcal{F}^0_{m,n}\}-E\{\exp i\langle u, U_{m,n}\rangle\}|+$$
$$+|E\{\exp i\langle u, U_{m,n}\rangle\}-E\{\exp i\langle u, \tilde{U}_{m,n}\rangle\}| = Q_1+Q_2, \quad \text{say.}$$

We estimate first Q_1. In the first case of (5.7) we get, using Lemma 1 and our assumption (1.2) with $\gamma\geqq 4098/\delta$,

$$Q_1\leqq 2\pi\varrho\Big(H_{m,1}\bigcup_{(i,j)\prec(m,1)}H_{i,j}\Big)\ll p'^{-4098/\delta}_{m-1}\ll p'^{-1}_m$$

for all u. Similarly, in the third case of (5.7) we get, using $m>[n^{\alpha/3}]$, $n>[m^{\alpha/3}]$,

$$Q_1\leqq 2\pi\varrho\Big(H_{m,n},\bigcup_{(i,j)\neq(m,n)}H_{i,j}\Big)\ll(p'_{m-1}\wedge p'_{n-1})^{-4098/\delta}\ll$$
$$\ll p'^{-2}_m\vee p'^{-2}_n\ll(p'_{[m^{\alpha/3}]}p'_{[n^{\alpha/3}]})^{-1}$$

for all u. Turning to the estimate of Q_2, we get in the first case of (5.7), using Lemma 5, $Q_2\ll p_m^{-\tau}\ll p'^{-1}_m$ provided that $|u|(q_m+m^{\alpha/3})\leqq p_m^\tau$ which is certainly satisfied (by (5.3)) if $|u|\ll m^6$. Similarly we get in the third case of (5.7), using the first statement of the Corollary to Lemma 5 and $m>[n^{\alpha/3}]$, $n>[m^{\alpha/3}]$,

$$Q_2\ll(p_m\wedge p_n)^{-\varrho}\ll p'^{-2}_m\vee p'^{-2}_n\ll(p'_{[m^{\alpha/3}]}p_{[n^{\alpha/3}]})^{-1}$$

provided that $|u|\leqq(p_m\wedge p_n)^\varrho$ which is certainly satisfied if $|u|\ll(mn)^6$. Hence Lemma 10 is proved.

We next observe that

$$P\{|\tilde{U}_{m,n}|\geqq\text{const}\cdot(mn)^6\}\ll(mn)^{-6}\tag{5.9}$$

as one can readily verify in each of the four cases of (5.7) by observing that the components of $\tilde{U}_{m,n}$ are all normal r.v.'s with mean zero and variance $\ll 1$.

We apply now Theorem 1 of [2] to the sequence $U_{m,n}$ in the above ordering and with $T_{m,n}=\text{const}\cdot(mn)^6$. Using (5.8), (5.9) it follows that the field $\{\xi_\nu, \nu\in N^2\}$ can be redefined on a suitable probability space together with a sequence $\{\hat{U}_{m,n}, m\geqq 1, n\geqq 1\}$ of independent random vectors such that the vectors $\hat{U}_{m,n}$ and $\tilde{U}_{m,n}$ have the same distribution and $P\{|U_{m,n}-\hat{U}_{m,n}|\geqq\alpha_{m,n}\}\leqq\alpha_{m,n}$ where

$$\alpha_{m,n}\ll d_{m,n}(mn)^{-6}\log mn+(p'_{[m^t]}p'_{[n^t]})^{-1/2}(mn)^{6d_{m,n}}+(mn)^{-6}$$

and $t=1$ in the first and second case of (5.7) and $t=\alpha/3$ in the third and fourth case of (5.7). Substituting the definition of p_k into the last estimate and noting that $d_{m,n}\sim(mn)^{\alpha/3}$ and $d_{m,n}=1$ in the first two and in the last two cases of (5.7), respectively, we get by a simple calculation that

$$\alpha_{m,n}\ll(mn)^{-5} \tag{5.10}$$

in each of the four cases of (5.7). As we have already noted, the random vectors $\tilde{U}_{m,n}$ are independent and thus the two sequences $\{\tilde{U}_{m,n}, m\geqq 1, n\geqq 1\}$ and $\{\hat{U}_{m,n}, m\geqq 1, n\geqq 1\}$ have the same distribution Hence by enlarging the probability space we can define on this space also the field $\{\zeta_\nu^{(H)}, \nu\in N^2\}$ (retaining its original distribution) such that the quantities $\tilde{U}_{m,n}$ belonging to this field are identical with the $\hat{U}_{m,n}$. Then by the above relations we have

$$P\{|U_{m,n}-\tilde{U}_{m,n}|\geqq\alpha_{m,n}\}\leqq\alpha_{m,n}$$

and thus

$$|U_{m,n}-\tilde{U}_{m,n}|\ll(mn)^{-5}\quad\text{a.s.} \tag{5.11}$$

by (5.10) and the Borel—Cantelli lemma. We claim that the fields $\{\xi_\nu, \nu\in N^2\}$ and $\{\zeta_\nu^{(H)}, \nu\in N^2\}$ satisfy (5.5).

As a first step in proving (5.5) we show the somewhat weaker inequality

$$\sum_{\nu\leqq n}\xi_\nu-\sum_{\nu\leqq n}\zeta_\nu^{(H)}\ll([n]\log\log[n])^{1/2}\big((\log n_1)^{-(1-\alpha)/2\alpha}+(\log n_2)^{-(1-\alpha)/2\alpha}\big)\quad\text{a.s.} \tag{5.12}$$

where $n=(n_1, n_2)$. To obtain (5.12) we observe first that

$$|Z_{m,n}-\tilde{Z}_{m,n}|\ll(mn)^{-3}\quad\text{a.s.}\quad(m\geqq 1,\ n\geqq 1). \tag{5.13}$$

Relation (5.13) is obvious from (5.11) if $m>[n^{\alpha/3}]$, $n>[m^{\alpha/3}]$; to get it e.g. for $1\leqq n\leqq[m^{\alpha/3}]$ we observe that (5.11) yields for $n=1$

$$\sum_{i=1}^{q_m}|X_m^{(i)}-\tilde{X}_m^{(i)}|\ll m^{-5}q_m\ll m^{-4}\quad\text{a.s.} \tag{5.14}$$

and

$$\sum_{i=r_m+1}^{[m^{\alpha/3}]}|Z_{m,i}-\tilde{Z}_{m,i}|\ll m^{-5}m^{\alpha/3}\ll m^{-4}\quad\text{a.s.}$$

Evidently, the last relations imply (5.13) for $1\leqq n\leqq r_m$ and $r_m+1\leqq n\leqq[m^{\alpha/3}]$, respectively. Next we state

LEMMA 11. *We have*

$$\Big|\sum_{\substack{1\le i\le m\\ 1\le j\le n}} \varepsilon_{i,j}\Delta_{i,j}\Big| \ll (t_m t_n \log\log t_m t_n)^{1/2}(t_m^{-1/4}+t_n^{-1/4}) \quad a.s. \tag{5.15}$$

for any fixed sequence $\{\varepsilon_{i,j},\ i\ge 1, j\ge 1\}$ *where* $\varepsilon_{i,j}=0$ *or* 1. *(The exceptional zero set in* (5.15) *may depend on the* $\varepsilon_{i,j}$.) *A similar statement holds for the* $\tilde{\Delta}_{i,j}$.

PROOF. Obviously

$$\sum_{\substack{1\le i\le m\\ 1\le j\le n}} \varepsilon_{i,j}\Delta_{i,j} = S_1+S_2-S_3$$

where

$$S_1 = \sum_{\nu\in[1,t_m]\times G_n} \varepsilon_\nu \xi_\nu, \quad S_2 = \sum_{\nu\in G_m\times[1,t_n]} \varepsilon_\nu\xi_\nu, \quad S_3 = \sum_{\nu\in G_m\times G_n} \varepsilon_\nu\xi_\nu.$$

Here all the ε_ν are equal to 0 or 1 ($\varepsilon_\nu=\varepsilon_{i,j}$ if $\nu\in N^2$ belongs to the index set of the L-shaped sum $\Delta_{i,j}$) and $G_k=\bigcup\limits_{1\le i\le k}(t_i', t_i]$. The sum S_1 can evidently be considered as a rectangular sum of a (non-stationary) random field $\{\xi_\nu^*, \nu\in N^2\}$ satisfying (1.1) and (1.2) with $\gamma\ge 4098/\delta$ and thus the second statement of the Corollary to Lemma 5 yields

$$P\{|S_1| \ge A(t_m T_n \log\log t_m T_n)^{1/2}\} \ll (\log t_m T_n)^{-BA^2}+$$
$$+(t_m T_n)^{-\varrho} \ll (m^\alpha + n^{\alpha/2})^{-BA^2} \ll (mn)^{-BA^2\alpha/4}$$

for any fixed $A>0$ where $T_n=\sum\limits_{i=1}^{n} p_i'$ and B, ϱ are the constants appearing in (3.5). Choosing A large and observing that $T_n\ll t_n^{1/2}$ the above estimate and the Borel—Cantelli lemma imply that $|S_1|$ is majorized a.s. by the right side of (5.15). Estimating $|S_2|$ and $|S_3|$ similarly, we get (5.15).

The proof for $\tilde{\Delta}_{i,j}$ is the same; we only have to observe that the assumptions made on $\{\xi_\nu, \nu\in N^2\}$ imply $c_k\ll k^{-4}$, $d_k\ll k^{-4}$ (see (2.1)) and thus the Gaussian field $\{\zeta_\nu^{(H)}, \nu\in N^2\}$ satisfies conditions (3.7) and (3.8) of Lemma 8. Consequently, the field satisfies also (3.10) uniformly in G; the latter relation now replaces (3.5) in the exponential estimates above.

Next we show

LEMMA 12. *With probability one,*

$$\sup_{\substack{t_{m-1}\le i\le t_m\\ t_{n-1}\le j\le t_n}} \Big(\Big|\sum_{\nu=1,1}^{i,j}\xi_\nu - \sum_{\nu=1,1}^{t_{m-1},t_{n-1}}\xi_\nu\Big| \ll \tag{5.16}$$
$$\ll (t_m t_n \log\log t_m t_n)^{1/2}\big((\log t_m)^{-(1-\alpha)/2\alpha}+(\log t_n)^{-(1-\alpha)/2\alpha}\big).$$

The same relation holds for $\zeta_\nu^{(H)}$ *instead of* ξ_ν.

PROOF. Set, for any rectangle I,

$$M(I) = \max_{\substack{I'\subset I\\ I' \text{ is rectangle}}} \Big|\sum_{\nu\in I'}\xi_\nu\Big|.$$

Obviously, the left side of (5.16) cannot exceed Y_1+Y_2 where

$$Y_1 = M([1, t_m]\times[t_{n-1}, t_n]), \quad Y_2 = M([t_{m-1}, t_m]\times[1, t_{n-1}]).$$

By Lemma 7 we have

$$P\{|Y_1| \geqq 4A(t_m T_n^* \log\log t_m T_n^*)^{1/2}\} \ll (\log t_m T_n^*)^{-DA^2} \ll (m^\alpha+n^\alpha)^{-DA^2} \ll (mn)^{-DA^2\alpha/2}$$

for any fixed $A>0$ where $T_n^*=t_n-t_{n-1}$ and D is the constant appearing in (3.6). Choosing A large, using the Borel—Cantelli lemma and observing that $T_n^*\sim p_n\sim$ $\sim\text{const}\cdot t_n(\log t_n)^{-(1-\alpha)/\alpha}$ by (5.2), it follows that $|Y_1|$ is majorized a.s. by the right side of (5.16). Repeating the argument for $|Y_2|$ we get (5.16). For the field $\{\zeta_\nu^{(H)}, \nu\in N^2\}$ the proof is the same, just instead of Lemma 7 we use Lemma 8 (see the last paragraph of the proof of Lemma 11).

Relations (5.13), (5.2) and Lemma 11 with $\varepsilon_{i,j}\equiv 1$ obviously imply (5.12) for vectors $n\in N^2$ of the form $n=(t_i, t_j)$. Hence, using Lemma 12 and $t_m/t_{m-1}\to 1$ we get (5.12) for all $n\in N^2$.

After these preparations we can now prove the validity of (5.5) as follows. Let

$$D = \bigcup_{m=1}^{\infty} \{(i,j)\colon\ t_{m-1} < i \leqq t_m,\ 1 \leqq j \leqq q_m\}, \quad d_j = \min\{i\colon (i,j)\in D\} \quad (j = 1, 2, \ldots),$$

$$k_n = d_{t_{n-1}}, \quad e_n = \min\{i\colon (t_i, t_n)\in D\},$$

$$V_n = \max_{t_{n-1}\leqq j\leqq t_n} \left| \sum_{\nu=1,t_{n-1}}^{k_n, j} \xi_\nu \right|, \quad T_{m,l} = \max_{t_{m-1}\leqq j\leqq t_m} \left| \sum_{\nu=t_{m-1},1}^{j,l} \xi_\nu \right|,$$

$$R_{m,l} = \max_{1\leqq i\leqq l} \left| \sum_{\nu=t_m',1}^{t_m, i} \xi_\nu \right|,$$

and denote by $\tilde V_n, \tilde T_{m,l}, \tilde R_{m,l}$ the analogues of $V_n, T_{m,l}, R_{m,l}$ for the field $\{\zeta_\nu^{(H)}, \nu\in N^2\}$. From Lemma 7 and the Borel—Cantelli lemma it follows that

$$V_n \ll (k_n p_n \log\log k_n p_n)^{1/2} \quad \text{a.s.} \tag{5.17}$$

and the same estimate holds for $\tilde V_n$. (Recall that $\{\zeta_\nu^{(H)}, \nu\in N^2\}$ satisfies (3.7) and (3.8) and thus Lemma 8 applies.) Further, by Lemma 7 and (5.3) we have

$$P\Big\{\max_{1\leqq l\leqq q_m} (p_m l \log\log p_m l)^{-1/2} T_{m,l} \geqq C\Big\} \ll q_m (\log p_m l)^{-DC^2} \ll m^{-2}$$

for large enough C and thus

$$T_{m,l} \ll (p_m l \log\log p_m l)^{1/2} \quad \text{a.s. for} \quad m \geqq 1, \quad 1 \leqq l \leqq q_m. \tag{5.18}$$

The same argument yields

$$R_{m,l} \ll (p_m' l \log\log p_m' l)^{1/2} \quad \text{a.s. for} \quad m \geqq 1, \quad 1 \leqq l \leqq q_m. \tag{5.19}$$

The analogues of the last two estimates are also valid for the quantities $\tilde T_{m,l}, \tilde R_{m,l}$. Consider now a point $(k, l)\in D$ and assume $t_{m-1}<k\leqq t_m$, $t_{n-1}<l\leqq t_n$. Then by (5.2), (5.13), (5.14), the relation $t_m/t_{m-1}\to 1$, Lemmas 11, 12, (5.17)—(5.19) and their

analogues for the $\sim$ quantities we get, setting $\beta=(1-\alpha)/2\alpha$,

$$(5.20)\quad \left|\sum_{\nu=1,1}^{k,l}\xi_\nu-\sum_{\nu=1,1}^{k,l}\zeta_\nu^{(H)}\right|\leqq\sum_{s=1}^{m-1}p_s^{1/2}\sum_{i=1}^{q_s}|X_s^{(i)}-\tilde X_s^{(i)}|+\sum_{i=1}^{e_n}(R_{i,q_i}+\tilde R_{i,q_i})+$$

$$+\sum_{i=e_n+1}^{m-1}(R_{i,l}+\tilde R_{i,l})+T_{m,l}+\tilde T_{m,l}+\sum_{j=1}^{n-1}\sum_{i=1}^{e_j}(p_ip_j)^{1/2}|Z_{i,j}-\tilde Z_{i,j}|+$$

$$+\sum_{j=1}^{n-1}\sum_{i=1}^{e_j}(\Delta_{i,j}+\tilde\Delta_{i,j})+V_n+\tilde V_n\ll\sum_{s=1}^{m-1}p_s^{1/2}s^{-4}+\sum_{i=1}^{e_n}(p_i'l\log\log p_i'l)^{1/2}+$$

$$+\sum_{i=e_n+1}^{m-1}(p_i'l\log\log p_i'l)^{1/2}+(p_ml\log\log p_ml)^{1/2}+$$

$$+\sum_{j=1}^{n-1}\sum_{i=1}^{m}(p_ip_j)^{1/2}(ij)^{-3}+(k_nt_n\log\log k_nt_n)^{1/2}+(k_np_n\log\log k_np_n)^{1/2}\ll$$

$$\ll p_m^{1/2}+(p_ml\log\log p_ml)^{1/2}+(p_mp_n)^{1/2}+(k_nt_n\log\log k_nt_n)^{1/2}+$$

$$+(k_np_n\log\log k_np_n)^{1/2}\ll(\log t_m)^{-\beta}t_m^{1/2}+(\log t_m)^{-\beta}(t_ml\log\log t_ml)^{1/2}+$$

$$+(\log t_m)^{-\beta}(\log t_n)^{-\beta}(t_mt_n)^{1/2}+(k_nt_n\log\log k_nt_n)^{1/2}\ll$$

$$\ll(\log k)^{-\beta}(kl\log\log kl)^{1/2}+(kl\log\log kl)^{1/2}(k_n/k)^{1/2}\quad\text{a.s.}$$

Let $D_1\subset D$ denote the set of those $(k,l)\in D$ such that $k\geqq d_l^2$. Observing that (5.1) and (5.3) imply $d_l\ll\exp(l^{\gamma_1})$ for some $\gamma_1>0$, it follows that $D_1\supset\{(k,l)\colon k\geqq1;\ 1\leqq l\leqq\leqq\text{const}\cdot(\log k)^{\gamma_2}\}$ for a suitable $\gamma_2>0$. Now, if $(k,l)\in D_1$ and $t_{n-1}<l\leqq t_n$ then $k\geqq l$, $k\geqq d_l^2\geqq d_{t_{n-1}}^2=k_n^2$ and thus (5.20) gives

$$\left|\sum_{\nu=1,1}^{k,l}\xi_\nu-\sum_{\nu=1,1}^{k,l}\zeta_\nu^{(H)}\right|\ll(kl\log\log k^2)^{1/2}((\log k)^{-\beta}+k^{-1/4})\ll(kl)^{1/2}(\log kl)^{-\beta/2}\quad\text{a.s.}$$

The same estimate can be obtained in a domain D_2 where $D_2\supset\{(k,l)\colon l\geqq1,$ $1\leqq k\leqq\text{const}\cdot(\log l)^{\gamma_2}\}$. Now if $(k,l)\notin D_1\cup D_2$ then $k>\text{const}\cdot(\log l)^{\gamma_2}$, $l>\text{const}\cdot\cdot(\log k)^{\gamma_2}$ i.e. $k\wedge l\gg(\log kl)^{\gamma_2}$ and thus we get, using (5.12),

$$\left|\sum_{\nu=1,1}^{k,l}\xi_\nu-\sum_{\nu=1,1}^{k,l}\zeta_\nu^{(H)}\right|\ll(kl\log\log kl)^{1/2}((\log k)^{-\beta}+(\log l)^{-\beta})\ll$$

$$\ll(kl)^{1/2}(\log\log kl)^{-(\beta-1)/2}\big((\log k)^{-\beta/2}+(\log l)^{-\beta/2}\big).$$

Hence (5.5) is proved in each of the domains $D_1, D_2, N^2\setminus(D_1\cup D_2)$ and thus Theorem 4 is proved in the mixing case.

Let us see now how the above proof should be modified in the case when $\{\xi_\nu,\nu\in N^2\}$ is a Gaussian field. As an inspection of the proof shows, we made direct use of mixing condition (1.2) only in the proof of Lemma 10; at all other places we used only the central limit theorem (Lemma 5) and oscillation inequalities for the field $\{\xi_\nu,\nu\in N^2\}$ which, by Lemma 8, remain valid also in the case when $\{\xi_\nu,\nu\in N^2\}$ is a centered stationary Gaussian field with covariance function r satisfying (2.2). Hence all what we have to prove is the analogue of Lemma 10 in the Gaussian case.

We shall do this by using the Kolmogorov—Rozanov theorem (see [7]) and the following two simple lemmas.

LEMMA 13. *Let* $\{\xi_\nu, \nu\in N^2\}$ *be a stationary Gaussian field with mean zero and covariance function* $r(\nu)$ *satisfying* (2.2). *Let* $H_1, H_2\subset N^2$ *be finite sets whose distance is* $d>0$. *Let* $\eta_i=|H_i|^{-1/2}\sum_{\nu\in H_i}\xi_\nu$ $(i=1, 2)$ *then* $|E\eta_1\eta_2|\ll d^{-\varepsilon/2}$ *where the constant implied by* $\ll$ *depends only on the covariance function* r.

PROOF. A simple calculation shows that

$$\sum_{\{\mu\in Z^2:\,|\mu|\geqq d\}}|\mu|^{-(2+\varepsilon)}\ll d^{-\varepsilon/2} \tag{5.21}$$

where the constant implied by $\ll$ depends only on ε. Evidently

$$E\eta_1\eta_2=|H_1|^{-1/2}|H_2|^{-1/2}\sum_{\{\mu\in Z^2:\,|\mu|\geqq d\}}r(\mu)\sum_{\substack{\nu_1\in H_1,\,\nu_2\in H^2\\ \nu_2-\nu_1=\mu}}1$$

and here the inner sum is clearly $\leqq|H_1|\wedge|H_2|\leqq|H_1|^{1/2}|H_2|^{1/2}$ for any fixed $\mu\in Z^2$. Using (2.2) and (5.21) we get the statement of the lemma.

LEMMA 14. *Let* $\{\xi_\nu, \nu\in N^2\}$ *be a stationary Gaussian field with mean zero and covariance function* $r(\nu)$ *satisfying* (2.2), (2.7). *Let* $H_1, H_2, \dots, H_k$ *be disjoint finite subsets of* N^2 *and set* $\eta_i=|H_i|^{-1/2}\sum_{\nu\in H_i}\xi_\nu$ $(i=1, \dots, k)$. *Then for any real numbers* $c_1, \dots, c_k$ *we have*

$$E\left(\sum_{i=1}^k c_i\eta_i\right)^2\geqq\sigma_0\left(\sum_{i=1}^k c_i^2\right).$$

PROOF. It suffices to show that for any finite $H\subset N^2$ and any real numbers $\{c_\nu, \nu\in N\}$ we have

$$E\Big(\sum_{\nu\in H}c_\nu\xi_\nu\Big)^2\geqq\sigma_0\Big(\sum_{\nu\in H}c_\nu^2\Big). \tag{5.22}$$

To verify (5.22) we extend the sequence $\{c_\nu\}$ for all $\nu\in N^2$ by setting $c_\nu=0$ for $\nu\notin H$ and note that

$$E\Big(\sum_{\nu\in H}c_\nu\xi_\nu\Big)^2\geqq r(0)\Big(\sum_{\nu\in H}c_\nu^2\Big)-\sum_{\substack{\mu\in Z^2\\ \mu\neq 0}}|r(\mu)|\Big(\sum_{\nu\in H}|c_\nu c_{\nu+\mu}|\Big)=I_1-I_2.$$

By the Cauchy—Schwarz inequality the absolute value of the inner sum in I_2 is $\leqq\sum_{\nu\in H}c_v^2$ and thus (5.22) is valid.

We can now prove the analogue of Lemma 10 for Gaussian fields $\{\xi_\nu, \nu\in N^2\}$. Using the same notations as in the mixing case, we shall show

LEMMA 10 (*). *Let* $\{\xi_\nu, \nu\in N^2\}$ *satisfy the conditions of Lemma* 14. *Then for* $|u|\ll(mn)^6$ *we have*

$$\lambda_{m,n}(u)\ll\begin{cases}(p'_m p'_n)^{-\varepsilon/5} & \text{in the first and second case of (5.7),}\\ (p'_{[m^{\alpha/3}]}p'_{[m^{\alpha/3}]})^{-\varepsilon/5} & \text{in rhe third and fourth case of (5.7).}\end{cases} \tag{5.23}$$

Notice that (5.23) differs slightly from (5.8); the exponent $\varepsilon/5$, however, does not cause any trouble in the sequel.

PROOF. As in the proof of Lemma 10, we have $\lambda_{m,n}(u)\leqq Q_1+Q_2$; we have to estimate only Q_1 since the estimate for Q_2 is the same as in the mixing case (cf. Lemma 8). Let $\mathscr{F}^0_{m,n}$ denote, as in the mixing case, the σ-field generated by those random vectors $U_{i,j}$ such that $(i,j)\prec(m,n)$; let further $\mathscr{A}_{m,n}$ denote the σ-field generated by the vector $U_{m,n}$. Clearly, $\mathscr{F}^0_{m,n}$ is generated by the one-dimensional random variables $x_1, x_2, \dots, x_s$ where $x_1, x_2, \dots, x_s$ are obtained by arranging the components of the vectors $U_{i,j}\colon (i,j)\prec(m,n)$ in an arbitrary manner. A trivial calculation shows (separating again the four cases in (5.7) and using (5.3)) that $s\ll(mn)^2$. On the other hand, $\mathscr{A}_{m,n}$ is generated by $y_1, \dots, y_t$, the components of the vector $U_{m,n}$; evidently $t\ll(mn)^{\alpha/3}$ in all four cases of (5.7). Now, the r.v.'s $x_1, \dots, x_s, y_1, \dots, y_t$ are all of the form $|H|^{-1/2}\sum_{v\in H}\xi_v$ for some finite sets $H\subset N^2$, say $H_1, \dots, H_s, H_1', \dots, H_t'$. Denoting by d_0 the distance of $\bigcup_{i=1}^{s} H_i$ and $\bigcup_{j=1}^{t} H_j'$, we obviously have

$$d_0 \approx \begin{cases} p_m' p_n' & \text{in the first two cases of (5.7),} \\ p_{m-1}'\wedge p_{n-1}' \gg (p'_{[m^{\alpha/3}]}p'_{[n^{\alpha/3}]})^{1/2} & \text{in the third case of (5.7),}\end{cases}$$

where $a\approx b$ means $a\ll b\ll a$. Since the vector $(x_1, \dots, x_s, y_1, \dots, y_t)$ is Gaussian and $\mathscr{F}^0_{m,n}=\sigma\{x_1, \dots, x_s\}$, $\mathscr{A}_{m,n}=\sigma\{y_1, \dots, y_t\}$, the Kolmogorov—Rozanov theorem (see [7]) implies

(5.24)
$$\sup_{A\in\mathscr{F}^0_{m,n},\, B\in\mathscr{A}_{m,n}} |P(AB)-P(A)P(B)| \leqq \sup_{\substack{c_1, \dots, c_s \\ d_1, \dots, d_t}} \frac{E\left(\left(\sum_{i=1}^{s} c_i x_i\right)\left(\sum_{j=1}^{t} d_j y_j\right)\right)}{E^{1/2}\left(\sum_{i=1}^{s} c_i x_i\right)^2 E^{1/2}\left(\sum_{j=1}^{t} d_j y_j\right)^2}.$$

From Lemma 13 and the Cauchy—Schwarz inequality it follows that the numerator on the right side of (5.24) is

$$\ll \sum_{i=1}^{s}\sum_{j=1}^{t}|c_i|\,|d_j| d_0^{-\varepsilon/2} \leqq \left(\sum_{i=1}^{s} c_i^2\right)^{1/2}\left(\sum_{j=1}^{t} d_j^2\right)^{1/2}(st)^{1/2}d_0^{-\varepsilon/2}.$$

Since the denominator is $\geqq\sigma_0\left(\sum_{i=1}^{s} c_1^2\right)^{1/2}\left(\sum_{j=1}^{t} d_j^2\right)^{1/2}$ by Lemma 14, (5.24) implies

(5.25)
$$\sup_{A\in\mathscr{F}^0_{m,n},\, B\in\mathscr{A}_{m,n}} |P(AB)-P(A)P(B)| \leqq \text{const}\cdot(st)^{1/2}d_0^{-1/2}.$$

In view of Lemma 1, the right side of (5.25) is an upper estimate for the quantity Q_1 in the proof of Lemma 10 and hence using (5.25) and the estimate obtained for s, t and d_0 we get (5.23). This completes the proof of Theorem 4 in the Gaussian case.

Proof of Theorem 1. Since statement (B) obviously implies statement (A), it suffices to prove the implications (A)⇒(C), (C)⇒(B) and the equivalence of (C) and (D). The implication (C)⇒(B) is an immediate consequence of Theorem 4. Indeed, observe that the field $\{\zeta_\nu^{(H)}, \nu\in N^2\}$ in Theorem 4 is completely determined by the quantities c_k, d_l in (1.8) where r is the covariance function of the field $\{\xi_\nu, \nu\in N^2\}$ (note that $\sigma^2=\sum_{k=-\infty}^{+\infty} c_k$). Hence if $\{\xi_\nu, \nu\in N^2\}$ and $\{\zeta_\nu, \nu\in N^2\}$ satisfy the conditions of Theorem 1 and condition (C) is valid then we have

$$\sum_{\nu\leqq n}\xi_\nu-\sum_{\nu\leqq n}\zeta_\nu^{(H)} \ll [n]^{1/2}(\log\log[n])^{-\lambda}\big((\log n_1)^{-\lambda}+(\log n_2)^{-\lambda}\big) \quad \text{a.s.,} \tag{5.26}$$

$$\sum_{\nu\leqq n}\zeta_\nu-\sum_{\nu\leqq n}\zeta_\nu^{(H)} \ll [n]^{1/2}(\log\log[n])^{-\lambda}\big((\log n_1)^{-\lambda}+(\log n_2)^{-\lambda}\big) \quad \text{a.s.} \tag{5.27}$$

with the same field $\{\zeta_\nu^{(H)}, \nu\in N^2\}$ in (5.26) and (5.27). (Here $n=(n_1, n_2)$ and λ is a positive constant.) (5.26) and (5.27) obviously imply (1.20) and consequently (1.12). (Strictly speaking, the probability space supporting the fields ξ_ν and $\zeta_\nu^{(H)}$ may be different from the space supporting the ζ_ν and $\zeta_\nu^{(H)}$. However, using the Kolmogorov existence theorem and Lemma A1 at the end of [2] it follows that the three fields $\xi_\nu, \zeta_\nu, \zeta_\nu^{(H)}$ can be redefined on the same probability space such that the joint distribution of the fields $\{\xi_\nu, \nu\in N^2\}$, $\{\zeta_\nu^{(H)}, \nu\in N^2\}$ and also the joint distribution of the fields $\{\zeta_\nu, \nu\in N^2\}$, $\{\zeta_\nu^{(H)}, \nu\in N^2\}$ remains unchanged. Obviously, in this case (5.26), (5.27) remain also valid.

Next we show the equivalence of (C) and (D). Observe that both $\{\xi_\nu, \nu\in N^2\}$ and $\{\zeta_\nu, \nu\in N^2\}$ satisfy the conditions of Lemma 4 and thus, using the Corollary of the same lemma we get, setting $R_n=E(\sum_{\nu\leqq n}\xi_\nu)^2/E(\sum_{\nu\leqq n}\zeta_\nu)^2$,

$$\lim_{n_1=k,\, n_2\to\infty} R_n=\frac{a_k}{a_k^*},\qquad \lim_{n_1\to\infty,\, n_2=k} R_n=\frac{b_k}{b_k^*},\qquad \lim_{n_1\wedge n_2=\infty} R_n=\frac{\sigma^2}{(\sigma^*)^2}$$

where a_k, b_k, σ^2 are defined by (2.5) and $a_k^*, b_k^*, (\sigma^*)^2$ are the analogous quantities for the field $\{\zeta_\nu, \nu\in N^2\}$. (Note that (1.9), (1.10) and Lemma 14 imply $E(\sum_{\nu\leqq n}\zeta_\nu)^2\geqq\alpha_0[n]$ for all $n\in N^2$ with a positive constant α_0 and thus $a_k^*>0, b_k^*>0, (\sigma^*)^2>0$.) The above limit relations show that condition (D) is equivalent to $a_k=a_k^*$, $b_k=b_k^*$ $(k=1, 2, \dots)$, $\sigma^2=(\sigma^*)^2$ which, in turn, is equivalent to (C) since the numbers a_k, b_k $(k=1, 2, \dots)$ in (2.5) uniquely determine the numbers c_k, d_k $(k=0, \pm 1, \dots)$ (note $c_k=c_{-k}, d_k=d_{-k}$).

Finally, to show the implication (A)⇒(C) assume that (1.11) holds. By the law of the iterated logarithm for one-parameter mixing and Gaussian processes (see e.g. [12], [13]) we get, using also the Corollary of Lemma 4,

$$\limsup_{n_1=k,\, n_2\to\infty}\Big(\sum_{\nu\leqq n}\xi_\nu\Big)\Big/(2[n]\log\log[n])^{1/2}=a_k^{1/2} \tag{5.28}$$

$$\limsup_{n_1=k,\, n_2\to\infty}\Big(\sum_{\nu\leqq n}\zeta_\nu\Big)\Big/(2[n]\log\log[n])^{1/2}=(a_k^*)^{1/2} \tag{5.29}$$

for every fixed k where a_k, a_k^* denote the same quantities as in the previous paragraph. From (1.11), (5.28), (5.29) it follows that $a_k=a_k^*$ and a similar argument yields $b_k=b_k^*$. As in the previous paragraph, this implies that (C) is valid.

PROOF OF THEOREM 2. Let $\{\xi_\nu, \nu\in N^2\}$ be a weakly stationary random field satisfying (1.1) and (1.2) with $\gamma \geqq 4098/\delta$. Let $H=(H_1, H_2, H_3)$ be a partition defined by (5.4) and $\{\zeta_\nu^{(H)}, \nu\in N^2\}$ the corresponding Gaussian field defined by (3.1). Let $\sigma^2=\sum_{\nu\in Z^2} r(\nu)$ where $r(\nu)$ is the covariance function of the field $\{\xi_\nu, \nu\in N^2\}$, set $v_l=\max_{(k,l)\in H_1} k$ and consider the finite Gaussian processes $\{x_k^{(l)}, 1\leqq k\leqq v_l\}$ $(l=1, 2, \ldots)$ where $x_k^{(l)}=\zeta_{(k,l)}^{(H)}$ Each of these processes is an initial segment of a stationary Gaussian process with mean zero and covariance sequence $\{c_n, n\geqq 0\}$ where c_n is defined by (1.8). Since $c_n \ll n^{-2}$ by (2.1) and $\sigma^2=\sum_{k=-\infty}^{+\infty} c_k=c_0+2\sum_{k=1}^{\infty} c_k$, it follows from Lemma 9 that there exist processes $\{y_k^{(l)}, 1\leqq k\leqq v_l\}$, $l=1, 2, \ldots$ such that, for every fixed l, the r.v.'s $y_k^{(l)}, 1\leqq k\leqq v_l$ are independent $N(0,\sigma^2)$ r.v.'s and

$$\left\|\sum_{k=1}^{N}(x_k^{(l)}-y_k^{(l)})\right\|_3 \ll N^{1/2}(\log N)^{-(1-\alpha)/2\alpha} \tag{5.30}$$

for each $1\leqq N\leqq v_l$ which is of the form $N=t_i$. (Of course, an enlargement of the probability space may be necessary for constructing all these sequences on it.) Moreover, since the processes $\{x_k^{(l)}, 1\leqq k\leqq v_l\}$ are independent, the approximating processes $\{y_k^{(l)}, 1\leqq k\leqq v_l\}$ can be chosen in such a way that the σ-fields $\mathcal{A}_l= =\sigma\{x_k^{(l)}, y_k^{(l)}, 1\leqq k\leqq v_l\}$, $l=1, 2, \ldots$ be independent. Analogously, considering the independent Gaussian processes $\{\bar{x}_l^{(k)}, 1\leqq l\leqq \bar{v}_k\}$, $k=1,2, \ldots$ where $\bar{v}_k=\max_{(k,l)\in H_2} l$ and $\bar{x}_l^{(k)}=\zeta_{(k,l)}^{(H)}$ it follows that there exist processes $\{\bar{y}_l^{(k)}, 1\leqq l\leqq \bar{v}_l\}$, $k=1, 2, \ldots$, each composed of independent $N(0,\sigma^2)$ r.v.'s such that the σ-fields $\mathcal{B}_k=\sigma\{\bar{x}_l^{(k)}, \bar{y}_l^{(k)}, 1\leqq l\leqq \bar{v}_k\}$, $k=1, 2, \ldots$ are independent and

$$\left\|\sum_{l=1}^{N}(\bar{x}_l^{(k)}-\bar{y}_l^{(k)})\right\|_3 \ll N^{1/2}(\log N)^{-(1-\alpha)/2\alpha} \tag{5.31}$$

for each $1\leqq N\leqq \bar{v}_k$ which is of the form $N=t_j$. Since the processes $\{x_k^{(l)}, 1\leqq k\leqq v_l\}$, $l=1, 2, \ldots$, $\{\bar{x}_l^{(k)}, 1\leqq l\leqq \bar{v}_k\}$, $k=1, 2, \ldots$ are independent both of each other and of the field $\{\zeta_\nu^{(H)}, \nu\in H_3\}$, we can also guarantee that the σ-fields $\mathcal{A}_l$, $l\leqq 1$ and $\mathcal{B}_k$, $k\geqq 1$ are independent of each other and of $\sigma\{\zeta_\nu^{(H)}, \nu\in H_3\}$. Set

$$\zeta_\nu=\begin{cases} y_k^{(l)} & \text{if} \quad \nu=(k,l)\in H_1,\\ \bar{y}_k^{(l)} & \text{if} \quad \nu=(k,l)\in H_2,\\ \zeta_\nu^{(H)} & \text{if} \quad \nu\in H_3,\end{cases}$$

then evidently $\zeta_\nu, \nu\in N^2$ are independent $N(0,\sigma^2)$ r.v.'s. We claim that $\zeta_\nu^{(H)}$ and ζ_ν satisfy

(5.32)
$$\sum_{\nu\leqq n}\zeta_\nu^{(H)}-\sum_{\nu\leqq n}\zeta_\nu \ll ([n]\log\log[n])^{1/2}\left((\log n_1)^{-(1-\alpha)/2\alpha}+(\log n_2)^{-(1-\alpha)/2\alpha}\right) \quad \text{a.s.}$$

where $n=(n_1, n_2)$. Since $W(n)=\sum_{\nu\leqq n}\zeta_\nu$ is a Wiener process, (5.32) and (5.5) together imply (1.14) i.e. the statement of Theorem 2. In view of Lemma 12 (which is also valid for the $\zeta_\nu^{(H)}$and ζ_ν) it suffices to show (5.32) for the values $n=(t_i, t_j)$. Observe now that

$$(5.33)\quad \sum_{\nu\le(P,Q)}(\zeta_\nu^{(H)}-\zeta_\nu)=\sum_{k=2}^{P}\sum_{\substack{\nu_1=k\\ 1\le\nu_2\le\bar v_k\wedge Q}}(\zeta_\nu^{(H)}-\zeta_\nu)+\sum_{l=2}^{Q}\sum_{\substack{\nu_2=l\\ 1\le\nu_1\le v_l\wedge P}}(\zeta_\nu^{(H)}-\zeta_\nu)=$$

$$=\sum_{k=2}^{P}\tau_k^{(Q)}+\sum_{l=2}^{Q}\varrho_l^{(P)},\quad \text{say}$$

and the terms of the last two sums are independent random variables. Now, both $\bar v_k$ and v_l are of the form t_i (cf. (5.4)) hence if (P, Q) is of the form (t_i, t_j) then by (5.31) we have

$$\|\tau_k^{(Q)}\|_3 \ll Q^{1/2}(\log Q)^{-(1-\alpha)/2\alpha},\quad 2\le k\le P.$$

Setting $z_k=Q^{-1/2}(\log Q)^{(1-\alpha)/2\alpha}\,\tau_k^{(Q)}$ we then have $Ez_k=0$, $\|z_k\|_3\le 1$ and thus

$$(5.34)\qquad P\left\{\left|\sum_{k=2}^{P} z_k\right|\ge A(P\log\log PQ)^{1/2}\right\}\ll(\log PQ)^{-A^2/2}+P^{-1/8}$$

as one readily sees by using the central limit theorem with Ljapunov's remainder term (see [9], p. 288) or the Chebisev inequality according as $\sum_{k=1}^{P} Ez_k^2$ exceeds $P^{3/4}$ or not. (Obviously the last sum is $\le P$.) Now, if $(P, Q)=(t_i, t_j)$ then $P\gg\exp(i^\alpha)$, $\log PQ\gg i^\alpha+j^\alpha\ge(ij)^{\alpha/2}$, further for any integer $P\ge 2$ there exists at most const $\cdot\, P^{1/16}$ different integers Q satisfying $(P, Q)\in H_2$. Hence adding up the right side of (5.34) for all $(P, Q)\in H_2$ which is of the form $(P, Q)=(t_i, t_j)$ we get a convergent sum provided A is large enough. Thus with probability one we have

$$(5.35)\qquad \sum_{k=2}^{P}\tau_k^{(Q)}\ll(PQ\log\log PQ)^{1/2}(\log Q)^{-(1-\alpha)/2\alpha}\quad \text{a.s.}$$

for $(P, Q)=(t_i, t_j)\in H_2$. Observe now that though by its definition the sum $\sum_{k=2}^{P}\tau_k^{(Q)}$ in (5.33) depends both on P and Q, for $(P, Q)\notin H_2$ this sum actually does not depend on Q i.e. its value does not change if we replace (P, Q) by (P, Q') where $Q'=\bar v_P$. Since $\bar v_P$ is of the form t_i and the right side of (5.35) is an increasing function of Q for large enough Q, it follows that (5.35) holds also for $(P, Q)=(t_i, t_j)\notin H_2$. Using the same argument for the $\varrho_l^{(P)}$ we get

$$\sum_{l=2}^{Q}\varrho_l^{(P)}\ll(PQ\log\log PQ)^{1/2}(\log P)^{-(1-\alpha)/2\alpha}\quad \text{a.s.}$$

for (P, Q) of the form (t_i, t_j) and thus (5.33) implies (5.32) for values n of the form $n=(t_i, t_j)$, This completes the proof of Theorem 2.

Proof of Theorem 3. Let $H=(H_1, H_2, H_3)$ be the partition of N^2 given by (5.4) and define the function $f(x)$ $(x\ge 0)$ by

$$f(x)=\begin{cases}1/2 & \text{if}\quad 0\le x\le t_1,\\ t_{[m^\alpha/3]}+1/2 & \text{if}\quad t_{m-1}<x\le t_m\quad (m=2,3,\ldots),\end{cases}$$

where $0<\alpha<1$ and t_m is the sequence defined at the beginning of Section 5. Then obviously

$$H_1 = \{n\in N^2\colon\ n_1 < f(n_2)\},\quad H_2 = \{n\in N^2\colon\ n_2 < f(n_1)\},$$

$$H_3 = \{n\in N^2\colon\ n_1 \geqq f(n_2),\ n_2 \geqq f(n_1)\}.$$

From (5.1) it follows that $f(x)\ll\exp((\log x)^\alpha)$ $(x\geqq 2)$. Let $r(v)$ be the covariance function of the field $\{\xi_v,\ v\in N^2\}$ and set $\sigma^2 = \sum_{v\in Z^2} r(v)$; let further $\{\zeta_v,\ v\in N^2\}$ be the stationary Gaussian field with mean zero and covariance function r. In view of Theorem 4, Theorem 3 will follow if we show that the fields $\{\zeta_v^{(f,\sigma^2)},\ v\in N^2\}$ and $\{\zeta_v^{(H)},\ v\in N^2\}$ can be jointly defined on a suitable probability space such that

$$\sum_{v\leqq n}\zeta_v^{(f,\sigma^2)} - \sum_{v\leqq n}\zeta_v^{(H)} \ll [n]^{1/2}(\log\log[n])^{-\tau}\big((\log n_1)^{-\tau}+(\log n_2)^{-\tau}\big) \quad \text{a.s.} \tag{5.36}$$

where $n=(n_1,n_2)$ and $\tau=(1-3\alpha)/8\alpha$. The proof of (5.36) is almost identical with that of Theorem 4 in the Gaussian case and thus can be omitted.

6. Proof of Lemmas 5, 7 and 9

To complete the proof of our theorems, in this section we give the proof of Lemmas 5 and 7 which we postponed until now because they all use standard but rather tedious calculations. We also give a simple direct proof of Lemma 9.

PROOF OF LEMMA 5. To simplify the formulas we assume $G=[1,m]$; the proof requires only notational changes in the general case. Let $0<\beta\leqq 1/2$ be a sufficiently small constant and put, for $t=1,2,\ldots$

$$K_t = \{j\colon\ (t-1)m^{1-\beta} < j \leqq tm^{1-\beta}\},\quad L_t = \{j\colon\ tm^{1-\beta}-\sqrt{m} < j \leqq tm^{1-\beta}\},$$

$$G_t = K_t\setminus L_t.$$

Also, set

$$S_{k,t} = \sum_{v\in I_k\times G_t}\xi_v \quad (1\leqq k\leqq r,\ t=1,2,\ldots),\quad Z_{k,t} = \sum_{v\in I_k\times L_t}\xi_v,$$

$$R_k = \sum_{v\in I_k\times(G\setminus \bigcup_{1\leqq t\leqq m^\beta} K_t)}\xi_v,$$

and

$$f_{i,j} = (u_i u_j)^{-1/2}E(S_1^{(i)}S_j^{(1)}) \quad (1\leqq i,\ j\leqq r). \tag{6.1}$$

Since the vector $(S_1^{(1)},\ldots,S_r^{(1)})$ is Gaussian with mean zero, we have

$$E\left\{\exp\left(\frac{i\lambda_1}{\sqrt{u_1}}S_1^{(1)}+\ldots+\frac{i\lambda_r}{\sqrt{u_r}}S_r^{(1)}\right)\right\} = \exp\left(-\frac{1}{2}\sum_{i,j=1}^{r} f_{ij}\lambda_i\lambda_j\right).$$

Hence the left side of (3.2) cannot exceed $|\varepsilon_1|+|\varepsilon_2|+|\varepsilon_3|$ where

$$\varepsilon_1 = E\left\{\exp\left(\sum_{1\leqq k\leqq r}\frac{i\lambda_k}{\sqrt{u_k}}S_k\right)\right\}-E\left\{\exp\left(\sum_{1\leqq t\leqq m^\beta}\sum_{1\leqq k\leqq r}\frac{i\lambda_k}{\sqrt{u_k}}S_{k,t}\right)\right\},$$

$$\varepsilon_2 = E\left\{\exp\left(\sum_{1\leqq t\leqq m^\beta}\sum_{1\leqq k\leqq r}\frac{i\lambda_k}{\sqrt{u_k}}S_{k,t}\right)\right\}-\prod_{1\leqq t\leqq m^\beta}E\left\{\exp\left(\sum_{1\leqq k\leqq r}\frac{i\lambda_k}{\sqrt{u_k}}S_{k,t}\right)\right\},$$

$$\varepsilon_3 = \prod_{1\leqq t\leqq m^\beta}E\left\{\exp\left(\sum_{1\leqq k\leqq r}\frac{i\lambda_k}{\sqrt{u_k}}S_{k,t}\right)\right\}-\exp\left(-\frac{1}{2}\sum_{i,j=1}^{r}f_{i,j}\lambda_i\lambda_j\right).$$

We estimate each of ε_1, ε_2 and ε_3 in turn. Using the inequality $|\exp(ia)-\exp(ib)|\leqq \leqq|a-b|$, valid for real numbers a, b, one readily observes that

$$(6.2)\qquad |\varepsilon_1| \leqq \sum_{k=1}^{r}\frac{|\lambda_k|}{\sqrt{u_k}}E\Big|\sum_{1\leqq t\leqq m^\beta}Z_{k,t}+R_k\Big| \leqq$$

$$\leqq \sum_{k=1}^{r}\frac{|\lambda_k|}{\sqrt{u_k}}\left(E^{1/(2+\alpha)}|R_k|^{2+\alpha}+\sum_{1\leqq t\leqq m^\beta}E^{1/(2+\alpha)}|Z_{k,t}|^{2+\alpha}\right)$$

where α is the constant appearing in Lemma 3. Estimating $E|R_k|^{2+\alpha}$ and $E|Z_{k,t}|^{2+\alpha}$ from Lemma 3, (6.2) gives

$$|\varepsilon_1| \leqq \left(\sum_{k=1}^{r}|\lambda_k|\right)m^{-\beta/2} \leqq |\lambda|\sqrt{r}\,m^{-\beta/2}$$

provided $\beta\leqq 1/6$. To estimate ε_2 we apply Lemma 2 repeatedly with $p_1=p_2=2+\delta$, $p_3=1+2/\delta$ and take (1.2), $0<\delta\leqq 1$ and $\gamma\geqq 82/\delta$ into account; one gets $|\varepsilon_2|\ll \ll(\sqrt{m})^{-\gamma\delta/(2+\delta)}m^\beta\leqq m^{-1}$ using $\beta\leqq 1/2$. Finally, to estimate ε_3 we use the well known expansion of the characteristic function of a r.v. with a $(2+\alpha)$-th moment (see [9] p. 199) to get

$$(6.3)\qquad E\left\{\exp\left(\sum_{k=1}^{r}\frac{i\lambda_k}{\sqrt{u_k}}S_{k,t}\right)\right\} = 1-\frac{1}{2}E\left(\sum_{k=1}^{r}\frac{\lambda_k}{\sqrt{u_k}}S_{k,t}\right)^2+T_\alpha$$

where α is the constant appearing in Lemma 3 and

$$|T_\alpha| \leqq c_\alpha E\left(\left|\sum_{k=1}^{r}\frac{\lambda_k}{\sqrt{u_k}}S_{k,t}\right|^{2+\alpha}\right)$$

with a constant c_α depending only on α. By Lemma 3 and the Minkowski inequality,

$$(6.4)\qquad |T_\alpha|^{1/(2+\alpha)} \ll \sum_{k=1}^{r}|\lambda_k|\,m^{-\beta/2} \leqq |\lambda|\sqrt{r}\,m^{-\beta/2}.$$

Applying Lemma 4 for $E(S_{i,t}S_{j,t})$ and Lemma 6 for $E(S_i^{(1)}S_j^{(1)})$ (notice that (2.1) and $\gamma\geqq 82/\delta$ imply (2.2) with $\varepsilon=1$) we get, using also (6.1) and assuming $\beta\leqq 1/4$,

$$(6.5)\qquad (u_i u_j)^{-1/2}E(S_{i,t}S_{j,t}) = m^{-1}|G_t|(f_{i,j}+\text{const}\cdot\theta\cdot m^{-1/8}) =$$

$$= m^{-\beta}(f_{i,j}+\text{const}\cdot\theta\cdot m^{-1/8})$$

where in the last step we used the fact that $|f_{i,j}| \ll 1$ (by Lemma 6 and $|\varphi_{I,J}(k)| \leqq \leqq |I| \wedge |J| \leqq |I|^{1/2} |J|^{1/2}$). Hence

$$E\left(\sum_{k=1}^{r} \frac{\lambda_k}{\sqrt{u_k}} S_{k,t}\right)^2 = m^{-\beta}\left(\sum_{i,j=1}^{r} f_{i,j}\lambda_i\lambda_j + \text{const}\cdot\theta\cdot|\lambda|^2 r m^{-1/8}\right). \tag{6.6}$$

Now, using (6.3), (6.4), (6.6) and the relation $1+x=\exp(x+D(x^2))$, valid for $x=O(1)$ we get, setting $\varrho=\min(1/8, \beta\alpha/2)$,

$$\prod_{1\leqq t\leqq m^\beta} E\left\{\exp\left(\sum_{1\leqq k\leqq r} \frac{i\lambda_k}{\sqrt{u_k}} S_{k,t}\right)\right\} =$$

$$= \prod_{1\leqq t\leqq m^\beta}\left(1-\frac{1}{2m^\beta}\sum_{i,j=1}^{r} f_{i,j}\lambda_i\lambda_j + \text{const}\cdot\theta\cdot|\lambda|^3 r^2 m^{-\beta-\varrho}\right) =$$

$$= \prod_{1\leqq t\leqq m^\beta}\exp\left(-\frac{1}{2m^\beta}\sum_{i,j=1}^{r} f_{i,j}\lambda_i\lambda_j + \text{const}\cdot\theta\cdot|\lambda|^6 r^6 m^{-\beta-\varrho}\right) =$$

$$= \exp\left(-\frac{1}{2}\sum_{i,j=1}^{r} f_{i,j}\lambda_i\lambda_j\right)(1+\text{const}\cdot\theta\cdot|\lambda|^6 r^6 m^{-\varrho})$$

provided that $|\lambda| r \leqq m^{\varrho/6}$. Since $(f_{i,j})^{r\times r}$ is a covariance matrix and hence non-negative definite, it follows that $|\varepsilon_3| \ll m^{-\varrho/2}$ for $|\lambda| r \leqq m^{\varrho/12}$. Collecting the estimates for $|\varepsilon_1|, |\varepsilon_2|, |\varepsilon_3|$ we get the first statement of our lemma.

Observe now that the only place in the above proof where the weak stationarity of $\{\xi_\nu, \nu\in N^2\}$ was used is relation (6.5) where we applied Lemmas 4 and 6. If we modify the definition of $f_{i,j}$ to $f_{i,j}=m^\beta(u_i u_j)^{-1/2}E(S_{i,t}S_{j,t})$ then everything remains valid in the proof above even without weak stationarity and hence the same estimates hold for $|\varepsilon_1|, |\varepsilon_2|, |\varepsilon_3|$. It remains now to notice that for the new $f_{i,j}$ we have $|f_{i,j}| \ll 1$ by the Cauchy—Schwarz inequality and Lemma 3.

To prove the Corollary we observe that (3.4) and (3.5) are symmetric with respect to n_1 and n_2 and thus there is no loss of generality in assuming $n_1 \leqq n_2$. From Lemma 5 it follows

$$\left|E\left\{\exp\left(\frac{i\lambda}{[n]^{1/2}} S_n\right)\right\} - \exp\left(-\frac{1}{2}\sigma_n^2\lambda^2\right)\right| \ll n_2^{-\tau} \leqq [n]^{-\tau/2} \quad \text{for} \quad |\lambda| \leqq [n]^{\tau/2} \tag{6.7}$$

where $\sigma_n^2=[n]^{-1}E(\sum_{\nu\leqq n}\zeta_\nu^{(1)})^2$ if $\{\xi_\nu, \nu\in N^2\}$ is weakly stationary and σ_n^2 is some number $\ll 1$ if weak stationarity is not assumed. In the latter case (3.5) follows immediately from (6.7) via an Esseen type lemma (see e.g. Lemma (2.2) of [2]). In the weakly stationary case we observe that Lemma 4 and its corollary are valid for the field $\{\zeta_\nu^{(1)}, \nu\in N^2\}$ as well (by Lemma 6), hence by the third relation of (2.4) we have $|\sigma_n^2-\sigma^2| \ll (n_1\wedge n_2)^{-1/4}$ and thus (3.4) also follows from (6.7).

PROOF OF LEMMA 7. We start with showing the following

LEMMA 7A. *Let* $\{\xi_\nu, \nu\in N^2\}$ *be a (not necessarily stationary) random field satisfying* (1.1) *and* (1.2) *with* $\gamma \geqq 4098/\delta$, *let* $S_\nu=\sum_{\mu\leqq\nu}\xi_\mu$ *and let* $n=(n_1, n_2)\in N^2$

with $n_1 \geqq n_2$. *Then for every real* x *we have*

$$P\{\max_{\nu \leqq n} S_\nu \geqq x\} \leqq 2P\{\max_{\nu \leqq n,\, \nu_1 = n_1} S_\nu \geqq x - L[n]^{1/2}\} + \mathrm{const} \cdot n_1^{-\beta} \tag{6.8}$$

and

$$P\{\max_{\nu \leqq n} S_\nu \geqq x\} \leqq 4P\{S_n \geqq x - L[n]^{1/2}\} + \mathrm{const} \cdot n_2^{-\beta} \tag{6.9}$$

where L, β *are positive constants.*

PROOF. We first show (6.8). Let $\prec$ be the lexicographic ordering of N^2 i.e. $\mu \prec \nu$ iff $\mu_1 < \nu_1$ or $\mu_1 = \nu_1$ and $\mu_2 < \nu_2$ where $\mu = (\mu_1, \mu_2)$, $\nu = (\nu_1, \nu_2)$. Set, for any $L > 0$,

$$A = \{\max_{\nu \leqq n} S_\nu \geqq x, \quad D = \{\max_{\nu \leqq n,\, \nu_1 \leqq n_1} S_\nu \geqq x - L[n]^{1/2}\},$$

$$A_\nu = \{S_\nu \geqq x \quad \text{and} \quad S_\mu < x \quad \text{for} \quad \mu \prec \nu\},$$

$$B_\nu = \left\{S_{\nu^{(p)}} - S_\nu \geqq -\frac{L}{2}[n]^{1/2}\right\} \quad (\nu \leqq n), \quad C_\nu = \left\{S_{\nu^*} - S_{\nu^{(p)}} \geqq -\frac{L}{2}[n]^{1/2}\right\}$$

where, for any $\nu = (\nu_1, \nu_2) \in N^2$ the vectors ν^*, $\nu^{(p)}$ are defined by $\nu^* = (n_1, \nu_2), \nu^{(p)} = ((\nu_1 + p) \wedge n_1, \nu_2)$; here $p = n_1^\varrho$ where ϱ is a number satisfying $2/\gamma < \varrho < \alpha/(2+\alpha)$ with γ and α appearing in (1.2) and Lemma 3. Since $\alpha = \delta/1024$ and $\gamma \geqq 4098/\delta$ by our assumption, such a number ϱ always exists. By Lemma 3 and the Markov inequality we have $P(\bar{C}_\nu) \leqq 1/2$ if L is large enough ($\bar{C}_\nu$ denotes the complement of C_ν). On the other hand, the events A_ν, $\nu \leqq n$ are disjoint and their union in A hence we have

$$P(A) = \sum_{\nu \leqq n} P(A_\nu) \leqq 2 \sum_{\nu \leqq n} P(A_\nu) P(C_\nu) \leqq 2 \sum_{\nu \leqq n} \{P(A_\nu C_\nu) + \mathrm{const} \cdot p^{-\gamma}\} \tag{6.10}$$

where in the last step we used (1.2). Now, by the disjointness of the A_ν's and $A_\nu B_\nu C_\nu \subset D$ we have

$$\sum_{\nu \leqq n} P(A_\nu C_\nu) = P(\bigcup_{\nu \leqq n} A_\nu B_\nu C_\nu) + P(\bigcup_{\nu \leqq n} A_\nu \bar{B}_\nu C_\nu) \leqq P(D) + P(\bigcup_{\nu \leqq n} \bar{B}_\nu) \leqq \tag{6.11}$$

$$\leqq P(D) + \sum_{l=1}^{n_1} P(\bigcap_{\nu \leqq n,\, \nu_1 = l} \bar{B}_\nu) \leqq P(D) + \sum_{l=1}^{n_1} P\Big\{\max_{\substack{I \subset I_l \\ I \text{ is rectangle}}} \Big|\sum_{\mu \in I} \xi_\mu\Big| \geqq \frac{L}{2}[n]^{1/2}\Big\}$$

where $I_l = \{\nu = (\nu_1, \nu_2) \in N^2 \colon \nu \leqq n,\ l \leqq \nu_1 \leqq l + p\mathrm{I}$ and "rectangle" means rectangle with sides parallel to the coordinate axes. From Lemma 3 and a maximal inequality of Móricz (see [10], Theorem 7 or the first inequality in Lemma 7 of [1]) it follows that the expression $P\{\max \dots\}$ in the last sum of (6.11) is

$$\ll (p/n_1)^{(1+\alpha/2)} \ll n_1^{-(1+\alpha/2)(1-\varrho)} = n_1^{-(1+\lambda)}$$

where $\lambda > 0$ by $\varrho < \alpha/(2+\alpha)$. Thus, by (6.11),

$$\sum_{\nu \leqq n} P(A_\nu C_\nu) \leqq P(D) + \mathrm{const} \cdot n_1^{-\lambda}. \tag{6.12}$$

Observe also that $[n] p^{-\gamma} \leqq n^{2 - \gamma\delta}$ by $n_1 \geqq n_2$ and here the exponent of n_1 is negative by $\varrho > 2/\gamma$. Consequently, relations (6.10) and (6.12) imply (6.8). Using a similar (in fact simpler) argument we get

$$P\{\max_{\nu\leqq n,\, \nu_1=n_1} S_\nu \geqq y\} \leqq 2P\{S_n \geqq y-L^*[n]^{1/2}\}+\text{const}\cdot n_2^{-\beta}$$

for any real y and suitable positive constants L^*, β. Hence (6.9) is also proved.

We can now easily prove Lemma 7. Clearly, it suffices to prove (3.6) without the absolute value sign on the left hand side. Also, there is no loss of generality in assuming $n_1\geqq n_2$ where $n=(n_1, n_2)$. Let $a_{n,t}$ denote the left hand side of (3.6) (without the absolute value sign) then by Lemma 7A and (3.5) we have the two estimates

$$a_{n,t} \ll \begin{cases} n_2(\exp(-Bt^2/4)+n_1^{-\varrho})+n_1^{-\beta} \\ \exp(-Bt^2/4)+n_2^{-\beta}\end{cases} \tag{6.13}$$

both of which are valid for all n and $t\geqq 2L$ where L, β are constants of Lemma 7A and B, ϱ are the constants in (3.5). Now, if $0\leqq t\leqq D\log^{1/2}[n]$ with a small D then evidently $n_1^\varrho\wedge n_1^\beta\geqq\exp(Bt^2)$ (since $[n]\leqq n_1^2$) and thus using the upper line of (6.13) for $n_2\leqq\exp(Bt^2/8)$ and the lower line for $n_2>\exp(Bt^2/8)$ we get the desired bound for $a_{n,t}$. Hence (3.6) is proved for $t\geqq 2L$; obviously it is true also for $0\leqq t\leqq 2L$.

Proof of Lemma 9. Note first that

$$E\left(\sum_{i=1}^k \xi_i\right)^2 = \sigma^2 k+O(k^{1-\varepsilon/2}) \tag{6.14}$$

as one can easily verify by a simple calculation. If $\sigma^2=0$ then (6.14) shows that the variance of $\sum_{i=1}^k \xi_i$ is $\ll k^{1-\varepsilon/2}$ and since this sum is normally distributed with mean zero, its fourth moment is $\ll k^{2-\varepsilon}$. Hence in this case (3.12) holds with $\zeta_i=0$ $(i=1, 2, \dots)$ i.e. the statement of the lemma is valid for $\sigma^2=0$. Assume now $\sigma^2>0$ and set, with the notations introduced at the beginning of Section 5,

$$S^{(k)} = \sum_{t_{k-1}<i\leqq t'_k} \xi_i,\quad T^{(k)} = \sum_{t'_k<i\leqq t_k} \xi_i,\quad x_k=p_k^{-1/2}S^{(k)},\quad y_k=(p'_k)^{-1/2}T^{(k)}.$$

By (6.14) and stationarity we have

$$\sigma^2/2 \leqq Ex_i^2 \leqq 2\sigma^2 \quad \text{for} \quad i\geqq i_0. \tag{6.15}$$

Further, by the one-parameter version of Lemma 13 we have

$$|Ex_i x_j| \ll (p'_{j-1})^{-\varepsilon} \ll j^{-40} \quad (1\leqq i<j). \tag{6.16}$$

Now let $a_1, \dots, a_k$ be arbitrary real numbers and set $a_{k+1}=a_{k+2}=\dots=0$. Then we get, using (6.15), (6.16) and the Cauchy—Schwarz inequality,

$$E\left(\sum_{i=i_1}^k a_i x_i\right)^2 \geqq \frac{\sigma^2}{2}\left(\sum_{i=i_1}^k a_i^2\right)-\text{const}\cdot\sum_{l=1}^{k=1}\left(\sum_{i=i_1}^k |a_i a_{i+l}|\right)(i_1+l)^{-40} \geqq \tag{6.17}$$

$$\geqq \frac{\sigma^2}{2}\left(\sum_{i=i_1}^k a_i^2\right)-\text{const}\cdot\left(\sum_{i=i_1}^k a_i^2\right)\sum_{l=1}^k (i_1+l)^{-40} \geqq \frac{\sigma^2}{4}\left(\sum_{i=i_1}^k a_i^2\right)$$

if i_1 is large enough. Now (6.15), (6.16), (6.17) and the Cauchy—Schwarz inequality imply

$$E\left(\left(\sum_{i=i_1}^{k} d_i x_i\right) x_{k+1}\right)/E^{1/2}\left(\sum_{i=i_1}^{k} d_i x_i\right)^2 E^{1/2}(x_{k+1}^2) \ll$$

$$\ll \left(\sum_{i=i}^{k} |d_i|\right) k^{-40} \left(\sum_{i=i_1}^{k} d_i^2\right)^{-1/2} \leqq k^{-39} \quad (k \geqq i_1)$$

for any real numbers $d_{i_1}, \ldots, d_k$. Hence, denoting by $\mathscr{F}_k$ and $\mathscr{A}_{k+1}$ the σ-fields generated by $x_{i_1}, \ldots, x_k$ and x_{k+1}, respectively, the theorem of Kolmogorov and Rozanov (see [7]) yields

$$\sup_{A\in\mathscr{F}_k, B\in\mathscr{A}_{k+1}} |P(AB)-P(A)P(B)| \ll k^{-39} \quad (k \geqq i_1).$$

In view of Lemma 1 and the trivial estimate $P\{|x_k|\geqq k^8/4 \ll \exp(-ck^{16}) \ll k^{-16}$ this shows that Theorem 1 of [2] applies to the sequence $\{x_k, k\geqq i_1\}$ with $T_k = k^8$ and $g_k(u)=\exp(-\sigma^2 u^2/2)$ and we obtain that there exist independent $N(0, \sigma^2)$ r.v.'s $\{\eta_k, k\geqq i_1\}$ such that

$$P\{|S^{(k)} - p_k^{1/2}\eta_k| \geqq p_k^{1/2}k^{-8}\} \ll k^{-8} \quad (k \geqq i_1). \tag{6.18}$$

Choosing η_k, $1\leqq k < i_1$ in such a way that they are independent $N(0, \sigma^2)$ r.v.'s and are independent of the sequence $\{\eta_k, k\geqq i_1\}$, (6.18) will hold for all $k\geqq 1$. If the probability space is large enough, there exist independent $N(0, \sigma^2)$ r.v.'s $\{\zeta_k, k\geqq 1\}$ such that $p_k^{1/2}\eta_k = \sum_{t_{k-1}<i\leqq t_k'} \zeta_i$. With exponential tail estimates for $T'^{(k)}$ and its analogue for the sequence ζ_k, we get from (6.18)

$$P\{|\sum_{t_{k-1}<i\leqq t_k} (\xi_i - \zeta_i)| \geqq p_k^{1/2}k^{-8}\} \ll k^{-8}. \tag{6.19}$$

Obviously the L_6 norm of the sum in (6.19) is $\ll (t_k - t_{k-1})^{1/2}$ and thus using the inequality

$$E|X|^3 \leqq (EX^6)^{1/2}P\{|X| \geqq t\}^{1/2} + t^3$$

valid for any r.v. X with $EX^6 < \infty$ and any $t \geqq 0$ we get from (6.19), using also (5.2,)

$$\|\sum_{t_{k-1}<i\leqq t_k} (\xi_i - \zeta_i)\|_3 \ll (t_k - t_{k-1})^{1/2} k^{-4/3} \ll t_k^{1/2}(\log t_k)^{-(1-\alpha)/2\alpha} k^{-4/3}$$

whence the statement of the lemma follows immediately.

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NORM ESTIMATES FOR (C, δ) MEANS OF HERMITE EXPANSIONS AND BOUNDS FOR δ_{eff}

C. MARKETT (Aachen)

1. Introduction and main results

Mean convergence of Hermite expansions in $L^p(-\infty, \infty)$, $4/3<p<4$, has been proved by Askey–Wainger [3]; Muckenhoupt [14] has shown that the range of convergence can be enlarged by introducing more general weight functions. In 1965 already, Freud and Knapowski [7] have solved the (C, δ) summability problem for $\delta=1$ and $p=\infty$. The $(C, 1)$ summability for $1\leqq p\leqq\infty$ has been established independently by Freud [5], [6], and Poiani [15], more general weight functions being admitted in [6], [15], too. The problem is unsolved as yet for $0<\delta<1$.

The purpose of this paper is to give norm estimates from above and below for the (C, δ) means of Hermite expansions for any $\delta\geqq 0$, $1\leqq p\leqq\infty$, which will, among other things, imply that the Hermite expansion is (C, δ) summable for each $1\leqq p\leqq\infty$ if $\delta>1/2$.

The following notations will be used. Denote the Hermite polynomials and functions by

$$(1.1)\qquad \begin{cases} H_n(x) = (-1)^n e^{x^2}(d/dx)^n e^{-x^2} \\ \mathfrak{H}_n(x) = (\pi^{1/2} 2^n n!)^{-1/2} H_n(x)\, e^{-x^2/2} \end{cases} \qquad (x\in \mathbf{R},\ n\in \mathbf{P} = \{0, 1, 2\ldots\}),$$

respectively, and let $L^p(-\infty, \infty)$ be the Lebesgue space with norm $\|f\|_{L^p(-\infty,\infty)}=\left\{\int_{-\infty}^{\infty} |f(x)|^p\,dx\right\}^{1/p}$ for $1\leqq p<\infty$, $\|f\|_{L^\infty(-\infty,\infty)}=\operatorname{ess\,sup}_{x\in\mathbf{R}} |f(x)|$ for $p=\infty$. The Cesàro means of order $\delta\geqq 0$ of the Hermite expansion of a function $f\in L^p(-\infty, \infty)$, $1\leqq p\leqq\infty$, are defined by

$$(1.2)\qquad (\mathfrak{C}, \delta)_n^H(f, x) = (A_n^\delta)^{-1} \int_{-\infty}^{\infty} f(t) \sum_{k=0}^{n} A_{n-k}^\delta\, \mathfrak{H}_k(x)\, \mathfrak{H}_k(t)\, dt \quad (x\in\mathbf{R},\ n\in\mathbf{P}),$$

where

$$(1.3)\qquad A_n^\delta = \binom{n+\delta}{n} \sim n^\delta \quad (\delta \geqq 0,\ n\to\infty).$$

(Here $\sim$ stands for $A_n^\delta=O(n^\delta)$ and $n^\delta=O(A_n^\delta)$ as $n\to\infty$.) Instead of an $\mathfrak{H}_n$-expansion in $L^p(-\infty, \infty)$, we will consider the (formal) H_n-expansion

$$(1.4)\qquad \begin{cases} f \sim \displaystyle\sum_{k=0}^{\infty} f^\wedge(k, H)\, H_k(x), \\ f^\wedge(k, H) = (\pi^{1/2} 2^k k!)^{-1} \displaystyle\int_{-\infty}^{\infty} f(t)\, H_k(t)\, e^{-t^2}\, dt \end{cases} \qquad (k\in\mathbf{P})$$

of a function f in the weighted Lebesgue space

$$(1.5)\qquad L^p_{u(H)} = \{f;\ \|f(x)\,u(x)\|_{L^p(-\infty,\infty)} < \infty,\ u(x) = e^{-x^2/2}\}\quad (1 \leqq p \leqq \infty),$$

and investigate the corresponding Cesàro means of order $\delta \geqq 0$

$$(1.6)\qquad (C,\delta)^H_n(f,x) = (A^\delta_n)^{-1} \sum_{k=0}^{n} A^\delta_{n-k} f^\wedge(k,H)\,H_k(x) =$$

$$= (A^\delta_n)^{-1} \int_{-\infty}^{\infty} f(t) \sum_{k=0}^{n} A^\delta_{n-k} (\pi^{1/2} 2^k k!)^{-1} H_k(x)\,H_k(t)\,e^{-t^2}\,dt \quad (x\in\mathbf{R},\ n\in\mathbf{P}).$$

The investigation of (1.6) in the space (1.5) is equivalent to that of (1.2) in the spaces $L^p(-\infty,\infty)$ since the following relation between the respective operator norms is obvious:

$$(1.7)\qquad \|(\mathfrak{C},\delta)^H_n\|_{[L^p(-\infty,\infty)]} = \|(C,\delta)^H_n\|_{[L^p_{u(H)}]}\quad (1 \leqq p \leqq \infty).$$

An expansion in Hermite functions $\mathfrak{H}_n$ can also be regarded as an expansion in eigenfunctions of the Hermite differential equation in its normal form (cf. [16; (5.5.2)])

$$(1.8)\qquad (d^2/dx^2)\,y(x) + (2n+1-x^2)\,y(x) = 0\quad (n\in\mathbf{P}),$$

so that the distinction between expansions of the second and third type made in [13], i.e., with respect to $\mathfrak{L}^\alpha_n$ or φ^α_n (cf. [13; (1.10), (1.12)]), does not arise here.

Our main results are the following.

THEOREM 1. *The* $(C,\delta)^H_n$ *means of the Hermite expansion satisfy,*

i) *for* $\delta=0$:

$$(1.9)\qquad \|(C,0)^H_n\|_{[L^p_{u(H)}]} \leqq C \begin{cases} n^{2/(3p)-1/2}, & 1 \leqq p < 4/3 \\ \log(n+1), & p = 4/3 \\ 1, & 4/3 < p < 4 \\ \log(n+1), & p = 4 \\ n^{1/6-2/(3p)}, & 4 < p \leqq \infty; \end{cases}$$

ii) *for* $0 < \delta \leqq 1/2$:

$$(1.10)\qquad \|(C,\delta)^H_n\|_{[L^p_{u(H)}]} \leqq \begin{cases} B(n)\,n^{2/p-3/2-\delta}, & 1 \leqq p \leqq 4/(2\delta+3) \\ C, & 4/(2\delta+3) < p < 4/(1-2\delta) \\ B(n)\,n^{1/2-2/p-\delta}, & 4/(1-2\delta) \leqq p \leqq \infty; \end{cases}$$

iii) *for* $\delta > 1/2$:

$$(1.11)\qquad \|(C,\delta)^H_n\|_{[L^p_{u(H)}]} \leqq C,\quad 1 \leqq p \leqq \infty.$$

Here $n\in\mathbf{N}=\{1, 2, \ldots\}$, *and* C *denotes a constant, independent of* n, *which may be different at each occurrence, and* $B(n)=B(n,p,\delta)=o(n^\tau)$ *as* $n\to\infty$ *for each* $\tau>0$. *In particular, if* $p=1$ *or* $p=\infty$,

$$(1.12)\qquad B(n) = \begin{cases} C, & 0 < \delta < 1/2 \\ C\log(n+1), & \delta = 1/2. \end{cases}$$

Comparing the bounds in (1.9) and (1.10), it follows immediately (cf. also [13; (1.20)]) that, e.g. for $p=1$, (1.10) can be sharpened to

$$(1.13)\qquad \|(C,\delta)_n^H\|_{[L^1_{u(H)}]} \leqq C\begin{cases} n^{1/6}, & 0\leqq\delta<1/3\\ n^{1/2-\delta}, & 1/3\leqq\delta<1/2\\ \log(n+1), & \delta=1/2\\ 1, & \delta>1/2.\end{cases}$$

THEOREM 2. *Let* $0\leqq\delta\leqq 1/6$, $1\leqq p\leqq\infty$. *Then there exists a sequence* $\{n_k\}_{k\in\mathbf{N}}\subset\mathbf{N}$ *and a constant* $C>0$, *such that*

$$(1.14)\qquad \|(C,\delta)_{n_k}^H\|_{[L^p_{u(H)}]} \geqq C\begin{cases} n_k^{2/(3p)-1/2-\delta}, & 1\leqq p\leqq 4/(3+6\delta)\\ 1, & 4/(3+6\delta)\leqq p\leqq 4/(1-6\delta)\\ n_k^{1/6-2/(3p)-\delta}, & 4/(1-6\delta)<p\leqq\infty.\end{cases}$$

For $\delta=0, p=1$, (1.14) *is even valid without restricting* n *to a subsequence of* $\mathbf{N}$.

Thus our upper and lower bounds match when $\delta=0$, whereas for $0<\delta<1/2$ there is still a gap between them. It follows that

$$(1.15)\qquad 1/6\leqq\delta_{\text{eff}}\leqq 1/2,$$

and it remains an open question whether $\delta_{\text{eff}}=1/2$ or not for the Hermite expansion. Here δ_{eff} is the largest $\delta>0$ with the property that the Hermite expansion is not (C,δ) summable for at least one $f\in L^1_{u(H)}$ (i.e., the (C,δ) means are "effective" in $L^p_{u(H)}$, $1\leqq p\leqq\infty$, for all $\delta>\delta_{\text{eff}}$). Nevertheless, the present results already imply that there is an essential distinction between Laguerre and Hermite expansions. Let p_0 denote the largest $p\in[1,2]$ such that a given orthogonal expansion diverges for at least one $f\in L^{p_0}$, and let γ be determined by $\|(C,0)_n\|_{[1]}\sim n^\gamma$, $n\to\infty$, thus

$$(1.16)\qquad \gamma=1/6,\ p_0=4/3,\ \delta_{\text{eff}}\in[1/6,1/2]$$

for the Hermite case; and in [12] it has been shown that

$$(1.17)\qquad \gamma=1/2,\ p_0=4/3,\ \delta_{\text{eff}}=1/2$$

for the Laguerre case. According to a conjecture of Lorch [11; p. 756] one should have $\gamma=\delta_{\text{eff}}$ for both cases (and others), which is true in (1.17) and would imply $\delta_{\text{eff}}=1/6$ in (1.16). On the other hand, Askey [1; p. 812] noted that in several known cases the line in the $(1/p,\delta)$ plane which connects the points $(1,\delta_{\text{eff}})$ and $(1/p_0,0)$ always meets the point $(1/2,-1/2)$; cf. also [2; p. 81]. The latter principle applies to (1.17), but it would apply to (1.16) only if $\delta_{\text{eff}}=1/2$. So, in contrast to the Laguerre case, in the Hermite case either Lorch's conjecture or Askey's principle fails.

Concerning the method of proof, one may either treat the Hermite case parallel to the Laguerre case, making use of the fact that many essential features coincide for the two expansions (as carried out e.g. in [3], [14]), or one may derive assertions for the Hermite expansion directly from those for the Laguerre case since the two expansions are transformable into each other by (2.1) below. The latter approach was chosen e.g. by Szegö [16], [17; Chap. 9] (pointwise convergence and summability) and by Poiani [15] ($(C, 1)$ summability in the mean) and will essentially also be followed here (except for the proof of the case $\delta=0, 1<p<\infty$ below). This requires norm estimates for the Laguerre expansion in spaces with parameter shifted (see Theorem 3 below) as have been presented in [13; Sec. 4, 5]. The fact that the shifting produces an improvement of the exponent of divergence then leads to the $\gamma=1/6$ in (1.16) instead of the $\gamma=1/2$ in (1.17).

2. Preliminaries

The following properties of the Hermite polynomials and functions (1.1) will be used (cf. [17], [14]):

$$\text{(2.1)} \qquad \begin{cases} H_{2m}(x) = (-1)^m 2^{2m} m!\, L_m^{-1/2}(x^2) \\ H_{2m+1}(x) = (-1)^m 2^{2m+1} m!\, L_m^{1/2}(x^2)\, x \end{cases} \qquad (m\in\mathbf{P},\ x\in\mathbf{R}).$$

The Christoffel—Darboux formula reads (cf. [14; (2.9—15)]):

$$\text{(2.2)} \qquad \sum_{k=0}^{n} \mathfrak{H}_k(x)\,\mathfrak{H}_k(t) = b_n h_1 + c_n(h_2+h_3),$$

where

$$h_1(n, x, t) = \mathfrak{H}_n(x)\,\mathfrak{H}_n(t), \quad h_2(n, x, t) = \frac{n^{1/2}\,\mathfrak{H}_n(t)\,[\mathfrak{H}_{n+1}(x)-\mathfrak{H}_{n-1}(x)]}{x-t},$$

$$h_3(n, x, t) = h_2(n, t, x), \quad 1/3 \leqq b_n, c_n \leqq 1 \quad (n\in\mathbf{N}).$$

Following [3] and [14], there exist positive constants C and γ, independent of x and n, such that

$$\text{(2.3)} \qquad |\mathfrak{H}_n(x)| \leqq C\begin{cases} (N^{1/3}+|x^2-N|)^{-1/4}, & x^2 \leqq 2N \\ \exp(-\gamma x^2), & x^2 > 2N \end{cases} \quad (n\in\mathbf{P}),$$

$$\text{(2.4)} \qquad |\mathfrak{H}_{n+1}(x)-\mathfrak{H}_{n-1}(x)| \leqq C\begin{cases} N^{-1/2}(N^{1/3}+|x^2-N|)^{1/4}, & x^2 \leqq 2N \\ \exp(-\gamma x^2), & x^2 > 2N \end{cases} \quad (n\in\mathbf{N}),$$

where $N=N(n):=2n+1$. Obviously, (2.3) and (2.4) remain valid, if the transition point $x^2=2N$ is replaced e.g. by

$$\text{(2.5)} \qquad x^2 = 4N.$$

LEMMA 1. (Cf. also [10] for $p=1$, $p=\infty$, and [8] for $p=4$.) *The Hermite functions satisfy*

$$\text{(2.6)} \qquad \|\mathfrak{H}_n(x)\|_{L^p(-\infty,\infty)} \sim \begin{cases} n^{1/(2p)-1/4}, & 1 \leqq p < 4 \\ n^{-1/8}(\log n)^{1/4}, & p = 4 \\ n^{-1/(6p)-1/12}, & 4 < p \leqq \infty \end{cases} \quad (n\to\infty),$$

$$\text{(2.7)} \qquad \|\mathfrak{H}_{n+1}(x)-\mathfrak{H}_{n-1}(x)\|_{L^p(-\infty,\infty)} \sim n^{1/(2p)-1/4} \quad (n\to\infty).$$

PROOF. Since $\mathfrak{H}_n(x)$ is either an even or an odd function on $\mathbf{R}$ (cf. (2.1)) one has, by (2.3),

$$\|\mathfrak{H}_n(x)\|_{L^p(-\infty,\infty)} = \Big\{\int_{-\infty}^{\infty} |\mathfrak{H}_n(x)|^p\,dx\Big\}^{1/p} = 2^{1/p}\Big\{\int_0^{\infty} |\mathfrak{H}_n(x)|^p\,dx\Big\}^{1/p} \le$$

$$\le C\Big\{\Big(\int_0^{\sqrt{N}} + \int_{\sqrt{N}}^{\sqrt{2N}}\Big)(N^{1/3}+|x^2-N|)^{-p/4}\,dx + \int_{\sqrt{2N}}^{\infty} e^{-\gamma p x^2}\,dx\Big\}^{1/p} = C\Big\{\sum_{j=1}^{3} I_j\Big\}^{1/p},$$

say. Setting $N^{1/3}+N-x^2=(\sqrt{N^{1/3}+N}+x)(\sqrt{N^{1/3}+N}-x)$ in I_1 and $N^{1/3}+x^2-N=(x+\sqrt{N-N^{1/3}})(x-\sqrt{N-N^{1/3}})$ in I_2, the first factor is $\le CN^{1/2}$ in both cases, uniformly for $x\in[0,\sqrt{N}]$, $x\in[\sqrt{N},\sqrt{2N}]$, respectively. Evaluating the remaining integrals and comparing the three upper bounds obtained, the upper estimate in (2.6) follows. Analogously, the upper estimate in (2.7) is proved by means of (2.4). For the lower bounds we use the asymptotic expansions of $\mathfrak{H}_n(x)$ and of $\mathfrak{H}_{n+1}(x)-\mathfrak{H}_{n-1}(x)$ from [14; (7.2−3)] and restrict the range of integration in (2.6) to $[1, N^{1/2}-bN^{-1/6}]$, and in (2.7) to $[1, b\sqrt{N}/2]$ where N is chosen sufficiently large to be able to apply Lemma 15 of [14] to the principal term. Moreover, in the first case $b>1$ has to be large enough, and in the second case $0<b<1$ has to be small enough, in order that the remaining terms can be neglected. Then the assertions follow.

3. A relation between the operator norms of the Cesàro means of Laguerre and Hermite expansions

We make use of the connection (2.1) between Laguerre and Hermite polynomials. But instead of comparing the (C,δ) means of the two orthogonal expansions directly, as done in [15], we first pass to the corresponding Riesz means since these are easier to handle in case δ is not an integer. Moreover, we want to set up an explicit relation between the operator norms in order to make also the rate of divergence transformable from one expansion to the other. A slight extension of a general equivalence theorem of Butzer, Nessel, and Trebels [4] (cf. also Trebels [18]) for Cesàro and Riesz means of Fourier expansions in Banach spaces is needed for this purpose.

Denoting by $[X]$ the set of bounded linear operators from a Banach space X into X, we assume that there exists a sequence $\{P_k\}_{k\in\mathbf{P}}\subset[X]$ of projections which are mutually orthogonal and total, i.e.,

$$P_jP_k=\delta_{jk}P_k \quad (j,k\in\mathbf{P}), \tag{3.1}$$

$$f\in X \text{ and } P_kf=0 \text{ for all } k\in\mathbf{P} \text{ imply } f=0. \tag{3.2}$$

The Cesàro and Riesz means of order $\delta\ge 0$ of the orthogonal expansion of $f\in X$ are then defined by (cf. (1.3))

$$(\mathscr{C},\delta)_nf=(A_n^\delta)^{-1}\sum_{k=0}^{n} A_{n-k}^\delta P_kf \quad (n\in\mathbf{P}), \tag{3.3}$$

$$\mathscr{R}_{1,\delta,\varrho}f=\sum_{0\le k<\varrho}(1-k/\varrho)^\delta P_kf \quad (\varrho>0). \tag{3.4}$$

LEMMA 2. *Let* $X, \{P_k\}_{k\in\mathbf{P}}$ *satisfy* (3.1) *and* (3.2). *For each* $f\in X$ *and each* $\delta\geqq 0$ *there is a constant* C, *independent of* f *and* n *or* ϱ, *such that*

$$\|(\mathscr{C},\delta)_n f\|_X \leqq C \sup_{0<\varrho\leqq n+1} \|\mathscr{R}_{1,\delta,\varrho} f\|_X \quad (n\in\mathbf{P}), \tag{3.5}$$

$$\|\mathscr{R}_{1,\delta,\varrho} f\|_X \leqq C \sup_{0\leqq k<\varrho} \|(\mathscr{C},\delta)_k f\|_X \quad (\varrho>0). \tag{3.6}$$

PROOF. The proof is an immediate extension of [18; Theorem 3.19]. Indeed, (3.5) follows from (cf. (1.3))

$$\|(\mathscr{C},\delta)_n f\|_X \leqq \int_0^{n+1} (\varrho^\delta/A_n^\delta)\|\mathscr{R}_{1,\delta,\varrho} f\|_X\,|u(n+1-\varrho)|\,d\varrho \leqq$$

$$\leqq C \sup_{0<\varrho\leqq n+1} \|\mathscr{R}_{1,\delta,\varrho} f\|_X \int_0^\infty |u(\varrho)|\,d\varrho,$$

where u denotes a certain $L^1(0,\infty)$-function.

For (3.6) one defines a multiplier sequence $\eta(\varrho,\delta)$ by

$$\eta_k(\varrho,\delta) = \begin{cases} (1-k/\varrho)^\delta, & k<\varrho \\ 0, & k\geqq\varrho \end{cases} \quad (\varrho>0),$$

and its fractional difference by

$$\Delta^{\delta+1}\eta_k = \sum_{m=0}^\infty A_m^{-\delta-2}\eta_{k+m} \quad (k\in\mathbf{P}).$$

The Riesz means can then be written as (cf. [18; proof of Theorem 3.3])

$$\mathscr{R}_{1,\delta,\varrho} f = \sum_{k=0}^\infty \eta_k(\varrho,\delta) P_k f = \sum_{0\leqq k<\varrho} A_k^\delta[\Delta^{\delta+1}\eta_k(\varrho,\delta)](\mathscr{C},\delta)_k f \quad (f\in X). \tag{3.7}$$

The assertion now follows in view of

$$\|\mathscr{R}_{1,\delta,\varrho} f\|_X \leqq \sup_{0\leqq k<\varrho} \|(\mathscr{C},\delta)_k f\|_X \,\|\eta(\varrho,\delta)\|_{bv_{\delta+1}}, \tag{3.8}$$

since $\eta\in bv_{\delta+1}$ (cf. [18; Theorem 3.18]).

In the sequel Lemma 2 will be applied to three particular cases. As in [13] we define the (formal) expansion of a function f into Laguerre polynomials $L_n^\alpha(x)$, $\alpha>-1$ ($R_n^\alpha := L_n^\alpha(x)/L_n^\alpha(0)$, $L_n^\alpha(0)=A_n^\alpha$) by

$$f\sim\sum_{k=0}^\infty f^\wedge(k,\alpha)L_k^\alpha(x),\quad f^\wedge(k,\alpha) = \big(\Gamma(\alpha+1)\big)^{-1}\int_0^\infty f(t)R_k^\alpha(t)e^{-t}t^\alpha\,dt \tag{3.9}$$

and its Cesàro means by ($x\geqq 0$, $\alpha>-1$, $\delta\geqq 0$, $n\in\mathbf{P}$)

$$(C,\delta)_n^\alpha(f,x) = (A_n^\delta)^{-1}\sum_{k=0}^n A_{n-k}^\delta f^\wedge(k,\alpha)L_k^\alpha(x) = \tag{3.10}$$

$$= \big(\Gamma(\alpha+1)A_n^\delta\big)^{-1}\int_0^\infty f(t)\sum_{k=0}^n A_{n-k}^\delta L_k^\alpha(x)R_k^\alpha(t)e^{-t}t^\alpha\,dt.$$

Here the function f belongs to some weighted Lebesgue space:

(3.11) $L^p_{u(\gamma)} = \{f;\ \|f(x)\,u(\gamma, x)\|_{L^p(0,\infty)} < \infty,\ u(\gamma, x) = e^{-x/2}x^{\gamma/2}\}\quad (1 \leqq p \leqq \infty)$.

COROLLARY 1. *Inequalities* (3.5—6) *of Lemma* 2 *hold in the following particular cases* (cf. (3.9—11), (1.4—6)).

i) $X = L^p_{u(\alpha+1/2-1/p)}$, $P_k f = f\hat{\ }(k,\alpha)L^\alpha_k$, *where* $k \in \mathbf{P}$, $f \in L^p_{u(\alpha+1/2-1/p)}$, $\alpha = \pm 1/2$, $1 \leqq p < \infty$. *Here*

(3.12)
$$(\mathfrak{C}, \delta)_n f = (C, \delta)^\alpha_n f,$$
$$\mathfrak{R}_{1,\delta,\varrho}(f, x) = R^\alpha_{1,\delta,\varrho}(f, x) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(t) \sum_{0 \leqq k < \varrho} (1-k/\varrho)^\delta L^\alpha_k(x)\, R^\alpha_k(t)\, e^{-t}\, t^\alpha\, dt \qquad (x \geqq 0)$$

ii) $X = L^p_{u(H)}$, $P_k f = f\hat{\ }(k, H)H_k$, *where* $k \in \mathbf{P}$, $f \in L^p_{u(H)}$, $1 \leqq p < \infty$. *Here*

(3.13)
$$(\mathfrak{C}, \delta)_n f = (C, \delta)^H_n f,$$
$$\mathfrak{R}_{1,\delta,\varrho}(f, x) = R^H_{1,\delta,\varrho}(f, x) = \int_{-\infty}^\infty f(t) \sum_{0 \leqq k < \varrho} (1-k/\varrho)^\delta (\pi^{1/2}2^k k!)^{-1} H_k(x)\, H_k(t)\, e^{-t^2}\, dt \qquad (x \in \mathbf{R}).$$

PROOF. The assumptions of Lemma 2 are fulfilled since X is always a Banach space and $\{P_k\}_{k\in\mathbf{P}}$ a sequence of bounded, linear, and mutually orthogonal projections from X into itself. The totality property (3.2) follows, e.g., from the fact that the $(C, 1)$ means form an approximation process, its operator norms being uniformly bounded ([13], [5], [15]), and the Laguerre and Hermite polynomials being dense in the respective spaces (cf. [14; Lemma 1, 2]).

Now to the proof of the main result of this section.

THEOREM 3. *Let* $1 \leqq p \leqq \infty$, $\delta \geqq 0$, *and* $n \in \mathbf{P}$. *The following relations hold between the* (C, δ) *means of Laguerre and Hermite expansions:*

(3.14)
$$\|(C, \delta)^H_n\|_{[L^p_{u(H)}]} \leqq C\Big\{\sup_{0\leqq k<(n+1)/2} \|(C, \delta)^{-1/2}_k\|_{[L^p_{u(-1/p)}]} + \sup_{0\leqq k<n/2} \|(C, \delta)^{1/2}_k\|_{[L^p_{u(1-1/p)}]}\Big\},$$

(3.15)
$$\|(C, \delta)^{-1/2}_n\|_{[L^p_{u(-1/p)}]} \leqq C \sup_{0\leqq k<2n+2} \|(C, \delta)^H_k\|_{[L^p_{u(H)}]},$$

(3.16)
$$\|(C, \delta)^{1/2}_n\|_{[L^p_{u(1-1/p)}]} \leqq C \sup_{0\leqq k<2n+3} \|(C, \delta)^H_k\|_{[L^p_{u(H)}]}.$$

PROOF. It suffices to consider the cases $0 \leqq \delta < 1$. The result for $p = \infty$ follows from that for $p = 1$ by duality, so that we can confine ourselves to $1 \leqq p < \infty$.

First we set up a relation between the Riesz means of Hermite and Laguerre expansions and then we transform them to the corresponding Cesàro means by virtue of Lemma 2 and Corollary 1. By (2.1) one obtains for $\varrho>1$ after some calculation

$$\text{(3.17)}\qquad \sum_{0\leqq k<\varrho}\left(1-\frac{k}{\varrho}\right)^{\delta}(\pi^{1/2}2^k k!)^{-1}H_k(x)H_k(t)=$$

$$=\frac{1}{\Gamma(1/2)}\sum_{0\leqq j<\varrho/2}\left(1-\frac{j}{\varrho/2}\right)^{\delta}L_j^{-1/2}(x^2)R_j^{-1/2}(t^2)+$$

$$+\left(\frac{\varrho-1}{\varrho}\right)^{\delta}\frac{1}{\Gamma(3/2)}\sum_{0\leqq j<(\varrho-1)/2}\left(1-\frac{j}{(\varrho-1)/2}\right)^{\delta}L_j^{1/2}(x^2)R_j^{1/2}(t^2)\,xt.$$

Observing the symmetry properties of these two terms with respect to t, we have for $f\in L^p_{u(H)}$

$$R^H_{1,\delta,\varrho}(f,x)=\int_0^{\infty}[f(t)+f(-t)]\left\{\frac{1}{\Gamma(1/2)}\sum_{0\leqq j<\varrho/2}\left(1-\frac{j}{\varrho/2}\right)^{\delta}L_j^{-1/2}(x^2)R_j^{-1/2}(t^2)\right\}e^{-t^2}dt+$$

$$+\int_0^{\infty}[f(t)-f(-t)]\times$$

$$\times\left\{\left(\frac{\varrho-1}{\varrho}\right)^{\delta}\frac{1}{\Gamma(3/2)}\sum_{0\leqq j<(\varrho-1)/2}\left(1-\frac{j}{(\varrho-1)/2}\right)^{\delta}L_j^{1/2}(x^2)R_j^{1/2}(t^2)\,xt\right\}e^{-t^2}\,dt.$$

Substituting $t=s^{1/2}$, and setting

$$f(x)=g_1(x^2)+g_2(x^2)x\quad(x\in\mathbf{R}),\ g_1(x):=[f(\sqrt{x})+f(-\sqrt{x})]/2,\quad x\geqq 0,$$

$$g_2(x):=\begin{cases}[f(\sqrt{x})-f(-\sqrt{x})]/(2\sqrt{x}), & x>0\\ 0, & x=0,\end{cases}$$

one obtains for $x\in\mathbf{R}$

$$\text{(3.18)}\qquad R^H_{1,\delta,\varrho}(f,x)=R^{-1/2}_{1,\delta,\varrho/2}(g_1,x^2)+\left(\frac{\varrho-1}{\varrho}\right)^{\delta}R^{1/2}_{1,\delta,(\varrho-1)/2}(g_2,x^2)\,x.$$

Here one has, if $f\in L^p_{u(H)}$, $1\leqq p<\infty$, $g_1\in L^p_{u(-1/p)}$, $g_2\in L^p_{u(1-1/p)}$. Indeed, the substitution $s=x^2$ and the inequality $|A+B|^p\leqq 2^{p-1}(|A|^p+|B|^p)$, $p\geqq 1$, yield

$$\text{(3.19)}\qquad \|g_1\|_{L^p_{u(-1/p)}}=\left\{\int_0^{\infty}|g_1(s)e^{-s/2}s^{-1/(2p)}|^p\,ds\right\}^{1/p}=$$

$$=\left\{\int_0^{\infty}2^{1-p}|[f(x)+f(-x)]e^{-x^2/2}|^p\,dx\right\}^{1/p}\leqq$$

$$\leqq\left\{\int_0^{\infty}(|f(x)e^{-x^2/2}|^p+|f(-x)e^{-x^2/2}|^p)\,dx\right\}^{1/p}=\left\{\int_{-\infty}^{\infty}|f(x)e^{-x^2/2}|^p\,dx\right\}^{1/p}=\|f\|_{L^p_{u(H)}}.$$

Analogously one can show that

$$\text{(3.20)}\qquad \|g_2\|_{L^p_{u(1-1/p)}}\leqq\|f\|_{L^p_{u(H)}}.$$

Concerning the norms of the Riesz means, one obtains by (3.18), substituting $x=s^{1/2}$ and estimating the factor $[(\varrho-1)/\varrho]^\delta$ by 1,

$$\|R^H_{1,\delta,\varrho}(f,x)\|_{L^p_{u(H)}} \leqq \|R^{-1/2}_{1,\delta,\varrho/2}g_1\|_{L^p_{u(-1/p)}} + \|R^{1/2}_{1,\delta,(\varrho-1)/2}g_2\|_{L^p_{u(1-1/p)}}. \tag{3.21}$$

Now we are able to prove (3.14) of Theorem 3 (for $1\leqq p<\infty$, $0\leqq\delta<1$). Starting with the estimation (3.5) of Lemma 2 in the special case (3.13), we use relation (3.21) of the Riesz means, and return to the Cesàro means by (3.6) of Lemma 2 in the cases (3.12). Indeed, given any $f\in L^p_{u(H)}$,

$$\begin{aligned}\|(C,\delta)^H_n f\|_{L^p_{u(H)}} &\leqq C \sup_{0<\varrho\leqq n+1} \|R^H_{1,\delta,\varrho}f\|_{L^p_{u(H)}} \leqq \\ &\leqq C \sup_{0<\varrho\leqq n+1} \{\|R^{-1/2}_{1,\delta,\varrho/2}g_1\|_{L^p_{u(-1/p)}} + \|R^{1/2}_{1,\delta,(\varrho-1)/2}g_2\|_{L^p_{u(1-1/p)}}\} \leqq \\ &\leqq C \sup_{0\leqq k<(n+1)/2} \|(C,\delta)^{-1/2}_k\|_{[L^p_{u(-1/p)}]}\|g_1\|_{L^p_{u(-1/p)}} + \\ &\quad + C \sup_{0\leqq k<n/2} \|(C,\delta)^{1/2}_k\|_{[L^p_{u(1-1/p)}]}\|g_2\|_{L^p_{u(1-1/p)}}.\end{aligned} \tag{3.22}$$

By means of (3.19—20) one immediately obtains (3.14).

For the proof of (3.15) we define for any $g\in L^p_{u(-1/p)}$ an even function f by $f(x)=g(x^2)$, $x\in\mathbf{R}$. Obviously,

$$\|f\|_{L^p_{u(H)}} = \|g\|_{L^p_{u(-1/p)}} < \infty,$$

and by applying (3.18) to this f (thus $g_1=g$, $g_2=0$), one has

$$\|R^H_{1,\delta,2\varrho}f\|_{L^p_{u(H)}} = \|R^{-1/2}_{1,\delta,\varrho}g\|_{L^p_{u(-1/p)}}. \tag{3.23}$$

Using Lemma 2 again, i.e. (3.5) in the case $\alpha=-1/2$ of (3.12) as well as (3.6) in the case (3.13), this yields (3.15).

The estimate (3.16) can be proved analogously. In this case, for any $g\in L^p_{u(1-1/p)}$ we define an odd function f by $f(x)=g(x^2)x$ for $x\neq0$, $=0$ for $x=0$. Then we have (cf. (3.18) with $g_1=0$, $g_2=g$)

$$\|f\|_{L^p_{u(H)}} = \|g\|_{L^p_{u(1-1/p)}} < \infty, \quad \|R^H_{1,\delta,2\varrho+1}f\|_{L^p_{u(H)}} = \left(\frac{2\varrho}{2\varrho+1}\right)^\delta \|R^{1/2}_{1,\delta,\varrho}g\|_{L^p_{u(1-1/p)}}.$$

By Lemma 2 and Corollary 1 this leads to (3.16). Thus Theorem 3 is proved completely.

COROLLARY 2. *Let $\delta\geqq0$, $1\leqq p\leqq\infty$. The Hermite operator norm $\|(C,\delta)^H_n\|_{[L^p_{u(H)}]}$ is uniformly bounded as $n\to\infty$ if and only if both the Laguerre operator norms $\|(C,\delta)^{-1/2}_n\|_{[L^p_{u(-1/p)}]}$ and $\|(C,\delta)^{1/2}_n\|_{[L^p_{u(1-1/p)}]}$ are uniformly bounded as $n\to\infty$.*

4. Norm estimates

The results given in [13] can now be used to derive upper and lower norm estimates for the Cesàro means of Hermite expansions.

PROOF OF THEOREM 1. By [13; Theorem 1] the Laguerre partial sum operators satisfy for all $\alpha\geqq -1/2$ and $n\in\mathbf{P}$

$$\|(C,0)_n^\alpha\|_{[L^1_{u(\alpha-1/2)}]}=\|(C,0)_n^\alpha\|_{[L^\infty_{u(\alpha+1/2)}]}\leqq C\max\{1,n^{1/6}\}. \tag{4.1}$$

Inserting $\delta=0$ and the particular cases $\alpha=\pm 1/2$ of (4.1) into (3.14) of Theorem 3, the assertion of Theorem 1 i) for $p=1$ and $p=\infty$ follows.

For $1<p<\infty$, we estimate the norms of the Hermite partial sums in the equivalent setting $\|(\mathfrak{C},0)_n^H\|_{[L^p(-\infty,\infty)]}$ (cf. (1.2), (1.7)), for convenience, proceeding along the lines of [14] without going back to the Laguerre case. The proof is essentially based on representing the kernel of the partial sums in terms of the Christoffel—Darboux formula (2.2), splitting up the double integral, and estimating each of its parts separately. Here some basic inequalities of [14] are used; these are variants of the well-known Hardy inequality and of the Hilbert transform theorem. We only sketch the proof. By (2.2) one has

$$\|(\mathfrak{C},0)_n^H g\|_{L^p(-\infty,\infty)}=\Big\{\int\limits_{-\infty}^{\infty}\Big|\int\limits_{-\infty}^{\infty}\sum_{k=0}^{n}\mathfrak{H}_k(x)\mathfrak{H}_k(t)\,dt\Big|^p dx\Big\}^{1/p}\leqq \tag{4.2}$$

$$\leqq\sum_{k=1}^{3}\Big\{\int\limits_{-\infty}^{\infty}\Big|\int\limits_{-\infty}^{\infty}g(t)\,h_k(x,t,n)\,dt\Big|^p dx\Big\}^{1/p}.$$

The first term can easily be estimated by Lemma 1, (2.6). The other two terms have to be treated more carefully since the double integrals now have to be split up according to the singularities of the $h_k(x,t,n)$, $k=2,3$, at $x=t$ and to the bounds of $\mathfrak{H}_n$, $\mathfrak{H}_{n+1}-\mathfrak{H}_{n-1}$ as given by (2.3—4). One of the 13 terms occurring reads, for example, $(N=2n+1)$

$$I=\Big\{\int\limits_{\sqrt{N}/4}^{7\sqrt{N}/4}\Big|\int\limits_{\sqrt{N}/4}^{7\sqrt{N}/4}g(t)\frac{n^{1/2}\mathfrak{H}_n(t)\big(\mathfrak{H}_{n+1}(x)-\mathfrak{H}_{n-1}(x)\big)}{x-t}\,dt\Big|^p dx\Big\}^{1/p}. \tag{4.3}$$

By (2.3—5) we can substitute into I

$$\mathfrak{H}_n(t)=(N^{1/3}+|t^2-N|)^{-1/4}\varphi(t,N),$$

$$\mathfrak{H}_{n+1}(x)-\mathfrak{H}_{n-1}(x)=N^{-1/2}(N^{1/3}+|x^2-N|)^{1/4}\psi(x,N),$$

where the functions $\varphi(t,N)$ and $\psi(x,N)$ are uniformly bounded on $[\sqrt{N}/4,\,7\sqrt{N}/4]$ with respect to N and t or x. Since for all such x

$$(N^{1/3}+|x^2-N|)^{1/4}=(N^{1/3}+|x+\sqrt{N}|\,|x-\sqrt{N}|)^{1/4}\sim$$

$$\sim\{\sqrt{N}(N^{-1/6}+|x-\sqrt{N}|)\}^{1/4}\quad(N\to\infty)$$

(and correspondingly for t), it follows that

$$I \leqq C\left\{\int\limits_{\sqrt{N}/4}^{7\sqrt{N}/4}\left|\int\limits_{\sqrt{N}/4}^{7\sqrt{N}/4}\frac{g(t)\varphi(t,N)}{x-t}\left(\frac{N^{-1/6}+|x-\sqrt{N}|}{N^{-1/6}+|t-\sqrt{N}|}\right)^{1/4}dt\right|^{p}dx\right\}^{1/p}.$$

Substituting now $x=\sqrt{N}+N^{-1/6}\xi$, $t=\sqrt{N}+N^{-1/6}\tau$, the double integral can be transformed in such a way that [14; Lemma 10, 11] is applicable. After some calculation one deduces

$$I \leqq C\|g\|_{L^p(-\infty,\infty)}\begin{cases} n^{2/(3p)-1/2}, & 1<p<4/3 \\ \log n, & p=4/3 \\ 1, & 4/3<p<\infty. \end{cases} \tag{4.4}$$

The remaining terms can be estimated similarly, using (2.7) too.

Finally, assertions ii) and iii) of Theorem 1 are easily obtained by inserting into (3.14) the bounds obtained in [13; Theorem 2] for the Laguerre operators $(C,\delta)_k^\alpha$ with $\alpha=\pm 1/2$, and (1.12) follows by [13; (1.17)].

Proof of Theorem 2. By (3.15) and [13; Theorem 3 with $\alpha=-1/2$] there exist a constant $C>0$ and a sequence $\{n_m\}_{m\in\mathbf{N}}\subset\mathbf{N}$ such that, for each $m\in\mathbf{N}$,

$$\sup_{0\leqq k<2n_m+2}\|(C,\delta)_k^H\|_{[L_u^p(H)]} \geqq C\begin{cases} n_m^{2/(3p)-1/2-\delta}, & 1\leqq p<4/(3+6\delta) \\ 1, & 4/(3+6\delta)\leqq p\leqq 4/(1-6\delta) \\ n_m^{1/6-2/(3p)-\delta}, & 4/(1-6\delta)<p\leqq\infty. \end{cases} \tag{4.5}$$

Using the same argument as in the proof of [13; Theorem 3, (5.8) ff] one obtains a (possibly different) sequence $\{n_k\}_{k\in\mathbf{N}}\subset\mathbf{N}$ with the properties asserted in (1.4).

Concerning the last assertion of Theorem 2, by (3.23),

$$\|(C,0)_n^{-1/2}\|_{[L_{u(-1/p)}^p]} \leqq \|(C,0)_{2n+1}^H\|_{[L_{u(H)}^p]}, \tag{4.6}$$

since Cesàro and Riesz means coincide for $\delta=0$. According to [13; (5.9)], for $p=1$ the left hand side of (4.6) has $Cn^{1/6}$ as a lower bound, and the proof is complete.

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LEHRSTUHL A FÜR MATHEMATIK
AACHEN TECHNICAL UNIVERSITY
TEMPLERGRABEN 55, 5100 AACHEN, FRG

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FAMILIES CLOSE TO DISJOINT ONES

P. KOMJÁTH (Budapest)

Introduction

The aim of this paper is to define and investigate two properties of systems of equicardinal sets, both are strengthenings of almost-disjointness and having the transversal property, but they have nice properties: they indicate property B and compactness is available. However we shall show, that under certain conditions, strong almost-disjointness implies these properties. So they fit naturally in the series of known transversal-like properties.

DEFINITION 1. A system $\mathscr{H}$ of sets, each of cardinality μ, is *sparse* if there is a function f with $f(H)\in[H]^{<\mu}$ $(H\in\mathscr{H})$ such that the sets $\{H-f(H)\colon H\in\mathscr{H}\}$ are mutually disjoint.

DEFINITION 2. A system $\mathscr{H}$ of sets, each of cardinality μ, has the large subset property if there is a function f with $f(H)\in[H]^{\mu}$ such that the sets $\{f(H)\colon H\in\mathscr{H}\}$ are mutually disjoint.

DEFINITION 3. A system $\mathscr{H}$ of infinite sets has the transversal property, if there is a function f with $f(H)\in H$ $(H\in\mathscr{H})$ and f is one-to-one.

DEFINITION 4. A system $\mathscr{H}$ of sets is 2-chromatic (has property B) if there is a set X such that $\emptyset\neq X\cap H\neq H$ for every $H\in\mathscr{H}$.

Any of these definitions implies the next (the only non-trivial implication is Def 3$\to$Def. 4, which is the main result of [10]). However, it is easy to see that Def. 2$\to$Def. 4 holds.

All notations are standard. Ordinals are identified with the set of their predecessors, cardinals with initial ordinals. $[S]^{<\mu}, [S]^{\mu}$ denote $\{X\subseteq S\colon |X|<\mu\}$ and $\{X\subseteq S\colon |X|=\mu\}$, respectively. If $\mathscr{H}$ is a system of sets, each of cardinality $\mu, \mathscr{H}$ is almost-disjoint, if $|A\cap B|<\mu$ for $A, B\in\mathscr{H}$, $A\neq B$. GCH denotes the generalized continuum hypothesis, $V=L$ the axiom of constructibility (see [11]).

General statements

PROPOSITION 1. (a) *If $\mathscr{H}$ is a system of sets, $|\mathscr{H}|\leqq\mu=|H|$ for $H\in\mathscr{H}$, then $\mathscr{H}$ has the large subset property.*

(b) *If $\mathscr{H}$ is an almost-disjoint system of sets, $|\mathscr{H}|\leqq\mu=|H|$ for $H\in\mathscr{H}$ and μ is regular, then $\mathscr{H}$ is sparse.*

(c) *The statement in* (b) *is not true for any singular μ.*

PROOF. (a) This is Bernstein's theorem [3].

(b) If $\mathscr{H}=\{H_\alpha: \alpha<\varkappa\}$, $\varkappa\leqq\mu$, choose

$$f(H_\alpha) = \bigcup_{\beta<\alpha} (H_\alpha \cap H_\beta).$$

(c) Put $\tau=\mathrm{cf}(\varkappa)$, assume that $\varkappa=\sup_{\xi<\tau}\varkappa_\xi$. The ground set will be $\tau^+\times\varkappa$. If $\alpha<\tau^+$ is limit with $\mathrm{cf}(\alpha)=\tau$, choose a sequence $\langle f_\xi(\alpha): \xi<\tau\rangle$ converging to α, and put $H_\alpha=\bigcup_{\xi<\tau} f_\xi(\alpha)\times[\varkappa_\xi, \varkappa_{\xi+1})$. Clearly $\{H_\alpha: \alpha<\tau^+, \mathrm{cf}(\alpha)=\tau\}$ is almost-disjoint. Assume that $g(H_\alpha)\in[H_\alpha]^{<\varkappa}$ holds for every appropriate α. Let us define $X_\xi=\{\alpha<\tau^+: \mathrm{cf}(\alpha)=\tau,\ |g(H_\alpha)|\leqq\varkappa_\xi\}$. As $\mathrm{cf}(\varkappa)=\tau<\tau^+$ there is a $\xi<\tau$ such that X_ξ is stationary. f_ξ is regressive on X_ξ, so there are α, α' with $\alpha, \alpha'\in X_\xi$ and $f_\xi(\alpha)=f_\xi(\alpha')=\gamma$. Then $\gamma\times[\varkappa_\xi, \varkappa_{\xi+1})\subseteqq H_\alpha\cap H_{\alpha'}$, and $g(H_\alpha)\cup g(H_{\alpha'})$ can not cover this set.

In connection with the last counter-example let me note that in [7] a similar technique is used to construct an almost-disjoint system of sets of singular cardinality without strong property B. As this proof is not easy to read and further progress was made concerning large chromatic almost-disjoint families, we involve a variant of that result.

PROPOSITION 2. *Assume that* $\varkappa=\mathrm{cf}(\mu)<\mu$, *then there exists an almost-disjoint system* $\mathscr{H}$ *of sets of cardinality* μ, *such that there does not exist a set* X *with* $1\leqq|X\cap H|<\mu$ *for every* $H\in\mathscr{H}$, *i.e.* $\mathscr{H}$ *does not possess strong property* B, $|\mathscr{H}|=2^\varkappa\eta$.

PROOF. Choose an almost-disjoint system $\mathscr{G}$ of sets of cardinality $\varkappa=\mathrm{cf}(\mu)$ with chromatic number $>\varkappa$, that is, if we colour the ground set S with $\varkappa$ colours there always exists a monochromatic member of $\mathscr{G}$. This can be done by a result of G. Elekes and G. Hoffmann [6] (a full description of the proof for every $\varkappa$ is given in [12]). This is the point changed in the proof, the original proof used a result of Hajnal [8] stating that large chromatic almost-disjoint systems exist under GCH. Notice that by [12] $|\mathscr{G}|$ can be made $\leqq2^\varkappa$.

Let us denote $\mathscr{G}=\{G_\alpha: \alpha<|\mathscr{G}|\}$, $\mu=\sup_{\xi<\varkappa}\mu_\xi$, $G_\alpha=\{g_\alpha(\xi): \xi<\varkappa\}$. The ground set of $\mathscr{H}$ will be $\mu^2\times S$, the members of $\mathscr{H}$ are the sets of form either $\mu^2\times\{x\}$ (if $x\in S$) or $T_{\alpha,\delta}=\bigcup\{[\mu\delta+\mu_\xi, \mu\delta+\mu_{\xi+1})\times g_\alpha(\xi): \xi<\varkappa\}$ (here $\delta<\mu$ and $\alpha<|\mathscr{G}|$).

First we prove that $\mathscr{H}$ is almost-disjoint. Clearly $\mu\times\{x\}\cap\mu\times\{y\}=\emptyset$ if $x\neq y$, $|\mu\times\{x\}\cap T_{\alpha,\delta}|=\mu_{\xi+1}$ (or 0) if $x=g_\alpha(\xi)$ (or $x\notin G_\alpha$); $|T_{\alpha,\delta}\cap T_{\alpha,\delta'}|=0$ and $|T_{\alpha,\delta}\cap T_{\alpha',\delta'}|\leqq\mu_{\xi+1}$ if $G_\alpha\cap G_{\alpha'}$ is covered by $\{g_\alpha(\zeta): \zeta<\xi\}$.

Next, assume that X is a subset of $\mu^2\times S$, meeting each member of $\mathscr{H}$ in less than μ points. As $|X\cap\mu^2\times\{x\}|<\mu$ for every $x\in S$, we can colour S by $\varkappa$ colours: x is coloured by ξ if $|X\cap\mu^2\times\{x\}|<\mu_\xi$. There is a monochromatic set G_α, and $T_{\alpha,\delta}$, $\delta<\mu$ are *disjoint* subsets of $\mu^2\times G_\alpha$, so one of them is disjoint from X, as $|X\cap(\mu^2\times G_\alpha)|<\mu$.

PROPOSITION 3. *If* $\mathscr{H}$ *is a system of sets each of cardinality* μ, *then* $\mathscr{H}$ *has the large subset property if and only if the following holds: for every* $g: \mathscr{H}\to[\bigcup\mathscr{H}]^{<\mu}$ *with* $g(H)\in[H]^{<\mu}$ *the system* $\{H-g(H) : H\in\mathscr{H}\}$ *has the transversal property.*

PROOF. Clearly, if f states the large subset property and g is given as above, we can select an element $t(H)$ from $f(H)-g(H)$. t proves the transversal property.

On the other hand, if $\mathscr{H}$ has the property described above, we can inductively define for $\alpha<\mu, f_\alpha(H)$ so that $f_\alpha(H)$ is a transversal of $\{H-\{f_\beta(H): \xi<\alpha\}: H\in\mathscr{H}\}$. That is, $f_\alpha(H)\neq f_\alpha(H')$ if $H\neq H'$ and $f_\alpha(H)\neq f_\beta(H)$ if $\beta<\alpha$. Let us define a graph G as follows: the vertex set is $\mathscr{H}$ and H, H' are connected if and only if there are $\alpha, \beta<\mu$ with $f_\alpha(H)=f_\beta(H')$. Observe that $\alpha=\beta$ is impossible and for a given pair $\langle\alpha, \beta\rangle$ and given H there is at most one appropriate H'. From this it follows that in G, every vertex has a degree at most μ, and we can deduce that every (connected) component of G has cardinality at most μ. Assume that $\mathscr{K}$ is the set of the components of G. If $C\in\mathscr{K}$ $|C|\leqq\mu$ and so by Proposition 1 the system $\{\{f_\alpha(H): \alpha<\mu\}: H\in C\}$ has the large subset property: assume that f_C witnesses this. Having defined an f_C for every C, let us define f as the union of them. f will witness the large subset property for $\mathscr{H}$: if $H, H'\in\mathscr{H}$ and $H, H'\in C$ by the choice of f_C, if $H\in C$, $H'\in C'$ with $C\neq C'$, $\{f_\alpha(H): \alpha<\mu\}$ and $\{f_\alpha(H'): \alpha<\mu\}$ are disjoint, so are $f_C(H)$ and $f_{C'}(H')$.

PROPOSITION 4. *Assume that $\mathscr{H}$ is a system of sets of cardinality μ, $|\mathscr{H}|=\lambda>\mu$ is singular, and every subsystem of smaller cardinality has the large subset property. Then so has $\mathscr{H}$.*

PROOF. By Proposition 3 we have to prove, that if $\mathscr{H}$ is given as above and g is a function with $g(H)\in[H]^{<\mu}$ for $H\in\mathscr{H}$, then $\mathscr{G}=\{H-g(H): H\in\mathscr{H}\}$ has the transversal property. By hypothesis, and the easier part of Proposition 3, every smaller subsystem of $\mathscr{G}$ has a transversal, so by the famous Shelah Singular Cardinal Compactness theorem (see [15], newer proofs were given subsequently in [2], [9]) $\mathscr{G}$ has a transversal.

PROPOSITION 5. *Assume that $\mathscr{H}$ is a system of sets of cardinality μ, $|\mathscr{H}|=\lambda>\mu$ is singular, and every subsystem of smaller cardinality is sparse. Then so is $\mathscr{H}$.*

PROOF. Assume that $\mathscr{H}=\{H_\alpha: \alpha<\lambda\}$. Choose a sequence $\{\lambda_\xi: \xi<\mathrm{cf}(\lambda)\}$ converging to λ. By hypothesis, $\{H_\alpha: \alpha<\lambda_\xi\}$ is sparse, choose an f_ξ witnessing this fact. We are going to define a graph G as follows: the ground set is λ, $\alpha<\beta<\lambda$ are connected in G if and only if there are $\xi, \eta<\mathrm{cf}(\lambda)$ such that $(H_\alpha-f_\xi(H_\alpha))\cap \cap(H_\beta-f_\eta(H_\beta))\neq\emptyset$. Observe, that necessarily $\xi\neq\eta$ and if α, ξ, η are fixed, the number of good β's is at most $|H_\alpha-f_\xi(H_\alpha)|=\mu$. So in G every vertex has a degree at most $\mu+\mathrm{cf}(\lambda)$. We can deduce again, that every connected component has size at most $\mu+\mathrm{cf}(\lambda)$. Let us denote by K the set of components in G. If $C\in K$ choose an f_C witnessing $\{H_\alpha: \alpha\in C\}$ is sparse. Let us define f as follows: if C is the (unique) component containing α and ξ is minimal with $\alpha<\lambda_\xi$, then $f(H_\alpha)=f_\xi(H)\cup f_C(H_\alpha)$. We prove that f is good. Assume that $\alpha<\beta<\lambda$ are given, $f(H_\alpha)=f_\xi(H_\alpha)\cup f_C(H_\alpha)$, $f(H_\beta)=f_\eta(H_\beta)\cup f_{C'}(H_\beta)$. If $C=C'$ or $\xi=\eta$, we are done. If neither possibility holds, α and β are not connected in G, so already $H_\alpha-f_\xi(H_\alpha)$ and $H_\beta-f_\eta(H_\beta)$ are disjoint.

PROPOSITION 6. *Assume that $\varkappa$ is regular, $\mathscr{H}=\{H_\alpha: \alpha<\varkappa\}$ is a system of sets of cardinality $\mu<\varkappa$. Assume moreover that every subfamily with smaller cardinality is sparse. Let us define $X=X(\mathscr{H})=\{\alpha<\varkappa$: there is a $\beta\geqq\alpha$ with $|H_\beta\cap \cap(\bigcup_{\xi<\alpha} H_\xi)|=\mu\}$. Then $\mathscr{H}$ is sparse if and only if $X(\mathscr{H})$ is not stationary.*

PROOF. Assume first that X is non-stationary, C is a closed, unbounded subset of $\varkappa$ with $0\in C\subseteq\varkappa-X$. Put $C=\{c_\xi\colon \xi<\varkappa\}$ in increasing order. By hypothesis, for fixed $\xi<\varkappa$ $\{H_\alpha\colon \alpha\in[c_\xi, c_{\xi+1})\}$ is sparse, so there is an f_ξ witnessing this, and moreover we can prescribe $f_\xi(H_\alpha)\supseteq H_\alpha\cap(\bigcup_{\tau<c_\xi} H_\tau)$, as $c_\xi\notin X$. Clearly $f=\bigcup_{\xi<\varkappa} f_\xi$ works.

On the other hand, assume that $\mathscr{H}$ is sparse and f witnesses this. Let us define for $\xi<\varkappa$ $g(\xi)=\sup\{\zeta\colon (H_\zeta-f(H_\zeta))\cap(\bigcup_{\tau<\xi} H_\tau)\neq\emptyset\}$.

As $\left|\bigcup_{\tau<\xi} H_\tau\right|\leqq\mu\cdot|\xi|<\varkappa$ and $\{H_\zeta-f(H_\zeta)\colon \zeta<\varkappa\}$ is disjointed, $g(\xi)<\varkappa$. Choose $C=\{\gamma\colon \text{if } \xi<\gamma \text{ then } g(\xi)<\gamma\}$. Let us denote by C' the limit points of C. We shall prove $X\cap C'=\emptyset$. If not, $\alpha\in X\cap C'$ and β witnesses $\alpha\in X$, there is a point $x\in(H_\beta-f(H_\beta))\cap(\bigcup_{\xi<\alpha} H_\xi)$, so $x\in H_\xi$ for a $\xi<\alpha$. By our definitions, $\xi<\alpha$, so $\xi+1, g(\xi+1)<\alpha$ and $\beta\leqq g(\xi+1)<\alpha$, a contradiction.

Proposition 7. *Assume μ is regular, $\mathscr{H}$ is a system of sets of cardinality μ. $\mathscr{H}$ is sparse if and only if the following two conditions hold:*

(a) *$\mathscr{H}$ is almost disjoint,*

(b) *$\mathscr{H}$ has the hereditary transversal property, that is, if h is a function with $h(H)\in[H]^\mu$ for $H\in\mathscr{H}$, then $\{h(H)\colon H\in\mathscr{H}\}$ has a transversal (has the large subset property).*

Proof. Clearly the conditions on transversal or large subset are equivalent. One direction of the proposition is also trivial. We are going to prove that if $\mathscr{H}$ fulfils (a) and (b) $\mathscr{H}$ is sparse, by induction on $\varkappa=|\mathscr{H}|$. If $\varkappa\leqq\mu$ $\mathscr{H}$ is sparse by (a) and Proposition 1 (b). If $\varkappa$ is singular we can use the inductive hypothesis and Proposition 5, as both (a) and (b) are hereditary for subsystems.

Assume $\varkappa=|\mathscr{H}|$ is regular, every subfamily of smaller cardinality is sparse, but $\mathscr{H}$ is not. Then, by Proposition 6, $X(\mathscr{H})$ is stationary. By the definition of $X(\mathscr{H})$, if $\alpha\in X(\mathscr{H})$ there is a $\beta(\alpha)$ with $\left|\bigcup_{\xi<\alpha} H_\xi\cap H_\beta\right|=\mu$. As $C=\{\gamma<\varkappa\colon \text{if } \alpha<\gamma \text{ then } \beta(\alpha)<\gamma\}$ is a closed unbounded set, $Y=X\cap C$ is stationary again.

Let us define h as follows: if $\beta<\varkappa$ is given and there is an α with $\alpha\in Y$, $\beta=\beta(\alpha)$ put $h(H_\beta)=(\bigcup_{\xi<\alpha} H_\xi)\cap H_\beta$. Clearly, α is unique. If no such α exists, $h(H_\beta)=H_\beta$. As $|h(H_\beta)|=\mu$ always holds, by (b) there is a transversal f for $\{h(H_\beta)\colon \beta<\varkappa\}$. For $\alpha\in Y$ there is a $g(\alpha)<\alpha$ with $f(h(H_{\beta(\alpha)}))\in H_{g(\alpha)}$. By Fodor's theorem, as $\varkappa\geqq\mu^+$, there are μ^+ $\alpha\in Y$ with the same $g(\alpha)$ which is impossible as $|H_{g(\alpha)}|=\mu$.

Compactness properties

In order to construct systems $\mathscr{H}$ of sets of cardinality $\mu=\mathrm{cf}(\mu)$ without the transversal property, but so that every smaller subfamily has a transversal the most direct way is to seek sets from $\{\alpha<\varkappa\colon \mathrm{cf}(\alpha)=\mu\}$ stationary in $\varkappa$, but nonstationary at every smaller ordinal. Clearly such a set exists if $\varkappa=\mu^+$ and for $\varkappa>\mu^+$, not weakly compact, Jensen constructed such sets from $V=L$. From this it is easy to construct a system $|\mathscr{H}|=\varkappa$ with $|H|=\mu$ if $H\in\mathscr{H}$, so that every smaller subfamily is sparse but $\mathscr{H}$ has no transversal. However, for $\varkappa=\mu^{++}$ and the transversal property (or the large subset property) there is an example by J. Truss (see

[14]) as follows: let us define for $\mu\leqq\alpha<\mu^+\leqq\beta<\mu^{++}$, $H_{\alpha,\beta}=\alpha\times\{\alpha,\beta\}$. Then every subfamily of $\mathscr{H}=\{H_{\alpha,\beta}\}$ of cardinality μ^+ has the large subset property while $\mathscr{H}$ has no transversal either. It is not the case when speaking of sparseness, we shall show that compactness results are available.

THEOREM 1.[1] *If $\mu<\varkappa$ are regular and $\varkappa$ is weakly compact ($\mu=\omega$ and $\varkappa$ is supercompact) then there is a generic extension in which $\varkappa=\mu^{++}$ and the following holds: GCH+if $\mathscr{H}$ is a system of cardinality $\varkappa$ (of arbitrary cardinality) which is not sparse, then there is a subsystem of cardinality μ^+ which is not sparse, either.*

PROOF. We shall use the Lévy-collapsing P making $\varkappa=\mu^{++}$. The elements of P are functions of cardinality μ, and P is μ^+-closed, i.e. if $p_\xi\in P$ is decreasing for $\xi<\mu$, there is a common extension of them. Assume $\varkappa$ is weakly compact, and $\mathscr{H}=\{H_\alpha: \alpha<\varkappa\}$ is a counterexample in $V[\mathscr{G}]$, the generic model.

By Proposition 6, $X=X(\mathscr{H})$ is stationary in $\varkappa$. We must be careful as Proposition 6 works only for stationary sets in cardinals. By well-known properties of P (see [1]) there is an $\alpha<\varkappa$ with the following properties: α is strongly inaccesible, if P_α, P^α is the set of functions from P with support $\alpha\times\mu$, $(\varkappa-\alpha)\times\mu$, respectively, $\langle H_\xi: \xi<\alpha\rangle$, $X\cap\alpha$ are equally defined by $\mathscr{G}\cap P_\alpha$ and $Y=X\cap\alpha$ is stationary in α (where α is the actual μ^{++}). Work in $V[\mathscr{G}\cap P_\alpha]$. Notice, that in $V[\mathscr{G}]$ there is a function f witnessing that $\langle H_\xi: \xi<\alpha\rangle$ is sparse in $V[\mathscr{G}]$. Choose a name $\underline{f}$ over P^α in $V[\mathscr{G}\cap P_\alpha]$. By induction we can define for every $\xi<\alpha$ an ordinal $\gamma_\xi<\alpha$ and a subset $\mathscr{K}_\xi\subseteq P^\alpha$ with $|\mathscr{K}_\xi|<\alpha$ and the following properties hold: if $p\in\mathscr{K}_\xi$, $x<\gamma_\xi$ then either $p\Vdash x\notin H_\beta-\underline{f}(H_\beta)$ for every β, or there is a $q\in\mathscr{K}_{\xi+1}$, $q\leqq p$, $\beta<\gamma_{\xi+1}$ with $q\Vdash x\in H_\beta-\underline{f}(H_\beta)$, moreover $\mathscr{K}_\xi$ is closed for decreasing limits of length $<\mu$. For limit ξ take unions. Put $C=\{\gamma_\xi: \xi<\alpha\}$, $\gamma_\xi\in Y\cap C'$. As Y is stationary, such γ_ξ does exist and it is easy to prove that if $X(\mathscr{H})$ is stationary, then those elements of it having cofinality μ constitute a set stationary, too. So we can assume that $\operatorname{cf}(\gamma_\xi)=\mu$. As $\gamma_\xi\in Y$, there is a $\beta\geqq\gamma_\xi$, $\beta<\alpha$ with $\big|H_\beta\cap\big(\bigcup_{\tau<\gamma_\xi} H_\tau\big)\big|=\mu$. Put $\xi=\lim_{\tau<\mu}\xi_\tau$, $H_\beta=\{x_\tau: \tau<\mu\}$, $x_\tau\in\gamma_{\xi_\tau}$. Choose p_τ as follows: p_τ $(\tau<\mu)$ is decreasing, $p_\tau\in\mathscr{K}_{\xi_\tau}$, p_τ forces either $x_\tau\notin H_\beta-f(H_\beta)$ for every β or $x_\tau\in H_{\beta_\tau}-f(H_{\beta_\tau})$ for $\beta_\tau<\gamma_{\xi_\tau}$. If p is the union of p_τ $(\tau<\mu)$, and $q\leqq p$ forces $x_\tau\notin f(H_\beta)$ for a certain $\tau<\mu$, then $q\Vdash x_\tau\in\big(H_{\beta_\tau}-f(H_{\beta_\tau})\big)\cap\big(H_\beta-f(H_\beta)\big)$, a contradiction.

In the second case, $\mu=\omega$, $\varkappa$ is supercompact. If we collapse $\varkappa$ to μ^{++}, and $\mathscr{H}=\{H_\alpha: \alpha<\lambda\}$ is a system in $V[\mathscr{G}]$, which is not sparse, and is of minimal cardinality, then λ is regular, and $X(\mathscr{H})$ is stationary in λ. We can equally assume that $X(\mathscr{H})$ has only points of cofinality μ.

As $\varkappa$ is supercompact, there is a normal ultrafilter on $P([\lambda]^{<\varkappa})$. Almost all $P\in[\lambda]^{<\varkappa}$ have the following properties: $\varkappa\cap P$ is inaccesible, $\operatorname{otp}(P)$ is regular, if $x\in P\cap H_x$ and the status of $x\in X$ is completely decided by $\mathscr{P}_{\varkappa\cap P}$, if $p\in\mathscr{P}_{\varkappa\cap P}$ forces $\big|H_\beta\cup\big(\bigcup_{\xi\in S} H_\xi\big)\big|=\mu$ then $\beta, S\in V[\mathscr{G}\cap\mathscr{P}_{\varkappa\cap P}]$.

So our system restricted to P has the exact properties described as above. That proof can be copied again, noting that, as $\mu=\omega$, no closure against decreasing sequence of conditions is needed.

[1] See the note at the end of the paper.

Miller's theorem revisited

In his paper [13] E. W. Miller proved that if $\mathscr{H}$ is a system of infinite sets, any two of its members intersect in at most n elements, then $\mathscr{H}$ has property B. P. Erdős and A. Hajnal [7] proved that if every member of $\mathscr{H}$ is countable then $\mathscr{H}$ has strong property B, i.e. there is a set X with $1 \leqq |X \cap H| < \omega$ for every $H \in \mathscr{H}$. We generalize both results.

THEOREM 2. [2] *Assume that $\mathscr{H}$ is a system of countable, infinite sets, $n<\omega$, $\mathscr{H}$ is almost disjoint, and $|\bigcap \{H: H \in \mathscr{H}'\}| < n$ if $\mathscr{H}' \subseteq \mathscr{H}$ and $|\mathscr{H}'| \geqq \aleph_1$. Then $\mathscr{H}$ is sparse.*

PROOF. We prove the statement by transfinite induction on $|\mathscr{H}|$. If $|\mathscr{H}| \leqq \aleph_0$, $\mathscr{H} = \{H_i: i<k \leqq \omega\}$. As $\mathscr{H}$ is almost-disjoint, $f(H_i) = \bigcup_{j<i} (H_j \cap H_i)$ $(i<k)$ will work.

If $|\mathscr{H}| = \varkappa > \aleph_0$, $\mathscr{H} = \{H_\alpha: \alpha < \varkappa\}$ and $n<\omega$ are given, let us define a tower $\langle X_\alpha: \alpha<\varkappa\rangle$ from subsets of $\bigcup \mathscr{H}$ with the following stipulations:

(i) $|X_\alpha| < \varkappa$,

(ii) $X_\alpha \subset X_\beta$ for $\alpha<\beta<\varkappa$,

(iii) if $|H \cap X_\alpha| \geqq n$ then $H \subseteq X_\alpha$ $(H \in \mathscr{H})$,

(iv) $\bigcup_{\alpha<\varkappa} X_\alpha = \bigcup \mathscr{H}$.

To get (i—iv) put $Y_{\alpha,0} = \bigcup \{H_\beta: \beta<\alpha\}$, $Y_{\alpha,k+1} = Y_{\alpha,k} \cup \bigcup \{H \in \mathscr{H}: |H \cap Y_{\alpha,k}| \geqq n\}$, $X_\alpha = \bigcup_{k<\omega} Y_{\alpha,k}$. As every n-tuple of $\bigcup \mathscr{H}$ is contained in only countably many elements of $\mathscr{H}$, we get $|Y_{\alpha,k+1}| \leqq \aleph_0 |Y_{\alpha,k}|^n$, so we have $|X_\alpha| \leqq \aleph_0 |\bigcup \{H_\beta: \beta<\alpha\}| \leqq \aleph_0 \cdot |\alpha| < \varkappa$. Next, let us define

$$T_\alpha = X_\alpha - \Big(\bigcup_{\beta<\alpha} X_\beta\Big), \quad \mathscr{H}_\alpha = \{H \in \mathscr{H}: |H \cap T_\alpha| = \aleph_0\}.$$

By (i—iv) $\mathscr{H} = \bigcup \mathscr{H}_\alpha$, and if $H \in \mathscr{H}_\alpha$, $H \subseteq X_\alpha$ and $g(H) = H \cap \big(\bigcup_{\beta<\alpha} X_\beta\big)$ has at most $n-1$ elements. By induction, as $\mathscr{G}_\alpha = \{H - g(H): H \in \mathscr{H}_\alpha\}$ $(\alpha<\varkappa)$ fulfils the assumptions of the theorem, and $|\mathscr{H}_\alpha| \leqq |X_\alpha|^n \cdot \aleph_1$, they are sparse, so there is an f_α witnessing this. Put $f(H) = g(H) \cup f_\alpha\big(H - g(H)\big)$ if $H \in \mathscr{H}_\alpha$. By our earlier remarks $f(H)$ is defined and finite for every $H \in \mathscr{H}_\alpha$. If $H, H' \in \mathscr{H}$, $H \neq H'$ and $H, H' \in \mathscr{H}_\alpha$ then $H \cap H' \subseteq g(H) \cup g(H') \cup f_\alpha\big(H - g(H)\big) \cup f_\alpha\big(H' - g(H')\big) = f(H) \cup f(H')$ surely holds, if $H \in \mathscr{H}_\alpha$, $H' \in \mathscr{H}_\beta$ with $\alpha<\beta$, $H \cap H' \subseteq X_\alpha \cap H' \subseteq g(H') \subseteq f(H') \subseteq f(H) \cup f(H')$, too. Q.E.D.

THEOREM 3. *If $\mathscr{H}$ is a system of infinite sets, $n<\omega$, and the intersection of uncountably many members of $\mathscr{H}$ has less than n elements, then there is an $f: \mathscr{H} \to \bigcup \mathscr{H}$ with $|f(H)| = |H|$, $f(H) \subseteq H$ for $H \in \mathscr{H}$, such that $\{f(H): H \in \mathscr{H}\}$ is disjointed. Hence, if $\mathscr{H}$ is uniform as well, $\mathscr{H}$ has the large subset property.*

PROOF. Put $\mathscr{H} = \{H_\alpha: \alpha<\varkappa\}$. Assume that $|H_\alpha| = \mu_\alpha$, $H_\alpha = \bigcup \{H_{\alpha,\xi}: \xi<\mu_\alpha\}$ where $|H_{\alpha,\xi}| = \aleph_0$, and $\{H_{\alpha,\xi}: \xi<\mu_\alpha\}$ is a disjoint system. We want to apply

[2] See the note at the end of the paper.

Theorem 2 to $\mathcal{G}=\{H_{\alpha,\xi}: \xi<\mu_\alpha, \alpha<\varkappa\}$, by it, $\mathcal{G}$ has a transversal g. $f(H_\alpha)=\{g(H_{\alpha,\xi}): \xi<\mu_\alpha\}$ $(\alpha<\varkappa)$ are large subsets for $\mathcal{H}$.

Notice that $\mathcal{G}$ is not almost-disjoint, so for the handling of the $\varkappa=\omega$ case we have to use Bernstein's lemma (Prop. 1), and then exactly the same proof as in Theorem 2.

THEOREM 4 (MA). *If $\mathcal{H}$ is a system of infinite sets, $n<\omega$, and $|\bigcap\{H: H\in\mathcal{H}'\}|<n$ if $\mathcal{H}'\subseteq\mathcal{H}$, $|\mathcal{H}'|\geqq 2^{\aleph_0}$ then $\mathcal{H}$ has property* B.

PROOF. Assume $\mathcal{H}=\{H_\alpha: \alpha<\varkappa\}$. Choose $H'_\alpha\in[H_\alpha]^{\aleph_0}$. It is enough to prove that $\mathcal{H}'=\{H'_\alpha: \alpha<\varkappa\}$ has property B. For $\varkappa<2^{\aleph_0}$ this is a well-known consequence of Martin's axiom. If $\varkappa\geqq 2^{\aleph_0}$ we define $Y_{\alpha,k}$ $(k<\omega)$ and X_α $(\alpha<\varkappa)$ as in the proof of Theorem 2. We want to show that $|X_\alpha|<\varkappa$ if $\alpha<\varkappa$. Clearly $|Y_{\alpha,0}|\leqq|\bigcup\{H'_\beta: \beta<\alpha\}|\leqq|\alpha|+\aleph_0<\varkappa$.

Assume first that $\varkappa=2^{\aleph_0}$. If $|Y_{\alpha,k}|<\varkappa$, $Y_{\alpha,k+1}$ sets, each of cardinality $<2^{\aleph_0}$, so as MA holds, $2^{\aleph_0}$ is regular, $|Y_{\alpha,k+1}|<2^{\aleph_0}$. By König's theorem, $|X_\alpha|\leqq\sum_{k<\omega}|Y_{\alpha,k}|<2^{\aleph_0}=\varkappa$. If $\varkappa>2^{\aleph_0}$, $|Y_{\alpha,k+1}|\leqq|Y_{\alpha,k}|^n\cdot 2^{\aleph_0}$ and $|X_\alpha|<|\alpha|\cdot 2^{\aleph_0}<\varkappa$.

THEOREM 5. *Let $\mathcal{H}$ be a system of sets of cardinality μ (μ is regular). Assume that $\mathcal{H}$ is almost-disjoint and the intersection of μ^+ members of the system has power $<\tau$. Then $\mathcal{H}$ is sparse, assuming either*

(a) $\mu\geqq\tau^{++}$ *and* GCH *holds, or*

(b) $\mu\geqq\tau^+$, GCH *and* $|\mathcal{H}|\leqq\mu^{+\omega}$ *holds, or*

(c) $\mu\geqq\tau^+$ *and* $V=L$ *holds.*

Let me note, that property B, in case (c), was proved first by Donder [4].

PROOF. We can give an inductive proof, as in Theorem 2. For $|\mathcal{H}|\leqq\mu$ the statement is true. Assume $|\mathcal{H}|=\varkappa>\mu$. The construction of Theorem 2 can be copied out, the only problem is when there are sets $S\subseteq\varkappa$ with $|S|<\varkappa$ and $|S|^\tau=\varkappa$. As GCH holds, this is possible only if $\varkappa=\lambda^+$ with $\mathrm{cf}(\lambda)\leqq\tau$.

In case (a), however it is true that for a set of cardinality λ a system of sets of cardinality τ^+ such that every set of cardinality τ is contained in only μ sets, has cardinality at most λ. For a proof assume that $\mathcal{H}\subseteq[\lambda]^{\tau^+}$ is as described, and $\lambda=\sup\{\lambda_\xi: \xi<\mathrm{cf}(\lambda)\}$, for every $H\in\mathcal{H}$ there is a $\xi<\mathrm{cf}(\lambda)$ with $|H\cap\lambda_\xi|=\tau^+$ as $\mathrm{cf}(\lambda)\leqq\tau$. For a given $\xi<\mathrm{cf}(\lambda)$ the set of H's as described above can be at most $2^{\lambda_\xi}\mu$ as $\lambda_\xi\geqq\tau^+$. So $|\mathcal{H}|\leqq\mu\sum_{\xi<\mathrm{cf}(\lambda)}2^{\lambda_\xi}=\lambda$. This observation enables us to define X_α $(\alpha<\varkappa)$ with

(i) $\bigcup X_\alpha=\bigcup\mathcal{H}$,

(ii) $X_\alpha\subset X_\beta$,

(iii) $|X_\alpha|<\varkappa$,

(iv) if $|H\cap X_\alpha|\geqq\tau^+$ then $H\subseteq X_\alpha$.

We can finish the proof exactly as in Theorem 2. For case (b), $\varkappa<\mu^{+\omega}$ we can prove the result without any problem, Proposition 4 gives the result for $\varkappa=\mu^{+\omega}$.

For handling case (c) we can assume that $\bigcup\mathcal{H}=\mu=\tau^+$, $\varkappa=\lambda^+$, $\mathrm{cf}(\lambda)\leqq\tau$ and $\square_\lambda$ holds. $\square_\lambda$ denotes the following axiom (introduced by Jensen [11]): for every $\alpha<\lambda^+$, limit there is a subset $C_\alpha\subseteq\alpha$, with

(a) C_α is closed unbounded in α,

(b) otp (order-type) $C_\alpha \leqq \lambda$,

(c) if γ is a limit point of C_α, then $C_\gamma = C_\alpha \cap \gamma$. Assume that $0 \in C_\alpha$ for every limit α, and $\langle(\xi, \alpha): \xi < \text{otp}(\alpha)\rangle$ is an increasing enumeration of C_α. For every $\alpha < \beta < \lambda^+$ let us fix $[\alpha, \beta) = \bigcup_{\xi < \text{cf}(\lambda)} S(\alpha, \beta, \xi)$ with $|S(\alpha, \beta, \xi)| \leqq \lambda_\xi$ where $\langle \lambda_\xi : \xi < \text{cf}(\lambda)\rangle$ is a fixed sequence with $\sum \lambda_\xi = \lambda$. For $\alpha < \lambda^+$, limit let us define (for $\xi < \text{cf}(\lambda)$)

$$T_\xi^\alpha = \bigcup_{\eta < \text{otp}\, C_\alpha} S(c(\eta, \alpha), c(\eta+1, \alpha), \xi).$$

Now we are able to define a sequence $\{d_\gamma : \gamma < \lambda^+\} \subseteqq \lambda^+$ with

(i) d_γ is increasing,

(ii) if $|H \cap d_\gamma| = \mu$ then $H \subseteqq d_\gamma$ for $H \in \mathcal{H}$.

If $\alpha < \lambda^+$ is given, as $\alpha = \bigcup_{\xi < \text{cf}(\lambda)} T_\xi^\alpha$, $|H \cap \alpha| = \tau^+$ holds and α is minimal with this property, then $\text{cf}(\alpha) = \tau^+$ so for $\xi < \text{cf}(\lambda)$, $|T_\xi^\alpha| \leqq \lambda_\xi \cdot \tau^+ < \lambda$, and there is a $\xi < \text{cf}(\lambda)$ with $|T_\xi^\alpha \cap H| = \tau^+$. So if α is given, then $f(\alpha) = \sup\{\sup H :$ there is a ξ with $|H \cap T_\xi^\alpha| \geqq \tau\}$ is defined and we just now proved that $f(\alpha) < \lambda^+$ whenever $\text{otp}(\alpha) \neq \lambda$ (this is the only case when $|T_\xi^\alpha| < \lambda$ is not guaranteed). From this we can inductively define $\alpha_0 = \alpha$, $\alpha_{\beta+1} = f(\alpha_\beta)$, $\alpha_\eta = \sup\{\alpha_\beta : \beta < \eta\}$ if η is limit. The point is, that α_{τ^+} has the property that if $|H \cap \alpha_{\tau^+}| = \tau^+$ then $H \subseteqq \alpha_{\tau^+}$. For the proof of this fact, assume $|H \cap \alpha_{\tau^+}| = \tau^+$. As $\text{cf}(\alpha_{\tau^+}) = \tau^+$, there is a $\xi < \text{cf}(\lambda)$ with $|H \cap T_\xi^{\alpha_{\tau^+}}| \geqq \tau$. If γ is an element of $C_{\alpha_{\tau^+}} \cap \{\alpha_\beta : \beta < \tau^+\}$ large enough, $|H \cap T_\xi^\gamma| \geqq \tau$ so H is included in $\alpha_{\gamma+1}$.

This proves that for every $\alpha < \lambda^+$ there is a $g(\alpha) > \alpha$ with $g(\alpha) < \lambda^+$, limit and if $|H \cap g(\alpha)| = \tau^+$ or even there is a $\xi < \text{cf}(\lambda)$ with $|H \cap T_\xi^{g(\alpha)}| \geqq \tau$ then $H \subseteqq g(\alpha)$.

From this we can define a closed unbounded $C \subseteqq \lambda^+$ with the property: if $H \in \mathcal{H}$ then there is a $\gamma \in C$ with $|\gamma \cap H| = \tau$, $H \subseteqq \gamma'$ where γ' is the next element of C. Using this "tower" we can finish the proof.

Added in proof (December 19, 1983). The author has found a proof for the unrestricted case of Theorem 1 if $\mu > \omega$ is regular. This result, along with some substantial generalizations of Theorems 2, 3, and 5 will soon be published.

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EÖTVÖS LORÁND UNIVERSITY
MATHEMATICAL INSTITUTE
BUDAPEST, MÚZEUM KRT. 6-8.
H—1088

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ON GENERALIZED UNIQUENESS THEOREMS FOR WALSH SERIES

K. YONEDA (Sakai)

Introduction

Let

$$\mu \sim \sum_{k=0}^{\infty} \hat{\mu}(k)\, w_k(x) \tag{1}$$

be a *Walsh series.*

Vilenkin [6] proved that *the empty set is a set of uniqueness in the classical sense.* Fine [2] and Shneider [3] generalized the result just cited and established independently that *a countable set is a set of uniqueness in the classical sense.* Crittenden and Shapiro [1] generalized their result such as *if*

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) = 0$$

except on a countable set and

$$\lim_{n\to\infty} 1/2^n \sum_{k=0}^{2^n-1} \hat{\mu}(k)\, w_k(x) = 0$$

everywhere then $\hat{\mu}(k)=0$ *for all* k. From an other point of view different from one of Crittenden and Shapiro, Skvortsov [5] gave a generalization such as *if some* $\{S_{2^{n_i}}(x)\}_i$, $n_1<n_2<\ldots$, *of partial sums of* (1) *such that* $\lim_{k\to\infty} \hat{\mu}(k)=0$ *converges to zero everywhere except perhaps on a countable set, then* $\hat{\mu}(k)=0$ *for all* k.

By Skvortsov's theorem in [4], it follows that *if a Walsh series* (1) *satisfies*

$$\liminf_{n\to\infty} \sum_{k=1}^{2^n-1} \hat{\mu}(k)\, w_k(x) \leqq 0 \leqq \limsup_{n\to\infty} \sum_{k=0}^{2^n-1} \hat{\mu}(k)\, w_k(x) \tag{2}$$

except perhaps on a countable set and $\lim_{k\to\infty} \hat{\mu}(k)=0$, *then* $\hat{\mu}(k)=0$ *for all* k. Wade [7] proved that *if a Walsh series* (1) *satisfies* (2) *everywhere and*

$$\liminf_{n\to\infty} 1/2^n \sum_{k=0}^{2^n-1} \hat{\mu}(k)\, w_k(x) \leqq 0 \leqq \limsup_{n\to\infty} 1/2^n \sum_{k=0}^{2^n-1} \hat{\mu}(k)\, w_k(x)$$

except on a certain countable set, then $\hat{\mu}(k)=0$ *for all* k.

On the other hand, the author [10] proved that *a set of Haar measure zero is a set of uniqueness for the class of Walsh series* (1) *satisfying*

$$\sup_n \left| \sum_{k=0}^{2^n-1} \hat{\mu}(k)\, w_k(x) \right| < \infty \quad \textit{everywhere.} \tag{3}$$

Wade and the author [8] extended the result just cited for the group of integers of a p-series field.

The purpose of this paper is to extend the Crittenden—Shapiro and Skvortsov theorems, Skvortsov and Wade theorems and the author's theorem.

When $\mathscr{B}$ is a certain class of Walsh series, a set E is said to be a *set of generalized uniqueness of the first kind for* $\mathscr{B}$, or a *U_1-set for* $\mathscr{B}$, if each Walsh series $\mu\in\mathscr{B}$ which satisfies

$$\liminf_{n\to\infty}\left|\sum_{k=0}^{2^n-1}\hat{\mu}(k)\,w_k(x)\right| = 0 \quad \text{except on} \quad E \tag{4}$$

vanishes identically. A set which is not U_1 is called an *M_1-set for* $\mathscr{B}$. And a set E is said to be a *set of generalized uniqueness of the second kind for* $\mathscr{B}$, or a *U_2-set for* $\mathscr{B}$, if each Walsh series $\mu\in\mathscr{B}$ which satisfies (2) except on E vanishes identically. A set which is not U_2 is called an *M_2-set for* $\mathscr{B}$.

The main theorems of this paper are as follows.

THEOREM 1. *A countable set is a U_1-set for the class of Walsh series* (1) *satisfying*

$$\liminf_{n\to\infty}\left|1/2^n\sum_{k=0}^{2^n-1}\hat{\mu}(k)\,w_k(x)\right| = 0 \quad \textit{everywhere.} \tag{5}$$

THEOREM 2. *A countable set is a U_2-set for the class of Walsh series* (1) *satisfying* (5) *everywhere.*

THEOREM 3. *A set of Haar measure zero is a U_1-set for the class of Walsh series* (1) *satisfying* (3) *everywhere except perhaps on a countable set and* (5) *everywhere.*

Preliminaries

The *dyadic group* [2] is the set of all $0-1$ series $(t_1, t_2, \ldots)$. For convenience, we shall identify $(t_1, t_2, \ldots)$, $\lim_{k\to\infty} t_k\neq 1$, with $\left(\sum_{k=1}^{\infty} t_k/2^k\right)$ and when $\lim_{k\to\infty} t_k=1$, we shall write as $(t_1, t_2, \ldots)=\left(\sum_{k=1}^{\infty} t_k/2^k\right)^-$. A *dyadic interval of rank n*, $[p/2^n, (p+1)^-/2^n]$, is the set of all $(t_1, t_2, \ldots)$ such that $\sum_{k=1}^{n} t_k/2^k=p/2^n$. Obviously $[0, 1^-]$ denotes the dyadic group.

A *dyadic measure* or a *quasi measure* m is a set function which satisfies

$$m[p/2^n, (p+1)^-/2^n] = m[2p/2^{n+1}, (2p+1)^-/2^{n+1}]+m[(2p+1)/2^{n+1}, (2p+2)^-/2^{n+1}]$$

for $n=0, 1, \ldots$ and $p=0, 1, \ldots, 2^n-1$. It is already known that *each Walsh series is a Walsh—Fourier series of some dyadic measure and each dyadic measure has its Walsh—Fourier series.* We shall write the *dyadic measure associated with* μ by m_μ. The relation between m_μ and its k-th *Walsh—Fourier coefficient* $\hat{\mu}(k)$ is as follows: for $k<2^n$,

$$\hat{\mu}(k) = \sum_{p=0}^{2^n-1} m_\mu[p/2^n, (p+1)^-/2^n]\,w_k(p/2^n).$$

It is easy to see that *a Walsh series* μ *is the null series if and only if* $m_\mu(I)=0$ *for all dyadic intervals* I. In this paper I and I' are dyadic intervals.

For details of dyadic measures and the dyadic group we refer the reader to [8], [9] and [10].

Proof of Theorem 1

In order to prove Theorem 1 we need three lemmas.

LEMMA 1. *The empty set is a* U_1*-set for the class of all Walsh series.*

PROOF. Let m_μ be the dyadic measure associated with μ. Then it follows that

$$|m_\mu[0,1^-]| \leqq |m_\mu[0,1^-/2]|+|m_\mu[1/2,1^-]| \leqq$$
$$\leqq 2\max\{|m_\mu[p/2,(p+1)^-/2]|:\ p=0,1\} = 2\,|m_\mu[p_1/2,(p_1+1)^-/2]|$$

where $p_1=0$ or 1.

Continuing in this way we obtain a sequence of dyadic intervals $\{[p_n/2^n, (p_n+1)^-/2^n]\}_n$ such that

$$[0,1^-] \supseteqq [p_1/2,(p_1+1)^-/2] \supseteqq \ldots \supseteqq [p_n/2^n,(p_n+1)^-/2^n] \supseteqq \ldots \tag{6}$$
$$\ldots \supseteqq |m_\mu[0,1^-]| \leqq 2\,|m_\mu[p_1/2,(p_1+1)^-/2]| \leqq \ldots \leqq 2^n\,|m_\mu[p_n/2^n,(p_n+1)^-/2^n]| \leqq \ldots.$$

Each dyadic interval is a closed and open set. Then we have $[p_n/2^n, (p_n+1)^-/2^n]= =I_n(x_0)$ where $I_n(x)$ *is the dyadic interval of rank* n *containing* x and

$$\bigcap_{n=1}^{\infty} [p_n/2^n,(p_n+1)^-/2^n] = \{x_0\}.$$

From the hypothesis there exists a sequence of integers $\{n_k\}_k$ tending to infinity such that

$$\lim_{k\to\infty} 2^{n_k}\left|m_\mu\big(I_{n_k}(x_0)\big)\right| = 0.$$

Therefore it follows that $m_\mu[0,1^-]=0$.

In the same way we obtain that $m_\mu(I)=0$ for all I. This completes the proof.

Lemma 1 implies Vilenkin's theorem in the dyadic case.

LEMMA 2. *A set of one point is a* U_1*-set for the class of Walsh series* (1) *satisfying* (5) *everywhere.*

PROOF. Let x_0 be the point where (4) does not hold. If $x_0\notin I$, then by Lemma 1 $m_\mu(I')=0$ for $I'\subset I$. Hence for $0\leqq k<2^n$ we have

$$\hat\mu(k) = \sum_{p=0}^{2^n-1} m_\mu[p/2^n,(p+1)^-/2^n]\,w_k(p/2^n) = m_\mu\big(I_n(x_0)\big)\,w_k(x_0) =$$
$$= \sum_{p=0}^{2^n-1} m_\mu[p/2^n,(p+1)^-/2^n]\,w_k(x_0) = m_\mu[0,1^-]\,w_k(x_0).$$

On the other hand it is obvious that

$$m_\mu[0, 1^-] = m_\mu(I_n(x_0)) = 1/2^n \sum_{k=0}^{2^n-1} \hat{\mu}(k)\, w_k(x_0).$$

Making use of (6), we have $m_\mu[0, 1^-]=0$. Lemma 2 is proved.

REMARK. $\sum_{k=0}^{\infty} w_k(x)$ *satisfies*

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} w_k(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \infty & \text{for } x = 0 \end{cases}$$

and (5) *everywhere but at* $x=0$.

LEMMA 3. *When a Walsh series* (1) *satisfies* (5) *everywhere, set*

$$D = \left\{x\colon \liminf_{n\to\infty} \left|\sum_{k=0}^{2^n-1} \hat{\mu}(k)\, w_k(x)\right| > 0\right\}.$$

If D *is a countable set, then it is nowhere dense.*

PROOF. At first we shall prove that (i) *if* x *satisfies for some* n_0

$$2^n\left|m_\mu(I_n(x))\right| > \varepsilon \quad (n \geqq n_0), \tag{7}$$

then it follows that

$$2^n\left|m_\mu(I_n'(x))\right| > \varepsilon \tag{8}$$

for countably many n *where* $I_n'(x)=I_{n-1}(x)\setminus I_n(x)$.

Suppose (8) does not hold. Then there exists n_1 such that $2^n\left|m_\mu(I_n'(x))\right|\leqq\varepsilon$ for $n\geqq n_1$. Let N be a number satisfying $N=\max(n_0, n_1)$ and

$$\begin{cases} 2^N\left|m_\mu(I_N(x))\right| > \varepsilon \\ 2^n\left|m_\mu(I_n'(x))\right| \leqq \varepsilon \quad \text{for} \quad n \geqq N. \end{cases}$$

Set $\theta=\varepsilon-2^N\left|m_\mu(I_N(x))\right|>0$. Hence we have $\left|m_\mu(I_N(x))\right|=(\varepsilon+\theta)/2^N$. From the definition of dyadic measures, we have

$$\left|m_\mu(I_{N+1}(x))\right|+\left|m_\mu(I_{N+1}'(x))\right| \geqq (\varepsilon+\theta)/2^N.$$

Then

$$\left|m_\mu(I_{N+1}(x))\right| \geqq (\varepsilon+\theta)/2^N-\left|m_\mu(I_{N+1}'(x))\right| \geqq (\varepsilon+\theta)/2^N-\varepsilon/2^{N+1}.$$

Continuing in this way we obtain

$$\left|m_\mu(I_{N+k}(x))\right| \geqq (\varepsilon+\theta)/2^N-\varepsilon/2^{N+1}-\ldots-\varepsilon/2^{N+k} >$$

$$> (\varepsilon+\theta)/2^N-\varepsilon/2^{N+1}-\ldots-\varepsilon/2^{N+k}-\ldots = \theta/2^N.$$

It is clear that

$$\liminf_{n\to\infty} \left|m_\mu(I_n(x))\right| \geqq \theta/2^N > 0.$$

This contradicts (5).

From (5) it is obvious that (ii) *if*

$$|m_\mu[p_0/2^N, (p_0+1)^-/2^N]| > \eta,$$

then there exists $x\in[p_0/2^N, (p_0+1)^-/2^N]$ *such that* $2^n|m_\mu(I_n(x))|>2^N\eta$ *for* $n\geqq N$.

At last we shall prove the lemma. Suppose that D is dense and set $D=\{x_0, x_1, \dots\}$. Let x_0 satisfy

$$\liminf_{n\to\infty} 2^n|m_\mu(I_n(x_0))| > \varepsilon.$$

There exists n_0 such that

$$2^n|m_\mu(I_n(x_0))| > \varepsilon$$

for $n\geqq n_0$. From (i) there exists n_1 which satisfies $n_1>n_0$ and $2^{n_1}|m_\mu(I'_{n_1}(x_0))|>\varepsilon$. From (ii) there exists at least one x contained in $I'_{n_1}(x_0)$ and satisfying (7) for $n\geqq n_0$. Let x_{m_1} be the first x in D satisfying the above relation. Then $2^n|m_\mu(I_n(x_{m_1}))|>\varepsilon$ for $n\geqq n_0$.

Continuing in this way we obtain $\{x_{m_k}\}_k$ and $\{n_k\}_k$, $n_0<n_1<\dots$, satisfying

$$\dots\supseteqq I'_{n_k}(x_{m_{k-1}})\supseteqq I'_{n_{k+1}}(x_{m_k})\supseteqq\dots$$

where $x_{m_0}=x_1$ and

$$\begin{cases} 2^{n_k}|m_\mu(I'_{n_k}(x_{m_{k-1}}))| > \varepsilon \\ 2^n|m_\mu(I_n(x_{m_{k-1}}))| > \varepsilon \quad \text{for} \quad n\geqq n_0. \end{cases}$$

Set

$$\bigcap_{k=1}^{\infty} I'_{n_k}(x_{m_{k-1}}) = \{x'\}.$$

It is obvious that $2^n|m_\mu(I_n(x'))|>\varepsilon$ for $n\geqq n_0$ and $x'\neq x_n$ for all n. This contradicts the hypothesis.

On each dyadic interval, similarly the same result holds. This proves Lemma 3.

REMARK. *There exists a Walsh—Fourier series of a positive Radon measure which satisfies*

$$\sum_{k=2^n}^{2^{n+1}-1} |\hat\mu(k)|^2 = o(1)$$

as $n\to\infty$ *and*

$$\lim_{n\to\infty}\sum_{k=0}^{2^n-1} \hat\mu(x)\,w_k(x) = 0$$

everywhere except on a dense set of Haar measure zero [10].

PROOF OF THEOREM 1. By Lemma 3 D is nowhere dense. Set $D=D_0=D_1\cup D'_1$ where D_1 is the set of accumulating points of D_0 and D'_1 is the set of isolated points of D_0. For $x\in D'_1$ there exists I such that $I\cap D_0=\{x\}$. By Lemma 2 $m_\mu(I')=0$ for $I'\subset I$. Then (4) holds on D'_1.

Set analogously $D_1=D_2\cup D'_2$ as we did on D_0. Then (4) holds on D'_2.

In general set $D_\alpha=D_{\alpha+1}\cup D'_{\alpha+1}$ where α is an ordinal number such that $\alpha<\Omega$ and Ω is the first ordinal number of the third class. D is a countable set, then there exists an ordinal number $\beta<\Omega$ such that $D_\beta=\emptyset$. If follows that

$D=\bigcup_{\alpha<\beta} D'_\alpha$ and (4) holds on each D'_α. That is, (4) holds on D. Therefore (4) holds everywhere. Consequently by Lemma 1 Theorem 1 is proved.

The following results are corollaries to Lemma 1.

COROLLARY 1. *For a dyadic interval I, if* (4) *holds everywhere on I, then $m_\mu(I')=0$ for all $I'\subset I$.*

COROLLARY 2. *Let $\{\lambda_n\}_n$ be a sequence of positive numbers tending to infinity. When E is a closed set such that $1/\lambda_n\cdot N_n(E)=O(1)$ as $n\to\infty$ where $N_n(E)$ is the number of dyadic intervals of rank n containing some elements of E, E is a U_1-set for the class* of *of Walsh series* (1) *satisfying*

$$\liminf_{n\to\infty}\left|\lambda_n/2^n\sum_{k=0}^{2^n-1}\hat{\mu}(k)\,w_k(x)\right|=0 \quad \textit{everywhere.}$$

We can extend Lemma 1 for double Walsh series.

COROLLARY 3. *The empty set is a U_1-set for the class of all double Walsh series.*

Proof of Theorem 2

To prove Theorem 2 we need following lemmas.

LEMMA 4. *The empty set is a U_2-set for the class of all Walsh series.*

PROOF. Similarly to the proof of Lemma 1 we can easily find two sequences of dyadic intervals $\{[p_n/2^n,(p_n+1)^-/2^n]\}_n$ and $\{[q_n/2^n,(q_n+1)^-/2^n]\}_n$ such that

$$m_\mu[0,1^-]\leqq 2m_\mu[p_1/2,(p_1+1)^-/2]\leqq\ldots\leqq 2^n m_\mu[p_n/2^n,(p_n+1)^-/2^n]\leqq\ldots,$$

$$[0,1^-]\supseteqq[p_1/2,(p_1+1)^-/2]\supseteqq\ldots\supseteqq[p_n/2^n,(p_n+1)^-/2^n]\supseteqq\ldots,$$

$$m_\mu[0,1^-]\geqq 2m_\mu[q_1/2,(q_1+1)^-/2]\geqq\ldots\geqq 2^n m_\mu[q_n/2^n,(q_n+1)^-/2^n]\geqq\ldots,$$

$$[0,1^-]\supseteqq[q_1/2,(q_1+1)^-/2]\supseteqq\ldots\supseteqq[q_n/2^n,(q_n+1)^-/2^n]\supseteqq\ldots.$$

Set $\bigcap_{n=1}^{\infty}[p_n/2^n,\ (p_n+1)^-/2^n]=\{x_0\}$ and $\bigcap_{n=1}^{\infty}[q_n/2^n,\ (q_n+1)^-/2^n]=\{x_0'\}$. From (2) we have

$$m_\mu[0,1^-]\leqq\liminf_{n\to\infty}2^n m_\mu\big(I_n(x_0)\big)\leqq 0,\ \ m_\mu[0,1^-]\geqq\limsup_{n\to\infty}2^n m_\mu\big(I_n(x_0')\big)\geqq 0$$

where $I_n(x_0)=[p_n/2^n,(p_n+1)^-/2^n]$ and $I_n(x_0')=[q_n/2^n,(q_n+1)^-/2^n]$. Consequently we get $m_\mu[0,1^-]=0$.

Quite similarly we can prove that $m_\mu(I)=0$ for all I. This proves Lemma 4. We state two lemmas without proofs.

LEMMA 5. *A set of one point is a U_2-set for the class of Walsh series* (1) *satisfying* (5).

LEMMA 6. *When a Walsh series* (1) *satisfies* (5) *everywhere, set*

$$D_1 = \left\{x\colon \liminf_{n\to\infty} \sum_{k=0}^{2^n-1} \hat{\mu}(k)\, w_k(x) < 0\right\},\quad D_2 = \left\{x\colon \limsup_{n\to\infty} \sum_{k=0}^{2^n-1} \hat{\mu}(k)\, w_k(x) > 0\right\}.$$

If D_1 and D_2 are countable sets, then both of D_1 and D_2 are nowhere dense.

We can prove these lemmas similarly to Lemma 2 and Lemma 3 respectively. Theorem 2 is proved quite similarly to Theorem 1.

Proof of Theorem 3

To prove Theorem 3 we need the following three lemmas.

LEMMA 7. *A set of Haar measure zero is a U_1-set for the class of Walsh series* (1) *satisfying* (3) *everywhere.*

PROOF. Let $0<\eta<1$. From (4) and Egorov theorem, there exists a closed set E_1 such that

$$\operatorname{mes} E_1 > 1-\eta,\ \lim_{N\to\infty} \inf_{n\ge N} \left|\sum_{k=0}^{2^n-1} \hat{\mu}(k)\, w_k(x)\right| = 0 \quad \text{uniformly on } E_1.$$

For each $\varepsilon>0$, there exists an integer $N_1=N_1(\varepsilon)$ such that

$$\inf_{n\ge N_1} \left|\sum_{k=0}^{2^n-1} \hat{\mu}(k)\, w_k(x)\right| = \inf_{n\ge N_1} \left|2^n m_\mu(I_n(x))\right| < \varepsilon \quad \text{on } E_1.$$

For each $x\in E_1$, let $k_1(x)$ be an integer satisfying $k_1(x)\ge N_1$ and

$$2^{k_1(x)} \left|m_\mu(I_{k_1(x)}(x))\right| < \varepsilon.$$

Since

$$\bigcup_{x\in E_1} I_{k_1(x)}(x) \supseteq E_1,$$

there exist $x_1^{(1)}, x_2^{(1)}, \dots, x_{s_1}^{(1)}\in E_1$ satisfying

$$(9)\qquad I_{k_1(x_i^{(1)})}(x_i^{(1)}) \cap I_{k_1(x_j^{(1)})}(x_j^{(1)}) = \varnothing \quad (i\ne j),\qquad \sum_{i=1}^{s_1} 1/2^{k_1(x_i^{(1)})} > 1-\eta.$$

Let k_1 be the largest $k_1(x_i^{(1)})$ for $1\le i\le s_1$ and B_1 be the set of all p such that

$$[p/2^{k_1}, (p+1)^-/2^{k_1}] \cap \bigcup_{i=1}^{s_1} I_{k_1(x_i^{(1)})}(x_i^{(1)}) = \varnothing.$$

Then we get

$$|m_\mu[0, 1^-]| \le \sum_{i=1}^{s_1} \left|m_\mu\left(I_{k_1(x_i^{(1)})}(x_i^{(1)})\right)\right| + \sum_{p\in B_1} |m_\mu[p/2^{k_1}, (p+1)^-/2^{k_1}]| \le$$

$$\le \varepsilon \sum_{i=1}^{s_1} 1/2^{k_1(x_i^{(1)})} + \# B_1 |m_\mu[p_1/2^{k_1},\ (p+1)^-/2^{k_1}]|$$

where $\# B_1$ is the number of elements of B_1 and on B_1 $|m_\mu[p/2^{k_1}, (p+1)^-/2^{k_1}]|$ takes the maximum at p_1. From (9) we have $\# B_1 \cdot 1/2^{k_1} < \eta$. Thus we have

$$|m_\mu[0, 1^-]| \leqq \varepsilon + \eta \cdot 2^{k_1} |m_\mu[p_1/2^{k_1}, (p_1+1)^-/2^{k_1}]|.$$

Continuing in this way, there exists a sequence of dyadic intervals

$$\{[p_j/2^{k_j}, (p_j+1)^-/2^{k_j}]\}_{j=1}^\infty$$

such that

$$[p_1/2^{k_1}, (p_1+1)^-/2^{k_1}] \supsetneqq [p_2/2^{k_2}, (p_2+1)^-/2^{k_2}] \supsetneqq \dots$$

$$|m_\mu[p_j/2^{k_j}, (p_j+1)^-/2^{k_j}]| \leqq \varepsilon/2^{j+k_j} + \eta/2^{k_j} 2^{k_{j+1}} |m_\mu[p_{j+1}/2^{k_{j+1}}, (p_{j+1}+1)^-/2^{k_{j+1}}]|.$$

Thus we get

$$|m_\mu[0, 1^-]| \leqq \varepsilon + \varepsilon/2 + \dots + \varepsilon/2^{j-1} + \eta^j 2^{k_j} |m_\mu[p_j/2^{k_j}, (p_j+1)^-/2^{k_j}]|.$$

Set

$$\bigcap_{j=1}^\infty [p_j/2^{k_j}, (p_j+1)^-/2^{k_j}] = \{x_0\}.$$

Therefore

$$|m_\mu[0, 1^-]| \leqq \varepsilon + \varepsilon/2 + \dots + \varepsilon/2^{j-1} + \eta^j 2^{k_j} \big|m_\mu\big(I_{k_j}(x_0)\big)\big| < 2\varepsilon + \eta^j 2^{k_j} \big|m_\mu\big(I_{k_j}(x_0)\big)\big|.$$

From the hypothesis the second term of the last formula tends to zero as $j \to \infty$, so that we have $|m_\mu[0, 1^-]| < 2\varepsilon$. Consequently we have $m_\mu[0, 1^-] = 0$.

Similarly we can prove that $m_\mu(I) = 0$ for each dyadic interval I, that is, $\hat\mu(k) = 0$ for all k. This completes the proof of Lemma 7.

COROLLARY 4. *If a Walsh series* (1) *satisfies* (3) *everywhere on* I *and* (4) *almost everywhere on* I, *then* $m_\mu(I') = 0$ *for all* $I' \subset I$.

LEMMA 8. *If a Walsh series* (1) *satisfies* (4) *everywhere on* I *but* x_0 *and* (5) *everywhere, then* $m_\mu(I') = 0$ *for all* $I' \subset I$.

The proof of Lemma 8 depends on Lemma 2.

LEMMA 9. *Let* D *be the set of all* x *satisfying*

$$\sup_n \left| \sum_{k=0}^{2^n-1} \hat\mu(k) w_k(x) \right| = \infty.$$

If D *is a countable set, then it is nowhere dense.*

PROOF. Suppose that D is dense on $[0, 1^-]$. Set $D = \{x_1, x_2, \dots\}$ and $x_i = (t_1^{(i)}, t_2^{(2)}, \dots)$. Let n_1 be the first number n such that $2^n \big|m_\mu\big(I_n(x_1)\big)\big| > 1$. Let x_{i_1} be the first element of D contained in $I_{n_1}(x_1) \setminus \{x_1\}$. Therefore $x_{i_1} = (t_1^{(1)}, t_2^{(1)}, \dots, t_{n_1}^{(1)}, t_{n_1+1}^{(i_1)}, \dots)$. Let n_2 be the first number n satisfying

$$2^n \big|m_\mu\big(I_n(x_{i_1})\big)\big| > 2, \ (t_{n_1+1}^{(1)}, \dots, t_{n_2}^{(1)}) \neq \big(t_{n_1+1}^{(i_1)}, \dots, t_{n_2}^{(i_1)}\big)$$

and x_{i_1} be the first element of D contained in $I_{n_2}(x_{i_1}) \setminus \{x_{i_1}\}$.

Continue in this way and set

$$y_0 = (t_1^{(1)}, \dots, t_{n_1}^{(1)}, t_{n_1+1}^{(i_1)}, \dots, t_{n_2}^{(i_1)}, t_{n_2+1}^{(i_2)}, \dots).$$

Obviously we have $2^{n_k}|m_\mu(I_{n_k}(y_0))| > k$, that is,

$$\sup_n 2^n |m_\mu(I_n(y_0))| = \infty.$$

Then $y_0 \in D$. But this contradicts the hypothesis.

It is obvious that a similar result is valid on each dyadic interval.

The proof of Theorem 3 is quite similar to the proof of Theorem 1.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OSAKA PREFECTURE
SAKAI, OSAKA, JAPAN

Obviously, we have [illegible], that is,

[illegible]

[illegible] But this [illegible] the hypothesis [illegible] the proof of [illegible] quite similar to the proof of [illegible] theorem [illegible]

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[1] [illegible]

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SATURATION FOR BERNSTEIN TYPE RATIONAL FUNCTIONS

V. TOTIK (Szeged)

§ 1.

Soon after Bernstein's fundamental work on Bernstein polynomials ([5]) it has begun the extension of these to infinite intervals (see e.g. [7, p. 36]). However, uniform approximation by polynomials on infinite intervals cannot be expected, so it is natural to seek a modification where the range of the operator consists of rational functions. In [1] C. Balázs introduced the operators

$$R_n(f;\, x) = \frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} f\left(\frac{k}{b_n}\right)\binom{n}{k}(a_n x)^k \quad (x \geqq 0)$$

and proved that $R_n(f)$ converges to f uniformly on compact intervals under the conditions: $a_n = b_n/n \to 0$, $b_n \to \infty$, f bounded and continuous on $(0, \infty)$. In [2] weighted estimates were given in the case $a_n = n^{\beta-1}$, $b_n = n^\beta$, $0 < \beta \leqq 2/3$ and certain questions of the uniform convergence of $R_n(f)$ on $(0, \infty)$ were also treated.

The aim of the present paper is to settle the saturation properties of $R_n(f)$ and to prove a general convergence theorem for R_n-like rational functions.

First let us consider the saturation case. We can solve the saturation problem on intervals (a, ∞), $a > 0$ when $a_n = n^{\beta-1}$, $b_n = n^\beta$, $0 < \beta < 1$, i.e. a neighbourhood of zero is omitted. Before formulating our theorem let us see what sort of results can be expected. If f is smooth on the interval (a, b) then we can hope a good approximation by $R_n(f)$ only on subsets of (a, b) "far" from a or b since the values of f outside (a, b) can spoil the order of approximation near a or b. Conversely, if we know the order of $R_n(f) - f$ on (a, b) then, clearly, we can infer smoothness properties of f only on parts of (a, b) "far from the endpoints". These remarks justify the introduction of the following definition.

Let T be a property, I an interval. We shall write $\{f \mid T\}_I$ for the class of functions f having property T uniformly on I.

Definition. $\{R_n\}$ is said to be *saturated* on (a, b) with order $\{\Phi_n\}$ if $\sup_{x \in (a,b)} |R_n(f;\, x) - f(x)| = o(\Phi_n)$ implies that f is constant on (a, b) and there exists a function f_0 not identically constant on (a, b) such that $\sup_{x \in (a,b)} |R_n(f_0;\, x) - f_0(x)| = O(\Phi_n)$. We say that the *saturation class* is

$$S(R_n) = \{g \mid T\}^*_{(a,b)}$$

if for every $\varepsilon > 0$

(i) $$|R_n(f;\, x) - f(x)| \leqq K\Phi_n, \quad x \in (a, b), \; n = 1, 2, \ldots$$

implies

$$f \in \{g|T\}_{(a+\varepsilon,\, b-\varepsilon)},^{1}$$

(ii) $$f \in \{g|T\}_{(a,b)}$$

implies

$$|R_n(f;\, x)-f(x)| \leqq K\Phi_n, \quad x\in(a+\varepsilon,\, b-\varepsilon),\ n=1, 2, \dots.$$

After these let us consider the operators

$$R_n(f;\, x) = R_n(\beta;\, f;\, x) = \sum_{k=0}^{n} f\left(\frac{k}{n^\beta}\right) r_{n,k}(x)$$

where

$$r_{n,k}(x) = \binom{n}{k} \frac{(n^{\beta-1}x)^k}{(1+n^{\beta-1}x)^n} \equiv \binom{n}{k} \frac{(a_n x)^k}{(1+a_n x)^n} \quad (0 \leqq k \leqq n). \tag{1.1}$$

Throughout the paper the terms β, a_n and $r_{n,k}(x)$ will be used always in this sense. With the notations

$$\Delta_h^1(f;\, x) = f(x+h)-f(x),\ \Delta_h^2(f;\, x) = f(x+2h)-2f(x+h)+f(x) \quad (h \geqq 0)$$

we shall prove

THEOREM 1. *Let* $a>0$.

(i) *If* $\frac{2}{3} \leqq \beta < 1$ *then* $\{R_n\}$ *is saturated on* (a, ∞) *with order* $\{n^{\beta-1}\}$ *and saturation class* $S(R_n) = \{f\,|\,f$ *absolutely continuous,* $x^2 f'(x) = O(1)\}^*_{(a,b)}$.

(ii) *If* $\frac{1}{2} < \beta < \frac{2}{3}$ *then* $\{R_n\}$ *is saturated on* (a, ∞) *with order* $\{n^{\beta-1}\}$ *and saturation class*

$$S(R_n) = \{f\,|\,f \text{ abs. cont., } x^2 f'(x) = O(1),\ x^{\frac{1-\beta}{\beta}} \Delta_h^2(f;\, x) = O\big(h^{\frac{2(1-\beta)}{\beta}}\big)\}^*_{(a,\infty)}.$$

(iii) *If* $\beta = \frac{1}{2}$ *then the saturation order of* $\{R_n\}$ *on* (a, ∞) *is* $\{n^{-1/2}\}$ *and the saturation class is* $S(R_n) = \{f\,|\,f'$ *abs. cont.,* $x^2 f'(x) = O(1)$, $xf''(x) = O(1)\}^*_{(a,\infty)}$.

(iv) *If* $0 < \beta < \frac{1}{2}$ *then the saturation order of* $\{R_n\}$ *on* (a, ∞) *is* $\{n^{-\beta}\}$ *and the saturation class is*

$$S(R_n) = \{f\,|\,f' \text{ abs. cont., } f(\infty) = \lim_{t\to\infty} f(t) \text{ exists,}$$

$$x^{1/(2-3\beta)} f'(x) = O(1),\ x^{2\beta/(1-\beta)} \Delta_h^1(f;\, x) = O(h^{\beta/(1-\beta)}),$$

$$xf''(x) = O(1),\ f(\infty)-f(x) = O(x^{-\beta/(1-\beta)})\}^*_{(a,\infty)}.$$

By localization we shall obtain the saturation conditions on finite intervals. Here, however, a so called trivial class $T(R_n)$ can also arise. This consists of all

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[1] For $b=\infty$ let $b-\varepsilon=\infty$.

functions for which $R_n(f)-f=o(\Phi_n)$ ($\{\Phi_n\}$ is the saturation order), (see [10, p. 123]). We have

COROLLARY 1. *Let* $0<a<b<\infty$ *be any two numbers.*

(i) *If* $\frac{2}{3}\leqq\beta<1$ *then* $\{R_n\}$ *is saturated on* (a, b) *with order* $\{n^{\beta-1}\}$, *with trivial class*

$$T(R_n) = \{f|f \ \ const\}^*_{(a,b)} \tag{1.2}$$

and saturation class $S(R_n)=\{f|f\in\mathrm{Lip}\ 1\}^*_{(a,b)}$.

(ii) *If* $\frac{1}{2}<\beta<\frac{2}{3}$ *then* $\{R_n\}$ *is saturated on* (a, b) *with order* $\{n^{\beta-1}\}$, *with trivial class* (1.2) *and saturation class* $S(R_n)=\left\{f\,|\,f'\in\mathrm{Lip}\dfrac{2-3\beta}{\beta}\right\}^*_{(a,b)}$.

(iii) *If* $\beta=\frac{1}{2}$ *then* $\{R_n\}$ *is saturated on* (a, b) *with order* $\{n^{-1/2}\}$, *with trivial class*

$$T(R_n) = \Big\{f|f(x) = c+d\int_0^x e^{\tau^2}\,d\tau\Big\}^*_{(a,b)}$$

and saturation class

$$S(R_n) = \{f|f'\in\mathrm{Lip}\ 1\}^*_{(a,b)}. \tag{1.3}$$

(iv) *If* $0<\beta<\frac{1}{2}$ *then* $\{R_n\}$ *is saturated on* (a, b) *with order* $\{n^{-\beta}\}$, *with trivial class*

$$T(R_n) = \{f|f(x) = c+dx\}^*_{(a,b)}$$

and saturation class (1.3).

REMARKS. 1. At first glance the dependence of the saturation order on β may be surprising, namely with the decrease of β, i.e. with the increase of the distance between the nodes $\frac{k}{n^\beta}$, the saturation order decreases (!) for $\beta>\frac{1}{2}$ and increases for $\beta<\frac{1}{2}$. The explanation is the following: in the approximation the dominating factor for $\beta>\frac{1}{2}$ is the distance of x from the weight point of the weight system:

$$\left\{\text{the weight } r_{n,k}(x)=\binom{n}{k}\frac{(a_nx)^k}{(1+a_nx)^n} \text{ is put into the point } \frac{k}{n^\beta}\right\}$$

and this distance decreases together with β. On the other hand, for $\beta<\frac{1}{2}$ the dominating factor is the "second moment" $R_n\big((t-x)^2; x\big)$ which increases when $1/\beta$ does so.

2. The form of the saturation classes is also odd, since most "natural" operators have Lip 1 or $\{f\,|\,f'\in\mathrm{Lip}\ 1\}$ as their saturation class.

3. Our theorem shows that from the point of view of best possible approximation the optimal value of β is $\frac{1}{2}$ (compare [1, p. 124]).

4. It would be very interesting to solve the saturation problem around $x=0$.

5. Let

$$\gamma_n = \begin{cases} n^{\beta-1} & \text{for} \quad 1 > \beta \geqq 1/2 \\ n^{-\beta} & \text{for} \quad 0 < \beta < 1/2. \end{cases} \tag{1.4}$$

If $0<\alpha<1$ and $a>0$ then one may be interested in determining the function classes

$$R^{\alpha}_{(a,\infty)} = \{f | R_n(f;\, x) - f(x) = O(\gamma_n^{\alpha})\}_{(a,\infty)}$$

i.e. the conditions under which the non-optimal approximation order $\{\gamma_n^{\alpha}\}$ can be achieved (for related problems see e.g. [4, 8, 9]). We conjecture the following: Let us write $f(\infty) = \lim_{t\to\infty} f(t)$ and when the term $f(\infty)$ is used then it is also understood that f is a function for which the limit exists. With the convention accepted in Theorem 1 the following hold very likely:

1. For $\frac{2}{3} \leqq \beta < 1$

$$R^{\alpha}_{(a,\infty)} = \{f | x^{2\alpha} \Delta_h^1(f;\, x) = O(h^{\alpha});\ f(x) - f(\infty) = O(x^{-\alpha})\}^{*}_{(a,\infty)}.$$

2. For $\frac{1}{2} \leqq \beta < \frac{2}{3}$

$$R^{\alpha}_{(a,\infty)} = \{f | x^{\frac{\alpha(1-\beta)}{\beta}} \Delta_h^2(f;\, x) = O\big(h^{\frac{2\alpha(1-\beta)}{\beta}}\big);\ x^{2\alpha}\Delta_h^1(f;\, x) = O(h^{\alpha}),$$
$$f(x) - f(\infty) = O(x^{-\alpha})\}^{*}_{(a,\infty)}.$$

3. For $0 < \beta < \frac{1}{2}$

$$R^{\alpha}_{(a,\infty)} = \{f | x^{\alpha} \Delta_h^2(f;\, x) = O(h^{2\alpha}),\ x^{\frac{\alpha}{2-3\beta}} \Delta_h^1(f;\, x) = O(h^{\alpha}),$$
$$x^{\frac{2\alpha\beta}{1-\beta}} \Delta_h^1(f;\, x) = O\big(h^{\frac{\alpha\beta}{1-\beta}}\big);\ f(x) - f(\infty) = O\big(x^{-\frac{\beta}{1-\beta}}\big)\}^{*}_{(a,\infty)}.$$

The sufficiency of these conditions can be proved similarly as in Theorem 1 and also much of the necessity proof of Theorem 1 can be transferred to the above non-optimal case. One major difficulty is the proof of $x^{2\alpha}\Delta_h^1(f;\, x) = O(h^{\alpha})$ when $R_n(f) - f = O(n^{\alpha(\beta-1)})$, $\beta \geqq 1/2$.

After these we turn to the uniform approximation problem for Bernstein type operators. Let $a = \{a_n\}_1^{\infty}$ be a positive sequence and $B = (b_{n,k})_{0 \leqq k \leqq n\, (n=1,2,\dots)}$ a matrix with entries

$$0 \leqq b_{n,0} < b_{n,1} < \dots < b_{n,n}. \tag{1.5}$$

We shall consider the operators

$$R_n(f;\, x) = R_n(B;\, a;\, f;\, x) = \frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} f(b_{n,k}) \binom{n}{k} (a_n x)^k = \sum_{k=0}^{n} f(b_{n,k})\, r_{n,k}(x).$$

Let S be the set of functions $f\in C[0,\infty)$ for which $f(\infty)=\lim_{t\to\infty} f(t)$ exists. Clearly, since $\lim_{x\to\infty} R_n(B;a;f;x)=f(b_{n,n})$ exists, uniform approximation on $[0,\infty)$ by the operators R_n can be achieved only for functions belonging to S.

DEFINITION. We say that $\{R_n\}$ has the *approximation property* if $\lim_{n\to\infty} R_n(B;a;f;x)=f(x)$ uniformly on $[0,\infty)$ for every $f\in S$.

We have

THEOREM 2. *Let B, a and R_n be as above. The following assertions are equivalent:*

(i) *$\{R_n\}$ has the approximation property,*

(ii) *$\lim_{n\to\infty} R_n(f;x)=f(x)$ uniformly on compact subsets of $[0,\infty)$ for every continuous and bounded f,*

(iii) *$\lim_{n\to\infty} R_n(1/(1+t);x)=1/(1+x)$ for every rational $x\geqq 0$,*

(iv) *$\lim_{n\to\infty} b_{n,[na_n x/(1+a_n x)]}=x$ for every $x\geqq 0$.*

E.g. we obtain from (iv) that the operators

$$\frac{1}{(1+x)^n}\sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right)\binom{n}{k}x^k$$

have the approximation property (cf. [6]).

Note that (iii) is a very strong Korovkin type characterization of the approximation property (for an application see Corollary 5 below).

COROLLARY 2. *If $a=\{a_n\}$ is a given sequence of positive numbers then there is a matrix B such that $\{R_n(B;a)\}$ has the approximation property if and only if $\lim_{n\to\infty} na_n=\infty$ and $\lim_{n\to\infty} a_n/n=0$.*

COROLLARY 3. *If $a=\{a_n\}$ and $b=\{b_n\}$ are two positive sequences then*

$$\left\{R_n(a;b;f;x)=\frac{1}{(1+a_n x)^n}\sum_{k=0}^{n} f\left(\frac{k}{b_n}\right)(a_n x)^k\right\}$$

has the approximation property if and only if $na_n/b_n\to 1$, $a_n\to 0$ and $b_n\to\infty$ as $n\to\infty$.

COROLLARY 4. *Let $B=(b_{n,k})$ be a matrix with property (1.5), and let $k_n^-(x)$ and $k_n^+(x)$ be the number of the $b_{n,k}$ which satisfy $b_{n,k}\leqq x$ and $b_{n,k}>x$, respectively. Then there is a sequence $a=\{a_n\}$ of positive numbers such that $\{R_n(B;a)\}$ has the approximation property if and only if for all $x,y>0$*

$$\lim_{n\to\infty}\frac{k_n^-(x)}{k_n^+(x)}\Bigg/\frac{k_n^-(y)}{k_n^+(y)}=\frac{x}{y}. \tag{1.6}$$

COROLLARY 5. *Let $\{a_n\}$ and $(b_{n,k})$ be as above and let us assume that $b_{n,0}>0$ for every n. Then $\{R_n((b_{n,k}),\{a_n\})\}$ has the approximation property if and only if it has the sequence of operators*

$$\left\{R_n\left(\left(\frac{1}{b_{n,n-k}}\right)_{0\leqq k\leqq n},\left(\frac{1}{a_n}\right)\right)\right\}.$$

Thus, e.g. we can conclude that

$$\lim_{n\to\infty}\frac{1}{(1+n^{1-\beta}x)^n}\left\{\sum_{k=1}^{n} f\left(\frac{n^\beta}{k}\right)\binom{n}{k}(n^{1-\beta}x)^{n-k}+f(n^\beta)(n^{1-\beta}x)^n\right\}=f(x)$$

uniformly on $[0, \infty)$ for every $f\in S$ $(0<\beta<1)$.

REMARK. Much of Theorem 2 could be extended to other type of operators, e.g. to the Bernstein type operators

$$\sum_{k=0}^{n} f(b_{n,k})\binom{n}{k}x^k(1-x)^{n-k},\quad 0\leqq x\leqq 1,\ 0\leqq b_{n,0}<b_{n,1}<\ldots\leqq 1,$$

or to the Szász—Mirakjan type ones:

$$\sum_{k=0}^{n} f(b_{n,k})e^{-nx}\frac{(nx)^k}{k!},\quad x\geqq 0,\ 0\leqq b_{n,0}<b_{n,1}<\ldots .$$

The paper is organized as follows: In § 2 and § 3 we prove Theorems 1 and 2, respectively, and § 4 contains the proof of our lemmas.

§ 2. Proof of Theorem 1

PROOF OF SUFFICIENCY, i.e. the given conditions ensure the stated approximation order.

α. The case $\beta\geqq 2/3$. By our assumption $|f(t)-f(x)|\leqq Kx^{-2}|t-x|$ for $x\geqq a+\varepsilon$ and $\frac{x}{4}\leqq t\leqq 2x$, so we get for $a+\varepsilon\leqq x\leqq\frac{1}{2}n^{1-\beta}$

$$|R_n(f;\,x)-f(x)|\leqq O(n^{-1})+\frac{K}{x^2}R_n(|t-x|;\,x)=O(n^{\beta-1})$$

where we used (i) and (ii) from Lemma 1 and the inequality $\beta/2\geqq 1-\beta$.

Since $f'(x)=O(x^{-2})$, the limit $f(\infty)=\lim_{x\to\infty} f(x)$ exists and $f(x)-f(\infty)=O(x^{-1})$, hence for $x\geqq\frac{1}{2}n^{1-\beta}$ we obtain from Lemma 1 (ii)

$$|R_n(f;\,x)-f(x)|\leqq R_n(|f-f(\infty)|;\,x)+|f(x)-f(\infty)|=$$
$$=O(n^{-1})+O(n^{\beta-1})+O(n^{\beta-1})=O(n^{\beta-1}).$$

β. The case $\frac{1}{2}<\beta<\frac{2}{3}$. For $x\geqq\frac{1}{2}n^{1-\beta}$ argue as above in α. For $n^{2/3-\beta}\leqq$ $\leqq x<\frac{1}{2}n^{1-\beta}$ we get by the argument of Case α

$$|R_n(f;\,x)-f(x)|=O(n^{-1})+\frac{K}{x^2}R_n(|t-x|;\,x)=$$
$$=O(n^{-1})+O\big(x^{-2}(n^{\beta-1}x^2+n^{-\beta/2}x^{1/2})\big)=O(n^{\beta-1})$$

since $n^{1-3/2\beta}x^{-3/2}\leqq 1$.

Finally, for $a+\varepsilon \leqq x \leqq n^{(2/3)-\beta}$ let $\delta=\sqrt{\dfrac{x}{n^\beta}}$,

$$\omega(\delta)=\sup_{0<h\leqq\delta,\, x\geqq a} |\Delta^2_{h\sqrt{x}}(f;x)|$$

and

$$f_\delta(t)=\left(\frac{2}{\delta}\right)^2 \iint\limits_0^{\delta/2} \bigl(2f(t+u+v)-f(t+2(u+v))\bigr)\,du\,dv$$

(see e.g. [8]). For these we have (see [8], especially formulas (3.2) and (3.3))

$$|f(t)-f_\delta(t)| \leqq K\omega(n^{-\beta/2}),\quad |f_\delta''(t)| \leqq K\frac{n^\beta}{x}\,\omega(n^{-\beta/2}) \qquad \left(\frac{x}{4}\leqq t\leqq 2x\right)$$

and by $x^2 f'(x)=O(1)$: $|f_\delta'(x)|\leqq K/x^2$. Now our assumption $x^{(1-\beta)/\beta}\Delta_h^2(f;x)= =O(h^{2(1-\beta)/\beta})$ says that $\omega(\delta)=O(\delta^{2(1-\beta)/\beta})$, hence the relations $|f_\delta(t)|\leqq \sup|f|$,

$$|R_n(f;x)-f(x)| \leqq R_n(|f-f_\delta|,x)+|f(x)-f_\delta(x)|+|R_n(f_\delta;x)-f_\delta(x)|$$

and

$$|f_\delta(t)-f_\delta(x)-f_\delta'(x)(t-x)| \leqq Kf_\delta''(\xi)(t-x)^2 \leqq K\frac{n^\beta}{x}\,\omega(n^{-\beta/2})(t-x)^2$$

$$\left(\frac{x}{4}\leqq t\leqq 2x,\ \xi\geqq\frac{x}{4}\right)$$

give together with Lemma 1

$$|R_n(f;x)-f(x)| \leqq O(n^{-1})+O\bigl(\omega(n^{-\beta/2})\bigr)+O\bigl(\omega(n^{-\beta/2})\bigr)+$$
$$+|f_\delta'(x)|\left(|R_n(t-x;x)|+\frac{1}{n}\right)+K\frac{n^\beta}{x}\,\omega(n^{-\beta/2})\,R_n\bigl((t-x)^2;x\bigr)\leqq$$
$$\leqq O(n^{-1})+O(n^{\beta-1})+O(n^{\beta-1})+Kn^{\beta-1}\frac{n^\beta}{x}\left(n^{2\beta-2}x^4+\frac{x}{n^\beta}\right)=O(n^{\beta-1})$$

since $n^{3\beta-2}x^3\leqq 1$.

γ. *The case* $\beta=1/2$. The proof given in Case β works also in our case because $xf''(x)=O(1)$ implies

$$|\Delta^2_{h\sqrt{x}}(f;x)|=\left|\iint\limits_0^{h\sqrt{x}} f''(x+u+v)\,du\,dv\right| \leqq K\iint\limits_0^{h\sqrt{x}} (x+u+v)^{-1}\,du\,dv \leqq Kh^2.$$

δ. *The case* $0<\beta<1/2$. For $k\geqq\dfrac{1}{4}n$ we have by $f(x)-f(\infty)=O(x^{\beta/(\beta-1)})$: $f\left(\dfrac{k}{n^\beta}\right)-f(\infty)=O(n^{-\beta})$ hence (see Lemma 1, (ii))

$$|R_n(f;x)-f(x)| \leqq O(n^{-1})+Kn^{-\beta}\sum_{k>\frac{1}{4}n} r_{n,k}(x)=O(n^{-\beta}) \qquad \left(x\geqq\frac{1}{2}n^{1-\beta}\right).$$

For $n^{2/3-\beta}<x\leqq\frac{1}{2}n^{1-\beta}$ we use

$$|f(t)-f(x)|\leqq Kx^{-2\beta/(1-\beta)}|t-x|^{\beta/(1-\beta)}\quad\left(x\geqq a+\varepsilon,\ \frac{x}{4}\leqq t\leqq 2x\right)$$

by which

$$|R_n(f;\ x)-f(x)|\leqq O(n^{-1})+Kx^{-2\beta/(1-\beta)}R_n(|t-x|^{\beta/(1-\beta)};\ x)\leqq$$

$$\leqq O(n^{-1})+Kn^{-\beta}\left(1+x^{-\frac{3}{2}\frac{\beta}{1-\beta}}n^{\beta-\frac{\beta^2}{2(1-\beta)}}\right)\leqq Kn^{-\beta}$$

where we used Lemma 1, (i), (ii) and $x\geqq n^{2/3-\beta}$.

Finally, when $a+\varepsilon\leqq x\leqq n^{2/3-\beta}$ then we use $f'(x)=O(x^{1/(3\beta-2)})$, $f''(x)=O(x^{-1})$, Lemma 1 and the estimate

$$|f(t)-f(x)-f'(x)(t-x)|\leqq\frac{K}{x}(t-x)^2\quad\left(\frac{x}{4}\leqq t\leqq 2x\right)$$

to obtain

$$|R_n(f;\ x)-f(x)|\leqq O(n^{-1})+|f'(x)|\left(|R_n(t-x;\ x)|+\frac{1}{n}\right)+\frac{K}{x}R_n((t-x)^2;\ x)\leqq$$

$$\leqq O(n^{-1})+O\left(x^{\frac{1}{3\beta-2}}n^{\beta-1}x^2\right)+O(n^{2\beta-2}x^3+n^{-\beta})=O(n^{-\beta})$$

by which the sufficiency of our conditions is proved.

Proof of Necessity, i.e. $R_n(f)-f=O(\gamma_n)$ (cf. §1) ensures the stated structural properties of f. The proofs below with minor modifications yield also that the saturation order is $\{\gamma_n\}$, we omit the details.

α. *The case* $2/3\leqq\beta<1$. First we show the absolute continuity of f on (a,∞). It is enough to do this on every finite interval (a, b). We may suppose that f is not linear in any left neighbourhood of b.

First of all let us prove that if f is bounded, continuous and

$$\limsup_{n\to\infty} n^{1-\beta}(R_n(f;\ x)-f(x))>0 \tag{2.1}$$

for every $x\in(a, b)$ then there is a point $x_0\in[a, b]$ such that f decreases on (a, x_0) and increases and convex on (x_0, b). Let x_0 be a point where f attains its minimum on $[a, b]$. If $a\leqq x<y<x_0$ and $f(x)\leqq f(y)$ then, by the choice of x_0, there is a z between x and x_0 where f has a local maximum. Thus, for a $\delta>0$ $f(t)\leqq f(z)$ for $z-\delta\leqq t\leqq z+\delta$ and then Lemma 5 yields $R_n(f;\ z)\leqq f(z)+O(n^{-1})$ by which

$$\limsup_{n\to\infty} n^{1-\beta}(R_n(f;\ z)-f(z))\leqq 0$$

and this contradicts (2.1). This contradiction shows that we cannot have $f(y)\geqq f(x)$, i.e. f strictly decreases on (a, x_0). The same argument gives that f increases on (x_0, b). Finally, let us suppose that for some $x_0\leqq x_1<x_2<x_3\leqq b$ we have

$$f(x_2)\geqq f(x_1)+\frac{f(x_3)-f(x_1)}{x_3-x_1}(x_2-x_1)\overset{\text{def}}{=}f(x_1)+\alpha(x_2-x_1).$$

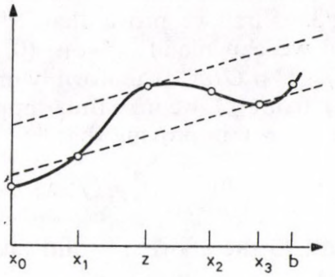

Fig. 1

Then (see also the figure) there is a $z\in(x_1, x_3)$ for which

$$f(t) \leqq f(z)+\alpha(t-z)$$

for all t sufficiently close to z, and so Lemma 5 gives

$$R_n(f; z)-f(z) \leqq \alpha R_n(t-z; t)+O(n^{-1}) = -\frac{\alpha z_0^2 n^{\beta-1}}{1+n^{\beta-1}z_0}+O(n^{-1})$$

which contradicts (2.1) because α is positive. Hence f is convex on (x_0, b).

Now let us turn back to our f on (a, b). Since $R_n(f)-f=O(n^{\beta-1})$ and $R_n(t; x)-x=-\dfrac{n^{\beta-1}x^2}{1+n^{\beta-1}x}$, there is a $C>0$ such that for the functions $f^{\pm}(t)= =-Ct\pm f(t)$ we have $\limsup\limits_{n\to\infty} (R_n(f^{\pm}; x)-f^{\pm}(x))>0$ for every $x\in(a, b)$. Let x_0^+ and x_0^- be points where f^+ and f^- attain their minima on $[a, b]$, respectively. According to what we have proved above both f^+ and f^- are convex on the interval $(\max(x_0^+, x_0^-), b)$. Since we assumed the non-linearity of f around b we can conclude that $\max(x_0^+, x_0^-)=b$. Let e.g. $x_0^-=b$ and $x_0^+\leqq b'<b$ arbitrary (if $x_0^+=b$ is also satisfied then the consideration below becomes even simpler). Since f^- is decreasing on (a, b), we have for $x, x+h\in(a, b), h>0$

$$f^-(x+h)-f^-(x)=-Ch-(f(x+h)-f(x)) \leqq 0. \tag{2.2}$$

Also, f^+ is decreasing on (a, x_0^+) and increasing and convex on (x_0^+, b), hence

$$-Ch+(f(x+h)-f(x)) \leqq 0 \tag{2.3}$$

for $x, x+h\in(a, x_0^+], h>0$ and

$$-Ch+(f(x+h)-f(x)) \leqq dh \tag{2.4}$$

for $x, x+h\in[x_0, b'], h>0$ where d denotes the right derivative of f^+ at the point b'. By (2.2), (2.3) and (2.4) f is in Lip 1 on (a, b') with Lipschitz constant $C+d$, and since $b'<b$ was arbitrary, the absolute continuity of f is proved. For later applications let us note that our argument is valid for all $\beta\geqq 1/2$.

Now at every point x where f is differentiable we have $\lim\limits_{n\to\infty} n^{1-\beta}(R_n(f; x)-f(x))= =x^2f'(x)$ (see Lemma 2, (i)) and so the boundedness of $n^{1-\beta}(R_n(f; x)-f(x))$ on (a, ∞) implies that of $x^2f'(x)$.

β. The case $1/2<\beta<2/3$. First we prove that $x^{(1-\beta)/\beta}\Delta_h^2(f;x)=O(h^{2(1-\beta)/\beta})$ on $(a+\varepsilon,\infty)$. By Lemma 6 we can modify f on $(0,a+\varepsilon)$ so that the resulting function f^* satisfies $R_n(f^*)-f^*=O(n^{\beta-1})$ uniformly on $[0,\infty)$ and f^* is constant on $(0,a)$. Subtracting $f^*(0)$ from f^* we may thus suppose that f is zero on $(0,a)$ and $R_n(f)-f=O(n^{\beta-1})$ on $(0,\infty)$ in proving that

$$\omega(\delta)=\omega(f;\delta)=\sup_{0<h\leqq\delta,\,x\geqq 0}|\Delta^2_{h\sqrt{x}}(f;x)|=O(h^{2(1-\beta)/\beta}). \tag{2.5}$$

In fact, if we apply this result to the above f^* and take into account that $f(x)=$ $=f^*(x)$ for $x\geqq a+\varepsilon$ we get

$$\sup_{x\geqq a+\varepsilon}|\Delta^2_{h\sqrt{x}}(f;x)|\leqq\omega(f^*;h)\leqq K(h^{2(1-\beta)/\beta})$$

which is equivalent to our assertion.

Almost everything from the proof is contained in

PROPOSITION 1. *If* $1/2<\beta<2/3$, $R_n(f)-f=O(n^{\beta-1})$ *uniformly on* $(0,\infty)$ *and* $f(x)=0$ *for* $x\in(0,a)$ *(*$a>0$ *is fixed) then*

$$|R_n''(f;x)|\leqq K\frac{n^\beta}{x}(n^{\beta-1}+\omega(n^{-\beta/2}))\quad(x>0,\ n=1,2,\ldots).$$

PROOF. Let $x>0$ and $\delta=\sqrt{x/n^\beta}$. Then

$$|R_n''(f;x)|\leqq|R_n''(f-f_\delta;x)|+|R_n''(f_\delta;x)|=I_1(x)+I_2(x)$$

where

$$f_\delta(t)=\left(\frac{2}{\delta}\right)^2\iint_0^{\delta/2}(2f(t+u+v)-f(t+2(u+v)))\,du\,dv$$

is the function already used in the sufficiency part of the proof.

We shall estimate $I_1(x)$ and $I_2(x)$ separately.

I. Estimation of $I_1(x)$. An easy computation gives (compare e.g. [4, p. 705]) that

$$\begin{aligned}R_n''(f;x)&=\frac{n^2}{x^2}\sum_{k=0}^n\left[\left(\frac{k}{n}-\frac{a_nx}{1+a_nx}\right)^2-\left(1-2\frac{a_nx}{1+a_nx}\right)\frac{k}{n^2}-\left(\frac{a_nx}{1+a_nx}\right)^2\frac{1}{n}\right]\times\\&\quad\times f\left(\frac{k}{n^\beta}\right)r_{n,k}(x)-\frac{2a_nn}{x(1+a_nx)}\sum_{k=0}^n f\left(\frac{k}{n^\beta}\right)\left(\frac{k}{n}-\frac{a_nx}{1+a_nx}\right)r_{n,k}(x)\stackrel{\text{def}}{=}\\&\stackrel{\text{def}}{=}\frac{n^2}{x^2}\sum_{k=0}^n f\left(\frac{k}{n^\beta}\right)p_{n,k}(x)-\frac{2a_nn}{x(1+a_nx)}\sum_{k=0}^n f\left(\frac{k}{n^\beta}\right)\left(\frac{k}{n}-\frac{a_nx}{1+a_nx}\right)r_{n,k}(x)=\\&=I_{11}(f;x)-I_{1,2}(f;x).\end{aligned}\tag{2.6}$$

First, let us consider the case $x\leqq(a/2)^2$. We have $f\left(\frac{k}{n^\beta}\right)-f_\delta\left(\frac{k}{n^\beta}\right)=0$ for $k\leqq\frac{1}{2}n^\beta a$ because $f(x)=0$ on $(0,a)$, furthermore for $k>\frac{1}{2}n^\beta a$ we get from the definition of ω (see also [7])

$$\left|f\left(\frac{k}{n^\beta}\right)-f_\delta\left(\frac{k}{n^\beta}\right)\right| \leqq \omega\left(\sqrt{\frac{x}{k}}\right) \leqq \omega\left(\sqrt{\frac{a^2}{4\cdot\frac{1}{2}n^\beta a}}\right) \leqq \omega(n^{-\beta/2})$$

where we assumed, as we may, $a<1$.

In the above expression of $R_n''(f-f_\delta; x)$ we have

$$\sum_{k=0}^{n} |p_{n,k}(x)| \leqq \frac{2a_n x}{(1+a_n x)^2 n} \tag{2.7}$$

(see [4, (1.3)]) and $\left|\frac{k}{n}-\frac{a_n x}{1+a_n x}\right| \leqq 1$, hence

$$|R_n''(f-f_\delta; x)| \leqq \frac{n^2}{x^2}\omega(n^{-\beta/2})\frac{2n^{\beta-1}x}{n}+\frac{2n^\beta}{x}\omega(n^{-\beta/2})\sum_{k=0}^{n} r_{n,k}(x) \leqq \frac{4n^\beta}{x}\omega(n^{-\beta/2}).$$

After this let $x>(a/2)^2$. We use that $\omega(\lambda\delta)\leqq K\lambda^2\omega(\delta)$ $(\lambda\geqq 1)$ (see [8]) by which

$$\left|f\left(\frac{k}{n^\beta}\right)-f_\delta\left(\frac{k}{n^\beta}\right)\right| \leqq K\omega\left(\sqrt{\frac{x}{k}}\right) \leqq K\begin{cases}\omega(n^{-\beta/2})\dfrac{n^\beta x}{k} & \text{for} \quad 0<k\leqq n^\beta x\\ \omega(n^{-\beta/2}) & \text{for} \quad k>n^\beta x.\end{cases} \tag{2.8}$$

We have also $f(0)-f_\delta(0)=0$ for $x\leqq n^\beta a^2$ and $|f(0)-f_\delta(0)|\leqq 2\sup|f|$ for all x, hence with

$$h(x)=\begin{cases}0 & \text{for} \quad x\leqq n^\beta a^2\\ 1 & \text{for} \quad x>n^\beta a^2\end{cases} \tag{2.9}$$

we get by the positivity of $\left(1-2\frac{a_n x}{1+a_n x}\right)\frac{k}{n^2}+\left(\frac{a_n x}{1+a_n x}\right)^2\frac{1}{2}$ that

$$|I_{11}(f-f_\delta; x)| \leqq K\frac{n^2}{x^2}\left[\left(\frac{a_n x}{1+a_n x}\right)^2-\left(\frac{a_n x}{1+a_n x}\right)^2\frac{1}{n}\right]h(x)r_{n,0}(x)+$$

$$+K\frac{n^2}{x^2}\omega(n^{-\beta/2})\sum_{k=1}^{[n^\beta x]}\left(\frac{k}{n}-\frac{a_n x}{1+a_n x}\right)^2\frac{n^\beta x}{k}r_{n,k}(x)+K\frac{n^2}{x^2}\omega(n^{-\beta/2})\times$$

$$\times\sum_{k=1}^{[n^\beta x]}\left[\left(1-2\frac{a_n x}{1+a_n x}\right)\frac{k}{n^2}+\left(\frac{a_n x}{1+a_n x}\right)^2\frac{1}{n}\right]\frac{n^\beta x}{k}r_{n,k}(x)+$$

$$+\frac{n^2}{x^2}\omega(n^{-\beta/2})\sum_{k=[n^\beta x]+1}^{n}|p_{n,k}(x)| = S_1(x)+S_2(x)+S_3(x)+S_4(x).$$

Since $\beta>\frac{1}{2}$ and $h(x)=0$ for $x\leqq n^\beta a^2$,

$$S_1(x)\leqq Kn^{2\beta}h(x)\,r_{n,0}(x)=K\frac{n^\beta}{x}n^{\beta-1}\left(nx\left(\frac{1}{1+n^{\beta-1}x}\right)^n h(x)\right)\leqq$$

$$\leqq K\frac{n^\beta}{x}n^{\beta-1}\left(n\cdot n^\beta a^2\left(\frac{1}{n^{2\beta-1}a^2}\right)^n\right)\leqq K\frac{n^\beta}{x}n^{\beta-1}.$$

By (2.7)

$$S_4(x)\leqq K\frac{n^2}{x^2}\,\omega(n^{-\beta/2})\frac{n^{\beta-1}x}{n}\leqq K\frac{n^\beta}{x}\,\omega(n^{-\beta/2}).$$

To estimate $S_2(x)$ let

$$B_n(g;\,x)=\sum_{k=0}^{n} g\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} \tag{2.10}$$

be the n-th Bernstein polynomial of $g\in C[0,1]$. We have (see [7, p. 14]) $B_n((t-x)^2;\,x)=\frac{x(1-x)}{n}$. Now

$$\frac{n^\beta x}{k}\,r_{n,k}(x)\leqq K(1+a_nx)\frac{n+1}{k+1}\,\frac{a_nx}{1+a_nx}\,r_{n,k}(x)=K(1+a_nx)\,r_{n+1,k+1}(x)$$

and

$$\left(\frac{k}{n}-\frac{a_nx}{1+a_nx}\right)^2\leqq 4\left(\frac{k+1}{n+1}-\frac{a_nx}{1+a_nx}\right)^2 \quad\text{when}\quad \left(\frac{k}{n}-\frac{a_nx}{1+a_nx}\right)^2\geqq\frac{4}{n^2}$$

so

$$S_2(x)\leqq K\frac{n^2}{x^2}(1+a_nx)\,\omega(n^{-\beta/2})\left\{O(n^{-2})+\sum_{k=1}^{n}\left(\frac{k+1}{n+1}-\frac{a_nx}{1+a_nx}\right)^2 r_{n+1,k+1}(x)\right\}\leqq$$

$$\leqq K\frac{1+a_nx}{x^2}\,\omega(n^{-\beta/2})+K\frac{n^2}{x^2}(1+a_nx)\,\omega(n^{-\beta/2})\,B_{n+1}\left(\left(t-\frac{a_nx}{1+a_nx}\right)^2;\ \frac{a_nx}{1+a_nx}\right)\leqq$$

$$\leqq K\frac{n^\beta}{x}\,\omega(n^{-\beta/2})+K\frac{n^2\,\omega(n^{-\beta/2})\,n^{\beta-1}x}{x^2(1+n^{\beta-1}x)(n+1)}\leqq K\frac{n^\beta}{x}\,\omega(n^{-\beta/2})$$

where we used also the inequality $x\geqq\left(\frac{a}{2}\right)^2$.

A similar argument gives

$$S_3(x)=O\left(\frac{n^2}{x^2}\,\omega(n^{-\beta/2})\left\{\frac{1}{n^2}+\frac{1}{n}\left|B_{n+1}\left(\left(1-\frac{2a_nx}{1+a_nx}\right)t+\left(\frac{a_nx}{1+a_nx}\right)^2;\ \frac{a_nx}{1+a_nx}\right)\right|\right\}\right)=$$

$$=O\left(\frac{n^\beta}{x}\,\omega(n^{-\beta/2})+\frac{n}{x^2}\,\frac{a_nx}{1+a_nx}\,\omega(n^{-\beta/2})+\frac{n}{x^2}\,\frac{a_nx}{1+a_nx}\,\frac{a_nx}{1+a_nx}\,\omega(n^{-\beta/2})\right)=$$

$$=O\left(\frac{n^\beta}{x}\,\omega(n^{-\beta/2})\right),$$

and so the estimate

$$I_{11}(f-f_\delta;\ x) = O\left(\frac{n^\beta}{x}\left(n^{\beta-1}+\omega(n^{-\beta/2})\right)\right)$$

has been proved for $x \geqq (a/2)^2$.

To estimate $I_{1,2}(f-f_\delta, x)$ (see (2.6)) for $\frac{1}{2}n^{1-\beta} \geqq x \geqq (a/2)^2$ we use Lemma 1 (ii) and (2.8) by which

$$|I_{1,2}(f-f_\delta;\ x)| \leqq K\left(\frac{n^\beta}{x}\left(n^{-1}+\sum_{\frac{x}{4}\leqq\frac{k}{n^\beta}\leqq 2x}\left|f\left(\frac{k}{n^\beta}\right)-f_\delta\left(\frac{k}{n^\beta}\right)\right| r_{n,k}(x)\right) \leqq\right.$$

$$\leqq K\frac{n^\beta}{x}\left(n^{-1}+\omega(n^{-\beta/2})\sum_{\frac{x}{4}\leqq\frac{k}{n^\beta}\leqq 2x}\frac{n^\beta x}{k}r_{n,k}(x)\right) \leqq K\frac{n^\beta}{x}\left(n^{\beta-1}+\omega(n^{-\beta/2})\right).$$

Let now $x > \frac{1}{2}n^{1-\beta}$. First of all let us notice that $\lim\limits_{t\to\infty} R_n(f;\ t) = f(n^{1-\beta})$ exists, so, by $R_n(f)-f=o(1)$, the limit $f(\infty)=\lim\limits_{t\to\infty} f(t)$ also exists and we have

$$|f(\infty)-f(n^{1-\beta})| = \left|\lim_{t\to\infty}\left(f(t)-R_n(f;\ t)\right)\right| \leqq Kn^{\beta-1}.$$

Now putting $v(\delta)=\omega\left(\delta/\sqrt{n^{1-\beta}}\right)$, $\varrho=(x-n^{1-\beta})n^{2\beta-1}$, $a=n^{1-\beta}$, $b=n^{1-\beta}+n^{1-2\beta}$ in Lemma 4 and taking into account that $(n+1)^{1-\beta}-n^{1-\beta}\leqq n^{-\beta}$ we obtain for $x\in\left(n^{1-\beta}, (n+1)^{1-\beta}\right)$:

$$|f(n^{1-\beta})-f(x)| = \left|f(n^{1-\beta})-f\left(n^{1-\beta}+\varrho(b-a)\right)\right| \leqq$$

$$\leqq K\left(v\left(\sqrt{\varrho}(b-a)\right)+\varrho\right) \leqq K\left(v\left(n^{\frac{\beta}{2}-\frac{1}{2}}n^{1-2\beta}\right)+n^{\beta-1}\right) \leqq K\left(\omega(n^{-\beta})+n^{\beta-1}\right).$$

Since this is true for all n and $n^{1-\beta}\leqq x\leqq(n+1)^{1-\beta}$, we get for $x\geqq n^{1-\beta}$:

$$f(x)-f(\infty) = O\left(n^{\beta-1}+\omega(n^{-\beta})\right)$$

by which

$$\left|f\left(\frac{k}{n^\beta}\right)-f_\delta\left(\frac{k}{n^\beta}\right)\right| \leqq 2K\sup_{x\geqq\frac{1}{4}n^{1-\beta}}|f(x)-f(\infty)| \leqq K\left(n^{\beta-1}+\omega(n^{-\beta})\right)$$

for $k\geqq 1/4n$.

Using this and Lemma 1 (ii) we obtain for $x\geqq n^{1-\beta}$:

$$|I_{12}(f-f_\delta;\ x)| \leqq K\frac{n^\beta}{x}\left(n^{-1}+\sum_{k\geqq\frac{1}{4}n}\left|f\left(\frac{k}{n^\beta}\right)-f_\delta\left(\frac{k}{n^\beta}\right)\right| r_{n,k}(x)\right) \leqq$$

$$\leqq K\frac{n^\beta}{x}\left(n^{-1}+n^{\beta-1}+\omega(n^{-\beta})\right) \leqq K\frac{n^\beta}{x}\left(n^{\beta-1}+\omega(n^{-\beta/2})\right).$$

Collecting our inequalities we obtain

$$|I_1(x)| \leqq K\frac{n^\beta}{x}(n^{\beta-1}+\omega(n^{-\beta/2}))$$

with a K independent of n and x.

II. Estimation of $I_2(x)$. A simple calculation gives (compare [7, p. 12])

$$R_n''(f;\,x) = n(n-1)\left(\sum_{k=0}^{n-2}\Delta^2_{n^{-\beta}}\left(f;\,\frac{k}{n^\beta}\right)\binom{n-2}{k}\frac{(a_nx)^k}{(1+a_nx)^{n-2}}\right)\frac{a_n^2}{(1+a_nx)^4}-$$

$$-\frac{2na_n^2}{(1+a_nx)^3}\sum_{k=0}^{n-1}\left(f\left(\frac{k+1}{n^\beta}\right)-f\left(\frac{k}{n^\beta}\right)\right)\binom{n-1}{k}\frac{(a_nx)^k}{(1+a_nx)^{n-1}} = I_{21}(f;\,x)-I_{22}(f;\,x).$$

By the definition of $f_\delta(t)$ and ω we have (see also [8])

$$\left|\Delta^2_{n^{-\beta}}\left(f_\delta;\,\frac{k}{n^\beta}\right)\right| \leqq K\begin{cases}\dfrac{1}{k}\,\omega(n^{-\beta/2}) & \text{for } 1\leqq k\leqq n^\beta x\\[2mm] \dfrac{1}{n^\beta x}\,\omega(n^{-\beta/2}) & \text{for } k>n^\beta x.\end{cases} \tag{2.11}$$

Furthermore, $\Delta^2_{n^{-\beta}}(f_\delta;\,0)=0$ for $n^{-\beta}\leqq a/8$ and $x\leqq n^\beta(a/2)^2$ and $|\Delta^2_{n^{-\beta}}(f_\delta,\,0)|\leqq$ $\leqq 4\sup|f|$ otherwise, hence using the function (2.9) we obtain

$$|I_{21}(f_\delta;\,x)| \leqq Kn^{2\beta}\left(\frac{1}{1+n^{\beta-1}x}\right)^{n+1}h(x)+K\frac{n^{2\beta}\omega(n^{-\beta/2})}{(1+a_nx)^4}\times \tag{2.12}$$

$$\times\sum_{k=1}^{[n^\beta x]}\frac{1}{k}\binom{n-2}{k}\frac{(a_nx)^k}{(1+a_nx)^{n-2}}+K\frac{n^\beta\omega(n^{-\beta/2})}{x}\sum_{k=[n^\beta x]+1}^{n-2}\binom{n-2}{k}\frac{(a_nx)^k}{(1+a_nx)^{n-2}}\leqq$$

$$\leqq K\frac{n^\beta}{x}n^{\beta-1}+K\frac{n^\beta}{x}\omega(n^{-\beta/2})\sum_{k=1}^{[n^\beta x]}\frac{n-1}{k+1}\,\frac{a_nx}{1+a_nx}\binom{n-2}{k}\frac{(a_nx)^k}{(1+a_nx)^{n-2}}+$$

$$+K\frac{n^\beta}{x}\omega(n^{-\beta/2})\leqq K\frac{n^\beta}{x}n^{\beta-1}+K\frac{n^\beta}{x}\omega(n^{-\beta/2})\sum_{k=1}^{[n^\beta x]}\binom{n-1}{k+1}\frac{(a_nx)^{k+1}}{(1+a_nx)^{n-1}}\leqq$$

$$\leqq K\frac{n^\beta}{x}(n^{\beta-1}+\omega(n^{-\beta/2})).$$

For $x\leqq 1$ the estimate of $I_{22}(f_\delta;\,x)$ is easy:

$$|I_{22}(f_\delta;\,x)|\leqq Kn^{2\beta-1}\leqq K\frac{n^\beta}{x}n^{\beta-1}.$$

If $1\leqq x\leqq \frac{1}{2}n^{1-\beta}$ we argue as follows. Since $|f_\delta''(t)|\leqq K(n^\beta/x)\omega(\sqrt{x/n^\beta t})$ (see

[8]) and $\omega(\lambda h) \leqq K\lambda^2\omega(h)$ $(\lambda \geqq 1)$ we have

$$|\Delta_h^2(f_\delta;\, \xi)| \leqq K \begin{cases} \dfrac{n^{2\beta}}{k}\,\omega(n^{-\beta/2})h^2 & \text{for} \quad \xi \geqq \dfrac{k}{n^\beta},\ 1 \leqq k \leqq n^\beta x \\ \dfrac{n^\beta}{x}\,\omega(n^{-\beta/2})h^2 & \text{for} \quad \xi \geqq \dfrac{k}{n^\beta},\ k \geqq n^\beta x. \end{cases}$$

Thus, putting $\varrho = \sqrt{\omega(n^{-\beta/2})/k}$, $a = k/n^\beta$, $b-a = \sqrt{k/n^{2\beta}\,\omega(n^{-\beta/2})}$ and $v(h) = (n^{2\beta}/k)\,\omega(n^{-\beta/2})h^2$ when $1 \leqq k \leqq n^\beta x$ and $\varrho = \sqrt{\omega(n^{-\beta/2})/n^\beta x}$, $a = k/n^\beta$, $b-a = \sqrt{x/n^\beta\,\omega(n^{-\beta/2})}$ and $v(h) = (n^\beta/x)\,\omega(n^{-\beta/2})h^2$ when $k > n^\beta x$ we obtain from Lemma 4

$$(2.13)\quad \left| f_\delta\left(\frac{k+1}{n^\beta}\right) - f_\delta\left(\frac{k}{n^\beta}\right)\right| \leqq K \begin{cases} \sqrt{\dfrac{\omega(n^{-\beta/2})}{k}} \leqq Kn\omega(n^{-\beta/2})/k & \text{for} \quad 1 \leqq k \leqq n^\beta x \\ \sqrt{\dfrac{\omega(n^{-\beta/2})}{n^\beta x}} \leqq K\dfrac{n^{1-\beta}}{x}\,\omega(n^{-\beta/2}) & \text{for} \quad k > n^\beta x \end{cases}$$

where at the last steps we used that $\omega(\lambda h) \leqq K\lambda^2\omega(h)$ implies $\omega(h) \geqq ch^2$ $(c>0)$ (we may assume that $\omega(1) \neq 0$ i.e. that f is not constant) and so

$$k/\omega(n^{-\beta/2}) \leqq n^\beta x/n^{-\beta} \leqq n^{2\beta} n^{1-\beta} \leqq n^2 \quad \left(1 \leqq k \leqq n^\beta x,\ x \leqq \frac{1}{2}\, n^{1-\beta}\right),$$

$$x/n^\beta\,\omega(n^{-\beta/2}) \leqq n^{1-\beta} \leqq n^{2-2\beta} \quad \left(x \leqq \frac{1}{2}\, n^{1-\beta}\right).$$

Using (2.13) the estimate

$$|I_{22}(f_\delta;\, x)| \leqq K\frac{n^\beta}{x}\left(n^{\beta-1} + \omega(n^{-\beta/2})\right)$$

can be proved exactly as we got (2.12) from (2.11).

Finally, for $x \geqq \frac{1}{2}\, n^{1-\beta}$ we use Lemma 1 (ii) and the above (in Part I) proved fact

$$f(x) - f(\infty) = O\left(n^{\beta-1} + \omega(n^{-\beta})\right) \quad \left(x \geqq \frac{1}{4}\, n^{1-\beta}\right)$$

by which the method of Part I gives

$$|I_{22}(f_\delta;\, x)| \leqq K\frac{n^{2\beta-1}}{1+n^{\beta-1}x}\left(\frac{1}{n} + n^{\beta-1} + \omega(n^{-\beta})\right) \leqq$$

$$\leqq K\frac{n^\beta}{x}\,\frac{n^{\beta-1}x}{1+n^{\beta-1}x}\left(n^{\beta-1} + \omega(n^{-\beta/2})\right) \leqq K\frac{n^\beta}{x}\left(n^{\beta-1} + \omega(n^{-\beta/2})\right).$$

Collecting our inequalities we obtain

$$|I_2(x)| \leqq K\frac{n^\beta}{x}\left(n^{\beta-1} + \omega(n^{-\beta/2})\right)$$

and the proof of Proposition 1 is complete.

Now let us turn back to (2.5). For every $h>0$ and $n\geqq 1$ an application of Proposition 1 gives

$$|\Delta^2_{h\sqrt{x}}(f;x)| \leqq |\Delta^2_{h\sqrt{x}}(f-R_n(f);x)|+|\Delta^2_{h\sqrt{x}}(R_n(f);x)| \leqq$$

$$\leqq Kn^{\beta-1}+\Big|\iint_0^{h\sqrt{x}} R_n''(f;x+u+v)\,du\,dv\Big| \leqq Kn^{\beta-1}+$$

$$+\iint_0^{h\sqrt{x}} \frac{n^\beta}{x}\left(n^{\beta-1}+\omega(n^{-\beta/2})\right)du\,dv \leqq Kn^{\beta-1}+K\frac{h^2}{(n^{-\beta/2})^2}\left(n^{\beta-1}+\omega(n^{-\beta/2})\right)$$

i.e.

$$\omega(h) \leqq Kn^{\beta-1}+K\frac{h^2}{(n^{-\beta/2})^2}\left(n^{\beta-1}+\omega(n^{-\beta/2})\right). \tag{2.14}$$

Let now $0<\delta\leqq 1$ be arbitrary. We can find an $n\geqq 2$ with $n^{-\beta/2}<\delta\leqq(n-1)^{-\beta/2}$ and then (2.14) implies

$$\omega(h) \leqq K\left(\delta^{2(1-\beta)/\beta}+\frac{h^2}{\delta^2}\left(\delta^{2(1-\beta)/\beta}+\omega(\delta)\right)\right)$$

for all $0\leqq h,\ \delta\leqq 1$. Since $2(1-\beta)/\beta<2$, we obtain (2.5) from Lemma 3.

It has remained to show the boundedness of $x^2f'(x)$. First of all let us note that $2(1-\beta)/\beta>1$ and $\Delta_h^2(f;x)=O\big(x^{\frac{\beta-1}{\beta}}h^{\frac{2(1-\beta)}{\beta}}\big)$ $(x\geqq a+\varepsilon)$ imply that f is continuously differentiable on $(a+\varepsilon,\infty)$ and $|f'(x)-f'(x+h)|\leqq Kx^{\frac{\beta-1}{\beta}}|h|^{\frac{2(1-\beta)}{\beta}-1}$ $\left(x\geqq a+\varepsilon,\ -\frac{x}{2}\leqq h\leqq x\right)$ (see [9, p. 6]). This and the mean value theorem yield

$$|f(t)-f(x)-f'(x)(t-x)| = |f'(\xi)(t-x)-f'(x)(t-x)| \leqq$$

$$\leqq Kx^{\frac{\beta-1}{\beta}}|\xi-x|^{\frac{2(1-\beta)}{\beta}-1}|t-x| \leqq Kx^{\frac{\beta-1}{\beta}}|t-x|^{\frac{2(1-\beta)}{\beta}} \quad (\xi\in(t,x))$$

by which (see Lemma 1 (iii))

$$\left|R_n(f;x)-f(x)-f'(x)\left(R_n(t-x;x)+O\left(\frac{1}{n}\right)\right)\right| \leqq Kx^{\frac{\beta-1}{\beta}}R_n\big(|t-x|^{\frac{2(1-\beta)}{\beta}};x\big).$$

Applying Lemma 1 we get from $R_n(f)-f=O(n^{\beta-1})$

$$O(n^{\beta-1})+|f'(x)|\left(\frac{n^{\beta-1}x^2}{1+n^{\beta-1}x}+O(n^{-1})\right) = O\big(x^{3\frac{1-\beta}{\beta}}n^{-\frac{2(1-\beta)^2}{\beta}}+n^{\beta-1}\big).$$

Multiplying this by $n^{1-\beta}$ and taking the limit $n\to\infty$ we obtain $|f'(x)|x^2=O(1)$ as we claimed above.

The proof of necessity in the case $\frac{1}{2}<\beta<\frac{2}{3}$ is thus complete.

γ. *The case* $\beta=\frac{1}{2}$. First we prove that f has an absolutely continuous derivative on (a, ∞). Let (a, b) be any finite interval. It is sufficient to show that f has an absolutely continuous derivative on (a, b).

The proof used in α shows that f is in Lip 1 on (a, b). Let e.g.

$$|f(x+h)-f(x)| \leqq M|h| \quad (x, x+h\in(a, b))$$

and

$$|R_n(f;\, x)-f(x)| \leqq Mn^{-1/2} \quad (x\in(a, b)). \tag{2.15}$$

We claim that

$$|\Delta_h^2(f;\, x)| \leqq Ch^2 \quad (a+h \leqq x \leqq b-h,\ h>0) \tag{2.16}$$

where $C=2(M+5Mb^2)/a$. (2.16) already implies the absolute continuity of f' (see [10, p. 6]).

Let us suppose on the contrary that e.g. $2f(x_0)-f(x_0+h)-f(x_0-h)=C_1h^2$ for some $x_0\in(a+h, b-h)$, $h>0$ and $C_1>C$. Let $C<C_2<C_1$. For the function

$$g(t)=f(t)+C_2(t-x_0)^2-\frac{f(x_0+h)-f(x_0-h)}{2h}(t-x_0+h)$$

we have

$$g(x_0)-g(x_0-h) = g(x_0)-g(x_0+h) = \frac{1}{2}\bigl(2g(x_0)-g(x_0+h)-g(x_0-h)\bigr) =$$

$$= C_1h^2-C_2h^2>0,$$

hence, if $m=\sup\limits_{x_0-h\leqq t\leqq x_0+h} g(t)$, and z is a point where the supremum is attained then $x_0-h<z<x_0+h$. Now since $m-g$ is nonnegative on (x_0-h, x_0+h) and $g(z)-m=0$ we have

$$f(t) \leqq m-C_2(t-x_0)^2+\frac{f(x_0+h)-f(x_0-h)}{2h}(t-x_0+h)$$

for $x_0-h\leqq t\leqq x_0+h$ and

$$f(z) = m-C_2(z-x_0)^2+\frac{f(x_0+h)-f(x_0-h)}{2h}(z-x_0+h),$$

which yield easily

$$f(t) \leqq f(z)-C_2(t-z)^2+\left(-2C_2(z-x_0)+\frac{f(x_0+h)-f(x_0-h)}{2h}\right)(t-z) =$$

$$\stackrel{\text{def}}{=} f(z)-C_2(t-z)^2+\alpha(t-z)$$

for $|t-z|<\min\bigl(z-(x_0-h), x_0+h-z\bigr)$. Applying Lemma 5 we get

$$R_n(f;\, z)-f(z) \leqq -C_2R_n\bigl((t-z)^2;\, z\bigr)+\alpha R_n(t-z;\, z)+O(n^{-1}) \leqq$$

$$\leqq -C_2n^{-1/2}\frac{z}{1+n^{-1/2}z}-\alpha n^{-1/2}\frac{z^2}{1+n^{-1/2}z}+O(n^{-1}).$$

Since $|f(x_0+h)-f(x_0-h)|\leqq M2h$;

$$C_2 h<\frac{1}{h}C_1h^2=\frac{1}{h}(2f(x_0)-f(x_0+h)-f(x_0-h))\leqq\frac{1}{h}2Mh=2M$$

and $|z-x_0|\leqq h$, we get $|\alpha|\leqq 5M$, and so

$$R_n(f;\,z)-f(z)\leqq n^{-1/2}\Big(-C_2\frac{a}{2}+5Mb^2\Big)<n^{-1/2}\Big(-C\frac{a}{2}+5Mb^2\Big)=-Mn^{-1/2}$$

for large n and this contradicts (2.15). Thus, the absolute continuity of f' is proved.

Applying Lemma 2 we get that at every point x where $f''(x)$ exists, i.e. almost everywhere on (a,∞)

$$|-2x^2f'(x)+xf''(x)|=\big|2\lim_{n\to\infty}n^{1/2}(R_n(f;\,x)-f(x))\big|\leqq A$$

independently of x. We show using this that $|x^2f'(x)|\leqq\frac{3}{2}A$ and $|xf''(x)|\leqq\frac{5}{2}A$, by which the proof will be complete. In fact, let us assume e.g. $x_0^2f'(x_0)>\frac{3}{2}A$ for some $x_0\in(a,\infty)$. Then $x^2f'(x)>\frac{3}{2}A$ in a neighborhood U of x_0 and so at every $x\in U$ where $f''(x)$ exists we have necessarily $f''(x)>\frac{A}{x}$. This implies that f' strictly increases in U and we can conclude easily that $x^2f'(x)>\frac{3}{2}A$ for every $x\geqq x_0$. But then $f''(x)>A/x$ for every $x\geqq x_0$ by which $f'(x)\geqq\text{const}+A\log x$ $(x\geqq x_0)$ and this contradicts the boundedness of f. Thus, $|x^2f'(x)|\leqq\frac{3}{2}A$ and so $|xf''(x)|\leqq A+\frac{3}{2}A$ for $x>a$. The proof is complete.

δ. *The case* $0<\beta<\frac{1}{2}$. The fact that f' is absolutely continuous follows from [10, Theorem 5.1], and the boundedness of $xf''(x)$ on (a,∞) can be proved exactly as above (the argument for $\beta<\frac{1}{2}$ is even simpler since $n^{\beta}(R_n(f;\,x)-f(x))\to\to\frac{1}{2}xf''(x)$ where $f''(x)$ exists, see Lemma 2).

For $x\geqq a+\varepsilon$ and $\frac{x}{4}\leqq t\leqq 2x$ we have by $xf''(x)=O(1)$:

$$f(t)-f(x)-\frac{K}{x}(t-x)^2\leqq f'(x)(t-x)\leqq f(t)-f(x)+\frac{K}{x}(t-x)^2$$

and so we get from Lemma 1 and $R_n(f)-f=O(n^{-\beta})$ the relation

$$|f'(x)|\left(\frac{n^{\beta-1}x^2}{1+n^{\beta-1}x}+O(n^{-1})\right)\leqq Kn^{-\beta}+K\frac{1}{x}(n^{2\beta-2}x^4+n^{-\beta}x).$$

If here $n^{2/3-\beta}\leqq x\leqq(n+1)^{2/3-\beta}$ then we obtain

$$|f'(x)|\leqq Kn^{-1/3}\leqq Kx^{-\frac{1}{2-3\beta}}. \tag{2.17}$$

Since $R_n(f)-f=o(1)$ uniformly on (a,∞) the limit $f(\infty)=\lim\limits_{t\to\infty}f(t)$ exists and for every n

$$|f(n^{1-\beta})-f(\infty)|=\left|\lim_{t\to\infty}(R_n(f,t)-f(t)\right|\leqq Kn^{-\beta}.$$

This, together with $f'(x)=O(1)$ easily imply $f(x)-f(\infty)=O\left(x^{-\frac{\beta}{1-\beta}}\right)$.

Finally we prove

$$x^{\frac{2\beta}{1-\beta}}|\Delta_h^1(f\colon x)|\leqq Kh^{\frac{\beta}{1-\beta}}\quad(a<x<\infty,\ h>0), \tag{2.18}$$

in several steps.

(2.18) is clearly equivalent to

$$|f(x+hx^2)-f(x)|\leqq Kh^{\frac{\beta}{1-\beta}}\quad(x\geqq a,\ h>0), \tag{2.19}$$

and $f'(x)=O(1)$ shows that we may assume in (2.19) $x\geqq 2$.

Step 1. For $n=1,2,\ldots$ and $n^{2/3-\beta}\leqq x\leqq\frac{1}{2}n^{1-\beta}$ we have

$$\left|f\left(x-\frac{n^{\beta-1}x^2}{1+n^{\beta-1}x}\right)-f(x)\right|\leqq Kn^{-\beta}.$$

In fact, by (2.17)

$$\left|f(t)-f\left(\frac{x}{1+n^{\beta-1}x}\right)\right|\leqq Kx^{-\frac{1}{2-3\beta}}\left|t-\frac{x}{1+n^{\beta-1}x}\right|\quad\left(t\in\left(\frac{x}{8},2x\right)\right)$$

thus (see Lemma 1)

$$\left|R_n(f;x)-f\left(\frac{x}{1+n^{\beta-1}x}\right)\right|\leqq Kx^{-\frac{1}{2-3\beta}}R_n\left(\left|t-\frac{x}{1+n^{\beta-1}x}\right|;x\right)+Kn^{-1}. \tag{2.20}$$

On the other hand, using the Bernstein polynomials B_n (see (2.10)), the fact

$$B_n((t-\alpha)^2;\alpha)=\frac{\alpha(1-\alpha)}{n}\quad(\alpha\in(0,1))$$

and Schwarz's inequality for positive functionals we obtain

$$R_n\left(\left|t-\frac{x}{1+n^{\beta-1}x}\right|;\, x\right) = B_n\left(\left|n^{1-\beta}t-\frac{x}{1+n^{\beta-1}x}\right|;\, \frac{n^{\beta-1}x}{1+n^{\beta-1}x}\right) =$$

$$= n^{1-\beta}B_n\left(\left|t-\frac{n^{\beta-1}x}{1+n^{\beta-1}x}\right|;\, \frac{n^{\beta-1}x}{1+n^{\beta-1}x}\right) \leqq$$

$$\leqq n^{1-\beta}\sqrt{B_n\left(\left(t-\frac{n^{\beta-1}x}{1+n^{\beta-1}x}\right)^2;\, \frac{n^{\beta-1}x}{1+n^{\beta-1}x}\right)} \leqq n^{1-\beta}\sqrt{n^{\beta-1}x/n} \leqq n^{-\beta/2}\sqrt{x}$$

by which (see (2.20))

$$\left|R_n(f;\, x)-f\left(\frac{x}{1+n^{\beta-1}x}\right)\right| \leqq Kn^{-\beta}\left(n^{\frac{\beta}{2}}x^{-\frac{1}{2-3\beta}+\frac{1}{2}}\right)+Kn^{-1} \leqq Kn^{-\beta}$$

where we used that $\frac{1}{2}-\frac{1}{2-3\beta}<0$ and $x \geqq n^{2/3-\beta}$.

Since we have also $|R_n(f;\, x)-f(x)| \leqq Kn^{-\beta}$, we can conclude

$$\left|f\left(x-\frac{n^{\beta-1}x^2}{1+n^{\beta-1}x}\right)-f(x)\right| = \left|f\left(\frac{x}{1+n^{\beta-1}x}\right)-f(x)\right| \leqq \left|f\left(\frac{x}{1+n^{\beta-1}x}\right)-R_n(f;\, x)\right|+$$

$$+|R_n(f;\, x)-f(x)| \leqq Kn^{-\beta}$$

as was stated above.

Step 2. For $x \geqq 1$ and $0 \leqq h \leqq \frac{1}{4}x^{-1}$ we have

$$|f(x-hx^2)-f(x)| \leqq Kh^{\frac{\beta}{1-\beta}}.$$

Indeed, if $0 \leqq h \leqq x^{-\frac{3(1-\beta)}{2-3\beta}}$ then (2.17) gives

$$|f(x-hx^2)-f(x)| \leqq Khx^2x^{-\frac{1}{2-3\beta}} = Kh^{\frac{\beta}{1-\beta}}h^{\frac{1-2\beta}{1-\beta}}x^{\frac{3(1-2\beta)}{2-3\beta}} \leqq$$

$$\leqq Kh^{\frac{\beta}{1-\beta}}\left(x^{-\frac{3(1-\beta)}{2-3\beta}}\right)^{\frac{1-2\beta}{1-\beta}}x^{\frac{3(1-2\beta)}{2-3\beta}} \leqq Kh^{\frac{\beta}{1-\beta}}.$$

If, conversely,

$$x^{-\frac{3(1-\beta)}{2-3\beta}} \leqq h \leqq \frac{1}{4}x^{-1} \tag{2.21}$$

then there is an n for which $n^{2/3-\beta} \leqq x \leqq \frac{1}{2}n^{1-\beta}$ and

$$\frac{(n+1)^{\beta-1}}{1+(n+1)^{\beta-1}x} \leqq h \leqq \frac{n^{\beta-1}}{1+n^{\beta-1}x}. \tag{2.22}$$

The existence of an n satisfying (2.22) follows from the fact that $n^{\beta-1}/(1+n^{\beta-1}x)$

decreases from $1/(1+x)$ to 0 as n increases from 1 to infinity, and then the inequality $n^{2/3-\beta} \leqq x \leqq \frac{1}{2} n^{1-\beta}$ is an easy consequence of (2.21) and (2.22).

Thus, if we apply Step 1 and the estimate (2.17) we can get

$$|f(x-hx^2)-f(x)| \leqq \left| f\left(x-\frac{n^{\beta-1}x^2}{1+n^{\beta-1}x}\right)-f(x)\right| + \left| f\left(x-\frac{n^{\beta-1}x^2}{1+n^{\beta-1}x}\right)-f(x-hx^2)\right| \leqq$$

$$\leqq Kn^{-\beta}+K\left|\frac{n^{\beta-1}x^2}{1+n^{\beta-1}x}-hx^2\right| \leqq Kn^{-\beta}+Kx\left(\frac{n^{\beta-1}x}{1+n^{\beta-1}x}-\frac{(n+1)^{\beta-1}x}{1+(n+1)^{\beta-1}x}\right) \leqq$$

$$\leqq Kn^{-\beta}+Kx^2\big((n+1)^{\beta-1}-n^{\beta-1}\big) \leqq Kn^{-\beta}(1+x^2n^{2\beta-2}) \leqq Kn^{-\beta} \leqq Kh^{\beta/(1-\beta)}$$

where at the last but one step we used that $x \leqq \frac{1}{2} n^{1-\beta}$.

Step 3. For any $x \geqq 1$ and $h>0$ the inequality (2.19) holds.

If $h \geqq \frac{1}{4} x^{-1}$ then by $f(x)-f(\infty)=O(x^{-\beta/(1-\beta)})$:

$$|f(x+hx^2)-f(x)| \leqq |f(x+hx^2)-f(\infty)|+|f(x)-f(\infty)| \leqq Kx^{-\frac{\beta}{1-\beta}} \leqq Kh^{\frac{\beta}{1-\beta}}.$$

For $h \leqq \frac{1}{4} x^{-1}$ let $y=x+hx^2$ and $h^*=h\left(\frac{x}{y}\right)^2$. Then $h^* \leqq h,\ y-h^*y^2=x$ and

$$h^* = h\frac{x^2}{y^2} \leqq \frac{1}{4}x^{-1}\frac{x^2}{y^2} = \frac{1}{4}y^{-1}\frac{x}{y} \leqq \frac{1}{4}y^{-1}.$$

Thus, we can apply the inequality proved in Step 2 at the point y and we can infer

$$|f(x+hx^2)-f(x)| = |f(y)-f(y-h^*y^2)| \leqq K(h^*)^{\beta/(1-\beta)} \leqq Kh^{\beta/(1-\beta)}$$

by which (2.19) has been proved.

We have completed the proof of Theorem 1.

THE PROOF OF COROLLARY 1. The statements concerning the local saturation classes follow easily from Theorem 1 by Lemma 6 and by the remark made after it (note that the sufficiency proof of Theorem 1 works also locally). All what we have to mention is that the conditions

$$\Delta_h^2(f;\, x) \leqq Kh^{\frac{2(1-\beta)}{\beta}} \quad \big(x \in (a,\, b-2h)\big)$$

and $f' \in \operatorname{Lip}\frac{2-3\beta}{\beta}$ on (a,b) are equivalent when $\frac{1}{2}<\beta<\frac{2}{3}$ (see e.g. [10, p. 6]).

It has remained to prove the statements concerning the trivial classes.

Following the proof of $x^2f'(x)=O(1)$ and $xf''(x)=O(1)$ in Theorem 1 one can easily see that $R_n(f)-f=o(\gamma_n)$ on (a,b) $\big($see (1.4)$\big)$ implies

1. $f(x)=\text{const}$ when $\beta>\frac{1}{2}$,

2. $-x^2 f'(x)+\frac{x}{2} f''(x)=\lim_{n\to\infty} n^{1/2}(R_n(f;x)-f(x))=0$ (a.e.) when $\beta=\frac{1}{2}$,

3. $\frac{1}{2} x f''(x)=\lim_{n\to\infty} n^{\beta}(R_n(f;x)-f(x))=0$ (a.e.) when $0<\beta<\frac{1}{2}$, and so f

is of the form c, $c+d\int_0^x e^{\tau^2}d\tau$, $c+dx$, respectively.

Conversely, if f has the form

$$\begin{cases} c & \text{when } \frac{1}{2}<\beta<1 \\ c+d\int_0^x e^{\tau^2}\,d\tau & \text{when } \beta=\frac{1}{2} \\ c+dx & \text{when } 0<\beta<\frac{1}{2} \end{cases}$$

on (a, b) then the asymptotic formula (see [1, p. 127] and Lemma 5)

$$R_n(f;\, x)=f(x)-\frac{n^{\beta-1}x^2}{1+n^{\beta-1}x}\, f'(x)+\frac{n^{2\beta-2}x^4+n^{-\beta}x}{2(1+n^{\beta-1}x)^2}\, f''(x)+o(n^{\beta-1})$$

valid uniformly on compact subintervals of (a, b) and Lemma 1 show that $R_n(f;x)-f(x)=o(\gamma_n)$ uniformly on compact subsets of (a, b).

§ 3. Proof of Theorem 2

Clearly, it is enough to show the implications (iii)$\Rightarrow$(iv), (iv)$\Rightarrow$(ii) and (iv)$\Rightarrow$(i).

I. PROOF OF (iii)$\Rightarrow$(iv). First we verify

PROPOSITION 2. *If $R_n(1/(1+t);\, x)\to 1/(1+x)$ $(n\to\infty)$ for every rational $x\geqq 0$ then $\lim_{n\to\infty} a_n/n=0$ and $\lim_{n\to\infty} na_n=\infty$.*

The proof is divided into three steps.

Step 1. $a_n=O(n)$. Indeed, let us suppose that for a sequence $n_i\to\infty$ we have $a_{n_i}/n_i\to\infty$. Then, because of $r_{n,k+1}(x)/r_{n,k}(x)=\frac{n-k}{k+1}\, a_n x$, we have

$$R_{n_i}(1/(1+t);\, x)=\big(1/(1+b_{n_i,n_i})\big)\, r_{n_i,n_i}(x)+o_x\big(r_{n_i,n_i}(x)\big)$$

and here

$$\lim_{i\to\infty} r_{n_i,n_i}(x)=\lim_{i\to\infty}\left(1-\frac{1}{1+a_{n_i}x}\right)^{n_i}=1$$

which proves that $R_{n_i}(1/(1+t);\, x)\to 1/(1+x)$ cannot hold.

Step 2. $a_n=o(n)$. Let $a_n=\alpha_n n$, and let us suppose that $a_n\neq o(n)$, i.e. $\alpha_n\neq o(1)$. Then, taking into account also Step 1 there is a sequence $\{n_i\}$ and a number $0<\alpha<\infty$ such that $\alpha_{n_i}\to\alpha$ as $i\to\infty$. For the sake of simplicity we shall suppose that $\alpha_n\to\alpha$ (otherwise we would have to use a fixed subsequence of the natural numbers instead of the whole set).

We claim that the limits

$$c_k = \lim_{n\to\infty} \frac{1}{1+b_{n,n-k}}$$

exist for each $k \geqq 0$ and that these c_k satisfy

$$c_k = (-1)^{k+1}\left\{k!\,\alpha^k + \sum_{i=1}^{k-1} (-1)^i \binom{k}{i} c_i\right\}. \tag{3.1}$$

This already contradicts our assumption $\alpha > 0$ since each c_i is between 0 and 1 but for odd k the right side in (3.1) is at least $k!\alpha^k - 2^k$ which tends to infinity if k does so.

Since

$$r_{n,k}(x)/r_{n,k}(1) = x^k \left(\frac{1+\alpha_n n}{1+\alpha_n nx}\right)^n \leqq x^{k-n} \left(\frac{1+\alpha_n n}{\alpha_n n}\right)^n \leqq cx^{k-n}$$

with a c independent of $x \geqq 1$ and n (note that $\alpha_n \to \alpha$), we have for every k

$$R_n(1/(1+t);\, x) = \sum_{i=0}^{k} \frac{1}{1+b_{n,n-i}} \binom{n}{i} \frac{1}{(\alpha_n nx)^i} \left(\frac{\alpha_n nx}{1+\alpha_n nx}\right)^n + O(x^{-k-1}) \tag{3.2}$$

where the "O" is independent of $x \geqq 1$ and n.

Let here $k=0$. Then

$$R_n(1/(1+t), x) = \frac{1}{1+b_{n,n}} \left(\frac{\alpha_n nx}{1+\alpha_n nx}\right)^n + O(x^{-1})$$

with "O" independent of $x \geqq 1$ and n and so we cannot have

$$\limsup_{n\to\infty} \frac{1}{1+b_{n,n}} = \varrho > 0$$

because choosing a large rational x we would get

$$\limsup_{n\to\infty} R_n(1/(1+t);\, x) \geqq \varrho e^{-\frac{1}{\alpha x}} - Kx^{-1} > \frac{2}{1+x}$$

which contradicts $\lim\limits_{n\to\infty} R_n(1/(1+t);\, x) = 1/(1+x)$. Thus, $c_0 = 0$.

In the proof of the existence of the c_i we proceed, by induction. If the existence of $c_0, c_1, \ldots, c_{k-1}$ has already been proved then taking into account

$$\lim_{n\to\infty} \binom{n}{n-i} \frac{1}{(n\alpha_n x)^i} \left(\frac{\alpha_n nx}{1+\alpha_n nx}\right)^n = \frac{1}{i!\,\alpha^i x^i} e^{-\frac{1}{\alpha x}},$$

we get from $R_n(1/(1+t);\, x) \to 1/(1+x)$ and (3.2) the formula

$$\frac{1}{1+x} = \sum_{i=0}^{k-1} \frac{c_i}{i!\,\alpha^i x^i} e^{-\frac{1}{\alpha x}} + \left(\limsup_{n\to\infty} \frac{1}{1+b_{n,n-k}}\right) \frac{1}{k!\,\alpha^k x^k} e^{-\frac{1}{\alpha x}} + O(x^{-k-1}) \tag{3.3}$$

i.e., expanding $e^{-\frac{1}{\alpha x}}$ into a series of powers of x^{-1},

$$\frac{1}{1+x} = \sum_{i=0}^{k} d_i \frac{1}{x^i} + \left(\limsup_{n\to\infty} \frac{1}{b_{n,n-k}}\right) \frac{1}{k!\,\alpha^k x^k} + O(x^{-k-1}) \tag{3.4}$$

for some $d_0, \ldots, d_k$ and with large "O" independent of $x \geqq 1$. By

$$\frac{1}{x^i(x+1)} - \frac{1}{x^{i+1}} = -\frac{1}{x^{i+1}(x+1)} \quad (i = 0, 1, 2, \ldots) \tag{3.5}$$

(3.4) implies easily that $d_0=0$, $d_1=1$, $d_2=-1, \ldots, d_{k-1}=(-1)^k$ and

$$d_k + \left(\limsup_{n\to\infty} \frac{1}{1+b_{n,n-k}}\right) \frac{1}{k!\,\alpha^k} = (-1)^{k+1}.$$

The same argument gives that

$$d_k + \left(\liminf_{n\to\infty} \frac{1}{1+b_{n,n-k}}\right) \frac{1}{k!\,\alpha^k} = (-1)^{k+1}$$

by which the existence of $\lim_{n\to\infty} 1/(1+b_{n,n-k})$ is proved.

By (3.3)

$$\frac{1}{1+x} = \sum_{i=0}^{k} \frac{c_i}{i!\,\alpha^i x^i} e^{-\frac{1}{\alpha x}} + O(x^{-k-1}).$$

Clearly, here (as we have already mentioned) $c_0=0$, and $c_1=\alpha$. Using this we get from the expansion of $e^{-1/\alpha x}$

$$\frac{1}{1+x} = \frac{1}{x} - \frac{c_1}{\alpha^2 x^2} + \frac{c_2}{2!\,\alpha^2 x^2} + O(x^{-3})$$

and hence (see also (3.5)) $-\frac{c_1}{\alpha} + \frac{c_2}{2!\,\alpha^2} = -1$ i.e. $c_2 = -2!\,\alpha^2 + \binom{2}{1} c_1$. Proceeding similarly we obtain

$$\frac{(-1)^{k+1}}{x^{k-1}(1+x)} = \frac{(-1)^{k-1} c_1}{\alpha^k x^k (k-1)!} + \frac{(-1)^{k-2} c_2}{2!\,\alpha^k x^k (k-2)!} + \ldots + \frac{c_k}{k!\,\alpha^k x^k} + O(x^{-k-1})$$

which cannot hold unless

$$\sum_{i=1}^{k} \frac{(-1)^{k-i} c_i}{\alpha^k i!\,(k-i)!} = (-1)^{k+1}$$

i.e.

$$c_k = (-1)^{k+1} \left\{ k!\,\alpha^k + \sum_{i=1}^{k-1} (-1)^i \binom{k}{i} c_i \right\}$$

and our proof is complete.

Step 3. Clearly, we may suppose $b_{n,0}>0$ for every n. Let $b^*_{n,k}=1/b_{n,n-k}$ and $a^*_n=1/a_n$. For every rational $x>0$ we have by our assumption

$$\lim_{n\to\infty} R_n((b^*_{n,k});\ \{a^*_n\};\ 1/(1+t);\ x) =$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\dfrac{1}{a_n}\right)^n} \sum_{k=0}^{n} \frac{1}{1+\dfrac{1}{b_{n,n-k}}} \binom{n}{n-k} \left(\frac{x}{a_n}\right)^k =$$

$$= \lim_{n\to\infty} \left(\frac{x/a_n}{1+x/a_n}\right)^n \sum_{k=0}^{n} \left(1-\frac{1}{1+b_{n,n-k}}\right) \binom{n}{n-k} \left(\frac{a_n}{x}\right)^{n-k} =$$

$$= 1-\lim_{n\to\infty} \left(\frac{1}{1+a_n/x}\right)^n \sum_{k=0}^{n} \frac{1}{1+b_{n,k}} \binom{n}{k} \left(\frac{a_n}{x}\right)^k = 1-\frac{1}{1+1/x} = \frac{1}{1+x}.$$

Thus, we can apply Step 2 to the sequence $\{a^*_n\}$ to obtain $\lim\limits_{n\to\infty} na_n = \lim\limits_{n\to\infty} n/a^*_n=\infty$.

The proof of Proposition 2 is complete.

Now let us turn back to the implication (iii)$\Rightarrow$(iv).

Let us suppose on the contrary that (iv) does not hold, e.g. for an $x_0>0$

$$\limsup_{n\to\infty} b_{n,[na_n x_0/(1+a_n x_0)]} > x_0 .$$

Then there is a $\delta>0$ and infinitely many n with

$$b_{n,[na_n x_0/(1+a_n x_0)]} \geqq x_0+\delta. \tag{3.6}$$

Let $y\in(x_0, x_0+\delta)$ be a rational number, say $y=x+\varepsilon$ where $\varepsilon<\delta/2$ is sufficiently small.

$$\frac{na_n y}{1+a_n y} - \frac{na_n x_0}{1+a_n x_0} = \frac{na_n(y-x_0)}{(1+a_n y)(1+a_n x_0)} > \frac{na_n\varepsilon}{(1+a_n y)^2}$$

and so if $z=\frac{1}{2}\sqrt{na_n}\,\varepsilon/(\sqrt{y}\,(1+a_n y))$ and $u=\dfrac{a_n y}{1+a_n y}$ in Lemma 5 (for small ε we have $z<\frac{3}{2}\sqrt{na_n y}\,/(1+a_n y))$ then we get

$$\sum_{k=0}^{\left[\frac{na_n x_0}{1+a_n x_0}\right]} r_{n,k}(y) \leqq \sum_{\left|k-\frac{na_n y}{1+a_n y}\right|>2z\frac{\sqrt{na_n y}}{1+a_n y}} q_{n,k}\left(\frac{a_n y}{1+a_n y}\right) \leqq 2e^{-\frac{\varepsilon^2}{4}\frac{na_n}{(1+a_n y)^2 y}} = o(1)$$

where at the last step we applied Proposition 2.

Thus, for the n's satisfying (3.6) we have

$$R_n(1/(1+t);y) = \left(\sum_{k=0}^{\left[\frac{na_nx_0}{1+a_nx_0}\right]} + \sum_{k=\left[\frac{na_nx_0}{1+a_nx_0}\right]+1}\right)\frac{1}{1+b_{n,k}}r_{n,k}(y) \leqq$$

$$\leqq o(1)+\frac{1}{1+x_0+\delta}\sum_{k=\left[\frac{na_nx_0}{1+a_nx_0}\right]+1} r_{n,k}(y) \leqq o(1)+\frac{1}{1+y+\varepsilon}$$

i.e. we cannot have $R_n(1/(1+t);y)\to 1/(1+y)$, and this proves that (iii) implies (iv).

II. Proof of (iv)$\Rightarrow$(ii). First of all let us remark that (iv) implies

$$\lim_{n\to\infty} na_n = \infty, \qquad \lim_{n\to\infty} a_n/n = 0. \tag{3.7}$$

In fact, if we had e.g. $a_n \geqq cn$ $(c>0)$ for infinitely many n then for every $x\geqq\frac{1}{c}$ and for infinitely many n we would have $b_{n,[na_nx/(1+a_nx)]}=b_{n,n-1}$ which contradicts (iv) (the limit of the left hand side must depend on x).

Let f be bounded, say $|f|\leqq M$, and continuous on $[0,\infty)$ and let $A>0$ be arbitrary. We have to prove that $R_n(f;x)\to f(x)$ uniformly on $[0,A]$. Let $A/8>\varepsilon>0$ be arbitrary small. Since

$$\lim_{n\to\infty} b_{n,[na_nk\varepsilon/(1+a_nk\varepsilon)]} = k\varepsilon$$

for $k=0,1,\dots,[A/\varepsilon]+1$, an easy consideration gives that there is a number N such that for $n\geqq N$ we have

$$x-2\varepsilon \leqq b_{n,[na_nx/(1+a_nx)]} \leqq x+2\varepsilon \quad (x\in[0,A]). \tag{3.8}$$

Clearly,

$$\left|\frac{na_nx}{1+a_nx}-\frac{na_ny}{1+a_ny}\right| \geqq \frac{na_n|x-y|}{(1+2a_nA)^2} \tag{3.9}$$

for $x,y\in[0,2A]$. Now for every $0\leqq k<n$ there is a y with $k=[na_ny/(1+a_ny)]$, so (3.7), (3.8) and (3.9) yield that for large n the relations $x\in[0,A]$, $b_{n,k}\in[0,2A]$ and $|b_{n,k}-x|>4\varepsilon$ imply

$$\left|k-\frac{na_nx}{1+a_nx}\right| \geqq \frac{na_n\varepsilon}{(1+2a_nA)^2}. \tag{3.10}$$

Since for n large enough we have also

$$A+4\varepsilon < b_{n,\left[na_n\frac{3}{2}A/\left(1+a_n\frac{3}{2}A\right)\right]} < 2A,$$

it follows that (3.10) holds for every $x\in[0,A]$ and k with $|b_{n,k}-x|>4\varepsilon$ provided n is sufficiently large.

If we put $z=\frac{1}{2}\sqrt{na_n/x}\,\frac{1+a_nx}{(1+2a_nA)^2}\varepsilon$ and $u=\frac{a_nx}{1+a_nx}$ in Lemma 5 then we get, by (3.10), for large n

$$(3.11)\qquad \sum_{|b_{n,k}-x|>4\varepsilon} r_{n,k}(x) \leqq e^{-\frac{1}{4}\frac{na_n}{x}\frac{(1+a_nx)^2\varepsilon^2}{(1+2a_nA)^4}} \leqq e^{-\frac{\varepsilon^2}{4A}\frac{na_n}{(1+2a_nA)^2}\left(\frac{1+a_n\varepsilon}{1+a_nA}\right)^2} = o(1)$$

uniformly in $x\in[\varepsilon, A]$ $\left(x\geqq\varepsilon \text{ is needed for } z\leqq\frac{3}{2}(nu(1-u))^{1/2}\right)$.

Since $r_{n,k}(x)$ increases for $na_nx/(1+a_nx)\leqq k$ we get easily from (3.11) that for $0\leqq x\leqq\varepsilon$ and large n

$$(3.12)\qquad \sum_{|b_{n,k}-x|>5\varepsilon} r_{n,k}(x) \leqq \sum_{b_{n,k}\geqq 5\varepsilon} r_{n,k}(x) \leqq \sum_{b_{n,k}\geqq 5\varepsilon} r_{n,k}(\varepsilon) = o(1)$$

holds, since for large n, $b_{n,k}\geqq 5\varepsilon$ implies $k\geqq na_n\varepsilon/(1+a_n\varepsilon)$ (take into account that for n large enough $b_{n,[na_n2\varepsilon/(1+a_n2\varepsilon)]}<3\varepsilon$ and $[na_n2\varepsilon/(1+a_n2\varepsilon)]>na_n\varepsilon/(1+a_n\varepsilon)$).

Using (3.11) and (3.12) the proof of the uniform convergence of R_n on $[0, A]$ is easy:

$$|R_n(f;x)-f(x)| \leqq \Big|\Big(\sum_{|b_{n,k}-x|\leqq 5\varepsilon} + \sum_{|b_{n,k}-x|>5\varepsilon}\Big)(f(b_{n,k})-f(x))r_{n,k}(x)\Big| \leqq$$

$$\leqq \sup_{\substack{x\in[0,A]\\|x-y|\leqq 5\varepsilon}} |f(x)-f(y)|+o(1)$$

uniformly for $x\in[0, A]$, and, by the continuity of f, the right hand side can be made arbitrary small by suitable choice of ε.

III. Proof of (iv)$\Rightarrow$(i). Let f be continuous with finite limit $f(\infty)$ at the infinity. We may suppose $f(\infty)=0$. If $\varepsilon>0$ is given, there is an A such that $|f(x)|<\varepsilon$ for $x\geqq A$. Since $R_n(f;x)\to f(x)$ uniformly on $[0, 2A]$ (see II above) it is enough to show that $|R_n(f;x)|\leqq 2\varepsilon$ for $x\geqq 2A$ and for sufficienty large n. However, this is easy: $r_{n,k}(x)$ decreases for $na_nx/(1+a_nx)\geqq k$ and for large n, $b_{n,k}\leqq A$ implies $k\leqq 2na_nA/(1+2a_nA)$, hence

$$|R_n(f;x)| \leqq M\sum_{b_{n,k}\leqq A} r_{n,k}(x)+\Big|\sum_{b_{n,k}>A} f(b_{n,k})\,r_{n,k}(x)\Big| \leqq$$

$$\leqq M\sum_{b_{n,k}\leqq A} r_{n,k}(2A)+\varepsilon\sum_{b_{n,k}>A} r_{n,k}(x) \leqq Mo(1)+\varepsilon \quad (x\geqq 2A)$$

(see also the proof in II).

The proof is complete.

Corollaries 2 and 3 follow easily from Theorem 2. For Corollary 5 see Step 2 above in the proof of Proposition 2. Finally, (iv) of Theorem 2 is equivalent to $\lim_{n\to\infty}\frac{1}{a_n}\frac{k_n^-(x)}{n-k_n^-(x)}=\lim_{n\to\infty}\frac{1}{a_n}\frac{k_n^-(x)}{k_n^+(x)}=x$, hence (iv)$\Rightarrow$(1.6) hold. The opposite implication (1.6)$\Rightarrow$(iv) is also easy if we put $a_n=k_n^-(1)/k_n^+(1)$ $(n\geqq n_0)$.

§ 4. Lemmas

In the following lemmas $R(f; x)=R_n(\beta; f; x)$.

LEMMA 1. (i) *For* $0<\alpha\leqq 2$

$$R_n(|t-x|^\alpha;\ x)\leqq n^{\alpha(\beta-1)}x^{2\alpha}+x^{\alpha/2}n^{-\beta\alpha/2}\quad (x\geqq 0),$$

(ii) *If* $a>0$ *is fixed then*

$$R_n(f;\ x)=\sum_{\frac{x}{4}\leqq\frac{k}{n^\beta}\leqq 2x} f\left(\frac{k}{n^\beta}\right)r_{n,k}(x)+O(n^{-1})$$

uniformly for $a\leqq x\leqq n^{1-\beta}$. *We have also*

$$R_n(f;\ x)=\sum_{k\geqq\frac{1}{4}n} f\left(\frac{k}{n^\beta}\right)r_{n,k}(x)+O(n^{-1})$$

uniformly for $x\geqq n^{1-\beta}$.

(iii)
$$R_n(t-x;\ x)=-\frac{n^{\beta-1}x^2}{1+n^{\beta-1}x}$$

and

$$R_n(t-x;\ x)=\sum_{\frac{x}{4}\leqq\frac{k}{n^\beta}\leqq 2x}\left(\frac{k}{n^\beta}-x\right)r_{n,k}(x)+O(n^{-1})$$

uniformly for $a\leqq x\leqq n^{1-\beta}$ *(*$a>0$ *fixed).*

PROOF. By

$$R_n((t-x)^2;\ x)=\frac{n^{2\beta-2}x^4+n^{-\beta}x}{(1+n^{\beta-1}x)^2}$$

(see [1, (2.4)]), (i) follows from Hölder's inequality

$$R_n(|t-x|^\alpha;\ x)=\sum_{k=0}^{n}\left|\frac{k}{n^\beta}-x\right|^\alpha r_{n,k}(x)\leqq\left\{\sum_{k=0}^{n}\left(\frac{k}{n^\beta}-x\right)^2 r_{n,k}(x)\right\}^{\alpha/2}=$$

$$=\left\{R_n((t-x)^2;\ x)\right\}^{\alpha/2}\leqq n^{\alpha(\beta-1)}x^{2\alpha}+x^{\alpha/2}n^{-\beta\alpha/2}.$$

(ii) and (iii) easily follow from [1, (2.3)] and Lemma 5 below.

LEMMA 2. *Let* f *be a bounded function.* (i) *If* $\beta\geqq\frac{2}{3}$ *and* f *is differentiable at the point* x_0 *then*

$$\lim_{n\to\infty} n^{1-\beta}(R_n(f;\ x_0)-f(x_0))=-x_0^2f'(x_0).$$

(ii) *If the second derivative of* f *exists at the point* x_0 *then we have*

$$\lim_{n\to\infty} n^{\beta}(R_n(f;\ x_0)-f(x_0))=\begin{cases}-x_0^2f'(x_0)+\dfrac{x}{2}f''(x_0) & \text{for}\quad \beta=\dfrac{1}{2}\\[2ex] \dfrac{x_0}{2}f''(x_0) & \text{for}\quad 0<\beta<\dfrac{1}{2}.\end{cases}$$

PROOF. For (ii) see [1, Theorem II]. In this theorem we used the assumption $n^{1/2}/b_n \to 0$ $(n\to\infty)$, however a careful examination of the proof together with the estimate

$$\sum_{\left|\frac{k}{n^\beta}-x\right|>\delta} r_{n,k}(x) \leqq Kn^{-1}$$

(see Lemma 5) show that for bounded f it can be omitted.

If the assumption of (i) holds then $f(t)=f(x_0)+f'(x_0)(t-x_0)+\lambda(t)(t-x_0)$ where $\lambda(t)\to 0$ as $t\to x_0$ and $\lambda(t)$ is bounded by the boundedness of f. Given $\varepsilon>0$ we can find a $\delta>0$ such that $|t-x_0|<\delta$ implies $|\lambda(t)|<\varepsilon$, and so we obtain for $n\geqq n_\varepsilon$ from Lemma 5 and Lemma 1 (i)

$$|R_n(f;\, x_0)-f(x_0)-f'(x_0)R_n(t-x_0;\, x_0)| \leqq R_n(|\lambda(t)|\,|t-x_0|;\, x_0) =$$

$$= \sum_{\left|\frac{k}{n^\beta}-x\right|<\delta} \left|\lambda\left(\frac{k}{n^\beta}\right)\right| \left|\frac{k}{n^\beta}-x_0\right| + O(n^{-1}) \leqq \varepsilon R_n(|t-x_0|;\, x_0)+O(n^{-1}) \leqq$$

$$\leqq K\varepsilon(n^{-\beta/2}+n^{\beta-1})+O(n^{-1}) \leqq K\varepsilon n^{\beta-1}.$$

Since $\varepsilon>0$ was arbitrary all we have to mention is that $\lim\limits_{n\to\infty} n^{1-\beta}R_n(t-x_0;\, x_0)= =-x_0^2$ (see Lemma 1, (iii)).

LEMMA 3. *Let Ω be monotonically increasing on* $[0, 1]$. *If for* $0<\alpha<2$ *one has for all* h *and* $\delta\in(0, 1]$

$$\Omega(h) \leqq M\left(\delta^\alpha+\left(\frac{h}{\delta}\right)^2(\delta^\alpha+\Omega(\delta))\right)$$

then $\Omega(\delta)=O(\delta^\alpha)$ $(\delta\to 0+)$.

This is [3, Lemma 2.1].

LEMMA 4. *If f is continuous on* $[a, b+2(b-a)]$, $|f|\leqq 1$ *and* $\sup\limits_{0<h'\leqq h} |\Delta^2_{h'}(f;\, x)| \leqq$ $\leqq v(h)$ $\big(x\in[a, b+(b-a)],\, 0<h\leqq b-a\big)$ *then*

$$\big|f(x)-f\big(x+\varrho(b-a)\big)\big| \leqq K\big(v\big(\sqrt{\varrho}\,(b-a)\big)+\varrho\big) \quad (x\in[a, b],\ 0<\varrho\leqq 1).$$

See [9, Lemma 2].

LEMMA 5. *Let* $q_{n,k}(u)=\binom{n}{k}u^k(1-u)^{n-k}$ $(0\leqq u\leqq 1,\ 0\leqq k\leqq n)$. *If* $0\leqq z\leqq$ $\leqq \frac{3}{2}(nu(1-u))^{1/2}$ *then*

$$\sum_{|k-nu|>2z(nu(1-u))^{1/2}} q_{n,k}(u) \leqq 2e^{-z^2} \quad (0\leqq u\leqq 1).$$

Especially, if $1\geqq\delta>0$ *and* $a>0$ *are given then* (see (1.1))

$$\sum_{\left|\frac{k}{n^\beta}-x\right|>\delta x} r_{n,k}(x)\left(x+\frac{k}{n^\beta}\right) \leqq Kn^{-1} \quad (n=1, 2, \ldots)$$

uniformly in $x\in[a, \delta n^{1-\beta}]$.

PROOF. The first estimate is well-known, see [7, p. 18].

Clearly, $r_{nk}(x)=q_{nk}(u)$ with $u=\dfrac{n^{\beta-1}}{1+n^{\beta-1}x}$ or $x=n^{1-\beta}u/(1-u)$ and for $x\in[a, \delta n^{1-\beta}]$, $\delta\leqq 1$ we have

$$\frac{1}{2}an^{\beta-1}\leqq u\leqq \delta/(1+\delta),\quad 1/(1+\delta)\leqq 1-u\leqq 1.$$

Furthermore,

$$k-nu = n^{\beta}\left(\frac{k}{n^{\beta}}-x\right)+un^{\beta}x,$$

hence it follows that $|k/n^{\beta}-x|>\delta x$ implies

$$|k-nu|\geqq \delta n^{\beta}x-(\delta/(1+\delta))n^{\beta}x = 2\sqrt{nu(1-u)}\frac{\sqrt{nu}}{(1-u)^{3/2}}\frac{\delta^2}{2(1+\delta)}.$$

For $\delta\leqq 1$

$$z=\frac{\sqrt{nu}}{(1-u)^{3/2}}\frac{\delta^2}{2(1+\delta)}\leqq\frac{3}{2}(nu(1-u))^{1/2}$$

and so the first inequality gives

$$\sum_{\left|\frac{k}{n^{\beta}}-x\right|>\delta x} r_{n,k}(x)\left(x+\frac{k}{n^{\beta}}\right)\leqq 2n^{1-\beta}\sum_{|k-nu|>2z(nu(1-u))^{1/2}} q_{n,k}(u)\leqq$$

$$\leqq 2n^{1-\beta}2\exp\left(-\frac{1}{2}a\delta^4 n^{\beta}/(4(1+\delta)^2(1-u)^3)\right)\leqq K_{a,\delta}n^{-1}$$

uniformly in $x\in[a, \delta n^{1-\beta}]$ which proves the second estimate of the lemma.

LEMMA 6. *Let* $\beta\geqq\frac{1}{2}$, $a>0$ *and* $R_n(f)-f=O(n^{\beta-1})$ *uniformly on* (a,∞). *Then for every* $\varepsilon>0$ *we can modify* f *on* $(0, a+\varepsilon)$ *so that the modified function* f^* *is constant on* $(0, a)$ *and* $R_n(f^*)-f^*=O(n^{\beta-1})$ *holds uniformly on* $(0,\infty)$.

By the same technique we could prove that if $R_n(f)-f=O(n^{\beta-1})$ on the finite interval (a, b), $a>0$ then f can be modified on $(0, a+\varepsilon)\cup(b-\varepsilon,\infty)$ so that the resulting function f^* satisfies: $R_n(f^*)-f^*=O(n^{\beta-1})$ uniformly on $(0,\infty)$, f^* is constant on $(0, a)$ and on (b,∞).

PROOF. First let us show that if $\eta>0$, $R_n(f)-f=O(n^{\beta-1})$ uniformly on $(x_0-\eta,\infty)$ and f has a local minimum at x_0 then for the function

$$f^*(x)=\begin{cases} f(x) & \text{for}\quad x\geqq x_0\\ f(x_0) & \text{for}\quad x\leqq x_0\end{cases}$$

we have $R_n(f^*)-f^*=O(n^{\beta-1})$ uniformly on $(0,\infty)$. We may assume $f(x_0)=0$. Let $\varepsilon<\eta/3$ be so small that $f(t)\geqq 0$ is satisfied on the interval $(x_0-\varepsilon, x_0+\varepsilon)$. Since $f^*(x)=0$ for $x<x_0$ $f^*(x)-f(x)=0$, $R_n(f;x)-f(x)=O(n^{\beta-1})$ for $x>x_0$, an easy consideration gives (see e.g. Lemma 5) that $R_n(f^*;x)-f^*(x)=O(n^{\beta-1})$ uniformly on $\left(0, x_0-\frac{\varepsilon}{2}\right)\cup\left(x_0+\frac{\varepsilon}{2}, \infty\right)$. Thus, it has remained to show the same relation for $x_0-\frac{\varepsilon}{2}\leqq x\leqq x_0+\frac{\varepsilon}{2}$.

The weight

$$r_{n,k}(x)=\binom{n}{k}\frac{(a_nx)^k}{(1+a_nx)^n}$$

increases for $x\leqq\frac{k}{n^\beta}\frac{n}{n-k}$ and decreases for $x>\frac{k}{n^\beta}\frac{n}{n-k}$, so for $x_0-\varepsilon/2\leqq x\leqq x_0$ we obtain by Lemma 5 and the inequality $f^*(t)\geqq 0$ $(x_0-\varepsilon\leqq t\leqq x_0+\varepsilon)$ the estimate

$$(4.1)\qquad -Kn^{\beta-1}\leqq R_n(f^*;x)=\sum_{x_0n^\beta\leqq k\leqq(x_0+\varepsilon/2)n^\beta} f\left(\frac{k}{n^\beta}\right)r_{n,k}(x)+O(n^{-1})\leqq$$

$$\leqq\sum_{x_0n^\beta\leqq k\leqq(x_0+\varepsilon/2)n^\beta} f\left(\frac{k}{n^\beta}\right)r_{n,k}(x_0)+O(n^{-1})\leqq R_n(f;x_0)+O(n^{-1})\leqq Kn^{\beta-1}.$$

Similarly, since for $k\leqq x_0n^\beta$, $r_{n,k}(x)$ attains its maximum at

$$\frac{k}{n^\beta}\frac{n}{n-k}\leqq\frac{k}{n^\beta}\left(1+c_1\frac{k}{n}\right)\leqq x_0+c_2n^{\beta-1}$$

we have for $x_0+c_2n^{\beta-1}\leqq x\leqq x_0+\varepsilon/2$

$$(4.2)\qquad -Kn^{\beta-1}\leqq R_n(f-f^*;x)=\sum_{(x_0-\varepsilon/2)n^\beta\leqq k\leqq x_0n^\beta} f\left(\frac{k}{n^\beta}\right)r_{n,k}(x)+O(n^{-1})\leqq$$

$$\leqq\sum_{(x_0-\varepsilon/2)n^\beta\leqq k\leqq x_0n^\beta} f\left(\frac{k}{n^\beta}\right)r_{n,k}(x_0+c_2n^{\beta-1})+O(n^{-1})\leqq R_n(f;x_0+c_2n^{\beta-1})+O(n^{-1}).$$

Now we have to use the fact that f is in Lip 1 on $(x_0-\varepsilon, x_0+\varepsilon)$. For the proof see that of the absolute continuity of f in Theorem 1 in the case $\beta\geqq\frac{2}{3}$, $R_n(f;x)-$ $-f(x)=O(n^{\beta-1})$ $(x\in(x_0-\eta,\infty))$ (this proof does not use Lemma 6). By

$$R_n'(f;x)=\frac{n^\beta}{(1+n^{\beta-1}x)^2}\sum_{k=0}^{n-1}\left[f\left(\frac{k}{n^\beta}\right)-f\left(\frac{k+1}{n^\beta}\right)\right]\binom{n-1}{k}\frac{(n^{\beta-1}x)^k}{(1+n^{\beta-1}x)^{n-1}}$$

(see [7, p. 12]) the relation "$f\in$Lip 1 on $(x_0-\varepsilon, x_0+\varepsilon)$" and Lemma 5 give

$$(4.3)\qquad |R_n'(f;x)|\leqq Kn^\beta(Kn^{-\beta}+O(n^{-1}))\leqq K$$

uniformly in $x_0-\varepsilon/2\leqq x\leqq x_0+\varepsilon/2$. Thus

$$R_n(f;x_0+c_2n^{\beta-1})\leqq R_n(f;x_0)+Kn^{\beta-1}\leqq Kn^{\beta-1}$$

and so (see (4.2))

$$(4.4)\qquad R_n(f-f^*;x)=O(n^{\beta-1})$$

uniformly in $x \in \left[x_0+c_2n^{\beta-1},\ x_0+\dfrac{\varepsilon}{2}\right)$. For $x \in (x_0;\ x_0+c_2n^{\beta-1})$ we obtain (4.4) from (4.3).

Now (4.4) gives for $x_0 \leqq x \leqq x_0+\varepsilon/2$

$$R_n(f^*;\ x)-f^*(x) = -R_n(f-f^*;\ x)+R_n(f;\ x)-f(x) = O(n^{\beta-1})$$

which, together with (4.1), prove our assertion.

Let us turn back to our lemma. For the function

$$g(t) = \begin{cases} (t-a)^2(t-a+\varepsilon)^2 & \text{if} \quad a \leqq t \leqq a+\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

we have $R_n(g;\ x)-g(x)=O(n^{\beta-1})$ uniformly on $(0, \infty)$ (see the sufficiency part of Theorem 1 and — for small x — the monotonicity argument used above).

Thus, if M is large enough then for $f_M(t)=f(t)-Mg(t)$ we have $R_n(f_M)-f_M= =O(n^{\beta-1})$ uniformly on (a, ∞) and $f_M(t)=f(t)$ for $t \geqq a+\varepsilon$, moreover $f_M(t)$ has a local minimum at a point $x_0 \in (a, a+\varepsilon)$. According to what we have proved above a possible suitable modification of f is

$$f^*(t) = \begin{cases} f(t)-Mg(t) & \text{for} \quad t \geqq x_0 \\ f(x_0)-Mg(x_0) & \text{for} \quad 0 \leqq t \leqq x_0. \end{cases}$$

The lemma is proved.

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A. JÓZSEF UNIVERSITY
BOLYAI INSTITUTE
SZEGED, H—6720
ARADI VÉRTANÚK TERE 1

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THE CONTINUITY OF SYMMETRIC AND SMOOTH FUNCTIONS

M. J. EVANS (Raleigh) and L. LARSON (Macomb)

Introduction

For a function $f\colon \mathbf{R}\to\mathbf{R}$ denote

$$\Delta^2 f(x,h)=f(x+h)+f(x-h)-2f(x).$$

If $\Delta^2 f(x,h)=o(1)$ as $h\to 0$, then f is said to be symmetric at x and if $\Delta^2 f(x,h)==o(h)$ as $h\to 0$, then f is said to be smooth at x. The function f is simply called symmetric (smooth) if it is symmetric (smooth) at each x. Both symmetric and smooth functions can be extremely badly behaved. For example, there are solutions to Cauchy's functional equation

$$f(x+y)=f(x)+f(y)$$

which are non-measurable, fail to possess the Baire property and assume every real value uncountably many times on every perfect set [4]. However, symmetric functions which are measurable are known to be quite nice in that they are of Baire class one and are continuous almost everywhere [5]. As might be expected, measurable smooth functions are even better behaved in that they belong to the class Baire* one [7] and have only a countable number of discontinuities [6].

In this work the set of points on which symmetric and smooth functions are continuous is studied. First, it is shown that an arbitrary function with the Baire property which is symmetric on a residual set of points is also continuous on a residual set. In then follows easily that any symmetric function with the Baire property is in Baire class one. Second, the set of points at which a measurable smooth function is discontinuous is characterized as a separated set in the sense of Hausdorff [3].

First, some notation must be introduced.

If $A\subset\mathbf{R}$, $\bar{A}$ will denote the closure of A, A' will denote the set of limit points of A, and A^c will denote the complement of A. The distance between a point x and the set A will be denoted by $d(x,A)$.

The set $A\subset\mathbf{R}$ is separated if A has no subset which is dense in itself.

All functions are finite-valued with domains contained in $\mathbf{R}$. If f is a function, then the oscillation of f at x is written $\omega_f(x)$. The set of points at which f is continuous is written $C(f)$ and the set of points at which f is discontinuous is written $D(f)$.

A function $f\colon\mathbf{R}\to\mathbf{R}$ has the Baire property if there is a set A residual in $\mathbf{R}$ such that the restriction of f to A, $f|_A$, is continuous.

If f is symmetric at x and $\varepsilon>0$, then $\delta(x,\varepsilon)$ denotes a positive number such that $|\Delta^2 f(x,h)|<\varepsilon$ for $0<h<\delta(x,\varepsilon)$.

If $A\subset\mathbf{R}$, $f|_A$ is continuous at $x\in A$ and $\varepsilon>0$, then $\delta^*(x,\varepsilon,A)$ denotes a positive number such that

$$|f(y)-f(x)|<\varepsilon \quad \text{for} \quad |x-y|<\delta^*(x,\varepsilon,A),\ y\in A.$$

If f is symmetric at $x\in A$ and $f|_A$ is continuous at x, then

$$\delta'(x,\varepsilon,A)=\min\{\delta(x,\varepsilon),\delta^*(x,\varepsilon,A)\}.$$

Symmetric functions

It is noted above that symmetric functions which are measurable exhibit a considerable degree of continuity. In this section it is shown that the assumption of the Baire property gives rise to the same nice behavior. First, a preliminary lemma is needed.

LEMMA 1.1. *Suppose $f:\mathbf{R}\to\mathbf{R}$ is symmetric at each point of a set A and that $f|_A$ is continuous at $x_0\in A$. Let $\eta>0$ be arbitrary. For each $x_1\in A$ satisfying*

$$0<x_1-x_0<\delta'\left(x_0,\frac{\eta}{4},A\right)$$

there exists a positive number $h<x_1-x_0$ such that

$$\left|f(x+(x_1-x_0))-f(x-(x_1-x_0))\right|<\eta$$

whenever $x_0<x<x_0+h$.

PROOF. Let x_0 and x_1 be as described. Set $h=\min\{x_1-x_0,\delta(x_1,\eta/4)\}$ and let x be any number satisfying $x_0<x<x_0+h$. Since $x+(x_1-x_0)=x_1+$ $+(x-x_0)$ and $0<x-x_0<\delta(x_1,\eta/4)$, we have

$$|f[x+(x_1-x_0)]+f[x_1-(x-x_0)]-2f(x_1)|<\eta/4. \tag{1}$$

Also, since $x_1-(x-x_0)=x_0+(x_1-x)$ with $0<x_1-x<\delta(x_0,\eta/4)$ we have

$$|f[x_1-(x-x_0)]+f[x_0-(x_1-x)]-2f(x_0)|<\eta/4. \tag{2}$$

Next, since $x_0-(x_1-x)=x-(x_1-x_0)$, it follows from (1), (2) and the triangle inequality that

$$|f[x+(x_1-x_0)]-f[x-(x_1-x_0)]|<\eta/2+2|f(x_1)-f(x_0)|. \tag{3}$$

Finally, since $0<x_1-x_0<\delta^*(x_0,\eta/4,A)$, the desired inequality follows from (3).

THEOREM 1.2. *If $f:\mathbf{R}\to\mathbf{R}$ has the Baire property and is symmetric at a residual set of points, then $C(f)$ is residual.*

PROOF. Since f has the Baire property, there is a residual set B such that $f|_B$ is continuous. Consequently, there is a residual set A at each point of which f is symmetric and $f|_A$ is continuous.

Suppose f is discontinuous at a second category set of points. Then there is an interval (a, b) and an $\alpha>0$ such that the oscillation of f at each point of (a, b) is $\geqq\alpha$. For $n=1, 2, \ldots$ set

$$A_n = \{x\in A\colon\ \delta(x, \alpha/24) \geqq 1/n\}.$$

Since $A=\bigcup A_n$, there must be an n_0 and a subinterval (c, d) of (a, b) of length $<1/n_0$ such that A_{n_0} is dense in (c, d).

Let $x_0\in(c, d)\cap A$. We shall now verify the existence of two points x_2 and x_3 satisfying the following:

$$c < x_0 < x_2 < x_3 < d, \tag{4}$$

$$|f(x_3)-f(x_2)| \geqq \alpha/6, \tag{5}$$

$$x_3-x_2 < \min\,[2\delta'(x_0, \alpha/48, A),\ x_2-x_0], \tag{6}$$

$$x_1 \equiv x_0+(x_3-x_2)/2\in A. \tag{7}$$

Since the oscillation at each point of (a, b) is $\geqq\alpha$, we can clearly find points $\bar{x}_2$ and $\bar{x}_3$ with $x_0<\bar{x}_2<\bar{x}_3<d$ such that

$$|f(\bar{x}_3)-f(\bar{x}_2)| \geqq \alpha/6 \quad\text{and}\quad 0 < \bar{x}_3-\bar{x}_2 < \min\,[2\delta'(x_0, \alpha/48, A), \bar{x}_2-x_0].$$

If $x_0+(\bar{x}_3-\bar{x}_2)/2\in A$, then (4)—(7) are satisfied with $x_2=\bar{x}_2$ and $x_3=\bar{x}_3$.

On the other hand, if $x_0+(\bar{x}_3-\bar{x}_2)/2\notin A$, then choose a positive $\varepsilon<(\bar{x}_3-\bar{x}_2)/3$ and let C denote the set of points in $(\bar{x}_3-\varepsilon, \bar{x}_3)$ for which $|f(x)-f(\bar{x}_2)|\geqq\alpha/6$.

If C is of the second category, then, since A is residual, there is a point $x_3\in C$ such that $x_0+(x_3-\bar{x}_2)/2\in A$. Then this x_3 and $x_2=\bar{x}_2$ would satisfy (4)—(7).

If C is of the first category we set $D=(\bar{x}_3-\varepsilon, \bar{x}_3)-C$ and note that

$$|f(x)-f(\bar{x}_2)| < \alpha/6 \quad\text{for each } x\in D. \tag{8}$$

Again, by considering the oscillation of f on (a, b) we see that there must be a point $x_2\in(\bar{x}_2, \bar{x}_2+\varepsilon)$ with

$$|f(\bar{x}_2)-f(x_2)| \geqq \alpha/3. \tag{9}$$

By (8) and (9) we have $|f(x)-f(x_2)|\geqq\alpha/6$ for each $x\in D$. Since D is of the second category, there must be an $x_3\in D$ for which $x_0+(x_3-x_2)/2\in A$. Once again, these points x_2 and x_3 satisfy (4)—(7).

Having considered all possible cases, the claim concerning x_2 and x_3 is established.

Now, by the Lemma, since $x_1-x_0=(x_3-x_2)/2$, there exists a positive $h<(x_3-x_2)/2$ such that

$$|f[x+(x_3-x_2)/2]-f[x-(x_3-x_2)/2]| < \alpha/12 \quad\text{for}\quad x_0 < x < x_0+h. \tag{10}$$

Set $\xi=(x_2+x_3)/2$. Since A_{n_0} is dense in (c, d), there is an $x'\in A_{n_0}$ such that $x\equiv 2x'-\xi\in(x_0, x_0+h)$. By (10) it follows that

$$|f(2x'-x_2)-f(2x'-x_3)| < \alpha/12. \tag{11}$$

Since $x_2=x'+(x_2-x')$ and $0<x_2-x'<1/n_0$, we have

$$|f(x_2)+f(2x'-x_2)-2f(x')| < \alpha/24. \tag{12}$$

Similarly, since $x_3=x'+(x_3-x')$ and $0<x_3-x'<1/n_0$, we have

(13) $$|f(x_3)+f(2x'-x_3)-2f(x')| < \alpha/24.$$

From (12) and (13) we have

(14) $$|f(x_3)-f(x_2)+f(2x'-x_3)-f(2x'-x_2)| < \alpha/12,$$

and from (14) and (11) we obtain

$$|f(x_3)-f(x_2)| < \alpha/6,$$

which contradicts (5) and completes the proof.

Neugebauer [5, Theorem 1] states that if a function f is symmetric and measurable, then $C(f)$ has full measure. Theorem 1.2 of the present paper implies that if f is symmetric and possesses the Baire property, then $C(f)$ is residual. Using this residual set of continuity points in place of Neugebauer's full measure set, it is easy to see that both of his Theorems 3 and 4 still hold with the new assumption. This implies the following two corollaries.

COROLLARY 1.3. *If* $f: \mathbf{R}\to\mathbf{R}$ *is a symmetric function with the Baire property, then* f *is in Baire class one.*

COROLLARY 1.4. *A symmetric function* $f: \mathbf{R}\to\mathbf{R}$ *is measurable if and only if it has the Baire property.*

It should be noted that a method similar to the proof of Theorem 1.2 was used by H. Fried [2] to show that if f is symmetrically continuous on a residual set, then $C(f)$ is residual. Also, D. Preiss [8] has shown that every symmetrically continuous function is continuous almost everywhere. However, it is not known whether symmetrically continuous functions are in Baire class one or not.

Smooth functions

The goal of this section is to characterize $D(f)$ when f is measurable and smooth.

THEOREM 2.1. *If* f *is smooth and measurable, then* $D(f)$ *is separated.*

PROOF. By definition it suffices to show that $D(f)$ contains no subset which is dense in itself [3, p. 136]. To do this let E denote the set of points x such that there exists a sequence of numbers $x_n\neq x$ such that

(1) $$x_n \to x$$

and

(2) $$\lim_{n\to\infty}\frac{\omega_f(x_n)}{|x_n-x|}=\infty.$$

To establish that $D(f)$ is separated, the following two statements will be established.

A) E is countable.

B) If $D(f)$ contains a subset which is dense in itself, then E is uncountable.

To establish A, suppose $x_0 \in E$ and let $\{x_n\}$ be a sequence disjoint from x_0 which satisfies (1) and (2). Since f is smooth at x_0 there is a number k and a $\delta>0$ such that

$$|f(x)+f(2x_0-x)-2f(x_0)| < k|x-x_0| \quad \text{for} \quad |x-x_0| < \delta. \tag{3}$$

From (1) and (2) it follows that there exists an integer N such that if $n \geqq N$, then $0<|x_n-x_0|<\delta$ and

$$\omega_f(x_n) > 3k|x_n-x_0|. \tag{4}$$

Fix an $n \geqq N$. For any number y satisfying

$$|y-x_n| < \min\{\delta-|x_n-x_0|, |x_n-x_0|\}$$

it follows from (3) that

$$|f(y)+f(2x_0-y)-2f(x_0)| < k|y-x_0| < 2k|x_n-x_0|. \tag{5}$$

(3) and (5) imply that for any such y,

$$|f(2x_0-y)-f(2x_0-x_n)| \geqq |f(y)-f(x_n)|-3k|x_n-x_0|.$$

Therefore,

$$\omega_f(2x_0-x_n) \geqq \omega_f(x_n)-3k|x_n-x_0|,$$

and an application of (4) yields $\omega_f(2x_0-x_n)>0$. Hence $2x_0-x_n \in D(f)$. Now A follows by noting that x_0 is the arithmetic mean of $2x_0-x_n$ and x_n and since $D(f)$ is countable [6], only countably many such arithmetic means exist.

To establish B, suppose that $D(f)$ contains a subset D_0 which is dense in itself. Let $\xi_1, \xi_2, \dots$ be an arbitrary sequence of real numbers. If it can be shown that there is an $x \in E$ which does not belong to this sequence, then B follows.

Since D_0 is dense in itself, there exists a sequence $\{x_n\} \subset D_0$ and a sequence of positive numbers $\{\varepsilon_n\}$ such that $x_1 \in D_0$, $x_1 \neq \xi_1$ and

$$\varepsilon_1 < \min\{\omega_f(x_1), 1, |x_1-\xi_1|\}$$

and for $n>1$, $x_n \in D_0$, $x_n \neq \xi_n$, $|x_n-x_{n-1}|<\xi_{n-1}$ and

$$\varepsilon_n < \min\left\{x_n-x_{n-1}+\varepsilon_{n-1}, x_{n-1}+\varepsilon_{n-1}-x_n, \frac{\omega_f(x_n)}{n}, \frac{1}{n}, |x_n-\xi_n|\right\}.$$

The sequence of intervals $[x_n-\varepsilon_n, x_n+\varepsilon_n]$ is nested and, consequently, there is a point x_0 in their intersection. This point does not belong to $\{\xi_n\}$ because $x_n-\varepsilon_n< <x_0<x_n+\varepsilon_n$ and either $\xi_n<x_n-\varepsilon_n$ or $\xi_n>x_n+\varepsilon_n$. Moreover,

$$\frac{\omega_f(x_n)}{|x_n-x_0|} > \frac{\omega_f(x_n)}{\varepsilon_n} > n.$$

Therefore,

$$\lim_{n\to\infty} \frac{\omega_f(x_n)}{|x_n-x_0|} = \infty$$

which implies that $x_0 \in E$.

The theorem follows at once from A and B.

THEOREM 2.2. *If D is an arbitrary separated set, then there is a non-decreasing and smooth function f such that $D=D(f)$.*

PROOF. It may be assumed without loss of generality that $D\subset(0,1)$. Let $D_0=D$ and for each ordinal number inductively define

$$D_\alpha = D_{\alpha-1}\cap D'_{\alpha-1} \text{ when } \alpha \text{ is not a limit ordinal}$$

and

$$D_\alpha = \bigcap_{\beta<\alpha} D_\beta \text{ when } \alpha \text{ is a limit ordinal.}$$

Any separated set is countable [4, p. 147], so D may be written as $D=\{x_1, x_2, \dots\}$ and since D contains no subset which is dense in itself it follows that

$$D = \bigcup_{\alpha<\Omega} (D_\alpha - D_{\alpha+1}).$$

If $x_n\in D$, then there is an ordinal number α such that $x_n\in D_\alpha - D_{\alpha+1}$. Define

$$f_n(x) = \begin{cases} 0 & \text{if } x < x_n \\ \dfrac{1}{2}(d(x_n, D'_\alpha))^2 2^{-n} & \text{if } x = x_n \\ (d(x_n, D'_\alpha))^2 2^{-n} & \text{if } x > x_n \end{cases}$$

and let

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

Then $\sum_{n-1}^{m} f_n(x)$ converges uniformly to f so that $D^c\subset D(f)^c$. If $x_n\in D$, then it is clear that

$$\omega_f(x_n) = \big(d(x, D'_\alpha)\big)^2 2^{-n} > 0;$$

so $x_n\in D(f)$. Therefore $D(f)=D$. Since f is locally constant on $\bar{D}^c$, it is obviously smooth on $\bar{D}^c$. It is clear that since each f_n is nondecreasing, f is also non-decreasing.

The theorem follows if it can be shown that f is smooth on $\bar{D}$. To do this, three cases are considered: 1) $x\in D-D'$; 2) $x\in D'\cap D$; and, 3) $x\in D'-D$.

1) Suppose $x\in D-D'$. Then there is an $h>0$ such that $(x-h, x+h)\cap D=\{x\}$ and therefore it is clear that on $(x-h, x+h)$, $f(x)=f_n(x)+C$, where C is some constant and $x=x_n$. Because f_n is smooth, it is clear that f is smooth at x.

2) Suppose $x\in D'\cap D=D_1$. Let α be the least ordinal with $x\notin D_\alpha$. From the definition of D_α it is clear that α cannot be a limit ordinal, so there is an ordinal β such that $\alpha=\beta+1$, $x\in D_\beta$ and $x\notin D'_\beta$. Hence there is an $\varepsilon>0$ such that $(x-\varepsilon, x+\varepsilon)\cap D_\beta=\{x\}$. This implies that if $y\in D\cap(x-\varepsilon, x+\varepsilon)$ and $y\neq x$, then $y\in D_\gamma - D_{\gamma+1}$, where $\gamma<\beta$. Therefore, $d(y, D'_\gamma)\leqq d(y, x)$ because $x\in D_\beta\subset \subset D_{\gamma+1}\subset D'_\gamma$.

Choose any y such that $y\in(x, x+\varepsilon)$. From the definition of f it follows that

$$f(y)-f(x) \leqq f_n(x) + \sum_{x_i\in D\cap(x,y]} 2f_i(x_i) \leqq \tag{6}$$

$$\leqq f_n(x) + \sum_{x_i\in D\cap(x,y]} (d(x_i, x))^2 2^{-i} < f_n(x) + \sum_{x_i\in D\cap(x,y]} (d(x,y))^2 2^{-i} < f_n(x) + |x-y|^2.$$

Similarly, if $y\in(x-\varepsilon, x)$, then

$$f(x)-f(y)<f_n(x)+|x-y|^2. \tag{7}$$

It follows from (6) and (7) that if $0<h<\varepsilon$, then

$$f(x)+f_n(x)<f(x+h)<f(x)+f_n(x)+h^2$$

and

$$f(x)-f_n(x)-h^2<f(x-h)<f(x)-f_n(x).$$

Adding these two inequalities and rearranging the terms yields

$$-h^2<f(x+h)+f(x-h)-2f(x)<h^2. \tag{8}$$

It follows at once that f is smooth at x.

3) Suppose $x\in D'-D$. Let α be the least ordinal such that $x\notin D'_\alpha$. Then there is an $\varepsilon>0$ such that $(x-\varepsilon, x+\varepsilon)\cap D_\alpha=\emptyset$. If $y\in(x-\varepsilon, x+\varepsilon)\cap D$, then $y\in D_\beta-D_{\beta+1}$ for some $\beta<\alpha$. Thus, $d(y, D'_\beta)\leqq d(y, x)$ because $x\in D'_\beta$. Proceeding as in case 2 it follows that if $0<h<\varepsilon$, then

$$f(x)-h^2<f(x-h)<f(x) \quad\text{and}\quad f(x)<f(x+h)<f(x)+h^2.$$

Adding these two inequalities and rearranging the terms yields (8) once again. Therefore, f is smooth at x.

Theorems 2.1 and 2.2 give a characterization of $D(f)$ for any smooth and measurable function f which interestingly is the same as the characterization of $D(f)$ when f has a finite symmetric derivative everywhere, a result which was obtained by Z. Charzynski [1] and E. Szpilrajn [9].

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DEPARTMENT OF MATHEMATICS
NORTH CAROLINA STATE UNIVERSITY
RALEIGH, NORTH CAROLINA 27695
USA

DEPARTMENT OF MATHEMATICS
WESTERN ILLINOIS UNIVERSITY
MACOMB, ILLINOIS 61455
USA

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MULTIPLICATIVE FUNCTIONS WITH REGULARITY PROPERTIES. III

I. KÁTAI (Budapest), corresponding member of the Academy

1. Introduction

This paper is a continuation of [1], [2]. $\mathcal{M}$ and $\mathcal{M}^*$ denote the set of multiplicative and completely multiplicative functions with complex values, respectively. $\mathcal{L}\subseteq\mathcal{M}$ denotes the set of those $f(n)$ for which

$$\sum_{n=1}^{\infty}\frac{|f(n)|}{n}<\infty \tag{1.1}$$

holds.

Let $p, q, \pi, \ldots$ denote general primes. Let

$$R(f;\, p)=\sum_{\alpha=1}^{\infty}\frac{|f(p^{\alpha})|}{p^{\alpha}}. \tag{1.2}$$

It is obvious that $f\in\mathcal{L}$ if and only if $R(f, p)<\infty$ for every p, and

$$\sum_{p}R(f, p)<\infty. \tag{1.3}$$

Furthermore, if $f\in\mathcal{M}^*$, then $f\in\mathcal{L}$ if and only if

$$|f(p)|<p \tag{1.4}$$

holds for every prime p, and

$$\sum_{p}\frac{|f(p)|}{p}<\infty. \tag{1.5}$$

Given a subset $\mathcal{S}$ of natural numbers, we shall write

$$\mathcal{F}(n|\mathcal{S})=\sum_{n\in\mathcal{S}}\frac{f(n)}{n};\quad \mathcal{G}(n|\mathcal{S})=\sum_{n\in\mathcal{S}}\frac{g(n)}{n},$$

permitting that the series do not converge.

Let us consider those functions $f, g\in\mathcal{M}$, for which

$$\sum_{n=1}^{\infty}\frac{|g(n+K)-f(n)|}{n}<\infty \tag{1.6}$$

holds. Here K is a positive integer.

We are interested in all solutions of (1.6). The case $K=1$ has been treated and completely solved in [2]. If $f, g\in\mathcal{L}$, then f, g is a solution of (1.6). If $f(n)=$ $=g(n)=\chi(n)n^{\sigma+i\tau}$, $0\leqq\sigma<1$, and $\chi(n)$ is a multiplicative character $\bmod K$, then it is a solution of (1.6). To avoid some technical difficulties we shall assume that $f, g \in \mathcal{M}^*$. We shall solve the problem for $K=2$ and for odd K.

First we shall prove Theorem 1 which will serve as the main lemma in the proof of the assertion above.

THEOREM 1. *Let $f, g \in \mathcal{M}^*$, C be a nonzero constant, $K \geqq 1$ be an integer,*

$$\sum_{n=1}^{\infty} \frac{|g(n+K)-Cf(n)|}{n} < \infty. \tag{1.7}$$

Assume that there exists a prime p coprime to K for which $f(p)=0$ or $g(p)=0$ holds. Then $f \in \mathcal{L}$, $g \in \mathcal{L}$.

Let $f \in \mathcal{M}^*$. Assume that there exists a suitable polynomial $P(z)=a_0+a_1z+\dots$ $\dots+a_kz^k$ $(a_k=1, a_0 \neq 0)$ over the field of complex numbers for which

$$\sum_{n \geqq 1} \frac{1}{n} |P(E)f(n)| < \infty \tag{1.8}$$

is satisfied. Here the operators E, Δ, Δ_B, I are defined as follows: $Ex_n=x_{n+1}$, $Ix_n=x_n$, $\Delta=E-I$, $\Delta_B=E^B-I$, $\Delta^k=(E-I)^k$, $\Delta_B^k=(E^B-I)^k$ (see [1]).

Let $\mathcal{A}_f=\mathcal{A}$ denote the set of all polynomials P satisfying (1.8). It is obvious that $\mathcal{A}$ is an ideal. The constant polynomials belong to $\mathcal{A}$ if and only if $f \in \mathcal{L}$. We shall prove that if $\mathcal{A}$ is not empty, then it contains some element of type $(z^B-1)^k$ with suitable positive integers B, k. To prove this, we may assume that $f \notin \mathcal{L}$. From (1.8) we deduce that there exist no more than $k-1$ primes p_i for which $f(p_i)=0$. Assume in contrary that $p_1, \dots, p_k$ are distinct primes such that $f(p_i)=0$. Let N_0 be such a positive integer for which $N_0+i-1 \equiv 0 \pmod{p_i}$ $(i=1, \dots, k)$. Then $f(n+j)=0$ for $j=0, 1, \dots, k-1$, $n \equiv N_0 \pmod{B}$, $B=p_1, \dots, p_k$, and so, from (1.8),

$$\sum_{n \equiv N_0 \pmod{B}} \frac{|f(n)|}{n} < \infty.$$

From (1.8) the following assertion follows immediately: if there exist k residue classes $\bmod B$ with consecutive residues $l=r, r+1, \dots, r+k-1$, such that the series $\mathcal{F}(n \mid n \equiv l \pmod{B})$ $(l=r, r+1, \dots, r+k-1)$ are absolute convergent, then $\mathcal{F}(n \mid n \equiv r+k \pmod{B})$ is absolutely convergent too, consequently by a repeated application of this argumentation we get that $f \in \mathcal{L}$.

Let us assume now that P is a minimal degree polynomial in $\mathcal{A}$. Let $P(z)=$ $=\prod_{i=1}^{k}(z-\theta_i)$, and $Q_m(z)$ be defined by $Q_m(z)=\prod_{i=1}^{k}(z-\theta_i^m)$.

Since $Q_m(z^m)$ is a multiple of $P(z)$, therefore $Q_m(z^m) \in \mathcal{A}$, and so

$$\sum_n \frac{|Q_m(E^m)f(n)|}{n} < \infty.$$

Extending the summation only for the integers that are multiples of m, and observing that $Q_m(E^m)f(nm)=f(m)Q_m(E)f(n)$, under the assumption $f(m) \neq 0$

we deduce that

$$\sum_n \frac{|Q_m(E)f(n)|}{n} < \infty,$$

i.e. $Q_m \in \mathscr{A}$. Since $\delta(z) = \text{g.c.d.}\ (P(z), Q_m(z)) \in \mathscr{A}$, therefore $\deg \delta(z) \geqq k$, and so $P(z) = Q_m(z)$, consequently

$$\{\theta_1, \ldots, \theta_k\} = \{\theta_1^m, \ldots, \theta_k^m\} \quad (\text{if } f(m) \neq 0!). \tag{1.9}$$

(1.9) is valid, if m does not contain any prime factor p with $f(p)=0$. Since the set of such primes is finite and $a_0 = \theta_1 \ldots \theta_k (-1)^k \neq 0$, we get that $|\theta_j| = 1$ and $\varphi_j = \dfrac{\arg \theta_j}{2\pi}$ are rational numbers. Let $\varphi_j = \dfrac{a_j}{B}$ $(j=1, \ldots, k)$, $(a_1, \ldots, a_k) = 1$. B contains only such prime factors p for which $f(p)=0$. So

$$P(z) = \prod_{j=1}^{k} \left(z - \exp\left(\frac{2\pi i a_j}{B}\right)\right) \tag{1.10}$$

and

$$Q_B(z^B) = (z^B - 1)^k \in \mathscr{A}. \tag{1.11}$$

We have proved the following

THEOREM 2. *Let $f \in \mathscr{M}^*, f \notin \mathscr{L}$, assume that P is a minimal degree polynomial for which* (1.8) *holds. Then*

$$\sum_{n \geqq 1} \frac{|(E^B - I)^k f(n)|}{n} < \infty \tag{1.12}$$

where $B=1$, or $B = p_1^{\alpha_1} \ldots p_j^{\alpha_j}$ and $f(p_1) = \ldots = f(p_j) = 0$. There exist at most $k-1$ primes p for which $f(p)=0$.

If $B|C$ and (1.12) is satisfied with B, then it is satisfied with C instead of B, too. This is obviously true, since $(z^B-1)^k$ is a divisor of $(z^C-1)^k$. Consequently, looking for the solutions of (1.12) we may assume that B contains all the primes p for which $f(p)=0$.

THEOREM 3. *Let $B=1$, or $B = p_1^{\alpha_1} \ldots p_j^{\alpha_j}$, $p_1, \ldots, p_j$ be distinct primes. Let $f \in \mathscr{M}^*$. Assume that $f(p_l) = 0$ $(l=1, \ldots, j)$, and $f(p) \neq 0$, if $p \nmid B$. If* (1.12) *holds, then $f \in \mathscr{L}$, or $f(n) = \chi_B(n) n^{\sigma + i\tau}$, $0 \leqq \sigma < k$, and $\chi_B(n)$ is a suitable multiplicative character* mod B. *Conversely, for these functions* (1.12) *is satisfied.*

The second part of this theorem is obvious, the first part will be proved in Section 3.

Finally in Section 4 we shall prove

THEOREM 4. *Let $f, g \in \mathscr{M}^*$, $f \notin \mathscr{L}$, $g \notin \mathscr{L}$, and assume that* (1.6) *is satisfied.*

(1) *Let K be an arbitrary odd integer. Then $f(n) = g(n) = \chi_{K_1}(n)\, n^{\sigma + i\tau}$, $0 \leqq \sigma < 1$, K_1 is a divisor of K, χ_{K_1} is a multiplicative character* mod K_1.

For these functions (1.6) *holds.*

(2) *Let $K=2$. Then all the solutions are:*

(a) $f(n) = g(n) = n^{\sigma + i\tau}$, $0 \leqq \sigma < 1$ $(n = 1, 2, \ldots)$;

(b) $f(n) = g(n) = n^{\sigma + i\tau}$, $0 \leqq \sigma < 1$ *for odd n and* $f(2) = g(2) = 0$;

(c) $f(n)=\chi_8(n)n^{\sigma+i\tau}$, $g(n)=\chi_4(n)f(n)$, $0\leqq\sigma<1$, *where* χ_4 *is the nonprincipal character* mod 4, *and* χ_8 *is the following character* mod 8: $\chi_8(1)=\chi_8(7)=+1$, $\chi_8(5)=\chi_8(3)=-1$.

REMARK. We hope to solve this problem for every even K in a subsequent paper.

2. Proof of Theorem 1

First we formulate some remarks.

1. Since $\big||g(n+K)|-|C|\,|f(n)|\big|\leqq|g(n+K)-Cf(n)|$, and the relation $f\in\mathcal{L}$ depends only on the absolute value of f, we may assume that $g\geqq 0, f\geqq 0, C>0$.

2. If $f\in\mathcal{L}$, then $g\in\mathcal{L}$ obviously holds, and conversely.

3. We may assume that $f(q)=0$ for every prime-divisor q of K.

We have to prove only the last assertion. By putting $n\to qn$, $K=qK_1$, from (1.7) we get

$$\sum_n \frac{1}{n}\,|g(q)g(n+K_1)-Cf(q)f(n)|<\infty. \tag{2.1}$$

If $f(q)\neq 0$ and $g(q)\neq 0$, then (2.1) is a relation similar to (1.7) with K_1 instead of K, $K_1<K$. If $f(q)=0$ and $g(q)\neq 0$, then $g\in\mathcal{L}$, while for $g(q)=0$, $f(q)\neq 0$ we have $f\in\mathcal{L}$.

We shall assume $f, g\geqq 0$, and $f(q)=0$ for every $q|K$. Assume the condition of Theorem 1 holds.

Let $\mathcal{P}$ denote the set of all primes p coprime to K for which $f(p)=0$. Similarly, $\mathcal{R}$ denotes the set of q, $(q,K)=1$, $g(q)=0$. The sets $\mathcal{P}_1\subseteq\mathcal{P}$, $\mathcal{R}_1\subseteq\mathcal{R}$ are defined as follows

$$\mathcal{P}_1=\{p\in\mathcal{P}\,|\,\exists n_0\equiv K \pmod p),\ g(n_0)\neq 0\}, \tag{2.2}$$

$$\mathcal{R}_1=\{q\in\mathcal{R}\,|\,\exists n_0\equiv -K \pmod q),\ f(n_0)\neq 0\}. \tag{2.3}$$

$\mathcal{P}_1$, $\mathcal{R}_1$, and one of $\mathcal{P}$ and $\mathcal{R}$ may be empty.

First we observe that if $\mathcal{P}_1=\emptyset$ and $\mathcal{R}_1=\emptyset$, then the following condition (COND) holds.

(COND): $f(n)=0$ *if and only if* $g(n+K)=0$.

Indeed, "$f(n)=0$ iff $g(n+K)=0$" is true if $(n,K)>1$. Assume that $\mathcal{P}\neq\emptyset$, $p\in\mathcal{P}$. Then, from (2.2) we get $g(n)=0$ for $n\equiv K \pmod p)$. Similarly, for $\mathcal{R}\neq\emptyset$, $q\in\mathcal{R}$, from (2.3) we have $f(n)=0$ for $n\equiv -K \pmod q)$.

If $f(n)=0$, $(n,K)=1$, then n has a prime divisor p in $\mathcal{P}$, and so $g(n+K)=0$. Similarly, for $g(n)=0$, $(n,K)=1$, there exists a $q|n$, $q\in\mathcal{R}$, and so $n-K\equiv -K \pmod q)$, $f(n-K)=0$.

Step 1. $\mathcal{P}_1=\emptyset$, $\mathcal{R}_1=\emptyset$. If $p\in\mathcal{P}$, then $\mathcal{R}$ contains all the primes $q\equiv K \pmod p)$. If $q\in\mathcal{R}$, then $\mathcal{P}$ contains all π, $\equiv -K \pmod q)$. Since at least one of $\mathcal{P}$ and $\mathcal{R}$ is not empty, therefore both of them contain infinitely many elements.

Let Q be a large (odd) prime, $Q\nmid K-1$, $Q\in\mathcal{R}$.

Let Z_Q be the set of all reduced residue classes mod Q, $\mathscr{A}=Z_Q\setminus\overline{\mathscr{A}}$, $\mathscr{B}=Z_Q\setminus\overline{\mathscr{B}}$, and $\overline{\mathscr{A}}, \overline{\mathscr{B}}$ be defined as follows:

$$\overline{\mathscr{A}}=\{l_1, \dots, l_R \mid f(n)=0 \text{ if } n\equiv l_i (\text{mod } Q)\}, \ \overline{\mathscr{B}}=\{k_1, \dots, k_S \mid g(n)=0 \text{ if } n\equiv k_i (\text{mod } Q)\}.$$

In other words, a reduced residue class $l (\text{mod } Q)$ belongs to $\overline{\mathscr{A}}$ if and only if $f(n)=0$ for every $n\equiv l (\text{mod } Q)$, and $k (\text{mod } Q)$ belongs to $\overline{\mathscr{B}}$ if and only if $g(n)=0$ for every $n\equiv k (\text{mod } Q)$.

Let $l_i\in\overline{\mathscr{A}}$, i.e. $f(n)=0$ for $n\equiv l_i (\text{mod } Q)$. Hence COND is true, therefore $g(n+K)=0$ for every such n, i.e. $g(m)=0$ if $m\equiv K+l_i (\text{mod } Q)$ and $m>K$. We shall prove that the condition $m>K$ is superfluous. Let $m\equiv K+l_i (\text{mod } Q)$. t be so large that $N=m^{1+t(Q-1)}>K$. Then $N\equiv K+l_i (\text{mod } Q)$, so $g(N)=0$ which involves that $g(m)=0$ ($g\in\mathscr{M}^*$!). So we have proved the following assertion: if $l_i\in\overline{\mathscr{A}}$, then $l_i+K\equiv 0 (\text{mod } Q)$, or $l_i+K\in\overline{\mathscr{B}}$. Similarly, we can see the validity of the following assertion: if $k_i\in\overline{\mathscr{B}}$, then $k_i-K\in\overline{\mathscr{A}}$ or $k_i-K\equiv 0 (\text{mod } Q)$.

Since $Q\in\mathscr{R}$, $\mathscr{R}_1=\emptyset$, we get $-K\in\overline{\mathscr{A}}$, so $\overline{\mathscr{A}}$ cannot be empty.

We shall distinguish two cases according to $K\in\overline{\mathscr{B}}$ or $K\notin\overline{\mathscr{B}}$.

Case I:

$$\text{(2.4)} \qquad \begin{cases} \overline{\mathscr{A}} = \{l_1, \dots, l_{R-1};\ l_R = -K\} \\ \overline{\mathscr{B}} = \{k_1, \dots, k_{R-1};\ \emptyset\} \\ k_j \equiv l_j + K (\text{mod } Q) \quad (j = 1, \dots, R-1). \end{cases}$$

Case II:

$$\text{(2.5)} \qquad \begin{cases} \overline{\mathscr{A}} = \{l_1, \dots, l_{R-1};\ l_R = -K\} \\ \overline{\mathscr{B}} = \{k_1, \dots, k_{R-1};\ k_R = K\} \\ k_i \equiv l_i + K (\text{mod } Q) \quad (i = 1, \dots, R-1). \end{cases}$$

In Case I we have

$$\text{(2.6)} \qquad \begin{cases} \mathscr{A} = \{s_1, \dots, s_H\}, \quad H = Q-1-R \\ \mathscr{B} = \{t_1, \dots, t_H, t_{H+1}\} \\ t_j = s_j + K \quad (j = 1, \dots, H), \ t_{H+1} = K. \end{cases}$$

Assume that $\mathscr{A}$ is not empty. Since $\mathscr{A}$ and $\mathscr{B}$ are subgroups in Z_Q, therefore $H|Q-1$, $H+1|Q-1$, and so $H(H+1)|Q-1$, $H\leqq\sqrt{Q}$.

We know that $\mathscr{R}$ has infinitely many elements. Assume that Case I occurs for infinitely many Q. Since all the primes in $\overline{\mathscr{A}}\cap\overline{\mathscr{B}}$ are in $\mathscr{P}\cap\mathscr{R}$, and $H\leqq\sqrt{Q}$, therefore there exists an arbitrary large prime π for which $f(\pi)=g(\pi)=0$. But in this case for $Q=\pi$ only Case II can occur.

So we have proved that there exists infinitely many $Q\in\mathscr{R}$, for which Case II occurs.

In Case II we have

$$(2.7)\qquad \begin{cases} \mathcal{A} = \{s_1, \dots, s_H\}, \ H = Q-1-R \\ \mathcal{B} = \{t_1, \dots, t_H\} \quad (K \notin \mathcal{B}, -K \notin \mathcal{A}) \\ t_j = s_j + K \quad (j = 1, \dots, H) \end{cases}$$

assuming that $\mathcal{A}$ and $\mathcal{R}$ are not empty.

Furthermore, $\mathcal{A}$ and $\mathcal{B}$ are subgroups of the same order of Z_Q, and so $\mathcal{A}=\mathcal{B}$. From (2.7) we get that $a\in\mathcal{A}$ involves that $a+K\in\mathcal{R}=\mathcal{A}$, and so $a+tK\in \in\mathcal{A}(t=1, 2, \dots)$, thus $\mathcal{A}=Z_Q$. This is impossible, since $-K\notin\bar{\mathcal{A}}$. Consequently $\mathcal{A}=\mathcal{B}=\emptyset$, and so $f(n)=g(n)=0$ if $n\geqq 2$, $(n, Q)=1$. Since $g(Q)=0$, therefore $g(n)=0$ for every $n\geqq 2$, and so by COND we get $f(n)=0$ for $n\geqq 2$. ∎

Step 2. $\mathcal{P}_1\neq\emptyset$. Let $p\in\mathcal{P}_1$. Then $f(p)=0$, $(p, K)=1$, and there exists an $n_0\equiv K \pmod p$ such that $g(n_0)\neq 0$. Then $\mathcal{G}(n|n\equiv K \pmod p)<\infty$.

Since $nn_0\equiv K \pmod p$ if $n\equiv 1 \pmod p$, and $g(n_0)\neq 0$, we have $\mathcal{G}(n|n\equiv \equiv 1 \pmod p)<\infty$.

Since $n^{\varphi(p)}\equiv 1 \pmod p$ for $(n, p)=1$, so $g(n^{t(p-1)})=o(n^{t(p-1)})$ as $t\to\infty$, consequently $0\leqq g(n)<n$ holds for every n, $(n, p)=1$.

If $q_1, \dots, q_{p-1}$ are belonging to the same reduced residue class $\bmod p$, then $q_1, \dots, q_{p-1}\equiv 1 \pmod p$, therefore

$$\mathcal{G}(q|q\equiv l \pmod p))^{\varphi(p)} \ll \mathcal{G}(n|n\equiv 1 \pmod p)) < \infty,$$

if $l\neq 0$. So we have proved that

$$(2.8)\qquad \mathcal{G}(q|q\neq p)<\infty, \ g(q)<q \quad \text{if} \quad q\neq p.$$

Hence we get

$$(2.9)\qquad \mathcal{G}(n|(n, p)=1)<\infty \Rightarrow \mathcal{F}(n|n+K\not\equiv 0 \pmod p))<\infty.$$

If $\mathcal{P}_1$ contains at least two elements, p_1, p_2 say, then we are ready, since from (2.8), applying it with $p=p_1, p_2$ we deduce that $g\in\mathcal{L}$.

Assume now that $\mathcal{P}_1=\{p\}$ and $p\nmid K+1$. Starting from the second inequality in (2.9), we deduce that

$$(2.10)\qquad \mathcal{F}(\pi|\pi\not\equiv -K \pmod p))<\infty.$$

Since $p\nmid K+1$, therefore $(-K)^2\not\equiv -K \pmod p$, and so

$$(2.11)\qquad \mathcal{F}(\pi|\pi\equiv -K \pmod p))^2 \ll \sum_{n\equiv K^2 \pmod p} \frac{f(n)}{n}<\infty.$$

Furthermore, if $(\pi, p)=1$, then $\pi^\alpha\not\equiv -K \pmod p$ if $\alpha\equiv 0 \pmod{(p-1)}$, and so $f(\pi^\alpha)=o(\pi^\alpha)$ as $\alpha\to\infty$, whence $0\leqq f(\pi)<\pi$.

Thus we have

$$(2.10)\qquad \mathcal{F}(|\pi|\pi\neq q)<\infty, \quad 0\leqq f(\pi)<\pi,$$

which by $f(p)=0$ gives $f\in\mathcal{L}$.

Let us assume that $\mathcal{P}_1=\{p\}$ and $p|K+1$. Consider first the case $p=2$. This can occur if K is odd. Let $K+1\equiv 2^\beta m$, $(m, 2)=1$. Then, from (2.9) we have $\mathcal{G}(n|(n, 2)=1)<\infty$.

Let $\gamma>\beta$. Since $2^\gamma\|n-1$ implies $2^\beta\|n+K$, therefore

$$\mathscr{F}(n|n\equiv 1 \ (\mathrm{mod}\ 2^\gamma))\ll\mathscr{G}(n|2^\beta\|n)+1\ll\mathscr{G}(n|(n,2)+1<\infty. \tag{2.11}$$

By repeating the argument used above, we get immediately that $\mathscr{F}(\pi|\pi\neq 2)<\infty$, $f(\pi)<\pi$, which by $f(2)=0$ gives $f\in\mathscr{L}$.

Let now $p>2$. Assume first that there exists a k, $k\not\equiv 1 \ (\mathrm{mod}\ p)$ for which $f(k)\neq 0$. Since $kn\not\equiv 1 \ (\mathrm{mod}\ p)$ for $n\equiv 1 (\mathrm{mod}\ p)$, therefore, after substituting kn in place of n in the second series in (2.9) we deduce that $\mathscr{F}(n|n\equiv 1(\mathrm{mod}\ p))<\infty$, which leads to $f\in\mathscr{L}$, by repeating the consideration used earlier.

It has remained to consider the proof under the following condition:

(A) $\quad \mathscr{P}_1=\{p^*\},\ p^*|K+1,\ p^*>2,\ f(k)=0 \quad$ for every $k\not\equiv 1 \ (\mathrm{mod}\ p^*)$.

We shall consider this later.

Step 3. $\mathscr{R}_1\neq\emptyset$. By repeating the argument used in Step 2, we can execute the proof easily in the following cases:

(1) $\mathscr{R}_1$ contains at least two elements,

(2) $\mathscr{R}_1=\{q\}$, $q\nmid K-1$,

(3) $\mathscr{R}_1=\{2\}$, $2|K-1$,

(4) $\mathscr{R}_1=\{q\}$, $q|K-1$, $q>2$, $\exists k$, $k\not\equiv 1 \ (\mathrm{mod}\ q)$, $q(k)\neq 0$.

It has remained to prove it under the condition:

(B) $\quad \mathscr{R}_1=\{q^*\},\ q^*|K+1,\ q^*>2,\ f(k)=0 \quad$ for every $k\not\equiv 1(\mathrm{mod}\ q^*)$.

Step 4. Let us assume now that (A) holds. Then $\mathscr{P}\supseteq\{\pi|\pi\not\equiv 1 \ (\mathrm{mod}\ p^*)\}$. Hence we deduce that $g(n)=0$ if $(n,p^*)=1$. Let $k\equiv l(\mathrm{mod}\ p^*)$, $l\equiv 2,\dots,p^*-1$. Since $f(k)=0$, there exists a $\pi|k$, $\pi\in\mathscr{P}$. Since $\pi\neq p^*$, $\pi\notin\mathscr{P}_1$, and so $g(k+K)=0$ by the definition (see (2.2)). Since $K\equiv -1(\mathrm{mod}\ p^*)$, $k+K\equiv l-1(\mathrm{mod}\ p)$. We have proved that $g(n)=0$ if $n>K$ and $n\equiv 1,2,\dots,p^*-2 \ (\mathrm{mod}\ p^*)$. Since $n^l\equiv n(\mathrm{mod}\ p)$ if $l-1\equiv 0(\mathrm{mod}\ (p^*-1))$ and $g(n^l)=g(n)^l$, the condition $n>K$ can be substituted by $n>1$. So, $g(n)=0$ if $n>1$, $n\not\equiv 0,\ -1(\mathrm{mod}\ p^*)$. Let $n\equiv\equiv -1(\mathrm{mod}\ p^*)$. Since $n^2\equiv 1(\mathrm{mod}\ p^*)$, $g(n^2)=0$, consequently $g(n)=0$.

We have proved the following assertion:

(AS 1): If (A) holds then $g(n)=0$ for every $n\geqq 2$ that is not a power of p^*.

We shall prove now

(AS 2): If (B) holds then $f(n)=0$ for every $n\geqq 2$ that is not a power of q^*.

From (B) we get that $\mathscr{R}\supseteq\{\pi|\pi\not\equiv 1(\mathrm{mod}\ q^*)\}$. Let $k\equiv l(\mathrm{mod}\ q^*)$, $l\in\in\{2,\dots,q^*-1\}$. Since $g(k)=0$, therefore there exists a $\pi|k$ such that $g(\pi)=0$, and $\pi\neq q^*$. Consequently $\pi\in\mathscr{R}$ and $\pi\notin\mathscr{R}_1$, and so by (2.3) we get $f(k-K)=0$. So from the assumption (B) we get $f(n)=0$ for every $n\geqq 2$, $n\equiv\{1,\dots,q^*-2\}$ $(\mathrm{mod}\ q^*)$. Since for $(N,q^*)=1$ we may choose an exponent t such that $N^t\equiv 1(\mathrm{mod}\ q^*)$, so $f(N)^t=f(N^t)=0$. This proves the assertion.

Hence we can finish the proof easily. The case $\mathscr{P}_1=\emptyset$, $\mathscr{R}_1=\emptyset$ has been considered in Step 1. Assume that $\mathscr{P}_1=\emptyset$ and $\mathscr{R}_1=\emptyset$. Then (AS 1) and (AS 2) hold. We have to prove in this case only that $g(p^*)<p^*$, or $f(q^*)<q^*$. As it is known, $p^{*\alpha}-K=q^{*\beta}$ has at most a finite number of solutions in α,β, therefore $f(p^{*\alpha}-K)=0$

for every large α, which by the assumption of the theorem gives $R(g, p^*)<\infty$, i.e. $g(p^*)<p^*$.

Assume now that (A) holds and $\mathscr{R}_1=\emptyset$. Since $\mathscr{R}$ contains all the primes except at most p^*, and $\mathscr{R}_1=\emptyset$, $f(m)=0$ if $m+K\neq p^{*\gamma}$. But then $f(\pi)$ has to be zero for every π. This is an obvious consequence of the fact that $\pi^\alpha+K$ is not a power of π^* if α is large. Then $f\in\mathscr{L}$.

Finally we assume that $\mathscr{P}_1=\emptyset$ and (B) holds. Since $\mathscr{P}$ contains all the primes except at most q^* and $\mathscr{P}_1=\emptyset$, $g(n)=0$ if $n-K\neq q^{*\gamma}$. But then $g(\pi)$ has to be zero for every π, since $\pi^\alpha-K$ is not a power of q^* if α is large enough. Then $g\in\mathscr{L}$. The proof has been completed.

3. Proof of Theorem 3

We may assume that $f\notin\mathscr{L}$.

We shall conduct the proof by making the following steps.

(A) $|f(n)|=n^\lambda$, $0\leqq\lambda<k$, for every n coprime to B.

(B) Let $k>1$. If $\lambda\geqq k-1$, then (1.12) holds with $v(n)=n^{-1}f(n)$ instead of n and with $k-1$ instead of k. If $\lambda<k-1$, then (1.12) holds with $k-1$ instead of k.

(C) Theorem 3 is true for $k=1$.

Hence the theorem will follow immediately.

PROOF OF (A). Let $H(n)=f(n)$ if $k=1$, and $H(n)=(E^B-I)^{k-1}f(n)$ for $k\geqq 2$. From (1.12) we get

$$\sum_{n\geqq 1}\frac{1}{n}\max_{|l_n|\leqq\mathscr{D}}|H(n+l_nB)-H(n)|<\infty. \tag{3.1}$$

where $\mathscr{D}$ is an arbitrary constant.

Let $q>1$, $(q, B)=1$ be an arbitrary integer. Let

$$(1+z+\ldots+z^{q-1})^{k-1}=\alpha_0+\alpha_1 z+\ldots+\alpha_h z^h;\quad h=(q-1)(k-1).$$

Then $\sum_{j=0}^{h}\alpha_j=q^{k-1}$. Hence we may deduce immediately that

$$f(q)H(n)=(E^{Bq}-I)^{k-1}f(qn)=\sum_{j=0}^{h}\alpha_j H(qn+jB),$$

and by (3.1)

$$\sum_{n}\frac{1}{n}\max_{|l_n|\leqq\mathscr{D}}\left|H(qn+l_nB)-\frac{f(q)}{q^{k-1}}H(n)\right|<\infty. \tag{3.2}$$

For a positive integer N_0 coprime to B, a_0 denotes the least nonnegative integer for which $N_0-a_0B\equiv 0 \pmod q$, and N_1 being defined by $N_0=a_0B+qN_1$. It is obvious that $(N_1, B)=1$. After replacing $qn+l_nB\to N_0$, $n\to N_1$, from (3.2) we get

$$\sum_{qB\leqq N_0}\frac{1}{N_0}\left|H(N_0)-\frac{f(q)}{q^{k-1}}H(N_1)\right|<\infty. \tag{3.3}$$

Assume first that $|f(q)|<q^{k-1}$. Then, by repeating the argument used at the proof of Theorem 1 in [2], we can deduce that

$$\sum_{n\geqq 1}\frac{H(n)}{n}<\infty. \tag{3.4}$$

If $k=1$, then (3.4) implies $f\in\mathcal{L}$, while in the case $k>1$, (1.16) holds with $k-1$ instead of k. Assume now that $|f(q)|\geqq q^{k-1}$, $|f(q)|=q^{k-1+\eta}$. Let

$$A(x)=\sum_{N\leqq x}\frac{|H(N)|}{N}. \tag{3.5}$$

Since $N_1<\frac{N_0}{q}$ and $N_1<\frac{x}{q}$ occurs at most for q distinct $N_0\leqq x$, from (3.3) we get

$$A(x)\leqq\frac{|f(q)|}{q^{k-1}}A\left(\frac{x}{q}\right)+c_1, \tag{3.6}$$

c_1 is a suitable positive constant.

Hence it follows easily that

$$A(x)\ll x^{\eta}. \tag{3.7}$$

Every integer $N_1\leqq\frac{x}{q}-B$ occurs as a component for exactly q distinct $N_0\leqq x$, therefore, by (3.4) we get

$$A(x)\geqq\frac{|f(q)|}{q^{k-1}}A\left(\frac{x}{q}-B\right)-c_2,$$

with a suitably chosen $c_2>0$. Hence we can deduce easily

$$A(x)\gg x^{\eta-\varepsilon}, \tag{3.8}$$

for every constant $\varepsilon>0$, (3.7) and (3.8) give

$$\frac{\log A(x)}{\log x}\to\eta\quad(x\to\infty), \tag{3.9}$$

consequently η does not depend on q, and so $|f(n)|=n^{\eta+(k-1)}$ if $(n,B)=1$.

Now we prove that $\eta<1$.

Since $I=E^{-B}(E^B-I)+E^{-B}$,

$$\Big|\sum_{n\leqq x}(E^B-I)^{j-1}f(n)\Big|\leqq\Big|\sum_{n\leqq x}(E^B-I)^{j}f(n-B)\Big|+\Big|\sum_{n\leqq x}(E^B-I)^{j-1}f(n-B)\Big|. \tag{3.10}$$

If

$$U_l(x)=\sum_{n\leqq x}(E^B-I)^l f(n),$$

then from (3.10) we get immediately

$$U_{l-1}(x)\leqq U_l(x-B)+U_{l-1}(x-B). \tag{3.11}$$

(1.16) gives $U_k(x)=o(x)$. By a repeated application of (3.12) we deduce that $U_{k-1}(x)=o(x^2)$, ..., $U_0(x)=o(x^{k+1})$. The last relation implies $\eta<1$.

PROOF OF (B). Since $|f(q)|<q^{k-1}$ implies (3.4), the assertion is true for $\lambda<k-1$.

Let $\lambda\geqq k-1$, $v(n)=n^{-1}f(n)$. We have $|v(n)|=n^{k-2+\eta}$, $0\leqq\eta<1$. By an easy computation we get

$$\Delta_B^k f(n) = n\Delta_B^k v(n)+Bk\Delta_B^{k-1}v(n+B).$$

Let $g(n)=\Delta_B^{k-1}v(n)$. Hence, and from (1.16) we get

$$(n+Bk)g(n+B)-ng(n) = \lambda_n, \tag{3.12}$$

$$\sum_n \frac{|\lambda_n|}{n}<\infty. \tag{3.13}$$

From (3.12) we get

$$|g(n)| \leqq |g(n-B)|+\frac{|\lambda_n|}{n} \quad (n > B). \tag{3.14}$$

If we put $B(x)=\sum_{n\leqq x}|g(n)|$ then from (3.14) and (3.13) we can deduce that $B(x)\leqq$ $\leqq B(x-B)+c_3$, and so $B(x)=O(x)$. We have

$$\sum_{n\leqq x}\frac{|\Delta_B^k v(n)|}{n} \leqq \sum_{n\leqq x}\frac{|n\Delta_B^k v(n)+Bkg(n)|}{n^2}+\sum_{n\leqq x}\frac{|Bkg(n)|}{n^2}.$$

Both sums on the right hand side are convergent. Consequently v satisfies (1.16). But $|v(q)|<q^{k-1}$, so using the proved part of (B) we get $\sum_n \frac{|g(n)|}{n}<\infty$.

PROOF OF (C). We may assume that $|f(n)|=n^\eta$ $(\eta\geqq 0)$ for $(n, B)=1$. Let $f(n)=n^\eta t(n)$, $|t(n)|=1$ for $(n, B)=1$. Observing that $f(n+B)-f(n)=n^\eta\Delta_B t(n)+$ $+\eta n_\xi^{\eta-1}t(n, B)$, $n\leqq n_\xi\leqq n+B$, from (1.16) we deduce immediately that

$$\sum_{n=1}^{\infty}\frac{|t(n+B)-t(n)|}{n}<\infty. \tag{3.15}$$

It has remained to prove only that $t(n)=\chi_B(n)\cdot n^{i\tau}$ under the condition (3.15). This has been proved for $B=1$ in [2] (see Theorem 3). The case $B>1$ needs only a little change in the proof, therefore we shall only sketch it.

Let $f\in\mathcal{M}^*$, $|f(n)|=1$ for $(n, B)=1, f(n)=0$ for $(n, B)>1$, and

$$\sum_{n=1}^{\infty}\frac{|f(n+B)-f(n)|}{n}<\infty. \tag{3.16}$$

Let $q_1\equiv 1 \pmod B$, $q_2\equiv 1 \pmod B$ be arbitrary integers such that $q_2^2<q_1$, and $\frac{\log q_1}{\log q_2}$ be an irrational number. We define the decompositions according to the following rules.

Let N_0 be a general positive integer. We define a_0 to be the least nonnegative integer for which $N_0-a_0B\equiv 0 \pmod{q_1}$, N_1 is defined by $N_0=q_1N_1+a_0B$, a_1, N_2, $a_2, N_3, \ldots$ are defined similarly. Let us do it with q_2 instead of q_1: $N_0=q_2n_1+\tilde a_0B$, $n_1=q_2n_2+\tilde a_1B, \ldots$. Since $q_j\equiv 1 \pmod B (j=1, 2)$, $N_0\equiv N_\nu \pmod B$, $N_0\equiv n_\mu \pmod B$

and so $N_\nu - n_\mu \equiv 0 \pmod{B}$ holds for every ν and μ. Repeating the arguments used in the proof of the cited theorem, the last remark allows to prove that

$$q_2^{-M_2} \sum_{N_0 \in \mathscr{A}_{M_2}^{(q_2)}} |f(N_{\nu_1}) - f(n_{\nu_2})| \to 0 \quad \text{as} \quad H_2 \to \infty.$$

Continuing the proof nearly word for word, we deduce that $f(n) = e^{i\tau \log n}$ for every $n \equiv 1 \pmod{B}$. Let now $R(n) = e^{i\tau \log n}$ $(n=1, 2, \ldots)$, $u(n) = f(n)\bar{R}(n)$.

Since (3.16) is valid for f and $\bar{R}$, so it is true for $u(n)$ as well. Since $|R(n)|=1$ for every n, $|u(n)|=1$ if $(n, B)=1$, and $u(n)=0$ if $(n, B)>1$. Since $f(n)=R(n)$ for $n \equiv 1 \pmod{B}$, therefore $u(n)=1$ if $n \equiv 1 \pmod{B}$. Furthermore $u \in \mathscr{M}^*$. But then u has to be a character mod B. So we have $f(n) = \chi_B(n) n^{i\tau}$.

4. Proof of Theorem 4

We may assume that $f(p)=g(p)=0$ for every $p|K$. If it were not true then we could change K with a proper divisor K_1 of it.

Let $\Delta_n = g(n+K) - f(n)$. Let $H(n) = \dfrac{f(n)}{g(n)}$ be defined when $g(n) \neq 0$, i.e. for $(n, K)=1$ (see Theorem 1).

First we consider the case when K is odd. We are starting from the relations

$$g(2n+2K) = f(2n+K) + \Delta_{2n+K}; \; f(2n) = g(2n+K) - \Delta_{2n}.$$

Since $2 \nmid K$, from Theorem 1 we get $f(2) \neq 0$, $g(2) \neq 0$, and so

$$\Delta_n = g(n+K) - f(n) = \frac{1}{g(2)} [f(2n+K) + \Delta_{n2+K}] - \frac{1}{f(2)} [g(2n+K) - \Delta_{2n}],$$

whence, by (1.6) we get

$$\sum_n \frac{1}{n} \left| \frac{1}{g(2)} f(2n+K) - \frac{1}{f(2)} g(2n+K) \right| < \infty,$$

i.e.

$$\sum_{(n,2)=1} \frac{1}{n} |Cf(n) - g(n)| < \infty, \quad C = H(2). \tag{4.1}$$

Assume for a moment that $\mathscr{F}(n|(n,2)=1)$ is absolutely convergent. Then, by (4.1), $\mathscr{G}(n|(n,2)=1)$ is absolutely convergent, and by (1.6) we get $\mathscr{G}(2^\beta|\beta=1,2,\ldots)$ is absolutely convergent, which implies $g \in \mathscr{L}$, $f \in \mathscr{L}$. So we have

$$\sum_{(n,2)=1} \frac{|f(n)|}{n} = \infty. \tag{4.2}$$

Let now m be odd, $(m, K)=1$. Then $g(m) \neq 0$. Putting mn in place of n in (4.1), we get

$$\sum_{(n,2)=1} \frac{1}{mn} |Cf(m)f(n) - g(m)g(n)| < \infty, \tag{4.3}$$

i.e.

$$\sum_{(n,2)=1} \frac{1}{n} |CH(m) f(n) - g(n)| < \infty. \tag{4.4}$$

Taking into account (4.2), we get $H(m)=1$. Furthermore $f(n)=g(n)=0$ if $(n,K)>1$. Replacing $g(n)$ by $f(n)$ in (4.1), from (4.2) we get $C=1$, consequently $f(n)=g(n)$ for every n, and so the conditions of Theorem 3 are satisfied. We are ready.

Let now $K=2$. First we prove the following assertion:

(A): If $\mathscr{D}$ is an arbitrary odd integer such that

$$\sum_{n\equiv 1 \,(\mathrm{mod}\, 2\mathscr{D})} \frac{|f(n)|}{n} < \infty, \tag{4.5}$$

then $f, g \in \mathscr{L}$.

Let l be an integer, $(l, 2\mathscr{D})=1$. Since $f(p)\neq 0$ for $p\neq 2$ (see Theorem 1), we get $f(l)\neq 0$. Furthermore $nl^{\varphi(2\mathscr{D})-1}\equiv 1 \,(\mathrm{mod}\, 2\mathscr{D})$ holds for $n\equiv l \,(\mathrm{mod}\, 2\mathscr{D})$, thus we get

$$\sum_{n\equiv l \,(\mathrm{mod}\, 2\mathscr{D})} \frac{|f(n)|}{n} < \infty,$$

and so

$$\sum_{(n, 2\mathscr{D})=1} \frac{|f(n)|}{n} < \infty. \tag{4.6}$$

Hence we get

$$\sum_{(n-2, 2\mathscr{D})=1} \frac{|g(n)|}{n} < \infty,$$

and by putting $n=\mathscr{D}\nu$, $\big(g(\mathscr{D})\neq 0\big)$,

$$\sum_{\nu\equiv 1 \,(\mathrm{mod}\, 2)} \frac{|g(\nu)|}{\nu} < \infty,$$

consequently

$$\sum_{\nu\equiv 1 \,(\mathrm{mod}\, 2)} \frac{|f(\nu)|}{\nu} < \infty.$$

Furthermore, from (1.6) we have

$$\sum_{\alpha=1}^{\infty} \frac{|f(2^\alpha)|}{2^\alpha} \ll \sum \frac{|g(2^\alpha+2)|}{2^\alpha+2} + 1 \ll |g(2)| \sum_{\nu\equiv 1 \,(\mathrm{mod}\, 2)} \frac{|g(\nu)|}{\nu} + 1 \ll 1,$$

and so $f\in\mathscr{L}$, $g\in\mathscr{L}$.

Let n be an odd integer. We consider first the identities:

$$(n+2)(1+2tn) - n\big(1+2t(n+2)\big) = 2, \tag{4.7}$$

$$(n+2)(-1+2tn) - n\big(-1+2t(n+2)\big) = -2. \tag{4.8}$$

Hence we deduce that

$$\sum_{t=1}^{\infty}\frac{1}{t}\left|g(n+2)\,g(1+2tn)-f(n)\,f\big(1+2t(n+2)\big)\right|<\infty, \tag{4.9}$$

$$\sum_{t=1}^{\infty}\frac{1}{t}\left|f(n+2)\,f(-1+2tn)-g(n)\,g\big(-1+2t(n+2)\big)\right|<\infty. \tag{4.10}$$

Furthermore, by (1.6), $K=2$, we get

$$\sum_{t=1}^{\infty}\frac{1}{t}\left|g(1+2tn)-f(-1+2tn)\right|<\infty, \tag{4.11}$$

and so, from (4.11),

$$\sum_{t=1}^{\infty}\frac{1}{t}\left|g(1+2tn)-\frac{g(n)}{f(n+2)}g\big(-1+2t(n+2)\big)\right|<\infty. \tag{4.12}$$

Comparing this with (4.9), we deduce that

$$\sum_{t}\frac{1}{t}\left|\frac{g(n)}{f(n+2)}g\big(-1+2t(n+2)\big)-\frac{f(n)}{g(n+2)}f\big(1+2t(n+2)\big)\right|<\infty,$$

whence it follows immediately that

$$\sum\frac{1}{t(n+2)}\left|H(n+2)\,H(n)\,f\big(1+2t(n+2)\big)-g\big(-1+2t(n+2)\big)\right|<\infty. \tag{4.13}$$

By putting $d=n+2$, $m\equiv 0 \pmod{d}$, we get

$$\sum_{m\equiv 0 \,(\mathrm{mod}\, d)}\frac{1}{m}\left|H(d-2)\,H(d)\,f(1+2m)-g(-1+2m)\right|<\infty. \tag{4.14}$$

If we put $d=d_1, d=d_2$ and m runs over the integers $m\equiv 0\,(\mathrm{mod}\,[d_1, d_2])$, we deduce that

$$\sum_{m\equiv 0\,(\mathrm{mod}\,[d_1,d_2])}\left|H(d_1)\,H(d_1-2)-H(d_2)\,H(d_2-2)\right|\frac{1}{m}\left|f(1+2m)\right|<\infty.$$

We have that $\nu=1+2m$ runs over the integers $\equiv 1\,(\mathrm{mod}\,2[d_1, d_2])$ while m runs over $m\equiv 0\,(\mathrm{mod}\,[d_1, d_2])$. By using the assertion (A), and assuming that $f\notin\mathcal{L}$, we get

$$H(d_1)\,H(d_1-2)=H(d_2)\,H(d_2-2) \tag{4.15}$$

if d_1, d_2 are arbitrary odd integers.

By putting $d_2=d_1+2$, we deduce that $H(d+4)=H(d)$ for $d=1, 3, 5, \ldots$. Since $H(1)=1$, we get immediately $H(n)=1$ for $n\equiv 1\,(\mathrm{mod}\ 4)$. Furthermore, from (4.15) we get $H(n)=\mathrm{const}$ for $n\equiv -1\,(\mathrm{mod}\ 4)$. Since $n^2\equiv 1\,(\mathrm{mod}\,4)$, $H(n^2)=H(n)^2$, therefore $H(n)=\pm 1\,(\mathrm{mod}\ 4)$. Assume that $H(n)=-1$ when $n\equiv -1\,(\mathrm{mod}\ 4)$. By using (1.6) and $g(n)=(-1)^{\frac{n-1}{2}}f(n)$ for $(n, 2)=1$, we can deduce easily that

$$\sum_n \frac{1}{n} |f(n+4)+f(n)| < \infty, \tag{4.16}$$

and so $\sum_n \frac{1}{n} |f(n+8)-f(n)| < \infty$.

Since $f \notin \mathcal{L}$, therefore by Theorem 3 we get $f(n)=\chi_8(n) \cdot n^{\sigma+i\tau}$, $0 \leqq \sigma < 1$, where χ_8 is a suitable character mod 8. Observing that $(-1)^{\frac{n-1}{2}} = \chi_4(n)$, where χ_4 is the non-principal character mod 4, and substituting $f(n)=\chi_8(n)n^{\sigma+i\tau}$, $g(n)= =\chi_4(n)f(n)$ into (1.6), we deduce that

$$\chi_4(n+2)\chi_8(n+2)-\chi_8(n) = 0 \tag{4.17}$$

is satisfied indentically. This gives $\chi_8(5)=\chi_8(3)=-1$, $\chi_8(1)=\chi_8(7)=+1$. It is obvious that this pair is indeed a solution of (1.6).

Assume now that $H(n)=1$ for every odd n. Since $f(2)=g(2)=0$, therefore $f(n)=g(n)$ for every n, and so the conditions of Theorem 3 are satisfied, consequently $f(n)=g(n)=n^{\sigma+i\tau}$ for odd n and $f(n)=g(n)$ if n is even.

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L. EÖTVÖS UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE
BUDAPEST, MÚZEUM KRT. 6—8
H—1088

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ON GLOBS

J. CEDER (Santa Barbara)

Agronsky [1], [2; p. 29] showed that each F_σ bilaterally c-dense-in-itself subset of R can be expressed in the form $\bigcup\{A(\alpha)\colon \alpha\in[1,\infty)\}$ where each $A(\alpha)$ is closed and whenever $\alpha<\beta$ each point of $A(\alpha)$ is a bilateral c-limit point of $A(\beta)$. (Any set expressible in this form will be called a *linear glob*.) Then according to Agronsky [1] the function defined by

$$f(x)=\begin{cases}\dfrac{1}{\inf\{\alpha\colon x\in A(\alpha)\}} & \text{if} \quad x\in A\\ 0 & \text{if} \quad x\notin A\end{cases}$$

becomes a Darboux, upper semi-continuous function such that $A=f^{-1}(0,\infty)$).

Therefore, the following are true

(1) *each bilaterally c-dense-in-itself F_σ is a linear glob;*

(2) *a set is a linear glob if and only if it is the inverse image of a non-void open set under a Darboux semi-continuous function.*

In this paper we attempt to extend these two results to a particular two-dimensional setting. First we must extend the notions of a bilateral c-limit point and a Darboux function to two dimensions.

A point z is called a *panoramic c-limit point* of a planar set A if each non-degenerate closed triangle containing z also contains $2^{\aleph_0}$ points of A. We write $A\subseteqq pB$ if $A\subseteqq B$ and each point of A is a panoramic c-limit point of B. A family of closed sets $\{F(\alpha)\colon \alpha\in[1,\infty)\}$ is called a *hierarchy* if $F(\alpha)\subseteqq pF(\beta)$ whenever $\alpha<\beta$. The union of a hierarchy is called a *glob*. A function $f\colon R^2\to R$ is said to be *Darboux* if the image of each non-degenerate closed triangle is an interval. The study of such functions was initiated in [4] (see also [3]).

In the sequel we will show, in contrast to the one dimensional case, that a panoramically c-dense-in-itself F_σ set is not necessarily a glob. Thus, (1) does not extend to our two dimensional setting. As for as (2) is concerned, it is true that each glob is the inverse image of an open set under a Darboux semi-continuous function; however, the converse remains an unsolved problem.

Our unsuccessful attempt to extend (1) and (2) satisfactorily then leads to studying globs in detail. We obtain some interesting results, one of which, (e) below, is of independent interest. Many open questions remain. The main additional results can be summarized as follows:

(a) *globness is not a topological invariant;*

(b) *a panoramically c-dense-in-itself F_σ which is "locally" a glob is a glob;*

(c) *a product of two linear globs is not necessarily a glob* (it is if one set is open in the density topology);

(d) *any non-empty open set minus a null set contains a glob* (it is a glob if the difference is an F_σ);

(e) *if A is a null subset (resp. of first category) of the unit square I^2, then there exists a non-void perfect subset P of I and a subset Q of I which has full measure (resp. is residual) such that $P\times Q$ misses A.*

LEMMA 1. *Let E be an analytic set which is panoramically c-dense-in itself. If A is a compact subset of E, then there exists a compact set B such that $A\subseteqq pB\subseteqq pE$.*

PROOF. Let $\mathscr{W}$ consist of all open arcs in the unit circle. For x and $W\in\mathscr{W}$ define $W(x)=\{x+re^{i\theta}: r>0, e^{i\theta}\in W\}$. For $W\in\mathscr{W}$ let $|W|$ denote the length of W.

First we prove the following statement: For any $\varepsilon>0$ and $\alpha\in(0,\pi)$ there exists a finite set $F(\varepsilon,\alpha)\subseteqq E\cap\{y: \operatorname{dist}(y,A)<\varepsilon\}$ such that $W(x)\cap F(\varepsilon,\alpha)\neq\emptyset$ whenever $x\in A$ and $|W|>\alpha$. To show this we proceed as follows: For any $y\in A$ we can find a finite set $D(y)$ in $E\cap\{z: \operatorname{dist}(z,A)<\varepsilon\}$ such that $W(y)$ hits $D(y)$ whenever $|W|>\frac{\alpha}{2}$. By continuity we may find an open set $O(y)$ containing y such that $W(z)$ hits $D(y)$ whenever $z\in O(y)$ and $|W|>\alpha$. Since A is compact we can find $O(z_1), O(z_2), \ldots, O(z_m)$ which cover A. Now put $F(\varepsilon,\alpha)=\bigcup_{i=1}^{m} D(z_m)$.

Next define a sequence $\{C_i\}_{i=1}^{\infty}$ of subsets of E by $C_n=F\left(\frac{1}{n},\frac{1}{n}\right)$. Put $C=\bigcup_{n=1}^{\infty} C_n$. Enumerate C by $\{c_i\}_{i=1}^{\infty}$. Clearly A is panoramically dense in $A\cup C$ and $\bar{C}-C\subseteqq A$ and $A\cup C$ is compact.

Next select an open sphere S_i centered at c_i with radius r_i in such a way that $\bar{S}_i\cap\bar{S}_j=\emptyset$ when $i\neq j$. Let $\{J_n\}_{n=1}^{\infty}$ be a dense subset of $\mathscr{W}$. For fixed i define $P(i,n)$ to be a non-empty perfect subset of

$$E\cap S_i\cap\left\{z: |z-c_i|<\frac{1}{n}\right\}\cap J_n(c_i).$$

Put $B=\left(\bigcup_{i=1}^{\infty}\bigcup_{n=1}^{\infty}\right)P(i,n)\cup A$. Then clearly B is a dense-in-itself compact set for which $A\subseteqq pB\subseteqq pE$.

The next result says that a panoramically c-dense-in-itself F_σ which is "locally" a glob is a glob.

THEOREM 1. *Let E be a panoramically c-dense-in-itself F_σ subset of the plane. If each non-empty relatively open subset of E contains a glob, then E is a glob.*

PROOF. Let $E=\bigcup_{n=1}^{\infty} A_n$ where each A_n is compact. By Lemma 1 we may choose a compact set B_1 such that $A_1\subseteqq pB_1\subseteqq pE$. In fact by modifying the proof of Lemma 1 in place of the sets $P(i,n)$ we may choose the closed set F_2 where $\{F_\alpha: \alpha\geqq 1\}$ is a hierarchy whose union is contained in

$$E\cap S_i\cap\left\{z: |z-c_i|<\frac{1}{n}\right\}\cap J_n(c_i).$$

Hence there exists a sequence of globs $\{G^1_m\}_{m=1}^\infty$ in E with associated hierarchies $\{G^1_{m,\alpha}\colon \alpha\geqq 1\}$ such that $B_1-A_1=\bigcup\limits_{m=1}^\infty G^1_{m,2}$.

Now put $E_1=A_1$ and $E_2=B_1\cup A_2$. Then $E_1\subseteqq pE_2\subseteqq pE$. By the same argument as above we may find a compact set B_2 such that $E_2\subseteqq pB_2\subseteqq pE$ and a sequence of globs $\{G^2_m\}_{m=1}^\infty$ with associated hierarchies $\{G^2_{m,\alpha}\colon \alpha\geqq 1\}$ such that $B_2-E_2=$ $=\bigcup\limits_{m=1}^\infty G^2_{m,3}$. Then put $E_3=B_2\cup A_3$.

Continuing in this way we obtain sequences $\{E_n\}_{n=1}^\infty$ and $\{B_n\}_{n=1}^\infty$ of compact sets together with sequences of globs $\{G^k_m\}_{m=1}^\infty$ with associated hierarchies $\{G^k_{m,\alpha}\colon \alpha\geqq 1\}$ such that $E_n\subseteqq pE_{n+1}\subseteqq pE$ and $B_n-E_n=\bigcup\limits_{m=1}^\infty G^n_{m,n+1}$ for all n.

Let us now define a family $\{E(\lambda)\colon \lambda\geqq 1\}$ as follows: $E(\lambda)=E_n\cup\left(\bigcup\limits_{m=1}^\infty G^n_{m,\lambda}\right)$ if $n\leqq\lambda<n+1$. It easily follows that each $E(\lambda)$ is closed. It is also clear that $E(\alpha)\subseteqq pE(\beta)$ whenever $\alpha<\beta$. Since $E=\bigcup\limits_{n=1}^\infty E_n$ it follows that E is the glob $\cup\{E(\lambda)\colon \lambda\geqq 1\}$.

THEOREM 2. *A union of a family of globs is a glob if and only if the union is an F_σ set.*

PROOF. In view of Theorem 1 it suffices to show that $S\cap A$ is a glob where A is any glob which intersects the open unit disk S. Let A be the union of a hierarchy $\{A(\alpha)\colon \alpha\geqq 1\}$. For $0<r<1$ let $S(r)$ be the closed disk of radius r centered at the origin. Pick δ such that $S(\delta)\cap A\neq\emptyset$. Let f be an increasing function from $[1,\infty)$ onto $[\delta,1)$. For $\alpha\geqq 1$ define $F(\alpha)=A(\alpha)\cap S(f(\alpha))$. It is easily checked that $\{F(\alpha)\colon \alpha\geqq 1\}$ is a hierarchy whose union is $A\cap S$.

COROLLARY 1. *The non-empty intersection of any open set with a glob is a glob.*

COROLLARY 2. *The union of countably many globs is a glob.*

PROOF. It follows trivially from Theorem 2. However there is also an easy direct proof: Let $\{\{G^k_\alpha\colon \alpha\geqq 1\}\}_{k=1}^\infty$ be a sequence of hierarchies. For $n\leqq\alpha<n+1$ define $F(\alpha)=\bigcup\limits_{k=1}^n G^k_\alpha$. Then $\{F(\alpha)\colon \alpha\geqq 1\}$ is a hierarchy whose union is the union of the family of globs.

THEOREM 3. *Any open set is a glob.*

PROOF. Since each open disk is obviously a glob the result follows from Corollary 1.

Agronsky's proof that a linear glob is the inverse image of a non-void open set under a Darboux semi-continuous function can be easily carried over to two dimensions.

THEOREM 4 (Ceder [4]). *If A is a glob, then there exists a Darboux, upper-semi-continuous function f such that $A=f^{-1}(0,\infty)$.*

The converse of Theorem 4 is true, namely:

THEOREM 5. *If f is an upper-semi-continuous (resp. lower semi-continuous) Darboux function, then any non-empty $f^{-1}(a, \infty)$ [resp. $f^{-1}(-\infty, a)$] is a glob.*

PROOF. Assume f is upper-semi-continuous and $f^{-1}(a, \infty) \neq \emptyset$. Choose $b \in (a, \infty) \cap \operatorname{rng} f$. Let g be a homeomorphism at $(a, b]$ onto $[1, \infty)$. For each $\alpha \in [1, \infty)$ put $F(\alpha) = \{x : f(x) \geqq g^{-1}(\alpha)\}$. Then it is easily checked that $\{F(\alpha); \alpha \geqq 1\}$ is a hierarchy of closed sets whose union is $f^{-1}(a, \infty)$.

In the case of a Darboux Baire 1 function $f: R \to R$ it is always true that the inverse image of an open set is a linear glob. It is unknown whether the analogue of this result is true for functions from R^2 to R, except in the special case of Theorem 5.

I conjecture that

CONJECTURE. *If f is a Darboux Baire* 1 *function and G is any non-void open set, then $f^{-1}(G)$ is a glob.*

In general if f is Darboux Baire 1 and G is an open set, then $f^{-1}(G)$ is a panoramically c-dense-in-itself F_σ set. However, such a set may not be a glob as shown by the following example.

EXAMPLE 1. *There exists a panoramically c-dense in-itself F_σ set which does not contain any glob.*

CONSTRUCTION. Let $A = \{(x, y): 0 < y < 1,\ 0 < x < 1,\ x \text{ is rational}\}$. Clearly A is a panoramically c-dense-in-itself F_σ set.

Suppose A contains the union of some hierarchy $\{F(\alpha): \alpha \geqq 1\}$. Pick $(x_\alpha, y_\alpha) \in F(\alpha)$ such that $x_\alpha = \inf(\operatorname{dom} F(\alpha))$ by the compactness of $F(\alpha)$. Since $F(\alpha) \subseteq pF(\beta)$ when $\alpha < \beta$ it follows that $x_\beta \neq x_\alpha$. This contradicts the countability of the rationals. Hence, A contains no glob.

By Theorem 5 it follows that $A \neq f^{-1}(a, \infty)$ for any a and lower-semi-continuous Darboux function f.

In view of Example 1 it would be interesting to find a reasonable characterization of those panoramically c-dense-in-itself F_σ sets which are globs. In this vein we do have the following

THEOREM 6. *Any panoramically c-dense-in-itself F_σ which is also of second category everywhere is a glob.*

PROOF. Obviously any F_σ set of second category has non-void interior and hence, contains a glob. Now apply Theorem 1.

The converse of Theorem 6 is not true by the following example.

EXAMPLE 2. *There exists a glob of* 1^{st} *category and measure* 0.

CONSTRUCTION. Choose $\{P_n\}_{n=1}^\infty$ be a sequence of nowhere-dense, non-void, null perfect sets in $(0, 1)$ such that any open subset of $(0, 1)$ contains some P_n. Then for each n the set $A_n = P_n \times \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ is a closed, nowhere-dense planar set. Hence $\bigcup_{n=1}^\infty A_n$ is an F_σ null set of first category. It is easy to see that $\bigcup_{n=1}^\infty A_n$ is a glob using Theorem 1 and Theorem 8.

It is unknown whether the measure analogue of Theorem 6 is valid: that is, any panoramically c-dense-in-itself F_σ of positive measure is a glob or more simply, *does any closed set of positive measure contain a glob?*

Next we proceed to investigate what product sets are globs. First we have a couple of lemmas.

A linear, non-void, nowhere dense bounded perfect set will be called a *Cantor* set.

LEMMA 2. *Each linear, uncountable analytic set contains a linear glob.*

PROOF. Let E be a linear, uncountable analytic set and choose P to be a Cantor set in E. Let P' be the set of bilateral limit points of P. Let $\{B_n\}_{n=1}^{\infty}$ be a countable base for the relative topology on P'. For each n pick P_n to be a Cantor set in $B_n \cap P'$. Then, $\bigcup_{n=1}^{\infty} P_n$ is a bilaterally c-dense-in-itself F_σ set, hence a glob.

The two-dimensional analogue of Lemma 2 is false by Example 1.

For a set A let $d(A)$ denote the distance set of A, i.e., $\{|x-y|: x, y \in A\}$.

LEMMA 3. *Let J be any open interval centered at* 1 *with length less than* 2. *There exist null Cantor sets P and Q such that*

$$d(P) \cap \big(d(Q) \cdot J\big) = \{0\}.$$

PROOF. Without loss of generality let us assume that $J=(.9, 1.1)$. Pick sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ in $(0, 1)$ converging to 0 such that for all n, $a_{n+1}<b_{n+1}<a_n<b_n$ and $11a_n<9b_n$. Let $d^+(P)=d(P)-\{0\}$ for any set P.

Let $G_n=(b_{n+1}, a_n)$ for each n. We will construct a Cantor set P such that $d^+(P) \subseteqq \bigcup_{n=1}^{\infty} \bar{G}_n$. First pick $x_1, x_2 \in I$ with $x_1<x_2$ and $x_2-x_1 \in G_1$. Choose $\varepsilon>0$ so that $z \in [x_1, x_1+\varepsilon]$ and $w \in [x_2-\varepsilon, x_2]$ imply that $w-z \in G_1$. Pick $a_k<\varepsilon$ and $y_1 \in (x_1, x_1+\varepsilon)$ and $y_2 \in (x_2-\varepsilon, x_2)$ such that y_1-x_1 and x_2-y_2 belong to G_k. Put $F_0=[x_1, y_1]$ and $F_1=[y_2, x_2]$ and $E_1=\{x_1, y_1, y_2, x_2\}$. Then $d^+(E_1) \subseteqq G_1 \cup G_k$.

Now consider the interval $[x_1, y_1]$. In a similar way we may find z_1 and z_2 such that $x_1<z_1<z_2<y_1$ and $d^+(\{x_1, z_1, z_2, y_1\}) \subseteqq \bigcup_{n=1}^{\infty} G_n$. Likewise we may find w_1 and w_2 such that $y_2<w_1<w_2<x_2$ and

$$d^+(\{x_1, x_2, y_1, y_2, z_1, z_2, w_1, w_2\}) \subseteqq \bigcup_{n=1}^{\infty} G_n.$$

Let $F_{00}=[x_1, z_1]$, $F_{01}=[z_2, y_1]$, $F_{10}=[y_2, w_1]$ and $F_{11}=[w_2, x_2]$. Letting E_2 be the set of endpoints of these four intervals we then have $d^+(E_2) \subseteqq \bigcup_{n=1}^{\infty} G_n$.

We may continue this process by induction. Letting $\mathscr{A}$ consist of all sequences of 0's and 1's, we obtain a system of closed intervals $\{F_{i_1, i_2, \dots, i_k}: i \in \mathscr{A},\ k \geqq 1\}$, such that for each $i \in \mathscr{A}$

(1) $F_{i_1, \dots, i_k, m} \cap F_{i_1, \dots, i_k, n} = \emptyset$ whenever $m \neq n$;

(2) $F_{i_1, \dots, i_k, m} \subseteqq F_{i_1, \dots, i_k}$ for all m;

(3) if E_k consists of the endpoints of all $F_{i_1,\ldots,i_k}$ as i ranges through $\mathscr{A}$, then $d^+(E_k)\subseteq \bigcup_{n=1}^{\infty} G_n$;

(4) $\lambda(\bigcup\{F_{i_1,\ldots,i_k}: i\in\mathscr{A}\})<\frac{1}{k}$ for each k.

Put $P=\bigcup\{\bigcap_{k=1}^{\infty} F_{i_1,\ldots,i_k}: i\in\mathscr{A}\}$. Then P is a Cantor set and $d^+(P)=$ $=\mathrm{cl}\,(\bigcup_{k=1}^{\infty} d^+(E_k))\subseteq \mathrm{cl}\,(\bigcup_{n=1}^{\infty} G_n)=(\bigcup_{n=1}^{\infty} \bar{G}_n)\cup\{0\}$. Hence, $d^+(P)\subseteq \bigcup_{n=1}^{\infty} \bar{G}_n$.

Let $U_n=(a_n, b_n)$. Then $a_n<\frac{9}{10}\frac{(a_n+b_n)}{2}<\frac{11}{10}\frac{(a_n+b_n)}{2}<b_n$. Hence for each n we may choose an open interval V_n centered at $\frac{a_n+b_n}{2}$ such that $\bar{V}_n\cdot J\subseteq U_n$.

Now proceeding as before we can find a Cantor set Q such that $d^+(Q)\subseteq \bigcup_{n=1}^{\infty} \bar{V}_n$. Therefore, $d^+(Q)\cdot J\subseteq(\bigcup_{n=1}^{\infty} \bar{V}_n)\cdot J=\bigcup_{n=1}^{\infty} (\bar{V}_n\cdot J)\subseteq \bigcup_{n=1}^{\infty} V_n$. Since $(\bigcup_{n=1}^{\infty} V_n)\cap(\bigcup_{n=1}^{\infty} \bar{G}_n)=\varnothing$ we must have $d(P)\cap(d(Q)\cdot J)=\{0\}$. Moreover, it is clear from (4) that P and Q both have measure zero.

THEOREM 7. *There exist null Cantor sets P and Q such that $P\times Q$ contains no panoramic c-limit points of itself.*

PROOF. Apply Lemma 3 to $J=(0.9, 1.1)$ to obtain null Cantor sets P and Q such that $d(P)\cap d(Q)\cdot J=\{0\}$. For any point $(p,q)\in P\times Q$ let $T(p,q)$ be that closed triangle having (p,q) as a vertex and having adjacent sides of length 2 and slopes 0.95 and 1.05. If $(p_1, q_1)\in T(p,q)-\{(p,q)\}$ then $\frac{q-q_1}{p-p_1}\in J$ and hence $d(P)\cap(d(Q)\cdot J)\neq\{0\}$. Therefore, $T(p,q)\cap(P\times Q)=\{(p,q)\}$ for all (p,q) in $P\times Q$.

Neither Lemma 3 nor Theorem 7 can be improved to assert that both P and Q have positive measure 0, because in that case $P\times Q$ would have positive measure and therefore, have a point of density which is obviously a panoramic c-limit point.

As an interesting consequence of Theorem 7 we have

COROLLARY 3. *There exists a continuous function from R into R whose graph contains a product of two Cantor sets.*

PROOF. Let A be the rotation of the set $P\times Q$ in Theorem 7 by 45 degrees. Then A is a closed set having no two points with the same first coordinate. Obviously we can find a continuous function $f: R\to R$ such that A is a subset of the graph of f.

COROLLARY 4. *A product of linear globs is not necessarily a glob.*

PROOF. Let P and Q be specified as in Theorem 7. By Lemma 2 let A and B be linear globs in P and Q respectively. By Theorem 7 $P\times Q$, and hence $A\times B$ does not have any panoramic c-limit points of itself. Therefore, $A\times B$ cannot contain a glob.

Although a product of linear globs may not be a glob in general, it is under certain conditions as shown by

THEOREM 8. *The product of two linear globs is a glob if one of them is open in the density topology.*

PROOF. Suppose A and B are linear globs and B is open in the density topology. According to [8] or [2, p. 29] we can write B as a union of a hierarchy $\{B(\alpha)\colon \alpha\geqq 1\}$ such that whenever $\alpha<\beta$ and $x\in B(\alpha)$, the density at x with respect to $B(\beta)$ is 1, written $\delta(x, B(\beta))=1$. Let A be the union of the hierarchy $\{A(\alpha)\colon \alpha\geqq 1\}$. For each $\alpha\geqq 1$ put $C(\alpha)=A(\alpha)\times B(\alpha)$.

It suffices to show that each point of $C(\alpha)$ is a panoramic limit point of $C(\beta)$ for any $\alpha<\beta$. Let $(a, b)\in C(\alpha)$ and W be any wedge with vertex at (a, b). Without loss of generality we may assume that W is determined by two lines given by $y=b+\lambda(x-a)$ and $y=b+\mu(x-a)$ where $0<\lambda<\mu$. Suppose (a, b) is not a limit point of $W\cap C(\beta)$. Then for each $\varepsilon>0$ and $x\in(a, a+\varepsilon)\cap A(\beta)$ the closed interval $[\lambda(x-a)+b, \mu(x-a)+b]$ misses $B(\beta)$. Then $\dfrac{B(\beta)\cap[b, b+\mu(x-a)]}{\mu(x-a)}\leqq\dfrac{\lambda(x-a)}{\mu(x-a)}=\dfrac{\lambda}{\mu}<1$. Hence $\delta(b, B(\beta))\leqq\lambda/\mu<1$, a contradiction. Therefore, $C(\alpha)\subseteqq pC(\beta)$.

As a consequence of Theorem 8 it follows that the intersection of two globs may fail to be a glob even if it is non-empty. For example, let A_1 and A_2 be linear globs where $A_1\cap A_2=\{0\}$. Then $A_1\times R$ and $A_2\times R$ are both globs, but $(A_1\times R)\cap(A_2\times R)=\{(x, y)\colon x=0\}$.

Example 3 in the sequel shows, surprisingly, that the product of two linear globs each dense in some open interval can fail to contain a glob. In this case both sets are null, first category linear globs.

Before reaching Example 3 we need a few preliminary results.

THEOREM 9. *Suppose* $\{A_n\}_{n=1}^{\infty}$ *is a sequence of non-void, closed, nowhere-dense planar sets such that* $\operatorname{diam} A_n\to 0$, $\operatorname{card}(A_n\cap A_m)\leqq\aleph_0$ *whenever* $n\neq m$, *and no* A_n *contains a glob then,* $\bigcup_{n=1}^{\infty} A_n$ *is not a glob.*

PROOF. Put $A=\bigcup_{n=1}^{\infty} A_n$ and let us suppose that A is the union of a hierarchy $\{F_\alpha\colon \alpha\geqq 1\}$ where $F_1\neq\emptyset$. Put $B=\operatorname{cl}\big(\bigcup_{\alpha<2} F_\alpha\big)$ and $B_i=B\cap A_i$. Then $\emptyset\neq B\subseteqq F_2\subseteqq A$.

Next suppose 0 is any open set hitting B. Since $\big(\bigcup_{\alpha<2} F_\alpha\big)\cap 0\subseteqq\bigcup_{i=1}^{\infty}(B_i\cap 0)$ and $\big(\bigcup_{\alpha<2} F_\alpha\big)\cap 0$ is a glob (by Corollary 1), hence $\operatorname{card}\big(\bigcup_{i=1}^{\infty}(B_i\cap 0)\big)=c$ and there exists an i such that $\operatorname{card}(B_i\cap 0)=c$. Moreover, since $\operatorname{card}(B_i\cap B_j)\leqq\aleph_0$ for all $j\neq i$ it follows that $\big(B_i-\bigcup_{j\neq i} B_j\big)\cap 0\neq\emptyset$.

Next there exists a $j\neq i$ for which $(B_j-B_i)\cap 0\neq\emptyset$. If this were not true then $0\cap\bigcup_{n\neq i} B_n\subseteqq B_i\cap 0$ and consequently $\big(\bigcup_{\alpha<2} F_\alpha\big)\cap 0\subseteqq B_i\subseteqq A_i$ and A_i would contain a glob, a contradiction. Since B_i is closed there exists an open set $N\subseteqq 0$ such that $N\cap B_i=\emptyset$ and $N\cap B_j\neq\emptyset$. Now repeating the argument in the previous paragraph there exists $m\neq i$ such that $\big(B_m-\bigcup_{n\neq m} B_n\big)\cap N\neq\emptyset$.

Now pick $x\in(B_i-\bigcup_{j\neq i} B_j)\cap 0$ and $y\in(B_m-\bigcup_{n\neq m} B_n)\cap N$. Since $\operatorname{diam} B_k\to 0$ there exists an M such that $k\geqq M$ implies $\operatorname{diam} B_k\leqq 1/3|x-y|$. There exists a sphere S_1 of radius less than $1/3|x-y|$ centered at x which misses $\bigcup\{B_k\colon k<M, k\neq i\}$. Likewise there exists a sphere S_2 of radius less than $1/3|x-y|$ centered at y which misses $\bigcup\{B_k\colon k<M, k\neq m\}$.

It follows then that there exists two closed spheres T_0 and T_1 each of diameter less than 1 such that $T_0\cap B\neq\emptyset$ and $T_1\cap B\neq\emptyset$ and moreover, such that no B_k hits both T_0 and T_1.

Next apply the previous argument again to the interior of T_0 (resp. T_1) which plays the role of 0. Then we obtain two closed spheres T_{00} and T_{01} (resp. T_{10} and T_{11}) each of diameter less than 1/2 and each hitting B such that no B_k hits both T_{00} and T_{01} (resp. T_{10} and T_{11}).

Proceeding in this way we obtain a system of closed sets, where consists of all sequences of 0's and 1's, $\{T_{i_1,\dots,i_n}\colon i\in\mathscr{A}, n\geqq 1\}$ with the following properties: For each i and n $T_{i_1,\dots,i_{n+1}}\subseteq T_{i_1,\dots,i_n}$, $T_{i_1,\dots,i_n,0}\cap T_{i_1,\dots,i_n,1}=\emptyset$, $T_{i_1,\dots,i_n}\cap B\neq\emptyset$, $\operatorname{diam} T_{i_1,\dots,i_n}<2^{-n}$ and no B_k hits both $T_{i_1,\dots,i_n,0}$, and $T_{i_1,\dots,i_n,1}$.

Finally put $T=\left\{\bigcap_{k=1}^{\infty} T_{i_1,\dots,i_k}\cap B\colon i\in\mathscr{A}\right\}$. Obviously $T\subseteq B$ and $\operatorname{card} T=c$. Moreover, for each i $\bigcap_{k=1}^{\infty} T_{i_1,\dots,i_k}\cap B$ consists of a single point. But by the last property above distinct points of T cannot lie in the same B_k. This is a contradiction.

It follows from Theorem 9 that some sets which are unions of countably many planar arcs (eg. all closed line segments each of whose endpoints are rational) are not globs. However, it is unsolved whether or not a union of an arbitrary countable set of planar arcs can be a glob. In particular, *can an arc contain a glob?*

It is unknown whether or not the condition that $\operatorname{diam} A_n\to 0$ is essential in Theorem 9.

LEMMA 4. *Let* $\{A_n\}_{n=1}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$ *be sequences of disjoint Cantor sets such that* $\operatorname{diam} B_n<1/n$ *and* $\operatorname{diam} A_n<1/n$. *Then there exists a sequence of planar, closed disjoint nowhere-dense sets* $\{Q_k\}_{k=1}^{\infty}$ *such that*

(1) $\left(\bigcup_{n=1}^{\infty} A_n\right)\times\left(\bigcup_{m=1}^{\infty} B_m\right)=\bigcup_{k=1}^{\infty} Q_k$,

(2) $\operatorname{diam} Q_k\to 0$,

(3) *for each* k *there exist* n *and* m *and open intervals* I *and* J *for which*

$$Q_k=(A_n\cap I)\times(B_m\cap J).$$

PROOF. For each n and m it is clear that we can decompose A_n and B_m into a finite number of disjoint portions (i.e., sets of the form A_n intersect an open interval) $\{A^m_{ni}\}_{i=1}^{k(m,n)}$ and $\{B^n_{mj}\}_{j=1}^{k(m,n)}$ such that $\operatorname{diam}(A^m_{ni}\times B^n_{mj})<\dfrac{1}{n+m}$ for each $i,j\leqq k(m,n)$. Choose $\{Q_k\}_{k=1}^{\infty}$ to be any enumeration of all the sets $A^m_{ni}\times B^n_{mj}$.

LEMMA 5. *Let* $\{I_n\}_{n=1}^{\infty}$ *be any sequence of closed intervals such that* $0<\sup I_{n+1}<\inf I_n$ *and* $\lim_{n\to\infty}(\sup I_n)=0$.

If W is any open subinterval of $(0, 1)$, *there exists a Cantor set* $P \subseteq W$ *for which* $d(P) \cap \left(\bigcup_{n=1}^{\infty} I_n \right) = \emptyset$.

PROOF. Pick distinct points x and y in W such that $|x-y| \notin \bigcup_{n=1}^{\infty} I_n$. Clearly we can find disjoint non-degenerate closed intervals T_0 and T_1 containing x and y respectively and having diameters less than 1 such that $|z-w| \notin \bigcup_{n=1}^{\infty} I_n$ whenever $z \in T_0$ and $w \in T_1$.

Repeating this process in T_0 (with int T_0 playing the role of W) we can find disjoint non-degenerate closed intervals T_{00} and T_{01} in int T_0 each of diameter less than $1/2$ such that $|z-w| \notin \bigcup_{n=1}^{\infty} I_n$ whenever $z \in T_{00}$ and $w \in T_{01}$.

Continuing this process by induction we obtain a system of closed intervals $\{T_{i_1, \dots, i_n} : i \in \mathscr{A}, n \geqq 1\}$ where $\mathscr{A}$ is the set of all sequences of 0's and 1's, having the following properties for each i and n

$$T_{i_1, \dots, i_{n+1}} \subseteq T_{i_1, \dots, i_n} \subseteq W, \quad T_{i_1, \dots, i_n, 0} \cap T_{i_1, \dots, i_n, 1} = \emptyset, \text{ diam } T_{i_1, \dots, i_n} < 2^{-n}$$

and $|z-w| \notin \bigcup_{k=1}^{\infty} I_k$ whenever $z \in T_{i_1, \dots, i_n, 0}$ and $w \in T_{i_1, \dots, i_n, 1}$. Put

$$P = \bigcup \left\{ \bigcap_{n=1}^{\infty} T_{i_1, \dots, i_n} : i \in \mathscr{A} \right\}.$$

Then P is a Cantor set contained in W and $d(P) \cap \bigcup_{n=1}^{\infty} I_n = \emptyset$.

EXAMPLE 3. *There exist null, first category linear globs each a dense subset of* $(0, 1)$, *whose product is not a glob.*

PROOF. Let $\{W_i\}_{i=1}^{\infty}$ be an open base of intervals for $(0, 1)$ with diam $W_i < 1/i$. Let $J = (.9, 1.1)$. Pick sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that $0 < (11/9) b_{n+1} < a_n < b_n < 1$ for all n. Put $I_n = [a_n, b_n]$.

Using Lemma 5 and induction we can find disjoint Cantor sets $\{B_i\}_{i=1}^{\infty}$ such that $B_i \subseteq W_i$ and $d(B_i) \cap \left(\bigcup_{n=1}^{\infty} \bar{J} \cdot I_n \right) = \emptyset$. Thus $\left(\bigcup_{i=1}^{\infty} d(B_i) \right) \cap \left(\bigcup_{n=1}^{\infty} \bar{J} \cdot J_n \right) = \emptyset$ and $\left(\bigcup_{i=0}^{\infty} \bar{J}^{-1} \cdot d(B_i) \right) \cap \left(\bigcup_{n=1}^{\infty} I_n \right) = \emptyset$. Letting $T_n = [b_{n+1}, a_n]$ we have $\bigcup_{n=1}^{\infty} \bar{J}^{-1} \cdot d(B_n) \subseteq \bigcup_{k=1}^{\infty} T_k \cup \{0\}$.

Again using Lemma 5 we can find disjoint Cantor sets $\{A_i\}_{i=1}^{\infty}$ such that $A_i \subseteq W_i$ and $d(A_i) \cap \left(\bigcup_{k=1}^{\infty} T_k \right) = \emptyset$. Put $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{m=1}^{\infty} B_m$. Then for each n and m $d(A_m) \cap J^{-1} \cdot d(B_n) = \{0\}$ so according to the argument in Theorem 7, $A_m \times B_n$ has no panoramic limit points and hence, cannot contain any glob.

Now applying Lemma 4 there exists a sequence $\{Q_k\}_{k=1}^{\infty}$ of disjoint closed, nowhere dense sets such that diam $Q_k \to 0$ and no Q_k contains a glob and $\bigcup_{k=1}^{\infty} Q_k =$

$=A\times B$. According to Theorem 9, $A\times B$ is not a glob. Obviously A and B are null, and of first category (from Theorems 7 and 8) and linear globs dense in $(0, 1)$.

THEOREM 10. *Globness is not preserved under product homeomorphisms.*

PROOF. Let P and Q be Cantor sets as specified in Theorem 7. Let C be a Cantor set which has positive measure in each relative open interval of C. Let h and g be homeomorphisms of R such that $h(C)=P$ and $q(C)=Q$. From the proof (and notations of that) of Lemma 2 it is clear that we can find a linear glob A with $A\subseteqq C$ such that $A=\bigcup_{n=1}^{\infty} A_n$ with A_n a perfect subset of $B_n\cap C'$ having measure greater than $\frac{1}{2}|B_n\cap C'|$. Then each point of A has density 1. Therefore by Theorem 8, $A\times A$ is a glob. However, $h(A)\times g(A)$ being a subset of $P\times Q$ is not a glob.

Note that linear globs are preserved under homeomorphisms of R. Also in contrast to one-dimensional Darboux function we have the following consequence of Theorem 10.

EXAMPLE 4. *There exists a Darboux upper semi-continuous function f and an homeomorphism k of R^2 such that $f\circ k$ is not Darboux.*

CONSTRUCTION. Let k be the product homeomorphism of Theorem 10 given by $k^{-1}(x, y)=\langle h(x), g(x)\rangle$. Let A be the set in Theorem 11. Since $A\times A$ is a glob, by Theorem 4 there exists a Darboux upper semi-continuous function f such that $A\times A=f^{-1}(0, \infty)$. However, $(f\circ k)^{-1}(0, \infty)=k^{-1}(f^{-1}(0, \infty))=h(A)\times \times g(A)$ which is not a glob. Hence, $f\circ k$ cannot be a Darboux upper semi-continuous function by Theorem 5. Therefore, $f\circ k$ is not Darboux since $f\circ k$ is upper semi-continuous.

We have seen that an F_σ set of second category contains a glob whereas it is unknown whether or not an F_σ set of positive measure contains a glob.

A related question is: does an open set minus a first category or a null set contain a glob. A natural approach to resolving this question is to try to find "fat" products of linear globs missing a given negligible set. We pursue this approach in the sequel and are able to conclude that an open set minus a null set does contain a product glob (Theorem 17). But the problem of finding a product glob in an open set minus a first category set remains unsolved.

To begin with we need to strengthen the following well-known theorem.

THEOREM 11 (Kuratowski—Ulam [6, 7]). *If A is a planar first category set, then there exists a residual subset B of the line such that for each $x\in B$ $\{y: (x, y)\in A\}$ is of first category.*

If A is a planar set, and $x\in R$, then $A(x)$ will denote the set $\{y: (x, y)\in A\}$. The stronger version of the Kuratowski—Ulam theorem is the following:

THEOREM 12. *If A is a planar set of first category, then there exists a residual G_δ set B in R such that $\bigcup_{x\in P} A(x)$ is of first category whenever P is a F_σ subset of B of first category.*

PROOF. It suffices to replace R^2 by $[0,1]^2$. Let $\mathcal{G}$ consist of all closed nowhere dense subsets of $[0,1]$. Let $\{B_n\}_{n=1}^{\infty}$ be a countable base for $[0,1]$.

Let us first show the theorem with A being closed and nowhere dense. Then, for each n there exists a residual G_δ set G_n such that for all $P\in\mathcal{G}$, $P\subseteq G_n$ implies $B_n \nsubseteq \bigcup_{x\in P} A(x)$. If this is not the case then there exists some n such that for all residual G_δ sets G there exists $P\in\mathcal{G}$ with $P\subseteq G$ and $B_n\subseteq \bigcup_{x\in P} A(x)$. By Theorem 11 there exists a $y\in B_n$ such that $L=\{x\colon (x,y)\in A\}$ is of first category. Since A is closed L is closed too and hence L is nowhere dense also. Hence, taking $G=[0,1]-L$ we obtain a contradiction.

Now put $B=\bigcap_{n=1}^{\infty} G_n$. Then B is residual and if $P\subseteq B$ with $P\in\mathcal{G}$, we have $B_n\nsubseteq \bigcup_{x\in P} A(x)$ for all n. Since A and P are closed it follows that $\bigcup_{x\in P} A(x)$ is also closed. Hence, $\bigcup_{x\in P} A(x)$ is nowhere dense. Therefore, we have shown that if A is closed and nowhere dense there exists a residual G_δ set B such that for all closed and nowhere dense subsets P of B, $\bigcup_{x\in P} A(x)$ is closed and nowhere dense.

Suppose $A=\bigcup_{n=1}^{\infty} A_n$ where each A_n is nowhere dense. Let $A'=\bigcup_{n=1}^{\infty} \bar{A}_n$. For each n there exists a residual G_δ set Γ_n such that $P\in\mathcal{P}$ and $P\subseteq\Gamma_n$ imply that $\bigcup_{x\in P} \bar{A}_n(x)\in\mathcal{P}$. Since $\bigcup_{x\in P} A(x)\subseteq \bigcup_{x\in P} A'(x)=\bigcup_{n=1}^{\infty}\bigcup_{x\in P} \bar{A}_n(x)$ we have that $P\in\mathcal{G}$ and $P\subseteq B=\bigcap_{n=1}^{\infty}\Gamma_n$ imply $\bigcup_{x\in P} A(x)$ is of first category.

Next suppose $P=\bigcup_{n=1}^{\infty} P_n$ where each P_n is closed and nowhere dense. Since $\bigcup_{x\in P} A(x)=\bigcup_{n=1}^{\infty}\bigcup_{x\in P_n} A(x)$ we have for each first category F_σ set $P\subseteq B$ that $\bigcup_{x\in P} A(x)$ is of first category.

It is unknown whether or not the F_σ requirement can be deleted from the above theorem.

THEOREM 13. *If A is a planar set of first category, there exists an F_σ set P of first category dense in R and a residual G_δ set Q such that $P\times Q$ misses A.*

PROOF. By Theorem 12 pick a F_σ set P of first category dense in B. Let Q be a residual G_δ missing $\bigcup_{x\in P} A(x)$. Then $P\times Q$ misses A.

Neither Theorem 12 nor Theorem 13 can be improved to assert that P is of second category. For example, take $A=\{(x,x)\colon x\in R\}$.

It is unknown whether or not the sets P and Q in Theorem 13 can be chosen so that $P\times Q$ contains a glob. Thus, the question of whether an open set minus a first category set contains a glob or not remains unsolved.

The well-known correlation between measure and category [7] and Theorem 11 suggests that Fubini's Theorem can also be extended in the manner of Theorem 12. We are unable to prove this and therefore offer it as a conjecture.

CONJECTURE. *If A is a null set in the plane, then there exists a set B of full measure in the line such that $\bigcup_{x\in P} A(x)$ has measure zero whenever P is a G_δ null subset of B.*

Nevertheless the measure analogues of Theorems 13, remain true. In order to show it we only need a weaker extension of Fubini's theorem which we now proceed to establish. Let λ_1 and λ_2 denote the Lebesgue measure in R and R^2 respectively.

THEOREM 14 (Eggleston [5]). *If E is an F_σ subset of I^2 and $\varepsilon>0$, then there exists a non-void perfect set $P\subseteqq I$ such that*

$$\lambda_1\Big(\bigcup_{x\in P} E(x)\Big)<\lambda_2(E)+\varepsilon.$$

LEMMA 6. *Suppose N is a G_δ set for which $\lambda_2(N)=0$ and A is a Cantor set in I such that $\lambda_1(N(x))=0$ whenever $x\in A$. Then, for each $\varepsilon>0$ there exists a closed set $Q\subseteqq I$ with $\lambda_1(Q)>1-\varepsilon$ and a non-void perfect set $P\subseteqq A$ such that $(P\times Q)\cap N=\emptyset$.*

PROOF. Let h be a homeomorphism of I onto I such that $\lambda_1(h(A))=1-\varepsilon/3$. Let $M=\{(x,y)\colon (h^{-1}(x),y)\in N\}$. Then $\lambda_2(M)=0$ and $\lambda_2(M\cup(I-h(A)\times I))=\varepsilon/3$. Let G be an open set containing $M\cup((I-h(A))\times I)$ with $\lambda_2(G)<2\varepsilon/3$. Applying Theorem 14 we obtain a perfect set $P\neq\emptyset$ such that $\lambda_1\big(\bigcup_{x\in P} G(x)\big)<\lambda_2(G)+\varepsilon/3<\varepsilon$. Put $Q=I-\bigcup_{x\in P} G(x)$. Then Q is closed and $\lambda_1(Q)>1-\varepsilon$. Since $x\notin h(A)$ implies $G(x)=I$ we must have $P\subseteqq h(A)$. Let $P^*=h^{-1}(P)$ then $P^*\subseteqq A$ and $P^*\times Q$ misses N.

The main idea of the proof of the next theorem is due to M. Laczkovich.

THEOREM 15. *If $E\subseteqq I^2$ and $\lambda_2(E)=0$, then there exists a non-void perfect set P and an F_σ set Q with $\lambda_1(Q)=1$ such that $(P\times Q)\cap E=\emptyset$.*

PROOF. Clearly it suffices to assume E is a G_δ set. Let $\{I_n\}_{n=1}^\infty$ be an enumeration of all open subintervals of $(0,1)$ with rational endpoints. Let $M=\{x\in I\colon \lambda_1(E(x))=0\}$. By Fubini's theorem $\lambda_1(M)=1$ and hence we can choose a Cantor set $A\subseteqq M$. Then, according to Lemma 6 there exists a closed subset Q_1 of I_1 with $\lambda_1(Q_1)>1/2\,\lambda_1(I_1)$ and a Cantor set $P_1\subseteqq A$ such that $P_1\times Q_1$ misses E.

Now split P_1 into two disjoint non-void portions F_0 and F_1 each having diameter less than $2/3$. For each $i\in\{0,1\}$ apply Lemma 6 again to obtain a closed subset S_i of I_2 with $\lambda_1(S_i)>3/4\,\lambda_1(I_2)$ and a Cantor set T_i in F_i such that $T_i\times S_i$ misses E. Then put $Q_2=S_0\cap S_1$ and $P_2=T_0\cup T_1$. Then $P_2\times Q_2$ misses E and $\lambda_1(Q_2)>1/2\,\lambda_1(I_2)$.

Now split T_0 (resp. T_1) into two portions F_{00} and F_{01} (resp. F_{10} and F_{11}) each of diameter less than $1/3$. Apply Lemma 6 to each F_{ij} to obtain a closed subset S_{ij} of I_3 with $\lambda_1(S_{ij})>7/8\,\lambda_1(I_3)$ and a Cantor set T_{ij} in F_{ij} such that $T_{ij}\times S_{ij}$ misses E. Put $Q_3=\bigcap\{S_{ij}\colon i,j\in\{0,1\}\}$ and $P_3=\bigcup\{T_{ij}\colon i,j\in\{0,1\}\}$. Then $P_3\times Q_3$ misses E and $\lambda_1(Q_3)>1/2\,\lambda_1(I_3)$.

Next we split each T_{ij} into 2 portions and apply Lemma 6 again. Continuing the inductive process (as done in the proofs of Lemmas 3 and 5 and Theorem 9) we will obtain a sequence of closed sets $\{Q_n\}_{n=1}^\infty$ such that $\lambda_1(Q_n)>1/2\,\lambda_1(I_n)$ and a sequence of closed sets $\{P_n\}_{n=1}^{\infty}$ such that $P_n \times Q_n$ misses E. Putting $P = \bigcap_{n=1}^{\infty} P_n$ it is easily seen that P is a non-void Cantor set. Putting $Q = \bigcup_{n=1}^{\infty} Q_n$ we see that $\lambda_1(Q \cap J) > 1/2\, \lambda_1(J)$ for each open subinterval J of I. Hence, $I - Q$ has no density points and thus $\lambda_1(Q) = 1$. Moreover, it follows that $P \times Q$ misses E.

The next result is the measure analogue of Theorem 13.

THEOREM 16. *If A is a planar null set, there exists an F_σ set P dense in R and an F_σ set Q of full measure such that $P \times Q$ misses A.*

PROOF. First of all it is clear that we can modify the proof of Theorem 15 to conclude that there exists an F_σ set Q of full measure (i.e. $\lambda_1(Q \cap J) = \lambda_1(J)$ for each finite interval J) such that $P \times Q$ misses E. Let $\{B_n\}_{n=1}^{\infty}$ be a countable base for R. Then applying this result we obtain a Cantor subset P_n of B_n and an F_σ set Q_n of full measure such that $P_n \times Q_n$ misses E. Now put $P = \bigcup_{n=1}^{\infty} P_n$ and let Q be an F_σ set of full measure which is a subset of $\bigcap_{n=1}^{\infty} Q_n$, a $F_{\sigma\delta}$ set of full measure. Then, $P \times Q$ misses E.

Theorem 16 cannot be improved to assert that the set P has measure > 0. For example, let A be the union of all lines of rational slope passing through the origin. Select closed subsets P' and Q' of positive measure of P and Q respectively. If $(P' \times Q') \cap A = \emptyset$, then the set of all ratios p/q, $p \in P'$ and $q \in Q' - \{0\}$ would be nowhere dense in R. But this contradicts the fact that the ratio set of two closed non-null sets contains an interval.

THEOREM 17. *If G is a planar open set and A is a planar null set, then $G - A$ contains a glob dense in $G - A$.*

PROOF. Note that the product $P \times Q$ in Theorem 16 is a glob because Q is open in the density topology by Theorem 8. Now let $\{S_n\}_{n=1}^{\infty}$ be a sequence of open rectangles forming a base for G. Then $S_n - A$ contains a glob B_n. Then $\bigcup_{n=1}^{\infty} B_n$ is a glob, by Corollary 2, which is dense in $G - A$.

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Note. Mustafa, in a work to be published, has shown that (1) a glob can be the union of countably many arcs and (2) each planar set of positive measure contains a glob.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
SANTA BARBARA, CA 93106
USA

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SMALL IDEALS IN RADICAL THEORY*

B. J. GARDNER (Hobart)

Introduction

It was proved by Armendariz [2] that a radical class of associative rings is hereditary if and only if its semi-simple class is closed under essential extensions. This result parallels an analogous one for modules obtained by Dickson [6]. Furthermore, it is possible to modify the argument of Armendariz to obtain the well-known result that hereditary classes of associative rings determine hereditary lower radical classes; the resulting argument closely resembles one used by Dickson [7] to obtain the corresponding result for modules. The moral is that in a suitably "nice" environment, the analogues of the module results referred to can be obtained without the necessity of having injective envelopes.

For modules over perfect rings, the results referred to can be dualized. Thus a semi-simple class is homomorphically closed if and only if its radical class is closed under essential covers and a homomorphically closed class always generates a homomorphically closed semi-simple class. These facts can be proved by means of arguments using projective covers [5].

In view of the "superfluity" of injective envelopes referred to above, one is led to ask about the possibility of obtaining analogues of these results on homomorphically closed semi-simple classes in contexts where there are no projective covers. Of course, such results are false in the class of associative rings.

An examination of the argument used by Armendariz in [2] reveals that two crucial facts are (i) the product of two normal epimorphisms is normal and (ii) for an ascending chain $\{I_\lambda \mid \lambda \in \Lambda\}$ of ideals and an ideal J of a ring A, we have $J \cap \sum I_\lambda = \sum (J \cap I_\lambda)$. Thus for the dual argument to work, we need transitivity for normality of subobjects plus the lattice identity $J + \bigcap I_\lambda = \bigcap (J + I_\lambda)$ for a descending chain of normal subobjects I_λ and a normal subobject J. (Cf. Proposition 2.10 below.) These latter conditions are met, for example, by the category of compact abelian groups, though the latter, by the dualization of some results of Gabriel [9], has projective covers, so we can use the same argument as for modules over a perfect ring. (Alternatively, we can exploit Pontryagin Duality.) Incidentally, by a remark of Grothendieck [10] (for a proof see, e.g., Mitchell [13], p. 86) an abelian category satisfying both the conditions on subobject lattices and with direct sums and products must be trivial, so the example of modules over a perfect ring makes it clear that the results on homomorphically closed semi-simple classes do not depend on the lattice condition. Whether there exist situations in which these results hold in the absence

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of both projective covers and the lattice conditions seem to be a question worthy of further investigation. (For that matter, one can make an analogous comment concerning hereditary radical classes and essentially closed semi-simple classes.)

The homomorphically closed semi-simple classes (i.e. the semi-simple radical classes) of associative rings are completely known. A generalization, wherein semi-simple classes $\mathscr{S}$ were required to satisfy the weaker condition

$$I \lhd S \in \mathscr{S} \quad \text{and} \quad I^2 = 0 \Rightarrow S/I \in \mathscr{S}$$

has recently been studied by Anderson and Wiegandt [1] and Sands [14].

Motivated by the above considerations, we consider small ideals in a similar way. An ideal I of a ring A is *small* if the following holds:

$$J \lhd A \quad \text{and} \quad I+J = A \Rightarrow J = A.$$

Under these conditions we write $I \ll\lhd A$. A ring R is an *essential cover* of a ring B if there is a surjective homomorphism $f\colon R \to B$ with $\operatorname{Ker}(f) \ll\lhd R$. (This latter is the dual of the notion of essential extension.)

Let $\mathscr{R}$ be a radical class with semi-simple class $\mathscr{S}$. We shall consider the following two conditions

(*) $I \ll\lhd S \in \mathscr{S} \Rightarrow S/I \in \mathscr{S}$;

(**) $I \ll\lhd A$ and $A/I \in \mathscr{R} \Rightarrow A \in \mathscr{R}$.

It turns out that (*) implies (**) but not conversely. After some general remarks about small ideals we consider semi-simple classes satisfying (*) and radical classes satisfying (**). Semi-simple classes satisfying (*) are not easy to find. The only ones we can give are those which are contained in the Brown—McCoy semi-simple class; in these (and they include, of course, the homomorphically closed semi-simple classes) there are no non-zero small ideals. There are plenty of examples of radical classes satisfying (**); which of these have semi-simple classes satisfying (*) is largely unknown. Condition (*) for a semi-simple class $\mathscr{S}$ is equivalent to the following weak "right exactness" condition for the corresponding radical class $\mathscr{R}$:

$$I \ll\lhd A \Rightarrow (\mathscr{R}(A)+I)/I = \mathscr{R}(A/I).$$

Throughout the paper, all rings considered are associative.

Small ideals

We first present some examples of small ideals.

EXAMPLE 1.1. Let A be subdirectly irreducible with heart $H(A)$. Then $H(A) \ll\lhd A$.

EXAMPLE 1.2. Let A be a ring in which the ideals form a chain. Then every proper ideal I of A is small. In particular, every proper ideal of the ring of all linear transformations of an arbitrary vector space is small.

PROPOSITION 1.3. *Let A be a ring, M a maximal ideal of A. If $I \ll\lhd A$, then $I \subseteq M$.*

Let $\mathcal{G}$ denote the Brown—McCoy radical class. There are important connections between $\mathcal{G}$ and small ideals. We have immediately

COROLLARY 1.4. *If $I \ll\lhd A$, then $I \subseteq \mathcal{G}(A)$.*

COROLLARY 1.5. *If the intersection of the maximal ideals of A is zero, in particular if $\mathcal{G}(A)=0$, then $I \ll\lhd A$ implies $I=0$.*

A ring A is called *strongly regular,* if for every $a \in A$ there is an $x \in A$ such that $a=a^2x$.

COROLLARY 1.6. *If A is strongly regular, then $I \ll\lhd A$ implies $I=0$.*

THEOREM 1.7. *Let A be a ring in which every ideal is contained in a maximal ideal. If $I \lhd A$, then $I \ll\lhd A$ if and only if $I \subseteq \cap \{M \mid M$ is a maximal ideal of $A\}$.*

PROOF. If I is not small, let $I+J=A, J \lhd A, J \neq A$. Let M be a maximal ideal such that $J \subseteq M$. Then $A=I+J \subseteq I+M$, so $I \subsetneqq M$. The converse is just Proposition 1.3. □

COROLLARY 1.8. *Let A be a ring with a left or right identity, $I \lhd A$. Then $I \ll\lhd A$ if and only if $I \subseteq \mathcal{G}(A)$.*

PROOF. For every maximal ideal M of A, the simple ring A/M has a one-sided, and therefore a two-sided, identity [4]. Thus $\mathcal{G}(A)= \cap \{M \mid M$ is a maximal ideal of $A\}$. □

COROLLARY 1.9. *Let A be a commutative ring with identity, $I \lhd A$. Then $I \ll\lhd A$ if and only if I is quasiregular.* □

Folowing the example of Leonard [11] we call a ring A *small* if $A \ll\lhd B$ for some ring B. The small rings can now be easily described.

PROPOSITION 1.10. *A ring A is small if and only if $A \in \mathcal{G}$.*

PROOF. Since $\mathcal{G}$ is hereditary, all small rings are in $\mathcal{G}$ by Corollary 1.4. Conversely, if $A \in \mathcal{G}$, then $A=\mathcal{G}(A^*Z)$, where A^*Z is the standard unital extension. By Corollary 1.8, $A \ll\lhd A^*Z$. □

Of course Brown—McCoy radical ideals are not always small; consider direct sums, for instance.

For further results on small ideals and the analogous concept of small normal subgroups, see the papers of Baer [3] and Michler [12].

Radical and semi-simple classes

In this section we examine conditions (*) and (**) of the Introduction. The following result will be useful.

PROPOSITION 2.1. *If $I \ll\lhd R$ and $J \lhd R$ then $(I+J)/J \ll\lhd R/J$.*

PROOF. If $(I+J)/J+K/J=R/J$, then $R=I+J+K=I+K$, so $K=R$ and $K/J=R/J$. $\square$

THEOREM 2.2. *Let $\mathcal{R}$ be a radical class, $\mathcal{S}$ the corresponding semi-simple class. Then $\mathcal{S}$ satisfies (*) if and only if $\mathcal{R}(R/I)=(\mathcal{R}(R)+I)/I$ whenever $I<\!\!\lhd R$.*

PROOF. Suppose $\mathcal{S}$ satisfies (*), and let I be a small ideal of some ring R. Then by Proposition 2.1, $(\mathcal{R}(R)+I)/\mathcal{R}(R)<\!\!\lhd R/\mathcal{R}(R)$, so $\mathcal{S}$ contains

$$\frac{(R/\mathcal{R}(R))}{((\mathcal{R}(R)+I)/\mathcal{R}(R))}\cong R/(\mathcal{R}(R)+I)\cong\frac{(R/I)}{((\mathcal{R}(R)+I)/I)}.$$

Hence $\mathcal{R}(R/I)\subseteq(\mathcal{R}(R)+I)/I$. But $(\mathcal{R}(R)+I)/I$ is in $\mathcal{R}$, so we have the desired equality.

Conversely, if the stated condition holds, let $J<\!\!\lhd S\in\mathcal{S}$. Then $\mathcal{R}(S/J)=$ $=(\mathcal{R}(S)+J)/J=0$, i.e. $S/J\in\mathcal{S}$. $\square$

COROLLARY 2.3. *Let $\mathcal{R}$ be a radical class with semi-simple class $\mathcal{S}$. If $\mathcal{S}$ satisfies (*), then $\mathcal{R}$ satisfies (**).*

PROOF. If $I<\!\!\lhd R$ and $R/I\in\mathcal{R}$, then $R/I=(\mathcal{R}(R)+I)/I$, so $\mathcal{R}(R)+I=R$' whence $\mathcal{R}(R)=R$. $\square$

This result is also a consequence of

PROPOSITION 2.4. *Let $\mathcal{X}$ be a regular class satisfying the condition*

$$J<\!\!\lhd R\in\mathcal{X}\Rightarrow R/J\in\mathcal{X}.$$

Let $\mathcal{R}$ denote the upper radical class defined by $\mathcal{X}$. Then $\mathcal{R}$ satisfies (**).

PROOF. Let $I<\!\!\lhd A$, with $A/I\in\mathcal{R}$. If $J\lhd A$ and $A/J\in\mathcal{X}$, then since $(J+J)/J<\!\!\lhd$ $<\!\!\lhd A/J$, we have $A/(I+J)\in\mathcal{X}$. But also $A/(I+J)\in\mathcal{R}$, so $A=I+J$, whence $J=A$, and so A has no non-zero homomorphic images in $\mathcal{X}$, i.e. A is in $\mathcal{R}$. $\square$

COROLLARY 2.5. *Let $\mathcal{X}$ be a homomorphically closed regular class. Then the upper radical class defined by $\mathcal{X}$ satisfies (**).*

COROLLARY 2.6. *The upper radical class defined by any class of simple rings satisfies (**).*

We next present an example to show that (*) and (**) are not equivalent.

EXAMPLE 2.7. Let G be the subgroup $\{m/n \mid n \text{ odd}\}$ of the additive group of rational numbers, A the zeroring on G. Let Z^0 denote the zeroring on the additive group of integers. Then $Z^0, 2Z^0\lhd A$. If $J\lhd A$ and $2Z^0+J=A$, then $Z^0+J=A$. Let J^+ denote the additive group of J. Then $Z+J^+=G$, so

$$J^+/(J^+\cap Z)\cong(J^++Z)/Z=G/Z\cong\oplus Z(p^\infty)\ (p\text{ odd}),$$

whence it follows that $J^+\cong G$ (see, e.g., [8], § 42). In particular, $pJ^+=J^+$ for every odd prime p. Now $2Z+J^+=G$. Let $1=2n+g$, where $g\in J^+$. Then $g=1-2n$ is an odd integer, so $J^+=gJ^+$. Hence $1=g/(1-2n)\in J^+$, so $Z\subseteq J^+$ and $J^+=G$, i.e.

$J=A$. This proves that $2Z^0 < \lhd A$. Now $A/2Z^0$ is the zeroring on $G/2Z \cong Z(2) \oplus \oplus \bigoplus Z(p^\infty)$ (p odd) ([8], § 42).

Now let $\mathscr{D}$ be the (radical) class consisting of all rings with divisible additive groups. Clearly $\mathscr{D}$ is the upper radical class defined by the class of all rings with bounded additive groups. The latter class is homomorphically closed, so by Corollary 2.5, $\mathscr{D}$ satisfies (**). However, the ring A described above is $\mathscr{D}$-semi-simple and $2Z^0 << \lhd A$, while $\mathscr{D}(A/2Z^0) \neq 0$. Thus the semi-simple class of $\mathscr{D}$ does not satisfy (*).

There is a very weak partial converse to Corollary 2.5, which we prove by a dualization of the proof of the theorem of Armendariz mentioned in the Introduction.

We shall be interested in the following lattice-theoretic condition.

For every chain $\{a_\lambda \mid \lambda \in \Lambda\}$ and every b,

$$b \vee \bigwedge a_\lambda = \bigwedge (b \vee a_\lambda).$$

A lattice is said to be *continuous* if it satisfies this condition and its dual. The latter is, of course, satisfied by the ideal lattice of every ring.

EXAMPLE 2.8. Any ring with DCC on ideals has a continuous ideal lattice.

EXAMPLE 2.9. Any ring whose ideals form a chain has a continuous ideal lattice.

PROPOSITION 2.10. *Let $\mathscr{R}$ be a radical class satisfying (**). If A is a hereditarily idempotent $\mathscr{R}$-semi-simple ring with continuous ideal lattice, then every homomorphic image of A is $\mathscr{R}$-semi-simple.*

PROOF. Let $I \lhd A$ and let $\mathscr{R}(A/I) = B/I$. Let

$$\mathscr{I} = \{J \mid J \lhd B \quad \text{and} \quad I + J = B\}.$$

Let $\mathscr{C}$ be a chain in $\mathscr{I}$, $C = \bigcap \mathscr{C}$. Then we have

$$I + C = I + \bigcap \{J \mid J \in \mathscr{C}\} = \bigcap \{I + J \mid J \in \mathscr{C}\} = B.$$

By Zorn's Lemma, $\mathscr{I}$ has a minimal element, K. Then $B/I = (I+K)/I \cong K/K \cap I$. Let $L \lhd K$ be such that $K = (K \cap I) + L$. Then

$$B = I + K = I + (K \cap I) + L = I + L.$$

By the minimality of K and the fact that $L \lhd K \lhd B \lhd A$ and A is hereditarily idempotent, we have $L = K$. Thus $(K \cap I) << \lhd K$, while $K/(K \cap I) \cong B/I \in \mathscr{R}$. By (**) we have $K \in \mathscr{R}$, so, since $K \lhd A$, we have $K = 0$. Thus $B/I \cong K/K \cap I = 0$, so that $\mathscr{R}(A/I) = 0$. □

Using Corollary 2.5, we get

COROLLARY 2.11. *Let $\mathscr{R}$ be the upper radical class defined by a homomorphically closed regular class (in particular, by a class of simple rings). If A is hereditarily idempotent and $\mathscr{R}$-semi-simple and has a continuous ideal lattice, then every homomorphic image of A is $\mathscr{R}$-semi-simple.*

A related result is

PROPOSITION 2.12. *Let $\mathcal{R}$ be a radical class satisfying* (**). *Let A be subdirectly irreducible with $\mathcal{R}(A)=0$. Then $\mathcal{R}(A/H(A))=0$, where $H(A)$ is the heart of A.*

PROOF. Let $\mathcal{R}(A/H(A))=B/H(A)$; if this is non-zero, then B is subdirectly irreducible with heart $H(A)$, so $H(A)<\!\!\lhd B$, whence $B\in\mathcal{R}$ — contradiction. □

Hereditary radical classes

In this section we obtain some information about hereditary radical classes whose semi-simple classes satisfy (*).

Let A be any ring. The ring $A^0 * A$ ("split null extension") is defined on the direct sum of two copies of the additive group of A by

$$(a, b)(c, d) = (ad+bc, db).$$

PROPOSITION 3.1. *If A is idempotent, then $A^0 * A$ is idempotent.*

PROOF. If $a\in A$, we can write $a=\sum xy$, $x, y\in A$, and then $(a, 0)=(\sum xy, 0)=$ $=\sum (x, 0)(0, y)$. Similarly if $b\in A$, we can write $b=\sum zw$ and then $(0, b)=$ $=\sum (0, z)(0, w)$. Thus

$$(a, b) = \sum (x, 0)(0, y)+\sum (0, z)(0, w). \quad \square$$

PROPOSITION 3.2. *Let $\mathcal{X}$ be a homomorphically closed class of rings. If a ring A has no non-zero homomorphic images in $\mathcal{X}$ and if $I\lhd A$, $I\in\mathcal{X}$, then $I<\!\!\lhd A$.*

PROOF. Let $I+J=A$, $J\lhd A$. Then $A/J\cong(I+J)/J\cong I/(I\cap J)\in\mathcal{X}$, so $J=A$. □

COROLLARY 3.3. *If A is idempotent, $I\lhd A$ and I is nilpotent, then $I<\!\!\lhd A$.*

We shall identify the zeroring A^0 on the additive group of a ring A with its "first component copy" in $A^0 * A$.

COROLLARY 3.4. *If A is idempotent, then $A^0<\!\!\lhd A^0 * A$.*

PROOF. By Proposition 3.1, $A^0 * A$ is idempotent. □

THEOREM 3.5. *Let $\mathcal{R}\neq\{0\}$ be a hereditary radical class whose semi-simple class $\mathcal{S}$ satisfies* (*). *Then $\mathcal{R}$ is supernilpotent.*

PROOF. Suppose $\mathcal{R}$ is not supernilpotent. Then $\mathcal{R}(Z^0)=0$, where Z^0 is the zeroring on the integers. The ring A of Example 2.7 is an essential extension of Z^0, so A is in $\mathcal{S}$. Since $2Z^0<\!\!\lhd A$, we have $A/2Z^0\in\mathcal{S}$, by $(*)$, whence $\mathcal{S}$ contains the zeroring on $Z(p^\infty)$ for every odd p. By an analogous argument, using, in place of A, the zeroring on

$$\{m/n\in Q \mid n \text{ is not divisible by } 3\},$$

we see that $\mathcal{S}$ contains the zeroring on $Z(p^\infty)$ for all $p \neq 3$. Thus each zeroring on a quasicyclic p-group is in $\mathcal{S}$ and it follows that $\mathcal{S}$ contains all zerorings.

Let S be an idempotent simple ring. Then S^0 is a maximal ideal of the idempotent (Proposition 3.1) ring $S^0 * S$, so S^0 is an essential ideal of $S^0 * S$. Since $S^0 \in \mathcal{S}$, we have $S^0 * S \in \mathcal{S}$. But by Corollary 3.4, $S^0 < \lhd S^0 * S$, so $S \cong \cong (S^0 * S)/S^0 \in \mathcal{S}$.

Hence $\mathcal{S}$ contains all simple rings as well as all zerorings, so $\mathcal{S}$ contains the heart of every subdirectly irreducible ring. Since $\mathcal{S}$ is closed under essential extensions, $\mathcal{S}$ contains all subdirectly irreducible rings and hence, being closed under subdirect products, all rings. □

We conclude with a result which gives a bit of an indication of which supernilpotent hereditary radical classes have semi-simple classes which satisfy (*). In what follows, $\mathcal{J}$ and $\mathcal{G}$ are, respectively, the Jacobson and Brown—McCoy radical classes.

THEOREM 3.6. *Let $\mathcal{R}$ be a hereditary supernilpotent radical class, $\mathcal{S}$ the corresponding semi-simple class.*

(i) *If $\mathcal{R} \subseteq \mathcal{J}$, then $\mathcal{S}$ does not satisfy* (*).

(ii) *If $\mathcal{G} \subseteq \mathcal{R}$, then $\mathcal{S}$ satisfies* (*).

PROOF. (i) Let V be a vector space of countably infinite dimension, $\{u_1, v_1, u_2, v_2, \dots\}$ a basis for V. Let f be the linear transformation defined by $f(u_i)=v_i; f(v_i)=0$. Let S be the ring of linear transformations of finite rank, R the ring generated by $S \cup \{f\}$. Then R is subdirectly irreducible with heart S, so $S < \lhd R$, while $(R/S)^2=0$.

If $\mathcal{R}$ is a hereditary supernilpotent radical class whose semi-simple class satisfies (*), then $R/S \in \mathcal{R}$ and hence $R \in \mathcal{R}$. Since $S \notin \mathcal{R}$, it follows that $\mathcal{R}$ is not hereditary.

(ii) If $\mathcal{G} \subseteq \mathcal{R}$, then every $\mathcal{R}$-semi-simple ring is $\mathcal{G}$-semi-simple, so by Corollary 1.5 the semi-simple class of $\mathcal{R}$ satisfies (*) trivially. □

Note that (ii) of Theorem 3.6 holds for non-hereditary radical classes also.

In view of Corollary 1.4, one possible way of obtaining semi-simple classes $\mathcal{S}$ satisfying (*) is *via* an investigation, analogous to that in [1], [14], of the partial homomorphic closure condition $I \lhd A, I \in \mathcal{G}, A \in \mathcal{S} \Rightarrow A/I \in \mathcal{S}$.

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MATHEMATICS DEPARTMENT
UNIVERSITY OF TASMANIA
G.P.O. BOX 252C
HOBART, TASMANIA 7001
AUSTRALIA

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ON B^*-PURE SEMIGROUPS

N. KUROKI (Niigata-ken)

1. A semigroup S is called normal ([4]) if $aS=Sa$ for all a of S. As is well-known ([3], II. 4.10 Corollary), a semigroup S is normal and regular if and only if it is a semilattice of groups. A subsemigroup A of a semigroup S is called a bi-ideal of S if $ASA\subseteqq A$. A bi-ideal A of a semigroup S is called B-pure if $A\cap xS=xA$ and $A\cap Sx=Ax$ for all x of S. A semigroup S is called B^*-pure if every bi-ideal of it is B-pure. It is easily seen that a normal regular semigroup is B^*-pure. In this note we shall give some properties of B^*-pure semigroups.

2. An element a of a semigroup S is called completely regular if there exists an element x in S such that $a=axa$ and $ax=xa$. We denote by $E(S)$ the set of all idempotents of a semigroup S.

LEMMA 1. *Let S be a B^*-pure semigroup. Then S has the following properties:*

(1) *$aS=a^2S$ and $Sa=Sa^2$ for all a of S.*
(2) *For every a of S, a^2 is completely regular.*
(3) *S is normal.*
(4) *$E(S)$ is contained in the center of S.*
(5) *$E(S)$ is a semilattice.*

PROOF. (1) Let a be any element of S. Then, since S is B^*-pure, the bi-ideal aS is B-pure. Then we have

$$aS=aS\cap aS=a(aS)=a^2S.$$

Similarly, $Sa=Sa^2$.

(2) Let a be any element of S. Then by (1) we have

$$a^2\in aS\cap Sa = a^2S\cap Sa^2 = (a^2)^2S\cap S(a^2)^2.$$

Then it follows from [3, IV. 1.2 Proposition] that a^2 is completely regular.

(3) Let a be any element of S. Then, since Sa is a B-pure bi-ideal of S, by (1) we have

$$aS = a^2S\subseteqq (Sa)S = Sa\cap SS\subseteqq Sa.$$

It can be seen in a similar way that $Sa\subseteqq aS$. Thus we have $aS=Sa$.

(4) This follows from (3) and [4, Lemma 1].

(5) This follows from (2) and (4).

3. A semigroup S is called archimedean if, for each elements a and b of S, there exists a positive integer n such that $a^n\in SbS$. As is easily seen, a normal semigroup is weakly commutative. Thus it follows from Lemma 1 (3) and [3, II. 5.6 Corollary] that a B^*-pure semigroup is a semilattice of archimedean semigroups.

THEOREM 2. *For a B^*-pure semigroup S the following conditions are equivalent.*
(1) *S is archimedean.*
(2) *$SaS=SbS$ for all a, b of S.*
(3) *$aS=bS$ for all a, b of S.*
(4) *$aSa=bSb$ for all a, b of S.*
(5) *S has exactly one idempotent.*
(6) *Every bi-ideal of S is archimedean.*

PROOF. (1)⇒(2). Let a and b be any elements of S. Then, since S is archimedean, there exists a positive integer n such that $a^n \in SbS$. Then by Lemma 1 (1) we have

$$SaS = Sa^n S \subseteqq S(SbS)S = (SS)b(SS) \subseteqq SbS.$$

Similarly, we have $SbS \subseteqq SaS$. Thus $SaS=SbS$. It follows from Lemma 1 (1), (3) that (2)⇒(3)⇒(4.)

(4)⇒(5) Let e and f be any idempotents of S. Then, since $eSe=fSf$, there exist elements x and y in S such that $e=fxf$ and $f=eye$. Then we have

$$e = fxf = ffxf = fe = eyee = eye = f.$$

Since $E(S)$ is nonempty by Lemma 1 (2), S has exactly one idempotent.

(5)⇒(6) Let A be any bi-ideal of S, and let a and b be any elements of A. Then, since a^2 and b^2 are regular by Lemma 1 (2), there exist elements x and y in S such that $a^2=a^2xa^2$ and $b^2=b^2yb^2$. Since a^2x and b^2y are idempotent, we have $a^2x=b^2y$. Then

$$a^3 = aa^2 = a(a^2xa^2) = a(b^2y)a^2 = ab(bya^2) \in Ab(ASA) \subseteqq AbA.$$

This means that A is archimedean.

(6)⇒(1) Obvious.

4. A semigroup S is called weakly commutative if, for all a, b of S, there exists a positive integer n such that $(ab)^n \in bSa$.

THEOREM 3. *Let S be a semigroup such that $aS=a^2S$ and $Sa=Sa^2$ for all a of S. Then the following conditions are equivalent.*
(1) *$E(S)$ is contained in the center of S.*
(2) *S is normal.*
(3) *S is weakly commutative.*

PROOF. (1)⇒(2) Let a be any element of S. Then, as is stated in the proof of Lemma 1 (2), a^2 is regular. Thus there exists an element x in S such that $a^2=a^2xa^2$. Let a^2y be any element of $aS(=a^2S)$. Then, since xa^2 is idempotent, we have

$$a^2y = (a^2xa^2)y = a^2((xa^2)y) = a^2(y(xa^2)) = (a^2ya^2)a^2 \in Sa^2 = Sa,$$

and so we have $aS \subseteqq Sa$. Similarly, $Sa \subseteqq aS$. Thus we obtain that $aS=Sa$, and that S is normal.

(2)⇒(3) Let a and b be any elements of S. Then, since S is normal,

$$(ab)^2 \in (Sb)(aS) = (bS)(Sa) = b(SS)a \subseteq bSa.$$

Thus S is weakly commutative.

(3)⇒(1) Let a be any element of S and e any idempotent of S. Since S is weakly commutative, we have $(ae)^n \in eSa$ for some positive integer n. Then we have

$$ae = aee \in aeS = (ae)^n S \subseteq (eSa)S \subseteq eS.$$

This implies that there exists an element x in S such that $ae=ex$. Similarly, there exists an element y in S such that $ea=ye$. Then we have

$$ae = ex = eex = eae = yee = ye = ea.$$

Thus $E(S)$ is contained in the center of S. This completes the proof.

THEOREM 4. *For a semigroup S the following conditions are equivalent.*
(1) *S is B^*-pure.*
(2) *S is normal and $Sa=Sa^2$ for all a of S.*

PROOF. It follows from Lemma 1 (1), (3) that (1) implies (2). Conversely, assume that (2) holds. Let A be any bi-ideal of S, and x any element of S. Let $a=x^2s$ ($a \in A$, $s \in S$) be any element of $A \cap xS$ $(=A \cap x^2S)$. Then, as is easily seen, x^2 is regular. Thus there exists an element y in S such that $x^2=x^2yx^2$. Since $ya \in Sa=Sa^2$, there exists an element z in S such that $ya=za^2$. Then, since S is normal, we have

$$a = x^2 s = (x^2 yx^2)s = (x^2 y)(x^2 s) = (x^2 y)a =$$

$$= x^2(ya) = x^2(za^2) = x((xz)a)a \in x(Sa)a = x(aS)a \subseteq x(ASA) \subseteq xA,$$

and so we have $A \cap xS \subseteq xA$. Let xa ($a \in A$) be any element of xA. Then we have

$$xa \in Sa = Sa^2 = aSa \subseteq ASA \subseteq A,$$

and so $xA \subseteq A$. Since $xA \subseteq xS$, we have $xA \subseteq A \cap xS$. Thus $A \cap xS = xA$. It can be seen in a similar way that $A \cap Sx = Ax$. Thus we obtain that S is B^*-pure and that (2) implies (1).

5. A semigroup S is called a semilattice of groups if it is the set-theoretical union of a family of mutually disjoint subgroups G_i ($i \in M$) such that, for each i, j in M, the products G_iG_j and G_jG_i are contained in the same group G_k ($k \in M$). The following is due to [3, II. 4.10 Corollary] and [2, Theorem 1].

LEMMA 5. *For a semigroup S the following conditions are equivalent.*
(1) *S is a semilattice of groups.*
(2) *S is normal and regular.*
(3) *The set of all bi-ideals of S is a semilattice under the multiplication of subsets.*

Now we give our main result.

THEOREM 6. *For a semigroup S the following conditions are equivalent.*
(1) *S is B^*-pure.*
(2) *S^2 is a semilattice of groups.*

PROOF. (1)⇒(2) Since S is normal by Theorem 4, S^2 is normal. Let $a=xy$ (x, y in S) be any element of S^2. Since by Theorem 4, $xy\in xS=x^2S$, there exists an element u in S such that $xy=x^2u$. Since x^2 is regular, there exists an element v in S such that $x^2=x^2vx^2$. Then, by Theorem 4, we have

$$a = xy = x^2u = (x^2vx^2)u = (x^2v)(x^2u) = (x^2v)a\in S^2a = S^2a^2 = aS^2a.$$

This means that S^2 is regular. Then it follows from Lemma 5 that S^2 is a semilattice of groups.

(2)⇒(1) Let a be any element of S. Then, since Sa is a bi-ideal of S^2, Sa is globally idempotent by Lemma 5. Since S^2 is normal by Lemma 5, we have

$$Sa = (Sa)^2 \subseteqq S^2a = aS^2 \subseteqq aS.$$

Similarly, we have $aS\subseteqq Sa$, and so $aS=Sa$. On the other hand,

$$Sa = (Sa)^2 = S(aS)a = S(Sa)a = S^2a^2 \subseteqq Sa^2 \subseteqq Sa,$$

and so $Sa=Sa^2$. Then it follows from Theorem 4 that S is B^*-pure. This completes the proof.

COROLLARY 7. *For a semigroup S the following conditions are equivalent.*
(1) *S is a semilattice of groups.*
(2) *S is regular and B^*-pure.*
(3) *S is completely regular and B^*-pure.*
(4) *S is intra-regular and B^*-pure.*
(5) *S is completely regular and weakly commutative.*
(6) *S is completely regular and $E(S)$ is contained in the center of S.*

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DEPARTMENT OF MATHEMATICS
JOETSU UNIVERSITY OF EDUCATION
JOETSU-SHI, NIIGATA-KEN
JAPAN

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ÜBER L. FEJES TÓTHS WURSTVERMUTUNG IN KLEINEN DIMENSIONEN

U. BETKE und P. GRITZMANN (Siegen)

1. Einleitung

Seien $B_1^d, \dots, B_k^d$ k Translate der Einheitskugel B^d im d-dimensionalen euklidischen Raum E^d, die höchstens Randpunkte gemeinsam haben. C_k bezeichne die konvexe Hülle der Kugelmittelpunkte und S_k eine Strecke der Länge $2(k-1)$.

Ferner sei V^d das d-dimensionale Volumen.

L. Fejes Tóth vermutete in [2], daß für $d \geqq 5$

$$V^d(S_k+B^d) \leqq V^d(C_k+B^d) \tag{1}$$

gilt. Da S_k+B^d einer d-dimensionalen Wurst ähnelt, nannte Fejes Tóth (1) die Wurstvermutung.

In [1] wurde gezeigt, daß (1) richtig ist, falls

$$\dim C_k \leqq \frac{7}{12}(d-1) \tag{2}$$

oder falls $\dim C_k \leqq 3$ und $d \geqq \dim C_k+1$ gilt. In der vorliegenden Arbeit wird folgender Satz bewiesen:

SATZ. *Für* $\dim C_k \leqq 9$ *und* $d \geqq \dim C_k+1$ *gilt* (1). *Gleichheit tritt genau für* $C_k = S_k$ *auf.*

Der Satz löst die Wurstvermutun gfür beliebige unterdimensionale Lagerungen von Einheitskugeln bis zur Dimension $d=10$; es bleibt demnach zum vollständigen Beweis von (1) in den Dimensionen 5 bis 10 nur der volldimensionale Fall $\dim C_k = d$ übrig. Das benutzte Beweisverfahren versagt bei n-dimensionalen Kugellagerungen im E^{n+1} für $n \geqq 10$. Die Methode ist für alle Dimensionen weiterreichend als die in [1] verwendete, liefert aber asymptotisch keine Verbesserung der Konstante 7/12 in (2).

Da der Beweis des Satzes auf einer Verallgemeinerung der Methoden aus [1] beruht, wird, um Wiederholungen zu vermeiden, bisweilen auf [1] zurückgegriffen.

Im folgenden Paragraphen wird die Behauptung umgeformt, und es werden hinreichende Bedingungen für die Gültigkeit von (1) angegeben. Paragraph 3 enthält analytische Lemmata zur Berechnung auftretender Integrale. Der Beweis des Satzes wird schließlich in Paragraph 4 geführt, wobei nach der Dimension der auftretenden Seiten von C_k unterschieden wird.

2. Hinreichende Bedingungen für die Wurstvermutung

Sei $n = \dim C_k$, und es bezeichne $F_i(C_k)$ die Menge der i-Seiten f von C_k und $\alpha(f)$ den auf 1 normierten äußeren Winkel von f bez. C_k. Ferner sei $\omega_i = \pi^{i/2}/\Gamma(1+i/2)$ und V_i das i-te innere Volumen gemäß [3], S. 253.

Aus der Steiner-Formel

$$V^d(C_k+B^d) = \sum_{i=0}^{d} \omega_{d-i} V_i(C_k)$$

ergibt sich wegen $V_{n+1}(C_k) = \ldots = V_d(C_k) = 0$ mit $V_{ij}(f) = \alpha(f) V^i(f \cap B_j^d)$

$$\sum_{j=1}^{k} \sum_{i=0}^{n} \sum_{f \in F_i(C_k)} \omega_{d-i} V_{ij}(f) \leqq V^d(C_k+B^d).$$

Hierbei gilt Gleichheit genau für $C_k = S_k$.

Mit Hilfe der Steiner-Formel für $V^d(S_k+B^d)$ und $V_0 = 1$ ergibt sich somit als hinreichende Bedingung für (1):

$$(3) \qquad (k-1)\,\omega_n \leqq \sum_{j=1}^{k} \sum_{i=1}^{n} \sum_{f \in F_i(C_k)} \frac{\omega_n \omega_{d-i}}{2\omega_{d-1}} V_{ij}(f).$$

Eine ausführlichere Herleitung der Bedingung (3) findet sich in [1]. Zum Beweis des Satzes bewerten wir die Punkte der n-dimensionalen Einheitskugel mit Hilfe einer geeigneten Gewichtungsfunktion. Dann zerlegen wir die so gewichtete Kugel, um die Summanden von (3) einzeln vergleichen zu können.

Als Bewertungsfunktion wählen wir $g(x) = \frac{n+\beta}{n} \|x\|^\beta$, wobei $\beta = \beta(n)$ folgenden Werten entspricht: $\beta(4) = \beta(5) = 2$, $\beta(6) = 4$, $\beta(7) = 8$, $\beta(8) = 18$, $\beta(9) = 60$. Es gilt $\omega_n = \int_{B^n} g(x)\,dx$. Für i-Seiten f von C_k sei $K(f)$ der Kegel mit Spitze 0 der äußeren Normalen von C_k, die an einen relativ inneren Punkt von f angetragen werden.

Ferner sei mit $B_j^n = B_j^d \cap \operatorname{aff} C_k$ (bei geeigneter Wahl des Ursprungs)

$$V'_{ij}(f) = \int_{(f+K(f)) \cap B_j^n} g(x)\,dx.$$

Da die Mengen $(f+K(f)) \cap B_j^n$ die Kugel B_j^n zerlegen, gilt

$$\sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{f \in F_i(C_k)} V'_{ij}(f) = (k-1)\,\omega_n,$$

und man erhält als hinreichende Bedingung für (1):

$$(4) \qquad \sum_{i=1}^{k} \sum_{f \in F_i(C_k)} \sum_{j=1}^{k} V'_{ij}(f) \leqq \sum_{i=1}^{n} \sum_{f \in F_i(C_k)} \sum_{j=1}^{k} \frac{\omega_n \omega_{d-i}}{2\omega_{d-1}} V_{ij}(f).$$

Wir erhalten für $V'_{ij}(f)$ $(i<n)$ folgende Abschätzung:

$$V'_{ij}(f) = \int\limits_{(f+K(f))\cap B_j^n} g(x)\,dx = \int\limits_{f\cap B_j^n} \int\limits_{(y+K(f))\cap B_j^n} g(x)\,dx\,dy \leqq$$

$$\leqq \alpha(f)\frac{n+\beta}{n}(n-i)\,\omega_{n-i} \int\limits_{f\cap B_j^n} \int\limits_0^{\sqrt{1-\|y\|^2}} (\|y\|^2+r^2)^{\beta/2}\, r^{n-i-1}\,dr\,dy.$$

Wir bezeichnen im folgenden die rechte Seite mit $\overline{V}_{ij}(f)$ und setzen $\overline{V}_{nj}(f)=V'_{nj}(f)$.

Wegen

$$\sqrt{\frac{d}{2\pi}} < \frac{\omega_{d-1}}{\omega_d} < \sqrt{\frac{d+1}{2\pi}} \quad (d \geqq 1)$$

(vgl. [1]) folgt $\frac{\omega_{n-i+1}}{2} \leqq \frac{\omega_n\omega_{d-i}}{2\omega_{d-1}}$. Insgesamt erhalten wir damit aus (4) die Bedingung

$$\sum_{i=1}^n \sum_{f\in F_i(C_k)} \sum_{j=1}^k \overline{V}_{ij}(f) \leqq \sum_{i=1}^n \sum_{f\in F_i(C_k)} \sum_{j=1}^k \frac{\omega_{n-i+1}}{2} V_{ij}(f).$$

Für $i\neq 2$ und in dem Fall, daß keine Ecke von f in den Mittelpunkt von B_j^n fällt, wird

$$\overline{V}_{ij}(f) \leqq \frac{\omega_{n-i+1}}{2} V_{ij}(f) \tag{5}$$

bewiesen. Im verbleibenden Fall wird

$$\sum \overline{V}_{2j}(f) \leqq \sum \frac{\omega_{n-1}}{2} V_{2j}(f) \tag{6}$$

gezeigt, wobei sich die Summation über alle j erstreckt, für die der Mittelpunkt von B_j^n eine Ecke von f ist.

3. Berechnung einiger Integrale

Im folgenden berechnen wir die $\overline{V}_{ij}(f)$ definierenden Integrale und erhalten insbesondere eine für den Beweis des Satzes wichtige Monotonieaussage.

Lemma 1. *Seien j und m nicht-negative ganze Zahlen und $0\leqq t\leqq 1$. Dann gilt*

$$\int\limits_0^{\sqrt{1-t}} (t+r^2)^m r^j\,dr = (1-t)^{\frac{j+1}{2}} \sum_{s=0}^m 2^s \frac{m!}{(m-s)!}\left(\prod_{l=m-s}^m (j+2l+1)\right)^{-1} t^s.$$

BEWEIS. Es gilt

$$\int_0^{\sqrt{1-t}} (t+r^2)^m r^j \, dr = \sum_{l=0}^{m} \binom{m}{l} t^l \int_0^{\sqrt{1-t}} r^{2(m-l)+j} \, dr =$$

$$= (1-t)^{\frac{j+1}{2}} \sum_{l=0}^{m} \binom{m}{l} \frac{1}{2(m-l)+j+1} t^l (1-t)^{m-l} =$$

$$= (1-t)^{\frac{j+1}{2}} \sum_{s=0}^{m} \binom{m}{s} \sum_{q=0}^{s} \binom{s}{q} \frac{(-1)^q}{2(m-s+q)+j+1} t^s =$$

$$= (1-t)^{\frac{j+1}{2}} \sum_{s=0}^{m} 2^s \frac{m!}{(m-s)!} \left(\prod_{l=m-s}^{m} (j+2l+1) \right)^{-1} t^s .$$

Der letzte Schritt ergibt sich unter Ausnutzung der Identität

$$\binom{s+1}{q} = \binom{s}{q-1} + \binom{s}{q}$$

durch vollständige Induktion nach s.

LEMMA 2. *Sei m eine nicht-negative ganze Zahl, j eine natürliche Zahl und $0 \leqq t \leqq 1$. Dann ist die durch*

$$f(t) = \int_0^{\sqrt{1-t}} (t+r^2)^m r^j \, dr$$

definierte Funktion in t monoton fallend.

BEWEIS. Nach Lemma 1 gilt:

$$f(t) = (1-t)^{\frac{j+1}{2}} \sum_{s=0}^{m} 2^s \frac{m!}{(m-s)!} \left(\prod_{l=m-s}^{m} (j+2l+1) \right)^{-1} t^s =$$

$$= (1-t)^{\frac{j-1}{2}} \left\{ \frac{1}{j+2m+1} - 2^m m! \left(\prod_{l=0}^{m} (j+2l+1) \right)^{-1} t^{m+1} + \right.$$

$$\left. + \sum_{s=1}^{m} \left[2^s \frac{m!}{(m-s)!} \left(\prod_{l=m-s}^{m} (j+2l+1) \right)^{-1} - 2^{s-1} \frac{m!}{(m-s+1)!} \left(\prod_{l=m-s+1}^{m} (j+2l+1) \right)^{-1} \right] t^s \right\} =$$

$$= (1-t)^{\frac{j-1}{2}} \left\{ \frac{1}{j+2m+1} - 2^m m! \left(\prod_{l=0}^{m} (j+2l+1) \right)^{-1} t^{m+1} - \right.$$

$$\left. - \sum_{s=1}^{m} (j-1) 2^{s-1} \frac{m!}{(m-s+1)!} \left(\prod_{l=m-s}^{m} (j+2l+1) \right)^{-1} t^s \right\} .$$

Hieraus folgt die Behauptung.

In Lemma 3 führen wir ein bei der Berechnung von $\overline{V}_{ij}(f)$ auftretendes Doppelintegral auf die Betafunktion

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \quad (x, y > 0)$$

zurück. Wegen

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

läßt sich das betrachtete Doppelintegral in den für diese Arbeit relevanten Fällen dann explizit bestimmen.

LEMMA 3. *Seien j, m und p nicht-negative ganze Zahlen. Dann gilt:*

$$\int_1^2 t^p \int_0^{\sqrt{1-(2-t)^2}} [(2-t)^2+r^2]^m r^j\,dr\,dt =$$

$$= \sum_{s=0}^{m} \sum_{q=0}^{p} (-1)^q 2^{p-1+s-q} \frac{m!}{(m-s)!} \binom{p}{q} \left(\prod_{l=m-s}^{m} (j+2l+1) \right)^{-1} B\left(s+\frac{q+1}{2}, \frac{j+3}{2}\right).$$

BEWEIS. Mit Hilfe von Lemma 1 ergibt sich:

$$\int_1^2 t^p \int_0^{\sqrt{1-(2-t)^2}} [(2-t)^2+r^2]^m r^j\,dr\,dt =$$

$$= \sum_{s=0}^{m} 2^s \frac{m!}{(m-s)!} \left(\prod_{l=m-s}^{m} (j+2l+1) \right)^{-1} \int_1^2 t^p [1-(2-t)^2]^{\frac{j+1}{2}} (2-t)^{2s}\,dt =$$

$$= \sum_{s=0}^{m} 2^s \frac{m!}{(m-s)!} \left(\prod_{l=m-s}^{m} (j+2l+1) \right)^{-1} \int_0^1 (2-u)^p (1-u^2)^{\frac{j+1}{2}} u^{2s}\,du =$$

$$= \sum_{s=0}^{m} 2^s \frac{m!}{(m-s)!} \left(\prod_{l=m-s}^{m} (j+2l+1) \right)^{-1} \sum_{q=0}^{p} \binom{p}{q} (-1)^q 2^{p-q} \int_0^1 u^{2s+q} (1-u^2)^{\frac{j+1}{2}}\,du =$$

$$= \sum_{s=0}^{m} \sum_{q=0}^{p} (-1)^q 2^{p-1+s-q} \frac{m!}{(m-s)!} \binom{p}{q} \left(\prod_{l=m-s}^{m} (j+2l+1) \right)^{-1} B\left(s+\frac{q+1}{2}, \frac{j+3}{2}\right).$$

LEMMA 4. *Seien $1 \leqq A \leqq B \leqq A+1$ und $z(t) = 1-(B-A)^2+(t-B)^2$. Dann gilt die Ungleichung*

$$\int_A^B t^p \int_0^{\sqrt{1-z(t)}} (z(t)+r^2)^m r^j\,dr\,dt \leqq \frac{1}{2} \frac{1}{j+1} \frac{1}{p+1} \frac{n}{n+2m} \frac{\omega_{j+2}}{\omega_{j+1}} (B^{p+1}-A^{p+1})$$

für Parameterwerte gemäß folgender Tabelle:

n	m	$j=1$	2	3	4	5	6	7
4	1	0	0	—	—	—	—	—
5	1	2	0	0	—	—	—	—
6	2	3	2	0	0	—	—	—
7	4	4	3	2	0	0	—	—
8	9	5	4	3	2	0	0	—
9	30	6	5	4	3	2	0	0

$p = p(n, j)$

BEWEIS. Zum Beweis von Lemma 4 nehmen wir an, daß die angegebene Ungleichung nicht gilt.

Wegen $z(t)-(1+A-t)^2=2(t-A)(1+A-B)\geqq 0$ folgt aus der Monotonie des inneren Integrals gemäß Lemma 2:

$$\int_A^B t^p \int_0^{\sqrt{1-z(t)}} (z(t)+r^2)^m r^j \, dr \, dt \leqq \int_A^B t^p \int_0^{\sqrt{1-(1+A-t)^2}} [(1+A-t)^2+r^2]^m r^j \, dr \, dt.$$

Erneute Anwendung von Lemma 2 liefert mit Hilfe der Widerspruchsannahme die Ungleichung

$$\int_A^B t^p \left\{ \int_0^{\sqrt{1-(1+A-t)^2}} [(1+A-t)^2+r^2]^m r^j \, dr - \frac{1}{2} \frac{1}{j+1} \frac{n}{n+2m} \frac{\omega_{j+2}}{\omega_{j+1}} \right\} dt \leqq$$

$$\leqq \int_A^{A+1} t^p \left\{ \int_0^{\sqrt{1-(1+A-t)^2}} [(1+A-t)^2+r^2]^m r^j \, dr - \frac{1}{2} \frac{1}{j+1} \frac{n}{n+2m} \frac{\omega_{j+2}}{\omega_{j+1}} \right\} dt.$$

Nach Lemma 2 und den Eigenschaften der Gewichtung t^p folgt wegen $A\geqq 1$:

$$2(j+1)(p+1)\frac{1}{2^{p+1}-1}\frac{n+2m}{n}\frac{\omega_{j+1}}{\omega_{j+2}}\int_1^2 t^p \int_0^{\sqrt{1-(2-t)^2}} [(2-t)^2+r^2]^m r^j \, dr \, dt > 1.$$

Berechnet man das Doppelintegral gemäß Lemma 3, so erhält man für die linke Seite folgende Tabelle:

n \ j	1	2	3	4	5	6	7
4	0,9000	1	—	—	—	—	—
5	0,9257	0,9333	1	—	—	—	—
6	0,7867	0,9611	0,9259	1	—	—	—
7	0,6295	0,8474	0,9821	0,9183	1	—	—
8	0,4847	0,7072	0,8819	0,9924	0,9100	1	—
9	0,3705	0,5733	0,7575	0,9041	0,9980	0,9020	1

Da kein Wert in der Tabelle größer als 1 ist, ergibt sich ein Widerspruch zur Annahme.

Beweis des Satzes

Im folgenden setzen wir — ohne Einschränkung — voraus, daß C_k so trianguliert ist, daß alle Ecken der Triangulierung bereits Ecken von C_k sind.

LEMMA 5. *Sei f eine 2-Seite von C_k, und das Zentrum von B_j^n sei keine Ecke von f. Dann gilt* (5).

BEWEIS. Wie in [1] gezeigt wurde, können höchstens zwei Kanten von f das Innere von B_j^n schneiden. Sei v eine Ecke von f, so daß alle Kanten, die das Innere von B_j^n schneiden, v enthalten. Wir führen nun Polarkoordinaten in $\operatorname{aff}(f)$ mit v als Ursprung ein. Dann ist (5) äquivalent zu:

(7)

$$\int_{\varphi_1}^{\varphi_2}\int_{R_1(\varphi)}^{R_2(\varphi)}\int_0^{\sqrt{1-z(R)}}(z(R)+r^2)^{\beta/2}r^{n-3}R\,dr\,dR\,d\varphi \leqq \frac{1}{2}\frac{1}{n-2}\frac{n}{n+\beta}\frac{\omega_{n-1}}{\omega_{n-2}}\int_{\varphi_1}^{\varphi_2}\int_{R_1(\varphi)}^{R_2(\varphi)}R\,dR\,d\varphi$$

mit

$$z(R) = 1-\left(\frac{R_2(\varphi)-R_1(\varphi)}{2}\right)^2+\left(R-\frac{R_2(\varphi)+R_1(\varphi)}{2}\right)^2.$$

Da für symmetrische Funktionen h $\int_{-a}^{a} xh(x)dx=0$ gilt, folgt (7) aus:

$$\int_{R_1(\varphi)}^{\frac{1}{2}(R_1(\varphi)+R_2(\varphi))}\int_0^{\sqrt{1-z(R)}}(z(R)+r^2)^{\beta/2}r^{n-3}\,dr\,dR \leqq \frac{1}{2}\frac{1}{n-2}\frac{n}{n+\beta}\frac{\omega_{n-1}}{\omega_{n-2}}\frac{R_2(\varphi)-R_1(\varphi)}{2}.$$

Damit haben wir die Situation von Lemma 4 mit $j=n-3$ und $m=\beta/2$ erreicht, und es folgt die Behauptung.

LEMMA 6. *Für jede 2-Seite f von C_k gilt* (6).

BEWEIS. Da sich die Innenwinkel von f zu π ergänzen, können wir die Durchschnitte $f\cap B_j^n$ mit den drei Kugeln, deren Mittelpunkte die Ecken von f sind, durch einen Teil eines Kreises ersetzen, der durch einen Durchmesser oder durch zwei disjunkte und nicht-parallele Segmente, von denen eines ein Durchmesser ist, begrenzt wird. Der Beweis ergibt sich dann analog Lemma 5. (Vgl. [1].)

LEMMA 7. *Sei $i=n$ und $f=C_k$. Dann gilt* (5).

BEWEIS. Durch Einführung von Polarkoordinaten (mit der Funktionaldeterminante $c(\varphi)r^{n-1}$) ergibt sich:

$$\bar{V}_{nj}(f) = \int_{C_k\cap B_j^n} g(x)\,dx = \int_{\Phi} c(\varphi)\int_0^{R(\varphi)}\frac{n+\beta}{n}r^{\beta+n-1}\,dr\,d\varphi = \int_{\Phi} c(\varphi)\frac{1}{n}R^{\beta+n}(\varphi)\,d\varphi \leqq$$

$$\leqq \int_{\Phi} c(\varphi)\frac{1}{n}R^n(\varphi)\,d\varphi = \int_{\Phi} c(\varphi)\int_0^{R(\varphi)} r^{n-1}\,dr\,d\varphi = V_{nj}(f).$$

LEMMA 8. *Sei $i=n-1$ und f eine Facette von C_k. Dann gilt* (5).

BEWEIS. Die Beziehung (5) ist äquivalent mit:

$$\int_{f\cap B_j^n} \left\{ \int_0^{\sqrt{1-\|y\|^2}} (\|y\|^2+r^2)^{\beta/2}\,dr - \frac{\pi}{4}\frac{n}{n+\beta} \right\} dy \leqq 0.$$

Wir zeigen, daß der Integrand des äußeren Integrals nicht positiv ist.

Dieses ist gleichwertig mit

$$\frac{4}{\pi}\frac{n+\beta}{n}\int_0^{\sqrt{1-\|y\|^2}} (\|y\|^2+r^2)^{\beta/2}\,dr \leqq 1.$$

Es gilt mit Lemma 1:

(8)
$$\frac{4}{\pi}\frac{n+\beta}{n}\int_0^{\sqrt{1-\|y\|^2}} (\|y\|^2+r^2)^{\beta/2}\,dr =$$

$$= \frac{4}{\pi}\frac{n+\beta}{n}\sum_{s=0}^{\beta/2} 2^s \frac{(\beta/2)!}{(\beta/2-s)!}\left(\prod_{l=\beta/2-s}^{\beta/2}(2l+1)\right)^{-1} \|y\|^{2s}(1-\|y\|^2)^{1/2}.$$

Für $n=4$ und $n=5$ wird das Maximum dieses Ausdrucks für $\|y\|^2=\frac{1}{2}$ angenommen; es ist $\frac{2\sqrt{2}}{\pi}$ bzw. $\frac{28\sqrt{2}}{15\pi}$. Beide Werte sind kleiner als 1.

Für $n=6$, 7 und 8 schätzen wir (8) durch Einsetzen des Maximums $\frac{(2s)^s}{(2s+1)^{s+1/2}}$ — mit der Setzung $0^0=1$ — von $\|y\|^{2s}(1-\|y\|^2)^{1/2}$ ab. Es ergibt sich:

$$\frac{4}{\pi}\frac{n+\beta}{n}\int_0^{\sqrt{1-\|y\|^2}} (\|y\|^2+r^2)^{\beta/2}\,dr \leqq$$

$$\leqq \frac{4}{\pi}\frac{n+\beta}{n}\sum_{s=0}^{\beta/2} 2^s \frac{(\beta/2)!}{(\beta/2-s)!}\left(\prod_{l=\beta/2-s}^{\beta/2}(2l+1)\right)^{-1} \frac{(2s)^s}{(2s+1)^{s+1/2}}.$$

Auswertung der rechten Seite für $n=6, 7, 8$ liefert die Werte 0,9662; 0,9182; 0,9466. Im Fall $n=9$ liefert numerische Auswertung von (8) als Maximum der linken Seite den Wert 0,9592.

Insgesamt ist damit Lemma 8 bewiesen.

Zum Beweis der Behauptung in den restlichen Fällen benutzen wir eine in [1] definierte spezielle Simplexzerlegung, deren Eigenschaften im folgenden Lemma zusammengefaßt werden. Zum Beweis vgl. [1].

LEMMA 9. *Sei S_i ein i-Simplex im E^n und $v_0, \dots, v_i$ seine Ecken. Sei $p\in E^n$ und B^n eine Kugel mit p als Mittelpunkt.*

Es seien $v_1, \dots, v_i \notin B^n$ und $v_0 \notin B^n$ oder $v_0=p$. Für alle $v_s\neq p$ bezeichne $\bar{S}_i(v_s)$ die Menge aller Punkte x von $S_i\cap B^n$, deren Abstand von v_s nicht größer ist als der des Mittelpunktes der Sehne $B^n\cap \operatorname{aff}\{v_s, x\}$.

Dann gibt es eine Zerlegung von S_i in konvexe Mengen $S_i(v_s)$ $(v_s \neq p)$ mit $v_s \in S_i(v_s)$ und $S_i(v_s) \cap B^n \subset \bar{S}_i(v_s)$.

LEMMA 10. *Sei $i=1$ oder $3 \leqq i \leqq n-2$. Dann gilt* (5).

BEWEIS. Sei f eine i-Seite von C_k, für die (5) nicht erfüllt ist. Dann gilt:

$$\int\limits_{f \cap B_j^n} \left\{ \int\limits_0^{\sqrt{1-\|y\|^2}} (\|y\|^2+r^2)^{\beta/2} r^{n-i-1}\, dr - \frac{1}{2} \frac{1}{n-i} \frac{n}{n+\beta} \frac{\omega_{n-i+1}}{\omega_{n-i}} \right\} dy > 0.$$

Zerlegen wir f gemäß Lemma 9, so gibt es eine Ecke v_s von f mit $v_s \notin B_j^n$ und

$$\int\limits_{S_i(v_s) \cap B_j^n} \left\{ \int\limits_0^{\sqrt{1-\|y\|^2}} (\|y\|^2+r^2)^{\beta/2} r^{n-i-1}\, dr - \frac{1}{2} \frac{1}{n-i} \frac{n}{n+\beta} \frac{\omega_{n-i+1}}{\omega_{n-i}} \right\} dy > 0.$$

Wir führen nun Polarkoordinaten mit v_s als Ursprung ein und erhalten mit $c(\varphi)r^{i-1}$ als Funktionaldeterminante und dem Winkelbereich Φ:

$$\int\limits_{\Phi} c(\varphi) \int\limits_{R_1(\varphi)}^{R_2(\varphi)} R^{i-1} \left\{ \int\limits_0^{\sqrt{1-z(R)}} (z(R)+r^2)^{\beta/2} r^{n-i-1}\, dr - \right.$$
$$\left. - \frac{1}{2} \frac{1}{n-i} \frac{n}{n+\beta} \frac{\omega_{n-i+1}}{\omega_{n-i}} \right\} dR\, d\varphi > 0,$$

wobei mit $\varrho(\varphi)$ als halber Sehnenlänge $z(R) = 1-\varrho^2(\varphi)+(R-R_1(\varphi)-\varrho(\varphi))^2$ ist.

Dann gibt es ein φ_0 mit

$$\int\limits_{R_1(\varphi_0)}^{R_2(\varphi_0)} R^{i-1} \left\{ \int\limits_0^{\sqrt{1-z(R)}} (z(R)+r^2)^{\beta/2} r^{n-i-1}\, dr - \frac{1}{2} \frac{1}{n-i} \frac{n}{n+\beta} \frac{\omega_{n-i+1}}{\omega_{n-i}} \right\} dR > 0.$$

Aus der Monotonie des Integranden gemäß Lemma 2 folgt mit Lemma 9:

$$\int\limits_{R_1(\varphi_0)}^{R_1(\varphi_0)+\varrho(\varphi_0)} R^{i-1} \int\limits_0^{\sqrt{1-z(R)}} (z(R)+r^2)^{\beta/2} r^{n-i-1}\, dr\, dR >$$
$$> \frac{1}{2} \frac{1}{n-i} \frac{1}{i} \frac{n}{n+\beta} \frac{\omega_{n-i+1}}{\omega_{n-i}} \left[(R_1(\varphi_0)+\varrho(\varphi_0))^i - R_1^i(\varphi_0) \right].$$

Mit $A = R_1(\varphi_0)$, $B = R_1(\varphi_0)+\varrho(\varphi_0)$, $j=n-i-1$, $p=i-1$ und $m=\beta/2$ ergibt sich ein Widerspruch zu Lemma 4.

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MATH. INST. UNIV. SIEGEN
HÖLDERLINSTR. 3
D—5900 SIEGEN

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LIMITS OF STRONG UNICITY CONSTANTS FOR CERTAIN C^∞ FUNCTIONS

M. S. HENRY (Mount Pleasant) and J. J. SWETITS (Norfolk)

1. Introduction

Let $C(\mathbf{X})$ denote the space of real-valued, continuous functions on the compact set $\mathbf{X}$, and let $\mathcal{P}_{n+1} \subseteq C(\mathbf{X})$ be a Haar subspace of dimension $n+1$. Denote the uniform norm on $C(\mathbf{X})$ by $\|\cdot\|$.

For each $f \in C(\mathbf{X})$ with best approximation $B_n(f)$ from $\mathcal{P}_{n+1}$, there is a smallest positive constant $M_n(f)$ such that for any $p \in \mathcal{P}_{n+1}$,

$$\|p-B_n(f)\| \leqq M_n(f)[\|f-p\|-\|f-B_n(f)\|].$$

This inequality is the strong unicity theorem [4, p. 80], and $M_n(f)$ is the strong unicity constant. A number of recent papers have considered the asymptotic behavior of $M_n(f)$ as a function of changing dimension [1—3, 6—10, 12, 14, 15].

For $f \in C(\mathbf{X})$, let

$$e_n(f)(x) = f(x) - B_n(f)(x), \quad x \in \mathbf{X},$$

and

$$E_n(f) = \{x \in \mathbf{X} \colon |e_n(f)(x)| = \|e_n(f)\|\}. \tag{1.1}$$

Let $|E_n(f)|$ be the cardinality of the extremal set (1.1). The following result is noted in [3, 11].

THEOREM 1. *Let* $f \in C(\mathbf{X})$. *If* $|E_n(f)| = n+2$, *then*

$$n+1 \leqq M_n(f). \tag{1.2}$$

Hereafter $\mathbf{X}$ is the interval $I=[-1, 1]$, and $\mathcal{P}_{n+1}$ is the space Π_n of polynomials of degree at most n. Under certain hypotheses, (1.2) can be improved.

THEOREM 2. [9] *Suppose that* $f^{(n+2)} \in C(I)$, *and that* $f^{(n+1)}(x) \cdot f^{(n+2)}(x) \neq 0$, $x \in I$. *Then*

$$2n+1 < M_n(f). \tag{1.3}$$

We observe that the hypotheses of Theorem 2 insure that $|E_n(f)| = n+2$.

Now let P_{n+1} be a polynomial of degree precisely $n+1$. Then it is known [5, 9] that

$$M_n(P_{n+1}) = 2n+1. \tag{1.4}$$

Since the left hand side of (1.3) does not involve the strong unicity constant of the same function for every n, (1.4) is of a slightly different character than (1.2) and (1.3).

However, the following characterization of the strong unicity constant, due to Bartelt and Schmidt [2], demonstrates that the strong unicity constant actually depends only on the elements of the extremal set (1.1) and a set of associated signs.

THEOREM 3. *If $f\in C(I)$, then*

$$M_n(f) = \max_{p\in \Pi_n} \{\|p\|: \operatorname{sgn} e_n(f)(x)\, p(x) \leqq 1 \text{ for } x\in E_n(f)\}. \tag{1.5}$$

Because of characterization (1.5), it is reasonable to expect that expressions like (1.4) may be useful in analyzing upper and lower bounds for $M_n(f)$ (e.g., (1.2) and (1.3)).

From (1.4) we immediately obtain

$$\lim_{n\to\infty} \frac{M_n(P_{n+1})}{n} = 2. \tag{1.6}$$

Considering (1.6) and the conclusions of Theorems 1 and 2, it would be of interest to find classes of functions for which

$$\lim_{n\to\infty} \frac{M_n(f)}{n} \tag{1.7}$$

can be determined. Sections 3 and 4 below are primarily devoted to determining (1.7) for a class of non-rational functions $\mathbf{F}$ defined in [10].

In the next section we analyze $M_n(f)$ for certain rational functions to obtain results consistent with both (1.6) and (1.7).

2. Rational functions

Let

$$r(x) = \frac{1}{a-x}, \quad -1 \leqq x \leqq 1, \tag{2.1}$$

where $a\geqq 2$. A principal objective of the current section is to determine the limit (1.7) for the rational function defined by (2.1). This result will subsequently be used to determine $\lim_{n\to\infty} \frac{M_n(r_n)}{n}$ for particular sequences of rational functions $\{r_n\}_{n=1}^{\infty}$, which will in turn prove useful in Section 3.

As usual let

$$E_n(r) = \{x\in I: |e_n(r)(x)| = \|e_n(r)\|\} = \{x_0, x_1, \dots, x_{n+1}\}, \tag{2.2}$$

where

$$-1 = x_0 < x_1 < \dots < x_{n+1} = 1. \tag{2.3}$$

Here each x_i, $i=0, 1, \dots, n+1$ actually depends on n, but since this dependence is clear in what follows we suppress notational reference to this dependence.

As in [6, (2.9)], now define $Q_{n+1}\in \Pi_{n+1}$ by

$$Q_{n+1}(x_k) = \operatorname{sgn} e_n(r)(x_k) = (-1)^{n+k}, \quad k = 0, 1, \dots, n+1. \tag{2.4}$$

Then it is known [6, (2.24)] that the $\{x_k\}_{k=0}^{n+1}$ given in (2.2) are the zeros of

$$w(x) = \frac{(x^2-1)[n(a^2-1)^{1/2}C_n(x)+(ax-1)C_n'(x)]}{n2^{n-1}[(a^2-1)^{1/2}+a]}, \tag{2.5}$$

where C_n is the Chebyshev polynomial of degree n.

LEMMA 1. *Let Q_{n+1} be defined by* (2.4). *Then*

$$Q_{n+1}(x) = \frac{a}{n\sqrt{a^2-1}}(1-x^2)C_n'(x)-xC_n(x). \tag{2.6}$$

PROOF. Let $H(x)$, $x\in I$, be the right hand side of (2.6). Evaluating H at the extremal points (2.2) yields

$$H(x_k) = \frac{a}{n\sqrt{a^2-1}}(1-x_k^2)C_n'(x_k)-x_kC_n(x_k). \tag{2.7}$$

But from [6, p. 284] we have that

$$C_n'(x_k) = (-1)^{n+k}\frac{n(a^2-1)^{1/2}}{a-x_k}, \quad k=1, \dots, n. \tag{2.8}$$

From (2.5) we observe that

$$n(a^2-1)^{1/2}C_n(x_k)+(ax_k-1)C_n'(x_k) = 0, \quad k=1, 2, \dots, n$$

(see also [6, p. 284]). Using (2.8) in this equation and then solving for $C_n(x_k)$ yields

$$C_n(x_k) = (-1)^{n+k}\frac{ax_k-1}{x_k-a}. \tag{2.9}$$

Substituting (2.8) and (2.9) into (2.7) and simplifying reveals that

$$H(x_k) = (-1)^{n+k}, \quad k=1, \dots, n. \tag{2.10}$$

On the other hand, (2.7) implies that

$$H(-1) = C_n(-1) = (-1)^n, \quad \text{and} \quad H(1) = -C_n(1) = -1.$$

These two equalities and (2.10) now imply that $H(x_k)=(-1)^{n+k}$, $k=0, 1, \dots, n+1$, and hence uniqueness in interpolation implies that $H(x)\equiv Q_{n+1}(x)$, $x\in I$. □

We next define $q_{jn}\in\Pi_n$ by

$$q_{jn}(x_k) = \operatorname{sgn} e_n(r)(x_k) = (-1)^{n+k}, \; k=0, \dots, n+1; \; k\neq j; \; j=0, \dots, n+1. \tag{2.11}$$

Since from (2.2), $|E_n(r)|=n+2$, finding $\lim\limits_{n\to\infty}\dfrac{M_n(r)}{n}$ is equivalent to determining [7]

$$\lim_{n\to\infty}\max_{0\leqq j\leqq n+1}\frac{\|q_{jn}\|}{n}. \tag{2.12}$$

If a_{n+1} is the leading coefficient of Q_{n+1} in Lemma 1, then it is known [6, (2.10) and (2.24)] that

$$q_{jn}(x) = Q_{n+1}(x) - a_{n+1}\frac{w(x)}{(x-x_j)}, \tag{2.13}$$

where $w(x)$ is given by (2.5), $j=0, \dots, n+1$. From [6, (2.25)],

$$2^{n-1}[(a-1)^{1/2}+a]|w_n'(x)| \leqq n(a+1)+2(a^2-1)^{1/2}+|2ax^2-x-a|\,n. \tag{2.14}$$

By Lemma 1 we have that

$$a_{n+1} = -\frac{2^{n-1}[a+(a^2-1)^{1/2}]}{\sqrt{a^2-1}}. \tag{2.15}$$

Therefore (2.15) and (2.14) imply that

$$|a_{n+1}w'(x)| \leqq 2n\frac{(a+1)^{1/2}}{(a-1)^{1/2}}+2. \tag{2.16}$$

Thus, (2.13) implies that

$$|q_{jn}(x)| \leqq |Q_{n+1}(x)|+2n\frac{(a+1)^{1/2}}{(a-1)^{1/2}}+2. \tag{2.17}$$

Hence

$$\|q_{jn}\| \leqq 2n\frac{(a+1)^{1/2}}{(a-1)^{1/2}}+2+\|Q_{n+1}\|,$$

and consequently

$$\frac{\|q_{jn}\|}{n} \leqq 2\frac{(a+1)^{1/2}}{(a-1)^{1/2}}+\frac{1}{n}(2+\|Q_{n+1}\|), \quad j=0, 1, \dots, n+1. \tag{2.18}$$

From (2.5), (2.6), (2.13), and (2.15) we obtain

$$q_{0n}(x) = -C_n(x) - \frac{(x-1)}{n\sqrt{a^2-1}}(2ax+a-1)C_n'(x).$$

Therefore

$$|q_{0n}(-1)| = 2n\frac{(a+1)^{1/2}}{(a-1)^{1/2}}+1. \tag{2.19}$$

Combining (2.18) and (2.19) yields

$$2\frac{(a+1)^{1/2}}{(a-1)^{1/2}}+\frac{1}{n} \leqq \max_{0\leqq j\leqq n+1}\frac{\|q_{jn}\|}{n} \leqq 2\frac{(a+1)^{1/2}}{(a-1)^{1/2}}+\frac{1}{n}(2+\|Q_{n+1}\|). \tag{2.20}$$

We have nearly established the following theorem.

THEOREM 4. *Let* $r(x)=\dfrac{1}{a-x}$, $a\geqq 2$, $x\in I$. *Then*

$$\lim_{n\to\infty}\frac{M_n(r)}{n} = 2\frac{(a+1)^{1/2}}{(a-1)^{1/2}}. \tag{2.21}$$

PROOF. The theorem follows from inequality (2.20) and the observation above (2.12). □

To the authors' knowledge (2.21) is the first non-polynomial limit of the type described by (1.7) to be determined. As already noted, non-rational functions will be considered in Section 3 below. The following corollary will be very useful in the succeeding section and parallels (1.6) for a sequence of rational functions.

COROLLARY. *If* $r_n(x)=\frac{1}{\lambda_n-x}$, $\lambda_n \geqq 2$, $x\in I$, *then*

(i) *If* $\lim_{n\to\infty} \lambda_n=\lambda$, *then*

$$\lim_{n\to\infty} \frac{M_n(r_n)}{n} = \frac{2(\lambda+1)^{1/2}}{(\lambda-1)^{1/2}}. \tag{2.22}$$

(ii) *If* $\lim_{n\to\infty} \lambda_n=+\infty$, *then*

$$\lim_{n\to\infty} \frac{M_n(r_n)}{n} = 2. \tag{2.23}$$

PROOF. Both (2.22) and (2.23) follow immediately from (2.20) and (2.12). □

The results of the Corollary to Theorem 4 are perhaps not unexpected when compared to (1.6), but the analysis needed to establish Theorem 4 is considerably more complex that than required to prove (1.4) and hence (1.6). It is thus not surprising that the arguments needed to determine (1.7) for a class of non-rational functions are even more complex.

3. A class of non-rational functions

In this section we determine (1.7) for a class of non-rational functions **F**. This class is introduced and analyzed in [10], but only upper and lower bounds to $M_n(f)$, $f\in\mathbf{F}$, are obtained.

DEFINITION. Let **F** be the set of all functions $f\in C^\infty(I)$ satisfying

(a) $f^{(n+1)}(x)\neq 0$ on I,

and

(b) $\frac{1}{\alpha} \leqq \left|\frac{f^{(n+2)}(x)}{f^{(n+1)}(x)}\right| \leqq \frac{1}{\beta}$ on I,

for all n sufficiently large, where $\alpha \geqq \beta > 0$ are constants depending on f but not on n.

THEOREM 5. *If* $f\in\mathbf{F}$, *then*

$$\lim_{n\to\infty} \frac{M_n(f)}{n} = 2. \tag{3.1}$$

COMMENT. Although we now proceed directly to the proof of Theorem 5, two lemmas needed in the proof will be stated where they are used and established subsequent to the proof of Theorem 5.

PROOF OF THEOREM 5. Let $f \in \mathbf{F}$. Then part (a) of the Definition implies that $|E_n(f)|=n+2$. As before, let $E_n(f)=\{x_0, x_1, \ldots, x_{n+1}\}$ where

$$-1 = x_0 < x_1 < \ldots < x_n < x_{n+1} = 1. \tag{3.2}$$

As in (2.2), each x_i, $i=0, \ldots, n+1$, actually depends on n and hence could be labeled x_i^n, $i=0, \ldots, n+1$. However, no confusion results from suppressing the superscripts, and hence for the sake of notational simplicity the superscript is deleted.

Define $q_{jn} \in \Pi_n$ by

$$q_{jn}(f)(x_i) = \operatorname{sgn} e_n(f)(x_i), \quad i = 0, \ldots, n+1;\ i \neq j;\ j = 0, \ldots, n+1. \tag{3.3}$$

Again by appealing to [7] we may select $k \in \{0, 1, \ldots, n+1\}$ such that

$$M_n(f) = \|q_{kn}(f)\|. \tag{3.4}$$

Paralleling (2.4), let $Q_{n+1}(f) \in \Pi_{n+1}$ be defined by

$$Q_{n+1}(f)(x_i) = \operatorname{sgn} e_n(f)(x_i), \quad i = 0, \ldots, n+1. \tag{3.5}$$

If $a_{n+1}(f)$ is the leading coefficient of x^{n+1} in $Q_{n+1}(f)$, then as in (2.13)

$$q_{kn}(f)(x) = Q_{n+1}(f)(x) - a_{n+1}(f) \prod_{\substack{j=0 \\ j \neq k}}^{n+1} (x-x_j). \tag{3.6}$$

Therefore (3.4) and (3.6) imply that

$$M_n(f) \leqq \|Q_{n+1}(f)\| + |a_{n+1}(f)| \prod_{\substack{j=0 \\ j \neq k}}^{n+1} |x-x_j|. \tag{3.7}$$

We shall shortly consider

$$\prod_{\substack{j=0 \\ j \neq k}}^{n+1} |x-x_j|, \quad k = 0, 1, \ldots, n+1. \tag{3.8}$$

First let

$$U_n(x) = \frac{1}{\alpha(n+2)+2-x}, \quad x \in I, \tag{3.9}$$

and denote the extreme points of $e_n(U_n)$ by

$$-1 = u_0 < u_1 < \ldots < u_n < u_{n+1} = 1. \tag{3.10}$$

Similarly, let

$$V_n(x) = \frac{1}{\beta(n+2)-x-2}, \quad x \in I, \tag{3.11}$$

and label the extreme points of $e_n(V_n)$ by

$$-1 = v_0 < v_1 < \ldots < v_n < v_{n+1} = 1. \tag{3.12}$$

Then it can be shown [10] that

$$z_i < u_i < x_i < v_i < \zeta_i, \quad i = 1, 2, \ldots, n, \tag{3.13}$$

where

$$(3.14)\qquad \begin{cases} z_i = \cos\dfrac{n+1-i}{n+1}\pi, & i = 0, 1, \ldots, n+1, \\ \zeta_i = \cos\dfrac{n-i}{n}\pi, & i = 0, 1, \ldots, n, \end{cases}$$

are the extreme points of the Chebyshev polynomials C_{n+1} and C_n, respectively, and where u_i, v_i, and x_i, $i=0, 1, \ldots, n+1$ are given by (3.10), (3.12), and (3.2), respectively.

We now estimate the size of (3.8). Four cases are considered.

Case I. Suppose $k=0$. Then it is known [16, Lemma 2] that there exists an $\bar{x}\in I$ such that $-1=x_0\leqq\bar{x}<x_1$ and such that

$$(3.15)\qquad \|q_{0n}(f)\| = |q_{0n}(f)(\bar{x})|,$$

where $q_{0n}(f)$ is defined by (3.3) with $k=0$. Now (3.13) implies that

$$(3.16)\qquad \prod_{j=1}^{n+1} |\bar{x}-x_j| \leqq \prod_{j=1}^{n+1} |\bar{x}-v_j|.$$

Case II. Next assume that $1\leqq k\leqq n-1$. By again appealing to [16, Lemma 2], we know for each k that there exists an $\bar{x}$ such that $x_{k-1}<\bar{x}<x_{k+1}$ and such that

$$(3.17)\qquad \|q_{kn}(f)\| = |q_{kn}(f)(\bar{x})|,$$

where q_{kn} is defined by (3.3), $k=1, \ldots, n-1$. For Case II first suppose that $x_{k-1}<$ $<\bar{x}\leqq x_k$. Then (3.13) implies that

$$(3.18)\qquad \prod_{\substack{j=0\\ j\neq k}}^{n+1} |\bar{x}-x_j| \leqq \prod_{j=0}^{k-1} |\bar{x}-u_j| \prod_{j=k+1}^{n+1} |\bar{x}-u_j+u_j-x_j| =$$

$$= \prod_{\substack{j=0\\ j\neq k}}^{n+1} |\bar{x}-u_j| \prod_{j=k+1}^{n+1} \left|1+\frac{u_j-x_j}{\bar{x}-u_j}\right| \leqq$$

$$\leqq \prod_{\substack{j=0\\ j\neq k}}^{n+1} |\bar{x}-u_j| \exp\left[\sum_{j=k+1}^{n+1} \frac{x_j-u_j}{u_j-\bar{x}}\right] \leqq \prod_{\substack{j=0\\ j\neq k}}^{n+1} |\bar{x}-u_j| \exp\left[\sum_{j=k+1}^{n+1} \frac{v_j-u_j}{z_j-\zeta_k}\right] =$$

$$= \prod_{\substack{j=0\\ j\neq k}}^{n+1} |\bar{x}-u_j| \exp\left[\sum_{j=k+1}^{n+1} \left(\frac{z_j-\zeta_{j-1}}{z_j-\zeta_k}\right)\left(\frac{v_j-u_j}{z_j-\zeta_{j-1}}\right)\right].$$

From [10, Theorem 5]

$$(3.19)\qquad \frac{v_j-u_j}{z_j-\zeta_{j-1}} \leqq \frac{A}{n}, \quad j = 1, \ldots, n,$$

where A is independent of j and n. Therefore (3.18) and (3.19) imply that

$$\prod_{\substack{j=0\\ j\neq k}}^{n+1} |\bar{x}-x_j| \leqq \prod_{\substack{j=0\\ j\neq k}}^{n+1} |\bar{x}-u_j| \exp\left[\frac{A}{n}\sum_{j=k+1}^{n+1}\frac{z_j-\zeta_{j-1}}{z_j-\xi_k}\right]. \tag{3.20}$$

To obtain an upper bound for $\prod_{\substack{j=0\\ j\neq k}}^{n+1} |\bar{x}-x_j|$, $k=1, \ldots, n-1$, similar to that obtained in (3.16) for Case I $(k=0)$ we will need to estimate $\sum_{j=k+1}^{n+1}\frac{z_j-\zeta_{j-1}}{z_j-\zeta_k}$. In Lemma 3 below we will show that

$$\sum_{j=k+1}^{n+1}\frac{z_j-\zeta_{j-1}}{z_j-\zeta_k} \leqq M\sqrt{n+1}\ln(n+1), \quad k=1, \ldots, n-1, \tag{3.21}$$

where M is independent of n.

Applying (3.21) to (3.20) yields

$$\prod_{\substack{j=0\\ j\neq k}}^{n+1} |\bar{x}-x_j| \leqq \exp\left\{\hat{M}\frac{\sqrt{n+1}}{n}\ln(n+1)\right\} \prod_{\substack{j=0\\ j\neq k}}^{n+1} |\bar{x}-u_j|, \quad k=1, \ldots, n-1, \tag{3.22}$$

where $\hat{M}$ does not depend on n. Inequality (3.22) is similar to (3.16) where u_j replaces v_j, $j\neq k$.

Now assume that $x_k<\bar{x}<x_{k+1}$. Then from (3.13) we obtain

$$\prod_{\substack{j=0\\ j\neq k}}^{n+1} |\bar{x}-x_j| \leqq \prod_{j=0}^{k-1} |\bar{x}-x_j| \prod_{j=k+1}^{n+1} |\bar{x}-v_j|. \tag{3.23}$$

If $k=1$, then (3.22) reduces to

$$\prod_{\substack{j=0\\ j\neq k}}^{n+1} |\bar{x}-x_j| \leqq \prod_{\substack{j=0\\ j\neq k}}^{n+1} |\bar{x}-v_j|, \tag{3.24}$$

which is in the spirit of (3.16).

For $k>1$, (3.13) implies that

$$\prod_{j=0}^{k-1} |\bar{x}-x_j| = \prod_{j=0}^{k-1} |\bar{x}-v_j+v_j-x_j| = \prod_{j=0}^{k-1} |\bar{x}-v_j|\left|1+\frac{v_j-x_j}{\bar{x}-v_j}\right| \leqq$$

$$\leqq \prod_{j=0}^{k-1} |\bar{x}-v_j| \exp\left[\sum_{j=0}^{k-1}\frac{v_j-u_j}{\bar{x}-v_j}\right] \leqq \prod_{j=0}^{k-1} |\bar{x}-v_j| \exp\left[\sum_{j=0}^{k-1}\frac{v_j-u_j}{z_k-\zeta_j}\right] =$$

$$= \prod_{j=0}^{k-1} |\bar{x}-v_j| \exp\left[\sum_{j=0}^{k-1}\left(\frac{z_{j+1}-\zeta_j}{z_k-\zeta_j}\right)\left(\frac{v_j-u_j}{z_{j+1}-\zeta_j}\right)\right].$$

From (3.19) we may now infer that

$$\prod_{j=0}^{k-1} |\bar{x}-x_j| \leqq \prod_{j=0}^{k-1} |\bar{x}-v_j| \exp\left[\frac{\hat{A}}{n}\sum_{j=0}^{k-1}\frac{z_{j+1}-\zeta_j}{z_k-\zeta_j}\right], \tag{3.25}$$

where $\hat{A}$ is independent of n.

Now Lemma 3 below will also assert that

$$\sum_{j=0}^{k-1}\frac{z_{j+1}-\zeta_j}{z_k-\zeta_j} \leqq M\sqrt{n+1}\ln(n+1), \quad k=1, \dots, n, \tag{3.26}$$

where M is again independent of n.

Applying (3.26) to (3.25) yields

$$\prod_{j=0}^{k-1} |\bar{x}-x_j| \leqq \prod_{j=0}^{k-1} |\bar{x}-v_j| \exp\left[\bar{M}\frac{\sqrt{n+1}}{n}\ln(n+1)\right].$$

This inequality and (3.23) now imply that

$$\prod_{\substack{j=0\\ j\neq k}}^{n+1} |\bar{x}-x_j| \leqq \exp\left[\bar{M}\frac{\sqrt{n+1}}{n}\ln(n+1)\right]\prod_{\substack{j=0\\ j\neq k}}^{n+1} |\bar{x}-v_j|, \quad k=2, \dots, n-1, \tag{3.27}$$

which resembles (3.22).

Case III. We next assume that $k=n$. Then [16, Lemma 2] implies that $\|q_{nn}(f)\|=|q_{nn}(f)(\bar{x})|$ where $x_{n-1}<\bar{x}<x_{n+1}=1$. We may therefore conclude from (3.13) that

$$\prod_{\substack{j=0\\ j\neq n}}^{n+1} |\bar{x}-x_j| \leqq \prod_{\substack{j=0\\ j\neq n}}^{n+1} |\bar{x}-u_j|. \tag{3.28}$$

Finally, for $k=n+1$, [16, Lemma 2] applied to $\|q_{n+1,n}(f)\|=|q_{n+1,n}(f)(\bar{x})|$ asserts that $x_n<\bar{x}\leqq x_{n+1}=1$. Therefore

$$\prod_{j=0}^{n} |\bar{x}-x_j| \leqq \prod_{j=0}^{n} |\bar{x}-u_j|. \tag{3.29}$$

Together, (3.16), (3.22), (3.27), (3.28), and (3.29) establish that

$$\prod_{\substack{j=0\\ j\neq k}}^{n+1} |\bar{x}-x_j| \leqq \exp\left[\frac{\tilde{M}\ln(n+1)}{\sqrt{n}}\right]\max\left\{\prod_{\substack{j=0\\ j\neq k}}^{n+1} |\bar{x}-v_j|, \prod_{\substack{j=0\\ j\neq k}}^{n+1} |\bar{x}-u_j|\right\} \tag{3.30}$$

where $\tilde{M}>0$ is independent of n and where again k is such that $\|q_{kn}(f)\|=M_n(f)$.

Assume for the sake of simplicity that

$$\prod_{\substack{j=0\\ j\neq k}}^{n+1} |\bar{x}-x_j| \leqq \exp\left[\tilde{M}\frac{\ln(n+1)}{\sqrt{n}}\right]\prod_{\substack{j=0\\ j\neq k}}^{n+1} |\bar{x}-u_j|.$$

Utilizing this inequality in (3.7) yields

$$(3.31)\quad M_n(f) \leqq \|Q_{n+1}(f)\| + \frac{|a_{n+1}(f)|}{|a_{n+1}(U_n)|}\exp\left[\frac{\tilde{M}\ln(n+1)}{\sqrt{n}}\right]|a_{n+1}(U_n)| \prod_{\substack{j=0\\ j\neq k}}^{n+1}|\bar{x}-u_j|,$$

where $a_{n+1}(U_n)$ is the coefficient of x^{n+1} in the $Q_{n+1}(U_n)$ defined by (2.6) with $a=\lambda_n=\alpha(n+2)+2$. By applying (2.13) to (3.31) we find that

$$(3.32)\quad M_n(f) \leqq \|Q_{n+1}(f)\| + \frac{|a_{n+1}(f)|}{|a_{n+1}(U_n)|}\exp\left[\tilde{M}\frac{\ln(n+1)}{\sqrt{n}}\right][M_n(U_n)+\|Q_{n+1}(U_n)\|].$$

Now from [10, Theorems 3 and 10],

$$(3.33)\qquad 1+K_1\frac{\|e_{n+1}(f)\|}{\|e_n(f)\|} \geqq \|Q_{n+1}(f)\| \geqq \left|\frac{a_{n+1}(f)}{2^n}\right| \geqq 1,$$

where K_1 does not depend on n.

At this juncture in the proof of Theorem 5 we assume the conclusion of Lemma 2 below: if $f\in\mathbf{F}$, then

$$(3.34)\qquad \frac{\|e_{n+1}(f)\|}{\|e_n(f)\|} \leqq \frac{K_2}{n}.$$

Using (3.33) and (3.34) in (3.32) we obtain

$$(3.35)\qquad M_n(f) \leqq \|Q_{n+1}(f)\| + \left[1+\frac{\hat{K}}{n}\right]\frac{2^n}{|a_{n+1}(U_n)|}\times$$
$$\times\exp\left[\tilde{M}\frac{\ln(n+1)}{\sqrt{n}}\right][M_n(U_n)+\|Q_{n+1}(U_n)\|],$$

where $\hat{K}=K_1\cdot K_2$. Thus from (2.15) and (3.9) we have that

$$(3.36)\qquad \frac{M_n(f)}{n} \leqq \frac{\|Q_{n+1}\|}{n} + 2\left[1+\frac{\hat{K}}{n}\right]\frac{\sqrt{[\alpha(n+2)+2]^2-1}}{\alpha(n+2)+2+[(\alpha n+2\alpha+2)^2-1]^{1/2}}\times$$
$$\times\exp\left[\tilde{M}\frac{\ln(n+1)}{\sqrt{n}}\right]\left[\frac{M_n(U_n)}{n}+\frac{\|Q_{n+1}(U_n)\|}{n}\right].$$

Finally, (2.23) implies that $\lim\limits_{n\to\infty}\dfrac{M_n(f)}{n}\leqq 2$. Since every $f\in\mathbf{F}$, also satisfies the hypotheses of Theorem 2, (1.3) now implies (3.1). □

The conclusion of Theorem 5 is mildly surprising in that it coincides with the limits given in (2.23) and (1.6).

4. Lemmas

We conclude this paper by considering two lemmas already used in Section 3. In the first of these two lemmas we reconsider (3.34).

LEMMA 2. *If* $f \in \mathbf{F}$, *then*

$$\frac{\|e_{n+1}(f)\|}{\|e_n(f)\|} \leqq \frac{C}{n}, \quad n = 1, 2, \ldots, \tag{4.1}$$

where C *is a positive constant independent of* n.

PROOF. It is sufficient to show that if $f \in \mathbf{F}$, then

$$\left|\frac{f^{(n+2)}(\eta)}{f^{(n+1)}(\varepsilon)}\right| \tag{4.2}$$

is bounded for $-1 \leqq \eta, \varepsilon \leqq 1$ independent of n. In fact, if (4.2) holds, then [13, p. 78] implies that there exist η and ε in I such that

$$\frac{\|e_{n+1}(f)\|}{\|e_n(f)\|} = \left|\frac{f^{(n+2)}(\eta)}{f^{(n+1)}(\varepsilon)}\right| \cdot \frac{1}{2(n+2)} = O\left(\frac{1}{n}\right). \tag{4.3}$$

If $f \in \mathbf{F}$, then

$$\left|\frac{f^{(n+2)}(\eta)}{f^{(n+1)}(\varepsilon)}\right| = \left|\frac{f^{(n+2)}(\eta)}{f^{(n+1)}(\eta)}\right| \cdot \left|\frac{f^{(n+1)}(\eta)}{f^{(n+1)}(\varepsilon)}\right|,$$

and consequently we have

$$\left|\frac{f^{(n+2)}(\eta)}{f^{(n+1)}(\varepsilon)}\right| \leqq A \left|\frac{f^{(n+1)}(\eta)}{f^{(n+1)}(\varepsilon)}\right|, \tag{4.4}$$

where A is a positive constant not depending on ε, η, or n. On the other hand, for $f \in \mathbf{F}$ it is known [10, Lemma 1] that

$$\left|\frac{f^{(n+1)}(\eta)}{f^{(n+1)}(\varepsilon)}\right| \leqq B, \quad -1 \leqq \varepsilon, \eta \leqq 1,$$

where B does not depend on ε, η, or n. This inequality, (4.4), and (4.3) now combine to imply the conclusion of Lemma 2. □

The proof of the next lemma is more complex than the proof of Lemma 2. Lemma 3 is of interest in its own right.

LEMMA 3. *Let* $\{z_j\}_{j=0}^{n+1}$ *and* $\{\zeta_j\}_{j=0}^{n}$ *be the extreme points of the Chebyshev polynomials* C_{n+1} *and* C_n, *respectively (see* (3.14), (3.21), *and* (3.26)*). Then there exists a constant* A *not depending on* n *such that*

$$\sum_{j=k+1}^{n+1} \frac{z_j - \zeta_{j-1}}{z_j - \zeta_k} \leqq A \sqrt{n+1} \ln (n+1), \quad 1 \leqq k \leqq n-1, \; n = 2, 3, \ldots; \tag{4.5}$$

and

$$\text{(4.6)}\qquad \sum_{j=0}^{k-1} \frac{z_{j+1}-\zeta_j}{z_k-\zeta_j} \leqq A\sqrt{n+1}\ln(n+1), \quad 1 \leqq k \leqq n,\ n = 1, 2, \ldots.$$

We prove only part (4.5) of Lemma 3. The proof of part (4.6) is similar.

PROOF OF (4.5). *Case I.* $k \leqq [\![\sqrt{n+1}]\!]$. Here $[\![\cdot]\!]$ is the greatest integer function. For n sufficiently large, the left side of (4.5) may be written as

$$\text{(4.7)}\qquad \sum_{j=k+1}^{n+1} \frac{z_j-\zeta_{j-1}}{z_j-\zeta_k} = \sum_{j=k+1}^{[\![\sqrt{n+1}+1]\!]} \frac{z_j-\zeta_{j-1}}{z_j-\zeta_k} + $$

$$+ \sum_{j=[\![\sqrt{n+1}+2]\!]}^{n-[\![\sqrt{n+1}]\!]} \frac{z_j-\zeta_{j-1}}{z_j-\zeta_k} + \sum_{j=n+1-[\![\sqrt{n+1}]\!]}^{n+1} \frac{z_j-\zeta_{j-1}}{z_j-\zeta_k}.$$

Label the sums on the right side of equality (4.7) by S_1, S_2, and S_3, respectively. Since $\dfrac{z_j-\zeta_{j-1}}{z_j-\zeta_k} \leqq 1$ for $j \geqq k+1$, S_1 satisfies

$$\text{(4.8)}\qquad S_1 \leqq [\![\sqrt{n+1}]\!]+1-(k+1)+1 \leqq [\![\sqrt{n+1}]\!] \leqq \sqrt{n+1}.$$

For S_2, $[\![\sqrt{n+1}+2]\!] \leqq j \leqq n-[\![\sqrt{n+1}]\!]$. Therefore from (3.13) we have that

$$\text{(4.9)}\qquad \left|\frac{z_j-\zeta_{j-1}}{z_j-\zeta_k}\right| \leqq \left|\frac{z_j-z_{j-1}}{z_j-z_{k+1}}\right| \leqq \left|\frac{z_j-z_{j-1}}{z_j-z_{[\![\sqrt{n+1}+1]\!]}}\right|.$$

From (3.14) and the mean value theorem,

$$\text{(4.10)}\qquad z_j-z_{j-1} = |\sin\theta_j|\frac{\pi}{n+1},$$

where $\dfrac{j-1}{n+1}\pi < \theta_j < \dfrac{j}{n+1}\pi$, and hence where

$$\frac{[\![\sqrt{n+1}+1]\!]}{n+1}\pi < \theta_j < \left(1-\frac{[\![\sqrt{n+1}+1]\!]}{n+1}\right)\pi.$$

Also for $[\![\sqrt{n+1}+2]\!] \leqq j \leqq n-[\![\sqrt{n+1}]\!]$,

$$\text{(4.11)}\qquad z_j-z_{[\![\sqrt{n+1}+1]\!]} = |\sin\mu_j|\frac{j-[\![\sqrt{n+1}+1]\!]}{n+1}\pi$$

where $\dfrac{[\![\sqrt{n+1}+1]\!]}{n+1}\pi < \mu_j < \dfrac{j}{n+1}\pi$, and hence where

$$\text{(4.12)}\qquad \frac{1}{\sqrt{n+1}}\pi < \mu_j < \left(1-\frac{1}{\sqrt{n+1}}\right)\pi.$$

For $\frac{1}{\sqrt{n+1}}\pi<\varphi<\left(1-\frac{1}{\sqrt{n+1}}\right)\pi$,

$$|\sin\varphi| \geqq \left|\sin\frac{\pi}{\sqrt{n+1}}\right|. \tag{4.13}$$

Applying (4.13), (4.12), (4.11), and (4.10) to (4.9) yields

$$\left|\frac{z_j-\zeta_{j-1}}{z_j-\zeta_k}\right| \leqq \frac{1}{(j-[\![\sqrt{n+1}+1]\!])\left|\sin\frac{\pi}{\sqrt{n+1}}\right|}, \tag{4.14}$$

$$[\![\sqrt{n+1}+2]\!] \leqq j \leqq n-[\![\sqrt{n+1}]\!].$$

Utilizing (4.14) to bound S_2 gives

$$S_2 = \sum_{j=[\![\sqrt{n+1}+2]\!]}^{n-[\![\sqrt{n-1}]\!]} \frac{z_j-\zeta_{j-1}}{z_j-\zeta_k} \leqq \frac{1}{\left|\sin\frac{\pi}{\sqrt{n+1}}\right|} \sum_{j=[\![\sqrt{n+1}+2]\!]}^{n-[\![\sqrt{n+1}]\!]} \frac{1}{j-[\![\sqrt{n+1}+1]\!]} = \tag{4.15}$$

$$= \frac{1}{\left|\sin\frac{\pi}{\sqrt{n+1}}\right|} \sum_{l=1}^{n+1-2[\![\sqrt{n+1}+1]\!]} \frac{1}{l} \leqq \frac{1}{\left|\sin\frac{\pi}{\sqrt{n+1}}\right|}\ln(n+1) \leqq M\sqrt{n+1}\ln(n+1),$$

where n is sufficiently large and M is independent of n.

To estimate S_3, we recall that $\frac{z_j-\zeta_{j-1}}{z_j-\zeta_k} \leqq 1$ for $j \geqq k+1$. Therefore

$$S_3 = \sum_{j=n+1-[\![\sqrt{n+1}]\!]}^{n+1} \frac{z_j-\zeta_{j-1}}{z_j-\zeta_k} \leqq \tag{4.16}$$

$$\leqq n+2-(n+1)+[\![\sqrt{n+1}]\!] = [\![\sqrt{n+1}]\!]+1 \leqq \sqrt{n+1}+1.$$

Finally, inequalities (4.16), (4.15), and (4.8) combined to (4.7) imply the conclusion of Lemma 3 for Case I.

Case II. $[\![\sqrt{n+1}+1]\!] \leqq k \leqq n-[\![\sqrt{n+1}+2]\!]$. In this case we write the left side of (4.5) as

$$\sum_{j=k+1}^{n+1} \frac{z_j-\zeta_{j-1}}{z_j-\zeta_k} = \sum_{j=k+1}^{n-[\![\sqrt{n+1}]\!]} \frac{z_j-\zeta_{j-1}}{z_j-\zeta_k} + \sum_{j=n+1-[\![\sqrt{n+1}]\!]}^{n+1} \frac{z_j-\zeta_{j-1}}{z_j-\zeta_k}. \tag{4.17}$$

Denote the terms on the right side of (4.17) by $\bar{S}_1$ and $\bar{S}_2$, respectively. Then it is clear that (4.16) serves as an upper bound for $\bar{S}_2$.

For $\bar{S}_1$, $k \geqq [\![\sqrt{n+1}+1]\!]$ and (3.13) imply that

$$\bar{S}_1 = 1+\sum_{j=k+2}^{n+[\![\sqrt{n+1}]\!]} \frac{z_j-\zeta_{j-1}}{z_j-\zeta_k} \leqq 1+\sum_{j=k+2}^{n-[\![\sqrt{n+1}]\!]} \frac{z_j-z_{j-1}}{z_j-z_{k+1}}. \tag{4.18}$$

Again using the mean value theorem and (3.14) we obtain

$$z_j - z_{j-1} = |\sin \bar{\theta}_j| \frac{\pi}{n+1}, \quad \frac{j-1}{n+1}\pi \leqq \bar{\theta}_j \leqq \frac{j}{n+1}\pi, \tag{4.19}$$

and

$$z_j - z_{k+1} = |\sin \bar{\mu}_j| \left[\frac{j-(k+1)}{n+1}\right]\pi, \quad \frac{k+1}{n+1}\pi < \bar{\mu}_j < \frac{n-[\![\sqrt{n+1}]\!]}{n+1}\pi, \tag{4.20}$$

where $k+2 \leqq j \leqq n-[\![\sqrt{n+1}]\!]$. But the inequality below (4.20) and $k \geqq [\![\sqrt{n+1}+1]\!]$ imply that

$$\frac{[\![\sqrt{n+1}+1]\!]+1}{n+1}\pi < \bar{\mu}_j < \frac{n+1-[\![\sqrt{n+1}+1]\!]}{n+1}\pi$$

which in turn implies

$$\frac{1}{\sqrt{n+1}}\pi < \bar{\mu}_j < \left(1-\frac{1}{\sqrt{n+1}}\right)\pi. \tag{4.21}$$

We find that (4.21) is identical to (4.12). This observation (4.13), and (4.20) yield

$$z_j - z_{k+1} \geqq \left|\sin \frac{\pi}{\sqrt{n+1}}\right| \frac{[j-(k+1)]}{n+1}\pi. \tag{4.22}$$

Using (4.22), (4.20), (4.19), and (4.18) produces

$$\bar{S}_1 \leqq 1 + \frac{1}{\left|\sin\frac{\pi}{\sqrt{n+1}}\right|} \sum_{j=k+2}^{n-[\![\sqrt{n+1}]\!]} \frac{1}{j-(k+1)} = \tag{4.23}$$

$$= 1 + \frac{1}{\left|\sin\frac{\pi}{\sqrt{n+1}}\right|} \sum_{j=1}^{n-([\![\sqrt{n+1}+1]\!]+k)} \frac{1}{j} \leqq 1 + \frac{1}{\left|\sin\frac{\pi}{\sqrt{n+1}}\right|} \ln(n+1) \leqq$$

$$\leqq \bar{M}\sqrt{n+1}\ln(n+1),$$

where n is sufficiently large and $\bar{M}$ is independent of n. Thus (4.5) is established for Case II.

Case III. $n-[\![\sqrt{n+1}+1]\!] \leqq k \leqq n-1$. In this case

$$\sum_{j=k+1}^{n+1} \frac{z_{j+1}-\zeta_j}{z_k-\zeta_j} \leqq n+1-(k+1)+1 \leqq \tag{4.23}$$

$$\leqq n+2-\left(n+[\![\sqrt{n+1}+1]\!]\right) \leqq [\![\sqrt{n+1}]\!]+3 \leqq M\sqrt{n+1}.$$

Thus Lemma 3 (4.5) is proven. □

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DEPARTMENT OF MATHEMATICS
CENTRAL MICHIGAN UNIVERSITY
MOUNT PLEASANT, MICHIGAN 48859
USA

DEPARTMENT OF MATHEMATICAL SCIENCES
OLD DOMINION UNIVERSITY
NORFOLK, VIRGINIA 23508
USA

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SOME TYPICAL RESULTS ON BOUNDED BAIRE 1 FUNCTIONS

A. M. BRUCKNER (Santa Barbara) and G. PETRUSKA (Budapest)

0. Introduction. Notations

The word "typical" in the title refers to properties which hold for most of the Baire 1 functions in the sense of category. Results of such type were proved in [2], where the authors dealt with Darboux Baire 1 functions and some subclasses of those. We denote by $\mathcal{A}$, Δ, $\mathcal{D}\mathcal{B}^1$, $\mathcal{B}^1$ the set of approximately continuous functions, derivatives, Darboux Baire 1 functions and Baire 1 functions, resp., all defined on [0, 1]. Taking the corresponding bounded classes $b\mathcal{A}$, $b\Delta$, $b\mathcal{D}\mathcal{B}^1$, $b\mathcal{B}^1$ one has the advantage, that all these are Banach spaces with norm $\|f\|=\sup|f|$ and these four spaces form a strictly increasing system of closed subspaces in each other. All of our results, however make sense in the unbounded cases as well.

For any function f we denote by C_f, A_f, R_f the set of continuity points, the set of approximate continuity points and the range of f, resp.

Lebesgue's measure is denoted by λ and we repeatedly make use of the fact that an arbitrary Borel measure μ (always supposed to be finite) on [0, 1] is (outer) regular: $\mu(H)=\inf\{\mu(U), H\subset U, U \text{ open}\}$.

The set of real numbers is denoted by $\mathbf{R}$.

1. Level sets

In this section we study the typical properties of the level sets $f^{-1}(y)(y\in\mathbf{R})$.

LEMMA 1.1. *Let I be an open interval, and put*

$$\mathcal{B}_I^1=\{f\in\mathcal{B}^1:\ \exists y\in\mathbf{R}, \operatorname{cl} f^{-1}(y)\supset I\}.$$

If a sequence $f_n\in\mathcal{B}_I^1$ is (pointwise) convergent to f, then also f is constant on an everywhere dense subset of I. In particular, if $f\in\mathcal{B}^1$, then $f\in\mathcal{B}_I^1$.

PROOF. Since $f_n\in\mathcal{B}_I^1$, it is constant on a G_δ set H_n everywhere dense in I $(n=1, \ldots)$. Hence $H=\bigcap_{n=1}^{\infty} H_n$ is a G_δ set everywhere dense in I, and the limit f is obviously constant on H.

LEMMA 1.2. *Let μ be an arbitrary Borel measure on* [0, 1]. *For a given $\delta>0$ we put*

$$\mathcal{B}_{\mu,\delta}^1=\{f\in\mathcal{B}^1, \exists y\in\mathbf{R}, \mu(f^{-1}(y))\geqq\delta\}.$$

If a sequence $f_n\in\mathcal{B}_{\mu,\delta}^1$ is (pointwise) convergent to f, then $\mu(f^{-1}(y))\geqq\delta$ for a suitable $y\in\mathbf{R}$.

PROOF. By our assumption we can find y_n such that for $L_n=f_n^{-1}(y_n)$ we have $\mu(L_n)\geqq\delta$. Putting $L=\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}L_n$ we also have $\mu(L)\geqq\delta$. We show first that $\{y_n\}$ contains a convergent subsequence. Fix $x_0\in L$, then $x_0\in L_n$ for infinitely many n, say, $n=n_1, n_2, \ldots$. Now we have

$$f(x_0)=\lim_{n\to\infty}f_n(x_0)=\lim_{k\to\infty}f_{n_k}(x_0)=\lim_{k\to\infty}y_{n_k}$$

because of $x_0\in L_{n_k}$ $(k=1, \ldots)$. Denoting $L^*=\bigcap_{N=1}^{\infty}\bigcup_{k=N}^{\infty}L_{n_k}$ we obtain again $\mu(L^*)\geqq\delta$ and for any $x\in L^*\subset L$

$$f(x)=\lim_{k\to\infty}f_{n_k}(x)=\lim_{k\to\infty}y_{n_k}=f(x_0).$$

That is, for $y=f(x_0)$ the level set $f^{-1}(y)$ satisfies $f^{-1}(y)\supset L^*$, thus $\mu(f^{-1}(y))\geqq\geqq\mu(L^*)\geqq\delta$.

COROLLARY 1.3. *Let* $\mathscr{B}_I^1$ *and* $\mathscr{B}_{\mu,\delta}^1$ *be as in Lemmas* 1.1 *and* 1.2, *respectively. Then,* $\mathscr{B}_I^1$ *and* $\mathscr{B}_{\mu,\delta}^1$ *are uniformly closed classes. If* $\mathscr{F}$ *is a uniformly closed subfamily in* $\mathscr{B}^1$, *then* $\mathscr{F}_I=\mathscr{F}\cap\mathscr{B}_I^1$ *and* $\mathscr{F}_{\mu,\delta}=\mathscr{F}\cap\mathscr{B}_{\mu,\delta}^1$ *are also uniformly closed.*

PROOF. This is immediate from the preceding lemmas.

LEMMA 1.4. *Let* $\mathscr{F}\subset\mathscr{B}^1$ *be uniformly closed, and* $\mathscr{F}+g\subset\mathscr{F}$ *if* g *is a piecewise linear continuous function. Then, for any given interval* I, $\mathscr{F}_I=\mathscr{F}\cap\mathscr{B}_I^1$ *is a uniformly closed and nowhere dense subset of* $\mathscr{F}$.

PROOF. By Corollary 1.3, only the nowhere dense property of $\mathscr{F}_I$ has to be verified. Let $f\in\mathscr{F}_I$, and $\varepsilon>0$. Let $g(x)=0$ outside of I, $g(c)=\varepsilon$ at the midpoint c of I, and g connects 0 and ε linearly on both halves of I. Then $f+g\in\mathscr{F}$, but $f+g\notin\mathscr{F}_I$. Indeed, f takes a fixed value on a dense subset of I, therefore it takes the same value on its continuity points in I. This property can not hold for $f+g$, because this sum has the same continuity points as f, and $f+g$ is not constant on $C_f\cap I$.

THEOREM 1.5. *Let* $\mathscr{F}\subset\mathscr{B}^1$ *be a class as in Lemma* 1.4. *Then*

$$\mathscr{L}=\{f\in\mathscr{F}: f^{-1}(y) \text{ is nowhere dense for every } y\in\mathbf{R}\}$$

is an everywhere dense G_δ *subset of* $\mathscr{F}$.

PROOF. $\mathscr{F}\setminus\mathscr{L}=\bigcup\mathscr{F}_I$, where I ranges over the open intervals with rational endpoints. Thus the result follows from Lemma 1.4.

COROLLARY 1.6. *For the elements* f *of a residual* G_δ *in* $b\mathscr{A}$, $b\Delta$, $b\mathscr{D}\mathscr{B}^1$, $b\mathscr{B}^1$ *the level set* $f^{-1}(y)$ *is nowhere dense for every* $y\in\mathbf{R}$.

THEOREM 1.7. *Let* μ *be an arbitrary continuous Borel measure on* [0, 1]. *Then* $\{f\in b\mathscr{B}^1: \mu(f^{-1}(y))=0$ *for every* $y\in\mathbf{R}\}$ *is a residual* G_δ *in* $b\mathscr{B}^1$.

PROOF. We prove that $b\mathscr{B}_{\mu,\delta}^1$ is nowhere dense in $b\mathscr{B}^1$. This and Corollary 1.3 give the result. Since $b\mathscr{B}_{\mu,\delta}^1$ is closed, we have to show that it can not contain a ball. Let $f\in b\mathscr{B}_{\mu,\delta}^1$ and $\varepsilon>0$. We can approximate f by a $g\in b\mathscr{B}^1$ having finite range: $R_g=\{y_1, y_2, ..., y_n\}$, and $\|f-g\|<\frac{\varepsilon}{2}$. Let $[0, 1]$ be divided into N equal subintervals, where N is large enough to ensure $\mu\left(\left[\frac{k-1}{N}, \frac{k}{N}\right]\right)<\delta$ $(k=1, ..., N)$. Now we slightly modify g so as to get a function h which takes any of its finitely many values in at most one subinterval $\left[\frac{k-1}{N}, \frac{k}{N}\right]$. To this end we choose nN different values $0<y(j, k)$ $(j=1, ..., n;\ k=1, ..., N)$ all smaller than

$$\min\left(\frac{\varepsilon}{2}, \frac{1}{3}|y_\alpha-y_\beta|, \beta\neq\alpha\right).$$

Let $\frac{k-1}{N}\leqq x<\frac{k}{N}$ and $g(x)=y_j$. Then we put $h(x)=y_j+y(j, k)$. Let x_1 and x_2 belong to different subintervals. If $g(x_1)\neq g(x_2)$, then

$$|h(x_1)-h(x_2)| = |h(x_1)-g(x_1)+g(x_1)-g(x_2)+g_2(x_2)-h(x_2)| \geqq$$

$$\geqq |g(x_1)-g(x_2)|-|h(x_1)-g(x_1)|-|h(x_2)-g(x_2)| \geqq \frac{1}{3}\min_{\beta\neq\alpha}|y_\beta-y_\alpha| > 0.$$

If $g(x_1)=g(x_2)=y_j$, then $h(x_1)-h(x_2)=y(j, k)-y(j, l)\neq 0$, because of $k\neq l$. Therefore, we have $\mu(h^{-1}(y))<\delta$ for any $y\in\mathbf{R}$, and by

$$\|f-h\| \leqq \|f-g\|+\|g-h\| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

we obtain that the ε-ball around f can not belong to $b\mathscr{B}^1_{\mu,\delta}$.

THEOREM 1.8. *For any continuous Borel measure* μ *the set*

$$\{f\in b\mathscr{B}^1\colon \mu(\operatorname{cl} f^{-1}(y))=0 \quad \textit{for every}\ y\in\mathbf{R}\}$$

is a residual subset in $b\mathscr{B}^1$.

We do not know that this set is a G_δ or not.

PROOF. The function h constructed in the proof of Theorem 1.7 has finite range and any of its level sets $h^{-1}(y)$ is a subset of a small interval $\left[\frac{k-1}{N}, \frac{k}{N}\right]$. But then, this property also holds for any $f\in b\mathscr{B}^1$ with $\|f-h\|<\varepsilon$ if $\varepsilon<\frac{1}{2}\min|y_i-y_j|$, $y_i\neq y_j$; $y_i, y_j\in\operatorname{rng} h$. Since such type of functions h form an everywhere open dense subset in $b\mathscr{B}^1$, we obtain immediately that

$$\{f\in b\mathscr{B}^1\colon \exists y,\ \mu(\operatorname{cl} f^{-1}(y))\geqq\delta\}$$

is nowhere dense in $b\mathscr{B}^1$, and hence the result follows.

If we restrict ourselves to Lebesgue's measure λ, then we can extend Theorem 1.7 (but not 1.8) to the subclasses $b\mathscr{A}, b\Delta, b\mathscr{D}\mathscr{B}^1$. For the cases $b\mathscr{A}, b\Delta$ we make use of a lemma of D. Preiss.

LEMMA 1.9 (D. Preiss, private communication). *For any measurable f and $\varepsilon>0$ there exists $g\in b\mathcal{A}$ such that $\|g\|<\varepsilon$ and $\lambda((f+g)^{-1}(y))<\varepsilon$ for any $y\in\mathbf{R}$.*

PROOF. We consider all the numbers $y=y_1, \dots, y_n$ with $\lambda(f^{-1}(y))\geqq\frac{\varepsilon}{3}$. (If no such y exists, we can choose $g\equiv 0$.) We can find a finite system S of mutually disjoint closed sets such that

(i) for $F\in S$ we have $F\subset f^{-1}(y_j)$ for some j, moreover each $x\in F$ is a point of density of $f^{-1}(y_j)$;

(ii) $\lambda(F)<\frac{\varepsilon}{3}$ for any $F\in S$;

(iii) $\lambda\left[\left(\bigcup_{j=1}^{n} f^{-1}(y_j)\right)\setminus\bigcup_{F\in S} F\right]<\frac{\varepsilon}{3}$.

For each $F\in S$ we choose a number $c(F)$ with the properties

(iv) $0<c(F)<\min_{k\neq j}\left(\varepsilon, \frac{1}{3}|y_k-y_j|\right)$

and

(v) $c(F_1)\neq c(F_2)$ if $F_1\neq F_2$.

Applying a theorem of Zahorski ([3], Theorem 7) we can find $g\in b\mathcal{A}$ satisfying

(vi) $g(x)=0 \quad \left(x\notin \bigcup_{j=1}^{n} f^{-1}(y_j)\right)$;

(vii) $0\leqq g(x)\leqq\varepsilon$ $(x\in[0, 1])$;

(viii) $g(x)=c(F)$ $(x\in F\in S)$.

Consider now $(f+g)^{-1}(y)$ for a given $y\in\mathbf{R}$. By (vi) we have

$$(f+g)^{-1}(y)=\left[f^{-1}(y)\setminus\bigcup_{j=1}^{n} f^{-1}(y_j)\right]\cup$$

$$\cup\left[(f+g)^{-1}(y)\cap\left[\bigcup_{j=1}^{n} f^{-1}(y_j)\setminus\cup S\right]\right]\cup[(f+g)^{-1}(y)\cap\cup S]\stackrel{\text{def}}{=}H_1\cup H_2\cup H_3.$$

If $y\neq y_j$ $(j=1, \dots, n)$ then $\lambda(f^{-1}(y))<\frac{\varepsilon}{3}$. If $y=y_j$ for some j then $H_1=\emptyset$, thus $\lambda(H_1)<\frac{\varepsilon}{3}$ in any case. $\lambda(H_2)<\frac{\varepsilon}{3}$ also holds by (iii). The level set $(f+g)^{-1}(y)$ can not intersect more than one closed set $F\in S$. Indeed, if

$$f(x_1)+g(x_1)=f(x_2)+g(x_2),\quad x_1\in F_1,\ x_2\in F_2,\ F_1\neq F_2,$$

then

$$f(x_1)-f(x_2)=g(x_2)-g(x_1)=c(F_2)-c(F_1),$$

that is

$$0<|f(x_1)-f(x_2)|<\min_{j\neq k}|y_j-y_k|,$$

a contradiction, because $f(x_1)=y_j$, $f(x_2)=y_k$ for some j and k. Thus by (ii) we have $\lambda(H_3)<\frac{\varepsilon}{3}$ and the lemma is proved.

THEOREM 1.10. *The set of functions* f *such that* $f^{-1}(y)$ *is a nowhere dense (Lebesgue) null set for all* $y\in\mathbf{R}$ *is a residual* G_δ *in any of* $b\mathcal{A}, b\Delta$ *and* $b\mathcal{D}\mathcal{B}^1$.

PROOF. Referring to Corollary 1.6 it remains to prove that $f^{-1}(y)$ is a null set for every $y\in\mathbf{R}$ for the elements of a residual G_δ in $b\mathcal{A}, b\Delta, b\mathcal{D}\mathcal{B}^1$, respectively. Applying Corollary 1.3, $\mathcal{F}\cap\mathcal{B}^1_{\lambda,\delta}=\{f\in\mathcal{F}: \exists y,\ \lambda(f^{-1}(y))\geqq\delta\}$ is uniformly closed in $\mathcal{F}=b\mathcal{A}, b\Delta, b\mathcal{D}\mathcal{B}^1$. Thus it is enough to show that $\mathcal{F}\cap\mathcal{B}^1_{\lambda,\delta}$ is nowhere dense. In the cases $\mathcal{F}=b\mathcal{A}, b\Delta$ this follows immediately from Lemma 1.9. For $\mathcal{F}=b\mathcal{D}\mathcal{B}^1$ we need a different method, because the sum of a Darboux function and an approximately continuous function is not necessarily Darboux. Suppose that $b\mathcal{D}\mathcal{B}^1\cap\mathcal{B}^1_{\lambda,\delta}$ contains the ball

$$S = \{h\in b\mathcal{D}\mathcal{B}^1: \|h-h_0\|<\varepsilon\},\ h_0\in b\mathcal{D}\mathcal{B}^1 \text{ is fixed.}$$

By Theorem 1.7 we can choose a function $f\in b\mathcal{B}^1$ such that $\lambda(f^{-1}(y))=0$ $(y\in\mathbf{R})$ and $\|f-h_0\|<\frac{\varepsilon}{5}$. Now, according to Theorem 3 of [2] we can find $g\in b\mathcal{D}\mathcal{B}^1$ such that $\|g-f\|\leqq 4\frac{\varepsilon}{5}$ and $g=f$ almost everywhere. Therefore

$$\lambda(g^{-1}(y)) = \lambda(f^{-1}(y)) = 0 \quad (y\in\mathbf{R})$$

and

$$\|g-h_0\| \leqq \|g-f\|+\|f-h_0\| < \varepsilon,$$

that is $g\in S$, a contradiction. Thus $\mathcal{F}\cap\mathcal{B}^1_{\lambda,\delta}$ is nowhere dense in all the three cases $\mathcal{F}=b\mathcal{A}, b\Delta, b\mathcal{D}\mathcal{B}^1$ and hence the result follows.

We remark that the typical behaviour of the closure of level sets has not been cleared up yet. In contrast to Corollary 1.3 the set $\{f: \lambda(\operatorname{cl} f^{-1}(y))\geqq\delta \text{ for some } y\in\mathbf{R}\}$ is not uniformly closed in $b\mathcal{A}, b\Delta, b\mathcal{D}\mathcal{B}^1, b\mathcal{B}^1$. This can be seen as follows. Let g denote the function of Lemma 2.2 (see below), and put

$$f_n(x) = \begin{cases} g(x), & x\notin \bigcup_{j=1}^{n} I_j \\ \min\left(g(x),\ 1-\dfrac{1}{2^j}\right), & x\in I_j \quad (1\leqq j\leqq n). \end{cases}$$

Then, $f_n\in b\mathcal{A}, f_n\to f$ uniformly and $f^{-1}(1)=\emptyset$, but $\operatorname{cl}(f_n^{-1}(1))\supset[0,1]\setminus\bigcup_{k=1}^{\infty} I_k$ for $n=1,2,\dots$. We can of course choose $\lambda\left([0,1]\setminus\bigcup_{k=1}^{\infty} I_k\right)\geqq\delta>0$.

2. Continuity points

It is well known that C_f is an everywhere dense G_δ for any $f\in\mathcal{B}^1$, thus C_f is large in the sense of category. We show, that, on the other hand, it is typically small in the sense of measure.

LEMMA 2.1. *Let* μ *be an arbitrary Borel measure on* $[0,1]$ *and* $\delta>0$. *The family*

$$\mathcal{M}_{\mu,\delta} = \{f\in\mathcal{B}^1,\ \mu(C_f)\geqq\delta\}$$

is uniformly closed.

PROOF. Suppose that $f_n \in \mathcal{M}_{\mu,\delta}$, $f_n \to f$ uniformly. We consider $C = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} C_{f_n}$. For any $x \in C$ we have $x \in C_{f_n}$ for infinitely many indices n, hence $C \subset C_f$. Thus $\lambda(C_f) \geqq \lambda(C) \geqq \delta$ and this proves $f \in \mathcal{M}_{\mu,\delta}$.

LEMMA 2.2. *Let $I_n = (a_n, b_n) = (c_n - h_n, c_n + h_n)$ be a sequence of pairwise disjoint open intervals in $[0, 1]$ such that the open set $H = \bigcup_{n=1}^{\infty} I_n$ is everywhere dense in $[0, 1]$. Let g_n $(n=1, \dots)$ be the piecewise linear continuous function, for which*

$$g_n(x) = \begin{cases} 1, & x = c_n \\ 0, & x \leqq c_n - \frac{1}{n} h_n \quad \text{or} \quad x \geqq c_n + \frac{1}{n} h_n \end{cases}$$

and g_n connects 0 and 1 linearly on $\left[c_n - \frac{1}{n} h_n, c_n\right]$ and $\left[c_n, c_n + \frac{1}{n} h_n\right]$. Let $g = \sum_{n=1}^{\infty} g_n$. Then

(i) *$g|_{[a_n, b_n]}$ is continuous on $[a_n, b_n]$ $(n=1, \dots)$, $C_g = H$ and $g(x) = 0$ for $x \notin H$;*
(ii) *g is approximately continuous on $[0, 1]$.*

PROOF. For any x there exists at most one n with $g_n(x) \neq 0$, thus the definition of g makes sense. Obviously $g|_{[a_n, b_n]} = g_n|_{[a_n, b_n]}$, hence it is continuous on $[a_n, b_n]$. Therefore $C_g \supset H$. For $x \notin H$, $g_n(x) = 0$ $(n=1, \dots)$, this implies $g(x) = 0$; since H is an everywhere dense open set, any point $x \notin H$ is the limit point of some subsequence of $\{c_n\}$, thus $x \notin C_g$ and this proves $C_g = H$. The approximate continuity is to be verified for $x \notin H$ only. We can suppose that every right hand side neighbourhood $(x, x+h)$ meets infinitely many intervals I_n (otherwise $x = a_n$ for some n and g is continuous from the right at a_n). Let $n = N$ be the smallest index with $I_n \cap (x, x+h) \neq \emptyset$. Then

$$\frac{1}{h} \lambda(\{t \colon x < t < x+h,\ g(t) > 0\}) \leqq$$

$$\leqq \begin{cases} \dfrac{\sum \frac{2}{n} h_n}{\sum 2h_n}, & \text{if} \quad x+h \notin H, \text{ or } x+h \in I_\nu, \text{ but } a_\nu < x+h \leqq c_\nu - \frac{1}{\nu} h_\nu \\[2ex] \dfrac{\sum \frac{2}{n} h_n + \frac{2}{\nu} h_\nu}{\sum 2h_n + \frac{\nu - 1}{\nu} h_\nu}, & \text{if} \quad c_\nu - \frac{1}{\nu} h_\nu \leqq x+h \leqq b_\nu \end{cases}$$

where the summation is always extended to the indices n such that $I_n \subset (x, x+h)$. Since $n \geqq N$, $\nu \geqq N$, $2 \geqq \frac{N-1}{N}$, $\frac{\nu - 1}{\nu} \geqq \frac{N-1}{N}$ in both the above cases

$$\frac{1}{h} \lambda(\{t \colon x < t < x+h,\ g(t) > 0\}) < \frac{2}{N-1}$$

and we obtain $\lim_{y \to x+0} \operatorname{app} g(y) = 0 = g(x)$, because $h \to 0$ implies $N \to \infty$. Similar argument applies to the left hand side neighbourhoods $(x-h, x)$ and hence the proof is complete.

LEMMA 2.3. *Let $\mathscr{F} = b\mathscr{A}, b\Delta, b\mathscr{D}\mathscr{B}^1, b\mathscr{B}^1$. In all cases the set $\mathscr{F}^* = \{f \in \mathscr{F}$: C_f does not contain any open interval$\}$ is an everywhere dense G_δ in $\mathscr{F}$.*

PROOF. Let $\mathscr{F}_I = \{f \in \mathscr{F}: C_f \supset I\}$ where I is a given open interval. It is trivial that $\mathscr{F}_I$ is uniformly closed and the complement $\mathscr{F} \setminus \mathscr{F}_I$ is everywhere dense in $\mathscr{F}$. Let J range over the open intervals with rational endpoints, then $\bigcup_J \mathscr{F}_J$ is a first category F_σ and hence the result follows.

THEOREM 2.4. *Let μ be an arbitrary Borel measure on $[0, 1]$ and $\mathscr{F} = b\mathscr{A}, b\Delta, b\mathscr{D}\mathscr{B}^1, b\mathscr{B}^1$. Then $\{f \in \mathscr{F}: \mu(C_f) = 0\}$ is an everywhere dense G_δ in $\mathscr{F}$.*

PROOF. By Lemma 2.1 the family $\mathscr{F} \cap \mathscr{M}_{\mu,\delta}$ is uniformly closed. We show that it is nowhere dense in $\mathscr{F}$, that is $\mathscr{F} \cap \mathscr{M}_{\mu,\delta}$ can not contain a ball. If there were a ball in $\mathscr{F} \cap \mathscr{M}_{\mu,\delta}$ then, by Lemma 2.3, we could also suppose that it is centered around a function $f \in \mathscr{F}^* \cap \mathscr{M}_{\mu,\delta}$ with radius ε, say. Now we cover the everywhere dense set $[0, 1] \setminus C_f$ by an open set H such that $\mu(C_f \cap H) < \delta$. Let g denote the function of Lemma 2.2 applied to H. Then $h = f + \frac{\varepsilon}{2} g \in \mathscr{F}$ and $\|h - f\| \leqq \frac{\varepsilon}{2} < \varepsilon$.

Though we cannot generally add $\frac{\varepsilon}{2} g$ to an element of $b\mathscr{D}\mathscr{B}^1$ without destroying the Darboux property, still we have now $h \in \mathscr{F}$ because f and g have no discontinuity points in common (see [1], p. 9, property (6)).

Furthermore, g is discontinuous out of H, whereas f is continuous there, and hence $C_h \subset C_f \cap H$. This implies $\mu(C_h) < \delta$, that is $h \notin \mathscr{M}_{\mu,\delta}$, a contradiction, and the proof is complete.

3. The range

This section deals with the typical range. The full range $f(I)$ on an interval I is to be studied only for $f \in \mathscr{B}^1$ because Darboux functions map intervals onto intervals. However, the restricted range taken on C_f or A_f is interesting for the subclasses as well.

LEMMA 3.1. *Let μ be an arbitrary Borel measure and $\delta > 0$. The set*

$$\mathscr{R}_{\mu,\delta} = \{f: \mu(\operatorname{cl} R_f) \geqq \delta\}$$

is a nowhere dense uniformly closed set in $b\mathscr{B}^1$.

PROOF. Let $f_n \in \mathscr{R}_{\mu,\delta}$ and $f_n \to f$ uniformly. Suppose $\mu(\operatorname{cl} R_f) < \delta$. We choose the open set U such that $\operatorname{cl} R_f \subset U$ and $\mu(U) < \delta$. Since $\operatorname{cl} R_f$ is compact, we can choose U to be the union of finite number of intervals: $U = U_1 \cup \ldots \cup U_n$. For each $j = 1, \ldots, n$, let V_j be an open interval such that $\operatorname{cl} U_j \subset V_j$ and $\mu\left(\bigcup_{j=1}^n V_j\right) < \delta$. We put $V = \bigcup_{j=1}^n V_j$ and $\varepsilon = \operatorname{dist}(\operatorname{cl} U, [0, 1] \setminus V)$. Then we fix k_0

such that $\|f_k - f\| < \varepsilon$ for $k \geqq k_0$. We have now $\mathrm{cl}\, R_{f_k} \subset V$ $(k \geqq k_0)$ and hence $\mu(\mathrm{cl}\, R_{f_k}) \leqq \mu(V) < \delta$, a contradiction. Thus $\mathscr{R}_{\mu,\delta}$ is closed. We know that the functions with finite range are everywhere dense in $b\mathscr{B}^1$. It follows immediately, that if D is a given everywhere dense subset of $\mathbf{R}$, then the family $\{f \in b\mathscr{B}^1 : R_f$ finite, $R_f \subset D\}$ is also everywhere dense in $b\mathscr{B}^1$. If we put $D = \{y : \mu(\{y\}) = 0\}$ then D is everywhere dense (D has countable complement). Applying the remark to this set D, we see that

$$\{f \in b\mathscr{B}^1 : \mu(\mathrm{cl}\, R_f) = 0\} \supset \{f \in b\mathscr{B}^1 : R_f \text{ finite, } R_f \subset D\}$$

is everywhere dense in $b\mathscr{B}^1$, thus the same holds for the complement of $\mathscr{R}_{\mu,\delta}$. That is, $\mathscr{R}_{\mu,\delta}$ is nowhere dense and the assertion is proved.

THEOREM 3.2. *Let μ be an arbitrary Borel measure. Then*

$$\mathscr{R} = \{f \in b\mathscr{B}^1 : \mu(\mathrm{cl}\, R_f) = 0\}$$

is an everywhere dense G_δ in $b\mathscr{B}^1$.

PROOF. By $b\mathscr{B}^1 \setminus \mathscr{R} = \bigcup\limits_{n=1}^{\infty} \mathscr{R}_{\mu,1/n}$ the result immediately follows from Lemma 3.1.

We recall some results on $f(C_f)$ and $f(A_f)$ with respect to the subclasses $\mathscr{F} = b\mathscr{A}, b\Delta, b\mathscr{D}\mathscr{B}^1$.

THEOREM 3.3. *Let μ be an arbitrary Borel measure. The family*

$$\{f \in \mathscr{F} : \mu(\mathrm{cl}\, f(C_f)) = 0\}$$

is an everywhere dense G_δ in both $\mathscr{F} = b\mathscr{A}$ and $b\Delta$.

For the case $\mathscr{F} = b\mathscr{D}\mathscr{B}^1$ an even stronger result holds:

THEOREM 3.4. *Let μ be an arbitrary Borel measure. The family*

$$\{f \in b\mathscr{D}\mathscr{B}^1 : \mu(\mathrm{cl}\, f(A_f)) = 0\}$$

is an everywhere dense G_δ in $b\mathscr{D}\mathscr{B}^1$.

These results were stated and proved in [2] for the special case $\mu = \lambda$. The general case can be proved without essential change of the original argument, thus we omit the proof here. Theorem 3.4 can not be extended to $b\Delta$: it was recently proved by the second named author that $f(A_f) = R_f$ for any derivative f, that is, in this respect derivatives stand closer to approximate continuous functions than to Darboux Baire 1 functions.

The results of Theorems 3.2, 3.3, 3.4 would be trivial if the underlying sets were countable. This is not the case, as shown by the next theorem.

THEOREM 3.5. *Let $\mathscr{F} = b\mathscr{A}, b\Delta, b\mathscr{D}\mathscr{B}^1, b\mathscr{B}^1$. Then the family*

$$\{f \in \mathscr{F} : f(C_f) \text{ is of power of continuum}\}$$

is an everywhere dense G_δ set in $\mathscr{F}$.

PROOF. $f(C_f)$ is an analytic set, therefore its power is either countable or of continuum. If $f(C_f)=\{y_1, ...\}$ is a countable set, then by the category theorem there exist an index n and an open interval I such that $f^{-1}(y_n)\cap C_f=I\cap C_f$. Thus $\operatorname{cl} f^{-1}(y_n)\supset I$. Therefore Theorem 1.5 gives the result.

4. Problems

4.1. What are the typical properties of $\mu(f^{-1}(y))$ in $b\mathscr{A}, b\Delta, b\mathscr{D}\mathscr{B}^1$ for Borel measures μ?

4.2. What are the typical properties of $\lambda(\operatorname{cl} f^{-1}(y))$ or $\mu(\operatorname{cl} f^{-1}(y))$ in the three subclasses considered?

4.3. Is it true, that there is a "large" set $Y\subset R_f$ such that $f^{-1}(y)$ is countable (finite or singleton) for typical $f\in\mathscr{F}$, $\mathscr{F}=b\mathscr{A}, b\Delta, b\mathscr{D}\mathscr{B}^1, b\mathscr{B}^1$ and $y\in Y$?

4.4. How large is the Hausdorff measure of $f^{-1}(y)$, C_f, $f(C_f)$ for typical f?

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UNIVERSITY OF CALIFORNIA
DEPARTMENT OF MATHEMATICS
SANTA BARBARA, CALIFORNIA
USA

EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF MATHEMATICS
BUDAPEST, MÚZEUM KRT. 6—8
H—1088

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ON MEASURES OF INTERSECTIVITY

I. Z. RUZSA (Budapest)

0. Notations

If $A, B, \ldots$ are sets of integers, $A(x), B(x), \ldots$ is the corresponding counting function, $d(A)$ the asymptotic density, $\underline{d}(A), \overline{d}(A)$ the lower and upper density;

$$\{A \pm B = a \pm b\colon\ a \in A,\ b \in B,\}$$

$$nA = \{na\colon\ a \in A\}, \quad A \pm n = \{a \pm n\colon\ a \in A\}$$

(n a number).

1. Introduction

A set A of positive integers is called (difference) *intersective,* if $A \cap (B-B) \neq \emptyset$ whenever B has positive upper density. Here instead of upper density we might equally naturally write lower or asymptotic density; we are going to show that these definitions lead to the same concept, even quantitatively. This is used to show that intersectivity is a "finitary" property. We also show that the situation becomes more complicated if we intend to distinguish between intersective sets, or if we consider sums rather than differences. A denser set than the previously known ones will also be constructed whose differences do not contain any prime-minus-one (this is known to be intersective).

2. Measures of intersectivity

Write

$$\delta_1(A) = \sup\{d(B)\colon\ A \cap (B-B) = \emptyset\},$$

where the choice of B is restricted to sets having an asymptotic density. Remark that $A \cap (B-B) = \emptyset$ is equivalent to $B \cap (B+a) = \emptyset$ for all $a \in A$. We can replace this by the weaker condition

$$d\big(B \cap (B+a)\big) = 0 \quad (a \in A) \tag{2.1}$$

and thus define

$$\delta_2(A) = \sup\{\overline{d}(B)\colon B \text{ satisfies } (2.1)\}.$$

So δ_1 is the smallest and δ_2 the biggest of the six possible definitions of "measure of intersectivity" that we get by combining lower, upper and asymptotic density and the above strict and weak disjointness conditions.

Put

$$D(A, x) = \max\{|T|\colon T \subset [1, x],\ (T-T) \cap A = \emptyset\},$$

a finite version of these concepts. $D(A, x)$ is obviously a subadditive function of x, thus there exists

$$\delta(A) = \lim \frac{D(A, x)}{x} = \inf \frac{D(A, x)}{x}, \tag{2.2}$$

where, of course, x runs over only positive integers.

THEOREM 1. *For every set* A, $\delta_1(A) = \delta_2(A) = \delta(A)$.

PROOF. $\delta_1 \leqq \delta_2$ is obvious. To show $\delta_2 \leqq \delta$ let B be a set satisfying (2.1) and fix an x. In $(kx, (k+1)x]$ B can have at most $D(A, x)$ elements, with $o(y)$ exceptions of $k \leqq y$, thus $\bar{d}(B) \leqq D(A, x)/x$; taking first the limit in x and then supremum in B we obtain the desired inequality.

Finally we have to show $\delta_1 \geqq \delta$; to this end we need some concepts and results from [2].

A set H of finite sets of integers is called a *homogeneous system*, if for every $S \in H$ all the subsets and translates of S belong to H as well. The counting function of a h. s. H is defined by

$$H(x) = \max_{T \in H} |T \cap [1, x]|$$

and its density by

$$d(H) = \lim H(x)/x = \inf H(x)/x.$$

Given our A, let H be the collection of all finite sets T such that $(T-T) \cap A = \emptyset$. Then obviously

$$H(x) = D(A, x), \quad d(H) = \delta(A).$$

Now Theorem 4 of [2] asserts that for every homogeneous system H there is a sequence of natural numbers B all whose finite subsets belong to H and such that $d(B) = d(H)$. Take this B for our H; in this case the above properties become

$$(B-B) \cap A = \emptyset, \quad d(B) = \delta(A)$$

as wanted. (As a by-product we learned that the supremum in the definition of δ_1 is actually a maximum.)

REMARK. $\delta_1 = \delta_2$ was essentially proved by Stewart and Tijdeman [4], Theorem 5.

3. Finitariness of intersectivity

Here we show that if a set is intersective, it is because some of its finite subsets are already almost so.

THEOREM 2. *For every set* A

$$\delta(A) = \inf \{\delta(T) : T \subset A, |T| < \infty\}.$$

PROOF. Write $A_x = A \cap [1, x]$. Obviously $\delta(A_x) \geqq \delta(A)$ and

$$D(A, x) = D(A_x, x) \geqq x\delta(A_x)$$

by (2.2), thus

$$\lim \delta(A_x) \leqq \lim D(A, x)/x = \delta(A).$$

4. The case $D(A, x) = o(x)$

In case $\delta(A)=0$ we can ask (i) how fast $D(A, x)/x$ tends towards zero, (ii) how dense an infinite set B can be if $(B-B)\cap A=\emptyset$. We shall see that, in contrast to the case $\delta(A)>0$, these questions may have a totally different answer. Authors on these subjects have generally ignored the difference between (i) and (ii), though it quite possibly occurs also at "ordinary" sequences, cf. the next section.

THEOREM 3. *Let F be any positive-valued function on natural numbers such that $F(x)\nearrow\infty$ but $F(x)/x\searrow 0$ as $x\to\infty$. There is a set A such that $D(A, x)\asymp F(x)$, but there is no infinite B for which $A\cap(B-B)=\emptyset$.*

(Recall that $f\asymp g$ means that both $f\ll g$ and $g\ll f$.)

PROOF. Let n be a natural number, $2^k\| n$. We set $n\in A$ if

$$2^k F(n) < n. \tag{4.1}$$

First we show that an infinite difference set cannot avoid A. Suppose $(B-B)\cap A=\emptyset$ and let $b_1<b_2$ be the first two elements of B, $2^k\| (b_2-b_1)$; let b be any other element. For $j=1$ or 2 we have $2^{k+1}\nmid(b-b_j)$, thus by (4.1) $2^k F(b-b_j)\geqq b-b_j$, which by $F(x)/x\to 0$ can have only a finite number of solutions.

Now we establish $D(A, x)\geqq F(x)/2$. Let $2^k\leqq x/F(x)<2^{k+1}$ and

$$T = \{n\colon n \leqq x,\ n\equiv 1 \pmod{2^{k+1}}\}.$$

Then $|T|\geqq x/2^{k+1}\geqq F(x)/2$ and if $b, b'\in T$, then $2^{k+1}|b-b'$ and

$$2^{k+1} > \frac{x}{F(x)} \geqq \frac{b-b'}{F(b-b')},$$

which shows $b-b'\notin A$.

Finally we show $D(A, x)\leqq 2F(x)+2$. Suppose

$$(T-T)\cap A = \emptyset,\quad T\subset[1, x].$$

Let u be the minimal and v the maximal element of T. For an arbitrary $b\in T$ we know $b-u\notin A$, $v-b\notin A$. Suppose first $b-u\geqq v-b$. If $2^k\| b-u$, then $2^k F(b-u)\geqq b-u$, i.e.

$$2^k \geqq \frac{b-u}{F(b-u)} \geqq \frac{(v-u)/2}{F\big((v-u)/2\big)}.$$

This leaves for $b-u$ at most

$$1+\frac{(v-u)/2}{2^k} \leqq 1+F\big((v-u)/2\big)$$

possibilities. The other half-interval can be estimated in the same way, thus we have obtained

$$|T| \leqq 2+2F((v-u)/2) \leqq 2+2F(x).$$

REMARK. $D(A, x)/x$ may not be monotonically decreasing, but it is almost so; namely subadditivity implies that for $y>x$

$$D(A, y)/y \leqq 2D(A, x)/x.$$

5. Variations on shifted primes

Let

$$P^{+} = \{p+1, p \text{ prime}\}, \quad P^{-} = \{p-1, p \text{ prime}\}.$$

That P^{+} and P^{-} are intersective (other translates of the set of primes are evidently not) was proved by Sárközy [3]. He even proved

$$D(P^{\pm}, x) \ll x(\log\log x)^{-2+\varepsilon}.$$

From below, Erdős and Sárközy [1] improved the trivial $D \gg \log x$ to

$$D(P^{-}, x) \gg \log x \log\log x.$$

$D(P^{\pm}, x) = O(x^{\varepsilon})$ seems certain but hopeless; it would imply that the first prime $p \equiv 1 \pmod{k}$ is $< k^{1+\varepsilon}$. We are going to improve the lower bound.

THEOREM 4. a) *There is a set* B *such that*

$$B(x) = \exp\left(\frac{\log 2}{2} + o(1)\right) \frac{\log x}{\log\log x}$$

and

$$(B-B) \cap (P^{+} \cup P^{-}) = \varnothing.$$

b) *If* $A = P^{+}$ *or* $A = P^{-}$, *then*

$$D(A, x) \gg \exp\left(\log 2 + o(1)\right) \frac{\log x}{\log\log x}.$$

PROOF. a) Let p_k denote the kth prime. First we construct auxiliary numbers r_k with the properties

$$r_{k+1} > 2r_k, \quad r_k > p_{2k}+1, \quad r_k \equiv 0 \pmod{p_1 p_2 \dots p_{2k-2}},$$
$$r_k \equiv -1 \pmod{p_{2k-1}}, \quad r_k \equiv 1 \pmod{p_{2k}}.$$

If we always choose the smallest possible value for r_k, then obviously

$$p_1 p_2 \dots p_{2k} \leqq r_{k+1} \leqq 2r_k + p_{2k} + p_1 p_2 \dots p_{2k},$$

whence easily

$$r_k = e^{(2+o(1))k \log k},$$

thus there are $\sim (\log x)/(2 \log\log x)$ r_k's up to x. Now let B be the set of numbers of the form

$$\sum \varepsilon_k r_k, \quad \varepsilon_k = 0 \text{ or } 1.$$

$r_{k+1} > 2r_k$ guarantees that all sums of this form are distinct, hence

$$B(x) = \exp\left(\frac{\log 2}{2} + o(1)\right) \frac{\log x}{\log\log x}$$

as wanted. Now let $b, b' \in B$; we have to show that $b - b' \pm 1$ is not a prime. Let

$$b = \sum \varepsilon_k r_k, \quad b' = \sum \varepsilon'_k r_k,$$

and let j be the first suffix for which $\varepsilon_j \neq \varepsilon'_j$. Then by the defining congruences

$$b-b' \equiv \varepsilon_j-\varepsilon'_j \pmod{p_{2j}},$$
$$\equiv \varepsilon'_j-\varepsilon_j \pmod{p_{2j-1}},$$

i.e. one of $b-b'+1$ and $b-b'-1$ is divisible by p_{2j} and the other by p_{2j-1}. Now if either of them is a prime, it can be only this one. But

$$b-b'-1 \geqq r_j-1 > p_{2j}$$

by assumption, thus this cannot happen either.

b) Consider P^+; P^- can be treated similarly. Again we define auxiliary numbers $r_1, \dots, r_k$, but now in dependence on x. Namely let

$$k=(1-\varepsilon)(\log x)/\log\log x$$

and let r_j $(j=1, \dots, k)$ satisfy

$$r_{j+1} > 2r_j, \quad r_j \equiv 1 \pmod{p_j}, \quad r_j \equiv 0 \pmod{p_{j+1}p_{j+2} \dots p_k}.$$

If again we choose the smallest possible r_k, then

$$r_1 \leqq Q = p_1p_2 \dots p_k = \exp\big(1+o(1)\big)k\log k$$

and $r_{j+1} \leqq 2r_j+Q$, whence by induction

$$r_j \leqq (2^j-1)Q, \quad r_k \leqq (2^k-1)Q < x$$

for $x > x_0(\varepsilon)$.

Define B as in a) and let again

$$b = \sum \varepsilon_i r_i, \quad b' = \sum \varepsilon'_i r_i$$

be two different elements of B, $b<b'$. Let j be now the suffix of the *greatest* different digit, i.e. $\varepsilon_j=0$, $\varepsilon'_j=1$. Then

$$b'-b-1 = (r_j-1)+\sum_{i=1}^{j-1} (\varepsilon'_i-\varepsilon_i)\, r_i \equiv 0 \pmod{p_j}.$$

The value p_j is excluded by

$$b'-b-1 \geqq r_j-1 \geqq r_1-1 \geqq p_2p_3 \dots p_k-1 > p_k \geqq p_j$$

for $k \geqq 3$. This concludes the proof.

REMARK. I have included this particular case because it is a "natural" sequence for which the phenomenon dealt with in the previous section likely occurs. At least with the present construction it seems inevitable that if we want to use the congruences "economically", then we must construct our sequence in dependence on x; if we want to know which of b and b' is the greater, we need the last different digit, and the smaller ones must not spoil it.

In our constructions $n=b'-b-1$ was not only composite but always had a prime divisor $\ll \log n$. I can show that under this stronger condition we obtained (logarithmically) the correct order of magnitude.

6. A note on sum-intersectivity

So far we have considered only the "difference-intersective" property; there are analogous problems for sums. Given a set A, consider the sets B such that $A\cap(B+B)=\emptyset$ and put

$$\sigma_1(A)=\sup\bar{d}(B),\quad \sigma_2(A)=\sup\underline{d}(B),\quad \sigma_3(A)=\sup d(B).$$

We may also define

$$S(A,x)=\max\{|T|:(T+T)\cap A=\emptyset,\ T\subset[1,x]\}$$

and put

$$\sigma_4(A)=\limsup S(A,x)/x,\quad \sigma_5(A)=\liminf S(A,x)/x.$$

Unlike the case of differences, these are in general five different numbers. E.g. let

$$A=\bigcup_{k=1}^{\infty}[10^k, 11\cdot 10^{k-1}].$$

In this case $\sigma_1=0.87$, $\sigma_3=0$, $\sigma_4=0.89$, $\sigma_5=0.45$. I do not know the exact value of σ_2 but $39/99\leqq\sigma_2<0.45$ (I think the lower bound is correct). Obviously we have always

$$\sigma_3\leqq\sigma_2\begin{Bmatrix}\leqq\sigma_1\\ \leqq\sigma_5\end{Bmatrix}\leqq\sigma_4,$$

and I am quite sure that this is the only connection between them. By varying the above construction one can show that these quantities really induce five different concepts of sum-intersectivity. Erdős and Sárközy [1] chose $\sigma_1=0$ for this purpose; I think none of them is more natural than the others.

To contrast Theorem 2 we may note that the omission or inclusion of a finite number of elements does not affect the sum-intersective properties (nor does it even modify any of $\sigma_1,\ldots,\sigma_5$).

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST, REÁLTANODA U. 13—15
H—1053

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EQUIVALENT PROBLEMS IN THE CALCULUS OF VARIATIONS WHOSE FUNDAMENTAL FUNCTIONS INVOLVE SECOND-ORDER DERIVATIVES

MAGDALEN SZ. KIRKOVITS (Sopron)

1. Introduction

Let $F(x^i, \dot{x}^i, \ddot{x}^i)$ and $F^*(x^i, \dot{x}^i, \ddot{x}^i)$ denote the fundamental functions of two n-dimensional variational problems, where $x^i(t)$ are of class C^4 and F, F^* are of class C^3 in their $3n$ variables.

It is well-known that the two variational problems are said to be equivalent in the sense of Carathéodory [1], if their respective fundamental functions differ by a total derivative that is (H. Rund [4] page 204)

$$F^*(x^i, \dot{x}^i, \ddot{x}^i) = F(x^i, \dot{x}^i, \ddot{x}^i) - \frac{\partial S}{\partial x^i}\dot{x}^i - \frac{\partial S}{\partial \dot{x}^i}\ddot{x}^i \quad (S = S(x, \dot{x})). \tag{1.1}$$

The Euler—Lagrange equations are

$$\mathscr{E}_i(F) = 0 \quad \text{and} \quad \mathscr{E}_i(F^*) = 0, \tag{1.2}$$

where the operator ε_i denotes

$$\mathscr{E}_i := \partial_i - \frac{d}{dt}\dot{\partial}_i + \frac{d^2}{dt^2}\ddot{\partial}_i; \quad \partial_i := \frac{\partial}{\partial x^i}, \; \dot{\partial}_i := \frac{\partial}{\partial \dot{x}^i}, \; \ddot{\partial}_i := \frac{\partial}{\partial \ddot{x}^i}. \tag{1.3}$$

If (1.1) holds, then obviously $\mathscr{E}_i(F^*) = \mathscr{E}_i(F)$. However, if a solution of the Euler—Lagrange equations of a variational problem is a solution of another one, too, it does not necessarily imply that the two problems are equivalent in the terminology of Carathéodory.

In this paper we furnish a discussion of the case, when the following relations hold

$$\mathscr{E}_i\big(F^*(x, \dot{x}, \ddot{x})\big) \equiv \lambda(x, \dot{x}, \ddot{x})\,\mathscr{E}_i\big(F(x, \dot{x}, \ddot{x})\big), \quad \lambda(x, \dot{x}, \ddot{x}) \neq 0. \tag{1.4}$$

The proportionality function λ is the same for each of these n equations and *it is dependent not only on positional coordinates* as proved by A. Moór [2], *but on their first and second derivatives $\dot{x}^i$ and $\ddot{x}^i$, too.* Our considerations are in some degree analogous to those of H. Rund [3], but it will be supposed that the fundamental functions depend on $\ddot{x}^i$, too.

In Section 2 we shall prove our fundamental result that if relation (1.4) holds identically, further λ satisfies certain conditions (see Theorem 1), then both fundamental functions are necessarily linear in $\ddot{x}^k$.

In Section 3 we are going to show that the function λ could be reduced to a constant under further conditions (2.10), (2.13) in which $A_k \neq \dot{\partial}_k A$.

Finally we shall examine a special case, when our problem reduces to the Finsler case which was investigated by H. Rund in [3].

2. The form of the fundamental functions F and F^*

THEOREM 1. *If $F(x, \dot{x}, \ddot{x})$ and $F^*(x, \dot{x}, \ddot{x})$ denote the fundamental functions of a pair of variational problems and the relations (1.4) hold, where the function $\lambda(x, \dot{x}, \ddot{x})$ is such that $\lambda \neq 0$ and $\frac{d\lambda}{dt} \neq 0$, then the functions F and F^* are necessarily linear in $\ddot{x}^k$.*

PROOF. (1.4) has the explicit form

$$\partial_i F^* - \frac{d}{dt}\partial_{\dot{i}} F^* + \frac{d^2}{dt^2}\partial_{\ddot{i}} F^* - \lambda(x, \dot{x}, \ddot{x})\left(\partial_i F - \frac{d}{dt}\partial_{\dot{i}} F + \frac{d^2}{dt^2}\partial_{\ddot{i}} F\right) \equiv 0. \tag{2.1}$$

If we calculate the derivatives with respect to t, we get an expression of the form

$$P^{(1)}_{ik}(x, \dot{x}, \ddot{x})\overset{(4)}{x}{}^k + P^{(0)}_{ijk}(x, \dot{x}, \ddot{x})\dddot{x}^j\dddot{x}^k + P^{(2)}_{ik}(x, \dot{x}, \ddot{x})\dddot{x}^k + P^{(3)}_i(x, \dot{x}, \ddot{x}) \equiv 0. \tag{2.2}$$

(2.2) is an identity if and only if the coefficients $P^{(\alpha)}(x, \dot{x}, \ddot{x})$ $(\alpha = 0, 1, 2, 3)$ vanish which yield the following conditions

$$P^{(1)}_{ik} := \partial_{\ddot{i}}\partial_{\ddot{k}} F^* - \lambda\partial_{\ddot{i}}\partial_{\ddot{k}} F \equiv 0, \tag{2.3}$$

$$P^{(0)}_{ijk} := \partial_{\ddot{i}}\partial_{\ddot{k}}\partial_{\ddot{j}} F^* - \lambda\partial_{\ddot{i}}\partial_{\ddot{k}}\partial_{\ddot{j}} F \equiv 0, \tag{2.4}$$

$$\begin{aligned} P^{(2)}_{ik} := {} & \partial_{\ddot{i}}\partial_{\dot{k}} F^* - \partial_{\dot{i}}\partial_{\ddot{k}} F^* - \lambda(\partial_{\ddot{i}}\partial_{\dot{k}} F - \partial_{\dot{i}}\partial_{\ddot{k}} F) + \\ & + 2[\partial_{\ddot{i}}\partial_{\ddot{k}}\partial_j F^* \dot{x}^j + \partial_{\ddot{i}}\partial_{\ddot{k}}\partial_{\dot{j}} F^* \ddot{x}^j - \lambda(\partial_{\ddot{i}}\partial_{\ddot{k}}\partial_j F\dot{x}^j + \partial_{\ddot{i}}\partial_{\ddot{k}}\partial_{\dot{j}} F\ddot{x}^j)] \equiv 0, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} P^{(3)}_i := {} & \partial_i F^* - \partial_{\dot{i}}\partial_k F^* \dot{x}^k - \partial_{\dot{i}}\partial_{\dot{k}} F^* \ddot{x}^k + \partial_{\ddot{i}}\partial_k\partial_j F^* \dot{x}^j\dot{x}^k + \partial_{\ddot{i}}\partial_k\partial_{\dot{j}} F^* \dot{x}^k\ddot{x}^j + \\ & + \partial_{\ddot{i}}\partial_k F^* \ddot{x}^k + \partial_{\ddot{i}}\partial_{\dot{k}}\partial_j F^* \dot{x}^j\ddot{x}^k + \partial_{\ddot{i}}\partial_{\dot{k}}\partial_{\dot{j}} F^* \ddot{x}^j\ddot{x}^k - \lambda(\partial_i F - \partial_{\dot{i}}\partial_k F\dot{x}^k - \partial_{\dot{i}}\partial_{\dot{k}} F\ddot{x}^k + \\ & + \partial_{\ddot{i}}\partial_k\partial_j F\dot{x}^j\dot{x}^k + \partial_{\ddot{i}}\partial_k\partial_{\dot{j}} F\dot{x}^k\ddot{x}^j + \partial_{\ddot{i}}\partial_k F\ddot{x}^k + \partial_{\ddot{i}}\partial_{\dot{k}}\partial_j F\dot{x}^j\ddot{x}^k + \partial_{\ddot{i}}\partial_{\dot{k}}\partial_{\dot{j}} F\ddot{x}^j\ddot{x}^k) \equiv 0. \end{aligned} \tag{2.6}$$

Let us construct the symmetric-part in i and k of (2.5). It follows that

$$\partial_{\ddot{i}}\partial_{\ddot{k}}\partial_j F^* \dot{x}^j + \partial_{\ddot{i}}\partial_{\ddot{k}}\partial_{\dot{j}} F^* \ddot{x}^j - \lambda(\partial_{\ddot{i}}\partial_{\ddot{k}}\partial_j F\dot{x}^j + \partial_{\ddot{i}}\partial_{\ddot{k}}\partial_{\dot{j}} F\ddot{x}^j) \equiv 0 \tag{2.7}$$

which could be written in the following form

$$\frac{d}{dt}(\partial_{\ddot{i}}\partial_{\ddot{k}} F^* - \lambda\partial_{\ddot{i}}\partial_{\ddot{k}} F) - (\partial_{\ddot{i}}\partial_{\ddot{k}}\partial_{\ddot{j}} F^* - \lambda\partial_{\ddot{i}}\partial_{\ddot{k}}\partial_{\ddot{j}} F)\dddot{x}^j + \frac{d\lambda}{dt}\partial_{\ddot{i}}\partial_{\ddot{k}} F \equiv 0. \tag{2.8}$$

Because of (2.3) and (2.4), respectively, the first as well as the second term on the left-hand side of (2.8) are identically zero, thus (2.8) reduces to:

$$\frac{d\lambda}{dt}\partial_{\ddot{i}}\partial_{\ddot{k}} F \equiv 0. \tag{2.9}$$

Since by supposition $\frac{d\lambda}{dt} \neq 0$, we obtain

$$F(x, \dot{x}, \ddot{x}) = A_k(x, \dot{x})\ddot{x}^k + B(x, \dot{x}). \tag{2.10}$$

Secondly, since $\lambda \neq 0$ for any value of its variables, the relation (1.4) is also true in the following form:

$$\mathcal{E}_i(F) \equiv \lambda^{-1}\mathcal{E}_i(F^*). \tag{2.11}$$

In this case evidently the same conditions are fulfilled for the coefficients of $\overset{(4)}{x}{}^k$, $\dot{\ddot{x}}^j \dot{\ddot{x}}^k$ and $\dot{\ddot{x}}^k$. Thus we can find an identity similar to (2.9) for λ^{-1} and F^*:

$$\frac{d\lambda^{-1}}{dt}\,\partial_{\ddot{i}}\,\partial_{\ddot{k}}\,F^* \equiv 0 \tag{2.12}$$

and since $\dfrac{d\lambda^{-1}}{dt} \neq 0$, consequently we also have

$$F^*(x, \dot{x}, \ddot{x}) = A_k^*(x, \dot{x})\,\ddot{x}^k + B^*(x, \dot{x}). \tag{2.13}$$

Q. e. d.

3. The form of the function λ

Let us suppose that (2.10) and (2.13) are satisfied, that is F and F^* are linear in $\ddot{x}^k$, $\lambda \neq 0$ $\left(\text{it is not supposed in the following that } \dfrac{d\lambda}{dt} \neq 0\right)$. Let these expressions be substituted in (2.2), from which it follows that

$$P_{ik}^{(2)}(x, \dot{x}, \ddot{x})\,\dot{\ddot{x}}^k + P_i^{(3)}(x, \dot{x}, \ddot{x}) \equiv 0. \tag{3.1}$$

In order that (3.1) hold identically, $P_{ik}^{(2)}$ and $P_i^{(3)}$ have to vanish necessarily for arbitrary values of their variables; so we get on account of (2.5), (2.6), (2.10) and (2.13)

$$P_{ik}^{(2)} := \partial_{\dot{k}} A_i^* - \partial_{\dot{i}} A_k^* - \lambda(x, \dot{x}, \ddot{x})\,(\partial_{\dot{k}} A_i - \partial_{\dot{i}} A_k) \equiv 0, \tag{3.2}$$

$$\begin{aligned} P_i^{(3)} := {} & (\partial_{\dot{k}}\partial_{\dot{j}} A_i^* - \partial_{\dot{i}}\partial_{\dot{k}} A_j^*)\,\ddot{x}^j \ddot{x}^k + (\partial_i A_k^* + \partial_k A_i^* + \partial_{\dot{k}}\partial_j A_i^*\,\dot{x}^j - \partial_{\dot{i}}\partial_j A_k^*\,\dot{x}^j + \\ & + \partial_{\dot{k}}\partial_j A_i^*\,\dot{x}^j - \partial_{\dot{i}}\partial_{\dot{k}} B^*)\,\ddot{x}^k + \partial_k\partial_j A_i^*\,\dot{x}^j \dot{x}^k + \partial_i B^* - \partial_{\dot{i}}\partial_k B^*\,\dot{x}^k - \\ & - \lambda(x, \dot{x}, \ddot{x})\,[(\partial_{\dot{k}}\partial_{\dot{j}} A_i - \partial_{\dot{i}}\partial_{\dot{k}} A_j)\,\ddot{x}^j \ddot{x}^k + (\partial_i A_k + \partial_k A_i + \partial_{\dot{k}}\partial_j A_i\,\dot{x}^j - \\ & - \partial_{\dot{i}}\partial_j A_k\,\dot{x}^j + \partial_{\dot{k}}\partial_j A_i\,\dot{x}^j - \partial_{\dot{i}}\partial_{\dot{k}} B)\,\ddot{x}^k + \partial_k\partial_j A_i\,\dot{x}^j \dot{x}^k + \partial_i B - \partial_{\dot{i}}\partial_k B\,\dot{x}^k] \equiv 0. \end{aligned} \tag{3.3}$$

Let us differentiate (3.2) with respect to $\ddot{x}^j$. Since A_k^* and A_k are independent of $\ddot{x}^i$, it follows that

$$(\partial_{\ddot{j}}\lambda)(\partial_{\dot{k}} A_i - \partial_{\dot{i}} A_k) \equiv 0. \tag{3.4}$$

From this we have

$$\partial_{\ddot{j}}\lambda \equiv 0, \tag{3.5}$$

or

$$\partial_{\dot{k}} A_i - \partial_{\dot{i}} A_k \equiv 0. \tag{3.6}$$

First we examine the case, when λ is independent of $\ddot{x}^j$.

THEOREM 2. *If the fundamental functions F and F^* are linear in $\ddot{x}^k$ and $A_k \neq \partial_{\dot{k}} A$, then the function λ in (3.1) must be constant.*

PROOF. If $A_k \neq \partial^{\cdot}_k A$, then from (3.4) we have $\partial^{\cdot\cdot}_j \lambda \equiv 0$, consequently λ is independent of $\ddot{x}^j$. Because of (3.2) the coefficient of $\dddot{x}^k$ in (3.1) is

$$P^{(2)}_{ik} := \partial^{\cdot}_k A^*_i - \partial^{\cdot}_i A^*_k - \lambda(x, \dot{x})(\partial^{\cdot}_k A_i - \partial^{\cdot}_i A_k) \equiv 0. \tag{3.7}$$

Furthermore, on account of (3.3) $P^{(3)}_i$ is a polynomial of $\ddot{x}^i$ and it vanishes identically, that is $P^{(3)}_i$ has the form

$$P^{(3)}_i := P^{(1)}_{ijk}(x, \dot{x})\ddot{x}^j \ddot{x}^k + P^{(4)}_{ik}(x, \dot{x})\ddot{x}^k + P^{(5)}_i(x, \dot{x}) \equiv 0. \tag{3.8}$$

We have $P^{(1)}_{ijk} \equiv P^{(1)}_{ikj}$ and by (3.3) it follows:

$$P^{(1)}_{ijk} := \frac{1}{2}(\partial^{\cdot}_k \partial^{\cdot}_j A^*_i - \partial^{\cdot}_i \partial^{\cdot}_j A^*_k) + \frac{1}{2}(\partial^{\cdot}_j \partial^{\cdot}_k A^*_i - \partial^{\cdot}_i \partial^{\cdot}_k A^*_j) - \tag{3.9}$$

$$-\frac{\lambda}{2}(\partial^{\cdot}_k \partial^{\cdot}_j A_i - \partial^{\cdot}_i \partial^{\cdot}_j A_k) - \frac{\lambda}{2}(\partial^{\cdot}_j \partial^{\cdot}_k A_i - \partial^{\cdot}_i \partial^{\cdot}_k A_j),$$

$$P^{(4)}_{ik} := \partial_i A^*_k + \partial_k A^*_i + (\partial^{\cdot}_k \partial_j A^*_i - \partial^{\cdot}_i \partial_j A^*_k)\dot{x}^j + \tag{3.10}$$

$$+(\partial^{\cdot}_k \partial_j A^*_i - \lambda \partial^{\cdot}_k \partial_j A_i)\dot{x}^j + \lambda(\partial^{\cdot}_k \partial_j A_i - \partial^{\cdot}_i \partial_j A_k)\dot{x}^j - \lambda(\partial_i A_k + \partial_k A_i) - \partial^{\cdot}_i \partial^{\cdot}_k B^* + \lambda \partial^{\cdot}_i \partial^{\cdot}_k B,$$

$$P^{(5)}_i := \partial_i B^* - \lambda \partial_i B - (\partial^{\cdot}_i \partial_k B^* - \lambda \partial^{\cdot}_i \partial_k B)\dot{x}^k + (\partial_k \partial_j A^*_i - \lambda \partial_k \partial_j A_i)\dot{x}^j \dot{x}^k. \tag{3.11}$$

(3.8) is an identity in $(x, \dot{x}, \ddot{x})$ if and only if $P^{(1)}_{ijk}$, $P^{(4)}_{ik}$ and $P^{(5)}_i$ identically vanish; thus we get

$$\text{(a) } P^{(1)}_{ijk} \equiv 0, \quad \text{(b) } P^{(4)}_{ik} \equiv 0, \quad \text{(c) } P^{(5)}_i \equiv 0. \tag{3.12}$$

If we differentiate (3.7) with respect to $\dot{x}^j$ then

$$\partial^{\cdot}_k \partial^{\cdot}_j A^*_i - \partial^{\cdot}_i \partial^{\cdot}_j A^*_k \equiv (\partial^{\cdot}_j \lambda)(\partial^{\cdot}_k A_i - \partial^{\cdot}_i A_k) + \lambda(\partial^{\cdot}_k \partial^{\cdot}_j A_i - \partial^{\cdot}_i \partial^{\cdot}_j A_k). \tag{3.13}$$

If we substitute the right-hand side of (3.13) in the first term of (3.12) (a) we have after multiplication by 2:

$$\partial^{\cdot}_j \partial^{\cdot}_k A^*_i - \partial^{\cdot}_i \partial^{\cdot}_k A^*_j + (\partial^{\cdot}_j \lambda)(\partial^{\cdot}_k A_i - \partial^{\cdot}_i A_k) - \lambda(\partial^{\cdot}_j \partial^{\cdot}_k A_i - \partial^{\cdot}_i \partial^{\cdot}_k A_j) \equiv 0.$$

The skew-symmetric part in i and k of the last identity gives

$$\frac{1}{2}(\partial^{\cdot}_k \partial^{\cdot}_j A^*_i - \partial^{\cdot}_i \partial^{\cdot}_j A^*_k) + \partial^{\cdot}_j[\lambda(\partial^{\cdot}_k A_i - \partial^{\cdot}_i A_k)] - \frac{3}{2}\lambda(\partial^{\cdot}_k \partial^{\cdot}_j A_i - \partial^{\cdot}_i \partial^{\cdot}_j A_k) \equiv 0. \tag{3.14}$$

We substitute the term $\lambda \partial^{\cdot}_{[k} A_{i]}$ from (3.7), and so we get

$$\partial^{\cdot}_j(\partial^{\cdot}_k A^*_i - \partial^{\cdot}_i A^*_k) - \lambda \partial^{\cdot}_j(\partial^{\cdot}_k A_i - \partial^{\cdot}_i A_k) \equiv 0 \tag{3.15}$$

which could be written in the form

$$\partial^{\cdot}_j[\partial^{\cdot}_k A^*_i - \partial^{\cdot}_i A^*_k - \lambda(\partial^{\cdot}_k A_i - \partial^{\cdot}_i A_k)] + (\partial^{\cdot}_j \lambda)(\partial^{\cdot}_k A_i - \partial^{\cdot}_i A_k) \equiv 0. \tag{3.16}$$

Because of (3.7) this identity reduces to

$$(\partial^{\cdot}_j \lambda)(\partial^{\cdot}_k A_i - \partial^{\cdot}_i A_k) \equiv 0. \tag{3.17}$$

Since $A_k \neq \dot\partial_k A$ the second factor does not vanish, consequently $\dot\partial_j \lambda \equiv 0$, thus λ is independent of $\dot x^j$, too.

Furthermore let us construct the skew-symmetric part in i and k of (3.12) (b) and (3.10), respectively. After division by 3/2 we get

$$(3.18) \qquad [\dot\partial_k \partial_j A_i^* - \dot\partial_i \partial_j A_k^* - \lambda(x)(\dot\partial_k \partial_j A_i - \dot\partial_i \partial_j A_k)]\dot x^j \equiv 0.$$

This relation can be written in the following form:

$$(3.19) \qquad \{\partial_j [\dot\partial_k A_i^* - \dot\partial_i A_k^* - \lambda(x)(\dot\partial_k A_i - \dot\partial_i A_k)] + (\partial_j \lambda)(\dot\partial_k A_i - \dot\partial_i A_k)\}\dot x^j \equiv 0.$$

Because of (3.7) we also get

$$(3.20) \qquad \frac{d\lambda}{dt}(\dot\partial_k A_i - \dot\partial_i A_k) \equiv 0.$$

From the condition $A_k \neq \dot\partial_k A$ it follows that

$$(3.21) \qquad \frac{d\lambda}{dt} \equiv 0$$

thus $\lambda = \text{const.}$ Q. e. d.

REMARK. The case $\lambda = \text{const.}$ has been also discussed by A. Moór ([2], §3). In his discussion the equality $\lambda = \text{const.}$ was not a direct consequence of the identity $\mathscr{E}_i(F^*) = \lambda(x)\mathscr{E}_i(F)$, but was stated as a separate condition. In this way he obtained for F^* the form:

$$F^* = \lambda F + \frac{d^* S}{dt}.$$

Furthermore we examine the case (3.6). Then $A_k = \dot\partial_k A$ and from (3.2) $A_k^* = \dot\partial_k A^*$ is also satisfied, so

$$(3.22) \qquad F^*(x, \dot x, \ddot x) = \dot\partial_k A^*(x, \dot x)\ddot x^k + B^*(x, \dot x)$$

and

$$(3.23) \qquad F(x, \dot x, \ddot x) = \dot\partial_k A(x, \dot x)\ddot x^k + B(x, \dot x).$$

Let (3.22) and (3.23) be substituted in (1.4), then it follows that

(3.24)

$$(\dot\partial_k \partial_i A^* + \dot\partial_i \partial_k A^* + \dot\partial_i \dot\partial_k \partial_j A^* \dot x^j - \dot\partial_i \dot\partial_k B^*)\ddot x^k + \partial_i B^* - \dot\partial_i \partial_k B^* \dot x^k + \dot\partial_i \partial_k \partial_j A^* \dot x^j \dot x^k -$$
$$-\lambda(x, \dot x, \ddot x)\{(\dot\partial_k \partial_i A + \dot\partial_i \partial_k A + \dot\partial_i \dot\partial_k \partial_j A \dot x^j - \dot\partial_i \dot\partial_k B)\ddot x^k + \partial_i B - \dot\partial_i \partial_k B \dot x^k + \dot\partial_i \partial_k \partial_j A \dot x^j \dot x^k\} \equiv 0,$$

and this corresponds to (1.4).

Finally we shall consider the following special case: $A_k = \dot\partial_k A(\dot x)$, $A_k^* = \dot\partial_k A^*(\dot x)$. Because of (3.23)

$$(3.25) \qquad F(x, \dot x, \ddot x) = B(x, \dot x) + \frac{dA(\dot x)}{dt},$$

so the extremals of the two fundamental integrals $\int F(x, \dot x, \ddot x)dt$ and $\int B(x, \dot x)dt$ are identical, since the addition of an exact differential to the integrand obviously cannot affect any extremals.

Furthermore, if $\lambda=\lambda(x, \dot{x})$ and $A_k=\partial^{\bullet}_k A(\dot{x})$, $A^*_k=\partial^{\bullet}_k A^*(\dot{x})$ then from (3.24) follows directly

$$\mathscr{E}_i(B^*) \equiv \lambda(x, \dot{x})\, \mathscr{E}_i(B), \tag{3.26}$$

therefore our equivalent problem reduces to that case which was examined by H. Rund in [3].

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ERDÉSZETI ÉS FAIPARI EGYETEM
MATEMATIKAI TANSZÉK
H—9401 SOPRON, PF. 132

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SYNTOPOGENOUS SPACES WITH PREORDER. I (CONVEXITY)

K. MATOLCSY (Debrecen)

The present paper is a member of a sequence of notices dealing with syntopogenous spaces equipped with a preorder. Starting out from the analogous properties of topological ([11]), proximity ([14]) and uniform ([11], [12]) ordered spaces, this direction of the research was taken the initiative by Burgess—Fitzpatrick ([1]—[3]). The examinations of this publication connect with the results of these authors, and study two general types of convexity of preordered syntopogenous spaces.

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1. Increasing and decreasing spaces

A *preorder* $\leqq$ on a set E is a reflexive and transitive relation on E, the pair $(E, \leqq)$ is called *preordered space*. The *graph* of the preorder $\leqq$ is the set $G(\leqq)$ defined by

$$G(\leqq) = \{(x, y) \in E \times E\colon\ x \leqq y\}.$$

A mapping f of the preordered space $(E, \leqq)$ into a preordered space $(E', \leqq')$ is said to be *preorder preserving (inversing)*, if $x \leqq y$ implies $f(x) \leqq' f(y)$ $\big(f(y) \leqq' f(x)\big)$ for every $x, y \in E$. The *product* of the preordered spaces $(E_i, \leqq_i)$ $(i \in I \neq \emptyset)$ is a preordered space $(E, \leqq)$, where $E = \bigtimes_{i \in I} E_i$, and $(x_i) \leqq (y_i)$ iff $x_i \leqq_i y_i$ for each $i \in I$.

A *preordered syntopogenous space* is a triplet $(E, \mathscr{S}, \leqq)$ consisting of a set E, a syntopogenous structure $\mathscr{S}$ (see [5]) and a preorder $\leqq$ on E.

Defining, for any $< \in \mathscr{S}$, the set

$$G(<) = \{(x, y) \in E \times E\colon\ x \nless E - y\},$$

we obtain the graph

$$G(\mathscr{S}) = \bigcap \{G(<)\colon\ < \in \mathscr{S}\}$$

of the *preorder generated by* $\mathscr{S}$ ([1], 3.1).

A preordered syntopogenous space $(E, \mathscr{S}, \leqq)$ $\big($or the syntopogenous structure $\mathscr{S}$ on $(E, \leqq)\big)$ will be called *increasing (decreasing)* iff, for $G = G(\leqq)$,

$$G \subset G(\mathscr{S}) \quad \big(G^{-1} \subset G(\mathscr{S})\big).$$

(Instead of the term "increasing" in [1]—[3] $\mathscr{S}$ was said to be "$\leqq$-inclusive".)

(1.1) THEOREM.

(1.1.1) *A syntopogenous structure coarser than an increasing (decreasing) syntopogenous structure is also increasing (decreasing).*

(1.1.2) *The syntopogenous structure* $\mathscr{S}$ *is increasing (decreasing) iff so is each of the structures* $\mathscr{S}^p, \mathscr{S}^b, \mathscr{S}^t, \mathscr{S}^{tp}$ *and* $\mathscr{S}^{tb}$.

(1.1.3) $\mathscr{S}$ *is increasing iff* $\mathscr{S}^c$ *is decreasing.*

(1.1.4) *Let* f *be a preorder preserving mapping of a preordered space* $(E, \leqq)$ *into the preordered syntopogenous space* $(E', \mathscr{S}', \leqq')$. *If the latter space is increasing (decreasing), then* $(E, f^{-1}(\mathscr{S}'), \leqq)$ *is also increasing (decreasing).*

(1.1.5) *If under the remaining conditions of* (1.1.4) f *is preorder inversing, then* $(E, f^{-1}(\mathscr{S}'), \leqq)$ *is decreasing (increasing).*

(1.1.6) *The supremum of any number of increasing (decreasing) syntopogenous structures is also increasing (decreasing).*

(1.1.7) *The product of any number of increasing (decreasing) preordered syntopogenous spaces is also increasing (decreasing).*

Proof. (1.1.1)—(1.1.3) is obvious by [1], 3.2.

(1.1.4). Suppose that $(E', \mathscr{S}', \leqq')$ is increasing. If $x, y \in E$, $x < E - y$ for $< = f^{-1}(<')$, where $<' \in \mathscr{S}'$, then $f(x) <' E' - f(y)$ (see [5], (6.1)), therefore $f(x) \not\geqq' f(y)$, hence $x \not\geqq y$. This shows that $G(\leqq) \subset G(f^{-1}(\mathscr{S}'))$. If $(E', \mathscr{S}', \leqq')$ is decreasing, the proof is similar.

(1.1.5) is analogous to (1.1.4).

(1.1.6): Put $\mathscr{S} = \bigvee_{i \in I} \mathscr{S}_i$, where $\mathscr{S}_i$ is an increasing syntopogenous structure on the preordered space $(E, \leqq)$ for any $i \in I$. Then $\mathscr{S}_i < \mathscr{S}$ implies $G(\mathscr{S}) \subset G(\mathscr{S}_i)$ for every $i \in I$, so that $G(\mathscr{S}) \subset \bigcap_{i \in I} G(\mathscr{S}_i)$ (cf. [1], 3.2). Conversely, if $(x, y) \notin G(\mathscr{S})$, then $x < E - y$ for some $< \in \mathscr{S}$. Putting $< = (\bigcup_{j=1}^{n} <_j)^q$, where $<_j \in \mathscr{S}_{i_j}$, $i_j \in I$ $(1 \leqq j \leqq n)$, we obtain $x <_j E - y$ for some index j, so that $(x, y) \notin G(\mathscr{S}_{i_j})$, therefore $\bigcap_{i \in I} G(\mathscr{S}_i) \subset G(\mathscr{S})$. In view of $G(\leqq) \subset G(\mathscr{S}_i)$ $(i \in I)$, from here $G(\leqq) \subset \bigcap_{i \in I} G(\mathscr{S}_i) = G(\mathscr{S})$ follows. The decreasing case is similar.

(1.1.7): The projections of the product onto its components are preorder preserving, hence (1.1.4) and (1.1.6) can be applied (cf. [5], (11.4)). ∎

In a preordered space $(E, \leqq)$ a set $A \subset E$ is said to be *increasing (decreasing)* iff $x \in A$ and $x \leqq y$ $(y \leqq x)$ imply $y \in A$. The smallest increasing (decreasing) set containing an arbitrary $X \subset E$ is $i(X) = \{y \in E : x \leqq y, x \in X\}$ $(d(X) = \{y \in E : y \leqq x, x \in X\})$.

Let us denote by $<'$ $(<'')$ the biperfect topogenous order generated by the system of all increasing (decreasing) sets of $(E, \leqq)$, and put $\mathscr{U}_{\leqq} = \{<'\}$ and $\mathscr{L}_{\leqq} = \{<''\}$ (cf. [5], (2.1)). Then we have:

(1.2) Theorem (cf. [1], 3.5).

(1.2.1) $\mathscr{U}_{\leqq}$ $(\mathscr{L}_{\leqq})$ *is an increasing (decreasing) biperfect topogenous structure on* $(E, \leqq)$, *and* $\mathscr{U}^c_{\leqq} = \mathscr{L}_{\leqq}$.

(1.2.2) *A syntopogenous structure* $\mathscr{S}$ *is increasing (decreasing) on* $(E, \leqq)$ *iff* $\mathscr{S} < \mathscr{U}_{\leqq}$ $(\mathscr{S} < \mathscr{L}_{\leqq})$.

Proof. By [1], 3.5 $\mathscr{U}_{\leqq}$ generates $\leqq$, that is $G(\leqq) = G(\mathscr{U}_{\leqq})$. A set $A \subset E$ is increasing iff $E - A$ is decreasing, therefore $\mathscr{U}^c_{\leqq} = \mathscr{L}_{\leqq}$, and from (1.1.3) it follows that $\mathscr{L}_{\leqq}$ is decreasing. If $\mathscr{S}$ is increasing (decreasing), then $< \in \mathscr{S}$, $A < B$ implies

$i(A)\subset B\,\big(d(A)\subset B\big)$, hence $A<'B\,(A<''B)$, so that $\mathcal{S}<\mathcal{U}_{\leqq}\,(\mathcal{S}<\mathcal{L}_{\leqq})$. The converse statement follows from (1.2.1) and (1.1.1). ∎

(1.3) REMARKS. For the "classical" topological structures we have the following connections (see also [1], [2], [9], [10], [13], [14], and [11], p. 58).

Let $(E,\mathcal{S},\leqq)$ be a preordered syntopogenous space, and let τ, δ and $\mathcal{U}$ denote the classical topology, the quasi-proximity and the quasi-uniformity associated with $\mathcal{S}^{tp}$, $\mathcal{S}^{t}$ and $\mathcal{S}^{b}$, respectively. Then

(1.3.1) $G(\mathcal{S})=\{(x,y)\in E\times E\colon x\in\bar{y}^{\tau}\}=\{(x,y)\in E\times E\colon \{x\}\delta\{y\}\}=\bigcap\{U\colon U\in\mathcal{U}\}$.

The following statements are equivalent:

(1.3.2) $(E,\mathcal{S},\leqq)$ is increasing (decreasing).

(1.3.3) Every τ-open set is increasing (decreasing).

(1.3.4) Every τ-closed set is decreasing (increasing).

(1.3.5) $x\leqq y\;(x\geqq y)$ implies $x\in\bar{y}^{\tau}$.

(1.3.6) $A,B\subset E$, $i(A)\cap d(B)\neq\emptyset$ (or $d(A)\cap i(B)\neq\emptyset$) implies $A\delta B$.

(1.3.7) $x\leqq y\;(x\geqq y)$ implies $\{x\}\,\delta\,\{y\}$.

(1.3.8) $x\leqq y\;(x\geqq y)$ implies $(x,y)\in U$ for any $U\in\mathcal{U}$. ∎

(1.4) EXAMPLE. Let us consider the syntopogenous structure $\mathcal{I}$ on the naturally ordered real line $(\mathbf{R},\leqq)$ (see [5], (7.12)). Then $(\mathbf{R},\mathcal{I},\leqq)$ is decreasing, and $(\mathbf{R},\mathcal{I}^{c},\leqq)$ is increasing. ∎

(1.5) LEMMA (cf. [5], ch. 12). *Let φ be a functional family on the preordered space $(E,\leqq)$ consisting of preorder preserving (inversing) functions. Then $(E,\mathcal{S}_{\varphi},\leqq)$ is decreasing (increasing).*

PROOF. Let us now denote by $\leqq_r$ the order of the real line. If $(x,y)\notin G(\mathcal{S}_{\varphi})$, then $x<_{\varphi,\varepsilon}E-y$ for some $\varepsilon>0$, which means that $f(x)+\varepsilon\leqq_r f(y)$ for a suitable $f\in\varphi$. $f(y)\not\leqq_r f(x)$ implies $x\not\leqq y\;(x\not\leqq y)$, so that $G(\leqq)^{-1}\subset G(\mathcal{S}_{\varphi})\;\big(G(\leqq)\subset G(\mathcal{S}_{\varphi})\big)$. ∎

(1.6) THEOREM. *The preordered syntopogenous space $(E,\mathcal{S},\leqq)$ is decreasing (increasing) iff each $(\mathcal{S},\mathcal{I})$-continuous real function is preorder preserving (inversing).*

PROOF. Suppose that $(E,\mathcal{S},\leqq)$ is decreasing and f is an $(\mathcal{S},\mathcal{I})$-continuous function. If $f(x)\not\leqq_r f(y)$ for some $x,y\in E$, then $f(y)<_{\varepsilon}\mathbf{R}-f(x)$ with a suitable $\varepsilon>0$. In this case $yf^{-1}(<_{\varepsilon})E-x$, consequently $(y,x)\notin G\big(f^{-1}(\mathcal{I})\big)\supset G(\mathcal{S})\supset G(\leqq)^{-1}$ by $f^{-1}(\mathcal{I})<\mathcal{S}$. Thus $x\not\leqq y$, i.e. f is preorder preserving. Conversely, assume that every $(\mathcal{S},\mathcal{I})$-continuous function is preorder preserving. Then there is an ordering structure Φ on E such that $\mathcal{S}\sim\mathcal{S}_{\Phi}$ (see [5], (12.37)). $f\in\varphi\in\Phi$ implies that f is $(\mathcal{S},\mathcal{I})$-continuous, and because of our condition it is preorder preserving. Hence $\mathcal{S}_{\varphi}$ is decreasing on $(E,\leqq)$ for each $\varphi\in\Phi$, and by (1.1.6) so is the structure $\mathcal{S}\sim\mathcal{S}_{\Phi}=\bigvee_{\varphi\in\Phi}\mathcal{S}_{\varphi}$. The other statement is analogous. ∎

In [2], Burgess—Fitzpatrick observed that, for any preordered syntopogenous space $(E,\mathcal{S},\leqq)$,

$$\mathcal{S}^{u}=\bigvee\{\mathcal{S}'\colon (E,\mathcal{S}',\leqq)\text{ is increasing and }\mathcal{S}'<\mathcal{S}\}$$

is the finest of all increasing, and

$$\mathcal{S}^l = \bigvee\{\mathcal{S}' : (E, \mathcal{S}', \leqq) \text{ is decreasing and } \mathcal{S}' < \mathcal{S}\}$$

is the finest of all decreasing syntopogenous structures coarser than $\mathcal{S}$. They will be called the *upper* and *lower syntopogenous structures* of $(E, \mathcal{S}, \leqq)$. We have the following generalization of [2], 3.1 and 3.2 for arbitrary ordinary operations ([5], p. 74):

(1.7) THEOREM. *Let ^{k} be an ordinary operation such that $\mathcal{S}^k < \mathcal{S}^{tb}$ for any syntopogenous structure $\mathcal{S}$. Then, for each preordered syntopogenous space $(E, \mathcal{S}, \leqq)$, we have $\mathcal{S}^{uk} < \mathcal{S}^{ku}$ and $\mathcal{S}^{lk} < \mathcal{S}^{kl}$. If $\mathcal{S} \sim \mathcal{S}^k$, then $\mathcal{S}^u \sim \mathcal{S}^{uk}$ and $\mathcal{S}^l \sim \mathcal{S}^{lk}$.*

PROOF. $\mathcal{S}^u < \mathcal{S}$ implies $\mathcal{S}^{uk} < \mathcal{S}^k$. By (1.1.2) $\mathcal{S}^{utb}$ is increasing, therefore from $\mathcal{S}^{uk} < \mathcal{S}^{utb}$ and from (1.1.1) we get that $\mathcal{S}^{uk}$ is also increasing, thus $\mathcal{S}^{uk} < \mathcal{S}^{ku}$. If $\mathcal{S} \sim \mathcal{S}^k$, then $\mathcal{S}^{uk} < \mathcal{S}^{ku} \sim \mathcal{S}^u$, on the other hand $\mathcal{S}^u < \mathcal{S}^{uk}$, thus $\mathcal{S}^u \sim \mathcal{S}^{uk}$. The case of ^{l} is similar. ∎

(1.8) THEOREM (cf. [2], 3.1). *For any preordered syntopogenous space $(E, \mathcal{S}, \leqq)$, we have $\mathcal{S}^{lc} \sim \mathcal{S}^{cu}$ and $\mathcal{S}^{uc} \sim \mathcal{S}^{cl}$.*

PROOF. $\mathcal{S}^{lc} < \mathcal{S}^{cu}$ and $\mathcal{S}^{uc} < \mathcal{S}^{cl}$ by [2], 3.1. Let us apply these inequalities for $\mathcal{S}' = \mathcal{S}^c$. Then $\mathcal{S}^{cu} = \mathcal{S}'^u = \mathcal{S}'^{ucc} < \mathcal{S}'^{clc} = \mathcal{S}^{cclc} = \mathcal{S}^{lc}$, and similarly $\mathcal{S}^{cl} = \mathcal{S}'^l = \mathcal{S}'^{lcc} < \mathcal{S}'^{cuc} = \mathcal{S}^{ccuc} = \mathcal{S}^{uc}$. ∎

(1.9) THEOREM. *For any preordered syntopogenous space $(E, \mathcal{S}, \leqq)$, $\mathcal{S}^{tu} \sim \mathcal{S}^{ut}$ and $\mathcal{S}^{tl} \sim \mathcal{S}^{lt}$.*

PROOF. Assume $\mathcal{T} = \mathcal{S}^{tut}$. There exists a totally bounded syntopogenous structure $\mathcal{S}_0$ on E such that $\mathcal{S}_0^t = \mathcal{T}$ (see [5], (19.38)). $\mathcal{T}$ is increasing on $(E, \leqq)$, thus so is $\mathcal{S}_0$, too (cf. (1.1.1)). Since $\mathcal{S}^t$ is topogenous, $\mathcal{T} = \mathcal{S}^{tut} \sim \mathcal{S}^{tu}$ by (1.7). From this $\mathcal{S}_0^t \sim \mathcal{S}^{tu} < \mathcal{S}^t$, hence owing to [5], (19.39) $\mathcal{S}_0 < \mathcal{S}$. But this means $\mathcal{S}_0 < \mathcal{S}^u$, hence $\mathcal{S}^{tu} \sim \mathcal{S}_0^t < \mathcal{S}^{ut}$. The inverse inequality can be found in (1.7). For ^{l} the proof is analogous. ∎

(1.10) EXAMPLE. For $^k = ^p$ or ^{b}, $\mathcal{S}^{ku} \sim \mathcal{S}^{uk}$ and $\mathcal{S}^{kl} \sim \mathcal{S}^{lk}$ cannot be always true. In fact, we give an example for a symmetrical topogenous structure $\mathcal{T}$ on the naturally ordered real line $(\mathbf{R}, \leqq)$ such that, for $^k = ^p$ or ^{b}, $\mathcal{T}^{ku} = \mathcal{U}_{\leqq}$, $\mathcal{T}^{kl} = \mathcal{L}_{\leqq}$ (see (1.2)), but $\mathcal{T}^{uk} = \mathcal{T}^{lk}$ is equal to the indiscrete syntopogenous structure $\mathcal{O}_{\mathbf{R}}$ on $\mathbf{R}$ (see [5], p. 86), which is not finer than $\mathcal{U}_{\leqq}$ and $\mathcal{L}_{\leqq}$.

Put, for a given natural number n,

$$H_n = (-\infty, -n] \cup [n, +\infty),$$

and define, for $A, B \subset \mathbf{R}$, $A <_{(n)} B$ iff $A \subset B$ and $A \cap H_n \neq \emptyset$ implies $H_n \subset B$. Then $\mathcal{S} = \{<_{(n)} : n \in \mathbf{N}\}$ is a symmetrical syntopology on $\mathbf{R}$ (in fact, $<_{(n)} \cup <_{(m)} \subset <_{(k)}$, where $k = \max\{n, m\}$, and $<_{(n)} \subset \overset{2}{<}_{(n)}$), consequently $\mathcal{T} = \mathcal{S}^t$ is a symmetrical topogenous structure. For any $x \in \mathbf{R}$ there exists $n \in \mathbf{N}$ such that $x \notin H_n$, and in this case $\{x\} <_{(n)} \{x\}$, thus $\mathcal{T}^p = \mathcal{D}_{\mathbf{R}}$ is the discrete syntopogenous structure of $\mathbf{R}$, $\mathcal{T}^b = \mathcal{T}^{pb} = \mathcal{D}_{\mathbf{R}}$ is also true, and from (1.2) it follows that $\mathcal{T}^{ku} = \mathcal{D}_{\mathbf{R}}^u = \mathcal{U}_{\leqq}$ and $\mathcal{T}^{kl} = \mathcal{D}_{\mathbf{R}}^l = \mathcal{L}_{\leqq}$.

For the determination of $\mathscr{T}^u$, let us consider an increasing syntopogenous structure $\mathscr{S}' \prec \mathscr{T}$ on $(\mathbf{R}, \leqq)$. Put $<' \in \mathscr{S}'$, $<'_1 \in \mathscr{S}'$, $<' \subset <'^3_1$ and $A <' B$. Disregarding the trivial case of $A = \emptyset$, suppose $A <'_1 C <'_1 D <'_1 B$. From (1.2.1) $i(A) \subset C$ and $i(D) \subset B$. For some $n \in \mathbf{N}$ we have $C <_{(n)} D$, hence $\emptyset \neq i(A) \cap H_n \subset C \cap H_n$ implies $H_n \subset D$. But in this case $\mathbf{R} = i(H_n) \subset i(D) \subset B$, that is $B = \mathbf{R}$. This shows $<' = _{\emptyset, \mathbf{R}}$, hence $\mathscr{S}' = \mathscr{O}_{\mathbf{R}}$, thus $\mathscr{T}^u = \mathscr{O}_{\mathbf{R}}$. The proof of $\mathscr{T}^l = \mathscr{O}_{\mathbf{R}}$ is very similar. As $\mathscr{O}_{\mathbf{R}}$ is biperfect, we have $\mathscr{T}^{uk} = \mathscr{O}_{\mathbf{R}} = \mathscr{T}^{lk}$ for ${}^k = {}^p$ or ^{b}. ∎

In the following theorem another construction of $\mathscr{S}^u$ and $\mathscr{S}^l$ will be given with the help of ordering structures defined on $(E, \mathscr{S}, \leqq)$. First of all it is convenient to formulate a lemma based upon the results of ch. 12 of [5]. The definition of the used notions can be found in [5] (p. 158, 160, 161, 165 and 167).

(1.11) LEMMA. *For an arbitrary syntopogenous space $(E, \mathscr{S})$ the ordering structure Φ consisting of all $(\mathscr{S}, \mathscr{I})$-continuous ordering families is saturated, and we have* $\mathscr{S} \sim \mathscr{S}_\Phi \sim \bigcup_{\varphi \in \Phi} \mathscr{S}_\varphi = \{<_{\varphi, \varepsilon} : \varphi \in \Phi,\ \varepsilon > 0\}$.

PROOF. Assume $\varphi_1, \varphi_2, \ldots, \varphi_n \in \Phi$. Then $\mathscr{S}_{\varphi_i} \prec \mathscr{S}$ by [5], (12.33). Putting $\varphi = [\varphi_1, \varphi_2, \ldots, \varphi_n]$, from [5], (12.21) we get $\mathscr{S}_\varphi \sim \bigvee_{i=1}^{n} \mathscr{S}_{\varphi_i} \prec \mathscr{S}$, thus from [5], (12.33) $\varphi \in \Phi$ issues, hence Φ is saturated. In view of [5], (12.10) $\Phi = \Phi^v$, therefore $\mathscr{S}_\Phi \sim \bigcup_{\varphi \in \Phi} \mathscr{S}_\varphi$ by [5], (12.27). Finally because of [5], (12.35)—(12.37) $\mathscr{S} \sim \mathscr{S}_\Phi$ holds. ∎

(1.12) THEOREM. *Let $(E, \mathscr{S}, \leqq)$ be a preordered syntopogenous space. Then the set Φ' (Φ'') of all $(\mathscr{S}, \mathscr{I})$-continuous ordering families consisting of preorder preserving (inversing) functions is a saturated ordering structure, and $\mathscr{S}^l \sim \mathscr{S}_{\Phi'}$ $(\mathscr{S}^u \sim \mathscr{S}_{\Phi''})$.*

PROOF. If $\varphi \in \Phi'$, then $\mathscr{S}_\varphi$ is decreasing by (1.5), and owing to [5], (12.33) we have $\mathscr{S}_\varphi \prec \mathscr{S}$. Thus $\mathscr{S}_\varphi \prec \mathscr{S}^l$, consequently φ is $(\mathscr{S}^l, \mathscr{I})$-continuous. On the other hand, if φ is $(\mathscr{S}^l, \mathscr{I})$-continuous, then any $f \in \varphi$ is $(\mathscr{S}^l, \mathscr{I})$-continuous, hence it is preorder preserving by (1.6). Further $\mathscr{S}^l \prec \mathscr{S}$ implies that φ is $(\mathscr{S}, \mathscr{I})$-continuous. Summing up, Φ' is identical with the ordering structure of all $(\mathscr{S}^l, \mathscr{I})$-continuous ordering families, hence (1.11) can be applied. For Φ'' the proof is analogous. ∎

2. Two general types of convexity

In a preordered space $(E, \leqq)$ a subset C is said to be *convex* iff $x, y \in C$, $z \in E$, $x \leqq z \leqq y$ imply $z \in C$. The smallest convex set containing an arbitrary subset X of E is $c(X) = \{z \in E : x \leqq z \leqq y$ for some $x, y \in X\}$. We have $c(X) = i(X) \cap d(X)$.

Let us now consider the family

$$\mathscr{E}_c = \{<_{X,C} : X \subset C \subset E,\ C \text{ is convex}\}$$

of elementary topogenous orders (see [5], p. 42), and let ^{a} be an arbitrary elementary operation ([5], p. 69).

The preordered syntopogenous space $(E, \mathscr{S}, \leqq)$ will be called *weakly ^{a}-convex* iff, for any $< \in \mathscr{S}$, there exists a family $\mathscr{E} \subset \mathscr{E}_c$ such that

$$\{<\} \prec \mathscr{E}^{ta} \prec \mathscr{S}. \tag{2.1}$$

On the other hand, $(E, \mathcal{S}, \leqq)$ will be said to be ^{a}*-convex* iff

$$\mathcal{S} \sim (\mathcal{S}^u \vee \mathcal{S}^l)^a. \tag{2.2}$$

(For ${}^a={}^i, {}^p$ or ^{b} the latter definition is similar (but not equivalent) to that of Burgess—Fitzpatrick [2].) We shall use the term "(weakly) ^{a}-convex" for a syntopogenous structure on a preordered space, too.

(2.3) LEMMA. *If $(E, \mathcal{S}, \leqq)$ is (weakly) ^{a}-convex, then $\mathcal{S} \sim \mathcal{S}^a$.*

PROOF. $\mathcal{S} < \mathcal{S}^a$ is always true. If $(E, \mathcal{S}, \leqq)$ is weakly ^{a}-convex, then from (2.1) $\{<^a\} < \mathcal{E}^{taa} = \mathcal{E}^{ta} < \mathcal{S}$ follows for any $< \in \mathcal{S}$ (with a suitable $\mathcal{E} \subset \mathcal{E}_c$), thus $\mathcal{S}^a < \mathcal{S}$. If the space is ^{a}-convex, then $\mathcal{S}^a \sim (\mathcal{S}^u \vee \mathcal{S}^l)^{aa} = (\mathcal{S}^u \vee \mathcal{S}^l)^a \sim \mathcal{S}$. ∎

(2.4) LEMMA (cf. [2], p. 21). *The preordered syntopogenous space $(E, \mathcal{S}, \leqq)$ is ^{a}-convex iff $\mathcal{S} \sim (\mathcal{S}_1 \vee \mathcal{S}_2)^a$, where $\mathcal{S}_1$ ($\mathcal{S}_2$) is an increasing (decreasing) syntopogenous structure on $(E, \leqq)$.*

PROOF. The necessity is obvious. Conversely, if this condition is fulfilled by $\mathcal{S}$, then $\mathcal{S}_1 < \mathcal{S}^u$ and $\mathcal{S}_2 < \mathcal{S}^l$ (in fact, $\mathcal{S}_i < (\mathcal{S}_1 \vee \mathcal{S}_2) < (\mathcal{S}_1 \vee \mathcal{S}_2)^a < \mathcal{S}$ for $i=1, 2$), therefore $\mathcal{S} \sim (\mathcal{S}_1 \vee \mathcal{S}_2)^a < (\mathcal{S}^u \vee \mathcal{S}^l)^a < \mathcal{S}^a$. But $\mathcal{S}^a \sim (\mathcal{S}_1 \vee \mathcal{S}_2)^{aa} = (\mathcal{S}_1 \vee \mathcal{S}_2)^a \sim \mathcal{S}$, so that $(\mathcal{S}^u \vee \mathcal{S}^l)^a \sim \mathcal{S}$. ∎

(2.5) LEMMA. *If the preordered syntopogenous space $(E, \mathcal{S}, \leqq)$ is (weakly) ^{a}-convex, and ${}^{a'}$ is an elementary operation such that ${}^{aa'}$ is also an elementary operation, then $(E, \mathcal{S}^{a'}, \leqq)$ is (weakly) ${}^{aa'}$-convex.*

PROOF. If $\mathcal{E} \subset \mathcal{E}_c$ satisfies (2.1), then $\{<^{a'}\} < \mathcal{E}^{taa'} < \mathcal{S}^{a'}$ is also true. Analogously, if $\mathcal{S} \sim (\mathcal{S}^u \vee \mathcal{S}^l)^a$, then $\mathcal{S}^{a'} \sim (\mathcal{S}^u \vee \mathcal{S}^l)^{aa'}$ (cf. (2.4)). ∎

(2.6) THEOREM. *Let $(E, \mathcal{S}, \leqq)$ be a (weakly) ^{a}-convex preordered syntopogenous space. Then $(E, \mathcal{S}^{ta}, \leqq)$ is also (weakly) ^{a}-convex.*

PROOF. If $(E, \mathcal{S}, \leqq)$ is weakly ^{a}-convex, then we can choose a family $\mathcal{E}_< \subset \mathcal{E}_c$ for every $< \in \mathcal{S}$ such that

$$\{<\} < \mathcal{E}^{ta}_< < \mathcal{S}.$$

Putting $\mathcal{E} = \cup \{\mathcal{E}_<: < \in \mathcal{S}\}$, we get $\mathcal{S} < \cup \{\mathcal{E}^{ta}_<: < \in \mathcal{S}\} < \mathcal{S}$, so that $\mathcal{S}^{ta} = \mathcal{E}^{ta}$ by [5], (8.49). In view of $\mathcal{E} \subset \mathcal{E}_c$, it issues from this that $\mathcal{S}^{ta}$ is weakly ^{a}-convex.

If $(E, \mathcal{S}, \leqq)$ is ^{a}-convex, then from [5], (8.50), (8.101) it follows that $\mathcal{S}^{ta} = (\mathcal{S}^u \vee \mathcal{S}^l)^{ata} = (\mathcal{S}^u \vee \mathcal{S}^l)^{ta} \sim (\mathcal{S}^{ut} \vee \mathcal{S}^{lt})^a \sim (\mathcal{S}^{tu} \vee \mathcal{S}^{tl})^a$ (see (1.9)). ∎

(2.7) THEOREM. *Let $\{\mathcal{S}_i: i \in I \neq \emptyset\}$ be a family of (weakly) ^{a}-convex syntopogenous structures on the preordered space $(E, \leqq)$. Then $(\bigvee_{i \in I} \mathcal{S}_i)^a$ is also (weakly) ^{a}-convex on $(E, \leqq)$.*

PROOF. Let $\mathcal{S}_i$ be weakly ^{a}-convex on $(E, \leqq)$. If $\mathcal{S} = (\bigvee_{i \in I} \mathcal{S}_i)^a$, and $< \in \mathcal{S}$ is arbitrary, then for a natural number n

$$< = \Big(\bigcup_{j=1}^{n} <_j\Big)^{qa} \quad (<_j \in \mathcal{S}_{i_j},\ i_j \in I,\ 1 \leqq j \leqq n).$$

Put $\mathscr{E}=\bigcup_{j=1}^{n}\mathscr{E}_j$, where the families $\mathscr{E}_j\subset\mathscr{E}_c$ are chosen in accordance with $\{<_j\}<$ $<\mathscr{E}_j^{ta}<\mathscr{S}_{i_j}$ for $1\leqq j\leqq n$. Then $\mathscr{E}\subset\mathscr{E}_c$, further

$$\mathscr{E}^{ta}=\left(\bigcup_{j=1}^{n}\mathscr{E}_j^{ta}\right)^{ta}\sim\left(\bigcup_{j=1}^{n}\mathscr{E}_j^{ta}\right)^{ga}=\mathscr{E}',$$

hence

$$\{<\}<\mathscr{E}^{ta}\sim\mathscr{E}'<\left(\bigcup_{j=1}^{n}\mathscr{S}_{i_j}\right)^{ga}<\mathscr{S},$$

i.e. $\mathscr{S}$ is in fact weakly ^{a}-convex.

Let $\mathscr{S}_i$ be ^{a}-convex on $(E,\leqq)$ for any $i\in I$, and put $\mathscr{S}=(\bigvee_{i\in I}\mathscr{S}_i)^a$. Suppose $\mathscr{S}_1=\bigvee_{i\in I}\mathscr{S}_i^u$ and $\mathscr{S}_2=\bigvee_{i\in I}\mathscr{S}_i^l$. Then $\mathscr{S}_1$ is increasing, $\mathscr{S}_2$ is decreasing on $(E,\leqq)$ by (1.1.6). Applying [5], (8.99) and (8.97), we obtain

$$\mathscr{S}\sim(\bigvee_{i\in I}(\mathscr{S}_i^u\vee\mathscr{S}_i^l)^a)^a\sim(\mathscr{S}_1\vee\mathscr{S}_2)^a,$$

so that $\mathscr{S}$ is ^{a}-convex by (2.4). ∎

(2.8) Corollary. *Any ^{a}-convex preordered syntopogenous space is weakly ^{a}-convex.*

Proof. Let $(E,\mathscr{S},\leqq)$ be an ^{a}-convex space and $\mathscr{S}_1$ $(\mathscr{S}_2)$ be an increasing (decreasing) syntopogenous structure on $(E,\leqq)$ such that $\mathscr{S}\sim(\mathscr{S}_1\vee\mathscr{S}_2)^a$. If $<\in\mathscr{S}_1$ $(\mathscr{S}_2)$ and $<'\in\mathscr{S}_1$ $(\mathscr{S}_2)$ are such that $<\subset<'^2$, then $A<B$ implies $A<'D<'B$, and we can find an increasing (decreasing) set C in $(E,\leqq)$ for which $D\subset C\subset B$ (see (1.2)). Then $A<'C$, hence $\mathscr{E}=\{<_{X,C}\colon X<'C,\ C$ is increasing (decreasing)$\}$ is a family of elementary topogenous orders such that $\mathscr{E}\subset\mathscr{E}_c$ (increasing or decreasing sets are convex), and $\{<\}<\mathscr{E}^t<\{<'\}$. This means that $\mathscr{S}_1$ and $\mathscr{S}_2$ are weakly ^{i}-convex, consequently so is $\mathscr{S}_1\vee\mathscr{S}_2$, too (see (2.7)). Finally $\mathscr{S}\sim(\mathscr{S}_1\vee\mathscr{S}_2)^a$ is weakly ^{a}-convex by (2.5). ∎

(2.9) Theorem. *Let f be a preorder preserving mapping of the preordered space $(E,\leqq)$ into a (weakly) ^{a}-convex preordered syntopogenous space $(E',\mathscr{S}',\leqq')$. Then $(E,f^{-1}(\mathscr{S}'),\leqq)$ is also (weakly) ^{a}-convex.*

Proof. Assume that $(E',\mathscr{S}',\leqq')$ is weakly ^{a}-convex. If $<'\in\mathscr{S}'$, then there exists a family $\mathscr{E}'\subset\mathscr{E}'_c$ such that $\{<'\}<\mathscr{E}'^{ta}<\mathscr{S}'$. Because of [5], (6.19) we have

$$f^{-1}(\mathscr{E}')=\{<_{f^{-1}(X),f^{-1}(C)}\colon <_{X,C}\in\mathscr{E}'\}.$$

If C is convex in $(E',\leqq')$, then $f^{-1}(C)$ is also convex in $(E,\leqq)$, therefore $f^{-1}(\mathscr{E}')\subset\mathscr{E}_c$. From here by $f^{-1}(\mathscr{E}')^{ta}=f^{-1}(\mathscr{E}'^{ta})$, we get $\{f^{-1}(<')\}<f^{-1}(\mathscr{E}')^{ta}<$ $<f^{-1}(\mathscr{S}')$.

Let $(E',\mathscr{S}',\leqq')$ be ^{a}-convex. Then $f^{-1}(\mathscr{S}')\sim f^{-1}((\mathscr{S}'^u\vee\mathscr{S}'^l)^a)=f^{-1}(\mathscr{S}'^u\vee$ $\vee\mathscr{S}'^l)^a=(f^{-1}(\mathscr{S}'^u)\vee f^{-1}(\mathscr{S}'^l))^a$ (see [5], (9.10)). Since $f^{-1}(\mathscr{S}'^u)$ is increasing, $f^{-1}(\mathscr{S}'^l)$ is decreasing on $(E,\leqq)$ by (1.1.4), from (2.4) it follows that $f^{-1}(\mathscr{S}')$ is ^{a}-convex on $(E,\leqq)$. ∎

(2.10) THEOREM. *Suppose that* $(E_i, \mathcal{S}_i, \leqq_i)$ *is a (weakly)* ^{a}*-convex preordered syntopogenous space for* $i \in I \neq \emptyset$. *Put* $E = \bigtimes_{i \in I} E_i$, $\mathcal{S} = (\bigtimes_{i \in I} \mathcal{S}_i)^a$ *and let* $\leqq$ *be the product preorder on* E. *Then* $(E, \mathcal{S}, \leqq)$ *is also (weakly)* ^{a}*-convex.*

PROOF. The projections $\mathrm{pr}_i \colon E \to E_i$ are preorder preserving, therefore (2.9) and (2.7) can be applied. ∎

Let us recall that an elementary operation ^{a} is said to be *symmetrical* iff $^{ca} = {}^{ac}$ (cf. [5], (8.1), (8.2)).

(2.11) THEOREM (for $^a = {}^i$ see [1], 4.2). *Let* ^{a} *be a symmetrical elementary operation. Then* $(E, \mathcal{S}, \leqq)$ *is* ^{a}*-convex iff so is* $(E, \mathcal{S}^c, \leqq)$. *In this case* $(E, \mathcal{S}^{sa}, \leqq)$ *is also* ^{a}*-convex.*

PROOF. $\mathcal{S} \sim (\mathcal{S}^u \vee \mathcal{S}^l)^a$ implies $\mathcal{S}^c \sim (\mathcal{S}^u \vee \mathcal{S}^l)^{ac} = (\mathcal{S}^u \vee \mathcal{S}^l)^{ca} = (\mathcal{S}^{uc} \vee \mathcal{S}^{lc})^a \sim$ $\sim (\mathcal{S}^{cl} \vee \mathcal{S}^{cu})^a = (\mathcal{S}^{cu} \vee \mathcal{S}^{cl})^a$ (see (1.8) and [5], (8.102)).

If $(E, \mathcal{S}, \leqq)$ is ^{a}-convex, then because of $\mathcal{S}^{sa} \sim (\mathcal{S} \vee \mathcal{S}^c)^a$ (2.7) gives that $\mathcal{S}^{sa}$ is ^{a}-convex. ∎

(2.12) THEOREM. *Let* $(E, \mathcal{S}, \leqq)$ *be a symmetrical preordered syntopogenous space. Then the following statements are equivalent:*

(2.12.1) $\mathcal{S}$ *is* ^{a}*-convex on* $(E, \leqq)$.

(2.12.2) $\mathcal{S} \sim \mathcal{S}_0^{sa}$, *where* $\mathcal{S}_0$ *is increasing on* $(E, \leqq)$.

(2.12.3) $\mathcal{S} \sim \mathcal{S}_0^{sa}$, *where* $\mathcal{S}_0$ *is decreasing on* $(E, \leqq)$.

PROOF. (2.12.1)⇒(2.12.2): By (1.8) $\mathcal{S} \sim (\mathcal{S}^u \vee \mathcal{S}^l)^a \sim (\mathcal{S}^u \vee \mathcal{S}^{uc})^a \sim \mathcal{S}^{usa}$.
(2.12.2)⇒(2.12.1): If $\mathcal{S} \sim \mathcal{S}_0^{sa}$, where $\mathcal{S}_0$ is increasing on $(E, \leqq)$, then $\mathcal{S}_0^c$ is decreasing, and $\mathcal{S} \sim \mathcal{S}_0^{sa} \sim (\mathcal{S}_0 \vee \mathcal{S}_0^c)^a$, hence $\mathcal{S}$ is ^{a}-convex by (2.4).
(2.12.2)⇔(2.12.3) by (1.1.3) and $^s = {}^{cs}$. ∎

3. Weakly convex and locally convex spaces

In this section we study the corollaries of the general theory of weakly ^{a}-convex spaces in the cases of $^a = {}^i$, ^{p} or ^{b}.

(3.1) LEMMA. *The preordered syntopogenous space* $(E, \mathcal{S}, \leqq)$ *is weakly* ^{i}*-convex iff for every* $< \in \mathcal{S}$ *there exists* $<_1 \in \mathcal{S}$ *such that* $A < B$ *implies* $A \subset \bigcup_{i=1}^m X_i$, $\bigcup_{i=1}^m C_i \subset B$, *where* m *is a suitable natural number,* $X_i <_1 C_i$, *and* C_i *is convex in* $(E, \leqq)$ *for each* $1 \leqq i \leqq m$.

PROOF. Suppose that $(E, \mathcal{S}, \leqq)$ is weakly ^{i}-convex, and put $< \in \mathcal{S}$. There exists $\mathcal{E} \subset \mathcal{E}_c$ and $<_1 \in \mathcal{S}$ such that $\{<\} < \mathcal{E}^t < \{<_1\}$. Disregarding the trivial cases of $A = \emptyset$ or $B = E$, form $A < B$

$$A \subset \bigcup_{i=1}^{m} \bigcap_{j=1}^{n_i} X_{ij}, \quad \bigcup_{i=1}^{m} \bigcap_{j=1}^{n_i} C_{ij} \subset B,$$

where C_{ij} is convex and $X_{ij} <_1 C_{ij}$ $(1 \leqq i \leqq m, 1 \leqq j \leqq n_i)$ by [5], (5.54). Then $C_i = \bigcap_{j=1}^{n_i} C_{ij}$ is convex, and with $X_i = \bigcap_{j=1}^{n_i} X_{ij}$ we have $A \subset \bigcup_{i=1}^{m} X_i$, $\bigcup_{i=1}^{m} C_i \subset B$, further $X_i <_1 C_i$, because $<_1$ is topogenous.

Conversely, if $<_1 \in \mathcal{S}$ is chosen for $< \in \mathcal{S}$ in accordance with the condition, then the family

$$\mathcal{E} = \{<_{X,C} \colon C \text{ is convex and } X <_1 C\}$$

(once more by [5], (5.54)) has the properties $\mathcal{E} \subset \mathcal{E}_c$ and $\{<\} < \mathcal{E}^t < \{<_1\}$. ∎

(3.2) LEMMA. *The preordered syntopological space $(E, \mathcal{S}, \leqq)$ is weakly ^{p}-convex iff, for each $< \in \mathcal{S}$, there exists $<_1 \in \mathcal{S}$ such that $x \in E$, $x < B$ imply $x <_1 C \subset B$, where C is a suitable convex set in $(E, \leqq)$.*

PROOF. Assume that $(E, \mathcal{S}, \leqq)$ is weakly ^{p}-convex, $< \in \mathcal{S}$, and choose $<_1 \in \mathcal{S}$ such that $\{<\} < \mathcal{E}^{tp} < \{<_1\}$ for some $\mathcal{E} \subset \mathcal{E}_c$. This means $< \subset <_0^p \subset <_1$, where $< \; = (\bigcup \mathcal{E})^q$. Put $x < B$. Then $x <_0 B$ is also true, consequently

$$x \in \bigcup_{i=1}^{m} \bigcap_{j=1}^{n_i} X_{ij}, \quad \bigcup_{i=1}^{m} \bigcap_{j=1}^{n_i} C_{ij} \subset B, \quad X_{ij} <_1 C_{ij},$$

where C_{ij} is convex $(1 \leqq i \leqq m, 1 \leqq j \leqq n_i)$. Since, for at least one i, $x \in \bigcap_{j=1}^{n_i} X_{ij}$, we obtain that with the notation $C = \bigcap_{j=1}^{n_i} C_{ij}$ the set C is convex and $x <_1 C \subset B$.

Conversely, suppose $< \in \mathcal{S}$, and let us choose $<_1 \in \mathcal{S}$ for $<$ in accordance with the condition. Then considering the family

$$\mathcal{E} = \{<_{X,C} \colon C \text{ is convex and } X <_1 C\},$$

we have

$$< \subset (\bigcup \mathcal{E})^p \subset (\bigcup \mathcal{E})^{qp} \subset <_1^{qp} = <_1,$$

that is $\{<\} < \mathcal{E}^{tp} < \mathcal{S}$, so that $(E, \mathcal{S}, \leqq)$ is weakly ^{p}-convex. ∎

(3.3) THEOREM. *Any weakly ^{b}-convex space is weakly ^{p}-convex.*

PROOF. If $(E, \mathcal{S}, \leqq)$ is a weakly ^{b}-convex preordered syntopogenous space, then $\mathcal{S} \sim \mathcal{S}^b$ by (2.3). For an arbitrary $< \in \mathcal{S}$ let us consider $\mathcal{E} \subset \mathcal{E}_c$ and $<' \in \mathcal{S}$ such that with the notation $<_0 = (\bigcup \mathcal{E})^q$, $< \subset <_0^b \subset <'$, finally put $<'^b \subset <_1 \in \mathcal{S}$. $x < B$ implies $x <_0^b B$, which means $x <_0 E - y$ for any $y \in E - B$. Because of $^{qb} = {}^b$ there exists $<_{X_y, C_y} \in \mathcal{E}$ so that $x \in X_y <' C_y \subset E - y$. Then $C = \bigcap \{C_y \colon y \in E - B\}$ is convex, and by $<'^b \subset <_1$ we have $x <_1 C \subset B$. On the basis of (3.2) this gives that $\mathcal{S}$ is ^{p}-convex on $(E, \leqq)$. ∎

(3.4) THEOREM. *Let $(E, \mathcal{S}, \leqq)$ be a preordered biperfect syntopological space and $\mathcal{U}$ be the quasi-uniformity associated with $\mathcal{S}$. $(E, \mathcal{S}, \leqq)$ is weakly ^{p}-convex iff*

(3.4.1) *for every $U \in \mathcal{U}$ there exists $U_1 \in \mathcal{U}$ such that*

$$c(U_1(x)) \subset U(x) \quad \textit{for any} \quad x \in E.$$

If $\mathcal{S}$ is symmetrical (i.e. $\mathcal{U}$ is a uniformity), then (3.4.1) *is equivalent to*

(3.4.2) *for every* $U \in \mathcal{U}$ *there exists* $U_1 \in \mathcal{U}$ *such that*

$$(x, y) \in U_1,\ x \leqq z \leqq y \text{ imply } (x, z) \in U.$$

PROOF. Let $(E, \mathcal{S}, \leqq)$ be weakly ^{p}-convex and $U \in \mathcal{U}$. For $< = <_U$ there exists an order $<_1 \in \mathcal{S}$ satisfying the condition of (3.2). Put $U_1 = U_{<_1}$. Then $y \notin U(x)$ means $x < E - y$, thus $x <_1 C \subset E - y$ for some convex C. We have $U_1(x) \subset C$, therefore $c(U_1(x)) \subset C$, hence $y \notin c(U_1(x))$. Conversely, suppose that (3.4.1) is satisfied by $\mathcal{U}$. Then, for an arbitrary $< \in \mathcal{S}$, let us choose $U_1 \in \mathcal{U}$ corresponding to (3.4.1) with $U = U_<$. Put $<_1 = <_{U_1}$. $x < B$ means $U(x) \subset B$, but because of (3.4.1) the convex set $C = c(U_1(x))$ has the property $x <_1 U_1(x) \subset C \subset U(x) \subset B$, therefore $\mathcal{S}$ is weakly ^{p}-convex by (3.2).

It is obvious that (3.4.1)$\Rightarrow$(3.4.2). Now assume that $\mathcal{U}$ is a uniformity satisfying (3.4.2). For $U \in \mathcal{U}$ find $U' \in \mathcal{U}$ such that $U' \circ U' \subset U$, $U'' \in \mathcal{U}$ for U' by (3.4.2), finally consider $U_1 \in \mathcal{U}$ with the property $U_1 \circ U_1 \subset U' \cap U''$. Then $z \in c(U_1(x))$ implies $u \in U_1(x)$, $v \in U_1(x)$, $u \leqq z \leqq v$. $(u, v) \in U''$ and (3.4.2) give $(u, z) \in U'$. $(x, u) \in U_1 \subset U'$ is also true, hence $(x, z) \in U$, that is $c(U_1(x)) \subset U(x)$. ∎

(3.5) EXAMPLES. A preordered topological space is weakly ^{p}-convex iff it is associated with a Nachbin's locally convex "classical" topology (see (3.2) and [11], p. 26; cf. also [3], 3.6).

In [7], Fedorchuk studied proximity relations δ on a linearly ordered set E, for which $A \bar{\delta} B$ implies $A \subset \bigcup_{i=1}^{m} O_i \subset E - B$, where $O_1, O_2, \ldots, O_m$ are closed intervals. This property is similar to that of the proximity relation associated with a weakly ^{i}-convex symmetrical topogenous structure (cf. (3.1)).

In [4], Carruth investigated ordered metric spaces with the condition that $x \leqq z < y$ implies $\varrho(x, z) < \varrho(x, y)$. The uniformity of such a space satisfies (3.4.2), therefore the associated symmetrical syntopology is weakly ^{p}-convex. ∎

In [3], Burgess—Fitzpatrick introduced the notion of a *locally convex preordered syntopogenous space* $(E, \mathcal{S}, \leqq)$ as follows ([3], 3.1):

(i) for each $< \in \mathcal{S}$ there exist $<' \in \mathcal{S}$ and $A \subset E \times E$ such that $G(<') \subset A \subset G(<)$, and $A(x)$ is convex for every $x \in E$;

(ii) for $< \in \mathcal{S}$ there is $<'' \in \mathcal{S}$ such that $x < B$ implies $x <'' C <'' B$ for a suitable convex set C.

This definition can be essentially simplified.

(3.6) LEMMA. *For an arbitrary preordered syntopogenous space* $(E, \mathcal{S}, \leqq)$ *the implication* (ii)$\Rightarrow$(i) *always holds. The space in question is locally convex iff, for each* $< \in \mathcal{S}$, *there is* $<_1 \in \mathcal{S}$ *such that* $x < B$ *implies* $x <_1 C \subset B$ *for a convex set* C.

PROOF. Suppose (ii) for $(E, \mathcal{S}, \leqq)$. Define $A \subset E \times E$ as follows: $(x, y) \in A$ iff $y \in C$ for every convex set C such that $x <'' C$. One can show that $G(<'') \subset A \subset G(<)$ and $A(x)$ is convex for $x \in E$. In fact, $(x, y) \notin G(<)$ means $x < E - y$, therefore $x <'' C \subset E - y$ for a convex set C. Since $y \notin C$, $(x, y) \notin A$. If $(x, y) \notin A$, then there exists a convex set C such that $x <'' C$, and $y \notin C$. Then $x <'' E - y$, thus $(x, y) \notin G(<'')$. Finally, if $y, z \in A(x)$ and $y \leqq u \leqq z$, then for each convex set

$C \subset E$, $x <'' C$ implies $y, z \in C$, consequently $u \in C$, but this yields $(x, u) \in A$, that is $u \in A(x)$.

Passing over to our second statement, let us remark that (ii) is obviously stronger than the present condition. Conversely, if $< \in \mathcal{S}$, and $<' \in \mathcal{S}$ with $< \sqsubset <'^2$, further $<_1 \in \mathcal{S}$ is chosen for $<'$ so that

$$x <' B_0 \Rightarrow x <_1 C \subset B_0, \text{ where } C \text{ is convex,}$$

finally if $<' \cup <_1 \sqsubset <'' \in \mathcal{S}$, then $x < B$ implies $x <' B_0 <' B$. From this $x <_1 C \subset B_0$, so that $x <'' C <'' B$, where C is convex. ∎

Comparing (3.6) with (3.2) we obtain a characterization of locally convex spaces by weakly convex spaces.

(3.7) THEOREM. *A preordered syntopogenous space* $(E, \mathcal{S}, \leqq)$ *is locally convex iff* $(E, \mathcal{S}^p, \leqq)$ *is weakly* ^{p}*-convex.* ∎

(3.8) COROLLARY. *A preordered syntopogenous space* $(E, \mathcal{S}, \leqq)$ *is locally convex iff so is* $(E, \mathcal{S}^p, \leqq)$. ∎

4. Convex spaces

Let us observe that a preordered syntopogenous space $(E, \mathcal{S}, \leqq)$ is *convex* (in the sense of [2]) iff it is ^{i}- or ^{p}- or ^{b}-convex. Now we can describe the fundamental properties of a larger class of preordered syntopogenous spaces containing every convex space (cf. [2], 4.2, 4.3, 4.6, and [1], 4.2, 5.2, 6.2, 7.6).

(4.1) THEOREM. *Let* $(E, \mathcal{S}, \leqq)$ *be a preordered syntopogenous space such that* $\mathcal{S} \sim \mathcal{S}^k$, *where* $^k = {}^a$ *or* $^k = {}^{ta}$ *for an elementary operation* ^{a}, *which fulfils* $<^a \sqsubset <^b$ *for every semi-topogenous order* $<$. *Then*

(4.1.1) $(E, \mathcal{S}, \leqq)$ *is* ^{a}*-convex iff there exist syntopogenous structures* $\mathcal{S}_1 = \mathcal{S}_1^k$ *and* $\mathcal{S}_2 = \mathcal{S}_2^k$ *such that* $\mathcal{S}_1$ $(\mathcal{S}_2)$ *is increasing (decreasing) on* $(E, \leqq)$, *and* $\mathcal{S} \sim (\mathcal{S}_1 \vee \mathcal{S}_2)^k$.

(4.1.2) *If* $\mathcal{S}$ *is symmetrical, then* $(E, \mathcal{S}, \leqq)$ *is* ^{a}*-convex iff* $\mathcal{S} \sim \mathcal{S}_0^{sk}$ *for an increasing or decreasing syntopogenous structure* $\mathcal{S}_0 = \mathcal{S}_0^k$ *on* $(E, \leqq)$.

(4.1.3) *In the case of* $\mathcal{S} = \mathcal{S}^{ta}$ *the symbol* $=$ *can be written instead of* $\sim$.

PROOF. (4.1.1): The sufficiency of the condition is clear even in the case of $^k = {}^{ta}$ (see (2.4), (2.5), (2.6) and [5], (8.101)). Conversely, if $(E, \mathcal{S}, \leqq)$ is ^{a}-convex, then $\mathcal{S} \sim \mathcal{S}^k \sim (\mathcal{S}^u \vee \mathcal{S}^l)^{ak} = (\mathcal{S}^u \vee \mathcal{S}^l)^k = (\mathcal{S}^{uk} \vee \mathcal{S}^{lk})^k$ (see [5], (8.50), (8.99) and (8.100)). Put $\mathcal{S}_1 = \mathcal{S}^{uk}$ and $\mathcal{S}_2 = \mathcal{S}^{lk}$. ^{k} is an ordinary operation, for which $\mathcal{S}^k < \mathcal{S}^{tb}$ (cf. [5], (8.23), (8.26)), thus from (1.7) $\mathcal{S}_1 \sim \mathcal{S}^u$ and $\mathcal{S}_2 \sim \mathcal{S}^l$, so that $\mathcal{S}_1$ is increasing and $\mathcal{S}_2$ is decreasing. Finally from (K_2) of [5], p. 74: $\mathcal{S}_1 = \mathcal{S}_1^k$ and $\mathcal{S}_2 = \mathcal{S}_2^k$.

(4.1.2): Suppose that the condition is satisfied by $\mathcal{S}$. For $^k = {}^a$ $\mathcal{S}$ is ^{a}-convex by (2.12). For $^k = {}^{ta}$ we have $\mathcal{S} \sim \mathcal{S}_0^{sta} = \mathcal{S}_0^{tsa}$ (cf. [5], (8.51)), thus we can refer to (2.12) and (1.1.2). Conversely, let $(E, \mathcal{S}, \leqq)$ be ^{a}-convex and $\mathcal{S} \sim \mathcal{S}_0^{sa}$ e.g. for an increasing $\mathcal{S}_0$ (see (2.12)). Then $\mathcal{S} \sim \mathcal{S}_0^{sa} < \mathcal{S}^{usa} < \mathcal{S}^{sa} = \mathcal{S}^a \sim \mathcal{S}$ (see (2.3)), therefore $\mathcal{S} \sim \mathcal{S}^k \sim \mathcal{S}^{usak} = \mathcal{S}^{usk}$. By the train of thought followed in the proof of (4.1.1), from (1.7) $\mathcal{S}^u \sim \mathcal{S}^{uk}$ can be obtained, thus $\mathcal{S}_1 = \mathcal{S}^{uk}$ is increasing, $\mathcal{S}_1 = \mathcal{S}_1^k$ and $\mathcal{S} \sim \mathcal{S}^{usk} \sim \mathcal{S}^{uksk} = \mathcal{S}_1^{sk}$.

(4.1.3) is clear. ∎

We can complete the above theorem as follows:

(4.2) COROLLARY (cf. (3.3)). *Any ^{b}-convex space is ^{p}-convex.*

PROOF. If $(E, \mathcal{S}, \leqq)$ is ^{b}-convex, then $\mathcal{S} \sim \mathcal{S}^b$ by (2.3). Because of (4.1.1) $\mathcal{S} \sim (\mathcal{S}_1 \vee \mathcal{S}_2)^b \sim (\mathcal{S}_1 \vee \mathcal{S}_2)^p$, where $\mathcal{S}_1$ ($\mathcal{S}_2$) is an increasing (decreasing) biperfect syntopology on $(E, \leqq)$ (see [5], (8.102)). ∎

(4.3) EXAMPLE. We show an easy example for a locally convex but non convex topology. Let $(\mathbf{R}, \mathcal{T}, \leqq)$ be the naturally ordered and topologized real line. Then

$$\underset{\sim}{G} = \{\emptyset, \mathbf{R}\} \cup \{V \subset \mathbf{R}: V \text{ is } \mathcal{T}\text{-open}, \ 0 \notin V\}$$

is the system of the open sets for another topology $\mathcal{T}'$ on $\mathbf{R}$ that is obviously locally convex. Suppose $x, y, z \in \mathbf{R}$ and $0<x<y<z$. Then the open interval (x, z) is a $\mathcal{T}'$-neighbourhood of y. If $\mathcal{T}'$ is convex, then $y \in I \cap D \subset (x, z)$, where I (D) is an increasing (decreasing) $\mathcal{T}'$-open set (cf. e.g. [2], p. 21). $y \in D$ implies $0 \in D$, but in view of that the only $\mathcal{T}'$-neighbourhood of 0 is $\mathbf{R}$ itself, we have $D = \mathbf{R}$. Thus $I \subset (x, z)$, which is impossible. ∎

REMARK. Convex classical structures were studied in [9]—[14]. Their connections with convex syntopogenous spaces were cleared by Burgess—Fitzpatrick in [1]—[2]. ∎

In order to show a remarkable categorical property of ^{a}-convex spaces, let us consider the following lemma:

(4.4) LEMMA. *Let ^{a} be an elementary operation and $(E, \mathcal{S}, \leqq)$ be a preordered syntopogenous space with $\mathcal{S} \sim \mathcal{S}^a$. Then $(\mathcal{S}^u \vee \mathcal{S}^l)^a$ is the finest of all ^{a}-convex syntopogenous structures on $(E, \leqq)$ coarser than $\mathcal{S}$.*

PROOF. $(\mathcal{S}^u \vee \mathcal{S}^l)^a$ is in fact ^{a}-convex by (2.4). $\mathcal{S}^u < \mathcal{S}$ and $\mathcal{S}^l < \mathcal{S}$ imply $(\mathcal{S}^u \vee \mathcal{S}^l)^a < \mathcal{S}^a \sim \mathcal{S}$. Let $\mathcal{S}_1$ be another ^{a}-convex syntopogenous structure on $(E, \leqq)$ such that $\mathcal{S}_1 < \mathcal{S}$. Then $\mathcal{S}_1^u < \mathcal{S}^u$ and $\mathcal{S}_1^l < \mathcal{S}^l$, consequently $\mathcal{S}_1 \sim (\mathcal{S}_1^u) \vee \mathcal{S}_1^l)^a < < (\mathcal{S}^u \vee \mathcal{S}^l)^a$. ∎

Let us recall that in the theory of the categories a full subcategory $\mathbf{B}$ of the category $\mathbf{A}$ is said to be *(epi)reflective* in $\mathbf{A}$ iff for any object $A \in \mathbf{A}$ there exists an object $R \in \mathbf{B}$ with an (epi)morphism $r: A \to R$ of $\mathbf{A}$ such that whenever $B \in \mathbf{B}$ and $f: A \to B$ is a morphism of $\mathbf{A}$, there is a unique morphism $g: R \to B$ of $\mathbf{B}$ for which $f = g \circ r$. R is called the *(epi)reflection* of A in $\mathbf{B}$ with the *(epi)reflector* r.

The preordered syntopogenous spaces (as objects) and the preorder preserving continuous mappings (as morphisms) form a category denoted by $\mathbf{Ps}$. For an elementary operation ^{a} let $\mathbf{Ps}^{(a)}$ be the subcategory of $\mathbf{Ps}$ consisting of those preordered syntopogenous spaces $(E, \mathcal{S}, \leqq)$ and their morphisms, for which $\mathcal{S} \sim \mathcal{S}^a$. Let us denote by $\mathbf{C}^{(a)}$ the subcategory of the ^{a}-convex preordered syntopogenous spaces in $\mathbf{Ps}^{(a)}$ (cf. (2.3)).

(4.5) THEOREM. $\mathbf{C}^{(a)}$ *is an epireflective subcategory of* $\mathbf{Ps}^{(a)}$.

PROOF. $\mathbf{C}^{(a)}$ is a full subcategory of $\mathbf{Ps}^{(a)}$, i.e. every morphism in $\mathbf{Ps}^{(a)}$ between the objects of $\mathbf{C}^{(a)}$ is a morphism of $\mathbf{C}^{(a)}$. Suppose $(E, \mathcal{S}, \leqq) \in \mathbf{Ps}^{(a)}$, and put $\mathcal{S}^* = (\mathcal{S}^u \vee \mathcal{S}^l)^a$. We prove that $(E, \mathcal{S}^*, \leqq)$ is the epireflection of

$(E, \mathscr{S}, \leqq)$ in $\mathbf{C}^{(a)}$, and the corresponding epireflector is the identity mapping of E. In fact, $(E, \mathscr{S}^*, \leqq)\in\mathbf{C}^{(a)}$, and $\mathrm{id}_E: (E, \mathscr{S}, \leqq)\to(E, \mathscr{S}^*, \leqq)$ is an epimorphism of $\mathbf{Ps}^{(a)}$ by (4.4). If $(E', \mathscr{S}', \leqq')\in\mathbf{C}^{(a)}$, and $f: (E, \mathscr{S}, \leqq)\to(E', \mathscr{S}', \leqq')$ is a morphism of $\mathbf{Ps}^{(a)}$, then there exists a unique mapping $g: E\to E'$ such that $f=g\circ\mathrm{id}_E$ (namely $f=g$). $g^{-1}(\mathscr{S}')=f^{-1}(\mathscr{S}')$ is ^{a}-convex, and it is coarser than $\mathscr{S}$, hence $g^{-1}(\mathscr{S}')<\mathscr{S}^*$ (see (2.9) and (4.4)). Thus $g: (E, \mathscr{S}^*, \leqq)\to(E', \mathscr{S}', \leqq')$ is a morphism of $\mathbf{Ps}^{(a)}$. ∎

An *arbitrary* preordered syntopogenous space $(E, \mathscr{S}, \leqq)$ will be called *symmetrizable* iff there exists a symmetrical ^{i}-convex syntopogenous structure $\mathscr{S}_0$ on $(E, \leqq)$ such that $\mathscr{S}_0<\mathscr{S}<\mathscr{S}_0^p$. This is a generalization of the notion of a symmetrizable syntopological space introduced by Császár [6] (cf. [8]). (In fact, let us consider a syntopology $\mathscr{S}$ on the set E equipped with the trivial preorder $=$. Then every syntopogenous structure on $(E, =)$ is both increasing and decreasing, consequently it is ^{i}-convex. Thus $(E, \mathscr{S}, =)$ is symmetrizable iff $\mathscr{S}_0<\mathscr{S}<\mathscr{S}_0^p$, i.e. $\mathscr{S}\sim\mathscr{S}_0^p$ for an arbitrary symmetrical syntopogenous structure $\mathscr{S}_0$ on E, which is equivalent to that $\mathscr{S}$ is a symmetrizable syntopology.)

(4.6) EXAMPLE. Let $(\mathbf{R}, \mathscr{T}, \leqq)$ be the preordered symmetrical topogenous space of (1.10). Then $(\mathbf{R}, \mathscr{T}^p, \leqq)$ is symmetrizable, but $(\mathbf{R}, \mathscr{T}, \leqq)$ is not. In (1.10) we showed that $\mathscr{T}^p=\mathscr{D}_{\mathbf{R}}$, $\mathscr{T}^{pu}=\mathscr{U}_{\leqq}$ and $\mathscr{T}^{pl}=\mathscr{L}_{\leqq}$. Let us now observe that $\mathscr{S}_0=(\mathscr{U}_{\leqq}\vee\mathscr{L}_{\leqq})\sim\mathscr{U}_{\leqq}^s$ is a symmetrical ^{i}-convex structure on $(\mathbf{R}, \leqq)$ such that $\mathscr{T}^p=\mathscr{S}_0^p$. On the other hand, if $\mathscr{S}_0'$ is an ^{i}-convex syntopogenous structure on $(\mathbf{R}, \leqq)$, for which $\mathscr{S}_0'<\mathscr{T}$, then $\mathscr{S}_0'^u<\mathscr{T}^u=\mathscr{O}_{\mathbf{R}}$ and $\mathscr{S}_0'^l<\mathscr{T}^l=\mathscr{O}_{\mathbf{R}}$ (see (1.10)), thus $\mathscr{S}_0'\sim\mathscr{S}_0'^u\vee\mathscr{S}_0'^l=\mathscr{O}_{\mathbf{R}}$ implies that $\mathscr{T}<\mathscr{S}_0'^p=\mathscr{O}_{\mathbf{R}}$ is impossible. ∎

(4.7) LEMMA.

(4.7.1) *Any symmetrizable preordered syntopological space is ^{p}-convex.*

(4.7.2) *Any ^{i}- or ^{p}-convex symmetrical preordered syntopogenous space is symmetrizable.*

PROOF. (4.1.7): If $(E, \mathscr{S}, \leqq)$ is symmetrizable and $\mathscr{S}=\mathscr{S}^p$, then, for a suitable ^{i}-convex syntopogenous structure $\mathscr{S}_0$ on $(E, \leqq)$, $\mathscr{S}_0<\mathscr{S}<\mathscr{S}_0^p$ implies $\mathscr{S}_0^p<\mathscr{S}^p= =\mathscr{S}<\mathscr{S}_0^p$, so that $\mathscr{S}\sim\mathscr{S}_0^p$, and from (2.5) we get that $\mathscr{S}$ is ^{p}-convex.

(4.7.2): If $(E, \mathscr{S}, \leqq)$ is symmetrical and ^{i}-convex, then it is trivially symmetrizable. If it is symmetrical and ^{p}-convex, then on the basis of (2.12.2) $\mathscr{S}\sim\mathscr{S}_0^{sp}$, where $\mathscr{S}_0$ is increasing, consequently $\mathscr{S}_0^s$ is ^{i}-convex on $(E, \leqq)$. ∎

We show that the notion of a symmetrizable preordered syntopogenous space is a generalization of that of Nachbin's uniformizable topological ordered space. (In the following theorem we use the notation $\mathscr{H}=\mathscr{I}^{sb}$ of [5].)

(4.8) THEOREM (cf. [11], pp. 52—53). *The preordered syntopogenous space $(E, \mathscr{S}, \leqq)$ is symmetrizable iff $<\in\mathscr{S}$ implies the existence of an $(\mathscr{S}, \mathscr{I}^s)$-continuous functional family φ on E, for which if $x<V$, then there are functions $f, g\in\varphi$ such that f is preorder preserving, g is preorder inversing, $f(E), g(E)\subset[0, 1]$, $f(x)=g(x)=0$ and $\max\{f(y), g(y)\}=1$ for $y\in E-V$.*

In this case denoting by Φ the set of all $(\mathscr{S}, \mathscr{I}^s)$-continuous ordering families consisting of preorder preserving functions, we have $\mathscr{S}_\Phi^s<\mathscr{S}<\mathscr{S}_\Phi^{sp}$. Furthermore, $\mathscr{S}_\Phi^s$ is the finest of all ^{i}-convex symmetrical syntopogenous structures on $(E, \leqq)$ coarser than $\mathscr{S}$. If $\mathscr{S}\sim\mathscr{S}^p$, then instead of $\mathscr{I}^s$ one can write $\mathscr{H}$.

For the sake of the verification of the theorem let us mention the following simple fact:

(4.9) LEMMA. *Let f be a real function on E and c be a constant. Then, for the function $g=c-f$, and for any $\varepsilon>0$, we have $g^{-1}(<_\varepsilon)=f^{-1}(<_\varepsilon)^c$.*

PROOF. For $A, B\subset E$, $Ag^{-1}(<_\varepsilon)B$ iff there exists $p\in\mathbf{R}$ such that $A\subset g^{-1}((-\infty, p])$ and $g^{-1}((-\infty, p+\varepsilon))\subset B$. This is equivalent to $E-B\subset f^{-1}((-\infty, c-p-\varepsilon])$ and $f^{-1}((-\infty, c-p))\subset E-A$ (cf. [5], p. 157). ∎

PROOF OF THE THEOREM. Let $(E, \mathcal{S}, \leqq)$ be symmetrizable. Then $\mathcal{S}_0^s < \mathcal{S} < \mathcal{S}_0^{sp}$ for a decreasing syntopogenous structure $\mathcal{S}_0$ on $(E, \leqq)$ by (2.12). Suppose $<\in\mathcal{S}$, and put $<\subset<_0^{sp}$ for a suitable $<_0\in\mathcal{S}_0$. In view of [5], (12.41) we can find an $(\mathcal{S}_0, \mathcal{I})$-continuous functional family φ_0 on E such that $A<_0B$ implies $f(E)\subset[0, 1]$, $f(A)=\{0\}$ and $f(E-B)=\{1\}$ for some $f\in\varphi_0$. By (1.6) the elements of φ_0 are preorder preserving. Further denote $\psi_0=\{1-f: f\in\varphi_0\}$. ψ_0 is $(\mathcal{S}_0^c, \mathcal{I})$-continuous by (4.9), consequently the family $\varphi=\varphi_0\cup\psi_0$ is $(\mathcal{S}_0^s, \mathcal{I}^s)$-continuous, and because of $\mathcal{S}_0^s<\mathcal{S}$ it is $(\mathcal{S}, \mathcal{I}^s)$-continuous. Now assume $x<V$. Then $x<_0^{sp}V$, that is $x<_0^sV$. With the help of [5], (3.44) it is easy to verify that sets $B, B'\subset E$ can be found such that $x<_0B$, $x<_0^cB'$ and $B\cap B'\subset V$. Because of the choice of φ_0 there are $f, f'\in\varphi_0$ for which $f(E), f'(E)\subset[0, 1]$, $f(x)=0$, $f(E-B)=\{1\}$, further $f'(E-B')=\{0\}$ and $f'(x)=1$. Let us consider the function $g=1-f'$. Then $f, g\in\varphi$, f is preorder preserving, g is preorder inversing, furthermore $f(E), g(E)\subset[0, 1]$ and $f(x)=g(x)=0$. If $y\in E-V$, then $y\in E-B$ implies $f(y)=1$, or $y\in E-B'$ implies $g(y)=1$, so that $\max\{f(y), g(y)\}=1$ is also true.

Conversely, let the condition be satisfied by $\mathcal{S}$, and let Φ denote the ordering structure described in the theorem ($\Phi\neq\emptyset$, since it contains any family φ consisting of constant functions). Every $\varphi\in\Phi$ is $(\mathcal{S}, \mathcal{I}^s)$-continuous, hence so is the family $-\varphi=\{-f: f\in\varphi\}$, too. By (4.9) we have $\mathcal{S}_\Phi^c=\mathcal{S}_{-\varphi}$. Since $\mathcal{I}<\mathcal{I}^s$, these families are $(\mathcal{S}, \mathcal{I})$-continuous, so that $\mathcal{S}_\varphi<\mathcal{S}$ and $\mathcal{S}_{-\varphi}<\mathcal{S}$. From here $\mathcal{S}_\varphi^s\sim\mathcal{S}_\varphi\vee\mathcal{S}_\varphi^c\sim\mathcal{S}_\varphi\vee\mathcal{S}_{-\varphi}<\mathcal{S}$, and $\mathcal{S}_\Phi^s=(\bigvee_{\varphi\in\Phi}\mathcal{S}_\varphi)^s=\bigvee_{\varphi\in\Phi}\mathcal{S}_\varphi^s<\mathcal{S}$ (see [5], (8.102)). By (1.5), (1.1.6) and (2.12.3) $\mathcal{S}_\Phi^s$ is symmetrical and ^{i}-convex.

After this we show that $\mathcal{S}<\mathcal{S}_\Phi^{sp}$. In fact, for a given $<\in\mathcal{S}$, let φ be the functional family mentioned in the condition. For each $\varepsilon>0$ choose an order $<_{(\varepsilon)}\in\mathcal{S}$ with

$$(*) \qquad f^{-1}(<_\varepsilon^s)\subset<_{(\varepsilon)}$$

for each $f\in\varphi$. Denote by ψ the family of all bounded, preorder preserving real functions f, which have the property $(*)$ for every $\varepsilon>0$. Then following the train of thought to be found under the formula (12.38) of [5], p. 168, one can easily verify that ψ is an $(\mathcal{S}, \mathcal{I}^s)$-continuous ordering family, hence $\psi\in\Phi$. Now suppose $x<V$. Then there are $f, g\in\varphi$ such that f (g) is preorder preserving (inversing), $f(x)=g(x)=0$ and $\max\{f(y), g(y)\}=1$ for $y\in E-V$. Lemma (4.9) and [5], (6.8) give that $f'=1-g$ fulfils $(*)$ iff so does g, therefore $f, f'\in\psi$. Put $B=f^{-1}((-\infty, 1))$ and $B'=g^{-1}((-\infty, 1))$. Then $xf^{-1}(<_1)B$ and

$$E-B' = f'^{-1}((-\infty, 0])f'^{-1}(<_1)f'^{-1}((-\infty, 1))\subset E-x,$$

that is $xf'^{-1}(<_1)^cB'$. This shows $x<_{\psi,1}^sB\cap B'\subset V$. From here $<\subset<^p\subset$

$\sqsubset <^{sp}_{\psi,1} \in \mathscr{S}^{sp}_{\psi}$, thus $\mathscr{S} \boldsymbol{<} \bigvee_{\varphi\in\Phi} \mathscr{S}^{sp}_{\varphi} \boldsymbol{<} (\bigvee_{\varphi\in\Phi} \mathscr{S}^{sp}_{\varphi})^p = (\bigvee_{\varphi\in\Phi} \mathscr{S}^{s}_{\varphi})^p = (\bigvee_{\varphi\in\Phi} \mathscr{S}_{\varphi})^{sp} = \mathscr{S}^{sp}_{\Phi}$ (cf. [5], (8.99) and (8.102)).

Further let $\mathscr{S}_0$ be a symmetrical ^{i}-convex syntopogenous structure on $(E, \leqq)$ such that $\mathscr{S}_0 \boldsymbol{<} \mathscr{S}$. Suppose $< \in \mathscr{S}_0^l$, and let φ be an $(\mathscr{S}_0, \mathscr{I})$-continuous ordering family consisting of preorder preserving functions on $(E, \leqq)$, for which $< \sqsubset <_{\varphi,\varepsilon}$ for some $\varepsilon > 0$ (see (1.12) and (1.11)). Since $\mathscr{S}_0$ is symmetrical, φ is $(\mathscr{S}_0, \mathscr{I}^s)$-continuous, and by $\mathscr{S}_0 \boldsymbol{<} \mathscr{S}$ it is $(\mathscr{S}, \mathscr{I}^s)$-continuous. We have $\varphi \in \Phi$, hence $\mathscr{S}_0^l \boldsymbol{<} \mathscr{S}_{\Phi}$. $\mathscr{S}_0$ is ^{i}-convex, thus by (1.8) $\mathscr{S}_0 \sim \mathscr{S}_0^u \vee \mathscr{S}_0^l \sim \mathscr{S}_0^{lc} \vee \mathscr{S}_0^l \sim \mathscr{S}_0^{ls} \boldsymbol{<} \mathscr{S}^s_{\Phi}$, so that $\mathscr{S}_0 \boldsymbol{<} \mathscr{S}^s_{\Phi}$.

Finally, if $\mathscr{S} \sim \mathscr{S}^p$ and if φ is an $(\mathscr{S}, \mathscr{I}^s)$-continuous functional family, then it is $(\mathscr{S}^p, \mathscr{I}^{sp})$-, i.e. $(\mathscr{S}, \mathscr{H})$-continuous. Conversely, if φ is $(\mathscr{S}, \mathscr{H})$-continuous, then by $\mathscr{I}^s \boldsymbol{<} \mathscr{H}$ we get its $(\mathscr{S}, \mathscr{I}^s)$-continuity. ∎

5. Convexity of compact and totally bounded spaces

It is an immediate consequence of (2.5) and (2.6) that if a preordered syntopogenous space $(E, \mathscr{S}, \leqq)$ is (weakly) ^{i}-convex, then $(E, \mathscr{S}^p, \leqq)$ and $(E, \mathscr{S}^b, \leqq)$ are (weakly) ^{p}-convex (see also (3.3) and (4.2)), $(E, \mathscr{S}^t, \leqq)$ is (weakly) ^{i}-convex, consequently $(E, \mathscr{S}^{tp}, \leqq)$ and $(E, \mathscr{S}^{tb}, \leqq)$ are (weakly) ^{p}-convex, too. The inverse statements are not always true.

(5.1) EXAMPLES. On the naturally ordered real line $(\mathbf{R}, \leqq)$ $\mathscr{H} = \mathscr{I}^{sb}$ is ^{p}-convex but it is not weakly ^{i}-convex. $\mathscr{H}^{tp}$ is ^{p}-convex, but $\mathscr{H}^t$ is not weakly ^{i}-convex.

In fact, $\mathscr{I}^s$ is ^{i}-convex by (1.4) and (2.12), thus $\mathscr{H} = \mathscr{I}^{sp}$ is ^{p}-convex. $\mathscr{H}^{tp} = \mathscr{I}^{stp}$ (see [5], p. 89), so that $\mathscr{H}^{tp}$ is ^{p}-convex. Finally put $\mathscr{H}^t = \{<\}$. Then, for the set $\mathbf{N}$ of the natural numbers and for $B = \bigcup_{n\in\mathbf{N}} (n-1/2, n+1/2)$, we have $\mathbf{N} < B$, but $\mathbf{N} \subset \bigcup_{i=1}^{m} C_i \subset B$ (where the sets C_i are convex) is impossible, therefore $\mathscr{H}^t$ (and *a fortiori* $\mathscr{H}$) is not weakly ^{i}-convex by (3.1) and (2.6). ∎

(5.2) THEOREM. *A compact symmetrical preordered syntopogenous space $(E, \mathscr{S}, \leqq)$ is weakly ^{i}-convex, provided $(E, \mathscr{S}^p, \leqq)$ is weakly ^{p}-convex.*

PROOF. Let $<$ be an arbitrary element of $\mathscr{S}$, and $<_0 \in \mathscr{S}$, $< \sqsubset <_0^2$. For $<_0$ let us choose and order $<_1 \in \mathscr{S}$ in accordance with (3.7) and (3.6), further suppose $<_1 \sqsubset <_2^2$, where $<_2 \in \mathscr{S}$. If $A < B$, then $A <_0 V <_0 B$ for some $V \subset E$, thus $x \in V$ implies $x <_1 C_x \subset B$ for a suitable convex set C_x. Let $H_x \subset E$ be defined so that $x <_2 H_x <_2 C_x$. Because of the inequality $E - V <_0 E - A$, from the system $\{H_x, E-A \colon x \in V\}$ we can choose a finite subsystem $\{H_{x_1}, \ldots, H_{x_m}, E-A\}$ which covers E. Then

$$A \subset \bigcup_{i=1}^{m} H_{x_i}, \quad \bigcup_{i=1}^{m} C_{x_i} \subset B, \quad H_{x_i} <_2 C_{x_i} \ (1 \leqq i \leqq m),$$

therefore $(E, \mathscr{S}, \leqq)$ is weakly ^{i}-convex by (3.1). ∎

I can prove the corresponding statement for convex spaces only under an additional condition, and I do not know whether this condition is necessary or not (cf. (1.10)).

(5.3) LEMMA. *Let* $(E, \mathcal{S}, \leqq)$ *be a compact symmetrical preordered syntopogenous space such that* $\mathcal{S}^{up} \sim \mathcal{S}^{pu}$ *and* $\mathcal{S}^{lp} \sim \mathcal{S}^{pl}$. *If* $(E, \mathcal{S}^p, \leqq)$ *is* ^{p}*-convex, then* $(E, \mathcal{S}, \leqq)$ *is* ^{i}*-convex.*

PROOF. $\mathcal{S}^p \sim (\mathcal{S}^{pu} \vee \mathcal{S}^{pl})^p \sim (\mathcal{S}^{up} \vee \mathcal{S}^{lp})^p = (\mathcal{S}^u \vee \mathcal{S}^l)^p$. Since the structure $\mathcal{S}^u \vee \mathcal{S}^l$ is also compact by $\mathcal{S}^u \vee \mathcal{S}^l < \mathcal{S}$, from Lemma 8 of [6] it follows that $\mathcal{S} < \mathcal{S}^u \vee \mathcal{S}^l$, i.e. $\mathcal{S} \sim \mathcal{S}^u \vee \mathcal{S}^l$. ∎

The condition of the lemma is satisfied by any compact symmetrical preordered syntopological space (see (1.7)), but since these spaces are totally bounded ([5], (19.19)), we can state more than (5.3):

(5.4) THEOREM. *For a totally bounded preordered syntopogenous space* $(E, \mathcal{S}, \leqq)$ *the following statements are equivalent:*

(5.3.1) $(E, \mathcal{S}, \leqq)$ *is (weakly)* ^{i}*-convex.*

(5.4.2) $(E, \mathcal{S}, \leqq)$ *is (weakly)* ^{p}*-convex.*

(5.4.3) $(E, \mathcal{S}^t, \leqq)$ *is (weakly)* ^{i}*-convex.*

PROOF. (5.4.1)⇒(5.4.2) by (2.5) and [5], (19.13).

(5.4.2)⇒(5.4.1): Suppose that $(E, \mathcal{S}, \leqq)$ is weakly ^{p}-convex. In view of $\mathcal{S} \sim \mathcal{S}^p$, for $< \in \mathcal{S}$ let us choose an order $<_1 \in \mathcal{S}$ in accordance with (3.2), finally assume $<_1 \subset <_2^2$, $<_2 \in \mathcal{S}$. If $A < B$, then there is a convex set C_x with $x <_1 C_x \subset B$ for every $x \in A$. Suppose $x <_2 H <_2 C_x$. Let $\mathfrak{P}(<_2)$ denote the system of those sets $P \subset E$, for which $X <_2 Y$, $X \cap P \neq \emptyset$ imply $P \subset Y$ (see [5], p. 220). By [5], (19.17) a finite subsystem $\mathfrak{P}'$ of $\mathfrak{P}(<_2)$ covers E. Assume that $P_1, \ldots, P_n$ are the members of $\mathfrak{P}'$ intersecting A. Then, for any index $1 \leqq j \leqq n$, we have $P_j \subset H_{x_j}$, where x_j is a suitable element of A. Thus

$$A \subset \bigcup_{j=1}^{n} P_j, \quad \bigcup_{j=1}^{n} C_{x_j} \subset B \quad \text{and} \quad P_j <_2 C_{x_j} \ (1 \leqq j \leqq n),$$

hence $(E, \mathcal{S}, \leqq)$ is weakly ^{i}-convex by (3.1).

Let $(E, \mathcal{S}, \leqq)$ be ^{p}-convex. Owing to $\mathcal{S}^u \vee \mathcal{S}^l < \mathcal{S}$, the structure $\mathcal{S}^u \vee \mathcal{S}^l$ is also totally bounded, therefore $\mathcal{S} \sim (\mathcal{S}^u \vee \mathcal{S}^l)^p \sim \mathcal{S}^u \vee \mathcal{S}^l$, that is $(E, \mathcal{S}, \leqq)$ is ^{i}-convex (see [5], (19.6) and (19.13)).

(5.4.1)⇒(5.4.3) by (2.6).

(5.4.3)⇒(5.4.1): Let $(E, \mathcal{S}^t, \leqq)$ be weakly ^{i}-convex. Then, for $< \in \mathcal{S}$, there exists a finite system $\mathfrak{S} = \{S_1, \ldots, S_n\}$ of the subsets of E and $<_1 \in \mathcal{S}$ such that $A < B$ implies $A \subset S_i <_1 S_j \subset B$ for suitable sets $S_i, S_j \in \mathfrak{S}$. In this case one can find an order $<_{ij} \in \mathcal{S}$ and a natural number m_{ij}, for which

$$S_i \subset \bigcup_{k=1}^{m_{ij}} X_k^{ij}, \quad \bigcup_{k=1}^{m_{ij}} C_k^{ij} \subset S_j, \quad X_k^{ij} <_{ij} C_k^{ij}$$

and C_k^{ij} is convex $(1 \leqq k \leqq m_{ij})$. Then $<_0$ satisfies the condition given in (3.1), therefore $(E, \mathcal{S}, \leqq)$ is weakly ^{i}-convex.

Let $(E, \mathcal{S}^t, \leqq)$ be ^{i}-convex. Then $\mathcal{S}^t \sim \mathcal{S}^{tu} \vee \mathcal{S}^{tl} \sim \mathcal{S}^{ut} \vee \mathcal{S}^{lt} \sim (\mathcal{S}^u \vee \mathcal{S}^l)^t$ (see (1.9) and [5], (8.101)). Since $\mathcal{S}$ is totally bounded, we have $\mathcal{S} < \mathcal{S}^u \vee \mathcal{S}^l$ by [5], (19.39), that is $\mathcal{S} \sim \mathcal{S}^u \vee \mathcal{S}^l$, hence $(E, \mathcal{S}, \leqq)$ is ^{i}-convex. ∎

(5.5) COROLLARY. *Let* $(E, \mathcal{S}, \leqq)$ *be a compact symmetrical preordered syntopological space. If* $(E, \mathcal{S}^{tp}, \leqq)$ *is weakly* ^{p}*-convex, then* $(E, \mathcal{S}, \leqq)$ *is weakly* ^{i}*-convex.*

PROOF. $(E, \mathcal{S}^t, \leqq)$ is weakly ^{i}-convex by (5.2). $\mathcal{S}$ is totally bounded, therefore $(E, \mathcal{S}, \leqq)$ is also weakly ^{i}-convex. ∎

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NOTE TO "ON BUNDLE-LIKE CONFORM DEFORMATION OF A RIEMANNIAN SUBMERSION"*

P. T. NAGY (Szeged)

The fourth equation of (13) is

$$\tilde{\varphi}_\gamma^\alpha = \varphi_\gamma^\alpha - \frac{1}{2}(\dots)\dots$$

The transformation of the forms φ_γ^α by the change of the bundles $O_M(P)$ and $O_M(P_\varrho)$ is calculated in the formulas (11). Thus instead of (14) we have $\varphi_\gamma^\alpha + \varphi_\alpha^\gamma = 0$ and we get the following correct equations in Theorem 2:

$$\tilde{\varphi}_c^a = \psi_c^a + \frac{1}{2}\exp\left(\frac{1}{2}\varrho\right)\sum_\beta A_\beta{}^a{}_c\theta^\beta,$$

$$\tilde{\varphi}_\gamma^a = \frac{1}{2}\exp\left(\frac{1}{2}\varrho\right)\sum_b A_\gamma{}^a{}_b\theta^b + \frac{1}{2}\sum_\beta (T_\gamma{}^a{}_\beta - \varrho_a\delta_\gamma^\beta)\theta^\beta,$$

$$\tilde{\varphi}_c^a = -\frac{1}{2}\exp\left(\frac{1}{2}\varrho\right)\sum_b A_\alpha{}^c{}_b\theta^b - \frac{1}{2}\sum_\beta (T_\alpha{}^c{}_\beta\varrho_c\delta_\alpha^\beta)\theta^\beta,$$

$$\tilde{\varphi}_\gamma^\alpha = \varphi_\gamma^\alpha + \frac{1}{2}\sum_\beta (\varrho_\gamma\delta_\beta^\alpha - \varrho_\alpha\delta_\beta^\gamma)\theta^\beta.$$

COROLLARY. *The fundamental tensors $\tilde{A}$ and $\tilde{T}$ of the submersion $\{P_\varrho, \pi, M\}$ have the form*

$$\tilde{A}_\gamma{}^a{}_b = \exp\left(\frac{1}{2}\varrho\right)A_\gamma{}^a{}_b, \quad \tilde{T}_\gamma{}^a{}_\beta = T_\gamma{}^a{}_\beta - \varrho_a\delta_\gamma^\beta.$$

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JÓZSEF ATTILA UNIVERSITY
BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

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