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ÜBER LINEAR KOMPAKTE RINGE [1]

Von

DINH VAN HUYNH (Hanoi)

§ 1. Einleitung

Unter einem Ring verstehen wir in dieser Arbeit stets einen assoziativen Ring, unter dem Radikal des Jacobsonsche.

In der Theorie der (nicht kommutativen) Ringe spielen die artinschen Ringe, d. h. Ringe mit Minimalbedingung für Rechtsideale, eine zentrale Rolle. Deshalb versucht man, die Klasse der artinschen Ringe durch gewisse Verallgemeinerung der Minimalbedingung zu erweitern und die möglichst guten Eigenschaften der artinschen Ringe auf die Untersuchung der erweiterten Klasse anzuwenden. Als eine derartige Verallgemeinerung wurde der Begriff „linear kompakte Ringe" eingeführt. Die linear kompakten Ringe wurden von mehreren Autoren weitgehend untersucht (z. B. ZELINSKY [12], [13], LEPTIN [5], [6], WIEGANDT [10], [11], WIDIGER [8]).

In dieser Arbeit betrachten wir zweiseitig im engeren Sinne linear kompakte Ringe (kurz. K^*-Ringe). Als Hauptergebnis zeigen wir, daß jeder K^*-Ring ein maximales streng linear kompaktes Ideal enthält, das sich aus dem Ring algebraisch und topologisch im Sinne einer ringtheoretischen direkten Zerlegung abspalten läßt. Der andere Summand ist ein K^*-Ring mit einem Einselement und enthält kein von Null verschiedenes streng linear kompaktes Rechts- oder Linksideal.

Als Folgerungen des Hauptergebnisses werden in §4 andere auch an sich interessante Ergebnisse über die Struktur der K^*-Ringe angeben, darunter das Hauptergebniss von WIDIGER [8].

§ 2. Vorbereitungen

Unter einem topologischen Ring verstehen wir einen solchen Ring, der zugleich ein Hausdorff-Raum mit stetigen Substraktion und stetiger Multiplikation bezüglich dieser Topologie ist. Ein System $\mathscr{F}=\{F_\nu\}$ von nicht leeren Untermengen F_ν eines topologischen Ringes R heißt ein Filter, wenn es für je zwei F_ν, F_μ aus $\mathscr{F}$ ein F_λ aus $\mathscr{F}$ mit $F_\lambda \subseteqq F_\nu \cap F_\mu$ gibt. Der Filter $\mathscr{F}$ heißt ein Basisfilter, wenn dessen Elemente ein Fundamentalsystem für die Umgebungen des Nullelementtes bilden.

Ein topologischer Ring heißt *rechtslinear* topologisch, wenn er einen Basisfilter aus *Rechtsidealen* besitzt (einen solchen Basisfilter nennen wir kurz Rechtsidealbasisfilter). Ein rechtslinear topologischer Ring R heißt *rechtslinear kompakt,* wenn jeder Filter $\mathscr{F}=\{x_\alpha+F_\alpha\}$ von Restklassen nach abgeschlossenen Rechtsidealen F_α von R einen nicht leeren Durchschnitt hat. Der Kürze halber bezeichnen

[1] Der Verfasser hat die Ergebnisse dieser Arbeit in der mathematischen Tagung in Hanoi im August 1977 bekanntgegeben.

wir solche Ringe als K_r-Ringe. Ein topologischer Ring R heißt streng linear kompakt, wenn R einen Idealbasisfilter besitzt und wenn jeder Filter $\mathscr{F}=\{x_\alpha+U_\alpha\}$ von Restklassen nach abgeschlossenen additiven Untergruppen U_α von R einen nichtleeren Durchschnitt hat. Diese Ringe bezeichnen wir kurz SK-Ringe.

Es sei φ ein stetiger Homomorphismus von einem K_r-Ring R als ein linear kompakter R-Rechtsmodul in einen linear topologischen R-Rechtsmodul M, so ist φ im allgemeinen nicht offen. LEPTIN [5] hat einen K_r-Ring R *im engeren Sinne* rechtslinear kompakt (kurz K_r^*-Ring) genannt, wenn jeder stetige Homomorphismus des R-Rechtsmoduls R in einen beliebigen linear topologischen R-Rechtsmodul M offen ist.

Analog kann man linkslinear kompakte Ringe (oder diese im engeren Sinne) definieren und bezeichnen sie als K_1-Ringe (bzw. K_1^*-Ringe). Ein K^*-Ring ist ein K_r^*-Ring, der auch ein K_1^*-Ring ist.

In dieser Arbeit werden wir die folgenden Bezeichnungen benutzen. Für einen Ring R sei $J(R)$ das Radikal von R. Ist n eine natürliche Zahl, so bezeichnet R_n der Ring aller quadratischen n-reihigen Matrizen mit Elementen aus R. $\boxplus$ sei eine ringtheoretische direkte Zerlegung. Mit Σ bezeichnen wir ringtheoretische komplette direkte Summe. Als Topologie in Σ wählen wir stets die Tychonoffsche, was andere Terminologie betrifft, halten wir an diejenige der Bücher [2] und [4].

§ 3. Struktur der K^*-Ringe

Es sei R ein K^*-Ring. Ist die Topologie in R die diskrete, so ist R nach [5] zweiseitig artinsch, d. h. R genügt der Minimalbedingung für Rechts- und Linksideale. Nach unserem Hauptergebnis aus [7] ist das maximale SK-Ideal von R ein ringtheoretischer direkter Summand von R. Wir betrachten nun den allgemeinen Fall. Es gilt der

SATZ 1. *Es sei R ein K^*-Ring. Dann besitzt R die folgende direkte Zerlegung $R=A\boxplus B$ im algebreischen und topologischen Sinne, wobei A das maximale SK-Ideal von R ist und B als ein K^*-Ring die folgenden Eigenschaften besitzt:*

(a) B enthält kein SK-Rechts- (und Links-) ideal $\neq(0)$,

(b) Ist $B\neq(0)$, so besitzt B ein Einselement,

(c) Es gilt $B/J(B)\cong\sum_\nu S_{m_\nu}^{(\nu)}$ im algebreischen und topologischen Sinne, wobei m_ν natürliche Zahlen und $S^{(\nu)}$ unendliche diskrete Schiefkörper sind.

BEWEIS. Es sei R ein K^*-Ring. Dann besitzt R nach der Definition einen Rechtsidealbasisfilter $\mathscr{F}_r=\{A_\alpha\}$ und einen Linksidealbasisfilter $\mathscr{F}_l=\{B_\lambda\}$. Wir zeigen zuerst, daß R dann einen Idealbasisfilter enthält.

Jedes $A_\alpha\in\mathscr{F}_r$ enthält ein $B_{\alpha_\lambda}\in\mathscr{F}_l$. Folglich ist $I_{\alpha_\lambda}\overset{\text{def}}{=}B_{\alpha_\lambda}+B_{\alpha_\lambda}R$ ein offenes Ideal von R in A_α. Es sei $\mathscr{F}$ die Familie sämtlicher auf dieser Weise gebildeten offenen Ideale I_{α_λ} für jedes α. Dann ist $\mathscr{F}$ offenbar ein Filter mit $\bigcap\mathscr{F}=(0)$. Es ist klar, daß jedes I aus $\mathscr{F}$ die mengentheoretische Vereinigung gewisser $A_\alpha\in\mathscr{F}_r$ ist. Umgekehrt, sei A_α ein festes Element aus $\mathscr{F}_r$. Dann gibt es gewisse B_{α_λ} aus $\mathscr{F}_l$ mit $A_\alpha=\bigcup_\lambda B_{\alpha_\lambda}$. Daher gilt $A_\alpha\supseteqq\bigcup_\lambda(I_{\alpha_\lambda}\overset{\text{def}}{=}B_{\alpha_\lambda}+B_{\alpha_\lambda}R)\supseteqq\bigcup_\lambda B_{\alpha_\lambda}$, d. h. $A_\alpha=\bigcup_\lambda I_{\alpha_\lambda}$ mit

$I_{\alpha_\lambda} \in \mathscr{F}$. Das bedeutet, daß $\mathscr{F}$ einen Basisfilter von R ist. Mit anderen Worten, R besitzt einen Idealbasisfilter. Wir bezeichnen ihn wieder mit $\mathscr{F}=\{I_\alpha\}$.

Wir setzen $\alpha \geqq \beta$, wenn $I_\alpha \subseteqq I_\beta$. Dann ist die Indexmenge von I_α eine gerichtete Menge. Es sei $R_\alpha \overset{\text{def}}{=} R/I_\alpha$ und Π_α der natürliche Homomorphismus von R auf R_α. Dann ist $\Pi^\alpha_\beta \overset{\text{def}}{=} \Pi_\alpha^{-1}\,\Pi_\beta$ ein natürlicher Homomorphismus von R_α auf R_β. Die diskreten Ringe R_α und die stetigen Homomorphismus Π^α_β bilden ein inverses System $[R_\alpha, \Pi^\alpha_\beta]$ (vgl. [13]). Nach [13] gilt

$$R \cong \varprojlim\, [R_\alpha, \Pi^\alpha_\beta] \tag{1}$$

im algebreischen und topologischen Sinne.

Da jedes stetige homomorphe Bild eines K^*-Ringes wieder ein K^*-Ring ist (vgl. [5]), ist jedes R_α zweiseitig artinsch. Nach dem oben erwähnten Ergebnis des Verfassers [7] gilt die direkte Zerlegung

$$R_\alpha = A_\alpha \boxplus B_\alpha, \tag{2}$$

wobei A_α das maximale streng artinsche Ideal (kurz, *SA*-Ideal) von R_α ist (d. h. A_α genügt der Minimalbedingung für seine additive Untergruppen) und B_α die folgenden Eigenschaften besitzt:

(a*) B_α enthält kein *SA*-Rechts- (und Links-) ideal $\neq (0)$,

(b*) Ist $B_\alpha \neq (0)$, so besitzt B_α ein Einselement,

(c*) Es gilt $B_\alpha/J(B_\alpha) = S^{(1)}_m \boxplus \ldots \boxplus S^{(n)}_m$ mit unendlichen Schiefkörpern $S^{(i)}$ und natürlichen Zahlen m_i.

$\Pi^\alpha_\beta(B_\alpha)$ ist ein zweiseitig artinscher Ring und besitzt offenbar die Eigenschaften (b*) und (c*) wie B_α. Daher können wir ähnlich wie in unserer Arbeit [7] schließen, daß $\Pi^\alpha_\beta\,(B_\alpha)$ auch die Eigenschaft (a*) besitzt. Es folgt dann $\Pi^\alpha_\beta(B) \subseteqq B_\beta$. Andererseits gilt $\Pi^\alpha_\beta(A_\alpha) \subseteqq A_\beta$, denn jedes homomorphe Bild eines *SA*-Ringes ist wieder ein *SA*-Ring. Da Π^α_β eine Abbildung von R_α auf R_β ist, gilt dann $\Pi^\alpha_\beta(A_\alpha) = A_\beta$ und $\Pi^\alpha_\beta(B_\alpha) = B_\beta$. Wenn wir mit $\Pi^\alpha_\beta\,|\,A_\alpha$ die Einschränkung von Π^α_β auf A_α und mit $\Pi^\alpha_\beta\,|\,B_\alpha$ die Eingeschränkung von Π^α_β auf B_α bezeichnen, so gilt nach [13] die direkte Zerlegung

$$R \cong \varprojlim\, [R_\alpha, \Pi^\alpha_\beta] \cong \lim\, [A_\alpha, \Pi^\alpha_\beta|A_\alpha] \boxplus \lim\, [B_\alpha, \Pi^\alpha_\beta|B_\alpha] \tag{3}$$

im algebreichen und topologischen Sinne.

Wir setzen $A \overset{\text{def}}{=} \varprojlim\, [A_\alpha, \Pi^\alpha_\beta\,|\,A_\alpha]$ und $B \overset{\text{def}}{=} \varprojlim\, [B_\alpha, \Pi^\alpha_\beta\,|\,B_\alpha]$. Dann besitzt B auch ein Einselement, es gilt also (b). Wegen (3) ist B ein K^*-Ring mit einem Idealbasisfilter. Folglich ist $B/J(B)$ $\big(J(B)$ ist in B abgeschlossen [5]$\big)$ auch ein K^*-Ring mit Idealbasisfilter, ist also die komplette direkte Summe von vollen Matrizenringen über diskreten Schiefkörpern $S^{(\nu)}$ (vgl. Struktursatz für halbeinfache linear kompakte Ringe von LEPTIN in [5]). Wegen der Bildung von B ist es klar, daß jedes $S^{(\nu)}$ unendlich ist. Es gilt also (c). Daß B auch (a) besitzt, sehen wir so ein: Es sei $I'_\alpha \overset{\text{def}}{=} I_\alpha \cap B$. So bilden diese I'_α einen Idealbasisfilter von B. Es sei N ein *SK*-Rechtsideal von B. Angenommen sei $N \subsetneqq I'_\alpha$ für ein I'_α. Dann ist $\overline{N} \overset{\text{def}}{=} N/(N \cap I'_\alpha)$ ein von Null verschiedenes *SA*-Rechtsideal in $B_\alpha \cong B/I'_\alpha$. Das ist ein Widerspruch zu (a*). Daher liegt N in allen I'_α. Folglich ist $N = (0)$. Analog gilt auch für *SK*-Linksideale.

A ist der inverse Limes von SA-Ringen. Analog wie in [13] kann man zeigen, daß A dann ein SK-Ring ist. Wegen (a) ist A ein maximales SK-Ideal von R, das jedes SK-Rechts- und SK-Linksideal von R umfaßt.

Damit ist der Satz bewiesen.

Aus dem Satz 1 ergibt sich, daß man beim Studium der K^*-Ringe auf die Untersuchung der K^*-Ringe mit Einselement und der SK-Ringe beschränken kann. Mit einer Methode von WIEGANDT [11] hat WIDIGER [8] sämtliche SK-Ringe bestimmt. Die Struktur der SK-Ringe ist also im gewissen Maße bekannt.

§ 4. Folgerungen

Es sei R ein K^*-Ring und I ein K^*-Ideal von R (d. h. das Ideal I ist selbst auch ein K^*-Ring). Mit Hilfe des Satzes 1 zeigen wir, daß I in einem „ein wenig größeren" ringtheoretischen direkten Summand I^* von R liegt, d. g. es gilt der

SATZ 2. *Es sei R ein K^*-Ring und I sei ein K^*-Ideal von R. Dann gilt* $R=R^*\boxplus I^*$ *im algebreischen und topologischen Sinne, wobei* $I^*\supseteqq I$ *und der Faktorring* I^*/I *ein SK-Ring ist, und* R^* *die Eigenschaften* (a), (b) *und* (c) *des Satzes* 1 *erfüllt. Enthält I das Radikal von R, so ist* $R^*\cong\Sigma\, S^{(\nu)}_{m_\nu}$ *mit unendlichen diskreten Schiefkörpern* $S^{(\nu)}$.

BEWEIS. Es sei R ein K^*-Ring und I ein K^*-Ideal von R. Nach Satz 1 gilt $R=A\boxplus B$ im algebreischen und topologischen Sinne, wobei A und B diesselbe wie im Satz 1 sind. Entsprechend gilt auch $I=A'\boxplus B'$, wobei A' und B' dasselbe wie A bzw. B im Satz 1 ist. Man kann leicht bestätigen, daß $A\supseteqq A'$ und $B\supseteqq B'$ gilt. Da B' ein Ideal mit Einselement ist, läßt es sich aus B abspalten: $B=B'\boxplus B''$. Diese direkte Zerlegung ist sogar topologisch (vgl. [10]). Nun sei $R^*\overset{\text{def}}{=\!=}B''$, $I^*\overset{\text{def}}{=\!=}B'+A$. Dann gilt $R=R^*\boxplus I^*$ im algebraischen und topologischen Sinne. Daß I^*/I ein SK-Ring ist, ist klar.

Ist $I\supseteqq J(R)$, so gilt $J(R^*)=(0)$. Da R^* die Eigenschaft (c) besitzt, gilt die letzte Behauptung.

Der Satzt ist damit bewiesen.

In [8] hat WIDIGER sämtliche K_r^*-Ringe R mit einem Idealbasisfilter und mit K_r^*-Radikal $J(R)$ charakterisiert. Da jeder solche Ring ein K^*-Ring ist, folgt aus Satz 1 (oder Satz 2) das Hauptergebnis aus [8]:

SATZ 3 (vgl. [8]). *Es sei R ein K_r^*-Ring mit Idealbasisfilter und mit K_r^*-Radikal $J(R)$. Dann gilt*

$$R\cong\sum S^{(\nu)}_{m_\nu}\boxplus A \tag{4}$$

im algebreischen und topologischen Sinne, wobei A ein SK-Ring ist, $S^{(\nu)}$ *unendliche diskrete Schiefkörper und* m_ν *natürliche Zahlen sind. Umgekehrt, jeder Ring mit der Struktur* (4) *ist ein K_r^*-Ring mit einem Idealbasisfilter und dem K_r^*-Radikal $J(R)$.*

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ON THE CONVERGENCE OF FOURIER SERIES

By
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1. Introduction

Our goal is to gain a general formula for the difference $S_n(f; x)-f(x)$ from where we prove some new and older convergence tests for Fourier series.

2. Definitions and preliminary results

Let f be a 2π-periodic integrable function and consider the n-th partial sum of its Fourier series

$$(2.1)\quad S_n(f; x)=\frac{1}{\pi}\int_0^{2\pi} f(x+t)\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\,dt=\frac{a_0}{2}+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx).$$

Define

$$(2.2)\qquad T_n(x; p)=\sum_{k=0}^{n-1}{}' \frac{f\left(x+\frac{k+1}{n}p\right)-f\left(x+\frac{k}{n}p\right)}{k+1}\qquad (n=1, 3, 5, \ldots).$$

Here and later $\sum\limits_{k=0}^{l}{}'$ means that $0\leq k\leq l$ and k is even.

In his paper [2] R. I. SALEM proved that for $f\in\tilde{C}$ ($=f$ is of 2π periodic and continuous on $[0, 2\pi]$), $S_n(f; x)$ uniformly tends to $f(x)$ in $[0, 2\pi]$ whenever $T_n(x; \pi)$ and $T_n(x; -\pi)$ uniformly tends to zero on the same interval, if $n\to\infty$.

The importance of the condition is that by this one can obtain the well-known Dini—Lipschitz and Dirichlet—Jordan tests; further some convergence criteria can be obtained for functions of Φ-bounded variation (see, e.g. [1], IV, § 5).

Now, by generalizing the Salem's criterion, we gain some new convergence tests for Fourier series, and at the same time we obtain new proofs for certain older ones.

3. Results

First we prove

THEOREM 3.1. *Suppose that the expression*

$$(3.1)\qquad n\sum_{k=0}^{n-4}{}' \frac{\int_0^{\pi/n}\left[f\left(x+\frac{k+1}{n}p+\tau\right)-f\left(x+\frac{k}{n}p+\tau\right)\right]d\tau}{k+1}\stackrel{\text{def}}{=} I_n(x; p)$$

for the values $p=\pi$ *and* $p=-\pi$ *uniformly tends to zero whenever* $x\in[0, 2\pi]$ $(n\to\infty)$. *Then*

$$\lim_{n\to\infty} \|S_n(f; x)-f(x)\| = 0 \quad \text{if} \quad f\in\tilde{C}.\text{[1]} \tag{3.2}$$

Here $\|g\| = \max_{0\leqq x\leqq 2\pi} |g(x)|$ *for* $g\in\tilde{C}$.

Using (3.1), we get

THEOREM 3.2. (One sided generalized Dini—Lipschitz test.) *Suppose*

$$\int_0^h [f(x+t)-f(x+t-h)]\,dt \geqq -\varepsilon(h)\frac{h}{|\ln h|} \quad \text{uniformly in} \quad x\in[0, 2\pi] \tag{3.3}$$

for certain $h_0\geqq h>0$ *and* $\varepsilon(h)\geqq 0$ *where* $\lim_{h=0}\varepsilon(h)=0$. *Then* (3.2) *is true; more exactly*

$$\|S_n(f; x)-f(x)\| = O(1)\left[\sum_{k=1}^{n}\omega\left(f; \frac{k}{n}\right)\frac{1}{k^2}+\varepsilon\left(\frac{1}{n}\right)\right] \tag{3.4}$$

for any function $f\in\tilde{C}$.

(Remark that $\Sigma\omega(k\cdot n^{-1})k^{-2}=O(1)\omega(n^{-1}\ln n)$, i.e. it tends to zero; $\omega(f; t)=\omega(t)$ is the modulus of continuity of f.)

The one sided Dini—Lipschitz test (i.e., when $f(x+h)-f(x)\geqq-\varepsilon(h)\,|\ln h|^{-1}$) was proved, using different method, by G. P. NÉVAI [5].

From Theorem 3.2, by

$$\int_0^h [f(x+t)-f(x-t)]\,dt = \int_0^h [f(x+t)-f(x+t-h)]\,dt, \tag{3.5}$$

we immediately obtain the well known

COROLLARY 3.3. (Generalized Dini—Lipschitz test.) *If* $f\in\tilde{C}$ *and*

$$\int_0^h [f(x+t)-f(x-t)]\,dt = o\left(\frac{h}{|\ln h|}\right) \quad \text{uniformly in} \quad [0, 2\pi] \tag{3.6}$$

(if $h\to 0$*) then* (3.2) *holds.*

This statement was proved by R. I. SALEM [3] and (independently) by S. B. STECKIN, later by V. I. CHEREISKAIA [4] (see further [1], IV, § 8).

Another new convergence test, coming from Theorem 3.2, is

COROLLARY 3.4. (Generalized Dirichlet—Jordan test.) *If* $f\in\tilde{C}$ *moreover* $f=f_1-f_2$ *where* $f_i\in\tilde{C}$ *and*

$$\int_0^h [f_i(x+t)-f_i(x+t-h)]\,dt \geqq 0 \quad \text{uniformly in} \quad x\in[0, 2\pi] \quad (i=1, 2), \tag{3.7}$$

then (3.2) *is true.*

[1] For the order of convergence, see (4.8).

DEFINITION. f is of integral Φ-bounded variation on $[a, b]$ (shortly $f\in \mathrm{IV}_\Phi^{[a,b]}$) if for arbitrary $a\leqq x_1<x_2<\ldots<x_s\leqq b$,

$$\sum_i \Phi\left(\left|\frac{1}{h_i}\int_0^{h_i}[f(y_i+t)-f(y_i-t)]\,dt\right|\right) \overset{\text{def}}{=} V_\Phi(\{y_i\}) \tag{3.8}$$

is uniformly bounded. Here $\Phi(x)$ is continuous, monotonically increasing for $x\geqq 0$ and $\Phi(0)=0$; moreover $2y_i=x_i+x_{i+1}$ and $x_{i+1}-x_i=2h_i$.

THEOREM 3.5. *Let* $f\in\tilde{C}$ *and* $f\in \mathrm{IV}_\Phi^{[0,2\pi]}$ *with* $\Phi(x)=x^p$ $(p\geqq 1)$. *Then the relation* (3.2) *holds.*

This part generalizes the end of [1], IV, § 5.

4. Proofs

PROOF OF THEOREM 3.1. As it is well known, for $f\in\tilde{C}$

$$S_n(f;\,x)-f(x) = \frac{1}{\pi}\int_0^\pi [f(x+t)-2f(x)+f(x-t)]\frac{\sin nt}{t}\,dt+ \tag{4.1}$$

$$+O(1)\,\omega\left(\frac{1}{n}\right) \quad \text{uniformly in} \quad x\in[0,\, 2\pi],$$

where $\omega_k(f;\,t)=\omega_k(t)$ stands for the k-th modulus of smoothness of f ([1], I, § 32).

Using the argument applied in [4] we obtain from (4.1)

$$S_n(f;\,x)-f(x) = \frac{1}{4}\int_{2\pi/n}^{\pi}\Phi_x(t)\left(\frac{\sin nt}{t}\right)'dt+O(1)\left[\omega\left(\frac{1}{n}\right)+\sum_{k=1}^{n}\omega_2\left(\frac{k}{n}\right)\frac{1}{k^2}\right], \tag{4.2}$$

where

$$\Phi_x(t) = \int_0^{\pi/n}[f(x+t+\tau)-f(x+t-\tau)-f(x-t+\tau)+f(x-t-\tau)]\,d\tau.\,[2] \tag{4.3}$$

Consider that

$$\left(\frac{\sin nt}{t}\right)' = \frac{n\cos nt}{t}-\frac{\sin nt}{t^2}. \tag{4.4}$$

[2] To get (4.2), we use in [3], (5) $\omega\left(\frac{1}{n}\right)$ instead of ε; at [3], (8) observe that

$$\frac{\pi}{n}\int_{2\pi/n}^{\pi}\left[f\left(x+t\pm\frac{\pi}{n}\right)-2f(x)+f\left(x-t\mp\frac{\pi}{n}\right)\right]\left(\frac{1}{t\pm\frac{\pi}{n}}-\frac{1}{t}\right)\sin nt\,dt = O(1)\sum_{k=1}^{n}\omega_2\left(\frac{k}{n}\right)\frac{1}{k^2};$$

finally (4.3) comes by a simple calculation from the definition of Φ_x and $F(x)=\int_0^x f(s)\,ds$ (see [3] and [1], I, §. 8).

Because of (4.3)

$$(4.5)\qquad \left|\int_{2\pi/n}^{\pi}\Phi_x(t)\frac{\sin nt}{t^2}\,dt\right| = O(1)\frac{\omega\left(\frac{1}{n}\right)}{n}\int_{2\pi/n}^{\pi}\frac{dt}{t^2} = O(1)\,\omega\left(\frac{1}{n}\right),$$

it is enough to estimate the expression

$$(4.6)\qquad n\int_{2\pi/n}^{\pi}\Phi_x(t)\frac{\cos nt}{t}\,dt \overset{\text{def}}{=} I_1.$$

Now, using that

$$(4.7)\qquad \begin{cases} |n\Phi_x(t)| = O(1), \quad \left|n\Phi_x\left(\frac{u+k\pi}{n}\right)\right| = O(1)\,\omega\left(\frac{k+1}{n}\right) \quad (|u|\leqq 3\pi),\\ \left|n\Phi_x\left(\frac{u+k\pi}{n}\right) - n\Phi_x\left(\frac{u+(k+1)\pi}{n}\right)\right| = O(1)\,\omega\left(\frac{1}{n}\right),\end{cases}$$

we have, by analogous arguments as in [1], IV, § 5,

$$(4.8)\qquad I_1 = \frac{1}{\pi}\int_{2\pi}^{3\pi}\left\{n\sum_{k=0}^{n-4}{}'\frac{\Phi_x\left(\frac{u+k\pi}{n}\right)-\Phi_x\left(\frac{u+(k+1)\pi}{n}\right)}{k+1}\right\}\cos u\,du +$$

$$+\left[O(1)\,\omega\left(\frac{1}{n}\right)+\sum_{k=1}^{n}\omega\left(\frac{k}{n}\right)\frac{1}{k^2}\right] \overset{\text{def}}{=} I_2+I_3.$$

By (4.3) we obtain our theorem.

PROOF OF THEOREM 3.2. Using Theorem 3.1, we have to investigate expressions like

$$(4.9)\qquad I_4 \overset{\text{def}}{=} n\sum_{k=0}^{n-4}{}'\frac{\int_0^{\pi/n}\left[f\left(x+\frac{k+1}{n}\pi+\tau\right)-f\left(x+\frac{k}{n}\pi+\tau\right)\right]d\tau}{k+1}.$$

To estimate I_4, we write

$$(4.10)\qquad I_4 = n\sum_{k=0}^{n-4}{}'\frac{\int_0^{\pi/n}\left[f\left(x+\frac{k+1}{n}\pi+\tau\right)-f\left(x+\frac{k}{n}\pi+\tau\right)+\frac{\varepsilon\left(\frac{\pi}{n}\right)}{\ln\frac{n}{\pi}}\right]d\tau}{k+1}-$$

$$-\pi\frac{\varepsilon\left(\frac{\pi}{n}\right)}{\ln\frac{n}{\pi}}\sum_{k=0}^{n-4}{}'\frac{1}{k+1} \overset{\text{def}}{=} I_5+I_6.$$

Here

$$I_6 = O(1)\varepsilon\left(\frac{1}{n}\right). \tag{4.11}$$

As for I_5, by (3.1) we obtain that the numerators in the sum are nonnegative; this is true for the odd values of k, too. So we can write

(4.12)

$$0 \leqq I_5 \leqq n \sum_{k=0}^{n-4} = n\left[\sum_{k=0}^{2} + \sum_{r=1}^{s}\sum_{k=2^r+1}^{2^{r+1}} + \sum_{k=2^s+1}^{n-4}\right] \quad \text{where} \quad 2^s < n-4 \leqq 2^{s+1}.$$

Here

$$n\sum_{k=2^r+1}^{2^{r+1}} \frac{\int_0^{\pi/n}[\dots]\,d\tau}{k+1} \leqq \frac{n}{2^r}\sum \int_0^{\pi/n}[\dots]\,d\tau =$$

$$= \frac{n}{2^r}\int_0^{\pi/n}\left[f\left(x+\frac{2^{r+1}+1}{n}\pi+\tau\right) - f\left(x+\frac{2^r+1}{n}\pi+\tau\right)\right]d\tau + O(1)\frac{\varepsilon(n^{-1})}{\ln n} =$$

$$= O(1)\left[\frac{\omega\left(\frac{2^r}{n}\right)}{2^r} + \frac{\varepsilon(n^{-1})}{\ln n}\right],$$

i.e.

$$|I_5| = O(1)\left[\sum_{r=1}^{s}\frac{\omega\left(\frac{2^r}{n}\right)}{2^r} + \sum_{r=1}^{s}\frac{\varepsilon(n^{-1})}{\ln n}\right] = \tag{4.13}$$

$$= O(1)\left[\sum_{r=1}^{s}\sum_{i=2^r}^{2^{r+1}}\frac{\omega\left(\frac{i}{n}\right)}{i^2} + \varepsilon(n^{-1})\right] = O(1)\left[\sum_{k=1}^{n}\omega\left(\frac{k}{n}\right)\frac{1}{k^2} + \varepsilon(n^{-1})\right].$$

Using similar arguments for the remaining parts, we obtain by (4.1)—(4.13)

$$|S_n(f;\, x) - f(x)| = O(1)\left[\sum_{k=1}^{n}\omega\left(\frac{k}{n}\right)\frac{1}{k^2} + \varepsilon\left(\frac{1}{n}\right)\right], \tag{4.14}$$

which is (3.4).

Proof of Theorem 3.4. The proofs run as in [1], IV, § 5. So we only sketch them. If $p=1$, let the even m be such that $\omega(n^{-1})m \to 0$ and $m \to \infty$ (with n). By (3.1), (3.8), (3.5), $y_k = x + \frac{k+1}{n}\pi$ and $h_i = h$,

$$|I_n(x;\, \pi)| = \left|n\left(\sum_{k=0}^{m-1}{}' + \sum_{k=m}^{n-4}{}'\right)\right| = O(1)\left[\omega\left(\frac{1}{n}\right)\ln m + \frac{V(\{y_k\})}{m+1}\right] = o(1) \quad \text{if} \quad n \to \infty.$$

Now let $p>1$. By $p^{-1}+q^{-1}=1$ we have for the second sum

$$\left|\frac{n}{\pi}\sum_{k=m}^{n-4}{}'\dots\right| \leqq \sum_{k=m}^{n-4}{}' \frac{\left|\frac{n}{\pi}\int\limits_0^{\pi/n}[f(y_k+t)-f(y_k-t)]\,dt\right|}{k+1} \leqq [V(\{y_k\})]^{1/p}\left(\sum_{k=m}^{\infty}\frac{1}{k^q}\right)^{1/q} = o(1),$$

which was stated.

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ALMOST PARACONTACT MANIFOLD WITH AN AFFINE CONNEXION

By
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Recently, SATO [1] defined an almost paracontact structure on a differentiable manifold which may be considered to be analogous to an almost contact structure on one hand and to an almost product structure on the other. In the present paper, we have introduced an affine connexion in an almost paracontact manifold and have studied some of its properties.

1. Introduction

Let M be an n-dimensional differentiable manifold endowed with a $(1, 1)$-tensor field F, a vector field t and a 1-form A such that the conditions

$$A(t) = 1, \tag{1.1}$$

and

$$F^2 = I - A\otimes t \tag{1.2}$$

are satisfied. Then M is called an almost paracontact manifold. From (1.1) and (1.2) we have

$$F(t) = 0, \quad A(\overline{X}) = 0, \quad \text{where} \quad \overline{X} \overset{\text{def}}{=} F(X),$$

moreover $\operatorname{rank}(F) = n-1$.

2. Affine connexion

We introduce an affine connexion D in an almost paracontact manifold satisfying

$$(D_X F)(Y) = 0, \tag{2.1}$$

and call it an F-connexion. Then we have

$$D_X\overline{Y} = \overline{D_X Y}, \tag{2.2}$$

further

$$A(Y)D_X t + (D_X A)(Y)t = 0. \tag{2.3}$$

Note. The equation (2.1) is the condition for an almost product metric manifold to be an almost product almost decomposable manifold, where D is the Riemannian connexion [4]. The following relations can easily be obtained:

THEOREM (2.1). *In the almost paracontact manifold we have*

$$(2.4)\quad \begin{cases} \text{(a)} & A(D_X\bar{Y}) = -(D_X A)(\bar{Y}) = 0,\\ \text{(b)} & \overline{D_X t} = 0,\\ \text{(c)} & D_{\bar{X}} Y = \overline{D_{\bar{X}}\bar{Y}},\\ \text{(d)} & D_X t = A(D_X t)t,\\ \text{(e)} & (D_X A)(Y) - A(Y)(D_X A)(t) = 0. \end{cases}$$

We observe that when (2.1) and (2.2) hold, (2.3) also holds but the converse is not necessarily true. In view of this, we have

THEOREM (2.2). *In an almost paracontact manifold let D be an arbitrary connexion satisfying the equation* (2.3). *Then the connexion* **B** *defined by*

$$(2.5)\quad B_X Y = \alpha(D_X Y + \overline{D_X\bar{Y}}) + \beta(D_{\bar{X}} Y + \overline{D_{\bar{X}}\bar{Y}}) + \gamma(D_X\bar{Y} + \overline{D_X Y}) + \delta(D_{\bar{X}}\bar{Y} + \overline{D_{\bar{X}} Y}),$$

is the most general F-connexion of this type.

PROOF. Let us define

$$(2.6)\quad B_X Y = \alpha D_X Y + \beta D_{\bar{X}} Y + \gamma D_X\bar{Y} + \delta D_{\bar{X}}\bar{Y} + \theta\overline{D_X Y} + \varphi\overline{D_{\bar{X}} Y} + \varrho\overline{D_X\bar{Y}} + \sigma\overline{D_{\bar{X}}\bar{Y}}.$$

Barring (2.6) and using (1.2) and (2.4) (a), we get

$$(2.7)\quad \overline{B_X Y} = \alpha\overline{D_X Y} + \beta\overline{D_{\bar{X}} Y} + \gamma\overline{D_X\bar{Y}} + \delta\overline{D_{\bar{X}}\bar{Y}} +$$
$$+\theta D_X Y - \theta A(D_X Y)t + \varphi D_{\bar{X}} Y - \varphi A(D_{\bar{X}} Y)t + \varrho D_X\bar{Y} + \sigma D_{\bar{X}}\bar{Y}.$$

Again, barring Y in (2.6) and using (1.2), (2.3) and (2.4) (b), we get

$$(2.8)\quad B_X\bar{Y} = \alpha D_X\bar{Y} + \beta D_{\bar{X}}\bar{Y} + \gamma D_X Y + \delta D_{\bar{X}} Y + \theta\overline{D_X\bar{\bar{Y}}} + \varphi\overline{D_{\bar{X}}\bar{\bar{Y}}} +$$
$$+\varrho\overline{D_X Y} + \sigma\overline{D_{\bar{X}} Y} - \{\gamma A(D_X Y) + \delta A(D_{\bar{X}} Y)\}t.$$

Subtracting (2.7) from (2.8) we have

$$(B_X F)(Y) =$$
$$= (\alpha-\varrho)(D_X F)(Y) + (\beta-\sigma)(D_{\bar{X}} F)(Y) + (\theta-\gamma)\overline{(D_X F)(Y)} + (\varphi-\delta)\overline{(D_{\bar{X}} F)(Y)}.$$

As D is an arbitrary connexion satisfying (2.3), the necessary and sufficient condition for B to be an F-connexion is

$$\alpha = \varrho,\quad \beta = \sigma,\quad \theta = \gamma\quad \text{and}\quad \varphi = \delta.$$

Substituting these in (2.6) we get (2.5).
This proves the theorem.

3. Curvature tensor

Let the curvature tensor K of the connexion D be given by

(3.1) $$K(X,Y,Z) \overset{\text{def}}{=} D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z.$$

THEOREM (3.1). *In an almost paracontact manifold the curvature tensor of the affine connexion D satisfies the following identities*

(3.2) $$\begin{cases} \text{(a)} & K(X,Y,\bar{Z}) = \overline{K(X,Y,Z)}, \\ \text{(b)} & \overline{K(X,Y,t)} = 0 = A(K(X,Y,\bar{Z})), \\ \text{(c)} & A(Z)A(K(X,Y,t)) = A(K(X,Y,Z)), \\ \text{(d)} & K(X,Y,t) = A(K(X,Y,t))t, \\ \text{(e)} & K(X,Y,t)A(D_Z t) = (D_Z t)A(K(X,Y,t)), \\ \text{(f)} & K(X,Y,\bar{Z}) + K(Y,Z,\bar{X}) + K(Z,X,\bar{Y}) = 0, \\ \text{(g)} & K(t,Y,\bar{Z}) = K(t,Z,\bar{Y}), \\ \text{(h)} & (D_X t)A(K(Y,Z,t)) + (D_Y t)A(K(Z,X,t)) + (D_Z t)A(K(X,Y,t)) + \\ & + \{(D_X A)(K(Y,Z,t)) + (D_Y A)(K(Z,X,t) + (D_Z A)(K(X,Y,t))\} t = 0, \\ \text{(i)} & A(D_X t)A(K(Y,Z,t)) + A(D_Y t)A(K(Z,X,t)) + A(D_Z t)A(K(X,Y,t)) + \\ & + (D_X A)(K(Y,Z,t)) + (D_Y A)(K(Z,X,t)) + (D_Z A)(K(X,Y,t)) = 0. \end{cases}$$

PROOF. The proof of (3.2) (a), (b), (c), (d), (e), (f) and (g) is straightforward. For (3.2) (h), we have from (3.2) (d)

$$(D_Z K)(X,Y,t) = (D_Z t)A(K(X,Y,t)) + (D_Z A)(K(X,Y,t))t + A((D_Z K)(X,Y,t))t.$$

By cyclic permutation of X, Y, Z; adding the three equations and using Bianchi's second identities and (3.2) (d), we get (3.2) (h). (3.2) (i) follows from (3.2) (h) and (2.4) (d).

Following MISHRA [2] an n-dimensional manifold is said to be

(i) (1)-recurrent if

(3.3) $$(D_X K)(\bar{Y},Z,U) + K((D_X F)(Y),Z,U) = \beta(X)K(\bar{Y},Z,U),$$

(ii) (12)-recurrent if

(3.4) $$(D_X K)(\bar{Y},\bar{Z},U) + K((D_X F)(Y),\bar{Z},U) + K(\bar{Y},(D_X F)(Z),U) = \beta(X)K(\bar{Y},\bar{Z},U),$$

(iii) (13)-recurrent if

$$(3.5)\ (D_X K)(\bar{Y}, Z, \bar{U}) + K((D_X F)(Y), Z, \bar{U}) + K(\bar{Y}, \bar{Z}, (D_X F)(Z)) = \beta(X) K(\bar{Y}, \bar{Z}, U),$$

(iv) (4)-recurrent if

$$(3.6)\qquad \overline{(D_X K)(Y, Z, U)} + (D_X F)(K(Y, Z, U)) = \beta(X)\overline{K(Y, Z, U)},$$

where D is a symmetric connexion and β is a 1-form.

Keeping in view the symmetry and the skew-symmetry of K we can similarly define other recurrences as well.

From these definitions the following theorems are easily obtained.

THEOREM (3.2). *Let the F-connexion* **D** *be symmetric. Then the necessary and sufficient condition for an almost paracontact manifold to be* (1)*-recurrent is*

$$(3.7)\qquad (D_X K)(\bar{Y}, Z, U) = \beta(X) K(\bar{Y}, Z, U),$$

equivalently

$$(D_X K)(Y, Z, U) - \beta(X) K(Y, Z, U) = A(Y)\{(D_X K)(t, Z, U) - \beta(X) K(t, Z, U)\}.$$

THEOREM (3.3). *Let the F-connexion D be symmetric. Then the recurrent almost paracontact manifold is* (1)*-recurrent, but the converse is not necessarily true, where* (1) *stands for any one of the cases of recurrence.*

4. Nijenhuis tensor

The Nijenhuis tensor N with respect to F is defined by

$$N(X, Y) \stackrel{\text{def}}{=} [\bar{X}, \bar{Y}] + \overline{\overline{[X, Y]}} - \overline{[X, \bar{Y}]} - \overline{[\bar{X}, Y]}$$

or equivalently

$$N(X, Y) = (D_{\bar{X}} F)(Y) - D_{\bar{Y}} F)(X) - \overline{(D_X F)(Y)} + \overline{(D_Y F)(X)}.$$

Then we have

THEOREM (4.1). *In an almost paracontact manifold the Nijenhuis tensor vanishes.*

In an almost paracontact manifold [1] three other tensors can be formed. These are

$$P(X, Y) = (D_{\bar{X}} A)(Y) - (D_{\bar{Y}} A)(X) + (D_X A)(\bar{Y}) - (D_Y A)(\bar{X})$$

$$Q(X) = D_{\bar{X}} t + (D_X F)(t) - (D_t F)(X),$$

$$R(X) = (D_X A)(t) - (D_t A)(X).$$

For these tensors the following identities can easily be obtained.

THEOREM (4.2). *In the almost paracontact manifold we have*

(a) $P(X, Y) = (D_X A)(Y) - (D_Y A)(X) = A(X)A(D_Y t) - A(Y)A(D_X t) =$
$= A(X)A(Q(Y)) - A(Y)A(Q(X)),$

(b) $P(\bar{X}, Y) + P(X, \bar{Y}) = A(X)A(D_Y t) - A(Y)A(D_X t),$

(c) $P(\bar{X}, \bar{Y}) = 0,$

(d) $P(\bar{X}, t) = -A(Q(X)) = R(\bar{X}),$

(e) $\overline{Q(X)} = 0,$

(f) $Q(t) = R(t) = 0.$

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SOME INEQUALITIES FOR ERGODIC POWER FUNCTIONS

By
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Given a measure preserving transformation T acting on a probability space $(X, \mathcal{B}, m)$, let $f_0=0$, $f_n(x)=\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x)$ $(n\geqq 1)$ for $f\in L_1(X, \mathcal{B}, m)$. Let $d_0=f_0$, $d_1=f_1-f_0$, $d_2=f_2-f_1, \ldots$, so that $f_n=\sum_{k=0}^{n} d_k$ for $n\geqq 0$. Then the square function Sf, for such T and f, is defined by $(Sf)(x)=\left[\sum_{n=0}^{\infty}|d_n(x)|^2\right]^{1/2}$. This square function Sf, arisen from the study of certain martingales, is an obvious generalization of the martingale square function, considering that the sequence $(f_0, f_1, \ldots)$ defined as above forms a martingale* in the special case that T is an independent shift.

R. L. Jones has shown in his recent paper [4] that the operator S is of weak type (1, 1), namely, for any $\lambda>0$ and every $f\in L_1(X, \mathcal{B}, m)$ with $\|f\|_1<\lambda$, there holds $m\{Sf>\lambda\}\leqq\frac{C}{\lambda}\|f\|_1$, where C is a constant independent of f and λ. Moreover since $\|Sf\|_\infty\leqq K\|f\|_\infty$ for $f\in L_\infty(X, \mathcal{B}, m)$ (K is constant), it follows immediately from the Marcinkiewicz's interpolation theorem that S is of strong type (p, p) with $1<p<\infty$, that is, $\|Sf\|_p\leqq C_p\|f\|_p$ for $f\in L_p(X, \mathcal{B}, m)$, where p is a constant depending only on p.

In this paper we introduce the ergodic power function including the above-mentioned square function in more general operator theoretic setting. Then the above facts on S are extended and generalized to ergodic power functions. Incidentically, our proofs can be used to study the ergodic Hilbert transform and the ergodic theorems.

In what follows we consider a σ-finite measure space $(X, \mathcal{B}, m)$ and a reflexive Banach space $(\mathcal{X}, |||\ |||)$. Equalities and inequalities are meant in the almost everywhere sense from now on. Let $L_p(X; \mathcal{X})=L_p(X, \mathcal{B}, m, \mathcal{X})$, $1\leqq p\leqq\infty$ denote the usual Banach spaces of strongly measurable $\mathcal{X}$-valued functions f defined on X. We denote by $L_p(X; \mathcal{X})+L_\infty(X; \mathcal{X})$ the class of all functions f such that $f=g+h$, $g\in L_p(X; \mathcal{X})$, $h\in L_\infty(X; \mathcal{X})$. An operator T defined on $L_p(X; \mathcal{X})+L_\infty(X; \mathcal{X})$ is called L_∞-bounded if $\|Tf\|_\infty\leqq K\|f\|_\infty$ for $f\in L_\infty(X; \mathcal{X})$, where K is a constant with $K\geqq 1$. Let $L^p(X; \mathcal{X})[\log^+ L(X; \mathcal{X})]^\alpha$ $(1\leqq p<\infty, 0\leqq\alpha<\infty)$ denote the class

* Recall that a martingale is a sequence of integrable functions $f=(f_0, f_1, \ldots)$ such that $E(f_n|f_0, f_1, \ldots, f_{n-1})=f_{n-1}$ a.e. for $n\geqq 1$. The martingale square function $S_M f$ is then defined by

$$(S_M f)(x)=\left[\sum_{n=0}^{\infty}(f_{n+1}(x)-f_n(x))^2\right]^{1/2}.$$

of all functions f for which

$$\int_X |||f(x)|||^p [\log^+ |||f(x)|||]^\alpha \, dm(x) < \infty,$$

where $\log^+ u = \log \max (u, 1)$ for $u \geqq 0$. In our consideration the basic setting is the function class $M_p^\alpha(X; \mathscr{X})$ $(1 \leqq p < \infty, 0 \leqq \alpha < \infty)$ which is defined as that consisting of all functions f such that

$$\int_{\{|||f|||>t\}} \left(\frac{|||f(x)|||}{t}\right)^p \left[\log^+ \frac{|||f(x)|||}{t}\right]^\alpha dm(x) < \infty$$

for every $t>0$. If $\mathscr{X}$ is the linear space of real or complex numbers we remove $\mathscr{X}$ from the notations $L_p(X; \mathscr{X})$, $L^p(X; \mathscr{X})[\log^+ L(X; \mathscr{X})]^\alpha$ and $M_p^\alpha(X; \mathscr{X})$. Concerning the properties of $M_p^\alpha(X; \mathscr{X})$ we have

THEOREM 1. *Let* $1 \leqq p, q < \infty$ *and* $0 \leqq \alpha, \beta < \infty$. *Then*

(1) $M_p^\alpha(X; \mathscr{X})$ *is a linear space.*

(2) $L_p(X; \mathscr{X}) \subset M_p^0(X; \mathscr{X}) \subset L_p(X; \mathscr{X}) + L_\infty(X; \mathscr{X})$.

(3) $M_p^\alpha(X; \mathscr{X}) \subset M_p^\beta(X; \mathscr{X})$ *if* $\beta < \alpha$.

(4) $M_p^\alpha(X; \mathscr{X}) \subset M_q^\alpha(X; \mathscr{X})$ *if* $q < p$.

(5) $L_q(X; \mathscr{X}) \subset M_p^\alpha(X; \mathscr{X})$ *if* $p < q$.

(6) $M_p^\alpha(X; \mathscr{X}) \subset L^p(X; \mathscr{X})[\log^+ L(X; \mathscr{X})]^\alpha \subset L_p(X; \mathscr{X}) + L_\infty(X; \mathscr{X})$.

(7) $M_p^\alpha(X; \mathscr{X}) = L^p(X; \mathscr{X})[\log^+ L(X; \mathscr{X})]^\alpha$ *if and only if* $m(X) < \infty$.

(8) *The linear span of* $\bigcup_{q>p} L_q(X; \mathscr{X}) \subset M_p^\alpha(X; \mathscr{X})$.

PROOF. Simple calculation. (Cf. [5], [6], [7], [8].)

Now let T be a linear operator on $L_1(X; \mathscr{X}) + L_\infty(X; \mathscr{X})$ with $\|T\|_1 \leqq 1$ and $\sup \{\|T^n\|_\infty : n \geqq 0\} \leqq K$ for some constant $K \geqq 1$. It follows from the Riesz's convexity theorem that $\sup \{\|T^n\|_p : n \geqq 0\} \leqq K$ for $1 \leqq p \leqq \infty$. Let $1 < p < \infty$ and $f \in L_1(X; \mathscr{X}) + L_\infty(X; \mathscr{X})$. For such T and f the ergodic pth power function $H^p f$ is defined by

$$(H^p f)(x) = \left[\sum_{n=0}^{\infty} |||d_n(x)|||^p\right]^{1/p}$$

which may be finite or infinite, where $\{d_n : n \geqq 0\}$ is the difference sequence of f_n, $n \geqq 0$, given by $f_0 = 0$ and $f_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k f(x)$ for $n \geqq 1$.

THEOREM 2. A. *For* $1 < p < \infty$ *the operator* H^p *is an* L_∞*-bounded sublinear operator of strong type* (p, p):

(i) *If* $f \in L_p(X; \mathscr{X})$ *there exists a constant* C *depending only on* p *such that* $H^p f \in L_p(X)$ *and* $\|H^p f\|_p \leqq C \|f\|_p$.

(ii) *If* $f \in L_\infty(X; \mathscr{X})$ *there exists a constant* $M \geqq 1$ *such that* $\|H^p f\|_\infty \leqq M \|f\|_\infty$.

(iii) *For* $f, g \in L_1(X; \mathscr{X})+L_\infty(X; \mathscr{X})$ *there hold* $|H^p(f+g)| \leqq |H^p f|+|H^p g|$ *and* $|H^p(\alpha f)|=|\alpha|\,|H^p f|$.

B. *Let* $f \in M_p^0(X; \mathscr{X})$, $1<p<\infty$. *Then with* M *appearing in* (ii)

$$m\{H^p f > 2Mt\} \leqq \frac{C}{t^p} \int_{|||f|||>t} |||f(x)|||^p \, dm(x)$$

for every $t>0$, *where* C *is a constant independent of* f *and* t.

PROOF. A—(i): Using the Minkowski's inequality we have

$$H^p f(x) = \left[\sum_{n=0}^{\infty} \left|\left|\left| \frac{T^n f(x)}{n+1} - \frac{f_n(x)}{n+1} \right|\right|\right|^p\right]^{1/p} \leqq \left[\sum_{n=0}^{\infty} \left|\left|\left| \frac{T^n f(x)}{n+1} \right|\right|\right|^p\right]^{1/p} + \left[\sum_{n=0}^{\infty} \left|\left|\left| \frac{f_n(x)}{n+1} \right|\right|\right|^p\right]^{1/p}$$

for almost all $x \in X$. Let us put for $n \geqq 0$

$$\xi_n(x) = \left|\left|\left| \frac{T^n f(x)}{n+1} \right|\right|\right|^p, \quad \eta_n(x) = \left|\left|\left| \frac{f_n(x)}{n+1} \right|\right|\right|^p$$

and

$$f^*(x) = \sup\{|||f_n(x)|||: n \geqq 0\}.$$

Obviously it follows that

$$\sum_{n=0}^{\infty} \|\xi_n\|_1 \leqq K^p \|f\|_p^p \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right)^p < \infty, \quad \sum_{n=0}^{\infty} \|\eta_n\|_1 \leqq \frac{(2K)^p p}{p-1} \|f\|_p^p \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right)^p < \infty$$

since by the dominated ergodic theorem ([6], Theorem 1)

$$\|f^*\|_p^p \leqq \frac{(2K)^p p}{p-1} \|f\|_p^p.$$

Accordingly, both $\sum_{n=0}^{\infty} \xi_n$ and $\sum_{n=0}^{\infty} \eta_n$ converge almost everywhere, belong to $L_1(X)$ and

$$\left\| \sum_{n=0}^{\infty} \xi_n \right\|_1 \leqq \sum_{n=0}^{\infty} \|\xi_n\|_1, \quad \left\| \sum_{n=0}^{\infty} \eta_n \right\|_1 \leqq \sum_{n=0}^{\infty} \|\eta_n\|_1.$$

This fact is enough to establish the desired one.

A—(ii): Let $f \in L_\infty(X; \mathscr{X})$. Then excepting a set of measure zero

$$\|H^p f\|_\infty \leqq 2K \|f\|_\infty \left[\sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right)^p\right]^{1/p} = M \|f\|_\infty.$$

A—(iii): Clear from the definition of H^p.

B: If the right hand side of the inequality appearing in **B** is infinite then the assertion holds trivially. So we may consider the case that $|||f|||$ is integrable over the set where $|||f(x)|||>t$. Define

$$f^{(t)} = f\chi_{\{|||f|||>t\}}, \quad f_{(t)} = f\chi_{\{|||f|||\leqq t\}}$$

for $t>0$, where χ_A indicates the characteristic function of the set A. Clearly $f=f^{(t)}+f_{(t)}$ and $H^p f \leqq H^p f^{(t)} + Mt$ by **A**—(iii).

Noting that the operator H^p is of weak type (p, p) on account of A—(i), we get with $f^{(t)}$

$$m\{H^p f > 2Mt\} \leqq m\{H^p f^{(t)} > Mt\} \leqq \frac{C}{t^p} \int_X |||f^{(t)}(x)|||^p\, dm(x) =$$
$$= \frac{C}{t^p} \int_{|||f|||>t} |||f(x)|||^p\, dm(x)$$

as required. Hence the proof of Theorem 2 is complete.

REMARK. Some obvious facts from the proof of Theorem 2: (I) Let $f \in L_p(X; \mathscr{X})$, $1 \leqq p < \infty$, and $\alpha > 1/p$. Then (i) $\sum_{n=0}^{\infty} \left|\left|\left| \frac{T^n f}{(n+1)^\alpha} \right|\right|\right|^p < \infty$ a.e. (ii) $\sum_{n=0}^{\infty} \left|\left|\left| \frac{f_n}{(n+1)^\alpha} \right|\right|\right|^p < \infty$ a.e. (iii) $\sum_{n=0}^{\infty} |||f_{n+1} - f_n|||^p < \infty$ a.e. if $p > 1$. Especially (iii) allows, for example, the consideration of the ergodic theorem of Chacon's type. (II) Let $f \in L_p(X; \mathscr{X})$, $1 < p < \infty$. Then $\lim_{n\to\infty} \frac{1}{(n+1)^\alpha} \sum_{k=0}^{n} \beta_k T^k f = 0$ a.e. for $\{\beta_n\} \subset l^q$ with $1 < q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

We remark that if $\mathscr{X}$ is the complex linear space and T is positive then Theorem 2 is also valid with the condition $\|T\|_p \leqq 1$ for $1 < p < \infty$ in place of $\|T\|_1 \leqq 1$, using the AKCOGLU's dominated ergodic theorem [1].

The following theorem is a further generalization of both Theorem 3.1 of JONES [4] and Theorem 3.1 of BURKHOLDER [2].

THEOREM 3. *Assume that there exists a constant $L \geqq 1$ such that for every $f \in L_1(X; \mathscr{X})$ and any $\beta > 0$*

$$m\{|||T^n f(x)||| > \beta\} \leqq Lm\{|||f(x)||| > \beta\}^*, \quad n \geqq 0.$$

Then for $1 < p < \infty$ the operator H^p is of weak type (1, 1). *That is, for every $f \in L_1(X; \mathscr{X})$ and any $\lambda > 0$, there exists a constant C independent of f and λ, such that*

$$m\{H^p f > \lambda K\} \leqq \frac{C}{\lambda} \|f\|_1.$$

PROOF. Put $E_n = \{x: |||T^n f(x)||| \leqq \lambda(n+1)\}$ for $n \geqq 0$, and denote the characteristic functions of E_n and its complement E_n^c by χ_n and $\bar{\chi}_n$ respectively. We set

$$I(x) = \left[\sum_{n=0}^{\infty} \left|\left|\left| \frac{f_n(x)}{n+1} \right|\right|\right|^p\right]^{1/p}$$

$$II(x) = \left[\sum_{n=0}^{\infty} \left|\left|\left| \frac{\chi_n T^n f(x)}{n+1} \right|\right|\right|^p\right]^{1/p}$$

$$III(x) = \left[\sum_{n=0}^{\infty} \left|\left|\left| \frac{\bar{\chi}_n T^n f(x)}{n+1} \right|\right|\right|^p\right]^{1/p}$$

* This condition holds trivially whenever T is induced by a measure preserving transformation. The method used here can also be useful to study the related problem for non-singular transformations.

and estimate the distribution functions for I, II and III since $H^p f \leqq I+II+III$ by the Minkowski's inequality. The maximal ergodic lemma ([6], Lemma 1) shows that

$$m\{f^* > \lambda K\} \leqq \frac{2}{\lambda} \int_{|||f|||>\lambda/2} |||f(x)|||\, dm(x) \leqq \frac{2}{\lambda} \int_X |||f(x)|||\, dm(x) = \frac{2}{\lambda} \|f\|_1,$$

so that as to I we obtain

$$m\left\{I > \frac{\lambda K}{3}\right\} \leqq \frac{6}{\lambda} \left(\frac{2p-1}{p-1}\right)^{1/p} \|f\|_1.$$

The estimate for II is as follows:

$$m\left\{II > \frac{\lambda K}{3}\right\} \leqq \left(\frac{3}{\lambda K}\right)^p \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right)^p \int_X |||\chi_n T^n f(x)|||^p\, dm(x) =$$

$$= p\left(\frac{3}{\lambda K}\right)^p \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right)^p \int_0^{\infty} \alpha^{p-1} m\{|||\chi_n T^n f(x)||| > \alpha\}\, d\alpha =$$

$$= p\left(\frac{3}{\lambda K}\right)^p \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right)^p \int_0^{\lambda(n+1)} \alpha^{p-1} m\{|||T^n f(x)||| > \alpha\}\, d\alpha \leqq$$

$$\leqq pL\left(\frac{3}{\lambda K}\right)^p \int_0^{\infty} \alpha^{p-1} \sum_{n=[\alpha/\lambda]}^{\infty} \left(\frac{1}{n+1}\right)^p m\{|||f(x)||| > \alpha\}\, d\alpha \leqq$$

$$\leqq pL\left(\frac{3}{\lambda K}\right)^p \frac{2p-1}{p-1} \int_0^{\infty} \alpha^{p-1} \left(\frac{\lambda}{\alpha}\right)^{p-1} m\{|||f(x)||| > \alpha\}\, d\alpha =$$

$$= \frac{3^p pL}{\lambda K^p} \frac{2p-1}{p-1} \int_0^{\infty} m\{|||f(x)||| > \alpha\}\, d\alpha = \frac{3^p pL}{\lambda K^p} \frac{2p-1}{p-1} \|f\|_1,$$

where $[u]$ is the integral part of u. As for the estimate of III we have

$$m\left\{III > \frac{\lambda K}{3}\right\} \leqq \sum_{n=0}^{\infty} m\{|||\bar{\chi}_n T^n f(x)||| > 0\} \leqq L \sum_{n=0}^{\infty} m\{|||f(x)||| > \lambda(n+1)\} \leqq$$

$$\leqq L \int_0^{\infty} m\{|||f(x)||| > \lambda\alpha\}\, d\alpha = \frac{L}{\lambda} \int_0^{\infty} m\{|||f(x)||| > \alpha\}\, d\alpha = \frac{L}{\lambda} \|f\|_1.$$

Consequently, combining the above three parts establishes our proposed task.

The following theorem is an extension of Theorem 3.2 of JONES [4].

THEOREM 4. *On the hypothesis of Theorem* 3, *let* $1<p<\infty$ *and* $1<q\leqq\infty$. *Then for every* $f\in L_q(X; \mathfrak{X})$ *there exists a constant* C *independent of* f *such that* $\|H^p f\|_q \leqq C\|f\|_q$. *Namely, the operator* H^p *is of strong type* (q, q).

PROOF. According to Theorem 2 and Theorem 3, the sublinear operator H^p is simultaneously of weak types $(1, 1)$ and (∞, ∞). Therefore we may apply the Marcinkiewicz's interpolation theorem to obtain the desired result.

Here it is worth while to note that the ergodic theorem of Chacon's type can be deduced from our results. Unfortunately, as of now, we cannot say anything about the question whether Theorem 3 holds or not without assuming the condition on the distribution function of $T^n f$ unless T is induced by a measure preserving transformation.

REMARK. If we suppose $\|T\|_1<1$ instead of both $\|T\|_1\leqq 1$ and $\sup\{\|T^n\|_\infty : n\geqq 0\}\leqq K$, then for $f\in L_1(X; \mathfrak{X})$ the series $\sum_{n=0}^{\infty}\frac{T^n f(x)}{n+1}$ and $\sum_{n=0}^{\infty}\frac{f_n(x)}{n+1}$ converge strongly in $\mathfrak{X}$ for almost all $x\in X$. In fact, we have

$$\int_X \sum_{n=0}^{\infty}\left\|\frac{T^n f(x)}{n+1}\right\| dm(x) = \sum_{n=0}^{\infty}\frac{1}{n+1}\|T^n f\|_1 \leqq \|f\|_1 \sum_{n=0}^{\infty}\|T^n\|_1 = \frac{\|f\|_1}{1-\|T\|_1} < \infty$$

and

$$\int_X \sum_{n=0}^{\infty}\left\|\frac{f_n(x)}{n+1}\right\| dm(x) \leqq \sum_{n=1}^{\infty}\frac{1}{n(n+1)}\sum_{k=0}^{n}\|T^k f\|_1 \leqq \sum_{n=1}^{\infty}\frac{\|f\|_1}{n(n+1)}\sum_{k=0}^{n}\|T\|_1^k =$$

$$= \sum_{n=1}^{\infty}\frac{\|f\|_1}{n(n+1)}\,\frac{1-\|T\|_1^{n+1}}{1-\|T\|_1} \leqq \frac{\|f\|_1}{1-\|T\|_1}\left\{3+\frac{\|T\|_1^2}{1-\|T\|_1}\right\} < \infty.$$

From this we see that the operator H^1 is of strong type (1, 1) (and hence of weak type (1, 1)).

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VERALLGEMEINERUNG EINES SATZES ÜBER GLEICHMÄßIGE APPROXIMATION IN EINEM UNENDLICHEN INTERVALL

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§ 1. Einleitung

Unter den Verallgemeinerungen bzw. den Analoga der Bernstein-Polynome ([3]) nehmen die Reihen

$$S_n(f;\, x) = e^{-nx} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!},$$

die im unendlichen Intervall eine ähnliche Rolle spielen wie die Bernstein-Polynome im endlichen, einen wichtigen Platz ein. (Hierbei sind f eine im Intervall $[0, \infty)$ definierte reellwertige Funktion; n eine positive reelle Zahl.) Die Einführung der Operatoren S_n und die Untersuchung ihrer Approximationseigenschaften ist mit den Namen von P. L. BUTZER [1], M. J. FAVARD [2], G. M. MIRAKYAN [4], O. SZÁSZ [6] und anderen verbunden.

In [6] untersucht O. SZÁSZ die Fragen der Approximation im Intervall $[0, \infty)$ sowohl vom Gesichtspunkt der punktweisen als auch der gleichmäßigen Konvergenz. Wir wollen hier die zum Kreis des letzteren Themas gehörigen Ergebnisse weiterentwickeln. Um unsere Zielsetzung zu konkretisieren, zitieren wir hier den in der Überschrift erwähnten Satz von SZÁSZ ([6], Seite 241):

„Wenn $f(x)$ die folgende Lipschitz-Bedingung erfüllt:

$$(1) \qquad |f(x_2)-f(x_1)| \leqq \gamma \frac{|x_2-x_1|^{\varrho}}{(x_2+x_1)^{\varrho/2}} \quad (0 < x_1 < x_2 < \infty),$$

wobei γ und ϱ Konstanten sind, $0<\varrho\leqq 1$, dann ist

$$(2) \qquad |f(x)-S_n(f;\, x)| \leqq \frac{\gamma}{\sqrt{n^{\varrho}}}$$

gleichmäßig im Intervall $0<x<\infty$."

(Der Satz ist dem sich auf Bernstein-Polynome beziehenden Satz von T. POPOVICIU ([5], [3]) analog.)

Wir werden zeigen, daß die Bedingung (1) zu streng ist: es kann eine wichtige Klasse von Funktionen angegeben werden, für deren Elemente (1) nicht gilt, die Konvergenz $S_n(f;\, x) \to f(x)$ $(n\to\infty)$ im Intervall $[0, \infty)$ aber gleichmäßig ist. Andererseits geben wir eine hinreichende Bedingung dafür an, daß die Ordnung der Näherung besser ist als $1/\sqrt{n}$.

Zur Illustration des Gesagten betrachten wir die durch die Gleichung $f(t)=t^{\alpha}$ $(t\geqq 0)$ definierte Funktion f, wobei $\alpha>0$ ist. Es ist leicht zu verifizieren, daß für

$\alpha=2$

$$S_n(f;\, x) = x^2+\frac{x}{n}$$

gilt und daher die Funktion $f(t)=t^2$ in bezug auf die punktweise Konvergenz im Intervall $[0, \infty)$ mit Hilfe von S_n approximierbar ist; von einer gleichmäßigen Näherung im Intervall $[0, \infty)$ kann allerdings nicht die Rede sein. Für welche α ist also eine gleichmäßige Approximation möglich? Was behauptet diesbezüglich der obige Satz von Szász? Nehmen wir an, daß für irgendwelche geeigneten Zahlen α, ϱ und γ die Bedingung (1) erfüllt ist, d. h.

$$|x_2^\alpha - x_1^\alpha| \leqq \gamma \frac{|x_2-x_1|^\varrho}{(x_2+x_1)^{\varrho/2}} \quad (0<x_1<x_2<\infty).$$

Daraus würde für $x_1=1$ folgen:

$$|x_2^\alpha - 1| \leqq \gamma \frac{|x_2-1|^\varrho}{(x_2+1)^{\varrho/2}} \quad (0<x_2<\infty).$$

Diese Ungleichung kann allerdings im Falle von $\alpha>\frac{1}{2}$ wegen der Bedingung $0<\varrho\leqq 1$ für keine Zahl γ erfüllt werden. Der Satz von Szász ist also im Falle $\alpha>\frac{1}{2}$ für die Funktion $f(t)=t^\alpha$ nicht verwendbar.

Im folgenden Paragraphen formulieren wir einen Satz, der eine hinreichende Bedingung für die Gleichmäßigkeit von $S_n(f;\, x)\to f(x)$ $(n\to\infty)$ im Intervall $[0, \infty)$ angibt. Im §3 zeigen wir, daß die Funktion $f(t)=t^\alpha$ den Bedingungen des Satzes genügt, und zwar für $0<\alpha\leqq 1$. Der Beweis des Satzes erfolgt in §5, nach einigen Hilfssätzen, die §4 beeinhaltet.

An dieser Stelle möchte ich János Balázs für viele nützliche Ratschläge Dank sagen.

§ 2. Der Satz und einige Korollare

Im weiteren sei f immer eine im Intervall $[0, \infty)$ definierte reellwertige Funktion. Führen wir die folgende Bezeichnung ein:

$$\Delta(f;\, x, y) = \frac{f(y)-f(x)}{y-x} \quad (y\neq x).$$

SATZ. *Voraussetzungen. Nehmen wir an, daß solche Zahlen* A, B, C, D, b, ϱ *und* σ *existieren* ($A, B, C, D, b\geqq 0$, $0<\varrho\leqq 1$, $0\leqq\sigma\leqq 1$), *mit denen die folgenden Ungleichungen gelten:*

$$\text{(i)} \qquad |f(y)-f(w)| \leqq A\frac{(y-w)^\varrho}{(y+w)^{\varrho/2}} \quad (0\leqq w<y<b),$$

$$\text{(ii)} \qquad |f(y)-f(w)| \leqq B|y-w| \quad (w\geqq 0,\ y\geqq b),$$

$$\text{(iii)} \qquad |\Delta(f;\, y, x)-\Delta(f;\, x, w)| \leqq C\frac{(y-w)^\sigma}{\sqrt{(x+1)^{1+\sigma}}}+D\frac{y-w}{x+1}$$

$$\left(x\geqq b,\ 0<\frac{x}{2}<w<x<y<\frac{3x}{2}<\infty\right).$$

Behauptung. Für $n \geqq 1$ *gilt*

$$|f(x)-S_n(f;\,x)| \leqq \frac{A+B\sqrt{b}}{\sqrt{n^{\varrho}}}+\frac{K}{\sqrt{n^{1+\sigma}}} \quad (0 \leqq x < \infty),$$

wobei $K=4B+5C+5D$ *ist.*

BEMERKUNGEN. 1. Der in § 1 zitierte Satz von Szász kann als Grenzfall des obigen Satzes aufgefaßt werden, denn wenn $b=\infty$ ist, dann ist die Bedingung (i) äquivalent (1); die Bedingungen (ii) und (iii) dagegen bedeuten für kein Zahlentripel (w, x, y) eine Bindung, so treten diese Bedingungen auch nicht auf, womit die B, C und D enthaltenden Glieder aus der Behauptung enfernt werden können.

2. Würden wir in (iii) anstelle von $\frac{x}{2}$ bzw. $\frac{3x}{2}$ den Ausdruck $(1-\vartheta)x$ bzw. $(1+\vartheta)x$ schreiben (wobei $0<\vartheta<1$ ist), müßten wir in der Behauptung höchstens K ändern.

3. Es sei $x \geqq b$. Die Bedingung (iii) fordert nicht in jedem Falle, daß f an der Stelle x differenzierbar sei, denn wenn $C>0$ und $\sigma=0$ ist, muß ja $|\Delta(f;\,y, x)-\Delta(f;\,x, w)|$ nicht gegen Null konvergieren, da $|y-w|$ gegen Null strebt.

4. Es sei f im Intervall $\left(\frac{b}{2}, \infty\right)$ (wobei $b \geqq 0$ ist) differenzierbar, weiter seien C, D und σ Zahlen $(C, D \geqq 0,\ 0 \leqq \sigma \leqq 1)$, mit denen

$$|f'(x_2)-f'(x_1)| \leqq C\frac{(x_2-x_1)^{\sigma}}{\sqrt{(x+1)^{1+\sigma}}}+D\frac{x_2-x_1}{x+1} \tag{3}$$

$$\left(x \geqq b,\ 0<\frac{x}{2}<x_1<x<x_2<\frac{3x}{2}<\infty\right)$$

besteht. Dann ist die Bedingung (iii) erfüllt, denn mit Hilfe des Mittelwertsatzes von Lagrange und der Ungleichung (3) ergibt sich

$$|\Delta(f;\,y, x)-\Delta(f;\,x, w)| = |f'(\xi_2)-f'(\xi_1)| \leqq$$

$$\leqq C\frac{(\xi_2-\xi_1)^{\sigma}}{\sqrt{(x+1)^{1+\sigma}}}+D\frac{\xi_2-\xi_1}{x+1} \leqq C\frac{(y-w)^{\sigma}}{\sqrt{(x+1)^{1+\sigma}}}+D\frac{y-w}{x+1}$$

$$\left(x \geqq b,\ 0<\frac{x}{2}<w<x<y<\frac{3x}{2}<\infty\right),$$

wobei $w<\xi_1<x<\xi_2<y$ ist.

5. Es sei f im Intervall $\left(\frac{b}{2}, \infty\right)$ (wobei $b \geqq 0$ ist) zweimal differenzierbar, ferner sei C' eine Zahl, mit der die Ungleichung

$$|f''(t)| \leqq \frac{C'}{t+1} \quad \left(\frac{b}{2}<t<\infty\right)$$

gilt. Dann ist die Bedingung (iii) erfüllt, und zwar für $\sigma=1$ und $C=D=C'$. Zum

Beweis nehmen wir an, daß die Ungleichungen $x \geqq b, \frac{x}{2} < x_1 < x < x_2 < \frac{3x}{2}$ gelten. Dann gilt

$$|f'(x_2) - f'(x_1)| = |f''(\xi)|(x_2 - x_1) \leqq \frac{C'(x_2 - x_1)}{\xi + 1} \leqq \frac{C'(x_2 - x_1)}{\frac{x}{2} + 1},$$

wobei $x_2 < \xi < x_1$ ist. Da $\frac{x}{2} + 1 > \frac{x}{2} + \frac{1}{2}$ ist, bekommen wir aus den Vorangegangenen, daß die Ungleichung

$$|f'(x_2) - f'(x_1)| < \frac{2C'}{x+1}(x_2 - x_1)$$

gilt, so daß aus der 4. Bemerkung unsere Behauptung folgt.

Wir wollen noch einige Korollare des Satzes formulieren.

KOROLLAR 1. *Seien* A^*, B, C, D, b, ϱ *und* σ *Zahlen* $(A^*, B, C, D \geqq 0, b > 0, 0 < \varrho \leqq 1, 0 \leqq \sigma \leqq 1)$, *mit denen die Ungleichung*

$$\text{(i*)} \qquad |f(y) - f(w)| \leqq A^*(y-w)^\varrho \quad (0 \leqq w < y < b)$$

sowie die Bedingungen (ii) *und* (iii) *erfüllt sind. Dann besteht für* $n \geqq 1$ *die Abschätzung*

$$|f(x) - S_n(f; x)| \leqq \frac{K^*}{\sqrt{n^\varrho}} + \frac{K}{\sqrt{n^{1+\sigma}}} \quad (0 \leqq x < \infty),$$

wobei $K^* = A^* \sqrt{(2b)^\varrho} + B\sqrt{b}$ *und* $K = 4B + 5C + 5D$ *sind.*

BEWEIS. Es sei $0 \leqq w < y < b$. Dann ist $2b > y + w$, so daß

$$A^*(y-w)^\varrho \leqq A^*(y-w)^\varrho \left(\frac{2b}{y+w}\right)^{\varrho/2} = A^* \sqrt{(2b)^\varrho}\, \frac{(y-w)^\varrho}{(y+w)^{\varrho/2}}$$

gilt, und so folgt aus der Bedingung (i*), daß die Bedingung (i) erfüllt ist, und zwar für $A = A^* \sqrt{(2b)^\varrho}$. Aus dem Satz folgt also unsere Behauptung.

KOROLLAR 2. *Nehmen wir an, daß solche Zahlen* B, C, D *und* σ *existieren, mit denen die folgenden zwei Bedingungen erfüllt sind:*

$$\text{(ii*)} \qquad |f(y) - f(w)| \leqq B|y - w| \quad (w \geqq 0,\ y \geqq 0)$$

$$\text{(iii*)} \qquad |\Delta(f; y, x) - \Delta(f; x, w)| \leqq C \frac{(y-w)^\sigma}{\sqrt{(x+1)^{1+\sigma}}} + D \frac{y-w}{x+1}$$

$$\left(0 < \frac{x}{2} < w < x < y < \frac{3x}{2} < \infty\right).$$

Dann gilt für $n \geqq 1$

$$\text{(4)} \qquad |f(x) - S_n(f; x)| \leqq \frac{K}{\sqrt{n^{1+\sigma}}} \quad (0 \leqq x < \infty),$$

wobei $K = 4B + 5C + 5D$ *ist.*

BEWEIS. Im Falle $b=0$ bedeutet die Bedingung (i) für kein einziges Zahlenpaar (y, w) eine Bindung, das heißt, die Bedingung (i) ist auch dann erfüllt, wenn $A=0$ ist. Die Bedingung (ii) bzw. (iii) ist schließlich im Falle $b=0$ mit der Bedingung (ii*) bzw. (iii*) identisch. Danach ist unsere Behauptung offensichtlich.

KOROLLAR 3. *Sei f im Intervall* $(0, \infty)$ *differenzierbar, weiter seien B, C, D und σ Zahlen* $(B, C, D\geqq 0,\ 0\leqq\sigma\leqq 1)$, *mit denen* (ii*) *und die folgende Bedingung erfüllt sind:*

$$|f'(x_2)-f'(x_1)| \leqq C\frac{(x_2-x_1)^\sigma}{\sqrt{(x+1)^{1+\sigma}}}+D\frac{x_2-x_1}{x+1} \quad \left(0<\frac{x}{2}<x_1<x<x_2<\frac{3x}{2}<\infty\right).$$

Dann besteht für $n\geqq 1$ die Ungleichung (4).

BEWEIS. Gemäß der Annahme folgt aus der 4. Bemerkung, daß die Bedingung (iii) bei $b=0$ erfüllt ist, also gilt (iii*), und so folgt die Behauptung sofort aus Korollar 2.

KOROLLAR 4. *Es sei f im Intervall* $(0, \infty)$ *zweimal differenzierbar, und es seien B, C' Zahlen, mit denen* (ii*) *und die Bedingung*

$$|f''(x)| \leqq \frac{C'}{x+1} \quad (0<x<\infty)$$

erfüllt sind. Dann gilt für $n\geqq 1$:

$$|f(x)-S_n(f;\, x)| \leqq \frac{K'}{n} \quad (0\leqq x<\infty)$$

wobei $K'=4B+10C'$ ist.

BEWEIS. Gemäß der Annahme folgt aus der 5. Bemerkung, daß die Bedingung (iii) bei $b=0, \sigma=1$ und $C=D=C'$ erfüllt ist, also gilt (iii*) im Falle $\sigma=1$, $C=D=C'$, und so folgt die Behauptung sofort aus Korollar 2.

§ 3. Ein Beispiel

Im § 1 haben wir erhalten, daß die Funktion $f(t)=t^\alpha$ der Bedingung (1) des Satzes von Szász nicht genügt wenn $\alpha>\frac{1}{2}$ ist.

Es sei jetzt $0<\alpha\leqq 1$. Wir beweisen, daß die Bedingungen unseres Satzes für die Funktion $f(t)=t^\alpha$ erfüllt sind.

Unter Benutzung des Hilfssatzes (H1) aus § 4 ergibt sich:

(5) $$|f(y)-f(w)| = |y^\alpha-w^\alpha| \leqq (y-w)^\alpha \quad (0\leqq w\leqq y<\infty).$$

Daraus folgt offensichtlich

(6) $$|f(y)-f(w)| \leqq 2^\alpha\frac{(y-w)^\alpha}{(y+w)^{\alpha/2}} \quad (0\leqq w<y<2),$$

wegen $y+w<4$. Aus (5) ergibt sich auch

(7) $$|f(y)-f(w)| \leqq y-w \quad (0\leqq w\leqq 1,\ y\geqq 2),$$

denn wenn $y-w\geqq 1$ ist, dann ist $(y-w)^\alpha \leqq y-w$. Schließlich besteht die Beziehung

$$(8)\qquad |f(y)-f(w)| = f'(\xi)|y-w| = \alpha\xi^{\alpha-1}|y-w| \leqq$$
$$\leqq \alpha|y-w| \leqq |y-w| \quad (w>1,\ y\geqq 2,\ w\neq y).$$

(Hier ist $w<\xi<y$ oder $y<\xi<w$, also $\xi>1$.) Aus (6) ist zu sehen, daß die Bedingung (i), aus (7) und (8), daß die Bedingung (ii) erfüllt ist, und zwar bei $A=2^\alpha$, $B=1$, $\varrho=\alpha$ und $b=2$.

Aus den Ungleichungen $2-\alpha>1$ und $x>1$ folgt, daß $x^{2-\alpha}>x$ ist, und so gilt

$$|f''(x)| = \frac{\alpha(1-\alpha)}{x^{2-\alpha}} < \frac{\alpha(1-\alpha)}{x} < \frac{2\alpha(1-\alpha)}{x+1} \quad (x>1),$$

denn aus $x>1$ folgt $x>\dfrac{x+1}{2}$. Im Falle $b=2$ erhält man also

$$|f''(x)| < \frac{2\alpha(1-\alpha)}{x+1} \quad \left(x>\frac{b}{2}=1\right),$$

und gemäß der 5. Bemerkung des 2. Paragraphen ist die Bedingung (iii) bei $b=2$, $\sigma=1$ und $C=D=2\alpha(1-\alpha)$ erfüllt.

Aufgrund des Satzes können wir also behaupten, daß für die Funktion $f(t)=t^\alpha$ im Falle von $n\geqq 1$ die Ungleichung

$$|S_n(f;\,x)-f(x)| \leqq \frac{M}{\sqrt{n^\alpha}} \quad (0\leqq x<\infty)$$

besteht, wobei $0<\alpha\leqq 1$ und $M=2^\alpha+\sqrt{2}+4+20\alpha(1-\alpha)$ ist.

§ 4. Hilfssätze

$$\text{(H1)}\qquad |b^\alpha-a^\alpha| \leqq |b-a|^\alpha \quad (a\geqq 0,\ b\geqq 0,\ 0<\alpha\leqq 1).$$

Beweis. Für $\alpha=1$ ist die Behauptung offensichtlich. Es seien $0<\alpha<1$ und $g(t)=|1-t|^\alpha-|1-t^\alpha|$ $(t\geqq 0)$. Dann gilt

$$g'(t) = \alpha[t^{\alpha-1}-(1-t)^{\alpha-1}] \quad (0<t<1),$$

also ist $g'(t)>0$ $\left(0<t<\dfrac{1}{2}\right)$ und $g'(t)<0$ $\left(\dfrac{1}{2}<t<1\right)$. Da $g(0)=g(1)=0$ ist, folgt $g(t)>0$ $(0<t<1)$. Andererseits haben wir

$$g'(t) = \alpha[(t-1)^{\alpha-1}-t^{\alpha-1}] > 0 \quad (t>1),$$

also ist g im Intervall $(1,\infty)$ streng monoton steigend. Da $g(1)=0$ ist, folgt $g(t)>0$ $(t>1)$. Zusammengefaßt:

$$|1-t|^\alpha \geqq |1-t^\alpha| \quad (0\leqq t<\infty).$$

Setzen wir $t=\dfrac{a}{b}$, so bekommen wir schon den Hilfssatz (H1).

$$\text{(H2)}\qquad (a+b)^\alpha \leqq a^\alpha+b^\alpha \quad (a\geqq 0,\ b\geqq 0,\ 0<\alpha\leqq 1).$$

BEWEIS. Für $\alpha=1$ ist die Behauptung offensichtlich. Es seien $0<\alpha<1$ und $h(t)=1+t^\alpha-(1+t)^\alpha$ $(t\geqq 0)$. Dann gilt

$$h'(t)=\alpha[t^{\alpha-1}-(1+t)^{\alpha-1}]>0 \quad (t>0),$$

also ist h im Intervall $(0, \infty)$ streng monoton steigend. Da $h(0)=0$ ist, folgt $h(t)\geqq 0$ $(t\geqq 0)$, also $1+t^\alpha\geqq(1+t)^\alpha$ $(t\geqq 0)$. Daraus folgt (H2).

(H3) $$\sum_{k=0}^{\infty}(\lambda-k)\frac{\lambda^k}{k!}=0 \quad (\lambda>0).$$

(H4) $$\sum_{k=0}^{\infty}(\lambda-k)^2\frac{\lambda^k}{k!}=\lambda e^\lambda \quad (\lambda>0).$$

Die Richtigkeit der Behauptungen (H3) und (H4) kann durch direkte Rechnungen nachgeprüft werden.

(H5) $$\sum_{k=0}^{\infty}|\lambda-k|^\delta\frac{\lambda^k}{k!}\leqq\sqrt{\lambda^\delta}e^\lambda \quad (\lambda>0, 0<\delta\leqq 2).$$

BEWEIS. Es sei $\lambda>0$, $0<\delta<2$. (Für $\delta=2$ s. (H4).) Wir verwenden die Ungleichung von Hölder:

$$\sum_k a_k b_k \leqq \Big(\sum_k a_k^\alpha\Big)^{1/\alpha}\Big(\sum_k b_k^\beta\Big)^{1/\beta} \quad \left(a_k, b_k>0, \alpha>1, \frac{1}{\alpha}+\frac{1}{\beta}=1\right),$$

und zwar mit folgender Besetzung:

$$a_k=\left(|\lambda-k|^2\frac{\lambda^k}{k!}\right)^{\delta/2}, \quad b_k=\left(\frac{\lambda^k}{k!}\right)^{(2-\delta)/2}, \quad \alpha=\frac{2}{\delta}, \quad \beta=\frac{2}{2-\delta}.$$

Gemäß der Ungleichung ist dann

$$\sum_{k=0}^{\infty}|\lambda-k|^\delta\frac{\lambda^k}{k!}\leqq\left(\sum_{k=0}^{\infty}|\lambda-k|^2\frac{\lambda^k}{k!}\right)^{\delta/2}\left(\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\right)^{(2-\delta)/2}=(\lambda e^\lambda)^{\delta/2}(e^\lambda)^{(2-\delta)/2}=\lambda^{\delta/2}e^\lambda$$

(unterdessen haben wir auch (H4) angewendet), und danach ist die Behauptung schon offensichtlich.

(H6) $$\sum_{|k-\lambda|\geqq\lambda/2}\frac{\lambda^k}{k!}\leqq\frac{4}{\lambda}e^\lambda \quad (\lambda>0).$$

BEWEIS. Aus (H4) ist zu sehen:

$$\lambda e^\lambda=\sum_{k=0}^{\infty}(k-\lambda)^2\frac{\lambda^k}{k!}\geqq\sum_{|k-\lambda|\geqq\lambda/2}(k-\lambda)^2\frac{\lambda^k}{k!}\geqq\frac{\lambda^2}{4}\sum_{|k-\lambda|\geqq\lambda/2}\frac{\lambda^k}{k!}.$$

(H7) $$\sum_{|k-\lambda|\geqq\lambda/2}|k-\lambda|\frac{\lambda^k}{k!}\leqq 2e^\lambda \quad (\lambda>0).$$

BEWEIS. Verwenden wir die Ungleichung von Bunjakowski—Schwarz, (H4) und (H6):

$$\sum_{|k-\lambda|\geqq\lambda/2} |k-\lambda| \frac{\lambda^k}{k!} \leqq \left(\sum_{|k-\lambda|\geqq\lambda/2} (k-\lambda)^2 \frac{\lambda^k}{k!}\right)^{1/2} \left(\sum_{|k-\lambda|\geqq\lambda/2} \frac{\lambda^k}{k!}\right)^{1/2} \leqq$$

$$\leqq (\lambda e^\lambda)^{1/2} \left(\frac{4}{\lambda} e^\lambda\right)^{1/2} = 2e^\lambda.$$

(H8) $$\left|\sum_{|k-\lambda|<\lambda/2} (k-\lambda)\frac{\lambda^k}{k!}\right| \leqq 2e^\lambda \quad (\lambda>0).$$

BEWEIS. Gehen wir von (H3) aus:

$$0 = \sum_{k=0}^{\infty} (k-\lambda)\frac{\lambda^k}{k!} = \sum_{|k-\lambda|\geqq\lambda/2} + \sum_{|k-\lambda|<\lambda/2}.$$

Daraus und aus (H7) folgt die Behauptung sofort, da

$$\left|\sum_{|k-\lambda|<\lambda/2}\right| = \left|\sum_{|k-\lambda|\geqq\lambda/2}\right| \leqq \sum_{|k-\lambda|\geqq\lambda/2} |k-\lambda| \frac{\lambda^k}{k!} \leqq 2e^\lambda.$$

§ 5. Der Beweis des Satzes

Für $x=0$ ist die Behauptung offensichtlich.

1° Es sei $0<x\leqq b$. (Wenn $b=0$ ist, kann der Teil 1° des Beweises weggelassen werden.)

$$e^{nx}|f(x)-S_n(f;\,x)| = \left|\sum_{k=0}^{\infty}\left[f(x)-f\left(\frac{k}{n}\right)\right]\frac{(nx)^k}{k!}\right| \leqq$$

$$\leqq \sum_{k<nb}\left|f(x)-f\left(\frac{k}{n}\right)\right|\frac{(nx)^k}{k!} + \sum_{k\geqq nb}\left|f(x)-f\left(\frac{k}{n}\right)\right|\frac{(nx)^k}{k!} \overset{\text{def}}{=} \Sigma_1+\Sigma_2.$$

Aufgrund der Bedingung (i) und unter Anwendung des Hilfssatzes (H5) (mit $\lambda=nx$ und $\delta=\varrho$) erhalten wir

$$\Sigma_1 = A\sum_{k<nb}\frac{\left|x-\frac{k}{n}\right|^\varrho}{\left(x+\frac{k}{n}\right)^{\varrho/2}}\frac{(nx)^k}{k!} \leqq \frac{A}{x^{\varrho/2}}\sum_{k=0}^{\infty}\left|x-\frac{k}{n}\right|^\varrho\frac{(nx)^k}{k!} =$$

$$= \frac{A}{(n\sqrt{x})^\varrho}\sum_{k=0}^{\infty}|nx-k|^\varrho\frac{(nx)^k}{k!} \leqq \frac{A}{(n\sqrt{x})^\varrho}\sqrt{(nx)^\varrho}\,e^{nx} = \frac{A}{\sqrt{n^\varrho}}e^{nx}.$$

Aus der Bedingung (ii) und dem Hilfssatz (H5) ergibt sich

$$\Sigma_2 \leqq B\sum_{k\geqq nb}\left|x-\frac{k}{n}\right|\frac{(nx)^k}{k!} = \frac{B}{n}\sqrt{nx}\,e^{nx} \leqq \frac{B\sqrt{b}}{\sqrt{n}}e^{nx}.$$

Zusammengefaßt bedeutet dies

$$(13)\qquad |f(x)-S_n(f;x)| \leqq \frac{A}{\sqrt{n^{\varrho}}}+\frac{B\sqrt{b}}{\sqrt{n}} \quad (0\leqq x<b).$$

2° Es sei $x\geqq b$, $x>0$.

$$e^{nx}[f(x)-S_n(f;x)] = \sum_{\left|\frac{k}{n}-x\right|\geqq\frac{x}{2}} \left[f(x)-f\left(\frac{k}{n}\right)\right]\frac{(nx)^k}{k!}+$$

$$+\sum_{\left|\frac{k}{n}-x\right|<\frac{x}{2}} \left[f(x)-f\left(\frac{k}{n}\right)\right]\frac{(nx)^k}{k!} \stackrel{\text{def}}{=} T_1+T_2.$$

Aufgrund der Bedingung (ii) und des Hilfssatzes (H7) ist zu sehen:

$$(14)\qquad |T_1| \leqq B \sum_{\left|\frac{k}{n}-x\right|\geqq\frac{x}{2}} \left|x-\frac{k}{n}\right| \frac{(nx)^k}{k!} \leqq \frac{2B}{n}e^{nx}.$$

Die Abschätzung von T_2 führen wir zuerst für die Fälle durch, bei denen $0<nx<1$ ist. Dann kann es höchstens eine solche ganze Zahl k geben, für die $|k-nx|<$ $<\frac{1}{2}nx$ ist, und zwar $k=1$. Dann gilt also

$$|T_2| \leqq \left|f(x)-f\left(\frac{1}{n}\right)\right| nx.$$

Wir nutzen (ii) und den Umstand aus, daß $te^{-t}<1$ $(0<t<1)$ ist:

$$(15)\qquad |T_2| \leqq B\left|x-\frac{1}{n}\right|(nxe^{-nx})e^{nx} < B\left|\frac{nx-1}{n}\right| \leqq \frac{B}{n}e^{nx} \quad (0<nx<1).$$

Im weiteren können wir schon annehmen, daß $nx\geqq 1$ ist. Wir spalten T_2 in zwei Teile:

$$T_2 = \sum_{\frac{x}{2}<\frac{k}{n}<x} \left[f(x)-f\left(\frac{k}{n}\right)\right]\frac{(nx)^k}{k!}+\sum_{x<\frac{k}{n}<\frac{3x}{2}} \left[f(x)-f\left(\frac{k}{n}\right)\right]\frac{(nx)^k}{k!} \stackrel{\text{def}}{=} T_{21}+T_{22}.$$

Schätzen wir zuerst T_{21} ab; k sei also ein solcher Index, für den $\frac{x}{2}<\frac{k}{n}<x$ ist. Setzen wir zur Abkürzung $m=[nx]$ $(=\text{entier}\,(nx))$. Da $\frac{k}{n}<x<\frac{m+1}{n}$ und $x\geqq b$ ist, bekommen wir unter Benutzung der Bedingung (iii)

$$(16)\qquad \left|\Delta\left(f;\frac{m+1}{n},x\right)-\Delta\left(f;x,\frac{k}{n}\right)\right| \leqq C\frac{\left(\frac{m+1}{n}-\frac{k}{n}\right)^{\sigma}}{\sqrt{(x+1)^{1+\sigma}}}+D\frac{\frac{m+1}{n}-\frac{k}{n}}{x+1}.$$

Wegen $m=nx-r$ (mit $0\leqq r<1$) gilt

$$\frac{m+1-k}{n} = \frac{nx-r+1-k}{n} \leqq x-\frac{k}{n}+\frac{2}{n}.$$

Benutzen wir den Hilfssatz (H2), demnach gilt

$$\left(x-\frac{k}{n}+\frac{2}{n}\right)^{\sigma} \leqq \left(x-\frac{k}{n}\right)^{\sigma}+\frac{2}{n^{\sigma}}. \tag{17}$$

Aus (16) erhalten wir also

$$\left|\Delta\left(f;\frac{m+1}{n},x\right)-\Delta\left(f;x,\frac{k}{n}\right)\right| \leqq \left\{\frac{C}{\sqrt{(x+1)^{1+\sigma}}}\left(\left|x-\frac{k}{n}\right|^{\sigma}+\frac{2}{n^{\sigma}}\right)+\frac{D}{x+1}\left(\left|x-\frac{k}{n}\right|+\frac{2}{n}\right)\right\}$$

(die rechte Seite bezeichnen wir im weiteren mit $\{\dots\}$), und daraus folgt, daß zu jedem in Frage kommenden Zahlentripel (x, n, k) eine Zahl $\tau(x, n, k)$ mit $-1 \leqq \tau(x, n, k) \leqq 1$ existiert, für die die folgende Gleichung erfüllt ist:

$$\Delta\left(f;x,\frac{k}{n}\right)=\Delta\left(f;\frac{m+1}{n},x\right)+\tau(x,n,k)\{\dots\}. \tag{18}$$

Schreiben wir T_{21} in der Form

$$T_{21}=\sum_{\frac{x}{2}<\frac{k}{n}<x}\Delta\left(f;x,\frac{k}{n}\right)\left(x-\frac{k}{n}\right)\frac{(nx)^k}{k!}.$$

Aufgrund des in (18) Zusammengefaßten erhält man

$$T_{21}=\sum_{\frac{x}{2}<\frac{k}{n}<x}\left[\Delta\left(f;\frac{m+1}{n},x\right)+\tau(x,n,k)\{\dots\}\right]\left(x-\frac{k}{n}\right)\frac{(nx)^k}{k!}. \tag{19}$$

Gehen wir zu T_{22} über; k sei also ein solcher Index, für den die Ungleichung $x<\frac{k}{n}<\frac{3x}{2}$ besteht.

Da $\frac{m-1}{n}<x<\frac{k}{n}$ und $x \geqq b$ ist, bekommen wir mit Hilfe der Bedingung (iii)

$$\left|\Delta\left(f;\frac{k}{n},x\right)-\Delta\left(f;x,\frac{m-1}{n}\right)\right| \leqq C\frac{\left(\frac{k}{n}-\frac{m-1}{n}\right)^{\sigma}}{\sqrt{(x+1)^{1+\sigma}}}+D\frac{\frac{k}{n}-\frac{m-1}{n}}{x+1}.$$

$\left(\Delta\left(f;x,\frac{m-1}{n}\right)\right.$ ist definiert, denn wegen der Annahme $nx \geqq 1$ ist $\left.m-1 \geqq 0.\right)$
Wegen $m=nx-r$ (wobei $0 \leqq r<1$) ist, ergibt sich

$$\frac{k-(m-1)}{n}=\frac{k-nx+r+1}{n} \leqq \left|x-\frac{k}{n}\right|+\frac{2}{n}.$$

Ebenso wie im Falle von T_{21} erhält man

$$T_{22}=\sum_{x<\frac{k}{n}<\frac{3x}{2}}\left[\Delta\left(f;x,\frac{m-1}{n}\right)+\tau(x,n,k)\{\dots\}\right]\left(x-\frac{k}{n}\right)\frac{(nx)^k}{k!}, \tag{20}$$

wobei $|\tau(x, n, k)| \leqq 1$ ist.

Aus (19) und (20) ergibt sich

$$T_2 = T_{21}+T_{22} = \Delta\left(f; \frac{m+1}{n}, x\right) \sum_{\frac{x}{2}<\frac{k}{n}<x} \left(x-\frac{k}{n}\right)\frac{(nx)^k}{k!}+$$

$$+\Delta\left(f; \frac{m-1}{n}, x\right) \sum_{x<\frac{k}{n}<\frac{3x}{2}} \left(x-\frac{k}{n}\right)\frac{(nx)^k}{k!}+ \sum_{\frac{x}{2}<\frac{k}{n}<\frac{3x}{2}} \tau(n, x, k)\left(x-\frac{k}{n}\right)\{\dots\}\frac{(nx)^k}{k!} \stackrel{\text{def}}{=\!=}$$

$$\stackrel{\text{def}}{=\!=} T_{2,a}+T_{2,b}+T_{2,c}.$$

$T_{2,a}$ läßt sich in der folgenden Form schreiben:

$$T_{2,a} = \Delta\left(f; \frac{m+1}{n}, x\right) \sum_{\frac{x}{2}<\frac{k}{n}<\frac{3x}{2}} \left(x-\frac{k}{n}\right)\frac{(nx)^k}{k!} - \Delta\left(f; \frac{m+1}{n}, x\right) \sum_{x<\frac{k}{n}<\frac{3x}{2}} \left(x-\frac{k}{n}\right)\frac{(nx)^k}{k!}.$$

Danach sieht man

$$T_{2,a}+T_{2,b} = \Delta\left(f; \frac{m+1}{n}, x\right)\frac{1}{n} \sum_{|nx-k|<\frac{1}{2}nx} (nx-k)\frac{(nx)^k}{k!}+$$

$$+\left[\Delta\left(f; x, \frac{m-1}{n}\right) - \Delta\left(f; \frac{m+1}{n}, x\right)\right]\frac{1}{n} \sum_{nx<k<\frac{3}{2}nx} (nx-k)\frac{(nx)^k}{k!} \stackrel{\text{def}}{=\!=} U_1+U_2.$$

Gemäß (ii) ist

$$\left|\Delta\left(f; \frac{m+1}{n}, x\right)\right| \leqq B,$$

deshalb erhalten wir nach Hilfssatz (H8)

$$|U_1| \leqq \frac{2B}{n} e^{nx}. \tag{21}$$

Zur Abschätzung von U_2 wenden wir nun (iii) und (H5) an:

$$e^{-nx}|U_2| \leqq \left[\frac{C\left(\frac{m+1}{n}-\frac{m-1}{n}\right)^{\sigma}}{\sqrt{(x+1)^{1+\sigma}}} + D\frac{\frac{m+1}{n}-\frac{m-1}{n}}{x+1}\right]\frac{1}{n}\sqrt{nx} \leqq \tag{22}$$

$$\leqq \frac{2^{\sigma}C}{n^{\sigma}\sqrt{n}}\sqrt{\frac{x}{(x+1)^{1+\sigma}}} + \frac{2D}{n\sqrt{n}}\frac{\sqrt{x}}{x+1} \leqq \frac{2C}{n^{\sigma}\sqrt{n}} + \frac{2D}{n\sqrt{n}}.$$

Es bleibt noch die Abschätzung von $T_{2,c}$. Wir nutzen aus, daß $|\tau(x, n, k)| \leqq 1$ ist.

$$|T_{2,c}| \leqq \sum_{k=0}^{\infty} \left|x-\frac{k}{n}\right| \left\{ \frac{C}{\sqrt{(x+1)^{1+\sigma}}} \left(\left|x-\frac{k}{n}\right|^{\sigma} + \frac{2}{n^{\sigma}} \right) + \frac{D}{x+1} \left(\left|x-\frac{k}{n}\right| + \frac{2}{n} \right) \right\} \frac{(nx)^k}{k!} =$$

$$= \frac{C}{\sqrt{(x+1)^{1+\sigma}}} \frac{1}{n^{1+\sigma}} \sum_{k=0}^{\infty} (|nx-k|^{1+\sigma} + 2|nx-k|) \frac{(nx)^k}{k!} +$$

$$+ \frac{D}{x+1} \frac{1}{n^2} \sum_{k=0}^{\infty} (|nx-k|^2 + 2|nx-k|) \frac{(nx)^k}{k!}.$$

Wir verwenden wieder (H5) (mit $\lambda = nx$ und $\delta = 1+\sigma, 1, 2, 1$):

$$e^{-nx} |T_{2,c}| \leqq \frac{C}{\sqrt{(x+1)^{1+\sigma}}} \frac{1}{n^{1+\sigma}} \left(\sqrt{(nx)^{1+\sigma}} + 2\sqrt{nx}\right) + \tag{23}$$

$$+ \frac{D}{x+1} \frac{1}{n^2} (nx + 2\sqrt{nx}) \leqq \frac{C}{\sqrt{n^{1+\sigma}}} + \frac{2C}{n^{\sigma}\sqrt{n}} + \frac{D}{n} + \frac{2D}{n\sqrt{n}}.$$

Erinnern wir uns daran, daß wir die Bezeichnungen

$$e^{nx}[f(x) - S_n(f; x)] = T_1 + T_2 =$$

$$= T_1 + (T_{2,a} + T_{2,b}) + T_{2,c} = T_1 + (U_1 + U_2) + T_{2,c}$$

gebrauchten, demnach können wir im Falle von $0 < nx < 1$ aufgrund von (14) und (15), im Falle von $nx \geqq 1$ unter Beachtung von (14), (21), (22) und (23) behaupten, daß die folgende Beziehung besteht:

$$|f(x) - S_n(f; x)| \leqq \frac{4B}{n} + \frac{4C}{n^{\sigma}\sqrt{n}} + \frac{C}{\sqrt{n^{1+\sigma}}} + \frac{4D}{n\sqrt{n}} + \frac{D}{n} \quad (x \geqq b). \tag{24}$$

3° Die Behauptung des Satzes folgt sofort aus (13) und (24).

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RADICALS, SEMISIMPLE CLASSES AND TORSION THEORIES

By

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1. Introduction

In the light of the recent results of [7], [9] and [13], the problem arose to clarify the connections among the algebraic properties characterizing radical, co-radical, semisimple, torsion and torsionfree classes of rings. It is the purpose of this paper to give various characterizations of the classes mentioned above and to point out some logical independence among them. Theorem 5 and Propositions 7 and 8 are analogous to some results of R. MLITZ [10] gained for multioperator-groups by a different approach.

Throughout this paper **A** will denote a *universal class* (i.e. non-empty, hereditary and homomorphically closed class) of *not necessarily associative rings*. All considerations done in **A** remain valid in any universal class of multioperator groups (thus, for instance, in the variety of near-rings), or in categories satisfying some additional requirements. Nevertheless, sometimes we have to confine ourselves to associative rings. We also assume that all classes considered are abstract classes and contain the one-element ring. As usual, let us define the following operators U and S acting on classes of rings by

$$U\mathbf{X} = \{A\in\mathbf{A} \mid A \text{ has no nonzero homomorphic image in } \mathbf{X}\}$$

and

$$S\mathbf{X} = \{A\in\mathbf{A} \mid A \text{ has no nonzero ideal in } \mathbf{X}\}.$$

Further, let us associate for any class **X** and to any ring A the ideals

$$\mathbf{X}(A) = \sum_{\alpha} (I_\alpha \lhd A \mid I_\alpha \in \mathbf{X})$$

and

$$(A)\mathbf{X} = \bigcap_{\beta} (K_\beta \lhd A \mid A/K_\beta \in \mathbf{X}).$$

We shall frequently refer to the following conditions a subclass **X** of **A** may satisfy.

(A) If $A\in\mathbf{X}$, then every nonzero homomorphic image B of A has a nonzero ideal in **X**.

(B) If every nonzero homomorphic image B of a ring A has a nonzero ideal in **X**, then $A\in\mathbf{X}$.

(A*) **X** is *a regular class,* that is, if $A\in\mathbf{X}$, then every nonzero ideal B of A has a nonzero homomorphic image in **X**.

(B*) If every nonzero ideal B of a ring A has a nonzero homomorphic image in **X**, then $A\in\mathbf{X}$.

(a) **X** is homomorphically closed.

(a*) $\mathbf{X}$ is hereditary, that is, $I \lhd A \in \mathbf{X}$ implies $I \in \mathbf{X}$.
(b) $\mathbf{X}(A) \in \mathbf{X}$ for every $A \in \mathbf{A}$.
(b*) $A/(A)\mathbf{X} \in \mathbf{X}$ for every $A \in \mathbf{A}$ (or equivalently: $\mathbf{X}$ is closed under subdirect sums).
(b_0) $\mathbf{X}$ has the *inductive property*, that is, if $I_1 \subseteq \ldots \subseteq I_\alpha \subseteq \ldots$ is an ascending chain of ideals of any ring A such that $I_\alpha \in \mathbf{X}$ for each I_α then $\bigcup_\alpha I_\alpha \in \mathbf{X}$.
(b_0^*) $\mathbf{X}$ has the *co-inductive property*, that is, if $A \supseteq \ldots \supseteq I_\alpha \supseteq \ldots$ is a descending chain of ideals of a ring A such that each A/I_α is in $\mathbf{X}$, then also $A/\bigcap_\alpha I_\alpha \in \mathbf{X}$.
(c)=(c*) $\mathbf{X}$ is closed under extensions, that is, if $I \lhd A$ and I, $A/I \in \mathbf{X}$, then also $A \in \mathbf{X}$.
(c_0) $\mathbf{X}(A/\mathbf{X}(A)) = 0$ for every $A \in \mathbf{X}$.
(1) $\mathbf{X}(I) \subseteq \mathbf{X}(A)$ for every ideal I of any ring $A \in \mathbf{A}$.
(1*) $(I)\mathbf{X} \subseteq (A)\mathbf{X}$ for every ideal I of any ring $A \in \mathbf{A}$.
(2) $\mathbf{X}(A) = (A)S\mathbf{X}$ for every ring $A \in \mathbf{A}$.
(2*) $U\mathbf{X}(A) = (A)\mathbf{X}$ for every ring $A \in \mathbf{A}$.
(3) $((A)\mathbf{X})\mathbf{X} = (A)\mathbf{X}$ for every ring $A \in \mathbf{A}$.
(4) $((A)\mathbf{X})\mathbf{X} \lhd A$ for every ring $A \in \mathbf{A}$.

2. Radical classes

A subclass $\mathbf{R}$ of $\mathbf{A}$ is called a *radical class* in the sense of Kurosh and Amitsur, if $\mathbf{R}$ satisfies conditions (A) and (B).

Assume that a subclass $\mathbf{P}$ of $\mathbf{A}$ satisfies condition (A). This means that every nonzero homomorphic image B of a ring $A \in \mathbf{P}$ is not in $S\mathbf{P}$, that is, $A \in US\mathbf{P}$ holds.

Suppose that a subclass $\mathbf{Q}$ of $\mathbf{A}$ satisfies condition (B). This means the implication: if for every nonzero homomorphic image B of a ring A we have $B \notin S\mathbf{Q}$ then $A \in \mathbf{Q}$, that is, if A has no nonzero homomorphic image in $S\mathbf{Q}$ then $A \in \mathbf{Q}$, that is, if $A \in US\mathbf{Q}$ then $A \in \mathbf{Q}$.

Thus we have got

PROPOSITION 1. *A subclass* $\mathbf{P}$ *of* $\mathbf{A}$ *satisfies condition* (A) *if and only if* $\mathbf{P} \subseteq US\mathbf{P}$. *A subclass* $\mathbf{Q}$ *of* $\mathbf{A}$ *satisfies condition* (B) *if and only if* $US\mathbf{Q} \subseteq \mathbf{Q}$. *A subclass* $\mathbf{R}$ *of* $\mathbf{A}$ *is a radical class if and only if* $\mathbf{R} = US\mathbf{R}$.

It is well known that radical classes are characterized by conditions (a), (b) and (c), and here condition (c) can be substituted by condition (c_0).

PROPOSITION 2. *Let* $\mathbf{P}$ *be a subclass of* $\mathbf{A}$ *satisfying conditions* (a) *and* (c). *Then the following are equivalent:*

i) $\mathbf{P}$ *satisfies* (b) *(and hence* $\mathbf{P}$ *is a radical class);*

ii) *For every ring* $A \in \mathbf{A}$ *the sum* $\sum_\alpha I_\alpha$ *of any set of* $\mathbf{P}$*-ideals of* A *is again a* $\mathbf{P}$*-ideal;*

iii) $\mathbf{P}$ *satisfies* (b_0);

iv) *For every ring* $A \in \mathbf{A}$ *the set of all* $\mathbf{P}$*-ideals of* A *has a largest element.*

The equivalence of i) and iii) is well known. The other implications are straightforward.

Next, we give two characterizations of radical classes which are not surprising but seem to be interesting in the context of studying semisimple classes.

THEOREM 3. *A subclass* $\mathbf{R}$ *of* $\mathbf{A}$ *is a radical class if and only if* $\mathbf{R}$ *satisfies conditions* (A), (b) *and* (c).

PROOF. All what we have to prove is that a class $\mathbf{R}$ with properties (A), (b) and (c) is homomorphically closed. Suppose that $\mathbf{R}$ is not homomorphically closed. Then there is a ring $A\in\mathbf{R}$ and an ideal I of A such that $A/I\notin\mathbf{R}$. By (A) the ring A/I has a nonzero $\mathbf{R}$-ideal, hence by (b) we get $0\neq\mathbf{R}(A/I)=K/I$ where K is a suitable ideal of A. Again by (A) and (b) the ring A/K has a nonzero radical $0\neq\mathbf{R}(A/K)=L/K$, further $L/K\cong\dfrac{L/I}{K/I}$ holds. Hence (c) implies $L/I\in\mathbf{R}$ which means that $L/I\subseteqq\mathbf{R}(A/I)=K/I$, that is $L\subseteqq K$. Thus $\mathbf{R}(A/K)=L/K=0$ follows, a contradiction.

THEOREM 4. *A subclass* $\mathbf{R}$ *of* $\mathbf{A}$ *is a radical class if and only if* $\mathbf{R}$ *satisfies conditions* (A), (b_0) *and* (c).

PROOF. Again, we have to see that (A), (b_0) and (c) imply (a). Suppose that $\mathbf{R}$ is not homomorphically closed. Then there is a ring $A\in\mathbf{R}$, and an ideal I of A such that $A/I\notin\mathbf{R}$. By (A) A/I has a nonzero $\mathbf{R}$-ideal K_1/I. Further, again by (A), A/K_1 has a nonzero ideal $K_2/K_1\in\mathbf{R}$. Now we have

$$K_2/K_1\cong\frac{K_2/I}{K_1/I}$$

and condition (c) implies $K_2/I\in\mathbf{R}$. Continuing this procedure we get an ascending chain $K_1/I\subseteqq\ldots\subseteqq K_n/I\subseteqq\ldots$ of ideals of A/I such that K_n/I is in $\mathbf{R}$ for every $n=1,2,\ldots$. Defining K_ω/I as $K_\omega/I=\bigcup_{n=1}^{\infty}K_n/I$, condition ($b_0$) implies $K_\omega/I\in\mathbf{R}$. Hence we get a transfinite ascending chain $K_1/I\subseteqq\ldots\subseteqq K_\alpha/I\subseteqq\ldots$ of ideals of A which must terminate at an ordinal ζ (card $\zeta\leqq$ card $2^{|A/I|}$). Hence

$$\frac{K_{\zeta+1}/I}{K_\zeta/I}\cong K_{\zeta+1}/K_\zeta=0,$$

and by (b_0) we get $K_\zeta/I\in\mathbf{R}$. If $K_\zeta\neq A$, then condition (A) would imply $\dfrac{K_{\zeta+1}}{K_\zeta}\neq0$ which is impossible. Therefore $K_\zeta=A$ and $A/I=K_\zeta/I\in\mathbf{R}$ hold, contradicting the assumption.

The following characterization of radical classes is not of the usual kind.

THEOREM 5 (cf. MLITZ [10]). *A subclass* $\mathbf{R}$ *of* $\mathbf{A}$ *is a radical class if and only if* $\mathbf{R}$ *satisfies conditions* (b) *and* (2).

PROOF. Assume that $\mathbf{R}$ is a radical class, and consider any ring A. By condition (c_0) we get $A/\mathbf{R}(A)\in S\mathbf{R}$ yielding $(A)S\mathbf{R}\subseteqq\mathbf{R}(A)$. Let K be any ideal of the ring A such that $A/K\in S\mathbf{R}$. Then we have

$$\mathbf{R}(A)/(\mathbf{R}(A)\cap K)\cong(\mathbf{R}(A)+K)/K\lhd A/K\in S\mathbf{R}.$$

Since by (b) $\mathbf{R}(A)\in\mathbf{R}$ and $\mathbf{R}$ is homomorphically closed, we have also $(\mathbf{R}(A)+K)/K\in\mathbf{R}$. Hence $(\mathbf{R}(A)+K)/K=0$, that is $\mathbf{R}(A)\subseteq K$, and the definition of $(A)S\mathbf{R}$ yields $\mathbf{R}(A)\subseteq(A)S\mathbf{R}$. Thus condition (2) is satisfied.

Conversely, assume that a class $\mathbf{X}$ satisfies conditions (b) and (2) and let $A\in US\mathbf{X}$. Then A has no nonzero homomorphic image in $S\mathbf{X}$. Hence condition (2) implies $A=(A)S\mathbf{X}=\mathbf{X}(A)$ and by condition (b) we get $A\in\mathbf{X}$. Hence $US\mathbf{X}\subseteq\mathbf{X}$ holds. If $B\in\mathbf{X}$, then we have by (2) $B=\mathbf{X}(B)=(B)S\mathbf{X}$ and so B has no nonzero homomorphic image in $S\mathbf{X}$. Hence $\mathbf{X}\subseteq US\mathbf{X}$ is valid, implying $\mathbf{X}=US\mathbf{X}$.

Thus by Proposition 1 $US\mathbf{X}$ is a radical class.

3. Semisimple classes

Semisimple classes are usually defined via radical classes as follows: the *semisimple class* of a radical class $\mathbf{R}$ in the universal class $\mathbf{A}$, is the class $S\mathbf{R}$. This way of defining semisimple classes suggests the priority of radical classes, which is misleading.

Let us recall that in a universal class $\mathbf{A}$ of rings a subclass $\mathbf{S}$ is a semisimple class of some radical class $\mathbf{R}$ if and only if $\mathbf{S}$ satisfies conditions (A^*) and (B^*), and then $\mathbf{R}=U\mathbf{S}$ (cf. [5] Theorems 6 and 7, or [14] Proposition 6.3 and Theorem 7.4). Moreover, if $\mathbf{M}$ is a regular class in $\mathbf{A}$, then $U\mathbf{M}$ is a radical class, the so-called *upper radical* of $\mathbf{M}$, and $U\mathbf{M}$ is the largest radical class whose semisimple class $SU\mathbf{M}$ contains $\mathbf{M}$ (cf. [5] Theorem 8 or [14] Theorem 7.2). Since conditions (A^*) and (B^*) are dual to (A) and (B), radical classes and semisimple classes may be regarded as dual notions, which is again misleading.

Analogously to Proposition 1 we can prove

PROPOSITION 6. *A subclass* $\mathbf{P}$ *of* $\mathbf{A}$ *satisfies condition* (A^*) *if and only if* $\mathbf{P}\subseteq SU\mathbf{P}$. *A subclass* $\mathbf{Q}$ *of* $\mathbf{A}$ *satisfies condition* (B^*) *if and only if* $SU\mathbf{Q}\subseteq\mathbf{Q}$. *A subclass* $\mathbf{S}$ *of* $\mathbf{A}$ *is a semisimple class if and only if* $\mathbf{S}=SU\mathbf{S}$.

Every semisimple class $\mathbf{S}$ satisfies also conditions (b^*), (b_0^*), and (c) (cf. [5] Theorem 12 or [14] Theorem 22.8 and [5] Theorem 11 or [14] Proposition 8.5).

PROPOSITION 7 (cf. MLITZ [10]). *Let* $\mathbf{S}$ *be a semisimple class in* $\mathbf{A}$. *Then* (2^*) *holds.*

PROOF. Since by (c_0) we have $A/U\mathbf{S}(A)\in SU\mathbf{S}=\mathbf{S}$, so by the definition of $(A)\mathbf{S}$ it follows that $(A)\mathbf{S}\subseteq U\mathbf{S}(A)$. Further, ($b^*$) implies $A/(A)\mathbf{S}\in\mathbf{S}$. Since $U\mathbf{S}(A)/(A)\mathbf{S}\in U\mathbf{S}$ by (a), from (A^*) it follows $U\mathbf{S}(A)/(A)\mathbf{S}=0$, that is $U\mathbf{S}(A)\subseteq(A)\mathbf{S}$.

PROPOSITION 8 (cf. MLITZ [10]). *Let* $\mathbf{Q}$ *be a subclass of* $\mathbf{A}$ *satisfying conditions* (A^*), (b^*) *and* (c). *Then conditions* (2^*), (3) *and* (4) *are equivalent on* $\mathbf{Q}$.

PROOF. Suppose (2^*). Since by (A^*) the class $U\mathbf{Q}$ is a radical class, $U\mathbf{Q}$ has property (b). Hence by (2^*) and (b) we get

$$((A)\mathbf{Q})\mathbf{Q}=U\mathbf{Q}(U\mathbf{Q}(A))=U\mathbf{Q}(A)=(A)\mathbf{Q}.$$

If (3) is satisfied, then by definition $((A)\mathbf{Q})\mathbf{Q}=(A)\mathbf{Q}\lhd A$ holds.

Assume the validity of (4) and consider the isomorphism

$$\frac{A/((A)\mathbf{Q})\mathbf{Q}}{(A)\mathbf{Q}/((A)\mathbf{Q})\mathbf{Q}} \cong A/(A)\mathbf{Q}.$$

By (b*) we have $A/(A)\mathbf{Q} \in \mathbf{Q}$ and also $(A)\mathbf{Q}/((A)\mathbf{Q})\mathbf{Q} \in \mathbf{Q}$. Hence by (c) we have $A/((A)\mathbf{Q})\mathbf{Q} \in \mathbf{Q}$. Thus the definition of $(A)\mathbf{Q}$ yields $(A)\mathbf{Q} \subseteq ((A)\mathbf{Q})\mathbf{Q} \subseteq (A)\mathbf{Q}$, consequently, $(A)\mathbf{Q}$ has no nonzero homomorphic image in $\mathbf{Q}$, that is $(A)\mathbf{Q} \in U\mathbf{Q}$, which implies $(A)\mathbf{Q} \subseteq U\mathbf{Q}(A)$. As in the proof of Proposition 7, we can again conclude $U\mathbf{Q}(A) \subseteq (A)\mathbf{Q}$, too.

COROLLARY 9. *If* $\mathbf{S}$ *is a semisimple class of* $\mathbf{A}$, *then* $\mathbf{S}$ *satisfies conditions* (3) *and* (4).

THEOREM 10. *A subclass* $\mathbf{S}$ *of* $\mathbf{A}$ *is a semisimple class if and only if* $\mathbf{S}$ *satisfies conditions* (A*), (b*), (c) *and* (4).

PROOF. In view of the previous considerations and Corollary 9, conditions (A*), (b*), (c) and (4) are necessary.

For the sufficiency let us assume that a subclass $\mathbf{S}$ of $\mathbf{A}$ satisfies conditions (A*), (b*), (c) and (4). We have to prove that $SU\mathbf{S}=\mathbf{S}$. The inclusion $\mathbf{S} \subseteq SU\mathbf{S}$ is obvious by Proposition 6. To prove the inclusion $SU\mathbf{S} \subseteq \mathbf{S}$, let us consider a ring $A \in SU\mathbf{S}$. Now $U\mathbf{S}(A)=0$ holds, so Proposition 8 implies $(A)\mathbf{S}=U\mathbf{S}(A)=0$. Hence by the definition of $(A)\mathbf{S}$ and by (b*) it follows $A \in \mathbf{S}$.

REMARK. Condition (b*) implies (b_0^*) immediately. It is an open question whether conditions (A*), (b_0^*), (c) and (4) imply the validity of (b*) (cf. Proposition 12).

In the context of semisimple classes there are two crucial conditions, namely (4) and (a*). Let us observe that condition (A*) is a trivial consequence of (a*), further by [1] Lemma 2 a semisimple class has (a*) if and only if it satisfies (1).

The dual assertion to that of Theorem 5 characterizes the semisimple classes. This result is a sharpening of [7] Corollary 1.

THEOREM 11. *A subclass* $\mathbf{S}$ *of* $\mathbf{A}$ *is a semisimple class if and only if* $\mathbf{S}$ *satisfies conditions* (b*) *and* (2*).

PROOF. Proposition 7 yields the necessity. For the sufficiency, assume that $\mathbf{S}$ satisfies conditions (b*) and (2*). If $A \in SU\mathbf{S}$, then A has no nonzero ideal in $U\mathbf{S}$. Hence by (2*) we have $(A)\mathbf{S}=U\mathbf{S}(A)=0$, and now by (b*) we get $A \in \mathbf{S}$. Hence $SU\mathbf{S} \subseteq \mathbf{S}$. Let $B \in \mathbf{S}$. Then $(B)\mathbf{S}=0$ and (2*) implies $U\mathbf{S}(B)=0$, that is B has no nonzero ideal in $U\mathbf{S}$. Thus $B \in SU\mathbf{S}$ holds, implying $\mathbf{S} \subseteq SU\mathbf{S}$.

A subclass $\mathbf{Q}$ of $\mathbf{A}$ is called a *co-radical class,* if $\mathbf{Q}$ satisfies conditions (a*), (b_0^*) and (c). The dual assertion to that of Proposition 2 characterizes the co-radical classes (cf. [8] Theorem 1).

PROPOSITION 12. *Let* $\mathbf{Q}$ *be a subclass of* $\mathbf{A}$ *satisfying conditions* (a*) *and* (c). *The following are equivalent:*

i) $\mathbf{Q}$ *satisfies* (b*);

ii) $\mathbf{Q}$ *is closed under subdirect sums;*

iii) $\mathbf{Q}$ *satisfies* (b_0^*) *(and hence* $\mathbf{Q}$ *is a co-radical class);*

iv) *For every ring* $A\in\mathbf{A}$ *there exists a smallest ideal* K *of* A *relative to the property* $A/K\in\mathbf{Q}$.

PROOF. The proof of the equivalence of i) and iii) is the same as that of [9] Theorem 3. Conditions (i), (ii) and (iv) are equivalent for any class of rings.

COROLLARY 13. *In the universal class of all rings conditions* (a*) *and* (4) *are logically independent in the following sense: neither* (a*), (b*) *and* (c) *imply* (4), *nor* (A*), (b*), (c) *and* (4) *imply* (a*).

PROOF. In virtue of Proposition 12, classes with properties (a*), (b*), (c) are co-radical classes, meanwhile conditions (A*), (b*), (c) and (4) characterize semisimple classes. In [6], [12] and [7] it has been exhibited that a co-radical class need not be a semisimple class and vice versa.

In this context let us mention that in the universal class of all associative or alternative rings every semisimple class is hereditary and every co-radical class satisfies condition (4) (cf. [1], [13] and [9]). Further, recently NIKITIN [11] has announced that in the class of Jordan algebras over an associative and commutative ring with 1/2, every semisimple class is hereditary and hence is a co-radical class.

Concerning *associative rings* we have the following two theorems. In what follows, $Z(n)$ will denote the zero-ring over the cyclic group of n elements, where $n=1, 2, \dots, \infty$.

THEOREM 14. $\mathbf{S}$ *is the semisimple class of a not necessarily hereditary supernilpotent radical if and only if* $\mathbf{S}$ *satisfies conditions* (A*), (b*), (c) *and* $Z(\infty)\notin\mathbf{S}$.

PROOF. The necessity is trivial.

We claim that a class $\mathbf{S}$ satisfying conditions (A*), (b*), (c) and $Z(\infty)\notin\mathbf{S}$, does not contain nonzero nilpotent rings. Assume that $\mathbf{S}$ contains a zero-ring A. By condition (A*), the cyclic zero-ring (r) generated by an element $r\neq 0$ of A, has a nonzero homomorphic image in $\mathbf{S}$. Hence $Z(n)\in\mathbf{S}$ for some $n=1, 2, \dots, \infty$. By the hypothesis, n must be finite. Let p be a prime factor of n, then $Z(p)$ can be embedded as an ideal into $Z(n)$ in the natural way, so again by (A*) we obtain that $Z(p)\in\mathbf{S}$. Applying condition (c) it follows that $Z(p^k)\in\mathbf{S}$ for every $k=1, 2, \dots$. Since $\bigcap_{k=1}^{\infty}(p^k)=0$, condition (b*) implies

$$Z(\infty)\cong\sum_{\text{subdirect}} Z(p^k)\in\mathbf{S},$$

contradicting the assumption. Thus $\mathbf{S}$ does not contain nonzero zero-rings. One can easily see that the class $\mathbf{S}$ must not contain nonzero nilpotent rings either, from which it follows that the class $\mathcal{U}\mathbf{S}$ is supernilpotent.

Next, we show that the class $\mathbf{S}$ satisfies also condition (B*) and so $\mathbf{S}$ is a semisimple class. Suppose that every nonzero ideal of a ring A has a nonzero homomorphic image in $\mathbf{S}$, but $A\notin\mathbf{S}$. Consider the ideal $J=(A)\mathbf{S}$ of A. Since $A\notin\mathbf{S}$, we have $J\neq 0$. By the hypothesis the ideal J has a nonzero homomorphic image J/K in $\mathbf{S}$ with a suitable ideal K of J. Let $\bar{K}$ denote the ideal of A generated by K. If $K=\bar{K}$, then K is an ideal of A and taking into account that by (b*) $A/J\in\mathbf{S}$ and

$J/K \in \mathbf{S}$ hold, the isomorphism

$$\frac{A/K}{J/K} \cong A/J$$

and condition (c) yield $A/K \in \mathbf{S}$. Hence $J \subseteq K \subseteq J$ holds, implying $J/K = 0$, which contradicts the assumption on K. Thus $K \neq \bar{K}$ and $\bar{K}/K$ is a nonzero ideal of the ring $J/K \in \mathbf{S}$. Hence condition (A*) implies that $\bar{K}/K$ has a nonzero homomorphic image B in $\mathbf{S}$. But by the lemma of Andrunakievich we have $\bar{K}^3 \subseteq K$, hence $\bar{K}/K$ is a nilpotent ring and so is its homomorphic image B. This contradicts the fact that $\mathbf{S}$ does not contain nonzero nilpotent rings. Thus $\mathbf{S}$ satisfies condition (B*).

THEOREM 15. *$\mathbf{S}$ is a semisimple class of a (not necessarily hereditary) subidempotent radical if and only if $\mathbf{S}$ satisfies conditions* (A*), (b*), (c) *and the rings $Z(p^\infty)$ are in $\mathbf{S}$ for all primes p.*

PROOF. The necessity is again obvious. For the sufficiency first we show that $\mathbf{S}$ contains all nilpotent rings. Note that by condition (A*) we have $Z(p) \in \mathbf{S}$, for $Z(p)$ is simple and is contained in $Z(p^\infty)$ as an ideal. Let C be any zero-ring. Now C is a subdirect sum of subdirectly irreducible zero-rings. As is well known, the subdirectly irreducible zero-rings are precisely the rings $Z(p^k)$, $k = 1, 2, \dots, \infty$, for all primes p. By the assumption and by condition (c) all these rings are in $\mathbf{S}$. Thus condition (b*) yields $C \in \mathbf{S}$. By induction it follows easily from condition (c) that $\mathbf{S}$ contains all nilpotent rings, whence the class $U\mathbf{S}$ is subidempotent.

Next, we exhibit the validity of condition (B*). Suppose that every nonzero ideal of A has a nonzero homomorphic image in $\mathbf{S}$. By this hypothesis A has a nonzero homomorphic image A/I in $\mathbf{S}$. Let us consider the ideal $J = (A)\mathbf{S}$. Since $J \subset I$, we have $J \neq A$. Further, condition (b*) implies $A/J \in \mathbf{S}$. We claim that $J = 0$. Assume that $J \neq 0$. Now by the assumption the ideal J of A has a nonzero homomorphic image J/K in $\mathbf{S}$. Let $\bar{K}$ denote the ideal of A generated by K. By the lemma of Andrunakievich we have $\bar{K}^3 \subseteq K$ and so $J/\bar{K}^3 \neq 0$. Further we have

$$\frac{J/\bar{K}^3}{K/\bar{K}^3} \cong J/K \in \mathbf{S}$$

and the ring $K/\bar{K}^3$ is nilpotent. Thus $K/\bar{K}^3 \in \mathbf{S}$ holds, and condition (c) yields $J/\bar{K}^3 \in \mathbf{S}$. Taking into consideration the isomorphism

$$\frac{A/\bar{K}^3}{J/\bar{K}^3} \cong A/J \in \mathbf{S},$$

again by condition (c) we obtain $A/\bar{K}^3 \in \mathbf{S}$. Hence by the definition of J we have $J \subseteq \bar{K}^3 \subseteq K$, contradicting the choice of K. Thus $J = 0$ holds, implying $A \in \mathbf{S}$. Thus $\mathbf{S}$ satisfies condition (B*), too.

Let us remark that the first parts of the proofs of Theorems 14 and 15 are similar to those of [8] Lemmas 3 and 7, respectively. We do not know whether conditions (A*), (b*), (c) characterize the semisimple classes of associative rings.

4. Torsion theories

Following [7], a pair $(\mathbf{T}, \mathbf{F})$ of subclasses $\mathbf{T}$ and $\mathbf{F}$ of a universal class $\mathbf{A}$ is called a *torsion theory,* if $\mathbf{T}$ and $\mathbf{F}$ satisfy the following conditions:

(I) $\mathbf{T}\cap\mathbf{F}$ consists of one-element rings,
(II) $\mathbf{T}$ satisfies condition (a),
(III) $\mathbf{F}$ satisfies condition (a*),
(IV) For every ring $A\in\mathbf{A}$ there is an ideal $I\lhd A$ such that $I\in\mathbf{T}$ and $A/I\in\mathbf{F}$.

The classes $\mathbf{T}$ and $\mathbf{F}$ will be referred to as a *torsion* and a *torsionfree class*, respectively. In view of [7] Theorem 1 the torsion classes are exactly the radical classes with property (1) and the torsionfree classes are precisely the hereditary semisimple classes (or equivalently: coradical classes with property (4)). A radical class $\mathbf{R}$ is called a *strict radical,* if for every ring $A\in\mathbf{A}$ the radical $\mathbf{R}(A)$ contains every subring $B\subseteqq A$ with $B\in\mathbf{R}$. Gardner called a radical $\mathbf{R}$ an *A-radical,* if $\mathbf{R}$ has the following property:

$$A\in\mathbf{R} \quad \text{and} \quad A^+\cong B^+ \quad \text{imply} \quad B\in\mathbf{R}$$

where X^+ denotes the additive group of the ring X. In [4] it has been proved that *in the universal class of all rings the notions of torsion classes, strict radicals and A-radicals coincide,* though for associative or alternative rings these three notions are different ones.

From our previous results we can deduce new characterizations of torsion theories.

THEOREM 16. *The following conditions are equivalent:*
(α) $(\mathbf{T}, \mathbf{F})$ *is a torsion theory,*
(β) $\mathbf{F}$ *is a torsionfree class and* $\mathbf{T}=U\mathbf{F}$,
(γ) $\mathbf{F}$ *has properties* (b*), (2*) *and* (1*) *and* $\mathbf{T}=U\mathbf{F}$,
(δ) $\mathbf{T}$ *is a torsion class and* $\mathbf{F}=S\mathbf{T}$,
(ε) $\mathbf{T}$ *has properties* (b), (2) *and* (1) *and* $\mathbf{F}=S\mathbf{T}$.

PROOF. $(\alpha)\Rightarrow(\beta)$ is trivial in view of [7] Theorem 1.

$(\beta)\Rightarrow(\gamma)$. Since a torsionfree class is always a semisimple class, $\mathbf{F}$ enjoys properties (b*) and (2*) by Theorem 11. Moreover, $\mathbf{T}=U\mathbf{F}$ is a radical class satisfying properties (1) and (2) and $\mathbf{F}=S\mathbf{T}$ holds. Hence for every ideal I of any ring $A\in\mathbf{A}$ we have

$$(I)\mathbf{F} = (I)S\mathbf{T} = \mathbf{T}(I) \subseteqq \mathbf{T}(A) = (A)S\mathbf{T} = (A)\mathbf{F},$$

that is $\mathbf{F}$ satisfies condition (1*).

$(\gamma)\Rightarrow(\delta)$. Applying (2*) and (1*) and again (2*) to $\mathbf{T}=U\mathbf{F}$ and to each ideal I of every ring $A\in\mathbf{A}$, we get

$$\mathbf{T}(I) = U\mathbf{F}(I) = (I)\mathbf{F} \subseteqq (A)\mathbf{F} = U\mathbf{F}(A) = \mathbf{T}(A),$$

that is $\mathbf{T}$ satisfies condition (1), and by [7] Theorem 1, a radical class satisfying (1), is a torsion class.

$(\delta)\Rightarrow(\varepsilon)$ is trivial in view of [7] Theorem 1 and our Theorem 5.

$(\varepsilon)\Rightarrow(\alpha)$. By Theorem 5 **T** is a radical class. Since **T** has also property (1), [7] Theorem 1 yields the assertion.

COROLLARY 17. *A semisimple class is a torsionfree class if and only if it has property* (1*). *A semisimple class* **S** *is a torsionfree class if and only if* $(I)\mathbf{S}\subseteq I\cap(A)\mathbf{S}$ *holds for every ideal* I *of any ring* $A\in\mathbf{A}$.

The first equivalence follows from Theorems 11 and 16. The second one is straightforward.

Taking into account that torsionfree classes are exactly the hereditary semisimple classes, Theorem 11 yields

COROLLARY 18. *A subclass* **S** *of* **A** *is a torsionfree class if and only if* **S** *satisfies conditions* (a*), (b*) *and* (2*).

A subring B of a ring A is called an *accessible subring* of A, if there exist a natural number n and subrings $A_1, \dots, A_n$ of A, such that $B=A_n\lhd\dots\lhd A_1=A$. Let **M** be a hereditary class in **A** and let us form the class

$$\overline{\mathbf{M}}=\{A\in\mathbf{A}|\text{ every accessible subring of } A \text{ is in } S U\mathbf{M}\}.$$

By [6] Lemma 1 $\overline{\mathbf{M}}$ is the largest hereditary subclass of $SU\mathbf{M}$ and $U\overline{\mathbf{M}}=U\mathbf{M}$ holds. Moreover, by [7] Theorem 2 $\overline{\mathbf{M}}$ is a co-radical class, though $\overline{\mathbf{M}}$ need not be a torsionfree class (cf. [7] Corollary 5).

PROPOSITION 19. *Let* **M** *be a hereditary subclass of* **A**. *Then the following four conditions are equivalent:*

i) $\overline{\mathbf{M}}$ *is a torsionfree class;*
ii) $\overline{\mathbf{M}}=SU\mathbf{M}$;
iii) $\overline{\mathbf{M}}$ *satisfies condition* (2*);
iv) $SU\mathbf{M}$ *is hereditary.*

The proof is straightforward in view of $U\overline{\mathbf{M}}=U\mathbf{M}$.

5. Small ideals and semisimple classes

An ideal L of a ring A is said to be *large* in A, if $L\cap I=0$ implies $I=0$ for any ideal I of A. In view of [9] Theorems 1 and 2, the assertion of [8] Theorem 8 can be easily sharpened as follows:

THEOREM 20. *A subclass* **P** *of associative or alternative rings is a semisimple class and* $U\mathbf{P}$ *is a hereditary radical class if and only if* **P** *satisfies conditions* (a*), (b*) *and*

(λ) *If* L *is large in* A *and* $L\in\mathbf{P}$, *then also* $A\in\mathbf{P}$.

Dually, an ideal K of a ring A is said to be *small* in A, if $K+I=A$ implies $I=A$ for any ideal I of A. The dual condition of (λ) is now

(λ^*) *If* K *is small in* A *and* $A/K\in\mathbf{P}$, *then* $A\in\mathbf{P}$.

The next statement asserts that condition (λ^*) is incompatible with (a) and (b*).

THEOREM 21.[1] *Let* **V** *denote one of the varieties of all associative rings. If a subclass* **P** *of* **V** *satisfies conditions* (a), (b*) *and* (λ^*), *then either* **P** *coincides with* **V** *or* **P** *consists of one-element rings only.*

PROOF. Conditions (a) and (b*) imply that **P** is a variety (cf. for instance [14] Theorem 32.1). Next, we are going to show that **P** has property (c). For this purpose let I be an ideal of a ring A such that both I and A/I are in **P**. If I is small in A, then by (λ^*) we are done. Assume that I is not small in A. Then there exists an ideal K_1 of A such that $I+K_1=A, K_1\neq A$. Since $I\in\mathbf{P}$, by (a) we get $I/(I\cap K_1)\in\mathbf{P}$. Also by

$$K_1/(I\cap K_1)\cong(K_1+I)/I\lhd A/I\in\mathbf{P}$$

we have $K_1/(I\cap K_1)\in\mathbf{P}$, since **P** is hereditary. Then we have

$$A/(I\cap K_1)\cong(I+K_1)/(I\cap K_1)=I/(I\cap K_1)+K_1/(I\cap K_1)\in\mathbf{P}$$

by condition (b*). If $I\cap K_1$ is small in A, then we are done by (λ^*). If not, we may use the same argument and we get $A/((I\cap K_1)\cap K_2)\in\mathbf{P}$ for some ideal K_2 of A such that $K_2\neq A$ and that $(I\cap K_1)+K_2=A$. So putting $J_1=I, J_2=I\cap K_1, J_3=(I\cap K_1)\cap K_2$ and so on, we get a strictly descending chain

$$J_1\supset J_2\supset J_3\supset\ldots$$

of non-small **P**-ideals of A such that $A/J_n\in\mathbf{P}$. If this chain terminates at a finite number $k\geqq1$, then by condition (λ^*) it follows that also A is in **P**. If this chain does not terminate within finitely many steps, then by condition (b*) we conclude $A/J_\omega\in\mathbf{P}$ for $J_\omega=\bigcap_{n=1}^{\infty}J_n$. Moreover, the hereditariness of **P** implies $J_\omega\in\mathbf{P}$. If J_ω is small in A, then again by (λ^*) we are done. If not, continue the procedure. Thus we get a strictly descending transfinite chain of **P**-ideals J_α of A such that $A/J_\alpha\in\mathbf{P}$. This chain, however, must terminate in at most $2^{|A|}$ steps. Clearly, either the last ideal J_ζ is small in A, or $J_\zeta=0$. In the first case (λ^*), in the second one (b*) infers that $A\in\mathbf{P}$. Thus **P** is a variety closed under extensions.

Suppose that **P** contains a ring $A\neq0$. We claim that **P** contains also a zero-ring of at least two elements. The ring A is a subdirect sum of subdirectly irreducible rings. Let B be a subdirectly irreducible component of A with heart H. Since **P** is a variety, $A\in\mathbf{P}$ implies $H\in\mathbf{P}$. As it is well-known, H is either a zero-ring, or a simple idempotent ring. In the first case we are done. If H is a simple idempotent ring, then let us consider the *split-null extension*

$$C=H*H=\{(\alpha,a)|\alpha,a\in H\}$$

where the addition is defined componentwise and the multiplication by the rule

$$(\alpha,a)(\beta,b)=(\alpha\beta,a\beta+\alpha b).$$

A straightforward calculation shows that C is a subdirectly irreducible ring with heart $0\neq D=\{(0,a)|a\in H\}$ and $C/D\cong H\in\mathbf{P}$ holds. Since the heart of a subdirectly irreducible ring is always small in that ring, condition (λ^*) is applicable

[1] Thanks are due to L. Márki, who has kindly pointed out that by a slight modification of the proof the assertion was valid not only for associative rings.

and implies $C \in \mathbf{P}$. Taking into account that $\mathbf{P}$ is a variety, it follows $D \in \mathbf{P}$ and in addition $D^2=0$ holds.

A special case of Corollary 1.9 of [3] states that if a variety $\mathbf{P}$ of rings is contained in the variety $\mathbf{V}$, is closed under extensions and contains a zero-ring $\neq 0$, then $\mathbf{P}$ coincides with the variety $\mathbf{V}$.

REMARK. In the proof we used the facts that the heart of a subdirectly irreducible ring of $\mathbf{V}$ is either a zero-ring or a simple idempotent ring and that the split-null extension $C=H*H$ is in $\mathbf{V}$ (see e.g. [2]). Nevertheless a similar proof and Corollary 1,9 of [3] yield the following: *Let* $\mathbf{V}$ *be the variety of alternative, Jordan, power-associative or arbitrary algebras. If a subclass* $\mathbf{P}$ *of* $\mathbf{V}$ *satisfies conditions* (a), (b*) *and* (λ*) *and* $\mathbf{P}$ *contains a zero-algebra* ($\neq 0$), *then* $\mathbf{P}$ *coincides with* $\mathbf{V}$.

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[illegible] being inside [illegible] that [illegible] it follows D_2P and it [illegible] D_2P holds.

A [illegible] case of Corollary [illegible] (3) [illegible] variety P of [illegible] is contained in a variety V [illegible] closed under extensions and [illegible] then P coincides with the variety V.

Remark. In the proof we used the fact that the [illegible] of a [illegible] irreducible [illegible] of V is either [illegible] or a simple [illegible] and that the [illegible] extension [illegible] V (see [illegible] [2]) [illegible] a similar [illegible] Corollary 7 of [3] [illegible] power [illegible] If [illegible] of V [illegible] condition (b) [illegible] (b') [illegible] and P contains [illegible] then P coincides with V.

References

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ON THE 2-DISTRIBUTIVITY OF SUBLATTICE LATTICES

By
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I. Introduction

The concept of *n-distributivity* was introduced by HUHN (cf. [4] and [6]). A lattice is said to be n-distributive ($n \geqq 1$) if it satisfies the identity

$$x \wedge \bigvee_{i=0}^{n} y_i \leqq \bigvee_{j=0}^{n} \Bigl(x \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n} y_i \Bigr).$$

The n-distributivity of subalgebra lattices of universal algebras proved to be an important property in several cases (cf., e.g., HUHN [4, 5] and NATION [9]). Sublattice lattices were investigated by FILIPPOV [2]. Lattices having modular and (upper) semi-modular sublattice lattices were characterized by KOH [7] and LAKSER [8], respectively. In [1] we have given a structure theorem for distributive lattices having 2-distributive sublattice lattices. In this paper lattices having 2-distributive sublattice lattices will be characterized. A necessary and sufficient condition for distributive lattices to have n-distributive sublattice lattices will be also given. A structure theorem for modular lattices having 2-distributive sublattice lattices will be deduced from the mentioned result of [1].

In what follows, for a lattice L and a subset H of L, let $\mathrm{Su}(L)$ and $[H]$ denote the lattice of sublattices of L and the sublattice generated by H, respectively. ($\mathrm{Su}(L)$ contains the empty set.)

II. Distributive lattices having *n*-distributive sublattice lattices

We intend to prove the following

THEOREM 1. *For an arbitrary distributive lattice L and integer* $n \geqq 1$ *the following two conditions are equivalent:*

(i) $\mathrm{Su}(L)$ *is n-distributive;*

(ii) *for any* $(n+1)$*-element subset H of L we have*

$$[H] = \bigcup_{h \in H} [H \setminus \{h\}].$$

REMARK. Since every finitely generated free distributive lattice is finite, this theorem makes it, at least theoretically, possible to list finitely many finite distributive lattices for each n so that a distributive lattice L has an n-distributive sublattice lattice iff none of the listed lattices is a sublattice of L.

For example, in case $n=1$ only the four-element lattice which is not a chain has to be listed. In case $n=2$, as it follows from Theorem 2 (stated later), S_1 and S_4 (defined in Theorem 2) can be listed.

PROOF. By Lemma 1 in [1] it is enough to show that for any non-negative integer k, the following implication holds:

I_k: for an arbitrary lattice L, if L satisfies (ii) then $[H] = \bigcup_{\substack{G\subset H\\ |G|=n}} [G]$ holds for any $(n+k+1)$-element subset H of L. So, the proof goes via induction on k. I_0 is evident. Now suppose $I_0, I_1, \dots, I_{k-1}$ hold for some $k\geqq 1$, but I_k does not hold. Then there exist a lattice L, $H=\{h_1, h_2, \dots, h_{n+k+1}\}\subseteq L$ and an $(n+k+1)$-ary lattice polynomial p such that $p(h_1, \dots, h_{n+k+1})\notin {}^nH$, where ${}^nH = \bigcup_{\substack{G\subset H\\ |G|=n}} [G]$. By the distributivity of L, p can be supposed to be of disjunctive normal form

(1) $p(x_1, \dots, x_{n+k+1}) = p_1(x_1, \dots, x_{n+k+1}) \vee p_2(x_1, \dots, x_{n+k+1}) \vee \dots \vee p_d(x_1, \dots, x_{n+k+1})$

where $p_2, \dots, p_d$ are conjunctions of (some of their) variables, p_1 is of disjunctive normal form or is omitted (i.e., $p=p_2\vee \dots \vee p_d$), p_1 does not depend on all the $n+k+1$ variables, and all the $n+k+1$ variables occur in $p_1\vee p_2$ (or in p_2, if p_1 is omitted). Suppose both p and its disjunctive normal form (1) are chosen so that d is minimal. By the induction hypothesis, $p_1(h_1, \dots, h_{n+k+1}) = p_0(g_1, \dots, g_n)$ for some $\{g_1, \dots, g_n\}\subset H$ and n-ary polynomial p_0. At least two elements of H, say h_{n+k} and h_{n+k+1}, do not belong to $\{g_1, \dots, g_n\}$. Let $h_0 = h_{n+k}\wedge h_{n+k+1}$, $H_0 = \{h_0, h_1, \dots, h_{n+k-1}\}$ and observe that

$$p_1(h_1, \dots, h_{n+k+1}) \vee p_2(h_1, \dots, h_{n+k+1}) =$$

$$= p_0(g_1, \dots, g_n) \vee p_2(h_1, \dots, h_{n+k-1}, h_{n+k}, h_{n+k+1}) = q(h_0, h_1, \dots, h_{n+k-1})$$

for some polynomial q. By applying the induction hypothesis (first to H_0 and then once more if necessary) we obtain $q(h_0, \dots, h_{n+k-1}) = r(h_1, h_2, \dots, h_{n+k+1})$ where r is a polynomial depending on at most n variables. It can be assumed that r is of disjunctive normal form, whence either all the $n+k+1$ variables occur on the right hand side of the equation $p(h_1, \dots, h_{n+k+1}) = r(h_1, \dots, h_{n+k+1}) \vee \vee p_3(h_1, \dots, h_{n+k+1}) \vee \dots \vee p_d(h_1, \dots, h_{n+k+1})$, which contradicts the minimality of d, or $p(h_1, \dots, h_{n+k+1}) \in {}^nH$ is obtained from the induction hypothesis, which is a contradiction again. Q.E.D.

III. Lattices having 2-distributive sublattice lattices

We intend to prove the following

THEOREM 2. *For an arbitrary lattice L the following three conditions are equivalent:*

(i) Su(L) *is 2-distributive;*

(ii) *None of the lattices* $S_1, S_2, \dots, S_8$ *(see below) is a sublattice of* L*;*

(iii) *For any three elements a, b, c in L, if $a \parallel b$, $a \parallel c$ and $b \parallel c$ then $[a, b, c]$ is isomorphic to one of the lattices $P_1, P_2, \ldots, P_7$, while if $a \parallel b$, $a \parallel c$ and $b<c$ then $[a, b, c]$ is isomorphic to R_1 or R_2.*

We define the lattices occurring in Theorem 2 by their diagrams as follows (S^d stands for the dual of S):

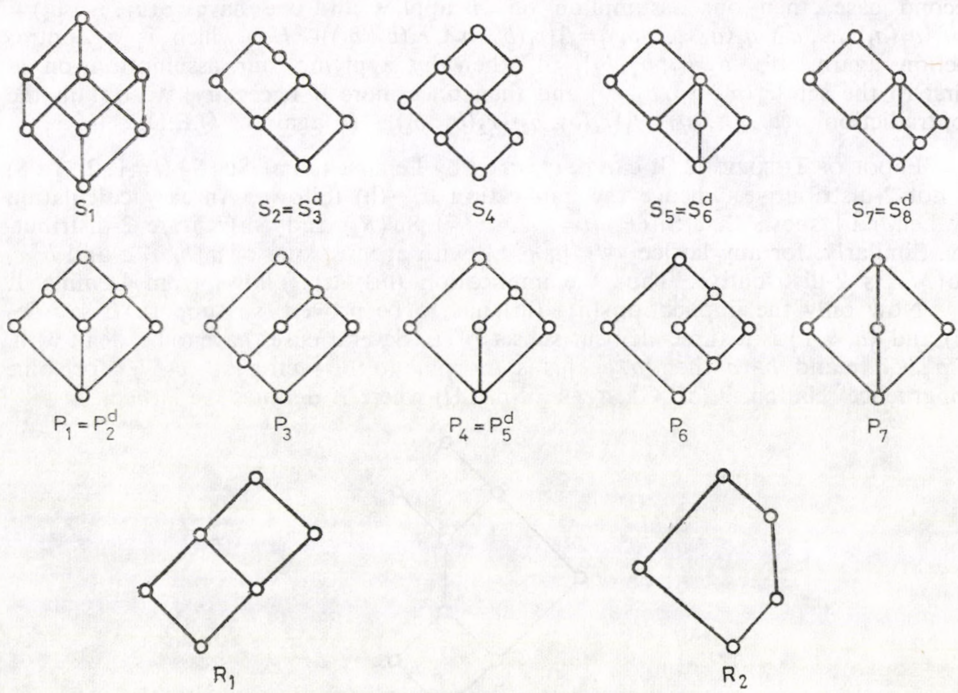

Fig. 1

In order to prove Theorem 2 we need the following

LEMMA 1. *For an arbitrary idempotent algebra A with at most binary fundamental operations, $\mathrm{Su}(A)$ is 2-distributive if and only if $[H]=\bigcup_{a\in H}[H\setminus\{a\}]$ for any three-element subset H of A.*

PROOF. Let us write ${}^2H$ instead of $\bigcup_{a,b\in H}[a, b]$. Consider an idempotent algebra A with at most binary fundamental operations and suppose $[H]={}^2H$ for any three-element subset H of A. By Lemma 1 of [1] it is enough to show that $[H]={}^2H$ holds for any subset H of A. In other words, it is enough to show that any k-ary polynomial p (in the similarity type of A) has the following property:

For any k-element subset $G=\{a_1, \ldots, a_k\}$ of A, there exist a binary polynomial q and elements b_1, b_2 in G such that $p(a_1, \ldots, a_k)=q(b_1, b_2)$.

Let us assume that this is not true and p is a polynomial of minimal length $|p|$ not having the above property. Then, for any k-element subset $H=\{a_1, \ldots, a_k\}$ of A we have either $p(a_1, \ldots, a_k)=f_0(q_0(a_1, \ldots, a_k))$ or $p(a_1, \ldots, a_k)=$

$=f_1(q_1(a_1, \dots, a_k), q_2(a_1, \dots, a_k))$, where q_0, q_1 and q_2 are k-ary polynomials, f_0 and f_1 are fundamental operations, f_1 is binary and f_0 is at most unary. In both cases $|q_i|<|p|$ implies $q_i(a_1, \dots, a_k)=r_i(b_{2i}, b_{2i+1})$ $(i=0, 1, 2)$ for some binary polynomial r_i and elements b_{2i}, b_{2i+1} in H. Thus in the first case $p(a_1, \dots, a_k)= =f_0(q_0(a_1, \dots, a_k))=f_0(r_0(b_0, b_1))\in {}^2H$, a contradiction. If $|\{b_2, b_3, b_4, b_5\}|\leqq 3$ in the second case, then our assumption on A applies and we have $p(a_1, \dots, a_k)= =f_1(q_1(a_1, \dots, a_k), q_2(a_1, \dots, a_k))=f_1(r_1(b_2, b_3), r_2(b_4, b_5))\in {}^2H$, which is a contradiction again. If $|\{b_2, b_3, b_4, b_5\}|=4$, then by applying our assumption on A (first to the set $\{r_1(b_2, b_3), b_4, b_5\}$ and then once more if necessary) we obtain the contradiction $p(a_1, \dots, a_k)=f_1(r_1(b_2, b_3), r_2(b_4, b_5))\in {}^2H$ again. Q.E.D.

PROOF OF THEOREM 2. It can be checked by Lemma 1 that $\mathrm{Su}(S_i)$ $(i=1, 2, \dots, 8)$ is not 2-distributive, whence the implication (i)→(ii) follows. An easy calculation by Lemma 1 shows that $\mathrm{Su}(P_i)$ $(i=1, 2, \dots, 7)$, $\mathrm{Su}(R_1)$ and $\mathrm{Su}(R_2)$ are 2-distributive. Similarly, for any lattice $M=[a, b, c]$ with at most one of $a\parallel b$, $a\parallel c$ and $b\parallel c$, $\mathrm{Su}(M)$ is 2-distributive. Thus the implication (iii)→(i) follows from Lemma 1.

Now only the implication (ii)→(iii) has to be proved, so suppose L satisfies (ii) and $\{a, b, c\}$ is a three-element subset of L. Several cases have to be dealt with. If $a\parallel b$, $a\parallel c$ and $b<c$ then $[a, b, c]$ is isomorphic to the factor lattice K/Θ for some congruence relation Θ (cf. GRÄTZER [3, p. 11]) where K denotes the lattice

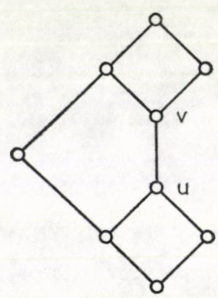

Fig. 2

Now it is not hard to check that $[a, b, c]\cong R_1$ in case $(u, v)\in\Theta$, and $[a, b, c]\cong R_2$ otherwise.

So we have proved that in case $a\parallel b$, $a\parallel c$ and $b<c$ $[a, b, c]$ is isomorphic to R_1 or R_2. This has an important consequence *that will be often used in the proof*: Suppose $a\parallel b$, $a\parallel c$ and $b<c$ $(a, b, c\in L)$. Then $a\vee b\neq a\vee c$ implies $a\wedge b=a\wedge c$ and $(a\vee b)\wedge c=b$. Similarly, $a\wedge b\neq a\wedge c$ implies $a\vee b=a\vee c$ and $(a\wedge c)\vee b=c$. Having three elements a, b, c in L with $a\parallel b$, $a\parallel c$ and $b<c$, *we shall refer to this consequence by* $R_{a,b,c}$.

In what follows let us assume $a\parallel b$, $a\parallel c$ and $b\parallel c$. We have

(2) $$a\vee b\vee c\in\{a\vee b, a\vee c, b\vee c\} \quad\text{and}\quad a\wedge b\wedge c\in\{a\wedge b, a\wedge c, b\wedge c\}$$

since otherwise $[a\vee b, a\vee c, b\vee c]$ or $[a\wedge b, a\wedge c, b\wedge c]$ would be isomorphic to S_1 by Lemma 9 in GRÄTZER [3, p. 38]. Let $j=|\{\{x, y\}\colon \{x, y\}\subseteqq\{a, b, c\}$ and $x\vee y= =a\vee b\vee c\}|$ and $m=|\{\{x, y\}\colon \{x, y\}\subseteqq\{a, b\ c\}$ and $x\wedge y=a\wedge b\wedge c\}|$. Then (2) states that $j\geqq 1$ and $m\geqq 1$.

Claim 1. If $j=1$ then $[a, b, c]\cong P_1$.

To prove this claim suppose $j=1$ and $a\vee c=a\vee b\vee c$. We obtain $a\wedge(b\vee c)=$ $=a\wedge b$ and $(a\vee b)\wedge c=b\wedge c$ from $R_{a,b,b\vee c}$ and $R_{c,b,a\vee b}$, respectively. If we had $(a\vee b)\wedge(b\vee c)>b$, then $\{a, b, a\wedge b, a\vee b, a\vee c, b\vee c, (a\vee b)\wedge(b\vee c)\}$ could be easily shown to be a sublattice of L isomorphic to S_3. Thus $(a\vee b)\wedge(b\vee c)=b$.

Case 1.1: $a\wedge b\parallel c$. We obtain $a\wedge b\wedge c=(a\vee b)\wedge c$ from $R_{c,a\wedge b,a\vee b}$. We have $(a\wedge b)\vee c<b\vee c$ since otherwise $\{a\wedge b, b, a\vee b, a\vee c, b\vee c, c, a\wedge b\wedge c\}$ would be a sublattice of L isomorphic to S_3. From $a\wedge b=a\wedge(a\wedge b)\leqq a\wedge((a\wedge b)\vee c)\leqq a\wedge(b\vee c)=$ $=a\wedge b$ we obtain $a\wedge((a\wedge b)\vee c)=a\wedge b$. Hence $\{a, c, b\vee c, a\wedge b, (a\wedge b)\vee c, a\wedge c,$ $a\vee c\}\cong S_2$, which is a contradiction. Therefore Case 1.1 is impossible.

Case 1.2: $a\wedge b\nparallel c$. We obtain $a\wedge b\wedge c=a\wedge b=a\wedge c$ from $a\wedge b<c$ and $a\wedge(b\vee c)=a\wedge b$.

Case 1.2.1: $b\wedge c\neq a\wedge b\wedge c$. If we had $a\vee b\neq a\vee(b\wedge c)$ then $\{a, b\wedge c, a\vee(b\wedge c),$ $a\vee b, a\wedge b, c, a\vee c\}$ would be isomorphic to S_2. Therefore $a\vee b=a\vee(b\wedge c)$ and so $\{a, b, a\vee b, b\wedge c, a\wedge b, b\vee c, a\vee c\}$ is isomorphic to S_3. This contradiction shows that Case 1.2.1 is impossible.

Case 1.2.2: $b\wedge c=a\wedge b\wedge c$. Then the earlier equalities yield $[a, b, c]\cong P_1$. This completes the proof of Claim 1.

Claim 2. If $j=2$ and $m=3$ then $[a, b, c]\cong P_4$.

To prove this claim suppose $b\vee c\neq a\vee b\vee c$.

Case 2.1: $a\wedge(b\vee c)\neq a\wedge b\wedge c$. Since $[a, b, c]\not\cong S_5$, we have either $(a\wedge(b\vee c))\vee$ $\vee b\neq b\vee c$ or $(a\wedge(b\vee c))\vee c\neq b\vee c$. Thus, e.g., $(a\wedge(b\vee c))\vee b\neq b\vee c$ can be assumed. But then $\{a, b, b\vee c, a\vee b, a\wedge b, a\wedge(b\vee c), (a\wedge(b\vee c))\vee b\}\cong S_2$, a contradiction, showing that Case 2.1 is impossible.

Case 2.2: $a\wedge(b\vee c)=a\wedge b\wedge c$. Then $[a, b, c]$ is isomorphic to P_4, which completes the proof of Claim 2.

For $j=m=2$ the only essentially different cases are

$$a\wedge b\neq a\wedge b\wedge c \quad\text{and}\quad b\vee c\neq a\vee b\vee c;$$

$$b\wedge c\neq a\wedge b\wedge c \quad\text{and}\quad b\vee c\neq a\vee b\vee c.$$

Claim 3. If $j=m=2, a\wedge b\neq a\wedge b\wedge c$ and $b\vee c\neq a\vee b\vee c$ then $[a, b, c]\cong P_3$.

To prove this claim suppose $[a, b, c]\not\cong P_3$. Then either $a\wedge(b\vee c)\neq a\wedge b$ or $(a\wedge b)\vee c\neq b\vee c$. By the lattice theoretical Duality Principle $a\wedge(b\vee c)\neq a\wedge b$ can be assumed. Then we have $(a\wedge b)\vee c\neq b\vee c$ as well, since otherwise $\{a, c, b\vee c,$ $a\wedge b, a\wedge(b\vee c), a\wedge c, a\vee c\}$ would be isomorphic to S_3. Similarly, we have $a\wedge(b\vee c)\not\equiv$ $\not\equiv(a\wedge b)\vee c$ since otherwise $\{a, c, b\vee c, (a\wedge b)\vee c, a\wedge(b\vee c), a\wedge c, a\vee c\}$ would be isomorphic to S_2. Therefore $\{a\wedge(b\vee c), b, (a\wedge b)\vee c\}$ is an antichain (i.e., a set of pairwise incomparable elements). Clearly, we have $a\wedge((a\wedge b)\vee c)=$ $=(a\wedge(b\vee c))\wedge((a\wedge b)\vee c)$. Therefore $(a\wedge(b\vee c))\vee((a\wedge b)\vee c)=b\vee c$ since otherwise $\{a, b\vee c, (a\wedge b)\vee c, a\wedge(b\vee c), a\vee c, a\wedge((a\wedge b)\vee c), (a\wedge(b\vee c))\vee((a\wedge b)\vee c)\}$ would be isomorphic to S_2. The dual argument shows $(a\wedge(b\vee c))\wedge((a\wedge b)\vee c)=a\wedge b$. From $R_{a,b,b\vee c}$ and $R_{c,a\wedge b,b}$ we get $(a\wedge(b\vee c))\vee b=b\vee c$ and $((a\wedge b)\vee c)\wedge b=a\wedge b$. On the other hand, it is clear that $(a\wedge(b\vee c))\wedge b=a\wedge b$ and $((a\wedge b)\vee c)\vee b=b\vee c$. It follows from the equations we have obtained that $\{a, b, a\wedge b, b\vee c, a\vee c, a\wedge(b\vee c), (a\wedge b)\vee c\}$ is isomorphic to S_5. This contradiction completes the proof of Claim 3.

Claim 4. If $j=m=2$, $b\wedge c\neq a\wedge b\wedge c$ and $b\vee c\neq a\vee b\vee c$ then $[a, b, c]$ is isomorphic to P_6.

Suppose $[a, b, c]\not\cong P_6$. Then we have $a\wedge(b\vee c)\neq a\wedge c$ or $a\vee(b\wedge c)\neq a\vee c$, so $a\wedge(b\vee c)\neq a\wedge c$ can be assumed by the Duality Principle. From $R_{a,b\wedge c,b\vee c}$ we obtain $a\vee(b\wedge c)=a\vee c$. We have $(a\wedge(b\vee c))\vee(b\wedge c)=b\vee c$ since otherwise $\{a, a\wedge(b\vee c), a\wedge c, a\vee c, b\vee c, b\wedge c, (a\wedge(b\vee c))\vee(b\wedge c)\}$ would be isomorphic to S_2. Therefore $[a, b, c]$ is isomorphic to S_7, which is a contradiction, completing the proof of Claim 4.

Clearly, if $j=m=3$, then $[a, b, c]\cong P_7$. Hence (2) together with Claims 1, 2, 3, 4 and their dual statements complete the proof of Theorem 2.

IV. Structure theorem for modular lattices having 2-distributive sublattice lattices

First we recall the notion of the *special sum of lattices* from [1]. Let a set of indices I which is a chain and lattices L_i $(i\in I)$ be given. Define the following binary relation ϑ on the ordinal sum $\sum_{i\in I} L_i$ of the lattices L_i:

$(a, b)\in\vartheta$ iff there exist $i, j\in I$ such that j covers i, a is the greatest element of L_i and b is the lowest element of L_j.

Let Θ be the equivalence relation generated by ϑ. Then Θ is a congruence relation. The factor lattice $\sum_{i\in I} L_i/\Theta$ will be called the special sum of the lattices L_i and will be denoted by $\sum'_{i\in I} L_i$.

For example, if $I=\{1, 3, 7\}$ with $1<3<7$, then $\sum'_{i\in I} S_i$ is the following lattice:

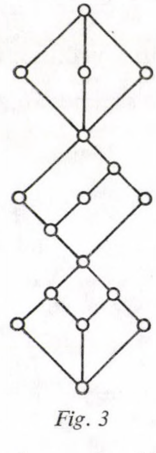

Fig. 3

Now we can state the following theorem, which generalizes the main result of [1].

THEOREM 3. *For an arbitrary modular lattice L the following four conditions are equivalent:*

(i) $\mathrm{Su}(L)$ *is 2-distributive;*

(ii) *None of* S_1, S_4, S_5 *and* S_6 *is a sublattice of* L*;*

(iii) S_4 *is not a sublattice of* L *and any three-element antichain in* L *generates a sublattice isomorphic to* P_7*;*

(iv) L *is isomorphic to a special sum* $\sum_{i\in I}' L_i$ *where* I *is a chain, and for all* $i\in I$ *one of the following three conditions is satisfied:*

(a) L_i *is a chain;*

(b) L_i *is the direct product of a chain and the two-element lattice;*

(c) L_i *has lowest and greatest elements (in notation* 0_i *and* 1_i*), and* $L_i\setminus\{0_i, 1_i\}$ *is an antichain.*

PROOF. The equivalence of (i), (ii) and (iii) immediately follows from Theorem 2 and the well-known criterion of modularity (cf. GRÄTZER [3, page 59]). The implication (iv)$\to$(iii) is straightforward. So, only the implication (iii)$\to$(iv) has to be shown. For a lattice M let us define $C'(M)=\{x\in M: x\neq 0_M, x\neq 1_M$ and $x\not\parallel y$ for any $y\in M\}$. By [1, Lemma 2] and [1, Theorem] it is enough to show that whenever M is a modular, non-distributive lattic efor which (iii) and $C'(M)=\emptyset$ hold then $M\setminus\{0_M, 1_M\}$ is an antichain. Suppose M is a modular, non-distributive lattice for which (iii) and $C'(M)=\emptyset$ hold. By the well-known criterion of distributivity (cf. GRÄTZER [3, page 59]), M contains a three-element antichain. Thus, by Zorn's Lemma, M contains an at least three-element *maximal* antichain A. Set $B=A\cup\{a\vee b, a\wedge b\}$ where a and b are distinct elements in A. By (iii) B is a sublattice of M, so it is enough to show that $B=M$. Suppose $a\vee b$ is not the greatest element of M. Then $x\parallel a\vee b$ for some $x\in M$. Choosing two distinct elements y, z from A such that $y\parallel x$ and $z\parallel x$, $x\vee y\vee z=x\vee a\vee b\neq a\vee b=y\vee z$ contradicts (iii). Therefore $a\vee b=1=1_M$ and, similarly, $a\wedge b=0=0_M$. Suppose x is an element in $M\setminus B$. Since $A\cup\{x\}$ is not an antichain, $x\not\parallel y$, say $x<y$, for some y in A. Choose two distinct elements d, e in $A\setminus\{y\}$. Since $\{x, d, e\}$ is an antichain, by (iii) we have $x\vee d=d\vee e=1$, $x\wedge d=d\wedge e=0$. Hence $\{0, 1, x, y, d\}$ is isomorphic to R_2, which contradicts the modularity of M. Therefore $M=B$. Q.E.D.

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ON UNEQUIVOCAL RINGS

By

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The aim of this paper is to investigate some new properties and some natural subclasses of the class of unequivocal rings defined in [3, 4]; these are associative rings A such that for any radical S in the class of associative rings either $S(A)=0$ or $S(A)=A$.

For any ring A by L_A we shall denote the lower radical determined by A.

The following obvious ([3]) characterization of unequivocal rings will be used in the paper: A is an unequivocal ring iff $A\in L_I$ for any ideal $I\neq 0$ of A.

All terms undefined in the paper and used facts may be found in [2, 7].

1. It is an important and fundamental property of any radical S in the class of associative rings that if A is an ideal of a ring B then also $S(A)$ is an ideal of B. Following this property we shall say that I is a $*$-ideal of a ring A if I is an ideal of any ring B containing A as an ideal.

Therefore if S is a radical then $S(A)$ is a $*$-ideal of A. Other examples of $*$-ideals are idempotent ($I=I^2$) and, as it follows from Andrunakievic Lemma, semiprime ideals.

Rings without non-trivial $*$-ideals will be called $*$-simple. Any such ring is unequivocal.

THEOREM 1. *A ring is $*$-simple if and only if it is either simple or a zero-algebra over a field.*

PROOF. Let A be a non-simple and $*$-simple ring. If I is an ideal of A then AIA is a $*$-ideal, so if $I\neq A$ then $AIA=0$. Thus $J=\{x\in A\,|\,AxA=0\}\neq 0$ and since J is a $*$-ideal then $J=A$. Therefore $A^3=0$. This and the fact that A^2 is a $*$-ideal of A implies that $A^2=0$. Since for any prime p, pA is a $*$-ideal of A then $pA=0$ or $pA=A$. If $pA=0$ then A is an algebra over the field of p elements. If $pA=A$ for any prime p then $A(p)=\{x\in A\,|\,px=0\}\neq A$. Since $A(p)$ is a $*$-ideal of A, $A(p)=0$. This implies that A is an algebra over the field of rational numbers.

Now let us assume that A is a zero-algebra over a field K. Let I be a non-trivial ideal of A. If I is a K-subspace of A then there exists a linear endomorphism f of A such that $f(I)\not\subset I$. Since $A^2=0$, f is a ring endomorphism. If I is not a K-subspace of A then there exists $\alpha\in K$ such that $\alpha I\not\subset I$. Then the map $f\colon A\to A$ defined by $f(x)=\alpha x$ is also a ring endomorphism of A such that $f(I)\not\subset I$. So in both cases there exists a ring endomorphism f of A such that $f(I)\not\subset I$. Now let F be the subring of the ring of endomorphisms of A generated by f. The set B of matrices of the form $\begin{pmatrix} g & a \\ 0 & 0 \end{pmatrix}$, where $g\in F$, $a\in A$, with natural operations is a ring.

We can identify A with the ideal $\begin{pmatrix}0 & A\\ 0 & 0\end{pmatrix}$ of B. In this identification I is an ideal of A and as $f(I)\not\subseteq I$, it is not an ideal of B. This proves the theorem.

For any ring A by A^0 we will denote the zero-ring on the additive group of A.

Since any simple ring is an algebra over a field, theorem 1 implies that if A is a simple ring then A^0 is $*$-simple. In particular we get

COROLLARY 1 ([3], Lemma 4). *If A is a simple ring then A^0 is an unequivocal ring.*

The Corollary above may be extended to the following

PROPOSITION 1. *If a ring A is unequivocal then so is A^0.*

PROOF. It is not difficult to check that if S is a radical then S^0 defined by the condition: $B\in S^0$ iff $B^0\in S$, is also a radical. Now the proof follows from the fact that for any ring A and any radical S, $(S^0(A))^0=S(A^0)$.

Another generalization of Corollary 1 is the following

PROPOSITION 2. *If K is a field then any L_{K^0}-radical K-algebra A which does not contain non-trivial idempotent ideals is an unequivocal ring.*

PROOF. Since A is K-algebra, for any radical S so is $S(A)$. If $S(A)$ is a non-trivial ideal of A then $B=S(A)/(S(A))^2$ can be homomorphically mapped onto K^0. Thus $L_{K^0}\subseteqq S$ and hence $A\in S$. This contradicts the assumption $S(A)\neq A$.

As it will be shown in the next section (Theorem 2) not every L_{K^0}-ring is unequivocal, where K is a field. Nevertheless there are some radicals purely contained in the class of unequivocal rings.

2. PROPOSITION 3. *If a radical $S\neq 0$ is contained in the class of unequivocal rings then S is an atom in the lattice of all radicals* ([6]).

If S is a hereditary radical which is an atom in the lattice of all radicals then S is contained in the class of unequivocal rings.

PROOF. Let $S\neq 0$ be a radical contained in the class of unequivocal rings and let A, B be $\neq 0$ S-rings. Then $A\oplus B$ is also S-ring and hence $A\oplus B$ is an unequivocal ring. Thus $L_A(A\oplus B)=A\oplus B$, so $B\in L_A$. Therefore $S\subseteqq L_A$ for any $0\neq A\in S$ and S is an atom.

Now let S be a hereditary radical. If $0\neq A\in S$ and $S_1(A)\neq 0$ for some radical S_1, then $S_1(A)\in S$. Thus $S_1\cap S\neq 0$. So if S is an atom in the lattice of all radicals then $S\subseteqq S_1$ and hence $A\in S_1$. This completes the proof.

Since the class of idempotent rings is radical, any atom in the lattice of all radicals is either subidempotent radical or it does not contain $\neq 0$ idempotent rings. The full characterization of subidempotent atoms is not known to the author. We only have

PROPOSITION 4. *If P is a simple ring with unity then L_P is an atom in the lattice of all radicals, so by Proposition 3 any L_P-ring is unequivocal.*

PROOF. If $0\neq A\in L_P$ then A contains an ideal I isomorphic to P. Since P has unity then $A\approx I\oplus I'$. Thus $P\in L_A$ and L_P is an atom in the lattice of all radicals.

The additive group of a ring A will be denoted by A^+.

LEMMA 1. *If* $S\neq 0$ *is not subidempotent radical then it contains either* K^0 *where* K *is a field or the zero-ring* $Z_p^0(\infty)$ *on the additive group* p^∞ *for some prime* p ([2], pp. 14—15).

PROOF. Let $A\neq 0$ be a non-idempotent S-ring. Then $B=A/A^2\neq 0$ is a zero S-ring. If $pB\neq B$ for some prime p then $B/pB\neq 0$ is an S-algebra over a field K. Since $(B/pB)^2=0$, there exists an epimorphism of B/pB onto K^0. Hence $K^0\in S$. If $pB=B$ for any prime p then B^+ is a divisible group. Therefore Q^0 or $Z_p^0(\infty)$ is a homomorphic image of B, where Q is the field of rational numbers.

The next theorem describes non-subidempotent atoms in the lattice of all radicals.

THEOREM 2. *For any prime p the class* S_p *of all zero-rings the additive groups of which are divisible p-groups, is a radical contained in the class of unequivocal rings.*

Any non-subidempotent atom in the lattice of all radical is equal to S_p *for somes prime p.*

PROOF. The first part of the theorem follows from [3] (Theorem 5) and the fact that the class of zero-rings the additive groups of which are divisible p-groups is radical.

Now suppose that $S\neq 0$ is a non-subidempotent atom. By Lemma 1, S contains either K^0 where K is a field or the ring $Z_p^0(\infty)$ for some prime p. But if $K^0\in S$ then the Zassenhaus algebra Z (Example 3 of [2] or Example 13.5 of [7]) over K with basis $\{x_r, 0<r<1\}$, is also in S. This is impossible because Z is an idempotent ring. Therefore by Proposition 3, $S=L_{Z_p^0(\infty)}$ for some prime p. Now the rest of the proof follows from the fact that the class of rings with additive groups that are both divisible and p-groups is radical and consists of zero-rings.

3. Similarly to Proposition 3 one can prove

PROPOSITION 5. *If a semisimple class M is contained in the class of unequivocal rings then the upper radical determined by M is a dual atom in the lattice of all radicals* ([6]).

It has been shown by R. L. SNIDER [6] that dual atoms in the lattice of hereditary radicals do not exist. His proof may be adapted to show that this holds also with the lattice of all radicals. For this it is enough to change part I of Snider's proof. It issues from the following

LEMMA 2. *If B is the Baer radical and S is a proper radical then the lower radical* $L_{B\cup S}$ *determined by* $B\cup S$ *is proper.*

PROOF. Of course the result holds if $B\subseteq S$. So let $B\nsubseteq S$. Then there exists an S-semisimple and B-radical ring A. Since the radical B and the class M of S-semisimple rings are hereditary then $S(C)=0$, where C is the zero-ring on a cyclic additive group. But the infinite cyclic group is a subdirect sum of cyclic groups of indices $n, n^2, \ldots$ for any integer $n>1$. So by the fact that the class M is closed on subdirect sums and extensions it follows that $S(Z^0)=0$, where Z is the ring of integers. Now let P be the ideal of the polynomial ring $Z[x]$ of indeterminate x generated by x. It is clear that P is a subdirect sum of rings $P_m=P/x^m P$ for

$m=1, 2, \ldots$. But $P_{m+1}/P_m \approx Z^0$ for $m=1, 2, \ldots$. Therefore $S(P)=0$. Since also $B(P)=0$ then $P \notin L_{S\cup B}$.

The essential role in the proof of Lemma 2 plays the fact that any semisimple class is subdirectly closed. In the next any subdirectly closed class contained in the class of unequivocal rings will be called unequivocal-residual or shortly *u-r*-class. Now we will describe such classes.

The cardinality of a set X will be denoted by $|X|$.

LEMMA 3. *Any* $\neq 0$ *ring of u-r-class contains nilpotent elements.*

PROOF. Let $R\neq 0$ be a ring without nilpotents of an *u-r*-class. Since the product of any copies of rings without nilpotents is also a ring without nilpotents then we can assume that R is infinite of the cardinality 2^α for some cardinal number α. Let ΠR be the product of $|R|$ copies of R and S be the set of sequences (x_β) of ΠR such that $|\{\beta | \{x_\beta \neq 0\}| < |R|$. Of course S is an ideal of ΠR. The fact that R is an infinite ring of cardinality 2^α implies that $|S|=|R|$. We will prove that $L_R(\Pi R/S)=0$. If it were not so then $\Pi R/S$ would contain an accessible subring $P/S\neq 0$ which is a homomorphic image of R. Then there exists an integer n such that $P^n \cdot \Pi R \subseteq P$. Let $(x_\alpha)\in P \setminus S$. Then $|\{\alpha | x_\alpha \neq 0\}|=|R|$. Since R does not contain nilpotents, $(x_\alpha)^n \cdot \Pi R$ contains $2^{|R|} > |R|$ elements. Hence $|P/S|>|R|$ and P/S can not be a homomorphic image of R. Therefore we have proved that $L_R(\Pi R/S)=0$. Since $L_R(\Pi R)\neq 0$, ΠR is not an unequivocal ring. This proves the lemma.

LEMMA 4. *If B is a subring of a ring A of an u-r-class M then* $B\in L_A$.

PROOF. Let ΠA and $\oplus A$ denote the product and the discrete direct sum of $\aleph_0$-copies of A respectively. Let $\bar{B}$ be the image of B by the map which sends $a\in A$ on the sequence $(a, a, \ldots)$. Then $B\approx \bar{B}$ and $\oplus A+\bar{B}$ is a subdirect sum of $\aleph_0$-copies of A, so $\oplus A+\bar{B}\in M$. Since $L_A(\oplus A+\bar{B})\neq 0$ then $L_A(\oplus A+\bar{B})= = \oplus A+\bar{B}$. This implies that $B\approx \bar{B}=\bar{B}/\bar{B}\cap \oplus A \approx \oplus A+\bar{B}/\oplus A \in L_A$.

For any prime p and any integers n, k, $0(n, p, k)$ will denote the class of all nil-algebras of the degree $\leqq n$ without $\neq 0$ idempotent ideals the additive groups of which are p-groups of the exponent $\leqq p^k$. For any integer m, $N(m)$ will denote the class of all nilpotent rings of the degree $\leqq m$ the additive groups of which are torsion-free and their $\neq 0$ ideals can be homomorphically mapped onto the zero ring on the infinite cyclic group.

Of course any class $0(n, p, k)$ and $N(m)$ is subdirectly closed. Since rings of $0(n, p, k)$ are nil and satisfy polynomial identity [5], they are Baer radical. This and conditions on ideals of rings of $0(n, p, k)$ and $N(m)$ implies that $0(n, p, k)$ and $N(m)$ are *u-r*-classes. On the other hand we have

THEOREM 3. *Any u-r-class is contained in* $0(n, p, k)$ *or* $N(m)$.

PROOF. Let M be an *u-r*-class and let $A\in M$. If $x\in A$ is a non-nilpotent element then the subring $\langle x\rangle$ of A generated by x can be homomorphically mapped onto a field K. Then Lemma 4 implies that $K\in L_A$. Thus K is a homomorphic image of A. This and the fact that A is an unequivocal ring imply that $U_K(A)=0$ and hence A is a subdirect sum of copies of K. So A does not contain nilpotents. This contradicts Lemma 3. Therefore any ring of M is nil. Now since the class M is closed under products, there exists an integer n such that any ring of M is nil of the degree $\leqq n$. Theorem 4 of [3] and the fact that the class M is closed under products implies that the additive groups of rings of M are either torsion-free or p-groups of the exponent $\leqq p^k$ for some prime p and some integer k. In the first case the Nagata—Higman theorem ([5]) implies that all rings of M are nilpotent. Their degree of nilpotency is $\leqq m$ for some integer m because the class M is closed under products. Since rings of M are nil and satysfy polynomial identity, they are Baer radical. This and Lemma 4 imply that rings of M do not contain $\neq 0$ idempotent ideals and if their additive groups are torsion-free then $\neq 0$ ideals can be homomorphically mapped onto the zero-ring on the infinite cyclic group. This ends the proof.

4. In earlier parts of the paper we had some examples of unequivocal rings which are algebras over a field. The following proposition gives another one.

PROPOSITION 6. *Any unequivocal ring with unity is an algebra over a field.*

PROOF. Let A be an unequivocal ring with unity. If $A \neq pA$ for some prime p then A can be homomorphically mapped onto a simple algebra P with unity over the field of p elements. Since A is an unequivocal ring, A is U_P-semisimple, where U_P is the upper radical determined by P. But this implies that A is a subdirect sum of copies of P. Hence A is an algebra over the field of p-elements. If $pA = A$ for any prime p then A^+ is a divisible group. Now by Theorem 4 of [3] A^+ is torsion-free or a p-group. The second case is impossible because otherwise A would be a zero-ring. Thus A^+ is a divisible torsion-free group, so A is an algebra over the field of rational numbers.

Now let A be an algebra over a commutative ring F with unity. Like in the definition of unequivocal rings we can say that A is an unequivocal algebra if $S(A)=0$ or $S(A)=A$ for any radical S in the class of F-algebras.

If A is an F-algebra and S is a radical in the class of rings then $S(A)$ is an F-algebra. Therefore any unequivocal algebra is also an unequivocal ring.

In the sequal we will consider only associative F-algebras over a commutative ring F with unity. All radicals will be in the class of F-algebras.

We shall say that the algebra K is normal if for any radical S $S(A \otimes_F K) = I \otimes_F K$ where I is an ideal of A.

For example, as it was proved in [6], the algebra F_n of all $n \times n$-matrices over F is normal. Also if F is an infinite field then the polynomial algebra over F of any set of commutative indeterminates is normal ([1]).

PROPOSITION 7. *If K is a normal algebra then for any radical S there exists a radical KS such that* $S(A \otimes_F K) = KS(A) \otimes_F K$.

PROOF. One can check that the class KS defined by the condition: $B \in KS$ iff $B \otimes_F K \in S$, satisfies the conditions of the proposition.

COROLLARY 4. *If K is a normal algebra and A is an unequivocal algebra then $A \otimes_F K$ is a nunequivocal algebra. In particular if A is an unequivocal algebra then so is the algebra A_n of all $n \times n$-matrices over A and if K is an infinite field then the polynomial algebra over A of any set of commutative indeterminates is unequivocal.*

COROLLARY 5. *If F is a field then any normal F-algebra is unequivocal.*

A normal algebra K will be called invariant if for any radical S, $KS=S$.

PROPOSITION 8. *If F is a field then the class of all invariant F-algebras is equal to L_F.*

PROOF. If $S=KS$ for any radical S then $L_F=KL_F$, so $K\in L_F$.

Now let $0\neq K\in L_F$ and let I be the intersection of the kernels of all homomorphisms of K onto F. If $I\neq 0$ then $I\in L_F$ implies $I=I_1\oplus I_2$, where $I_1\approx F$. Therefore there exists an epimorphism $f\colon I\to F$. Since F has the unity, f may be extended to an epimorphism $\bar{f}\colon K\to F$. Now $\operatorname{Ker}\bar{f}\not\supseteq I$. This contradiction yields $I=0$. Therefore K is a subdirect sum of copies of F. Hence if $S(A)=0$ for some radical S then also $S(A\otimes K)=0$. It is easy to see that also $A\in S$ implies $A\otimes K\in S$. This ends the proof.

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ON A SPECIAL ALGEBRA WITH TWO OPERATIONS

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In [1] J. SZÉP suggested the study of a special algebra; the aim of this work is to analyze such an algebra.

Let $S(\cdot,\times)$ be an algebra where "$\cdot$" is a group operation (instead of $a\cdot b$ we shall write ab) and "$\times$" is a semigroup operation with an idempotent element e; moreover

$$(\alpha)\quad \forall a, b, c\in S\colon (a\times b)c = ac\times bc, \quad c(a\times b) = ca\times c^{-1}a$$

where c^{-1} is the inverse of c in $S(\cdot)$.

The following results hold:

1) $S(\times)$ is an idempotent semigroup. In fact, let 1 denote the unit element of $S(\cdot)$; since $e\times e=e$, we have $a=1\cdot a=ee^{-1}a=(e\times e)e^{-1}a=(1\times 1)a=a\times a$ for every $a\in S$.

2) For every $a\in S$: $a^2=a\cdot a=a(a\times a)=a^2\times a^{-1}a=a^2\times 1$.

3) Let $G_1=\{x\in S\,|\,x\times 1=x\}$, $G_2=\{x\in S\,|\,1\times x=x\}$; then the following result holds: $u\in G_1\Rightarrow 1\times u=1$, $u\in G_2\Rightarrow u\times 1=u^2$. In fact

$$u\in G_1\Rightarrow u\times 1 = u\Rightarrow u^{-1}(u\times 1) = u^{-1}u\Rightarrow 1\times u = 1;$$

$$u\in G_2\Rightarrow 1\times u = u\Rightarrow u(1\times u) = u^2\Rightarrow u\times 1 = u^2.$$

4) For every $u\in S$: $u\in G_2\Rightarrow u^2=1$. In fact (see 3), 1))

$$u\in G_2\Rightarrow u^2\times u = (u\times 1)\times u = u\times(1\times u) = u\times u = u$$

and hence

$$u\in G_2\Rightarrow u^2\times u = u\Rightarrow (u\times 1)u = u\Rightarrow u^2\cdot u = u\Rightarrow u^2 = 1.$$

5) $$G_1\cap G_2 = \{1\}\quad ((\text{see } 3)).$$

6) The sets G_1, G_2 are normal subgroups of $S(\cdot)$.

The following steps prove the assertion:

(i) if $u, v\in G_1$ then $u\times 1=u$, $v\times 1=v$ so $uv=(u\times 1)v=uv\times v=uv\times(v\times 1)=(u\times 1)v\times 1=uv\times 1$ and hence $uv\in G_1$. (Similarly $u, v\in G_2\Rightarrow uv\in G_2$).

(ii) $u\in G_1\Rightarrow 1\times u=1\Rightarrow(1\times u)u^{-1}=u^{-1}\Rightarrow u^{-1}\times 1=u^{-1}\Rightarrow u^{-1}\in G_1$; $u\in G_2\Rightarrow u^{-1}\in G_2$ ((see 4)).

(iii) For every $a\in G_1, b\in S$: $b^{-1}ab\times 1=b^{-1}(a\times 1)b\times 1=(b^{-1}a\times b)b\times 1=(b^{-1}ab\times b^2)\times 1=b^{-1}ab\times(b^2\times 1)=b^{-1}ab\times b^2=b^{-1}ab$ and hence $b^{-1}ab\in G_1$. For every $a\in G_2, b\in S$: $b^{-1}ab=b^{-1}(1\times a)b=(b^{-1}\times ba)b=1\times bab\in G_2$.

7) $S=G_1\cdot G_2$. In fact for $x\in S$ we consider the element $a=x\times 1\in G_1$. So it follows that $a=(1\times x^{-1})x$, i.e. $1\times x^{-1}=ax^{-1}$ and hence, since $1\times x^{-1}\in G_2$, $x=(1\times x^{-1})^{-1}\cdot a=a\cdot(1\times x^{-1})^{-1}\in G_1\cdot G_2$.

From 5), 6), 7) it follows that

8) $G=G_1\otimes G_2$ (where $\otimes$ denotes the direct product)

9) $1\times g_1g_2=g_2,\ g_1g_2\times 1=g_1$ where $g_1\in G_1,\ g_2\in G_2$. In fact, since

$$(g_1^{-1}g_2^{-1}\times 1)\,g_1g_2=(1\times g_1g_2)\in G_2,\quad \text{it follows}\quad g_1^{-1}g_2^{-1}\times 1\in g_1^{-1}G_2.$$

But $g_1^{-1}g_2^{-1}\times 1\in G_1$ and so (by 5) $g_1^{-1}g_2^{-1}\times 1=g_1^{-1}$. This means that $1\times g_1g_2==(g_1^{-1}g_2^{-1}\times 1)g_1g_2=g_1^{-1}g_1g_2=g_2$ (Likewise for $g_1g_2\times 1=g_1$.)

10) If $a=g_1g_2,\ b=h_1h_2$, where $g_1,h_1\in G_1,\ g_2,h_2\in G_2$ then $a\times b=g_1g_2\times h_1h_2==(1\times h_1h_2g_2^{-1}g_1^{-1})g_1g_2=h_2g_2^{-1}g_1g_2=h_2g_1=g_1h_2$.

So we have the following

THEOREM 1. *For the algebra $S(\cdot,\times)$ the following properties hold*

1) $S=G_1\otimes G_2$ *and* $g^2=1$ *for any* $g\in G_2$.

2) $g_1g_2\times h_1h_2=g_1h_2$ $(g_1,h_1\in G_1,\ g_2,h_2\in G_2)$.

Conversely we can prove

THEOREM 2. *Let the group $S(\cdot)$ be the direct product of two subgroups G_1, G_2 such that $g^2=1$ for every $g\in G_2$. A semigroup operation "$\times$" with an idempotent element exists in S such that for any $a, b, c\in S$: $(a\times b)c=ac\times bc$, $c(a\times b)=ca\times c^{-1}b$.*

PROOF. A few calculations show that the required operation is defined as follows:

$$g_1g_2\times h_1h_2=g_1h_2 \quad \text{for every} \quad g_1,h_1\in G_1,\quad g_2,h_2\in G_2.$$

REMARK. Finally we observe that theorems analogous to Theorems 1 and 2 can be proved if in place of (α) one has

(β) for any $a, b, c\in S$: $(a\times b)c=ac\times bc$, $c(a\times b)=c^{-1}a\times cb$.

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THE PARA-CANCELLATION LAW IN COMMUTATIVE SEMIGROUPS

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Introduction

In § 1 of this paper we consider a commutative semigroup $S(\cdot)$ and give a generalization of the notion of cancellative element: the notion of "para-cancellative element" (see definition 1). Afterwards we consider commutative semigroups such that every element is para-cancellative (the "para-cancellative semigroups" or, in other words, "semigroups with para-cancellation law") and give some meaningful properties. In § 3 we consider $Z_n(\cdot)$, the factor semigroup of $Z(\cdot)$ (where Z is the set of the integers) with respect to the congruence determined by $n \in Z$, and prove that if $n=p^\alpha$ (where p is a prime integer) then $Z_n(\cdot)$ is para-cancellative, otherwise the only para-cancellative elements of Z_n are the cancellative elements.

The definition of para-cancellative element derives from a natural extension (see § 2) of the concept of unique factorization semigroup.

§ 1. Para-cancellative elements

Let $S(\cdot)$ be a commutative semigroup. If x and y are elements of S, we shall write $x \sim y$ iff $x=y$ or x is a divisor of y and y is a divisor of x (i.e. $y=s\cdot x$ and $x=t\cdot y$, where s and t are suitable elements of S). Clearly $\sim$ is a congruence such that if $0 \in S$ then $[0]_\sim=\{0\}$.

DEFINITION 1. If $S(\cdot)$ is a commutative semigroup (possibly with 0) then we say that an element $b \in S$ is a "para-cancellative element" iff for any $x, y \in S$

$$0 \neq b\cdot x \sim b\cdot y \Rightarrow x \sim y. \tag{1}$$

As an immediate consequence of the foregoing definition the following theorem holds:

THEOREM 2. *The subset of all para-cancellative elements of a commutative semigroup is a subsemigroup.*

REMARK 3. Obviously if $S(\cdot)$ is a semigroup with 0 then 0 and every annihilator are para-cancellative elements; however there are examples of non-zero, non-annihilator and non-cancellative elements which are para-cancellative (e.g. the element $2+(4) \in Z_4$). Furthermore we observe that if in the multiplicative semigroup of a ring $A(+, \cdot)$ the property (1) is replaced by the following one: for any $x, y \in S$

$$b\cdot x \sim b\cdot y \Rightarrow x \sim y \tag{2}$$

then b is a cancellative element. In fact $b\cdot x=b\cdot y$ entails $b(x-y)=0\sim b\cdot 0$, then $x-y\sim 0$ and hence $x-y=0$.

THEOREM 4. *Let $S(\cdot)$ be a commutative semigroup and $b\in S$. Then the following properties are equivalent:*

(i) *for any* $x, y\in S$: $b\cdot x=b\cdot y\neq 0\Rightarrow x\sim y$.

(ii) *for any* $x, y\in S$: $(b\cdot x\cdot s=b\cdot y\neq 0$ *for some* $s\in S)\Rightarrow(x\cdot t=y$ *for some* $t\in S)$;

(iii) *b is para-cancellative.*

PROOF. One can easily verify that (i)⇒(ii), (ii)⇒(iii) and (iii)⇒(i). Q.E.D.

DEFINITION 5. We shall call a commutative semigroup $S(\cdot)$ a "para-cancellative semigroup" (or, in other words, a "semigroup with para-cancellation law") iff every element of S is para-cancellative.

We recall that if $S(\cdot)$ is a semigroup (with identity or not) we can always adjoin to S an identity by taking the set $S'=S\cup\{e\}$, where e is an element not contained in S, and putting $a*b=a\cdot b$ iff $a, b\in S$ and $a*e=e*a=a$ iff $a\in S\cup\{e\}$. Now we can give the next

THEOREM 6. *Let $S(\cdot)$ be a para-cancellative semigroup and $S'(*)$ the semigroup obtained adjoing to S the identity e. Then the following properties are equivalent:*

(j) *there exist* $b, h\in S$ *such that* $b\cdot h=b\neq 0$;

(jj) *there exist* $b, h\in S$ *such that* $b\cdot h\sim b\neq 0$;

(jjj) *there exists* $h\in S$ *such that* $h^2\sim h\neq 0$;

(jjjj) *$S'(*)$ is not para-cancellative.*

PROOF. Clearly (j) entails (jj).

Moreover if $b\cdot h\sim b\neq 0$ then $b\cdot h^2\sim b\cdot h\neq 0$ and hence $h^2\sim h\neq 0$, thus (jj) entails (jjj).

Now let $h\in S$ such that $h^2\sim h\neq 0$; then in $S'(*)$ $h*h\sim h*e\neq 0$ and $h\nsim e$. This assures that h is not para-cancellative in $S'(*)$, and hence (jjj) entails (jjjj).

Finally if $S'(*)$ is not para-cancellative then there exists a non para-cancellative element $b\in S'$, then $b\neq 0$ and $b\neq e$. As a consequence there exist $h, k\in S'$ such that $b*h=b*k\neq 0$ and $h\nsim k$; thus, since $S(\cdot)$ is para-cancellative, $h\in S$ and $k=e$ or vice-versa. This assures that (jjjj) entails (j). Q.E.D.

From here to the end of § 1, since $\sim$ is a congruence and $[0]_\sim=\{0\}$, for simplicity's sake we shall tacitly suppose that in a para-cancellative semigroup the relation $\sim$ is the equality relation. Anyhow one can easily extend the results to the more general case.

THEOREM 7. *Let $S(\cdot)$ be a para-cancellative semigroup with 0, h an element of S such that $h^2=h\neq 0$ and $S\cdot h$ the subset $\{z\in S\colon z=z'\cdot h$ for some $z'\in S\}$, which is equal to the subset $\{z\in S\colon z=z\cdot h\}$. Then the following properties hold:*

(1) *for any* $x, y\in S$: $x\in S\cdot h$ *and* $y\in S-S\cdot h\Rightarrow x\cdot y=0$;

(2) *for any* $a, b, c\in S$: $0\neq a=b\cdot c\in S\cdot h\Rightarrow b\in S\cdot h$.

PROOF. (1) Let us assume, ab absurdo, that $x\cdot y\neq 0$; then $x\cdot y=x\cdot h\cdot y\neq 0$, thus $y=y\cdot h\in S\cdot h$. This contradicts the hypothesis.

(2) In fact $a=b\cdot c\cdot h\neq 0$, then $b\cdot h\neq 0$ and hence, as a consequence of (1), $b\in S\cdot h$. Q.E.D.

Then, as a corollary, the next theorem holds.

THEOREM 8. *If $S(\cdot)$ is a para-cancellative semigroup with 0 and h, k are elements of S such that $h^2=h\neq 0, k^2=k\neq 0$ and $h\neq k$ then $h\cdot k=0$.*

PROOF. Let us assume, ab absurdo, that $h \cdot k \neq 0$; then $h \cdot h \cdot k = h \cdot k = h \cdot k \cdot k \neq 0$, and hence $h \cdot k = k$ and $h = h \cdot k$. This contradicts the hypothesis that $h \neq k$. Q.E.D.

Now let $S(\cdot)$ be a semigroup with 0 such that $S = \bigcup_{A \in \alpha} A$, where the elements of α are subsemigroups of $S(\cdot)$. Then we shall call S an "almost-disjoint union" of the subsemigroups belonging to α iff $A_1 \neq A_2$ entails $A_1 \cap A_2 = \{0\}$ for every $A_1, A_2 \in \alpha$. Thus from Theorem 8 the next theorem follows.

THEOREM 9. *If $S(\cdot)$ is a para-cancellative semigroup with 0 and H is the set $\{h \in S : h^2 = h \neq 0\}$ then S is an almost-disjoint union of the semigroup $(S - \bigcup_{h \in H} S \cdot h) \cup \{0\}$ and of the semigroups of the type $S \cdot h$.*

PROOF. Indeed if $x \in S \cdot h \cap S \cdot k$, where $h, k \in H$ and $h \neq k$, then $x = x \cdot h \cdot k = x \cdot 0 = 0$. Q.E.D.

§ 2. A short account of para-unique factorization semigroups

Let $S(\cdot)$ be a commutative semigroup (possibly with 0). We shall say that an element $h \in S$ is a "para-invertible" element or a "para-unity" for $b \in S$ iff $b \neq 0$ and $b \sim b \cdot h$ (or, equivalently, $b = b \cdot h \cdot k$, where k is a suitable element of S). One can easily verify that every divisor of a para-unity is a para-unity too. As a consequence the subset of all elements of S which are not para-unity is an ideal of $S(\cdot)$.

In the following an element $b \in S$ will be called "irriducible" iff the next property holds:

(3) for any $c, d \in S$: $b \sim c \cdot d \Rightarrow b \sim c$ (and hence d is a quasi-unity for b and c) or $b \sim d$ (and hence c is a quasi-unity for b and d); moreover b will be said to be "para-prime" iff the next property holds:

(4) for any $a, c, d \in S$: $0 \neq a \cdot b \sim c \cdot d \Rightarrow b$ is a divisor of c or b is a divisor of d. We observe that if $S(\cdot)$ verifies the cancellation law (cf. the multiplicative semigroup of a Euclidean ring) or $\sim$ is equal to equality relation (cf. the two semigroups of a lattice) the foregoing definition of irreducible element is equivalent to the classical ones.

Now we agree that an element c occurs in $b_1 \cdot \ldots \cdot b_n$ s times (where $s \geqq 0$) iff $c \sim b_i$ exactly for s b_i in $b_1 \cdot \ldots \cdot b_n$. Thus we recall that a commutative semigroup with cancellation law $S(\cdot)$ is a unique factorization semigroup iff for every not invertible element $b \in S$ the following property holds:

(5) $b = b_1 \cdot \ldots \cdot b_n$ (with b_i irreducible and not invertible) and if $b = \bar{b}_1 \cdot \ldots \cdot \bar{b}_m$ (with $\bar{b}_i$ irreducible and not invertible) then every element occurs s times in $b_1 \cdot \ldots \cdot b_n$ iff it occurs s times in $b_1 \cdot \ldots \cdot \bar{b}_m$.

Now then we can say that a commutative semigroup $S(\cdot)$ is a "weak unique factorization semigroup" iff for every not para-invertible element $b \in S - \{0\}$ the property (5) holds.

One can easily prove that if $S(\cdot)$ is a weak unique factorization semigroup then the ideal S_1 of all the elements of $S(\cdot)$ that are not para-invertible is a para-cancellative semigroup.

If $S(\cdot)$ is a para-cancellative and weak unique factorization semigroup we shall say that it is a "para-unique factorization semigroup". It is easy to verify that in a para-unique factorization semigroup every irriducible element is para-prime.

It would be interesting to extend the notions introduced in [1] to para-cancellative semigroups. A. Letizia is actively investigating this problem.

§ 3. The quasi-cancellation law for the elements of $Z_n(\cdot)$

For simplicity's sake we shall say that an element $b\in Z$ is "(n)-para-cancellative" iff the class $b+(n)$ is para-cancellative in $Z_n(\cdot)$. In particular we shall say that b is "(n)-cancellative" iff the class $b+(n)$ is cancellative in $Z_n(\cdot)$; in this case, and only in this case, an element $b'\in Z$ exists such that $b\cdot b'+(n)=1+(n)$, then we shall say that b' is an "(n)-inverse" of b. Moreover we shall call $b\in Z$ an "(n)-divisor" of $a\in Z$ iff $a+(n)=b\cdot c+(n)$ for some $c\in Z$. In the sequel we shall consider without loss of generality only integers that belong to N.

REMARK 10. We observe that if $\text{G.C.D.}(b_1, n)=1$ then b_1 has an (n)-inverse b_1', thus b_1 and b_1' are (n)-cancellative ad hence b_1 and b_1' are (n)-para-cancellative. Because of this and because of the product of two (n)-para-cancellative elements is (n)-para-cancellative too (cf. Theorem 4), if $a=b\cdot b_1$ and $\text{G.C.D.}(b_1, n)=1$ then a is (n)-para-cancellative iff b is (n)-para-cancellative.

LEMMA 11. *Let $b, n\in N$ be such that $b\not\equiv 0 \pmod n$ and $0\neq n=p_1^{\alpha_1}\cdot\ldots\cdot p_l^{\alpha_l}$, where $l\geqq 2$ and for every $i=1, \ldots, l$, p_i is a prime factor of n. Then* $\text{G.C.D.}(b, n)\neq 1$ *entails that b is not (n)-para-cancellative.*

PROOF. Let $\text{G.C.D.}(b, n)\neq 1$; then, by remark 10, we can suppose that $b=p_1^{\beta_1}\cdot\ldots\cdot p_l^{\beta_l}$, where for some $i=1, \ldots, l$, $\beta_i>0$. Now let us consider $c=p_1^{\gamma_1}\cdot\ldots\cdot p_l^{\gamma_l}$ where for every $i=1, \ldots, l$, $\gamma_i=\alpha_i-\beta_i$ iff $\alpha_i>\beta_i$ and $\gamma_i=0$ iff $\alpha_i\leqq\beta_i$. Then $b\cdot c\equiv 0 \pmod n$ and, since $b\not\equiv 0 \pmod n$, $\beta_i<\alpha_i$ for some $i=1, \ldots, l$ and hence $\gamma_i>0$. Now we consider the following two cases:

CASE 1. For some (but not for every, since $b\not\equiv 0 \pmod n$) $j=1, \ldots, l$, $\alpha_j\leqq\beta_j$, so $\gamma_j=0$. Then we put $d=p_1^{\delta_1}\ldots\cdot p_l^{\delta_l}$, where $\delta_i=1$ iff $\gamma_i=0$, on the contrary $\delta_i=0$. Therefore $\text{G.C.D.}(c+d, n)=1$ and $\text{G.C.D.}(d, n)\neq 1$, then $c+d$ is (n)-invertible but d is not, and hence d is not an (n)-divisor of $c+d$; moreover $b\cdot(c+d)\equiv$ $\equiv b\cdot d \pmod n$ and $b\cdot d\not\equiv 0 \pmod n$. As a consequence b is not (n)-para-cancellative.

CASE 2. For every $i=1, \ldots, l$, $\beta_i<\alpha_i$ and hence $\gamma_i\neq 0$. Now, without loss of generality, let us suppose $\beta_1>0$. Thus $b(p_1^{\alpha_1}+c)\equiv bp_1^{\alpha_1} \pmod n$ and $b\cdot p_1^{\alpha_1}\not\equiv 0 \pmod n$ since $\beta_i<\alpha_i$. Now then $p_1^{\alpha_1}$ is not an (n)-divisor of $p_1^{\alpha_1}+c$, because on the contrary there exist $s, t\in N$ such that $p_1^{\alpha_1}\cdot s=p_1^{\alpha_1}+c+tn$; but in this case $p_1^{\alpha_1}\cdot s$, $p_1^{\alpha_1}$ and $t\cdot n$ are divisible by $p_1^{\alpha_1}$ and hence c is divisible by $p_1^{\alpha_1}$. This is absurd since $c=p_1^{\gamma_1}\cdot\ldots\cdot p_l^{\gamma_l}$ and $\gamma_1<\alpha_1$ as $\beta_1>0$. Then b is not (n)-para-cancellative also in this case. Q.E.D.

LEMMA 12. *Let $b, n\in N$ be such that $0\neq n=p^\alpha$, where p is the only prime factor of n and $\alpha\geqq 2$. Then b is (n)-para-cancellative.*

PROOF. If $b\equiv 0 \pmod n$ the theorem is trivial since 0 is (n)-para-cancellative; then let $b\not\equiv 0 \pmod n$. By Remark 10 we can suppose that $b=p^{\beta}$; moreover, as the product of two (n)-para-cancellative elements is (n)-para-cancellative too and $p^0=1$ is (n)-cancellative, let us consider $\beta=1$.

Now let $0\not\equiv px\equiv py \pmod n$ (where $x\geqq y$) then $p(x-y)=p^{\alpha}\cdot h$ (where $h\in N$) and hence $x-y=p^{\alpha-1}\cdot h$. Moreover let $x=z\cdot p^{\varepsilon}$, where G.C.D. $(z, p)=1$ and ε is a natural number less than $\alpha-1$ (since $p\cdot x\not\equiv 0 \pmod n$) thus $y=z_1\cdot p^{\varepsilon}$, where G.C.D. $(z_1, p)=1$, since $y=x-p^{\alpha-1}\cdot h$.

As a consequence of the previous considerations z and z_1 are (p^{α})-invertible, thus z is a (p^{α})-divisor of z_1, z_1 is a (p^{α})-divisor of z and hence x is a (p^{α})-divisor of y and vice-versa. Then b is (n)-para-cancellative. Q.E.D.

As an immediate consequence of the previous two lemmata we can conclude with the following

THEOREM 13. *Let $0\neq n\in Z$. If $n=p^{\alpha}$, where p is a prime element, then every element of Z is (n)-para-cancellative and hence $Z_n(\cdot)$ is a semigroup with the para-cancellation law. No element of Z is (n)-para-cancellative iff it is (n)-cancellative and hence $Z_n(\cdot)$ is not a semigroup with the para-cancellation law.*

Reference

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ON THE ALMOST EVERYWHERE DIVERGENCE OF LAGRANGE INTERPOLATORY POLYNOMIALS FOR ARBITRARY SYSTEM OF NODES

By

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Dedicated to the memory of John Curtiss

1. Introduction

In a previous paper P. ERDŐS [1] stated without proof that if $X=\{x_{in}\}$, $n=1, 2, \ldots;\ 1\leqq i\leqq n$,

$$(1.1)\qquad -1 \leqq x_{nn} < x_{n-1,n} < \ldots < x_{1n} \leqq 1 \quad (n = 1, 2, \ldots)$$

is a triangular matrix then there is a continuous function $F(x)$, $-1\leqq x\leqq 1$, so that the sequence of Lagrange interpolation polynomials

$$L_n(F, X, x) = L_n(F, x) = \sum_{k=1}^{n} F(x_{kn})\, l_{kn}(x)$$

diverges almost everywhere in $[-1, 1]$, and in fact

$$\overline{\lim_{n\to\infty}}\, |L_n(F, X, x)| = \infty$$

for almost all x. (Here, as usual,

$$(1.2)\qquad l_{kn}(x) = \frac{\omega_n(x)}{\omega_n'(x_{kn})(x-x_{kn})} \quad \left(k = 1, 2, \ldots, n;\ \omega_n(x) = \prod_{k=1}^{n} (x-x_{kn})\right)$$

are the corresponding fundamental polynomials,

$$(1.3)\qquad \lambda_n(x) = \sum_{k=1}^{n} |l_{kn}(x)|, \quad \lambda_n = \max_{-1\leqq x\leqq 1} \lambda_n(x) \quad (n = 1, 2, \ldots)$$

are the Lebesgue functions and Lebesgue constants of the interpolation, respectively.

We now prove this statement in full detail. The detailed proof turned out to be quite complicated and several unsuspected difficulties had to be overcome.

In the same paper P. Erdős also stated, that there is a pointgroup $\{x_{kn}\}$ so that for every continuous $f(x)$ $(-1\leqq x\leqq 1)$ $L_n(f, x_0)\to f(x_0)$ holds for at least one x_0 for which $\overline{\lim_{n\to\infty}} \sum_{k=1}^{n} |l_{kn}(x_0)| = \infty$. This is perhaps true, but at this moment we cannot prove it (the original "proof" was probably incomplete). We hope to settle it on another occasion.

2. Preliminary results

In his classical paper [2] G. FABER proved that for any matrix X

$$\overline{\lim_{n\to\infty}}\ \lambda_n = \infty$$

from where we immediately obtain that for every point group there exists a continuous function $f_1(x)$, $-1 \leqq x \leqq 1$ (shortly $f_1 \in C$) so that

$$\overline{\lim_{n\to\infty}}\ \|L_n(f_1, x)\| = \infty.$$

(Henceforward $\|g(x)\| = \|g\| = \max_{-1 \leqq x \leqq 1} |g(x)|$ for $g \in C$.) Almost twenty years later, in 1931, S. BERNSTEIN [3] showed that for every X with (1.1) there is an $f_2 \in C$ and an x_0, $-1 \leqq x_0 \leqq 1$, such that

$$\overline{\lim_{n\to\infty}}\ |L_n(f_2, x_0)| = \infty.$$

Another problem is to prove divergence theorem on a set *of positive measure.*

In his paper [14] S. BERNSTEIN proved, that for the "bad" matrix $E = \{-1 + 2(k-1)/(n-1)\}$ and the function $|x|$

$$\overline{\lim_{n\to\infty}}\ |L_n(|t|, E, x)| = \infty \quad \text{if} \quad x \in (-1, 1), \quad x \neq 0.$$

Then, using the "good" Chebyshev matrix

$$T = \left\{ x_{kn} = \cos \frac{2k-1}{2n} \pi;\ k = 1, 2, \ldots, n;\ n = 1, 2, \ldots \right\} \tag{2.1}$$

G. GRÜNWALD [4] got that there exists a function $f_3 \in C$, for which

$$\overline{\lim_{n\to\infty}}\ |L_n(f_3, T, x)| = \infty \tag{2.2}$$

holds for almost all x in $[-1, 1]$. Later he and (independently) J. MARCINKIEWICZ proved that for a suitable $f_4 \in C$, (2.2) is true for every x from $[-1, 1]$ (see [5] and [6]).

Very recently A. A. PRIVALOV [7] settled the case of Jacobi matrices

$$X^{(\alpha,\beta)} = \{x_{kn}^{(\alpha,\beta)},\ k = 1, 2, \ldots, n;\ n = 1, 2, \ldots\}, \quad \alpha, \beta > -1$$

(see e.g. [8], Part 2), showing that with a certain $f_4 \in C$

$$\overline{\lim_{n\to\infty}}\ |L_n(f_4, X^{(\alpha,\beta)}, x)| = \infty \quad \text{a.e. on } [-1, 1], \tag{2.3}$$

where "a.e." stands for almost everywhere. (He considered some further point groups, too.) His proof strongly depends on the properties of the Jacobi roots $x_{kn}^{(\alpha,\beta)}$. Finally, he proved (2.3) for the whole $(-1, 1)$ (see [13]).

3. Result

As indicated above we are going to prove (2.2) for *any* fixed point group X, i.e. we state

THEOREM. *For any matrix X with* (1.1) *one can find a function $F(x)\in C$ such that*

$$\overline{\lim_{n\to\infty}}\, |L_n(F, X, x)| = \infty \quad \textit{for almost all } x \textit{ in } [-1, 1]. \tag{3.1}$$

On the other hand, considering the special matrix

$$\begin{array}{c} x_1 \\ x_1, x_2 \\ x_1, x_2, x_3 \\ \dots\dots\dots \end{array}$$

we can say that (3.1) generally is not true for *all* $x\in[-1, 1]$ (see P. TURÁN [9], Problem III).

Finally, let us remark that the "$\overline{\lim}$" cannot be replaced by "lim" or "$\underline{\lim}$". Indeed, as P. ERDŐS showed, one can construct a point group so that for every $f\in C$ and every $x_0\in[-1, 1]$ there exists a sequence n_k (depending on f and x_0) so that

$$\lim_{k\to\infty} L_{n_k}(f, x_0) = f(x_0)$$

(see [1], p. 384).

4. Proof

4.1. In what follows, sometimes omitting the superfluous notations, let $x_{0n}=1$, $x_{n+1,n}=-1$ and

$$\Delta x_{kn} = x_{kn} - x_{k+1,n} \quad (k = 0, 1, \dots, n;\ n = 1, 2, \dots). \tag{4.1}$$

Let us define the index-sets K_{1n} and K_{2n} and sets D_{1n} and D_{2n} by

$$\begin{cases} \Delta x_{kn} \begin{cases} \le \dfrac{1}{\ln n} \overset{\text{def}}{=} \delta_n & \text{iff} \quad k\in K_{1n}, \\ > \delta_n & \text{iff} \quad k\in K_{2n}, \end{cases} \\ D_{1n} = \bigcup_{k\in K_{1n}} [x_{k+1}, x_k], \quad D_{2n} = [-1, 1]\setminus D_{1n}. \end{cases} \tag{4.2}$$

If $\Delta x_k \le \delta_n$ (which means $k\in K_{1n}$, $[x_{k+1}, x_k]\subset D_{1n}$) we say that the interval $[x_{k+1}, x_k]$ is *short*; the other ones are *long*.

The fact that for any given positive numbers A and ε the measure of those x $(-\infty<x<\infty)$ for which

$$\lambda_n(x) \le A$$

holds if $n \geqq n_0(\varepsilon, A)$, is less than ε, was shown by the first of us in [1]. But here we need a stronger statement. Namely, if

$$I_{lm} = \left[-1+\frac{2(l-1)}{m}, -1+\frac{2l}{m}\right) \quad (l = 1, 2, \ldots, m),$$

then for the short intervals we prove

LEMMA 4.1. *Let $A>0$ be an arbitrary fixed number. Then with arbitrary $m \geqq \max\,[\exp(8A^3), \exp(\exp 100)] \stackrel{\text{def}}{=} m_0(A)$, for any $n \geqq n_0(m)$ there exists a set $H_{1n} \subset D_{1n}$ for which $\mu(H_{1n}) \leqq 1/\ln\ln m$. Further, whenever $x \in D_{1n} \setminus H_{1n}$,*

$$\sum_{\substack{1 \leqq k \leqq n \\ x_{kn} \in D_{1n} \\ x_{kn} \notin I_{j(x), m} \\ k \notin K_{3n}}} |l_{kn}(x)| \geqq (\ln m)^{1/3} \geqq 2A \quad \textit{if} \quad n \geqq n_0(m). \tag{4.3}$$

Here $x \in I_{j(x), m}$ $(1 \leqq j \leqq m)$, K_{3n} is a certain index-set having $\sqrt{\ln n}$ elements at most, $\mu(\ldots)$ stands for the Lebesgue measure.

4.1.1. The proof of this lemma, which is one of the most important parts of our theorem, consists of severals steps.

First we settle Lemma 4.2 regarding both short and long intervals.

Let us introduce the following notation.

$$J_k(q) = J_{kn}(q) = [x_{k+1}+q\Delta x_k, x_k - q\Delta x_k], \quad J_k = J_k(0) = [x_{k+1}, x_k],$$

for $0 \leqq q \leqq 1/2$, $0 \leqq k \leqq n$. If $z_k = z_{kn}(q)$ is defined by

$$|\omega_n(z_k)| = \min_{x \in J_k(q)} |\omega_n(x)|, \quad k = 0, 1, \ldots, n \tag{4.4}$$

(obviously, z_k is one of the endpoints of $J_k(q)$), we state

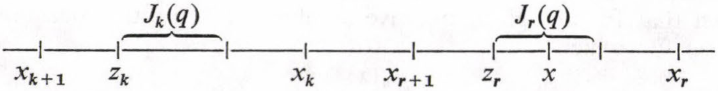
LEMMA 4.2. *If $x_k \leqq x_{r+1}$ $(1 \leqq r < k < n)$ then for arbitrary $0 < q \leqq 1/2$*

$$|l_k(x)| + |l_{k+1}(x)| \geqq q^2 \frac{|\omega_n(z_r)|}{|\omega_n(z_k)|} \frac{\Delta x_k}{x_r - x_{k+1}} \quad \textit{if} \quad x \in J_r(q). \tag{4.5}$$

To prove (4.5), first we use

$$|l_s(x)| = \left|\frac{\omega(x)}{\omega'(x_s)(x-x_s)}\right| = \frac{|\omega(x)|}{|\omega(z_r)|} \frac{|z_r - x_s|}{|x-x_s|} |l_s(z_r)| \geqq q\,|l_s(z_r)| \tag{4.6}$$

if $s = k, k+1$ and $x \in J_r(q)$



(because $z_r-x_s \geqq q\Delta x_r+(x_{r+1}-x_s) \geqq q(\Delta x_r+x_{r+1}-x_s) \geqq q(x-x_s)$], from where

$$|l_k(x)|+|l_{k+1}(x)| \geqq q[|l_k(z_r)|+|l_{k+1}(z_r)|] = \tag{4.7}$$

$$= q\frac{|\omega(z_r)|}{|\omega(z_k)|}\left[|l_k(z_k)|\frac{x_k-z_k}{z_r-x_k}+|l_{k+1}(z_k)|\frac{z_k-x_{k+1}}{z_r-x_{k+1}}\right] \geqq$$

$$\geqq q\frac{|\omega(z_r)|}{|\omega(z_k)|}\frac{\Delta x_k}{x_r-x_{k+1}}\frac{\min(z_k-x_{k+1}, x_k-z_k)}{\Delta x_k}[|l_k(z_k)|+|l_{k+1}(z_k)|] \geqq$$

$$\geqq q^2\frac{|\omega(z_r)|}{|\omega(z_k)|}\frac{\Delta x_k}{x_r-x_{k+1}} \quad (x\in J_r(q)),$$

using that $l_k(u)+l_{k+1}(u) \geqq 1$ if $u\in J_k$ (see [11], Lemma IV).

Similar estimation holds when $x_r \leqq x_{k+1}$.

4.1.2. We construct the set H_{1n} for $n \geqq n_0(m)$.

a) Any of J_{0n}, J_{nn} contained in D_{1n} should belong to H_{1n}. Further, if $J_{kn}\subset D_{1n}$ intersecting two I_{lm} $(1 \leqq l \leqq m)$ or whenever either k or $k+1\in K_{3n}$, it should also belong to H_{1n}. The measure of these intervals J_{kn} is $\leqq 2\delta_n+(m-1)\delta_n+\sqrt{\ln n}\,\delta_n \leqq$ $\leqq (\ln\ln m)^{-2} \overset{\text{def}}{=} \varepsilon_m^2$, if, e.g. $n \geqq \exp(m^2)=n_0(m)$.

b) Let $q=q_m=\varepsilon_m/8$. The intervals $J_{kn}(q)$ or J_{kn} from D_{1n} not considered at a) will be called *exceptional* if there exists an $x=x(k,n)\in J_{kn}(q)$ for which the estimation (4.3) does not hold. The exceptional J_{kn}'s should also belong to H_{1n}. If $\sum_k' \mu(J_{kn}(q))=2c$ (where the dash indicates that the summation is extended only over the exceptional J_{kn}'s), we state that $c=c(n,m) \leqq \varepsilon_m^2$ if $n \geqq n_0(m)$ (whence the aggregate measure of the exceptional intervals J_{kn} is $<3\varepsilon_m^2$).

Indeed, supposing $c>\varepsilon_m^2$ we shall obtain a contradiction.

Let us order the ψ_n exceptional $\bar{J}_1(q), \bar{J}_2(q), \ldots, \bar{J}_{\psi_n}(q)$ such that

$$|\omega(\bar{z}_i)| \geqq \omega(\bar{z}_k) \quad (1 \leqq i \leqq k \leqq \psi_n),$$

where $\bar{z}_k$ stands for the corresponding minimum in $\bar{J}_k(q)$ (see (4.4)). Then for a certain φ_n, $1 \leqq \varphi_n \leqq \psi_n$,

$$\begin{cases} |\omega(\bar{z}_1)| \geqq |\omega(\bar{z}_k)| \geqq (\ln m)^{-1/2}|\omega(\bar{z}_1)| & \text{if} \quad 1 \leqq k \leqq \varphi_n, \\ |\omega(\bar{z}_1)| > (\ln m)^{1/2}|\omega(\bar{z}_k)| & \text{if} \quad \varphi_n < k \leqq \psi_n. \end{cases} \tag{4.8}$$

By a simple computation

$$\sum_{i=\varphi_n+1}^{\psi_n} \mu(\bar{J}_i(q)) \leqq c \quad \text{if} \quad n \geqq n_0(m) \tag{4.9}$$

(if, of course, $\varphi_n<\psi_n$).

Indeed, otherwise, using that

$$\sum_{i=\varphi_n+1}^{\psi_n} = \sum_{\substack{i=\varphi_n+1 \\ J_i\cap I_{jm}=\emptyset}}^{\psi_n} + \sum_{\substack{i=\varphi_n+1 \\ J_i\cap I_{jm}\neq\emptyset}}^{\psi_n} \overset{\text{def}}{=} \sum{}^{(1)}+\sum{}^{(2)}$$

where $\bar{z}_1 \in I_{jm} = I_{j(\bar{z}_1),m}$, we obtain

$$\sum{}^{(2)} \mu(\bar{J}_i(q)) \leqq 2m^{-1} < \varepsilon_m^2/2 < c/2,$$

from where $\sum^{(1)} \mu(\bar{J}_i(q)) > c/2$. Then by (4.5) and (4.8) for *any* $x \in \bar{J}_1(q)$ the sum (4.3) can be estimated as follows

$$\sum_{\dots} |l_k(x)| \geqq \frac{1}{2} \sum{}^{(1)} [|l_i(x)| + |l_{i+1}(x)|] \geqq \frac{q^2}{2} \sum{}^{(1)} \frac{|\omega(\bar{z}_1)|}{|\omega(\bar{z}_i)|} \frac{\Delta \bar{x}_i}{|\bar{x}_{i+1} - \bar{x}_1|} \geqq$$

$$\geqq \frac{q^2}{4} (\ln m)^{1/2} \sum{}^{(1)} \Delta \bar{x}_i \geqq \frac{q^2 c}{8} (\ln m)^{1/2} > \frac{\varepsilon_m^4}{8^3} (\ln m)^{1/2} > (\ln m)^{1/3}$$

which is a contradiction, i.e. (4.9) is true. (Here $\bar{x}_{i+1}$ and $\bar{x}_1$ are the "farthest" points of the corresponding intervals.)

4.1.3. Consequently, using the fact that the total measure γc $(1 \leqq \gamma \leqq 2)$ of the exceptional $\bar{J}_1(q), \dots, \bar{J}_{\varphi_n}(q)$ is bigger than ε_m^2, we should obtain a contradiction. Notice that for $\bar{J}_l$ we have (4.8), each $\bar{J}_l$ is in exactly one I_{km}; if $i = 0, n$, or when i or $i+1 \in K_{3n}$, then J_i cannot be exceptional. Obviously $\varphi_n \geqq c \ln n$.

Dropping $\bar{J}_l$ containing the middle point of $[-1, 1]$ and bisecting the same interval $[-1, 1]$, we have (say) in $[0, 1]$ a set of measure $\geqq [c - \mu(\bar{J}_t)]/2 \geqq (c - \delta_n)/2$, consisting of certain $\bar{J}_l(q)$'s $(1 \leqq l, t \leqq \varphi_n)$.

At the k-th bisection we obtain that interval of length 2^{1-k} which contains certain $\bar{J}_l(q)$'s $(1 \leqq l \leqq \varphi_n)$ of aggregate measure $\geqq 2^{-k} c - \delta_n \geqq 2^{-k-1} c$, if e.g. $1 \leqq k \leqq [\breve{\log}\, m] + 2 \overset{\text{def}}{=} p = p_m$.

Consider these intervals $L_1^*, L_2^*, \dots, L_p^*$.

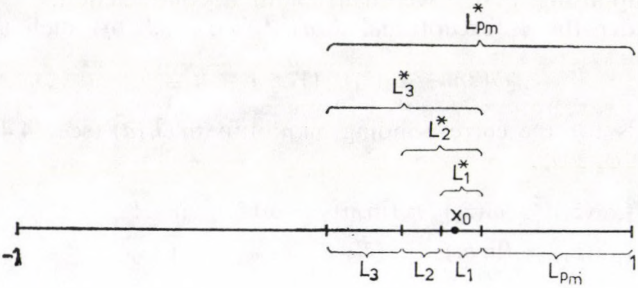


Fig. 1

Obviously $\mu(L_k^*) = 2^{k-p}$, each L_k^* contains at least k exceptional $\bar{J}_l(q)$'s, further

$$\sum_{J_l(q) \subset L_k^*} \mu(\bar{J}_l(q)) \geqq 2^{k-p-2} c \quad (1 \leqq k \leqq p_m,\ 1 \leqq l \leqq \varphi_n).$$

Let $L_1 = L_1^*$, further $L_k = L_k^* \setminus L_{k-1}^*$ $(2 \leqq k \leqq p_m)$ (see Fig. 1). It is easy to see that $(2m)^{-1} \leqq \mu(L_1) \leqq m^{-1}$. Let us choose any fixed point x from any exceptional

$\bar{J}_{l_0,n}(q)$ contained in L_1 $(1 \leqq l_0 \leqq \varphi_n)$. Then, by (4.5) and (4.8) the sum (4.3) can be estimated as follows

$$(4.10)\quad \sum |l_t(x)| \geqq \frac{1}{2} \sum [|l_t(x)|+|l_{t+1}(x)|] \geqq \sum_{k=1}^{p}{}' \sum_{\substack{l \\ J_l(q)\subset L_k}} q^2 \frac{|\omega(\bar{z}_{l_0})|}{|\omega(\bar{z}_l)|} \cdot \frac{\Delta \bar{x}_l}{|\bar{x}_{l+1}-\bar{x}_0|} \geqq$$

$$\geqq q^2 (\ln m)^{-1/2} \sum_{k=1}^{p}{}' \sum_{\substack{l \\ J_l(q)\subset L_k}} \frac{\Delta \bar{x}_l}{|\bar{x}_{l+1}-\bar{x}_0|} \overset{\text{def}}{=} q^2 (\ln m)^{-1/2} B,$$

where $\bar{x}_{l+1}$ and $\bar{x}_0$ are the "farthest" points of the corresponding intervals, $1 \leqq l$, $l_0 \leqq \varphi_n$, the dash means that we exclude k whenever $I_{j(x),m} \cap L_k \neq \emptyset$. To estimate B, let

$$(4.11)\qquad \sum_{\substack{l \\ J_l(q)\subset L_k}} \Delta \bar{x}_l \overset{\text{def}}{=} c\alpha_k .$$

Using the construction, it is easy to see that

$$(4.12)\qquad c \sum_{k=1}^{i} \alpha_k \geqq 2^{i-p-2} c \quad (1 \leqq i \leqq p),$$

$$(4.13)\qquad |\bar{x}_0 - \bar{x}_i| \leqq 2^{k-p} \quad \text{if} \quad \bar{x}_0 \in L_1 \quad \text{and} \quad \bar{x}_i \in L_k \quad (1 \leqq k \leqq p).$$

By induction

$$(4.14)\qquad \alpha_k \leqq 2^{k-2} \alpha_1 \quad (2 \leqq k \leqq p).$$

(Indeed, by construction $\alpha_2 \leqq \alpha_1, \alpha_3 \leqq \alpha_1 + \alpha_2 \leqq 2\alpha_1, \dots,$ from where we get (4.14).)

Now, by (4.13), (4.11), (4.12), the Abel transformation and (4.14) we can write

$$B \geqq c2^p \sum_{k=1}^{p}{}' 2^{-k} \alpha_k \geqq c2^p \left[\sum_{k=1}^{p} 2^{-k} \alpha_k - 4 \max_{1 \leqq k \leqq p} \frac{\alpha_k}{2^k} \right] \geqq$$

$$\geqq c2^p \left[\sum_{k=1}^{p-1} \left(\sum_{i=1}^{k} \alpha_i \right) \frac{1}{2^{k+1}} + \left(\sum_{i=1}^{p} \alpha_i \right) \frac{1}{2^p} - 4\alpha_1 \right] \geqq$$

$$\geqq c2^p \left[\sum_{k=1}^{p-1} \frac{2^{k-p-2}}{2^{k+1}} + \frac{1}{2^{p+2}} - \frac{4}{mc} \right] \geqq c \frac{\overset{2}{\log} m}{16} - 16 \geqq \frac{c \ln m}{20},$$

i.e., in virtue of (4.10),

$$\sum_{\dots} |l_t(x)| \geqq \frac{\varepsilon_m^4 (\ln m)^{1/2}}{8^2 \cdot 20} > (\ln m)^{1/3} \quad (n \geqq n_0(m)),$$

i.e. for *any* $x \in \bar{J}_{l_0}(q)$ we have (4.3). But then $\bar{J}_{l_0}(q)$ is not exceptional which is a contradiction. So $c \leqq \varepsilon_m^2$, as it was stated.

4.1.4. c) Clearly, *for any point* $x \in J_{kn}(q)$ $(J_{kn} \subset D_{1n})$ *considered neither at* a) *nor at* b), *the estimation* (4.3) *will be true*. For these J_{kn} the sets $J_{kn} \setminus J_{kn}(q)$

of aggregate measure c_1 should belong to H_{1n}, too. Obviously, c_1 can be estimated as follows

$$c_1 \leqq \sum_{k\in K_{1n}} [\mu(J_{kn})-\mu(J_{kn}(q))] = 2q \sum_{k\in K_{1n}} \Delta x_k \leqq \frac{\varepsilon_m}{2}.$$

So by a), b) and c)

$$\mu(H_{1n}) \leqq \varepsilon_m^2+3\varepsilon_m^2+\varepsilon_m/2 \leqq \varepsilon_m$$

which proves Lemma 4.1.

4.2. Here we introduce an important definition. The interval I_{km} and its index k will be called *good* for a certain $n\geqq n_0(m)$ if

$$\sum_{J_{in}\subset H_{1n}}' \mu(I_{km}\cap J_{in}) \leqq \frac{\varepsilon_m}{2m} \quad (n \geqq n_0(m)),$$

where the dash means that we take only such J_{in}'s which were considered in a) or b) $(1\leqq k\leqq m)$. (Observe that I_{km} is good whenever $I_{km}\cap D_{1n}=\emptyset$.) Using that

$$\sum_{J_{in}\subset H_{1n}}' \mu(J_{in}) \leqq 4\varepsilon_m^2,$$

for any $n\geqq n_0(m)$ *at most* $8m\varepsilon_m$ *intervals* I_{km} *are not good* (m is fixed).

If we can choose a subsequence $\{n_i\}_{i=1}^\infty$ such that I_{1m} is good whenever $n\in\{n_i\}$ we take it. Otherwise, let us define $\{n_i\}$ so that I_{1m} is *not* good if $n\in\{n_i\}$. Starting from $\{n_i\}$ let us make the analogous process for I_{2m}. So after the m-th step we essentially derive the following statement.

LEMMA 4.3. *For every fixed* $m\geqq m_0(A)$ *and sequence* $\{l_r\}_{r=1}^\infty$ $(l_r\geqq n_0(m)$ *are integers) one can select a subsequence* $\{n_i\}_{i=1}^\infty\subset\{l_r\}_{r=1}^\infty$ *such that for any* $n\in\{n_i\}_{i=1}^\infty$ *the intervals* $I_{1m}, I_{2m}, \dots, I_{mm}$ *are good, apart from* $I_{k_1,m}, I_{k_2,m}, \dots, I_{k_j,m}$. *Here* $1\leqq k_1<k_2<\dots<k_j\leqq m$, $j=j(m)\leqq 8m\varepsilon_m$ *and, which is very important, the indices* k_s $(1\leqq s\leqq j)$ *depend only on* m. *(If* $j=0$, *every* I_{km} *is good.)*

4.3. Now we shall treat the long intervals, i.e. the case when $\Delta x_k>\delta_n$ or what is the same, $k\in K_{2n}$, $(x_{k+1}, x_k)\subset D_{2n}$.

The following estimation plays a similar role as Lemma 4.1.

LEMMA 4.4. *Let* $\Delta x_{kn}>\delta_n$ *(k is fixed,* $0\leqq k\leqq n$*). Then for any fixed* $0<q<1/2$ *we can define the index* $t=t(k,n)$ *and the set* $h_{kn}\subset J_{kn}$ *so that* $\mu(h_{kn})\leqq 4q\Delta x_{kn}$, *moreover*

$$|l_t(x)| \geqq 3^{n\delta_n^5} \overset{\text{def}}{=} \eta_n \quad \textit{if} \quad x\in J_{kn}\setminus h_{kn} \quad \textit{and} \quad n \geqq n_1(q). \tag{4.15}$$

In the proof we refine some ideas of the papers by ERDŐS and TURÁN [11] and ERDŐS and SZABADOS [12]. Take those roots $y_{in}=\cos\vartheta_{in}$ $(1\leqq i\leqq n)$ of the n-th Chebychev polynomials $T_n(x)=\cos n\vartheta=2^{n-1}x^n+\dots$ $(x=\cos\vartheta)$ which are in $J_{kn}(q)$. Their number is not less than $(1-2q)n\delta_n/\pi$ because of $\vartheta_{i+1}-\vartheta_i=\pi/n$ $(1\leqq i\leqq n-1$; see (2.1)) and $\Delta x_k>\delta_n$. If

$$h_k = [J_k\setminus J_k(q)]\cup\left\{\bigcup_{y_i\in J_k(q)}\left[\cos\left(\vartheta_i+q\frac{\pi}{n}\right), \cos\left(\vartheta_i-q\frac{\pi}{n}\right)\right]\right\},$$

then $\mu(h_k)\leqq 4q\Delta x_k$ and for arbitrary $y\in J_k\setminus h_k=J_k(q)\setminus h_k$ we can write $|T_n(y)|\geqq$ $\geqq|\sin n\vartheta_i \sin q\pi|\geqq 2q$. Consider now the interval $M=M(y)=\left[y-\frac{q}{4}\delta_n, y+\frac{q}{4}\delta_n\right]\subset$ $\subset J_k$ which contains at least $\frac{q}{2\pi}n\delta_n>n\delta_n^2$ roots of $T_n(x)$ $(n\geqq n_0(q))$. Then the polynomial $p(y,n;\, x)=p(x)=\prod_{y_{in}\notin M(y)}(x-y_{in})$ of degree less than n, can be estimated at any $x\notin(x_{k+1}, x_k)$ as follows

$$|p(x)|=\frac{|T_n(x)|}{2^{n-1}\prod\limits_{y_i\in M}|x-y_i|}=\left|p(y)\frac{T_n(x)}{T_n(y)}\right|\prod_{y_i\in M}\frac{|y-y_i|}{|x-y_i|}\leqq$$

$$\leqq\frac{|p(y)|}{2q}\prod_{y_i\in M}\frac{1}{3}\leqq\frac{|p(y)|}{2q}\frac{1}{3^{n\delta_n^2}}<\frac{|p(y)|}{3^{n\delta_n^3}}.$$

Now, using the Lagrange interpolatory formula,

$$|p(y)|\leqq\sum_{i=1}^{n}|p(x_i)||l_i(y)|\leqq|p(y)|\,3^{-n\delta_n^3}\sum_{i=1}^{n}|l_i(y)|$$

from where $\sum_{i=1}^{n}|l_i(y)|\geqq 3^{n\delta_n^3}$ if $n\geqq n_1(q)$, because $|p(y)|\neq 0$.

So for any fixed $y\in J_k(q)\setminus h_k$ there exists an index $t=t(y,k,n)$ such that

$$|l_t(y)|=\left|\frac{\omega(y)}{\omega'(x_t)(y-x_t)}\right|\geqq 3^{n\delta_n^4}\quad(n\geqq n_1(q)). \tag{4.16}$$

Let us choose the point $y=u_k$ such that

$$|\omega(u_k)|=\min_{x\in J_k(q)\setminus h_k}|\omega(x)|.$$

Then, for arbitrary $y\in J_k(q)\setminus h_k$

$$|l_t(y)|=|l_t(u_k)|\frac{|\omega(y)|}{|\omega(u_k)|}\frac{|u_k-x_t|}{|y-x_t|}.$$

If $t\neq k, k+1$ we obtain as in (4.6) that $|u_k-x_t|\geqq q|y-x_t|$. (This inequality is trivial if $t=k$ or $t=k+1$.)

I.e., in both cases for $n\geqq n_1(q)$

$$|l_t(y)|\geqq q|l_t(u_k)|>3^{n\delta_n^5}\quad\text{if}\quad y\in J_k(q)\setminus h_k,$$

which means that in (4.15) the index t *does not depend on* x.

4.4. In the following part we shall construct the function $F(x)$.

4.4.1. Let us consider the short intervals, the sequences $\{A_t\}_{t=1}^{\infty}$, $\{m_t\}_{t=1}^{\infty}$ satisfying $A_t\nearrow\infty$, $m_t=[m_0(A_t)]+1$ and the intervals I_{j,m_t} (I_j, for short) of length $2/m_t$ $(1\leqq j\leqq m_t)$.

Let $t=1$. Let us choose the subsequence Q fulfilling the requirements of Lemmas 4.1 and 4.3. If $n_{11}\in Q$, let us define $g_1(x)$ *only on the nodes* as follows.

$$g_1(x_{k,n_{11}})=\begin{cases}(-1)^{k+1} & \text{if } x_{k,n_{11}}\in D_{1,n_{11}}\setminus I_1,\\ 0 & \text{otherwise.}\end{cases} \tag{4.17}$$

Then, in virtue of Lemma 4.1

$$|L_{n_{11}}(g_1,x)|=\sum_{x_k\in D_1\setminus I_1}|l_k(x)|\geqq(\ln m_1)^{1/3}\geqq 2A_1 \tag{4.18}$$

if $x\in(I_1\cap D_{1,n_{11}})\setminus H_{1,n_{11}}\overset{\text{def}}{=}T_1$. (Generally, if $f(x)$ is defined only for certain $x_k=$ $=x_{kn}$, $k=k_1,k_2,\ldots,k_s$, then let $L_n(f,x)\overset{\text{def}}{=}\sum_{i=1}^{s}f(x_{k_i})l_{k_i}(x)$. If $T_1=\emptyset$, (4.18) is meaningless.)

4.4.2. Let $n_{12}>n_{11}$ $(n_{11},n_{12}\in Q)$ satisfy $\sqrt{\ln n_{12}}>n_{11}$. Let us define the set $\mathcal{T}_2$ by

$$2|L_{n_{12}}(g_1,x)|>(\ln m_1)^{1/3} \quad \text{if} \quad x\in\mathcal{T}_2\subset(I_2\cap D_{1,n_{12}})\setminus H_{1,n_{12}}\overset{\text{def}}{=}T_2. \tag{4.19}$$

Moreover, if $x\in T_2\setminus\mathcal{T}_2$, (4.19) should not hold.

a) *If* $2\mu(\mathcal{T}_2)\geqq\mu(T_2)$ or $T_2=\emptyset$ let $g_1(x_{k,n_{12}})=0$ *at* $x_{k,n_{12}}$ *not considered in* (4.17) (i.e. those for which does not exist l $(1\leqq l\leqq n_{11})$ such that $x_{k,n_{12}}=x_{l,n_{11}}$).

b) If $2\mu(\mathcal{T}_2)<\mu(T_2)$ then *for* $x_{k,n_{12}}$ *not considered in* (4.17) let, with $[a_j,a_{j+1})=I_j$,

$$g_1(x_{k,n_{12}})=\begin{cases}(-1)^k & \text{if } x_{k,n_{12}}\in D_{1,n_{12}}\setminus I_2 \text{ and } x_k<a_2,\\ (-1)^{k+1} & \text{if } x_{k,n_{12}}\in D_{1,n_{12}}\setminus I_2 \text{ but } x_k\geqq a_3,\\ 0 & \text{otherwise.}\end{cases} \tag{4.20}$$

By (4.19) and (4.3) if $x\in T_2\setminus\mathcal{T}_2$, then

$$|L_{n_{12}}(g_1,x)|\geqq\left|\pm\sum\nolimits^{(1)}|l_{k,n_{12}}(x)|+\left|\sum\nolimits^{(2)}g_1(x_{k,n_{12}})l_{k,n_{12}}(x)\right|\right|\geqq$$

$$\geqq(\ln m_1)^{1/3}-\frac{1}{2}(\ln m_1)^{1/3}=\frac{1}{2}(\ln m_1)^{1/3}\geqq A_1 \quad (x\in T_2\setminus\mathcal{T}_2).$$

Here $\sum^{(1)}$ is extended over the x_k's considered in (4.20); for them Lemma 4.1 can be applied (because $\sqrt{\ln n_{12}}>n_{11}$); in $\sum^{(2)}$ we take those k's for which $x_{k,n_{12}}=x_{l,n_{11}}$ at certain $1\leqq l\leqq n_{11}$. So, by (4.19) $2|\sum^{(2)}|\leqq(\ln m_1)^{1/3}$, because $x\in T_2\setminus\mathcal{T}_2$.

Consequently, in both cases we can define the set $R_2\subset T_2$ and the function $g_1(x)$ such that $2\mu(R_2)\geqq\mu(T_2)$. Moreover

$$|L_{n_{12}}(g_1,x)|\geqq A_1 \quad \text{whenever} \quad x\in R_2\subset T_2. \tag{4.21}$$

(At a) $R_2=\mathcal{T}_2$; at b) $R_2=T_2\setminus\mathcal{T}_2$; if $T_2=\emptyset$, the statement (4.21) is meaningless.)

4.4.3. By the above method we can obtain the sets $T_i = T_{1i} = (I_{i,m_1} \cap D_{1n_{1i}}) \setminus H_{1n_{1i}}$, the subsets $R_i = R_{1i} \subset T_{1i}$ $(i=1, 2, \ldots, m_1;\ R_1 \equiv T_1)$ and the function $g_1(x)$ such that $2\mu(R_{1i}) \geqq \mu(T_{1i})$ and

(4.22) $$|L_{n_{1i}}(g_1, x)| \geqq A_1 \quad \text{if} \quad x \in R_{1i} \subset T_{1i}, \quad 1 \leqq i \leqq m_1.$$

Let

(4.23) $$G_1 \overset{\text{def}}{=} \bigcup_{i=1}^{m_1} R_{1i}.$$

4.4.4. Now consider the *polynomial* $\varphi_1(x) = \varphi_1(g_1, x)$ satisfying $\varphi_1(x_{k,n_{1i}}) = g_1(x_{k,n_{1i}})$ $(1 \leqq k \leqq n_{1i};\ 1 \leqq i \leqq m_1)$ and $\|\varphi_1\| \leqq 2$. Here $\deg \varphi_1 \leqq N_1$, where N_1 depends only on the distribution of the nodes defining $g_1(x)$ (see [8], Part 3, II/§ 3).

4.4.5. Generally, starting from the subsequence obtained in the $(t-1)$-th step, let us make the above construction for (A_t, m_t) $(t=2, 3, \ldots)$. We can suppose

(4.24) $$n_{t-1, m_{t-1}} < N_{t-1} < n_{t1} \quad (t = 2, 3, \ldots).$$

We successively get the sets T_{ti}, their parts R_{ti} with $2\mu(R_{ti}) \geqq \mu(T_{ti})$ $(i=1, 2, \ldots, m_t)$, the functions $g_t(x)$ for which

(4.25) $$|L_{n_{ti}}(g_t, x)| \geqq A_t \quad \text{if} \quad x \in R_{ti} \subset T_{ti}, \quad 1 \leqq i \leqq m_t,$$

further the sets

(4.26) $$G_t = \bigcup_{i=1}^{m_t} R_{ti}.$$

We can also construct the corresponding polynomials $\varphi_t(x)$, taking the values $g_t(x_{k,n_{ti}})$ $(1 \leqq k \leqq n_{ti};\ 1 \leqq i \leqq m_t)$ for which $\|\varphi_t\| \leqq 2$ and $\deg \varphi_t \leqq N_t$ $(t=2, 3, \ldots)$.

Supposing

(4.27) $$A_t > t^3 \lambda^2_{N_{t-1}} \quad (\lambda_{N_0} \equiv 1,\ t = 1, 2, \ldots),$$

let us define the set

(4.28) $$G = \bigcap_{k=1}^{\infty} \left(\bigcup_{t=k}^{\infty} G_t \right)$$

and the function

(4.29) $$f(x) = \sum_{t=k}^{\infty} \frac{\varphi_t(x)}{t^2 \lambda_{N_{t-1}}}.$$

We state that

(4.30) $$\varlimsup_{n \to \infty} |L_n(f, x)| = \infty \quad \text{whenever} \quad x \in G.$$

(Clearly $f \in C$, moreover $\|f\| \leqq 4$ can be attained.) If $G = \emptyset$, we have nothing to prove. Otherwise, if $x \in G$ there exists an index-set $\{r_k\}_{k=1}^{\infty}$ depending on x for which $x \in G_{r_k}$ $(k=1, 2, \ldots)$. Then, by (4.26), for any fixed r_k we can find an s such that $x \in R_{r_k, s}$. By (4.29)

$$L_{n_{r_k, s}}(f, x) = \sum_{i=1}^{\infty} \frac{L_{n_{r_k, s}}(\varphi_i, x)}{i^2 \lambda_{N_{i-1}}} = \sum_{i<r_k} + \sum_{i=r_k} + \sum_{i>r_k}.$$

Here by (4.24) $L_{n_{r_k,s}}(\varphi_i, x) \equiv \varphi_i(x)$ if $i < r_k$, so

$$\Big|\sum_{i<r_k}\Big| \leqq 2 \sum_{i=1}^{\infty} i^{-2}\lambda_{N_{i-1}}^{-1} \leqq c_1,$$

further, by (4.25) and (4.27)

$$\left|\frac{L_{n_{r_k,s}}(\varphi_{r_k}, x)}{r_k^2 \lambda_{N_{r_k-1}}}\right| \geqq \frac{A_{r_k}}{r_k^2 \lambda_{N_{r_k-1}}} > r_k \lambda_{N_{r_k-1}}.$$

Finally, supposing $\lambda_l > \lambda_j$ if $l > j$, $l, j \in \{n_{ti}\} \cup \{N_t\}$, we can write

$$\Big|\sum_{i>r_k}\Big| \leqq 2\lambda_{n_{r_k,s}} \sum_{i=r_k+1}^{\infty} i^{-2}\lambda_{N_{i-1}}^{-1} \leqq 2 \sum_{i=1}^{\infty} i^{-2} \leqq c_2,$$

because $\lambda_{n_{r_k,s}} < \lambda_{N_{r_k}}$ (see (4.24)). Consequently,

$$|L_{n_{r_k,s}}(f, x)| \geqq r_k \quad (k = 2, 3, \dots;\ x \in G)$$

which actually is more than (4.30).

4.4.6. Let us now take the sets $T_{ti}^{[2]} = T_{ti}^{[1]} \setminus R_{ti}^{[1]}$ $(i = 1, 2, \dots, m_t;\ t = 1, 2, \dots;$ $T_{ti}^{[1]} = T_{ti}, R_{ti}^{[1]} = R_{ti})$ given by the previous steps. If, e.g. $t = 1$, let us begin the construction of $g_1^{[2]}(x)$ exactly as we did for $g_1(x) = g_1^{[1]}(x)$ in **4.4.1** (i.e., we use the *same* A_1, m_1, T_1 and nodes; compare (4.17)), but the distinctions a) and b) in **4.4.2** should be defined by the measure of $\mathcal{T}_{12}^{[2]}$ instead of $\mathcal{T}_2 = \mathcal{T}_{12}^{[1]}$ where $\mathcal{T}_{12}^{[2]}$ collects those points of the set $T_{12}^{[2]} = T_{12}^{[1]} \setminus R_{12}^{[1]}$ for which $2|L_{n_{12}}(g_1^{[2]}, x)| >$ $> (\ln m_1)^{1/3}$ (see (4.19)). Consequently, by the method analogous to **4.4.1—4.4.5** (using the same $\{n_{ti}\}$) we can construct the corresponding sets $R_{ti}^{[2]}, G_t^{[2]}$, the polynomials $\varphi_t^{[2]}(x)$ of degree $\leqq N_t$ and the continuous function

$$f^{[2]}(x) = \sum_{t=1}^{\infty} \frac{\varphi_t^{[2]}(x)}{t^2 \lambda_{N_{t-1}}} \tag{4.31}$$

with $\|f^{[2]}\| \leqq 4$ such that on the set

$$G^{[2]} = \bigcap_{k=1}^{\infty}\left(\bigcup_{t=k}^{\infty} G_t^{[2]}\right) \tag{4.32}$$

we have

$$\overline{\lim_{n\to\infty}}\, |L_n(f^{[2]}, x)| = \infty \quad (x \in G^{[2]}). \tag{4.33}$$

By the same considerations starting from the sets $T_{ti}^{[l]} = T_{ti}^{[l-1]} \setminus R_{ti}^{[l-1]}$ $(l = 3, 4, \dots, p$ where p will be defined later), we can successively define the functions $f^{[l]} \in C$, $\|f^{[l]}\| \leqq 4$ and the sets $G^{[l]}$ such that

$$\overline{\lim_{n\to\infty}}\, |L_n(f^{[l]}, x)| = \infty \quad (x \in G^{[l]}) \tag{4.34}$$

$$(l = 1, 2, \dots, p;\ f^{[1]} = f,\ G^{[1]} = G).$$

Later we shall apply the fact that for *any* t and i

$$\mu(R_{ti}^{[l]}) \leqq \frac{1}{2^{l-2} m_t} \quad (l = 1, 2, \dots, p) \tag{4.35}$$

and for any fixed t and i

$$R_{ti}^{[l_1]} \cap R_{ti}^{[l_2]} = \varnothing \quad (l_1 \neq l_2) \tag{4.36}$$

(see the definition of the sets $R_{ti}^{[l]}$).

Now let $\varrho > 0$ be arbitrarily small and $p = p_\varrho$ the smallest positive integer so that

$$\mu(R_{ti}^{[p_\varrho]}) \leqq \frac{\varrho}{m_t} \quad (i = 1, 2, \dots, m_t;\ t = 1, 2, \dots). \tag{4.37}$$

It is easy to see that $1 \leqq p_\varrho \leqq 3 + |\overset{2}{\log}\, \varrho|$.

4.4.7. To define the proper (linear) combination of the functions $f^{[1]}, f^{[2]}, \dots, f^{[p]}$ on $G^{[1]} \cup G^{[2]} \cup \dots \cup G^{[p]}$ we prove the following statement, which generalizes an idea of G. GRÜNWALD [4].

LEMMA 4.5. *If* $r_1(x), r_2(x) \in C$, *moreover*

$$\overline{\lim_{n\to\infty}}\, |L_n(r_1, x)| = \infty \quad \textit{if} \quad x \in B_1,\ \mu(B_1) < \infty, \tag{4.38}$$

$$\overline{\lim_{n\to\infty}}\, |L_n(r_2, x)| = \infty \quad \textit{if} \quad x \in B_2,\ \mu(B_2) < \infty, \tag{4.39}$$

then any fixed interval (β_1, β_2) $(\beta_1 < \beta_2)$ *contains an* α *such that*

$$\overline{\lim_{n\to\infty}}\, |L_n(\alpha r_1 + r_2, x)| = \infty \quad \textit{a.e. on } B_1 \cup B_2. \tag{4.40}$$

REMARK. An interesting special case can be obtained by $B_2 = \varnothing$. To prove the lemma let $\tilde{B}_1$ be the part of $B_1 \cup B_2$ fulfilling (4.38). Clearly $B_1 \subset \tilde{B}_1$. If

$$E_\lambda = \{x\colon x \in \tilde{B}_1,\ \overline{\lim_{n\to\infty}}\, |L_n(\lambda r_1 + r_2, x)| < \infty\} \quad (\beta_1 < \lambda < \beta_2)$$

then $E_\lambda \cap E_\mu = \varnothing$ $(\lambda \neq \mu)$. Indeed, otherwise we can write for $x \in E_\lambda \cap E_\mu$

$$\infty = \overline{\lim_{n\to\infty}}\, |(\lambda - \mu) L_n(r_1, x)| = \overline{\lim_{n\to\infty}}\, |L_n(\lambda r_1 + r_2, x) - L_n(\mu r_1 + r_2, x)| \leqq$$

$$\leqq \overline{\lim_{n\to\infty}}\, \big(|L_n(\lambda r_1 + r_2, x)| + |L_n(\mu r_1 + r_2, x)|\big) < \infty,$$

a contradiction. Using $\mu(\tilde{B}_1) < \infty$ and that only countable E_λ's have positive measure $(\beta_1 < \lambda < \beta_2)$, there exists $\alpha \in (\beta_1, \beta_2)$ such that $\mu(E_\alpha) = 0$ from where (4.40) is true a.e. on $\tilde{B}_1$. If $x \in (B_1 \cup B_2) \setminus \tilde{B}_1$ (when $x \in B_2$, too) both $|L_n(\alpha r_1, x)| \leqq K(x)$ $(0 \leqq K(x) < \infty)$, and $\overline{\lim\limits_{n\to\infty}}\, |L_n(r_2, x)| = \infty$ hold which mean (4.40) for x. These prove the lemma.

4.4.8. Choosing $\beta_1 = 0$ and $\beta_2 = 0.5$, consider that $\alpha \in (0, 0.5)$ for which, with $e_2 = \alpha_1 f^{[1]} + f^{[2]}$,

$$\overline{\lim_{n\to\infty}}\, |L_n(e_2, x)| = \infty \quad \text{a.e. on} \quad G^{[1]} \cup G^{[2]}.$$

Obviously $\|e_2\| \leqq 2+4<8$. By this construction we succesively get the values $\alpha_{i-1}\in(0, 0.5)$ and the continuous functions $e_i=\alpha_{i-1}e_{i-1}+f^{[i]}$ satisfying

$$\overline{\lim_{n\to\infty}}\, |L_n(e_i, x)| = \infty \quad \text{a.e. on} \quad G^{[1]}\cup G^{[2]}\cup\ldots\cup G^{[i]}$$

and $\|e_i\| \leqq 0.5\,\|e_{i-1}\| + \|f^{[i]}\| < 8$ $(i=3, 4, \ldots, p_\varrho)$.

I.e., if $i=p_\varrho$, we can say that *for every fixed* $\varrho>0$ *there exists a function* $f_\varrho\in C$, $\|f_\varrho\| \leqq 8$ *so that*

$$\overline{\lim_{n\to\infty}}\, |L_n(f_\varrho, x)| = \infty \quad \textit{a.e. on } G_\varrho \tag{4.41}$$

where $G_\varrho = \bigcup_{i=1}^{p_\varrho} G^{[i]}$.

4.4.9. We go on with the construction of $F(x)$ for the long intervals ($\Delta x_{kn}>\delta_n$ i.e. $k\in K_{2n}$) employing the same A_t, m_t, n_{ti} and I_{im_t} $(i=1, 2, \ldots, m_t;\ t=1, 2, \ldots)$ as for the short intervals. First a simple note. If

$$H_{2n} \overset{\text{def}}{=} \bigcup_{k\in K_{2n}} h_{kn} \quad (n=1, 2, \ldots) \tag{4.42}$$

and

$$q = q_t = \frac{\varepsilon_{m_t}}{8m_t} \tag{4.43}$$

then by Lemma 4.4 for any t and i

$$\mu(H_{2,n_{ti}}) \leqq 2\cdot 4q_t = \frac{\varepsilon_{m_t}}{m_t}, \quad \text{if} \quad n_{ti} \geqq n_1(m_t); \tag{4.44}$$

the latter should be supposed.

For simplicity's sake let $(D_{2,n_{11}}\setminus H_{2,n_{11}})\cap I_{1,m_1}\neq\emptyset$, say, for the indices $j_1, j_2, \ldots, j_s\in K_{2,n_{11}}$

$$(J_{i,n_{11}}\setminus h_{i,n_{11}})\cap I_{1,m_1}\neq\emptyset \quad (i=j_1, j_2, \ldots, j_s;\ s\geqq 1). \tag{4.45}$$

We take the indices $t(i, n_{11})$ $(i=j_1, j_2, \ldots, j_s)$ guaranteed by Lemma 4.4 and define the function $u_i(x)$ as follows.

$$u_i(x_{k,n_{11}}) = \begin{cases} 1 & \text{when} \quad k=t(i, n_{11}),\\ 0 & \text{otherwise,}\end{cases}$$

$|u_i(x)|\leqq 1$, $u_i\in C$. Then clearly

$$|L_{n_{11}}(u_i, x)| \geqq \eta_{n_{11}} \quad \text{if} \quad x\in J_{i,n_{11}}\setminus h_{i,n_{11}} \quad (i=j_1, j_2, \ldots, j_s). \tag{4.46}$$

4.4.10. To combine the at most $2m_1^{-1}\ln n_{11}$ functions $u_i(x)$ we need the following

LEMMA 4.6. *Let* $r_1, r_2\in C$, *moreover*

$$|L_n(r_1, x)| \geqq M_1 \quad \textit{if} \quad x\in B_1, \quad \mu(B_1)<\infty, \tag{4.47}$$

$$|L_n(r_2, x)| \geqq M_2 \quad \textit{if} \quad x\in B_2, \quad \mu(B_2)<\infty. \tag{4.48}$$

Consider the fixed real numbers $\beta_1<\beta_2$ *and the positive integer* k. *Further take*

$$\alpha_i = (\beta_2-\beta_1)\frac{i}{k}+\beta_1 \quad (i=0, 1, \ldots, k). \tag{4.49}$$

Then, if $M_2 \geqq M_1$ *and* $0 \leqq \beta_1 < \beta_2 \leqq 0.5$, *there exists an* α_j $(0 \leqq j \leqq k)$ *and E of measure at least* $\left(1-\frac{1}{k+1}\right)\mu(B_1 \cup B_2)$, $E \subset B_1 \cup B_2$, *so that*

$$(4.50) \qquad |L_n(\alpha_j r_1 + r_2, x)| \geqq \frac{\beta_2-\beta_1}{2k} M_1 \quad if \quad x \in E.$$

To prove this, we verify at first a statement which is slightly more than the special case corresponding to $B_2=0$.

Namely, *if we have only* (4.47), *then there exist* P_1 *of measure* $\geqq \left(1-\frac{1}{k+1}\right)\mu(B_1)$, $P_1 \subset B_1$ *and* α_j $(0 \leqq j \leqq k)$ *such that* (4.50) *is true for* $x \in P_1$.

Indeed, let

$$C_i = \left\{x\colon x \in B_1 \text{ and } |L_n(\alpha_i r_1 + r_2, x)| \geqq \frac{\beta_2-\beta_1}{2k} M_1\right\} \quad (i = 0, 1, \dots, k).$$

It is easy to see that any $x \in B_1$ can be contained in at most one $B_1 \setminus C_i$ (see (4.47), (4.50) and the similar part of **4.4.7**), from where $(B_1 \setminus C_i) \cap (B_1 \setminus C_l) = \emptyset$ $(i \neq l)$. By $B_1 \setminus C_i \subset B_1$, for certain $0 \leqq j \leqq k$ $\mu(B_1 \setminus C_j) \leqq \mu(B_1)(k+1)^{-1}$, which gives the special case with $P_1 = C_j$.

Now let $\tilde{B}_1$ be that part of $B_1 \cup B_2$ where (4.47) is satisfied. Take that α_j, for which (4.50) is true on certain $\tilde{P}_1 \subset \tilde{B}_1$. If $x \in (B_1 \cup B_2) \setminus \tilde{B}_1$ then by (4.48),

$$|L_n(\alpha_j r_1 + r_2, x)| \geqq |M_2 - 0.5 M_1| \geqq 0.5 M_2 > (\beta_2 - \beta_1) M_1$$

from where we obtain the lemma by $E = \tilde{P}_1 \cup ((B_1 \cup B_2) \setminus \tilde{B}_1)$.

4.4.11. Using this lemma with the cast

$$r_i(x) = u_{j_i}(x), \quad B_i = (J_{j_i, n_{11}} \setminus h_{j_i, n_{11}}) \cap I_{1, m_1}, \quad M_i = \eta_{n_{11}} \quad (i = 1, 2),$$

$$\beta_1 = 0, \quad \beta_2 = 0.5 \quad \text{and} \quad k = [\ln^2 n_{11}]$$

(see **4.4.9** and **4.4.10**), we obtain a $v_2(x) \in C$ for which

$$|L_{n_{11}}(v_2, x)| \geqq \frac{\eta_{n_{11}}}{4k} \quad \text{if} \quad x \in E_2$$

where (with the above cast)

$$0 \leqq \mu(B_1 \cup B_2) - \mu(E_2) \leqq \frac{\mu(B_1 \cup B_2)}{k+1} \leqq \frac{2\delta_{n_{11}}^2}{m_1},$$

$\|v_2\| \leqq \beta_2 \|r_1\| + \|r_2\| < 2$. At the next step, by $r_1 = v_2$, $B_1 = E_2$, $r_2 = u_{j_3}$ and $B_2 = (J_{j_3} \setminus h_{j_3}) \cap I_1$ we get the function $v_3(x) \in C$ and the set E_3. Finally, the $(s-1)$-th step gives the function $v_s(x) \overset{\text{def}}{=} w_1(x) \in C$, the set $E_s \overset{\text{def}}{=} W_1 \subset I_1$ so that

$$(4.51) \qquad |L_{n_{11}}(w_1, x)| \geqq \frac{\eta_{n_{11}}}{(4k)^{s-1}} \geqq \frac{\eta_{n_{11}}}{(4 \ln^2 n_{11})^{\ln n_{11}}} \overset{\text{def}}{=} \gamma_{n_{11}} \quad \text{if} \quad x \in W_1$$

where

$$\sum_{i=1}^{s} \mu[(J_{j_i} \setminus h_{j_i}) \cap I_1] - \mu(W_1) \leqq \frac{2s\delta_{n_{11}}^2}{m_1} \leqq \frac{2\delta_{n_{11}}}{m_1}, \tag{4.52}$$

because $s < \ln n_{11}$. Further notice that $\|w_1\| \leqq 2$. By definition $\gamma_n \nearrow \infty$ (e.g. $\gamma_n \gg 3^{\sqrt{n}}$) and

$$\mu[(D_{2,n_{11}} \setminus H_{2,n_{11}}) \cap I_{1,m_1}] - \mu(W_1) \leqq \frac{\varepsilon_{m_1}}{m_1} \tag{4.53}$$

if $n_{11} > n_1(m_1)$. (It is easy to see that the left hand sides of (4.52) and (4.53) are the same.)

Now consider the polynomial $\psi_1(x)$ for which $\|\psi_1\| \leqq 4$ and $\psi_1(x_{kn}) = w_1(x_{kn})$ $(k=1, 2, \ldots, n;\ n=n_{11})$. Clearly we can suppose $n_{12} > \deg \psi_1$, too (compare with **4.4.4**).

By this construction one successively obtains the polynomials $\psi_i(x) = \psi_{1i}(x)$ and the sets $W_i = W_{1i}$ $(i=1, 2, \ldots, m_1)$, then generally the polynomials $\psi_{ti}(x)$ and the sets W_{ti} $(i=1, 2, \ldots, m_t,\ t=1, 2, \ldots)$ such that $\|\psi_{ti}\| \leqq 4$, $\deg \psi_{ti} < n_{t,i+1}$ (where $n_{t,m_t+1} \equiv n_{t+1,1}$) and

$$|L_{n_{ti}}(\psi_{ti}, x)| \geqq \gamma_{n_{ti}} \quad \text{if} \quad x \in W_{ti} \subset I_{i,m_t}, \tag{4.54}$$

$$\mu[(D_{2,n_{ti}} \setminus H_{2,n_{ti}}) \cap I_{i,m_t}] - \mu(W_{ti}) \leqq \frac{\varepsilon_{m_t}}{m_t}. \tag{4.55}$$

(If $(D_{2,n_{ti}} \setminus H_{2,n_{ti}}) \cap I_{i,m_t} = \emptyset$ then the corresponding $W_{ti} = \emptyset$, further $w_{ti}(x) = \psi_{ti}(x) = 0$.)

4.4.12. We can define the sequence $\{n_{ti}\}$ (satisfying *all* the requirements mentioned above) such that

$$\gamma_{n_{ti}} > m_t^2 t^3 \lambda_{n_{t,i-1}}.$$

Consider the function

$$h(x) = \sum_{t=1}^{\infty} \frac{1}{t^2 m_t^2} \sum_{i=1}^{m_t} \frac{\psi_{ti}(x)}{\lambda_{n_{t,i-1}}} \tag{4.56}$$

(where $\lambda_{n_{10}} = 1$ and $\lambda_{n_{t,0}} = \lambda_{n_{t-1,m_{t-1}}}$) on the set

$$W = \bigcup_{k=1}^{\infty} \bigcup_{t=k}^{\infty} \left(\bigcup_{i=1}^{m_t} W_{ti} \right). \tag{4.57}$$

By the method applied in **4.4.5** we get

$$\overline{\lim_{n\to\infty}} |L_n(h, x)| = \infty \quad \text{if} \quad x \in W. \tag{4.58}$$

Moreover it is easy to fulfil the condition $\|h\| \leqq 8$. Now, using Lemma 4.5 for $f_\varrho \in C$ and the set G_ϱ (see **4.4.8**), further for $h \in C$ and W, we obtain as follows.

For arbitrary fixed $\varrho > 0$ *there exists a continuous function* $F_\varrho(x)$, $\|F_\varrho\| \leqq 16$ *(if, e.g.* $[\beta_1, \beta_2] = [0, 1]$*) such that*

$$\overline{\lim_{n\to\infty}} |L_n(F_\varrho, x)| = \infty \quad \text{a.e. on } P_\varrho, \quad \mu(P_\varrho) \geqq 2 - \varrho, \tag{4.59}$$

where $P_\varrho = G_\varrho \cup W \subset [-1, 1]$.

Here the only thing we have to prove is that $\mu(P_\varrho)\geqq 2-\varrho$. For this aim let us see the definitions made in **4.4.6** and **4.4.8.** We can write

$$G_\varrho \cup W = \Big(\bigcup_{j=1}^{p_\varrho} G^{[j]}\Big)\cup W = \Big(\bigcup_{j=1}^{p_\varrho}\bigcap_{k=1}^{\infty}\bigcup_{t=k}^{\infty}\bigcup_{i=1}^{m_t} R_{ti}^{[j]}\Big)\cup$$

$$\cup\Big(\bigcap_{k=1}^{\infty}\bigcup_{t=k}^{\infty}\bigcup_{i=1}^{m_t} W_{ti}\Big) = \bigcap_{k=1}^{\infty}\bigcup_{t=k}^{\infty}\bigcup_{i=1}^{m_t}\Big[\Big(\bigcup_{j=1}^{p_\varrho} R_{ti}^{[j]}\Big)\cup W_{ti}\Big] \overset{\text{def}}{=} \bigcap_{k=1}^{\infty}\Big(\bigcup_{t=k}^{\infty} V_t\Big) \overset{\text{def}}{=} \bigcap_{k=1}^{\infty} Q_k.$$

(Indeed, by $W=G^{[0]}$ and $\bigcup_{i=1}^{m_t} R_{ti}^{[j]} = A_{tj}$,

$$G_\varrho \cap W = \bigcup_{j=0}^{p} G^{[j]} = \Big\{\bigcup_{j=0}^{p}\bigcap_{k=1}^{\infty}\bigcup_{t=k}^{\infty} A_{tj}\Big\}_1 = \Big\{\bigcap_{k=1}^{\infty}\bigcup_{t=k}^{\infty}\bigcup_{j=0}^{p} A_{tj}\Big\}_2$$

because $x\in\{\dots\}_s$ if and only if for a certain j there exist infinitely many t such that $x\in A_{tj}$ $(s=1,2)$. Of course, $\{\dots\}_2=\bigcap_{k=1}^{\infty} Q_k$.)

Let us see the measure of [...] for a good interval I_{i,m_t} if $n=n_{ti}$.

The sets $R_{ti}^{[j]}$ $(j=1, 2, \dots, p_\varrho)$ overlap $(D_{1,n_{ti}}\setminus H_{1,n_{ti}})\cap I_{i,m_t}$ apart from a part of measure not exceeding ϱm_t^{-1} (see (4.35)—(4.37)). Moreover, the sets of type a) and b) from $H_{1,n_{ti}}\cap I_{i,m_t}$ have the measure not exceeding $\varepsilon_{m_t}(2m_t)^{-1}$ altogether (since i is good); the same is true for the parts of type c) (see **4.1.4** and **4.2**).

Further, by (4.55) the set W_{ti} contains the set $(D_{2,n_{ti}}\setminus H_{2,n_{ti}})\cap I_{i,m_t}$ excluding a part of measure not exceeding $\varepsilon_{m_t}(m_t)^{-1}$.

Using that $D_1\cap D_2=\emptyset$, $H_1\subset D_1$, $H_2\subset D_2$ and $D_1\cup D_2=[-1, 1]$, by the above considerations and (4.44) we can estimate as follows $(I_i=I_{i,m_t})$.

$$\mu([\dots]) \geqq \mu((D_1\setminus H_1)\cap I_i)-\frac{\varrho}{m_t}+\mu((D_2\setminus H_2)\cap I_i)-\frac{\varepsilon_{m_t}}{m_t} =$$

$$= \mu(I_i\cap(D_1\cup D_2)\setminus(I_i\cap H_1)\setminus(I_i\cap H_2))-\frac{\varrho}{m_t}-\frac{\varepsilon_{m_t}}{m_t} \geqq \frac{2}{m_t}-\frac{1}{m_t}(3\varepsilon_{m_t}+\varrho).$$

By the construction and Lemma 4.3, *the good intervals I_{i,m_t} are uniquely determined by m_t, i.e. by t whenever $n=n_{Tk}$ $(k=1, 2, \dots, m_T;\ T\geqq t)$, its number is* $\geqq m_t-$ $-8m_t\varepsilon_{m_t}$. So we can write

$$\mu(V_t) = \sum_{i=1}^{m_t}\mu([\dots]) \geqq \sum_i{}' \mu([\dots]) \geqq (m_t-8m_t\varepsilon_{m_t})\frac{1}{m_t}(2-3\varepsilon_{m_t}-\varrho) =$$

$$= (1-8\varepsilon_{m_t})(2-3\varepsilon_{m_t}-\varrho) > 2-19\varepsilon_{m_t}-\varrho \quad (t = 1, 2, \dots),$$

where $\sum'$ means that we consider only the good indices i (t is fixed).

By this we obtain

$$\mu(Q_k) = \mu\Big(\bigcup_{t=k}^{\infty} V_t\Big) \geqq \mu(V_k) > 2-19\varepsilon_{m_k}-\varrho.$$

On the other hand, $Q_1\supset Q_2\supset \dots$ from where, as it is well-known, $\mu(Q_k)\to\mu(P_\varrho)$, which gives $\mu(P_\varrho)\geqq 2-\varrho$.

4.4.13. Now we state the following

LEMMA 4.7. *If* $g_1, g_2, \ldots \in C$ *and* $\overline{\lim_{n\to\infty}}\, g_n(x)=\infty$ *on* B, *then for arbitrary fixed* A, ε *and* M *there exist the set* $H \subseteq B$ *and the index* N *such that* $\mu(H) \leqq \varepsilon$; *moreover if* $x \in B \setminus H$ *then for a certain* $u(x)$ *we have*

$$g_{u(x)}(x) \geqq A \quad \text{where} \quad M \leqq u(x) \leqq N. \tag{4.60}$$

Indeed, let

$$H_t = \{x : x \in B,\ g_{M+i}(x) < A,\ i = 0, 1, \ldots, t\} \quad (t = 0, 1, \ldots).$$

If for a certain $t=s$, $\mu(H_s) \leqq \varepsilon$, then we can choose $N=M+s$, because if $x \in B \setminus H_s$ then with suitable $u(x)$, $M \leqq u(x) \leqq N$, we obtain (4.60). On the other hand, if $\mu(H_t) > \varepsilon$ $(t=0, 1, \ldots)$ then using $H_t \supseteq H_{t+1}$ we get $\mu\left(\bigcup_{t=0}^{\infty} H_t\right) \geqq \varepsilon$ wich means that for $x \in \bigcap_{t=0}^{\infty} H_t \subseteq B$, $\overline{\lim_{t\to\infty}}\, g_t(x) \leqq A$ holds, a contradiction.

4.4.14. Now we construct the function $F(x)$. For this aim let $m_1=\lambda_{N_0}=1$, $A_1=2$ and $\varrho_1=2^{-1}$. By (4.59) and the previous lemma we can find an $f_1 \in C$, $\|f_1\| \leqq 16$, the index n_1 and the set $S_1 \subset [-1,1]$, $\mu(S_1) \geqq 2-2\varrho_1$ so that

$$|L_{u_1(x)}(f_1, x)| \geqq A_1 > 1^3 \lambda_{N_0}^2 \quad \text{whenever} \quad x \in S_1 \quad (\text{see } \mathbf{4.4.4}).$$

Generally, let $\delta_k = 2^{-k}$, $A_k > k_8 \lambda_{N_{k-1}}^2$ and choose $m_k = N_{k-1}+1$. As above, we obtain the polynomial $\varphi_k(x)$ of degree $\leqq N_k$, $\|\varphi_k\| \leqq 32$, the set $S_k \subset [-1,1]$, $\mu(S_k) \geqq 2-2\delta_k$, and the index n_k so that

$$|L_{u_k(x)}(\varphi_k, x)| \geqq A_k > k^3 \lambda_{N_{k-1}}^2 \quad \text{if} \quad x \in S_k$$

with $m_k \leqq u_k(x) \leqq n_k$ $(k=2, 3, \ldots)$. Choosing N_k large enough compared to n_k, we obtain, using the arguments of **4.4.4-4.4.5**, that for the continuous function

$$F(x) = \sum_{k=1}^{\infty} \frac{\varphi_k(x)}{k^2 \lambda_{N_{k-1}}}$$

and for the set $S = \bigcap_{k=1}^{\infty} \bigcap_{i=k}^{\infty} S_i$ of measure 2

$$\overline{\lim_{n\to\infty}}\, |L_n(F, x)| = \infty \quad \text{on } S,$$

which is the statement of the theorem.

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ON BOX PRODUCTS OF SYNTOPOGENOUS SPACES

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1. Introduction

The syntopogenous structure is a common generalization of topology, proximity and quasi-uniformity. Actually, the category **Snt** of syntopogenous spaces contains the category **Top** of topological spaces, the category **Prox** of proximity spaces and the category **q-Unif** of quasi-uniform spaces as bicoreflective subcategories. The category **Snt** has product which induces the product in **Top** and in **q-Unif**, of course. The aim of this paper is to define and investigate another kind of product which gives the box product in **Top** and in **q-Unif**.

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Our basic references are [4] on box products and the monograph [2] on syntopogenous spaces. For convenience we recall the main definitions.

A binary relation $<$ defined on the subsets of a set X is said to be a topogenous order provided that the following conditions are satisfied:

(01) $\emptyset < \emptyset$, $X < X$,

(02) $A < B$ implies $A \subset B$,

(03) $A \subset A' < B' \subset B$ implies $A < B$,

(Q) $A < B$ and $A' < B'$ imply $A \cup A' < B \cup B'$ and $A \cap A' < B \cap B'$.

A pair (X, S) consisting of a set X and a nonempty family S of topogenous orders on X is called syntopogenous space provided the following axioms hold:

(S1) for ${}^1{<}, {}^2{<} \in S$ there is a ${}^3{<} \in S$ with the property ${}^1{<} \cup {}^2{<} \subset {}^3{<}$,

(S2) for ${}^1{<} \in S$ there is a ${}^2{<} \in S$ with the property ${}^2{<} \circ {}^2{<} \supset {}^1{<}$.

If S and S' are syntopogenous structures on the set X then S and S' are equivalent provided that for any $<\in S$ there is a $<' \in S'$ such that $< \subset <'$ and for any $<' \in S'$ there is a $< \in S$ such that $<' \subset <$.

A syntopogenous space (X, S) is topological if $S = \{<\}$ and $<$ satisfies the following axiom:

(P1) $$A_i < B_i \ (i \in I) \quad \text{implies} \quad \bigcup_{i \in I} A_i < \bigcup_{i \in I} B_i.$$

A syntopogenous space (X, S) is quasi-uniform if the axioms (P1) and (P2) hold for every $< \in S$:

(P2) $$A_i < B_i \ (i \in I) \quad \text{implies} \quad \bigcap_{i \in I} A_i < \bigcap_{i \in I} B_i.$$

We shall speak about the topology of an arbitrary syntopogenous space (X, S). By this topology we mean the topology associated with the **Top**-coreflection of (X, S). The set G is open with respect to the topology of (X, S) if, for any $x \in G$, there are $\{B_i\colon 1\leqq i\leqq n\}$ and $\{<_i\colon 1\leqq i\leqq n\}\subset S$ such that $\bigcap_{i=1}^{n} B_i\subset G$ and $\{x\}<_i B_i$ $(1\leqq i\leqq n)$.

Similarly, the quasi-uniformity of (X, S) is the quasi-uniformity associated with the **q-Unif**-coreflection space of (X, S). This quasi-uniformity consists of the entourages $\{U_<\colon <\in S\}$ where $U_<$ is defined in the following way:

$$(x, y)\in U_< \quad \text{if} \quad \{x\} < B \quad \text{implies} \quad y\in B.$$

2. Box products

Let K be an infinite cardinal, $X=\Pi\{X_i\colon i\in I\}$, $J\subset I$ and $|J|<K$. Suppose that for $j\in J$, $<_j$ is a topogenous order on X_j. For two subsets A and B of X we say that $A<B$ if there are A^α and B^α $(\alpha<m<K)$ with the following properties:

(i) $$A\subset \bigcup_{\alpha<m} A^\alpha\subset \bigcup_{\alpha<m} B^\alpha\subset B,$$

(ii) $$A^\alpha=\prod_{i\in I} A_i^\alpha,\quad B^\alpha=\prod_{i\in I} B_i^\alpha,$$

(iii) $$A_i^\alpha=B_i^\alpha=X_i \quad \text{for} \quad i\notin J,$$

(iv) $$A_j^\alpha <_j B_j^\alpha \quad \text{for} \quad j\in J,$$

(v) for every $i\in I$ the sets $\{A_i^\alpha\colon \alpha<m\}$ and $\{B_i^\alpha\colon \alpha<m\}$ are finite.

We shall denote $<$ by $\overset{J}{\boxed{K}}<_j$. Routine check shows that

(vi) $$\overset{J}{\boxed{K}}<_j \quad \text{is a topogeneous order on } X,$$

(vii) $$\text{if } \leqslant_j\subset <_j \text{ for any } j\in J \text{ then } \left(\overset{J}{\boxed{K}}\leqslant_j\right)\subset\left(\overset{J}{\boxed{K}}<_j\right),$$

(viii) $$\text{if } J\subset L \text{ and } \leqslant_j=<_j \text{ for any } j\in J \text{ then } \left(\overset{J}{\boxed{K}}\leqslant_j\right)\subset\left(\overset{L}{\boxed{K}}<_k\right),$$

(ix) $$\text{if } <_j\circ<_j\supset\leqslant_j \text{ for any } j\in J \text{ then } \left(\overset{J}{\boxed{K}}\leqslant_j\right)\subset\left(\overset{J}{\boxed{K}}<_j\right)\circ\left(\overset{J}{\boxed{K}}<_j\right).$$

Let $\{(X_i, S)\colon i\in I\}$ be a family of syntopogenous spaces and K an infinite cardinal. (X, S) is called the K-box product of the family $\{(X_i, S_i)\colon i\in I\}$ provided that

(x) $X=\Pi\{X_i\colon i\in I\}$,

(xi) S consists of all topogenous orders $\overset{J}{\boxed{K}}<_j$ where $|J|<K, J\subset I$ and $<_j\in S_j$ $(j\in J)$.

According to the properties (vi)—(ix) it is easy to see that (X, S) defined above is a syntopogenous structure, i.e. the axioms (S1) and (S2) are fulfilled. We denote S by $\underset{i\in I}{\boxed{K}}(X_i, S_i)$.

Evidently, if S_i and S_i' are equivalent syntopogenous structures on the set X_i $(i\in I)$, then the structures $\overline{\underline{|K|}}_{i\in I}(X_i, S_i)$ and $\overline{\underline{|K|}}_{i\in I}(X_i, S_i')$ are also equivalent.

If (X_i, S_i) is a syntopogenous space $(i\in I)$ then the box product $\overline{\underline{|\;\;|}}_{i\in I}(X_i, S_i)$ of the spaces (X_i, S_i) $(i\in I)$ is defined as $\overline{\underline{|K|}}_{i\in I}(X_i, S_i)$ where $K=|I|^+\cdot\omega$.

THEOREM 1. *Let* $\{(X_i, \{<_i\})\colon i\in I\}$ *be a family of topological syntopogenous spaces and K an infinite cardinal. Then the topology of* $\overline{\underline{|K|}}_{i}(X_i, \{<_i\})$ *is determined by the topological basis consisting of the sets* $\bigcap_{j\in J} p_j^{-1}(G_j)$ *where* G_j *is open with respect to the topology* $(X_j, \{<_j\})$ *for any* $j\in J$ *and* $J\subset I$, $|J|<K$.

PROOF. Let G be in the topology associated with $\overline{\underline{|K|}}_{i\in I}(X_i, \{<_i\})=(X, S)$ and $x\in G$. Then there are $<^k\in S$ $(k=1, 2, \dots, n)$ and $B^k\subset X$ such that $\{x\}<^k B^k$ and $\bigcap_1^n B^k\subset G$. We may suppose that $<^k=<=\overline{\underline{|K|}}^J<_j$. So $\{x\}<G$. This implies the existence of A and B such that $A=\bigcap_{j\in J} p_j^{-1}(A_j)$, $B=\bigcap_{j\in J} P_j^{-1}(B_j)$, $A_j<_j B_j$ $(j\in J)$ and $x\in A\subset B\subset G$. There is a $G_j\subset X_j$ such that $A_j\subset G_j\subset B_j$ and G_j is open with respect to $(X_j, \{<_j\})$ $(j\in J)$. Hence $x\in\bigcap_{j\in J} P_j^{-1}(G_j)\subset G$.

On the other hand the set $G=\bigcap_{j\in J} p_j^{-1}(G_j)$ is open with respect to the topology of (X, S) provided that G_j is open in $(X_j, \{<j\})$ for any $j\in J$. Indeed, for $x\in G$ we have $\{x\}<G$ if we put $<=\overline{\underline{|K|}}^J<_j$.

By this theorem the K-box product of syntopogenous spaces gives the usual K-box product for topological spaces and the similar theorem for box product is also true (see [4]).

THEOREM 2. *Let* $\{(X_i, S_i)\colon i\in I\}$ *be a family of quasi-uniform syntopogenous spaces, K an infinite cardinal and* $(X, S)=\overline{\underline{|K|}}_{i\in I}(X_i, S_i)$. *Then the entourages of the form* $(x, y)\in U$ *if and only if* $(p_j(x), p_j(y))\in U_j$ $(j\in J)$ *form a basis for the quasi-uniformity associated with* (X, S) *where* $|J|<K$, $J\subset I$ *and* U_j *is an entourage from the quasi-uniformity* (X_j, S_j) *for any* $j\in J$.

PROOF. Let U' be an entourage from the quasi-uniformity of (X, S). Then there is a $<\in S$ such that $(x, y)\in U'$ if and only if $\{x\}<H$ implies $y\in H$. Suppose that $<=\overline{\underline{|K|}}^J<_j$. Let $(x, y)\in U$ if and only if $(p_j(x), p_j(y)\in U_{<_j}$ (i.e. $\{p_j(x)\}<F$ implies $P_j(y)\in F)$ for any $j\in J$. Let us show that $U'=U$.

Assume that $(x, y)\notin U'$. Then there is a set H such that $\{x\}<H$ and $y\notin H$. We may suppose that $H=\bigcap_{j\in J} p_j^{-1}(H_j)$. So there is a $j_0\in J$ with the property $p_{j_0}(y)\notin H_{j_0}$. Since $(p_{j_0}(x), p_{j_0}(y))\notin U_{<_{j_0}}$ we have $(x, y)\notin U$.

Conversely, let $(x, y)\notin U$. Then there is $F\subset X_{j_0}$ for some $j_0\in J'$ such that $\{p_{j_0}(x)\}<_{j_0}F$ but $p_{j_0}(y)\notin F$. These imply that $\{x\}<p_{j_0}^{-1}(F)$ and $y\notin p_{j_0}^{-1}(F)$. Hence we obtain that $(x, y)\notin U'$.

We remark that if (X_i, S_i) is a uniform syntopogenous space for any $i\in I$ then the entourages described in Theorem 2 and associated with $\mathop{\boxed{K}}_{i\in I}(X_i, S_i)$ are symmetrical so they form a basis for a uniformity. This uniformity is the same as the K-box product of the uniformities (X_i, S_i) $(i\in I)$ which was defined in [4]. The similar assertion holds for box product instead of K-box product. (We refer to [1] p. 429 about the box product of uniform spaces).

3. Completeness of K-box products

We say that a filter $\mathscr{R}$ converges in the syntopogenous space (X, S) to a point $x\in X$ if $\{x\}<V$ and $<\in S$ imply that an $R_0\in\mathscr{R}$ can be found with the property $R_0\subset V$.

A filter $\mathscr{R}$ is said to be a Cauchy filter provided that to every $<\in S$ there corresponds an $R_0\in\mathscr{R}$ such that $A<B$, $R_0\cap A\neq\emptyset$ imply $R_0\subset B$.

The syntopogenous space (X, S) is complete if every Cauchy filter converges in (X, S). A syntopogenous space is complete if and only if the associated quasi-uniformity is complete (in the uniform sense) (cf. [2] p. 229).

LEMMA 3. *Let* $(X, S)=\mathop{\boxed{K}}_{i\in I}(X_i, S_i)$ *and* $P=\bigcap_{j\in J}p_j^{-1}(P_j)$ *where* $P_j\subset X_j$ *for any* $j\in J$ *and* $|J|<K$. *If* $<=\overset{J}{\boxed{K}}<_j\in S$ *and* $P<B$ *then there exists a set* $Q=\bigcap_{j\in J}p_j^{-1}(T_j)$ *such that* $P_j<_jT_j$ *for any* $j\in J$ *and* $Q\subset B$.

PROOF. By the definition of $<$ there are A^α and B^α $(\alpha<m, m<K)$ such that $A^\alpha=\Pi\{A_i^\alpha:\ i\in I\}$, $B^\alpha=\Pi\{B_i^\alpha:\ i\in I\}$, $P\subset\bigcup_{\alpha<m}A^\alpha\subset\bigcup_{\alpha<m}B^\alpha\subset B$, $A_j^\alpha<_jB_j^\alpha$ $(\alpha<m, j\in J)$, $A_i^\alpha=B_i^\alpha=X_i^\alpha$ for any $i\in I\setminus J$, furthermore the sets $\{A_i^\alpha:\ \alpha<m\}$ and $\{B_i^\alpha:\ \alpha<m\}$ are finite for any $i\in I$.

Let $C_i^s=\bigcap\{B_i^\alpha:\ s\in A_i^\alpha, \alpha<m\}$ for $s\in P_i$. So we have $P_j<_j\bigcup\{C_j^s:\ s\in P_j\}=T_j$ $(j\in J)$. Hence $Q=\bigcap_{j\in J}p_j^{-1}(T_j)=\bigcup\{\bigcap_{j\in J}p_j^{-1}(C_j^{s_j}):\ s_j\in P_j\ (j\in J)\}$. We show that $\bigcap_{j\in J}p_j^{-1}(C_j^{s_j})\subset\bigcup_{\alpha<m}B^\alpha$ for any choice of $s_j\in P_j$ $(j\in J)$. We can find $s\in P$ such that $p_j(s)=s_j$ $(j\in J)$ and there is $\alpha_0<m$ with the property $s\in A$. Consequently $s_j\in A_j^{\alpha_0}$ $(j\in J)$ and $C_p^{s_j}\subset B_j^{\alpha_0}$ $(j\in J)$. Therefore $\bigcap_{j\in J}p_j^{-1}(C_j^{s_j})\subset B^{\alpha_0}$.

The proof of the lemma is complete.

Now we can prove our main theorem on the completeness of box products.

THEOREM 4. *Let* $(X, S)=\mathop{\boxed{K}}_{i\in I}(X_i, S_i)$. *Then* (X, S) *is complete if and only if* (X_i, S_i) *is complete for any* $i\in I$.

PROOF. Assume that (X, S) is complete and let $i_0\in I$. Fixing a point $x_i\in X_i$ for any $i\in I$ we define $H'=\{x\in X:\ p_{i_0}(x)\in H, p_i(x)=x_i\ (i\in I\setminus\{i_0\})\}$ provided that $H\subset X_{i_0}$. Let $\mathscr{R}^*$ be a Cauchy filter in (X_{i_0}, S_{i_0}). It is a routine to check that $\mathscr{R}'=$

$=\{R'\colon R\in\mathscr{R}^*\}$ is a base of a Cauchy filter in (X, S) hence $\mathscr{R}'\to q$ for some $q\in X$. So we have $\mathscr{R}^*\to p_{i_0}(q)$.

Now let $\mathscr{R}^*$ be a Cauchy filter in (X, S). We show that $p_i(\mathscr{R}^*)$ is a Cauchy filter in (X_i, S_i) for any $i\in I$. Taking $<_{i_0}\in S_{i_0}$ we can find $R\in\mathscr{R}^*$ such that $A<B$, $R\cap A\neq\emptyset$ imply $R<B$ where $<=\overset{\{i_0\}}{\boxed{K}}<_{i_0}$. Suppose that $C<_{i_0}D$ and $p_{i_0}(R)\cap\cap C\neq\emptyset$. These imply that $p_{i_0}^{-1}(C)<p_{i_0}^{-1}(D)$ and $R\cap p_{i_0}^{-1}(C)\neq\emptyset$. By the choice of R we obtain that $R\subset p_{i_0}^{-1}(D)$ or equivalently $p_{i_0}(R)\subset D$. Thus $p_{i_0}(\mathscr{R})$ is a Cauchy filter in (X_{i_0}, S_{i_0}) for any $i_0\in I$.

Consequently, there is a $q\in X$ such that $p_i(\mathscr{R})\to p_i(q)$ for any $i\in I$. Let $<\in S$ and $\{q\}<V$. Then we have $<_1\in S$ and $V_1\subset X$ such that $\{q\}<_1V_1<_1V$. For $<_1\in S$ there is an $R_0\in\mathscr{R}^*$ such that $A<_1B, A\cap R_0\neq\emptyset$ imply $R_0\subset B$.

By the definition of $<_1=\overset{J}{\boxed{K}}<_j$ there are $C=\cap\{p_j^{-1}(C_j)\colon j\in J\}$ and $D=\cap\{p_j^{-1}(D_j)\colon j\in J\}$ such that $q\in C\subset D\subset V_1$ and $C_j<_jD_j$ for any $j\in J$. Using the lemma we have $E=\cap\{p_j^{-1}(E_j)\colon j\in J\}$ such that $E\subset V$ and $D_j<_jE_j$ for any $j\in J$.

Since $p_j(q)<_jD_j$ and $p_j(\mathscr{R}^*)\to p_j(q)$, we have $p_j(R_0)\cap D_j\neq\emptyset$ $(j\in J)$. So $R_0\cap p_j^{-1}(D_j)\neq\emptyset$ and $p_j^{-1}(D_j)<_1p_j^{-1}(E_j)$ $(j\in J)$. Hence $R_0\subset p_j^{-1}(E_j)$ for any $j\in J$. We obtained that $R_0\subset\cap\{p_j^{-1}(E_j)\colon j\in J\}=E\subset V$ which shows that $\mathscr{R}^*\to q$, indeed.

COROLLARY 5. *For any infinite cardinal K any K-box product of complete (quasi-) uniform spaces is complete.*

PROOF. This is evident for quasi-uniform spaces according to Theorem 2 and in the case of uniformities one can use the remark after Theorem 2.

COROLLARY 6. *For any infinite cardinal K any K-box product of topologically complete topological spaces is topologically complete.*

The latter assertion was proved in [5].

4. Normality of *K*-box products

For a topogenous order $<$ on the set X we define its complement $<^C$ by the rule: $A<^CB$ if and only if $X\setminus B<X\setminus A$.

LEMMA 7. *Let K be an infinite cardinal such that $\lambda<K$ implies $2^\lambda<K$. If* $(X, S)=$ $=\underset{I}{\boxed{K}}(X_i, S_i)$, $<=\overset{J}{\boxed{K}}<_j\in S$ *and* $<'=\overset{J}{\boxed{K}}<_j^C$ *then* $<^C=<'$.

PROOF. Assume that $A<^CB$. Then $X\setminus B\subset\cup A^\alpha\subset\cup B^\alpha\subset X\setminus A$ for some $m<K$, $A^\alpha=\bigcap_{j\in J}p_j^{-1}(A_j^\alpha)$, $B^\alpha=\bigcap_{j\in J}p_j^{-1}(B_j^\alpha)$ such that $A_j^\alpha<_jB_j^\alpha$ $(j\in J, \alpha<m)$ moreover $\{A_j^\alpha\colon \alpha<m\}$ and $\{B_j^\alpha\colon \alpha<m\}$ are finite. So $A\subset X\setminus\bigcup_{\alpha<m}B^\alpha=\bigcap_{\alpha<m}(X\setminus B^\alpha)=$ $=\bigcap_{\alpha<m}\bigcup_{j\in J}p_j^{-1}(X_j\setminus B_j^\alpha)=\bigcup_{f\in J^m}\big(\bigcap_{\alpha<m}p_{f(\alpha)}^{-1},(X_{f(\alpha)}\setminus B_{f(\alpha)}^\alpha)\big)\subset\bigcup_{f\in J^m}\big(\bigcap_{\alpha<m}p_{f(\alpha)}^{-1}(X_{f(\alpha)}\setminus A_{f(\alpha)})\big)\subset$ $\subset B$. Define $C^f=\bigcap_{\alpha<m}p_{f(\alpha)}^{-1}(X_{f(\alpha)}\setminus B_{f(\alpha)}^\alpha)$ and $D^f=\bigcap_{\alpha<m}p_{f(\alpha)}^{-1}(X_{f(\alpha)}\setminus A_{f(\alpha)}^\alpha)$ for every

function $f: m \to J$. We have $C^f = \bigcap_{k\in J} p_k^{-1}(C_k^f)$ and $D^f = \bigcap_{k\in J} p_k^{-1}(D_k^{-1})$ where $C_k^f = \bigcap\{(X_k \setminus B_k^\alpha): f(\alpha)=k\}$ and $D_k^f = \bigcap\{X_k \setminus A_k^\alpha): f(\alpha)=k\}$ $(k\in J)$. Using the conditions of finiteness we obtain that $\{C_k^f: f\in J^m\}$ and $\{D_k^f: f\in J^m\}$ are finite and $C_k^f <_k^C D_k^f$ for any $k\in J$. Since $|J|<K$ and $m<K, |J|^m<K$ and so $A<B$.

Applying this result for $<_j^c$ instead of $<_j$, we get $<'^C \subset <$, hence $<' = <'^{CC} \subset <^C$ and finally $<' = <^C$ which was to be proved.

A topogenous order $<$ is said to be symmetrical provided that $<$ coincides with its complement $<^C$. A syntopogenous space is called symmetrical if it consists of symmetrical orders.

THEOREM 8. *If K is a cardinal with the property that $\lambda<K$ implies $2^\lambda<K$ then the K-box product of any family of symmetrical syntopogenous spaces is symmetrical.*

PROOF. This is a consequence of Lemma 7.

A topogenous order $<$ is said to be normal if it satisfies $<^C \cdot < \subset < \cdot <^C$ (i.e. $A<^C B<C$ implies that there is a $D\subset X$ such that $A<D<^C C$). The syntopogenous structure S and the space (X, S) are said to be normal if S is equivalent to a syntopogenous structure S' such that each $<\in S'$ is normal. This definition was given by CSÁSZÁR ([3]). He proved that product preserves normality. The following theorem generalizes this assertion.

THEOREM 9. *Let K be an infinite cardinal such that $\lambda<K$ implies $2^\lambda<K$. Then the K-box product of any family of normal syntopogenous spaces is normal.*

PROOF. Suppose that (X_i, S_i) is normal $(i\in I)$. We may assume that S_i consists of normal topogenous orders $(i\in I)$ and it is sufficient to show that $\overline{|K|}_{i\in I} (X_i, S_i)$ is normal.

Let $(X, S) = \overline{|K|}_{i\in I} (X_i, S_i)$ and pick $< = \overline{|K|}^J <_j \in S$. Suppose that $A<^C B<C$. Then there are $m<K, n<K, A^\alpha, B^\alpha (\alpha<m), C^\beta, D^\beta (\beta<n)$ such that

$$\text{(i)} \qquad A\subset \bigcup_{\alpha<m} A^\alpha \subset \bigcup_{\alpha<m} B^\alpha \subset B \subset \bigcup_{\beta<n} C^\beta \subset \bigcup_{\beta<n} D^\beta \subset C,$$

$$\text{(ii)} \qquad A^\alpha = \bigcap p_j^{-1}(A_j^\alpha), \quad B^\alpha = \bigcap p_j^{-1}(B_j^\alpha), \quad C^\beta = \bigcap p_j^{-1}(C_j^\beta),$$

$$D^\beta = \bigcap p_j^{-1}(D_j^\beta) \quad (\alpha<m, \beta<n),$$

$$\text{(iii)} \qquad A_j^\alpha <_j^c B_j^\alpha, \quad C_j^\beta <_j D_j^\beta \quad (\alpha<m, \beta<n, j\in J),$$

(iv) $\{A_j^\alpha: \alpha<m\}$, $\{B_j^\beta: \alpha<m\}$, $\{C_j^\beta: \beta<n\}$ and $\{D_j^\beta: \beta<n\}$ are finite.

Using the construction in the proof of Lemma 3 we can find $Q^\alpha \subset C$ $(\alpha<m)$ such that $Q^\alpha = \bigcap p_j^{-1}(T_j^\alpha)$, $B_j^\alpha <_j T_j^\alpha$ $(\alpha<m, j\in J)$ and T_j^α is contained in the finite Boolean algebra generated by $\{D_j^\beta: \beta<n\}$ in X_j. Hence $\{T_j^\alpha: \alpha<m\}$ is finite. Since $A_j^\alpha <_j^C B_j^\alpha <_j T_j^\alpha$ $(j\in J)$ and (X_j, S_j) is normal $(j\in J)$ there is V_j^α such that $A_j^\alpha <_j V_j^\alpha <_j^C T_j^\alpha$ $(j\in J, \alpha<m)$.

We may assume that in the case of $(A_j^{\alpha_1}, T_j^{\alpha_1})=(A_j^{\alpha_2}, T_j^{\alpha_2})$ we have $V_j^{\alpha_1}=V_j^{\alpha_2}$ $(j\in J, \alpha_1, \alpha_2\in m)$. So $\{V_j^\alpha: \alpha<m\}$ is also finite. Let $V^\alpha=\bigcap p_j^{-1}(V_j^\alpha)$. Then $A<\bigcup_{\alpha<m} V^\alpha$ and $\bigcup_{\alpha<m} V^\alpha <^c C$.

The proof is complete.

COROLLARY 10. *GCH implies that the K-box product of normal syntopogenous spaces is normal provided that K is a limit cardinal.*

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ZUR KONVERGENZ VON FUNKTIONENREIHEN

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0. In dieser Arbeit wird aus der Beschränktheit von Lebesgue-Funktionen bzg. eines Summierungsverfahrens auf Fastüberall-Konvergenz von Teilfolgen der Partialsummen von Funktionenreihen mit Koeffizienten aus l^2 geschlossen. Sei $E\subset[0, 1]$ Lebesgue-meßbar, $\mu(E)>0$, und sei $F=\{f_k(x)\}$ ein System reellwertiger Funktionen $f_k\in L^1(E)$. Ist dann $T=(\alpha_{nk})$ $(n, k=1, 2, \dots)$ eine zeilenfinite Matrix, also $\alpha_{nk}=0$ für $k\geqq k(n)$, so definieren wir die Lebesgue-Funktionen von F bzg. T durch

$$L_n(T, F; x) = \int\limits_E \Big|\sum_{k=1}^{\infty} \alpha_{nk} f_k(x) f_k(t)\Big|\, dt.$$

Weiter seien

$$L_n(F; x) = \int\limits_E \Big|\sum_{k=1}^{n} f_k(x) f_k(t)\Big|\, dt$$

die Lebesgue-Funktionen von F.

Wir werden zeigen, daß unter gewissen Voraussetzungen an die Matrix T und aus der Beschränktheit der Lebesgue-Funktionen von F bzg. T, d. h.

(1) $$L_n(T, F; x) = O_x(1) \quad (n\to\infty) \text{ f. ü. auf } E,$$

folgt, daß Indexfolgen $1\leqq\nu_1<\nu_2<\dots$ existieren, so daß $\lim\limits_{n\to\infty} S_{\nu_n}(x)$ mit $S_n(x)=$ $=\sum\limits_{k=1}^{n} c_k f_k(x)$ f. ü. auf E existiert für alle reellen Zahlenfolgen $\{c_k\}$ aus l^2, also $\sum\limits_{k=1}^{\infty} c_k^2<\infty$.

Im ersten Teil der Arbeit wird zunächst im wesentlichen aus (1) die T-Summierbarkeit der Reihe $\sum\limits_{k=1}^{\infty} c_k f_k(x)$ f. ü. auf E für alle $\{c_k\}\in l^2$ gefolgert. Hieraus wird dann (einem Hinweis von G. Alexits folgend) auf die Fastüberall-Konvergenz gewisser Teilfolgen $S_{\nu_n}(x)$ geschlossen. Dabei werden Sätze von KACZMARZ [3] sowie NIKIŠIN und TANDORI [6], [7] verwendet.

In Teil 2 wird ein Beweis angegeben, der die T-Summierbarkeit, insbesondere die Sätze von Kaczmarz und Nikišin—Tandori nicht benutzt, sondern nur auf einem Satz von ALEXITS [2] beruht. In diesen Beweis wird aus dem ursprünglichen System F durch „Auffüllen" ein neues Funktionensystem $\Phi=\{\varphi_k(x)\}$ konstruiert, so daß aus (1) für das System Φ für fast alle $x\in E$ folgt, daß $L_{\nu_n}(\Phi; x)=O_x(1)$ $(n\to\infty)$ gilt. Auch wenn das Ergebnis dieses Abschnitts etwas schwächer ist als das in Teil 1, erscheint uns die Beweismethode unabhängig vom Ergebnis interessant, weil hier durch die Anwendung der von Alexits vorgeschlagenen Methode des „Auffüllens“ die Tragweite seines Satzes besonders hervortritt.

1. Wir beweisen in diesem Abschnitt

Satz 1. *Sei* $T=(\alpha_{nk})$ *eine Matrix mit*

$$\alpha_{n+1,k} \geqq \alpha_{nk} \geqq 0 \quad \text{für} \quad n, k = 1, 2, \dots; \tag{2}$$

$$\alpha_{nk} = 0 \quad \text{für} \quad k \geqq k(n); \tag{3}$$

$$\lim_{n\to\infty} \alpha_{nk} = 1 \quad \text{für} \quad k \in \mathbf{N}. \tag{4}$$

Weiter sei $F=\{f_k(x)\}$ *ein System reellwertiger Funktionen* $f_k \in L^1(E)$, *die* (1) *erfüllen. Dann gibt es eine, nur von* T *abhängige, Funktion* $H(n)$ $(n\in\mathbf{N})$, *so daß für alle Indexfolgen* $1 \leqq \nu_1 < \nu_2 < \dots$ *mit* $\nu_{n+1} \geqq H(\nu_n)$ *gilt:* $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{\nu_n} c_k f_k(x)$ *existiert f. ü. auf* E *für alle* $\{c_k\}\in l^2$.

Beweis. Zunächst folgt aus (2) und (4), daß

$$0 \leqq \alpha_{nk} \leqq 1 \quad \text{für alle } n, \ k. \tag{5}$$

a) Wir stellen zunächst die im Beweis benötigen Sätze von Kaczmarz, Nikišin—Tandori sowie aus [4] in der hier benötigten Form zusammen:

Satz A ([4]). *Unter der Voraussetzung von Satz* 1 *ist* $\sum\limits_{k=1}^{\infty} c_k f_k(x)$ *f. ü. T-summierbar auf* E, *d. h.* $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{\infty} \alpha_{nk} c_k f_k(x)$ *existiert f. ü. auf* E, *für alle Folgen* $\{c_k\}\in l^2$.

Satz B (Tandori [7]). *Die Matrix* $T=(\alpha_{nk})$ *erfülle* (3) *und* (4), *und es sei* $F=\{f_k(x)\}$ *eine Folge reeller, auf der meßbaren Menge* $E\subset[0, 1]$ *fast überall endlicher, meßbarer Funktionen mit:* $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{\infty} \alpha_{nk} c_k f_k(x)$ *konvergiert dem Maß nach auf* E *gegen eine auf* E *meßbare, fast überall endliche Funktion für alle* $\{c_k\}\in l^2$. *Dann ist das System* F *fast-orthonormiert, d. h. für alle* $\varepsilon>0$ *existieren eine meßbare Menge* $E_\varepsilon \subset E$, *ein System* $\psi_\varepsilon=\{\psi_n(\varepsilon, x)\}$, *und eine Zahl* $M_\varepsilon>0$ *mit* $\mu(E_\varepsilon)<\varepsilon$, $f_k(x)=M_\varepsilon \cdot \psi_k(\varepsilon, x)$ *für alle* $x\in E\setminus E_\varepsilon$, $k\in\mathbf{N}$, *wobei* ψ_ε *ein Orthonormalsystem auf* [0, 1] *ist.*

Der folgende Satz von Kaczmarz wurde nicht direkt aus [3] übernommen, sondern ergibt sich in der vorliegenden Formulierung direkt aus dem Kaczmarz'schen Beweis des entsprechenden Satzes in [3].

Satz C (Kaczmarz). *Sei* $T=(\alpha_{nk})$ *eine Matrix mit* $|\alpha_{nk}|\leqq K$, $n, k\in\mathbf{N}$ *und* (3), *und für die Indexfolge* $1\leqq \nu_1<\nu_2<\dots$ *gelte:*

$$\sum_{\substack{n=1\\ \nu_n\geqq k}}^{\infty} (1-\alpha_{\nu_n k})^2 \leqq K_1, \quad k\in\mathbf{N} \quad \textit{und} \tag{6}$$

$$\sum_{\substack{n=1\\ \nu_n< k}}^{\infty} \alpha_{\nu_n k}^2 \leqq K_1, \quad k\in\mathbf{N}. \tag{7}$$

Ist dann $\Phi=\{\varphi_k(x)\}$ *ein Orthonormalsystem auf* $[0,1]$, *so gilt* $\lim_{n\to\infty}\left(S_{\nu_n}(x)-t_{\nu_n}(x)\right)=0$ *f. ü. auf* $[0,1]$ *für alle* $\{c_k\}\in l^2$, *wobei* $S_n(x)=\sum_{k=1}^{n} c_k\varphi_k(x)$, $t_n(x)=\sum_{k=1}^{\infty}\alpha_{nk}c_k\varphi_k(x)$ *ist.*

b) Nach Satz A ist zunächst $\sum_{k=1}^{\infty} c_k f_k(x)$ f. ü. T-summierbar auf E für $\{c_k\}\in l^2$, und daher ist F fast-orthonormiert nach Satz B. Wegen (3) und (4) ist dann

$$H(n):=\min\{a\in\mathbf{N}\colon\ a>n,\ |1-\alpha_{bk}|\leqq 2^{-n},\ 1\leqq k\leqq n,\ b\geqq a,\ |\alpha_{nk}|\leqq 2^{-n},\ k\geqq a\}$$

für $n\in\mathbf{N}$ wohldefiniert; und für jede Indexfolge mit $\nu_1\geqq 1$, $\nu_{n+1}\geqq H(\nu_n)$ für $n\in\mathbf{N}$ gelten dann (6) und (7); denn aus $\nu_{n+1}\geqq H(\nu_n)$ folgt zusammen mit $|\alpha_{nk}|\leqq 1$ nach (5):

$$\sum_{\nu_n\geqq k}(1-\alpha_{\nu_n k})^2\leqq 4+\sum_{\nu_{n-1}\geqq k}(1-\alpha_{\nu_n k})^2\leqq 4+\sum_{n=0}^{\infty}2^{-\nu_n}\leqq 6;$$

$$\sum_{\nu_n<k}\alpha_{\nu_n k}^2\leqq 1+\sum_{\nu_{n+1}<k}\alpha_{\nu_n k}^2\leqq 1+\sum_{n=0}^{\infty}2^{-\nu_n}\leqq 3.$$

Sei (ν_n) eine Indexfolge mit $\nu_1\geqq 1$, $\nu_{n+1}\geqq H(\nu_n)$, und sei $\varepsilon>0$. Ist dann $\psi_\varepsilon=\{\psi_n(\varepsilon,x)\}$ ein Orthonormalsystem gemäß Satz B, so folgt mit Satz A, daß $\lim_{n\to\infty}\sum_{k=1}^{\infty}\alpha_{nk}c_k\psi_k(\varepsilon,x)$ f. ü. auf $E\setminus E_\varepsilon$ existiert falls $\{c_k\}\in l^2$. Mit Satz C folgt somit, daß $\lim_{n\to\infty}\sum_{k=1}^{\nu_n}c_kf_k(x)=M_\varepsilon\cdot\lim_{n\to\infty}\sum_{k=1}^{\nu_n}c_k\psi_k(\varepsilon,x)$ f. ü. auf $E\setminus E_\varepsilon$ existiert. Nun war $\varepsilon>0$ beliebig, $\mu(E_\varepsilon)<\varepsilon$, also folgt die Behauptung von Satz 1.

BEMERKUNGEN 1. Die Bedingungen (2)—(4) an die Matrix T sind bei einer großen Klasse geläufiger Summierungsverfahren erfüllt. Gilt $\|f_k\|_\infty\leqq\lambda_k$ für alle k, so kann (3) durch die schwächere Bedingung

$$\text{(3')}\qquad \sum_{k=1}^{\infty}|\alpha_{nk}|\,\lambda_k<\infty\quad\text{für alle } n,$$

ersetzt werden.

2. Für die Cèsaro-Verfahren C_α mit $\alpha>0$ bzw. die unstetigen Riesz-Verfahren (R^*,λ,α) mit $\alpha>0$ sind (6) und (7) für alle Indexfolgen (ν_n) mit $\frac{\nu_{n+1}}{\nu_n}>1+\varepsilon$ für ein $\varepsilon>0$, $n\in\mathbf{N}$ bzw. $\frac{\lambda_{\nu_{n+1}}}{\lambda_{\nu_n}}\geqq 1+\varepsilon$, $\varepsilon>0$, $n\in\mathbf{N}$ erfüllt, und damit folgt Fastüberall-Konvergenz von $S_{\nu_n}(x)$ in Satz 1 bei diesen speziellen Verfahren für alle derartigen Folgen $\{\nu_n\}$.

(Vergleiche dazu die bekannten Sätze über C_α bzw. (R^*,λ,α)-Summierbarkeit und die Konvergenz von Teilfolgen $S_{\nu_n}(x)$ bei Orthogonalreihen in [1]).

3. In diesem Abschnitt wird ein etwas schwächeres Ergebnis als Satz 1 allein aus dem folgenden Satz von G. Alexits hergeleitet:

SATZ D (ALEXITS [2]). *Ist* $F=\{f_k(x)\}$ *ein System reeller, Lebesgue-integrierbarer Funktionen auf der meßbaren Menge* $E\subset[0,1]$, *und gilt*

$$L_{\nu_n}(F;x)=O_x(1)\quad (n\to\infty)\quad f.\,ü.\ auf\ E \tag{8}$$

für eine Indexfolge $1\leqq\nu_1<\nu_2<\dots$, *so folgt, daß* $\lim_{n\to\infty} S_{\nu_n}(x)$ *f. ü. auf E für alle* $\{c_k\}\in l^2$ *existiert.*

BEMERKUNG. Es genügt hier (8) anstelle von $L_{\nu_n}(F;x)=O(1)$ $(n\to\infty)$ gleichmäßig für $x\in E$ vorauszusetzen nach einem Argument von F. MÓRICZ [5] (vergleiche auch [4]).

SATZ 2. *Unter den Voraussetzungen von Satz* 1 *und der zusätzlichen Annahme*

$$\alpha_{n,k+1}\leqq\alpha_{nk}\quad für\quad n,k\in\mathbf{N} \tag{9}$$

gibt es eine Indexfolge $1\leqq\nu_1<\nu_2<\dots$, *nur von T abhängig, so daß* $\lim_{n\to\infty} S_{\nu_n}(x)$ *f. ü. auf E existiert für alle* $\{c_k\}\in l^2$.

BEWEIS. (Mit Satz D.) Sei $0<\delta<\frac{1}{2}$ $\left(\text{etwa } \delta=\frac{1}{4}\right)$. Wir definieren ein neues System $\Phi=\{\varphi_k(x)\}$ von reellen Funktionen $\varphi_k\in L^1(E)$, auf die Satz D anwendbar ist, wie folgt: Das neue System Φ soll von der Form

$$\Phi=(\varphi_1,\varphi_2,\dots)=(f_1,\dots,f_{\nu_1},g_1,\dots,g_{\mu_1},f_{\nu_1+1},\dots,f_{\nu_2},g_{\mu_1+1},\dots) \tag{10}$$

sein (d. h. das ursprüngliche System $F=\{f_k(x)\}$ wird durch gewisse noch zu definierende Funktionen $g_k(x)$ „aufgefüllt"), so daß die folgende zentrale Identität erfüllt ist:

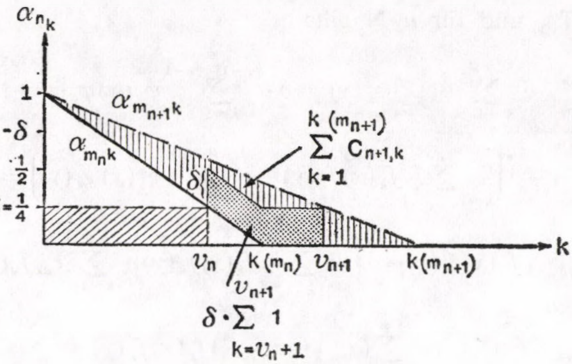
$$\delta\sum_{k=1}^{\nu_n+\mu_n}\varphi_k(x)\varphi_k(y)=\sum_{k=1}^{\infty}\alpha_{m_n k}f_k(x)f_k(y) \tag{11}$$

für $x,y\in E$, $n\in\mathbf{N}$, und eine gewisse Indexfolge $1\leqq m_1<m_2<\dots$. Aus (11) folgt dann offenbar

$$L_{\nu_n+\mu_n}(\Phi;x)=\frac{1}{\delta}L_{m_n}(T,F;x)=O_x(1)\quad (n\to\infty)\ \text{f. ü. auf } E,$$

und somit existiert $\lim_{n\to\infty}\sum_{k=1}^{\nu_n+\mu_n} c_k\varphi_k(x)$ fast überall auf E, wenn $\{c_k\}\in l^2$ nach Satz D. Insbesondere existiert $\lim_{n\to\infty}\sum_{k=1}^{\nu_n} c_k f_k(x)$ fast überall auf E für $\{c_k\}\in l^2$, denn man setze die Koeffizienten der $g_k(x)$ einfach alle gleich Null (siehe 10)).

Die Konstruktion der $g_k(x)$ besteht nun darin, diese als geeignete Vielfache der gegebenen Funktionen $f_j(x)$ $(j=j(k))$ anzusetzen, so daß (11) erfüllt ist. An der ff. Figur ist die Konstruktion der $g_k(x)$ $(g_{\mu_n+k}(x)=d_{n,k}f_k(x))$ und der Indexfolgen $\{\nu_n\}$, $\{\mu_n\}$, $\{m_n\}$ illustriert (am Beispiel des C_1-Verfahrens).



Alle Folgen werden induktiv definiert. Sei dazu $\nu_0=\mu_0=m_0=0, \alpha_{n0}=1$ für $n\in\mathbf{N}_0$, $\alpha_{0k}=0$ für $k\in\mathbf{N}$. Die Indexfolgen $\{\nu_n\}$, $\{m_n\}$ (für $n\in\mathbf{N}_0$) werden induktiv definiert, so daß gilt (siehe obige Figur!):

(12) $$\alpha_{m_n k} \geqq \delta \text{ für } 1 \leqq k \leqq \nu_n, \quad \alpha_{m_n k} < 1-\delta \text{ für } k > \nu_n;$$

(13) $$\alpha_{m_{n+1}k} \geqq \delta+\alpha_{m_n k} \quad \text{für} \quad \nu_n < k \leqq k(m_n);$$

(14) $$m_{n+1} > m_n, \; \nu_n < k(m_n) \leqq \nu_{n+1}, \quad \text{falls} \quad n\in\mathbf{N}_0 \; (k(0)=1 \text{ per Definition}).$$

Offenbar ist (12) für $n=0$ erfüllt. Es gelte (12) für n. Wegen $\alpha_{m_n k}<1-\delta$ für $k>\nu_n$, $\lim_{m\to\infty}\alpha_{mk}=1$ nach (4) gibt es $m_{n+1}>m_n$ mit $\alpha_{m_{n+1}k}\geqq\delta+\alpha_{m_n k}$ für $\nu_n<k\leqq \leqq k(m_n)$. Also gilt (13). Wegen $\alpha_{m_{n+1},k(m_n)}\geqq\delta$ (nach (13), und $\alpha_{nk}\geqq 0$ nach (2)), $\alpha_{m_n k+1}\leqq\alpha_{m_n k}$ für $k\in\mathbf{N}$ nach (9), $\alpha_{m_{n+1}k}=0$ für $k\geqq k(m_{n+1})$ nach (3) gibt es $\nu_{n+1}\geqq k(m_n)$ $(\nu_{n+1}>\nu_n$ wegen (12)) mit $\alpha_{m_{n+1}\nu_{n+1}}\geqq\delta$ und $\alpha_{m_{n+1}\nu_{n+1}+1}<\delta<1-\delta$, da $\delta<\dfrac{1}{2}$. Mit (2) $(\alpha_{n,k+1}\leqq\alpha_{nk})$ folgt $\alpha_{m_{n+1}k}\geqq\delta$ für $1\leqq k\leqq\nu_{n+1}$, und $\alpha_{m_{n+1}k}< <1-\delta$ für $k>\nu_{n+1}$, also (12) für $n+1$ anstelle von n.

Die Indexfolge (μ_n) und die Funktionen $g_k(x)$ werden schließlich definiert durch

$$\mu_{n+1} = \mu_n+k(m_{n+1})-1 \quad \text{für} \quad n\in\mathbf{N}_0,$$

und

$$g_{\mu_n+k}(x) = \sqrt{c_{n,k}}\, f_k(x) \quad \text{für} \quad 1 \leqq k \leqq k(m_n),$$

wobei (siehe Figur!)

$$c_{nk} = \begin{cases} \dfrac{1}{\delta}(\alpha_{m_n k}-\alpha_{m_{n-1}k}) & \text{für} \quad 1 \leqq k \leqq \nu_{n-1}, \\ \dfrac{1}{\delta}(\alpha_{m_n k}-\alpha_{m_{n-1}k})-1 & \text{für} \quad \nu_{n-1}+1 \leqq k \leqq \nu_n, \quad \text{und} \\ \dfrac{1}{\delta}(\alpha_{m_n k}-\alpha_{m_{n-1}k}) & \text{für} \quad \nu_{n+1} \leqq k \leqq k(m_n). \end{cases}$$

Wegen (2) $(\alpha_{n+1,k}\geqq\alpha_{nk})$ und (13) gilt $c_{nk}\geqq 0$ für alle k, so daß die $g_k(x)$ wohldefiniert sind. Nun folgt die zentrale Identität (11) durch Induktion nach n $(n=0$

trivial nach Def.), und für $n\in\mathbf{N}$ gilt:

$$\delta \sum_{k=1}^{\nu_n+\mu_n} \varphi_k(x)\varphi_k(y) = \delta \cdot \sum_{k=1}^{\nu_{n-1}+\mu_{n-1}} \varphi_k(x)\varphi_k(y) +$$

$$+\delta\left(\sum_{k=\nu_{n-1}+1}^{\nu_n} f_k(x)f_k(y) + \sum_{k=\mu_{n-1}+1}^{\mu_n} g_k(x)\,g_k(y)\right) =$$

$$= \sum_{k=1}^{\infty} \alpha_{m_{n-1}k} f_k(x)f_k(y) + \delta\left(\sum_{k=\nu_{n-1}+1}^{\nu_n} f_k(x)f_k(y) + \sum_{k=1}^{k(m_n)} c_{nk} f_k(x)f_k(y)\right) =$$

$$= \sum_{k=1}^{\infty} \alpha_{m_{n-1}k} f_k(x)f_k(y) + \sum_{k=1}^{\infty} (\alpha_{m_n k} - \alpha_{m_{n-1}k}) f_k(x)f_k(y) = \sum_{k=1}^{\infty} \alpha_{m_n k} f_k(x)f_k(y).$$

Damit ist unsere Behauptung gezeigt.

BEMERKUNG. Mit einer zusätzlichen Annahme über die Matrix (α_{nk}) kann man mit dieser Beweismethode auch zeigen, daß $\lim_{n\to\infty} S_{\nu_n}(x)$ f. ü. existiert für alle genügend schnell wachsenden Folgen (ν_n) $\left(\nu_{n+1} \geqq H(\nu_n)\right)$.

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ON DECOMPOSING FINITE ABELIAN GROUPS

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I.

Since F. KÁRTESZI had set the problem for crosses $(1, n)$, the n-dimensional mosaics of this type have often been examined ([1], [2], [3], [4], [5]). S. K. STEIN [8] formulated the same problem in a more general way for crosses (k, n). It was also motivated by [4] and the famous proof by G. HAJÓS [7] who solved the well-known conjecture of H. Minkowski.

In this note we shall prove that a lattice-like mosaic of crosses $(2, n)$ exists if and only if the order of $2 \bmod (p)$ is a multiple of 4 for any prime p in the prime factor decomposition of the number $4n+1$.

THEOREM 0. *A lattice-like mosaic consisting of crosses* $(2, n)$ *exists if and only if the number* $4n+1$ *has only prime divisors of the form* $8m+5$ *or* $8m+1$. *In this latter case the number* 2 *is non-residue of order* $2^{l-1} \bmod (8m+1)$, *where* 2^l *is defined by the condition* $8m+1=2^l q+1, q=odd$ *(i.e. there is no solution of the congruence*

$$x^{2^{l-1}} \equiv 2 \mod (8m+1);$$

e.g. this congruence has no solution for primes 17, 41, 97, *but it has solution for* 73 *and* 89*).*

This criterion shows that a lattice-like mosaic of crosses $(2, n)$ exists for an infinite number of n's and does not exist for another sequence of n's.

II.

Let 0 be a fixed origin and $e_1, e_2, \dots, e_n$ an orthonormed basis in an n-dimensional Euclidean space. Further let K be an n-dimensional unit cube with centre 0 and of the position $e_1, e_2, \dots, e_n$. Unification of all shifts generated by replacement of K by vectors je_i where $i=1, 2, \dots, n, j=0, \pm 1, \pm 2, \dots, \pm k$ will be called cross (k, n).

Throughout this note we shall consider only fillings with cubes consisting of adjacent crosses (k, n) joining along the side of an $n-1$ dimensional cube.

The possibility of decomposing an n-dimensional Euclidean space by crosses (k, n) in a lattice-like arrangement is connected with a certain mode of producing finite Abelian groups. Let G be a finite Abelian group. Then the possibility of representing G in the form

$$\begin{aligned} G = \{0, & g_1, 2g_1, \dots, kg_1, -g_1, -2g_1, \dots, -kg_1, \\ & g_2, 2g_2, \dots, kg_2, -g_2, -2g_2, \dots, -kg_2, \\ & \vdots \\ & g_n, 2g_n, \dots, kg_n, -g_n, -2g_n, \dots, -kg_n\} \end{aligned}$$

where $g_1, g_2, \ldots, g_n \in G \setminus \{0\}$, will be indicated briefly by

$$G = (\{\pm 1, \pm 2, \ldots, \pm k\}: \{g_1, g_2, \ldots, g_n\}).$$

THEOREM 1. *An n-dimensional Euclidean space can be filled by crosses (k, n) in a lattice-like arrangement if and only if there exists an Abelian group G of order $2kn+1$ with representation*

$$G = (\{\pm 1, \pm 2, \ldots, \pm k\}: \{g_1, g_2, \ldots, g_n\}),$$

where $g_1, g_2, \ldots, g_n \in G \setminus \{0\}$.

The proof of this theorem can be found in [8]. (The ideas of the proof appear already in [6].)

The main result of this paper will be proved if we verify the following statement: for any n satisfying the condition of Theorem 0 there exists an Abelian group of order $4n+1$ which can be represented in the form

$$G = (\{\pm 1, \pm 2\}: \{g_1, g_2, \ldots, g_n\}), g_1, g_2, \ldots, g_n \in G \setminus \{0\}$$

and this is not the case for the other n-s.

THEOREM 2. *An Abelian group G of order $4n+1$ can be represented in the form*

$$G = \{\pm 1, \pm 2\}: \{g_1, g_2, \ldots, g_n\}, g_1, g_2, \ldots, g_n \in G \setminus \{0\}$$

if and only if $4n+1$ has only prime divisors of the form $8m+5$ or $8m+1$. In this latter case the number 2 is non-residue of order 2^{l-1} mod $(8m+1)$ where 2^l is defined by the condition $8m+1 = 2^l q + 1$, $q = odd$.

Theorem 2 gives an interesting result for the fillings. According to considerations of [5], it leads to a lower estimate for $f(n)$, the number of not congruent fillings by lattice-like crosses $(2, n)$. Indeed, $f(n) \geqq 1$ implies that $f(n) \geqq g(\alpha_1) g(\alpha_2) \ldots g(\alpha_t)$ where $4n+1 = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t}$ is the prime factor decomposition of $4n+1$, $g(\alpha)$ is the number of essentially different additive decompositions of the natural number α.

Theorem 2 will be proved by means of the fundamental theorem for finite Abelian groups which states that any Abelian group can be represented as the direct sum of cyclic groups of prime power order. Elements of the Abelian group represented as direct sum of cyclic groups of prime power orders $q_1^{\beta_1}, q_2^{\beta_2}, \ldots, q_s^{\beta_s}$ will be denoted by vector columns of s components, where $q_1, q_2, \ldots, q_s$ are not absolutely different. Components of all vectors are integer numbers. Further we say, that

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_s \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_s \end{bmatrix}$$

if and only if $a_i \equiv b_i \bmod (q_i^{\beta_i})$, $i=1, 2, \dots, s$. The operation is defined by

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_s \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_s \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_s \end{bmatrix}$$

where $c_i \equiv a_i + b_i \bmod (q_i^{\beta_i})$ $i=1, 2, \dots, s$.

III.

Proof of Theorem 2 follows in 8 steps.

1) If $G=\{0, g_1, g_2, \dots, g_{4n}\}$ is an Abelian group, then the elements $2g_1, 2g_2, \dots, \dots, 2g_{4n}$ will be a permutation of the elements $g_1, g_2, \dots, g_{4n}$.

In fact, $2g_i = 2g_j$ implies that $2(g_i - g_j) = 0$. For $i \neq j$, $g_i - g_j \neq 0$, thus $g_i - g_j$ is a second order element of G. But according to the Lagrange theorem, G has no elements of order 2.

2) Let $G=\{0, g_1, g_2, \dots, g_{4n}\}$ be an Abelian group, Γ a graph with nodes g_i and edges $\{g_i, 2g_i\}$, where $i=1, 2, \dots, 4n$. Then G can be represented in the form

$$G = (\{\pm 1, \pm 2\}\colon \{g_{i_1}, \dots, g_{i_n}\}),\ g_{i_1}, \dots, g_{i_n} \in G \setminus \{0\}$$

if and only if Γ has a 1-factor (i.e. Γ has a set of disjoint edges containing all the nodes of Γ) such that its edges can be pairwise matched by $\{g_i, 2g_i\} \leftrightarrow \leftrightarrow \{-g_i, -2g_i\}$.

If $G=(\{\pm 1, \pm 2\}\colon \{g_{i_1}, \dots, g_{i_n}\}),\ g_{i_1}, \dots, g_{i_n} \in G \setminus \{0\}$ then by definition

$$\begin{aligned} G = \{0, g_{i_1}, &2g_{i_1}, -g_{i_1}, -2g_{i_1} \\ \vdots\ &\ \vdots \\ g_{i_n}, &2g_{i_n}, -g_n, -2g_{i_n}\}. \end{aligned}$$

Thus the edges

$$\begin{matrix} \{\ g_{i_1}, & 2g_{i_1}\} \dots \{\ g_{i_n}, & 2g_{i_n}\} \\ \{-g_{i_1}, & -2g_{i_1}\} \dots \{-g_{i_n}, & -2g_{i_n}\} \end{matrix}$$

are a 1-factor of Γ where edges in one column form pairs.

But by this 1-factor, a desired production of G can be directly obtained.

3) If G can be represented in the form

$$G = (\{\pm 1, \pm 2\}\colon \{g_{i_1}, \dots, g_{i_n}\}),\ g_{i_1}, \dots, g_{i_n} \in G \setminus \{0\}$$

then according to 2) Γ has a 1-factor. In conformity with 1) Γ consists of disjoint circuits and all circuits of Γ are of even length. That is, the length of all cycles of the permutation

$$\begin{pmatrix} g_1, \dots, & g_{4n} \\ 2g_1, \dots, & 2g_{4n} \end{pmatrix}$$

is also an even number.

4) If $(a_1, \ldots, a_h)$ is a cycle of the permutation

$$\begin{pmatrix} g_1, \ldots, & g_{4n} \\ 2g_1, \ldots, & 2g_{4n} \end{pmatrix}$$

including $-a_j$ for some $1 \leqq j \leqq h$, then there exists $0<t<h$ such that

(1) $$-a_i = a_r, \quad \text{where} \quad r = i+t \mod (h), \quad i = 1, 2, \ldots, h.$$

The cycle $(a_1, \ldots, a_h)$ is identical to $(a_1, 2a_1, \ldots, 2^{h-1}a_1)$ where

(2) $$2^h a_1 = a_1 \quad \text{and}$$

(3) $$2^{h'} a_1 \neq a_1 \quad \text{for} \quad 0 < h' < h.$$

$-a_j$ was assumed to be equal to one of the elements of the cycle. Therefore

(4) $$-a_j = a_x \quad \text{and} \quad 1 \leqq x \leqq h \quad \text{that is,}$$

(5) $$-a_1 2^{j-1} = a_1 2^{x-1}.$$

Multiplying (5) by 2^{h-j+1} and taking into consideration relation (2), we obtain

(6) $$-a_1 = a_1 2^{x-j} = a_1 2^t = a_{1+t},$$

where we put $t=x-j$. Of course, $0 \leqq t<h$ but $t \neq 0$, because a_1 can not be of order 2. Multiplying (6) by 2 we obtain the needed statement.

In particular from (6) follows that

(7) $$a_1 = -a_{t+1} = a_{2t+1}$$

and hence $h=2t$.

5) Let $G=\{[0], [1], \ldots, [4r]\}$ be a cyclic group of order p^α, where p is an odd prime number. G can be represented in the form:

(8) $$G = (\{\pm 1, \pm 2\}: A), \quad A \subset G \setminus \{0\}$$

if and only if the length of any cycle in the permutation

(9) $$\begin{pmatrix} [1], \ldots, & [4r] \\ 2[1], \ldots, & 2[4r] \end{pmatrix}$$

is a multiple of 4.

a) Let $([a_1], \ldots, [a_{2h}])$ be a cycle of permutation (9), then

$$-[a_i] = [a_{h+i}], \quad i = 1, 2, \ldots, h.$$

Namely a_1 can be uniquely represented in the form $a_1 = xp^{\alpha-\beta}$ where x and p are relative primes, $\beta>0$. The cycle $([a_1], \ldots, [a_{2h}])$ has the following representation

$$([xp^{\alpha-\beta}], [xp^{\alpha-\beta}2], \ldots, [xp^{\alpha-\beta}2^{2h-1}]).$$

Evidently

(10) $$xp^{\alpha-\beta}2^{2h} \equiv xp^{\alpha-\beta} \mod (p^\alpha), \quad \text{but}$$

(11) $$xp^{\alpha-\beta}2^{2h'} \not\equiv xp^{\alpha-\beta} \mod (p^\alpha), \quad \text{if} \quad 0 < h' < 2h.$$

(10) may also be written in the following way

$$xp^{\alpha-\beta}(2^h-1)(2^h+1) \equiv 0 \mod (p^\alpha)$$

which implies that either $p|(2^h-1)$ or $p|(2^h+1)$ since $\beta>0$. Both cases can not hold, because it would imply that $p|2$ and it is impossible. For $p|(2^h-1), p\nmid(2^h+1)$ hence $p^\beta|(2^h-1)$ i.e. $2^h\equiv 1 \bmod (p^\beta)$.

This implies that

$$xp^{\alpha-\beta}2^h \equiv xp^{\alpha-\beta} \bmod (p^\alpha)$$

hence we obtained a contradiction with (11). Thus $p\nmid(2^h-1)$ hence $p|(2^h+1)$ or more precisely $p^\beta|(2^h+1)$, i.e., $2^h\equiv -1 \bmod (p^\beta)$ and $xp^{\alpha-\beta 2h}\equiv -xp^{\alpha-\beta} \bmod (p^\alpha)$. Thus $[xp^{\alpha-\beta 2h}]$ is the converse of $[xp^{\alpha-\beta}]$. In general $[xp^{\alpha-\beta}2^{h+i-1}]$ is the converse of $[xp^{\alpha-\beta}2^{i-1}]$, $i=1, 2, \dots, h$.

b) *Necessity*. Evidently Γ consists of disjoint circuits. According to 3, circuits of Γ and the cycles of permutation (9) are of even length. Thus according to a), any cycle of (9) contains the converse of some of its element. According to 2) Γ has a 1-factor where edges may be pairwise matched by

$$\{g_i, 2g_i\} \leftrightarrow \{-g_i, -2g_i\}.$$

Thus, every circuit is decomposed into 1-factors having edges matched into pairs belonging to the same circuit, hence in fact, all circuit lengths are multiples of 4.

Sufficiency. Assume that every cycle length of (9) is divisible by 4. Let

$$([a_1], \dots, [a_{4h}]) \tag{12}$$

be a cycle of 9. According to a) it can be written in the following way

$$([a_1], \dots, [a_{2h}], -[a_1], \dots, -[a_{2h}]).$$

Edges

$$\{[a_1], \quad [a_2]\}, \dots, \quad \{[a_{2h-1}], \quad [a_{2h}]\}$$
$$\{-[a_1], -[a_2]\}, \dots, \{-[a_{2h-1}], -[a_{2h}]\}$$

of Γ give a 1-factor of the circuit corresponding to cycle (12) and are matched into pairs by

$$\{[a_i], [2a_i]\} \leftrightarrow \{-[a_i], -[2a_i]\}.$$

Doing this operation for every circuit of Γ we obtain a convenient 1-factor of Γ, hence, by 2) we arrive to (8).

6) The cyclic group G of order p^α can be represented in the form (8) if and only if the order of the number 2 mod (p) is a multiple of 4.

At first we prove that the cycle lengths of permutation (9) are multiples of the order of 2 mod (p). Indeed, let us consider the cycle of (9) containing $z=yp^{\alpha-\gamma}$, where y and p are relative primes, $\gamma>0$. The length of the cycle is h if and only if

$$yp^{\alpha-\gamma}2^h \equiv yp^{\alpha-\gamma} \bmod (p^\alpha) \tag{13}$$

but

$$yp^{\alpha-\gamma}2^{h'} \not\equiv yp^{\alpha-\gamma} \bmod (p^\alpha) \tag{14}$$

for $0<h'<h$. Denoting the order of 2 mod (p) by d, we have

$$2^h \equiv 1 \bmod (p) \tag{15}$$

and

(16) $$2^{d'} \not\equiv 1 \mod (p)$$

for $0<d'<d$. (13) can also be written in the form $2^h \equiv 1 \mod (p^\gamma)$, hence

(17) $$2^h \equiv 1 \mod (p).$$

Then by (15), (16), (17), $d|h$.

If the order of 2 mod (p) can be divided by 4, then the length of every cycle of (9) is a multiple of 4 also. Thus, according to 5) G can be represented in the form (8).

If the order of 2 mod (p) cannot be divided by 4, then the length of the cycle of (9) containing $[p^{\alpha-1}]$ which is equal to the order of the number 2 mod (p) cannot be divided by 4 also. Thus, according to 5) G cannot be written in the form (8).

7) The order of 2 mod (p) is a multiple of 4 if and only if p has the form $8m+5$ or $8m+1$, where 2 is a non-residue of order 2^{l-1} mod (p), and l is defined by relation $8m+1=2^l q+1$ (q is an odd number).

Let d be equal to the order of 2 mod (p).

Assume first that p has the form $8m+7$ or $8m+3$. Then $p-1$ is not a multiple of 4. According to Fermat's theorem, $2^{p-1} \equiv 1 \mod (p)$. Hence, $d|(p-1)$ and $4\nmid d$.

Further assume that p has the form $8m+5$. Then $p-1$ can be written in the form $4r$ where r is odd. According to Fermat's theorem $2^{4r} \equiv 1 \mod (p)$. Hence $d|4r$ and for $4\nmid d$, $d|2r$. Thus $2^d \equiv 1 \mod (p)$ and this implies that

(18) $$2^{2r} \equiv 1 \mod (p).$$

Since $p=8m+5$, 2 is a quadratic non-residue mod (p). Thus, according to Euler's lemma $2^{2r} \equiv -1 \mod (p)$ which contradicts (18). Thus, $4|d$.

Now let p be of the form $8m+1$. Writing $p-1$ in the form $2^l q$, where q is odd we obtain that 2 is a residue of order 2^{l-1} mod (p) if and only if

$$2^{2^l q/2^{l-1}} \equiv 2^{2q} = 1 \mod (p).$$

Let 2 be a residue of order 2^{l-1} mod (p), then $2^{2q} \equiv 1 \mod (p)$. Hence, $d|2q$, i.e. $4\nmid d$. According to Fermat's Theorem $2^{2^l q} \equiv 1 \mod (p)$. Hence, $d|2^l q$, and for $4\nmid d$, $d|2q$. Thus, by $2^d \equiv 1 \mod (p)$ we have

(19) $$2^{2q} \equiv 1 \mod (p).$$

If 2 is a non-residue of order 2^{l-1} mod (p) then $2^{2q} \equiv 1 \mod (p)$ and this contradicts (19). Thus, if 2 is a non-residue of order 2^{l-1} then $4|d$.

8) Let $G=\{0, g_1, \dots, g_{4n}\}$ be a direct sum of cyclic groups $G_1, \dots, G_s$ of prime power order. Set

$$G_1 = \{0, g_{1,1}, \dots, g_{1,v_1}\}$$
$$\vdots$$
$$G_s = \{0, g_{s,1}, \dots, g_{s,v_s}\}.$$

Then G can be represented in the form

$$G = (\{\pm 1, \pm 2\}\colon A), \quad A \subset G \setminus \{0\}$$

if and only if G_i has the form:

$$G_i = (\{\pm 1, \pm 2\}\colon A_i), \quad A_i \subset G_i \setminus \{0\}$$

for any $i=1, 2, \ldots, s$.

Necessity. G contains elements of the form

$$\begin{bmatrix} g_{1,1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} g_{1,2} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} g_{1,v_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Their doubles are of the same form, hence cycles of permutation

$$\begin{pmatrix} g_1, \ldots, g_{4n} \\ 2g_1, \ldots, 2g_{4n} \end{pmatrix}$$

containing these elements correspond to cycles of permutation

$$\begin{pmatrix} g_{1,1}, \ldots, g_{1,v_1} \\ 2g_{1,1}, \ldots, 2g_{1,v_1} \end{pmatrix} \tag{20}$$

in matching

$$\begin{bmatrix} g_{1,j} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \to g_{1,j}.$$

Γ_1 is the graph of permutation, (20) is a partial graph of Γ. If $g_i \in G$ is a point of Γ_1 then $-g_1, 2g_i, -2g_i$ are points of Γ_1 also. Thus, if $\{g_i, 2g_i\}$ is an edge of Γ_1 then $\{-g_i, -2g_i\}$ is also an edge of it. The existence of a convenient 1-factor of Γ follows from its existence in Γ_1. Thus representation of G in the form

$$G = (\{\pm 1, \pm 2\}\colon A), \quad A \subset G \setminus \{0\}$$

follows from the representation

$$G_1 = (\{\pm 1, \pm 2\}\colon A_1), \quad A_1 \subset G_1 \setminus \{0\}.$$

The same is true for the other G_i-s.

Sufficiency. Assume that

$$G_i = (\{\pm 1, \pm 2\}\colon A_i), \quad A_i \subset G_i \setminus \{0\} \quad (i = 1, 2, \ldots, s).$$

Then the lengths of all cycles of permutation

$$\begin{pmatrix} g_1, \ldots, g_{4n} \\ 2g_1, \ldots, 2g_{4n} \end{pmatrix} \tag{21}$$

are multiples of 4. Namely, if among $g_{1,i_1}, g_{2,i_2}, \ldots, g_{s,i_s}$ the elements $g_{j_1,i_{j_1}}, g_{j_2,i_{j_2}}, \ldots, g_{j_z,i_{j_z}}$ are not equal to zero then the length of the cycle of permutation (21) containing the element

$$\begin{bmatrix} g_{1,i_1} \\ g_{2,i_1} \\ \vdots \\ g_{s,i_n} \end{bmatrix}$$

is equal to the least common multiple of the length of cycles containing elements $g_{j_1,i_{j_1}}, \ldots, g_{j_z,v_{i_z}}$ of permutations

$$\begin{pmatrix} g_{j_1,1}, \ldots, & g_{j_1,v_{j_1}} \\ 2g_{j_1,1}, \ldots, & 2g_{j_1,v_{j_1}} \end{pmatrix}, \ldots, \begin{pmatrix} g_{j_z,1}, \ldots, & g_{j_z,v_{j_z}} \\ 2g_{j_z,1}, \ldots, & 2g_{j_z,v_{j_z}} \end{pmatrix}$$

respectively. By 5, the lengths of these latter cycles are divisible by 4. Cycles of

$$\begin{pmatrix} g_1, \ldots, & g_{4n} \\ 2g_1, \ldots, & 2g_{4n} \end{pmatrix}$$

may be of two kinds. Either they contain the converse of some element or not. (We do not intend to justify whether the latter case may occur or not.)

If the cycle $(a_1, \ldots, a_{4h})$ is of second kind, then it can be complemented by cycle $(-a_1, \ldots, -a_{4h})$, In this case the edges

$$\{a_1, a_2\},\ \{a_3, a_4\}, \ldots, \{a_{4h-1}, a_{4h}\}$$

$$\{-a_1, -a_2\},\ \{-a_3, -a_4\}, \ldots, \{-a_{4h-1}, -a_{4h}\}$$

constitute a 1-factor of two circuits corresponding to cycles, and the edges of the 1-factor are matched by $\{g_i, 2g_i\} \leftrightarrow \{-g_i, -2g_i\}$.

If the cycle $(a_1, \ldots, a_{4h})$ is of the first kind, then according to 4) $-a_i = a_{2h+i}$, $i = 1, 2, \ldots, 2h$. Thus $(a_1, \ldots, a_{2h}, a_{2h+1}, \ldots, a_{4h}) = (a_1, \ldots, a_{2h}, -a_1, \ldots, -a_{2h})$ and the edges

$$\{a_1, a_2\}, \ldots, \{a_{2h-1}, a_{2h}\}$$

$$\{-a_1, -a_2\}, \ldots, \{-a_{2h-1}, -a_{2h}\}$$

constitute a 1-factor of the corresponding circuit of graph Γ, with pairwise matched

$$\{g_i, 2g_i\} \leftrightarrow \{-g_i, -2g_i\}.$$

Thereby, circuits of the graph Γ are decomposed into convenient 1-factors. Thus, according to 2) G really has the form:

$$G = (\{\pm 1, \pm 2\}\colon A), \quad A \subset G \setminus \{0\}.$$

IV.

We shall consider in details the case of crosses (2, 6). Let $2kn+1 = 25 = 5 \cdot 5$. Then a mosaic consisting of crosses (2, 6) does exist. What is more, according to the remark following Theorem 2, there are at least two incongruent mosaics. There are two Abelian groups of order 25, one being a cyclic group, and the other the direct sum of two cyclic groups of order 5.

1. Set $G=\{[0],[1],[2],\dots,[23],[24]\}$. The permutation is as follows

$$\begin{pmatrix}[1][2][3][4]\,[5]\,\,[6]\,\,[7]\,\,[8]\,\,[9]\,[10][11][12][13][14][15][16][17][18][19][20][21][22][23][24]\\ [2][4][6][8][10][12][14][16][18][20][22][24]\,[1]\,\,[3]\,\,[5]\,[7]\,[9]\,[11][13][15][17][19][21][23]\end{pmatrix}=$$

$$=([1][2][4][8][16][7][14][3][6][12][24][23][21][17][9][18][11][22][19][13])([5][10][20][15]).$$

Thus:

$$G=(\{\pm 1,\pm 2\}\colon\ \{[1],[4],[16],[14],[6],[5]\}).$$

Using the decomposition of the group G we can construct the basic vectors $r_1,\dots,r_6$ but we do not intend to deal with it. For details see [5].

	r_1	r_2	r_3	r_4	r_5	r_6
e_1	25	4	16	14	6	5
e_2		-1				
e_3			-1			
e_4				-1		
e_5					-1	
e_6						-1

Fig. 1

2. Set

$$G=\left\{\begin{bmatrix}0\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix}\begin{bmatrix}0\\3\end{bmatrix}\begin{bmatrix}0\\4\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}\cdots\begin{bmatrix}4\\3\end{bmatrix}\begin{bmatrix}4\\4\end{bmatrix}\right\}.$$

Then the permutation is as follows

$$\begin{pmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix}\begin{bmatrix}0\\3\end{bmatrix}\begin{bmatrix}0\\4\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}1\\4\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix}\begin{bmatrix}3\\0\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}\begin{bmatrix}3\\3\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix}\begin{bmatrix}4\\0\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix}\begin{bmatrix}4\\3\end{bmatrix}\begin{bmatrix}4\\4\end{bmatrix}\\ \begin{bmatrix}0\\2\end{bmatrix}\begin{bmatrix}0\\4\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0\\3\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}\begin{bmatrix}4\\0\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix}\begin{bmatrix}4\\4\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix}\begin{bmatrix}4\\3\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}1\\4\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}3\\0\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}3\\3\end{bmatrix}\end{pmatrix}=$$

$$\left(\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix}\begin{bmatrix}0\\4\end{bmatrix}\begin{bmatrix}0\\3\end{bmatrix}\right)\left(\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}\begin{bmatrix}4\\0\end{bmatrix}\begin{bmatrix}3\\0\end{bmatrix}\right)\left(\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix}\begin{bmatrix}4\\4\end{bmatrix}\begin{bmatrix}3\\3\end{bmatrix}\right)\left(\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix}\begin{bmatrix}4\\3\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}\right)\left(\begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix}\right)\left(\begin{bmatrix}1\\4\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}\right).$$

Thus $G=\left(\{\pm 1,\pm 2\}\colon \left\{\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}1\\4\end{bmatrix}\right\}\right).$

The basic vectors $r_1, \ldots, r_6$ of the mosaic lattice are given by the table

	r_1	r_2	r_3	r_4	r_5	r_6
e_1	5		1	1	1	1
e_2		5	1	2	3	4
e_3			-1			
e_4				-1		
e_5					-1	
e_6						-1

Fig. 2

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NONCOMPLETE BASES ON DISCRETE SPACES

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The word space will refer to Tychonoff spaces. In [2] the author considers a particular class of bases* $\mathscr{D}$ on a space X, called complete. They are characterized by the relation $\beta(\nu(X,\mathscr{D}))=\omega(X,\mathscr{D})$ between their associated Wallman spaces. Obviously, $Z(X)$ is always a complete base on X, while the σ-algebra of all Lebesgue measurable sets in the real line $\mathbf{R}$ is an example of noncomplete base on the discrete space $\mathbf{R}$ ([2], Corollary 4.1).

We shall describe here a method of generating noncomplete bases $\mathscr{D}$ on an uncountable discrete space X which are "nearly all of $Z(X)$", in the sense of $X=\nu(X,\mathscr{D})$.

1. Definitions and basic results

As usual, $C(X)$ will denote the ring of all continuous real-valued functions on X and $Z(X)$ will denote the family of all zero-sets in X, $Z(X)=\{f^{-1}(0): f\in C(X)\}$.

For a base $\mathscr{D}$ on a space X let $\hat{\mathscr{D}}$ be the trace on X of all zero-sets in the Wallman realcompactification $\nu(X,\mathscr{D})$. Then $\hat{\mathscr{D}}$ is a base on X containing $\mathscr{D}$ ([5], 1.4). A base $\mathscr{D}$ on X is called complete if $\mathscr{D}=\hat{\mathscr{D}}$. It follows from [2] that:

(i) $\hat{\mathscr{D}}$ is the largest base on X such that $\nu(X,\mathscr{D})=\nu(X,\hat{\mathscr{D}})$. **

(ii) $\hat{\mathscr{D}}$ is the smallest complete base on X containing $\mathscr{D}$.

Then if $\mathscr{D}$ is a base on X, distinct from $Z(X)$, such that $\nu X=\nu(X,\mathscr{D})$, it follows that $\mathscr{D}$ is not complete and $\hat{\mathscr{D}}=Z(X)$.

Note. Corollary 2.3 in [5] shows that the correspondence between the family $\mathscr{L}(X)$ of all bases on a space X and the associated Wallman compactifications is one-to-one. This is not the case between $\mathscr{L}(X)$ and the associated Wallman realcompactifications ([5], 3.13; [2], Corollary 4.1). From (i) and (ii), if $\nu(X,\mathscr{D})=\nu(X,\mathscr{F})$ then $\mathscr{D}\subset\hat{\mathscr{F}}$ and $\mathscr{F}\subset\hat{\mathscr{D}}$. Therefore $\hat{\mathscr{F}}=\hat{\mathscr{D}}$. So the correspondence between the complete bases on X and the associated Wallman realcompactifications is one-to-one.

* This notion is due to E. F. STEINER [4] who uses the term separating nest generated intersection ring. An equivalent concept is the strong delta normal base due to ALÒ and SHAPIRO [1]. We adopt our notation and terminology from [1], [5] and our earlier paper [2].

** Two extensions T_1 and T_2 of a space X are said to be equivalent if they are homeomorphic via a map that leaves X pointwise fixed. In this case we write $T_1=T_2$.

2. A technique for constructing noncomplete bases

We denote by X the union of countably many copies $D^n, n=1, 2, \ldots$ of an uncountable discrete space D of nonmeasurable cardinal. We consider $\{D^n: n=1, 2, \ldots\}$ as forming a discrete open covering of X. We denote $A\subset\beta D$ in βD^n by A^n. As each set D^n is clopen in X, we have $\mathrm{cl}_{\beta X} D^n=\beta D^n$. Let us denote by C the set $\beta D\sim D$.

Let n be a fixed positive integer. We consider a decomposition $\mathscr{A}_n$ of βX consisting of $\{p^{2n-1}, p^{2n}\}, p\in C$ and the other points of βX. Consider the quotient space $K_n=\beta X(\mathscr{A}_n)$. Since $\mathscr{A}_n$ is easily verified to be upper semicontinuous, K_n is a Hausdorff compactification of X. If ψ is the canonical quotient map from βX onto K_n, we do not distinguish notationally between X and $\psi(X)$.

Now, we shall discuss some properties of the obtained compactification.

A) *The base* $\mathscr{D}_n=\{Z\cap X\colon Z\in Z(K_n)\}$ *is distinct from* $Z(X)$.

PROOF. The zero-set (in X) D^{2n-1} is not the trace of a zero-set in K_n on X. For, if there were a zero-set $Z\in Z(K_n)$ such that $Z\cap X=D^{2n-1}$, then $\psi^{-1}(Z)$ would be a zero-set in βX and $\psi^{-1}(Z)\cap\beta D^{2n}$ would be a zero-set containing C^{2n} but missing X. This is a contradiction because every zero-set in βD^{2n} which contains C^{2n} meets D^{2n}. Therefore $D^{2n-1}\in Z(X)\sim\mathscr{D}_n$.

B) *The correspondence* $n\to\mathscr{D}_n$ *is one-to-one.*

PROOF. Indeed, if $n\neq m$ then $D^{2n-1}\in\mathscr{D}_m\sim\mathscr{D}_n$.

C) $X=v(X, \mathscr{D}_n)$.

PROOF. First, we shall prove that the intersection of all cozero sets in K_n which contain X coincides with X. Let p be a point in $\beta X\sim\{X\cup C^{2n-1}\cup C^{2n}\}$. Since X is realcompact there is a function $f\in C(\beta X)$ such that $f(p)=0, Z(f)\cap X=\varnothing$ and $0\leqq f(y)\leqq 1$ for every $y\in\beta X$. Let g be the function in $C(\beta X)$ such that $g(\beta D^{2n-1}\cup\beta D^{2n})=\{1\}$ and vanishes in the other points. If $h=f\vee g\in C(\beta X)$ we have that $h(p)=0$ and $Z(h)\cap X=\varnothing$. The function s defined on K_n by the equality $s_0\psi=h$ is continuous, $s(\psi(p))=0$ and $Z(s)\cap X=\varnothing$.

If y is a point in $\beta D\sim D$, since D is realcompact there is a function $f\in C(\beta D)$ such that $f(y)=0$ and $Z(f)\cap D=\varnothing$. Let g be the function in $C(\beta X)$ whose value in $\beta X\sim\{\beta D^{2n-1}\cup\beta D^{2n}\}$ is 1 and whose restriction to βD^{2n-1} and βD^{2n} coincides with f. As $g(q^{2n-1})=g(q^{2n})$ for every $q\in\beta D$, it follows that the function h defined on K_n by the equality $h_0(\psi)=g$ belongs to $C(K_n)$. Furthermore, $\psi(y^{2n})\in Z(h)$ and $Z(h)\cap X=\varnothing$.

From ([3], 4.2) (see ([5], 3.9) also) we have that $X=v(X, \mathscr{D}_n)$. Moreover $\mathscr{D}_n$ is not complete and $\hat{\mathscr{D}}_n=Z(X)$.

From the properties A), B) and C) we obtain the following result.

THEOREM. *If Y is an uncountable discrete space of nonmeasurable cardinal, there are infinitely many noncomplete bases $\mathscr{D}$ on Y such that $Y=v(Y, \mathscr{D})$.*

NOTE. By an algebra on a space X is meant a subalgebra of $C(X)$ which contains the constants, separates points and closed sets, and is closed under uniform convergence and inversion in $C(X)$. If A is an algebra on X, we write $\mathscr{Z}(A)$ for the set $\{Z(f): f\in A\}$. In [5] it is proved that the mapping $A\to\mathscr{Z}(A)$ is a one-to-one correspondence between the family of all algebras on X and $\mathscr{L}(X)$. On the other hand, from ([2], Theorem 2.5) a base on X is complete if and only if its associated algebra on X is isomorphic to $C(Z)$ for some space Z. Thus, from the above Theorem it follows that if Y is an uncountable discrete space of nonmeasurable cardinal, there are infinitely many algebras on Y which are isomorphic to no $C(Z)$.

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A CLASS OF TRANSLATION-INVARIANT CHARACTERS OF $M(G)$

By

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1. Introduction

Let G be a locally compact non discrete abelian group; we shall denote by $M(G)$ the (convolution) Banach algebra of all complex valued, bounded, countable additive, regular Borel measures on G. The point mass of $x \in G$ will be written by $\delta(x)$ and the maximal ideal space of $M(G)$ by Δ.

A subset P of G is said to be independent if whenever $x_1, x_2, \ldots, x_k$ are distinct elements of P and $n_1, n_2, \ldots, n_k$ are integers then either $n_1x_1+\ldots+n_kx_k \neq 0$ or $n_1x_1=\ldots=n_kx_k=0$. It is well known that there are compact perfect independent subsets P of G. Throughout this work we shall denote by P subsets as above and by $M(P)$ $(M_c(P))$ the space of all (continuous) measures of $M(G)$ concentrated on P.

HEWITT and KAKUTANI [1] show the following theorem: "if L is a linear functional on $M_c(P)$ of norm non exceeding 1, then there is a multiplicative linear functional (character) h on $M(G)$ which agrees with L in $M_c(P)$". They prove that the set $\{\mu - L_1(\mu)\cdot\delta(0)\colon \mu \in M(P)\}$ is contained in some maximal ideal $h^{-1}(0)$ where L_1 is a linear functional on $M(P)$ and $\|L_1\| \leqq 1$. For any continuous or discontinuous character χ of G they consider

$$(1)\qquad L_1(\mu) = L(\mu_c)+\sum a_k\chi(x_k) \quad \text{for any} \quad \mu\in M(P) \ \left(\mu=\mu_c+\sum a_k\delta(x_k)\right),$$

thus since L_1 agrees with L in $M_c(P)$, h extends L.

W. RUDIN ([2] Theorem 5.4.1) givex the same result by proving that the compact sets $H(\mu, \varepsilon)=\{h\in\Delta\colon |h(\mu)-L(\mu)|\leqq\varepsilon\}$ where $\mu\in M_c(P)$ and $\varepsilon>0$, have the finite intersection property.

In § 2 we use methods as in [2] and [4] to show that if L is as above it has an extension to an $h\in\Delta$ which is translation-invariant, that is $h(\mu*\delta(x))=h(\mu)$, for each $x\in G$. This result is not contained in the above theorems, which do not suffice to show that h is translation invariant. Even if we take $\chi=1$ in (1) it is not assured that h will have this property.

As an application of it we show in § 3 that there is a translation-invariant character of $M(G)$ which is neither idempotent character, nor produced by any Raikov system or by Sreider's method.

2. Multiplicative translation-invariant extension of certain linear functional

LEMMA (2.1). *Let P be a compact independent set in G and $\mu_1, \mu_2, \ldots, \mu_k$ nonnegative continuous measures of norm one concentrated on the disjoint subsets $E_1, \ldots, E_k$ of respectively. If $z_1, z_2, \ldots, z_k$ are complex numbers with $|z_i|\leqq 1$, $i=1, 2, \ldots, k$ and $\delta(x_1), \ldots, \delta(x_r)$ the point masses at the distinct points $x_1, x_2, \ldots, x_r$ of G, then* *there is an $h\in\Delta$ such that*

$$h(\mu_i)=z_i \quad \textit{and} \quad h(\delta(x_j))=1 \quad (i=1,2,\ldots,k,\ j=1,2,\ldots,r).$$

PROOF. We use the following steps.

Step 1. Let $\mu=\mu_1+\ldots+\mu_k$. We show that μ^n and $(\mu^m)_z$ are mutually singular; where $m<n$ and z arbitrary or $m=n$ and $z\neq 0, (\mu^m)_z=\mu^m*\delta(z)$ and $(z\in G)$.

The measures μ^n and $(\mu^m)_z$ are concentrated on $(n)P$ and $(m)P-z$ respectively. If $(n)P$ and $(m)P-z$ are disjoint there is nothing to prove; otherwise we have

$$x_1+\ldots+x_n=y_1+\ldots+y_m-z \tag{1}$$

for some $x_i, y_j, i=1,\ldots,n, j=1,\ldots,m$ belonging to P.

Denote by S the set of $(x_1,\ldots,x_n)$ with $(x_1+\ldots+x_n)\in(m)P-z$. We prove that

$$\mu_n(S)=(\mu\times\mu\times\ldots\times\mu)(S)=0.$$

Let $z=y_1'+\ldots+y_m'-x_1'-\ldots-x_n'$ for fixed x_i', y_j' in P. If $x_1,\ldots,x_n$ in (1) were all different and different from $x_1',\ldots,x_n'$, then we should have a linear relation between the elements of P, contradicting the independence of P.

Hence S is contained in a finite union of sets of the form

$$P_{ij}=\{x|x_i=x_j\}\ (i\neq j),\quad P_{ij}'=\{x|x_i=x_j'\}\ (\text{any } i,j).$$

Since μ is continuous Fubini's theorem shows that $\mu_n(P_{ij})=\mu_n(P_{ij}')=0$ and hence $\mu_n(S)=0$.

Step 2. Let $\lambda=\mu_1^{r_1}*\ldots*\mu_k^{r_k}, \nu=\mu_1^{s_1}*\ldots*\mu_k^{s_k}$; then λ and ν_z are mutually singular unless $(r_1,\ldots,r_k)=(s_1,\ldots,s_k)$ and $z=0$.

Let $r=r_1+\ldots+r_k, s=s_1+\ldots+s_k$, since λ and ν_z are absolutely continuous with respect to μ^r and $(\mu^s)_z$ respectively and since $\mu^r\perp(\mu^s)_z$ it follows that $\lambda\perp(\nu)_z$ for $r\neq s$ and $r=s$ with $z\neq 0$.

For the remainder case $r=s$ but $(r_1,\ldots,r_k)\neq(s_1,\ldots,s_k)$ see [2] Lemma 5.4.2.

Step 3. We prove that there exists $h\in\Delta$ with the required properties. Let $\sigma=\bar z_1\mu_1+\bar z_2\mu_2+\ldots+\bar z_k\mu_k+\delta(x_1)+\ldots+\delta(x_s)$ where $|z_i|=1, i=1,2,\ldots,k$.

We express σ^n as a polynomial in $\mu_1,\ldots,\mu_k,\delta(x_1),\ldots,\delta(x_s)$

$$\sigma^n=\sum_{r_1+\ldots+r_k+r_1'+\ldots+r_s'=n}\alpha(r_1,\ldots,r_k,r_1',\ldots,r_s')\mu_1^{r_1}*\ldots*\mu_k^{r_k}*\delta(r_1'x_1+\ldots+r_s'x_s).$$

According to step 2 it is easy to see that all $\mu_1^{r_1}*\ldots*\mu_k^{r_k}*\delta(r_1'x_1+\ldots+r_s'x_s)$ are mutually singular.

Hence $\|\sigma^n\|=\sum|\alpha(r_1,\ldots,r_s')|=(k+s)^n, n=1,2,\ldots$ and so the spectral radius of σ is $k+s$.

It follows that there is a complex homomorphism h of $M(G)$ such that

$$|h(\sigma)|=\left|z_1h(\mu_1)+\ldots+z_kh(\mu_k)+h(\delta(x_1))+\ldots+h(\delta(x_r))\right|=k+r.$$

Since $|h(\mu_i)|\leqq 1, |h(\delta(x_j))|=1\ (i=1,\ldots,k;\ j=1,\ldots,r)$ it follows that $h(\mu_i)=z_i$ and $h(\delta(x_j))=1$.

We remove the assumption $|z_i|=1$ as in [2].

THEOREM (2.2). *For each L in $M_c(P)^*$ with $\|L\| \leqq 1$ we can find a translation invariant character of $M(G)$ whose restriction to $M_c(P)$ is L.*

PROOF. For $\varepsilon > 0$, $\mu_i \in M_c(P)$ and $x_i \in G$ $(i = 1, 2, \ldots, k)$ we consider the sets

$$H(\varepsilon_i, x_i, \mu_i) = \{h \in \Delta : |h(\mu_i) - L(\mu_i)| < \varepsilon_i \text{ and } h(\delta(x_i)) = 1\};$$

by use of (2.1) we modify (5.4.1) in [2] to prove that they are non-empty.

3. Applications

We note that each element of Δ can be represented as a generalized character (CH. Y. ŠREIDER [3]) and we determine the semigroup structure of Δ by the usual multiplication of generalized characters. We observe that any idempotent character of $M(G)$ is translation-invariant.

In fact, let $(\chi_\mu)_{\mu \in M(G)} \in \prod_{\mu \in M(G)} L^\infty(G)$ be the generalized character which corresponds to $h \in \Delta$ and suppose that $h^2 = h$, then for any $x \in G$ we obtain

$$\langle h^2, \delta(x) \rangle = \int \chi^2_{\delta(x)}(t)\, d\delta_{(x)}(t) = \chi^2_{\delta(x)}(x) = \chi_{\delta(x)}(x)$$

it follows that $h(\delta(x)) = 1$ for any $x \in G$.

PROPOSITION (3.1). *There is a translation-invariant character which is not idempotent. In fact, let $L(\mu) = \frac{1}{2}\mu(P)$ for any $\mu \in M_c(P)$ and let h be the corresponding translation-invariant character; then for $\mu \in M_c(P)$*

$$\langle h^2, \mu \rangle = \int \chi^2_\mu(t)\, d\mu(t) = \int \chi_\mu(t)\, d\mu(t) \neq \langle h, \mu \rangle.$$

COROLLARY (3.2). *There exists a character of $M(G)$ which cannot be described by the Raikov systems or by Šreider's methods.*

We note from [3] and [5] the following. Given a Raikov system $\mathcal{J}$ (a non empty collection of σ-compact subsets of G satisfying conditions R1 to R4 in [5]) and a character $\chi(t)$ of G which is measurable with respect to all measures concentrated on $\mathcal{J}$, then

$$h_{\mathcal{J}}(\mu) = \int \chi(t)\, d\mu_{\mathcal{J}}(t), \quad \mu \in M(G)$$

is a character of $M(G)$; $\mu_{\mathcal{J}}$ is the projection of μ on $\mathcal{J}$. Suppose that a set H of characters of G is given; denote by S_H the set of all measures of $M(G)$ with respect to which every character in H is measurable. Y. Šreider proved that S_H is a subalgebra of $M(G)$ and if $\chi \in H$ then the formula

$$h_S(\mu) = \int \chi(t)\, d\mu_S(t), \quad \mu \in M(G)$$

where μ_S denotes the projection of μ on the subalgebra S_H, determines a character of $M(G)$.

To show (3.2) it is not difficult to see that if $h_{\mathcal{J}}$ or h_S are translation-invariant, then they are idempotent characters.

Y. ŠREIDER constructed in [3] an h_S which differs from $h_{\mathscr{J}}$ for any Raikov system $\mathscr{J}$. Thus corollary (3.2) says also that the set of Raikov's characters $h_{\mathscr{J}}$, the set of Šreider's characters h_S and the maximal ideal space Δ are different

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TO THE MATHEMATICAL THEORY OF "FIFTEENTH-PUZZLE" *

By

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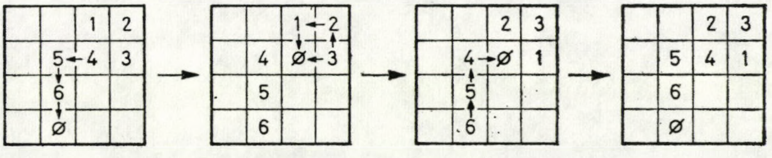
In this paper we give a short, although not constructive proof for the fundamental theorem of "Fifteenth-puzzle" making use of a group theoretical result. As for the description and the mathematical theory of this game we refer to the book of AHRENS [1, pp. 226—260]. From the theory of permutation groups we need the following result.

LEMMA ([2], p. 22). *A connected system of cycles of order* 3 *generates the alternating group on the set of elements permuted by the* 3-*cycles. (Let* $K_1, K_2, \dots, K_r$ *be the three element sets permuted by the cycles, then we call this system connected if there does not exist a set K, such that* $\emptyset \subset K \subset \bigcup_{i=1}^{r} K_i$ *and for any* $i=1, 2, \dots, r$, *either* $K_i \subseteq K$ *or* $K_i \cap K = \emptyset$ *holds.)*

THEOREM. *If two arrangements of the "Fifteenth-puzzle" having the same place empty can be obtained from each other by an even permutation of the pieces, then they can be transformed to each other by the rules of the game.*

PROOF. Take the 2×2 squares and omit from them the fields nearest to the empty square in the given arrangements. We shall show that the 3-cycle permuting the pieces in the remaining three squares of such a 2×2 square can be accomplished by the rules of the game.

Indeed, first let us move away the empty square to the omitted field of the given 2×2 square in the shortest way, then permute cyclically the three pieces, and finally move back the empty field to its original place. For example:



These 3-cycles obviously form a connected system and permute all the fifteen pieces. Therefore, by the lemma, we are done.

REMARKS. The necessity of the condition given in the theorem is more or less obvious. The sufficiency is generally proved by distinction of cases. Our proof

* To the centenary of its invention (1978).

clearly works for the game of arbitrary size $k_1 \times k_2 \times \ldots \times k_n$, $n \geqq 2$, $k_i \geqq 2$ for $1 \leqq i \leqq n$.

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CONVERGENCE OF SEQUENCES OF MEASURABLE FUNCTIONS

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E. MARCZEWSKI in [3] has proved, that if (X, S, μ) is a finite measure space and μ is an atomic measure, then the convergence a.e. of a sequence of measurable real functions is equivalent to the convergence in measure. GRIBANOV in [2] has observed that the converse theorem is also true, i.e. this equivalence implies the fact that μ is atomic. In this note we shall prove that a similar theorem holds for more general kind of convergence considered in [5]. In the proof we shall assume Souslin's hypothesis.

Suppose that (X, S) is a measurable space. Let $I \subset S$ be a proper σ-ideal of sets. We shall say that some property holds I-almost everywhere if and only if the set of elements, which do not possess this property, belongs to I. According to [5] we shall say that a sequence $\{f_n\}_{n\in\mathbf{N}}$ of S-measurable real functions defined on X converges to a function f (which is also S-measurable) with respect to the σ-ideal I if and only if for every subsequence $\{f_{m_n}\}_{n\in\mathbf{N}}$ there exists a subsequence $\{f_{p_{m_n}}\}_{n\in\mathbf{N}}$ convergent I-a.e. to f.

We shall say that the pair (S, I) fulfils the countable chain condition (C.C.C.) if and only if every disjoint family included in $S-I$ is at most denumerable. An element $A\neq\Lambda$ of a Boolean algebra $\mathscr{A}$ is said to be an atom of $\mathscr{A}$ (see [4]) provided that for every $B\in\mathscr{A}$ the inclusion $B\subset A$ implies $B=\Lambda$ or $B=A$. A Boolean algebra $\mathscr{A}$ is said to be atomic if and only if for every $B\neq\Lambda$ $(B\in\mathscr{A})$ there exists an atom $\Lambda\neq A\subset B$. We shall consider the equivalence relation defined on S in the following way: for every $A, B\in S$ $A\sim B$ if and only if $A\triangle B\in I$. This equivalence yields a quotient Boolean algebra S/I in which the inclusion $[A]\subset[B]$ means that $A-B\in I$. If S/I is atomic, then from the C.C.C. it follows immediately that $X=\bigcup_k A_k$ (the union is finite or denumerable), where all A_k-s are disjoint and all $[A_k]$-s are atoms.

Recall basic facts connected with the Souslin's hypothesis (see [1]). A tree is a partially ordered set $(T, \prec)$ such that for every $x\in T$ the set $\{y\in T\colon y\prec x\}$ is well ordered. A branch of T is a maximal linearly ordered subset of T. An antichain in T is a set of pairwise incomparable elements of T. A tree T is called a Souslin's tree if card $(T)=\aleph_1$ and every branch and antichain T is at most denumerable. Souslin's hypothesis is equivalent to the non-existence of a Souslin's tree.

THEOREM. *If the pair (S, I) fulfills C.C.C. then the convergence I-a.e. is equivalent to the convergence with respect to the σ-ideal I if and only if S/I is atomic.*

PROOF. Suppose that S/I is atomic. Let $\{f_n\}_{n\in\mathbf{N}}$ converges to f with respect to I. We shall prove that $\{f_n\}_{n\in\mathbf{N}}$ converges to f I-a.e. (observe that the convergence

I-a.e. always implies convergence with respect to I). The idea of the proof is essentially the same as in [2].

If $\{[A_k]\}_{k\in N}$ denotes the sequence of all atoms of S/I, then it is not difficult to observe that every S-measurable real function g defined on X is equal I-a.e. to the function of the form $\sum_{k=1}^{\infty} c_k \chi_{A_k}$, that is g is I-a.e. constant on every A_k. Hence f is equal I-a.e. to $\sum_{k=1}^{\infty} a_k \chi_{A_k}$ and for every n f_n is equal I-a.e. to $\sum_{k=1}^{\infty} a_k^{(n)} \chi_{A_k}$. From the assumption and from the definition of the convergence with respect to I it follows immediately that for every k and for every subsequence $\{a_k^{(m_n)}\}_{n\in N}$ of $\{a_k^{(n)}\}_{n\in N}$ there exists a subsequence $\{a_k^{(p_{m_n})}\}_{n\in N}$ convergent to a_k, hence $a_k^{(n)} \to a_k$ $(n\to\infty)$ for every k. This means the convergence I-a.e. of $\{f_n\}_{n\in N}$ to f.

Suppose now that S/I is not atomic. We shall construct a sequence $\{f_n\}_{n\in N}$ of S-measurable real functions, which is convergent to zero with respect to I, but does not converge to zero I-a.e.

From the assumption it follows that there exists $[A]\in S/I$ not including any atoms, i.e. for every $B\subset A$, if $B\in S-I$, then there exists a pair of sets $B_1, B_2\in S-I$ such that $B_1\cap B_2=\emptyset$ and $B_1\cup B_2=B$. The pair of sets $\{B_1, B_2\}$ we shall call a proper subdivision of B. We shall prove now that if $[A]$ is an element of S/I which does not include atoms, then there exists a sequence $\{d_n\}_{n\in N}$ of subdivisions of A (by a subdivision we shall mean as usual a finite disjoint family of subsets of A with the union equal to A) such that

a) for every n d_{n+1} is a refinement of d_n;

b) for every sequence $\{A_n\}_{n\in N}$ of sets such that $A_n\in d_n$ we have $\bigcap_{n=1}^{\infty} A_n\in I$;

c) for every n $d_n\subset S-I$.

Let $\{d_n^{(1)}\}_{n\in N}$ be a sequence of subdivisions defined in the following way: $d_1^{(1)}=\{A_{1,1}^{(1)}, A_{1,2}^{(1)}\}$ is a proper subdivision of A and for every natural n let $d_{n+1}^{(1)}=\{A_{n+1,1}^{(1)}, \ldots, A_{n+1,2^n}^{(1)}\}$, where $\{A_{n+1,2k-1}^{(1)}, A_{n+1,2k}^{(1)}\}$ is a proper subdivision of $A_{n,k}^{(1)}$ for $k=1, \ldots, 2^{n-1}$. It is not difficult to verify that $\{d_n^{(1)}\}_{n\in N}$ has the properties a) and c). Denote $I_1=\{\bigcap_{n=1}^{\infty} B_n\colon B_n\in d_n^{(1)}\}$. If $I_1\subset I$, then put $d_n=d_n^{(1)}$ for every natural n. If $I_1-I\neq\emptyset$, then we shall construct next sequences of subdivisions to obtain the desired result.

Suppose that for all ordinal numbers $\alpha<\eta$, where $\eta<\Omega$ we have already defined the sequences $\{d_n^{(\alpha)}\}_{n\in N}$ of subdivisions of A and related families $I_\alpha=\{\bigcap_{n=1}^{\infty} B_n\colon B_n\in d_n^{(\alpha)}\}$ having the properties a) and c) and the following two properties for all ordinal numbers $\alpha, \beta<\eta, \alpha<\beta$:

i) there exists a number N such that for every $n\geqq N$ $d_n^{(\beta)}$ is a refinement of $d_n^{(\alpha)}$;

ii) the family I_β is essentially finer than the family I_α (it means that for every pair of sets C, D such that $C\in I_\alpha-I, D\in I_\beta-I$ and $D\subset C$ we have $C-D\in S-I$).

We shall define a sequence of subdivisions $\{d_n^{(\eta)}\}_{n\in N}$. Consider two cases.

The first case. A number η has the predecessor $(\eta-1)$. Suppose that $I_{\eta-1}-I\neq\emptyset$, because in the contrary case the construction would be finished at earlier stage. From the C.C.C. it follows that $I_{\eta-1}-I=\{D_1, D_2, \ldots\}$. Let for every natural n $\{D_n', D_n''\}$

be a proper subdivision of D_n. Let $d_1^{(\eta)}$ be a common refinement of $d_1^{(\eta-1)}$ and $\{A-D_1, D_1', D_1''\}$. If we have defined $d_k^{(\eta)}$ for $k=1, ..., n$, then let $d_{n+1}^{(\eta)}$ be a common refinement of $d_{n+1}^{(\eta-1)}$, $d_n^{(\eta)}$ and $\{A-D_{n+1}, D_{n+1}', D_{n+1}''\}$. It is not difficult to verify that the sequence $\{d_n^{(\eta)}\}_{n\in N}$ has the properties a) and c) and that now i) and ii) are fulfilled for $\alpha, \beta \leqq \eta$.

The second case. The number η is a limit number. Suppose that for every $\alpha<\eta$ $I_\alpha - I \neq \emptyset$, because in the contrary case the construction would be finished at earlier stage. Let $\{\alpha_k\}_{k\in N}$ be an increasing sequence of ordinal numbers tending to η. From the property i) it follows that for every k there exists N_k such that for each $n \geqq N_k$ $d_n^{(\alpha_k+1)}$ is simultaneously a refinement of $d_n^{(\alpha_1)}, d_n^{(\alpha_2)}, ..., d_n^{(\alpha_k)}$. Obviously we can suppose that $\{N_k\}_{k\in N}$ is an increasing sequence of natural numbers. Put $d_1^{(\eta)}=d_1^{(\alpha_1)}$ and for every $k\geqq 2$ $d_k^{(\eta)}=d_{N_{k-1}}^{(\alpha_k)}$. It is not difficult to verify that the sequence $\{d_n^{(\eta)}\}_{n\in N}$ has the properties a) and c) and that now i) and ii) are again fulfilled for $\alpha, \beta \leqq \eta$.

Our aim is now to prove that for some $\eta<\Omega$ we obtain $I_\eta \subset I$. Suppose the contrary. Then by transfinite induction we obtain a transfinite sequence $\{\{d_n^{(\eta)}\}_{n\in N}\}_{\eta<\Omega}$ of sequences of subdivisions such that for every $\alpha, \beta<\Omega$ if $\alpha<\beta$, then I_β is essentially finer than I_α.

Put $T=\bigcup_{\eta<\Omega}(I_\alpha - I)$. Consider the partial ordering $\prec$ defined in the following way: $A \prec B$ if and only if $A \supset B$ for $A, B\in T$. It is easy to prove that $(T, \prec)$ is a tree. From the assumption $I_\eta - I \neq \emptyset$ for every $\eta<\Omega$ we conclude that card $(T)=\chi_1$. We assume the Souslin's hypothesis, so $(T, \prec)$ is not a Souslin's tree. Hence it has a nondenumerable branch or a nondenumerable antichain. In the first case let $\{A_\alpha\}_{\alpha<\Omega}$ be this branch. Then $\{A_\alpha - A_{\alpha+1}\}_{\alpha<\Omega}$ is a nondenumerable family of disjoint sets from $S-I$ — a contradiction with C.C.C. In the second case let $\{B_i\}_{i\in I}$ be an antichain in T such that card $(I)=\aleph_1$. From the construction it follows that the sets belonging to the antichain are pairwise disjoint — again a contradiction with C.C.C.

Hence there exists an ordinal number $\eta<\Omega$ such that $I_\eta \subset I$. Put $d_n = d_n^{(\eta)}$ for every n. The sequence $\{d_n\}_{n\in N}$ has obviously the properties a), b) and c).

Let $\{B_n\}_{n\in N}$ be a sequence of sets in which every set from every partition of the sequence $\{d_n\}_{n\in N}$ arrives exactly one time. Put $f_n=\chi_{B_n}$ for every n. It is nearly obvious that for every $x\in A$ the sequence $\{f_n(x)\}_{n\in N}$ is divergent, so the sequence of functions $\{f_n\}_{n\in N}$ does not converge I-a.e. (because $A\notin I$). We shall show that $\{f_n\}_{n\in N}$ converges to zero with respect to I. Let $\{f_{m_n}\}_{n\in N}$ be a subsequence of $\{f_n\}_{n\in N}$. We can suppose (taking suitable subsequence of $\{f_{m_n}\}_{n\in N}$ if necessary) that a corresponding sequence of sets $\{B_{m_n}\}_{n\in N}$ has the following property: $B_{m_n}\in d_{k_n}$, where $\{k_n\}_{n\in N}$ is increasing sequence of natural numbers. Two cases are possible:

1. there exists an increasing subsequence $\{p_{m_n}\}_{n\in N}$ of $\{m_n\}_{n\in N}$ such that $B_{p_{m_1}} \supset B_{p_{m_2}} \dots$;
2. there is no such sequence.

In the first case we take a subsequence $\{f_{p_{m_n}}\}_{n\in N}$. This subsequence converges to zero I-a.e. (more precisely, except on $\bigcap_{n=1}^{\infty} B_{m_{p_n}}$), because from condition b) it follows that $\bigcap_{n=1}^{\infty} B_{m_{p_n}} \in I$.

In the second case we proceed as follows: let $\{B_{m_1}, \ldots, B_{p_{m_1}}\}$ be a maximal (i.e. improlongable) descending subsequence of $\{B_{m_n}\}_{n\in N}$ (in this case such sequence does exist). We observe that every set in the sequence $\{B_{p_{m_1}+1}, B_{p_{m_1}+2}, \ldots\}$ is disjoint with $B_{p_{m_1}}$ (in the contrary case the above sequence would not be maximal). For the set $B_{p_{m_1}+1}$ we also take a maximal descending sequence $\{B_{p_{m_1}+1}, \ldots, B_{p_{m_2}}\}$. Now every set in the sequence $\{B_{p_{m_2}+1}, B_{p_{m_2}+2}, \ldots\}$ is disjoint with $B_{p_{m_1}}$ and with $B_{p_{m_2}}$. Proceeding further by induction we obtain a sequence $\{B_{p_{m_n}}\}_{n\in N}$ of disjoint sets. The subsequence $\{f_{p_{m_n}}\}_{n\in N}$ of $\{f_{m_n}\}_{n\in N}$ converges to zero at every point.

So in both cases it is possible to find a subsequence $\{f_{p_{m_n}}\}_{n\in N}$ convergent I-a.e. to zero, hence $\{f_n\}_{n\in N}$ converges to zero with respect to I.

This ends the proof.

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A NOTE ON DISCRETE ČEBYSEV APPROXIMATION

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Let $C_0[0, 2\pi]$ be the space of 2π-periodic real-valued continuous functions; $C_0^r[0, 2\pi]=\{f\in C_0[0,2\pi]: f^{(r)}\in C_0[0, 2\pi]\}$ $(r\in\mathbf{N})$; T_n be the space of trigonometric polynomials of order at most n $(n\in\mathbf{Z}_+)$. By $p_n(f)\in T_n$ and $E_n(f)$ we denote the polynomial and error of best Čebysev approximation to $f\in C_0[0, 2\pi]$, respectively. That is

$$E_n(f)=\inf_{q_n\in T_n}\|f-q_n\|=\|f-p_n(f)\|,$$

where

$$\|g\|=\max_{x\in[0,2\pi]}|g(x)|\quad (g\in C_0[0, 2\pi]).$$

One of the simplest "approximate" methods of determining $p_n(f)$ and $E_n(f)$ consists in selecting a finite subset Y in $[0, 2\pi]$ and solving the approximation problem on Y instead of $[0, 2\pi]$. (We assume, of course, that Y consists of at least $2n+2$ points and is 2π-periodically extended to the real axis.) Let us denote by $p_n(f)_Y$ and $E_n(f)_Y$ the polynomial and error of best Čebysev approximation to $f\in C_0[0, 2\pi]$ on Y, respectively:

$$E_n(f)_Y=\inf_{q_n\in T_n}\|f-q_n\|_Y=\|f-p_n(f)_Y\|_Y\quad (p_n(f)_Y\in T_n),$$

where

$$\|g\|_Y=\max_{x\in Y}|g(x)|\quad (g\in C_0[0, 2\pi]).$$

Further, define the density of Y in $[0, 2\pi]$ by

$$\varrho(Y)=\max_{x\in[0,2\pi]}\min_{y\in Y}|x-y|.$$

Then we are interested in following questions: does $p_n(f)_Y$ tend uniformly to $p_n(f)$ as $\varrho(Y)\to 0$? Does $\|f-p_n(f)_Y\|\to E_n(f)$ hold as $\varrho(Y)\to 0$? How can the rates of convergence be estimated?

In this note we shall study the convergence of $\|f-p_n(f)_Y\|$ to $E_n(f)$. This matter was discussed in a number of papers (see [1]—[4]). The different estimations given in these papers can be summarized in the following way:

If $\varrho(Y)$ is small enough, then for arbitrary $f\in C_0[0, 2\pi]$

$$\|f-p_n(f)_Y\|\leqq E_n(f)+C_n(f)\,\omega_2(f, \varrho(Y)), \tag{1}$$

where $\omega_2(f, \delta)$ denotes the modulus of continuity of f of order 2 (the modulus of continuity of f will be denoted by $\omega(f, \delta)$), the constant $C_n(f)$ depends only on f and n. Moreover from the estimations given in [1]—[4] it follows that $C_n(f)=O(n)$. The rate of convergence in (1) can not be improved in general but the estimation for the constant $C_n(f)=O(n)$ is far from the best. In the present note we shall prove that $C_n(f)$ in (1) can be replaced by an absolute constant and this estimation for $C_n(f)$ is optimal in general. Furthermore it will be shown that $C_n(f)$ even tends to 0 as $n\to\infty$ if the function f is smooth enough.

In what follows $C_i(\dots)$ denote constants depending only on quantities specified in the brackets, while C_i denote absolute constants.

New results

THEOREM 1. *Let $n\in\mathbf{N}$, $f\in C_0[0, 2\pi]$. Then for any $Y\subset[0, 2\pi]$ with $0<\varrho(Y)\leqq n^{-1}$*

$$\|f-p_n(f)_Y\| \leqq E_n(f)+\left(1+\frac{13\pi^2}{2}\right)\omega_2(f, \varrho(Y)) \tag{2}$$

holds.

REMARK. If $n=0$ then for any $Y\subset[0, 2\pi]$, $\|f-p_0(f)_Y\|\leqq E_0(f)+\omega_2(f, \varrho(Y))$; therefore we consider only the nontrivial case $n\geqq 1$.

COROLLARY 1. *Let $n\in\mathbf{N}$, $f\in C_0^1[0, 2\pi]$. Then for any $Y\subset[0, 2\pi]$ with $0<\varrho(Y)\leqq n^{-1}$*

$$\|f-p_n(f)_Y\| \leqq E_n(f)+\left(1+\frac{13\pi^2}{2}\right)\varrho(Y)\omega(f', \varrho(Y)) \tag{3}$$

holds.

COROLLARY 2. *If $n\in\mathbf{N}$ and $f\in C_0^r[0, 2\pi]$, where $r\geqq 2$ then for any $Y\subset[0, 2\pi]$ with $0<\varrho(Y)\leqq n^{-1}$*

$$\|f-p_n(f)_Y\| \leqq E_n(f)+\frac{\pi}{2}\left(1+\frac{13\pi^2}{2}\right)(n+1)^{-r+2}\omega\left(f^{(r)}, \frac{\pi}{n+1}\right)\varrho^2(Y). \tag{4}$$

Thus by (2) the constant $C_n(f)$ in (1) can be estimated by an absolute constant. Moreover, if $f\in C_0^r[0, 2\pi]$, where $r\geqq 2$, then by (4) $C_n(f)$ even tends to 0 as $n\to\infty$.

In our next theorem we show that inequalities (2)—(4) are in general the best possible from the point of view of rate of convergence (as $\varrho(Y)\to 0$) and degree of the constants (as $n\to\infty$).

THEOREM 2. *For any $n\in\mathbf{N}$ and $r\in\mathbf{Z}_+$ there exists a function $g_r\in C_0^r[0, 2\pi]$ such that for arbitrary $0<d\leqq C_1 n^{-1}$ $(0<C_1<1)$ we can find a suitable finite subset $\subset[0Y, 2\pi]$ with $\varrho(Y)=d$ such that*

$$\|g_r-p_n(g_r)_Y\| \geqq E_n(g_r)+C_2(r)\begin{cases} d^r\omega(g_r^{(r)}, d), & \text{for } r=0 \text{ or } 1;\\ n^{-r+2}\omega(g_r^{(r)}, n^{-1})\,d^2, & \text{for } r\geqq 2.\end{cases} \tag{5}$$

Proof of Theorem 1

We shall need some lemmas.

LEMMA 1. *For any $f\in C_0[0, 2\pi]$ and $n\in \mathbf{Z}_+$*

$$E_n(f) \leqq \omega_2\left(f, \frac{\pi}{2(n+1)}\right) \tag{6}$$

holds.

PROOF. The proof is based on standard method of Stekloff transforms. Set

$$f_{h,2}(x) = (2h)^{-2} \int_{-h}^{h} \int_{-h}^{h} f(x+t_1+t_2)\, dt_1\, dt_2.$$

It follows by easy calculations that

$$\|f-f_{h,2}\| \leqq \frac{1}{2}\omega_2(f, 2h), \quad \|f''_{h,2}\| \leqq (2h)^{-2}\omega_2(f, 2h).$$

Using these inequalities and the theorem of FAVARD and AHIEZER—KREIN (see [6], p. 302) we have

$$E_n(f) \leqq \|f-f_{h,2}\| + E_n(f_{h,2}) \leqq \frac{1}{2}\omega_2(f, 2h)+$$

$$+\frac{\pi^2}{8(n+1)^2}\|f''_{h,2}\| \leqq \omega_2(f, 2h)\left(\frac{1}{2}+\frac{\pi^2}{32h^2(n+1)^2}\right).$$

Setting in this inequality $h=\pi/(4(n+1))$ we obtain (6).

The proof of the following lemma is based on an idea used in [5].

LEMMA 2. *Let $n\in \mathbf{N}$, $f\in C_0[0, 2\pi]$. Then*

$$\omega_2(p_n(f), \delta) \leqq \frac{5\pi^2}{2}\omega_2(f, \delta) \quad (\delta > 0). \tag{7}$$

Furthermore, for any $Y\subset[0, 2\pi]$ with $0<\varrho(Y)\leqq n^{-1}$

$$\omega_2(p_n(f)_Y, \delta) \leqq \frac{13\pi^2}{2}\omega_2(f, \delta) \quad (\delta > 0). \tag{8}$$

PROOF. Let us prove (7). Assume at first that $\delta\geqq\pi/(2(n+1))$. Then (6) implies

$$\omega_2(p_n(f), \delta) \leqq \omega_2(f, \delta)+4E_n(f) \leqq \tag{9}$$

$$\leqq \omega_2(f, \delta)+4\omega_2\left(f, \frac{\pi}{2(n+1)}\right) \leqq 5\omega_2(f, \delta).$$

By STEČKIN's inequality ([5], p. 227) for arbitrary $t_n\in T_n$

$$\|t''_n\| \leqq \frac{n^2}{2}\omega_2\left(t_n, \frac{\pi}{2n}\right). \tag{10}$$

Further, it is known (see [6], p. 116) that for any $0<h_1<h_2$

$$4\frac{\omega_2(f, h_1)}{h_1^2} \geqq \frac{\omega_2(f, h_2)}{h_2^2}. \tag{11}$$

Let $0<\delta<\pi/(2(n+1))$. Then using (9) for $\bar{\delta}=\pi/(2n)>\pi/(2(n+1))$, (10) and (11) we obtain

$$\omega_2(p_n(f), \delta) \leqq \delta^2 \|p_n''(f)\| \leqq \frac{\delta^2 n^2}{2}\,\omega_2\left(p_n(f), \frac{\pi}{2n}\right) \leqq$$

$$\leqq \frac{5\delta^2 n^2}{2}\,\omega_2\left(f, \frac{\pi}{2n}\right) \leqq \frac{5\pi^2}{2}\,\omega_2(f, \delta).$$

This and (9) imply (7).

Now we shall prove (8). Set $\bar{f}=f-p_n(f)$. Then $p_n(\bar{f})_Y=p_n(f)_Y-p_n(f)$. It is known (see [1]), that for any $t_n\in T_n$, $\|t_n\|\leqq 2\|t_n\|_Y$ if $0<\varrho(Y)\leqq n^{-1}$. Thus and by (6)

$$\omega_2(p_n(\bar{f})_Y, \delta) \leqq \delta^2 \|p_n''(\bar{f})_Y\| \leqq 2\delta^2 n^2 \|p_n(\bar{f})_Y\|_Y \leqq \tag{12}$$

$$\leqq 4\delta^2 n^2 \|\bar{f}\| = 4\delta^2 n^2 E_n(f) \leqq 4\delta^2 n^2 \omega_2\left(f, \frac{\pi}{2(n+1)}\right).$$

If $\delta<\pi/(2(n+1))<\pi/(2n)$, then (11) and (12) imply

$$\omega_2(p_n(\bar{f})_Y, \delta) \leqq 4\delta^2 n^2 \omega_2\left(f, \frac{\pi}{2n}\right) \leqq 4\pi^2 \omega_2(f, \delta). \tag{13}$$

On the other hand if $\delta\geqq\pi/(2(n+1))$, then using again (6) we have

$$\omega_2(p_n(\bar{f})_Y, \delta) \leqq 4\|p_n(\bar{f})_Y\| \leqq 8\|p_n(\bar{f})_Y\|_Y \leqq$$

$$\leqq 16\|\bar{f}\| = 16E_n(f) \leqq 16\omega_2\left(f, \frac{\pi}{2(n+1)}\right) \leqq 16\omega_2(f, \delta).$$

Evidently, this last inequality is contained in (13), hence (13) holds for any $\delta>0$. Thus by (7)

$$\omega_2(p_n(f)_Y, \delta) \leqq \omega_2(p_n(\bar{f})_Y, \delta)+\omega_2(p_n(f), \delta) \leqq$$

$$\leqq 4\pi^2 \omega_2(f, \delta)+\frac{5\pi^2}{2}\omega_2(f, \delta) = \frac{13\pi^2}{2}\omega_2(f, \delta) \quad (\delta>0).$$

The proof of the Lemma is complete.

We now can prove Theorem 1. Let $f\in C_0[0, 2\pi]$ and $Y\subset[0, 2\pi]$ satisfy $0<\varrho(Y)\leqq n^{-1}$. Set $f^*=f-p_n(f)_Y$. Then by (8)

$$\omega_2(f^*, \delta) \leqq \omega_2(f, \delta)+\omega_2(p_n(f)_Y, \delta) \leqq \left(1+\frac{13\pi^2}{2}\right)\omega_2(f, \delta). \tag{14}$$

For some $\xi\in[0, 2\pi]$, $\|f^*\|=|f^*(\xi)|$. Further, there exists $\eta\in Y$ such that $|\eta-\xi|\leqq\varrho(Y)$. Hence and by (14)

$$\|f-p_n(f)_Y\| = \|f^*\| = |f^*(\xi)| \leqq |f^*(\eta)|+|f^*(\xi)-f^*(\eta)| \leqq$$
$$\leqq \|f^*\|_Y+\left|f^*(\xi)-f^*(\xi+|\xi-\eta|)+f^*(\xi)-f^*(\xi-|\xi-\eta|)\right| \leqq$$
$$\leqq E_n(f)_Y+\omega_2(f^*, \varrho(Y)) \leqq E_n(f)+\left(1+\frac{13\pi^2}{2}\right)\omega_2(f, \varrho(Y)).$$

Theorem 1 is proved.

It is known that for $f\in C_0^1[0, 2\pi]$, $\omega_2(f, \delta)\leqq\delta\omega(f', \delta)$ $(\delta>0)$. Thus Corollary 1 follows from Theorem 1 and this remark.

Let us prove Corollary 2. If $f\in C_0^r[0, 2\pi]$, where $r\geqq 2$, then $\omega_2(f, \delta)\leqq\delta^2\|f''\|$. Hence and by (2)

$$\|f-p_n(f)_Y\| \leqq E_n(f)+\left(1+\frac{13\pi^2}{2}\right)\|f''\|\,\varrho^2(Y).$$

Replacing in this inequality f by $f-\int_0^x\int_0^y p_n(f'', t)\,dt\,dy$ we obtain

$$\|f-p_n(f)_Y\| \leqq E_n(f)+\left(1+\frac{13\pi^2}{2}\right)E_n(f'')\,\varrho^2(Y). \tag{15}$$

But it is known (see [7], p. 231), that for any $f\in C_0^r[0, 2\pi]$, $E_n(f)\leqq\frac{\pi}{2(n+1)^r}\,\omega\left(f^{(r)}, \frac{\pi}{n+1}\right)$, hence

$$E_n(f'')\leqq\frac{\pi}{2}(n+1)^{-r+2}\omega\left(f^{(r)}, \frac{\pi}{n+1}\right).$$

Combining this with (15) we obtain (4).

Proof of Theorem 2

Let us construct the counter-example.

Set $N=28n$; $h=2\pi/N$ $\left(0<h<1/(4n)\right)$; $0<d<h/2$;

$$g_0(x)=\begin{cases} x, & x\in[0, h/2];\\ -x+h, & x\in[h/2, h];\\ -g_0(x-h), & x\in[h, 2h],\end{cases}\qquad g_r(x) = n^{-r-1}\sin\frac{\pi x}{h}\quad (r\geqq 1),$$

where $g_0(x)$ is extended to $[0, 2\pi]$ as a $2h$-periodic function. Then evidently $g_r\in C_0^r[0, 2\pi]$ and $\omega(g_r^{(r)}, \delta)\leqq C_3(r)\delta$ $(r\in\mathbf{Z}_+)$. Further

$$g_r(x_i) = (-1)^i\|g_r\| \quad (i = 0, 1, \dots, N-1), \tag{16}$$

where $x_i=h/2+ih$ $(i=0, 1, \dots, N-1)$. Thus $E_n(g_r)=\|g_r\|$.

Set $Y=\{a_j+\frac{i}{k}(b_j-a_j);\ i=0,1,\dots,k;\ j=0,1,\dots,N-1\}\cup\{x_{N-1}\}$, where $a_0=0$; $a_j=x_{j-1}+d$ $(j=1,2,\dots,N-1)$; $b_j=x_j-d$ $(j=0,1,\dots,N-2)$; $b_{N-1}=2\pi$; $k=[\pi/d]+1$. Then $\varrho(Y)=d$. Further, denoting $\Delta_r=g_r(h/2)-g_r(h/2-d)$ we have

$$\Delta_0=d,$$

$$C_5n^{-r+1}d^2\leqq\Delta_r\leqq C_4n^{-r+1}d^2\quad(r\geqq1). \tag{17}$$

It can be easily shown that

$$E_n(g_r)_Y=\|g_r\|-\Delta_r+\mu_r, \tag{18}$$

where $0<\mu_r\leqq\Delta_r/2$. Assume that $\|p_n(g_r)_Y\|=\gamma p_n(g_r,\xi)_Y$ $(\gamma=\pm1,\ \xi\in[0,2\pi])$. It follows by construction that there exists $\eta\in Y$ such that $|\xi-\eta|\leqq h$ and

$$-\gamma g_r(\eta)\geqq\|g_r\|-\Delta_r.$$

Then by (18) $\gamma p_n(g_r,\eta)_Y\leqq\mu_r$, hence (using that $nh<1/4$)

$$\mu_r\geqq\gamma p_n(g_r,\xi)_Y+\gamma(p_n(g_r,\eta)_Y-p_n(g_r,\xi)_Y)\geqq \tag{19}$$

$$\geqq\|p_n(g_r)_Y\|-nh\|p_n(g_r)_Y\|\geqq\frac{3}{4}\|p_n(g_r)_Y\|.$$

Moreover $x_{N-1}\in Y$, thus by (18) and (19) we have

$$\|g_r\|-\frac{4}{3}\mu_r\leqq|(g_r-p_n(g_r)_Y)(x_{N-1})|\leqq\|g_r\|-\Delta_r+\mu_r$$

i.e.

$$\mu_r\geqq\frac{3}{7}\Delta_r. \tag{20}$$

Further, (19) and (17) imply that

$$\|p_n(g_r)_Y\|\leqq\frac{4}{3}\mu_r\leqq\frac{2}{3}\Delta_r\leqq C_6n^{-r-1}(nd)^{s(r)},$$

where $s(0)=1$, $s(r)=2$ $(r\geqq1)$, while

$$\|g_r\|-\Delta_r\geqq C_7n^{-r-1}-C_8n^{-r-1}(nd)^{s(r)}\geqq C_7n^{-r-1}(1-C_9(nd)^{s(r)}).$$

Thus

$$\|p_n(g_r)_Y\|<\|g_r\|-\Delta_r<E_n(g_r)_Y$$

for $nd<C_{10}$.

But $g_r\leqq0$ on $[2\pi-h,2\pi]$, hence $g_r-p_n(g_r)_Y<E_n(g_r)_Y$ while $x\in[2\pi-h,2\pi]$. Therefore

$$(g_r-p_n(g_r)_Y)(t)=E_n(g_r)_Y$$

for some $t\in[0,2\pi-h]\cap Y$. Using that $|g_r|\leqq\|g_r\|-\Delta_r$ on $[0,2\pi-h]\cap Y$, and (18) we obtain

$$p_n(g_r,t)_Y=g_r(t)-E_n(g_r)_Y\leqq\|g_r\|-\Delta_r-\|g_r\|+\Delta_r-\mu_r=-\mu_r. \tag{21}$$

Evidently for some even j $(0 \leqq j \leqq N-2)$, $|t-x_j| \leqq h$, hence (19) and (21) yield

$$p_n(g_r, x_j)_Y \leqq p_n(g_r, t)_Y + |p_n(g_r, x_j)_Y - p_n(g_r, t)_Y| \leqq$$
$$\leqq -\mu_r + nh \| p_n(g_r)_Y \| \leqq -\mu_r + \mu_r/3 = -2\mu_r/3.$$

Thus combining (16), (20) and (17) we have

$$(g_r - p_n(g_r)_Y)(x_j) \geqq \|g_r\| + 2\mu_r/3 \geqq E_n(g_r) + 2\Delta_r/7 \geqq$$
$$\geqq E_n(g_r) + C_{11}(r) \begin{cases} d^r \omega(g_r^{(r)}, d), & \text{for} \quad r = 0 \text{ or } 1; \\ n^{-r+2} \omega(g_r^{(r)}, n^{-1}) \, d^2, & \text{for} \quad r \geqq 2. \end{cases}$$

The proof of Theorem 2 is complete.

REMARK. Let $C[-1, 1]$ be the space of real-valued continuous functions on $[-1, 1]$, P_n the set of algebraic polynomials of order at most n. Then taking a finite subset $Y \subset [-1, 1]$ we can consider an analogous problem in the algebraic case. As a corollary from Theorem 1 we obtain that for any $f \in C[-1, 1]$ and $Y \subset [-1, 1]$ with $0 < \bar{\varrho}(Y) \leqq n^{-1}$

$$\|f - p_n^*(f)_Y\|_* \leqq E_n(f)_* + C_{12} \omega(f, \bar{\varrho}(Y)),$$

where $\bar{\varrho}(Y) = \max_{x \in [-1, 1]} \min_{y \in Y} |\cos^{-1} x - \cos^{-1} y|$, C_{12} is an absolute constant, and the stars indicate that the corresponding notations are introduced with Čebysev norm on $[-1, 1]$.

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A NOTE ON REFLEXIVE A_n-MODULES

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Let k be a field of characteristic zero. The Weyl algebra $A_n = A_n(k)$ is the associative k-algebra with 1 generated by the $2n$ elements $x_1, y_1, \ldots, x_n, y_n$ subject to:

$$x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i, \quad \text{and} \quad x_i y_j - y_j x_i = \delta_{ij}.$$

Our aim in this short note is to characterize reflexive A_n-modules. Reflexive ideals of A_n have been dealt with in [1], [2] and [3]. We begin with making some observations. Let R be any ring, M an R-modules. The abelian group $\mathrm{Hom}_R(M, R)$ endowed with the usual R-module structure is denoted by M^*. If the natural homomorphism $\theta\colon M \to M^{**}$ is a monomorphism we call M torsion-less, and if θ is an isomorphism M is called refiexive. Now suppose that R is a Noetherian integral domain with full ring of quotients Q and let I be a non-zero finitely generated left (right) R submodule of Q. Then I^* is naturally isomorphic with I^{-1} where $I^{-1} = \{q \in Q\colon Iq \subseteqq R\}$ $(I^{-1} = \{q \in Q\colon qI \subseteqq R\})$. Thus we shall identify I^{-1} with I^* and so $I \subseteqq I^{**}$, $I^* = I^{***}$.

Let now M be a finitely generated left R-module and suppose that R has a two-sided partial quotient ring R_S. Then there is an isomorphism of right R_S-modules

$$\sigma\colon \mathrm{Hom}_R(M, R) \otimes R_S \to \mathrm{Hom}_{R_S}(R_S \otimes M, R_S),$$

given by $\sigma(f \otimes s_1^{-1})(s^{-1} \otimes m) = s^{-1} f(m) s_1^{-1}$, where $f \in \mathrm{Hom}_R(M, R)$, $m \in M$ and $s_1, s \in S$. Consequently

$$R_S \otimes M^{**} \simeq (R_S \otimes M)^{**}. \tag{1}$$

Throughout n is a fixed positive integer and W will denote $A_n(k)$. For each $i = 1, \ldots, 2n$, let B_i be the $2n-1$ element subset obtained by deleting the i-th element of $x_1, y_1, \ldots, x_n, y_n$ and let S_i be the set of nonzero elements of the subalgebra (with 1) of W generated by B_i. Then W has a two-sided partial quotient ring W_{S_i} which is a principal ideal domain [2]. We shall need the following lemma which is proved in [1, Lemma 4.2].

LEMMA. *Let I be a non-zero left ideal of W. Then $I^* = A_n$ if and only if $I \cap S_i \neq \varnothing$ for $i = 1, \ldots, 2n$.*

We are now ready to state and prove:

THEOREM. *Let M be a finitely generated left W-module. The following statements are equivalent.*

(1) *M is reflexive.*

(2) *There is a finitely generated free left W-module F containing M with the property that* $\mathrm{Hom}(W/I, F/M) = 0$ *whenever I is a left ideal of W with $I^* = W$.*

(3) *M is torsion-free and* $\mathrm{Ext}^1(W/I, M) = 0$ *for any left ideal I of W for which $I^* = W$.*

PROOF. (2)⇒(1). Suppose that F as described in (2) exists. Then M, being a submodule of a free module, is torsion-free and hence torsion-less since W is a Noetherian integral domain. Consider the exact sequence $0\to M\to M^{**}\to \text{Coker}\,\theta\to 0$. Let $N=\text{Coker}\,\theta$; we must show that $N=0$. Let W_i $(1\leqq i\leqq 2n)$ denote the localisation W_{S_i} and tensor the above sequence with W_i. Since W_i is hereditary, torsionless W_i-modules are reflexive and thus by (1) we obtain $W_i\otimes N=0$, $i=1, \ldots, 2n$. Suppose that there is a non-zero element $u\in N$ and let I be the left annihilator of u. Clearly $I\cap S_i\neq\emptyset$, hence by Lemma $I^*=W$. The exact sequence $0\to M\to M^{**}\to$ $\to N\to 0$ yields

$$\text{Hom}\,(W/I, M^{**}) \to \text{Hom}\,(W/I, N) \to \text{Ext}^1\,(W/I, M),$$

in which the first term is zero since M^{**} is torsion-free as W is a domain. Clearly $\text{Hom}\,(W/I, N)\neq 0$, and we reach a contradiction by showing that $\text{Ext}^1\,(W/I, M)=0$. Now from $0\to M\to F\to F/M\to 0$ we obtain the exact sequence

$$\text{Hom}\,(W/I, F/M) \to \text{Ext}^1\,(W/I, M) \to \text{Ext}^1\,(W/I, F),$$

in which the first term vanishes by (2). Since F is finitely generated free over W we have:

$$\text{Ext}^1\,(W/I, F) \cong \otimes\,\text{Ext}^1\,(W/I, W) \cong \otimes\, I^*/W = 0.$$

Thus $\text{Ext}^1\,(W/I, M)=0$, and it follows that M is reflexive.

(1)⇒(2). Since M is reflexive there is an exact sequence, $0\to M\to F\to G$, where F and G are some finitely generated free left W-modules. Thus for any non-zero left ideal I of W, and in particular for I with $I^*=W$, we have $\text{Hom}\,(W/I, F/M)=0$.

The equivalence of (3) and (2) is easy.

By taking M, in the Theorem, to be a left ideal we have:

COROLLARY. *The following statements are equivalent for a non-zero left ideal I of W.*

(1) *is reflexive.*

(2) $\text{Hom}\,(W/J, W/I)=0$ *for all left ideals J of W with $J^*=W$.*

(3) *If J is a left ideal of W for which $J^*=W$, then* $\text{Ext}^1\,(W/J, I)=0$.

PROOF. We only show that (1)⇒(2). Suppose there exists a left ideal J such that $J^*=W$ and $\text{Hom}\,(W/J, W/I)\neq 0$. Then there is an element $w\in W/I$ for which $Jw\subseteq I$. This gives $w^{-1}J^*\supseteq I^*$, hence $J^{**}w\subseteq I^{**}$. But $I=I^{**}$ and $J^{**}=W$ by the assumptions; so we get the contradiction $w\in I$.

Finally the Theorem gives the following result.

COROLLARY. *Let $0\to A\to B\to C\to 0$ be an exact sequence with A and C finitely generated reflexive W-modules. Then B is reflexive.*

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SUR CERTAINES SUITES PSEUDO-ALÉATOIRES. II

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Introduction

Dans l'article [1] de même titre, dont nous reprenons les notations et définitions, nous avons étudié la corrélation de certaines suites arithmétiques $(g(n))_{n\in\mathbf{N}}$ définies de la manière suivante:

$$g(n)=e(f(n)) \quad \text{où} \quad f(n)=\sum_{k=1}^{+\infty} b_k\left[\frac{n}{s_k}\right],$$

$(b_k)_{k\in\mathbf{N}^*}$ désignant une suite de nombres réels et $(s_k)_{k\in\mathbf{N}^*}$ une suite de nombres réels >0 telle que: $\sum_{k=1}^{+\infty}\frac{1}{s_k}<+\infty$.

Nous avons obtenu en particulier le résultat suivant:

TÉHORÈME 1. *Dans le cas où les nombres s_k sont des entiers naturels $\geqq 2$ et deux à deux premiers entre eux, la suite g est pseudo-aléatoire si et seulement si* $\sum_{k=1}^{+\infty}\|b_k\|^2=+\infty$.

Nous nous proposons ici de généraliser ce résultat dans deux directions: d'une part, remplacer les entiers s_k par des nombres rationnels soumis à certaines conditions, d'autre part, remplacer la suite g par une suite h de la forme:

$$h(n)=g(un)\,g'(vn),$$

où g et g' sont des suites du type décrit plus haut et où u et v sont des entiers naturels distincts. Plus précisément, nous obtenons le:

THÉORÈME 2. *Soient $(s_k)_{k\in\mathbf{N}^*}$ une suite d'entiers $\geqq 2$, deux à deux premiers entre eux, $(\sigma_k)_{k\in\mathbf{N}^+}$ une suite d'entiers $\geqq 1$ tels que $(s_k,\sigma_k)=1$ pour tout $k\in\mathbf{N}^*$, $(b_k)_{k\in\mathbf{N}^*}$ et $(c_k)_{k\in\mathbf{N}^*}$ deux suites de nombres réels, u et v deux entiers $\geqq 1$ et distinct. On suppose que:*

$$\sum_{k=1}^{+\infty}\frac{\sigma_k}{s_k}<+\infty.$$

On pose $f(n)=\sum_{k=1}^{+\infty} b_k\left[\frac{n\sigma_k}{s_k}\right]$ *et* $f'(n)=\sum_{k=1}^{+\infty} c_k\left[\frac{n\sigma_k}{s_k}\right]$ *et on définit les suites g, g' et h par:* $\forall n\in\mathbf{N},\ g(n)=e(f(n)),\ g'(n)=e(f'(n))$ *et* $h(n)=g(un)\,g'(vn)$.

Les trois assertions suivantes sont équivalentes:

(A) *h est pseudo-aléatoire,*

(B) *l'une au moins des suites g et g' est pseudo-aléatoire,*

(C) $\sum_{k=1}^{+\infty} \|b_k\|^2 = +\infty$ *ou* $\sum_{k=1}^{+\infty} \|c_k\|^2 = +\infty.$

REMARQUES. 1. La motivation du théorème 2 réside dans un résultat d'équirépartition modulo 1 dû à HALÁSZ et VAUGHAN [2] et dont l'idée revient essentiellement à Daboussi:

PROPOSITION. *Soit* $(f(n))_{n\in\mathbf{N}}$ *une suite de réels et* $\mathscr{E}$ *un ensemble de nombres premiers tel que* $\sum_{p\in\mathscr{E}} \frac{1}{p} = +\infty$. *Si, quels que soient u et v appartenant à* $\mathscr{E}$ *et distincts, la suite* $(f(un)-f(vn))_{n\in\mathbf{N}}$ *est équirépartie modulo* 1, *alors la suite* $(f(n)+\lambda(n))_{n\in\mathbf{N}}$ *est équirépartie modulo* 1 *quelle que soit la suite additive* λ.

2. Il est intéressant de remarquer que g et g' jouent dans le théorème 2 des rôles «indépendants» alors que, si l'on pose $l(n)=g(n+u)g'(n+v)$, l est pseudo-aléatoire si et seulement si (voir [1]): $\sum_{k=1}^{+\infty} \|b_k+c_k\|^2 = +\infty$, ce qui fait que g et g' peuvent être pseudo-aléatoires sans que l le soit.

3. Enfin, on peut généraliser le théorème 2 à un produit de plus de deux suites $(g_i(u_i n))_{n\in\mathbf{N}}$ où les entiers u_i sont distincts.

Preuve du théorème 2

D'après le théorème 1, il suffit de prouver l'équivalence de (A) et (C). Nous conviendrons que $u>v$.

La démonstration de l'existence de la corrélation γ (resp. γ') de g (resp. g') est faite dans [1]. De la même manière, on prouve l'existence de la corrélation χ de h.

Pour tout $k\in\mathbf{N}^*$, on pose $\varphi_k(n)=e\left(b_k\left[\frac{n\sigma_k u}{s_k}\right]+c_k\left[\frac{n\sigma_k v}{s_k}\right]\right)$.

Pour tout $r\in\mathbf{N}^*$, on pose $h_r(n)=\prod_{k=1}^{r} \varphi_k(n)$.

On constate que h_r a une corrélation χ_r de période $s_1, \dots, s_r$ et on montre comme dans l'article [1], que: $\chi(t)=\lim_{r\to\infty} \chi_r(t)$ pour tout $t\in\mathbf{N}$.

D'après les hypothèses faites, pour k assez grand (disons $k>\varrho$),

$$(s_k, u) = (s_k, v) = 1 \quad \text{et} \quad s_k > u > v \quad \text{et} \quad s_k > \frac{uv}{u-v}.$$

Posons $h'(n) = \frac{h(n)}{h_\varrho(n)} = \prod_{k>\varrho} \varphi_k(n)$ et, pour $r>\varrho$, $h'(n) = \frac{h_r(n)}{h_\varrho(n)} = \prod_{\varrho<k\leqq r} \varphi_k(n)$.

Soit χ' la corrélation de h' et χ'_r celle de h'_r. La suite $(h'_r(n+t)\overline{h'_r(n)})_{n\in\mathbf{N}}$ a pour période $\prod_{\varrho<k\leqq r} s_k$ et la suite $(h_\varrho(n+t)\overline{h_\varrho(n)})_{n\in\mathbf{N}}$ a pour période $\prod_{k\leqq\varrho} s_k$.

Ces périodes étant premières entre elles, on a, pour tout $t\in\mathbf{N}$, $\chi_r(t)=\chi_\varrho(t)\chi'_r(t)$ et, par passage à la limite sur r: $\chi(t)=\chi_\varrho(t)\chi'(t)$. Puisque $|\chi(t)|\leqq|\chi'(t)|$, h est pseudo-aléatoire dès que h' l'est. Réciproquement, puisque les périodes de χ_ϱ et χ'_r sont premières entre elles,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n<N}|\chi_r(t)|^2=\lim_{N\to\infty}\frac{1}{N}\sum_{n<N}|\chi_\varrho(t)|^2\lim_{N\to\infty}\frac{1}{N}\sum_{n<N}|\chi'_r(t)|^2.$$

Il en résulte, compte tenu de $\chi_\varrho(0)=1$ que:

$$\lim_{N\to\infty}\sup\frac{1}{N}\sum_{n<N}|\chi'(t)|^2\leqq\lim_{N\to\infty}\frac{1}{N}\sum_{n<N}|\chi'_r(t)|^2\leqq\Big(\prod_{k\leqq\varrho}s_k\Big)\lim_{N\to\infty}\frac{1}{N}\sum_{n<N}|\chi_r(t)|^2.$$

Or, nous verrons au paragraphe II.5.3), que, lorsque h est pseudoaléatoire, $\lim_{r\to\infty}\left(\lim_{N\to\infty}\frac{1}{N}\sum_{n<N}|\chi_r(t)|^2\right)=0.$

Finalement, h et h' sont simultanément pseudo-aléatoires. *A partir de maintenant,* nous ferons l'hypothèse: $(s_k,u)=(s_k,v)=1$ et $s_k>u>v$ et $s_k>\dfrac{uv}{u-v}$, pour tout $k\in\mathbf{N}^*$.

Soit ω_k la corrélation de $\varphi_k\cdot\omega_k$ a pour période s_k. D'après l'hypothèse faite sur les s_k, on a: $\chi_r(t)=\prod_{k=1}^{r}\omega_k(t)$ pour tout $t\in\mathbf{N}$ et, $\lim_{n\to\infty}\frac{1}{N}\sum_{t<N}|\chi_r(t)|^2=\prod_{k=1}^{r}\mu_k$ où

$$\mu_k=\frac{1}{s_k}\sum_{0\leqq n<s_k}|\omega_k(t)|^2. \tag{1}$$

D'autre part, $\omega_k(t)=\dfrac{1}{s_k}\sum_{0\leqq n<s_k}\varphi_k(n+t)\,\overline{\varphi_k(n)}$ donc,

$$\omega_k(t)=\frac{1}{s_k}e\left(b_k\left[\frac{\sigma_k ut}{s_k}\right]+c_k\left[\frac{\sigma_k vt}{s_k}\right]\right)\big(A_k(t)+B_k(t)\,e(b_k)+C_k(t)\,e(c_k)+D_k(t)e(b_k)e(c_k)\big) \tag{2}$$

où

$$A_k(t)=\mathrm{Card}\left\{n|0\leqq n<s_k,\left\{\frac{un\sigma_k}{s_k}\right\}<1\left\{\frac{ut\sigma_k}{s_k}\right\}\ \text{ et }\ \left\{\frac{vn\sigma_k}{s_k}\right\}<1-\left\{\frac{vt\sigma_k}{s_k}\right\}\right\},$$

$$B_k(t)=\mathrm{Card}\left\{n|0\leqq n<s_k,\left\{\frac{un\sigma_k}{s_k}\right\}\geqq1\left\{\frac{ut\sigma_k}{s_k}\right\}\ \text{ et }\ \left\{\frac{vn\sigma_k}{s_k}\right\}<1-\left\{\frac{vt\sigma_k}{s_k}\right\}\right\},$$

$$C_k(t)=\mathrm{Card}\left\{n|0\leqq n<s_k,\left\{\frac{un\sigma_k}{s_k}\right\}<1-\left\{\frac{ut\sigma_k}{s_k}\right\}\ \text{ et }\ \left\{\frac{vn\sigma_k}{s_k}\right\}\geqq1-\left\{\frac{vt\sigma_k}{s_k}\right\}\right\},$$

$$D_k(t)=\mathrm{Card}\left\{n|0\leqq n<s_k,\left\{\frac{un\sigma_k}{s_k}\right\}\geqq1-\left\{\frac{ut\sigma_k}{s_k}\right\}\ \text{ et }\ \left\{\frac{vn\sigma_k}{s_k}\right\}\geqq1-\left\{\frac{vt\sigma_k}{s_k}\right\}\right\}.$$

Un calcul simple montre que:

(3) $$|\omega_k(t)|^2 = 1-\frac{4}{s_k^2}\big((A_k(t)B_k(t)+C_k(t)D_k(t))\sin^2\pi b_k+(A_k(t)C_k(t)+$$

$$+B_k(t)D_k(t))\sin^2\pi c_k+A_k(t)D_k(t)\sin^2\pi(b_k+c_k)+B_k(t)C_k(t)\sin^2\pi(b_k-c_k)\big).$$

Preuve De (A)⇒(C). On fait l'hypothèse que $\sum_{k=1}^{+\infty}(\|b_k\|^2+\|c_k\|^2)<+\infty$ et on montre que h n'est pas pseudo-aléatoire.

D l'égalité (3), il résulte que:

(4) $$1 \geqq |\omega_k(t)|^2 \geqq 1-K(\|b_k\|^2+\|c_k\|^2)$$

où K est une constante absolue >0. Par conséquent:

(5) $$0 \leqq |\chi_r(t)|^2-|\chi(t)|^2 \leqq 1-\prod_{k>r}|\omega_k(t)|^2 \leqq K\sum_{k>r}(\|b_k\|^2+\|c_k\|^2).$$

La relation précédente, compte tenu de la convergence de $\sum_{k=1}^{+\infty}(\|b_k\|^2+\|c_k\|^2)$, montre que $|\chi|^2$ est limite uniforme de suites périodiques.

Donc $|\chi|^2$ a une valeur moyenne $\mathcal{M}(|\chi|^2)$ qui est égale à:

$$\lim_{r\to\infty}\mathcal{M}(|\chi_r|^2) = \lim_{r\to\infty}\left(\prod_{k=1}^{r}\mu_k\right).$$

Les relations (1) et (4) donnent:

$$0 \leqq 1-\mu_k \leqq K(\|b_k\|^2+\|c_k\|^2),$$

de sorte que le produit infini $\prod_{k=1}^{+\infty}\mu_k$ converge. Et, puisque $\mu_k>0$ pour tout $k\in\mathbf{N}^*$, $\mathcal{M}(|\chi|^2)>0$.

Preuve De (C)⇒(A). De la relation (3) il résulte que:

(6)

$$\mu_k \leqq 1-\frac{K'}{s_k^3}\Big(\|b_k\|^2\sum_{0\leqq t<s_k}\big(A_k(t)B_k(t)+C_k(t)D_k(t)\big)+\|c_k\|^2\big(A_k(t)C_k(t)+B_k(t)D_k(t)\big)\Big)$$

où K' est une constante absolue positive.

1. *Minoration de* $\sum_{0\leqq t<s_k}A_k(t)B_k(t)$. Puisque $(u\sigma_k, s_k)=(v\sigma_k, s_k)=1$,

$$\sum_{0\leqq t<s_k}A_k(t)B_k(t) = \sum_{0\leqq t<s_k}A_k^*(t)B_k^*(t)$$

où

$$A_k^*(t) = \text{Card}\left\{m \,|\, 0 \leqq m < s_k, \left\{\frac{um}{s_k}\right\} < 1 - \left\{\frac{ut}{s_k}\right\} \text{ et } \left\{\frac{vm}{s_k}\right\} < 1 - \left\{\frac{vt}{s_k}\right\}\right\},$$

$$B_k^*(t) = \text{Card}\left\{m \,|\, 0 \leqq m < s_k, \left\{\frac{um}{s_k}\right\} \geqq 1 - \left\{\frac{ut}{s_k}\right\} \text{ et } \left\{\frac{vm}{s_k}\right\} < 1 - \left\{\frac{vt}{s_k}\right\}\right\}.$$

Pour $t < \frac{s_k}{u}$,

$$A_k^*(t) = \text{Card}\left\{m \,|\, 0 \leqq m < s_k, \left\{\frac{um}{s_k}\right\} < 1 - \frac{ut}{s_k} \text{ et } \left\{\frac{vm}{s_k}\right\} < 1 - \frac{vt}{s_k}\right\} \geqq$$

$$\geqq \text{Card}\{m \geqq 0 \,|\, um < s_k - ut \text{ et } vm < s_k - vt\} \geqq \frac{s_k}{u} - t.$$

Pour $t < \frac{s_k}{u}$,

$$B_k^*(t) = \text{Card}\left\{m \,|\, 0 \leqq m < s_k, \left\{\frac{um}{s_k}\right\} \geqq 1 - \frac{ut}{s_k} \text{ et } \left\{\frac{vm}{s_k}\right\} < 1 - \frac{vt}{s_k}\right\} \geqq$$

$$\geqq \text{Card}\{m \geqq 0 \,|\, s_k - ut \leqq um < s_k \text{ et } vm < s_k - vt\} =$$

$$= \text{Card}\left\{m \geqq 0 \,|\, m < \frac{s_k}{u} \text{ et } \frac{s_k}{u} - t \leqq m < \frac{s_k}{v} - t\right\} \geqq \text{Inf}\left(t, \left[\frac{s_k}{v} - \frac{s_k}{u}\right]\right).$$

Donc

$$\sum_{0 \leqq t < s_k} A_k^*(t) B_k^*(t) \geqq \sum_{0 \leqq t < \frac{s_k}{u}} \left(\frac{s_k}{u} - t\right) . \text{Inf}\left(t, \left[\frac{s_k}{v} - \frac{s_k}{u}\right]\right) \geqq \sum_{0 \leqq t < \text{Inf}\left(\frac{s_k}{u}, \left[\frac{s_k}{v} - \frac{s_k}{u}\right]\right)} t\left(\frac{s_k}{u} - t\right),$$

de sorte qu'il existe une constante $C(u, v) > 0$ telle que:

(7) $$\sum_{0 \leqq t < s_k} A_k(t) B_k(t) \geqq C(u, v) s_k^3.$$

2. *Minoration de* $\sum_{0 \leqq t < s_k} B_k(t) D_k(t)$. $\sum_{0 \leqq t < s_k} B_k(t) D_k(t) = \sum_{0 \leqq t < s_k} B_k^*(t) D_k^*(t)$ où

$$D_k^*(t) = \text{Card}\left\{m \,|\, 0 \leqq m < s_k, \left\{\frac{um}{s_k}\right\} \geqq 1 - \left\{\frac{ut}{s_k}\right\} \text{ et } \left\{\frac{vm}{s_k}\right\} \geqq 1 - \left\{\frac{vt}{s_k}\right\}\right\}.$$

Pour $t < \frac{s_k}{u}$,

$$D_k^*(t) = \text{Card}\left\{m \,|\, 0 \leqq m < s_k, \left\{\frac{um}{s_k}\right\} \geqq 1 - \frac{ut}{s_k} \text{ et } \left\{\frac{vm}{s_k}\right\} \geqq 1 - \frac{vt}{s_k}\right\} \geqq$$

$$\geqq \text{Card}\{m \,|\, s_k - t \leqq m < s_k\} = t.$$

Donc

$$\sum_{0 \leqq t < s_k} B_k^*(t) D_k^*(t) \geqq \sum_{0 \leqq t < \frac{s_k}{u}} t \,\text{Inf}\left(t, \left[\frac{s_k}{v} - \frac{s_k}{u}\right]\right) \geqq \sum_{0 \leqq t < \text{Inf}\left(\frac{s_k}{u}, \left[\frac{s_k}{u} - \frac{s_k}{u}\right]\right)} t^2,$$

de sorte qu'il existe une contante $C'(u, v) > 0$ telle que:

$$\sum_{0 \leqq t < s_k} B_k(t) D_k(t) \geqq C'(u, v) s_k^3. \tag{8}$$

3. *Fin de la preuve.* D'après les relations (6), (7) et (8), il existe une constante $C''(u, v) > 0$ (indépendante de k) telle que: $\mu_k \leqq 1 - C''(u, v)(\|b_k\|^2 + \|c_k\|^2)$. Il en résulte que, pour tout $r \in \mathbf{N}^*$

$$\overline{\lim_{N \to \infty}} \frac{1}{N} \sum_{t > N} |\chi(t)|^2 \leqq \lim_{N \to \infty} \frac{1}{N} \sum_{t < N} |\chi_r(t)|^2 \leqq \prod_{k=1}^{r} \left(1 - C''(u, v)(\|b_k\|^2 + \|c_k\|^2)\right).$$

La divergence de $\sum_{k=1}^{+\infty} (\|b_k\|^2 + \|c_k\|^2)$ entraine que: $\overline{\lim_{N \to \infty}} \frac{1}{N} \sum_{t < N} |\chi(t)|^2 = 0$, ce qui signifie que h est pseudo-aléatoire.

Compléments

Dans l'article [1], nous avions étudié aussi les suites du type g dans le cas où la suite $(s_k)_{k \in \mathbf{N}^*}$ est formée de réels dont les inverses sont $\mathbf{Q}$-linéairement indépendants. Un énoncé analogue au théorème 2 vaut pour ces suites:

THÉORÈME 3. *Soit $(s_k)_{k \in \mathbf{N}^*}$ une suite de nombres irrationnels positifs telle que les nombres $\frac{1}{s_k}$, $k \in \mathbf{N}^*$, soient $\mathbf{Q}$-linéairement indépendants et que*

$$\sum_{k=1}^{+\infty} \frac{1}{s_k} < +\infty.$$

Soient $(b_k)_{k \in \mathbf{N}^*}$ et $(c_k)_{k \in \mathbf{N}^*}$ deux suites de nombres réels, u et v deux entiers $\geqq 1$ distincts. On pose, pour tout $n \in \mathbf{N}$:

$$g(n) = e\left(\sum_{k=1}^{+\infty} b_k \left[\frac{n}{s_k}\right]\right), \quad g'(n) = e\left(\sum_{k=1}^{+\infty} c_k \left[\frac{n}{s_k}\right]\right) \quad et \quad h(n) = g(un)\, g'(vn).$$

Les trois assertions suivantes sont équivalentes:
(A′) *h est pseudo-aléatoire*
(B′) *L'une au moins des suites g et g' est pseudo-aléatoire*

(C′) $\sum_{k=1}^{+\infty} \|b_k\|^2 = +$ *ou* $\sum_{k=1}^{+\infty} \|c_k\|^2 = +\infty.$

L'équivalence entre (B′) et (C′) est prouvée dans [1]. La preuve de (A′)⇔(C′) est analogue à celle de (A)⇔(C).

On pose $\varphi_k(n) = e\left(b_k \left[\frac{nu}{s_k}\right] + c_k \left[\frac{nv}{s_k}\right]\right)$ et, pour tout $r \in \mathbf{N}^*$,

$$h_r(n) = \prod_{1 \leqq k \leqq r} \varphi_k(n).$$

h (resp. h_r) a une corrélation χ (resp. χ_r) et $\chi(t)=\lim_{r\to\infty}\chi_r(t)$ pour tout $t\in\mathbf{N}$. Comme dans la preuve du théorème 2, on peut supposer que: $s_k>u>v$ pour tout $k\in\mathbf{N}^*$.

L'hypothèse faite sur les nombres $\frac{1}{s_k}$ permet d'affirmer (voir [1]) que: $\chi_r(t)= =\prod_{1\leqq k\leqq r}\omega_k(t)$ pour tout $t\in\mathbf{N}$, ω_k désignant la corrélation de φ_k.

On obtient

$$\omega_k(t)=\frac{1}{s_k}\varphi_k(t)\big(\alpha_k(t)+\beta_k(t)e(b_k)+\gamma_k(t)e(c_k)+\delta_k(t)e(b_k)e(c_k)\big)$$

avec

$$\alpha_k(t)=d\left\{n\in\mathbf{N}\,\middle|\,\left\{\frac{un}{s_k}\right\}<1-\left\{\frac{ut}{s_k}\right\}\text{ et }\left\{\frac{vn}{s_k}\right\}<1-\left\{\frac{vt}{s_k}\right\}\right\}$$

et des notations évidentes pour $\beta_k(t)$, $\gamma_k(t)$ et $\delta_k(t)$.

On établit facilement une inégalité analogue à (4) et ainsi l'implication (A')⇒(C'). La preuve de (C')⇒(A') est basée sur la minoration des quantités $\mathscr{M}(\alpha_k\beta_k)= =\lim_{N\to\infty}\frac{1}{N}\sum_{t<N}\alpha_k(t)\beta_k(t)$ et $\mathscr{M}(\beta_k\delta_k)=\lim_{N\to\infty}\frac{1}{N}\sum_{t<N}\beta_k(t)\delta_k(t)$.

Par exemple, $\alpha_k(t)\geqq\operatorname{Inf}\left(\frac{1}{u}\left(1-\left\{\frac{ut}{s_k}\right\}\right),\ \frac{1}{v}\left(1-\left\{\frac{vt}{s_k}\right\}\right)\right)$ de sort que, si $\left\{\frac{t}{s_k}\right\}< <\frac{1}{u}$, $\alpha_k(t)\geqq\operatorname{Inf}\left(\frac{1}{u}-\left\{\frac{t}{s_k}\right\},\ \frac{1}{v}-\left\{\frac{t}{s_k}\right\}\right)=\frac{1}{u}-\left\{\frac{t}{s_k}\right\}$. Et si $\left\{\frac{t}{s_k}\right\}<\frac{1}{u}$,

$$\beta_k(t)=d\left\{m\in\mathbf{N}\,\middle|\,\left\{\frac{um}{s_k}\right\}\geqq1-u\left\{\frac{t}{s_k}\right\}\text{ et }\left\{\frac{vm}{s_k}\right\}<1-v\left\{\frac{t}{s_k}\right\}\right\}\geqq$$

$$\geqq d\left\{m\in\mathbf{N}\,\middle|\,\left\{\frac{m}{s_k}\right\}<\frac{1}{u}\text{ et }\frac{1}{u}-\left\{\frac{t}{s_k}\right\}\leqq\left\{\frac{m}{s_k}\right\}<\frac{1}{v}-\left\{\frac{t}{s_k}\right\}\right\}\geqq\operatorname{Inf}\left(\frac{1}{v}-\frac{1}{u},\left\{\frac{t}{s_k}\right\}\right).$$

On obtient alors

$$\mathscr{M}(\alpha_k\beta_k)\geqq\int_0^{1/u}\left(\frac{1}{u}-x\right)\operatorname{Inf}\left(x,\frac{1}{v}-\frac{1}{u}\right)dx=C^*(u,v)>0.$$

On majore alors $|\chi|^2$ en moyenne comme dans l'inégalité (9).

En reprenant les notations du théorème 2, on peut établir le résultat suivant:

THÉORÈME 4. *Soient u^* et v^* deux nombres réels >0 distincts. On pose $h^*(n)= =g([u^*n])g'([v^*n])$. h^* est pseudo-aléatoire si et seulement si*

$$\sum_{k=1}^{+\infty}(\|b_k\|^2+\|c_k\|^2)=+\infty.$$

La démonstration est basée sur l'indépendance statistique des ensembles $E_k = \left\{n\in\mathbf{N}\left|\left\{\frac{[u^* n]\sigma_k}{s_k}\right\}\leqq x_k\right.\right\}$ entre eux ainsi que des ensembles $E'_j = \left\{n\in\mathbf{N}\left|\left\{\frac{[v^* n]\sigma_j}{s_j}\right\}\leqq y_j\right.\right\}$ entre eux.

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ON COCYCLIC MODULES

By
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1. Introduction

The object of this paper is to study some properties of cocyclic modules. We mainly follow [2] for notation and terminology. Thus the symbol R throughout the paper denotes a ring with identity. R is not assumed to be commutative, unless otherwise specified. By an R-module we always mean a unitary left R-module. When the ring R is understood, we call an R-module simply a module.

The notion of a cocyclic module as the dual of a cyclic module was first introduced by MARANDA [1]. In [4], it was introduced the concept of a finitely cogenerated abelian group as dual to that of a finitely generated abelian group, and showed that a finitely cogenerated abelian group is a direct product of a finite number of cocyclic abelian groups. P. VÁMOS [3] calls a module finitely embedded if it is a submodule of a direct product of a finite number of injective cocyclic modules. Finitely embedded modules serve as dual of finitely generated modules, and the class of finitely embedded abelian groups is identical with that of finitely cogenerated abelian groups. Cocyclic modules play a fundamental role in the theory of modules, particularly of injective modules.

We recall that a module M is cocyclic if it is cogenerated by a single non-zero element, i.e., there exists a non-zero element x in M such that every non-zero submodule of M contains x. It is clear that the cyclic submodule generated by x is a simple submodule of M. Thus a module is cocyclic if and only if it is an essential extension of a simple module. Moreover, a cocyclic module is always a submodule of an injective cocyclic module, which is its maximal essential extension.

The class of cocyclic abelian groups is completely known. An abelian group A is cocyclic if and only if $A\cong Z(p^k)$ with $k=1, 2, \ldots$ or ∞, where p is a prime. Any element of order p is a cogenerator of A. Note that a subgroup and a non-zero factor group of a cocyclic abelian group are again cocyclic.

In this paper we study cocyclic modules over an arbitrary ring R. In section 2 we discuss some of their elementary properties. Among other results it is shown that a cocyclic module is isomorphic to a submodule of $\mathrm{Hom}_Z(R, Z(p^\infty))$ for some prime p. We observe that a submodule of a cocyclic module is also a cocyclic module. However, in general, a non-zero factor module of a cocyclic module need not be cocyclic. In section 3 we describe the structure of a module M, of which every non-zero homomorphic image is again a cocyclic module. We also characterize those commutative rings R for which every R-cocyclic module has this structure. In particular, it turns out that if R is a commutative integral domain, then every nonzero homomorphic image of a cocyclic R-module is again cocyclic if and only if the localization R_J is a discrete valuation ring for each non-zero maximal ideal J of R. The class of such integral domains includes that of Dedekind domains.

Proofs, which are either trivial or otherwise evident, have been omitted for the sake of brevity.

2. Some elementary results

First, we prove a lemma.

LEMMA 2.1. *Let J be a maximal left ideal of R. If there exists a least positive integer m such that $m1\in J$, then m is a prime.*

PROOF. If m is not a prime, then $m=m_1m_2$, where $m_1, m_2>1$. Now $m_2 1\notin J$. Consider the left ideal J' generated by J and $m_2 1$. Then $1\notin J'$, for if $1=x+m_2r$, where $x\in J$ and $r\in R$, then $m_1 1=m_1x+m_1m_2r=m_1x+mr=m_1x+r(m1)\in J$, which is a contradiction. Thus $J'\neq R$, contradicting the maximality of J.

COROLLARY 2.2. *Let M be a simple module. Then the order of each non-zero element of M, considered as an abelian group, is either infinite or a fixed prime.*

PROPOSITION 2.3. *Let M be a cocyclic module, cogenerated by a. If M, regarded as an abelian group, is a torsion group, then it is a p-group for some prime p and a is of order p.*

PROOF. Note that a p-component of M is also a submodule of M. Since M is an indecomposable module it has exactly one p-component for some fixed prime p. That a is of order p follows from Corollary 2.2.

COROLLARY 2.4. *If R is a ring of characteristic $m\neq 0$, then the cocyclic module M is a p-group for some prime factor p of m.*

COROLLARY 2.5. *If the cocyclic module M, regarded as an abelian group, has non-zero elements of finite order, then a is of order p for some prime p.*

PROOF. The torsion subgroup of M is a cocyclic submodule of M, cogenerated by a.

If G is an abelian group, then the abelian group $\mathrm{Hom}_Z(R, G)$ can be given the structure of an R-module: if $r\in R$ and $f\in \mathrm{Hom}_Z(R, G)$, we define $rf\in \mathrm{Hom}_Z(R, G)$ by

$$(rf)(r')=f(r'r),\quad r'\in R.$$

Furthermore, it can be shown that if G is injective, then $\mathrm{Hom}_Z(R, G)$ is an injective R-module (see 2; Lemma 2.13).

We shall now prove one of the main results of this section.

THEOREM 2.6. *Let M be a cocyclic module, then there exists a prime p such that M can be embedded in the module* $\mathrm{Hom}_Z(R, Z(p^\infty))$.

PROOF. Let a be a cogenerator of M. If a is of infinite order in M (considered as an abelian group), then we choose p to be any prime, otherwise we take p to be the order of a. In either case there exists an abelian group homomorphism θ from the cyclic group $\langle a\rangle$ to $Z(p^\infty)$, given by $\theta(a)=c$, where c is some element of order p in $Z(p^\infty)$. By the injectivity of $Z(p^\infty)$ it follows that θ can be extended to an abelian group homomorphism $\varphi\colon M\to Z(p^\infty)$. We define $\psi\colon M\to \mathrm{Hom}_Z(R, Z(p^\infty))$ by setting $(\psi(x))(r)=\varphi(rx)$, where $x\in M$ and $r\in R$. It can be easily verified that ψ is a module homomorphism. Moreover, $\psi(a)\neq 0$, for $(\psi(a))(1)=\varphi(1a)=\varphi(a)=c\neq 0$. Hence $a\notin \mathrm{Ker}\,\psi$, whence $\mathrm{Ker}\,\psi=0$. Thus ψ is a monomorphism.

We now identify M with its isomorphic image in $\mathrm{Hom}_Z(R, Z(p^\infty))$.

COROLLARY 2.7. *M is contained in an injective cocyclic direct summand of* $\mathrm{Hom}_Z(R, Z(p^\infty))$. *Moreover,* $\mathrm{Hom}_Z(R, Z(p^\infty))$ *is itself cocyclic if it is indecomposable.*

PROOF. Since $\mathrm{Hom}_Z(R, Z(p^\infty))$ is injective, it contains the injective envelope $E(M)$ of M, which is a direct summand of $\mathrm{Hom}_Z(R, Z(p^\infty))$.

COROLLARY 2.8. *Let p be a given prime. If R contains a maximal left ideal J such that either $p1\in J$ or no non-zero multiple of* 1 *belongs to J, then* $\mathrm{Hom}_Z(R, Z(p^\infty))$ *has an injective cocyclic direct summand.*

REMARK. If R is such that it satisfies the condition of Corollary 2.8 for each prime p and if $\mathrm{Hom}_Z(R, Z(p^\infty))$ is indecomposable, then the class of cocyclic R-modules is completely determined. It consists precisely of the modules $\mathrm{Hom}_Z(R, Z(p^\infty))$ and their submodules. An example of such a ring is Z itself, for $\mathrm{Hom}_Z(Z, Z(p^\infty))\cong Z(p^\infty)$.

In the rest of this section we examine the structure of the module $M=\mathrm{Hom}_Z(R, Z(p^\infty))$ for a given prime p. We assume that the choice of p is such that $M\neq 0$.

PROPOSITION 2.9. *The module* $M=\mathrm{Hom}_Z(R, Z(p^\infty))$ *is an essential extension of its submodule generated by those homomorphisms which send* 1 *to c, where c is a fixed cogenerator of* $Z(p^\infty)$.

PROOF. Let N be any submodule of M. We first show that if $\theta(1)=0$ for all $\theta\in N$, then $N=0$. Suppose $\theta(1)=0$ for all $\theta\in N$. Let $\varphi\in N$ and $r\in R$, then $\varphi(r)=\varphi(1r)=(r\varphi)(1)=0$, for $r\varphi\in N$. Hence $\varphi=0$, so $N=0$. Thus if $N\neq 0$, then it contains a homomorphism θ such that $\theta(1)\neq 0$. Let $\theta(1)=x$ say. Since $x\neq 0$, there is a positive integer m such that $mx=c$. Then $(m\theta)1=c$, and $m\theta\in N$.

PROPOSITION 2.10. *Let* $\theta\in M=\mathrm{Hom}_Z(R, Z(p^\infty))$ *be such that* $\theta(1)=c$, *where c is a fixed cogenerator of* $Z(p^\infty)$. *Then the cyclic submodule $R\theta$ is simple if and only if* $\mathrm{Ker}\,\theta$ *contains a maximal left ideal of R.*

PROOF. Let $R\theta$ be simple. Then the kernel of the homomorphism $\varphi\colon R\to R\theta$, given by $\varphi(r)=r\theta$, is a maximal left ideal J of R. Let $r\in J$, then $\theta(r)=\theta(1r)=(r\theta)(1)=0$, for $r\theta=0$. Hence $J\subseteq \mathrm{Ker}\,\theta$. Conversely, suppose there is a maximal left ideal $J\subseteq \mathrm{Ker}\,\theta$. Now $r\in \mathrm{Ker}\,\varphi$ if and only if $Rr\subseteq \mathrm{Ker}\,\theta\Leftrightarrow Rr\subseteq J$ by the maximality of J. Hence $\mathrm{Ker}\,\varphi=J$, so $R\theta$ is simple.

THEOREM 2.11. *If every abelian subgroup of R maximal with respect to missing* 1 *contains $p1$ and a maximal left ideal of R, and if R is Noetherian, then* $M=\mathrm{Hom}_Z(R, Z(p^\infty))$ *is a direct sum of injective cocyclic modules.*

PROOF. First, we observe that if $\theta\colon R\to Z(p^\infty)$ is such that $\theta(1)=c$, where c is a cogenerator of $Z(p^\infty)$, then $\mathrm{Ker}\,\theta$ is an abelian subgroup of R, which is maximal with respect to missing 1 and containing $p1$. Conversely, a subgroup of R maximal with respect to missing 1 and containing $p1$ gives rise to such a homomorphism. The result then follows from Propositions 2.9, 2.10 and [2; Theorem 4.4].

THEOREM 2.12. *If R is Artinian, then $M=\mathrm{Hom}_Z(R, Z(p^\infty))$ is a direct sum of injective cocyclic modules.*

PROOF. Since M is injective, the result follows from [2; Theorem 4.5].

3. Homomorphic images of a cocyclic module

In this section we describe the structure of a cocyclic module of which every non-zero homomorphic image is again cocyclic, and we characterize those commutative rings R for which every cocyclic R-module has this property. First, we construct and study a particular class of cocyclic modules. Let J be a maximal left ideal of R, and let N be the set of natural numbers. Consider the descending cahin

$$R=J^0\supseteqq J\supseteqq J^2\supseteqq\ldots\supseteqq J^\omega=\bigcap_{n\in N} J^n.$$

The chain may be finite, i.e., $J^k=J^{k+1}=\ldots$ for some $k\in N$. Suppose that the following conditions hold:

(i) J^{k+1} is maximal in J^k for each $k\in N$ if $J^{k+1}\neq J^k$, that is, if I is any ideal of R such that $J^{k+1}\subseteqq I\subseteqq J^k$, then $I=J^k$ or J^{k+1}.

(ii) Each J^k, $k>0$, is an irreducible ideal of R.

Then it is not difficult to prove the following lemma.

LEMMA 3.1. *If $J^{j+k}\neq J^{j+k-1}$, $j, k\in N$, then J^j/J^{j+k} is a cyclic module, isomorphic to R/J^k. Moreover, it is generated by any coset $x+J^{j+k}$, where $x\in J^j/J^{j+1}$.*

Note that if R is a Dedekind domain, then conditions (i) and (ii) certainly hold for every maximal ideal J of R.

By the above lemma we can construct the following ascending chain of cyclic modules C_n:

$$0=C_0\subseteqq C_1\subseteqq C_2\subseteqq\ldots\subseteqq C_\omega=\bigcup_{n\in N} C_n,$$

where $C_n\cong R/J^n$ and it is generated by any element $x\in C_n\setminus C_{n-1}$. Note that the ascending chain may be finite, i.e., $C_k=C_{k+1}=\ldots$ for some $k\in N$. This happens when $J^k=J^{k+1}=\ldots$. In this case $C_\omega=C_k$, otherwise C_ω is not a cyclic module. We shall show that C_ω is a cocyclic module.

THEOREM 3.2. *C_ω is a cocyclic module.*

PROOF. First, we observe that C_1 is a simple module. Next, we shall show that C's are precisely all the submodules of C_ω. Let M be a proper submodule of C_ω, then not all C_n's are contained in M. Let C_{k+1} be the first that is not contained in M. We claim that $M=C_k$. Otherwise, there exists an element $x\in M\setminus C_k$. Let C_m be the first containing x, $m>k$. But the module C_m is generated by x, so $C_m\subseteqq M$, which is a contradiction.

COROLLARY 3.3. *Each C_k, $1\leqq k\leqq\omega$, is a cocyclic module.*

COROLLARY 3.4. *Every non-zero homomorphic image of* C_k, $1 \leqq k \leqq \omega$, *is again a cocyclic module.*

PROOF. Note that C_{k+1}/C_k is a simple module if $C_k \neq C_{k+1}$.

PROPOSITION 3.5. *If R is commutative, then* C_ω *is divisible by every* $r \in R \setminus J^\omega$, *where* $C_\omega \neq C_n$ *for any* $n \in N$.

PROOF. Let $0 \neq x \in C_\omega$, then $x \in C_{k+1} \setminus C_k$ for some $k \in N$. Let $r \in R \setminus J^\omega$. If $r \notin J$, then rx generates C_{k+1}, so $x = r'(rx) = r(r'x)$ for some $r' \in R$. If $r \in J$, then $r \in J^m \setminus J^{m+1}$ for some $m \in N$. Let $y \in C_{k+1+m} \setminus C_{k+m}$, then $ry \in C_{k+1} \setminus C_k$ and ry generates C_{k+1}. Hence $x = r''(ry) = r(r''y)$ for some $r'' \in R$.

COROLLARY 3.6. *Let R be commutative. If* $C_\omega \neq C_n$ *for any* $n \in N$ *and* J^ω *consists of zero-divisors only, then* C_ω *is a divisible module. In particular, the result holds if* $J^\omega = 0$.

COROLLARY 3.7. *If R is a Dedekind domain, then every cocyclic R-module is isomorphic to* C_ω *or* C_k *for some* $k \in N$, *corresponding to some maximal ideal J of R.*

PROOF. The result follows from the following observation. If R is a Dedekind domain, then $J^\omega = 0$, $C_\omega \neq C_n$ for any $n \in N$, and any divisible R-module is injective.

COROLLARY 3.8. *If R is a Dedekind domain, then every non-zero homomorphic image of any cocyclic R-module is again cocyclic.*

The following theorem gives the structure of a cocyclic module of which every non-zero homomorphic image is again cocyclic.

THEOREM 3.9. *The following statements for a module M are equivalent:*

(i) *Every non-zero homomorphic image of M is a cocyclic module.*

(ii) (a) *M is Artinian;*

(b) *the submodules of M are totally ordered with respect to inclusion.*

(iii) *The submodules of M are well-ordered with respect to inclusion.*

(iv) (a) *The submodules of M are totally ordered with respect to inclusion;*

(b) *for every proper submodule* M_α *of M there exists a unique submodule* $M_{\alpha+1}$ *such that* $M_{\alpha+1}/M_\alpha$ *is simple (or* M_α *is maximal in* $M_{\alpha+1}$*).*

(v) *The submodules of M can be arranged in an ascending series (which may be finite)*

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_\alpha \subset M_{\alpha+1} \subset \ldots \subset M_\gamma = M$$

such that

(a) $M_{\alpha+1}$ *is cyclic, being generated by any element* $x \in M_{\alpha+1} \setminus M_\alpha$, *and* $M_{\alpha+1}/M_\alpha$ *is simple for each ordinal* α;

(b) $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ *if* α *is a limit ordinal.*

PROOF. (i)⇒(ii). Since every homomorphic image of M is finitely embedded M is Artinian by [2; Theorem 3.21]. Let M_α, M_β be two submodules of M such that none is contained in the other. Then $N' = M_\alpha \cap M_\beta \neq M_\alpha, M_\beta$, and M/N' is not cocyclic, which is a contradiction.

(ii)⇒(i). Every non-zero homomorphic image of M is finitely embedded by [2; Theorem 3.21] and it has an indecomposable socle by (ii) (b), so it is cocyclic.

(ii)$\Leftrightarrow$(iii). Immediate.

(iii)$\Rightarrow$(iv). (iv) (a) is trivial. In (iv) (b) $M_{\alpha+1}$ is the immediate successor of M_α.

(iv)$\Rightarrow$(v). By (iv) (b) we can build an ascending series of submodules beginning at 0 and ending at M such that condition (v) (b) is satisfied, and that $M_{\alpha+1}/M_\alpha$ is simple. We have yet to show that all the submodules of M are included in the series, and that $M_{\alpha+1}$ is cyclic, being generated by any element $x \in M_{\alpha+1} \setminus M_\alpha$. Consider the cyclic submodule Rx of $M_{\alpha+1}$, where $x \in M_{\alpha+1} \setminus M_\alpha$. By (iv) (a) it follows that $M_\alpha \subset Rx \subseteq M_{\alpha+1}$, whence $Rx = M_{\alpha+1}$. Let M' be any proper submodule of M. Let M_α be the first member of the series not contained in M'. Then it is easy to see that α is not a limit ordinal and $M' = M_\beta$, where $\alpha = \beta + 1$.

(v)$\Rightarrow$(iii). Immediate.

Corollary 3.10. *If R is commutative, then every non-zero homomorphic image of M is cocyclic if and only if M is of the form C_α, $\alpha \leqq \omega$.*

Proof. In this case it is easy to see that $M_{\alpha+1}/M_\alpha \cong M_1 \cong R/J$ for all α, where J is some maximal ideal of R. We show now that the series in (v) above terminates at or before M_ω. Let x be a generator of $M_{\omega+1}$ and let $r \in J \setminus J^2$. Then $rx \in M_\omega$, and so $rx \in M_k$ for some $k \in N$. Hence there exists an element $x_1 \in M_{k+1}$ such that $rx = rx_1$, so $r(x - x_1) = 0$. Hence $x - x_1 \in M_1$, and so $x \in M_{k+1}$. This implies that $M_{\omega+1} \subseteq M_{k+1}$.

Now we are in a position to characterize those commutative rings R for which every cyclic R-module has the property that its each non-zero homomorphic image is again a cocyclic module. First, we establish a lemma. For this we recall that a local ring is a commutative ring which has a unique maximal ideal.

Lemma 3.11. *Let R be a local ring, I its unique maximal ideal. Then the injective envelope $E = E(R/I)$ is of the form C_α, $\alpha \leqq \omega$, if and only if every non-zero ideal of R is a power of I. (I is a principal ideal in this case.)*

Proof. Let every non-zero ideal of R be a power of I. Then there exists a cocyclic module of the form C_α, $\alpha \leqq \omega$, such that $C_{\alpha+1} = C_\alpha$. C_α can be embedded in E. Since R is also Noetherian, E is Artinian by [2; Theorem 4.30]. If $E \neq C_\alpha$, there exists a minimal submodule M of E containing C_α properly. Then M/C_α is simple, so M coincides with C_α by an argument similar to that given in the proof of Corollary 3.10. Hence $E = C_\alpha$. Suppose now that E is of the form C_α. Let K be any non-zero ideal of R, and let $K^* = \{x \in E \mid Kx = 0\}$. Then K^* is a proper submodule of E by [2; Proposition 2.26, Corollary 2]. Hence $K^* = C_k$ for some $k \in N$, so $K \subseteq I^k$, whence it follows that $K = I^k$ (see the proof of Theorem 2 in [3]).

Corollary 3.12. *Let R be a local ring. Then every cocyclic R-module is of the form C_α, $\alpha \leqq \omega$, if and only if every non-zero ideal of R is a power of its maximal ideal.*

Corollary 3.13. *Let R be a local ring. Then every cocyclic R-module is of the form C_k, $k \leqq$ some fixed $n \in N$, if and only if R is Artinian and every nonzero ideal of R is a power of its maximal ideal.*

Corollary 3.14. *Let R be a local ring. Then each non-zero homomorphic image of every cocyclic R-module is again cocyclic if and only if each non-zero ideal of R is a power of its maximal ideal.*

Now we prove one of our main theorems.

THEOREM 3.15. *Let R be a commutative ring. Then the following statements are equivalent:*

(i) *Each non-zero homomorphic image of every cocyclic R-module is again a cocyclic module.*

(ii) *For each maximal ideal J of R the localization R_J has the property that its every non-zero ideal is a power of its unique maximal ideal.*

PROOF. (i) holds if and only if for each maximal ideal J of R the injective envelope $E=E(R/J)$ is of the form C_α, $\alpha\leqq\omega$ (see Corollary 3.10). But E, regarded as an R_J-module, has the same structure as it has as an R-module (see [2; 5.1]). The result then follows from Lemma 3.11.

COROLLARY 3.16. *Let R be a commutative integral domain. Then each non-zero homomorphic image of every cocyclic R-module is cocyclic if and only if for each non-zero maximal ideal J of R the localization R_J is a discrete valuation ring.*

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Now we prove one of our main theorems.

THEOREM 5.15. *Let R be a commutative ring. Then the following statements are equivalent:*

(i) *Each non-zero [illegible] R-module [illegible]*

(ii) *For each maximal ideal J of R the localization R_J has the property that every non-zero ideal is a power of the unique maximal ideal.*

PROOF. (i) holds if and only if for each maximal ideal J of R the [illegible] is of the form [illegible] (see Corollary 5.10). [illegible] as an R_J-module, has the same structure [illegible] as an R-module (see [illegible]). The result then follows from Lemma 5.1.

COROLLARY [illegible]. *Let R be a commutative [illegible] domain. Then [illegible] R-module [illegible]*

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DETERMINATION OF CERTAIN ASYMPTOTIC CONSTANTS RELATED WITH THE POST-WIDDER INVERSION OF LAPLACE TRANSFORM

By

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1. Introduction

Let $\mathbf{N}, \mathbf{R}^+$ and $\bar{f}$, respectively, denote the set of natural numbers, the interval $(0, \infty)$ and the Laplace transform of a function f. Let p be a fixed integer. The n-th Post—Widder operator $L_{n,x}$ is defined by

$$L_{n,x}\bar{f} = \frac{(-1)^{n+p}}{(n+p)!}\bar{f}^{(n+p)}\left(\frac{n}{x}\right)\left(\frac{n}{x}\right)^{n+p+1}, \quad x\in\mathbf{R}^+ \tag{1}$$

and it exists for all sufficiently large $n\in\mathbf{N}$, if $\bar{f}(t)$ exists for some $t\in\mathbf{R}^+$. Following WIDDER [7, p. 288] an integral representation for $L_{n,x}$ is as follows:

$$L_{n,x}\bar{f} = \frac{1}{(n+p)!}\left(\frac{n}{x}\right)^{n+p+1}\int_0^\infty f(t)\,t^{n+p}e^{-nt/x}\,dt. \tag{2}$$

The inversion formula

$$\lim_{n\to\infty} L_{n,x}\bar{f} = f(x), \tag{3}$$

which holds at each continuity point x of f, was investigated by WIDDER [7] for $p=0$ and by MAY [2] for $p=-1$. Using the representation (2), the result for a general p can be established analogously.

The error estimate

$$|L_{n,x}\bar{f}-f(x)| \leqq \left\{1+x^2\left(1+\frac{(p+1)(p+2)}{n}\right)\right\}w(f;\, n^{-1/2}), \quad x\in\mathbf{R}^+, \tag{4}$$

where $w(f;\,\delta)$ denotes the modulus of continuity of f, can be easily verified from (2). It follows that if $f\in\mathrm{Lip}\,\alpha$ the order of convergence in (3) is $O(n^{-\alpha/2})$.

This paper concerns with a determination of the best asymptotic factor with $w(f;\,\delta)$ in (4) and the Lipschitz—Nikolskiĭ constants of the operators $L_{n,x}\bar{f}$. Pertinent references on Lipschitz—Nikolskiĭ constants can be found in [3].

2. The Lipschitz—Nikolskiĭ constants

Lipschitz—Nikolskiĭ constants of the operators $L_{n,x}\bar{f}$ are determined in the following results:

THEOREM 1. *If* $E_n(\alpha, x) = \sup_f |L_{n,x}\bar{f}-f(x)|$, $x\in\mathbf{R}^+$, *where the supremum is taken over all functions of the class* $\mathrm{Lip}_1\,\alpha\,(0<\alpha\leqq 1)$, *then*

$$\lim_{n\to\infty} n^{\alpha/2}E_n(\alpha, x) = \frac{\Gamma\left(\frac{1+\alpha}{2}\right)}{\sqrt{\pi}}(2x^2)^{\alpha/2}. \tag{5}$$

THEOREM 2. *If* $E_n(\alpha;\ r, x)=\sup_f |L_{n,x}\bar{g}|$ $(x\in\mathbf{R}^+;\ r=1, 2, \dots)$, *where* $g(t)=$ $=f(t)-\sum_{k=0}^{r}\frac{(t-x)^k}{k!}f^{(k)}(x)$ *and the supremum is taken over all functions with* $f^{(r)}\in \mathrm{Lip}_1\,\alpha$ $(0<\alpha\leqq 1)$, *then*

$$(6)\qquad \frac{2^{(r+3\alpha-2)/2}\Gamma\left(\frac{r+\alpha+1}{2}\right)x^{r+\alpha}}{(1+\alpha)(2+\alpha)\dots(r+\alpha)\sqrt{\pi}} \leqq \varliminf_{n\to\infty} n^{(r+\alpha)/2}E_n(\alpha;\ r, x) \leqq$$

$$\leqq \varlimsup_{n\to\infty} n^{(r+\alpha)/2}E_n(\alpha;\ r, x) \leqq \frac{2^{(r+\alpha)/2}\Gamma\left(\frac{r+\alpha+1}{2}\right)x^{r+\alpha}}{(1+\alpha)(2+\alpha)\dots(r+\alpha)\sqrt{\pi}}.$$

COROLLARY 1. *We have*

$$(7)\qquad \lim_{n\to\infty} n^{(r+1)/2}E_n(1;\ r, x) = \frac{2^{(r+1)/2}\Gamma\left(\frac{r}{2}+1\right)x^{r+1}}{(r+1)!\sqrt{\pi}} \quad (x\in\mathbf{R}^+).$$

First, we obtain an asymptotic evaluation of $L_{n,x}\bar{f}$ where $f(t)=|t-x|^{\alpha}$ $(x\in\mathbf{R}^+,\ \alpha>0)$. For this, we have

$$L_{n,x}\bar{f} = \frac{n^{n+p+1}}{(n+p)!}\int_0^{\infty}|xt-x|^{\alpha}t^{n+p}e^{-nt}\,dt = \frac{n^{n+p+1}}{(n+p)!}x^{\alpha}I,$$

where

$$I = \int_0^{\infty}|t-1|^{\alpha}t^{n+p}e^{-nt}\,dt.$$

For the values of $t\in\mathbf{R}^+$ satisfying $|t-1|<n^{-\gamma}$, where $1/3<\gamma<1/2$, one has $t=1+\theta n^{-\gamma}$, where θ lies between -1 and 1. We define

$$I_1 = \int_{1-n^{-\gamma}}^{1+n^{-\gamma}} |\theta|^{\alpha}n^{-\gamma\alpha}(1+\theta n^{-\gamma})^{n+p}e^{-n(1+\theta n^{-\gamma})}\,dt,$$

where θ is related to t by $t=1+\theta n^{-\gamma}$, and put $I_2=I-I_1$. Thus

$$L_{n,x}\bar{f} = x^{\alpha}\frac{n^{n+p+1}}{(n+p)!}\{I_1+I_2\}.$$

Now

$$(1+\theta n^{-\gamma})^{n+p} = \exp\left\{(n+p)\left(\theta n^{-\gamma}-\frac{\theta^2n^{-2\gamma}}{2}+\frac{\theta^3n^{-3\gamma}}{3}-\dots\right)\right\} =$$

$$= \exp\left\{\theta n^{1-\gamma}-\frac{\theta^2}{2}n^{1-2\gamma}+o(1)\right\}$$

(uniformly in $\theta\in[-1, 1]$). Therefore

$$I_1 = 2n^{-\gamma(\alpha+1)}e^{-n+o(1)}\int_0^1|\theta|^{\alpha}\exp\left\{-\frac{\theta^2}{2}n^{1-2\gamma}\right\}d\theta.$$

Putting $n^{1-2\gamma}\theta^2=2t$, we have

$$I_1=\sqrt{2}\,n^{-\gamma(\alpha+1)}e^{-n+o(1)}2^{\alpha/2}\int_0^{n^{1-2\gamma}/2} n^{(1+\alpha)(\gamma-1/2)}t^{\frac{\alpha-1}{2}}e^{-t}\,dt\cong$$

$$\cong 2^{\alpha/2}\sqrt{2}\,n^{-\frac{1+\alpha}{2}}e^{-n}\Gamma\left(\frac{1+\alpha}{2}\right).$$

Thus

$$I_1\cdot\frac{x^\alpha n^{n+p+1}}{(n+p)!}\cong\frac{\Gamma\left(\frac{1+\alpha}{2}\right)}{\sqrt{\pi}}\left(\frac{2x^2}{n}\right)^{\alpha/2}\quad\text{(using Stirling's formula).}$$

In order to deal with I_2, we require the following result which can be easily obtained from (2): For a fixed $x\in\mathbf{R}^+$ and $k\in\mathbf{N}^0$ (the set of non-negative integers), if we define $\mu_{n,k}(x)=L_{n,x}\bar{f}$ where $f(t)=(t-x)^k$ then

$$\mu_{n,k+1}(x)=\frac{x}{n}\{(p+k+1)\mu_{n,k}(x)+kx\mu_{n,k-1}(x)\}\quad(k=1,2,\ldots). \tag{8}$$

Since $\mu_{n,0}(x)=1$ and $\mu_{n,1}(x)=\dfrac{x(p+1)}{n}$, using (8) and an induction on k, we have

$$x^{-k}\mu_{n,k}(x)=O\left(n^{-\left[\frac{k+1}{2}\right]}\right) \tag{9}$$

and moreover that the left hand side is independent of x.

Using (9) we have

$$x^\alpha\frac{n^{n+p+1}}{(n+p)!}I_2=x^\alpha\frac{n^{n+p+1}}{(n+p)!}\int_{|t-1|\geqq n^{-\gamma}}|t-1|^\alpha t^{n+p}e^{-nt}\,dt\leqq$$

$$\leqq x^\alpha\frac{n^{n+p+1}}{(n+p)!}\int_0^\infty|t-1|^{2s}t^{n+p}e^{-nt}n^{\gamma(2s-\alpha)}\,dt=x^\alpha n^{\gamma(2s-\alpha)}O(n^{-s})=o(n^{-\alpha/2}),$$

where s is an integer greater than $\alpha/2$.

Hence

$$L_{n,x}\bar{f}\cong\frac{\Gamma\left(\frac{1+\alpha}{2}\right)}{\sqrt{\pi}}\left(\frac{2x^2}{n}\right)^{\alpha/2}. \tag{10}$$

Theorems 1—2 and Corollary 1 now follow from the estimate (10) and proceeding in the manner of the proofs of analogous results of [3, Ths. 1—2, Cor. 1].

3. Best asymptotic constants

In the following theorem, we obtain the best asymptotic constant with modulus of continuity, where in addition to the dependence on n, the argument in the modulus of continuity also depends on x.

THEOREM 3. *Let φ be a positive function on* $\mathbf{R}^+$. *If*

(11) $$C_n(x) = \inf\{C\colon |L_{n,x}\bar{f} - f(x)| \leqq Cw(f;\, \varphi(x)n^{-1/2}) \text{ for all continuous functions } f\} \quad (x > 0),$$

then

(12) $$C_n(x) \cong C_\infty(x) = 2\sum_{j=0}^{\infty}(j+1)\left\{\operatorname{erf}\left(\frac{\varphi(x)}{x}(j+1)\right) - \operatorname{erf}\left(\frac{\varphi(x)}{x}j\right)\right\},$$

where

$$\operatorname{erf}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-s^2/2}\, ds.$$

PROOF. Using a well known property of the modulus of continuity, we have

$$|L_{n,x}\bar{f} - f(x)| \leqq \{1 + L_{n,x}\bar{e}_n\}w(f;\, \varphi(x)n^{-1/2}) = C_n^*(x)w(f;\, \varphi(x)n^{-1/2}),$$

where $e_n(t) = \left[\frac{|t-x|}{\varphi(x)n^{-1/2}}\right]$ ($[\cdot]$ stands for the integral part) and $C_n^*(x) = 1 + L_{n,x}\bar{e}_n$. Thus $C_n(x) \leqq C_n^*(x)$.

Let ε be a positive number less than $\varphi(x)n^{-1/2}$. Consider the function $f_{\varepsilon,\varphi(x)n^{-1/2}}(t)$ defined on $\mathbf{R}^+$ as follows:

(i) $f_{\varepsilon,\varphi(x)n^{-1/2}}(x) = 0$,

(ii) $f_{\varepsilon,\varphi(x)n^{-1/2}}(t) = k, \quad t \in [x + (k-1)\varphi(x)n^{-1/2} + \varepsilon,\ x + k\varphi(x)n^{-1/2}]$ $(k = 1, 2, \ldots)$,

(iii) $f_{\varepsilon,\varphi(x)n^{-1/2}}(t)$ is linear if $t \in [x + (k-1)\varphi(x)n^{-1/2},\ x + (k-1)\varphi(x)n^{-1/2} + \varepsilon]$

and

(iv) $f_{\varepsilon,\varphi(x)n^{-1/2}}(t)$ is symmetric about $t = x$.

It is easy to see that

$$w(f_{\varepsilon,\varphi(x)n^{-1/2}};\, \varphi(x)n^{-1/2}) = 1,$$

and

$$\lim_{\varepsilon\to\infty} f_{\varepsilon,\varphi(x)n^{-1/2}}(t) = \begin{cases} 1 + e_n(t), & t \neq x \pm k\varphi(x)n^{-1/2} \\ k, & t = x \pm k\varphi(x)n^{-1/2} \end{cases} \quad (k = 0, 1, 2, \ldots).$$

Since the points of the form $x \pm k\varphi(x)n^{-1/2}$, $k = 0, 1, 2, \ldots$ constitute a set of measure zero, using Lebesgue's dominated convergence theorem, we have

(13) $$\lim_{\varepsilon\to 0} L_{n,x}\bar{f}_{\varepsilon,\varphi(x)n^{-1/2}} = 1 + L_{n,x}\bar{e}_n = C_n^*(x).$$

According to the definition of $C_n(x)$,

$$|L_{n,x}\bar{f}_{\varepsilon,\varphi(x)n^{-1/2}}| \leqq C_n(x)w(f_{\varepsilon,\varphi(x)n^{-1/2}};\ \varphi(x)n^{-1/2}) = C_n(x).$$

Hence, by (13),

$$C_n^*(x) = \lim_{\varepsilon\to 0} L_{n,x}\bar{f}_{\varepsilon,\varphi(x)n^{-1/2}} \leqq C_n(x).$$

If follows that $C_n^*(x)=C_n(x)$. Now we determine the asymptotic value of $C_n(x)= =L_{n,x}\bar{f}_0$, say, where $f_0(t)=1+e_n(t)$.

First, we observe that with $1/3<\gamma<1/2$,

$$L_{n,x}\bar{f}_0 = L_{n,x}\overline{f_0\chi_\gamma}, \tag{14}$$

where χ_γ is the characteristic function of the interval $[x-\varphi(x)n^{-\gamma}, x+\varphi(x)n^{-\gamma}]$. For, if m is a positive integer, using (10),

$$L_{n,x}\overline{(1-\chi_\gamma)f_0} \leqq L_{n,x}\bar{f} = O(n^{-m+2m\gamma}) = o(1), \tag{15}$$

where $f(t)=\left(\dfrac{t-x}{\varphi(x)n^{-\gamma}}\right)^{2m}\left(1+\dfrac{|t-x|}{\varphi(x)n^{-1/2}}\right)$. Thus (14) follows from (15) and the fact that $L_{n,x}\bar{f}_0\geqq 1$.

In view of (14) and the result of (15), we have

$$L_{n,x}\bar{f}_0 = \frac{1}{(n+p)!}\left(\frac{n}{x}\right)^{n+p+1}\left[\left(\int_0^x+\int_x^\infty\right)f_0(t)\,t^{n+p}e^{-nt/x}\,dt\right] \cong$$

$$\cong \sum_{j=-[n^{1/2-\gamma}]-1}^{0}(-j+1)\frac{1}{(n+p)!}\left(\frac{n}{x}\right)^{n+p+1}\cdot\int_{x+(j-1)\varphi(x)n^{-1/2}}^{x+j\varphi(x)n^{-1/2}} t^{n+p}e^{-nt/x}\,dt+$$

$$+\sum_{j=0}^{[n^{1/2-\gamma}]+1}(j+1)\frac{1}{(n+p)!}\left(\frac{n}{x}\right)^{n+p+1}\cdot\int_{x+j\varphi(x)n^{-1/2}}^{x+(j+1)\varphi(x)n^{-1/2}} t^{n+p}e^{-nt/x}\,dt=$$

$$=\sum\nolimits_1+\sum\nolimits_2.$$

We consider Σ_1 and Σ_2 separately.

$$\sum\nolimits_1 = \sum_{j=-[n^{-1/2-\gamma}]-1}^{0}(-j+1)\frac{1}{(n+p)!}\frac{\varphi(x)}{2x}n^{n+p-1/2}\cdot$$

$$\cdot\int_{2j-2}^{2j}\left(1+\frac{\varphi(x)}{2x}sn^{-1/2}\right)^{n+p}e^{-n\left(1+\frac{\varphi(x)}{2x}sn^{-1/2}\right)}ds=$$

$$=\sum_{j=-[n^{-1/2-\gamma}]-1}^{0}(-j+1)\frac{1}{(n+p)!}\frac{\varphi(x)}{2x}n^{n+p-1/2}\int_{2j-2}^{2j}\exp\left\{\frac{\varphi(x)}{2x}sn^{1/2}-\frac{\varphi^2(x)s^2}{8x^2}+o(1)\right\}\cdot$$

$$\cdot e^{-n\left(1+\frac{\varphi(x)}{2x}sn^{-1/2}\right)}ds \cong \sum_{j=-[n^{1/2-\gamma}]-1}^{0}(-j+1)\frac{\varphi(x)}{2x\sqrt{2\pi}}\int_{2j-2}^{2j}e^{-\varphi^2(x)s^2/8x^2}\,ds$$

(o-term holding uniformly in s and j). In a similar fashion we can show that

$$\sum\nolimits_2 \cong \sum_{j=0}^{[n^{1/2-\gamma}]+1}(j+1)\frac{\varphi(x)}{2x\sqrt{2\pi}}\int_{2j}^{2j+2}e^{-\frac{\varphi^2(x)s^2}{8x^2}}\,ds.$$

Notice that the above asymptotic values of Σ_1 and Σ_2 are equal. Therefore

$$L_{n,x}\bar{f}_0 \cong 2 \sum_{j=0}^{[n^{1/2-\gamma}]+1} (j+1)\frac{\varphi(x)}{2x\sqrt{2\pi}} \int_{2j}^{2j+2} e^{-\frac{\varphi^2(x)s^2}{8x^2}}\, ds \cong$$

$$\cong 2\sum_{j=0}^{\infty}(j+1)\frac{\varphi(x)}{2x\sqrt{2\pi}} \int_{2j}^{2j+2} e^{-\frac{\varphi^2(x)s^2}{8x^2}}\, ds =$$

$$= 2\sum_{j=0}^{\infty}(j+1)\left\{\operatorname{erf}\left(\frac{\varphi(x)}{x}(j+1)\right)-\operatorname{erf}\left(\frac{\varphi(x)}{x}j\right)\right\}.$$

This completes the proof of Theorem 3.

As a consequence of Theorem 3, we have the following results:

COROLLARY 2. *If*

(16)

$B_n(x) = \inf\{C\colon |L_{n,x}\bar{f}-f(x)| \leqq Cw(f;\, n^{-1/2})$ *for all continuous functions* $f\}$ $(x>0)$,

then

$$B_n(x) \cong B_\infty(x) = 2\sum_{j=0}^{\infty}(j+1)\left\{\operatorname{erf}\left(\frac{j+1}{x}\right)-\operatorname{erf}\left(\frac{j}{x}\right)\right\}. \tag{17}$$

COROLLARY 3. *If*

(18)

$D_n = \inf\{C\colon |L_{n,x}\bar{f}-f(x)| \leqq Cw(f;\, 2xn^{-1/2})$ *for all continuous functions* $f\}$ $(x>0)$,

then

$$D_n \cong D_\infty = 2\sum_{j=0}^{\infty}(j+1)\{\operatorname{erf}(2j+2)-\operatorname{erf}(2j)\} = 1.045\,564\,\ldots. \tag{19}$$

Notice that in the last result the constants D_n and D_∞ are independent of x and moreover that D_∞ is the same constant as occurs in the work of ESSEEN [1] for Bernstein polynomials.

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REGULAR LOCALLY TESTABLE SEMIGROUPS AS SEMIGROUPS OF QUASI-IDEALS

By

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J. LUH has shown that a semigroup S is regular if and only if the set $\mathcal{Q}(S)$ of quasi-ideals of S forms a regular semigroup [6], [7], [4]. We show that the semigroups $\mathcal{Q}(S)$ which can be so obtained must be regular locally testable semigroups and that every regular locally testable semigroup can be faithfully represented by a semigroup of quasi-ideals. This will lead us to a structure theorem of Rees type for regular locally testable semigroups.

We assume that the reader is familiar with the standard notation and terminology of semigroup theory as established in [1] and [12].

A non-empty subset Q of a semigroup S is called a *quasi-ideal* of S if $QS\cap SQ\subseteq Q$ [11]. It is easy to show that a non-empty subset Q of a semigroup S is a quasi-ideal of S if and only if Q is the intersection of a left ideal and a right ideal of S (Corollary 2.7 of [12]). The set of quasi-ideals of the semigroup S will be denoted by $\mathcal{Q}(S)$.

The following result (Theorem 9.3 of [12]) compiles results of [2], [4], [6], [7].

THEOREM 1. *The following conditions on a semigroup S are equivalent:*

(i) *S is regular;*

(ii) *for every right ideal R and every left ideal L of S $RL=R\cap L$;*

(iii) *$\mathcal{Q}(S)$ forms a regular semigroup;*

(iv) *every quasi-ideal of S has the form $Q=QSQ$.*

From the foregoing it follows that in a regular semigroup S every quasi-ideal Q can be written as $Q=QS\cap SQ=QSQ$; in particular, the quasi-ideal $Q(a)= =aS\cap Sa=aSa$, $a\in S$ is the smallest quasi-ideal containing a; $Q(a)$ will be called the *principal quasi-ideal* generated by a. The elements a and b of the regular semigroup S generate the same principal quasi-ideal if and only if they are $\mathcal{H}$-related ([3], [12] Proposition 4.1).

A *locally testable semigroup* S is a semigroup which is locally finite and which satisfies the condition that eSe is a semilattice for all $e=e^2\in S$ [13], [14]. NAMBOORIPAD [9] has shown that a regular semigroup S is locally testable if and only if eSe is a semilattice for all $e=e^2\in S$. The following theorem determines a variety of algebras of type $\langle 2, 0\rangle$ which gives rise to a class of regular locally testable semigroups.

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THEOREM 2. *Let S be a semigroup which contains an idempotent* $e=e^2$, *such that*

$$xex = x \tag{1}$$

$$exeye = eyexe \tag{2}$$

for all $x, y\in S$. *Then S is a regular idempotent-generated locally testable semigroup.*

PROOF. From (1) it follows that eSe consists of idempotents, and from (2) we have that eSe must be a semilattice. Let $f=f^2$ be any idempotent of S. Since the mappings

$$fSf \to efSfe, \quad fxf \to efxfe$$

$$efSfe \to fSf, \quad efxfe \to fefxfef = fxf$$

are mutually inverse isomorphisms between fSf and $efSfe$, we conclude that fSf must be isomorphic to a subsemilattice of eSe. Hence fSf is a semilattice for all $f=f^2\in S$. By (1) S is a regular semigroup. We conclude that S is a regular locally testable semigroup. If $x\in S$, then $x=(xe)(ex)$ is the product of the idempotents xe and ex. Thus, S is idempotent-generated.

THEOREM 3. *Let S be a regular semigroup. Then* $\mathcal{Q}(S)$ *is a regular idempotent-generated locally testable semigroup. The set* $I[\Lambda]$ *of right [left] ideals of S forms a subsemigroup of* $\mathcal{Q}(S)$ *which is a left [right] normal band, and* $T=I\cap\Lambda$ *is the (complete) semilattice of the two-sided ideals of S which intersects each* $\mathcal{D}$*-class of* $\mathcal{Q}(S)$ *in precisely one element.*

PROOF. We show that $\mathcal{Q}(S)$ is a semigroup which satisfies the conditions of Theorem 2. Clearly $S^2=S$ is an idempotent of $\mathcal{Q}(S)$, and for every $Q\in\mathcal{Q}(S)$ we must have $QSQ=Q$ by Theorem 1. We show that $SQ_1SQ_2S=SQ_2SQ_1S$ for all $Q_1, Q_2\in\mathcal{Q}(S)$. Indeed,

$$SQ_1SQ_2S = (SQ_1S)(SQ_2S) = (SQ_1S)\cap(SQ_2S) = (SQ_2S)(SQ_1S) = SQ_2SQ_1S.$$

Thus $\mathcal{Q}(S)$ is a regular idempotent-generated locally testable semigroup. I consists of the elements QS, $Q\in\mathcal{Q}(S)$. Clearly $(QS)^2=QS$ for all $Q\in\mathcal{Q}(S)$, and so I must be a band. It follows from Theorem 5 of [14] that $I=\mathcal{Q}(S)S$ is a left normal band. Analogously, $\Lambda=S\mathcal{Q}(S)$ forms a right normal band.

Let Q be any element of $\mathcal{Q}(S)$. Then $Q\mathcal{L}SQ\mathcal{R}SQS$ in $\mathcal{Q}(S)$. Hence every $\mathcal{D}$-class of $\mathcal{Q}(S)$ contains an element of $T=I\cap\Lambda=S\mathcal{Q}(S)S$. A $\mathcal{D}$-class cannot contain two different elements of T since $T=S\mathcal{Q}(S)S$ is a semilattice.

COROLLARY 4. *Let S be a regular semigroup. Then* $\mathcal{J}=\mathcal{D}$ *for* $\mathcal{Q}(S)$. *The poset of* $\mathcal{D}$*-classes of* $\mathcal{Q}(S)$ *is isomorphic to the complete lattice of the two-sided ideals of S.*

PROOF. Since $\mathcal{Q}(S)$ is a periodic regular semigroup, $\mathcal{J}=\mathcal{D}$ is immediate (Corollary 2.56 of [1]).

COROLLARY 5. *If S is a regular semigroup, then* $\mathcal{Q}(S)$ *is a normal band if and only if S is intra-regular;* $\mathcal{Q}(S)$ *is a rectangular band if and only if S is simple.*

PROOF. Since $\mathcal{Q}(S)$ is locally testable, $\mathcal{Q}(S)$ is a band if and only if $\mathcal{Q}(S)$ is a normal band (Theorem 5 of [14]). This is the case if and only if S is intra-regular (Corollary 9.10 of [12]).

A regular semigroup S is called a *pseudo-inverse semigroup* if eSe is an inverse semigroup for all $e=e^2\in S$ [8], [9], [10]. If S is a pseudo-inverse semigroup, then the relation $\leqq$ on S defined by

$$x \leqq y \quad \text{if and only if} \quad xS \subseteq yS \quad \text{and} \quad x = ey \quad \text{for some} \quad e = e^2, \quad e\mathcal{R}x$$

is a partial order which is compatible with the multiplication [8], [9]. One can show that the above given definition for $\leqq$ is self-dual, and that $\leqq$ induces the usual natural partial order on the set of idempotents of S. The relation $\leqq$ is called the *natural partial order* on S. If S is a regular semigroup, then $\mathcal{Q}(S)$ must be a pseudo-inverse semigroup. Therefore $\mathcal{Q}(S)$ is endowed with two partial orders $\leqq$ and $\subseteq$ which are both compatible with the multiplication. If Q_1 and Q_2 are any two quasi-ideals such that $Q_1\leqq Q_2$ in $\mathcal{Q}(S)$, then $SQ_1S\leqq SQ_2S$, and so $SQ_1S\subseteq SQ_2S$; we also have $SQ_1\leqq SQ_2$, from which $SQ_1=SQ_1SQ_2\subseteq SQ_2SQ_2=SQ_2$; we conclude that $SQ_1\subseteq SQ_2$, and dually $Q_1S\subseteq Q_2S$; thus $Q_1=Q_1S\cap SQ_1\subseteq Q_2S\cap SQ_2=Q_2$. Hence the inclusion $\subseteq$ extends the natural partial order $\leqq$ on $\mathcal{Q}(S)$. The reader may verify that $\leqq$ and $\subseteq$ coincide if and only if S is a semilattice of groups.

THEOREM 6. *For a regular semigroup S the following are equivalent.*

(i) *S is locally testable,*

(ii) *S is a pseudo-inverse semigroup and $Q(a)=\{b\,|\,b\leqq a\}$ for all $a\in S$;*

(iii) *S is combinatorial and $Q(a)Q(b)=Q(ab)$ for all $a, b\in S$.*

PROOF. (i)$\Rightarrow$(ii). If $b\leqq a$, then $b\in aS\cap Sa=Q(a)$. Let us now consider any element $b=axa$ of $Q(a)=aSa$. Let a' be any inverse of a, and let $aa'=e$. It is easy to see that the mapping $aSa\to eSe$, $aya\to ayaa'$ is an $\mathcal{R}$-class preserving isomorphism of aSa onto eSe. Since eSe must be a semilattice this implies that $f=axaa'$ is an idempotent in the $\mathcal{R}$-class of $b=axa$. From $b=(axaa')a=fa$ then follows $b\leqq a$. We conclude that $Q(a)=\{b|b\leqq a\}$ for all $a\in S$. Since $\mathcal{Q}(S)$ is a regular locally testable semigroup, $\mathcal{Q}(S)$ is obviously a pseudo-inverse semigroup.

(ii)$\Rightarrow$(iii). Every element of $Q(ab)$ is of the form $abxab=(abxa)b\in Q(a)Q(b)$, and therefore $Q(ab)\subseteq Q(a)Q(b)$ always holds. Furthermore, if $x\in Q(a)$ and $y\in Q(b)$, then $x\leqq a$ and $y\leqq b$ implies $xy\leqq ab$, and so $xy\in Q(ab)$. We conclude that $Q(a)Q(b)=Q(ab)$ for all $a, b\in S$. Let a and b be any $\mathcal{H}$-related elements of S. Then $Q(a)=Q(b)$, and so $b\leqq a$ in S. Clearly $b\leqq a$ and $b\mathcal{H}a$ implies $a=b$. Thus S has trivial $\mathcal{H}$-classes.

(iii)$\Rightarrow$(i). The mapping $\varphi\colon S\to\mathcal{Q}(S)$, $a\to Q(a)$ is a homomorphism of S into $\mathcal{Q}(S)$. Since $Q(a)=Q(b)$ if and only if $a\mathcal{H}b$ in S, and since the $\mathcal{H}$-relation on S is trivial, the mapping φ is injective ([3], [12] Proposition 4.1). Thus S can be isomorphically embedded into a locally testable semigroup. Hence S is locally testable.

COROLLARY 7. *Every regular locally testable semigroup can be faithfully represented by a semigroup of quasi-ideals.*

COROLLARY 8. *Every regular locally testable semigroup can be isomorphically embedded in a regular idempotent-generated locally testable semigroup.*

COROLLARY 9. *Let S be a regular semigroup. The mapping $\varphi: S \to \mathcal{Q}(S)$, $a \to Q(a)$ is a homomorphism if and only if S is a subdirect product of completely 0-simple and completely simple semigroups.*

PROOF. If φ is a homomorphism, then S is a coextension of the regular locally testable semigroup $S\varphi$. Since $Q(a)=Q(b)$ if and only if $a\mathcal{H}b$, S must be a $\mathcal{H}$-coextension of $S\varphi$. It follows from Theorem 6 of [14] that S must be a subdirect product of completely 0-simple and completely simple semigroups.

Conversely, if S is a subdirect product of completely 0-simple and completely simple semigroups, then the $\mathcal{H}$-relation on S must be a congruence [5], and $S/\mathcal{H}$ must be a regular locally testable semigroup (Theorem 6 of [14]). It follows that φ is a homomorphism.

If T is a semilattice, I and Λ index sets and $P=(p_{\lambda i})$ a $\Lambda \times I$-matrix with entries $p_{\lambda i} \in T$, then $\mathcal{M}(T; P; I, \Lambda)$ will denote the set of all elements of the form (i, a, λ). $i \in I$, $\lambda \in \Lambda$, $a \in T$. On $\mathcal{M}(T; P; I, \Lambda)$ we define a multiplication by

$$(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j}b, \mu).$$

It is easy to see that $\mathcal{M}(T; P; I, \Lambda)$ is a semigroup.

THEOREM 10. *The semigroup $\mathcal{M}(T; P; I, \Lambda)$ is a locally testable semigroup which contains a greatest regular locally testable subsemigroup.*

If S is any regular locally testable semigroup, then S divides the locally testable semigroup $\mathcal{M}(T; P; I, \Lambda)$, where $I[\Lambda]$ is the set of right [left] ideals of S, T the complete semilattice of two-sided ideals of S, and P the $\Lambda \times I$-matrix which has the element $LR \in T$ on the (L, R)-position for every $(R, L) \in I \times \Lambda$.

PROOF. Let us consider a finite subset $X=\{(i_k, a_k, \lambda_k) \mid 1 \leqq k \leqq n\}$ of $\mathcal{M}(T; P; I, \Lambda)$. Every element of the subsemigroup which is generated by X is of the form (i, a, λ), where $i \in \{i_k \mid 1 \leqq k \leqq n\}$, $\lambda \in \{\lambda_k \mid 1 \leqq k \leqq n\}$, and where a belongs to the finite subsemilattice of T which is generated by the elements $a_k, p_{\lambda_k i_m}$, $1 \leqq k \leqq n, 1 \leqq m \leqq n$. This means that the subsemigroup of $\mathcal{M}(T; P; I, \Lambda)$ which is generated by X is finite. Thus, $\mathcal{M}(T; P; I, \Lambda)$ is locally finite. An element (i, a, λ) of $\mathcal{M}(T; P; I, \Lambda)$ is an idempotent if and only if $a \leqq p_{\lambda i}$; if this is the case then $(i, a, \lambda)\mathcal{M}(T; P; I, \Lambda)(i, a, \lambda)=\{(i, b, \lambda) \mid b \leqq a\}$ forms a semilattice. Thus $\mathcal{M}(T; P; I, \Lambda)$ is locally testable.

Let (i, a, λ) and (j, b, μ) be any regular elements of $\mathcal{M}(T; P; I, \Lambda)$, and consider $(i, a, \lambda)(j, b, \mu)=(i, ap_{\lambda j}b, \mu)$. Any idempotent in the $\mathcal{R}$-class of (i, a, λ) is of the form $(i, a, \varkappa)$, with $a \leqq p_{\varkappa i}$, and any idempotent in the $\mathcal{L}$-class of (j, b, μ) is of the form (k, b, μ), with $b \leqq p_{\mu k}$. It follows that $(k, ap_{\lambda j}b, \varkappa)$ is an inverse of $(i, ap_{\lambda j}b, \mu)$ in $\mathcal{M}(T; P; I, \Lambda)$. We conclude that the set of the regular elements of $\mathcal{M}(T; P; I, \Lambda)$ forms a subsemigroup of $\mathcal{M}(T; P; I, \Lambda)$.

Let us now consider a regular locally testable semigroup S, and let the locally testable semigroup $\mathcal{M}(T; P; I, \Lambda)$ be as prescribed by the theorem. We must show that S is a homomorphic image of a subsemigroup of $\mathcal{M}(T; P; I, \Lambda)$.

Let $\mathcal{S}$ be the set which consists of the elements $(R, Q(a), L)$, where $R \in I$, $L \in \Lambda$, $a \in S$ and $Q(a)S \leqq R$, $SQ(a) \leqq L$ in $\mathcal{Q}(S)$. On $\mathcal{S}$ we define a multiplication by

$$(R_1, Q(a_1), L_1)(R_2, Q(a_2), L_2) = (R_1, Q(a_1 a_2), L_2).$$

We must show that this multiplication is well-defined. Indeed,

$$(Q(a_1)S)(Q(a_1a_2)S) = Q(a_1)SQ(a_1)Q(a_2)S = Q(a_1)Q(a_2)S = Q(a_1a_2)S,$$

$$(Q(a_1a_2)S)(Q(a_1)S) = Q(a_1)Q(a_2)SQ(a_1)S = Q(a_1)SQ(a_1)Q(a_2)SQ(a_1)S =$$
$$= Q(a_1)(SQ(a_1)Q(a_2)S\cap SQ(a_1)S) = Q(a_1)SQ(a_1)SQ(a_1)Q(a_2)S =$$
$$= Q(a_1)Q(a_2)S = Q(a_1a_2)S$$

implies that $Q(a_1a_2)S\subseteq Q(a_1)S\subseteq R_1$, and dually, $SQ(a_1a_2)\subseteq SQ(a_2)\subseteq L_2$. By Theorem 6 and Corollary 7 the mapping $\varphi\colon \mathcal{S}\to S, (R, Q(a), L)\to a$ is a homomorphism of $\mathcal{S}$ onto S.

Let us now consider the mapping $\psi\colon \mathcal{S}\to\mathcal{M}(T;\ P;\ I, \Lambda)$, $(R, Q(a), L)\to \to(R, SQ(a)S, L)$. Let a_1 and a_2 be any elements of S such that

$$(R_1, Q(a_1), L_1)\psi = (R_1, SQ(a_1)S, L_1) = (R_2, S(Q(a_2)S, L_2)) = (R_2, Q(a_2), L_2)\psi.$$

Then $Q(a_1)S\subseteq R_1=R_2$, $Q(a_2)S\subseteq R_1=R_2$ and $Q(a_1)S\mathcal{L}SQ(a_1)S=SQ(a_2)S\mathcal{L}Q(a_2)S$ implies that $Q(a_1)S=Q(a_2)S$ since $I=\mathcal{Q}(S)S$ forms a left normal band. Dually, $SQ(a_1)=SQ(a_2)$. Thus, $Q(a_1)=SQ(a_1)\cap Q(a_1)S=SQ(a_2)\cap Q(a_2)S=Q(a_2)$. We conclude that the mapping $\psi\colon \mathcal{S}\to\mathcal{M}(T;\ P;\ I, \Lambda)$ is injective. Furthermore, if $a_1, a_2\in S$, then

$$(R_1, Q(a_1), L_1)\psi(R_2, Q(a_2), L_2)\psi = (R_1, SQ(a_1)S, L_1)(R_2, SQ(a_2)S, L_2) =$$
$$= (R_1, SQ(a_1)SL_1R_2SQ(a_2)S, L_2) = (R_1, SQ(a_1)L_1R_2Q(a_2)S, L_2) =$$
$$= (R_1, SQ(a_1)Q(a_2)S, L_2) = (R_1, SQ(a_1a_2)S, L_2) = (R_1, Q(a_1a_2), L_2)\psi =$$
$$= ((R_1, Q(a_1), L_1)(R_2, Q(a_2), L_2))\psi$$

shows that ψ is an isomorphism of $\mathcal{S}$ onto a subsemigroup of $\mathcal{M}(T;\ P;\ I, \Lambda)$. We conclude that S is a homomorphic image of the subsemigroup $\mathcal{S}\psi$ of $\mathcal{M}(T;\ P;\ I, \Lambda)$.

Remark 1. If S is a finite regular locally testable semigroup, then S divides a finite locally testable semigroup $\mathcal{M}(T;\ P;\ I, \Lambda)$.

Remark 2. Let us consider the semigroup $\mathcal{S}$ of the proof of Theorem 10. If $(R, Q(a), L)$ is any element of $\mathcal{S}$, and if a' is any inverse of a in S, then it is easy to show that $(Q(a')S, Q(a'), SQ(a'))$ is an inverse of $(R, Q(a), L)$ in $\mathcal{S}$. Hence $\mathcal{S}\psi$ is a regular subsemigroup of $\mathcal{M}(T;\ P;\ I, \Lambda)$. If $e=e^2$ is any idempotent of S, then the subsemigroup $e\varphi^{-1}$ consists of the elements of $\mathcal{S}$ which are of the form $(R, Q(e), L)$; it is easy to see that these elements form a rectangular band. Hence the $\varphi\varphi^{-1}$-classes which contain idempotents form rectangular bands, and we can say that $\mathcal{S}$ is a coextension of S by rectangular bands. This indicates that the division considered in Theorem 10 is rather special: if S is a regular locally testable semigroup, then there exists a coextension of S by rectangular bands which can be embedded as a regular locally testable subsemigroup into the locally testable semigroup $\mathcal{M}(T;\ P;\ I, \Lambda)$ which is constructed in Theorem 10.

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A NOTE ON STRONG CESÀRO SUMMABILITY

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1. This note is concerned with sectional convergence properties of strong Cesàro summability $[C, \alpha]_p$, $\alpha>0, p>0$ (see section 2 for definitions). The main results of the paper characterise the range of values of α and p for which the series to sequence version of strong Cesàro summability has AK and where the sequence to sequence version of strong Cesàro summability (with limit 0) has AK. This generalizes and extends results of PEYERIMHOFF [12] for ordinary Cesàro summability and of WILANSKY and ZELLER [13] for strong Cesàro summability $[C, 1]_p$. As an application, a convergence factor result is given.

2. In this section we give the notation and some basic definitions. We shall throughout let $\underset{\sim}{s}=\{s_n\}_{n\geqq 1}$ be a sequence of complex numbers whose terms are the n^{th} partial sums of the series $\sum_{n=1}^{\infty} a_n$ i.e. $s_n=\sum_{k=1}^{n} a_k$ so that in vector notation $\underset{\sim}{s}=S(\underset{\sim}{a})$ where $\underset{\sim}{a}=\{a_n\}_{n\geqq 1}$ and S denotes the summation matrix. For any real α let A_n^α be defined by the identity

$$\sum_{n=1}^{\infty} A_{n-1}^{\alpha} x^{n-1} = (1-x)^{-\alpha-1}$$

and define the α^{th} Cesàro sum of the sequence $\underset{\sim}{s}$ by

$$S_n^\alpha = S_n^\alpha(\underset{\sim}{s}) = \sum_{k=1}^{n} A_{n-k}^{\alpha-1} s_k = \sum_{k=1}^{n} A_{n-k}^{\alpha} a_k .$$

For $\alpha>-1$ and n a positive integer let

$$\sigma_n^\alpha = S_n^\alpha / A_n^\alpha$$

denote the (C, α) transform of the sequence $\underset{\sim}{s}=\{s_n\}_{n\geqq 1}$ or the series $\sum_{n=1}^{\infty} a_n$. Let C_α denote the sequence to sequence (C, α) transformation and C_α^R denote the series to sequence (C, α) transform.

For $\alpha>0, p>0$ let

$$[C, \alpha]_p = \left\{\underset{\sim}{s} \mid \exists l \in \mathrm{C} \quad \text{with} \quad \sum_{k=1}^{n} |\sigma_k^{\alpha-1} - l|^p = o(n)\right\}$$

denote the space of sequences strongly Cesaro summable of order α and index p, and $[C, \alpha]_{0,p}$ denote the subspace with $l=0$. Let ${}_R[C, \alpha]_p$ denote the corresponding series space to $[C, \alpha]_p$ i.e. $\underset{\sim}{a} \in {}_R[C, \alpha]_p$ if and only if $\underset{\sim}{s} \in [C, \alpha]_p$.

We shall use the usual notation for differences i.e. for any real α

$$\Delta^\alpha s_n = \sum_{k=n}^{\infty} A_{k-n}^{-\alpha-1} s_k$$

whenever the series converges, (see [1]).

A linear sequence space X with a given topology is called a K-space if the coordinate functionals $\underline{x} \mapsto x_k$ are continuous for every $k \geqq 1$. A p-normed or normed sequence space X has AK if X contains all sequences with finitely many non zero coordinates and if for every $\underline{x} \in X$, $\|\underline{x} - \underline{x}^{(n)}\| \to 0$ as $n \to \infty$ where $\underline{x}^{(n)}$ denotes the n^{th} section of $\underline{x}$ i.e. $\underline{x}^{(n)} = \sum_{k=1}^{n} x_k \underline{\delta}^k$ where $\underline{\delta}^k$ denotes the sequence with 1 in the k^{th} coordinate and 0's elsewhere. X has SAK if X contains all sequences with finitely many non zero coordinates and if for every $\underline{x} \in X$, $f(\underline{x}^{(n)}) \to f(\underline{x})$ as $n \to \infty$ for all $f \in X^*$, the continuous dual of X. If X is a Banach K-space then ZELLER showed in [14] that X has AK if and only if X has SAK. In the p-normed case (where $0<p<1$) we shall show that this need not necessarily hold (see the corollary to Theorem 8).

Σ_r denotes a sum taken over the range $2^r \leqq k < 2^{r+1}$ and $\max_r$ denotes the maximum of a function of the integer k over the same range.

We use $u_n \asymp v_n$ to mean that u_n lies between two positive constant multiples of v_n, and $M(a, b, \ldots)$ to denote a positive constant, depending on $(a, b, \ldots)$, that may be different at each occurrence.

If A and B are two summability methods then A$\Rightarrow$B means that any sequence limitable A is limitable B to the same value.

If X and Y are sequence spaces then $\underline{\varepsilon} \in (X; Y)$ means that $\{\varepsilon_n x_n\}_{n \geqq 1} \in Y$ for every sequence $\underline{x} \in X$.

δ denotes the constant sequence consisting of all 1's.

If $p>0$ then q denotes the conjugate index i.e. q is determined by $\frac{1}{p}+\frac{1}{q}=1$, and this is interpreted with the usual conventions of $q=\infty$ if $p=1$ and $q=1$ if $p=\infty$.

3. It was observed in [9] that $[C, 1]_p$ could be made into a Banach K-space if $p \geqq 1$ and a complete p-normed K-space if $0<p<1$. For $\alpha>0$, since $C_{\alpha-1}\colon [C, \alpha]_p \to [C, 1]_p$ and $C_{\alpha-1}^R\colon {}_R[C, \alpha]_p \to [C, 1]_p$ are bijections, we can lift the norm structure on $[C, 1]_p$ back to $[C, \alpha]_p$ and ${}_R[C, \alpha]_p$. Thus, if $p \geqq 1$, both are Banach spaces under the norm

$$\|\underline{a}\| = \|\underline{s}\| = \sup_{r \geqq 0} \Big(2^{-r} \sum_r |\sigma_k^{\alpha-1}|^p\Big)^{1/p} \tag{1}$$

where $\underline{s} \in [C, \alpha]_p$, $\underline{a} \in {}_R[C, \alpha]_p$, and complete p-normed spaces, if $0<p<1$, under the p-norm

$$\|\underline{a}\| = \|\underline{s}\| = \sup_{r \geqq 0} \Big(2^{-r} \sum_r |\sigma_k^{\alpha-1}|^p\Big). \tag{2}$$

With these norms, $C_{\alpha-1}$ and $C_{\alpha-1}^R$ are isometries between $[C, \alpha]_p$, resp. ${}_R[C, \alpha]_p$, and $[C, 1]_p$, and it is easy to check that both are K-spaces. For example, if $\underline{s} \in [C, \alpha]_p$

with $p \geqq 1$ then

$$s_k = \sum_{\nu=1}^{k} A_{k-\nu}^{-\alpha} A_{\nu-1}^{\alpha-1} \sigma_\nu^{\alpha-1} \quad \text{so that} \quad |s_k| \leqq \sum_{\nu=1}^{k} |A_{k-\nu}^{-\alpha} A_{\nu-1}^{\alpha-1} \sigma_\nu^{\alpha-1}|$$

and $|\sigma_\nu^{\alpha-1}| \leqq M(\nu) \|\underline{s}\|$ since $[C, 1]_p$ is a K-space. Hence

$$|s_k| \leqq \left(\sum_{\nu=1}^{k} |A_{k-\nu}^{-\alpha} A_{\nu-1}^{\alpha-1} M(\nu)| \right) \|\underline{s}\|$$

and so the coordinate functions are continuous.

We can use the isometries $C_{\alpha-1}$ and $C_{\alpha-1}^R$ and the known representation (see [10]) of $[C, 1]_p^*$ to obtain the continuous duals of $[C, \alpha]_p$ and ${}_R[C, \alpha]_p$. For example, $f \in {}_R[C, \alpha]_p^*$ if and only if there is an $F \in [C, 1]_p^*$ such that $f(\underline{a}) = F(C_{\alpha-1}^R(\underline{a}))$ for all $\underline{a} \in {}_R[C, \alpha]_p$ and (from [10])

$$F(\underline{x}) = l\left(F(\underline{\delta}) - \sum_{k=1}^{\infty} F(\underline{\delta}^k)\right) + \sum_{k=1}^{\infty} x_k F(\underline{\delta}^k)$$

where $\underline{x} - l\underline{\delta} \in [C, 1]_{0,p}$ and where

$$\|F\| = \begin{cases} |F(\underline{\delta})| + \sum_{r=0}^{\infty} 2^{r/p} \max_r |F(\underline{\delta}^k)| & \text{if} \quad 0 < p \leqq 1, \\ |F(\underline{\delta})| + \sum_{r=0}^{\infty} 2^{r/p} \left(\sum_r |F(\underline{\delta}^k)|^q\right)^{1/q} & \text{if} \quad p > 1. \end{cases} \tag{3}$$

Thus, if l is the ${}_R[C, \alpha]_p$ limit of the series $\sum_{n=1}^{\infty} a_n$, the mapping defined by $f(\underline{a}) = l$ is a continuous linear functional on ${}_R[C, \alpha]_p$ since it corresponds to taking $F(\underline{\delta}^k) = 0$ for $k \geqq 1$ and $F(\underline{\delta}) = 1$.

It was pointed out in [9] that $[C, 1]_{0,p}$ has AK for all $p > 0$; but even though there is an isometry $C_{\alpha-1}$ between $[C, \alpha]_{0,p}$ and $[C, 1]_{0,p}$, it does not follow that $[C, \alpha]_{0,p}$ has AK for all $p > 0$ and it is the investigation of this question that concerns us here. As a preliminary to this we obtain a limitation theorem for $[C, \alpha]_p$. For $p \geqq 1$ it was proved in [5] that $[C, \alpha]_p \Rightarrow (C, \alpha + \frac{1}{p} - 1 + \delta)$ for every $\delta > 0$ and so we have a limitation theorem for $[C, \alpha]_p$ from the known one for (C, α). However we can improve this. We first state a known result.

LEMMA 1. *If* $x_\nu \geqq 0$ *and* λ, μ *are real constants with* $\lambda + \mu > 0$ *and if* $\sum_{\nu=1}^{n} x_\nu = o(n^\lambda)$ *then* $\sum_{\nu=1}^{n} \nu^\mu x_\nu = o(n^{\lambda+\mu})$.

PROOF. This result may easily be verified by partial summation.

THEOREM 1. *Let* $\alpha > 0$ *and* $1 \leqq p < \infty$. *If* $s_n \to l[C, \alpha]_p$ *then*
(i) $S_n^\beta = o(n^{\alpha+1/p-1})$ *if* $\beta < \alpha + 1/p - 1$,
(ii) $S_n^\beta - lA_{n-1}^\beta = o((\log n)^{1/q} n^\beta)$ *if* $\beta = \alpha + 1/p - 1$,
(iii) $S_n^\beta - lA_{n-1}^\beta = o(n^\beta)$ *if* $\beta > \alpha + 1/p - 1$.

Moreover these results are best possible in the sense that they become false if we insert a factor $\chi(n)$ *on the right in* (i), (ii) *or* (iii), *where* $\chi(n)$ *is any given positive function decreasing to* 0.

PROOF. Suppose $s_n \to l[C, \alpha]_p$. If $p=1$ then $[C, \alpha]_1 \Rightarrow (C, \alpha)$ and (i), (ii), (iii) follow from the known results for (C, α) (see [6]). If $p>1$ then write

$$S_n^\beta - lA_{n-1}^\beta = \sum_{\nu=1}^{n} A_{n-\nu}^{\beta-\alpha} A_{\nu-1}^{\alpha-1}(\sigma_\nu^{\alpha-1} - l) \tag{4}$$

and apply Hölder's inequality to get

$$|S_n^\beta - lA_{n-1}^\beta| \leqq \left(\sum_{\nu=1}^{n} |A_{n-\nu}^{\beta-\alpha}|^q A_{n-1}^{\alpha-1}\right)^{1/q} \left(\sum_{\nu=1}^{n} A_{\nu-1}^{\alpha-1} |\sigma_\nu^{\alpha-1} - l|^p\right)^{1/p}.$$

Using Lemma 1 on the second term on the right and a simple calculation on the first term, gives the result. (Note that we may omit the second term on the left in (ii) if $p>1$).

To show these are best possible, let $\varphi(n)$ denote the expression inside the bracket of the 'o' term in the theorem. From (4)

$$\frac{S_n^\beta - lA_{n-1}^\beta}{\varphi(n)\chi(n)} = \frac{1}{\varphi(n)\chi(n)} \sum_{\nu=1}^{n} A_{n-\nu}^{\beta-\alpha} A_{\nu-1}^{\alpha-1}(\sigma_\nu^{\alpha-1} - l) = \sum_{\nu=1}^{n} \gamma_{n\nu}(\sigma_\nu^{\alpha-1} - l),$$

say, so that if the results are not best possible, then for some $\chi(n)$, $(\gamma_{n\nu})$ has the property that it transforms every $[C, 1]_{0,p}$ summable sequence into a sequence converging to 0. Such matrices were investigated by MADDOX in [10] and a necessary condition is that

$$\sup_{n\geqq 1} \sum_{r=0}^{N} 2^{r/p}(\Sigma_r |\gamma_{nk}|^q)^{1/q} < \infty$$

where $2^N \leqq n < 2^{N+1}$. Taking $n = 2^{N+1}-1$ and the term $r=N$ in the sum we see, in particular, that it is necessary that

$$2^{N/p}\Big(\sum_N |\gamma_{nk}|^q\Big)^{1/q} = O(1). \tag{5}$$

This is to be interpreted in accordance with the usual conventions when $p=1$. But it is easily seen that, for $n=2^{N+1}-1$,

$$2^{N/p}\Big(\sum_N |\gamma_{nk}|^q\Big)^{1/q} \asymp \frac{1}{\chi(n)}.$$

Thus (5) is false, since $\chi(n) \to 0$ as $n \to \infty$ and so the results are best possible.

THEOREM 2. *Let* $\alpha>0$ *and* $0<p<1$. *If* $s_n \to l[C, \alpha]_p$ *then*
(i) $S_n^\beta = o(n^{\alpha+1/p-1})$ *if* $\beta \leqq \alpha$,
(ii) $S_n^\beta = o(n^{\beta+1/p-1})$ *if* $\beta > \alpha$.

These results are best possible in the sense of Theorem 1. *Moreover they are best possible even if we restrict* $\{s_n\}_{n\geqq 1}$ *so that* $S_n^{\alpha-1} \geqq 0$.

PROOF. Since (i) and (ii) are unaffected if we change s_n by a constant we may suppose $l=0$. Now

(6) $$S_n^\beta = \sum_{\nu=1}^{n} A_{n-\nu}^{\beta-\alpha} S_\nu^{\alpha-1}$$

so that if $\beta \leqq \alpha$, $A_{n-\nu}^{\beta-\alpha}$ is bounded and

(7) $$|S_n^\beta| \leqq M \sum_{\nu=1}^{n} |S_\nu^{\alpha-1}|.$$

Since $1+(\alpha-1)p>0$, $s_n \to 0\ [C, \alpha]_p$ implies by Lemma 1 that

$$\sum_{\nu=1}^{n} |S_\nu^{\alpha-1}|^p = o(n^{1+(\alpha-1)p})$$

and hence

$$(\sup_{1\leqq\nu\leqq n} |S_\nu^{\alpha-1}|)^p = o(n^{1+(\alpha-1)p}).$$

Thus

$$\sum_{\nu=1}^{n} |S_\nu^{\alpha-1}| \leqq (\sup_{1\leqq\nu\leqq n} |S_\nu^{\alpha-1}|)^{1-p} \sum_{\nu=1}^{n} |S_\nu^{\alpha-1}|^p = o(n^{\alpha+1/p-1})$$

and so (i) follows from (7).

If $\beta>\alpha$,

$$S_n^\beta = \sum_{\nu=1}^{n} A_{n-\nu}^{\beta-\alpha-1} S_\nu^\alpha$$

and so by the result for $\beta=\alpha$, we see that (ii) holds.

To prove the best possible clause in the case $\beta \leqq \alpha$, suppose that a positive $\chi(n)$ decreasing to 0 has been given; choose the sequence $\{s_n\}_{n\geqq 1}$ so that

$$S_n^{\alpha-1} = \begin{cases} 2n^{\alpha+1/p-1}\chi(n) \quad \text{or} \quad 0 & \text{if} \quad n=2^r,\ r=0,1,\ldots \\ 0 \text{ otherwise,} \end{cases}$$

where the choice in the case $n=2^r$ is to be made inductively. Whatever choice is made, we have $s_n \to 0\ [C, \alpha]_p$. If $S_n^{\alpha-1}$ has been chosen for $n=2^\varrho$, $\varrho=0, 1, \ldots, r-1$, then the two possible values of S_n^β for $n=2^r$ will differ by $2n^{\alpha+1/p-1}\chi(n)$, so for at least one of these we must have

(8) $$|S_n^\beta| \geqq n^{\alpha+1/p-1}\chi(n).$$

Choose $S_n^{\alpha-1}$ so that (8) holds. This shows that (i) does not hold with a factor $\chi(n)$ on the right, and so (i) is best possible even when $S_n^{\alpha-1} \geqq 0$.

If $\beta>\alpha$, choose $\{s_n\}_{n\geqq 1}$ so that

$$S_n^{\alpha-1} = \begin{cases} n^{\alpha+1/p-1}\chi(2n) & \text{if} \quad n=2^r,\ r=0,1,\ldots, \\ 0 \text{ otherwise.} \end{cases}$$

Then $s_n \to 0\ [C, \alpha]_p$ and for $n=2^{r+1}$, (6) gives

$$S_n^\beta \geqq A_{2^r}^{\beta-\alpha} S_{2^r}^{\alpha-1} \geqq M n^{\beta+1/p-1}\chi(n)$$

which shows that (ii) is best possible even when $S_n^{\alpha-1} \geqq 0$.

4. In this section we determine the AK properties of ${}_R[C, \alpha]_p$. We begin with a useful lemma concerning sums of binomial coefficients.

LEMMA 2. (i) *If* $\nu \leqq n < k$ *then*

$$\sum_{\mu=\nu}^{n} A_{k-\mu}^{\alpha-2} A_{\mu-\nu}^{-\alpha} = -\frac{1-\alpha}{k-\nu} A_{n-\nu}^{1-\alpha} A_{k-1-n}^{\alpha-1}. \tag{9}$$

(ii) *If* $\nu < m \leqq k$ *then*

$$\sum_{\mu=m}^{k} A_{k-\mu}^{\alpha-1} A_{\mu-\nu}^{-\alpha-1} = \frac{-\alpha}{k-\Lambda} A_{m-1-\nu}^{-\alpha} A_{k-m}^{\alpha}. \tag{10}$$

PROOF. (i) A short calculation shows that for any real α

$$(1-\alpha)(A_{\mu-\nu}^{1-\alpha} A_{k-1-\mu}^{\alpha-1} - A_{\mu-1-\nu}^{1-\alpha} A_{k-\mu}^{\alpha-1}) = -(k-\nu) A_{k-\mu}^{\alpha-2} A_{\mu-\nu}^{-\alpha} \tag{11}$$

and so (9) follows on summing (11) from $\mu=\nu$ to n.

(ii) We deduce from (i) that, for $\nu < m \leqq k$

$$\sum_{\mu=m}^{k} A_{k-\mu}^{\alpha-2} A_{\mu-\nu}^{-\alpha} = \frac{1-\alpha}{k-\nu} A_{m-1-\nu}^{1-\alpha} A_{k-m}^{\alpha-1}.$$

Since this is valid for any value of α, we obtain (10) on replacing α by $\alpha+1$.

THEOREM 3. *Let* $1 \leqq p < \infty$. *If* $0 < \alpha < 1 - \dfrac{1}{p}$ *then* ${}_R[C, \alpha]_p$ *has* AK.

PROOF. Since it makes no difference if we alter the first term of the series, we may suppose throughout that the series is summable $[C, \alpha]_p$ to 0. Let $\underline{a} \in {}_R[C, \alpha]_p$, $\underline{t} = C_{\alpha-1}^R(\underline{a})$ and $\underline{t}^n = C_{\alpha-1}^R(\underline{a}^{(n)})$ where $\underline{a}^{(n)}$ is the n^{th} section of $\underline{a}$ (note that $\underline{t}^n$ is *not* the n^{th} section of $\underline{t}$). Thus ${}_R[C, \alpha]_p$ has AK if and only if

$$2^{-r} \sum_r |t_k|^p \to 0 \tag{12}$$

as $r \to \infty$ implies that

$$2^{-r} \sum_r |t_k - t_k^n|^p \to 0 \tag{13}$$

as $n \to \infty$, uniformly in r. For $k \leqq n$, $t_k - t_k^n = 0$ and for $k > n$,

$$\begin{aligned} t_k - t_k^n &= \frac{1}{A_{k-1}^{\alpha-1}} \sum_{\mu=n+1}^{k} A_{k-\mu}^{\alpha-1} a_\mu = \frac{1}{A_{k-1}^{\alpha-1}} \sum_{\mu=n+1}^{k} A_{k-\mu}^{\alpha-1} \sum_{\nu=1}^{\mu} A_{\mu-\nu}^{-\alpha-1} A_{\nu-1}^{\alpha-1} t_\nu = \\ &= \frac{1}{A_{k-1}^{\alpha-1}} \sum_{\nu=1}^{k} A_{\nu-1}^{\alpha-1} t_\nu \sum_{\mu=\max(\nu, n+1)}^{k} A_{k-\mu}^{\alpha-1} A_{\mu-\nu}^{-\alpha-1} = \\ &= t_k + \frac{1}{A_{k-1}^{\alpha-1}} \sum_{\nu=1}^{n} A_{\nu-1}^{\alpha-1} t_\nu \sum_{\mu=n+1}^{k} A_{k-\mu}^{\alpha-1} A_{\mu-\nu}^{-\alpha-1} = t_k + v_k^{(n)}, \end{aligned} \tag{14}$$

say. Since (12) holds, the first term on the right of (14) gives no trouble and it follows from Minkowski's inequality that (12) implies (13) if and only if (12) implies that

$$2^{-r} \sum_r |v_k^{(n)}|^p \to 0 \tag{15}$$

as $n\to\infty$, uniformly in r. Also (15) is equivalent to (see [11] page 171).

$$\frac{1}{N}\sum_{k=n+1}^{N}|v_k^{(n)}|^p\to 0 \tag{16}$$

as $n\to\infty$, uniformly in $N>n$. We note that, by Lemma 2 (ii)

$$v_k^{(n)}=-\alpha\frac{A_{k-n-1}^{\alpha}}{A_{k-1}^{\alpha-1}}\sum_{\nu=1}^{n}\frac{1}{k-\nu}A_{\nu-1}^{\alpha-1}A_{n-\nu}^{-\alpha}t_\nu . \tag{17}$$

In what follows $\sum_{k=x}^{y}$, where x and y are not necessarily integers, is to be taken to mean the sum over those integers k for which $x\leqq k\leqq y$. We consider first the contribution to (16) of those terms with $k\geqq 3n/2$. By (17),

$$\frac{1}{N}\sum_{k=3n/2}^{N}|v_k^{(n)}|^p\leqq\frac{M}{N}\sum_{k=3n/2}^{N}\left\{\sum_{\nu=1}^{n}A_{\nu-1}^{\alpha-1}A_{n-\nu}^{-\alpha}|t_\nu|\right\}^p\leqq$$

$$\leqq M\left\{\sum_{\nu=1}^{n}A_{\nu-1}^{\alpha-1}A_{n-\nu}^{-\alpha}|t_\nu|\right\}^p=o(1)$$

by the case $\beta=0,\ l=0$ of Theorem 1(iii), since it is clear that $\underline{t}\in[C,1]_{0,p}$ if and only if $\{|t_n|\}\in[C,1]_{0,p}$.

Thus it remains to show that (12) implies that

$$\sum_{k=n+1}^{3n/2}|v_k^{(n)}|^p=o(n). \tag{18}$$

Let $v_k^{(n)}=x_k^{(n)}+y_k^{(n)}+z_k^{(n)}$, where $x_k^{(n)}$ denotes the contribution to (17) of those terms with $\nu\leqq\frac{1}{2}n$, $y_k^{(n)}$ the contribution of those terms with $\frac{1}{2}n<\nu\leqq 2n-k$, and $z_k^{(n)}$ the contribution of those terms with $2n-k<\nu\leqq n$. By Minkowski's inequality, it is enough to show that (18) holds with $v_k^{(n)}$ replaced by each of $x_k^{(n)}, y_k^{(n)}, z_k^{(n)}$.

Consider first $x_k^{(n)}$. We have

$$\frac{1}{n}\sum_{k=n+1}^{3n/2}|x_k^{(n)}|^p\leqq Mn^{-1-2\alpha p}\sum_{k=n+1}^{3n/2}(k-n)^{\alpha p}\left\{\sum_{\nu=1}^{(1/2)n}\nu^{\alpha-1}|t_\nu|\right\}^p\leqq \tag{19}$$

$$\leqq Mn^{-\alpha p}\left\{\sum_{\nu=1}^{n}\nu^{\alpha-1}|t_\nu|\right\}^p.$$

But $[C,1]_p\Rightarrow[C,1]_1$ (see [5]), and thus, by Lemma 1, the expression in curly brackets in (19) is $o(n^\alpha)$. Hence (18) holds with $v_k^{(n)}$ replaced by $x_k^{(n)}$.

For $y_k^{(n)}$, we use the fact that, for relevant values of the parameters, $k-\nu\geqq\geqq n+1-\nu$. Hence, by (17),

$$\frac{1}{n}\sum_{k=n+1}^{3n/2}|y_k^{(n)}|^p\leqq\frac{M}{n}\sum_{k=n+1}^{3n/2}(k-n)^{\alpha p}\left\{\sum_{\nu=1/2(n+1)}^{2n-k}(n+1-\nu)^{-\alpha-1}|t_\nu|\right\}^p. \tag{20}$$

But, by Hölder's inequality,

$$\sum_{\nu=1/2(n+1)}^{2n-k}(n+1-\nu)^{-\alpha-1}|t_\nu| \leqq \left\{\sum_{\nu=1/2(n+1)}^{2n-k}(n+1-\nu)^{-1}|t_\nu|^p\right\}^{1/p}\left\{\sum_{\nu=1/2(n+1)}^{2n-k}(n+1-\nu)^{-\alpha q-1}\right\}^{1/q} \leqq$$

$$\leqq M(k-n)^{-\alpha}\left\{\sum_{\nu=1/2(n+1)}^{2n-k}(n+1-\nu)^{-1}|t_\nu|^p\right\}^{1/p}.$$

Thus, by (20),

$$\frac{1}{n}\sum_{k=n+1}^{3n/2}|y_k^{(n)}|^p \leqq \frac{M}{n}\sum_{k=n+1}^{3n/2}\sum_{\nu=(n+1)/2}^{2n-k}(n+1-\nu)^{-1}|t_\nu|^p =$$

$$= \frac{M}{n}\sum_{\nu=1/2(n+1)}^{n-1}(n+1-\nu)^{-1}|t_\nu|^p\sum_{k=n+1}^{2n-\nu}1 \leqq \frac{M}{n}\sum_{\nu=1/2(n+1)}^{n-1}|t_\nu|^p = o(1)$$

by (12).

For $z_k^{(n)}$, using $k-\nu \geqq k-n$, we deduce from (17) that

$$\frac{1}{n}\sum_{k=n+1}^{3n/2}|z_k^{(n)}|^p \leqq \frac{M}{n}\sum_{k=n+1}^{3n/2}(k-n)^{\alpha p-p}\left(\sum_{\nu=2n-k+1}^{n}(n+1-\nu)^{-\alpha}|t_\nu|\right)^p. \tag{21}$$

We choose η so that $0<\eta<1-\alpha-\frac{1}{p}$. Noting that the second inequality is equivalent to $(-\alpha-\eta)q>-1$, we see by Hölder's inequality that

$$\sum_{\nu=2n-k+1}^{n}(n+1-\nu)^{-\alpha}|t_\nu| \leqq \left(\sum_{\nu=2n-k+1}^{n}(n+1-\nu)^{\eta p}|t_\nu|^p\right)^{1/p}\cdot$$

$$\cdot\left(\sum_{\nu=2n-k+1}^{n}(n+1-\nu)^{(-\alpha-\eta)q}\right)^{1/q} \leqq M(k-n)^{-\alpha-\eta+1/q}\left(\sum_{\nu=2n-k+1}^{n}(n+1-\nu)^{\eta p}|t_\nu|^p\right)^{1/p}.$$

Substituting in (21), we obtain

$$\frac{1}{n}\sum_{k=n+1}^{3n/2}|z_k^{(n)}|^p \leqq \frac{M}{n}\sum_{k=n+1}^{3n/2}(k-n)^{-\eta p-1}\sum_{\nu=2n-k+1}^{n}(n+1-\nu)^{\eta p}|t_\nu|^p =$$

$$= \frac{M}{n}\sum_{\nu=(1/2)n+1}^{n}(n+1-\nu)^{\eta p}|t_\nu|^p\sum_{k=2n-\nu+1}^{3n/2}(k-n)^{-\eta p-1} \leqq \frac{M}{n}\sum_{\nu=(1/2)n+1}^{n}|t_\nu|^p = o(1)$$

by (12). This completes the proof.

We remark that, using the "best possible" clause of Theorem 1(iii), the treatment of the range $k \geqq 3n/2$ could easily be adapted to prove that ${}_R[C,\alpha]_p$ does not have AK when $p \geqq 1, \alpha>0, 1-1/p<\alpha<1$. We have not done this, because this result is included in the following theorem.

THEOREM 4. *If* $\alpha>0, 0<p<\infty$ *and* $\alpha \geqq 1-\frac{1}{p}$ *then* ${}_R[C,\alpha]_p$ *does not have* SAK.

PROOF. If ${}_R[C,\alpha]_p$ has SAK and $\underline{a}\in {}_R[C,\alpha]_p$, then for any $f\in {}_R[C,\alpha]_p^*$, we have $f(\underline{a})=\sum_{k=1}^{\infty}a_k f(\underline{\delta}^k)$.

We construct an f for which this is false for some $\underline{a}\in {}_R[C,\alpha]_p$. Take f to be the limit functional i.e. if $\sum_{k=1}^{\infty} a_k = l[C,\alpha]_p$ define $f(\underline{a})=l$. By the remarks after (3), $f\in {}_R[C,\alpha]_p^*$ and $f(\underline{\delta}^k)=1$ for all $k\geqq 1$. Thus a necessary condition for ${}_R[C,\alpha]_p$ to have SAK is that $\sum_{k=1}^{\infty} a_k$ converges for all $\underline{a}\in {}_R[C,\alpha]_p$. If $0<p<1$, then Theorem 2(i) with $\beta=-1$ gives the best possible limitation theorem as $a_n=o(n^{\alpha+1/p-1})$ and so a_n is not necessarily $o(1)$. If $p\geqq 1$, then Theorem 1(i) with $\beta=-1$ gives the same result as long as $\alpha>1-\dfrac{1}{p}$. If $p\geqq 1$ and $\alpha=1-\dfrac{1}{p}$ then a result of FLETT in [5] shows that ${}_R[C,\alpha]_p \not\Rightarrow {}_R\left(C,\alpha+\dfrac{1}{p}-1\right)$ and so there exists $\underline{a}\in {}_R[C,\alpha]_p$ with $\sum_{k=1}^{\infty} a_k$ divergent. Hence the result.

It follows à fortiori that ${}_R[C,\alpha]_p$ does not have AK in the range $\alpha>0$, $p>0$, $\alpha\geqq 1-\dfrac{1}{p}$.

5. In this section we determine the AK properties of $[C,\alpha]_{0,p}$. The main result is

THEOREM 5. *Let* $1\leqq p<\infty$. *If* $1-\dfrac{1}{p}<\alpha<2-\dfrac{1}{p}$ *then* $[C,\alpha]_{0,p}$ *has* AK.

PROOF. The result for $\alpha=1$ (and all $p>0$) was pointed out in [9] and so in what follows we assume $\alpha\neq 1$. Let $\underline{s}\in[C,\alpha]_{0,p}$, $\underline{t}=C_{\alpha-1}(\underline{s})$, and $\underline{t}^n=C_{\alpha-1}(\underline{s}^{(n)})$ where $\underline{s}^{(n)}$ is the n^{th} section of $\underline{s}$ (note $\underline{t}^n$ is *not* the n^{th} section of $\underline{t}$). For $k\leqq n$, $t_k-t_k^n=0$ and for $k>n$,

$$t_k-t_k^n=\frac{1}{A_{k-1}^{\alpha-1}}\sum_{\mu=n+1}^{k} A_{k-\mu}^{\alpha-2}s_\mu=\frac{1}{A_{k-1}^{\alpha-1}}\sum_{\mu=n+1}^{k} A_{k-\mu}^{\alpha-2}\sum_{\nu=1}^{\mu} A_{\mu-\nu}^{-\alpha}A_{\nu-1}^{\alpha-1}t_\nu=$$

$$=t_k-\frac{1}{A_{k-1}^{\alpha-1}}\sum_{\nu=1}^{n} A_{\nu-1}^{\alpha-1}t_\nu\sum_{\mu=\nu}^{n} A_{\mu-\nu}^{-\alpha}A_{k-\mu}^{\alpha-2}=t_k-v_k^{(n)},$$

say. As in Theorem 3, $[C,\alpha]_{0,p}$ has AK if and only if (12) implies (16) for $v_k^{(n)}$ just defined. We note that, by Lemma 2(i),

$$v_k^{(n)}=-\frac{(1-\alpha)A_{k-1-n}^{\alpha-1}}{A_{k-1}^{\alpha-1}}\sum_{\nu=1}^{n}\frac{1}{k-\nu}A_{n-\nu}^{1-\alpha}A_{\nu-1}^{\alpha-1}t_\nu=\frac{u_k^{(n)}}{A_{k-1}^{\alpha-1}},\tag{22}$$

say. By an argument similar to that giving Lemma 1, we see (since $\alpha>1-1/p$) that (16) is equivalent to the assertion that

$$N^{(1-\alpha)p-1}\sum_{k=n+1}^{N}|u_k^{(n)}|^p\to 0\tag{23}$$

as $n\to\infty$, uniformly in $N>n$. Write $u_k^{(n)}=x_k^{(n)}+y_k^{(n)}$ where $x_k^{(n)}$ denotes the contribution of those terms in the sum (22) for which $\nu\leqq 2n-k$, and $y_k^{(n)}$ the contribution of the remaining terms; note that $x_k^{(n)}=0$ when $k\geqq 2n$. By Minkowski's inequality, it is enough to prove that (23) holds with $u_k^{(n)}$ replaced by each of $x_k^{(n)}$, $y_k^{(n)}$; and, in the case of $x_k^{(n)}$, it is enough to consider $N=2n$.

For $x_k^{(n)}$, we use the result that $k-\nu \geqq n+1-\nu$. Choose η so that $0<\eta<$ $<\alpha-(1-1/p)$, and note that the second inequality is equivalent to $(\eta-\alpha)q<-1$. If $p>1$ we deduce from Hölder's inequality that

$$(24)\qquad |x_k^{(n)}| \leqq MA_{k-1-n}^{\alpha-1} \sum_{\nu=1}^{2n-k} (n+1-\nu)^{-\alpha} \nu^{\alpha-1}|t_\nu| \leqq$$

$$\leqq MA_{k-1-n}^{\alpha-1} \left(\sum_{\nu=1}^{2n-k} (n+1-\nu)^{-\eta p}(\nu^{\alpha-1}|t_\nu|)^p\right)^{1/p} \left(\sum_{\nu=1}^{2n-k} (n+1-\nu)^{(\eta-\alpha)q}\right)^{1/q} \leqq$$

$$\leqq M(k-n)^{\eta-1/p} \left(\sum_{\nu=1}^{2n-k} (n+1-\nu)^{-\eta p}(\nu^{\alpha-1}|t_\nu|)^p\right)^{1/p}.$$

Moreover, it is easily verified that (24) holds also when $p=1$. Thus

$$n^{(1-\alpha)p-1} \sum_{k=n+1}^{2n} |x_k^{(n)}|^p \leqq Mn^{(1-\alpha)p-1} \sum_{k=n+1}^{2n} (k-n)^{\eta p-1} \sum_{\nu=1}^{2n-k} (n+1-\nu)^{-\eta p}(\nu^{\alpha-1}|t_\nu|)^p =$$

$$= Mn^{(1-\alpha)p-1} \sum_{\nu=1}^{n-1} (n+1-\nu)^{-\eta p}(\nu^{\alpha-1}|t_\nu|)^p \sum_{k=n+1}^{2n-\nu} (k-n)^{\eta p-1} \leqq$$

$$\leqq Mn^{(1-\alpha)p-1} \sum_{\nu=1}^{n-1} (\nu^{\alpha-1}|t_\nu|)^p = o(1)$$

by (12) and Lemma 1, since $1+(\alpha-1)p>0$.

For $y_k^{(n)}$, we use $k-\nu \geqq k-n$. Choose η so that $0<\eta<2-\alpha-1/p$; we note that the second inequality is equivalent to $(1-\alpha-\eta)q>-1$. If $p>1$ we have, by Hölder's inequality,

$$(25)\qquad |y_k^{(n)}| \leqq M(k-n)^{\alpha-2} \sum_{\nu=\max(1,2n-k+1)} (n-\nu+1)^{1-\alpha} \nu^{\alpha-1}|t_\nu| \leqq$$

$$\leqq M(k-n)^{\alpha-2} \left(\sum_{\nu=\max(1,2n-k+1)}^{n} (n-\nu+1)^{\eta p}(\nu^{\alpha-1}|t_\nu|)^p\right)^{1/p} \cdot$$

$$\cdot \left(\sum_{\nu=\max(1,2n-k+1)}^{n} (n-\nu+1)^{(1-\alpha-\eta)q}\right)^{1/q} \leqq$$

$$\leqq M(k-n)^{-\eta-1/p} \left(\sum_{\nu=\max(1,2n-k+1)}^{n} (n-\nu+1)^{\eta p}(\nu^{\alpha-1}|t_\nu|)^p\right)^{1/p}.$$

Moreover, it is easily verified that (25) holds also when $p=1$. Thus

$$N^{(1-\alpha)p-1} \sum_{k=n+1}^{N} |y_k^{(n)}|^p \leqq$$

$$\leqq MN^{(1-\alpha)p-1} \sum_{k=n+1}^{N} (k-n)^{-\eta p-1} \sum_{\nu=\max(1,2n-k+1)}^{n} (n-\nu+1)^{\eta p}(\nu^{\alpha-1}|t_\nu|)^p =$$

$$= MN^{(1-\alpha)p-1} \sum_{\nu=\max(1,2n-N+1)}^{n} (n-\nu+1)^{\eta p}(\nu^{\alpha-1}|t_\nu|)^p \sum_{k=2n-\nu+1}^{N} (k-n)^{-\eta p-1} \leqq$$

$$\leqq MN^{(1-\alpha)p-1} \sum_{\nu=1}^{n} (\nu^{\alpha-1}|t_\nu|)^p = o(1)$$

by (12) and Lemma 1.

THEOREM 6. *If* $1\leqq p<\infty$ *and* $0<\alpha\leqq 1-\frac{1}{p}$ *then* $[C,\alpha]_{0,p}$ *does not have* AK.

PROOF. Suppose $[C,\alpha]_{0,p}$ does have AK for $0<\alpha\leqq 1-\frac{1}{p}$. Then by Corollary 2(b) to Theorem 3 in [8] with $F=[C,1]_{0,p}$, $A=C_{\alpha-1}$ we have for $\underline{s}\in[C,\alpha]_{0,p}$

$$s_n=o(\|\underline{\delta}^n\|^{-1}). \tag{26}$$

From (1), if $2^N\leqq n<2^{N+1}$

$$\|\delta^n\|=\max\left[\sup_{r>N}\left\{2^{-r}\sum_r\left|\frac{A_{k-n}^{\alpha-2}}{A_{k-1}^{\alpha-1}}\right|^p\right\}^{1/p},\left\{2^{-N}\sum_{k=n}^{2^{N+1}-1}\left|\frac{A_{k-n}^{\alpha-2}}{A_{k-1}^{\alpha-1}}\right|^p\right\}^{1/p}\right]\asymp n^{1-\alpha-1/p}. \tag{27}$$

so that (26) would give $s_n=o(n^{\alpha+1/p-1})$. But this contradicts Theorem 1 (ii), (iii) with $\beta=0, l=0$ and so if $0<\alpha\leqq 1-\frac{1}{p}$, $[C,\alpha]_{0,p}$ does not have AK.

We need the following lemma (an analogous result for functions has been given in [7]).

LEMMA 3. *If* $0<\beta<1$ *and* $\Delta^{-\beta}s_n$ *exists, then for* $n>k$

$$\left|\sum_{\mu=n}^{\infty}A_{\mu-k}^{\beta-1}s_\mu\right|\leqq\sup_{\mu\geqq n}|\Delta^{-\beta}s_\mu|. \tag{28}$$

We remark that (28) is the transpose of an inequality obtained by BOSANQUET in [3].

PROOF. Let $t_n=\Delta^{-\beta}s_n$. Since $\beta>0$ it follows from Saetning 1 of [2] that $s_n=\Delta^{\beta}t_n$. Hence

$$\sum_{\mu=n}^{\infty}A_{\mu-k}^{\beta-1}s_\mu=\sum_{\mu=n}^{\infty}A_{\mu-k}^{\beta-1}\sum_{\nu=\mu}^{\infty}A_{\nu-\mu}^{-\beta-1}t_\nu=\sum_{\nu=n}^{\infty}t_\nu\sum_{\mu=n}^{\nu}A_{\mu-k}^{\beta-1}A_{\nu-\mu}^{-\beta-1} \tag{29}$$

provided that the interchange in order of summation can be justified. The result would be valid if $n=k$, since the expression on the left would reduce to $\Delta^{-\beta}s_k$ and that on the right to t_k. Hence the inversion can be justified for $n>k$ by writing

$$\sum_{\mu=n}^{\infty}=\sum_{\mu=k}^{\infty}-\sum_{\mu=k}^{n-1}.$$

Thus we have

$$\sum_{\mu=n}^{\infty}A_{\mu-k}^{\beta-1}s_\mu=-\sum_{\nu=n}^{\infty}t_\nu\sum_{\mu=k}^{n-1}A_{\nu-\mu}^{-\beta-1}A_{\mu-k}^{\beta-1}$$

and so (28) follows since

$$\sum_{\nu=n}^{\infty}\left|\sum_{\mu=k}^{n-1}A_{\nu-\mu}^{-\beta-1}A_{\mu-k}^{\beta-1}\right|=1$$

for $0<\beta<1$.

THEOREM 7. *If* $0<p\leqq 1$ *and* $0<\alpha\leqq 1$ *then* $[C,\alpha]_{0,p}$ *has* SAK.

PROOF. We can assume that $0<\alpha<1$ since the case $\alpha=1$ was given in [9]. Since $C_{\alpha-1}\colon [C,\alpha]_{0,p}\to[C,1]_{0,p}$ is an isometry, $f\in[C,\alpha]_{0,p}^*$ if and only if $F\in[C,1]_{0,p}^*$ where $f(\underline{s})=F(\underline{t})$ for $\underline{s}\in[C,\alpha]_{0,p}$ and $\underline{t}=C_{\alpha-1}(\underline{s})$. Also $[C,1]_{0,p}$ has AK and thus $F(\underline{t})=\sum_{k=1}^{\infty} t_k F_k$ for $\underline{t}\in[C,1]_{0,p}$ where $F_k=F(\delta^k)$ (see [10]). Using this we have

$$f_k=f(\underline{\delta}^k)=F(C_{\alpha-1}(\underline{\delta}^k))=\Delta^{1-\alpha}\left(\frac{F_k}{A_{k-1}^{\alpha-1}}\right). \tag{30}$$

Now, for $\underline{s}\in[C,\alpha]_{0,p}$ and $f\in[C,\alpha]_{0,p}^*$, using the notation above

$$\sum_{k=1}^{n} s_k f_k=\sum_{k=1}^{n} f_k\sum_{\mu=1}^{k} A_{k-\mu}^{-\alpha}A_{\mu-1}^{\alpha-1}t_\mu=\sum_{\mu=1}^{n} t_\mu A_{\mu-1}^{\alpha-1}\sum_{k=\mu}^{n} A_{k-\mu}^{-\alpha}f_k. \tag{31}$$

From (3) we know that $F_k=o(k^{-1/p})$, and so $F_k=o(k^{\alpha-1})$. Thus by a result of ANDERSEN [1] we have from (30) that $F_k=A_{k-1}^{\alpha-1}\Delta^{\alpha-1}f_k$. Putting this in (31) we get

$$\sum_{k=1}^{n} s_k f_k=\sum_{\mu=1}^{n} t_\mu F_\mu-\sum_{\mu=1}^{n} t_\mu A_{\mu-1}^{\alpha-1}\sum_{k=n+1}^{\infty} A_{k-\mu}^{-\alpha}f_k,$$

and so $[C,\alpha]_{0,p}$ has SAK if and only if for every $\underline{t}\in[C,1]_{0,p}$ and every $f\in[C,\alpha]_{0,p}^*$

$$\sum_{\mu=1}^{n} t_\mu A_{\mu-1}^{\alpha-1}\sum_{k=n+1}^{\infty} A_{k-\mu}^{-\alpha}f_k\to 0 \tag{32}$$

as $n\to\infty$. By Lemma 3 the left side of (32) is in modulus

$$\leqq\sum_{\mu=1}^{n}|t_\mu|A_{\mu-1}^{\alpha-1}\sup_{k\geqq n+1}|\Delta^{\alpha-1}f_k|=\sup_{k\geqq n+1}\frac{|F_k|}{A_{k-1}^{\alpha-1}}\sum_{\mu=1}^{n}|t_\mu|A_{\mu-1}^{\alpha-1}.$$

From (3), if $2^N\leqq n<2^{N+1}$

$$\sup_{k\geqq n+1}\frac{|F_k|}{A_{k-1}^{\alpha-1}}\leqq\sum_{r=N}^{\infty}\max_r\frac{|F_k|}{A_{k-1}^{\alpha-1}}\leqq M\sum_{r=N}^{\infty}2^{r(1-\alpha)}\max_r|F_k|=o(n^{1-\alpha-1/p}).$$

Thus to prove (32) it is sufficient to prove that

$$\sum_{\mu=1}^{n}|t_\mu|A_{k-1}^{\alpha-1}=o(n^{\alpha+1/p-1})$$

and this follows by Lemma 1, since $\underline{s}\in[C,\alpha]_{0,p}$ implies that $\sum_{\mu=1}^{n}|t_\mu|=o(n^{1/p})$ in the case $0<p\leqq 1$. Hence the result.

The case $p=1$ and $0<\alpha<1$ in Theorem 7 is contained in Theorem 5 since AK and SAK are equivalent for Banach spaces (see [14]). By contrast we have

THEOREM 8. *If* $0<p<1$ *and* $0<\alpha<1$ *then* $[C,\alpha]_{0,p}$ *does not have* AK.

PROOF. Using the same notation as the proof of Theorem 5 we see that $[C,\alpha]_{0,p}$ has AK if and only if (12) implies (16) for the $v_k^{(n)}$ defined as in the proof of Theorem 5. In particular it is necessary though not sufficient, that (12) implies that

$$\frac{1}{3n}\sum_{k=2n}^{3n}|v_k^{(n)}|^p\to 0 \tag{33}$$

as $n\to\infty$. We now show that there is $\underline{s}\in[C,\alpha]_{0,p}$ with $t_n\geqq 0$ for all $n\geqq 1$ but such that (33) does not hold. If $t_n\geqq 0$, it follows from (22) that, uniformly in $2n\leqq k\leqq 3n$

$$v_k^{(n)}\asymp\frac{1}{n}\sum_{\nu=1}^{n}A_{n-\nu}^{1-\alpha}A_{\nu-1}^{\alpha-1}t_\nu=\frac{S_n^{(1)}}{n}.$$

Thus (33) would imply that $S_n^{(1)}=o(n)$ and this contradicts the best possible clause of Theorem 2(ii) with $\beta=1$. Hence $[C,\alpha]_{0,p}$ does not have AK if $0<\alpha<1, 0<p<1$.

COROLLARY. *If* $0<p<1$ *and* $0<\alpha<1$ *then* $[C,\alpha]_{0,p}$ *has* SAK *but does not have* AK.

PROOF. Theorems 7 and 8.

This result is in contrast to the Banach space case and the case $\alpha=1$. We remark here that the natural embedding of $[C,1]_{0,p}$ for $0<p<1$ into its double dual $[C,1]_{0,p}^{**}$ is not an isometry and the metrics are not equivalent. For, given any $\varepsilon>0$ there is $\underline{t}\in[C,1]_{0,p}$ such that for all $F\in[C,1]_{0,p}^{*}$

$$|F(\underline{t})|\leqq\varepsilon\|F\|\,\|\underline{t}\|^{1/p}. \tag{34}$$

To show this, define $t_n=1$ for $2^N\leqq n<2^{N+1}$ where N is a fixed integer, and $t_n=0$ otherwise. Then, by (2), $\|\underline{t}\|=1$ and for $F\in[C,1]_{0,p}^{*}$, $F(\underline{t})=\sum_{k=1}^{\infty}F_k t_k$ where

$$\|F\|=\sum_{r=0}^{\infty}2^{r/p}\max_r|F_k|\geqq 2^{N/p}\max_N|F_k|$$

by (3). Hence $|F(\underline{t})|=|\sum_N F_k|\leqq 2^{N-N/p}\|F\|$ and since $0<p<1$, (34) holds by suitable choice of N. The same remarks apply to $[C,\alpha]_{0,p}$ and ${}_R[C,\alpha]_p$ for $0<p<1$ since they are isometric to $[C,1]_{0,p}$. We now investigate the case $\alpha>1$.

THEOREM 9. *If* $0<p\leqq 1$ *and* $\alpha>1$ *then* $[C,\alpha]_{0,p}$ *does not have* SAK.

PROOF. Suppose $\underline{s}\in[C,\alpha]_{0,p}$. It follows from Theorems 1(i) and 2(i) with $\beta=0$, that $s_n=o(n^{\alpha+1/p-1})$. The proofs of the best possible clauses of these theorems can be adapted to show that, in order that $\varepsilon_n s_n\to 0$ as $n\to\infty$ for every $\underline{s}\in[C,\alpha]_{0,p}$ it is necessary that $\varepsilon_n=O(n^{1-\alpha-1/p})$. A necessary condition for $[C,\alpha]_{0,p}$ to have SAK is that for every $\underline{s}\in[C,\alpha]_{0,p}$ and every $f\in[C,\alpha]_{0,p}^{*}$ $f(\underline{\delta}^n)s_n\to 0$ as $n\to\infty$. By the previous remarks, to show that $[C,\alpha]_{0,p}$ does not have SAK, it is enough to show that there exists $f\in[C,\alpha]_{0,p}^{*}$ such that $f(\underline{\delta}^n)\neq O(n^{1-\alpha-1/p})$. To prove this, suppose on the contrary that, for all $f\in[C,\alpha]_{0,p}^{*}$,

$$f(\underline{\delta}^n)=O(n^{1-\alpha-1/p}),$$

and define $\underline{\psi}^n=n^{\alpha+1/p-1}\underline{\delta}^n$. Thus $f(\underline{\psi}^n)=O(1)$ for all $f\in[C,\alpha]_{0,p}^{*}$ and if we use $\hat{}:[C,\alpha]_{0,p}\to[C,\alpha]_{0,p}^{**}$ to denote the natural map into the double dual we have

$$\hat{\underline{\psi}}^n(f)=f(\underline{\psi}^n)=O(1)$$

i.e. $\{\hat{\underline{\psi}}^n\}_{n\geqq 1}$ is a pointwise bounded system of continuous linear functionals on the Banach space $[C,\alpha]_{0,p}^{*}$ and so is uniformly bounded i.e. $\|\hat{\underline{\psi}}^n\|=O(1)$ so

$$\|\hat{\underline{\delta}}^n\|=O(n^{1-\alpha-1/p}). \tag{35}$$

From (2), if $2^N \leqq n < 2^{N+1}$

$$\|\underline{\delta}^n\| = \max\left[\sup_{r>N}\left(2^{-r}\sum_r\left|\frac{A_{k-n}^{\alpha-2}}{A_{k-1}^{\alpha-1}}\right|^p\right), \left(2^{-N}\sum_{k=n}^{2^{N+1}-1}\left|\frac{A_{k-n}^{\alpha-2}}{A_{k-1}^{\alpha-1}}\right|^p\right)\right] \asymp n^{-p}. \tag{36}$$

In the case $p=1$, $[C, \alpha]_{0,1}$ is a Banach space so $\|\underline{\delta}^n\| = \|\hat{\underline{\delta}}^n\|$ and thus (35) is contradicted by (36).

In the case $0<p<1$,

$$\|\hat{\underline{\delta}}^n\| = \sup_{\|f\|=1} |\hat{\underline{\delta}}^n(f)| = \sup_{\|f\|=1} |f(\underline{\delta}^n)| = \sup_{\|F\|=1} \left|F(C_{\alpha-1}(\underline{\delta}^n))\right| = \sup_{\|F\|=1}\left|\sum_{k=n}^{\infty}\frac{A_{k-n}^{\alpha-2}F_k}{A_{k-1}^{\alpha-1}}\right|$$

where $F\in[C, 1]_{0,p}^*$ and $F(\delta^k)=F_k$.

If $2^N\leqq n<2^{N+1}$, let $F_k=2^{-N/p}$ for $2^N\leqq k<2^{N+1}$, and $F_k=0$ otherwise so that by (3), $F\in[C, 1]_{0,p}^*$ and $\|F\|=1$. Hence

$$\|\hat{\underline{\delta}}^n\| \geqq 2^{-N/p}\sum_{k=n}^{2^{N+1}-1}\frac{A_{k-n}^{\alpha-2}}{A_{k-1}^{\alpha-1}}$$

and so $\|\hat{\underline{\delta}}^{2^N}\|\geqq M2^{-N/p}$ which implies that (35) does not hold since $\alpha>1$.

Thus if $0<p\leqq 1$ and $\alpha>1$ then $[C, \alpha]_{0,p}$ does not have SAK and à fortiori does not have AK.

To complete the results for $[C, \alpha]_{0,p}$ we now prove

THEOREM 10. *If* $1<p<\infty$ *and* $\alpha\geqq 2-\dfrac{1}{p}$ *then* $[C, \alpha]_{0,p}$ *does not have* SAK.

PROOF. We use the same ideas as in the proof of the case $p=1$ in Theorem 9. To calculate $\|\underline{\delta}^n\|$ in this case we use (1) instead of (2) and (36) is replaced by

$$\|\underline{\delta}^n\| \asymp \begin{cases} n^{-1} & \text{if } \alpha > 2-\dfrac{1}{p}, \\ (\log n)^{1/p}\, n^{-1} & \text{if } \alpha = 2-\dfrac{1}{p}, \\ n^{1-\alpha-1/p} & \text{if } \alpha < 2-\dfrac{1}{p}. \end{cases} \tag{37}$$

From Theorem 1(i) with $\beta=0$ we see that the result that $\underline{s}\in[C, \alpha]_{0,p}$ implies $s_n=o(n^{\alpha+1/p-1})$ is best possible. Repeating the argument in Theorem 9, we find that a necessary condition for $[C, \alpha]_{0,p}$ to have SAK is that $\|\underline{\delta}^n\|=O(n^{1-\alpha-1/p})$. Thus (37) shows that $[C, \alpha]_{0,p}$ does not have SAK if $\alpha\geqq 2-\dfrac{1}{p}$.

Before giving an application of the results we make a few remarks concerning the case $p=\infty$. It is familiar that with the usual conventions ${}_R[C, \alpha]_\infty = {}_R(C, \alpha-1)$ and $[C, \alpha]_{0,\infty}=(C, \alpha-1)_0$. Now the translation in α (by 1) between the hypotheses in Theorems 3 and 5 correspond to the results for ordinary Cesàro summability, where $(C, \alpha)_0$ has AK if $0\leqq\alpha\leqq 1$ and ${}_R(C, \alpha)$ has AK if $-1\leqq\alpha\leqq 0$, (see [12] for the first of these and [4] for the second). From these remarks the changes needed for the case $p=\infty$ in Theorems 3, 4, 5, 6 and 10 should be obvious.

6. In this section we give one application of our results to convergence factors for ${}_R[C, \alpha]_p$.

THEOREM 11. *Let* $1 \leqq p < \infty$ *and* $0 < \alpha < 1 - \frac{1}{p}$. *Then* $\underline{\varepsilon} \in ({}_R[C, \alpha]_p;\ {}_R(C, 0))$ *if and only if*

$$\sum_{r=0}^{\infty} 2^{r(\alpha+1/p-1)} \Big(\sum_r |\Delta^\alpha \varepsilon_k|^q\Big)^{1/q} < \infty; \tag{38}$$

where (38) *is taken to include the assertion that the series defining* $\Delta^\alpha \varepsilon_k$ *converges.*

PROOF. Suppose $\underline{\varepsilon} \in ({}_R[C, \alpha]_p;\ {}_R(C, 0))$. Then $\underline{a} \mapsto \sum_{k=1}^{\infty} a_k \varepsilon_k$ defines a continuous linear functional on ${}_R[C, \alpha]_p$, say f, and so there is $F \in [C, 1]_p^*$ such that $f(\underline{a}) = F(C_{\alpha-1}^R(\underline{a}))$. Choosing $\underline{a}$ so that $C_{\alpha-1}^R(\underline{a}) = \underline{\delta}^k$ we have $\underline{a} \in {}_R[C, \alpha]_p$ and

$$F(\underline{\delta}^k) = A_{k-1}^{\alpha-1} \Delta^\alpha \varepsilon_k. \tag{39}$$

Thus (38) follows from (3). Notice that the argument works for all $\alpha > 0$ and $1 \leqq p < \infty$, and so (38) is necessary for $\underline{\varepsilon} \in ({}_R[C, \alpha]_p;\ {}_R(C, 0))$ in all these cases.

For the sufficiency, suppose (38) holds and define $F_k = F(\underline{\delta}^k)$ by (39). By (3), $\{F_k\}_{k \geqq 1} \in [C, 1]_{0,p}^*$ and so for each constant $c \in \mathbf{C}$ there is an $F^c \in [C, 1]_p^*$ such that $F^c(\underline{t}) = \sum_{k=1}^{\infty} t_k F_k + cl$ where $\underline{t} - l\underline{\delta} \in [C, 1]_{0,p}$. Consequently there is an f^c such that $f^c \in {}_R[C, \alpha]_p^*$ and $f^c(\underline{a}) = F^c(C_{\alpha-1}^R(\underline{a}))$ for all $\underline{a} \in {}_R[C, \alpha]_p$. Since ${}_R[C, \alpha]_p$ has AK by Theorem 3, this means (if $\underline{t} = C_{\alpha-1}^R(\underline{a})$)

$$f^c(\underline{a}) = \sum_{k=1}^{\infty} a_k f^c(\underline{\delta}^k) = \sum_{k=1}^{\infty} t_k F_k + cl = \sum_{k=1}^{\infty} t_k A_{k-1}^{\alpha-1} \Delta^\alpha \varepsilon_k + cl. \tag{40}$$

Choosing $\underline{a} = \underline{\delta}^k \in {}_R[C, \alpha]_p$ gives

$$f(\underline{\delta}^k) = \Delta^{-\alpha}(\Delta^\alpha \varepsilon_k) + c. \tag{41}$$

By a result of ANDERSEN in [2] since $\Delta^{-\alpha}(\Delta^\alpha \varepsilon_n)$ exists and $0 < \alpha < 1$, $\varepsilon_k = \Delta^{-\alpha}(\Delta^\alpha \varepsilon_k) + d$ where d is a constant. Combining this with (41) we see $f^c(\underline{\delta}^k) = \varepsilon_k + (c + d)$. Now ${}_R[C, \alpha]_p \Rightarrow {}_R(C, 0)$ if $0 < \alpha < 1 - \frac{1}{p}$ (see [5]), and we find that (40) implies that $\underline{\varepsilon} \in ({}_R[C, \alpha]_p;\ {}_R(C, 0))$. Hence the result.

In conclusion we add that we have obtained convergence factor results for all values of $\alpha > 0$, $p > 0$ for both ${}_R[C, \alpha]_p$ and $[C, \alpha]_{0,p}$ and that the second author has extended Theorem 5 to strong Riesz summability $[R, \lambda, k]_p$. These results will appear elsewhere.

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ÜBER DIE LEBESGUESCHEN FUNKTIONEN. III

Von

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1. In einer vorigen Arbeit [3] haben wir den Einfluß der Lebesgueschen Funktionen von orthonormierten Systeme auf die Konvergenz der Orthogonalreihen untersucht.

Für ein orthonormiertes System $\varphi=\{\varphi_k(x)\}_1^\infty$ im Intervall (0, 1) seien

$$L_n(\varphi;\, x)=\int_0^1\left|\sum_{k=1}^n \varphi_k(x)\,\varphi_k(t)\right| dt \quad (x\in(0,1);\ n=1,2,\ldots)$$

die Lebesgueschen Funktionen. Es sei $\lambda=\{\lambda_k\}_1^\infty$ eine nichtabnehmende Folge von positiven Zahlen mit $\lambda_k\to\infty$ $(k\to\infty)$. Für eine meßbare Menge $E(\subseteqq(0,1))$ mit positivem Maß führen wir die folgenden Klassen der orthonormierten Systeme φ in (0, 1) und die folgenden Klassen der reellen Zahlenfolgen $a=\{a_k\}_1^\infty$ ein:

$$\Omega_E=\bigl\{\varphi:\ \sup_n L_n(\varphi;\,x)/\lambda_n\in L^\infty(E)\bigr\},\quad \Omega_E^{**}=\bigl\{\varphi:\ \sup_n L_n(\varphi;\,x)/\lambda_n<\infty \text{ f. ü. in } E\bigr\},$$

$$\Omega_E^{*}=\bigl\{\varphi:\ \int_E \sup_n L_n(\varphi;\,x)/\lambda_n\,dx<\infty\bigr\},\quad \Omega_E^{*}(1)=\bigl\{\varphi:\ \int_E \sup_n L_n(\varphi;\,x)/\lambda_n\,dx\leqq 1\bigr\},$$

und es sei M die Klasse der Folgen a, für die die Reihe

$$\sum_{k=1}^\infty a_k\varphi_k(x) \tag{1}$$

bei jedem $\varphi\in\Omega_{(0,1)}$ in (0, 1) fast überall konvergiert. M_0, bzw. M_0^{**} ist die Klasse der Folgen a, für die die Reihe (1) bei jeder meßbare Menge $E\bigl(\subseteqq(0,1)\bigr)$ und bei jedem System φ aus Ω_E, bzw. aus Ω_E^{**} in E fast überall konvergiert. Weiterhin sei M^* die Klasse der Folgen a, für die die Reihe (1) bei jedem System $\varphi\in\Omega_{(0,1)}^*$ in (0, 1) fast überall konvergiert. (Diese Koeffizientenklassen hängen von der Folge λ ab. Da in dieser Arbeit λ dieselbe Folge ist, die Abhängigkeit werden wir nicht bezeichnen.)

Offensichtlich gilt $M\supseteqq M_0\supseteqq M_0^{**}$. In der Arbeit [3] haben wir bewiesen, daß $M_0=M^*=M_0^{**}$ ist, und $a\in M^*$ gilt dann, und nur dann, wenn

$$\|a;\lambda\|^*=\sup_{\varphi\in\Omega^*_{(0,1)}}\int_0^1 \sup_{1\leqq i<j}\left|\sum_{k=i}^j a_k\varphi_k(x)\right| dx<\infty$$

besteht.

Es ist ein Problem, ob die Gleichung $M=M_0$ besteht.

2. Nach den Sätzen von G. ALEXITS und A. SHARMA [1] und von Verf. [4] ist es bekannt, daß die Lebesgueschen Funktionen in nichtorthonormiertem Fall auf die Konvergenz der Entwicklungen einwirken. Die obigen Resultaten werden wir für nichtorthonormierten Fall erweitern.

Wir betrachten die Systeme der Funktionen $\varphi=\{\varphi_k(x)\}_1^\infty$ der Funktionen $\varphi_k(x)\in L(0,1)$ $(k=1,2,\ldots)$, die Konvergenzsysteme für l^2 dem Maß nach im Intervall $(0,1)$ sind. Für solche Systeme führen wir die entsprechenden Klassen $\bar\Omega_E$, $\bar\Omega_E^*$, $\bar\Omega_E^{**}$ und die entsprechenden Klassen der Koeffizientfolgen $\bar M$, $\bar M_0$, $\bar M^*$, $\bar M_0^{**}$ ein. Für diese Klassen gilt $\bar M\supseteq\bar M_0\supseteq\bar M_0^{**}$.

In dieser Arbeit werden wir den folgenden Satz beweisen.

SATZ. *Es gilt* $\bar M=\bar M_0=\bar M^*=\bar M_0^{**}=M^*$.

BEWEIS. Wir werden erstens $\bar M=\bar M_0=\bar M_0^{**}=\bar M^*$ beweisen.

Es sei $a\in\bar M$ und $\varphi\in\Omega^*_{(0,1)}$. Dann gibt es eine Folge der meßbaren Untermengen von $(0,1)$ $E_1\subseteq\ldots\subseteq E_m\subseteq\ldots$ mit $\operatorname{mes} E_m\to1$, und

$$\sup_n\frac{L_n(\varphi;x)}{\lambda_n}\leqq m\quad(x\in E_m;\ m=1,2,\ldots). \tag{2}$$

Nach einem Satz von E. M. NIKIŠIN [2] für beliebige Zahl $\varepsilon>0$ gibt es eine meßbare Menge $F_\varepsilon(\subseteq(0,1))$ mit $\operatorname{mes} F_\varepsilon\geqq1-\varepsilon$, eine positive Zahl M_ε und ein orthonormiertes System $\psi(\varepsilon)=\{\psi_k(\varepsilon;x)\}_1^\infty$ in $(0,1)$ mit

$$\varphi_k(x)=M_\varepsilon\psi_k(\varepsilon;x)\quad(x\in F_\varepsilon;\ k=1,2,\ldots). \tag{3}$$

Wir setzen $\bar E_m=E_m\cap F_\varepsilon$ $(m=1,2,\ldots)$ und

$$\varphi_k(m;x)=\begin{cases}\varphi_k(x), & x\in\bar E_m,\\ 0 & \text{sonst}\end{cases}\quad(k=1,2,\ldots;\ m=1,2,\ldots). \tag{4}$$

Nach (3) ist das System $\varphi(m)=\{\varphi_k(m;x)\}_1^\infty$ ein Konvergenzsystem für l^2 dem Maß nach in $(0,1)$ und nach (2) gilt

$$\sup_n\frac{L_n(\varphi(m);x)}{\lambda_n}\leqq m\quad(x\in(0,1)). \tag{5}$$

Also ist $\varphi(m)\in\bar\Omega_{(0,1)}$, und so konvergiert die Reihe

$$\sum_{k=1}^\infty a_k\varphi_k(m;x) \tag{6}$$

in $(0,1)$ fast überall. Wegen (4) konvergiert aber auch die Reihe (1) in $\bar E_m$ fast überall $(m=1,2,\ldots)$. Da $\operatorname{mes}(\lim\limits_{m\to\infty}\bar E_m)\geqq1-\varepsilon$ und $\varepsilon>0$ beliebig ist, ergibt sich, daß die Reihe (1) in $(0,1)$ fast überall konvergiert. D. h. ist $a\in\bar M^*$. Damit haben wir

$$\bar M\subseteq\bar M^*. \tag{7}$$

Es sei nun $a\in\bar M^*$, $E(\subseteq(0,1))$ eine meßbare Menge mit positivem Maß und $\varphi\in\bar\Omega_E^{**}$. Dann gibt es eine Folge der meßbaren Mengen in E $E_1\subseteq\ldots\subseteq E_m\subseteq\ldots\subseteq E$ mit $\operatorname{mes} E_m\to\operatorname{mes} E$, für die (2) besteht. Es sei $\varepsilon>0$, und mit der nach dem Nikišinschen Satz existierenden Menge F_ε bilden wir $\bar{E}_m = E_m \cap F_\varepsilon$ $(m=1, 2, \ldots)$ und die Funktionen (4). Das System $\varphi(m) = \{\varphi_k(m;\, x)\}_1^\infty$ ist ein Konvergenzsystem für l^2 dem Maß nach in $(0, 1)$ und es gilt (5). Also ist $\varphi(m) \in \bar{\Omega}^*_{(0,1)}$. So konvergiert die Reihe (6) in $(0, 1)$ fast überall und so konvergiert die Reihe (1) in $\bar{E}_m$ fast überall $(m=1, 2, \ldots)$. Da $\operatorname{mes}(\lim_{m\to\infty} E_m) \geqq \operatorname{mes} E - \varepsilon$, und $\varepsilon > 0$ beliebig ist, ergibt sich, daß die Reihe (1) in E fast überall konvergiert. D. h. ist $a \in \bar{M}_0^{**}$. Damit haben wir $\bar{M}_0^{**} \supseteqq \bar{M}^*$ bewiesen. Daraus und aus (7) folgt: $\bar{M} = \bar{M}_0 = \bar{M}_0^{**} = \bar{M}^*$.

Zweitens beweisen wir $M^* = \bar{M}^*$. Da $M^* \supseteqq \bar{M}^*$ ist klar, ist es genügend $M^* \subseteqq \bar{M}^*$ zu beweisen. Es sei $a \in M^*$ und $\varphi \in \bar{\Omega}^*_{(0,1)}$. Es sei $\varepsilon (<1)$ eine positive Zahl und seien F_ε, M_ε und $\psi(\varepsilon)$ nach dem erwähnten Nikišinschen Satz gewählt. Für die Menge $E = (0, 1) \setminus F_\varepsilon$ gilt dann

$$\operatorname{mes} E \geqq 1 - \varepsilon. \tag{8}$$

Nach $\varphi \in \bar{\Omega}^*_{(0,1)}$ gilt

$$\int_E \sup_n \frac{1}{\lambda_n} \int_E \left| \sum_{k=1}^n \varphi_k(x)\, \varphi_k(t) \right| dt \leqq K(<\infty) \tag{9}$$

mit einer Konstante K. Weiterhin für die Zahlen

$$\alpha_{ij} = \int_E \varphi_i(x)\, \varphi_j(x)\, dx \left(= M_\varepsilon^2 \int_E \psi_i(\varepsilon;\, x)\, \psi_j(E;\, x)\, dx\right)$$

besteht

$$|\alpha_{ij}| \leqq M_\varepsilon^2 \quad (i, j = 1, 2, \ldots). \tag{10}$$

Es seien E_{ij} $(i, j=1, 2)$ paarweise disjunkte Intervalle im Intervall $(1, 2)$ mit

$$\operatorname{mes} E_{ij} = \frac{1}{2 i^4 j^4} \quad (i, j = 1, 2, \ldots). \tag{11}$$

Wir definieren ein Funktionensystem $\Phi = \{\Phi_k(x)\}_1^\infty$ folgenderweise. Es sei

$$\Phi_k(x) = \begin{cases} \varphi_k(x)/\sqrt{2}\, M_\varepsilon, & x \in E, \\ \beta_k/\sqrt{\operatorname{mes} E_{kk}}, & x \in E_{kk}, \\ \dfrac{1}{M_\varepsilon} \sqrt{\dfrac{|\alpha_{ki}|}{4 \operatorname{mes} E_{ki}}}, & x \in E_{ki} \ (i \neq k;\ i = 1, 2, \ldots), \\ -\dfrac{1}{M_\varepsilon} \sqrt{\dfrac{|\alpha_{ki}|}{4 \operatorname{mes} E_{ki}}} \operatorname{sign} \alpha_{ki}, & x \in E_{ik} \ (i \neq k;\ i = 1, 2, \ldots), \\ 0, & \text{sonst} \end{cases}$$

$(k=1, 2, \ldots)$, wobei die Konstante β_k mit der Bedingung

$$\int_0^2 \Phi_k^2(x)\, dx = 1 \tag{12}$$

bestimmt ist. Da nach (10), und nach der Besselschen Ungleichung

$$\int_0^2 \Phi_k^2(x)\,dx = \frac{1}{2M_\varepsilon^2}\int_E \varphi_k^2(x)\,dx + \beta_k^2 + \frac{1}{2M_\varepsilon^2}\sum_{\substack{i=1\\ i\neq k}}^{\infty} \alpha_{ki}^2 =$$

$$= \frac{1}{2}\int_E \psi_k^2(\varepsilon;\,x)\,dx + \beta_k^2 + \frac{1}{2}\sum_{\substack{i=1\\ i\neq k}}^{\infty}\Big(\int_E \psi_k(\varepsilon;\,x)\,\psi_i(\varepsilon;\,x)\,dx\Big)^2 \leqq$$

$$\leqq \frac{1}{2}\int_0^1 \psi_k^2(\varepsilon;\,x)\,dx + \beta_k^2 + \frac{1}{2}\int_E \psi_k^2(\varepsilon;\,x)\,dx \leqq 1 + \beta_k^2 < \infty$$

ist, gilt $\Phi_k(x) \in L^2(0,1)$ $(k=1,2,\ldots)$, und gibt es eine positive Zahl $\beta_k \leqq 1$, für die (12) erfüllt wird. Durch einfacher Rechnung folgt, daß Φ ein orthonormales System in (0, 2) ist.

Wir schätzen die Lebesgueschen Funktionen $L_n(\Phi;\,x)$ ab. Ist $x \in E$, dann gilt nach (3), (10) und (11)

$$L_n(\Phi;\,x) = \frac{1}{2M_\varepsilon^2}\int_E \Big|\sum_{k=1}^{n} \varphi_k(x)\,\varphi_k(t)\Big|\,dt +$$

$$+ \frac{1}{\sqrt{2}}\sum_{k=1}^{n} \beta_k\,|\psi_k(\varepsilon;\,x)|\,\sqrt{\operatorname{mes} E_{kk}} +$$

$$+ \frac{1}{2\sqrt{2}\,M_\varepsilon}\sum_{k=1}^{n}\sum_{\substack{i=1\\ i\neq k}}^{\infty} |\psi_k(\varepsilon;\,x)|\,\sqrt{|\alpha_{ki}|}\,\big(\sqrt{\operatorname{mes} E_{ik}} + \sqrt{\operatorname{mes} E_{ki}}\big) \leqq$$

$$\leqq \frac{1}{M_\varepsilon^2}\int_E \Big|\sum_{k=1}^{n} \varphi_k(x)\,\varphi_k(t)\Big|\,dt + \sum_{k=1}^{\infty} |\psi_k(\varepsilon;\,x)|/k^4,$$

und so ist nach (9)

$$\int_E \sup_n \frac{L_n(\Phi;\,x)}{\lambda_n}\,dx \leqq \frac{K}{M_\varepsilon^2} + \frac{1}{\lambda_1}\sum_{k=1}^{\infty}\frac{1}{k^4}\int_E |\psi_k(\varepsilon;\,x)|\,dx = C_1(<\infty). \tag{13}$$

Es sei $x \in E_{kk}$. Dann gilt nach (3)

$$L_n(\Phi;\,x) = \beta_k^2 + \frac{M_\varepsilon \beta_k}{\sqrt{2}}\int_E |\psi_k(\varepsilon;\,x)|\,dx/\sqrt{\operatorname{mes} E_{kk}},$$

und so ist

$$\int_{E_{kk}} \sup_n \frac{L_n(\Phi;\,x)}{\lambda_n}\,dx \leqq \frac{1}{\lambda_1}\big(\sqrt{\operatorname{mes} E_{kk}} + M_\varepsilon \sqrt{\operatorname{mes} E_{kk}}\big),$$

und folglich nach (11)

$$\int_{\bigcup_{k=1}^{\infty} E_{kk}} \sup_n \frac{L_n(\Phi;\,x)}{\lambda_n}\,dx \leqq \frac{1}{\lambda_1}\Big(\sum_{k=1}^{\infty}\frac{1}{k^8} + M_\varepsilon \sum_{k=1}^{\infty}\frac{1}{k^4}\Big) = C_2(<\infty).$$

Es sei endlich $x \in E_{ij}$. Dann gilt nach (3)

$$L_n(\Phi;\, x) = \frac{1}{2M_\varepsilon^2}|\alpha_{ij}| + \frac{1}{\sqrt{2}\, 2M_\varepsilon^2}|\alpha_{ij}| \Big(\int_E |\psi_i(\varepsilon;\, x)|\, dx + \int_E |\psi_j(\varepsilon;\, x)|\, dx\Big) / \sqrt{\operatorname{mes} E_{ij}} \leqq$$

$$\leqq 1 + 1/\sqrt{\operatorname{mes} E_{ij}},$$

und nach (11) ergibt sich:

$$(15) \qquad \int\limits_{\bigcup\limits_{\substack{i,j=1\\ i\neq j}}^{\infty} E_{ij}} \sup_n \frac{L_n(\Phi;\, x)}{\lambda_n}\, dx \leqq \frac{1}{\lambda_1} \sum_{\substack{i,j=1\\ i\neq j}}^{\infty} \big(\operatorname{mes} E_{ij} + \sqrt{\operatorname{mes} E_{ij}}\big) \leqq$$

$$\leqq \frac{1}{\lambda_1} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\frac{1}{i^4 j^4} + \frac{1}{i^2 j^2}\right) = C_3 (< \infty).$$

Aus (13), (14) und (15) erhalten wir

$$(16) \qquad \int\limits_0^2 \sup_n \frac{L_n(\Phi;\, x)}{\lambda_n}\, dx < \infty.$$

Es sei

$$\psi_k(x) = \sqrt{2}\, \Phi_k(2x) \quad (x \in (0,1);\ k = 1, 2, \ldots).$$

Das System $\psi = \{\psi_k(x)\}_1^\infty$ ist ein orthonormiertes System in (0, 1), weiterhin nach (16) gilt $\psi \in \Omega^*_{(0,1)}$. Auf Grund der Voraussetzung $a \in M^*$ konvergiert also die Reihe $\sum\limits_{k=1}^{\infty} a_k \psi_k(x)$ in (0, 1) fast überall. Auf Grund der Definitionen der Funktionen $\psi_k(x)$ und $\Phi_k(x)$ folgt, daß die Reihe (1) in E fast überall konvergiert. Nach (8), da $\varepsilon > 0$ beliebig ist, folgt, daß die Reihe (1) in (0, 1) fast überall konvergiert. Also ist $a \in \overline{M}^*$. Damit haben wir $M^* \subseteqq \overline{M}^*$ bewiesen.

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A UNIFIED APPROACH TO MONOTONE EXTENSIONS OF MAPPINGS

By

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Introduction

Several authors have considered the possibility of improving the behaviour of maps by extending their domains. The problem is of fairly recent vintage and has been investigated by several authors (see for example, WHYBURN [14] [15], BAUER [1], KROLEVEC [10], DICKMAN [5], FRANKLIN and KOHLI [6], CAIN [2] [3], KOHLI [7] [8], DELHAN and STRECKER [4]). WHYBURN [14] developed a method of unifying the domain and range of a mapping, so as to achieve a compact extension of the mapping. A modification of the "unified space technique" of Whyburn was developed by DICKMAN [5] to obtain a perfect extension of the mapping. Another modification of the same was devised by the author [7] to yield a meaningful open extension of the mapping. In view of the fact that every mapping is the restriction of a monotone mapping [8], a natural question arises: can a modification of the "unified space technique" be developed so as to yield a meaningful monotone extension of the mapping? Although this does not seem plausible in general, here we develop a variant of Whyburn's "unified space technique" of unifying the domain and range of a mapping so as to yield a meaningful monotone extension of the mapping, thus showing that answer is affirmative in a certain sense. In section 1 of this paper, we construct a monotone extension and relate the topological properties of the domain of extension with the topological properties of the domain of the mapping. This method is then applied to the constructions of WHYBURN [14] [15] and DICKMAN [5] to obtain a monotone extension with additional properties. Section 3 is devoted to applications of these ideas to the theories of semiconnected and weak semiconnected functions. Some open questions are also raised.

1. Monotone extensions

Let $f\colon X\to Y$ be a function, not necessarily continuous, from a topological space X into a topological space Y. We say that f is *monotone* if for each $y\in Y$ the fibre $f^{-1}(y)$ is either connected or empty. We say that a point $y\in Y$ is a *non-monotone point relative to* f if $f^{-1}(y)$ is not connected. Let M_f denote the set of all non-monotone points relative to f. The function f is monotone just in case the set M_f is empty.

Without any loss of generality we may assume that the spaces X and Y are disjoint. In particular, X and M_f are disjoint. Let X^* denote the set theoretic union of X and M_f and let τ denote the collection of all subsets U of X^* which satisfy the following two conditions:

(i) The set $U\cap X$ is open in X.

(ii) $f(U\cap X)\cap M_f\subset U$.

Then it is easily verified that the collection τ is a topology for X^*. From hereonward X^* is always assumed to be endowed with the topology τ.

PROPOSITION 1.1. *The space X is embedded as a closed subspace in X^* and M_f (as a subspace of X^* and not as a subspace of Y) is an open discrete set in X^*.*

PROOF. Since every subset of M_f satisfies conditions (i) and (ii), it is open in X^*. In particular, M_f is an open discrete subspace of X^*. The set X being the complement of M_f is closed in X^*. Thus every set which is closed in X is also closed in X^*. So, the inclusion map $i\colon X\to X^*$ is a closed embedding of X into X^*.

PROPOSITION 1.2. *The function $f^*\colon X^*\to Y$ from X^* into Y defined by $f^*(z)=f(z)$ for $z\in X$ and $f^*(z)=z$ for $z\in M_f$ is a monotone function. If f is continuous, so is f^*.*

PROOF. Let $y\in Y$, if $y\notin M_f$, then $f^{*-1}(y)=f^{-1}(y)$ and by the definition of M_f the set $f^{-1}(y)$ is either empty or connected. If $y\in M_f$ then $f^{*-1}(y)=f^{-1}(y)\cup\{y\}$ and in this case any open set (in X^*) containing a point of $f^{-1}(y)$ contains the point y. Thus the set $f^{*-1}(y)$ is connected. So, the function f^* is monotone.

Now, suppose f is continuous and let V be any open set in Y. Then $f^{-1}(V)$ is open in X and $f^{*-1}(V)=f^{-1}(V)\cup(V\cap M_f)$. The set $f^{*-1}(V)$ satisfies the conditions (i) and (ii) and is therefore open in X^*. So, the function f^* is continuous.

PROPOSITION 1.3. *If the function f is closed, then the function f^* as defined in Proposition 1.2 is a closed function.*

PROOF. Let F be any closed subset of X^*. Then $F\cap X$ is closed in X and since f is closed, the set $f(F\cap X)$ is closed in Y. If $y\in F\cap M_f$, then $f^{-1}(y)\subset \mathrm{Cl}_{X^*}(y\}\subset F$ and so, $y\in f(X\cap F)$. Thus $f^*(F\cap M_f)=F\cap M_f\subset f(X\cap F)$. Now, the proof follows since

$$f^*(F)=f^*(F\cap X)\cup f^*(F\cap M_f)=f(F\cap X)\cup(F\cap M_f)=f(F\cap X).$$

PROPOSITION 1.4. *If X is a T_0-space, so is X^*.*

PROOF. Let $x,y\in X^*$ with $x\neq y$. If at least one of the points x and y is in M_f, the proof is immediate in view of Proposition 1.1. If $x,y\in X^*-M_f$, then since X is T_0, there is an open set U (in X) containing one of the point x and y but not the other. The set $U\cup M_f$ is an open set (in X^*) containing one of the points x and y but not the other.

PROPOSITION 1.5. *The space X^* is never T_1 and hence never T_2 or T_3 (except in the case when f is monotone).*

PROOF. If $y\in M_f$ and $x\in f^{-1}(y)$, then any open set (in X^*) containing x also contains y.

PROPOSITION 1.6. *The space X^* is connected, whenever X is.*

PROOF. Let $X^*=A\cup B$, where A and B are disjoint open sets in X^*. Then $X=(A\cap X)\cup(B\cap X)$ and since X is connected either $A\cap X=\emptyset$ or $B\cap X=\emptyset$. Suppose $B\cap X=\emptyset$. Then $X\subset A$. Let $y\in M_f$. Since A is an open set (in X^*) containing $f^{-1}(y)$, $y\in A$. So, $X^*=A$ and $B=\emptyset$.

PROPOSITION 1.7. *The space X^* is locally connected whenever X is.*

PROOF. Let $x \in X^*$ and let U be an open neighbourhood of x in X^*. If $x \in M_f$, then $\{x\}$ is an open connected neighbourhood of x contained in U. If $x \notin M_f$, then $x \in X$ and $U \cap X$ is an open neighbourhood of x (in X). There exists an open connected set V in X such that $x \in V \subset U \cap X$. Then $W = V \cup (f(V) \cap M_f)$ is an open connected set in X^* and $x \in W \subset U$.

PROPOSITION 1.8. *Local weight is preserved in X^*. In particular, if X is first countable, so is X^*.*

PROOF. Let $x \in X^*$. If $x \in M_f$, then $\{x\}$ is open in X^* and thus forms a local base at x. If $x \in X$ and if $\mathscr{B}$ is a local base at x in X of cardinality m, then the sets of the form $\tilde{B} = B \cup (f(B) \cap M_f)$ with $B \in \mathscr{B}$ form a local base at x in X^* of cardinality m.

PROPOSITION 1.9. *The space X^* is locally compact, whenever X is.*

PROOF. Let $z \in X^*$. If $z \in M_f$, then $\{z\}$ is a compact neighbourhood of z. If $z \in X$, then since X is locally compact, there is a compact neighbourhood N of z in X. The set $N_1 = N \cup (f(N) \cap M_f)$ is a neighbourhood of z in X^*. We shall show that N_1 is compact. Let $N_1 \subset \bigcup_{i \in I} U_i$, U_i open in X^* for $i \in I$. Then $N \subset \bigcup_{i \in I} (U_i \cap X)$ and $U_i \cap X$ is open in X for each i. Hence a finite number of the sets $U_i \cap X$ cover N and the corresponding sets U_i cover N_1. This completes the proof.

PROPOSITION 1.10. *The space X^* is compact, whenever X is compact.*

PROOF. In the proof of Proposition 1.9, replace the role of N_1 by X^* and that of N by X. Then the same argument suffices for the compactness of X^*.

Proposition 1.5 shows that any topological property which implies T_1 is not preserved in X^*. However, X^* is a T_D-space [13, p. 92] whenever X is a T_D-space, Y is T_1 and f is continuous. Furthermore, simple examples can be given to show that X^* need not preserve weight even if each of the spaces X and Y is the closed unit interval I. For, let $f\colon X \to I$ be defined by $f(x) = |2x-1|$ or $f(x) = (2x-1)^2$. Then each point, except the point $\frac{1}{2}$, is a non-monotone point relative to f. The space X^* is not second countable in this case.

2. Closed and compact monotone extensions

The technique of the preceding section can be used to obtain monotone extensions of a map with additional properties such as compactness of the fibres or closedness of the extension or both.

Let W denote the set theoretic union of X and Y (X and Y are assumed to be disjoint). Let $Q \subset W$ satisfy the following two conditions.

(iii) The sets $Q \cap X$ and $Q \cap Y$ are open in X and Y respectively.

(iv) For each $y \in Q \cap Y$, the set $f^{-1}(y) \cap (X - Q \cap X)$ is compact.

Let τ_1 denote the collection of all subsets Q of W satisfying (iii) and (iv) above.

PROPOSITION 2.1. *The collection τ_1 is a topology for W.*

From hereonward, let W be assumed to be equipped with the topology τ_1.

PROPOSITION 2.2. *The spaces X and Y are embedded in W as open and closed subspaces respectively.*

PROPOSITION 2.3. *Let a retraction map r from W onto Y be defined by $r(z)=z$ for each $z\in Y$ and $r(z)=f(z)$ for each $z\in X$. Then $r^{-1}(y)$ is compact for each $y\in Y$. Moreover, if f is continuous or open, so also is r.*

The proofs of the above three propositions are exact analogues of their counterparts in ([14] [15]) where (iv) is replaced by (iv)′: For each compact set $K\subset Q\cap Y$, the set $f^{-1}(K)\cap(X-Q\cap X)$ is compact.

Now, let $r^*\colon W^*\to Y$ be the monotone extension of r as constructed in the last section. Then $r^{*-1}(y)$ *is a continuum (=compact connected set)* for each $y\in Y$. In general r^* is not a closed function. However, the above technique can be modified to obtain a closed monotone extension.

Let $f_0\colon X_0\to Y$ be the r-extension of f constructed in [5]. Then f_0 is a closed and compact function and X is an open dense subspace of X_0. Furthermore, f_0 is continuous, whenever f is. Let $f_0^*\colon X_0^*\to Y$ be the monotone extension of f_0 constructed in the last section. Then in view of Proposition 1.3 f_0^* *is a closed function such that* $f_0^{*-1}(y)$ *is* a *continuum* for each $y\in Y$. Thus by [11, Theorem 2] it follows that for each connected set $C\subset Y$, $f_0^{*-1}(C)$ is connected.

In view of preceding paragraphs it seems interesting to ask: can any function $f\colon X\to Y$ in general be extended to an open monotone function? This question bears an affirmative answer in case f is continuous [8]. It would also be interesting to discover a closed monotone extension which does not rely on constructions of Whyburn and Dickman.

3. Semiconnected and weak semiconnected functions

A function $f\colon X\to Y$ is called *weak semiconnected (semiconnected)* if for each closed connected set $B\subset Y$, $f^{-1}(B)$ is closed (closed and connected) (see [11, p. 117] and, [9]). Every continuous and every semiconnected function is weak semiconnected and it is easy to construct examples of weak semiconnected functions which are neither continuous nor semiconnected. Further, it is well known and easily shown that the concepts of continuity and semiconnectedness are independent of each other.

A function $f\colon X\to Y$ is said to be a *connected function* if for each connected, set $A\subset X$, $f(A)$ is connected, and a *connectivity function* if the function $G\colon X\to X\times Y$ defined by $G(x)=(x, f(x))$ is a connected function.

Throughout the section the symbols f^* and X^* will have the same meaning as in Section 1.

LEMMA 3.1. *A set $F\subset X^*$ is closed if and only if it satisfies the following two conditions:*

(a) *$F\cap X$ is closed in X.*

(b) *For each $y\in F\cap M_f$, $f^{-1}(y)\subset F$.*

PROOF. If F is closed in X^*, then it is easily verified that F satisfies the conditions (a) and (b). Conversely, suppose F satisfies the conditions (a) and (b). It suffices to show that X^*-F is open in X^*. Now $(X^*-F)\cap X=X-(X\cap F)$ and therefore by condition (a) $(X^*-F)\cap X$ is open in X. Let $y\in f((X^*-F)\cap X)\cap M_f= =f(X-(X\cap F))\cap M_f$. Then $f^{-1}(y)\cap(X-(X\cap F))\neq\emptyset$. Thus X^*-F satisfies the conditions (i) and (ii) (as defined in section 1) and is therefore open in X^*.

PROPOSITION 3.2. *If $f\colon X\to Y$ is a (weak) semiconnected function, then so is f^*.*

PROOF. Let $B\subset Y$ be any closed and connected set. Since f is weak semiconnected, $f^{-1}(B)$ is closed in X. The set $f^{*-1}(B)=f^{-1}(B)\cup(B\cap M_f)$ satisfies the conditions (a) and (b) of Lemma 3.1 and is therefore, closed in X^*. Thus f^* is weak semiconnected.

If f is semiconnected, then $f^{-1}(B)$ is connected. Now, an argument similar to that used to prove the connectedness of X^* in Proposition 1.6 suffices for the connectedness of $f^{*-1}(B)$.

We point out that in contrast to Proposition 3.2, the monotone extension of a connected function need not be connected. For, let $X=Y=\mathbf{R}$, the real line and let $f\colon X\to Y$ be given by $f(x)=\sin\frac{1}{x}$ if $x\neq 0$ and $f(0)=0$. Then f is connected but its monotone extension $f^*\colon X^*\to Y$ is not a connected function. In fact, $f^{*-1}\left(Y-\left\{\frac{1}{2}\right\}\right)$ is connected without its image under f^* being connected.

QUESTION. What (nontrivial) conditions on a function $f\colon X\to Y$ ensure the connectedness of its monotone extension $f^*\colon X^*\to Y$?

PROPOSITION 3.3. *If $f\colon X\to Y$ is continuous, then the function f_0^* as defined in Section 2 is a semiconnected function.*

PROOF. Since f_0^* is a continuous closed monotone function, the result follows from [11, Theorem 2].

REMARK 3.4. Proposition 3.3 points out a significant difference between semiconnected and weak semiconnected functions that the restriction of former is not necessarily semiconnected while (it is easily verified that) the restriction of a weak semiconnected function is again a weak semiconnected function.

The results of the preceding paragraphs also provide a convenient method of constructing certain examples and counterexamples in the theory of non-continuous functions. For example, consider the following theorem.

THEOREM 3.5 ([12, p. 234]). *Let X be a connected, locally connected T_1-space, and let $f\colon X\to Y$ be a connectivity function. If $V\subset Y$ is open, then each point of $f^{-1}(V)$ is a limit point of $f^{-1}(V)$.*

The results of the preceding sections can be utilized to show that the hypothesis on X of being a T_1-space cannot be weakened even to a T_0-space in Theorem 3.5. For, let $X=Y=\mathbf{R}$, the real line and let $f\colon X\to Y$ be defined by $f(x)=x^2$. Then the monotone extension $f^*\colon X^*\to Y$ is a continuous closed monotone map and X^* is a connected locally connected T_0-space. But not for every open set $V\subset Y$ every point of $f^{*-1}(V)$ is a limit point of $f^{*-1}(V)$, e.g., $f^{*-1}(Y)$ contains isolated points.

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SELECTIVE DERIVATIVES AND THE M_2 OR DENJOY—CLARKSON PROPERTIES

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In [3], the concept of a function f: $[0, 1]\to R$ having a selective derivative was introduced and examined. In this paper, we show that a selective derivative has the Denjoy—Clarkson property. A measurable function g: $[0, 1]\to R$ is said to have the D.C. property if, for every pair of real numbers $a<b$, the set $\{x: a<f(x)<b\}$ has positive measure in every one-sided neighborhood of any of its points. The proof that selective derivatives have this property follows from a study of the relation between the D.C. property and a modified version of Zahorski's M_2 property. As defined by Zahorski in [4], a function h: $[0, 1]\to R$ has the M_2 property if for every a the sets $\{x: h(x)>a\}$ and $\{x: h(x)<a\}$ are F_σ and have positive measure in any one-sided neighborhood of any of their points. Our modification is a natural one. Namely, we delete the condition that the sets $\{x: f(x)>a\}$ and $\{x: f(x)<a\}$ should be F_σ. The modified property we label m_2. Two obvious facts should be noted. First, if a function has the D.C. property it also has the m_2 property. Second, the M_2 property forces a function to be Baire class 1. For Baire class 1 — Darboux functions, M_2 and the D.C. property are equivalent properties [2]. Here we establish an equivalence between the m_2 and D.C. properties for a class of functions, E, containing the selective derivatives and the Baire class 1, Darboux functions.

It follows from a theorem in [3] that selective derivatives have the m_2 property. Therefore, if all selective derivatives were of Baire class 1 it would immediately follow that selective derivatives have the D.C. property. However, we are only guaranteed that they are of Baire class 2; see [3] and [1].

We begin by presenting several definitions and theorems which can be deduced from [3]. Let $[a, b]$ be a fixed subinterval of $[0, 1]$. We select one point from the interior of $[a, b]$ and label it $p_{[a,b]}$. The collection of p's thus obtained is called a selection. For notational convenience we will use $[a, b]$ to describe the interval having a and b as end-points irregardless of whether $a<b$ or $b<a$.

DEFINITION. For a given selection S, a function f: $[0, 1]\to R$ has at a point x a selective derivative if there exists a real number α such that

$$\lim_{h\to 0}\frac{f(p_{[x,x+h]})-f(x)}{p_{[x,x+h]}-x}=\alpha.$$

The value α is denoted as $sf'(x)$.

THEOREM A. *Let f: $[0, 1]\to R$ and let S be a fixed selection for which f has a selective derivative for each x in* $[0, 1]$. *Then:*

(i) *The function f is Darboux.*
(ii) *If* $sf'(x)\geqq N$ *on* (a,b), *then* f *is differentiable a.e. in* $[a,b]$.
(iii) *If* $sf'(x)\geqq N$ *for a.e.* x *in* (a,b), *then* $sf'(x)\geqq N$ *on* $[a,b]$.
(iv) *The function* $sf'(x)$ *is Darboux.*
(v) *For any interval* $[a,b]$,

$$\inf\left\{\frac{f(x)-f(y)}{x-y}:\ a\leqq x<y\leqq b\right\}=\inf[sf'(x):\ a\leqq x\leqq b]$$

and

$$\sup\left\{\frac{f(x)-f(y)}{x-y}:\ a\leqq x<y\leqq b\right\}=\sup[sf'(x):\ a\leqq x\leqq b].$$

From (iii) it readily follows that selective derivatives have the m_2 property.

We will need a lemma which is similar in nature and proof to Lemma 3 of [3]. The proof will not be presented.

LEMMA. *Let* $f\colon [0,1]\to R$ *and let* S *be a selection for which* f *has a selective derivative* $sf'(x)$ *for all* x *in* $[0,1]$. *Let* $a<b$ *be fixed and* $\varepsilon>0$ *be given. Define*

$$A=\left\{x:\ a\leqq\frac{f(p_{[x,x+h]})-f(x)}{p_{[x,x+h]}-x}\leqq b;\ |h|<\varepsilon\right\}.$$

Let A^* *be the closure of* A. *Then, if* x *and* y *are any two points of* A^* *with* $|x-y|<\varepsilon$,

$$a\leqq\frac{f(x)-f(y)}{x-y}\leqq b.$$

Throughout this paper, for any set V the notation V^* will be used for the closure of V. Further, for fixed $a<b$ the set $\{x\colon a<f(x)<b\}$ will be indicated as $E(a,b)$.

Next we describe the functions comprising class E. For the considerations we have in mind it becomes natural to require that the functions be Darboux.

DEFINITION. A Darboux function $f\colon [0,1]\to R$ is in class E if for any two numbers $a<b$ and every open interval I the sets $\{x\colon f(x)\leqq a\}$ and $\{x\colon f(x)\geqq b\}$ are not simultaneously dense in $E^*(a,b)\cap I$. (Assuming that $E^*(a,b)\cap I\neq\emptyset$.)

The difference between class E functions and Baire class 1 functions, besides the Darboux property, is apparent in this definition. Basically it consists of the fact that for a Baire class 1 function f the set $E^*(a,b)$ can be replaced by an arbitrary perfect set. Further, it is easy to construct examples of class E functions which are not even measurable.

The above definition is useful for illustrating the connection between class E and Baire class 1 functions, but an equivalent definition will be more useful in showing selective derivatives are in class E. Namely:

THEOREM 1. *A Darboux function f is in class E if and only if for every $a<b$ and open interval I, with* $I\cap E^*(a,b)\neq\emptyset$, *there is an open subinterval J of I with* $J\cap E^*(a,b)\neq\emptyset$ *such that either:*

(1) $f(x)<b$ *for all* x *in* J *or,*
(2) $f(x)>a$ *for all* x *in* J.

PROOF. ($\rightarrow$) Since $\{x: f(x)\leqq a\}$ and $\{x: f(x)\geqq b\}$ are not simultaneously dense in $E^*(a, b)\cap I$ there must be an open subinterval J of I with $E^*(a, b)\cap J\neq\emptyset$ and either $f(x)>a$ on $E^*(a, b)\cap J$, or $f(x)<b$ on $E^*(a, b)\cap J$. The set $U=J\setminus E^*(a, b)$ is an open set and each component of U has at least one end point in $E^*(a, b)$. Since f is Darboux, on each component (r, s) of U we have $f(x)\geqq b$ on $[r, s]$ or $f(x)\leqq a$ on $[r, s]$ because $(r, s)\cap E(a, b)=\emptyset$. However, if, for example, $f(x)>a$ on $E^*(a, b)\cap J$, then we must have $f(x)\geqq b$ on (r, s) because either r or s belongs to $E^*(a, b)\cap J$. Therefore, $f(x)>a$ on J if $f(x)>a$ on $E^*(a, b)\cap J$ and $f(x)<b$ on J if $f(x)<b$ on $E^*(a, b)\cap J$.
($\leftarrow$) Follows immediately from definitions.

THEOREM 2. *Let* $f\colon [0, 1]\to R$ *have a selective derivative* sf' *at every point of* $[0, 1]$. *Then* sf' *is a class* E *function.*

PROOF. Since we already know that sf' is a Darboux function, it will suffice to show that sf' satisfies the alternate definition contained in Theorem 1. Further, we assume that $I=[0, 1]$. Let $a<c<d<b$ be given. We define:

$$D_n=\left\{x\colon -n\leqq\frac{f(p_{[x, x+h]})-f(x)}{p_{[x, x+h]}-x}\leqq d,\ |h|<1/n\right\}$$

and

$$C_n=\left\{x\colon c\leqq\frac{f(p_{[x, x+h]})-f(x)}{p_{[x, x+h]}-x}\leqq n,\ |h|<1/n\right\}.$$

(Note that $E^*(a, b)$ is a perfect set.) Clearly, $\bigcup_{n=1}^{\infty} C_n\cup\bigcup_{n=1}^{\infty} D_n=[0, 1]$. Therefore by the Baire category theorem there is an open interval (u, v) and an integer N such that $\emptyset\neq(u, v)\cap E^*(a, b)$ is a subset of C_N^* or D_N^*.

We will consider only the case where $v-u<\dfrac{1}{N}$, u and v both belong to C_N^* and $(u, v)\cap E^*(a, b)\subset C_N^*$. It will be seen that $sf'(x)\geqq c$ on $[u, v]$. By (v) of Theorem A, we need only establish that

$$\inf\left\{\frac{f(x)-f(y)}{x-y}\colon u\leqq x<y\leqq v\right\}\geqq c.$$

We have

$$\frac{f(x)-f(y)}{x-y}\geqq c$$

if both x and y belong to $C_N^*\cap[u, v]$.

Of the remaining possible locations for x and y, we consider only the case when x and y both belong to $(u, v)\setminus E^*(a, b)$. The rest will then be clear. Suppose

x and y are both in the same component (r, s) of $(u, v)\setminus E^*(a, b)$. Then $sf'(x)\geqq b$ on $[r, s]$ or $sf'(x)\leqq a$ on $[r, s]$. However, since r and s belong to C_N^*, we have

$$\frac{f(r)-f(s)}{r-s} \geqq c > a.$$

This, by (v) of Theorem A, means we must have $sf'\geqq b$ on $[r, s]$. Therefore,

$$\frac{f(x)-f(y)}{x-y} \geqq b > c.$$

Next let $x<y$ be in different components. These components do not have a common end point. Let x_0 be the right end point of x's component and y_0 the left end point of y's component. Then, as above,

$$f(x_0)-f(x) \geqq b(x_0-x) > c(x_0-x),$$

$$f(y)-f(y_0) \geqq b(y-y_0) > c(y-y_0),$$

$$f(y_0)-f(x_0) \geqq c(y_0-x_0)$$

so that $f(y)-f(x)>c(y-x)$ which completes the proof.

THEOREM 3. *Let f: $[0, 1]\to R$ be a measurable class E function. Then f has the* D.C. *property if and only if f has the m_2 property.*

PROOF. $(\to)$ Obvious.

$(\leftarrow)$ Let $I=[0, 1]$. Let $a<b$ be given. Suppose that $\emptyset\neq E(a, b)$ and $m(E(a, b))=0$. Select two values c and d such that $a<c<d<b$ and $E(c, d)\neq\emptyset$. There is an interval (u, v) with $(u, v)\cap E^*(c, d)\neq\emptyset$, and either

(i) $f(x)>c>a$ on (u, v), or

(ii) $f(x)<d<b$ on (u, v).

Assume (i) is true. Now in addition we have

$$m(\{x\colon b>f(x)\geqq c\}\cap[u, v]) = 0,$$

so $f(x)\geqq b$ for a.e. x in $[u, v]$ yielding $f\geqq b$ on $[u, v]$. This contradicts $E(c, d)\cap\cap(u, v)\neq\emptyset$.

COROLLARY 1. *All selective derivatives have the Denjoy—Clarkson property.*

As a final note, we remark that the proof that $sf'(x)$ is in class E can be used to establish a stronger result than that stated in the above corollary. Namely:

COROLLARY 2. *Let s be a selection relative to which f has a selective derivative. Let $Df=\{x\colon f$ is differentiable at $x\}$. Then for every $a<b$ and for every open interval I, if $I\cap\{x\colon a<sf'(x)<b\}\neq\emptyset$ then $I\cap Df\cap\{x\colon a<sf'(x)<b\}$ has positive measure.*

PROOF. Let $a<b$ be given and let I be any open interval with $I\cap\cap\{x\colon a<sf'(x)<b\}\neq\emptyset$. Then since sf' is a class E function there is an open

interval J, $J \subset I$, with $J \cap \{x: a < sf'(x) < b\} \neq \emptyset$ and either $sf' > a$ on J or $sf' < b$ on J. By (ii) of Theorem A, f is differentiable a.e. in J. Since $J \cap \{x: a < sf' < b\}$ has positive measure, this implies that $J \cap \{x: a < sf' < b\} \cap Df$ has positive measure. This completes the proof and the paper.

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interval $I \subset I_k$ with $\lambda^*\{x \in I: f(x)-f(c)\} > 0$ and either $s' > a$ on I or $s' < b$ on I. By (proof of Theorem A) f' is differentiable a.e. in I. Since $\lambda^*\{x \in I: f'(x)=f'_s(x)\} > 0$ has positive measure, this implies that $\lambda^*\{x \in I: a < s' < b\} \cap D$ has positive measure. This completes the proof and the paper.

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DERIVATIVE FUNCTIONS AND STRONG APPROXIMATION OF FOURIER SERIES

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1. Let $f(x)$ be a continuous 2π-periodic function, and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

be its Fourier series. Denote by $s_n(x) = s_n(f;\, x)$ the nth partial sum of (1), furthermore let $\tilde{f}(x)$ and $\tilde{s}_n(x)$ denote the conjugate functions of $f(x)$ and $s_n(x)$, respectively. Moreover, let $f^{(r)}(x)$ denote the rth derivative of $f(x)$, and $\|\cdot\|$ denote the usual supremum norm.

Generalizing a result of FREUD [1] we proved in [6] among others (see also Theorems 1 and 3) the following

THEOREM A. *If* $0<\alpha<1, p>0$ *and*

$$\left\| \sum_{n=1}^{\infty} n^{\alpha p-1} |s_n - f|^p \right\| < \infty$$

then $f \in \operatorname{Lip} \alpha$; *furthermore*

$$\lim_{h \to 0} h^{-\alpha}\big(f(x+h) - f(x)\big) = 0 \tag{2}$$

holds almost everywhere. These statements are best possible in general; (2) *cannot be extended to every x.*

An analogous result for the derivative $f^{(r)}$ has not been proved. For the sake of accuracy we recall two theorems showing into this direction but the conditions of these theorems claim more than those of our new Theorem to be presented here.

THEOREM B ([6], Corollary 2.3). *If* $0<\alpha<1, p>0$, *r is a nonnegative integer and*

$$\left\| \sum_{n=m}^{\infty} n^{(r+\alpha)p-1} |s_n - f|^p \right\| \to 0 \quad as \quad m \to \infty, \tag{3}$$

then

$$\lim_{h \to 0} h^{-\alpha}\big(f^{(r)}(x+h) - f^{(r)}(x)\big) = 0$$

holds everywhere.

THEOREM C ([5], Theorem 4). *If* $0<\alpha<1, p>0$, *r is a nonnegative integer and* $p^* = \min\left(p, \frac{1}{p}\right)$ *then*

$$\left\| \sum_{m=0}^{\infty} \left\{ \sum_{n=2^m+1}^{2^{m+1}} n^{(r+\alpha)p-1} |s_n - f|^p \right\}^{p^*} \right\| < \infty$$

implies

(4) $$\lim_{h\to 0} h^{-\alpha}(f^{(r)}(x+h)-f^{(r)}(x)) = 0 \quad \textit{for even } r,$$

and

(5) $$\lim_{h\to 0} h^{-\alpha}(\tilde{f}^{(r)}(x+h)-\tilde{f}^{(r)}(x)) = 0 \quad \textit{for odd } r$$

almost everywhere. These statements cannot be extended to all x.

2. We shall prove the following

THEOREM. *Let* $0<\alpha<1, p>0,$ *and r be a nonnegative integer. Then*

(6) $$\left\| \sum_{n=1}^{\infty} n^{(r+\alpha)p-1}|s_n-f|^p \right\| < \infty$$

implies $f^{(r)}\in \mathrm{Lip}\,\alpha$; *furthermore* (4) *and* (5) *hold almost everywhere. These statements are best possible.*

It is clear that this Theorem includes Theorems A and C, but not Theorem B, namely condition (6) does not imply (3).

3. We require some lemmas to prove our theorem.

LEMMA 1 ([6], a special case of the first part of Theorem 2). *If* $0<\alpha<1, p>0$ *and r is a nonnegative integer, then* (6) *implies* $f^{(r)}\in \mathrm{Lip}\,\alpha$.

LEMMA 2 ([6], the first part of Corollary 3.1). *Let* $0<\alpha<1, p>0$ *and r be a nonnegative integer. Then there exists a function* $F(x)=F(r, \alpha;\ x)$ *such that*

$$\left\| \sum_{n=1}^{\infty} n^{(r+\alpha)p-1}|s_n(F)-F|^p \right\| < \infty,$$

but

$$\omega(F^{(r)};\ \delta) \geqq C\delta^{\alpha},$$

where $C>0$ *and* $\omega(f;\ \delta)$ *denotes the modulus of continuity of f.*

LEMMA 3 ([4], Lemma). *Let* $\{\varrho_n\}$ *be a nonincreasing sequence of positive numbers such that* $\sum_{n=1}^{\infty} \varrho_n n^{-1}<\infty$, *and let*

$$\varrho(x) = \sum_{n=1}^{\infty} \varrho_n n^{-1} \sin nx.$$

Then for any $m>2^9$

$$\varrho\left(\frac{\pi}{m}\right) > \frac{1}{2m}\sum_{n=1}^{\infty} \varrho_n.$$

LEMMA 4 ([2], Lemma 4). *Let* $0<\alpha\leqq 1, p>0$ *and r be a nonnegative integer. Then the function*

$$R(x) = R(r, \alpha;\ x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^{1+r+\alpha}}$$

satisfies the inequality

$$\left\| \sum_{n=1}^{\infty} n^{(r+\alpha)p-1} |s_n(R)-R|^p \right\| < \infty.$$

LEMMA 5 ([5], Lemma 7). *If* $0<\alpha<1$ *and*

$$K \geqq n^{\alpha-1} \sum_{k=n+1}^{2n} |s_k(x)-f(x)| \to 0$$

almost everywhere, then

$$\lim_{h\to 0} h^{-\alpha}(f(x+h)-f(x)) = 0$$

holds almost everywhere.

LEMMA 6 ([3], Theorem 5). *We have for any positive p and natural number n the estimation*

$$\left\| \left\{ \frac{1}{n} \sum_{k=n+1}^{2n} |s_k - f|^p \right\}^{1/p} \right\| = O(E_n(f)),$$

where $E_n(f)$ *denotes the best approximation of f by trigonometric polynomials of order at most n.*

4. PROOF of Theorem. By Lemma 1, (6) implies that $f^{(r)} \in \operatorname{Lip} \alpha$; furthermore Lemma 2 shows that this statement in respect to the modulus of continuity is best possible.

To prove (4), by Lemma 5, it is enough to verify that

$$n^{\alpha-1} \sum_{k=n+1}^{2n} |s_k^{(r)}(x)-f^{(r)}(x)| \tag{4.1}$$

tends to zero almost everywhere. Since for even r

$$|s_n^{(r)}(x)-f^{(r)}(x)| = \left| \sum_{k=n+1}^{\infty} k^r (a_k \cos kx + b_k \sin kx) \right|,$$

so using Abel's transformation we obtain that

$$|s_n^{(r)}(x)-f^{(r)}(x)| \leqq K\Big(n^r |s_{n+1}(x)-f(x)| + \sum_{k=n+2}^{\infty} k^{r-1} |s_k(x)-f(x)|\Big). \tag{4.2}$$

Setting $R_n(x) = |s_n(x)-f(x)|$, by (4.1) and (4.2), the statement (4) holds if

$$n^{r+\alpha-1} \sum_{k=n+1}^{2n} R_k(x) \to 0 \quad \text{as} \quad n\to\infty \tag{4.3}$$

and

$$n^{\alpha} \sum_{k=n}^{\infty} k^{r-1} R_k(x) \to 0 \quad \text{as} \quad n\to\infty \tag{4.4}$$

are verifiable almost everywhere. Obviously (4.4) implies (4.3) thus we only have to prove (4.4).

Let $2^m < n \leqq 2^{m+1}$. Then

$$(4.5)\qquad n^\alpha \sum_{k=n}^{\infty} k^{r-1} R_k(x) \leqq n^\alpha \sum_{l=m}^{\infty} 2^{-\alpha l} \sum_{k=2^l+1}^{2^{l+1}} k^{r+\alpha-1} R_k(x).$$

In the estimation of

$$\sum_{k=n+1}^{2n} k^{r+\alpha-1} R_k(x)$$

we distinguish two cases according $p \geqq 1$ or $p<1$.

If $p \geqq 1$ we can use Hölder's inequality in the following usual form:

$$(4.6)\qquad \sum_{k=n+1}^{2n} k^{r+\alpha-1} R_k(x) \leqq \left\{\sum_{k=n+1}^{2n} k^{(r+\alpha)p-1} R_k^p(x)\right\}^{1/p} \left\{\sum_{k=n+1}^{2n} \frac{1}{k}\right\}^{1-1/p}.$$

If $0<p<1$, we also use Hölder's inequality but with the exponents $1/p$ and $1/(1-p)$. Then we obtain that

$$(4.7)\qquad \sum_{k=n+1}^{2n} k^{r+\alpha-1} R_k(x) = \sum_{k=n+1}^{2n} k^{p[(r+\alpha)p-1]} R_k^{p^2}(x) R_k^{1-p^2}(x) k^{r+\alpha-1-p[(r+\alpha)p-1]} \leqq$$

$$\leqq \left\{\sum_{k=n+1}^{2n} k^{(r+\alpha)p-1} R_k^p(x)\right\}^p \left\{\sum_{k=n+1}^{2n} R_k^{1+p}(x) k^{(r+\alpha)(p+1)-1}\right\}^{1-p}.$$

As we have shown (6) implies $f^{(r)} \in \mathrm{Lip}\,\alpha$, thus $E_n(f) = O(n^{-r-\alpha})$, whence by Lemma 6 the inequalities

$$\sum_{k=n+1}^{2n} R_k^{1+p}(x) \leqq KnE_n^{1+p} \leqq K_1 n^{1-(r+\alpha)(1+p)}$$

follow. Using this and (4.7) we obtain that

$$(4.8)\qquad \sum_{k=n+1}^{2n} k^{r+\alpha-1} R_k(x) \leqq K_2 \left\{\sum_{k=n+1}^{2n} k^{(r+\alpha)p-1} R_k^p(x)\right\}^p.$$

Consequently, (4.6) and (4.8) show that for any positive p we have the estimation with $p^* = \min\left(p, \frac{1}{p}\right)$

$$\sum_{k=n+1}^{2n} k^{r+\alpha-1} R_k(x) \leqq K \left\{\sum_{k=n+1}^{2n} k^{(r+\alpha)p-1} R_k^p(x)\right\}^{p^*}.$$

Hence, by (6), at any point x

$$\sum_{k=n+1}^{2n} k^{r+\alpha-1} R_k(x) \to 0 \quad \text{as} \quad n \to \infty,$$

whence, in respect to (4.5), (4.4) follows everywhere; and this ends the proof of (4), namely (4.4) implies (4.1), what is a sufficient condition of (4).

In the proof of (5) we can run parallel to the previous line, the only difference occurring is that for odd r

$$|\tilde{s}_n^{(r)}(x)-\tilde{f}^{(r)}(x)| = \left|\sum_{k=n+1}^{\infty} k^r (a_k \cos kx + b_k \sin kx)\right|,$$

and thus the use of Lemma 5 gives (5).

In order to verify that (4) and (5) cannot be extended to all x we consider the function

$$F(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^{r+\alpha+1}}.$$

By Lemma 4 this function satisfies (6); furthermore, using Lemma 3, we obtain that

$$\left|F^{(r)}\left(\frac{\pi}{m}\right) - F^{(r)}(0)\right| \geqq \frac{1}{2m} \sum_{n=1}^{m} n^{-\alpha} \quad \text{for even } r$$

and

$$\left|\tilde{F}^{(r)}\left(\frac{\pi}{m}\right) - \tilde{F}^{(r)}(0)\right| \geqq \frac{1}{2m} \sum_{n=1}^{m} n^{-\alpha} \quad \text{for odd } r$$

hold.

This completes the proof.

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[illegible]

The proof of [illegible] is quite parallel to the previous one; the only difference occurring is that [illegible]

$$[illegible]$$

and thus the statement [illegible].

In order to verify that (4) and (5) cannot be extended [illegible] we consider the function

$$[illegible]$$

By Lemma 1 this function satisfies (6) for [illegible]; furthermore using Lemma 3, we obtain that

$$[illegible]$$

and

$$[illegible]$$

hold.

This completes the proof.

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ON THE STABILITY OF QUASIDOUBLE STEP SPLINE FUNCTION APPROXIMATIONS FOR SOLUTIONS OF INITIAL VALUE PROBLEMS

By

S. SALLAM (Assiut) *

1. Introduction

The purpose of this paper is to investigate the stability of spline function approximations to the Cauchy problems $y'=f(x,y), y(0)=y_0$ and $y''=f(x,y,y')$, $y(0)=y_0, y'(0)=y'_0$, respectively. The constructed approximation is a spline $s(x)$ of degree m and continuity class C^{m-2} and the step is of length $2h$. In [3], LOSCALZO and TALBOT present a method which uses a spline of degree $m (m\geqq 2)$ and continuity class C^{m-1} to approximate (1.1). Their method is unstable for $m\geqq 4$.

We shall show that the change of step length together with relaxing the continuity condition will assure better stability.

Let

(1.1) $$y'=f(x,y),\quad 0\leqq x\leqq b,$$

and $f\in C^m$ in some domain $T, T=\{(x,y)\,|\,0\leqq x\leqq b\}$ and that satisfies a Lipschitz condition.

It can be proved that there exists a unique spline $s(x)\in C^{m-2}$ which approximates (1.1) and also that for any spline $s(x)$ of degree m and continuity class C^{m-2} with knots $x_{2\nu}=2\nu h$, the quantities s_{2N+i} and $s^{(p)}_{2N+i}, i=O(1)m+1$ are linearly dependent, i.e., there exist constants $A_{i,m}$ and $B^p_{i,m}$ such that

$$\sum_{i=0}^{m+1} A_{i,m}s_{2N+i}=\sum_{i=0}^{m+1} B^p_{i,m}s^{(p)}_{2N+i},\quad p=1,\dots,m-2.$$

Similarly, for second order equations

(1.2) $$y''=f(x,y,y'),\quad y(0)=y_0,\quad y'(0)=y'_0,$$

and $f\in C^{m-1}$ in T and which satisfies a Lipschitz condition, it can be shown that there exists a unique spline $s(x)\in C^{m-2}$ $(m\geqq 3)$ which approximates (1.2).

MICULA [4] made use of spline functions of full continuity, i.e., $s\in C^{m-1}$, in approximating the equation $y''=f(x,y)$ where he shows that the method is equivalent to a linear multistep method and is unstable for $m\geqq 5$.

The present paper is an answer to the following

PROPOSITION 1. *The quasidouble step spline function approximations* $s(x)\in C^{m-2}$, $m\geqq 2$, to (1.1) *are divergent for* $m\geqq 5$.

PROPOSITION 2. *The quasidouble step spline function approximations* $s(x)\in C^{m-2}$, $m\geqq 3$, *to* (1.2) *are divergent for* $m\geqq 6$.

* On leave from the Department of Mathematics, Assiut University, Egypt.

In the interval $[2vh, 2(v+1)h]$, $s(x)$ is defined by

$$s(x) = \sum_{j=0}^{m-2} \frac{1}{j!}(x-2vh)^j s_{2v}^{(j)} + \sum_{j=1}^{2} (x-2vh)^{m-2+j} a_j^{(v)}, \tag{1.3}$$

where $\frac{c_{i,v}}{i!} = a_{i-m+2}^{(v)}$, $i=m-1, m$, and the parameters $a_j^{(v)}$ are determined according to the relations

$$s'((2v+r)h) = f((2v+r)h), \quad r = 1, 2, \tag{1.4}$$

and

$$s''((2v+k)h) = f((2v+k)h, s((2v+k)h), s'((2v+k)h)) \quad (k = 1, 2) \tag{1.5}$$

for first and second order equations, respectively.

2. A stability criterion

To study the behaviour of the method and hence its stability we need the following

DEFINITION 2.1. A quasidouble step method is said to be stable if all solutions $\{s_{2v}\}$ remain bounded, as $v \to \infty$, $h \to 0$ while $x_{2v} = 2vh$ remains fixed when the method is applied to any differential equation of the form

$$y' = \lambda y, \quad y(0) = 1, \tag{2.1}$$

λ being a constant [1] *.

Let

$$\begin{cases} \underline{a}_v = (h^{m-1} a_1^{(v)}, h^m a_2^{(v)})^T, \\ \underline{s}_v = \left(s_{2v}, \frac{h}{1!} s_{2v}^{(1)}, \dots, \frac{h^{m-2}}{(m-2)!} s_{2v}^{(m-2)}\right)^T. \end{cases} \tag{2.2}$$

Applying the method to equation (2.1), thus (1.4), using (1.3) and (2.2), can be written in the matrix form

$$B\underline{a}_v = C\underline{s}_v, \tag{2.3}$$

where B is a 2×2 matrix whose elements b_{ij} are given by

$$b_{ij} = (m-2+j-\lambda hi)\, i^{m-3+j}, \quad i, j = 1, 2, \tag{2.4}$$

and C is a $2\times(m-1)$ matrix whose elements c_{ij} are

$$c_{ij} = \begin{cases} \lambda h, & j = 1, i = 1, 2; \\ (\lambda hi - (j-1))\, i^{j-2}; & j = 2, \dots, m-1; \; i = 1, 2. \end{cases} \tag{2.5}$$

Also differentiation of (1.3) $m-2$ times and putting $x = 2(v+1)h$ gives

$$\underline{s}_{v+1} = D\underline{s}_v + E\underline{a}_v, \tag{2.6}$$

* This definition is equivalent to the root condition for instability to the linear multistep methods (see [2]).

where D is an $(m-1)\times(m-1)$ upper triangular matrix whose elements d_{ij} are

(2.7) $$d_{ij}=\begin{cases}\binom{j}{i}2^{j-i}, & i=0,1,\dots,m-2, j\geqq i,\\ 0, & j<i,\end{cases}$$

and E is the $(m-1)\times 2$ matrix with elements

(2.8) $$e_{ij}=\binom{m-2+j}{i}2^{m-2+j-i},\quad i=0,1,\dots,m-2, j=1,2.$$

Eliminating $\underline{a}_\nu$ between (2.3) and (2.5) we have

(2.9) $$\underline{s}_{\nu+1}=A\underline{s}_\nu,$$

where $A=D+EB^{-1}C$. Let A_0 denote the matrix A when $h=0$, and let μ_i, $i=1,\dots,m-1$, be its eigenvalues. By definition (2.1), $|\mu_i|\leqq 1$ for stable solutions and $|\mu|_i>1$ for unstable ones. It can be easily seen that the diagonal elements a_{ii} of A_0 are given by

$$a_{11}=1,\quad a_{22}=0,\quad a_{ii}=1+\binom{m-2}{i-3}\left[2^{m-i+1}-\frac{m+i-3}{i-2}\right],\quad i=3,\dots,m-1.$$

Using the property that the trace of a matrix is equal to the sum of its eigenvalues, we have

$$\sum_{i=1}^{m-1}\mu_i=\operatorname{tr}(A_0)=3^{m-2}-3\cdot 2^{m-2}+m+1.$$

Hence

(2.10) $$\sum_{i=1}^{m-1}\mu_i=3^{m-2}-3\cdot 2^{m-2}+m+1.$$

Taking the moduli of both sides of (2.10) we have

(2.11) $$\sum_{i=1}^{m-1}|\mu_i|\geqq\left|\sum_{i=1}^{m-1}\mu_i\right|=|m+1+3^{m-2}-3\cdot 2^{m-2}|>m$$

for $m\geqq 5$. Let $\mu_{\max}=\max_i|\mu_i|$, then (2.11) becomes $(m-1)\mu_{\max}>m$, and hence $\mu_{\max}>1$ for $m\geqq 5$. Thus we have proved Proposition 1.

Next we turn to complete the proof of Proposition 2, a similar terminology of stability, which is an algebriac concept, to that of first order equations will be considered.

DEFINITION 2.2. A quasidouble step method is said to be stable if all solutions $\{s_{2\nu}\}$ remain bounded, as $\nu\to\infty, h\to 0$ while $x_{2\nu}=2\nu h$ remains fixed when the method is applied to any differential equation of the form

(2.12) $$y''=\lambda^2 y,\quad y(0)=1,\quad y'(0)=\lambda,$$

λ being a constant. *

* This definition is equivalent to the root condition for instability to the linear multistep methods (see [2]).

Actually, we follow similar steps to that for the proof of Proposition 1. Applying the method to (2.12), we arrive to the same equations (2.3), (2.6) and (2.9) with

$$b_{ij} = [(m+j-2)(m+j-3)-(\lambda hi)^2]\, i^{m+j-4}, \quad i,j = 1, 2,$$

$$c_{ij} = \begin{cases} \lambda^2 h^2, & j = 1;\ i = 1, 2, \\ [(\lambda hi)^2-(j-1)(j-2)]\, i^{j-3}, & j = 2, \dots, m-1;\ i = 1, 2, \end{cases}$$

$$d_{ij} = \begin{cases} \binom{j}{i} 2^{j-i}, & i = 0, 1, \dots, m-2;\ j \geqq i, \\ 0, & j < i, \end{cases}$$

and

$$e_{ij} = \binom{m-2+j}{i} 2^{m-2+j-i}, \quad i = 0, 1, \dots, m-2;\ j = 1, 2.$$

Also, the elements a_{ij} of the matrix A_0 will be

$$a_{11} = 1;\ a_{22} = 1,\ a_{33} = 0,\ a_{ii} = 1+\binom{m-3}{i-4}\left[2^{m-i+1}-\frac{m+i-5}{i-3}\right], \quad i = 4, \dots, m-1,$$

and (2.10) will take the form

$$\sum_{i=1}^{m-1} \mu_i = 3^{m-3}-3\cdot 2^{m-3}+m. \tag{2.13}$$

Taking the moduli of both sides of (2.13), we have

$$\sum_{i=1}^{m-1} |\mu_i| \geqq \left|\sum_{i=1}^{m-1} \mu_i\right| = |3^{m-3}-3\cdot 2^{m-3}+m| > m$$

for $m \geqq 6$. This completes the proof of Proposition 2.

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A PEXIDER EQUATION FOR FUNCTIONS DEFINED ON A SEMIGROUP

By

M. A. TAYLOR (Wolfville) *

0. In this paper the general solution of the Pexider equation, $f(xy)=g(x)*h(y)$, is given for functions f, g, h defined on a semigroup and mapping into a group. VINCZE [2] and ACZÉL [1] have given the solution of this equation; the former on semigroups with a solvability condition with the functions mapping into a group, the latter for functions mapping an abelian semigroup into a group.

1. Let $(S, \cdot)$ be a semigroup and $(G, *)$ be a group. Suppose f, g and h are functions from S into G satisfying

$$f(xy) = g(x)*h(y), \quad \text{for all} \quad x, y\in S. \tag{1}$$

If we set $g(x)=f(x)*k(x)$ and $h(x)=l(x)*f(x)$, where l, k are functions from S into G, then for all $x, y, z\in S$,

$$f(x(yz)) = g(x)*h(yz)=f(x)*k(x)*l(yz)*f(y)*k(y)*l(z)*f(z)$$

and

$$f((xy)z)=f(xy)*k(xy)*l(z)*f(z)=f(x)*k(x)*l(y)*f(y)*k(xy)*l(z)*f(z).$$

The associativity of $(S, \cdot)$ and the cancellativity of $(G, *)$ give

$$l(yz)*f(y)*k(y) = l(y)*f(y)*k(xy).$$

This equation can be written as

$$l(y)^{-1}*l(yz)*f(y)=f(y)*k(xy)*k(y)^{-1} = m(y) \tag{2}$$

where a^{-1} denotes the inverse of a in the group $(G, *)$, and $m\colon S\to G$. This leads to the equation

$$l^{-1}(y)*l(yz) = m(y)*f(y)^{-1} = q'(y) \quad \text{(say)}$$

$$k(xy)*k(y)^{-1}=f(y)^{-1}*m(y) = r'(y) \quad \text{(say)}.$$

Brief calculations show that $q'(xy)=r'(xy)=e$ for all $x, y\in S$, where e is the identity for $(G, *)$.

The functions $q, r\colon S\to G$ defined by

$$q(x) = l(x)*q'(x), \quad r(x) = r'(x)*k(x), \quad \text{for all} \quad x\in S$$

* This work was supported in part by N.R.C. grant A8144.

satisfy, respectively,

$$q(xy) = q(x) \tag{3}$$

$$r(xy) = r(y), \quad \text{for all} \quad x, y \in S. \tag{4}$$

We now define $p\colon S \to G$ by $p(x) = l(x) * m(x) * k(x)$. Using (2) this leads to $p(x) = l(x) * f(x) * r(x) = h(x) * r(x)$ and $p(x) = q(x) * f(x) * k(x) = q(x) * g(x)$. A straightforward calculation shows that p is a homomorphism. The functions f, g, h can then be expressed as

$$f(x) = q(x)^{-1} * p(x) * r(x)^{-1} * r'(x),$$

$$h(x) = p(x) * r(x)^{-1}, \quad g(x) = q(x)^{-1} * p(x)$$

where q, r satisfy (3) and (4) respectively and $r'(x) = e$ for all $x \in S \cdot S$.

Conversely, if f, g, h are defined, with functions p, q, r, r' given, as above, then $f(xy) = g(x) * h(y)$.

2. We now investigate equations (3) and (4).

Let $(S, \cdot)$ be a semigroup and $x \in S$. Define

$$L_0(x) = xS' \quad \text{where} \quad xS' = xS \cup \{x\},$$

and

$$L_k(x) = \bigcup_{i_k} x_{i_k} \quad \text{where} \quad x_{i_k} S' \cap L_{k-1}(x) \neq \varnothing, \quad k = 1, 2, 3, \ldots.$$

Finally, we define

$$L(x) = \bigcup_{k=1}^{\infty} L_k(x).$$

Dually we start with $R_0(x) = S'x = Sx \cup \{x\}$, and define

$$R(x) = \bigcup_{k=0}^{\infty} R_k(x).$$

LEMMA. *Let $x, y \in S$. Then $y \in L(x)$ if and only if there exists a finite set of equations*

$$xs_0 = x_1 t_1, \; x_1 s_1 = x_2 t_2, \; \ldots, \; x_k s_k = y t_{k+1}$$

with $s_i, t_i \in S$, $i = 0, 1, \ldots, k$.

PROOF. Clearly if a finite set of equations exists then $x_1 \in L_1(x)$, $x_2 \in L_2(x), \ldots, y \in L_k(x)$ and consequently $y \in L(x)$. On the other hand, if $y \in L(x)$ then $y \in L_n(x)$ for a least $n \geqq 0$ and the required result follows from a finiteness argument.

Noticing that $y \in L(x)$ implies $x \in L(y)$ leads us to

COROLLARY. $L(x) = L(y)$ *if and only if* $y \in L(x)$.

The family of sets $\mathscr{L} = \{L(x) \mid x \in S\}$ *partitions* S, *as does* $\mathscr{R} = \{R(x) \mid x \in S\}$.

THEOREM 1. *The general solution of the equation* (3) $q(xy) = q(x)$, *among functions defined on a semigroup onto a set T, is $q = q_2 \circ q_1$, where q_1 is the canonical surjection $q_1(x) = L(x)$, and q_2 is an arbitrary function from $\mathscr{L}$ onto T.*

PROOF. Suppose q satisfies (3). Let $y \in L(x)$, then there is a finite set of equations

$$xs_0 = x_1t_1, \quad x_1s_1 = x_2t_2, \quad \dots, \quad x_ks_k = yt_{k+1}.$$

Applying q to these equations and using (3) we find

$$q(x) = q(x_1), \; q(x_1) = q(x_2), \dots, q(x_k) = q(y).$$

Hence $q(x)=q(y)$ if $L(x)=L(y)$, and the required factorization of q follows immediately.

It is trivial that a function $q=q_2 \circ q_1$, with q_1 and q_2 as described above, satisfies (3).

Because the solution to $r(xy)=r(y)$ is given by the dual of Theorem 1, we can now proceed to the general solution of (1).

THEOREM 2. *The general solution of the equation* $f(xy)=g(x)*h(y)$, *among functions* f, g, h *defined on a semigroup* $(S, \cdot)$ *into a group* $(G, *)$, *is*

$$f(x) = q(x)*p(x)*r(x)*m(x), \quad g(x) = q(x)*p(x), \quad h(x) = p(x)*r(x)$$

where p *is a homomorphism,* $q=q_2 \circ q_1$ *and* $q_1(x)=L(x), q_2\colon \mathscr{L} \to G, r=r_2 \circ r_1$ *and* $r_1(x)=R(x), r_2\colon \mathscr{R} \to G$, *and* $m(x)=e$ for $x \in S \cdot S$.

3. EXAMPLE. Consider the semigroup S given by

	a	b	c	d
a	a	b	a	a
b	a	b	a	a
c	a	b	a	a
d	a	b	c	d

Then $L(a)=L(b)=L(c)=L(d)=\{a, b, c, d\}$ and $R(a)=R(c)=R(d)=\{a, c, d\}$, $R(b)=\{b\}$. Define functions p, q, r, m from S into the additive reals by $p(x)=$ $=m(x)=0$, $q(x)=1$ for all $x \in S$, $r(a)=r(c)=r(d)=2$, $r(b)=3$. The functions f, g, h given by $f(x)=1+r(x)$, $g(x)=1$, $h(x)=r(x)$, satisfy (1).

REMARK. If S has a non void centre then $R(x)=L(x)=S$ for all $x \in S$ and consequently q and r are constant mappings. Also if S has a left (right) identity then $L(x)=S(R(x)=S)$ for all $x \in S$ and $q(r)$ is a constant map.

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ON THE CONCENTRATION OF ADDITIVE FUNCTIONS

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1. Introduction

Let f be an additive arithmetical function,

$$R(a, b, x) = x^{-1}|\{n\colon n \leqq x,\ a \leqq f(n) < b\}| \tag{1.1}$$

and

$$Q_h(x) = \sup_a R(a, a+h, x). \tag{1.2}$$

In 1946 ERDŐS [2] proved the following results (they are stated in his paper quite differently):

THEOREM A. *$Q_1(x)\to 0$ as $x\to\infty$, unless there is a number λ such that*

$$\sum_p \frac{\min\left(1, (f(p)-\lambda \log p)^2\right)}{p} < \infty. \tag{1.3}$$

THEOREM B. *$Q_h(x)\to 0$ as $x\to\infty$ and $h\to 0$, unless*

$$\sum_{f(p)\neq 0} 1/p < \infty. \tag{1.4}$$

Theorem B has the following consequence:

THEOREM C. *The sequence of solutions of $f(n)=a$ has asymptotic density* 0 *for every a, unless* (1.4) *holds.*

It is easy to see that if (1.3) holds, then f has a distribution with a suitable centralization, and if (1.4) holds, then f has a discrete distribution, so these conditions are necessary and sufficient.

More than 20 years later HALÁSZ [6] proved the following quantitative version of Theorem C:

THEOREM D. *We have*

$$|\{n\colon n \leqq x, f(n) = a\}| \ll \frac{x}{\sqrt{E(x)}}, \tag{1.5}$$

where

$$E(x) = \sum_{\substack{p \leqq x \\ f(p)\neq 0}} 1/p. \tag{1.6}$$

The implied constant is absolute.

By a slight modification of Halász's method one can prove the following result:

THEOREM E. *We have*

$$Q_1(x) \ll \frac{1}{\sqrt{U(x)}}, \tag{1.7}$$

where

$$U(x) = \min U(x, \lambda), \tag{1.8}$$

$$U(x, \lambda) = \sum_{p \leqq x} \frac{\min\left(1, (f(p)-\lambda \log p)^2\right)}{p}. \tag{1.9}$$

Theorems A—D all can be deduced form Theorem E. (I do not describe how. It needs some tricks, and the deduction will be immediate from the following stronger results.) However, it has the defect that for the logarithmic function

$$f(n) = c \log n, \tag{1.10}$$

where c is a large number, it gives no nontrivial bound. The real order of $Q_1(x)$ in this case is, of course, $1/c$.

In this direction Halász proved (oral communication, unpublished):

THEOREM F. *We have*

$$Q_1(x) \ll \frac{1}{\sqrt{V(x)}}, \tag{1.11}$$

where

$$V(x) = \min\left(\frac{\lambda^2}{\log^2(|\lambda|+2)} + U(x, \lambda)\right). \tag{1.12}$$

For the function (1.10) this gives

$$Q_1(x) \ll (\log c)/c.$$

Our aim is to prove a result, which gives the right order in this case, namely:

THEOREM. *For every additive function we have*

$$Q_1(x) \ll \frac{1}{\sqrt{W(x)}}, \tag{1.13}$$

where

$$W(x) = \min\left(\lambda^2 + U(x, \lambda)\right). \tag{1.14}$$

The implied constant is absolute.

Of course, for the function (1.10) our theorem gives $Q_1(x) \ll 1/c$ only if c is not too large compared with x, namely for $c \ll (\log\log x)^{1/2}$. This can be extended up to $c \ll \log^{1/2} x$, but it needs a much more complicated formulation; I think the above form is the most easily applicable. — Sometimes the following version is the most convenient:

REFORMULATION OF THE THEOREM. *If f is additive and $f(n)\in[a, a+h)$ for qx values of $n, n\leqq x$, then there is a λ, $|\lambda|\leqq ch/q$ such that*

$$\sum_{p\leqq x}\frac{\min\left(h^2, (f(p)-\lambda\log p)^2\right)}{p}\leqq ch^2q^{-2}, \tag{1.15}$$

c being an absolute constant.

NOTE. The theorem has a probabilistic background. We have

$$f=\sum_p f_p,$$

where f_p is defined by $f_p(n)=f(p^k)$ if $p^k\| n$. If the f_p's were independent, by the Kolmogorov—Rogozin inequality (see ROGOZIN [10]; for stronger results see ESSEEN [3, 4]) we could get $Q_1(x)\ll U(x, 0)^{-1/2}$. Of course, they are far from being independent, as the logarithmic function shows. But in some sense this is the only kind of exception. If we regard a decomposition

$$f=\lambda\log+f'=\lambda\log+\sum_p f_p'$$

and suppose that the functions "$\lambda\log$" and f_p' are independent, then the same inequality yields $Q_1(x)\ll(\lambda^2+U(x, \lambda))^{-1/2}$. Our theorem just states that this inequality is indeed true for a suitably chosen value of λ.

2. An outline

We shall deduce our Theorem from Theorem E and the following statement:

MAIN LEMMA (2.1). *Let f be an additive function and suppose*

$$U(x, \lambda)\leqq\frac{1}{10}\log\log x \tag{2.2}$$

for some λ. Then we have

$$Q_1(x)\ll|\lambda|^{-1}+\log^{-3/10}x. \tag{2.3}$$

The constant 1/10 in (2.2) and the exponent 3/10 in (2.3) can be improved. However, the second term of (2.3) cannot be improved over $\log^{-1}x$, as the following example shows. Let λ be large,

$$f(p)=\begin{cases}\lambda\log p & \text{for}\quad p\leqq x/2,\\ 0 & \text{for}\quad x/2<p\leqq x,\end{cases}$$

f completely additive. Then we have

$$U(x, \lambda)=O(\log^{-1}x),$$

while

$$Q_1(x)\geqq x^{-1}R(0, 1, x)\gg\log^{-1}x.$$

Deduction of the Theorem from Theorem E and the Main Lemma. We have always $W(x) \leqq U(x, 0) \ll \log\log x$, so the bound for $Q_1(x)$ cannot be of smaller order than $(\log\log x)^{-1/2}$. Let λ_0 be any of the values minimizing $U(x, \lambda)$. If $U(x, \lambda_0) \geqq \frac{1}{10} \log\log x$, then we get this order from Theorem E; if

$$U(x, \lambda_0) < \frac{1}{10} \log\log x, \quad |\lambda_0| \geqq \sqrt{\log\log x},$$

then we get it from the Main Lemma. Finally, if

$$U(x, \lambda_0) < \frac{1}{10} \log\log x, \quad |\lambda_0| < \sqrt{\log\log x},$$

then $Q_1(x) \ll U(x, \lambda_0)^{-1/2}$ by Theorem E and

$$Q_1(x) \ll |\lambda_0|^{-1}$$

by the Main Lemma; taking the (-2)-th power mean, we get the desired bound.

Section 3 contains a general inequality on concentration. In sections 4—7 we prove Theorem E. The proof is almost the same as that of Theorem D in HALÁSZ's paper [6], but we include it for sake of completeness. In sections 8—10 we prove the Main Lemma and in section 11 we show how Theorems A—D can be deduced from ours.

3. A probabilistic lemma

Let ξ be a random variable, $Q = \sup_a P(a \leqq \xi < a+1)$ its concentration and $\chi(t) = M(e^{it\xi})$ its characteristic function.

LEMMA (3.1). *For every a we have*

$$P(|\xi - a| \leqq 1) \leqq 2 \int_{-1}^{1} \chi(t) e^{ita}(1-|t|)\, dt.$$

LEMMA (3.2). *We have*

$$Q \leqq 2 \sup_a \int_{-1}^{1} \chi(t) e^{ita}(1-|t|)\, dt \leqq 2 \int_{-1}^{1} |\chi(t)|\, dt.$$

Lemma (3.2) is an immediate consequence of Lemma (3.1). They are essentially due to ROSÉN [11], though he stated only the inequality with the integral of the modulus. Their proof is based on the equality

$$\int_{-1}^{1} \chi(t) e^{ita}(1-|t|)\, dt = M\big(F(\xi - a)\big), \tag{3.3}$$

where

$$F(y) = \left(\frac{\sin y/2}{y/2}\right)^2$$

is Fejér's kernel function; the reader can easily complete the proof.

LEMMA (3.4). *For all ξ and a we have*

$$\int_{-1}^{1} \chi(t) e^{ita}(1-|t|)\, dt \leqq 9Q.$$

That is, the first inequality of (3.2) always gives the right order of Q.

PROOF. We start with (3.3) and continue as follows:

$$M\big(F(\xi-a)\big) \leqq \sum_{k=-\infty}^{\infty} P(a+k \leqq \xi < a+k+1) \max_{k \leqq y \leqq k+1} F(y) \leqq$$

$$\leqq Q \sum_{k=-\infty}^{\infty} \max_{k \leqq y \leqq k+1} F(y) \leqq 9Q,$$

since $F(y) \leqq \min(1, 4y^{-2})$.

These lemmas will be applied to the variables $f^{(x)} = f|_{N_x}$, where f is an additive function and N_x is the probability space of natural numbers $1, \dots, x$, each having probability $1/x$. Then we have $Q = Q_1(x)$ and

$$\chi(t) = x^{-1} \sum_{n \leqq x} e^{itf(n)}.$$

The integral of the modulus in (3.2) is sufficient to prove Theorem E or even Halász's Theorem F, but it cannot yield our Main Lemma. Indeed, for $f \doteq \lambda \log$ we have

$$\chi(t) = x^{-1} \sum_{n \leqq x} n^{it\lambda} = \frac{x^{it\lambda}}{1+it} + O\left(\frac{|\lambda t|}{\log x}\right),$$

thus the modular integral gives only

$$Q \ll \int_{-1}^{1} |\chi(t)|\, dt \asymp \frac{\log |\lambda|}{|\lambda|}.$$

4. Proof of Theorem E: outline

Using the notations of the previous section, we have

$$Q_1(x) \ll \int_{-1}^{1} |\chi(t)|\, dt, \tag{4.1}$$

where

$$\chi(t) = x^{-1} \sum_{n \leqq x} h_t(n), \qquad h_t(n) = e^{itf(n)}. \tag{4.2}$$

h_t is a multiplicative function; we shall estimate its sum by the following theorem of HALÁSZ [6].

LEMMA (4.3). *Let h be a multiplicative function,* $|h(n)| = 1$,

$$m(u) = \sum_{p \leqq x} \frac{1-\operatorname{Re}\big(h(p)p^{-iu}\big)}{p}, \tag{4.4}$$

and

$$M = \min_{|u| \leq \log x} m(u). \tag{4.5}$$

We have

$$\sum_{n \leq x} h(n) \ll x e^{-cM}, \quad c = 1/16. \tag{4.6}$$

(4.1) and (4.6) imply

$$Q_1(x) \ll \int_{-1}^{1} e^{-cM(t)}\, dt, \tag{4.7}$$

where we use the symbols $M(t)$ and $m(u, t)$ for denoting the quantities got by (4.4) and (4.5) for $h = h_t$.

Let

$$X_k = \{x \in [-1, 1]: \; M(t) \leq k\}. \tag{4.8}$$

(7) gives

$$Q_1(x) \ll \sum_{k=1}^{\infty} e^{-ck} \mu(X_k), \tag{4.9}$$

where μ denotes the Lebesgue measure.

LEMMA (4.10). *We have*

$$\mu(X_k) \ll \sqrt{k/U(x)}.$$

(4.9) and (4.10) clearly imply Theorem E:

$$Q_1(x) \ll U(x)^{-1/2} \sum_{k=0}^{\infty} e^{-ck} \sqrt{k} \ll U(x)^{-1/2},$$

as the series is convergent. (4.10) will be proved in the next three sections.

5. Proof of Lemma (4.10)

LEMMA (5.1). *If $a_1, \ldots, a_k$ are unimodular complex numbers, then we have*

$$1 - \mathrm{Re} \prod_{j=1}^{\infty} a_j \leq k \sum_{j=1}^{k} (1 - \mathrm{Re}\, a_j). \tag{5.1}$$

(See HALÁSZ [6].)

LEMMA (5.2). *Let $t_1, \ldots t_k, u_1, \ldots u_k$ be real numbers. We have*

$$m\left(\sum u_j, \sum t_j\right) \leq k \sum_{j=1}^{k} m(u_j, t_j). \tag{5.2}$$

PROOF. Immediate consequence of the previous lemma.

For a set $X \subset \underline{R}$ let

$$X^{(r)} = \{x_1 + \ldots + x_r: \; x_1, \ldots, x_r \in X\}.$$

LEMMA (5.3). *If* $X \subset [-1, 1]$ *is a set of positive Lebesgue measure, symmetric to the origin and containing it, then we have*

$$X^{(r)} \supset [-1, 1], \quad r = \left[\frac{12}{\mu(X)}\right]. \tag{5.4}$$

(Proof in the next section.)

The sets X_k satisfy the conditions of this lemma, thus we get

$$X_k^{(r)} \supset [-1, 1], \quad r = \left[\frac{12}{\mu(X_k)}\right]. \tag{5.5}$$

Our aim is to prove $\mu(X_k) \ll \sqrt{k/U(x)}$, that is,

$$k^2 r \gg U(x). \tag{5.6}$$

Let

$$t = t_1 + \ldots + t_r, \quad t_j \in X_k. \tag{5.7}$$

For each t_j there is a u_j, $|u_j| \leqq \log x$, such that $m(u_j, t_j) \leqq k$; applying Lemma (5.2) we get

$$m(u, t) \leqq kr^2, \quad u = \sum_{j=1}^{r} u_j, \quad |u| \leqq r \log x. \tag{5.8}$$

By (5.5) every $t \in [-1, 1]$ has a decomposition of type (5.7); let $u(t)$ denote any of the u's satisfying (5.8). We have

$$m\big(u(t), t\big) \leqq kr^2, \quad |u(t)| \leqq r \log x \quad (t \in [-1, 1]). \tag{5.9}$$

We may assume $r \leqq \log x$ and hence

$$|u(t)| \leqq \log^2 x, \tag{5.10}$$

since in the other case (5.6) is trivial, as $|U(x)| \ll \log\log x$.

Now we shall show that u is nearly linear; what we can directly do is to estimate the difference $u(t_1+t_2) - u(t_1) - u(t_2)$.

LEMMA (5.11). *Let u be a bounded real function, defined on the interval* $[-1, 1]$. *Suppose*

$$|u(t_1+t_2) - u(t_1) - u(t_2)| \leqq K$$

whenever $t_1, t_2, t_1+t_2 \in [-1, 1]$. *Then we have*

$$|u(t) - u(1)t| \leqq 3K.$$

(Proof in Section 7.)

Now let $K = \sup |u(t_1+t_2) - u(t_1) - u(t_2)|$ and choose numbers t_1, t_2, for which $L = u(t_1+t_2) - u(t_1) - u(t_2)$ satisfies $|L| \geqq K/2$. Using (5.9), Lemma (5.2) and the fact that $m(-u, -t) = m(u, t)$, we get

$$m(L, 0) \leqq 9kr^2. \tag{5.12}$$

On the other hand, $m(L, 0)$ does not depend on f:

$$m(L, 0) = \sum_{p \leqq x} \frac{1 - \cos(L \log p)}{p}; \tag{5.13}$$

therefore we have (see HALÁSZ [6])

$$m(L,0)=\log\log x+\log|\zeta(1+\log^{-1}x+iL)|+O(1),$$

which easily yields

$$m(L,0)=\begin{cases}\log(|L|\log x+2)+O(1), & |L|\leqq 1,\\ \log\log x+O(\log\log 3|L|), & |L|>1.\end{cases} \tag{5.14}$$

From (5.12) and (5.14) we could easily estimate L, but we do not actually need to. Applying Lemma (5.11) we get

$$|u(t)-\lambda t|\leqq 3K\leqq 6|L|\leqq c_1\log^2 x \tag{5.15}$$

with $\lambda=u(1)$, the last inequality being valid because of (5.10). Now (5.14) implies

$$m(L_1,0)\leqq 2m(L,0)+c_2 \tag{5.16}$$

if $|L_1|\leqq c_3|L|\leqq\log^{c_4}x$ for some constants c_3 and c_4, with another constant c_2 depending only on c_3 and c_4.

From (5.15), (5.16) and (5.12) we conclude

$$m\big(u(t)-\lambda t,0\big)\leqq 2m(L,0)+c_2\leqq c_5kr^2,$$

and now using (5.9) and again Lemma (5.2) we get

$$m(\lambda t,t)\leqq c_6kr^2. \tag{5.17}$$

We have

$$m(\lambda t,t)=\sum_{p\leqq x}\frac{1-\cos t\big(f(p)-\lambda\log p\big)}{p}.$$

Since

$$\int_0^1(1-\cos\alpha t)\,dt=1-\frac{\sin\alpha}{\alpha}\geqq c_7\min(1,\alpha^2),$$

(5.17) implies

$$kr^2\gg\int_0^1 m(\lambda t,t)\,dt\gg\sum_{p\leqq x}\frac{\min\big(1,(f(p)-\lambda\log p)^2\big)}{p}=U(x,\lambda)\geqq U(x).$$

6. Addition of sets

In this section I shall present a simple proof of Lemma (5.3), whose idea has been communicated to me by Mr. G. Halász.

The constant 12 is far from the best possible, which is, I think, 3. This would follow from the following statement:

CONJECTURE. If $A,B\subset[-1,1]$ are measurable sets, symmetric to the origin and $0\in B$, then we have

$$\mu\big((A+B)\cap[-1,1]\big)\geqq\min\big(2,\mu(A)+\mu(B)/2\big). \tag{6.1}$$

(6.1) can be given the following discrete reformulation: if $A, B\subset[-N, N]$ are sets of integers, symmetric to the origin, $0\in B$ and $C=((A+B)\cup(A+B+1))\cap\cap[-N, N]$, then $|C|\geqq\min(2N, |A|+|B|/2)$.

I can improve the constant 12 to 4, using the fact that $\mu(A+B)\geqq\geqq\min(t, \mu(A)+\mu(B))$ if addition of sets is interpreted (mod t). This is a very special case of the deep theorem of KNESER [8].

PROOF OF LEMMA (5.3). Let $Y_r=X^{(r)}\cap[-1, 1]$. Evidently (Y_r) is an increasing sequence of sets. Since X has positive measure, $X+X=X-X$ contains a neighbourhood of 0 and hence $Y_r=[-1, 1]$ for large r. Now we show that $Y_j=Y_{j-1}$ implies $Y_{j+1}=Y_j$. Regard a number $z=x_1+\ldots+x_{j+1}\in Y_{j+1}$. There is an i which has the same sign as z, and thus

$$z' = z-x_i\in[-1, 1] \Rightarrow z'\in Y_j \Rightarrow z'\in Y_{j-1}\Rightarrow z = z'+x_i\in Y_j.$$

Thus (Y_j) is strictly increasing until it becomes equal to the whole interval $[-1, 1]$.

Let $a_0=0$ and for $k\geqq1$ let a_k be an arbitrary element of $Y_k\setminus Y_{k-1}$, as far as $Y_{k-1}\neq[-1, 1]$. For $j\geqq i+3$ we have $(X+a_i)\cap(X+a_j)=\emptyset$. Namely suppose the contrary, that is $x+a_i=x'+a_j$, $x, x'\in X$. This would imply $a_j==a_i+x-x'\in X^{(i+2)}\in X^{(j-1)}$, a contradiction to the definition of a_j.

The sets $X+a_0, X+a_3, \ldots$ are all disjoint and they are contained in $[-2, 2]$ therefore there can be at most $4/\mu(X)$ of them; consequently, $Y_r=[-1, 1]$ must happen for an $r\leqq12/\mu(X)$.

7. The quasi-Cauchy equation

In this section we shall discuss Lemma (5.11) and some related problems. Knowing the inequality

$$|u(x+y)-u(x)-u(y)| \leqq 1 \tag{7.1}$$

for a function u, we want to conclude that u is near to a solution v of the Cauchy equation

$$v(x+y) = v(x)+v(y), \tag{7.2}$$

which in our case must be linear because of the boundedness. Varying the domain and the range of v, we get different problems, some of which have already been investigated. E.g. HYERS [7] investigated the case of functions, mapping from a Banach space into another, the absolute value being replaced by norm; MEYER [9, p. 72] dealt with the case of real-valued functions, defined on the set of integers and our Lemma (5.11) requires real-valued functions, defined on an interval. It would be desirable to find a common generalization, which I have not succeeded in.

Hyers' and Meyer's theorems can be given the following common generalization:

STATEMENT (7.3). *Let G be an Abelian group (written additively), B a Banach space and u: $G\to B$ a function. If*

$$\|u(x+y)-u(x)-u(y)\| \leqq K \tag{7.4}$$

holds for all $x, y \in G$, then we can find a homomorphism $v: G \to B$ such that

$$\|u(x)-v(x)\| \leqq K \tag{7.5}$$

for all $x \in G$.

Note (7.6). It is of secondary importance whether u maps into a Banach space or simply into the set of reals; the essential role is played by the domain.

Problem (7.7). Does the above statement hold for noncommutative groups?

Proof. (Same as Hyers' and Meyer's.) Set

$$v(x) = \lim_{n\to\infty} 2^{-n} u(2^n x). \tag{7.8}$$

The existence of the limit and inequality (5) follow from the inequality

$$\left\|\frac{u(2^{n-1}x)}{2^{n-1}} - \frac{u(2^n x)}{2^n}\right\| = \frac{\|2u(2^{n-1}x)-u(2^n x)\|}{2^n} \leqq 2^{-n}K.$$

(8) and (4) yield

$$\|v(x+y)-v(x)-v(y)\| = \lim \frac{\|u(2^n(x+y))-u(2^n x)-u(2^n y)\|}{2^n} \leqq \lim K/2^n = 0,$$

which shows that v is indeed a homomorphism.

Deduction of Lemma (5.11) *from the previous statement.* Let u be defined on $[-1, 1]$; put

$$w(2k+t) = 2ku(1)+u(t), \quad t \in [-1, 1) \tag{7.9}$$

for integers k. w is defined on the whole line and it is an extension of u, except at 1, where $w(1)=2u(1)+u(-1)$. We are going to estimate the difference $d=w(x+y)-w(x)-w(y)$. As $w(x)-u(1)x$ is 2-periodical, we may assume $x, y \in [-1, 1)$. If $x+y \in [-1, 1)$, we get $|d| \leqq K$. If $x+y \geqq 1$, we have

$$d = u(x+y-2)+2u(1)-u(x)-u(y) = [u(x+y-2)-u(x-1)-u(y-1)]+$$
$$+[u(x-1)+u(1)-u(x)]+[u(y-1)+u(1)-u(y)],$$

thus $|d| \leqq 3K$; for $x+y<-1$ we can get $|d| \leqq 3K$ similarly.

Now the above statement yields $|w(x)-v(x)| \leqq 3K$ with $v(x)=\lim 2^{-n}w(2^n x)=u(1)x$, whence $|u(x)-u(1)x| \leqq 3K$ follows for $x \in [-1, 1)$, and it holds obviously for $x=1$.

By a small modification of the method I can prove the same for functions, defined on a convex set $T \subset R^n$, containing the origin; however, the coefficient of K will depend on the dimension, it is of order $\log n$, which makes the generalization for infinite dimension impossible. This is inevitable: e.g. on the set

$$T = \{(x_1, \ldots, x_n): \sum |x_j| \leqq 1\}$$

(the "generalized octahedron") the function

$$u(\underline{x}) = u(x_1, \ldots, x_n) = \sum x_j \log |x_j|$$

satisfies $|u(\underline{x}+\underline{y})-u(\underline{x})-u(\underline{y})|\leqq 2$, while $\max_{\underline{x}\in T}|u(\underline{x})-v(\underline{x})|\geqq(\log n)/2$ for every linear v. Nevertheless, a dimension-independent estimate can be proved for some special T, e.g. for the ball. I cannot decide whether the cube is a "good" or a "bad" set in this connexion.

8. Proof of the Main Lemma: outline

This time our task is quite different from the previous; so far we estimated the concentration of f under the assumption that it is not too near to a logarithmic function, and now we do it assuming it is near. Let

$$f=\lambda\log+g, \tag{8.1}$$

where

$$\sum_{p\leqq x}\frac{\min(1, g^2(p))}{p}\leqq 0.1\log\log x. \tag{8.2}$$

The characteristic function of $g|_{N_x}$ is

$$\psi(t)=x^{-1}\sum_{n\leqq x}e^{itg(n)}, \tag{8.3}$$

and that of $(\lambda\log)|_{N_x}$ is

$$\vartheta(t)=\sum_{n\leqq x}n^{it\lambda}\approx\frac{x^{it\lambda}}{1+it\lambda}. \tag{8.4}$$

We expect that $\lambda\log$ and g are almost independent. If they were independent, we should have $\chi=\vartheta\psi$; we hope that $\chi(t)$ will be near to $\vartheta(t)\psi(t)$. Indeed, we shall prove

$$\chi(t)=\frac{x^{it\lambda}}{1+it\lambda}\psi(t)+O(\log^{-0.3}x) \tag{8.5}$$

for $|\lambda|\leqq\log x$ and $|t|\leqq 1$. (Instead of ϑ it is more convenient to use its approximate value.)

Now we deduce (8.5) from the following lemma, which will be proved in the next two sections.

LEMMA (8.6). *Let h be a unimodular multiplicative function (the term "unimodular" means $|h(n)|=1$ for all n), b a real number and*

$$L=\sum_{p\leqq x}\frac{|h(p)-1|}{p}. \tag{8.7}$$

Then we have

$$\sum_{n\leqq x}h(n)n^{ib}=\frac{x^{ib}}{1+ib}\sum_{n\leqq x}h(n)+O\left(xe^{L/2}\log^{-1/2}x\log^2(|b|+\log x)\right). \tag{8.8}$$

We apply this lemma for $b=\lambda t$ and $h(n)=e^{itg}$. We have

$$|h(p)-1|=|e^{itg(p)}-1|\leqq\min(2, |g(p)|).$$

We have to estimate by $\min(1, g^2(p))$, so we use the inequality $\min(2, |x|) \leqq 2\min(1, x^2)+1/8$. This yields

$$L \leqq 2 \sum_{p\leqq x} \frac{\min(1, g^2(p))}{p} + \frac{1}{8} \sum_{p\leqq x} 1/p \leqq 0.33 \log\log x + O(1).$$

As $|b| \leqq |\lambda| \leqq \log x$, (8.8) gives

$$\chi(t) - \frac{x^{it\lambda}}{1+it\lambda} \psi(t) \ll \log^{-.335} x (\log\log x)^2 \ll \log^{-.3} x$$

as wanted.

According to Lemma (3.2), we have to prove

$$\text{(8.9)} \qquad \int_{-1}^{1} \chi(t)(1-|t|)e^{ita}\,dt \ll |\lambda|^{-1} + \log^{-.3} x$$

for all a. Because of (7.5) it is sufficient to prove that

$$\text{(8.10)} \qquad J = \int_{-1}^{1} \frac{x^{it\lambda} e^{ita}}{1+it\lambda} \psi(t)(1-|t|)\,dt \ll |\lambda|^{-1}.$$

$\psi(t)$ is the characteristic function of $g|_{N_x}$, $1/(1+it\lambda)$ is the characteristic function of a random variable ξ with the distribution

$$P(\xi \leqq u) = \begin{cases} e^{u/\lambda} & u < 0, \\ 1 & u \geqq 0 \end{cases} \quad (\lambda > 0), \quad P(\xi \leqq u) = \begin{cases} 0 & u < 0, \\ 1 - e^{u/\lambda} & u \geqq 0 \end{cases} \quad (\lambda < 0),$$

respectively. Hence $\psi(t)/(1+it\lambda)$ is thech aracteristic function of their sum (assumed they are independent) and $x^{it\lambda}e^{ita} = e^{itb}$ with $b = a + \lambda \log x$, thus according to Lemma (3.4) J can be estimated by the concentration of $\xi + g|_{N_x}$. This is not greater than the concentration of ξ, which is equal to

$$\max_{u\leqq 0} (e^{u/\lambda} - e^{(u-1)/\lambda}) = 1 - e^{-1/\lambda} \leqq \lambda^{-1}$$

for $\lambda > 0$, and similarly $\leqq |\lambda|^{-1}$ for $\lambda < 0$. (The case $\lambda = 0$ is obvious.)

We have proved (8.10) for all λ, and thus (8.9), which is a reformulation of the Main Lemma, for $|\lambda| \leqq \log x$. If $|\lambda| > \log x$, put $\lambda_0 = \log x$, $f_0 = \frac{\lambda_0 f}{\lambda}$, $g_0 = f_0 - \lambda_0 \log$. Then (8.1) and (8.2) hold with f_0, g_0 and λ_0 in the place of f, g and λ, respectively, thus from the proved part of the Main Lemma we conclude that the concentration of f_0 is $\ll \log^{-.3} x$. But as $f = (\lambda/\lambda_0) f_0$, the concentration of f is not greater than that of f_0 and we are ready.

9. Estimate for the sum of a completely multiplicative function

In this section we prove Lemma (8.6) for completely multiplicative functions. This will be done by a variant of the analytic techniques developed by HALÁSZ [5, 6]. We shall use the Dirichlet series

$$\text{(9.1)} \qquad D(s) = \sum_{n=1}^{\infty} h(n) n^{-s}$$

of h. We wish to express

$$H(x)=\sum_{n\leqq x} h(n) \tag{9.2}$$

through D. There are a number of such expressions; we choose one which gives only an approximate value, but is easy to handle.

LEMMA (9.3). *If h is a unimodular arithmetical function, then with the above notations we have*

$$H(x)=\frac{-1}{2\pi i\log x}\int_{\sigma-iT_1}^{\sigma+iT_2}\frac{D'(s)x^s}{s}\,ds+O\left(\frac{x}{\log x}\right), \tag{9.4}$$

where $\sigma=1+1/\log x$ *and* $T_1,\ T_2\geqq\log x\log\log x$.

PROOF. We start with the formula

$$\frac{1}{2\pi i}\int_{\sigma-iT_1}^{\sigma+iT_2}\frac{y^s}{s}\,ds=\delta(y)+O\left(y^\sigma\min\left(1,\ \frac{1}{T\log y}\right)\right), \tag{9.5}$$

where $T=\min(T_1,\ T_2)$ and

$$\delta(y)=\begin{cases}0 & \text{if } y<1,\\ 1 & \text{if } y\geqq 1.\end{cases}$$

This formula with $T_1=T_2$ is well-known and can be found in most books on analytic number theory, see e.g. DAVENPORT [1]. The case $T_1\neq T_2$ can be proved just in the same way.

As $D'(s)=-\sum_{n=1}^{\infty}h(n)(\log n)n^{-s}$, applying (9.5) we get

$$\frac{1}{2\pi i}\int_{\sigma-iT_1}^{\sigma+iT_2}\frac{D'(s)x^s}{s}\,ds=$$

$$=-\sum_{n=1}^{\infty}h(n)(\log n)\,\delta(x/n)+O\left(\sum_{n=1}^{\infty}(\log n)\frac{x^\sigma}{n^\sigma}\min\left(1,\ \frac{1}{T\left|\log\frac{x}{n}\right|}\right)\right)=$$

$$=-\sum_{u\leqq x}h(n)\log n+O\left(\frac{x\log x\log(T\log x)}{T}\right)=-\sum_{n\leqq x}h(n)\log n+O(x),$$

supposing $T\geqq\log x\log\log x$.

The main term is equal to

$$\cdot\log x\sum_{n\leqq x}h(n)+O\Big(\sum_{n\leqq x}\log(x/n)\Big)=-\log x\sum_{n\leqq x}h(n)+O(x);$$

collecting our estimates we obtain (9.4).

This lemma will be applied to the function h of Lemma (8.6) and also for $h_1(n)=n^{ib}h(n)$. We use the notations (9.1) and (9.2), and we use $D_1(s)$ and $H_1(s)$ to denote the quantities got by replacing h by h_1.

We want to express

$$A = H_1(x) - \frac{x^{ib}}{1+ib} H(x). \tag{9.6}$$

We use the formula (9.4); for h_1 it yields

$$H_1(x) = \frac{-1}{2\pi i \log x} \int_{\sigma - iT_3}^{\sigma + iT_4} \frac{D_1'(s) x^s}{s} ds + O(x/\log x). \tag{9.7}$$

D and D_1 are connected by the equality

$$D_1(s) = \sum_{n=1}^{\infty} \frac{h(n) n^{ib}}{n^s} = \sum_{n=1}^{\infty} \frac{h(n)}{n^{s-ib}} = D(s-ib), \tag{9.8}$$

hence the integral in (9.7) can be written as

$$\int_{\sigma - iT_3}^{\sigma + iT_4} \frac{D'(s-ib) x^s}{s} ds = \int_{\sigma - iT_3 - ib}^{\sigma + iT_4 - ib} \frac{D'(s) x^{s+ib}}{s+ib} ds. \tag{9.9}$$

In order that the paths of integration should coincide we choose

$$T_1 = T_2 = T = |b| + \log^2 x, \quad T_3 = T - b, \quad T_4 = T + b, \tag{9.10}$$

thus we get

$$A = \frac{1}{2\pi i \log x} \int_{\sigma - iT}^{\sigma + iT} D'(s) \left(\frac{x^s}{s} \frac{x^{ib}}{1+ib} - \frac{x^{s+ib}}{s+ib} \right) ds + O\left(\frac{x}{\log x} \right).$$

The above integral is equal to

$$B = \int_{\sigma - iT}^{\sigma + iT} D'(s) \frac{x^{s+ib}(1-s) ib}{s(1+ib)(s+ib)} ds, \tag{9.11}$$

and to verify (8.6) it is sufficient to prove

$$|B| \ll x e^{L/2} \log^{1/2} x \log^2 T. \tag{9.12}$$

We shall show

$$\int_{\sigma + ik}^{\sigma + ik + i} \left| D'(s) \frac{s-1}{s} \right| |ds| \ll e^{L/2} \log^{1/2} x \log(|k| + \log x) \tag{9.13}$$

for every k. Since

$$|x^{s+ib}| = x^\sigma = ex, \quad \left| \frac{ib}{1+ib} \right| \leqq 1,$$

with $K = [T] + 2$ (9.13) implies

$$|B| \ll x \sum_{k=-K}^{K} \max_{k \leqq t \leqq k+1} \frac{1}{|\sigma + it + ib|} \int_{\sigma + ik}^{\sigma + ik + i} \left| D'(s) \frac{s-1}{s} \right| |ds| \ll$$

$$\ll x e^{L/2} \log^{1/2} x \log T \sum_{k=-K}^{K} \max_{k \leqq t \leqq k+1} \frac{1}{|\sigma + it + ib|} \ll x e^{L/2} \log^{1/2} x \log^2 T$$

as wanted. Now we have to prove (9.13).

Due to complete multiplicativity, we have

$$D(s) = \prod_p (1-h(p)p^{-s})^{-1};$$

taking the logarithm we get

(9.14) $$\log D(s) = \sum_{p,k} \frac{h(p)}{p^{ks}} = \sum_{p \leqq x} \frac{h(p)p^{-it}}{p} + O(1)$$

if

(9.15) $$s = \sigma + it, \quad \sigma = 1 + 1/\log x,$$

since

$$\sum_{k \geqq 2} p^{-k} = O(1), \quad \sum_{p>x} p^{-\sigma} = O(1), \quad \sum_{p \leqq x} (p^{-1} - p^{-\sigma}) = O(1).$$

Applying (9.14) for the identically one function and subtracting we obtain

$$\log \frac{D(s)}{\zeta(s)} = \sum_{p \leqq x} \frac{(h(p)-1)p^{-it}}{p} + O(1);$$

hence

$$\log \left| \frac{D(s)}{\zeta(s)} \right| \leqq \sum_{p \leqq x} \frac{|h(p)-1|}{p} + O(1),$$

that is,

(9.16) $$|D(s)| \ll e^L |\zeta(s)|.$$

Now we turn to (9.13). We begin with Schwarz's inequality:

(9.17) $$\int_I \left| D'(s) \frac{s-1}{s} \right| dt \leqq \left(\int_I \left| \frac{D'}{D}(s) \right|^2 dt \int_I \left| D(s) \frac{s-1}{s} \right|^2 dt \right)^{1/2}$$

where we use the notations of (9.15) and also $I = [\sigma + ik, \sigma + ik + i]$.

For any unimodular completely multiplicative function and $a \in (0, 1]$ we have

(9.18) $$\int_I \left| \frac{D'}{D}(s) \right|^2 dt \ll \frac{1}{\sigma - 1} = \log x$$

and

(9.19) $$\int_I |D(s)|^{1+a} dt \ll \int_{(\sigma)} \frac{|D(s+ik)|^{1+a}}{|s|^2} dt \leqq$$

$$\leqq \int_{(\sigma)} \frac{|\zeta(s)|^{1+a}}{|s|^2} ds \ll a^{-1}(\sigma-1)^a = a^{-1} \log^a x$$

(cf. HALÁSZ [5, pp. 337—338]). In (9.19) the first inequality is obvious, the third easily follows from the well-known estimates for the zeta function and the second is due to Halász. In his paper it is worked out only for $a = 1/2$, but it can be done similarly for every $a > 0$.

For the second term of (9.17) we have

(9.20) $$\int_I \left| D(s) \frac{s-1}{s} \right|^2 dt \leqq \int_I |D(s)|^{1+a} dt \cdot \max_{s \in I} |D(s)|^{1-a} \left| \frac{s-1}{s} \right|^2.$$

To estimate the second factor, we apply (9.16):

$$(9.21)\ |D(s)|^{1-a}\left|\frac{s-1}{s}\right|^2 \ll e^{L(1-a)}\left|\zeta(s)\frac{s-1}{s}\right|^{1-a}\left|\frac{s-1}{s}\right|^{1+a} \ll e^{L(1-a)}\log(|k|+2).$$

Combining (9.19), (9.20) and (9.21) with $a=1/\log\log x$ we get

$$(9.22)\qquad \int_I \left|D(s)\frac{s-1}{s}\right|^2 dt \ll (\log(|k|+2))e^L a^{-1}(e^{-L}\log x)^a \ll$$

$$\ll e^L \log(|k|+2)\log\log x \ll e^L \log^2(|k|+\log x).$$

(9.17), (9.18) and (9.22) just yield (9.13) and thus we have completed the proof of Lemma (8.6) assuming h is completely multiplicative.

10. The case h is not completely multiplicative

In order to prove Lemma (8.6) for all multiplicative h we represent h as a convolution $h=h_1 * w$, i.e.

$$(10.1)\qquad h(n) = \sum_{d|n} h_1(n/d)\,w(d),$$

where h_1 is the completely multiplicative function defined by $h_1(p)=h(p)$ for all primes p. w is easily seen to be multiplicative and

$$(10.2)\qquad w(p^k) = h(p^k) - h(p)h(p^{k-1}) \quad (k \geqq 1).$$

This yields

$$|w(p^k)| \begin{cases} = 0 & (k=1), \\ \leqq 2 & (k \geqq 2); \end{cases}$$

hence for $s>1/2$

$$(10.3)\qquad \sum_{n=1}^{\infty} |w(n)|\,n^{-s} = \prod_p \sum_{j=0}^{\infty} |w(p^j)|\,p^{-js} \leqq \prod_p (1+2p^{-2s}+2p^{-3s}+\dots) < \infty.$$

We have to estimate the sum

$$(10.4)\qquad H(x,b) = \sum_{n\leqq x} h(n)\left(n^{ib} - \frac{x^{ib}}{1+ib}\right).$$

Using (10.1) we get

$$(10.5)\qquad H(x,b) = \sum_{d\leqq x} w(d)\,d^{ib} H_1(x/d, b),$$

where $H_1(y,b)$ is got by (10.4) via replacing h and x by h_1 and y, respectively.

Putting $A = xe^{L/2}\log^{-1/2}x\log^2(|b|+\log x)$, for $d\leqq\sqrt{x}$ we have $H_1(x/d, b) \ll \ll A/d \leqq Ad^{-2/3}$ by the already proved part of Lemma (8.6), and for $d>\sqrt{x}$

$$|H_1(x/d, b)| \leqq 2x/d \ll Ad^{-2/3}$$

obviously. For the sum in (10.9) this gives

$$H(x, b) \ll A \sum_{d=1}^{\infty} |w(d)|\, d^{-2/3} \ll A$$

according to (10.3) and the proof is complete.

11. Deduction of Theorems A, B and D

Here we show how these theorems follow from ours. I note that they follow similarly from Halász's Theorem F and in a more sophisticated manner they can be deduced from Theorem E. We shall use the Reformulation, given at the end of the first section.

Deduction of Theorem A. Suppose $Q_1(x) \nrightarrow \infty$. This means that arbitrarily large values of x can be found for which there is a λ such that

$$\sum_{p \leqq x} \frac{\min\left(1, (f(p) - \lambda \log p)^2\right)}{p} \leqq c, \quad |\lambda| \leqq c. \tag{11.1}$$

For a given x let Λ_x denote the set of λ's satisfying (11.1). Λ_x is evidently compact and $\Lambda_{x+1} \subset \Lambda_x$, hence $\Lambda = \bigcap \Lambda_x \neq \emptyset$ by Cantor's theorem. Let λ be any element of Λ. With this λ (11.1) holds for every x, which is equivalent to (1.3).

Deduction of Theorem B. Suppose that there are sequences $h_n \to 0$ and $x_n \to \infty$ such that $Q_{h_n}(x_n) > c_1 > 0$. Our theorem yields the existence of numbers λ_n such that

$$|\lambda_n| \leqq c_2 h_n, \tag{11.2}$$

$$\sum_{p \leqq x_n} \frac{1}{p} \min\left(h_n^2, (f(p) - \lambda_n \log p)^2\right) \leqq c_2 h_n^2, \tag{11.3}$$

with a constant c_2 depending on c_1. Now choose an x and a positive ε. If $p \leqq x$ and $|f(p)| > \varepsilon$, then $|f(p) - \lambda_n \log p| \geqq h_n$ for large n by (11.2), thus (11.3) implies

$$\sum_{p \leqq x, |f(p)| > \varepsilon} 1/p \leqq c_2.$$

This being valid for all x and ε, we get (1.4).

Deduction of Theorem D. Let $|\{n: n \leqq x,\ f(n) = a\}| = qx$. Applying the Reformulation with an arbitrary positive h, we obtain a λ_h such that $|\lambda_h| \leqq ch/q$ and

$$\sum_{p \leqq x} \frac{1}{p} \min\left(h^2, f(p) - \lambda_h \log p)^2\right) \leqq ch^2 q^{-2}. \tag{11.4}$$

Choosing h so small that $h(1 + cq^{-1} \log p) \leqq |f(p)|$ for all $p \leqq x$, $f(p) \neq 0$, the minimum in (11.4) will be h^2 for all such primes. This means

$$\sum_{p \leqq x, f(p) \neq 0} 1/p \leqq cq^{-2},$$

which is a reformulation of (1.6).

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SOME REMARKS ON THE COMMUTATIVITY OF RINGS

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JOHNSON, OUTCALT and YAQUB [6] showed that a ring R with unity satisfying $(xy)^2=x^2y^2$ for all $x, y\in R$ is necessarily commutative. In this direction we prove the following theorem.

THEOREM 1. *Let R be a semiprime ring (not necessarily having a unity) satisfying $(xy)^2-x^2y^2\in Z(R)$ for all $x, y\in R$, then R is commutative.*

In [1] BELL has proved that if a ring R satisfies a polynomial identity and the identity $[x^n, y]=[x, y^n]$ for all $x, y\in R$ and a fixed integer $n>1$, then R is commutative. Recently HARMANCI [2] has proved that if a ring R with unity satisfies the identities (i) $[x^n, y]=[x, y^n]$ (ii) $[x^{n+1}, y]=[x, y^{n+1}]$ for all $x, y\in R$ and a fixed integer $n>1$, then R is commutative. In this direction we prove the following two theorems.

THEOREM 2. *Let R be a semiprime ring (not necessarily having a unity) satisfying $[x^2, y]-[x, y^2]\in Z(R)$ for all $x, y\in R$, then R is commutative.*

THEOREM 3. *Let R be a semiprime ring with unity satisfying*

(i) $[x^n, y]-[x, y^n]\in Z(R)$ (ii) $[x^{n+1}, y]-[x, y^{n+1}]\in Z(R)$

for all $x, y\in R$ and a fixed integer $n>1$, then R is commutative.

Throughout this paper R is an associative ring, $Z(R)$ denotes the centre of R and $[a, b]$ denotes the commutator $ab-ba$.

We start with the following Lemma which will be used frequently in the subsequent study.

LEMMA 1. *Let R be a prime ring and $x\neq 0$ and y be elements of R. If x and $xy\in Z(R)$, then $y\in Z(R)$.*

PROOF. Let x and $xy\in Z(R)$. Then $xyz=zxy=xzy$ for all $z\in R$. From this we have $Rx(yz-zy)=0$. Since $x\in Z(R)$, we get $xR(yz-zy)=0$. Since R is a prime ring and $x\neq 0$, we get $yz-zy=0$ for all $z\in R$. Thus $y\in Z(R)$.

LEMMA 2. *Let R be a prime ring (not necessarily having a unity) satisfying $(xy)^2-x^2y^2\in Z(R)$ for all $x, y\in R$, then R is commutative.*

PROOF. First we will show that $Z(R)\neq 0$. Let us suppose that $Z(R)=0$ then for all $x, y\in R$

(1) $$(xy)^2 = x^2y^2.$$

Now replacing x by $x+y$ in (1) we get

(2) $$y(xy-yx)y=0.$$

If $y(xy-yx)\neq 0$ then $y(xy-yx)$ is a non zero nilpotent element of index 2. By (1) we have $(y(xy-yx)r)^2=(y(xy-yx))^2r^2=0$ for all $r\in R$. Hence $y(xy-yx)R$ is a right ideal in which the square of every element is zero. By Lemma 1.1 of [4] R will have a non zero nilpotent ideal which contradicts the hypothesis. Hence

(3) $$y(xy-yx)=0.$$

Now replacing x by r in (3) we get

(4) $$y(ry-yr)=0.$$

Again replacing x by rx in (3) we get

(5) $$y(rxy-yrx)=0.$$

Using (4) we get for all $x, y, r\in R$.

(6) $$yr(xy-yx)=0.$$

Since R is a prime ring, we obtain $y=0$ or $xy=yx$ for all $x\in R$. In either case $y=0$ which is a contradiction. Hence $Z(R)\neq 0$.

Let $0\neq r\in Z(R)$. Let for all $x, y\in R$

(7) $$(xy)^2-x^2y^2\in Z(R).$$

Replacing x by $x+r$ in (7) we get,

(8) $$r(yx-xy)y\in Z(R).$$

Since R is a prime and $0\neq r\in Z(R)$, by Lemma 1 we have

(9) $$(yx-xy)y\in Z(R).$$

Replacing x by xy in (9) we get

(10) $$(yx-xy)y^2\in Z(R).$$

Since R is prime and using (9) and (10), we get by Lemma 1 that $y\in Z(R)$ unless $(yx-xy)y=0$. If $(yx-xy)y=0$ then we can use the argument given in (3) onward to prove that $y\in Z(R)$ for all $y\in R$. In either case $y\in Z(R)$. Hence R is commutative.

Proof of Theorem 1. Let R be a semiprime ring then it is isomorphic to the subdirect sum of prime rings R_α each of which is a homomorphic image of R and satisfies the identity of the hypothesis. By Lemma 2 R_α is commutative and hence R is commutative.

Lemma 3. *Let R be a prime ring (not necessarily having a unity) satisfying* (i) $[x^2, y]-[x, y^2]\in Z(R)$ *for all* $x, y\in R$, *then R is commutative.*

Proof. First we will show that $Z(R)\neq 0$. Let us suppose that $Z(R)=0$, then $[x^2, y]=[x, y^2]$ for all $x, y\in R$. Replacing x by $x+y$ we get $y^2x=xy^2$ for all $x, y\in R$.

This shows that $y^2 \in Z(R)=0$ for all $y \in R$. Then $(x+y)^2=0$ will imply that $xy+yx=0$. This will show that $xyx=0$ for all $y \in R$. Since R is prime, we have $x=0$ which is a contradiction. Hence $Z(R) \neq 0$.

Let R be a ring of characteristic $\neq 2$. Let $0 \neq r \in Z(R)$. Replace x by $x+r$ in the condition (i) of the Lemma and then subtract (i) to get

$$\{[(x+r)^2, y]-[(x+r), y^2]\}-\{[x^2, y]-[x, y^2]\} \in Z(R). \tag{1}$$

Using $r \in Z(R)$, after simplification we get

$$2r[x, y] \in Z(R). \tag{2}$$

Since R is of characteristic $\neq 2$, it can be seen that for all $x, y \in R$

$$r[x, y] \in Z(R). \tag{3}$$

Since $0 \neq r \in Z(R)$ and R is a prime ring, by Lemma 1 we get

$$[x, y] \in Z(R) \tag{4}$$

for all $x, y \in R$. Replace x by xy in (4) to get

$$[x, y]y \in Z(R) \tag{5}$$

for all $x, y \in R$. Using (4) and (5) and again by Lemma 1 we get $y \in Z(R)$ unless $[x, y]=0$. If $[x, y]=0$ then $y \in Z(R)$. In either case we get $y \in Z(R)$ for all $y \in R$. Hence R is commutative.

If the characteristic of R is 2, then replace x by $x+y$ in the condition (i) of the Lemma and then subtract (i) to get for all $x, y \in R$

$$xy^2-y^2x \in Z(R). \tag{6}$$

Replacing x by xy^2 we get

$$(xy^2-y^2x)y^2 \in Z(R) \tag{7}$$

for all $x, y \in R$. Using (6) and (7) and by Lemma 1, we get $y^2 \in Z(R)$ unless $xy^2-y^2x=0$. In either case $y^2 \in Z(R)$ for all $y \in R$. Now $(x+y)^2 \in Z(R)$ for all $x, y \in R$. From this we get $xy+yx \in Z(R)$ for all $x, y \in R$. Replace x by xy to get $(xy+yx)y \in Z(R)$. By Lemma 1 we have $y \in Z(R)$ unless $xy+yx=0$. In either case $y \in Z(R)$ for all $y \in R$. Hence R is commutative.

PROOF OF THEOREM 2. It is obvious by Lemma 3 and by the argument given in the proof of Theorem 1.

REMARK 1. The assumption that R is semiprime in Theorems 1 and 2 is not superfluous. The ring R of strictly upper triangular 3×3 matrices over some field, which is not semiprime, satisfies the identities in the hypothesis of Theorems 1 and 2 but R is not commutative.

LEMMA 4. *Let R be a prime ring with unity satisfying*

(i) $[x^n, y]-[x, y^n] \in Z(R)$ (ii) $[x^{n+1}, y]-[x, y^{n+1}] \in Z(R)$

for all $x, y \in R$ and a fixed integer $n>1$, then R is commutative.

PROOF. We use the technique given in the proof of Theorem B of [2]. Replace x by $1+x$ in the condition (i) of the hypothesis of the Lemma and then substract (i) we obtain

$$(1)\qquad n[x, y]+\sum_{k=2}^{n-1}\binom{n}{k}[x^k, y]\in Z(R).$$

Replace x by $1+x$ in the condition (ii) of the hypothesis of the Lemma and then subtract (ii) to get

$$(2)\qquad (n+1)[x, y]+\sum_{j=2}^{n}\binom{n+1}{j}[x^j, y]\in Z(R).$$

Subtract (2) from (1) to get

$$(3)\qquad \left[\sum_{k=2}^{n-1}\binom{n}{k}x^k-\sum_{j=2}^{n}\binom{n+1}{j}x^j, y\right]-[x, y] = [x^2p(x)-x, y]\in Z(R)$$

for all $x, y\in R$. Now replacing y by yx in (3) we get

$$(4)\qquad [x^2p(x)-x, y]x\in Z(R)$$

for all $x, y\in R$. Since R is prime and using (3) and (4), we get by Lemma 1 that $x\in Z(R)$ unless $[x^2p(x)-x, y]=0$. In all situations, it can be seen that $[x^2p(x)-x, y]=0$ for all $x, y\in R$. By the result of HERSTEIN [3], R is commutative.

PROOF OF THEOREM 3. Now it is obvious.

REMARK 2. The assumption that R is semiprime in Theorem 3 is not superfluous. However, it is not possible to replace semiprime ring by primary ring in Theorem 3. LUH [7, Example 2] has given an example of a primary ring with unity. One can verify that it satisfies (i) $[x^2, y]-[x, y^2]\in Z(R)$ and (ii) $[x^3, y]-[x, y^3]\in Z(R)$ for all $x, y\in R$, but R is not commutative.

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NUMBERS CONTRAVENING A CONDITION IN DENSITY MODULO 1

By

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Let $(q_n)_{n\in\mathbf{N}^*}$ be a sequence of real positive numbers, for which there exists $\lambda>1$ such that, for every n, $q_{n+1}/q_n\geqq\lambda$. It was proved by P. ERDŐS and S. J. TAYLOR [1] that the set of the real numbers x belonging to any interval $[a, b]$ $(a<b)$ such that the sequence $(q_n x)_{n\in\mathbf{N}^*}$ is not equidistributed mod 1, has Hausdorff dimension 1 (although it has measure zero).

P. Erdős has asked if there exists a real number $x\in[a, b]$ such that the sequence $(q_n x)_{n\in\mathbf{N}^*}$ is not *everywhere dense* mod 1. The answer is obviously affirmative if $\lambda>2$. We prove that it is so for any $\lambda>1$, and moreover that the set of $x\in[a, b]$ such that the sequence $(q_n x)_{n\in\mathbf{N}^*}$ is not everywhere dense mod 1, has Hausdorff dimension 1. Indeed we obtain a more precise result which, for instance, can be also applied to the sequence $(x^n)_{n\in\mathbf{N}^*}$. It was already known that for every $v\in\mathbf{R}$, and any interval $[a, b]$, the set of $x\in[a, b]$ such that the sequence $(vx^n)_{n\in\mathbf{N}^*}$ is not equidistributed mod 1, has Hausdorff dimension 1 ([2]).

THEOREM 1. *Let* $[a, b]$ *be an interval of* $\mathbf{R}$ $(a<b)$, *and let* $(\varphi_n)_{n\in\mathbf{N}^*}$ *be a sequence of continuous functions from* $[a, b]$ *to* $\mathbf{R}$. *Suppose that the functions* φ_n *are monotonic and differentiable on* (a, b), *with non-vanishing derivatives. Suppose also that there exist real numbers* λ, μ, $1<\lambda\leqq\mu$, *such that for every* $\xi\in(a, b)$ *and* $n\in\mathbf{N}^*$

$$\lambda \leqq |\varphi'_{n+1}(\xi)|/|\varphi'_n(\xi)| \leqq \mu. \tag{1}$$

Then there exists $x\in[a, b]$ *such that the sequence* $\big(\varphi_k(x)\big)_{n\in\mathbf{N}^*}$ *is not everywhere dense* mod 1.

If moreover there exists a real number $\tau\geqq 0$ *such that for every pair* $(\xi, \xi')\in\in(a, b)\times(a, b)$ *and every* $n\in\mathbf{N}^*$

$$\big(\mathrm{Log}\,|\varphi'_n(\xi)|-\mathrm{Log}\,|\varphi'_n(\xi')|\big) \leqq \tau|\varphi_n(\xi)-\varphi_n(\xi')| \tag{2}$$

then the set of $x\in[a, b]$ *such that the sequence* $\big(\varphi_n(x)\big)_{n\in\mathbf{N}^*}$ *is not everywhere dense* mod 1, *has Hausdorff dimension* 1.

COROLLARY 1. *Let* $(q_n)_{n\in\mathbf{N}^*}$ *be a sequence of real positive numbers such that there exists* $\lambda>1$ *with* $q_{n+1}/q_n\geqq\lambda$ *for all* n, *and let* $[a, b]$ *be an interval in* $\mathbf{R}$.

Then the set of real numbers $x\in[a, b]$ *such that the sequence* $(q_n x)$ *is not everywhere dense* mod 1, *has Hausdorff dimension* 1.

COROLLARY 2. *Let* v *be a real number. The set of real numbers* x *belonging to any interval* $[a, b]$ *such that the sequence* $(vx^n)_{n\in\mathbf{N}^*}$ *is not everywhere dense* mod 1, *has Hausdorff dimension* 1.

Note that the condition $q_{n+1}/q_n \geqq \lambda$ of Corollary 1 does in place of the condition (1) of Theorem 1, because we can if necessary refine the sequence (q_n) so that $\lambda \leqq q_{n+1}/q_n \leqq \lambda^2$ for all n.

PROOF OF THEOREM 1. We shall prove that there exists a real number $\varepsilon>0$ and a real number $x\in[a, b]$ such that $\|\varphi_n(x)\| \geqq \varepsilon$ for every $n\in\mathbf{N}^*$ sufficiently large (the symbol $\|\cdot\|$ denotes for any $z\in\mathbf{R}$, $\|z\|=\min_{k\in\mathbf{Z}} |z-k|$). Indeed, for such an ε and for every sequence $(H_n)_{n\in\mathbf{N}^*}$ of closed intervals of $\mathbf{R}/\mathbf{Z}$ with radius ε, there exists $x\in[a, b]$ such that $\varphi_n(x)\notin H_n \pmod 1$ for n sufficiently large. To show this, one replaces φ_n by $\varphi_n+\alpha_n$ for some $\alpha_n\in\mathbf{R}$.

Let n_0 be a positive integer such that

$$\lambda^{n_0} \geqq 2n_0+1 \tag{3}$$

and put $\varepsilon=\mu^{1-2n_0}/2$. As for every n, there exists $\xi\in(a, b)$ such that

$$\frac{\varphi_n(b)-\varphi_n(a)}{\varphi_1(b)-\varphi_1(a)} = \frac{\varphi_n'(\xi)}{\varphi_1'(\xi)}$$

we can suppose by removing at most a finite number of terms of the sequence (φ_n), that $|\varphi_{n_0}(b)-\varphi_{n_0}(a)| \geqq 2$. We assume that the functions φ_n are increasing (putting $-\varphi_n$ for φ_n if necessary).

Assuming this, we shall now show that there is an $x\in[a, b]$ such that $\|\varphi_n(x)\| \geqq \varepsilon$ for all n.

For each integer $n\in\mathbf{N}^*$, let F_n be the set of $x\in[a, b]$ such that $\|\varphi_n(x)\| \geqq \varepsilon$, and for any $N\in\mathbf{N}^*$ put $G_N = \bigcap_{1\leqq n\leqq N} F_n$ ($G_0=[a, b]$). We construct inductively a sequence $(K_s)_{s\in\mathbf{N}}$ of integers such that for all $s\geqq 0$

$$[K_s, K_s+1] \subset \varphi_{(s+1)n_0}([a, b]), \quad \varphi_{(s+1)n_0}^{-1}([K_s, K_s+1]) \subset G_{sn_0}$$

and

$$\varphi_{(s+1)n_0}^{-1}([K_s, K_s+1]) \subset \varphi_{sn_0}^{-1}([K_{s-1}, K_{s-1}+1]) \quad \text{if} \quad s>0$$

(the symbol φ^{-1} denotes the inverse of the bijection φ).

The intersection $\bigcap_s \varphi_{(s+1)n_0}^{-1}([K_s, K_{s+1}])$ will then be non-empty, and hence there will exist $x\in\bigcap_N G_N$, that is an $x\in[a, b]$ such that for every n, $\|\varphi_n(x)\| \geqq \varepsilon$.

We can find a $K_0\in\mathbf{Z}$ such that $[K_0, K_0+1]\subset\varphi_{n_0}([a, b])$. Let s be a positive integer, suppose we have found a $K_{s-1}\in\mathbf{Z}$ such that $[K_{s-1}, K_{s-1}+1]\subset\varphi_{sn_0}([a, b])$ and $\varphi_{sn_0}^{-1}([K_{s-1}, K_{s-1}+1])\subset G_{(s-1)n_0}$. Denote by I_{s-1} the interval $\varphi_{sn_0}^{-1}([K_{s-1}+\varepsilon, K_{s-1}+1-\varepsilon])$. We have $I_{s-1}\subset G_{(s-1)n_0}\cap F_{sn_0}$. On the other hand, we have for n such that $(s-1)n_0<n<sn_0$ and for some $\xi\in(a, b)$

$$\frac{|\varphi_n(I_{s-1})|}{|\varphi_{sn_0}(I_{s-1})|} = \frac{\varphi_n'(\xi)}{\varphi_{sn_0}'(\xi)}.$$

(The symbol $|\cdot|$ denotes the length of an interval.) Hence

$$|\varphi_n(I_{s-1})| < 1-2\varepsilon$$

and so, $\varphi_n(I_{s-1}-I_{s-1}\cap F_n)$, the set of real numbers $z\in\varphi_n(I_{s-1})$ such that $\|z\|<\varepsilon$, is an interval of length at most 2ε (and if it is of length 2ε, it is open). We have

$$\frac{|\varphi_{(s+1)n_0}(I_{s-1}-I_{s-1}\cap F_n)|}{|\varphi_n(I_{s-1}-I_{s-1}\cap F_n)|}=\frac{\varphi'_{(s+1)n_0}(\xi)}{\varphi'_n(\xi)}\leqq\mu^{2n_0-1}\quad(\xi\in(a,b)).$$

the interval $\varphi_{(s+1)n_0}(I_{s-1}-I_{s-1}\cap F_n)$ has length at most 1 (since $\varepsilon=\mu^{1-2n_0}/2$), being open if it has length 1. Thus there are at most two integers $K\in\mathbf{Z}$ such that $[K,K+1]\cap\varphi_{(s+1)n_0}(I_{s-1}-I_{s-1}\cap F_n)\neq\emptyset$. Since $I_{s-1}-I_{s-1}\cap G_{sn_0}=I_{s-1}-I_{s-1}\cap\cap(\bigcup_{(s-1)n_0<n<sn_0}F_n)$, there exist in total at most $2(n_0-1)$ integers K such that $[K,K+1]\subset\varphi_{(s+1)n_0}(I_{s-1})$ and $[K,K+1]\not\subset\varphi_{(s+1)n_0}(G_{sn_0})$. Now, as

$$|\varphi_{(s+1)n_0}(I_{s-1})|/|\varphi_{sn_0}(I_{s-1})|=\varphi'_{(s+1)n_0}(\xi)/\varphi'_{sn_0}(\xi)\geqq\lambda^{n_0}\quad(\text{for some }\xi\in(a,b)),$$

$\varphi_{(s+1)n_0}(I_{s-1})$ is an in terval of length at least $\lambda^{n_0}(1-2\varepsilon)\geqq\lambda^{n_0}-1$. Thus the number of intervals $[K,K+1]$ ($K\in\mathbf{Z}$) which are included in $\varphi_{(s+1)n_0}(I_{s-1})$, is (strictly) greater than $\lambda^{n_0}-3$. Now, from (3), $\lambda^{n_0}-3\geqq2(n_0-1)$. Hence, we can find an interval $[K_s,K_s+1]$ ($K_s\in\mathbf{Z}$) which is included in $\varphi_{(s+1)n_0}(I_{s-1}\cap G_{sn_0})$. So the interval $\varphi^{-1}_{(s+1)n_0}([K_s,K_s+1])$ is included in $\varphi^{-1}_{sn_0}([K_{s-1},K_{s-1}+1])\cap G_{sn_0}$. This proves the first part.

To demonstrate the assertion concerning Hausdorff dimension, we shall use a result similar to that of [2] (p. 165, IV, th. I‴). We give a very simple proof.

LEMMA 1. *Let a, b be real numbers, $a<b$, and for any integer $s\in\mathbf{N}$ let $\mathcal{I}_s$ be a finite family of closed intervals in* **R**, *which are non-empty and mutually disjoint. Assume the following conditions:*

(4) $\mathcal{I}_0=\{[a,b]\}$.

(5) *For $s\in\mathbf{N}^*$, each interval belonging to $\mathcal{I}_s$ is included in an interval belonging to $\mathcal{I}_{s-1}$, and each interval of $\mathcal{I}_{s-1}$ contains at least two intervals of $\mathcal{I}_s$.*

(6) *There is a real number δ, $0<\delta<1$, such that for any $s\in\mathbf{N}^*$, for each interval $I\in\mathcal{I}_{s-1}$, and for any pair (J,J') of distinct intervals of $\mathcal{I}_s$, which are included in I*

$$d(J,J')\geqq\delta|I|\quad\left(d(J,J')=\inf_{\xi\in J,\xi'\in J'}|\xi-\xi'|\right).$$

Let $\alpha\in(0,1)$. Suppose that for any $s\in\mathbf{N}^*$ and for each interval $I\in\mathcal{I}_{s-1}$

(7)
$$|I|^\alpha\leqq\sum_{J\in\mathcal{I}_s,J\subset I}|J|^\alpha.$$

Then the set $C=\bigcap_{s\in\mathbf{N}}(\bigcup_{J\in\mathcal{I}_s}J)$ has Hausdorff dimension at least α.

PROOF. We prove that the measure of C in the dimension α is not zero. More precisely:

LEMMA 2. *Under the conditions of Lemma 1, if (Ω_i) is a family of open intervals covering C, then*

(8)
$$\sum|\Omega_i|^\alpha\geqq\delta^\alpha(b-a)^\alpha.$$

PROOF. Put $C_s=\bigcup_{J\in\mathcal{I}_s}J$. Any open covering of C is a covering of some C_s (by compactness). We proceed to prove (8) for any covering of C_s by a family of open intervals, by induction on s. The result is obvious if $s=0$, so suppose $s>0$

and let (Ω_i) be a covering of C_s by open intervals. If there is an i such that $|\Omega_i| \geqq \delta(b-a)$, the result is obvious. If not, then each Ω_i meets at most one interval of $\mathcal{I}_1$, and so

$$\sum |\Omega_i|^\alpha \geqq \sum_{J\in\mathcal{I}_1} \Big(\sum_{\Omega_i \cap J \neq \emptyset} |\Omega_i|^\alpha \Big).$$

By the induction hypothesis, one can suppose that for each J of $\mathcal{I}_1$

$$\sum_{\Omega_i \cap J \neq \emptyset} |\Omega_i|^\alpha \geqq \delta^\alpha |J|^\alpha$$

therefore by the condition (7) $\sum |\Omega_i|^\alpha \geqq \delta^\alpha (b-a)^\alpha$. This proves Lemma 2 and hence Lemma 1.

We can now conclude the proof of Theorem 1. We resume the proof of the first part and we assume to begin with, that $\lambda^{n_0} \geqq 2n_0+2$. Put $\mathcal{I}'_0 = \varphi_{n_0}^{-1}([K_0, K_0+1])$ (for some $K_0 \in \mathbf{Z}$ such that $[K_0, K_0+1] \subset \varphi_{n_0}([a, b])$. For $s \in \mathbf{N}^*$, we define inductively $\mathcal{I}'_s$, to be the family of intervals of the form $\varphi_{(s+1)n_0}^{-1}([K, K+1])$ $(K \in \mathbf{Z})$, which are included in G_{sn_0} and in an interval of $\mathcal{I}'_{s-1}$. Now let for any $s \in \mathbf{N}$, $\mathcal{I}_s$ be the family of intervals of the form $\varphi_{(s+1)n_0}^{-1}([K+\varepsilon, K+1-\varepsilon])$ for any $K \in \mathbf{Z}$ such that $\varphi_{(s+1)n_0}^{-1}([K, K+1]) \in \mathcal{I}'_s$. It is clear that each interval of $\mathcal{I}_s$ is included in an interval of $\mathcal{I}_{s-1}$. Since $\lambda_{n_0} \geqq 2n_0+2$, the first part of the proof shows that each interval of $\mathcal{I}_{s-1}$ contains at least two intervals of $\mathcal{I}_s$. For each x belonging to the set $C = \bigcap_s \big(\bigcup_{J \in \mathcal{I}_s} J \big)$, the sequence $(\varphi_n(x))_{n \in \mathbf{N}}$ is not everywhere dense mod 1.

Consider the condition (6). Note that for every interval $I \in \mathcal{I}_{s-1}$ and any interval $L \subset \varphi_{(s+1)n_0}(I)$

$$|\varphi_{(s+1)n_0}^{-1}(L)| = |L| / \varphi'_{(s+1)n_0}(\xi)$$

for some interior point ξ of I. Now $|I| = (1-2\varepsilon)/\varphi'_{sn_0}(\xi')$ for some interior point ξ' of I. Since $|\varphi_{sn_0}(\xi) - \varphi_{sn_0}(\xi')| \leqq 1$, the conditions (1) and (2) imply that, if we put $e^\tau = c$

$$\lambda^{n_0} \varphi'_{sn_0}(\xi)/c \leqq \varphi_{(s+1)n_0}^{-1}(\xi') \leqq c\mu^{n_0} \varphi'_{sn_0}(\xi)$$

hence

$$|I|\,|L|/c\mu^{n_0} \leqq |\varphi'_{(s+1)n_0}(L)| \leqq c|I|\,|L|/(1-2\varepsilon)\lambda^{n_0}. \tag{9}$$

Let J and J' be distinct intervals of $\mathcal{I}_s$ included in I. Since

$$d\big(\varphi_{(s+1)n_0}(J),\ \varphi_{(s+1)n_0}(J')\big) \geqq 2\varepsilon,$$

it follows from (9) that $d(J, J') \geqq 2\varepsilon |I| / c\mu^{n_0}$ and so the condition (6) is satisfied.

We proceed to condition (7). Let $\alpha \in (0, 1)$. For each interval I of $\mathcal{I}_{s-1}$, the interval $\varphi_{(s+1)n_0}(I)$ is a union of intervals of the form $[K, K+1]$, $K \in \mathbf{Z}$, together with at most two intervals of length smaller than 1. We showed that there are at most $2(n_0-1)$ intervals $[K, K+1]$, $K \in \mathbf{Z}$, included in $\varphi_{(s+1)n_0}(I)$ which are not included in $\varphi_{(s+1)n_0}(G_{sn_0})$. Hence the set $I - \bigcup_{J \subset I, J \in \mathcal{I}_s} J$ is the union of a family of at most $2n_0$ intervals whose images under $\varphi_{(s+1)n_0}$ have length 1 at most, together with the intervals $\varphi_{(s+1)n_0}^{-1}([K, K+\varepsilon])$ and $\varphi_{(s+1)n_0}^{-1}([K+1-\varepsilon, K+1])$, for those $K \in \mathbf{Z}$ such that $[K, K+1] \subset \varphi_{(s+1)n_0}(I)$. Now for any integer K such that $[K, K+1] \subset \varphi_{(s+1)n_0}(I)$, we have $|\varphi_{(s+1)n_0}^{-1}([K, K+1])| = 1/\varphi'_{(s+1)n_0}(\xi_K)$ and $|\varphi_{(s+1)n_0}^{-1}([K, K+\varepsilon])| = \varepsilon/\varphi'_{(s+1)n_0}(\xi'_K)$

for some ξ_K, ξ'_K in $\varphi^{-1}_{(s+1)n_0}((K, K+1))$. Hence by (2): $\varphi'_{(s+1)n_0}(\xi_K) \leqq c\varphi'_{(s+1)n_0}(\xi'_K)$ thus

(10) $$|\varphi^{-1}_{(s+1)n_0}([K, K+\varepsilon])| \leqq c\varepsilon|\varphi^{-1}_{(s+1)n_0}([K, K+1])|$$

and the same holds for the interval $\varphi^{-1}_{(s+1)n_0}([K+1-\varepsilon, K+1])$. It follows from inequalities (9) and (10) that the (Lebesgue) measure of the set $I - \bigcup_{J\subset I, J\in\mathscr{I}_s} J$ satisfies

$$|I| - \sum_{J\subset I, J\in\mathscr{I}} |J| \leqq \big(2n_0/((1-2\varepsilon)\lambda^{n_0}) + 2\varepsilon\big)c|I|.$$

Thus, since $\varepsilon \leqq \lambda^{-n_0}/2$:

$$\sum_{J\in I, J\in\mathscr{I}_s} |J| \geqq \big(1-(2n_0+1)c/(\lambda^{n_0}-1)\big)|I|.$$

Now for each interval J of $\mathscr{I}_s$ included in I $|J| \leqq c|I|/\lambda^{n_0}$ so

$$\sum_{J\subset I, J\in\mathscr{I}_s} |J|^\alpha \geqq (\lambda^{n_0}/c)^{1-\alpha}\big(1-(2n_0+1)c/(\lambda^{n_0}-1)\big)|I|^\alpha.$$

But, for every $\alpha\in(0, 1)$ it is possible to choose n_0 sufficiently large so that

$$(\lambda^{n_0}/c)^{1-\alpha}\big(1-(2n_0+1)c/(\lambda^{n_0}-1)\big) \geqq 1.$$

The condition (7) is then satisfied, and thus the set C has Hausdorff dimension at least α. We conclude that the set of real numbers $x\in[a, b]$ such that the sequence $(\varphi_n(x))_{n\in\mathbf{N}^*}$ is not everywhere dense mod 1 (and, indeed, the set of $x\in[a, b]$ such that the sequence $(\varphi_n(x))_{n\in\mathbf{N}^*}$ does not have zero as a point of accumulation mod 1) has Hausdorff dimension 1.

Added in proof (April 8, 1980). Similar results concerning sequences $(q_n x)$ were obtained independently by A. D. POLLINGTON, *Illinois J. Math.*, **23** (1979), 511—515.

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COMPLEX CUBIC SPLINE INTERPOLATION

By

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1. Introduction

The convergence properties of complex cubic splines which interpolate to a given function at the nodal points have been studied in details in [1]. The object of the present paper is to investigate the convergence properties of periodic complex cubic splines interpolating at two points of each subarc. Since the number of interpolating points is more than one the deficiency of the spline increases, that is the order of differentiability at the nodal points decreases (see [2], p. 7). However, this deficiency is well compensated by the increase in smoothness at the points of interpolation (cf. [4]).

2. Existence and uniqueness

Let $\Delta=\{t_1, t_2, \dots, t_n\}$ be the set of distinct points of a rectifiable Jordan arc K, arranged in counter clock wise order. For $j=1, \dots, n$, K_j denotes the subarc of K from t_{j-1} to t_j with $t_n=t_0$ and $t_j-t_{j-1}=h_j$. The existence and uniqueness of cubic splines $q_\Delta(t)\in C^2(K)$ for the subdivision Δ of K, with the interpolating condition $q_\Delta(t_j)=f(t_j)$, $j=1, \dots, n$, where f is a given function has been shown in [1]. The corresponding error bound has also been obtained ([1], Theorem 1).

Considering $q_\Delta(t)\in C^1(K)$ with the interpolatory conditions

$$q_\Delta(\alpha_j)=f(\alpha_j), \quad \text{with} \quad \alpha_j=t_{j-1}+\frac{1}{3}h_j, \quad j=1, \dots, n, \tag{2.1}$$

and

$$q_\Delta(\beta_j)=f(\beta_j), \quad \text{with} \quad \beta_j=t_{j-1}+\frac{2}{3}h_j, \quad j=1, \dots, n, \tag{2.2}$$

we shall prove the following:

THEOREM 1. *If K is a sufficiently smooth Jordan curve, then there exists a unique complex cubic spline $q_\Delta(t)\in C^1(K)$ satisfying the interpolatory conditions* (2.1) *and* (2.2).

Writing $P(t)=(t-t_{j-1})(t-t_j)(t-\alpha_j)(t-\beta_j)$ we suppose that in the subarc K_j,

$$q_\Delta(t)=AP_{j-1}(t)-BP_j(t)+CP_j(t,\alpha)-DP_{j-1}(t,\beta) \tag{2.3}$$

where $P_i(t)$ $(i=j, j-1)$ is $P(t)$ without $(t-t_i)$ and $P_i(t,\alpha)$ (or $P_i(t,\beta)$) is $P_i(t)$

with β_j replaced by α_j (or α_j replaced by β_j) (cf. [3, p. 7]). Thus,

$$P_j(t,\alpha) = (t-t_{j-1})(t-\alpha_j)^2;$$

$$P'_{j-1}(t_j) = P'_j(t_{j-1}) = \frac{2}{9}h_j^2;\quad P'_{j-1}(t_{j-1}) = P'_j(t_j) = \frac{11}{9}h_j^2;$$

$$P'_{j-1}(t_j,\beta) = P'_j(t_{j-1},\alpha) = \frac{1}{9}h_j^2;\quad P'_j(t_j,\alpha) = P'_{j-1}(t_{j-1},\beta) = \frac{16}{9}h_j^2.$$

Using the interpolatory conditions (2.1) and (2.2), we have

$$2h_j^3 D = 27f(\alpha_j) \tag{2.4}$$

$$2h_j^3 C = 27f(\beta_j). \tag{2.5}$$

Setting $m_j = q'_\Delta(t_j)$, $j=1,\dots,n$ and using (2.4)—(2.5), we have from (2.3),

$$m_j = \frac{2}{9}Ah_j^2 - \frac{11}{9}Bh_j^2 + 24h_j^{-1}f(\beta_j) - \frac{3}{2}h_j^{-1}f(\alpha_j) \tag{2.6}$$

$$m_{j-1} = \frac{11}{9}Ah_j^2 - \frac{2}{9}Bh_j^2 + \frac{3}{2}h_j^{-1}f(\beta_j) - 24h_j^{-1}f(\alpha_j). \tag{2.7}$$

Hence using (2.4)—(2.7), we have another expression for $q_\Delta(t)$:

$$13h_j^3 q_\Delta(t) = m_j h_j[-2P_{j-1}(t)+11P_j(t)] + m_{j-1}h_j[11P_{j-1}(t)-2P_j(t)] + \tag{2.8}$$

$$+\frac{9}{2}f(\beta_j)[7P_{j-1}(t)-58P_j(t)+39P_j(t,\alpha)] + \frac{9}{2}f(\alpha_j)[58P_{j-1}(t)-7P_j(t)-39P_{j-1}(t,\beta)].$$

Since $q_\Delta(t)\in C^1(K)$, the quantities m_j are to be chosen in such a manner that $q_\Delta(t_j)$ obtained by approaching t_j on K_j and K_{j+1} are equal for $j=1,\dots,n$. Thus,

$$22m_jh_j - 4m_{j-1}h_j + 180f(\beta_j) - 63f(\alpha_j) = \tag{2.9}$$

$$= 4m_{j+1}h_{j+1} - 22m_jh_{j+1} - 63f(\beta_{j+1}) + 180f(\alpha_{j+1})$$

and, therefore,

$$\gamma_j m_{j-1} - \frac{11}{2}m_j + \delta_j m_{j+1} = F_j(\alpha,\beta), \tag{2.10}$$

say, where we set $\delta_j = h_{j+1}/(h_j+h_{j+1})$, $\gamma_j = 1-\delta_j$ and

$$4(h_j+h_{j+1})F_j(\alpha,\beta) = 63\big(f(\beta_{j+1})-f(\alpha_j)\big) - 180\big(f(\alpha_{j+1})-f(\beta_j)\big). \tag{2.11}$$

The existence and uniqueness of $q_\Delta(t)$ depends on the existence of a unique solution of the equations (2.10) in m_j's. This follows if the coefficient matrix of the equations has the diagonal dominant property. That is if,

$$\frac{11}{2}|h_j+h_{j+1}| > |h_j| + |h_{j+1}|. \tag{2.12}$$

This condition is equivalent to the assumption that for each j, t_j lies within the ellipse of eccentricity (2/11) and with foci as the dividing points t_{j-1}, t_{j+1} (cf. [1, p. 392]). In particular, if K is a smooth Jordan curve then the foregoing condition is easily seen to be satisfied for sufficiently small mesh norm

$$\|\Delta\| = \max_{1 \le j \le n} |h_j|. \tag{2.13}$$

This completes the proof of Theorem 1.

Throughout, we now assume that $\|\Delta\|$ is small enough so that (2.12) holds and A is a nonsingular matrix giving a unique solution of (2.10). Thus there exists a unique $q_\Delta(t) \in C^1(K)$ satisfying the interpolatory conditions (2.1) and (2.2).

It may be observed that for $F_j(\alpha, \beta)$ defined by (2.11), we have

$$\max_k |F_k(\alpha, \beta)| \le \max_k \big[|4(h_k+h_{k+1})|^{-1}(63|f(\beta_{k+1})-f(\alpha_{k+1})| + \tag{2.14}$$

$$+117|f(\beta_k)-f(\alpha_{k+1})|+63|f(\beta_k)-f(\alpha_k)|)\big] \le \max_k \left[\frac{243}{4}|h_k+h_{k+1}|^{-1}\right] w(\|\Delta\|; f)$$

where $w(\|\Delta\|; f)$ is the modulus of continuity of the function $f(t)$.

Interior to the arc K we may define the complex spline $S_\Delta(z)$ by the Cauchy integral

$$S_\Delta(z) = (1/2\pi i) \int_K (t-z)^{-1} q_\Delta(t)\, dt \tag{2.15}$$

with $S_\Delta(t)$, $t \in K$, as the limiting value for approach from within K (cf. [1, p. 393] and [5]),

$$S_\Delta(t) = \frac{1}{2} q_\Delta(t) + \frac{1}{2\pi i} \int_K (\theta - t)^{-1} q_\Delta(\theta)\, d\theta. \tag{2.16}$$

3. Convergence

Let us consider a sequence of meshes $\{\Delta_r\}$ such that $\|\Delta_r\| \to 0$ as $r \to \infty$. Suppose $\Delta_r = \{t_{r,1}, t_{r,2}, \dots, t_{r,n_r}\}$ and the inverse of the coefficient matrix corresponding to the equations (2.10) for Δ_r is $B_r = (b_{r;i,j})$. Then for B_r the row norm

$$\sum_k |b_{r;j,k}| \le \left\{\min_j \left(\frac{11}{2} - |\delta_{r,j}| - |\gamma_{r,j}|\right)\right\}^{-1} \tag{3.1}$$

(cf. [2, p. 21]). We also observe that the bound in (3.1) can be made not to exceed $2(1+\eta)/9$ for arbitrary $\eta > 0$ by taking $\|\Delta_r\|$ sufficiently small.

For convenience, we drop in the following the index r on the meshes Δ_r. Considering the subarc K_j and setting $t = \frac{1}{2}(t_{j-1}+t_j)+\varepsilon$ with $|\varepsilon| \le |h_j|/2$, we have from (2.8)

$$(3.2)\quad q_\Delta(t)-f(t)=\frac{1}{13}h_j^{-2}\left(\frac{1}{36}h_j^2-\varepsilon^2\right)\left[m_j\left(-\frac{13}{2}h_j-9\varepsilon\right)+m_{j-1}\left(\frac{13}{2}h_j-9\varepsilon\right)\right]+$$

$$+h_j^{-3}f(\beta_j)\left[\frac{1}{2}h_j^3+\frac{81}{26}h_j^2\varepsilon-\frac{54}{13}\varepsilon^3\right]+h_j^{-3}f(\alpha_j)\left(\frac{1}{2}h_j^3-\frac{81}{26}h_j^2\varepsilon+\frac{54}{13}\varepsilon^3\right]-f(t)=$$

$$=\frac{1}{13}h_j^{-2}\left(\frac{1}{36}h_j^2-\varepsilon^2\right)\left[m_j\left(-\frac{13}{2}h_j-9\varepsilon\right)+m_{j-1}\left(\frac{13}{2}h_j-9\varepsilon\right)\right]+$$

$$+\left[\frac{1}{2}(f(\beta_j)+f(\alpha_j))-f(t)\right]+\frac{27}{13}\varepsilon h_j^{-1}\left(\frac{3}{2}-2\varepsilon^2h_j^{-2}\right)(f(\beta_j)-f(\alpha_j)).$$

Since B is the inverse of the coefficient matrix corresponding to the equations (2.10), we have

$$(3.3)\qquad m_j=\sum_k b_{j,k}F_k(\alpha,\beta)$$

where of course, $\Sigma_k\,|b_{j,k}|\leqq 2(1+\eta)/9$ by virtue of (3.1) and the observations following it.

Thus, in view of (2.14) we have from (3.2),

$$|q_\Delta(t)-f(t)|\leqq\left|\frac{1}{2}(f(\beta_j)+f(\alpha_j))-f(t)\right|+$$

$$+\frac{1}{26}|h_j|-2(13|h_j|+18|\varepsilon|)\left(\frac{1}{36}|h_j|^2+|\varepsilon|^2\right)\frac{4}{9}(1+\eta)\cdot$$

$$\cdot\max_k\left\{\frac{243}{4}|h_k+h_{k+1}|^{-1}\right\}w(\|\Delta\|;f)+\frac{27}{13}\left|\frac{3}{2}\varepsilon h_j^{-1}-2\varepsilon^3h_j^{-3}\right||f(\beta_j)-f(\alpha_j)|.$$

Now $|\gamma_j|<1$ for sufficiently small $\|\Delta\|$, therefore, we have

$$(3.4)\qquad |q_\Delta(t)-f(t)|\leqq\left[\frac{165}{26}(1+\eta)+\frac{40}{13}\right]w(\|\Delta\|;f)$$

since the right hand side of (3.4) is independent of j, we have proved the following:

THEOREM 2. *Let $f(t)$ be continuous on K. Let $\{\Delta_r\}$ be a sequence of subdivisions of K with $\lim_{r\to\infty}\|\Delta_r\|=0$. Let $q_{\Delta_r}(t)$ be the deficient complex cubic spline on K, interpolating to $f(t)$ on Δ_r at the points $\alpha_{r,j}$ and $\beta_{r,j}$, $j=1,2,\dots,n_r$. Then $\{q_{\Delta_r}(t)\}\to f(t)$ uniformly as $\|\Delta_r\|\to 0$. Further if $f(t)$ satisfies a Hölder condition of order p on K $(0<p\leqq 1)$, then $|q_{\Delta_r}(t)-f(t)|=O(\|\Delta_r\|^p)$.*

It may be observed that suitable Hölder conditions for $q_\Delta(t)-f(t)$ or its derivatives are involved in the definition of the Cauchy Principal Value of the integral in (2.16). Thus, in order to study convergence properties of the complex spline $S_\Delta(z)$ we investigate such properties for $q_\Delta(t)-f(t)$.

Writing $t^*=\frac{1}{2}(t_j+t_{j-1})$ we see from (2.8) that for t and θ on K_j, we have

$$(3.5)\quad q_\Delta(t)-q_\Delta(\theta)=(t-\theta)\Big\{\frac{1}{2}h_j^{-1}(m_j-m_{j-1})(t+\theta-2t^*)+3h_j^{-1}(f(\beta_j)-f(\alpha_j))+$$

$$+[m_j+m_{j-1}-6h_j^{-1}(f(\beta_j)-f(\alpha_j))]\Big[\frac{9}{13}h_j^{-2}((t-t^*)^2+(t-t^*)(\theta-t^*)+(\theta-t^*)^2)-\frac{1}{52}\Big]\Big\}.$$

If $f(t)$ satisfies a Hölder condition of order p $(0<p\leqq 1)$ and if $0<s\leqq p$, it follows when $\|\Delta\|$ is sufficiently small that for t and θ on K_j

$$(3.6)\qquad |[q_\Delta(t)-f(t)]-[q_\Delta(\theta)-f(\theta)]|\leqq|t-\theta|^s|t-\theta|^{p-s}(\|\Delta\|\,\|\Delta\|^{-1})^{p-s}\cdot$$

$$\cdot\Big\{\frac{1}{2}(|m_j+m_{j-1}|+|m_j-m_{j-1}|+12|h_j|^{-1}|f(\beta_j)-f(\alpha_j)|)|t-\theta|^{1-p}+|t-\theta|^{-p}|f(\theta)-f(t)|\Big\}.$$

Since the matrix B_r has a finite row norm, and since the hypothesis that $f(t)$ satisfies a Hölder condition of order p, implies that $\max_k|F_k(\alpha,\beta)|=O(\|\Delta\|^{p-1})$ we have, for all j, $|m_j\pm m_{j-1}|=O(\|\Delta\|^{p-1})$. Thus, we finally see from (3.6) that the following holds.

THEOREM 2′. *Suppose the conditions of Theorem 2 hold and $f(t)$ satisfies a Hölder condition of order p $(0<p\leqq 1)$. Then the function $(q_{\Delta_r}(t)-f(t))/\|\Delta_r\|^{p-1}$ satisfies a Hölder condition of order s, $0<s\leqq p$, uniformly with respect to r.*

4. Case when $f(t)\in C^1(K)$

We next consider an estimate of $q_\Delta(t)-f(t)$ under the assumption that $f(t)\in C^1$ on K.

Writing the equation (2.10) as $A_r m_r=-\frac{9}{2}e_r$ where e_r is the n_r-vector $(e_{r,1}, e_{r,2}, \dots, e_{r,n_r})^T$ with $e_{r,j}=-\frac{2}{9}F_{r,j}(\alpha,\beta)$, A_r is the coefficient matrix corresponding to the equation (2.10) and m_r the corresponding vector of spline first derivatives, $(m_{r,1}, m_{r,2}, \dots, m_{r,n_r})^T$, we observe that

$$(4.1)\qquad A_r(m_r-e_r)=\Big(-\frac{9}{2}I_r-A_r\Big)e_r.$$

Here I_r is the $n_r\times n_r$ identity matrix. Thus, the right hand member of (4.1) has the jth row as

$$-\gamma_{r,j}e_{r,j-1}+e_{r,j}-\delta_{r,j}e_{r,j+1}=-[\gamma_{r,j}(e_{r,j-1}-e_{r,j})+\delta_{r,j}(e_{r,j+1}-e_{r,j})].$$

It may be easily seen that by virtue of the assumption that $f(t)\in C^1$ on K, $\max_j|f'(t_{r,j})-e_{r,j}|$ can be made arbitrarily small, uniformly with respect to r, by taking $\|\Delta_r\|$ sufficiently small. Thus, the row-max norm

$$\Big\|\Big(-\frac{9}{2}I_r-A_r\Big)e_r\Big\|=\max_j|\gamma_{r,j}(e_{r,j-1}-e_{r,j})+\delta_{r,j}(e_{r,j+1}-e_{r,j})|$$

can be made arbitrarily small uniformly with respect to r. Hence using the fact that the inverse of the coefficient matrix A_r that is B_r has a bounded row norm, it follows that

(4.2) $$\max_j |m_{r,j}-f'(t_{r,j})| \to 0 \quad \text{as} \quad \|\Delta_r\| \to 0.$$

Writing $t=\frac{1}{2}(t_{r,j-1}+t_{r,j})+\varepsilon$, for t on $K_{r,j}$, we have from (2.8)

$$q'_{\Delta_r}(t) = \varepsilon h_{r,j}^{-1}(m_{r,j}-m_{r,j-1})+\left(\frac{27}{13}\varepsilon^2 h_{r,j}^{-2}-\frac{1}{52}\right)(m_{r,j}+m_{r,j-1})+$$

$$+h_{r,j}^{-1}\left\{f(\beta_{r,j})\left(\frac{81}{26}-\frac{162}{13}\varepsilon^2 h_{r,j}^{-2}\right)+f(\alpha_{r,j})\left(-\frac{81}{26}+\frac{162}{13}\varepsilon^2 h_{r,j}^{-2}\right)\right\}.$$

Thus,

(4.3) $$\left|q'_{\Delta_r}(t)-3h_{r,j}^{-1}(f(\beta_{r,j})-f(\alpha_{r,j}))\right| \leqq |\varepsilon h_{r,j}^{-1}|\,|m_{r,j}-m_{r,j-1}|+$$

$$+\frac{27}{13}|\varepsilon^2 h_{r,j}^{-2}|\,\left|m_{r,j}+m_{r,j-1}-6h_{r,j}^{-1}(f(\beta_{r,j})-f(\alpha_{r,j}))\right|$$

and, therefore, it follows directly from (4.2) that for sufficiently small $\|\Delta_r\|$, $\{q'_{\Delta_r}(t)\}$ converges uniformly to $f'(t)$. This property in its turn, implies that $q_{\Delta_r}(t)-f(t)= =o(\|\Delta_r\|)$ when we observe that for t on $K_{r,j}$

$$q_{\Delta_r}(t)-f(t) = \int_{\alpha_{r,j}}^{t} [q'_{\Delta_r}(t)-f'(t)]\,dt.$$

Thus, we have proved the following:

THEOREM 3. *Let $f(t)$ be of class C^1 on K. Let $\Delta_r=\{t_{r,1}, t_{r,2}, \dots, t_{r,n_r}\}$, $(r=1, 2, \dots)$ represent a sequence of subdivisions of K with $\lim_{r\to\infty} \|\Delta_r\|=0$ and let $q_{\Delta_r}(t)$ be the deficient complex cubic spline on K of Theorem 2 for Δ_r. Then $\{q'_{\Delta_r}(t)\}$ converges uniformly on K to $f'(t)$ and $(q_{\Delta_r}(t)-f(t))=o(\|\Delta_r\|)$.*

Since, for any t on $K_{r,j}$, we may write

$$f'(t)-3h_{r,j}^{-1}(f(\beta_{r,j})-f(\alpha_{r,j})) = 3h_{r,j}^{-1}\int_{\alpha_{r,j}}^{\beta_{r,j}} [f'(t)-f'(\theta)]\,d\theta$$

hence it follows from (4.3) that $q'_{\Delta_r}(t)-f'(t)=O(\|\Delta_r\|^p)$ if $f'(t)$ satisfies a Hölder condition of order p on K with $0<p\leqq 1$. Thus, we have proved

COROLLARY 1. *Let $f'(t)$ satisfy a Hölder condition of order p on K $(0<p\leqq 1)$ and let the conditions of Theorem 3 concerning $f(t)$, $\{\Delta_r\}$ and $\{q_{\Delta_r}(t)\}$ hold, then for $u=0, 1$, we have*

$$f^{(u)}(t)-q^{(u)}_{\Delta_r}(t) = O(\|\Delta_r\|^{1+p-u}).$$

Writing $t^*=\frac{1}{2}(t_{j-1}+t_j)$ (dropping the index r), and using (2.8) we have:

$$q'_\Delta(t)-q'_\Delta(\theta) = (t-\theta)\{[(m_j-3h_j^{-1}(f(\beta_j)-f(\alpha_j)))+$$

$$+(m_{j-1}-3h_j^{-1}(f(\beta_j)-f(\alpha_j)))]\frac{27}{13}h_j^{-2}(t+\theta-2t^*)+h_j^{-1}(m_j-m_{j-1})\}.$$

From the proof of Theorem 3, it follows that

(4.4) $m_j - 3h_j^{-1}(f(\beta_j) - f(\alpha_j))/\|\Delta\|^p$ and $m_{j-1} - 3h_j^{-1}(f(\beta_j) - f(\alpha_j))/\|\Delta\|^p$

are uniformly bounded. These of course imply that $m_j - m_{j-1}$ is $O(\|\Delta\|^p)$. Hence for sufficiently small $\|\Delta\|$, we have

$$|[q'_\Delta(t) - f'(t)] - [q'_\Delta(\theta) - f'(\theta)]| \leqq |t-\theta|^s \|\Delta\|^{p-s} \Big\{|f'(\theta) - f'(t)|\,|t-\theta|^{-p} +$$

$$+ \left[\frac{27}{13}\left|m_j - 3h_j^{-1}(f(\beta_j) - f(\alpha_j))\right| + \frac{27}{13}\left|m_{j-1} - 3h_j^{-1}(f(\beta_j) - f(\alpha_j))\right| + |m_j - m_{j-1}|\right] \cdot$$

$$\cdot |h_j|^{-1} \|\Delta\|^{s-p} |t-\theta|^{1-s}\Big\}.$$

Thus, we have proved

COROLLARY 2. *Under the conditions of Corollary* 1, $[q'_{\Delta_r}(t) - f'(t)]/\|\Delta_r\|^{p-s}$ *satisfies a uniform Hölder condition of order* s, $0 < s \leqq p$, *provided*

(4.5) $$\max_j \|\Delta_r\|/|h_{r,j}| \leqq C_1 < \infty.$$

We notice that for uniform boundedness of expression in (4.4), we do not use the restriction (4.5). Thus, it follows from (3.5) that

COROLLARY 3. *Under the conditions of Theorem* 3, *and for arbitrary* s, *such that* $0 < s \leqq 1$, *the quantity* $[q_{\Delta_r}(t) - f(t)]/\|\Delta_r\|^{1-s}$ *satisfies a Hölder condition of order* s *on* K *uniformly with respect to* r.

5. Conclusion

In the results obtained in the paper we have proved the convergence of deficient complex cubic spline and its derivatives for the situation in which the interpolating function $f(t)$ belongs to $C(K)$ or $C^1(K)$. It will be interesting to investigate the convergence problem when $f(t)$ belongs to $C^2(K), C^3(K)$ or $C^4(K)$.

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LACUNARY INTERPOLATION BY SPLINES

(0; 0, 2, 3) AND (0; 0, 2, 4) CASES

By
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P. Turán and his associates initiated the study of (0, 2) lacunary interpolatory polynomials, which was a subject of further investigation by A. Sharma, R. B. Saxena and A. K. Varma. For the history of the problem one is referred to the interesting papers [2, 3, 4]. The idea of lacunary interpolation through polynomials was exploited in an interesting paper of I. J. SCHOENBERG [6], where he studied existence and uniqueness of the so called g-splines in connection with the problem of lacunary interpolation by splines. Later on A. MEIR and A. SHARMA [4] obtained error bounds for some classes of quintic splines which interpolate to (0, 2) data on equidistant knots. Recently A. K. VARMA [12] has obtained deficient quintic splines interpolating (0, 2), (0, 4) and (0, 1, 3) data, when the nodes of interpolation and the knots of the spline do not coincide.

In one of the papers I. J. Schoenberg remarked that whole of the theory of interpolation through polynomials can be translated into the theory of interpolation through splines. The object of this paper is to investigate the existence, uniqueness and error bounds of six degree deficient splines interpolating (0; 0, 2, 3) and (0; 0, 2, 4) data, which as far as we know has no counter part in the theory of interpolation through polynomials.

We shall denote by $\mathcal{R}^{(2)}_{n,6}$ the class of six degree splines $R_n(x)$ on [0, 1] having the following two properties:

(1.1) (i) $R_n(x)\in C^2[0, 1]$,
(ii) $R_n(x)$ is of six degree in each piece $[x_{2i}, x_{2i+2}]$, where $x_i=\frac{i}{2m}$, $i=0, 2, \dots, 2m$ and $n=2m+1$.

Here we shall prove the following theorems.

THEOREM 1. *For given arbitrary numbers* $f(x_0), f(x_2), \dots, f(x_{2m})$; $f(t_0), f(t_2), \dots, f(t_{2m-2})$; $f''(t_0), f''(t_2), \dots, f''(t_{2m-2})$; $f'''(t_0), f'''(t_2), \dots, f'''(t_{2m-2})$; $f'(x_0), f'(x_{2m})$, *there exists a unique spline* $R_n(x)\in\mathcal{R}^{(2)}_{n,6}$ *such that*

$$(1.2)\qquad \begin{cases} R_n(x_{2i}) = f(x_{2i}), & i=0, 1, \dots, m\\ R^q_n(t_{2i}) = f^q(t_{2i}), & i=0, 1, \dots, m-1, \ q=0, 2, 3,\\ R'_n(x_0)=f'(x_0), & R'_n(x_{2m})=f'(x_{2m}), \end{cases}$$

where $t_{2i}=x_{2i}+\frac{2}{3}h$, $i=0, 1, \dots, m-1$, $2mh=1$.

THEOREM 2. *Let* $f\in C^5[0, 1]$ *and* $R_n(x)\in\mathcal{R}^{(2)}_{n,6}$ *be a unique spline satisfying the conditions of Theorem* 1, *then*

$$(1.3)\quad \|R^{(q)}_n(x)-f^{(q)}(x)\|_\infty \leqq 293m^{q-5}\omega_5\left(\frac{1}{m}\right)+4m^{q-5}\|f^{\mathrm{v}}\|_\infty \quad (q=0, 1, 2, 3, 4).$$

THEOREM 3. *For given arbitrary numbers* $f(x_0), f(x_2), \ldots, f(x_{2m}); f(t_0), f(t_2), \ldots, f(t_{2m-2}); \quad f''(t_0), f''(t_2), \ldots, f''(t_{2m-2}); \quad f^{\mathrm{iv}}(t_0), f^{\mathrm{iv}}(t_2), \ldots, f^{\mathrm{iv}}(t_{2m-2});$ $f'(x_0), f'(x_{2m})$, *there exists a unique spline* $S_n(x)\in\mathscr{R}^{(2)}_{n,6}$ *such that*

$$(1.4)\qquad \begin{cases} S_n(x_{2i}) = f(x_{2i}), & i = 0, 1, \ldots, m \\ S_n^q(t_2) = f^q(t_{2i}), & i = 1, \ldots, m-1, \; q = 0, 2, 4 \\ S'(x_0) = f'(x_0), & S_n'(x_{2m}) = f'(x_{2m}), \end{cases}$$

where $t_{2i} = x_{2i} + \frac{2}{3}h,\; i = 0, 1, \ldots, m-1,\; 2h = \frac{1}{m}$.

THEOREM 4. *Let* $f\in C^5[0, 1]$ *and* $S_n(x)\in\mathscr{R}^{(2)}_{n,6}$ *be the unique spline satisfying the conditions of Theorem* 3, *then*

$$(1.5)\quad \|S_n^{(q)}(x) - f^{(q)}(x)\|_\infty \leqq 2245 m^{q-5}\omega_5\left(\frac{1}{m}\right) + 4m^{q-5}\|f^{\mathrm{v}}\|_\infty \quad (q = 0, 1, 2, 3, 4).$$

The proofs of Theorems 1 and 2 are given in part I and that of Theorems 3 and 4 are sketched in Part II.

Preliminaries

If $P(x)$ is a polynomial of degree six on [0, 1], then we have

$$(2.1)\qquad P(x) = P(0)\lambda_0(x) + P\left(\frac{1}{3}\right)\lambda_1(x) + P(1)\lambda_2(x) + P'(0)\mu_0(x) +$$

$$+ P'(1)\mu_2(x) + P''\left(\frac{1}{3}\right)\nu_1(x) + P'''\left(\frac{1}{3}\right)\xi_1(x),$$

where

$$(2.2)\quad \xi_1(x) = \frac{1}{14}(9x^6 - 27x^5 + 29x^4 - 13x^3 + 2x^2) = \frac{1}{14}x^2(x-1)^2(3x-1)(3x-2);$$

$$(2.3)\quad \nu_1(x) = \frac{1}{56}(243x^6 - 540x^5 + 342x^4 - 36x^3 - 9x^2) = \frac{9}{56}x^2(x-1)^2(3x-1)(3x+1);$$

$$(2.4)\quad \mu_2(x) = \frac{1}{56}(243x^6 - 540x^5 + 426x^4 - 148x^3 + 19x^2) =$$

$$= \frac{1}{56}x^2(x-1)(3x-1)(81x^2 - 72x + 19);$$

$$(2.5)\quad \mu_0(x) = \frac{1}{7}(-81x^6 + 243x^5 - 282x^4 + 166x^3 - 53x^2 + 7x) =$$

$$= -\frac{x}{7}(x-1)^2(3x-1)(27x^2 - 18x + 7);$$

$$(2.6)\quad \lambda_2(x) = \frac{1}{112}(-3321x^6 + 7884x^5 - 6438x^4 + 2284x^3 - 297x^2) =$$

$$= \frac{1}{112}x^2(3x-1)(-1107x^3 + 2259x^2 - 1393x + 297);$$

(2.7) $\lambda_1(x) = \frac{1}{112}(12393x^6+32076x^5+30294x^4-13932x^3+3321x^2) =$

$= \frac{1}{112}x^2(x-1)^2(12393x^2-7290x+3321);$

(2.8) $\lambda_0(x) = -81x^6+216x^5-213x^4+104x^3-27x^2+1 =$
$= (x-1)^2(3x-1)(-27x^3+9x^2-5x-1).$

For the later references we note that

$$(2.9)\quad \begin{cases}
\xi_1''(0) = \frac{2}{7}, \quad \xi_1^{\mathrm{iv}}(0) = \frac{348}{7}, \quad \xi_1^{\mathrm{v}}(0) = -\frac{1620}{7}, \quad \xi_1^{\mathrm{iv}}\left(\frac{1}{3}\right) = -\frac{12}{7}, \\
\xi_1''(1) = \frac{2}{7}, \quad \xi_1^{\mathrm{iv}}(1) = \frac{348}{7}, \quad \xi_1^{\mathrm{v}}(1) = \frac{1620}{7}, \\
\nu''(0) = -\frac{9}{28}, \quad \nu_1^{\mathrm{iv}}(0) = \frac{1026}{7}, \quad \nu_1^{\mathrm{v}}(0) = -\frac{8100}{7}, \quad \nu_1^{\mathrm{iv}}\left(\frac{1}{3}\right) = -\frac{459}{7}, \\
\nu_1''(1) = \frac{45}{7}, \quad \nu_2^{\mathrm{iv}}(1) = \frac{3861}{7}, \quad \nu_1^{\mathrm{v}}(1) = \frac{13770}{7}, \\
\mu_2''(0) = \frac{19}{28}, \quad \mu_2^{\mathrm{iv}}(0) = \frac{1278}{7}, \quad \mu_2^{\mathrm{v}}(0) = -\frac{8100}{7}, \quad \mu_2^{\mathrm{iv}}\left(\frac{1}{3}\right) = -\frac{207}{7}, \\
\mu_2''(1) = \frac{94}{7}, \quad \mu_2^{\mathrm{iv}}(1) = \frac{4113}{7}, \quad \mu_2^{\mathrm{v}}(1) = \frac{13770}{7}, \\
\mu_0''(0) = -\frac{106}{7}, \quad \mu_0^{\mathrm{iv}}(0) = -\frac{6768}{7}, \quad \mu_0^{\mathrm{v}}(0) = \frac{29160}{7}, \quad \mu_0^{\mathrm{iv}}\left(\frac{1}{3}\right) = -\frac{288}{7}, \\
\mu_0''(1) = -\frac{64}{7}, \quad \mu_0^{\mathrm{iv}}(1) = -\frac{6768}{7}, \quad \mu_0^{\mathrm{v}}(1) = -\frac{29160}{7}, \\
\lambda_2''(0) = -\frac{297}{56}, \quad \lambda_2^{\mathrm{iv}}(0) = -\frac{9657}{7}, \quad \lambda_2^{\mathrm{v}}(0) = \frac{59130}{7}, \quad \lambda_2^{\mathrm{iv}}\left(\frac{1}{3}\right) = \frac{3501}{14}, \\
\lambda_2''(1) = -\frac{381}{7}, \quad \lambda_2^{\mathrm{iv}}(1) = -\frac{50499}{14}, \quad \lambda_2^{\mathrm{v}}(1) = -\frac{90315}{7}, \\
\lambda_1''(0) = \frac{3321}{56}, \quad \lambda_1^{\mathrm{iv}}(0) = \frac{45441}{7}, \quad \lambda_1^{\mathrm{v}}(0) = -\frac{240570}{7}, \quad \lambda_1^{\mathrm{iv}}\left(\frac{1}{3}\right) = -\frac{7533}{14}, \\
\lambda_1''(1) = \frac{1053}{7}, \quad \lambda_1^{\mathrm{iv}}(1) = \frac{167427}{14}, \quad \lambda_1^{\mathrm{v}}(1) = \frac{317115}{7}, \\
\lambda_0''(0) = -54, \quad \lambda_0^{\mathrm{iv}}(0) = -5112, \quad \lambda_0^{\mathrm{v}}(0) = 25920, \quad \lambda_0^{\mathrm{iv}}\left(\frac{1}{3}\right) = 288, \\
\lambda_0''(1) = -96, \quad \lambda_0^{\mathrm{iv}}(1) = -8352, \quad \lambda_0^{\mathrm{v}}(1) = -32400.
\end{cases}$$

For $f \in C^5[0, 1]$, we have the expansions:

$$
(2.10)\quad \left\{
\begin{aligned}
& f(x_{2i+2}) = f(x_{2i}) + 2hf'(x_{2i}) + 2h^2 f''(x_{2i}) + \frac{4}{3} h^3 f'''(x_{2i}) + \\
& + \frac{2}{3} h^4 f^{\mathrm{iv}}(x_{2i}) + \frac{4}{15} h^5 f^{\mathrm{v}}(\eta_{1,2i});\ x_{2i} < \eta_{1,2i} < x_{2i+2}, \\
& f(x_{2i-2}) = f(x_{2i}) - 2hf(x_{2i}) + 2h^2 f''(x_{2i}) - \frac{4}{3} h^3 f'''(x_{2i}) + \\
& + \frac{2}{3} h^4 f^{\mathrm{iv}}(x_{2i}) - \frac{4}{15} h^5 f^{\mathrm{v}}(\eta_{2,2i}),\ x_{2i-2} < \eta_{2,2i} < x_{2i}, \\
& f(t_{2i}) = f(x_{2i}) + \frac{2}{3} hf'(x_{2i}) + \frac{2}{9} h^2 f''(x_{2i}) + \frac{4}{81} h^3 f'''(x_{2i}) + \\
& + \frac{2}{243} h^4 f^{\mathrm{iv}}(x_{2i}) + \frac{4}{3645} h^5 f^{\mathrm{v}}(\eta_{3,2i}),\ x_{2i} < \eta_{3,2i} < t_{2i}, \\
& f(t_{2i-2}) = f(x_{2i}) - \frac{4h}{3} f'(x_{2i}) + \frac{8}{9} h^2 f''(x_{2i}) - \frac{32}{81} h^3 f'''(x_{2i}) + \\
& + \frac{32}{243} h^4 f^{\mathrm{iv}}(x_{2i}) - \frac{128}{3645} f^{\mathrm{v}}(\eta_{4,2i}),\ t_{2i-2} < \eta_{4,2i} < x_{2i}, \\
& f'(x_{2i+2}) = f'(x_{2i}) + 2hf''(x_{2i}) + 2h^2 f'''(x_{2i}) + \frac{4}{3} h^3 f^{\mathrm{iv}}(x_{2i}) + \frac{2}{3} h^4 f^{\mathrm{v}}(\eta_{5,2i}), \\
& \qquad\qquad x_{2i} < \eta_{5,2i} < x_{2i+2}, \\
& f'(x_{2i-2}) = f'(x_{2i}) - 2hf''(x_{2i}) + 2h^2 f'''(x_{2i}) - \frac{4}{3} h^3 f^{\mathrm{iv}}(x_{2i}) + \frac{2}{3} h^4 f^{\mathrm{v}}(\eta_{6,2i}), \\
& \qquad\qquad x_{2i-2} < \eta_{6,2i} < x_{2i}, \\
& f''(t_{2i-2}) = f''(x_{2i}) - \frac{4h}{3} f'''(x_{2i}) + \frac{8}{9} h^2 f^{\mathrm{iv}}(x_{2i}) - \frac{32}{81} h^3 f^{\mathrm{v}}(\eta_{7,2i}),\ t_{2i-2} < \eta_{7,2i} < x_{2i}, \\
& f''(t_{2i}) = f''(x_{2i}) + \frac{2h}{3} f'''(x_{2i}) + \frac{2}{9} h^2 f^{\mathrm{iv}}(x_{2i}) + \frac{4}{81} h^3 f^{\mathrm{v}}(\eta_{8,2i}),\ x_{2i} < \eta_{8,2i} < t_{2i}, \\
& f'''(t_{2i-2}) = f'''(x_{2i}) - \frac{4h}{3} f^{\mathrm{iv}}(x_{2i}) + \frac{8}{9} h^2 f^{\mathrm{v}}(\eta_{9,2i}),\ t_{2i-2} < \eta_{9,2i} < x_{2i}, \\
& f'''(t_{2i}) = f'''(x_{2i}) + \frac{2h}{3} f^{\mathrm{iv}}(x_{2i}) + \frac{2}{9} h^2 f^{\mathrm{v}}(\eta_{10,2i}),\ x_{2i} < \eta_{10,2i} < t_{2i}.
\end{aligned}
\right.
$$

Part I

Proof of Theorem 1. The proof depends on the following representation of $R_n(x)$.

For $2ih \leqq x \leqq (2i+2)h$, $i=0, 1, \dots, m-1$, we have

$$(3.1)\quad R_n(x) = f(x_{2i})\lambda_0\left(\frac{x-2ih}{2h}\right) + f(x_{2i+2})\lambda_2\left(\frac{x-2ih}{2h}\right) + f(t_{2i})\lambda_1\left(\frac{x-2ih}{2h}\right) +$$

$$+2hR_n'(x_{2i})\mu_0\left(\frac{x-2ih}{2h}\right) + 2hR_n'(x_{2i+2})\mu_2\left(\frac{x-2ih}{2h}\right) +$$

$$+4h^2 f''(t_{2i})\nu_1\left(\frac{x-2ih}{2h}\right) + 8h^3 f'''(t_{2i})\xi_1\left(\frac{x-2ih}{2h}\right);$$

on using (2.1) and

$$(3.2)\quad R_n'(0) = f'(0), \quad R_n'(1) = f'(1).$$

It is easy to see that $R_n(x)$ as given by (3.1) indeed satisfies the second condition of (1.1). We still need to decide whether it is possible to determine $R_n'(x_{2i})$ $(i=1, 2, \dots, m-1)$ uniquely. For this purpose we use the fact that $R_n(x) \in C^2[0, 1]$ and therefore the conditions

$$(3.3)\quad R_n''(x_{2i}+) = R_n''(x_{2i}-), \quad i = 1, 2, \dots, m-1$$

with the help of (3.1) and (2.9) reduce to

$$(3.4)\quad \frac{32}{7}hR_n'(x_{2i-2}) - \frac{100}{7}hR_n'(x_{2i}) + \frac{19}{56}hR_n'(x_{2i+2}) =$$

$$= -24f(x_{2i-2}) - \frac{3}{28}f(x_{2i}) + \frac{297}{224}f(x_{2i+2}) + \frac{1053}{28}f(t_{2i-2}) -$$

$$-\frac{3321}{224}f(t_{2i}) + \frac{45}{7}h^2 f''(t_{2i-2}) + \frac{9}{28}h^2 f''(t_{2i}) +$$

$$+\frac{4}{7}h^3 f'''(t_{2i-2}) - \frac{4}{7}h^3 f'''(t_{2i}), \quad i = 1, 2, \dots, m-1.$$

But (3.4) is a strictly tridiagonal dominant system, which has a unique solution. Thus $R_n'(x_{2i})$, $i=1, 2, \dots, m-1$ can be obtained uniquely which establishes Theorem 1.

In order to prove Theorem 2, we require the following.

LEMMA 1. *Let* $A_{2i}=|f(x_{2i})-R_n'(x_{2i})|$, *then*

$$\max_{1\le i\le m-1} A_{2i} \leqq \frac{6113}{7875}\, h^4\omega_5\left(\frac{1}{m}\right), \tag{4.1}$$

where $\omega_5(\cdot)$ *is the modulus of continuity of* $f^{\mathrm{v}}(\cdot)$.

PROOF. From (3.4) and (2.10) it follows that

$$\frac{32}{7}h\big(R_n'(x_{2i-2})-f'(x_{2i-2})\big)-\frac{100}{7}h\big(R_n'(x_{2i})-f'(x_{2i})\big)+\frac{19}{56}h\big(R_n'(x_{2i+2})-f'(x_{2i+2})\big)=$$

$$=-\frac{64}{21}h^5f^{\mathrm{v}}(\eta_{6,2i})-\frac{19}{84}h^5f^{\mathrm{v}}(\eta_{5,2i})+\frac{32}{5}h^5f^{\mathrm{v}}(\eta_{2,2i})+$$

$$+\frac{99}{280}h^5f^{\mathrm{v}}(\eta_{1,2i})-\frac{416}{315}h^5f^{\mathrm{v}}(\eta_{4,2i})-\frac{41}{2520}h^5f^{\mathrm{v}}(\eta_{3,2i})-$$

$$-\frac{160}{63}h^5f^{\mathrm{v}}(\eta_{7,2i})+\frac{1}{63}h^5f^{\mathrm{v}}(\eta_{8,2i})+\frac{32}{63}h^5f^{\mathrm{v}}(\eta_{9,2i})-$$

$$-\frac{8}{63}h^5f^{\mathrm{v}}(\eta_{10,2i})=\frac{6113}{840}h^5\theta_0\omega_5\left(\frac{1}{m}\right),\quad |\theta_0|<1.$$

The lemma follows on using the property of diagonal dominance.

LEMMA 2. *Let* $f\in C^5\,[0,1]$, *then*

$$|R_n^{\mathrm{iv}}(x_{2i}+)-f^{\mathrm{iv}}(x_{2i})| \leqq 135h\omega_5\left(\frac{1}{m}\right), \tag{4.2}$$

$$|R_n^{\mathrm{iv}}(x_{2i}-)-f^{\mathrm{iv}}(x_{2i})| \leqq 214h\omega_5(1/m), \tag{4.3}$$

$$|R_n^{\mathrm{iv}}(t_{2i})-f^{\mathrm{vi}}(t_{2i})| \leqq 12h\omega_5(1/m), \tag{4.4}$$

$$|R_n^{\mathrm{v}}(x_{2i}+)-f^{\mathrm{v}}(x_{2i})| \leqq 330\omega_5(1/m), \tag{4.5}$$

$$|R_n^{\mathrm{v}}(x_{2i}-)-f^{\mathrm{v}}(x_{2i})| \leqq 407\omega_5(1/m), \tag{4.6}$$

$$|R_n^{\mathrm{vi}}(t_{ri})| \leqq 367h^{-1}\omega_5(1/m). \tag{4.7}$$

For the proof we refer to A. K. VARMA [12].

PROOF OF THEOREM 2. For $0\leqq y\leqq 1$, we have

$$\lambda_0(y)+\lambda_1(y)+\lambda_2(y)\equiv 1. \tag{5.1}$$

Let $x_{2i} \leqq x \leqq x_{2i+2}$, on using (5.1) and (3.1) we obtain

(5.2) $$R^{\mathrm{iv}}(x)-f^{\mathrm{iv}}(x)=\left(R_n^{\mathrm{iv}}(x_{2i}+)-f^{\mathrm{iv}}(x)\right)\lambda_0\left(\frac{x-x_{2i}}{2h}\right)+$$

$$+\left(R_n^{\mathrm{iv}}(x_{2i+2}-)-f^{\mathrm{iv}}(x)\right)\lambda_2\left(\frac{x-x_{2i}}{2h}\right)+\left(R_n^{\mathrm{iv}}(t_{2i})-f^{\mathrm{iv}}(x)\right)\lambda_1\left(\frac{x-x_{2i}}{2h}\right)+$$

$$+2hR_n^{\mathrm{v}}(x_{2i}+)\mu_0\left(\frac{x-x_{2i}}{2h}\right)+2hR_n^{\mathrm{v}}(x_{2i+2}-)\mu_2\left(\frac{x-x_{2i}}{2h}\right)+$$

$$+4h^2R_n^{\mathrm{vi}}(t_{2i})\nu_1\left(\frac{x-x_{2i}}{2h}\right)=E_1+E_2+E_3+E_4+E_5+E_6.$$

From (2.3) to (2.8) one can easily see that for $0 \leqq x \leqq 1$,

(5.3) $$\begin{cases} |\lambda_0(x)| \leqq 1, \quad |\lambda_1(x)| \leqq 2, \quad |\lambda_2(x)| \leqq 1, \\ |\mu_0(x)| \leqq \dfrac{1}{14}, \ |\mu_2(x)| \leqq \dfrac{1}{20}, \ |\nu_1(x)| \leqq \dfrac{1}{12}. \end{cases}$$

Since $f^{\mathrm{iv}}(x)=f^{\mathrm{iv}}(x_{2i})+(x-x_{2i})f^{\mathrm{v}}(\eta_{\prime\prime,2i})$ $(x_{2i}<\eta_{\prime\prime,2i}<x)$ and $|x-x_{2i}| \leqq 2h$, on using Lemma 2 and (5.3), we have

$$|E_1| \leqq \left(135h\omega_5(1/m)+2h\|f^{\mathrm{v}}\|_\infty\right), \quad |E_2| \leqq \left(214h\omega_5(1/m)+2h\|f^{\mathrm{v}}\|_\infty\right),$$

$$|E_3| \leqq \left(24h\omega_5(1/m)+\frac{8}{3}h\|f^{\mathrm{v}}\|_\infty\right), \quad |E_4| \leqq \frac{1}{7}h\left(330\omega_5(1/m)+\|f^{\mathrm{v}}\|_\infty\right),$$

$$|E_5| \leqq \frac{1}{10}h\left(407\omega_5(1/m)+\|f^{\mathrm{v}}\|_\infty\right), \quad |E_6| \leqq \frac{367}{3}h\omega_5(1/m).$$

Thus from (5.2)

$$|R_n^{\mathrm{iv}}(x)-f^{\mathrm{iv}}(x)| \leqq \frac{293}{m}\omega_5(1/m)+\frac{8}{m}\|f^{\mathrm{v}}\|_\infty.$$

This is the result for $q=4$. On using the usual device

$$R_n'''(x)-f'''(x)=\int_{t_{2i}}^{x}[R_n^{\mathrm{iv}}(t)-f^{\mathrm{iv}}(t)]\,dt$$

we obtain estimates for $\|R_n'''(x)-f'''(x)\|_\infty$, which prove (1.3) for $q=3$.

For $q=0, 1, 2$, the proof follows using the same technique, this completes the proof of Theorem 2.

Part II

The following results are required for the proof of Theorems 3 and 4. If $P(x)$ is a polynomial of degree six on [0, 1] then we have

$$(6.1)\qquad P(x) = P(0)\alpha_0(x)+P\left(\frac{1}{3}\right)\alpha_1(x)+P(1)\alpha_2(x)+P'(0)\beta_0(x)+$$

$$+P'(1)\beta_2(x)+P''\left(\frac{1}{3}\right)\gamma_1(x)+P^{\mathrm{iv}}\left(\frac{1}{3}\right)\delta_1(x),$$

where

$$(6.2)\quad \delta_1(x) = \frac{1}{24}(-9x^6+27x^5-29x^4+13^3-2x^2) = \frac{1}{24}x^2(x-1)^2(3x-1)(-3x+2),$$

$$(6.3)\qquad \gamma_1(x) = \frac{1}{8}(-162x^6+513x^5-585x^4+279x^3-45x^2) =$$

$$= \frac{9}{8}x^2(x-1)^2(3x-1)(-6x+5),$$

$$(6.4)\qquad \beta_2(x) = \frac{1}{8}(-54x^6+189x^5-225x^4+107x^3-17x^2) =$$

$$= \frac{1}{8}x^2(x-1)(3x-1)(-18x^2+39x-17),$$

$$(6.5)\qquad \beta_0(x) = -27x^6+81x^5-90x^4+46x^3-11x^2+x = -x(x-1)^2(3x-1)^3,$$

$$(6.6)\qquad \alpha_2(x) = \frac{1}{16}(1026x^6-3375x^5+3915x^4-1841x^3+291x^2) =$$

$$= \frac{1}{16}x^2(3x-1)(342x^3-1011x^2+968x-291),$$

$$(6.7)\qquad \alpha_1(x) = \frac{1}{16}(-1458x^6+5103x^5-6075x^4+2673x^3-243x^2) =$$

$$= \frac{1}{16}243x^2(x-1)^2(-6x^2+9x-1),$$

$$(6.8)\qquad \alpha_0(x) = 27x^6-108x^5+135x^4-52x^3-3x^2+1 =$$

$$= (x-1)^2(3x-1)(9x^3-15x^2-5x-1).$$

We need also

$$\delta_1''(0) = -\frac{1}{6},\quad \delta_1^{\text{iv}}(0) = -29,\quad \delta_1^{\text{v}}(0) = 135,\quad \delta_1^{\text{vi}} = -270,$$

$$\delta_1''(1) = -\frac{1}{6},\quad \delta_1^{\text{iv}}(1) = -29,\quad \delta_1^{\text{v}}(1) = -135,$$

$$\gamma_1''(0) = -\frac{45}{4},\quad \gamma_1^{\text{iv}}(0) = -1755,\quad \gamma_1^{\text{v}}(0) = 7695,\quad \gamma_1^{\text{vi}} = -14580,$$

$$\gamma_1''(1) = -\frac{9}{2},\quad \gamma_1^{\text{iv}}(1) = -1350,\quad \gamma_1^{\text{v}}(1) = -6885,$$

$$\beta_2''(0) = -\frac{17}{4},\quad \beta_2^{\text{iv}}(0) = -675,\quad \beta_2^{\text{v}}(0) = 2835,\quad \beta_2^{\text{vi}} = -4860,$$

$$\beta_2''(1) = \frac{17}{2},\quad \beta_2^{\text{iv}}(1) = -270,\quad \beta_2^{\text{v}}(1) = -2025,$$

$$\beta_0''(0) = -22,\quad \beta_0^{\text{iv}}(0) = -2160,\quad \beta_0^{\text{v}}(0) = 9720,\quad \beta_0^{\text{vi}}(0) = 19440,$$

$$\beta_0''(1) = -16,\quad \beta_0^{\text{iv}}(1) = -2160,\quad \beta_0^{\text{v}}(1) = -9720,$$

$$\alpha_2''(0) = \frac{291}{8},\quad \alpha_2^{\text{iv}}(0) = \frac{11745}{2},\quad \alpha_2^{\text{v}}(0) = -\frac{50625}{2},\quad \alpha_2^{\text{vi}} = 46170,$$

$$\alpha_2''(1) = -\frac{51}{4},\quad \alpha_2^{\text{iv}}(1) = 3645,\quad \alpha_2^{\text{v}}(1) = \frac{41715}{2},$$

$$\alpha_1''(0) = -\frac{243}{8},\quad \alpha_1^{\text{iv}}(0) = -\frac{18225}{2},\quad \alpha_1^{\text{v}}(0) = \frac{76545}{2},\quad \alpha_1^{\text{vi}}(0) = -65610,$$

$$\alpha_1''(1) = \frac{243}{4},\quad \alpha_1^{\text{iv}}(1) = -3645,\quad \alpha_1^{\text{v}}(1) = -\frac{54675}{2},$$

$$\alpha_0''(0) = -6,\quad \alpha_0^{\text{iv}}(0) = 3240,\quad \alpha_0^{\text{v}}(0) = -12960,\quad \alpha_0^{\text{vi}}(0) = 19440,$$

$$\alpha_0''(1) = -48,\quad \alpha_0^{\text{iv}}(1) = 0,\quad \alpha_0^{\text{v}}(1) = 6480.$$

PROOF OF THEOREM 3. To prove existence and uniqueness of $S_n(x)\in\mathcal{R}_{n,6}^{(2)}$ satisfying the requirement of Theorem 3, we express

$$S_n(x) = f(x_{2i})\alpha_0\left(\frac{x-2ih}{2h}\right) + f(x_{2i+2})\alpha_2\left(\frac{x-2ih}{2h}\right) + f(t_{2i})\alpha_1\left(\frac{x-2ih}{2h}\right) +$$

$$+2hS_n'(x_{2i}+)\beta_0\left(\frac{x-2ih}{2h}\right) + 2hS_n'(x_{2i+2}-)\beta_2\left(\frac{x-2ih}{2h}\right) +$$

$$+4h^2 f''(t_{2i})\gamma_1\left(\frac{x-2ih}{2h}\right) + 16h^4 f^{\text{iv}}(t_{2i})\delta_1\left(\frac{x-2ih}{2h}\right),\quad 2ih \leqq x \leqq (2i+2)h,$$

on using $S_n'(0)=f'(0)$, $S_n'(1)=f'(1)$.

As $S_n(x)\in C^2\,[0,1]$, we have $S_n''(x_{2i}+)=S_n''(x_{2i}-)$, $i=1,2,\dots,m-1$, therefore,

$$8hS_n'(x_{2i-2})-\frac{61}{4}hS_n'(x_{2i})-\frac{17}{8}hS_n'(x_{2i+2})=-12f(x_{2i-2})-\frac{27}{16}f(x_{2i})-\frac{291}{32}f(x_{2i+2})+$$

$$+\frac{243}{16}f(t_{2i-2})-\frac{243}{32}f(t_{2i})-\frac{9}{2}h^2f''(t_{2i-2})+\frac{45}{4}h^2f''(t_{2i})-\frac{2}{3}h^4f^{\text{iv}}(t_{2i-2})+\frac{2}{3}h^4f^{\text{iv}}(t_{2i}),$$
$$1\leqq i\leqq m-1.$$

But this is obviously a tridiagonal dominant system, which will prove the required assertion of Theorem 3.

To prove Theorem 4 we require the following estimates:

Let $f\in C^5\,[0,1]$ and set $B_{2i}=|f'(x_{2i})-S_n'(x_{2i})|$, then

$$\max_{1\leqq i\leqq m-1} B_{2i}\leqq\frac{199}{123}h^4\omega_5(1/m), \tag{7.1}$$

$$|S_n^{\text{iv}}(x_{2i}+)-f^{\text{iv}}(x_{2i})|\leqq 672h\omega_5(1/m), \tag{7.2}$$

$$|S_n^{\text{iv}}(x_{2i}-)-f^{\text{iv}}(x_{2i})|\leqq 553h\omega_5(1/m), \tag{7.3}$$

$$|S_n^{\text{v}}(x_{2i}+)-f^{\text{v}}(x_{2i})|\leqq 1482\omega_5(1/m), \tag{7.4}$$

$$|S_n^{\text{v}}(x_{2i}-)-f^{\text{v}}(x_{2i})|\leqq 1362\omega_5(1/m), \tag{7.5}$$

$$|S_n^{\text{vi}}(t_{2i})|\leqq 1422h^{-1}\omega_5(1/m). \tag{7.6}$$

The proof of Theorem 4 follows on the lines of Theorem 2, on using (7.1)—(7.6).

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INTEGER SOLUTIONS IN ARITHMETIC PROGRESSION FOR $y^2-k=x^3$

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Recently it was pointed out in [3] that an interesting problem like finding the number of k's having a given number of solutions of a given type of the equation $y^2-k=x^3$ has been over looked so far by all. In [3] the author concentrated on the problem of finding out the number of k's having a given number of consecutive integer solutions for y or x or both. The problem was completely solved except in one case, that is, finding out the number of k's having four consecutive integer solutions for y. The conjecture in [3] that there are only finite number of such k's (most probably none except $k=1025$) has roused a great deal of interest in many number theorists.

In this paper we discuss the problem of investigating the number of integers k for which the equation $y^2-k=x^3$ has integer solutions in arithmetic progression either for x or for y or for both. Though the problem may look some what artificial, it has its own beauty and hence should not be overlooked.

Again in course of our discussion it will be clear that this problem presents more difficulties than the earlier problem discussed in [3]. For existence of solutions in A.P. for $y^2-k=x^3$ one can readily check that for every positive integer d we may take $k=\left(\frac{d(d-1)}{2}\right)^2$, $x=0$ and $y=\frac{d(d-1)}{2}$ and then both $y^2-k=x^3$ and $(y+d)^2-k=(x+d)^3$ hold.

Let the number of k's for which $y^2-k=x^3$, k an integer has integer solutions given by $(x_1, y_1), (x_1-d, y_1-d), \ldots, \big(x_1-(i-1)d, y_1-(i-1)d\big)$ be denoted by $N_{i,d}^{(x,y)}$. $N_{i,d}^{(x)}$ and $N_{i,d}^{(y)}$ have the usual meaning. Below we state and prove some theorems. Some of the results of the earlier paper [3] can be made corollaries by taking $d=1$.

THEOREM 1. $N_{2,d}^{(x,y)}=\infty$ *for every integer d.*

PROOF. Let (x_1, y_1) and (x_1-d, y_1-d) be two integer solutions for $y^2-k=x^3$. Then from $y_1^2-k=x^3$ and $(y_1-d)^2-k=(x_1-d)^3$ we get

$$y_1 = \frac{1}{2}(3x_1^2-3x_1d+d^2+d)$$

and hence

$$k = \frac{1}{4}(3x_1^2-3x_1d+d^2+d)^2-x_1^3.$$

It is now easy to see that y_1 and k are both integers if x_1 is an even integer or x_1 and d are of same parity.

Thus we can get an infinite number of k's having two integer solutions for $y^2-k=x^3$ in the same A.P.

THEOREM 2. *If* $y^2-k=x^3$, *k an integer has integer solutions given by* (x_1, y_1) *and* (x_1-d, y_1-d) *then k must be positive except the case* $x_1=y_1=d=1$.

PROOF. If x is zero or a negative integer, then $k=y^2-x^3$ is always positive. So we may assume that x takes only positive integer values. Let (x_1, y_1) and (x_1-d, y_1-d) be two integer solutions for $y^2-k=x^3$. By the above assumption x_1 and $x_1-d=e$ are both positive integers. Now we have

$$k=\frac{1}{4}(3x_1^2-3x_1d+d^2+d)^2-x_1^3.$$

If d is negative, then $3x_1^2-3x_1d+d^2+d>3x_1^2$ and $k>\frac{9}{4}x_1^4-x_1^3$. If d is positive, then

$$k=\frac{1}{4}[(9e^4-4e^3)+6e^2d(3e-1)+3d^2e(2d+3e)+6d^2(e^2-e)+(d^2-d)^2].$$

In either case, k is a positive integer and the theorem is proved.

THEOREM 3. $N_{3,d}^{(x)}=0$ *for every* $d\not\equiv 0 \pmod 3$.

PROOF. Suppose that $y^2-k=x^3$ has integer solutions given by (x_1, y_1), (x_1-d, y_2) and (x_1-2d, y_3). Then we have $y_1^2-y_2^2\equiv y_2^2-y_3^2\equiv d^3 \pmod 3$. Since $3\nmid d$, $d\equiv 1$ or $2 \pmod 3$. Hence $y_1^2-y_2^2\equiv y_2^2-y_3^2\equiv 1$ or $2 \pmod 3$ according as $d\equiv 1$ or $2 \pmod 3$. From $y_1^2-y_2^2\equiv 1 \pmod 3$ we get $y_2^2\equiv 0 \pmod 3$. Then $y_2^2-y_3^2\equiv 1 \pmod 3$ would imply that $y_3^2\equiv 2 \pmod 3$ which is quite absurd. Again from $y_1^2-y_2^2\equiv 2 \pmod 3$ we obtain $y_2^2\equiv 1 \pmod 3$ whence $y_3^2\equiv 2 \pmod 3$, an impossible congruence. Hence the theorem is proved.

For $d\equiv 0 \pmod 3$, we give below some examples.

(i) $15^2-6^3=6^2-3^3=3^2-0^3=9$, $d=3$,

(ii) $25^2-8^3=11^2-2^3=7^2-(-4)^3=113$, $d=6$.

(iii) $21^2-6^3=15^2-0=3^2-(-6)^3=225$, $d=6$,

(iv) $355^2-50^3=95^2-20^3=5^2-(-10)^3=1025$, $d=30$.

Now the question arises whether for every $d\equiv 0 \pmod 3$ there exists an integer k for which $y^2-k=x^3$ has three integer solutions for x in an A.P. with common difference d. The answer to this equation is given by Theorem 4.

THEOREM 4. $N_{3,d}^{(x)}\geqq 1$ *for every* $d\equiv 0 \pmod 3$.

PROOF. The following identity proves the theorem

$$\left[\frac{3a}{2}(1+3a^2)+9a^2\right]^2-(3a^2+3a)^3=\left[\frac{3a}{2}(1+3a^2)\right]^2-(3a^2)^3=$$

$$=\left[\frac{3a}{2}(1+3a^2)-9a^2\right]^2-(3a^2-3a)^3=\frac{9}{4}a^2(1+6a^2-3a^4).$$

Now $d=3a$ and $k=\frac{9a^2}{4}(1+6a^2-3a^4)$. Since $\frac{3a}{2}(1+3a^2)$ is an integer whether a is odd or even, our theorem is proved.

COROLLARY. *We have a k having $(x_1, y_1), (x_1-d, y_1-e)$ and (x_1-2d, y_1-2e) as solutions for $y^2-k=x^3$ for every $d\equiv 0 \pmod 3$ and $e=d^2$.*

We state three problems below.

PROBLEM 1. Is $N^{(x)}_{3, d\equiv 0 (\mathrm{mod}\, 3)}$ finite or infinite for a fixed d?

PROBLEM 2. Is it true that $N^{(x)}_{4,d}\geqq 1$ for every $d\equiv 0 \pmod 3$?

PROBLEM 3. Is it true that $N^{(x)}_{5, d\equiv 0 (\mathrm{mod}\, 3)}\geqq 1$ for some d?

Let $y^2-k=x^3$ possess three solutions given by $(x_1, y_1), (x_2, y_1-d)$ and (x_3, y_1-2d). Then we get $2y_1 d=x_1^3-x_2^3+d^2=x_2^3-x_3^3+3d^2$ whence $x_1^3+x_3^3=2(d^2+x_2^3)$. For $d=1$, one can show without much difficulty that the general solution for $x_1^3+x_3^3=2(1+x_2^3)$ is given by

$$x_1=\frac{c}{2}\{2(3b-a)(a^2+3b^2)+1\},\quad x_2=\frac{c}{2}\{(3b-a)+2(a^2+3b^2)^2\},$$

$$x_3=\frac{c}{2}\{2(3b+a)(a^2+3b^2)-1\},\quad 1=\frac{c}{2}\{(3b+a)-2(a^2+3b^2)^2\},$$

where c, a, b are rational numbers and $c\neq 0$. We note that the values of x_1 and x_3 can be interchanged and values for 1 and x_2 can be interchanged. The problem for finding all values of c, a and b so that x_1, x_2, x_3 would have integral values is very difficult. Four parametric solutions for this equation were given in [3]. We refer to [4, 5, 6] for detailed information about the important equation $ax^3+by^3+cz^3+d=0$, $abcd\neq 0$ where a, b, c, d are integers. $x_1^3+x_3^3=2(d^2+x_2^3)$ is a particular case of the above equation. Though we are unable to find the general solution for this equation we could get two parametric solutions which we give below.

(a) $x_1 = 2tu,\quad x_2 = -t^2,\quad x_3=-2tu,\quad d = t^3$

(b) $x_1 = 12t^2+6ut,\quad x_2 = 12t^2,\quad x_3 = 12t^2-6ut,\quad d = 36ut^2.$

Then we have

(a′) $(4u^3+t^3)^2-(2tu)^3 = (4u^3)^2-(-t^2)^3 = (4u^3-t^3)^2-(-2tu)^3 = 16u^6+t^6$

(b′) $(3u^2t+36ut^2+36t^3)^2-(12t^2+6ut)^3 = (3u^2t+36t^3)^2-(12t^2)^3 =$

$= (3u^2t^2-36ut^2+36t^3)^2-(12t^2-6ut)^3 = 9t^2(u^4-48t^4+24u^2t^2).$

Our above discussion proves the following theorem.

THEOREM 5. *There is an infinite number of k's having three integer solutions in A.P. for y in $y^2-k=x^3$ when the common difference $d=t^3$* or $36ut^2$.

We note that $x_2=x_3$ and $y_2=-y_3$ if $t=2u$ in (a′) and from (b′) we see that we have an infinite number of k's having $(x_1, y_1), (x_1-e, y_1-d)$ and (x_1-2e, y_1-2d) as solutions for $y^2-k=x^3$ where $e=6ut$ and $d=36ut^2$.

We show below the existence of k's having three solutions in A.P. for y where the common difference d is not of the form $t^3, 9t^2$ or $36ut^2$.

(i) $17^2-4^3=10^2-(-5)^3=3^2-(-6)^3=225, d=7,$
(ii) $60^2-15^3=35^2-10^3=10^2-(-5)^3=225, d=25,$
(iii) $269^2-42^3=134^2-27^3=(-1)^2-12^3=-1727, d=135.$

Now we have some problems:

(i) For what other forms of d (i) $N_{3,d}^{(y)}\neq 0$ (ii) $N_{3,d}^{(y)}=\infty$?
(ii) Is it true that $N_{4,d}^{(y)}\geqq 1$ for some $d>1$?

THEOREM 6. $N_{3,d}^{(x,y)}=0.$

PROOF. If $y^2-k=x^3$ possesses three solutions $(x_1, y_1), (x_2, y_1-d)$ and (x_3, y_1-2d) then we have $x_1^3+x_3^3=2(d^2+x_2^3)$. If $x_2=x_1-d$ and $x_3=x_1-2d$ then $x_1^3+(x_1-2d)^3=2[d^2+(x_1-d)^3]$. On simplification we get $2d^2=2d^2(3x_1-3d)$ whence $3x_1=3d+1$ which is impossible mod 3. This proves the theorem.

THEOREM 7. *If there is a k for which $y^2-k=x^3$ has five integer solutions for y in an A.P. with common difference d then* $d\equiv 0, \pm 3 \pmod{21}$.

PROOF. We claim that $N_{5,d\not\equiv 0 \pmod 3}^{(y)}=0$. Suppose that there exists a k for which $y^2-k=x^3$ has solutions given by $(x_1, y_1), (x_2, y_1-d), (x_3, y_1-2d), (x_4, y_1-3d)$ and (x_5, y_1-4d) where $d\not\equiv 0 \pmod 3$. Then we must have

$$x_1^3-2x_2^3+x_3^3 = x_2^3-2x_3^3+x_4^3 = x_3^3-2x_4^3+x_5^3 = 2d^2.$$

Now $d^2\equiv 1, 4$ or $7 \pmod 9$ and cube of an integer is congruent to $0, 1, -1 \pmod 9$. Hence $x_1^3-2x_2^3+x_3^3\equiv x_2^3-2x_3^3+x_4^3\equiv x_3^3-2x_4^3+x_5^3\equiv 2, 5$ or $8 \pmod 9$.

From $x_1^3-2x_2^3+x_3^3\equiv 2 \pmod 9$ we have the following possibilities:

(i) $x_1^3\equiv 1, \quad x_2^3\equiv 0, \quad x_3^3\equiv 1$

(ii) $x_1^3\equiv 0, \quad x_2^3\equiv -1, \quad x_3^3\equiv 0$

(iii) $x_1^3\equiv 1, \quad x_2^3\equiv -1, \quad x_3^3\equiv -1$

(iv) $x_1^3\equiv -1, \quad x_2^3\equiv -1, \quad x_3^3\equiv 1.$

We see that $x_1^3-2x_2^3+x_3^3\equiv x_2^3-2x_3^3+x_4^3\equiv 2 \pmod 9$ is satisfied only for $x_1^3\equiv 1$, $x_2^3\equiv -1, x_3^3\equiv -1, x_4^3\equiv 1$. But then $x_3^3-2x_4^3+x_5^3\equiv 2 \pmod 9$ is impossible.

Again for $8 \pmod 9$ we have the following possibilities

(i) $x_1^3\equiv -1, \quad x_2^3\equiv 0, \quad x_3^3\equiv 0$

(ii) $x_1^3\equiv 0, \quad x_2^3\equiv 0, \quad x_3^3\equiv -1$

(iii) $x_1^3\equiv 1, \quad x_2^3\equiv 1, \quad x_3^3\equiv 0$

(iv) $x_1^3\equiv 0, \quad x_2^3\equiv 1, \quad x_3^3\equiv 1.$

Then $x_1^3-2x_2^3+x_3^3\equiv 8$ and $x_2^3-2x_3^3+x_4^3\equiv 8$ are simultaneously satisfied only for $x_1^3\equiv -1$, $x_2^3\equiv 0, x_3^3\equiv 0$ and $x_4^3\equiv -1$. But then $x_3^3-2x_4^3+x_5^3\equiv 1, 2$ or $3 \pmod 9$ contradicts the fact that $x_3^3-2x_4^3+x_5^3\equiv 8 \pmod 9$. If $x_1^3-2x_2^3+x_3^3\equiv 5 \pmod 9$

then $x_1^3\equiv -1$, $x_2^3\equiv 1$ and $x_3^3\equiv -1$ whence $x_2^3-2x_3^3+x_4^3\equiv 2$, 3 or 4 (mod 9), a contradiction.

In a similar fashion we can show that $N_{5,d}^{(y)}\equiv 0$ for $d\equiv 1, 2, 5$ or 6 (mod 7). Therefore to have a k with five integer solutions for y in A.P. with common difference d we must have $d\equiv 0$ (mod 3) and $d\equiv 0, \pm 3$ (mod 7). Combining these two facts we get $d\equiv 0, \pm 3$ (mod 21).

THEOREM 8. $N_{6,d}^{(y)}=0$ *for every* $d\not\equiv 0$ (mod 21).

PROOF. $N_{5,d}^{(y)}=0$ for $d\equiv 1, 2, 5$ or 6 (mod 7). If $d\equiv \pm 3$ (mod 7) then $d^2\equiv 2$ (mod 7). To have six solutions we must have

$$x_1^3-2x_2^3+x_3^3 \equiv x_2^3-2x_3^3+x_4^3 \equiv x_3^3-2x_4^3+x_5^3 \equiv x_4^3-2x_5^3+x_6^3 \equiv 4 \pmod 7.$$

This is impossible (simultaneously) for all the possibilities of $x_1^3-2x_2^3+x_3^3\equiv 4$ (mod 7). This can be seen from the following

	x_1^3	x_2^3	x_3^3	x_4^3	x_5^3	x_6^3
(i)	1	-1	1	0	X	
(ii)	0	1	-1	1	0	X
(iii)	-1	1	0	X		

Therefore, to have six solutions $d\equiv 0$ (mod 7). Hence the result.

We conclude this paper with the following conjectures.

CONJECTURE 1. $N_{5,d}^{(y)}=0$ for all d.

CONJECTURE 2. $N_{5,d}^{(x)}=0$ for all d.

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A HÁJEK—RÉNYI INEQUALITY FOR *U*-STATISTICS

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1. Introduction

Let $X_1, X_2, \ldots, X_n$ be n independent identically distributed random variables and let $h(x_1, \ldots, x_m)$ be a symmetric function of m arguments. The U-statistics are defined by (see HOEFFDING [3]):

$$U_n = \binom{n}{m}^{-1} \sum_C h(X_{i_1}, \ldots, X_{i_m}) \tag{1.1}$$

where $\sum_C$ extends over all indices $i_1, \ldots, i_m$ such that $1 < i_1 \leqq \ldots < i_m \leqq n$. Many of the well-known statistics such as the sample mean and sample variance are special cases of U-statistics. Many asymptotic properties of U_n are known, e.g., asymptotic normality, consistency, and unbiasedness of U_n as estimator of $Eh(X_1, \ldots, X_m)$ are in HOEFFDING [3] while strong consistency is in HOEFFDING [4], and further other properties are in SPROULE [5] who also gives a Kolmogorov-type inequality for U_n.

The following notations and definitions are needed in the sequel. Let $\theta = Eh(X_1, \ldots, X_m)$ exist, then define

$$h_c(x_1, \ldots, x_c) = E[h(X_1, \ldots, X_m) | X_1 = x_1, \ldots, X_n = x_c], \quad c = 1, 2, \ldots, m \tag{1.2}$$

and define $\eta_c = \text{Var}\,[h_c(X_1, \ldots, X_c)]$, $c = 1, \ldots, m$. Then HOEFFDING [3] showed that

$$\text{Var}\,(U_n) = \binom{n}{m}^{-1} \sum_{c=1}^{m} \binom{m}{c} \binom{n-m}{m-c} \eta_c. \tag{1.3}$$

The following decomposition, attributed to Hoeffding, appears in SPROULE [5]. Let

$$k^{(1)}(x_1) = h_1(x_1) - \theta, \tag{1.4}$$

and for $r = 2, \ldots, m$

$$k^{(r)}(x_1, \ldots, x_r) = f_r(x_1, \ldots, x_r) - \theta - \sum_{j=1}^{v-1} \sum_{C(j)} k^{(j)}(x_{i_1}, \ldots, x_{i_j}), \tag{1.5}$$

where $\sum_{C(j)}$ denotes the summation over all combinations $1 \leqq i_1 < \ldots < i_j \leqq r$. Set

$$V_n^{(r)} = \binom{n}{r}^{-1} \sum_{C(r)} k^{(r)}(X_{i_1}, \ldots, X_{i_r}), \quad r = 1, 2, \ldots, m. \tag{1.6}$$

Then HOEFFDING [3] proved that

$$U_n = \theta + \sum_{r=1}^{m} \binom{m}{r} V_n^{(r)}, \tag{1.7}$$

and that $S_n^{(r)}=\binom{n}{r}V_n^{(r)}$ forms a martingale sequence (in n) for $r=1, 2, \dots, m$. It is this result that we use to obtain a Hájek—Rényi-type inequality for U_n. The result reported here should prove useful in establishing the strong law of large numbers for U_n.

2. Main result

THEOREM. *Let* $E(h(X_1, \dots, X_m))^2<\infty$ *and assume that* $0<\delta_r=\binom{n}{r} E(V_n^{(r)})^2$, $r=1, \dots, m$. *Then for any* $\varepsilon>0$, *and a sequence of nonincreasing constants* $\{C_n\}$,

$$(2.1)\qquad P\left[\max_{m\le\alpha\le n}\binom{\alpha}{m}C_\alpha|U_\alpha-\theta|\geqq\varepsilon\right]\leqq 2^{2m}\varepsilon^{-2}\sum_{r=1}^{m}\delta_r\left[C_r^2+r\sum_{\alpha=r+1}^{n}\binom{\alpha}{r}\alpha^{-1}C_\alpha^2\right].$$

PROOF. We split the proof into two steps; first we prove that for any $\gamma>0$

$$(2.2)\qquad P\left[\max_{r\le\alpha\le n}C_\alpha|S_\alpha^{(r)}|\geqq\gamma\right]\leqq\frac{\delta_r}{\gamma^2}\left[C_r^2+r\sum_{\alpha=r+1}^{n}\binom{\alpha}{r}\alpha^{-1}C_\alpha^2\right].$$

Note that $E(S_n^{(r)})^2=\binom{n}{r}\delta_r$, $r=1, \dots, m$, by HOEFFDING [4]. Since $\{S_n^{(r)}\}_{n=1}^{\infty}$, $r=1, \dots, m$ are martingales; then by the second inequality of CHOW [1] we get

$$(2.3)\qquad P\left[\max_{r\le\alpha\le n}C_\alpha|S_n^{(r)}|\geqq\gamma\right]\leqq\gamma^{-2}\sum_{\alpha=r}^{n-1}(C_\alpha^2-C_{\alpha-1}^2)E(S_\alpha^{(r)})^2+C_n^2E(S)^2=$$

$$=\gamma^{-2}\left[\sum_{\alpha=r}^{n-1}(C_\alpha^2-C_{\alpha+1}^2)\binom{\alpha}{r}\delta_r+C_n^2\binom{n}{r}\delta_r\right]=$$

$$=\frac{\delta_r}{\gamma^2}\left\{\binom{r}{r}C_r^2+\binom{r+1}{r}C_{r+1}^2\left[1-\binom{r}{r}\Big/\binom{r+1}{r}\right]+\dots+\right.$$

$$\left.+\binom{n}{r}C_n^2\left[1-\binom{n-1}{r}\Big/\binom{n}{r}\right]\right\}=\frac{\delta_r}{\gamma^2}\left[\binom{r}{r}C_r^2+\frac{r}{r+1}\binom{r+1}{r}C_{r+1}^2+\dots+\frac{r}{n}\binom{n}{r}C_n^2\right].$$

Hence (2.2) follows from (2.3). In the second step we define

$$A=\left\{\max_{m\le\alpha\le n}C_\alpha\binom{\alpha}{m}|U_\alpha-\theta|\geqq\varepsilon\right\}\quad\text{and}\quad A_r=\left\{\max_{r\le\alpha\le n}C_\alpha|S_\alpha^{(r)}|\geqq\varepsilon_m\right\}$$

with $\varepsilon_m=2^{-m}\varepsilon$, $r=1, 2, \dots, m$. Let further A^c denote the complement of A. If each of $A_1^c, \dots, A_m^c$ occurs, then

$$(2.4)\qquad c_\alpha\binom{\alpha}{m}|U_\alpha-\theta|\leqq c_\alpha\binom{\alpha}{m}\sum_{r=1}^{m}\left[\binom{m}{r}\Big/\binom{\alpha}{m}\right[S_\alpha^{(r)}<(2^{-m}\varepsilon)\sum_{r=1}^{m}\binom{m}{r}=\varepsilon.$$

Hence $A\subset\bigcup_{r=1}^{m}A_r$. Therefore,

$$(2.5)\qquad P(A)\leqq\sum_{r=1}^{m}P(A_r)\leqq\sum_{r=1}^{m}(2^{-m}\varepsilon)^{-2}\delta_r\left[C_r^2+r\sum_{\alpha=r+1}^{n}\binom{\alpha}{r}\alpha^{-1}C_\alpha^2\right].$$

The proof is now complete.

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NECESSARY AND SUFFICIENT CONDITIONS FOR THE CONVERGENCE OF THE EXTENDED HERMITE—FEJÉR INTERPOLATION PROCESS

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1. The polynomial $H_n(f, x)$ of Hermite—Fejér interpolation based on the zeros $x_{kn}=\cos\left(\frac{2k-1}{2n}\pi\right)$, $k=1, \ldots, n$ of the Chebyshev polynomial $T_n(x)= =\cos(n \arc\cos x)$, is defined by

$$H_n(f, x_{kn}) = f(x_{kn}), \quad H_n'(f, x_{kn}) = 0, \quad k = 1, \ldots, n \tag{1.1}$$

or, more explicitly, by

$$H_n(f, x) = \sum_{k=1}^{n} f(x_{kn})(1-xx_{kn})\left(\frac{T_n(x)}{n(x-x_{kn})}\right)^2. \tag{1.2}$$

It was proved by L. FEJÉR [1] in 1916 that for a continuous function f on $[-1, 1]$ $\lim\limits_{n\to\infty} H_n(f, x)=f(x)$ uniformly on $[-1, 1]$. A survey of various quantitative estimates of the rate of convergence can be found in [2], where it was proved that

$$|H_n(f, x)-f(x)| \leqq \frac{C}{n}\sum_{k=1}^{n} w_f\left(\frac{1}{k}\right). \tag{1.3}$$

Here C is a constant and w_f is the modulus of continuity of f defined for $h \geqq 0$ by

$$w_f(h) = \max\{|f(x)-f(y)|:\ x, y\in[-1, 1],\ |x-y| \leqq h\}.$$

R. B. SAXENA [3] improved further the inequality (1.3) by showing that

$$|H_n(f, x)-f(x)| \leqq \frac{C}{n}\sum_{k=1}^{n} w_f\left(\frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\right). \tag{1.4}$$

The simplest modification of the process of Hermite—Fejér interpolation consists in the inclusion of the end points -1 and 1 in the interpolation process. The resulting polynomial $Q_n(f, x)$ is defined by

$$Q_n(f, x_{kn}) = f(x_{kn}), \quad Q_n(f, \pm 1) = f(\pm 1), \quad Q_n'(f, x_{kn}) = 0, \tag{1.5}$$

$k=1, \ldots, n,$ or, more explicitly, by

$$Q_n(f, x) = \left(\frac{1+x}{2}f(1)+\frac{1-x}{2}f(-1)\right)T_n^2(x)+ \tag{1.6}$$

$$+(1-x^2)\sum_{k=1}^{n} f(x_{kn})\frac{1+xx_{kn}-2x_{kn}^2}{1-x_{kn}^2}\left(\frac{T_n(x)}{n(x-x_{kn})}\right)^2.$$

This polynomial, however, is not much different from the polynomial $H_n(f, x)$, as far as its approximation theoretical properties are concerned. In view of the formula

$$Q_n(f, x) = H_n(f, x)+(f(-1)-H_n(f, -1))\frac{1-x}{2}T_n^2(x)+(f(1)-H_n(f, 1))\frac{1+x}{2}T_n^2(x)$$

and inequality (1.4) it is clear that we have

$$(1.7)\qquad |Q_n(f, x)-f(x)| \leqq \frac{3C}{n}\sum_{k=1}^{n} w_f\left(\frac{\sqrt{1-x^2}}{k}+\frac{1}{k^2}\right).$$

A more interesting modification of the process of Hermite—Fejér interpolation was studied by D. L. BERMAN ([4], [5], [6], [7]). Berman considers the polynomial $R_n(f, x)$ defined by

$$(1.8)\qquad \begin{cases} R_n(f, x_{kn}) = f(x_{kn}), & R_n(f, \pm 1) = f(\pm 1) \\ R_n'(f, x_{kn}) = 0, & R_n'(f, \pm 1) = 0, \end{cases}$$

$k=1, \dots, n$. The important discovery of Berman was that the inclusion of the end-points -1 and 1 in the process of Hermite—Fejér interpolation may completely change the nature of the process. He showed in [4], [5] that $(R_n(|t|, 0))$ is a divergent sequence. Later, in [6], [7] he showed that for $f(x)=x^2$ the sequence $(R_n(f, x))$ does not converge to x^2 at any point of $(-1, 1)$ and that a similar result is true for the function $f(x)=x$ with the exception of the point 0.[1]

The study of the properties of the polynomial $R_n(f, x)$ is quite involved since

$$(1.9)\qquad R_n(f, x) = f(1)(1+(2n^2+1)(1-x))\left(\frac{1+x}{2}T_n(x)\right)^2 +$$
$$+f(-1)(1+(2n^2+1)(1+x))\left(\frac{1-x}{2}T_n(x)\right)^2 +$$
$$+\sum_{k=1}^{n} f(x_{kn})\frac{1+3xx_{kn}-4x_{kn}^2}{(1-x_{kn}^2)^2}\left(\frac{(1-x^2)T_n(x)}{n(x-x_{kn})}\right)^2.$$

The aim of this paper is to give a simpler method for the study of the properties of the polynomials $R_n(f, x)$ of the extended Hermite—Fejér interpolation. This method is based on the formula

(1.10)

$$R_n(f, x) = Q_n(f, x)+(1-x)\left(\frac{1+x}{2}T_n(x)\right)^2 Q_n'(f, 1)-(1+x)\left(\frac{1-x}{2}T_n(x)\right)^2 Q_n'(f, -1)$$

and on the fact that $Q_n(f, x)\to f(x)$ $(n\to\infty)$ uniformly on $[-1, 1]$.

[1] It is interesting to observe that Berman's proof of the divergence of $R_n(t^2, x)$ in [7] is based on formula (13) which is incorrect, the correct formula being

$$R_n(1-t^2, \cos\theta) = \sin^2\theta - 3\sin^2\theta\cos^2 n\theta + \frac{\sin 2\theta \sin 2n\theta}{2n}.$$

The error comes from a misprint in the formula corresponding to (1.9). The formula (22) for $x-R_n(t, x)$ is correct. The same remarks apply to Berman's paper [6].

Our first result is an extension of the results Berman proved for the special functions $f(x)=|x|$, $f(x)=x$ and $f(x)=x^2$.

THEOREM 1. *Let f be a continuous function on $[-1, 1]$ and let $R_n(f, x)$ be the polynomial of extended Hermite—Fejér interpolation defined by* (1.8). *If the left and right derivatives $f_L'(1)$, $f_R'(-1)$ exist, then*

$$\limsup_{n\to\infty} |R_n(f, x)-f(x)| = \frac{3}{4}(1-x^2)|(1+x)f_L'(1)-(1-x)f_R'(-1)|. \tag{1.11}$$

The asymptotic relation (1.11) explains not only why $R_n(f, x)$ does not converge to $f(x)$ when $f(x)=|x|$, $f(x)=x$ or $f(x)=x^2$ but it indicates also that for a continuous function f on $[-1, 1]$ which has derivatives at the endpoints -1 and 1 the necessary and sufficient conditions for the convergence of the sequence $(R_n(f, x))$ to $f(x)$ on $(-1, 1)$ are $f_R'(-1)=0$ and $f_L'(1)=0$.

THEOREM 2. *Let f be a continuous function on $[-1, 1]$ such that $f_L'(1), f_R'(-1)$ exist and let $R_n(f, x)$ be defined by* (1.8). *In order that $\lim_{n\to\infty} R_n(f, x)=f(x)$ uniformly on $[-1, 1]$ it is necessary and sufficient that*

$$f_L'(1) = 0 \quad \textit{and} \quad f_R'(-1) = 0. \tag{1.12}$$

As far as arbitrary continuous functions on $[-1, 1]$ are concerned, we have the following result.

THEOREM 3. *Let f be a continuous function on $[-1, 1]$ and let $R_n(f, x)$ be defined by* (1.8). *In order that $\lim_{n\to\infty} R_n(f, x)=f(x)$ uniformly on $[-1, 1]$ it is necessary and sufficient that*

$$\lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n} \frac{f(1)-f(x_{kn})}{(1-x_{kn})^2} = 0 \tag{1.13}$$

and

$$\lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n} \frac{f(-1)-f(x_{kn})}{(1+x_{kn})^2} = 0. \tag{1.14}$$

The rate of convergence of $R_n(f, x)$ to $f(x)$ depends on how fast the sequences

$$A_n(f) = \frac{1}{n^2}\sum_{k=1}^{n} \frac{f(1)-f(x_{kn})}{(1-x_{kn})^2} \tag{1.15}$$

and

$$B_n(f) = \frac{1}{n^2}\sum_{k=1}^{n} \frac{f(-1)-f(x_{kn})}{(1+x_{kn})^2} \tag{1.16}$$

converge to zero.

2. In this section we shall give proofs of Theorems 1, 2 and 3. As we have pointed out earlier, these proofs are based on the relation (1.10). Our first lemma is a transformation of (1.10) into a more suitable form.

LEMMA 1. *If the polynomials $H_n(f,x)$, $Q_n(f,x)$ and $R_n(f,x)$ are defined by* (1.1), (1.5) *and* (1.8) *respectively, then*

$$R_n(f,x) = Q_n(f,x) + \frac{1}{4}(1-x^2)T_n^2(x)\big(H_n(f,1)-f(1)+H_n(f,-1)-f(-1)\big)+$$

$$+\frac{3}{4}(1-x^2)T_n^2(x)\big((1+x)A_n(f)+(1-x)B_n(f)\big)$$

where the sequences $(A_n(f))$ *and* $(B_n(f))$ *are defined by* (1.15) *and* (1.16).

PROOF. It is easy to see, by differentiating (1.6), that

$$Q_n'(f,1) = \left(2n^2+\frac{1}{2}\right)f(1) - \frac{1}{2}f(-1) - \frac{2}{n^2}\sum_{k=1}^{n} f(x_{kn})\frac{1+2x_{kn}}{1+x_{kn}}\frac{1}{(1-x_{kn})^2}.$$

Since

$$\frac{1+2x_{kn}}{1+x_{kn}}\frac{1}{(1-x_{kn})^2} = \frac{3}{2}\frac{1}{(1-x_{kn})^2} - \frac{1}{4}\frac{1}{1-x_{kn}} - \frac{1}{4}\frac{1}{1+x_{kn}}$$

it follows that

$$\frac{1}{n^2}\sum_{k=1}^{n} f(x_{kn})\frac{1+2x_{kn}}{1+x_{kn}}\frac{1}{(1-x_{kn})^2} = \frac{3}{2n^2}\sum_{k=1}^{n}\frac{f(x_{kn})}{(1-x_{kn})^2} - \frac{1}{4}\big(H_n(f,1)+H_n(f,-1)\big).$$

Hence

$$Q_n'(f,1) = \left(2n^2+\frac{1}{2}\right)f(1) - \frac{1}{2}f(-1) - \frac{3}{n^2}\sum_{k=1}^{n}\frac{f(x_{kn})}{(1-x_{kn})^2} + \frac{1}{2}\big(H_n(f,1)+H_n(f,-1)\big)$$

or

$$Q_n'(f,1) = (2n^2+1)f(1) - \frac{3}{n^2}\sum_{k=1}^{n}\frac{f(x_{kn})}{(1-x_{kn})^2} + \frac{1}{2}\big(H_n(f,1)-f(1)+H_n(f,-1)-f(-1)\big).$$

Since

$$\sum_{k=1}^{n}\frac{T_n^2(x)}{(x-x_{kn})^2} = T_n'^2(x) - T_n''(x)T_n(x)$$

it follows that

$$\sum_{k=1}^{n}\frac{1}{(1-x_{kn})^2} = n^4 - \frac{1}{3}n^2(n^2-1) = \frac{2}{3}n^4 + \frac{1}{3}n^2$$

or

$$\frac{3}{n^2}\sum_{k=1}^{n}\frac{1}{(1-x_{kn})^2} = 2n^2+1.$$

Hence

$$Q_n'(f,1) = \frac{3}{n^2}\sum_{k=1}^{n}\frac{f(1)-f(x_{kn})}{(1-x_{kn})^2} + \frac{1}{2}\big(H_n(f,1)-f(1)+H_n(f,-1)-f(-1)\big) \tag{2.1}$$

or

$$Q_n'(f,1) = 3A_n(f) + \frac{1}{2}\big(H_n(f,1)-f(1)+H_n(f,-1)-f(-1)\big).$$

We find likewise that

$$(2.2)\quad Q_n'(f,-1) = -\frac{3}{n^2}\sum_{k=1}^{n}\frac{f(-1)-f(x_{kn})}{(1+x_{kn})^2} - \frac{1}{2}\big(H_n(f,1)-f(1)+H_n(f,-1)-f(-1)\big)$$

or

$$Q_n'(f,-1) = -3B_n(f) - \frac{1}{2}\big(H_n(f,1)-f(1)+H_n(f,-1)-f(-1)\big).$$

A substitution of these expressions for $Q_n'(f,1)$ and $Q_n'(f,-1)$ into (1.10) completes the proof of Lemma 1.

While Lemma 1 shows why conditions (1.13) and (1.14) are necessary and sufficient for the uniform convergence of the sequence $\big(R_n(f,x)\big)$ to $f(x)$ for every continuous function f on $[-1,1]$, the following lemma explains why $f_R'(-1)=0$ and $f_L'(1)=0$ are necessary and sufficient for the uniform convergence of the sequence $\big(R_n(f,x)\big)$ to $f(x)$ for every continuous function f on $[-1,1]$ which has derivatives at the end points -1 and 1.

LEMMA 2. *If $f_L'(1)$ exists, then*

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\frac{f(1)-f(x_{kn})}{(1-x_{kn})^2} = f_L'(1).$$

Likewise, if $f_R'(-1)$ exists, then

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\frac{f(-1)-f(x_{kn})}{(1+x_{kn})^2} = -f_R'(-1).$$

Here, $x_{kn}=\cos\left(\frac{2k-1}{2n}\pi\right)$, $k=1,\dots,n$ are the zeros of $T_n(x)$.

PROOF. It is clearly sufficient to prove only the first of these relations. We note first that

$$\sum_{k=1}^{n}\frac{1}{1-x_{kn}} = \frac{T_n'(1)}{T_n(1)} = n^2.$$

We have, for all $0<1-h<h_\varepsilon<\frac{\pi^2}{2}$, the inequality

$$\left|\frac{f(1)-f(h)}{1-h} - f_L'(1)\right| \leqq \varepsilon.$$

Next, if $\alpha_n=\frac{\sqrt{2h_\varepsilon}}{\pi}n$ we have

$$1-x_{kn} = 1-\cos\left(\frac{2k-1}{2n}\pi\right) = 2\sin^2\left(\frac{2k-1}{2n}\frac{\pi}{2}\right) \leqq \frac{\pi^2}{2}\frac{k^2}{n^2} \leqq h_\varepsilon$$

for $1\leqq k\leqq\frac{\sqrt{2h_\varepsilon}}{\pi}n$. We can now prove the first part of Lemma 2 as follows. We have

$$\Delta_n(f) = \left|\frac{1}{n^2}\sum_{k=1}^{n}\frac{f(1)-f(x_{kn})}{(1-x_{kn})^2} - f'_L(1)\right| = \frac{1}{n^2}\left|\sum_{k=1}^{n}\left(\frac{f(1)-f(x_{kn})}{1-x_{kn}} - f'_L(1)\right)\frac{1}{1-x_{kn}}\right| \leqq$$

$$\leqq \frac{1}{n^2}\Big(\sum_{1\leqq k\leqq \alpha_n} + \sum_{\alpha_n<k\leqq n}\Big)\left|\frac{f(1)-f(x_{kn})}{1-x_{kn}} - f'_L(1)\right|\frac{1}{1-x_{kn}} = S_1+S_2.$$

In view of the definition of α_n we have

$$\left|\frac{f(1)-f(x_{kn})}{1-x_{kn}} - f'_L(1)\right| \leqq \varepsilon$$

for $1\leqq k\leqq \alpha_n$ and so

$$S_1 \leqq \frac{\varepsilon}{n^2}\sum_{1\leqq k\leqq \alpha_n}\frac{1}{1-x_{kn}} \leqq \varepsilon. \tag{2.3}$$

On the other hand, for $\alpha_n<k\leqq n$ we have

$$\left|\frac{f(1)-f(x_{kn})}{1-x_{kn}} - f'_L(1)\right| \leqq \frac{2\|f\|}{1-x_{kn}} + |f'_L(1)|$$

where $\|f\| = \max\{|f(x)|:\ -1\leqq x\leqq 1\}$ and so

$$S_2 \leqq \frac{2\|f\|}{n^2}\sum_{\alpha_n\leqq k\leqq n}\frac{1}{(1-x_{kn})^2} + \frac{|f'_L(1)|}{n^2}\sum_{\alpha_n\leqq k\leqq n}\frac{1}{1-x_{kn}}.$$

Since

$$1-x_{kn} = 2\sin^2\left(\frac{2k-1}{2n}\,\frac{\pi}{2}\right) \geqq 2\left(\frac{2k-1}{2n}\right)^2 \geqq \frac{1}{2}\left(\frac{k}{n}\right)^2$$

and $\alpha_n = \frac{\sqrt{2h_\varepsilon}}{\pi}\,n$, it follows that

$$S_2 \leqq \frac{8\|f\|}{n^2}\sum_{\alpha_n\leqq k\leqq n}\left(\frac{n}{k}\right)^4 + \frac{2|f'_L(1)|}{n^2}\sum_{\alpha_n\leqq k\leqq n}\left(\frac{n}{k}\right)^2 \leqq \tag{2.4}$$

$$\leqq 8n^2\|f\|\sum_{\alpha_n\leqq k<\infty}\frac{1}{k^4} + 2|f'_L(1)|\sum_{\alpha_n\leqq k<\infty}\frac{1}{k^2} \leqq \frac{C(f,\varepsilon)}{n}.$$

From (2.3) and (2.4) follows that for every $\varepsilon>0$ we have $\Delta_n(f)\leqq\varepsilon+\frac{C(f,\varepsilon)}{n}$ and the lemma is proved.

Proof of Theorem 1. Suppose that f is a continuous function on $[-1, 1]$ and that the derivatives $f'_R(-1)$ and $f'_L(1)$ exist. We have, by Lemma 1,

$$R_n(f,x)-f(x) = Q_n(f,x)-f(x)+ \tag{2.5}$$

$$+\frac{1}{4}(1-x^2)T_n^2(x)\big(H_n(f,1)-f(1)+H_n(f,-1)-f(-1)\big)+$$

$$+\frac{3}{4}(1-x^2)T_n^2(x)\big((1+x)A_n(f)+(1-x)B_n(f)\big).$$

Hence

(2.6) $$|R_n(f,x)-f(x)| \leqq \|Q_n(f)-f\| + \frac{1}{2}\|H_n(f)-f\| + \\ + \frac{3}{4}(1-x^2)|(1+x)A_n(f)+(1-x)B_n(f)|$$

and by Lemma 2 it follows that

(2.7) $$\limsup_{n\to\infty} |R_n(f,x)-f(x)| \leqq \frac{3}{4}(1-x^2)|(1+x)f_L'(1)-(1-x)f_R'(-1)|.$$

On the other hand

$$|R_n(f,x)-f(x)| \geqq \frac{3}{4}(1-x^2)T_n^2(x)|(1+x)A_n(f)+(1-x)B_n(f)| - \\ -\|Q_n(f)-f\| - \frac{1}{2}\|H_n(f)-f\|.$$

Now, for each fixed x in $(-1, 1)$ there exists a sequence (n_r) of integers such that $n_r\to\infty$ $(r\to\infty)$ and $T_{n_r}^2(x)\to 1$ $(r\to\infty)$. If we put $x=\cos\theta, 0<\theta<\pi$, then $1-T_n^2(\cos\theta)=1-\cos^2 n\theta=\sin^2 n\theta$ and it is clearly sufficient to find an increasing sequence of integers (n_r) such that $\sin n_r\theta\to 0$ $(r\to\infty)$ for every fixed θ in $\left(0, \frac{\pi}{2}\right)$. Details of the proof of this well-known result are given in [7]. It follows then, by Lemma 2, that

(2.8) $$\limsup_{n\to\infty} |R_n(f,x)-f(x)| \geqq \limsup_{r\to\infty} |R_{n_r}(f,x)-f(x)| \geqq \\ \geqq \frac{3}{4}(1-x^2)|(1+x)f_L'(1)-(1-x)f_R'(-1)|$$

and Theorem 1 is proved in view of (2.7) and (2.8).

Proofs of Theorems 2 and 3. The sufficiency of conditions (1.13) and (1.14) for the uniform convergence of the sequence $(R_n(f,x))$ to $f(x)$ on $[-1, 1]$ for every continuous function f follows clearly from the inequality

$$|R_n(f,x)-f(x)| \leqq \|Q_n(f)-f\| + \frac{1}{2}\|H_n(f)-f\| + \frac{3}{2}(|A_n(f)|+|B_n(f)|)$$

since the first two terms on the right-hand side of this inequality converge to zero. The same inequality, in connection with Lemma 2 shows that $R_n(f,x)\to f(x)$ $(n\to\infty)$ uniformly on $[-1, 1]$ for every continuous function f on $[-1, 1]$ whose derivative vanishes at the points -1 and 1.

To see that conditions (1.13) and (1.14) are necessary suppose that $(R_n(f,x))$ converges uniformly to $f(x)$ on $[-1, 1]$. Let $x_n=\cos\left(\frac{[n/4]}{n}\pi\right)$. We have then $T_n^2(x_n)=1$ and $\lim_{n\to\infty} x_n=\cos(\pi/4)=1/\sqrt{2}$. It follows then from (2.5), with x replaced by x_n, that

$$\lim_{n\to\infty}\left((1+x_n)A_n(f)+(1-x_n)B_n(f)\right)=0.$$

On the other hand, if we replace x_n by $-x_n$, we find that

$$\lim_{n\to\infty}\left((1-x_n)A_n(f)+(1+x_n)B_n(f)\right)=0.$$

The preceding relations can be written also as follows:

$$(1+x_n)A_n(f)+(1-x_n)B_n(f)=\varrho_n,\quad (1-x_n)A_n(f)+(1+x_n)B_n(f)=\sigma_n$$

where $\varrho_n,\sigma_n\to 0$ $(n\to\infty)$. From these relations it follows that

$$A_n(f)=\frac{1}{4x_n}\left((1+x_n)\varrho_n+(1-x_n)\sigma_n\right)\to 0,\quad B_n(f)=\frac{1}{4x_n}\left(-(1-x_n)\varrho_n+(1+x_n)\sigma_n\right)\to 0$$

as $n\to\infty$ and Theorem 2 is proved. In view of this result and Lemma 2 the necessity of conditions (1.12) is obvious.

3. The same method is useful in the process of extended Hermite—Fejér interpolation when only one of the end points -1 or 1 is added to the points $x_{kn}=\cos\left(\frac{2k-1}{2n}\pi\right)$, $k=1,\dots,n$. This process was studied by D. L. Berman [8] and R. B. Saxena [9].

Let $M_n(f,x)$ be the polynomial defined by

$$(3.1)\qquad M_n(f,x_{kn})=f(x_{kn}),\quad M_n'(f,x_{kn})=0\quad M_n(f,1)=f(1),\quad M_n'(f,1)=0.$$

The explicit representation of the polynomial $M_n(f,x)$ is

$$M_n(f,x)=$$

$$=f(1)\left(1+2n^2(1-x)\right)T_n^2(x)+\sum_{k=1}^{n}f(x_{kn})\frac{1-x_{kn}^2+(2+x_{kn})(x-x_{kn})}{(1-x_{kn})^2}\left(\frac{(1-x)T_n(x)}{n(x-x_{kn})}\right)^2.$$

Using this representation Berman proved in [8] that the sequence $\left(M_n(|t|,0)\right)$ is divergent. An error in Berman's proof was corrected by Saxena [9] who showed that $\limsup_{n\to\infty} M_n(|t|,0))=3$.

A comparison of conditions (3.1) and (1.5) shows that we must have

$$M_n(f,x)=Q_n(f,x)+(1-x)T_n^2(x)Q_n'(f,1).$$

Using formula (2.1) we find that

$$M_n(f,x)=Q_n(f,x)+\frac{1}{2}(1-x)T_n^2(x)\left(H_n(f,1)-f(1)+H_n(f,-1)-f(-1)\right)+$$

$$+\frac{3}{n^2}(1-x)T_n^2(x)\sum_{k=1}^{n}\frac{f(1)-f(x_{kn})}{(1-x_{kn})^2}.$$

From this formula, Lemma 2 and the fact that $Q_n(f,x)$ converges uniformly to $f(x)$, it follows immediately that for every continuous function f on $[-1,1]$ for which $f_L'(1)$ exists,

$$\limsup_{n\to\infty}|M_n(f,x)-f(x)|=3(1-x)|f_L'(1)|.$$

It follows also that for such a continuous function we have $\lim_{n\to\infty} M_n(f, x)=f(x)$ uniformly on $[-1, 1]$ if and only if $f_L'(1)=0$. Finally, for an arbitrary continuous function f on $[-1, 1]$, the sequence $(M_n(f, x))$ converges uniformly to $f(x)$ on $[-1, 1]$ if and only if

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} \frac{f(1)-f(x_{kn})}{(1-x_{kn})^2} = 0.$$

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BOOLEAN SKEW ALGEBRAS

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1. Introduction

In this paper we introduce a class of skew lattices which generalizes relatively complemented distributive lattices with a smallest element. A member $(A;\ \wedge, \vee, 0)$ of this class can be considered as an algebra of type (2, 2, 0) satisfying the identities: $a\wedge a=a$, $a\wedge(b\wedge c)=(a\wedge b)\wedge c$, $a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$, $(b\vee c)\wedge a= =(b\wedge a)\vee(c\wedge a)$, $(b\wedge a)\vee a=a$, and the condition: for all $a, b\in A$, there exists $c\in A$ such that $a=(a\wedge b)\vee c$ and $c\wedge b=0$. The element c in this last condition is uniquely determined by a and b and is denoted by $r(a, b)$. In this way, the class gives rise to a variety of algebras $(A;\ \wedge, \vee, r, 0)$ of type (2, 2, 2, 0); we call it the variety of *Boolean skew algebras*. Consequences of our axioms are the identities $a\vee a=a$, $a\vee(b\vee c)=(a\vee b)\vee c$, $a=a\vee(a\wedge b)=(a\wedge b)\vee a=a\wedge(a\vee b)=a\wedge(b\vee a)=(b\vee a)\wedge \wedge a$, $(b\wedge c)\vee a=(b\vee a)\wedge(c\vee a)$, $a\wedge 0=0=0\wedge a$, and $a\vee 0=a=0\vee a$. Thus, we really are considering a class of skew lattices and it turns out that each of the identities: $a\wedge b=b\wedge a$, $a\vee b=b\vee a$, $a=a\vee(b\wedge a)$, $a=(a\vee b)\wedge a$, and $a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c)$, defines the same proper subvariety.

On any Boolean skew algebra, the maps $x\to x\vee a$ and $x\to x\wedge a$ are actually endomorphisms. Using this observation it turns out that, up to isomorphism, there are two subdirectly irreducible Boolean skew algebras, viz. $\mathbf{3}=\{0, 1, 2\colon 1\wedge 2=1, 2\wedge 1=2, 1\vee 2=2, 2\vee 1=1$ are the non-trivial relations$\}$, which is the cogenerator of the variety, and its subalgebra $\mathbf{2}=\{0, 1;\ \wedge, \vee, 0\}$, which is the two element lattice considered as a relatively complemented distributive lattice. Thus, the lattice of subvarieties of Boolean skew algebras is the three-chain.

In the last section of the paper, we show how Boolean skew algebras arise from rings which possess central idempotent covers and from quasiprimal varieties of universal algebras.

2. Fundamentals

A *Boolean skew lattice* is an algebra $(A;\ \wedge, \vee, 0)$ of type (2, 2, 0) satisfying the identities

(2.1) $$a\wedge a = a,$$

(2.2) $$a\wedge(b\wedge c) = (a\wedge b)\wedge c,$$

(2.3) $$a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c),$$

(2.4) $$(b\vee c)\wedge a = (b\wedge a)\vee(c\wedge a),$$

(2.5) $$(b\wedge a)\vee a = a,$$

and the first order sentence

(2.6) for all $a, b \in A$, there exists $c \in A$ such that $a=(a \wedge b) \vee c$ and $c \wedge b=0$.

The laws (2.1) and (2.2) say that $(A;\ \wedge)$ is a *band* (idempotent semigroup). In section 3 we will see that the $\vee$-operation is associative so that a Boolean skew lattice is in effect a special type of semiring, i.e. an algebra $(A;\ \cdot, +)$ such that both $(A;\ \cdot)$ and $(A;\ +)$ are semigroups and $\cdot$ distributes over $+$ from both the left and the right. The skew lattice nature is indicated only by the vital absorption law (2.5). The precise relation of our algebras to skew lattices is considered in the next section; in the present section, we consider the properties of the $\wedge$-operation and the role of the element 0.

PROPOSITION 2.1 *A Boolean skew lattice satisfies the identities:* $a \vee a=a$; $a \wedge 0=0=0 \wedge a$; $a \vee 0=a=0 \vee a$.

PROOF. Because of identities (2.1) and (2.5), $a \vee a=(a \wedge a) \vee a=a$. By (2.6) $a=(a \wedge a) \vee t$ for some t such that $t \wedge a=0$. Hence, $a=a \wedge a=(a \vee t) \wedge a=(a \wedge a) \vee (t \wedge a)$, by 2.4, and so $a=a \vee 0$. Also, $0=(0 \wedge a) \vee z$ for some z such that $z \wedge a=0$. Hence, $0 \wedge a=(z \wedge a) \wedge a=z \wedge a=0$. This leads to $0 \vee a=(0 \wedge a) \vee a=a$ via (2.5). Finally, $a \wedge 0=(a \wedge 0) \vee 0=0$.

Thus, $(A;\ \wedge, 0)$ is a band with 0 as its zero element. In this connection we have the important:

LEMMA 2.2. *In a band $(A;\ \wedge, 0)$ with zero, $a \wedge b=0$ if and only if $b \wedge a=0$. Also, $a \wedge b=0$ implies $a \wedge x \wedge b=0$ for any $x \in A$.*

PROPOSITION 2.3. *For any elements a and b in a Boolean skew lattice $(A;\ \wedge, \vee, 0)$, the element c such that $a=(a \wedge b) \vee c$ and $c \wedge b=0$, arising from* (2.6), *is unique and is denoted by $r(a, b)$.*

PROOF. Suppose $a=(a \wedge b) \vee c=(a \wedge b) \vee d$ and $c \wedge b=0=d \wedge b$. Using (2.3) and Proposition 2.1, and Lemma 2.2, $c \wedge a=(c \wedge a \wedge b) \vee (c \wedge c)=0 \vee c=c$. But $c \wedge a=c \wedge ((a \wedge b) \vee d)=(c \wedge a \wedge b) \vee (c \wedge d)=c \wedge d$. Also, $a \wedge d=(a \wedge b \wedge d) \vee (d \wedge d)=d$ and $a \wedge d=(a \wedge b \wedge d) \vee (c \wedge d)=c \wedge d$. Hence, $c=c \wedge a=c \wedge d=a \wedge d=d$, as required.

Because of the proposition, we can introduce a new binary operation r on any Boolean skew lattice, obtain a variety and yet not affect homomorphisms and congruences. More precisely, a *Boolean skew algebra* $(A;\ \wedge, \vee, r, 0)$ is an algebra of type (2, 2, 2, 0) such that the reduct $(A;\ \wedge, \vee, 0)$ satisfies the identities (2.1)—(2.5) and (2.6) is replaced by the identities

$$(2.6)' \qquad a=(a \wedge b) \vee r(a, b), \quad r(a, b) \wedge b=0.$$

Also, if $(A_1;\ \wedge, \vee, r, 0)$ and $(A_2;\ \wedge, \vee, r, 0)$ are Boolean skew algebras and $f: (A_1;\ \wedge, \vee, 0) \to (A_2;\ \wedge, \vee, 0)$ is a homomorphism between the underlying Boolean skew lattices then for any $a, b \in A_1$, $f(a)=f((a \wedge b) \vee r(a, b))=(f(a) \wedge f(b)) \vee f(r(a, b))$ and $f(r(a, b)) \wedge f(b)=f(r(a, b) \wedge b)=f(0)=0$. By Proposition 2.3, $f(r(a, b))=r(f(a), f(b))$ and so f is a homomorphism of Boolean skew algebras. It follows that if $(A;\ \vee, \wedge, r, 0)$ is a Boolean skew algebra and Θ is a congruence on the underlying Boolean skew lattice $(A;\ \wedge, \vee, 0)$ then the quotient A/Θ is a Boolean skew algebra, Θ has the substitution property for the r-operation, and the associated projection of A onto A/Θ is homomorphism of Boolean skew algebras. These observations will be used whenever we consider congruences and homomorphisms in the variety of Boolean skew algebras, which will be henceforth denoted by **BSA**. For the sake of brevity, we will refer to a Boolean skew algebra as a **BSA**-algebra.

The next result summarizes the most important properties of the $\wedge$-operation.

PROPOSITION 2.4. *Any* **BSA**-*algebra satisfies the identities:*

$$\text{(i) } a\wedge b\wedge a = a\wedge b, \quad \text{(ii) } (a\wedge b)\wedge c = (a\wedge c)\wedge(b\wedge c), \quad \text{(iii) } a\wedge b\wedge c = a\wedge c\wedge b,$$

$$\text{(iv) } c\wedge(a\wedge b) = (c\wedge a)\wedge(c\wedge b), \quad \text{(v) } (a\wedge b)\wedge(c\wedge d) = (a\wedge c)\wedge(b\wedge d).$$

PROOF. (i) $a\wedge b\wedge a=(a\wedge b)\wedge((a\wedge b)\vee r(a, b))=((a\wedge b)\wedge(a\wedge b))\vee((a\wedge b)\wedge r(a, b))= a\wedge b$, by Proposition 2.1 and Lemma 2.2.

(ii) $a=(a\wedge c)\vee r(a, c)$ and so $a\wedge(b\wedge c)=((a\wedge c)\wedge(b\wedge c))\vee(r(a, c)\wedge b\wedge c)=(a\wedge c)\wedge \wedge(b\wedge c)$, by Lemma 2.2.

(iii) $a\wedge b\wedge c=a\wedge b\wedge c\wedge b$ (by (i))$=(a\wedge b)\wedge(c\wedge b)=(a\wedge(c\wedge b))\wedge((b\wedge(c\wedge b))$ (by (ii))$=a\wedge(c\wedge b)\wedge b\wedge(c\wedge b)=a\wedge(c\wedge b)\wedge b$ (by (i))$=a\wedge c\wedge b$.

(iv) $c\wedge(a\wedge b)=c\wedge(b\wedge a)$ (by (iii))$=c\wedge(b\wedge a)\wedge c$ (by (i))$=(c\wedge(c\wedge b))\wedge(a\wedge c)= =(c\wedge a\wedge c)\wedge(c\wedge b)$ (by (iii))$=(c\wedge a)\wedge(c\wedge b)$ (by (i)).

(v) $(a\wedge b)\wedge(c\wedge d)=(a\wedge b\wedge c)\wedge d=(a\wedge c\wedge b)\wedge d$ (by (iii))$=(a\wedge c)\wedge(b\wedge d)$.

It may be worthwhile to make some remarks about the above identities. Firstly, a band $(A;\ \wedge)$ satisfying the identity (iii) of Proposition 2.4 necessarily satisfies (i) and (ii) and hence all of the identities of the proposition. Moreover, it is not hard to see that if the law $a\wedge b\wedge a=a\wedge b$ holds on $(A;\wedge)$ then the laws (ii), (iii) and (v) are equivalent. Bands satisfying (iii) and (v) have been studied extensively by semigroup theorists, see PETRICH [9] for detailed information. The identity $a\wedge b\wedge a=a\wedge b$ holds in any skew lattice $(A;\ \wedge, \vee)$, see for example JORDAN [6] and GERHARDTS [4]; the role of (iii) was also considered by JORDAN in the same paper and GERHARDTS in another paper [5].

3. Skew lattices

Here we will take implicit advantage of the results of the previous section.

PROPOSITION 3.1. *Any* **BSA**-*algebra satisfies the identities:*

$$\text{(i) } a\vee(a\wedge b) = a, \quad \text{(ii) } (a\wedge b)\vee a = a, \quad \text{(iii) } a\wedge(a\vee b) = a, \quad \text{(iv) } a\wedge(b\vee a) = a.$$

PROOF. Clearly (i) and (iii) are equivalent, as are (ii) and (iv).

(iii) $a\wedge(a\vee b) = ((a\wedge b)\vee r(a, b))\wedge(a\vee b) = (a\wedge b\wedge(a\vee b))\vee(r(a, b)\wedge(a\vee b)) =$
$= ((a\wedge b\wedge a)\vee(a\wedge b))\wedge((r(a, b)\wedge a)\vee(r(a, b)\wedge b)) = ((a\wedge b)\wedge(a\wedge b))\vee((r(a, b)\wedge a)\vee$
$\vee 0) = (a\wedge b)\vee(r(a, b)\wedge a) = (a\wedge b\wedge a)\vee(r(a, b)\wedge a) = ((a\wedge b)\vee r(a, b))\wedge a = a\wedge a = a.$

(iv) $a\wedge(b\vee a) = ((a\wedge b)\vee r(a, b))\wedge(b\vee a) = ((a\wedge b)\wedge(b\vee a))\vee(r(a, b)\wedge(b\vee a)) =$
$= (a\wedge b\wedge b)\vee(a\wedge b\wedge a)\vee(r(a, b)\wedge a) = (a\wedge b)\vee((r(a, b)\wedge a) = a,$ as above.

PROPOSITION 3.2. *Any* **BSA**-*ylgebra satisfies the associative law:*

$$a\vee(b\vee c)=(a\vee b)\vee c.$$

PROOF. Let $x=a\vee(b\vee c)$ and $y=(a\vee b)\vee c$. Then, $x\wedge c=(a\wedge c)\vee((b\vee c)\wedge c)=(a\wedge c)\vee c=c$, by repeated application of the identity (2.5). Hence, $x=c\vee r(x,c)$. Also, $y\wedge c=c$, by (2.5) and (2.4) and so $y=c\vee r(y,c)$.

But $x\wedge r(x,c)=r(x,c)$ i.e. $(a\vee(b\vee c))\wedge r(x,c)=r(x,c)$. Expanding and simplifying, we obtain $(a\vee b)\wedge r(x,c)=r(x,c)$.

Also, $a\wedge x=a\wedge(a\vee(b\vee c))=a$ and $b\wedge x=b\wedge(a\vee(b\vee c))=(b\wedge a)\vee b=b$, by the previous proposition. Hence, $(a\vee b)\wedge x=a\vee b$. Using the previous paragraphs, we obtain: $a\vee b=(a\vee b)\wedge x=(a\vee b)\wedge(c\vee r(x,c))=((a\vee b)\wedge c)\vee((a\vee b)\wedge r(x,c))=((a\vee b)\wedge c)\vee r(x,c)$.

On the other hand, $y\wedge r(y,c)=r(y,c)$, i.e. $((a\vee b)\vee c)\wedge r(y,c)=r(y,c)$ and hence $(a\vee b)\wedge r(y,c)=r(y,c)$.

But $(a\vee b)\wedge y=(a\vee b)\wedge((a\vee b)\vee c)=a\vee b$, by Proposition 3.1. Thus $a\vee b=(a\vee b)\wedge y=(a\vee b)\wedge((y\wedge c)\vee r(y,c))=(a\vee b)\wedge(c\vee r(y,c))=((a\vee b)\wedge c)\vee((a\vee b)\wedge r(y,c))=((a\vee b)\wedge c)\vee r(y,c)$.

Hence, $a\vee b=((a\vee b)\wedge c)\vee r(x,c)=((a\vee b)\wedge c)\vee r(y,c)$ and $r(x,c)\wedge c=0=r(y,c)\wedge c$. By Proposition 2.3, $r(x,c)=r(y,c)$. Thus, $x=c\vee r(x,c)=c\vee r(y,c)=y$, as required.

PROPOSITION 3.3. *A* **BSA**-*algebra satisfies the distributive law:*

$$(b\wedge c)\vee a=(b\vee a)\wedge(c\vee a).$$

PROOF. $(b\vee a)\wedge(c\vee a)=(b\wedge(c\vee a))\vee(a\wedge(c\vee a))=((b\wedge c)\vee(b\wedge a))\vee a$ (by Proposition 3.1) $=(b\wedge c)\vee((b\wedge a)\vee a)$ (by Proposition 3.2) $=(b\wedge c)\vee a$ (by identity (2.5).

We are finally in a position to describe **BSA**-algebras as skew lattices.

According to GERHARDTS [4] and SLAVIK [10], [11], a *skew lattice* is an algebra $(A;\ \wedge,\vee)$ of type $(2,2)$ satisfying the identities:

(3.1) $$a\wedge(b\wedge c)=(a\wedge b)\wedge c \quad\text{and}\quad a\vee(b\vee c)=(a\vee b)\vee c,$$

(3.2) $$a\wedge(b\vee a)=a \quad\text{and}\quad (a\wedge b)\vee a=a$$

(3.3) $$a\wedge(a\vee b)=a \quad\text{and}\quad (b\wedge a)\vee a=a.$$

It should be noted that with these identities, the usual lattice-duality between $\wedge$ and $\vee$ is extended by the dualities: $a\wedge b\to b\vee a$ and $a\vee b\to b\wedge a$. This duality was built into the subject by its founder P. JORDAN, see [6] for an extensive bibliography. According to GERHARDTS [4] a skew lattice $(A;\ \wedge,\vee)$ is *distributive* if the dual identities

(3.4) $$(a\vee b)\wedge c=(a\wedge c)\vee(b\wedge c) \quad\text{and}\quad a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c),$$

are satisfied. Also, in [10], [11], SLAVIK gives the necessary and sufficient condition for the reflection (maximal homomorphic image) of a skew lattice in the variety of lattices to be a distributive lattice; it is the satisfaction of the identity $(a\wedge(b\vee c))\wedge((a\wedge b)\vee(a\wedge c))=a\wedge(b\vee c)$. Thus, when it comes to distributivity, it is not clear what the appropriate notion of a "distributive skew lattice" should be; certainly, our identities (2.3) and (2.4) offer an alternative which is consistent with the work of Slavik even if they do not conform with Jordan's inbuilt notion of duality. Bearing in mind (3.1)—(3.3) and our propositions so far, we can summarize as follows:

THEOREM 3.4. *Let* $(A;\ \wedge, \vee, r, 0)$ *be a Boolean skew algebra. Then,* $(A;\ \wedge, \vee)$ *is a skew lattice satisfying:*

(i) *the additional absorption law* $a\wedge(a\vee b)=a$, *and*

(ii) *the distributive laws* $(a\vee b)\wedge c=(a\wedge c)\vee(b\wedge c)$, $a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$, $(a\wedge b)\vee c=(a\vee c)\wedge(b\vee c)$.

Of all possible absorption laws and distributive laws, we are missing: $a\vee(b\wedge a)=a=(a\vee b)\wedge a$ and $a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c)$; we shall decide their occurrence in the next section.

4. Lattice of subvarieties

We begin this section with a technical result; of course, the associativity of the $\vee$-operation is presumed from now on.

LEMMA 4.1. *In any* **BSA**-*algebra,* $a\wedge b=0$ *implies* $a\vee b=b\vee a$. *In particular,* $a=r(a, b)\vee(a\wedge b)$ *is an identity on any* **BSA**-*algebra. More generally,* $b\vee a=a\vee b\vee(a\wedge b)$ *is an identity.*

PROOF. Suppose $a\wedge b=0$. Then, $(a\vee b)\wedge(b\vee a)=((a\vee b)\wedge b)\vee((a\vee b)\wedge a)=b\vee a$ (by identities 2.4, 2.5 and Lemma 2.2). Because of Lemma 2.2, we may validly interchange the roles of a and b to obtain $(b\vee a)\wedge(a\vee b)=a\vee b$. But Proposition 2.4(i) implies that $(a\vee b)\wedge(b\vee a)=(a\vee b)\wedge((b\vee a)\wedge(a\vee b))$, and so $(a\vee b)\wedge(b\vee a)=(a\vee b)\wedge(a\vee b)=a\vee b$. Hence, $a\vee b=b\vee a$.

Now $a=(a\wedge b)\vee r(a, b)$ and so $b\vee a=b\vee((a\wedge b)\vee r(a, b))=b\vee(a\wedge b)\vee r(a, b)$. In addition $a\vee b=((a\wedge b)\vee r(a, b))\vee b=(a\wedge b)\vee b\vee r(a, b)$ (as $r(a, b)\wedge b=0$) $=b\vee r(a, b)$. Hence, $b\vee a=b\vee(r(a, b)\vee(a\wedge b))=(b\vee r(a, b))\vee(a\wedge b)=(a\vee b)\vee(a\wedge b)=a\vee b\vee(a\wedge b)$, as required.

We also need a well-known consequence of Theorem 3.4:

LEMMA 4.2. *In any skew lattice,* $a\vee b=b\vee a\vee b$ *is an identity.*

PROOF. $b\vee a\vee b=b\vee(a\vee b)=(b\wedge(a\vee b))\vee(a\vee b)$ (by identity (3.2), of Proposition 3.1) $=a\vee b$ (by identity (3.3)).

PROPOSITION 4.3. *The following conditions on any two fixed elements* a *and* b *of a* **BSA**-*algebra are equivalent.*

(i) $a\wedge b=b\wedge a$,

(ii) $a\vee b=b\vee a$,

(iii) $a\vee b=a\vee b\vee a$,

(iv) $b\vee a=b\vee a\vee b$,

(v) $a\vee(b\wedge a)=a$ *and* $b\vee(a\wedge b)=b$,

(vi) $(a\vee b)\wedge a=a$ *and* $(b\vee a)\wedge b=b$.

PROOF. (i)⇒(ii) By (i) and Lemma 4.1, $a\vee b=b\vee a\vee(b\wedge a)=b\vee a\vee(a\wedge b)=b\vee(a\vee(a\wedge b))=b\vee a$.

(ii)⇒(iii) By (iii) $a\vee b=a\vee b\vee a$. But $a\vee b\vee a=b\vee a$ due to Lemma 4.2. Hence (ii) holds.

(iv)⇒(ii) follows in a similar fashion.

Of course, (v) and (vi) are equivalent and it is easy to see that (ii) implies (v).

It remains to establish (v)⇒(i). Assume that $a\vee(b\wedge a)=a$ and $b\vee(a\wedge b)=b$. Hence, $(a\vee(b\wedge a))\wedge b=a\wedge b$, i.e. $(a\wedge b)\vee(b\wedge a)=a\wedge b$. Also, $(b\vee(a\wedge b))\wedge a=b\wedge a$, i.e. $(b\wedge a)\vee(a\wedge b)=b\wedge a$. By Lemma 4.2, $a\wedge b=(a\wedge b)\vee(b\wedge a)=((b\wedge a)\vee(a\wedge b))\vee\vee(b\wedge a)=(b\wedge a)\vee(b\wedge a)=(b\wedge a)$, as required.

From this we obtain

THEOREM 4.4. *Each of the following identities defines the same subvariety of the variety* **BSA**

(i) $a\wedge b=b\wedge a$

(ii) $a\vee b=b\vee a$

(iii) $a\vee b=a\vee b\vee a$

(iv) $a\vee(b\wedge a)==a$

(v) $(a\vee b)\wedge a=a$

(vi) $a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c)$

(vii) $a\vee(b\vee c)=(a\vee b)\vee(a\vee c)$.

PROOF. Because of Proposition 4.3, we can assume that (i)—(v) are equivalent.

Of course, (ii)⇒(vii). But (vii) implies (iii). Indeed, in (vii) put $c=0$, to obtain $a\vee b=a\vee b\vee a$.

Due to Proposition 3.3, (ii) implies (vi). But (vi) implies (iv). Indeed, put $c=0$ in (vi) to obtain $a=a\vee 0=(a\vee b)\wedge a$.

The subvariety defined by Theorem 4.4 is nothing more than the variety of generalized Boolean algebras (relatively complemented distributive lattices with 0), wherein the relative complement of $a\wedge b$ in the interval $[0, a]$ is taken as a fundamental operation, namely $r(a, b)$. In this context, we may also describe Boolean algebras.

COROLLARY 4.5. *Let* $(A;\ \wedge, \vee, r, 0)$ *be a* **BSA***-algebra and suppose there exists an element* $1\in A$ *such that* $1\wedge a=a$ *for each* $a\in A$. *Then,* $1\vee a=a$ *for all* $a\in A$, *both* $\wedge$ *and* $\vee$ *are commutative, and* $(A;\ \wedge, \vee, ', 0, 1)$ *is a Boolean algebra, wherein* $a'=r(1, a)$ *and* $r(a, b)=a\wedge b'$ *for all* $a, b\in A$.

PROOF. Firstly, $1\vee a=1\vee(1\wedge a)=1$. Secondly, $a\vee(b\wedge a)=(1\wedge a)\vee(b\wedge a)==(1\vee b)\wedge a=1\wedge a=a$. The remainder follows from Theorem 4.4.

As a contrast to (vii) of Theorem 4.4, we have the following positive consequence of Lemma 4.1.

PROROSITION 4.6. *Any skew lattice satisfies the identity* $(b\vee c)\vee a=(b\vee a)\vee(c\vee a)$.

PROOF. This is an easy consequence of Lemma 4.2.

Combining this proposition with (ii) of Proposition 2.4 and Proposition 3.3, we are led to the following important results.

THEOREM 4.7. *Let a be a fixed element of a* **BSA***-algebra* $(A;\ \wedge, \vee, r, 0)$. *Then, the maps* $x\to x\vee a$ *and* $x\to x\wedge a$ *are endomorphisms of the algebra. Moreover, if*

Θ_a and Ψ_a denote the respective associated congruences, whereby for $x, y \in A$

$$x \equiv y(\Theta_a) \text{ if and only if } x \vee a = y \vee a, \text{ and}$$

$$x \equiv y(\Psi_a) \text{ if and only if } x \wedge a = y \wedge a,$$

then $\Theta_a \cap \Psi_a = \omega$, the smallest congruence on A and $\Theta_a \vee \Psi_a = \iota$, the largest congruence on A.

PROOF. It remains to prove our claims about the congruences.

Let $x, y \in A$ be arbitrary. Then, $x \equiv x \wedge a$ and $y \wedge a \equiv y(\Psi_a)$. Also, $(x \wedge a) \vee a = a = (y \wedge a) \vee a$ so that $x \wedge a \equiv y \wedge a(\Theta_a)$. It follows that $x \equiv y(\Theta_a \vee \Psi_a)$ and $\Theta_a \vee \Psi_a = \iota$.

Let $c, d \in A$ be such that $c \equiv d(\Theta_a \cap \Psi_a)$. In other words $c \vee a = d \vee a$ and $c \wedge a = d \wedge a$. Then $(c \vee a) \wedge r(c, a) = (d \vee a) \wedge r(d, a)$ and so $c \wedge r(c, a) = d \wedge r(c, a)$. But, $c \wedge r(c, a) = r(c, a)$ and so $r(c, a) = d \wedge r(c, a) = ((d \wedge a) \vee r(d, a)) \wedge r(c, a) = r(d, a) \wedge r(c, a)$. Similarly, $r(d, a) = r(c, a) \wedge r(d, a)$. Hence, $r(d, a) = r(c, a) \wedge r(d, a) = (r(d, a) \wedge r(c, a)) \wedge r(d, a) = r(d, a) \wedge r(c, a)$ (by Proposition 2.4(i)) $= r(c, a)$, i.e. $r(c, a) = r(d, a)$. Then, $c = (c \wedge a) \vee r(c, a) = (d \wedge a) \vee r(d, a) = d$. Hence, $\Theta_a \cap \Psi_a = \omega$.

We are now in a position to determine the subdirectly irreducible members of **BSA**. To do this, we introduce an important subclass of **BSA**-algebras and briefly study them.

Let B be any non-empty set and 0 be an element which is not in B. Put $A = B \cup \{0\}$ and endow A with the operations $\wedge$, $\vee$ and r defined as follows:

$$a \wedge b = \begin{cases} a & \text{if } b \neq 0 \\ 0 & \text{if } b = 0, \end{cases} \quad a \vee b = \begin{cases} b & \text{if } b \neq 0 \\ a & \text{if } b = 0, \end{cases} \quad r(a, b) = \begin{cases} 0 & \text{if } b \neq 0 \\ a & \text{if } b = 0. \end{cases}$$

Also, treat 0 as the constant associated with a nullary operation on A. Then, it is readily verifiable that $(A; \wedge, \vee, r, 0)$ is a **BSA**-algebra; it will be called the *smooth* **BSA**-algebra generated by the set $B = A \setminus \{0\}$.

LEMMA 4.8. *Let $(A; \wedge, \vee, r, 0)$ be a smooth* **BSA**-*algebra generated by the set $B = A \setminus \{0\}$. Then,*

i. *for $b_1, b_2 \in B$ with $b_1 \neq b_2$, the smallest congruence on A which identifies b_1 and b_2 is given by $x \equiv y(\Theta(b_1, b_2))(x, y \in A)$ if and only if $x = y$ or $\{x, y\} = \{b_1, b_2\}$; in other words, $\Theta(b_1, b_2)$ is the smallest equivalence relation on A which identifies b_1 and b_2;*

ii. *for $b \in B$, i.e. $b \neq 0$, the smallest congruence on A which identifies b and 0 is $\Theta(b, 0) = \iota$;*

iii. *the congruence lattice of A is isomorphic to the lattice of equivalence relations of the set $B = A \setminus \{0\}$, together with a new largest element $1 = \Theta(b, 0)$ for any $b \in B$, adjoined.*

PROOF. (i) An examination of the possibilities shows that whenever $x \equiv y(\Theta(b_1, b_2))$, where $\Theta(b_1, b_2)$ is as claimed, $x \wedge t \equiv y \wedge t$, $t \wedge x \equiv t \wedge y$, $x \vee t \equiv y \vee t$ and $t \vee x \equiv t \vee y(\Theta(b_1, b_2))$ for any $t \in A$. It follows that $\Theta(b_1, b_2)$ is a congruence, and it must have the desired properties.

(ii) Let $a \in A$ and $b \neq 0$. As $b \equiv 0(\Theta(b, 0))$, $a = a \wedge b \equiv a \wedge 0 = 0(\Theta(b, 0))$. It follows that $\Theta(b, 0) = \iota$.

(iii) is an immediate consequence of (i) and (ii).

Because of Theorem 1 of WHITMAN [12], any lattice is isomorphic to a sublattice of the lattice of equivalence relations on a suitable chosen set. Hence, part (iii) of Lemma 4.8 shows

COROLLARY 4.9. *The congruence lattices of all (smooth)* **BSA**-*algebras do not satisfy any particular lattice-identity.*

On the other hand, we have

THEOREM 4.10. *Up to isomorphism, the only subdirectly irreducible* **BSA**-*algebras are the two-element and three-element smooth algebras* **2** *and* **3**, *described in Section* 1.

PROOF. Suppose $(A;\ \wedge, \vee, r, 0)$ is subdirectly irreducible. Let $b\in A$ be such that $b\neq 0$. As $0\vee b=b=b\vee b$, $0\equiv b(\Theta_b)$ and so $\Theta_b\neq\omega$. Because of Theorem 4.7, $\Psi_b=\omega$. But for any $a\in A$, $a\wedge b\equiv a(\Psi_b)$ and consequently $a\wedge b=a$. By absorption, $a\vee b=(a\wedge b)\vee b=b$. Also, $r(a, b)\wedge b=0$, so $r(a, b)=0$. When $b=0$, the results $a\wedge b=0=b$, $a\vee b=a\vee 0=a$ and $a=(a\wedge b)\vee r(a, b)=0\vee r(a, b)=r(a, b)$ are forced. Hence, $(A;\ \wedge, \vee, r, 0)$ is smooth and generated by $B=A\setminus\{0\}$; B is not empty as $(A;\ \wedge, \vee, r, 0)$ is subdirectly irreducible and so A has at least two elements. But, if B possessed at least three distinct elements b_1, b_2 and b_3, Lemma 4.8 (i) would produce the impossibility $\Theta(b_1, b_2)\wedge\Theta(b_1, b_3)=\omega$, yet $\Theta(b_1, b_2)\neq\omega\neq\Theta(b_1, b_3)$. Thus, there are at most two non-zero elements; an easy computation shows that **2** and **3** are subdirectly irreducible.

COROLLARY 4.11. *The lattice of varieties of* **BSA**-*algebras is the three-chain. The only non-trivial variety of* **BSA**-*algebras is the variety of generalised Boolean algebras, which is described by any of the identities of Theorem* 4.4.

5. The occurrence of Boolean skew lattices

(1) *Rings with a central idempotent covers*

Let R be an associative ring. Let $E(R)$ be its generalized Boolean algebra of central idempotents. The order on $E(R)$ is given by: $e\leqq f$ $(e, f\in E(R))$ if and only if $e=ef$. The infimum and supremum of $e, f\in E(R)$ are $e\wedge f=ef$ and $e\vee f=e+f-ef$, respectively. Moreover, $r(e, f)=e-ef$ for any $e, f\in E(R)$. An element $e\in E(R)$ is called a central idempotent cover of $a\in R$ if $a=ae$ and e is the smallest element in $E(R)$ with this property, i.e. if $a=af$ for $f\in E(R)$ also, then $e\leqq f$. A ring R is a *ring with central idempotent covers,* or more briefly a *C-ring* if each element $a\in R$ possesses a central idempotent cover denoted by $C(a)$. This class of rings was briefly considered by the author in Section 4.1.2 of [3]. However, PENNING [8] seems to have been the first to have explicitly discussed these rings; they are his "minimal duplicator rings".

LEMMA 5.1. *In any C-ring R both* $C(C(a)b)=C(a)C(b)$ *and* $C(a+b-aC(b))=C(a)\vee C(b)(=C(a)+C(b)-C(a)C(b))$ *hold for any* $a, b\in R$.

PROOF. The first assertion is well known and vital to the study of C-rings; it is Lemma 2.13 of PENNING [8]. However, we will include a proof.

Firstly, if $e\in E(R)$ and $xe=0$ then $C(x)e=0$. Indeed, $x\big(C(x)-C(x)e\big)=x$ and $C(x)-C(x)e\in E(R)$ and so $C(x)\leqq C(x)-C(x)e$, i.e. $C(x)\big(C(x)-C(x)e\big)==C(x)-C(x)e$, and so $C(x)e=0$.

Secondly, let $x\in R$ and $e\in E(R)$ be arbitrary. Then, $x\big(e-eC(ex)\big)=0$ and $e-C(ex)\in E(R)$. From the previous paragraph, we can infer that $C(x)\big(e-eC(ex)\big)=0$. Hence, $C(x)e=eC(ex)$. But $e\in E(R)$ and $(ex)e=ex$ so $C(ex)\leqq e$. Hence, $C(x)e==eC(ex)=C(ex)$. Finally, for any $a, b\in R, C(a)\in E(R)$ and so $C\big(C(a)b\big)==C\big(C(a)\big)C(b)=C(a)C(b)$, as required.

We now turn to the second identity. Now, $\big(a+b-aC(b)\big)\big(C(a)\vee C(b)\big)==a\big(C(a)\vee C(b)\big)+b\big(C(a)\vee C(b)\big)-aC(b)\big(C(a)\vee C(b)\big)=a+b-aC(b)$ since $C(a)$, $C(b)\leqq C(a)\vee C(b)$. Hence, $C\big(a+b-aC(b)\big)\leqq C(a)\vee C(b)$.

On the other hand, let $e\in E(R)$ be any central idempotent such that $\big(a+b-aC(b)\big)e=a+b-aC(b)$. Multiply both sides by $C(b)$ and simplify to obtain $be=b$. Then, we must have $C(b)e=C(b)$. But $ae+be-aC(b)e=a+b-aC(b)$. Hence, $ae+b-aC(b)=a+b-aC(b)$ and so $a=ae$ and $C(a)\leqq e$. But we already know that $C(b)\leqq e$. Hence, $C(a)\vee C(b)\leqq e$. It now follows that $C\big(a+b-aC(b)\big)==C(a)\vee C(b)$.

Using the first identity of Lemma 5.1 it is not hard to see that by introducing a new unary operation C, it is possible to turn the class of C-rings into a quasivariety of algebras. It is the quasivariety **CR** of algebras $(R;\ +,\ -,\ \cdot, 0, c)$ of type (2, 1, 2, 0, 1), whose defining relations are

(i) the identities saying that the reduct $(R;\ +,\ -,\ \cdot, 0)$ is a ring,

(ii) the identities $C(a)b=bC(a)$, $aC(a)=a, C\big(C(a)b\big)=C(a)C(b)$, and

(iii) the quasi-identity (universal Horn sentence)

$$(a^2 = a)\&(ab = ba) \Rightarrow a = C(a).$$

We now turn to the relationship with Boolean skew algebra.

THEOREM 5.2. *Let* $(R;\ +,\ -,\ \cdot, 0, C)$ *be member of the quasivariety* **CR** *of* C*-rings. Then, the algebra* $(R;\ \wedge, \vee, r, 0)$, *whose operations are defined by:*

$$a\wedge b = aC(b),\ a\vee b = a+b-aC(b),\ r(a, b) = a-aC(b)$$

is a Boolean skew algebra. Moreover, the map $a\mapsto C(a)$ *is a* **BSA**-*retraction of* $(R;\ \wedge, \vee, r, \mathbf{0})$ *onto the generalized Boolean algebra* $\big(E(R);\ \wedge, \vee, r, 0\big)$ *of central idempotents of the ring* $(R;\ +,\ -,\ \cdot, 0)$.

PROOF. Identity (2.1) holds as $aC(a)=a$. Identity (2.5) holds because $C(a)^2==C(a)$. Identities (2.2) and (2.4) are easy consequences of $C\big(C(a)b\big)=C(a)C(b)$. Identity (2.3) holds because of $C(a\vee b)=C(a)\vee C(b)$ (the second identity of Lemma 5.1). As $C(b)\ \big(a-aC(b)\big)=0, C(b)\,C\big(a-aC(b)\big)=0$. Hence $(a\wedge b)C\big(r(a, b)\big)==aC(b)C\big(a-aC(b)\big)=0$. It follows that $a=(a\wedge b)\vee r(a, b)$. Of course, $r(a, b)\wedge b==\big(a-aC(b)\big)C(b)=0$, so the identity (2.6) also holds. The final assertion is clear.

Of course, Theorem 5.2 yields a faithful functor $\mathscr{F}$: **CR**$\to$**BSA** which preserves products. If Z_2 and Z_3 denote the fields with two and three elements, respectively, viewed as **CR**-algebras then the subdirectly irreducible Boolean skew algebras are $\mathbf{2}=F(Z_2)$ and $\mathbf{3}=F(Z_3)$. Hence, Theorem 4.10 says that each **BSA**-algebra is isomorphic to a **BSA**-subalgebra of $F(R)$ for some suitable C-ring R.

(2) *Quasiprimal algebras*

Let A be a non-empty set. Then, the functions $t\colon A^3 \to A$ and $q\colon A^4 \to A$, defined by

$$t(a, b, c) = \begin{cases} a & \text{if} \quad a \neq b \\ c & \text{if} \quad a = b, \end{cases} \qquad q(a, b, c, d) = \begin{cases} c & \text{if} \quad a = b \\ d & \text{if} \quad a \neq b \end{cases}$$

are respectively called the *ternary* and *quaternary discriminators* on A. These functions are related by

$$t(a, b, c) = q(a, b, c, a) \text{ and } q(a, b, c, d) = t(t(a, b, c), t(a, b, d), d).$$

A universal algebra A is called *quasiprimal* if it is finite, not trivial and the ternary (quaternary) discriminator on the underlying set is a polynomial over A. A variety **V** of universal algebras is called quasiprimal if it is generated by a finite set of quasiprimal algebras such the ternary (quaternary) discriminator is represented by a common polynomial on each of these generators. Quasiprimal varieties abound; an excellent survey is contained in BULMAN-FLEMING and WERNER [2].

Let A be a set with at least two elements, 0 be any element of A and $B = A \setminus \{0\}$. In terms of the discriminator functions on A, we may define binary operations by

$$a \wedge b = q(0, b, 0, a) = t(0, t(0, b, a), a), \quad a \vee b = q(0, b, a, b) = t(b, 0, a)$$

and

$$r(a, b) = q(0, b, a, 0) = t(0, b, a).$$

Then, the resulting algebra $(A;\ \wedge, \vee, r, 0)$ is nothing more than the smooth **BSA**-algebra generated by B. From this it follows that there is a faithful functor from any quasiprimal variety into the variety of Boolean skew algebras.

It should be mentioned that on page 64 of [7], KEIMEL and WERNER define the derived operations $\wedge$, $\vee$, and r (their notation for $r(a, b)$ is $a \setminus b$) on any algebra in a quasiprimal variety. Thus, our remarks provide a characterization of their derived algebra.

On any non-empty set A it is possible to define other functions of interest besides t, q, $\wedge$, $\vee$, and r. Some authors, e.g. KEIMEL and WERNER [7] and BIGNALL [1] prefer to replace t by the function $d\colon A^3 \to A$, defined by

$$d(a, b, c) = \begin{cases} c & \text{if} \quad b \neq c \\ a & \text{if} \quad b = c. \end{cases}$$

It is a matter of choice; the two functions are related by $t(a, b, c) = d(c, b, a)$. In [1], BIGNALL introduced the function $/\colon A^2 \to A$ given by

$$a/b = d(0, a, b) = q(a, b, 0, b) = \begin{cases} 0 & \text{if} \quad a = b \\ b & \text{if} \quad a \neq b. \end{cases}$$

Here, as before, 0 is a fixed element of A. Of course, each of $\wedge$, $\vee$, r and $/$ are given in terms of d (and t and q). The importance of $/$ is that d can be put in terms of $\wedge$, $\vee$ and $/$. Bignall's equation is $d(a, b, c) = ((a \wedge b)/a) \vee (a \wedge c) \vee (b/c)$. Notice also that $r(a, b) = b/(a \vee b)$. In this way Bignall showed that the variety of algebras

$(B: d, 0)$ of type $(3, 0)$ generated by any algebra $(A; d, 0)$ where A is an infinite set and d is the above discriminator on A is definitionally equivalent to a variety (his variety of quasi-Boolean skew lattices or **QBSL**'s) of algebras $(B; \vee, \wedge, /, 0)$ of type $(2, 2, 2, 0)$, where $(B; \wedge, \vee)$ is a certain skew lattice. His work has important applications to quasiprimal varieties and is to be published elsewhere. His equational base for **QBSL** contains twelve identities which we will not state explicitly. As each **QBSL** yields a derived Boolean skew algebra, some of his axioms are redundant, for example it follows that there is no need to postulate the associativity of the $\vee$-operation. In this way, we obtain applications of our work, which needless to say was greatly inspired by my student Bignall.

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ON COMMUTATIVITY OF PERIODIC RINGS AND NEAR-RINGS

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A ring or near-ring R is called *periodic* if for each $x \in R$, there exist distinct positive integers n, m for which $x^n = x^m$. This paper continues the study of commutativity in such rings and near-rings which was previously undertaken in [1]—[5].

The first section is devoted to new proofs of two basic results — Chacron's result that co-algebraic rings are necessarily periodic, and Herstein's result that periodic rings with central nilpotent elements are commutative. The next section deals with commutativity of periodic rings with constraints on certain commutators involving nilpotent elements, and the third is a discussion of structure and commutativity of certain of the δ-rings introduced by PUTCHA and YAQUB in [17]. The final section presents related results on commutativity of periodic near-rings.

Throughout the paper Z will denote the ring of integers, Z^+ the set of positive integers, and $Z[X]$ the ring of polynomials in one indeterminate with integer coefficients. The (multiplicative) center of R will be denoted by C, the set of nilpotent elements by N, and the additive group by $(R, +)$. For each subset S of R, the left, right and two-sided annihilators will be denoted respectively by $A_l(S)$, $A_r(S)$, and $A(S)$; the subring or sub-near-ring generated by S will be denoted by $\langle S \rangle$.

As usual, the symbol $[x, y]$ will stand for the commutator $xy - yx$; and the symbol $\mathscr{C}(R)$ will indicate the commutator ideal of R. Generalized commutators are defined by letting $[x, y]_1 = [x, y]$ and extending inductively by the formula $[x, y]_n = \big[[x, y]_{n-1}, y\big]$, $n \geqq 2$.

We shall make frequent use of the following properties of periodic rings [4, Lemma 1]:

(P_1) For each $x \in R$, some power of x is idempotent.

(P_2) For each $x \in R$, there exists an integer $n(x) > 1$ for which $x - x^{n(x)} \in N$.

(P_3) Each $x \in R$ can be expressed in the form $y + w$, where $w \in N$ and $y^n = y$ for some $n = n(y) > 1$.

(P_4) If I is an ideal of R and $x + I$ is a non-zero nilpotent element of R/I, then R contains a nilpotent element u such that $u \equiv x \pmod I$.

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1. Two basic theorems

Our first theorem, due to CHACRON [6], is a useful one which deserves to be better known. The proof resembles that in [7], which appears to be incomplete.

THEOREM 1. *Let R be an associative ring; and suppose that for each $x \in R$, there exists $n = n(x) \in Z^+$ and $p_x(X) \in Z[X]$ for which $x^n = x^{n+1} p_x(x)$. Then R is periodic.*

PROOF. Let x be an arbitrary element of R, and assume without loss of generality that x generates R as a ring. Choose $n \in Z^+$ and $p(X) \in Z[X]$ such that $x^n = x^{n+1}p(x)$; and note that $x - x^2 p(x) \in A(x^{n-1})$. Letting $\bar{R} = R/A(x^{n-1})$ and $\bar{x}$ the canonical image of x in $\bar{R}$, we see that $\bar{x} = \bar{x}^2 p(\bar{x})$, so that $\bar{e} = \bar{x}p(\bar{x})$ is idempotent and $\bar{x} = \bar{x}\bar{e}$. If $\bar{e} = \bar{0}$, then $\bar{x} = \bar{0}$ and $x^n = 0$, so there is nothing more that need be proved about x. If $\bar{e}$ has infinite additive order in $\bar{R}$, then $\bar{R}$ contains Z as a subring — a contradiction, since Z does not satisfy our hypotheses. Thus, $\bar{e}$ has finite additive order, and so does $\bar{x}$.

Let k be the order of $\bar{x}$. Then $k\bar{R} = \bar{0}$; and if $\bar{N}$ is the ideal of nilpotent elements of $\bar{R}$, the factor ring $\tilde{R} = \bar{R}/\bar{N}$ has all its elements of square-free order, hence is a finite direct sum $\tilde{I}_1 \oplus \ldots \oplus \tilde{I}_t$, where each $\tilde{I}_i$ has prime characteristic and satisfies our original hypotheses. Thus if $\tilde{x}$ is the image of $\bar{x}$ in $\tilde{R}$ and $\tilde{x} = \tilde{x}_1 + \ldots + \tilde{x}_t$ is its direct sum decomposition, each $\tilde{x}_i$ is algebraic over Z (in the usual sense of satisfying a polynomial equation with leading coefficient 1) and hence generates a finite subring of $\tilde{I}_i$. It follows that $\tilde{x}$ generates a finite subring of $\tilde{R}$, so that there exist distinct $n, m \in Z^+$ for which $\tilde{x}^n = \tilde{x}^m$ — that is, $\bar{x}^n - \bar{x}^m \in \bar{N}$. But this forces $\bar{x}$ to be algebraic over Z, so that $\bar{x}$ generates a finite subring of $\bar{R}$. Consequently, there exist distinct $j, k \in Z^+$ such that $x^j - x^k \in A(x^{n-1})$ or $x^{j+n-1} = x^{k+n-1}$.

Our second theorem, due to HERSTEIN [13], is well-known. The proof we include is simple, and is the first proof of this theorem which does not require transfinite methods.

THEOREM 2. *If R is a periodic ring all of whose nilpotent elements are central, then R is commutative.*

PROOF. Let e be any idempotent element of R. For each $x \in R$, $ex - exe$ and $xe - exe$ are both nilpotent, hence central; and it follows that e must be central.

The set N is clearly an ideal, and by (P_2) the factor ring $\bar{R} = R/N$ has the $a^n = a$ property of Jacobson. Therefore $\bar{R}$ is commutative by Jacobson's "$a^n = a$ theorem" (which has a number of non-transfinite proofs — e.g. [15]); hence $\mathscr{C}(R) \subseteq N \subseteq C$. Thus, for each $x, y \in R$ and $k \in Z^+$ we have

$$[x^k, y] = kx^{k-1}[x, y]. \tag{1}$$

Suppose temporarily that R has 1. Since there exist $m, n \in Z^+$ for which $(1+1)^n = (1+1)^m$, there must be $q \in Z^+$ for which $qR = 0$, hence R may be expressed as a finite direct sum of primary components, each of bounded additive order. Each of these components inherits our original hypothesis, hence we may assume that $p^k R = 0$ for some prime p.

Let $x \in R$ and $\bar{x} = x + N \in R/N$. Then the subring of $\bar{R}$ generated by $\bar{x}$ is finite without nilpotent elements, hence a direct sum of finite fields $F_1, \ldots, F_n$ of orders $p^{\alpha_1}, \ldots, p^{\alpha_n}$. Let $\bar{x} = \bar{x}_1 + \ldots + \bar{x}_n$ be the direct-sum decomposition of $\bar{x}$. For fixed

$i\in\{1, \dots, n\}$, choose m such that $m\alpha_i \geqq k$ and let $t=p^{m\alpha_i}$. Since $\bar{x}_i^t=\bar{x}_i$, we have $x_i^t-x_i\in N\subseteqq C$ for any pre-image x_i of $\bar{x}_i$; moreover, by (1), $[x_i^t, y]=tx_i^{t-1}[x_i, y]=0$ for each $y\in R$. Therefore, $x_i\in C$ for $i=1, 2, \dots, n$; and since $x=x_1+\dots+x_n+u$ for some $u\in N$, $x\in C$ as well. This finishes the case of R with 1.

Now consider the general case. For each idempotent $e\in R$, eR has an identity, hence is commutative; thus $e[x, y]=0$ for all $x, y\in R$ and all idempotents $e\in R$. Moreover, if $u\in N$, we have $u[x, y]=[ux, y]=0$. Now by property (P_3) each $x\in R$ can be written in form $w+u$, where $u\in N$ and $w^k=w$ for some $k\in Z^+$; and since w^{k-1} is idempotent, we now have $x[y, z]=0$ for all $x, y, z\in R$. In particular, $x[x, y]=0$ for all $x, y\in R$, and it follows from (1) that $x^k\in C$ for all $k\geqq 2$. Recalling that for each $x\in R$, there exists $k(x)>1$ such that $x^{k(x)}-x\in N\subseteqq C$, we have $x\in C$ and our proof is complete.

2. Periodic rings with restrictions on certain commutators

In [2], we presented a variety of restrictions on nilpotent elements which imply some measure of commutativity in periodic rings. The theorems of this section have similar character.

THEOREM 3. *Let R be a periodic ring with the property that for each $x\in R$ and $u\in N$, there is a positive integer $n=n(x, u)>1$ such that $[u, x]^n=[u, x]$. Then R is commutative.*

PROOF. It is immediate that if $x\in R$ and $u\in N$ are such that either of ux and xu is 0, so is the other; and it follows that R has the *modified insertion-of-factors property* (MIFP) — that is, if at least one of x, y is in N and $xy=0$, then $xry=0$ for all $r\in R$. But MIFP yields the result that if $u^j=v^k=0$, then any product of ring elements having at least j factors of u or k factors of v must be trivial; therefore, $u-v\in N$, and both ux and xu are in N for arbitrary $x\in R$. We now have $[u, x]\in N$ for all $u\in N$ and $x\in R$; and since $[u, x]=[u, x]^n$ for some $n>1$, $N\subseteqq C$ and R is commutative by Theorem 2.

Theorem 3, which is motivated by HERSTEIN's well-known theorem in [14], has the following extension.

THEOREM 4. *Let R be a periodic ring. Suppose that for each nilpotent element u and zero divisor d, there exists a positive integer $n=n(u, d)>1$ such that $[u, d]^n=[u, d]$. Then N is an ideal; and either R is commutative, or R has 1 and R/N is a field.*

PROOF. If R is a nil ring, it is obviously commutative; thus, by property (P_1), we may assume that R contains non-zero idempotents. Suppose that there exists an idempotent zero divisor $e\neq 0$. Then for arbitrary $x\in R$, $(ex-exe)^2=0$; and since $e(ex-exe)-(ex-exe)e=ex-exe=(ex-exe)^n$ for some $n>1$, we have $ex-exe=0$. Similarly, $xe-exe=0$, so e must be central; and $R=eR\oplus A(e)$. Each of eR and $A(e)$ consists of zero divisors in R, hence each satisfies the hypotheses of Theorem 3; therefore R must be commutative.

On the other hand, if no non-zero idempotent is a zero divisor, the condition that $e(x-ex)=(x-xe)e=0$ for each idempotent e and each $x\in R$ guarantees that

R has 1, which is the unique non-zero idempotent. It follows that every non-nilpotent element is invertible. It is now clear that for $x \in R$ and $u \in N$, one of xu and ux is 0 if and only if the other is; and we argue as in the proof of Theorem 3 that N is an ideal. The factor ring R/N is commutative by the "$a^n = a$ theorem"; and since every non-zero element is invertible, it must be a field.

In view of the results in [2], it is natural to inquire whether the conclusion of Theorem 4 or the conclusion that $\mathscr{C}(R)$ is nil will follow if we merely assume

$$[u, v]^{n(u,v)} = [u, v] \tag{2}$$

for all nilpotents u, v and appropriate integers $n(u, v) > 1$. That the answer is negative is clear from examining $M_2(GF(2))$, which has the property that $[u, v] = 1$ for all distinct $u, v \in N$; moreover, as a more detailed analysis shows, $M_2(F)$ satisfies (2) for all finite fields F. However, we can identify two modifications of the hypotheses of Theorem 3 which yield positive results.

THEOREM 5. *Let R be a periodic ring with the property that for each $u \in N$ and $x \in R$, there exist positive integers $j \geqq 1$ and $k > 1$ such that $([u, x]_j)^k = [u, x]_j$. Then $\mathscr{C}(R)$ is nil, and the nilpotent elements form an ideal.*

PROOF. Let $J(R)$ denote the Jacobson radical, which in a periodic ring must be a nil ideal; and note that by (P_4), $\bar{R} = R/J(R)$ inherits the original hypothesis. Thus, it will suffice to establish commutativity under the additional hypothesis that $J(R) = 0$, hence under the assumption that R is primitive.

If R is such a primitive ring, either R is a division ring, or for some division ring Δ, the ring of 2×2 matrices over Δ must also satisfy our hypotheses. Take $u = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; then for every $j \geqq 1$, $[u, x]_j = (-1)^j u$, so that for each $n > 1$, we have $0 = ([u, x]_j)^n \neq [u, x]_j$. Therefore R must be a periodic division ring, hence be commutative by the "$a^n = a$ theorem."

THEOREM 6. *Let R be a periodic ring; and suppose that for each $u \in N$, $A_r(u) = A_l(u)$. Then $\mathscr{C}(R)$ is nil and the nilpotent elements form an ideal.*

PROOF. It follows as in the proof of Theorem 3, that in any R satisfying our hypothesis,

$$[u, x] \in N \quad \text{for all} \quad u \in N \quad \text{and} \quad x \in R. \tag{3}$$

Now (3) persists under the taking of subrings and homorphic images, so as in the proof of Theorem 5, we need only establish that primitive rings R satisfying (3) are division rings. But this is clear once we note that if $u = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $x = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, then $[u, x] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \notin N$. The proof of Theorem 6 is now complete.

3. On certain rings of Putcha and Yaqub

PUTCHA and YAQUB in [17] call a ring R a δ-ring if for some finite subset S of R, each $x \in R$ has the property that $x - x^2 p_x(x) \in S$ for some polynomial $p_x(X) \in Z[X]$. They prove that a semi-simple δ-ring is a subdirect sum of a commutative ring and a finite ring, and ask whether arbitrary δ-rings have similar structure. CHANDRAN [8] has recently announced that, in general, the answer is negative; however, positive results are obtainable if the elements of S are assumed to commute with each other. We shall call a δ-ring with this additional property a δ_*-ring.

THEOREM 7. *Every δ_*-ring is periodic, and can be expressed as a subdirect product of periodic fields and finite rings of bounded order.*

PROOF. The periodicity of δ_*-rings (indeed of arbitrary δ-rings) follows from Lemma 2 of [17] and our Theorem 1. Moreover, a slight modification of the proof of Theorem 1 of [17] shows that any δ_*-ring has only a finite number of nilpotent elements, which must commute with each other. Thus, by Theorem 1 of [2], the commutator ideal $\mathscr{C}(R)$ must be nil and hence R contains only finitely many commutators. We shall therefore say that R has the *finite-commutator property* (FCP).

Write R as a subdirect product of a family $\{R_\alpha\}$ of subdirectly irreducible images of R, each of which is a δ_*-ring. The crucial property of subdirectly irreducible rings for our purposes is that any non-zero central idempotent must be a multiplicative identity element.

Let T be any of the R_α. If $N=\{0\}$, then T is commutative by (P_2) and Jacobson's "$a^n=a$ theorem", and (P_1) shows that every non-zero element is invertible. Thus, suppose that $N \neq \{0\}$. For a fixed commutator c, let φ_c denote the inner derivation of T induced by c. Then φ_c is an endomorphism of the additive group $(T, +)$ such that $\varphi_c(T)$ is finite, and its kernel is equal to the centralizer $C_T(c)$ of c in T; thus, by the first isomorphism theorem, $C_T(c)$ is of finite index in $(T, +)$. Now define H to be the intersection of all $C_T(c)$, and invoke FCP to show that H is also of finite index in $(T, +)$.

By FCP, there must exist $n \in Z$ for which $n[x, y]=0$ for all $x, y \in T$; moreover, elements of H commute with all commutators of T. Hence for each $x \in H$ and $y \in T$, we have $[x^n, y]=nx^{n-1}[x, y]=0$; in particular, idempotent elements of H must be in the center of T and H has no non-zero idempotents except possibly 1.

Now H has non-zero nilpotent elements — indeed H contains all nilpotent elements of T; if it has no non-nilpotent elements, it is clearly finite. The other possibility is that T has 1, which must belong to H, and that every non-nilpotent element of H is invertible. In this case, H has only finitely many zero divisors, hence is finite by a theorem of GANESAN [11, 12]. Since $(T, +)$ has a finite subgroup of finite index, T must be finite.

Now Ganesan's theorem actually states that a ring having a finite number $n>1$ of divisors of zero (including 0) cannot have more than n^2 elements. Thus, the number of elements of H is bounded by k^2, where k is the number of nilpotent elements of T; moreover the index of H is bounded by k^{k-1}, so T has at most k^{k+1} elements. But, by (P_4), if T is any R_α in the subdirect-product representation of R, the number of nilpotent elements of T is bounded by the number of nilpotent elements of R; and the boundedness assertion in our theorem is now established.

THEOREM 8. *Let R be a δ_*-ring having only a finite number of central idempotents. Then R is a direct sum of a finite ring and finitely many periodic fields.*

PROOF. Let $t(R)$ denote the sum of the number of non-zero central idempotents of R and the number of non-zero nilpotent elements of R, and use induction on $t(R)$. Note that if R has no non-zero central idempotents except perhaps 1, the argument for T in the proof of Theorem 7 works. In particular, if $t(R)=1$, R is either a finite ring or a periodic field.

Suppose now that our conclusion holds for all δ_*-rings T with $t(T)<n$, and let R be a δ_*-ring with $t(R)=n$. We may assume R contains non-zero nilpotent elements — otherwise R is readily shown to be a direct sum of finitely many periodic fields; and we may also assume R contains a non-zero central idempotent e which is not a multiplicative identity element. Write $R=Re\oplus A(e)$ and note that since $e\notin A(e)$, $t(A(e))<n$. As for Re, either it contains all the nilpotent elements of R (in which case $A(e)$ is central and contains a non-zero idempotent), or it has fewer nilpotent elements than R does; in either case, $t(Re)<n$, so our inductive hypothesis now yields the desired decomposition for R.

4. Some commutativity results for near-rings

We have obtained in [3] and [5] respectively the following commutativity results for near-rings.

(NR1) If R is a distributively-generated (briefly: d-g) near-ring with $N\subseteq C$, then N is an ideal; moreover, if R/N is periodic, then R is commutative.

(NR2) If R is a periodic near-ring with 1 and $N\subseteq C$, then $(R, +)$ is abelian.

The following theorem extends (NR1) and Theorem 3.

THEOREM 9. *Let R be a periodic d-g near-ring such that to each $x\in R$ and $u\in N$, there exist integers $n=n(u, x)$ and $m=m(u, x)$, each greater than 1, for which $[u, x]^n=[u, x]$ and $[x, u]^m=[x, u]$. Then R is commutative.*

PROOF. We shall assume R has left distributivity, and note that d-g near-rings are zero-symmetric — that is, have the property that $0x=x0=0$ for all $x\in R$.

Suppose $u\in N$ and $x\in R$ are such that $ux=0$; then $[u, x]^2=(ux-xu)ux-(ux-xu)xu=-(-xu)xu$. Now $[(-xu)x, u]=(-xu)xu-u(-xu)x=(-xu)xu-(-uxu)x=(-xu)xu$, hence for some $k>1$, $(-xu)xu=((-xu)xu)^k$. But $((-xu)xu)^2=(-xu)xu(-xu)xu=(-xu)(-xuxu)xu=0$, hence $(-xu)xu=0$; therefore $[u, x]^2=0$ and consequently $[u, x]=0$. Thus, if either of ux and xu is equal to 0, so is the other; and it follows that, as in the case of rings, we have MIFP.

Suppose that u is an arbitrary nilpotent element and $u^n=0$. Let x be an arbitrary element of R. Then, using left distributivity and MIFP repeatedly, we get $0=u^n=u^{n-1}u=u^{n-1}xu=u^{n-1}ux=u^{n-1}[u, x]=u^{n-2}[u, x]u=u^{n-2}[u, x]xu=u^{n-2}[u, x]ux=u^{n-2}[u, x]^2=\ldots=[u, x]^n$. It follows that $[u, x]=0$, hence R is commutative by (NR1).

The computations shown above used the hypothesis of distributive generation only to obtain the result that R is zero-symmetric (though the d-g property was used extensively in the proof in [3]). Thus, combining these computations with (NR2) gives the following result.

THEOREM 10. *Let R be a periodic zero-symmetric near-ring with* 1; *and suppose that for each $x\in R$ and $u\in N$ there exist integers $n, m\in Z^+$ such that $[u, x]^n=[u, x]$ and $[x, u]^m=[x, u]$. Then $(R, +)$ is abelian.*

It is natural to ask whether there is a version of Theorem 1 for near-rings or at least for d-g near-rings, but I have been unable to answer that question. However, the extension of (NR1) which would be immediate from such a theorem is certainly true, as Theorem 12 shows. Before stating it, we note that there are various possible interpretations of $p(x)$ in the near-ring context — for example, $p(x)$ may be assumed to be an element of $\langle x\rangle$ (or of $\langle 1, x\rangle$ if R has 1), or $p(x)$ may be assumed to be a finite sum of powers of x. In the event that x is a distributive element, these two notions are equivalent.

It is not easy, notationally at least, to give an elementwise description of $\langle x\rangle$ or $\langle 1, x\rangle$. The following lemma allows us to circumvent this difficulty.

LEMMA 11. *Let R be an arbitrary near-ring with* 1. *Then for every $x\in R$, $\langle x\rangle=x\langle 1, x\rangle$.*

PROOF. Define an increasing sequence $L_0, L_1, \dots$ of subsets of $\langle x\rangle$ as follows: let L_0 be the set of all finite sums of (positive) powers of x and their negatives; and for $n\geqq 1$, let L_n be the set of finite sums, each term of which is a finite product of elements of L_{n-1} or the negative of such a product. Let $S=\bigcup_{n=0}^{\infty} L_n$. It is clear that for $x, y\in L_n$, $x-y\in L_{n+1}$ and $xy\in L_{n+1}$, so S is a subnear-ring; and since S must obviously be contained in any subnear-ring containing x, we have $S=\langle x\rangle$. Obviously $L_0\subseteqq x\langle 1, x\rangle$, and an easy induction shows that $L_n\subseteqq x\langle 1, x\rangle$ for all n; thus $\langle x\rangle\subseteqq x\langle 1, x\rangle$.

This inclusion is, in fact, the only one we shall use; however, if we write $\langle 1, x\rangle$ as the union of the analogous sequence of subsets $T_0, T_1, \dots$, another induction shows that $\langle x\rangle T_m\subseteqq\langle x\rangle$ for all m, hence $x\langle 1, x\rangle\subseteqq\langle x\rangle$.

THEOREM 12. *Let R be a d-g near-ring with the property that for each $x\in R$, there exists $n=n(x)\in Z^+$ and $p(x)\in\langle x\rangle$ such that $x^n=x^n p(x)$. If $N\subseteqq C$, then R is periodic and commutative.*

PROOF. Since $N\subseteqq C$, N is an ideal by (NR1); and we consider the near-ring $\bar{R}=R/N$. Suppose we can show that $(\bar{R}, +)$ is abelian. Then by FRÖHLICH's theorem [10], $\bar{R}$ is a ring, and hence is periodic by Theorem 1; and commutativity of R follows from (NR1). Another application of Fröhlich's theorem shows that $(R^2, +)$ is abelian, so that R^2 is a ring, which is periodic by Theorem 1. But R^2 is periodic if and only if R is periodic, so the conclusions of our theorem are established.

It remains only to show that $(\bar{R}, +)$ is abelian; to do this we note that $\bar{R}$ can be written as a subdirect product of near-rings without zero divisors, so we may assume $\bar{R}$ has no non-zero divisors of zero. Let d be any non-zero distributive element of $\bar{R}$, and note that $d=dp(d)$ for some $p(d)\in\langle d\rangle$. It follows that $e=p(d)$

is a non-zero idempotent; and since the distributivity of d allows us to write e as a sum of (positive) powers of d and negatives of such powers, e must commute with d. Considering arbitrary $x, y \in R$ and using the fact that $d=de$, we now get

$$0 = (x+y)de-(xde+yde) = (x+y)ed-(xed+yed) =$$
$$= (x+y)ed-(xe+ye)d = ((x+y)e-(xe+ye))d;$$

and cancelling d shows that e is distributive. Since $\bar{R}$ has no zero divisors, e must then be central, and it follows easily that e is a multiplicative identity element.

Now let x be an arbitrary non-zero element of $\bar{R}$; then $x=xp(x)$, where $p(x)\in\langle x\rangle$. Use Lemma 11 to write $p(x)=xq(x)$ for some $q(x)\in\langle 1, x\rangle$, so $x=x^2q(x)$ and $xq(x)=1$. Thus, $\bar{R}$ is a division near-ring, hence is additively commutative by the ZASSENHAUS—NEUMANN theorem [16, 18]. This completes the proof.

THEOREM 13. *Let R be a near-ring with 1. Suppose that for each $x, y \in R$, there exists $p(x)\in\langle x\rangle$ which is a sum of positive powers of x and their negatives, such that*

$$xy = yxp(x). \tag{4}$$

Then R is periodic and $(R, +)$ is abelian.

PROOF. It follows from (4) that R is zero-symmetric and zero-commutative — the latter term means that $ab=0$ if and only if $ba=0$ — and that all nilpotent elements square to zero. This final observation, together with (4), implies that $N=A(R)$ and hence $N\subseteq C$; and we can argue as in [5] that N is an ideal.

As usual, we write $\bar{R}=R/N$ as a subdirect product of near-rings $\bar{R}_\alpha$ with no non-zero divisors of zero; since each $\bar{R}_\alpha$ satisfies (4), taking $x=y\neq 0$ and cancelling yields the result that each $\bar{R}_\alpha$ is a division near-ring. Hence $(\bar{R}, +)$ is abelian. Moreover, as we see by taking $x=y=1+1$ in (4) and noting that $\bar{R}$ is free of nilpotent elements, $(\bar{R}, +)$ is a periodic group of bounded square-free order. Let $(\bar{R}_1, +), \dots, (\bar{R}_n, +)$ be the non-trivial primary components of $(\bar{R}, +)$, corresponding to primes $p_1, p_2, \dots, p_n$; note that $(\bar{R}, +)$ is the direct sum of the $(\bar{R}_i, +)$ and that $p_i\bar{R}_i=0$ for $i=1, \dots, n$. In fact, each $\bar{R}_i$ can be regarded as $A(p_i)$ in $\bar{R}$, hence is an ideal of $\bar{R}$; thus $\bar{R}$ is a near-ring direct sum of the $\bar{R}_i$. It is our immediate object to show that $\bar{R}$ is periodic, for which purpose it will suffice to show that each $\bar{R}_i$ is periodic; thus we assume that $p\bar{R}=0$ for some prime p.

Operating in $\bar{R}$, we take $x=y$ in (4) to obtain

$$x^2 = \sum_{i=3}^{m} k_i x^i, \quad \text{where} \quad k_m \not\equiv 0 \bmod p, \tag{5}$$

and where k_ix^i is to be thought of as the sum of k_i summands each equal to x^i. (Note that the sum on the right side of (5) is unambiguous because $(\bar{R}, +)$ is abelian.) Choosing a positive integer j_m such that $k_mj_m\equiv 1 \pmod p$ and adding (5) to itself j_m times expresses x^m as a sum of lower powers of x; thus the additive subgroup generated by the powers of x is finite, and hence $x^n=x^m$ for some distinct $n, m\in Z^+$. (If $\bar{R}$ were a ring, $\langle x\rangle$ would be finite; whether that is the case in this context is not clear.)

Interpreting our latest results for $\bar{R}$ in the original near-ring R, we see that for each $x\in R$, there exist distinct $m, n\in Z^+$ such that $x^n-x^m\in N$; and recalling that

$N=A(R)$, we have $x^{n+1}=x^{m+1}$, so that R is periodic. Since R has 1 and $N\subseteqq C$, (NR2) now yields our final conclusion — that $(R, +)$ must be abelian.

In [4] we called a ring a G-T-P-W ring if for each $x, y\in R$, either $[x, y]=0$ or $xy=y\pi$ for some monomial π in x and y with x-degree at least 2. This notion obviously extends to near-rings; and our final two theorems, the proofs of which will be handled simultaneously, extend Theorem 6 of [4].

THEOREM 14. *Let R be a distributively-generated periodic G-T-P-W near-ring. Then R is commutative.*

THEOREM 15. *If R is a periodic zero-symmetric G-T-P-W near-ring with 1, then $(R, +)$ is abelian.*

PROOF. In view of (NR1) and (NR2) and the fact that all d—g near-rings are zero-symmetric, we need only show that $N\subseteqq C$ for any zero-symmetric G-T-P-W near-ring R.

Note first that such R have the property that $xy=0$ if and only if $yx=0$. It follows easily that R must have the insertion-of-factors property — that is, any product equal to 0 remains so on the insertion of additional factors between any existing factors; in particular, if $u^k=0$, any product of ring elements having at least k factors equal to u is 0.

Let $u\in N$ and $x\in R$, and suppose $u^k=0$. Either $[u, x]=0$, in which case there is nothing further to do, or $ux=x\pi$, where π is a monomial in x and u having u-length at least 2. If π contains ux as a submonomial, then we can write

$$ux=\pi_1 ux\pi_2, \tag{6}$$

where π_1 and π_2 are monomials in u and x, at least one of which has u as a factor. (Here π_2 may be empty.) On the other hand, if π is of the form $x^j u^k$ for $j\geqq 0$ and $k\geqq 2$, then $ux=x^{j+1}u^k=(u^k)^t(x^{j+1})^s\dots$, and we again have a representation of ux of the form (6). It follows that $ux=\pi_1^n ux\pi_2^n$ for all $n\in Z^+$, so that ux is a product with k or more factors equal to u — that is, $ux=0=xu$. Thus $N\subseteqq C$ and we are finished.

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