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INDEX

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PSEUDOFUNCTIONS AND UNIQUENESS ON THE GROUP OF INTEGERS OF A p-SERIES FIELD

By

G. E. LIPPMAN (Hayward) and W. R. WADE (Knoxville)

§ 1. Introduction

Let G denote the group of integers of a p-series field, where p is prime. Thus an element $x\in G$ can be represented as $x=(x_0, x_1, \dots)$ with $0\leqq x_i<p$ for each $i\geqq 0$. Moreover, the dual group Γ can be identified with the functions $\{\psi_0, \psi_1, \dots\}$ which are defined by the following process. If m is a non-negative integer with $m=\sum_{k=0}^{\infty}\alpha_k p^k$, $0\leqq \alpha_k<p$ for each integer $k\geqq 0$, and if $x\in G$, then

$$\psi_m(x) = \prod_{k=0}^{\infty} \varphi_k^{\alpha_k}(x), \tag{1}$$

where for each integer $k\geqq 0$ and each $x=(x_0, x_1, \dots)\in G$, the function φ_k is defined by

$$\varphi_k(x) = \exp(2\pi i\, x_k/p). \tag{2}$$

The subgroups $G_n\equiv\{x\in G\colon x_0=x_1=\dots=x_{n-1}=0\}$ for $n=1, 2, \dots$ form a base for the neighbourhoods of $0\in G$. In the case that $p=2$, the group G is the compact dyadic group introduced by FINE [3] and the functions $\{\psi_0, \psi_1, \dots\}$ coincide with the Walsh—Paley system. Consult [6] and [7] for basic properties of such a group G.

If ξ and ζ are real numbers, then we shall denote their sum modulo p by $\xi\oplus\zeta$. Recall that if $x=(x_0, x_1, \dots)$ and $y=(y_0, y_1, \dots)$ belong to G then their sum is given by

$$x\dotplus y = (x_0\oplus y_0, x_1\oplus y_1, \dots).$$

Since each ψ_m is a character for G, we have that

$$\psi_m(x\dotplus y) = \psi_m(x)\psi_m(y) \tag{3}$$

for $x, y\in G$ and $m=0, 1, \dots$.

Define the p-sum of two non-negative integers n and l as follows. If $m=\sum_{i=0}^{\infty}\alpha_i p^i$ and if $l=\sum_{i=0}^{\infty}\beta_i p^i$, with $0\leqq\alpha_i, \beta_i<p$, then $m\dotplus l=\sum_{i=0}^{\infty}(\alpha_i\oplus\beta_i)p^i$. Observe by (1) and (2) that

$$\psi_{m\dotplus l}(x) = \psi_m(x)\psi_l(x) \tag{4}$$

for $x\in G$ and $m, l=0, 1, \dots$.

Let E be a subset of the group G. The set E is said to be a *set of uniqueness for* G if $\sum_{k=0}^{\infty} a_k\psi_k(x)=0$ for $x\in G\setminus E$ and $a_k\in(-\infty, \infty)$ implies that $a_k=0$ for

$k=0, 1, \ldots$. The set E is said to be a *set of uniqueness for G in the wide sense* if $\sum_{k=0}^{\infty} a_k \psi_k(x)=0$ for $x \in G \setminus E$ and $a_k = \int_G \psi_k(x)\, d\mu$ for some real Borel measure μ on G, implies that $a_k=0$ for $k=0, 1, \ldots$. It is clear, then, that every set of uniqueness for G is a set of uniqueness for G in the wide sense.

The purpose of this paper is to show that there exist sets of uniqueness for G in the wide sense which are not sets of uniqueness for G. Our proof follows the outline given in the trigonometric case by PYATESKIĬ—SHAPIRO [5], but most of the computational details take on a flavor distinctive to the group G.

§ 2. Elementary sets of uniqueness

Denote by $\mathscr{A}(G)$ the collection of all series $f=\sum_{k=0}^{\infty} a_k \psi_k$ whose coefficients satisfy $\sum_{k=0}^{\infty} |a_k| < \infty$. Since each function ψ_k is continuous on G and since each series in $\mathscr{A}(G)$ necessarily converges absolutely and uniformly, the series f converges everywhere to a continuous function, which we shall also denote by f. In particular, $\mathscr{A}(G)$ is a subspace of the Banach space of functions which are continuous on G.

A set $E \subseteq G$ is said to be an *elementary set of uniqueness for G* if there exists a sequence of continuous functions $f_1, f_2, \ldots$, each belonging to $\mathscr{A}(G)$, which satisfies three properties:

$$f_n(x) = 0 \qquad \text{for} \quad x \in E \quad \text{and} \quad n = 1, 2, \ldots; \tag{5}$$

$$\sum_{k=0}^{\infty} |a_k^{(n)}| \leqq C < \infty \quad \text{for} \quad n = 1, 2, \ldots, \tag{6}$$

where for each integer $n \geqq 1$ the coefficients $a_k^{(n)}$ are defined by the identity $f_n(x) = \sum_{k=0}^{\infty} a_k^{(n)} \psi_k(x)$ for $x \in G$;

$$\lim_{n\to\infty} a_0^{(n)} = 1 \quad \text{and} \quad \lim_{n\to\infty} a_k^{(n)} = 0 \quad \text{for} \quad k = 1, 2, \ldots. \tag{7}$$

In this section we shall show that elementary sets of uniqueness for G generate the closed sets of uniqueness for G in the following sense.

THEOREM 1. *Let E be a closed set of uniqueness for G. Then E is a countable union of elementary sets of uniqueness for G.*

Before we begin to prove this theorem, we observe that if $f=\sum_{k=0}^{\infty} a_k \psi_k$ belongs to $\mathscr{A}(G)$ then the correspondence $\|f\| = \sum_{k=0}^{\infty} |a_k|$ provides a norm for the vector space $\mathscr{A}(G)$. Moreover, since the map which takes a sequence $\{a_k\}_{k=0}^{\infty}$ is an isometric isomorphism of l^1 onto $\mathcal{A}(G)$, the space $\mathcal{A}(G)$ is actually a Banach space whose dual is isometrically isomorphic to l^∞. In view of this, we say that a continuous linear functional T on $\mathcal{A}(G)$ is a *pseudofunction* if the corresponding sequence in l^∞ actually converges to zero. We shall denote the collection of pseudofunctions by $\mathcal{P}$.

Observe by definition that $T(\psi_n)\to 0$ as $n\to\infty$ for all $T\in\mathcal{P}$. Furthermore, since the dual of c_0 is l^1, we know that the dual of $\mathcal{P}$ is isometrically isomorphic to $\mathcal{A}(G)$. In particular, a sequence $f_1, f_2, \ldots$ of functions belonging to $\mathcal{A}(G)$ converges to f in the *weak* topology* if given any pseudofunction T, we have $T(f_n-f)\to 0$ as $n\to\infty$.

As usual, we say that a pseudofunction T vanishes on an open $U\subseteq G$ if $Tf=0$ for each $f\in\mathcal{A}(G)$ which is supported in U. And, that the *support* of T, denoted by spt T, is the complement of the union of all open sets in G on which T vanishes.

LEMMA 1. *Let T be a pseudofunction and E be a closed subset of G. If E is a set of uniqueness for G and if* spt $T\subset E$, *then* $T\equiv 0$.

To prove this lemma, we suppose to the contrary that T does not vanish on $\mathcal{A}(G)$. For each integer $k\geqq 0$ set $c_k=T(\psi_k)$ and observe that at least one of these numbers must be non-zero (because the character series are dense in $\mathcal{C}(G)$.) We shall obtain a contradiction, then, by showing that the series $\sum_{k=0}^{\infty} c_k\psi_k$ converges to zero on the complement of E.

In view of (3), we shall show that if $0\notin E$, then $\sum_{k=0}^{\infty} c_k=0$. Toward this, let U be a neighbourhood of $0\in G$ which is disjoint from E. By hypothesis, then, T vanishes on U. Consider the Dirichlet kernel $D_n(x)=\sum_{k=0}^{n-1}\psi_n(x)$ for $x\in G$ and $n=1, 2, \ldots$. It is well known [7] that for large j, $D_{p^j}(x)\equiv 0$ when $x\notin U$. Fix such an integer j and let q be any nonnegative integer. By the definition of $\dotplus$, the sequence $\{lp^j\dotplus\tau\colon 0\leqq l<q, 0\leqq\tau<p^j\}$ is a rearrangement of the integers $\{0, 1, \ldots, qp^j-1\}$. Combining this observation with equation (4), we conclude that

$$D_{qp^j}=\sum_{l=0}^{q-1}\sum_{\tau=0}^{p^j-1}\psi_{lp^j}\psi_\tau.$$

Similarly, we verify that if $n=qp^j+r$, with $0\leqq r<p^j$, then

$$D_n=\sum_{l=0}^{q-1}\sum_{\tau=0}^{p^j-1}\psi_{lp^j}\psi_\tau+\sum_{\tau=0}^{r-1}\psi_{qp^j}\psi_\tau.$$

Interchanging the order of summation in the first term, we conclude that

$$D_n=D_{p^j}\left(\sum_{l=0}^{q-1}\psi_{lp^j}\right)+\psi_{qp^j}D_r.$$

By the choice of j, the linearity of T, the fact that T vanishes on U, and equation (4), we are led to the following statement. If $n=qp^j+r$, with $0\leqq r<p^j$, then

$$T(D_n)=\sum_{i=0}^{r-1}T(\psi_{qp^j+i}).$$

Since j is fixed, and since $r<p^j$, the number of terms in this sum does not exceed a prefixed number, namely p^j. Moreover, as $n\to\infty$ both q and qp^j+j tend to infinity for each fixed integer i. Since T is a pseudofunction, it follows that $T(D_n)\to 0$ as $n\to\infty$. However, $T(D_n)\equiv\sum_{k=0}^{n-1} c_k$ by definition. We have reached the desired contradiction, completing the proof of Lemma 1.

For each set $E\subset G$, let $J(E)$ represent the collection of all functions $f\in\mathcal{A}(G)$ which vanish in a neighbourhood of E. It is clear that $J(E)$ is a subspace of $\mathcal{A}(G)$. We intend to show that $J(E)$ is actually an ideal, under pointwise multiplication. All that need be verified is that $\mathcal{A}(G)$ is closed under pointwise multiplication. Toward this suppose that $f=\sum_{k=0}^{\infty} a_k\psi_k$ and $g=\sum_{l=0}^{\infty} b_l\psi_l$ belong to $\mathcal{A}(G)$. Since both these series converge absolutely and uniformly we can manipulate them like polynomials. By equation (4), then, we have

$$f(x)\,g(x)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_k b_l \psi_{k\dotplus l}.$$

By uniqueness [7], then, the jth G-Fourier coefficient of the function fg is given by $\sum\{a_k b_l\colon j=k\dotplus l\}$. It follows that

$$\|fg\|_{\mathcal{A}(G)}\leqq\|f\|_{\mathcal{A}(G)}\|g\|_{\mathcal{A}(G)}$$

and, in particular, that fg belongs to $\mathcal{A}(G)$. Notice, then, that $\mathcal{A}(G)$ is a Banach Algebra.

LEMMA 2. *Let E be a closed subset of G. If E is a set of uniqueness for G then $J(E)$ is weak* dense in $\mathcal{A}(G)$.*

To prove this lemma, we suppose to the contrary, that there exists a function $f_0\in\mathcal{A}(G)$ which does not belong to the weak* closure of $J(E)$. By the Hahn—Banach theorem, then, we can choose a pseudofunction T which vanishes on $J(E)$ and which satisfies $T(f_0)=1$. But if T vanishes on $J(E)$, then spt $T\subset E$. Hence the property $T(f_0)=1$ is clearly in contradiction to Lemma 1. The proof of Lemma 2 is therefore complete.

LEMMA 3. *Let E be a subset of the group G. A necessary and sufficient condition that E be an elementary set of uniqueness for G is the existence of a sequence of functions $f_1, f_2, \ldots$ in $\mathcal{A}(G)$ which vanish on E, and which converge to 1 in the weak* topology.*

Toward sufficiency, let $f_1, f_2, \ldots$ be such a sequence which converges to 1 in the weak* topology. We must verify (5), (6) and (7). It is clear that (5) holds. Toward (6), observe that if $T\in\mathcal{P}$ then $T(f_n-1)\to 0$ as $n\to\infty$. Hence by the Banach—Steinhaus theorem, there exists a constant $M<\infty$ such that $|T(f_n-1)|\leqq \leqq M\|T\|$, for $T\in\mathcal{P}$.

If we take the supremum of this inequality over all pseudofunctions T which satisfy $\|T\|=1$, we conclude that $\|f_n-1\|_{\mathcal{A}(G)}\leqq M$.

It follows that (6) holds with $C=M+1$. To verify that (7) holds, choose $T\in\mathcal{P}$ such that $T(\psi_0)=1$ but $T(\psi_k)=0$ for all integers $k\geqq 1$. Then $T(f_n)=a_0^{(n)}$ for

$n=1, 2, \ldots$ and $T(f_n)\to T(1)$ imply that $a_0^{(n)}\to 1$, as $n\to\infty$. The remainder of (7) is obtained by the same reasoning applied to the pseudofunction S which satisfies $S(\psi_0)=S(\psi_k)=1$ but $S(\psi_n)=0$ for all integers $n>0$ with $n\neq k$.

To show necessity, suppose that $f_1, f_2, \ldots$ is a sequence of functions in $\mathscr{A}(G)$ which satisfy (5), (6) and (7). We need to show that $f_n\to 1$ in the weak* topology, as $n\to\infty$. Toward this, let $\varepsilon>0$ and let $T\in P$. Since $T(\psi_n)\to 0$ we can use (6) to choose an integer M so large that

$$\sum_{k=M}^{\infty} |a_k^{(n)}T(\psi_k)| < \varepsilon/2. \tag{8}$$

Moreover, by (7) we can choose an integer N so large that $n\geqq N$ implies

$$\sum_{k=1}^{M-1} |a_k^{(n)}T(\psi_k)| < \varepsilon/2. \tag{9}$$

However, since T is both continuous and linear, we have that $T(f_n)=\sum_{k=0}^{\infty} a_k^{(n)}T(\psi_k)$ for $n=1, 2, \ldots$. In view of (8) and (9), then, we conclude that $|T(f_n-a_0^{(n)})|\leqq\varepsilon$ for $n\geqq N$. By (7), then, and the fact that $T\in\mathscr{P}$ was arbitrary, we have established the weak* convergence of f_n to 1, as $n\to\infty$.

LEMMA 4. *If f_0 belongs to $\mathscr{A}(G)$ and if $f_0(x)\neq 0$ for all $x\in G$, then $1/f_0$ belongs to $\mathscr{A}(G)$.*

In the trigonometric case, this result is known as Weiner's theorem. In the Walsh case, this result was obtained by AGAEV [1]. His proof, briefly outlined in the next paragraph, also works in the p-adic case.

Begin by observing that by (2), if $\varepsilon_1, \varepsilon_2, \ldots$ is a sequence of pth roots of unity then there exists an $x\in G$ (just specify its coefficients!) such that $\varphi_k(x)=\varepsilon_k$ for $k=1, 2, \ldots$. Now, if h is any complex-valued algebra homomorphism on $\mathscr{A}(G)$ then $h^p(\varphi_k)\equiv 1$. Hence we can choose an $x\in G$ such that $h(\varphi_k)=\varphi_k(x)$ for $k=1, 2, \ldots$. Since each ψ_n is a product of φ_k's, and since the character polynomials are dense in $\mathscr{C}(G)$, it follows that given any complex-valued algebra homomorphism h on $\mathscr{A}(G)$, there exists a point $x\in G$ such that $h(f)=f(x)$ for all $f\in\mathscr{A}(G)$.

Applying this remark to the function f_0, given in Lemma 4, we conclude by hypothesis, that $h(f_0)\neq 0$ for all complex-valued algebra homomorphisms h on $\mathscr{A}(G)$. It is well known [2] that this condition is necessary and sufficient to conclude that f_0 is invertible in $\mathscr{A}(G)$, because $\mathscr{A}(G)$ is a Banach Algebra. This completes the proof of Lemma 4.

Let B be any separable Banach space and C be a convex subset of B^*. Recall [2] that C_1 denotes the limit points in B^* of sequences in C, and that for any ordinal $\xi>1$, the set C_ξ is recursively defined by

$$C_\xi = \begin{cases} (C_{\xi-1})_1 & \text{if } \xi \text{ is not a limit ordinal} \\ \bigcup\limits_{\eta<\xi} C_\eta & \text{otherwise.} \end{cases}$$

It is well-known that given such a set C there exists a countable ordinal ξ_0 such that $C_{\xi_0}=C_\eta$ for all ordinals $\eta>\xi_0$. Hence, we can define the *index* of C to be the smallest ordinal ξ_0 such that C_{ξ_0} is closed in B^*.

To prove Theorem 1, apply these remarks to the convex set $J(E)$ and the Banach space $\mathscr{P}$. Since E is a closed set of uniqueness for G, we apply Lemma 2 to conclude that

$$J(E)_{\xi_0} = \mathscr{A}(G), \tag{10}$$

where ξ_0 is the index of $J(E)$ in $\mathscr{A}(G)$. Our proof is by induction on ξ_0. If $\xi_0=1$, then by (10) E is itself an elementary set of uniqueness for G, and there is nothing to prove.

Suppose, then, that the theorem holds for all closed sets of uniqueness whose fundamental ideal space has index $\eta<\xi_0$, and suppose that (10) holds. We observe that ξ_0 cannot be a limit ordinal. Indeed, if this were the case, then since $J(E)_{\xi_0}= = \bigcup_{\eta<\xi_0} J(E)_\eta$, equation (10) would force us to conclude that $1\in J(E)_\eta$ for some $\eta<\xi_0$. But $J(E)_\eta$ is an ideal, and therefore, $J(E)_\eta=\mathscr{A}(G)$. This contradicts the fact that ξ_0 was the index of $J(E)$.

Hence we may suppose that ξ_0 has a predecessor ξ_0-1. By (10), then, there is a sequence of functions $f_1, f_2, \dots$ in $J(E)_{\xi_0-1}$ which converges to 1 in the weak* topology. Set $E=\{x\in G\colon f(x)=0 \text{ for all } f\in J(E)_{\xi_0-1}\}$. According to Lemma 3, E is an elementary set of uniqueness for G. Now the ordinal ξ_0 is countable and any open subset of G is a countable union of closed sets in G (actually, cosets of the subgroups G_n). Hence, the proof of the theorem will be complete if we can show that given any closed set F which satisfies $F\subseteqq E\setminus E_0$ and whose fundamental ideal space has index α_0, necessarily satisfies $\alpha_0<\xi_0-1$.

To begin on this last leg of the proof, we observe that $J(E)_{\xi_0-1}\subset J(F)_{\xi_0-1}$ since $J(E)\subset J(F)$. Hence

$$F_0 \subset E_0, \tag{11}$$

where $F_0=\{x\in G\colon f(x)=0 \text{ for all } f\in J(F)_{\xi_0-1}\}$. However, characteristic functions of the basic open sets in G (that is the cosets of G_n) are character polynomials [7]. Moreover, polynomials belong to $\mathscr{A}(G)$. Hence, by considering basic open sets in the complement of F, we conclude that $F_0\subset F$. By the choice of F, then, we also have $F_0\subset E\setminus E_0$. In view of (11), the only logical conclusion is that F_0 is empty. In particular, given $x\in G$ we can choose a function $f_x\in J(F)_{\xi_0-1}$ and an open set V_x containing x such that $f_x\neq 0$ on V_x. Since G is compact, we can choose a finite sequence of points $x_1, \dots, x_N$ in G such that G is covered by the sets V_{x_i}, $i=1, 2, \dots, N$. Set $f(x)=\sum_{i=1}^{N} f_{x_i}^2(x)$, for $x\in G$. Since $J(F)_{\xi_0-1}$ is an ideal, it contains the function f. Moreover, $f(x)>0$ for all $x\in G$. It follows from Lemma 4, then, that $1/f$ belongs to $\mathscr{A}(G)$, and therefore that $1\in J(F)_{\xi_0-1}$. Hence $J(F)_{\xi_0-1}=\mathscr{A}(G)$ is weak* closed which implies that its index ξ_0 is no greater than ξ_0-1. The proof of Theorem 1 is complete.

We close this section by showing that the class elementary sets of uniqueness for G is closed under translations and dilations. This result will be used in the next section to verify that a certain set is not a set of uniqueness for G.

THEOREM 2. *Let E be a subset of the group G.*

a) *If E is an elementary set of uniqueness for G, and if $x_0\in G$, then $x_0\dotplus E$ is an elementary set of uniqueness for G.*

b) *If* $E_k=\{x\in G\colon x_1=x_2=\ldots=x_k=0$ *and* $(x_{k+1}, x_{k+2}, \ldots)\in E\}$ *for* $k=1, 2, \ldots,$ *and if* E_{k_0} *is an elementary set of uniqueness for G, for some integer* k_0*, then E is an elementary set of uniqueness for G.*

To prove a) apply Lemma 3 to choose a sequence of functions $f_1, f_2, \ldots$ in $\mathcal{A}(G)$ which converges to 1 in the weak* topology, and such that each f_n vanishes on E. For each integer $n\geqq 1$, let $a_k^{(n)}$ $(k=0, 1, \ldots)$ be the G-Fourier coefficients of f_n and set $g_n(x)=f_n(x\dot{-}x_0)$, for $x\in G$. Then each g_n vanishes on $x_0\dot{+}E$ and its G-Fourier coefficients are given by $\psi_k(x_0)a_k^{(n)}$, for $k=0, 1, \ldots$. Hence the functions $g_1, g_2, \ldots$ and the set $x_0\dot{+}E$ satisfy (5), (6) and (7), i.e., $x_0\dot{+}E$ is an elementary set of uniqueness for G.

To prove b), apply Lemma 3 to choose a sequence of functions $f_1, f_2, \ldots$ in $\mathcal{A}(G)$ which converges to 1 in the weak* topology, such that each f_n vanishes on E_{k_0}. Set $g_n(x)=f_n(0, \ldots, 0, x_1, x_2, \ldots)$ for each $x\in G$, where the first k_0 places of the argument of f_n are zero. For each integer $n\geqq 1$ let $a_k^{(n)}$ (respectively, $b_k^{(n)}$), for $k=0, 1, \ldots,$ represent the G-Fourier coefficients of f_n, (respectively, g_n).

We shall complete the proof of Theorem 2 by showing that the functions $g_1, g_2, \ldots$ and the set E satisfy (5), (6) and (7). That (5) is satisfied is clear. To verify (6) and (7), we begin by observing that for each pair of integers $n\geqq 1, k\geqq 0$, the following formula subsists:

$$b_k^{(n)} = \sum_{l=0}^{\infty} a_l^{(n)} \int_G \psi_l(0, \ldots, 0, x)\psi_k(x)\,dx.$$

If we make the change of variables $y=(0, \ldots, 0, x)$ and recall that the Haar measure of G_{k_0} is p^{-k_0}, we obtain that

$$b_k^{(n)} = \sum_{l=0}^{\infty} a_l^{(n)}\, p^{k_0} \int_{G_{k_0}} \psi_l(y)\psi_{kp^{k_0}}(y)\,(dy).$$

Apply (4), noticing that for each integer $l\geqq 0$, the integral on the right is zero unless $l\dot{+}kp^{k_0}<p^{k_0}$. Hence

$$b_k^{(n)} = \sum_l \Big\{a_l^{(n)}\, p^{k_0} \int_{G_{k_0}} \psi_{l\dot{+}kp^{k_0}}(y)\,dy\colon\ l\dot{+}kp^{k_0} < p^{k_0}\Big\}.$$

By the definition of $\dot{+}$, a necessary and sufficient condition that $l\dot{+}kp^{k_0}$ be less than p^{k_0} is that the p-adic expansions of l and kp^{k_0} be identical from the k_0th component onward. In particular, the above expression reduces to

$$b_k^{(n)} = \sum_{l=kp^{k_0}}^{(k+1)p^{k_0}-1} a_l^{(n)}\, p^{k_0} \int_{G_{k_0}} \psi_{l\dot{+}kp^{k_0}}(y)\,dy.$$

Both (6) and (7) are easily verified from this identity. Hence E is an elementary set of uniqueness, and the proof of Theorem 2 is complete.

§ 3. The main theorem

In the next two sections we shall prove the following theorem.

THEOREM 3. *There exists a set* $E\subseteqq G$ *which is a set of uniqueness for* G *in the wide sense, but which is not a set of uniqueness for* G.

This set will be given by

$$E=\left\{(x_0, x_1, \ldots): \sum_{k=0}^{n} x_k \leqq \gamma n \text{ for } n=1,2,\ldots\right\} \tag{11}$$

where γ is any real number which satisfies $0<\gamma<p^{-1}$.

We begin with a preliminary result.

LEMMA 5. *Let* $\mathcal{M}_0$ *represent the collection of finite Borel measures* μ *on* G *which satisfy*

$$\lim_{n\to\infty} \int_G \psi_n(x)\, d\mu = 0.$$

If $\mu\in\mathcal{M}_0$ *then* $|\mu|\in\mathcal{M}_0$.

The lemma will be verified if we can show that given $\varepsilon>0$ there exists a measure $\nu\in\mathcal{M}_0$ such that $\eta=\nu-|\mu|$ implies $\int_G d|\eta|<\varepsilon$.

The measure ν will be the indefinite integral of simple function $s(x)=\sum c_i\chi_{E_i}(x)$, where each E_i is a coset of one of the subgroups $G_1, G_2, \ldots$ which form a neighbourhood base at the origin. Such simple functions will be called *step functions on* G. It is well-known [7] that step functions on G are also character polynomials on G, and thus, if s is a step function on G and $\mu\in\mathcal{M}_0$, then

$$\lim_{n\to\infty} \int_G s(x)\psi_n(x)\, d\mu = 0. \tag{12}$$

Let $\varepsilon>0$ and choose disjoint fundamental open sets $E_1, E_2, \ldots, E_r$ so that $G=\bigcup_{j=1}^{r} E_j$ and that

$$|\mu|(G)-\sum_{j=1}^{r} |\mu(E_j)| < \varepsilon. \tag{13}$$

Let $s(x)=\delta_j$ when $x\in E_j$ where

$$\delta_j=\begin{cases} +1 & \text{if } \mu(E_j)\geqq 0 \\ -1 & \text{if } \mu(E_j)<0 \end{cases}$$

$(j=1, 2, \ldots, r)$, and set $\nu(E)=\int_E s(x)\, d\mu$ for each Borel set $E\subseteqq G$. By (12), the measure ν belongs to $\mathcal{M}_0$. Moreover, if $F_1, F_2, \ldots, F_t$ is a disjoint sequence of fundamental open sets in G which satisfies $G=\bigcup_{i=1}^{t} F_i$ and refines $E_1, E_2, \ldots, E_r$, then for each integer $i\in[1, t]$, $\eta(F_i)=|\mu|(F_i)-\delta_j\mu(F_i)>0$, where j is determined by $F_i \subset E_j$. Hence,

$$\sum_{i=0}^{t} |\eta(F_i)| = |\mu|(G) - \sum_{i=1}^{t} \delta_j \mu(F_i) \equiv |\mu|(G) - \sum_{j=1}^{r} |\mu(E_j)|.$$

It follows from (13) that $\int_G d|\eta| < \varepsilon$. Hence Lemma 5 is established.

The rest of this section will be devoted to proving that the set E given by (11) is a set of uniqueness for G in the wide sense. Let f be a polynomial on G which satisfies three properties:

(14) $$f(x) = 1 \quad \text{for} \quad x \in G_1;$$

there exists a constant $c \in (0, (1-\gamma)/\gamma)$ such that

(15) $$-c \leqq f(x) \leqq 1 \quad \text{for} \quad x \in G;$$

(16) $$\int_G f(x)\, dx = 0.$$

Fix an integer $n \geqq 1$ and set $f_n(x) = n^{-1} \sum_{k=0}^{n-1} f(p^k x)$ where for each integer $k \geqq 0$ and each $x = (x_0, x_1, \ldots) \in G$, the symbol $p^k x$ represents the point $(x_k, x_{k+1}, \ldots)$ in G. We intend to show that

(17) $$f_n(x) \geqq (1-\gamma) - c\gamma, \quad x \in E, \quad n = 1, 2, \ldots.$$

Toward this inequality, let $x = (x_0, x_1, \ldots) \in E$ and observe by (11) that there are at least $n - [n\gamma]$ zeroes in the set $\{x_0, \ldots, x_{n-1}\}$. Yet, if $x_i = 0$ for some i, then $p^i x \in G_1$, i.e., $f(p^i x) = 1$. It follows (see (15)) that

$$\sum_{k=0}^{n-1} f(p^k x) \geqq (n - [n\gamma]) - c[n\gamma].$$

Since $[n\gamma] \leqq n\gamma$, we have verified (17).

Suppose, for the purposes of a contradiction, that E is not a set of uniqueness for G in the wide sense. Then there exists a non-zero measure μ such that $\sum_{k=0}^{\infty} c_k \psi_k(x) = 0$ for $x \notin E$, where $c_k = \int_G \psi_k(x)\, d\mu$ for each integer $k \geqq 0$.

Set $a_k = \int_G f(x) \psi_k(x)\, dx$ and $b_k = \int_G \psi_k(x)\, d|\mu|$ for $k = 0, 1, \ldots$. Observe that

$$\int_G f_n(x)\, d|\mu| = n^{-1} \int_G \sum_{k=0}^{n-1} \sum_{j=0}^{\infty} a_j \psi_j(p^k x)\, d|\mu|.$$

Since $\psi_j(p^k x) = \psi_{jp^k}(x)$ for each pair of integers $j, k \geqq 0$ and each $x \in G$, we can rewrite this equation as follows:

(18) $$\int_G f_n(x)\, d|\mu| = n^{-1} \sum_{k=0}^{n-1} \sum_{j=1}^{\infty} a_j b_{jp^k}.$$

The term $j=0$ can be omitted because (16) holds. We know that the measure μ belongs to $\mathscr{M}_0$, because the series $\sum_{k=0}^{\infty} c_k \psi_k(x)$ converges for at least one p-adic rational x. Hence, by Lemma 5, the measure $|\mu|$ belongs to $\mathscr{M}_0$, i.e., the coefficients

$b_l \to 0$ as $l \to \infty$. But for $j \neq 0$, the indices $jp^k \to \infty$ as $k \to \infty$. It follows from (18), then, that $\int_G f_n(x)\, d|\mu| \to 0$ as $n \to \infty$. This is clearly impossible, however, since (17) and the fact that $|\mu|$ is supported on E implies that

$$\int_G f_n(x)\, d|\mu| \geqq (1-\gamma-c\gamma)|\mu|(E) > 0.$$

This contradiction proves that E is a set of uniqueness for G in the wide sense.

§ 4. The set E is not a set of uniqueness for G

We begin this section by constructing measures $\{\mu_{N,t}\colon N=0, 1, \ldots, |t|<p\}$ and choosing a $\gamma'<\gamma$ such that

$$\lim_{N\to\infty} \mu_{N,t}(E) = 1 \quad \text{uniformly for} \quad t\in[1-p\gamma', 1]. \tag{19}$$

Fix an integer $N \geqq 0$ and a real number $|t|<p$. Set $r=((p-1)+t)/p$ and $s=1-r$. Furthermore, for each integer $n \geqq 0$ and for $k=0, 1, \ldots, p^n-1$ set $I(k, n)=e(k)\dot{+}G_{N+n}$ where $e(k)=(\alpha_1, \alpha_2, \ldots)\in G$ satisfies $k/p^{N+n}=\sum_{j=1}^{\infty} \alpha_j p^{-j}$. Let $\mu_{N,t}$ denote the measure supported in G which satisfies $\mu_{N,t}(I(0, 0)=1)$ and

$$\mu_{N,t}(I(k, n)) = \binom{p-1}{\tilde{k}} r^{(p-1)-\tilde{k}} s^{\tilde{k}} \mu_{N,t}(I(k', n-1))$$

for each integer $n \geqq 1$ and for $k=0, 1, \ldots, p^n-1$, where $0 \leqq \tilde{k} < p$ satisfies $k \equiv \tilde{k} \pmod p$ and where k' is chosen so that $I(k, n)$ is a subset of $I(k', n-1)$. Observe that if $k/p^n=\sum_{j=1}^{n} x_j p^{-j}$ and if $q=\sum_{j=1}^{n} x_j$ then there exist constants $\theta_{k,n}$ which are products of certain binomial coefficients, such that

$$\mu_{N,t}(I(k, n)) = \theta_{k,n} r^{(p-1)n-q} s^q \tag{20}$$

for $n=1, 2, \ldots$ and for $k=0, 1, \ldots, p^n-1$.

To show that these measures satisfy (19), we set $C=G\setminus E$, $B_N=C\cap G_N$ and

$$B_{N,n} = \Big\{(\underbrace{0, \ldots, 0}, x_0, x_1, \ldots)\colon \sum_{j=0}^{n-1} x_j > \alpha(n+N)\Big\},$$

and observe that $B_N=\bigcup_{n=1}^{\infty} B_{N,n}$. Hence by identity (20), we have

$$\mu_{N,t}(B_{N,n}) = \sum \{\theta_{k,n} r^{(p-1)n-q} s^q\colon q > \gamma(n+N)\}. \tag{21}$$

Set $r_1=1-\gamma$, $s_1=\gamma$ and $\beta=r^{1-\gamma}s^{1-\gamma}r^{1-\gamma}r_1^{\gamma-1}s_1^{\gamma-1}$. We notice that a necessary and sufficient condition that $t>1-\gamma$ is that $r>r_1$. Moreover, since the function $f^{1-\gamma}s^{\gamma}$ attains its maximum when and only when $r=1-\gamma$, we conclude that $\beta<1$. Bearing in mind that $r>r_1$, $s<s_1$ and $\gamma(n+N)<q$, we can therefore dominate the right hand side of (21) by

$$\beta^{N+n} \sum \{\theta_{k,n} r_1^{1-\gamma} s_1^{\gamma})^{N+n} r_1^{\gamma(n+N)-q-N} s_1^{q-\gamma(n+N)}\}$$

where the sum is taken over $q>(n+N)$. Since the coefficients $\theta_{k,n}$ were chosen so that the measures $\mu_{N,k}$ had total mass 1, we know that the sum itself is less than or equal to 1. In particular, $\mu_{N,t}(B_{N,n})\leqq\beta^{N+n}$. Summing over n, we conclude that $\mu_{N,t}(C)\leqq\beta^N/(1-\beta)$. Since $\beta<1$, we have verified (19) for any $\gamma_1<\gamma$.

Next, we shall compute the G-Fourier Stieltjes coefficients of the measures constructed above. Using our previous notation, fix an integer $l\geqq 0$ and observe that

$$\int_G \psi_l(x)\,d\mu_{N,t}(x) = \sum_{k=0}^{p^N-1} \int_{I(k,N)} \psi_l(x)\,d\mu_{N,t}(x) =$$

$$= \lim_{n\to\infty}\left\{\sum_{x_1=0}^{p-1}\cdots\sum_{x_N=0}^{p-1}\psi_l(p^{-N}(x_1,x_2,\ldots,x_n,0,0,\ldots))\theta_{k,n}r^{(p-1)n-1}s^q\right\}.$$

An induction on N establishes the fact that this last expression can be rewritten as the following infinite product:

$$R_l(t) = \prod_{k=1}^{\infty}\left\{\sum_{j=0}^{p-1}\binom{p-1}{j}r^{p-1-j}s^j\psi_l(je_{k+N})\right\}$$

where e_{k+N} is that element of G whose $k+N$th component is 1, and all other components are zero.

Since $r=1-s$ and $s=(1-t)/p$, each factor of $R_l(t)$ can be written in the form $1+(1-t)/pQ_{l,k}$. Since $\psi_l(ve_{k+N})=\psi_l^v(e_{k+N})$ for $0\leqq v<p$, a routine calculation verifies that the coefficient $Q_{l,k}$ has a factor $(1-\psi_l(e_{k+N}))$. In particular, $Q_{l,k}\equiv 0$ for k large. It follows that

$$\int_G \psi_l(x)\,d\mu_{N,t} = \prod_{k=1}^{\infty}(1+(1-t)/pQ_{l,k}) \tag{22}$$

is a finite product for each l, N and that the number of factors in that product does not depend upon t.

Recall that we are out to show that the set E defined by (11) is not a set of uniqueness for G. Assume, to the contrary, that E is a set of uniqueness for G. Then by Theorem 1 there exists a sequence $E_1, E_2, \ldots$ of elementary sets of uniqueness for G such that $E=\bigcup_{i=1}^{\infty} E_i$. Since the closure of an elementary set of uniqueness is also an elementary set of uniqueness, we may suppose that each E_i is closed. By the Baire category theorem, then, there exists an index i and an open set $U\subseteqq G$ such that $U\cap E_i$ is dense in $U\cap E$.

Let $y\in U\cap E$. The topology on G allows us to choose an n so large that $(y_1,\ldots,y_n,x_1,x_2,\ldots)\in U$ for any choice of the coordinates $x_1, x_2, \ldots$. By the definition of E, then, it follows that the set $H=\{(y_1,\ldots,y_n,x_1,x_2,\ldots):(x_1,x_2,\ldots)\in E\}$ is a subset of $U\cup E$. Since E_i is closed and $U\cap E_i$ is dense in $U\cap E$, we conclude that H is a subset of E_i. In particular, H is also an elementary set of uniqueness for G. By Theorem 2, then, so is E.

We can therefore choose a sequence of functions $f_1, f_2, \ldots$ in $\mathcal{A}(G)$ which converges to 1 in the weak* topology such that each f_n vanishes on E. We shall use this sequence and the measures $\mu_{N,t}$ to obtain a contradiction.

For each complex number z and each integer $l \geqq 0$ set $R_l(z) = \prod_{k=1}^{\infty} \{1+(1-z)/pQ_{l,k}\}$, and $g_{n,N}(z) = \sum_{l=0}^{\infty} a_l^{(n)} R_l(z)$, for $n, N = 1, 2, \ldots$, where for each $n \geqq 1$, the numbers $a_l^{(n)}$ $(l=0, 1, \ldots)$ represent the G-Fourier coefficients of f_n. Both of these functions are well defined on the interior of the unit disc. Indeed, the product $R_l(z)$ is finite and by equation (22) we have that $R_l(t) = \int_G \psi_l(x)\, d\mu_{N,t}$ for $l=0, 1, \ldots$ and $-1<t<1$. Since the total mass of $\mu_{N,t}$ is 1, it follows that $|R_l(t)| \leqq 1$ for $l=0, 1, \ldots$ and $-1<t<1$. In particular, the sequence $R_1(z), R_2(z), \ldots$ is uniformly bounded and thus, since each f_n belongs to $\mathscr{A}(G)$, we know that the functions $g_{n,N}(z)$ form a normal family of holomorphic functions which are uniformly bounded on the open unit disc. Hence, given any pair of sequences $n_1, n_2, \ldots$ and $N_1, N_2, \ldots$ we may choose a subsequence of $\{g_{n_j, N_j}\}_{j=1}^{\infty}$ which converges to a function h which is holomorphic on the unit disc. We shall reach our contradiction by choosing the sequences $\{n_j\}$ and $\{N_j\}$ so that h vanishes on the segment $(1-p\gamma', 1)$ and also satisfies $|h(0)| \geqq \frac{1}{2}$.

Toward this, multiply equation (22) by $a_l^{(n)}$ and sum over l to conclude that $g_{n,N}(t) = \int_G f_n(x)\, d\mu_{N,t}$. Since each function f_n vanishes on E, and since $\mu_{N,t}(E)$ converges uniformly to 1 for $1-p\gamma' \leqq t \leqq 1$, we can therefore choose integers $N_1, N_2, \ldots$ such that $|g_{n,N_j}(t)| < 1/j$ for $t \in [1-p\gamma', 1]$ and for $n, j = 1, 2, \ldots$. This forces h to vanish on $(1-p\gamma', 1)$. On the other hand, since f_n converges to 1 in the weak* topology, as $n \to \infty$, and since $\mu_{N,0}$ has total variation 1, we know that for each integer $j \geqq 1$, $\lim_{n\to\infty} \int_G f_n(x)\, d\mu_{N,0} = 1$.

Hence, we can choose integers n_j such that $|g_{n_j, N_j}(0)| > \frac{1}{2}$. Since this implies $|h(0)| \geqq \frac{1}{2}$, we have reached the desired contradiction, thereby completing the proof that E is not a set of uniqueness for G.

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О ПОВЕДЕНИИ ТЕМПЕРАТУРНЫХ ФРОНТОВ В СРЕДАХ С НЕЛИНЕЙНОЙ ТЕПЛОПРОВОДНОСТЬЮ ПРИ НАЛИЧИИ ПОГЛОЩЕНИЯ. II

Р. КЕРШНЕР (Будапешт)

§ 1. Введение. Предлагаемая работа посвящена изучению первой краевой задачи для уравнения

$$Lu \equiv -u_t + (u^\mu)_{xx} - cu^\nu = 0 \tag{1}$$

в четверти плоскости $\mathbf{R}^2_+ = \{(t, x):\ t \geqq 0,\ x \geqq 0\}$ с начальным и граничным условиями

$$\begin{aligned} u(0, x) &= u_0(x), \quad 0 \leqq x < \infty \\ u(t, 0) &= u_1(t), \quad 0 \leqq t < \infty. \end{aligned} \tag{2}$$

Здесь $\mu > 1$, $\nu > 0$ и $c > 0$ — постоянные. Относительно функций $u_0(x)$ и $u_1(t)$ мы будем предполагать следующее:

Условие А. $u_0(x) \in C([0, \infty))$, $u_0(x) > 0$ при $0 \leqq x < l$, $u_0(x) = 0$ при $x \geqq l$, $u_0(x)$ имеет ограниченную обобщенную производную $(u_0^\sigma)_x$, где $\sigma = \max\left(\mu - 1, \dfrac{\mu - \nu}{2}\right)$.

Условие Б. $u_1(t) \in C^1([0, \infty))$, $u_1(t) > 0$ при $0 \leqq t < \infty$, $u_1(0) = u_0(0) > 0$.

Уравнение (1) — вырождающееся квазилинейное параболическое уравнение. Оно описывает, например, продесс теплопередачи с тепловым поглощением. При этом коэффициент теплопроводности и интенсивность поглощения степенным образом зависят от температуры.

Имеют место следующие факты (см. напр. [2], [3]):

1. при каждом $t_0 > 0$ существует такое x_0, что $u(t_0, x) = 0$ при $x \geqq x_0$ («тепловые возмущения распространяются с конечной скоростью»).

2. задача (1), (2), вообще говоря, не имеет классического решения.

Определение 1. Неотрицательная в $\mathbf{R}^2_+$ функция $u(t, x)$, удовлетворяющая условия Гельдера и ограниченная во всякой полуполосе $\mathbf{R}^2_+ \cap \{t \leqq T < \infty\}$, называется обобщенным решением уравнения (1), если для $u(t, x)$ выполняется интегральное тождество

$$I(u, f;\, t_0, t_1;\, x_0, x_1) = \int_{t_0}^{t_1} \int_{x_0}^{x_1} (uf_t + u^\mu f_{xx} - cu^\nu f)\, dx\, dt - \int_{x_0}^{x_1} uf\, dx \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} u^\mu f_x\, dt \Big|_{x_0}^{x_1} = 0,$$

где $t_0 < t_1$, $x_0 < x_1$ — произвольные числа, такие, что прямоугольник $P = [t_0, t_1] \times [x_0, x_1]$ содержится в $\mathbf{R}^2_+$, а $f(t, x) \in C^{1,2}_{t,x}(P)$ — произвольная функция, равная нулю при $x = x_0$ и $x = x_1$.

Определение 2. Обобщенным решением задачи (1), (2) называется обобщенное решение уравнения (1), удовлетворяющее условиям (2).

Определение 3. Неотрицательная в $\mathbf{R}_+^2$ функция $v(t, x)$, удовлетворяющая условию Гельдера и ограниченная при ограниченных t, называется обобщенным суперрешением уравнения (1), если для $v(t, x)$ выполняется интегральное неравенство

$$I(v, f;\ t_0, t_1;\ x_0, x_1) \leqq 0,$$

каковы бы ни были числа $t_0 < t_1$, $x_0 < x_1$, такие, что $P \subset \mathbf{R}_+^2$ и неотрицательная функция $f(t, x) \in C_{t,x}^{1,2}(P)$, равная нулю при $x = x_0$, $x = x_1$.

Замечание. Если $v(t, x)$ является гладкой вне некоторой непрерывной кривой вида $x = \xi(t)$, удовлетворяет там неравенству $Lv \leqq 0$ и производная $\partial v^\mu / \partial x$ непрерывна при $x = \xi(t)$, то с помощью интегрирования по частям легко убедиться, что $v(t, x)$ является обобщенным суперрешением уравнения (1) в $\mathbf{R}_+^2$.

Существование и единственность обобщенного решения задачи (1), (2) доказаны в [3]. Там же можно найти доказательства следующих утверждений, используемых в дальнейшем.

Пусть $u(t, x)$ — обобщенное решение задачи (1), (2).

Лемма 1. *Если $y(t, x)$ — обобщенное суперрешение уравнения* (1), $u_0(x) \leqq$ $\leqq y(0, x)$ *и* $u_1(t) \leqq y(t, 0)$, *то* $u(t, x) \leqq y(t, x)$ *всюду в* $\mathbf{R}_+^2$.

Обозначим через D область $\{0 < t < T,\ 0 < x < \xi(t)\}$, где $\xi(t) \in C([0, T])$. Пусть Γ — ее граница.

Лемма 2. *Пусть* $z(t, x) \in C_{t,x}^{1,2}(D) \cap C(\bar{D})$, $z(t, x) > 0$, $Lz > 0$ *в* D. *Тогда если* $u(t, x) \geqq z(t, x)$ *на* Γ, *то* $u(t, x) \geqq z(t, x)$ *всюду в* D.

В первой части работы изучается глобальное поведение носителя обобщенного решения задачи (1), (2). Известно [4], что если $u_1(t) \geqq c(t+1)^{-\frac{1}{\mu-1}}$, то в случае $\nu \geqq \mu$ «отсутствует локализация возмущений», то есть для каждого $x_0 \geqq 0$ существует такое $t_0 > 0$, что $u(t, x_0) > 0$ при $t > t_0$. При более сильном убывании функции $u_1(t)$ ответ неизвестен.

Аналогично тому, как это сделано в случае задачи Коши в работе [2], можно доказать, что при $\nu \geqq 1$ «не происходит движение обратного фронта волны», то есть если $u_0(x_0) > 0$, то $u(t, x_0) > 0$ при всех $t > 0$.

Нас будет интересовать случай «сильного поглощения», и в основном будем предполагать, что $\nu < 1$. В § 2 мы покажем, например, что если $u_1(t)$ достаточно регулярно стремится к нулю, то в задаче (1), (2) «происходит полное проникание волны охлаждения», то есть для каждого $x_0 \in (0, l)$ существует такое $t_0 > 0$, что $u(t, x_0) = 0$ при $t \geqq t_0$, хотя $u(t, x_0) > 0$ при $t < t_0$. Если $u_1(t)$ стремится к бесконечности, то в задаче (1), (2) даже нет «локалозации возмущений».

Во второй части работы исследуется вопрос о поведении носителя обобщенного решения задачи (1), (2) вблизи точки $x = l$. Оказывается, что в зависимости от порядка обращения в нуль функции $u_0(x)$ в точке $x = l$, носитель может и расширяться и ссужаться.

Объединяя результаты §§ 2 и 3 мы видим, что носитель обобщенного решения задачи (1), (2) может сначала сужаться и потом расширяться, а также сначала расширяться и потом сужаться, то есть функция $x=\xi(t)$, задающая границу носителя, вообще говоря, не является монотонной.

Настоящая работа является продолжением работы [5], которая посвящена изучению задачи Коши для уравнения (1) в случае $\nu<1$ (случай «полного остывания за конечное время»).

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§ 2. Глобальное поведение носителя решений. Обозначин через E множество положительных, непрерывно дифференцируемых, монотонных и стремящихся к бесконечности при $t\to\infty$ функций, для которых при всех $t\geqq 0$ и $\delta>0$ выполнено неравенство

$$f^{-1-\delta}(t)f'(t)\leqq C_1. \tag{3}$$

Замечание 1. К множеству E принадлежат, например, степенные функции $(t+1)^n$, $n>0$, а также $\exp(t+1)^n$, $\ln(t+2)$, сумма и произведение таких функций.

Вот пример положительной, непрерывно дифференцируемой, монотонной и стремящееся к бесконечности при $t\to\infty$ функции, не удовлетворяющей неравенству (3).

Пример 1. Обозначим через $h(t)$ следующую кусочно-линейную функцию: $h(n)=n$, $h(t)=2n^4(t-n)+n$ при $t\in\left(n-\dfrac{1}{2n^3},\, n\right)$, $h(t)=2n^4(n-t)+n$ при $t\in\left(n,\, n+\dfrac{1}{2n^3}\right)$ $(n=1, 2, \ldots)$ и $h(t)=0$ вне этих множеств. Очевидно, что $\int\limits_0^\infty h(t)\,dt<+\infty$. Положим $f(t)=\left[e^{-t}+\int\limits_t^\infty h(s)\,ds\right]^{-\frac{1}{\delta}}$. Ясно, что $f'(t)>0$ и $\lim\limits_{t\to\infty}f(t)=\infty$. Однако $f^{-1-\delta}(n)f'(n)=\dfrac{1}{\delta}(e^{-n}+n)\to\infty$ при $n\to\infty$.

Теорема 1. *Пусть* $\nu<1$, *а функция* $u_1(t)$ *такая, что* $u_1(t)\leqq f^{-1}(t)$ *при всех* $t\geqq 0$, *где* $f(t)\in E$. *Тогда для каждой точки* x_0 *из* $(0, l)$ *существует такое* $t_0>0$, *что* $u(t, x_0)=0$ *при* $t\geqq t_0$.

Иными словами: в задаче (1), (2) происходит полное проникание волны охлаждения.

Доказательство. Рассмотрим в $\mathbf{R}^2_+$ вспомогательную функцию

$$v(t, x)=\begin{cases}\sigma f^{-\alpha}(\varepsilon t+a)A^{\omega} & \text{при}\quad A\equiv L-x\ln f^{\alpha}(\varepsilon t+a)>0\\ 0 & \text{при}\quad A\leqq 0,\end{cases}$$

где положительные постоянные α, ε, ω, σ, и L будут выбраны ниже. Постоянная $a>0$ выбрана так, чтобы было $f(a)>1$.

Пусть $M = \max u_0(x)$. Нетрудно видеть, что неравенство $v(0, x) \geqq u_0(x)$ при $\sigma > 1$ вытекает из неравенства

$$(4) \qquad L\sigma^{1/\omega} \geqq M^{1/\omega}[f(a)]^{\alpha/\omega} + lf^{\alpha}(a) \equiv M_1,$$

и что неравенство $v(t, 0) \geqq u_1(t)$ при $\alpha < 1$, $\varepsilon \leqq \varepsilon_1 < 1$ есть следствие неравенства

$$(5) \qquad L\sigma^{1/\omega} \geqq f^{\frac{\alpha-1}{\omega}}(a) \equiv M_2.$$

Положим $M_3 = 1 + \max(M_1, M_2)$ и будем считать, что L и σ связаны соотношением $L\sigma^{1/\omega} = M_3$. При $A > 0$, учитывая, что $\alpha x < L \ln^{-1} f$, находим

$$(6) \quad Lv = \sigma\alpha\varepsilon f^{-\alpha-1} f' A^{\omega} + x\varepsilon\sigma\alpha\omega f^{-\alpha-1} f' A^{\omega-1} + \omega\mu(\omega\mu-1)\alpha^2\sigma^{\mu} f^{-\alpha\mu} \ln^2 f A^{\omega\mu-2} -$$
$$- c\sigma^{\nu} f^{-\alpha\nu} A^{\omega\nu} \leqq v^{\nu}(-c + \sigma^{1-\nu}\alpha\varepsilon f^{-\alpha(1-\nu)-1} f' A^{\omega(1-\nu)} + \sigma^{1-\nu}\varepsilon\omega f^{-\alpha(1-\nu)-1} \ln^{-1} f \cdot$$
$$\cdot f' L A^{\omega(1-\nu)-1} + \omega\mu(\omega\mu-1)\alpha^2\sigma^{\mu-\nu} f^{-\alpha(\mu-\nu)} \ln^2 f A^{\omega(\mu-\nu)-2}).$$

Обозначим величину в скобках через B. Фиксируем ω таким образом, чтобы было

$$(7) \qquad \omega > \max\left(\frac{2}{\mu-\nu}, \frac{1}{1-\nu}\right).$$

В этом случае имеем

$$B \leqq -c + \alpha\varepsilon f^{-\alpha(1-\nu)-1} f' M_3^{\omega(1-\nu)} + \varepsilon\omega f^{-\alpha(1-\nu)-1} f' \ln^{-1} f M_3^{\omega(1-\nu)} +$$
$$+ \omega\mu(\omega\mu-1)\alpha^2 f^{-\alpha(\mu-\nu)} \ln^2 f M_3^{\omega(\mu-\nu)} L^{-2} \equiv -c + I_1 + I_2 + I_3.$$

Так как $f \in E$, то величины $f^{-\alpha(1-\nu)-1}(\varepsilon t + a) f'(\varepsilon t + a)$ и $f^{-\alpha(\mu-\nu)}(\varepsilon t + a) \ln 2f(\varepsilon t + a)$ ограничены постоянной, не зависящей от ε. Сначала выберем $L > 1$ таким образом, чтобы $3I_3 < c$, потом $\sigma < 1$ выберем из равенства $L\sigma^{1/\omega} = M_3$, наконец $\varepsilon > 0$ так, чтобы выполнялись неравенства $\varepsilon < \varepsilon_1$, $3I_1 < c$ и $3I_2 < c$. При таком выборе параметров, входящих в $v(t, x)$, при $A > 0$ имеем $\mathscr{L}v < 0$. Так как $\mathscr{L}v = 0$ при $A < 0$ и вследствие (7) производная $\partial v^{\mu}/\partial x$ непрерывна при $A = 0$, то $v(t, x)$ является обобщенным суперрешением уравнения (I) в $\mathbf{R}^2_+$ и утверждение теоремы следует из Леммы 1.

Теорема 2. *Пусть $\nu < \mu$, а функция $u_1(t)$ такая, что $u_1(t) \geqq f(t)$ при всех $t \geqq 0$, где $f(t) \in E$. Тогда для каждой точки $x_0 \geqq 0$ существует такое $t_1 > 0$, что $u(t, x_0) > 0$ при $t > t_1$.*

Иными словами: в задаче (1), (2) отсутствует локализация возмущений.

Замечание 2. Эта теорема не исключает того, что из точки $x = l$ может выходить волна охлаждения, то есть что носитель $u(t, x)$ сначала сужается. Но при возрастании t фронт волны обязательно должен повернуть и неограниченно удаляться от начала координат.

Доказательство Теоремы 2. Рассмотрим в области

$$H = \{(t, x) \in \mathbf{R}^2_+ : A \equiv \varrho - xf^{-\beta}(t+a) > 0\}$$

функцию $y(t, x)=f^{\alpha}(t+a)[\varrho-xf^{-\beta}(t+a)]^{\omega}$, где положительные постоянные β, $\varrho<1$, и ω мы выберем позже, $0<\alpha<1$, постоянная $a>0$ выбрана так, чтобы $f(a)>1$.

Неравенства $y(t, 0)\leqq u_1(t)$ и $y(0, x)\leqq u_0(x)$ выполнены, если ϱ достаточно мало ($\varrho\leqq\varrho_1<1$), так как $f^{\alpha}(t+a)f^{-1}(t)$ ограничено. Находим $\big(f=f(t+a)\big)$

$$Ly=-\alpha f^{\alpha-1}f'A^{\omega}-\beta\omega xf^{\alpha-\beta-1}f'A^{\omega-1}-cf^{\alpha\nu}A^{\omega\nu}+\omega\mu(\omega\mu-1)f^{\alpha\mu-2\beta}A^{\omega\mu-2}.$$

Так как $x<\varrho f^{\beta}$ в H, то

$$Ly\geqq f^{\alpha\mu-2\beta}A^{\omega\mu-2}[\omega\mu(\omega\mu-1)-\alpha f^{2\beta-\alpha(\mu-1)-1}f'A^{2-\omega(\mu-1)}- \tag{8}$$

$$-\varrho\beta\omega f^{2\beta-\alpha(\mu-1)-1}f'A^{1-\omega(\mu-1)}-cf^{2\beta-\alpha(\mu-\nu)}A^{2-\omega(\mu-\nu)}].$$

Обозначим величину в квадратных скобках через B и фиксируем число ω следующим образом:

$$\mu^{-1}<\omega<\min\left(\frac{1}{\mu-1},\,\frac{2}{\mu-\nu}\right). \tag{9}$$

В этом случае из (8) имеем

$$B\geqq\omega\mu(\omega\mu-1)-(\alpha+\beta\omega)f^{-1-\alpha(\mu-1)+2\beta}f'\varrho^{2-\omega(\mu-1)}-cf^{-\alpha(\mu-\nu)+2\beta}\varrho^{2-\omega(\mu-\nu)}.$$

Если $2\beta<\alpha\min(\mu-1, \mu-\nu)$, то величины $f^{-\alpha(\mu-1)-1+2\beta}f'$ и $f^{-\alpha(\mu-\nu)+2\beta}$ равномерно ограничены (т. к. $f\in E$), и за счет достаточно малого ϱ ($\varrho\leqq\varrho_2$) можно достичь неравенства $B>0$ при $A>0$. Положим $\varrho=\min(\varrho_1, \varrho_2)$. Из Леммы 2 следует, что $u(t, x)\geqq y(t, x)$ в H. Теорема доказана.

Для иллюстрации теоремы 2 мы приведем два примера.

Пример 2 (ср. [1]). Пусть $\mu+\nu=2$, $u_1(t)=[a(\mu-1)t+b]^{\frac{1}{\mu-1}}$,

$$u_0(x)=\begin{cases}\left[b-\dfrac{(\mu-1)\sqrt{c+a}}{\sqrt{\mu}}x\right]^{\frac{1}{\mu-1}} & \text{при}\quad [\dots]>0,\\ 0 & \text{при}\quad [\dots]\leqq 0,\end{cases}$$

где положительные постоянные a и b любые. Тогда обощенным решением задачи (1), (2) является функция

$$u(t, x)=\begin{cases}\left[a(\mu-1)t+b-\dfrac{(\mu-1)\sqrt{c+a}}{\sqrt{\mu}}x\right]^{\frac{1}{\mu-1}} & \text{при}\quad [\dots]>0,\\ 0 & \text{при}\quad [\dots]\leqq 0.\end{cases}$$

Пример 3. Пусть $\mu+2\nu=3$, $u_1(t)=\left[\dfrac{2(1-\nu)c}{3}t+b\right]^{\frac{1}{1-\nu}}$,

$$u_0(x)=\begin{cases}b^{\frac{1}{1-\nu}}1-ab^{-3/2}x]^{\frac{1}{1-\nu}} & \text{при}\quad x<b^{3/2}a^{-1},\\ 0 & \text{при}\quad x\geqq b^{3/2}a^{-1},\end{cases}$$

где $a=(1-\nu)(5c)^{1/2}[3(3-2\nu)(2-\nu)]^{-1/2}$, $b>0$ — любое. Тогда обощенное решение задачи (1), (2) дается формулой

$$u(t,x)=\begin{cases}\left(\dfrac{2c(1-\nu)}{3}t+b\right)^{\frac{1}{1-\nu}}\left[1-ax\left(\dfrac{2c(1-\nu)}{3}t+b\right)^{-3/2}\right]^{\frac{1}{1-\nu}} & \text{при}\quad [\ldots]>0\\ 0 & \text{при}\quad [\ldots]\leqq 0.\end{cases}$$

Теорема 3. *Пусть* $\nu<\mu$, $u_1(t)\geqq c_2>0$, $c_2=\text{const}$. *Тогда существует такая положительная постоянная* $l_1<l$, *что* $u(t,x)>0$ *при* $x<l_1$.

Иными словами: в этом случае полного проникания волны охлаждения (при $\nu<1$) не происходит.

Доказательство. Нетрудно проверить, что функция

$$y(t,x)=y(x)=\begin{cases}[l_1^{\frac{\mu-\nu}{2}}-ax]^{\frac{2}{\mu-\nu}} & \text{при}\quad x<a^{-1}l_1^{\frac{\mu-\nu}{2}},\\ 0 & \text{при}\quad x\geqq a^{-1}l_1^{\frac{\mu-\nu}{2}},\end{cases}$$

где $a^2=\dfrac{c(\mu-\nu)^2}{2\mu(\mu+\nu)}$, $a>0$, при любом $l_1>0$ является обобщенным решением уравнения (1) в $\mathbf{R}_+^2$. Неравенство $y(t,0)\leqq u_1(t)$ выполнено, если $l_1\leqq c_2$. За счет дальнейшего уменьшения l_1 можно достичь того, чтобы было $y(0,x)\leqq u_0(x)$, и утверждение теоремы следует из Леммы 2.

В некоторых случаях можно утверждать больше. Например, имеет место следующая

Теорема 4. *Пусть* $\nu<\mu$, $u_1(t)\geqq C_2>0$, C_2 — *постоянная*. $u_0(x)\geqq\sigma(l-x)^\omega$, *где* $\mu^{-1}<\omega\leqq 2(\mu-\nu)^{-1}$, $\sigma l^\omega\leqq C_2$. *Тогда если* $c<\sigma^{\mu-\nu}\omega\mu(\omega\mu-1)l^{\omega(\mu-\nu)-2}$, *то* $u(t,x)>0$ *при* $0\leqq x<l$ *и при всех* $t\geqq 0$.

Доказательство. Сравнивая функции $u(t,x)$ и

$$y(t,x)=\begin{cases}\sigma(l-x)^\omega & \text{при}\quad x<l,\\ 0 & \text{при}\quad x\geqq l,\end{cases}$$

при $x=0$ и $t=0$, мы видим, что там $u(t,x)\geqq y(t,x)$. Далее,

$$Ly=\sigma^\nu(l-x)^{\omega\nu}[\sigma^{\mu-\nu}\omega\mu(\omega\mu-1)(l-x)^{\omega(\mu-\nu)-2}-c].$$

В силу условий теоремы $Ly>0$ при $0\leqq x<l$. В силу Леммы 2 $u(t,x)\geqq y(t,x)$ всюду в $\mathbf{R}_+^2$.

Теорема 5. *Пусть* $\nu<\mu$, $u_1(t)\leqq C_2>0$, — *постоянная*. $u_0(x)\leqq\sigma(l-x)^\omega$, $\omega\geqq 2(\mu-\nu)^{-1}$, $\sigma l^\omega\geqq C_2$. *Тогда если* $c>\sigma^{\mu-\nu}\omega\mu(\omega\mu-1)l^{\omega(\mu-\nu)-2}$, *то* $u(t,x)=0$ *при* $x\geqq l$.

Доказательство. Сравниваем решение $u(t,x)$ с функцией $y(t,x)$ из доказательства предыдущей теоремы. При $x=0$ и $t=0$ $u(t,x)\leqq y(t,x)$. По условиям теоремы $Ly<0$ при $0<x<l$. В силу Леммы 1 $u(t,x)\leqq y(t,x)$ всюду в $\mathbf{R}_+^2$. Теорема доказана.

§ 3. Локальное поведение носителя решения. Две последних теоремы предыдующего параграфа, помимо того, что они имеют глобальный характер, содержат и информации локального характера. А именно, при условиях Теоремы 4 из точки $x=l$ не выходит волна охлаждения (то есть носитель $u(t, x)$ сначала не сужается), при условиях Теоремы 5 — волна разогрева (то есть носитель $u(t, x)$ сначала не расширяется).

Определение 4. Будем говорить, что из точки $x=l$ выходит волна охлаждения, если $u(t, l)=0$ при $0<t\leqq t_0$.

Определение 5. Будем говорить, что из точки $x=l$ выходит волна разогрева, если существует такое $t_0>0$, что $u(t, l)>0$ при $t\in(0, t_0)$.

В дальнейщем мы увидим, что поведение носителя обобщенного решения задачи (1), (2) вблизи точки $x=l$ в основном зависит от порядка обращения в нуль функции $u_0(x)$ в этой точке. При некоторых граничных значениях этого параметра на поведение носителя может влиять и максимум $u_0(x)$.

В течение этого параграфа через $\varkappa$ мы будем обозначать порядок обращения в нуль функции $u_0(x)$ в точке $x=l$: $\lim\limits_{x\to l-0}\dfrac{u_0(x)}{(l-x)^\varkappa}=C_0$, где $0<C_0<\infty$. Будем считать, что $\varkappa\mu>1$. Как и раньше, положим $M=\max u_0(x)$.

Теорема 6. *Пусть* $\nu<1$, $u(t, x)$ — *обобщенное решение задачи* (1), (2). *Тогда если* $\varkappa\leqq\dfrac{1}{1-\nu}$, $\varkappa<\min\left(\dfrac{1}{\mu-1}, \dfrac{2}{\mu-\nu}\right)$, *то из точки* $x=l$ *выходит волна разогрева.*

Доказательство. Рассмотрим в области

$$H=\{(t, x)\colon\ 0<t<\varepsilon,\ A\equiv 1-xl^{-1}(t+1)^{-1}\}$$

функцию $y(t, x)=\sigma(t+1)^{-\gamma}A^\varkappa$, где положительные постоянные σ, γ, и $\varepsilon<1$ будут выбраны ниже. Выберем σ таким образом, чтобы выполнялись неравенства $y(0, x)\leqq u_0(x)$ и $y(t, 0)\leqq u_1(t)$ (скажем при $t\leqq 1$) — это очевидно возможно. Покажем, что $Ly>0$ в H при подходящем выборе γ и ε. Находим

$$Ly=\gamma\sigma(t+1)^{-\gamma-1}A^\varkappa-\varkappa\sigma(t+1)^{-\gamma-1}A^{\varkappa-1}xl^{-1}(t+1)^{-1}+ \tag{10}$$

$$+\sigma^\mu l^{-2}\varkappa\mu(\varkappa\mu-1)(t+1)^{-\gamma\mu-2}A^{\varkappa\mu-2}-c\sigma^\nu(t+1)^{-\gamma\nu}A^{\varkappa\nu}\equiv I_1+\ldots+I_4.$$

Расмотрим сначала случай, когда $x\leqq l(1-\varepsilon)$. В этом случае $A\geqq\varepsilon$. Так как $I_3\geqq 0$ и $\varkappa<(\mu-1)^{-1}$, то

$$Ly\geqq\sigma(t+1)^{-\gamma-1}A^\varkappa\left[\gamma-\frac{\varkappa}{\varepsilon}-c\sigma^{\nu-1}(t+1)^{1+\gamma(1-\nu)}\varepsilon^{-\varkappa(1-\nu)}\right].$$

Обозначим величину в квадратных скобках через B_1. Рассмотрим теперь случай, когда $x>l(1-\varepsilon)$. Здесь $A<2\varepsilon$. Так как $I_1>0$, $\varkappa<(\mu-1)^{-1}$ и $\varkappa<2(\mu-\nu)^{-1}$ то

$$Ly\geqq(t+1)^{-\gamma\mu-2}A^{\varkappa\mu-2}[\sigma^\mu l^{-2}\varkappa\mu(\varkappa\mu-1)-\varkappa\sigma(t+1)^{\gamma(\mu-1)}(2\varepsilon)^{1-\varkappa(\mu-1)}-$$

$$-c\sigma^\nu(t+1)^{2+\gamma(\mu-\nu)}(2\varepsilon)^{2-\varkappa(\mu-\nu)}].$$

Обозначим величину в квадратных скобках через B_2. Имеем

$$B_2 \geqq C_1 - C_2(1+\varepsilon)^{\gamma(\mu-1)}\varepsilon^{1-\varkappa(\mu-1)} - C_3(1+\varepsilon)^{2+\gamma(\mu-\nu)}\varepsilon^{2-\varkappa(\mu-\nu)} \equiv B_3,$$

где константы C_i $(i=1, 2, 3)$ от ε не зависят. Так как $\varkappa \leqq (1-\nu)^{-1}$, то существует такая не зависящая от ε постоянная $k>0$, что при $\gamma = k\varepsilon^{-1}$ выполняется неравенство $B_1>0$ для всех $\varepsilon \in (0, 1)$, имея ввиду, что $\lim\limits_{\varepsilon\to 0}(1+\varepsilon)^{k\varepsilon^{-1}} = \exp(k)$. В силу этого же равенства можно так выбрать $\varepsilon>0$, чтобы выполнялось и неравенство $B_3>0$. При так выбранных ε и γ: $Ly>0$ в H, и утверждение теоремы вытекает из Леммы 2.

Теорема 7. *Пусть $\nu<1$, $u(t, x)$ — обобщенное решение задачи* (1), (2). *Тогда если $\varkappa \geqq \max\left(2(\mu-1)^{-1}, (1-\nu)^{-1}\right)$, то из точки $x=l$ выходит волна охлаждения.*

Доказательство. Рассмотрим в $\mathbf{R}^2_+ \cap \{t \leqq \varepsilon\}$ функцию

$$y(t, x) = \begin{cases} 0 & \text{при} \quad A \equiv \varrho - t - \varrho l^{-1} x \leqq 0 \\ K(\varepsilon\varrho^{\varkappa})^{-1}(t+\varepsilon)\cdot A^{\varkappa} & \text{при} \quad A>0, \end{cases}$$

где A, ε и ϱ — положительные постоянные, которые мы выберем позже. Пусть $K \geqq K_0$ таково, что $y(0, x) \geqq u_0(x)$. Неравенство $y(t, 0) = K(\varepsilon\varrho^{\varkappa})^{-1}(t+\varepsilon)(\varrho - t)^{\varkappa} \geqq \geqq u_1(t)$ при $t \leqq \varepsilon < 1$ выполнено, если $\varrho>2$ и $K>u_1(0)2^{\varkappa} = K_1$. Положим $K = \max(K_0, K_1)$. При $A>0$ находим

$$Ly = -K(\varepsilon\varrho^{\varkappa})^{-1}A^{\varkappa} + \varkappa K(t+\varepsilon)(\varepsilon\varrho^{\varkappa})^{-1}A^{\varkappa-1} +$$
$$+\varkappa\mu(\varkappa\mu-1)\varrho^2 l^{-2}K^{\mu}(\varepsilon\varrho^{\varkappa})^{-\mu}(t+\varepsilon)^{\mu}A^{\varkappa\mu-2} - cK^{\nu}(\varepsilon\varrho^{\varkappa})^{-\nu}(t+\varepsilon)^{\nu}A^{\varkappa\nu} \equiv I_1 + \ldots + I_4.$$

Далее, так как $A \leqq \varrho$, $\varkappa(\mu-1) \geqq 2$ и $\varepsilon^{-1}(t+\varepsilon) \leqq 2$, получаем

$$I_1 + I_3 \leqq K(\varepsilon\varrho^{\varkappa})^{-1}A^{\varkappa}[-1 + 2^{\mu}\varkappa\mu(\varkappa\mu-1)l^{-2}K^{\mu-1}\varepsilon].$$

Так как $\varkappa(1-\nu) \geqq 1$, то

$$I_2 + I_4 \leqq K^{\nu}(\varepsilon\varrho^{\varkappa})^{-\nu}(t+\varepsilon)^{\nu}A^{\varkappa\nu}[-c + \varkappa K^{1-\nu}2^{1-\nu}\varrho^{-1}].$$

Отсюда видно, что за счет достаточно малого $\varepsilon>0$ и достаточно большого ϱ можно достичь того, чтобы было $Ly<0$ при $A>0$. При $A<0$ $Ly=0$, а при $A=0$ производная $\partial y^{\mu}/\partial x$ непрерывна. Поэтому $y(t, x)$ является обобщенным суперрешением уравнения (1). Ввиду Леммы 1, теорема доказана.

Теорема 8. *Пусть $\nu<1$ и константа $K>0$ такова, что выполнены неравенства*

$$K \geqq 2u_1(t) \quad (t \leqq 1), \quad u_0(x) \leqq K(1 - xl^{-1})^{\varkappa}, \tag{11}$$

где $\varkappa \geqq \max\left(\dfrac{1}{\mu-1}, \dfrac{2}{\mu-\nu}\right)$. Тогда если $K^{\mu-\nu} < \dfrac{cl^2}{\varkappa\mu(\varkappa\mu-1)}$, то из точки $\varkappa = l$ выходит волна охлаждения.

Доказательство. Рассмотрим в $\mathbf{R}^2_+ \cap \{t \leqq \varepsilon\}$ функцию

$$y(t, x) = \begin{cases} 0 & \text{при} \quad A \equiv 1 - \dfrac{x}{l}(t+1)^{\beta} < 0, \\ K(t+1)^{-\alpha}A^{\varkappa} & \text{при} \quad A \geqq 0, \end{cases}$$

где положительные постоянные α, β и ε будут выбраны ниже. По предположению $y(0, x) \geqq u_0(x)$, а $y(t, 0) = K(t+1)^{-\alpha} \geqq K(t+1)^{-1} \geqq K(\varepsilon+1)^{-1} > \frac{K}{2} \geqq u_1(t)$ при $\varepsilon < 1$, $\alpha < 1$ и $t \leqq \varepsilon < 1$. При $A > 0$, то есть при $x(t+1)^\beta < l$ находим

$$Ly \leqq \alpha K(t+1)^{-\alpha-1}A^\varkappa + \varkappa\beta K(t+1)^{-\alpha-1}A^{\varkappa-1} + K^\mu l^{-2}\varkappa\mu(\varkappa\mu-1)(t+1)^{-\alpha\mu+2\beta}A^{\varkappa\mu-2} -$$
$$-cK^\nu(t+1)^{-\alpha\nu}A^{\varkappa\nu} = K^\nu(t+1)^{-\alpha\nu}A^{\varkappa\nu}[-c + \alpha K^{1-\nu}(t+1)^{-\alpha(1-\nu)-1}A^{\varkappa(1-\nu)} +$$
$$+ l^{-2}K^{\mu-\nu}\varkappa\mu(\varkappa\mu-1)(t+1)^{-\alpha(\mu-\nu)+2\beta}A^{\varkappa(\mu-\nu)-2} + K^{1-\nu}\varkappa\beta(t+1)^{-\alpha(1-\nu)-1}A^{\varkappa(1-\nu)-1}].$$

Так как $t+1 < 2$, то за счет достаточно малых α и β величину в квадратных скобках можно сделать отрицательной. Доказательство теперь можно закончить так же, как и в предыдущей теореме.

Замечание 3. Пусть $u_1(t)$ ограничена. Тогда в доказательстве Теоремы 8 можно взять $\varepsilon > 0$ как угодно большим, если подчинить α и β дополнительному условию $2\beta \leqq \alpha(\mu-\nu)$. Таким образом, в этом случае мы получаем глобальную оценку для носителя обобщенного решения задачи (1), (2).

Замечание 4. Из условий Теоремы 8 вытекает, что

$$\max u_0(x) \leqq K \leqq \{cl^2[\varkappa\mu(\varkappa\mu-1)]^{-1}\}^{1/\mu-\nu}.$$

Такое ограничение на величину $\max u_0(x)$ связано с существом дела. Чтобы убедиться в этом, рассмотрим следующий пример.

Пусть $\mu+\nu=2$. Тогда $(1-\nu)^{-1} = (\mu-1)^{-1} = 2(\mu-\nu)^{-1}$. Пусть $\varkappa = (1-\nu)^{-1}$ и $u(t, x)$ — обобщенное решение краевой задачи

$$Lu = -u_t + (u^\mu)_{xx} - cu^{2-\mu},$$

$$u_0(x) = \begin{cases} \alpha^\varkappa \left(1 - \dfrac{x}{l}\right)^\varkappa & \text{при} \quad x < l \\ 0 & \text{при} \quad x \geqq l, \end{cases} \qquad u_1(t) = \alpha^\varkappa(\beta t+1)^{-\varkappa},$$

где $\alpha > 0$ и β — постоянные. Введем функцию

$$v(t, x) = \begin{cases} y(t, x) = \alpha^\varkappa(\beta t+1)^{-\varkappa}A^\varkappa & \text{при} \quad A \equiv 1 - \dfrac{x}{l}(\beta t+1)^\varkappa > 0 \\ 0 & \text{при} \quad A \leqq 0. \end{cases}$$

Нетрудно проверить, что при $y(t, x) > 0$

$$Lv = \varkappa y^{\varkappa(2-\mu)}\left[\frac{\alpha\beta}{(\beta t+1)^2} + \frac{\alpha^2\mu}{l^2(\mu-1)} - c(\mu-1)\right]. \tag{12}$$

Поэтому из Лемм 1, 2 и равенства (12) вытекает следующая

Теорема 9. а) *если* $\alpha^2 < cl^2(\mu-1)^2\mu^{-1}$, *то при достаточно малом* $\beta > 0$ *справедливо неравенство* $u(t, x) \leqq v(t, x)$, *то есть из точки* $x = l$ *выходит волна охлаждения.*

б) *елси* $\beta > 0$ *и* α *достаточно большое, то* $u(t, x) \geqq y(t, x)$ *для* $t < -\beta^{-1}$, *то есть при достаточно большом* $\max u_0(x)$ *из точки* $x = l$ *выходит волна разогрена.*

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LAGRANGE INTERPOLATION FOR CONTINUOUS FUNCTIONS OF BOUNDED VARIATION

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1. Introduction

We investigate the Lagrange interpolation for continuous functions of bounded variation on the Jacobi abscissas. Uniform convergence theorems will be established on the whole interval $[-1, 1]$. We give estimations for the order of convergence when f is continuous and monotone.

2. Notations and preliminary results

Let

$$(2.1) \qquad -1 < x_{nn}^{(\alpha,\beta)} < x_{n-1,n}^{(\alpha,\beta)} < \ldots < x_{2n}^{(\alpha,\beta)} < x_{1n}^{(\alpha,\beta)} < 1 \quad (\alpha, \beta > -1)$$

be the roots of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ $(\alpha, \beta > -1, n = 1, 2, \ldots;$ see, e.g. G. SZEGŐ [1]). Denote, as usual,

$$(2.2) \qquad L_n^{(\alpha,\beta)}(f; x) = \sum_{k=1}^{n} f(x_{kn}^{(\alpha,\beta)})\, l_{kn}^{(\alpha,\beta)}(x) \quad (n = 1, 2, \ldots)$$

the Lagrange interpolatory polynomials of degree $\leqq n-1$ based on the nodes (2.1), i.e. $l_{kn}^{(\alpha,\beta)}(x)$ are the k-th fundamental polynomials of the Lagrange interpolation. If $f \in C$ $(= f$ is continuous on $[-1, 1])$, then $L_n^{(\alpha,\beta)}(f; x)$ generally does not tend uniformly to $f(x)$, but supposing $f \in BC$ $(= f \in C$ and is of bounded variation on $[-1, 1])$ we can state positive results.

E.g., D. L. BERMAN [2] proved that if $-1 < \alpha, \beta < 0$, the corresponding Lagrange parabolas tend to $f(x)$ for arbitrary $x \in [-1, 1]$ whenever $f \in BC$.

For arbitrary α and β (> -1), using a general theorem due to J. L. GERONIMUS [3], we can state the *uniform* convergence, too, but only for arbitrary fixed $[a, b] \subset \subset (-1, 1)$.

Now we prove uniform convergence theorems for the *whole interval* $[-1, 1]$, using an estimation dealing with the order of convergence, for $f \in MC$ $(= f \in C$ and f is monotone on $[-1, 1])$. Our method is different from the above mentioned ones.

3. Results

3.1. Using the above notations we state

THEOREM 3.1. *Supposing* $-1<\alpha,\beta<0{,}5$, *we have*

$$\lim_{n\to\infty}\|L_n^{(\alpha,\beta)}(f;\,x)-f(x)\|=0 \tag{3.1}$$

whenever $f\in BC$.

Here $\|g\|_{[a,b]}=\max_{a\le x\le b}|g(x)|$ and $\|g\|=\|g\|_{[-1,1]}$.

3.2. The above theorem is a simple consequence of

THEOREM 3.2. *For arbitrary* $\alpha,\beta>-1$ *and* $f\in MC$,

$$|L_n^{(\alpha,\beta)}(f;\,x)-f(x)|=\begin{cases} O(1)\displaystyle\sum_{i=1}^{n}\omega\left(f;\,\frac{\sin\vartheta}{n}i+\frac{i^2}{n^2}\right)\frac{1}{i^\gamma} & \text{uniformly in } x\in[-1+\varepsilon,1],\\ O(1)\displaystyle\sum_{i=1}^{n}\omega\left(f;\,\frac{i}{n}\right)\frac{1}{i^2} & \text{uniformly in } x\in[a,b]\subset(-1,1)\end{cases} \tag{3.2}$$

where $\gamma=\min(2;\,1.5-\alpha)$ *and* $x=\cos\vartheta$. ($\omega(f;\,t)$ *is the modulus of continuity of* $f(x)$.)

To obtain from (3.2) the relation (3.1), let

$$\varepsilon_n=n^{-1}\ln n\quad\text{if}\quad -1<\alpha,\beta\le-0{,}5$$

and

$$\varepsilon_n=n^{\delta-\frac{1}{2}}\quad\text{if}\quad\max(\alpha,\beta)=\delta>-0{,}5.$$

We have by (3.2)

$$\|L_n(f;\,x)-f(x)\|=O(1)\,\omega(f;\,\varepsilon_n)\quad\text{if}\quad f\in MC.$$

If $f\in BC$, then $f=f_1-f_2$, where $f_1,f_2\in MC$, from where we obtain (3.1).

3.3. Another consequence of (3.2) is

COROLLARY 3.3. *If* $-1<\alpha,\beta\le-0{,}5$ *and* $\omega(f;\,t)\sim t^\varrho$ $(0<\varrho<0.5)$ *then for* $f\in MC$

$$|L_n^{(\alpha,\beta)}(f;\,x)-f(x)|=O(1)\left[\left(\frac{\sqrt{1-x^2}}{n}\right)^{\varrho}+\frac{1}{n^{2\varrho}}\right] \tag{3.3}$$

uniformly for $|x|\le1$, *further when* $\alpha,\beta>-1$ *and* $\omega(f;\,t)\sim t^\varrho$ $(0<\varrho<1)$ *then*

$$\|L_n^{(\alpha,\beta)}(f;\,x)-f(x)\|_{[a,b]}=O(1)\,\omega\left(f;\,\frac{1}{n}\right)\quad\textit{if}\quad[a,b]\subset(-1,1),\quad f\in MC. \tag{3.4}$$

These formulae of Timan and Jackson type can be obtained by simple calculation.

3.4. It is interesting to compare (3.2) to

$$|H_n^{(\alpha,\beta)}(f;\,x)-f(x)| = O(1)\sum_{i=1}^{n}\omega\left(f;\,\frac{i\sin\vartheta}{n}+\frac{i^2}{n^2}\right)i^{2\eta-1}\quad (x\in[-1+\varepsilon,1])$$

where $H_n^{(\alpha,\beta)}(f;\,x)$ is the Hermite—Fejér interpolatory polynomial of degree $\leqq 2n-1$ defined by

$$H_n(f;\,x_k^{(\alpha,\beta)})=f(x_k^{(\alpha,\beta)}),\quad H_n'(f;\,x_k^{(\alpha,\beta)})=0\quad (k=1,2,\dots,n);\ f\in C$$

and $\eta=\max(-0{,}5,\alpha)$ (see [6], (2.1)).

3.5. Let us consider the Lagrange interpolatory polynomials $R_{n+1}^{(\alpha+2,\beta)}(f;\,x)$ [or $Q_{n+2}^{(\alpha+2,\beta+2)}(f;\,x)$] of degree $\leqq n$ [or $\leqq n+1$] defined by

$$R_{n+1}^{(\alpha+2,\beta)}(f;\,x_{kn}^{(\alpha+2,\beta)})=f(x_{kn}^{(\alpha+2,\beta)})\quad (k=0,1,\dots,n)$$

$$[\text{or } Q_{n+2}^{(\alpha+2,\beta+2)}(f;\,x_{kn}^{(\alpha+2,\beta+2)})=f(x_{kn}^{(\alpha+2,\beta+2)})\quad (k=0,1,\dots,n+1)]$$

where $x_{on}\equiv 1$ and $x_{n+1,n}\equiv -1$ (see, e.g. [5]). Then, using the methods of [5] and of this paper, we can prove for $R_{n+1}^{(\alpha+2,\beta)}(f;\,x)$ [or $Q_{n+2}^{(\alpha+2,\beta+2)}(f;\,x)$] the statements (3.1)—(3.3), using the functions and values of α and β defined in the quoted theorems. We omit the details.

3.6. Theorems 3.1—3.3 and the Part 3.5 are valid for $\min(\alpha,\beta)=-1$, using [1], (4.22.2) and the ideas of [5]. We omit the further details.

3.7. In the cases considered in 3.5 and 3.6, we often can get estimations of Teliakovskii—Gopengauz type instead of (3.3).

3.8. As for the trigonometric case, see e.g. G. NÉVAI [7].

4. Proof of Theorem 3.2

4.1. To make the method more clear, first we investigate the case $\alpha=\beta=-1/2$, when $P_n^{(-1/2,-1/2)}=c_nT_n(x)=c_n\cos n\vartheta$. As it is well known (omitting the superfluous notations),

$$l_{kn}(x)=\frac{(-1)^{k-1}T_n(x)\sin\vartheta_k}{n(\cos\vartheta-\cos\vartheta_k)}\quad (k=1,2,\dots,n),\tag{4.1}$$

where $x_{kn}^{(\alpha,\beta)}=\cos\vartheta_{kn}^{(\alpha,\beta)}$ and for $\alpha=\beta=-1/2$, $\vartheta_k=(2k-1)\pi(2n)^{-1}$. Let $\min\limits_{1\leqq k\leqq n}|x-x_k|=|x-x_j|$, further denote

$$\begin{cases} I_0=\left[\vartheta-\dfrac{\pi}{n},\,\vartheta\right), & I_r=\left[\vartheta-2^r\dfrac{\pi}{n},\,\vartheta-2^{r-1}\dfrac{\pi}{n}\right)\quad (r=1,2,\dots),\\[2mm] K_0=\left[\vartheta,\,\vartheta+\dfrac{\pi}{n}\right), & K_r=\left[\vartheta+2^{r-1}\dfrac{\pi}{n},\,\vartheta+2^r\dfrac{\pi}{n}\right)\quad (r=1,2,\dots).\end{cases}\tag{4.2}$$

We have

$$(4.3)\qquad |L_n(f;\,x)-f(x)| = \left|\sum_{k=1}^{n}\{[f(x_k)-f(x)]\,l_k(x)\}\right| \leqq$$

$$\leqq \sum_{r=0}^{q_1}\Big|\sum_{\vartheta_k\in I_r}\{\dots\}\Big| + \sum_{r=0}^{q_2}\Big|\sum_{\vartheta_k\in K_r}\{\dots\}\Big|,$$

where $0\leqq q_1,\, q_2\leqq \overset{2}{\log}\, n < c\cdot \ln n$.

4.2. To go further we mention that

$$(4.4)\qquad |f(x)-f(x_k)| = O(1)\,\omega\left(\frac{i\sin\vartheta}{n}+\frac{i^2}{n^2}\right),$$

where $i=|k-j|$ if $k\neq j$ and $i=1$ for $k=j$ (see [4], (2.2)).

4.3. If $r=0$ we obtain

$$(4.5)\qquad |l_k(x)| = O(1)\quad \text{if}\quad \vartheta_k\in I_0\cup K_0$$

(see e.g. (4.1)). Using (4.4) and (4.5) we get

$$(4.6)\qquad \Big|\sum_{\vartheta_k\in I_0\cup K_0}\{\dots\}\Big| = O(1)\,\omega\left(\frac{\sin\vartheta}{n}+\frac{1}{n^2}\right).$$

4.4. We suppose $f(x)$ is monotone increasing, further let $0\leqq\vartheta\leqq\pi/2$. Then, using the Abel inequality three times, the relations (4.4), $|T_n(x)|\leqq 1$, we get

$$(4.7)\qquad \left|\sum_{\vartheta_k\in I_r}\left\{[f(x_k)-f(x)](-1)^{k-1}\frac{T_n(x)\sin\vartheta_k}{n(\cos\vartheta_k-\cos\vartheta)}\right\}\right| \leqq$$

$$\leqq \omega\left(\frac{\sin\vartheta}{n}\,2^r+\frac{2^{2r}}{n^2}\right)\max\left|\sum(-1)^{k-1}\frac{T_n(x)}{n}\,\frac{\sin\vartheta_k}{(\cos\vartheta-\cos\vartheta_k)}\right| \leqq$$

$$\leqq \omega\left(\frac{\sin\vartheta}{n}\,2^r+\frac{2^{2r}}{n^2}\right)\sin\left(\vartheta-2^{r-1}\frac{\pi}{n}\right)\max\left|\sum(-1)^{k-1}\frac{T_n(x)}{n(\cos\vartheta-\cos\vartheta_k)}\right| \leqq$$

$$\leqq \omega\left(\frac{\sin\vartheta}{n}\,2^r+\frac{2^{2r}}{n^2}\right)\frac{\sin\left(\vartheta-2^{r-1}\dfrac{\pi}{n}\right)}{2\sin\left(\vartheta-2^{r-2}\dfrac{\pi}{n}\right)\sin 2^{r-2}\dfrac{\pi}{n}}\max\left|\sum(-1)^{k-1}\frac{T_n(x)}{n}\right| \leqq$$

$$\leqq \frac{\omega\left(\dfrac{\sin\vartheta}{n}\,2^r+\dfrac{2^{2r}}{n^2}\right)}{2^{r-1}}\qquad (r=1,\,2,\,\dots)$$

where, supposing that the ϑ_k's in I_r are $\vartheta_t,\vartheta_{t+1},\dots,\vartheta_{t+p}$, the notation $|\sum\dots|$ stands for $\max\limits_{0\leqq q\leqq p}\left|\sum\limits_{k=t}^{t+q}\dots\right|$.

[Remark, that here and generally when $\alpha, \beta \leqq 0$ we could have used the monotonicity properties of the Jacobi polynomials proved by D. L. BERMAN (see e.g. [2], Theorem 1, Property A)].

4.5. Using similar argument, we obtain

$$(4.8) \qquad \Big|\sum_{\vartheta_k \in K_r} \{[f(x_k)-f(x)]\, l_k(x)\}\Big| = O(1)\frac{\omega\left(\frac{\sin\vartheta}{n}2^r+\frac{2^{2r}}{n^2}\right)}{2^r} \qquad (r=1,2,\ldots).$$

4.6. Now, by (4.3) and (4.6)—(4.8) we obtain

$$|L_n(f;x)-f(x)| = O(1)\sum_{r=1}^{[\ln n]}\left[\frac{\omega\left(\frac{\sin\vartheta}{n}2^r+\frac{2^{2r}}{n^2}\right)}{2^{2r}}\sum_{i=2^{r-1}}^{2^r}1\right] =$$

$$= O(1)\sum_{r=1}^{[\ln n]}\sum_{i=2^{r-1}}^{2^r}\omega\left(\frac{\sin\vartheta}{n}i+\frac{i^2}{n^2}\right)\frac{1}{i^2}$$

as we stated. Using the symmetry of $T_n(x)$, we get the statement for $\pi/2 \leqq \vartheta \leqq \pi$, i.e. for the whole $[-1, 1]$.

4.7. Now we pass over to the general case $\alpha, \beta > -1$. First let us see some tools. To estimate $P_n(x)$, we shall use

$$(4.9) \qquad |P_n(x)| \sim |\vartheta-\vartheta_j|\,\vartheta_j^{-\alpha-1/2}n^{1/2} \sim |x-x_j|\,\vartheta_j^{-\alpha-3/2}n^{1/2}$$

uniformly in $x \in [-1+\varepsilon, 1]$ (see e.g. [5], Lemma 3.2) which, with other formulas, can be applied for the interval $[-1, 1-\varepsilon]$ by the symmetry-relation $P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$.

We often use

$$(4.10) \qquad \vartheta_{k+1}-\vartheta_k \sim \frac{1}{n} \qquad (k=0,1,2,\ldots,n).$$

(Here $\vartheta_0 \equiv 0$, $\vartheta_{n+1} \equiv \pi$; see e.g. [6], Lemma 3.1; for the symbol "$\sim$" see [1], 1.1.) By (4.10) we can verify (4.4) for arbitrary $\alpha, \beta > -1$.

4.8. Here we state the following

LEMMA 4.1. *For certain integers $M = M(\alpha,\beta)$ and $M_1 = M_1(\alpha,\beta)$*

$$(4.11) \qquad P_n'^{(\alpha,\beta)}(x_{k+\mu}) = (-1)^{k-1}\sqrt{\frac{n}{\pi}}\frac{1+O(k^{-1})}{2\left(\sin\frac{\vartheta_{k+\mu}}{2}\right)^{\alpha+3/2}\left(\cos\frac{\vartheta_{k+\mu}}{2}\right)^{\beta+3/2}} \overset{\text{def}}{=\!=}$$

$$\overset{\text{def}}{=\!=} F_{k+\mu}[1+O(k^{-1})]$$

uniformly in n if $k \geqq M$ and $\vartheta_{k+\mu} \leqq \pi-\varepsilon$. Here $0 \leqq |\mu| \leqq M_1$, $\varepsilon > 0$ and $n \geqq n_0(\alpha,\beta)$.

Indeed, using the considerations of [1], Theorem 8.9.1, one can obtain that for each fixed $\varepsilon>0$ the interval

$$B_k=\left[\frac{(k-1/2)\pi-\gamma-\varrho}{N}, \frac{(k-1/2)\pi-\gamma+\varrho}{N}\right], \quad \text{for} \quad k\geqq M(\alpha,\beta),$$

contains exactly one zero of $P_n(\cos\vartheta)$ if $B_k\in[0,\pi-\varepsilon]$ and $\varrho=c_1(\alpha,\beta,\varepsilon)k^{-1}$ where c_1 is big enough (as usual $N=n+(\alpha+\beta+1)/2$, $\gamma=-(\alpha+1/2)\pi/2$). Denote this zero by $x_{k+\mu}$. Considering the symmetry and that $\bigcup_{k=1}^{M} B_k$ contains a bounded number of zeros (see (4.10)), we obtain $0\leqq|\mu|\leqq M_1$. Now using [1], (8.8.1) with $\cos\vartheta=\cos\vartheta_{k+\mu}$ we get (4.11).

If in (4.11) M is big enough then $1+O(k^{-1})\geqq 0.5$, i.e., $1+O(k^{-1})=\left[1+O\left(\frac{1}{k+\mu}\right)\right]^{-1}$.

Sometimes we use [1] (8.9.2) which states

$$|F_k|\sim|P_n'^{(\alpha,\beta)}(x_k)|\sim k^{-\alpha-3/2}n^{\alpha+2}, \quad 0<\vartheta_k\leqq\pi-\varepsilon. \tag{4.12}$$

4.9. As above, we get

$$\begin{aligned}|L_n(f;x)-f(x)|\leqq&\left|\sum_{\vartheta_k\in I_0\cup K_0}\{\dots\}\right|+\sum_{r=1}^{q_1}\left|\sum_{\substack{\vartheta_k\in I_r\\ k>M_1}}\{\dots\}\right|+\\ &+\sum_{r=1}^{q_2}\left|\sum_{\substack{\vartheta_k\in K_r\\ k<n-M_2}}\{\dots\}\right|+\left|\sum_{k=1}^{M_1}\{\dots\}\right|+\left|\sum_{k=n-M_2}^{n}\{\dots\}\right|\equiv\\ &=\textstyle\sum_1+\sum_2+\sum_3+\sum_4+\sum_5.\end{aligned} \tag{4.13}$$

Here at I_r and K_r, N stands for n; the number M_2 corresponds to M_1 for the interval (ε,π) (see Lemma 4.1).

4.10. Let $0\leqq\vartheta\leqq\pi/8$. If in $\sum_1$ we apply (4.9), (4.10) and (4.4), $\sum_1$ can be estimated by the term $i=1$ of the first estimation of (3.2).

In $\sum_2$, using the disintegration $P_n'(x_{k+\mu})=F_{k+\mu}\left[1+O\left(\frac{1}{k+\mu}\right)\right]^{-1}$ and (4.9)—(4.12), we get, using for the first part the Abel inequality, as in 4.4

$$\left|\sum_{\substack{\vartheta_k\in I_r\\ k>M_1}}[f(x_k)-f(x)]\,l_k(x)\right|\leqq\left|\sum[f(x_k)-f(x)]\frac{P_n(x)}{(x-x_k)F_k}\right|+$$

$$+O(1)\sum|f(x_k)-f(x)|\left|\frac{P_n(x)}{(x-x_k)kF_k}\right|=O(1)\,\omega\left(\frac{\sin\vartheta}{n}2^r+\frac{2^{2r}}{n^2}\right)\frac{\vartheta_j^{-\alpha-0,5}}{n}\cdot$$

$$\cdot\frac{\left[\sin\left(\vartheta-2^{r-1}\frac{\pi}{N}\right)\right]^{\alpha+1,5}}{\cos\vartheta-\cos\left(\vartheta-2^{r-1}\frac{\pi}{N}\right)}+O(1)\frac{\omega\left(\frac{\sin\vartheta}{n}2^r+\frac{2^{2r}}{n^2}\right)\vartheta_j^{-\alpha-0,5}}{\left[\cos\vartheta-\cos\left(\vartheta-2^{r-1}\frac{\pi}{n}\right)\right]n^{\alpha+2,5}}\sum k^{\alpha+0,5}\equiv S_r.$$

First take such r's for which $(\vartheta_k <)\vartheta \leqq 2\vartheta_k$. By $\vartheta \approx \vartheta_k \approx jn^{-1}$

$$S_r = O(1)\omega\left(\frac{\sin\vartheta}{n}2^r + \frac{2^{2r}}{n^2}\right)2^{-r} + O(1)\omega\left(\frac{\sin\vartheta}{n}2^r + \frac{2^{2r}}{n^2}\right)\frac{\vartheta_j^{-\alpha-0,5}}{\vartheta_j 2^r n^{\alpha+1,5}}\cdot$$

$$\cdot\sum_{k=j}^{2j} k^{\alpha+0,5} = O(1)\omega\left(\frac{\sin\vartheta}{n}2^r + \frac{2^{2r}}{n^2}\right)2^{-r}$$

which can be treated analogously to 4.5.

Now we consider the I_r's where $\vartheta > 2\vartheta_k$. We have by $\cos\vartheta - \cos\vartheta_k \sim \sim \sin^2\vartheta \sim \vartheta^2 \sim \vartheta_j^2$

$$\sum\nolimits_2 = \sum S_r = O(1)\sum_{r=[c\ln j]}^{[\ln j]} \omega\left(\frac{\sin\vartheta}{n}2^r + \frac{2^{2r}}{n^2}\right)\frac{\left(\vartheta_j - 2^r\frac{\pi}{N}\right)^{\alpha+1,5}}{n\vartheta_j^{\alpha+2,5}} +$$

$$+O(1)\sum_{k=1}^{[j/2]} |f(x)-f(x_k)|\,|l_k(x)|\,k^{-1}.$$

Here for the first part we have

$$\sum_{r=[c\ln j]}^{[\ln j]} = O(1)\frac{n^{\alpha+1,5}}{j^{\alpha+2,5}}\sum_{r=[c\ln j]}^{[\ln j]}\sum_{i=2^r}^{2^{r+1}} \omega\left(\frac{\sin\vartheta}{n}i + \frac{i^2}{n^2}\right)\frac{j-i}{in} =$$

$$= O(1)\left(\frac{j}{n}\right)^{-\alpha-0,5}\frac{\omega\left(\frac{\sin\vartheta}{n}j + \frac{j^2}{n^2}\right)}{j}.$$

If $-1<\alpha\leqq -0,5$, $(jn^{-1})^{-\alpha-0,5}\leqq 1$, the remaining part can be included in $\sum_{i=[j/2]}^{j}\omega\left(\frac{\sin\vartheta}{n}i + \frac{i^2}{n^2}\right)i^{-2}$.

On the other hand, if $\alpha \geqq -0,5$, by $(\sin\vartheta)^{-\alpha-0,5} \leqq (\sin\vartheta_k)^{-\alpha-0,5}$, we get that the first part equals $\omega(\sin\vartheta\cdot n^{-1}j + j^2n^{-2})\,j^{-1}$.

Considering the second part, we can write

$$\sum_{k=1}^{[j/2]}\ldots = O(1)\omega\left(\frac{\sin\vartheta}{n}j + \frac{j^2}{n^2}\right)\frac{\vartheta_j^{-\alpha-0,5}}{n^{0,5}}\frac{\sum_{k=1}^{j} k^{\alpha+0,5}}{\vartheta_j^2 n^{\alpha+2}} =$$

$$= \frac{O(1)\omega\left(\frac{\sin\vartheta}{n}j + \frac{j^2}{n^2}\right)}{j}.$$

Let us consider $\sum_3$. If we choose K_r such that $2\vartheta \geqq \vartheta_k$ ($r\geqq 1$ and $k > M_1$), we can argue as above.

Let us consider those K_r for which $2\vartheta<\vartheta_k\leqq\pi/2$. Using that now $\cos\vartheta_k-$ $-\cos\vartheta\sim\vartheta_k^2$ we obtain as before

$$\Big|\sum_{k\in K_r}[f(x)-f(x_k)]\,l_k(x)\Big| = O(1)\frac{n^{\alpha-0,5}}{j^{\alpha+0,5}}\frac{\omega\left(\dfrac{\sin\vartheta}{n}2^r+\dfrac{2^{2r}}{n^2}\right)}{\vartheta_k^{-\alpha+0,5}}+$$

$$+O(1)\sum_{k\in K_r}|f(x)-f(x_k)|\,|l_k(x)|\,k^{-1} = S_r+Q_r.$$

I.e., we obtain for the corresponding intervals

$$\sum_{r=[c\ln j]}^{[\ln n]} S_r = O(1)\sum_{i=j}^{n}\omega\left(\frac{i\sin\vartheta}{n}+\frac{i^2}{n^2}\right)\frac{j^{-\alpha-0,5}}{i^{1,5-\alpha}}.$$

If $\alpha\leqq-0,5$ then by $j\leqq i$ we get $j^{-\alpha-0,5}\leqq i^{-\alpha-0,5}$, i.e. $\sum S_r=O(1)\sum\omega(\dots)i^{-2}$. On the other hand, for $\alpha\geqq-0,5$, $j^{-\alpha-0,5}\leqq1$. Finally we have

$$\sum_{k=2j}^{[n/2]}|f(x)-f(x_k)|\,|l_k(x)|\,k^{-1} = O(1)\sum_{k=2j}^{n}\omega\left(\frac{\sin\vartheta}{n}k+\frac{k^2}{n^2}\right)\frac{j^{-\alpha-0,5}}{k^{1,5-\alpha}}$$

which can be handled as above.

If for $\vartheta_k\in K_r$ we have $\vartheta_k\geqq\pi/2$ $(k\leqq n-M_2)$, one can apply the above used relations and the symmetry of $P_n^{(\alpha,\beta)}(x)$. We obtain

$$\Big|\sum_{k\in K_r}[f(x)-f(x_k)]\,l_k^{(\alpha,\beta)}(x)\Big| = \Big|\sum_{k\in I_r}[f(-x)-f(x_k)]\,l_k^{(\beta,\alpha)}(-x)\Big| =$$

$$=O(1)\frac{(n-j+1)^{-\alpha-0,5}}{n^{-\alpha-0,5}}\left(\sin\frac{\vartheta_k}{2}\right)^{\beta+3/2}+$$

$$+O(1)\sum_{k\in I_k}|f(-x)-f(x_k)|\,|l_k^{(\beta,\alpha)}(-x)|\,k^{-1} = S_r+Q_r,$$

where now $-1\leqq-x\approx\cos\vartheta_j\leqq\cos 7\pi/8$ and $I_r\subset[0,\pi/2]$. Using the above formulae we get

$$\sum_{\substack{r=1\\ \vartheta_k\geqq M_1}}^{[c\ln n]} S_r = O(1)(n-j+1)^{-\alpha-0,5}\,n^{\alpha-0,5}.$$

If $-1<\alpha\leqq0,5$, then $(n-j+1)^{-\alpha-0,5}\leqq n^{-\alpha-0,5}$, i.e. $\sum S_r=O(n^{-1})$ which corresponds to

$$\sum_{i=[n/2]}^{n}\omega(in^{-1}\sin\vartheta+i^2n^{-2})\,i^{-2}\sim n^{-1}.$$

Similarly, when $\alpha\geqq-0,5$, $(n-j+1)^{-\alpha-0,5}\leqq1$, i.e. $\sum S_r=O(n^{\alpha-0,5})$ which corresponds to

$$\sum_{i=[n/2]}^{n}\omega(\dots)\,i^{\alpha-1,5}\sim n^{\alpha-0.5}.$$

Finally using suitable formulae

$$\sum_{k=1}^{[n/2]} |f(-x)-f(x_k)|\,|l_k^{(\beta,\alpha)}(-x)|\,k^{-1} = O(1)\,\frac{(n-j+1)^{-\alpha-0,5}}{n^{-\alpha}} \sum_{k=1}^{[n/2]} \frac{k^{\beta+1,5}}{kn^{\alpha+2}},$$

which can be handled as above.

To estimate $\sum_4+\sum_5$ we have, using (4.12) and the symmetry,

$$\sum\nolimits_4+\sum\nolimits_5 = O(1)\,\frac{\omega\left(\dfrac{\sin\vartheta}{n}j+\dfrac{j^2}{n^2}\right)}{j^{\alpha+2,5}}+O(1)\,\frac{j^{-\alpha-0,5}}{n^{\beta-\alpha+2}},$$

which can be handled as above.

It may occur that for certain p, q and r, e.g. $\vartheta_p\in I_r$, $\vartheta_q\in I_r$ but $\vartheta_p<2\vartheta$ and $\vartheta_q>2\vartheta$. These cases are analogous to the above ones.

4.11. Let now $\varepsilon\leqq\vartheta\leqq\pi-\varepsilon$. Using that $j\sim n$, analogous estimations give the second formula of (3.2).

4.12. Finally, 4.10 and 4.11 give the estimation (3.2).

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CRITERIA FOR CONSTANCY OF FUNCTIONS WITH ALMOST TOTALLY DISCONNECTED RANGE OR DOMAIN

By

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In [1] J. F. CHEW and P. H. DOYLE improved a result of C. A. COPPIN [2] and proved other results giving conditions under which a function $f\colon X\to Y$ is constant. The first was what they called a "companion theorem to the theorem of Coppin", presumably because it partially reversed the roles of X and Y. By making use of the generality and symmetry of multifunctions and their inverses, Coppin's result can be improved still more and one can more readily reverse the role of domain and range to investigate "companion" theorems. In this sense it seems that Theorem 4 (rather than Theorem 2) of [1] is closer to being a companion to Coppin's result and the "true" companions of each of the latter two can be proved. The goal of this paper is then to improve the results in the above references, clarify their interrelations and give a negative answer to the question posed at the end of [1].

DEFINITIONS. A multifunction $F\colon X\to Y$ assigns to each point of X a non-empty subset of Y. As usual, if $A\subset X$ the image of A is $F(A)=\bigcup\limits_{x\in A} F(x)$ and point inverses $F^{-1}(y)=\{x\in X\mid y\in F(x)\}$ define the inverse multifunction $F^{-1}\colon F(X)\to X$. The following standard definitions are useful ("connected" may be replaced by another property such as "closed"):

(i) F is connected-valued if $F(c)$ is connected for any point $c\in X$.
(ii) F is connected if $F(C)$ is connected for any connected set $C\subset X$.
(iii) F^{-1} preserves connected sets if $F^{-1}(C)$ is connected for any connected $C\subset Y$.

If $F(X)\neq Y$, (iii) may be much stronger than F^{-1} being connected. Chew and Doyle overlooked this and, as a result, their Corollary 2 is false (e.g., if f is inclusion of X in Y where X is in the xy-plane and Y is the union of X and all points above the plane). They call a space LW if each point of an open set U is contained in a non-degenerate connected subset of U (i.e., components of non-empty open sets are non-degenerate). A space is totally disconnected iff no connected subset contains more than one point (note that empty and degenerate spaces are both connected and totally disconnected!).

THEOREM 1. *If $F\colon X\to Y$ is a multifunction and Y is totally disconnected, then*
(a) *F is a function if it has connected values,*
(b) *F is a locally constant function (hence continuous with closed-valued inverse) if it is connected and X is locally connected,*
(c) *F is a constant function if it is connected and X is connected.*

PROOF. All are immediate from the definitions: (a) since one-point sets map onto one-point sets, (b) since each point has a connected neighbourhood which must have a one-point image and (c) because X must have a one-point image.

The following corollary shows that Y need not be T_1, connected nor F single-valued in Chew and Doyle's improvement of Coppin's result.

COROLLARY 1. *A connected multifunction $F: X \to Y$ with closed point inverses is a constant function if X is connected, locally connected and for some $p \in Y$, $Y-p$ is totally disconnected and $F^{-1}(p) \neq X$.*

PROOF. By Theorem 1b, F is locally constant on the open (hence locally connected) subspace $X-F^{-1}(p)$. Since the latter is not empty it contains a nonempty open and closed point inverse which must be all of the connected space X. Thus $F^{-1}(p)$ is empty and F is a constant function by Theorem 1c.

If the roles of domain and range are completely reversed as suggested in the opening paragraph, the result, Corollary 2 below, is not only a companion but a consequence of Corollary 1 (simply set $F=G^{-1}$).

COROLLARY 2. *A closed-valued multifunction $G: Y \to Z$ with connected inverse is constant if $X=G(Y)$ is connected, locally connected and for some $p \in Y$, $Y-p$ is totally disconnected and $G(p) \neq X$.*

Note that G^{-1} being a constant function makes $Y=\{p\}$ and G constant. For non-degenerate Y the Corollary then implies no such multifunction maps Y onto a connected, locally connected space. Thus Corollary 2 is very close to Theorem 4 of [1] (Corollary 3 below) since LW is similar to local connectedness and a function with T_1 range is closed-valued. In this sense, the latter is a closer companion to Coppin's result than Theorem 2 of [1] (Corollary 4 below).

Theorem 1 has the following near-companion:

THEOREM 2. *If $f: Y \to X$ is a function and Y is totally disconnected, then*

(a) *f is $1-1$ if it is monotone (f^{-1} is connected-valued),*

(b) *$f(Y)$ is totally disconnected if f^{-1} is connected,*

(c) *$f(Y)$ is discrete (and f^{-1} is a continuous function) if f^{-1} preserves connected sets and X is locally connected,*

(d) *Y is discrete (and f an embedding) if f is continuous, f^{-1} preserves connected sets and X is locally connected.*

PROOF. Part (a) is immediate (also follows from Theorem 1a) and implies (b) since the inverse of a non-degenerate connected subset of $f(Y)$ would be non-degenerate and connected which is impossible. For (c), if $x \in f(Y)$ then $x \in U \subset X$ where U is open and connected so $f^{-1}(U)$ is a point and $U \cap f(Y)=\{x\}$ (and f^{-1} is a function, by (a), with discrete domain therefore continuous). Part (d) is a direct consequence of (c).

Theorems 3 (4, parenthetically) and 2 of [1] follow as Corollaries 3 and 4, respectively. Condition (iii), assumed by Chew and Doyle, is replaced in Corollary 3 by the weaker condition that f^{-1} be connected.

COROLLARY 3 (Chew—Doyle). *If $f: Y \to X$ is a function, f^{-1} is connected and $Y-p$ is totally disconnected then $f(Y)-f(p)$ is totally disconnected (hence $f(Y)$ cannot be LW and T_1).*

PROOF. Theorem 2b applies to $f|Y-f^{-1}(f(p))$: $Y-f^{-1}(f(p))\to X-f(p)$ since this function still has connected inverse and $Y-f^{-1}(f(p))\subset Y-p$ is totally disconnected. Observe that p could be a set rather than a point as prescribed in Theorem 3 of [1] (but perhaps not in Theorem 4).

COROLLARY 4 (Chew—Doyle). *A continuous function f: $Y\to X$ whose inverse preserves connected sets is constant if X is a locally connected T_1-space and for some $p\in Y$, $Y-p$ is totally disconnected but contains no one-point open subset of Y.*

PROOF. Since $Y-f^{-1}(f(p))$ is open, it is either empty or non-discrete. Also $X-f(p)$ is open, hence locally connected. Thus, restricting f as in the previous proof, Theorem 2d implies Corollary 4 (observe that if $f(p)$ is closed the T_1 condition can be omitted).

As previously noted, Corollary 4 was called a companion to Coppin's result (Corollary 1) by Chew and Doyle and in a sense Corollary 3 is a closer companion. As far as reversing roles of domain and range, however, Corollary 2 is the closest and perhaps should be called the twin of Coppin's result. Corollary 4 has its own twin, Corollary 5, which completes this "family" of results.

COROLLARY 5. *A connected open function f: $X\to Y$ with closed-valued inverse is constant if X is locally connected and for some $p\in Y$, $Y-p$ is totally disconnected but contains no one-point open subset of Y.*

PROOF. By Theorem 1b f is locally constant on the open (hence locally connected) set $X-f^{-1}(p)$ so if $x\in X-f^{-1}(p)$, $\{f(x)\}$ is open in Y. This contradiction implies $f(X)=\{p\}$. (Note that if $f^{-1}(p)$ is closed the closed-valued inverse condition is unnecessary, in fact it then follows from Theorem 1b.)

At the end of [1], CHEW AND DOYLE ask if a continuous function f: $X\to Y$ must be constant if Y is connected, locally connected, T_2 and X is a countable connected T_2-space with $X-p$ totally disconnected. An example given by G. G. MILLER [3] provides a negative answer. His quasimetric extension X of the rational subspace Q^2 of the plane R^2 satisfies the above conditions and is Urysohn (distinct points have disjoint closed neighbourhoods). Miller obtains X by adding the integers Z and the point $p=(\pi, \sqrt{2})$ to Q^2. He chooses a map F of Z onto a dense subset of R^2-Q^2-p such that no two points map onto the same line through p, and $G(z)$ to be the point halfway between p and $F(z)$. The subspace topology is given to $X\cap R^2$ and a neighbourhood of $z\in Z$ consists of z together with the points of Q^2 in an open subset of R^2 containing $F(z)$ and $G(z)$. If $Y=X\cup(R^2-F(Z)-G(Z))$ with the usual topology on $Y\cap R^2$ (and neighbourhoods of $z\in Z$ including z and points of $Y\cap R^2$ in an open set containing $F(z)$ and $G(z)$ as before) then Y is connected, locally connected and T_2. Since X is a subspace of Y, the inclusion map f: $X\to Y$ is continuous.

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RESIDUALLY SPECTRAL OPERATORS

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1. Introduction

Let X be a complex Banach space and $B(X)$ the set of bounded linear operators on X. In his fundamental paper on spectral operators [6], N. DUNFORD gave the following definition. An operator $T \in B(X)$ is a spectral operator of class (F, G) if

1° F is a Boolean algebra of sets in the complex plane C;

2° G is a total linear manifold in X^*, the dual of X;

3° there is a bounded spectral measure E in X with domain F such that $TE(f) = E(f)T$ and $\sigma(T|E(f)X) \subset \bar{f}$ for f in F;

4° for every (x, g) in $(X \times G)$, the function $gE(f)x$ is countably additive on F; (see notations below).

In most subsequent works it was assumed that $F = B$, the σ-field of all Borel sets (prespectral operators; see, e.g., [4], [8]), and that, in addition, $G = X^*$ (spectral operators proper).

Let S be a compact set contained in the spectrum of T. Let B_S be the σ-field of all Borel sets that contain S or are disjoint from S. T is said to be S-spectral if it is spectral of class (B_S, X^*). Such operators have been studied by I. BACALU [2]. He has shown that quotients and, under certain conditions, restrictions of spectral operators on invariant subspaces are in this class. We prove here that an S-spectral operator has a uniquely determined S-resolution of the identity and is S-decomposable in the sense of [1]. Though it need not have the single-valued extension property, it satisfies a commutativity theorem, well-known for spectral operators. Theorem 2 shows that the family of all sets S for which a given T is S-spectral, is directed with respect to the relation $\supset$. Theorem 3 gives necessary and sufficient conditions that this family have a smallest set; such operators will be called residually spectral. Another characterization of residually spectral operators is given in Theorem 4 as direct sums of spectral and essentially nonspectral operators. We show that this result fails to hold if essentially nonspectral is replaced by completely nonspectral (see definitions later).

The proof of the commutativity theorem (Theorem 1) will be a generalization of the proof of the respective result for spectral operators [7]. This proof will also employ notions and methods characteristic of the theory of residually decomposable operators [10]. In Section 4 we shall make use of a characterization of S-spectral operators, due to I. BACALU [2].

2. Preliminaries

For any T in $B(X)$ $\sigma(T)$ will denote its spectrum and $T|Y$ will denote the restriction of T to the invariant closed subspace Y in X. A homomorphic mapping E of the Bololean algebra B_S (S compact in C) into a Boolean algebra of projections in $B(X)$ such that $E(C)=I$ and $x^*E(b)x$ is countably additive on B_S for each $x\in X$, $x^*\in X^*$, will be called an S-spectral measure. By [7; XV. 2.4], E is then countably additive in the strong operator topology and is bounded. If $S\subset\sigma(T)$, and T commutes with $E(f)$ and $\sigma(T|E(f)X)\subset\bar{f}$ for each f in B_S, then T is called S-spectral and we say that E is an S-resolution of the identity for T. If $e\in B_S$, then T_e will denote $T|E(e)X$, and $E|E(e)X$ the restriction of E to $E(e)X$. For any Borel set b in C, $\bar{b}$ denotes its closure and b' its complement.

We recall some concepts and facts from [10]. An open set G in C is a set of analytic uniqueness for $T\in B(X)$ if for each open $H\subset G$ and each holomorphic function $h\colon H\to X$ such that $(z-T)h(z)=0$ for z in H, we have $h(z)=0$ on H. For any T in $B(X)$ there is a unique maximal open set of analytic uniqueness, denoted O_T. S_T will denote O_T'. A holomorphic function $f_x\colon G\to X$ such that $(z-T)f_x(z)=x$ for z in G is called a T-associated function of $x\in X$ on the open set $G\subset C$. $\delta_T(x)$ denotes the open set of points z in C such that z has a neighbourhood where a T-associated function of x exists. Define $\gamma_T(x)=\delta_T(x)'$, $\varrho_T(x)=\delta_T(x)\cap O_T$ and $\sigma_T(x)=\varrho_T(x)'$. Then there exists a unique T-associated function of x on $\varrho_T(x)$, which will be denoted by $x(z)$. For any $H\subset C$ set

$$X_T(H)=\{x\in X;\ \sigma_T(x)\subset H\}.$$

$X_T(H)$ is a linear manifold in X. If F is a closed set in C, the family $I_{T,F}$ consists of those closed T-invariant subspaces Y of X, for which $\sigma(T|Y)\subset F$. If $I_{T,F}$ has a largest element with respect to the relation $\subset$, then this largest element is denoted by $X_{T,F}$.

A spectral maximal space [5] of T is a closed T-invariant subspace Y of X such that for any closed T-invariant subspace Z of X the relation $\sigma(T|Z)\subset\sigma(T|Y)$ implies $Z\subset Y$. It is clear that if F is closed in C and $X_{T,F}$ exists, then $X_{T,F}$ is a spectral maximal space of T. Conversely, if Y is a spectral maximal space of T and $\sigma(T|Y)=F$, then $Y=X_{T,F}$. Finally, for concepts concerning S-decomposable operators we refer to [1].

REMARK. The assumption $S\subset\sigma(T)$ is only a matter of convenience (e.g. to ensure that $\sigma(T)\in B_S$). If in the definition of S-spectral operators we assume only that S is a compact subset of C and put $S_1=S\cap\sigma(T)$, then T is S-spectral if and only if T is S_1-spectral. Indeed, $B_S\subset B_{S_1}$ proves the "if" part. Conversely, if E is an S-resolution of the identity for T, then set

$$E_1(b)=\begin{cases}E(b\cup S), & b\supset S_1\\ E(b\cap S'), & b\subset S_1'\end{cases}\quad (b\in B_{S_1}).$$

Then E_1 is an S_1-resolution of the identity for T. We show here, e.g., that $\sigma(T|E_1(b)X)\subset\bar{b}$ for $b\in B_{S_1}$, $b\supset S_1$. We have $\sigma(T|E_1(b)X)=\sigma(T|E(b\cup S)X)\subset\subset\bar{b}\cup S$. Since $E_1(b)$ commutes with T, we obtain $\sigma(T|E_1(b)X)\subset\sigma(T)$. Hence

$$\sigma(T|E_1(b)X)\subset\bar{b}\cup S_1=\bar{b}.$$

3. S-spectral operators

LEMMA 1. *If E is an S-resolution of the identity for T, further $z \in S'$ and $(z-T)x=0$, then $E(\{z\})x=x$.*

PROOF. Let d be a closed set such that $z \notin d \in B_S$. If T_d denotes $T|E(d)X$ then $z \in \varrho(T_d)$, hence

$$E(d)x = (z-T_d)^{-1}(z-T)E(d)x = (z-T_d)^{-1}E(d)(z-T)x = 0.$$

Set $d_n=\{v \in C;\ |v-z| \geqq n^{-1}\}$, then $d_n \in B_S$ for n large enough. By the countable additivity of E, $E(\{z\}')x=\lim_n E(d_n)x=0$, hence $E(\{z\})x=x$.

LEMMA 2. *If T is S-spectral, then $S_T \subset S$.*

PROOF. Denote by $\sigma_p^0(T)$ the set of all $z \in C$ such that there exist a connected open neighbourhood U of z and an X-valued holomorphic function f, not identically 0 and satisfying $(u-T)f(u)=0$ on U. Then S_T is the closure of $\sigma_p^0(T)$, thus it suffices to show that $\sigma_p^0(T)$ is contained in S. Assume the contrary, then there are a sequence $\{z_n;\ n=0, 1, 2, \ldots\} \subset S'$ and a holomorphic function f on a neighbourhood of z_0 such that $\lim_n z_n=z_0$, $z_n \neq z_0$ for $n \neq 0$ and $(z_n-T)f(z_n)=0$ though $f(z_n) \neq 0$ for $n=0, 1, 2, \ldots$. By Lemma 1, we obtain for any S-resolution of the identity E of T

$$0 = E(\{z_0\} \cap \{z_n\})f(z_n) = E(\{z_0\})f(z_n) \to E(\{z_0\})f(z_0) = f(z_0),$$

a contradiction.

LEMMA 3. *If E is an S-resolution of the identity for T, c is a compact set in C, disjoint from S, $x \in E(c)X$ and $\sigma_T(x) \cap c = \emptyset$, then $x=0$.*

PROOF. Since T is S-spectral, the resolvent $(z-T_c)^{-1}$ exists on c' and satisfies $(z-T)(z-T_c)^{-1}x=x$ for $z \in c'$. On $\varrho_T(x)$, there exists the unique T-associated function $x(z)$ of x, for which $(z-T)x(z)=x$ for $z \in \varrho_T(x)$. The nonvoid open set $c' \cap \varrho_T(x)$ is in O_T hence the holomorphic function

$$f(z) = \begin{cases} (z-T_c)^{-1}x, & z \in c' \\ x(z), & z \in \varrho_T(x) \end{cases}$$

is well-defined on all of C and satisfies $\lim_{z\to\infty} f(z) = \lim_{z\to\infty} (z-T_c)^{-1}x=0$. By Liouville's theorem, $f(z) \equiv 0$, hence $x=0$.

LEMMA 4. *If E is an S-resolution of the identity for T and the closed set e contains S, then $E(e)X=X_T(e)$.*

PROOF. The resolvent $(z-T_e)^{-1}$ exists on e' and if $x \in E(e)X$, then $(z-T)(z-T_e)^{-1}x = x$ for $z \in e'$. By Lemma 2, we have $S_T \subset e$, hence $\sigma_T(x) \subset e$. Conversely, suppose $\sigma_T(x) \subset e$, choose an arbitrary compact set c, disjoint from e and set $y=E(c)x$. Then clearly $\delta_T(y) \supset c'$, further $(z-T)x(z)=x$ for $z \in e'$. Since E commutes with T, we obtain $(z-T)E(c)x(z)=y$ for $z \in e'$. Hence $\delta_T(y) \supset e'$, thus $\gamma_T(y)=\emptyset$ and $\sigma_T(y)=S_T \subset S$. By Lemma 3, we have $y=0$. If $\{c_n\}$

is an increasing sequence of compact sets with union e', then the countable additivity of E yields $E(e')x = \lim_n E(c_n)x=0$, hence $E(e)x=x$.

COROLLARY. *If E is an S-resolution of the identity for T, then* $E(\sigma(T))=I$.

Now we prove a commutativity theorem, which is well-known for spectral operators but, remarkably, fails for prespectral operators, see FIXMAN [8] and BERKSON and DOWSON [4].

THEOREM 1. *If E is an S-resolution of the identity for T and $A\in B(X)$ commutes with T, then $AE(b)=E(b)A$ for every $b\in B_S$.*

PROOF. Let the compact set d be a subset of S', and put $Y=E(d\cup S)X$ and $V=T|Y$. Then $\sigma(V)\subset d\cup S$. If W is a closed T-invariant subspace of X such that $\sigma(T|W)\subset d$, then for every w in W we have $\gamma_T(w)\subset d$, and Lemma 2 implies $\sigma_T(w)\subset d\cup S$. By Lemma 4, we obtain $W\subset Y$, thus $\sigma(V|W)\subset d$. Choose a bounded Cauchy domain D [9; pp. 288—289] such that $d\subset D$, $\bar{D}\subset S'$, and denote its positively oriented boundary by $B(D)$. The positions of the spectra of the occurring operators yield that for every w in W

$$w = k\int_{B(D)}(z-V|W)^{-1}w\,dz = k\int_{B(D)}(z-V)^{-1}E(d\cup S)w\,dz =$$

$$= k\int_{B(D)}(z-T|E(S)X)^{-1}E(S)w\,dz+k\int_{B(D)}(z-T|E(d)X)^{-1}E(d)w\,dz = E(d)w,$$

where $k=(2\pi i)^{-1}$. Hence $W\subset E(d)X$, thus $E(d)X=X_{T,d}$ is a spectral maximal space of T. By [5; 1.3.2], any spectral maximal space of T is hyperinvariant for T, therefore $E(d)AE(d)=AE(d)$ for any compact $d\subset S'$.

$(z-T)f(z)=x$ implies $(z-T)Af(z)=Ax$, hence $\gamma_T(Ax)\subset\gamma_T(x)$ and $\sigma_T(Ax)\subset\sigma_T(x)$. Thus Lemma 4 yields $E(c)AE(c)=AE(c)$ for any closed $c\supset S$. Choose any compact set $d\subset S'$, an increasing sequence of closed sets $c_n\supset S$, converging to d', and $x\in X$, then

$$E(d)AE(d')x = \lim_n E(d)AE(c_n)x = \lim_n E(d)E(c_n)AE(c_n)x = 0.$$

Hence $E(d)A = E(d)A(E(d)+E(d')) = AE(d)$ for any compact $d\subset S'$. Since the σ-field of Borel subsets of S' is generated by the class of compact subsets of S', the countable additivity of E implies $E(b)A=AE(b)$ for every Borel set $b\subset S'$. Taking complements we end the proof.

This result has the following important corollaries.

COROLLARY 1. *If T is S-spectral, then its S-resolution of the identity is uniquely determined for every b in B_S.*

PROOF. If E and F are two S-resolutions of the identity for T and c is a closed set containing S then, by Lemma 4, $E(c)F(c)=F(c)$ and $F(c)E(c)=E(c)$. By Theorem 1, $E(c)$ and $F(c)$ commute, hence $E(c)=F(c)$. The proof ends like that of Theorem 1.

COROLLARY 2. *If T is S_i-spectral with S_i-resolutions of the identity E_i with domains B_i and $b_i \in B_i$ $(i=1, 2)$, then $E_1(b_1)$ and $E_2(b_2)$ commute.*

COROLLARY 3. *If T is S-spectral, then T is S-decomposable.*

PROOF. Let $(G_1, \ldots, G_n, G_S)$ be an open S-covering of $\sigma(T)$ (see [1]) and let E denote the S-resolution of the identity of T. The set $U = \bigcup_{i=1}^{n} \bar{G}_i \cup \bar{G}_S$ contains $\sigma(T)$, thus $E(U)=I$. By Lemma 4 and the proof of Theorem 1, $E(\bar{G}_S)X = X_T(\bar{G}_S)$ and $E(\bar{G}_i)X = X_{T,\bar{G}_i}$ for $i=1, \ldots, n$. Hence

$$I = E(U) = E(\bar{G}_1) + E(\bar{G}_2 \cap \bar{G}_1') + \ldots + E(\bar{G}_S \cap \bar{G}_1' \cap \ldots \cap \bar{G}_n')$$

implies

$$X = X_{T,\bar{G}_1} + \ldots + X_{T,\bar{G}_n} + X_T(\bar{G}_S).$$

Since $X_T(\bar{G}_S)$ is closed, [10; Proposition 3.4] yields that it is also a spectral maximal space of T. Thus T is S-decomposable.

THEOREM 2. *If T is S_i-spectral $(i=1, 2)$, then T is $S_1 \cap S_2$-spectral.*

PROOF. Let E_i denote the S_i-resolutions of the identity for T with domains B_i and put $S = S_1 \cap S_2$. For any $b \in B_S$ set

$$b^0 = b \cap S, \quad b^1 = b \cap (S_2 \setminus S_1), \quad b^2 = b \cap (S_1 \setminus S_2), \quad b^3 = b \cap S_1' \cap S_2'.$$

Set $E(S) = E_1(S_1) E_2(S_2)$, $E(\emptyset)=0$ and

$$(*) \qquad E(b) = E(b^0) + E_1(b^1) + E_2(b^2) + E_i(b^3) \quad (i = 1 \text{ or } 2),$$

then the projection-valued mapping E is well-defined on B_S. Indeed, by Corollary 2 above, $E(S)$, hence $E(b^0)$ are projections. Since T is clearly $S_1 \cup S_2$-spectral, $E_1(b^3) = E_2(b^3)$, by Corollary 1 above. To see that $E(b)$ is a projection, it suffices to prove that any two terms on the right side of $(*)$ have product 0. We show that $E_1(b^1)E_2(b^2)=0$, the remaining cases being evident.

Let c_i be compact sets belonging to S_i' $(i=1, 2)$. By the proof of Theorem 1, the spectral maximal spaces X_{T,c_i} $(i=1, 2)$ and $X_{T,c_1 \cap c_2}$ then exist. By the same proof and [10; Proposition 3.2], we obtain

$$\bigcap_{i=1}^{2} E_i(c_i)X = \bigcap_{i=1}^{2} X_{T,c_i} = X_{T,c_1 \cap c_2} = E_k(c_1 \cap c_2)X \quad (k = 1, 2).$$

Since all the occurring projections commute, we have

$$E_1(c_1)E_2(c_2)X = E_k(c_1 \cap c_2)X,$$

hence

$$E_1(c_1)E_2(c_2) = E_k(c_1 \cap c_2) \quad (k = 1, 2).$$

Let $\{c_{in};\ n=1, 2, \ldots\}$ be increasing sequences of compact sets converging to $S_k \setminus S_i$ $(i, k=1, 2;\ k \neq i)$. Then $E_1(c_{1n})E_2(c_{2r})=0$ for each pair n, r, hence countable additivity implies $E_1(S_2 \setminus S_1)E_2(S_1 \setminus S_2)=0$. Therefore

$$E_1(b^1)E_2(b^2) = E_1(b^1)E_1(S_2 \setminus S_1)E_2(S_1 \setminus S_2)E_2(b^2) = 0.$$

In order to see that E is an S-resolution of the identity for T, we first show that $E_1(S_1')E_2(S_2')=E_k(S_1'\cap S_2')$ for $k=1, 2$. Let $\{d_{in};\ n=1, 2, \ldots\}$ be increasing sequences of compact sets converging to S_i' $(i=1, 2)$. By the preceding paragraph, then $E_1(d_{1n})E_2(d_{2r})=E_k(d_{1n}\cap d_{2r})$ for each pair n, r, hence countable additivity yields the stated equality. Thus for every $b\in B_S$

$$E(b)+E(b') = E(S)+E_1(S_2\setminus S_1)+E_2(S_1\setminus S_2)+E_2(S_1'\cap S_2') =$$
$$= (I-E_1(S_1'))(I-E_2(S_2'))+E_1(S_2\setminus S_1)+E_2(S_2') = I.$$

If $b_1, b_2\in B_S$ then $(b_1\cap b_2)^k=b_1^k\cap b_2^k$ for $k=0, \ldots, 3$ and similarly for unions. Hence we easily obtain that E is an S-spectral measure, commuting with T. Now for any $b\in B_S$ we prove that $\sigma(T|E(b)X)\subset\bar{b}$. Suppose first that $b=S$. Then

$$E(S)X = E_1(S_1)X\cap E_2(S_2)X = X_T(S_1)\cap X_T(S_2) = X_T(S),$$

hence $\sigma(T|E(S)X)\subset S$, by [10; Proposition 2.4]. If $b\in B_S$ is arbitrary, then

$$E(b)X = E(b^0)X+E_1(b^1)X+E_2(b^2)X+E_1(b^3)X,$$

and $T|E(b)X$ is completely reduced by the four subspaces on the right side. By the above remark (if $b^0=S$) and since T is S_i-spectral $(i=1, 2)$,

$$\sigma(T|E(b)X)\subset \bigcup_{k=0}^{3} \overline{b^k} = \bar{b},$$

and the proof is complete.

4. Residually spectral operators

For any S-spectral measure E set

$$Z(E, S) = \int_{S'} zE(dz)|E(S')X.$$

It is clear that $T\in B(X)$ is S-spectral if and only if there is an S-spectral measure E, commuting with T and such that

$$T = Z(E, S)+N(E, S)\oplus T|E(S)X,$$

where $N(E, S)$ is quasinilpotent in $B(E(S')X)$, commuting with $Z(E, S)$, and $\sigma(T|E(S)X)\subset S$. In this case the operators $Z(S)=Z(E, S)$, hence $N(S)=N(E, S)$ are uniquely determined, by Corollary of Theorem 1, and we shall call them the S-scalar and the S-radical parts of T, respectively. If $N(S)=0$, we shall say that T is S-spectral of scalar type, or simply S-scalar.

Let $S(T)$ denote the family of all compact sets R such that $S_T\subset R\subset\sigma(T)$ and T is R-spectral.

DEFINITION 1. If there exists $S\in S(T)$ such that $S\subset R$ for every $R\in S(T)$, then T is called residually spectral with residuum S.

DEFINITION 2. A family $\{N_a;\ a\in A\}$ of operators in Banach spaces $\{X_a;\ a\in A\}$ is uniformly quasinilpotent, if $\lim_k |N_a^k|^{1/k}=0$ uniformly on A.

In the following theorem E_n denotes the S_n-resolution of the identity for T and N_n denotes $N(S_n)$ for $n=1, 2, \ldots$.

THEOREM 3. *An operator $T\in B(X)$ is residually spectral if and only if there is a decreasing sequence $\{S_n\}$ in $S(T)$ with intersection $S=\bigcap\{R;\ R\in S(T)\}$ such that*
1° for every $b\in B_S$, $b\subset S'$ and $x\in X$ there exists $E(b)x=\lim_n E_n(b\cap S_n')x$,
2° the sequence $\{N_n\}$ is uniformly quasinilpotent.

PROOF. *Sufficiency.* For $b\in B_S$, $b\supset S$ and $x\in X$ set $E(b)x=x-E(b')x$. Define the X-valued countably additive measures m_n on B_S by

$$m_n(b)=\begin{cases} E_n(b\cap S_n')x, & b\subset S',\\ E_n(b\cup S_n)x, & b\supset S,\end{cases}\qquad (b\in B_S).$$

By condition 1° we have $E(b)x=\lim_n m_n(b)$ for $b\in B_S$. [7; IV. 10.6] yields that $E(b)x$ is a countably additive X-valued measure on B_S. By [7; IV. 10.2], the set $\{E(b)x;\ b\in B_S\}$ is bounded. The operators $E(b)$, being strong limits of increasing (decreasing) sequences of continuous projections, are continuous projections. The uniform boundedness principle implies that the set $\{E(b);\ b\in B_S\}$ is bounded. For each positive integer n, $b\in B_{S_n}$ implies $E_n(b)=E(b)$, by Corollary 1 to Theorem 1. Hence there is a positive K such that for every n and $b_n\in B_{S_n}$ we have $|E_n(b_n)|\leq K$.

Now if $b_1, b_2\in B_S$ and b_1, $b_2\subset S'$, then for each $x\in X$, $E(b_1\cap b_2)x= =\lim_n E_n(b_1\cap S_n')E_n(b_2\cap S_n')x$. On the other hand, 1° implies for every $\varepsilon>0$

$$|E(b_1)E(b_2)x-E_n(b_1\cap S_n')E(b_2)x|+|E_n(b_1\cap S_n')|\,|E(b_2)x-E_n(b_2\cap S_n')x|<\varepsilon,$$

if n is large enough. Hence $E(b_1\cap b_2)=E(b_1)E(b_2)$, and this multiplicative property can be proved similarly in the remaining cases, too. We have $(E(b)+E(b')=I$ for every $b\in B_S$; thus we have seen that E is an S-spectral measure, which obviously commutes with T.

Since T is S_n-spectral, by Lemma 4 we have $E(S_n)X=E_n(S_n)X=X_T(S_n)$. It is easily seen that $E(S)X=\bigcap_{n=1}^{\infty} E(S_n)X$, hence $E(S)X=X_T(S)$ and $\sigma(T|E(S)X)\subset S$, by [10; Proposition 2.4].

Define the operators Z and N on $E(S')X$ by $Z=Z(E, S)$, $N=T|E(S')X-Z$. Then the restrictions $Z_n=Z|E(S_n')X$ and $N_n=N|E(S_n')X$ are the S_n-scalar and S_n-radical parts of T, respectively. N commutes with the restriction of E to $E(S')X$, hence with Z. Further, for any $x\in E(S')X$ and $k=1, 2, \ldots$

$$N^k x=N^k E(S')x=\lim_n N_n^k E(S_n')x.$$

We have, by condition 2°,

$$\lim_k |N_n^k E(S_n')x|^{1/k}=\lim_k |N_n^k|^{1/k}(K|x|)^{1/k}=0$$

uniformly for n. According to [7; I. 7.6], then

$$\lim_k |N^k x|^{1/k}=\lim_k \lim_n |N_n^k E(S_n')x|^{1/k}=0.$$

Hence N is quasinilpotent in $B(E(S')X)$, thus T is S-spectral with $Z=Z(S)$ and $N=N(S)$.

Necessity. If T is S-spectral, then Corollary 1 to Theorem 1 and countable additivity imply 1°. If N is the S-radical part of T, the inequality $|N_n^k|\leqq|N^k|$ for each pair of positive integers n, k implies 2°. The proof is complete.

REMARK 1. Since the topological space S' has a countable base and Theorem 2 is valid, a decreasing sequence $\{S_n\}$ in $S(T)$ with intersection S always exists. Further, since E has been shown to be the S-resolution of the identity for T, the limits in 1° are independent of the choice of the sequence $\{S_n\}$.

REMARK 2. If X is weakly complete, then condition 1° is equivalent to the condition that the Boolean algebra of projections $Q=\{E_n(b_n);\ b_n\in B_{S_n},\ n=1, 2, \dots\}$ be bounded. In fact, the necessity of this condition was proved in the course of the preceding proof. Conversely, if X is weakly complete, any bounded Boolean algebra of projections may be imbedded in a σ-complete Boolean algebra of projections, by [3; 2.9]. According to [7; XVII. 3.4], the limits occurring in condition 1° then exist.

As a consequence of this remark, in a weakly complete X an operator T with uniformly bounded S_n-resolutions of the identity and of S_n-scalar type for a decreasing sequence $\{S_n\}$ with intersection $S=\bigcap\{R;\ R\in S(T)\}$ is always residually spectral (of scalar type).

EXAMPLE 1. Let $X=m$, the space of bounded sequences, and for $x=(x_1, x_2, \dots)$ let $Tx=(c_1x_1, c_2x_2, \dots)$ with $c_k=k^{-1}$. Set $S_n=\{0, c_{n+1}, c_{n+2}, \dots\}$ and $F_nx=(0, \dots, 0, \underbrace{x_n}^{n}, 0, \dots)$ for $n=1, 2, \dots$. Define $E_n(S_n)x=(0, \dots, \underbrace{0}^{n}, x_{n+1}, x_{n+2}, \dots)$; for $b\in B_{S_n}$, $b\subset S_n'$ put $E_n(b)x=\sum_{c_k\in b} F_kx$, and extend E_n to B_{S_n} in an obvious way. Then T is S_n-spectral of scalar type with S_n-resolution of the identity E_n for each positive integer n. We have $S=\bigcap\{R;\ R\in S(T)\}=\{0\}$. If we set $y=(1, 1, \dots)$, we see that the sequence $E_n(S_n')y=(1, \dots, \underbrace{1}^{n}, 0, \dots)$ does not converge in X, hence T is not residually spectral. This shows that uniform boundedness of the S_n-resolutions of the identity does not ensure the existence of the S-resolution of the identity, if X is not weakly complete, even if T is S_n-scalar for each n.

EXAMPLE 2. Let $Y=l_1$ and for $y=(y_1, y_2, \dots)$ let $T_1y=(y_2, y_3, \dots)$. Then $\sigma(T_1)=S_{T_1}=(z\in C;\ |z|\leqq 1\}$ (see [11; p. 1238]). Let $C_2=\{z\in C;\ |z|\leqq 2\}$, $Z=L_2(C_2)$ and for $f\in L_2(C_2)$ set $(T_2f)(z)=zf(z)$. Then T_2 is spectral of scalar type with $\sigma(T_2)=C_2$ and resolution of the identity $E_2(e)f=k_ef$, where k_e denotes the characteristic function of the Borel set $e\subset C_2$. On $X=Y\oplus Z$ set $T=T_1\oplus T_2$, then $\sigma(T)=C_2$, $S_T=S_{T_1}$ (by [11; Proposition 2.2]). T is clearly residually spectral with residuum S_T and S_T-resolution of the identity

$$E(e)=\begin{cases} I_1\oplus E_2(e), & e\supset S_T \\ 0\oplus E_2(e), & e\subset S_T' \end{cases}\qquad (e\in B_{S_T}),$$

where I_1 is the unit in $B(Y)$.

DEFINITION 3. $T\in B(X)$ is essentially nonspectral, if there is no pair (X_1, X_2) of closed subspaces of X, completely reducing T ([9; p. 268]) and such that $X_1\neq\{0\}$, $T|X_1$ is spectral and X_2 is a spectral maximal space of T.

THEOREM 4. *$T\in B(X)$ is residually spectral if and only if there is a pair (X_1, X_2) of closed subspaces of X, completely reducing T and such that $T|X_1$ is spectral and $T|X_2$ is essentially nonspectral. In this case the residuum S is $\sigma(T|X_2)$.*

PROOF. *Necessity.* If T is residually spectral with residuum S and S-resolution of the identity E, then

$$T = T|E(S')X\oplus T|E(S)X.$$

Here $T_1=T|E(S')X$ is spectral and, with the notation $T_2=T|E(S)X$, we have $\sigma(T_2)=S$. Indeed, if $\sigma(T_2)$ were a proper subset S_0 of S, then a remark in [2; p. 377] would yield that T is S_0-spectral, a contradiction.

We shall show that T_2 is essentially nonspectral. Assume the contrary, with corresponding decompositions $E(S)X=Y_1\oplus Y_2$ and $T_2=V_1\oplus V_2$. If F is the resolution of the identity for V_1, then $G(b)=E(b)|E(S')X\oplus F(b)$ for each Borel set b defines the resolution of the identity for $T|\big(E(S')X\oplus Y_1\big)$, hence this operator is spectral. By assumption, Y_2 is a spectral maximal space of T_2. Hence $\sigma(V_2)=S$ would imply $E(S)X\subset Y_2$, though Y_2 is, by assumption, properly contained in $E(S)X$. Thus $\sigma(V_2)$ is a proper subset S_2 of S, and [2; p. 377] shows again that

$$T = T|\big(E(S')X\oplus Y_1\big)\oplus V_2$$

is S_2-spectral, a contradiction.

Sufficiency. Suppose that $T=T|X_1\oplus T|X_2$, where $T|X_1$ is spectral, $T|X_2$ is essentially nonspectral and let S denote $\sigma(T|X_2)$. Let E be the resolution of the identity for $T|X_1$, then ([2]) T is S-spectral and

$$F(b)=\begin{cases}E(b)\oplus 0, & b\subset S'\\ E(b)\oplus I_2, & b\supset S\end{cases}\quad (b\in B_S),$$

is the S-resolution of the identity for T.

If T is not residually spectral with residuum S then, by Theorem 2, T is S_1-spectral with S_1-resolution of the identity F_1 for some proper compact subset S_1 of S. Thus

$$T = T|F_1(S_1')X\oplus T|F_1(S_1)X$$

where $T|F_1(S_1')X$ is spectral. If $P_2\in B(X)$ denotes the projection operator on X_2 parallel to X_1 then, by Theorem 1, P_2 commutes with F_1. Therefore $X_2=F(S)X_2=$ $=F_1(S\setminus S_1)X_2\oplus F_1(S_1)X_2$, and

$$T|X_2 = T|F_1(S\setminus S_1)X_2\oplus T|F_1(S_1)X_2.$$

Since [7; XV. 3.10] can be proved in the same way for S_1-spectral operators, we see that $T|X_2$ is S_1-spectral with S_1-resolution of the identity $F_1|X_2$. Hence $F_1(S_1)X_2$ is a spectral maximal space of $T|X_2$ (by Lemma 4 and [10; Proposition 3.4]), and $T|F_1(S\setminus S_1)X_2$ is spectral. Further, the subspace $F_1(S\setminus S_1)X_2$ is not $\{0\}$. If it were $\{0\}$, then we would have $\sigma(T|X_2)=\sigma(T|F_1(S_1)X_2)\subset S_1$, which contradicts $\sigma(T|X_2)=S$. Thus $T|X_2$ is not essentially nonspectral, contrary to assumption, and the proof is complete.

REMARK. Any decomposition $T=T|X_1 \oplus T|X_2$ into spectral and essentially nonspectral parts satisfies

$$X_1 \supset E(S')X \quad \text{and} \quad X_2 \subset E(S)X,$$

where S is the residuum and E is the S-resolution of the identity for T. Indeed, the preceding proof of the sufficiency yields $X_2 \subset E(S)X$. The projection P_2 commutes with T, hence with $E(S)$, thus $P_2E(S)=E(S)P_2=P_2$. Therefore $(I-P_2)E(S')=E(S')$.

Decompositions of T of the above type are, in general, not unique. Set $X=l_1$ and for $x=(x_1, x_2, \ldots)$ put $T_1x=(x_1, 2^{-1}x_2, \ldots)$ and $T_2x=(x_2, x_3, \ldots)$. Set $Y=X\oplus X$ and $T=T_1\oplus T_2$. Since $\sigma(T)=\{z\in C;\ |z|\leqq 1\}=S_{T_2}=S_T$, therefore T is residually spectral with residuum S_T, by Lemma 2. If F is the resolution of the identity for T_1 and $F(b)\neq 0$ for a Borel set b, then

$$T = T_1|F(b)X_1 \oplus (T_1|F(b')X_1 \oplus T_2)$$

is a decomposition of T into spectral and essentially nonspectral parts. There are infinitely many different of them.

DEFINITION 4. $T\in B(X)$ is completely nonspectral if there is no pair (X_1, X_2) of closed subspaces of X, completely reducing T and such that $X_1\neq\{0\}$ and $T|X_1$ is spectral.

By Theorem 4, if $T\in B(X)$ is the direct sum of a spectral and a completely nonspectral operator, then T is residually spectral. In order to show that the converse is false, we exhibit an essentially nonspectral operator that is not the direct sum of a spectral and a completely nonspectral operator.

EXAMPLE 3. Let $Y=m$ and for $y=(y_1, y_2, \ldots)\in Y$ set $T_1y=(c_1y_1, c_2y_2, \ldots)$ where $c_k=k^{-1}$ for $k=1, 2, \ldots$. Let $W=l_1$ and for $w=(w_1, w_2, \ldots)\in W$ set $T_2w=(0, w_1, w_2, \ldots)$. Put $X=Y\oplus W$ and $T=T_1\oplus T_2$. Let Q denote the projection of X on Y parallel to W, and $Q_2=I-Q$. First we show that T is essentially nonspectral.

We have $\sigma(T_2)=\{z;\ |z|\leqq 1\}=C_1$, and the point spectrum of T_2 is void. Hence T_2 has the single-valued extension property and, since T_1 is prespectral, so has T_1. Thus S_T is void and for $x=y\oplus w$ we have $\sigma_T(x)=\sigma_{T_1}(y)\cup\sigma_{T_2}(w)$ (cf. [11; p. 1192]). Since $\sigma_{T_2}(w)=C_1$ for $w\neq 0$ ([7; XV. 15. 37]), we obtain

$$\sigma_T(x) = \begin{cases} \sigma_{T_1}(y) & \text{if } w=0 \\ C_1 & \text{otherwise.} \end{cases}$$

Let Z be an arbitrary spectral maximal space of T and $\sigma(T|Z)=c$. For any $x\in Z$ we have $\sigma_T(x)\subset c$, so that $Z=X$ or, if c is a proper subset of C_1 then $Z\subset Y$. In the latter case it is easily seen that Z is a spectral maximal space of T_1. Since T_1 is prespectral, hence decomposable, we obtain that $Z=Y_{T_1}(c)=E(c)Y$, where E denotes any resolution of the identity for T_1. Thus, if T is not essentially nonspectral, then there are projections $P\in B(X)$ and $P_2=I-P$ such that $TP=PT$, $PX=E(c)Y$ and $T=T|PX\oplus T|P_2X$ where $V=T|P_2X$ is spectral. Since $\sigma(T)=$ $=\sigma(T_1|E(c)Y)\cup\sigma(V)$, the situation of $\sigma(T_1|E(c)Y)$ yields $\sigma(V)=C_1$.

For any x in P_2X we clearly have $\varrho_{T|P_2X}(x)\subset\varrho_T(x)$. Conversely, since P_2 commutes with T, the relation $(z-T)x(z)=x$ implies $(z-T)P_2x(z)=P_2x=x$. Hence $\sigma_{T|P_2X}(x)=\sigma_T(x)$. Thus, if F is the resolution of the identity for V and e is closed in C, then $F(e)P_2X=\{x\in P_2X;\ \sigma_T(x)\subset e\}$. Put $e=\left\{z\in C;\ \left|z+\frac{1}{2}\right|\leqq\frac{1}{4}\right\}$. For any $x\in P_2X$ we have $x=Qx\oplus Q_2x$, hence $\sigma_T(x)=\sigma_{T_1}(Qx)\cup\sigma_{T_2}(Q_2x)\subset e$ implies $x=Qx\in Y$. The situation of $\sigma(T|Y)$ and $\sigma_{T|Y}(x)=\sigma_T(x)$ together imply $\sigma_T(x)=\emptyset$, hence $x=0$. Thus $F(e)=0$, which contradicts the facts that $e\subset\sigma(V)$ and V is spectral. So T is essentially nonspectral.

Now we show that T is not the direct sum of a spectral and a completely nonspectral operator. Suppose that, on the contrary, there is a projection $P\in B(X)$ such that $PT=TP$, $T|PX$ is spectral and $T|P_2X$ is completely nonspectral, where $P_2=I-P$. Let e_i be the element of Y with ith coordinate 1 and the others 0, and put $P_1=P$. Then for $r=1,2$ and $i=1,2,\dots$ we have

$$TP_re_i=P_rTe_i=P_rc_ie_i=c_iP_re_i.$$

Hence $P_re_i\in N(T-c_i)$, the null space of $T-c_i$. Since the point spectrum of T_2 is void, we obtain $(T_1-c_i)P_re_i=0$, thus $P_re_i=k_{ri}e_i$. Here $k_{ri}=1$ or 0 according as $e_i\in R(P_r)$ or $e_i\in R(I-P_r)$, where R denotes operator range. For any subset c of $\{c_i;\ i=1,2,\dots\}$ and $x=(y_1,y_2,\dots)\oplus w$ define $F(c)x=(d_1y_1,\ d_2y_2,\dots)\in Y$ where $d_i=1$ or 0 according as $c_i\in c$ or not. Then $F(c)\in B(X)$ is a projection commuting with T and we have (limits in the norm of m)

$$TP_rF(c)x=P_rF(c)(c_1y_1,c_2y_2,\dots)=P_rF(c)\lim_n(c_1y_1,\dots,c_ny_n,0,\dots)=$$

$$=\lim_n(k_{r1}d_1c_1y_1,\dots,k_{rn}d_nc_ny_n,0,\dots)=(c_1k_{r1}d_1y_1,c_2k_{r2}d_2y_2,\dots)\in Y.$$

Hence $P_rF(c)x=(d_1k_{r1}y_1,\ d_2k_{r2}y_2,\dots)\in F(c)X$, thus

$$P_rF(c)=F(c)P_rF(c)\quad (r=1,2).$$

As a consequence, $P_rF(c)$ and $P_rF(c)P_r$ are projections commuting with T. Let f be a finite subset of $\{c_1,c_2,\dots\}$. The operator

$$(*)\qquad (T|P_2X)|P_2F(f)P_2X=T|F(f)P_2F(f)P_2X$$

is the restriction of the completely nonspectral $T|P_2X$ to the range of the projection $P_2F(f)|P_2X\in B(P_2X)$, which commutes with $T|P_2X$. Further, this range belongs to $F(f)X$, hence is finite dimensional. Thus $(*)$ is spectral, which implies $P_2F(f)P_2=0$, and so $F(f)P_2=PF(f)P_2$. Now set $h_r=\{c_i;\ e_i\in R(P_r)\}$ $(r=1,2)$. Then, as we have seen above, for every x in X

$$PF(h_1)x=(d_1y_1,d_2y_2,\dots)=F(h_1)x.$$

Further, there is an increasing sequence $\{f_n\}$ of finite subsets with union h_2. We have $PF(f_n)=0$, thus $F(f_n)P_2=PF(f_n)P_2=0$.

The restrictions $E(c)=F(c)|QX$ belong to the resolution of the identity of class l_1 for T_1, which is prespectral of class l_1. Hence σ-additivity in the l_1-topology of m gives for every $x\in X$, $v\in l_1$

$$(F(h_2)P_2x)v = (E(h_2)QP_2x)v = \lim_n (E(f_n)QP_2x)v = \lim_n (F(f_n)P_2x)v = 0.$$

Thus $F(h_2)P_2=0$. Since $F(h_1)P_2=PF(h_1)P_2$, we obtain $QP_2=(F(h_1)+F(h_2))P_2= =PF(h_1)P_2$, hence $QP_2=PQP_2$. So $Q-QP=PQ-PQP$. Multiplying on the right by Q and observing $PQ=QPQ$, we obtain $Q=PQ$. Hence $QP=PQP$ is a projection. Further, the restriction $Q_p=Q|PX$ belongs to $B(PX)$ and $Q_p^2=Q_p$. By assumption, $T_p=T|PX$ is spectral and

$$Q_pT_p = QT|PX = TQ|PX = T_pQ_p.$$

Hence $T_p|Q_pPX$ is spectral. Since $Q=PQ$, we have $R(Q)\subset R(P)$, thus $Q_pPX= =QPX=QX$. So $T_1=T|QX$ ought to be a spectral operator, which it is not, a contradiction. Thus the operator T has the stated properties.

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A REMARK ON RESTRICTED SERIAL RINGS[1]

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1. In [1], IWANAGA posed the problem whether there exists a restricted uniserial ring which is neither left nor right noetherian.

In this note we give an example for such a ring. Moreover we show that we can drop the assumption of Proposition 2 of [1] that A is a noetherian ring. Thus we give a characterization of non-prime, non-noetherian restricted serial rings (for its definition see below) even without identity. This is essentially based on the main result of [3]. A (unitary) module is called serial if the lattice of submodules with respect to inclusion is linearly ordered. A ring with identity is called left serial (right serial) if it is a direct sum of serial left (right) modules and a left-right serial ring is said to be serial. As in [1] we call a ring A restricted serial provided that every proper homomorphic image of A is a left and right artinian serial ring. $J(A)$ denotes the Jacobson radical of the ring A, A_n the $n\times n$ matrix ring over A.

2. By symmetry we can restrict ourselves to the case where A is not right noetherian.

THEOREM. *Let A be a non-prime non right noetherian ring. Then A is restricted serial if and only if A is precisely one of the following*

(I) $A=\begin{bmatrix} S_n & B \\ 0 & 0 \end{bmatrix}$, *$S$ an infinite division ring, B a simple unitary S_n-left module.*

(II) *A is a full matrix ring over a completely primary ring R, $J(R)^2=(0)$. $J(R)$ has infinite dimension as $R/J(R)$-right module, and either*

(a) $J(R)$ is, on each side, a direct sum of a minimal two-sided ideal of R and a simple one-sided ideal or

(b) $J(R)$ is a minimal two-sided ideal of R

(III) $A=\begin{bmatrix} S_n & M \\ 0 & K_m \end{bmatrix}$, *$M$ a unitary S_n-K_m-bimodule, S, K division rings, M has infinite dimension as K_m-right module, and either*

(a) $J(A)$ is, on each side, a direct sum of a minimal two-sided ideal of A and a simple one-sided ideal or

(b) $J(A)$ is a minimal two-sided ideal of A.

PROOF. The "if" part is almost trivial. To prove the "only if" part we show first that A is not right artinian. If it were then, since it is not right noetherian, it

[1] The paper was written during a study leave in Budapest.

would contain a Prüfer subgroup $Z(p^\infty)$ ([2], Satz 10.10). Then clearly there exists a proper factorring of A without identity since $Z(p^\infty)$ is in the annihilator of A ([2], Satz 10.3) contrary to our assumption.

Thus A is not right artinian but every proper factor ring is right artinian. Since A cannot be nilpotent, A is of type (II), (III) or (IV) of Satz 5 of [3].

If in case (II) or (III) of the theorem, $J(R)$ or M is a simple bimodule, it is clearly a minimal ideal of A.

Assume that $J(R)$ is not a minimal ideal of R in (II). Let B be an ideal of R, $B \subsetneqq J(R)$. Then $J(R)=B\oplus K=B\oplus L$ with right or left ideals K or L, resp., since $J(R)$ is a completely reducible module. Since R/B is serial, K is a simple right ideal and L is a simple left ideal of R.

Moreover, if C is another ideal of R such that $(0)\neq C\subsetneqq J(R)$, then so is $C\cap B$. If $C\cap B=(0)$, then

$$B \cong B/B\cap C \cong (B+C)/C$$

would be a right artinian A-module since A/C is right artinian. Since A/B is itself an artinian A-module, A would be right artinian contradicting our first statement. Therefore $C\cap B\neq(0)$. Again $B=C\cap B\oplus L_1$ with right ideal L_1 and $R/(B\cap C)$ is not serial if $L_1\neq(0)$. Thus $L_1=(0)$, $C\cap B=B$ and similarly $C\cap B=C$. Hence B is a minimal ideal.

The same procedure is applicable to (III) and the theorem is proved.

3. Example. Let Q be the field of rational numbers, x transcendental over Q and consider the chain

$$Q\subset Q(\sqrt{2})\subset Q(\sqrt[4]{2})\subset\ldots\subset Q(2^{2^{-n}})\subset\ldots.$$

Define

$$K=\bigcup_{n=1}^{\infty} Q(2^{2^{-n}}).$$

Then K is a field, in fact an infinite dimensional algebraic extension of Q. Clearly x is transcendental over K too, $K(x)$ is an infinite dimensional algebraic extension of $Q(x)$. Moreover, there is a chain

$$Q(x)\subset Q(x)(\sqrt{2})\subset Q(x)(\sqrt[4]{2})\subset\ldots$$

of subfields of $K(x)$ with

$$K(x)=\bigcup_{n=1}^{\infty} Q(x)(2^{2^{-n}}).$$

Let

$$A=\begin{bmatrix} Q(x) & K(x)\\ 0 & K\end{bmatrix}.$$

Clearly A is neither left nor right noetherian. To prove that

$$B=\begin{bmatrix} 0 & K(x)\\ 0 & 0\end{bmatrix}$$

is a minimal ideal of A let $0 \neq b \in K(x)$. Then $b \in Q(x)(2^{1/k})$ for some k and $1/b \in Q(x)(2^{1/k})$ is a linear combination of powers of $2^{1/k}$ with coefficients in $Q(x)$;

$$1/b = f_0(x) + f_1(x)2^{1/k} + \dots + f_{k-1}(x)2^{(k-1)/k}.$$

Then

$$\sum_{i=0}^{k-1} \begin{bmatrix} f_i(x) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2^{i/k} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in B.$$

Therefore we have

$$\begin{bmatrix} 0 & Q(x)(2^{2^{-n}}) \\ 0 & 0 \end{bmatrix} \subseteqq B.$$

Hence $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ generates B. The only proper ideals of A are

$$\begin{bmatrix} 0 & K(x) \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} Q(x) & K(x) \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & K(x) \\ 0 & K \end{bmatrix}$$

and the factor rings are semisimple artinian.

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THE CATEGORY OF UNARY ALGEBRAS CONTAINING A GIVEN SUBALGEBRA. II

By

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I. Introduction

This paper continues the study of certain categories of unary algebras which were examined in the first part of the present paper [2].

Recall that a category is *binding*, if every category of algebras has a full embedding in it. For B a $\varkappa$-unary algebra let $\mathcal{A}_\varkappa(B)$ denote the category of $\varkappa$-unary algebras having a subalgebra isomorphic to B. In [2] we have proved the following (Theorem 1.3).

Let $\varkappa \geqq 2$ and B a $\varkappa$-unary algebra having more than one element. The followings are equivalent:

(i) $\mathcal{A}_\varkappa(B)$ is binding.

(ii) There exists a rigid $C \in \mathcal{A}_\varkappa(B)$ (i.e. End $C \cong 1$).

(iii) There exists a $C \in \mathcal{A}_\varkappa(B)$ such that $|C| > 2^{\varkappa \cdot \omega}$ and Aut $C =$ End C.

Although this theorem gives quite a good answer from the point of view of testing categories (see [3]), no substantial information is given about the structure of B. The aim of the present note is to characterise those algebras B for which $\mathcal{A}_\varkappa(B)$ is binding, in terms of forbidden subalgebras. In our result $\varkappa$ (the number of the operations) is finite. It is an open question whether an analogous characterisation can be given for infinite $\varkappa - s$.

Henceforth let $2 \leqq \varkappa < \omega$. We shall omit the subscript $\varkappa$ if no confusion will arise.

THEOREM. *There exists a unique $\varkappa$-unary algebra A such that for any $\varkappa$-unary algebra B the followings are equivalent:*

(i) *$\mathcal{A}_\varkappa(B)$ is binding.*

(iv) *B has no one-element subalgebra and no subalgebra isomorphic to A.*

We shall prove that A is determined by the following properties;

(α) *A has no one element subalgebra.*

(β) *Every proper homomorphic image of A has a one-element subalgebra.*

(γ) *Any algebra C having no one-element subalgebra has a homomorphism into A.*

The unique algebra A, possessing properties (α), (β) and (γ) has been constructed in [2, Lemma 5.2].

For every prime number p there are only finitely many algebras of order p whose automorphism group is transitive, and the algebra contains no one-element subalgebra. We order these algebras somehow, and the i^{th} in this ordering will be denoted by $C_p(i)$. Let further C denote the disjoint union of the algebras $C_p(i)$ for all possible p and i.

II. Construction of A-critical algebras

Let λ be a cardinal number. We shall say that an algebra C is *A-critical in cardinality λ* if C contains no subalgebra isomorphic to A or to the one-element algebra, but if $\varphi\colon C\to B$ is any homomorphism and $|B|\leqq\lambda$ then B has such a subalgebra.

The existence of many A-critical algebras is needed for the proof of our theorem.

DEFINITION. Let L and K be unary algebras, H a subalgebra of K, H' a subset of L, and h a bijection between the sets H and H'. By the phrase '*we glue L and K together by h*' we mean the following procedure:

We define an algebra on $L\cup(K-H)$ in the following way. L will be a subalgebra in the natural way. If f is an operation and $k\in K$ then $f(k)$ remains the same if it is in $K-H$, and will be $h(f(k))$ otherwise.

Now let F denote the subalgebra of A which consists of those elements of A that remain fixed under all automorphisms of A. Let further $D=C_2(1)$ and let $G=(V,E)$ be an undirected graph without isolated points, loops and multiple edges.

Let finally $P(G)=(D\times V)\cup F\cup(C-D)$ be endowed with the structure of a unary algebra in the natural way. Let $f_1,\dots,f_\varkappa$ be our operations.

If R is any algebra containing $P(G)$, and we glue to it an algebra S by $h\colon H\to H'$, then this glueing is called to be *superfluous* if there is a homomorphism $\varphi\colon R\cup\cup(S-H)\to R$ which is identical on R.

This glueing is called *permitted* if the following conditions are satisfied:

(i) $R\cup(S-H)$ contains no one-element subalgebra.

(ii) If $\varphi\colon R\cup(S-H)\to B$ is any onto homomorphism, and $\varphi(a)$ generates a subalgebra isomorphic to a $C_p(i)$ then a itself generates a subalgebra isomorphic to $C_p(i)$, and $a\in P(G)$.

(iii) If $a\in R\cup(S-H)$ and there exist $u,v\in V$ such that $f_1(a)\in D\times\{u\}$ and $f_2(a)=\dots=f_\varkappa(a)\in D\times\{v\}$ then $(u,v)\in E$.

This definition perhaps needs some explanation. Condition (iii) assures that we shall be able to recognize the graph structure from certain algebras. The role of (ii) is twofold. First, it will help us to recognize the vertices of the graph, and it guarantees that (iii) is preserved under certain homomorphic images. More exactly we have

LEMMA 2.1. *Assume that R is obtained by permitted glueings from $P(G)$. Let Θ be a congruence of R and assume that Θ is the identity on $P(G)$. Then R/Θ can be obtained by a permitted glueing from $P(G)$ provided it has no one-element subalgebras.*

PROOF. Only condition (iii) needs verification. Assume that a/Θ has the required property. Then $f_i(a)$ generates a subalgebra isomorphic to D, hence by (ii) $f_i(a)\in P(G)$ for all i. But Θ is the identity on $P(G)$, hence $f_2(a)=\dots=f_\varkappa(a)\in D\times\{u\}$ for some u, hence (iii) is also satisfied in the factor algebra.

Now for a given G we define a series of algebras $\{R_\alpha\colon\alpha<\omega_1\}$ in the following way.

Let $R_0=P(G)$. If R_α is already defined, then we define $R_{\alpha+1}$ as follows. Two glueings $R_\alpha\cup(S_1-H_1)$ and $R_\alpha\cup(S_2-H_2)$ are said to be isomorphic if there is an isomorphism $\varphi\colon R_\alpha\cup(S_1-H_1)\to R_\alpha\cup(S_2-H_2)$ which is identical on R_α. Now take a glueing from each isomorphy class, where $S-H$ can be generated by one element (as a partial algebra). Let $\{R_\alpha\cup(S_\beta-H_\beta)\colon\beta<\delta\}$ be the set of these selected algebras. Assume that the sets $S_\beta - H_\beta$: $\beta < \delta$ are mutually disjoint. Then one can consider the union of these selected algebras as a $\varkappa$-unary algebra in the natural way. This algebra will be denoted by $R_{\alpha+1}$.

For limit ordinals α let R_α be the union of the ascending chain of the $R_\beta - s$, $\beta < \alpha$.

Now we can finish our construction: let Θ denote a congruence of R_{ω_1} which is the identity on $P(G)$ and the corresponding factor algebra contains no one-element subalgebras, and let Θ be maximal with respect to these properties. We shall denote the corresponding factor algebra by $E(G)$. Our aim is to prove that $E(G)$ is A-critical if the graph G satisfies certain properties.

LEMMA 2.2. *$E(G)$ admits no permitted non-superfluous glueings.*

PROOF. If $E(G)$ admits such a glueing then so does by an algebra, generated by one element. If S is generated by one element we have $|H| \leqq |S| \leqq \omega$. For each element of H we specify a co-image of this element in R_{ω_1} and the set of these co-images will be denoted by H_1. Since H_1 is countable there exists an $\alpha < \omega_1$ such that $H_1 \subseteq R_\alpha$. Now we can glue R_α and S by the natural mapping h': $H \to H_1$, and this is permitted. By the construction of $R_{\alpha+1}$ we have a homomorphism φ: $R_\alpha \cup (S-H) \to R_{\alpha+1}$ and this induces a homomorphism $E(G) \cup (S-H) \to E(G)$, contradicting the assumption that $E(G) \cup (S-H)$ was not superfluous.

LEMMA 2.3. *$E(G)$ contains no subalgebra isomorphic to A.*

PROOF. We have seen in [2, Lemma 5.2] that A satisfies the common coimage property (i.e. if $a_1, \dots, a_\varkappa \in A$ then there exists an $a \in A$ such that $f_i(a) = a_i$ for all i). Apply this to the special case where $a_1 \in D$ and $a_2 = \dots = a_\varkappa \in D$. But if A were a subalgebra of $E(G)$ then this would yield a non permitted glueing since G has no loops by the assumption.

LEMMA 2.4. *Let $T = (\{u, v\}, \{(u, v)\})$, and let φ: $E(T) \to Z$ be any homomorphism for which $\varphi(D \times \{u\}) = \varphi(D \times \{v\})$. Then Z contains either a one-element subalgebra or a subalgebra isomorphic to A.*

PROOF. We consider A as the ascending union of its subalgebras $\{A_\alpha$: $\alpha < \gamma\}$ in the following way. Let $A_0 = C \cup F$. If A_α is already constructed then let $A_{\alpha+1}$ be such a subalgebra of A that contains only one element more than A_α if this is possible. Otherwise take an arbitrary $a_\alpha \in A - A_\alpha$ and let $A_{\alpha+1} = \langle A_\alpha, a_\alpha \rangle$. For limit ordinals α let A_α be the ascending union of the subalgebras A_β; $\beta < \alpha$.

Now assume that Z contains no one-element subalgebra. We construct a homomorphism ψ: $A \to Z$ step by step using the constructed chain of subalgebras of A as follows.

Z must contain a subalgebra isomorphic to A_0, and let ψ_0: $A_0 \to Z$ be this isomorphism.

If ψ_α: $A_\alpha \to Z$ is already constructed then we distinguish two cases. If $A_{\alpha+1} - A_\alpha$ has more than one element then let $S = \langle a_\alpha \rangle$ and $H = S \cap A_\alpha$. For each $b \in H$ we specify a co-image in $E(T)$, and let H'' be the set of these co-images. Now $E(T) \cup (S-H)$ is a permitted glueing, hence by Lemma 2.2 it is superfluous, and we have a homomorphism χ: $S \to Z$ which agrees by ψ_α on H. Therefore $\psi_\alpha \cup \chi$ defines a homomorphism $A_{\alpha+1} \to Z$, and this $\psi_{\alpha+1}$ is a continuation of ψ_α.

If $A_{\alpha+1}-A_\alpha$ has only one element, then we do the same if the constructed glueing $E(T)\cup(S-H)$ is permitted. If it is not permitted then for $S-H=\{a\}$ we have $f_i(a)\in D\times\{u\}$ for all i or $f_i(a)\in D\times\{v\}$ for all i since these are the only possibilities for this glueing not to be permitted. But $\varphi(D\times\{u\})=\varphi(D\times\{v\})$ so we can specify an other set of co-images for the elements of H such that it will define a permitted glueing so we are able to define $\psi_{\alpha+1}$ in both cases.

For limit ordinals α let ψ_α be the union of the $\psi_\beta-s$ for $\beta<\alpha$.

At the end we get a homomorphism $\varphi\colon A\to Z$. But Z contains no one-element subalgebra, hence ψ must be one-to-one and this completes the proof of the Lemma.

LEMMA 2.5. *Assume that G contains the complete graph on λ^+ vertices as a subgraph. Then $E(G)$ is A-critical in cardinality λ.*

PROOF. By Lemma 2.3, $E(G)$ contains no subalgebra isomorphic to A. On the other hand, if $\varphi\colon E(G)\to B$ is any homomorphism and $|B|\leqq\lambda$ then φ cannot be one-to-one on the set $D\times K$ where K denotes the vertex set of the mentioned complete subgraph of G. Hence for some different $u, v\in K$ we necessarily have $\varphi(D\times\{u\})=\varphi(D\times\{v\})$, and we can apply Lemma 2.4 to obtain a prohibited subalgebra in B.

III. The main result

First we need a description of the endomorphism semigroup of the algebra $E(G)$ for various graphs G.

LEMMA 3.1. *All the endomorphisms of $P(G)$ are induced by automorphisms of the $C_p(i)-s$ different from D and by an endomorphism of $D\times V$.*

PROOF. This is quite clear using the fact that F has no endomorphisms, and none of the mentioned subalgebras can be mapped homomorphically into an other one.

LEMMA 3.2. *Assume that G is rigid. Then* End $E(G)=$ Aut $E(G)$.

PROOF. Let χ be an endomorphism of $E(G)$. Then χ induces an endomorphism of $D\times V$. Moreover, by property (iii) of the permitted glueing we are able to recognize the edges of the graph G from the structure of $E(G)$ and therefore χ induces a graph endomorphism of G on V. Therefore $\chi(D\times\{u\})\subseteqq D\times\{u\}$ for all $u\in V$. Specially χ is one-to-one on $D\times V$ and clearly it is also one-to-one on $C-D$ and on F. Hence it must be one-to-one on the whole of $E(G)$, since by the construction of $E(G)$, if T is a proper homomorphic image of $E(G)$ and this homomorphism is one-to-one on $P(G)$, then T contains a one element subalgebra.

On the other hand, using the fact that $E(G)$ admits no permitted non-superfluous glueings, one can see that if χ is an endomorphism of $P(G)$ that induces the trivial endomorphism on G then χ can be extended to an endomorphism of the whole $E(G)$.

Further, one can repeat the proof of [2, Lemma 5.5] to obtain the result that if χ is an endomorphism of $E(G)$, and χ induces the trivial endomorphism of $(D\times V)\cup(C-D)$ then χ itself is the identity.

Putting all these together we obtain that there is a one-to-one and onto correspondence between the endomorphisms of $E(G)$ and those endomorphisms of $P(G)$

which induce the identity mapping on V. Moreover this correspondence is a group isomorphism, hence $\mathrm{End}\, E(G) = \mathrm{Aut}\, E(G)$ and it is easy to write out this group explicitly using the fact that $\mathrm{Aut}\, C_p(i) \cong Z_p$ (the cyclic group of order p).

Using again the fact that one can recognize the structure of G from the structure of $E(G)$ we obtain the following

LEMMA 3.3. *Let G_1 and G_2 be two graphs and assume that G_1 has no homomorphism into G_2. Then $E(G_1)$ has no homomorphism into $E(G_2)$.*

Now we are in the position to prove our Theorem.

Let H satisfy condition (4). Further let λ be the cardinality of H and $H = \{h_\alpha : \alpha < \lambda\}$. Let $\tau = \max(\lambda^+, (2^\omega)^+)$ and $\{G_\alpha : \alpha < \lambda\}$ be a set of graphs such that each G_α contains the complete graph on τ vertices as a subgraph, each G_α is rigid and none of them has a homomorphism into any other one. The existence of such graphs is assured by a theorem of BABAI—NEŠETŘIL [1].

Specify an element $d \in F$. The corresponding element in $E(G_\alpha)$ will be denoted by d_α. Now we construct an algebra H'' as follows.

The underlying set of H'' will be the disjoint union of the underlying sets of the algebras H and $E(G_\alpha)$: $\alpha < \lambda$ and a new set $\{c_\alpha : \alpha < \lambda\}$.

H and the algebras $E(G_\alpha)$: $\alpha < \lambda$ will be subalgebras of H'' and we define the operations on the $c_\alpha - s$ as follows: $f_1(c_\alpha) = h_\alpha$, $f_2(c_\alpha) = \ldots = f_\varkappa(c_\alpha) = d_\alpha$.

Clearly, H'' contains H as a subalgebra, and $|H| > 2^\omega$. Hence, if we can prove that $\mathrm{End}\, H'' = \mathrm{Aut}\, H''$, then the proof will be complete.

To prove this, first we remark that none of the $E(G_\alpha) - s$ can be decomposed into the disjoint union of its proper subalgebras and that for each $x \in E(G_\alpha)$ there is a $y \in E(G_\alpha)$ such that $f_1(y) = x$, and this later property is preserved under homomorphisms.

Now let φ be an endomorphism of H''. Let $\alpha < \lambda$ be fixed. By the above remark $c_\beta \notin \varphi(E(G_\alpha))$, hence either $\varphi(E(G_\alpha)) \subseteqq E(G_\gamma)$ for some γ or $\varphi(E(G_\alpha)) \subseteqq H$. The latter is impossible since $E(G_\alpha)$ is A-critical in cardinality λ, and the first case is possible iff $\gamma = \alpha$. Hence for all α we have $\varphi(E(G_\alpha) \subseteqq E(G_\alpha)$.

This yields immediately that $\varphi(h_\alpha) = h_\alpha$ or $\varphi(h_\alpha) \in E(G_\alpha)$ holds for all α. This latter would yield that $f_i(h_\alpha) \in E(G_\alpha)$ holds for all i, but for at least one value of i we have $f_i(h_\alpha) \neq h_\alpha$ hence the second possibility never occurs, therefore $\varphi(h_\alpha) = h_\alpha$ and also $\varphi(c_\alpha) = c_\alpha$ for all $\alpha < \lambda$. This means that any endomorphism of H'' is induced by endomorphisms of the $E(G_\alpha) - s$. Now from Lemma 3.2 we conclude that $\mathrm{End}\, H'' = \mathrm{Aut}\, H''$ and this completes the proof of the theorem.

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EXTENDING COMMUTATIVITY RESULTS TO ALTERNATIVE RINGS

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A recent paper of BELL [2] contains some results concerning periodic rings whose commutator ideals are nil. Some of these results are proved for associative and alternative rings, some for associative rings only. We present here a simple result (Theorem 2) which enables the alternative generalizations of Bell's "associative only" results, and other similar ones, to be easily deduced.

A commutator $ab-ba$ will be written as $[a, b]$ in what follows, and $C(R)$ will denote the commutator ideal of a ring R, the ideal generated by all elements $[a, b]$. The first result is well known, at least in the associative case.

THEOREM 1. *The following conditions are equivalent for an alternative ring R.*

(i) *$C(R)$ is nil.*

(ii) *The nilpotent elements of R form an ideal N, and R/N is commutative.*

(iii) *The nilpotent elements of R form an ideal N and N contains $[a, b]$ for all $a, b\in R$.*

PROOF. (i)$\Rightarrow$(ii): Let N be the largest nil ideal of R. Then $C(R)\subseteq N$, so R/N is commutative. If $u, v\in R/N$ and $u^n=0$, then $(uv)^n=u^n v^n=0=(vu)^n$, while if $v^m=0$ then $(u+v)^{mn}=0$ (by the usual argument — the subring of R/N generated by u and v is associative). The nilpotent elements of R/N thus form an ideal, so R/N has no nilpotent elements and therefore N contains all nilpotent elements of R.

(ii)$\Rightarrow$(iii)$\Rightarrow$(i): Obvious.

If a, b are elements of an alternative ring, we shall denote the subring generated by a and b by the symbol $\langle a, b\rangle$. We can now prove our transfer theorem.

THEOREM 2. *Let K be a class of alternative rings such that a ring R belongs to K if and only if $\langle a, b\rangle\in K$ for all $a, b\in R$. The following conditions are equivalent.*

(i) *$C(R)$ is nil for each $R\in K$.*

(ii) *$C(R)$ is nil for each associative ring $R\in K$.*

(iii) *$C(\langle a, b\rangle)$ is nil for each $a, b\in R\in K$.*

PROOF. Since $\langle a, b\rangle\in K$ for each $a, b\in R\in K$ and every such $\langle a, b\rangle$ is associative, we need only show that (iii)$\Rightarrow$(i). Let $a, b\in R\in K$. If a is nilpotent, then, since the nilpotent elements form an ideal in $\langle a, b\rangle$, ab and ba are nilpotent. If a and b are nilpotent, then $a-b$ is nilpotent for the same reason. Hence the nilpotent elements of R form an ideal. Finally, $[a, b]$ is in $\langle a, b\rangle$, where all commutators are nilpotent. Hence $[a, b]$ belongs to the ideal of all nilpotent elements of R. By Theorem 1, $C(R)$ is nil.

In each of the following corollaries, the associative version is known, and can be found in the paper referred to. The alternative version follows from Theorem 2.

Let $[x_1, \ldots, x_n]$ be defined inductively by $[x_1, \ldots, x_{k+1}]=[[x_1, \ldots, x_k], x_{k+1}]$.

COROLLARY 3 (cf. [1] and [3]). *If R is alternative and if for each $a, b \in R$ there exist integers $m, n \geqq 1$, depending on a and b, such that $a^m b^n = b^n a^m$, then $C(R)$ is nil.*

COROLLARY 4 (cf. [2]). *Let R be a periodic, 2-torsion-free, alternative ring in which for each pair a, b of nilpotent elements, some $[a, b, a, b, \ldots, a, b]$ vanishes. Then $C(R)$ is nil.*

The next corollary follows from Corollary 4.

COROLLARY 5 (cf. [2]). *Let R be alternative, periodic and 2-torsion-free, such that for some n, every $[u_1, \ldots, u_n]$, with nilpotent u_i, vanishes. Then $C(R)$ is nil.*

COROLLARY 6 (cf. [2]). *If R is alternative and periodic and if for each $a, b \in R$ some $[a, b, b, \ldots, b]$ vanishes, then $C(R)$ is nil.*

COROLLARY 7 (cf. [2]). *If R is alternative and periodic, and $[a, b]$ is nilpotent for all nilpotent elements a, b, then $C(R)$ is nil.*

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ON THE APPROXIMATION OF LACUNARY SERIES BY BROWNIAN MOTION

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Introduction. We consider series $\sum f(n_k x)$, in which $1 \leqq n_1 \leqq n_2 \leqq \ldots \leqq n_k < \ldots$ and f satisfies these conditions

$$f(x+1) = f(x), \quad \int_0^1 f(x)\,dx = 0, \quad \int_0^1 f^2(x)\,dx = 1, \tag{1}$$

$$\int_0^1 |f - s_m|^2\,dx \leqq Cm^{-2\alpha} \tag{2}$$

for a certain $\alpha > 0$, s_m being the m-th Fourier sum of f. Our aim is to find sequences (n_k) so that $\sum f(n_k x)$ admits an approximation by Brownian motion X in the following sense: $\left|\sum_1^N f(n_k x) - X(N)\right| = O(N^{1/2-\delta})$ for almost all x in $(0, 1)$ with a certain $\delta > 0$. In contrast to [2, 3, 6], we require the correct variance, i.e. $E(X^2(N)) = N$, and we require the approximation for all sufficiently smooth functions f. Sequences (n_k) allowing this approximation are called (illogically) "strongly independent".

THEOREM 1. *Let $\theta > 1$ be an algebraic number. Then the sequence $(\theta^k)_1^\infty$ is strongly independent unless it contains a rational number.*

THEOREM 2. *Let $m_1 > 1$ and $m_{k+1} > m_k(1 + k^{-c})$ for a certain c in* (0, 1). *Then the sequence (m_k^t) is strongly independent for almost all $t > 0$.*

1. PROOF OF THEOREM 1. It is easy to prove that the irrationality condition is necessary. Indeed, let $\theta^r = pq^{-1}$ for some $r \geqq 1$ and let $f(x) = \cos(2\pi px) - \cos(2\pi qx)$. Then $\sum_1^N \cos(2\pi p\theta^k x) = \sum_1^N \cos(2\pi q\theta^{k+r} x)$, whence $\left|\sum_1^N f(\theta^k x)\right| \leqq 2r$. The proof of the sufficiency consists (apart from the analytical machinery) of an estimate of certain polynomials in θ. In all other details, we follow [3, 6].

a) First of all we break the series $\sum f(n_k x)$ into long blocks T_j and short blocks S_j. T_j is the sum over the integers k in the range $j^4 + \log^2 j < k \leqq (j+1)^4$, and S_j extends over $(j+1)^4 < k \leqq (j+1)^4 + \log^2(j+1)$. As we intend to neglect the blocks S_j, we carry out the analysis for T_j only. Let r_j be integers such that $r_j \log 2 \leqq \left(j^4 + \frac{1}{2}\log^2 j\right) \log\theta < (r_j + 1)\log 2$, and let F_j be the field of subsets of $[0, 1]$ generated by intervals $[m2^{-r_j}, (m+1)2^{-r_j}]$, $0 \leqq m < 2^{r_j}$.

To obtain a bound for the L^2-norm of $f(n_k x)-E(f(n_k x)|F_{j+1})$, where $k \leqq (j+1)^4$, we write

$$f(n_k x) = s_q(n_k x)+[f(n_k x)-s_q(n_k x)]$$

for a certain q to be chosen later. The L^2-norm of the difference is at most $2\|f-s_q\|_2=O(q^{-\alpha})$. Moreover, since $f \in L^2$, we find $s_q'=O(q^{3/2})$ so $|s_q(n_k x)-$ $-E(s_q(n_k x)|F_{j+1})|=O(q^{3/2})\,n_k 2^{-r_{j+1}}$. But $n_k 2^{-r_{j+1}} \leqq (1-\eta)^{\log^2 j}$ for a certain $\eta>0$, and we can choose q to be very large, for example $q=\exp \log^{3/2} j$.

An equally crude analysis yields a uniform estimate for $E(s_q(n_k x)|F_j)$, where $k \geqq j^4+\log^2 j$, and therefore an estimate for $E(T_j|F_j)$.

The first place in which our method differs from the standard ones is in calculating $E(T_j^2|F_j)$. Here again we use the approximation of f by $s_q(f)$, but we choose q more cautiously, for example $q=j^{30}$, or $q=j^M$ for a large, fixed M. We describe in detail the treatment of $E(s_q(n_k x)s_q(n_l x)|F_j)$, with $j^2+\log^2 j \leqq k<l$. We write

$$s_q(n_k x) = \sum{}' a_\nu e(2\pi n_k \nu x), \quad s_q(n_l x) = \sum{}' a_\mu e(2\pi n_l \mu x),$$

wherein $|a_\nu| \leqq 1$, $|a_\mu| \leqq 1$, $a_0=0$, and the sums extend to $1 \leqq |\nu| \leqq q$, $1 \leqq |\mu| \leqq q$. Now $n_k \nu - n_l \mu = \theta^k \mu - \theta^l \nu = \theta^k(\mu-\theta^l \nu)$. We shall prove later that $|\mu-\theta^l \nu| \geqq j^{-M_1}$, and from this we obtain the upper bound, $|E(s_q(n_k x)s_q(n_l x)|F_j)|<(1-\eta)^{\log^2 j}$. To estimate $E(s_q^2(n_k x)|F_j)$, we have only to use the lower bound for $n_k 2^{-r_j}$ and the relation $\|s_q\|^2=1+O(q^{-2\alpha})$.

b) To prove Theorem 1 we must bound the maximal function T_j^* of the block T_j, and in fact we prove that $T_j^*=0(j^{2-\varepsilon})$, a.e., with an $\varepsilon>0$ depending only on α. Unfortunately, even the estimate for T_j seems inaccessible via the L^2-theory, so we use norms in L^u for a certain $u=u(\alpha)>2$; the norm inequalities found below are useful in controlling the stopping times in the Skorokhod representation. In the Fourier expansion $f(x)=\sum c_n e(2\pi nx)$, with $|c_n|=|c_{-n}|$, condition (2) becomes $\sum_q^{2q} |c_n|^2 \leqq Cq^{-2\alpha}$ and Hölder's inequality leads to $\sum_q^{2q} |c_n|^s \leqq C' q^{-s\alpha} q^{1-s/2}$ if $1<s\leqq 2$. As soon as $-s\alpha+1-s/2<0$, that is, $s(\alpha+1/2)>1$, we obtain $\sum |c_n|^s<+\infty$, and therefore [8, p. 101] f is in L^u with $u=s/s-1$; we fix a number u in the interval $2<u<4$. (In case $\alpha>1/2$ f fulfills a uniform Hölder condition).

The sum $T_j=\sum f(n_k x)$: $j^4+\log^2 j \leqq k \leqq (j+1)^4$ is first approximated by a sum $\sum s_q(n_k x)$, and q is chosen to be a large power of j. Then the indices k are divided into approximately $\log^2 j$ arithmetic progressions, of difference $[\log^2 j]$, containing $O(j^4)$ terms. Each of these admits an approximation by a sum of martingale differences, with a uniformly small remainder. Let $\sum g_r$ be a sum of martingale differences; by an inequality of BURKHOLDER [4],

$$\left\|\sum g_r\right\|_u^2 \leqq C_u \left\|\sum |g_r|^2\right\|_{u/2} \leqq C_u \sum \|g_r\|_u^2$$

by Minkowski's inequality (and $u>2$). By the maximal inequality, a similar inequality holds for the maximal function [4]. Summing all these inequalities, in L^u, we find that $\|T_j^{**}\|_u=O(j^{3/2}\log j)$, where T_j^{**} is the maximal function of the sum $\sum s_q(n_k x)$. Then $\sum m(T_j^{**}>j^{2-\varepsilon})<\infty$, if $(2-\varepsilon)u>1+3u/2$ or $0<\varepsilon<1/2-u^{-1}$. In view of the choice of q, we find in addition $\sum |T_j^*-T_j^{**}|<+\infty$ a.e., so $T_j^*=O(j^{2-\varepsilon})$ a.e.

Before passing to the Skorokhod representation, we remark that the approximation of T_j by $s_q(n_k x)$ is also valid in L^u, as seen by applying the inequality of Hausdorff—Young to the remainder $f-s_q$.

c) In this paragraph we follow [6, § 3.5] in presenting the Skorokhod representation. We can find a Brownian motion (on a new probability space) and stopping times $\tau_1, \tau_2, \tau_3, \dots, \tau_j, \dots$ so that $X(\tau_1+\dots+\tau_j)$ and $T_1+\dots+T_j$ have the same distribution. Let L_j be the σ-field of the variables $\tau_1, \dots, \tau_j$ and $X(\tau_1+\dots+\tau_j)$. Then $E(\tau_j|L_{j-1})=E(T_j^2|T_1, \dots, T_{j-1})$, and the latter is $(j+1)^4-j^4+O(j^{1/2})$ uniformly. Moreover $E(\tau_j^{u/2})\ll E(T_j^u)=O(j^{3u/2}\log^4 j)$, again because $u>2$. Now $\sum \tau_j-E(\tau_j|L_{j-1})$ is a sum of martingale differences, whose growth could be controlled if we had estimates in $L^{u/2}$. Unfortunately Burkholder's inequality cannot be applied directly because $2<u<4$. We therefore combine Minkowski's inequality with an L^2-estimate. Let $3<\beta<4$ and then set $\mu_j=\tau_j$ on the set $\tau_j\leqq j^\beta$, and $\mu_j=0$ on the set $\tau_j>j^\beta$. Then

$$E(\tau_j-\mu_j)\leqq j^{\beta(1-u/2)}E(\tau_j^{u/2})=O(j^{\beta(1-u/2)})j^{3u/2}\log^4 j,$$

and we obtain the inequality

$$\sum_1^N E(\tau_j-\mu_j)=O(N^{c+1}\log^4 N),$$

with $c=\beta(1-u/2)+3u/2$. Hence, by a standard estimation $(\tau_1-\mu)+\dots+(\tau_N-\mu_N)=$ $=O(N^{c+1}\log^6 N)$ a.e. As to μ_j, we observe that

$$E(\mu_j^2)\leqq j^{\beta(2-u/2)}E(\tau_j^{u/2})=O(j^{c+\beta}\log^4 j).$$

Thus $\sum \mu_j-E(\mu_j|L_{j-1})$ is a series of martingale differences, and the sum $\sum\limits_1^N$ has variance $O(N^{c+\beta+1}\log^4 N)$.

By the maximal inequality we find

$$\sum_1^N \mu_j-E(\mu_j|L_{j-1})=O(N^{c+\beta+1}\log^6 N)^{1/2}\quad\text{a.e.}$$

Our choice of β leads to the inequalities $c+1<4$ and $c+\beta+1<8$. (The most efficient choice of β is determined by the equality $2(c+1)=c+\beta+1$, and here we find $c+1=3+2/u$.) We also note that $\sum E(\tau_j-\mu_j|L_{j-1})$ has the same growth as $\sum \tau_j-\mu_j$, whence finally $\tau_1+\dots+\tau_N=(N+1)^4+O(N^{4-\delta})$ a.e., with a $\delta>0$ depending only on $\alpha>0$. By well known inequalities on Gaussian variables, we have

$$X(\tau_1+\dots+\tau_N)-X\big((N+1)^4\big)=O(N^{1/2-\delta/2})\log N\quad\text{a.e.}$$

Until now we neglected the short blocks S_j, but clearly they can be estimated (from above) much more easily. Assembling all the almost everywhere inequalities, we obtain Theorem 1.

d) *Proof of the lower bound for* $p\theta^k-q\theta^l$. This has to be evaluated when k, l, p, q fulfill the inequalities $k<l$, $1\leqq|p|\leqq k^M$, $1\leqq|q|\leqq k^M$, for a certain constant $M>1$. The problem is then to minorize $q\theta^r-p$, with $r=l-k>0$. If $|q\theta^r-p|<1$ then $|q\theta^r|<1+|p|\leqq 2k^M$. Then $\theta^r<2k^M$, and $r\log\theta<\log 2+M\log k$, or $r\leqq M_1\log k$. We note again that $|q|<2k^M$.

Let $\theta=\theta_1, \theta_2, \dots, \theta_s$ be a complete set of conjugates of θ, and let D be a rational integer such that $D\theta$ is an algebraic integer. $D^r(q\theta_i^r+p)$ is an algebraic integer for $i=1, 2, \dots, s$ and $\prod_1^s D^r(q\theta_i^r+p)$ is a rational integer and not 0, hence $\prod_1^s |D^r(q\theta_i^r+p)|\geqq 1$. Setting $H=\max |\theta_1|, |\theta_2|, \dots, |\theta_s|$, we see that $|q\theta_i^r+p|\leqq \leqq 2k^M H^r+k^M\leqq k^{M_2}$, $D^r\leqq k^{M_2}$, whence $|q\theta^r-p|\geqq k^{-M_3}$. (For an introduction to algebraic integers, see [8, pp. 34—37].)

3. Proof of Theorem 2. This construction differs from the foregoing in two major points. More care is necessary in defining the minor blocks, as the gaps in the sequence (m_k) are very small. More important, to control the large blocks T_j, we cannot use an approximation by a sum of a small number of martingales, for the same reason. Therefore we use fourth moments, and this requires lower bounds on combinations of four of the powers m_k^t. Obtaining lower bounds for these exponential polynomials, for almost all $t>0$, is the main part of the argument.

a) Let L be an integer so large that $L>cL+3c$, or $L>3c(1-c)^{-1}$ and let T_j be the sum $\sum f(m_k^t x)$ over the range $j^L+j^{L+3}<k\leqq(j+1)^{L+3}$ and S_j over the range $j^{L+3}<k<j^{L+3}+j^L$. Let r_j be an integer such that

$$|r_j \log 4-t\log m(j^{L+3})-t\log m(j^{L+3}+j^L)|<2.$$

We observe that when $k\geqq j^{L+3}+j^L$,

$$2+t\log m_k\geqq[\log m(j^{L+3}+j^L)-\log m(j^{L+3})]\,t/2+r_j\log 2,$$

and

$$\log m(j^{L+3}+j^L)-\log m(j^{L+3})\geqq Bj^L j^{-c(L+3)}>Bj^\delta,$$

for a certain $B>0$ and $\delta>0$. With $k\leqq j^{L+3}$, the inequalities are reversed. These inequalities allow us to replace the series $\sum T_j$ by a sequence of martingale differences $\sum R_j$, in which $E(R_j\,|\,F_j)=0$ and R_j is F_{j+1}-measurable. F_j is the field of dyadic intervals of length 2^{-r_j}. (This is true for any $t>0$). In constructing R_j we replace f by s_q, with $q=j^M$ for an appropriate M.

Next we must verify that R_j is nearly an orthogonal sum over the field F_j, and in effect this requires lower bounds on the sums $qm_k^t+pm_l^t$, wherein $1\leqq|p|$, $|q|\leqq j^M$ and $j^{L+3}\leqq k<l\leqq 2j^{L+3}$. An effective lower bound is, for example, $|qm_k^t+pm_l^t|\geqq m_k^t\exp(-j^\beta)$, where β is smaller than the number δ of the previous paragraph. Suppose, then, that $t>0$ and $|q+p(m_l m_k^{-1})^t|<\exp(-j^\beta)<1/2$. Then

$$|t(\log m_l-\log m_k)-\log qp^{-1}|=O\big(\exp(-j^\beta)\big).$$

Here $\log m_l-\log m_k\geqq Bj^{-c(L+3)}$. Taking into account the number of choices of k, l, p, q (a power of j), we see that the lower bound on $qm_k^t+pm_l^t$ is valid for almost all $t>0$, for $j>J(t)$.

b) In this and the following sections we use a complicated procedure to obtain upper bounds for R_j^* and also for the corresponding stopping times. Section c) contains an almost-everywhere lower bound, to replace the one derived in d) of the last proof.

Recall that R_j was formed from $\sum f(m_k^t x)$ by replacing f by s_q. Now we replace s_q by s_j, and use the inequality $\|s_q-q_j\|_2=O(j^{-\alpha})$. The sum $\sum s_q(m_k^t x)-$ $-s_j(m_k^t x)$ is nearly orthogonal, for almost all $t>0$ and $j>J(t)$ Hence

$$\left\|\sum a_k s_q(m_k^t x)-a_k s_j(m_k^t x)\right\|_2^2 = O\left(\sum a_k^2\right) j^{-2\alpha}$$

for any set of scalars. By an inequality of Rademacher—Menshoff [8, p. 193], which in fact is proved to be valid for 'quasi-orthogonal' sums, the maximal function of the sum $\sum_k s_q(m_k^t x)-s_j(m_q x)$ has second moment $O(j^{-2\alpha})(j^{L+2})\log^2 j$. Thus the maximal function, for almost all x, is $O(j^{-\alpha/2})(j^{L/2+3/2})$. This is an allowable error as the approximation is to be within $O(j^{L/2+3/2-\delta})$ of $T_1+\ldots+T_j$. Moreover, the error in the variance is $O(j^{-2\alpha})j^{L+2}$, and if we sum this for $j=1, \ldots, N$ the error, is $O(N^{-2\alpha})N^{L+3}$, also allowable. Henceforth we work entirely with $P_j=\sum s_j(m_k^t x)$.

We proved before that $\sum |\hat{f}(n)|^s<+\infty$ for a certain $s<2$, and from this we find by Hölder's inequality that $|s_j|=O(j^\beta)$ for $\beta=1-s^{-1}<1/2$. Our objective is to find bounds for expected values $E\left(\left|\sum a_k s_j(m_k^t x)\right|^4 \middle| F_j\right)$ for almost all $t>0$. In expanding the fourth power we encounter integrals

$$E\left(s_j(m_k^t x)s_j(m_l^t x)s_j(m_\mu^t x)s_j(m_\nu^t x)\middle| F_j\right)$$

wherein $k\leqq l\leqq \mu\leqq \nu$. In case $k=l$, $\mu=\nu$ we use the bound $\|s_j\|_\infty^2=O(j^{2\beta})$ and the relation $E(s_j^2(m_k^t x)\,|\,F_j)=1+O(1)$ for almost all $t>0$, proved above. Assuming, as we shall demonstrate in a moment, that all other integrals are extremely small (for almost all $t>0$) we find, by the classical method for obtaining moments of Rademacher sums, the bound

$$E\left(\left|\sum a_k s_j(m_k^t x)\right|^4 \middle| F_j\right) = O\left(\sum |a_k|^2\right) j^{2\beta}.$$

By an inequality of Erdös—Stechkin—Serfling [1, p. 257; 7], the maximal function, M_j, of $\sum s_j(m_k^t x)$, is controlled by an inequality $\operatorname{mes}(x\colon M_j>Y)=O(Y^{-4})j^{2L+4+2\beta}$. Because $\beta<1/2$, we have $2L+4+2\beta<2L+5$. Choosing γ so that $2L+5+2\beta<$ $<4\gamma<2L+6$, and $Y=j^\gamma$, we obtain $M_j=O(Y)\leqq j^{L/2+3/2}j^{-\delta}$, a.e.

c) In this section we show how to bound the integrals encountered before, when $k<l<\mu<\nu$. Other cases are handled similarly, with less algebra. We require lower bounds on sums $c_1 m_k^t+c_2 m_l^t+c_3 m_\mu^t+c_4 m_\nu^t$; after dividing by m_k^t and changing notation somewhat, we obtain a sum $p(t)=c_1+\sum c_k e^{v_k t}$, in which the numbers v_2, v_3-v_2, v_4-v_3 are bounded below by a fixed negative power of j, and $1\leqq |c_1|\leqq j$. We observe that each of the sets $|p(t)|<\delta$ has at most 7 component intervals. For at each boundary point we have $p(t)-\delta=0$ or $p(t)+\delta=0$, and each of these functions admits at most 3 zeroes (Rolle's theorem). To find the measure of a set $|p(t)|<\delta$ it is sufficient, therefore, to find a bound for the length of an interval entirely contained in this set. Let $t_j=t_0+j\lambda$ by elements of our set, with $\lambda>0$, $j=0, 1, 2, 3$. We make the change of variables $d_1=c_1$, $d_2=c_2 e^{t_0 v_2}$, $\ldots$ $d_4=$ $c_4 e^{t_0 v_4}$, so the system of inequalities can be written

$$d_1+\sum d_k e^{j\lambda v_k} = \delta_j, \quad |\delta_j| < \delta \quad (j = 0, 1, 2, 3).$$

The determinant of this system is a Vandermonde $V(1, e^{\lambda v_2}, \ldots, e^{\lambda v_4})$, and d_1 can be found by Cramer's rule. The formula is

$$Vd_1 = \delta_1 R_1(e^{\lambda v_2}, \ldots, e^{\lambda v_4})+\ldots+\delta_4 R_4(e^{\lambda v_2}, \ldots, e^{\lambda v_4}),$$

with certain polynomials $R_1, \ldots, R_4$. By inspection we find that $R_j(z_1, z_2, z_3)$ is divisible by $V(z_1, z_2, z_3)$ and is homogeneous of weight $7-j$ in z_1, z_3, z_3. Therefore $R_j(z_1, z_2, z_3) = V(z_1, z_2, z_3)U_j$, wherein $U_j(z_1, z_2, z_3)$ is homogeneous of weight $4-j$. We obtain by inspection $U_1 = z_1z_2z_3$ and $U_4=1$, and after some calculation $U_3 = z_1+z_2+z_3$, $U_2 = z_1z_2+z_1z_3+z_2z_3$. In the case at hand, we note that $z_1>1$, $z_2>1$, $z_3>1$ so $1<U_j\leqq 3z_1z_2z_3$.

The denominator in Cramer's rule is $V(1, z_1, z_2, z_3) = V(z_1, z_2, z_3)(z_1-1)\cdot\cdot(z_2-1)(z_3-1) = V(z_1, z_2, z_3)z_1z_2z_3(1-z_1^{-1})(1-z_2^{-1})(1-z_3)^{-1}$. Putting $h=\min(\lambda v_2, 1)$, we obtain for the denominator a lower bound $V(z_1, z_2, z_3)\cdot z_1z_2z_3\cdot h^3/8$. But $|d_1|\geqq 1$, and we find $h^3=0(\delta)$. In passing we remark that this depends only on the minimum of the numbers v_2, v_3, v_4 and not their mutual separation. We now know a bound for the measure of our set.

Returning to the lower bound we were seeking, we choose $\delta=\exp(-\log^2 j)$ for example. Taking account of the inequalities $|c_1|\leqq j, \ldots, |c_4|\leqq j$, and $1\leqq|k|\leqq\leqq l\leqq\ldots\leqq(j+1)^{L+3}$, we obtain for almost all $t>0$ and $j>J(t)$, a workable lower bound for $(c_1m_k^t+\ldots+c_4m_v^t)2^{-r_j}$, and from this we obtain the conditional fourth moments used in b)

The stopping times $\tau_1, \ldots, \tau_j$ are introduced as before, but the proof that $\tau_1+\tau_2+\ldots+\tau_N = N^{L+3}+O(N^{L+3-\delta})$ is somewhat simpler, with the aid of our estimates of fourth moments, and the maximal inequality of martingales.

In conclusion, we remark that the case $c>1/2$ is treated in [2] by combinatorial methods; only cosine series are treated.

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CONTINUOUS VERSIONS OF SOME EXTREMAL HYPERGRAPH PROBLEMS. II

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Introduction

In the previous paper [1] the simplest kind of extremal hypergraph problems was considered: given a condition on the hypergraph, minimize the number of edges (the number of vertices is fixed). In the present paper we treat another class of problems. A good representative of this class is the following theorem of KRUSKAL [2]. (By some kind authors it is called Kruskal—Katona theorem, based on [3]; for simple proofs see [10], [11] and [12].): Given the number of vertices and edges (g-tuples with a fixed g) of the hypergraph, the theorem determines the minimum number of $(g-1)$-tuples contained by at least one edge. To have a "continuous" version of this we take first all the "oriented" copies of the edges and of the $(g-1)$-tuples. In this way and edge of the hypergraph becomes simply $g!$ elements of the direct product X^g of the vertex set X. We have to minimize the size of the set of projections of these elements on X^{g-1}. It is easy to find a continuous analogue of this problem; Take a measure space $M=(X, \sigma, \mu)$ and choose a measurable and symmetric set $E\subset M^g$ with a prescribed measure $\mu_g(E)$, so that the (outer) measure of the projection of E on M^{g-1} is minimal.

It is not hard to prove the continuous version of this problem by approximating E with finitely many cubes (if it has a good shape), as it was made by DAYKIN [4] who proved the continuous version of this problem, independently. However, if we take a more complicated mapping in place of projection or we take some assumptions on E, then we need a more complex proof. BOLLOBÁS [6] suggested a way, simpler than ours, which works for a wider class of problems. But their methods fail to work in the generality of the present paper.

We try to make the continuous version of the following extremal combinatorial problems. The number of vertices and edges of a hypergraph is fixed. The hypergraph satisfies certain conditions. Given a transformation which makes a new hypergraph on the same vertex-set, but the sizes of the edges may be new. The number of edges of this transformed hypergraph is to be minimized.

The condition is of the following form: all finite spanned (induced) subhypergraphs belong to a prescribed family. The transformation maps any family of g-tuples into a family of h-tuples ($h\leqq g$) in a hereditary manner.

Although the paper is a continuation of [1], we shall repeat the necessary definitions to make the paper selfcontained.

The methods of the proof are very similar to that of [1].

Definitions, results

Let X be a finite or infinite set. $G=(X, E)$ is called a *directed g-graph (hypergraph)*, where $E \subset X^g$, that is, E consists of some ordered sequences of form $e=(x_{i_1}, \dots, x_{i_g})$. The elements of X and E are called *vertices* and *edges*, respectively. Multiple edges are excluded. The edges having less than g different vertices are called *loops*. If $Y \subset X$ then the *spanned subgraph* $G_Y=(X, E)_Y=(Y, E_Y)$ consists of all those edges of G which satisfy $x_{i_j} \in Y$ for all j. If $X' \subset X$, $E' \subset E$, then (X', E') is a *subgraph* of (X, E).

Let $\mathcal{G}$ be a set of finite directed g-graphs. We say that $\mathcal{G}$ is *hereditary* if for any spanned subgraph G_1 of $G \in \mathcal{G}$, $G_1 \in \mathcal{G}$ holds. If $\mathcal{G}$ is not *hereditary* the *hereditary kernel* $\hat{\mathcal{G}}$ of $\mathcal{G}$ can be produced in the following way: $G \in \hat{\mathcal{G}}$ if and only if all the spanned subgraphs (including G) are in $\mathcal{G}$. It is easy to see that $\hat{\mathcal{G}}$ is always hereditary.

Let $M=(X, \sigma, \mu)$ be a measure space with a finite measure. (In this paper we shall consider only finite measures.) Furthermore, let $E \subset X^g$ be a measurable set in the product space $(X, \sigma, \mu)^g=(X^g, \sigma_g, \mu_g)$. We define the measure of a graph $G=(X, E)$ in the following way: $\mu(G)=\mu_g(E)$.

Let φ_X be a function which maps the finite directed g-graphs to directed h-graphs ($h \leqq g$, fixed integer) with the same vertex-set. In other words φ_X maps the subsets of X^g ($|X|<\infty$) for subsets of X^h in such a way that

(1) $\qquad \varphi_X(E)$ is invariant under the permutations of X

and

(2) $$\varphi_X(E) = \varphi_{X_1}(E) \quad \text{when} \quad X \subset X_1 \quad \text{and} \quad E \subset X^g.$$

(1) and (2) imply that φ_X does not depend on X, just on the "configuration" E. It means that if φ is determined on a set X, then it is determined in any set of a smaller cardinality. Thus, we write simply φ rather than φ_X. We call φ *hereditary* if it satisfies (1), (2) and the condition

(3) $$\varphi(E) \subset \varphi(E_1) \quad \text{if} \quad E \subset E_1.$$

For infinite X's φ is defined in the following way:

(4) $$\varphi(E) = \varphi_X(E) = \bigcup_{\substack{Y \subset X \\ Y \text{ finite}}} \varphi(E_Y).$$

Let us introduce the next notation:

(5) $$\beta(\alpha, \mathcal{G}, \varphi, M) = \inf \frac{\bar{\mu}_h(\varphi(E))}{\mu(X)^h},$$

where $\bar{\mu}_h$ is the outer measure generated by μ_h and inf is taken subject to the following conditions:

(6) $\qquad E$ is measurable in M^g,

(7) $$\frac{\mu_g(E)}{\mu(X)^g} \geqq \alpha,$$

(8) all the finite spanned subgraphs of $G=(X, E)$ are isomorphic to some graph in $\mathcal{G}$.

It is easy to see that in condition (8) and in definition (5) $\hat{\mathcal{G}}$ can be used equivalently in place of $\mathcal{G}$. Thus, in the future we can always assume that $\mathcal{G}$ is hereditary without any loss of generality.

If there is no graph with $|X|$ vertices satisfying (6)—(8), then $\beta(\alpha, \mathcal{G}, \varphi, M)$ is undefined. We assume throughout this paper, that this is not the case; when $\beta(\alpha, \mathcal{G}, \varphi, M)$ is used, it is always understood that there exists such a graph with vertex-set X.

If X is finite, σ will always be the family of all subsets of X. M_n denotes the measure space $(\{x_1, \dots, x_n\}, \sigma, \mu)$, where $\mu(x_i)=1/n$ $(1\leqq i\leqq n)$. M is called *atomless* if for any $A\in\sigma$, $\mu(A)>0$ there is a set $B\subset A$, $B\in\sigma$ such that $0<\mu(B)<\mu(A)$.

THEOREM. *If $\mathcal{G}$ is a class of directed g-graphs and φ is a hereditary function then the limit*

$$\lim_{n\to\infty} \beta(\alpha, \mathcal{G}, \varphi, M_n) = \beta_1(\alpha, \mathcal{G}, \varphi)$$

exists for all but countable many values of α $(0\leqq\alpha\leqq 1)$; it exists for $\alpha=0$. $\beta_2(\alpha, \mathcal{G}, \varphi)$ is defined to be equal to $\beta_1(\alpha, \mathcal{G}, \varphi)$, where the latter is continuous and $\beta_2(\alpha, \mathcal{G}, \varphi)=\lim_{\varepsilon\to 0}\beta_1(\alpha-\varepsilon, \mathcal{G}, \varphi)$ otherwise $(\alpha>0)$; $\beta_2(0, \mathcal{G}, \varphi)=\lim_{n\to\infty}\beta(0, \mathcal{G}, \varphi, M_n)$. Then for any atomless measure space M

$$\beta_2(\alpha, \mathcal{G}, \varphi) \leqq \beta(\alpha, \mathcal{G}, \varphi, M)$$

holds.

Examples, remarks

EXAMPLE 1. We call a g-graph *symmetric* if it contains all the permutations of its edges $(x_{i_1}, \dots, x_{i_g})$. Let $\mathcal{G}$ be the class of all symmetric g-graphs without loops. Choose $h=g-1$ and define φ by $\varphi(E)=\{(x_1, \dots, x_{g-1}):(x_1, \dots, x_g)\in E\}$. Then for a graph $G=(X, E)$ $\varphi(E)$ denotes the set of ("oriented") non-loop $(g-1)$-tuples being subsets of some edge in E. It is known (see [2], [3], simple proofs: [10], [11] and [12]) that $\binom{N}{g}$ non-oriented g-edges contain at least $\binom{N}{g-1}$ non-oriented $(g-1)$-tuples (N is an integer). This means, that

$$\beta\left(\frac{\binom{N}{g}g!}{n^g}, \mathcal{G}, \varphi, M_n\right) = \frac{(g-1)!\binom{N}{g-1}}{n^{g-1}}$$

holds. As $\beta(\alpha, \mathcal{G}, \varphi, M_n)$ is a monotonic function of α, the inequality

$$\frac{(g-1)!\binom{N_1}{g-1}}{n^{g-1}} \geqq \beta(\alpha, \mathcal{G}, \varphi, M_n) \geqq \frac{(g-1)!\binom{N_2}{g-1}}{n^{g-1}} \tag{9}$$

follows from

$$\frac{\binom{N_1}{g}g!}{n^g} \geqq \alpha \geqq \frac{\binom{N_2}{g}g!}{n^g}.$$

It is easy to see that the latter inequalities can be satisfied with $N_1=n\alpha^{1/g}+o_1(n)$ and $N_2=n\alpha^{1/g}+o_2(n)$. Consequently,

$$\lim_{n\to\infty} \beta(\alpha, \mathcal{G}, \varphi, M_n) = \alpha^{(g-1)/g}$$

follows from (9). Using the theorem, we obtain that

$$\beta(\alpha, \mathcal{G}, \varphi, M) \geqq \alpha^{(g-1)/g}$$

for any atomless M. The equality can be shown by the "cube", that is the direct product (g times with itself) of a set with measure $\alpha\mu(X)$.

This result was independently deduced from the discrete case by DAYKIN [4]. On the other hand, it can be proved directly using the Hölder-inequality, as it was observed by LOOMIS and WHITNEY [5], and A. MEIR [17].

EXAMPLE 2. Let $\mathcal{G}$ consist of the symmetric graphs without loops, containing no empty $(g+1)$-tuple ($g+1$ vertices containing no g-edge). Choose an h $(1\leqq h\leqq g)$ and φ as in the above example: $\varphi(E)=\{(x_1, \ldots, x_h):(x_1, \ldots, x_g)\in E\}$. Even the discrete problem $(\beta(\alpha, \mathcal{G}, \varphi, M_n))$ is unsolved if $g\geqq 3$ and $h\geqq 2$. The case $g=h=2$ is the well known TURÁN theorem [8]. The case $g=2$, $h=1$ is a consequence of it.

EXAMPLE 3. The discrete question is the following open problem due to P. FRANKL [15]. Given the number of vertices and g-edges of a symmetric g-graph. Determine the minimal number of $(g-1)$-tuples contained by any union of two edges. It is easy to see that this also fits to our conditions. $h=g-1$, and

$$\varphi(E) = \big\{(x_1, \ldots, x_{g-1}) \colon x_i \neq x_j \ (i \neq j);\ \exists (y_1, \ldots, y_g),\ (z_1, \ldots, z_g)\in E,$$
$$\{x_1, \ldots, x_{g-1}\} \subset \{y_1, \ldots, y_g,\ z_1, \ldots, z_g\}\big\}.$$

EXAMPLE 4. Put $g=3$, $h=2$. $\mathcal{G}$ consists of the graphs without loops, containing (x_3, x_2, x_1) and (x_1, x_2, x_3) simultaneously, φ as in Example 1. It is known (see [9]) that

$$\beta\left(\frac{(n-1)(n-2)(n-m)+m(n-m)(n-m-1)}{n^3}, \mathcal{G}, \varphi, M_n\right) = \frac{n(n-1)-m(m-1)}{n^2}$$

and

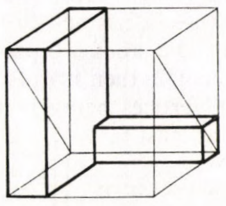
$$\beta\left(\frac{m(m-1)(m-2)}{n^3}, \mathcal{G}, \varphi, M_n\right) = \frac{m(m-1)}{n^2}$$

if $m(m-1) > \dfrac{n(n-1)}{2}+n$. It is easy to deduce

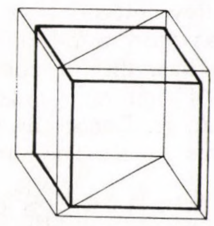
$$\lim_{n\to\infty} \beta(\alpha, \mathcal{G}, \varphi, M_n) = \begin{cases} f^{-1}(\alpha) & \text{if } \alpha \leqq \dfrac{1}{\sqrt{8}} \leqq 1 \\ \alpha^{2/3} & \text{if } \alpha \geqq \dfrac{1}{\sqrt{8}} \geqq 0, \end{cases} \tag{10}$$

where $f^{-1}(\alpha)$ is the inverse function of $f(x)=(1-x)^{3/2}+2x-1$.

What is now the continuous variant of this problem for the case when X is (e.g.) the [0, 1] interval and μ is the Lebesgue measure? Given a measurable set E with volume α in the unit cube. E is symmetric on the plane connecting two opposite edges of the cube, minimize the area of the projection on the side-plane which is not cut by the above plane. It follows from our theorem, that the right hand side of (10) is a lower estimation on $\beta(\alpha, \mathcal{G}, \varphi, M)$. The contructions of Figure 1 show that this estimation is the best possible. For this example see also [13].



$\alpha \leqq 1/\sqrt{8}$



$\alpha \geqq 1/\sqrt{8}$

Fig. 1

EXAMPLE 5. The following problem of G. HALÁSZ [14] is not solved even in the discrete case: In an undirected graph, the number of circuits of length g is given. What is the minimal number of edges? It does fit to our model:

Let $\mathcal{G}$ consist of graphs (with non-loop edges) G in which $(x_1, \dots, x_g) \in G$ is followed by $(x_i, x_{i+1}, \dots, x_g, x_1, \dots, x_{i-1}) \in G$ and $(x_i, x_{i-1}, \dots, x_1, x_g, \dots, x_{i+1}) \in G$ for all $1 \leqq i \leqq g$. $h=2$ and φ is as in Example 2.

REMARK 1. Sometimes it is easier to prove the continuous version than the discrete one. The aim of our theorem is not necessarily to show a way of proof for the continuous cases. Its aim is only to show the connection. However, it can happen that there is only an inductional proof and in this case our theorem gives a good way to the continuous through the discrete. On the other hand, the continuous version can be better visualized and this geometric picture can give a hint for the proof of both cases.

REMARK 2. It is very concievable that we have equality in the theorem. However, we were not able to prove it.

REMARK 3. We did not work out here the case when M has "atoms". However [1] shows how it could be made.

The last example shows that the limit $\lim_{n\to\infty} \beta(\alpha, \mathcal{G}, \varphi, M_n)$ does not exist in general, and $\overline{\lim}_{n\to\infty} \beta(\alpha, \mathcal{G}, \varphi, M_n)$ is not necessarily a continuous function of α.

EXAMPLE 6. Let $\mathcal{G}$ consist of the symmetric 2-graphs. Let further E be a set of (2-) edges, then

$$\varphi(E) = \begin{cases} \text{the set of vertices contained by the edges in } E \\ \text{if it contains a triangle (3 vertices with all the} \\ \text{6 non-loop edges) or a loop} \\ \emptyset \text{ otherwise.} \end{cases}$$

It is easy to see that φ satisfies (1), (2) and (3) that is, φ is hereditary. Choose $\alpha=1/2$. If n is even, then $\beta(1/2, \mathcal{G}, \varphi, M_n)=0$, since the complete bipartitie graph with $n/2$, $n/2$ vertices has exactly $n^2/2$ edges, contains no loop or triangle, consequently its $\varphi=\emptyset$. On the other hand,

$$\beta(1/2, \mathcal{G}, \varphi, M_n) = \left\{\sqrt{\frac{n^2+1}{2}}\right\}\Big/ n$$

($\{a\}$ is the smallest integer $\geqq a$) if n is odd. This follows from Turán's theorem: If the graph has more non-loop edges than the $(n-1)/2$ times $(n+1)/2$ complete bipartite graph has, then it contains a triangle. The above bipartite graph has $(n^2-1)/2$ edges. Therefore, if E satisfies (7) with $\alpha=1/2$ then it must contain either a loop or a triangle. Denote by v the number of vertices being in the edges of E (the "real vertices" of E). The number of edges is at least $(n^2+1)/2$ if n is odd. Thus the inequality $(n^2+1)/2\leqq v^2$ is obvious, its consequence is $\left\{\sqrt{\frac{n^2+1}{2}}\right\}\leqq v$. On the other hand, $\left\{\sqrt{\frac{n^2+1}{2}}\right\}$ can be easily constructed by a complete graph.

We have obtained

$$\varliminf_{n\to\infty} \beta(1/2, \mathcal{G}, \varphi, M_n) = 0$$

and

$$\varlimsup_{n\to\infty} \beta(1/2, \mathcal{G}, \varphi, M_n) = 1/\sqrt{2},$$

i.e. the limit does not exist.

It is easy to see, using similar ideas, that

$$\lim_{n\to\infty} \beta(\alpha, \mathcal{G}, \varphi, M_n) = \begin{cases} 0 & \text{if} \quad \alpha < 1/2 \\ \sqrt{\alpha} & \text{if} \quad \alpha > 1/2 \end{cases}$$

(see Fig. 2). The limit "function" is not continuous.

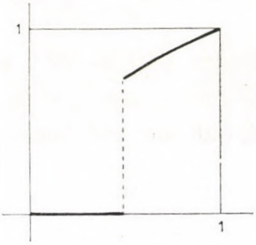


Fig. 2

On the other hand, it is easy to construct an example, when $\beta(1/2, \mathcal{G}, \varphi, M)=0$, that is, the $\varliminf$ is the exact estimation. Let M be the [0, 1] interval with the Lebesgue measure and let E be the set of pairs (x, y) where $(1/2<x$ and $y<1/2)$ or $(x<1/2$ and $1/2<y)$. Then E satisfies conditions (6)—(8), but it does not contain a complete triangle, consequently $\varphi(E)=\emptyset$ and $\beta(1/2, \mathcal{G}, \varphi, M)=0$.

REMARK 4. Condition (8) could be substituted by $\varphi(E)=X^h$ when (8) is not satisfied. In this case we would not use $\mathscr{G}$ just φ. However, in this case it is more complicated to formulate the conditions assumed on φ. This is why we choose this way of formulation. (See problem 3.)

The proof

For the proof of the theorem we need a lemma which is a special case of the law of the large numbers. We did not find it in the same form, but there are many close versions (see e.g. [16]).

LEMMA 1. *Let $\xi_1, \xi_2, \ldots$ be identically distributed random variables with existing expectation M_1 and variance D_1. Denote by $f(n)$ the number of pairs ξ_i, ξ_j $(1\leqq i, j\leqq n)$ such that ξ_i and ξ_j are not independent. If*

$$\frac{f(n)}{n^2}\to 0 \tag{11}$$

then for any $\varepsilon>0$ and $\delta>0$

$$\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i-M_1\right|<\varepsilon \tag{12}$$

with probability $1-\delta$ when n is large enough.

PROOF. Let us consider the variance of the random variable $\zeta_n=\frac{1}{n}\sum_{i=1}^{n}\xi_i$. The equalities

$$D^2(\zeta_n)=M((\zeta_n-M_1)^2)=M\left(\left(\frac{1}{n}\sum_{i=1}^{n}(\xi_i-M_1)\right)^2\right)= \tag{13}$$

$$=\frac{1}{n^2}M\left(\left(\sum_{i=1}^{n}(\xi_i-M_1)\right)^2\right)=\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}M((\xi_i-M_1)(\xi_j-M_1))$$

are obvious.

Observe that $M((\xi_i-M_1)(\xi_j-M_1))=0$ if ξ_i and ξ_j are independent, and

$$|M((\xi_i-M_1)(\xi_j-M_1))|\leqq\sqrt{M((\xi_i-M_1)^2)M((\xi_j-M_1)^2)}=D_1^2$$

otherwise. We have

$$D^2(\zeta_n)\leqq\frac{1}{n^2}f(n)D_1^2$$

by (13). Hence

$$\lim_{n\to\infty}D^2(\zeta_n)=0 \tag{14}$$

follows from (11). Apply the well-known Čebyšev-inequality:

$$P(|\zeta_n-M_1|>\varepsilon)<\frac{D^2(\zeta_n)}{\varepsilon^2}. \tag{15}$$

If n is large enough, $D^2(\xi_n)/\varepsilon^2<\delta$ holds by (14) and (15) gives the statement of the lemma.

Now we give another form of this lemma, closer to our needs.

LEMMA 2. *Let* $M=(X, \sigma, \mu)$ *be a finite measure space, E be a measurable subset of* X^g. *Then*

$$\left| \frac{|E_{\{y_1, \dots, y_n\}}|}{n^g} - \frac{\mu_g(E)}{\mu(X)^g} \right| < \varepsilon$$

with a measure $\mu(X)^n(1-\delta)$.

PROOF. We may suppose that $\mu(X)=1$. Let $y_1, \dots, y_n$ be independently chosen elements of X, and define the random variables $\xi(i_1, \dots, i_g; y_1, \dots, y_n)$ by

$$\xi(i_1, \dots, i_g; y_1, \dots, y_n) = \begin{cases} 1 & \text{if } (y_{i_1}, \dots, y_{i_g}) \in E \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

It is easy to see that their expectations and variances do not depend on $i_1, \dots, i_g$, that is, they are identical, $M_1=\mu_g(E)$. Two variables of form (16) can be dependent only when there is an equal number among $i_1, \dots, i_g$ and $i_1', \dots, i_g'$. The number of such pairs is equal to

$$\sum_{i_1, \dots, i_g} (\text{the number of sequences } i_1', \dots, i_g' \text{ non-disjoint to } i_1, \dots, i_g). \tag{17}$$

One term here is equal to n^g-(the number of disjoint sequences). The latter term is at least $(n-g)^g$ and this gives an upper estimation for (17):

$$\sum_{i_1, \dots, i_g} (n^g-(n-g)^g) = n^{2g}-n^g(n-g)^g.$$

Since the total number of pairs is n^{2g} and $(n^{2g}-n^g(n-g)^g)/n^{2g} \to 0$ when $n \to \infty$, (11) is satisfied. We may apply Lemma 1. $\frac{1}{n}\sum_{i=1}^{n} \xi_i$ in (12) becomes $|E_{\{y_1, \dots, y_n\}}|/n^g$ in our case. The lemma is proved.

PROOF OF THE THEOREM 1. Suppose $M=(X, \sigma, \mu)$ is an atomless measure space and $G=(X, E)$ is a graph satisfying (6), (7) and (8). Let us introduce the following functions:

$$I(i_1, \dots, i_h; y_1, \dots, y_n) = \begin{cases} 1 & \text{if } (yi_1, \dots, yi_h \in \varphi(E_{\{y_1, \dots, y_n\}}) \\ 0 & \text{otherwise,} \end{cases} \tag{18}$$

and

$$f(y_1, \dots, y_n) = \sum_{\substack{1 \leq i_j \leq n \\ (1 \leq j \leq h)}} I(i_1, \dots, i_h; y_1, \dots, y_n). \tag{19}$$

In other words $f(y_1, \dots, y_n)$ is the number of the ordered sequences $(y_{i_1}, \dots, y_{i_h})$ being in $\varphi(E_{\{y_1, \dots y_n\}})$. Note that the functions (18) and (19) are not necessarily measurable. For this reason we introduce $\bar{I}(i_1, \dots, i_h; y_1, \dots, y_n)$ as the indicating function of a measurable set containing the support (denoted by supp) of $I(i_1, \dots, i_h; y_1, \dots, y_n)$ while

$$\mu(\operatorname{supp} \bar{I}(i_1, \dots, i_h; y_1, \dots, y_n)) = \bar{\mu}(\operatorname{supp} I(i_1, \dots, i_h; y_1, \dots, y_n)). \tag{20}$$

Similarly to (20) we have

$$\bar{f}(y_1, \dots, y_n) = \sum_{\substack{1 \leqq i_j \leqq n \\ (1 \leqq j \leqq h)}} \bar{I}(i_1, \dots, i_h;\ y_1, \dots, y_n). \tag{21}$$

After these, the notation $\bar{\varphi}(E)$ is obvious.

$I(1, 2, \dots, h;\ y_1, \dots, y_n)$ is necessarily zero if $(y_1, \dots, y_h) \notin \varphi(E)$ by (3) and (4). In other words, $I(1, 2, \dots, h;\ y_1, \dots, y_n)$ as a function of $y_1, \dots, y_n$ can be one only if $(y_1, \dots, y_h) \in \varphi(E)$. Consequently, $\operatorname{supp} I(1, 2, \dots, h;\ y_1, \dots, y_n) \subset \varphi(E) \times X^{n-h}$ and $\operatorname{supp}(I(1, 2, \dots, h;\ y_1, \dots, y_n)) \subset \bar{\varphi}(E) \times X^{n-h}$ follow, where $\bar{\varphi}(E) \times X^{n-h}$ is measurable. Thus $(\bar{\varphi}(E) \times X^{n-h}) \cap \operatorname{supp}(\bar{I}(1, 2, \dots, h;\ y_1, \dots, y_n))$ is measurable and it contains $\operatorname{supp}(I(1, 2, \dots, h;\ y_1, \dots, y_n)$. This means, that we could choose this set in place of $\bar{I}(1, \dots, h;\ y_1, \dots, y_n)$. Suppose, that $\bar{I}(1, \dots, h;\ y_1, \dots, y_n)$ is chosen in this way. Then we have

$$\operatorname{supp}(\bar{I}(1, \dots, h;\ y_1, \dots, y_n)) \subset \bar{\varphi}(E) \times X^{n-h}. \tag{22}$$

$$\int_{X^n} \bar{f}(y_1, \dots, y_n)\, d\mu_n = \sum_{\substack{1 \leqq i_j \leqq n \\ (1 \leqq j \leqq h)}} \int_{X^n} \bar{I}(i_1, \dots, i_h;\ y_1, \dots, y_n)\, d\mu_n =$$

$$= n^h(1+o(n)) \int_{X^n} \bar{I}(1, \dots, h;\ y_1, \dots, y_n)\, d\mu_n \leqq n^h(1+o(n)) \int_{\bar{\varphi}(E) \times X^{n-h}} 1\, d\mu_n =$$

$$= n^h \bar{\mu}_h(\varphi(E)) \mu(X)^{n-h}(1+o(n))$$

follow from (21), (22) and the definition of $\bar{\varphi}(E)$. We shall use the inequality

$$\int_{X^n} \bar{f}(y_1, \dots, y_n)\, d\mu_n \leqq n^h \bar{\mu}_h(\varphi(E)) \mu(X)^{n-h}(1+o(n)). \tag{23}$$

2. Assume that $y_1, \dots, y_n$ are different. $f(y_1, \dots, y_n)$ is simply the number of elements in $\varphi(E_{\{y_1, \dots y_n\}})$. Thus

$$\beta\left(\frac{|E_{\{y_1, \dots, y_n\}}|}{n^g}, \mathcal{G}, \varphi, M_n\right) \leqq \frac{|\varphi(E_{\{y_1, \dots, y_n\}})|}{n^h} = \frac{f(y_1, \dots, y_n)}{n^h} \tag{24}$$

follows from (5), since (6)—(8) are satisfied. As β is a monotonic function of α, we obtain

$$\beta\left(\frac{\mu_g(E)}{\mu(X)^g} - \varepsilon, \mathcal{G}, \varphi, M_n\right) \leqq \frac{\bar{f}(y_1, \dots, y_n)}{n^h} \tag{25}$$

from (24), lemma 2 and the definition of $\bar{f}$. (25) holds with a measure $\mu(X)^n(1-\delta)$ on the basis of Lemma 2 and the fact that in an atomless measure $y_1, \dots, y_n$ are almost surely different (see Lemma 4 of [1]).

Take the integral of (25) over X^n and use (23):

$$\mu(X)^n(1-\delta)\beta\left(\frac{\mu_g(E)}{\mu(X)^g} - \varepsilon, \mathcal{G}, \varphi, M_n\right) \leqq \frac{1}{n^h} \int_{X^n} \bar{f}(y_1, \dots, y_n)\, d\mu_n \leqq \tag{26}$$

$$\leqq \bar{\mu}_h(\varphi(E)) \mu(X)^{n-h}(1+o(n)).$$

Since E is supposed to satisfy (7) we can write

$$\beta(\alpha-\varepsilon, \mathcal{G}, \varphi, M_n) - \delta \leqq (1-\delta)\beta(\alpha-\varepsilon, \mathcal{G}, \varphi, M_n) \leqq \frac{\bar{\mu}_h(\varphi(E))}{\mu(X)^h}(1+o(n)) \tag{27}$$

instead of (26). By (5), E can be chosen in this way:

$$\frac{\bar{\mu}_h(\varphi(E))}{\mu(X)^h} \leqq \beta(\alpha, \mathcal{G}, \varphi, M)+\delta.$$

From this and (27)

$$\beta(\alpha-\varepsilon, \mathcal{G}, \varphi, M_n)-\delta \leqq (\beta(\alpha, \mathcal{G}, \varphi, M)+\delta)(1+o(n))$$

is a consequence of (5). Hence

$$\lim_{\varepsilon\to 0}\overline{\lim_{n\to\infty}}\, \beta(\alpha-\varepsilon, \mathcal{G}, \varphi, M_n) \leqq \beta(\alpha, \mathcal{G}, \varphi, M) \tag{28}$$

easily follows when $\alpha>0$.

3. Apply the proof of Section 2 for M_m in place of M (m is $\gg n$). In Section 2, we used that M is atomfree only in one place, namely, that $y_1, \dots, y_n$ are different with measure $\mu(X)^n$. If we use M_m in place of M, this is no longer true, but the measure of sequences $y_1, \dots, y_n$ with two identical members is small; at most $\binom{n}{2}/m$. Consequently, (25) holds with a measure $\mu(X)^n(1-\delta)-\binom{n}{2}/m$ (where $\mu(X)=1$) in place of $\mu(X)^n(1-\delta)$. If $m>\binom{n}{2}/\delta$ then the new term is less than δ, thus we can simply write $1-2\delta$ in place of $\mu(X)^n(1-\delta)$ into the formulas (26) and (27):

$$\beta(\alpha-\varepsilon, \mathcal{G}, \varphi, M_n)-2\delta \leqq \mu_h(\varphi(E))(1+o(n)).$$

That is,

$$\beta(\alpha-\varepsilon, \mathcal{G}, \varphi, M_n)-2\delta \leqq \beta(\alpha, \mathcal{G}, \varphi, M_m)(1+o(n))$$

follows if n is large enough depending on ε, δ and $m>\binom{n}{2}/\delta$. Hence

$$\overline{\lim_{n\to\infty}}\, \beta(\alpha-\varepsilon, \mathcal{G}, \varphi, M_n) \leqq \underline{\lim}_{m\to\infty}\, \beta(\alpha, \mathcal{G}, \varphi, M_m). \tag{29}$$

Denote the interval $[\underline{\lim}\, \beta(\alpha, \mathcal{G}, \varphi, M_n), \overline{\lim}\, \beta(\alpha, \mathcal{G}, \varphi, M_n)]$ by I_α. It follows by (29) that these intervals are disjoint (and, of course, they lie in $[0, 1]$), therefore the length of I_α is positive only for a countable set of values α. For the other values of α, I_α is a single point, that is $\overline{\lim}=\underline{\lim}$; the limit $\lim_{n\to\infty} \beta(\alpha, \mathcal{G}, \varphi, M_n)$ exists. The function $\beta_1(\alpha, \mathcal{G}, \varphi)$ is defined in all but countably many places. Note that β_1 is monotonically increasing. An increasing function is continuous with countably many exceptions. Thus $\beta_1(\alpha, \mathcal{G}, \varphi)$ is defined and continuous on a set $[0, 1]-A$ where A is a countable set. The left hand side of (28) equals

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \beta(\alpha-\varepsilon, \mathcal{G}, \varphi, M_n) \quad (\alpha-\varepsilon\in[0, 1]-A) \quad \text{for any} \quad \alpha\in[0, 1],$$

that is, $\lim_{\varepsilon\to 0} \beta_1(\alpha-\varepsilon, \mathcal{G}, \varphi)$ (for $\alpha-\varepsilon\in[0, 1]-A$). This is, by definition, equal to $\beta_2(\alpha, \mathcal{G}, \varphi)$. The inequality of the theorem follows from (28).

4. The case $\alpha=0$ can be settled by an easy modification of sections 2 and 3. The proof is complete.

Open problems

1. Is equality in the theorem?
2. What happens if we allow the existence of "atoms"?
3. For what general class of φ's can a similar theorem be proved? (See Remark 4.)
4. Under what conditions on $\mathcal{G}$ and φ can we state that $\beta_1(\alpha, \mathcal{G}, \varphi)$ is a) continuous, b) continuous from left (right) hand side, c) defined everywhere?
5. The papers [4] and [5], where the product of the volumes of the different projections is considered, suggest a more general concept of our function φ. It would be nice to work out the right concept and prove a more general theorem.

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FINSLERSCHE PROJEKTIVGEOMETRIE. I

EINE GLOBALE BEGRÜNDUNG DER FINSLERSCHEN PROJEKTIVZUSAMMENHÄNGE

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Einleitung

In der klassischen Auffassung ist ein allgemeiner Bahnraum auf einer Mannigfaltigkeit M durch die Berwaldschen Zusammenhangsobjekte $G^i_{jk}(x, y)$ definiert, wobei diese Objekte in y positiv homogen von 0-ter Ordnung sind und die Form $G^i_{jk}=\partial^2 G^i/\partial y^j \partial y^k$ haben, mit in y positiv homogenen Objekten $G^i(x, y)$ von 2-ter Ordnung. Die Bahnen $x^i(t)$ werden danach mit Hilfe der Differentialgleichungen

$$\frac{d^2x^i}{dt^2}+G^i_{jk}\big(x(t), x'(t)\big)\frac{dx^j}{dt}\frac{dx^k}{dt}=0$$

eingeführt. Ungeachtet der Parametrisationen bestimmen die Objekte G^i_{jk} bzw. $\bar{G}^i_{jk}$ genau dann denselben Bahnraum, wenn zwischen ihnen die Relation:

$$\bar{G}^i_{jk}-\frac{1}{n+1}\bar{G}^s_{sjk}y^i = G^i_{jk}-\frac{1}{n+1}G^s_{sjk}y^i+p_j\delta^i_k+p_k\delta^i_j, \tag{1}$$

mit $p_j=\partial p/\partial y^j$ besteht, wobei $p(x, y)$ eine globale Funktion (in y positiv homogen von 1-ter Ordnung) ist [14]. Die projektiven Zusammenhangsobjekte

$$\Pi^i_{jk}=G^i_{jk}-\frac{1}{n+1}(G^s_{sj}\delta^i_k+G^s_{sk}\delta^i_j+G^s_{sjk}y^i)$$

bleiben bei den Projektiven Deformationen (1) ungeändert, und ungeachtet der Parametrisationen bestimmen sie den Bahnraum eindeutig.

In dieser Arbeit leiten wir diesen Zusammenhangsbegriff auf dem „Ehresmannschen Weg" ab. Wir behandeln auch solche Finslerschen Projektivzusammenhänge die die klassische Theorie nicht geprüft hat (siehe Cartansche und Runosche Projektivzusammenhänge), und zwischen ihnen charakterisieren wir die sogenannten Berwaldschen Projektivzusammenhänge, die das eigentliches Objekt der klassischen Untersuchungen waren.

In dem gewöhnlichen Fall (ist Π^i_{jk} die Funktion nur des Ortes) haben die Verfasser S. KOBAYASHI und T. NAGANO [9], N. TANAKA [12], S. ISHARA [5] eine globale Begründung der Projektivzusammenhänge gegeben. In dieser Arbeit verallgemeinern wir mehrere Ergebnisse von S. KOBAYASHI und T. NAGANO [9].

Mein aufrichtiger Dank gilt Herrn Prof. A. Rapcsák für seine Hilfe.

1. § Finslersche Hauptfaserbündel

Um die weiteren Überlegungen nicht unterbrechen zu müssen, führen wir einige Begriffe ein. Die einzelnen Teile werden wir mit röhmischen Zahlen bezeichnen.

I. Es bezeichnet M im folgenden immer eine n-dimensionale, parakompakte C^∞-Mannigfaltigkeit. In einem Punkt $p\in M$ definieren wir den $\mathbf{R}$-Raum $T_p(M)$ der Tangentenvektoren bzw. den $\mathbf{R}$-Raum $T_p^{(2)}(M)$ der Vektoren von zweiter Ordnung, wie folgt: Es stehe $U_p^{(1)}$ für die $\mathbf{R}$-Algebra der gewöhnliche Funktionskeimen im Punkt $p\in M$ ([4], 5 S.), die die natürliche $\mathbf{R}$-Algebra der folgenden Funktionenklassen ist: Zwei reelwertige C^∞-Funktionen, die um p definiert sind, sind in eine Klasse eingereiht, wenn sie in einer Umgebung des Punktes p übereinstimmen. Betrachten wir jetzt die Menge $U_p^{(2)*}$ der reellwertigen C^∞-Funktionen f, die in den Umgebungen des Punktes definiert sind, ferner in jeder Karte (U, x^i) um p den Bedingungen $\frac{\partial f}{\partial x^i}(p)=0$ genügen. Sind zwei Funktionen aus $U_p^{(2)*}$ in eine Klasse eingereiht, wenn sie in einer Umgebung des Punktes p übereinstimmen, so bilden auch diese Funktionenklassen eine kanonische $\mathbf{R}$-Algebra, die wir die $\mathbf{R}$-Algebra der 2-Funktionskeim im p nennen, und mit $U_p^{(2)}$ bezeichnen. Es gilt offensichtlich $U_p^{(2)}\subset U_p^{(1)}$.

Definition 1. Ein Vektor $v\in T_p^{(2)}(M)$ von zweiter Ordnung im Punkt $p\in M$ ist eine Abbildung $v\colon U_p^{(1)}\to\mathbf{R}$ mit den folgenden Eigenschaften:

1. $v(\alpha f+\beta g)=\alpha v(f)+\beta v(g), \quad \alpha;\ \beta\in\mathbf{R},\quad f;\ g\in U_p^{(1)}$,
2. $v(fg)=v(f)\,g(p)+f(p)\,v(g), \quad f\in U_p^{(1)};\quad g\in U_p^{(2)}$.

Es gilt die Relation $T_p(M)\subset T_p^{(2)}(M)$, da $T_p(M)$ der Raum der Derivationen der $\mathbf{R}$-Algebra $U_p^{(1)}$ ist ([4], S. 7). Ist (x^i) eine Karte um p, so definieren wir die Vektoren $\partial_{ij}(p);\ \partial_i(p)\in T_p^{(2)}(M)$ durch:

$$\partial_{ij}(p)(f)\overset{\text{def}}{=}\partial^2 f^*/\partial x^i\partial x^j(p),\qquad \partial_i(p)(f)\overset{\text{def}}{=}\partial f^*/\partial x^i(p),$$

mit $f\in U_p^{(1)}$, $f^*\in f$. Diese Vektoren bilden eine Basis im Raum $T_p^{(2)}(M)$, da sich jeder Vektor $v\in T_p^{(2)}(M)$ in der Form:

$$v=v(x^i)\partial_i(p)+v\big((x^i-x^i(p))(x^j-x^j(p))\big)\partial_{ij}(p)$$

darstellen läßt. Damit gilt $\big(\text{wegen}\ \ \partial_{ij}(p)=\partial_{ji}(p)\big)\dim T_p^{(2)}(M)=2n+\binom{n}{2}$. Das Vektorbündel $T^{(2)}(M)=\bigcup_{p\in M}T_p^{(2)}(M)$ heißt das Vektorbündel zweiter Ordnung von M.

Wir haben gesehen, daß $T_p(M)$ ein n-dimensionaler Unterraum im $T_p^{(2)}(M)$ ist. Ein $n+\binom{n}{2}$-dimensionaler Unterraum ω im $T_p^{(2)}(M)$ heißt ein ergänzender Unterraum, wenn $T_p^{(2)}(M)$ die direkte Summe der Räume ω bzw. $T_p(M)$ ist: $\omega\cap T_p(M)=0$, $\omega\oplus T_p(M)=T_p^{(2)}(M)$. Ist ein ergänzender Unterraum ω im $T_p^{(2)}(M)$ gegeben, so läßt sich jedes Vektor $X\in T_p^{(2)}(M)$ in eine ω-Komponente $X_\omega\in\omega$ und in eine $T_p(M)$-Komponente X_T zerlegen: $X=X_\omega+X_T$. Ist (U, x^i) eine Karte um p, so definieren wir die Koordinaten Γ^i_{jk} bezüglich ω durch

$$\partial_{ij}(p)_T = \Gamma^k_{ij}\partial_k(p) \quad \text{oder} \quad \partial_{ij}(p)_\omega = \partial_{ij}(p)-\Gamma^k_{ij}\partial_k(p).$$

Es gilt offensichtlich $\Gamma^i_{jk}=\Gamma^i_{kj}$, und die Transformationsregeln lauten;

$$\bar{\Gamma}^i_{jk} = \Gamma^l_{qr}\frac{\partial x^q}{\partial \bar{x}^j}_{|p}\frac{\partial x^r}{\partial \bar{x}^k}_{|p}\frac{\partial \bar{x}^i}{\partial x^l}_{|p}+\frac{\partial \bar{x}^i}{\partial x^r}_{|p}\frac{\partial^2 x^r}{\partial x^j\partial x^k}_{|p}.$$

Es stehe $E_p(M)$ für die Menge der ergänzenden Unterräume im $T^{(2)}_p(M)$. Das Faserbündel $E(M)=\bigcup_{p\in M} E_p(M)$, mit der Projektion $\varrho\colon E(M)\to M$, $\varrho\colon E_p\to p$ und mit den Bündelkarten der Form $(x^i, -\Gamma^i_{jk})$ (die durch die Karten (U, x^i) auf M induziert sind) heißt das Faserbündel der ergänzenden Unterräume.

Das Hauptfaserbündel der n-Beine auf M beschreiben wir durch $\{L(M), M, l, GL(n, \mathbf{R})\}$, wobei die Abbildung $l\colon L(M)\to M$ die Projektion ist, und $L_p(M)= =l^{-1}(p)$ für einen Punkt $p\in M$ die Faser auf p bezeichnet.

Das Hauptfaserbündel $\{L^{(2)}(M), M, l^{(2)}, G^{(2)}(u, \mathbf{R})\}$ von zweiter Ordnung beschreiben wir folgendermaßen: $L^{(2)}(M)$ ist die Whitney-Summe der Bündel $L(M)$ bzw. $E(M)$:

$$L^{(2)}(M) = \{(u, \omega)\in L(M)\times E(M)\,|\, l(u) = \varrho(\omega)\},$$

und die Abbildung $l^{(2)}\colon L^{(2)}(M)\to M$ mit $l^{(2)}(u, \omega)=l(U)=\varrho(\omega)$ ist die Projektion. Eine Karte (U, x^i) auf M induziert eine Bündelkarte der Form $(x^i, u^i_j, -\Gamma^i_{jk})$ auf $L^{(2)}(M)$, wobei für $(u, \omega)\in L^{(2)}_p(M)$ mit $p\in U$ die Koordinaten $(x^i, -\Gamma^i_{jk})$ sich auf $\omega\in E_p(M)$ beziehen, und die Koordinaten u^i_j aus der Darstellung $u_i=u^j_i\partial_j(p)$, $(u=(u_1, \dots, u_n))$, stammen.

Bevor wir die Strukturgruppe $G^{(2)}(n, \mathbf{R})$ des Bündels $L^{(2)}(M)$ beschreiben, beweisen wir noch, daß für jedes Element $(u, \omega)\in L^{(2)}_p(M)$ auch im Unterraum ω eine Basis eindeutig ausgezeichnet ist. In der Tat, für $(u, \omega)\in L^{(2)}_p(M)$ bezeichne $\Omega(u, \omega)$ die Menge der Karten (U, x^i) auf M um p mit den folgenden Eigenschaften:

1. Das System $\{\partial_1(p), \partial_2(p), \dots, \partial_n(p)\}$ ist genau die Basis u.
2. Die Vektoren $\partial_{ij}(p)$ sind im Unterraum ω erhalten. Wir bemerken, daß solche Karten um p existieren. Da $(\bar{U}, \bar{x}^i)$ eine beliebige Karte um p und $(\bar{x}^i(p), \bar{u}^i_j, -\bar{\Gamma}^i_{jk})$ die Bündelkoordinaten des Elementes $(u, \omega)\in L^{(2)}_p(M)$ bezüglich $(\bar{U}, \bar{x}^i)$ sind, und so besitzt die Karte (x^i), die durch die Koordinatentransformation

$$x^i = \bar{x}^i(p)+\bar{u}^i_k(\bar{x}^k-\bar{x}^k(p))-\frac{1}{2}\Gamma^i_{kl}(\bar{x}^k-\bar{x}^k(p))(\bar{x}^l-\bar{x}^l(p))$$

definiert ist, die Eigenschaften 1 bzw. 2.

Aus den Transformationsregeln:

$$\partial_i = \partial_i(\bar{x}^j)\bar{\partial}_j, \quad \partial_{ij} = \partial_{ij}(\bar{x}^l)\bar{\partial}_l+\partial_i(\bar{x}^k)\partial_j(\bar{x}^l)\bar{\partial}_{kl}$$

sind auch die folgenden Behauptungen klar; Ist die Karte (U, x^i) in $\Omega(u, \omega)$, so ist die Karte $(\bar{U}, \bar{x}^i)$ genau dann in $\Omega(u, \omega)$, wenn $\partial_i(\bar{x}^j)|_p=\delta^j_i$ und $\partial_{ij}(\bar{x}^k)|_p=0$ gelten. Damit stimmen auch die Vektoren $\underset{(u,\omega)}{\partial_{ij}}(p)\in\omega$ (induziert durch die Karten (U, x^i) aus $\Omega(u, \omega)$) überein. Diese eindeutig bestimmten Vektoren in ω bezeichnen wir durch ∂_{ij}. Es ist klar, daß diese Vektoren in ω eine Basis bilden, und es gilt noch in einer beliebigen Karte (U, x^i) um p die folgende Darstellung:

$$\overset{(u,\omega)}{\partial_{ij}} = (\partial_{pq} - \Gamma^l_{pq}\partial_l)u_i^p u_j^q.$$

Wir sind schon in der Lage die Strukturgruppe $G^{(2)}(u, \mathbf{R})$ zu definieren. Betrachtet man für jedes Element $(u, \omega) \in L_0^{(2)}(\mathbf{R}^n)$ mit $u=(u_1, u_2, \dots, u_n)$ die nichtentartete lineare Abbildung $g^{(u,\omega)}\colon T_0^{(2)}(\mathbf{R}^n) \to T_0^{(2)}(\mathbf{R}^n)$ definiert durch:

$$g^{(u,\omega)}\colon \partial_i(0) \to u_i, \quad g^{(u,\omega)}\colon \partial_{ij}(0) \to \overset{(u,\omega)}{\partial_{ij}}.$$

So ist $G^{(2)}(n, \mathbf{R}) = \{g^{(u,\omega)} \mid (u, \omega) \in L_0^{(2)}(\mathbf{R}^n)\}$ eine Liesche Untergruppe der nichtentarteten linearen Abbildungen des Raumes $T_0^{(2)}(\mathbf{R}^n)$. Bestehen die natürlichen Koordinaten $(u^i_j, -\Gamma^i_{jk})$ eines Elements $(u, \omega) \in L_0^{(2)}(\mathbf{R}^n)$ auch für die natürlichen Koordinaten der Transformation $g^{(u,\omega)}$, so ist das Produkt in diesen Koordinaten, wie folgt:

$$(u^i_j, -\Gamma^i_{jk})(\bar u^i_j, -\bar\Gamma^i_{jk}) = (u^i_l \bar u^l_j, -(\Gamma^i_{jk} + u^i_l v^q_j v^r_k \bar\Gamma^l_{qr})), \tag{1.1}$$

wobei $\operatorname{mat}(v^i_j) = \operatorname{mat}(u^i_j)^{-1}$.

Die Aktion der Gruppe $G^{(2)}(n, \mathbf{R})$ auf $L^{(2)}(M)$ ist, wie folgt: Identifiziert man jedes Element $(u, \omega) \in L_p^{(2)}(M)$ mit der linearen Abbildung $f^{(u,\omega)}\colon T_0^{(2)}(\mathbf{R}^n) \to T_p^{(2)}(M)$ definiert durch:

$$f^{(u,\omega)}\colon \partial_i(0) \to u_i, \quad f^{(u,\omega)}\colon \partial_{ij}(0) \to \overset{(u,\omega)}{\partial},$$

wobei $u=(u_1, u_2, \dots, u_n)$, so ist die Aktion $\Phi\colon L^{(2)}(M) \times G^{(2)}(u, \mathbf{R}) \to L^{(2)}(M)$ durch $\Phi(f, g) = fg$ definiert.

Es bezeichne $H^{(2)}(n, \mathbf{R})$ die Liesche Untergruppe der Gruppe $G^{(2)}(n, \mathbf{R})$ die aus den Elementen der Form $(u^i_j, -\Gamma^i_{jk})$ mit $\Gamma^i_{jk} = \delta^i_j v^l_k u_l + \delta^i_k v^l_j u_l$ besteht. Nach S. Kobayashi und T. Nagano ist die Gruppe $H^{(2)}(n, \mathbf{R})$ zur isotropische Untergruppe der projektiven Transformationen $PL(n, \mathbf{R})$ des n-dimensionalen reelen projektiven Raumes isomorph, bei denen der Punkt 0 fest bleibt. ([9]. S. 216.)

Definition 2. Ein gewöhnliches Projektivbündel $P(M)$ auf M is ein Unterfaserbündel des Bündels $L^{(2)}(M)$ mit der Strukturgruppe $H^{(2)}(u, \mathbf{R})$. ([9], S. 227.)

II. Es stehe V für die Untermannigfaltigkeit des Tangentenbündels $T(M)$, die alle Nichtnullvektoren aus $T(M)$ enthält. Die natürliche Projektion $t\colon V \to M$ induziert die folgenden Finslerschen Faserbündel $F(M) = t^* L(M)$, $F^{(2)}(M) = t^* L^{(2)}(M)$, $E_F(M) = t^* E(M)$, deren Faserräume durch

$$F(M) = \{(v, u) \in V \times L(M) \mid t(v) = l(u)\},$$
$$F^{(2)}(M) = \{(v, w) \in V \times L^{(2)}(M) \mid t(v) = l^{(2)}(w)\},$$
$$E_F(M) = \{(v, z) \in V \times E(M) \mid t(v) = \varrho(z)\}$$

definiert sind, und bei denen die Mannigfaltigkeit V der Basisraum ist. Die Projektionen: $F(M) \to V$ mit $(v, u) \to v$, $F^{(2)}(M) \to v$ mit $(v, w) \to v$, $E_F(M) \to V$ mit $(v, z) \to v$ werden wir mit l_F, $l_F^{(2)}$ bzw. mit ϱ_F bezeichnen. Damit sind $F(M)$ bzw. $F^{(2)}(M)$ Finslersche Hauptfaserbündel mit der Strukturgruppe $GL(n, \mathbf{R})$ bzw.

$G^{(2)}(n, \mathbf{R})$. Ist (x^i, y^i) eine natürliche Bündelkarte auf V, (induziert durch die Karte (U, x^i) auf M) so induziert diese Karte eine natürliche Bündelkarte der Form (x^i, y^i, u^i_j) auf $F(M)$, und die Karte $(x^i, y^i, u^i_j, -\Gamma^i_{jk})$ auf $F^{(2)}(M)$, bzw. die Bündelkarte $(x^i, y^i, -\Gamma^i_{jk})$ auf $E_F(M)$.

Wir benötigen später auch die Abbildungen

$$\alpha\colon F^{(2)}(M) \to F(M), \quad \delta\colon F^{(2)}(M) \to L^{(2)}(M), \quad \lambda\colon F(M) \to L(M),$$

die durch $\alpha((v, u))=(v, w)$ (wobei $u=(w, z)$), $\delta(v, w)=w$, $\lambda(v, u)=u$ definiert sind. Diese Abbildungen können wir im folgenden kommutativen Diagramm zusammenfassen:

$$\begin{tikzcd}
L^{(2)}(M) \arrow[d, "l^{(2)}"] \arrow[rrr, bend left] & F^{(2)}(M) \arrow[l, "\delta"'] \arrow[r, "\alpha"] \arrow[d, "l^{(2)}_F"] & F(M) \arrow[r, "\lambda"] \arrow[d, "l_F"] & L(M) \arrow[d, "l"] \\
M \arrow[rrr, leftrightarrow, bend right, "\mathrm{id}"'] & V \arrow[l, "t"'] \arrow[r, leftrightarrow, "\mathrm{id}"] & V \arrow[r, "t"] & M
\end{tikzcd}$$

S. Kobayashi [8] folgend führen wir die kanonischen Formen des Finslerschen Hauptfaserbündels $F^{(2)}(M)$ ein. Um es kurz zu machen, werden wir diese Formen in den Bündelkarten $(x^i, y^i, u^i_j, -\Gamma^i_{jk})$ aufschreiben, doch sind diese Formen wegen der Transformationsregeln auf dem ganzen Bündel $F^{(2)}(M)$ eindeutig bestimmt. Sind $(e_1, e_2, \dots, e_n)$ die natürliche Basis in $\mathbf{R}^n$ und e^i_j, $i, j=1; 2; \dots; n$ die natürliche Basis in $\mathrm{gl}\,(n, \mathbf{R})$, so definieren wir die $\mathbf{R}^n$-wertige Funktion $\bar{y}^i e_i$, die $\mathbf{R}^n$-wertigen 1-Formen $\theta^i e_i$, $\overset{(v)}{\theta^i} e_i$ bzw. die $\mathrm{gl}\,(n, \mathbf{R})$-wertige 1-Form $\theta^i_j e^j_i$ auf $F^{(2)}(M)$ durch:

$$\bar{y}^i = y^j v^i_j \quad \text{mit} \quad \mathrm{mat}\,(v^i_j) = \mathrm{mat}\,(u^i_j)^{-1}, \tag{1.2}$$

$$\theta^i = v^i_k\, dx^k, \tag{1.3}$$

$$\theta^i_j = v^i_k (du^k_j + u^n_j \Gamma^k_{mn}\, dx^m), \tag{1.4}$$

$$\overset{(v)}{\theta^i} = d\bar{y}^i + \bar{y}^j \theta^i_j = v^i_j (dy^j + y^n \Gamma^j_{mn}\, dx^m). \tag{1.5}$$

Man rechnet mühelos die folgenden Strukturgleichunges aus:

$$d\theta^i = -\theta^i_k \wedge \theta^k. \tag{1.6}$$

Wir bemerken, daß die Formeln (1, 2) und (1, 3) die kanonischen Formen auch des Hauptfaserbündels $F(M)$ definieren, und wir bezeichnen diese Formen auch auf $F(M)$ mit $\bar{y}^i e_i$ bzw. mit $\theta^i e_i$.

III. Wir beschreiben jetzt die verschiedenen Finslerschen Affinzusammenhänge, nämlich die Cartanschen-, Rundschen- bzw. Berwaldschen Affinzusammenhänge. Wir schicken voraus, daß wir unter einem Finslerschen Affinzusammenhang immer einen positiv homogenen, deflexionsfreien [10] Affinzusammenhang vestehen.

Definition 3. Ein Rundscher Affinzusammenhang Γ ist eine positiv homogene C^∞-Abbildung: $\Gamma\colon V \to E_F(M)$ mit $\Gamma(p)$ in $E_{F/p} = \varrho_F^{-1}(p)$. Die positiv Homogenität bedeutet, daß für jeden Vektor $v \in V$ und für jede positive Zahl $\varepsilon \in \mathbf{R}_+$ die invariante Eigenschaft $\varkappa(\Gamma(v)) = \varkappa(\Gamma(\varepsilon v))$ gilt, wobei die Abbildung $\varkappa\colon E_F(M) \to E(M)$ die natürliche Projektion ist: $\varkappa(v, u)=u$.

In einer Bündelkarte $(x^i, y^i, -\Gamma^i_{jk})$ auf $E_F(M)$ läßt sich die Abbildung Γ durch

$$\Gamma\colon (x^i, y^i) \to (x^i, y^i, -\Gamma^i_{jk}(x, y))$$

beschreiben, wobei wir die Funktionen $\Gamma^i_{jk}(x, y)$ die Zusammenhangsobjekte des Zusammenhangs Γ nennen. Diese Objekte sind in den unteren Indizes symmetrisch, ferner in y positivhomogen von 0-ter Ordnung.[1]

Ein Rundscher Affinzusammenhang Γ induziert eine natürliche Einbettung $\gamma\colon F(M) \to F^{(2)}(M)$, definiert durch $\gamma\colon (v, u) \to (v, (u, \Gamma(v)))$, die in den Bündelkarten die folgenden Form besitzt;

$$\gamma\colon (x^i, y^i, u^i_j) \to (x^i, y^i, u^i_j, -\Gamma^i_{jk}(x, y)). \tag{1.7}$$

Es ist leicht zu zeigen daß die Formen θ^i, $\gamma^* \theta^i_j$, $\gamma^* \overset{(v)}{\theta^i}$ auf $F(M)$ die Zusammenhangsformen (im Sinne von M. Matsumoto) eines positiv homogenen, deflexionsfreien Rundschen Affinzusammenhangs sind.

Im folgenden verstehen wir unter einer Cartanschen affinen Zusammenhangsform eine gl $(u, \mathbf{R})$-wertige Form $\omega = \omega^i_j e^j_i$ auf $F(M)$, die in den Bündelkarten (x^i, y^i, u^i_j) des $F(M)$ die folgene Form besitzt:

$$\omega^i_j = v^i_k(du^k_j + u^n_j \Gamma^k_{mn}\, dx^m + u^n_j C^k_{mn}(dy^m + y^l \Gamma^m_{rl}\, dx^r)), \tag{1.8}$$

dabei sind die Funktionen $\Gamma^k_{mn}(x, y)$ die Zusammenhangsobjekte eines Rundschen Affinzusammenhangs, ferner die Funktionen $C^k_{mn}(x, y)$ die Komponente eines Finslerschen (1.2)-Tensors, die in den unteren Indizes symmetrisch und in y positiv homogen von (-1)-ter Ordnung sind und den Bedingungen $y^m C^i_{jm} = 0$ genügen.[2] In diesem Fall sind die Strukturgleichungen der Zusammenhangsformen θ^i, ω^i_j, $\overset{(v)}{\omega} \overset{\text{def}}{=} d\bar{y}^i + \bar{y}^j \omega^i_j$ die Folgenden:

$$d\theta^i = -\omega^i_k \wedge \theta^k - C^{*i}_{jk} \overset{(v)}{\omega}{}^j \wedge \theta^k, \tag{1.9}$$

$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j - \frac{1}{2} R^{*\,i}_{j\;\,kl} \theta^k \wedge \theta^l - P^{*\,i}_{j\;\,kl} \theta^k \wedge \overset{(v)}{\omega}{}^l - \frac{1}{2} S^{*i}_{jkl} \overset{(v)}{\omega}{}^k \wedge \overset{(v)}{\omega}{}^l, \tag{1.10}$$

$$d\overset{(v)}{\omega}{}^i = -\omega^i_k \wedge \overset{(v)}{\omega}{}^k - R^{*i}_{kl} \theta^k \wedge \theta^l - P^{*i}_{kl} \theta^k \wedge \overset{(v)}{\omega}{}^l, \tag{1.11}$$

wobei die Formen $-C^{*i}_{jk} \overset{(v)}{\omega}{}^j \wedge \theta^k$; $-P^{*i}_{jk} \theta^j \wedge \overset{(v)}{\omega}{}^k$; $-R^{*\,i}_{\;\,jk} \theta^j \wedge \theta^k$ Torsionsformen, und die Formen $-\frac{1}{2} R^{*\,i}_{\;\,jkl} \theta^k \wedge \theta^l$, $-P^{*\,i}_{\;\,jkl} \theta^k \wedge \overset{(v)}{\omega}{}^l$, $-\frac{1}{2} S^{*\,i}_{\;\,jkl} \overset{(v)}{\omega}{}^k \wedge \overset{(v)}{\omega}{}^l$ Krümmungsformen des Affinzusammenhangs heißen.

[1] Ein Rundscher Affinzusammenhang Γ ist durch die Zusammenhangsobjekte Γ^i_{jk} eindeutig bestimmt, also bezeichnen wir einen Rundschen Affinzusammenhang manchmal mit Γ^i_{jk}.

[2] Ein deflexionsfreier Cartanscher Affinzusammenhang ist durch die Zusammenhangsobjekte Γ^i_{jk}, C^i_{jk} eindeutig bestimmt, da die Zusammenhangsobjekte G^i_j der nichtlinearen Zusammenhang (der Finsler—Triad [10]) durch $G^i_j = y^l \Gamma^i_{jl}$ definiert sind. Später werden wir ein Cartanscher Affinzusammenhang auch mit $\{\Gamma^i_{jk}, C^i_{jk}\}$ bezeichnen.

Ein Cartanscher Affinzusammenhang heißt vom Rundschen-Typ, wenn für ihn die Gleichung $C^{*i}_{jk}=0$ gilt, und er heißt vom Berwaldschen-Typ, wenn $C^{*i}_{jk}=0$, $P^{*i}_{jk}=0$ gelten.

Aus den Funktionen C^{*i}_{jk}, P^{*i}_{jk}, R^{*i}_{jk}, R^{*i}_{jkl}, P^{*i}_{jkl}, S^{*i}_{jkl} lassen sich auch die klasischen Komponenten der verschiedenen Torsions- und Krümmungstensoren darstellen, da (x^i, y^i, u^i_j) eine Bündelkarte auf $F(M)$ ist; so besitzt z. B. R^{*i}_{jkl} in dieser Karte die folgenden Form:

$$R^{*i}_{j\ kl} = v^i_p u^q_j u^r_k u^s_l R_q{}^p{}_{rs}(x, y),$$

wobei $R^i_{jkl}(x, y)$ die klassischen Komponenten eines Finslerschen (1.3)-Tensors sind. Ähnlicherweise bestimmt man die anderen klassischen Komponenten der Torsions- bzw. Krümmungstensoren.

IV. Wir benötigen später auch die Strukturgleichungen der Gruppe $PL(n, \mathbf{R})$ d. h. der Gruppe der projektiven Transformationen der n-dimensionalen reellen projektiven Raumes. Die Liesche Algebra $\mathrm{pl}\,(n, \mathbf{R})$ der Gruppe $\mathrm{PL}\,(n, \mathbf{R})$ ist mit der direkten Summe $\mathrm{pl}\,(n, \mathbf{R})=\mathbf{R}^n+\mathrm{gl}\,(n, \mathbf{R})+\mathbf{R}^{n*}$ identisch [9]. Sind (e_j, e^i_j, e^i), $i; j=1; \dots; n$ die natürliche Basis in $\mathrm{pl}\,(n, \mathbf{R})$ und $(\psi^i; \psi^i_j; \psi_j)$ die Dualbasis (bezüglich (e_j, e^i_j, e^i)) in $\mathrm{pl}^*\,(n, \mathbf{R})$, so sind die Strukturgleichungen die Folgenden

$$d\psi^i = -\psi^i_k \wedge \psi^k, \tag{1.12}$$

$$d\psi^i_j = -\psi^i_k \wedge \psi^k_j - \psi^i \wedge \psi_j + \delta^i_j \psi_k \wedge \psi^k, \tag{1.13}$$

$$d\psi_j = -\psi_k \wedge \psi^k_j. \tag{1.14}$$

2. § Finslersche Projektivbündel und Projektivzusammenhänge von Cartan-Typ

DEFINITION 4. Ein Finslersches Projektivbündel $\{P_F(M), V, \pi, H^2(n, \mathbf{R})\}$ ist ein positiv-homogenes Unterhauptfaserbündel des Hauptfaserbündels $F^{(2)}(M)$, wobei V der Basisraum, die Abbildung $\pi;\ P_F(M) \to V$ die Projektion und $H^{(2)}(n, \mathbf{R})$ die Strukturgruppe sind. Der Faser auf einem Punkt $v \in V$ ist mit $P_F(M)/v$ bezeichnet.

Der Faserraum $P_F(M)$ wird als positiv-homogen vorausgesetzt. Dies bedeutet, daß die invariante Eigenschaft: $\delta(P_F(M)/\varepsilon v)=\delta(P_F(M)/v)$ für jeden Vektor $v\in V$ und für jede positive Zahl $\varepsilon \in \mathbf{R}_+$ gilt, wobei die Abbildung $\delta\colon F^{(2)}(M) \to L^{(2)}(M)$ die natürliche Projektion ist.

Für einen Punkt $p \in P_F(M)$ bezeichne $T^v_p(P_F(M))$ den (n^2+n)-dimensionalen Unterraum in $T_p(P_F(M))$, der alle zur Faser $P_F(M)/\pi(p)$ tangente Vektoren enthält. Aus der Theorie der Hauptfaserbündel folgt, daß der Raum $T^v_p(P_F(M))$ mit der Lieschen Algebra $h^{(2)}(n, \mathbf{R})$ der Gruppe $H^{(2)}(n, \mathbf{R})$ identifizierbar ist. Doch ist $h^{(2)}(n, \mathbf{R})=\mathbf{R}^{n*}\oplus \mathrm{gl}\,(n, \mathbf{R})$ (siehe [9]), und so bezeichnen wir den n^2-dimensionalen, bzw. den n-dimensionalen Unterraum in $T^v_p(P_F(M))$, der mit $\mathrm{gl}\,(n, \mathbf{R})$ bzw. mit $\mathbf{R}^{n*}$ identifiziert ist, mit $\mathrm{gl}\,(n, \mathbf{R})/p$, bzw. mit $\mathbf{R}^{n*}/p$. Damit ist $T^v_p(P_F(M))=\mathrm{gl}\,(u, \mathbf{R})/p \oplus \mathbf{R}^{n*}/p$. Führen wir noch durch

$$T^Q_p(P_F(M)) = \{X \in T_p(P_F(M)) | \delta_*(X) = 0\}$$

den n-dimensionalen Unterraum $T_p^Q(P_F(M))$ in $T_p(P_F(M))$ ein, so ist $T_p^Q(P_F(M))\oplus$ $\oplus\,\mathrm{gl}\,(n, \mathbf{R})/p$ ein (n^2+n)-dimensinaler Unterraum in $T_p(P_F(M))$.

DEFINITION 5. Ein Cartanscher Zusammenhang C auf einem Projektivbündel $P_F(M)$ ist ein C^∞-Feld der n-dimensionalen Unterräume

$$C\colon (p\in P_F(M)) \to C(p) \subset T_p^Q(P_F(M))\oplus \mathrm{gl}\,(n, \mathbf{R})/p$$

derart, daß

1. der Raum $T_p^Q(P_F(M))\oplus\mathrm{gl}\,(n, \mathbf{R})/p$ die direkte Summe der Räume $C(p)$ und $\mathrm{gl}\,(n, \mathbf{R})/p$ ist:

$$T_p^Q(P_F(M))\oplus \mathrm{gl}\,(n, \mathbf{R})/p = C(p)\oplus \mathrm{gl}\,(n, \mathbf{R})/p;$$

2. $R_{a*}\, C(p)=C(R_a(p))$, $a\in H^{(2)}(n, \mathbf{R})$, wobei die Abbildung $R_a\colon P_F(M)\to P_F(M)$ die recht-Translation bezüglich $a\in H^{(2)}(n, \mathbf{R})$ ist;
3. $C(p)$ positiv homogen ist.

Die letzte Eigenschaft kann man folgendermaßen erklären. Für eine Zahl $\varepsilon\in\mathbf{R}_+$ induziert die Abbildung $A_\varepsilon\colon V\to V$ mit $A_\varepsilon\colon (v\in V)\to\varepsilon v$ eine Abbildung $B_\varepsilon\colon P_F(M)\to P_F(M)$, definiert durch: $B_\varepsilon\colon (p\in P_F(M)/v)\to(B_\varepsilon(p)\in P_F(M)/\varepsilon v)$, $\delta\circ B_\varepsilon(p)= =\delta(p)$. Hier ist $P_F(M)$ positiv homogen, so $C(p)$ heißt gerade dann positiv homogen, wenn $B_{\varepsilon*}\, C(p)=C(B_\varepsilon(p))$ für jede Zahl $\alpha\in R_+$ gilt.

DEFINITION 6. Das System $\{P_F(M), C\}$ mit den obigen Eigenschaften heißt einen Cartanschen Projektivzusammenhang auf M. Ein Projektivzusammenhang $\{P_F(M), C\}$, für die die Identität $C(p)\equiv T_p^Q(P_F(M))$ für jeden Punkt $p\in P_F(M)$ gilt, heißt einen Rundschen Projektivzusammenhang. Es ist klar, daß für ein Projektivbündel $P_F(M)$ der Rundsche Projektivzusammenhang. $\{P_F(M), T^Q(P_F(M))\}$ eindeutig bestimmt ist, und so nennt man ein Projektivbündel $P_F(M)$ manchmal auch einen Rundschen Projektivzusammenhang.

Wir leiten jetzt die Zusammenhangsformen eines Projektivzusammenhangs $\{P_F(M), C\}$ her. Beschränken wir zunächst die Formen θ^i, θ^i_j, $\overset{(v)}{\theta}{}^i$ (siehe (1, 3)—(1, 5)) auf $P_F(M)$, und bezeichnen wir diese Formen mit ω^i, ω^i_j, $\overset{(v)}{\omega}{}^i$. Sie sind weiterhin linear unabhängig. Die $\mathrm{gl}\,(n, \mathbf{R})$-wertige 1-Form $\Omega^i_j e^j_i$ bzw. die $\mathbf{R}^n$-wertige 1-Form $\overset{(v)}{\Omega}{}^i e_i$ definieren wir auf $P_F(M)$ durch:

$$\Omega^i_j(X) \overset{\mathrm{def}}{=\!=} \omega^i_j(X), \quad \text{wenn} \quad \overset{(v)}{\omega}{}^i(X)=0 \quad \text{und} \quad \Omega^i_j(Y)\overset{\mathrm{def}}{=\!=} 0,$$

wenn $Y\in C(p)$, weiterhin $\overset{(v)}{\Omega}{}^i \overset{\mathrm{def}}{=\!=} d\bar{y}^i+\bar{y}^j\Omega^i_j$.

In einer Bündelkarte $(x^i, y^i, u^i_j, -\Gamma^i_{jk})$ auf $P_F(M)$ lassen sich diese Formen so darstellen:

$$\overset{(v)}{\Omega}{}^i = \overset{(v)}{\omega}{}^i+\bar{y}^j u^n_j C^k_{mn}\overset{(v)}{\omega}{}^m = (\delta^i_m+y^n C^k_{mn})\overset{(v)}{\omega}{}^m,\text{[3]} \tag{2.1}$$

$$\Omega^i_j = v^i_k\big(du^k_j+u^n_j\Gamma^k_{mn}\,dx^m+u^n_j C^k_{mn}(dy^m+y^l\Gamma^m_{rl}\,dx^r)\big), \tag{2.2}$$

wobei die Objekte C^i_{jk} nur von den Koordinaten x^i und y^i abhängen und die Komponenten eines Finslerschen (1, 2)-Tensors sind.

[3] Im folgenden wird auch vorausgesetzt, daß für einen $\{P_F(M), C\}$ die Gleichungen $C^k_{mn}=C^k_{nm}$ und $y^nC^k_{mn}=0$ gelten.

Ergänzen wir jetzt das System $(\omega^i, \Omega^i_j, \overset{(v)}{\Omega^i_j})$ durch eine $\mathbf{R}^{n*}$-wertige Form $\Omega_j e^{*j}$ derart, dass die Formen ω^i/p, Ω^i_j/p, Ω_j/p, $\overset{(v)}{\Omega^i}/p$ für jeden Punkt $p \in P_F(M)$ in dem Dualraum $T^*_p(P_F(M))$ eine Basis bilden, ferner für die $\mathrm{pl}\,(n, \mathbf{R}) = \mathbf{R}^n \oplus \mathrm{gl}\,(n, \mathbf{R}) \oplus \oplus \mathbf{R}^{n*}$-wertigen Form $\Omega \overset{\text{def}}{=} \omega^i e_i + \Omega^i_j e^j_i + \Omega_j e^{*j}$ die folgenden Eigenschaften gelten:

1. $R^*_a \Omega = \mathrm{ad}\,(a^{-1})\,\Omega, \quad a \in H^{(2)}(n, \mathbf{R}) \subset PL(n, \mathbf{R}),$

2. $\Omega(Z) = 0$ für jeden Vektor $Z \in C(p)$, $p \in P_F(M)$, ferner $\Omega(X^* + Y^*) = X + Y$, wobei $X^* + Y^*$ ein fundamentales Vektorfeld auf $P_F(M)$, induziert durch $X + Y \in \in \mathrm{gl}\,(n, \mathbf{R}) \oplus \mathbf{R}^{*n}$, mit $X \in \mathrm{gl}\,(n, \mathbf{R})$ und $Y \in \mathbf{R}^{n*}$ ist.

Man beweist ähnlich wie in [9], S. 222, daß solche Formen auf $P_F(M)$ existieren.

Aus den Strukturgleichungen der Gruppe $\mathrm{PL}(n, \mathbf{R})$ (siehe (1.12)—(1.14)) erhält man offensichtlich die folgenden Strukturgleichungen des Systems $(\omega^i, \Omega^i_j, \Omega_j, \overset{(v)}{\Omega^i})$:

$$d\omega^i = -\Omega^i_k \wedge \omega^k - C^i, \tag{2.3}$$

$$d\Omega^i_j = -\Omega^i_k \wedge \Omega^k_j - \omega^i \wedge \Omega_j + \delta^i_j \Omega_k \wedge \omega^k - \Lambda^i_j, \tag{2.4}$$

$$d\Omega_j = -\Omega_k \wedge \Omega^k_j - \Lambda_j, \tag{2.5}$$

$$d\overset{(v)}{\Omega^i} = -\Omega^i_k \wedge \overset{(v)}{\Omega^k} - \bar{y}^j \omega^i \wedge \Omega_j + \bar{y}^i \Omega_k \wedge \omega^k - \bar{y}^j \Lambda^i_j, \tag{2.6}$$

mit

$$C^i = C^{*\,i}_{j\;k}\, \omega^j \wedge \overset{(v)}{\Omega^k}, \tag{2.7}$$

$$\Lambda^i_j = \frac{1}{2} W^{*\,i}_{j\;kl}\, \omega^k \wedge \omega^l + D^{*\,i}_{j\;kl}\, \omega^k \wedge \overset{(v)}{\Omega^l} + \frac{1}{2} S^{*\,i}_{j\;kl}\, \overset{(v)}{\Omega^k} \wedge \overset{(v)}{\Omega^l}, \tag{2.8}$$

$$\bar{y}^j \Lambda^i_j = \frac{1}{2} W^{*i}_{kl}\, \omega^k \wedge \omega^l + D^{*i}_{kl}\, \omega^k \wedge \overset{(v)}{\Omega^l} + \frac{1}{2} S^{*i}_{kl}\, \overset{(v)}{\Omega^k} \wedge \overset{(v)}{\Omega^l},^{4} \tag{2.9}$$

$$\Lambda_j = \frac{1}{2} W^*_{jik}\, \omega^i \wedge \omega^k + D^*_{jik}\, \omega^i \wedge \overset{(v)}{\Omega^k} + \frac{1}{2} S^*_{jik}\, \overset{(v)}{\Omega^i} \wedge \overset{(v)}{\Omega^k}, \tag{2.10}$$

wobei die Formen C^i, Λ^i_j, $\bar{y}^j \Lambda^i_j$, Λ_j kein solches Glied enthalten, in dem die Formen Ω^i_j und Ω_j vorkommen.

Wir wählen jetzt zu jedem Projektivzusammenhang $\{P_F(M), C\}$ eine natürliche Form Ω_j aus.

SATZ 1. *Für einen Projektivzusammenhang* $\{P_F(M), C\}$ *gibt es genau eine Form* Ω_j *mit den Eigenschaften* 1 *und* 2, *und es gelten noch die Gleichungen:*

3. $W^{*\,i}_{i\;jk} = W^{*\,i}_{j\;ik} = D^{*\,i}_{i\;jk} = 0.$

[4] Ist $C^i_{jk} = C^i_{kj}$, so ist $S^{*i}_{kl} = 0$.

BEWEIS. Der Beweis dieser Tatsache erfolgt wiederum ganz ähnlich wie in [9], S. 221. Es sind Ω_j und $\bar{\Omega}_j$ die Formen auf $P_F(M)$ mit 1, 2 und mit

$$\Omega_j - \bar{\Omega}_j = A_{jk}\omega^k + B_{jk}\overset{(v)}{\Omega^k},$$

und im Hinblick auf die Strukturgleichungen erhalten wir daher, daß für das System $(\omega^i, \Omega^i_j, \bar{\Omega}_j, \overset{(v)}{\Omega^i})$ die Gleichungen $\bar{W}^{*i}_{i\ jk} = \bar{W}^{*i}_{j\ ik} = \bar{D}^{*i}_{i\ jk} = 0$ genau dann gelten, wenn

$$A_{jk} = \frac{1}{n-1} W^{*i}_{j\ ik} + \frac{1}{n^2-1} W^{*i}_{i\ jk}, \quad B_{jk} = \frac{1}{n+1} D^{*i}_{i\ jk}.$$

W. z. b. w.

Für einen Projektivzusammenfang $\{P_F(M), C\}$ heißen die obigen, in eindeutiger Weise Bestimmten Formen $(\omega^i, \Omega^i_j, \Omega_j, \overset{(v)}{\Omega^i})$ die projektiven Zusammenhangsformen bezüglich $\{P_F(M), C\}$.

Der Operator d wirkt an den Strukturgleichungen, womit wir die projektiven Bianchi-Identitäten erhalten:

$$dC^i - \Lambda^i_k \wedge \omega^k + \Omega^i_k \wedge C^k = 0, \tag{2.11}$$

$$d\Lambda^i_j + \omega^i \wedge \Lambda_j + \delta^i_j \omega^k \wedge \Lambda_k + \Lambda^k_j \wedge \Omega^i_k - \Lambda^i_k \wedge \Omega^k_j$$
$$- C^i \wedge \Omega_j - \delta^i_j \Omega_k \wedge C^k = 0, \tag{2.12}$$

$$d\Lambda_j - \Lambda_k \wedge \Omega^k_j + \Lambda^k_j \wedge \Omega_k = 0. \tag{2.13}$$

Wir leiten noch in einer Bündelkarte (U, x^i, y^i) auf V die projektiven Zusammenhangsobjekte $\Pi^i_{jk}(x, y)$, $C^i_{jk}(x, y)$ eines Projektivzusammenhangs $\{P_F(M), C\}$ her.

SATZ 2. *Sind $\{P_F(M), C\}$ ein Projektivzusammenhang und (U, x^i, y^i) eine Bündelkarte auf V, so gibt es genau einen C^∞-Schnitt $\sigma\colon U \to P_F(M)$ mit den Eigenschaften:*

$$\sigma^* \omega^i = dx^i, \quad \sigma^* \Omega^i_j = \Pi^i_{jk}\, dx^k + C^i_{jk}(dy^k + y^l \Pi^k_{lj}\, dx^j),$$

und mit $\Pi^i_{jk} = \Pi^i_{kj}$, $\Pi^i_{ij} = 0$.

Die Objekte Π^i_{jk} und C^i_{jk} sind Funktionen der Koordinaten x^i und y^i, ferner sind sie in y positiv homogen von 0-ter bzw. von (-1)-ter Ordnung. Die Funktionen C^i_{jk} sind die Komponenten eines Finslerschen (1, 2)-Tensorfeldes, und die Transformationsregeln der Objekte Π^i_{jk} stimmen mit den Transformationsregeln der gewöhnlichen projektiven Zusammenhangsobjekte (Π^i_{jk} ist Funktion nur des Ortes) überein [3].

BEWEIS. Ist $(x^i, y^i) \to (x^i, y^i, \delta^i_j, -\Gamma^i_{jk})$ ein beliebiger Schnitt in $P_F(M)$ auf U, so besitzt der Schnitt

$$(x^i, y^i) \to \left(x^i, y^i, \delta^i_j, -\left(\Gamma^i_{jk} - \frac{1}{n+1}\delta^i_j \Gamma^h_{hk} - \frac{1}{n+1}\delta^i_k \Gamma^h_{hj}\right)\right) \tag{2.14}$$

die in dem Satz formulierte Eigenschaften.

Umgekehrt, haben die Schnitte σ bzw. $\bar{\sigma}$ die obigen Eigenschaften, so sind sie von der Form

$$\sigma\colon (x^i, y^i) \to (x^i, y^i, \delta^i_j, -\Pi^i_{jk}),$$
$$\bar{\sigma}\colon (x^i, y^i) \to (x^i, y^i, \delta^i_j, -\bar{\Pi}^i_{jk}),$$

mit $\Pi^i_{jk}=\bar{\Pi}^i_{jk}+\psi_j(x,y)\delta^i_j+\psi_k(x,y)\delta^i_j$. Mit der Kontraktion $i\to j$ erhalten wir $(n+1)\psi_k=0$ und daher $\Pi^i_{jk}=\bar{\Pi}^i_{jk}$. W. z. b. w.

Wir beweisen noch, daß jeder Cartansche Affinzusammenhang mit den Zusammenhangsobjekten $\{\Gamma^i_{jk}, C^i_{jk}\}$ in eindeutiger Weise einen natürlichen Projektivzusammenhang $\{P_F(M), C\}$ bestimmt, die wir wie folgt, beschreiben. Es bezeichne Γ den Rundschen Affinzusammenhang mit den Zusammenhangsobjekten Γ^i_{jk}, und $\gamma\colon F(M)\to F^{(2)}(M)$ die natürliche Einbettung bezüglich Γ (siehe 1.7). So gibt es genau ein Projektivbündel $P_F(M)$ derart, daß die Bildmenge $\gamma(F(M))$ in $P_F(M)$ liegt. Wir definieren noch den Cartanschen Zusammenhang C auf $P_F(M)$. Sind $\theta^i, \omega^i_j, \overset{(v)}{\omega}{}^i \overset{\text{def}}{=} d\bar{y}^i+\bar{y}^j\omega^i_j$ die Zusammenhangsformen des Cartanschen Affinzusammenhangs auf $F(M)$, so führen wir nach M. Matsumoto den vertikalen Zusammenhang Γ^v durch die folgende Formel ein:

$$\Gamma^v_{/p} = \{X\in T_p(F(M))|\omega^i(X)=0,\ \omega^i_j(X)=0\}.$$

Es folgt leicht, daß für jeden Punkt $p\in P_F(M)$ genau ein n-dimensionaler Unterraum $C(p)$ in $T^Q_p(P_F(M))\oplus\mathrm{gl}\,(n,\mathbf{R})/p\subset T_p(P_F(M))$ existiert derart, daß $\alpha_*(C(p))=\Gamma^v_{/p}$ gilt, wobei $\alpha\colon P_F(M)\to F(M)$ die natürliche Projektion ist. Damit ist der Zusammenhang C auf $P_F(M)$ bestimmt. Aus den Formeln (1.1)—(1.6), (1.7), (1.8) und aus den Strukturgleichungen (1.9)—(1.11), (2.3)—(2.6) erhalten wir den

SATZ 3. 1. *Die Cartanschen Affinzusammenhänge mit den Zusammenhangsobjekten* $\{\Gamma^i_{jk}, C^i_{jk}\}$ *und* $\{\bar{\Gamma}^i_{jk}, \bar{C}^i_{jk}\}$ *bestimmen dieselben Projektivzusammenhang* $\{P_F(M), C\}$ *genau dann, wenn zwischen ihnen die Relationen*

$$\bar{\Gamma}^i_{jk}=\Gamma^i_{jk}+p_j\delta^i_k+p_k\delta^i_j,\quad \bar{C}^i_{jk}=C^i_{jk} \tag{2.15}$$

gelten, wobei $p_j(x,y)$ *ein Finschlersches kovariantes Vektorfeld, positiv homogen von* 0*-ter Ordnung ist.*

2. *Induziert der Cartansche Affinzusammenhang mit den Zusammenhangsobjekten* $\{\Gamma^i_{jk}, C^i_{jk}\}$ *den Projektivzusammenhang* $\{P_F(M), C\}$, *so sind die Formen* $\gamma^*\theta^i=\theta^i$, $\gamma^*\Omega^i_j$, $\gamma^*\overset{(v)}{\Omega}{}^i$ *auf* $F(M)$ *die Zusammenhangsformen des gegebenen Cartanschen Affinzusammenhangs ferner gilt*

$$\gamma^*\Omega_j=R^*_{jk}\theta^k+P^*_{jk}\gamma^*\overset{(v)}{\Omega}{}^k \quad mit \quad R^*_{jk}=\frac{1}{n^2-1}(nR^{*i}_{j\ ik}+R^{*i}_{k\ ij}) \tag{2.16}$$

und $P^*_{jk}=\dfrac{1}{n+1}P^{*i}_{i\ jk}$, *wobei die Funktionen* $R^{*i}_{j\ kl}$, $P^{*i}_{j\ kl}$, $S^{*i}_{i\ kl}$ *auf* $F(M)$ *in* (1.10) *definiert sind.*

Es gelten auch die folgenden Formeln:

$$W^{*i}_{j\ kl}\circ\gamma=R^{*i}_{j\ kl}-R^*_{jl}\delta^i_k+R^*_{jk}\delta^i_l+(R^*_{lk}-R^*_{kl})\delta^i_j, \tag{2.17}$$

$$D^{*i}_{j\ kl}\circ\gamma=P^{*i}_{j\ kl}-P^*_{jl}\delta^i_k-P^*_{kl}\delta^i_j, \tag{2.18}$$

(2.19) $S^{*i}_{j\ kl}\circ\gamma$ *sind genau die Komponenten der vertikalen Krümmungsforme des gegebenen Cartanschen Affinzusammenhangs in* (1.10).

3. § Projektivzusammenhänge vom Rundschen und Berwaldschen Typ

Ein Projektivzusammenhang $\{P_F(M), C\}$ heißt von Rundschen Typ, wenn für jeden Punkt $p \in P_F(M)$ die Identität $C(p) = T_p^{\Omega}(M)$ gilt. Dies trifft genau dann zu, wenn der Tensor C^i_{jk} verschwindet, und in diesem Fall sind die Zusammenhangsformen Ω^i_j, $\overset{(v)}{\Omega^i}$ keine anderen als die Beschränkungen der Formen θ^i_j, $\overset{(v)}{\theta^i}$ auf $P_F(M)$. Induziert der Rundsche Affinzusammenhang Γ das Projektivbündel $P_F(M)$, so sagt man, daß Γ in $P_F(M)$ ist.

Wir fragen jetzt, unter welchen Bedingungen die Krümmungsformen Λ^i_j eines Projektivzusammenhangs $\{P_F(M), C\}$ Finslersche Affintensoren sind. Die Form Λ^i_j heißt ein Finslerscher Affinentensor, wenn Λ^i_j die Form $\Lambda^i_j = \alpha^* \bar{\Lambda}^i_j$ hat, wobei $\bar{\Lambda} \overset{\text{def}}{=} \bar{\Lambda}^i_j e^j_i$ eine gl $(n, \mathbf{R})$-wertige Form auf $F(M)$ mit $R^*_a \bar{\Lambda} = \operatorname{ad}(a^{-1})\bar{\Lambda}$, $a \in \mathrm{GL}(n, \mathbf{R})$ ist, ferner die Abbildung $\alpha\colon F^{(2)}(M) \to F(M)$ die natürliche Projektion bezeichnet.

Satz 4. *Λ^i_j ist genau dann eine Finslerscher Affintensor, wenn der Projektivzusammenhang $\{P_F(M), C\}$ vom Rundschen Typ ist. Sind $\{P_F(M), T^{\Omega}\}$ ein Rundscher Projektivzusammenhang und Γ ein Rundscher Affinzusammenhang in $P_F(M)$, bezeichnet ferner $\gamma\colon F(M) \to P_F(M)$ die natürliche Einbettung bezüglich Γ, so erhalten wir auf $F(M)$:*

$$(3.1)\qquad \bar{\Lambda}^i_j = \gamma^* \Lambda^i_j = \frac{1}{2} \bar{W}_j{}^i{}_{kl} \theta^k \wedge \theta^l + \bar{D}_j{}^i{}_{kl} \theta^k \wedge \gamma^* \overset{(v)}{\theta^l},$$

wobei die Funktionen $\bar{W}_j{}^i{}_{kl}$, $\bar{D}_j{}^i{}_{kl}$ in einer Bündelkarte (x^i, y^i, u^i_j) auf $F(M)$ die folgenden Formen haben:

$$\bar{W}_j{}^i{}_{kl} = v^i_p u^q_j u^r_k u^s_l W_q{}^p{}_{rs}(x, y),$$

$$\bar{D}_j{}^i{}_{kl} = v^i_p u^q_j u^r_k u^s_l D_q{}^p{}_{rs}(x, y),$$

damit sind die Funktionen $W_q{}^p{}_{rs}(x, y)$, $D_q{}^p{}_{rs}(x, y)$ die klassischen Komponenten der Finslerschen (1.3)-Tensoren mit

$$(3.2)\qquad W_q{}^p{}_{rs} = R_q{}^p{}_{rs} + R_{qr}\delta^p_s - R_{qs}\delta^p_r + (R_{sr} - R_{rs})\delta^p_q,$$

$$(3.3)\qquad D_q{}^p{}_{rs} = P_q{}^p{}_{rs} - P_{qs}\delta^p_r - P_{rs}\delta^p_q = \frac{\partial \Pi^p_{qr}}{\partial y^s},$$

so $D_q{}^p{}_{rs} = D_r{}^p{}_{qs}$.

Der Tensor $D_j{}^i{}_{kl}(x, y)$, den wir den Douglasschen Tensor des Projektivbündels $P_F(M)$ nennen, ist unabhängig von der Wahl des Affinzusammenhangs Γ aus $P_F(M)$. Der Tensor $W_j{}^i{}_{kl}$ ist von der Wahl der Einbettung γ genau dann unabhängig, wenn der Tensor $D_j{}^i{}_{kl}$ in den Indizes k und l symmetrisch ist.

Bemerkung. Später werden wir beweisen, daß der Tensor $W_j{}^i{}_{kl}$ genau dann eindeutig Bestimmt ist, wenn das Projektivbündel $P_F(M)$ von Berwald-Typ ist (siehe die folgende Definition). In diesem Fall nennen wir den Tensor $W_j{}^i{}_{kl}$ den Weylschen Krümmungstensor des Bündels $P_F(M)$. Diesen Tensor hat zuerst L. Berwald [1] eingeführt.

Beweis des Satzes. Es bezeichne M^* den Kern der natürlichen Projektion $H^{(2)}(n, \mathbf{R}) \to \mathrm{GL}(n, \mathbf{R})$, und m^* sei die Liesche Algebra der Gruppe M^*. Λ^i_j ist genau dann ein Finslerscher Affintensor, wenn $\mathscr{L}_{A^*}\Lambda^i_j=0$ für jedes fundamentale Vektorfeld A^* auf $P_F(M)$ mit $A\in m^*$ gilt. Mit Rücksicht auf die Strukturgleichungen und auf die Bianchi-Identitäten erhalten wir:

$$\mathscr{L}_{A^*}\Lambda^i_j = A^* \lrcorner\, d\Lambda^i_j + d(A^* \lrcorner\, \Lambda^i_j) = A^* \lrcorner\, C^i \wedge \Omega_j = A_j C^i,$$

damit ist die erste Behauptung bewiesen.

Die Formeln (3.2) und (3.3) erhält man aus den Strukturgleichungen durch einfache Rechnung.

Wir beweisen, daß der Tensor $D_j{}^i{}_{kl}$ für $\{P_F(M), T^Q\}$ eindeutig bestimmt ist. Sind $\overset{(1)}{\gamma}$ und $\overset{(2)}{\gamma}$ zwei natürliche Einbettungen bezüglich $\overset{(1)}{\Gamma}$ und $\overset{(2)}{\Gamma}$, zwischen denen die Relation

$$\overset{(2)}{\Gamma}{}^i_{jk} = \overset{(1)}{\Gamma}{}^i_{jk} + p_j\delta^i_k + p_k\delta^i_j$$

gilt, so folgt aus (1, 5): $\overset{(2)}{\gamma}{}^*\overset{(v)}{\theta}{}^l = \overset{(1)}{\gamma}{}^*\overset{(v)}{\theta}{}^l + y^l p_l\theta^k + \bar{y}^k u^m_l\theta^l$. Wegen $y^m D_k{}^i{}_{lm} = y^m\partial\Pi^i_{kl}/\partial y^m = 0$ erhalten wir:

$$\overset{(1)}{\gamma}{}^*\Lambda^i_j = \overset{(2)}{\gamma}{}^*\Lambda^i_j = \frac{1}{2}\overset{(2)}{\overline{W}}_j{}^i{}_{kl}\theta^k\wedge\theta^l + \overset{(2)}{\overline{D}}_j{}^i{}_{kl}\theta^k\wedge\overset{(2)}{\gamma}{}^*\overset{(v)}{\theta}{}^l =$$

$$= \left(\frac{1}{2}\overset{(2)}{\overline{W}}_j{}^i{}_{kl} + y^l p_l\overset{(2)}{\overline{D}}_j{}^i{}_{kl}\right)\theta^k\wedge\theta^l + \overset{(2)}{\overline{D}}_j{}^i{}_{kl}\theta^k\wedge\overset{(1)}{\gamma}{}^*\overset{(v)}{\theta}{}^l,$$

und so bekommt man:

$$\overset{(1)}{\overline{D}}_j{}^i{}_{kl} = \overset{(2)}{\overline{D}}_j{}^i{}_{kl}, \quad \overset{(2)}{\overline{W}}_j{}^i{}_{kl} = \overset{(1)}{\overline{W}}_j{}^i{}_{kl} + y^l p_l(\overset{(2)}{\overline{D}}_j{}^i{}_{kl} - \overset{(2)}{\overline{D}}_j{}^i{}_{lk}).$$

W. z. b. w.

DEFINITION 7. Ein Projektivbündel $P_F(M)$ heißt vom Berwaldschen Typ, wenn der Douglassche Torsionstensor $D^i_{jk} = y^l D_l{}^i{}_{jk}$ verschwindet. Eine Rundsche Projektivzusammenhang $\{P_F(M), T^Q\}$ heißt vom Berwaldschen Typ, wenn $P_F(M)$ vom Berwaldschen Typ ist. In Folgenden werden wir die Berwaldschen Projektivbündel mit $P_B(M)$ bezeichnen.

Um die Berwaldschen Projektivbündel näher beschreiben zu können, führen wir die folgenden Begriffe ein.

DEFINITION 8. Eine Rundsche Affinzusammenhang Γ (mit den Zusammenhangsobjekten $\Gamma^i_{jk}(x, y)$) heißt projizierbar, wenn auch die Berwaldschen Zusammenhangsobjekte $G^i_{jk} = \partial(y^r\Gamma^i_{jr})/\partial y^k = \Gamma^i_{jk} + y^r\partial\Gamma^i_{jr}/\partial y^k$ in den indizes j und k symmetrisch sind. Dies ist der Fall genau dann, wenn der Torsionstensor $P^i_{jk} = y^r\partial\Gamma^i_{jr}/\partial y^k$ in j und k symmetrisch ist. Ein Projektivbündel $P_F(M)$ heißt projizierbar, wenn in $P_F(M)$ mindestens ein projizierbarer Rundscher Affinzusammenhang existiert.

SATZ 5. *Ein Projektivbündel $P_F(M)$ ist genau dann projizierbar, wenn für $P_F(M)$ die Gleichung*

$$D^i_{jk} - D^i_{kj} + \frac{1}{n}D^s_{ks}\delta^i_j - \frac{1}{n}D^s_{js}\delta^i_k + \frac{1}{n}(D_r{}^s{}_{ksj} - D_r{}^s{}_{jsk})y^i = 0 \tag{3.4}$$

mit $D_r{}^s{}_{jkl} \overset{\text{def}}{=} \partial D_r{}^s{}_{jk}/\partial y^l$ *besteht.*

BEWEIS. Es sei Γ ein Rundscher Affinzusammenhang (mit den Zusammenhangsobjekten Γ^i_{jk}) in $P_F(M)$. Aus (2.14) und aus (3.3) folgt die Gleichung:

$$P^i_{jk}-P^i_{kj}=D^i_{jk}-D^i_{kj}+\frac{1}{n+1}(P^s_{sk}\delta^i_j-P^s_{sj}\delta^i_k)+\frac{1}{n+1}(P_s{}^s{}_{jk}-P_s{}^s{}_{kj})\,y^i. \tag{3.5}$$

Ist Γ projizierbar, so erhalten wir mit der Kontraktion $i\to j$ die folgenden Gleichungen:

$$D^s_{ks}=\frac{n}{n+1}P^s_{sk}, \tag{3.6}$$

$$y^r(D_r{}^s{}_{ksj}-D_r{}^s{}_{jsk})=\frac{n}{n+1}(P_s{}^s{}_{jk}-P_s{}^s{}_{kj}). \tag{3.7}$$

Substituiert man (3.6) und (3.7) in (3.5), erhält man (3.4). Umgekehrt sei vorausgesetzt, daß für $P_F(M)$ die Gleichung (3.4) gilt. Komponiert man (3.4) mit y^j, so erhält man mit der Kontraktion $i\to k$ $y^jD^s_{js}=0$, und aus dieser Relation folgt:

$$\frac{\partial y^jD_j{}^s{}_s}{\partial y^l}=2y^jD_j{}^s{}_{ls}+y^r y^j D_r{}^s{}_{jsl}=0, \quad \text{so} \quad y^r y^j D_r{}^s{}_{jsl}=-2D^s_{ls}. \tag{3.8}$$

Es bezeichne (U, x^i) eine Karte auf M und $(t^{-1}(U), x^i, y^i)$ die induzierte Bündelkarte auf V. Betrachten wir auf der Menge $t^{-1}(U)$ den lokalen Rundschen Affinzusammenhang Γ mit den Zusammenhangsobjekten

$$\Gamma^i_{jk}=\Pi^i_{jk}-\frac{1}{n}D_j{}^s{}_s\delta^i_k-\frac{1}{n}D^s_{ks}\delta^i_j, \tag{3.9}$$

die offensichtlich in $P_F(M)$ ist, und beweisen wir noch, daß auch projizierbar ist. Aus (3.9) und (3.8) folgt:

$$P^i_{kl}=D^i_{kl}+\frac{1}{n}D^s_{ls}\delta^i_k-\frac{1}{n}(D_l{}^s{}_{ks}+y^rD_r{}^s{}_{ksl})\,y^i,$$

damit ist wegen (3.4) der Torsionstensor P^i_{kl} symmetrisch.

Es bezeichne jetzt $(U_\alpha, \varphi_\alpha)$, $\alpha\in I$ eine beliebige Zerlegung der Einheit auf M, wobei die Umgebungen U_α Koordinatenumgebungen sind. In diesem Fall ist $(l_F^{-1}\circ t^{-1}(U_\alpha), \varphi_\alpha\circ t\circ l_F)$ eine Zerlegung der Einheit auf $F(M)$ die wir mit $(\overline{U}_\alpha, \overline{\varphi}_\alpha)$ bezeichnen. Es stehe $\omega_{(\alpha)}{}^i_j$ für die Zusammenhangsform des Rundschen Affinzusammenhangs definiert durch (3.9) auf $t^{-1}(u_\alpha)$. Damit definiert die Form $\sum\limits_\alpha \varphi_\alpha\omega_{(\alpha)}{}^i_j$ einen globalen projizierbaren Rundschen Affinzusammenhang in $P_F(M)$. W. z. b. w.

Durch eine elementare Rechnung bekommen wir den

SATZ 6. *Ist Γ^i_{jk} ein projizierbarer Rundscher Affinzusammenhang in $P_F(M)$, so ist der Rundsche Affinzusammenhang $\overline{\Gamma}^i_{jk}$ (die auch in $P_F(M)$ ist) genau dann projizierbar, wenn die Relation*

$$\overline{\Gamma}^i_{jk}=\Gamma^i_{jk}+p_j\delta^i_j+p_k\delta^i_j$$

mit $p_j=\partial p/\partial y^j$ gilt, wobei p eine globale reellwertige Funktion (positiv homogen von 1-ter Ordnung) auf V ist.

Wir charakterisieren jetzt die Projektivbündel vom Berwald Typ mit den folgenden

SATZ 7. 1. *Ein Berwaldsches Projektivbündel $P_B(M)$ ist ein projizierbares Projektivbündel, und seine projizierbaren Rundschen Affinzusammenhänge (die in $P_B(M)$ sind) sind der Form*

$$G^i_{jk} - \frac{1}{n+1} G_s{}^s{}_{jk} y^i \quad (mit\ G_j{}^i{}_{kl} = \partial G^i_{jk}/\partial y^l),$$

wobei die Funktionen $G^i_{jk}(x, y)$ die Zusammenhangsobjekte eines Berwaldschen Affinzusammenhangs sind.

2. *Ist der Affinzusammenhang $G^i_{jk} - \frac{1}{n+1} G_s{}^s{}_{jk} y^i$ in $P_B(M)$, so sind*

$$\Pi^i_{jk} = G^i_{jk} - \frac{1}{n+1} (G_s{}^s{}_j \delta^i_k + G^s_{sk} \delta^i_j + G_s{}^s{}_{jk} y^i), \tag{3.10}$$

$$D_l{}^i{}_{jk} = G_l{}^i{}_{jk} - \frac{1}{n+1} (G_s{}^s{}_{jl} \delta^i_k + G_s{}^s{}_{kl} \delta^i_j + G_s{}^s{}_{jk} \delta^i_l + G_s{}^s{}_{jkl} y^i). \tag{3.11}$$

3. *Für einen Rundschen Projektivzusammenhang $\{P_F(M), T^\Omega\}$ ist der Tensor $W_j{}^i{}_{kl}$ im Satz 4 genau dann eindeutig bestimmt, wenn $\{P_F(M), T^\Omega\}$ vom Berwaldschen Typ ist. In diesem Fall gelten die folgenden Formeln:*

$$W_j{}^i{}_{kl} = K_j{}^i{}_{kl} - K_{jl} \delta^i_k + K_{jk} \delta^i_l + (K_{lk} - K_{kl}) \delta^i_j - \tag{3.12}$$

$$- \frac{\partial (K_{kl} - K_{lk})}{\partial y^j} y^i + \frac{1}{n^2-1} y^s \frac{\partial K_j{}^r{}_{rs}}{\partial y^k} \delta^i_l - \frac{1}{n^2-1} y^s \frac{\partial K_j{}^r{}_{rs}}{\partial y^l} \delta^i_k,$$

mit

$$K_{jk} = \frac{1}{n^2-1} (n K_j{}^s{}_{sk} + K_k{}^s{}_{sj}),$$

wobei $K_j{}^i{}_{kl}$ der Hauptkrümmungstensor des Berwaldschen Zusammenhangs G^i_{jk} ist. Aus (3.12) *folgt auch daß $W_j{}^i{}_{kl}$ genau der von Berwald bestimmte Weylsche Krümmungstensor ist* [1].

BEWEIS. Für ein Berwaldsches Projektivbündel $P_B(M)$ verschwindet jedes Glied in der Gleichung (3.4), und daher ist $P_B(M)$ projizierbar. Es sei Γ^i_{jk} ein projizierbarer Affinzusammenhang in $P_B(M)$, und man betrachte die Berwaldschen Zusammenhangsobjekte $G^i_{jk} = \Gamma^i_{jk} + P^i_{jk}$ mit $P^i_{jk} = y^l \partial \Gamma^i_{lj}/\partial y^k$. Aus (3.6) folgt die Gleichung $P^s_{sk} = 0$, so $G^s_{sk} = \Gamma^s_{sk}$, damit erhalten wir aus (3.3):

$$D_j{}^i{}_{kl} = P_j{}^i{}_{kl} - \frac{1}{n+1} (G_s{}^s{}_{jl} \delta^i_k + G_s{}^s{}_{kl} \delta^i_j).$$

Komponiert man diese Gleichung mit y^j, so ergibt sich

$$P^i_{kl} = \frac{1}{n+1} G_s{}^s{}_{kl} y^i,$$

womit Punkt 1 beweisen ist. Die Formeln (3.10), (3.11), (3.13)[5] berechnet man unmittelbar aus (2.14), (3.3) und (3.2).

Wir beweisen noch den Punkt 3. Ist $P_B(M)$ von Berwaldschen Typ, so folgt aus (3.11), daß der Douglassche Tensor $D_j{}^i{}_{kl}$ in den unteren Indizes totalsymmetrisch ist, und damit ist $W_j{}^i{}_{kl}$ im Satz 4 eindeutig bestimmt. Umgekehrt, gilt für $P_F(M)$ die Gleichung $D_j{}^i{}_{kl}=D_j{}^i{}_{lk}$ so $D^i_{kl}=y^j D_j{}^i{}_{kl}=y^j D_k{}^i{}_{jl}=y^j D_k{}^i{}_{lj}=0$, somit ist $P_F(M)$ von Berwaldschen Typ. W. z. b. w.

Jetzt Projizieren wir jedes projizierbare Projektivbündel $P_F(M)$ auf ein Berwaldsches Projektivbündel $P_B(M)$. Es sei Γ^i_{jk} ein projizierbarer Rundscher Affinzusammenhang in $P_F(M)$, und definieren wir den Rundschen Affinzusammenhang $B\Gamma^i_{jk}$ durch

$$B\Gamma^i_{jk} = \Gamma^i_{jk} + P^i_{jk} + \frac{1}{n+1}\frac{\partial(P^s_{sj}+\Gamma^s_{sj})}{\partial y^k} = G^i_{jk} + \frac{1}{n+1} G_s{}^s{}_{jk} y^i, \tag{3.13}$$

wobei $G^i_{jk}=\Gamma^i_{jk}+P^i_{jk}$, $(P^i_{jk}=y^l\partial\Gamma^i_{lj}/\partial y^k)$, die Zusammenhangsobjekte eines Berwaldschen Affinzusammenhangs sind. Damit ist $B\Gamma^i_{jk}$ in einen Berwaldschen Projektivzusammenhang. Es sei $\bar\Gamma^i_{jk}=\Gamma^i_{jk}+p_j\delta^i_k+p_k\delta^i_j$ (mit $p_j=\partial p/\partial y^j$) ein anderer projizierbarer Affinzusammenhang in $P_F(M)$; dann rechnet man leicht die Folgenden aus:

$$B(\Gamma^i_{jk}+p_j\delta^i_k+p_k\delta^i_j) = B\Gamma^i_{jk}+p_j\delta^i_k+p_k\delta^i_j, \tag{3.14}$$

und damit sind $B\Gamma^i_{jk}$ und $B\bar\Gamma^i_{jk}$ in demselben Berwaldschen Projektivbündel, das wir mit $\beta(P_F(M))$ bezeichnen. Wir beschreiben jetzt den natürlichen Bündelisomorphismus

$$\beta\colon\ P_F(M) \to \beta(P_F(M)) \tag{3.15}$$

wie folgt. Zuerst wählt man einen festen projizierbaren Affinzusammenhang Γ^i_{jk} aus $P_F(M)$, und dann ergänzt man die Zuordnung

$$(x^i, y^i, \delta^i_j, -\Gamma^i_{jk}(x, y)) \to (x^i, y^i, \delta^i_j, -B\Gamma^i_{jk}(x, y))$$

zum Bündelisomorphismus β: $P_F(M)\to\beta(P_F(M))$.

Aus (3.14) folgt, daß der Isomorphismus β von der Wahl des projizierbaren Affinzusammenhangs Γ^i_{jk} aus $P_F(M)$ unabhängig ist.

Es bezeichne Π^i_{jk} bzw. D^i_{jk} die projektiven Zusammenhangsobjekte bzw. den Douglasschen Torsionstensor des Bündels $P_F(M)$, und es stehe $\overset{(B)}{\Pi}{}^i_{jk}$ für die projektiven Zusammenhangsobjekte des Bündels $\beta(P_F(M))$. Wir erhalten die folgenden Relationen mittels einer elementaren Rechnung:

$$\overset{(B)}{\Pi}{}^i_{jk} = \frac{1}{2}\partial^2(y^q y^r \Pi^i_{qr})/\partial y^j\partial y^k = \Pi^i_{jk}+D^i_{jk}+\frac{1}{2}\partial y^r D^i_{rj}/\partial y^k. \tag{3.16}$$

Ist $\{P_F(M), C\}$ ein Cartanscher Projektivzusammenhang, wobei $P_F(M)$ projizierbar ist, so folgt aus (3.16) daß dieser Projektivzusammenhang durch die Objekte $\overset{(B)}{\Pi}{}^i_{jk}$, D^i_{jk}, C^i_{jk} eindeutig bestimmt ist.

[5] Bei der Rechnung der Formel (3.12) (aus (3.2)) wendet man an, daß der Krümmungstenso $R_j{}^i{}_{kl}$ des Rundschen Affinzusammenhängs $G^i_{jk}-\dfrac{1}{n+1}G_s{}^s{}_{jk}y^i$ die Form $R_j{}^i{}_{kl}=K_j{}^i{}_{kl}-\partial(K_{kl}-K_{lk})/ /\partial y^j y^i$ besitzt.

4. § Bahnen der Projektivzusammenhänge

Ist $\{P_F(M), C\}$ ein Projektivzusammenhang mit den Zusammenhangsformen $\omega^i, \Omega^i_j, \Omega_j, \overset{(v)}{\Omega^i}$ so läßt sich für jeden Vektor $\xi \in \mathbf{R}^n$ und für jeden Punkt $p \in P_F(M)$ ein Vektor $\xi^{(h)}/p \in T_p(P_F(M))$ durch $\Omega^i_j(\xi^{(h)}_{/p}) = \Omega_j(\xi^{(h)}_{/p}) = \overset{(v)}{\Omega^i}(\xi^{(h)}_{/p}) = 0$, $\omega^i(\xi^{(h)}_{/p}) = \xi^i$ zuordnen. Das Vektorfeld $\overline{Y}^{(h)}: p \to \overline{Y}^{(h)}_{/p}$ auf $P_F(M)$, (wobei die $\mathbf{R}^n$-wertige Funktion $\overline{Y} = \bar{y}^i e_i$ auf $P_F(M)$ in (1.2) definiert ist), heißt das fundamentale Vektorfeld bezüglich $\{P_F(M), C\}$.

DEFINITION. Eine Kurve $x(t)$ auf M heißt eine Bahn des Projektivzusammenhangs $\{P_F(M), C\}$, wenn die Kurve $(x(t), x'(t))$ auf V die Projektion einer Integralkurve des fundamentalen Vektorfeldes $\overline{Y}^{(h)}$ ist.

Es gilt der

SATZ 8. 1. *Induziert der Cartansche Affinzusammenhang $\{\Gamma^i_{jk}, C^i_{jk}\}$ den Projektivzusammenhang $\{P_F(M), C\}$ so stimmen (ungeachtet der Parametrisation) die Bahnen des Affinzusammenhangs $\{\Gamma^i_{jk}, C^i_{jk}\}$ und des Projektivzusammenhangs $\{P_F(M), C\}$ überein.*

2. *Die projizierbaren Projektivzusammenhänge $\{P_F(M), C\}$ bzw. $\{\overline{P}_F(M), \overline{C}\}$ bestimmen genau dann das gleiche Bahnsystem, wenn die Berwaldschen Projektivzusammenhänge $\beta(P_F(M))$ bzw. $\beta(\overline{P}_F(M))$ übereinstimmen.*

BEWEIS. Sind $\theta^i, \omega^i_j, \overset{(v)}{\omega^i} = d\bar{y}^i + \bar{y}^j \omega^i_j$ die Zusammenhangsformen des Cartanschen Zusammenhangs $\{\Gamma^i_{jk}, C^i_{kj}\}$ auf $F(M)$, so heißt das Vektorfeld Z auf $F(M)$, definiert durch $\theta^i(Z) = \bar{y}^i$, $\omega^i_j(Z) = \overset{(v)}{\omega^i}(Z) = 0$, das fundamentale Vektorfeld bezüglich $\{\Gamma^i_{jk}, C^i_{jk}\}$. Eine Kurve $x(t)$ auf M heißt eine Bahn des Zusammenhangs $\{\Gamma^i_{jk}, C^i_{jk}\}$, wenn die Kurve $(x(t), x'(t))$ auf V die Projektion einer Integralkurve des Vektorfeldes Z ist. In einer Karte (U, x^i) ist eine Kurve $x^i(t)$ genau dann eine Bahn, wenn für sie die Differentialgleichungen:

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}(x(t), x'(t)) \frac{dx^j}{dt} \frac{dx^k}{dt} = 0$$

bestehen. Mit dieser Bemerkung wird der Beweis von Punkt 1 ganz ähnlich, wie in [9] S. 230.

Der Beweis von Punkt 2 folgt offensichlich aus Punkt 1 und aus der folgenden Bemerkung. Ist der Cartansche Zusammenhang $\{\Gamma^i_{jk}, C^i_{jk}\}$ mit dem projizierbaren Rundschen Affinzusammenhang Γ^i_{jk} in $\{P_F(M), C\}$, so stimmen die Bahnen des Cartanschen Zusammenhangs $\{\Gamma^i_{jk}, C^i_{jk}\}$ und der Rundschen Zusammenhänge Γ^i_{jk} bzw. $B\Gamma^i_{jk}$ überein. W. z. b. w.

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FINSLERSCHE PROJEKTIVGEOMETRIE. II

ÜBER FINSLERSCHE PROJEKTIVBÜNDEL MIT WEYLSCHER PROJEKTIVKRÜMMUNG NULL

Von

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Einleitung

Ein allgemeiner affiner Raum läßt sich genau dann durch bahntreue Abbildung in allgemeine affine Räume mit der Krümmung Null abbilden, wenn der Weylsche Krümmungstensor des Raumes verschwindet. Das hat L. BERWALD [2] bewiesen, aber der Beweis dieses Satzes ist unserer Meinung nach nicht befriedigend. Um dieses Problem von näher zubeleichten, werden wir diesen Beweis kurz zurückrufen.

Die allgemeinen affinen Räume, definiert durch die Berwaldschen Übertragungsparameter G^i_{jk} bzw. $\bar{G}^i_{jk}$, haben genau dann dasselbe Bahnsystem (ungeachtet der Parametrisation), wenn zwischen ihnen die Relation $\bar{G}^i = G^i + py^i$ besteht, wobei $\bar{G}^i = \frac{1}{2} y^k y^j \bar{G}^i_{jk}$, $G^i = \frac{1}{2} y^k y^j G^i_{jk}$ ist. Bildet man einen affinen Raum durch bahntreue Abbildung in einen allgemeinen affinen Raum mit Krümmung Null ab, so muß man die Differentialgleichung

$$(1) \qquad E_k(p) - p\frac{\partial p}{\partial y^k} = -\frac{1}{n^2-1}(nH_k + y^m H_{km})$$

bezüglich der unbekannten Funktion p auflösen, wobei $E_k = \frac{\partial}{\partial x^k} - G^r_k \frac{\partial}{\partial y^r}$, $G^r_k = y^j G^r_{jk}$ besteht [2].

Bei der Prüfung der Integrierbarkeit der Gleichung (1) hat L. Berwald nur bewiesen, daß die Vertauschungsformeln

$$\frac{\partial^2 p}{\partial x^i \partial x^j} = \frac{\partial^2 p}{\partial x^j \partial x^i}$$

bezüglich eines allgemeinen affinen Raumes mit $W_i{}^l{}_{jk} = 0$ automatisch erfüllt sind.

Aus der Theorie von T. Y. Thomas und O. Veblen folgt aber nicht offensichtlich, daß zur Auflösung der Gleichung (1) die Prüfung nur dieser einzigen Vertauschungsformel ausreichend wäre, da die Funktion p auch von der Veränderlichen y^i abhängt und die Gleichung (1) auch die Glieder der Form $\frac{\partial p}{\partial y^j}$ enthält.

Im ersten Teil dieser Arbeit geben wir einen genauen Beweis dieses schönen Satzes von L. Berwald. Aus dem folgenden Beweis folgt leicht der

SATZ. *Hat der allgemeine affine Raum einen verschwindenen Weylschen Krümmungstensor, so gibt es für jeden festen Punkt $\overset{(0)}{x}$ und für jede feste, in y differenzierbare Funktion $\bar{p}(\overset{(0)}{x}, y)$ (die in y positiv homogen von 1-ter Ordnung ist) genau* *eine Lösung $p(x, y)$ der Gleichung* (1) *derart, daß $p(x, y)$ in den Linienelementen $(\overset{(0)}{x}, y)$ den Anfangswert $\bar{p}(\overset{(0)}{x}, y)$ hat.*

Im zweiten Teil werden wir die Finslerschen Räume von skalarer bzw. konstanter Krümmung (im Sinne von L. Berwald bzw. von Akbar Zadeh) mit Hilfe des Weylschen Krümmungstensors charakterisieren.

Diese Arbeit ist eine direkte Fortsetzung unserer Arbeit [7], so sind die dortigen Bezeichnungen, Definitionen und Sätze als bekannt vorausgesetzt.

1. § Der Beweis des Berwaldschen Satzes

Es sei $P_B(M)$ ein Berwaldsches Projektivbündel mit den natürlichen projektiven Zusammenhangsformen (oder Normalformen) $(\omega^i, \omega^i_j, \omega_j, \overset{(v)}{\omega}{}^i)$. Sie bestimmen einen absoluten Parallelismus auf $P_B(M)$, und ihre Strukturgleichungen lauten:

$$\text{(1.1)} \quad \begin{cases} d\omega^i = -\sum \omega^i_k \wedge \omega^k, \\ d\omega^i_j = -\sum \omega^i_k \wedge \omega^k_j - \omega^i \wedge \omega_j + \delta^i_j \sum \omega_r \wedge \omega^r - \Lambda^i_j, \\ d\omega_j = -\sum \omega_k \wedge \omega^k_j - \Lambda_j, \\ d\overset{(v)}{\omega}{}^i = -\sum \omega^i_k \wedge \overset{(v)}{\omega}{}^k - \sum \bar{y}^j \omega^i \wedge \omega_j + \bar{y}^i \sum \omega_j \wedge \omega^j - \sum \bar{y}^j \Lambda^i_j, \end{cases}$$

mit

$$\text{(1.2)} \quad \begin{cases} \Lambda^i_j = \dfrac{1}{2} W^{*i}_{j\ kl}\, \omega^k \wedge \omega^l + D^{*i}_{j\ kl} \omega^k \wedge \overset{(v)}{\omega}{}^l, \\ \Lambda_j = \dfrac{1}{2} W^*_{jkl} \omega^k \wedge \omega^l + D^*_{jkl} \omega^k \wedge \overset{(v)}{\omega}{}^l, \\ \bar{y}^j \Lambda^i_j = \dfrac{1}{2} W^{*i}_{kl} \omega^k \wedge \omega^l, \end{cases}$$

und damit

$$\text{(1.3)} \quad W^{*i}_{kl} = \bar{y}^j W^{*i}_{j\ kl}, \quad \bar{y}^j D^{*\ i}_{j\ kl} = 0.$$

Die Krümmungsfunktionen $W^{*i}_{j\ kl}$, $D^{*i}_{j\ kl}$ sind auf $P_B(M)$ definiert, und aus diesen läßt sich der Weylsche bzw. der Douglassche Krümmungstensor herleiten. Wir haben ja früher gesehen, daß die Funktionen $W^{*i}_{j\ kl}$ bzw. $D^{*i}_{j\ kl}$ in einer Bündelkarte $(x^i, y^i, u^i_j, -\Gamma^i_{jk})$ auf $P_B(M)$ die Form

$$\text{(1.3')} \quad \begin{cases} W^{*i}_{j\ kl} = u^q_j u^r_k u^s_l v^i_m W_q{}^m{}_{rs}(x, y), \\ D^{*\ i}_{j\ kl} = u^q_j u^r_k u^s_l v^i_m D_q{}^m{}_{rs}(x, y) \end{cases}$$

haben, damit sind die Funktionen $W_j{}^i{}_{kl}$ bzw. $D_j{}^i{}_{kl}$ die klassischen Komponenten des Weylschen bzw. Douglasschen Tensors. Früher haben wir auch die folgende Charackterisierung gegeben. Die Berwaldschen Projektivzusammenhänge sind genau die projizierbaren Rundschen Projektivzusammenhänge, deren projizierbare Rundsche Affinzusammenhänge die Übertragungsparameter der Form

$$\text{(1.4)} \quad \Gamma^i_{jk} = G^i_{jk} - \frac{1}{n+1} G_s{}^s{}_{jk} y^i$$

haben, wobei die Funktionen $G^i_{jk}(x, y)$ die Übertragungsparameter eines Berwaldschen Affinzusammenhangs sind, dh. wenn

$$G^i \stackrel{\text{def}}{=} \frac{1}{2} y^j y^k G^i_{jk}, \quad \text{so} \quad G^i_j = \partial G^i / \partial y^j, \quad G^i_{jk} = \partial G^i_j / \partial y^k \quad \text{u.s.w.}$$

Es sei $H_j{}^i{}_{kl}$ der Krümmungstensor des Berwaldschen Affinzusammenhangs G, definiert durch die Übertragungsparameter G^i_{jk}, damit haben wir auch ausgerechnet:

$$\text{(1.5)} \qquad W_j{}^i{}_{kl} = H_j{}^i{}_{kl} + (H_{kl} - H_{lk})\delta^i_j + \frac{1}{n^2-1}\left(nH_{jl} + H_{lj} + y^m \frac{\partial H_{lm}}{\partial y^j}\right)\delta^i_k -$$

$$- \frac{1}{n^2-1}\left(nH_{jk} + H_{kj} + y^m \frac{\partial H_{km}}{\partial y^j}\right)\delta^i_l + \frac{1}{n+1}\frac{\partial(H_{kl} - H_{lk})}{\partial y^j} y^i,$$

$$\text{(1.6)} \qquad D_j{}^i{}_{kl} = G_j{}^i{}_{kl} - \frac{1}{n+1}\left(G_s{}^s{}_{jk}\delta^i_l + G_s{}^s{}_{kl}\delta^i_j + G_s{}^s{}_{kl}\delta^i_k + G_s{}^s{}_{jkl} y^i\right),$$

wobei $H^i_{jk} = y^r H_r{}^i{}_{jk}$, $H_j = H^r_{jr}$, $H_{hj} = H_h{}^r{}_{jr}$.

Der Operator d wirkt auf den Strukturgleichungen, und damit bekommen wir die Bianchischen Identitäten (1,7), (1.8) und (1.9).

$$\text{(1.7)} \qquad \Lambda^i_k \wedge \omega^k = 0,$$

woraus

$$W_j{}^i{}_{kl} + W_k{}^i{}_{lj} + W_l{}^i{}_{jk} = 0, \quad D_j{}^i{}_{kl} = D_k{}^i{}_{jl}.$$

Aus der Formel (1.6) folgt offensichtlich, daß der Douglassche Tensor in den unteren Indizes totalsymmetrisch ist, weiterhin folgt aus (1.5):

$$W^i_{jk} = y^r W_r{}^i{}_{jk}, \quad W^i_k \stackrel{\text{def}}{=} y^r W_r{}^i{}_k, \quad W_j{}^i{}_{kl} = \frac{\partial W^i_{kl}}{\partial y^j}, \quad W^i_{jk} = \frac{\partial W^i_k}{\partial y^j} - \frac{\partial W^i_j}{\partial y^k}.$$

Die Eigenschaften $W_i{}^i{}_{jk} = W_j{}^i{}_{ik} = W_j{}^i{}_{ki} = D_i{}^i{}_{jk} = 0$ sind auch offensichtlich.

$$\text{(1.8)} \qquad d\Lambda^i_j + \omega^i \wedge \Lambda_j + \Lambda^k_j \wedge \omega^i_k - \Lambda^i_k \wedge \omega^k_j = 0.$$

Mit der Kontraktion $i \to j$ bekommen wir $\omega^j \wedge \Lambda_j = 0$, und daraus folgt

$$W^*_{ijk} + W^*_{jki} + W^*_{kij} = 0, \quad D^*_{ijk} = D^*_{jik}.$$

$$\text{(1.9)} \qquad d\Lambda_j - \Lambda_k \wedge \omega^k_j + \Lambda^k_j \wedge \omega_k = 0.$$

LEMMA 1. *Verschwindet der Tensor $W_j{}^i{}_{kl}$ (bzw. $D_j{}^i{}_{kl}$) des Bündels $P_B(M)$, so verschwinden auch die Funktionen W^*_{ijk} (bzw. D^*_{ijk}).*

BEWEIS. Es bestehe bezüglich $P_B(M)$ die Gleichung $W_j{}^i{}_{kl} = 0$ dann bekommen wir aus (1.8)

$$(dD^{*i}_j{}_{kl}) \wedge \omega^k \wedge \overset{(v)}{\omega}{}^l + D^*{}_j{}^i{}_{kl}\, d\omega^k \wedge \overset{(v)}{\omega}{}^l - D^{*i}_j{}_{kl}\, \omega^k \wedge d\overset{(v)}{\omega}{}^l +$$

$$+ \frac{1}{2} W^*_{jkl}\omega^i \wedge \omega^k \wedge \omega^l + D^*_{jkl}\omega^i \wedge \omega^k \wedge \overset{(v)}{\omega}{}^l + D^*{}_j{}^k{}_{lm}\omega^l \wedge \overset{(v)}{\omega}{}^m \wedge \omega^i_k - D^*{}_k{}^i{}_{lm}\omega^l \wedge \overset{(v)}{\omega}{}^m \wedge \omega^k_j = 0.$$

Substituiert man anstelle von $d\omega^i$ bzw. $d\overset{(v)}{\omega}{}^i$ die Formeln (1.1), so wird das einzige Glied, in dem die Formen $\omega^i\wedge\omega^k\wedge\omega^l$ vorkommen, $\frac{1}{2}W^*_{jkl}\omega^i\wedge\omega^k\wedge\omega^l$ sein. Damit gilt auch die Gleichung $W^*_{jkl}=0$. Ganz ähnlich läßt sich auch die andere Behaptung bewiesen. W. z. b. w.

Jedem Vektor $\zeta=(\zeta^1,\zeta^2,\dots,\zeta^n)\in\mathbf{R}^n$ läßt sich ein sogenanntes horizontales Vektorfeld $\overset{(h)}{\zeta}$ und ein quasivertikales Vektorfeld $\overset{(v)}{\zeta}$ auf $P_B(M)$ zuordnen:

$$\omega^i(\overset{(h)}{\xi})=\xi^i,\quad \omega^i_j(\overset{(h)}{\xi})=\omega_j(\overset{(h)}{\xi})=\overset{(v)}{\omega}{}^i(\overset{(h)}{\xi})=0, \tag{1.10}$$

$$\overset{(v)}{\omega}{}^i(\overset{(v)}{\xi})=\xi^i,\quad \omega^i(\overset{(v)}{\xi})=\omega^i_j(\overset{(v)}{\xi})=\omega_j(\overset{(v)}{\xi})=0. \tag{1.11}$$

Sind noch $(e^{*1},e^{*2},\dots,e^{*n})$ die natürliche Basis im Dualraum $\mathbf{R}^{n*}$ und (e^i_j), $i,j=1,\dots,n$ die natürliche Basis in $\mathrm{gl}(n,\mathbf{R})$, und definieren wir die fundamentalen Vektorfelder $\overset{(v)}{e}{}^{*i}$, $\overset{(v)}{e}{}^i_j$ auf $P_B(M)$ durch

$$\omega_j(\overset{(v)}{e}{}^{*i})=\delta^i_j,\quad \omega^j(\overset{(v)}{e}{}^{*l})=\omega^i_j(\overset{(v)}{e}{}^{*l})=\overset{(v)}{\omega}{}^i(\overset{(v)}{e}{}^{*l})=0, \tag{1.12}$$

$$\omega^i_j(\overset{(v)}{e}{}^l_m)=\delta^i_m\delta^l_j,\quad \omega^i(\overset{(v)}{e}{}^l_m)=\omega_j(\overset{(v)}{e}{}^l_m)=\overset{(v)}{\omega}{}^i(\overset{(v)}{e}{}^l_m)=0, \tag{1.13}$$

so bestimmen die Vektorfelder $(\overset{(h)}{e}_i,\overset{(v)}{e}_i,\overset{(v)}{e}{}^i_j,\overset{(v)}{e}{}^{*i})$, $i,j=1,\dots,n$, auf $P_B(M)$ einen absoluten Parallelismus, wobei $(e_1,\dots,e_n)$ die natürliche Basis in $\mathbf{R}^n$ ist. Aus den Strukturgleichungen erhalten wir

$$[\overset{(h)}{e}_i,\overset{(h)}{e}_j]=-W_l^{*k}{}_{ij}\overset{(v)}{e}{}^l_k-W^*_{kij}\overset{(v)}{e}{}^{*k}-W^{*k}_{ij}\overset{(v)}{e}_k, \tag{1.14}$$

und so bekommen wir aus Lemma 1 den

Satz 1. *Die horizontalen Vektorfelder* $\overset{(h)}{\zeta}$ $(\zeta\in\mathbf{R}^n)$, *bilden eine Liesche Algebra (und daher eine Abelsche—Liesche Algebra) genau dann, wenn der Weylsche Tensor* $W_j{}^i{}_{kl}$ *des Bündels* $P_B(M)$ *verschwindet.*

Es bezeichne $T^{(h)}(P_B(M))$ das Feld der n-dimensionalen Unterräume auf $P_B(M)$, das durch die Vektorfelder der Form $\overset{(h)}{\zeta}$, $\zeta\in\mathbf{R}^n$ aufgespannt wird. So haben wir

$$T^{(h)}(P_B(M))=\{X\in T(P_B(M))\,|\,\omega^i_j(X)=\omega_j(X)=\overset{(v)}{\omega}{}^i(X)=0\}.$$

Aus dem vorigen Satz und aus dem Satz von Frobenius folgt leicht der

Satz 2. *Die n-dimensionale horizontale Distribution* $T^{(h)}(P_B(M))$ *auf* $P_B(M)$ *ist genau dann vollständig integrabel, wenn die Weylsche Tensor des Bündels* $P_B(M)$ *verschwindet.*

Ist U eine offene Menge der Grundmannigfaltigkeit M, so bezeichne $P_B(U)$ die Beschränkung eines Berwaldschen Projektivbündels $P_B(M)$ auf U. Der Basisraum des Faserraumes $P_B(U)$ ist $t^{-1}(U)$, wobei die Abbildung $t\colon V\to M$ die Projektion des Bündels der nichtverschwindenden Tangentenvektoren (der M) auf M ist.

DEFINITION. Ein Berwaldsches Projektivbündel $P_B(M)$ heißt 0-projektiv, wenn für jeden Punkt $q \in M$ eine Umgebung U um p, und in U mindestens ein projizierbarer Rundscher Affinzusammenhang Γ, mit den Übertragungsparameteren

$$\Gamma^i_{jk} = G^i_{jk} - \frac{1}{n+1} G_s{}^s{}_{jk} y^i \tag{1.15}$$

existiert, derart daß der Krümmungstensor $H_j{}^i{}_{kl}$ des Berwaldschen Affinzusammenhangs G (definiert durch die Übertragungsparameter G^i_{jk}) verschwindet.

SATZ 3 (von Berwald). *Ein Berwaldsches Projektivbündel $P_B(M)$ ist gerade dann 0-projektiv, wenn der Weylsche Tensor des Bündels $P_B(M)$ verschwindet.*

BEWEIS. Ist $P_B(M)$ 0-projektiv, so verschwindet der Tensor $W_j{}^i{}_{kl}$. Diese Tatsache folgt offensichtlich aus der obigen Definition und aus der Formel (1.5).

Umgekehrt nehmen wir an, daß für $P_B(M)$ der Tensor $W_j{}^i{}_{kl}$ verschwindet. Wir beweisen zuerst, daß für jeden Punkt $q \in M$ eine Umgebung U um q, und in $P_B(U)$ ein Rundscher Affinzusammenhang Γ existiert, für den der Krümmungstensor $R_j{}^i{}_{kl}$ verschwindet.

Wegen des Satzes 2 ist $T^{(h)}(P_B(M))$ vollständig integrierbar. Man betrachte für einen beliebigen, aber festen Punkt $q \in M$ den Raum V_q der nichtverschwindenden Tangentenvektoren auf q, und auf V_q einen beliebigen (in y) differenzierbaren Schnitt σ_q: $V_q \to P_B(M)$, $\sigma_q(v) \in P_B(M)/v$, positiv homogen von 0-ter Ordnung. Man betrachte auch für jeden Vektor $v \in V_q$ die maximale Integralmannigfaltigkeit $I_{\sigma(v)}$ der Distribution $T^{(h)}(P_B(M))$, die durch $\sigma_q(v)$ geht. Wegen der Homogenität und der vollständigen Integrierbarkeit gelten die folgenden Behauptungen selbstverständlich:

1. Ist $v \neq w$, so ist $I_{\sigma(v)} \cap I_{\sigma(w)} = 0$.
2. Es existiert eine Umgebung U um q, derart daß jede Faser $P_B(U)_{/w}$, $w \in t^{-1}(U)$ genau eine Integralmannigfaltigkeit der Form $I_{\sigma(v)}$, $v \in V_q$, genau in einem (mit $\sigma(w)$ bezeichneten) Punkt schneidet.

Dadurch ist σ: $w \to \sigma(w)$ eine differenzierbare Abbildung:

$$\sigma: t^{-1}(U) \to P_B(U). \tag{1.16}$$

Es bezeichne Γ den eindeutig bestimmten Rundschen Affinzusammenhang in $P_B(U)$, der diese Integralmannigfaltigkeiten $I_{\sigma(v)}$, $v \in V_q$ erhält, dh. $\sigma(t^{-1}(U)) \subset \subset \gamma(F(U))$, wobei γ: $F(U) \to P_B(U)$ die natürliche Einbettung bezüglich Γ ist. Wir beweisen, daß der Krümmungstensor $R_j{}^i{}_{kl}$ des Affinzusammenhangs Γ verschwindet.

Die Normalformen des Affinzusammenhangs Γ auf $F(U)$ sind $\gamma^* \omega^i$, $\gamma^* \omega^i_j$, $\gamma^* \overset{(v)}{\omega}{}^i$. Mit der Bezeichnung $\Gamma^h \subset T(F(U))$ für das Feld der horizontalen Unterräume von Γ:

$$\Gamma^h = \{X \in T(F(U)) \,|\, \gamma^* \omega^i_j(X) = \gamma^* \overset{(v)}{\omega}{}^i(X) = 0\}, \tag{1.17}$$

folgt aus der obigen Konstruktion offensichtlich

$$\gamma_*(\Gamma^h) \subset T^{(h)}(P_B(U)). \tag{1.18}$$

Damit $\gamma^* \omega_j(X)=0$ besteht, wenn $X \in \Gamma^h$. Da aber die Formel

$$\gamma^* \omega_j = R^*{}_{jk} \gamma^* \omega^k + P^*_{jk} \gamma^* \overset{(v)}{\omega}{}^k \tag{1.19}$$

(aus [7]) gilt, wobei

$$\begin{cases} R^*_{jk} = \dfrac{1}{n^2-1} (nR^*{}_l{}^s{}_{sj} + R^*{}_j{}^s{}_{sl}), \\ R^*{}_j{}^i{}_{kl} = v^i_p u^q_j u^r_k u^s_l R_q{}^p{}_{rs}(x, y), \end{cases} \tag{1.20}$$

folgt wegen (1.18) die Identität $R^*_{jk}=0$. Aus einer anderen Formel von [7] gilt:

$$W^*{}_j{}^i{}_{kl} = R^*{}_j{}^i{}_{kl} - R^*_{jl} \delta^i_k + R^*_{jk} \delta^i_l + (R^*_{lk} - R^*_{kl}) \delta^i_j, \tag{1.21}$$

daher folgt wegen $W^*{}_j{}^i{}_{kl}=0$ auch, daß der Krümmungstensor $R_j{}^i{}_{kl}$ des Affinzusammenhangs Γ verschwindet.

Wir müssen noch beweisen, daß in $P_B(U)$ auch ein projizierbarer Rundscher Affinzusammenhang Γ, mit den Übertragungsparametern

$$\Gamma^i_{jk} = G^i_{jk} - \frac{1}{n+1} G_s{}^h{}_{jk} y^i$$

existiert, für die der Krümmungstensor $R_j{}^i{}_{kl}$ verschwindet. Damit wird der Satz vollständig bewiesen sein. In der Tat, bezeichnet $H_j{}^i{}_{kl}$ den Krümmungstensor der Berwaldschen Affinzusammenhang G, definiert durch die Übertragungsparameter G^i_{jk}, so folgt aus den wohlbekanten Relationen

$$H_j{}^i{}_{kl} = \partial y^r R_r{}^i{}_{kl} / \partial y^j, \quad H^i_{jk} = y^r R_r{}^i{}_{jk}, \tag{1.22}$$

daß auch der Tensor $H_j{}^i{}_{kl}$ verschwindet.

Zuerst beweisen wir das

LEMMA 2. *Sind Γ bzw. $\tilde{\Gamma}$ Rundsche Affinzusammenhänge in $P_B(U)$, Γ projizierbar, zwischen denen die Beziehung*

$$\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + p_j \delta^i_k + p_k \delta^i_j$$

besteht, so besteht zwischen den Krümmungstensoren $R^i_{jk} = y^r R_r{}^i{}_{jk}$ bzw. $\tilde{R}^i_{jk} = y^r \tilde{R}_r{}^i{}_j{}^k$ die Relation

$$\begin{aligned} \tilde{R}^i_{kl} = R^i_{kl} &+ \big(E_k(p) - pF_k(p)\big)\delta^i_l - \big(E_l(p) - pF_l(p)\big)\delta^i_k + \\ &+ \big(E_k(p_l)) - E_l(p_k) - pF_k(p_l) + pF_l(p_k)\big) y^i, \end{aligned} \tag{1.23}$$

wobei $E_k \overset{\text{def}}{=} \partial/\partial x^k - y^r \Gamma^l_{kr} \partial/y^l$, $F_l \overset{\text{def}}{=} \partial/\partial y^l$, $p \overset{\text{def}}{=} y^r p_r$ ist.

BEWEIS. Der Beweis des Lemmas folgt aus den wohlbekannten Formeln

$$\tilde{R}^i_{kl} = \tilde{E}_k(\tilde{G}^i_l) - \tilde{E}_l(\tilde{G}^i_k), \quad \tilde{G}^i_l \overset{\text{def}}{=} y^r \tilde{\Gamma}^i_{lr}, \quad \tilde{G}^i_l = G^i_l + p\delta^i_l + p_l y^i$$

mittels einer elementaren Rechnung. W. z. b. w.

LEMMA 3. *Ist Γ ein projizierbarer Rundscher Affinzusammenhang in $P_B(U)$, so läßt sich der Tensor R^i_{jk} des Affinzusammenhangs Γ eindeutig in der Form*

$$R^i_{jk} = A_j \delta^i_k - A_k \delta^i_j - B_{jk} y^i \tag{1.24}$$

darstellen. In dieser kanonischen Darstellung ist A_j ein Finslersches kovariantes Vektorfeld, positiv homogen von 1-ter Ordnung, ferner gilt immer die Relation

$$B_{jk} = \frac{\partial A_k}{\partial y^j} - \frac{\partial A_j}{\partial y^k}. \tag{1.25}$$

BEWEIS. Die Eindeutigkeit der Darstellung (1.24) ist offensichtlich. Zum Beweis der Existenz der Formeln (1.25), (1.24) bemerken wir, daß wir diese Behauptung bezüglich der Finslerschen Räume schon bewiesen haben [8]. Der dortige Beweis läßt sich nun aber ohne Schwierigkeit auf den obige Fall übertragen, da die Übertragungsparameter von Γ haben die Form (1.15), damit ist $R^i_{jk} = H^i_{jk}$, wobei $H^i_{jk} = y^r H_r{}^i{}_{jk}$ und $H_r{}^i{}_{jk}$ ist der Krümmungstensor des durch G^i_{jk} definierten Berwaldschen Affinzusammenhang. W. z. b. w.

Wir kehren jetzt zum Beweis des Satzes 3 zurück. Es sei $\tilde{\Gamma}$ ein Rundscher Affinzusammenhang in $P_B(U)$, deren Krümmungstensor $\tilde{R}_j{}^i{}_{kl}$ Null ist, und Γ sei ein beliebiger projizierbarer Rundscher Affinzusammenhang in $P_B(U)$. Zwischen ihnen gilt die Relation

$$\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + \bar{p}_j \delta^i_k + \bar{p}_k \delta^i_j. \tag{1.26}$$

Wir definieren jetzt den projizierbaren Rundschen Affinzusammenhang $\tilde{\tilde{\Gamma}}$ in $P_B(U)$ durch

$$\tilde{\tilde{\Gamma}}^i_{jk} = \Gamma^i_{jk} + p_j \delta^i_k + p_k \delta^i_j, \tag{1.27}$$

wobei $p_j = \partial p / \partial y^j$, $p \overset{\text{def}}{=} y^r \bar{p}_r$ sind. Wegen der Bemerkungen vor (1.22) müssen wir noch beweisen, daß der Tensor $\tilde{\tilde{R}}^i_{jk}$ des Affinzusammenhangs $\tilde{\tilde{\Gamma}}$ verschwindet. Wegen (1.23) gelten die Gleichungen:

$$\begin{aligned} 0 = R^i_{kl} &+ (E_k(p) - pF_k(p))\delta^i_l - (E_l(p) - pF_l(p))\delta^i_k + \\ &+ (E_k(\bar{p}_l) - E_l(\bar{p}_k) - pF_k(\bar{p}_l) + pF_l(\bar{p}_k))\, y^i, \end{aligned} \tag{1.28}$$

$$\begin{aligned} \tilde{\tilde{R}}^i_{kl} = R^i_{kl} &+ (E_k(p) - pF_k(p))\delta^i_l - (E_l(p) - pF_l(p))\delta^i_k + \\ &+ (E_k(p_l) - E_l(p_k))\, y^i. \end{aligned} \tag{1.29}$$

Kontrahiert man aus (1.29) die Gleichung (1.28), so ergibt sich

$$\tilde{\tilde{R}}^i_{kl} = (E_l(\bar{p}_k) - E_k(\bar{p}_l) + E_k(p_l) - E_l(p_k) + pF_k(\bar{p}_l) - pF_l(\bar{p}_k))\, y^i \overset{\text{def}}{=} -B_{kl} y^i.$$

Mit Rücksicht auf die kanonische Darstellung (1.24) des Tensors $\tilde{\tilde{R}}^i_{jk}$ sieht man, daß der Tensor A_j verschwindet, und damit verschwindet wegen (1.25) auch der Tensor B_{jk}. So besteht die Gleichung $\tilde{\tilde{R}}^i_{jk} = 0$ und damit ist der Satz vollständig bewiesen.

2. § Über Finslersche Räume mit verschwindenem Weylschen Krümmungstensor

Es sei $\mathscr{L}(x, y)$ die metrische Grundfunktion eines n-dimensionalen Finslerschen Raumes F_n. Der metrische Grundtensor $g_{ij}=\frac{1}{2}\partial^2\mathscr{L}^2/\partial y^i\partial y^j$ wird als positiv definit, und in y als positiv homogen von 0-ter Ordnung vorausgesetzt.

Es bezeichne G^i_{jk} die Berwaldschen Übertragungsparameter des Raumes und $H_j{}^i{}_{kl}$ den Berwaldschen Krümmungstensor. Wir werden später die folgenden Formeln benötigen

$$H^r_k = y^j H_j{}^r{}_k, \quad H_{hj} = \partial H_j/\partial y^h, \quad H_{jk}-H_{kj} = -H_r{}^r{}_{jk}, \quad H = \frac{1}{n-1} H^s_s.$$

Der Skalar $R=H/\mathscr{L}^2$ ist der Krümmungsskalar des Raumes F_n.

L. Berwald hat das Krümmungsmaß $R(x, y, \eta)$ im Linienelement (x, y) nach der 2-Richtung (y, η) durch

$$R(x, y, \eta) = H_{ikhm} y^i y^h \eta^k \eta^m/(g_{ih}g_{km}-g_{im}g_{hk}) y^i y^h \eta^k \eta^m$$

definiert [2]. Der Raum heißt von skalarer Krümmung, wenn $R(x, y, \eta)$ von der Wahl des Vektors η unabhängig ist. In diesem Fall gilt die Gleichung $R(x, y, \eta)=R$.

Ein Finslerscher Raum ist gerade dann von skalarer Krümmung, wenn der Tensor $(1/\mathscr{L}^2)H^i_j$ die folgende Form

$$(2.1) \qquad (1/\mathscr{L}^2)H^i_j = R(\delta^i_j - l^i l_j)$$

hat, wobei l^i bzw. l_j den kontravarianten bzw. kovarianten Einheitsvektor bezeichnet.

Der Satz von Schur lautet: Ist der Raum von skalarer Krümmung, und ist $R(x, y)$ eine Funktion nur des Ortes, so ist $R(x, y)$ konstant. Diese Räume sind die Räume von konstanter Krümmung. Der Verfasser dieser Arbeit hat schon an anderer Stelle den folgenden Satz bewiesen [8]:

SATZ 4. *Ein Finslerscher Raum F_n mit* $\dim F_n > 2$ *ist genau dann von skalarer Krümmung, wenn sein Weylscher Krümmungstensor verschwindet.*

Wir geben auch jetzt mit Hilfe gewisser Weylscher Tensor notwendige und hinreichende Bedingungen dafür, daß ein Finslerscher Raum von konstanter Krümmung im Sinne von L. Berwald, bzw. von Akbar Zadeh ist.

Später werden wir auch die folgenden Relationen benötigen [8]. Ist $H^i_{jk}= =A_j\delta^i_k - A_k\delta^i_j - B_{jk}y^i$ die kanonische Darstellung in einem Raum von skalarer Krümmung, so

$$(2.2) \qquad A_j = R\mathscr{L}_{\|j} + (1/3)\mathscr{L}R_{\|j},$$

$$(2.3) \qquad B_{jk} = (1/3)(l_j R_{\|k} - l_k R_{\|j}) = -\frac{1}{n+1} H_s{}^s{}_{jk},$$

wobei die Operation $\| j$ durch $\mathscr{L}\partial/\partial y^j$ definiert ist. Es gilt auch die folgende Behauptung [8]:

Ein Raum von skalarer Krümmung ist gerade dann von konstanter Krümmung, wenn die Gleichung $B_{jk}=0$ besteht.

Betrachten wir jetzt bezüglich eines Finslerschen Raumes F_n das Rundschen Projektivbündel $P_F(M)$, das den (mit G bezeichnete) Berwaldschen Affinzusammenhang des Raumes enthält. Es bezeichne $\gamma\colon F(M)\to P_F(M)$ die kanonische Einbettung bezüglich G, und betrachten wir auch noch den $\mathrm{gl}(n, \mathbf{R})$-wertigen Tensor

$$\gamma^* \Lambda^i_j = \frac{1}{2}\tilde{W}_j{}^i{}_{kl}\gamma^*\omega^k\wedge\gamma^*\omega^l+\tilde{D}_j{}^i{}_{kl}\gamma^*\omega^k\wedge\gamma^*\overset{(v)}{\omega}{}^l \tag{2.4}$$

auf $F(M)$. In einer Bündelkarte (x^i, y^i, u^i_j) auf $F(M)$ sind die Funktionen $\tilde{W}_j{}^i{}_{kl}$ bzw. $\tilde{D}_j{}^i{}_{kl}$ der Form (1.3′) und damit sind $W_j{}^i{}_{kl}$ bzw. $D_j{}^i{}_{kl}$ die klassischen Komponenten Finslerscher (1.3) Tensoren. Doch sind diese Tensoren in allgemeinen keine projektiven Invarianten, da $P_F(M)$ im allgemeinen nicht von Berwald-Typ ist [7]. Um Mißverständnissen vorzubeugen nennen wir den Tensor $W_j{}^i{}_{kl}$ den Berwald—Weylschen Krümmungstensor des Raumes. Es gilt die Formel [7]:

$$W_j{}^i{}_{kl} = H_j{}^i{}_{kl}-K_{jl}\delta^i_k+K_{jk}\delta^i_l+(K_{lk}-K_{kl})\delta^i_j, \tag{2.5}$$

wobei

$$K_{lj} = \frac{1}{n^2-1}(nH_l{}^s{}_{sj}+H_j{}^s{}_{sl}), \tag{2.6}$$

und damit

$$H = -y^l y^j K_{lj}, \quad K_{lk}-K_{kl} = -\frac{1}{n+1}H_s{}^h{}_{kl}. \tag{2.7}$$

SATZ 5. *Ein Finslerscher Raum F_n, mit $\dim F_n>2$ ist gerade dann von konstanter Krümmung, wenn der Berwald—Weylsche Krümmungstensor (2.5) des Raumes F_n verschwindet.*

BEWEIS. Ist der Raum von konstanter Krümmung, so ist der Tensor $H_j{}^i{}_k{}^l$ von der Form

$$H_j{}^i{}_{kl} = R(g_{jk}\delta^i_l-g_{jl}\delta^i_k). \tag{2.8}$$

Substituiert man (2.8) in (2.5), so ergibt sich, daß der Tensor $W_j{}^i{}_{kl}$ verschwindet.

Umgekehrt sei vorausgesetzt, daß der Tensor $W_j{}^i{}_{kl}$ verschwindet. Zuerst beweisen wir, daß der Raum von skalarer Krümmung ist. Aus (2.5) bekommen wir

$$H^i_l = H\delta^i_l+(2y^rK_{rl}-y^rK_{lr})\,y^i. \tag{2.9}$$

Komponiert man diese Gleichung mit l_i, so hat man

$$2y^rK_{rl}-y^rK_{lr} = -(H/\mathcal{L})\,l_l. \tag{2.10}$$

Substituiert man (2.10) in (2.9), so ergibt sich

$$(1/\mathcal{L}^2)\,H^i_l = R(\delta^i_l-l_l l^i),$$

und somit ist der Raum von skalarer Krümmung.

Wir müssen noch beweisen, daß der Tensor $K_{lk}-K_{kl}=-\dfrac{1}{n+1}H_s{}^s{}_{kl}$ verschwindet (siehe den Satz nach der Formel (2.3)).

Aus (2.5) folgt auch:

$$H^i_{kl} = y^r K_{rl}\delta^i_k - y^r K_{rk}\delta^i_l - (K_{lk} - K_{kl})y^i. \tag{2.11}$$

Substituieren wir (2.5) und (2.11) in die meistens wohlbekannte Gleichung $H_j{}^i{}_{kl} = \partial H^i_{kl}/\partial y^j$:

$$y^r \partial K_{rl}/\partial y^j \delta^i_k - y^r \partial K_{rk}/\partial y^j \delta^i_l - \partial(K_{lk} - K_{lk})/\partial y^j y^i = 0, \tag{2.12}$$

und aus dieser Identität folgt leicht

$$\partial(K_{lk} - K_{kl})/\partial y^j = 0, \tag{2.13}$$

d. h., wegen (2.7) und (2.3):

$$\begin{aligned} &\partial(\mathscr{L}R_i l_j - \mathscr{L}R_j l_i)/\partial y^k = \\ &= l_k R_i l_j + \mathscr{L}R_{ik} l_j + \mathscr{L}R_i l_{jk} - l_k R_j l_i - \mathscr{L}R_{jk} l_i - \mathscr{L}R_j l_{ik} = 0, \end{aligned} \tag{2.14}$$

wobei $R_i = \partial R/\partial y^i$, $R_{ij} = \partial R_i/\partial y^j$, $l_{ij} = \partial l_i/\partial y^j$ u. s. w. Komponiert man (2.14) mit y^i, so ergibt sich

$$R_{jk} = -(1/\mathscr{L})(R_k l_j + l_k R_j). \tag{2.15}$$

Substituiert man (2.15) in (2.14)

$$\mathscr{L}R_i l_{jk} - \mathscr{L}R_j l_{ik} = 0, \tag{2.16}$$

so reduziert sich (2.14) auf die Gleichung

$$l_k R_i l_j + \mathscr{L}R_{ik} l_j - l_k R_j l_i - \mathscr{L}R_{jk} l_i = 0. \tag{2.17}$$

Vertauscht man die Indizes i und k, und subtrahiert man dann diese Gleichung aus (2.17), so erhält man mit Rücksicht auf (2.15)

$$R_k l_j l_i - R_j l_i l_k = 0.$$

Komponiert man diese Gleichung mit y^i, so hat man

$$R_{\|k} l_j - R_{\|j} l_k = 0,$$

und somit ist der Raum von konstanter Krümmung. W. z. b. w.

Akbar Zadeh [1] hat das Krümmungsmaß $R(x, y, \zeta, \eta)$ im Linienelement (x, y) nach der 2-Richtung (ζ, η) durch

$$R(x, y, \zeta, \eta) = R_{ikhm}\zeta^i\zeta^h\eta^k\eta^m/(g_{ih}g_{km} - g_{im}g_{kh})\zeta^i\zeta^h\eta^k\eta^m$$

definiert, wobei $R_j{}^i{}_{kl}$ der Cartansche erste Krümmungstensor des Finslerschen Raumes F_n ist.

Der Satz von Schur [1] lautet: Ist $R(x, y, \zeta, \eta)$, $(n > 2)$, von der Wahl der Vektoren (ζ, η) unabhängig so ist $R(x, y, \zeta, \eta) = R$ konstant.

Diese Räume sind die Räume von konstanter Krümmung im Sinne von Akbar Zadeh. In diesem Fall verschwindet auch der dritte Cartansche Krümmungstensor $S_j{}^i{}_{kl}$ (siehe [1]), und so bekommen wir nach einem Satz von Brickel [3] den

SATZ 6. *Ist* F_n $(n>2)$ *von konstanter Krümmung im Sinne von Akbar Zadeh, und ist die metrische Grundfunktion* $\mathscr{L}(x, y)$ in y symmetrisch: $\mathscr{L}(x, y)=\mathscr{L}(x, -y)$, *so ist* F_n *ein Riemannscher Raum von konstanter Krümmung.*

Der Raum F_n mit dim $F_n>2$ ist gerade dann von konstanter Krümmung im Sinne von Akbar Zadeh, wenn der Tensor $R_j{}^i{}_{kl}$ die Form:

$$R_j{}^i{}_{kl} = R(g_{jk}\delta^i_l - y_{jl}\delta^i_k) \tag{2.18}$$

hat.

Betrachten wir jetzt den Cartanschen Affinzusammenhang (gegeben durch die Übertragungsparameter $\{\Gamma^i_{jk}, C^i_{jk}\}$) des Finslerschen Raumes F_n. Dieser Affinzusammenhang bestimmt auch einen Cartanschen Projektivzusammenhang $(P_F(M), C)$. Es sei γ: $F(M)\to P_F(M)$ die kanonische Einbettung bezüglich $\{\Gamma^i_{jk}, C^i_{jk}\}$, und betrachten wir die Form

$$\sigma^* \Lambda^i_j = \frac{1}{2}\tilde{W}_j{}^i{}_{kl}\gamma^*\omega^k\wedge\gamma^*\omega^l + \tilde{D}_j{}^i{}_{kl}\gamma^*\omega^k\wedge\gamma^*\overset{(v)}{\omega}{}^l + \frac{1}{2}\tilde{S}_j{}^i{}_{kl}\gamma^*\overset{(v)}{\omega}{}^k\wedge\gamma^*\overset{(v)}{\omega}{}^l$$

auf $F(M)$. Die Funktionen $\tilde{W}_j{}^i{}_{kl}, \tilde{D}_j{}^i{}_{kl}, \tilde{S}_j{}^i{}_{kl}$ auf $F(M)$ sind in einer Bündelkarte (x^i, y^i, u^i_j) der Form (1.3′), wobei $W_j{}^i{}_{kl}, D_j{}^i{}_{kl}, S_j{}^i{}_{kl}$ die klassischen Komponenten Finslerscher (1.3)-Tensoren sind. Auch diese sind keine projektiven Invarianten. Den Tensor

$$W_j{}^i{}_{kl} = R_j{}^i{}_{kl} - \overset{*}{R}_{jl}\delta^i_k + \overset{*}{R}_{jk}\delta^i_l + (\overset{*}{R}_{lk} - \overset{*}{R}_{kl})\delta^i_j, \tag{2.19}$$

wobei

$$\overset{*}{R}_{lj} = \frac{1}{n^2-1}(nR_l{}^s{}_{sj} + R_j{}^s{}_{sl}), \quad (\text{also } H = -y^l y^j \overset{*}{R}_{lj}),$$

nennen wir den Cartan—Weylschen Krümmungstensor des Raumes.

SATZ 7. *Ein Finslerscher Raum* F_n *mit* dim $F_n>2$ *ist genau dann von konstanter Krümmung im Sinne von Akbar Zadeh, wenn der Cartan—Weylsche Krümmungstensor* (2.19) *des Raumes verschwindet.*

BEWEIS. Ist der Raum von konstanter Krümmung im Sinne von Akbar Zadeh, so erhalten wir aus (2.18) mühelos, daß der Tensor $W_j{}^i{}_{kl}$ unter (2.19) verschwindet.

Umgekehrt sei angenommen, daß der Tensor $W_j{}^i{}_{kl}$ verschwindet. Für einen Finslerschen Raum F_n verschwindet auch der Tensor

$$\overset{*}{R}_{lk} - \overset{*}{R}_{kl} = \frac{1}{n+1}R_s{}^s{}_{kl},$$

somit erhalten wir aus (2.19)

$$R_j{}^i{}_{kl} = \overset{*}{R}_{jl}\delta^i_k - \overset{*}{R}_{jk}\delta^i_l. \tag{2.20}$$

Für den Tensor $R_{jikl}=g_{ir}R_j{}^r{}_{kl}$ sind die Eigenschaften:

$$R_{jikl} = -R_{ijkl}, \quad R_{jikl} = R_{klji}$$

wohlbekannt, und so folgt aus (2.20)

$$\overset{*}{R}_{jl}\, g_{ik} - \overset{*}{R}_{jk}\, g_{li} = -\overset{*}{R}_{il}\, g_{jk} + \overset{*}{R}_{ik}\, g_{lj} = -\overset{*}{R}_{li}\, g_{kj} + \overset{*}{R}_{lj}\, g_{ik},$$

daher

$$\overset{*}{R}_{jk}\, \delta^i_l = \overset{*}{R}{}^i_l\, g_{kj}, \tag{2.21}$$

mit

$$\overset{*}{R}{}^i_l = g^{ri} \overset{*}{R}_{lr}.$$

Mit der Kontraktion $i \to l$ in (2.21) erhalten wir:

$$\overset{*}{R}_{jk} = \frac{1}{n} \overset{*}{R}{}^s_s\, g_{kj} = R g_{kj},$$

und so folgt aus (2.20) daß der Raum von konstanter Krümmung im Sinne von Akbar Zadeh ist. W. z. b. w.

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ÜBER APPROXIMATION DURCH POLYNOME MIT BELEGUNGSFUNKTION

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§ 1. Einleitung

Außer den Approximationseigenschaften sind auch die engen Beziehungen bekannt, die die Bernsteinsche Operatoren [1]

$$B_n(f;\, x) = \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right)\binom{n}{\nu} x^\nu (1-x)^{n-\nu}$$

zur Binomialverteilung der Wahrscheinlichkeitstheorie binden. Ähnliche Beziehungen findet man z. B. in den Betrachtungen der Operatoren

$$M_n(f;\, x) = (1-x)^{n+1} \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{\nu+n}\right)\binom{n+\nu}{\nu} x^\nu,$$

mit denen man eine im Intervall $[0, 1]$ stetige Funktion f in $[0, a]$ $(a<1)$ gleichmäßig approximieren kann [2], oder z. B. auch im Fall der Gamme-Operatoren [3]

$$G_n(f;\, x) = \frac{x^{n+1}}{n!} \int_0^\infty e^{-xt} t^n f\left(\frac{n}{t}\right) dt.$$

Die M_n sind mit der Pascal-Verteilung, die G_n mit der Gamma-Verteilung verbunden.

Betrachten wir jetzt — ohne diese Parallelität ausführlich darzulegen — die entsprechenden Beziehungen der Poisson-Verteilung zur Approximationstheorie. Mit diesem Problem beschäftigten sich O. SZÁSZ [4], G. M. MIRAKYAN [5] und auch andere Mathematiker. Die Operatoren von SZÁSZ—Mirakyan ordnen einer, im Intervall $[0, \infty)$ gegebenen Funktion f Potenzreihen mit Belegungsfunktion:

$$S_n(f;\, x) = e^{-nx} \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{n}\right) \frac{(nx)^\nu}{\nu!}.$$

In seiner eben zitierten Arbeit prüfte O. Szász die zu den bekannten Approximationseigenschaften der Bernsteinschen Polynome analogen Behauptungen im Fall des Operators S_n.

Die Approximation irgendeiner Funktion durch M_n, G_n bzw. S_n ist im Fall numerischer Rechnungen im allgemeinen problematisch, wegen des Integrals bzw. der unendlichen Reihen. Es scheint also nützlich die Frage zu behandeln (bleiben wir im weiteren beim Operator S_n), ob man nicht statt der unendlichen Reihen mit ihrer Partialsummen rechnen kann. Genauer gesagt, wenn man die folgende Bezeichnung einführt:

$$S_{n,N}(f;\, x) = e^{-nx} \sum_{\nu=0}^{N} f\left(\frac{\nu}{n}\right) \frac{(nx)^\nu}{\nu!},$$

so lautet unsere Frage: Welche Approximationseigenschaften des Operators S_n beliben auch für $S_{n,N}$ gültig, und wie muß man dann N angeben? Ferner: ist es nicht möglich, in einem beliebigen Punkt bzw. beliebig langen Intervall der reellen Achse auch solche Funktionen durch $S_{n,N}$ beliebig genau zu approximieren bzw. gleichmäßig zu approximieren, für welche die Transformation S_n gar nicht erklärt ist. Diese Fragen wollen wir in der vorliegenden Arbeit beantworten.

§ 2. Sätze 1—3.

O. SZÁSZ bewies in [4]; „Ist $f(x)$ eine, in allen endlichen Intervallen beshränkte Funktion, und gilt $f(x)=O(x^k)$ $(k>0, x\to\infty)$, ferner, ist $f(x)$ stetig an der Stelle x_0, so gilt die Behauptung: $\lim\limits_{n\to\infty} S_n(f;\, x_0)=f(x_0)$." Unsere obige Frage lautet also: Ist dieser Satz auch für $S_{n,N}$ gültig, wenn N in geeigneter Weise angegeben wird? Die Frage können wir positiv beantworten, wir werden aber viel mehr beweisen: statt der Klasse der Funktionen, die durch die Bedingung $f(x)=O(x^k)$ zugelassen werden, können wir eine viel breitere Klasse angeben, deren Funktionen durch $S_{n,N}$ approximiert werden können. Vor der genauen Formulierung unserer Behauptung betrachten wir z. B. die Funktion $f(x)=e^{x^k}$, wobei $k>0$ ist. Für diese Funktion lautet die Transformation S_n:

$$S_n(e^{t^k};\, x) = e^{-nx}\sum_{\nu=0}^{\infty}\exp\left[\left(\frac{\nu}{n}\right)^k\right]\frac{(nx)^\nu}{\nu!}.$$

Bezeichnen wir das allgemeine Glied der unendlichen Reihe mit $a_{n,\nu}(x)$, so bekommt man durch Anwendung der Stirling-Formel die asymptotische Gleichung:

$$a_{n,\nu}(x)\sim\frac{1}{\sqrt{2\pi\nu}}\exp\left[\left(\frac{\nu}{n}\right)^k+\nu\ln(nx)-\nu\ln\nu+\nu\right]\quad(\nu\to\infty),$$

aus der sich ergibt, daß für $k>1$ die Reihe für beliebige fixe $x>0$, $n>0$ divergent ist. Die Funktion $f(x)=e^{x^k}$ $(k>1)$ wird also für keinen positiven Wert von x durch S_n approximiert. Laut des nachstehenden Satzes kann aber diese Funktion, sogar für beliebige $x>0$, durch $S_{n,N}$ approximiert werden.

SATZ 1. (A) *Es sei f in allen (in $[0,\infty)$ liegenden) endlichen Intervallen beschränkt und $f(x)=O\big(g_m(x)\big)$ $(x\to\infty,\ m\geqq 1$ ganze Zahl), mit*

$$g_0(x)=x;\quad g_i(x)=\exp[g_{i-1}(x)]\quad(i\geqq 1);$$

es sei ferner $N=N(n)$ so angegeben, daß

$$\lim_{n\to\infty}\frac{N(n)}{n}=\infty$$

gilt, wobei aber eine Schwellenzahl n_0 existiert, derart, daß für $n>n_0$ die Bedingung $N(n)\leqq nh_{m-1}(n)$ mit $h_0(x)=x$; $h_i(x)=\ln[h_{i-1}(x)]$ $(i\geqq 1)$ erfüllt ist. Dann gilt die folgende Behauptung für jede nicht-negative Stelle x_0, wo f stetig ist:

$$\lim_{n\to\infty} S_{n,N}(f;\, x_0)=f(x_0).$$

(B) *Existiert zur Funktion f eine Schwellenzahl x' derart, daß für $x>x'$ die Ungleichung $f(x)>\gamma g_m(x)$ $(m\geqq 2,\ \gamma>0)$ gilt, ferner eine Zahl n_1 derart, daß für $n>n_1$ $N(n)\geqq nh_{m-2}(n)$ zutrifft, so ist die Folge $\{S_{n,N}(f;x)\}_{n=1}^{\infty}$ für alle $x>0$ divergent.*

BEWEIS. Es sei x_0 eine beliebige nichtnegative Zahl. Teilen wir die Summe $S_{n,N}(f;x_0)$ in zwei Teile, die wir mit T_1 bzw. T_2 bezeichnen werden:

$$S_{n,N}(f;\ x_0) = e^{-nx_0}\sum_{\nu=0}^{[3nx_1]} f\left(\frac{\nu}{n}\right)\frac{(nx_0)^\nu}{\nu!} + e^{-nx_0}\sum_{\nu=[3nx_1]+1}^{N} f\left(\frac{\nu}{n}\right)\frac{(nx_0)^\nu}{\nu!} = T_1+T_2.$$

Mit x_1 wurde x_0+1 bezeichnet. Führen wir jetzt die Folgende Funktion ein:

$$f^*(x) = \begin{cases} f(x) & \text{für} \quad 0\leqq x\leqq 3x_1 \\ 0 & \text{für} \quad x>3x_1. \end{cases}$$

Es ist evident, daß

$$S_{n,[3nx_1]}(f;\ x) \equiv S_n(f^*;\ x)$$

ist und deshalb

$$T_1 = S_n(f^*;\ x_0) \tag{1}$$

gilt. Nehmen wir jetzt T_2. Aus den Bedingungen des Satzes folgt:

$$T_2 = O\left[e^{-nx_0}\sum_{\nu=[3nx_1]+1}^{N} g_m\left(\frac{\nu}{n}\right)\frac{(nx_0)^\nu}{\nu!}\right] \quad (n\to\infty). \tag{2}$$

Bezeichnet man den Ausdruck hinter dem Σ-Zeichen mit $b_{n,\nu}$, so bekommt man nach Anwendung der Stirling-Formel:

$$b_{n,\nu} = g_m\left(\frac{\nu}{n}\right)\frac{(nx_0)^\nu}{\sqrt{2\pi\nu}\,\nu^\nu e^{-\nu}(1+q_\nu)} \leqq \frac{1}{\sqrt{2\pi\nu}}\, g_m\left(\frac{\nu}{n}\right)\left(\frac{enx_0}{\nu}\right)^\nu.$$

Mit der Bezeichnung

$$\tau_{n,\nu} = g_m\left(\frac{\nu}{n}\right)\left(\frac{enx_0}{\nu}\right)^\nu$$

erhalten wir

$$b_{n,\nu} \leqq \frac{1}{\sqrt{2\pi\nu}}\,\tau_{n,\nu}.$$

Es folge jetzt die Abschätzung der Glieder $\tau_{n,[3nx_1]}$ und $\tau_{n,N}$:

$$\tau_{n,[3nx_1]} \leqq g_m(3x_1)\left(\frac{ex_0}{3x_1-n^{-1}}\right)^{[3nx_1]}.$$

Da für $n>1$ die Ungleichung $ex_0<3x_1-n^{-1}$ gilt, erhalten wir

$$\tau_{n,[3nx_1]} = O(e^{-\lambda nx_1}) \quad (n\to\infty,\ \lambda>0). \tag{3}$$

Betrachten wir jetzt das letzte Glied in T_2. Nach den Bedingungen existiert eine Zahl n_0 derart, daß

$$\tau_{n,N} \leqq g_m(h_{m-1}(n))\left(\frac{enx_0}{N}\right)^N$$

für alle $n>n_0$ gültig ist. Da aber $g_m(h_{m-1}(n))=e^n$ ist, nimmt unsere Ungleichung die folgende Gestalt an:

$$\tau_{n,N} \leqq \exp\left[n\left(1-\frac{N}{n}\ln\frac{N}{enx_0}\right)\right].$$

Wegen der Bedingung $\lim_{n\to\infty}(Nn^{-1})=\infty$, gilt unbedigt die Gleichung

$$\tau_{n,N} = O(e^{-n}) \quad (n\to\infty). \tag{4}$$

Zur Abschätzung der anderen Glieder der Summe T_2 betrachten wir die Funktion

$$G_n(t) = g_m\left(\frac{t}{n}\right)\left(\frac{enx_0}{t}\right)^t \quad (t>0).$$

Differenzieren wir G_n. Im Fall $m\geqq 2$ ergibt sich

$$G_n'(t) = g_m\left(\frac{t}{n}\right)\left(\frac{enx_0}{t}\right)^t\left[g_{m-1}\left(\frac{t}{n}\right)g_{m-2}\left(\frac{t}{n}\right)\dots g_1\left(\frac{t}{n}\right)\frac{1}{n}-\ln\frac{t}{nx_0}\right].$$

Die Funktion in der eckigen Klammer wird mit $\Gamma_n(t)$ bezeichnet. Es ist evident, daß die Funktion Γ_n 0, 1 oder 2 Nullstellen hat. Da die Werte $g_i([3nx_1]n^{-1})$ unter eine von n unabhängigen Schranke bleiben, existiert eine Schwellenzahl n' derart, daß für $n>n'$ $\Gamma_n([3nx_1])<0$ ist, woraus folgt, daß Γ_n für $n>n'$ zwei Nullstellen besitzt. Seien diese Nullstellen mit $z_{n,1}$ bzw. $z_{n,2}$ bezeichnet. Da aus $\Gamma_n(t)<0$ $z_{n,1}<t<z_{n,2}$ folgt, gelten auch die Ungleichungen $z_{n,1}<[3nx_1]<z_{n,2}$. So erhielten wir also, daß $G_n(t)$ im Intervall $([3nx_1], z_{n,2})$ monoton abnimmt, bzw. in $(z_{n,2}, N)$ monoton wächst. Im Fall $m=1$ bekommt man, daß $G_n(t)$ im ganzen Intervall $([3nx_1], N)$ monoton abnimmt. Folglich gilt in den beiden Fällen $\tau_{n,\nu}\leqq$ $\leqq\max(\tau_{n,[3nx_1]}, \tau_{n,N})$ $(\nu=[3nx_1], \dots, N)$. Die Anzahl der Glieder der endlichen Folge $\tau_{n,[3nx_1]}, \dots, \tau_{n,N}$, ferner (3) und (4) in Betracht ziehend bekommt man die folgende Gleichung: $T_2=O(e^{-\lambda_1 n})$ $(n\to\infty, \lambda_1>0)$. Dann gilt mit Rücksicht auf (1)

$$S_{n,N}(f;x_0) = S_n(f^*;x_0)+O(e^{-\lambda_1 n}) \quad (n\to\infty). \tag{5}$$

Da im Fall der Funktion f^* die Bedingungen des zitierten Satz von SzÁsz erfüllt sind, und $f^*(x_0)=f(x_0)$ ist, folgt unmittelbar die Behauptung (A).

Zum Beweis des Teils (B) genügt es nur soviel zu beweisen, daß die Folge $\{S_{n,N}(g_m;x)\}_{n=1}^{\infty}$ divergent ist. Betrachten wir nun das allgemeine Glied der Folge:

$$S_{n,N}(g_m;x) = \sum_{\nu=0}^{N} e^{-nx}g_m\left(\frac{\nu}{n}\right)\frac{(nx)^\nu}{\nu!}.$$

Wird das letzte Glied der Summe mit $\tau_n^*(x)$ bezeichnet, so erhält man laut der Stirling-Formel die folgende Gleichung:

$$\tau_n^*(x) = e^{-nx}g\left(\frac{N}{n}\right)\frac{(nx)^N}{\sqrt{2\pi N}N^N e^{-N}(1+q_N)},$$

wobei $\lim\limits_{N\to\infty} q_N=0$ ist. Wenn man jetzt N folgenderwise angibt: $N(n)=[nh_{m-2}(n)+1]$, und in Betracht nimmt, daß $g_m\left(\frac{N}{n}\right)\geqq g_m(h_{m-2}(n))=g_2(n)=e^{e^n}$ ist, so gilt auch

$$\tau_n^*(x)\geqq \frac{1}{\sqrt{2\pi N}(1+q_N)}\exp\left[-nx+e^n+N(\ln enx-\ln N)\right],$$

woraus schon unmittelbar folgt, daß $\tau_n^*(x)$ und auch $S_{n,N}(g_m;\ x)$ unbegrenzt wächst, wenn $n\to\infty$.

Mit Rücksicht auf den Beweis, möchten wir unsere Gedanken folgenderweise zusammenfassen: Auf gleiche Weise, wie bei den Bernsteinschen Polynomen, wurzeln die Approximationseigenschaften auch bei den Potenzreihen von Szász—Mirakyan in Lokalisationseigenschaften, die durch Betrachtung der entsprechenden Verteilung der Wahrscheinlichkeitstheorie anschaulich werden: Obwohl zum Wert $S_n(f;\ x_0)$ alle Werte $f\left(\frac{\nu}{n}\right)$ beitragen, werden die Werte $f\left(\frac{\nu}{n}\right)$, deren Argument nahe zu x_0 liegt, mit großem, die anderen mit bedeutungslosem Gewicht in Betracht gezogen. Wir haben aber schon gesehen, daß im Fall der Funktionen, deren Ordnungsgröße gleich e^{x^k} $(x\to\infty,\ k>1)$, oder größer ist, obiges nicht mehr gilt; das Gewicht des Wertes $f\left(\frac{\nu}{n}\right)$ nimmt vergeblich ab, wenn $f\left(\frac{\nu}{n}\right)$ viel „schneller" wächst. Im Fall dieser Funktionen ist also S_n zur Approximation ungeeignet. Obwohl unser Ausgangspunkt war, daß es — um die numerischen Rechnungen zu ermöglichen — zweckmäßig wäre statt der unendlichen Reihen bloß durch deren Partialsummen zu approximieren, erhielten wir, daß man — wenn die Ordnungsgröße der zu approximierenden Funktion genügend groß ist — dies sogar tun muß, um die Approximationseigenschaften zu bewahren.

Die Bedingung $m\geqq 2$ im Satz 1 (B), ferner die Tatsache, daß in der Praxis haupsächlich Funktionen vorkommen, deren Ordnungsgröße kleiner als die der Funktion e^{e^x} ist, motivieren unsere weitere Zielsetzung: den Satz „zwischen" $m=2$ und $m=1$ zu verfeinern, ferner uns auch mit dem Fall $m=1$ ausführlicher zu beschäftigen.

SATZ 2. (A) *Es sei f in allen (in $[0,\infty)$ liegenden) endlichen Intervallen beschränkt und $f(x)=O(e^{x^k})$ $(x\to\infty,\ k>0$ konst.), es sei ferner $N=N(n)$ derart angegeben, daß $\lim\limits_{n\to\infty}(N(n)/n)=\infty$ erfüllt ist, aber für $k>1$ eine solche Schwellenzahl n_0 existiert, daß für $n>n_0$ die Ungleichung $N(n)\leqq \alpha n^\beta$ gilt, wobei α, β positive Konstanten sind, und für β die Ungleichung $\beta<\frac{k}{k-1}$ erfüllt ist. Dann gilt für jede nicht-negative Stelle x_0 wo f stetig ist, die Limesbeziehung*

$$\lim_{n\to\infty} S_{n,N}(f;\ x_0)=f(x_0).$$

(B) *Existiert für die Funktion f eine Schwellenzahl x' derart, daß für $x>x'$ die Ungleichung*

$$f(x)>\gamma e^{x^k}\quad (k>1,\ \gamma>0)$$

erfüllt ist, ferner, existiert eine Zahl n_1 *derart, daß*

$$N(n) > \alpha n^{\beta} \quad \left(\beta > \frac{k}{k-1}\right)$$

zutrifft, falls $n > n_1$ *gilt, so divergiert die Folge* $\{S_{n,N}(f;\,x)\}_{n=1}^{\infty}$ *für alle* $x > 0$.

Da der Beweis analog dem Beweis des Satzes 1 ist, lassen wir ihn weg.

Nach der Behandlung des ersten Satzes motivierte die Bedingung $m \geqq 2$ die Aussage eines analogen Satzes für $m < 2$. Jetzt kann, in Anbetracht der Voraussetzung $k > 1$ im Satz 2 (B) eine ähnliche Frage gestellt werden, und zwar, ob es nicht einen analogen Satz für $k = 1$ gibt (oder mit der Bezeichnung von Satz 1 ausgedrückt: für $m = 1$)? Nun, diese Frage müßen wir mit nein beantworten, man kann keine den Sätzen (B) analoge Behauptung aussagen. (Darum wird jetzt der Teil (A) selbstverständlich einfacher.) Unsere Behauptung lautet also:

SATZ 3. *Es sei f in allen (in* $[0, \infty)$ *liegenden) endlichen Intervallen beschränkt und* $f(x) = O(e^{cx})$ $(x \to \infty,$ *c beliebig konst.), es sei ferner* $N = N(n)$ *derart angegeben, daß* $\lim_{n \to \infty} (N(n)/n) = \infty$ *erfüllt ist; dann gilt für jede nicht-negative Stelle* x_0, *wo f stetig ist:*

$$\lim_{n \to \infty} S_{n,N}(f;\, x_0) = f(x_0).$$

BEWEIS. Wir können einige Schritte aus dem vorangehenden Beweis übernehmen. Die Summe T_2^*, die jetzt abgeschätzt werden muß, ist der unter (2) stehenden Summe T_2 analog:

$$T_2^* = e^{-nx_0} \sum_{\nu=[3nx_1]+1}^{N} e^{c\frac{\nu}{n}} \frac{(nx_0)^{\nu}}{\nu!}.$$

Schreiben wir T_2^* in folgender Form;

$$T_2^* = \exp\,[nx_0(e^{c/n}-1)] \sum_{\nu=[3nx_1]+1}^{N} \exp\,[-nx_0 e^{c/n}] \frac{(nx_0 e^{c/n})^{\nu}}{\nu!}. \tag{6}$$

Da für alle Indizes ν die Relationen

$$\nu > [3nx_1] > 3nx_1 - 1 = 3nx_0 + 3n - 1$$

erfüllt sind, gilt für $n \geqq 1$ die Ungleichung $\nu - 3nx_0 > 2n$. Es sei $n \geqq c$, dann ist auch $\nu - nx_0 e^{c/n} > 2n$, sowie

$$|\nu - nx_0 e^{c/n}| > 2n \tag{7}$$

gültig. Im nächsten Schritt wird die Tschebyscheffsche Ungleichung der Wahrscheinlichkeitstheorie angewandet, die im Falle der Poisson-Verteilung folgenderweise lautet:

$$\sum_{|\nu-\lambda| > \delta\sqrt{\lambda}} e^{-\lambda} \frac{\lambda^{\nu}}{\nu!} < \frac{1}{\delta^2}. \tag{8}$$

Es ist nur die Besetzung zu erledigen. Führen wir die folgenden Bezeichnungen ein:

$$\lambda = nx_0 e^{c/n}, \quad \delta = 2e^{-c/2n} \sqrt{\frac{n}{x_0}}. \tag{9}$$

Wird nun in (6) unter das Σ-Zeichen die in (7) stehende Ungleichung geschrieben und werden die eben eingeführten Bezeichnungen (9) angewendet, so bekommt man:

$$T_2^* \leqq \exp\left[nx_0(e^{c/n}-1)\right] \sum_{|\nu-\lambda|>2n} e^{-\lambda}\frac{\lambda^\nu}{\nu!}.$$

Es ist leicht zu sehen, daß

$$\lim_{n\to\infty}\left[nx_0(e^{c/n}-1)\right] = cx_0$$

ist. Da ferner laut unserer Bezeichnungen sich $\delta\sqrt{\lambda}=2n$ ergibt, ist der Beweis nach Anwendung der Ungleichung (8) beendet.

§ 3. Bemerkungen

1. Es kann sich die Frage erheben, ob die Ordnung der Approximation durch $S_{n,N}$ nicht wesentlich schlechter ist, alls im Falle von S_n. Die Frage kann sofort beantwortet werden, man muß nur beachten, daß aus (5)

$$|S_{n,N}(f;x_0)-f(x_0)| \leqq |S_n(f^*;x_0)-f^*(x_0)|+O(e^{-\lambda n})$$

folgt ($n\to\infty$, $\lambda>0$). Wie bedeutungslos im allgemeinen $O(e^{-\lambda n})$ neben der anderen Glied ist, darauf weist z. B. die Tatsache hin, daß für eine so „gute" Funktion, wie $f(x)=x^2$, die Ordnung der Approximation nur $\frac{1}{n}$ beträgt. $\left(\text{Es ist nähmlich } S_n(t^2;\ x)=x^2+\frac{x}{n}.\right)$

2. O. Szász befaßte sich in [4] auch mit dem folgenden Problem: Es sei die Funktion f im Intervall $(0,\infty)$ r-mal differenzierbar, und $f^{(r)}$ an der Stelle x_0 stetig. Es ist die Frage zu beantworten, ob die r-te Ableitung von $S_n(f;\ x)$ an der Stelle x_0 mit $n\to\infty$ gegen $f^{(r)}(x_0)$ konvergiert. Szász bewies, daß die Frage positiv beantwortet werden kann, falls $f^{(r)}(x)=O(x^k)$ $(x\to\infty,\ k>0)$ erfüllt ist. Im Zusammenhang damit erwähnen wir unsere folgende Behauptung: Wird $S_{n,N}(f;\ x)$ statt $S_n(f;\ x)$ in Betrachtung genommen, ferner, werden für $f^{(r)}(x)$ bzw. $N(n)$ dieselben Bedingungen vorgeschrieben, wie in den Sätzen 1.—3. für $f(x)$ bzw. $N(n)$, so gilt die Gleichung:

$$\lim_{n\to\infty}\frac{d^r}{dx^r}S_{n,N}(f;\ x)\Big|_{x=x_0}=f^{(r)}(x_0).$$

Wir übergehen die Beweise, da sie langwierig sind und nur ähnliche Abschätzungen wie die Beweise der Sätze 1.—3. enthalten.

3. Aus dem Beweis des dritten Satzes stellt sich heraus, daß die Gedankenfolge der Abschätzung von T_2^* auch dann ebenso durchgeführt werden könnte, wenn statt T_2^* der Ausdruck

$$e^{-nx_0}\sum_{\nu=[3nx_1]+1}^{\infty} e^{c\frac{\nu}{n}}\frac{(nx_0)^\nu}{\nu!}$$

abzuschätzen wäre. Das bedeutet aber, daß der im Anfang von § 2 zitierte Szászsche Satz verschärft werden kann: statt $f(x)=O(x^k)$ genügt eine schwächere Bedingung: $f(x)=O(e^{cx})$.

Hier möchte ich Herrn Prof. Dr. J. BALÁZS für das Thema und für seine wertvolle Hilfe danken. Ich schulde Dank ferner Herrn Dozenten Dr. L. G. PÁL, der meine Arbeit durchgesehen und mich auf manche Probleme, die sich dem Thema anschließen, aufmerksam gemacht hat.

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NEW SEQUELS OF A PROBLEM OF ALEXITS

By

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1. The problem of the strong approximation of Fourier series is due to G. ALEXITS. He and his colleague D. KRÁLIK in some papers (cf. [1], [2] and [3]) investigated this problem and proved several interesting results. Before formulating some known theorems and the aim of our paper we give a few definitions.

Let $f(x)$ be a continuous and 2π-periodic function and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

be its Fourier series. Denote $s_n(x) = s_n(f, x)$ the n-th partial sum of (1), furthermore let $f^{(r)}(x)$ denote the r-th derivative of $f(x)$. For any positive β and p we define the following strong mean:

$$h_n(f, \beta, p) := \left\| \left\{ \frac{1}{(n+1)^\beta} \sum_{k=0}^{n} (k+1)^{\beta-1} |s_k(x) - f(x)|^p \right\}^{1/p} \right\|,$$

where $\|\cdot\|$ denotes the usual supremum norm.

Let $\omega(\delta)$ be a nondecreasing continuous function on the interval $[0, 2\pi]$ having the properties:

$$\omega(0) = 0, \quad \omega(\delta_1 + \delta_2) \leqq \omega(\delta_1) + \omega(\delta_2)$$

for any $0 \leqq \delta_1 \leqq \delta_2 \leqq \delta_1 + \delta_2 \leqq 2\pi$. Such functions will be called moduli of continuity.

Let $E_n(f)$ denote the best approximation of f by trigonometric polynomials of order at most n.

We define the following classes of functions:

$$\left\{\begin{aligned} H(\beta, p, r, \omega) &= \left\{ f \colon h_n(f, \beta, p) = O\left(n^{-r} \omega\left(\frac{1}{n}\right)\right) \right\}, \\ E_r^\omega &= \left\{ f \colon E_n(f) = O\left(n^{-r} \omega\left(\frac{1}{n}\right)\right) \right\}, \\ W^r E^\omega &= \left\{ f \colon E_n(f^{(r)}) = O\left(\omega\left(\frac{1}{n}\right)\right) \right\}, \\ W^r H^\omega &= \{ f \colon \omega(f^{(r)}; \delta) = O(\omega(\delta)) \}, \\ W^r H^* &= \{ f \colon f^{(r)} \in \Lambda_* \}, \end{aligned}\right. \tag{2}$$

where Λ_* denotes the class of ZYGMUND (see [9], p. 43), and $\omega(f;\delta)$ is the modulus of continuity of f. In the case $\omega(\delta)=\delta^\alpha$ we write W^rH^α and $H(\beta,p,r,\alpha)$ instead of $W^rH^{\delta^\alpha}$ and $H(\beta,p,r,\delta^\alpha)$, respectively; and if $r=0$, H^ω stands for W^0H^ω.

One of the results of ALEXITS and KRÁLIK [2] reads as follows:

If $f\in H^\alpha$, $0<\alpha<1$, *then* $h_n(f,1,1)=O(n^{-\alpha})$; thus we have $H^\alpha\subset H(1,1,0,\alpha)$.

By the well-known Bernstein theorem, which states that if $E_n(f)=O(n^{-r-\alpha})$, $0<\alpha<1$, then $f^{(r)}\in H^\alpha$; the inclusion $H(1,1,0,\alpha)\subset H^\alpha$ also holds, so the equivalence

$$H(1,1,0,\alpha)\equiv H^\alpha \tag{3}$$

is valid for any $0<\alpha<1$.

In [6] (Theorem 2) we extended the equivalence relation (3) to the following cases:

If β, p and α are positive numbers and r is a nonnegative integer such that $\beta>(r+\alpha)p$ then

$$H(\beta,p,r,\alpha)\equiv W^rH^\alpha \quad \textit{if} \quad \alpha<1, \tag{4}$$

and

$$W^rH^1\subset H(\beta,p,r,1)\equiv W^rH^* \quad (\alpha=1).$$

It is also shown in [6] that if $\beta=(r+\alpha)p$ then these equivalences fail to hold, and we only have the following proper inclusions:

If $\beta=(r+\alpha)p$ *then* $H(\beta,p,r,\alpha)\subset W^rH^\alpha$ *for* $\alpha<1$; *and* $H(\beta,p,r,1)\subset W^rH^*$ $(\alpha=1)$.

The aim of the paper is to give conditions which imply similar imbedding relations for the classes defined under (2).

In the course of the proof besides some known results we shall use the following recent result of V. TOTIK [8] (see also [4], Theorem 1):

If $f\in W^rH^\omega$ *and* $p>0$ *then for any* β

$$h_n(f,\beta,p)\leqq K\left\{n^{-\beta}\sum_{k=1}^{n}k^{\beta-1}\left(\omega\left(\frac{1}{k}\right)k^{-r}\right)^p\right\}^{1/p}.[1] \tag{5}$$

We prove the following

THEOREM. *Let β and p be positive numbers, r a nonnegative integer, and let $\omega(\delta)$ denote a modulus of continuity satisfying the condition*

$$\int_0^\delta\frac{\omega(x)}{x}\,dx=O(\omega(\delta)). \tag{6}$$

If there exists a natural number μ such that the inequalities

$$2^\mu\omega(2^{-n-\mu})>2\omega(2^{-n}) \tag{7}$$

hold for all n, then

$$H(\beta,p,r,\omega)\subset E_r^\omega\subset W^rH^\omega. \tag{8}$$

[1] $K, K_1, K_2, \ldots$ will always denote positive constants not necessarily the same at each occurrence.

If instead of (7) *we have*

(9) $$2^{\mu(\beta/p-r)}\omega(2^{-n-\mu}) > 2^{1/p}\omega(2^{-n})$$

then

(10) $$W^r H^\omega \subset H(\beta, p, r, \omega).$$

Consequently (7) *and* (9) *together imply*

(11) $$H(\beta, p, r, \omega) \equiv W^r H^\omega.$$

It is clear that if $\omega(\delta)=\delta^\alpha$ with $0<\alpha<1$ and $\beta>(r+\alpha)p$ then conditions (6), (7) and (9) are satisfied, thus statement (11) includes (4).

On account of the well-known inequalities

$$E_n(f) \leqq K\omega\left(f, \frac{1}{n}\right) \quad \text{and} \quad E_n(f) \leqq Kn^{-r}E_n(f^{(r)})$$

the following imbedding relations

(12) $$W^r H^\omega \subset W^r E^\omega \subset E_r^\omega$$

obviously hold.

In [5] we proved that for any positive β and p

(13) $$E_n(f) \leqq Kh_n(f, \beta, p),$$

so it is also clear that

(14) $$H(\beta, p, r, \omega) \subset E_r^\omega.$$

Under (6) the inequality (see [7], p. 61. Th. 8)

$$E_n(f^{(r)}) \leqq K \sum_{k=\left[\frac{n}{2}\right]}^{\infty} k^{r-1}E_k(f)$$

implies

(15) $$E_r^\omega \subset W^r E^\omega.$$

Hence, assuming (6), by (12) $E_r^\omega \equiv W^r E^\omega$ follows; but $H(\beta, p, r, \omega) \subset W^r E^\omega$ and $W^r H^\omega \subset W^r E^\omega$ are proper inclusions; and, in general, neither $H(\beta, p, r, \omega) \subset W^r H^\omega$ nor $W^r H^\omega \subset H(\beta, p, r, \omega)$ hold (see [6], Theorem 2).

Using our Theorem, (12) and (15), we can see that (6) and (7) imply

$$H(\beta, p, r, \omega) \subset W^r H^\omega \equiv W^r E^\omega \equiv E_r^\omega,$$

furthermore (6) and (9) imply

$$W^r H^\omega \subset H(\beta, p, r, \omega) \subset W^r E^\omega \equiv E_r^\omega.$$

Summing up our conclusions we have obtained the following

COROLLARY. *Assuming* (6) *and* (7) *we have* $W^r H^\omega \equiv W^r E^\omega \equiv E_r^\omega$; *and if in addition* (9) *is also fulfilled then* $H(\beta, p, r, \omega) \equiv W^r H^\omega \equiv W^r E^\omega \equiv E_r^\omega$.

2. To prove our theorem we require the following

LEMMA ([7], pp. 59 and 61). *For any nonnegative integer* r

$$\omega\left(f^{(r)};\frac{1}{n}\right)\leqq K\left[\frac{1}{n}\sum_{k=1}^{n}k^{r}E_k+\sum_{k=n+1}^{\infty}k^{r-1}E_k\right].$$

3. PROOF OF THE THEOREM. First we prove the implication (7)⇒(8). By (13) it is clear that if $f\in H(\beta,p,r,\omega)$ then

$$E_n(f)=O\left(n^{-r}\omega\left(\frac{1}{n}\right)\right), \tag{3.1}$$

i.e. (14) holds, so the first part of (8) is proved. Therefore it is sufficient to show that

$$E_r^{\omega}\subset W^rH^{\omega}, \tag{3.2}$$

or equivalently, that (3.1) implies $f\in W^rH^{\omega}$.

Using Lemma, by (6) and (3.1), we obtain that

$$\omega\left(f^{(r)},\frac{1}{n}\right)\leqq K\left\{\frac{1}{n}\sum_{k=1}^{n}\omega\left(\frac{1}{k}\right)+\sum_{k=n+1}^{\infty}\frac{1}{k}\,\omega\left(\frac{1}{k}\right)\right\}\leqq K\left\{\frac{1}{n}\sum_{k=1}^{n}\omega\left(\frac{1}{k}\right)+K_1\omega\left(\frac{1}{n}\right)\right\}.$$

If we can show that (7) implies

$$\frac{1}{n}\sum_{k=1}^{n}\omega\left(\frac{1}{k}\right)\leqq K\omega\left(\frac{1}{n}\right) \tag{3.3}$$

then (3.2) is proved.

A standard calculation gives by (7) that for any n $(m\mu\leqq n<(m+1)\mu)$

$$\sum_{i=1}^{n}2^i\omega(2^{-i})\leqq\sum_{k=1}^{m+1}\sum_{i=(k-1)\mu+1}^{k\mu}2^i\omega(2^{-i})\leqq \tag{3.4}$$

$$\leqq 2\sum_{i=m\mu+1}^{(m+1)\mu}2^i\omega(2^{-i})\leqq 2\mu 2^{(m+1)\mu}\omega(2^{-m\mu})\leqq \mu 2^{\mu+1}2^n\omega(2^{-n})=O\big(2^n\omega(2^{-n})\big),$$

whence (3.3) follows, consequently (3.2) and (8) are proved.

Next we verify the implication (9)⇒(10). If $f\in W^rH^{\omega}$ then by (5) it remains to prove that

$$\left\{n^{-\beta}\sum_{k=1}^{n}k^{\beta-1}\big(k^{-r}\omega(k^{-1})\big)^p\right\}^{1/p}\leqq Kn^{-r}\omega\left(\frac{1}{n}\right). \tag{3.5}$$

If $2^{m-1}<n\leqq 2^m$ then we have

$$\sum_{k=1}^{n}k^{\beta-1}\big(k^{-r}\omega(k^{-1})\big)^p\leqq K\sum_{i=0}^{m}2^{i(\beta-rp)}\omega(2^{-i})^p. \tag{3.6}$$

By (9) we have for any n the inequality

$$2^{(n+\mu)(\beta-rp)}\omega(2^{-n-\mu})^p>2\,2^{n(\beta-rp)}\omega(2^{-n})^p,$$

whence, using the same method as before in (3.4), we obtain that

$$\sum_{i=0}^{m} 2^{i(\beta-rp)}\omega(2^{-i})^p \leqq K2^{m(\beta-rp)}\omega(2^{-m})^p.$$

Hence and from (3.6) the following inequality

$$n^{-\beta}\sum_{k=1}^{n} k^{\beta-1}\big(k^{-r}\omega(k^{-1})\big)^p \leqq K_1 n^{-rp}\omega\left(\frac{1}{n}\right)^p$$

holds, which proves (3.5), i.e.

$$f\in W^r H^\omega \Rightarrow f\in H(\beta, p, r, \omega);$$

and this verifies (10).

Hereby we have completed the proof.

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Hence, using the same method as before in (3.4), we obtain that

[illegible]

Hence [illegible] the following inequality

[illegible]

holds, which proves (3.3)

[illegible]

and the result follows.

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[illegible]

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OPERATORS ON BANACH SPACES WITH COMPLEMENTED RANGES

By

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This paper is concerned with when the closure of the range, $\overline{R(A)}$, of a bounded linear operator A, acting on a Banach space X, is complemented. Several sets of sufficient conditions will be given. Examples will show that some type of assumptions, similar to those made here, are necessary.

1. Introduction. The concept of a generalized inverse has been useful in a variety of settings. The uses of generalized inverses in finite dimensional spaces include solving linear systems, least squares analysis, solving differential equations [5], Markov Chains [5], and so on.

The usefulness of the concept in Banach spaces has been studied in [1], [2], [3], [6], [8], [10], [11], [15]. A 1,775 item bibliography on generalized inverses may be found in [13].

An operator T is called a (1, 2)-inverse of A if $ATA=A$, $TAT=T$. ((1, 2)-inverses are among the more important generalized inverses, especially in solving consistent linear equations.) An operator A will have a bounded (1, 2)-inverse if and only if $\overline{R(A)}$ is complemented and the nullspace of A, $N(A)$, has a complement on which A is bounded below ($\|Ax\| \geqq m\|x\|$, $m>0$). In particular, if A is bounded below, then A has a bounded (1, 2)-inverse if and only if $R(A)$ is complemented.

Thus conditions which guarantee that $\overline{R(A)}$ be complemented are not only of obvious theoretical interest but have immediate application to the theory of generalized inverses in Banach spaces.

There are two types of results that deserve mention here. There exist many results, see [7], [17], [18], for examples, which give that $\overline{R(A)}$ is complemented (in fact by $N(A)$) if 0 is a boundary point of the spectrum of A, $\sigma(A)$, and the resolvent of A satisfies an appropriate growth condition. The other sometimes related results try to mimic the result that for an operator on a Hilbert space $R(A)\oplus \oplus R(A^*)=X$. Our results will not be of this type.

We shall denote the set of all bounded linear operators from X into X with $\overline{R(A)}$ complemented by $\tilde{\zeta}(X)$. The closed linear span of a set $S\subseteq X$ is denoted $[S]$. Subspaces are assumed closed. Uniform, strong, and weak limits are denoted by $\Rightarrow$, $\rightarrow$, and $\rightharpoonup$, respectively.

2. Limits of complemented operators. Our first results show that if A is the limit of $A_n\in\tilde{\zeta}(X)$ and the $\overline{R(A_n)}$ and their complements have limits in the appropriate sense, then $A\in\tilde{\zeta}(X)$ under reasonable assumptions. First we need a definition and a technical result.

DEFINITION 1. Let $\{M_n\}$ be a sequence of subspaces of X. If

$$\bigcap_{n\geqq 1}\Big[\bigcup_{k\geqq n} M_k\Big] = \overline{\bigcup_{n\geqq 1}\bigcap_{k\geqq n} M_k} \tag{1}$$

holds, the common value of the two sets is denoted $\lim M_n$.

If $\{M_n\}$ is monotonic, then $\lim M_n$ exists. In fact $\lim M_n=\overline{\bigcup_n M_n}$ for increasing sequences and $\lim M_n=\bigcap_n M_n$ for decreasing sequences. However, $\{M_n\}$ may have a limit without being monotonic. For example, $\lim_n [\{e_1, e_n\}]=[e_1]$ if $\{e_k\}$ is the standard basis in l^p, $1\leqq p<\infty$.

THEOREM 1. *Let X be a reflexive Banach space, M_k, N_k subspaces such that $X=M_k\oplus N_k$, and P_k the projection onto M_k along N_k. That is, $P_kX=M_k$, $(I-P_k)X=N_k$. If $\|P_k\|\leqq K$ for all k and $\lim M_k$, $\lim N_k$ exist, then $X=\lim M_k\oplus\lim N_k$.*

PROOF. Let $z\in X$ with $z=m_k+n_k$, $m_k\in M_k$, $n_k\in N_k$. Since $K\|z\|=K\|m_k+n_k\|\geqq \|m_k\|$, $m_{k_j}\rightharpoonup m$ for some subsequence m_{k_j} of m_k. Since $m_{k_j}\in[\bigcup_{k\geqq n} M_k]$ for all but finitely many terms, and $[\bigcup_{k\geqq n} M_k]$ is weakly closed, it follows that $m\in\lim M_k$. Also we have $z-m_{k_j}=n_{k_j}\rightharpoonup z-m$. Since $P_{k_j}(z-m_{k_j})=0$, we also have $z-m\in\lim_{n\to\infty} N_n$. Thus $\lim M_k+\lim N_k=X$. To see the sum is direct, suppose $z\in\lim M_k\cap\lim N_k$. Then there exists $w_j\in\bigcup_{n\geqq 1}\bigcap_{k\geqq n} M_k$ and $v_j\in\bigcup_{n\geqq 1}\bigcap_{k\geqq n} N_k$ with $w_j\to z$ and $v_j\to z$. Now $w_j\in\bigcap_{k\geqq n_j} M_k$ and $v_j\in\bigcap_{k\geqq m_j} N_k$ for some m_j, n_j. Let $r_j=\max\{m_j, n_j\}$. Then $P_{r_j}(m_j-n_j)=m_j$ and $P_{r_j}(n_j)=0$. But $w_j-v_j\to 0$, so that $\|P_{r_j}(w_j-v_j)\|\leqq K\|w_j-v_j\|\to 0$ and hence $\|w_j\|\to 0$. Since $w_j\to z$ we must have $z=0$. Q.e.d.

THEOREM 2. *Let X be a reflexive Banach space, A_n, A operators on X. Suppose that $A_n\to A$ strongly and $A_n\in\xi(X)$. Let $M_n=\overline{R(A_n)}$. Suppose there exist projections P_n such that $P_n(X)=M_n$, $(I-P_n)(X)=N_n$, $\|P_n\|\leqq K$, and $M=\lim M_n$ and $N=\lim N_n$ exist with $M\subseteq\overline{R(A)}$. Then $A\in\xi(X)$.*

PROOF. By Theorem 1, M is complemented in X and $M\subseteq\overline{R(A)}$. Suppose now that $y\in R(A)$ i.e. $Ax=y$. It follows that $A_nx\to Ax=y$. But for each n, $A_nx\in M_n$, and thus $A_nx\in[\bigcup_{k\geqq m} M_k]$ for $m\leqq n$. Thus $Ax=y\in[\bigcup_{k\geqq m} M_k]$ for each m, so that $y\in\bigcap_n[\bigcup_{k\geqq n} M_k]=M$. Hence $A\in\xi(X)$ since $M=\overline{R(A)}$ is complemented. Q.e.d.

The assumptions of Theorem 2 appear quite strong. However, it is very difficult to get a much more general theorem.

EXAMPLE 1. Let A be a bounded below operator in l^p, $1<p<\infty$, $p\neq 2$, such that $R(A)$ is not complemented [16]. Let $\{e_i\}_{i=1}^{\infty}$ be the standard basis for l^p. Define $A_ne_i=Ae_i$ if $i\leqq n$, $A_ne_i=0$ if $i>n$. Then $A_n\to A$. Since $R(A_n)$ is finite dimensional it has a complement N_n. Since $R(A_n)$ is monotonically increasing and the codimension of $R(A_n)$ in $R(A_{n+1})$ is one, it is possible to inductively define the N_n so they are decreasing. That is, $N_n\supseteq N_{n+1}$. Define P_n by $P_nM_n=M_n$, $P_nN_n=0$. Then $A_n\to A$, $A_n\in\xi(X)$, $\lim M_n$, $\lim N_n$ exist, $\lim M_n\subseteq\overline{R(A)}$, and $A\notin\xi(K)$.

Even if one strengthens the convergence to uniform, some condition on the P_n is needed.

EXAMPLE 2. Take the same A, P_n of Example 1. Define $\hat{A}$ by $\hat{A}e_i=Ae_i/i$, and $\hat{A}_n e_i=A_n e_i/i$. Then $\hat{A}_n\Rightarrow\hat{A}$, $\hat{A}_n\in\zeta(X)$, $\lim M_n$, $\lim N_n$ exist and $\lim M_n\subseteq\overline{R(A)}$, but $\hat{A}\notin\zeta(K)$.

We can remove the assumption that the complements N_n have a limit, provided, of course, some additional assumptions hold.

DEFINITION 2. A sequence of bounded linear operators A_n converges weakly unifomly to the bounded operator A provided that $x^*(A_n y)=f_n(y)$ converges uniformly to $x^*(Ay)=f(y)$ on the unit ball of X for each $x^*\in X^*$. If A_n converges weakly uniformly to A, we write $A_n \overset{u}{\rightharpoonup} A$.

Note that uniform convergence implies weakly uniform convergence and weakly uniform convergence implies weak convergence. However, weakly uniform convergence is independent of strong convergence.

EXAMPLE 3. Let S be the unilateral shift on l^2. Then $S^n\overset{u}{\rightharpoonup}0$ but not $S^n\to0$. Also $S^{*n}\to0$ but S^{*n} does not converge weakly uniformly to zero.

For two sets Σ, Σ' in X, let $\varrho(\Sigma,\Sigma')=\inf\{\|\sigma-\sigma'\|:\ \sigma\in\Sigma,\ \sigma'\in\Sigma'\}$.

THEOREM 3. *Let X be uniformly convex Banach space. Suppose that $A_n\overset{u}{\rightharpoonup}A$, $R(A)\subseteq\overline{R(A_n)}$ and there exist norm one projections P_n onto the $\overline{R(A_n)}$. If for each $y\in\bigcap\overline{R(A_n)}$ there exists M_y independent of n such that $\varrho(\{A_n^{-1}(y)\},\{x\,\|x\|\leqq M_y\})\to0$ as $n\to\infty$, then $A\in\zeta(X)$ and $R(A)$ is closed.*

PROOF. We first show that $\bigcap\overline{R(A_n)}$ is complemented. Let $Q_k=\lim_{m\to\infty}(P_1P_2\dots P_k)^m$. Then Q_k is a norm one projection onto $\bigcap_{n=0}^{k}R(P_n)$ since $\|P_i\|=1$ and X is uniformly convex [4]. Note that $R(Q_k)\supseteq R(Q_{k+1})$. Let $S_0=Q_0$ and define S_k inductively by $S_k=Q_kS_{k-1}$ for $k\geqq1$. S_0 is a norm one projection and $R(S_0)=R(Q_0)$. It is an easy inductive argument that S_k is a norm one projection and $R(S_k)=R(Q_k)$. But $N(S_{k-1})\subseteq N(S_k)$. Thus $X=R(S_k)\oplus N(S_k)$, $\{R(S_k)\}$ is decreasing, $\{N(S_k)\}$ is increasing, and $\|S_k\|\leqq1$. Hence $\bigcap R(S_k)=\bigcap R(Q_k)=\bigcap R(P_n)=\bigcap\overline{R(A_n)}$ is complemented by Theorem 1. By assumption $R(A)\subseteq\bigcap\overline{R(A_n)}$.

It suffices then to show $\bigcap\overline{R(A_n)}\subseteq R(A)$. Suppose $y\in\overline{R(A_n)}$ for all n. Then there exists x_n with $A_nx_n\to y$ and $\|x_n\|\leqq M_y$ by assumption. Being uniformly convex, X is reflexive so there is a subsequence x_{n_j} such that $x_{n_j}\rightharpoonup x$. We shall show $Ax=y$ so that $y\in R(A)$. For $x^*\in X^*$ we have

$$|x^*(Ax-y)|\leqq|x^*(Ax-Ax_{n_j})|+|x^*(Ax_{n_j}-A_{n_j}x_{n_j})|+|x^*(A_{n_j}x_{n_j}-y)|.$$

But $|x^*(Ax-Ax_{n_j})|\to0$ since $x_{n_j}\rightharpoonup x$, $|x^*(Ax_{n_j}-A_{n_j}x_{n_j})|\to0$ since $A_{n_j}\overset{u}{\rightharpoonup}A$, and $|x^*(A_{n_j}x_{n_j}-y)|\to0$ since $A_nx_n-y\to0$. Thus $y=Ax$. Q.e.d.

The existence of the M_y in Theorem 3 is needed in some form.

EXAMPLE 4. Let $\hat{A}$ be as in Example 2. Since $\hat{A}$ is compact, there exist scalars $\lambda_n\to0$ such that $\lambda_n-\hat{A}$ is invertible. Let $\hat{A}_n=\lambda_n-\hat{A}$. Then $\hat{A}_n\Rightarrow\hat{A}$, $R(\hat{A})\subseteq\overline{R(\hat{A}_n)}=X$, and $P_n = I$ so that $\|P_n\| = 1$. Thus all conditions of Theorem 3 are met except for the existence of an M_y and $\hat{A} \notin \tilde{\zeta}(X)$.

Note that, in some sense, Theorem 2, considers A as a limit from below of $A_n \in \tilde{\zeta}(X)$ while Theorem 3 discusses limits from above.

3. Perturbations. While the results of Section 2 are of interest, the following result is probably more useful for the applications discussed in the Introduction. We shall give sufficient conditions that $A \in \tilde{\zeta}(X)$ implies $A + K \in \tilde{\zeta}(X)$.

First we need to recall two facts.

LEMMA 1. *If X is a Banach space and M a complemented subspace of X, then for every finite dimensional subspace $F \subseteqq X$, $M + F$ is complemented.*

THEOREM 4 (Nikol'ski). *A total sequence $\{x_n\}$ in a Banach space X is a basis for X if and only if there is a constant $M \geqq 1$ such that*

$$\left\| \sum_{n=1}^{m} a_i x_i \right\| \leqq M \left\| \sum_{n=1}^{m+p} a_i x_i \right\|$$

for arbitrary coefficients $a_1, \ldots, a_{m+p}$. The smallest such M is the basis constant for $\{x_n\}$.

A proof of Theorem 4 may be found in [12, p. 57].

THEOREM 5. *Let X be a Banach space with a basis $\{x_n\}$ satisfying $0 < l \leqq \|x_n\| \leqq \leqq L < \infty$. Let $A: X \to X$ be bounded below and in $\tilde{\zeta}(X)$. If $K: X \to X$ satisfies $\sum_{n=1}^{\infty} \|Kx_n\| < \infty$, then $A + K \in \tilde{\zeta}(X)$.*

PROOF. Suppose $m\|x\| \leqq \|Ax\|$. Since $ml \leqq \|Ax_n\| \leqq \|A\| L$, $y_n = \dfrac{x_n}{\|Ax_n\|}$ is again a base for X. Also since A is bounded below $\{Ay_n\}$ is a normalized basis for the complemented subspace $\overline{R(A)} = R(A)$. If M is the basis constant for $\{y_n\}$, then M_1, the basis constant for $\{Ay_n\}$, satisfies $M_1 \leqq m\|A\| M$. For each n the subspace $R_n = [\{Ay_n\colon k \geqq n\}]$ is complemented in $R(A)$ with projection P_n having $\|P_n\| \leqq \leqq 1 + M_1$. Now $\sum \|Ky_{n_\infty}\| \leqq \dfrac{1}{ml} \sum \|Kx_n\| < \infty$ so that we may choose N such that $k \geqq N$ implies that $\sum_{n=k}^{\infty} \|Ky_n\| < \dfrac{1}{8M_1(1+M_1)}$. Since R_N has a normalized base $\{Ay\}_{n=N}^{\infty}$ with basis constant $M_2 \leqq M_1$, and R_N is complemented in X, we observe that $[\{(A+K)y_n\}_{n=N}^{\infty}]$ is complemented in X since

$$\sum_{n=N}^{\infty} \|Ay_n - (A+k)y_n\| = \sum_{n=N}^{\infty} \|Ky_n\| < \frac{1}{8M_1(1+M_1)} \leqq \frac{1}{8M_1\|P_N\|} \leqq \frac{1}{8M_2\|P_N\|}$$

(cf. [9, p. 14]). But $\overline{R(A+K)} = F + [\{(A+K)y_n\colon n \geqq N\}]$ where F is finite dimensional. Thus $\overline{R(A+K)}$ is complemented in X and $A + K \in \tilde{\zeta}(X)$. Q.e.d.

EXAMPLE 5. Define K as $\hat{A}$ of Example 2, except that $Ke_i = \hat{A}e_i/i$. Then $\sum \|Ke_i\| < \infty$. But $0 \in \tilde{\zeta}(X)$ and $0 + K \notin \tilde{\zeta}(X)$. Thus some type of additional assumption, such as A bounded below, is needed in Theorem 4.

Note that if K is as defined in Example 5, then by Theorem 5 $A+K\in\tilde{\zeta}(X)$ for all $A\in\tilde{\zeta}(X)$ with A bounded below, even though $K\notin\tilde{\zeta}(X)$.

4. Applications and comments. For $X=l^p$, the standard unilateral shift S of finite multiplicity is in $\tilde{\zeta}(X)$ since $R(S)$ has finite codimension. The next result generalizes this fact to shift like operators of possibly infinite multiplicity.

THEOREM 6. *Suppose that X is a reflexive Banach space and that* $X=\sum_{i=0}^{\infty}\oplus M_i$ *where for any index set* $\sigma\subseteqq\{1,2,\ldots\}$, $\|\sum_{i\in\sigma}P_i\|\leqq M$, *where P_i is the projection onto M_i with nullspace* $\sum_{j\neq i}\oplus M_j$. *Suppose also that there exists a base $\{x_i\}$ for X, such that* $A\colon X\to X$ *satisfies* $Ax_i\in M_i$. *Then* $A\in\tilde{\zeta}(K)$.

PROOF. For each i, $\{Ax_i\}$ is complemented in M_i. Hence there exists a projection Q_i in M_i with range $[Ax_i]$ such that $\|Q_i\|\leqq 1$. Define $R_n=\sum_{i=0}^{n}Q_iP_i$, and $A_n=\sum_{i=0}^{n}Q_iP_iA$. Now R_n is a projection onto $R(A_n)$ and $\|R_n\|\leqq M$. But $\lim R_nX$ and $\lim(I-R_n)X$ exist since these sequences of subspaces are monotonically increasing and decreasing, respectively. Thus $A\in\tilde{\zeta}(X)$ by Theorem 2 since $A_n\to A$. Q.e.d.

COROLLARY 1. *Suppose* $X=l^p$, $1\leqq p<\infty$, *and* $\{e_k\}$ *is the standard basis for* X. *If there exists an increasing sequence* $\{n_k\}$ *of integers such that* $Ae_k\in[\{e_{n_k},\ldots,e_{n_{k+1}-1}\}]$, *then* $A\in\tilde{\zeta}(X)$.

Some classes of operators are always in $\tilde{\zeta}(X)$. For example, isometries of l^p, L^p for $1<p<\infty$ [14, p. 217] and scalar type operators [18] are always in $\tilde{\zeta}(X)$. However, spectral operators are not necessarily in $\tilde{\zeta}(X)$ for general X.

EXAMPLE 6. Let $X=l^p, 1<p<\infty, p\neq 2$. Define A as in Example 1. Define T on $X\oplus X$ by the operator matrix $\begin{bmatrix}0 & A\\ 0 & 0\end{bmatrix}$. Since $T^2=0$, T is nilpotent and hence spectral. To show $T\notin\tilde{\zeta}(X\oplus X)$ it suffices to show $R(T)=R(A)\oplus 0$ is not complemented in $X\oplus X$. Suppose $R(A)\oplus 0$ is complemented in $X\oplus X$. Let P be a bounded projection onto $R(A)\oplus 0$. Let i be the natural imbedding $X\to X\oplus 0$. Then $Q=i^{-1}Pi$ is a projection of X onto $R(A)$ which is a contradiction. Hence $R(A)\oplus 0$ is not complemented in $X\oplus X$.

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ÜBER DIE EINZIGKEIT DER TERNIONENALGEBRA UND LINKSALTERNATIVE ALGEBREN KLEINEN RANGES

Von

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Bei der Bestimmung der rangniedrigsten nichtkommutativen Algebren über einem (kommutativen) Körper K stößt man auf die Tatsache, daß es lediglich eine nichtkommutative assoziative K-Algebra mit Einselement vom Rang 3 gibt, nämlich die *Ternionenalgebra*

$$T_K = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \middle| \, \alpha, \beta, \gamma \in K \right\}$$

der zweireihigen (o. B. d. A. oberen) Dreiecksmatrizen über K. Dieser Sachverhalt scheint einerseits Folklore, andererseits weitgehend unbekannt zu sein, obwohl er sich für $K = \mathbf{R}$ im wesentlichen bereits bei Study [9], [10], [11] und Scheffers [8] im Zusammenhang mit Fragen aus der Theorie der Transformationsgruppen findet. Die Bezeichnung „Ternionen" tritt bei Beck [1], [2] auf. [1]

In § 1 dieser Arbeit verallgemeinern wir diese Aussage in Satz 1 für linksalternative (und damit auch für rechtsalternative) Algebren. Der Beweis ist elementar, wird aber bei Spezialisierung auf den wichtigen Fall assoziativer Algebren nicht einfacher. Wir geben daher für letztere einen kürzeren zweiten Beweis mit Hilfe des Satzes von Wedderburn—Artin und einfachsten Radikalaussagen. Die sich aus Satz 1 ergebenden Korollare, die insbesondere T_K als die rangniedrigste nichtkommutative linksalternative bzw. assoziative K-Algebra mit Einselement kennzeichnen, ermöglichen zusammen mit Überlegungen des eben genannten zweiten Beweises von Satz 1 die Angabe aller linksalternativen K-Algebren mit Einselement, deren Rang höchstens 3 ist (vgl. Satz 2).

Gemäß Satz 3 von § 2 lassen sich die Voraussetzungen aller Ergebnisse von § 1 (mit Ausnahme von Kor. 2) kaum wesentlich abschwächen. Zur Konstruktion der erforderlichen Beispiele verwenden wir zwei wohl auch allgemein nützliche Hilfssätze. Der erste gibt notwendige und hinreichende Bedingungen für die Linksalternativität bzw. Flexibilität einer beliebigen K-Algebra A mit Hilfe der Assoziatoren einer Basis von A. Der zweite beschreibt ein konstruktives Verfahren, welches diese Assoziatoren für K-Algebren endlichen Ranges n (jedenfalls für nicht zu große n) in Verallgemeinerung des Light'schen Assoziativitätstests für Gruppoide (vgl. [4]; § 1, 2) in übersichtlicher Weise effektiv zu berechnen gestattet.

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§ 1

Unter einer K-Algebra A verstehen wir einen Vektorraum über einem (kommutativen) Körper K mit einer bilinearen Multiplikation. Mit Hilfe des Assoziators $[x, y, z]=(xy)z-x(yz)$ definiert man A als *links-* bzw. *rechtsalternativ* bzw. *flexibel*, wenn

$$[x, x, y] = 0 \quad \text{bzw.} \quad [x, y, y] = 0 \quad \text{bzw.} \quad [x, y, x] = 0$$

für alle $x, y \in A$ gilt. Eine links- und rechtsalternative Algebra heißt *alternativ* und ist dann bekanntlich auch flexibel. Zur Unterscheidung von modultheoretischer und ringtheoretischer direkter Summe verwenden wir die Zeichen $\oplus$ und $\boxplus$.

SATZ 1. *Jede nichtkommutative linksalternative K-Algebra A mit Einselement e vom Rang 3 ist isomorph zur Ternionenalgebra T_K.*

BEWEIS. Für jede Basis $B=\{e, a_1, a_2\}$ von A mit $e \in B$ gilt

(1) $$a_i^2 = \alpha_{i0} e + \alpha_{ii} a_i \quad (i = 1, 2;\ \alpha_{ik} \in K),$$

da die lineare Unabhängigkeit von e, a_i, a_i^2 wegen $[a_i, a_i, a_i]=0$, also $a_i^2 a_i = a_i a_i^2$ die Kommutativität von A nach sich ziehen würde. Weiter sei

(2) $$a_i a_j = \beta_{i0} e + \beta_{i1} a_1 + \beta_{i2} a_2 \quad (i \neq j;\ \beta_{1k}, \beta_{2k} \in K).$$

Da für beliebige $\xi, \eta \in K$

$$\begin{aligned}(a_1+\xi e)(a_2+\eta e) &= \ldots + (\beta_{11}+\eta) a_1 + \quad \ldots \\ (a_2+\eta e)(a_1+\xi e) &= \ldots + \quad \ldots \quad + (\beta_{22}+\xi) a_2\end{aligned}$$

gilt, können wir die Basis $\{e, a_1, a_2\}$ so wählen, daß (2) mit $\beta_{11}=\beta_{22}=0$ erfüllt ist. Erneute Verwendung der Linksalternativität

$$\begin{aligned}0 = [a_i, a_i, a_j] &= (\alpha_{i0} e + \alpha_{ii} a_i) a_j - a_i (\beta_{i0} e + \beta_{ij} a_j) = \\ &= \alpha_{i0} a_j + \alpha_{ii} (\beta_{i0} e + \beta_{ij} a_j) - \beta_{i0} a_i - \beta_{ij} (\beta_{i0} e + \beta_{ij} a_j)\end{aligned}$$

für $i \neq j$ zeigt $\beta_{10}=\beta_{20}=0$ und

(3) $$\alpha_{i0} + \alpha_{ii} \beta_{ij} = \beta_{ij}^2.$$

Entsprechend betrachten wir wieder für $i \neq j$

$$0 = [a_i+a_j, a_i+a_j, a_i] = [a_i, a_j, a_i] + [a_j, a_i, a_i] =$$

$$= (\beta_{ij} a_j) a_i - a_i (\beta_{ji} a_i) + (\beta_{ji} a_i) a_i - a_j (\alpha_{i0} e + \alpha_{ii} a_i) = \beta_{ij} \beta_{ji} a_i - \alpha_{i0} a_j - \alpha_{ii} \beta_{ji} a_i.$$

Daraus folgt $\alpha_{10}=\alpha_{20}=0$ bei (1) und (3) sowie

(4) $$\alpha_{ii} \beta_{ji} = \beta_{ij} \beta_{ji}.$$

Da aus $\beta_{12}=\beta_{21}=0$ die Kommutativität von A folgen würde, dürfen wir $\beta_{12} \neq 0$ annehmen. Dann ergibt sich $\alpha_{11}=\beta_{12}$ aus (3) und $\alpha_{22}=\beta_{21}$ aus (4). Die damit erhaltenen Multiplikationsformeln

(1′) $$a_1^2 = \beta_{12} a_1 \qquad a_2^2 = \beta_{21} a_2$$

(2′) $$a_1 a_2 = \beta_{12} a_2 \quad a_2 a_1 = \beta_{21} a_1$$

zeigen unmittelbar, daß durch

$$e \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a_1 \to \begin{pmatrix} \beta_{12} & 0 \\ 0 & 0 \end{pmatrix}, \quad a_2 \to \begin{pmatrix} \beta_{21} & 1 \\ 0 & 0 \end{pmatrix}$$

der gewünschte Isomorphismus von A auf T_K definiert wird.

Zweiter Beweis von Satz 1 *im assoziativen Falle.* Das Radikal N von A ist das größte nilpotente Ideal von A und wegen $e \in A$ gilt $N \neq A$. Nach dem Satz von Wedderburn—Artin folgt aus Ranggründen

$$(5) \qquad A/N \cong \boxplus_{i=1}^{r} K_i \quad \text{mit } r \leqq 3 \text{ und (kommutativen) Körpern } K_i \supseteqq K.$$

Die Nichtkommutativität von A schließt also $N=0$, aber auch $\mathrm{Rg}\, N=2$ aus, denn aus letzterem folgte $A=Ke \oplus N$ und entweder

$$(6) \qquad N = Ka \oplus Ka^2 \quad \text{mit } a^3 = 0 \quad \text{falls} \quad N^2 \neq 0, \quad \text{oder}$$

$$(7) \qquad N = Ka \oplus Kb \quad \text{mit } a^2 = ab = ba = b^2 = 0 \quad \text{falls} \quad N^2 = 0.$$

Es gilt also $\mathrm{Rg}\, N=1$, d. h.

$$A = Ke \oplus Ka \oplus Kb \quad \text{mit} \quad N = Kb \quad \text{und} \quad b^2 = 0.$$

Dabei können wir $ab=0$ annehmen, da man für $ab=\gamma b \neq 0$ statt a das Basiselement $a-\gamma e$ verwenden kann. Aus der Nichtkommutativität von A folgt nun einmal

$$(8) \qquad a^2 = \lambda e + \alpha a \quad \text{mit} \quad \lambda = 0 \quad \text{wegen} \quad ab = 0,$$

also $\alpha \neq 0$ da $a \notin N$, zum anderen $ba = \beta b \neq 0$. Dabei gilt $\alpha = \beta$ gemäß

$$\beta^2 b = \beta ba = ba^2 = b\alpha a = \alpha\beta b,$$

und einen Isomorphismus von A auf T_K erhält man gemäß

$$e \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a \to \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}, \quad b \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

KOROLLAR 1. *Es sei A eine linksalternative K-Algebra mit Einselement e vom Rang n. Dann gilt:*

a) *Für $n \leqq 3$ ist A assoziativ.*
b) *Für $n \leqq 2$ ist A kommutativ.*
c) *Für $n=3$ ist A kommutativ oder $A \cong T_K$.*

BEWEIS. Es sei B eine e enthaltende Basis von A. Ist A kommutativ, so auch rechtsalternativ; wegen $n \leqq 3$ assoziieren dann also die Elemente von B untereinander, was die Assoziativität von A bedingt. Da für $n \leqq 2$ die Elemente von B kommutieren, ist dann auch A kommutativ und unser Korollar nur noch für nichtkommutative Algebren A mit $n=3$ fraglich. Für solche Algebren gilt jedoch $A \cong T_K$ nach Satz 1.

Da T_K Nullteiler enthält, ergibt sich insbesondere folgende Verallgemeinerung des bekannten Satzes über den Rang eines Schiefkörpers über seinem Zentrum:

KOROLLAR 2. *Jede nullteilerfreie nichtkommutative linksalternative K-Algebra mit Einselement hat mindestens den Rang* 4.

Schließlich ergibt sich aus unseren Überlegungen die Struktur aller möglichen linksalternativen K-Algebren mit Einselement, deren Rang höchstens 3 ist. Zur Vereinfachung geben wir nur diejenigen an, die sich nicht in die ringtheoretische direkte Summe echter Unteralgebren zerlegen lassen:

SATZ 2. *Jede unzerlegbare linksalternative K-Algebra A mit Einselement und* $\mathrm{Rg}\, A \leqq 3$ *ist bis auf Isomorphie entweder*

a) *ein Körper L über K mit* $1 \leqq [L\colon K] \leqq 3$, *oder*

b) *eine der Laguerre-Algebren* (*vgl.* [3])

1) $K[x]/(x^2)$ 2) $K[x]/(x^3)$ 3) $K[x, y]/(xy, x^2, y^2)$ *oder*

c) *die Ternionenalgebra* T_K.

BEWEIS. Die unter a) bis c) angegebenen K-Algebren haben ersichtlich die behaupteten Eigenschaften und sind paarweise nichtisomorph. Sei umgekehrt A eine unzerlegbare linksalternative K-Algebra mit Einselement e und $\mathrm{Rg}\, A \leqq 3$. Dann ist A nach Kor. 1 assoziativ, so daß wir die Überlegungen im zweiten Beweis von Satz 1 (mit „A unzerlegbar" statt „A nichtkommutativ" und $\mathrm{Rg}\, A \leqq 3$) aufgreifen können: Für $N=0$ ergibt sich $A \cong L$ wegen (5), also a). Für $\mathrm{Rg}\, N=2$ gilt $\mathrm{Rg}\, A=3$, und die Fälle (6) und (7) entsprechen ersichtlich b2) bzw. b3). Für $\mathrm{Rg}\, N=1$ und $\mathrm{Rg}\, A=3$ haben wir bei (8) ff. gezeigt, daß für eine nichtkommutative K-Algebra dieser Art $A \cong T_K$, also c) gilt. Im kommutativen Fall wird sich eine zerfallende Algebra A, also ein Widerspruch ergeben: Es gilt dann nämlich mit $ab=0$ auch $ba=0$, und statt (8) liefert der Ansatz

$$a^2 = \lambda e + \alpha a + \beta b \tag{8'}$$

zunächst wieder $\lambda=0$ wegen $ab=bb=0$ und dann

$$(a^2)^2 = \alpha^2 a^2 \quad \text{mit} \quad \alpha \neq 0 \quad \text{wegen} \quad a \notin N.$$

Also ist $\alpha^{-2}a^2$ ein von e verschiedenes Idempotent von A, was im kommutativen Fall zum Zerfallen (und zwar gemäß $A \cong K \boxplus K[x]/(x^2)$) führt. Es bleibt $\mathrm{Rg}\, N=1$ und $\mathrm{Rg}\, A=2$ zu betrachten, wofür sich $A=Ke \oplus Ka$ mit $a^2=0$, d. h. b1) ergibt.

§ 2

In den Sätzen 1 und 2 kann weder die Voraussetzung der Existenz eines Einselementes weggelassen, noch die Voraussetzung „linksalternativ" zu „flexibel" abgewandelt werden. Das gleiche gilt für die Aussagen a) und c) von Kor. 1. Diese Feststellungen entsprechen den Existenzaussagen im folgenden Satz 3.

Bezüglich den analogen Fragestellungen für die verbleibenden Behauptungen von § 1 bemerken wir zunächst zu Aussage b) von Kor. 1: Es gibt trivialerweise sogar assoziative K-Algebren vom Rang 2, die nicht kommutativ sind, während eine K-Algebra vom Rang 2 mit Einselement stets kommutativ ist. Dagegen folgt in Kor. 2 gerade die Existenz eines Einselementes aus den übrigen Voraussetzungen (vgl. [6], Satz 4b), und es hat sogar jede flexible nullteilerfreie nichtkommutative K-Algebra mindestens den Rang 4 (vgl. Satz 4 und [6], Satz 13).

SATZ 3. *Über jedem Körper K gibt es nichtkommutative und sogar unzerlegbare K-Algebren vom Rang 3;*

a) *die linksalternativ, aber nicht flexibel (also auch nicht rechtsalternativ) sind;*

b) *mit Einselement, die flexibel, aber weder links- noch rechtsalternativ sind.*

Dem Beweis dieses Satzes schicken wir zwei allgemeine Hilfsmittel zur Konstruktion solcher K-Algebren voraus:

HILFSSATZ 1. *Eine K-Algebra A mit einer Basis B beliebiger Mächtigkeit ist genau dann linksalternativ bzw. rechtsalternativ bzw. flexibel, wenn*

$$[x, x, z] = 0 \quad \textit{und} \quad [x, y, z]+[y, x, z] = 0 \quad \textit{bzw.} \tag{9}$$

$$[x, z, z] = 0 \quad \textit{und} \quad [x, y, z]+[x, z, y] = 0 \quad \textit{bzw.} \tag{10}$$

$$[x, y, x] = 0 \quad \textit{und} \quad [x, y, z]+[z, y, x] = 0 \tag{11}$$

jeweils für alle $x, y, z \in B$ gilt.

BEWEIS. Wegen der Ähnlichkeit der Schlüsse genügt es, die Behauptung für den linksalternativen Fall zu führen. Die Notwendigkeit von $[x, x, z]=0$ ist klar, und aus

$$[x, y, z]+[y, x, z] = [x+y, x+y, z] = 0$$

folgt, daß auch die zweite Bedingung (9) für linksalternative Algebren erfüllt ist. Wir setzen umgekehrt (9) voraus und betrachten beliebige Elemente

$$u = \sum_{i=1}^{r} \alpha_i a_i, \quad v = \sum_{i=1}^{r} \beta_i a_i$$

von A, die wir ohne Beschränkung der Allgemeinheit als Linearkombinationen der gleichen Elemente $a_1, \ldots, a_r \in B$ schreiben können. Dann ist

$$[u, u, v] = \left[\sum \alpha_i a_i, \sum \alpha_j a_j, \sum \beta_k a_k\right]$$

eine Summe von Ausdrücken der Form

$$\alpha_i \alpha_i \beta_k [a_i, a_i, a_k] \quad \text{und} \quad \alpha_i \alpha_j \beta_k ([a_i, a_j, a_k]+[a_j, a_i, a_k]),$$

die gemäß (9) einzeln gleich 0 sind.

HILFSSATZ 2. *Eine K-Algebra A vom Rang n sei durch eine Basis $B=\{a_1, \ldots, a_n\}$ und die Multiplikationstabelle*

$$\begin{array}{c|ccc} \cdot & a_1 & a_j & a_n \\ \hline a_1 & & \vdots & \\ & & \vdots & \\ a_i & \cdots\cdots & a_i a_j & \\ a_n & & & \end{array} \qquad \textit{mit} \quad a_i a_j = \sum_{k=1}^{n} \gamma_{ij}^k a_k \in \bigoplus_{k=1}^{n} K a_k \tag{12}$$

dieser Basiselemente gegeben. Zur Vereinfachung der Sprechweise betrachten wir eine solche Tabelle auch als $n\times n$-Matrix über A, mit den Zeilenvektoren $\mathfrak{a}_i = (a_i a_1, \dots, a_i a_n)$ und entsprechenden Spaltenvektoren $\mathfrak{a}^j$. Für jedes feste $a_m \in B$ lassen sich die Assoziatoren $[a_i, a_m, a_j]$ für alle $a_i, a_j \in B$ nach folgendem Verfahren gewinnen: Man bildet einmal eine Tabelle

(13)
$$\begin{array}{c|c|ccc}
a_m & & a_1 & a_j & a_n \\
\hline
a_1 & a_1 a_m & & \vdots & \\
a_i & a_i a_m & \cdots\cdots & (a_i a_m) a_j & \\
a_n & a_n a_m & & &
\end{array}
\quad \leftarrow \quad \sum_{k=1}^{n} \gamma_{im}^{k} \mathfrak{a}_k \quad \text{als Zeile}$$

indem man jeweils als i-ten Zeilenvektor die angegebene Linearkombination der Zeilenvektoren $\mathfrak{a}_k$ von (12) *einträgt. Das Element*

$$(a_i a_m) a_j = \sum_{k=1}^{n} \gamma_{im}^{k} a_k a_j$$

erscheint dann ersichtlich jeweils im Schnittpunkt der i-ten Zeile mit der j-ten Spalte Zum anderen bildet man analog eine Tabelle

(14)
$$\begin{array}{c|ccc}
 & a_1 & a_j & a_n \\
a_m & & & \\
\cline{2-4}
 & a_m a_1 & a_m a_j & a_m a_n \\
\hline
a_1 & & \vdots & \\
a_i & \cdots\cdots & a_i(a_m a_j) & \\
a_n & & &
\end{array}$$

$$\uparrow$$

$$\sum_{k=1}^{n} \gamma_{mj}^{k} \mathfrak{a}^k \quad \text{als Spalte}$$

aus Spaltenvektoren, die wie angegeben aus den Spaltenvektoren $\mathfrak{a}^k$ von (12) *linear kombiniert werden, im Einklang mit*

$$a_i(a_m a_j) = \sum_{k=1}^{n} \gamma_{mj}^{k} a_i a_k.$$

Der Vergleich von (13) *und* (14) *bzw. die Differenz dieser Matrizen liefert $(a_i a_m) a_j - a_i(a_m a_j) = [a_i, a_m, a_j]$ jeweils an der Stelle (i, j).*

Das Verfahren läuft also für jedes $a_m \in B$ als „mittleres Element" auf eine Tabellierung der Funktionen $f_m(a_i, a_j) = (a_i a_m) a_j$ und $g_m(a_i, a_j) = a_i(a_m a_j)$ für alle $a_i, a_j \in B$ mit Hilfe von (12) hinaus, und man hat sich nur davon zu überzeugen, daß dies bei (13) und (14) geschieht. Insgesamt sind also $2n$ Tabellen bzw. Matrizen aufzuschreiben und zu vergleichen, um alle Assoziatoren der Basiselemente zu erhalten; natürlich wird man dabei Elemente $a_m \in B$ weglassen, für die $[a_i, a_m, a_j] = 0$ ohnehin klar ist, wie etwa für das Einselement e. Wir verweisen auch auf die folgenden Beispiele, und bemerken insbesondere im Hinblick auf das erste:

Besonders einfach wird dieses Verfahren für monomiale Algebren (vgl. [12]), bei denen für eine geeignete Basis B für alle Paare (i, j) jeweils höchstens eine der Strukturkonstanten $\gamma_{i,j}^k$ von 0 verschieden ist, insbesondere also wenn B bezüglich (12) ein Gruppoid ist. In diesem Falle besteht (13) einfach aus skalaren Vielfachen gewisser Zeilen von (12) bzw. solchen Zeilen selbst, und entsprechend für (14) mit Spalten. Der Gruppoidfall entspricht dabei dem bereits in der Einleitung genannten Light'schen Assoziativitätstest, wenn man einfach nur nach der Übereinstimmung von (13) und (14) für jedes a_m des Gruppoids fragt, wofür übrigens nur jeweils eine dieser Tabellen hingeschrieben zu werden braucht.

Beispiel zu Satz 3, a) Für einen beliebigen Körper K konstruieren wir eine K-Algebra $A=Ka\oplus Kb\oplus Kc$ als „Gruppoid-Algebra", indem wir für die Basis $B=\{a, b, c\}$ eine Multiplikation gemäß

$$\text{(12')}\qquad \begin{array}{c|ccc} \cdot & a & b & c \\ \hline a & a & b & b \\ b & a & c & c \\ c & a & c & c \end{array}$$

definieren. Für a als mittleres Element entstehen die Tabellen

$$\text{(13')}\qquad \begin{array}{c|c|ccc} & a & a & b & c \\ \hline a & a & a & b & b \\ b & a & a & \underline{b} & \underline{b} \\ c & a & a & \underline{b} & \underline{b} \end{array} \qquad\qquad \text{(14')}\qquad \begin{array}{c|ccc} & a & b & c \\ \cline{2-4} a & a & b & b \\ \hline a & a & b & b \\ b & a & \underline{c} & \underline{c} \\ c & a & \underline{c} & \underline{c} \end{array}$$

aus (12') einfach durch Abschreiben der a-Zeile [3 mal] bzw. der a-Spalte [1 mal] und der b-Spalte [2 mal]. Gemäß Hilfssatz 2 gilt also für die Assoziatoren der Basiselemente

$$[x, a, z] = 0 \qquad \text{für} \quad x = a \quad \text{oder} \quad z = a$$

$$[x, a, z] = b-c \quad \text{für} \quad x, z \in \{b, c\}.$$

Da sich weiterhin die Elemente b und c bezüglich der Multiplikation völlig gleich verhalten, können wir die entsprechenden Tabellen zusammenfassen:

$$\text{(13')}\qquad \begin{array}{c|c|ccc} & b=c & a & b & c \\ \hline a & b & a & \underline{c} & \underline{c} \\ b & c & a & c & c \\ c & c & a & c & c \end{array} \qquad\qquad \text{(14')}\qquad \begin{array}{c|ccc} & a & b & c \\ \cline{2-4} b=c & a & c & c \\ \hline a & a & \underline{b} & \underline{b} \\ b & a & c & c \\ c & a & c & c \end{array}$$

Wir lesen ab, daß alle Assoziatoren der Basiselemente mit b oder c als mittlerem Element verschwinden, mit Ausnahme von

$$[a, y, z] = c-b \quad \text{für} \quad y, z \in \{b, c\}.$$

Man übersieht jetzt sofort, daß die Bedingung (9), nicht aber die Bedingung (11) von Hilfssatz 1 erfüllt ist. Damit haben wir eine nichtkommutative linksalternative K-Algebra A vom Rang 3 konstruiert, die jedoch nicht flexibel ist. Wir beweisen sogleich, daß diese Algebra unzerlegbar ist, indem wir die Annahme einer Zerlegung gemäß

$$A = U \boxplus V = Ku \boxplus V \tag{15}$$

(einander annullierender) Unteralgebren U und V mit $\operatorname{Rg} U=1$, $\operatorname{Rg} V=2$ zum Widerspruch führen. Ersichtlich müßte das Basiselement u von U im Zentrum von A liegen. Aus dem Ansatz $u=\alpha a+\beta b+\gamma c$ folgt aber

$$\alpha = 0 \quad \text{wegen} \quad bu = ub$$

und

$$\beta+\gamma = 0 \quad \text{wegen} \quad au = ua.$$

Statt $u=\beta b-\beta c$ können wir $u=b-c$ als Basiselement wählen. Die (15) entsprechende Zerlegung der Elemente a und b gemäß

$$a = \lambda_1 u+v_1, \quad b = \lambda_2 u+v_2 \quad \text{mit} \quad v_i \in V$$

führt wegen der aus (12′) ersichtlichen Annullatoreigenschaft von $u=b-c$ zu

$$a = a^2 = v_1^2 \in V \quad \text{und} \quad c = b^2 = v_2^2 \in V,$$

woraus sich auch $ac=b\in V$ im Widerspruch zu $\operatorname{Rg} V=2$ ergibt.

Beispiel zu Satz 3, b) Für einen beliebigen Körper K definieren wir eine K-Algebra $A=Ke\oplus Ka\oplus Kb$ durch die Multiplikationstabelle

(12″)

$\cdot$	e	a	b
e	e	a	b
a	a	$e+a$	$e+a$
b	b	$e+b$	$e+b$

und wenden wieder Hilfssatz 2 an. Das Einselement brauchen wir wegen $[x, e, z]=0$ nicht als mittleres Element zu betrachten, und für a ergibt sich

(13″)

	a	e	a	b
e	a	a	$e+a$	$e+a$
a	$e+a$	$e+a$	$e+2a$	$\underline{e+a+b}$
b	$e+b$	$e+b$	$\underline{e+a+b}$	$e+2b$

(14″)

a	e	a	b
	a	$e+a$	$e+a$
e	a	$e+a$	$e+a$
a	$e+a$	$e+2a$	$\underline{e+2a}$
b	$e+b$	$\underline{e+2b}$	$e+2b$,

also das Verschwinden aller Assoziatoren der Basiselemente der Form $[x, a, z]$ außer

$$[a, a, b] = b-a \quad \text{und} \quad [b, a, a] = a-b.$$

Da die Vertauschung von a und b ersichtlich einen Automorphismus von A bewirkt, folgt daraus das Verschwinden aller $[x, b, z]$ mit Ausnahme von

$$[b, b, a] = a-b \quad \text{und} \quad [a, b, b] = b-a.$$

Damit ist diese nichtkommutative K-Algebra A mit Einselement und $\mathrm{Rg}\, A=3$ nach Hilfssatz 1 flexibel, aber weder links- noch rechtsalternativ. Die Unzerlegbarkeit von A gemäß $A=U_1 \boxplus U_2$ mit $\mathrm{Rg}\, U_i \leqq 2$ folgt hier einfach daraus, daß die entsprechende Zerlegung $e=e_1+e_2$ des Einselementes e von A ersichtlich Einselemente e_i von U_i liefert, so daß diese Unteralgebren von A und damit A selbst kommutativ sein müßten.

Satz 4. *Eine K-Algebra A von Rang 2 ist genau dann flexibel, wenn A kommutativ oder antikommutativ ist oder eine Basis $B=\{a, b\}$ von A mit der Multiplikationstafel*

$$(12''') \qquad \begin{array}{c|cc} & a & b \\ \hline a & 0 & \lambda a \\ b & \mu a & b \end{array} \qquad \textit{mit} \quad \lambda, \mu \in K;\ \lambda+\mu=1$$

existiert. In letzterem Falle ist A genau für $\lambda=\pm\mu$ (anti)kommutativ und genau für $\lambda\mu=0$ assoziativ, sonst aber weder links- noch rechtsalternativ. Umgekehrt definiert (12''') mit beliebigen $\lambda, \mu \in K$ stets eine solche flexible K-Algebra A vom Rang 2, und für Körper K mit mindestens 4 Elementen lassen sich λ und $\mu=1-\lambda$ stets so wählen, daß

$$\lambda \neq \pm\mu \quad \textit{und} \quad \lambda\mu \neq 0 \quad \textit{gilt.}$$

Beweis. Die Vollständigkeit unserer Aufzählung ergibt sich durch Nachprüfung aller wesentlich verschiedenen Möglichkeiten, was wir hier nicht ausführen wollen. Zum Nachweis, daß (12''') eine K-Algebra mit den angegebenen Eigenschaften definiert, bilden wir gemäß Hilfssatz 2

$$(13''') \qquad \begin{array}{c|c|cc} & a & a & b \\ \hline a & 0 & 0 & 0 \\ b & \mu a & 0 & \mu\lambda a \end{array} \qquad\qquad (14''') \qquad \begin{array}{c|cc} & & a & b \\ a & & 0 & \lambda a \\ \hline a & & 0 & 0 \\ b & & 0 & \lambda\mu a \end{array}$$

$$\begin{array}{c|c|cc} & b & a & b \\ \hline a & \lambda a & 0 & \underline{\lambda^2 a} \\ b & b & \underline{\mu a} & b \end{array} \qquad\qquad \begin{array}{c|cc} & & a & b \\ b & & \mu a & b \\ \hline a & & 0 & \underline{\lambda a} \\ b & & \underline{\mu^2 a} & b \end{array}$$

Daraus folgt, daß alle Assoziatoren der Basiselemente mit Ausnahme von

$$[a, b, b] = \lambda^2 a - \lambda a = -\lambda\mu a, \quad [b, b, a] = \mu a - \mu^2 a = \lambda\mu a$$

verschwinden. Also ist A nach Hilfssatz 1 flexibel, aber weder links- noch rechtsalternativ für $\lambda\mu \neq 0$.

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TWO THEOREMS ABOUT TOPOLOGIES ON COUNTABLY GENERATED $Op*$-ALGEBRAS

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0. Introduction

The present paper deals with the topologization of unbounded operator algebras ($Op*$-algebras) in Hilbert space. We consider two possible topologies, the so-called uniform topology $\tau_{\mathscr{D}}$ introduced in [2] and the strong operator topology $\sigma^{\mathscr{D}}$. We characterize the countably generated [closed] $Op*$-algebras $\mathscr{A}$ for which $\tau_{\mathscr{D}}$ [resp. $\sigma^{\mathscr{D}}$] agrees with the strongest locally convex topology on $\mathscr{A}$. Our main theorems contain some known result for concrete $Op*$-algebras ([2], [3], [4], [6]).

In [5], theorem 1 was used in proving that for certain $Op*$-algebras $\mathscr{A}$ (for example, the $Op*$-algebra of all differential operators with polynomial coefficients on the Schwartz space $\mathscr{S}(R_n)$) *all* linear functionals f on $\mathscr{A}$ are trace functionals, i.e. they can be given by $f(a)=\operatorname{Tr} ta$, $a\in\mathscr{A}$, t an appropriate nuclear operator.

1. Definitions and notations

First we repeat some basic definitions and facts about unbounded operator algebras from [2]. Let $\mathscr{D}$ be a dense linear subspace of a Hilbert space $\mathscr{H}$. An *$Op*$-algebra* $\mathscr{A}$ on $\mathscr{D}$ is a $*$-algebra of unbounded operators leaving the domain $\mathscr{D}$ invariant. We assume that the identity map is in $\mathscr{A}$ and denote it by 1. The *graph topology* $t_{\mathscr{A}}$ on $\mathscr{D}$ is the locally convex topology defined by the seminorms $\|\varphi\|_a:=\|a\varphi\|$, $\varphi\in\mathscr{D}$, $a\in\mathscr{A}$. For each bounded subset $\mathfrak{M}$ of $\mathscr{D}[t_{\mathscr{A}}]$ we put $p_{\mathfrak{M}}(a)=$ $=\sup\limits_{\varphi,\psi\in\mathfrak{M}}|\langle a\varphi,\psi\rangle|$. The *uniform topology* $\tau_{\mathscr{D}}$ on $\mathscr{A}$ is generated by the family $\{p_{\mathfrak{M}}\}$ of these seminorms. $\mathscr{A}[\tau_{\mathscr{D}}]$ is always a topological $*$-algebra. The *strong operator topology* $\sigma^{\mathscr{D}}$ on $\mathscr{A}$ is given by the seminorms $\|a\|_\varphi:=\|a\varphi\|$, $\varphi\in\mathscr{D}$, $a\in\mathscr{A}$.

Let $\mathscr{D}(\mathscr{A}):=\bigcap\limits_{a\in\mathscr{A}}\mathscr{D}(\bar a)$. The operators $\underline{a}:=\bar a\restriction\underline{\mathscr{D}}$ form an $Op*$-algebra $\underline{\mathscr{A}}$ on $\underline{\mathscr{D}}=\underline{\mathscr{D}}(\mathscr{A})$ which will be called the closed extension of $\mathscr{A}$. $\mathscr{A}$ is said to be *closed* if $\mathscr{A}=\underline{\mathscr{A}}$, i.e. $\mathscr{D}=\underline{\mathscr{D}}(\mathscr{A})$. An Op-$*$ algebra $\mathscr{A}$ is closed on $\mathscr{D}$ if and only if the space $\mathscr{D}[t_{\mathscr{A}}]$ is complete.

Furthermore we use the following notations throughout the paper (adapted from [1]):

$$\mathscr{N}_x:=\{a\in\mathscr{A}\colon |\langle a\varphi,\varphi\rangle|\leqq C_{a,x}\|x\varphi\|^2\,\forall\,\varphi\in\mathscr{D}\},$$

$$\mathscr{M}_x:=\{a\in\mathscr{A}\colon \|a\varphi\|\leqq C_{a,x}\|x\varphi\|\,\forall\,\varphi\in\mathscr{D}\},$$

$$\varrho_x(a):=\sup_{\varphi\in\mathscr{D}}\frac{|\langle a\varphi,\varphi\rangle|}{\|x\varphi\|^2},\quad \lambda_x(a):=\sup_{\varphi\in\mathscr{D}}\frac{\|a\varphi\|}{\|x\varphi\|}\quad\text{for}\quad a,x\in\mathscr{A}.$$

Here we make the convention that $\frac{C}{0}=+\infty$ for $C>0$ and $\frac{0}{0}=0$. Clearly, $\mathcal{N}_x$ and $\mathcal{M}_x$ are vector spaces.

By τ_{st} we always denote the strongest locally convex topology on $\mathcal{A}$.

3. The results

THEOREM 1. *For each countably generated $Op*$-algebra $\mathcal{A}$ on $\mathcal{D}$ the following are equivalent:*

(1.1) *For all operators $x\in\mathcal{A}$ the vector space $\mathcal{N}_x$ is finite dimensional.*

(1.2) *There are operators $x_n\in\mathcal{A}$, $n\in\mathbf{N}$, such that $\mathcal{A}=\bigcup_{n\in\mathbf{N}}\mathcal{N}_{x_n}$ and the vector spaces $\mathcal{N}_{x_n}$ are finite dimensional.*

(1.3) $\tau_{\mathcal{D}}=\tau_{st}$.

THEOREM 2. *Let $\mathcal{A}$ be a countably generated $Op*$-algebra on $\mathcal{D}$. Consider the following conditions:*

(2.1) *For all operators $x\in\mathcal{A}$ the vector space $\mathcal{M}_x$ is finite dimensional.*

(2.2) *There are operators $x_n\in\mathcal{A}$, $n\in\mathbf{N}$, such that $\mathcal{A}=\bigcup_{n\in\mathbf{N}}\mathcal{M}_{x_n}$ and the vector spaces $\mathcal{M}_{x_n}$ are finite dimensional.*

(2.3) $\sigma^{\mathcal{D}}=\tau_{st}$.

Then we have (2.3)$\to$(2.2)$\leftrightarrow$(2.1). *If $\mathcal{A}$ is a closed $Op*$-algebra on $\mathcal{D}$, then all three conditions are equivalent.*

The proofs of Theorems 1 and 2 will be given in Sections 4 and 5. Here we note a corollary only.

COROLLARY 1. *If $\mathcal{A}$ is a countably generated $Op*$-algebra on $\mathcal{D}$ and if $\tau_{\mathcal{D}}=\tau_s^t$ on $\mathcal{A}$, then $\sigma^{\underline{\mathcal{D}}}=\tau_{st}$ on $\underline{\mathcal{A}}$.*

PROOF. Let $x\in\underline{\mathcal{A}}$. Since $\tau_{\mathcal{D}}=\tau_{st}$, Theorem 1 implies that the space $\mathcal{N}_{x^+x+1}$ is finite dimensional. Moreover, $\mathcal{M}_x\subseteqq\mathcal{N}_{x^+x+1}$ by the Cauchy—Schwarz inequality. Therefore, $\mathcal{M}_x$ is finite dimensional. Since all operators $a\in\mathcal{A}$ are $t_{\mathcal{A}}$-continuous on $\mathcal{D}$ and $\mathcal{D}$ is $t_{\mathcal{A}}$-dense in $\underline{\mathcal{D}}$, it is clear that $a\in\mathcal{M}_x$ if and only if $\underline{a}\in\mathcal{M}_{\underline{x}}$. Hence, $\mathcal{M}_{\underline{x}}$ is a finite dimensional vector space. Thus, condition (2.1) is fulfilled and we have $\sigma^{\underline{\mathcal{D}}}=\tau_{st}$.

REMARK. In [1], $Op*$-algebras satisfying condition (1.1) are called hyperfinite.

4. Some examples

In this section we mention some examples of $Op*$-algebras satisfying the assumptions of our theorems.

EXAMPLE 1. Let $\mathcal{A}_1$ be the $Op*$-algebra $\mathcal{P}(T)$ of all polynomials in a symmetric linear operator T on a dense invariant domain $\mathcal{D}_1$ in a Hilbert space. Suppose that the operator T is not bounded on $\mathcal{D}_1$.

EXAMPLE 2. Denote by $\mathscr{A}_2$ the $Op*$-algebra generated by the position and momentum operators $q_j=t_j, p_j=i\dfrac{\partial}{\partial t_j}, j=1, \dots, n,$ on the domain $\mathscr{D}_2 := C_0^\infty(\mathbf{R}_n)$. In other words, $\mathscr{A}_2$ is the $*$-algebra of all differential operators with polynomial coefficients.

Now we pass to a more general class of examples which give a greater variety of $Op*$-algebras fulfilling our conditions.

EXAMPLE 3. Let G be a Lie group and $\mathscr{E}(G)$ be the universal enveloping algebra of the Lie algebra of G. Suppose $\mathscr{U}$ is a (fixed) neighbourhood of the unit element in G. If we realize $\mathscr{E}(G)$ as an algebra of left invariant differential operators acting on the Lie group G with the domain $\mathscr{D}_3 := C_0^\infty(\mathscr{U})$ in the Hilbert space $L_2(\mathscr{U}, \mu)$, μ the right Haar measure of G, then we obtain an $Op*$-algebra $\mathscr{A}_3$ depending on G.

With this notations we have the following theorem.

THEOREM 3. *The uniform topologies on the $Op*$-algebras $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ and the strong operator topologies on $\underline{\mathscr{A}}_1, \underline{\mathscr{A}}_2, \underline{\mathscr{A}}_3$ coincide with the strongest locally convex topologies τ_{st} on $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ and $\underline{\mathscr{A}}_1, \underline{\mathscr{A}}_2, \underline{\mathscr{A}}_3$, resp.*

REMARKS. 1. Most of these results are already known. For $\mathscr{A}_1$ both assertions were first proved in [3] (for the uniform topology partial results were obtained in [2]). In the case $\mathscr{A}_2$ both statements are shown in [6]. For $\mathscr{A}_3$ the assertion concerning the uniform topology was proved in [4]. The methods applied in the proofs for $\mathscr{A}_2$ and $\mathscr{A}_3$ in [6] and [4] are different from the method used in the present paper. Some basic arguments of the proofs are drawn from the proofs of Theorems B and C in [3].

2. By considering unitary representations of Lie groups (more precisely, the associated representations of the enveloping algebras), algebras of differential operators (for example, the $Op*$-algebra generated by $a=t^{-1}$ and $p^2=-\dfrac{d^2}{dt^2}$ on $\mathscr{D}=C_0^\infty(0, 1)$), sequence spaces etc., it is not difficult to construct further examples satisfying the conditions of Theorems 1 and 2.

LEMMA 1. *Suppose that $\mathscr{A}$ is an $Op*$-algebra on $\mathscr{D}$, $a, x\in\mathscr{A}$ and $a\in\mathscr{N}_x$. Then there is a constant $K_{a,x}$ such that*

$$\|a\varphi\|^2 \leqq K_{a,x}\|x\varphi\|\,\|xa\varphi\|, \quad \forall\,\varphi\in\mathscr{D}. \tag{1}$$

PROOF. Let $\mathscr{U}_x := \{\varphi\in\mathscr{D}: \|x\varphi\|\leqq 1\}$. Since $|\langle a\varphi, \varphi\rangle| \leqq C_{a,x}\|x\varphi\|^2\,\forall\,\varphi\in\mathscr{D}$ by $a\in\mathscr{N}_x$, we have $\sup\limits_{\varphi\in\mathscr{U}_x}|\langle a\varphi, \varphi\rangle|\leqq C_{a,x}$. Using $\langle a\varphi, \psi\rangle = 1/4\,\{\langle a(\varphi+\psi), \varphi+\psi\rangle - \langle a(\varphi-\psi), \varphi-\psi\rangle - i\langle a(\varphi+i\psi), \varphi+i\psi\rangle + i\langle a(\varphi-i\psi), \varphi-i\psi\rangle\}$ it follows $\sup\limits_{\varphi,\psi\in\mathscr{U}_x}|\langle a\varphi, \psi\rangle|\leqq 4C_{a,x}$ because the elements $1/2(\varphi+\psi), \dots, 1/2(\varphi-i\psi)$ are in the absolutely convex set $\mathscr{U}_x$. Hence, we get $|\langle a\varphi, \psi\rangle|\leqq 4C_{a,x}\|x\varphi\|\,\|x\psi\|\,\forall\,\varphi, \psi\in\mathscr{D}$. Putting $\psi=a\varphi$, this gives (1).

Now let us turn to the proof of Theorem 3. By Corollary 1, we only have to prove the assertions about the uniform topologies. In view of Theorem 1, it is sufficient to show that condition (1.1) is fulfilled for $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$, that is, the vector spaces $\mathscr{N}_x$ are finite dimensional for all operators x of the $Op*$-algebras $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$.

Here we only carry out the proof of this fact in the case of $\mathcal{A}_2$. For $\mathcal{A}_1$ and $\mathcal{A}_3$ condition (1.1) could be verified by repeating parts of the arguments used in [3] and [4].

Proof for the Op$*$-algebra $\mathcal{A}_2$. For simplicity in notation we restrict us to the case $n=1$. The subset $\{y_{ij}:=q^i p^j,\ i\in\mathbf{N},\ j\in\mathbf{N}\}$ is a Hamel base for $\mathcal{A}_2$. Take a fixed vector $\varphi\in\mathcal{D}_2$, $\varphi\neq 0$, with $\operatorname{supp}\varphi\subseteq[0,1]$ and put $\varphi_{\alpha,\beta}(t):=\beta\varphi(\beta(t-\alpha))$ for $\alpha,\beta\in\mathbf{R}_1$. A simple calculation leads to

$$\alpha^i\beta^j\|p^j\varphi\|\leqq\|y_{ij}\varphi_{\alpha,\beta}\|\leqq(\alpha+1)^i\beta^j\|p^j\varphi\| \tag{2}$$

where $\|p^j\varphi\|\neq 0\,\forall j\in\mathbf{N}$. Let $a=\sum_{i,j}\alpha_{ij}y_{ij}$, $x=\sum_{i,j}\beta_{ij}y_{ij}$ and $xa=\sum_{i,j}\gamma_{ij}y_{ij}$. The degree of an element a is defined by $d(a):=\operatorname{Max}\{i+j\colon \alpha_{ij}\neq 0\}$. Let (k,l) be the lexicographic largest tupel for which this maximum will be attained, i.e. $k=\operatorname{Max}\{i\colon \alpha_{ij}\neq 0 \text{ for } j=d(a)-i\geqq 0\}$ and $l=d(a)-j$. Denote by (r,s) the corresponding tupel for the element x.

Now we show that $a\in\mathcal{N}_x$ implies $d(a)\leqq 2d(x)+1$. In particular, this means that $\mathcal{N}_x$ is a finite dimensional vector space. We assume that $d(a)\geqq 1$ and $d(x)\geqq 1$ (otherwise the assertion is trivial).

Let $\beta=\alpha^{1-1/(k+l+r+s)}$. Then

$$\alpha^i\beta^j=\alpha^{i+j-j/(k+l+r+s)}\,\forall\, i,j\in\mathbf{N}. \tag{3}$$

We write, $f(\alpha)=\mathcal{O}(\alpha^m)$, $m\in\mathbf{R}_1$, for a function $f(\alpha)$ if $|f(\alpha)\alpha^{-m'}|$ is bounded for sufficiently large α if and only if $m'\geqq m$. m is called the order of the function f with respect to α. From (2) and (3) it follows that $\|y_{kl}\varphi_{\alpha,\beta}\|=\mathcal{O}(\alpha^{k+1-j/(k+l+r+s)})$ and for all other elements y_{ij} with $\alpha_{ij}\neq 0$ the functions $\|y_{ij}\varphi_{\alpha,\beta}\|$ have smaller orders with respect to α. By the triangle inequality, this implies

$$\|a\varphi_{\alpha,\beta}\|^2=\mathcal{O}(\alpha^{2[k+l-1/(k+l+r+s)]}). \tag{4}$$

Further we have

$$\|x\varphi_{\alpha,\beta}\|=\mathcal{O}(\alpha^{r+s-s/(k+l+r+s)}). \tag{5}$$

By the commutation rules it is clear that $\gamma_{ij}=0$ for $j>l+s$ and $\gamma_{k+r,l+s}\equiv\equiv\alpha_{kl}\beta_{rs}\neq 0$. Hence, if there is an $i\in\mathbf{N}$ such that $\gamma_{ij}\neq 0$, then $j/(k+l+r+s)<1$. Using these two facts, we get

$$\|xa\varphi_{\alpha,\beta}\|=\mathcal{O}(\alpha^{k+l+r+s-(l+s)/(k+l+r+s)}). \tag{6}$$

Putting (4), (5) and (6) into (1), it follows

$$2\left(k+l-\frac{1}{k+l+r+s}\right)\leqq r+s-\frac{s}{k+l+r+s}+k+l+r+s-\frac{l+s}{k+l+r+s}.$$

Therefore, $d(a)=k+l\leqq 2r+2s+1=2d(x)+1$ which finishes the proof.

4. Proof of Theorem 1

(1.1)→(1.2): Trivial.

(1.2)→(1.1): Let $x\in\mathscr{A}$. Since $\mathscr{A}=\bigcup_{n\in\mathbf{N}}\mathscr{N}_{x_n}$, there exists a number $n\in\mathbf{N}$ such that $x^+x\in\mathscr{N}_{x_n}$. This implies $\mathscr{N}_x\subseteqq\mathscr{N}_{x_n}$. Hence $\mathscr{N}_x$ is finite dimensional.

(1.3)→(1.1): We suppose that $\tau_{\mathscr{D}}=\tau_{\mathrm{st}}$. Let us assume that (1.1) is not true, i.e. there is an element $x\in\mathscr{A}$ such that $\mathscr{N}_x$ contains an infinite set $\{a_n, n\in\mathbf{N}\}$ of linear independent operators $a_n\in\mathscr{A}$. We supply elements $b_m\in\mathscr{A}$ such that the system $\{a_n, b_m\}$ is a Hamel base of $\mathscr{A}$. Without restriction of generality we may assume that $\varrho_x(a_n)\leqq 1/4\ \forall n\in\mathbf{N}$. Then $|\langle a\varphi,\psi\rangle|\leqq\|x\varphi\|\,\|x\psi\|\,\forall\varphi,\psi\in\mathscr{D}$ by polarization (cf. lemma 5 in Section 3). For each positive sequence $\gamma=\{\gamma_n\}$ we define the seminorm $p_\gamma(a)=\sum_n\gamma_n|\alpha_n|$ for $a=\sum_n\alpha_n a_n+\sum_m\beta_m b_m$. Since $\tau_{\mathscr{D}}=\tau_{\mathrm{st}}$, there is a bounded subset $\mathfrak{M}$ of $\mathscr{D}[t_{\mathscr{A}}]$ such that $p_\gamma(a)\leqq p_{\mathfrak{M}}(a)\quad\forall a\in\mathscr{A}$. Putting $a=\alpha_1a_1+\ldots+\alpha_ka_k$ we get

$$p_\gamma(a)\equiv\sum_n\gamma_n|\alpha_n|\leqq p_{\mathfrak{M}}(a)\equiv\sup_{\varphi,\psi\in\mathfrak{M}}|\langle(\alpha_1a_1+\ldots+\alpha_ka_k)\varphi,\psi\rangle|\leqq$$

$$\leqq\sum_n|\alpha_n|(\sup_{\varphi,\psi\in\mathfrak{M}}\|x\varphi\|\,\|x\psi\|)=C\sum_n|\alpha_n|$$

whereby $C=\sup_{\varphi,\psi\in\mathfrak{M}}\|x\varphi\|\,\|x\psi\|<+\infty$. Since $\alpha_1,\ldots,\alpha_k,\ldots$ are arbitrary complex numbers, this is a contradiction if $\sup_{n\in\mathbf{N}}\gamma_n=+\infty$.

Now we turn to the main part in the proof of Theorem 1.

(1.1)→(1.3): First we note a simple lemma. We shall need it only for finite dimensional Hilbert spaces $\mathscr{H}_1$.

LEMMA 2. *Let $\mathscr{H}_1$ be a Hilbert subspace of $\mathscr{H}$ with $\mathscr{H}_1\subseteqq\mathscr{D}$. Let P_1 be the orthogonal projection on $\mathscr{H}_1$ and $\mathscr{D}_1:=(1-P_1)\mathscr{D}\equiv\mathscr{D}\ominus\mathscr{H}_1$. Suppose $\mathscr{A}$ is an $Op*$-algebra on $\mathscr{D}$, $a, x\in\mathscr{A}$ and $\|\varphi\|\leqq\|x\varphi\|$ for all $\varphi\in\mathscr{D}$.*

If

$$\varrho_x(a)\equiv\sup_{\varphi\in\mathscr{D}}\frac{|\langle a\varphi,\varphi\rangle|}{\|x\varphi\|^2}=+\infty,$$

then

$$\sup_{\psi\in\mathscr{D}_1}\frac{|\langle a\psi,\psi\rangle|}{\|x\psi\|^2}=+\infty.$$

PROOF. Since the operators $a, a^+, x\in\mathscr{A}$ have dense defined adjoint operators in $\mathscr{H}$, their restrictions to $\mathscr{H}_1$ are closed and hence bounded by the closed graph theorem. Thus $\|a\eta\|\leqq C\|\eta\|$, $\|a^+\eta\|\leqq C\|\eta\|$, $\|x\eta\|\leqq C\|\eta\|\ \forall\eta\in\mathscr{H}_1$. Let us assume that $\sup_{\psi\in\mathscr{D}_1}\frac{|\langle a\psi,\psi\rangle|}{\|x\psi\|^2}=C_1<+\infty$, i.e. $|\langle a\psi,\psi\rangle|\leqq C_1\|x\psi\|^2\,\forall\psi\in\mathscr{D}_1$. For each $\varphi\in\mathscr{D}$, $\varphi=\psi+\eta$, $\psi\in\mathscr{D}_1$, $\eta\in\mathscr{H}_1$, we get $|\langle a\varphi,\varphi\rangle|=|\langle a(\psi+\eta),\psi+\eta\rangle|\leqq|\langle a\psi,\psi\rangle|+|\langle a\eta,\psi\rangle|+$ $+|\langle\psi,a^+\eta\rangle|+|\langle a\eta,\eta\rangle|\leqq C_1\|x\psi\|^2+2C\|\eta\|\,\|\psi\|+C\|\eta\|^2\leqq C_1\|x(\varphi-\eta)\|^2+2C\|\varphi\|^2+$ $+C\|\varphi\|^2\leqq C_1(\|x\varphi\|+C\|\eta\|)^2+3C\|\varphi\|^2\leqq[C_1(1+C)^2+3C]\,\|x\varphi\|^2$ because $\|\varphi\|\leqq\|x\varphi\|$. Therefore $\varrho_x(a)<+\infty$ which is a contradiction.

Now suppose that condition (1.1) is fulfilled. To prove that $\tau_{\mathscr{D}}=\tau_{st}$, we need some preparations and notations. Let us take a sequence $\{x_n, n\in\mathbf{N}\}$ of operators $x_n\in\mathscr{A}$ such that

(i) $\|\varphi\|\leqq\|x_n\varphi\|\leqq\|x_{n+1}\varphi\|\ \forall\varphi\in\mathscr{D},\ n\in\mathbf{N}$,

(ii) $\mathscr{N}_{x_n}\subseteqq\mathscr{N}_{x_{n+1}}$ and (iii) $\mathscr{A}=\bigcup_{n\in\mathbf{N}}\mathscr{N}_{x_n}$.

It is very easy to see that such a sequence exists. The vector space $\mathscr{A}$ has a countable Hamel basis $\{y_n, n\in\mathbf{N}\}$. Let $z_n=1+y_1^+y_1+\ldots+y_n^+y_n$. Then $y_i\in\mathscr{N}_{z_n}$ for $i\leqq n$. Take $x_1=z_1$. Because $\mathscr{N}_{x_1}$ is finite dimensional, there is a number, hence a smallest number $n_2\in\mathbf{N}$ such that $y_{n_2}\notin\mathscr{N}_{x_1}$. Putting $x_2=x_1^2+z_{n_2}^2+1$ we have

$$\mathscr{N}_{z_{n_2}}\cup\mathscr{N}_{x_1}\subseteqq\mathscr{N}_{x_2},\quad \mathscr{N}_{x_1}\neq\mathscr{N}_{x_2}\quad\text{and}\quad y_i\in\mathscr{N}_{x_2}\quad\forall i=1,\ldots,n_2.$$

Continuing this procedure, we get a sequence $\{x_n\}$ with desired properties.

Since each vector space $\mathscr{N}_{x_n}$, $n\geqq 2$, is finite dimensional, $\mathscr{N}_{x_n}$ can be decomposed as a direct sum of $\mathscr{N}_{x_{n-1}}$ and a certain vector space $\mathscr{A}_n\subseteqq\mathscr{N}_{x_n}$. Let $\mathscr{A}_1=\mathscr{N}_{x_1}$. Then $\mathscr{A}=\sum_n\mathscr{A}_n$ (direct sum of vector spaces). Let d_n be the dimension of $\mathscr{A}_n$ and let $a_1^n, a_2^n, \ldots, a_{d_n}^n$ be a basis of $\mathscr{A}_n$. $\mathscr{A}_n$ is $*$-invariant because $\mathscr{N}_{x_n}$ obviously is $*$-invariant. Without loss of generality we suppose that the operators a_i^n are symmetric (which is possible since $\mathscr{A}_n$ is $*$-invariant) and $\varrho_{x_n}(a_i^n)=1$ for $i=1,\ldots,d_n$. By $a_\alpha^n=\alpha_1a_1^n+\ldots+\alpha_{d_n}a_{d_n}^n$, $\alpha=(\alpha_1,\ldots,\alpha_{d_n})$, we shall denote the elements of $\mathscr{A}_n$. Further we use the norm $\|a_\alpha^n\| := \sum_{i=1}^{d_n}|\alpha_i|$ on $\mathscr{A}_n$. Let S_n be the unit sphere in this norm. For each sequence $\gamma=\{\gamma_n, n\in\mathbf{N}\}$ of positive real numbers γ_n we define the seminorm $p_\gamma(a):=\sum_n\gamma_n\|a_\alpha^n\|$, $a=\sum_n a_\alpha^n\in\mathscr{A}$, on $\mathscr{A}$. Clearly, all seminorms of this kind give the strongest locally convex topology τ_{st} on $\mathscr{A}$. Let us take a fixed sequence $\gamma=\{\gamma_n\}$.

STATEMENT 1. *For each $n\in\mathbf{N}$ there exists a finite set of vectors $\psi_1^n,\ldots,\psi_{r_n}^n$ having the following properties:*

(a) $\max\limits_{i=1,\ldots,r_n}|\langle a_\alpha^n\psi_i^n,\psi_i^n\rangle|\geqq\|a_\alpha^n\|\{\gamma_n+1+\sum\limits_{k,m<n}\max\limits_{j,l,s}|\langle a_s^n\psi_j^k,\psi_l^m\rangle|$,

(b) $\|x_k\psi_i^n\|\leqq 2^{-n}\quad\forall k<n\quad$ *and* $\quad i=1,\ldots,r_n$,

(c) $\langle a\psi_i^n,\psi_j^m\rangle=0\quad\forall a\in\mathscr{N}_{x_m},\quad 1\leqq n<m,\ i=1,\ldots,r_n,\ j=1,\ldots,r_m$.

PROOF. We choose the sequence $\psi_1^n,\ldots,\psi_{r_n}^n$ by induction on n. We postpone the proof that $\psi_1^1,\ldots,\psi_{r_1}^1$ exist because it requires parts of the following arguments.

Suppose for $k=1,\ldots,n-1$ sequences ψ_i^k, $i=1,\ldots,r_k$, are chosen such that the conditions (a), (b), (c) are satisfied. Let $\mathscr{H}_1$ be the linear span of all elements $a\psi_i^k$ where $a\in\mathscr{N}_{x_n}$, $k=1,\ldots,n-1$, $i=1,\ldots,r_k$. $\mathscr{H}_1$ is finite dimensional because $\mathscr{N}_{x_n}$ is finite dimensional. Put $\mathscr{D}_1=\mathscr{D}\ominus\mathscr{H}_1$. Let C_n be a fixed positive number. If $a_\alpha^n\in S_n$, then $a_\alpha^n\notin\mathscr{N}_{x_{n-1}}$ by construction. This means that $\varrho_{x_{n-1}}(a_\alpha^n)=+\infty$. In view of Lemma 2, this implies $\sup\limits_{\psi\in\mathscr{D}_1}\dfrac{|\langle a_\alpha^n\psi,\psi\rangle|}{\|x_{n-1}\psi\|^2}=+\infty$. Hence there exists a vector $\psi_\alpha^n\in\mathscr{D}_1$ (depending on a_α^n) such that

$$|\langle a_\alpha^n\psi_\alpha^n,\psi_\alpha^n\rangle|>C_n\|x_{n-1}\psi_\alpha^n\|^2. \tag{7}$$

Since inequality (1) remains valid if we multiply ψ_α^n with a factor, we may assume that

(8) $$\|x_{n-1}\psi_\alpha^n\| = 2^{-n}.$$

By $U(a_\alpha^n)$ we denote the set of all elements $a_\beta^n \in S_n$ with

$$|\langle a_\beta^n \psi_\alpha^n, \psi_\alpha^n\rangle| \equiv |\beta_1\langle a_1^n\psi_\alpha^n, \psi_\alpha^n\rangle + \ldots + \beta_{d_n}\langle a_{d_n}^n\psi_\alpha^n, \psi_\alpha^n\rangle| > C_n 2^{-n}.$$

Clearly, $U(a_\alpha^n)$ is an open subset of the sphere S_n. Furthermore, $a_\alpha^n \in U(a_\alpha^n)$ according to (7) and (8). By the Heine—Borel theorem the open cover $\{U(a_\alpha^n)\}$ of S_n has a finite subcover $\{U(a_{\alpha i}^n), i=1, \ldots, r_n$. Put $\psi_i^n = \psi_{\alpha^i}^n$, $i=1, \ldots, r_n$. Then we have

$$\operatorname*{Max}_{i=1,\ldots,r_n} |\langle a_\alpha^n \psi_i^n, \psi_i^n\rangle| > C_n 2^{-n} \quad \text{for all} \quad a_\alpha^n \in S_n.$$

By norming elements $a_\alpha^n \in \mathscr{A}_n$ it follows that

$$\operatorname*{Max}_i |\langle a_\alpha^n \psi_i^n, \psi_i^n\rangle| \geqq C_n 2^{-2n} [\![a_\alpha^n]\!] \quad \text{for each} \quad a_\alpha^n \in \mathscr{A}_n.$$

Putting now $C_n = 2^{2n}\{\gamma_n + 1 + \sum_{k,\,m<n} \operatorname*{Max}_{j,l,s} |\langle a_j^n \psi_1^k, \psi_s^m\rangle|\}$, this is just condition (a). Because $\|x_1\varphi\| \leqq \|x_{n-1}\varphi\| \; \forall \varphi \in \mathscr{D}$, $l \leqq n-1$, condition (b) is fulfilled by (2). The vectors ψ_i^n, $i=1, \ldots, r_n$, are in $\mathscr{D}_1 = \mathscr{D} \ominus \mathscr{H}_1$ by construction. Hence, (c) is also true. Consequently, the induction hypothesis is proved.

We have to say some words about the construction of $\psi_1^1, \ldots, \psi_{r_1}^1$. In this case we only have to check condition (a), i.e.

$$\operatorname*{Max}_i |\langle a_\alpha^1 \psi_i^1, \psi_i^1\rangle| \geqq [\![a_\alpha^1]\!](\gamma_1 + 1).$$

This can be done by using the covering argument of the preceding proof. Now the proof of Statement 1 is complete.

Next we regard the following subset $\mathfrak{M}$ of the domain $\mathscr{D}$:

$$\mathfrak{M} := \left\{\eta = \sum_{n=1}^{q} \varepsilon_n \psi_{i_n}^n \colon q \in \mathbf{N},\ \varepsilon_n \in \mathbf{C}_1,\ |\varepsilon_n| = 1\right\}.$$

STATEMENT 2. *$\mathfrak{M}$ is a bounded subset of the locally convex space $\mathscr{D}[t_{\mathscr{A}}]$.*

PROOF. The seminorms $\|\varphi\|_{x_n} := \|x_n\varphi\|$, $n \in \mathbf{N}$, already define the topology $t_{\mathscr{A}}$ because $\mathscr{A} = \bigcup_{n \in \mathbf{N}} \mathscr{N}_{x_n}$. Take a fixed operator x_k, $k \in \mathbf{N}$. The boundedness of $\mathfrak{M}$ follows from

$$\sup_{\eta \in \mathfrak{M}} \|x_k \eta\| \leqq \sup_{i_n = 1, \ldots, r_n} \sum_{n=1}^{\infty} \|x_k \psi_{i_n}^n\| =$$

$$= \sup_i \left(\sum_{n=1}^{k} \|x_k \psi_i^n\| + \sum_{n=k+1}^{\infty} 2^{-n}\right) \leqq 1 + \sup_i \sum_{n=1}^{k} \|x_k \psi_i^n\| < +\infty.$$

STATEMENT 3. $p_{\mathfrak{B}}(a) \geqq p_\gamma(a)$ for all $a \in \mathscr{A}$.

PROOF. Let $a=\sum_{n=1}^{q} a_\alpha^n$ be an arbitrary element of $\mathscr{A}$. Depending on a, we take elements $\eta_1=\sum_{n=1}^{q}\varepsilon_n\psi_{i_n}^n$ and $\eta_2=\sum_{n=1}^{q}\psi_{i_n}^n$ of $\mathfrak{M}$. Here the vectors $\psi_{i_n}^n$ are chosen in such a way that in condition (a) of Statement 1 the maximum will be attained for a_α^n, i.e.

$$\operatorname*{Max}_i |\langle a_\alpha^n\psi_i^n, \psi_i^n\rangle| = |\langle a_\alpha^n\psi_{i_n}^n, \psi_{i_n}^n\rangle|. \tag{9}$$

The complex numbers ε_n, $|\varepsilon_n|=1$, are taken so that

$$\varepsilon_n\langle a_\alpha^n\psi_{i_n}^n, \psi_{i_n}^n\rangle = |\langle a_\alpha^n\psi_{i_n}^n, \psi_{i_n}^n\rangle|. \tag{10}$$

Applying (9) and (10) we estimate

$$p_{\mathfrak{M}}(a) = \sup_{\varphi,\psi\in\mathfrak{M}} |\langle a\varphi,\psi\rangle| \geqq |\langle a\eta_1,\eta_2\rangle| = \left|\sum_{n=1}^{q}\left\langle a_\alpha^n\left(\sum_{k=1}^{q}\varepsilon_k\psi_{i_k}^k\right), \sum_{m=1}^{q}\psi_{i_m}^m\right\rangle\right| \geqq$$

$$\geqq \left|\sum_{n=1}^{q}\varepsilon_n\langle a_\alpha^n\psi_{i_n}^n, \psi_{i_n}^n\rangle\right| - \sum_{\substack{n,k,m=1\\(k,m)\neq(n,n)}}^{q} |\langle a_\alpha^n\psi_{i_k}^k, \psi_{i_m}^m\rangle| \geqq$$

$$\geqq \sum_{n=1}^{q}|\langle a_\alpha^n\psi_{i_n}^n, \psi_{i_n}^n\rangle| - \sum_{\dots}\|a_\alpha^n\| \operatorname*{Max}_{s=1,\dots,r_n} |\langle a_s^n\psi_{i_k}^k, \psi_{i_m}^m\rangle| \geqq$$

$$\geqq \sum_{n=1}^{q}\{\operatorname*{Max}_i |\langle a_\alpha^n\psi_i^n, \psi_i^n\rangle|\} - \|a_\alpha^n\| \sum_{\substack{n,k,m=1\\(k,m)\neq(n,n)}}^{q} \operatorname*{Max}_s |\langle a_s^n\psi_{i_k}^k, \psi_{i_m}^m\rangle|.$$

Since the operators a_s^n are symmetric, condition (c) in Statement 1 immediately implies that $\langle a_s^n\psi_{i_k}^k, \psi_{i_m}^m\rangle=0$ if $k\neq m$ and $k\geqq n$ or $m\geqq n$ is fulfilled. This observation combined with $\varrho_{x_n}(a_s^n)=1$ gives us

$$\sum_{\substack{k,m=1\\(k,m)\neq(n,n)}}^{q} \operatorname*{Max}_{s=1,\dots,r_n} |\langle a_s^n\psi_{i_k}^k, \psi_{i_m}^m\rangle| =$$

$$= \sum_{m\geqq n+1}\operatorname*{Max}_s |\langle a_s^n\psi_{i_m}^m, \psi_{i_m}^m\rangle| + \sum_{k,m<n}\operatorname*{Max}_s |\langle a_s^n\psi_{i_k}^k, \psi_{i_m}^m\rangle| \leqq$$

$$\leqq \sum_{m\geqq n+1}\|x_n\psi_{i_m}^m\|^2 + \sum_{k,m<n}\operatorname*{Max}_{j,l,s} |\langle a_s^n\psi_j^k, \psi_l^m\rangle| \leqq$$

$$\leqq \sum_{m\geqq n+1} 2^{-2m} + \sum\dots \leqq 1 + \sum_{k,m<n}\operatorname*{Max}_{j,l,s} |\langle a_s^n\psi_j^k, \psi_l^m\rangle|.$$

Putting both estimations together we get

$$p_{\mathfrak{M}}(a) \geqq \sum_{n=1}^{q}\{\operatorname*{Max}_i |\langle a_\alpha^n\psi_i^n, \psi_i^n\rangle| - \|a_\alpha^n\|\,[1 + \sum_{k,m<n}\operatorname*{Max}_{j,l,s} |\langle a_s^n\psi_j^k, \psi_l^m\rangle|]\} \geqq$$

$$\geqq \sum_{n=1}^{q}\|a_\alpha^n\|\gamma_n \equiv p_\gamma(a).$$

5. Proof of Theorem 2

(2.1)→(2.2): Trivial.

(2.2)→(2.1): Obvious because $x \in \mathcal{M}_{x_n}$ implies $\mathcal{M}_x \subseteq \mathcal{M}_{x_n}$.

(2.3)→(2.1): This follows in a similar way as (1.3)→(1.1) in the proof of Theorem 1. We only have to replace $p_{\mathfrak{W}}(a)$ by $\|a\varphi\|$.

(2.1)→(2.3): Suppose $\mathcal{A}$ is a closed $Op*$-algebra on $\mathcal{D}$ satisfying (2.1). Since the set $\{\mathcal{M}_x, x \in \mathcal{A}\}$ of vector spaces is directed, we can take a sequence $\{x_n, n \in \mathbf{N}\}$, $x_n \in \mathcal{A}$, such that

$$\text{(i)} \quad \mathcal{M}_{x_n} \subsetneqq \mathcal{M}_{x_{n+1}} \quad \forall n \in \mathbf{N} \quad \text{and} \quad \text{(ii)} \quad \mathcal{A} = \bigcup_{n \in \mathbf{N}} \mathcal{M}_{x_n}.$$

We choose a vector space $\mathcal{A}_n \subseteq \mathcal{M}_{x_n}$, $n \geqq 2$, such that $\mathcal{M}_{x_n}$ is the direct sum of $\mathcal{M}_{x_{n-1}}$ and $\mathcal{A}_n$. Let $\mathcal{A}_1 = \mathcal{M}_{x_1}$. Then $\mathcal{A} = \sum_n \mathcal{A}_n$. Furthermore, we may assume that in addition the following is true:

$$\text{(iii)} \qquad \lambda_a(x_n) \equiv \sup_{\varphi \in \mathcal{D}} \frac{\|x_n \varphi\|}{\|a\varphi\|} < +\infty \quad \text{for all} \quad a \in \mathcal{A}_n, n \in \mathbf{N}.$$

Let us verify this assertion. If $\lambda_a(x_n) = +\infty$ for a certain operator $a \in \mathcal{A}_n$, then the new sequence $x_1, \ldots, x_{n-1}, a, x_n, \ldots$ gives a "finer" decomposition of $\mathcal{A}$ which satisfies (i) and (ii). (ii) is obvious. Since $a \notin \mathcal{M}_{x_{n-1}}$ implies $\mathcal{M}_{x_{n-1}} \neq \mathcal{M}_a$ and $\lambda_a(x_n) = +\infty$ implies $\mathcal{M}_a \neq \mathcal{M}_{x_n}$, (i) is also true. According to (2.1), all vector spaces $\mathcal{M}_x$, $x \in \mathcal{A}$, are finite dimensional. Consequently, by an induction argument this procedure can be continued until (iii) is fulfilled.

Without loss of generality, we can assume that $\lambda_{x_{n+1}}(x_n) \leqq 1$ and $\lambda_{x_n}(a_i^n) \leqq 1$. Further we use the following notations from the proof of Theorem 1: $a_i^n, d_n, a_\alpha^n, \|a_\alpha^n\|, S_n$.

STATEMENT 4. $C_n := \sup_{a_\alpha^n \in S_n} \lambda_{a_\alpha^n}(x_n) < +\infty$ *for each* $n \in \mathbf{N}$.

PROOF. Assume that the contrary is true. Then there exist sequences $a_{\alpha_k}^n \in S_n$, $k \in \mathbf{N}$, (for brevity we write a_k instead of $a_{\alpha_k}^n$ and x for x_n) and $\varphi_k \in \mathcal{D}$, $k \in \mathbf{N}$, such that $\|a_k \varphi_k\| \geqq k \|x \varphi_k\|$. We may assume that $\|x\varphi_k\| = 1 \; \forall k \in \mathbf{N}$ (otherwise we multiply by a suitable factor). Then we have $\lim_{k \to \infty} \|a_k \varphi_k\| = 0$. By the compactness of the unit sphere S_n there is a subsequence of $\{a_k\}$ converging to an element $a \in S_n$. For simplicity suppose that $\lim_{k \to \infty} \|a_k - a\| = 0$. Let $a_k - a = \alpha_{1k} a_1^n + \ldots + \alpha_{d_n k} a_{d_n}^n$. Then

$$\|(a_k - a)\varphi_k\| \leqq \sum_{i=1}^{d_n} |\alpha_{ik}| \, \|a_i^n \varphi_k\| \leqq \sum_{i=1}^{d_n} |\alpha_{ik}| \, \|x\varphi_k\| = \|a_k - a\| \to 0 \text{ for } k \to \infty.$$

By $\|a\varphi_k\| \leqq \|(a_k - a)\varphi_k\| + \|a_k \varphi_k\|$ this gives $\lim_{k \to \infty} \|a\varphi_k\| = 0$.

On the other hand, we have $\lambda_a(x) < +\infty$ by condition (iii). In particular, this implies that $1 = \|x\varphi_k\| \leqq \lambda_a(x) \|a\varphi_k\|$. This contradicts $\lim_{k \to \infty} \|a\varphi_k\| = 0$.

An immediate consequence of Statement 4 is

STATEMENT 5. *There are constants* $C_n>0, n\in\mathbf{N}$, *with*

$$\|a_\alpha^n\|\,\|x_n\varphi\| \leqq C_n\|a_\alpha^n\varphi\| \quad \forall\varphi\in\mathcal{D},\ a_\alpha^n\in\mathcal{A}_n,\ n\in\mathbf{N}.$$

Let $\gamma=\{\gamma_n\}$ be a sequence of positive numbers and q_γ be the seminorm on $\mathcal{A}$ defined by $q_\gamma(a)=\{\sum_n \gamma_n\|a_\alpha^n\|^2\}^{1/2}$ for $a=\sum_n a_\alpha^n\in\mathcal{A}$. Our goal is to prove that $\sigma^{\mathcal{D}}=\tau_{\mathrm{st}}$ on $\mathcal{A}$. Since all seminorms q_γ define the topology τ_{st}, it is enough to show that for each sequence γ there exists a vector $\varphi\in\mathcal{D}$ (depending on γ) such thta $q_\gamma(a)\leqq\|a\varphi\|\ \forall a\in\mathcal{A}$. Now fix $\gamma=\{\gamma_n\}$.

STATEMENT 6. *There exist a sequence* $\{\delta_{ij}, i, j\in\mathbf{N}\}$ *of positive numbers and a sequence* $\{\varphi_n, n\in\mathbf{N}\}$ *of vectors* $\varphi_n\in\mathcal{D}$ *satisfying the following conditions:*

(a) $$\|x_n\varphi_n\| = \sqrt{\delta_{nn}^2+\gamma_n C_n^2}+1+\sum_{i=1}^{n-1}\|x_n\varphi_i\| \leqq \delta_{nn} \quad \forall n\in\mathbf{N}.$$

(b) $$\|x_n\varphi_n\| \leqq \delta_{nn}^{-1}2^{-n} \quad \forall k<n,\quad k, n\in\mathbf{N}.$$

(c) $$\langle a_i^k\varphi_n, a_j^m\varphi_1\rangle = 0 \quad \forall k, m = 1, \dots, n,\quad l = 1, \dots, n-1,\quad i = 1, \dots, d_k,$$
$j = 1, \dots, d_m$.

(d) $$\delta_{mn} = \delta_{nm} = \sum_{r,s<n}\|x_n\varphi_r\|\,\|x_m\varphi_s\|+1 \quad \forall m<n,\ m, n\in\mathbf{N}.$$

(e) *The determinants*

$$D_n = \begin{vmatrix} C_1^{-2}\delta_{11} & -\delta_{12} & \dots -\delta_{1n} \\ -\delta_{21} & C_2^{-2}\delta_{22} & \dots -\delta_{2n} \\ \vdots & \vdots & \vdots \\ -\delta_{n1} & -\delta_{2n} & C_n^{-2}\delta_{nn} \end{vmatrix}$$

are positive.

PROOF. In the case $n=1$ we take a positive number δ_{11} with $\delta_{11}\geqq\sqrt{\delta_{11}+\gamma_1 C_1^2}+1$ and a vector $\varphi_1\in\mathcal{D}$ with $\|x_1\varphi_1\|=\sqrt{\delta_{11}+\gamma_1 C_1^2}+1$. Now suppose that δ_{ij}, φ_i, $i, j=1, \dots, n-1$ are already chosen so that (a)—(e) are fulfilled. We define $\delta_{mn}=\delta_{nm}$, $m=1, \dots, n-1$, by (d). Since $D_{n-1}>0$ by induction assumption, δ_{nn} may be taken so large that $D_n>0$ and

$$\delta_{nn}\geqq\sqrt{\delta_{nn}+\gamma_n C_n^2}+1+\sum_{i=1}^{n-1}\|x_n\varphi_i\| =: M_n.$$

Further, we assumed that $\mathcal{M}_{x_{n-1}}\neq\mathcal{M}_{x_n}$, i.e. $\lambda_{x_{n-1}}(x_n)=+\infty$. Thus there is a vector $\varphi_n\in\mathcal{D}$ such that $\|x_n\varphi_n\|\geqq M_n\delta_{nn}2^n\|x_{n-1}\varphi_n\|$. Similarly as in the proof of Theorem 1 we may suppose that φ_n is orthogonal to the vectors $(a_i^k)^+a_j^m\varphi_1$, $k, m\leqq$ $\leqq n$, $l<n-1$, $i=1, \dots, d_k$, $j=1, \dots, d_m$. After norming of φ_n we obtain $\|x_n\varphi_n\|=M_n$. Consequently, $\|x_{n-1}\varphi_n\|\leqq\delta_{nn}^{-1}2^{-n}$. Since $\|x_l\varphi\|\leqq\|x_{n-1}\varphi\|\ \forall l\leqq n-1$, the conditions (a)—(e) are satisfied for $\delta_{ij}, \varphi_i, i, j=1, \dots, n$. By induction, Statement 6 is proved.

From $\mathcal{A}=\bigcup_{n\in N}\mathcal{M}_{x_n}$ it is clear that the topology $t_{\mathcal{A}}$ on $\mathcal{D}$ can be given by the seminorms $\|\varphi\|_{x_n}:=\|x_n\varphi\|$, $n\in\mathbf{N}$. Therefore condition (b) of Statement 6 implies that the sequence $\psi_n:=\sum_{i=1}^{n}\varphi_i$ is a Cauchy sequence in $\mathcal{D}[t_{\mathcal{A}}]$. Since the *Op*∗-

algebra $\mathscr{A}$ was assumed to be closed on $\mathscr{D}$, the space $\mathscr{D}[t_{\mathscr{A}}]$ is complete. Consequently, the sequence $\{\psi_n, n\in\mathbf{N}\}$ is converging to an element $\varphi=\sum_{i=1}^{\infty}\varphi_i\in\mathscr{D}$.

STATEMENT 7. *For all* $a\in\mathscr{A}$ *we have* $\|a\varphi\|\geqq q_\gamma(a)$.

PROOF. Applying (a) and (b) we obtain

$$\|x_n\varphi\|\geqq\|x_n\varphi_n\|-\sum_{i=1}^{n-1}\|x_n\varphi_i\|-\sum_{i=n+1}^{\infty}\|x_n\varphi_i\|\geqq\sqrt{\delta_{nn}+\gamma_n C_n^2},$$

i.e.

$$\|x_n\varphi\|^2-\gamma_n C_n^2\geqq\delta_{nn}. \tag{11}$$

For $n>m$, (b), (c) and (d) give us

$$|\langle a_i^n\varphi, a_j^m\varphi\rangle|\leqq\sum_{r,s<n}|\langle a_i^n\varphi_r, a_j^m\varphi_s\rangle|+\sum_{r=n}^{\infty}|\langle a_i^n\varphi_r, a_j^m\varphi_r\rangle|\leqq \tag{12}$$

$$\leqq\sum_{r,s<n}\|x_n\varphi_r\|\,\|x_m\varphi_s\|+\|x_n\varphi_n\|\,\|x_m\varphi_n\|+\sum_{r=n+1}^{\infty}\|x_n\varphi_r\|\,\|x_m\varphi_r\|\leqq$$

$$\leqq\ldots+\delta_{nn}\delta_{nn}^{-1}2^{-n}+\sum_{r=n+1}^{\infty}2^{-n}\leqq\delta_{nm}.$$

Now we make use of condition (e). It implies that the quadratic form $Q(t):=$ $:=\sum_n t_n\bar{t}_n C_n^{-2}\delta_{nn}^2-\sum_{n\neq m}t_n\bar{t}_m\delta_{nm}$ is positive definite. In particular, this means that

$$\sum_n [\![a_\alpha^n]\!]^2 C_n^{-2}\delta_{nn}-\sum_{n\neq m}[\![a_\alpha^n]\!]\,[\![a_\alpha^m]\!]\,\delta_{nm}\geqq 0.$$

By the estimations (11) and (12) we get

$$\sum_n [\![a_\alpha^n]\!]^2 C_n^{-2}(\|x_n\varphi\|^2-\gamma_n C_n^2)-\sum_{n\neq m}[\![a_\alpha^n]\!]\,[\![a_\alpha^m]\!]\max_{i,j}|\langle a_i^n\varphi, a_j^m\varphi\rangle|\geqq 0. \tag{13}$$

The triangle inequality leads to

$$|\langle a_\alpha^n\varphi, a_\alpha^m\varphi\rangle|\leqq\|a_\alpha^n\|a_\alpha^m\|\|\max_{i,j}|\langle a_i^n\varphi, a_j^m\varphi\rangle|.$$

Further, we have $C_n^{-2}[\![a_\alpha^n]\!]^2\|x_n\varphi\|^2\leqq\|a_\alpha^n\varphi\|^2$ by Statement 5. Putting these two inequalities into (13) it follows that

$$\|a\varphi\|^2-q_\gamma(a)^2\geqq\sum_n(\|a_\alpha^n\varphi\|^2-\gamma_n[\![a_\alpha^n]\!]^2)-\sum_{n\neq m}|\langle a_\alpha^n\varphi, a_\alpha^m\varphi\rangle|\geqq 0$$

which completes the proof.

6. Concluding remarks

The preceding proofs of our Theorems 1 and 2 show that the multiplication in the $Op*$-algebra $\mathcal{A}$ was used only to ensure that the families of vector spaces $\{\mathcal{N}_x, x\in\mathcal{A}\}$ and $\{\mathcal{M}_x, x\in\mathcal{A}\}$ are directed. In fact, our proofs yield the following more general results.

Let $\mathcal{A}$ be a vector space of linear operators on a dense domain $\mathcal{D}$ in a Hilbert space (we do not assume that the operators map $\mathcal{D}$ into itself). Suppose, $\{x_n, n\in\mathbf{N}\}$ is a sequence of linear operators defined on $\mathcal{D}$ so that $x_1=1$ and $\|x_n\varphi\|\leqq\|x_{n+1}\varphi\|\ \forall$ $\forall\varphi\in\mathcal{D}, n\in\mathbf{N}$. By the seminorms $\|\varphi\|_{x_n}:=\|x_n\varphi\|, n\in\mathbf{N}$, we define a locally convex topology t_+ on $\mathcal{D}$. Let

$$\mathcal{N}_{x_n}:=\{a\in\mathcal{A}\colon |\langle a\varphi,\varphi\rangle|\leqq C_{a,n}\|x_n\varphi\|^2\ \forall\varphi\in\mathcal{D}\}$$

and

$$\mathcal{M}_{x_n}:=\{a\in\mathcal{A}\colon \|a\varphi\|\leqq C_{a,n}\|x_n\varphi\|\ \forall\varphi\in\mathcal{D}\}.$$

THEOREM 1′. *Suppose that for each $a\in\mathcal{A}$ the operator a^* is defined on $\mathcal{D}$ and $a^+:=a^*\restriction\mathcal{D}\in\mathcal{A}$. Suppose $\mathcal{A}=\bigcup_{n\in\mathbf{N}}\mathcal{N}_{x_n}$. The uniform topology $\tau_{\mathcal{D}}$ on $\mathcal{A}$ will be defined by the seminorms $p_{\mathfrak{M}}(a)=\sup_{\varphi,\psi\in\mathfrak{M}}|\langle a\varphi,\psi\rangle|$ taken for all bounded subsets $\mathfrak{M}$ of the locally convex space $\mathcal{D}[t_+]$. Then, $\tau_{\mathcal{D}}=\tau_{\mathrm{st}}$ if and only if all vector spaces $\mathcal{N}_{x_n}, n\in\mathbf{N}$, are finite dimensional.*

THEOREM 2′. *Suppose $\mathcal{A}=\bigcup_{n\in\mathbf{N}}\mathcal{M}_{x_n}$ and the space $\mathcal{D}[t_+]$ is complete. Let $\sigma^{\mathcal{D}}$ be the locally convex topology on $\mathcal{A}$ generated by the seminorms $\|a\|_\varphi:=\|a\varphi\|, \varphi\in\mathcal{D}$. Then, $\sigma^{\mathcal{D}}=\tau_{\mathrm{st}}$ of and only if all vector spaces $\mathcal{M}_{x_n}$ are finite dimensional.*

Notice that the assumption $\mathcal{A}=\bigcup_{n\in\mathbf{N}}\mathcal{N}_{x_n}$ implies that $p_{\mathfrak{M}}(a)<+\infty\,\forall a\in\mathcal{A}$. If $a\in\mathcal{N}_{x_n}$, then $|\langle a\varphi,\psi\rangle|\leqq4\varrho_{x_n}(a)\|x_n\varphi\|\,\|x_n\psi\|$ by polarization; hence $p_{\mathfrak{M}}(a)\leqq$ $\leqq4\varrho_{x_n}(a)\sup_{\varphi,\psi\in\mathfrak{M}}\|x_n\varphi\|\,\|x_n\psi\|<+\infty$ because $\mathfrak{M}$ is bounded in $\mathcal{D}[t_+]$.

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ON THE STRONG APPROXIMATION OF FOURIER SERIES

By

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1. Let f be a continuous and 2π-periodic function. Denote $\omega(f;\delta)$ and $s_n(x)= =s_n(f;\ x)$ the modulus of continuity of f and the n-th partial sum of its Fourier series, respectively.

If ω is a modulus of continuity and r is a natural number we define $W^r H^\omega$ as the class of all functions f, for which

$$\omega(f^{(r)};\delta) \leqq K_f\omega(\delta) \quad (0 \leqq \delta \leqq 2\pi)$$

where K_f is a constant (depending generally on f).

The first result on strong approximation is due to ALEXITS and KRÁLIK [1]. In 1965 LEINDLER [2] proved a very general theorem about the order of strong approximation. This result can be applied to the most important strong means

$$h_n(f, p, \beta;\ x) = \left\{\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_k(x)-f(x)|^p\right\}^{1/p} \quad (\beta, p > 0),$$

$$\tilde{h}_n(f, p, \beta;\ x) = h_n(\tilde{f}, p, \beta;\ x),$$

$$\sigma_n^\gamma|f, p;\ x| = \left\{\frac{1}{A_n^\gamma}\sum_{k=0}^{n}A_{n-k}^{\gamma-1}|s_k(x)-f(x)|^p\right\}^{1/p} \quad \left(\gamma, p>0,\ A_n^\gamma = \binom{n+\gamma}{n}\right),$$

and

$$\tilde{\sigma}_n^\gamma|f, p;\ x| = \sigma_n^\gamma|\tilde{f}, p;\ x|.$$

One can get e.g. the following theorem (see [2]).

THEOREM A. *Let us suppose that* $f^{(r)}\in \mathrm{Lip}\,\alpha$ $(0<\alpha\leqq 1)$ *and* $p>0$. *If* $\beta>(r+\alpha)p$ *then*

$$h_n(f, p, \beta;\ x) = O(n^{-r-\alpha})$$

and

$$\tilde{h}_n(f, p, \beta;\ x) = O(n^{-r-\alpha})$$

while if $\beta=(r+\alpha)p$ *then we have only*

$$h_n(f, p, \beta;\ x) = O(n^{-r-\alpha}(\log n)^{1/p})$$

and

$$\tilde{h}_n(f, p, \beta;\ x) = O(n^{-r-\alpha}(\log n)^{1/p}).$$

Moreover, there are functions f_1 *and* f_2, *so that* $f_i^{(r)}\in \mathrm{Lip}\,\alpha$ $(i=1, 2)$, *but*

$$h_n(f_1, p, \beta;\ 0) \geqq cn^{-r-\alpha}(\log n)^{1/p} \quad (c>0)$$

and

$$\tilde{h}_n(f_2, p, \beta;\ 0) \geqq cn^{-r-\alpha}(\log n)^{1/p}.$$

Leindler [5] proved also that in the case $\alpha=1, \beta=(r+1)p$ the additional assumption $\tilde{f}^{(r)}\in \mathrm{Lip}\,\alpha$ does not improve the above estimation.

In [3] we can find the analogue of Theorem A for the means $\sigma_n^\gamma|f, p; x|$ and $\tilde{\sigma}_n^\gamma|f, p; x|$.

In the first part of the present paper we give the exact approximation order, which can be achieved by the above means, if f is taken from a class $W^r H^\omega$. After that we shall deal with the so-called generalized strong de la Vallée Poussin means.

Let ω be an arbitrary modulus of continuity. We define

$$\omega^*(\delta) = \int_0^\delta \frac{\omega(t)}{t}\, dt.$$

It is possible to see that in the case $\int_0^1 \frac{\omega(t)}{t}\, dt<\infty$, $\omega^*(\delta)$ is between a modulus of continuity and its twofold, so we may regard $\omega^*(\delta)$ as a modulus of continuity.

Let ω_0 be the infimum of those α, for which

$$\sum_{k=0}^{n} 2^{k\alpha}\omega\left(\frac{1}{2^k}\right) \leqq K_\alpha 2^{n\alpha}\omega\left(\frac{1}{2^n}\right) \quad (n=0, 1, 2, \dots) \tag{1}$$

is true with a constant K_α.

By Lemma 2 it is clear that if (1) holds for a certain α then it holds for any $\alpha'>\alpha$ and there exists a positive $\varepsilon=\varepsilon(\alpha)$ such that (1) holds for $\alpha-\varepsilon$, too.

Thus by the definition of ω_0 it is clear that (1) holds if and only if $\alpha>\omega_0$. With the above notations we prove

Theorem 1. *Let $f\in W^r H^\omega$ and $p>0$. We have*

$$h_n(f;\, p, \beta;\, x) = O(H_{r,\omega}^{p,\beta,n}) \tag{2}$$

where

$$H_{r,\omega}^{p,\beta,n} = \left\{\frac{1}{(n+1)^\beta}\sum_{k=1}^{n}(k+1)^{\beta-1}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p\right\}^{1/p}.$$

Moreover if $r>0$ then

$$\tilde{h}_n(f, p, \beta;\, x) = O(H_{r,\omega}^{p,\beta,n}), \tag{3}$$

while if $r=0$ then

$$\tilde{h}_n(f, p, \beta;\, x) = O(H_{0,\omega^*}^{p,\beta,n}) \tag{4}$$

are true.

Furthermore, there are functions f_r $(r=0, 1, \dots)$ and f_0^* so that $f_r, \tilde{f}_r\in W^r H^\omega$ $(r=0, 1, \dots)$, $f_0^*\in H^\omega$, and

$$h_n(f_r, p, \beta;\, 0) \geqq cH_{r,\omega}^{p,\beta,n} \tag{5}$$

$$\tilde{h}_n(f_r, p, \beta;\, 0) \geqq cH_{r,\omega}^{p,\beta,n} \tag{6}$$

$$\tilde{h}_n(f_0^*, p, \beta;\, 0) \geqq cH_{0,\omega^*}^{p,\beta,n} \tag{7}$$

are true with a positive constant c independent of n.

REMARK. *We have for every* $f \in W^r H^\omega$

$$h_n(f, p, \beta; x) = O\left(\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\right)$$

if and only if $\beta > (r+\omega_0)p$.

By the Remark, the means h_n may give worse approximation order than $\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)$ (which is the order of the best trigonometric approximation for the whole class $W^r H^\omega$) in the case $\beta \leqq (r+\omega_0)p$.

THEOREM 2. *Let* $f \in W^r H^\omega$ *and* $p > 0$. *We have*

$$\sigma_n^\gamma |f, p; x| = O(H_{r,\omega}^{p,1,n}), \tag{8}$$

moreover

$$\tilde{\sigma}_n^\gamma |f, p; x| = O(H_{r,\omega}^{p,1,n}) \tag{9}$$

if $r > 0$, *while if* $r = 0$ *then*

$$\tilde{\sigma}_n^\gamma |f, p; x| = O(H_{0,\omega^*}^{p,1,n}). \tag{10}$$

Furthermore, for the functions occurring in Theorem 1 *we have*

$$\sigma_n^\gamma |f_r, p; 0| \geqq c H_{r,\omega}^{p,1,n} \tag{11}$$

$$\tilde{\sigma}_n^\gamma |f_r, p; 0| \geqq c H_{r,\omega}^{p,1,n} \quad (c > 0) \tag{12}$$

$$\tilde{\sigma}_n^\gamma |f_0^*, p; 0| \geqq c H_{0,\omega^*}^{p,1,n}. \tag{13}$$

REMARK. *We have for every* $f \in W^r H^\omega$

$$\sigma_n^\gamma |f, p; x| = O\left(\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\right)$$

if and only if $(r+\omega_0)\ p < 1$.

Especially, if $\omega_\alpha(\delta) = \delta^\alpha$ $(0 < \alpha \leqq 1)$ then $(\omega_\alpha)_0 = \alpha$, and we get Theorem A from Theorem 1.

Going over to more general means we shall prove

THEOREM 3. *To every fixed r and* ω *there are functions* f_1 *and* f_2 *so that* $f_i \in W^r H^\omega$, $\tilde{f}_i \in W^r H^\omega$ $(i = 1, 2)$ *and for every n either*

$$\begin{cases} |s_n(f_1; 0) - f_1(0)| \geqq c\,\dfrac{1}{n^r}\,\omega\left(\dfrac{1}{n}\right) \\ |\tilde{s}_n(f_1; 0) - \tilde{f}_1(0)| \geqq c\,\dfrac{1}{n^r}\,\omega\left(\dfrac{1}{n}\right) \end{cases} \quad (c > 0) \tag{14}$$

or

$$\begin{cases} |s_n(f_2; 0) - f_2(0)| \geqq c\,\dfrac{1}{n^r}\,\omega\left(\dfrac{1}{n}\right) \\ |\tilde{s}_n(f_2; 0) - \tilde{f}_2(0)| \geqq c\,\dfrac{1}{n^r}\,\omega\left(\dfrac{1}{n}\right) \end{cases} \quad (c > 0) \tag{15}$$

are satisfied.

REMARK. *To every ω there are four functions f_i $(i=1, 2, 3, 4)$ so that $f_i \in H^\omega$ $(i=1, 2, 3, 4)$, and for every n we can find an $i=i(n)$ for which*

$$|\tilde{s}_n(f_i;\, 0)-\tilde{f}_i(0)| \geqq c\omega^*\left(\frac{1}{n}\right).$$

We mention two corollaries of Theorem 3. In order to simplify the writing we introduce the following notations. If $T=(t_{nk})_{n,k=1}^{\infty}$ is a nonnegative matrix and ω is a modulus of continuity, let

$$T_n(\omega, r, p) = \left\{\sum_{k=1}^{\infty} t_{nk}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p\right\}^{1/p} \qquad (p>0,\ r=0, 1, \ldots)$$

and

$$T_n^*(\omega, r, p) = \begin{cases} T_n(\omega, r, p) & \text{if } r>0 \\ T_n(\omega^*, 0, p) & \text{if } r=0. \end{cases}$$

COROLLARY 1. *For every nonnegative matrix $T=(t_{nk})$ and for every r, ω there exist functions f_1, f_2 so that $f_1, \tilde{f}_1, f_2 \in W^r H^\omega$ and*

$$\limsup_{n\to\infty}\left\{\sum_{k=1}^{\infty} t_{nk}|s_k(f_1;\, 0)-f_1(0)|^p\right\}^{1/p} \Big/ T_n(\omega, r, p) > 0,$$

$$\limsup_{n\to\infty}\left\{\sum_{k=1}^{\infty} t_{nk}|\tilde{s}_k(f_2;\, 0)-\tilde{f}_2(0)|^p\right\}^{1/p} \Big/ T_n^*(\omega, r, p) > 0.$$

COROLLARY 2. *If for the sequence $\{\varrho_n\}$ we have $T_n(\omega, r, p) \neq O(\varrho_n)$ or $T_n^*(\omega, r, p) \neq O(\varrho_n)$ then there is a function $f_0 \in W^r H^\omega$ for which*

$$\left\{\sum_{k=1}^{\infty} t_{nk}|s_k(f_0;\, 0)-f_0(0)|^p\right\}^{1/p} \neq O(\varrho_n)$$

or

$$\left\{\sum_{k=1}^{\infty} t_{nk}|\tilde{s}_k(f_0;\, 0)-\tilde{f}_0(0)|^p\right\}^{1/p} \neq O(\varrho_n),$$

respectively.

Corollaries 1 and 2 say that we minorate the order of the strong approximation (taking into consideration the whole class $W^r H^\omega$) if we replace in the means $|s_k-f|$ by $\dfrac{1}{k^r}\,\omega\left(\dfrac{1}{k}\right)$.

Now we prove two theorems about the generalised strong de la Vallée Poussin means. These were defined by LEINDLER [4] as follows: if $\lambda=\{\lambda_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of integers such that $\lambda_1=1$ and $\lambda_{n+1}-\lambda_n \leqq 1$, then let

$$V_n(f, \lambda, p;\, x) = \left\{\frac{1}{\lambda_n}\sum_{k=n-\lambda_n+1}^{n} |s_k(x)-f(x)|^p\right\}^{1/p}.$$

In [6] LEINDLER proved:

THEOREM B. *For every positive p we have*

$$V_n(f, \lambda, p; x) = O\left(\left(\frac{n}{\lambda_n}\right)^{1/p} E_{n-\lambda_n+1}(f)\right).$$

THEOREM C. If $f^{(r)} \in \operatorname{Lip} \alpha$ $(0<\alpha\leqq 1)$ *and* $p>0$ *then*

$$V_n(f, \lambda, p; x) = O(\Lambda_n^{r,\alpha})$$

where

$$\Lambda_n^{r,\alpha} = \begin{cases} \left(\dfrac{n}{\lambda_n}\right)^{1/p} n^{-r-\alpha} & \text{if} \quad (r+\alpha)p<1 \\ \lambda_n^{-r-\alpha}\left(1+\log\dfrac{n}{n-\lambda_n+1}\right)^{1/p} & \text{if} \quad (r+\alpha)p=1 \\ \lambda_n^{-1/p}(n-\lambda_n+1)^{(1/p)-r-\alpha} & \text{if} \quad (r+\alpha)p>1. \end{cases}$$

Similar statement is true for $\tilde{V}_n(f, \lambda, p;\; x) = V_n(\tilde{f}, \lambda, p;\; x)$.

Furthermore, if $n=O(\lambda_n)$ *then there is a function* f_0 *such that* $f_0^{(r)}, \tilde{f}_0^{(r)} \in \operatorname{Lip}\alpha$, *but*

$$\limsup_{n\to\infty} V_n(f_0, \lambda, p;\; 0)/\Lambda_n^{r,\alpha} > 0.$$

Generalizing these results we prove

THEOREM 4. *For every* $p>0$ *we have*

$$V_n(f, \lambda, p; x) = O\left(E_{n-\lambda_n+1}(f)\log\frac{2n}{\lambda_n}\right). \tag{16}$$

More generally, there exists a constant K, depending only on p, for which

$$\left\{\frac{1}{r}\sum_{i=1}^{r}|s_{k_i}(f;\; x)-f(x)|^p\right\}^{1/p} \leqq KE_{k_1}(f)\log\frac{2n}{r}, \tag{17}$$

where $0<k_1<k_2<\ldots<k_r\leqq n$ *are arbitrary indices.*

THEOREM 5. *If* $f\in W^r H^\omega$, $p>0$ *then we have*

$$V_n(f, \lambda, p; x) = O(\Lambda_n(r, p, \omega)) \tag{18}$$

and

$$\tilde{V}_n(f, \lambda, p;\; x) = O(\Lambda_n^*(r, p, \omega)) \tag{19}$$

where

$$\Lambda_n(r, p, \omega) = \left\{\frac{1}{\lambda_n}\sum_{k=n-\lambda_n+1}^{n}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p\right\}^{1/p}\log\frac{2n}{\lambda_n}$$

and

$$\Lambda_n^*(r, p, \omega) = \begin{cases} \Lambda_n(r, p, \omega) & \text{if} \quad r>0 \\ \Lambda_n(0, p, \omega) + \left\{\dfrac{1}{\lambda_n}\displaystyle\sum_{k=n-\lambda_n+1}^{n}\left(\omega^*\left(\frac{1}{k}\right)\right)^p\right\}^{1/p} & \text{if} \quad r=0. \end{cases}$$

Furthermore, there are functions f_1, f_2 *for which* $f_1, f_2 \in W^r H^\omega$ *and*

(20) $$\limsup_{n\to\infty} V_n(f_1, \lambda, p; 0)/\Lambda_n(r, p, \omega) > 0,$$

(21) $$\limsup_{n\to\infty} \tilde{V}_n(f_2, \lambda, p; 0)/\Lambda_n^*(r, p, \omega) > 0.$$

We mention that even in the case $\omega(\delta)=\delta^\alpha$ $(0<\alpha\leqq 1)$ Theorem 5 is a slight generalization of Theorem C, because we did not assume $n=O(\lambda_n)$.

2. To prove our theorems we require some lemmas.

LEMMA 1. *Let* $\omega(\delta)$ *be an arbitrary modulus of continuity. For the functions*

$$f_0(x) = \sum_{k=1}^{\infty} (-1)^k \omega\left(\frac{1}{2^k}\right) \sum_{l=2^{k-1}+1}^{2^k} \left(\frac{\cos(5\cdot 2^k - l)x}{l} - \frac{\cos(5\cdot 2^k + l)x}{l}\right)$$

and

$$g_0(x) = \sum_{k=1}^{\infty} (-1)^k \omega\left(\frac{1}{2^k}\right) \sum_{l=2^{k-1}+1}^{2^k} \left(\frac{\sin(5\cdot 2^k - l)x}{l} - \frac{\sin(5\cdot 2^k + l)x}{l}\right)$$

we have

(2.1) $$\omega(f_0; \delta) \leqq K\omega(\delta)$$

and

(2.2) $$\omega(g_0; \delta) \leqq K\omega(\delta)$$

where K *depends only on* ω.

PROOF. The proof is similar to that of Theorem 1 of [5].
First we prove (2.1). Let

$$R_k(x) = \sum_{l=2^{k-1}+1}^{2^k} \left(\frac{\cos(5\cdot 2^k - l)x}{l} - \frac{\cos(5\cdot 2^k + l)x}{l}\right) = 2\sin 5\cdot 2^k x \sum_{l=2^{k-1}+1}^{2^k} \frac{\sin lx}{l}$$

and

$$h_n(x) = \sum_{k=1}^{n} (-1)^k \omega\left(\frac{1}{2^k}\right) R_k(x).$$

It is enough to show that there is a constant K[1] for which

(2.3) $$|f_0(x) - f_0(x+h)| \leqq K\omega(h)$$

is true for every $0<h<\frac{1}{2}$ and $0\leqq x\leqq \pi-h$.

Let us fix h, $\frac{1}{2^n}\leqq h<\frac{1}{2^{n-1}}$, and for the moment let $h\leqq x\leqq \pi-h$. By the well-known inequality

(2.4) $$\left|\sum_{k=p}^{q} a_k \sin kx\right| \leqq \frac{4}{x} a_p \quad (a_p \geqq a_{p+1} \geqq \dots,\ x\in(0, \pi])$$

we have $|R_k(x)| \leqq 8\, 2^{-(k-1)} x^{-1}$.

[1] K, C and these with indices denote constants not necessarily the same at each occurrence.

Using this we get

$$|f_0(x)-f_0(x+h)| \leqq |h_n(x)-h_n(x+h)|+\sum_{k=n+1}^{\infty}\omega\left(\frac{1}{2^k}\right)\frac{8}{2^{k-1}}\left(\frac{1}{x}+\frac{1}{x+h}\right) \leqq$$

$$\leqq |h_n'(\vartheta)|\,h+\omega\left(\frac{1}{2^n}\right)\frac{16}{h}\sum_{k=n+1}^{\infty}\frac{1}{2^{k-1}} \leqq |h_n'(\vartheta)|\,h+32\omega(h),$$

where $\vartheta\in(x, x+h)$.

Thus it is enough to prove

$$|h_n'(\vartheta)| \leqq K\omega\left(\frac{1}{2^n}\right)2^n \quad (0\leqq\vartheta\leqq\pi). \tag{2.5}$$

We have

$$h_n'(\vartheta) = 10\sum_{k=1}^{n}(-1)^k\omega\left(\frac{1}{2^k}\right)2^k\cos 5\cdot 2^k\vartheta\sum_{l=2^{k-1}+1}^{2^k}\frac{\sin l\vartheta}{l}+ \tag{2.6}$$

$$+2\sum_{k=1}^{n}(-1)^k\omega\left(\frac{1}{2^k}\right)2^k\frac{\sin 5\cdot 2^k\vartheta}{2^k}\sum_{l=2^{k-1}+1}^{2^k}\cos l\vartheta = A_1(\vartheta)+A_2(\vartheta).$$

Let now ϑ be fixed, m be the least natural number for which $\frac{1}{2^m}\leqq\vartheta$ and $\mu=\min(m, n)$. By an elementary calculation we get that if

$$C_k(\vartheta) = \cos 5\cdot 2^k\vartheta\sum_{l=2^{k-1}+1}^{2^k}\frac{\sin l\vartheta}{l}$$

and

$$D_k(\vartheta) = \frac{\sin 5\cdot 2^k\vartheta}{2^k}\sum_{l=2^{k-1}+1}^{2^k}\cos l\vartheta$$

then

$$C_k(\vartheta)\leqq C_{k+1}(\vartheta),\quad D_k(\vartheta)\leqq D_{k+1}(\vartheta)\quad (1\leqq k\leqq m-7),$$

$$|C_k(\vartheta)|\leqq 1;\quad |D_k(\vartheta)|\leqq 1\quad (k=1,2,).$$

From the subadditivity of ω it follows that $2^k\omega\left(\frac{1}{2^k}\right)\leqq 2^{k+1}\omega\left(\frac{1}{2^{k+1}}\right)$. This, (2.4), and the previous inequalities give [2]

$$\frac{1}{10}A_1(\vartheta)\leqq\left|\sum_{k=1}^{\mu-7}(-1)^k\omega\left(\frac{1}{2^k}\right)2^kC_k(\vartheta)\right|+\sum_{k=\mu-6}^{n}\omega\left(\frac{1}{2^k}\right)2^k|C_k(\vartheta)|\leqq \tag{2.7}$$

$$\leqq\left|\omega\left(\frac{1}{2^{\mu-7}}\right)2^{\mu-7}C_{\mu-7}(\vartheta)\right|+7\omega\left(\frac{1}{2^n}\right)2^n+\omega\left(\frac{1}{2^n}\right)2^n\sum_{k=m-6}^{n}\frac{4}{2^{k-1}\vartheta}\leqq K\omega\left(\frac{1}{2^n}\right)2^n$$

and

$$\frac{1}{2}|A_2(\vartheta)|\leqq\left|\sum_{k=1}^{\mu-7}(-1)^k\omega\left(\frac{1}{2^k}\right)2^kD_k(\vartheta)\right|+\sum_{k=\mu-6}^{n}\omega\left(\frac{1}{2^k}\right)2^k|D_k(\vartheta)|\leqq \tag{2.8}$$

$$\leqq K\omega\left(\frac{1}{2^n}\right)2^n+7\omega\left(\frac{1}{2^n}\right)2^n+\omega\left(\frac{1}{2^n}\right)2^n\sum_{k=m-6}^{n}\frac{4}{2^k\vartheta}\leqq K\omega\left(\frac{1}{2^n}\right)2^n,$$

[2] If $m\geqq n$ then $\sum_{m-6}^{n}$ does not occur below.

where we used the well-known estimation

$$\left|\sum_{l=p}^{q} \cos lx\right| \leqq \frac{4}{x} \quad (0 < x \leqq \pi).$$

(2.6), (2.7) and (2.8) verify (2.5), and so we proved (2.3) for $h \leqq x \leqq \pi - h$. Let now $0 < x \leqq h$. It is clear that

$$|f_0(x) - f_0(x+h)| \leqq 2 \sup_{0 \leqq x \leqq 2h} |f_0(x)|, \tag{2.9}$$

and for $2^{-n} \leqq x < 2^{-(n-1)}$ we have

$$|f_0(x)| \leqq |h_n(x) - h_n(0)| + \omega\left(\frac{1}{2^n}\right) \sum_{k=n+1}^{\infty} \frac{8}{2^{k-1}x} \leqq$$

$$\leqq |h_n'(\vartheta)|x + 16\omega\left(\frac{1}{2^n}\right) \quad (\vartheta \in (0, x)).$$

Using again (2.5) we get that

$$|f_0(x)| \leqq K2^n \omega\left(\frac{1}{2^n}\right) x + 16\omega\left(\frac{1}{2^n}\right) \leqq K\omega(h).$$

This and (2.9) prove (2.3) for $0 < x \leqq h$, and so the proof of (2.3) is complete. The proof of (2.2) is similar, therefore we omit the details.

LEMMA 2. *Let* $a_n \geqq 0$. *Then*

$$\sum_{i=0}^{n} a_i \leqq K a_n \quad (n = 1, 2, \ldots)$$

holds if and only if there exist two numbers $c_1 > 0$, $c_2 > 1$, *and a natural number* μ *such that for any* n, $a_{n+1} \geqq c_1 a_n$ *and* $a_{n+\mu} > c_2 a_n$ *are valid.*

The proof can be given by an elementary calculation.

Hence, in respect to the fact that (1) holds if and only if $\alpha > \omega_0$, we obtain immediately

LEMMA 3. *For any* $p > 0$

$$\sum_{k=0}^{n} 2^{k\alpha p} \left[\omega\left(\frac{1}{2^k}\right)\right]^p \leqq K_\alpha^{(p)} 2^{n\alpha p} \left[\omega\left(\frac{1}{2^n}\right)\right]^p \quad (n = 1, 2, \ldots)$$

holds if and only if $\alpha > \omega_0$.

LEMMA 4. *Let* ω *be a concave modulus of continuity, and*

$$a_k = \omega\left(\frac{1}{k}\right) - \frac{k-1}{k}\, \omega\left(\frac{1}{k-1}\right) \quad (k = 2, 3, \ldots).$$

For the function

$$f(x) = \sum_{k=2}^{\infty} a_k \sin kx$$

we have

$$\omega(f; \delta) = O(\omega(\delta)).$$

PROOF. We show that

$$\text{(i)}\quad a_k \geqq a_{k+1} \qquad \text{(ii)}\quad 0 \leqq a_k k \leqq \omega\left(\frac{1}{k}\right) \quad (k=2, 3, \ldots).$$

From the relations

$$\frac{1}{k} = \frac{k+1}{2k+1}\cdot\frac{1}{k+1} + \frac{(k+1)(k-1)}{k(2k+1)}\,\frac{1}{k-1}$$

and

$$\frac{k+1}{2k+1} + \frac{(k+1)(k-1)}{k(2k+1)} < 1,$$

using the concavity of ω we obtain

$$\omega\left(\frac{1}{k}\right) \geqq \frac{k+1}{2k+1}\,\omega\left(\frac{1}{k+1}\right) + \frac{(k+1)(k-1)}{k(2k+1)}\,\omega\left(\frac{1}{k-1}\right)$$

i.e.

$$\omega\left(\frac{1}{k}\right) - \frac{k-1}{k}\,\omega\left(\frac{1}{k-1}\right) \geqq \omega\left(\frac{1}{k+1}\right) - \frac{k}{k+1}\,\omega\left(\frac{1}{k}\right),$$

and this is exactly (i).

$$k\omega\left(\frac{1}{k}\right) \geqq (k-1)\,\omega\left(\frac{1}{k-1}\right),$$

and so $a_k \geqq 0$. The right side of (ii) follows from the monotonicity of ω.

In order to prove the relation $f \in H^\omega$, it is enough to show (see the proof of Lemma 1) that

$$|f(0)-f(h)| \leqq K\omega(h) \tag{2.10}$$

and

$$|f(x)-f(x+h)| \leqq K\omega(h) \quad (h \leqq x \leqq \pi - h) \tag{2.11}$$

are satisfied with a constant K.

First we show (2.11). Let $\frac{1}{n} \leqq h < \frac{1}{n-1}$. By (i) and (ii) we obtain

$$|f(x)-f(x+h)| \leqq \left|\sum_{k=2}^{n} a_k(\sin kx - \sin k(x+h))\right| + \left|\sum_{k=n+1}^{\infty} a_k \sin kx\right| +$$

$$+\left|\sum_{k=n+1}^{\infty} a_k \sin k(x+h)\right| \leqq h\left|\sum_{k=2}^{n} a_k k\cos k\vartheta\right| + \frac{4}{x}\,a_{n+1} + \frac{4}{x+h}\,a_{n+1} \leqq$$

$$\leqq h\sum_{k=2}^{n} a_k k + 4na_{n+1} + 4na_{n+1} \leqq h\left(n\omega\left(\frac{1}{n}\right) - \omega(1)\right) + 8\omega\left(\frac{1}{n}\right) \leqq K\omega(h)$$

where we used (2.4).

The verification of (2.10) follows the same lines:

$$|f(0)-f(h)| \leqq h\left|\sum_{k=2}^{\infty} ka_k \cos k\vartheta\right| + \frac{4}{h}\,a_{n+1} \leqq K\omega(h).$$

The proof of Lemma 4 is now complete.

LEMMA 5. *To every r and ω there exists a function $f \in W^r H^\omega$ such that*

$$|s_{n\pm\lambda}(f;\,0)-f(0)| > 10^{-2}\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\log\frac{n}{\lambda} \tag{2.12}$$

holds for infinitely many n and $\lambda < ne^{-100}$.

We mention that the existence of $f \in W^r H^\omega$ for which

$$|\tilde{s}_{n\pm\lambda}(f;\,0)-\tilde{f}(0)| \geqq 10^{-2}\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\log\frac{n}{\lambda} \quad \left(\frac{n}{\lambda} > e^{100}\right)$$

hold for infinitely many n could be proved similarly.

PROOF. First we show that to every $\frac{\pi}{2} \geqq a > 0$ and M there exist a function $f = f_{a,M}$ and a natural number $n = n(a, M)$ for which $n > M$, and

(i) f is $(r+2)$-times continuously differentiable, and

$$|f^{(i)}| \leqq K_r\omega\left(\frac{1}{n}\right)n^{i-r} \quad (i = 0, 1, \ldots, r+1),$$

(ii) $f(x) = 0$ if $x \notin \left(\frac{1}{n};\, a\right)$,

(iii) if $\lambda < ne^{-100}$ then

$$|s_{n\pm\lambda}(f;\,0)-f(0)| > \frac{1}{30}\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\log\frac{n}{\lambda}.$$

Let $h(x)$ be an $(r+2)$-times continuously differentiable function for which

$$0 \leqq h(x) \leqq 1 \quad \text{and} \quad h(x) = \begin{cases} 0 & \text{if} \quad x \leqq 0 \\ 1 & \text{if} \quad x \geqq 1, \end{cases}$$

and let

$$K = \max_{0 \leqq i \leqq r+1}\ \max_{0 \leqq x \leqq 1} |h^{(i)}(x)|.$$

Let $n = n(a, M) > \left(M + \frac{1}{a}\right)$ be a natural number, which we shall choose later.

By a suitable choice of n the function

$$f(x) = f_{a,M}(x) = h\left(n\left(x - \frac{1}{n}\right)\right)h\big(n(a-x)\big)\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\sin nx$$

will satisfy our requirements.

(ii) is clear for f. It is also clear that f is $(r+2)$-times continuously differentiable. For the i-th derivative we get

$$|f^{(i)}(x)| = \frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\left|\sum_{\substack{i_1+i_2+i_3=i\\ i_1,i_2,i_3\geqq 0}} \frac{i!}{i_1!\,i_2!\,i_3!}\left(h\left(n\left(x-\frac{1}{n}\right)\right)\right)^{(i_1)}(h(n(a-x)))^{(i_2)}\cdot\right.$$

$$\left.\cdot(\sin nx)^{(i_3)}\right| \leqq \frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)n^i \sum_{\substack{i_1+i_2+i_3=i\\ i_1,i_2,i_3\geqq 0}} \frac{i!}{i_1!\,i_2!\,i_3!}K^i \leqq (3K)^i\,\omega\left(\frac{1}{n}\right)n^{i-r}$$

and this proves (i).

Finally for $\lambda<n$

$$|s_{n\pm\lambda}(f;0)-f(0)| = \left|\frac{1}{\pi}\int_{1/n}^{a}\frac{f(t)\sin(n\pm\lambda+\frac{1}{2})t}{2\sin\frac{t}{2}}\,dt\right| \geqq \tag{2.13}$$

$$\geqq \left|\frac{1}{\pi}\int_{2/n}^{a-1/n}\frac{f(t)\sin(n\pm\lambda+\frac{1}{2})}{2\sin\frac{t}{2}}\,dt\right| - \frac{1}{\pi}\int_{1/n}^{2/n}\frac{|f(t)|}{2\sin\frac{t}{2}}\,dt -$$

$$-\frac{1}{\pi}\int_{a-1/n}^{a}\frac{|f(t)|}{2\sin\frac{t}{2}}\,dt = I_1-I_2-I_3.$$

Because of $|f|\leqq\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)$ and $\sin x\geqq\frac{2}{\pi}x\left(0\leqq x\leqq\frac{\pi}{2}\right)$ one gets

$$I_2+I_3 \leqq 2\frac{1}{\pi}\frac{1}{n}\left(\frac{2}{\pi}\frac{1}{n}\right)^{-1}\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right) = \frac{1}{n^r}\,\omega\left(\frac{1}{n}\right) \tag{2.14}$$

easily.

If $x\in\left(\frac{2}{n},\,a-\frac{1}{n}\right)$ then $f(x)=\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\sin nx$, and so

$$I_1 = \frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\frac{1}{\pi}\left|\int_{2/n}^{a-1/n}\frac{\frac{1}{2}(\cos(\lambda\pm\frac{1}{2})t-\cos(2n+\frac{1}{2}\pm\lambda)t)}{2\sin\frac{t}{2}}\,dt\right| \geqq \tag{2.15}$$

$$\geqq \frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\frac{1}{2\pi}\left|\int_{2/n}^{a-1/n}\frac{\cos(\lambda\pm\frac{1}{2})t}{2\sin\frac{t}{2}}\,dt\right| -$$

$$-\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\frac{1}{2\pi}\left|\int_{2/n}^{a-1/n}\frac{\cos(2n+\frac{1}{2}\pm\lambda)t}{2\sin\frac{t}{2}}\,dt\right| = \frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)I_{11}-\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)I_{12}.$$

If $\dfrac{\pi}{4(\lambda+\frac{1}{2})}\geqq a-\dfrac{1}{n}$, then

$$I_{11}\geqq \frac{1}{2\pi}\int_{2/n}^{a-1/n}\frac{1/2}{2\sin\frac{t}{2}}\,dt>\frac{1}{20}\log n$$

provided that n is large enough, e.g. $n\geqq n_a$. If however $\dfrac{\pi}{4\left(\lambda+\dfrac{1}{2}\right)}<a-\dfrac{1}{n}$, then the second mean value theorem gives for $\dfrac{n}{\lambda}>e^{100}$

$$I_{11}\geqq \frac{1}{2\pi}\left|\int_{2/n}^{\pi/4(\lambda+\frac{1}{2})}\frac{\cos(\lambda\pm\frac{1}{2})t}{2\sin\frac{t}{2}}\,dt\right|-\frac{1}{2\pi}\left|\int_{\pi/4(\lambda+\frac{1}{2})}^{a-1/n}\frac{\cos(\lambda\pm\frac{1}{2})}{2\sin\frac{t}{2}}\,dt\right|\geqq$$

$$\geqq \frac{1}{4\pi}\int_{2/n}^{\pi/4(\lambda+\frac{1}{2})}\frac{1}{t}\,dt-\frac{1}{2\pi}\,\frac{1}{2\sin\dfrac{\pi}{8(\lambda\pm\frac{1}{2})}}\left|\int_{\pi/4(\lambda+\frac{1}{2})}^{\xi}\cos\left(\lambda\pm\frac{1}{2}\right)t\,dt\right|\geqq$$

$$\geqq \frac{1}{4\pi}\left(\log\frac{n}{\lambda}+\log\frac{\lambda\pi}{8(\lambda+\frac{1}{2})}\right)-\frac{6}{\pi}\geqq\frac{1}{20}\log\frac{n}{\lambda}.$$

Thus we get that if $n\geqq n_a$ and $\dfrac{n}{\lambda}>e^{100}$ then

$$I_{11}\geqq\frac{1}{20}\log\frac{n}{\lambda}. \tag{2.16}$$

Using again the second mean value theorem we obtain

$$I_{12}\leqq\frac{1}{2\pi}\left(2\frac{2}{\pi}\frac{1}{2}\frac{2}{n}\right)^{-1}\frac{2}{2n+\frac{1}{2}\pm\lambda}\leqq\frac{1}{4}. \tag{2.17}$$

By (2.15), (2.16) and (2.17), if $n\geqq n_a$, $\dfrac{n}{\lambda}>e^{100}$ then

$$I_1\geqq\frac{1}{22}\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\log\frac{n}{\lambda}$$

and this with (2.13), (2.14) prove (iii).

Thus, if we choose n greater than n_a and $\left(M+\dfrac{1}{a}\right)$, the above $f_{a,M}$ and n satisfy (i)—(iii).

Now we turn to the proof of Lemma 5.

First we define four sequences, $\{g_m\}$, $\{n_m\}$, $\{a_m\}$, $\{M_m\}$ as follows: Let $a_1 = \frac{\pi}{2}$, $M_1=1$, $g_1=f_{a_1,M_1}$ and $n_1=n(a_1, M_1)$. Let us suppose that g_k, n_k, a_k, M_k are already defined for $1\leqq k\leqq m-1$, $a_k>0$ and the g_k's are taken from the above functions $f_{a,M}$. The function $\sum_{k=1}^{m-1} g_k$ is (by (i)) $(r+2)$-times continuously differentiable, therefore

$$\sum_{k=1}^{m-1} g_k(x) - s_n\left(\sum_{k=1}^{m-1} g_k;\, x\right) = O\left(\frac{\log n}{n^{r+2}}\right)$$

and thus there is an M_m so that in the case $n>M_m$, $\frac{n}{\lambda}>2$ we have

$$\left|\sum_{k=1}^{m-1} g_k(0) - s_{n\pm\lambda}\left(\sum_{k=1}^{m-1} g_k;\, 0\right)\right| < \frac{1}{n^r}\,\omega\left(\frac{1}{n}\right) \tag{2.18}$$

and

$$\frac{1}{n^r}\,\omega\left(\frac{1}{n}\right)\log 2n_k < \frac{1}{3}\,\frac{1}{n_k^r}\,\omega\left(\frac{1}{n_k}\right) \quad (k=1,2,\dots,m-1). \tag{2.19}$$

Now let $a_m>0$ be a number for which $g_k(x)=0$ if $x\in(0, a_m)$ for every $1\leqq k\leqq m-1$.

Let

$$g_m(x) = f_{a_m, M_m}(x) \quad\text{and}\quad n_m = n(a_m, M_m).$$

We obtain hereby the required sequences.

Let

$$f(x) = \sum_{m=1}^{\infty} g_m(x).$$

We show that f satisfies Lemma 5.

Taking into account that the supports of the functions g_m (where they are not 0) are disjoint, (i) gives that the series $\sum_{m=1}^{\infty} g_m^{(i)}(x)$ $(i=0, 1, \dots, r)$ are uniformly convergent from which the r-times differentiability of f follows. By (i) and (ii) we have that if $0<h\leqq\frac{1}{n_1}$ is arbitrary and $\frac{1}{n_{m+1}}<h\leqq\frac{1}{n_m}$, then for every x

$$|f^{(r)}(x+h)-f^{(r)}(x)| \leqq \left|\sum_{k=1}^{m} g_k^{(r)}(x+h) - \sum_{k=1}^{m} g_k^{(r)}(x)\right| + 2\max_{m+1\leqq k<\infty}\ \max_{-\pi\leqq x\leqq\pi} |g_k^{(r)}(x)| \leqq$$

$$\leqq h\left|\sum_{k=1}^{m} g_k^{(r+1)}(x+\vartheta h)\right| + 2\omega\left(\frac{1}{n_{m+1}}\right) \leqq Kh\left(\max_{1\leqq k\leqq m} n_k\,\omega\left(\frac{1}{n_k}\right)\right) + 2\omega(h) \leqq$$

$$\leqq Kh\left(2\frac{\omega(h)}{h}\right) + 2\omega(h) \leqq K\omega(h)$$

i.e. $f^{(r)}\in H^\omega$ $\left(\text{here we used that if } h'>h \text{ then } \frac{\omega(h')}{h'}\leqq 2\,\frac{\omega(h)}{h}\right)$.

Finally if $n=n_m$ and $\frac{n}{\lambda}>e^{100}$ then the known inequality

$$|s_k(g;x)-g(x)|<3\,(\max|g|)\log k$$

as well as (2.18), (2.19) and (iii) give

$$|f(0)-s_{n\pm\lambda}(f;0)|\geqq|g_m(0)-s_{n\pm\lambda}(g_m;0)|-\left|\sum_{k=1}^{m-1}g_k(0)-s_{n\pm\lambda}\left(\sum_{k=1}^{m-1}g_k;0\right)\right|-$$

$$-\left|\sum_{k=m+1}^{\infty}g_k(0)-s_{n\pm\lambda}\left(\sum_{k=m+1}^{\infty}g_k;0\right)\right|\geqq\frac{1}{30}\frac{1}{n^r}\omega\left(\frac{1}{n}\right)\log\frac{n}{\lambda}-$$

$$-\frac{1}{n^r}\omega\left(\frac{1}{n}\right)-3\max\left|\sum_{k=m+1}^{\infty}g_k\right|\log(n\pm\lambda)\geqq 3\cdot 10^{-2}\frac{1}{n^r}\omega\left(\frac{1}{n}\right)\log\frac{n}{\lambda}-$$

$$-\frac{1}{n^r}\omega\left(\frac{1}{n}\right)-\frac{1}{n^r}\omega\left(\frac{1}{n}\right)>10^{-2}\frac{1}{n^r}\omega\left(\frac{1}{n}\right)\log\frac{n}{\lambda},$$

i.e. (2.12) holds for any $n=n_m$.

Thus the proof of Lemma 5 is complete.

3. PROOF OF THEOREM 1. We use the following inequality (see [2, p. 260]): if $2^{m_0-1}<n\leqq 2^{m_0}$ then

$$h_n(f,p,\beta;x)\leqq K\left\{\frac{1}{n^\beta}\sum_{m=2}^{m_0}2^{m\beta}\left(E_{2^{m-2}}(f)\right)^p\right\}^{1/p}. \tag{3.1}$$

If $f\in W^rH^\omega$ then $E_m(f)=O\left(\frac{1}{m^r}\omega\left(\frac{1}{m}\right)\right)$, and so

$$h_n(f,p,\beta;x)\leqq K\left\{\frac{1}{n^\beta}\sum_{m=2}^{m_0}2^{m(\beta-1)}2^m\left(\frac{1}{2^{mr}}\omega\left(\frac{1}{2^m}\right)\right)^p\right\}^{1/p}=$$

$$=O\left(\left\{\frac{1}{n^\beta}\sum_{k=1}^{2^{m_0}}(k+1)^{\beta-1}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p\right)=O(H_{r,\omega}^{p,\beta,n}),$$

and this is (2).

In the proof of (3) and (4) we use the inequality

$$E_n(\tilde f)\leqq K\left(E_n(f)+\sum_{\nu=n+1}^{\infty}\frac{1}{\nu}E_\nu(f)\right)$$

(see [8, p. 320]), by which if $f\in W^rH^\omega$ and $r>0$ then $E_n(\tilde f)=O\left(\frac{1}{n^r}\omega\left(\frac{1}{n}\right)\right)$, while $E_n(\tilde f)=O\left(\omega^*\left(\frac{1}{n}\right)\right)$ if $r=0$.

Using this we obtain (3) and (4) from (3.1) similarly as we have got (2).

In order to prove (5) and (6) let us consider the functions

$$f_r(x) = \sum_{\nu=1}^{\infty} (-1)^\nu \omega\left(\frac{1}{2^\nu}\right) \sum_{l=2^{\nu-1}+1}^{2^\nu} \left(\frac{\cos(5\cdot 2^\nu - l)x}{(5\cdot 2^\nu - l)^r l} - \frac{\cos(5\cdot 2^\nu + l)x}{(5\cdot 2^\nu + l)^r l}\right) +$$

$$+ \sum_{\nu=1}^{\infty} (-1)^\nu \omega\left(\frac{1}{2^\nu}\right) \sum_{l=2^{\nu-1}+1}^{2^\nu} \left(\frac{\sin(5\cdot 2^\nu - l)x}{(5\cdot 2^\nu - l)^r l} - \frac{\sin(5\cdot 2^\nu + l)x}{(5\cdot 2^\nu + l)^r l}\right).$$

One may integrate Fourier series term by term, thus Lemma 1 shows that $f_r \in W^r H^\omega$ and $\tilde{f}_r \in W^r H^\omega$. Let

$$R_\nu = \sum_{l=2^{\nu-1}+1}^{2^\nu} \left(\frac{1}{(5\cdot 2^\nu - l)^r l} - \frac{1}{(5\cdot 2^\nu + l)^r l}\right) = \sum_{l=2^{\nu-1}+1}^{2^\nu} r(\nu, l).$$

It is clear that

$$r(\nu, l+1) \geqq r(\nu, l) \quad \text{and} \quad r(\nu+1, 2l) = \frac{1}{2^{r+1}} r(\nu, l),$$

and this implies

$$R_{\nu+1} = \sum_{l=2^{\nu-1}+1}^{2^\nu} (r(\nu+1, 2l-1) + r(\nu+1, 2l)) \leqq \sum_{l=2^{\nu-1}+1}^{2^\nu} 2r(\nu+1, 2l) =$$

$$= 2\frac{1}{2^{\nu+1}} \sum_{l=2^{\nu-1}+1}^{2^\nu} r(\nu, l) \leqq R_\nu$$

from which

$$\omega\left(\frac{1}{2^\nu}\right) R_\nu \geqq \omega\left(\frac{1}{2^{\nu+1}}\right) R_{\nu+1} \tag{3.2}$$

follows.

Now if $5\cdot 2^\nu - 2^{\nu-1} \leqq k \leqq 5\cdot 2^\nu + 2^{\nu-1}$, then

$$f_r(0) - s_k(f_r; 0) = (-1)^{\nu+1} \omega\left(\frac{1}{2^\nu}\right) \sum_{l=2^{\nu-1}+1}^{2^\nu} \frac{1}{(5\cdot 2^\nu + l)^r l} +$$

$$+(-1)^{\nu+1}\left(\omega\left(\frac{1}{2^{\nu+1}}\right) R_{\nu+1} - \omega\left(\frac{1}{2^{\nu+2}}\right) R_{\nu+2} + \ldots\right),$$

and so, by (3.2), we obtain

$$|s_k(f_r; 0) - f_r(0)| \geqq \omega\left(\frac{1}{2^\nu}\right) \sum_{l=2^{\nu-1}+1}^{2^\nu} \frac{1}{(5\cdot 2^\nu + l)^r l} \geqq c\frac{1}{(2^\nu)^r} \omega\left(\frac{1}{2^\nu}\right) \tag{3.3}$$

$$(\nu = 1, 2, \ldots, 5\cdot 2^\nu - 2^{\nu-1} \leqq k \leqq 5\cdot 2^\nu + 2^{\nu-1}),$$

where c is a positive constant independent of ν and k.

(3.3) already gives (5), namely if $6\cdot 2^{n_0-1} \leqq n < 6\cdot 2^{n_0}$ then

$$h_n(f_r, p, \beta; 0) \geqq \left\{\frac{1}{(n+1)^\beta} \sum_{\nu=0}^{n_0-1} \sum_{k=5\cdot 2^\nu + 2^{\nu-1}}^{5\cdot 2^\nu + 2^{\nu-1}} (k+1)^{\beta-1} |s_k(0) - f_r(0)|^p\right\}^{1/p} \geqq$$

$$\geqq c\left\{\frac{1}{(n+1)^\beta} \sum_{\nu=0}^{n_0-1} \sum_{k=5\cdot 2^\nu + 2^{\nu-1}}^{5\cdot 2^\nu + 2^{\nu-1}} (2^\nu)^{\beta-1} \left(\frac{1}{2^{\nu r}} \omega\left(\frac{1}{2^\nu}\right)\right)^p\right\}^{1/p} \geqq$$

$$\geqq c\left\{\frac{1}{(n+1)^\beta} \sum_{\nu=0}^{n_0-1} \sum_{k=5\cdot 2^\nu + 2^{\nu-1}}^{5\cdot 2^\nu + 2^{\nu-1}} (k+1)^{\beta-1} \left(\frac{1}{k^r} \omega\left(\frac{1}{k}\right)\right)^p\right\}^{1/p} \geqq cH_{r,\omega}^{p,\beta,n}.$$

The above proof of (5) — taking into account the form of $\tilde{f}_r$ — simultaneously proves (6), too.

To prove (7), it is enough to show that there is a function $f_0^* \in H^\omega$ such that if ν is even and $5\cdot 2^\nu - 2^{\nu-1} \leqq k \leqq 5\cdot 2^\nu + 2^{\nu-1}$, then

$$|\tilde{s}_k(f_0^*;0) - f_0^*(0)| \geqq c\omega^*\left(\frac{1}{k}\right) \tag{3.4}$$

(see the proof of (5)).

To every modulus of continuity ω one can find a concave $\bar{\omega}$, for which

$$\omega(\delta) \leqq \bar{\omega}(\delta) \leqq 2\omega(\delta) \quad (\delta \in [0, 2\pi])$$

(see [7, p. 45]). From our view-point ω and $\bar{\omega}$ are equivalent, so we may suppose that ω is a concave modulus of continuity.

Let us consider the function

$$g(x) = \sum_{l=2}^{\infty} a_l \sin lx \tag{3.5}$$

where

$$a_l = \omega\left(\frac{1}{l}\right) - \frac{l-1}{l}\,\omega\left(\frac{1}{l-1}\right),$$

and let

$$f_0^* = -g(x) + 2g_0(x).$$

By Lemmas 1 and 4, $f_0^* \in H^\omega$. If ν is even and $5\cdot 2^\nu - 2^{\nu-1} \leqq k \leqq 5\cdot 2^\nu + 2^{\nu-1}$, then

$$\tilde{f}_0^*(0) - \tilde{s}_k(f_0^*;0) = \sum_{l=k+1}^{\infty} a_l + 2\omega\left(\frac{1}{2^\nu}\right) \sum_{l=2^{\nu-1}+1}^{2^\nu} \frac{1}{l} \geqq \sum_{l=k+1}^{\infty} \frac{1}{l+1}\,\omega\left(\frac{1}{l}\right) -$$

$$-\frac{k}{k+1}\,\omega\left(\frac{1}{k}\right) + 2\frac{1}{2}\,\omega\left(\frac{1}{2^\nu}\right) \geqq \frac{1}{2}\int_0^{1/k} \frac{\omega(t)}{t}\,dt = \frac{1}{2}\,\omega^*\left(\frac{1}{k}\right),$$

what is exactly (3.4).

The proof of Theorem 1 is now complete.

The Remark follows at once from Theorem 1 by Lemma 3 and by the estimation

$$c_1 H_{r,\omega}^{p,\beta,n} \leqq \left\{\frac{1}{n^\beta} \sum_{k=0}^{\log n} (2^k)^{\beta - rp} \left(\omega\left(\frac{1}{2^k}\right)\right)^p\right\}^{1/p} \leqq c_2 H_{r,\omega}^{p,\beta,n} \quad (c_1, c_2 > 0).$$

Proof of Theorem 2. Using the last formula of [2], p. 260, we have for $2^{m_0-1} < n \leqq 2^{m_0}$

$$\sigma_n^\gamma|f,p;x| \leqq K\left\{\frac{1}{A_n^\gamma}\left(\sum_{m=2}^{m_0-2} + \sum_{m=m_0-1}^{m_0}\right)\left(\sum_{k=2^{m-2}+1}^{2^m} A_{n-k}^{\gamma-1}\right)(E_{2^{m-2}}(f))^p\right\}^{1/p} \leqq$$

$$\leqq K\left\{\frac{1}{n}\sum_{m=2}^{m_0-2} 2^m \left(\frac{1}{2^{mr}}\,\omega\left(\frac{1}{2^m}\right)\right)^p + \left(\frac{1}{2^{m_0 r}}\,\omega\left(\frac{1}{2^{m_0}}\right)\right)^p\right\}^{1/p} \leqq K H_{r,\omega}^{p,1,n},$$

(we used the estimation

$$c_1(\gamma)n^\gamma \leqq A_n^\gamma \leqq c_2(\gamma)n^\gamma \quad (n=1,2,\dots,\ \gamma>-1,\ c_1(\gamma),c_2(\gamma)>0))$$

and this proves (8). The proofs of (9) and (10) are similar.

(11), (12) and (13) follow from Theorem 1, e.g. if we apply (5) with $\beta=1$, we obtain [3]

$$\sigma_n^\gamma|f_r,p;0| \geqq \left\{\frac{1}{A_n^\gamma}\sum_{k=0}^{\left[\frac{n}{2}\right]} A_{n-k}^{\gamma-1}|s_k(0)-f_r(0)|^p\right\}^{1/p} \geqq$$

$$\geqq c\left\{\frac{1}{\left[\frac{n}{2}\right]}\sum_{k=0}^{\left[\frac{n}{2}\right]}|s_k(0)-f_r(0)|^p\right\}^{1/p} \geqq cH_{r,\omega}^{p,1,[n/2]} \geqq cH_{r,\omega}^{p,1,n}$$

and this is (11).

The proof of Theorem 2 is thus complete.

Proof of Theorem 3. Let

$$A_\nu(r,a,l;x) = \frac{\cos(a2^\nu-l)x}{(a2^\nu-l)^r\,l} - \frac{\cos(a2^\nu+l)x}{(a2^\nu+l)^r\,l}$$

and let us consider the functions

$$g_1(r;x) = g_1(x) = \sum_{\nu=1}^{\infty}(-1)^\nu\omega\left(\frac{1}{2^\nu}\right)\sum_{l=3\cdot2^{\nu-1}+1}^{2^\nu+1}(A_\nu(r,7,l;x)+\tilde{A}_\nu(r,7,l;x))$$

and

$$g_2(r;x) = g_2(x) = \sum_{\nu=1}^{\infty}(-1)^\nu\omega\left(\frac{1}{2^\nu}\right)\sum_{l=5\cdot2^{\nu-1}+1}^{3\cdot2^\nu}(A_\nu(r,10,l;x)+\tilde{A}_\nu(r,10,l;x)).$$

If after r-times differentiation we repeat the proof of Lemma 1 word by word we obtain that the functions $g_1, \tilde{g}_1, g_2, \tilde{g}_2$ are in W^rH^ω.

Furthermore, exactly as in the proof of Theorem 1, we get that if

$$7\cdot2^\nu-3\cdot2^{\nu-1} \leqq k \leqq 7\cdot2^\nu+3\cdot2^{\nu-1} \tag{3.6}$$

then

$$|s_k(g_1;0)-g_1(0)| \geqq c\left(\frac{1}{2^\nu}\right)^r\omega\left(\frac{1}{2^\nu}\right) \geqq c\frac{1}{k^r}\omega\left(\frac{1}{k}\right)$$

and

$$|\tilde{s}_k(g_1;0)-\tilde{g}_1(0)| \geqq c\frac{1}{k^r}\omega\left(\frac{1}{k}\right),$$

while if

$$10\cdot2^\nu-5\cdot2^{\nu-1} \leqq k \leqq 10\cdot2^\nu+5\cdot2^{\nu-1} \tag{3.7}$$

[3] $[y]$ denotes the integral part of y.

is true then

$$|s_k(g_2; 0)-g_2(0)| \geqq c\frac{1}{k^r}\omega\left(\frac{1}{k}\right)$$

and

$$|\tilde{s}_k(g_2; 0)-\tilde{g}_2(0)| \geqq c\frac{1}{k^r}\omega\left(\frac{1}{k}\right)$$

hold.

Now every $k>10$ satisfies either (3.6) or (3.7) for some v, and from this it follows easily that the functions

$$f_1(x) = -(\cos 11x+\sin 11x)+g_1(x); \quad f_2(x) = g_2(x)$$

satisfy Theorem 3. Q.E.D.

One can see similarly that the functions $g(x)\pm 6\big(g_1(0; x_m)-\omega(1)\sin 11x\big)$ and $g(x)\pm 6g_2(0; x)$ (where $g(x)$ is the function (3.5)) satisfy the requirements of the Remark.

The two corollaries follow at once from Theorem 3 (and from the Remark), namely it gives e.g. that

$$\sum_{k=1}^{\infty} t_{nk}\big(|s_k(f_1; 0)-f_1(0)|^p+|s_k(f_2; 0)-f_2(0)|^p\big) \geqq c\sum_{k=1}^{\infty} t_{nk}\left(\frac{1}{k^r}\omega\left(\frac{1}{k}\right)\right)^p$$

is true for every n.

Proof of Theorem 4. It is enough to show (17). By Hölder's inequality we may assume that $p\geqq 2$.

Let $T_m(x)$ denote the trigonometric polinomial of best approximation of order not higher than m.

The Minkowski inequality gives

$$\left\{\frac{1}{r}\sum_{i=1}^{r}|s_{k_i}(f; x)-f(x)|^p\right\}^{1/p} \leqq \left\{\frac{1}{r}\sum_{i=1}^{r}|s_{k_i}(f-T_{k_1}; x)|^p\right\}^{1/p} + \left\{\frac{1}{r}\sum_{i=1}^{r}|T_{k_1}(x)-f(x)|^p\right\}^{1/p}.$$

Here the second member of the right side is at most $E_{k_1}(f)$ and so it is enough to show that if $|f|\leqq M$ then

$$\left\{\frac{1}{r}\sum_{i=1}^{r}|s_{k_i}(f; x)|^p\right\}^{1/p} \leqq K_p M\log\frac{2n}{r} \tag{3.8}$$

where K_p depends only on p.

Using Dirichlet's formula we obtain

$$\sum_{i=1}^{r}|s_{k_i}(x)|^p \leqq K_p\left\{\sum_{i=1}^{r}\Big(\int_{-1/n}^{1/n}|f(x+t)|\,|D_{k_i}(t)|\,dt\Big)^p+\right.$$

$$\left.+\sum_{i=1}^{r}\Big(\int_I|f(x+t)|\,|D_{k_i}(t)|\,dt\Big)^p+\sum_{i=1}^{r}\Big|\int_J f(x+t)D_{k_i}(t)\,dt\Big|^p\right\} = S_1+S_2+S_3,$$

where $I=\left[-\frac{1}{r},\ -\frac{1}{n}\right]\cup\left[\frac{1}{n},\frac{1}{r}\right]$, $J=\left[-\pi,\ -\frac{1}{r}\right]\cup\left[\frac{1}{r},\ \pi\right]$. $|D_{k_i}(t)|\leqq n+1$ if $k_i\leqq n$, thus $S_1\leqq K_p M^p r$.

To estimate S_2 we use the inequality $|D_k(t)|\leqq\frac{\pi}{2}\frac{1}{|t|}$:

$$S_2\leqq\sum_{i=1}^{r}\left(\frac{\pi}{2}\right)^p K_p\left(2M\int_{1/n}^{1/r}\frac{1}{t}\,dt\right)^p\leqq K_p M^p r\left(\log\frac{2n}{r}\right)^p.$$

Finally

$$S_3=K_p\sum_{i=1}^{r}\left|\int_J f(x+t)\operatorname{cotg}\frac{t}{2}\sin k_i t\,dt+\int_J f(x+t)\cos k_i t\,dt\right|^p\leqq$$

$$\leqq K_p\sum_{i=1}^{r}\left(\left|\int_J\left(f(x+t)\operatorname{cotg}\frac{t}{2}\right)\sin k_i t\,dt\right|^p+\left|\int_J f(x+t)\cos k_i t\,dt\right|^p\right)=S_{31}+S_{32}.$$

Applying the Hausdorff—Young inequality to the functions

$$f^*(t)=\begin{cases} f(x+t)\operatorname{cotg}\frac{t}{2} & \text{if } \frac{1}{r}\leqq|t|\leqq\pi\\ 0 & \text{otherwise}\end{cases}$$

and

$$g^*(t)=\begin{cases} f(x+t) & \text{if } \frac{1}{r}\leqq|t|\leqq\pi\\ 0 & \text{otherwise,}\end{cases}$$

respectively, we obtain with $q=\frac{p}{p-1}$

$$S_{31}\leqq K_p\left\{\int_J\left|f(x+t)\operatorname{cotg}\frac{t}{2}\right|^q dt\right\}^{p/q}\leqq K_p M^p\left(\int_{1/r}^{\pi}\frac{dt}{t^q}\right)^{p/q}\leqq K_p M^p r$$

and

$$S_{32}\leqq K_p\left\{\int_J|f(x+t)|^q\,dt\right\}^{p/q}\leqq K_p M^p.$$

Collecting the above estimations we get Theorem 4.

PROOF OF THEOREM 5. Let us first consider (18).

For a given n we distinguish two cases according as λ_n larger than $\frac{n}{2}$ or not.

(i) $\lambda_n>\frac{n}{2}$. Let μ and ν be the largest and least number of the form 2^k which is not larger than $n-\lambda_n+1$ and which is not less than n, respectively.

By Theorem 4 if $f\in W^r H^\omega$ then

$$\left\{\frac{1}{m}\sum_{k=m}^{2m}|s_k-f|^p\right\}^{1/p}=O\left(\frac{1}{m^r}\,\omega\left(\frac{1}{m}\right)\right),$$

and so

$$V_n(f,\lambda,p;x)\leqq\left\{\frac{1}{\lambda_n}\sum_{l=\mu}^{\nu-1}\sum_{k=2^l}^{2^{l+1}}|s_k(x)-f(x)|^p\right\}^{1/p}\leqq$$

$$\leqq K\left\{\frac{1}{\lambda_n}\sum_{l=\mu}^{\nu-1}2^l\left(\frac{1}{2^{lr}}\,\omega\left(\frac{1}{2^l}\right)\right)^p\right\}^{1/p}\leqq K\left\{\frac{1}{\lambda_n}\sum_{k=2^\mu}^{2^\nu}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p\right\}^{1/p}\leqq K\Lambda_n(r,p,\omega).$$

(ii) Let now $\lambda_n\leqq\dfrac{n}{2}$. By Theorem 4

$$V_n(f,\lambda,p;x)\leqq KE_{n-\lambda_n}(f)\log\frac{2n}{\lambda_n}\leqq K\frac{1}{(n-\lambda_n)^r}\,\omega\left(\frac{1}{n-\lambda_n}\right)\log\frac{2n}{\lambda_n}\leqq K\Lambda_n(r,p,\omega).$$

(i) and (ii) prove (18).

The proof of (19) is similar in the cases $\lambda_n>\dfrac{n}{2}$ or $r>0$ $\Big($namely if $f\in W^r H^\omega$ then $E_n(\tilde f)=O\left(\dfrac{1}{n^r}\,\omega\left(\dfrac{1}{n}\right)\right)$ for $r>0$ and $E_n(\tilde f)=O\left(\omega^*\left(\dfrac{1}{n}\right)\right)$ for $r=0\Big)$.

If however $\lambda_n\leqq\dfrac{n}{2}$ and $r=0$, (19) will follow from

$$\left\{\frac{1}{\lambda_n}\sum_{k=n-\lambda_n+1}^{n}|\tilde s_k(x)-\tilde f(x)|^p\right\}^{1/p}\leqq K\left(\omega\left(\frac{1}{n}\right)\log\frac{2n}{\lambda_n}+\int_0^{1/n}\frac{\omega(t)}{t}\,dt\right). \tag{3.9}$$

Our next aim is to prove (3.9). We may suppose that $\displaystyle\int_0^1\frac{\omega(t)}{t}\,dt<\infty$, and thus $\tilde f(x)$ is defined for every $f\in H^\omega$ at every point x.

With the notation $\psi_x(t)=\dfrac{1}{2}\left(f(x+t)-f(x-t)\right)$ we have

$$\tilde s_k(x)-\tilde f(x)=\frac{2}{\pi}\int_0^\pi\frac{\psi_x(t)\cos(k+\frac12)t}{2\sin\frac{t}{2}}\,dt=\frac{2}{\pi}\int_0^{1/n}\frac{\psi_x(t)\cos(k+\frac12)t}{2\sin\frac{t}{2}}\,dt+$$

$$+\frac{2}{\pi}\int_{1/n}^{\pi}\frac{\psi_x(t)\cos(k+\frac12)t}{2\sin\frac{t}{2}}\,dt=I_k^{(1)}(x)+I_k^{(2)}(x)\quad(n-\lambda_n<k\leqq n).$$

Evidently

$$|I_k^{(1)}(x)|\leqq K\int_0^{1/n}\frac{\omega(t)}{t}\,dt.$$

Using the function $\Psi_x(t)=n\int\limits_t^{t+1/n}\psi_x(\tau)\,d\tau$ we get by integration by parts

$$I_k^{(2)}(x)=\frac{2}{\pi}\int\limits_{1/n}^{\pi}\frac{(\psi_x(t)-\Psi_x(t))\cos(k+\frac12)t}{2\sin\frac{t}{2}}\,dt+\frac{2}{\pi}\frac{1}{k+\frac12}\frac{\Psi_x(t)\sin(k+\frac12)t}{2\sin\frac{t}{2}}\Bigg|_{\frac1n}^{\pi}-$$

$$-\frac{2}{\pi}\frac{n}{k+\frac12}\int\limits_{1/n}^{\pi}\frac{(\psi_x(t+1/n)-\psi_x(t))\sin(k+\frac12)t}{2\sin\frac{t}{2}}\,dt+$$

$$+\frac{2}{\pi}\frac{1}{k+\frac12}\int\limits_{1/n}^{1/\lambda_n}\frac{\Psi_x(t)\cos\frac{t}{2}}{\left(2\sin\frac{t}{2}\right)^2}\sin\left(k+\frac12\right)t\,dt+$$

$$+\frac{2}{\pi}\frac{1}{k+\frac12}\int\limits_{1/\lambda_n}^{\pi}\frac{\Psi_x(t)\cos\frac{t}{2}}{\left(2\sin\frac{t}{2}\right)^2}\sin\left(k+\frac12\right)t\,dt=I_k^{(3)}(x)+\left(I_k^{(4)}(x)-I_k^{(5)}(x)\right)-$$

$$-I_k^{(6)}(x)+I_k^{(7)}(x)+I_k^{(8)}(x).$$

As $|\Psi_x(t)|\leqq K\omega(t)$ $\left(t\in\left[\frac1n,\pi\right]\right)$, it is clear that $|I_k^{(4)}(x)|+|I_k^{(5)}(x)|\leqq K\omega\left(\frac1n\right)$,

$$|I_k^{(7)}(x)|\leqq K\frac1n\int\limits_{1/n}^{1/\lambda_n}\frac{\omega(t)}{t^2}\,dt\leqq K\frac1n\frac{\omega(1/n)}{1/n}\int\limits_{1/n}^{1/\lambda_n}\frac{dt}{t}\leqq K\omega\left(\frac1n\right)\log\frac{2n}{\lambda_n}$$

$$(n/2<k\leqq n).$$

Finally, exactly as in the proof of Theorem 4 we get

$$\left\{\frac{1}{\lambda_n}\sum_{k=n-\lambda_n+1}^{n}|I_k^{(3)}(x)|^p\right\}^{1/p}\leqq K\omega\left(\frac1n\right)\log\frac{2n}{\lambda_n},$$
$$\left\{\frac{1}{\lambda_n}\sum_{k=n-\lambda_n+1}^{n}|I_k^{(6)}(x)|^p\right\}^{1/p}\leqq K\omega\left(\frac1n\right)\log\frac{2n}{\lambda_n}$$

and (we may assume $p\geqq2$)

$$\left\{\frac{1}{\lambda_n}\sum_{k=n-\lambda_n+1}^{n}|I_k^{(8)}(x)|^p\right\}^{1/p}\leqq K\frac1n\lambda_n^{-1/p}\left(\int\limits_{1/\lambda_n}^{\pi}\left(\frac{\omega(t)}{t^2}\right)^q\right)^{1/q}\leqq$$

$$\leqq K\frac1n\lambda_n^{-1/p}\frac{\omega(1/n)}{1/n}\left(\int\limits_{1/\lambda_n}^{\pi}\frac{dt}{t^q}\right)^{1/q}\leqq K\omega\left(\frac1n\right)\quad\left(q=\frac{p}{p-1}\right)$$

uniformly in x $\left(\text{taking into account that } |\Psi_x(t)-\psi_x(t)|\leqq K\omega\left(\frac1n\right) \text{ and } \left|\psi_x\left(t+\frac1n\right)-\right.\right.$ $\left.\left.-\psi_x(t)\right|\leqq K\omega\left(\frac1n\right)\right)$.

Collecting the above inequalities we obtain (3.9).

If there are infinitely many n for which $\frac{n}{\lambda_n} \leqq e^{100}$ then (20) and (21) follow from Corollary 1 of Theorem 3.

Thus we may suppose that $\frac{n}{\lambda_n} > e^{100}$ for all large enough n. By Lemma 5 there is a function $f_1 \in W^r H^\omega$ for which

$$V_n(f_1, \lambda, p; 0) \geqq \left\{ \frac{1}{\lambda_n} \sum_{k=n-\lambda_n+1}^{n} \left(c \frac{1}{n^r} \omega\left(\frac{1}{n}\right) \log \frac{2n}{n-k+1} \right)^p \right\}^{1/p} \geqq$$

$$\geqq c \frac{1}{n^r} \omega\left(\frac{1}{n}\right) \log \frac{2n}{\lambda_n} \geqq c\Lambda_n(r, p, \omega)$$

holds for infinitely many n, and this is exactly (20).

Similarly we can give a function $g_1 \in W^r H^\omega$ and a sequence $\{n_k\}$ for which

$$\lim_{k\to\infty} \tilde{V}_{n_k}(g_1, \lambda, p; 0)/\Lambda_{n_k}(r, p, \omega) = c > 0 \tag{3.10}$$

(see the remark made after Lemma 4).

Let

$$\omega_r^*(\delta) = \begin{cases} \delta^r \omega(\delta) & \text{if} \quad r > 0 \\ \omega^*(\delta) & \text{if} \quad r = 0. \end{cases}$$

By Corollary 1 of Theorem 3 there is a $g_2 \in W^r H^\omega$ for which

$$\limsup_{k\to\infty} \tilde{V}_{n_k}(g_2, \lambda, p; 0)/\omega_r^*\left(\frac{1}{n_k}\right) > 0. \tag{3.11}$$

Now (3.10) and (3.11) give that (21) will be true either for g_1 or for g_2, namely

$$\Lambda_{n_k}(r, p, \omega) + \omega_r^*\left(\frac{1}{n_k}\right) \geqq c\Lambda_{n_k}^*(r, p, \omega)$$

$\left(\text{taking into account that } \lambda_n \leqq \frac{n}{2}\right)$.

Thus the proof of Theorem 5 is complete.

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ORTHOGONALITY IN MODULES

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I. Introduction

In [1, 2] Abian and McCrimmon develop a notion of order in a semiprime ring which is useful in describing rings which are products of prime rings or fields. In this note we show how to extend this notion to modules. Here the approach is more homological but hopefully will lend itself to wider application. Roughly, the idea is "two elements of a module are orthogonal if they lie in completely different parts of the module". Of course the critical point is "what is completely different?" We take the point of view that if $\mathrm{Hom}(A, B)=0=\mathrm{Hom}(B, A)$ then A and B are "completely different" provided, naturally, that one of them is not zero. This leads to a definition of orthogonality which, if linear, gives rise to a partial order, hence to atoms and completeness etc. It also gives rise to a concept of primeness, i.e. "a module M is prime if x and y are in M and x is orthogonal to y then either $x=0$ or $y=0$." Possibly a better adjective for such a module would be molecular, so that is what we will use.

II. Fundamental definitions and notations

Throughout R will denote an associative ring with a unit element; all modules will be left unitary modules over R and right modules over their endomorphism rings by having the R-endomorphism operate on the right.

Definition. For an R-module M, let $Z(M)$ be the singular submodule i.e. the elements annihilated by essential left ideals.

Definition. For an R-module M and N a subset of M let $l(N)=\{r\in R\colon rN=0\}$ and in a similar fashion define $r(x)=\{r\in R\colon Xr=0\}$.

Definition. For an R-module M let $\hat{M}$ be the injective hull of M, and for $S=\mathrm{Hom}_R(M, M)$ let $\hat{S}=\mathrm{Hom}_R(\hat{M}, \hat{M})$.

Definition. Let M be an R-module. We say, for x and y in M, that x is orthogonal to y if $RxS\cap RyS=0$ and write $x\perp_M y$ or $x\perp y$ if there is no possibility of ambiguity.

Note that if R is a semiprime ring and x and y are in R then $xRy=0$ iff $RxR\cap RyR=0$ so for semiprime rings our definition of orthogonal is the same as McCrimmon's.

As is pointed out in [2] a ring is prime iff $xRy=0$ implies $x=0$ or $y=0$ and semiprime iff $xRx=0$ implies $x=0$. The second condition translates to $RxS\cap RxS=0$ implies $x=0$ which is always true, but the first statement yields:

DEFINITION. An R-module M will be called a molecule if $x \perp y$ implies $x=0$ or $y=0$. Some examples of molecular modules are modules with simple essential socle, and von Neumann regular rings are molecular if and only if they are prime.

PROPOSITION. *If M is a quasi-injective module and x and y are in M, then $x \perp_M y$ iff $x \perp_{\hat{M}} y$.*

PROOF. Since M is an invariant essential submodule of $\hat{M}$, $RxS = Rx\hat{S}$.

For nonsingular modules there is a weaker notion than quasi-injective which yields the same result.

DEFINITION. A module M is locally pseudo-injective if for every pair of submodules A and B with $A \cap B = 0$ and $f\colon A \to B$, $f \neq 0$ there exists $g \in S$ such that $gf|_{A'} = 1_{A'}$ for some $0 \neq A' \subset A$.

LEMMA. *If M is nonsingular and quasi-injective then M is locally pseudo-injective.*

PROOF. If $f\colon A \to B$, f is locally a monomorphism i.e. there exists a non-zero submodule of A, A' say, such that $f|_{A'}$ is a monomorphism since the kernel of f cannot be essential. By quasi-injectivity we obtain the required g.

PROPOSITION. *Let M be a locally pseudo-injective nonsingular module. Then for every pair x and y in M, $x \perp_M y$ iff $x \perp_{\hat{M}} y$, where $\hat{M}$ is the injective hull of M.*

PROOF. If $f \in \operatorname{End}(M) = S$ and $\hat{f}_1$ and $\hat{f}_2$ are two extensions of f, then $\hat{f}_1 - \hat{f}_2$ vanishes on M, but $\hat{M}$ is nonsingular so $\hat{f}_1 - \hat{f}_2 = 0$. This says that S embeds in $\hat{S}$ by extension. Now $x \perp_{\hat{M}} y$ clearly implies $x \perp_M y$. Suppose $x \perp_M y$. Let $= Rx\hat{S} \cap Ry\hat{S}$ and take $z = \sum_{i=1}^{n_1} r_i x \hat{s}_i = \sum_{j=1}^{n_2} r_2 y \hat{s}_j$, where $\hat{s}_i$, $i=1, \dots, n_1$ and $\hat{s}_j$, $j=1, \dots, n_2$ are in $\hat{S}$.

If $z \neq 0$ choose the r_i and $\hat{s}_i$ so that n_1 is minimal. Let π_1 be the projection onto $Rx\hat{s}_1$. Then $z\pi_1 \neq 0$ and $z\pi_1 \in Rx\hat{s}_1$ so $Rx\hat{s}_1 \cap Ry\hat{S} \neq 0$. For some $r \neq 0$ in R, $\hat{s}_1$ induces a nonzero map of Rrx into $Ry\hat{S}$. Using the fact that M is essential in $\hat{M}$ and the fact that M is nonsingular there exists a submodule $0 \neq B$ of Rrx such that $\hat{s}_1|_B$ is a monomorphism into a submodule C of M. Clearly $C \subset Ry\hat{S}$. The local pseudo-injectivity of M says either $\hat{s}_1|_B$ extends to an endomorphism of M or $s_1|_B$ inverse on C extends to an endomorphism on M. In either case we see that $RxS \cap RyS \neq 0$, a contradiction, so $z=0$.

EXAMPLE. Let R be the 2×2 lower triangular matrices over a field. R is not quasi-injective but is non-singular and is locally pseudo-injective as a module over itself.

PROPOSITION. *Let M be a locally pseudo-injective nonsingular or injective module. Then $\perp$ is linear, i.e. for $x, y_1, y_2 \in M$ and $r_1, r_2 \in R$ if $x \perp y_1$ and $x \perp y_2$ then $x \perp (r_1 y_1 + r_2 y_2)$.*

PROOF. By the above propositions we can assume M is injective. Let $z \in R(r_1 y_1 + r_2 y_2) S \cap RxS$.

Let e be a projection onto $Ry_1 S$. We know $e\in S$. Since $x\perp y_1$, $ze=0$, but then $z=\sum_{i=1}^{n_1} r_i' r_1 y_1 s_i+\sum_{j=1}^{n_2} r_j' r_2 y_2 s_j$ says $\sum r_i' r_1 y s_i=\sum -r_j' r_2 y_2 s_j$ i.e. $z\in Ry_2 S$ so $Ry_2 S\cap \cap RxS\neq 0$ a contradiction unless $z=0$.

If R is a commutative ring then R is semi-prime iff $z(R)=0$ and in case $z(R)=0$, $Ra\cap Rb$ iff $\hat{R}a\cap \hat{R}b=0$ so the order is the same.

III. The ordering of Abian

DEFINITION. Let M be an R-module. Define $x\leqq y$ if $x\perp y-x$.

We would like this to be a partial order and whenever M is injective or locally pseudo-injective the order will be a partial order.

THEOREM. *If M is an injective R-module, then $\leqq$ is a partial order.*

PROOF. That $x\leqq x$ is clear since $x\perp 0$.

To see that $x\leqq y\leqq x$ implies $x=y$ we have $x\perp y-x$ and $y\perp x-y$ so by linearity $x-y\perp y-x$ which gives $x-y=0$. Now assume M is injective and $x\leqq y$ and $y\leqq z$, so we have $x\perp y-x$ and $y\perp y-z$. Suppose $w=x_1+x_2$ where $w\in RxS$, $x_1\in R(x-y)S$, $x_2\in R(y-z)S$. There exist $e^2=e\in S$ such that $we=w$ and $R(x-y)Se=0$ since $R(x-y)S\cap RxS=0$, so $w\in R(y-z)S$. Also there exists an $f^2=f\in S$ such that f acts as the identity on $R(y-z)S$ and $RySf=0$ for a similar reason. Since $w=\sum_{i=1}^{n} r_i x s_i=w-\sum_{i=1}^{n} r_i y s_i f=wf-\sum_{i=1}^{n} r_i(x-y)s_i$ is in $R(x-y)S\cap RxS$ we have that $w=0$.

DEFINITION. We call a module M a molecule if $x\perp y\to x=0$ or $y=0$.

DEFINITION. A submodule N of a module M is called a submolecule if $x\perp y\to \to x\in N$ or $y\in N$ and $NS\subset N$.

Zorn's lemma gives the existence of minimal submolecules so we say:

DEFINITION. An R-module M is semimolecular if the intersection of the minimal submolecules of M is zero and $\leqq$ is a linear partial order.

PROPOSITION. *If M is semi molecular, then $\perp$ is linear and $\leqq$ is a partial order.*

PROOF. If $x\perp y$ and N is any minimal submolecule and $x\notin N$ then $y\in N$. Now if for all minimal submolecules $x\notin N$ implies $y\in N$ and M is semimolecular then $x\perp y$ for if $RxS\cap RyS\neq 0$ let $z=\sum_{i=1}^{n} r_1 x s_i\in RxS\cap RyS$. Choose a minimal submolecule N such that $z\notin N$. Then $x\notin N$ so $y\in N$ implies $z\in N$ a contradiction. This says that in the case at hand to check if x is perpendicular to y we need only check, "If for all minimal submolecules, N, such that $x\notin N$, $y\in N$." Now take $x\perp y_1$, and $x\perp y_2$. Then for N a minimal submolecule such that $x\notin N$, y_1 and $y_2\in N$ so $y_1+ +y_2\in N$ so $\perp$ is easily seen to be linear. Next suppose $x\leqq y$ and $y\leqq z$. If N is a minimal submolecule such that $x\notin N$, $y-x\in N$ and $y\notin N$ so $y\perp y-z$ gives $y-z\in N$ too, hence $x-z\in N$. This means $x\perp x-z$, i.e. $x\leqq z$ and "$\leqq$" is a partial order.

DEFINITION. An R-module M is called molecular if it is the product of invariant molecules.

One of the aims is to describe the molecular modules so, of course, we need to look at some finer structure.

DEFINITION. Let M be an R-module and $a \in M_1$. We call a a prime atom if there exists a unique minimal submolecule N of M such that $a \notin N$.

PROPOSITION. *If M is semimolecular then a prime atom is an atom.*

PROOF. Suppose a is a prime atom and $0 \leqq x < a$. Then $a = x + x'$ where $x' = x - a$ so $x' \perp a$. But $x' \leqq a$ too for $x' \perp a - x'$. Now since M is semimolecular there exist minimal submolecules N_1 and N_2 such that $x \notin N_1$ and $x' \notin N_2$. But $x \leqq a$ implies $x \perp x - a$ and $x \notin N_1$ gives $a - x \in N_1$ which implies $a \notin N_1$. Similarly $x' \leqq a$ implies $a \notin N_2$ so $N_1 = N_2$, but then $a \perp x'$ gives $x' = 0$, i.e., $x = a$.

PROPOSITION. *Let M be a semimolecular module, $x \in M$, and $A(x) = \{a\colon a \leqq x$ and x is a prime atom$\}$. Then $A(x)$ is an orthogonal system.*

PROOF. Let a and b be in $A(x)$ and $a \neq b$. Suppose $a \notin N_1$ and $b \notin N_2$, where the N_i are minimal submolecules. Now if $N_1 \neq N_2$, then $RaS \cap RbS \subset N_1 \cap N_2$ and it must be zero since a and b are in all minimal submolecules other than N_1 and N_2. It follows that $a \perp b$ or $N_1 = N_2$. If $N_1 = N_2$ and $a \leqq x$ and $b \leqq x$, then $x - a \in N_1$ and $x - b \in N_2$. But then $a - b$ is in N_1 and certainly $a - b$ is in all the other minimal submolecules, i.e. $a = b$ a contradiction so $a \perp b$.

PROPOSITION. *Let N be a minimal submolecule of M, a semimolecular R-module. Let $M(N) = \{a\colon a$ is a prime atom and $a \notin N\} \cup \{0\}$. Then $M(N)$ is an invariant submodule which is a molecule.*

PROOF. Clearly if n_1 and $n_2 \notin N$ and $n_1 - n_2 \in N$ then $n_1 - n_2$ is in every minimal submolecule so is zero. Similarly for rn_1 and ns. Now if $n_1 \perp n_2$ then n_1 is in N or n_2 is in N so either $n_1 = 0$ or $n_2 = 0$.

In general if $M = \prod_{i \in I} M_i$ with $\{a_i\}_{i \in I}$ and $\{b_i\}_{i \in I}$ two elements of M with $a_i \perp_{m_i} b_i$, $i \in I$ we cannot say $a_i \perp_M b_i$ unless the M_i are invariant in M in which case we can work coordinatewise. This is reasonable for in the case of rings we are interested in two sided ideals and invariant submodules play the corresponding role. One can check $a \leqq b$ coordinatewise if the product is over homologically independent modules. Similar remarks hold for direct sums.

PROPOSITION. *If $M = \prod_{i \in I} N_i$, $\{N_i\}_{i \in I}$ a homologically independent family of molecules, then $P_i = \prod_{i \neq j} N_j$ is a minimal submolecule and if P is any minimal submolecule, then either $P = P_i$ for some i or $P \supset \oplus \sum_{i \in I} N_i$.*

PROOF. P_i is a submolecule for $P_i \oplus N_i = M$ and this is an S sum, so for $p_i + n_i \perp p_i' + n_i'$, $p_i, p_i' \in P_i$, $n_i, n_i' \in N_i$ either $n_i = 0$ or $n_i' = 0$.

Suppose P is a minimal submolecule, then for each $i \in I$ if $p_i \notin P$, $0 \neq p_i \in P_i$ and $p_i \notin P$. If $n_i \in N_i$, $n_i \perp p_i$ so $n_i \in P$ hence if $P \supset P_i$ for all i then $P \supset \oplus \sum N_i$ in particular $P \not\supset P_i$ for some i, hence the P_i are minimal.

DEFINITION. A module is atomically complete if every orthogonal system of atoms has a supremum in M.

DEFINITION. A module is called prime atomic if for each $0 \neq x \in M \exists a \in M$, a prime atom, and $a \leqq x$.

PROPOSITION. *If M is semimolecular, prime atomic, and atomically complete, then for each minimal submolecule P such that there exists a prime atom $b \notin P$ and $x \notin P$ there exists $a \leqq x$, $a \notin P$.*

PROOF. Let P be a minimal submolecule in M and $x \notin P$. Since $A(x)$ is an orthogonal system set $y = \sup A(x)$. Now $y \leqq a$ for any $a \in A(x)$ and $y \leqq x$ so $y - x \leqq x$ too. This yields $A(y-x) \leqq A(x)$. Take $z = \sup A(y-x) \leqq y$. $z \leqq x - y$ so $z \perp z - y$ and $z \perp x - y - z$ says $z \perp z$ hence $z = 0$. But then $A(y-x) = \emptyset$ so $y - x = 0$ i.e., $y = x$. Now if all $a \in A(x)$ are in P each $a \notin Q_a$ where Q_a is a minimal submolecule not equal to P. Let b be a prime atom and $b \notin P$. For each $a \in A(x)$ we have $a \perp b$ so $a \perp (x-a) + b$ i.e., $a \leqq x + b$. Since $x = \sup A(x)$, $x \leqq x + b$ giving $x \perp b$. But $x \notin P$ and $b \notin P$ a contradiction, hence at least one of $a \in A(x)$ is not in P.

We now have the ingredients to formulate the decomposition theorems.

THEOREM. *A module M is isomorphic to a direct product of invariant molecules iff it is*

i) *semimolecular*

ii) *prime atomic*

iii) *atomically complete.*

PROOF. Suppose $M = \prod_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a homologically independent family of molecules. Then M is semimolecular by a previous proposition so i) is established. For ii) let $0 \neq x = \{x_i\}_{i \in I}$. For some $i \in I$, $x_i \neq 0$ so $x_i \notin P_i$ and $x_i \in N$ for all other minimal submolecules and also $x_i \leqq x$. This demonstrates the validity of ii). Finally let $\{a_j\}_{j \in J}$ be an orthogonal set. Note that an atom is a prime atom in the case at hand and the prime atoms are of the form $\{b_i\}_{i \in I}$ with at most one $b_i \neq 0$. If $\{a_j\}_{j \in J}$ is an orthogonal set, by associating to each a_j the minimal submolecule to which it does not belong we obtain a bijection of J into I, hence the $\prod a_j$ exists in M. It is easy to check (working coordinatewise) that this product is a supremum of $\{a_j\}_{j \in J}$, so iii) holds.

Now assume i), ii) and iii) hold. Using i) and ii) we have that the intersection of all the minimal submolecules to which some prime does not belong must be zero. Let $\{P_i\}_{i \in I}$ be the set of minimal submolecules such that for each i there is a prime atom $a \notin P_i$. Let $N_i = M(P_i)$. Let A be any module and $\alpha_i \colon A \to N_i$, $i \in I$, be module homomorphisms. Define $\alpha(a) = \sup_{i \in I} \{\alpha_i(a)\}$. Note that this can be done since $N_i \cap N_j = 0$ if $i \neq j$ and sups exist by hypothesis. We claim α is a module map and so $M \supset \prod N_i$. To back up our claim let $x = \sup \{a_i\}$ and $y = \sup \{b_i\}$, a_i, $b_i \in N_i$. We first look at $R(x + y - (a_i + b_i)) S \cap R(a_i + b_i) S$. If $a_i \neq 0$ and $b_i \neq 0$, then $(x+y) - (a_i + b_i) \in P_i$ for $x - a_i \perp a_i$ gives $x - a_i \in P_i$ and $y - b_i \perp b_i$ gives $y - b_i \in N_i$ so $(x + y - (a_i + b_i)) \in P_i$ if both a_i and $b_i \neq 0$. In case $a_i = 0$ and $x \notin P_i$ then $\exists\, 0 \neq$ $\neq x_i \leqq x$ and $x_i \notin P_i$, but $x_i \in P_j$ for all $j \neq i$. Also $x - x_i \in P_i$ and $x_i \perp a_j$ for all $j \neq i$ so $a_j \perp x - a_j - x_i$ for any $i \neq j$ hence $x - x_i \geqq a_j$ for any $j \in I$ so $x \geqq x - x_i$ and $x \perp x_i$ along with $x \perp x - x_i$ gives $x_i = 0$, so $x \in P_i$. Using the same argument if

$b_i=0$ to yield $y\in P_i$ we see that $x+y-(a_i+b_i)\in P_i$ for all i so $x+y\geqq a_i+b_i$ for all i. Now take $x+y\geqq z\geqq a_i+b_i$ for all i and assume $x+y-z\neq 0$. Then for some i there exists $0\neq w_i\in N_i$ such that $x+y-z\geqq w_i$. $(z-w_i)-(a_j+b_j)\perp a_j+b_j$ since $w_i\perp a_j+b_j$ for $j\neq i$. But for $j=i$, $x+y-z-w_i\perp w_i$ and $x+y-(a_i+b_i)\perp a_i+b_i$ so $x+y-z-w_i\in P_i$ and $x+y-a_i+b_i\in P_i$ hence $(z-w_i)+a_i+b_i\in P_i$ i.e. $z-w_i\geqq$ $\geqq a_i+b_i$ and $z-w_i\geqq a_j+b_j$ for all j. So $z\leqq z-w_i$, hence, $z\perp w_i$. This yields $x+y-w_i-z\perp z$ i.e., $x+y-w_i\geqq z\geqq a_j+b_j$ for all j. We already know $x+y-$ $-(a_i+b_i)\in P_i$ so if $a_i+b_i\neq 0$, $x+y-w_i-(a_i+b_i)\perp a_i+b_i$ says $w_i\in P_i$ a contradiction. If $a_i+b_i=0$, $x+y\in P_i$ and by the above $z-w_i\in P_i$. But $z\perp w_i$ so $z\in P_i$ a contradiction again, so it must be that $z=x+y$. That $\sup\alpha(ra_i)=\sup\alpha(ra_i)=$ $=r\sup\alpha(a_i)$ is easy. This all says that $M\supset\prod_{i\in I}N_i$ by identifying $\{a_i\}_{i\in I}$ with $\sup\{a_i\}$. There also is a map of $M\xrightarrow{f}\prod_{i\in I}N_i$ given by the natural maps of M to M/P_i followed by the inverse of these natural maps restricted to N_i. These natural maps restricted to N_i are epimorphisms by completeness and monomorphisms by the definition of the N_i. But f is an isomorphism for it is a monomorphism by semimolecularity and an epimorphism by the above.

Next we define a division atom as an element, a, in an R-module M such that if $ra\neq 0$ there exists $s\in S$ such that $ras=a$.

Proposition. *If M is semimolecular locally pseudo-injective, then division atoms are atoms.*

Proof. Let a be a division atom. Then $a\notin P$ for some minimal submolecule of M. Let Q be any other minimal submolecule of M. If $a\notin Q$ and $b\in Q$ and $b\notin P$, $a\perp b$ gives $a\in P$ so $RaS\cap RbS\neq 0$.

Without loss of generality assume M is injective.

By the injectivity of M we see that $Ra\cap RbS\neq 0$. To see this choose $0\neq\sum_{i=1}^n r_ias_i\in RbS$ so that n is as small as possible. Now let e be a projection onto $\widehat{Ras_1}$. Then $\sum_{i=1}^n r_ias_ie=0$ implies $\sum_{i=1}^n r_ias_i=\sum_{i=2}^n r_ia(s_i-s_ie)$ contradicting the minimality of n, so in fact $n=1$, because $(\sum_{i=1}^n r_ias_i)e\in\widehat{Ras_1}$ implies $Ras_1\cap Rbs\neq 0$. So for some $r, s, 0\neq ras\in Rbs$. Since M is locally pseudo-injective s is locally invertible so in fact $r'a\in RbS$ for some $r'\in R$. But now $r'as'=a$ for some s' so in fact $a\in RbS\subset Q$, hence a is in all other minimal submolecules and is a prime atom.

Remark. If M is such that for every $m\in M$ there is a division atom less than m then for each minimal submolecule P, $M(P)=RaS$ for $0\neq a\in M(P)$, in which case we call M hyper atomic.

Theorem. *A module M is the product of hyper atomic invariant molecules iff*
i) *M is semimolecular*
ii) *M is atomically complete*
iii) *M is hyperatomic.*

Proof. An easy modification of the previous theorem's proof.

IV. Molecular rings

As we have pointed out prime rings are molecular but the reverse is not true for take k any field and x an indeterminate over K. Form $k[x]/(x^2)$. This ring is molecular but not even semiprime.

THEOREM. *If the ring R is molecular, then $\hat{R}$ is molecular.*

PROOF. Let R be molecular. Then for $0 \neq x$ and $y \neq 0$ in $\hat{R}$ if $Rx\hat{S} \cap Ry\hat{S}=0$ for a fixed $y \neq 0$, the set of x such that $Rx\hat{S} \cap Ry\hat{S}=0$ is a submodule of $\hat{R}$. To see this if $Rx\hat{S} \cap Ry\hat{S}=0$ and $Rw\hat{S} \cap Ry\hat{S}=0$ and $R(x+w)\hat{S} \cap Ry\hat{S} \neq 0$ then z_1+z_2 in $Ry\hat{S}$ with z_1 in $Rx\hat{S}$ and z_2 in $Rw\hat{S}$. But if the projection onto RxS of z_1+z_2 is zero, then $z_1 \in Rw\hat{S}$ so $z_1+z_2 \in Rw\hat{S}$. Otherwise the projection is non-zero and hence yields a nonzero element of $Rx\hat{S} \cap Ry\hat{S}$. Since R is essential in $\hat{R}$ there is a $0 \neq x' \in R$ such that $Rx'\hat{S} \cap Ry\hat{S}=0$. Now, similarly, $y' \in R$, $y' \neq 0$, so that $Rx'\hat{S} \cap Ry'\hat{S}=0$. But clearly $Rx'S \subset Rx\hat{S}$ and $Ry'S \cap Ry'\hat{S}$ so $Rx'S \cap Ry'S=0$ which gives $x'=0$ or $y'=0$ a contradiction.

THEOREM. *If R is a prime ring and $\hat{S}=\operatorname{End}(\hat{R})$, then $\hat{S}$ is a prime ring and molecular.*

PROOF. We need only show $\hat{S}$ is a prime ring. Suppose A and B are two ideals of S and $AB=0$. Then $\hat{R}A \cap R=A$, and $A_2=\hat{R}B \cap R$ are two sided ideals of R, hence $RA \cap RB=0$ iff $A_1=0$ or $A_2=0$ iff $A_1A_2=0$. But if $a_1=\sum_{i=1}^{n} \hat{r}_i a_i$, $a_2= =\sum_{j=1}^{k} \hat{r}_j b_j$, $\hat{r}_i, \hat{r} \in \hat{R}$, $a_1 a_2=\sum_{j=1}^{k} a_1 \hat{r}_j b_j$. Now right multiplication by $\hat{r}_j$ extends to an element of $\hat{S}$ denoted by s_j so we have $a_1\hat{r}_j=a_1 s_j \in \hat{R}As_j \subset RA$. So $a_1\hat{r}_j b_j= =a_1 s_j b_j \in RAB=0$, hence $A_1A_2=0$ and $\hat{S}$ must be prime.

Using the same techniques we have:

THEOREM. *If R is a semiprime ring and $\hat{S}=\operatorname{End}(\hat{R})$, then $\hat{S}$ is a semiprime and semimolecular ring.*

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ON RIESZ REPRESENTATION THEOREM

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Let S denote a Hausdorff topological space and E, F denote locally convex Hausdorff topological vector spaces over the real or complex field. Let $C(S, E)$ be the space of continuous functions from S to E and let $C_0(S, E)$ be the subspace of continuous functions vanishing at infinity. If $E=\mathbf{C}$ we simply write $C(S)$ or $C_0(S)$. Let T be a continuous linear transformation from $C(S, E)$ or $C_0(S, E)$ to F. Riesz type representation theorem for T has been studied by numerous authors, for example, see BARTLE, DUNFORD and SCHWARTZ [2], DINCULEANU [4], and more recently GOODRICH [5], BROOKS and LEWIS [3] for the cases where S is compact and locally compact, respectively.

If S is locally compact and $C_0(S, E)$ is given the topology of uniform convergence, it is proved by BROOKS and LEWIS [3, Theorem 2.2] that for each continuous linear transformation $T\colon C_0(S, E)\to F$ there is a unique weakly regular set function $K\colon \mathscr{B}(S)\to L[E, F'']$ so that $T(f)=\int f dK$, $f\in C_0(S, E)$.

In the remainder of this paper S is a locally compact Hausdorff space and $E=F=A$ is a Banach algebra. We will give a necessary and sufficient condition for $K(e)$, $e\in\mathscr{B}(S)$, to be a multiplier. The advantage of knowing that $K(e)$ is a multiplier is that it permits considerable freedom in handling the repeated integrals.

It is well known that the second dual A'' of A is a Banach algebra with the Arens multiplication and that the algebraic tensor product $C_0(S)\otimes A$ is dense in $C_0(S, A)$ (see [4] or [6]).

DEFINITION. A continuous linear operator $T\colon C_0(S, A)\to A$ is called a *multiplier* of A if

$$T(f\otimes xy) = T(f\otimes x)\,y = xT(f\otimes y)$$

for $f\in C_0(S)$, $x, y\in A$.

The aim of this paper is to prove the following

THEOREM. *For each $e\in\mathscr{B}(S)$, $K(e)\colon A\to A''$ is a multiplier iff T is a multiplier of A.*

Before we prove the theorem we need the following lemma. For $e\in\mathscr{B}(S)$, $x\in A$, $f'\in C_0'(S, A)$, $1_e\otimes x$ can be viewed as an element of $C_0''(S, A)$ and there is a unique regular Borel measure $\mu(x, f')$ such that $(1_e\otimes x)(f')=\mu(x, f')(e)$ and $\int f d\mu(x, f')=\langle f\cdot x, f'\rangle$ for $f\in C_0(S)$ (see GOODRICH [5] or BROOKS and LEWIS [3]).

LEMMA. *Let $T\colon C_0(S, A)\to A$ be a multiplier of A. For $e\in\mathscr{B}(S)$, $x, y\in A$ we have*

$$T''(1_e\otimes xy) = T''(1_e\otimes x)\,y = xT''(1_e\otimes y),$$

where the multiplication in A'' is defined by the Arens product.

PROOF. For any $x' \in A'$,

$$T''(1_e \otimes xy)(x') = (1_e \otimes xy)(T'x') = \mu(xy, T'x')(e)$$

and

$$(xT''(1_e \otimes y))(x') = x([T''(1_e \otimes y), x']) =$$

$$= [T''(1_e \otimes y), x'](x) = (1_e \otimes y)(T'\langle x', x\rangle) = \mu(y, T'\langle x', x\rangle)(e).$$

For $f \in C_0(S)$,

$$\int f\, d\mu(xy, T'x') = \langle f \cdot xy, T'x'\rangle = \langle T(f \cdot xy), x'\rangle,$$

$$\int f\, d\mu(y, T'\langle x', x\rangle) = \langle f \cdot y, T\langle x', x\rangle\rangle = \langle T(f \cdot y), \langle x', x\rangle\rangle = \langle xT(f \cdot y), x'\rangle.$$

Since T is a multiplier of A, we conclude that $\mu(xy, T'x') = \mu(y, T'\langle x', x\rangle)$. This completes the proof of the lemma.

PROOF OF THE THEOREM. Suppose $K(e)$ is a multiplier for each $e \in \mathcal{B}(S)$. Then for any Borel partition $\{e_i\}$ of S,

$$\sum_{i=1}^{n} K(e_i)(xy) = \left(\sum_{i=1}^{n} K(e_i)x\right) y.$$

Hence

$$T(f \otimes xy) = \int (f \otimes xy)\, dK = T(f \otimes x)y = xT(f \otimes y)$$

for $f \in C_0(S)$ and $x, y \in A$.

On the other hand, suppose T is a multiplier of A. For $e \in \mathcal{B}(S), x, y \in A$ we have

$$K(e)(xy) = T''(1_e \otimes xy) = xT''(1_e \otimes y) = xK(e)y.$$

Similarly we have $K(e)(xy) = [K(e)x]y$. This completes the proof of the theorem.

Concluding remarks. If S is completely regular, JOHNSON [7] has identified $C'(S, E)$ with a certain space $M(S, E')$ of E'-valued measures. Making use of this result, Riesz representation theorem may be generalized to the case where S is completely regular and $C(S, E)$ is endowed with the topology of uniform convergence on the members of a sufficient family of compact sets in S. In particular for $e \in \mathcal{B}(S), x \in A, 1_e \otimes x$ can be viewed as an element in $C''(S, E)$ defined by $(1_e \otimes x)(m) = m(e)x$ for $m \in M(S, E')$. A proof of the results in this paper can be formulated for the case where S is completely regular if $1_e \otimes x$ is so defined. We should know that if S is compact, E is a Banach space, and the sufficient family of compact sets of S consists of S only, then the topology on $C(S, E)$ of uniform convergence on the members of the sufficient family is in fact the topology on $C(S, E)$ of uniform convergence.

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THE FUNCTIONAL CENTRAL LIMIT THEOREM FOR LACUNARY TRIGONOMETRIC SERIES

By
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1. Introduction. In this note let $\{n_m\}$ be a sequence of positive integers satisfying the gap condition

$$(1.1) \qquad n_{m+1}/n_m > 1+cm^{-\alpha} \quad (c>0 \text{ and } 0 \leqq \alpha \leqq 1/2),$$

and $\{a_m\}$ be a sequence of positive numbers such that

$$(1.2) \qquad A_k = \left(2^{-1} \sum_{m=1}^{k} a_m^2\right)^{1/2} \to +\infty \quad \text{and} \quad a_k = o(A_k k^{-\alpha}), \quad \text{as} \quad k \to +\infty.$$

Then we put, for any sequence $\{\alpha_m\}$ of real numbers

$$(1.3) \qquad \xi_m(\omega) = a_m \cos 2\pi (n_m \omega + \alpha_m) \quad \text{and} \quad S_k(\omega) = \sum_{m=1}^{k} \xi_m(\omega).$$

Consider ξ_m'S as random variables on a probability space $([0, 1), \mathcal{F}, P)$ where $\mathcal{F}$ is the σ-field of all Borel sets on $[0, 1)$ and P is the Lebesgue measure on $\mathcal{F}$. Further we write, for $\omega \in [0, 1)$, $t \in [0, 1]$ and every positive integer k,

$$(1.4) \qquad X_k(t) = X_k(t, \omega) = A_k^{-1} S_m(\omega), \quad \text{if} \quad A_m^2 A_k^{-2} \leqq t < A_{m+1}^2 A_k^{-2}.$$ [1]

Then $X_k(t)$ is a random element of $(D, \mathcal{D})$ defined on the probability space $([0, 1), \mathcal{F}, P)$ where D is the set of real-valued functions that are right continuous and have left-hand limits and $\mathcal{D}$ is the Skorohod σ-field in D (cf. [2], p. 111).

The purpose of the present paper is to prove the following

THEOREM. *We have $X_k(t) \Rightarrow X(t)$, in $(D, \mathcal{D})$, as $k \to +\infty$, where $\{X(t), 0 \leqq t \leqq 1\}$ is the standard Brownian motion.*

In [4] we proved that if μ is a probability measure on $([0, 1), \mathcal{F})$ such that $\mu \ll P$, then for any real number x

$$(1.5) \qquad \lim_{k\to\infty} \mu\{\omega;\, X_k(1, \omega) \leqq x\} = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp(-u^2/2)\, du.$$

Further, in [5] it is proved that if we replace the condition $a_k = o(A_k k^{-\alpha})$ in (1.2) by $a_k = O(A_k k^{-\alpha})$ as $k \to +\infty$, then (1.5) does not necessarily hold for the measure P. Therefore (1.2) is the best possible condition for the functional central limit theorem of lacunary trigonometric series.

[1] We put $A_0 = 0$.

In case $0 \leqq \alpha < 1/2$ and $a_k = 1$ for all k I. Berkes [1] proved an almost sure invariance principle for $X_k(t)$.

For the proof of our theorem we approximate $\{S_k(\omega)\}$ by a martingale and then apply a martingale version of the functional central limit theorem due to McLeish ([3], (3.2)).

2. Preliminaries. Let us put, for each integer k

$$\begin{cases} p(0) = 0, \quad p(k) = \max\{m;\, n_m < 2^k\} \\ \Delta_k = \sum_{m=p(k)+1}^{p(k+1)} \xi_m \quad \text{and} \quad B_k = A_{p(k+1)}.^2 \end{cases} \tag{2.1}$$

Then if $p(k)+1 < p(k+1)$, we have, by (1.1)

$$2 > n_{p(k+1)}/n_{p(k)+1} > \prod_{m=p(k)+1}^{p(k+1)-1} (1+cm^{-\alpha}) > 1 + c\{p(k+1)-p(k)-1\}p^{-\alpha}(k+1).$$

Hence we have

$$p(k+1)-p(k) = O\big(p^{\alpha}(k)\big), \quad \text{as} \quad k \to +\infty \tag{2.2}$$

and if $m_k = o\big(p^{1-\alpha}(k)\big)$ as $k \to +\infty$, then

$$p(k+m_k)/p(k) \to 1, \quad \text{as} \quad k \to +\infty. \tag{2.3}$$

From (1.2) and (2.2), it is easily seen that

$$b_k = \max_{p(k)<m\leqq p(k+1)} |a_m| = o\big(B_k p^{-\alpha}(k)\big), \quad \text{as} \quad k \to +\infty.$$

Hence we can take an increasing sequence $\{g(k)\}$ of real numbers such that

$$\begin{cases} g(k) \leqq \min\{p(k),\, B_k\, g(k-1)\, B_{k-1}^{-1}\} \quad \text{for} \quad k > 1 \quad \text{and} \\ g(k) \to +\infty, \quad b_k = O\big(B_k/p^{\alpha}(k)\, g(k)\big), \quad \text{as} \quad k \to +\infty. \end{cases} \tag{2.4}$$

Therefore, we have

$$\begin{cases} \sum_{m=p(k)+1}^{p(k+1)} |a_m| \leqq b_k\{p(k+1)-p(k)\} = O\big(B_k/g(k)\big), \\ E\Delta_k^2 \leqq b_k^2\{p(k+1)-p(k)\} = O\big(B_k^2/p^{\alpha}(k)\, g^2(k)\big), \quad \text{as} \quad k \to +\infty. \end{cases} \tag{2.5}$$

Lemma 1. *For any given integers k, j, q and h such that*

$$p(j)+1 < h \leqq p(j+1) < p(k)+1 < q \leqq p(k+1)$$

the number of solutions (n_r, n_i) of the equations

$$n_q - n_r = n_h \pm n_i$$

where $p(j) < i < h$ and $p(k) < r < q$, is at most $C2^{j-k}p^{\alpha}(k)$ where C is a positive constant which does not depend on k, j, q and h.

[2] $\Delta_k = 0$ if $p(k) = p(k+1)$.

PROOF. If $k<j+5$, the lemma is evident by (2.2). We assume that $k\geqq j+5$. Let m denote the smallest index r of the solutions (n_r, n_i), then the number of solutions is at most $q-m$. Since $(n_h\pm n_i)<2^{j+2}$, we have

$$n_m > n_q-2^{j+2} > n_q(1-2^{j+2-k}) \geqq n_q(1+2^{j-k}\cdot 5)^{-1}.$$

By (1.1), we have

$$1+2^{j-k}\cdot 5 > n_q/n_m > \prod_{s=m}^{q-1}(1+cs^{-\alpha}) > 1+c(q-m)p^{-\alpha}(k+1).$$

Therefore, by (2.2) we can prove the lemma.

LEMMA 2. *We have, for any M and N $(M<N)$*

$$E\left|\sum_{m=M}^{N}\{\Delta_m^2-E\Delta_m^2\}\right|^2 \leqq CB_N^2\sum_{m=M}^{N}E\Delta_m^2/\{g^2(N)\},$$

where C is a positive constant which does not depend on M and N.

PROOF. Let us put, for $k=1, 2, \ldots,$

$$U_k=\Delta_k^2-E\Delta_k^2-2^{-1}\sum_{m=p(k)+1}^{p(k+1)}a_m^2\cos 4\pi(n_m\omega+\alpha_m).$$

Then we have, by (1.2) and (2.5)

$$\left\{E\left|\sum_{m=M}^{N}(\Delta_m^2-E\Delta_m^2)\right|^2\right\}^{1/2} \leqq \left\{E\left(\sum_{m=M}^{N}U_m\right)^2\right\}^{1/2}+2^{-1}\left(\sum_{m=M}^{N}\sum_{j=p(m)+1}^{p(m+1)}a_j^4\right)^{1/2}=$$

$$=\left|2\sum_{k=M+1}^{N}\sum_{j=M}^{k-1}EU_kU_j\right|^{1/2}+O\left(\left\{\sum_{m=M}^{N}E\Delta_m^2B_N^2\right\}^{1/2}\Big/g(N)\right),\quad\text{as}\quad N\to+\infty.$$

Further by Lemma 1, (2.4) and (2.3), we have for $k>j$

$$|EU_kU_j|\leqq C2^{j-k}p^\alpha(k)\sum_{q=p(k)+1}^{p(k+1)}|a_q|\,b_k\sum_{h=p(j)+1}^{p(j+1)}|a_h|\,b_j=$$

$$=O\left(2^{j-k}\{E\Delta_k^2E\Delta_j^2p^\alpha(k)p^{-\alpha}(j)\}^{1/2}B_N^2(g(N))^{-2}\right),\quad\text{as}\quad N\to+\infty.$$

Since $p(j+1)/p(j)\to 1$ as $j\to+\infty$, we have for all k

$$\sum_{j=1}^{k-1}p^{-\alpha}(j)2^{j-k}\leqq Cp^{-\alpha}(k),\quad\text{for some}\quad C>0. \tag{2.6}$$

Therefore we have

$$\sum_{k=M+1}^{N}\sum_{j=M}^{k-1}2^{j-k}\{E\Delta_k^2E\Delta_j^2p^\alpha(k)p^{-\alpha}(j)\}^{1/2}\leqq C\sum_{k=M+1}^{N}\left\{E\Delta_k^2\sum_{j=M}^{k-1}E\Delta_j^2 2^{j-k}\right\}^{1/2}\leqq$$

$$\leqq C\left\{\sum_{k=M+1}^{N}E\Delta_k^2\right\}^{1/2}\left\{\sum_{k=M+1}^{N}\sum_{j=M}^{k-1}E\Delta_j^2 2^{j-k}\right\}^{1/2}\leqq$$

$$\leqq C\left\{\sum_{k=M+1}^{N}E\Delta_k^2\right\}^{1/2}\left\{\sum_{j=M}^{N-1}E\Delta_j^2\sum_{k=j+1}^{N}2^{j-k}\right\}^{1/2}\leqq C\sum_{k=M}^{N}E\Delta_k^2.$$

From the above relations we can complete the proof.

Also we need the following

LEMMA 3. *We have, for any M and N $(M<N)$*

$$E\left(\max_{M\leqq r\leqq N}\left|\sum_{k=M}^{r}\Delta_k\right|^4\right)\leqq C\sum_{k=M}^{N}E\Delta_k^2\left\{B_N^2(g(N))^{-2}+\sum_{k=M}^{N}E\Delta_k^2\right\},$$

where C is a positive constant independent of M and N.

PROOF. From the definition of Δ_m we obtain, for $p>1$

$$\text{(i)}\qquad E\left(\max_{M\leqq r\leqq N}\left|\sum_{k=M}^{r}\Delta_k\right|^p\right)\leqq C_pE\left|\sum_{k=M}^{N}\Delta_k\right|^p,$$

$$\text{(ii)}\qquad E\left|\sum_{k=M}^{N}\Delta_k\right|^4\leqq CE\left(\sum_{k=M}^{N}\Delta_k^2\right)^2.$$ [3]

Hence for the proof of the lemma it is sufficient to show that

$$E\left(\sum_{k=M}^{N}\Delta_k^2\right)^2\leqq C\sum_{k=M}^{N}E\Delta_k^2\left\{B_N^2(g(N))^{-2}+\sum_{k=M}^{N}E\Delta_k^2\right\}.$$

By Lemma 2, we have

$$\left\{E\left(\sum_{k=M}^{N}\Delta_k^2\right)^2\right\}^{1/2}\leqq\left\{E\left|\sum_{k=M}^{N}(\Delta_k^2-E\Delta_k^2)\right|^2\right\}^{1/2}+\sum_{k=M}^{N}E\Delta_k^2\leqq$$

$$\leqq\left[C\sum_{k=M}^{N}E\Delta_k^2\left\{B_N^2(g(N))^{-2}+\sum_{k=M}^{N}E\Delta_k^2\right\}\right]^{1/2}.$$

3. Dividing into blocks. Let us put $q(0)=1$ and for every $k\geqq 1$

$$\text{(3.1)}\qquad q(k)=\min\left\{m;\ B_m^2-B_{q(k-1)}^2\geqq B_{q(k-1)}^2/g^{\varepsilon}\big(q(k-1)\big)\right\}.$$

where ε is any fixed number such that $0<\varepsilon<1$. Then we have, by (2.5)

$$\text{(3.2)}\qquad \begin{cases}B_{q(k)}/B_{q(k-1)}\to 1, & \text{as}\quad k\to+\infty\\ q(k)-q(k-1)>Cp^{\alpha}\big(q(k-1)\big)\big\{g\big(q(k-1)\big)\big\}^{2-\varepsilon} & \text{for some}\quad C>0.\end{cases}$$

Putting $\psi(k)=\left[\left\{\alpha\log p\big(q(k-1)\big)+2\log g\big(q(k-1)\big)\right\}/\log 2\right]$ (2.4) implies that

$$\text{(3.3)}\qquad \psi(k)=\begin{cases}O\big(\log p(q(k-1))\big), & \text{if}\quad \alpha>0,\\ O\big(\log g(q(k-1))\big), & \text{if}\quad \alpha=0,\quad \text{as}\quad k\to+\infty.\end{cases}$$

Since $\psi(k)=o\big(q(k)-q(k-1)\big)$ as $k\to+\infty$, if we put

$$q'(k)=q(k-1)+\psi(k)+1$$

[3] (i) and (ii) are (4.4) and (2.7) in Chapter XV of [6] respectively.

then $q'(k) < q(k)$ for some $k > k_0$. Without loss of generality we may assume that $q'(k) < q(k)$ for all k, if otherwise we consider only those k's for which $q'(k) < q(k)$. We write

$$\text{(3.4)}\qquad \begin{cases} V_k = \sum_{m=q'(k)}^{q(k)-1} \Delta_m, \quad W_k = \sum_{m=q(k-1)}^{q'(k)-1} \Delta_m, \quad C_k^2 = \sum_{m=q(k-1)}^{q(k)-1} E\Delta_m^2 \\ D_N^2 = \sum_{k=1}^{N} C_k^2 \quad \text{and} \quad G_N = g(q(N)). \end{cases}$$

Then we obtain from (3.1), (3.3) and (2.5)

$$\text{(3.5)}\qquad C_k^2 = D_{k-1}^2 G_{k-1}^{-\varepsilon}(1+o(1))$$

and

$$\text{(3.6)}\qquad EW_k^2 = O(D_{k-1}^2 G_{k-1}^{-2} \psi(k) p^{-\alpha}(q(k-1))) = o(C_k^2), \quad \text{as} \quad k \to +\infty.$$

Estimating the maximum and minimum frequencies of terms of a trigonometric polynomial $\{V_k^2 - EV_k^2\}$ it is easily seen that

$$\text{(3.7)}\qquad E\{(V_n^2 - EV_n^2)(V_m^2 - EV_m^2)\} = 0, \quad \text{if} \quad n > m \geqq n_0.$$

LEMMA 4. *We have*

(i) $$E\big(\max_{1 \leqq k \leqq N} |V_k|^4\big) = o(D_N^4),$$

(ii) $$E\left(\left|\sum_{k=1}^{N} V_k^2 - D_N^2\right|^2\right) = o(D_N^4), \quad \text{as} \quad N \to +\infty.$$

PROOF. By Lemma 3 and (3.5) we have

$$\text{(3.8)}\qquad \sum_{k=1}^{N} EV_k^4 = o\left(D_N^2 \sum_{k=1}^{N} C_k^2\right) = o(D_N^4), \quad \text{as} \quad N \to +\infty.$$

Further, by (3.7) and the above relation we have

$$E\left|\sum_{k=1}^{N} (V_k^2 - EV_k^2)\right|^2 = \sum_{k=1}^{N} E|V_k^2 - EV_k^2|^2 \leqq \sum_{k=1}^{N} EV_k^4 = o(D_N^4), \quad \text{as} \quad N \to +\infty.$$

Since $\sum_{k=1}^{N} EV_k^2 \sim D_N^2$ as $N \to +\infty$, we can prove the lemma.

LEMMA 5. *We have*

(i) $\max_{1 \leqq r \leqq N} D_N^{-1} \max_{q(r-1) \leqq k < q(r)} \left|\sum_{m=q(r-1)}^{k} \Delta_m\right| \to 0$, *in probability,*

(ii) $\max_{1 \leqq r \leqq N} D_N^{-1} \left|\sum_{k=1}^{r} \sum_{m=q(k-1)}^{q'(k)-1} \Delta_m\right| \to 0$, *in probability, as* $N \to +\infty$.

PROOF. (i) We have, by Lemma 3 and (3.5)

$$E\left(\max_{1\le r\le N}\max_{q(r-1)\le k<q(r)}\left|\sum_{m=q(r-1)}^{k}\Delta_m\right|^4\right)\le C\sum_{r=1}^{N}E\left|\sum_{m=q(r-1)}^{q(r)-1}\Delta_m\right|^4=$$

$$=o\left(D_N^2\sum_{r=1}^{N}C_r^2\right)+O\left(\sum_{r=1}^{N}C_r^4\right)=o(D_N^4),\quad\text{as}\quad N\to+\infty.$$

(ii) By (i) in the proof of Lemma 3 and (3.6), we have

$$E\left(\max_{1\le r\le N}\left|\sum_{k=1}^{r}\sum_{m=q(k-1)}^{q'(k)-1}\Delta_m\right|^2\right)\le CE\left|\sum_{k=1}^{N}\sum_{m=q(k-1)}^{q'(k)-1}\Delta_m\right|^2=$$

$$=C\sum_{k=1}^{N}\sum_{m=q(k-1)}^{q'(k)-1}E\Delta_m^2=C\sum_{k=1}^{N}EW_k^2=o(D_N^2),\quad\text{as}\quad N\to+\infty.$$

4. Martingale approximation. For each positive integer k let $r(k)=q(k)+ +[(\alpha\log p(q(k))+2\log G_k)/2\log 2]$ and $\mathscr{F}_k$ be the σ-field generated by the intervals $[\nu 2^{-r(k)},(\nu+1)2^{-r(k)})$, $0\le\nu<2^{r(k)}$. Then we put

$$Z_k=V_k-E(V_k|\mathscr{F}_k)\quad\text{and}\quad Y_k=E(V_k|\mathscr{F}_k)-E(V_k|\mathscr{F}_{k-1}).$$

Clearly $\{Y_k,\mathscr{F}_k\}$ is a martingale difference sequence.

LEMMA 6. *We have*

(i) $$Z_k=o(C_k^2D_k^{-1}),\quad a.s.,$$

(ii) $$E(V_k|\mathscr{F}_{k-1})=o(C_k^2D_k^{-1}),\quad a.s.,\ as\quad k\to+\infty.$$

PROOF. (i) Since $|\xi_j-E(\xi_j|\mathscr{F}_k)|\le 2\pi|a_j|n_j2^{-r(k)}$ a.s., we have by (2.2)

$$|\Delta_m-E(\Delta_m|\mathscr{F}_k)|\le 2\pi\sum_{j=p(m)+1}^{p(m+1)}|a_j|\,n_j2^{-r(k)}=$$

$$=O\left(\{E\Delta_m^2p^\alpha(m)\}^{1/2}2^{m-r(k)}\right)\quad\text{a.s., as}\quad k\to+\infty.$$

On the other hand we have, by (3.5)

$$\sum_{m=q'(k)}^{q(k)-1}\{E\Delta_m^2p^\alpha(m)\}^{1/2}2^{m-r(k)}=O\left(C_kp^{\alpha/2}(q(k))2^{q(k)-r(k)}\right)=$$

$$=O\left(C_kp^{\alpha/2}(q(k))p^{-\alpha/2}(q(k))G_{k-1}^{-1/2}\right)=o(C_k^2D_k^{-1}),\quad\text{as}\quad k\to+\infty.$$

By the above two relations we can complete the proof of (i).

(ii) Since $|E(\xi_j|\mathscr{F}_{k-1})|\le 2(2\pi n_j)^{-1}|a_j|2^{r(k-1)}$ a.s., we have

$$|E(\Delta_m|\mathscr{F}_{k-1})|=O\left(\{E\Delta_m^2p^\alpha(m)\}^{1/2}2^{r(k-1)-m}\right)\quad\text{a.s., as}\quad k\to+\infty.$$

On the other hand by (2.6), (2.3), (3.5) and the definitions of $\{r(k)\}$ and $\{q'(k)\}$, we have

$$\sum_{m=q'(k)}^{q(k)-1} \{E\Delta_m^2 p^\alpha(m)\}^{1/2} 2^{r(k-1)-m} = O\Big(C_k \sum_{m=q'(k)}^{q(k)} p^{\alpha/2}(m) 2^{r(k-1)-m}\Big) =$$

$$= O\big(C_k p^{\alpha/2}(q'(k)) 2^{r(k-1)-q'(k)}\big) = O\big(C_k p^{\alpha/2}(q'(k)) p^{-\alpha/2}(q(k-1)) G_{k-1}^{-1/2}\big) =$$

$$= o(C_k^2 D_k^{-1}), \quad \text{as} \quad k \to +\infty.$$

By the above relations we can prove (ii).

LEMMA 7. *We have*

(i) $$\max_{1 \leqq r \leqq N} D_N^{-1} \sum_{m=1}^{r} |Y_m - V_m| \to 0, \quad a.s., \ as \quad N \to +\infty,$$

(ii) $$D_N^{-2} \Big| \sum_{m=1}^{N} Y_m^2 - D_N^2 \Big| \to 0, \quad in\ probability,\ as \quad N \to +\infty.$$

PROOF. (i) follows easily from Lemma 6.

(ii) By Lemma 4 (ii), it is sufficient to show that

$$D_N^{-2} \Big| \sum_{m=1}^{N} (Y_m^2 - V_m^2) \Big| \to 0, \quad \text{in probability, as} \quad N \to +\infty.$$

Since

$$\max_{1 \leqq r \leqq N} |Y_r + V_r| \leqq \max_{1 \leqq r \leqq N} \big(|Z_r| + |E(V_r|\mathcal{F}_{r-1})| + 2|V_r|\big),$$

Lemma 4 (i), Lemma 5 and the first part of this lemma prove (ii).

5. Proof of the theorem. For the proof we use the following

THEOREM OF MCLEISH. *Suppose for* $n \geqq 1$ *and* $t \in [0, 1]$ $\{k(n, t)\}$ *is a sequence of integer valued, non-decreasing right continuous functions such that* $k(n, 0) = 0$ *and* $\{X_{n,k}\}$ *is a martingale difference array satisfying*

(i) $$E\big(\max_{1 \leqq k \leqq k(n,t)} |X_{n,k}|^2\big) \to 0,$$

(ii) $$\sum_{k=1}^{k(n,t)} X_{n,k}^2 \to t, \quad in\ probability\ for\ each \quad t \in [0, 1], \quad as \quad n \to +\infty.$$

Then $\sum_{k=1}^{k(n,t)} X_{n,k} \Rightarrow X$ *in* $(D, \mathcal{D})$, *as* $n \to +\infty$.

For any positive integer n and $t \in [0, 1]$, $k(n, t)$ will denote the integer such that

$$D_{k(n,t)}^2 A_n^{-2} \leqq t < D_{k(n,t)+1}^2 A_n^{-2}.^{4}$$

[4] We put $A_0 = D_0 = 0$.

Then by (3.5), $D^2_{k(n,1)} A_n^{-1} \to 1$ and

$$\max_{0\le t\le 1} |t - D^2_{k(n,t)} A_n^{-2}| \to 0, \quad \text{as} \quad n \to +\infty. \tag{5.1}$$

Further if we put $\bar{X}_n(t) = A_n^{-1} \sum_{k=1}^{k(n,t)} Y_k$, then we have, by (2.5), (3.5) and Lemmas 5 and 7,

$$\max_{0\le t\le 1} |X_n(t) - \bar{X}_n(t)| \to 0, \quad \text{in probability as} \quad n \to +\infty. \tag{5.2}$$

Putting $X_{n,k} = A_n^{-1} Y_k$ we obtain a martingale difference array $\{X_{n,k}\}$ and $\bar{X}_n(t) = \sum_{k=1}^{k(n,t)} X_{n,k}$. On the other hand we have, by (3.8)

$$E\Big(\max_{1\le k\le k(n,1)} |X_{n,k}|^4\Big) \le \sum_{k=1}^{k(n,1)} EX^4_{n,k} = o\left(\sum_{k=1}^{k(n,1)} EV_k^4 A_n^{-4}\right) = o(D^4_{k(n,1)} A_n^{-4}) = o(1),$$
$$\text{as} \quad n \to +\infty.$$

Since $t > 0$ implies $\lim_{n\to+\infty} k(n,t) = +\infty$, we have by (5.1) and Lemma 7 (ii)

$$\sum_{k=1}^{k(n,t)} X^2_{n,k} \to t, \quad \text{in probability for each} \quad t \in [0, 1], \quad \text{as} \quad n \to +\infty.$$

By above two relations, (5.2), and McLeish's theorem we can complete the proof.

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SOME PROBABILISTIC METHODS IN THE THEORY OF APPROXIMATION OPERATORS

By

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1. Introduction. For a function $f \in C[0, 1]$ define the Bernstein polynomials

$$B_n(f, x) = \sum_{k=0}^{n} f(k/n)\, p_{n,k}(x), \tag{1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$. It is well known that $\lim_{n\to\infty} B_n(f, x) = f(x)$ uniformly on [0, 1]. T. Popoviciu ([9], p. 20) proved the stronger result:

$$|B_n(f, x) - f(x)| \leqq \frac{5}{4}\, \omega(n^{-1/2}), \tag{2}$$

where $\omega(\delta) = \sup \{|f(x) - f(y)|: \ |x-y| \leqq \delta, 0 \leqq x, y \leqq 1\}$, $\delta > 0$. Furthermore, Bernstein ([9], pp. 22—23) proved that if the derivative $f^{(2k)}(x)$ exists at x, then

$$\lim_{n\to\infty} n^k \left[B_n(f, x) - f(x) - \sum_{s=1}^{2k-1} \frac{n^{-s}}{s!} T_{ns}(x) f^{(s)}(x) \right] = \left(\frac{x(1-x)}{2} \right)^k \frac{f^{(2k)}(x)}{k!}, \tag{3}$$

where $T_{ns}(x) = \sum_{k=0}^{n} (k - nx)^s p_{n,k}(x)$, $n = 1, 2, \ldots$, $s = 0, 1, 2, \ldots$. Following Bernstein a number of new operators have been defined and studied. However, many of them are special cases of an operator due to Feller [4]. To define the Feller operator let $\{X_n, n \geqq 1\}$ be a sequence of random variables having distribution function (df) $F^*_{n,x}(t)$ with expectation $EX_n = x$ and variance $\sigma_n^2(x)$ where x is a real continuous parameter. For a continuous function f on the real line $R = (-\infty, \infty)$ define the operator

$$L_n(f, x) = Ef(X_n) = \int_{-\infty}^{\infty} f(t)\, dF^*_{n,x}(t) \quad \text{if} \quad E|f(X_n)| < \infty. \tag{4}$$

Using Chebyschev's inequality Feller [4] proved

Lemma 1. *If $\sigma_n^2(x) \to 0$ as $n \to \infty$, then $L_n(f, x) \to f(x)$ for every continuous bounded function f. The convergence is uniform if $\sigma_n^2(x) \to 0$ uniformly and $f(x)$ is uniformly continuous.*

Let the continuous parameter x take values in an interval I (possibly infinite) and let $G(x)$ be a df on I. The dominated convergence theorem combined with Feller's proof gives

LEMMA 2. *Suppose that $\sigma_n^2(x)\leqq g(x)$ where g is G-integrable, and that the conditions of Lemma 1 hold. Then*

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} L_n(f,x)\,dG(x)=\int_{-\infty}^{\infty} f(x)\,dG(x).$$

To specialize (4) let $Y_1, Y_2, \dots$ be iid (independent and identically distributed) random variables with mean $x\in I$ and variance $\sigma^2(x)$, and set $S_n=\sum_{i=1}^{n} Y_i$. Then (4) is equivalent to

$$L_n(f,x)=Ef(S_n/n)=\int_{-\infty}^{\infty} f\left(\frac{t}{n}\right)dF_{n,x}(t), \tag{5}$$

where $F_{n,x}(t)$ is the df of S_n. The following well known operators are special cases of (5) and are listed here for reference purposes.

(i) *Bernstein operator*. Letting $P(Y_1=1)=1-P(Y_1=0)=x$ $(0\leqq x\leqq 1)$ (5) defines the Bernstein polynomials given by (1).

(ii) *Szász operator*. Let $P(Y_1=k)=\dfrac{e^{-x}x^k}{k!}$, $k=0,1,\dots$ $(x\geqq 0)$. Then (5) reduces to the Szász operator

$$S_n(f,x)=e^{-nx}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\frac{(nx)^k}{k!}. \tag{6}$$

(iii) *Gamma operator*. Let the pdf of Y_1 be $f_x(y)=x^{-1}e^{-y/x}$, $y>0$ $(x>0)$. Then (5) defines the gamma operator

$$G_n(f,x)=\frac{x^{-n}}{(n-1)!}\int_{0}^{\infty} f\left(\frac{y}{n}\right)y^{n-1}e^{-y/x}\,dy. \tag{7}$$

(iv) *Weierstrass operator*. Let the pdf of Y_1 be $f_x(y)=(2\pi)^{-1/2}\exp\left(-\frac{1}{2}(y-x)^2\right)$, $-\infty<y,\ x<\infty$. Then (5) defines the Weierstrass transform

$$W_n(f,x)=\sqrt{(n/2\pi)}\int_{-\infty}^{\infty} f\left(x+\frac{u}{n}\right)\exp\left(-\frac{nu^2}{2}\right)du. \tag{8}$$

(v) *Baskakov operator*. If Y_i has geometric distribution $P(Y_i=k)=pq^k$, $k=0,1,2,\dots,$ $(0\leqq p\leqq 1,\ p+q=1)$, then $S_n=\sum_{i=1}^{n} Y_i$ has negative binomial distribution with probability function

$$P(S_n=k)=\binom{n+k-1}{k}p^n q^k,\quad k=0,1,\dots. \tag{9}$$

Taking $p=(1+x)^{-1}$ $(x\geqq 0)$ (5) defines the Baskakov operator

$$B_n^*(f,x)=(1+x)^{-n}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k. \tag{10}$$

(vi) *Meyer-König—Zeller operator*. An operator equivalent to the Baskakov operator was defined by MEYER-KÖNIG and ZELLER [10]. However, CHENEY and SHARMA [2] slightly modified it as follows:

$$M_n(f, y) = (1-y)^{n+1} \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k} y^k, \quad 0 \leqq y \leqq 1, \tag{11}$$

and note that $M_n(f, y) = Ef\left(\frac{S_{n+1}}{n+S_{n+1}}\right)$ where q in (9) is replaced by y. Some other operators derived from (5) can be found in FELLER [4].

The object of this paper is to generalize (2) and (3) for Feller operator (5) which is the subject matter of Section 2. For the Szász and Meyer-König—Zeller operators CHENEY and SHARMA [2] proved monotonic convergence when f is convex. In Section 3 we give an elementary probabilistic proof of monotonic convergence for the Feller operator (5) and Meyer-König—Zeller operator when f is convex. Section 4 deals with an extension of a result due to GRÓF [5] and HERMANN [6] for the SZÁSZ operator in the case of unbounded function. Finally, in Section 5 Lemma 1 for the Feller operator (4) is strengthened to provide a simple proof of uniform convergence for an operator due to CHENEY and SHARMA [2] which is defined in terms of Laguerre polynomials and generalizes (11).

2. Generalizations of (2) and (3). An analogue of (2) for the Feller operator (5) is given by

THEOREM 1. *Let f be a continuous bounded function on R, and let $Y_1, Y_2, \dots$ be iid random variables with mean $x \in I \subset R$ and variance $\sigma^2(x)$. Then for $x \in I$ the Feller operator* (5) *satisfies:*

$$|L_n(f, x) - f(x)| \leqq (1+A)\omega(f; n^{-1/2}), \tag{12}$$

where $\omega(f; \delta) = \sup\{|f(x)-f(y)|: |x-y| \leqq \delta, x, y \in R\}$ *and* $A = \sup_{x\in I} \sigma^2(x)$. *Moreover, for* $x \in [\alpha, \beta] \subset I$ *we have*

$$|L_n(f, x) - f(x)| \leqq \big(1 + \sup_{\alpha \leqq x \leqq \beta} \sigma^2(x)\big)\omega(f; n^{-1/2}). \tag{13}$$

PROOF. Let $S_n = \sum_{i=1}^{n} Y_i$ and $\lambda = \lambda(S_n/n) = \left[\left|\frac{S_n}{n} - x\right| \Big/ \delta\right]$ where $[r]$ denotes the greatest integer $\leqq r$. Clearly, $|f(S_n/n) - f(x)| \leqq \omega(f; \delta)(1+\lambda)$, and hence

$$|L_n(f, x) - f(x)| \leqq \omega(f; \delta)E(1+\lambda) \leqq \omega(f; \delta)(1+E\lambda^2) \leqq$$

$$\leqq \big(1 + E(S_n - nx)^2/n^2\delta^2\big)\omega(f; \delta) = \left(1 + \frac{\sigma^2(x)}{n\delta^2}\right)\omega(f; \delta).$$

The conclusion follows on taking $\delta = n^{-1/2}$.

EXAMPLES. Let $P(Y_1 = 1) = 1 - P(Y_1 = 0) = x$, $0 \leqq x \leqq 1$. Then $\sigma^2(x) = x(1-x)$ and $A = \sup_{0 \leqq x \leqq 1} \sigma^2(x) = \frac{1}{4}$, and (2) follows from (12). Taking Y_1 to be normally distributed with mean x and variance one we see from (12) that the Weierstrass operator (8) satisfies: $|W_n(f, x)-f(x)| \leqq 2\omega(f; n^{-1/2})$. Similar results for other operators can be obtained from (12) or (13).

In order to generalize (3) it is convenient to consider a general exponential distribution which includes Bernoulli, geometric, Poisson, and normal distribution, etc. In what follows we denote by $u^{(k)}(x)$ the kth derivative of $u(x)$ $(k=0, 1, 2, \ldots)$ where $u^{(0)}(x)=u(x)$. Let $Y_1, Y_2, \ldots$ be iid random variables with a common pdf $g(y, x)=\exp(yx-b(x))$ with respect to a σ-finite measure μ where x is a real continuous parameter taking values in an interval Ω of R. It is well known (cf. LEHMANN [7]) that $b(x)$ is analytic and $EY_1=b^{(1)}(x)$, $\operatorname{var}(Y_1)=b^{(2)}(x)>0$. The following lemma due to CHERNOFF [3] can easily be proved by Chebyshev's inequality.

LEMMA 3. *Let $Y_1, Y_2, \ldots$ be iid random variables having finite mgf (moment generating function) $\varphi(\theta)=E\exp(\theta Y_1)$ with $EY_1=0$, and set $S_n=\sum_{i=1}^{n} Y_i$. Then for $\delta>0$ there exists a number ϱ such that*

$$P(S_n \geqq n\delta) \leqq \varrho^n, \quad 0<\varrho<1, \tag{14}$$

where $\varrho=\inf_{\theta>0}\{\varphi(\theta)\exp(-\delta\theta)\}$. Moreover, there exists a number ϱ_1 such that

$$P(|S_n| \geqq n\delta) \leqq 2\varrho_1^n, \quad 0<\varrho_1<1. \tag{15}$$

In particular, if $Y_1, Y_2, \ldots$ are iid random variables with exponential density $g(y, x)=\exp(yx-b(x))$ relative to a σ-finite measure μ and $S_n=\sum_{i=1}^{n}(Y_i-b^{(1)}(x))$, then (14) and (15) hold.

The exponential bound (14) leads to

LEMMA 4. *Let $Y_1, Y_2, \ldots$ be iid non-negative random variables with mean $x\in[0, \infty)$ and assume that Y_1 has finite mgf. If f is continuous and bounded on $[0, \infty)$, then for $x\in[0, A]$ the Feller operator (5) satisfies:*

$$|L_n(f, x)-f(x)| = O(\omega_{2A}(f; n^{-1/2})),$$

where $\omega_A(f; \delta)=\sup\{|f(x+t)-f(x)|: |t|\leqq\delta, x\in[0, A]\}$.

PROOF. It follows from (5) that

$$|L_n(f, x)-f(x)| \leqq \int_{0\leqq t\leqq 2nA}\left|f\left(\frac{t}{n}\right)-f(x)\right| dF_{n,x}(t)+\int_{t>2nA}\left|f\left(\frac{t}{n}\right)-f(x)\right| dF_{n,x}(t).$$

Since f is bounded and $F_{n,x}(t)$ is the df of $S_n=\sum_{i=1}^{n} Y_i$, for $x\in[O, A]$ we have

$$|L_n(f, x)-f(x)| \leqq O(\omega_{2A}(f; \delta))+CP(S_n>2nA) \leqq O(\omega_{2A}(f; \delta))+CP(S_n-nx \geqq An),$$

and the result follows from the exponential bound (14).

The preceding result includes the Szász, Baskakov, and Meyer-König—Zeller operators. A similar result can be proved when $f\in C(-\infty, \infty)$ (e.g. Weierstrass operator). A generalization of (3) is based on

LEMMA 5. *Let $Y_1, Y_2, \dots$ be iid random variables with a pdf $g(y, x) = \exp(yx - b(x))$ relative to a σ-finite measure μ, and set*

$$T_{n,s} = T_{n,s}(b^{(1)}(x)) = E\left(\sum_{i=1}^{n}(Y_i - b^{(1)}(x))\right)^s, \quad s = 0, 1, 2, \dots.$$

Then

$$n^{-2k} T_{n,2k} = \frac{(2k)!}{k!}\left(\frac{b^{(2)}(x)}{2}\right)^k n^{-k} + O(n^{-k-1}). \tag{16}$$

PROOF. Let $Z = \sum_{i=1}^{n}(Y_i - b^{(1)}(x))$, and note that the mgf of Z is

$$\varphi(t) = E\exp(tZ) = \exp\left(-nb^{(1)}(x)t + n(b(t+x) - b(x))\right). \tag{17}$$

Differentiating (17) with respect to t we have $\varphi^{(1)}(t) = \varphi(t)\psi(t)$, where $\psi(t) = n(b^{(1)}(t+x) - b^{(1)}(x))$. Applying Leibniz formula we have

$$\varphi^{(s+1)}(t) = \varphi^{(s)}(t)\psi^{(0)}(t) + \binom{s}{1}\varphi^{(s-1)}(t)\psi^{(1)}(t) +$$

$$+ \binom{s}{2}\varphi^{(s-2)}(t)\psi^{(2)}(t) + \dots + \varphi^{(0)}\psi^{(s)}(t),$$

and hence

$$\varphi^{(s+1)}(0) = \varphi^{(s)}(0)\psi^{(0)}(0) + s\varphi^{(s-1)}(0)\psi^{(1)}(0) + \binom{s}{2}\varphi^{(s-2)}(0)\psi^{(2)}(0) + \dots + \varphi^{(0)}\psi^{(s)}(0).$$

Since $\varphi^{(0)}(0) = \varphi(0) = 1$, $\psi^{(0)}(0) = \psi(0) = 0$, $\psi^{(s)}(0) = nb^{(s+1)}(x)$, $s \geqq 1$, we have

$$\varphi^{(s+1)}(0) = s\varphi^{(s-1)}(0)\psi^{(1)}(0) + \binom{s}{2}\varphi^{(s-2)}(0)\psi^{(2)}(0) + \dots + \psi^{(s)}(0). \tag{18}$$

From (18) we obtain

$$\varphi^{(1)}(0) = 0, \quad \varphi^{(2)}(0) = nb^{(2)}(x), \quad \varphi^{(3)}(0) = nb^{(3)}(x),$$

$$\varphi^{(4)}(0) = 1\cdot 3n^2(b^{(2)}(x))^2 + nb^{(4)}(x), \quad \varphi^{(5)}(0) = 10n^2 b^{(2)}(x) b^{(3)}(x) + nb^{(5)}(x),$$

$$\varphi^{(6)}(0) = 1\cdot 3\cdot 5n^3(b^{(2)}(x))^3 + 15n^2 b^{(2)}(x) b^{(4)}(x) + 10n^2(b^{(3)}(x))^2 + nb^{(6)}(x), \dots.$$

Continuing in this manner we obtain

$$\varphi^{(2k)}(0) = T_{n,2k} = 1\cdot 3\cdot 5\cdot \dots (2k-1)n^k(b^{(2)}(x))^k + O(n^{k-1}),$$

and hence

$$n^{-2k} T_{n,2k} = \frac{(2k)!}{k!}\left(\frac{b^{(2)}(x)}{2}\right)^k n^{-k} + O(n^{-k-1}).$$

We can now prove the following generalization of (3).

THEOREM 2. *Let $Y_1, Y_2, \dots$ be iid random variables with pdf $g(y,x)=\exp(yx-b(x))$ relative to a σ-finite measure μ. Let $L_n(f,x)=Ef(S_n/n)$ be the Feller operator where $S_n=\sum_{i=1}^{n} Y_i$, and assume that $f^{(2k)}(\cdot)$ exists at $b^{(1)}(x)$. Then*

(19)

$$\lim_{n\to\infty} n^k\left[L_n(f,x)-f(b^{(1)}(x))-\sum_{s=1}^{2k-1}\frac{n^{-s}}{s!}T_{n,s}f^{(s)}(b^{(1)}(x))\right]=\left(\frac{b^{(2)}(x)}{2}\right)^k\frac{1}{k!}f^{(2k)}(b^{(1)}(x)),$$

where $T_{n,s}$ has been defined in Lemma 5.

PROOF. Writing $\overline{Y}_n=S_n/n$ and expanding $f(S_n/n)$ around $b^{(1)}(x)$ we have

$$f(S_n/n)=f(b^{(1)}(x))+\sum_{s=1}^{2k-1}\frac{f^{(s)}(b^{(1)}(x))}{s!}(\overline{Y}_n-b^{(1)}(x))^s+$$
$$+(\overline{Y}_n-b^{(1)}(x))^{2k}\left[\frac{f^{(2k)}(b^{(1)}(x))}{(2k)!}+\eta(\overline{Y}_n-b^{(1)}(x))\right],$$

where $|\eta|\leqq M$ and $\eta(h)\to 0$ as $h\to 0$. Taking expectation we obtain

$$L_n(f,x)=f(b^{(1)}(x))+\sum_{s=1}^{2k}\frac{n^{-s}f^{(s)}(b^{(1)}(x))}{s!}T_{n,s}+R_n,$$

where $R_n=E(\overline{Y}_n-b^{(1)}(x))^{2k}\eta(\overline{Y}_n-b^{(1)}(x))$. Choosing $\varepsilon>0$ such that $|\eta(h)|<\varepsilon$ when $|h|<\delta$ we have

$$|R_n|\leqq E(\overline{Y}_n-b^{(1)}(x))^{2k}\left|\eta(\overline{Y}_n-b^{(1)}(x))\right| I\{|\overline{Y}_n-b^{(1)}(x)|<\delta\}+$$
$$+E(\overline{Y}_n-b^{(1)}(x))^{2k}\left|\eta(\overline{Y}_n-b^{(1)}(x))\right| I\{|\overline{Y}_n-b^{(1)}(x)|\geqq\delta\}\leqq$$
$$\leqq \varepsilon n^{-2k}T_{n,2k}+Mn^{-2k}E(S_n-nb^{(1)}(x))^{2k}I\{|S_n-nb^{(1)}(x)|\geqq n\delta\},$$

where $I\{A\}$ denotes the indicator function of an event A. By Cauchy—Schwarz inequality we have

$$E(S_n-nb^{(1)}(x))^{2k}I\{|S_n-nb^{(1)}(x)|\geqq n\delta\}\leqq\left[T_{n,4k}P(|S_n-nb^{(1)}(x)|\geqq n\delta)\right]^{1/2}.$$

From (16) it is easy to see that

$$n^{-2k}\sqrt{T_{n,4k}}=n^{-k}O(1),$$

and it follows from Lemma 3 that

$$|R_n|\leqq\varepsilon n^{-2k}T_{n,2k}+Mn^{-k}\exp(-bn)O(1)$$

for some $b>0$ independent of n. Thus we proved that

$$L_n(f,x)=f(b^{(1)}(x))+\sum_{s=1}^{2k}\frac{f^{(s)}(b^{(1)}(x))}{s!}n^{-s}T_{n,s}+\varepsilon_n n^{-k},\quad \varepsilon_n\to 0 \text{ as } n\to\infty.$$

Hence the desired conclusion follows from Lemma 5.

REMARK. Included in Theorem 2 are the operators (1), (6), (7), (8) and (10), etc. One simply identifies $b^{(1)}(x)$ and $b^{(2)}(x)$ for different operators. The following gives the required identification.

Distribution	*Operator*	$b^{(1)}(x)$	$b^{(2)}(x)$
Bernoulli	Bernstein	x $(0\leqq x\leqq 1)$	$x(1-x)$
Poisson	Szász	x $(x\geqq 0)$	x
Gamma	Gamma	x $(x>0)$	x^2
Normal	Weierstrass	x $(-\infty<x<\infty)$	1
Geometric, $p=(1+x)^{-1}$	Baskakov	x $(x\geqq 0)$	$x(1+x)$

3. Monotonic convergence. By direct calculutions CHENEY and SHARMA [2] proved monotonic convergence for the Szász and Meyer-König—Zeller operators when the underlying function is convex. Here we give an elementary probabilistic proof of monotonic convergence for the Feller operator (5) which includes Szász and many other operators. A similar proof of monotonic convergence for the Meyer-König—Zeller operator (11) is also given. But first we need the following elementary lemma on conditional expectation.

LEMMA 6. *Let $Y_1, Y_2, \ldots$ be iid random variables with finite expectation, and let $S_n=\sum_{i=1}^{n} Y_i$. Then*

$$E\left(\frac{S_n}{n}\middle| S_{n+1}\right) = \frac{S_{n+1}}{n+1} \quad a.s.$$

PROOF. Since $E(Y_1 \mid S_{n+1})=E(Y_2 \mid S_{n+1})=\ldots=E(Y_{n+1} \mid S_{n+1})$, it follows that

$$E(Y_i|S_{n+1}) = E\left(\frac{\sum_{i=1}^{n+1} Y_i}{n+1}\middle| S_{n+1}\right) = \frac{S_{n+1}}{n+1}.$$

Hence

$$E\left(\frac{S_n}{n}\middle| S_{n+1}\right) = \frac{1}{n}\sum_{i=1}^{n} E(Y_i|S_{n+1}) = \frac{S_{n+1}}{n+1} \quad \text{a.s.}$$

We can now prove

THEOREM 3. *Let $Y_1, Y_2, \ldots$ be iid random variables with mean $x\in I$ and variance $\sigma^2(x)$. For a continuous convex and bounded function f on R define the Feller operator (5) by*

$$L_n(f, x) = Ef(S_n/n) = \int_{-\infty}^{\infty} f(t/n)\, dF_{n,x}(t),$$

where $F_{n,x}(t)$ is the df of $S_n=\sum_{i=1}^{n} Y_i$. Then $L_n(f, x)\geqq L_{n+1}(f, x)\geqq\ldots\geqq f(x)$, and $L_n(f, x)\downarrow f(x)$ uniformly on every bounded interval.

PROOF. Clearly, $L_n(f, x)=Ef(S_n/n)=E\big(E(f(S_n/n)|S_{n+1})\big)$. Since f is convex, the conditional version of Jensen's inequality and Lemma 6 give

$$L_n(f, x) \geqq Ef\left(E\left(\frac{S_n}{n}\middle| S_{n+1}\right)\right) = Ef\left(\frac{S_{n+1}}{n+1}\right) = L_{n+1}(f, x).$$

Hence $L_n(f, x)\geqq L_{n+1}(f, x)\geqq \ldots \geqq f(x)$, and the last statement follows from Lemma 1.

THEOREM 4. *Let $f\in C[0, 1]$ and assume that f is convex. Then the Meyer-König—Zeller operator $M_n(f, y)$ defined by* (11) *satisfies $M_n(f, y)\geqq M_{n+1}(f, y)\geqq \ldots \geqq f(y)$, and $M_n(f, y)\to f(y)$ uniformly on* [0, 1].

PROOF. All we need to prove is that $M_n(f, y)\geqq M_{n+1}(f, y)$. To this end, using (9) we obtain

$$P(S_{n+1} = k|S_{n+2} = k') = \binom{n+k}{k}\Big/\binom{n+k'+1}{k'}, \quad k = 0, 1, \ldots, k'.$$

From this conditional distribution it is easy to verify that

$$E\left(\frac{S_{n+1}}{n+S_{n+1}}\middle| S_{n+2} = k'\right) = \frac{k'}{n+1+k'},$$

and hence

$$E\left(\frac{S_{n+1}}{n+S_{n+1}}\middle| S_{n+2}\right) = \frac{S_{n+2}}{n+1+S_{n+2}} \quad \text{a.s.}$$

Now recall from (11) that $M_n(f, y)=Ef\left(\dfrac{S_{n+1}}{n+S_{n+1}}\right)$. Since f is convex, it follows from Jensen's inequality that

$$M_n(f, y) = E\left(E\left(f\left(\frac{S_{n+1}}{n+S_{n+1}}\right)\middle| S_{n+2}\right)\right) \geqq Ef\left(\frac{S_{n+2}}{n+1+S_{n+2}}\right) = M_{n+1}(f, y).$$

4. The case of unbounded function. Let $f(x)\in C[0, \infty)$ where f is not necessarily bounded. Let $Y_1, Y_2, \ldots$ be iid non-negative random variables with mean $x\in[0, \infty)$ and variance $\sigma^2(x)$, and assume that Y_1 has finite moment generating function. Define the operator

$$U_n(f, x) = Ef(S_n/n) = \int_0^\infty f(t/n)\, dF_{n,x}(t),$$

where $F_{n,x}(t)$ is the df of $S_n=\sum_{i=1}^n Y_i$. Assume that $f(x)=O\big(g(x)\big)$ as $x\to\infty$ where $g(x)$ is a positive continuous function on $[0, \infty)$. If $Eg^2(S_n/n)=O(1)$, then for $x\in[0, A]$

$$|U_n(f, x)-f(x)| \leqq O\big(\omega_{2A}(f;\, n^{-1/2})\big), \tag{20}$$

where $\omega_A(f;\, \delta)=\sup\{|f(x+t)-f(x)|\colon\ |t|\leqq\delta,\ x\in[0, A]\}$.

To establish (20) we note that

$$|U_n(f, x)-f(x)| \leqq \int_{t\leqq 2nA} |f(t/n)-f(x)|\, dF_{n,x}(t)+$$

$$+\int_{t\geqq 2nA} |f(t/n)-f(x)|\, dF_{n,x}(t) = R_1+R_2.$$

By Lemma 4, $R_1=O(\omega_{2A}(f;\, n^{-1/2}))$ when $x\in[0, A]$. Since $f(x)=O(g(x))$ as $x\to\infty$, there exists a constant A' such that $|f(x)|\leqq Cg(x)$ for $x\geqq A'$. Hence it follows that

$$R_2 \leqq \sup_{0\leqq x\leqq A} |f(x)|\, P(S_n\geqq 2nA)+\int_{2nA\leqq t\leqq nA'} |f(t/n)|\, dF_{n,x}(t)+$$

$$+C\int_{t\geqq 2nA,\, t\geqq nA'} g(t/n)\, dF_{n,x}(t) \leqq$$

$$\leqq MP(S_n-nx\geqq nA)+M_1P(S_n-nx\geqq nA)+CEg(S_n/n)\, I\{S_n\geqq 2nA\}.$$

By Lemma 3 and Cauchy—Schwarz inequality we have

$$R_2\leqq M_2\varrho^n+C[Eg^2(S_n/n)P(S_n\geqq 2nA)]^{1/2}\leqq M_2\varrho^n+C[O(1)P(S_n-nx\geqq nA)]^{1/2}\leqq$$
$$\leqq M_2\varrho^n+C\sqrt{O(1)\varrho^n},$$

and (20) follows.

EXAMPLE. Consider the Szász operator

$$S_n(f, x)=e^{-nx}\sum_{k=0}^{\infty} f(k/n)\frac{(nx)^k}{k!}.$$

Let $f(x)=O(e^{\alpha x})$ as $x\to\infty$ $(\alpha>0)$. Clearly,

$$Eg^2(S_n/n)=E\exp\left(\frac{2\alpha}{n}S_n\right)=\varphi^n(2\alpha/n)$$

where $S_n=\sum_{i=1}^{n} Y_i$ and $\varphi(t)$ is the mgf of a Poisson random variable Y_1 with parameter x. Hence

$$Eg^2(S_n/n)=\exp\left(nx(e^{2\alpha/n}-1)\right)=O(1),$$

and (20) holds which is due to GRÓF [5]. HERMANN [6] extended this result to the case when $g(x)=e^{\alpha x\ln x}$ $(\alpha>0)$. It is easy to show that $Eg^2(S_n/n)=O(1)$ and hence (20) holds. We remark in passing that a result similar to (20) holds for the Weierstrass operator for an unbounded continuous function f on R if it satisfies a suitable growth condition, e.g. $f(x)=O(e^{\alpha|x|})$ as $x\to\infty$. The details are omitted.

5. Weak convergence and approximation operators. Although most approximation operators are covered by (4) or (5) but for certain operators it may be difficult to prove uniform convergence via (4) or (5) (e.g. the Meyer-König—Zeller operator or its generalization by CHENEY and SHARMA [2]). A general simple approach can be based on the following concept. A sequence of random variables

$(Z_n, n \geqq 1)$ whose df depends on a continuous parameter $x \in I$ is said to converge weakly or in probability to x $(Z_n \xrightarrow{P} x)$ if $P_x(|Z_n - x| \geqq \delta) \to 0$ as $n \to \infty$ for all $\delta > 0$. The convergence $Z_n \xrightarrow{P} x$ is uniform if $\sup_{x \in I} P_x(|Z_n - x| \geqq \delta) \to 0$. Weak convergence leads to

THEOREM 5. *Let* $\{Z_n, n \geqq 1\}$ *be a sequence of random variables having df dependent on* $x \in [0, 1]$ *such that* $Z_n \xrightarrow{P} x$ *uniformly in* x. *Then* $R_n(f, x) = Ef(Z_n) \to f(x)$ *uniformly on* $[0, 1]$ *if* $f \in C[0, 1]$.

PROOF. Obviously,

$$|R_n(f, x) - f(x)| \leqq \int_{|z-x| \leqq \delta} |f(z) - f(x)| \, dF_{n,x}(z) + \int_{|z-x| \geqq \delta} |f(z) - f(x)| \, dF_{n,x}(z),$$

where $F_{n,x}(z)$ is the df of Z_n. Hence the result follows from

$$|R_n(f, x) - f(x)| \leqq \omega(\delta) + M \sup_{0 \leqq x \leqq 1} P_x(|Z_n - x| \geqq \delta),$$

where $\omega(\delta) = \sup \{|f(x) - f(z)| \colon |x - z| \leqq \delta, \ 0 \leqq x, \ z \leqq 1\}$.

From the preceding theorem we deduce a result of CHENEY and SHARMA [2] which generalizes (11). Let $L_\nu^{(\alpha)}$ $(\alpha > -1)$ denote the Laguerre polynomial of degree ν where α is a parameter. For $f \in C[0, 1]$ define the operator of CHENEY and SHARMA [2] by

$$P_n(f, x) = (1-x)^{n+1} \exp\left(\frac{tx}{1-x}\right) \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{\nu+n}\right) L_\nu^{(n)}(t) x^\nu, \quad 0 \leqq x \leqq a < 1,$$

where t is a parameter $\leqq 0$. They proved that $P_n(f, x) \to f(x)$ uniformly on $[0, a]$ $(a < 1)$ if $t/n \to 0$ as $n \to \infty$. To deduce this result from Theorem 5 define the probabilities

$$p_{n,j}(x) = P(Y_n = j) = (1-x)^{n+1} \exp\left(tx/(1-x)\right) L_j^{(n)}(t) x^j,$$

$$j = 0, 1, \ldots, \quad 0 \leqq x \leqq a < 1, \ t \leqq 0.$$

It is well known that $p_{n,j}(x) \geqq 0$ for all $x \in [0, a]$ and $\sum_{j=0}^{\infty} p_{n,j}(x) = 1$ (see SZEGŐ [12]). Clearly, $P_n(f, x) = Ef(Z_n)$ where $Z_n = Y_n/(n + Y_n)$ and the expectation is taken with respect to the above probability distribution. The result will follow if we can show that $Z_n \xrightarrow{P} x$ uniformly in $x \in [0, a]$. To this end, we note from [12] that the generating function of Y_n is given by

$$\varphi_n(s) = Es^{Y_n} = \sum_{j=0}^{\infty} s^j p_{n,j}(x) = (1-x)^{n+1} \exp\left(\frac{tx}{1-x}\right) (1-sx)^{-(n+1)} \exp\left(\frac{-tsx}{1-sx}\right),$$

where $0 < s \leqq 1$. Using $\varphi_n(s)$ we can easily compute the first two moments of Y_n and show that $Y_n/n \xrightarrow{P} x/(1-x)$ uniformly in $x \in [0, a]$ if $t/n \to 0$. From this we conclude that $Z_n \xrightarrow{P} x$ uniformly. To see this let $\varepsilon > 0$ and choose $\delta > \varepsilon/(a + \varepsilon)$.

Writing $U_n = n^{-1} Y_n - x(1-x)^{-1}$ we have

$$P_x(|Z_n - x| \geqq \varepsilon) = P_x(|Z_n - x| \geqq \varepsilon, \ |U_n| \geqq \delta) + P_x(|Z_n - x| \geqq \varepsilon, \ |U_n| < \delta) \leqq$$
$$\leqq P_x(|U_n| \geqq \delta) + P_x(|U_n| \geqq \eta, \ |U_n| < \delta),$$
$$\eta = a\delta - \varepsilon(1-\delta) \leqq P_x(|U_n| \geqq \delta) + P_x(|U_n| \geqq \eta) \to 0 \quad \text{as} \quad n \to \infty,$$

uniformly in $x \in [0, a]$ if $t/n \to 0$. Thus $Z_n \xrightarrow{P} x$ uniformly and hence by Theorem 5 $P_n(f, x) \to f(x)$ uniformly on $[0, a]$ provided $t/n \to 0$ as $n \to \infty$. Uniform convergence of various operators can be established by uniform weak convergence. Moreover, new operators can be generated by uniformly convergent sequence of random variables.

Finally, it is interesting to note that Lemma 2 applied to Bernstein polynomials $B_n(f, x)$ with $G(x)$ as uniform distribution on [0, 1] gives

$$\lim_{n \to \infty} \int_0^1 B_n(f, x)\, dx = \int_0^1 f(x)\, dx, \quad f \in C[0, 1].$$

Substituting the expression for $B_n(f, x)$ and using the well known property of Beta function we obtain

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n} f(j/n) = \int_0^1 f(x)\, dx,$$

a useful result for evaluating limits.

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Writing [illegible] we have

$$[illegible]$$

uniformly [illegible]. Thus [illegible] uniformly and hence by Theorem 5 [illegible] uniformly [illegible] provided [illegible] convergence of [illegible] can be established by uniform [illegible] convergence, [illegible] can be generalized [illegible] sequence of

Finally, it is interesting [illegible] Bernstein polynomials [illegible] in the form [illegible]

$$[illegible]$$

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ÜBER EIN GRAPHENTHEORETISCHES ERGEBNIS VON T. GALLAI

Von

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Im Jahre 1961 veröffentlichte T. GALLAI [4] eine hübsche und natürliche Verallgemeinerung des bekannten Existenzsatzes für f-Faktoren von W. T. TUTTE [10]. In [5] gelang es L. LOVÁSZ, auch umgekehrt diesen Satz von T. Gallai aus dem Existenzsatz für f-Faktoren von W. T. Tutte herzuleiten. Da ich das Ergebnis von T. Gallai aber als natürlichen Zugang zu den Faktorsätzen betrachte, erscheint es mir wünschenswert, einen einfachen, direkten Beweis hiervon zu finden. Während der ursprüngliche Beweis von T. Gallai auf der Methode der alternierenden Kantenzüge basierte und recht kompliziert war, möchte ich in der vorliegenden Arbeit einen einfacheren Induktionsbeweis darstellen. Zum Schluß möchte ich noch zwei simple, direkte Beweise des Spezialfalles des Satzes von T. Gallai skizzieren, welcher dem Existenzsatz für 1-Faktoren von W. T. TUTTE [9] entspricht

Wir betrachten hier Pseudographen, lassen also mehrfache Kanten und Schlingen zu. Die Menge aller Kanten zwischen den nicht unbedingt verschiedenen Ecken x und y im Pseudographen G bezeichnen wir mit $[x, y]^G$ und die Eckenmenge bzw. Kantenmenge von G mit $E(G)$ bzw. $K(G)$; weiterhin seien $|G| := |E(G)|$ und $\|G\| := |K(G)|$. Wir setzen immer $E(G) \cup K(G)$ als endlich voraus. Für $A, B \subseteq E(G)$ mit $A \cap B = \emptyset$ seien $K(A, B; G) := \bigcup_{\substack{a \in A \\ b \in B}} [a, b]^G$ und $\varkappa(A, B; G) := |K(A, B; G)|$; ferner bezeichne $G(A)$ den von A in G induzierten Pseudographen, und es sei $N(A; G) := \{x \in E(G) - A \mid \bigvee_{a \in A} [a, x]^G \neq \emptyset\}$. Sei $\mathfrak{L}(G)$ die Menge der (Zusammenhangs-) Komponenten von G und für $x \in E(G)$ sei $C(x; G)$ die Komponente von G, welche x enthält. Bei einem „Weg" W: $a_0, k_1, a_1, \dots, k_m, a_m$ $(a_i \in E(G), k_i \in [a_{i-1}, a_i]^G)$ komme keine Ecke a_i und keine Kante k_j zweimal vor, lediglich $a_0 = a_m$ sei zugelassen; im Falle $a_0 \neq a_m$ nennen wir W *offen*, im Falle $a_0 = a_m$ *geschlossen*. Ein geschlossener Weg ist also ein Kreis mit einer (als Anfangs- und Endpunkt) ausgezeichneten Ecke. Insbesondere betrachten wir eine Schlinge als (geschlossenen) Weg der Länge 1. Wir sagen, der Weg W: $a_0, k_1, a_1, \dots, k_m, a_m$ *durchläuft* $x \in E(G)$, wenn $x \in \{a_1, \dots, a_{m-1}\}$ ist, und er *endet in* x, wenn $x = a_0$ oder $x = a_m$ gilt. Die Ecken a_0 und a_m heißen die *Endpunkte von* W, und wir nennen W einen a_0, a_m-*Weg*.

Sei $N^+ := \{0, 1, 2, 3, \dots\} \cup \{\infty\}$, wobei wir mit dem Symbol ∞ wie üblich rechnen. Für eine Funktion f: $D \to N^+$ und für $X \subseteq D$ sei $f(X) := \sum_{x \in X} f(x)$ und $f|X$ bezeichne die Einschränkung von f auf X. Für den Pseudographen G sei ein Funktionenpaar (e, d) mit e: $E(G) \to N^+$ und d: $E(G) \to N^+$ gegeben; wir nennen dann $(G, (e, d))$ einen *bewerteten Pseudographen* und bezeichnen ihn mit G_e^d. Ein System von Wegen W_i $(i \in J)$ in G nennen wir *zulässig* in G_e^d, wenn die Wege

W_i die Länge $\geqq 1$ haben und paarweise kantendisjunkt sind und für jedes $x \in E(G)$ gilt: es laufen höchstens $d(x)$ der Wege W_i durch x und x tritt höchstens $e(x)$-mal als Endpunkt der Wege W_i auf. (Hierbei trete x zweimal als Endpunkt von W_i auf, wenn W_i ein geschlossener Weg mit dem Endpunkt x ist.) Bezeichne $\lambda(G_e^d)$ die Maximalzahl von Wegen, die in einem in G_e^d zulässigen Wegesystem auftreten kann. Ein zulässiges Wegesystem in G_e^d aus $\lambda(G_e^d)$-Wegen nennen wir ein *größtes zulässiges Wegesystem* von G_e^d. Als Hauptergebnis von [4] bestimmte T. GALLAI $\lambda(G_e^d)$ als Minimum des „Wertes" gewisser Partitionen von $E(G)$. Betrachten wir hierzu $\mathfrak{P}(G) := \{(A, B) \mid A \subseteqq E(G) \wedge B \subseteqq E(G) \wedge A \cap B = \emptyset\}$ und für $(A, B) \in \mathfrak{P}(G)$ definieren wir $s(A, B; G_e^d) := \|G(A)\| + e(B) + d(B) + \sum_{C \in \mathfrak{L}(G-(A \cup B))} \left[\frac{e(C) + \varkappa(C, A; G)}{2}\right]$, wobei $e(C) := e(E(C))$ und $\varkappa(C, A; G) := \varkappa(E(C), A; G)$ seien und $[r]$ die größte ganze Zahl $\leqq r$ bedeute. $\left(\text{Natürlich sei } \left[\frac{\infty}{2}\right] = \infty.\right)$ Hiermit können wir das Ergebnis von T. Gallai formulieren.

SATZ 1 (T. Gallai). *Für jeden endlichen, bewerteten Pseudographen G_e^d gilt*

$$\lambda(G_e^d) = \min_{(A, B) \in \mathfrak{P}(G)} s(A, B; G_e^d) =: s(G_e^d).$$

Zunächst noch einige Bezeichnungen. Für $f: D \to N^+$ und $\{x_1, \ldots, x_n\} \subseteqq D$ sei $f^m_{x_1, \ldots, x_n}$ definiert durch $f^m_{x_1, \ldots, x_n}(x) := f(x) + m$ für $x \in \{x_1, \ldots, x_n\}$ und $f^m_{x_1, \ldots, x_n}(x) := f(x)$ für $x \in D - \{x_1, \ldots, x_n\}$, wobei m eine ganze Zahl oder $m = \infty$ sei. Weiterhin seien $F(f) := \{x \in D \mid f(x) \text{ endlich}\}$ und $\bar{F}(f) := D - F(f)$. Sei G_d^e ein bewerteter Pseudograph. Für einen Teilpseudographen $H \subseteqq G$ bedeute H_e^d den bewerteten Pseudographen $H_{e'}^{d'}$ mit $e' := e \mid E(H)$ und $d' := d \mid E(H)$. Die Komponenten von $G - (A \cup B)$ nennen wir die *Komponenten von* $(A, B) \in \mathfrak{P}(G)$ *in* G, und eine Komponente C von (A, B) in G_e^d heiße *gerade (ungerade)*, wenn $e(C) + \varkappa(C, A; G)$ gerade (ungerade) ist. Ferner sei $\bar{\mathfrak{P}}(G_e^d) := \{(A, B) \in \mathfrak{P}(G) \mid s(A, B; G_e^d) = s(G_e^d)\}$. Seien $X \subseteqq E(G)$ und $x \in X$. Der Pseudograph $\bar{G}$ entstehe aus $G - X$ durch Hinzufügen der Ecke x, wobei die mit x in $\bar{G}$ inzidierenden Kanten festgelegt seien durch $[x, y]^{\bar{G}} := K(X, y; G)$ für $y \in E(G) - X$ und $[x, x]^{\bar{G}} := K(G(X))$. Weiterhin seien die Funktionen $e': E(\bar{G}) \to N^+$ und $d': E(\bar{G}) \to N^+$ definiert durch $e'(y) := e(y)$ und $d'(y) := d(y)$ für $y \in E(G) - X$ und $e'(x) := e(X)$, $d'(x) := d(X)$. Wir sagen, der bewertete Pseudograph $\bar{G}_{e'}^{d'}$ entstehe aus G_e^d *durch Identifizieren von* X *zu* x. — Für $H \subseteqq G$ und $X \subseteqq E(G)$ sei $H \cap X := E(H) \cap X$. Den Grad der Ecke x in G bezeichnen wir mit $\varkappa(x; G)$ und $\mathrm{I}(x; G)$ bedeute die Anzahl der Schlingen durch x in G. (Eine Schlinge durch x zählt für den Grad $\varkappa(x; G)$ doppelt.)

Wegen $s(E(G), \emptyset; G_e^d) = \|G\|$ gilt $s(G_e^d) \leqq \|G\|$ für jeden bewerteten Pseudographen G_e^d. Insbesondere ist $s(A, B; G_e^d)$ endlich für $(A, B) \in \bar{\mathfrak{P}}(G_e^d)$. Wenn $s(A, B; G_e^d)$ endlich ist, muß $\bar{F}(e) \subseteqq A$ sein und $\bar{F}(d) \cap B = \emptyset$ gelten. Dem Beweis von Satz 1 schicken wir einige Lemmata voraus.

LEMMA 1. (a) *Für alle* $(A, B) \in \mathfrak{P}(G)$ *gilt* $s(A, B; G_e^d) \geqq \lambda(G_e^d)$.

(b) *Sei* $(A, B) \in \mathfrak{P}(G)$ *mit* $s(A, B; G_e^d) = \lambda(G_e^d)$ *und sei* $\mathfrak{S}$ *ein größtes zulässiges Wegesystem von* G_e^d. *Dann läuft kein Weg von* $\mathfrak{S}$ *durch eine Ecke von* A. *Wenn* C *eine gerade Kompenente von* (A, B) *in* G_e^d *ist, dann ist* $K(W) \cap K(C, B; G) = \emptyset$ *für jedes* $W \in \mathfrak{S}$, *aber zu jedem* $k \in K(C, A; G)$ *existiert ein* $W \in \mathfrak{S}$ *mit* $k \in K(W)$. *Weiterhin* *existieren genau* $\frac{e(C)+\varkappa(C, A;\, G)}{2}$ *Wege* $W\in\mathfrak{S}$ *mit* $W\cap C\neq\emptyset$, *und jeder dieser Wege* W *wird in* $e(C)+\varkappa(C, A;\, G)$ *genau zweimal gezählt.* [1]

Beweis. Seien $(A, B)\in\mathfrak{P}(G)$, $\mathfrak{S}$ ein größtes zulässiges Wegesystem von G_e^d und $W\in\mathfrak{S}$. Da $\mathfrak{S}$ zulässig in G_e^d ist, gilt $|\{W\in\mathfrak{S}\,|\,K(W)\cap K(G(A))\neq\emptyset \vee W\cap B\neq\emptyset\}|\leqq \leqq\|G(A)\|+e(B)+d(B)$. Sei nun $W\in\mathfrak{S}$ mit $K(W)\cap K(G(A))=\emptyset$ und $W\cap B=\emptyset$. Wegen $\|W\|\geqq 1$ existiert eine Komponente C von (A, B) mit $C\cap W\neq\emptyset$. Dann gehören aber beide (nicht notwendig verschiedenen) Endpunkte von W zu C oder es gehört ein Endpunkt von W zu C und es ist $K(W)\cap K(C, A;\, G)\neq\emptyset$ oder es gilt $|K(W)\cap K(C, A; G)|\geqq 2$. Für $\mathfrak{S}_C := \{W\in\mathfrak{S}\,|\, W\cap B=\emptyset\wedge W\cap C\neq\emptyset\}$ ergibt sich hieraus $2|\mathfrak{S}_C|\leqq e(C)+\varkappa(C, A;\, G)$ wegen der Zulässigkeit von $\mathfrak{S}$. Somit folgt

$$\lambda(G_e^d)=|\mathfrak{S}|\leqq\|G(A)\|+e(B)+d(B)+\sum_{C\in\mathfrak{L}(G-(A\cup B))}\left[\frac{e(C)+\varkappa(C, A;\, G)}{2}\right]=s(A, B;\, G_e^d).$$

Nehmen wir nun $s(A, B;\, G_e^d)=\lambda(G_e^d)$ an und sei C eine gerade Komponente von (A, B) in G_e^d. Wie man aus obiger Abschätzung ersieht, muß dann auch die Gleichung $|\mathfrak{S}_C|=\frac{e(C)+\varkappa(C, A;\, G)}{2}$ gelten. Somit wird jedes $W\in\mathfrak{S}_C$ in $e(C)+ +\varkappa(C, A;\, G)$ genau zweimal gezählt, jedes $k\in K(C, A;\, G)$ gehört zu einem $W\in\mathfrak{S}_C$ und jedes $c\in E(C)$ tritt genau $e(c)$-mal als Endpunkt bei Wegen von $\mathfrak{S}_C$ auf. Wegen der Zulässigkeit von $\mathfrak{S}$ hat also kein $W\in\mathfrak{S}-\mathfrak{S}_C$ einen Endpunkt in C und es gilt $K(W)\cap K(C, A;\, G)=\emptyset$ für jedes $W\in\mathfrak{S}-\mathfrak{S}_C$. Nehmen wir die Existenz eines $W\in\mathfrak{S}-\mathfrak{S}_C$ mit $W\cap C\neq\emptyset$ an. Da W keinen Endpunkt in C hat und $K(W)\cap \cap K(C, A;\, G)=\emptyset$ gilt, folgt $|K(W)\cap K(C, B;\, G)|\geqq 2$. Dann wird aber W zweimal in $e(B)+d(B)$ gezählt, was nicht sein kann, da auch die Gleichung $|\{W\in\mathfrak{S}|W\cap \cap B\neq\emptyset\}|=e(B)+d(B)$ gelten muß. Also ist $\mathfrak{S}_C=\{W\in\mathfrak{S}\,|\,W\cap C\neq\emptyset\}$, und es ergeben sich alle Behauptungen von (b) bis auf die erste.

Sei nun W: $a_0, k_1, a_1, \ldots, k_m, a_m$ aus $\mathfrak{S}$ und es existiere ein $i\in\{1, \ldots, m-1\}$ mit $a_i\in A$. Betrachten wir den Weg W': $a_0, k_1, a_1, \ldots, a_i$. Wenn $K(W')\cap K(G(A))=\emptyset$ und $W'\cap B=\emptyset$ sind, existiert wie oben ein $C\in\mathfrak{L}(G-(A\cup B))$, so daß W' zu $e(C)+ +\varkappa(C, A;\, G)$ mindestens den Wert 2 beiträgt. Da dasselbe für W'': $a_i, k_{i+1}, a_{i+1}, \ldots \ldots, a_m$ gilt, wird also W in $\|G(A)\|+e(B)+d(B)+\sum_{C\in\mathfrak{L}(G-(A\cup B))}\left[\frac{e(C)+\varkappa(C, A;\, G)}{2}\right]= =s(A, B;\, G)$ mindestens zweimal gezählt, im Widerspruch zu $|\mathfrak{S}|=s(A, B;\, G)$.

Lemma 2. *Seien* G_e^d *und* $G_{e'}^{d'}$ *bewertete Pseudographen mit* $e'(x)\geqq e(x)$ *und* $d'(x)\geqq d(x)$ *für alle* $x\in E(G)$.

(a) *Dann sind* $s(G_{e'}^{d'})\geqq s(G_e^d)$ *und* $\lambda(G_{e'}^{d'})\geqq\lambda(G_e^d)$.

(b) *Wenn* $e(a)\geqq\varkappa(a;\, G)$ *für ein* $a\in E(G)$ *gilt, dann ist* $\lambda(G_{\bar e}^d)=\lambda(G_e^d)$ *für* $\bar e := e_a^\infty$.

(c) *Wenn* $d(a)\geqq\left[\frac{\varkappa(a;\, G)}{2}\right]$ *für ein* $a\in E(G)$ *gilt, dann ist* $\lambda(G_e^{\bar d})=\lambda(G_e^d)$ *für* $\bar d := d_a^\infty$.

[1] Genauer: W trägt zu $\sum_{x\in E(C)} i(x;\, \mathfrak{S})+\sum_{W'\in\mathfrak{S}}|K(W')\cap K(C, A;\, G)|$ genau den Wert 2 bei, wobei $i(x;\, \mathfrak{S})$ angebe, wie oft x als Endpunkt von Wegen aus $\mathfrak{S}$ vorkommt. Wegen der Zulässigkeit von $\mathfrak{S}$ gilt $\sum_{x\in E(C)} i(x;\, \mathfrak{S})+\sum_{W'\in\mathfrak{S}}|K(W')\cap K(C, A;\, G)|\leqq e(C)+\varkappa(C, A;\, G)$.

Der Beweis von (a) bis (c) ergibt sich unmittelbar.

LEMMA 3. *Sei* $(A, B)\in\overline{\mathfrak{P}}(G_e^d)$ *und sei* $a\in A$. *Sei* $d' := d_a^m$ *mit* $m\in\mathbf{N}^+$ *und es gelte* $\lambda(G_e^{d'})=s(G_e^{d'})$. *Dann gilt auch* $\lambda(G_e^d)=\lambda(G_e^{d'})=s(G_e^d)$.

BEWEIS. Nach den Lemmata 1a und 2a genügt es, $\lambda(G_e^d)\geqq\lambda(G_e^{d'})$ zu zeigen. Wegen $s(G_e^{d'})\leqq s(A, B;\ G_e^{d'})=s(A, B;\ G_e^d)=s(G_e^d)\leqq s(G_e^{d'})$ nach Lemma 2a folgt $(A, B)\in\overline{\mathfrak{P}}(G_e^{d'})$. Sei $\mathfrak{S}$ ein größtes zulässiges Wegesystem in $G_e^{d'}$. Wegen $\lambda(G_e^{d'})==s(A, B;\ G_e^{d'})$ geht nach Lemma 1b keiner der Wege von $\mathfrak{S}$ durch $a\in A$. Also ist $\mathfrak{S}$ auch in G_e^d zulässig.

LEMMA 4. *Sei C eine gerade Komponente von* $(A, B)\in\overline{\mathfrak{P}}(G_e^d)$ *mit* $|C|=1$, *etwa* $E(C)=\{c\}$, *und sei* $\mathfrak{l}(c;\ G)=0$. *Sei* $d' := d_c^m$ *mit* $m\in\mathbf{N}^+$ *und es gelte* $\lambda(G_e^{d'})==s(G_e^{d'})$. *Dann gilt auch* $\lambda(G_e^d)=\lambda(G_e^{d'})=s(G_e^d)$.

BEWEIS. Es genügt wieder, $\lambda(G_e^d)\geqq\lambda(G_e^{d'})$ zu zeigen. Wegen $s(A, B\cup\{c\};\ G_e^d)\geqq\geqq s(A, B;\ G_e^d)$ und C gerade gilt $e(c)+d(c)\geqq\dfrac{e(c)+\varkappa(c, A;\ G)}{2}=:k$. Sei $\mathfrak{S}$ ein größtes zulässiges Wegesystem in $G_e^{d'}$. Wegen $s(A, B;\ G_e^{d'})=s(A, B;\ G_e^d)$ folgt wieder $(A, B)\in\overline{\mathfrak{P}}(G_e^{d'})$. Wegen $s(A, B;\ G_e^{d'})=s(G_e^{d'})=\lambda(G_e^{d'})$ existieren nach Lemma 1b genau k Wege $W\in\mathfrak{S}$ mit $c\in W$, und jeder dieser Wege wird in $e(c)++\varkappa(c, A;\ G)$ genau zweimal gezählt. Wegen $K(W)\cap K(c, B;\ G)=\emptyset$ für $W\in\mathfrak{S}$ nach Lemma 1b und $\mathfrak{l}(c;\ G)=0$ haben also genau $e(c)$ Wege von $\mathfrak{S}$ c als Endpunkt und es laufen somit genau $k-e(c)$ Wege von $\mathfrak{S}$ durch c. Wegen $k-e(c)\leqq d(c)$ ist also $\mathfrak{S}$ auch in G_e^d zulässig.

LEMMA 5. *Seien* $a, a_1, \dots, a_m, b_1, \dots, b_m$ *ganze Zahlen mit* $a_i<b_i$ *für* $i=1, \dots, m$. *Dann ist* $\left[\dfrac{a+\sum_{i=1}^m a_i}{2}\right]\leqq\dfrac{a}{2}+\sum_{i=1}^m\left[\dfrac{b_i}{2}\right]$, *wobei das Gleichheitszeichen nur gelten kann, wenn* $b_i-1=a_i\equiv 0 \pmod 2$ *für alle* $i=1, \dots, m$ *gilt.*

Der Beweis von Lemma 5 ergibt sich unmittelbar aus

$$\left[\frac{a+\sum_{i=1}^m a_i}{2}\right]\leqq\frac{a}{2}+\sum_{i=1}^m\frac{a_i}{2}\leqq\frac{a}{2}+\sum_{i=1}^m\left[\frac{a_i+1}{2}\right]\leqq\frac{a}{2}+\sum_{i=1}^m\left[\frac{b_i}{2}\right].$$

LEMMA 6. *Sei* $(A, B)\in\mathfrak{P}(G_e^d)$. *Dann gilt* $\varkappa(a;\ G(A))\leqq e(a)$ *für alle* $a\in A$. *Sei* $a\in A$ *mit* $e(a)=0$ *und sei* $k\in K(a, \tilde{E}(G)-(A\cup B);\ G)$. *Dann ist* $(A-\{a\}, B)\in\overline{\mathfrak{P}}(G_e^d)$ *und k ist trennende Kante einer geraden Komponente von* $(A-\{a\}, B)$ *in* G_e^d.

BEWEIS. Sei $a\in A$ und seien $C_1, \dots, C_m$ die Komponenten C von (A, B) mit $\varkappa(a, C;\ G)>0$. Dann ist $C_0 := G\big(\bigcup_{i=1}^m E(C_i)\cup\{a\}\big)$ eine Komponente von $(A-\{a\}, B)$ in G und es gilt

$$s(A-\{a\}, B;\ G_e^d)=s(A, B;\ G_e^d)-\big(\varkappa(a;\ G(A))-\mathfrak{l}(a;\ G)\big)-$$
$$-\sum_{i=1}^m\left[\frac{e(C_i)+\varkappa(C_i, A;\ G)}{2}\right]+\left[\frac{e(C_0)+\varkappa(C_0, A-\{a\};\ G)}{2}\right].$$

Wegen $(A, B)\in\overline{\mathfrak{P}}(G_e^d)$ ist $s(A-\{a\}, B;\ G_e^d)\geqq s(A, B;\ G_e^d)$, also gilt

$$\text{(U)}\quad \left[\frac{e(C_0)+\varkappa(C_0, A-\{a\};\ G)}{2}\right] \geqq \varkappa(a;\ G(A))-\mathrm{I}(a;\ G)+\sum_{i=1}^{m}\left[\frac{e(C_i)+\varkappa(C_i, A;\ G)}{2}\right].$$

Nach Lemma 5 gilt andererseits

$$\text{(V)}\qquad \left[\frac{e(C_0)+\varkappa(C_0, A-\{a\};\ G)}{2}\right] =$$

$$= \left[\frac{e(a)+\varkappa(a;\ G(A))-2\mathrm{I}(a;\ G)+\sum\limits_{i=1}^{m}\left(e(C_i)+\varkappa(C_i, A-\{a\};\ G)\right)}{2}\right] \leqq$$

$$\leqq \frac{e(a)+\varkappa(a;\ G(A))}{2}-\mathrm{I}(a;\ G)+\sum_{i=1}^{m}\left[\frac{e(C_i)+\varkappa(C_i, A;\ G)}{2}\right].$$

Aus (U) und (V) ergibt sich $\varkappa(a;\ G(A))\leqq e(a)$.

Nehmen wir nun $e(a)=0$ an. Nach dem eben Bewiesenen ist dann $\varkappa(a;\ G(A))= =\mathrm{I}(a;\ G)=0$. Aus den Ungleichungen (U) und (V) folgt dann

$$\left[\frac{e(C_0)+\varkappa(C_0, A-\{a\};\ G)}{2}\right] = \sum_{i=1}^{m}\left[\frac{e(C_i)+\varkappa(C_i, A;\ G)}{2}\right].$$

Somit ist $s(A-\{a\}, B;\ G_e^d)=s(A, B;\ G_e^d)$, also auch $(A-\{a\}, B)\in\overline{\mathfrak{P}}(G_e^d)$. Mit Lemma 5 folgt außerdem $e(C_i)+\varkappa(C_i, A;\ G)-1=e(C_i)+\varkappa(C_i, A-\{a\};\ G)\equiv 0 \pmod 2$ für alle $i=1, \dots, m$. Somit ist C_0 eine gerade Komponente von $(A-\{a\}, B)$ in G_e^d. Außerdem ergibt sich $\varkappa(C_i, A;\ G)=\varkappa(C_i, A-\{a\};\ G)+1$, also $\varkappa(C_i, a;\ G)=1$ für $i=1, \dots, m$. Somit ist k eine trennende Kante von C_0.

LEMMA 7. *Sei $X\subseteqq E(G)$ und $H_{e'}^{d'}$ entstehe aus G_e^d durch Identifizieren von X zu $x_0\in X$. Dann gilt $s(H_{e'}^{d'})\geqq s(G_e^d)$.*

BEWEIS. Sei $(A, B)\in\overline{\mathfrak{P}}(H_{e'}^{d'})$. Es genügt, ein $(A', B')\in\mathfrak{P}(G)$ mit $s(A', B';\ G_e^d)\leqq \leqq s(A, B;\ H_{e'}^{d'})$ zu finden. Im Falle $x_0\in B$ gilt $s(A, B\cup X;\ G_e^d)=s(A, B;\ H_{e'}^{d'})$ nach Definition des Identifizierens. Im Falle $x_0\in A$ gilt analog $s(A\cup X, B;\ G_e^d)=s(A, B;\ H_{e'}^{d'})$ wegen $\mathrm{I}(x_0;\ H)=\|G(X)\|$ und $\varkappa(y, X;\ G)=\varkappa(y, x_0;\ H)$ für $y\in E(G)-X$. Nehmen wir nun $x_0\notin A\cup B$ an. Sei $C_0:=C(x_0;\ H-(A\cup B))$ und seien $C_1, \dots, C_m$ die Komponenten von $G(E(C_0)\cup X)$. Wegen $e'(C_0)+\varkappa(C_0, A;\ H)=\sum\limits_{i=1}^{m}(e(C_i)+ +\varkappa(C_i, A;\ G))$ und $\sum[r_i]\leqq[\sum r_i]$ gilt dann $s(A, B;\ G_e^d)\leqq s(A, B;\ H_{e'}^{d'})$.

Wenden wir uns nun dem Beweis des Satzes von T. Gallai zu.

BEWEIS VON SATZ 1. Wir nehmen an, daß Satz 1 falsch ist, und wählen unter den Gegenbeispielen zu Satz 1 mit möglichst kleiner Eckenzahl eines mit möglichst wenig Kanten aus, etwa G. Dann existieren also Funktionen e': $E(G)\to\mathbf{N}^+$ und d': $E(G)\to\mathbf{N}^+$ mit $\lambda(G_{e'}^{d'})\neq s(G_{e'}^{d'})$, also $\lambda(G_{e'}^{d'})<s(G_{e'}^{d'})$ nach Lemma 1a. Seien $P:=\{(e', d')\,|\,e'\colon\ E(G)\to\mathbf{N}^+\wedge d'\colon\ E(G)\to\mathbf{N}^+\wedge\lambda(G_{e'}^{d'})<s(G_{e'}^{d'})\}$ und $\bar{P}:=$

$:=\{(e', d')\in P \mid |\bar F(e')|+|\bar F(d')| = \max\limits_{(e'', d'')\in P} (|\bar F(e'')|+|\bar F(d'')|)\}$. Man wähle $(e, d)\in \bar P$ mit $e(F(e))+d(F(d)) = \max\limits_{(e', d')\in \bar P} (e'(F(e'))+d'(F(d')))$ und sei $\sigma := s(G_e^d)$. Wir leiten einige Eigenschaften von G_e^d her.

(1) *Sei* $(A, B)\in \mathfrak{P}(G_e^d)$. *Dann ist* $d(a)=\infty$ *für alle* $a\in A$.

Wäre nämlich $a\in F(d)$, so müßte für $d' := d_a^\infty$ wegen $|\bar F(d')|>|\bar F(d)|$ nach Wahl von (e, d) gelten $(e, d')\notin P$, also $\lambda(G_e^{d'})=s(G_e^{d'})$. Nach Lemma 3 wäre dann aber auch $\lambda(G_e^d)=s(G_e^d)$.

(2) *Für alle* $x\in E(G)$ *gilt* $\varkappa(x; G)>0$ *und* $e(x)+d(x)>0$.

Sei $x\in E(G)$ mit $\varkappa(x; G)=0$ oder $e(x)=d(x)=0$ und sei $(A, B)\in \mathfrak{P}((G-x)_e^d)$. Im Falle $\varkappa(x; G)=0$ ist $s(A, B; (G-x)_e^d)=s(A\cup\{x\}, B; G_e^d)\geqq\sigma$ und im Falle $e(x)=d(x)=0$ gilt $s(A, B; (G-x)_e^d)=s(A, B\cup\{x\}; G_e^d)\geqq\sigma$. Da G als „kleinstes" Gegenbeispiel gewählt wurde, ergibt sich somit der Widerspruch $\lambda(G_e^d)\geqq\lambda((G-x)_e^d)= =s((G-x)_e^d)=s(A, B; (G-x)_e^d)\geqq\sigma$.

(3) *Es gibt keine Schlinge in* G.

Sei $k\in K(G)$ eine Schlinge durch $x\in E(G)$ und sei $(A, B)\in\mathfrak{P}((G-k)_e^d)$. Da nach Wahl von G gilt $s((G-k)_e^d)=\lambda((G-k)_e^d)\leqq\lambda(G_e^d)<\sigma$, folgt $s(A, B; G_e^d)\geqq\sigma> >s(A, B; (G-k)_e^d)$, also $x\in A$ und $(A, B)\in\mathfrak{P}(G_e^d)$. Nach (1) ist dann $d(x)=\infty$ und nach Lemma 6 gilt $e(x)\geqq 2$. Sei $e' := e_x^{-2}$ und sei $(A', B')\in\mathfrak{P}((G-k)_{e'}^d)$. Wegen $d(x)=\infty$ ist $x\notin B'$, also $s((G-k)_{e'}^d)=s(A', B'; (G-k)_{e'}^d)=s(A', B'; G_e^d)- -1\geqq\sigma-1$, wie man in beiden Fällen $x\in A'$ und $x\notin A'\cup B'$ leicht sieht. Nach Wahl von G existiert also ein zulässiges System von $\sigma-1$ Wegen in $(G-k)_{e'}^d$, woraus wir durch Hinzufügen des geschlossenen Weges x, k, x ein in G_e^d zulässiges System von σ Wegen erhalten, im Widerspruch zu $\lambda(G_e^d)<\sigma$.

(4) *Sei* $k\in[x, y]^G\neq\emptyset$.

(a) $e(x)=\infty\to e(y)=0$.

(b) *Wenn* $e(x)=0$ *und* $d(y)=\infty$ *gilt, dann existiert ein* $(A, B)\in\mathfrak{P}(G_e^d)$, *so daß* k *trennende Kante einer geraden Komponente von* (A, B) *in* G_e^d *ist.*

(c) *Wenn* $d(x)=d(y)=\infty$ *gilt, dann existiert ein* $(A, B)\in\mathfrak{P}(G_e^d)$, *so daß* k *trennende Kante einer geraden Komponente von* (A, B) *in* G_e^d *ist.*

Beweis von (4). Nach (3) ist $x\neq y$. Nehmen wir zunächst $e(x)=0$ und $d(y)=\infty$ an. Dann ist $d(x)>0$ nach (2). Seien $e' := e_x^{+2}$ und $d' := d_x^{-1}$. Da $s(A', B'; G_{e'}^{d'})\geqq \geqq s(A', B'; G_e^d)$ für alle $(A', B')\in\mathfrak{P}(G)$ ist, gilt $s(G_{e'}^{d'})\geqq\sigma$. Es ist $|\bar F(e')|+|\bar F(d')|= =|\bar F(e)|+|\bar F(d)|$, aber $e'(F(e'))+d'(F(d'))>e(F(e))+d(F(d))$ wegen $e'(x)>e(x)$ im Falle $x\in\bar F(d)$ und $e'(x)+d'(x)>e(x)+d(x)$ im Falle $x\in F(d)$. Also ist $(e', d')\notin P$ und somit gilt $\lambda(G_{e'}^{d'})=s(G_{e'}^{d'})$. Nehmen wir $s(G_{e'}^{d'})>\sigma$ an. Dann existiert also ein zulässiges System $\mathfrak{S}$ von $\sigma+1$ Wegen in $G_{e'}^{d'}$. Es existieren zwei Wege $W_1\neq W_2$ in $\mathfrak{S}$, die in x enden, da wir sonst aus $\mathfrak{S}$ durch Fortlassen höchstens eines Weges ein in G_e^d zulässiges System von σ Wegen erhalten könnten. Sei etwa W_i ein x_i, x-Weg für $i=1, 2$. Wegen $K(W_1)\cap K(W_2)=\emptyset$ existiert ein x_1, x_2-Weg W der Länge $\geqq 1$ in $W_1\cup W_2$. Dann ist aber $(\mathfrak{S}-\{W_1, W_2\})\cup\{W\}$ ein in G_e^d zulässiges System von σ Wegen. Dieser Widerspruch zeigt $s(G_{e'}^{d'})=\sigma$. — Sei $(A, B)\in\mathfrak{P}(G_{e'}^{d'})$. Aus $\sigma=s(A, B; G_{e'}^{d'})\geqq s(A, B; G_e^d)\geqq\sigma$ folgt $x\in A$ und $(A, B)\in\mathfrak{P}(G_e^d)$. Wegen $d(y)=\infty$

ist $y \notin B$ und wegen $e(x)=0$ ist $y \notin A$ nach Lemma 6. Somit ist $k \in K(x, E(G)-(A \cup B); G)$ und Behauptung (b) ergibt sich aus Lemma 6.

Seien nun $e(x)>0$ und $e(y)>0$ und sei $e' := e_{x,y}^{-1}$. Es muß $s((G-k)_{e'}^d) \leqq \sigma-2$ sein, da sonst ein in $(G-k)_{e'}^d$ zulässiges System von $\sigma-1$ Wegen existierte, aus dem wir durch Hinzufügen des Weges x, k, y ein in G_e^d zulässiges System von σ Wegen erhielten. Sei $(A, B) \in \mathfrak{P}((G-k)_{e'}^d)$. Aus $s(A, B; G_e^d) \geqq \sigma \geqq s((G-k)_{e'}^d)+2 = s(A, B; (G-k)_{e'}^d)+2$ ersieht man leicht $\{x, y\} \cap A = \emptyset$, insbesondere $\{x, y\} \subseteqq F(e)$. Hieraus ergibt sich zunächst Behauptung (a). Nehmen wir nun noch zusätzlich $d(x)=d(y)=\infty$ an. Dann gilt auch $\{x, y\} \cap B = \emptyset$. Sei $C_z := C(z; (G-k)-(A \cup B))$ für $z=x, y$. Wiederum wegen $s(A, B; G_e^d) \geqq \sigma \geqq s(A, B; (G-k)_{e'}^d)+2$ sind C_x und C_y verschiedene ungerade Komponenten von (A, B) in $(G-k)_{e'}^d$ und weiterhin $(A, B) \in \mathfrak{P}(G_e^d)$. Dann ist aber $C_0 := G(E(C_x) \cup E(C_y)) = C_x \cup C_y \cup k$ eine gerade Komponente von (A, B) in G_e^d, die k als trennende Kante enthält. Hieraus ergibt sich Behauptung (c), da wir $e(x)>0$ und $e(y)>0$ wegen (b) voraussetzen können.

(5) *Sei $(A, B) \in \mathfrak{P}(G_e^d)$ und sei C eine Komponente von (A, B) mit* $1<|C|<|G|$. *Dann ist C ungerade.*

Beweis von (5). Nehmen wir an, es existiere ein $(A, B) \in \mathfrak{P}(G_e^d)$, zu dem es eine gerade Komponente C in G_e^d gibt mit $1<|C|<|G|$. Entstehe $\bar{G}_{\bar{e}}^{\bar{d}}$ aus G_e^d durch Identifizieren von $E(C)$ zu $c \in E(C)$. Wegen $s(A, B; \bar{G}_{\bar{e}}^{\bar{d}})=\sigma$ ist $s(\bar{G}_{\bar{e}}^{\bar{d}})=\sigma$ nach Lemma 7 und wegen $|\bar{G}|<|G|$ also $\lambda(\bar{G}_{\bar{e}}^{\bar{d}})=\sigma$. Somit existiert ein zulässiges System $\bar{\mathfrak{S}}$ von σ Wegen in $\bar{G}_{\bar{e}}^{\bar{d}}$. Da c eine gerade Komponente von (A, B) in $\bar{G}_{\bar{e}}^{\bar{d}}$ bildet, existieren nach Lemma 1b genau $m := \dfrac{\bar{e}(c)+\varkappa(c, A; \bar{G})}{2} = \dfrac{e(C)+\varkappa(C, A; G)}{2}$ Wege $W \in \bar{\mathfrak{S}}$ mit $c \in W$, etwa die Wege $W_1, \dots, W_m$. Weiterhin besteht nach Lemma 1b jeder Weg W_i aus ein oder zwei Kanten aus $K(c, A; \bar{G})$ oder aus einer Schlinge bei c, und es gilt $K(c, A; \bar{G}) \subseteqq \bigcup_{i=1}^{m} K(W_i)$.

Andererseits betrachten wir einen bewerteten Pseudographen $\bar{C}_{e'}^{d'}$. Im Falle $A=\emptyset$ sei $\bar{C}_{e'}^{d'} := C_e^d$. Im Falle $A \neq \emptyset$ entstehe $\bar{\bar{C}}_{\bar{e}}^{d'}$ aus $G(E(C) \cup A)_e^d$ durch Identifizieren von A zu $a \in A$ und es seien $\bar{C} := \bar{\bar{C}} - K(a, a; \bar{\bar{C}})$ und $e' := \tilde{e}_a^{\infty}$, wobei $K(a, a; \bar{\bar{C}})$ die Menge der Schlingen durch a in $\bar{\bar{C}}$ bezeichne. Um Fallunterscheidungen zu vermeiden, setzen wir im ersteren Fall $A_0 := \emptyset$ und im letzteren $A_0 := \{a\}$. Überlegen wir uns $s(\bar{C}_{e'}^{d'})=m$. Sei $(A', B') \in \mathfrak{P}(\bar{C}_{e'}^{d'})$. Wegen $e'(a)=\infty$ ist $A_0 \subseteqq A'$. Für $\bar{A} := A \cup A'$ und $\bar{B} := B \cup B'$ gilt

$$\sigma \leqq s(\bar{A},\bar{B}; G_e^d) = s(A, B; G_e^d) - m + \|G(A'-A_0)\| + \varkappa(A'-A_0, A; G) + e(B') + d(B') +$$
$$+ \sum_{C' \in \mathfrak{L}(C-(A' \cup B'))} \left[\frac{e(C')+\varkappa(C', A \cup A'; G)}{2}\right] = \sigma - m + s(A', B'; \bar{C}_{e'}^{d'}).$$

Hieraus folgt $s(\bar{C}_{e'}^{d'}) = s(A', B'; \bar{C}_{e'}^{d'}) \geqq m$. Wegen $s(A_0, \emptyset; \bar{C}_{e'}^{d'})=m$ ergibt sich also $s(\bar{C}_{e'}^{d'})=m$ und $(A_0, \emptyset) \in \mathfrak{P}(\bar{C}_{e'}^{d'})$. Betrachten wir zunächst den Fall $|C| \leqq |G|-2$ oder $B \neq \emptyset$. Dann ist $|\bar{C}|<|G|$, also $\lambda(\bar{C}_{e'}^{d'})=m$. Somit existiert ein zulässiges System $\mathfrak{S}''$ von m Wegen in $\bar{C}_{e'}^{d'}$. Da wegen $s(A_0, \emptyset; \bar{C}_{e'}^{d'})=m$ nach Lemma 1b kein Weg aus $\mathfrak{S}''$ durch eine Ecke von A_0 läuft, können wir ersichtlich $\mathfrak{S}''$ auch als ein System $\mathfrak{S}'$ von m kantendisjunkten Wegen in $G(E(C) \cup A) - K(G(A))$ auffassen. Das System $\mathfrak{S} := (\bar{\mathfrak{S}} - \{W_1, \dots, W_m\}) \cup \mathfrak{S}'$ besteht aus σ kantendisjunkten

Wegen von G. Wir wollen uns noch die Zulässigkeit von $\mathfrak{S}$ in G_e^d überlegen. Da $\bar{\mathfrak{S}}$ in $G_{\bar{e}}^{\bar{d}}$ und $\mathfrak{S}''$ in $C_{e'}^{d'}$ zulässig sind und da kein Weg aus $\mathfrak{S}'$ durch eine Ecke von A läuft, genügt es, nachzuweisen, daß jedes $x \in A$ höchstens $e(x)$-mal als Endpunkt von Wegen aus $\mathfrak{S}$ vorkommt. Dies ist aber klar, da $\bar{\mathfrak{S}}$ zulässig in $G_{\bar{e}}^{\bar{d}}$ ist und da $x \in A$ nach Lemma 1b genau $\varkappa(x, C; G)$-mal als Endpunkt bei den Wegen $W_1, \dots, W_m$ vorkommt, während es natürlich höchstens $\varkappa(x, C; G)$-mal als Endpunkt bei den Wegen von $\mathfrak{S}'$ auftritt. Somit erhalten wir den Widerspruch $\lambda(G_e^d) \geqq \sigma$.

Es bleibt noch der Fall $|C|=|G|-1$ und $B=\emptyset$ zu betrachten. Wegen (2) und (3) ist dann $|A|=1$, etwa $A=\{a\}$, und es existiert ein $x \in C$ mit $[a, x]^G \neq \emptyset$. Nach (1) gilt $d(a)=\infty$. Da C eine gerade Komponente von $(\{a\}, \emptyset)=(A, B) \in \mathfrak{P}(G_e^d)$ ist, folgt $e(a) \geqq \varkappa(a, C; G)=\varkappa(a; G)$ wegen (3) aus $s(\emptyset, \emptyset; G_e^d) \geqq s(\{a\}, \emptyset; G_e^d)$. Nach Lemma 2 (b) und (a) gilt dann $\lambda(G_{e_a^\infty}^d)=\lambda(G_e^d)<s(G_e^d) \leqq s(G_{e_a^\infty}^d)$, also $(e_a^\infty, d) \in P$, wegen $(e, d) \in \bar{P}$ also $|\bar{F}(e_a^\infty)| \leqq |\bar{F}(e)|$ und somit $e(a)=\infty$. Aus 4 (a) folgt dann weiter $e(x)=0$, und damit existiert nach 4 (b) ein $(\tilde{A}, \tilde{B}) \in \mathfrak{P}(G_e^d)$ mit $\{a, x\} \cap (\tilde{A} \cup \tilde{B})=\emptyset$, im Widerspruch zu $e(a)=\infty$.

(6) *Zu jedem $k \in K(G)$ existiert ein $(A, B) \in \bar{\mathfrak{P}}(G_e^d)$, so daß k trennende Kante einer geraden Komponente von (A, B) in G_e^d ist.*

Beweis von (6). Sei $k \in [x, y]^G$ gegeben. Wegen der Minimaleigenschaft von G ist $s((G-k)_e^d)=\lambda((G-k)_e^d) \leqq \lambda(G_e^d)<\sigma$. Sei $(A, B) \in \mathfrak{P}((G-k)_e^d)$. Wegen $s(A, B; G_e^d) \geqq \sigma \geqq s(A, B; (G-k)_e^d)+1$ muß einer der drei folgenden Fälle vorliegen: (α) *Es ist $k \in K(G(A))$*; (β) *Es existiert eine ungerade Komponente C von (A, B) in $(G-k)_e^d$ mit $k \in K(C, A; G)$*; (γ) *Es existieren zwei ungerade Komponenten $C \neq C'$ von (A, B) in $(G-k)_e^d$ mit $k \in K(C, C'; G)$*. In allen drei Fällen ergibt sich $s(A, B; G_e^d)=s(A, B; (G-k)_e^d)+1$, also $(A, B) \in \bar{\mathfrak{P}}(G_e^d)$. Im Fall γ sind wir somit fertig, da $G(E(C) \cup E(C'))$ eine gerade Komponente von (A, B) in G_e^d ist. Im Falle α gilt $d(x)=d(y)=\infty$ nach (1), womit sich die Behauptung aus 4 (c) ergibt. Wir haben noch den Fall β zu betrachten. Dann ist C eine gerade Komponente von (A, B) in G_e^d und wegen $\{x, y\} \cap A \neq \emptyset$ nach (5) somit $|C|=1$, etwa $E(C)=\{y\}$. Wäre $y \in F(d)$, dann wäre $(e, d_y^\infty) \notin P$ nach Wahl von (e, d), und wir erhielten nach Lemma 4 wegen (3) den Widerspruch $\lambda(G_e^d)=s(G_e^d)$. Also ist $d(y)=\infty$. Da nach (1) auch $d(x)=\infty$ gilt, liefert wiederum 4 (c) die Behauptung.

Nun können wir den Beweis von Satz 1 leicht zu Ende führen. Wegen $s(H_{e'}^{d'})=$ $=0=\lambda(H_{e'}^{d'})$ für den Graphen $H=(\emptyset, \emptyset)$ ist $K(G) \neq \emptyset$ nach (2). Sei $k \in K(G)$. Dann existieren nach (6) ein $(A, B) \in \bar{\mathfrak{P}}(G_e^d)$ und eine gerade Komponente C von (A, B) in G_e^d, so daß k trennende Kante von C ist. Wegen $|C| \geqq 2$ folgt $C=G$ und somit $A=B=\emptyset$ aus (5). Also ist G zusammenhängend, $\sigma=s(\emptyset, \emptyset; G_e^d)=$ $=\frac{e(G)}{2}$ und jede Kante $k \in K(G) \neq \emptyset$ trennt G. Somit ist G ein Baum mit $|G| \geqq 2$ und es existiert ein $x \in E(G)$ mit $\varkappa(x; G)=1$. Nehmen wir $e(x)=0$ an und sei $(A', B') \in \mathfrak{P}((G-x)_e^d)$. Dann erhielten wir aber den Widerspruch $\lambda(G_e^d) \geqq \lambda((G-x)_e^d)=$ $=s((G-x)_e^d)=s(A', B'; (G-x)_e^d)=s(A', B'; G_e^d) \geqq \sigma$. Also ist $e(x)>0$, somit $e(x) \geqq$ $\geqq \varkappa(x; G)=1$. Für $\bar{e}:=e_x^\infty$ ergibt sich nach Lemma 2 (b) und (a) dann $\lambda(G_{\bar{e}}^d)=$ $=\lambda(G_e^d)<s(G_e^d) \leqq s(G_{\bar{e}}^d)$, also $(\bar{e}, d) \in P$ und somit $e(x)=\infty$ nach Wahl von (e, d). Dies steht aber im Widerspruch zu $\frac{e(G)}{2}=\sigma<\infty$.

Wenden wir uns noch einem Spezialfall des Satzes von T. Gallai zu. Sei G ein endlicher Graph und sei $H \subseteqq E(G)$. Die Funktionen e, d seien definiert durch $e(x)=1$ und $d(x)=0$ für $x \in H$ und $e(x)=0$ und $d(x)=1$ für $x \in E(G)-H$. Dann besteht ein in G_e^d zulässiges Wegesystem aus disjunkten, offenen Wegen, deren beide (verschiedenen) Endpunkte zu H gehören und die sonst keine Ecke mit H gemeinsam haben, und umgekehrt ist jedes solche System von Wegen in G_e^d zulässig. Einen offenen x, y-Weg W der Länge $\geqq 1$ mit $\{x, y\} \subseteqq H$ und mit $(W-\{x, y\}) \cap H = \emptyset$ nennen wir einen *H-Weg* und die Maximalzahl (ecken-) disjunkter H-Wege in G bezeichnen wir mit $\mu^*(H;\, G)$. Mit obigen (speziellen) Funktionen e und d gilt dann also $\mu^*(H;\, G)=\lambda(G_e^d)$. In diesem Spezialfall läßt sich Satz 1 auf eine einfachere Gestalt bringen. Hierzu definieren wir $s_H(B;\, G) := |B| + \sum_{C \in \mathfrak{L}(G-B)} \left[\frac{|C \cap H|}{2}\right]$ für $B \subseteqq E(G)$ und $s(H;\, G) := \min_{B \subseteqq E(G)} s_H(B;\, G)$; weiterhin sei $\mathfrak{P}(H;\, G) := \{B \subseteqq E(G) | s_H(B;\, G) = s(H;\, G)\}$. Mit diesen Bezeichnungen lautet das Resultat von T. Gallai:

SATZ 2. ([4], Satz 12.3): *Für jeden endlichen Graphen G und jedes $H \subseteqq E(G)$ gilt $\mu^*(H;\, G)=s(H;\, G)$.*

Im Spezialfall $H=E(G)$ erhält man aus Satz 2, wie T. GALLAI in Satz (14.4) aus [4] zeigte, unmittelbar das Existenzkriterium für 1-Faktoren von W. T. TUTTE [9]. (Umgekehrt folgt auch Satz 2 aus der Formel von C. BERGE [2] für die Maximalzahl unabhängiger Kanten, worauf mich L. Lovász hinwies. Denn der Defekt des Graphen $\bar{G}$, welcher aus G durch „Aufspalten" jeder Ecke $x \in E(G)-H$ in zwei benachbarte Ecken x' und x'' entsteht, ist gleich $|H|-2\mu^*(H;\, G)$. Die genaue Ausführung dieses Gedankens ist aber wohl nicht wesentlich einfacher als der folgende direkte Beweis.) Da Satz 2 auch bei weiterführenden Untersuchungen eine Rolle spielt (siehe etwa [3]), erscheint es wünschenswert, einen einfachen, direkten Beweis zu finden. Wir wollen im folgenden einen solchen Beweis (als Spezialisierung des Beweises von Satz 1) ausführen und einen weiteren skizzieren.

BEWEIS VON SATZ 2. Sei $B \subseteqq E(G)$. Da ein H-Weg W mit $W \cap B = \emptyset$ ganz in einer Komponente von $G-B$ verläuft, gilt ersichtlich $\mu^*(H;\, G) \leqq s_H(B;\, G)$, also auch $\mu^*(H;\, G) \leqq s(H;\, G) =: \sigma$. Wir brauchen somit nur $\mu^*(H;\, G) \geqq \sigma$ nachzuweisen. Dies ist offensichtlich richtig für $\|G\|=0$. Sei also $\|G\|>0$. Wir nehmen an, daß $\mu^*(H';\, G') \geqq s(H';\, G')$, also $\mu^*(H';\, G')=s(H';\, G')$, schon für alle Graphen G' mit $|G'|<|G|$ oder $|G'|=|G|$ und $\|G'\|<\|G\|$ und für alle $H' \subseteqq E(G')$ bewiesen ist. Betrachten wir zunächst den folgenden Fall.

(Z) *Es existieren ein $B_0 \in \mathfrak{P}(H;\, G)$ und ein $C_0 \in \mathfrak{L}(G-B_0)$ mit $|C_0|<|G|$ und mit $|C_0 \cap H|$ gerade. Dann ist $\mu^*(H;\, G) \geqq \sigma$.*

Beweis von (Z): Sei $H_0 := C_0 \cap H$ und sei etwa $|H_0|=2k$. Für alle $B \subseteqq E(C_0)$ gilt

$$\sigma \leqq s_H(B_0 \cup B;\, G) = s_H(B_0;\, G) - k + s_{H_0}(B;\, C_0), \quad \text{also} \quad s(H_0;\, C_0) \geqq k.$$

Wegen $|C_0|<|G|$ existiert somit nach Induktionsannahme ein System $\mathfrak{S}_1$ von k disjunkten H_0-Wegen in C_0.

Sei $G_0 := \big(E(G), K(G) \cup \{[x, y] \mid x \neq y \wedge \{x, y\} \subseteq B_0\}\big)$ [2] und sei $G' := G_0 - E(C_0)$. Nehmen wir $s(H'; G') < \sigma - k$ für $H' := H - H_0$ an. Sei $B' \in \mathfrak{P}(H'; G')$. Da $G'(B_0)$ ein vollständiger Graph ist, gilt $B_0 \cap C \neq \emptyset$ für höchstens eine Komponente C von $G' - B'$. Also ist auch $N(C_0; G) \cap C \neq \emptyset$ für höchstens ein $C \in \mathfrak{L}(G' - B')$. Falls ein $C' \in \mathfrak{L}(G' - B')$ mit $N(C_0; G) \cap C' \neq \emptyset$ existiert, ist $G_0\big(E(C_0) \cup E(C')\big)$ eine Komponente von $G_0 - B'$, während $C_0 \in \mathfrak{L}(G_0 - B')$ gilt, wenn $N(C_0; G) \cap C = \emptyset$ für alle $C \in \mathfrak{L}(G' - B')$ ist. Somit ergibt sich $s_H(B'; G_0) = s_{H'}(B'; G') + k = s(H'; G') + k < \sigma$. Hieraus folgt $s_H(B'; G) < \sigma$, da für Graphen $F' \subseteq F$ und $\bar{H} \cup \bar{B} \subseteq E(F')$ gilt $s_{\bar{H}}(\bar{B}; F') \leqq s_{\bar{H}}(\bar{B}; F)$. Dieser Widerspruch zeigt $s(H'; G') = \sigma - k$ und $B_0 \in \mathfrak{P}(H'; G')$. Wegen $|G'| < |G|$ existiert nach Induktionsannahme ein System $\mathfrak{S}_2$ von $\sigma - k$ disjunkten H'-Wegen in G'. Wegen $s_{H'}(B_0; G') = \sigma - k$ gilt $|W \cap B_0| \leqq 1$ für alle $W \in \mathfrak{S}_2$, also sogar $W \subseteq G - E(C_0)$ für alle $W \in \mathfrak{S}_2$. Da somit $\mathfrak{S}_1 \cup \mathfrak{S}_2$ ein System von σ disjunkten H-Wegen in G ist, folgt $\mu^*(H; G) \geqq \sigma$.

Sei $[x, y] \in K(G)$. Wir können $s(H; G - [x, y]) < \sigma$ annehmen. Sei $B_0 \in \mathfrak{P}(H; G - [x, y])$. Wegen $s_H(B_0; G - [x, y]) < \sigma$, aber $s_H(B_0; G) \geqq \sigma$ ist $[x, y]$ eine trennende Kante einer Komponente C_0 von $G - B_0$ und für die beiden Komponenten $C_x := C(x; C_0 - [x, y])$ und $C_y := C(y; C_0 - [x, y])$ gilt $|C_x \cap H| \equiv \equiv |C_y \cap H| \equiv 1 \pmod 2$. Dann ist $B_0 \in \mathfrak{P}(H; G)$ und wegen $|C_0 \cap H| = |C_x \cap H| + |C_y \cap H| \equiv 0 \pmod 2$ können wir $C_0 = G$ nach (Z) annehmen. Insbesondere ist dann G zusammenhängend und $\sigma = \frac{1}{2}|H|$.

Wir können annehmen, daß jedes $[x, y] \in K(G) \neq \emptyset$ eine trennende Kante von G ist und für $C_x := C(x; G - [x, y])$ und $C_y := C(y; G - [x, y])$ gilt $|C_x \cap H| \equiv \equiv |C_y \cap H| \equiv 1 \pmod 2$. Dann ist G ein Baum. Sei $z \in E(G)$ mit $m := \varkappa(z; G) \geqq 2$. Dann besteht $G - z$ aus m Komponenten $C_1, \dots, C_m$, und nach obigem ist $|C_i \cap H|$ ungerade für alle $i = 1, \dots, m$. Somit folgt $\frac{1}{2}|H| = \sigma \leqq s_H(\{z\}; G) = 1 + \sum_{i=1}^{m} \frac{|C_i \cap H| - 1}{2} \leqq 1 + \frac{1}{2}|H| - \frac{m}{2}$, also $m \leqq 2$. Also ist G ein Weg (mit $|H| = 2$) und somit ersichtlich $\mu^*(H; G) = \frac{1}{2}|H| = \sigma$.

Zum Schluß möchte ich noch bemerken, daß man den Beweis von Satz 2 auch analog zu dem in [1] und [6] gegebenen Induktionsbeweis des Satzes von Tutte führen kann. Man betrachte hierzu ein $B_0 \in \mathfrak{P}(H; G)$, das bezüglich der Inklusion maximal in $\mathfrak{P}(H; G)$ ist. Seien $C_0 \in \mathfrak{L}(G - B_0)$ und $H_0 := H \cap C_0$. Aus der Maximaleigenschaft von B_0 ergibt sich leicht $H_0 = \emptyset$ oder $|H_0|$ ungerade. Nehmen wir $|H_0| \equiv 1 \pmod 2$ an, etwa $|H_0| = 2k + 1$. Wiederum nach Wahl von B_0 ergibt sich ohne Schwierigkeit in üblicher Weise (vgl. etwa (4) im Beweis des Hauptsatzes von [7]):

(1) *Für jedes* $x \in H_0$ *ist* $s(H_0 - \{x\}; C_0 - x) = k$ *und für jedes* $x \in E(C_0) - H_0$ *gilt* $s(H_0 \cup \{x\}; C_0) = k + 1$.

Wir definieren $f\colon B_0 \to \{1, 2\}$ durch $f(b) := 1$ für $b \in B_0 \cap H$ und $f(b) := 2$ für $b \in B_0 - H$. Weiterhin sei $\mathfrak{L}_0 := \{C \in \mathfrak{L}(G - B_0) \mid C \cap H \neq \emptyset\}$ und für $B \subseteq B_0$ sei $\mathfrak{N}(B) := \{C \in \mathfrak{L}_0 \mid C \cap N(B; G) \neq \emptyset\}$. Überlegen wir uns

[2] In Graphen G (d. h. $[x, x]^G = \emptyset$ und $|[x, y]^G| \leqq 1$ für alle x, y aus $E(G)$) betrachten wir $[x, y]^G \neq \emptyset$ als die Kante $[x, y]$ zwischen x und y in G.

(2) *Für alle* $B \subseteqq B_0$ *gilt* $|\mathfrak{N}(B)| \geqq f(B)$.

Nehmen wir die Existenz eines $B \subseteqq B_0$ mit $n := |\mathfrak{N}(B)| < f(B)$ an. Sei etwa $\mathfrak{N}(B) = \{C_1, \ldots, C_n\}$ und seien $C_{n+1}, \ldots, C_m$ die $C \in \mathfrak{L}(G - B_0) - \mathfrak{L}_0$ mit $C \cap N(B;\ G) \neq \emptyset$; ferner sei $G' := G\big(B \cup \bigcup_{i=1}^{m} E(C_i)\big)$. Dann gilt

$$\text{(A)}\quad s_H(B_0 - B;\ G) = s_H(B_0;\ G) - |B| - \sum_{i=1}^{n} \left[\frac{|C_i \cap H|}{2}\right] + \sum_{C \in \mathfrak{L}(G')} \left[\frac{|C \cap H|}{2}\right].$$

Aus $n < f(B) = |B \cap H| + 2|B - H|$ folgt

$$|G' \cap H| + B| \cap < 2 \sum_{i=1}^{n} \left(2\left[\frac{|C_i \cap H|}{2}\right] + 1\right) \sum_{i=1}^{n} \left[\frac{|C_i \cap H|}{2}\right] + 2|B|,$$

also

$$\sum_{C \in \mathfrak{L}(G')} \left[\frac{|C \cap H|}{2}\right] \leqq \frac{|G' \cap H|}{2} < \sum_{i=1}^{n} \left[\frac{|C_i \cap H|}{2}\right] + |B|.$$

Somit ergibt sich aus (A) der Widerspruch $s_H(B_0 - B;\ G) < s_H(B_0;\ G)$.

Nach einer einfachen Verallgemeinerung des Kriteriums von P. Hall (vgl. etwa Satz 3.3.1 in [8]) existiert wegen (2) ein System von Kanten $[b, x_1^b] \in K(G)$ für $b \in B_0 \cap H$ und $[b, x_1^b] \in K(G)$, $[b, x_2^b] \in K(G)$ für $b \in B_0 - H$ mit $x_i^b \in \bigcup_{C \in \mathfrak{L}_0} E(C)$ und mit $C(x_i^b;\ G - B_0) \neq C(x_{i'}^{b'};\ G - B_0)$ für alle in Frage kommenden $(b, i) \neq (b', i')$. Mit Hilfe von (1) kann man dieses Kantensystem zu einem System $\mathfrak{S}$ von $|B_0|$ disjunkten H-Wegen fortsetzen, und zwar so, daß zu jedem $C \in \mathfrak{L}_0$ ein System $\mathfrak{S}_C$ von $\frac{|C \cap H| - 1}{2}$ disjunkten H-Wegen in C existiert, die auch zu den Wegen aus $\mathfrak{S}$ paarweise disjunkt sind. Dann besteht $\mathfrak{S} \cup \bigcup_{C \in \mathfrak{L}_0} \mathfrak{S}_C$ aus $|B_0| + \sum_{C \in \mathfrak{L}_0} \frac{|C \cap H| - 1}{2} = s_H(B_0; G)$ disjunkten H-Wegen in G.

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FUNCTIONS WITH MEASURABLE DIFFERENCES

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1. Let $\mathbf{R}$ denote the set of real numbers and let F be a class of real valued functions defined on $\mathbf{R}$. We say that F has the *difference property*, provided that every function $f\colon \mathbf{R}\to\mathbf{R}$ for which $f(x+h)-f(x)\in F$ holds for every h, can be written in the form $f=g+H$ where $g\in F$ and H is additive, that is H satisfies the functional equation $H(x+y)=H(x)+H(y)$. The notion of difference property was introduced by N. G. de Bruijn who proved that a series of important classes have the difference property (e.g. the classes of continuous, differentiable, analytic, absolute continuous, Riemann-integrable functions, respectively; see [1] and [2]). The results of de Bruijn have been extended and generalized in various ways, see [3], [4], [5], [6], [11], [13].

However, the following example given by Erdős shows that the class of Lebesgue measurable functions does not have the difference property if we assume the continuum hypothesis. Indeed, the continuum hypothesis implies the existence of a bounded and non-measurable function $S\colon \mathbf{R}\to\mathbf{R}$ such that for every $h\in\mathbf{R}$, $S(x+h)-S(x)=0$ holds for all but countably many values of x (see [16], p. 27). Now S is not of the form $g+H$ where g is measurable and H is additive because otherwise $H=S-g$ would be bounded on a set of positive measure. By a theorem of Ostrowski, this implies that H is linear (see [15] or [12]) and thus S is measurable, a contradiction.

It was conjectured by Erdős that every function $f\colon \mathbf{R}\to\mathbf{R}$ for which $f(x+h)-f(x)$ is measurable for every h, is of the form $f=g+H+S$, where g is measurable, H is additive and S has the property that, for every h, $S(x+h)-S(x)=0$ for almost every x.

We say that a class F has the *weak difference property* if every function $f\colon \mathbf{R}\to\mathbf{R}$ for which $f(x+h)-f(x)\in F$ holds for every h admits a decomposition $f=g+H+S$ with $g\in F$, H additive, and S satisfying the condition that for every h, $S(x+h)-S(x)=0$ holds for a.e. x (see [6]). Let L denote the class of Lebesgue measurable functions defined on $\mathbf{R}$. Then Erdős' conjecture can be formulated as follows: the class L has the weak difference property. The main purpose of this paper is to prove this conjecture (Theorem 3).

We remark that the weak difference property has been established for the classes $L_p(0,1)$ if $p\geqq 1$ (see [4] and for a generalization, [13]). F. W. Carroll raised the question whether this is true for $0<p<1$. We give an affirmative answer in Theorem 4.

We prove Theorems 3 and 4 in Section 2, making use of the preparatory results of Lemmas 1, 2 and Theorem 2. In Section 3 we give some applications of Theorem 3. These applications will be based on Theorem 5 which states that the classes of measurable functions of one and two variables have a "double difference property" in the following sense.

Let F_1 be a class of real functions defined on $\mathbf{R}$ and let F_2 be a class of real functions defined on $\mathbf{R}^2$. We say that the pair (F_1, F_2) has the *double difference property* if whenever $f(x+y)-f(x)-f(y)\in F_2$ holds for a function $f\colon \mathbf{R}\to\mathbf{R}$ then f is of the form $f=g+H$, where $g\in F_1$ and H is additive.

For example, the pair of classes of bounded functions of one and two variables, respectively has the double difference property (see [1], Theorem 1.2, p. 196). First we prove that the same is true for the classes of functions whose limit equals zero at the origin. More precisely we prove

THEOREM 1. *If f is defined on* $\mathbf{R}$ *and*

$$\lim_{\substack{x\to 0\\ y\to 0}} \{f(x+y)-f(x)-f(y)\} = 0$$

holds then $f=g+H$ *where* H *is additive and* $\lim\limits_{x\to 0} g(x)=g(0)=0$.

PROOF. Let $\delta>0$ be such that

$$(1)\qquad |f(x+y)-f(x)-f(y)| \leqq \max(1, |f(0)|) \overset{\text{def}}{=} A \quad \text{for every} \quad |x|, |y| \leqq \delta.$$

We put $f^*(x)\overset{\text{def}}{=} f(x-k\delta)$ if $k\delta\leqq x<(k+1)\delta$ $(k=0, \pm 1, \dots)$. We show that $F(x, y)\overset{\text{def}}{=} |f^*(x+y)-f^*(x)-f^*(y)|$ is bounded on $\mathbf{R}^2$. Let $x, y\in\mathbf{R}$ be arbitrary and let $k=\left[\frac{x}{\delta}\right]$, $n=\left[\frac{y}{\delta}\right]$. If $\left[\frac{x+y}{\delta}\right]=k+n$ then we have

$$F(x, y) = \big|f(x+y-(k+n)\delta)-f(x-k\delta)-f(y-n\delta)\big| \leqq A$$

by (1). If $\left[\frac{x+y}{\delta}\right]=k+n+1$ then we have

$$F(x, y) = \big|f\big(x+y-(k+n+1)\delta\big)-f(x-k\delta)-f(y-n\delta)\big| \leqq$$
$$\leqq \big|f\big(x-k\delta+y-(n+1)\delta\big)-f(x-k\delta)-f\big(y-(n+1)\delta\big)\big|+$$
$$+\big|f\big(y-(n+1)\delta\big)-f(y-n\delta)-f(-\delta)\big|+|f(-\delta)| \leqq 2A+|f(-\delta)|$$

using (1) again. Hence $F(x, y)\leqq 2A+|f(-\delta)|$ for every $x, y\in R$. Thus, by the above mentioned theorem ([1], Theorem 1.2, p. 196), there exists an additive function H such that f^*-H is bounded. Hence the function $g(x)\overset{\text{def}}{=} f(x)-H(x)$ is bounded in $[0, \delta)$. If $x\in(-\delta, 0)$ then we have

$$|g(x)| = |f(x)-H(x)| \leqq |f(x)+f(-x)-f(0)|+|f(0)|+|H(-x)-f(-x)| \leqq$$
$$\leqq A+|f(0)|+|g(-x)|$$

and hence g is bounded in $(-\delta, \delta)$.

We show $\lim_{x\to+0} g(x)=0$ (the proof of $\lim_{x\to-0} g(x)=0$ is similar). We put

$$M_n = \sup\left\{g(x);\ \frac{\delta}{2^n} \leqq x < \frac{\delta}{2^{n-1}}\right\} \quad (n = 1, 2, \dots).$$

Let $\varepsilon>0$ be arbitrary and let N be such that

$$|g(x+y)-g(x)-g(y)| = |f(x+y)-f(x)-f(y)| < \varepsilon$$

holds for every $0<|x|, |y|<\frac{\delta}{2^N}$. Then we have

$$M_{n+1} \leqq \frac{1}{2} M_n + \frac{1}{2}\varepsilon \quad (n \geqq N). \tag{2}$$

Indeed, for every $x\in\left[\frac{\delta}{2^{n+1}}, \frac{\delta}{2^n}\right)$, $n\geqq N$, we have $|g(2x)-2g(x)|<\varepsilon$ from which

$$g(x) < \frac{1}{2} g(2x)+\frac{1}{2}\varepsilon \leqq \frac{1}{2} M_n+\frac{1}{2}\varepsilon$$

and hence $M_{n+1} \leqq \frac{1}{2} M_n+\frac{1}{2}\varepsilon$.

If $M_n\leqq\varepsilon$ holds for at least one $n\geqq N$ then by (2) we have $M_{n+1}\leqq\frac{1}{2}\varepsilon+ +\frac{1}{2}\varepsilon=\varepsilon$ and by induction $M_k\leqq\varepsilon$ for every $k\geqq n$. Thus in this case $\limsup_{n\to\infty} M_n\leqq\varepsilon$ holds.

If $M_n>\varepsilon$ holds for every $n\geqq N$ then $M_{n+1}<\frac{1}{2} M_n+\frac{1}{2} M_n=M_n$ for every $n\geqq N$ i.e. the sequence $\{M_n\}_{n=N}^{\infty}$ is decreasing. Let $M\overset{\text{def}}{=}\lim_{n\to\infty} M_n$. Then (2) implies $M\leqq\frac{1}{2} M+\frac{1}{2}\varepsilon$, $M\leqq\varepsilon$ and thus we have $\limsup_{n\to\infty} M_n\leqq\varepsilon$ again.

Since $\varepsilon>0$ was arbitrary we proved $\limsup_{x\to+0} g(x)\leqq \limsup_{n\to\infty} M_n\leqq 0$. A similar argument shows that $\liminf_{x\to+0} g(x)\geqq 0$ and hence $\lim_{x\to+0} g(x)=0$ as we stated.

Finally $\lim_{x\to 0} |g(0)-g(x)-g(-x)|=0$ gives $g(0)=0$, which completes the proof.

2. Let S denote the class of all functions defined on **R** which are Lebesgue measurable and periodic mod 1. For $f\in S$ we denote

$$I\{f \gtreqless c\} = \{x\in[0, 1];\ f(x) \gtreqless c\}$$

and

$$\|f\| = \inf\{a+\lambda(I\{|f| \geqq a\});\ a > 0\}$$

where λ denotes the Lebesgue measure.[1] The following properties of the "pseudo-norm" $\|\cdot\|$ are well-known (and can be easily verified).

(3) $0 \leqq \|f\| \leqq 1 \quad (f \in S)$,

(4) $\|f\| = 0$ iff $f(x) = 0$ for a.e. $x \in \mathbf{R}$,

(5) $\|f+g\| \leqq \|f\| + \|g\| \quad (f, g \in S)$,

(6) $\|f(x+h)\| = \|f(x)\| \quad (f \in S,\ h \in \mathbf{R})$,

(7) If $f \in S$ and $\|f\| < a$ then $\lambda(I\{|f| \geqq a\}) < a$,

(8) $\lim\limits_{h\to 0} \|f(x+h) - f(x)\| = 0 \quad (f \in S)$,

(9) $\lim\limits_{n\to\infty} \|f_n - f\| = 0$ iff the sequence f_n converges to f in measure $(f_n, f \in S)$.

LEMMA 1. *Let* $s(f) \overset{\text{def}}{=} \inf\{\|f(x)-c\|;\ c \in \mathbf{R}\}$. *Then for every* $f \in S$,
a) *there exists* $c_0 \in \mathbf{R}$ *such that* $\|f(x)-c_0\| = s(f)$,
b) *if* $s(f) = d$ *then there exists* $c_1 \in \mathbf{R}$ *such that* $\lambda(I\{f \leqq c_1\}) \geqq \frac{d}{3}$ *and* $\lambda\left(I\left\{f \geqq c_1 + \frac{d}{3}\right\}\right) \geqq \frac{d}{3}$.

PROOF. a) Let $f \in S$ be given. First we show that $c_n \in \mathbf{R}$, $|c_n| \to \infty$ implies $\|f - c_n\| \to 1$. For every $\varepsilon > 0$ and $n = 1, 2, \dots$ there exists $a_n > 0$ such that

$$a_n + \lambda(I\{|f-c_n| \geqq a_n\}) < \|f-c_n\| + \varepsilon. \tag{10}$$

Since $|f(x) - c_n| \to \infty$ for every $x \in \mathbf{R}$ hence $\lambda(I\{|f-c_n| \geqq 1\}) \to 1$, thus there exists $N > 0$ such that $\lambda(I\{|f-c_n| \geqq 1\}) > 1 - \varepsilon$ for $n \geqq N$. If $a_n > 1$ then $\|f - c_n\| > 1 - \varepsilon$ by (10). If $a_n \leqq 1$ and $n \geqq N$ then

$$\|f-c_n\| > \lambda(I\{|f-c_n| \geqq a_n\}) - \varepsilon \geqq \lambda(I\{|f-c_n| \geqq 1\}) - \varepsilon > 1 - 2\varepsilon.$$

That is, $\|f-c_n\| > 1 - 2\varepsilon$ for $n \geqq N$ and hence $\|f - c_n\| \to 1$.

Now let $\|f - c_n\| \to s(f)$. If the sequence $\{c_n\}$ is not bounded, then for a suitable subsequence we have $|c_{n_k}| \to \infty$, consequently $s(f) = \lim\limits_{k\to\infty} \|f - c_{n_k}\| = 1$. Then $1 = s(f) \leqq \|f - 0\| \leqq 1$ and we put $c_0 = 0$.

If $\{c_n\}$ is bounded then it has a convergent subsequence $c_{n_k} \to c_0$. We have $\|f - c_0\| = \lim\limits_{k\to\infty} \|f - c_{n_k}\| = s(f)$ which proves a).

b) We can suppose $d > 0$. Let $C = \left\{c \in \mathbf{R};\ \lambda(I\{f \leqq c\}) \leqq \frac{d}{3}\right\}$. C is non-empty and bounded from above. Indeed, $\bigcap\limits_{n=1}^{\infty} I\{f \leqq -n\} = \emptyset$ and hence $\lambda(I\{f \leqq -n\}) \to 0$; $\bigcup\limits_{n=1}^{\infty} I\{f \leqq n\} = [0, 1]$ from which $\lambda(I\{f \leqq n\}) \to 1 > \frac{d}{3}$. Consequently $-n \in C$ and

[1] This norm was introduced by M. Fréchet. See his book *Les Espaces Abstraits*, Gauthier-Villars (Paris, 1928), p. 92.

$C\subset(-\infty, n)$ if n is large enough. We put $c_1=\sup C$. Then $c_1+\frac{1}{n}\notin C$ from which $\lambda(I\{f\leqq c_1\})=\lim_{n\to\infty}\lambda\left(I\left\{f\leqq c_1+\frac{1}{n}\right\}\right)\geqq\frac{d}{3}$. On the other hand we have

$$d=s(f)\leqq\|f-c_1\|\leqq\frac{d}{3}+\lambda\left(I\left\{|f-c_1|\geqq\frac{d}{3}\right\}\right)=$$

$$=\frac{d}{3}+\lambda\left(I\left\{f\leqq c_1-\frac{d}{3}\right\}\right)+\lambda\left(I\left\{f\geqq c_1+\frac{d}{3}\right\}\right)\leqq\frac{2d}{3}+\lambda\left(I\left\{f\geqq c_1+\frac{d}{3}\right\}\right)$$

since $c_1-\frac{d}{3}\in C$. Hence $\lambda\left(I\left\{f\geqq c_1+\frac{d}{3}\right\}\right)\geqq\frac{d}{3}$ and b) is proved.

LEMMA 2. *Let A and B be measurable subsets of the interval* $[0, a]$ *with* $\lambda(A)\geqq c$, $\lambda(B)\geqq c$ $(c\geqq 0)$. *Then there exists* $|h|\leqq a$ *such that*

$$\lambda((A+h)\cap B)\geqq\frac{c^2}{2a}$$

where $A+h$ denotes the set $\{x+h;\ x\in A\}$.

PROOF. We put $D=\{(x, y);\ -a\leqq x\leqq a,\ y-x\in A\}\cap(\mathbf{R}\times B)$; it is easy to see that D is measurable. For every $y\in B$ we have

$$D^y\overset{\text{def}}{=}\{x;\ (x, y)\in D\}=(-A)+y$$

and hence $\lambda(D^y)=\lambda(A)$. By Fubini's theorem $\lambda_2(D)=\int_B\lambda(D^y)\,dy=\lambda(B)\cdot\lambda(A)$.

On the other hand $D_x\overset{\text{def}}{=}\{y;\ (x, y)\in D\}=(A+x)\cap B$ for every $x\in[-a, a]$ and thus we have

$$\int_{-a}^{a}\lambda((A+x)\cap B)\,dx=\int_{-a}^{a}\lambda(D_x)\,dx=\lambda(B)\lambda(A)\geqq c^2.$$

Hence $\lambda((A+x)\cap B)\geqq\frac{c^2}{2a}$ holds for at least one $|x|\leqq a$, q.e.d.

Our next theorem is a generalization of the simple fact that a function $f\in S$ is constant a.e. (that is $s(f)=0$) if and only if $\|f(x+h)-f(x)\|=0$ for every h.

THEOREM 2. *Let $\{f_n\}$ be an arbitrary sequence of functions belonging to S. Then* $\lim_{n\to\infty}s(f_n)=0$ *if and only if* $\lim_{n\to\infty}\|f_n(x+h)-f_n(x)\|=0$ *holds for every* $h\in\mathbf{R}$.

PROOF.[2] Suppose first $\lim_{n\to\infty}s(f_n)=0$. By Lemma 1 a), there exists a sequence $\{c_n\}$ such that $s(f_n)=\|f_n(x)-c_n\|$. Then, for every $h\in\mathbf{R}$ we have

$$\|f_n(x+h)-f_n(x)\|\leqq\|f_n(x+h)-c_n\|+\|c_n-f_n(x)\|=2s(f_n)\to 0.$$

[2] A simpler proof can be found in A. J. E. M. Janssen, Note on a paper by M. Laczkovich on functions with measurable differences (Erdős' conjecture) (to appear).

Now suppose indirectly that $\lim_{n\to\infty} \|f_n(x+h)-f_n(x)\|=0$ holds for every h but $s(f_n)\not\to 0$. Then, after selecting a suitable subsequence, we may assume that $s(f_n)\geqq\geqq 3d>0$ for $n=1, 2, \ldots$. We prove that for every non-degenerate interval $[a, b]$ and for every $N\geqq 0$ there exist $h\in(a, b)$ and $n>N$ such that

$$\|f_n(x+h)-f_n(x)\| \geqq \frac{d^2}{4}. \tag{11}$$

Let $\frac{p}{q}$ be a rational number such that $a<\frac{p-1}{p}<\frac{p+1}{p}<b$. By our assumption $\lim_{n\to\infty}\left\|f_n\left(x+\frac{1}{q}\right)-f_n(x)\right\|=0$; hence there exists $n>N$ such that $\left\|f_n\left(x+\frac{1}{q}\right)-f_n(x)\right\|<\eta$ where $\eta=\frac{d^2}{8q^2}$. Thus by (7) we have

$$\lambda\left(I\left\{\left|f_n\left(x+\frac{1}{q}\right)-f_n(x)\right| \geqq \eta\right\}\right) < \eta. \tag{12}$$

This easily implies

$$\lambda\left(I\left\{\left|f_n\left(x+\frac{k}{q}\right)-f_n(x)\right| \geqq q\eta\right\}\right) < q\eta \quad \text{for every} \quad k=1, 2, \ldots, q. \tag{13}$$

Indeed, $\left|f_n\left(x+\frac{k}{q}\right)-f_n(x)\right|\geqq q\eta$ implies

$$\left|f_n\left(x+\frac{i+1}{q}\right)-f_n\left(x+\frac{i}{q}\right)\right| \geqq \eta$$

for at least one of the values $i=0, 1, \ldots, k-1$. Thus

$$I\left\{\left|f_n\left(x+\frac{k}{q}\right)-f_n(x)\right| \geqq q\eta\right\} \subset \bigcup_{i=0}^{k-1}\left[I\left\{\left|f_n\left(x+\frac{1}{q}\right)-f_n(x)\right| \geqq \eta\right\} - \frac{i}{q}\right]; \tag{14}$$

and (13) follows from (12) and (14).

Since $s(f_n)\geqq 3d$, by Lemma 1 b), there is a $c\in\mathbf{R}$ such that for the level sets $A=I\{f_n\leqq c\}$, $B=I\{f_n\geqq c+d\}$ we have $\lambda(A)\geqq d$, $\lambda(B)\geqq d$. Then there are indices $1\leqq i\leqq q$, $1\leqq j\leqq q$ such that

$$\lambda\left(A\cap\left[\frac{i-1}{q}, \frac{i}{q}\right]\right) \geqq \frac{d}{q}, \quad \lambda\left(B\cap\left[\frac{j-1}{q}, \frac{j}{q}\right]\right) \geqq \frac{d}{q}.$$

Applying Lemma 2 for $a=\frac{1}{q}$ and for the sets

$$A' = \left(A\cap\left[\frac{i-1}{q}, \frac{i}{q}\right]\right)-\frac{i-1}{q}, \quad B' = \left(B\cap\left[\frac{j-1}{q}, \frac{j}{q}\right]\right)-\frac{j-1}{q}$$

we get $|h_1| \leqq \frac{1}{q}$ such that $\lambda((A'+h_1)\cap B') \geqq \frac{d^2}{2q}$. We put $h=\frac{p}{q}-h_1$. Obviously $h\in(a, b)$; we are going to show (11). Let $1\leqq k\leqq q$ be arbitrary and let

$$D = I\left\{\left|f_n\left(x+\frac{p}{q}-h_1\right)-f_n\left(x-\frac{k-i}{q}-h_1\right)\right| \geqq q\eta\right\}$$

and

$$E = I\left\{\left|f_n(x)-f_n\left(x-\frac{k-j}{q}\right)\right| \geqq q\eta\right\}.$$

It follows from (13) (and from the periodicity of f_n) that

(15) $$\lambda(D) < q\eta, \quad \lambda(E) < q\eta.$$

Let $F=[(A'+h_1)\cap B']+\frac{k-1}{q}$, then

(16) $$F\subset\left[\frac{k-1}{q}, \frac{k}{q}\right] \quad \text{and} \quad \lambda(F)\geqq\frac{d^2}{2q}.$$

If $x\in F\setminus(D\cup E)$ then $x-\frac{k-i}{q}-h_1\in A$ and $x-\frac{k-j}{q}\in B$. Therefore, by the definition of A and B we have $f_n\left(x-\frac{k-i}{q}-h_1\right)\leqq c$, $f_n\left(x-\frac{k-j}{q}\right)\geqq c+d$ from which

$$\left|f_n\left(x-\frac{k-i}{q}-h_1\right)-f_n\left(x-\frac{k-j}{q}\right)\right| \geqq d.$$

On the other hand $x\notin D\cup E$ and hence we get

$$\left|f_n\left(x+\frac{p}{q}-h_1\right)-f_n(x)\right| \geqq d-2q\eta.$$

Consequently

$$\lambda\left(\left\{x\in\left[\frac{k-1}{q}, \frac{k}{q}\right];\ |f_n(x+h)-f_n(x)| \geqq d-2q\eta\right\}\right) \geqq$$

$$\geqq \lambda(F\setminus(D\cup E)) \geqq \frac{d^2}{2q}-2q\eta = \frac{d^2}{4q}$$

by (15) and (16). This inequality holds for every $k=1, 2, \ldots, q$, therefore

$$\lambda(I\{|f_n(x+h)-f_n(x)| \geqq d-2q\eta\}) \geqq \frac{d^2}{4}.$$

Since $d-2q\eta>\frac{d^2}{4}$ this implies

$$\lambda\left(I\left\{|f_n(x+h)-f_n(x)| \geqq \frac{d^2}{4}\right\}\right) \geqq \frac{d^2}{4}$$

and hence we have (11) by (7).

Let $n_0=0$ and $[a_0, b_0]=[0, 1]$. Suppose that $k>0$, and the index n_{k-1} and the non-degenerate interval $[a_{k-1}, b_{k-1}]$ have been defined. Then, applying the foregoing argument with $[a, b]=[a_{k-1}, b_{k-1}]$ and $N=n_{k-1}$, we get an index $n_k>n_{k-1}$ and $h_k\in(a_{k-1}, b_{k-1})$ such that

$$\|f_{n_k}(x+h_k)-f_{n_k}(x)\| \geqq \frac{d^2}{4}.$$

It follows easily from (8) that there exists $\delta>0$ such that

$$[a_k, b_k] \overset{\text{def}}{=\!=} [h_k-\delta, h_k+\delta] \subset [a_{k-1}, b_{k-1}]$$

and

$$\|f_{n_k}(x+h)-f_{n_k}(x)\| > \frac{d^2}{8} \quad \text{holds for every} \quad h\in[a_k, b_k]. \tag{17}$$

Thus by induction we define the sequence $n_1, n_2, \ldots$ and the nested sequence of intervals $[a_k, b_k]$ such that (17) holds for every k. Let $h_0\in \bigcap_{k=1}^{\infty} [a_k, b_k]$. Then by (17) we have

$$\|f_{n_k}(x+h_0)-f_{n_k}(x)\| > \frac{d^2}{8} \quad (k=1, 2, \ldots).$$

This obviously contradicts our assumption $\|f_n(x+h_0)-f_n(x)\| \to 0$ and this contradiction proves Theorem 2.

Now we turn to prove our main result.

THEOREM 3. *The class L has the weak difference property.*

PROOF. Suppose that, for a function $f\colon \mathbf{R}\to\mathbf{R}$, $f(x+h)-f(x)\in L$ holds for every h. We have to prove that f can be written in the form $f=g+H+S$, where $g\in L$, H is additive and, for every h, $S(x+h)-S(x)=0$ holds for a.e. x. We may suppose that f is periodic mod 1. Indeed, let the periodic function f^* be defined by

$$f^*(x)=f(x) \quad (0\leqq x<1) \quad \text{and} \quad f^*(x+1)=f^*(x) \quad (x\in\mathbf{R}).$$

Then $f-f^*$ is measurable since for $n\leqq x<n+1$ we have $f^*(x)-f(x)=f(x-n)-f(x)$. On the other hand

$$f^*(x+h)-f^*(x)=[f^*(x+h)-f(x+h)]+[f(x+h)-f(x)]+[f(x)-f^*(x)]$$

is measurable for every h. Hence, if $f^*=g^*+H+S$ where g^*, H and S have the desired properties then we have $f=g+H+S$ where $g=(f-f^*)+g^*$ is measurable. (This argument is due to DE BRUIJN [1], § 1.)

Now suppose that f is periodic mod 1, then $f(x+h)-f(x)\in S$ for every h. By Lemma 1 a), for every h there exists a constant $c(h)$ such that

$$s\big(f(x+h)-f(x)\big)=\|f(x+h)-f(x)-c(h)\|.$$

We may suppose that $c(0)=0$ and the function $c(x)$ is periodic mod 1. We show that

$$\lim_{\substack{h\to 0\\ k\to 0}} \big(c(h+k)-c(h)-c(k)\big)=0. \tag{18}$$

First we prove that

$$\lim_{h\to 0} s\bigl(f(x+h)-f(x)\bigr) = 0. \tag{19}$$

Indeed, let h_n be an arbitrary sequence tending to zero and let $f_n(x)=f(x+h_n)-$ $-f(x)$. Then for every fixed $k\in R$ we have

$$\|f_n(x+k)-f_n(x)\| = \|f(x+h_n+k)-f(x+k)-f(x+h_n)+f(x)\| = \|F(x+h_n)-F(x)\|$$

where $F(x)=f(x+k)-f(x)\in S$. It follows from (8) that $\|F(x+h_n)-F(x)\|\to 0$, therefore $\|f_n(x+k)-f_n(x)\|\to 0$ for every $k\in\mathbf{R}$. Applying Theorem 2 we have $s(f_n)=s\bigl(f(x+h_n)-f(x)\bigr)\to 0$ which gives (19).

Now let $h_n\to 0$ and $k_n\to 0$ be arbitrary, then we have

$$s\bigl(f(x+h_n+k_n)-f(x)\bigr) = \|f(x+h_n+k_n)-f(x)-c(h_n+k_n)\| \to 0,$$

$$s\bigl(f(x+h_n)-f(x)\bigr) = \|f(x+h_n)-f(x)-c(h_n)\| = \|f(x+h_n+k_n)-f(x+k_n)-c(h_n)\| \to 0$$

and

$$s\bigl(f(x+k_n)-f(x)\bigr) = \|f(x+k_n)-f(x)-c(k_n)\| \to 0.$$

Hence

$$\|c(h_n+k_n)-c(h_n)-c(k_n)\| \leqq \|f(x+h_n+k_n)-f(x)-c(h_n+k_n)\| +$$

$$+\|c(h_n)-f(x+h_n+k_n)+f(x+k_n)\|+\|c(k_n)+f(x)-f(x+k_n)\| \to 0.$$

Since $\|c\|=\min(1,|c|)$ holds for every constant function c, therefore

$$|c(h_n+k_n)-c(h_n)-c(k_n)| \to 0$$

which proves (18).

Now we can apply Theorem 1 for the function $c(x)$ and get the functions $H(x)$ and $u(x)$ such that $c(x)=H(x)+u(x)$, H is additive and

$$\lim_{x\to 0} u(x) = u(0) = 0. \tag{20}$$

We may suppose $H(1)=0$ since otherwise we put

$$H_1(x) = H(x)-x\cdot H(1), \quad u_1(x) = u(x)+x\cdot H(1).$$

Then $H(x)$ (and thus $u(x)$ as well) is periodic mod 1. We put

$$K(x, y) \overset{\text{def}}{=} f(x+y)-f(x)-H(y).$$

Obviously, for every fixed y, $K(x, y)$ (as a function of x) belongs to S. We show that

$$\|K(x, y_n)-K(x, y_0)\| \to 0 \quad \text{whenever} \quad y_n\to y_0. \tag{21}$$

(Here and in the sequel the "norm" $\|\cdot\|$ of a function of x and y denotes the norm of that function as a function of x; the variable y is always fixed.)

Let $y_n\to 0$, then

$$\|K(x, y_n)\| = \|f(x+y_n)-f(x)-H(y_n)\| \leqq \|f(x+y_n)-f(x)-c(y_n)\|+\|c(y_n)-H(y_n)\| =$$

$$= s\bigl(f(x+y_n)-f(x)\bigr)+\min(1, |u(y_n)|) \to 0$$

by (19) and (20). If $y_n \to y_0$ then we have

$$\|K(x, y_n)-K(x, y_0)\| = \|f(x+y_n)-f(x+y_0)-H(y_n-y_0)\| =$$
$$= \|f(x+y_n-y_0)-f(x)-H(y_n-y_0)\| = \|K(x, y_n-y_0)\| \to 0$$

and hence (21) is proved.

The next step of our proof is the construction of a measurable function $G(x, y)$ satisfying the following condition:

(22) For every $y \in R$, $G(x, y) = K(x, y)$ for a.e. $x \in \mathbf{R}$.

Let $\varepsilon > 0$ be arbitrary. Then there exists $\delta > 0$ such that $\|K(x, y)-K(x, y')\| < \varepsilon$ whenever $|y-y'| < \delta$. Indeed, otherwise we could find two sequences y_n and y'_n such that $y_n - y'_n \to 0$ and $\|K(x, y_n)-K(x, y'_n)\| \geqq \varepsilon$. Since $K(x, y)$ is periodic in y, we may suppose $y_n, y'_n \in [0, 3]$ $(n=1, 2, \dots)$. Then, for a suitable subsequence n_k we have $y_{n_k} \to y_0$, $y'_{n_k} \to y_0$. By (21) we have $\|K(x, y_{n_k})-K(x, y'_{n_k})\| \to 0$ which is a contradiction.

Now let $\delta_n > 0$ be such that $|y-y'| < \delta_n$ implies $\|K(x, y)-K(x, y')\| < \frac{1}{2^n}$ $(n=1, 2, \dots)$ and put

$$G_n(x, y) \stackrel{\text{def}}{=} K(x, i\delta_n) \quad \text{if} \quad i\delta_n \leqq y < (i+1)\delta_n \quad (i = 0, \pm 1, \pm 2, \dots;\ n = 1, 2, \dots).$$

Then G_n is measurable for every n and

$$\|G_n(x, y)-K(x, y)\| = \|K(x, i\delta_n)-K(x, y)\| < \frac{1}{2^n} \tag{23}$$

holds for every $y \in \mathbf{R}$.

We define

$$G(x, y) = \begin{cases} \lim\limits_{n\to\infty} G_n(x, y), & \text{if the finite limit exists,} \\ 0, & \text{otherwise.} \end{cases}$$

G is measurable and satisfies (22). Indeed, let y be fixed. By (7) and (23) we have

$$\lambda\left(I\left\{|G_n(x, y)-K(x, y)| \geqq \frac{1}{2^n}\right\}\right) < \frac{1}{2^n}$$

and hence, by the Borel—Cantelli lemma we have

$$G(x, y) = \lim_{n\to\infty} G_n(x, y) = K(x, y) \quad \text{for a.e.} \quad x \in [0, 1].$$

Thus the periodicity of the functions G_n and K proves $G(x, y) = K(x, y)$ for a.e. $x \in \mathbf{R}$.

Let

$$S_1(x, y) \stackrel{\text{def}}{=} K(x, y)-G(x, y) = f(x+y)-f(x)-H(y)-G(x, y). \tag{24}$$

According to (22), for every fixed y we have

$$S_1(x, y) = 0 \quad \text{for a.e.} \quad x \in \mathbf{R}. \tag{25}$$

We shall prove that there exists a point $x_0 \in \mathbf{R}$ such that

(26) for every fixed h, $S_1(x_0, x+h) - S_1(x_0, x) = 0$ for a.e. $x \in \mathbf{R}$.

We have:

$$S_1(x, y+z) = f(x+y+z) - f(x) - H(y+z) - G(x, y+z);$$

$$-S_1(x+y, z) = -f(x+y+z) + f(x+y) + H(z) + G(x+y, z);$$

$$-S_1(x, y) = -f(x+y) + f(x) + H(y) + G(x, y).$$

By adding we get

$$S_1(x, y+z) - S_1(x+y, z) - S_1(x, y) = -G(x, y+z) + \tag{27}$$

$$+G(x+y, z) + G(x, y) \overset{\text{def}}{=} L(x, y, z).$$

The measurability of G implies that L is measurable, too. On the other hand it follows from (25) and (27) that for every fixed y and z, $L(x, y, z) = 0$ for a.e. x. Therefore $L(x, y, z) = 0$ for almost every $(x, y, z) \in \mathbf{R}^3$. Hence there exists a point x_0 such that $L(x_0, y, z) = 0$ for almost every pair $(y, z) \in \mathbf{R}^2$. Thus there exists a subset $Z \subset \mathbf{R}$ such that $\lambda\ (\mathbf{R} \setminus Z) = 0$ and for every $z \in Z$ we have $L(x_0, y, z) = S_1(x_0, y+z) - S_1(x_0+y, z) - S_1(x_0, y) = 0$ for a.e. y. However $S_1(x_0+y, z) = 0$ for a.e. y by (25) hence

$$S_1(x_0, y+z) - S_1(x_0, y) = 0 \quad \text{holds for a.e. } y. \tag{28}$$

Now let $h \in \mathbf{R}$ be arbitrary. Then there are $z_1, z_2 \in Z$ such that $h = z_1 + z_2$, since $Z \cap (h - Z) \neq \emptyset$. Therefore

$$S_1(x_0, x+h) - S_1(x_0, x) =$$

$$= [S_1(x_0, x+z_1+z_2) - S_1(x_0, x+z_2)] + [S_1(x_0, x+z_2) - S_1(x_0, x)] = 0$$

holds for a.e. x by (28) and hence (26) is proved.

Now we apply (24) by replacing x by x_0 and y by $x - x_0$:

$$S_1(x_0, x-x_0) = f(x) - f(x_0) - H(x) + H(x_0) - G(x_0, x-x_0)$$

from which

$$f(x) = [G(x_0, x-x_0) + f(x_0) - H(x_0)] + H(x) + S_1(x_0, x-x_0) \overset{\text{def}}{=} g(x) + H(x) + S(x).$$

It is easy to see from the construction of $G(x, y)$ that $G(x_0, x)$ is measurable for every fixed x_0. Hence $g(x) = G(x_0, x-x_0) + f(x_0) - H(x_0)$ is measurable. Furthermore, for every fixed h we have

$$S(x+h) - S(x) = S_1(x_0, x+h-x_0) - S_1(x_0, x-x_0) = 0$$

for a.e. $x \in \mathbf{R}$ by (26), q.e.d.

In our next theorem $L_p(0, 1)$ denotes the class of those functions $f \in L$ which are periodic mod 1 and for which $\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p} < \infty$.

THEOREM 4. *The classes $L_p(0,1)$ have the weak difference property for every $p>0$.*

PROOF. Suppose that for a function $f\colon \mathbf{R}\to\mathbf{R}$, $f(x+h)-f(x)\in L_p(0,1)$ holds for every h. Then, by our preceding theorem, $f=g+H+S$, where $g\in L$, H is additive and, for every h, $S(x+h)-S(x)=0$ holds for a.e. x. We may assume that g, H and S are periodic mod 1 since otherwise we consider the functions $g_1(x)=$ $=g(x-[x])+(x-[x])\cdot H(1)$, $H_1(x)=H(x)-H(1)\cdot x$ and $S_1(x)=S(x-[x])$ instead of g, H and S.

For every fixed h we have

$$g(x+h)-g(x)=[f(x+h)-f(x)]-H(h)-[S(x+h)-S(x)]=[f(x+h)-f(x)]-H(h)$$

for a.e. x and thus

$$N(h)\overset{\text{def}}{=}\|g(x+h)-g(x)\|_p^p<\infty$$

holds for every h.

We prove that the function N is bounded on $[0,1]$. First observe that

$$N(-h)=\|g(x-h)-g(x)\|_p^p=\|g(x)-g(x+h)\|_p^p=N(h)$$

holds for each $h\in\mathbf{R}$.

Furthermore, for every $f_1,f_2\in L_p(0,1)$ we have

$$\|f_1+f_2\|_p^p=\int_0^1|f_1+f_2|^p\,dx\leqq\int_0^1[2\max(|f_1|,|f_2|)]^p\,dx\leqq$$

$$\leqq 2^p\int_0^1(|f_1|^p+|f_2|^p)\,dx=2^p(\|f_1\|_p^p+\|f_2\|_p^p)$$

and thus

$$N(y_1+y_2)=\|g(x+y_1+y_2)-g(x)\|_p^p=$$

$$=\|[g(x+y_1+y_2)-g(x+y_2)]+[g(x+y_2)-g(x)]\|_p^p\leqq$$

$$\leqq 2^p(\|g(x+y_1+y_2)-g(x+y_2)\|_p^p+\|g(x+y_2)-g(x)\|_p^p)=$$

$$=2^p(\|g(x+y_1)-g(x)\|_p^p+N(y_2))=2^p(N(y_1)+N(y_2))$$

holds for every $y_1,y_2\in\mathbf{R}$. The function $G(x,y)\overset{\text{def}}{=}|g(x+y)-g(x)|^p$ is measurable on $[0,1]\times[0,1]$ hence $N(y)=\int_0^1 G(x,y)\,dx$ is measurable on $[0,1]$. Thus there exists $K>0$ such that the set $A=\{y\in[0,1];\ N(y)<K\}$ is of positive measure. By a theorem of Steinhaus (see [12], p. 145), there is $\delta>0$ such that if $|y|<\delta$ then $y=y_1-y_2$ for suitable $y_1,y_2\in A$ and so,

$$0\leqq N(y)\leqq 2^p(N(y_1)+N(-y_2))=2^p(N(y_1)+N(y_2))\leqq 2^{p+1}K.$$

Hence, if $1/2^n<\delta$ then for every $y\in[0,1]$ we have

$$0\leqq N(y)\leqq 2^{p+1}N\left(\frac{y}{2}\right)\leqq 2^{2(p+1)}\cdot N\left(\frac{y}{4}\right)\leqq\ldots\leqq 2^{n(p+1)}\cdot N\left(\frac{y}{2^n}\right)\leqq 2^{(n+1)(p+1)}\cdot K.$$

It follows by Fubini's theorem that

$$\int_0^1\Big(\int_0^1 G(x,y)\,dy\Big)\,dx = \int_0^1\Big(\int_0^1 G(x,y)\,dx\Big)\,dy = \int_0^1 N(y)\,dy < \infty$$

and hence, for at least one value of x we have

$$\int_0^1 G(x,y)\,dy = \int_0^1 |g(x+y)-g(x)|^p\,dy < \infty.$$

By the periodicity of g, this obviously implies $g\in L_p(0,1)$, q.e.d.

3. Our next theorem states that the pair $(L, L^{(2)})$ has the double difference property, where $L^{(2)}$ denotes the class of Lebesgue measurable functions defined on $\mathbf{R}^2$.

THEOREM 5. *If a function $f\colon \mathbf{R}\to\mathbf{R}$ is such that $f(x+y)-f(x)-f(y)$ is Lebesgue measurable (as a function of two variables), then f is of the form $g+H$ where $g\in L$ and $H\colon \mathbf{R}\to\mathbf{R}$ is additive.*

PROOF. There exists a subset $Y\subset\mathbf{R}$ such that $\lambda(\mathbf{R}\setminus Y)=0$ and, for every $y\in Y$, $f(x+y)-f(x)-f(y)$ is measurable, as a function of x. Let $h\in\mathbf{R}$ be arbitrary. Then there are $y_1, y_2\in Y$ such that $h=y_1+y_2$ since $Y\cap[(-Y)+h]\neq\emptyset$. Since

$$f(x+h)-f(x) = [f(x+y_1+y_2)-f(x+y_2)-f(y_1)]+$$
$$+[f(x+y_2)-f(x)-f(y_2)]+f(y_1)+f(y_2),$$

hence $f(x+h)-f(x)$ is measurable for every h. According to Theorem 3, $f=g+H+S$ where g is measurable, H is additive and, for every h, $S(x+h)-S(x)=0$ for a.e. x.

Let $F(x,y)\stackrel{\text{def}}{=}S(x+y)-S(x)-S(y)$, then

$$F(x,y) = [f(x+y)-f(x)-f(y)]-g(x+y)+g(x)+g(y)$$

and thus $F(x,y)$ is measurable. For every fixed x we have

$$-F(x,y) = S(x)-[S(x+y)-S(y)] = S(x) \quad \text{for a.e. } y.$$

Consequently $S(x)=-\int_0^1 F(x,y)\,dy$ holds for every x which proves that S is measurable, too. Hence $f=[g+S]+H$ is a sum of a measurable and an additive function, q.e.d.

For the analogous theorem concerning Borel measurable functions we need the following simple

LEMMA 3. *Let $f(x,y)\colon \mathbf{R}\times[0,1]\to\mathbf{R}$ be a bounded function of Baire class α. Then $F(x)=\int_0^1 f(x,y)\,dy$ is of Baire class α, too.*

PROOF. We prove by transfinite induction. If $\alpha=0$, that is if f is continuous, then the continuity of the function F is well-known.

Let $\alpha>0$ and suppose the assertion is true for $\beta<\alpha$. Let $|f(x,y)|\leqq M$ and let $f_n(x,y)$ be a sequence of functions of Baire class $\alpha_n<\alpha$ converging to f. We may suppose $|f_n(x,y)|\leqq M$ because otherwise we take the functions $\min(M, \max(f_n, -M))$ instead of f_n. Then $F_n(x)=\int_0^1 f_n(x,y)\,dy$ is of Baire class α_n by the induction hypothesis. Furthermore Lebesgue's theorem implies

$$F(x)=\int_0^1 f(x,y)\,dy=\lim_{n\to\infty}\int_0^1 f_n(x,y)\,dy=\lim_{n\to\infty}F_n(x)$$

which proves that F is of class α.

THEOREM 6. *If a function $f\colon \mathbf{R}\to\mathbf{R}$ is Lebesgue measurable and such that $f(x+y)-f(x)-f(y)$ is of Baire class α (as a function of two variables), then f is of Baire class α.*

PROOF. In the case of $\alpha=0$ the assertion is a simple consequence of de Bruijn's theorem on the difference property of the class of continuous functions. For, if $f(x+y)-f(x)-f(y)$ is continuous, then $f(x+h)-f(x)$ is continuous for every h and hence $f=g+H$ where g is continuous and H is additive. By our assumptions $H=f-g$ is measurable and thus H must be linear.

Now we suppose $\alpha\geqq 2$. It easily follows from Luzin's theorem that every measurable function equals almost everywhere to a Baire 2 function. Hence there exists a Baire 2 function $p(x)$ such that

$$q(x)\stackrel{\text{def}}{=}f(x)-p(x)=0 \quad \text{for a.e. } x. \tag{29}$$

$$q(x+y)-q(x)-q(y)=[f(x+y)-f(x)-f(y)]-p(x+y)+p(x)+p(y)$$

is a Baire α function since $f(x+y)-f(x)-f(y)$ is Baire α by our assumption, p is Baire 2 and $\alpha\geqq 2$. Hence

$$F(x,y)\stackrel{\text{def}}{=}-\operatorname{arctg}[q(x+y)-q(x)-q(y)]$$

is a bounded Baire α function. For every fixed x we have $F(x,y)=\operatorname{arctg}[q(x)]$ for a.e. y by (29). Hence by Lemma 3, $\operatorname{arctg}[q(x)]=\int_0^1 F(x,y)\,dy$ is Baire α and thus so is the function $f(x)=p(x)+q(x)$.

Finally suppose $\alpha=1$. By Luzin's theorem, for every natural number n there exists a closed subset $F_n\subset\mathbf{R}$ such that $\lambda(\mathbf{R}\setminus F_n)<\dfrac{1}{n}$ and the restriction $f|_{F_n}$ is continuous. Let

$$p_n(x)=\begin{cases} f(x) & \text{if } x\in F_n \\ 0 & \text{if } x\notin F_n, \end{cases} \qquad q_n(x)=\begin{cases} 0 & \text{if } x\in F_n \\ f(x) & \text{if } x\notin F_n. \end{cases}$$

Then $p_n(x)$ is Baire 1, $q_n(x)$ is measurable,

$$f(x) = p_n(x)+q_n(x) \quad (x\in \mathbf{R}), \tag{30}$$

$$\lambda(\{x;\ q_n(x) \neq 0\}) < \frac{1}{n} \tag{31}$$

and

$$|q_n(x)| \leqq |f(x)| \quad \text{for every} \quad x\in\mathbf{R}. \tag{32}$$

$$q_n(x+y)-q_n(x)-q_n(y) = [f(x+y)-f(x)-f(y)]-p_n(x+y)+p_n(x)+p_n(y)$$

is Baire 1 by our assumption. Hence

$$F_n(x, y) \stackrel{\text{def}}{=\!=} -\operatorname{arctg}[q_n(x+y)-q_n(x)-q_n(y)]$$

is a bounded Baire 1 function and thus by Lemma 3, $G_n(x) \stackrel{\text{def}}{=\!=} \int_0^1 F_n(x, y)\, dy$ is Baier 1, too. It follows from (31) that

$$\lambda(\{y;\ F_n(x, y) \neq \operatorname{arctg}[q_n(x)]\}) < \frac{2}{n}$$

for every $x\in\mathbf{R}$. Hence

$$|G_n(x)-\operatorname{arctg}[q_n(x)]| < \frac{2\pi}{n} \tag{33}$$

that is

$$G_n(x)-\frac{2\pi}{n} < \operatorname{arctg}[q_n(x)] < G_n(x)+\frac{2\pi}{n} \quad \text{for every } x. \tag{34}$$

Let

$$U_n = \left\{x;\ -\frac{\pi}{2} < G_n(x)-\frac{2\pi}{n} < G_n(x)+\frac{2\pi}{n} < \frac{\pi}{2}\right\}.$$

U_n is an F_σ set since G_n is Baire 1.

Let

$$a_n(x) \stackrel{\text{def}}{=\!=} p_n(x)+\operatorname{tg}\left[G_n(x)-\frac{2\pi}{n}\right], \quad b_n(x) \stackrel{\text{def}}{=\!=} p_n(x)+\operatorname{tg}\left[G_n(x)+\frac{2\pi}{n}\right] \quad (x\in U_n),$$

then by (30) and (34) we have

$$a_n(x) < f(x) < b_n(x) \quad (x\in U_n). \tag{35}$$

Since p_n and G_n are Baire 1 functions, hence a_n and b_n are Baire 1 functions, too (on the F_σ set U_n) and thus the level sets $\{x\in U_n;\ a_n(x)>c\}$ and $\{x\in U_n,\ b_n(x)<c\}$ are F_σ sets for every $c\in\mathbf{R}$ and $n=1, 2, \dots$.

We prove that

$$\{x;\ f(x) < c\} = \bigcup_{n=1}^{\infty} \{x\in U_n;\ b_n(x) < c\} \tag{36}$$

for every $c\in\mathbf{R}$. The inclusion

$$\{x;\, f(x)<c\}\supset\bigcup_{n=1}^{\infty}\{x\in U_n;\, b_n(x)<c\}$$

is obvious by (35). Suppose $f(x)<c$ and let $|f(x)|=A$. It follows from (32) that

$$|\operatorname{arctg}[q_n(x)]|\leqq\operatorname{arctg}A\overset{\text{def}}{=}\frac{\pi}{2}-\varepsilon$$

for every n. Hence by (33) we have

$$-\frac{\pi}{2}+\frac{\varepsilon}{2}<G_n(x)-\frac{2\pi}{n}<G_n(x)+\frac{2\pi}{n}<\frac{\pi}{2}-\frac{\varepsilon}{2} \tag{37}$$

if $n\geqq\frac{8\pi}{\varepsilon}$ and thus $x\in U_n$ for $n\geqq\frac{8\pi}{\varepsilon}$. (37) implies

$$\lim_{n\to\infty}(b_n(x)-a_n(x))=\lim_{n\to\infty}\left(\operatorname{tg}\left[G_n(x)+\frac{2\pi}{n}\right]-\operatorname{tg}\left[G_n(x)-\frac{2\pi}{n}\right]\right)=0$$

since $\operatorname{tg}(x)$ is uniformly continuous on the interval $\left[-\frac{\pi}{2}+\frac{\varepsilon}{2},\frac{\pi}{2}-\frac{\varepsilon}{2}\right]$. Hence $\lim_{n\to\infty}a_n(x)=\lim_{n\to\infty}b_n(x)=f(x)$ by (35), consequently $b_n(x)<c$ if n is large enough. Thus we have

$$\{x;\, f(x)<c\}\subset\bigcup_{n=1}^{\infty}\{x\in U_n;\, b_n(x)<c\}$$

and (36) is proved.

Hence $\{x;\, f(x)<c\}$ is an F_σ set for every c. The same argument shows that $\{x;\, f(x)>c\}$ is F_σ, too which proves that f is Baire 1, q.e.d.

Now Theorems 5 and 6 immediately imply

THEOREM 7. *Suppose that $f(x+y)-f(x)-f(y)$ is Baire α (as a function of two variables). Then there are a Baire α function $g\colon\mathbf{R}\to\mathbf{R}$ and an additive function H such that $f(x)=g(x)+H(x)$.*

For, by Theorem 5, f is of the form $g+H$ where g is measurable and H is additive. Since

$$g(x+y)-g(x)-g(y)=f(x+y)-f(x)-f(y)$$

is Baire α, Theorem 6 gives that g is Baire α, too.

We remark that de Bruijn's theorem on the class of continuous functions can be deduced from Theorem 7. Indeed, suppose that $f(x+h)-f(x)$ is continuous for every h. Then the function $f(x+y)-f(x)-f(y)$ is separately continuous in both variables. Then, by Baire's theorem, it is a Baire 1 function (as a function of two variables) and thus, by Theorem 7, there are a Baire 1 function g and an additive function H such that $f=g+H$. Let x_0 be a point of continuity of the function g. Then the continuity of $g(x+h)-g(x)=f(x+h)-f(x)-H(h)$ implies that g is continuous at x_0+h. Since h is arbitrary, g is continuous everywhere. Obviously, this proof is much more complicated than de Bruijn's. Yet, the same argument applies for the following theorems.

THEOREM 8. *The class of approximately continuous functions has the difference property.*

PROOF. Suppose that $f(x+h)-f(x)$ is approximately continuous for every h. Then $f(x+y)-f(x)-f(y)$ is separately approximately continuous and hence, by a theorem of R. O. DAVIES [7], it is measurable. Applying Theorem 5 we get $f=g+H$ where g is measurable and H is additive. It is well-known that every measurable function is approximately continuous almost everywhere (see [10], Theorem 5.9, p. 118). Let x_0 be a point at which g is approximately continuous. Then $g(x+h)==g(x)+[f(x+h)-f(x)]-H(h)$ implies that g is approximately continuous at x_0+h. Since h is arbitrary, g is approximately continuous everywhere.

THEOREM 9. *Let $f\colon \mathbf{R}\to\mathbf{R}$ be a bounded function for which $f(x+h)-f(x)$ is a derivative for every h. Then f is a derivative.*

PROOF. The function $F(x, y)=f(x+y)-f(x)-f(y)$ is bounded and for every fixed x_0, y_0, the functions $F(x_0, y)$ and $F(x, y_0)$ are derivatives. Then, according to a theorem of Z. GRANDE [9], F is measurable. Applying Theorem 5 we get $f=g+H$ where g is measurable and H is additive. Since $H=f-g$ is bounded on a set of positive measure, H is linear (see [12]) and thus f is measurable. Now the assertion of Theorem 9 immediately follows from the following

THEOREM 10. *Suppose that $f(x+h)-f(x)$ is a derivative for every h. If there exists an interval on which f is measurable and summable, then f is a derivative.*

PROOF. Suppose that f is summable on $[a, b]$ and let $F(x)=\int\limits_a^x f(t)\,dt$ $(x\in[a, b])$. We can choose a point $x_0\in(a, b)$ such that $F'(x_0)=f(x_0)$ (since $F'(x)==f(x)$ holds a.e. in (a, b)). Let $0<h<b-x_0$ be fixed and let $G(x)$ be a primitive of $f(x+h)-f(x)$. Then $G'(x)$ is summable on the interval $[a, b-h]$ and hence $G(x)-G(a)=\int\limits_a^x G'(t)\,dt$ for every $x\in[a, b-h]$ (see [10], Theorem 6.6, p. 143). That is, for $x\in[a, b-h]$ we have

$$G(x)-G(a)=\int\limits_a^x [f(t+h)-f(t)]\,dt=\int\limits_a^{x+h} f(t)\,dt-$$

$$-\int\limits_a^x f(t)\,dt-\int\limits_a^{a+h} f(t)\,dt=F(x+h)-F(x)-F(a+h).$$

This implies

$$F'(x_0+h)=G'(x_0)+F'(x_0)=[f(x_0+h)-f(x_0)]+f(x_0)=f(x_0+h).$$

Since $h\in(0, b-x_0)$ was arbitrary, we have $F'(x)=f(x)$ $(x\in(x_0, b))$. Hence $f(x+h)$ has a primitive on $(x_0-h, b-h)$ and the same holds for

$$f(x)=f(x+h)-[f(x+h)-f(x)].$$

That is, f has a primitive on every open interval of length $b-x_0$. This easily implies that f has a primitive everywhere, q.e.d.

REMARKS. Let B_α denote the class of Baire α functions defined on $\mathbf{R}$. Since every measurable function equals a Baire 2 function a.e., it follows from Theorem 3 that the classes B_α have the weak difference property for $\alpha \geqq 2$. On the other hand, assuming the continuum hypothesis, the classes B_α do not have the difference property for $\alpha \geqq 2$. This can be shown by the same example as in the case of measurable functions; for if the set $\{x;\ S(x+h) \neq S(x)\}$ is countable for every h then $S(x+h)-S(x) \in B_2$ for every h (see Section 1). This raises the following

PROBLEM 1. *Has the class B_1 the difference property?*[3]

We remark here that *if B_1 has the difference property, then so is the class of derivatives*. Indeed, suppose that $f(x+h)-f(x)$ is a derivative for every h. Then, by assumption $f=g+H$ where $g \in B_1$ and H is additive. Then $g(x+h)-g(x)= =[f(x+h)-f(x)]-H(h)$ is a derivative for every h. On the other hand, g has a point of continuity and in a sufficiently small neighbourhood of this point g is measurable and bounded. Hence, by Theorem 10, g is a derivative.

PROBLEM 2. Suppose that $f(x+h)-f(x)$ is Borel measurable for every h. Is it true that the functions $f(x+h)-f(x)$ belong to the *same* Baire class of order $\alpha<\omega_1$? (In the example above, $\alpha=2$.)

Now we prove that (assuming the continuum hypothesis) there exists a Lebesgue measurable function $S(x)$ such that $S(x+h)-S(x) \in B_2$ for every h and S is not Borel measurable. This means that Theorem 6 fails to remain valid if we replace the condition "$f(x+y)-f(x)-f(y)$ is Baire α" by "$f(x+h)-f(x)$ is Baire α for every h".

Let $\{a_\alpha\}_{\alpha<\omega_1}$ be a well-ordering of $\mathbf{R}$. Let U be an everywhere dense G_δ set of measure zero and let $\{P_\alpha\}_{\alpha<\omega_1}$ be a well-ordering of the family of perfect subsets of U. Let G_α denote the additive group generated by the set $\{a_\beta;\ \beta<\alpha\}$. Then G_α is countable for every $\alpha<\omega_1$ and $G_0=\{0\}$. Let $p_0 \in P_0$ and $x_0 \in U \setminus \{p_0\}$ be arbitrary and put $H_0=\{x_0\}$.

Suppose that $\alpha>0$ and the points p_β and the countable sets H_β have been defined for every $\beta<\alpha$. Then the set

$$A = \bigcup_{\beta<\alpha} H_\beta \cup \{p_\beta;\ \beta<\alpha\}$$

is countable. Let $p_\alpha \in P_\alpha \setminus A$. $V=U\setminus(A \cup \{p_\alpha\})$ is an everywhere dense G_δ set and hence so is $V'=\bigcap_{h\in G_\alpha}(V+h)$. Let $x_\alpha \in V'$ and define $H_\alpha=G_\alpha+x_\alpha$. Hence the points $p_\alpha \in P_\alpha$ and the countable sets H_α are defined for every $\alpha<\omega_1$. Let $X \overset{\text{def}}{=} \bigcup_{\alpha<\omega_1} H_\alpha$. It is easy to see that

(38) $X \subset U$ and hence $\lambda(X)=0$,
(39) $p_\alpha \notin X$ $(\alpha<\omega_1)$ and hence X does not contain any perfect set,
(40) X has the cardinality $\aleph_1$ and
(41) $(X+h)\setminus X$ is countable for every $h \in R$.

[3] Meanwhile I succeeded in giving an affirmative answer.

(We remark that a set with the properties (38), (40) and (41) was constructed in [16], Theorème I.) Then X is not a Borel set by (40) and (39) (see [14], p. 355). Hence the function

$$S(x)=\begin{cases}1, & x\in X\\ 0, & x\notin X\end{cases}$$

is not Borel measurable. On the other hand, S is Lebesgue measurable by (38) and $\{x;\ S(x+h)\neq S(x)\}$ is countable for every h by (41).

PROBLEM 3. *Let f be Borel measurable and suppose that $f(x+h)-f(x)$ is of class α for every h. Does it follow that f is of class α, too?*

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THE INTERCHANGING SEQUENCE BETWEEN TENSOR, TOR AND HOMOLOGY

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Introduction

Let R be a P.I.D. Denote by $\otimes$ the functor of tensor product and by $*$ the functor Tor, the first derived functor of $\otimes$. Then for any two R-complexes K^1 and K^2 we prove the following long exact sequence,

$$(1) \qquad \to H_{n-1}(K^1 * K^2) \to \big(H(K^1)\otimes H(K^2)\big)_n \oplus \big(H(K^1) * H(K^2)\big)_{n-1} \to$$
$$\to H_n(K^1 \otimes K^2) \xrightarrow{\sigma} H_{n-2}(K^1 * K^2) \to \ldots$$

where σ is natural in K^1 and K^2 and the other two maps are not necessarily so. This is a generalization of the Kunneth formula, which is obtained in the case that $K^1 * K^2$ is acyclic.

We call this sequence the interchanging sequence between tensor, tor and homology because it relates the tensor and tor of $H(K^1)$ and $H(K^2)$ with H of the tensor product and of the functor tor.

We shall proceed by showing that the interchanging sequence (1) and two additional requirements determine $\otimes$ and $*$ up to a natural equivalence, and thus characterise $\otimes$ and $*$ as the only couple of functors which obey (1) and the other two requirements.

We shall give here the notions of the categories used in this paper: for R fixed let RM denote the category of R-modules, FRM denote the category of free R modules and RC the category of R-chain complexes. All the functors considered are in two variables, covariant in each.

THEOREM 1 (The interchanging sequence). *Let K^1 and K^2 be in RC. Then we have the following long exact sequence:*

$$(1) \qquad \to H_{n-1}(K^1 * K^2) \to \big(H(K^1)\otimes H(K^2)\big)_n \oplus \big(H(K^1) * H(K^2)\big)_{n-1} \to$$
$$\to H_n(K^1 \otimes K^2) \xrightarrow{\sigma} H_{n-2}(K^1 * K^2) \to \ldots$$

where σ is natural in K^1 and K^2 and the other two maps are not necessarily so.

PROOF. The proof uses the idea of free approximation defined in [1], chapter 5.2. Let K be any object in RC. $\overline{K}$ in RC is called a free approximation to K if:

(i) there exists a surjection $a\colon \overline{K} \to K \to 0$,
(ii) $\overline{K}$ is a free chain complex,
(iii) a induces isomorphism $a_*\colon H(\overline{K}) \xrightarrow{\sim} H(K)$.

In [1], chapter 5.2 it was shown that for any $K \in RC$ $\overline{K}$ exists and is unique up to homotopy.

Let $K^1, K^2 \in RC$. Take $\bar{K}^1 \xrightarrow{a} K^1 \to 0$ a free approximation and define $\bar{\bar{K}}^1 = \ker a$. Then we have the following exact sequences:

$$(*) \qquad 0 \to \bar{\bar{K}}^1 \xrightarrow{b} \bar{K}^1 \xrightarrow{a} K^1 \to 0,$$

$$(**) \qquad 0 \to K^1 * K^2 \xrightarrow{c} \bar{\bar{K}}^1 \otimes K^2 \xrightarrow{b\otimes 1} \bar{K}^1 \otimes K^2 \xrightarrow{a \otimes 1} K^1 \otimes K^2 \to 0.$$

$\bar{K}^1$ is free and acyclic because a_* is a bijection. Therefore $\bar{K}^1$ is contractible and so is $\bar{K}^1 \otimes K^2$. Hence, $H(\bar{K}^1 \otimes K^2) = 0$. Let us denote $\mathrm{im}\,(b \otimes 1)$ by I. Then we have the following portions of $(**)$:

$$(***) \qquad 0 \to K^1 * K^2 \to \bar{\bar{K}}^1 \otimes K^2 \to I \to 0,$$

$$(****) \qquad 0 \to I \to \bar{K}^1 \otimes K^2 \to K^1 \otimes K^2 \to 0.$$

Then from $(***)$ $H_n(I) \cong H_{n-1}(K^1 * K^2)$ where the isomorphism is natural. Also, $\bar{K}^1 * K^2$ is null, and therefore the conditions of the classical Kunneth formula are satisfied and hence there is the following bijection which is not necessarily natural:

$$H_n(\bar{\bar{K}}^1 \otimes K^2) = \big(H(K^1) \otimes H(K^2)\big)_n \oplus \big(H(K^1) * H(K^2)\big)_{n-1}.$$

All this we insert into the long exact sequence of $(****)$ and get the interchanging sequence in which, any map with domain or range $\big(H(K^1) \otimes H(K^2)\big)_n \oplus \oplus \big(H(K^1) * H(K^2)\big)_{n-1}$ is not necessarily natural, and the other map $\sigma_n\colon H_n(K^1 \otimes K^2) \to \to H_{n-2}(K^1 * K^2)$ is natural.

COROLLARIES. (i) *For some n the Kunneth relation holds iff* $\mathrm{im}\,\sigma_n = 0$ *and* σ_{n+1} *is a surjection.*

(ii) *The Kunneth formula holds for any n iff $K^1 * K^2$ is acyclic.*

(iii) *If either of K^1 and K^2 is acyclic then* $H_n(K^1 \otimes K^2) \xrightarrow{\sim} H_{n-2}(K^1 * K^2)$ *naturally.*

From (ii) we see that the classical condition needed for the Kunneth formula to hold for all n is also necessary.

Assume that $T\colon (RM)^2 \to RM$ is additive in both variables. Then it is possible to extend T to another functor (also denoted by T) $T\colon (RC)^2 \to RC$ by letting $-T(K^1, K^2) = K$, $\quad K_n = \bigoplus_{i+j=n} T(K_i^1, K_j^2)$, $\quad \partial_i\colon\; K_n \to K_{n-1}\partial / T(K_i^1, K_j^2) = T(\partial_i^1, 1) + +(-1)^i T(1, \partial_j^2)$ where ∂^1 and ∂^2 are the derivatives of K^1 and K^2 respectively. It is easy to see using the additivity of T that $\partial_{n-1} \cdot \partial_n = 0$ for any n. If we are given T: $(RM)^2 \to RM$ not necessarily additive in its variables, then for $K^1, K^2 \in RC$ we can define similarly K and ∂_n, but generally we do not attain the equation $\partial_{n-1} \cdot \partial_n = 0$. Still, for $f_i\colon K^i \to L^i$, $i = 1, 2$, $T(f_1, f_2)\colon T(K^1, K^2) \to T(L^1, L^2)$ is a chain map. For such T we can generalize the meaning of $H(K)$ by letting $H_n(K) = \ker \partial_n / (\ker \partial_n) \cap \cap (\mathrm{im}\,\partial_{n+1})$. This definition applies in the following converse to Theorem 1.

THEOREM 2. *Let* $T, T_1\colon (RM)^2 \to RM$ *so that:*

(i) *there exists a bijection* $\theta\colon R \otimes R \xrightarrow{\sim} T(R, R)$ *which is natural in each* R.

(ii) $T_1(R, R) = 0$.

(iii) *For any* K^1 *and* $K^2 \in RC$ T *and* T_1 *obey the following exact interchaning sequence:*

$$(2) \quad \to \dots H_{n-1}(T_1(K^1, K^2)) \to (T(H(K^1), H(K^2))_n \oplus (T_1(H(K^1), H(K^2)))_{n-1} \to$$
$$\to H_n(T(K^1, K^2) \xrightarrow{\sigma} H_{n-2}(T_1(K^1, K^2))) \to \dots$$

where $H(T(K^1, K^2))$ *and* $H(T_1(K^1, K^2))$ *are interpreted in the generalized meaning and* σ *is natural in* K^1 *and* K^2. *Then, there exist natural equivalences:* $U\colon \otimes \xrightarrow{\sim} T$, $V\colon * \xrightarrow{\sim} T_1$.

PROOF. The proof will be made in several steps. Throughout T and T_1 are understood to have the properties assumed in Theorem 2.

LEMMA C-1. *Let* M *be any* R *module. Then* $T(0, M) = T(M, 0) = T_1(0, M) = T_1(M, 0) = 0$.

PROOF OF C-1. Let K^1 and K^2 be the following chain complexes:

$$n = 2, \qquad 1, \qquad 0, \quad -1, \quad -2$$
$$K^1 = 0 \xrightarrow{h} R \xrightarrow{1_R} R \xrightarrow{k} 0 \xrightarrow{1_0} 0$$
$$K^2 = 0 \xrightarrow{1_0} 0 \xrightarrow{f} M \xrightarrow{g} 0 \xrightarrow{1_0} 0.$$

Construct the following portion of the lattice $T(K_i^1, K_j^2)$, $i, j \in Z$:

$$\begin{array}{cccccccccc}
j = & 2, & & 1, & & 0, & & -1, & & -2 \\
i = 2 & T(0,0) & \to & T(0,0) & \to & T(0,M) & \to & T(0,0) & \to & T(0,0) \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & T(R,0) & \to & T(R,0) & \to & T(R,M) & \to & T(R,0) & \to & T(R,0) \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & T(R,0) & \to & T(R,0) & \to & T(R,M) & \to & T(R,0) & \to & T(R,0) \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
-1 & T(0,0) & \to & T(0,0) & \to & T(0,M) & \to & T(0,0) & \to & T(0,0) \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
-2 & T(0,0) & \to & T(0,0) & \to & T(0,M) & \to & T(0,0) & \to & T(0,0).
\end{array}$$

We want to show that ∂_1 is onto $(T(K^1, K^2))_0 \cdot T(K_0^1, K_0^2) = T(R, M)$ is in the image of ∂_1 because $T(K_1^1, K_0^2) \xrightarrow{T(1,1)} T(K_0^1, K_0^2)$ is the identity map by the functional properties of T.

We shall show now that $T(K_1^1, K_{-1}^2) = T(R, 0)$ is in im ∂_1. The map $T(K_1^1, K_0^2) \xrightarrow{T(1,g)} T(K_1^1, K_{-1}^2)$ is onto by the following argument: $g \cdot f = 1_0$, $T(1, g)T(1, f) = T(1_R, 1_0) = 1_{T(R,0)}$ any homomorphism of R modules which has a right inverse is onto.

For any other $T(K_i^1, K_j^2)$, $i+j=0$, we have that $T(K_i^1, K_j^2)=T(K_i^1, K_{j+1}^2)= =T(0, 0)$ and the map $T(1_0, 1_0)$; $T(0, 0)\to T(0, 0)$ is bijective and onto. Thus ∂_1: $(T(K^1, K^2))_1\to(T(K^1, K^2))_0$ is onto, and by definition of homology in the generalized sense $H_0(T(K^1, K^2))=\ker\partial_0/(\ker\partial_0)\cap(\operatorname{im}\partial_1)=0$. We now show in the same way that $H_{-1}(T(K^1, K^2))=0$. $T(K_{-1}^1, K_0^2)$ is in $\operatorname{im}\partial_0$, for again $kh=1_0$ and $T(k, 1_M)T(h, 1_M)=1_{T(0,M)}$. Thus, $T(k, 1_M)$: $T(K_0^1, K_0^2)\to T(K_{-1}^1, K_0^2)$ is onto. $T(K_0^1, K_{-1}^2)$ is in $\operatorname{im}\partial_0$ for $T(1, 1)=1$: $T(K_1^1, K_{-1}^2)\to T(K_0^1, K_{-1}^2)$ is bijective. For any other $T(K_i^1, K_j^2)$ $i+j=-1$, $T(K_i^1, K_j^2)=T(K_{i+1}^1, K_j^2)=T(0, 0)$ and $T(1_0, 1_0)= =1$: $T(K_{i+1}, K_j^2)\to T(K_i^1, K_j^2)$ is surjective. Thus $H_{-1}(T(K^1, K^2)))=0$.

As the proof of Lemma C-1 has used so far only the functorial properties of T, we deduce that $H_0(T_1(K^1, K^2))=H_{-1}T_1(K^1, K^2))=0$.

This we substitute in the following portion of the interchanging sequence (2) for K^1 and K^2:

$$\ldots\to H_{-1}(T_1(K^1, K^2))\to(T(H(K^1)), H(K^2)))_0\oplus(T_1(H(K^1), H(K^2)))_{-1}\to$$
$$\to H_0(T(K^1, K^2))\to\ldots.$$

The terms on the right and on the left vanish and we may deduce that $(T(H(K^1), H(K^2)))_0\oplus(T_1(H(K^1), H(K^2)))_{-1}=0$.

Especially $T(H_0(K^1), H_0(K^2))=T(0, M)=0$ and $T_1(H_{-1}(K^1), H_0(K^2))= =T_1(0, M,)=0$. The other two results are obtained similarly by changing the roles of K^1 and K^2.

Thus we can deduce that if f: $M\to N$ and g: $R\to S$ are arrows in RM so that $\operatorname{im} f=0$, then f can be factored through the module 0 and therefore $T(f, g)$, $T(g, f)$, $T_1(f, g)$ and $T_1(g, f)$ can be factored through 0 and are with $\operatorname{im}=0$. From this we may deduce the following corollary.

Corollary C-2. *For any K^1 and $K^2\in RC$, $T(K^1, K^2)$ and $T_1(K^1, K^2)\in RC$ and therefore their "generalized homologies" coincide with the homologies in the regular sense.*

Proof of C-2. This is very easy to check for:

$$\partial_{n-1}^{T(K,L)}\cdot\partial_n^{T(K,L)}/T(K_i^1, K_j^2)=T(\partial_{i-1}^1, 1)\cdot T(\partial_i^1, 1)+$$
$$+(-1)^i T(\partial_i^1, 1)T(1, \partial_j^2)+(-1)^{i-1}T(1, \partial_j^2)T(\partial_i^1, 1)+(-1)^{2i}T(1, \partial_{j-1}^2)T(1, \partial_j^2).$$

The two middle terms cancel each other, while each of the first and last terms are the zero function because of the considerations after the proof of C-1.

The proof for T_1 is the same.

Lemma C-3. *Let $0\to A\xrightarrow{f}B\xrightarrow{g}C\to 0$ be a short exact sequence of R modules and let D be any R module. Then there are the following two exact sequences, each of which is natural in $0\to A\to B\to C\to 0$ and in D:*

$$0\to T_1(A, D)\xrightarrow{T_1(f,1)}T_1(B, D)\xrightarrow{T_1(g,1)}T_1(C, D)\xrightarrow{\lambda}T(A, D)\xrightarrow{T(f,1)}$$
$$\xrightarrow{T(f,1)}T(B, D)\xrightarrow{T(g,1)}T(C, D)\to 0,$$

$$0\to T_1(D, A)\xrightarrow{T_1(1,f)}T_1(D, B)\xrightarrow{T_1(1,g)}T_1(D, C)\xrightarrow{\lambda}T(D, A)\xrightarrow{T(1,f)}$$
$$\xrightarrow{T(1,f)}T(D, B)\xrightarrow{T(1,g)}T(D, C)\to 0.$$

PROOF OF C-3. We shall prove the existence of only one of the two sequences. Let K^1 and $K^2 \in RC$ be the following:

$$\begin{array}{llllllll} n = & 4, & 3, & 2, & 1, & 0, & -1, & -2 \\ K^1 & 0 \to & 0 \to & A \xrightarrow{f} & B \xrightarrow{g} & C \to & 0 \to & 0 \\ K^2 & 0 \to & 0 \to & 0 \to & 0 \to & D \to & 0 \to & 0. \end{array}$$

Construct the complexes $T(K^1, K^2)$ and $T_1(K^1, K^2)$

$$\begin{array}{ll} n = & 4,\ 3,\quad 2,\qquad\qquad 1,\qquad\qquad 0,\qquad 1, \\ T(K^1, K^2) & 0 \to 0 \to T(A, D) \xrightarrow{T(f,1)} T(B, D) \xrightarrow{T(g,1)} T(C, D) \to 0 \\ T_1(K^1, K^2) & 0 \to 0 \to T_1(A, D) \xrightarrow{T_1(f,1)} T_1(B, D) \xrightarrow{T_1(g,1)} T_1(C, D) \to 0. \end{array}$$

$H(K^1)=0$ and by Lemma C-1 $(T(H(K^1), H(K^2)))_n \oplus (T_1(H(K^1), H(K^2)))_{n-1}=0$ this we insert into the interchanging sequence and obtain a natural bijection: σ: $H_n(T(K^1, K^2)) \xrightarrow{\sim} H_{n-2}(T_1(K^1, K^2))$. Therefore:

$$H_0(T(K^1, K^2)) = H_{-2}(T_1(K^1, K^2)) = 0,\quad H_1(T(K^1, K^2)) = H_{-1}(T_1(K^1, K^2)) = 0,$$

$$H_1(T_1(K^1, K^2)) = H_3(T(K^1, K^2)) = 0,\quad H_2(T_1(K^1, K^2)) = H_4(T(K^1, K^2)) = 0,$$

$$H_0(T_1(K^1, K^2)) \xrightarrow[\sigma]{\approx} H_2(T(K^1, K^2)).$$

By the first four equalities we get that the following are exact sequences:

$$T(A, D) \xrightarrow{T_1(f,1)} T(B, D) \xrightarrow{T(g,1)} T(C, D) \to 0,$$

$$0 \to T_1(A, D) \xrightarrow{T_1(f,1)} T_1(B, D) \xrightarrow{T_1(g,1)} T_1(C, D).$$

Define λ: $T_1(C, D) \to T(A, D)$ by

$$T_1(C, D) \to \frac{T_1(C, D)}{\operatorname{im} T_1(g, 1)} \equiv H_0(T_1(K^1, K^2)) \xrightarrow[\sigma]{\sim} H_2(T(K^1, K^2)) \equiv$$

$$\equiv \ker T(f, 1) \to T(A, D).$$

λ is a well defined composition of natural maps, and therefore is natural. Also, by definition of λ $\ker \lambda = \operatorname{im} T_1(g, 1)$ and $\operatorname{im} \lambda = \ker T(f, 1)$. Therefore we get the six term exact sequence:

$$0 \to T_1(A, D) \xrightarrow{T_1(f,1)} T_1(B, D) \xrightarrow{T_1(g,1)} T_1(C, D) \xrightarrow{\lambda} T(A, D) \xrightarrow{T(f,1)}$$

$$\xrightarrow{T(f,1)} T(B, D) \xrightarrow{T(g,1)} T(C, D) \to 0$$

If we are given the commutative diagram of short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "p"] & B \arrow[r, "g"] \arrow[d, "q"] & C \arrow[r] \arrow[d, "s"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0 \end{tikzcd}$$

and if $h\colon D\to D'$ is any morphism, we have to show that the following diagram is commutative:

$$\begin{array}{ccccccc}
0\to T_1(A,D) & \xrightarrow{T_1(f,1)} & T_1(B,D) & \xrightarrow{T_1(g,1)} & T_1(C,D) & \xrightarrow{\lambda} \\
\downarrow{\scriptstyle T_1(p,h)} & & \downarrow{\scriptstyle T_1(q,h)} & & \downarrow{\scriptstyle T_1(s,h)} & \\
0\to T_1(A',D') & \xrightarrow{T_1(f',1)} & T_1(B',D') & \xrightarrow{T_1(g',1)} & T_1(C',D') & \xrightarrow{\lambda}
\end{array}$$

$$\begin{array}{ccccccc}
\xrightarrow{\lambda} & T(A,D) & \xrightarrow{T(f,1)} & T(B,D) & \xrightarrow{T(g,1)} & T(C,D)\to 0 \\
& \downarrow{\scriptstyle T(p,h)} & & \downarrow{\scriptstyle T(q,h)} & & \downarrow{\scriptstyle T(s,h)} \\
\xrightarrow{\lambda} & T(A',D') & \xrightarrow{T(f',1)} & T(B',D') & \xrightarrow{T(g',1)} & T(C',D')\to 0.
\end{array}$$

The middle square commutes because λ was shown to be natural. All the other squares commute because of the functorial properties of T and T_1.

LEMMA C-4. *Suppose that $0\to A\xrightarrow{f} B\xrightarrow{g} C\to 0$ is a short exact sequence of R-modules that splits. Let D be any R module. Then any of the six term exact sequences of Lemma C-3 breaks into two portions of short exact sequences:*

$$0\to T_1(A,D)\xrightarrow{T_1(f,1)} T_1(B,D)\xrightarrow{T_1(g,1)} T_1(C,D)\to 0$$

$$0\to T(A,D)\xrightarrow{T(f,1)} T(B,D)\xrightarrow{T(g,1)} T(C,D)\to 0$$

$$0\to T_1(D,A)\xrightarrow{T_1(1,f)} T_1(D,B)\xrightarrow{T_1(1,g)} T_1(D,C)\to 0$$

$$0\to T(D,A)\xrightarrow{T(1,f)} T(D,B)\xrightarrow{T(1,g)} T(D,C)\to 0.$$

Any of those four exact sequences is split and natural in $0\to A\to B\to C\to 0$ and in D. We omit the proof.

LEMMA C-5. *Let $F_1, F_2\in FRM$. Then $T_1(F_1, F_2)=0$.*

PROOF by transfinite induction on rank F_1+rank $F_2=k$. For $k=2$ $F_1=F_2=R$ and by the second hypothesis of Theorem 2 we know that $T_1(R,R)=0$.

Assume that $k\geqq 3$ and that the lemma holds for any $i, i<k$. Let $F_1, F_2\in FRM$ be so that rank F_1+rank $F_2=k$.

Then without loss of generality rank $F_1\geqq 2$ and we can find a surjection

$$F_1\xrightarrow{y} R\to 0$$

then, if $A=\ker y$, we have a short exact sequence that split

$$0\to A\to F_1\xrightarrow{y} R\to 0.$$

By Lemma C-4 we get the following short exact sequence that split

$$0\to T_1(A,F_2)\to T_1(F_1,F_2)\to T_1(R,F_2)\to 0.$$

By the induction hypothesis the right and left terms vanish, and so $T_1(F_1,F_2)=0$.

The same kind of argument applies for the functor T, but here we should proceed carefully.

DEFINITION C-6. Let F_1 and F_2 be arbitrary in FRM. We define the set $U(F_1, F_2)$ by induction on rank F_1+rank F_2 as follows:

(1) $U(R, R)=\{\theta/\theta$ is the natural bijection: $\theta\colon R\otimes R\to T(R, R)$ given in the first assumption of Theorem 2$\}$

(2) Suppose that rank F_1+rank $F_2\geqq 3$. If rank $F_1\geqq 2$ we can take a sequence of the form

$$0\to A \underset{k}{\overset{f}{\rightleftarrows}} F_1 \underset{h}{\overset{g}{\rightleftarrows}} R\to 0$$

which is short exact and split. Any $b\in U(A, F_2)$ and $c\in U(R, F_2)$ define $u\colon F_1\otimes F_2\to$ $\to T(F_1, F_2)$ by $u=T(h, 1)c(g\otimes 1)+T(f, 1)b(k\otimes 1)$.

If rank $F_2\geqq 2$, we can take a sequence of the form: $0\to A \underset{g}{\overset{f}{\rightleftarrows}} F_2 \underset{h}{\overset{g}{\rightleftarrows}} R\to 0$ any $b\in U(F_1, A)$, $c\in U(F_1, R)$ define a unique $u\colon F_1\otimes F_2\to T(F_1, F_2)$ by $u=$ $=T(1, h)c(1\otimes g)+T(1, f)b(1\otimes k)$. Define $U(F_1, F_2)=\{u/u$ is obtained as above$\}$. Thus u ranges over all the possibilities of splitting $F_1(F_2)$ by an exact sequence and choosing $b\in U(A, F_2)$ $(\in U(F_1, A))$ and $c\in U(R, F_2)(\in U(F_1, R))$. $U(F_1, F_2)$ is a family of morphisms $v_i\colon F_1\otimes F_2\to T(F_1, F_2)$. By induction on rank F_1+rank F_2, it is clear that $U(F_1, F_2)$ is not empty and that any $u\in U(F_1, F_2)$ is an isomorphism. As for naturality, we should work harder.

LEMMA C-7. *Let F_1, F_2, G_1 and G_2 be free; let a_i; $F_i\to G_i$, $i=1, 2$ be arbitrarily chosen. Then the following diagram is commutative:*

$$\begin{array}{ccc} F_1\otimes F_2 & \xrightarrow{a_1\otimes a_2} & G_1\otimes G_2 \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ T(F_1, F_2) & \xrightarrow{T(a_1, a_2)} & T(G_1, G_2). \end{array}$$

PROOF. We prove on double induction.

Stage 1. $G_1=G_2=R$. Then $U(R, R)$ consists of the single element θ given in the first hypothesis of Theorem 2, and thus $U=\{\theta\}$. We prove that case by induction on rank F_1+rank $F_2=k$.

If $k=2$ then also $F_1=F_2=R$ and $u=\theta$. Then we get the following square which is commutative because of the naturality of θ:

$$\begin{array}{ccc} R\otimes R & \xrightarrow{a_1\otimes a_2} & R\otimes R \\ {\scriptstyle \theta}\downarrow & & \downarrow{\scriptstyle \theta} \\ T(R, R) & \xrightarrow{T(a_1, a_2)} & T(R, R). \end{array}$$

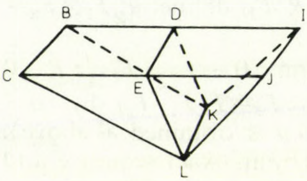
Suppose the lemma was proven for $G_1=G_2=R$ and for any F_1 and F_2 with rank F_1+rank $F_2<k$. Let F_1 and F_2 be free so that rank F_1+rank $F_2=k$, and take $a_i\colon F_i\to R$ and $u\in U(F_1, F_2)$. Then without loss of generality, there is the following short exact split sequence $0\to A \underset{k}{\overset{f}{\rightleftarrows}} F_1 \underset{h}{\overset{g}{\rightleftarrows}} R\to 0$ and there are $u_A\in U(A, F_2)$

and $u_R \in U(R, F_2)$ so that we get the following commutative diagram:

$$\begin{array}{ccccccc} 0 \to A\otimes F_2 & \overset{f\otimes 1}{\underset{k\otimes 1}{\rightleftarrows}} & F_1\otimes F_2 & \overset{g\otimes 1}{\underset{h\otimes 1}{\rightleftarrows}} & R\otimes F_2 \to 0 \\ \downarrow u_A & & \downarrow u & & \downarrow u_R \\ 0 \to T(A, F_2) & \overset{T(f,1)}{\underset{T(k,1)}{\rightleftarrows}} & T(F_1, F_2) & \overset{T(g,1)}{\underset{T(h,1)}{\rightleftarrows}} & T(R, F_2) \to 0 \end{array}$$

which defines u:

We shall get the following prism, with top that is the last diagram and with bottom corresponding to θ: $R\otimes R \to T(R, R)$ where: $B=A\otimes F_2$, $C=T(A, F_2)$, $D=F_1\otimes F_2$, $E=T(F_1, F_2)$, $I=R\otimes F_2$, $J=T(R, F_2)$, $K=R\otimes R$, $L=T(R, R)$. $BD=f\otimes 1$, $DB=k\otimes 1$, $DI=g\otimes 1$, $ID=h\otimes 1$, $CE=T(f, 1)$, $EC=T(k, 1)$, $EJ=T(g, 1)$, $JE=R(h, 1)$, $BC=u_A$, $DE=u$, $IJ=u_R$, $DK=a_1\otimes a_2$, $EL=T(a_1, a_2)$. We also define: $BK==BDK=(a_1\otimes a_2)\cdot(f\otimes 1)$ $IK=IDK=(a_1\otimes a_2)(h\otimes 1)$, $CL=CEL=T(a_1, a_2)\cdot T(f, 1)$, $JL=JEL=T(a_1, a_2)\cdot\cdot T(h, 1)$. We want to prove that $DEL=DKL$.



We know that $DE=DBCE+DIJE$ by the definition of u. $BCL=BKL$ and $IJL=IKL$ by the induction hypothesis on u_A and u_R respectively, and $CEL=CL$, $JEL=JL$, $BDK=BD$, $IDK=IK$ by definition.

Then: $DKL=(DID+DBD)DKL=DIDKL+DBDKL=DIKL+DBKL=DIJL++DBCL=DIJEL+DBCEL=(DIJE+DBCE)EL=DEL.$

Stage 2. We define the following class: $\chi=\{(G_1, G_2)/G_1, G_2\in FRM$ so that for every $v\in U(G_1, G_2)$ $F_1, F_2\in FRM$, $u\in U(F_1, F_2)$ and for any a_i: $F_i\to G_i$, there is the commutative diagram:

$$\begin{array}{ccc} F_1\otimes F_2 & \xrightarrow{a_1\otimes a_2} & G_1\otimes G_2 \\ \downarrow u & & \downarrow v \\ T(F_1, F_2) & \xrightarrow{T(a_1, a_2)} & T(G_1, G_2). \end{array}$$

In Stage 1 we proved that $(R, R)\in\chi$. Assume that $k>2$ and that for any $G_1, G_2\in FRM$ so that $\operatorname{rank} G_1+\operatorname{rank} G_2<k$ we have that $(G_1, G_2)\in\chi$. Take $G_1, G_2\in FRM$ so that $\operatorname{rank} G_1+\operatorname{rank} G_2=k$. Take $v\in U(G_1, G_2)$ which, without loss of generality, was defined by the following short exact split sequence when $v_A\in U(A, G_2)$, $v_R\in U(R, G_2)$

$$\begin{array}{ccccc} 0 \to A\otimes G_2 & \to & G_1\otimes G_2 & \to & R\otimes G_2 \to 0 \\ \downarrow v_A & & \downarrow & & \downarrow v_R \\ 0 \to T(A, G_2) & \to & T(G_1, G_2) & \to & T(R, G_2) \to 0. \end{array}$$

Take any $F_1, F_2\in FRM$, $u\in U(F_1, F_2)$ and a_i: $F_i\to G_i$, $i=1, 2$. We shall obtain a prism, similar to that of Stage 1, whose bottom is the last diagram and whose head is $-u\in U(F_1, F_2)$. As was shown in Stage 1, we obtain the needed commutative square.

Thus the family $U(F_1, F_2)$ has a kind of naturality property. We can show that as a matter of fact $U(F_1, F_2)$ consists of a single element and therefore $\otimes$ is equivalent to T on the category RFM.

For $F_1, F_2 \in FRM$, take any $u, v \in U(F_1, F_2)$. Take $a_i\colon F_i \to F_i$, $i=1, 2$ to be the identity map. Then the following square commutes

$$\begin{array}{ccc} F_1 \otimes F_2 & \xrightarrow{1\otimes 1} & F_1 \otimes F_2 \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ T(F_1, F_2) & \xrightarrow{T(1,1)} & T(F_1, F_2). \end{array}$$

The two horizontal arrows are the unit homomorphisms and thus $u=v$ and $|U(F_1, F_2)| \leqq 1$. As was shown, $U(F_1, F_2)$ is not empty, and thus consists of exactly one element. Then Lemma C-7 shows the following result:

LEMMA C-8. *On* $(FRM)^2$ $u\colon \otimes \to T$ *is a natural equivalence.*

We want now to extend this last result to $(RM)^2$.

LEMMA C-9. *On* $RM \times FRM$
(i) $T_1=0$,
(ii) *There is a natural equivalence* $w\colon \otimes \to T$.

PROOF OF C-9. Let $M \in RM$ and $F \in FRM$ be arbitrary. Take any free resolution of M $0 \to G_1 \xrightarrow{f} G_0 \xrightarrow{g} M \to 0$. From Lemmas C-3 and C-5 we have the following exact sequence: $0 \to T_1(M, F) \xrightarrow{\lambda} T(G_1, F) \xrightarrow{T(f,1)} T(G_0, F) \xrightarrow{T(g,1)} T(M, F) \to 0$. Using the properties of $\otimes$ we obtain:

$$\begin{array}{ccccccc} 0 \to & G_1 \otimes F & \xrightarrow{f\otimes 1} & G_0 \otimes F & \xrightarrow{g\otimes 1} & M \otimes F \to 0 \\ & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} & & \\ 0 \to T_1(M, F) \xrightarrow{\lambda} & T(G_1, F) & \xrightarrow{T(f,1)} & T(G_0, F) & \xrightarrow{T(g,1)} & T(M, F) \to 0 \end{array}$$

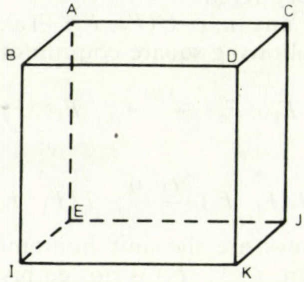
when $u\colon G_1 \otimes F \to T(G_1, F)$ and $u\colon G_0 \otimes F \to T(G_0, F)$ are those of Lemma C-8. Then the square commutes and

$$(1) \qquad T_1(M, F) \simeq \operatorname{im} \lambda \simeq \ker T(f, 1) \xleftarrow[\sim]{u} \ker f \otimes 1 = 0.$$

As for (2), the last diagram defines $w\colon M \otimes F \to T(M, F)$. It is easy to see that w is well defined and that it is bijective. Thus for any free resolution of M, there is a unique isomorphism w. Denote by $W(M, F)$ the family of all those w's. We shall show that W has the same naturality property as was proven in Lemma C-7. Let $w_i\colon M_i \otimes F_i \to T(M_i, F_i)$ be in $W(M_i, G_i)$, and take $a\colon M_1 \to M_2$ $b\colon F_1 \to F_2$. Let $0 \to G_1^1 \to G_0^1 \to M_1 \to 0$ and $0 \to G_1^2 \to G_0^2 \to M_2 \to 0$ be the free resolutions that define w_1 and w_2 respectively. Then $a\colon M_1 \to M_2$ extends to the following diagram with commutative squares:

$$\begin{array}{ccccccc} 0 \to & G_1^1 & \xrightarrow{f} & G_0^1 & \xrightarrow{g} & M_1 & \to 0 \\ & {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle a} & \\ 0 \to & G_1^2 & \xrightarrow{h} & G_0^2 & \xrightarrow{k} & M_2 & \to 0. \end{array}$$

Operating once with $-\otimes F$ and once with $T(-, F)$ we shall obtain two cubes. One of the cubes is the following:



where $A=G_0^1\otimes F_1$, $B=G_0^2\otimes F_2$, $C=M_1\otimes F_1$, $D=M_2\otimes F_2$, $E=T(G_0^1, F_1)$, $I=T(G_0^2, F_2)$, $J=T(M_1, F_1)$, $K=T(M_2, F_2)$, and: $AB=c\otimes b$, $EI=T(c, b)$, $AC=g\otimes 1$, $EJ=T(g, 1)$, $BD=k\otimes 1$, $IK=T(k, 1)$, $CD=a\otimes b$, $JK=T(a, b)$, $AE=u(G_0^1, F_1)$, $BI=u(G_0^2, F_2)$, $CJ=w_1$ and $DK=w_2$.

We have to show that $CDK=CJK$. All the other squares are commutative, because of Lemma C-7, of the definitions of w_i and because of the functorial properties of $\otimes$ and T. Using the fact that AC is surjective, we can easily deduce the needed result. Repeating the argument after the proof of Lemma C-7 we find that $|W(M, F)|=1$ for any object $(M, F)\in RM\times FRM$, and thus $W(M, F)$ is a natural equivalence.

LEMMA C-10. *In $(RM)^2$ there are natural equivalences $U\colon \otimes\to T$, $V\colon *\to T_1$.*

PROOF OF C-10. The proof is very similar to that of C-9 or of C-7 and we shall give it in brief:

For $M_1, M_2\in RM$, we take a free resolution of M_2 $0\to F_1\xrightarrow{f} F_0\xrightarrow{g} M_2\to 0$. Then we obtain the following diagram:

$$\begin{array}{ccccccccc}
0\to M_1 * M_2 & \xrightarrow{t} & M_1\otimes F_1 & \xrightarrow{1\otimes f} & M_1\otimes F_0 & \xrightarrow{1\otimes g} & M_1\otimes M_2\to 0 \\
 & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w} & & \\
0\to T_1(M_1, M_2) & \xrightarrow{\lambda} & T(M_1, F_1) & \xrightarrow{T(1,f)} & T(M_1, F_0) & \xrightarrow{T(1,g)} & T(M_1, M_2)\to 0.
\end{array}$$

This diagram defines a unique $u\colon M_1\otimes M_2\to T(M_1, M_2)$ and a unique $v\colon M_1 * M_2\to T_1(M_1, M_2)$, and it is easily seen that u and v are isomorphisms. Let the set of all the u's be $U(M_1, M_2)$ and the set of all v's be $V(M_1, M_2)$. To prove the property of naturality we proceed for U in the same fashion as in the proof of Lemma C-8, for the naturality of V, we have the same cube as in Lemma C-9, but we have to prove that $ABI=AEI$ and all the other squares are commutative. For this we use the fact that IK is injective and we show that $ABIK=AEIK$. Thus $|U|=|V|=1$ and U and V are natural equivalences.

This establishes the proof of Theorem 2.

We can summarize Theorems 1 and 2 in the following:

COROLLARY. *Let T and T_1 be functors $(RM)^2\to RM$. Then $T=\otimes$ and $T_1=*$ (up to natural equivalence) iff $R\otimes R\xrightarrow{\sim} T(R, R)$ naturally, $T_1(R, R)=0$ and T and T_1 obey the interchanging sequence.*

An example concerning an additional requirement

This example shows that without the condition $R\otimes R \not\to T(R, R)$ we cannot expect to prove the converse theorem.

Indeed, let $M\in RM$ be fixed and define:

$$T(A, B) = A\otimes B\otimes M, \quad T_1(A, B) = (A * B)\otimes M, \quad A, B\in RM.$$

Then $T(R, R)=M$ (naturally) and $T_1(R, R)=0$.

Also, for K^1 and $K^2\in RC$

$$\big(T(H(K^1), H(K^2))\big)_n = \bigoplus_{i+j=n} H_i(K^1)\otimes H_j(K^2)\otimes M,$$

$$\big(T_1(H(K^1), H(K^2))\big)_n = \bigoplus_{i+j=n} \big(H_i(K^1) * H_j(K^2)\big)\otimes M.$$

Assume further that M is flat. Then for any $K^1, K^2\in RC$ $H_n\big(T(K^1, K^2)\big)= =H_n(K^1\otimes K^2\otimes M)=\big(H_n(K^1\otimes K^2)\otimes M\big)\oplus\big(H_{n-1}(K^1\otimes K^2) * M\big)=\big(H_n(K^1\otimes K^2)\big)\otimes M$. Also $H_n\big(T_1(K^1, K^2)\big)=H_n\big((K^1 * K^2)\otimes M\big)=H_n(K^1 * K^2)\otimes M$.

M is a flat module and flat modules preserve exact sequences. The terms of the interchanging sequence of T and T_1 are those corresponding to them in the interchanging sequence of $\otimes$ and $*$, respectively, all tensor multiplied by M. Thus T and T_1 obey the interchanging sequence. $T_1(R, R)=0$ but $T(R, R)=M$. M is not necessarily R.

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THE REFLEXIVE DIMENSION OF AN R-SPACE

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Let X be a Banach space. Define $Q^0(X)=X$, $Q(X)=X^{**}/X$, and inductively, $Q^m(X)=Q\big(Q^{m-1}(X)\big)$. A Banach space X will be called an $R(a)$-space, if $Q^a(X)$ is a finite dimensional space; it is an *R-space,* if it is an $R(a)$-space for some $a\geqq 0$. If X is an $R(a)$-space but not an $R(a-1)$-space (there are no $R(-1)$-spaces), then a will be called the *reflexive dimension* of X. If X is an R-space, then the pair of natural numbers (a, b), where a is the reflexive dimension of X and b is the (ordinary) dimension of $Q^a(X)$, measures the complexity of the space X. This paper is devoted to the study of this invariant and related topics.

1. Preliminaries

Let $k=$ either the real field or the complex field; $B=$ the category whose objects are Banach spaces over k and whose morphisms are continuous linear operators; $B(X, Y)=$ the set of all continuous linear operators from X to Y. $|T|=\sup\limits_{|x|\leqq 1}|T(x)|$, where $T\in B(X, Y)$, $X^*=B(X, k)$, $T^*=B(T, k)$, $I_X=$ the identity operator on X.

The sequence of continuous operators

$$X_1 \xrightarrow{T_1} X_2 \xrightarrow{T_2} X_3 \xrightarrow{T_3} \dots X_n \xrightarrow{T_n} X_{n+1}$$

is *exact,* if $T_i(X_i)=\operatorname{Ker}(T_{i+1})$, $i=1, 2, \dots, n-1$.

For any X in B, there is a natural injection $n_X\colon X\to X^{**}$ defined by $\big(n_X(x)\big)(x^*)=x^*(x)$, for all $x\in X$ and $x^*\in X^*$. We shall identify X with $n_X(X)$. Let $Q(X)=X^{**}/X$. If $T\in B(X, Y)$, $Q(T)\in B\big(Q(X), Q(Y)\big)$ is uniquely defined by the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & X & \to & X^{**} & \to & Q(X) & \to & 0 \\ & & {\scriptstyle T}\downarrow & & {\scriptstyle T^{**}}\downarrow & & {\scriptstyle Q(T)}\downarrow & & \\ 0 & \to & Y & \to & Y^{**} & \to & Q(Y) & \to & 0. \end{array}$$

Theorems (1.1)—(1.5) in the following are all direct consequences of results in [7, Section 2]. $(m\geqq 0)$.

THEOREM (1.1). (i) $Q^m(I_X)=I_{Q^m(x)}$,

(ii) *if* $S\in B(X,Y)$ *and* $T\in B(Y,Z)$, *then* $Q^m(TS)=Q^m(T)Q^m(S)$,

(iii) $|Q^m(T)|\leqq |T|$,

(iv) *for any* s, t *in* k, *and any* S, T *in* $B(X,Y)$, $Q^m(sS+tT)=sQ^m(S)+tQ^m(T)$,

(v) *if* $0\to X\to Y\to Z\to 0$ *is exact, then* $0\to Q^m(X)\to Q^m(Y)\to Q^m(Z)\to 0$ *is exact.*

THEOREM (1.2). *If* $T\in B(X,Y)$ *has a closed range, then*

(i) $\operatorname{Ker}(Q^m(T))\simeq Q^m(\operatorname{Ker}(T))$,

(ii) $\operatorname{Coker}(Q^m(T))\simeq Q^m(\operatorname{Coker}(T))$.

THEOREM (1.3). *If* $S\in B(X,Y)$, $T\in B(Y,Z)$ *and* TS *have closed ranges, then the following sequence of Banach spaces is exact:*

$$0\to Q^m(\operatorname{Ker}S)\to Q^m(\operatorname{Ker}TS)\to Q^m(\operatorname{Ker}T)\to Q^m(\operatorname{Coker}S)\to Q^m(\operatorname{Coker}TS)\to$$
$$\to Q^m(\operatorname{Coker}T)\to 0.$$

THEOREM (1.4). *If* X *is a Banach space, then* $(Q^m(X))^*\simeq Q^m(X^*)$.

THEOREM (1.5). *If* $T\in B(X,X_1)$, $U\in B(Y,Y_1)$, $V\in B(Z,Z_1)$ *all have closed ranges, and if the following commutative diagram in* B *has exact rows*

$$\begin{array}{ccccccccc}
0 & \to & X & \to & Y & \to & Z & \to & 0\\
 & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle V} & & \\
0 & \to & X_1 & \to & Y_1 & \to & Z_1 & \to & 0
\end{array}$$

then there exists a natural continuous linear map d *such that* $0\to Q^m(\operatorname{Ker}T)\to$ $\to Q^m(\operatorname{Ker}U)\to Q^m(\operatorname{Ker}V)\xrightarrow{Q^m(d)} Q^m(\operatorname{Coker}T)\to Q^m(\operatorname{Coker}U)\to Q^m(\operatorname{Coker}V)\to 0$ *is exact.*

2. The generalized dimension of an R-space

Let X be an R-space with reflexive dimension a, and let b be the (ordinary) dimension of $Q^a(X)$. We shall call the pair of natural numbers (a,b) the *generalized dimension* of X and denote it by $\operatorname{Dim}(X)$ (ordinary dimension will be denoted by dim as usual). So, if $\operatorname{Dim}(X)=(0,n)$, X is a (finite) n-dimensional space. If $\operatorname{Dim}(X)=(1,0)$, X is an infinite dimensional reflexive space. If $\operatorname{Dim}(X)=(1,n)$, then X is a quasi-reflexive space in the sense of [1]. Banach spaces X with $\operatorname{Dim}(X)=(2,0)$ can be found in [2, Section 4, Proposition 1], [3], [5].

Now we define addition of generalized dimensions (a,b) and (a',b') in $\mathbf{N}\times\mathbf{N}$ ($\mathbf{N}=\{0,1,2,\dots\}$) by

$$(a,b)+(a',b')=\begin{cases}(a,b), & \text{if } a>a'\\ (a,b+b'), & \text{if } a=a'\\ (a',b'), & \text{if } a<a'.\end{cases}$$

THEOREM (2.1). *Let*

$$0 \to X \to Y \to Z \to 0$$

be an exact sequence in B. Then, Y is an R-space if and only if X and Z are R-spaces, and if Y is an R-space

$$\mathrm{Dim}\,(Y) = \mathrm{Dim}\,(X) + \mathrm{Dim}\,(Z).$$

PROOF. All except the generalized dimension formula follow from Theorem (1.1) part (v). If $\mathrm{Dim}\,(Y)=(a, b)$, then the same theorem implies that the sequence

$$0 \to Q^a(X) \to Q^a(Y) \to Q^a(Z) \to 0$$

is exact and we see the reflexive dimensions of X and Z are both $\leqq a$. If the reflexive dimension of X is $<a$, then $Q^a(X)=0$ and we have $\mathrm{Dim}\,(Y)=\mathrm{Dim}\,(Z)=\mathrm{Dim}\,(X)+ +\mathrm{Dim}\,(Z)$. If the reflexive dimension of Z is $<a$, then $Q^a(Z)=0$ and we have $\mathrm{Dim}\,(Y)=\mathrm{Dim}\,(X)=\mathrm{Dim}\,(X)+\mathrm{Dim}\,(Z)$. Finally, if the reflexive dimensions of X and Z are both $= a$, then the above exact sequence implies $\mathrm{Dim}\,(Y)=\mathrm{Dim}\,(X)+ +\mathrm{Dim}\,(Z)$. Hence in all cases the formula holds. Q.E.D.

Clearly, the above formula is a generalization of the usual dimension formula and [7, Theorem (3.2)] and [1, Corollary (4.2)].

If we define the (linear) order relation on $\mathbf{N}\times\mathbf{N}$ by declaring that $(a, b)\leqq(c, d)$, if there exists $(e, f)\in\mathbf{N}\times\mathbf{N}$ such that $(a, b)+(e, f)=(c, d)$, then we see $\mathrm{Dim}\,(X)\leqq \leqq\mathrm{Dim}\,(Y)$ and $\mathrm{Dim}\,(Z)\leqq\mathrm{Dim}\,(Y)$.

COROLLARY (2.2). *Let*

$$0 \to X \to Y \to Z \to 0$$

be an exact sequence in B. Then,

(i) *Y is quasi-reflexive if and only if X and Z are quasi-reflexive.*

(ii) *Y is reflexive if and only if X and Z are reflexive.*

PROOF. Both statements follow from the generalized dimension formula. Q.E.D.

THEOREM (2.3). *A Banach space X is an R-space if and only if X^* is an R-space. Furthermore,* $\mathrm{Dim}\,(X)=\mathrm{Dim}\,(X^*)$.

PROOF. This theorem is an immediate consequence of Theorem (1.4). Q.E.D.

COROLLARY (2.4). (i) *X is reflexive if and only if X^* is reflexive,*

(ii) *X is quasi-reflexive if and only if X^* is quasi-reflexive.*

PROOF. Both assertions follow from the equation $\mathrm{Dim}\,(X)=\mathrm{Dim}\,(X^*)$. Q.E.D. Corollary (2.4) part (ii) is the same as [1, Lemma (3.4)].

3. The generalized index of an F-operator

Let $T\in B(X, Y)$ be an operator with a closed range. T will be called an *$F(i;\, k, c)$-operator*, if $\mathrm{Dim}\,(\mathrm{Ker}\, T)\leqq(i, k)$ and $\mathrm{Dim}\,(\mathrm{Coker}\, T)\leqq(i, c)$. An operator is an *$F(i)$-operator* if it is an $F(i;\, k, c)$-operator for some k and c, and it is an *F-operator,* if it is an $F(i)$-operator for some $i\geqq 0$. If T is an $F(i)$-operator but not an $F(i-1)$-operator (there are no $F(-1)$-operators), then i will be called the *reflexive index* of T. Let T be an operator with reflexive index i. Let j=*index of* $Q^i(T)=\dim(\operatorname{Ker}(Q^i(T)))-\dim(\operatorname{Coker}(Q^i(T)))$. We shall call the pair (i, j) the *generalized index* of T and we shall denote it by $\operatorname{Ind}(T)$. Notice that an operator T is an $F(0)$-operator if and only if it is a Fredholm operator; its generalized index $\operatorname{Ind}(T)=(0,$ *(ordinary) index of* $T)$. An operator is an $F(1;\,0,0)$-operator if and only if it is a generalized Fredholm operator [7].

Addition of generalized indices mimics that of generalized dimensions; we define for (i, j) and (i', j') in $\mathbf{N}\times\mathbf{Z}$ ($\mathbf{Z}=\{0, \pm 1, \pm 2, \ldots\}$),

$$(i,j)+(i',j') = \begin{cases} (i,j), & \text{if } \; i>i' \\ (i,j+j'), & \text{if } \; i=i' \\ (i',j'), & \text{if } \; i<i'. \end{cases}$$

THEOREM (3.1). *Suppose $S\in B(X,Y)$, $T\in B(Y,Z)$ and TS are operators with closed ranges. Then, if S is an $F(a;\,k,c)$-operator and T is an $F(a;\,k',c')$-operator, then TS is an $F(a;\,k+k',c+c')$-operator. Furthermore, the equality*

$$\operatorname{Ind}(TS) = \operatorname{Ind}(S)+\operatorname{Ind}(T)$$

holds in all but the case [(reflexive index of S)=(reflexive index of T)>(reflexive index of TS)].

PROOF. By Theorem (1.3),

$$0 \to Q^a(\operatorname{Ker} S) \to Q^a(\operatorname{Ker} TS) \to Q^a(\operatorname{Ker} T) \to Q^a(\operatorname{Coker} S) \to Q^a(\operatorname{Coker} TS) \to$$
$$\to Q^a(\operatorname{Coker} T) \to 0$$

is exact. Hence if S is an $F(a;\,k,c)$-operator and T is an $F(a;\,k',c')$-operator, then

$$\dim(Q^a(\operatorname{Ker} S)) \leqq k, \quad \dim(Q^a(\operatorname{Ker} T)) \leqq k',$$

$$\dim(Q^a(\operatorname{Coker} S)) \leqq c, \quad \dim(Q^a(\operatorname{Coker} T)) \leqq c',$$

whence

$$\dim(Q^a(\operatorname{Ker} TS)) \leqq k+k', \quad \dim(Q^a(\operatorname{Coker} TS)) \leqq c+c'.$$

This shows that TS is an $F(a;\,k+k',c+c')$-operator.

To show the generalized index formula, we suppose $\operatorname{Ind}(S)=(i,j)$ and $\operatorname{Ind}(T)=(i',j')$. Let $m=\max(i,i')$. Then, by Theorems (1.2) and (1.3),

$$0 \to \operatorname{Ker}(Q^m(S)) \to \operatorname{Ker}(Q^m(TS)) \to \operatorname{Ker}(Q^m(T)) \to$$

$$\to \operatorname{Coker}(Q^m(S)) \to \operatorname{Coker}(Q^m(TS)) \to \operatorname{Coker}(Q^m(T)) \to 0$$

is exact.

Case 1. $i<m(=i')$. In this case, the reflexive index of TS is equal to the reflexive index of T (which is $=m$), and $\operatorname{Ker}(Q^m(S))=0$ and $\operatorname{Coker}(Q^m(S))=0$, whence $\operatorname{Ind}(TS)=\operatorname{Ind}(T)=\operatorname{Ind}(S)+\operatorname{Ind}(T)$.

Case 2. $i'<i(=m)$. As in Case 1, $\operatorname{Ind}(TS)=\operatorname{Ind}(S)=\operatorname{Ind}(S)+\operatorname{Ind}(T)$.

Case 3. $i=i'(=m)$. The exact sequence yields that the reflexive index of TS is $\leqq m$. If $m=0$, the reflexive index of TS is equal to 0, and the equality is just the (ordinary) index formula for Fredholm operators. If $m>0$, the equality needs to be shown only in case the reflexive index of TS is equal to m; but in this case it is an immediate consequence of the same exact sequence. Q.E.D.

Theorem (3.1) contains as special cases [7, Theorem (5.3)] and the usual index formula for Fredholm operators [7, (A.6)].

The case excluded by Theorem (3.1) does indeed occur as can be seen from the following example. Let G be an arbitrary Banach space and let H be an infinite dimensional reflexive Banach space (Hilbert space, for instance). Let $S\colon G\to G\oplus H$ be the operator defined by $S(g)=(g, 0)$ and $T\colon G\oplus H\to G$ be the operator $T(g, h)=g$. Then $S, T, TS\ (=I_G)$ all have closed ranges. The reflexive index of TS is equal to 0, while the reflexive indices of S and T are equal to 1.

We note that in case $i=i'=m$ (Theorem (3.1), Case 3), the reflexive index of TS *is* equal to m, if the reflexive dimension of Ker S is equal to m, or if the reflexive dimension of Coker T is equal to m.

4. W-operators and other operators

An operator $T\in B(X, Y)$ will be called a *$W(d)$-operator*, if $Q^d(T)=0$; it will be called a *W-operator*, if it is a $W(d)$-operator for some $d\geqq 0$. If T is a $W(d)$-operator but not a $W(d-1)$-operator (there are no $W(-1)$-operators), we shall write $d=\operatorname{Deg}(T)$. For instance, if $\operatorname{Deg}(T)=0$, then $T=0$; if $\operatorname{Deg}(T)=1$, then T is a non-zero weakly compact operator.

An operator $T\in B(X, Y)$ will be called an *$F(a;\ k,\ -)$-operator*, if $\operatorname{Dim}(\operatorname{Ker} T)\leqq (a, k)$. Note that an operator T is an $F(1;\ 0, -)$-operator if and only if it is a Tauberian operator [4].

An operator $T\in B(X, Y)$ will be called an *$F(a;\ -,\ c)$-operator* if it has a closed range and if $\operatorname{Dim}(\operatorname{Coker} T)\leqq (a, c)$. Note that an operator T is an $F(1;\ -, 0)$-operator if and only if it is a co-Tauberian operator [7, p. 322].

Clearly, an operator is an $F(a;\ k, c)$-operator if and only if it is an $F(a;\ k, -)$-operator and an $F(a;\ -, c)$-operator.

THEOREM (4.1). *If X is an R-space and* $\operatorname{Dim}(X)\leqq (d, 0)$, *or Y is an R-space and* $\operatorname{Dim}(Y)\leqq (d, 0)$, *then every $T\in B(X, Y)$ is a $W(d)$-operator.*

PROOF. In $Q^d(T)\colon Q^d(X)\to Q^d(Y)$, either $Q^d(X)=0$ or $Q^d(Y)=0$. Q.E.D.

Theorem (4.1) is a generalization of [7, Corollary (4.2)].

THEOREM (4.2). *Let $T\in B(X, Y)$ be an operator with a closed range. Then,* $\operatorname{Deg}(T)=d$ *if and only if* $\operatorname{Dim}(T(X))=(d, 0)$ *or* $(d-1, n)$ $(d\neq 0)$ *where* $n>0$.

PROOF. Since T has a closed range, we may assume without loss of generality that $T\colon X\to Y$ is onto. Then, by Theorem (1.2), $Q^d(T)\colon Q^d(X)\to Q^d(Y)$ is also onto. If $\operatorname{Dim}(T(X))=\operatorname{Dim}(Y)=(d, 0)$ or $=(d-1, n)$ $(d\neq 0)$ where $n>0$, then clearly $Q^d(T)=0$ and $Q^{d-1}(T)\neq 0$ (if $d\neq 0$), whence $\operatorname{Deg}(T)=d$. Conversely, suppose $\operatorname{Deg}(T)=d$. Then we have two cases:

Case 1. $d=0$. In this case, $T=0$ and $\mathrm{Dim}\,(T(X))=(0, 0)$.

CASE 2. $d>0$. In this case, $Q^d(T(X))=0$ but $Q^{d-1}(T(X))\neq 0$. If $Q^{d-1}(T(X))$ is infinite dimensional, then $\mathrm{Dim}\,(T(X))=(d, 0)$. If $Q^{d-1}(T(X))$ is (non-zero) finite dimensional, then $\mathrm{Dim}\,(T(X))=(d-1, n)$ where $n>0$. Q.E.D.

Theorem (4.2) is a generalization of [7, Theorem (4.5)].

COROLLARY (4.3). *For any Banach space X,* $\mathrm{Deg}\,(I_X)=d$ *if and only if* $\mathrm{Dim}\,(X)==(d, 0)$ *or* $=(d-1, n)$ $(d\neq 0)$ *where* $n>0$.

THEOREM (4.4). *Let T and K be operators in* $B(X, Y)$. *If T is an* $F(i;\, k, -)$-*operator (respectively, an* $F(i;\, -, c)$-*operator, or an* $F(i;\, k, c)$-*operator) and K is a* $W(j)$-*operator where* $j\leqq i$, *then* $T+K$ *is an* $F(i;\, k, -)$-*operator (respectively, an* $F(i;\, -, c)$-*operator, or an* $F(i;\, k, c)$-*operator, provided* $T+K$ *has a closed range).*

PROOF. $Q^i(T+K)=Q^i(T)$. Q.E.D.

Theorem (4.4) is a generalization of [7, Theorems (5.7), (6.2), (6.5)].

THEOREM (4.5). (i) *If* $S\in B(X, Y)$ *is an* $F(i;\, k, -)$-*operator and* $T\in B(Y, Z)$ *is an* $F(i;\, k', -)$-*operator, then* TS *is an* $F(i;\, k+k', -)$-*operator.*

(ii) *If* $S\in B(X, Y)$ *is an* $F(i;\, -, c)$-*operator, and* $T\in B(Y, Z)$ *is an* $F(i;\, -, c')$-*operator, then* TS *is an* $F(i;\, -, c+c')$-*operator, provided* TS *has a closed range.*

PROOF. (i) follows from the exactness of the sequence

$$0 \to Q^i(\mathrm{Ker}\, S) \to Q^i(\mathrm{Ker}\, TS) \to Q^i(\mathrm{Ker}\, T).$$

(ii) follows from the exactness of the sequence.

$$Q^i(\mathrm{Coker}\, S) \to Q^i(\mathrm{Coker}\, TS) \to Q^i(\mathrm{Coker}\, T) \to 0.$$

Q.E.D.

Theorem (4.5) is a generalization of [7, Theorems (6.3) and (6.6)].

THEOREM (4.6). *Let* $S\in B(X, Y)$, $T\in B(Y, Z)$ *be operators with closed ranges. Suppose TS is an* $F(i;\, k, c)$-*operators. Then,*

(i) *if S is an* $F(i;\, -, c')$-*operator, then T is an* $F(i;\, k+c', c)$-*operator,*

(ii) *if T is an* $F(i;\, k'', -)$-*operator, then S is an* $F(i;\, k, k''+c)$-*operator,*

(iii) *if* $\mathrm{Dim}\,(\ker T)\leqq(i, l)$, *then S is an* $F(i, k, c+l)$-*operator and T is an* $F(i;\, l, c)$-*operator,*

(iv) *if* $\mathrm{Dim}\,(\mathrm{Coker}\, T)\leqq(i, m)$, *then S is an* $F(i, k, m)$-*operator and T is an* $F(i;\, k+m, c)$-*operator.*

PROOF. Use Theorem (1.3) and reason as the first part of Theorem (3.1). Q.E.D.

Theorem (4.6) is a generalization of [7, Theorem (5.4) and (A.2)].

THEOREM (4.7). *Let* $T\in B(X, Y)$ *be an operator with a closed range. If there exist operators* $S, S'\in B(Y, X)$ *with closed ranges such that* TS' *is an* $F(i;\, -, c)$-*operator, and ST is an* $F(i, k, -)$-*operator, then T is an* $F(i;\, k, c)$-*operator.*

PROOF. Theorem (1.3). Q.E.D.

Theorem (4.7) is a generalization of [7, Theorem (5.5) and (A.3)].

THEOREM (4.8). *Same assumption as in Theorem* (1.5). *Then,*

(i) *if* T *is* $F(i\colon -, c)$ *and* U *is* $F(i;\ k', c')$, *then* V *is* $F(i;\ k'+c, c')$,
(ii) *if* U *is* $F(i;\ k', c')$ *and* V *is* $F(i;\ k'', -)$, *then* T *is* $F(i;\ k', k''+c')$,
(iii) *if* T *is* $F(i;\ k, c)$ *and* V *is* $F(i;\ k'', c'')$, *then* U *is* $F(i;\ k+k'', c+c'')$,
(iv) *if* U *is* $F(i;\ k', c')$, *then* T *is* $F(i;\ k', -)$ *and* V *is* $F(i;\ -, c')$.

PROOF. Theorem (1.5). Q.E.D.

Theorem (4.8) is a generalization of [7, Theorems (5.9), (6.8) and (A.11)].

THEOREM (4.9). *Let* $d \geqq 0$.

(i) *The subset of* $W(d)$*-operators in* $B(X, Y)$ *forms a norm-closed linear subspace of* $B(X, Y)$.

(ii) *The class of all* $W(d)$*-operators is a "two-sided ideal" in the sense of* [6, *p.* 17].

PROOF. (i) is implied by Theorem (1.1) parts (iii) and (iv). (ii) is a consequence of part (ii) of the same theorem. Q.E.D.

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THEOREM (4.8). *Same assumption as in Theorem (4.6). Then:*

(i) *If* $U \in F(i, [illegible])$ *and* $U' \in F(i', [illegible])$ *then* [illegible]

(ii) *If* $U \in F(i, [illegible])$ *and* $V \in F(j, [illegible])$ [illegible]

(iii) *If* $T \in F(i, [illegible])$ *and* [illegible] *then* [illegible]

(iv) *If* $U \in F(i, [illegible])$ *then* [illegible]

PROOF. Theorem (4.3). Q.E.D.

Theorem (4.8) is a generalization of [7, Theorem (5.4), (iv) and (v)].

THEOREM (4.9). *Let* [illegible]

(i) *The subset of* [illegible] *of* $B(X, Y)$ *forms a norm-closed* [illegible] *of* $B(X, Y)$.

(ii) *The* [illegible] *is a* "[illegible]" *in the sense of* [illegible]

PROOF. (i) is implied by Theorem (4.1) part (iii) and (iv); (ii) is a consequence of part (ii) of the same theorem. Q.E.D.

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[illegible]

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ON THE MAXIMAL VALUE OF ADDITIVE FUNCTIONS IN SHORT INTERVALS AND ON SOME RELATED QUESTIONS

By

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1. Let (a, b) and $[a, b]$ be the greatest common divisor and the least common multiple of a and b, respectively. p_n denotes the n'th prime; $p, q, q_1, q_2, \ldots$ are prime numbers. A sum $\sum_p$ and a product $\prod_p$ denote a summation and a multiplication, respectively, over primes indicated. The symbol $\#\{\ldots\}$ denotes the number of elements indicated in the bracket $\{\ \}$. P_μ is the product of the first μ primes.

The aim of this paper is to continue our investigation on the distribution of the maximal value of additive functions in small intervals.

In the sequel let $g(n)$ be a non-negative strongly additive function,

$$f_k(n) = \max_{j=1,\ldots,k} g(n+j). \tag{1.1}$$

Let

$$\varrho(k, \varepsilon) = \sup_{x \geqq 1} \frac{1}{x} \#\{n \leqq x \mid f_k(n) > (1+\varepsilon) f_k(0)\}, \tag{1.2}$$

$$\delta(k_0, \varepsilon) = \sup_{x \geqq 1} \frac{1}{x} \#\{n \leqq x \mid \exists k, k > k_0, f_k(n) > (1+\varepsilon) f_k(0)\}, \tag{1.3}$$

$$\theta(k, \varepsilon) = \limsup_{x=\infty} \frac{1}{x} \#\{n \leqq x \mid f_k(n) > f_k(0)(1+\varepsilon)\}.$$

It is obvious that

$$\theta(k, \varepsilon) \leqq \varrho(k, \varepsilon), \tag{1.4}$$

and that

$$\delta(k_0, \varepsilon) \geqq \sup_{k \geqq k_0} \varrho(k, \varepsilon). \tag{1.5}$$

In [1] we tried to determine those additive $g(n)$ for which the relation

$$\delta(k_0, \varepsilon) \to 0 \quad (k_0 \to \infty), \quad \forall \varepsilon > 0 \tag{1.6}$$

holds. There we noticed that (1.6) implies

$$\sum_p \frac{\min(1, g(p))}{p} < \infty, \tag{1.7}$$

but we could not decide if the condition

$$\sum_p \frac{g(p)}{p} < \infty \tag{1.8}$$

were necessary. Now we shall prove this. More exactly, we shall prove the following assertion.

THEOREM 1. *If*

$$\theta(k, \varepsilon) \to 0 \quad (k \to \infty) \tag{1.9}$$

for all $\varepsilon > 0$, *then*

$$\sum_p \frac{g(p)^r}{p} < \infty, \tag{1.10}$$

for every $r \geqq 1$.

Let $F(x)$ be the limit distribution function of $g(n)$, the existence of which is guaranteed by (1.7).

THEOREM 1′. *Assume that*

$$k\big(1 - F(f_k(0)(1+\varepsilon))\big) \to 0 \tag{1.11}$$

holds for every $\varepsilon > 0$. *Then* (1.10) *holds for every* $r \geqq 1$.

Theorem 1 is an immediate consequence of Theorem 1′. Indeed, (1.11) implies that the density of integers n, satisfying $g(n) > (1+\varepsilon) f_k(0)$ is $o(1/k)$, consequently (1.9) holds.

Perhaps (1.11) implies that

$$\sum_p \frac{e^{ug(p)} - 1}{p} < \infty \tag{1.12}$$

for every $u > 0$. We could not give a counter example.

THEOREM 2. *If for some constant* $A > 0$

$$k\big(1 - F(f_k(0) + A)\big) \to 0 \quad (k \to \infty), \tag{1.13}$$

thne (1.12) *holds for every* $u > 0$.

On the other hand, we shall prove that (1.6) does not imply $g(p) = O(1)$. This will follow easily from the following

THEOREM 3. *Let* $L(k)$ *be a function on* $[1, \infty)$ *tending to infinity arbitrary slowly. Then there exists a strongly additive non-negative* $g(n)$ *with* $\overline{\lim}\, g(p) = \infty$, *so that*

$$\sup_{x \geqq 1} \frac{1}{x} \# \{n \leqq x \mid \exists k \geqq k_0, f_k(n) > L(k)\} \to 0 \quad (k_0 \to \infty). \tag{1.14}$$

We are interested in the conditions that imply

$$\sup_{x \geqq 1} \frac{1}{x} \# \{n \leqq x \mid \exists k > k_0, f_k(n) > f_k(0) + A\} \to 0 \quad (k_0 \to \infty), \tag{1.15}$$

with some suitable constant A.

THEOREM 4. *If* $g(p) = \dfrac{1}{p}$, *then*

$$\sup_{x \geqq 1} \frac{1}{x} \# \{n \leqq x \mid \exists k > k_0, f_k(n) > f_k(0) + \lambda_k\} \to 0 \quad (k_0 \to \infty), \tag{1.16}$$

where $\lambda_k = 3/(\log \log k)$.

THEOREM 5. *If* $g(p)=1/p^\delta$, $0<\delta<1$, $\varrho>0$ *being an arbitrary constant, then*

$$(1.17)\qquad \lim_{k\to\infty}\varliminf_{x=\infty}\frac{1}{x}\ \#\{n\leqq x|f_k(n)>f_k(0)+(\log k)^{1-\delta-\varrho}\}=1.$$

By somewhat more trouble we could prove that

$$(1.18)\qquad \sup_{x\geqq 1}\frac{1}{x}\ \#\{n\leqq x|\exists k>k_0, f_k(n)<f_k(0)+(\log k)^{1-\delta-\varrho}\}\to 0,$$

as $k_0\to\infty$.

Let $F_\delta(x)$, $F_\gamma(x)$ denote the limit distribution functions corresponding to $g(p)=1/p^\delta$, $g(p)=(\log p)^{-\gamma}$, respectively; $G_\delta(x)=1-F_\delta(x)$, $G_\gamma(x)=1-F_\gamma(x)$.

We shall consider $G(x)$ for large $x(>0)$.

THEOREM 6. *We have for* $\delta=1$:

$$(1.19)\qquad \log\log\frac{1}{G_1(\tau)}\geqq e^{\tau-a}-c\tau^2e^{-\tau},$$

where $a=\gamma-\sum_{k\geqq 2}\sum_p\frac{1}{kp^k}$; γ *being Euler's constant, c denotes a suitable constant.*

Furthermore, if $0<\delta<1$,

$$(1.20)\qquad \log\frac{1}{G_\delta(\tau)}\geqq(\tau\log\tau)^{1/(1-\delta)}\big(1+O(\log\tau)^{-1}\big)\quad(\tau>1),$$

and

$$(1.21)\qquad \log\frac{1}{G_\gamma(\tau)}\geqq\tau(\log\tau)^{\gamma+1}-c_1\tau(\log\tau)^\gamma,$$

c_1 being a positive constant depending on γ.

REMARK. It is easy to see that the previous inequalities are quite sharp. Indeed, if g is monotonically decreasing on the set of primes p, then for $P_\mu\leqq k<P_{\mu+1}$ we have

$$1-F\big(g(P_\mu)\big)\geqq\frac{1}{P_\mu}\geqq\frac{1}{k}.$$

Hence, after some simple computation, we have the following inequalities for $\tau>1$:

(i) $\log\log\frac{1}{G_{\delta=1}(\tau)}\leqq e^{\tau-a}+O(e^{-B\tau})$, B being an arbitrary but fixed number;

(ii) $\log\frac{1}{G_\delta(\tau)}\leqq(\tau\log\tau)^{1/(1-\delta)}\big(1+O((\log\tau)^{-1})\big)$, if $0<\delta<1$;

(iii) $\log\frac{1}{G_\gamma(\tau)}\leqq\tau(\log\tau)^{\gamma+1}\big(1+O((\log\tau)^{-1})\big)$.

Let now

$$(1.22)\qquad \sum_p \frac{g(p)}{p} = \infty; \quad \sum_p \frac{g^2(p)}{p} < \infty,$$

$$(1.23)\qquad A_x = \sum_{p \leqq x} \frac{g(p)}{p};$$

$$(1.24)\qquad \psi(y) = \sum_{p \leqq y} g(p),$$

$$(1.25)\qquad F_k(n) = \max_{1 \leqq j \leqq k} \{g(n+j) - A_{n+j}\}.$$

THEOREM 7. *Let* $0 < t(x)$ *monotonically tend to zero in* $[1, \infty)$, *let* $g(n)$ *be strongly additive defined for primes* p *by* $g(p) = t(p)$. *If* (1.22) *holds, then for every fixed* k, $P_\mu \leqq k < P_{\mu+1}$, *we have*

$$(1.26)\qquad F_k(n) \geqq \psi(P_\mu) + A_{\log k} - \varepsilon_k$$

for every but $O(\delta_k x)$ *of* $n \leqq x$; $\varepsilon_k \to 0$, $\delta_k \to 0$ *as* $k \to \infty$.

Suppose, in addition, that

$$(1.27)\qquad \lim_{y \to \infty} \frac{\psi(y)}{yt(e^{e^{y^\delta}})} = \infty$$

for every $\delta > 0$, *and that*

$$(1.28)\qquad \sum_{p > y} \frac{t^2(p)}{p} \ll t^2(y)(\log \log y)^\gamma \quad (y \to \infty)$$

for a suitable $\gamma > 0$. *Then*

$$(1.29)\qquad \lim_{k_0 \to \infty} \sup_{x \geqq 1} \frac{1}{x} \# \left\{ n \leqq x \,|\, \exists k > k_0, \left| \frac{F_k(n)}{\psi(\log k)} - 1 \right| \geqq \varepsilon \right\} = 0,$$

for every $\varepsilon > 0$.

2. Asymptotic of distribution functions for large values. Let $g(n) \geqq 0$ be strongly additive. Then for every $u \geqq 0$

$$(2.1)\qquad \sum_{n \leqq x} e^{ug(n)} \leqq x \prod_{p \leqq x} \left(1 + \frac{e^{ug(p)} - 1}{p}\right).$$

As it is well known

$$(2.2)\qquad \frac{1}{x} \sum_{n \leqq x} e^{ug(n)} \to K(u) = \prod_p \left(1 + \frac{e^{ug(p)} - 1}{p}\right),$$

if the infinite product on the right hand side converges. Let $F(\tau)$ be the distribution function of $g(n)$. Then

$$(2.3)\qquad 1 - F(\tau) \leqq K(u) e^{-u\tau} \quad (0 < u < \infty).$$

By choosing u appropriately, we shall use (2.3) to give an upper estimate for $G(\tau) = 1 - F(\tau)$ for some special additive functions.

Let $t(x)$, $x\in[1,\infty)$, tend to zero monotonically, $g(p)=t(p)$ for primes p, $\psi(y)=\sum_{p\leqq y} t(p)$. Suppose that $t(x)$ is differentiable.

Let the values t_0, t_1 be defined by the relations

$$ut(t_0)=\log t_0+H;\quad ut(t_1)=\log t_1-H, \tag{2.4}$$

where $H>1$. Let

$$K(u)=K_1(u)K_2(u)K_3(u),$$

where in $K_i(u)$ $(i=1,2,3)$ the product is extended over the primes in the intervals $(1,t_0]$, $(t_0,t_1]$, (t_1,∞), respectively.

For $p\in(1,t_0)$ we use the inequality

$$\log\left(1+\frac{e^{ug(p)}-1}{p}\right)<\log\frac{e^{ug(p)}}{p}+e^{-ug(p)}p\leqq ug(p)-\log p+e^{-H},$$

and deduce

$$\log K_1(u)<u\psi(t_0)-\sum_{p\leqq t_0}\log p+\sum_{p\leqq t_0} pe^{-ug(p)}. \tag{2.5}$$

Since

$$1+\frac{e^{ug(p)}-1}{p}\leqq 1-\frac{1}{p}+e^H<e^{H+1}$$

in $p\in(t_0,t_1]$, therefore

$$\log K_2(u)<(H+1)\big(\pi(t_1)-\pi(t_0)\big). \tag{2.6}$$

Furthermore

$$\log K_3(u)<\sum_{p>t_1}\frac{e^{ug(p)}-1}{p}. \tag{2.7}$$

We shall give an upper estimate for the right hand side of the last inequality when $t(x)=x^{-\delta}$ $(0<\delta\leqq 1)$; $t(x)=(\log x)^{-\gamma}$. For this we use the prime number theorem in the form

$$\pi(x)=\operatorname{li} x+R(x),\quad |R(x)|\leqq c_2x(\log x)^{-c_3},$$

where c_3 is a large constant. Let

$$f(x)=\frac{e^{ut(x)}-1}{x}. \tag{2.8}$$

Then

$$\sum_{p>t_1}\frac{e^{ug(p)}-1}{p}=I_1+I_2,\quad I_1=\int_{t_1}^{\infty}\frac{f(x)}{\log x}\,dx,\quad I_2=\int_{t_1}^{\infty}f(x)\,dR(x).$$

For the estimation of I_2 we integrate by parts:

$$I_2=R(x)f(x)\Big|_{t_1}^{\infty}-\int_{t_1}^{\infty}R(x)f'(x)\,dx. \tag{2.9}$$

Suppose that

$$f'(x)=\frac{e^{ut(x)}\big(ut'(x)x-1\big)+1}{x^2}$$

changes its sign in $[t_1, \infty)$ at most once, for example at z_0. Then, by integrating by parts, we have

$$\int_{t_1}^{\infty} |R(x)|\,|f'(x)|\,dx \leqq c_2\left|\int_{t_1}^{z_0} \frac{x}{(\log x)^{c_3}} f'(x)\,dx\right| + c_2\left|\int_{z_0}^{\infty} \frac{x}{(\log x)^{c_3}} f'(x)\,dx\right| \ll$$

$$\ll f(t_1)\frac{t_1}{(\log t_1)^{c_3}} + \int_{t_1}^{\infty} \frac{f(x)}{(\log x)^{c_3}}\,dx.$$

So, observing that

$$f(t_1) = \frac{e^{-H}t_1 - 1}{t_1} \leqq e^{-H},$$

we get

$$I_2 \ll e^{-H}\frac{t_1}{(\log t_1)^{c_3}} + \frac{1}{(\log t_1)^{c_3-1}}\cdot I_1. \tag{2.10}$$

To estimate I_1, we write

$$I_1 = \int_{\log t_1}^{\infty} \frac{e^{ut(e^\lambda)}-1}{\lambda}\,d\lambda = \sum_{k=1}^{\infty}\frac{u^k}{k!}\int_{\log t_1}^{\infty}\frac{t(e^\lambda)^k}{\lambda}\,d\lambda \doteq \mathscr{H}(g;\, \log t_1). \tag{2.11}$$

For the integral

$$J(y, h) = \int_{y}^{\infty} \lambda^h e^{-\lambda}\,d\lambda$$

we have

$$J(y, h) = y^h e^{-y} + hJ(y, h-1).$$

Let now $t(p) = p^{-\delta}$ $(0 < \delta \leqq 1)$. Then

$$\int_{\log t_1}^{\infty} \frac{t(e^\lambda)^k}{\lambda}\,d\lambda = \int_{\log t_1}^{\infty} \frac{e^{-\lambda\delta k}}{\lambda}\,d\lambda = J(\delta k \log t_1, -1) < \frac{e^{-\delta k \log t_1}}{\delta k \log t_1},$$

and so

$$\mathscr{H}\left(\frac{1}{p^\delta};\, \log t_1\right) \leqq \sum_{k=1}^{\infty} \frac{(ut_1^{-\delta})^k}{k!\,k\delta \log t_1}.$$

Since $ut_1^{-\delta} = \log t_1 - H$, we have

$$I_1 \leqq \frac{4e^{-H}t_1}{\delta(\log t_1)^2}, \tag{2.12}$$

if $H < \frac{1}{2}\log t_1$.

Let now $t(p) = (\log p)^{-\gamma}$, $(\gamma > 0)$. Then, from (2.11),

$$\mathscr{H}\left((\log p)^{-\gamma};\, \log t_1\right) = \sum_{k=1}^{\infty}\frac{u^k}{k!}\int_{\log t_1}^{\infty} \lambda^{-k\gamma-1}\,d\lambda =$$

$$= \sum_{k\geqq 1} \frac{\left(u(\log t_1)^{-\gamma}\right)^k}{k!\,(k\gamma+1)} = \sum_{k\geqq 1}\frac{(\log t_1 - H)^k}{k!(k\gamma+1)} \leqq \frac{4e^{-H}t_1}{\gamma \log t_1},$$

if $H < \frac{1}{2}\log t_1$.

So for $t(p)=p^{-\delta}$ $(0<\delta\leqq 1)$

$$\log K_3(u) \leqq Be^{-H}\frac{t_1}{(\log t_1)^2}, \tag{2.13}$$

while for $t(p)=(\log p)^{-\gamma}$ $(\gamma>0)$

$$\log K_3(u) \leqq Be^{-H}\frac{t_1}{\log t_1},$$

B being a constant.

For the sake of brevity we shall write $u_1=\log u$, $u_2=\log u_1$, $u_3=\log u_2$.

Let us first consider the case $t(p)=p^{-1}$. By choosing $H=1$, and collecting our inequalities we have

$$\log K(u) < u \sum_{p\leqq t_0}\frac{1}{p}-t_0+O\left(\frac{t_0}{\log t_0}\right),$$

where

$$t_0=\frac{u}{\log t_0+1}, \quad t_1=\frac{u}{\log t_1-1}.$$

Since, from the prime number theorem

$$\sum_{p\leqq t_0}\frac{1}{p}=\log\log t_0+a+O(u_1^{-2}),$$

where

$$a=\gamma-\sum_{k\geqq 2}\sum_p\frac{1}{kp^k},$$

(γ being Euler's constant), and observing that

$$\log\log t_0=u_2-\frac{u_2}{u_1}+O(u_2u_1^{-2}), \quad t_0=\frac{u}{u_1}+O(uu_2u_1^{-2}),$$

we get

$$\log K(u) < u\left[u_2+a-\frac{u_2+1}{u_1}\right]+O(uu_2^2u_1^{-2}).$$

So, from (2.3),

$$\log(1-F(\tau)) \leqq u\left[u_2+a-\tau-\frac{u_2+1}{u_1}\right]+O(uu_2^2u_1^{-2}).$$

Let u be chosen according to the equation

$$\tau=u_2+a-u_2u_1^{-1}.$$

Then, by an easy calculation, we get

$$\log(1-F(\tau)) \leqq -\frac{u}{u_1}+O(uu_2^2u_1^{-2}),$$

$$\mathcal{L}\overset{\text{def}}{=}\log\log\frac{1}{1-F(\tau)} \geqq u_1-u_2+O(u_2^2u_1^{-1}).$$

Since

$$u_1 = e^{\tau-a}+\frac{u_2}{u_1} = e^{\tau-a}\left(1+\frac{u_2}{u_1}+O\left(\frac{u_2^2}{u_1^2}\right)\right) = e^{\tau-a}+u_2+O\left(\frac{u_2^2}{u_1}\right),$$

we have $\mathscr{L} \geqq e^{\tau-a}-c\tau^2 e^{-\tau}$, that is (1.19) holds.

Now we consider the case $t(p)=p^{-\delta}$, $0<\delta<1$. By choosing $H=1$, we have

$$t_0^\delta = \frac{u}{\log t_0+1} < \frac{u}{\log t_1-1} = t_1^\delta,$$

and so $t_1/t_0 \leqq e^2$. Consequently, by (2.3)

$$\log\frac{1}{1-F(\tau)} \geqq \tau u - u\psi(t_0)+t_0+O\big(t_0/(\log t_0)\big).$$

Since

$$\psi(t_0) = \sum_{p\leqq t_0} 1/p^\delta = \frac{t_0^{1-\delta}}{(1-\delta)\log t_0}\left(1+O\left(\frac{1}{\log t_0}\right)\right),$$

and $u=t_0^\delta(\log t_0+1)$, we have

$$u\psi(t_0) = \frac{t_0}{1-\delta}\left(1+O\left(\frac{1}{\log t_0}\right)\right),$$

and so

$$\log\frac{1}{1-F(\tau)} \geqq \tau u - \frac{\delta}{1-\delta}\, t_0+O\big(t_0/(\log t_0)\big).$$

By choosing t_0 to satisfy

$$\tau = \frac{t_0^{1-\delta}}{(1-\delta)\log t_0},$$

we have

$$\log\frac{1}{1-F(\tau)} \geqq t_0+O\left(\frac{t_0}{\log t_0}\right) = (\tau\log\tau)^{1/(1-\delta)}\left(1+O\left(\frac{1}{\log\tau}\right)\right),$$

and so (1.20) holds.

To prove (1.21), we observe that

$$\log\frac{1}{1-F(\tau)} \geqq \tau u - \log K(u) \geqq u\tau + t_0 - \frac{ut_0}{(\log t_0)^{\gamma+1}} - \frac{c_4 t_0}{\log t_0}.$$

By choosing $u=(\log\tau)^{\gamma+1}$, we have

$$\log\frac{1}{1-F(\tau)} \geqq \tau(\log\tau)^{\gamma+1} - c_1\tau(\log\tau)^\gamma$$

and this proves (1.21).

Now we shall prove Theorem 4. Let $g(p)=1/p$,

$$g_y(n) = \sum_{\substack{p\mid n\\ p<y}} g(p); \quad g(y;\, n) = g(n)-g_y(n).$$

Then

$$\mathcal{S}_\Delta \stackrel{\text{def}}{=} \frac{1}{x} \# \{n \leqq x \mid g_{t_0}(n) \geqq \psi(t_0)+\Delta\} \leqq e^{-u(\psi(t_0)+\Delta)} \prod_{p \leqq t_0} \left(1+\frac{e^{ug(p)}-1}{p}\right),$$

where $u=u_{t_0}$ is defined according to (2.4), i.e. $u_{t_0}=t_0(\log t_0+H)$. By using (2.5), we get

$$\log \mathcal{S}_\Delta < -\Delta u - t_0 + O\left(\frac{t_0}{(\log t_0)^c}\right) + \sum_{p \leqq t_0} p e^{-u/p},$$

where c is an arbitrary large constant. Since

$$\sum_{\frac{y}{2}<p<y} p e^{-u/p} < y\pi(y) e^{-u/y} \ll \frac{y^2}{\log y} e^{-u/y},$$

by choosing $y=y_k=\frac{t_0}{2^k}$ $(k=0, 1, 2, \ldots)$, we have

$$\sum_{p \leqq t_0} p e^{-u/p} \ll \frac{t_0^2 e^{-u/t_0}}{\log t_0} = \frac{e^{-H} t_0}{\log t_0}.$$

By choosing $H=c\log\log t_0$, with a fixed c,

$$\log \mathcal{S}_\Delta < -\Delta u_{t_0} - t_0 + B\frac{t_0}{(\log t_0)^c}, \tag{2.14}$$

B being a constant.

Let $u_{t_1}=t_1(\log t_1-H)$. Then, by choosing $H=c\log\log t_1$,

$$\frac{1}{x} \# \{n \leqq x \mid g(t_1, n) \geqq R\} \leqq \exp\left(-Ru_{t_1}+B\frac{t_1}{(\log t_1)^{c+2}}\right). \tag{2.15}$$

Let

$$t_0 = t_1 = (\log k)^{1+\varepsilon_k}, \quad \varepsilon_k = \frac{\log\log\log k}{\log\log k};$$

$$f_k^{(1)}(n) = \max_{j=1,\ldots,k} g_{t_0}(n+j); \quad f_k^{(2)}(n) = \max_{j=1,\ldots,k} g(t_0; n+j).$$

Let

$$H_k \stackrel{\text{def}}{=} \psi(t_0) - \log k = \log(1+\varepsilon_k) + O\left(\frac{1}{\log\log k}\right) = \frac{\log\log\log k}{\log\log k} + O\left(\frac{1}{\log\log k}\right).$$

Let k be so large that $H_k < 2\varepsilon_k$. Then, by (2.14),

$$a(x, k, 2\varepsilon_k) \stackrel{\text{def}}{=} \frac{1}{x} \# \{n \leqq x \mid f_k^{(1)}(n) \geqq \psi(\log k)+2\varepsilon_k\} \leqq \tag{2.16}$$

$$\leqq \left(1+\frac{k}{x}\right)\frac{k}{x+k} \# \{n \leqq x+k \mid g_{t_0}(n) \geqq \psi(t_0)\} \leqq$$

$$\leqq \left(1+\frac{k}{x}\right) k \exp\left(-t_0+B\frac{t_0}{(\log t_0)^c}\right) \leqq \left(1+\frac{k}{x}\right) k^{-\log\log k+c},$$

c being a constant. Similarly, from (2.15),

$$(2.17)\qquad b(x, k, \varepsilon_k) = \frac{1}{x} \,\#\, \{n \leqq x | f_k^{(2)}(n) \geqq \varepsilon_k\} \leqq$$

$$\leqq \left(1+\frac{k}{x}\right) k \exp\left(-\varepsilon_k u_{t_1} + O\left(\frac{t_1}{(\log t_1)^c}\right)\right) \leqq \left(1+\frac{k}{x}\right) k^{-\log\log k}.$$

So for $k \leqq x$ we have

$$(2.18)\qquad \frac{1}{x} \,\#\, \{n \leqq x | f_k(n) > \psi(\log k) + 3\varepsilon_k\} < 1/k^3,$$

if k is large. For $k > x$, $n \leqq x$ we have

$$f_k(0) \leqq f_k(n) \leqq f_{k+x}(0) = \psi(\log k) + O\left(\frac{1}{\log k}\right).$$

Hence it follows immediately that

$$\frac{1}{x} \,\#\, \{n \leqq x | \exists k > k_0, f_k(n) \geqq \psi(\log k) + 3\varepsilon_k\} < \frac{1}{k_0^2}.$$

By this, Theorem 4 has been proved.

3. Proof of Theorem 7. Suppose that the conditions of Theorem 7 are satisfied. Let $\tilde{g}(n)$ be strongly additive defined for primes by

$$\tilde{g}(p) = \begin{cases} g(p) & \text{if} \quad p > p_\mu \\ 0 & \text{if} \quad p \leqq p_\mu. \end{cases}$$

It is obvious that $g(P_\mu m) = g(P_\mu) + \tilde{g}(m)$. From the Turán—Kubilius inequality

$$\sum_{m \leqq x/P_\mu} \{\tilde{g}(m) - A'\}^2 \ll \frac{x}{P_\mu} \sum_{p > p_\mu} \frac{g^2(p)}{p},$$

if $P_\mu < x$; $A' = A_{x/P_\mu} - A_{p_\mu}$. Hence we get immediately

$$(3.1)\qquad M_B \overset{\text{def}}{=} \#\left\{m \leqq \frac{x}{P_\mu} \,\middle|\, |\tilde{g}(m) - A'| \geqq B\right\} \ll \frac{x}{P_\mu B^2} \sum_{p > p_\mu} \frac{g^2(p)}{p}.$$

If $\tilde{g}(m) - A' \geqq -B$, then

$$g(P_\mu m) = \psi(p_\mu) + \tilde{g}(m) \geqq \psi(p_\mu) + A' - B.$$

So for $P_\mu(m-1) < n < P_\mu m$ we get

$$(3.2)\qquad F_{P_\mu}(n) \geqq g(P_\mu m) - A_{(m+1)P_\mu} \geqq \psi(p_\mu) + A_{x/P_\mu} - A_{(m+1)P_\mu} - A_{p_\mu} - B.$$

Let now $x \to \infty$. For $m \geqq \sqrt{x}$ we have

$$A_{x/P_\mu} - A_{(m+1)p_\mu} \ll \left(\sum \frac{1}{p}\right)^{1/2} \left(\sum \frac{g^2(p)}{p}\right)^{1/2} \to 0 \quad (x \to \infty),$$

where the summation is over the primes in $\left[(m+1)p_\mu, \frac{x}{P_\mu}\right]$. By choosing

$$B_\mu = B = \left(\sum_{p>p_\mu} \frac{g^2(p)}{p}\right)^{1/4}$$

we obtain (1.26) immediately for $k=P_\mu$.

Let now $P_\mu<k<P_{\mu+1}$. To prove (1.26) it is enough to observe that $F_k(n)\geqq F_{P_\mu}(n)$, and that $A_{\log k}-A_{p_\mu}\to 0$ $(k\to\infty)$.

Now we assume that (1.27), (1.28) hold. If $P_\mu\leqq k<P_{\mu+1}$ then, $\psi(\log k)=\psi(p\mu)(1+o(1))=\psi(p_{\mu+1})(1+o(1))$ and $F_{P_{\mu+1}}(n)\geqq F_k(n)\geqq F_{P_\mu}(n)$, and so it is enough to prove (1.29) for $k=P_\mu$. From (1.28) we have

$$M_B \ll \frac{x}{P_\mu B^2} t^2(p_\mu)(\log\log p_\mu)^\gamma.$$

From the monotonicity of t we have

$$\frac{t^2(p_\mu)}{\psi^2(p_\mu)} \leqq 1/\mu^2,$$

so by choosing $B=\lambda_\mu\psi(p_\mu)$, $0<\lambda_\mu<1$, we have

$$M_B \ll \frac{x}{P_\mu\lambda_\mu^2}\frac{(\log\log\mu)^\gamma}{\mu^2}.$$

Let $x>P_\mu^3$. In the interval $n\in[1, x]$ we drop the n's for which $n\leqq x^{1/2}$. Observing that $A_{p_\mu}=o(\psi(p_\mu))$, and that $A_y-A_{y^\alpha}=O(1)$ $(0<\alpha<1)$, from (3.2) we get that

$$F_{P_\mu}(n) \geqq (1-2\lambda_\mu)\psi(p_\mu)$$

for all but $\frac{x(\log\log\mu)^\gamma}{\mu^2\lambda_\mu^2}$ of $n\leqq x$, if λ_μ tends to zero sufficiently slowly. Let $x<P_\mu^3$. Then, for every $n\leqq x$,

$$F_{P_\mu}(n) = \max_{j=1,\dots,P_\mu} (g(n+j)-A_{n+j}) \geqq \psi(p_\mu)-A_{x+P_\mu}.$$

Since

$$A_{x+P_\mu}-A_{P_\mu} \ll \left(\sum_{P_\mu<p<P_\mu+x}\frac{1}{p}\right)^{1/2}\left(\sum_{p>p_\mu}\frac{t^2(p)}{p}\right)^{1/2} \ll$$

$$\ll t(p_\mu)(\log\log p_\mu)^\gamma(\log p_\mu)^{1/2} \ll \frac{\psi(p_\mu)}{\mu}(\log\log p_\mu)^\gamma(\log p_\mu)^{1/2} = o(\psi(p_\mu)),$$

therefore

$$F_{P_\mu}(n) \geqq (1-2\lambda_\mu)\psi(p_\mu)$$

holds for every n if μ is large. Applying this argument for the sequence $x=2^\nu$, we get the relation:

$$\forall\varepsilon>0:\ \lim_{k_0\to\infty}\sup_{x\geqq 1}\frac{1}{x}\#\{n\leqq x|\exists k>k_0,\ F_k(n)<(1-\varepsilon)\psi(\log k)\} = 0.$$

To prove the second half of (1.29) we choose $\log\log t_0 = p_\mu^\delta$, where $0<\delta<\gamma$ (see (1.27), (1.28)), and define $g(t_0, n)$, $g_{t_0}(n)$ to be strongly additive satisfying

$$g(t_0; p) = \begin{cases} 0 & \text{if } p \leqq t_0, \\ g(p), & \text{if } p > t_0, \end{cases}$$

$$g_{t_0}(n) = g(n) - g(t_0; n).$$

Let $A_x^{t_0} = A_x - A_{t_0}$. For every $u \geqq 0$ we have

$$D(x, u) \overset{\text{def}}{=} \sum_{n \leqq x} e^{u(g(t,n) - A_x^{t_0})} \leqq x \prod_{t_0 < p \leqq x} \left(1 + \frac{e^{ug(p)} - 1}{p}\right) e^{-ug(p)/p},$$

whence it follows that

$$\frac{1}{x} \# \{n \leqq x \mid g(t_0, n) \geqq \Delta\} \leqq \exp\left(-\Delta u + u^2 \sum_{p > t_0} \frac{g^2(p)}{p}\right),$$

if $u = \dfrac{1}{2t(t_0)}$. Let $\Delta = \eta_\mu \psi(p_\mu)$, $\eta_\mu \to 0$ slowly. Then, from (1.27)

$$\Delta u = u \frac{\psi(p_\mu)}{2t(t_0)} > 4p_\mu,$$

if μ is large. Furthermore, from (1.28)

$$\frac{1}{4t^2(t_0)} \sum_{p > t_0} \frac{g^2(p)}{p} \ll (\log\log t_0)^\gamma = p_\mu^{\delta\gamma} = o(p_\mu),$$

since $\delta\gamma < 1$. Consequently

$$\# \{n \leqq x \mid g(t; n) \geqq \eta_\mu \psi(p_\mu)\} \ll x/P_\mu^3. \tag{3.3}$$

Let $C_r(x)$ be the number of those $n \leqq x$, that have at least r prime factors in $[1, t_0]$. We have by Stirling's formula,

$$C_r(x) \leqq x \cdot \frac{1}{r!} \left(\sum_{p < t_0} \frac{1}{p}\right)^r \leqq x \exp\left(-r \log \frac{r}{e(p_\mu^\delta + O(1))} + O(\log r)\right).$$

Let $r = [(1+4\varrho)\mu]$, ϱ being a small positive constant. Then,

$$r \log \frac{r}{e(p_\mu^\delta + O(1))} \geqq (1+4\varrho)(1-2\delta)\, p_\mu \geqq (1+2\varrho)\, p_\mu,$$

if δ is small enough. Consequently

$$C_r(x) \ll \frac{x}{P_\mu^{1+\varrho}}.$$

Let n be a such number that has $s(>\mu)$ prime factors in $[1, t_0]$. From the monotonicity of $t(y)$ we get

$$g_{t_0}(n) \leqq g(p_1 \dots p_s) \leqq \psi(p_\mu) + (s-\mu)\, t(p_\mu) \leqq \left(\frac{s}{\mu} - 1\right) \psi(p_\mu).$$

So, if $g_{t_0}(n) \geqq (1+4\varrho)\psi(p_\mu)$, then $s \geqq r$. Consequently

$$\# \{n \leqq x | g_{t_0}(n) > (1+4\varrho)\psi(p_\mu)\} \ll \frac{x}{P_\mu^{1+\varrho}}. \tag{3.4}$$

From (3.3) and (3.4) we get immediately that

$$\# \{n \leqq x | \max_{j=1,\dots,k} g(n+j) > (1+5\varrho)\psi(p_\mu)\} \ll \frac{x}{P_\mu^\varrho},$$

if $P_\mu < x$.

For $P_\mu > x$ we have

$$F_{P_\mu}(n) \leqq \max_{n \leqq x+P_\mu} g(n) \leqq \psi(p_{\mu+1}) = \psi(p_\mu) + o(1).$$

Applying this estimation for $x = 2^\nu$ $(\nu = 1, 2, \dots)$ and summing up for $\mu \geqq \mu_0$, we have

$$\sup_{x \geqq 1} \frac{1}{x} \{n \leqq x | \exists \mu > \mu_0, F_{P_\mu}(n) > (1+5\varrho)\psi(p_\mu)\} \ll \frac{1}{P_{\mu_0}^\varrho}.$$

By this we proved (1.29).

4. Proof of Theorem 1′ and Theorem 2. To prove Theorem 1′ we suppose that (1.11) holds. From the existence of the distribution function $F(x)$,

$$\sum_p \frac{\min(1, g(p))}{p} < \infty.$$

Let $\delta > 0$ be fixed, $\mathcal{P}_k$ be the set of those primes p, for which

$$(1+\delta) f_k(0) \leqq g(p) < (1+\delta) f_{2k}(0)$$

holds. Then

$$\sum_{p \in \mathcal{P}_k} 1/p < \infty,$$

if $f_k(0) \neq 0$. Let $b(n) = (n+1)\dots(n+k)$; $R_k = \prod_{p \in \mathcal{P}_k} p$.

From (1.11),

$$\sum_{\substack{n \leqq x \\ (b(n), R_k) = 1}} 1 \geqq (1-\varepsilon)x,$$

if $k > k_0(\delta, \varepsilon)$. Since $1 - F(f_k(0)) \geqq 1/k$ for every k, from (1.11) it follows that

$$f_{\nu k}(0) \leqq (1+\varepsilon) f_k(0)$$

for every fixed ν, if k is large. So $f_k(0) = O(k^\varepsilon)$ and for $p \in \mathcal{P}_k$ we have $p/k \to \infty$ $(k \to \infty)$. Consequently

$$\prod_{p \in \mathcal{P}_k} \left(1 - \frac{k}{p}\right) > 1 - \varepsilon,$$

and

$$\sum_{p \in \mathcal{P}_k} \frac{k}{p} < 2\varepsilon,$$

if k is sufficiently large.

So we have

$$\sum_{g(p)>(1+\delta)f_k(0)} \frac{g(p)^r}{p} < \sum_{2^\nu \geqq k_0} \frac{\varepsilon(1+\delta)^r f_{2^\nu}^r(0)}{2^\nu} \ll \sum \frac{2^{\varepsilon\nu}}{2^\nu} < \infty,$$

and Theorem 1' has been proved.

The proof of Theorem 2 is almost the same. We need to observe only that from (1.13)

$$f_k(0) = o(\log k) \tag{4.1}$$

follows. Since for fixed ν

$$\nu k(1-F(f_{\nu k}(0))) \geqq 1,$$

and

$$\nu k\big(1-F(f_k(0)+A)\big) \to 0 \quad (k\to\infty),$$

therefore $f_{\nu k}(0) < f_k(0)+A$ if k is large, that implies (4.1).

5. Proof of Theorem 3. Let $L(k)\nearrow\infty$ be given. We can give $L_1(k)\nearrow\infty$, so that $L_1(k)\leqq L(k)$, $L_1(k+k^2)\leqq 2L_1(k)$, $L_1(k)$ has integer values with jump 1. It is enough to prove our theorem for $L_1(k)$ instead of $L(k)$.

Let $\mathscr{P}=\{q_1<q_2<\ldots\}$ be a rare sequence of primes. We shall define $g(n)$ so that $g(q_i)\nearrow\infty$, and $g(p)=0$ for $p\notin\mathscr{P}$.

Let B_k be a sequence tending to infinity monotonically, $\mathscr{P}$ be so rare and the increase of $g(q_i)$ so slow that

$$\sum_{q_i>k} \frac{g(q_i)}{q_i} < \frac{B_k}{k}, \tag{i}$$

$$g\big(\prod_{q_i\leqq k} q_i\big) \leqq \frac{1}{4} L_1(k) \tag{ii}$$

hold for every $k\geqq 1$.

So $f_k(0)\leqq \frac{1}{4}L_1(k)$ for every $k\geqq 1$. Let $g_1(n), g_2(n)$ be strongly additive defined for primes as

$$g_1(p) = \begin{cases} 0, & p>k,\\ g(p), & p\leqq k,\end{cases}$$

$$g_2(p) = g(p)-g_1(p), \quad f_k^{(i)}(n) = \max_{j=1,\ldots,k} g_i(n+j).$$

It is obvious that

$$f_k^{(1)}(n) \leqq g\big(\prod_{q_i\leqq k} q_i\big) \leqq \frac{1}{4}L_1(k).$$

Furthermore

$$\sum_{n\leqq x} f_k^{(2)}(n) \leqq k \sum_{n\leqq x+k} g_2(n) \leqq k \sum_{q_i>k} g(q_i)\frac{x+k}{q_i},$$

and so for $x>k$,

$$\frac{1}{x} \sum_{\substack{n\leqq x\\ f_k^{(2)}(n)>C_k}} 1 \leqq \frac{1}{C_k}\sum_{n\leqq x} f_k^{(2)}(n) \leqq 2\frac{k}{C_k}\sum_{q_i>k}\frac{g(q_i)}{q_i} < \frac{2B_k}{C_k} \;(=\varrho_k).$$

Let $C_k=\frac{1}{4}L_1(k)$, $B_k=\frac{1}{8}\cdot\sqrt{L_1(k)}$. Then $\varrho_k=(\sqrt{L_1(k)})^{-1}$.
Since, for $k\geqq x$, $n\leqq x$,

$$f_k(n)\leqq f_{k+x}(0)\leqq\frac{1}{4}L_1(k+x)\leqq\frac{1}{4}L_1(2k)\leqq\frac{1}{2}L_1(k).$$

Since $f_k(n)\leqq f_k^{(1)}(n)+f_k^{(2)}(n)$, therefore

$$\sup_{x\geqq 1}\frac{1}{x}\#\left\{n\leqq x|f_k(n)>\frac{1}{2}L_1(k)\right\}\leqq\varrho_k.$$

Let now k_0 be fixed, the sequence $k_1<k_2<\dots$ be defined by

$$k_\nu=\min_{L_1(k)=2L_1(k_{\nu-1})}k.$$

It is clear that

$$\lambda(k_0)=\sum_{\nu=0}^{\infty}\varrho_{k_\nu}<\frac{c}{\sqrt{L_1(k_0)}},$$

$\lambda(k_0)\to 0$ $(k_0\to\infty)$.

Applying this argument for $x=2^\mu$ $(\mu=0,1,2,\dots)$ we deduce that

$$\sup_{x\geqq 1}\frac{1}{x}\#\left\{n\leqq x|\exists\nu: f_{k_\nu}(n)>\frac{1}{2}L_1(k)\right\}\leqq\lambda(k_0).$$

Let now n be such a number for which $f_{k_\nu}(n)<\frac{1}{2}L_1(k_\nu)$ $(\nu=0,1,2,\dots)$ holds. Then for every $k\in(k_{\nu-1},k_\nu)$,

$$f_k(n)\leqq f_{k_\nu}(n)\leqq\frac{1}{2}L_1(k_\nu)=L_1(k_{\nu-1})\leqq L_1(k).$$

This finishes the proof of Theorem 3.

6. Proof of Theorem 5. Let $\varepsilon>0$ and t be given, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ be the set of primes in the intervals $[1,(1-\varepsilon)t]$, $((1-\varepsilon)t,t,]$ $(t,(1+\varepsilon)t,]$ P_i be the product of the elements $\mathcal{P}_i$, i.e.

$$P_i=\prod_{p\in\mathcal{P}_i}p.$$

Let r, s be natural numbers. In this section b_r, $b_r^{(j)}$, $j=1,2,\dots$, denote a number that is a product of r distinct elements of $\mathcal{P}_2$. Similarily c_s, $c_s^{(1)}$, $c_s^{(2)}$, … denote such numbers that are the product of s distinct primes from $\mathcal{P}_3$. Let H and K be the number of elements in $\mathcal{P}_2$, and in $\mathcal{P}_3$, respectively.

Then the number of b_r's is $\binom{H}{r}$, and the number of c_s's is $\binom{K}{s}$.

From the prime number theorem

$$H=\frac{\varepsilon t}{\log t}+O\left(\frac{t}{(\log t)^2}\right),\quad K=\frac{\varepsilon t}{\log t}+O\left(\frac{t}{(\log t)^2}\right). \tag{6.1}$$

Let $\mathscr{A}$ be the set of those integers that have the form $n=\frac{P_2}{b_r}m$, where $(m, P_2)=1$, and that contains at least s prime factors from $\mathscr{P}_3$. Let

$$F(n) = \sum_{c_s|m} 1,$$

if $n\in\mathscr{A}$, and $F(n)=0$ otherwise.

Let $0<\delta<1$, $r=[t^\delta]$, $s=[cr]$, $c>1$ being a constant.

To prove our theorem we shall deduce a Turán—Kubilius' type inequality for the sum

$$\mathscr{E}(y) \overset{\text{def}}{=} \sum_{n\le y}\left[\sum_{i=1}^{P_2} F(n+i)-A\right]^2, \tag{6.1}$$

where

$$A = \left(\sum b_r\right)\left(\sum 1/c_s\right). \tag{6.2}$$

For the sake of simplicity we shall assume that r, s, t are large but temporarily fixed numbers, $y\to\infty$.

Let

$$S(y, i) = \sum_{n\le y} F(n)\,F(n+i). \tag{6.3}$$

Squaring out (6.1) we get easily that

$$\begin{aligned}\mathscr{E}(y) &= \sum_{i=1}^{P_2} 2(P_2-i)\,S(y, i)+P_2\sum_{n\le y}F^2(n)-2AP_2\sum_{n\le y}F(n)+\\ &\quad +A^2y+O(P_2^3y^{1/10}) =\\ &= \textstyle\sum^{(1)}+P_2\sum^{(2)}-2AP_2\sum^{(3)}+A^2y+O(P_2^3y^{1/10}).\end{aligned} \tag{6.4}$$

We shall use Eratosthenian sieve for some primes in $\mathscr{P}_2$. We note that

$$\prod_{p\in\mathscr{P}_2}\left(1-\frac{\gamma(p)}{p}\right) = 1+O\left(\frac{\varepsilon}{\log t}\right) \quad (t\to\infty)$$

if $\gamma(p)$ is bounded by an absolute constant.

Then

$$H(z) = \sum_{\substack{n\le z\\ (n, P_2)=1}} 1 = z\prod_{p\in\mathscr{P}_2}(1-1/p)+O(2^H).$$

Consequently

$$\textstyle\sum^{(3)} = \sum\limits_{b_r}\sum\limits_{\substack{m\le\frac{b_r y}{P_2}\\ (m, P_2)=1}}\sum\limits_{c_s|m} 1 = \sum\limits_{b_r, c_s} H\left(\frac{b_r y}{P_2 c_s}\right) = \frac{1}{P_2}\left(1+O\left(\frac{\varepsilon}{\log t}\right)\right)Ay+O_t(1), \tag{6.5}$$

where t in the order term denotes that the constant involved may depend on t.

We shall give an upper estimate for $\sum^{(2)}$. We have

$$\textstyle\sum^{(2)} = \sum\limits_{b_r}\sum\limits_{c_s^{(1)}, c_s^{(2)}}\sum\limits_{n\le\frac{b_r y}{P_2[c_s^{(1)}, c_s^{(2)}]}} 1 \le B\frac{y}{P_2}\left(\sum b_r\right), \tag{6.6}$$

where

$$B = \sum \frac{1}{[c_s^{(1)}, c_s^{(2)}]}. \tag{6.7}$$

Let ε_μ be a fixed product of μ prime factors from $\mathscr{P}_3$. The equation $\varepsilon_\mu = (c_s^{(1)}, c_s^{(2)})$ has

$$\binom{K-\mu}{2(s-\mu)}\binom{2(s-\mu)}{s-\mu}$$

solutions. For all of them $[c_s^{(1)}, c_s^{(2)}] \geqq t^{2s-\mu}$ holds. ε_μ can be chosen $\binom{K}{\mu}$ times Consequently

$$B \leqq \sum_{\mu=0}^{s} t^{\mu-2s}\binom{K}{\mu}\binom{K-\mu}{2(s-\mu)}\binom{2(s-\mu)}{s-\mu}. \tag{6.8}$$

Furthermore it is obvious that

$$\sum b_r \leqq t^r \binom{H}{r}.$$

So by the Stirling formula

$$\sum b_r < \frac{(tH)^r}{r!} < \exp\left(2r\log t - r\delta\log t + O(r)\right) = \exp\left((2-\delta)\,r\log t + O(r)\right).$$

Similarly, from (6.8),

$$B < \sum_{\mu=0}^{s} \frac{K^{2s-\mu}}{t^{2s-\mu}\mu!\,(s-\mu)!^2} < \sum_{\mu=0}^{s} \frac{1}{\mu!\,(s-\mu)!^2} < \exp\left(-s\delta\log t + O(r)\right).$$

Consequently

$$\sum^{(2)} \leqq \frac{y}{P_2}\exp\left([(2-\delta)\,r - \delta s]\log t + O(r)\right). \tag{6.9}$$

Now we estimate A. Counting the b_r's and c_s's we have

$$t^{r-s}\binom{H}{r}\binom{K}{s} \geqq A \geqq \frac{(1-\varepsilon)^r}{(1+\varepsilon)^s}\cdot t^{r-s}\binom{H}{r}\binom{K}{s}.$$

Since

$$\frac{(H-r)^r}{r!} < \binom{H}{r} < \frac{H^r}{r!},$$

from the Stirling formula we deduce easily that

$$\log A = (r-s)\log t + r\log H + O\left(\frac{r^2}{H}\right) + s\log K + O\left(\frac{s^2}{K}\right) - r\log r - s\log s + O(r),$$

and so by (6.1) that

$$\log A = [2r - (r+s)\,\delta]\log t + O(r\log\log t). \tag{6.10}$$

We choose c $(s=[cr])$ so that

$$\alpha = 2-(1+c)c > 0. \tag{6.11}$$

This guarantees that $A \gg 1$.

Let now consider the sum

$$\sum_B = \sum_{\Delta > P_2} \frac{b_r^{(1)} b_r^{(2)}}{c_s^{(1)} c_s^{(2)}}, \tag{6.12}$$

where

$$\Delta = \frac{P_2(c_s^{(1)}, c_s^{(2)})}{[b_r^{(1)}, b_r^{(2)}]}.$$

The condition $\Delta > P_2$ implies that $(c_s^{(1)}, c_s^{(2)}) \geqq [b_r^{(1)}, b_r^{(2)}]$.

Let $\delta_l, \varepsilon_\mu$ be fixed, where the index denotes the number of its prime divisors, and consider those $b_r^{(1)}, b_r^{(2)}, c_s^{(1)}, c_s^{(2)}$ for which $\delta_l = (b_r^{(1)}, b_r^{(2)})$, $\varepsilon_\mu = (c_s^{(1)}, c_s^{(2)})$. If $\Delta > P_2$, then

$$\{(1+\varepsilon)t\}^\mu \geqq \{(1-\varepsilon)t\}^{2r-l},$$

i.e.

$$\frac{1}{(1-\varepsilon)^{2r-(l+\mu)}} \geqq \frac{(1+\varepsilon)^\mu}{(1-\varepsilon)^{2r-l}} \geqq t^{2r-(l+\mu)},$$

whence

$$1 \geqq [(1-\varepsilon)t]^{2r-(l+\mu)},$$

i.e. $l+\mu \geqq 2r$.

For fixed l and μ the number of $b_r^{(1)}, b_r^{(2)}, c_s^{(1)}, c_s^{(2)}$ that satisfy $\omega\big((b_r^{(1)}, b_r^{(2)})\big) = l$, $\omega([c_s^{(1)}, c_s^{(2)}]) = \mu$ is

$$\binom{H}{l}\binom{H-l}{2(r-l)}\binom{2(r-l)}{r-l}\binom{K}{\mu}\binom{K-\mu}{2(s-\mu)}\binom{2(s-\mu)}{s-\mu} \leqq \frac{H^{r-l}}{l!(r-l)!^2} \cdot \frac{K^{s-\mu}}{\mu!(s-\mu)!^2}.$$

Since $\dfrac{b_r^{(1)} b_r^{(2)}}{c_s^{(1)} c_s^{(2)}} \leqq t^{2(r-s)}$ and $H<t, K<t$, therefore

$$\sum_B \ll t^{2(r-s)} \sum_{l+\mu \geqq 2r} \frac{t^{r+s-l-\mu}}{l!(r-l)!^2 \mu!(s-\mu)!^2} \ll t^{r-s+1}. \tag{6.13}$$

Consider now

$$\sum_C = \left(\sum (b_r^{(1)}, b_r^{(2)})\right)\left(\sum \frac{1}{[c_s^{(1)}, c_s^{(2)}]}\right). \tag{6.14}$$

Arguing as before, we have

$$\sum_C \leqq \left\{H^r \sum_{l=0}^{r} \frac{(t/H)^l}{l!(r-l)!^2}\right\}\left\{\sum_{\mu=0}^{s} \frac{(K/t)^{2s-\mu}}{\mu!(s-\mu)!^2}\right\} = \sum^{(b)} \cdot \sum^{(c)}.$$

By Stirling's formula

$$\frac{1}{l!(r-l)!^2} < \exp(-g(l) + O(\log r)),$$

where

$$g(l) = l\log l + 2(r-l)\log(r-l) - 2r + l.$$

By differentiating, we see that the smallest value is achieved at $l=l_0$, where l_0 is the solution of $l_0=(r-l_0)^2$. We have easily that

$$g(l_0)=r\log l_0-r+O(\sqrt{r})=r\delta\log t-r+O(\sqrt{r}).$$

Since $H^r(t/H)^l\leqq t^r$,

$$\sum{}^{(b)}<\exp\big(r(1-\delta)\log t-r+O(\sqrt{r})\big).$$

We have similarly that

$$\sum{}^{(c)}<\exp\big(-s\delta\log t+O(s\log\log t)\big).$$

Consequently

(6.15) $$\sum{}_C<\exp\big([r-\delta(r+s)]\log t+O(s\log\log t)\big).$$

Let now consider the sum $S(y,i)$. This is equal to the number of solutions of the equation

(6.16) $$\frac{P_2}{b_r^{(2)}}c_s^{(2)}v-\frac{P_2}{b_r^{(1)}}c_s^{(1)}u=i,\quad \frac{P_2}{b_r^{(1)}}c_s^{(1)}u\leqq y,$$

$(uv,P_2)=1$; in variable $b_r^{(1)}, b_r^{(2)}, c_s^{(1)}, c_s^{(2)}, u, v$. Let $b_r^{(j)}, c_s^{(j)}$ $(j=1,2)$ be fixed; $\delta=(b_r^{(1)}, b_r^{(2)})$; $\varepsilon=(c_s^{(1)}, c_s^{(2)})$; $\xi^{(j)}, f^{(j)}, \Delta$ $(j=1,2)$ be defined by

$$c_s^{(j)}=\xi^{(j)}\varepsilon,\quad \delta f^{(j)}=b_r^{(j)};\quad \Delta=\frac{P_2}{[b_r^{(1)}, b_r^{(2)}]}(c_s^{(1)}, c_s^{(2)}).$$

If (6.16) has a solution, then $\Delta\,|\,i$. Let $i=\Delta i_1$. Dividing by Δ we reduce (6.16) to

(6.17) $$\xi^{(2)}f^{(1)}v-\xi^{(1)}f^{(2)}u=i_1,\quad (uv,P_2)=1.$$

It has a solution if and only if $(i_1,\xi^{(2)}\xi'^{1)})=1$. The solutions u,v are of the forms

$$u=u_0+l\xi^{(2)}f^{(1)},\quad v=v_0+l\xi^{(1)}f^{(2)}\quad (l=0,1,2,\ldots).$$

To enumerate the l's for which $(uv,P_2)=1$, we sieve for primes $p\in\mathcal{P}_2$. Since the number $\gamma(p)$ of the solution of $uv\equiv 0\ (\mathrm{mod}\ p)$ is 1 or 2, we get

$$\prod_{p|P_2}\left(1-\frac{\gamma(p)}{p}\right)=1+O\left(\frac{\varepsilon}{\log t}\right).$$

On the previous assumptions (6.16) has

$$\frac{y(b_r^{(1)}, b_r^{(2)})}{P_2[c_s^{(1)}, c_s^{(2)}]}\left(1+O\left(\frac{\varepsilon}{\log t}\right)\right)+O_t(1)$$

solutions. O_t denotes that the constant involved by the order term may depend on t.

Hence we have

(6.18) $$\sum{}^*\stackrel{\mathrm{def}}{=}\sum_{i=1}^{P_2}S(y,i)=\frac{y}{P_2}\left(1+O\left(\frac{\varepsilon}{\log t}\right)\right)\sum\frac{(b_r^{(1)}, b_r^{(2)})}{[c_s^{(1)}, c_s^{(2)}]}\cdot\sum_{\substack{i_1\leqq P_2/\Delta\\ (i_1,\xi^{(1)}\xi^{(2)})=1}}1+O_t(1).$$

Since

$$\sum_{\substack{i_1 \leqq P_2/\Delta \\ (i_1, \xi^{(1)} \xi^{(2)})=1}} 1 = \begin{cases} \frac{P_2}{\Delta}\left(1+O\left(\frac{r}{t}\right)\right)+O(1), & \text{if} \quad \Delta \leqq P_2, \\ 0, & \text{if} \quad \Delta > P_2, \end{cases}$$

and $\frac{r}{t} \ll \frac{\varepsilon}{\log t}$ as $t \to \infty$, we have

$$\sum{}^* = \frac{y}{P_2}\left(1+O\left(\frac{\varepsilon}{\log t}\right)\right)(A^2 - \sum{}_B) + O\left(\frac{y}{P_2} \sum{}_C\right) + O_t(1),$$

i.e.

$$\sum{}^* = \frac{y}{P_2}\left(1+O\left(\frac{\varepsilon}{\log t}\right)\right) A^2 + O\left(\frac{y}{P_2}(\sum{}_B + \sum{}_C)\right) + O_t(1). \tag{6.19}$$

Similarly, for the sum

$$\sum{}^{**} \stackrel{\text{def}}{=} \sum_{i=1}^{P_2} iS(y, i) \tag{6.20}$$

we have

$$\sum{}^{**} = \frac{y}{P_2}\left(1+O\left(\frac{\varepsilon}{\log t}\right)\right) \sum \frac{(b_r^{(1)}, b_r^{(2)})}{[c_s^{(1)}, c_s^{(2)}]} \cdot \Delta\{ \sum_{\substack{i_1 \leqq P_2/\Delta \\ (i_1, \xi^{(1)} \xi^{(2)})=1}} \}.$$

Since

$$\sum_{\substack{i_1 \leqq P_2/\Delta \\ (i_1, \xi^{(1)} \xi^{(2)})=1}} i_1 = \frac{P_2^2}{2\Delta^2}\left(1+O\left(\frac{r}{t}\right)\right) + O\left(\frac{P_2}{\Delta}\right)$$

for $\Delta \leqq P_2$, we have, as earlier

$$\sum{}^{**} = \frac{y}{2}\left(1+O\left(\frac{\varepsilon}{\log t}\right)\right) A^2 + O\left(y(\sum{}_B + \sum{}_C)\right) + O_t(1).$$

Consequently for $\sum^{(1)}$ defined in (6.4) we have

$$\sum{}^{(1)} = 2\left(P_2 \sum{}^* - \sum{}^{**}\right) = y\left(1+O\left(\frac{\varepsilon}{\log t}\right)\right) A^2 + O\left(y(\sum{}_B + \sum{}_C)\right) + O_t(1). \tag{6.21}$$

So, by (6.21) and (6.5) we have

$$\mathscr{E}(y) \leqq B_1 \frac{\varepsilon}{\log t} A^2 y + B_2 y(\sum{}_B + \sum{}_C) + O(P_2 \sum{}_2) + O_t(1),$$

where B_1, B_2 are absolute constants. Now by (6.10), (6.13), (6.15) we get

$$\sum{}_C < t^{-r/2} A, \quad \sum{}_B < 1.$$

From (6.9) $P_2 \sum_2 \ll Ae^{O(r)}$, and so from (6.10), (6.11),

$$Ae^{O(r)} \ll \frac{\varepsilon}{\log t} A^2.$$

Consequently

$$\mathscr{E}(y) \leqq B \frac{\varepsilon}{\log t} A^2 y + O_t(1). \tag{6.22}$$

Let $M(y)$ be the number of $n \leqq y$, for which no one of $n+1, \ldots, n+P_2$ is belonging to $\mathscr{A}$. Then, from (6.22)

$$M(y) \leqq B \frac{\varepsilon}{\log t} y + O_t(1). \tag{6.23}$$

Since

$$\{P_1(n+1), \ldots, P_1(n+P_2)\} \subseteqq \{P_1 n+1, \ldots, P_1 n+P_1 P_2\},$$

we have immediately the following assertion.

THEOREM 8. *Let $\varepsilon>0$, $0<\delta<1$, c be fixed so that*

$$\alpha \overset{\text{def}}{=} 2-(1+c)\delta > 0,$$

t a large constant; $r=[t^\delta]$, $s=[ct^\delta]$. Let $\mathscr{B}$ be the set of those integers n for which there exist b_r and c_s so that

$$n \equiv 0 \left(\mathrm{mod}\, \frac{P_1 P_2}{b_r} c_s \right).$$

Let

$$N(x) = \#\{n \leqq x \,|\, \{n+1, \ldots, n+P_1 P_2\} \cap \mathscr{B} = \emptyset\}.$$

Then

$$\overline{\lim_x} \frac{N(x)}{x} \leqq B \frac{\varepsilon}{\log t},$$

where B is an absolute constant.

Hence we deduce easily Theorem 5. Indeed, if $n \equiv 0 \left(\frac{P_1 P_2}{b_r} c_s \right)$, then

$$g(n) \geqq g(P_1 P_2) + g(c_s) - g(b_r).$$

Let $g(p)=p^{-\delta}$. By choosing $r=[t^\gamma]$, $s=[ct^\gamma]$, $\gamma<1$,

$$g(c_s)-g(b_r) \geqq \frac{s}{[(1+\varepsilon)t]^\delta} - \frac{r}{[(1-\varepsilon)t)^\delta} \geqq t^{\gamma-\delta} \left\{ \frac{c}{1+\varepsilon} - \frac{1}{1-\varepsilon} \right\} > c_1 t^{\gamma-\delta}$$

$(c_1 > 0$ constant)

if ε is sufficiently small.

Let $P_1 P_2 = p_1 \ldots p_\mu \leqq k < P_1 P_2 p_{\mu+1}$. Then $f_k(0) = g(P_1 P_2)$. If we put $t=p_\mu$, we get immediately Theorem 5.

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THE LIPSCHITZ CONSTANT OF THE OPERATOR OF BEST APPROXIMATION

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Introduction

Let $C[a, b]$ be the space of real-valued continuous functions on $[a, b]$, $U_n \subset C[a, b]$ be an n-dimensional Cebysev subspace, and for $f \in C[a, b]$ denote by $p_n(f) \in U_n$ its polynomial of best approximation.

Then by a theorem of G. FREUD [1] the operator of best approximation satisfies a local Lipschitz condition, i.e. for a given $f \in C[a, b]$ and any $f_1 \in C[a, b]$

$$\|p_n(f) - p_n(f_1)\| \leqq C(a, b, f, U_n)\|f - f_1\|$$

holds.

This theorem initiated a wide investigation of the continuity properties of the operator of best approximation. (A number of papers studied the uniform continuity of the operator of best approximation on different sets of continuous functions (see [2]—[7]). Local and uniform continuity properties of best approximations in L_p spaces $(1 \leqq p < \infty)$ were also examined (see [8]—[10]).)

Recently, M. S. HENRY and J. A. ROULIER [11] raised an interesting question: let $f \in C[a, b]$ be given and define the Lipschitz constant $\gamma_n(f)$ of f by

$$\gamma_n(f) = \sup_{\substack{f_1 \in C[a,b] \\ f_1 \neq f}} \frac{\|p_n(f) - p_n(f_1)\|}{\|f - f_1\|}. \tag{1}$$

Then how does the Lipschitz constant $\gamma_n(f)$ vary when n tends to infinity? In [11] the authors considered the algebraic case and showed that there exists a function $f \in C[a, b]$ for which the $\gamma_n(f)$'s are unbounded. They also ask whether the $\gamma_n(f)$'s are unbounded for any nonpolynomial continuous function or not. In the present paper some new properties of $\gamma_n(f)$ in the algebraic case will be discussed. We prove that if $f \in C[a, b]$ has exactly $n+2$ points of Cebysev alternation (in the case of approximation by algebraic polynomials of order at most n), then $\gamma_n(f) \geqq c \ln n$. Answering a question of J. Szabados, we show that in general a "bad" subsequence of the $\gamma_n(f)$'s can tend to infinity arbitrarily quickly. On the other hand, it turns out that if $f \in \mathrm{Lip}\,\alpha$ with some $0 < \alpha \leqq 1$, then there exists a proper subsequence of $\gamma_n(f)$'s which increases at most exponentially.

The question of estimation of $\gamma_n(f)$ from above is closely connected with the question of distribution of the points of Cebysev alternation. In the first part of the present paper some new results in this direction will be obtained.

The distribution of the points of Cebysev alternation in the periodic case

Let f be a nonpolynomial 2π-periodic continuous function, denote $p_k(f)$ its trigonometric polynomial of best approximation of degree k, and let $E_k(f)$ be the error of approximation. Then in $[-\pi, \pi]$ there exist $2k+2$ points of Cebysev alternation $-\pi \leqq x_{1,k} < x_{2,k} < \ldots < x_{2k+2,k} < \pi$ satisfying the relations

$$f(x_{i,k}) - p_k(f, x_{i,k}) = \gamma_k(-1)^i E_k(f) \quad (i = 1, 2, \ldots, 2k+2;\ |\gamma_k| = 1). \tag{2}$$

The question of distribution of the points $\{x_{i,k}\}_{i=1}^{2k+2}$ was already studied by M. Kadec [12]. He considered the case of approximation by even trigonometric polynomials.

Theorem (M. Kadec). *Set*

$$\Delta_k = \max_{1 \leqq i \leqq 2k+2} \left| x_{i,k} - \frac{i-k-2}{k+1}\pi \right|.$$

Then for arbitrary $\varepsilon > 0$

$$\underline{\lim}_{k\to\infty} \Delta_k k^{1/2-\varepsilon} = 0. \tag{3}$$

Further B. O. Björnestål [9] proved that the $\underline{\lim}_{k\to\infty}$ in Kadec's theorem cannot be strengthened to $\lim_{k\to\infty}$.

We shall investigate the distribution of the points of Cebysev alternation from another point of view. Denote by $d_k(f)$ the minimal distance between the points of Cebysev alternation:

$$d_k(f) = \min_{1 \leqq i \leqq 2k+2} (x_{i+1,k} - x_{i,k}); \quad 0 < d_k(f) \leqq \frac{\pi}{k+1},$$

where $x_{2k+3,k} = x_{1,k} + 2\pi$. Then the general question is as follows: how does $d_k(f)$ tend to zero when $k \to \infty$? Obviously, (3) does not give any information about $d_k(f)$. Let T be the set of all trigonometric polynomials, $C_{2\pi}$ the space of 2π-periodic real valued continuous functions. Then we have the following

Theorem 1. *Let* $f \in C_{2\pi} \setminus T$, *then*

$$\overline{\lim}_{k\to\infty} k d_k(f) > 0 \tag{4}$$

if $f \in \mathrm{Lip}\,\alpha$ *for some* $0 < \alpha \leqq 1$, *and*

$$\overline{\lim}_{k\to\infty} k \log^\beta k \; d_k(f) = \infty \tag{5}$$

if $f \notin \mathrm{Lip}\,\alpha$ *for all* $0 < \alpha \leqq 1$, *where* $\beta > 1$ *is arbitrary.*

The proof of Theorem 1 will follow from two lemmas.

Lemma 1. *For any* $f \in C_{2\pi} \setminus T$; $k, m \in \mathbf{N}$, $m > k$

$$d_k(f) \geqq \frac{2\big(E_k(f) - E_m(f)\big)}{m\big(E_k(f) + E_m(f)\big)} \tag{6}$$

holds.

PROOF. For any $l, s \in \mathbf{N}$ define $r_s(f)=f-p_s(f)$; $\varphi_{s,l}(f)=r_s(f)-r_l(f)=p_l(f)-p_s(f)$. Let $\{x_{i,k}\}_{i=1}^{2k+2}$ be the system of Cebysev alternation of f of degree k, i.e. these points satisfy (2). Assume that $d_k(f)=x_{j+1,k}-x_{j,k}$ and $\gamma_k(-1)^j=1$ $(1 \leqq j \leqq 2k+2)$. (If $\gamma_k(-1)^j=-1$ the proof does not change.) Then

$$r_k(f, x_{j,k}) = E_k(f); \quad r_k(f, x_{j+1,k}) = -E_k(f)$$

and using that $\|r_m(f)\| = E_m(f) \leqq E_k(f)$ we obtain

$$\varphi_{k,m}(f, x_{j,k}) = r_k(f, x_{j,k}) - r_m(f, x_{j,k}) \geqq E_k(f) - E_m(f);$$

$$\varphi_{k,m}(f, x_{j+1,k}) = r_k(f, x_{j+1,k}) - r_m(f, x_{j+1,k}) \leqq -E_k(f) + E_m(f).$$

Hence

$$0 \leqq 2\big(E_k(f) - E_m(f)\big) \leqq \varphi_{k,m}(f, x_{j,k}) - \varphi_{k,m}(f, x_{j+1,k}).$$

But $\varphi_{k,m}(f)$ is a polynomial of degree at most m, therefore Bernstein's inequality implies

$$2\big(E_k(f) - E_m(f)\big) \leqq |\varphi_{k,m}(f, x_{j,k}) - \varphi_{k,m}(f, x_{j+1,k})| \leqq$$

$$\leqq \|\varphi'_{k,m}(f)\| \, d_k(f) \leqq m \|\varphi_{k,m}(f)\| \, d_k(f) \leqq m\big(E_k(f) + E_m(f)\big) d_k(f)$$

and the lemma is proved.

LEMMA 2. *For any* $f \in C_{2\pi} \setminus T$

$$\overline{\lim_{k \to \infty}} \left(1 - \frac{E_{2k}(f)}{E_k(f)}\right) \log^\beta k = \infty \tag{7}$$

holds, where $\beta > 1$ *is arbitrary. Moreover, if* $f \in \operatorname{Lip} \alpha$ *for some* $0 < \alpha \leqq 1$ *then*

$$\overline{\lim_{k \to \infty}} \left(1 - \frac{E_{2k}(f)}{E_k(f)}\right) > 0. \tag{8}$$

PROOF. Assume at first that $f \in \operatorname{Lip} \alpha$ for some $0 < \alpha \leqq 1$. Then for any $k \geqq 1$, $0 < E_k(f) \leqq A k^{-\alpha}$, $A > 0$. Let us prove that in this case $\underline{\lim}_{k \to \infty} \frac{E_{2k}(f)}{E_k(f)} < 1$. Assume the contrary. Then $\lim_{k \to \infty} \frac{E_{2k}(f)}{E_k(f)} = 1$, hence $E_k(f) < 2^{\alpha/2} E_{2k}(f)$ for $k \geqq k_0$ and iterating this inequality n times we obtain

$$0 < E_{k_0}(f) < 2^{\alpha/2} E_{2k_0}(f) < \ldots < 2^{n\alpha/2} E_{2^n k_0}(f) \leqq 2^{n\alpha/2} A (2^n k_0)^{-\alpha} = A k_0^{-\alpha} 2^{-n\alpha/2}.$$

But this is an obvious contradiction, because n can be chosen arbitrarily large. Therefore $\underline{\lim}_{k \to \infty} \frac{E_{2k}(f)}{E_k(f)} < 1$ if $f \in \operatorname{Lip} \alpha$ and (8) is verified.

Let us prove now (7). Assume again the contrary. Then for any $k \geqq 1$

$$\left(1 - \frac{E_{2k}(f)}{E_k(f)}\right) \log^\beta k \leqq M, \quad M > 0$$

or

$$E_k(f) \leqq \frac{E_{2k}(f)}{1-\dfrac{M}{\log^\beta k}} \quad (k \geqq k_1 > 1)$$

and iterating this inequality n times we have

$$0 < E_{k_1}(f) < \frac{E_{2^n k_1}(f)}{\prod\limits_{l=1}^{n}\left(1-\dfrac{M}{\log^\beta 2^{l-1} k_1}\right)}.$$

But $\prod\limits_{l=1}^{n}\left(1-\dfrac{M}{\log^\beta 2^{l-1} k_1}\right)$ tends to a positive number as $n \to \infty$ because

$$\sum_{l=1}^{\infty} M/\log^\beta 2^{l-1} k_1 = M \sum_{l=1}^{\infty} [(l-1)\log 2 + \log k_1]^{-\beta} \leqq$$

$$\leqq \frac{M}{\log^\beta 2} \sum_{l=2}^{\infty} 1/(l-1)^\beta < \infty \quad (\beta > 1).$$

By $E_{2^n k_1}(f) \to 0$ $(n \to \infty)$ we obtain a contradiction again. The lemma is proved.

Proof of Theorem 1. Setting $m=2k$ in (6) and combining it with (7) and (8) we obtain (5) and (4), respectively.

Example. It is known that

$$\frac{c_2}{k} \leqq E_k(|\cos x|) \leqq \frac{c_1}{k}$$

for any $k \geqq 1$. Then setting $m=\left(\left[\dfrac{2c_1}{c_2}\right]+1\right)k$ in (6) we obtain for any $k \geqq 1$

$$d_k(|\cos x|) \geqq \frac{c_3}{k}\left(1-\frac{E_m}{E_k}\right) \geqq \frac{c_4}{k}.$$

Theorem 1 shows that for a "good" subsequence $d_k(f)$ tends to zero essentially as $\dfrac{1}{k}$.

Remark. Replacing Bernstein's theorem by Markov's inequality we can get analogous results in the algebraic case.

The main lemma

Let us formulate now the main lemma which essentially is proved in [1]. We shall consider again the general case, when elements of $C[a, b]$ are approximated by polynomials belonging to an n-dimensional Cebysev subspace $U_n \subset C[a, b]$.

Take a system of points $(a \leqq) x_1 < x_2 < \ldots < x_{n+1} (\leqq b)$. Omitting the rth point $(1 \leqq r \leqq n+1)$ we can construct the Lagrange interpolatory operator corresponding to the system of points $x_1 < x_2 < \ldots < x_{r-1} < x_{r+1} < \ldots < x_{n+1}$, which projects $C[a, b]$ into U_n. Let $\lambda_{n,r}$ be the norm of this operator and set $\lambda_n = \max_{1 \leqq r \leqq n+1} \lambda_{n,r}$.

Then we have the following

MAIN LEMMA. *Let $q_n \in U_n$ satisfy*

$$(9) \qquad (-1)^{i+1} q_n(x_i) \leqq \mu \quad (i = 1, 2, \ldots, n+1)$$

with some $\mu > 0$, then

$$(10) \qquad \|q_n\| \leqq \lambda_n \mu.$$

Let $f \in C[a, b]$ and $\{x_i\}_{i=1}^{n+1}$ be its system of Cebysev alternation, $p_n(f)$ the polynomial of best approximation. Then for any $f_1 \in C[a, b]$

$$(11) \qquad \|f - p_n(f_1)\| \leqq \|f - f_1\| + E_n(f_1) \leqq E_n(f) + 2\|f - f_1\|$$

holds. Further using the relations $f(x_i) - p_n(f, x_i) = \gamma(-1)^i E_n(f)$ $(1 \leqq i \leqq n+1;$ $|\gamma| = 1)$ and (11) we have

$$\gamma(-1)^{i+1}[p_n(f_1) - p_n(f)](x_i) \leqq 2\|f - f_1\| \quad (i = 1, 2, \ldots, n+1),$$

thus by (10)

$$\|p_n(f) - p_n(f_1)\| \leqq 2\lambda_n(f)\|f - f_1\|$$

and by definition of $\gamma_n(f)$ (see (1))

$$(12) \qquad \gamma_n(f) \leqq 2\lambda_n(f)$$

holds.

This inequality gives us a possibility to estimate $\gamma_n(f)$ from above. $\lambda_n(f)$ is the "Lebesgue constant" of the system of points of Cebysev alternation and it depends, of course, on the minimal distance between the points of Cebysev alternation.

An upper estimation for $\lambda_n(f)$ in the periodic case was obtained in [17].

Let $f \in C_{2\pi}$, $\{x_{i,k}\}_{i=1}^{2k+2}$ be its system of Cebysev alternation of degree k, $d_k(f)$ the minimal distance between the points of Cebysev alternation and by the points $\{x_{i,k}\}_{i=1}^{2k+2}$ define $\lambda_k = \lambda_k(f)$ as above. Then by a lemma from [17]

$$(13) \qquad \lambda_k(f) \leqq \left(\frac{e\pi}{k d_k(f)}\right)^{2k}.$$

Hence and from (12)

$$(14) \qquad \bar{\gamma}_k(f) \leqq 2\left(\frac{e\pi}{k d_k(f)}\right)^{2k}$$

where $\bar{\gamma}_k(f)$ is the Lipschitz constant of degree k in the periodic case.

Now applying the results of the previous section we are able to obtain upper estimations for $\gamma_n(f)$.

Upper estimations for $\gamma_n(f)$ in the algebraic case

Denote by P_n the set of algebraic polynomials of degree at most n and set $P=\bigcup_{n=0}^{\infty} P_n$. In what follows $p_n(f)\in P_n$ denotes the best algebraic approximation of $f\in C[a, b]$, and $E_n(f)$ the error of approximation. For any $f\in C[a, b]\setminus P$ define $\gamma_n(f)$ as the Lipschitz constant of the operator of best approximation (see (1)).

THEOREM 2. *Let $f\in C[a, b]\setminus P$. Then*

$$\varlimsup_{n\to\infty} \frac{\sqrt[n]{\gamma_n(f)}}{\log^\beta n} = 0, \tag{15}$$

where $\beta>2$ is arbitrary. Moreover, if $f\in \mathrm{Lip}\,\alpha$ for some $0<\alpha\leqq 1$, then

$$\varlimsup_{n\to\infty} \sqrt[n]{\gamma_n(f)} < \infty. \tag{16}$$

PROOF. Set

$$F(y) = f\left(\frac{b-a}{2}\cos y + \frac{a+b}{2}\right),$$

then $F\in C_{2\pi}$ and evidently $\gamma_n(f)\leqq\bar{\gamma}_n(F)$, where $\bar{\gamma}_n(F)$ denotes the Lipschitz constant of degree n in the periodic case. Then by (14)

$$\gamma_n(f) \leqq \bar{\gamma}_n(F) \leqq 2\left(\frac{e\pi}{nd_n(F)}\right)^{2n}. \tag{17}$$

Further if $f\in \mathrm{Lip}\,\alpha$, then $F\in \mathrm{Lip}\,\alpha$, and combining (17) with (5) and (4) we get (15) and (16), respectively.

Lower estimations for $\gamma_n(f)$ in the algebraic case

In [13] it was proved that if the number of points of Cebysev alternation of $f\in C[a, b]$ is exactly $n+2$, then the operator of best approximation is differentiable by direction, i.e. for any $g\in C[a, b]$,

$$\lim_{t\to 0} \frac{p_n(f+tg)-p_n(f)}{t} = D_f p_n(g) \tag{18}$$

exists. Moreover, it turned out that the derivative is a linear operator of the variable g. Further, $D_f p_n$ projects $C[a, b]$ into P_n and if $g\in P_n$, then $D_f p_n(g)\equiv g$. Thus $D_f p_n$ is a linear projection from $C[a, b]$ to P_n and by a well-known theorem (see e.g. [14])

$$\|D_f p_n\|_* \geqq \frac{2}{\pi^2}\log n + O(1) \tag{19}$$

where $\|\cdot\|_*$ denotes the operator norm. It follows from (18) that for any $g \in C[a, b]$, $g \not\equiv 0$

$$\|D_f p_n(g)\| = \lim_{t \to 0} \frac{\|p_n(f+tg) - p_n(f)\|}{|t|} \leqq \gamma_n(f) \|g\|$$

or

$$\gamma_n(f) \geqq \frac{\|D_f p_n(g)\|}{\|g\|}.$$

Hence and from (19)

$$\gamma_n(f) \geqq \|D_f p_n\|_* \geqq \frac{2}{\pi^2} \log n + O(1).$$

Thus we proved the following

THEOREM 3. *Let $f \in C[a, b] \setminus P$ and assume that f has exactly $n+2$ points of Cebysev alternation of degree n. Then*

$$\gamma_n(f) \geqq \frac{2}{\pi^2} \log n + O(1).$$

EXAMPLES. The functions e^x and $\frac{1}{x-c}$ $(c \in \mathbf{R} \setminus [a, b])$ satisfy the condition of Theorem 3 for any $n \geqq 0$ therefore

$$\gamma_n(e^x) \to \infty; \quad \gamma_n\left(\frac{1}{x-c}\right) \to \infty \quad (n \to \infty),$$

thus Theorem 3 generalizes the result proved in [11].

Our next theorem shows that $\varliminf_{n \to \infty}$ in Theorem 2 cannot be strengthened to $\lim_{n \to \infty}$.

THEOREM 4. *For any sequence of positive numbers $\{a_n\}_{n=1}^{\infty}$, $a_n \to \infty$ $(n \to \infty)$ there exists a function $f \in C[a, b]$ such that*

$$\varlimsup_{n \to \infty} \frac{\sqrt[n]{\gamma_n(f)}}{a_n} = \infty \tag{20}$$

holds.

The proof of this theorem will follow from several lemmas.

For $f \in C[a, b]$ denote by $A_n(f)$ its set of all points of Cebysev alternation of degree n:

$$A_n(f) = \{x \in [a, b] \colon |f(x) - p_n(f, x)| = E_n(f)\}.$$

For the sets $B, C \subseteq [a, b]$ define the density of B in C by

$$\varrho(B, C) = \sup_{x \in C} \inf_{y \in B} |x - y|.$$

LEMMA 3. *Let f, $\{f_k\}_{k=1}^{\infty} \in C[a, b]$ and assume that the sequence $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f as $k \to \infty$. Then for all fixed n we have*

$$\varrho(A_n(f), A_n(f_k)) \to 0 \quad (k \to \infty).$$

PROOF. Assume the contrary. Then there exist a positive $\delta > 0$ and a subsequence k_i such that

$$\varrho(A_n(f), A_n(f_{k_i})) \geqq \delta \quad (i = 1, 2, \ldots).$$

Hence we can choose a sequence of points $\{x_{k_i}\}_{i=1}^{\infty}$ satisfying the following properties: $x_{k_i} \in A_n(f_{k_i})$ and $|y - x_{k_i}| \geqq \frac{\delta}{2}$ for any $y \in A_n(f)$ $(i=1, 2, \ldots)$. Without loss of generality we may assume that $x_{k_i} \to x_0 \in [a, b]$, then $|y - x_0| \geqq \frac{\delta}{2}$ for any $y \in A_n(f)$. Thus $x_0 \notin A_n(f)$, hence

$$|f(x_0) - p_n(f, x_0)| < E_n(f). \tag{21}$$

But $x_{k_i} \in A_n(f_{k_i})$ therefore

$$|f_{k_i}(x_{k_i}) - p_n(f_{k_i}, x_{k_i})| = E_n(f_{k_i}). \tag{22}$$

By continuity $p_n(f_{k_i}) \rightrightarrows p_n(f)$; $E_n(f_{k_i}) \to E_n(f)$ $(i \to \infty)$, moreover $x_{k_i} \to x_0$ $(i \to \infty)$ and this and (22) imply

$$|f(x_0) - p_n(f, x_0)| = E_n(f).$$

This contradiction completes the proof of the lemma.

LEMMA 4. *Let $f \in C[a, b]$, $A_n(f) = \{x_{i,n}\}_{i=0}^{n+1} \subset \left[a, \frac{a+b}{2}\right]$, where $x_{i,n} = x_{0,n} + ih_n$ $(i=1, 2, \ldots, n+1;\ h_n > 0)$. Then*

$$\gamma_n(f) \geqq c_5^n \left(\frac{b-a}{nh_n}\right)^n. \tag{23}$$

PROOF. Set $\bar{f} = f - p_n(f)$, then $\|\bar{f}\| = E_n(f)$, and let $\bar{x}$ be the zero of $\bar{f}$ nearest to $x_{n+1,n}$ and satisfying $\bar{x} < x_{n+1,n}$. Then for any $\varepsilon > 0$ define $f_1^{\varepsilon} \in C[a, b]$ by

$$f_1^{\varepsilon}(x) = \begin{cases} \dfrac{\|\bar{f}\| + \varepsilon}{\|\bar{f}\|} \bar{f}(x), & x \in [\bar{x}, b]; \\ \bar{f}(x), & x \in [a, \bar{x}]. \end{cases}$$

Obviously $\|\bar{f} - f_1^{\varepsilon}\| = \varepsilon$ and

$$E_n(f_1^{\varepsilon}) = E_n(\bar{f}) + \mu = \|\bar{f}\| + \mu \tag{24}$$

where $\mu = \mu(\varepsilon) > 0$. Without loss of generality we may assume that

$$\bar{f}(x_{i,n}) = (-1)^i \|\bar{f}\| \quad (i = 0, 1, \ldots, n+1). \tag{25}$$

Then (24) yields

$$(-1)^{i+1} p_n(f_1^{\varepsilon}, x_{i,n}) \leqq \mu \quad (i = 0, 1, \ldots, n+1)$$

and applying the main lemma on the interval $[x_{0,n}, x_{n+1,n}]$ we get

$$\|p_n(f_1^\varepsilon)\|_{[x_{0,n},x_{n+1,n}]} \leqq \mu \max_{0\leqq r\leqq n+1} \left\| \sum_{\substack{i=0\\ i\neq r}}^{n+1} \left| \frac{\omega_r(x)}{(x-x_{i,n})\omega_r'(x_{i,n})} \right| \right\|_{[x_{0,n},x_{n+1,n}]}$$

where $\omega_r(x) = \prod_{\substack{j=0\\ j\neq r}}^{n+1} (x-x_{j,n})$. Thus

$$\|p_n(f_1^\varepsilon)\|_{[x_{0,n},x_{n+1,n}]} \leqq \mu(x_{n+1,n}-x_{0,n})^n \max_{0\leqq r\leqq n+1} \sum_{\substack{i=0\\ i\neq r}}^{n+1} \frac{|i-r|}{i!(n+1-i)!\,h_n^n} \leqq \tag{26}$$

$$\leqq \mu(n+1)^n h_n^n \frac{n+1}{h_n^n} \sum_{i=0}^{n+1} \frac{1}{i!(n+1-i)!} = \mu(n+1)^{n+1} \frac{2^{n+1}}{(n+1)!} \leqq c_6^n \mu,$$

where $c_6 > 1$ is an absolute constant. Further by (24) and (26)

$$\|\bar f\| + \varepsilon - c_6^n \mu \leqq |f_1^\varepsilon(x_{n+1,n}) - p_n(f_1^\varepsilon, x_{n+1,n})| \leqq E_n(f_1^\varepsilon) = \|\bar f\| + \mu$$

i.e.

$$\mu \geqq \frac{\varepsilon}{1+c_6^n} \geqq c_7^n \varepsilon. \tag{27}$$

Let $\{x_i^\varepsilon\}_{i=0}^{n+1}$ be a system of Cebysev alternation of f_1^ε. By assumption f has exactly $n+2$ points $\{x_{i,n}\}_{i=0}^{n+1}$ of Cebysev alternation, therefore Lemma 3 implies that

$$x_i^\varepsilon \to x_{i,n} \quad (i = 0, 1, \dots, n+1)$$

as $\varepsilon \to 0$. Then for $0 < \varepsilon \leqq \varepsilon_0$, $\{x_i^\varepsilon\}_{i=0}^n \subset [a, \bar x]$. By construction $f_1^\varepsilon = \bar f$ on $[a, \bar x]$, hence

$$f_1^\varepsilon(x_i^\varepsilon) = \bar f(x_i^\varepsilon) \quad (i = 0, 1, \dots, n), \quad 0 < \varepsilon \leqq \varepsilon_0. \tag{28}$$

On the other hand, using (24) we obtain

$$f_1^\varepsilon(x_i^\varepsilon) - p_n(f_1^\varepsilon, x_i^\varepsilon) = \gamma_\varepsilon(-1)^i E_n(f_1^\varepsilon) = \gamma_\varepsilon(-1)^i(\|\bar f\| + \mu) \tag{29}$$

$$(i = 0, 1, \dots, n+1;\ |\gamma_\varepsilon| = 1).$$

Then by (28) and (29) for any $i = 0, 1, \dots, n$ and $0 < \varepsilon \leqq \varepsilon_0$

$$\|\bar f\| + \mu = \gamma_\varepsilon(-1)^i f_1^\varepsilon(x_i^\varepsilon) + \gamma_\varepsilon(-1)^{i+1} p_n(f_1^\varepsilon, x_i^\varepsilon) =$$

$$= \gamma_\varepsilon(-1)^i \bar f(x_i^\varepsilon) + \gamma_\varepsilon(-1)^{i+1} p_n(f_1^\varepsilon, x_i^\varepsilon)$$

holds and thus

$$\gamma_\varepsilon(-1)^{i+1} p_n(f_1^\varepsilon, x_i^\varepsilon) \geqq \mu \quad (i = 0, 1, \dots, n;\ 0 \leqq \varepsilon \leqq \varepsilon_0). \tag{30}$$

By Lagrange interpolatory formula

$$p_n(f_1^\varepsilon, x) = \sum_{i=0}^n p_n(f_1^\varepsilon, x_i^\varepsilon) \frac{\omega_\varepsilon(x)}{(x-x_i^\varepsilon)\omega_\varepsilon'(x_i^\varepsilon)}$$

where $\omega_\varepsilon(x)=\prod_{i=0}^{n}(x-x_i^\varepsilon)$. Then (30) implies

$$(-1)^{n+1}\gamma_\varepsilon p_n(f_1^\varepsilon, b)=\sum_{i=0}^{n}(-1)^{n+1}\gamma_\varepsilon p_n(f_1^\varepsilon, x_i^\varepsilon)\frac{\omega_\varepsilon(b)}{(b-x_i^\varepsilon)\omega_\varepsilon'(x_i^\varepsilon)}=$$

$$=\sum_{i=0}^{n}\gamma_\varepsilon(-1)^{i+1}p_n(f_1^\varepsilon, x_i^\varepsilon)\frac{\omega_\varepsilon(b)}{(b-x_i^\varepsilon)|\omega_\varepsilon'(x)_i^\varepsilon|}\geqq\mu\left(\frac{b-a}{2}\right)^n\sum_{i=0}^{n}|\omega_\varepsilon'(x_i^\varepsilon)|^{-1}\quad(0<\varepsilon\leqq\varepsilon_0),$$

hence and by (27)

$$\|p_n(f_1^\varepsilon)\|\geqq\varepsilon c_8^n(b-a)^n\sum_{i=0}^{n}|\omega_\varepsilon'(x_i^\varepsilon)|^{-1}\quad(0<\varepsilon\leqq\varepsilon_0). \tag{31}$$

On the other hand

$$\|p_n(f_1^\varepsilon)\|\leqq\gamma_n(\bar f)\|\bar f-f_1^\varepsilon\|=\gamma_n(\bar f)\varepsilon$$

thus combining this inequality with (31) we obtain

$$\gamma_n(\bar f)\geqq c_8^n(b-a)^n\sum_{i=0}^{n}|\omega_\varepsilon'(x_i^\varepsilon)|^{-1}\quad(0<\varepsilon\leqq\varepsilon_0). \tag{32}$$

Here the left hand side does not depend on ε, therefore we may set $\varepsilon\to 0$ on the right hand side and using that $x_i^\varepsilon\to x_{i,n}$ $(\varepsilon\to 0)$ we obtain

$$\gamma_n(\bar f)\geqq c_8^n(b-a)^n\frac{2^n}{n!\,h_n^n}\geqq c_5^n\left(\frac{b-a}{nh_n}\right)^n. \tag{33}$$

But evidently $\gamma_n(\bar f)=\gamma_n(f)$ hence (33) gives (23).

In order to prove Theorem 4 we must construct a function satisfying the conditions of Lemma 4 for a certain subsequence n_k, $k=1, 2, \ldots$ on which h_{n_k} tend to zero arbitrarily quickly.

The method of construction will be essentially the same as that of [9] and will be based on the following statement proved by S. W. YOUNG [15]:

For any $x_{i-1}<x_i$, $y_{i-1}\neq y_i$ $(i=1, 2, \ldots, n)$ there exists a polynomial p such that $p(x_i)=y_i$ $(i=0, 1, \ldots, n)$ and p is monotone on each subinterval $[x_{i-1}, x_i]$ $(i=1, 2, \ldots, n)$.

LEMMA 5. *For any sequence of positive numbers $\{\varepsilon_i\}_{i=0}^\infty$ monotonically converging to zero as $i\to\infty$, there exists a function $f\in C[a, b]$ and a sequence $\{n_k\}_{k=0}^\infty$ such that $A_{n_k}(f)$ consists of exactly n_k+2 points $\{x_{i,n_k}\}_{i=0}^{n_k+1}$ satisfying*

$$x_{i,n_k}=x_{0,n_k}+ih_{n_k}\quad(i=1, 2, \ldots, n_k+1);\ x_{n_k+1,n_k}\leqq\frac{b+a}{2},$$

where $0<h_{n_k}\leqq\varepsilon_{n_k}$ $(k=0, 1, \ldots,)$.

PROOF. We shall represent f as a sum $\sum_{i=0}^{\infty}q_{n_i}$ where q_{n_i} are polynomials of degree n_i, $n_{i+1}>n_i$ $(i=0, 1, \ldots)$. Let us construct these polynomials. Set $q_{n_0}\equiv q_0\equiv 1$ $(n_0=0)$; $a<c<\dfrac{a+b}{2}$. Choose y_0 so that $c<y_0<\dfrac{a+b}{2}$, $y_0-c<\varepsilon_0$ and take any points x_0, x_1 satisfying $c<x_0<x_1<y_0$. Then there exists a polynomial q_{n_1} such that

$$(34)\quad q_{n_1}(x_0)=-q_{n_1}(x_1)=\frac{1}{2};\quad q_{n_1}(a)=q_{n_1}(b)=q_{n_1}(c)=0;\quad q_{n_1}\left(\frac{a+c}{2}\right)=\frac{1}{4}$$

and q_{n_1} is monotone between any two consecutive points at which its value is prescribed in (34). Using that $q_{n_1}(c)=0$ we can choose y_1 such that $|q_{n_1}|<\frac{1}{4}$ for $x\in[c, y_1]$ and $0<y_1-c<\varepsilon_{n_1}$. Further take any x_2 satisfying $c<x_2<y_1$ and set $x_{2+i}=x_2-ih_1$ $(i=1, 2, \dots, n_1+2)$ where $h_1>0$ is so small that $c<x_{n_1+4}$. Then there exists a polynomial q_{n_2} such that

$$(35)\quad \begin{cases} q_{n_2}(x_i)=\operatorname{sign} q_{n_1}(x_i)/8 \quad (i=0, 1); \\ q_{n_2}(x_2)=0; \\ q_{n_2}(x_i)=(-1)^i/4 \quad (i=3, 4, \dots, n_1+4); \\ q_{n_2}(a)=q_{n_2}(b)=q_{n_2}(c)=0; \quad q_{n_2}\left(\frac{a+c}{2}\right)=\frac{1}{8} \end{cases}$$

and q_{n_2} is monotone between any two consecutive points at which its value is prescribed in (35). Evidently $n_2>n_1>n_0=0$ and we have constructed the sequences $\{x_i\}_{i=0}^{n_1+4}$, $\{y_i\}_{i=0}^{1}$ and $\{q_{n_i}\}_{i=0}^{2}$.

If we have already constructed the sequences $\{x_i\}_{i=0}^{\sum_{j=0}^{k-1} n_j+3k-2}$, $\{y_i\}_{i=0}^{k-1}$, $\{q_{n_i}\}_{i=0}^{k}$ $(k\geqq 2)$ and $q_{n_k}(c)=0$ then

A) choose y_k such that

$$(36)\quad |q_{n_k}|<\frac{1}{2^{k+1}},\ x\in[c, y_k] \quad\text{and}\quad 0<y_k-c<\varepsilon_{n_k};$$

B) take any $c<x_{\sum_{j=0}^{k-1} n_j+3k-1}<y_k$ and set

$$x_{\sum_{j=0}^{k-1} n_j+3k+i-1}=x_{\sum_{j=0}^{k-1}+3k-1}-ih_{n_k} \quad (i=1, 2, \dots, n_k+2),$$

where h_{n_k} is so small that $x_{\sum_{j=0}^{k} n_j+3k+1}>c$;

C) consider the polynomial $q_{n_{k+1}}$ satisfying the following relations

$$(37)\quad \begin{cases} q_{n_{k+1}}(x_i)=\dfrac{\operatorname{sign} q_{n_k}(x_i)}{2^{k+2}} \quad \left(i=0, 1, \dots, \sum_{j=0}^{k-1} n_j+3k-2\right); \\ q_{n_{k+1}}\left(x_{\sum_{j=0}^{k-1} n_j+3k-1}\right)=0; \\ q_{n_{k+1}}(x_i)=\dfrac{(-1)^i}{2^{k+1}} \quad \left(i=\sum_{j=0}^{k-1} n_j+3k, \dots, \sum_{j=0}^{k} n_j+3k+1\right); \\ q_{n_{k+1}}(a)=q_{n_{k+1}}(b)=q_{n_{k+1}}(c)=0; \quad q_{n_{k+1}}\left(\dfrac{a+c}{2}\right)=\dfrac{1}{2^{k+2}} \end{cases}$$

and monotone between any two consecutive points at which its value is prescribed in (37). Then by (37) $n_{n+1}>n_k$ and it can be easily proved by induction that this construction gives us sequences $\{x_i\}_{i=0}^{\infty}$, $\{y_j\}_{j=0}^{\infty}$ and $\{q_{n_k}\}_{k=0}^{\infty}$ $(n_0<n_1<\dots)$ satisfying A)—C) for any $k=1, 2, \dots$. Moreover by (36) and (37) for any $k=1, 2, \dots$

$$|q_{n_{k+1}}| \begin{cases} \leqq \dfrac{1}{2^{k+2}}, & x\in[x_{\sum_{j=0}^{k-1} n_j+3k-1}, b]\cup[a, c]; \\ < \dfrac{1}{2^{k+2}}, & x\in[c, x_{\sum_{j=0}^{k} n_j+3k+2}]; \\ \leqq \dfrac{1}{2^{k+1}}, & x\in[x_{\sum_{j=0}^{k} n_j+3k+2}, x_{\sum_{j=0}^{k-1} n_j+3k-1}] \end{cases} \tag{38}$$

holds.

Set $f=\sum_{i=2}^{\infty} q_{n_i}$, $p_{n_k}^*=\sum_{i=2}^{k} q_{n_i}$ $(k\geqq 2)$. Evidently $p_{n_k}^*$ are polynomials of degree at most n_k $(k=2, 3, \dots)$. Let us prove that $p_{n_k}^*$ are the polynomials of best approximation of f of degree n_k $(k=2, 3, \dots)$. Consider $f-p_{n_k}^*=\sum_{i=k+1}^{\infty} q_{n_i}$. Then by (38)

$$|f-p_{n_k}^*| \leqq \sum_{i=k+1}^{\infty} |q_{n_i}|$$

$$\begin{cases} \leqq \displaystyle\sum_{i=k+1}^{\infty} \frac{1}{2^{i+1}} = \frac{1}{2^{k+1}}, & x\in[a, c]\cup[x_{\sum_{j=0}^{k-1} n_j+3k-1}, b]; \\ < \dfrac{1}{2^{k+2}} + \displaystyle\sum_{i=k+2}^{\infty} \frac{1}{2^i} = \frac{3}{2^{k+2}}, & x\in[c, x_{\sum_{j=0}^{k} n_j+3k+2}]; \\ \leqq \dfrac{1}{2^{k+1}} + \displaystyle\sum_{i=k+2}^{\infty} \frac{1}{2^{i+1}} = \frac{3}{2^{k+2}}, & x\in[x_{\sum_{j=0}^{k} n_j+3k+2}, x_{\sum_{j=0}^{k-1} n_j+3k-1}]. \end{cases} \tag{39}$$

On the other hand

$$q_{n_{k+1}}(x_i) = (-1)^i/2^{k+1} \quad \left(i = \sum_{j=0}^{k-1} n_j+3k, \dots, \sum_{j=0}^{k} n_j+3k+1\right);$$

$$q_{n_m}(x_i) = (-1)^i/2^{m+1} \quad \left(i = \sum_{j=0}^{k-1} n_j+3k, \dots, \sum_{j=0}^{k} n_j+3k+1\right), \quad m \geqq k+2.$$

Therefore

$$(f-p_{n_k}^*)(x_i) = \sum_{j=k+1}^{\infty} q_{n_j}(x_i) = (-1)^i/2^{k+1} + \sum_{j=k+2}^{\infty} (-1)^i/2^{j+1} =$$

$$= (-1)^i \frac{3}{2^{k+2}} \quad \left(i = \sum_{j=0}^{k-1} n_j+3k, \dots, \sum_{j=0}^{k} n_j+3k+1\right), \quad k \geqq 2$$

and by (39) it is evident that $|f-p_{n_k}^*|=\dfrac{3}{2^{k+2}}$ only at these n_k+2 points. Thus $p_{n_k}^*\equiv$ $\equiv p_{n_k}(f)$ $(k=2, 3, \dots)$; $A_{n_k}(f)=\{x_i;\ i=\sum_{j=0}^{k-1} n_j+3k, \dots, \sum_{j=0}^{k} n_j+3k+1\}$ consists of exactly n_k+2 points which are equidistant and belong to the interval $[c, y_k]\subset \subset\left[a, \frac{a+b}{2}\right]$ $(k=2, 3, \dots)$. Further by (36) $y_k-c<\varepsilon_{n_k}$ and this completes the proof of the lemma.

Lemmas 4 and 5 verify the statement of Theorem 4.

REMARK. In 1963 D. J. NEWMAN and H. S. SHAPIRO [16] proved a strong unicity theorem which states that for any $f\in C[a, b]$ and $q_n\in P_n$

$$\|p_n(f)-q_n\| \leqq \bar{c}_n(a, b, f)\{\|f-q_n\|-E_n(f)\}$$

holds. This generalizes Freud's theorem and gives an idea of defining the strong unicity constant

$$\alpha_n(f)=\sup_{\substack{q_n\in P_n\\ q_n\neq p_n(f)}} \frac{\|p_n(f)-q_n\|}{\|f-q_n\|-E_n(f)}.$$

Evidently, for any $f\in C[a, b]$

$$\gamma_n(f)\leqq 2\alpha_n(f). \tag{40}$$

This immediately implies that for arbitrary $f\in C[a, b]$ with exactly $n+2$ points of Cebysev alternation

$$\alpha_n(f)\geqq\frac{\log n}{\pi^2}+O(1) \tag{41}$$

(Theorem 3), and in general a subsequence of the $\alpha_n(f)$-s can tend to infinity arbitrarily quickly (Theorem 4). Moreover, the upper estimations given in Theorem 2 remain true after replacing $\gamma_n(f)$ by $\alpha_n(f)$, the proof can be obtained analogously.

The behaviour of $\alpha_n(f)$ for varying n was first studied by S. J. POREDA [18] who proved that the sequence $\{\alpha_n(f)\}_{n=0}^{\infty}$ is in general unbounded. In [11] the authors extended this result for a certain class of functions. However the functions of their class have exactly $n+2$ points of Cebysev deviation, thus (41) gives a more general statement.

The author is indebted to J. Szabados for his valuable remarks.

Added in proof (June 2, 1980). Meanwhile D. Schmidt *(J. Approx. Th.,* **24** (1978), 216—223) proved that if for given $f\in C[a, b]$ the numce of points of Cebysev alternation is minimal for infinitely many n's, then $\overline{\lim\limits_{n\to\infty}}\, \alpha_n(f)=\infty$. This statement immediately follows form (41).

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CORRECTION TO OUR PAPER "ON MODULES OVER RINGS OF TYPE (n, k)"*

By

G. J. HAUPTFLEISCH (Johannesburg) and F. LOONSTRA (Delft)

Theorem 1 (b) should be deleted and inserted in corrected form just prior to Theorem 1 as a

LEMMA. *If for a ring R $\tau(R)=(n, k)$, then $\tau(M)\leqq(n, k)$ for every R-module M.*

The proof (a)$\Rightarrow$(b) in the proof of Theorem 1 proves the lemma.

The authors want to express their appreciation towards W. G. Leavitt for bringing the error to their attention.

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ON THE CAUCHY PROBLEM FOR NON-LINEAR SYSTEMS OF PARTIAL DIFFERENTIAL-FUNCTIONAL EQUATIONS OF THE FIRST ORDER

By
Z. KAMONT (Gdańsk)

Assume that functions $f^{(\tau)}(x, Y, z, Z, Q)$, $\alpha_\tau(x, Y)$, $\tau=1, \ldots, \bar{m}$, $\varphi_i(x, Y)$, $\Psi^{(i)}(x, Y)=(\psi_1^{(i)}(x, Y), \ldots, \psi_n^{(i)}(x, Y))$, $i=1, \ldots, m$, where $Y=(y_1, \ldots, y_n)$, $Z=(z_1, \ldots, z_m)$, $Q=(q_1, \ldots, q_n)$, are given. Suppose that I is the initial set of the form

$$I = \{(x, Y)\colon\ x\in(p_0, 0],\ Y\in\mathbf{R}^n\}$$

where $\mathbf{R}^n$ is n-dimensional Euclidean space and $p_0<0$ (in particular it may be $p_0=-\infty$).

In this paper we shall deal with the Cauchy problem for the non-linear system of differential equations of the first order with a retarded argument

$$\text{(1)}\qquad \begin{cases} \dfrac{\partial z_\tau(x, Y)}{\partial x} = f^{(\tau)}\left(x, Y, z_\tau(x, Y), Z(\varphi(x, Y), \Psi(x, Y)), \dfrac{\partial z_\tau(x, Y)}{\partial Y}\right) \\ z_\tau(x, Y) = \alpha_\tau(x, Y) \quad \text{for} \quad (x, Y)\in I, \quad \tau = 1, \ldots, \bar{m}, \end{cases}$$

where

$$Z(\varphi(x, Y), \Psi(x, Y)) = \big(z_1(\varphi_1(x, Y), \Psi^{(1)}(x, Y)), \ldots, z_1(\varphi_{k_1}(x, Y), \Psi^{(k_1)}(x, Y)),$$
$$z_2(\varphi_{k_1+1}(x, Y), \Psi^{(k_1+1)}(x, Y)), \ldots, z_2(\varphi_{k_2}(x, Y), \Psi^{(k_2)}(x, Y)),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$z_{\bar{m}}(\varphi_{k_{\bar{m}-1}+1}(x, Y), \Psi^{(k_{\bar{m}-1}+1)}(x, Y)), \ldots, z_{\bar{m}}(\varphi_{k_{\bar{m}}}(x, Y), \Psi^{(k_{\bar{m}})}(x, Y))\big),$$
$$0 \leqq k_1 \leqq k_2 \leqq \ldots \leqq k_{\bar{m}} = m,$$
$$\frac{\partial z_\tau(x, Y)}{\partial Y} = \left(\frac{\partial z_\tau(x, Y)}{\partial y_1}, \ldots, \frac{\partial z_\tau(x, Y)}{\partial y_n}\right).$$

The Cauchy problem for first order partial differential equations was considered by many authors; see [1], [5], [9]. Some results concerning first order partial differential-functional equations and inequalities can be found in the papers [2], [3], [6]—[8], [10], [13], [14].

In this paper we prove that there exists a unique solution of problem (1) and we give the estimation of the existence domain of solutions of (1). Our results are generalizations of some results of papers [6], [7], where theorems concerning the global existence of solutions of linear partial differential-functional equations of the first order have been established. The Cauchy problem for a single differential-functional equation was considered in [8].

I. Assumptions and notations

Let $\Delta=\{x, y_1, \dots, y_n\}$, $\tilde{\Delta}=\{x, y_1, \dots, y_n, z, z_1, \dots, z_m, q_1, \dots, q_n\}$. We introduce

ASSUMPTION H_1. We suppose that

1° the functions $f^{(\tau)}$, $\tau=1, \dots, \overline{m}$, of the variables (x, Y, z, Z, Q) are of class C^2 in the set

$$\Omega = \{(x, Y, z, Z, Q)\colon\ x\in[0, a),\ Y, Q\in \mathbf{R}^n,\ z\in\mathbf{R},\ Z\in\mathbf{R}^m\},$$

2° the functions φ_i, $\Psi^{(i)}=(\psi_1^{(i)}, \dots, \psi_n^{(i)})$, $i=1, \dots, m$, of the variables (x, Y) are of class C^2 in

$$E_0 = \{(x, Y)\colon\ x\in[0, a),\ Y\in\mathbf{R}^n\}$$

and

$$p_0 < \varphi_i(x, Y) \leqq x, \quad (x, Y)\in E_0, \quad i = 1, \dots, m, \tag{2}$$

3° the initial function $\alpha=(\alpha_1, \dots, \alpha_{\overline{m}})$ is of class C^2 in I,

4° let $Z(x, Y)=(z_1(x, Y), \dots, z_{\overline{m}}(z, Y))$ and

$$R_\tau(x, Y, Z(x, Y)) = f^{(\tau)}\left(x, Y, z_\tau(x, Y), Z(\varphi(x, Y), \Psi(x, Y)), \frac{\partial z_\tau(x, Y)}{\partial Y}\right);$$

we assume that the consistency condition

$$R_\tau(0, Y, \alpha(0, Y)) = \frac{\partial \alpha_\tau(0, Y)}{\partial x},$$

$$\left.\frac{\partial R_\tau(x, Y, Z(x, Y))}{\partial s}\right|_{\substack{Z(x, Y)=\alpha(x, Y)\\ x=0}} = \frac{\partial^2 \alpha_\tau(0, Y)}{\partial x\, \partial s}$$

is satisfied for $Y\in\mathbf{R}^n$, $\tau=1, \dots, \overline{m}$, $s\in\Delta$.

ASSUMPTION H_2. We assume that

1° there exists a constant $A>0$ such that for each $u, v\in\tilde{\Delta}$ we have

$$|f_u^{(\tau)}(x, Y, z, Z, Q)| \leqq A, \quad |f_{uv}^{(\tau)}(x, Y, z, Z, Q)| \leqq A \quad \text{on} \quad \Omega, \quad \tau = 1, \dots, \overline{m},$$

2° there exist constants $B, C\geqq 0$ such that for each $s, t\in\Delta$

$$\left|\frac{\partial \alpha_\tau(x, Y)}{\partial s}\right| \leqq B, \quad \left|\frac{\partial^2 \alpha_\tau(x, Y)}{\partial s\, \partial t}\right| \leqq B, \quad (x, Y)\in I,\ \tau = 1, \dots, \overline{m},$$

and

$$\left|\frac{\partial \varphi_i(x, Y)}{\partial s}\right|, \quad \left|\frac{\partial \psi_j^{(i)}(x, Y)}{\partial s}\right|, \quad \left|\frac{\partial^2 \varphi_i(x, Y)}{\partial s\, \partial t}\right|, \quad \left|\frac{\partial^2 \psi_j^{(i)}(x, Y)}{\partial s\, \partial t}\right| \leqq C,$$

$$(x, y)\in E_0, \quad i = 1, \dots, m, \quad j = 1, \dots, n.$$

We adopt the following notations. Let

$$0 < q < \frac{1}{N_0} \ln\left[1 + \frac{\ln 3}{2n(1+(1+n)B)}\right] \tag{3}$$

where

$$N_0 = A[1+m\tilde{r}C(1+n)]^2 + mAC(1+n)[\tilde{r}+\tilde{p}C(1+n)] \tag{4}$$

and the constants $\tilde{r}, \tilde{p}$ are defined by

$$\begin{cases} \tilde{r} = [mC(1+n)+1]^{-1}\left\{\left[B\big(mC(n+1)+1\big)+1\right]\exp\dfrac{1+mC(n+1)}{2r(1+rB)}-1\right\}, \\ \tilde{p} = 2[B+Ac(1+rB)] \end{cases} \tag{5}$$

and

$$\begin{cases} c = [2Ar(1+rB)]^{-1}, \\ r = \dfrac{1}{2}+(n+1)\left[1+\dfrac{1}{2}mC\big(3+C(n+1)\big)\right]. \end{cases} \tag{6}$$

Let $E=\{(x, Y)\colon x\in[0, b),\ Y\in\mathbf{R}^n\}$ where $b=\min(a, c, q)$.

In order to prove the existence of a solution of problem (1) we define the sequence $\{Z^{(k)}\}$, $Z^{(k)}=(z_1^{(k)}, \dots, z_{\overline{m}}^{(k)})$ in the following way:

$z_\tau^{(0)}$, $\tau=1, \dots, \overline{m}$, are arbitrary functions such that the functions

$$u_\tau^{(0)}(x, Y) = \begin{cases} z_\tau^{(0)}(x, Y) & \text{for} \quad (x, Y)\in E \\ \alpha_\tau(x, Y) & \text{for} \quad (x, Y)\in I \end{cases} \qquad \tau = 1, \dots, \overline{m} \tag{7}$$

are of class C^2 on $E\cup I$ and

$$\left|\frac{\partial z_\tau^{(0)}(x, Y)}{\partial s}\right|, \quad \left|\frac{\partial^2 z_\tau^{(0)}(x, Y)}{\partial s\,\partial t}\right| \le B, \quad (x, Y)\in E,\ s, t\in\Delta. \tag{8}$$

If $Z^{(k)}$ is a known function, then $z_\tau^{(k+1)}$ is a solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial z_\tau(x, Y)}{\partial x} = F^{(\tau k)}\left(x, Y, z_\tau(x, Y), \dfrac{\partial z_\tau(x, Y)}{\partial Y}\right), \\ z_\tau(0, Y) = \omega_\tau(Y) \end{cases} \tag{9}$$

where

$$F^{(\tau k)}(x, Y, z, Q) = f^{(\tau)}(x, Y, z, U^{(k)}(\varphi(x, Y), \Psi(x, Y)), Q) \tag{10}$$

and

$$\begin{aligned} U^{(k)}(x, Y) &= \big(u_1^{(k)}(x, Y), \dots, u_{\overline{m}}^{(k)}(x, Y)\big), \\ U^{(k)}&\big(\varphi(x, Y), \Psi(x, Y)\big) = \\ &= \big(u_1^{(k)}(\varphi_1(x, Y), \Psi^{(1)}(x, Y)), \dots, u_1^{(k)}(\varphi_{k_1}(x, Y), \Psi^{(k_1)}(x, Y)), \\ &\quad u_2(\varphi_{k_1+1}(x, Y), \Psi^{(k_1+1)}(x, Y)), \dots, u_2(\varphi_{k_2}(x, Y), \Psi^{(k_2)}(x, Y)), \\ &\quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ u_{\overline{m}}&(\varphi_{k_{\overline{m}-1}+1}(x, Y), \Psi^{(k_{\overline{m}-1}+1)}(x, Y)), \dots, u_{\overline{m}}(\varphi_{k_{\overline{m}}}(x, Y), \Psi^{(k_{\overline{m}})}(x, Y)\big), \quad k_{\overline{m}} = m, \end{aligned} \tag{11}$$

and

$$u_\tau^{(k)}(x, Y) = \begin{cases} z_\tau^{(k)}(x, Y) & \text{for} \quad (x, Y)\in E, \\ \alpha_\tau(x, Y) & \text{for} \quad (x, Y)\in I, \end{cases} \tag{12}$$

$$\omega_\tau(Y) = \alpha_\tau(0, Y). \tag{13}$$

II. The existence of the sequence of successive approximations

LEMMA 1. *If assumptions* H_1 *and* H_2 *are satisfied then for an arbitrary index* k *we have*

(I_k) $Z^{(k)}$ *is defined and is of class* C^2 *on* E,

(II_k) *for* $(x, Y)\in E$ *we have*

$$\left|\frac{\partial z_\tau^{(k)}(x, Y)}{\partial s}\right| \leqq \lambda(x), \quad s\in\Delta, \quad \tau = 1, \dots, \overline{m}, \tag{14}$$

where

$$\lambda(x) = [mC(n+1)+1]^{-1}\{[B(mC(n+1)+1)+1]\,e^{A(mC(n+1)+1)x}-1\} \tag{15}$$

and

$$\left|\frac{\partial z_\tau^{(k)}(x, Y)}{\partial s}\right| \leqq \tilde{r}, \ (x, Y)\in E, \quad \tau = 1, \dots, \overline{m}, \ s\in\Delta, \tag{16}$$

(III_k) *for* $(x, Y)\in E$ *we have*

$$\left|\frac{\partial^2 z_\tau^{(k)}(x, Y)}{\partial s\,\partial t}\right| \leqq \mu(x), \ s, t\in\Delta, \quad \tau = 1, \dots, \overline{m}, \tag{17}$$

where

$$\mu(x) = \frac{B+A(1+rB)x}{1-Ar(1+rB)x} \tag{18}$$

and

$$\left|\frac{\partial^2 z_\tau^{(k)}(x, Y)}{\partial s\,\partial t}\right| \leqq \tilde{p}, \ s, t\in\Delta, \quad \tau = 1, \dots, \overline{m}, \tag{19}$$

(IV_k) $F^{(\tau k)}$ *are of class* C^2 *in the set*

$$\Omega_0 = \{(x, Y, z, Q)\colon\ x\in[0, b),\ (Y, z, Q)\in\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n\}$$

and for each $u, v\in\{x, y_1, \dots, y_1, z, q_1, \dots, q_n\}$ *we have*

$$|F_u^{(\tau k)}(x, Y, z, Q)| \leqq N_0, \quad |F_{uv}^{(\tau k)}(x, Y, z, Q)| \leqq N_0, \quad (x, Y, z, Q)\in\Omega_0. \tag{20}$$

PROOF. It follows from Assumptions H_1, H_2 and from (7), (8), (10)—(12) that conditions (I_0)—(IV_0) are satisfied.

Let us suppose that for some $k\geqq 0$ the conditions (I_k)—(IV_k) hold. Now we shall prove that (I_{k+1})—(IV_{k+1}) hold too.

Let us consider the initial problem (9) where the functions $F^{(\tau k)}$ and ω_τ are defined by (10)—(13). From Assumptions H_1, H_2 and from (I_k)—(IV_k) it follows (see [5], [12]) that there exists a solution $z_\tau^{(k+1)}(x, Y)$ of problem (9). This solution is defined and is of class C^2 in the set

$$\tilde{E} = \{(x, Y)\colon\ x\in[0, \tilde{b}),\ Y\in\mathbf{R}^n\}$$

where $\tilde{b}=\min(b, q)$ and q is defined by (3). Since $\min(b, q)=b$, we have $\tilde{E}=E$ and the condition (I_{k+1}) is proved.

Now we prove (II_{k+1}). Let

$$T(b,\eta,A)=\{(x,Y)\colon\ x\in[0,b),\ |y_i|\leqq\eta-Ax,\ i=1,\dots,n\}\tag{21}$$

where $\eta>0,\ b<\dfrac{\eta}{A}$. It follows from Assumption H_2 and from (I_{k+1}) and (II_k) that for each $s\in\Delta$ the differential inequality

$$\left|\frac{\partial}{\partial x}\left(\frac{\partial z_\tau^{(k+1)}(x,Y)}{\partial s}\right)\right|\leqq A+mAC(n+1)\lambda(x)+A\left|\frac{\partial z_\tau^{(k+1)}(x,Y)}{s}\right|+\tag{22}$$

$$+A\sum_{j=1}^{n}\left|\frac{\partial}{\partial y_j}\left(\frac{\partial z_\tau^{(k+1)}(x,Y)}{\partial s}\right)\right|,\quad (x,Y)\in T(b,\eta,A),$$

and the initial inequality

$$\left|\frac{\partial z_\tau^{(k+1)}(0,Y)}{\partial s}\right|\leqq B,\quad (0,Y)\in T(b,\eta,A)\tag{23}$$

hold. Hence, by comparison theorems for partial differential inequalities we get (see [9])

$$\left|\frac{\partial z^{(k+1)}(x,Y)}{\partial s}\right|\leqq\bar u(x),\quad (x,Y)\in T(b,\eta,A),\tag{24}$$

where $\bar u$ is the solution of the initial problem

$$\frac{du(x)}{dx}=A+mAC(n+1)\lambda(x)+Au(x),$$
$$u(0)=B.$$

Since $\bar u(x)=\lambda(x)$ for $x\in[0,b)$ therefore

$$\left|\frac{\partial z_\tau^{(k+1)}(x,Y)}{\partial s}\right|\leqq\lambda(x)\tag{25}$$

for $(x,Y)\in T(b,\eta,A)$. Because for each point $(x,Y)\in E$ there exists an $\eta>0$ such that $(x,Y)\in T(b,\eta,A)$, then we have inequality (25) in E. Since $\lambda(c)=\tilde r$ and $b\leqq c$, it follows from (25) that

$$\left|\frac{\partial z_\tau^{(k+1)}(x,Y)}{\partial s}\right|\leqq\tilde r,\quad (x,Y)\in E,$$

which completes the proof of (II_{k+1}).

Now we prove (III_{k+1}). We will show first that

$$\left|\frac{\partial^2 z_\tau^{(k+1)}(x,Y)}{\partial y_i\partial y_j}\right|\leqq\mu(x),\quad i,j=1,\dots,n,\tag{26}$$

for $(x,Y)\in\overline{T}(b_0,\eta,A)$ where $0<b_0<b,\ b_0<\dfrac{\eta}{A}$ and

$$\overline{T}(b_0,\eta,A)=\{(x,Y)\colon\ x\in[0,b_0],\ |y_i|\leqq\eta-Ax,\ i=1,\dots,n\}.\tag{27}$$

Put

$$V_{ij}(x, Y, h) = \frac{1}{h}\left[\frac{\partial z_\tau^{(k+1)}(x, Y+\theta_i h)}{\partial y_j} - \frac{\partial z_\tau^{(k+1)}(x, Y)}{\partial y_j}\right] \tag{28}$$

where $\theta_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, 1 standing on the i-th place. We will prove that there exists a continuous and non-negative function $\delta(h)$ such that

$$\lim_{h\to 0} \delta(h) = 0 \tag{29}$$

and

$$\left|\frac{\partial}{\partial x} V_{ij}(x, Y, h)\right| \leqq A\left[1+\mu(x)(1+mC(1+n)) + \sum_{l=1}^{n} |V_{il}(x, Y, h)|\right] \cdot \tag{30}$$

$$\cdot\left[1+\mu(x)(1+mC(1+n)) + \sum_{l=1}^{n} |V_{jl}(x, Y, h)| + \delta(h)\right] +$$

$$+\mu(x)mAC(1+n)(1+C(n+1)) + A|V_{ij}(x, Y, h)| + A\sum_{l=1}^{n}\left|\frac{\partial}{\partial y_l}(V_{ij}(x, Y, h))\right|,$$

$$(x, Y) \in \overline{T}(b_0, \eta, A), \quad i, j = 1, \ldots, n.$$

Substituting $z_\tau^{(k+1)}(x, Y)$ and $z_\tau^{(k+1)}(x, Y+\theta_i h)$ into (9) and differentiating the identities thus obtained with respect to y_j we get

(31)

$$\left|\frac{\partial}{\partial x} V_{ij}(x, Y, h)\right| \leqq \left|\frac{1}{h}\left[F_{y_j}^{(\tau k)}\left(x, Y+\theta_i h, z_\tau^{(k+1)}(x, Y+\theta_i h), \frac{\partial z_\tau^{(k+1)}(x, Y+\theta_i h)}{\partial Y}\right) -\right.\right.$$

$$\left.\left. - F_{y_j}^{(\tau k)}\left(x, Y, z_\tau^{(k+1)}(x, Y), \frac{\partial z_\tau^{(k+1)}(x, Y)}{\partial Y}\right)\right]\right| +$$

$$+\left|\frac{1}{h}\right] F_z^{(\tau k)}\left(x, Y+\theta_i h, z_\tau^{(k+1)}(x, Y+\theta_i h), \frac{\partial z_\tau^{(k+1)}(x, Y+\theta_i h)}{\partial Y}\right) -$$

$$- F_z^{(\tau k)}\left(x, Y, z^{(k+1)}(x, Y), \frac{\partial z_\tau^{(k+1)}(x, Y)}{\partial Y}\right)\Big]\Big| \left|\frac{\partial z_\tau^{(k+1)}(x, Y+\theta_i h)}{\partial y_j}\right| +$$

$$+\left|\frac{1}{h}\left[\frac{\partial z_\tau^{(k+1)}(x, Y+\theta_i h)}{\partial y_j} - \frac{\partial z_\tau^{(k+1)}(x, Y)}{\partial y_j}\right]\right| \cdot$$

$$\cdot\left|F_z^{(\tau k)}\left(x, Y, z_\tau^{(k+1)}(x, Y), \frac{\partial z_\tau^{(k+1)}(x, Y)}{\partial Y}\right)\right| +$$

$$+\sum_{l=1}^{n}\left\{\left|\frac{1}{h}\left[F_{q_l}^{(\tau k)}\left(x, Y+\theta_i h, z_\tau^{(k+1)}(x, Y+\theta_i h), \frac{\partial z_\tau^{(k+1)}(x, Y+\theta_i h)}{\partial Y}\right) -\right.\right.\right.$$

$$\left.\left. - F_{q_l}^{(\tau k)}\left(x, Y, z_\tau^{(k+1)}(x, Y), \frac{\partial z_\tau^{(k+1)}(x, Y)}{\partial Y}\right)\right]\right| \left|\frac{\partial^2 z_\tau^{(k+1)}(x, Y+\theta_i h)}{\partial y_j \partial y_l}\right| +$$

$$+\left|\frac{1}{h}\left[\frac{\partial^2 z_\tau^{(k+1)}(x, Y+\theta_i h)}{\partial y_j \partial y_l} - \frac{\partial^2 z_\tau^{(k+1)}(x, Y)}{\partial y_j \partial y_l}\right]\right| \cdot$$

$$\cdot\left|F_{q_l}^{(\tau k)}\left(x, Y, z^{(k+1)}(x, Y), \frac{\partial z_\tau^{(k+1)}(x, Y)}{\partial Y}\right)\right|.$$

It follows from Assumption H_2 and from (10)—(12) and from (II_k) that for each $u\in\tilde{\Delta}$

$$\left|\frac{1}{h}\left[f_u^{(\tau)}\left(P_\tau^{(k)}(x, Y+\theta_i h)\right)-f_u^{(\tau)}\left(P_\tau^{(k)}(x, Y)\right)\right]\right| \leqq \tag{32}$$

$$\leqq A\left[1+\lambda(x)+mC(n+1)\lambda(x)+\sum_{l=1}^{n}|V_{il}(x, Y, h)|\right]$$

where

$$P_\tau^{(k)}(x, Y)=\left(x, Y, z_\tau^{(k+1)}(x, Y), U^{(k)}\left(\varphi(x, Y), \Psi(x, Y)\right), \frac{\partial z_\tau^{(k+1)}(x, Y)}{\partial Y}\right). \tag{33}$$

For each $s\in\Delta$ we define

$$\tilde{P}_{js}^{(k)}(x, Y)=\frac{\partial u_\tau^{(k)}\left(\varphi_j(x, Y), \Psi^{(j)}(x, Y)\right)}{\partial x}\frac{\partial\varphi_j(x, Y)}{\partial s}+ \tag{34}$$

$$+\sum_{l=1}^{n}\frac{\partial u_\tau^{(k)}\left(\varphi_j(x, Y), \Psi^{(j)}(x, Y)\right)}{\partial y_l}\frac{\partial\psi_l^{(j)}(x, Y)}{\partial s}, \quad j=1, \dots, m,\ k_{\tau-1}\leqq\tau\leqq k_\tau.$$

From (10), (32)—(34) and from (II_k)—(III_k) we obtain

$$\left|\frac{1}{h}\left[F_{y_j}^{(\tau k)}\left(x, Y+\theta_i h, z_\tau^{(k+1)}(x, Y+\theta_i h), \frac{\partial z_\tau^{(k+1)}(x, Y+\theta_i h)}{\partial Y}\right)-\right.\right. \tag{35}$$

$$\left.\left.-F_{y_j}^{(\tau k)}\left(x, Y, z_\tau^{(k+1)}(x, Y), \frac{\partial z_\tau^{(k+1)}(x, Y)}{\partial Y}\right)\right]\right| \leqq$$

$$\leqq\left|\frac{1}{h}\left[f_{y_j}^{(\tau)}\left(P_\tau^{(k)}(x, Y+\theta_i h)\right)-f_{y_j}^{(\tau)}\left(P_\tau^{(k)}(x, Y)\right)\right]\right|+$$

$$+\sum_{l=1}^{m}\left|\left\{\frac{1}{h}\left[f_{z_l}^{(\tau)}\left(P_\tau^{(k)}(x, Y+\theta_i h)\right)-f_{z_l}^{(\tau)}\left(P_\tau^{(k)}(x, Y)\right)\right]\right|\,|\tilde{P}_{ly_j}^{(k)}(x, Y+\theta_i h)|+\right.$$

$$+\sum_{l=1}^{m}\left\{\left|f_{z_l}^{(\tau)}\left(P_\tau^{(k)}(x, Y)\right)\right|\left|\frac{1}{h}\left[\tilde{P}_{ly_j}^{(k)}(x, Y+\theta_i h)-\tilde{P}_{ly_j}^{(k)}(x, Y)\right]\right|\right\}\leqq$$

$$\leqq A[1+mC(n+1)\lambda(x)]\left[1+\lambda(x)+mC(n+1)\lambda(x)+\sum_{l=1}^{n}|V_{il}(x, Y, h)|\right]+$$

$$+mAC(n+1)[(n+1)C\mu(x)+\lambda(x)].$$

It follows from (28) that there exists a continuous and non-negative function δ_0 such that

$$\lim_{h\to\infty}\delta_0(h)=0 \tag{36}$$

and

$$(37)\qquad \begin{cases} \left|\dfrac{\partial^2 z_\tau^{(k+1)}(x, Y)}{\partial y_i \partial y_j}\right| \leqq |V_{ij}(x, Y, h)| + \delta_0(h), \\ |V_{ij}(x, Y, h)| \leqq \left|\dfrac{\partial^2 z_\tau^{(k+1)}(x, Y)}{\partial y_i \partial y_j}\right| + \delta_0(h), \quad i, j = 1, \dots, n \end{cases}$$

for $(x, Y) \in \overline{T}(b_0, \eta, A)$.

Finally, by (10)—(12), (31), (35), (37) and by (III_k) we have

$$(38)\qquad \left|\frac{\partial}{\partial x} V_{ij}(x, Y, h)\right| \leqq$$

$$\leqq A[1+mC(n+1)\lambda(x)]\big[1+\lambda(x)+mC(n+1)\lambda(x)+\sum_{l=1}^{n} V_{il}(x, Y, h)|\big]+$$

$$+mAC(n+1)[(n+1)C\mu(x)+\lambda(x)]+A|V_{ij}(x, Y, h)|+$$

$$+A\big[1+\lambda(x)+mC(n+1)\lambda(x)+\sum_{l=1}^{n}|V_{il}(x, Y, h)|\big]\lambda(x)+$$

$$+A\big[1+\lambda(x)+mC(n+1)\lambda(x)+\sum_{l=1}^{n}|V_{il}(x, Y, h)|\big]\big[\sum_{l=1}^{n}|V_{jl}(x, Y, h)|+\delta(h)\big]+$$

$$+A\sum_{l=1}^{n}\left|\frac{\partial}{\partial y_l} V_{ij}(x, Y, h)\right|,$$

where $\delta(h)=n\delta_0(h)$. Since $\lambda(x) \leqq \mu(x)$ for $x \in [0, b)$, we get from (38) differential inequalities (30).

Because

$$(39)\qquad |V_{ij}(0, Y, h)| \leqq B+\delta_0(h), \quad (0, Y) \in \overline{T}(b_0, \eta, A),$$

then we obtain in virtue of theorems on differential inequalities that

$$(40)\qquad |V_{ij}(x, Y, h)| \leqq \bar{u}_{ij}^{(h)}(x), \quad i, j = 1, \dots, n,$$

for $(x, Y) \in \overline{T}(b_0, \eta, A)$, where the functions $\bar{u}_{ij}^{(h)}(x)$, $i, j=1, \dots, n$, are the solution of the initial problem

$$(41)\qquad \frac{du_{ij}(x)}{dx} = A\big[1+\mu(x)\big(1+mC(n+1)\big)+\sum_{l=1}^{n} u_{il}(x)\big]\cdot$$

$$\cdot\big[1+\mu(x)\big(1+mC(n+1)\big)+\sum_{l=1}^{n} u_{jl}(x)+\delta(h)\big]+\mu(x)mAC(n+1)[1+C(n+1)]+$$

$$+Au_{ij}(x),$$

$$u_{ij}(0) = B+\delta_0(h).$$

Let $\bar{u}_{ij}$, $i, j=1, \ldots, n$, be a solution of the initial problem

$$\frac{du_{ij}(x)}{dx} = A\left[1+\mu(x)(1+mC(n+1))+\sum_{l=1}^{n} u_{il}(x)\right]\cdot \tag{42}$$

$$\cdot\left[1+\mu(x)(1+mC(n+1))+\sum_{l=1}^{n} u_{jl}(x)\right]+\mu(x)mAC(n+1)[1+C(n+1)]+Au_{ij}(x),$$

$$u_{ij}(0) = B.$$

It follows from (29), (36) and from theorems on the continuous dependence of the solution of Cauchy problem on the initial values and on the right-hand sides that

$$\lim_{h\to 0} \bar{u}_{ij}^{(h)}(x) = \bar{u}_{ij}(x), \quad i, j = 1, \ldots, n,$$

uniformly with respect to $x\in[0, b_0]$.

In virtue of (28) and (40) we get (making h in (40) tend to zero)

$$\left|\frac{\partial^2 z_\tau^{(k+1)}(x, Y)}{\partial y_i\,\partial y_j}\right| \leqq \bar{u}_{ij}(x), \quad i, j = 1, \ldots, n, \ (x, Y)\in \bar{T}(b_0, \eta, A). \tag{43}$$

Since

$$\frac{d\bar{u}_{ij}(x)}{dx} \leqq A\left[1+\left(r-n-\frac{1}{2}\right)\mu(x)+\left(n+\frac{1}{2}\right)\bar{u}_{ij}(x)\right]^2, \quad i, j = 1, \ldots, n$$

we obtain

$$\bar{u}_{ij}(x) \leqq \tilde{u}(x), \quad x\in[0, b), \quad i, j = 1, \ldots, n, \tag{44}$$

where $\tilde{u}$ is the solution of the initial problem

$$\frac{du(x)}{dx} = A\left[1+\left(r-n-\frac{1}{2}\right)\mu(x)+\left(n+\frac{1}{2}\right)u(x)\right]^2, \quad u(0) = B.$$

Because $\tilde{u}(x)=\mu(x)$ for $x\in[0, c)$, then we have by (43), (44) that

$$\left|\frac{\partial^2 z_\tau^{(k+1)}(x, Y)}{\partial y_i\,\partial y_j}\right| \leqq \mu(x) \tag{45}$$

where $(x, Y)\in \bar{T}(b_0, \eta, A)$.

For each point $(x, Y)\in E$ we can choose b_0, η so large that $(x, Y)\in \bar{T}(b_0, \eta, A)$ and $0<b_0<b$, $b<\frac{\eta}{A}$. Therefore inequality (45) is satisfied in E.

In a similar way we can prove that

$$\left|\frac{\partial^2 z_\tau^{(k+1)}(x, Y)}{\partial x^2}\right| \leqq \mu(x), \quad \left|\frac{\partial^2 z_\tau^{(k+1)}(x, Y)}{\partial x\,\partial y_i}\right| \leqq \mu(x), \quad i = 1, \ldots, n, \quad x, Y\in E. \tag{46}$$

Because $\mu(c)=\tilde{p}$ and $b\leqq c$ then we obtain by (45), (46) the estimations

$$\left|\frac{\partial^2 z_\tau^{(k+1)}(x, Y)}{\partial s\,\partial t}\right| \leqq \tilde{p}, \quad (x, Y)\in E, \ \tau = 1, \ldots, \bar{m}, \ s, t\in \Delta.$$

Thus the proof of (III_{k+1}) is complete.

As an immediate corollary of (II_{k+1}), (III_{k+1}) we obtain (IV_{k+1}). Now, we obtain Lemma 1 by induction.

III. The convergence of the sequences $\{Z^{(k)}\}$ and $\left\{\dfrac{\partial z_\tau^{(k)}}{\partial Y}\right\}$

Let

$$\overline{T}(b_0, \eta, A_0) = \{(x, Y)\colon x\in[0, b_0],\ |y_i| \leqq \eta - A_0 x,\ i = 1, \dots, n\},$$

where $0<b_0<b$, $b<\dfrac{\eta}{A_0}$, $A_0=\max(A, C)$. Put

$$M = \max_{\tau} \max_{(x,Y)\in \overline{T}(b_0,\eta,A_0)} \left| \frac{\partial z_\tau^{(0)}(x, Y)}{\partial x} - \right.$$

$$\left. -f^{(\tau)}\left(x, Y, z_\tau^{(0)}(x, Y), U^{(0)}(\varphi(x, Y), \Psi(x, Y)), \frac{\partial z_\tau^{(0)}(x, Y)}{\partial Y}\right)\right|,$$

$$N = mAe^{Ab}, \quad \tilde{A} = A(Nb+m)\left\{be^{Ab}[1+\tilde{r}+n\tilde{r}C(n+1)+\tilde{p}n]+\frac{nCM}{m}\right\}+nC\tilde{B}(1+nA^2),$$

$$\tilde{B} = \max\left[\tilde{r}+B,\ nA\bigl(1+\tilde{r}+n\tilde{r}C(n+1)+\tilde{p}n\bigr)+A\right], \quad P = \max\,[N,\ \tilde{A}\tilde{B}^{-1}e^{Bb}].$$

LEMMA 2. *If Assumptions* H_1 *and* H_2 *are satisfied then*

$$|z_\tau^{(k)}(x, Y) - z_\tau^{(k-1)}(x, Y)| \leqq \frac{M}{mA}\frac{(Nx)^k}{k!}, \quad (x, Y)\in \overline{T}(b_0, \eta, A_0), \tag{47}$$

$$\tau = 1, \dots, \overline{m},\ k = 1, 2, \dots$$

and

$$\left|\frac{\partial z_\tau^{(k+1)}(x, Y)}{\partial y_i} - \frac{\partial z_\tau^{(k)}(x, Y)}{\partial y_i}\right| \leqq \tilde{B}\frac{(Px)^k}{k!}, \quad (x, Y)\in \overline{T}(b_0, \eta, A_0), \tag{48}$$

$$\tau = 1, \dots, \overline{m},\ k = 0, 1, 2, \dots.$$

PROOF. At first we prove (47) for $k=1$. It follows from Assumption H_2 that the function $z_\tau^{(1)}(x, Y)-z_\tau^{(0)}(x, Y)$ satisfies the differential inequality

$$\left|\frac{\partial}{\partial x}\bigl(z_\tau^{(1)}(x, Y) - z_\tau^{(0)}(x, Y)\bigr)\right| \leqq A\,|z_\tau^{(1)}(x, Y) - z_\tau^{(0)}(x, Y)| + M +$$

$$+A_0\sum_{j=0}^{n}\left|\frac{\partial}{\partial y_j}\bigl(z_\tau^{(1)}(x, Y) - z_\tau^{(0)}(x, Y)\bigr)\right|, \quad (x, Y)\in \overline{T}(b_0, \eta, A_0),$$

and the initial condition

$$z_\tau^{(1)}(0, Y) - z_\tau^{(0)}(0, Y) = 0, \quad (0, Y)\in \overline{T}(b_0, \eta, A_0).$$

In virtue of theorems on differential inequalities we have

$$|z^{(1)}(x, Y) - z^{(0)}(x, Y)| \leqq \bar{v}_0(x), \quad (x, Y)\in \overline{T}(b_0, \eta, A_0), \tag{49}$$

where $\bar{v}_0$ is the solution of the Cauchy problem

$$\frac{dv}{dx} = Av + M, \quad v(0) = 0.$$

Since $\bar{v}_0(x)=\dfrac{M}{A}(e^{Ax}-1)$ and $e^{Ax}-1\leqq Axe^{Ax}$ for $x\geqq 0$ it follows from (49) that

$$|z_\tau^{(1)}(x, Y)-z_\tau^{(0)}(x, Y)| \leqq \frac{M}{mA}\frac{Nx}{1!}, \quad (x, Y)\in \bar{T}(b_0, \eta, A_0), \ \tau = 1, \dots, \bar{m},$$

which completes the proof of (47) for $k=1$.

Let us suppose that for some $k\geqq 1$ inequalities (47) are true. It follows from Assumption H_2 and from (47) that the function $z_\tau^{(k+1)}(x, Y)-z_\tau^{(k)}(x, Y)$ satisfies the differential inequality

$$(50) \qquad \left|\frac{\partial}{\partial x}\left(z_\tau^{(k+1)}(x, Y)-z_\tau^{(k)}(x, Y)\right)\right| \leqq A|z_\tau^{(k+1)}(x, Y)-z_\tau^{(k)}(x, Y)|+$$

$$+M\frac{(Nx)^k}{k!}+A_0\sum_{j=0}^{n}\left|\frac{\partial}{\partial y_j}\left(z_\tau^{(k+1)}(x, Y)-z_\tau^{(k)}(x, Y)\right)\right|, \quad (x, Y)\in \bar{T}(b_0, \eta, A_0).$$

Since $z_\tau^{(k+1)}(0, Y)-z_\tau^{(k)}(0, Y)=0$ for $(0, Y)\in \bar{T}(b_0, \eta, A_0)$ we obtain

$$(51) \quad |z_\tau^{(k+1)}(x, Y)-z_\tau^{(k)}(x, Y)| \leqq \bar{v}_k(x), \quad (x, Y)\in \bar{T}(b_0, \eta, A_0), \ \tau = 1, \dots, \bar{m},$$

where $\bar{v}_k$ is the solution of the initial problem

$$\frac{dv}{dx} = Av+M\frac{(Nx)^k}{k!}, \quad v(0) = 0.$$

Because

$$\bar{v}_k(x) = \frac{MN^k}{A^{k+1}}\left[e^{Ax}-1-\frac{Ax}{1!}-\frac{(Ax)^2}{2!}-\dots-\frac{(Ax)^k}{k!}\right]$$

and

$$(53) \qquad e^x-1-\frac{x}{1!}-\dots-\frac{x^k}{k!} \leqq \frac{x^{k+1}}{(k+1)!}e^x \quad \text{for} \quad x\geqq 0$$

we conclude by (51) that

$$|z_\tau^{(k+1)}(x, Y)-z_\tau^{(k)}(x, Y)| \leqq \frac{MN^k}{A^{k+1}}\frac{(Ax)^{k+1}}{(k+1)!}e^{Ax} \leqq \frac{M}{mA}\frac{(Nx)^{k+1}}{(k+1)!}, \quad \tau = 1, \dots, \bar{m},$$

which completes the proof of (47) for $k+1$.

Now, we obtain the estimations (47) by induction.

We will prove (48). Since

$$\left|\frac{\partial z_\tau^{(1)}(x, Y)}{\partial y_i}-\frac{\partial z_\tau^{(0)}(x, Y)}{y_i}\right| \leqq \tilde{r}+B \leqq \tilde{B}, \quad \tau = 1, \dots, \bar{m}, \ i = 1, \dots, n,$$

for $(x, Y)\in \bar{T}(b_0, \eta, A_0)$, it follows that the estimations (48) are satisfied for $k=0$. Let us suppose that for some $k\geqq 1$ the inequalities

$$(54) \qquad \left|\frac{\partial z_\tau^{(k)}(x, Y)}{\partial y_i}-\frac{\partial z_\tau^{(k-1)}(x, Y)}{\partial y_i}\right| \leqq \tilde{B}\frac{(Px)^{k-1}}{(k-1)!}, \quad (x, Y)\in \bar{T}(b_0, \eta, A_0),$$

$$\tau = 1, \dots \bar{m}, \quad i = 1, \dots, n,$$

are true. To prove that (54) is satisfied for $k+1$ we will show first that the differential inequalities

$$(55)\qquad \left|\frac{\partial}{\partial x}\left(\frac{\partial z_\tau^{(k+1)}(x,Y)}{\partial y_i}-\frac{\partial z_\tau^{(k)}(x,Y)}{\partial y_i}\right)\right| \leqq$$

$$\leqq A[1+\tilde r+n\tilde r C(1+n)+\tilde p n]\sum_{j=1}^{n}\left|\frac{\partial z_\tau^{(k+1)}(x,Y)}{\partial y_j}-\frac{\partial z_\tau^{(k)}(x,Y)}{\partial y_j}\right|+$$

$$+A\left|\frac{\partial z_\tau^{(k+1)}(x,Y)}{\partial y_i}-\frac{\partial z_\tau^{(k)}(x,Y)}{\partial y_i}\right|+\tilde A\frac{(Px)^{k-1}}{(k-1)!}+$$

$$+A_0\sum_{j=1}^{n}\left|\frac{\partial}{\partial y_j}\left(\frac{\partial z_\tau^{(k+1)}(x,Y)}{\partial y_i}-\frac{\partial z_\tau^{(k)}(x,Y)}{\partial y_i}\right)\right|,\quad (x,Y)\in\overline{T}(b_0,\eta,A_0),$$

$$\tau=1,\dots,\overline{m},\; i=1,\dots,n,$$

hold.

It follows from Assumption H_2 and from (47), (54) that for each $u\in\{y_1,\dots,y_1,z,z_1,\dots,z_m\}$

$$(56)\qquad \left|f_u^{(\tau)}\big(P_\tau^{(k)}(x,Y)\big)-f_u^{(\tau)}\big(P_\tau^{(k-1)}(x,Y)\big)\right|\leqq \Lambda_k(x,Y)$$

where $P_\tau^{(k)}(x,Y)$ is defined by (33) and

$$(57)\quad \Lambda_k(x,Y)=\frac{M}{m}\frac{(Nx)^{k+1}}{(k+1)!}+M\frac{(Nx)^k}{k!}+A\sum_{j=1}^{n}\left|\frac{\partial z_\tau^{(k+1)}(x,Y)}{\partial y_j}-\frac{\partial z_\tau^{(k)}(x,Y)}{\partial y_j}\right|.$$

From (2), (12), (34) (47), (54) we obtain

$$(58)\qquad |\tilde P_{jy_i}^{(k)}(x,Y)-\tilde P_{jy_i}^{(k-1)}(x,Y)|\leqq$$

$$\leqq C\left[\frac{M}{m}\frac{(Nx)^k}{k!}+M\frac{(Nx)^{k-1}}{(k-1)!}+An\tilde B\frac{(Px)^{k-1}}{(k-1)!}\right]+Cn\tilde B\frac{(Px)^{k-1}}{(k+1)!}.$$

Because

$$\left|\frac{\partial}{\partial x}\left(\frac{\partial z_\tau^{(k+1)}(x,Y)}{\partial y_i}-\frac{\partial z^{(k)}(x,Y)}{\partial y_i}\right)\right|\leqq\left|f_{y_i}^{(\tau)}\big(P_\tau^{(k)}(x,Y)\big)-f_{y_i}^{(\tau)}\big(P^{(k-1)}(x,Y)\big)\right|+$$

$$+\left|f_z^{(\tau)}\big(P_\tau^{(k)}(x,Y)-f_z^{(\tau)}\big(P_\tau^{(k-1)}(x,Y)\big)\right|\left|\frac{\partial z_\tau^{(k+1)}(x,Y)}{\partial y_i}\right|+$$

$$+\left|f_z^{(\tau)}\big(P_\tau^{(k-1)}(x,Y)\big)\right|\left|\frac{\partial z_\tau^{(k+1)}(x,Y)}{\partial y_i}-\frac{\partial z_\tau^{(k)}(x,Y)}{\partial y_i}\right|+$$

$$+\sum_{j=1}^{m}\left[\left|f_{z_j}^{(\tau)}\big(P_\tau^{(k)}(x,Y)\big)-f_{z_j}^{(\tau)}\big(P_\tau^{(k-1)}(x,Y)\big)\right|\,|\tilde P_{jy_i}^{(k)}(x,Y)|\right]+$$

$$+\sum_{j=1}^{m}\left[|\tilde P_{yj_i}^{(k)}(x,Y)-\tilde P_{jy_i}^{(k-1)}(x,Y)|\,\left|f_{z_j}^{(\tau)}\big(P_\tau^{(k-1)}(x,Y)\big)\right|\right]+$$

$$+\sum_{j=1}^{n}\left[\left|f_{q_j}^{(\tau)}\big(P_\tau^{(k)}(x,Y)\big)-f_{q_j}^{(\tau)}\big(P_\tau^{(k-1)}(x,Y)\big)\right|\left|\frac{\partial^2 z_\tau^{(k+1)}(x,Y)}{\partial y_i\,\partial y_j}\right|\right]+$$

$$+\sum_{j=1}^{n}\left[\left|f_{q_j}^{(\tau)}\big(P_\tau^{(k-1)}(x,Y)\big)\right|\left|\frac{\partial}{\partial y_j}\left(\frac{\partial z_\tau^{(k+1)}(x,Y)}{\partial y_i}-\frac{\partial z_\tau^{(k)}(x,Y)}{\partial y_i}\right)\right|\right]$$

then we obtain (55) by Lemma 1 and by (56)—(58).

Since

$$\frac{\partial z_\tau^{(k+1)}(0, Y)}{\partial y_i} - \frac{\partial z_\tau^{(k)}(0, Y)}{\partial y_i} = 0, \quad i = 1, \dots, n, \quad \tau = 1, \dots, \overline{m},$$

for $(0, Y) \in \overline{T}(b_0, \eta, A_0)$, it follows from theorems on differential inequalities that

$$\left| \frac{\partial z_\tau^{(k+1)}(x, Y)}{\partial y_i} - \frac{\partial z_\tau^{(k)}(x, Y)}{\partial y_i} \right| \leqq \bar{v}_i(x), \quad (x, Y) \in \overline{T}(b_0, \eta, A_0), \tag{59}$$

$$i = 1, \dots, n, \quad \tau = 1, \dots, \overline{m},$$

where $(\bar{v}_1, \dots, \bar{v}_n)$ is the solution of the initial problem

$$\frac{dv_i}{dx} = A[1 + \tilde{r} + n\tilde{r}C(n+1) + \tilde{p}n] \sum_{j=1}^{n} v_j + Av_i + \tilde{A}\frac{(Px)^{k-1}}{(k-1)!}, \quad i = 1, \dots, n,$$

$$v_i(0) = 0.$$

Since

$$\bar{v}_i(x) \leqq \frac{\tilde{A}P^{k-1}}{\tilde{B}^k}\left[e^{\tilde{B}x} - 1 - \frac{\tilde{B}x}{1!} - \frac{(\tilde{B}x)^2}{2!} - \dots - \frac{(\tilde{B}x)^{k-1}}{(k-1)!}\right], \quad i = 1, \dots, n,$$

it follows from (53), (59) that

$$\left| \frac{\partial z_\tau^{(k+1)}(x, Y)}{\partial y_i} - \frac{\partial z_\tau^{(k)}(x, Y)}{y_i} \right| \leqq \frac{\tilde{A}P^{k-1}}{\tilde{B}^k} e^{\tilde{B}x} \frac{(\tilde{B}x)^k}{k!} \leqq \tilde{B}\frac{(Px)^k}{k!},$$

$$(x, Y) \in \overline{T}(b_0, \eta, A_0), \quad i = 1, \dots, n, \quad \tau = 1, \dots, \overline{m}.$$

Now, we obtain (48) by induction.
Thus the proof of Lemma 2 is complete.

IV. Theorems on the existence and uniqueness of solutions

We have

THEOREM 1. *If Assumptions* H_1 *and* H_2 *are satisfied then there exists on* E *a unique solution* $\overline{Z} = (\bar{z}_1, \dots, \bar{z}_m)$ *of the Cauchy problem* (1). *In an arbitrary closed and bounded domain enclosed in* E *the sequence* $\{Z^{(k)}\}$ *defined by* (7)—(13) *and the sequences of partial derivatives* $\left\{\frac{\partial z_\tau^{(k)}}{\partial x}\right\}$ *and* $\left\{\frac{\partial z_\tau^{(k)}}{\partial Y}\right\}$, $\tau = 1, \dots, \overline{m}$, *are uniformly convergent to the solution* $\overline{Z}$ *and its derivatives* $\frac{\partial \bar{z}_\tau}{\partial x}$, *and* $\frac{\partial \bar{z}_\tau}{\partial Y}$ *respectively.*

PROOF. In virtue of Lemma 1 it follows that the functions $Z^{(k)}$, $k = 0, 1, 2, \dots$, are defined on E. From Lemma 2 we obtain that the sequences $\{Z^{(k)}\}$, $\left\{\frac{\partial z_\tau^{(k)}}{\partial Y}\right\}$,

$\tau=1, \dots, \overline{m}$, are uniformly convergent in $\overline{T}(b_0, \eta, A_0)$. Because

$$\text{(60)}\quad \frac{\partial z_\tau^{(k+1)}(x, Y)}{\partial x} = f^{(\tau)}\left(x, Y, z_\tau^{(k+1)}(x, Y), U^{(k)}(\varphi(x, Y), \Psi(x, Y)), \frac{\partial z_\tau^{(k+1)}(x, Y)}{\partial Y}\right),$$

$$(x, Y) \in E, \quad \tau = 1, \dots, \overline{m}, \; k = 0, 1, 2, \dots$$

it follows from Assumptions H_1 and H_2 and from Lemma 2 that the sequences $\left\{\frac{\partial z_\tau^{(k)}}{\partial x}\right\}$, $\tau=1, \dots, \overline{m}$, are uniformly convergent in $\overline{T}(b_0, \eta, A_0)$.

Put

$$\overline{Z}(x, Y) = \lim_{k\to\infty} Z^{(k)}(x, Y), \quad (x, Y) \in \overline{T}(b_0, \eta, A_0).$$

Then, from (9), (13), (60) it follows that $\overline{Z}$ is the solution of (1) in $\overline{T}(b_0, \eta, A_0)$.

For an arbitrary closed and bounded domain E^* such that $E^* \subset E$ we can choose $b_0, \eta, 0<b_0<b, \eta>0, b_0<\frac{\eta}{A_0}$, so large that $E^* \subset \overline{T}(b_0, \eta, A_0)$. Hence the function $\overline{Z}$ is the solution of (1) in E.

The uniqueness of solution $\overline{Z}$ of (1) follows immediately from [13]. Thus the proof of Theorem 1 is complete.

The initial problem mentioned above is such that the initial set I is the $n+1$-dimensional zone and the initial function α is the function of $n+1$ variables. In the following Theorem 2 we shall consider a Cauchy problem for partial differential-functional equations with the initial set I_0 of the form

$$I_0 = \{(x, Y) \colon x = 0, \; Y \in \mathbf{R}^n\}.$$

The initial function $\tilde{\alpha}$ is a function of n variables.

THEOREM 2. *Assume that*

1° *the functions* $f^{(\tau)}$, $\tau=1, \dots, \overline{m}$, *of the variables* (x, Y, z, Z, Q) *satisfy condition* 1° *of Assumption* H_1 *and condition* 1° *of Assumption* H_2,

2° *the functions* φ_i, $\Psi^{(i)}=(\psi_1^{(i)}, \dots, \psi_n^{(i)})$, $i=1, \dots, m$, *are of class* C^2 *in* E_0 *and*

$$0 \leqq \varphi_i(x, Y) \leqq x, \quad (x, Y) \in E_0, \quad i = 1, \dots, m,$$

3° *the initial function* $\tilde{\alpha}=(\tilde{\alpha}_1, \dots, \tilde{\alpha}_m)$ *of the variable* Y *is of class* C^2 *for* $Y \in \mathbf{R}^n$,

4° *there exist constants* $B, C \geqq 0$ *such that*

$$\left|\frac{\partial \tilde{\alpha}_\tau(Y)}{\partial y_i}\right|, \; \left|\frac{\partial^2 \tilde{\alpha}_\tau(Y)}{\partial y_i \partial y_j}\right| \leqq B, \quad Y \in \mathbf{R}^n, \quad i, j = 1, \dots, n, \; \tau = 1, \dots, \overline{m},$$

and for $s, t \in \Delta$

$$\left|\frac{\partial \varphi_i(x, Y)}{\partial s}\right|, \; \left|\frac{\partial \psi_j^{(i)}(x, Y)}{\partial s}\right|, \; \left|\frac{\partial^2 \varphi_i(x, Y)}{\partial s\, \partial t}\right|, \; \left|\frac{\partial^2 \varphi_j^{(i)}(x, Y)}{\partial s\, \partial t}\right| \leqq C$$

on E_0, $i=1, \dots, m$, $j=1, \dots, n$.

Under these assumptions there exists on E *(see* I*) a unique solution* $\overline{Z}$ *of the initial problem*

$$\text{(61)}\quad \frac{\partial z_\tau(x, Y)}{\partial x} = f^{(\tau)}\left(x, Y, z_\tau(x, Y), Z(\varphi(x, Y), \Psi(x, Y)), \frac{\partial z_\tau(x, Y)}{\partial Y}\right), \quad \tau = 1, \dots, \overline{m},$$

$$z_\tau(0, Y) = \tilde{\alpha}_\tau(Y), \quad Y \in \mathbf{R}^n.$$

Let

$$Z^{(0)}(x, Y) = \tilde{\alpha}(Y), \quad (x, Y) \in E,$$

if $Z^{(k)}, Z^{(k)} = (z_1^{(k)}, \ldots, z_m^{(k)})$, *is a known function then* $z_\tau^{(k+1)}$ *is a solution of the Cauchy problem*

$$\frac{\partial z_\tau(x, Y)}{\partial x} = f^{(\tau)}\left(x, Y, z_\tau(x, Y), Z^{(k)}(\varphi(x, Y), \Psi(x, Y)), \frac{\partial z_\tau(x, Y)}{\partial Y}\right),$$

$$z_\tau(0, Y) = \tilde{\alpha}_\tau(Y), \quad Y \in \mathbf{R}^n.$$

In an arbitrary closed and bounded domain enclosed in E *the sequence* $\{Z^{(k)}\}$ *is uniformly convergent to the solution* $\bar{Z}$ *of* (61) *and the sequences of partial derivatives* $\left\{\frac{z_\tau^{(k)}}{\partial x}\right\}, \left\{\frac{\partial z_\tau^{(k)}}{\partial Y}\right\}$, $\tau = 1, \ldots, \bar{m}$, *are uniformly convergent to the derivatives* $\frac{\partial \bar{z}_\tau}{\partial x}$ *and* $\frac{\partial \bar{z}_\tau}{\partial Y}$ *respectively.*

The proof of this theorem is similar to the proof of Theorem 1.

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Let

$$z^{(0)}(x, t) \in [illegible], \quad t \in [illegible]$$

If $Z^{(1)}, Z^{(2)}, \ldots$ [illegible] is a convergent sequence [illegible] solution of the Cauchy problem

$$\frac{dz_i(t, [illegible])}{dt} = [illegible]$$

[illegible] the sequence [illegible] and [illegible] sequences of partial derivatives [illegible] converge to the derivatives [illegible]

The proof of this theorem is similar to the proof of Theorem [illegible]

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ASYMPTOTISCHE ENTWICKLUNG VON INTEGRALEN MIT ZWEI PARAMETERN

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1. Einleitung

Das Anliegen dieser Arbeit besteht in der asymptotischen Auswertung von Integralen der Form

$$\int f(t) e^{x\psi(t) - y\varphi(t)} \, dt,$$

wobei $f(t), \psi(t), \varphi(t)$ analytische Funktionen bezeichnen, die — etwa in der Umgebung des Nullpunktes — holomorph sind, und der Integrationsweg in der komplexen Ebene verläuft. Die Ausdehnung der Laplaceschen Methode für einparametrige Integrale auf Integrale obiger Gestalt, wobei der Integrationsweg aus einem Intervall der reellen Achse besteht, kann man in den Arbeiten von TRICOMI [8], FULKS [2], und PEDERSON [7] verfolgen. Für beliebige Integrationswege werden Methoden in den Arbeiten von ERDÉLYI—WYMAN [1] und URSELL [9] besprochen. Unser Vorhaben lehnt sich zu wesentlichen Teilen an den Abschnitt über asymptotische Entwicklungen nach Airy-Funktionen in [1] an.

Wir gehen von der Voraussetzung aus, daß der wesentliche Anteil bei der asymptotischen Entwicklung der Integrale von der Umgebung des Nullpunktes herrüht. Verhalten sich die Funktionen $\psi(t), \varphi(t)$ für $t \to 0$ wie

$$\varphi(t) \sim a_0 t^\nu, \quad \psi(t) \sim b_0 t^\mu$$

mit $\nu > \mu$, so streben wir asymptotische Entwicklungen für $x>0, y>0, x+y \to \infty$, $x = o(y)$ an. Dabei legen wir die in [1] dargelegte und in [3] spezifizierte Auffassung über die Asymptotik von Funktionen mehrerer Veränderlichen zugrunde. Der Fall $\nu = 2$, $\mu = 1$ (Satz 3 dieser Arbeit) wurde bereits in [5] entwickelt. Für $\nu = 3$, $\mu = 1$ oder 2 erhält man die in [1] abgehandelte Entwicklung nach Airy-Funktionen. Für beliebige ν und μ stellen wir in § 3 eine Methode vor, die ihren Ausgangspunkt in der Sattelpunktsmethode für einparametrige Integrale hat. Die Sätze 1 und 2 — die Hauptsätze unserer Arbeit — bilden die entscheidende Grundlage der Methode. In ihnen werden in x, y gleichmäßige Entwicklungen nach verallgemeinerten Airy—Hardyschen Integralen ausgesprochen. In § 2 stellen wir in den Hilfssätzen 1 bis 4 die notwendigen Grundlagen für diese Integrale zusammen. Die Hilfssätze 5 und 6 in § 3 sind für die Methode selbst von Bedeutung. Die Sätze 3 bis 5 in § 4 stellen wichtige Spezialfälle der Sätze 1 und 2 dar. In § 5 illustrieren wir unsere Methode an Beispielen verallgemeinerter Bessel-Funktionen.

Wir benutzen ständig die folgenden Bezeichnungsweisen: Sind $f, g, \varphi, \ldots$ Funktionen der beiden Veränderlichen x, y und gilt für $x>0, y>0, x+y \to \infty$, $x = o(y)$ gleichzeitig $f = O(g)$ und $g = O(f)$, so schreiben wir $f \asymp g$. $f \sim g$ bedeutet $\lim \dfrac{f}{g} = 1$. Eine unendliche Folge von Funktionen $\{\varphi_n\}$ heißt eine asymptotische Skala, wenn $\varphi_{n+1} = o(\varphi_n)$ für alle n gilt.

Es bezeichne $\sum_{\nu=0}^{\infty} g_\nu$ bezüglich der Skala $\{\varphi_\nu\}$ eine asymptotische Entwicklung der Funktion f, geschrieben

$$f \sim \sum_{\nu=0}^{\infty} g_\nu;\ \{\varphi_\nu\},$$

wenn für alle $N \geqq 0$

$$f - \sum_{\nu=0}^{N} g_\nu = o(\varphi_N)$$

gilt.

Ist $f = f_1 + f_2 + \ldots + f_n$ mit

$$f_k \sim \sum_{\nu=0}^{\infty} g_{k,\nu};\ \{\varphi_{k,\nu}\}$$

für $k = 1, 2, \ldots, n$, so werden wir für die Summe f die Bezeichnung

$$f \sim \sum_{k=1}^{n} \sum_{\nu=0}^{\infty} g_{k,\nu}; \{\varphi_{1,\nu}\}, \ldots, \{\varphi_{n,\nu}\}$$

benutzen und dies ebenfalls als asymptotische Entwicklung von f ansprechen.

2. Verallgemeinerte Airy—Hardysche Integrale

Definition. *Es seien* $x = (x_2, x_3, \ldots, x_\nu)$, $\nu \geqq 2$, $P_\nu(t, x) = -\sum_{k=2}^{\nu} x_k \frac{t^k}{k!}$, $x_\nu \neq 0$ *und* n *nichtnegativ ganz, dann heißt*

$$W_{n,\nu}(x; \alpha, \beta) = \int_{\mathfrak{L}_{\alpha,\beta}} t^n \exp\{P_\nu(t, x)\}\, dt \tag{1}$$

verallgemeinertes Airy—Hardysches Integral, sofern der Integrationsweg $\mathfrak{L}_{\alpha,\beta}$ *von* $e^{i\alpha}\infty$ *über* 0 *nach* $e^{i\beta}\infty$, *in verschiedenen Sektoren, die durch* $\cos(\nu\vartheta + \gamma_\nu) > 0$, $\cos(2\vartheta + \gamma_2)$ *mit* $\operatorname{arc} t = \vartheta$, $\operatorname{arc} x_k = \gamma_k$ *gekennzeichnet sind, verläuft.*

Wir betrachten zwei Spezialfälle von (1).

a) Der Fall $\nu = 2$. Wir bezeichnen aus Einfachheitsgründen $x = x_2$ und setzen ohne Einschränkung der Allgemeinheit $|\operatorname{arc} x| < \frac{\pi}{2}$ voraus. Dann lasser sich α bzw. β als Winkel in zwei Viertelebenen, mittels $\cos(2\vartheta + \gamma) > 0$ bestimmt, auswählen. Beispielsweise ist $\alpha = \pi$ bzw. $\beta = 0$ möglich. Setzen wir noch

$$W_{n,2}(x; \pi, 0) = W_{n,2}(x),$$

so ergibt sich für

$$W_{n,2}(x) = \int_{-\infty}^{\infty} t^n \exp\left(-\frac{x}{2} t^2\right) dt$$

unmittelbar
$$W_{n,2}(x) = 0, \quad \text{falls} \quad n \equiv 1\,(2) \quad \text{ist.}$$
Für
$$W_{2m,2}(x) = 2\int_0^\infty t^{2m} \exp\left(-\frac{x}{2}t^2\right) dt$$
folgt nach einfacher Rechnung
$$W_{2m,2}(x) = \Gamma\left(m+\frac{1}{2}\right)\left(\frac{x}{2}\right)^{-m-1/2}.$$

b) Der Fall $\nu=3$. Es ist bekannt, daß dann $W_{0,3}(x;\alpha,\beta)$ ein Airy—Hardysches Integral, siehe beispielsweise KRATZER—FRANZ [6], darstellt. Weiterhin lassen sich $W_{n,3}(x;\alpha,\beta)$, $n \geqq 1$, aus $W_{0,3}(x;\alpha,\beta)$, auf der Grundlage von Rekursionsformeln, vgl. ERDÉLYI—WYMAN [1], Seite 248, ermitteln.

BEMERKUNG. Wir können deshalb $W_{0,\nu}(x;\alpha,\beta)$ für $\nu>3$, als verallgemeinertes Airy—Hardysches Integral bezeichnen und behalten diese Terminologie auch bezüglich $W_{n,\nu}(x;\alpha,\beta)$, für $n \geqq 1$, bei.

Es gelten, wie man leicht beweisen kann, für $W_{n,\nu} = W_{n,\nu}(x;\alpha,\beta)$ die folgenden Rekursionsformeln
$$W_{1,\nu} = \frac{1}{x_2}\left\{x_3\frac{\partial W_{0,\nu}}{\partial x_2} + x_4\frac{\partial W_{0,\nu}}{\partial x_3} + \dots + x_\nu \frac{\partial W_{0,\nu}}{\partial x_{\nu-1}}\right\}$$
$$W_{2,\nu} = -2!\,\frac{\partial W_{0,\nu}}{\partial x_2}$$
$$\vdots$$
$$W_{\nu-1,\nu} = -(\nu-1)!\,\frac{\partial W_{0,\nu}}{\partial x_{\nu-1}}$$
$$\frac{x_\nu}{(\nu-1)!}W_{n+\nu-1,\nu} = -\frac{x_{\nu-1}}{(\nu-2)!}W_{n+\nu-2,\nu} - \dots - \frac{x_2}{1!}W_{n+1,\nu} + nW_{n-1,\nu} \quad (n \geqq 1).$$

Als nächstes untersuchen wir das asymptotische Verhalten von (1). Die hierbei geltenden Aussagen werden, wegen ihrer späteren Bedeutung in § 2, in Form von Hilfssätzen aufgeschrieben.

HILFSSATZ 1. *Unter den Voraussetzungen*

1) $\nu=2$,

2) $\nu>2$ *und*

   a) $x_k^\nu \asymp x_\nu^k$ *für* $k=2,\dots,\nu-1$

   b) $x_k^2 = o(x_2^k)$ *für* $k=3,\dots,\nu$

   c) $x_k^\nu = o(x_\nu^k)$ *für* $k=2,\dots,\nu-1$

gilt für $\varkappa = \sum_{k=2}^{\nu} |x_k|^{1/k} \to \infty$
$$W_{n,\nu}(x;\alpha,\beta) = O(\varkappa^{-n-1}).$$

BEWEIS. Da der Hilfssatz für 1) $\nu=2$ trivial ist, genügt der Beweis zu 2). Wir behandeln zunächst die Fälle a) und c) gemeinsam. Offensichtlich gilt hierfür $\varkappa\asymp x_\nu^{1/\nu}$. Die Substitution $t\to x_\nu^{-1/\nu}t$ bewirkt, daß in dem Polynom $P_\nu(x_\nu^{-1/\nu}t, x)$ die Koeffizienten von t^k $(2\leq k\leq \nu)$ beschränkt bleiben und der Koeffizient von t^ν verschieden von 0 ist. Das verbleibende Integral ist demzufolge beschränkt. Im Fall b) ist $\varkappa\sim x_2^{1/2}$. Die Substitution $t\to x_2^{-1/2}t$ bewirkt entsprechendes. Hier ist der Koeffizient von t^2 verschieden von 0. Somit ergibt sich die Behauptung in allen drei Fällen.

HILFSSATZ 2. *Entsprechend den Voraussetzungen über den Integrationsweg* $\mathfrak{L}_{\alpha,\beta}$ *in* (1) *existieren ganze Zahlen* l, l' *mit* $l\neq l'$, $0\leq l\leq l'\leq \nu-1$, *ferner seien* α, β *etwa durch die Einschränkungen*

$$\frac{\pi}{\nu}\left(2l-\frac{1}{2}\right)-\frac{\gamma_\nu}{\nu}<\alpha<\frac{\pi}{\nu}\left(2l+\frac{1}{2}\right)-\frac{\gamma_\nu}{\nu}$$

$$\frac{\pi}{\nu}\left(2l'-\frac{1}{2}\right)-\frac{\gamma_\nu}{\nu}<\beta<\frac{\pi}{\nu}\left(2l'+\frac{1}{2}\right)-\frac{\gamma_\nu}{\nu}$$

gegeben.

a) *Falls* $x_k^\nu=o(x_\nu^k)$ $(k=2, \dots, \nu-1)$, *gilt*

$$W_{n,\nu}(x;\alpha,\beta)=(e^{2\pi i l'(n+1)/\nu}-e^{2\pi i l(n+1)/\nu})Ax_\nu^{-(n+1)/\nu}+o(x_\nu^{-(n+1)/\nu}) \tag{2}$$

für $\varkappa\to\infty$.

b) *Unter der verschärften Annahme* $x_k^\nu=o(x_{k+1}^\nu x_\nu^{-1})$ $(k=2, \dots, \nu-1)$, *gilt*

$$\begin{aligned} W_{n,\nu}(x;\alpha,\beta)=&(e^{2\pi i l'(n+1)/\nu}-e^{2\pi i l(n+1)/\nu})Ax_\nu^{-(n+1)/\nu}-\\ &-(e^{2\pi i l' n/\nu}-e^{2\pi i l n/\nu})Bx_{\nu-1}x_\nu^{-(n+\nu)/\nu}+o(x_{\nu-1}x_\nu^{-(n+\nu)/\nu}) \end{aligned} \tag{3}$$

für $\varkappa\to\infty$.

Dabei sind $A=\dfrac{1}{\nu}(\nu!)^{(n+1)/\nu}\Gamma\left(\dfrac{n+1}{\nu}\right)$, $B=(\nu!)^{n/\nu}\Gamma\left(\dfrac{n+\nu}{\nu}\right)$.

BEWEIS. Wir schreiben an Stelle von (1)

$$W_{n,\nu}(x;\alpha,\beta)=\left\{\int_0^{e^{i\beta}\infty}-\int_0^{e^{i\alpha}\infty}\right\}t^n\exp\{P_\nu(t,x)\}\,dt=S-T$$

und betrachten zunächst nur das erste Integral S. Unter Beachtung der Definition von $P_\nu(t, x)$ gilt hierfür

$$S=\int_0^{e^{i\beta}\infty}t^n\exp\{P_\nu(t,x)\}\,dt=\int_0^{e^{i\beta}\infty}t^n\exp\left\{-x_\nu\frac{t^\nu}{\nu!}\right\}\sum_{m=0}^{\infty}\frac{1}{m!}\left\{P_\nu(t,x)+x_\nu\frac{t^\nu}{\nu!}\right\}^m dt$$

bzw. $S=I_1+I_2+I_3$ mit

$$I_1=\int_0^{e^{i\beta}\infty}t^n\exp\left\{-\frac{x_\nu}{\nu!}t^\nu\right\}dt,\qquad I_2=-\sum_{k=2}^{\nu-1}\frac{x_k}{k!}\int_0^{e^{i\beta}\infty}t^{n+k}\exp\left\{-\frac{x_\nu}{\nu!}t^\nu\right\}dt,$$

$$I_3=\sum_{m=2}^{\infty}\frac{1}{m!}\int_0^{e^{i\beta}\infty}t^n\left\{P_\nu(t,x)+x_\nu\frac{t^\nu}{\nu!}\right\}^m\exp\left\{-\frac{x_\nu}{\nu!}t^\nu\right\}dt$$

wobei nunmehr für I_1, I_2, I_3 asymptotische Darstellungen aufgestellt werden sollen. Durch Drehung des Integrationsweges erhält man

$$I_1 = \int_0^{e^{i\beta'}\infty} t^n \exp\left\{-\frac{x_\nu}{\nu!} t^\nu\right\} dt \quad \text{mit} \quad \beta' = \frac{2\pi i}{\nu} - \frac{\gamma_\nu}{\nu}$$

und hieraus

$$I_1 = e^{2\pi i l'(n+1)/\nu} \frac{1}{\nu} (\nu!)^{(n+1)/\nu} \Gamma\left(\frac{n+1}{\nu}\right) x_\nu^{-(n+1)/\nu}.$$

Analog zur Auswertung von I_1 lassen sich I_2 und I_3 behandeln. Wir erhalten

$$I_2 = -\sum_{k=2}^{\nu-1} \frac{1}{k!} e^{2\pi i l'(n+k+1)/\nu} \frac{1}{\nu} (\nu!)^{(n+k+1)/\nu} \Gamma\left(\frac{n+k+1}{\nu}\right) x_k x_\nu^{-(n+k+1)/\nu} = o(x_\nu^{-(n+1)/\nu})$$

und

$$I_3 = O(x_{\nu-1}^2 x_\nu^{-(n+2\nu-1)/\nu}) = o(x_{\nu-1} x_\nu^{-(n+\nu)/\nu}) = o(x_\nu^{-(n+1)/\nu}).$$

Ferner folgt unter Berücksichtigung der verschärften Annahme b) unseres Hilfssatzes für I_2 die genauere Aussage

$$I_2 = -e^{2\pi i l' n/\nu} (\nu!)^{n/\nu} \Gamma\left(\frac{n+\nu}{\nu}\right) x_{\nu-1} x_\nu^{-(n+\nu)/\nu} + o(x_{\nu-1} x_\nu^{-(n+\nu)/\nu}).$$

In völliger Analogie zum ersten Integral S läßt sich das zweite Intergal T behandeln, so daß sich damit die Behauptung ergibt.

HILFSSATZ 3. *Der Integrationsweg* $\mathfrak{L}_{\alpha,\beta}$ *in* (1) *sei so gewählt, daß er in zwei verschiedenen Viertelebenen verläuft.* α *und* β *mögen also etwa der Einschränkung*

$$\frac{3\pi}{4} - \frac{\gamma_2}{2} < \alpha < \frac{5\pi}{4} - \frac{\gamma_2}{2}, \quad -\frac{\pi}{4} - \frac{\gamma_2}{2} < \beta < \frac{\pi}{4} - \frac{\gamma_2}{2}$$

genügen.

a) *Falls* $x_k^2 = o(x_2^k)$ $(k=3, \ldots, \nu)$ *ist, so gilt*

$$W_{n,\nu}(x;\alpha,\beta) = (1+(-1)^n) C x_2^{-(n+1)/2} + o(x_2^{-(n+1)/2}) \tag{4}$$

für $\varkappa \to \infty$.

b) *Unter der verschärften Annahme* $x_k^2 = o(x_{k-1}^2 x_2)$ $(k=3, \ldots, \nu)$ *gilt*

$$W_{n,\nu}(x;\alpha,\beta) = (1+(-1)^n) C x_2^{-(n+1)/2} + (1+(-1)^n) D x_3 x_2^{-(n+4)/2} + o(x_3 x_2^{-(n+4)/2}) \tag{5}$$

für $\varkappa \to \infty$.

Dabei sind $C = 2^{(n-1)/2} \Gamma\left(\frac{n+1}{2}\right)$, $D = \frac{1}{3} 2^{n/2} \Gamma\left(\frac{n}{2} + 2\right)$.

BEWEIS. Wir schreiben an Stelle von (1)

$$W_{n,\nu}(x;\alpha,\beta) = \left\{\int_0^{t_0} + \int_{t_0}^{e^{i\beta}\infty} - \int_0^{t_0'} - \int_{t_0'}^{e^{i\alpha}\infty}\right\} t^n \exp\{P_\nu(t,x)\} dt = S_1 + S_2 - T_1 - T_2.$$

Wählt man nun $t_0=ax_2^{\nu/2}$, $t_0'=bx_2^{\nu/2}$ mit geeigneten a, b, so existiert sicherlich ein $K>0$ mit

$$S_2-T_2=O(e^{-K|x_2|^{\nu+1}}).$$

Im weiteren betrachten wir wiederum, wie im Beweis des Hilfssatzes 2, zunächst nur das erste Integral S_1. Hierfür gilt

$$S_1=\int_0^{t_0} t^n \exp\{P_\nu(t,x)\}\,dt=\int_0^{t_0} t^n \exp\left\{-x_2\frac{t^2}{2}\right\}\sum_{m=0}^{\infty}\frac{1}{m!}\left\{P_\nu(t,x)+x_2\frac{t^2}{2}\right\}^m dt=$$

$$=I_1+I_2+I_3$$

mit

$$I_1=\int_0^{t_0} t^n \exp\left\{-\frac{x_2}{2}t^2\right\}dt,\qquad I_2=-\sum_{k=3}^{\nu}\frac{x_k}{k!}\int_0^{t_0} t^{n+k}\exp\left\{-\frac{x_2}{2}t^2\right\}dt,$$

$$I_3=\sum_{m=2}^{\infty}\frac{1}{m!}\int_0^{t_0} t^n\left\{P_\nu(t,x)+x_2\frac{t^2}{2}\right\}^m \exp\left\{-\frac{x_2}{2}t^2\right\}dt.$$

Die asymptotische Darstellung von I_1, I_2 und I_3 erfolgt zweckmäßigerweise so, daß man den Integrationsweg von t_0 nach $e^{-i(\gamma_2/2)}\infty$ ziehen wird. Nach einfachen Rechnungen folgt

$$I_1=e^{-(n+1)\gamma_2/2}\int_0^{\infty} t^n\exp\left\{-\frac{|x_2|}{2}t^2\right\}dt+O(e^{-K|x_2|^{\nu+1}})=$$

$$=2^{(n-1)/2}\Gamma\left(\frac{n+1}{2}\right)x_2^{-(n+1)/2}+O(e^{-K|x_2|^{\nu+1}}).$$

Entsprechend ergeben sich

$$I_2=-\sum_{k=3}^{\nu}\frac{1}{k!}2^{(n+k-1)/2}\Gamma\left(\frac{n+k+1}{2}\right)x_k x_2^{-(n+k+1)/2}+O(e^{-K|x_2|^{\nu+1}})=o(x_2^{-(n+1)/2})$$

und

$$I_3=O(x_3^2x_2^{-(n+7)/2})=o(x_3x_2^{-(n+4)/2})=o(x_2^{-(n+1)/2})$$

sowie unter der verschärften Annahme b) unseres Hilfssatzes für I_2 die genauere Aussage

$$I_2=-\frac{1}{3}2^{n/2}\Gamma\left(\frac{n}{2}+2\right)x_3x_2^{-(n+4)/2}+o(x_3x_2^{-(n+4)/2}).$$

Da sich ganz analoge Ergebnisse für das Integral T_2 aufstellen lassen, folgt unmittelbar die Behauptung.

HILFSSATZ 4. *Es sei ein Integrationsweg* $\mathfrak{L}_{\alpha,\beta}$ — (1) *entsprechend* — *gegeben und* $\mathfrak{L}_0$ *ein Abschnitt dieses Weges, der von* t_1 *über* 0 *nach* t_2 *führt. Für* $\nu\geqq 3$ *und* $|x_\nu|\to\infty$ *sei*

$$\frac{x_k}{x_\nu}=c_k z^{\nu-k}\quad (k=2,3,\dots,\nu-1)\quad \textit{mit}\quad z=o(1).$$

Ferner möge auf dem gesamten Integrationsweg stets $\operatorname{Re}\{P_\nu(t,x)\}<0,$ *für* $|t|>0,$ *sein. Dann gibt es für alle* $\varepsilon>0$ *ein* $C>0$ *und ein* $K>0$*, so daß für* $|t_1|, |t_2|\geqq \geqq C|x_\nu|^{(\varepsilon-1)/\nu}$

$$\int_{\mathfrak{L}_0} t^n \exp\{P_\nu(t,x)\}\,dt = W_{n,\nu}(x;\alpha,\beta)+O(e^{-K|x_\nu|^\varepsilon}) \tag{6}$$

gilt.

Beweis. Wir schreiben $\int\limits_{\mathfrak{L}_0} t^n \exp\{P_\nu(t,x)\}\,dt = W_{n,\nu}(x;\alpha,\beta)-S$ mit $S=$ $=\left\{\int\limits_{t_2}^{e^{i\beta}\infty} - \int\limits_{t_1}^{e^{i\alpha}\infty}\right\} t^n \exp\{P_\nu(t,x)\}\,dt = I_2-I_1$ und $I_1=\int\limits_{t_1}^{e^{i\alpha}\infty} t^n \exp\left\{-x_\nu \sum\limits_{k=2}^{\nu-1} c_k z^{\nu-k}t^k - x_\nu t^\nu\right\}dt,$ $I_2=\int\limits_{t_2}^{e^{i\beta}\infty} t^n \exp\left\{-x_\nu \sum\limits_{k=2}^{\nu-1} c_k z^{\nu-k}t^k - x_\nu t^\nu\right\}dt,$ wobei nunmehr zu zeigen ist, daß I_1, $I_2=O(e^{-K|x_\nu|^\varepsilon})$ gilt.

Ohne Einschränkung der Allgemeinheit führen wir nur den Nachweis bezüglich I_1, da sich I_2 analog behandeln läßt. Es erweist sich dabei als zweckmäßig, mit einer Fallunterscheidung zu arbeiten.

1) Fall: $|z|\leqq |x_\nu|^{(\varepsilon-1)/\nu}$. Die Substitution $t=t'|x_\nu|^{(\varepsilon-1)/\nu}$ mit $t'=re^{i\vartheta}$ bewirkt, daß man

$$\operatorname{Re}\{P_\nu(t,x)\} = -|x_\nu|^\varepsilon \sum_{k=2}^{\nu} c_k'(|x_\nu|^{(1-\varepsilon)/\nu}|z|)^{\nu-k} r^k \tag{7}$$

mit reellem $c_k'=c_k'(\vartheta)$ erhält.

Beachtet man ferner die Voraussetzung $\operatorname{Re}\{P_\nu(t,x)\}<0$ des Hilfssatzes, so folgt mit geeignet gewähltem reellen $C, K'>0$ die für alle $r\geqq C$ geltende Ungleichung

$$\operatorname{Re}\{P_\nu(t,x)\} < -K'|x_\nu|^\varepsilon r^\nu$$

und damit

$$|I_1| \leqq \int_C^\infty r^n e^{-K'|x_\nu|^\varepsilon r^\nu}\,dr = O(e^{-K|x_\nu|^\varepsilon}).$$

2) Fall: $|z|>|x_\nu|^{(\varepsilon-1)/\nu}$. Auf Grund der Forderung $\operatorname{Re}\{P_\nu(t,x)\}<0$ ergibt sich aus (7) unmittelbar $c_2', c_\nu'>0$. Wegen $c_2'>0$ ist für hinreichend kleine r

$$-\sum_{k=2}^{\nu-1} c_k'\left(\frac{r}{|x_\nu|^{(1-\varepsilon)/\nu}|z|}\right)^k < 0.$$

Hieraus folgt, daß ein $r_1>0$ existiert, so daß für alle $r\leqq r_1|x_\nu|^{(1-\varepsilon)/\nu}|z|$

$$-\sum_{k=2}^{\nu} c_k'\left(\frac{r}{|x_\nu|^{(1-\varepsilon)/\nu}|z|}\right)^k \leqq 0.$$

gilt. Wir erhalten somit

$$\left|\int_{ce^{i\vartheta_0}}^{r_1|x_\nu|^{(1-\varepsilon)/\nu}|z|e^{i\vartheta_1}} t^n \exp\{P_\nu(t,x)\}\,dt\right| \leqq \int_c^{r_1|x_\nu|^{(1-\varepsilon)/\nu}|z|} r^n e^{-|x_\nu|^\varepsilon c_\nu'' r^\nu}\,dr$$

und wegen $|x_\nu|\,|z|^\nu > |x_\nu|^\varepsilon$ folglich

$$\int_{ce^{i\vartheta_0}}^{r_1|x_\nu|^{(1-\varepsilon)/\nu}|z|e^{i\vartheta_1}} t^n \exp\{P_\nu(t,x)\}\,dt = O(e^{-K|x_\nu|^\varepsilon}).$$

In entsprechender Weise ist wegen $c'_\nu>0$ für hinreichend große r

$$-\sum_{k=3}^{\nu} c'_k \left(\frac{r}{|x_\nu|^{(1-\varepsilon)/\nu}|z|}\right)^k < 0.$$

Es folgt nun wiederum, daß ein $r_2>0$ existiert, so daß für alle $r \geqq r_2|x_\nu|^{(1-\varepsilon)/\nu}|z|$

$$-\sum_{k=3}^{\nu} c'_k \left(\frac{r}{|x_\nu|^{(1-\varepsilon)/\nu}|z|}\right)^k \leqq 0$$

gilt. Wir bekommen deshalb unmittelbar

$$\int_{r_2|x_\nu|^{(1-\varepsilon)/\nu}|z|e^{i\vartheta_2}}^{e^{i\alpha}\infty} t^n \exp\{P_\nu(t,x)\}\,dt \leqq \int_{r_2|x_\nu|^{(1-\varepsilon)/\nu}|z|}^{\infty} r^n e^{-|x_\nu|^\varepsilon c_2(|x_\nu|^{(1-\varepsilon)/\nu}|z|)^{\nu-2}r^2}\,dr =$$

$$= O(e^{-K'|x_\nu|^\varepsilon |z|^\nu}) = O(e^{-K|x_\nu|^\varepsilon}).$$

Der Beweis des Hilfssatzes 4 ist erbracht, sofern die Konstanten r_1, r_2 so gewählt werden konnten, daß $r_1 \geqq r_2$ gilt. Andernfalls, daß heißt falls $r_1<r_2$ ist, muß noch das Integral

$$\int_{r_2|x_\nu|^{(1-\varepsilon)/\nu}|z|e^{i\vartheta_1}}^{r_2|x_\nu|^{(1-\varepsilon)/\nu}|z|e^{i\vartheta_2}} t^n \exp\{P_\nu(t,x)\}\,dt$$

geeignet abgeschätzt werden. Nach einfacher Rechnung ergibt sich

$$\int_{r_1|x_\nu|^{(1-\varepsilon)/\nu}|z|e^{i\vartheta_1}}^{r_2|x_\nu|^{(1-\varepsilon)/\nu}|z|e^{i\vartheta_2}} t^n \exp\{P_\nu(t,x)\}\,dt \leqq$$

$$\leqq (|x_\nu|^{(1-\varepsilon)/\nu}|z|)^{n+1}\int_{r_1}^{r_2} r^n \exp\left\{-|x_\nu|\,|z|^\nu \sum_{k=2}^{\nu} c'_k r_k\right\} dr = O(e^{-K'|x_\nu|\,|z|^\nu}) = O(e^{-K|x_\nu|^\varepsilon})$$

und insgesamt die Behauptung des Hilfssatzes 4.

3. Die Sattelpunktsmethode für zwei große Parameter

Das Ziel besteht in der asymptotischen Entwicklung von Integralen der Form

$$I(x,y) = \int_{\mathfrak{L}} f(t)\,e^{x\psi(t)-y\varphi(t)}\,dt$$

mit geeignetem Integrationsweg unter den folgenden Voraussetzungen:

A. Es seien $x>0$, $y>0$, und es gelte $x+y\to\infty$ unter der Einschränkung $x=o(y)$.

B. Die analytischen Funktionen $f(t)$, $\psi(t)$, $\varphi(t)$ seien holomorph in einer hinreichend großen Umgebung von $t=0$ mit den Potenzreihenentwicklungen

$$\psi(t) = t^{\mu} \sum_{j=0}^{\infty} b_j t^j, \quad b_0 \neq 0, \ \mu > 0 \text{ ganz,} \tag{8}$$

$$\varphi(t) = t^{\nu} \sum_{j=0}^{\infty} a_j t^j, \quad a_0 \neq 0, \ \nu > 0 \text{ ganz.} \tag{9}$$

Dabei sei immer $\nu > \mu$.

C. Mit τ werden solche Lösungen von

$$x\psi'(\tau) = y\varphi'(\tau)$$

bezeichnet, die für $x+y \to \infty$, $x=o(y)$ die Eigenschaften $\tau \neq 0$, $\tau = o(1)$ besitzen. Nachstehend aufgeführte Bezeichnungen werden ständig beibehalten:

$$\sigma = x\psi(\tau) - y\varphi(\tau), \tag{a}$$

$$\zeta_k = y\varphi^{(k)}(\tau) - x\psi^{(k)}(\tau) \quad (k = 2, 3, \ldots), \tag{b}$$

$$\zeta = (\zeta_2, \zeta_2, \ldots, \zeta_\nu), \tag{c}$$

$$P_\nu(t, \zeta) = -\sum_{k=2}^{\nu} \zeta_k \frac{t^k}{k!}, \tag{d}$$

$$\Delta_\nu(t, \zeta) = -\sum_{k=\nu+1}^{\infty} \zeta_k \frac{t^k}{k!}. \tag{e}$$

Beschreibung der Methode. Es soll beschrieben werden, wie unter bestimmten Annahmen die genannten Integrale asymptotisch entwickelt werden können, wobei das Verfahren durch die Sätze 1 und 2 gestützt wird. Zunächst werden entsprechend der Voraussetzung C die Sattelpunkte in der Umgebung von 0 bestimmt. Dann werde angenommen, daß der Integrationsweg über einen oder mehrere dieser Sattelpunkte gelegt werden kann. Weiter nehmen wir an, daß nur diese Sattelpunkte für das Integral wesentliche Beiträge liefern. Das bedeutet, es muß $I(x, y)$ so in Teilintegrale zerlegt werden können, daß diese genau einen Sattelpunkt enthalten. Die Restintegrale seien demgegenüber hinreichend klein. Die einen Sattelpunkt enthaltenden Teilintegrale müssen im gemeinsamen Holomorphiegebiet der drei Funktionen $f(t)$, $\varphi(t)$, $\psi(t)$ verlaufen. Nun können $f(t)$ und $x\psi(t) - y\varphi(t)$ in der Umgebung des Sattelpunktes τ in Potenzreihen entwickelt werden. Für die nunmehr auftretenden und nach der Substitution $t \to t+\tau$ über 0 laufenden Integrale

$$\int t^n \exp\{P_\nu(t, \zeta)\}\, dt$$

sei es möglich, sie so nach $e^{i\alpha}\infty$ und $e^{i\beta}\infty$ zu ziehen, daß für $|t|>0$ stets $\operatorname{Re}\{P_\nu(t, \zeta)\} < 0$ bleibt und die Restintegrale hinreichend klein ausfallen. Dann gibt dieser Sattelpunkt einen Anteil der asymptotischen Entwicklung nach Funktionen $W_{n,\nu}(\zeta; \alpha, \beta)$. Kann der Integrationsweg über alle in Frage kommenden Sattelpunkte gelegt werden, so erhält man eine in x, y gleichmäßige Entwicklung. Hierzu ist zu bemerken, daß einerseits nicht alle Lösungen von $x\psi'(\tau) = y\varphi'(\tau)$ Verwendung finden müssen, daß man andererseits aber auch mit ein oder zwei Sattelpunkten auskommt, sofern man gewisse ungleichmäßige Entwicklungen in Kauf nimmt. Die Benutzung der Funktionen $W_{n,\nu}(\zeta;\alpha,\beta)$ ist nur notwendig für $x^\nu \asymp y^\mu$. Für $x^\nu = o(y^\mu)$ beziehungsweise $y^\mu = o(x^\nu)$ können diese Funktionen entsprechend den Hilfssätzen 2 und 3 asymptotisch ausgewertet werden.

In Vorbereitung der beiden Sätze bringen wir noch zwei Hilfssätze über Eigenschaften der Sattelpunkte.

HILFSSATZ 5. *Unter der Voraussetzung* C *besitzt die Gleichung*

$$x\psi'(\tau) = y\varphi'(\tau) \tag{10}$$

$\nu-\mu$ *Lösungen* τ_r, *deren asymptotische Entwicklungen mit geeigneten Koeffizienten* α_n *durch*

$$\tau_{r+1} \sim \sum_{n=1}^{\infty} \alpha_n \left(e^{2\pi i r}\frac{\mu b_0}{\nu a_0}\frac{x}{y}\right)^{n/(\nu-\mu)}; \left\{\left(\frac{x}{y}\right)^{n/(\nu-\mu)}\right\} \tag{11}$$

$(r=0,1,\dots,\nu-\mu-1)$ *gegeben sind. Insbesondere sind* $\alpha_1 = 1$, $\alpha_2 = \dfrac{(\mu+1)b_1}{\mu(\nu-\mu)b_0} + \dfrac{(\nu+1)a_1}{\nu(\mu-\nu)a_0}$. *Ist gemäß* (a)

$$\sigma_r = x\psi(\tau_r) - y\varphi(\tau_r),$$

so gilt mit geeigneten Koeffizienten

$$\sigma_r \sim y \sum_{n=1}^{\infty} \beta_n \left(e^{2\pi i r \mu_0 b_0 x/(\nu a_0 y)}\right)^{(\nu+n-1)/(\nu-\mu)}; \left\{\left(\frac{x}{y}\right)^{(\nu+n-1)/(\nu-\mu)}\right\}. \tag{12}$$

Insbesondere sind

$$\beta_1 = \frac{\nu-\mu}{\mu} a_0, \quad \beta_2 = \frac{\nu a_0}{\mu b_0} b_1 - a_1.$$

BEWEIS. Aus (8), (9) und (10) ergibt sich

$$\sum_{j=0}^{\infty} (j+\nu) a_j \tau^{j+\nu-\mu} = \frac{x}{y} \sum_{j=0}^{\infty} (j+\mu) b_j \tau^j. \tag{13}$$

Wegen $x=o(y)$ und $\nu>\mu$ ist in erster Näherung

$$\nu a_0 \tau^{\nu-\mu} \sim \mu b_0 \frac{x}{y}.$$

Damit ergeben sich $\nu-\mu$ Lösungen mit der Eigenschaft $\tau=o(1)$. Unter Berücksichtigung auch der höheren Potenzen von τ in (13) kann τ in beliebiger Näherung berechnet werden und man gelangt zur Entwicklung (11). Die Entwicklung (12) erhält man durch Einsetzen von (11) in $\sigma_r = y\left(\dfrac{x}{y}\psi(\tau_r) - \varphi(\tau_r)\right)$.

HILFSSATZ 6. *Für die Lösungen* τ_r *von* (10) *des Hilfssatzes* 5 *gilt*

$$x\psi^{(k)}(\tau_r) \neq y\varphi^{(k)}(\tau_r)$$

mit $k=2,3,\dots,\nu$ *und* $r=1,2,\dots,\nu-\mu$.

BEWEIS. Wir nehmen an, daß für geeignete k und r

$$x\psi^{(k)}(\tau_r) = y\varphi^{(k)}(\tau_r)$$

ist. Durch Einsetzen von (8) und (9) ergibt sich

$$x\sum_{j=0}^{\infty}\binom{j+\mu}{k} b_j \tau_r^{j+\mu-k} = y\sum_{j=0}^{\infty}\binom{j+\nu}{k} a_j \tau_r^{j+\nu-k}.$$

Für $2 \leqq k \leqq \mu < \nu$ muß dann in erster Näherung

$$x\binom{\mu}{k} b_0 \tau_r^{\mu-k} \sim y\binom{\nu}{k} a_0 \tau_r^{\nu-k}$$

und für $1 \leqq \mu < k \leqq \nu$

$$x\binom{\mu}{k-\mu} b_{k-\mu} \sim y\binom{\nu}{k} a_0 \tau_r^{\nu-k}$$

sein. Beide Beziehungen sind aber nach (11) unmöglich.

SATZ 1. *Für das Integral*

$$I(x, y) = \int_{\mathfrak{L}^*} f(t) e^{x\psi(t)-y\varphi(t)}\, dt$$

seien die Voraussetzungen B *und* C *erfüllt.* τ *sei eine der Lösungen* τ_{r+1} *von* (10) *entsprechend dem Hilfssatz 5.* $\mathfrak{L}^*$ *sei ein Teil des Integrationsweges* $\mathfrak{L}^*_{\alpha,\beta}$ *von* $t_1+\tau$ *über* τ *nach* $t_2+\tau$ *mit* $|t_1|, |t_2| \geqq cy^{(\varepsilon-1)/\nu}$ $(c>0,\ \varepsilon>0)$. *Auf dem gesamten Weg* $\mathfrak{L}^*_{\alpha,\beta}$ *sei mit Ausnahme von* $t=\tau$

$$\operatorname{Re}\{P_\nu(t-\tau, \zeta)\} < 0.$$

Ferner liege $\mathfrak{L}^*$ *im Innern des gemeinsamen Holomorphiegebietes von* $f(t), \psi(t), \varphi(t)$. *Ist* $f(0) \neq 0$ *und*

$$f(t+\tau)\exp\{\Delta_\nu(t, \zeta)\} = \sum_{n=0}^{\infty} H_n(x, y) t^n \tag{14}$$

so gilt mit (1) *und*

$$\varkappa = \sum_{k=2}^{\nu} |\zeta_k|^{1/k}$$

unter der Voraussetzung A

$$I(x, y) \sim e^{\sigma} \sum_{n=0}^{\infty} H_n(x, y) W_{n,\nu}(\zeta; \alpha, \beta);\ \{e^{\sigma} y^{n/(\nu+1)} \varkappa^{-n-1}\}. \tag{15}$$

Dabei bezeichnet in (1) $\mathfrak{L}_{\alpha,\beta}$ *den durch die Substitution* $t \to t+\tau$ *verschobenen Weg* $\mathfrak{L}^*_{\alpha,\beta}$.

BEWEIS. Wir führen im Integral $I(x, y)$ die Substitution $t \to t+\tau$ aus und bezeichnen ebenso mit $\mathfrak{L}$ den hierdurch verschobenen Weg $\mathfrak{L}^*_0$. Im Integral

$$I(x, y) = \int_{\mathfrak{L}} f(t+\tau) e^{x\psi(t+\tau)-y\varphi(t+\tau)}\, dt$$

entwickeln wir den Exponenten mit den Bezeichnungen (a) bis (e) unter der Voraussetzung (10) nach Potenzen von t. Wir erhalten

$$x\psi(t+\tau)-y\varphi(t+\tau)=\sigma+P_\nu(t,\zeta)+\Delta_\nu(t,\zeta)$$

und

$$\exp\{\Delta_\nu(t,\zeta)\}=1+\sum_{n=\nu+1}^{\infty}G_n(x,y)t^n.$$

Da für $n>\nu$ $\zeta_n=O(y)$ ist, gilt

$$G_n(x,y)=O(y^{[n/(\nu+1)]}),$$

wobei $\left[\frac{n}{\nu+1}\right]$ das größte Ganze von $\frac{n}{\nu+1}$ bedeutet. Da in

$$f(t+\tau)=\sum_{n=0}^{\infty}\frac{f^{(n)}(\tau)}{n!}t^n$$

$f^{(n)}(\tau)=O(1)$ gilt, ist in der Entwicklung (14)

$$H_n(x,y)=\frac{f^{(n)}(\tau)}{n!}+\begin{cases}0 & \text{für } n\leqq\nu,\\ \sum_{k=\nu+1}^{n}G_k(x,y)\dfrac{f^{(n-k)}(\tau)}{(n-k)!} & \text{für } n>\nu,\end{cases} \tag{16}$$

und

$$H_n(x,y)=O(y^{[n/(\nu+1)]}). \tag{17}$$

Für $I(x,y)$ erhalten wir zunächst

$$I(x,y)=e^\sigma\sum_{n=0}^{\infty}H_n(x,y)\int_{\mathfrak{L}}t^n\exp\{P_\nu(t,\zeta)\}\,dt. \tag{18}$$

Nach Hilfssatz 6 ist für $k=2,3,\dots,\nu$ $\zeta_k\neq0$, und offensichtlich ist mit geeigneten Konstanten h_k

$$\zeta_k\sim h_k y\left(\frac{x}{y}\right)^{(\nu-k)/(\nu-\mu)}.$$

Beachtet man die Voraussetzungen des Satzes 1 und setzt in Hilfssatz 4 $\mathfrak{L}_0=\mathfrak{L}$, $x_k=\zeta_k$, $z^{\nu-\mu}=\frac{x}{y}$, so sind die Voraussetzungen dieses Hilfssatzes erfüllt. Damit folgt aus (6) die Entwicklung (15). Es verbleibt nur noch zu zeigen, daß

$$\{e^\sigma y^{n/(\nu+1)}\varkappa^{-n-1}\} \tag{19}$$

eine Skala und (15) eine asymptotische Entwicklung darstellen. Hierzu verwenden wir Hilfssatz 1 mit $x_k=\zeta_k$.

Fall a) bedeutet dann $x^\nu\asymp y^\mu$ und Fall c) $x^\nu=o(y^\mu)$. In beiden Fällen ist $\varkappa\asymp y^{1/\nu}$ und daher

$$y^{1/(\nu+1)}\varkappa^{-1}=o(1). \tag{20}$$

Demzufolge bildet (19) eine Skala. Fall b) bedeutet $y^\mu = o(x^\nu)$. Damit ist

$$\varkappa \sim \sum_{k=2}^{\nu} |h_k|^{1/k} y^{1/k} \left(\frac{x}{y}\right)^{\frac{1}{k}\frac{\nu-k}{\nu-\mu}}$$

und $\varkappa > \gamma y^{1/\nu}$, $\gamma > 0$.

Also ist auch hier (20) erfüllt, und (19) bildet eine Skala. Analog dazu ist nach (17)

$$H_{n+1}(x, y) W_{n+1,\nu}(\zeta; \alpha, \beta) = O(y^{(n+1)/(\nu+1)} \varkappa^{-n-2}) = o(y^{n/(\nu+1)} \varkappa^{-n-1}),$$

so daß (15) tatsächlich eine asymptotische Entwicklung darstellt, und sich insgesamt die Behauptung des Satzes ergibt.

Der Satz 1 läßt sich leicht auf den Fall $f(0)=0$ ausdehnen. Nehmen wir $f(t) = t^\lambda g(t)$ mit $\lambda > 0$ ganz, $g(0) \neq 0$ an, so haben wir (14) lediglich durch

$$f(t+\tau) \exp\{\Delta_\nu(t, \zeta)\} = (t+\tau)^\lambda \sum_{n=0}^{\infty} H_n(x, y) t^n \tag{21}$$

abzuändern. In der Entwicklung gemäß des Beweises des Satzes 1 erhalten wir an Stelle von (18)

$$I(x, y) = e^\sigma \sum_{n=0}^{\infty} H_n(x, y) \int_{\Omega} t^n (t+\tau)^\lambda \exp\{P_\nu(t, \zeta)\}\, dt.$$

Durch die Entwicklung des Binoms $(t+\tau)^\lambda$ werden wir auf die Funktion

$$W_{n,\nu,\lambda}(\zeta; \alpha, \beta) = \sum_{m=0}^{\lambda} \binom{\lambda}{m} \tau^{\lambda-m} W_{n+m,\nu}(\zeta; \alpha, \beta) \tag{22}$$

geführt. Nach Hilfssatz 1 ist

$$W_{n,\nu,\lambda}(\zeta; \alpha, \beta) = \sum_{m=0}^{\lambda} O(\tau^{\lambda-m} \varkappa^{-n-m-1}).$$

Für $x^\nu = O(y^\mu)$ ist $\tau\varkappa = O(1)$ und daher

$$W_{n,\nu,\lambda}(\zeta; \alpha, \beta) = O(\varkappa^{-n-\lambda-1}).$$

Für $y^\mu = o(x^\nu)$ ist $\dfrac{1}{\tau\varkappa} = o(1)$ und deshalb

$$W_{n,\nu,\lambda}(\zeta; \alpha, \beta) = O(\tau^\lambda \varkappa^{-n-1}).$$

Abgesehen von diesem Unterschied erfolgt die asymptotische Entwicklung analog zum Satz 1. Damit haben wir:

SATZ 2. *Es seien die Voraussetzungen des Satzes* 1 *außer* $f(0) \neq 0$ *erfüllt. Statt dessen sei* $f(t) = t^\lambda g(t)$, $\lambda > 0$ *ganz,* $g(0) \neq 0$. *Ersetzt man* (14) *durch* (21) *und bezeichnet* $\delta = \max(\tau^\lambda, \varkappa^{-\lambda})$, *so gilt mit* (22)

$$I(x, y) \sim e^\sigma \sum_{n=0}^{\infty} H_n(x, y) W_{n,\nu,\lambda}(\zeta; \alpha, \beta); \{e^\sigma \delta y^{n/(\nu+1)} \varkappa^{-n-1}\}. \tag{23}$$

4. Spezialfälle

4.1. Der Fall $\nu=2$, $\mu=1$. Wir beschränken uns auf den Fall des Satzes 1. Der einzige Sattelpunkt ist hier nach Hilfssatz 5 $\tau\sim\frac{b_0}{2a_0}\frac{x}{y}$ mit $\sigma\sim\frac{b_0^2}{4a_0}\frac{x^2}{y}$. Weiter ist $\zeta_2=y\varphi^{(2)}(\tau)-x\psi^{(2)}(\tau)\sim 2a_0y$ und $\varkappa=|\zeta_2|^{1/2}\sim(2a_0y)^{1/2}$. Fixieren wir $a_0>0$, dann können wir im Satz 1 $\operatorname{Re}(t_1)<0$ und $\operatorname{Re}(t_2)>0$ annehmen und die Funktionen $W_{n,2}(\zeta;\alpha,\beta)$ nach § 2 berechnen. Wir erhalten:

SATZ 3. *Es seien die Voraussetzungen des Satzes* 1 *erfüllt. Mit* $\nu=2$, $\mu=1$, $a_0>0$, $\operatorname{Re}(t_1)<0$, $\operatorname{Re}(t_2)>0$ *gilt für*

$$I(x,y)=\int_{t_1+\tau}^{t_2+\tau} f(t)\,e^{x\psi(t)-y\varphi(t)}\,dt$$

die asymptotische Entwicklung

$$I(x,y)\sim e^{\sigma}\sum_{n=0}^{\infty}\Gamma\left(n+\frac{1}{2}\right)H_{2n}(x,y)\left(\frac{\zeta_2}{2}\right)^{-n-1/2};\ \{e^{\sigma}y^{-n/3-1/2}\}.$$

BEMERKUNG. Dieser Satz lag der asymptotischen Entwicklung der Integrale in [5] zugrunde.

4.2. Der Fall $y^{\mu}=o(x^{\nu})$. Es soll eine asymptotische Darstellung von $I(x,y)$ in erster Näherung gegeben werden. In Satz 2 ist $\varkappa\sim|\zeta_2|^{1/2}$, $\delta=\tau^{\lambda}$ und somit

$$I(x,y)=e^{\sigma}\left\{H_0(x,y)\,W_{0,\nu,\lambda}(\zeta;\alpha,\beta)+O\left(\frac{\tau^{\lambda}}{\zeta_2}y^{1/(\mu+1)}\right)\right\}.$$

Auf $W_{0,\nu,\lambda}(\zeta;\alpha,\beta)=\sum_{m=0}^{\lambda}\binom{\lambda}{m}\tau^{\lambda-m}W_{m,\nu}(\zeta;\alpha,\beta)$ wenden wir Hilfssatz 3 an, in dem mit $x_k=\zeta_k$ die verschärfte Annehme $\zeta_k^2=o(\zeta_{k-1}^2\zeta_2)$ für $k=3,4,\dots,\nu$ in diesem Fall erfüllt ist. Damit ergibt sich aus (5)

$$W_{0,\nu,\lambda}(\zeta;\alpha,\beta)=\sqrt{2\pi}\frac{\tau^{\lambda}}{\sqrt{\zeta_2}}+O(\zeta_3\zeta_2^{-2}\tau^{\lambda})+O(\tau^{\lambda-1}\zeta_2^{-1}).$$

Wegen $f(\tau)=\tau^{\lambda}H_0(x,y)\asymp\tau^{\lambda}$ ist

$$I(x,y)=\sqrt{\frac{2\pi}{\zeta_2}}f(\tau)e^{\sigma}\left\{1+O\left(\frac{1}{\sqrt{\zeta_2}}y^{1/(\nu+1)}\right)+O(\zeta_3\zeta_2^{-3/2})+O\left(\frac{1}{\tau\sqrt{\zeta_2}}\right)\right\}.$$

Aus $\frac{\zeta_3}{\zeta_2}=O\left(\frac{1}{\tau}\right)$ und $y^{1/(\nu+1)}=o\left(\frac{1}{\tau}\right)$ ergibt sich:

SATZ 4. *Neben den Voraussetzungen des Satzes* 2 *sei noch* $y^{\mu}=o(x^{\nu})$ *erfüllt. Dann gilt*

$$I(x,y)=\sqrt{\frac{2\pi}{\zeta_2}}f(\tau)e^{\sigma}\left\{1+O\left(\frac{1}{\tau\sqrt{\zeta_2}}\right)\right\}.$$

4.3. Der Fall $x^\nu = o(y^\mu)$. Es sollen die ersten beiden Näherungen aus Satz 2 gewonnen werden, da die erste Näherung den Koeffizienten 0 haben kann. Wir haben $\varkappa \sim |\zeta_\nu|^{1/\nu} \asymp y^{1/\nu}$, $\delta = \varkappa^{-\lambda}$ und nach (23)

$$I(x, y) = e^\sigma \left\{ \sum_{n=0}^{\nu+1} H_n(x, y) W_{n,\nu,\lambda}(\zeta;\ \alpha, \beta) + O\left(\zeta^{\frac{1}{\nu+1} - \frac{\lambda+3}{\nu}}\right) \right\}.$$

Wir schreiben wieder $f(t) = t^\lambda g(t)$, so daß aus (21) $H_n(x, y) = O(1)$ für $n \leqq \nu$ zu erkennen ist. Nach (22) ist

$$W_{n,\nu,\lambda}(\zeta;\ \alpha, \beta) = O(\zeta_\nu^{-(n+\lambda+1)/\nu})$$

und demzufolge

$$H_n(x, y) W_{n,\nu,\lambda}(\zeta;\ \alpha, \beta) = O\left(\zeta^{\frac{1}{\nu+1} - \frac{\lambda+3}{\nu}}\right)$$

für $2 \leqq n \leqq \nu$. Für $n = 0, 1, \nu+1$ ergibt sich aus (21) entsprechend zu (16)

$$H_0(x, y) = g(\tau) = g(0)\{1 + O(\tau)\}, \quad H_1(x, y) = g'(\tau) = g'(0)\{1 + O(\tau)\},$$

$$H_{\nu+1}(x, y) = \frac{g^{(\nu+1)}(\tau)}{(\nu+1)!} - \frac{\zeta_{\nu+1}}{(\nu+1)!} g(\tau) = -\frac{\zeta_{\nu+1}}{(\nu+1)!} g(0)\{1 + O(\tau)\}.$$

Bei Verwendung von (3) aus Hilfssatz 2, der mit $x_k = \zeta_k$ hinsichtlich der verschärften Annahme erfüllt ist, folgt

$$H_0(x, y) W_{0,\nu,\lambda}(\zeta;\ \alpha, \beta) = A_1 \zeta_\nu^{-(\lambda+1)/\nu}\{1 + O(\tau\zeta_\nu^{1/\nu})\},$$

$$H_1(x, y) W_{1,\nu,\lambda}(\zeta;\ \alpha, \beta) = A_2 \zeta_\nu^{-(\lambda+2)/\nu}\{1 + O(\tau\zeta_\nu^{1/\nu})\},$$

$$H_{\nu+1}(x, y) W_{\nu+1,\nu,\lambda}(\zeta;\ \alpha, \beta) = A_3 \zeta_{\nu+1} \zeta_\nu^{-(\nu+\lambda+2)/\nu}\{1 + O(\tau\zeta_\nu^{1/\nu})\}.$$

Die Konstanten A_1, A_2, A_3 sind bei Benutzung der Bezeichnungen des Hilfssatzes 2 durch

$$A_1 = (e^{2\pi i l'(\lambda+1)/\nu} - e^{2\pi i l(\lambda+1)/\nu}) \frac{1}{\nu} (\nu!)^{(\lambda+1)/\nu} \Gamma\left(\frac{\lambda+1}{\nu}\right) g(0), \tag{24}$$

$$A_2 = (e^{2\pi i l'(\lambda+2)/\nu} - e^{2\pi i l(\lambda+2)/\nu}) \frac{1}{\nu} (\nu!)^{(\lambda+2)/\nu} \Gamma\left(\frac{\lambda+2}{\nu}\right) g'(0), \tag{25}$$

$$A_3 = -(e^{2\pi i l'(\lambda+2)/\nu} - e^{2\pi i l(\lambda+2)/\nu}) \frac{1}{\nu(\nu+1)} (\nu!)^{(\lambda+2)/\nu} \Gamma\left(\frac{\lambda+\nu+2}{\nu}\right) g(0) \tag{26}$$

gegeben. Nach (12) ist ferner $e^\sigma = 1 + O(\tau^\nu \zeta_\nu)$, so daß gilt:

SATZ 5. *Neben den Voraussetzungen des Satzes 2 sei noch $x^\nu = o(y^\mu)$ erfüllt. Mit den Bezeichnungen des Hilfssatzes 2 und (24), (25), (26) gilt*

$$I(x, y) = A_1 \zeta_\nu^{-(\lambda+1)/\nu} + \left(A_2 + A_3 \frac{\zeta_{\nu+1}}{\zeta_\nu}\right) \zeta_\nu^{-(\lambda+2)/\nu} + O\left(\zeta^{\frac{1}{\nu+1} - \frac{\lambda+3}{\nu}}\right) + O(\tau\zeta_\nu^{-\lambda/\nu}).$$

5. Verallgemeinerte Besselfunktionen

Wir betrachten die in [4] definierten verallgemeinerten Besselfunktionen

$$(27)\qquad J_\nu^{(2k)}(x) = \frac{1}{\sqrt{\pi}\,\Gamma\left(\nu+1-\dfrac{1}{2k}\right)}\left(\frac{x}{2}\right)^{k\nu}\int_{-1}^{1}(1-t^{2k})^{\nu-1/(2k)}\cos xt\,dt,$$

wobei k eine natürliche Zahl bezeichnet und $\mathrm{Re}\,(\nu) > \dfrac{1}{2k}-1$ vorausgesetzt ist.

Speziell für $k=1$ stellt (27) die wohlbekannte Besselfunktion $J_\nu^{(2)}(x)=J_\nu(x)$ dar. Das Ziel besteht darin, auf der Grundlage der in § 3 dargestellten Sattelpunktsmethode für zwei große Parameter, eine asymptotische Entwicklung für (27) zu erzeugen. Mit $\delta=e^{-y^{1+\varepsilon}}$, $\varepsilon>0$ und $y=\nu-1/(2k)$ erhalten wir aus (27) unmitelbar

$$(28)\qquad J_\nu^{(2k)}(x) = \frac{1}{\sqrt{\pi}\,\Gamma(y+1)}\left(\frac{x}{2}\right)^{ky+1/2}\{I_{2k}(x,y)+O(e^{-y^{1+\varepsilon}})\}$$

und

$$(29)\qquad I_{2k}(x,y) = \int_{-1+\delta}^{1-\delta}(1-t^{2k})^{y}e^{ixt}\,dt.$$

Wir wollen nunmehr bezüglich (28) für festes natürliches $k\geqq 1$ und unter der Vorausetzung A aus § 3 eine asymptotische Entwicklung ableiten, indem wir auf (29) die in § 3 entwickelte Methode anwenden. Zunächst ist offensichtlich, daß $I_{2k}(x,y)$ auch die Voraussetzung B aus § 3 erfüllt, weil wir für (29) auch

$$(30)\qquad I_{2k}(x,y) = \int_{\mathfrak{L}} f(t)\exp\{x\psi(t)-y\varphi(t)\}\,dt,$$

mit dem Integrationsweg $\mathfrak{L}$, als Geradenstück von $-1+\delta$ nach $1-\delta$ führend und $f(t)=1$, $\psi(t)=it$, $\varphi(t)=t^{2k}\sum\limits_{j=1}^{\infty}\dfrac{t^{2k(j-1)}}{j}$, schreiben können. Ferner gilt die Voraussetzung C aus § 3, weil $2k-1$ Sattelpunkte nach Hilfssatz 5 durch

$$(31)\qquad \tau_{r+1}\sim\sum_{n=1}^{\infty}\alpha_n\left(e^{2\pi ir}\frac{i}{2k}\frac{x}{y}\right)^{n/(2k-1)};\ \left\{\left(\frac{x}{y}\right)^{n/(2k-1)}\right\}$$

$\bigl(r=0,1,\dots,2(k-1)\bigr)$ gegeben sind.

Im weiteren kommt es nun — gemäß der Beschreibung der Methode in § 3 — darauf an, in (30) den Integrationsweg $\mathfrak{L}$ — in Abhängigkeit von k — über einen oder mehrere der Sattelpunkte τ_{r+1} derart zu verlegen, daß nur diese ausgewählten Sattelpunkte für das Integral (30) wesentliche Beiträge liefern, sowie die außerdem auftretenden Restintegrale hinreichend klein genug abzuschätzen. Zur Verdeutlichung dieser Problematik diskutieren wir deshalb zuerst die Spezialfälle $k=1$, $k=2$ und behandeln danach den allgemeinen Fall.

5.1. Der Fall $k=1$. Für (30) erhalten wir mit $k=1$

$$(32)\qquad I_2(x,y) = \int_{\mathfrak{L}} f(t)\exp\{x\psi(t)-\varphi(t)\}\,dt$$

wobei $\varphi(t)=t^2 \sum_{j=1}^{\infty} \frac{t^{2(j-1)}}{j}$ und $f(t)\,\psi(t)$, $\mathfrak{L}$ wie in (30) sind. Weiterhin folgt aus (31),

daß hier nur ein Sattelpunkt $\tau=i\frac{y}{x}\left(1-\sqrt{1-\frac{x^2}{y^2}}\right)$ mit

$$\tau \sim \sum_{n=1}^{\infty} \alpha_n \left(\frac{i}{2}\frac{x}{y}\right)^n ; \left\{\left(\frac{x}{y}\right)^n\right\} \tag{33}$$

und $\alpha_1=1$ existiert, so daß dadurch die Verältnisse relativ einfach werden. Unter Berücksichtigung dieses Sattelpunktes ergeben sich mit den obigen Funktionen ψ und φ unmittelbar:

(a) $\sigma=x\psi(\tau)-y\varphi(\tau)=ix\tau+y\log(1-\tau^2)$ und wegen Hilfssatz 5 folgt $\sigma\sim$ $\sim y\sum_{n=1}^{\infty}\beta_n\left(\frac{i}{2}\frac{x}{y}\right)^{n+1} ; \left\{\left(\frac{x}{y}\right)^{n+1}\right\}$ mit $\beta_1=1$,

(b) $$\zeta_k = y\varphi^{(k)}(\tau), \quad k=2,3,\ldots,$$

(c) $$\zeta=\zeta_2=2y\,\frac{1+2\tau-\tau^2}{(1-\tau^2)^2} \quad \text{mit} \quad \zeta_2\sim 2y,$$

(d) $$P_2(t,\zeta)=-\zeta_2\frac{t^2}{2!},$$

(e) $$\Delta_2(t,\zeta)=-\sum_{k=3}^{\infty}\zeta_k\frac{t^k}{k!}.$$

Als nächstes soll nun der Integrationsweg $\mathfrak{L}$ über den durch (33) charakterisierten Sattelpunkt verlagert werden. Auf Grund von (1) und Satz 1 bzw. 3 können jetzt $\mathfrak{L}^*$ und $\mathfrak{L}^*_{\alpha,\beta}$ fixiert werden. Nach einfacher Rechnung folgt $\frac{3\pi}{4}<\alpha<\frac{5\pi}{4}$ bzw. $-\frac{\pi}{4}<\beta<\frac{\pi}{4}$, so daß beispielsweise $\alpha=\pi$ bzw. $\beta=0$ und $t_1=e^{\pi i}y^{(\varepsilon-1)/2}$ bzw. $t_2=y^{(\varepsilon-1)/2}$, $0<\varepsilon<1$, sowie $\mathfrak{L}^*$ von $t_1+\tau$ über τ nach $t_2+\tau$ führend, als Teil des Integrationsweges $\mathfrak{L}^*_{\alpha,\beta}$, wählbar sind. Damit gelangen wir zu dem in der Abbildung 1 dargestellten Integrationsweg.

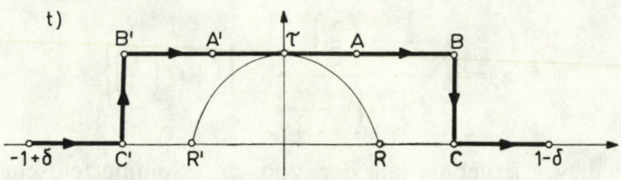

Abb. 1

Dabei wurde τ nur unter Verwendung der ersten Näherung eingezeichnet, und es bedeuten:

$$A=\tau+t_2,\quad B=\tau+\left(\frac{x}{y}\right)^{(1-\varepsilon)/2},\quad C=\left(\frac{x}{y}\right)^{(1-\varepsilon)/2},\quad R=\frac{x}{2y},\quad 0<\varepsilon<1;$$

A', B', C', R' zu A, B, C, R symmetrisch.

Unter Beachtung der Abbildung 1 erhalten wir deshalb für (32)

$$I_2(x,y)=\left\{\int_{\mathfrak{L}^*}+\int_{\mathfrak{A}}+\int_{\mathfrak{B}}\right\}e^{x\psi(t)-y\varphi(t)}\,dt \tag{34}$$

mit

$$\mathfrak{L}^*=[A',A],\quad \mathfrak{A}=[B',A']+[A,B],$$

$$\mathfrak{B}=[(-1+\delta),C']+[C',B']+[B,C]+[C,(1-\delta)],$$

und wir werden in der anschließenden Auswertung nachweisen, daß das erste Integral mit dem Integrationsweg $\mathfrak{L}^*$ den entscheidenden Beitrag in der asymptotischen Entwicklung liefert. Anwendung des Satzes 3 ergibt

$$\int_{\mathfrak{L}^*}e^{x\psi(t)-y\varphi(t)}\,dt\sim e^{\sigma}\sum_{n=0}^{\infty}\Gamma\left(n+\frac{1}{2}\right)H_{2n}(x,y)\left(\frac{\zeta_2}{2}\right)^{-n-1/2};\ \{e^{\sigma}y^{-n/3-1/2}\}, \tag{35}$$

wobei $H_{2n}(x,y)$ durch (14) gegeben ist.

Ferner gelten mit den Konstanten $K>0$, $L>0$, $1>\varepsilon>0$

$$S_1=\int_{\mathfrak{A}}e^{x\psi(t)-y\varphi(t)}\,dt\leqq\int_{\mathfrak{A}-\tau}e^{\sigma-\zeta_2t^2/2!}\,dt=O(e^{\sigma-Ky^{\varepsilon}}) \tag{36}$$

und

$$S_2=\int_{\mathfrak{B}}e^{x\psi(t)-y\varphi(t)}\,dt=O(e^{x\psi(C)-y\varphi(C)})=O(e^{-Lx(y/x)^{\varepsilon}}). \tag{37}$$

Insgesamt haben wir für $y=\nu-\frac{1}{2}$, $\tau=i\frac{y}{x}\left(1-\sqrt{1-\frac{x^2}{y^2}}\right)$, $\sigma=ix\tau+y\log(1-\tau^2)$, $\zeta_2=2y\,\frac{1+2\tau-\tau^2}{(1-\tau^2)^2}$ die asymptotische Entwicklung

$$\Gamma\left(\nu+\frac{1}{2}\right)\left(\frac{x}{2}\right)^{-\nu}J_\nu(x)\sim e^{\sigma}\sum_{n=0}^{\infty}\Gamma\left(n+\frac{1}{2}\right)H_{2n}(x,y)\left(\frac{\zeta_2}{2}\right)^{-n-1/2};\ \{e^{\sigma}y^{-n/3-1/2}\} \tag{38}$$

mit $x+\nu\to\infty$, $x=o(\nu)$. Speziell ist

$$J_\nu(x)=\sqrt{\frac{2}{\zeta_2}}\,\frac{\left(\frac{x}{2}\right)^{\nu}}{\Gamma\left(\nu+\frac{1}{2}\right)}\,e^{\sigma}\left\{1+O\left(\frac{1}{\nu}\right)\right\}.$$

Man vergleiche dieses Ergebnis mit der von der Sommerfeldschen Integraldarstellung der Besselfunktion ausgehenden Entwicklung in [6], Seite 318/319.

5.2. Der Fall $k=2$. Für (30) erhalten wir mit $k=2$

$$I_4(x,y)=\int_{\mathfrak{L}} f(t)\exp\{x\psi(t)-y\varphi(t)\}\,dt, \tag{39}$$

wobei $\varphi(t)=t^4\sum_{j=1}^{\infty}\frac{t^{4(j-1)}}{j}$ und $f(t)$, $\psi(t)$, $\mathfrak{L}$ wie in (30) sind. Aus (31) folgt, daß 3 Sattelpunkte τ_1, τ_2, τ_3, wofür

$$\tau_{r+1}\sim\sum_{n=1}^{\infty}\alpha_n\left(e^{2\pi ir}\frac{i}{4}\frac{x}{y}\right)^{n/3};\ \left\{\left(\frac{x}{y}\right)^{n/3}\right\}\quad (r=0,1,2) \tag{40}$$

und in erster Näherung

$$\tau_1\sim e^{\pi i/6}\left(\frac{x}{4y}\right)^{1/3},\quad \tau_2\sim e^{5\pi i/6}\left(\frac{x}{4y}\right)^{1/3},\quad \tau_3\sim e^{-\pi i/2}\left(\frac{x}{4y}\right)^{1/3}$$

gilt, existieren.

Später werden wir rechtfertigen, daß nur die Sattelpunkte τ_1 und τ_2 einen wesentlichen Beitrag in der asymptotischen Entwicklung liefern werden. Unter Verwendung dieser Sattelpunkte τ_r $(r=1,2)$ bekommen wir:

(a) $\sigma_r=x\psi(\tau_r)-y\varphi(\tau_r)$ und wegen Hilfssatz 5 folgt

$$\sigma_r\sim y\sum_{n=1}^{\infty}\beta_n\left(e^{2\pi(r-1)}\frac{i}{4}\frac{x}{y}\right)^{(n+3)/3};\ \left\{\left(\frac{x}{y}\right)^{(n+3)/3}\right\}\quad \text{mit}\quad \beta_1=3,$$

(b) $$\zeta_{k,r}=y\varphi^{(k)}(\tau_r),\quad k=2,3,\dots,$$

(c) $$\zeta_{(r)}=(\zeta_{2,r},\zeta_{3,r},\zeta_{4,r})\quad\text{mit}\quad \zeta_{2,r}\sim 3\sqrt[3]{4}\,e^{\pi i(1/3-2/3(r-1))}(yx^2)^{1/3},$$

$$\zeta_{3,r}\sim 12\sqrt[3]{2}\,e^{\pi i(1/6+2/3(r-1))}(y^2x)^{1/3},\quad \zeta_{4,r}\sim 24y,$$

(d) $$P_4(t,\zeta_{(r)})=-\sum_{k=2}^{4}\zeta_{k,r}\frac{t^k}{k!},$$

(e) $$\Delta_4(t,\zeta_{(r)})=-\sum_{k=5}^{\infty}\zeta_{k,r}\frac{t^k}{k!},$$

wobei in (a) bis (e) stets $r=1$ bzw. $r=2$ gemeint ist.

Als nächstes wollen wir den Integrationsweg $\mathfrak{L}$ über die Sattelpunkte τ_1 und τ_2 verlagern. Auf Grund von (1) und Satz 1 folgt nach einfachen Rechnungen

$$\frac{7\pi}{12}<\alpha_1<\frac{5\pi}{8}\quad\text{bzw.}\quad -\frac{\pi}{8}<\beta_1<\frac{\pi}{12}\quad\text{bezüglich } \tau_1,$$

$$\frac{11\pi}{12}<\alpha_2<\frac{9\pi}{8}\quad\text{bzw.}\quad \frac{3\pi}{8}<\beta_2<\frac{5\pi}{12}\quad\text{bezüglich } \tau_2,$$

so daß beispielsweise $\alpha_1=\frac{3\pi}{5}$, $\beta_1=0$, $\alpha_2=\pi$, $\beta_2=\frac{2\pi}{5}$ gewählt werden können.

Demzufolge ergeben sich als Integrationswege über die Sattelpunkte, $\mathfrak{L}_1^*$ von $t_{1,1}+\tau_1$ über τ_1 nach $t_{2,1}+\tau_1$, sowie $\mathfrak{L}_2^*$ von $t_{1,2}+\tau_2$ über τ_2 nach $t_{2,2}+\tau_2$ mit $t_{1,r}=e^{\alpha_r \pi i} y^{(\varepsilon-1)/4}$, $t_{2,r}=e^{\beta_r \pi i} y^{(\varepsilon-1)/4}$, $0<\varepsilon<1$ $(r=1, 2)$.

Wir gelangen damit in Abhängigkeit vom relativen Wachstum von x und y zu dem in der Abbildung 2 bzw. 3 dargestellten Integrationsweg.

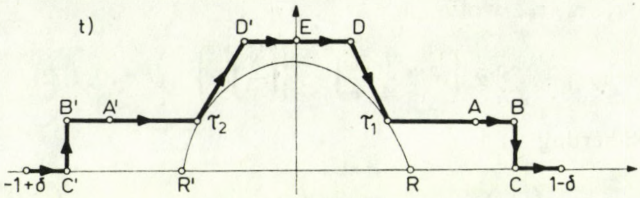

Abb. 2

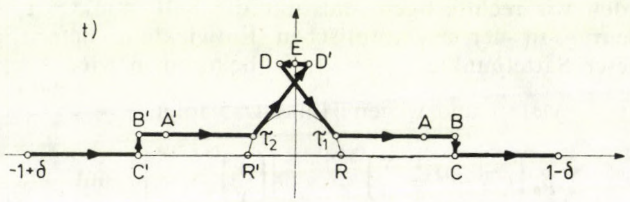

Abb. 3

Dabei wurden τ_1, τ_2 nur unter Verwendung der ersten Näherung eingezeichnet und es bedeuten:

$$A = \tau_1 + t_{2,1}, \quad B = \tau_1 + \left(\frac{x}{y}\right)^{(1-\varepsilon)/4}, \quad C = \left(\frac{x}{y}\right)^{(1-\varepsilon)/4}, \quad D = \tau_1 + t_{1,1}$$

$$E = i \operatorname{Im}(\tau_1 + t_{1,1}), \quad R = \left(\frac{x}{4y}\right)^{1/3},$$

$0<\varepsilon<1$; A', B', C', D', E', R' zu A, B, C, D, E, R symmetrisch.

Unter Berücksichtigung der Abbildung 2 bzw. 3 erhalten wir deshalb für (39)

$$I_4(x, y) = \left\{ \int_{\mathfrak{L}_1^*} + \int_{\mathfrak{L}_2^*} + \int_{\mathfrak{A}_1} + \int_{\mathfrak{A}_2} + \int_{\mathfrak{B}} \right\} e^{x\psi(t) - y\varphi(t)}\, dt \tag{41}$$

mit $\mathfrak{L}_1^* = [D, \tau_1] + [\tau_1, A]$, $\mathfrak{L}_2^* = [A', \tau_2] + [\tau_2, D']$, $\mathfrak{A}_1 = [E, D] + [A, B]$, $\mathfrak{A}_2 = [B', A'] + [D', E']$, $\mathfrak{B} = [(-1+\delta), C'] + [C', B'] + [B, C] + [C, (1-\delta)]$.

Anwendung des Satzes 1 ergibt

$$\int_{\mathfrak{L}_1^*} e^{x\psi(t) - y\varphi(t)}\, dt \sim e^{\sigma_1} \sum_{n=0}^{\infty} H_{n,1}(x, y) W_{n,4}\left(\zeta_{(1)};\ \frac{3\pi}{5},\ 0\right);\ \{e^{\sigma_1} y^{n/5} \varkappa_1^{-n-1}\} \tag{42}$$

und

$$\int_{\mathfrak{L}_2^*} e^{x\psi(t)-y\varphi(t)}\,dt \sim e^{\sigma_2}\sum_{n=0}^{\infty} H_{n,2}(x,y)W_{n,4}\left(\zeta_{(2)};\,\pi,\frac{2\pi}{5}\right);\ \{e^{\sigma_2}y^{n/5}\varkappa_2^{-n-1}\}, \tag{43}$$

wobei

$$\varkappa_r = |\zeta_{2,r}|^{1/2}+|\zeta_{3,r}|^{1/3}+|\zeta_{4,r}|^{1/4},\quad W_{n,4}(\zeta_{(r)};\,\alpha_r,\beta_r) = \int_{\mathfrak{L}_{\alpha_r,\beta_r}} t^n \exp\{P_4(t,\zeta_{(r)})\}\,dt$$

und $H_{n,r}(x, y)$ mittels (14) durch

$$\exp\{\Delta_4(t,\zeta_{(r)})\} = \sum_{n=0}^{\infty} H_{n,r}(x,y)\,t^n \quad (r=1,2),$$

bestimmt sind.

Weiterhin läßt sich — analog zu den Abschätzungen in (36) und (37) — zeigen, daß Konstanten $K>0$, $L>0$, $M>0$, $1>\varepsilon>0$ existieren, so daß

$$S_1 = \int_{\mathfrak{A}_1} e^{x\psi(t)-y\varphi(t)}\,dt = O(e^{\sigma_1-Ky^\varepsilon}), \tag{44}$$

$$T_1 = \int_{\mathfrak{A}_2} e^{x\psi(t)-y\varphi(t)}\,dt = O(e^{\sigma_2-Ly^\varepsilon}), \tag{45}$$

und

$$S_2 = \int_{\mathfrak{B}} e^{x\psi(t)-y\varphi(t)}\,dt = O(e^{-Mx(y/x)^\varepsilon}), \tag{46}$$

gelten.

Insgesamt haben wir damit für $y=\nu-\dfrac{1}{4}$ die asymptotische Entwicklung

$$\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{4}\right)\left(\frac{x}{2}\right)^{-2\nu} J_\nu^{(4)}(x) \sim e^{\sigma_1}\sum_{n=0}^{\infty} H_{n,1}(x,y)W_{n,4}\left(\zeta_{(1)};\,\frac{3\pi}{5},0\right)+ \tag{47}$$

$$+e^{\sigma_2}\sum_{n=0}^{\infty} H_{n,2}(x,y)W_{n,4}\left(\zeta_{(2)};\,\pi,\frac{2\pi}{5}\right);\ \{e^{\sigma_1}y^{n/5}\varkappa_1^{-n-1}\},\ \{e^{\sigma_2}y^{n/5}\varkappa_2^{-n-1}\}$$

mit $x+\nu\to\infty$, $x=o(\nu)$.

Außerdem erhalten wir unter Verwendung der Sätze 4 und 5, als Spezialfälle von (47), folgende asymptotische Darstellungen. Für $y=\nu-\dfrac{1}{4}$ gilt

$$\Gamma\left(\nu+\frac{3}{4}\right)\left(\frac{x}{2}\right)^{-2\nu} J_\nu^{(4)}(x) = \sum_{r=1}^{2}\sqrt{\frac{2}{\zeta_{2,r}}}\,e^{\sigma_r}\left\{1+O\left(\frac{1}{\tau_r\sqrt{\zeta_{2,r}}}\right)\right\} \tag{48}$$

mit $x+\nu\to\infty$, $x=o(\nu)$, $\nu=o(x^4)$.

Entsprechend gilt für $y=\nu-\dfrac{1}{4}$

$$\Gamma\left(\nu+\frac{3}{4}\right)\left(\frac{x}{2}\right)^{-2\nu} J_\nu^{(4)}(x) = \sum_{r=1}^{2}\left\{A_{1,r}\zeta_{4,r}+A_{3,r}\frac{\zeta_{5,r}}{\zeta_{4,r}}\zeta_{4,r}^{-1/2}+O(\tau_r)\right\} \tag{49}$$

mit $x+\nu\to\infty$, $x^4=o(\nu)$, wobei die Konstanten $A_{1,r}$ und $A_{3,r}$ auf Grund von (24) und (26) durch

$$A_{1,r}=\left(e^{(\pi/2)il'_r}-e^{(\pi/2)il_r}\right)\left(\frac{3}{8}\right)^{1/4}\Gamma\left(\frac{1}{4}\right) \quad \text{und} \quad A_{3,r}=\left(e^{\pi il'_r}-e^{\pi il_r}\right)\frac{6^{1/2}}{10}\Gamma\left(\frac{3}{2}\right)$$

gegeben sind.

5.3. Der allgemeine Fall. In Verallgemeinerung zum Fall $k=2$ liegt es nahe, den Integrationsweg $\mathfrak{L}$ in (30) über die k Sattelpunkte $\tau_1, \tau_2, \ldots, \tau_k$ von (31), die sich alle in der oberen Halbebene befinden, zu verlagern. Unter Beachtung dieser Sattelpunkte τ_r $(r=1, 2, \ldots, k)$ bekommen wir:

(a) $\sigma_r=x\psi(\tau_r)-y\varphi(\tau_r)$ und wegen Hilfssatz 5

$$\sigma_r \sim y\sum_{n=0}^{\infty}\beta_n\left(e^{2\pi ir}\frac{i}{2k}\frac{x}{y}\right)^{(2k+n-1)/(2k-1)};\left\{\left(\frac{x}{y}\right)^{(2k+n-1)/(2k-1)}\right\} \quad \text{mit} \quad \beta_1=2k-1,$$

(b) $$\zeta_{l,r}=y\varphi^{(l)}(\tau_r), \quad l=2, 3, \ldots,$$

(c) $$\zeta_{(r)}=(\zeta_{2,r}, \zeta_{3,r}, \ldots, \zeta_{2k,r}) \quad \text{mit}$$

$$\zeta_{2,r}\sim(2k-1)\sqrt[2k-1]{2k}\exp\left\{i\frac{k-1}{2k-1}\pi(4r-3)\right\}(x^{2(k-1)}y)^{1/(2k-1)}, \quad \zeta_{2k,r}\sim(2k)!\,y,$$

(d) $$P_{2k}(t, \zeta_{(r)})=-\sum_{l=2}^{2k}\zeta_{l,r}\frac{t^l}{l!},$$

(e) $$\Delta_{2k}(t, \zeta_{(r)})=-\sum_{l=2k+1}^{\infty}\zeta_{l,r}\frac{t^l}{l!},$$

wobei in (a) bis (e) stets $r=1, 2, \ldots, k$ bedeutet.

Durch Analogieschluß zu den Fällen $k=1, 2$ finden wir:

Für $y=\nu-\dfrac{1}{2k}$ gilt die asymptotische Entwicklung

(50) $$\sqrt{\pi}\,\Gamma\left(\nu+\frac{2k-1}{2k}\right)\left(\frac{x}{2}\right)J_\nu^{(2k)}(x)\sim$$

$$\sim\sum_{r=1}^{k}e^{\sigma_r}\sum_{n=0}^{\infty}H_{n,r}(x, y)W_{n,2k}(\zeta_{(r)};\alpha_r,\beta_r);\ \{e^{\sigma_1}y^{n/(2k+1)}\varkappa_1^{-n-1}\}, \ldots, \{e^{\sigma_k}y^{n/(2k+1)}\varkappa_k^{-n-1}\}$$

mit $x+\nu\to\infty$, $x=o(\nu)$.

Dabei sind in (50)

$$\varkappa_r=\sum_{l=2}^{2k}|\zeta_{l,r}|^{1/l}, \quad W_{n,2k}(\zeta_{(r)};\alpha_r,\beta_r)=\int_{\mathfrak{L}_{\alpha_r,\beta_r}}t^n\exp\{P_{2k}(t,\zeta_{(r)})\}\,dt$$

mit gewissen, wohlbestimmbaren, Winkeln α_r, β_r, die den Bedingungen $\cos(2\vartheta+\arg\zeta_{2,r})>0$ und $\cos(2k\vartheta)>0$ für $\vartheta=\alpha_r, \beta_r$, genügen müssen, sowie $H_{n,r}(x, y)$ durch (14)

$$\exp\{\Delta_{2k}(t,\zeta_{(r)})\}=\sum_{n=0}^{\infty}H_{n,r}(x, y)t^n$$

$(r=1, 2, \ldots, k)$ gegeben.

Beispielsweise sind in $W_{n,2k}(\zeta_{(r)};\,\alpha_r,\,\beta_r)$ für $k=3,\ \alpha_1=\frac{2\pi}{3},\ \beta_1=0,\ \alpha_2=\pi,\ \beta_2=0,$ $\alpha_3=\pi,\ \beta_3=\frac{\pi}{3}$ wählbar.

Unter Berücksichtigung der Sätze 4 und 5 könnten für (50) für die Spezialfälle $y=o(x^{2k})$ und $x^{2k}=o(y)$ noch asymptotische Darstellungen angegeben werden. Darauf soll jedoch verzichtet werden, da die Ergebnisse (48) und (49) sich in einfacher Weise auf (50) verallgemeinern lassen.

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MULTIPLICATIVE CANCELLATIVITY OF SEMIRINGS AND SEMIGROUPS

Von

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In this note, the usual ring-like definition of a multiplicatively cancellative semiring $S=(S, +, \cdot)$ (cf. [1], [2]) and some generalizations of this concept will be clarified by equivalent characterizations. As a matter of fact, the intrinsic parts of these considerations are consequences of statements concerning merely the multiplicative semigroup $(S, \cdot)$ of S, regarded as an arbitrary one, and our results on semirings become those on very general algebras $(S, +, \cdot)$. In this context, we give some applications on nearrings.

A *semiring* $S=(S, +, \cdot)$ is defined as an algebra such that $(S, +)$ and $(S, \cdot)$ are any semigroups, connected by ring-like distributivity. A neutral element o of $(S, +)$, if there is one, is called the *zero* of the semiring S, and in particular an *annihilating zero* iff, as for rings, $\{o\}$ is an ideal of $(S, \cdot)$. For simplicity, expressions like "$a\neq o$" or "$S\setminus\{o\}$" are used for any semiring S, being of no consequence if S has no zero.

We further denote by $\mathfrak{L}$ [$\mathfrak{R}$] the set of all elements of S being left [right] cancellable in $(S, \cdot)$. Clearly, both are subsemigroups of $(S, \cdot)$ if not empty, and for all $x, y\in S$ we have

$$xy\in\mathfrak{L} \Rightarrow y\in\mathfrak{L}; \quad xy\in\mathfrak{R} \Rightarrow x\in\mathfrak{R}. \tag{1}$$

In general, very few can be said about the multiplicative behaviour of a zero of a semiring S (cf. [2]). So it is notable that, as claimed in [4], the ring-like concept of multiplicative cancellability (in short: $a\neq o\Rightarrow a\in\mathfrak{L}\cap\mathfrak{R}$), applied to any semiring S, forces a zero of S to be either multiplicatively cancellable, too, or else annihilating. But even corresponding one-sided or mixed versions of this (replacing $\mathfrak{L}\cap\mathfrak{R}$ by $\mathfrak{L}$, $\mathfrak{R}$, or $\mathfrak{L}\cup\mathfrak{R}$) will prove correct, and we start with left and two-sided cancellability, treated simultaneously. To avoid trivial rubs, always $|S|\geqq 2$ is assumed.

DEFINITION 1. A semiring S is called *multiplicatively (left) cancellative* iff each element $a\neq o$ of S is (left) cancellable in $(S, \cdot)$.

PROPOSITION 1. *A semiring S is multiplicatively (left) cancellative iff one of the following conditions holds:*

α) S has no zero, and $(S, \cdot)$ is (left) cancellative.

β) S has a zero, and $(S, \cdot)$ is (left) cancellative.

γ) S has an annihilating zero, and $(S\setminus\{o\}, \cdot)$ is a subsemigroup of $(S, \cdot)$, which is (left) cancellative.[1]

[1] With obvious definitions, we have for each semiring S: $(S\setminus\{o\}, \cdot)$ is a subsemigroup $\Leftrightarrow$ S has no proper divisors of zero $\Leftrightarrow$ $\{o\}$ is a completely prime set. But even for semirings with an annihilating zero this does not imply anything concerning cancellability. We note in this context (cf. [2]): If $a\in S$ is (left) cancellable in $(S, \cdot)$, a is no (left) zero divisor, but no conversely.

Since α), β) and γ) clearly fit the definition, we merely have to prove that a multiplicative (left) cancellative semiring S with zero o satisfies β) or γ). But this will be a consequence of the following lemma, dealing only with a semigroup $(S, \cdot)$ and one special element $o \in S$, being exceptional for what reason ever.

LEMMA 1. *For any semigroup* $(S, \cdot)$ *and one element* $o \in S$ *holds:*

(2) *If* $S \setminus \{o\} \subseteq \mathfrak{L}$, *then either* $S = \mathfrak{L}$ *or* $So = oS = o$.

(3) *If* $S \setminus \{o\} \subseteq \mathfrak{L} \cap \mathfrak{R}$, *then either* $S = \mathfrak{L} \cap \mathfrak{R}$ *or* $So = oS = o$.

PROOF. Suppose $S \setminus \{o\} \subseteq \mathfrak{L}$ and $o \notin \mathfrak{L}$. Then we have $So \cap \mathfrak{L} = \varnothing$ by (1), hence $So = o$. In order to prove $oS = o$, suppose $oa = b \neq o$ for some $a \in S$. From this we obtain $b^2 = boa = b$ since $bo = o$, hence $b \in \mathfrak{L}$ should be a left identity of $(S, \cdot)$, implying the same for o since $ob = ooa = b$ by $oo = o$. This contradiction to $o \notin \mathfrak{L}$ proves (2), yielding (3) by duality.[2]

By our way of proceeding, these considerations also are applicable to each class $\mathfrak{S}$ of algebras $S = (S, +, \cdot)$ such that $(S, \cdot)$ is a semigroup, in particular to seminearrings and nearrings (cf. [3], [4]). In principle, there are 3 cases α), β) and γ) for each concept of multiplicative left, right and two-sided cancellability, and we denote the corresponding 9 subclasses of $\mathfrak{S}$ by

$$(4) \qquad \begin{array}{lll} (\mathfrak{L}, \alpha) & (\mathfrak{R}, \alpha) & (\mathfrak{L} \cap \mathfrak{R}, \alpha) \\ (\mathfrak{L}, \beta) & (\mathfrak{R}, \beta) & (\mathfrak{L} \cap \mathfrak{R}, \beta) \\ (\mathfrak{L}, \gamma) & (\mathfrak{R}, \gamma) & (\mathfrak{L} \cap \mathfrak{R}, \gamma). \end{array}$$

To contrast the clear situation with rings, we give informations concerning these subclasses for semirings and nearrings.

THEOREM. i) *For semirings, all* 9 *subclasses* (4) *are mutually distinct and not empty.*

ii) *For left distributive nearrings, only the subclasses* $(\mathfrak{L}, \beta)$, $(\mathfrak{L}, \gamma)$, $(\mathfrak{R}, \gamma)$, $(\mathfrak{L} \cap \mathfrak{R}, \gamma)$ *are not empty, we have* $(\mathfrak{R}, \gamma) = (\mathfrak{L} \cap \mathfrak{R}, \gamma)$, *and the remaining* 3 *cases are mutually distinct.*

In particular: A left distributive nearring S *which is (multiplicatively) right cancellative, is two-sided cancellative.*

iii) *All subclasses claimed not to be empty in* i) *and* ii) *contain as well finite as infinite semirings or nearrings, respectively, with one exception: There are no finite multiplicatively cancellative semirings with a multiplicatively cancellable zero* o.

PROOF. i) Let $(S, +)$ be any idempotent semigroup without a neutral element, and define a multiplication by $ab = b$ for all $a, b \in S$. It is easily checked, that $S = (S, +, \cdot)$ is a semiring and that $S \in (\mathfrak{L}, \alpha)$, $S \notin (\mathfrak{R}, \alpha)$. Corresponding examples of semirings with β) or γ) are obtained from S adjoining a neutral element o to $(S, +)$ and defining

$$ox = x, \quad xo = o \quad \text{or} \quad ox = xo = o, \quad \text{respectively,}$$

[2] The implication $o \notin \mathfrak{L} \Rightarrow So \cap \mathfrak{L} = \varnothing \Rightarrow So = o \Rightarrow o \notin \mathfrak{R}$ and its dual, both valid if $S \setminus \{o\} \subseteq \mathfrak{L} \cap \mathfrak{R}$, provide (3) directly.

for all $x \in S \cup \{o\}$. Since semirings belonging to $(\mathfrak{L} \cap \mathfrak{R}, \alpha)$ or $(\mathfrak{L} \cap \mathfrak{R}, \gamma)$ clearly exist, it remains to prove $(\mathfrak{L} \cap \mathfrak{R}, \beta) \neq \emptyset$. Now let $(S, \cdot)$ be an infinite cyclic semigroup with identity, noted as

$$S = \{a^0, a^1, a^2, \dots\}, \quad \text{and by} \quad a^i + a^j = a^{\max(i,j)}$$

define an addition on S. Clearly, $S = (S, +, \cdot)$ becomes a semiring, and S as well as each of its subsemirings belong to $(\mathfrak{L} \cap \mathfrak{R}, \beta)$; in particular, the zero a^0 of S is at the same time the identity of $(S, \cdot)$.

ii) Let $S = (S, +, \cdot)$ be a (left distributive) nearring. Then S has a zero o, and from $a(b+c) = ab + ac$ it follows $So = o$. Hence the 3 subclasses with α) are empty, and also $(\mathfrak{R}, \beta)$ and $(\mathfrak{L} \cap \mathfrak{R}, \beta)$ because of $o \notin \mathfrak{R}$. Nearrings $S = (S, +, \cdot)$ satisfying $S \in (\mathfrak{L}, \beta)$, $S \notin (\mathfrak{R}, \beta)$ or $S \in (\mathfrak{L}, \gamma)$, $S \notin (\mathfrak{R}, \gamma)$ are obtained as follows: Let $(S, +)$ be any group, $|S| \geqq 3$, and define a multiplication by $xy = y$ for all $x, y \in S$ or by

$$ay = y, \quad oy = o \quad \text{for all} \quad a \in S \setminus \{o\}, \quad y \in S,$$

respectively. Since nearrings which belong to $(\mathfrak{L} \cap \mathfrak{R}, \gamma)$ clearly exist, it remains to show $(\mathfrak{R}, \gamma) = (\mathfrak{L} \cap \mathfrak{R}, \gamma)$. Suppose $S \in (\mathfrak{R}, \gamma)$, and that $r \in S$ is not left cancellative, i.e. $ra = rb$ for some $a, b \in S$ such that $a - b \neq o$. Since $r(-b) = -rb$, we obtain $r(a-b) = o$. But we also have $o(a-b) = o$, hence $r = o$, both by $S \in (\mathfrak{R}, \gamma)$. This proves $S \in (\mathfrak{L} \cap \mathfrak{R}, \gamma)$.

iii) All examples discussed above, except the semirings contained in $(\mathfrak{L} \cap \mathfrak{R}, \beta)$, may be chosen finite or infinite. Hence it remains to prove that there is no finite semiring S in $(\mathfrak{L} \cap \mathfrak{R}, \beta)$. But each semiring S of this kind would be a semifield according to the far-reaching definition of this concept given in [2], § 2. Moreover, each semifield has either no or one annihilating zero (hence it belongs to $(\mathfrak{L} \cap \mathfrak{R}, \alpha)$ or $(\mathfrak{L} \cap \mathfrak{R}, \gamma)$), appart from exactly 4 exceptions. The latter are semifields consisting of a non annihilating zero o and one more element, but by tables given in [2], § 1, they are not multiplicatively cancellative.

In the final part of this note we are going to deal similarly with the following common generalization of our concepts.

Definition 2. A semiring $(S, +, \cdot)$ is called *multiplicatively weakly cancellative* iff each element $a \neq o$ of S is left or right cancellable in $(S, \cdot)$, i.e. iff $S \setminus \{o\} \subseteq \mathfrak{L} \cup \mathfrak{R}$. Analogously, a semigroup $(S, \cdot)$ is called *weakly cancellative* iff $S \subseteq \mathfrak{L} \cup \mathfrak{R}$.

Astonishing at a first view, a statement like Proposition 1 also holds with these concepts. But this is a consequence of the unexpected fact, that the local alternative "left or right", conceded to each element in Definition 2, turns out to be a global one, hitting all elements in the same way.

Proposition 2. *A semiring S is multiplicatively weakly cancellative iff one of the conditions α), β) or γ) of Proposition 1 holds with respect to weakly cancellativity. Moreover, such a semiring S is in fact multiplicatively left cancellative or right cancellative.*

Again, the essential part of the proof is a consequence of the following statement, concerning only semigroups. Hence Proposition 2 also applies to any class $\mathfrak{S}$ of algebras as discussed above, and the subclass consisting of all multiplicatively weakly cancellative algebras is just the union of the 9 subclasses (4).

LEMMA 2. *Let* $S=(S, \cdot)$ *be a semigroup such that each element of* S *with at most one exception, noted by* o, *is left or right cancellable in* S. *Then either*

i) *all elements of* S *have this property, i.e.* S *is a weakly cancellative semigroup, which implies that* S *is in fact a left or a right cancellative one, or*

ii) $\{o\}$ *is an ideal of* S *and* $(S\setminus\{o\}, \cdot)$ *a weakly cancellative semigroup to which* i) *applies.*

In short: $S\setminus\{o\}\subseteqq\mathfrak{L}\cup\mathfrak{R}$ *implies* $S=\mathfrak{L}$ *or* $S=\mathfrak{R}$ *or* $S\setminus\{o\}=\mathfrak{L}$ *or* $S\setminus\{o\}=\mathfrak{R}$ *with* $So=oS=o$ *in the latter two cases.*

PROOF. With respect to Lemma 1, we only prove that $S\setminus\{o\}\subseteqq\mathfrak{L}\cup\mathfrak{R}$ implies $S\setminus\{o\}\subseteqq\mathfrak{R}$ or $S\setminus\{o\}\subseteqq\mathfrak{R}$. By way of contradiction, assume that there are elements $x, y\in S$ such that $o\neq x\in\mathfrak{L}\setminus\mathfrak{R}$ and $o\neq y\in\mathfrak{R}\setminus\mathfrak{L}$. Using (1), we obtain $xy\notin\mathfrak{L}\cup\mathfrak{R}$, hence $xy=o$. Also by (1), we get $xxy=xo\notin\mathfrak{L}\cup\mathfrak{R}$, from which $xo=o$ follows, contradicting $xy=o$ and $x\in\mathfrak{L}$.

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REES MATRIX SEMIGROUPS WITH 4-DIMENSIONAL SANDWICH MATRICES

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1. Introduction

Rees' well-known theorem characterizes the completely 0-simple semigroups by means of regular Rees matrix semigroups over groups with zero. In his paper [7], O. STEINFELD introduces the notion of similarly decomposable semigroups which are generalizations of completely 0-simple semigroups and he proves that similarly decomposable semigroups are isomorphic to locally regular Rees matrix semigroups over semigroups with zero and identity elements. In their papers, G. LALLEMENT and M. PETRICH [5] and E. HOTZEL [3] investigate some further semigroup classes which can be characterized by means of Rees matrix semigroups over semigroups with zero and identity elements. In these characterizations the product of the Rees matrices is defined by a sandwich matrix. Our purpose is to give some semigroup classes which can be characterized by means of Rees matrix semigroups over semigroups with zero and with no identity element.

In his paper [2], H.-J. HOEHNKE further generalizes O. Steinfeld's result. He gives a general class of semigroups (that of the homogeneously decomposable semigroups) and proves that they are also isomorphic some Rees matrix semigroups. But the product of these matrices is not given by means of a sandvich matrix, the product is defined by means of a so-called pseudoheap. Since the characterization of a semigroup by means of a pseudoheap is more complicated than the characterization by means of a sandwich matrix our purpose is to give some semigroup classes which are between the similarly decomposable and the homogeneously decomposable semigroups and which have matrix representations with sandwich matrices.

In Section 2 we rewiew the results of O. Steinfeld and H.-J. Hoehnke and we introduce the definitions of the semigroup classes to be investigated. We start with the notion of similarly decomposable semigroups and, by weakening the conditions, we come to the notion of homogeneously decomposable semigroups.

In Section 3 we give a general matrix representation theorem in which the product of the matrices is defined by a 4-dimensional sandwich matrix. Evidently, the 4-dimensional sandwich matrix P^4 can be given by means of some 2-dimensional sandwich matrices. We shall examine (Theorem 3.8) under what conditions P^4 can be given in the simplest way, that means, with the fewest 2-dimensional sandwich matrices. Furthermore we shall examine under what conditions the sandwich matrix P^4 can be given as a 3-dimensional or as a 2-dimensional one as in the case of completely 0-simple and similarly decomposable semigroups.

In Section 4 we give the matrix characterizations of the semigroup classes defined in Section 2.

At the end of Section 4 a table is given to summarise the matrix characterizations which are proved in the present paper and in those I have referred to.

Some applications of the results in the present paper for primitive regular semigroups will be published elsewhere. For example, we shall show that certain primitive regular semigroups are isomorphic to generalized Rees matrix semigroups with 4-dimensional sandwich matrices. Moreover, every primitive regular semigroup will be proved to be embeddable in a generalized Rees matrix semigroup with a 4-dimensional sandwich matrix.

2. Preliminaries and definitions

Let S be a semigroup with zero and L_1, L_2 left ideals of S. By a *left translation of* L_1 *into* L_2 we mean a single valued mapping ψ of L_1 into L_2 such that $s(x\psi)=(sx)\psi$ for all $x\in L_1$ and $s\in S$.

In the case $L_1=L_2=S$ this notion gives the notion of the right translation of S in the sense of CLIFFORD—PRESTON [1].

Analogously, one can define the *right translation* φ of the right ideal R_1 into the right ideal R_2 of S to be a mapping φ with $(x\varphi)s=(xs)\varphi$ for all $x\in R_1$ and $s\in S$.

We say that the left ideals L_1, L_2 of S are *left similar* if there exists a one-to-one left translation ψ of L_1 onto L_2.

Dually, one can define the right similarity of right ideals.

In his paper [7], O. STEINFELD has proved that a semigroup S with zero is completely 0-simple if and only if it has the form $S=\bigcup_{i\in I} e_i S=\bigcup_{\lambda\in\Lambda} Se_\lambda$ with some idempotents $e_i(e_\lambda)$ where $1\in I\cap\Lambda$ and the $e_i S$'s (Se_λ's) are 0-minimal right (left) similar right (left) ideals of S. This observation led him to introducing the following notion.

The semigroup S with 0 is called a *similarly decomposable semigroup* if (2.1) $S=\bigcup_{i\in I} R_i=\bigcup_{\lambda\in\Lambda} L_\lambda$ where the R_i's $(i\in I)$, $\big(L_\lambda\text{'s}\,(\lambda\in\Lambda)\big)$ are right (left) similar right (left) ideals of S with

$$R_j\cap R_k=0 \quad (j,k\in I;\ j\neq k),\quad \big(L_\mu\cap L_\nu=0\ (\mu,\nu\in\Lambda;\ \mu\neq\nu)\big);$$

$$(2.2)\qquad R_i=e_iS\quad (e_i^2=e_i;\ i\in I),\quad L_\lambda=Se_\lambda\quad (e_\lambda^2=e_\lambda;\ \lambda\in\Lambda)$$

and $1\in I\cap\Lambda$.

If a semigroup S satisfies condition (2.1) and $1\in I\cap\Lambda$ then there exist one-to-one right translation φ_i of R_i onto R_1 (one-to-one left translations ψ_λ of L_λ onto L_1) by the right (left) similarity condition. This notion is used throughout the paper.

We say that a Rees matrix semigroup $M^0\big(H;I,\Lambda;P=(p_{\lambda i})\big)$ over a semigroup with identity element e and zero element 0 is (Q)-*regular* (quasi-regular in TRAN QUY TIEN [6]) if $P=(p_{\lambda i})$ has the following properties:

a) for every $\lambda\in\Lambda$ there exists an entry $P_{\lambda j(\lambda)}$ $\big(j(\lambda)\in I\big)$ in P which has a right inverse in H;

b) for every $i\in I$ there exists an entry $p_{\mu(i)i}$ $\big(\mu(i)\in\Lambda\big)$ in P which has a left inverse in H.

A Rees matrix semigroup $M^0(H; I, \Lambda; P=(p_{\lambda i}))$ over a semigroup H with identity element is called (R)-*regular* if there exists an entry $p_{\lambda i}$ in P which has a two-sided inverse in H.

We say that a Rees matrix semigroup $M^0(H; I, \Lambda; P)$ is *locally regular* if it is both (Q)-regular and (R)-regular.

THEOREM 2.1. (Theorem 4.1 in [7].) *A semigroup S with 0 is similarly decomposable if and only if it is isomorphic to a locally regular Rees matrix semigroup over a semigroup with zero and identity.*

Let $H, H_1, H_2, ..., H_n$ $(n\geqq 1)$ be arbitrary sets. The set of all mappings of $H_1\times H_2\times ...\times H_n$ into H is denoted by $F_H^{H_1, H_2, ..., H_n}$. If $\psi(\varphi)$ is a left (right, ⟨one-to-one left, one-to-one right⟩) translation of I_1 into [into, ⟨onto⟩] I_2 then we shall abbreviate this by writing $\psi\in F_{I_2}^{I_1}\cap\mathbf{\Lambda}$ $(\varphi\in F_{I_2}^{I_1}\cap\mathbf{P}$, $\langle\psi\in F_{I_2}^{I_1}\cap\overline{\mathbf{\Lambda}}$, $\varphi\in F_{I_2}^{I_1}\cap\overline{\mathbf{P}}\rangle)$.

In his paper [2], H.-J. HOEHNKE generalizes Theorem 2.1 by introducing the notion of a 4-pseudoheap.

Let $T_1, T_2, ..., T_n$ $(n\geqq 2)$ be arbitrary sets such that $T_1=T_n=T$. By an *n-pseudoheap* we mean a mapping $\Phi\in F_T^{T_1, T_2, ..., T_n}$,

$$(t_1, t_2, ..., t_n)\Phi = [t_1 t_2 ... t_n] \quad (t_i\in T_i,\ 1\leqq i\leqq n)$$

for which $[[t_1 t_2 ... t_n]t_2' ... t_n'] = [t_1 t_2 ... t_{n-1}[t_n t_2' ... t_n']]$ holds for all $t_i, t_i'\in T_i$ $(2\leqq i\leqq n)$ and $t_1\in T_1$.

REMARK. Clearly, every 2-pseudoheap is a semigroup and conversely, every semigroup is a 2-pseudoheap.

We say that the semigroup S with zero is a *homogeneously decomposable semigroup* (see H.-J. HOEHNKE [2]) if S satisfies condition (2.1) such that there exists an index $1\in I\cap\Lambda$ and there exist systems $\{\varphi_i\in F_{R_1}^{R_i}\cap\overline{\mathbf{P}}\}_{i\in I}$ and $\{\psi_\lambda\in F_{L_1}^{L_\lambda}\cap\overline{\mathbf{\Lambda}}\}_{\lambda\in\Lambda}$ fulfilling the following conditions:

$$(R_i\cap L_\lambda)\varphi_i \subseteqq R_1\cap L_\lambda, \quad (R_i\cap L_\lambda)\psi_\lambda \subseteqq R_i\cap L_1 \tag{2.3}$$

and $s\varphi_i\psi_\lambda = s\psi_\lambda\varphi_i$ for all $s\in R_i\cap L_\lambda$, $i\in I$, $\lambda\in\Lambda$.

THEOREM 2.2. (Theorem 6.1 in [2].) *The following two conditions on a semigroup S with zero are equivalent:*

i) *S is a homogeneously decomposable semigroup;*

ii) *S is isomorphic to a Rees matrix semigroup $M^0(H; I, \Lambda; [\ ])$ where the product of the matrices $(a)_{i\lambda}$, $(b)_{j\mu}$ is defined by $(a)_{i\lambda}\circ(b)_{j\mu} = ([a\lambda jb])_{i\mu}$ $(a, b\in H;$ $i, j\in I;$ $\lambda, \mu\in\Lambda)$ where the mapping $(a, \lambda, j, b)\to[a\lambda jb]$ is a 4-pseudoheap.*

In the decomposition of a similarly decomposable semigroup the left (right) ideals are generated by idempotent elements. We can give a new class of semigroups if we do not require the generating elements to be idempotent.

We say that a semigroup S is (r)-*decomposable* if S satisfies condition (2.1) and $L_\lambda = Sl_\lambda$ $(l_\lambda\in S, \lambda\in\Lambda)$, $R_i = r_i S$ $(r_i\in S_i, i\in I)$ such that there exists $1\in I\cap\Lambda$ with $r_1\in Sr_1$, $l_1\in l_1 S$. We shall prove that an (r)-decomposable semigroup is isomorphic to an (R)-regular Rees matrix semigroup (Theorem 4.5). It is easy to prove that if S is (r)-decomposable then $l_\lambda\in L_\lambda$ and $r_i\in R_i$ for all $\lambda\in\Lambda$, $i\in I$.

An alternative definition (see Theorem 4.4) of (r)-decomposability is that S satisfies condition (2.1) and there exists an element $e=e^2\in S$ and an index $1\in I\cap\Lambda$ such that e is a right identity element of L_1 and a left identity element of R_1.

The notion of similar decomposability can be further generalized such that instead of the existence of an identity element we require the existence of relative identity elements of every element in L_1 and R_1.

We say that a semigroup S is a *semigroup with relative right (left) identity elements* if, for every element a in S, there exists an element $d^a[e^a]$ in S with $a=ad^a[a=e^aa]$. If S is a semigroup with relative right and left identity elements then we say that S is a *semigroup with relative identity elements.*

An (r*)-*decomposable* semigroup is a semigroup with zero which satisfies condition (2.1) such that there exists an index $1\in I\cap\Lambda$ such that R_1 is a semigroup with relative left identity elements and $H=L_1\cap R_1$ is a semigroup with relative identity elements. (It is easy to prove that in this case L_1 is a semigroup with relative right identity elements.) In Theorem 4.2 we shall give a characterization of (r*)-decomposable semigroups by means of Rees matrix semigroups with 4-dimensional (R*)-regular sandwich matrices. The notion of (R*)-regularity is a generalization of (R)-regularity.

If S is (r*)-decomposable then the relations $(L_1\cap R_1)^2=L_1\cap R_1$ and $R_1=R_1S$, $L_1=SL_1$ are trivially satisfied. The notion of similar decomposability can be further generalized if we require these properties only.

We say that the semigroup S with zero is (r′)-*decomposable* if it satisfies condition (2.1) such that there exists an index $1\in I\cap\Lambda$ with $(R_1\cap L_1)^2=R_1\cap L_1$ and $R_1=R_1S$. (It is easy to prove that $L_1=SL_1$ is implied.) We shall prove that S is (r′)-decomposable if and only if it is isomorphic to a Rees matrix semigroup with a property which is a generalization of the notion of (R)-regularity, called (R′)-regularity, and the product is defined by a 4-dimensional sandwich matrix and four mappings of $R_1\cap L_1$ into $R_1\cap L_1$. (Theorem 3.1.)

We shall prove that the notion of (r′)-decomposability is a special case of the notion of homogeneous decomposability (Lemma 3.2).

Similarly to the above definitions we can get another generalization of the notion of local regularity by generalizing the notion of (Q)-regularity and so we can characterize another semigroup class (Theorem 4.3).

3. On (r′)-decomposable semigroups

Let H be a semigroup with zero and I, Λ two index sets. Suppose $\varrho_1, \varrho_2, \varrho_3, \varrho_4\in F_H^H$ and $P^4=(p[a, \lambda, j, b])_{\substack{a,b\in H\\ \lambda\in\Lambda, j\in I}}\in F_H^{H,\Lambda,I,H}$ is a 4-dimensional matrix. Define a multiplication on the set of all $I\times\Lambda$ Rees matrices over H in the following way:

$$(a)_{i\lambda}\circ(b)_{j\mu}=(a\varrho_1\cdot p\,[a\varrho_2, \lambda, j, b\varrho_3]\cdot b\varrho_4)_{i\mu}. \tag{3.1}$$

If this multiplication is associative then the semigroup obtained in this way is called a *generalized Rees matrix semigroup* and is denoted by $M^0=M^0(H; I, \Lambda; P^4; \varrho_1, \varrho_2, \varrho_3, \varrho_4)$.

REMARK. The associativity condition can be formulated but it is too complicated so we omit it.

An (R′)-*regular Rees matrix semigroup* is a generalized Rees matrix semigroup which satisfies the following condition: there exists an index $1 \in I \cap \Lambda$ and for all elements $a \in H$ there exist elements $a_1, a_2 \in H$ such that

$$a = a_1 \varrho_1 \cdot p[a_1 \varrho_2, 1, 1, a_2 \varrho_3] \cdot a_2 \varrho_4. \tag{3.2}$$

THEOREM 3.1. *A semigroup S with zero is* (r′)-*decomposable if and only if it is isomorphic to an* (R′)-*regular Rees matrix semigroup.*

For the proof we need the following

LEMMA 3.2. *If S is an* (r′)-*decomposable semigroup then it is homogeneously decomposable.*

PROOF. Let S be an (r′)-decomposable semigroup. Then $1 \in I \cap \Lambda$, $(L_1 \cap R_1)^2 = L_1 \cap R_1$ and $R_1 = R_1 S$ according to (2.1).

1) Consider an arbitrary non-zero element $a = \bar{a}\varphi_i$ of $(R_i \cap L_\lambda)\varphi_i$ where $\bar{a} \in R_i \cap L_\lambda$, $\varphi_i \in F_{R_1}^{R_i} \cap \overline{\mathbf{P}}$. Clearly $\bar{a} \neq 0$. Assume that $a \in R_1 \cap L_\mu$ $(\mu \in \Lambda)$.

Since $R_1 = R_1 S$ there exist elements $a_1 \in R_1$ and $a_2 \in S$ such that $a = a_1 a_2$. Then $a_2 \in L_\mu$ as $a_1 a_2 \in L_\mu$. Therefore $a\varphi_i^{-1} = \bar{a}\varphi_i \varphi_i^{-1} = \bar{a} \in R_i \cap L_\lambda$ and $a\varphi_i^{-1} = (a_1 a_2)\varphi_i^{-1} = (a_1 \varphi_i^{-1}) a_2 \in R_i \cap L_\mu$, since $\varphi_i^{-1} \in F_{R_i}^{R_1} \cap \overline{\mathbf{P}}$. The equality $\lambda = \mu$ follows from the facts that $L_\lambda \cap L_\mu = 0$ provided $\lambda \neq \mu$ and $\bar{a} = a\varphi_i^{-1} \neq 0$.

Therefore $(R_i \cap L_\lambda)\varphi_i \subseteq R_1 \cap L_\lambda$ for all $\lambda \in \Lambda$, $i \in I$.

2) Now consider a non-zero element a in $(R_i \cap L_\lambda)\psi_\lambda$ $(\psi_\lambda \in F_{L_1}^{L_\lambda} \cap \overline{\mathbf{\Lambda}})$ and suppose that $a \in R_j \cap L_1$. Then $a\varphi_j \in R_1 \cap L_1$ and there exist elements $a_1, a_2 \in R_1 \cap L_1$ such that $a\varphi_j = a_1 a_2$. Consequently, $a = a\varphi_j \varphi_j^{-1} = (a_1 a_2)\varphi_j^{-1} = (a_1 \varphi_j^{-1}) a_2 \in R_j \cap L_1$ and $a\psi_\lambda^{-1} = ((a_1\varphi_j^{-1})a_2)\psi_\lambda^{-1} = (a_1\varphi_j^{-1})(a_2\psi_\lambda^{-1}) \in R_j \cap L_\lambda$. The equality $i = j$ is implied by the facts that $a\psi_\lambda^{-1} \in R_i \cap L_\lambda$, $a \neq 0$ and $R_i \cap R_j = 0$ if $i \neq j$.

3) Let a be an arbitrary element of $R_i \cap L_\lambda$. Then $a\psi_\lambda \varphi_i \in R_1 \cap L_1$ and hence $a\psi_\lambda \varphi_i = a_1 a_2$ where $a_1, a_2 \in R_1 \cap L_1$. Thus we obtain that

$$a = (a_1 a_2)\varphi_i^{-1}\psi_\lambda^{-1} = (a_1\varphi_i^{-1})(a_2\psi_\lambda^{-1}) \quad \text{and} \quad a\varphi_i\psi_\lambda = [(a_1\varphi_i^{-1})(a_2\psi_\lambda^{-1})]\varphi_i\psi_\lambda =$$

$$= (a_1\varphi_i^{-1}\varphi_i)(a_2\psi_i^{-1}\psi_\lambda) = a_1 a_2 = a\psi_\lambda\varphi_i.$$

We have made use of (and shall do so) the fact that the inverse translation of a left (right) translation is also a left (right) translation.

PROOF OF THEOREM 3.1. Let S be an (r')-decomposable semigroup. Then Lemma 3.2 and Theorem 2.2 imply S to be isomorphic to a Rees matrix semigroup $M^0 = M^0(H; I, \Lambda; [\ ])$ where the product $(a)_{i\lambda} \circ (b)_{j\mu} = ([a\lambda jb])_{i\mu}$ is defined by a 4-pseudoheap. Denote by Φ $(\in F_{M^0}^S)$ the isomorphism given in the proof of Theorem 2.2 in [2]. The 4-pseudoheap is the following: $[a\lambda jb] = (a\psi_\lambda^{-1})(b\varphi_j^{-1})$ where $\psi_\lambda \in F_{L_1}^{L_\lambda} \cap \overline{\mathbf{\Lambda}}$ and $\varphi_j \in F_{R_1}^{R_j} \cap \overline{\mathbf{P}}$ are the translations in (2.1) and (2.3).

Since $H = R_1 \cap L_1$ we can define mappings $\varrho_1, \varrho_2, \varrho_3, \varrho_4 \in F_H^H$ such that for every element a in H we have $a = a\varrho_1 \cdot a\varrho_2 = a\varrho_3 \cdot a\varrho_4$. Of course, ϱ_1 may be equal to ϱ_3 and, similarly, ϱ_2 to ϱ_4. Then we have

$$(a)_{i\lambda}\circ(b)_{j\mu}=([a\lambda jb])_{i\mu}=\big((a\psi_\lambda^{-1})\cdot(b\varphi_j^{-1})\big)_{i\mu}=$$

$$=(a\varrho_1\cdot(a\varrho_2)\psi_\lambda^{-1}\cdot b\varrho_3)\varphi_j^{-1}\cdot b\varrho_4)_{i\mu}=(a\varrho_1\cdot p[a\varrho_2,\lambda,j,b\varrho_3]\cdot b\varrho_4)_{i\mu}$$

where $p[a\varrho_2,\lambda,j,b\varrho_3]=(a\varrho_2)\psi_\lambda^{-1}\cdot(b\varrho_3)\varphi_j^{-1}$.

Thus $M^0=M^0(H;I,\Lambda;P^4;\varrho_1,\varrho_2,\varrho_3,\varrho_4)$ is a generalized Rees matrix semigroup.

We still have to prove that M^0 is (R′)-regular. If $a\in H$ and $(a)_{11}=\bar{a}\Phi$ $(\bar{a}\in R_1\cap L_1)$ then there exist elements $\bar{a}_1,\bar{a}_2\in R_1\cap L_1$ such that $\bar{a}=\bar{a}_1\bar{a}_2$ since $(R_1\cap L_1)^2=R_1\cap L_1$. Then $(a)_{11}=\bar{a}\Phi=(\bar{a}_1\bar{a}_2)\Phi=\bar{a}_1\Phi\circ\bar{a}_2\Phi=(a_1)_{11}\circ(a_2)_{11}=$ $=(a_1\varrho_1\cdot p[a_1\varrho_2,1,1,a_2\varrho_3]\cdot a_2\varrho_4)_{11}$ where $\bar{a}_1\Phi=(a_1)_{11}$, $\bar{a}_2\Phi=(a_2)_{11}$. Consequently, condition (3.2) is fulfilled for the sandwich matrix P^4.

Conversely, let S be isomorphic to the (R′)-regular Rees matrix semigroup $M^0=M^0(H;I,\Lambda;P^4;\varrho_1,\varrho_2,\varrho_3,\varrho_4)$. Denote the elements of M^0 by $(a)_{i\lambda}$ where $a\in H$, $i\in I$ and $\lambda\in\Lambda$. Let $M_\lambda^l\,[M_i^r]$ be the set of the matrices $(a)_{i\lambda}$ for all $a\in H$ and $i\in I$ [for all $a\in H$ and $\lambda\in\Lambda$]. It follows from (3.1) that $M_\lambda^l\,[M_i^r]$ is a left [right] ideal of M^0 for every $\lambda\in\Lambda\,[i\in I]$.

The decomposition $M^0=\bigcup_{\lambda\in\Lambda}M_\lambda^l=\bigcup_{i\in I}M_i^r$ where $M_\lambda^l\cap M_\mu^l=0$ if $\lambda\neq\mu$ and $M_i^r\cap M_j^r=0$ if $i\neq j$ trivially holds.

Let $\varphi_j\in F_{M_1^r}^{M_j^r}\cap\mathbf{P}$ and $\psi_\lambda\in F_{M_1^l}^{M_\lambda^l}\cap\bar{\mathbf{\Lambda}}$ is defined by $(a)_{j\lambda}\varphi_j=(a)_{1\lambda}$, $j\in I$ and $(a)_{i\lambda}\psi_\lambda=(a)_{i1}$, $\lambda\in\Lambda$, respectively.

Denote by $1\in I\cap\Lambda$ the index which satisfies condition (3.2). For each $(a)_{11}\in M_1^r\cap M_1^l$ there exist elements $a_1,a_2\in H$ such that

$$(a)_{11}=(a_1\varrho_1\cdot p[a_1\varrho_2,1,1,a_2\varrho_3]\cdot a_2\varrho_4)_{11}=(a_1)_{11}\circ(a_2)_{11}\in(M_1^r\cap M_1^l)^2.$$

Hence $(M_1^r\cap M_1^l)^2=M_1^r\cap M_1^l$. Similarly we get $(a)_{1\lambda}=(a_1)_{11}\circ(a_2)_{1\lambda}$ whence $M_1^r=M_1^r\circ M^0$. Consequently, M^0 is an (r′)-decomposable semigroup which completes the proof.

REMARK. Though it seems now to be unnecessary to make use of four mappings instead of two later we shall need the original form of Theorem 3.1.

COROLLARY 3.3. *Let $M^0=M^0(H;I,\Lambda;P^4;\varrho_1,\varrho_2,\varrho_3,\varrho_4)$ be an* (R′)*-regular Rees matrix semigroup. Then there exists an* (R′)*-regular Rees matrix semigroup $\bar{M}^0=\bar{M}^0(\bar{H};I,\Lambda;\bar{P}^4;\bar{\varrho}_1,\bar{\varrho}_2,\bar{\varrho}_1,\bar{\varrho}_2)$ such that M^0 is isomorphic to $\bar{M}^0$ and $\bar{\varrho}_1,\bar{\varrho}_2\in F_{\bar{H}}^{\bar{H}}$ satisfy the condition $\bar{a}=\bar{a}\bar{\varrho}_1\cdot\bar{a}\bar{\varrho}_2$ for all $\bar{a}\in\bar{H}$.*

The following theorem concerns the connection between homogeneously decomposable semigroups and (r′)-decomposable semigroups.

THEOREM 3.4. *The Rees matrix semigroup $M^0(H;I,\Lambda;[\ ])$ where the product is defined by a 4-pseudoheap is isomorphic to an* (R′)*-regular Rees matrix semigroup if and only if there exists $1\in I\cap\Lambda$ such that for every element $a\in H$ there are elements $a_1,a_2\in H$ with $a=[a_1\,11\,a_2]$.*

PROOF. Let $M^0(H;I,\Lambda;[\ ])$ be a Rees matrix semigroup fulfilling the conditions of the theorem. Let $R_1=\{(a)_{1\lambda}|a\in H,\lambda\in\Lambda\}$ and $L_1=\{(a)_{i1}|a\in H,i\in I\}$. Then, for every element $(a)_{11}\in R_1\cap L_1$ we have $(a)_{11}=([a_1\ 11\ a_2])_{11}=(a_1)_{11}\circ(a_2)_{11}\in(R_1\cap L_1)^2$ and $(a)_{1\lambda}=(a_1)_{11}\circ(a_2)_{1\lambda}\in R_1\circ M^0$. Therefore $M^0(H; I, \Lambda; [\ ])$ is an (r′)-decomposable semigroup.

The converse part immediately follows from Lemma 3.2.

Our purpose in the next part of this section is to determine which semigroup classes can be obtained in the simplest way by means of the mappings $\varrho_1, \varrho_2, \varrho_3, \varrho_4$ and P^4.

Let $F_1\,[F_2]$ denote the subset of all mappings ϱ in F_H^H with the property that $0\varrho=0\ (\in H)$ and $a\in a\varrho\cdot H\ [a\in H\cdot a\varrho]$ for all $a\in H$.

If $P^4\in F_H^{H,I,\Lambda,H}$ then it can be given by means of $|H|\cdot|H|$ pieces of mappings in $F_H^{\Lambda,I}$. Here $|H|$ denotes the cardinality of H.

By the proof of Theorem 3.1 we infer

LEMMA 3.5. *Let S be an* (r′)*-decomposable semigroup. Then S is isomorphic to a Rees matrix semigroup $M^0(H; I, \Lambda; P^4; \varrho_1, \varrho_2, \varrho_3, \varrho_4)$ such that $\varrho_1, \varrho_3\in F_1$, $\varrho_2, \varrho_4\in F_2$ and P^4 can be given by means of $|H\varrho_2|\cdot|H\varrho_3|$ pieces of mappings from $F_H^{\Lambda,I}$.*

An important question is to answer when P^4 can be given by the fewest mappings from $F_H^{\Lambda,I}$, that is, by means of the fewest 2-dimensional sandwich matrices.

LEMMA 3.6. *Let H be a semigroup with zero. Assume that there exists a mapping $\varrho_2\in F_2$ such that $(sa)\varrho_2=a\varrho_2$ for all $a, s\in H$, $sa\neq 0$. Then every mapping $\varrho_2'\in F_2$ satisfies the condition*

$$|H\varrho_2|\leqq|H\varrho_2'|.$$

PROOF. If there exists a mapping $\varrho_2'\in F_2$ with $|H\varrho_2|>|H\varrho_2'|$ then H contains elements a, b such that $a\varrho_2'=b\varrho_2'$ but $a\varrho_2\neq b\varrho_2$. Then $a=a\varrho_1'\cdot a\varrho_2'$ since $\varrho_2'\in F_2$ and $b=b\varrho_1'\cdot b\varrho_2'=b\varrho_1'\cdot a\varrho_2'$. Thus $a, b\in Sa\varrho_2'$.

The equality $(st)\varrho_2=t\varrho_2$ holding for all $s, t\in H$ implies $a\varrho_2=a\varrho_2'\varrho_2$ and $b\varrho_2=a\varrho_2'\varrho_2$, whence we have $a\varrho_2=b\varrho_2$. This contradiction shows that $|H\varrho_2|\leqq|H\varrho_2'|$ for every $\varrho_2'\in F_2$.

Evidently, the dual of Lemma 3.6 also holds.

LEMMA 3.7. *Let S be an* (r′)*-decomposable semigroup. If $R_1\cap L_1=H=\bigcup_{\omega\in\Omega} Ha_\omega\ (a_\omega\in H)$ where $Ha_\omega\cap Ha_\mu=0$ is valid provided $\omega, \mu\in\Omega$, $\omega\neq\mu$ then there exists an* (R′)*-regular Rees matrix semigroup $M^0(H; I, \Lambda; P^4; \varrho_1, \varrho_2, \varrho_3, \varrho_4)$ which is isomorphic to S and $\varrho_1\in F_1$, $\varrho_2\in F_2$ satisfy the following conditions:*

i) $a\varrho_1\cdot a\varrho_2=a$ *for all* $a\in H$,

ii) $(sa)\varrho_2=a\varrho_2$ *for all* $a, s\in H$, $sa\neq 0$.

PROOF. Let $R_1\cap L_1=H=\bigcup_{\omega\in\Omega} Ha_\omega$. Suppose $a_\omega=a_\omega^{(1)}\cdot a_\omega^{(2)}$ is an arbitrary product-decomposition of the element a_ω.

(3.3) Let the mappings ϱ_1, ϱ_2 be defined by $b\varrho_2=a_\omega^{(2)}$ and $b\varrho_1=b'(a_\omega^{(1)})$ if $b\in Ha_\omega$ and b' is a fixed element of H with $b=b'a_\omega$, $b\neq 0$ and $0\varrho_1=0\varrho_2=0$. Then $b\varrho_1\cdot b\varrho_2=b'a_\omega^{(1)}\cdot a_\omega^{(2)}=b'a_\omega=b$. Furthermore, we have $(sb)\varrho_2=\big((sb')a_\omega\big)\varrho_2=a_\omega^{(2)}=b\varrho_2$ for all $s\in H$, $sb\neq 0$.

Note that the mapping ϱ_1 just defined is contained in $F_1 \cap \Lambda$.

Evidently, the dual of Lemma 3.7 holds for the mappings ϱ_3, ϱ_4 if $H = \bigcup_{\omega \in \Omega} a_\omega H$ is a zero-disjoint decomposition.

The following theorem is an immediate consequence of Lemmas 3.5, 3.6 and 3.7, their duals and Theorem 3.1.

THEOREM 3.8. *Let S be an* (r′)*-decomposable semigroup. Assume that $R_1 \cap L_1 = H = \bigcup_{\omega \in \Omega_1} Ha_\omega = \bigcup_{\omega \in \Omega_2} b_\omega H$ are zero-disjoint decompositions of H. Then by* (3.3) *and its dual we can define an* (R′)*-regular Rees matrix semigroup $M^0(H; I, \Lambda; P^4; \varrho_1, \varrho_2, \varrho_3, \varrho_4)$ such that among all the representations of S by an* (R′)*-regular Rees matrix semigroup the* 4*-dimensional sandwich matrix P^4 can be given by means of the fewest* 2*-dimensional sandwich matrices.*

COROLLARY 3.9. *Let S be an* (r′)*-decomposable semigroup. If there exists an element $a \in R_1 \cap L_1 = H$ such that $H = Ha$ then there exists an* (R′)*-regular Rees matrix semigroup $M^0(H; I, \Lambda; P^4; \varrho_1, \varrho_2, \varrho_3, \varrho_4)$ with ϱ_2 a constant mapping such that it is isomorphic to S. Consequently, P^4 does not depend on its first component, that is, it is essentially* 3*-dimensional. The product of matrices is defined by*

$$(a)_{i\lambda} \circ (b)_{j\mu} = (a\varrho_1 \cdot p[\lambda, j, b\varrho_3] \cdot b\varrho_4)_{i\mu}.$$

COROLLARY 3.10. *Let S be an* (r′)*-decomposable semigroup. If there exist elements $a, b \in R_1 \cap L_1 = H$ such that $H = Ha = bH$ then there exists an* (R′)*-regular Rees matrix semigroup $M^0(H; I, \Lambda; P^4; \varrho_1, \varrho_2, \varrho_3, \varrho_4)$ with ϱ_2 and ϱ_3 constant mappings such that it is isomorphic to S. Thus P^4 does not depend on its first and last component, that is, P^4 is essentially a* 2*-dimensional sandwich matrix and the product of matrices is defined by*

$$(a)_{i\lambda} \circ (b)_{j\mu} = (a\varrho_1 \cdot p_{\lambda j} \cdot b\varrho_4)_{i\mu}.$$

In their paper [3], S. LAJOS and J. SZÉP have proved the following theorem:

THEOREM 3.11. *A semigroup S is a semigroup with an identity element if and only if it contains elements a, b such that*

1) *$Sa = S$ but $Ta \neq S$ for all proper subsets T of S, and*
2) *$bS = S$ but $bT \neq S$ for all proper subsets T of S.*

COROLLARY 3.12. *Let S be an* (r′)*-decomposable semigroup. If there exist elements $a, b \in R_1 \cap L_1 = H$ such that $H = Ha = bH$ and $Ta \neq H$ and $bT \neq H$ for all proper subsets T of H (for example H is a finite semigroup) then S is an* (r)*-decomposable semigroup.*

Corollary 3.10 implies that in this case S can be represented by means of a Rees matrix semigroup with a 2-dimensional sandwich matrix (see Theorem 4.5).

In the next section we shall get some further simple characterizations of certain semigroups.

4. Some special cases

Let H be a semigroup with zero and relative identity elements and let $P^4 = (p[a, \lambda, j, b])_{\substack{a,b\in H\\ \lambda\in\Lambda, j\in I}} \in F_H^{H,I,\Lambda,H}$. Suppose $\varrho_1, \varrho_2 \in F_H^H$ with the property that

(4.1) for each element a, b, s, t of H and for each index $i\in I$ and $\lambda\in\Lambda$ the equality

$$sap[a\varrho_1, \lambda, j, b\varrho_2]bt = sap[(sa)\varrho_1, \lambda, j, (bt)\varrho_2]bt$$

is satisfied.

Define a multiplication on the set of all Rees matrices over H by

$$(a)_{i\lambda} \circ (b)_{j\mu} = (ap[a\varrho_1, \lambda, j, b\varrho_2]b)_{i\mu}. \tag{4.2}$$

The following lemma shows this multiplication to be associative. So we have obtained in this way a semigroup which we term a $(*)$*-generalized Rees matrix semigroup* and denote by $M^0 = M^0(H; I, \Lambda; P^4; \varrho_1, \varrho_2)$.

LEMMA 4.1. *The multiplication defined in* (4.2) *is associative.*

PROOF. Assume that $(a)_{i\lambda}, (b)_{j\mu}, (c)_{k\nu}$ are Rees matrices over H. By (4.2) we have

$$(a)_{i\lambda} \circ [(b)_{j\mu} \circ (c)_{k\nu}] = (a)_{i\lambda} \circ (bp[b\varrho_1, \mu, k, c\varrho_2]c)_{j\nu} =$$

$$= \big(ap\big[a\varrho_1, \lambda, j, (bp[b\varrho_1, \mu, k, c\varrho_2]c)\varrho_2\big]bp[b\varrho_1, \mu, k, c\varrho_2]c\big)_{i\nu} =$$

$$= (ap[a\varrho_1, \lambda, j, b\varrho_2]bp[b\varrho_1, \mu, k, c\varrho_2]c)_{i\nu} =$$

$$= \big(ap[a\varrho_1, \lambda, j, b\varrho_2]bp\big[(ap[a\varrho_1, \lambda, j, b\varrho_2]b)\varrho_1, \mu, k, c\varrho_2\big]c\big)_{i\nu} =$$

$$= (ap[a\varrho_1, \lambda, j, b\varrho_2]b)_{i\mu} \circ (c)_{k\nu} = [(a)_{i\lambda} \circ (b)_{j\mu}] \circ (c)_{k\nu}$$

which was to be proved.

We say that a $(*)$-generalized Rees matrix semigroup $M^0(H; I, \Lambda; P^4; \varrho_1, \varrho_2)$ is (R*)-*regular* if it has the following property:

(4.3) there exists an index $1\in I\cap\Lambda$ such that for each element $s\in H$ the equalities

$$q_1 p[q_1\varrho_1, 1, 1, s\varrho_2]s = sp[s\varrho_1, 1, 1, q_2\varrho_2]q_2 = s$$

hold for some $q_1, q_2 \in H$.

This generalizes the notion of (R)-regularity and is a special case of the notion of (R′)-regularity.

We obtain the following assertion from Theorem 3.1 as a special case.

THEOREM 4.2. *A semigroup S with zero is* (r*)-*decomposable if and only if it is isomorphic to an* (R*)-*regular Rees matrix semigroup over a semigroup with zero and relative identity elements.*

PROOF. If S is (r*)-decomposable it is (r′)-decomposable. Since $H = R_1 \cap L_1$ is a semigroup with relative identity elements S is isomorphic to an (R′)-regular

Rees matrix semigroup $M^0(H; I, \Lambda; P^4; \bar{\varrho}_1, \bar{\varrho}_2, \bar{\varrho}_3, \bar{\varrho}_4)$ such that $a\bar{\varrho}_1 = a\bar{\varrho}_4 = a$ for every $a \in H$. Define $\varrho_1, \varrho_2 \in F_H^H$ to be equal to $\bar{\varrho}_2$ and $\bar{\varrho}_3$, respectively. Then $M^0(H; I, \Lambda; P^4; \varrho_1, \varrho_2) = M^0(H; I, \Lambda; P^4; \bar{\varrho}_1, \bar{\varrho}_2, \bar{\varrho}_3, \bar{\varrho}_4)$ and the product of matrices is $(a)_{i\lambda} \circ (b)_{j\mu} = (ap[a\varrho_1, \lambda, j, b\varrho_2]b)_{i\mu}$.

Furthermore

$$sap[a\varrho_1, \lambda, j, b\varrho_2]bt = sa(a\varrho_1)\psi_\lambda \cdot (b\varrho_2)\varphi_j bt =$$

$$= [sa \cdot (a\varrho_1)]\psi_\lambda \cdot [(b\varrho_2)bt]\varphi_j = (sa)\psi_\lambda \cdot (bt)\varphi_j =$$

$$= [sa(sa)\varrho_1]\psi_\lambda \cdot [(bt)\varrho_2 bt]\varphi_j = sa[(sa)\varrho_1]\psi_\lambda \cdot [(bt)\varrho_2]\varphi_j bt =$$

$$= sap[(sa)\varrho_1, \lambda, j, (bt)\varrho_2]bt$$

for all $a, b, s, t \in H$; $\lambda \in \Lambda$; $j \in I$. Therefore condition (4.1) is fulfilled. Condition (4.3) follows from the fact that H is a semigroup with relative identity elements.

The converse part is easily proved.

We say that an (r*)-decomposable semigroup S is (r*q*)-*decomposable if* S is a semigroup with relative identity elements.

A (Q*)-*regular Rees matrix semigroup* is a $(*)$-generalized Rees matrix semigroup satisfying the following conditions:

a) for each $a \in H, \lambda \in \Lambda$ there exist $j \in I$ and $q \in H$ such that $a = ap[a\varrho_1, \lambda, j, q\varrho_2]q$;

b) for each $a \in H, j \in I$ there exist $\lambda \in \Lambda$ and $q \in H$ such that $a = qp[q\varrho_1, \lambda, j, a\varrho_2]a$.

THEOREM 4.3. *A semigroup S with zero is* (r*q*)-*decomposable if and only if it is isomorphic to a* (Q*)- *and* (R*)-*regular* $(*)$-*generalized Rees matrix semigroup over a semigroup with zero and relative identity elements.*

REMARK. In [6] and [9] some results are proved under what conditions the Rees matrix semigroup and its base semigroup H have similar properties (regularity, 0-D-simplicity, complete simplicity). In these investigations the concept of (Q)-regularity plays an important role. Analogous questions can be raised for $(*)$-generalized Rees matrix semigroups. In this case, (Q*)-regularity plays the role of (Q)-regularity.

Theorem 4.5 immediately follows from Theorem 4.3 if we observe the following property of semigroups with identity. This result is connected with Theorem 3.11.

THEOREM 4.4. *A semigroup S is a semigroup with identity element if and only if it contains elements a, b such that*

1) $Sa = S$ *and* $a \in aS$;

2) $bS = S$ *and* $b \in Sb$.

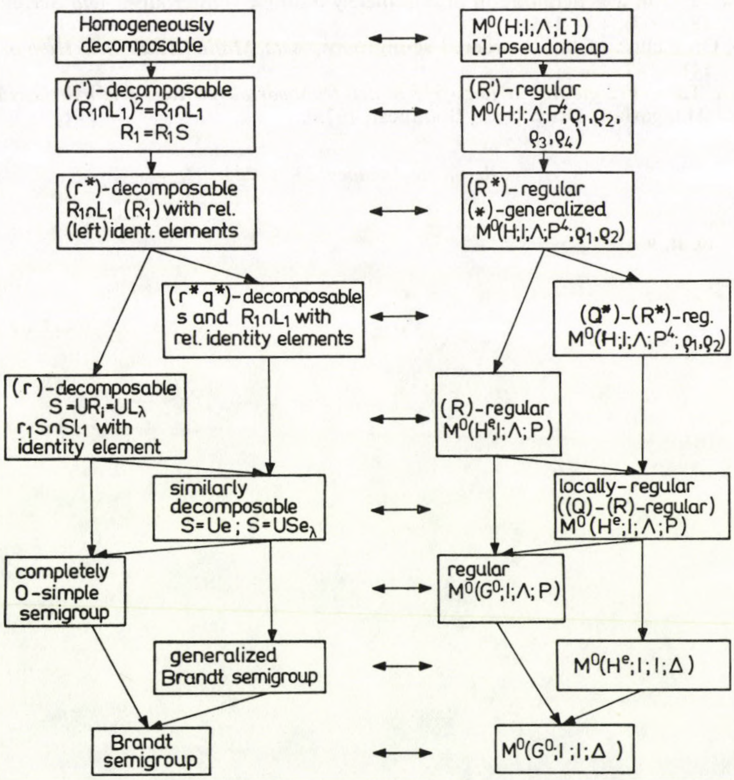
PROOF. Assume S satisfies conditions 1) and 2) for the elements $a, b \in S$. Then there exist elements e^a, e^b with $a = ae^a$ and $b = e^b b$. For each element c in S there exists an element $f^c \in S$ such that $c = f^c a = f^c ae^a = ce^a$. Therefore e^a is a right identity element of S. Similarly we obtain that e^b is a left identity element of S.

Therefore $e^a = e^b$ is an identity element of S.
The converse part is trivial.

THEOREM 4.5. *A semigroup S with zero is* (r)*-decomposable if and only if it is isomorphic to an* (R)*-regular Rees matrix semigroup over a semigroup with zero and identity element. The product of matrices is defined by a 2-dimensional sandwich matrix.*

From Theorem 4.5 we get O. Steinfeld's result stated in Theorem 2.1 as a special case.

In the next table we sum up the matrix characterizations of the semigroup classes which were defined in this paper and in those the references with the exception of some semigroup classes which can be characterized by means of Rees matrix semigroups over a semigroup with zero and identity defined in [3], [5]. The characterizations of Brandt semigroups, completely 0-simple semigroups are in [1]; of generalized Brandt semigroups and similarly decomposable semigroups are in [7]. In the paper [8] we can see that the matrix characterization of generalized Brandt semigroups can be further generalized to a certain class of lattice-ordered semigroups which plays an important role in the mathematical theory of codes and finite-state transducers.



NOTATIONS. $S=\bigcup_{i\in I} R_i=\bigcup_{\lambda\in\Lambda} L_\lambda$ is a semigroup having properties (2.1) and (2.3); H^e is a semigroup with zero and identity element; G^0 is a group with zero; P^4 [P] is a 4- [2-] dimensional sandwich matrix; $S\leftrightarrow M^0$ means that S is isomorphic to M^0; Δ is the identity matrix.

The arrows go from the general case to the special.

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FAMILIES OF FINITE SETS WITH PRESCRIBED CARDINALITIES FOR PAIRWISE INTERSECTIONS

By

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1. Introduction

In DEZA, ERDŐS, FRANKL [1] the following problem is considered.

Let n, k be integers, $n>k>0$, and let $L=\{l_1, \ldots, l_r\}$ be a set of integers $0 \leqq l_1 < l_2 < \ldots < l_r < k$.

Let X be a finite set of cardinality n and let $\mathcal{F}$ be a family of k-element subsets of X. We say that $\mathcal{F}$ is an (n, k, L)-system if for all $F, F' \in \mathcal{F}$, $F \neq F'$ we have

$$|F \cap F'| \in L. \tag{1}$$

In [1] it is proven that, denoting by $m(n, k, L)$ the maximum cardinality of an (n, k, L)-system, for $n>n_0(k)$ we have

$$m(n, k, L) \leqq \prod_{i=1}^{r} (n-l_i)/(k-l_i), \tag{2}$$

moreover

$$m(n, k, L) \leqq c(k) n^{r-1} \tag{3}$$

unless

$$(l_2-l_1)|(l_3-l_2)|\ldots|(l_r-l_{r-1})|(k-l_r). \tag{4}$$

However in many cases the estimations (2), (3) are very far from best possible (see [3]).

In this paper we are concerned with determining the correct order of magnitude of $m(n, k, L)$ for $k \leqq 7$. Actually we shall do it excepting the following cases: $k=7$, $L=\{0, 2, 3, 5\}$, $\{0, 2, 3, 5, 6\}$.

The proofs will heavily depend on the techniques developed in [1], [4], [3].

2. The methods and some reductions

Let $\mathcal{F}$ be an (n, k, L)-system and $n>n_0(k)$. Let us define (see [4]):

$$\mathcal{F}^* = \{G \subset X \mid \exists F_1, \ldots, F_{k^{|G|}} \in \mathcal{F}, F_i \cap F_j = G \quad \text{for} \quad 1 \leqq i < j \leqq k^{|G|}\},$$

$$\Delta(\mathcal{F}) = \{D \bar{\in} \mathcal{F}^* \mid \bar{\exists} G \in \mathcal{F}^*, G \subset D\}.$$

In [4] it is proved that $|\Delta(\mathcal{F})| < c(k)$, a constant depending only on k. As $\mathcal{F} \subset \mathcal{F}^*$ (for $F \in \mathcal{F}$, $G=F_1=\ldots=F_{k^{|G|}}=F$ satisfies the conditions) any member of $\mathcal{F}$ contains a member of $\Delta(\mathcal{F})$. Consequently for some $D \in \Delta(\mathcal{F})$

$$|\mathcal{F}' = \{F-D \mid F \in \mathcal{F}, D \subseteqq F\}| > \frac{1}{c(k)} |\mathcal{F}|. \tag{5}$$

This implies

PROPOSITION 1. *Let $l_1>0$. Then there exists a positive constant $c_1(k)$ depending only on k such that*

$$c_1(k)m(n,k,L) < m(n-l_1, k-l_1, \{0, l_2-l_1, \dots, l_r-l_1\}) \leqq m(n,k,L). \tag{6}$$

PROOF. The second part of (6) is trivial, since if we are given an $(n-l_1, k-l_1, \{0, l_2-l_1, \dots, l_r-l_1\})$-system $\mathcal{F}'$ then we can take an l_1-set A, disjoint to X, and define $\mathcal{F}=\{F\cup A \mid F\in\mathcal{F}'\}$. Then obviously $\mathcal{F}$ is an (n,k,L)-system, proving $m(n,k,L)\geqq m(n-l_1, k-l_1, \{0, l_2-l_1, \dots, l_r-l_1\})$.

To prove the first part let us choose $D\in\Delta(\mathcal{F})$ to satisfy (5) and let C be an l_1-subset of D. Let us set $\mathcal{F}'=\{F-C \mid C\subset F\in\mathcal{F}\}$.

Then $\mathcal{F}'$ is an $(n-l_1, k-l_1, \{0, l_2-l_1, \dots, l_r-l_1\})$-system of cardinality at least $\frac{1}{C(k)}|\mathcal{F}|$ and (6) follows.

Let us set $\mathcal{F}_0=\mathcal{F}$ and $\mathcal{C}_0=\emptyset$. Suppose $\mathcal{F}_i$ and $\mathcal{C}_i$ are defined yet, and let K be a set of maximal cardinality such that there exist $k+1$ different members $F_1, \dots, F_{k+1}$ of $\mathcal{F}_i$, satisfying

$$F_j\cap F_{j'} = K \quad \text{for} \quad 1\leqq j < j' \leqq k+1. \tag{7}$$

Let us set

$$\mathcal{F}_{i+1}=\mathcal{F}_i-\{F\in\mathcal{F}_i \mid F\supset K\}, \quad \mathcal{C}_{i+1}=\mathcal{C}_i\cup\{(K,F) \mid K\subset F\in\mathcal{F}_i\}.$$

Then obviously $|\mathcal{F}_{i+1}|+|\mathcal{F}_{i+1}|=|\mathcal{F}|$.

Let this procedure stop at the q-th step, i.e. in $\mathcal{F}_q$ we cannot find $K, F_1, \dots, F_{k+1}$ satisfying (7). Then in view of a result of ERDŐS, RADO [2]

$$|\mathcal{F}_q| \leqq k!\,k^k. \tag{8}$$

In [3] the following two propositions are proved.

PROPOSITION 2. *Let $(K_1, F_1), (K_2, F_2)\in\mathcal{C}_q$ then*

$$|K_1\cap K_2|\in L. \tag{9}$$

Moreover if $F_1-K_1=F_2-K_2$, then

$$|K_1\cap K_2|\in\{l_1-|F_1-K_1|, \dots, l_r-|F_1-K_1|\}. \tag{10}$$

PROPOSITION 3. *For $1\leqq i\leqq q$*

$$|\mathcal{F}_{i-1}-\mathcal{F}_i| \leqq k!\,k^k\cdot n. \tag{11}$$

We need the following

PROPOSITION 4. *Let us suppose that $\mathcal{F}$ does not contain a Δ-system of cardinality $k+1$ and with kernel of cardinality at least l_{s+1}, $s+1\leqq r$. Then we can find $\mathcal{F}'\subseteq\mathcal{F}$, such that $\mathcal{F}'$ is an $(n,k,\{l_1, \dots, l_s\})$-system and for some constant $c^*(k)$, depending only on k*

$$|\mathcal{F}'| \geqq c^*(k)\,|\mathcal{F}|. \tag{12}$$

PROOF. Let us set $\mathcal{G}_0=\mathcal{H}_0=\emptyset$. Suppose that $\mathcal{G}_i$ and $\mathcal{H}_i$ are already defined. Let $F\in(\mathcal{F}-(\mathcal{H}_i\cup\mathcal{G}_i))$, and let us set $\mathcal{G}_{i+1}=\mathcal{G}_i\cup\{F\}$.

Let us define $\mathcal{H}(F)=\{H\in(\mathcal{F}-(\mathcal{H}_i\cup\mathcal{G}_i))\,|\,|H\cap F|\geqq l_{s+1}\}$, and for $G\subset F$ $\mathcal{H}(F, G)=\{H\in\mathcal{H}(F)|H\cap F=G\}$.

We claim that $\mathcal{H}(F, G)$ does not contain a Δ-system of cardinality $k+1$, for any $G\subset F$. Indeed for $|G|<l_{s+1}$, $\mathcal{H}(F, G)=\emptyset$ and for $|G|\geqq l_{s+1}$ the kernel K of the Δ-system would contain G, contradicting $|K|<l_{s+1}$.

Now (8) implies $|\mathcal{H}(F, G)|\leqq k!\,k^k$. Consequently

$$\mathcal{H}(F)\leqq 2^k k!\,k^k. \tag{13}$$

Now let us define $\mathcal{H}_{i+1}=(\mathcal{H}_i\cup\mathcal{H}(F))-\{F\}$.

Continuing this way, after a finite number, say t, of steps we have $\mathcal{F}=\mathcal{G}_t\cup\mathcal{H}_t$. Then inequality (13) yields $|\mathcal{G}_t|\geqq(1/(2^k\cdot k!k^k))\,|\mathcal{F}|$.

As by the definitions for $G\neq G'\in\mathcal{G}_t$ $|G\cap G'|<l_{s+1}$ setting $\mathcal{F}'=\mathcal{G}_t$, $c^*(k)=$ $=1/(2^k\cdot k!k^k)$ the assertion of the proposition follows.

We need one more proposition.

PROPOSITION 5. *In proving* $m(n, k, L)\leqq c(k)n^q$ *we may suppose that for any* $Q\subset F\in\mathcal{F}$, $|Q|=q$ *there are at least* $M(k)$ *members of* $\mathcal{F}$ *containing* Q, *where* $M(k)$ *is any fixed number depending only on* k.

PROOF. Let us set $\mathcal{F}_0=\mathcal{F}$, $\mathcal{G}_0=\emptyset$. If $\mathcal{F}_i$ and $\mathcal{G}_i$ are defined then let $Q\subset F\in\mathcal{F}_i$, such that Q is contained in less than $M(k)$ members of $\mathcal{F}_i$. Let us define

$$\mathcal{F}_{i+1}=\mathcal{F}_i-\{F\in F_i|Q\subset F\},\quad \mathcal{G}_{i+1}=\mathcal{G}_i\cup\{Q\}.$$

Let this procedure stop after the p-th step. Then obviously $|\mathcal{G}_p|\leqq\binom{n}{q}$, $|\mathcal{F}-\mathcal{F}_p|\leqq$ $\leqq M(k)\binom{n}{q}\leqq\dfrac{M(k)}{q!}n^q$, and for $\forall Q\subset F\in\mathcal{F}_p$ there are at least $M(k)$ members of $\mathcal{F}_p$ containing Q.

Now if we prove $|\mathcal{F}_p|\leqq c'(k)n^q$, then it follows $|\mathcal{F}|\leqq\left(c'(k)+\dfrac{M(k)}{q!}\right)n^q=c(k)n^q$, as desired.

3. The upper bounds

In view of Proposition 1 we may always suppose $l_1=0$. In what follows $c(k)$ is a constant depending only on k. The following inequalities are consequences of (2) and (3).

$$m(n, k, \{0, 1, \dots, r-1\})\leqq c(k)n^r \tag{14}$$

for $k=1$, $r=1$; $k=2$, $r=1, 2, \dots$; $k=7$, $r=1, 2, \dots, 7$.

$$m(n, k, \{0, 1, \dots, r-1, s\}\leqq c(k)n^r \tag{15}$$

for $k=3$, $r=1$, $s=2$; $k=4$, $r=1, 2$, $s=3$; $k=5$, $r=1$, $s=2, 3, 4$; $k=5$, $r=2, 3$; $s=4$, $k=6$, $r=1$, $s=4, 5$; $k=6$, $r=2$, $s=3, 4, 5$; $k=6$, $r=3, 4$, $s=5$, $k=7$, $r=1$, $s=2, 3, 4, 5, 6$; $k=7$, $r=2$, $s=5, 6$; $k=7$, $r=3$, $s=4, 5, 6$; $k=7$, $r=4, 5$, $s=6$.

$$m(n, k, \{0, 2, \dots, 2(r-1)\})\leqq c(k)n^r \tag{16}$$

for $k=4, 6, r=2$; $k=6, r=3$.

(17) $$m(n, k, \{0, 2, s\}) \leqq c(k)n^2$$

for $k=4, s=3$; $k=6, s=5$.

(18) $$m(n, 6, \{0, 3\}) \leqq c(k)n^2.$$

(19) $$m(n, 6, \{0, 3, s\}) \leqq c(k)n^2$$

for $s=4, 5$.

(20) $$m(n, k, \{0, 2, 3, \dots, r\}) \leqq c(k)n^{r-1}$$

for $k=5, 6, 7, r=3, 4$; $k=6, 7, r=5$; $k=7, r=6$.

(21) $$m(n, k, \{0, 3, 4, \dots, r\} \leqq c(k)n^{r-2}$$

for $k=5, 7, r=4$; $k=6, 7, r=5$; $k=7, r=6$.

(22) $$m(n, k, \{0, 4, 5, \dots, r\} \leqq c(k)n^{r-3}$$

for $k=6, 7, r=5$; $k=7, r=6$.

(23) $$m(n, 7, \{0, 5, 6\}) \leqq c(k)n^2.$$

(24) $$m(n, 7, \{0, s, 5\}) \leqq c(k)n^2 \quad (s = 2, 3)$$

(25) $$m(n, 7, \{0, 1, 3\}) \leqq c(k)n^3.$$

(26) $$m(n, k, \{0, 1, 3, 4\}) \leqq c(k)n^3 \quad (k = 6, 7).$$

$$m(n, 6, \{0, 2, 4, 5\}) \leqq c(6)n^3.$$

These cover 86 of the 127 possible cases. For $k \leqq 4$ no cases remain. Let us now consider the cases $k=5$, $L=\{0, 2, 4\}$; $k=7, L \subseteqq \{0, 2, 4, 6\}$; $k=7, L=\{0, 3, 6\}$. In all of these cases we claim

(27) $$m(n, k, L) \leqq n.$$

To prove (27) let $X=\{x_1, \dots, x_n\}$ and let $e_1, e_2, \dots, e_n$ be an orthonormal basis for the n-dimensional vector space over $GF(2)$ in the first cases and $GF(3)$ in the last case.

To $F \in \mathcal{F}$ let us associate $v(F)=\sum_{x_i \in F} e_i$. We claim that the vectors $v(F)$, $F \in \mathcal{F}$ are linearly independent — which of course yields (27). Indeed $(v(F), v(F')) \neq 0$ iff $F=F'$, where (v_1, v_2) is the scalar product of v_1, v_2.

Now for $k=5$ two cases are left: $L=\{0, 1, 3\}$ and $L=\{0, 1, 3, 4\}$.

In these cases we prove

(28) $$m(n, k, L) \leqq c(k)n^2.$$

This was already proved in LARMAN [5], but we give a shorter argument.

Let us first consider the case $L=\{0, 1, 3\}$. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be one of the sets. In view of Proposition 5 we may suppose that there are 100 other sets containing $\{x_1, x_2\}$. By symmetry reasons we may assume then

$$\{x_1, x_2, x_3, x_6, x_7\} \in \mathcal{F}, \quad \{x_1, x_2, x_3, x_8, x_9\} \in \mathcal{F}, \quad \{x_1, x_2, x_3, x_{10}, x_{11}\} \in \mathcal{F}.$$

There are also at least 100 members of $\mathscr{F}$ containing $\{x_4, x_5\}$; by symmetry reasons we may assume that $\{x_1, x_4, x_5, x_{12}, x_{13}\}$, $\{x_1, x_4, x_5, x_{14}, x_{15}\}$, $\{x_1, x_4, x_5, x_{16}, x_{17}\}$ are 3 of them.

The set $\{x_2, x_4\}$ is contained as well in at least 100 members of $\mathscr{F}$. Let F be one of them which is different from $\{x_1, x_2, x_3, x_4, x_5\}$. As they intersect in 0, 1 or 3 elements there is exactly one of x_1, x_3 and x_5 which belongs to F.

If it is x_1 or x_3 then a similar argument yields $F\cap\{x_8, x_9\}\neq\emptyset$, $F\cap\{x_{10}, x_{11}\}\neq\emptyset$, $F\cap\{x_6, x_7\}\neq\emptyset$, yielding $|F|\geqq 6$, a contradiction.

If it is x_5 then we come to the same contradiction using $F\cap\{x_{12}, x_{13}\}\neq\neq\emptyset\neq F\cap\{x_{14}, x_{15}\}$, $F\cap\{x_{16}, x_{17}\}\neq\emptyset$. So the case $L=\{0, 1, 3\}$ is settled.

Essentially the same argument yields

$$m(n, 7, \{0, 1, 4\}) \leqq c(k)n^2. \tag{29}$$

Now let us consider the case $L=\{0, 1, 3, 4\}$. Let us define $s=\max\limits_{v\geqq 0} v$ such that $|F-K|=1$ for all $(K, F)\in\mathscr{C}_v$.

In view of Proposition 4 and the case $L=\{0, 1, 3\}$,

$$|\mathscr{F}_s| < c(k)n^2. \tag{30}$$

Let us choose a $y\in X$ such that $\{y\}=F-K$ holds for at least $\frac{|\mathscr{C}_s|}{n}$ times, and let $\mathscr{F}_y=\{K|F-K=\{y\},\ (K, F)\in\mathscr{C}_s\}$.

Then in view of Proposition 2 for $K\neq K'\in\mathscr{F}_y$ we have $|K\cap K'|\in\{0, 3\}$. Using (15) for this case $(k=4,\ r=1,\ s=3)$ we obtain

$$|\mathscr{C}_s| \leqq n|\mathscr{F}_y| \leqq c''(k)n^2. \tag{31}$$

As $|\mathscr{C}_s|+|\mathscr{F}_s|=|\mathscr{F}|$ we obtain from (30) and (31)

$$m(n, 5, \{0, 1, 3, 4\}) \leqq c(5)n^2. \tag{32}$$

Now let us consider the case $k=6$.

Let us first list the open cases in terms of the different L:

$$\{0, 1, 2, 4, 5\},\quad \{0, 1, 2, 4\},\quad \{0, 1, 3, 5\}$$
$$\{0, 1, 4, 5\},\quad \{0, 2, 3, 5\}$$
$$\{0, 1, 3, 4, 5\}.$$

In the cases of the first line let $x\in X$ be of maximal degree and let us set $\mathscr{F}_x=\{F-x|x\in F\in\mathscr{F}\}$. Then

$$|\mathscr{F}_x| \geqq (k/n)|\mathscr{F}|, \tag{33}$$

and $\mathscr{F}_x$ is an $(n-1, 5, \{l_2-1, \dots, l_r-1\})$-system.

Consequently in view of (28), (27), respectively (33) implies

$$m(n, 6, \{0, 1, 2, 4\}) \leqq c(6)n^3, \tag{34}$$

$$m(n, 6, \{0, 1, 2, 4, 5\}) \leqq c(6)n^3, \tag{35}$$

$$m(n, 6, \{0, 1, 3, 5\}) \leqq c(6)n^2. \tag{36}$$

For the cases of the second line let us define again

$$s = \max_{0 \leqq v} v\,|F-K| = 1 \quad \text{for} \quad (K, F) \in \mathscr{F}_v. \tag{37}$$

Using Proposition 4 we obtain

$$|\mathscr{F}_s| \leqq c''(6)\,n^2. \tag{38}$$

On the other hand for some $y \in X$ $F-K=\{y\}$ holds for at least $|\mathscr{C}_s|/n$ different $(K, F) \in \mathscr{C}_s$. Setting $\mathscr{F}_y = \{K \,|\, (K, F) \in \mathscr{C}_s,\ F-K=\{y\}\}$ Proposition 2 implies that $\mathscr{F}_y$ is an $(n, 5, \{0, 4\}$, $(n, 5, \{2\})$-system, respectively. Hence in both of the cases we obtain

$$|\mathscr{C}_s| \leqq c'(6)\,n^2. \tag{39}$$

Combining (38) and (39) we get

$$m(n, 6, \{0, 2, 3, 5\}) \leqq c(6)\,n^2, \tag{40}$$

$$m(n, 6, \{0, 1, 4, 5\}) \leqq c(6)\,n^2. \tag{41}$$

In the case $L = \{0, 1, 3, 4, 5\}$ we can proceed in the following way.

Let us define

$$\mathscr{D} = \{K \,|\, JF \in \mathscr{F},\ (K, F) \in \mathscr{C}_q\}. \tag{42}$$

In view of Proposition 3 we have

$$q = |\mathscr{D}| \geqq c_1(k)\,|\mathscr{F}|/n, \tag{43}$$

consequently for some $l \in L$

$$\mathscr{D}_l = \{D \in \mathscr{D} \,|\, |D| = l\}| \geqq c_2(k) \cdot |\mathscr{F}|/n. \tag{44}$$

But $\mathscr{D}_l$ is an $(n, l, \{l' \in L \,|\, l' < l\})$-system.

So in our case we obtain using (44)

$$m(n, 6, \{0, 1, 3, 4, 5\}) \leqq c(6)\,n^3. \tag{45}$$

Now we turn to the case $k=7$. The first case we consider is $L = \{0, 1, 2, 4, 5, 6\}$. Let us define s as in (37). Then we obtain

$$|\mathscr{F}_s| \leqq c'(7)\,m(n, 7, \{0, 1, 2, 4, 5\}). \tag{46}$$

Choosing $y \in X$ such that $F-K=\{y\}$ holds for at least $|\mathscr{C}_s|/n$ $(K, F) \in \mathscr{C}_s$, and setting $\mathscr{F}_y = \{K \,|\, (K, F) \in \mathscr{C}_s,\ F-K=\{y\}\}$ in view of Proposition 2 we obtain

$$|\mathscr{C}_s| \leqq c^*(7)\,n \cdot m(n, 6, \{0, 1, 4, 5\}) \leqq c^{**}(7)\,n^3. \tag{47}$$

Now

$$m(n, 7, \{0, 1, 2, 4, 5, 6\}) \leqq c(7)\,n^3 \tag{48}$$

will follow from

$$m(n, 7, \{0, 1, 2, 4, 5\}) \leqq c_1(7)\,n^3. \tag{49}$$

To prove (49) let us define

$$t = \max_{0 \leqq v} v \big| |F-K| = 2 \quad \text{for} \quad (F, K) \in \mathscr{C}_v.$$

Then Proposition 4 implies

$$|\mathscr{F}_t| \leqq c_2(7)\, m(n, 7, \{0, 1, 2, 4\}) \leqq c_3(7)\, n^3. \tag{50}$$

On the other hand for $\mathscr{C}_t$ let us choose $Z \subset X$, $|Z| = 2$ such that

$$|\mathscr{F}_z = \{K | (K, F) \in \mathscr{C}_t,\ F-K = Z\}|$$

is maximal. Then of course $|\mathscr{F}_z| \geqq |\mathscr{C}_t| / \binom{n}{2}$, and by Proposition 2 $\mathscr{F}_z$ is an $(n, 5, \{0, 2\})$-system. Using (27) we deduce

$$|\mathscr{C}_t| \leqq \binom{n}{2} |\mathscr{F}_z| \leqq \binom{n}{2} n. \tag{51}$$

Combining (50) and (51) we obtain (49) and (48).

The inequality (48) in turn implies

$$m(n, 7, L) \leqq c(7)\, n^3 \tag{52}$$

for ever $L \subseteqq \{0, 1, 2, 4, 5, 6\}$, $\{0, 1, 2\} \subseteqq L$ and for $L = \{0, 1, 4, 5, 6\}$, $\{0, 2, 4, 5, 6\}$.

Let us list the remaining cases.

$\{0, 1, 2, 3, 5, 6\}$, $\{0, 1, 2, 3, 5\}$, $\{0, 1, 3, 6\}$, $\{0, 1, 3, 4, 6\}$

$\{0, 1, 3, 4, 5, 6\}$, $\{0, 1, 3, 4, 5\}$, $\{0, 1, 3, 5\}$, $\{0, 1, 3, 5, 6\}$,

$\{0, 1, 4, 6\}$, $\{0, 1, 5, 6\}$, $\{0, 2, 3, 4, 6\}$, $\{0, 2, 3, 6\}$, $\{0, 2, 5, 6\}$, $\{0, 3, 4, 6\}$, $\{0, 3, 5, 6\}$

$\{0, 1, 4, 5\}$, $\{0, 2, 4, 5\}$,

$\{0, 2, 3, 5\}$, $\{0, 2, 3, 5, 6\}$.

For the cases listed in the first line let $x \in X$ be of maximal degree, and

$$\mathscr{F}_x = \{F - \{x\} | F \in \mathscr{F},\ x \in F\}.$$

Then $\mathscr{F}_x$ is an $(n-1, 6, \{l_2 - 1, \dots, l_r - 1\})$-system of cardinality at least $(k/n)|\mathscr{F}|$. Hence we obtain

$$m(n, 7, \{0, 1, 2, 3, 5, 6\}) \leqq \frac{n}{k}\, m(n, 6, \{0, 1, 2, 4, 5\}) \leqq c(7)\, n^4, \tag{53}$$

$$m(n, 7, \{0, 1, 2, 3, 5\}) \leqq m(n, 7, \{0, 1, 2, 3, 5, 6\}) \leqq c(7)\, n^4, \tag{54}$$

$$m(n, 7, \{0, 1, 3, 4, 6\}) \leqq \frac{n}{k}\, m(n, 6, \{0, 2, 3, 5\}) \leqq c(7)\, n^3 \tag{55}$$

$$m(n, 7, \{0, 1, 3, 6\}) \leqq m(n, 7, \{0, 1, 3, 4, 6\}) \leqq c(7)\, n^3. \tag{56}$$

For the cases of the second line let us define $\mathscr{D}, \mathscr{D}_l$ as in (42), (43) respectively. Then using (43) and considering separately the possibilities for the value of l we obtain:

$$m(n, 7, \{0, 1, 3, 4, 5, 6\}) \leqq c(7)n^4, \tag{57}$$

$$m(n, 7, \{0, 1, 3, 4, 5\}) \leqq c(7)n^3, \tag{58}$$

$$m(n, 7, \{0, 1, 3, 5\}) \leqq c(7)n^3, \tag{59}$$

$$m(n, 7, \{0, 1, 3, 5, 6\}) \leqq c(7)n^3. \tag{60}$$

For the cases listed in the third and fourth line we define s as in (37) and let $y \in X$ be such that $F-K=\{y\}$ holds for at least $|\mathscr{C}_s|/n$ members (K, F) of $\mathscr{C}_s$. Set

$$\mathscr{F}_y = \{K|(K, F) \in \mathscr{C}_s,\ F-K = \{y\}\}. \tag{61}$$

Then Proposition 4 yields

$$|\mathscr{F}_s| \leqq c'(7)\, m(n, 7, L-\{6\}). \tag{62}$$

On the other hand Proposition 2 yields

$$|\mathscr{C}_s| \leqq nm(n, 6, L \cap \{l-1|l \in L\}). \tag{63}$$

Combining (62), (63) and $|\mathscr{F}|=|\mathscr{F}_s|+|\mathscr{C}_s|$ we deduce

$$m(n, 7, L) \leqq c(7)n^2 \tag{64}$$

for $L=\{0, 1, 4, 6\}$, $\{0, 1, 5, 6\}$, $\{0, 2, 3, 6\}$, $\{0, 2, 5, 6\}$, $\{0, 3, 4, 6\}$, $\{0, 3, 5, 6\}$;

$$m(n, 7, L) \leqq c(7)n^3 \tag{65}$$

for $L=\{0, 2, 3, 4, 6\}$.

Now let $L=\{0, 1, 4, 5\}$ and let us define

$$t = \max_{v \geqq 0} v \,\big|\, |F-K| = 2, \quad (K, F) \in \mathscr{C}_v. \tag{66}$$

We assert

$$|\mathscr{C}_t| \leqq \binom{n}{2}. \tag{67}$$

Otherwise we can find (K_1, F_1), $(K_2, F_2) \in \mathscr{C}_t$ such that $F_1-K_1=F_2-K_2$. But in this case Proposition 2 implies $|K_1 \cap K_2| \in (L \cap \{l-2|l \in L\}) = \emptyset$, a contradiction.

For $\mathscr{F}_t$ Proposition 4 yields using (29)

$$|\mathscr{F}_t| \leqq c'(7)\, m(n, 7, \{0, 1, 4\}) \leqq c^*(7)n^2. \tag{68}$$

(68) and (67) yield together

$$m(n, 7, \{0, 1, 4, 5\}) \leqq c(7)n^2. \tag{69}$$

Now let $L=\{0, 2, 4, 5\}$. Let us define t by (66).

Then Proposition 4 implies

$$|\mathscr{F}_t| \leqq c'(7)n. \tag{70}$$

Let $\mathcal{D}=\{K_1, \ldots, K_t\}$ be the different 5-element sets which occur in some $(K, F)\in\mathcal{C}_t$. By Propositions 2 and 3 and inequality (27)

$$|\mathcal{C}_t| \leqq c^*(7)n\cdot t \leqq c^*(7)n^2. \tag{71}$$

From (70) and (71) we deduce

$$m(n, 7, \{0, 2, 4, 5\}) \leqq c(7)n^2. \tag{72}$$

Let now $L=\{0, 2, 3, 5, 6\}$. Defining s as in (37), we obtain as above ($|\mathcal{F}_8 \leqq \leqq c'm(n, 7, \{0, 2, 3, 5,\}$, $\mathcal{F}_y$ is an $(n, 6, \{2, 5\})$-system, consequently $|\mathcal{F}_y| \leqq c^*n$ and $|\mathcal{C}_s| \leqq c^*n^2$)

$$m(n, 7, \{0, 2, 3, 5, 6\}) \leqq c'(7)m(n, 7, \{0, 2, 3, 5\})+c^*(7)n^2. \tag{73}$$

For $L=\{0, 2, 3, 5\}$ (3) yields

$$m(n, 7, \{0, 2, 3, 5\}) \leqq c(7)n^3. \tag{74}$$

However we think that n^3 can be replaced by n^2 in (74) and consequently in (73).

4. The constructions

Let A be an arithmetic progression (i.e. for some integers $a, d, f \geqq 0, d>0$, $A=\{a+id \mid 0\leqq i \leqq f\}$) of maximal length (i.e. $f+1$ is maximal) such that $A\subseteqq L$, $d|(k-a)$. We define $a(L)=f+1$.

Let p be the greatest prime not exceeding $\dfrac{n-a}{k-a}$. Then we may choose in X pairwise disjoint sets $X_0, X_1, \ldots, X_{\bar{k}}$ where $|X_0|=a$, $|x_i|=dp$ for $1\leqq i\leqq \bar{k}$, and $\bar{k}=(k-a)/d$. Let $X_i=\bigcup_{j=0}^{p-1} Y_i^j$ where the Y_i^j's are pairwise disjoint, $|Y_i^j|=d$ for $j=0, \ldots, p-1$.

Let V be the $(f+1)$-dimensional linear space over $GF(p)$. Let $g_1, g_2, \ldots, g_{\bar{k}}$ be $\bar{k}$ elements of V^*, the dual space of V, in general position i.e. the g_i's are linear functions $V\to GF(p)$, such that any $f+1$ of them are linearly independent.

Let us define $\mathcal{F}$ by

$$\mathcal{F} = \left\{F \mid \exists v\in V, F = X_0 \cup \left(\bigcup_{i=1}^{\bar{k}} Y_i^{g_j(v)}\right)\right\}. \tag{75}$$

From the definition (75) it is obvious that for $F, F'\in\mathcal{F}$ $|F\cap F'|=a+bd$ for some $b\geqq 0$. Let us suppose $b\geqq f+1$ i.e. there are at least $f+1$ values of j, say $j_1, \ldots, j_{f+1}$ such that $g_{j_t}(v)=g_{j_t}(v')$ for $t=1, \ldots, f+1$. But the functions $\{g_{j_t} \mid 1\leqq t\leqq f+1\}$ are linearly independent, and consequently they generate V^*. It yields $g(v)=g(v')$ for any $g\in V^*$ i.e. $v=v'$. Thus we proved $\mathcal{F}$ to be an (n, k, L)-system:

$$m(n, k, L) \geqq p^{f+1} = p^{a(L)} > c'(k)n^{a(L)}. \tag{76}$$

The inequality (76) shows in 119 of the considered 127 cases that the order of magnitude of the upper bounds is best possible.

For $k=7$, $L=\{0, 2, 3, 5\}$ or $\{0, 2, 3, 5, 6\}$ it gives

$$m(n, k, L) \geqq c'(k)n^2, \tag{77}$$

which we think is best possible apart from the value of $c'(k)$.

Suppose we are given an (n_i, k_i, L_i)-system $\mathcal{F}_i$ on the set X_i, $i=1, \dots, t$. Suppose further the X_i-s are mutually disjoint. Let us define $\mathcal{F} = \{\bigcup_{i=1}^{t} F_i\} F_i \in \mathcal{F}_i$ for $i=1, \dots, t\}$.

Then $\mathcal{F}$ is a family of $\sum_{i=1}^{t} k_i$-element subsets of a set of $\sum_{i=1}^{t} n_i$ elements, and for $F, F' \in \mathcal{F}$ we have $|F \cap F'| = \sum_{i=1}^{t} |F_i \cap F_i'| = \sum_{i=1}^{t} l_i$ where $l_i \in (L_i \cup \{k_i\})$ for $i=1, \dots, t$.

This proves

$$(78) \qquad m\left(\sum_{i=1}^{t} n_i,\ \sum_{i=1}^{t} k_i,\ \sum_{i=1}^{t}\left(L_i \cup \{k_i\} - \left\{\sum_{i=1}^{t} k_i\right\}\right)\right) \geqq \prod_{i=1}^{t} m(n_i, k_i, L_i),$$

where $\sum_{i=1}^{t} B_i = \left\{\sum_{i=1}^{t} b_i | b_i \in B_i \text{ for } i = 1, \dots, t\right\}$.

Using (78) and (76) we obtain

$$(79) \qquad m(n, 7, \{0, 2, 5\}) \geqq m(n/2, 5, \{0\})\, m(n/2, 2, \{0\}) \geqq c'(7)\, n^2,$$

$$(80) \qquad m(n, 6, \{0, 1, 3, 4\}) \geqq m\left(\frac{n}{2}, 3, \{0, 1\}\right) m\left(\frac{n}{2}, 3, \{0\}\right) \geqq c'(6)\, n^3.$$

The last cases are $k=7$, $L=\{0, 1, 3\} \cup B$ where $B \subseteqq \{4, 6\}$. Let p be the greatest integer such that $2^p \leqq n+1$, and let $Y \subset X$, $|Y| = 2^p - 1$.

Let us consider the p-dimensional projective space on Y. Let $\mathcal{F}$ consist of the 2-dimensional subspaces.

Then $\mathcal{F}$ is a $(2^p-1, 7, \{0, 1, 3\})$-system of cardinaity $\frac{2^p-1}{7} \frac{2^p-2}{6} \frac{2^p-4}{4}$, proving

$$(81) \qquad m(n, 7, \{0, 1, 3\} \cup B) \geqq c'(7)\, n^3,$$

for every $B \subseteqq \{4, 6\}$.

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EXPONENTIAL ESTIMATES FOR THE MAXIMUM OF PARTIAL SUMS. II (RANDOM FIELDS)

By

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§ 1. Introduction and preliminaries

Let $\{\zeta_{kl}\}$ $(k, l=1, 2, \ldots)$ be a doubly infinite random field. The rv's ζ_{kl} need not be independent or identically distributed. Set

$$S(b, c;\ m, n) = \sum_{k=b+1}^{b+m} \sum_{l=c+1}^{c+n} \zeta_{kl}$$

and

$$M(b, c;\ m, n) = \max_{1\leqq p\leqq m} \max_{1\leqq q\leqq n} |S(b, c;\ p, q)|,$$

where $b, c\geqq 0$ and $m, n\geqq 1$ are integers. It is of practical use to define

$$S(b, c; m, n) = M(b, c; m, n)=0 \quad \text{if} \quad m=0 \quad \text{or} \quad n=0.$$

In the following, $f(b, c; m, n)$ will denote a non-negative function depending on the joint df of $\{\zeta_{kl}: k=b+1, \ldots, b+m;\ l=c+1, \ldots, c+n\}$, and possessing the following two properties of a rather general nature:

$$(1) \qquad f(b,c;\ h, n)+f(b+h, c;\ m-h, n) \leqq f(b, c;\ m, n)$$

and

$$(2) \qquad f(b, c;\ m, i)+f(b, c+i;\ m, n-i) \leqq f(b, c;\ m, n)$$

for all $b\geqq 0, 1\leqq h<m$ and for all $c\geqq 0, 1\leqq i<n$. In other words, (1) means that $f(b, c; m, n)$ as a function of the interval $[b+1, b+m]$ for any fixed c and n is "superadditive", and (2) expresses the "superadditivity" property of $f(b, c; m, n)$ as a function of the interval $[c+1, c+n]$ for any fixed b and m. Examples are $f(b, c; m, n)=m^{\alpha_1}n^{\alpha_2}$ with $\alpha_1, \alpha_2\geqq 1$ or $f(b, c; m, n)=\sum_{k=b+1}^{b+m} \sum_{l=c+1}^{c+n} \sigma_{kl}^2$, in the latter case assuming the existence of the finite variances σ_{kl}^2 of the rv's ζ_{kl}.

The subject of this paper is to provide bounds for the distribution $P\{M(b, c; m, n)\geqq\lambda\}$ in terms of given bounds for $P\{|S(b, c; m, n)|\geqq\lambda\}$, where $\lambda>0$. More precisely, our permanent assumption reads as follows

$$(3) \qquad P\{|S(b, c;\ m, n)| \geqq \lambda\} \leqq C\exp\left(-\frac{\lambda^2}{f(b, c;\ m, n)}\right),$$

where $\lambda\in(0, \Lambda)$, $0<\Lambda\leqq\infty$, C is a constant, $f(b, c; m, n)$ satisfies (1) and (2). Throughout the paper Λ denotes a fixed positive number or ∞, and $C_1, C_2, \ldots$ denote positive constants.

We note that if $f(b, c; m, n)=0$ for certain $b, c \geqq 0$ and $m, n \geqq 1$, then we take the right-hand side of (3) to be equal to 0 for all $\lambda>0$. In this case $P\{|S(b, c; m, n)|=0\}=1$, i.e. $S(b, c; m, n)=0$ a.s.

Let us introduce the following "partial" maxima, i.e. the maxima of partial sums taken with respect to only p or q:

$$M_1(b, c; m, n) = \max_{1 \leqq p \leqq m} |S(b, c; p, n)|$$

and

$$M_2(b, c; m, n) = \max_{1 \leqq q \leqq n} |S(b, c; m, q)|,$$

where $b, c \geqq 0$ and $m, n \geqq 1$ are integers.

The following auxiliary result will be useful in the sequel.

LEMMA 1. *Suppose that there exists a non-negative function $f(b, c; m, n)$ satisfying* (3) *for all $\lambda \in (0, \Lambda)$, $b, c \geqq 0$ and $m, n \geqq 1$. If* (2) *holds, then there exist positive constants $C_1 \geqq C$ and $C_2 < 1$ such that the inequality*

$$P\{M_2(b, c; m, n) \geqq \lambda\} \leqq C_1 \exp\left(-\frac{C_2 \lambda^2}{f(b, c; m, n)}\right) \tag{4}$$

holds for all $\lambda \in (0, \Lambda)$, $b, c \geqq 0$ and $m, n \geqq 1$. The constants C_1 and C_2 may be chosen as follows:

(i) $C_1 = \max(81, C)$ *and* $C_2 = 1/5$, *or*

(ii) C_2 *can be made as close to* 1 *as required* (*C_1 approaches ∞ as C_2 approaches* 1).

An analogous result is true for $M_1(b, c; m, n)$ under the assumptions (1) and (3).

This lemma can be obtained by a simultaneous application to all possible fixed values of $b \geqq 0$ and $m \geqq 1$ of recent results [3, Theorems 1 and 1*] in the case when $\xi_l = \sum_{k=b+1}^{b+m} \zeta_{kl}$, $g(c, n) = f(b, c; m, n)$, and $M(c, n) = M_2(b, c; m, n)$, where $l = c+1, \ldots$ $\ldots, c+n$ (the notations are the same as in the cited paper).

§ 2. The main result

THEOREM 1. *Suppose that there exists a non-negative function $f(b, c; m, n)$ satisfying* (1)—(2) *such that* (3) *holds for all $\lambda \in (0, \Lambda)$, $b, c \geqq 0$ and $m, n \geqq 1$. Then, for any $0 < \varepsilon < 1$, there exists a constant $C_3 = C_3(\varepsilon)$ such that the inequality*

$$P\{M(b, c; m, n) \geqq \lambda\} \leqq C_3 \exp\left(-\frac{(1-\varepsilon)\lambda^2}{f(b, c; m, n)}\right)$$

holds for all $\lambda \in (0, \Lambda)$, $b, c \geqq 0$ and $m, n \geqq 1$.

PROOF. The proof will be done in a similar way as that of [3, Theorem 1]. We are going to define two positive constants C_3 and C_4, the latter as close to 1 as we wish, such that the inequality

$$P\{M(b, c; k, n) \geqq \lambda\} \leqq C_3 \exp\left(-\frac{C_4 \lambda^2}{f(b, c; k, n)}\right) \tag{5}$$

holds for all $\lambda \in (0, \Lambda)$, $b, c \geqq 0$ and $k, n \geqq 1$.

The proof goes by induction on k. If $k=1$ and $n\geqq 1$ is an arbitrary integer, then (5) is an obvious consequence of Lemma 1, provided $C_3\geqq C_1$ and $0<C_4\leqq C_2$. In fact, we have $M(b, c; 1, n)=M_2(b, c; 1, n)$ for all $b, c\geqq 0$ and $n\geqq 1$.

Assume now, as induction hypothesis, that (5) holds for all $k<m$ (and for all $b, c\geqq 0, n\geqq 1$) and prove it for $k=m$ (and for all $b, c\geqq 0, n\geqq 1$).

If for certain $b, c\geqq 0$ and $m, n\geqq 1$, we have

$$f(b, c; m, n) = 0, \tag{6}$$

then by (1)—(2) we also have $f(b, c; k, l)=0$, and hence $S(b, c; k, l)=0$ a.s. for $k=1, 2, \ldots, m$; $l=1, 2, \ldots, n$. Thus $M(b, c; m, n)=0$ a.s., and (5) is clearly satisfied for all $\lambda>0$.

From now on we assume that $f(b, c; m, n)\neq 0$. Let α be a real number, determined later. For the moment we only assume $1/2<\alpha<1$. Since $f(b, c; m, n)$ is a non-decreasing function in m for any fixed $b, c\geqq 0$ and $n\geqq 1$, there exists an integer $h=h(\alpha)$, $1\leqq h\leqq m$, such that

$$f(b, c; h-1, n) \leqq \alpha f(b, c; m, n) < f(b, c; h, n), \tag{7}$$

where $f(b, c; h-1, n)$ on the left is 0, if $h=1$. Then (1) implies

$$f(b+h, c; m-h, n) \leqq f(b, c; m, n)-f(b, c; h, n) < (1-\alpha)f(b, c; m, n). \tag{8}$$

Since, for $1\leqq p<h$ and $1\leqq q\leqq n$, we have

$$|S(b, c; p, q)| \leqq M(b, c; h-1, n),$$

and, for $h\leqq p\leqq m$ and $1\leqq q\leqq n$,

$$|S(b, c; p, q)| \leqq M_2(b, c; h, n)+M(b+h, c; m-h, n),$$

hence, for all $\lambda>0$, we have

$$P\{M(b, c; m, n) \geqq \lambda\} \leqq P\{M(b, c; h-1, n) \geqq \lambda\}+$$
$$+P\{M_2(b, c; h, n)+M(b+h, c; m-h, n) \geqq \lambda\}.$$

Let λ_1 and λ_2 be positive numbers, $\lambda_1+\lambda_2=\lambda$. Then the last inequality implies

$$P\{M(b, c; m, n) \geqq \lambda\} \leqq P\{M(b, c; h-1, n) \geqq \lambda\}+ \tag{9}$$
$$+P\{M_2(b, c; h, n) \geqq \lambda_1\}+P\{M(b+h, c; m-h, n) \geqq \lambda_2\}.$$

Applying the induction hypothesis to $M(b, c; h-1, n)$, we get, for all $\lambda\in(0, \Lambda)$,

$$P\{M(b, c; h-1, n) \geqq \lambda\} \leqq C_3 \exp\left(-\frac{C_4\lambda^2}{f(b, c; h-1, n)}\right) \leqq \tag{10}$$

$$\leqq C_3 \exp\left(-\frac{C_4\lambda^2}{\alpha f(b, c; m, n)}\right),$$

the right-most inequality following from (7). Applying again the induction hypothesis now to $M(b+h, c; m-h, n)$ and using (8), we find, for all $\lambda_2\in(0, \Lambda)$,

$$P\{M(b+h, c; m-h, n) \geqq \lambda_2\} \leqq C_3 \exp\left(-\frac{C_4\lambda_2^2}{(1-\alpha)f(b, c; m, n)}\right). \tag{11}$$

Finally, by (4)

$$(12)\qquad P\{M_2(b, c; h, n) \geqq \lambda_1\} \leqq C_1 \exp\left(-\frac{C_2\lambda_1^2}{f(b, c; h, n)}\right) \leqq$$

$$\leqq C_1 \exp\left(-\frac{C_2\lambda_1^2}{f(b, c; m, n)}\right)$$

for all $\lambda_1 \in (0, \Lambda)$. Combining inequalities (10)—(12) with (9), we obtain that

$$(13)\qquad P\{M(b, c; m, n) \geqq \lambda\} \leqq C_3 \exp\left(-\frac{C_4\lambda^2}{\alpha f(b, c; m, n)}\right)+$$

$$+C_1 \exp\left(-\frac{C_2\lambda_1^2}{f(b, c; m, n)}\right)+C_3 \exp\left(-\frac{C_4\lambda_2^2}{(1-\alpha) f(b, c; m, n)}\right)$$

for all $\lambda \in (0, \Lambda)$ and for all positive λ_1 and λ_2, $\lambda_1+\lambda_2=\lambda$.

For a fixed $\alpha \in (1/2, 1)$, let

$$(14)\qquad \lambda_2^2 = \frac{1-\alpha}{\alpha}\lambda^2,$$

then let $\lambda_1 = \lambda - \lambda_2$, and finally, let

$$(15)\qquad C_4 = \frac{\alpha C_2 \lambda_1^2}{\lambda^2}.$$

It is clear that $0<\lambda_2<\lambda$ and $0<C_4<C_2$. Furthermore, it is not hard to check that if $\alpha \to 1-0$, then $\lambda_2 \to +0$, $\lambda_1 \to \lambda - 0$, and $C_4 \to C_2 - 0$. Since, by Lemma 1, C_2 may be taken as close to 1 as needed, hence C_4 can approach 1 with any prescribed accuracy $\varepsilon>0$. To sum up, we may and do fix the values of C_2 and α so that we have $C_4 \geqq 1-\varepsilon$.

Under the conditions (14)—(15), the three exponents on the right-hand side of (13) coincide. Thus we may write

$$(16)\qquad P\{M(b, c; m, n) \geqq \lambda\} \leqq 3C_3 \exp\left(-\frac{C_4\lambda^2}{\alpha f(b, c; m, n)}\right).$$

If

$$(17)\qquad f(b, c; m, n) \leqq \frac{(1-\alpha) C_4 \lambda^2}{\alpha \ln 3},$$

the right-hand side of (16) does not exceed $C_3 \exp\left(-\frac{C_4\lambda^2}{f(b, c; m, n)}\right)$, and this proves (5) in this case.

On the other hand, if (17) is not satisfied, i.e. if

$$(18)\qquad f(b, c; m, n) > \frac{(1-\alpha) C_4 \lambda^2}{\alpha \ln 3},$$

then by choosing

$$(19)\qquad C_3 = \max(C_2, 3^{\alpha/(1-\alpha)}),$$

the right-hand side of (5) will be greater than 1. Therefore, (5) will automatically be satisfied.

The cases (6), (17), and (18) together cover all possible values of $f(b, c; m, n)$. This completes the induction step and the proof of Theorem 1.

We are going to make two supplements to Theorem 1. Taking into account that in Lemma 1 we can choose the constants C_1 and C_2 as $C_1=\max(81, C)$ and $C_2=1/5$, putting $\alpha=4/5$ in the above proof of Theorem 1, (14), (15), and (19) will provide $C_3=\max(81, C)$ and $C_4=1/25$. The result obtained by this particular choice of C_3 and C_4 is weaker than the result stated in Theorem 1, but it is satisfactory in most applications.

THEOREM 1*. *Suppose that there exists a non-negative function* $f(b, c; m, n)$ *satisfying* (1)—(2) *such that* (3) *holds for all* $\lambda\in(0, \Lambda)$, $b, c\geqq 0$ *and* $m, n\geqq 1$. *Then*

$$P\{M(b, c; m, n)\geqq\lambda\}\leqq\max(81, C)\exp\left(-\frac{\lambda^2}{25f(b, c; m, n)}\right).$$

As in [3, Theorem 2] we can substitute λ^2 by a rather general function $\varphi(\lambda)$ in the exponents of the above inequalities.

THEOREM 2. *Suppose that there exist a function* $\varphi(\lambda)$ *defined on* $[0, \Lambda)$, *subject to the following three conditions:*

(i) $\varphi(0)=0$,

(ii) $\varphi(\lambda)$ *is strictly increasing on* $[0, \Lambda)$, *and*

(iii) $\varphi(\lambda)$ *is continuous on* $[0, \Lambda)$;

and a non-negative function $f(b, c; m, n)$ *satisfying* (1)—(2) *such that*

$$P\{|S(b, c; m, n)|\geqq\lambda\}\leqq C\exp\left(-\frac{\varphi(\lambda)}{f(b, c; m, n)}\right)$$

holds for all $\lambda\in(0, \Lambda)$, $b, c\geqq 0$ *and* $m, n\geqq 1$. *Then*

$$P\{M(b, c; m, n)\geqq\lambda\}\leqq C_3\exp\left(-\frac{C_4\varphi(\lambda)}{f(b, c; m, n)}\right),$$

and even C_4 *can be made (by increasing* C_3*) as close to* 1 *as we wish.*

§ 3. A two-parameter version of the LIL for multiplicative random fields

We say the random field $\{\zeta_{kl}\}$ to be *multiplicative* if for all positive integers r, and for all systems of mutually distinct pairs $(k_1, l_1), (k_2, l_2), \ldots, (k_r, l_r)$ with positive integers for k_j and l_j, we have

$$E\left\{\prod_{j=1}^{r}\zeta_{k_jl_j}\right\}=0;$$

two pairs (k_1, l_1) and (k_2, l_2) are distinct if $k_1\neq k_2$ or $l_1\neq l_2$. In particular, $E\zeta_{kl}=0$ $(k, l=1, 2, \ldots)$. This definition for sequences of rv's was introduced by ALEXITS [1, p. 186].

The following inequality constitutes the basis of obtaining an exponential bound for the tail distribution $P\{|S(b, c; m, n)|\geqq\lambda\}$.

LEMMA 2. *Let $\{\zeta_{kl}\}$ be multiplicative and uniformly bounded,*

$$|\zeta_{kl}| \leqq B \quad a.s. \quad (k, l = 1, 2, \ldots).$$

Then

$$E\{\exp(uS(b, c; m, n))\} \leqq \exp\left(\frac{1}{2} mnu^2B^2\right) \tag{20}$$

for all $u>0$, $b, c\geqq 0$ and $m, n\geqq 1$.

Putting here $u=\lambda/(mnB^2)$, by the Chebyshev inequality we obtain that

$$P\{|S(b, c; m, n)| \geqq \lambda\} \leqq 2\exp\left(-\frac{\lambda^2}{2mnB^2}\right) \tag{21}$$

for all $\lambda>0$, $b, c\geqq 0$ and $m, n\geqq 1$. Hence Theorem 1* implies that

$$P\{M(b, c; m, n) \geqq \lambda\} \leqq 81\exp\left(-\frac{\lambda^2}{50mnB^2}\right) \tag{22}$$

also for all $\lambda>0$, $b, c\geqq 0$ and $m, n\geqq 1$.

PROOF. We use an argument due to AZUMA [2]. Applying the elementary inequality

$$\exp(at) \leqq \exp\left(\frac{1}{2}a^2\right)\left\{1+\frac{at}{|a|}\tanh(|a|)\right\},$$

which is valid for every $a\neq 0$ and $|t|\leqq 1$, we find that

$$E\{\exp(uS(b, c; m, n))\} = E\left\{\prod_{k=b+1}^{b+m}\prod_{l=c+1}^{c+n}\exp\left(uB\frac{\zeta_{kl}}{B}\right)\right\} \leqq$$

$$\leqq \exp\left(\frac{1}{2}mnu^2B^2\right) E\left\{\prod_{k=b+1}^{b+m}\prod_{l=c+1}^{c+n}\left(1+\frac{\tanh(uB)}{B}\zeta_{kl}\right)\right\}.$$

Since the last expected value is equal to 1 owing to multiplicativity, the proof is complete.

In the following, we use the abbreviated notation

$$S(m, n) = S(0, 0; m, n) = \sum_{k=1}^{m}\sum_{l=1}^{n}\zeta_{kl},$$

by $\limsup_{m,n\to\infty} h(m, n)$ we mean

$$\lim_{\max(m,n)\to\infty} \sup\{h(s, t)\colon s > m \text{ or } t > n\},$$

and the phrase that a statement holds "if s or t is large enough" will mean that the statement in question holds for all s and t, except for a finite number of pairs (s, t).

After these preliminaries we can state a two-parameter version of the "$\leqq$" part of the LIL for multiplicative $\{\zeta_{kl}\}$ as follows.

THEOREM 3. *Let* $\{\zeta_{kl}\}$ *be multiplicative and uniformly bounded by* B. *Then*

$$P\left\{\limsup_{m,n\to\infty}\frac{|S(m,n)|}{(4B^2 mn\ln\ln mn)^{1/2}}\leqq 1\right\}=1. \tag{23}$$

The corresponding result for multiplicative sequences of rv's was proved by TAKAHASHI [7].

PROOF. We have to prove that, for any $\theta>4B^2$, we have

$$|S(m,n)|\leqq(\theta mn\ln\ln mn)^{1/2}\quad\text{a.s.},$$

if m or n is large enough. It is clear that this implies (23).

Step 1. Let $\delta>1$ be a fixed number, and set

$$m_s=n_s=[\delta^s]+1,\quad s=1,2,\ldots;\quad m_0=n_0=1,$$

where $[.]$ denotes the integral part. From (21) we obtain

$$P(s,t)=P\{|S(m_s,n_t)|\geqq(\theta m_s n_t\ln\ln m_s n_t)^{1/2}\}\leqq$$

$$\leqq 2\exp\left(-\frac{\theta\ln\ln m_s n_t}{2B^2}\right)=\frac{2}{(\ln m_s n_t)^{\gamma}}$$

with $\gamma=\theta/2B^2$. Since by assumption $\gamma>2$, hence

$$\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}P(s,t)\leqq\frac{2}{(\ln\delta)^{\gamma}}\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{1}{(s+t)^{\gamma}}\leqq\frac{2}{(\gamma-1)(\ln\delta)^{\gamma}}\sum_{s=1}^{\infty}\frac{1}{s^{\gamma-1}}<\infty.$$

By virtue of the Borel—Cantelli lemma, this yields

$$|S(m_s,n_t)|<(\theta m_s n_t\ln\ln m_s n_t)^{1/2}\quad\text{a.s.}, \tag{24}$$

if s or t is large enough.

Step 2. For arbitrary positive integers m and n, with appropriate s and t we have $m_s\leqq m<m_{s+1}$ and $n_t\leqq n<n_{t+1}$. Our goal is to show that

$$V(s,t)=\max_{m_s\leqq m<m_{s+1}}\ \max_{n_t\leqq n<n_{t+1}}\frac{|S(m,n)-S(m_s,n_t)|}{(\theta m_s n_t\ln\ln m_s n_t)^{1/2}}< \tag{25}$$

$$<5(\delta-1)+2(5(\delta-1))^{1/2}\quad\text{a.s.},$$

if s or t is large enough.

To this end, let us represent the difference $S(m,n)-S(m_s,n_t)$ as follows:

$$S(m,n)-S(m_s,n_t)=S(m_s,n_t;\ m-m_s,n-n_t)+$$

$$+S(m_s,0;\ m-m_s,n_t)+S(0,n_t;\ m_s,n-n_t).$$

Hence

$$\max_{m_s\leqq m<m_{s+1}}\ \max_{n_t\leqq n<n_{t+1}}|S(m,n)-S(m_s,n_t)|\leqq$$

$$\leqq M(m_s,n_t;\ \mu_s,\nu_t)+M_1(m_s,0;\ \mu_s,n_t)+M_2(0,n_t;\ m_s,\nu_t),$$

where $\mu_s = m_{s+1} - m_s - 1$ and $\nu_t = n_{t+1} - n_t - 1$. This immediately implies

$$(26) \qquad V(s,t) = \left(\frac{25\mu_s\nu_t}{m_s n_t}\right)^{1/2} \frac{M(m_s, n_t;\ \mu_s, \nu_t)}{(25\theta\mu_s\nu_t \ln\ln m_s n_t)^{1/2}} + $$
$$+\left(\frac{5\mu_s}{m_s}\right)^{1/2} \frac{M_1(m_s, 0;\ \mu_s, n_t)}{(5\theta\mu_s n_t \ln\ln m_s n_t)^{1/2}} + $$
$$+\left(\frac{5\nu_t}{n_t}\right)^{1/2} \frac{M_2(0, n_t;\ m_s, \nu_t)}{(5\theta m_s\nu_t \ln\ln m_s n_t)^{1/2}} = W_1 + W_2 + W_3,$$

where $W_i = W_i(s,t)$ $(i=1,2,3)$.

On the one hand, from the definition it follows that

$$\frac{\mu_s}{m_s} = \frac{m_{s+1}-1}{m_s-1} - 1 \leqq \delta - 1 \quad \text{and} \quad \frac{\nu_t}{n_t} \leqq \delta - 1.$$

On the other hand

$$(27) \qquad \frac{M(m_s, n_t;\ \mu_s, \nu_t)}{(25\theta\mu_s\nu_t \ln\ln m_s n_t)^{1/2}} < 1 \quad \text{a.s.},$$

if s or t is large enough. In fact, by virtue of Theorem 1, we get that

$$P\{M(m_s, n_t;\ \mu_s, \nu_t) \geqq (25\theta\mu_s\nu_t \ln\ln m_s n_t)^{1/2}\} \leqq$$
$$\leqq 81 \exp\left(-\frac{\theta \ln\ln m_s n_t}{2B^2}\right) = \frac{81}{(\ln m_s n_t)^{\gamma}}$$

with $\gamma = \theta/2B^2$, and we need only apply the Borel—Cantelli lemma, as in Step 1, in order to obtain (27). Summarizing the above reasonings, we can establish that

$$(28) \qquad W_1(s,t) < 5(\delta-1) \quad \text{a.s.},$$

if s or t is large enough.

Now consider W_2. On account of Lemma 1 (with $C_1=81$ and $C_2=1/5$),

$$P\{M_1(m_s, 0;\ \mu_s, n_t) \geqq (5\theta\mu_s n_t \ln\ln m_s n_t)^{1/2}\} \leqq$$
$$\leqq 81 \exp\left(-\frac{\theta \ln\ln m_s n_t}{2B^2}\right) = \frac{81}{(\ln m_s n_t)^{\gamma}},$$

then on account of the Borel—Cantelli lemma as $\mu_s/m_s \leqq \delta - 1$, we find that

$$(29) \qquad W_2(s,t) < \big(5(\delta-1)\big)^{1/2} \quad \text{a.s.},$$

if s or t is large enough.

In a quite similar way, we can see that

$$(30) \qquad W_3(s,t) < \big(5(\delta-1)\big)^{1/2} \quad \text{a.s.},$$

if s or t is large enough.

Collecting (26) and (28)—(30) together, we arrive at the wanted (25). Since θ may be chosen arbitrarily close to $4B^2$ and δ to 1, the assertion of Theorem 3 is a simple consequence of (24) and (25).

§ 4. A one-parameter version of the LIL for multiplicative random fields

We want to emphasize that in (23) m and n vary independently of each other. However, in case $m=n$, or more generally, if $m=m(p)$ and $n=n(p)$, where $1\leqq m(1)\leqq m(2)\leqq\ldots$ and $1\leqq n(1)\leqq n(2)\leqq\ldots$ are two sequences of integers and $m(p)n(p)\to\infty$ as $p\to\infty$, we have the following

THEOREM 4. *Let $\{\zeta_{kl}\}$ be multiplicative and uniformly bounded by B, and let $\{m(p)\}$ and $\{n(p)\}$ be two non-decreasing sequences of positive integers such that* $\max\{m(p), n(p)\}\to\infty$ *as $p\to\infty$. Then*

$$P\left\{\limsup_{p\to\infty}\frac{|S(m(p), n(p))|}{(2B^2 m(p)n(p)\ln\ln m(p)n(p))^{1/2}}\leqq 1\right\}=1. \tag{31}$$

PROOF. Theorem 4 is a simple consequence of a result of TAKAHASHI [7] concerning uniformly bounded and multiplicative sequences $\{\xi_j\}$ of rv's.

In fact, let us rearrange the rv's of the field $\{\zeta_{kl}\}$ so as they constitute a sequence $\{\xi_j\}$ as follows. Set $m(0)=n(0)=0$ and $N(p)=m(p)n(p)$ for $p=0, 1, 2, \ldots$. Since, for each p, the number of the ζ_{kl} for which $m(p-1)<k\leqq m(p)$ or $n(p-1)<l\leqq n(p)$ is equal to $N(p)-N(p-1)$, we can place the rv's of this pth "block" so as they form those ξ_j for which $N(p-1)<j\leqq N(p)$. Further, set

$$\tilde{S}(J)=\sum_{j=1}^{J}\xi_j \quad (J=1, 2, \ldots).$$

From the definition of the above rearrangement it is clear that $\tilde{S}(N(p))=$ $=S(m(p), n(p))$, and consequently, for each p,

$$\frac{S(m(p), n(p))}{(2B^2 m(p)n(p)\ln\ln m(p)n(p))^{1/2}}=\frac{\tilde{S}(N(p))}{(2B^2 N(p)\ln\ln N(p))^{1/2}}. \tag{32}$$

It is obvious that $\{\xi_j\}$, as a sequence, is also multiplicative and uniformly bounded by B. Therefore the theorem of Takahashi mentioned above results

$$P\left\{\limsup_{N\to\infty}\frac{|\tilde{S}(N)|}{(2B^2 N\ln\ln N)^{1/2}}\leqq 1\right\}=1, \tag{33}$$

and a fortiori

$$P\left\{\limsup_{p\to\infty}\frac{|\tilde{S}(N(p))|}{(2B^2 N(p)\ln\ln N(p))^{1/2}}\leqq 1\right\}=1.$$

By (32), this is equivalent to (31), which was to be proved.

§ 5. The LIL for equinormed multiplicative random fields

The factor B^2 in the denominator of the expressions (23) and (31) is generally not superfluous. The counterexample of a multiplicative sequence of uniformly bounded rv's, presented by RÉVÉSZ [6], can be simply modified so as to get, for any $B>1$, a multiplicative random field $\{\zeta_{kl}\}$ with the following properties:

$$E\zeta_{kl}^2 = 1, \quad |\zeta_{kl}| \leqq B \quad \text{a.s.},$$

and

$$P\left\{\limsup_{m,n\to\infty} \frac{S(m,n)}{(4B^2 mn \ln\ln mn)^{1/2}} = 1\right\} > 0.$$

Further, Theorem 4 can also be shown to be best possible in this sense.

However, the factor B^2 in the denominator of (23) and (31) becomes unnecessary if $\{\zeta_{kl}\}$ is equinormed multiplicative. A multiplicative random field $\{\zeta_{kl}\}$ is said to be *equinormed* if

(34) $E\zeta_{k_1 l_1}^2 = E(\zeta_{k_1 l_1}^2 \zeta_{k_2 l_2}^2) = 1 \quad (k_1, l_1, k_2, l_2 = 1, 2, \dots;\ \text{and } k_1 \neq k_2 \text{ or } l_1 \neq l_2).$

The notion of equinormality is used here in an essentially wider sense than it was introduced by ALEXITS [1, p. 187].

It is obvious that any random field $\{\zeta_{kl}\}$, consisting of independent rv's with means zero and variances $\sigma_{kl}^2 = 1$, is equinormed multiplicative. The next theorem contains as a special case the "$\leqq$" part of the LIL for uniformly bounded independent random fields (see the papers of ZIMMERMAN [8] and of PARK [4], where $\limsup_{m,n\to\infty} h(m,n)$ is defined in a more restricted sense as follows

$$\lim_{\min(m,n)\to\infty} \sup \{h(s,t)\colon s>m \text{ and } t>n\}).$$

THEOREM 5. *Let $\{\zeta_{kl}\}$ be equinormed multiplicative and uniformly bounded. Then*

$$P\left\{\limsup_{m,n\to\infty} \frac{|S(m,n)|}{(4mn \ln\ln mn)^{1/2}} \leqq 1\right\} = 1. \tag{35}$$

The corresponding result for equinormed multiplicative sequences was actually achieved by TAKAHASHI [7].

We begin with the following auxiliary result, which is interesting in itself.

LEMMA 3. *If a random field $\{\zeta_{kl}\}$ satisfies* (34) *and $E\zeta_{kl}^4 \leqq C$ $(k, l = 1, 2, \dots)$, then for $T(m,n) = \sum_{k=1}^{m} \sum_{l=1}^{n} \zeta_{kl}^2$ $(m, n = 1, 2, \dots)$ we have*

$$\lim_{\max(m,n)\to\infty} \frac{T(m,n)}{mn} = 1 \quad a.s. \tag{36}$$

PROOF. Indeed, $\{\zeta_{kl}^2 - 1\colon k, l = 1, 2, \dots\}$ is orthogonal owing to (34). Thus, for the arithmetic mean

$$\sigma(m,n) = \frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} (\zeta_{kl}^2 - 1) = \frac{T(m,n)}{mn} - 1$$

we have

$$E\sigma^2(m, n) = \frac{1}{m^2 n^2} \sum_{k=1}^{m} \sum_{l=1}^{n} E(\zeta_{kl}^2 - 1)^2 \leqq \frac{C}{mn}.$$

By Beppo Levi's theorem, this implies the a.s. convergence of $\sigma(p^2, q^2)$ to 0 as $\max(p, q) \to \infty$. This means that (36) holds true for the special case $m = p^2$ and $n = q^2$. The general case follows from the following inequality: If $p^2 \leqq m < (p+1)^2$ and $q^2 \leqq n < (q+1)^2$, then

$$\frac{T(p^2, q^2)}{(p+1)^2 (q+1)^2} \leqq \frac{T(m, n)}{mn} \leqq \frac{T((p+1)^2, (q+1)^2)}{p^2 q^2}.$$

We remark that in the proof of Theorem 5 the fulfilment of (36) is exploited only for $m_s = [\delta^s] + 1$, $n_t = [\delta^t] + 1$, where $\delta > 1$.

Proof of Theorem 5. We will show that, for any $\theta > 4$, we have

$$|S(m, n)| \leqq (\theta\, mn \ln \ln mn)^{1/2} \quad \text{a.s.},$$

if m or n is large enough, which obviously gives (35).

Step 1. Making use of the simple inequality

$$e^x \leqq (1+x) \exp\left(\frac{x^2}{2} + |x|^3\right) \text{ if } |x| \leqq \frac{1}{2},$$

we have

$$\exp(uS(m, n)) \leqq \prod_{k=1}^{m} \prod_{l=1}^{n} \left\{(1 + u\zeta_{kl}) \exp\left(\frac{u^2 \zeta_{kl}^2}{2} + |u\zeta_{kl}|^3\right)\right\},$$

whence

$$\exp\left(uS(m, n) - \frac{u^2}{2} T(m, n)\right) \leqq \prod_{k=1}^{m} \prod_{l=1}^{n} \{(1 + u\zeta_{kl}) \exp(uB)^3\},$$

provided $0 < uB \leqq 1/2$. If in addition $mn(uB)^3 \leqq 1$, then

$$E\left\{\exp\left(uS(m, n) - \frac{u^2}{2} T(m, n)\right)\right\} \leqq \exp(mn(uB)^3) \leqq e.$$

Hence, by the Chebyshev inequality, we have

$$P\left\{|S(m, n)| \geqq \frac{u}{2} T(m, n) + v\right\} \leqq 2e \cdot e^{-uv}.$$

Setting here $u = u_{st} = \{(\theta \ln \ln m_s n_t)/m_s n_t\}^{1/2}$ and $v = v_{st} = \frac{1}{2}(\theta m_s n_t \ln \ln m_s n_t)^{1/2}$, where $m_s = [\delta^s] + 1$, $n_t = [\delta^t] + 1$, $\delta > 1$, just as in the proof of Theorem 3, and $\theta > 4$, we obtain that

$$P(s, t) = P\left\{S(m_s, n_t)| \geqq \frac{u_{st}}{2} T(m_s, n_t) + v_{st}\right\} \leqq$$

$$\leqq 2e \cdot \exp\left(-\frac{\theta}{2} \ln \ln m_s n_t\right) = \frac{2}{(\ln m_s n_t)^{\gamma}}$$

with $\gamma=\theta/2>2$. Since again $\sum_{s=1}^{\infty}\sum_{t=1}^{\infty} P(s,t)<\infty$, the Borel—Cantelli lemma yields

$$|S(m_s, n_t)|<\frac{u_{st}}{2}\,T(m_s,n_t)+v_{st} \quad \text{a.s.},$$

or equivalently

$$\frac{|S(m_s,n_t)|}{(\theta m_s n_t \ln\ln m_s n_t)^{1/2}} \leqq \frac{T(m_s,n_t)}{2m_s n_t}+\frac{1}{2} \quad \text{a.s.},$$

if s or t is large enough. Taking into account (36) we get that, for every $\theta>4$,

$$|S(m_s,n_t)| \leqq (\theta m_s n_t \ln\ln m_s n_t)^{1/2} \quad \text{a.s.},$$

if s or t is large enough.

Step 2. As we saw in the proof of Theorem 3, only the assumption of multiplicativity and uniform boundedness ensures the fulfilment of inequality (25). This completes the proof.

In case $m=m(p)$ and $n=n(p)$, $m(p)n(p)\to\infty$ as $p\to\infty$, the following one-parameter version of the LIL holds.

THEOREM 6. *Let* $\{\zeta_{kl}\}$ *be equinormed multiplicative and uniformly bounded, and let* $\{m(p)\}$ *and* $\{n(p)\}$ *be two non-decreasing sequences of positive integers such that* $\max\{m(p),n(p)\}\to\infty$ *as* $p\to\infty$. *Then*

$$P\left\{\limsup_{p\to\infty}\frac{|S(m(p),n(p))|}{(2m(p)n(p)\ln\ln m(p)n(p))^{1/2}}\leqq 1\right\}=1. \tag{37}$$

Theorem 6 is a consequence of the result of TAKAHASHI [7], which can be seen just as in the case of Theorem 4.

Finally we mention a problem. It would be interesting to establish both two-parameter and one-parameter versions of the LIL for weighted sums $\sum_{k=1}^{m}\sum_{l=1}^{n}\sigma_{kl}\zeta_{kl}$, where $\{\zeta_{kl}\}$ is a multiplicative or an equinormed multiplicative random field, and $\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sigma_{kl}^2=\infty$. Although, Lemmas 2 and 3 can be generalized with ease in this more general setting, it is not clear for us how to treat Steps 1 and 2 in the above proofs (e.g., how to define m_s and n_t?).

§ 6. Convergence rates in the LIL

Turning to the rate of convergence in (23), we can state

THEOREM 7. *Let* $\{\zeta_{kl}\}$ *be multiplicative and uniformly bounded by* B. *Then, for any* $\theta>4B^2$, *we have*

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn(\ln mn)^2}\,P\left\{\sup_{k\geqq m\text{ or }l\geqq n}\frac{|S(k,l)|}{(\theta kl\ln\ln kl)^{1/2}}\geqq 1\right\}<\infty. \tag{38}$$

The proof of Theorem 7 is based on Lemma 2 and on the following

LEMMA 4. *For any* $\varepsilon>0$, *the numerical series*

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn(\ln mn)^2(\ln m)^{\varepsilon}}$$

converges.

PROOF OF LEMMA 4. An easy computation gives

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn(\ln mn)^2(\ln m)^{\varepsilon}}=\sum_{n=2}^{\infty}\frac{1}{n}\sum_{k=1}^{\infty}\sum_{m=2^k}^{2^{k+1}-1}\frac{1}{m(\ln mn)^2(\ln m)^{\varepsilon}}=$$

$$=O(1)\sum_{n=2}^{\infty}\frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k^{\varepsilon}(k+\ln n)^2}.$$

Now let us deal with the inner series:

$$\sum_{k=1}^{\infty}\frac{1}{k^{\varepsilon}(k+\ln n)^2}=\left\{\sum_{k=1}^{[\ln n]}+\sum_{k=[\ln n]+1}^{\infty}\right\}\frac{1}{k^{\varepsilon}(1+\ln n)^2}\leqq$$

$$\leqq\frac{1}{(\ln n)^2}\sum_{k=1}^{[\ln n]}\frac{1}{k^{\varepsilon}}+\sum_{k=[\ln n]+1}^{\infty}\frac{1}{k^{2+\varepsilon}}=O\left(\frac{1}{(\ln n)^{1+\varepsilon}}\right).$$

The proof is ready.

PROOF OF THEOREM 7. By virtue of Lemma 4 it is enough to demonstrate that

$$P(m,n)=P\left\{\sup_{k\geqq m\,\text{or}\,l\geqq n}\frac{|S(k,l)|}{(\theta kl\ln\ln kl)^{1/2}}\geqq 1\right\}=O\left\{\frac{1}{(\ln m)^{\varepsilon}}+\frac{1}{(\ln n)^{\varepsilon}}\right\}$$

with an appropriate $\varepsilon>0$.

To this effect, let us fix a number θ_1 so that $4B^2<\theta_1<\theta$, and let $s_0=s_0(m)$ and $t_0=t_0(n)$ be defined by $m_{s_0}\leqq m<m_{s_0+1}$ and $n_{t_0}\leqq n<n_{t_0+1}$, where $m_s=n_s=[\delta^s]+1$, $\delta>1$ (see Section 4). It is obvious that

$$P(m,n)\leqq\left\{\sum_{s=s_0}^{\infty}\sum_{t=t_0}^{\infty}+\sum_{s=s_0}^{\infty}\sum_{t=0}^{t_0-1}+\sum_{s=0}^{s_0-1}\sum_{t=t_0}^{\infty}\right\}Q(s,t)$$

with

$$Q(s,t)=P\left\{\max_{m_s\leqq k<m_{s+1}}\max_{n_t\leqq l<n_{t+1}}\frac{|S(k,l)|}{(\theta kl\ln\ln kl)^{1/2}}\geqq 1\right\}.$$

It can be easily checked that

$$Q(s,t)\leqq P\left\{\frac{|S(m_s,n_t)|}{(m_sn_t\ln\ln m_sn_t)^{1/2}}\geqq\theta_1^{1/2}\right\}+$$

$$+P\left\{\max_{m_s\leqq k<m_{s+1}}\max_{n_t\leqq l<n_{t+1}}\frac{|S(k,l)-S(m_s,n_t)|}{(m_sn_t\ln\ln m_sn_t)^{1/2}}\geqq\theta^{1/2}-\theta_1^{1/2}\right\}.$$

After these we have essentially to repeat the arguments that yielded (24) and (25) in the proof of Theorem 3. We do not enter into the details.

Now we turn to the question of the convergence rate in (31). For the sake of simplicity we treat the special case $m(p)=p$ and $n(p)=[\alpha p]$ with a fixed $\alpha>0$.

THEOREM 8. *Let* $\{\zeta_{kl}\}$ *be multiplicative and uniformly bounded by* B. *Then, for any* $\theta_1>2B^2$ *and for any* $\alpha>0$, *we have*

$$\sum_{m=2}^{\infty}\frac{1}{m\ln m}P\left\{\sup_{k\geqq m}\frac{|S(k,[\alpha k])|}{(\theta_1\alpha k^2\ln\ln k)^{1/2}}\geqq 1\right\}<\infty. \tag{39}$$

PROOF. It is sufficient to prove that

$$P(m)=P\left\{\sup_{k\geqq m}\frac{|S(k,[\alpha k])|}{(\theta_1\alpha k^2\ln\ln k)^{1/2}}\geqq 1\right\}=O\left(\frac{1}{(\ln m)^{\varepsilon}}\right)$$

with a suitable $\varepsilon>0$.

For this purpose, let $m_s=[\delta^s]+1$, $\delta>1$, and $m_{s_0}\leqq m<m_{s_0+1}$. Then we obviously have

$$P(m)\leqq\sum_{s=s_0}^{\infty}P\left\{\max_{m_s\leqq k<m_{s+1}}\frac{|S(k,[\alpha k])|}{(\theta_1\alpha k^2\ln\ln k)^{1/2}}\geqq 1\right\}=\sum_{s=s_0}^{\infty}Q(s)$$

and, for each s,

$$Q(s)\leqq P\left\{\frac{|S(m_s,[\alpha m_s])|}{(\alpha m_s^2\ln\ln m_s)^{1/2}}\geqq\theta_2^{1/2}\right\}+$$

$$+P\left\{\max_{m_s\leqq k<m_{s+1}}\frac{|S(k,[\alpha k])-S(m_s,[\alpha m_s])|}{(\alpha m_s^2\ln\ln m_s)^{1/2}}\geqq\theta_1^{1/2}-\theta_2^{1/2}\right\}=Q_1(s)+Q_2(s),$$

where θ_2 is chosen so that $2B^2<\theta_2<\theta_1$ and θ_2 be "sufficiently close" to $2B^2$.

Now we have to estimate these probabilities from above in a familiar way using (21) and (22), respectively. For example, by (21)

$$Q_1(s)\leqq 2\exp\left(-\frac{\theta_2\ln\ln m_s}{2B^2}\right)=\frac{2}{(\ln m_s)^{\gamma_2}},$$

where $\gamma_2=\theta_2/2B^2>1$. Thus

$$\sum_{s=s_0}^{\infty}Q_1(s)\leqq\frac{2}{(\ln\delta)^{\gamma_2}}\sum_{s=s_0}^{\infty}\frac{1}{s^{\gamma_2}}=\frac{O(1)}{s_0^{\gamma_2-1}}=\frac{O(1)}{(\ln m)^{\varepsilon}}$$

with $\varepsilon=\gamma_2-1$.

We note that the above convergence rates cannot be improved even for random fields consisting of independent rv's. Furthermore, it seems to be very likely that analogous results can be stated concerning the convergence rates in (35) and (37). The question is somewhat more complicated because of the fact that an appropriate estimate of the convergence rate in (36) for the special choice $m_s=[\delta^s]+1$ and $n_t=[\delta^t]+1$, $\delta>0$, is also needed here. We do not enter into details.

§ 7. Generalizations to the multi-parameter case

Let Z^d denote the set of all d-tuples of non-negative integers, and let Z^d_+ denote the set of all d-tuples of positive integers, where $d \geqq 1$ is a fixed integer. The points in Z^d are denoted by $\mathbf{k}, \mathbf{m}$ etc., or sometimes, when necessary, more explicitly by $(k_1, k_2, \ldots, k_d)$, $(m_1, m_2, \ldots, m_d)$ etc. Two d-tuples $\mathbf{k}$ and $\mathbf{m}$ are said to be distinct if for at least one j we have $k_j \neq m_j$ $(1 \leqq j \leqq d)$. Z^d is partially ordered by agreeing that $\mathbf{k} \leqq \mathbf{m}$ iff $k_j \leqq m_j$ for each j, $1 \leqq j \leqq d$. Consequently, $\mathbf{k} \not\leqq \mathbf{m}$ means that for at least one j we have $k_j > m_j$. We write $\mathbf{0}$ and $\mathbf{1}$ respectively for the tuples $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$ in Z^d.

Let $\{\zeta_{\mathbf{k}}\} = \{\zeta_{\mathbf{k}} : \mathbf{k} \in Z^d_+\}$ be a random field, i.e. a collection of rv's indexed by the set Z^d_+. Put

$$S(\mathbf{b}, \mathbf{m}) = \sum_{\mathbf{b}+\mathbf{1} \leqq \mathbf{k} \leqq \mathbf{b}+\mathbf{m}} \zeta_{\mathbf{k}} = \sum_{k_1=b_1+1}^{b_1+m_1} \cdots \sum_{k_d=b_d+1}^{b_d+m_d} \zeta_{k_1, \ldots, k_d}$$

and

$$M(\mathbf{b}, \mathbf{m}) = \max_{\mathbf{1} \leqq \mathbf{k} \leqq \mathbf{m}} |S(\mathbf{b}, \mathbf{k})| = \max_{1 \leqq k_1 \leqq m_1} \cdots \max_{1 \leqq k_d \leqq m_d} |S(\mathbf{b}, \mathbf{k})|,$$

where $\mathbf{b} \in Z^d$, $\mathbf{m} \in Z^d_+$, and $\mathbf{b}+\mathbf{1}$, $\mathbf{b}+\mathbf{m}$ are the usual coordinatewise sums. In case $\mathbf{b}=\mathbf{0}$ we use the abbreviated notation $S(\mathbf{m}) = S(\mathbf{0}, \mathbf{m})$ $(\mathbf{m} \in Z^d_+)$.

The following definitions arise in a quite natural way. The random field $\{\zeta_{\mathbf{k}}\}$ is said to be *multiplicative* if for all positive integers r and for all systems of pairwise distinct points $\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_r$ from Z^d_+, we have $E\{\prod_{j=1}^{r} \zeta_{\mathbf{k}_j}\} = 0$. If in addition

$$E\zeta^2_{\mathbf{k}} = E(\zeta^2_{\mathbf{k}} \zeta^2_{\mathbf{l}}) = 1 \quad (\mathbf{k}, \mathbf{l} \in Z^d_+,\ \mathbf{k} \neq \mathbf{l}),$$

then $\{\zeta_{\mathbf{k}}\}$ is said to be *equinormed multiplicative.*

If $\mathbf{m} = (m_1, m_2, \ldots, m_d)$, let $|\mathbf{m}|$ stand for the product $m_1 m_2 \ldots m_d$. In this paper by the limit $\mathbf{m} \to \infty$ we mean $\max_{1 \leqq j \leqq d} m_j \to \infty$, and by $\limsup_{\mathbf{m} \to \infty} h(\mathbf{m})$ we mean $\lim_{\mathbf{m} \to \infty} \sup \{h(\mathbf{s}) : \mathbf{s} \not\leqq \mathbf{m}\}$. The "$\leqq$" part of the LIL and convergence rates thereof can be extended as follows.

THEOREM 9. *Let $\{\zeta_{\mathbf{k}} : \mathbf{k} \in Z^d_+\}$ be multiplicative and uniformly bounded,*

$$|\zeta_{\mathbf{k}}| \leqq B \quad a.s. \quad (\mathbf{k} \in Z^d_+).$$

Then

$$P\left\{\limsup_{\mathbf{m} \to \infty} \frac{|S(\mathbf{m})|}{(2dB^2 |\mathbf{m}| \ln \ln |\mathbf{m}|)^{1/2}} \leqq 1\right\} = 1. \tag{40}$$

Furthermore, for any $\theta > 2dB^2$, we have

$$\sum_{\mathbf{m} \geqq \mathbf{2}} \frac{1}{|\mathbf{m}| (\ln |\mathbf{m}|)^d} P\left\{ \sup_{\substack{\mathbf{k}: k_j > m_j \\ \text{for at least one } j}} \frac{|S(\mathbf{k})|}{(\theta |\mathbf{k}| \ln \ln |\mathbf{k}|)^{1/2}} \geqq 1 \right\} < \infty.$$

We recall that $\sum_{\mathbf{m} \geqq \mathbf{2}}$ means the d-fold summation $\sum_{m_1=2}^{\infty} \sum_{m_2=2}^{\infty} \cdots \sum_{m_d=2}^{\infty}$. In case $k_i = k_i(p)$, $1 \leqq i \leqq d$, the following one-dimensional version of the LIL holds.

THEOREM 10. *Let* $\{\zeta_{\mathbf{k}}: \mathbf{k}\in Z^d\}$ *be multiplicative and uniformly bounded, and let* $\{\mathbf{k}(p): p=1, 2, \ldots\}$ *be a sequence from* Z^d_+ *such that*

(41) $$\mathbf{k}(1) \leqq \mathbf{k}(2) \leqq \ldots \quad \textit{and} \quad \mathbf{k}(p) \to \infty \quad \textit{as} \quad p \to \infty.$$

Then

(42) $$P\left\{\limsup_{p\to\infty} \frac{|S(\mathbf{k}(p))|}{(2B^2|\mathbf{k}(p)| \ln\ln |\mathbf{k}(p)|)^{1/2}} \leqq 1\right\} = 1.$$

In particular, for any $\theta_1 > 2B^2$ *and for any* $\alpha_i > 0$, $2 \leqq i \leqq d$, *we have*

$$\sum_{m=2}^{\infty} \frac{1}{m \ln m} P\left\{\sup_{k\geqq m} \frac{|S(k, [\alpha_2 k], \ldots, [\alpha_d k])|}{(\theta_1 \alpha_2 \ldots \alpha_d k^d \ln\ln k)^{1/2}} \geqq 1\right\} < \infty.$$

The factor B^2 in the denominator of (40) and (42) can be omitted for equinormed multiplicative random fields.

THEOREM 11. *Let* $\{\zeta_{\mathbf{k}}: \mathbf{k}\in Z^d_+\}$ *be equinormed multiplicative and uniformly bounded. Then*

$$P\left\{\limsup_{\mathbf{m}\to\infty} \frac{|S(\mathbf{m})|}{(2d\,|\mathbf{m}| \ln\ln |\mathbf{m}|)^{1/2}} \leqq 1\right\} = 1.$$

If $\{\mathbf{k}(p): p=1, 2, \ldots\}$ *is such that* (41) *is satisfied, then*

$$P\left\{\limsup_{p\to\infty} \frac{|S(\mathbf{k}(p))|}{(2|\mathbf{k}(p)| \ln\ln |\mathbf{k}(p)|)^{1/2}} \leqq 1\right\} = 1.$$

For multi-parameter Gaussian processes Theorem 11 (including also the "$\geqq$" part of the LIL) were proved by PARK [4] and by PYKE [5], respectively.

The common root of the validity of Theorems 9, 10, and 11 is that Lemmas 2 and 3 are true in the general multi-parameter case, too. To be more precise, on the one hand, if $\{\zeta_{\mathbf{k}}\}$ is multiplicative and uniformly bounded by B, then there exists an exponential bound for the tail distribution of $S(\mathbf{b}, \mathbf{m})$:

(43) $$P\{|S(\mathbf{b}, \mathbf{m})| \geqq \lambda\} \leqq 2 \exp\left(-\frac{\lambda^2}{2\,|\mathbf{m}|\, B^2}\right)$$

for all $\lambda > 0$, $\mathbf{b} \geqq \mathbf{0}$ and $\mathbf{m} \geqq \mathbf{1}$ from Z^d; on the other hand, if $\{\zeta_{\mathbf{k}}\}$ is equinormed and $E\zeta_{\mathbf{k}}^4 \leqq C$ $(\mathbf{k}\in Z^{4d}_+)$, then

$$\lim_{\mathbf{m}\to\infty} \frac{1}{|\mathbf{m}|} \sum_{\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{m}} \zeta_{\mathbf{k}}^2 = 1 \text{ a.s.}$$

Now from (43) it follows that the tail distribution of the "maximal" function $M(\mathbf{b}, \mathbf{m})$ has almost the same exponential bound as $S(\mathbf{b}, \mathbf{m})$ has in (43). This is ensured by Theorem 12 below. To formulate this, let $f(\mathbf{b}, \mathbf{m})$ denote a non-negative function depending on the joint df of $\{\zeta_{\mathbf{k}}: \mathbf{b}+\mathbf{1} \leqq \mathbf{k} \leqq \mathbf{b}+\mathbf{m}\}$. We require that the inequality

(44) $$\begin{aligned} &f(b_1, \ldots, b_j, \ldots, b_d;\ m_1, \ldots, h_j, \ldots, m_d) + \\ &+ f(b_1, \ldots, b_j + h_j, \ldots, b_d;\ m_1, \ldots, m_j - h_j, \ldots, m_d) \leqq \\ &\leqq f(b_1, \ldots, b_j, \ldots, b_d;\ m_1, \ldots, m_j, \ldots, m_d) = f(\mathbf{b}, \mathbf{m}) \end{aligned}$$

holds for all $\mathbf{b}\in Z^d$, $\mathbf{m}\in Z_+^d$, $1\leqq h_j<m_j$, and $1\leqq j\leqq d$. Condition (44) expresses that $f(\mathbf{b},\mathbf{m})$ as a function of the interval $[b_j+1, b_j+m_j]$ is "superadditive" for any fixed values of $b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_d$ and $m_1, \ldots, m_{j-1}, m_{j+1}, \ldots, m_d$, where j may equal $1, 2, \ldots, d$. An example is $f(\mathbf{b},\mathbf{m})=|\mathbf{m}|=m_1 m_2 \ldots m_d$.

THEOREM 12. *Suppose that there exists a non-negative function $f(\mathbf{b},\mathbf{m})$ satisfying* (44) *such that*

$$P\{|S(\mathbf{b},\mathbf{m})|\geqq\lambda\}\leqq C\exp\left(-\frac{\lambda^2}{f(\mathbf{b},\mathbf{m})}\right)$$

holds for all $\lambda\in(0,\Lambda)$, $\mathbf{b}\geqq\mathbf{0}$, and $\mathbf{m}\geqq\mathbf{1}$, where Λ is a fixed positive number or ∞. Then

$$P\{M(\mathbf{b},\mathbf{m})\geqq\lambda\}\leqq C_3\exp\left(-\frac{C_4\lambda^2}{f(\mathbf{b},\mathbf{m})}\right)$$

holds for all $\lambda\in(0,\Lambda)$, $\mathbf{b}\geqq 0$, and $\mathbf{m}\geqq 1$. For the constants C_3 and C_4 the following choices are possible:

(i) $C_3=\max(81, C)$ *and* $C_4=5^{-d}$, *or*

(ii) *by increasing C_3 we can make C_4 as close to 1 as we wish.*

We note that here λ^2 can be replaced by any function $\varphi(\lambda)$ satisfying the conditions (i)—(iii) enumerated in Theorem 2.

The proof of Theorem 12 may be carried out by induction on d in the same manner as we did it from $d=1$ to $d=2$ in Section 2. The simplest case $d=1$ was proved in [3].

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ÜBER HERMITE—FEJÉRSCHE INTERPOLATION

Von
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Es sei $\{X\}=\{-1\leqq x_{1n}<\ldots<x_{kn}<\ldots<x_{nn}\leqq 1\}$ $(n=1, 2, \ldots)$ eine im Intervall $[-1, +1]$ gegebene Punktmatrix.

Wir führen die folgenden Bezeichnungen ein:

$$\omega(x)=\prod_{k=1}^{n}(x-x_{kn}); \tag{1}$$

$$l_{kn}(x)=\frac{\omega(x)}{\omega'(x_{kn})(x-x_{kn})}; \tag{2}$$

$$v_{kn}(x)=1-\frac{\omega''(x_{kn})}{\omega'(x_{kn})}(x-x_{kn}); \tag{3}$$

$$h_{kn}(x)=v_{kn}(x)\,l_{kn}^2(x); \tag{4}$$

$$\mu_n(X)=\sup_{|x|\leqq 1}\sum_{1}^{n}|h_{kn}(x)|; \tag{5}$$

$$\mathfrak{H}_{kn}(x)=(x-x_{kn})\,l_{kn}^2(x); \tag{6}$$

$$\nu_n(X)=\sup_{|x|\leqq 1}\sum_{1}^{n}|\mathfrak{H}_{kn}(x)|; \tag{7}$$

$$\tilde{\mathfrak{H}}_{kn}(x)=\frac{(x-x_{kn})\,l_{kn}^2(x)}{\sqrt{1-x_{kn}^2}+\dfrac{1}{n}}; \tag{8}$$

$$\tilde{\nu}_n(X)=\sup_{|x|\leqq 1}\sum_{1}^{n}|\tilde{\mathfrak{H}}_{kn}(x)|. \tag{9}$$

Es sei für $f(x)\in C$ $\|f(x)\|=\max\limits_{-1\leqq x\leqq 1}|f(x)|$ und weiter sei $\omega(t)$ ein beliebiges Stetigkeitsmodul. Es bezeichne $C(\omega)$ die Klasse der stetigen Funktionen $f(x)$, für welche $\omega(f; h)=O(1)\omega(h)$ [1] besteht. Ferner sei

$$H_n(f; x, X)=\sum_{1}^{n}f(x_{kn})\,h_{kn}(x). \tag{10}$$

[1] Im Zeichen $O(1)$ ist eine von der Funktion $f(x)$ abhängige Konstante inbegriffen.

G. GRÜNWALD und J. BALÁZS haben in den Arbeiten [1] bzw. [2] folgendes bewiesen:

Sind die Bedingungen

$$\limsup_{n\to\infty} \mu_n(x) < \infty \tag{11}$$

und

$$\limsup_{n\to\infty} \nu_n(x) = 0 \tag{12}$$

erfüllt, so konvergieren die Interpolationspolynome $H_n(f; x, X)$ für $|x_{kn}| \leqq 1$ im abgeschlossenen Intervall $[-1, +1]$ gleichmäßig gegen $f(x) \in C$.

In unserer Arbeit geben wir notwendige und hinreichende Bedingungen für die Konvergenz der Interpolationsfolge $H_n(f; x, X)$ mit Hilfe des Stetigkeitsmoduls $\omega(t)$ bzw. der für die Punktmatrix $\{X\}$ bezeichnenden Zahlen $\mu_n(X)$ und $\tilde{\nu}_n(X)$.

LEMMA 1. *Es seien* $0 < m \leqq n$ *und* $f(x) \in C(\omega)$. *Dann gilt*

$$|f(x) - H_n(f; x, X)| = O(1)\,\omega\left(\frac{1}{m}\right)\left[\sum_1^n |h_{kn}(x)| + m \sum_1^n |\tilde{\mathfrak{H}}_{kn}(x)|\right]. \tag{13}$$

BEWEIS. Es sei $P_m(x)$ das beste Approximationspolynom m-ter Ordnung an $f(x)$ in $[-1, +1]$. Dann ist

$$|f(x) - P_m(x)| \leqq O(1)\,\omega\left(f; \frac{1}{m}\right). \tag{14}$$

S. B. STETSCHKIN hat in der Arbeit [3] die folgende Ungleichung gezeigt:

$$|P_m'(x)| = \begin{cases} O(1)\dfrac{m}{\sqrt{1-x^2}}\,\omega\left(f; \dfrac{1}{m}\right), & |x| < 1 \\ O(m^2)\,\omega\left(f; \dfrac{1}{m}\right), & |x| \leqq 1 \end{cases}$$

d. h.

$$|P_m'(x)| = O(m)\frac{\omega\left(f; \frac{1}{m}\right)}{\sqrt{1-x^2} + \frac{1}{m}} = O(m)\frac{\omega\left(f; \frac{1}{m}\right)}{\sqrt{1-x^2} + \frac{1}{n}}. \tag{15}$$

Da $m \leqq n$ ist, können wir das Polynom $P_m(x)$ in der Form

$$P_m(x) = \sum_1^n P_m(x_{kn}) h_{kn}(x) + \sum_1^n P_m'(x_{kn}) \mathfrak{H}_{kn}(x)$$

schreiben, wonach

$$|f(x) - H_n(f; x, X)| \leqq |f(x) - P_m(x)| +$$

$$+ \sum_1^n |P_m(x_{kn}) - f(x_{kn})|\,|h_{kn}(x)| + \sum_1^n |P_m'(x_{kn})|\,|\mathfrak{H}_{kn}(x)|$$

ist. Aus den Abschätzungen (14) und (15) folgt die Behauptung (13).

Es sei $m=n$. Nach (13), (5) und (9) gilt dann

$$\|f(x)-H_n(f;\,x,X)\| = O(1)\left[\omega\left(\frac{1}{n}\right)\mu_n(X)+\omega\left(\frac{1}{n}\right)n\tilde{v}_n(X)\right]. \tag{16}$$

Aus (15) ergibt sich der folgende Satz:

SATZ 1. *Es seien für* $\omega(t)$ *und* X *die Bedingungen*

$$\lim_{n\to\infty}\omega\left(\frac{1}{n}\right)\mu_n(X)=0 \tag{17}$$

und

$$\lim_{n\to\infty}\omega\left(\frac{1}{n}\right)n\tilde{v}_n(X)=0 \tag{18}$$

erfüllt. Dann gilt für jedes $f(x)\in C(\omega)$

$$\lim_{n\to\infty}\|f(x)-H_n(f;\,x,X)\|=0.$$

BEMERKUNGEN. P. ERDŐS und P. TURÁN haben in der Arbeit [4] bewiesen daß für jedes $\{X\}$

$$v_n(X)\geqq c_1\frac{\log n}{n}$$

(mit c_i, $i=1,2,\ldots$ bezeichnen wir von n und X unabhängige Konstanten) und sogar

$$\tilde{v}_n(X)\geqq c_1\frac{\log n}{n} \tag{19}$$

ist.

Wegen (18) und (19) folgt die Relation

$$\lim_{n\to\infty}\omega\left(\frac{1}{n}\right)\log n=0. \tag{20}$$

Es ist leicht zu zeigen, daß die Bedingung (20) für die Konvergenz der Polynome $H_n(f;\,x,X)$ im allgemeninen nicht notwendig ist. Es sei z.B. $\{X\}=\{T\}$, wo $\{T\}$ die Tschebischeffsche Punktmatrix erster Art bezeichnet. Es ist wohlbekannt, daß $\{T\}$ den Bedingungen (11) und (12) genügt.

SATZ 2. *Es seien für* $\omega(t)$ *und* $\{X\}$

$$\frac{\mu_n(X)}{\tilde{v}_n(X)}\geqq c_2 n \tag{21}$$

und

$$\lim_{n\to\infty}\omega\left(\frac{1}{n}\right)\mu_n(X)=0. \tag{22}$$

Dann gilt für jedes $f(x)\in C(\omega)$

$$\lim_{n\to\infty}\|f(x)-H_n(f;\,x,X)\|=0.$$

BEWEIS. Aus (21) folgt, daß $\omega\left(\frac{1}{n}\right)\mu_n(X) \leqq c_2\omega\left(\frac{1}{n}\right)\tilde{\mu}_n(X)n$ ist; so sind nach (19) die Bedingungen (17) und (18) des Satzes erfüllt.

SATZ 3. *Es seien für $\omega(t)$ und $\{X\}$ die Bedingungen*

$$0 < c_2^* \leqq \frac{\mu_n(X)}{\tilde{\nu}_n(X)} < n \tag{23}$$

und

$$\lim_{n\to\infty} \omega\left(\frac{\tilde{\nu}_n(X)}{\mu_n(X)}\right)\mu_n(X) = 0 \tag{24}$$

erfüllt. Dann gilt für jedes $f(x)\in C(\omega)$

$$\lim_{n\to\infty} \|f(x) - H_n(f;\, x, X)\| = 0,$$

BEWEIS. Wir ziechen die Beziehung (13) des Lemmas 1 heran und wählen

$$m = \left[\frac{1}{c_2^*}\frac{\mu_n(X)}{\tilde{\nu}_n(X)}\right].$$

Nach (13) gilt die Abschätzung

$$\|f(x) - H_n(f;\, x, X)\| \leqq O(1)\left[\omega\left(\frac{\tilde{\nu}_n(X)}{\mu_n(X)}\right)\mu_n(X) + \right.$$

$$\left. + \omega\left(\frac{\tilde{\nu}_n(X)}{\mu_n(X)}\right)\frac{\mu_n(X)}{\tilde{\nu}_n(X)}\tilde{\nu}_n(X)\right] = O(1)\,\omega\left(\frac{\tilde{\nu}_n(X)}{\mu_n(X)}\right)\mu_n(X),$$

woraus die Behauptung des Satzes 3 folgt.

Im weiteren beweisen wir Sätze über notwendige Bedingungen der Konvergenz der Interpolationsfolgen $H_n(f;\, x, X)$.

LEMMA 2. *Es sei für $\omega(t)$ und $\{X\}$*

$$\limsup_{n\to\infty} \omega\left(\frac{1}{n^2\mu_n(X)}\right)\mu_n(X) > 0. \tag{25}$$

Dann existiert eine Funktion $f(x)\in C(\omega)$ mit

$$\limsup_{n\to\infty} \|f(x) - H_n(f;\, x, X)\| > 0.$$

BEWEIS. P. VÉRTESI hat in der Arbeit [5] gezeigt, daß für jede Punktmatrix $\{X\}$ mit $\mu_n(X)=1$ eine Funktion und eine Folge $f(x)\in C(\omega)$ bzw. $n_i\to\infty$ $(i=1, 2, \ldots)$ existieren so daß

$$\|f(x) - H_{n_i}(f;\, x, X)\| \geqq \mu_{n_i}(X)\,\omega(d_{n_i})$$

mit $d_n = \min\limits_{1\leqq i\leqq n}(x_{i+1,n} - x_{in})$ gilt.[2]

[2] Das hier zitierte Lemma ist ein Spezialfall des Satzes von P. Vértesi.

Aus der Definition unter (4) der Polynome $h_{kn}(x)$ folgen die Relationen

$$h_{in}(x_{in}) = 1, \quad h_{i,n}(x_{i+1}) = 0, \quad i = 1, 2, \dots .$$

So gilt

$$\frac{1}{x_{i+1}-x_i} = \left| \frac{h_{in}(x_{i+1}) - h_{in}(x_i)}{x_{i+1}-x_i} \right| \leqq |h'_{in}(\xi)| \leqq$$

$$\leqq 4n^2 \sup_{|x| \leqq 1} \sum_{i=1}^{n} |h_{in}(x)| \leqq 4n^2 \mu_n(X),$$

woraus

$$d_n \geqq \frac{1}{4n^2 \mu_n(X)}. \tag{26}$$

Aus (26) und (24) folgt die Behauptung des Lemmas 2.

LEMMA 3. *Es seien für* $\omega(t)$ *und* $\{X\}$ *die Bedingungen*

$$\mu_n(X) \geqq c_3 \log n \tag{27}$$

und

$$\limsup_{n\to\infty} \omega\left(\frac{1}{n}\right) \log n > 0 \tag{28}$$

erfüllt. Dann existiert eine Funktion $f(x) \in C(\omega)$ *mit*

$$\limsup_{n\to\infty} \|f(x) - H_n(f;\, x, X)\| > 0.$$

LEMMA 4. *Es seien für* $\omega(t)$ *und* $\{X\}$

$$\liminf_{t=0} \frac{\omega(t^2)}{\omega(t)} = l > 0 \tag{29}$$

und

$$\limsup_{n\to\infty} \omega\left(\frac{1}{n}\right) \mu_n(X) > 0. \tag{30}$$

Dann existiert eine Funktion $f(x) \in C(\omega)$ *mit*

$$\limsup_{n\to\infty} \|f(x) - H_n(f;\, x, X)\| > 0.$$

O. KIS und J. SZABADOS haben in der Arbeit [6] (s. § 3. 1—3) neue Beweise bezüglich zwei Sätze von S. M. Losinski gegeben. Die Beweisgänge der Lemmata 3 und 4 sind diesselben wie jene von O. Kis und J. Szabados.

Es sei für $\{X\}$

$$\frac{\mu_n(X)}{\tilde{\nu}_n(X)} \geqq c_4 n.$$

Aus (19) folgt dann die Abschätzung

$$\mu_n(X) \geqq c_5 n \tilde{v}_n(X) \geqq c_6 \log n,$$

also ist die Bedingung (27) des Lemmas 3 erfüllt. Es gilt:

SATZ 4. *Es sei für* $\{X\}$ *die Bedingung* (21) *erfüllt. Dann ist die Relation*

$$\lim_{n\to\infty} \omega\left(\frac{1}{n}\right) \log n = 0$$

eine notwendige Bedingung der Konvergenz der Polynome $H_n(f; x, X)$.

Aus dem Satz 2 und aus dem Lemma 4 folgt:

SATZ 5. *Gelten für* $\{X\}$ *und* $\omega(t)$ (21) *und* (29), *so ist die Bedingung* (22) *für die Konvergenz der Polynome* $H_n(f; x, X)$ *notwendig und hinreichend.*

Es sei nun $\mu_n(X) \leqq c_6 \log n$; dann ist wegen (19)

$$\frac{\mu_n(X)}{\tilde{v}_n(X)} \leqq c_7 n.$$

Wir bekommem aus dem Lemma 4:

SATZ 6. *Gelten für* $\{X\}$ *und* $\omega(t)$ *die Beziehungen* (23) *und* (29), *so ist*

$$\lim_{n\to\infty} \omega\left(\frac{1}{n}\right) \mu_n(X) = 0$$

eine notwendige Bedingung der Konvergenz der Polynome $H_n(f; x, X)$.

Wir zeigen, daß Punktmatrizes mit der Eigenschaft (21) existieren.

P. VÉRTESI hat in der Arbeit [7] die folgende Punktmatrix definiert:

Es seien $r \geqq 0$, $s > 0$ ganze Zahlen, $s = \left[\frac{n+3}{2}\right]$, weiter

$$\{Y\} = \begin{cases} y_{ln} = \cos \dfrac{2l-1}{2n} \pi \\ y_{s+i,n} = \cos \left[\dfrac{2s+2r+1}{2n} - (r+1-i) \varrho_n\right] \pi, \quad i = 0, \dots, r, \ 0 < \varrho_n < \dfrac{1}{n}. \end{cases}$$

P. Vértesi hat bewiesen, daß

$$\mu_n(Y) \geqq \frac{c_8}{(n\varrho_n)^{2(r+1)}}$$

ist.

Nach den Beziehungen (5) und (9) und aus den Abschätzungen (4.1)—(4.17) der Arbeit [7] ergibt sich

$$\tilde{v}_n(Y) \leqq c_9 \left(\frac{\log n}{n} + \frac{1}{n} \frac{1}{(n\varrho_n)^{2(r+1)}}\right).$$

Für $0<\varrho_n<\dfrac{1}{n(\log n)^{\frac{1}{2(r+1)}}}$ ist nun

$$\tilde{v}_n(Y)\geqq c_{10}\frac{1}{n}\frac{1}{(n\varrho_n)^{2(r+1)}},$$

woraus die Abschätzung

$$\frac{\mu_n(Y)}{\tilde{v}_n(Y)}\geqq c_{11}n$$

folgt.

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TENSOR PRODUCTS OF DISTRIBUTIVE LATTICES AND THEIR PRIESTLEY DUALS

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0. Introduction

Tensor products of distributive lattices and of semilattices have been considered by several authors, namely in a series of papers by FRASER [3], [4], [5] and [6], but also in [7], [8] and [9]. Tensor products — defined as universal objects through which every bihomomorphism splits — exist in any variety of algebras. This note deals with tensor products in the variety **D** of distributive lattices and in the variety **S** of distributive join-semilattices. The former is called lattice tensor product and written $L_1 \otimes_L L_2$, the latter semilattice tensor product, written $S_1 \otimes_S S_2$. Existence and uniqueness are established in the usual way, taking suitable homomorphic images of the corresponding free structures generated by the cartesian product of the factors. See [3] and [4] for details.

It is somewhat surprising that for $L_1, L_2 \in \mathbf{D}$ their semilattice tensor product $L_1 \otimes_S L_2$ (obtained by considering the L_i as join-semilattices) is in fact even a (distributive) lattice (Theorem 2.6 in [3]). So there are two kinds of tensor product available within **D**. The main purpose of this note is to determine the relationship between $L_1 \otimes_L L_2$ and $L_1 \otimes_S L_2$. We find that $L_1 \otimes_S L_2$ is a quotient of $L_1 \otimes_L L_2$ modulo a simply characterizable congruence relation.

The main tool used is that of Priestley duality. As a byproduct, we obtain a representation of $L_1 \otimes_L L_2$ as a ring of sets which seems more manageable than that given by Corollary 2.5 in [4]. The paper is organized as follows: Section 1 contains a brief sketch of Priestley duality for distributive lattices without universal bounds. In Section 2 we determine the Priestley duals of free distributive lattices. These are used to characterize, in Section 3, the duals of lattice tensor products, which in turn gives the set ring representation mentioned above. In Section 4, the relationship between $L_1 \otimes_L L_2$ and $L_1 \otimes_S L_2$ is studied.

All lattices considered are distributive, and all semilattices are assumed to be join-semilattices. The general reference for lattice-theoretic concepts is [2]. **D** denotes the category of distributive lattices and lattice homomorphisms, $\mathbf{D}_{01}$ that of distributive lattices with universal bounds and homomorphisms preserving these bounds.

1. Priestley duals of general distributive lattices

The Priestley space $X(L)$ for any $L \in \mathbf{D}_{01}$ is described in detail in [10] and [11]. Both papers, however, touch only on the case of lattices lacking a zero or a unit or both.

Let $L \in \mathbf{D}$. We choose to construct $X(L)$ as a filter space. In detail: The carrier set of $X(L)$ consists of all proper nontrivial — i.e., different from L and $\varnothing$ —

prime filters on L. A topology is introduced on $X(L)$ by taking as an open subbase the empty set together with all sets of the form $\{A; x\in A\}$ or the form $\{A; x\notin A\}$, where x runs through L and A through $X(L)$. Under set inclusion $X(L)$ becomes an ordered space in the sense of Nachbin. See also § 10 of [11].

Given $L\in\mathbf{D}$, denote by L^b the lattice obtained from L by adjoining new elements as a zero and a unit to L, respectively (even if L already has such elements). Hence $L^b\in\mathbf{D}_{01}$. Writing $L^b=L\cup\{0, 1\}$, one checks readily that $X(L^b)=$ $=\{A\cup\{1\};\ A\in X(L)\}\cup\{L\cup\{1\}\}\cup\{\{1\}\}$. Moreover, $j\colon X(L)\to X(L^b)$ defined by $jA=A\cup\{1\}$ for $A\in X(L)$ embeds $X(L)$ as an order subspace into $X(L^b)$.

The dual lattice of L is now described most easily in terms of L^b. Since $L^b\in\mathbf{D}_{01}$, L^b is canonically isomorphic with the lattice of all clopen increasing subsets of $X(L^b)$. It follows at once that L is isomorphic with the lattice of all proper nonempty such sets. Actually, these sets may be characterized topologically independently from $X(L^b)$, see e.g. Proposition 25 of [11] for one of the cases arising, put such a description is not needed in what follows.

2. Free distributive lattices

The description of Priestley duals of free distributive lattices rests on the well-known structure theorems for the posets of prime filters of such lattices. See [1] and [2] for convenient summaries of the facts we use. For technical reasons we prefer to work with filters rather than with ideals.

Let G be some nonempty set. $F(G)$ stands for the free distributive lattice generated by G, and $F_{01}(G)$ for the corresponding free lattice in $\mathbf{D}_{01}$. It is well known that $F_{01}(G)\cong F(G)^b$. Denote, for any set G, its power set by PG and put $P'G=PG\setminus\{\varnothing, G\}$.

LEMMA 1. *As posets,* $X(F(G))\cong(P'G, \subseteqq)$ *and* $X(F_{01}(G))\cong(PG, \subseteqq)$.

PROOF. It is well known that a filter $A\subseteqq F(G)$ is prime iff A is generated by some $H\in P'G$. Let $g\in A\cap G$. It follows that $g\geqq g_1\wedge\ldots\wedge g_n$, $g_1, \ldots, g_n\in H$. Since G is a set of free generators for $F(G)$, this implies $g\in H$. Hence $A\cap G\subseteqq H$. The reverse inclusion is trivial, so $H=A\cap G$. The map $A\to A\cap G$ gives the order isomorphism required in the first half of the lemma. It follows (cf. Section 1) that $A\mapsto(A\cap G)\cup\{1\}$ is an isomorphism between $X(F_{01}(G))$ and PG.

We turn now to the topologies on $X(F(G))$ and $X(F_{01}(G))$, identifying the respective carrier sets with $P'G$ and PG, respectively.

LEMMA 2. $Z\subseteqq PG$ *is clopen increasing under Priestley topology iff* Z *belongs to the ring of sets generated by all sets of the form* $\{H\in PG;\ x\in H\cup\{1\}\}$, *where* x *runs through* $G\cup\{0, 1\}$.

PROOF. Clearly, $G\cup\{0, 1\}$ generates $F_{01}(G)$, and so the sets $\{A\in X(F_{01}(G));$ $x\in A\}$ and $\{A\in X(F_{01}(G));\ x\notin A\}$ with $x\in G\cup\{0, 1\}$ already constitute an open subbase for the Priestley topology (cf. Section 1). These sets are actually clopen, and since $X(F_{01}(G))$ is compact, a subset of this space is clopen iff it belongs to the set ring generated by the above family of sets. Using the fact that $X(F_{01}(G))$ is totally order disconnected, it is easy to check that a clopen set is increasing iff it belongs to the subring generated by all sets $\{A; x \in A\}$. The result now follows by the isomorphism given in the proof of Lemma 1.

LEMMA 3. *$F(G)$ is isomorphic to the ring of sets generated by all sets of the form $\{H \in PG; x \in H\}$ where x runs through G.*

PROOF. Using Lemma 2, $\{H \in PG, 0 \in H \cup \{1\}\} = \emptyset$ and $\{H \in PG; 1 \in H \cup \{1\}\} = PG$, and these are the zero and unit elements, respectively, of the dual lattice of $F_{01}(G)$. The lemma follows from $F_{01}(G) \cong F(G)^b$.

3. Lattice tensor products

Under Priestley duality, taking homomorphic images of lattices corresponds to passing to order subspaces. If $L \in \mathbf{D}_{01}$ and ϑ is a congruence on L, then $X(L/\vartheta)$ is a subspace of $X(L)$. $X(L/\vartheta)$ consists exactly of those prime filters on L which are set unions of whole ϑ-classes — for short, which are ϑ-closed.

Let $L_1, L_2 \in \mathbf{D}$. $L_1 \otimes_L L_2$ is constructed as usual as $F(L_1 \times L_2)$ modulo the smallest congruence which identifies

$$(1.1) \qquad (a, b_1 \vee b_2) \text{ with } (a, b_1) \vee (a, b_2)$$

$$(1.2) \qquad (a, b_1 \wedge b_2) \text{ with } (a, b_1) \wedge (a, b_2)$$

$$(1.3) \qquad (a_1 \vee a_2, b) \text{ with } (a_1, b) \vee (a_2, b)$$

$$(1.4) \qquad (a_1 \wedge a_2, b) \text{ with } (a_1, b) \wedge (a_2, b)$$

for a, a_1, a_2 in L_1 and b, b_1, b_2 in L_2.

We denote this congruence with ϑ. ϑ may be extended to a congruence $\bar{\vartheta}$ on $F_{01}(L_1 \times L_2)$ by taking $\{0\}$ and $\{1\}$ as additional congruence classes. It follows that $F_{01}(L_1 \times L_2)/\bar{\vartheta} \cong (F(L_1 \times L_2)/\vartheta)^b$.

We set out to determine the $\bar{\vartheta}$-closed prime filters of $F_{01}(L_1 \times L_2)$. Recall (Lemma 1) that a filter $A \subseteq F_{01}(L_1 \times L_2)$ is prime iff A is generated by $H \cup \{1\}$, $H \subseteq L_1 \times L_2$.

LEMMA 4. *A prime filter $A \subseteq F_{01}(L_1 \times L_2)$ is $\bar{\vartheta}$-closed iff A is generated by $H \cup \{1\}$, where $H \subseteq L_1 \times L_2$ satisfies*

$$(2.1) \qquad (a, b_1 \vee b_2) \in H \text{ iff } (a, b_1) \in H \text{ or } (a, b_2) \in H$$

$$(2.2) \qquad (a, b_1 \wedge b_2) \in H \text{ iff } (a, b_1) \in H \text{ and } (a, b_2) \in H$$

$$(2.3) \qquad (a_1 \vee a_2, b) \in H \text{ iff } (a_1, b) \in H \text{ or } (a_2, b) \in H$$

$$(2.4) \qquad (a_1 \wedge a_2, b) \in H \text{ iff } (a_1, b) \in H \text{ and } (a_2, b) \in H.$$

PROOF. Assume A is of the type considered. Elements of $F_{01}(L \times L_2)$ are lattice polynomials over $L_1 \times L_2$, together with $0, 1$. We define a binary relation ϑ' on $F_{01}(L_1 \times L_2)$ as follows: $0\vartheta'0$, $1\vartheta'1$ and for w, z $F(L_1 \times L_2)$ we put $w\vartheta'z$ iff there is a finite sequence $w = x_0, x_1, \ldots, x_{n-1}, x_n = z$, $x_i \in F(L_1 \times L_2)$ such that for $i=1, \ldots, n$ x_i arises from x_{i-1} by application of a single substitution of one of the following types

$$(3.1) \qquad (a, b) \leftrightarrow (a, b_1)\vee(a, b_2) \quad \text{provided} \quad b = b_1 \vee b_2$$

$$(3.2) \qquad (a, b) \leftrightarrow (a, b_1)\wedge(a, b_2) \quad \text{provided} \quad b = b_1 \wedge b_2$$

$$(3.3) \qquad (a, b) \leftrightarrow (a_1, b)\vee(a_2, b) \quad \text{provided} \quad a = a_1 \vee a_2$$

$$(3.4) \qquad (a, b) \leftrightarrow (a_1, b)\wedge(a_2, b) \quad \text{provided} \quad a = a_1 \wedge a_2.$$

It is obvious that ϑ' is a congruence on $F_{01}(L_1\times L_2)$. Moreover, the restriction of ϑ' to $F(L_1\times L_2)$ satisfies (1.1)—(.14), hence $\bar{\vartheta}\subseteqq\vartheta'$. The reverse inclusion is trivial and thus $\bar{\vartheta}=\vartheta$. Now let $z\in A$. We propose to show that any of the substitutions (3.1)—(3.4) applied to z produces a member of A. Hence $z\in A$ and $z\vartheta' w$ together imply $w\in A$, that is, A is ϑ'-closed and thus $\bar{\vartheta}$-closed.

We use induction on the ranks of the polynomials involved in such substitutions: Let $(a, b)\in A$ with $b=b_1\vee b_2$. Then $(a, b)\in H$ (see the proof of Lemma 1) and by (2.1) $(a, b_1)\in H$ or $(a, b_2)\in H$. Consequently, $(a, b_1)\vee(a, b_2)\in H\subseteqq A$. Conversely, let $(a, b_1)\vee(a, b_2)\in A$. Since A is prime, $(a, b_1)\in A$ or $(a, b_2)\in A$. Again, it follows that $(a, b_1)\in H$ or $(a, b_2)\in H$ and thus by (2.1) $(a, b_1\vee b_2)\in H\subseteqq A$. Similar arguments work for (3.2)—(3.4).

For the induction step, let $p(x_1, \ldots, x_n)\in A$, $x_i\in L_1\times L_2$, p a polynomial. If $p=p_1\vee p_2$, then, say, $p_1(x_1, \ldots, x_n)\in A$ since A is prime. So application of any of (3.1)—(3.4) keeps $p_1(x_1, \ldots, x_n)$ in A by induction hypothesis, thus also $p(x_1, \ldots, x_n)$. A similar argument works for $p=p_1\wedge p_2$.

Necessity of (2.1)—(2.4) is immediate, so the proof is complete.

To get a hold on $L_1\otimes_L L_2$, we need a workable description of the sets $H\subseteqq L_1\times L_2$ satisfying (2.1)—(2.4). We introduce the following notation: For $H\subseteqq L_1\times L_2$ and $(a, b)\in H$, put $H_1(a, b)=\{x\in L_1;\ (x, b)\in H\}$ and $H_2(a, b)=\{y\in L_2;\ (a, y)\in H\}$.

LEMMA 5. *$H\subseteqq L_1\times L_2$ satisfies (2.1)—(2.4) iff for any $(a, b)\in H$, $H_1(a, b)$ and $H_2(a, b)$ are prime filters in L_1 and L_2, respectively. H is then a subdirect product of two prime filters $A_1\subseteqq L_1$, $A_2\subseteqq L_2$.*

PROOF. Assume H satisfies (2.1)—(2.4). Let $x_1, x_2\in H_1(a, b)$. Hence $(x_1, b)\in H$ and $(x_2, b)\in H$ and by (2.4) $(x_1\wedge x_2, b)\in H$, that is, $x_1\wedge x_2\in H_1(a, b)$. If $x_1\in H_1(a, b)$ and $x_2\geqq x_1$, consider $x_2\wedge x_1=x_1$ and apply (2.4) to obtain $x_2\in H_1(a, b)$. If $x_1\vee x_2\in H_1(a, b)$, apply (2.3) to obtain $x_1\in H_1(a, b)$ or $x_2\in H_1(a, b)$. So $H_1(a, b)$ is a prime filter, and so is $H_2(a, b)$, using (2.1) and (2.2).

Conversely, assume that $H_1(a, b)$ and $H_2(a, b)$ are prime filters for any $(a, b)\in H$. Now $(a, b_1\vee b_2)\in H$ iff $b_1\vee b_2\in H_2(a, b_1\vee b_2)$ iff $b_1\in H_2(a, b_1\vee b_2)$ or $b_2\in H_2(a, b_1\vee b_2)$ iff $(a, b_1)\in H$ or $(a, b_2)\in H$; and similarly $(a, b_1\wedge b_2)\in H$, establishing (2.1) and (2.2). Dually for (2.3) and (2.4).

Let $A_1=\bigcup\{H_1(a, b);\ (a, b)\in H\}$. Consider $x_1, x_2\in A_1$. Hence $(x_1, b_1)\in H$ and $(x_2, b_2)\in H$ for suitable $b_1, b_2\in L_2$. We infer that $(x_1, b_1\vee b_2)\in H$ and $(x_2, b_1\vee b_2)\in H$ and thus $(x_1\wedge x_2, b_1\vee b_2)\in H$, that is, $x_1\wedge x_2\in A_1$. So A_1 is closed under meets, and the remaining properties of a prime filter are trivially satisfied. Defining $A_2=\bigcup\{H_2(a, b);\ (a, b)\in H\}$, H is a subdirect product of A_1 and A_2.

DEFINITION. $H\subseteqq L_1\times L_2$ is called *doubly prime* iff H satisfies (2.1)—(2.4). Put $D=\{H\subseteqq L_1\times L_2;\ H \text{ doubly prime}\}$.

The Priestley space of $L_1 \otimes_L L_2$ may now be described in terms of doubly prime sets:

PROPOSITION 6. *$X(L_1 \otimes_L L_2)$ is order homeomorphic to $(P'(L_1 \times L_2) \cap D, \subseteq)$, the latter space carrying the subspace topology induced by $X(F_{01}(L_1 \times L_2))$.*

PROOF. Combine Lemmata 1 and 2 with the results discussed in Section 1.
Our main goal here is however the representation of $L_1 \otimes_L L_2$ as a ring of sets:

THEOREM 7. *$L_1 \otimes_L L_2$ is isomorphic to the ring of sets generated by all sets of the form $\{H \in D; x \in H\}$ where x runs through $L_1 \times L_2$.*

PROOF. As for Lemma 3, using Proposition 6.

EXAMPLE. Let C_1 and C_2 be chains. We propose to show that $H \subseteq C_1 \times C_2$ is doubly prime iff H is an increasing subset of the direct product $C_1 \times C_2$.

Indeed, every doubly prime set is increasing by Lemma 5. Conversely, let $H \subseteq C_1 \times C_2$ be increasing and consider $(a, b) \in H$. If $c \in H_2(a, b)$, then $(a, c) \in H$ and so for any $c' \geqq c$, $(a, c') \in H$ since H is increasing. Consequently, $H_2(a, b)$ is increasing, and since C_2 is a chain, it is even a filter. But in a chain every filter is prime. Analogously for $H_1(a, b)$.

It follows that $C_1 \otimes_L C_2$ is isomorphic to the set ring generated by the families of increasing subsets of $C_1 \times C_2$ fixed by some element of $C_1 \times C_2$.

4. Semilattice vs. lattice tensor product

Basing on Theorem 3.5 of [3] it is easy to describe the Priestley space of $L_1 \otimes_S L_2$ (the presence of zeros as required there is not essential): $X(L_1 \otimes_S L_2)$, as a poset, is order isomorphic with $(X(L_1) \cup \{L_1\}) \times (X(L_2) \cup \{L_2\}) \setminus \{(L_1, L_2)\}$. Consider such (A_1, A_2): It is readily verified that $A_1 \times A_2$ is doubly prime, and since $(A_1, A_2) \neq (L_1, L_2)$, $(A_1, A_2) \in P'(L_1 \times L_2) \cap D$. Priestley topology is introduced as usual, and it follows that $f\colon X(L_1 \otimes_S L_2) \to X(L_1 \otimes_L L_2)$ defined by $(A_1, A_2) \mapsto A_1 \times A_2$ (using Proposition 6 and the order isomorphism given above) is an embedding of ordered spaces. Put $D_1 = \{H \in D;\ H = A_1 \times A_2 \text{ with } A_i \in X(L_i) \cup \{L_i\}\} \subseteq D$. Using the duality between order subspaces and homomorphic images of lattices, we may sum up as follows:

THEOREM 8. *$X(L_1 \otimes_S L_2)$ is order homeomorphic with $(P'(L_1 \times L_2) \cap D_1, \subseteq)$, the latter space carrying the subspace topology induced by $X(F_{01}(L_1 \times L_2))$.*

$L_1 \otimes_S L_2$ is isomorphic to the set ring generated by all sets of the form $\{H \in D_1; x \in H\}$ where x runs through $L_1 \times L_2$. $L_1 \otimes_S L_2$ is a homomorphic image of $L_1 \otimes_L L_2$, a canonical epimorphism being given by $\{H \in D; x \in H\} \mapsto \{H \in D_1; x \in H\}$.

PROOF. Proposition 6 and Theorem 7.

To get a more explicit relationship between the two tensor products, we introduce the following definition: Let η be the smallest congruence on $L_1 \otimes_L L_2$ which identifies $(a_1 \otimes b_1) \wedge (a_2 \otimes b_2)$ with $(a_1 \wedge a_2) \otimes (b_1 \wedge b_2)$, where $a_1, a_2 \in L_1$, $b_1, b_2 \in L_2$ and $a \otimes b$ stands for the ϑ-class of (a, b).

PROPOSITION 9. $L_1 \otimes_S L_2 \cong L_1 \otimes_L L_2 / \eta$.

PROOF. Note that $L_1 \otimes_L L_2/\eta \cong F(L_1 \times L_2)/\vartheta \vee \eta$. The right-hand side may be constructed as in Section 3, by characterizing those $\bar{\vartheta}$-closed prime filters in $F_{01}(L_1 \times L_2)$ which are also $\bar{\eta}$-closed (where $\bar{\eta}$ extends η as $\bar{\vartheta}$ extends ϑ). These are the filters generated in $F_{01}(L_1 \times L_2)$ by sets $H \cup \{1\}$, where $H \in D$ and H satisfies also

$$(4) \qquad (a_1, b_1) \in H \quad \text{and} \quad (a_2, b_2) \in H \quad \text{iff} \quad (a_1 \wedge a_2, b_1 \wedge b_2) \in H.$$

The proof follows that of Lemma 4 and will be omitted.

Let $H \in D$ satisfy (4), and consider $x \in A_1 = \bigcup \{H_1(a, b);\ (a, b) \in H\}$ and $y \in A_2 = \bigcup \{H_2(a, b);\ (a, b) \in H\}$. Hence $(x, b) \in H$ and $(a, y) \in H$ for some $a \in L_1$, $b \in L_2$. By (4) $(a \wedge x, b \wedge y) \in H$. But H is doubly prime, hence increasing, thus $(x, y) \in H$. It follows that $H = A_1 \times A_2$, that is, $H \in D_1$. Conversely, every $H \in D_1$ obviously satisfies (4), so the Priestley spaces of $F(L_1 \times L_2)/\vartheta \vee \eta$ and of $L_1 \otimes_S L_2$ are order homeomorphic.

In other words, $L_1 \otimes_S L_2$ is the largest image of $L_1 \otimes_L L_2$ in which the rule $(a_1 \otimes b_1) \wedge (a_2 \otimes b_2) = (a_1 \wedge a_2) \otimes (b_1 \wedge b_2)$ holds.

COROLLARY 10. *$L_1 \otimes_L L_2 \cong L_1 \otimes_S L_2$ iff at least one factor is trivial.*

PROOF. If one factor is trivial, every $H \in D$ satisfies (4). Conversely, let $a_1 > a_2$, $b_1 > b_2$, $a_i \in L_1$, $b_i \in L_2$. Choose prime filters $A_1 \subseteq L_1$, $A_2 \subseteq L_2$ such that $a_1 \in A_1$, $a_2 \notin A_1$, and $b_1 \in A_2$, $b_2 \notin A_2$. Then $(A_1 \times L_2) \cup (L_1 \times A_2)$ is doubly prime and thus ϑ-closed. Clearly, $(a_1, b_2), (a_2, b_1) \in H$. But $(a_1 \wedge a_2, b_1 \wedge b_2) = (a_2, b_2) \notin H$, and H is not η-closed.

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THE EDGE INDUCIBILITY OF GRAPHS

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1. Introduction

All graphs will be undirected, with no edge from a vertex to itself, no multiple edges and *no isolated vertices*. Let $G=(V, E)$ be a graph. A subset of vertices $A \subseteq V$ is said to *induce* the subgraph $G_A=(A, E_A)$ where $E_A=\{ab \in E | a, b \in A\}$. If A is the collection of vertices spanned by a subset of edges $F \subseteq E$, then we also say that F induces G_A. We sometimes call G an *(n, m)-graph* when $n=|V|$ and $m=|E|$.

Let G be an (n, m)-graph and H be a (p, q)-graph with $n \leqq p$ and $m \leqq q$. As in [6] we define $\mathscr{I}(G, H)$ to be the number of induced subgraphs of H that are isomorphic to G (where an induced subgraph is counted at most once, even if it is isomorphic in several ways). More formally, if X is the vertex set of H, then

$$\mathscr{I}(G, H) = \text{card}\, \{A \subseteq X | H_A \cong G\}.$$

This number lies between 0 and $\min\left\{\binom{p}{n}, \binom{q}{m}\right\}$; we normalize it by setting

$$I_e(G, H) = \mathscr{I}(G, H) \Big/ \binom{q}{m},$$

obtaining a number that lies between 0 and 1. We further define

$$\mathscr{I}_e(G, q) = \max\, \{\mathscr{I}(G, H) | H \text{ has } q \text{ edges}\},$$

$$I_e(G, q) = \mathscr{I}_e(G, q) \Big/ \binom{q}{m}.$$

We use the subscript e to emphasize that we are talking about *edge inducibility* of G in H as distinguished from what could be called the *vertex inducibility* which was discussed in PIPPENGER and GOLUMBIC [6]. The vertex inducibility is computed by maximizing $\mathscr{I}(G, H)$ over all p-vertex graphs and normalizing by $\binom{p}{n}$, namely,

$$\mathscr{I}_v(G, p) = \max\, \{\mathscr{I}(G, H) | H \text{ has } p \text{ vertices}\}, \quad I_v(G, H) = \mathscr{I}(G, H) \Big/ \binom{p}{n},$$

$$I_v(G, p) = \mathscr{I}_v(G, p) \Big/ \binom{p}{n}, \quad I_v(G) = \lim_{p \to \infty} I_v(G, p).$$

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These two notions are similar, yet surprisingly different. In addition, Galvin and Straus [unpublished] have examined a type of inducibility of permutations, and Golumbic [3] has studied the vertex inducibility of hypergraphs.

The following results were presented in [6] for the vertex inducibility of graphs, but their proofs can be easily modified for edge inducibility.

Proposition 1. *For any m-edge graph G and any $q \geqq m$,*

$$I_e(G, q+1) \leqq I_e(G, q).$$

Proposition 2. *For any m-edge graph G, m'-edge graph G', and q-edge graph H, where $m \leqq m' \leqq q$, we have $I_e(G, H) \geqq I_e(G, G') I_e(G', H)$.*

Proposition 1 shows that the sequence $I_e(G, q)$ is non-increasing, and since it is bounded below by zero, it converges to a definite limit as $q \to \infty$. We define $I_e(G) = \operatorname{limit}_{q \to \infty} I_e(G, q)$.

The invariant $I_e(G)$, and by abuse of language, $I_e(G, q)$ and $\mathscr{I}_e(G, q)$, we call the *edge inducibility* of G. A q-edge graph H *maximizes the edge inducibility of* G if $I_e(G, H) = I_e(G, q)$.

Consider the *star graph* $K_{1,m}$. For all $q \geqq m$, $\mathscr{I}(K_{1,m}, K_{1,q}) = \binom{q}{m}$, hence $I_e(K_{1,m}, q) = 1$ and $I_e(K_{1,m}) = 1$. Similarly, the graph mK_2, consisting of m disjoint copies of K_2, has edge inducibility $I_e(mK_2) = 1$. These are the only graphs having this property (see Theorem 6).

By choosing H in Proposition 2 to maximize the edge inducibility of G' and then passing to the limit as $q \to \infty$, we obtain the following:

Corollary 3. *For any m-edge graph G and m'-edge graph G', where $m \leqq m'$, we have $I_e(G) \geqq I_e(G, G') I_e(G')$.*

2. The behavior of the edge inducibility $I_e(G)$

Theorem 4. *For any graph G, $I_e(G) > 0$ if and only if G is the union of disjoint star graphs.*

Proof. If G is not the union of disjoint star graphs, then it must contain an induced subgraph F isomorphic to one of the graphs K_3, P_4 of C_4 (see Figure 1).

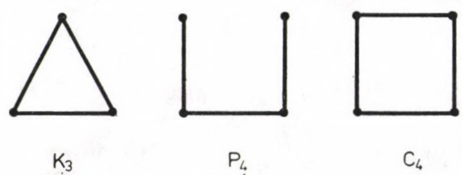

Fig. 1

By Corollary 3, $I_e(F) \geqq I_e(F, G) I_e(G)$, and since $I_e(F, G)$ in nonzero, it is sufficient to show that $I_e(F) = 0$.

Let H be a q-edge graph. An arbitrary choice of two edges of H may induce at most one copy of F, and every copy of F can be found in this way (twice for C_4 and thrice for K_3). In fact,

$$3\mathscr{I}(K_3, H)+2\mathscr{I}(C_4, H)+\mathscr{I}(P_4, H) \leqq \binom{q}{2}.$$

Thus, $\mathscr{I}(F, H)\leqq\binom{q}{2}$ implies that $I_e(F, q)\leqq\binom{q}{2}\Big/\binom{q}{r}$ where $r=3$ if $F\cong K_3$ or $F\cong P_4$ and $r=4$ if $F\cong C_4$. Taking the limit as $q\to\infty$ we have $I_e(F)=0$.

Conversely, suppose $G\cong K_{1,m_1}\cup\ldots\cup K_{1,m_t}$, $m=m_1+\ldots+m_t$. For any $s>0$ consider the sm-edge graph $H\cong K_{1,sm_1}\cup\ldots\cup K_{1,sm_t}$ consisting of the union of t disjoint stars of sizes proportional to those of G. Clearly

$$\mathscr{I}(G, H) \geqq \binom{sm_1}{m_1}\binom{sm_2}{m_2}\cdots\binom{sm_t}{m_t}$$

with equality if and only if the m_i are all equal. Dividing both sides of the inequality by $\binom{sm}{m}$ we obtain

$$I_e(G, sm) \geqq I_e(G, H) \geqq \left[\prod_i \binom{sm_i}{m_i}\right]\Big/\binom{sm}{m} > \frac{m!}{m^m}\prod_i\left(\frac{sm_i-m_i}{s}\right)^{m_i}\frac{1}{m_i!}.$$

Passing to the limit as $s\to\infty$, we obtain $I_e(G)\geqq\dfrac{m!}{m^m}\prod_i\dfrac{m_i^{m_i}}{m_i!}$ and Theorem 4 is proved.

The line graph $L(G)$ of a graph G is constructed as follows: The vertices of $L(G)$ correspond to the edges of G, and two vertices in $L(G)$ are connected if their corresponding edges in G share a common vertex. For any m-edge graph G and q-edge graph H, where $q\geqq m$,

(1) $$\mathscr{I}(G, H) \leqq \mathscr{I}(L(G), L(H)).$$

This follows from the that the edges of a copy of G in H induce one copy of $L(G)$ in $L(H)$, and different edge sets of H give different subgraphs of $L(H)$. By choosing H to maximize the edge inductibility of G in (1) and then dividing by $\binom{q}{m}$, we have $I_e(G, q)\leqq I_v(L(G), q)$. Taking the limit as $q\to\infty$ gives

(2) $$I_e(G) \leqq I_v(L(G)).$$

For most graphs (2) is too weak to be useful, since $I_e(G)$ is usually zero and $I_v(L(G))$ is always bounded away from zero. However, if G consists of two disjoint stars differing in size by at most one, we can calculate exactly its edge inductibility using a known result on vertex inductibility.

THEOREM 5. *Let* $\mu=\lfloor m/2\rfloor$ *and* $\psi=\lfloor q/2\rfloor$ *where* $3\leqq m\leqq q$.

$$\mathscr{I}_e(K_{1,\mu}\cup K_{1,m-\mu}, q) = \begin{cases} \binom{\psi}{\mu}\binom{q-\psi}{\mu} & m \text{ even} \\ \binom{\psi}{\mu}\binom{q-\psi}{\mu+1}+\binom{q-\psi}{\mu}\binom{\psi}{\mu+1} & m \text{ odd.}\end{cases}$$

Thus,

$$I_e(K_{1,\mu} \cup K_{1,m-\mu}) = \binom{m}{\mu}\Big/ 2^{2\mu}.$$

PROOF. The expression for $\mathscr{I}_e(K_{1,\mu} \cup K_{1,m-\mu}, q)$ in the theorem is easily seen to be equal to $\mathscr{I}(K_{1,\mu} \cup K_{1,m-\mu}, K_{1,\psi} \cup K_{1,q-\psi})$ and is therefore a lower bound. Since the line graph of $K_{1,\mu} \cup K_{1,m-\mu}$ is the graph $K_\mu \cup K_{m-\mu}$ consisting of two disjoint complete graphs, it follows from (2) and [6, Proposition 4 and Theorem 10] that the expression is also an upper bound.

By a similar proof technique, one can easily show the following.

THEOREM 6. *Let G be an m-edge graph. Then $I_e(G)=1$ if and only if $G=mK_2$ or $G=K_{1,m}$.*

3. Maximizing the edge inducibility of complete graphs

For a graph G we may well ask for which q-edge graph H does $\mathscr{I}(G, H) = \mathscr{I}_e(G, q)$, even if $I_e(G)=0$. In this section we show which graphs maximize the edge inducibility of the complete graph.

A graph G is *almost complete* if there is a vertex v such that $G-v$ is complete. The vertex v is called the *deficient* vertex, and the *deficiency* of G equals the number of vertices not adjacent to v (excluding v itself). The almost complete graph with $p+1$ vertices and deficiency r is denoted by $K_{p+1}^{[r]}$. Thus, $K_{p+1}^{[0]} = K_{p+1}$ and $K_{p+1}^{[p]} = K_p \cup K_1$. The graph $K_{p+1}^{[r]}$ has $\binom{p+1}{2} - r$ edges.

THEOREM 7. *The almost complete graph with q edges maximizes the edge inducibility of K_n. That is, writing $q = \binom{p+1}{2} - r$ with $0 \leqq r < p$, we have*

$$\mathscr{I}_e(K_n, q) = \binom{p}{n} + \binom{p-r}{n-1}.$$

PROOF. In $K_{p+1}^{[r]}$ there are $\binom{p}{n}$ copies of K_n which do not include the deficient vertex and $\binom{p-r}{n-1}$ copies of K_n which do include the deficient vertex. This gives, $\binom{p}{n} + \binom{p-1}{n-1} = \mathscr{I}(K_n, K_{p+1}^{[r]}) \leqq \mathscr{I}(K_n, q)$.

Let us assume that the almost complete graph with q'-edges maximizes the edge inducibility of K_n for all $q' < q$ which is certainly true for $q' = \binom{n}{2}$. Let $H=(V, E)$ be a q-edge graph such that $\mathscr{I}(K_n, H) = \mathscr{I}_e(K_n, q)$. We will show that H is almost complete. Assume that H is not complete, for otherwise we are done.

Choose a vertex x of H whose degree d_x is smallest possible. Clearly, $\mathscr{I}(K_n, H) = \mathscr{I}(K_n, H-x) + \mathscr{I}(K_{n-1}, H_{\mathrm{Adj}(x)})$. Since $\binom{d_x}{2} \leqq q - d_x$, we may assume by induction (and moving the edges of $H-x$ if necessary) that $H-x$ is almost complete in such a way that its deficient vertex y (of degree d_y) is not adjacent to x. Hence $C=V-\{x, y\}$ is complete.

If either d_x or d_y equals $|C|$, then H is almost complete. Suppose that $0<d_x\leqq d_y<|C|$, renaming x and y if necessary. Erase an edge adjacent to x and add an edge between y and any member of $C-\mathrm{Adj}\,(y)$. The net increase in the number of copies of K_n is

$$\left[\binom{d_x-1}{n-1}+\binom{d_y+1}{n-1}\right]-\left[\binom{d_x}{n-1}+\binom{d_y}{n-1}\right]=\binom{d_y}{n-2}-\binom{d_x-1}{n-2}$$

which is strictly greater than zero for $n-2\leqq d_y$, implying that H does not maximize the edge inducibility, a contradiction. Hence, H must be almost complete.

REMARK. Theorem 7 may be regarded as a particular instance of the KRUSKAL—KATONA Theorem, which first appeared in [5] and was rediscovered in [4]. A short proof can be found in [1]. We are indebted to G. Katona for mentioning this fact to us.

4. A comparison of two general construction techniques

Let G be an (n, m)-graph and let r be a positive integer. Consider the graph G^{*r} obtained from G by multiplying each vertex by r; i.e., we replace each vertex v of G by an r-set of new vertices $v_1, v_2, \dots, v_r$ and we connect v_i with w_j by an edge for $i, j=1, 2, \dots, r$ whenever v and w are adjacent in G. The graph G^{*r} has nr vertices, mr^2 edges and at least r^n copies of G, obtained by choosing one vertex from each of the r-sets. Equality holds for the cycles C_n $(n\geqq 5)$. Thus, C_5^{*698} has 2,436,020 edges and 1.6568×10^{14} copies of C_5.

THEOREM 8. *For any (n, m)-graph G and any $q\geqq m$, $\mathcal{I}_e(G, q)\geqq\lfloor\sqrt{q/m}\rfloor^n$.*

PROOF. Let $mr^2\leqq q<m(r+1)^2$ where r is an integer. Clearly $r=\lfloor\sqrt{q/m}\rfloor$. By the multiplication technique, $\mathcal{I}_e(G, q)\geqq\mathcal{I}_e(G, mr^2)\geqq\mathcal{I}(G, G^{*r})\geqq r^n$, which proves the theorem.

The process of repeated composition of a graph with itself was used in [6] to obtain lower bounds for the vertex inducibility. Let $G^{(t)}$ be the result of composing G with itself $t-1$ times; i.e., $G^{(1)}=G$ and $G^{(t)}=G[G^{(t-1)}]$. Applied to an arbitrary (n, m)-graph G, one can easily show by induction that $G^{(t)}$ has n^t vertices, $mn^{t-1}(n^t-1)/(n-1)$ edges and at least $n^{n(t-1)}(1-n^{(1-n)t})/(1-n^{1-n})$ copies of G, (see [6, p. 195]). Again, equality holds for certain graphs, including the cycles C_n $(n\geqq 5)$. So, for example, $C_5^{(5)}$ has 2,440,625 edges and 9.552×10^{13} copies of C_5. Thus, the multiplication technique is marginally better than composition for the example. We can show that multiplication always gives a better lower bound in the case of edge inducibility for an arbitrary graph.

PROPOSITION 9. *If $mr^2=mn^{t-1}(n^t-1)/(n-1)$ and $n>2$, then*

$$r^n>n^{n(t-1)}(1-n^{(1-n)t})/(1-n^{1-n}).$$

PROOF. On one hand, the hypothesis implies that

$$r^n = (r^2)^{n/2} = n^{n(t-1)/2}[1+n+n^2+\ldots+n^{t-1}]^{n/2} >$$

$$> n^{n(t-1)/2} \sum_{i=0}^{t-1} n^{(t-1-i)n/2} = n^{n(t-1)} \sum_{i=0}^{t-1} n^{-in/2}.$$

On the other hand,

$$L = n^{n(t-1)}(1-n^{(1-n)t})/(1-n^{1-n}) = n^{n(t-1)} \sum_{i=0}^{t-1} n^{i(1-n)}.$$

But, $n^{-in/2} > n^{i(1-n)}$ for $i=0, \ldots, t-1$ since both exponents are negative. Thus, $r^n > L$.

5. Conclusions

The multiplication technique has given us a lower bound on maximum number of copies of an arbitrary graph G that we can expect by "spending" q edges. We believe that this is best possible for the graphs C_n $(n>5)$, but are as yet unable to prove it. We also believe that C^{*r}_{2v+1} is the best graph in which to embed the path P_{2v} on $2v$ vertices for $v \geqq 2$ and large values of r.

For some graphs G, like the complete graphs (Theorem 7), $\mathscr{I}_e(G, q)$ is of the same order of magnitude as the bound in Theorem 8, namely $O(q^{n/2})$. On the other hand, Theorem 4 says that $\mathscr{I}_e(G, q) = O(q^m)$ iff G is a union of stars. Future research is needed for determining the growth rate of $\mathscr{I}_e(G, q)$ for other graphs. Some preliminary results for lopsided bipartite graphs are reported in [2].

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ON THE RADICAL CLASSES AND THE TRANSFREE-IMAGES OF RINGS

By

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1. The purpose of this note is to consider the relation between the transfree-images and the radical classes in category of associative rings. This stays in connection to the solution of problem 8 of [2].

The notion of transfree-images is dual to that of subdirect embedding (cf. [3]). An object A of the category $\mathscr{C}$ is said to be a transfree-image of the free product $\coprod_{i\in I} A_i(\varphi_i)$ if there exists an epimorphism $\gamma\colon \coprod_{i\in I} A_i(\varphi_i)\to A$ such that all maps $\gamma\colon \varrho_i\colon A_i\to A$, $i\in I$ are normal monomorphisms.

Instead of a transfree-image of the free product $\coprod_{i\in I} A_i(\varrho_i)$ we speak of a transfree-image of the objects A_i, $i\in I$.

A class M of rings is said to be a radical class in sense of Amitsur and Kurosh if the following conditions are satisfied:

(i) M is homomorphically closed.

(ii) The sum of all M-ideals of a ring A is an M-ideal.

(iii) M is closed under extensions, that is if B and $A/B\in M$ then also $A\in M$.

The lower radical class defined by the class M is the smallest radical class containing M.

2. Assume that the ring A is a transfree-image of rings A_i, $i\in I$, by an epimorphism γ. Following the definition all maps $\gamma\colon \varrho_i\colon A_i\to A$, $i\in I$ are normal monomorphisms, so they are embeddings and their images are ideals of the ring A.

Let M be an arbitrary abstract class of rings. We put

$$T_r(M)=\{A|A \text{ is a transfree-image of some } M\text{-rings}\}.$$

LEMMA 1. *Every nonzero $T_r(M)$-ring has a non-zero M-ideal.*

This statement follows immediately from the above remark.

DEFINITION. The class M is said to be closed under transfree-images if $T_r(M)=M$.

THEOREM 1. *For every class M of rings there always exists the smallest class $\overline{M}$ satisfying: $\overline{M}\supseteq M$ and $\overline{M}$ is closed under transfree-images.*

PROOF. The class of all rings is closed under transfree-images and it contains M. Since the intersection of classes being closed under transfree-images, is again such a class, we have $\overline{M}=\bigcap(M_\gamma|M_\gamma\supseteq M$ and M_γ is closed under transfree-images).

THEOREM 2. *If M is a homomorphically closed class of rings, then also the class $T_r(M)$ is homomorphically closed.*

PROOF. Let A be a transfree-image of M-rings A_i, $i\in I$, by an epimorphism γ, and $\bar{A}$ a homomorphic image of A by a homomorphism f. Then the images $\bar{A}_i$ of each A_i, $i\in I$ are M-ideals in $\bar{A}$. Further by the universal property of the free product there exist uniquely determined mappings

$$\bar{\gamma}\colon \coprod_{i\in I} \bar{A}_i(\bar{\varrho}_i) \to \bar{A} \quad \text{and} \quad \varphi\colon \coprod_{i\in I} A_i(\varrho_i) \to \coprod_{i\in I} \bar{A}_i(\bar{\varrho}_i)$$

such that the following diagrams are commutative

$$\begin{tikzcd} \bar{A}_i \arrow[r, "\bar{\varrho}_i"] \arrow[dr, "k_i"'] & \coprod_{i\in I} \bar{A}_i(\bar{\varrho}_i) \arrow[d, "\bar{\gamma}"] \\ & \bar{A} \end{tikzcd}, \qquad \begin{tikzcd} A_i \arrow[r, "\varrho_i"] \arrow[d, "f_i"] & \coprod_{i\in I} A_i(\varrho_i) \arrow[d, "\varphi"] \\ \bar{A}_i \arrow[r, "\bar{\varrho}_i"] & \coprod_{i\in I} \bar{A}_i(\bar{\varrho}_i) \end{tikzcd}$$

where $f_i = f\cdot\varrho_i$ and k_i is an embedding. Clearly, the mappings $\bar{\gamma}\colon \bar{\varrho}_i$, $i\in I$ are normal monomorphisms. We can easily show that the square

$$\begin{tikzcd} \coprod_{i\in I} A_i(\varrho_i) \arrow[r, "\gamma"] \arrow[d, "\varphi"] & A \arrow[d, "f"] \\ \coprod_{i\in I} \bar{A}_i(\bar{\varrho}_i) \arrow[r, "\bar{\gamma}"] & \bar{A} \end{tikzcd}$$

is commutative. Hence we have $\bar{\gamma}\varphi = f\cdot\gamma$. Since $f\cdot\gamma$ is an epimorphism, so is $\bar{\gamma}$. Thus $\bar{A}\in T_r(M)$ holds. The theorem is proved.

THEOREM 3. *Every radical class is closed under transfree-images.*

PROOF. Let M be a radical class and $A\in T_r(M)$. $M(A)$ denotes the sum of all M-ideals of A. Suppose $M(A)\neq A$. By Theorem 2 $\dfrac{A}{M(A)}$ is a nonzero $T_r(M)$-ring, and hence by Lemma 1 it contains a non-zero M-ideal $\dfrac{A'}{M(A)}$. By the condition (iii), A' is an M-ideal of A, so $A'\leqq M(A)$, a contradiction. Thus $A\in M$ holds. This completes the proof.

COROLLARY. *For every class M of rings the inclusion $T_r(M)\subseteqq \mathscr{L}(M)$ holds, where $\mathscr{L}(M)$ is the lower radical class defined by M.*

Let us consider the subclass of $T_r(M)$ defined as

$$T_{r_0}(M) = \left\{ A \,\middle|\, \begin{array}{l} A \text{ is a transfree-image of some } M\text{-rings} \\ \text{by an epimorphism which is a surjection} \end{array} \right\}.$$

LEMMA 2. *Assume that M is an abstract class of rings. The ring A belongs to $T_{r_0}(M)$ if and only if in A there exist M-ideals B_i, $i\in I$ such that $\sum_{i\in 1} B_i = A$.*

The proof is trivial.

THEOREM 4. *The class* M *of rings is a radical class if and only if the following conditions are satisfied:*
(A) M *is homomorphically closed.*
(B) M *is closed under transfree-images.*
(C) M *is closed under extensions.*

PROOF. Theorem 3 and the definition of radical yield the necessity.

For the sufficiency we only must show that condition (B) implies condition (ii). If condition (B) is valid, then $M \subseteq T_{r_0}(M) \subseteq T_r(M) = M$. By Lemma 2 it is clear that condition (ii) is satisfied.

Next, with the help of transfree-images we shall get a new construction which does give the lower radical. In order to do this we consider the following operator W acting on classes of rings by

$$W(M) = \left\{ A \,\middle|\, \frac{A}{B} \in M \text{ for some } M\text{-ideal } B \text{ of } A \right\}.$$

LEMMA 3. *If* M *is a homomorphically closed class then* $W(M)$ *is homomorphically closed, too.*

PROOF. Let A be in $W(M)$ and B any proper ideal of A. By the definition of $W(M)$ there exists an ideal C of A such that C and A/C both belong to M. Since the class M is homomorphically closed so we have

$$\frac{B+C}{B} \cong \frac{C}{B \cap C} \in M, \quad \frac{A}{B} \Big/ \frac{B+C}{B} \cong \frac{A}{B+C} \in M.$$

Thus A/B belongs to $W(M)$ and so $W(M)$ is homomorphically closed. The lemma is proved.

Now, let M be any class of rings. Define $K_1(M)$ to be the homomorphic closure of M. For every ordinal $\alpha > 1$, put

$$K_\alpha(M) = \begin{cases} T_r(K_{\alpha-1}(M)) & \text{if } \alpha \text{ is not a limit ordinal} \\ W\big(\bigcup\limits_{\beta<\alpha} K_\beta(M)\big) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Finally define $K(M) = \bigcup K_\alpha(M)$, where the union is taken over all ordinals α. Clearly, if α and β are ordinals with $\beta \leqq \alpha$ then $K_\alpha(M) \subseteq K_\alpha(M)$.

LEMMA 4. *For every ordinal* $\alpha \geqq 1$, $K(M)$ *is homomorphically closed. Hence* $K(M)$ *is a homomorphically closed class.*

PROOF. $K_1(M)$ is homomorphically closed. Let $\alpha > 1$ be an ordinal and suppose $K_\beta(M)$ is homomorphically closed for all $\beta < \alpha$.

If α is not a limit ordinal, then by Theorem 2 and the induction hypothesis, $K_\alpha(M) = T_r(K_{\alpha-1}(M))$ is homomorphically closed.

Let α be a limit ordinal. Clearly, by the induction hypothesis, the class $\bigcup\limits_{\beta<\alpha} K_\beta(M)$ is homomorphically closed. So by Lemma 3, the class $K_\beta(M) = W\big(\bigcup\limits_{\beta<\alpha} K_\beta(M)\big)$ is also homomorphically closed. Thus by transfinite induction $K_\alpha(M)$ is homomorphically closed for all ordinals α. It follows immediately that $K(M)$ is homomorphically closed.

THEOREM 5. $K(M)=\mathcal{L}(M)$.

PROOF. We use Theorem 4 to show that $K(M)$ is a radical class. By Lemma 4, the class $K(M)$ satisfies condition (A). Suppose that a ring A is a transfree-image of $K(M)$-rings A_i, $i\in I$. Then for $i\in I$ there exists an ordinal α_i such that $A_i\in K_{\alpha_i}(M)$. Let α be an ordinal greater than all α_i-s, $i\in I$. Hence every ring A_i, $i\in I$ belongs to $K_\alpha(M)$. So we have

$$A\in T_r(K_\alpha(M)) = K_{\alpha+1}(M) \subseteq K(M).$$

Thus, condition (B) is satisfied.

Now, let A have an ideal B such that both B and A/B are in $K(M)$. Then, there exist ordinals α_1 and α_2 such that $B\in K_{\alpha_1}(M)$, $A/B\in K_{\alpha_2}(M)$. We take a limit ordinal α greater than the ordinals α_i, $i=1, 2$. Then both B and A/B belong to the class $\bigcup\limits_{\beta<\alpha} K_\beta(M)$. So we have

$$A\in W\Big(\bigcup_{\beta<\alpha} K_\beta(M)\Big) = K_\alpha(M) \subseteq K(M).$$

Hence condition (C) is valid. Thus $K(M)$ is a radical class.

By the minimality of $\mathcal{L}$ among radical classes containing M, it is enough to show $K(M)\subseteq\mathcal{L}(M)$. This is accomplished by proving $K_\alpha(M)\subseteq\mathcal{L}(M)$ for every ordinal.

Clearly, $K_1(M)\subseteq\mathcal{L}(M)$. Let α be an ordinal exceeding one, and assume $K_\beta(M)\subseteq\mathcal{L}(M)$ for all ordinals $\beta<\alpha$. Suppose $A\in K_\alpha(M)$.

If α is not a limit ordinal, then we have

$$A\in K_\alpha(M) = T_r(K_{\alpha-1}(M)) \subseteq T_r(\mathcal{L}(M)) = \mathcal{L}(M).$$

Let α be a limit ordinal, then by the definition we have

$$A\in K_\alpha(M) = W\Big(\bigcup_{\beta<\alpha} K_\beta(M)\Big).$$

Therefore there exists an ideal B in A such that both B and A/B belong to $\bigcup\limits_{\beta<\alpha} K_\beta(M)$. By the induction hypothesis, it is clear that $\bigcup\limits_{\beta<\alpha} K_\beta(M)\subseteq\mathcal{L}(M)$. From this $A\in\mathcal{L}(M)$ follows by condition (C) of Theorem 4, and the theorem is proved.

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APPROXIMATION IN L_2-SPACE BY INTERPOLATORY TYPE OPERATORS

By

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The aim of this paper is giving interpolatory type operators which can approximate not only continuous functions. Namely, if we suppose that f is integrable in Lebesgue sense on the finite interval $[a, b]$, then we can define the following linear operator:

$$L_n^*(f; x) = \sum_{k=1}^{n} \frac{1}{h_k} \int_{x_k}^{x_k+h_k} f(t)\, dt\, l_k(x) \quad (n = 1, 2, \ldots)$$

where $l_k(x)$ $(k=1, 2, \ldots, n)$ are the fundamental polynomials of the Lagrange interpolation corresponding to the nodes $(a\leqq)x_n<\ldots<x_0(\leqq b)$, $0<h_k\leqq x_{k-1}-x_k$. Of course $x_k=x_k^{(n)}$ and $h_k=h_k^{(n)}$ depend on n, but we shall omit the upper indices, when they are not necessary.

$L_n^*(f; x)$ is formally similar to the Lagrange interpolating polynomial $L_n(f; x)= = \sum_{k=1}^{n} f(x_k)\, l_k(x)$, but it differs from $L_n(f; x)$ in having the mean value $\frac{1}{h_k} \int_{x_k}^{x_k+h_k} f(t)\, dt$ instead of the value $f(x_k)$. This modification is made because integrable functions are not uniquely characterized by their denumerable values in general, and operators defined by those values do not converge in L_p-norm $(1\leqq p<\infty)$ to the functions determining them.

We remark that J. SZABADOS [1] defined another interpolating type linear procedure in L_p-space. He investigated the L_p-convergence of this operator denoted by $L_n(f; x)$ and gave an estimation for $\|f(x)-L_n(f; x)\|_p$.

Returning to $L_n^*(f; x)$ the following problems arise naturally. In what spaces of functions are $L_n^*(f; x)$ convergent? How to choose the nodes $(a\leqq)x_n<\ldots< \ldots<x_0(\leqq b)$ and the values h_k in this case?

We shall show, that, for instance, choosing for nodes the zeros of orthogonal polynomials associated with some weight functions yields a convergent procedure.

Now let $w(x)$ be a weight function defined on $[a, b]$, that is $0< \int_a^b w(x)\, dx<\infty$ and $w(x)\geqq 0$. We shall assume that $w(x)>0$ almost everywhere in $[a, b]$. A consequence of this condition is that for the zeros $(a<)x_n^{(n)}<\ldots<x_1^{(n)}(<x_0=b)$ $(n=1, 2, \ldots)$ of the orthogonal polynomials $\{\omega_n(x)\}_{n=1}^{\infty}$ associated with the weight function $w(x)$ the relation $\max_k (x_{k-1}^{(n)}-x_k^{(n)})\to 0$ $(n\to\infty)$ holds. (In fact, it is sufficient to suppose the weaker condition that for every subinterval $[a', b']$ of $[a, b]$, $\int_{a'}^{b'} w(x)\, dx>0$; see Theorem 6.1.1 of SZEGŐ [2].)

Now define the linear operators

$$L_n^*(f; x) = \sum_{k=1}^{n} \frac{1}{h_k} \int_{x_k}^{x_{k-1}} f(t)\, dt\, l_k(x), \quad (n = 1, 2, \dots) \tag{1}$$

where $x_k = x_k^{(n)}$ $(k=1, 2, \dots; n=1, 2 \dots)$ are the zeros of the orthogonal polynomials $\omega_n(x)$ associated with the weight function $w(x)$, $a < x_n < \dots < x_1 < b$, $x_0 = b$, $0 < h_k = h_k^{(n)} \leqq x_{k-1} - x_k$, and

$$l_k(x) = \frac{\omega_n(x)}{\omega_n'(x)(x-x_k)} \quad (k = 1, 2, \dots, n)$$

are the fundamental polynomials of the Lagrange interpolation.

Denote

$$\lambda_k = \lambda_k^{(n)} = \int_a^b l_k(x) w(x)\, dx = \int_a^b l_k^2(x) w(x)\, dx \quad (k = 1, 2, \dots, n;\ n = 1, 2, \dots)$$

the Cotes-numbers and let $a_n = \max_k h_k^{(n)}$. Since $0 < a_n \leqq \max (x_{k-1}^{(n)} - x_k^{(n)})$, in our case the relation $\lim_{n\to\infty} a_n = 0$ is fulfilled.

In what follows, C will always denote different constants independent of x, n and k.

For the operator defined in (1) the following estimation is true.

THEOREM. *If $w(x) \leqq M$ ($M > 0$ is an arbitrary constant) and*

$$d_n = \sup_{1 \leqq k \leqq i \leqq n} \frac{\lambda_k^{(i)}}{h_k^{(i)}} = O(n^6) \quad (n = 1, 2, \dots) \tag{2}$$

then for arbitrary $f \in L_2[a, b]$ the inequality

$$\left\{ \int_a^b [f(x) - L_n^*(f; x)]^2 w(x)\, dx \right\}^{1/2} \leqq C \sqrt{d_n}\, \omega \left(f; \frac{\sqrt[4]{a_n}}{\sqrt[8]{d_n}} \right)_2 \tag{3}$$

holds, where $\omega(f; \delta)_2$ is the L_2-modulus of continuity of f.

Since $a_n = \max_k h_k \to 0$ $(n \to \infty)$, substituting (2) by the stronger condition (4) below, we obtain the following convergence theorem.

COROLLARY. *If $w(x) \leqq M$ $(M > 0)$ and*

$$\frac{\lambda_k^{(n)}}{h_k^{(n)}} \leqq d \quad (k = 1, 2, \dots, n;\ n = 1, 2, \dots) \tag{4}$$

where $d > 0$ is a constant independent of k and n, then for arbitrary $f \in L_2[a, b]$

$$\left\{ \int_a^b [f(x) - L_n^*(f; x)]^2 w(x)\, dx \right\}^{1/2} \leqq C \omega(f; \sqrt[4]{a_n})_2. \tag{5}$$

We shall see that there are some cases when the rate of convergence is better than (5). This is the case when $f\in\text{Lip}\,\alpha$ $(0<\alpha<1)$, where $\text{Lip}\,\alpha$ is defined by the L_2-modulus of continuity.

An important special case of the Corollary is when the nodes are the zeros of the Jacobi polynomials associated with $w(x)=(1-x)^\alpha(1+x)^\beta$, if $\alpha,\beta\geqq 0$. Choosing $h_k=K(x_{k-1}-x_k)$ where $0<K\leqq 1$ is a fixed constant (then condition (4) fulfils as we will show later) and taking into consideration that $a_n=O\left(\frac{1}{n}\right)$, we obtain by the Corollary that for every $f\in L_2[-1,1]$

$$\text{(6)}\qquad \left\{\int_{-1}^{1}[f(x)-L_n^*(f;x)]^2(1-x)^\alpha(1+x)^\beta\,dx\right\}^{1/2}\leqq C\omega(f;n^{-1/4})_2.$$

The most important case of (6) is when $w(x)\equiv 1$, and the nodes are the zeros of the Legendre polynomials. Then choosing $h_k=x_{k-1}-x_k$ we get that for every $f\in L_2[-1,1]$

$$\text{(7)}\qquad \left\{\int_{-1}^{1}\left[f(x)-\sum_{k=1}^{n}\left(\frac{1}{x_{k-1}-x_k}\int_{x_k}^{x_{k-1}}f(t)\,dt\right)l_k(x)\right]^2dx\right\}^{1/2}\leqq C\omega(f;n^{-1/4})_2.$$

We mention that, by the Theorem, convergence is possible when condition (4) of the Corollary is not satisfied, for example in the Jacobi case $\alpha<0$ or $\beta<0$, if $f\in\text{Lip}\,\gamma$ with suitable γ.

Proof of the Theorem. Let $P_m(x)$ $(m<n)$ be the m-th polynomial of best approximation to f in $L_2[a,b]$, that is

$$\left\{\int_a^b[f(x)-P_m(x)]^2\,dx\right\}^{1/2}=\|f-P_m\|_2=E_m(f)_2.$$

Using Minkowski's inequality we have

$$\text{(8)}\qquad \left\{\int_a^b[f(x)-L_n^*(f;x)]^2w(x)\,dx\right\}^{1/2}=\|f-L_n^*(f)\|_{2,w}\leqq$$

$$\leqq\|f-P_m\|_{2,w}+\|P_m-L_n^*(P_m)\|_{2,w}+\|L_n^*(P_m-L_n^*(f)\|_{2,w}=S_1+S_2+S_3.$$

It is obvious by the assumption $w(x)\leqq M$ that

$$\text{(9)}\qquad S_1=\left\{\int_a^b[f(x)-P_m(x)]^2w(x)\,dx\right\}^{1/2}\leqq\sqrt{M}E_m(f)_2.$$

Since $P_m(x)=\sum_{k=1}^{n}P_m(x_k)l_k(x)$ $(m<n)$, we can write

$$S_2^2=\int_a^b\left[P_m(x)-\sum_{k=1}^{n}\frac{1}{h_k}\int_{x_k}^{x_k+h_k}P_m(t)\,dt\,l_k(x)\right]^2w(x)\,dx=$$

$$=\int_a^b\left\{\sum_{k=1}^{n}\left[P_m(x_k)-\frac{1}{h_k}\int_{x_k}^{x_k+h_k}P_m(t)\,dt\right]l_k(x)\right\}^2w(x)\,dx.$$

By orthogonality and Lagrange's theorem

$$S_2^2 = \sum_{k=1}^{n} \left[P_m(x_k) - \frac{1}{h_k} \int_{x_k}^{x_k+h_k} P_m(t)\,dt \right]^2 \lambda_k =$$

$$= \sum_{k=1}^{n} [P_m(x_k) - P_m(\xi_k)]^2 \lambda_k = \sum_{k=1}^{n} P_m'^2(\eta_k)(x_k-\xi_k)^2 \lambda_k \leqq$$

$$\leqq \max_{a \leqq x \leqq b} P_m'^2(x) \max_k h_k^2 \sum_{k=1}^{n} \lambda_k. \quad (x_k < \eta_k < \xi_k < x_k + h_k).$$

Taking into consideration the facts $\max_k h_k = a_n$, $\sum_{k=1}^{n} \lambda_k = \int_a^b w(x)\,dx$, and the estimation

$$\max_{a \leqq x \leqq b} |P_m'(x)| \leqq Cm^2 \max_{a \leqq x \leqq b} |P_m(x)| \leqq Cm^3 \|P_m(x)\|_2$$

where Markov's inequality and another inequality (TIMAN [3], p. 251) were applied, we get

(10) $$S_2 \leqq Ca_n m^3 \|P_m(x)\|_2 \leqq Ca_n m^3 \|f\|_2 \leqq Ca_n m^3.$$

For the estimation of S_3 we shall use orthogonality, inequality of Schwarz and condition (2), so

$$S_3^2 = \|L_n^*(P_m - f)\|_{2,w}^2 = \int_a^b \left\{ \sum_{k=1}^{n} \int_{x_k}^{x_k+h_k} [P_m(t) - f(t)]\,dt\, l_k(x) \right\}^{1/2} w(x)\,dx =$$

$$= \sum_{k=1}^{n} \frac{1}{h_k^2} \left\{ \int_{x_k}^{x_k+h_k} [P_m(t) - f(t)]\,dt \right\}^2 \lambda_k \leqq \sum_{k=1}^{n} \frac{1}{h_k} \int_{x_k}^{x_k+h_k} [P_m(t) - f(t)]^2\,dt \lambda_k \leqq$$

$$\leqq d_n \sum_{k=1}^{n} \int_{x_k}^{x_k+h_k} [P_m(t) - f(t)]^2\,dt \leqq d_n \int_a^b [P_m(t) - f(t)]^2\,dt$$

that is

(11) $$S_3 \leqq \sqrt{d_n} E_m(f)_2.$$

Summarizing (8), (9), (10) and (11) it is clear that

(12) $$\|f - L_n^*(f)\|_{2,w} \leqq C\left(\sqrt{d_n} E_m(f)_2 + a_n m^3\right).$$

By Jackson's theorem in L_2-space

(13) $$\|f - L_n^*(f)\|_{2,w} \leqq C\left(\sqrt{d_n}\, \omega\left(f; \frac{1}{m}\right)_2 + a_n m^3\right).$$

Solving the equation $\frac{\sqrt{d_n}}{m} = a_n m^3$ suggested by (13) we can choose $m = \left[\frac{d_n^{1/8}}{a_n^{1/4}}\right]$ or $m = \left[C \frac{d_n^{1/8}}{a_n^{1/4}}\right]$ with a suitable constant C, which guarantees $m < n$, taking into consideration that $a_n \geqq \frac{b-a}{n+1}$ always and the condition $d_n = O(n^6)$. So we have

(14) $$\|f - L_n^*(f)\|_{2,w} \leqq C\sqrt{d_n}\,\omega\left(f;\ \frac{\sqrt[4]{a_n}}{\sqrt[8]{d_n}}\right)_2$$

which was to be proved.

Returning to the problem of the rate of convergence in the Corollary, if $f \in \mathrm{Lip}\,\alpha$ in L_2-sense $(0 < \alpha \leqq 1)$, then (5) is equivalent to

(15) $$\|f - L_n^*(f)\|_{2,w} \leqq C a_n^{\alpha/4}.$$

For $f \in \mathrm{Lip}\,\alpha$, $0 < \alpha < 1$, we can obtain from (13) a slightly better rate of convergence than (15). We have to solve the equation

$$\omega\left(f;\ \frac{1}{m}\right)_2 = \left(\frac{1}{m}\right)^\alpha = a_n m^3$$

from which $m = \left[\frac{1}{\sqrt[3+\alpha]{a_n}}\right]$ follows, and the rate of convergence is

(16) $$\|f - L_n^*(f)\|_{2,w} \leqq C a_n^{\alpha/(3+\alpha)},$$

for $f \in \mathrm{Lip}\,\alpha$ $(0 < \alpha < 1)$ in L_2-sense.

Now we return to the case when $w(x) = (1-x)^\alpha (1+x)^\beta$ $(\alpha, \beta \geqq 0,\ -1 \leqq x \leqq 1)$ and $-1 < x_n < \ldots < x_1 < 1$ are the zeros of Jacobi polynomials associated with $w(x)$. It is trivial that $0 < w(x) < 1$ $(-1 < x < 1)$. A consequence of SZEGŐ's [2] relation (15.3.14) is that

(17) $$\lambda_k \leqq C\frac{k}{n^2} \quad (k = 1, 2, \ldots, n).$$

Further

(18) $$h_k = C(x_{k-1} - x_k) \sim C\frac{k}{n^2},$$

where formula (8.9.1) of SZEGŐ [2] was used. Comparing (17) with (18) it is obvious that $\frac{\lambda_k}{h_k} < d$, so the Corollary can be applied and we get (6) and (7).

Consider now the case when the nodes are the zeros of Jacobi polynomials with $\alpha < 0$ or $\beta < 0$. Then condition (4) of the Corollary is not satisfied, but by the Theorem convergence is possible for some classes of functions. Suppose for the sake of simplicity that $\alpha \leqq \beta$. By SZEGŐ [2], (15.3.14)

$$\lambda_k \leqq C\frac{k}{n^2}\left(\frac{k}{n}\right)^{2\alpha} \quad (\alpha < 0,\ \alpha \leqq \beta)$$

so

$$\frac{\lambda_k}{h_k} \leqq C\left(\frac{k}{n}\right)^{2\alpha} \leqq C\left(\frac{1}{n}\right)^{2\alpha} = d_n \quad (\alpha < 0).$$

In our case $a_n = O\left(\frac{1}{n}\right)$, and let $f \in \operatorname{Lip} \gamma$ in L_2-sense $(0 < \gamma \leqq 1)$. So solving the appropriate equation $\frac{1}{n^\alpha m^\gamma} = \frac{m^3}{n}$ which we obtain from (13), we have $m = [Cn^{\frac{1-\alpha}{3+\gamma}}.]$ Substituting in (3) we get

$$\|f - L_n^*(f)\|_{2,w} \leqq C\left(\frac{1}{n}\right)^\alpha \left(\frac{1}{n}\right)^{\gamma(1-\alpha)/(3+\gamma)} \qquad (\alpha < 0,\ \alpha \leqq \beta). \tag{19}$$

The right-hand side of (19) tends to zero if $-\alpha < \frac{\gamma(1-\alpha)}{3+\gamma}$, that is $\gamma > -3\alpha$. This relation can be fulfilled if $-\frac{1}{3} < \alpha < 0$ only, because $0 < \gamma \leqq 1$. As we have seen, for fixed $-\frac{1}{3} < \alpha \leqq \beta$, $L_n^*(f; x)$ is convergent if $f \in \operatorname{Lip} \gamma$ in L_2-sense, where $\gamma > -3\alpha$.

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О НЕКОТОРЫХ МЕТОДАХ ПРИБЛИЖЕНИЯ ФУНКЦИЙ

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Введение

Условимся в следующих обозначениях: k неотрицательное, m и r положительное целое число; n целая часть числа $\frac{m}{2}$. R множество вещественных чисел; $t\in R$; C множество непрерывных и 2π-периодических функций, отображающих R в R. $\omega(g, \delta)$ модуль непрерывности функции $g\in C$;

$$H_\omega = \{g\in C\colon\ \omega(g, \delta) \leqq \omega(\delta),\ \delta \geqq 0\}, \tag{1}$$

где $\omega(\delta)$ модуль непрерывности некоторой функции из C.

$$d_m(t) = \begin{cases} \dfrac{1}{m}+\dfrac{2}{m}\sum\limits_{i=1}^{n}\cos it \quad (m = 1, 3, 5, \ldots), \\ \dfrac{1}{m}+\dfrac{1}{n}\sum\limits_{i=1}^{n-1}\cos it+\dfrac{1}{m}\cos nt \quad (m = 2, 4, 6, \ldots) \end{cases} \tag{2}$$

видоизмененное ядро Дирихле;

$$s_k(t) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} d_m\left(t+\frac{2j-k}{m}\pi\right), \tag{3}$$

$$t_i = \frac{2\pi i}{mr} \quad (i = 0, \pm 1, \pm 2, \ldots), \tag{4}$$

$$S_{m,k,r}(g, t) = \frac{1}{r}\sum_{i=1}^{mr} g(t_i)\, s_k(t-t_i) \tag{5}$$

тригонометрические многочлены n-того порядка, которые получаются методом наименьших квадратов, если $k=0$, и интерполируют функцию g в узлах t_i, если еще и $r=1$. Случай $k=1$ и $k=2$ впервые рассматривал С. Н. Бернштейн в статьях [1] и [2].

В работе [3] H. Brass и R. Günttner вычислили при $k=0$ и $r=1$ выражение

$$C_{m,k,r} = \sup_{t\in R}\ \sup_{\substack{g\in C \\ g(t)\neq \mathrm{const}}} \frac{|S_{m,k,r}(g, t)-g(t)|}{\omega\left(g, \dfrac{2\pi}{m}\right)}; \tag{6}$$

в статье [5] R. Günttner опубликовал неравенства

$$C_{m,0,1} \leqq 1+\frac{1}{\pi}\ln m \quad (m = 1, 3, 5, \dots),$$

$$C_{m,0,1} \leqq 1+\frac{1}{\pi}\ln n \quad (m = 2, 4, 6, \dots).$$

В [7]—[10] вычислялись числа $C_{m,k,r}$ при любых k и r. Были доказаны, например, следующие соотношения:

$$C_{m,1,1} \leqq 1+\frac{1}{\pi} \quad (m \neq 4, 8, 12, \dots);$$

$$C_{m,2,1} = \frac{5}{4} \quad (m = 3, 4, 5, \dots);$$

$$C_{m,3,1} \leqq \frac{5}{4}+\frac{2}{3\pi} \quad (m = 4, 8, 12, \dots);$$

$$C_{m,4,1} = \frac{23}{16} \quad (m = 6, 8, 10, \dots).$$

В [5] при $k=0$ и $r=1$ R. Günttner доказал следующий результат:

Теорема 1. *Если $\omega(\delta)$ выпуклый вверх модуль непрерывности и $g \in H_\omega$, то*

$$|S_{m,k,r}(g, t)-g(t)| \leqq \omega\left(\frac{\pi}{m}\right)+(C_{m,k,r}-1)\,\omega\left(\frac{2\pi}{m}\right). \tag{7}$$

Ниже мы докажем теорему 1 для всех k и r.

Тригонометрические многочлены n-того порядка

$$S_{m,k,\infty}(g, t) = \frac{m}{2\pi}\int_0^{2\pi} g(u)\,s_k(t-u)\,du \tag{8}$$

получаются из (5) при $r\to\infty$. Если m нечетно (четно) и $k=0$, то (8) обычная (модифицированная) n-тая частная сумма ряда Фурье функции g. Случай $k=1$ впервые рассматривал W. Rogosinski в статье [12], а случай $k>1$ первым изучал Ф. И. Харшиладзе в работе [13]. В [8] и [10] доказано, что выражение

$$C_{m,k,\infty} = \sup_{\substack{g\in C\\ g(t)\neq\text{const}}} \frac{|S_{m,k,\infty}(g, t)-g(t)|}{\omega\left(g, \dfrac{2\pi}{m}\right)}$$

получается из (6) при $r\to\infty$, и вычислялись его значения при $k=0$, 1 и 2. Получены неравенства

$$C_{m,1,\infty} < 1{,}15; \quad C_{m,2,\infty} < 1{,}21.$$

Из (7) при $r\to\infty$ очевидно получается следующий аналогичный результат:

$$|S_{m,k,\infty}(g,t)-g(t)|\leqq\omega\left(\frac{\pi}{m}\right)+(C_{m,k,\infty}-1)\,\omega\left(\frac{2\pi}{m}\right).$$

Введем следующие обозначения: $C[-1,1]$ множество непрерывных функций, отображающих отрезок $[-1,1]$ в R;

$$H_\omega[-1,1]=\{f\in C[-1,1]\colon\ \omega(f,\delta)\leqq\omega(\delta),\ 0\leqq\delta\leqq 2\},\tag{9}$$

где $\omega(\delta)$ модуль непрерывности некоторой функции из $C[-1,1]$;

$$x_i=\cos\frac{2\pi i}{mr}\quad(i=0,\pm1,\pm2,\dots);\tag{10}$$

$$P_{m,k,r}(f,x)=\frac{1}{r}\sum_{i=1}^{mr}f(x_i)\,s_k(\arccos x-t_i).\tag{11}$$

Это алгебраический многочлен n-той степени, который при $r=1$ и $k=0$ интерполирует функцию f в узлах x_i $(i=0,1,\dots,n)$; при $r=1$ и $k=1$ его впервые рассматривал G. GRÜNWALD в статье [4], при $r=1$ и $k=2$ С. Н. Бернштейн в [1]. Для четных m в [9] при $r=1$, $k=2$ и $k=4$, а в [11] для $r>1$ и $k=2$ доказано неравенство

$$|P_{m,k,r}(f,x)-f(x)|\leqq C_{m,k,r}\,\omega\left(f,\frac{2\pi}{m}\sqrt{1-x^2}\right)+\gamma_{m,k,r}\,\omega\left(f,\frac{2\pi^2}{m^2}|x|\right),\tag{12}$$

где $\gamma_{m,k,r}$ положительное число, определяемое ниже формулами (14)—(15) и (51). Например

$$\gamma_{m,2,1}=\frac{7}{4}\quad(m=4,6,8,\dots);$$

$$\gamma_{m,4,1}=\frac{39}{16}\quad(m=6,8,10,\dots).$$

Приведенные доказательства верны при всех k и r. Ниже доказывается

Теорема 2. *Если $\omega(\delta)$ выпуклый вверх модуль непрерывности и $f\in H_\omega[-1,1]$, то*

$$|P_{m,k,r}(f,x)-f(x)|\leqq\omega\left(\frac{\pi}{m}\sqrt{1-x^2}\right)+(C_{m,k,r}-1)\,\omega\left(\frac{2\pi}{m}\sqrt{1-x^2}\right)+$$

$$+\omega\left(\frac{\pi^2}{m^2}|x|\right)+(\gamma_{m,k,r}-1)\,\omega\left(\frac{2\pi^2}{m^2}|x|\right).$$

В [9] и [11] для случая $k=0$ и $k=1$ получены также оценки более сложные и более точные, чем (12). Если выполняются условия теоремы 2, то эти оценки можно улучшить также, как и (12).

Для многочленов $P_{m,k,\infty}(f,x)$, получаемых из (11) при $r\to\infty$, очевидно также выполняется (12) и теорема 2, где $\gamma_{m,k,\infty}$ предел последовательности $\gamma_{m,k,r}$ при $r\to\infty$.

Также, как доказывается теорема 2 и (12), можно доказать их аналоги для многочленов, отличающихся от (11) тем, что t_i и x_i заменяется на $\frac{2i-1}{mr}\pi$ и $\cos\frac{2i-1}{mr}\pi$.

Вспомогательные предложения

Введем следующие обозначения:

$$\vartheta_i = \frac{2\pi i}{m} \quad (i = 0, \pm 1, \pm 2, \ldots); \tag{13}$$

если m четное число, то

$$\sigma_i(t) = \sigma_{i,k}(t) = \begin{cases} \sum_{j=1-n}^{i} s_k(t-\vartheta_j) & (i = 0, -1, \ldots, 1-n); \\ \sum_{j=i}^{n} s_k(t-\vartheta_j) & (i = 1, 2, \ldots, n); \end{cases} \tag{14}$$

а если m нечетное число, то

$$\sigma_i(t) = \sigma_{i,k}(t) = \begin{cases} \frac{1}{2} s_k(t+\vartheta_n) + \sum_{j=1-n}^{i} s_k(t-\vartheta_j) & (i = 0, -1, \ldots, -n); \\ \frac{1}{2} s_k(t+\vartheta_n) + \sum_{j=i}^{n} s_k(t-\vartheta_j) & (i = 1, 2, \ldots, n+1). \end{cases} \tag{15}$$

Лемма 1. *Если* $0 \leqq t \leqq \frac{\pi}{m}$, *то* $\sigma_0(t) \geqq \sigma_1(t)$, *а если* $\frac{\pi}{m} \leqq t \leqq \frac{2\pi}{m}$, *то* $\sigma_1(t) \geqq \sigma_0(t)$.

Доказательство леммы 1. Ввиду (13)—(15) и (2)—(3)

$$\sigma_{1,0}(t) - \sigma_{0,0}(t) = \sum_{i=1}^{n} [d_m(t-\vartheta_i) - d_m(t+\vartheta_{i-1})] = \sum_{j=1}^{n} e_{j,m} \sin j\left(t - \frac{\pi}{m}\right), \tag{16}$$

где

$$e_{j,m} = \begin{cases} \frac{4}{m} \sum_{i=1}^{n} \sin\frac{2i-1}{m} j\pi & \left(1 \leqq j < \frac{m}{2};\ m = 3, 4, 5, \ldots\right); \\ \frac{2}{m} \sum_{i=1}^{n} \sin\frac{2i-1}{2}\pi & (j = n;\ m = 2, 4, 6, \ldots). \end{cases}$$

Используя тождество

$$2 \sum_{i=1}^{n} \sin(2i-1)t = (1-\cos 2nt) \operatorname{cosec} t,$$

получаем:

$$e_{j,m} = \begin{cases} \frac{2}{m}\left(1 - \cos\frac{2nj\pi}{m}\right) \operatorname{cosec}\frac{j\pi}{m} & \left(1 \leqq j < \frac{m}{2};\ m = 3, 4, 5, \ldots\right); \\ \frac{1}{m}[1-(-1)^n] & (j = n;\ m = 2, 4, 6, \ldots). \end{cases} \tag{17}$$

Из (3) следует:

$$2s_{k+1}(t) = s_k\left(t-\frac{\pi}{m}\right)+s_k\left(t+\frac{\pi}{m}\right) \quad (k = 0, 1, 2, \ldots).$$

Поэтому и ввиду (14)—(15)

$$2\sigma_{i,k+1}(t) = \sigma_{i,k}\left(t-\frac{\pi}{m}\right)+\sigma_{i,k}\left(t+\frac{\pi}{m}\right).$$

Отсюда и из (16) получаем:

$$\sigma_{1,k}(t)-\sigma_{0,k}(t) = \sum_{i=1}^{n} e_{j,m}\left(\cos\frac{j\pi}{m}\right)^k \sin j\left(t-\frac{\pi}{m}\right) \quad (k = 0, 1, 2, \ldots).$$

Так как ввиду (17) здесь $e_{j,m} \geqq 0$, то отсюда следует лемма 1.

Замечание. В [8] функции $\sigma_i(t)$ при нечетном m определялись не формулой (15), а равенством

$$\sigma_i(t) = \begin{cases} \sum\limits_{j=-n}^{i} s_k(t-\vartheta_j) & (i = 0, -1, \ldots, -n); \\ \sum\limits_{j=i}^{n} s_k(t-\vartheta_j) & (i = 1, 2, \ldots, n). \end{cases} \tag{18}$$

Однако для этих выражений лемма 1 не выполняется.

Лемма 2. *Если $\omega(\delta)$ выпуклый вверх модуль непрерывности функции из C, то*

$$\omega(t)\sigma_0(t)+\omega\left(\frac{2\pi}{m}-t\right)\sigma_1(t) \leqq \omega\left(\frac{\pi}{m}\right) \quad \left(0 \leqq t \leqq \frac{2\pi}{m}\right).$$

Доказательство леммы 2. Будем рассуждать также, как рассматривался в [5] случай $k=0$. В [7] и [8] доказаны соотношения

$$\sigma_0(t) \geqq 0, \quad \sigma_1(t) \geqq 0 \quad \left(0 \leqq t \leqq \frac{2\pi}{m}\right), \tag{19}$$

$$\sigma_0(t)+\sigma_1(t) = 1. \tag{20}$$

(В [8] рассматривался случай (18), но доказательство верно и для (15).) Если $\omega(\delta)$ выпуклая вверх функция, то ввиду (19)—(20)

$$\omega(t)\sigma_0(t)+\omega\left(\frac{2\pi}{m}-t\right)\sigma_1(t) \leqq \omega\left[t\sigma_0(t)+\left(\frac{2\pi}{m}-t\right)\sigma_1(t)\right]. \tag{21}$$

Здесь ввиду (20) и леммы 1

$$t\sigma_0(t)+\left(\frac{2\pi}{m}-t\right)\sigma_1(t) = \frac{\pi}{m}[\sigma_0(t)+\sigma_1(t)]+\left(t-\frac{\pi}{m}\right)[\sigma_0(t)-\sigma_1(t)] \leqq \frac{\pi}{m}. \tag{22}$$

Так как $\omega(\delta)$ неубывающая функция, то из (21) и (22) следует лемма 2.

Лемма 3. *Если $\omega(\delta)$ выпуклый вверх модуль непрерывности функции из $C[-1, 1]$, $0 \leqq t \leqq \pi$, $0 \leqq \vartheta \leqq \frac{2\pi}{m}$, $y = \cos(t-\vartheta) - \cos t$ и $z = \cos t - \cos\left(\frac{2\pi}{m} + t - \vartheta\right)$, то*

$$\omega(|y|)\sigma_0(\vartheta) + \omega(|z|)\sigma_1(\vartheta) \leqq \omega\left(\frac{\pi}{m}\sin t\right) + \omega\left(\frac{\pi^2}{m^2}|\cos t|\right). \tag{23}$$

Доказательство леммы 3. Из условия леммы и (19)—(20) как и (21) получаем:

$$\omega(|y|)\sigma_0(\vartheta) + \omega(|z|)\sigma_1(\vartheta) \leqq \omega[|y|\sigma_0(\vartheta) + |z|\sigma_1(\vartheta)]. \tag{24}$$

Здесь

$$y = \sin t \sin\vartheta - 2\cos t\left(\sin\frac{\vartheta}{2}\right)^2,$$

$$z = \sin t \sin\left(\frac{2\pi}{m} - \vartheta\right) + 2\cos t\left[\sin\left(\frac{\pi}{m} - \frac{\vartheta}{2}\right)\right]^2.$$

Поэтому

$$|y|\sigma_0(t) + |z|\sigma_1(t) \leqq \sin t\left[\sin\vartheta\sigma_0(\vartheta) + \sin\left(\frac{2\pi}{m} - \vartheta\right)\sigma_1(\vartheta)\right] + \tag{25}$$

$$+ 2|\cos t|\left|\left[\sin\left(\frac{\pi}{m} - \frac{\vartheta}{2}\right)\right]^2\sigma_1(\vartheta) - \left(\sin\frac{\vartheta}{2}\right)^2\sigma_0(\vartheta)\right|.$$

Очевидно

$$\sin\vartheta \leqq \vartheta, \quad \sin\left(\frac{2\pi}{m} - \vartheta\right) \leqq \frac{2\pi}{m} - \vartheta. \tag{26}$$

Если $0 \leqq \vartheta \leqq \frac{\pi}{m}$, то

$$\sin\left(\frac{\pi}{m} - \frac{\vartheta}{2}\right) \leqq \frac{\pi}{m}, \quad \sin\frac{\vartheta}{2} \leqq \frac{\pi}{2m} \tag{27}$$

и ввиду леммы 1 и (19)—(20)

$$\sigma_1(\vartheta) \leqq \frac{1}{2}, \quad \sigma_0(\vartheta) \leqq 1. \tag{28}$$

Если $\frac{\pi}{m} \leqq \vartheta \leqq \frac{2\pi}{m}$, то

$$\sin\left(\frac{\pi}{m} - \frac{\vartheta}{2}\right) \leqq \frac{\pi}{2m}, \quad \sin\frac{\vartheta}{2} \leqq \frac{\pi}{m}, \tag{29}$$

$$\sigma_1(\vartheta) \leqq 1, \quad \sigma_0(\vartheta) \leqq \frac{1}{2}. \tag{30}$$

Так как $\omega(\delta)$ неубывающая функция, то из (24)—(30) и (19) следует (23).

Доказательство теоремы 1

Введем следующее обозначение:

$$S_{m,k}^j(g,t)=\sum_{i=1}^{m} g(t_{ir-j})s_k(t-t_{ir-j}) \quad (j=0,1,\ldots,r-1). \tag{31}$$

Из формул (4)—(5) и (31) как и в [10] получаем:

$$S_{m,k,r}(g,t)=\frac{1}{r}\sum_{j=0}^{r-1} S_{m,k}^j(g,t). \tag{32}$$

Обозначим через l целое число, определяемое условием

$$t_l \leqq t < t_{l+1}, \tag{33}$$

и положим

$$\vartheta = t-t_{l-j}. \tag{34}$$

Так как g и s_k 2π-периодичны, то при четных m

$$S_{m,k}^j(g,t)=\sum_{i=1-n}^{n} g(t_{l+ir-j})s_k(\vartheta-\vartheta_i); \tag{35}$$

а для нечетных m имеет место равенство

$$S_{m,k}^j(g,t)=\frac{1}{2}g(t_{l-nr-j})s_k(\vartheta+\vartheta_n)+ \tag{36}$$
$$+\sum_{i=1-n}^{n} g(t_{l+ir-j})s_k(\vartheta-\vartheta_i)+\frac{1}{2}g(t_{l+nr+r-j})s_k(\vartheta+\vartheta_n).$$

Пусть $\nu=m-n$. Преобразованием Абеля из (35)—(36), (14)—(15) и (20), как и в [7] и [8], получаем:

$$S_{m,k}^j(g,t)-g(t)=\sum_{i=1-\nu}^{-1}[g(t_{l+ir-j})-g(t_{l+ir+r-j})]\sigma_i(\vartheta)+ \tag{37}$$
$$+[g(t_{l-j})-g(t)]\sigma_0(\vartheta)+[g(t_{l+r-j})-g(t)]\sigma_1(\vartheta)+$$
$$+\sum_{i=2}^{\nu}[g(t_{l+ir-j})-g(t_{l+ir-r-j})]\sigma_i(\vartheta).$$

Если $g\in H_\omega$, то отсюда, из (4) и из (33) следует:

$$|S_{m,k}^j(g,t)-g(t)|\leqq \omega\left(\frac{2\pi}{m}\right)\sum_{\substack{i=1-\nu\\ i\neq 0,1}}^{\nu}|\sigma_i(\vartheta)|+\omega(\vartheta)\sigma_0(\vartheta)+\omega\left(\frac{2\pi}{m}-\vartheta\right)\sigma_1(\vartheta). \tag{38}$$

Ввиду (4) и (33)—(34) здесь $0 \leqq \vartheta \leqq \frac{2\pi}{m}$. Если $\omega(\delta)$ выпуклый вверх модуль непрерывности, то из (32), (38) и леммы 2 следует:

$$|S_{m,k,r}(g,t)-g(t)| \leqq \omega\left(\frac{2\pi}{m}\right)\frac{1}{r}\sum_{j=0}^{r-1}\sum_{\substack{i=1-\nu\\ i\neq 0,1}}^{\nu}|\sigma_i(\vartheta)|+\omega\left(\frac{\pi}{m}\right). \tag{39}$$

В [7] и [8] при $r=1$, а в [10] при $r>1$ и $k=0$, 2 доказано соотношение

$$C_{m,k,r} = \max_{0\leqq t\leqq t_1}\frac{1}{r}\sum_{j=0}^{r-1}\sum_{i=1-\nu}^{\nu}|\sigma_i(t+t_j)| \tag{40}$$

для (14) и аналогичное неравенство для (18). Аналогичным образом (40) можно доказать для (14)—(15) при любых $k\geqq 0$ и $r\geqq 1$. Так как ввиду (33)—(34) $\vartheta=t-t_l+t_j$ и $0\leqq t-t_l\leqq t_1$, то из (39—(40) и (20) следует (7):

$$|S_{m,k,r}(g,t)-g(t)| \leqq \omega\left(\frac{\pi}{m}\right)+(C_{m,k,r}-1)\,\omega\left(\frac{2\pi}{m}\right),$$

что и требовалось доказать.

Доказательство теоремы 2

Пусть $f\in C[-1,1]$ и $x\in[-1,1]$. В дальнейщем мы будем пользоваться следующими обозначениями:

$$g = f\circ\cos, \tag{41}$$

$$t = \arccos x. \tag{42}$$

Ввиду (5) и (10)—(11)

$$P_{m,k,r}(f,x) = S_{m,k,r}(g,t). \tag{43}$$

Тождество (37) можно записать в виде

$$S^j_{m,k}(g,t)-g(t) = \sum_{1-\nu}^{-1}[f(x_{l+ir-j})-f(x_{l+ir+r-j})]\sigma_i(\vartheta)+$$

$$+[f(x_{l-j})-f(x)]\sigma_0(\vartheta)+[f(x_{i+r-j})-f(x)]\sigma_1(\vartheta)+\sum_{i=2}^{\nu}[f(x_{l+ir-j})-f(x_{l+ir-r-j})]\sigma_i(\vartheta).$$

Отсюда, как и в [11], следует:

$$|S^j_{m,k}(g,t)-f(x)| \leqq \sum_{i=1-\nu}^{-1}\omega(|x_{l+ir-j}-x_{l+ir+r-j}|)|\sigma_i(\vartheta)|+ \tag{44}$$

$$+\omega(|x_{l-j}-x|)\sigma_0(\vartheta)+\omega(|x-x_{l+r-j}|)\sigma_1(\vartheta)+\sum_{i=2}^{\nu}\omega(|x_{l+ir-r-j}-x_{l+ir-j}|)|\sigma_i(\vartheta)|.$$

Покажем, что

$$(45)\qquad |x_{l+ir-r-j}-x_{l+ir-j}| \leqq \frac{2\pi}{m}\sqrt{1-x^2}+\begin{cases}\dfrac{2\pi^2}{m^2}(2i-1)|x| & (i=2,3,\dots,\nu);\\[2mm] \dfrac{2\pi^2}{m^2}(2|i|+3)|x| & (i=-2,-3,\dots,-\nu).\end{cases}$$

Очевидно

$$(46)\qquad x_{l+ir-r-j}-x_{l+ir-j}=2\sin\frac{\pi}{m}\sin t_{l+ir-r/2-j}=$$

$$=2\sin\frac{\pi}{m}[\sin t\cos(t_{l+ir-r/2-j}-t)+\cos t\sin(t_{l+ir-r/2-j}-t)].$$

Здесь

$$(47)\qquad \sin\frac{\pi}{m}<\frac{\pi}{m},$$

$$(48)\qquad |\cos(t_{l+ir-r/2-j}-t)|\leqq 1,$$

$$(49)\qquad |\sin(t_{l+ir-r/2-j}-t)|\leqq\begin{cases}\pi\dfrac{2i-1}{m} & (i=2,3,\dots,\nu);\\[2mm] \pi\dfrac{2|i|+3}{m} & (i=-2,-3,\dots,-\nu).\end{cases}$$

Из (46)—(49) следует (45).

Полагая $f\in H_\omega[-1,1]$, где $\omega(\delta)$ выпуклый вверх модуль непрерывности и используя (41)—(45), (32) и лемму 3, получаем:

$$(50)\qquad |P_{m,k,r}(f,x)-f(x)|\leqq\omega\left(\frac{2\pi}{m}\sqrt{1-x^2}\right)\frac{1}{r}\sum_{j=0}^{r-1}\sum_{\substack{i=1-\nu\\ i\neq 0,1}}^{\nu}|\sigma_i(\vartheta)|+$$

$$+\omega\left(\frac{2\pi^2}{m^2}|x|\right)\frac{1}{r}(2|i|-1)[|\sigma_i(\vartheta)|+|\sigma_{1-i}(\vartheta)|]+\omega\left(\frac{\pi}{m}\sqrt{1-x^2}\right)+\omega\left(\frac{\pi^2}{m^2}|x|\right).$$

Пусть

$$(51)\qquad \gamma_{m,k,r}=\max_{0\leqq t\leqq t_1}\frac{1}{r}\sum_{j=0}^{r}\sum_{i=1}^{\nu}(2|i|-1)[|\sigma_i(t+t_j)|+|\sigma_{1-i}(t+t_j)|].$$

Из (50)—(51) и (40) следует

$$|P_{m,k,r}(f,x)-f(x)|\leqq\omega\left(\frac{\pi}{m}\sqrt{1-x^2}\right)+(C_{m,k,r}-1)\,\omega\left(\frac{2\pi}{m}\sqrt{1-x^2}\right)+$$

$$+\omega\left(\frac{\pi^2}{m^2}|x|\right)+(\gamma_{m,k,r}-1)\,\omega\left(\frac{2\pi^2}{m^2}|x|\right),$$

что и требовалось доказать.

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KERNELS OF FINITE OSCILLATIONS AND CONVOLUTION INTEGRALS

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1. Introduction. The aim of this note is to give a simple construction of singular convolution integrals of class S_{2m} for the approximation of 2π-periodic functions. Such an operator is said to belong to S_{2m}, $m\in\mathbf{N}$, provided the kernel is of finite oscillation of order $2m$, $p_n\in\mathcal{S}_{2m}$, i.e., the kernel, being an even trigonometric polynomial of degree n, has exactly $2m$ symmetric changes of sign (zeroes of odd multiplicity, i.e., of multiplicity 1 for the simplest case) in the fundamental period $[-\pi, \pi]$, this being so with m independent of the (approximation) parameter n.

A general theorem of P. P. Korovkin states that the optimal order of approximation for these operators may be increased up to $O(n^{-2-2m})$, $n\to\infty$; the trivial case $m=0$ is the earlier well-known theorem of P. P. Korovkin on the restriction of the optimal approximation order for singular integrals with positive (more exactly: non-negative) kernels to $O(n^{-2})$.

A series of papers is concerned with the class S_{2m} (as well as its algebraic analog); see e.g. [12]. In connection with this note the farthest reaching contributions are due to A. I. Kovalenko [6], P. L. Butzer—R. J. Nessel—K. Scherer [2], C. J. Hoff [5], and J. Szabados [13]; the most general investigation for the initial case $m=1$ may be found in [9].

Here it is the paper [13] which should be given greatest attention to: in order to avoid many cross references the reader is urged to compare the method of construction and proof given in this note with [13].

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2. Notations and auxiliary results. Let Π_n, $n\in\mathbf{N}_0:=\{0, 1, 2, \ldots\}$, denote the set of all normalized even trigonometric polynomials of degree n, i.e.,

$$p_n(x):=\frac{1}{2}+\sum_{k=1}^{n}\varrho_{k,n}(p)\cos kx; \tag{1}$$

Π_n^+ is used provided $p_n(x)\geqq 0$. The convergence factors in (1) are given by the Fourier coefficients, namely

$$\varrho_{k,n}(p):=\frac{1}{\pi}\int_{-\pi}^{\pi}p_n(t)\cos kt\,dt \quad (0\leqq k\leqq n). \tag{2}$$

As a fundamental characterization of the polynomials (1) the trigonometrical moments of order 2σ are used, i.e.,

$$T_{2\sigma} \equiv T_{2\sigma}(p) := \frac{1}{\pi} \int_{-\pi}^{\pi} \left(2 \sin \frac{t}{2}\right)^{2\sigma} p_n(t)\, dt \quad (\sigma \geqq 0). \tag{3}$$

The connection between the quantities (2) and (3) is given by

$$\varrho_{k,n}(p) = 1 - \sum_{\sigma=1}^{k} (-1)^{\sigma+1} \frac{k}{k+\sigma} \binom{k+\sigma}{2\sigma} T_{2\sigma}(p) \quad (k = 1, 2, 3, \ldots); \tag{4}$$

this depends upon the identity

$$\cos kx = 1 - \sum_{\sigma=1}^{k} (-1)^{\sigma+1} \frac{k}{k+\sigma} \binom{k+\sigma}{2\sigma} \left(2 \sin \frac{x}{2}\right)^{2\sigma} \quad (k \in \mathbf{N}) \tag{5}$$

(for an easy proof see e.g. [11, p. 67], cf. also [13, p. 185]).

Moreover, the following determinants built up from the trigonometric moments $T_{2\sigma}$ will play an essential role:

$$G_\sigma^{(m)} := \begin{vmatrix} T_{2\sigma} & T_{2\sigma+2} & \ldots & T_{2(\sigma+m-1)} \\ T_{2\sigma+2} & T_{2\sigma+4} & \ldots & T_{2(\sigma+m)} \\ \vdots & \vdots & & \vdots \\ T_{2(\sigma+m-1)} & & \ldots & T_{2(\sigma+2m-2)} \end{vmatrix} \quad (\sigma \in \mathbf{N}_0,\ m \in \mathbf{N}). \tag{6}$$

By Gram's inequality (cf. e.g. [4, p. 176]) it follows that $G_\sigma^{(m)} > 0$ for all parameters as indicated provided $p_n \in \Pi_n^+$. This is due to the fact that the set of linearly independent functions

$$f_{k,\sigma}(x) := \left(2 \sin \frac{x}{2}\right)^{2(k-1)+\sigma} \sqrt{p_n(x)} \quad (1 \leqq k \leqq m,\ \sigma \in \mathbf{N}_0)$$

form the Gram determinant (6) under the inner product

$$(f_{j,\sigma}, f_{k,\sigma}) := \frac{1}{\pi} \int_{-\pi}^{\pi} f_{j,\sigma}(t) f_{k,\sigma}(t)\, dt = T_{2(j+k+\sigma-2)} > 0 \quad (1 \leqq j,\ k \leqq m,\ \sigma \in \mathbf{N}_0).$$

The minors of $G_\sigma^{(m)}$, obtained by deleting the kth row and lth column of $G_\sigma^{(m)}$, will be denoted by $G_{\sigma,k,l}^{(m)}$.

Finally, for the purpose of approximating e.g. continuous functions by means of singular convolution integrals, a kernel $\{p_n\}_{n \in \mathbf{N}}$ of polynomials (1) is called an approximate identity provided

$$L_n(p) := \frac{1}{\pi} \int_{-\pi}^{\pi} |p_n(t)|\, dt \leqq M,\ M \geqq 1 \quad (n \in \mathbf{N}), \tag{7}$$

$$\lim_{n \to \infty} \varrho_{k,n}(p) = 1 \quad (k = 1, 2, 3, \ldots); \tag{8}$$

see e.g. [1, p. 31, 57, 59].

3. Kernels of finite oscillations. First of all, the general representation of a polynomial $q_{n+m}\in\mathcal{S}_{2m}\cap\Pi_{n+m}$ $(n, m\in\mathbf{N})$ is discussed. For $m\in\mathbf{N}$ let there be given m distinct points $\alpha_{j,n}$, $1\leq j\leq m$, such that

$$0<\alpha_{1,n}<\alpha_{2,n}<\ldots<\alpha_{m,n}<\pi; \tag{9}$$

then it is obvious that

$$Z_m(x):=\prod_{j=1}^{m}(\cos x-\cos\alpha_{j,n}) \tag{10}$$

is an even trigonometric polynomial of degree m having $2m$ symmetric changes of sign in $(-\pi,\pi)/\{0\}$. If, in addition $\{p_n\}_{n\in\mathbf{N}}$ is a positive (factor) kernel, then the polynomial

$$q_{n+m}(x):=A_nZ_m(x)p_n(x)=\frac{1}{2}+\sum_{k=1}^{n+m}\varrho_{k,n+m}(q)\cos kx \tag{11}$$

(with A_n being the normalization constant) belongs to $\mathcal{S}_{2m}\cap\Pi_{n+m}$. With respect to the application of trigonometric moments the following representation of (10) is more appropriate:

$$Z_m(x)=\frac{1}{2^m}\prod_{i=1}^{m}\left\{\left(2\sin\frac{\alpha_{i,n}}{2}\right)^2-\left(2\sin\frac{x}{2}\right)^2\right\}=\frac{1}{2^m}\sum_{i=1}^{m+1}(-1)^{i-1}C_i\left(2\sin\frac{x}{2}\right)^{2(i-1)} \tag{12}$$

with

$$C_i:=\sum_{1\leq j_0<j_1<\ldots<j_{m-i}\leq m}\ \prod_{l=0}^{m-i}\left(2\sin\frac{\alpha_{j_l,n}}{2}\right)^2\quad(1\leq i\leq m),$$

in particular,

$$C_1=\prod_{l=1}^{m}\left(2\sin\frac{\alpha_{l,n}}{2}\right)^2,\ \ldots,\ C_m=\sum_{j=1}^{m}\left(2\sin\frac{\alpha_{j,n}}{2}\right)^2,\ C_{m+1}\equiv 1. \tag{13}$$

(For e.g. (13) and later purposes it is appropriate to use the seemingly complicated indexing in (12).)

In order to generate a well-behaved kernel (11) the problem is as follows: starting from a given suitable positive factor kernel $\{p_n\}_{n\in\mathbf{N}}$ the set of zeroes $\{\alpha_{j,n}\}_{j=1}^m$ or, equivalently, the set of the m constants $\{C_j\}_{j=1}^m$ has to be determined such that (11) has good approximation properties, e.g. measured by the corresponding saturation limit; cf. (17).

THEOREM 1. *Let $m\in\mathbf{N}$ and $\{p_n\}_{n\in\mathbf{N}}$ be a positive approximate identity with trigonometric moments satisfying*

$$T_{2i}(p)=O(n^{-2i}),\quad i=1,2,\ldots,2m+1, \tag{14}$$

$$T_{2(2m+2)}(p)=o(n^{-2(2m+1)}),\qquad (n\to\infty). \tag{15}$$

Then the kernel $\{q_{n+m}\}_{n\in\mathbf{N}}\subset\Pi_{n+m}$ defined by

$$q_{n+m}(x):=\frac{1}{G_0^{(m+1)}}\sum_{j=1}^{m+1}(-1)^{j+1}G_{0,1,j}^{(m+1)}\left(2\sin\frac{x}{2}\right)^{2(j-1)}p_n(x)=$$

$$=\frac{1}{2}+\sum_{k=1}^{n+m}\varrho_{k,n+m}(q)\cos kx \tag{16}$$

is an approximate identity of class $\mathcal{S}_{2m}$. *For the corresponding convergence factors, explicitly given by* (27), *the saturation limit*

(17) $$\lim_{n\to\infty} n^{2+2m}(1-\varrho_{k,n+m}(q)) = S_m(q)\,\frac{\prod_{j=0}^{m}(k^2-j^2)}{(2m+2)!} \quad (k=1,2,3,\dots)$$

holds, the positive saturation constant $S_m(q)$ *being given by* (28).

PROOF. Using the representation (11) together with (12) the normalized oscillating kernel (i.e., $\in \Pi_{n+m}$) is calculated as

(18) $$q_{n+m}(x) = \frac{\sum_{j=1}^{m+1}(-1)^{j-1}C_j\left(2\sin\frac{x}{2}\right)^{2(j-1)}}{\sum_{j=1}^{m+1}(-1)^{j-1}C_jT_{2(j-1)}}\,p_n(x).$$

Concerning the expression

(19) $$N_{m,i}(p) := \sum_{j=1}^{m+1}(-1)^{j-1}C_jT_{2(j+i-1)} \quad (i\in\mathbf{N}_0)$$

it will be shown in (23) that indeed $N_{m,0}>0$ for all $m\in\mathbf{N}$, so that the denominator of (18) does not vanish. Using (5) it then follows straightforwardly that the new convergence factors are

(20) $$\varrho_{k,n+m}(q) = 1-\sum_{i=1}^{k}(-1)^{i+1}\frac{k}{k+i}\binom{k+i}{2i}\frac{N_{m,i}}{N_{m,0}} \quad (1\leqq k\leqq n+m).$$

Now one can choose C_j, $1\leqq j\leqq m$, in such a way that

(21) $$N_{m,i} = 0 \quad (1\leqq i\leqq m);$$

this is possible because the system of m linear equations

$$\sum_{j=1}^{m}(-1)^{j-1}C_jT_{2(j+i-1)} = (-1)^{m+1}T_{2(m+i)} \quad (1\leqq i\leqq m)$$

has the determinant $G_1^{(m)}>0$. The solution of this system is, by a painstaking application of Cramer's rule, given by

(22) $$C_j = \frac{G_{1,m+1,j}^{(m+1)}}{G_1^{(m)}} \quad (1\leqq j\leqq m).$$

Moreover, $G_{1,m+1,j}^{(m+1)}=G_{0,1,j}^{(m+1)}$ gives from (19) that

$$N_{m,0} = \frac{1}{G_1^{(m)}}\sum_{j=1}^{m+1}(-1)^{j-1}G_{0,1,j}^{(m+1)}T_{2(j-1)};$$

however, the latter sum is the Laplace expansion of the determinant $G_0^{(m+1)}$ with respect to the first column, which results in

(23) $$N_{m,0} = \frac{G_0^{(m+1)}}{G_1^{(m)}} > 0 \quad (m\in\mathbf{N}).$$

This finally proves via (18) the representation (16) of the oscillating kernel $\{q_{n+m}\}_{n\in\mathbf{N}}$. In order to investigate the convergence behaviour of the convergence factors (20) the quantities $N_{m,i}$ are now examined for $i\geqq m+1$. The representation of $N_{m,m+1}$ is settled using (19) by another Laplace expansion (with respect to the last column), namely

$$N_{m,m+1}=\frac{(-1)^m}{G_1^{(m)}}\sum_{j=1}^{m+1}(-1)^{m+j-1}G_{1,m+1,j}^{(m+1)}\,T_{2(j+m)}=(-1)^m\frac{G_1^{(m+1)}}{G_1^{(m)}}\quad(m\in\mathbf{N}).$$

Now definition (6), after inserting the order relations (14), (15), delivers

$$G_\sigma^{(m)}=g_\sigma^{(m)}\,n^{-2m(\sigma+m-1)}+o(n^{-2m(\sigma+m-1)})\quad(n\to\infty) \tag{24}$$

with $g_\sigma^{(m)}>0$ being defined by this asymptotic expansion. This yields

$$\frac{N_{m,m+1}}{N_{m,0}}=(-1)^m\frac{g_1^{(m+1)}}{g_0^{(m+1)}}\,n^{-2-2m}+o(n^{-2-2m})\quad(n\to\infty); \tag{25}$$

moreover, it easily follows that

$$\frac{N_{m,i}}{N_{m,0}}=o(n^{-2-2m}),\quad i>m+1\ (n\to\infty). \tag{26}$$

For the convergence factors (20), recalling (21), one now has that

$$1-\varrho_{k,n+m}(q)=\begin{cases}0, & 1\leqq k\leqq m,\\ \displaystyle\sum_{i=m+1}^{k}(-1)^{i+1}\frac{k}{k+i}\binom{k+i}{2i}\frac{N_{m,i}}{N_{m,0}}, & m+1\leqq k\leqq n+m.\end{cases} \tag{27}$$

Using the identity (see e.g. [9, p. 10])

$$\frac{k}{k+i}\binom{k+i}{2i}=\frac{1}{(2i)!}\prod_{j=0}^{i-1}(k^2-j^2)\quad(i,\,k\in\mathbf{N})$$

as well as (25), (26) one now obtains

$$1-\varrho_{k,n+m}(q)=\frac{g_1^{(m+1)}}{g_0^{(m+1)}}\frac{1}{(2m+2)!}\prod_{j=0}^{m}(k^2-j^2)\,n^{-2-2m}+o(n^{-2-2m})\quad(n\to\infty)$$

which gives (17) with

$$S_m(q)=\frac{g_1^{(m+1)}}{g_0^{(m+1)}}>0\quad(m\in\mathbf{N}). \tag{28}$$

Thus condition (8) as applied to the kernels (26) is obviously satisfied. It remains to show that (7) holds, too. But

$$L_{n+m}(q)\leqq\frac{1}{G_0^{(m+1)}}\sum_{j=1}^{m+1}|G_{0,1,j}^{(m+1)}|\,T_{2(j-1)}(p)=O(1)\quad(n\to\infty) \tag{29}$$

in view of (14), (15) as well as of $G_0^{(m+1)}=O(n^{-2(m+1)m})$, cf. (24), and $G_{0,1,j}^{(m+1)}==O(n^{-2(m+1)n+2(j-1)})$, $n\to\infty$.

It should be noted that the motivation leading to (21) is also seen by a comparison of (20) with (4). Indeed, (21) assures that the first m even trigonometric moments of the oscillating kernel will vanish on account of $T_{2i}(q)=N_{m,i}/N_{m,0}$ $(1 \leqq i \leqq m)$. This fact, in turn, is a nice characterization of the $\mathscr{S}_{2m}$-kernels under construction. — Secondly, at this point a remark concerning the growth of the zeroes (9) according to the construction of Theorem 1 is of interest. Knowing the quantities C_j of (22) the problem of solving the *non*linear system (13) for the quantities $\sin^2(\alpha_{j,n}/2)$ or $\alpha_{j,n}$, respectively, becomes intricate or impossible. However, it may be concluded from the easily verified relation

$$C_j = O(n^{-2(m-j+1)}) \quad (1 \leqq j \leqq m;\ n \to \infty)$$

by using (13) and an indirect argument, that $\sin^2(\alpha_{j,n}/2)=O(n^{-2})$, $\alpha_{j,n}=O(n^{-1})$, $1 \leqq j \leqq m$, $n \to \infty$, respectively. This together with the monotone ordering of (9) is in accordance with general statements of [2, p. 95 f] by which the zeroes of optimal oscillating kernels must tend to zero but not too rapidly.

4. Improved approximation. The construction of the foregoing section, in particular the basic limit relation (17), leads at once to the formulation of a general approximation and saturation theorem which reveals the improved approximation behaviour of the operators under consideration.

THEOREM 2. *Let the assumptions of Theorem* 1 *be satisfied, the singular convolution integral of class* S_{2m} *with kernel* (16) *being defined by*

$$I_{n+m}(q; f; x) := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t)\, q_{n+m}(t)\, dt \quad (f \in C_{2\pi}). \tag{30}$$

(i) *If* $f \in C_{2\pi}^{(2m+2)}$ *one has the Voronovskaja-type expansion*

$$\lim_{n \to \infty} n^{2+2m}\{f(x) - I_{n+m}(q; f; x)\} = \frac{S_m(q)}{(2m+2)!} \sum_{j=1}^{m+1} (-1)^j t(2m+2, 2j) f^{(2j)}(x) \tag{31}$$

where the coefficients are given by the central factorial numbers $t(2m+2, 2j)$ *as defined by* (32).

(ii) *The singular integral* (30) *is saturated with order* $O(n^{-2-2m})$, $n \to \infty$.

(iii) *One has* $\|f(x) - I_{n+m}(q; f; x)\| = o(n^{-2-2m})$, $n \to \infty$, *if and only if* f *is any trigonometric polynomial of degree* $\leqq m$.

(iv) *One has* $\|f(x) - I_{n+m}(q; f; x)\| = O(n^{-2-2m})$, $n \to \infty$, *if and only if* $f \in C_{2\pi}^{(2m+2)}$ *i.e., the saturation class of* (30) *is characterized by* $\{f; f^{(2m+1)} \in \text{Lip } 1\}$.

PROOF. (i) First of all, the polynomial representation of the product appearing in (17), namely

$$\prod_{j=0}^{m} (k^2 - j^2) = \sum_{j=1}^{m+1} t(2m+2, 2j) k^{2j} \quad (m \in \mathbf{N}_0,\ k \in \mathbf{N}), \tag{32}$$

(see [8, p. 233]) will play an essential role. The coefficients in (32) are the so-called central factorial numbers which satisfy ([8, p. 213 f])

$$t(2m+2, 2j) = t(2m, 2j-2) - m^2 t(2m, 2j), \quad t(m, 0) = \delta_{m,0}$$

and, in particular ([8, p. 233 f]),

$$t(2m+2,2)=(-1)^m(m!)^2,\quad t(2m+2,2m+2)=1.$$

The following part of a table ([8, p. 217]) may indicate the behaviour of these numbers:

	$j \backslash m$	0	1	2	3	4
$t(2m+2,2j)$:	1	1	−1	4	−36	576
	2		1	−5	49	−820
	3			1	−14	273
	4				1	−30
	5					1

Now, a trigonometric analog of Taylor's formula will be applied, namely

$$f(x+t)-f(x)=\sum_{j=1}^{2m+2}\frac{1}{j!}\left(2\sin\frac{t}{2}\right)^j\sum_{k=1}^{j}\sigma_{k,j}f^{(k)}(x)+ \tag{33}$$

$$+\frac{1}{(2m+2)!}\left(2\sin\frac{t}{2}\right)^{2m+2}\sum_{k=1}^{2m+2}\sigma_{k,2m+2}\{f^{(k)}(\eta_k)-f^{(k)}(x)\}$$

with $\eta_k=\eta_k(x,t)$ ranging between x and $x+t$. The coefficients $\sigma_{k,j}$ $(1\leqq k\leqq j,$ $1\leqq j\leqq 2m+2)$ are certain numbers which, as far as is needed here, satisfy

$$\begin{cases}\text{(i) } \sigma_{2j-1,2m+2}=0\\ \text{(ii) } \sigma_{2j,2m+2}=(-1)^{m+j+1}t(2m+2,2j)\end{cases}\quad (1\leqq j\leqq m+1). \tag{34}$$

A complete proof will be given elsewhere; for (34, (ii)) see (40) below. (A formula similar to (33) involving powers of $\sin t$ may be found for $1\leqq j\leqq 20$ with explicit numerical coefficients in [7]; cf. also [11, p. 65 f].)

After these preliminaries one easily deduces that

$$I_{n+m}(q;f;x)-f(x)=\frac{1}{(2m+2)!}T_{2m+2}(q)\sum_{k=1}^{2m+2}\sigma_{k,2m+2}f^{(k)}(x)+R_n(f) \tag{35}$$

$$=\frac{1}{(2m+2)!}T_{2m+2}(q)\sum_{k=1}^{m+1}\sigma_{2k,2m+2}f^{(2k)}(x)+R_n(f), \tag{36}$$

(37)

$$R_n(f):=\frac{1}{(2m+2)!}\sum_{j=1}^{m+1}\sigma_{2j,2m+2}\cdot\frac{1}{\pi}\int_{-\pi}^{\pi}\{f^{(2j)}(\eta_{2j})-f^{(2j)}(x)\}\left(2\sin\frac{t}{2}\right)^{2m+2}q_{n+m}(t)\,dt.$$

Indeed, all odd trigonometric moments appearing in (35) vanish since the underlying kernel is even, also $T_{2j}(q)=0$, $1\leqq j\leqq m$, by construction in view of (21); (36) and (37) are consequences of (34, (i)). Concerning the remainder term it has to be shown that (37) is of order $o(T_{2m+2}(q))$, $n\to\infty$.

However, introducing the modulus of continuity of $f^{(2j)}$, $1\leqq j\leqq m+1$, i.e., $\omega(f^{(2j)};\delta)$, together with the elementary property $\omega(f;\lambda\delta)\leqq(1+\lambda)\omega(f;\delta)$, $\lambda>0$, and choosing $\delta=\sqrt{T''_{2m+4}(q)/T''_{2m+2}(q)}$, where $T''_{2m}(q)=(1/\pi)\int_{-\pi}^{\pi}\left(2\sin\frac{t}{2}\right)^{2m}|q_{n+m}(t)|\,dt$

denotes the *absolute* trigonometric moment of order $2m$ of the oscillating kernel $\{q_{n+m}(x)\}_{n\in\mathbf{N}}$ and, finally, applying the Cauchy—Schwarz inequality, then routine arguments lead to the estimate

$$|R_n(f)| \leqq \frac{3}{(2m+2)!} T''_{2m+2}(q) \sum_{j=1}^{m+1} |\sigma_{2j,\,2m+2}|\, \omega\left(f^{(2j)};\ \sqrt{\frac{T''_{2m+4}(q)}{T''_{2m+2}(q)}}\right).$$

But $T''_{2m+2}(q)=O\big(T_{2m+2}(q)\big)$, $T''_{2m+4}(q)=o\big(T''_{2m+2}(q)\big)$, $n\to\infty$; this can be seen by the same arguments which lead to the estimate (29) for the Lebesgue constants of the oscillating kernel. Moreover, $\omega\big(f^{(2j)}, o(1)\big)=o(1)$, $n\to\infty$, since $f^{(2j)}\in C_{2\pi}$, $1\leqq j\leqq$ $\leqq m+1$. This finally proves that indeed

(38) $$|R_n(f)| = o\big(T_{2m+2}(q)\big) \quad (n\to\infty).$$

Last not least the representation of the coefficients in (31) is established. For the functions $f_k(x):=\cos kx$, $k\in\mathbf{N}$, one has for $x=0$ that

$$f_k^{(j)}(0) = \begin{cases} 0, & j=1,3,5,\dots \\ (-1)^{j/2}k^j, & j=2,4,6,\dots. \end{cases}$$

Applying this to (35) together with (38) yields

(39) $$\varrho_{k,n+m}(q)-1 = \frac{1}{(2m+2)!} T_{2m+2}(q) \sum_{j=1}^{m+1} (-1)^j \sigma_{2j,\,2m+2} k^{2j} + o\big(T_{2m+2}(q)\big) \ (n\to\infty).$$

Since, recalling (25),

$$T_{2m+2}(q) = (-1)^m \frac{S_m(q)}{n^{2+2m}} + o(n^{-2-2m}) \quad (n\to\infty),$$

the expansion (39) may be rewritten as

$$1-\varrho_{k,n+m}(q) = \frac{S_m(q)}{(2m+2)!} \sum_{j=1}^{m+1} (-1)^{m+j+1} \sigma_{2j,\,2m+2} k^{2j} \frac{1}{n^{2+2m}} + o(n^{-2-2m}) \quad (n\to\infty).$$

On the other hand, from (17) and (32) it follows that

$$1-\varrho_{k,\,n+m}(q) = \frac{S_m(q)}{(2m+2)!} \sum_{j=1}^{m+1} t(2m+2,2j)\, k^{2j} \frac{1}{n^{2+2m}} + o(n^{-2-2m}) \quad (n\to\infty)$$

so that

(40) $$\sigma_{2j,\,2m+2} = (-1)^{m+j+1} t(2m+2,2j).$$

This proves $(34,\ (\mathrm{ii}))$ as well as (31).

The proof of (ii) and (iii) is immediate; for the direct part of (iv) the Voronskaja-type theorem of (i) is used, whereas the indirect part of (iv) is a consequence of the general theorem given in [14].

5. Remarks and examples. Concerning the conditions (14), (15) upon the positive factor kernel, they are very restrictive; thus most of the classical positive kernels do not apply. On the other hand, there exists a number of suitable kernels generated by quite different methods: a main aspect in [13] is to show that certain powers of known kernels (e.g. Jackson, Fejér—Korovkin, modified de La Vallée

Poussin) satisfy a set of conditions which are equivalent to (14), (15). Whereas the polynomial degree is raised considerably by powering, there is another procedure given in [6] which leads to polynomials of class Π_{n-2} by using e.g. $\sin^{2m+1}\pi x$ as particular generating functions; this again generalizes the Fejér—Korovkin kernel ($m=0$). For further conditions equivalent to (14), (15) compare also [5], [9]: these are e.g. posed on the asymptotic expansions of the corresponding convergence factors.

It should be noted that by Theorems 1 and 2 just the optimal order of approximation for singular integrals of class S_{2m} is stressed. However, the approximation behaviour may also be improved by requiring — instead of (14), (15) — for any τ, $0<\tau\leqq 2$, that

$$\begin{aligned} T_{2i}(p) &= O(n^{-\tau i}), \quad i=1,2,\dots,2m+1, \\ T_{2(2m+2)}(p) &= o(n^{-\tau(2m+1)}), \end{aligned} \qquad (n\to\infty) \tag{41}$$

holds. Thus the main point is that the order of the trigonometric moments decreases linearly up to a certain degree determined by m. Then the modifications in the above statements are obvious, the proofs being completely parallel.

Finally, three examples may illustrate the range of applicability of the general theorems.

For the kernel of de La Vallée Poussin

$$V_n(x) = \frac{(n!)^2}{2(2n)!}\left(2\cos\frac{x}{2}\right)^{2n}, \quad \varrho_{k,n}(V) = \frac{(n!)^2}{(n-k)!(n+k)!} \tag{42}$$

the trigonometric moments admit the simple closed form (cf. [9, p. 69])

$$T_{2\sigma}(V) = \frac{(2\sigma)!\,n!}{\sigma!(n+\sigma)!} = \frac{(2\sigma)!}{\sigma!}n^{-\sigma}+o(n^{-\sigma}) \quad (\sigma\in\mathbf{N}_0;\ n\to\infty);$$

thus (41) is satisfied with $\tau=1$. For $m=1$ it follows that

$$\begin{aligned} V^*_{n+1}(x) &= \frac{(n+1)(n+2)}{2(2n+1)}\left\{\frac{6}{n+2}-\left(2\sin\frac{x}{2}\right)^2\right\}V_n(x) = \\ &= \frac{(n+1)(n+2)}{2n+1}\left\{\cos x-\frac{n-1}{n+2}\right\}V_n(x)\in\mathcal{S}_2, \end{aligned} \tag{43}$$

$$\begin{cases} \varrho_{k,n+1}(V^*) = \left(1+\dfrac{k^2}{n+1}\right)\varrho_{k,n+1}(V), \\ \lim\limits_{n\to\infty} n^2(1-\varrho_{k,n+1}(V^*)) = \dfrac{1}{2}k^2(k^2-1). \end{cases} \tag{44}$$

The kernel (43) is also discussed in [5 II, p. 131] and as an extremal case of another general construction in [9, p. 70]; for (44) see also [9, p. 71]. Concerning the decisive zero $\alpha_{1,1}(V,n)=\alpha_n$, from (43) one sees that

$$\left(2\sin\frac{\alpha_n}{2}\right)^2 = \frac{6}{n+2}, \quad \cos\alpha_n = \frac{n-1}{n+2}$$

respectively, giving the asymptotic expansion

$$\alpha_n = \sqrt{6}\, n^{-1/2} + o(n^{-1/2}) \quad (n \to \infty).$$

For another approach to the construction of kernels of class $\mathcal{S}_2$ where, with respect to (42), the zero is exactly determined by the limiting case $\bar{\alpha}_n = \sqrt{6/n}$ see [10, p. 343].

For $m=2$ one obtains

$$V^{**}_{n+2}(x) = \frac{(n+1)(n+2)(n+3)(n+4)}{4(2n+1)(2n+3)} \cdot$$

$$\cdot \left\{ \frac{60}{(n+3)(n+4)} - \frac{20}{(n+4)} \left(2 \sin \frac{x}{2}\right)^2 + \left(2 \sin \frac{x}{2}\right)^4 \right\} V_n(x),$$

$$\lim_{n\to\infty} n^3 (1 - \varrho_{k,n+2}(V^{**})) = \frac{1}{6} k^2(k^2-1)(k^2-4);$$

the corresponding Voronovskaja-type expansion reads

$$\lim_{n\to\infty} n^3 \{I_{n+2}(V^{**}; f; x) - f(x)\} = \frac{1}{6} \{4f^{(2)}(x) + 5f^{(4)}(x) + f^{(6)}(x)\} \quad (f \in C^{(6)}_{2\pi}).$$

The generalized kernel of Fejér—Korovkin, arising from $\sin^5 \pi x$ $(m=2)$, is given by ([9, p. 65 f])

$$K_{n-2}(x) = \frac{128}{63n} \sin^{10} \frac{\pi}{n} \cdot$$

$$\cdot \left\{ \frac{\cos^2 x + \cos x \cos \frac{\pi}{n}\left(16 \cos^2 \frac{\pi}{n} - 3\right) + \cos^2 \frac{\pi}{n}\left(24 \cos^2 \frac{\pi}{n} - 8\right)}{\left(\cos x - \cos \frac{\pi}{n}\right)\left(\cos x - \cos \frac{3\pi}{n}\right)\left(\cos x - \cos \frac{5\pi}{n}\right)} \cos \frac{nx}{2} \right\}^2;$$

for the trigonometric moments it can be shown that

$$T_2(K) = \frac{5^2}{3^2}\left(\frac{\pi}{n}\right)^2, \quad T_4(K) = \frac{5^3 \cdot 11}{3^2 \cdot 7}\left(\frac{\pi}{n}\right)^4, \quad T_6 = \frac{5^2 \cdot 97}{3^2}\left(\frac{\pi}{n}\right)^6,$$

$$T_8(K) = \frac{5^3 \cdot 317}{3^2}\left(\frac{\pi}{n}\right)^8, \quad T_{10} = \frac{5^2 \cdot 151 \cdot 1489}{3^2 \cdot 7}\left(\frac{\pi}{n}\right)^{10}.$$

This results in the optimal oscillating kernel of class $\mathcal{S}_4$ given by

$$K^*_n(x) = \frac{3^3}{2^2 \cdot 7 \cdot 677}\left(\frac{n}{\pi}\right)^4 \cdot \left\{ \frac{37}{3}\left(\frac{\pi}{n}\right)^4 - 2 \cdot 3 \cdot 5 \cdot 7 \left(\frac{\pi}{n}\right)^2 \left(2 \sin \frac{x}{2}\right)^2 + \left(2 \sin \frac{x}{2}\right)^4 \right\} K_{n-2}(x)$$

with saturation behaviour characterized by

$$\lim_{n\to\infty} n^6 (1 - \varrho_{k,n}(K^*)) = \frac{2^7 \cdot 5\pi^6}{7^2 \cdot 677} k^2(k^2-1)(k^2-4).$$

The Voronovskaja-type theorem holds as well. — The corresponding case $m=1$ is considered under a more general point of view in [9, p. 62 f], and the limiting case of a separated zero is investigated in [3].

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PROPERTIES OF S-CLOSED SPACES

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1. Introduction

In [12], T. THOMPSON has first introduced and investigated the concept of S-closed spaces. The purpose of the present paper is to improve upon some results concerning S-closed spaces due to T. THOMPSON [13] and to continue the investigations of such spaces. In § 3, we shall give a characterization of S-closed spaces and two sufficient conditions for a space to be S-closed. In § 4, we shall investigate which functions preserve sets S-closed relative to a space (or S-closed subspaces) and show that S-closedness is preserved under weakly-continuous almost-open surjections. In the final section, as an improvement of [13, Theorem 3.10], we shall show that a semi-continuous function of an S-closed Hausdorff space into a Hausdorff space is almost-closed. Moreover, the following characterization of S-closed spaces will be given: A Hausdorff space X is S-closed if and only if every semi-continuous function of X into any Hausdorff space is almost-closed.

2. Preliminaries

Throughout the present paper, spaces will mean topological spaces on which no separation axioms are assumed unless explicitely stated. Let S be a subset of a space X. The closure (interior) of S will be denoted by $\mathrm{Cl}_X(S)$ (resp. $\mathrm{Int}_X(S)$). A subset S of X is said to be *regular-closed (regular-open)* if $\mathrm{Cl}_X(\mathrm{Int}_X(S))=S$ (resp. $\mathrm{Int}_X \mathrm{Cl}_X(S))=S$). A subset S of X is said to be *semi-open* [4] if there exists an open set U of X such that $U\subset S\subset \mathrm{Cl}_X(U)$. The family of all semi-open sets of X will be denoted by $\mathrm{SO}(X)$. The complement of a semi-open set is said to be *semi-closed*. The intersection of all semi-closed sets containing $S\subset X$ is called the *semi-closure* of S and is denoted by $\mathrm{sCl}_X(S)$. A space X is said to be S-*closed* [12] if, for every semi-open cover $\{U_\alpha|\alpha\in\nabla\}$ of X, there exists a finite subfamily ∇_0 of ∇ such that $X=\bigcup\{\mathrm{Cl}_X(U_\alpha)|\alpha\in\nabla_0\}$. A subset S of X is said to be S-*closed* if it is S-closed as the subspace of X. A subset S of X is said to be S-*closed relative to* X [8] if, for every cover $\{U_\alpha|\alpha\in\nabla\}$ of S by semi-open sets in X, there exists a finite subfamily ∇_0 of ∇ such that $S\subset\bigcup\{\mathrm{Cl}_X(U_\alpha)|\alpha\in\nabla_0\}$. A space X is said to be *extremally disconnected* if the closure of every open set in X is open. The following lemmas will be used in the sequel.

LEMMA 2.1. *A subset G of a space X is* S-*closed relative to X if and only if every cover of G by regular-closed sets of X has a finite subcover.*

LEMMA 2.2 (NOIRI [8]). *An open set G of a space X is* S-*closed if and only if G is* S-*closed relative to X.*

3. S-closed spaces

In [8, Theorem 1.3], we have shown that a space X is S-closed if and only if every proper regular-open set of X is S-closed. Moreover, we shall obtain a characterization of S-closed spaces.

LEMMA 3.1 (NOIRI [9]). *Let A be a subset of a space X. If A is* S-*closed relative to X, then $\mathrm{Cl}_X(A)$ and $\mathrm{Int}_X(\mathrm{Cl}_X(A))$ are* S-*closed relative to X.*

THEOREM 3.2. *A space X is* S-*closed if and only if every proper regular-closed set of X is* S-*closed relative to X.*

PROOF. *Necessity*. Let F be any proper regular-closed set of X. Then, $\mathrm{Int}_X(F)$ is proper regular-open and hence $\mathrm{Int}_X(F)$ is S-closed [8, Theorem 1.3]. By Lemma 2.2, $\mathrm{Int}_X(F)$ is S-closed relative to X and hence, by Lemma 3.1, $F=\mathrm{Cl}_X(\mathrm{Int}_X(F))$ is S-closed relative to X.

Sufficiency. Let F be a proper regular-closed set of X. Then, by hypothesis' F is S-closed relative to X and hence by Lemma 3.1 $\mathrm{Int}_X(F)$ is S-closed relative to X. Since $X-\mathrm{Int}_X(F)$ is proper regular-closed, it is S-closed relative to X. Therefore, $X=\mathrm{Int}_X(F)\cup(X-\mathrm{Int}_X(F))$ is S-closed [9, Theorem 3.6].

In [1], it is stated that if a space X is written as the union of a finite number of S-closed clopen subsets, then X is S-closed. The following theorem shows that the condition "closed" on subsets in this results can be dropped.

THEOREM 3.3. *If a space X is the union of a finite number of* S-*closed open subsets, then X is* S-*closed.*

PROOF. By Lemma 2.2, open S-closed sets are S-closed relative to X. The union of a finite number of sets S-closed relative to X is *S*-closed relative to X [9, Theorem 3.6]. Therefore, X is S-closed.

THEOREM 3.4. *If there exists a dense subset G of X which is* S-*closed relative to X, then X is* S-*closed.*

PROOF. Let $\{F_\alpha|\alpha\in\nabla\}$ be any regular-closed cover of X. Since G is S-closed relative to X, by Lemma 2.1 there exists a finite subfamily ∇_0 of ∇ such that $G\subset\bigcup\{F_\alpha|\alpha\in\nabla_0\}$. Therefore, we have $X=\mathrm{Cl}_X(G)\subset\bigcup\{F_\alpha|\alpha\in\nabla_0\}$ which shows that X is S-closed [1, Theorem 2].

4. Sets S-closed relative to a space

A function $f\colon X\to Y$ is said to be *irresolute* [2] if $f^{-1}(V)\in\mathrm{SO}(X)$ for every $V\in\mathrm{SO}(Y)$. In [13], T. THOMPSON has showed that S-closedness is preserved under irresolute surjections. In this section we shall investigate the conditions on functions which preserve (inverse preserve) sets S-closed relative to a space.

LEMMA 4.1. *If X is an extremally disconnected space, then $\mathrm{sCl}_X(U)=\mathrm{Cl}_X(U)$ for each $U\in\mathrm{SO}(X)$.*

PROOF. In general, we have $\mathrm{sCl}_X(S)\subset \mathrm{Cl}_X(S)$ for every subset S of X. Thus, we shall show that $\mathrm{sCl}_X(U)\supset \mathrm{Cl}_X(U)$ for each $U\in \mathrm{SO}(X)$. Let $U\in \mathrm{SO}(X)$ and $x\notin \mathrm{sCl}_X(U)$, then there exists a $V\in \mathrm{SO}(X)$ such that $x\in V$ and $V\cap U=\emptyset$; hence $\mathrm{Int}_X(U)\cap \mathrm{Int}_X(V)=\emptyset$. Since X is extremally disconnected, we have $\mathrm{Cl}_X(\mathrm{Int}_X(U))\cap \mathrm{Cl}_X(\mathrm{Int}_X(V))=\emptyset$. Therefore, we have $x\notin \mathrm{Cl}_X(\mathrm{Int}_X(U))=\mathrm{Cl}_X(U)$.

THEOREM 4.2. *Let X be an extremally disconnected space and $f\colon X\to Y$ an irresolute function. If G is* S-*closed relative to X, then $f(G)$ is* S-*closed relative to Y.*

PROOF. Let $\{V_\alpha|\alpha\in\nabla\}$ be a cover of $f(G)$ and $V_\alpha\in \mathrm{SO}(Y)$ for each $\alpha\in\nabla$. Since f is irresolute, $f^{-1}(V_\alpha)\in \mathrm{SO}(X)$ for each $\alpha\in\nabla$ and $G\subset \bigcup\{f^{-1}(V_\alpha)|\alpha\in\nabla\}$. Therefore, there exists a finite subfamily ∇_0 of ∇ such that $G\subset\bigcup\{\mathrm{Cl}_X(f^{-1}(V_\alpha))|\alpha\in\nabla_0\}$. By Lemma 4.1, we have $G\subset \mathrm{sCl}_X(\bigcup\{f^{-1}(V_\alpha)|\alpha\in\nabla_0\})$. By using Theorem 1.5 of [2], we obtain

$$f(G)\subset f(\mathrm{s\,Cl}_X(\bigcup_{\alpha\in\nabla_0} f^{-1}(V_\alpha)))\subset \mathrm{s\,Cl}_Y(f(\bigcup_{\alpha\in\nabla_0} f^{-1}(V_\alpha)))\subset \bigcup_{\alpha\in\nabla_0}\mathrm{Cl}_Y(V_\alpha).$$

This shows that $f(G)$ is S-closed relative to Y.

THEOREM 4.3. *Let $f\colon X\to Y$ be an irresolute function. If G is an open* S-*closed set of X, then $f(G)$ is* S-*closed in Y.*

PROOF. Let $f_G\colon G\to f(G)$ be a function defined by $f_G(x)=f(x)$ for every $x\in G$. We shall show that f_G is irresolute. For any $V_0\in \mathrm{SO}(f(G))$, there exists a $V\in \mathrm{SO}(Y)$ such that $V_0=V\cap f(G)$ [10, Teorema 3.2]. Since f is irresolute and G is open in X, $f^{-1}(V)\cap G\in \mathrm{SO}(X)$ and hence, by Theorem 1 of [5], $f_G^{-1}(V_0)= =f^{-1}(V)\cap G\in \mathrm{SO}(G)$. This shows that $f_G\colon G\to f(G)$ is irresolute. Since G is S-closed, it follows from Theorem 3.5 of [13] that $f_G(G)=f(G)$ is S-closed.

A function $f\colon X\to Y$ is said to be *weakly-continuous* [3], if, for each point $x\in X$ and each open set $V\subset Y$ containing $f(x)$, there exists an open set $U\subset X$ containing x such that $f(U)\subset \mathrm{Cl}_Y(V)$. A function $f\colon X\to Y$ is said to be *almost-open* [14] if $f^{-1}(\mathrm{Cl}_Y(V))\subset \mathrm{Cl}_X(f^{-1}(V))$ for every open set V of Y.

LEMMA 4.4. *If a function $f\colon X\to Y$ is weakly-continuous and almost-open, then $f^{-1}(F)$ is regular-closed in X for every regular-closed set F of Y.*

PROOF. Let F be any regular-closed set of Y. Since f is weakly-continuous and almost-open, by Theorem 4 of [7], we have

$$\mathrm{Cl}_X(f^{-1}(\mathrm{Int}_Y(F)))=f^{-1}(\mathrm{Cl}_Y(\mathrm{Int}_Y(F)))=f^{-1}(F).$$

This shows that $f^{-1}(F)$ is closed in X. By using Theorem 1 of [3], we obtain $f^{-1}(\mathrm{Int}_Y(F))\subset \mathrm{Int}_X(f^{-1}(F))$ and hence $\mathrm{Cl}_X(f^{-1}(\mathrm{Int}_Y(F)))\subset \mathrm{Cl}_X(\mathrm{Int}_X(f^{-1}(F)))\subset \subset f^{-1}(F)$. Consequently, we obtain $f^{-1}(F)=\mathrm{Cl}_X(\mathrm{Int}_X(f^{-1}(F)))$. This shows that $f^{-1}(F)$ is regular-closed.

THEOREM 4.5. *Let $f\colon X\to Y$ be a weakly-continuous and almost-open function. If G is* S-*closed relative to X, then $f(G)$ is* S-*closed relative to Y.*

PROOF. Let $\{F_\alpha|\alpha\in\nabla\}$ be any cover of $f(G)$ by regular-closed sets of Y. Then, by Lemma 4.4, $\{f^{-1}(F_\alpha)|\alpha\in\nabla\}$ is a cover of G by regular-closed sets of X. It follows from Lemma 2.1 that $G\subset\bigcup\{f^{-1}(F_\alpha)|\alpha\in\nabla_0\}$ for some finite subfamily

∇_0 of ∇. Thus, we have $f(G) \subset \cup \{F_\alpha | \alpha \in \nabla_0\}$. This shows that $f(G)$ is S-closed relative to Y.

COROLLARY 4.6. *If X is an S-closed space and $f: X \to Y$ is a weakly-continuous almost-open surjection, then Y is S-closed.*

A function $f: X \to Y$ is said to be *semi-closed* [6] if $f(F)$ is semi-closed in Y for every closed set F of X.

THEOREM 4.7. *Let X be an extremally disconnected space, $f: X \to Y$ a semi-closed almost-open surjection and $f^{-1}(y)$ S-closed relative to X for each point $y \in Y$. If G is S-closed relative to Y, then $f^{-1}(G)$ is S-closed relative to X.*

PROOF. Let $\{F_\alpha | \alpha \in \nabla\}$ be a cover of $f^{-1}(G)$ by regular-closed sets of X. For each $y \in G$, by Lemma 2.1 there exists a finite subfamily $\nabla(y)$ of ∇ such that $f^{-1}(y) \subset \cup \{F_\alpha | \alpha \in \nabla(y)\}$. Since X is extremally disconnected, for each $\alpha \in \nabla$, $F_\alpha = \mathrm{Cl}_X(\mathrm{Int}_X(F_\alpha))$ is open in X. Now, put $U(y) = \cup \{F_\alpha | \alpha \in \nabla(y)\}$, then there exists a $V(y) \in \mathrm{SO}(Y)$ such that $y \in V(y)$ and $f^{-1}(V(y)) \subset U(y)$ [6, Theorem 5]. Since $\{V(y) | y \in G\}$ is a cover of G by semi-open sets of Y, there exists a finite number of points $y_1, y_2, \ldots, y_n$ in G such that $G \subset \cup \{\mathrm{Cl}_Y(V(y_j)) | j = 1, 2, \ldots, n\}$. By using the almost-openness of f, we obtain

$$f^{-1}(G) \subset \bigcup_{j=1}^{n} f^{-1}(\mathrm{Cl}_Y(V(y_j))) = \bigcup_{j=1}^{n} f^{-1}(\mathrm{Cl}_Y(\mathrm{Int}_Y(V(y_j)))) \subset$$

$$\subset \bigcup_{j=1}^{n} \mathrm{Cl}_X(f^{-1}(\mathrm{Int}_Y(V(y_j)))) \subset \bigcup_{j=1}^{n} \mathrm{Cl}_X(f^{-1}(V(y_j))) \subset \bigcup_{j=1}^{n} \mathrm{Cl}_X(U(y_j)) = \bigcup_{j=1}^{n} \bigcup_{\alpha \in \nabla(y_j)} F_\alpha.$$

It follows from Lemma 2.1 that $f^{-1}(G)$ is S-closed relative to X.

COROLLARY 4.8. *Let X be an extremally disconnected space and $f: X \to Y$ a closed open surjection with compact point inverses. If Y is an S-closed space, then X is S-closed.*

PROOF. This follows immediately from Lemma 2.1 and Theorem 4.7.

COROLLARY 4.9. *If X is compact, Y is S-closed and $X \times Y$ is extremally disconnected, then $X \times Y$ is S-closed.*

PROOF. Since X is compact, the natural projection $\pi_y: X \times Y \to Y$ is a closed open surjection with compact point inverses. Therefore, it follows from Corollary 4.8 that $X \times Y$ is *S*-closed.

REMARK 4.10. In Corollary 4.9, the condition "extremally disconnected" on $X \times Y$ cannot be dropped because βN is S-closed and compact, but $\beta N \times \beta N$ is not S-closed [1, Example 2; 8, Remark 3.2].

5. Semi-continuous images of S-closed spaces

In Theorem 3.10 of [13], T. THOMPSON showed that the irresolute image of any S-closed Hausdorff space in any Hausdorff space is closed. In this section we shall improve upon this result. A function $f\colon X \to Y$ is said to be *semi-continuous* [4] if $f^{-1}(V) \in \mathrm{SO}(X)$ for every open set V of Y. A function $f\colon X \to Y$ is said to be *almost-closed* [11] if $f(F)$ is closed in Y for every regular-closed set F of X.

LEMMA 5.1. *If $f\colon X \to Y$ is a semi-continuous function and G is an open set of X, then the function $f_G\colon G \to f(G)$, defined by $f_G(x) = f(x)$ for every $x \in G$, is semi-continuous.*

PROOF. This is proved in a similar way to Theorem 4.3.

The following theorem shows that the conditions "irresolute" and "S-closed Hausdorff" in Theorem 3.10 of [13] can be replaced by "semi-continuous" and "S-closed", respectively.

THEOREM 5.2. *The semi-continuous image of any* S-*closed space in any Hausdorff space is closed.*

PROOF. This follows immediately from Lemma 5.1 and Theorem 3.2 of [13].

THEOREM 5.3. *If X is an* S-*closed Hausdorff space, Y is a Hausdorff space and $f\colon X \to Y$ is a semi-continuous function, then f is almost-closed.*

PROOF. Since X is S-closed Hausdorff, it is extremally disconnected [12, Theorem 7]. Let F be any regular-closed set of X, then $F = \mathrm{Cl}_X(\mathrm{Int}_X(F))$ is open in X. Therefore, by Lemma 5.1, $f_F\colon F \to f(F)$ is semi-continuous. Since X is S-closed, F is S-closed [12, Lemma] and hence $f_F(F) = f(F)$ is H-closed [13, Theorem 3.2]. Since Y is Hausdorff, $f(F)$ is closed in Y. This shows that f is almost-closed.

COROLLARY 5.4. *A Hausdorff space X is* S-*closed if and only if every semi-continuous function of X into any Hausdorff space is almost-closed.*

PROOF. *Necessity*. See Theorem 5.3.

Sufficiency. Let Y be any Hausdorff space and $f\colon X \to Y$ any irresolute function. Then, f is semi-continuous and hence, by hypothesis, f is almost-closed. Thus, $f(X)$ is closed in Y and hence X is S-closed [13, Theorem 3.11].

COROLLARY 5.5. *If X is an S-closed Hausdorff space, Y is a Hausdorff space and $f\colon X \to Y$ is a semi-continuous bijection, then f is irresolute and Y is* S-*closed,*

PROOF. Let V be any semi-open set of Y. Since f is semi-continuous. $f^{-1}(\mathrm{Int}_Y(V)) \in \mathrm{SO}(X)$. Now, put $F = \mathrm{Cl}_X(\mathrm{Int}_X(f^{-1}(\mathrm{Int}_Y(V))))$, then we have $f^{-1}(\mathrm{Int}_Y(V)) \subset F$ and hence $\mathrm{Int}_Y(V) \subset f(F)$. By Theorem 5.3, f is almost-closed and hence $f(F)$ is closed in Y. Therefore, we have $\mathrm{Cl}_Y(\mathrm{Int}_Y(V)) \subset f(F)$. Consequently, we obtain

$$f^{-1}(V) \subset f^{-1}(\mathrm{Cl}_Y(\mathrm{Int}_Y(V))) \subset F \subset \mathrm{Cl}_X(\mathrm{Int}_X(f^{-1}(V))).$$

This shows that $f^{-1}(V) \in \mathrm{SO}(X)$. Therefore, f is irresolute and hence Y is S-closed [13, Theorem 3.5].

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STRONG SUBBAND-SEPARATING EXTENSIONS OF ORTHODOX SEMIGROUPS

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The greatest idempotent separating congruence and the least inverse semigroup congruence play a significant role in the theory of orthodox semigroups. On the one hand, a number of structure theorems have been proved for orthodox semigroups with a given band of idempotents (e.g. in [1], [6]—[9], [14], [16]—[19]) where an orthodox semigroup is described as an extension of a well-determined orthodox semigroup. The congruence relation corresponding to this extension is the greatest congruence in some sense, mostly the greatest idempotent separating congruence. On the other hand, M. YAMADA [21] has characterized the structure of orthodox semigroups by means of inverse semigroups and bands. Actually, he described an orthodox semigroup as the extension of an inverse semigroup to which the least inverse semigroup congruence corresponds. By investigating the common features of these congruences we came to the concept of subband-parcelling and strong subband-parcelling congruences which were defined and investigated in [11].

Let B be a band and δ a congruence on it with $\delta \subseteq \mathcal{D}$. Let $\bar{B}$ be a subband of B such that $\bar{B}$ is a union of δ-classes. Suppose S is an orthodox semigroup with band of idempotents B. A congruence $\varkappa$ on S is said to be a $(\bar{B}, \delta)$*-parcelling congruence* if the following conditions are satisfied:

(i) $\delta \subseteq \varkappa | B$,

(ii) every $\varkappa$-class containing an idempotent contains an element of $\bar{B}$,

(iii) the elements of $\bar{B}$ contained in a $\varkappa$-class form a δ-class which is the greatest one among the δ-classes belonging to this $\varkappa$-class. (The ordering of the δ-classes is the natural ordering of B/δ.)

If $\varkappa$ is a $(\bar{B}, \delta)$-parcelling congruence on S then

$$S_{\bar{B}} = \{s \in S \colon \text{there exist } e, f \in \bar{B} \text{ with } e \mathcal{R} s \mathcal{L} f\}$$

is an orthodox subsemigroup of S with band of idempotents $\bar{B}$ as it has been proved in [11]. By a *strong* $(\bar{B}, \delta)$-parcelling *congruence* $\varkappa$ we mean a $(\bar{B}, \delta)$-parcelling congruence with the property that every $\varkappa$-class contains an element in $S_{\bar{B}}$.

In particular, if δ is the equality relation then we call $\varkappa$ a $\bar{B}$*-separating* or a *strong* $\bar{B}$*-separating congruence*, respectively.

An orthodox semigroup T is said to be a *strong subband-parcelling* [*subband-separating*] *extension* of an orthodox semigroup S if there exists a strong subband-parcelling [subband-separating] congruence $\varkappa$ on T with $T/\varkappa \cong S$.

In [12] we described certain strong subband-separating extensions of 0-$\mathcal{D}$-simple orthodox semigroups. Applying this result, in [14] we proved structure theorems generalizing some of the above results.

In this paper we characterize all strong subband-separating extensions of orthodox semigroups. This result can be considered as a generalization of H. D'ALARCAO's theorem [3] concerning the idempotent separating extensions of inverse semigroups.

In [15] a more general problem is raised. We describe all strong subband-parcelling extensions of orthodox semigroups. However, it is worth dealing with the special case of strong subband-separating extensions separately because their characterization demands much simpler means.

Throughout the paper we adhere to the notations and terminology of [2]. For the basic properties of orthodox semigroups we refer to [5] or [22].

1. The construction

In this section we introduce the construction by means of which the strong subband-separating extensions of orthodox semigroups will be described.

Let S be an orthodox semigroup with band of idempotents E. Suppose E is a semilattice Y of rectangular bands $E_\alpha (\alpha\in Y)$; in notation, $E=\bigcup_{\alpha\in Y} E_\alpha$. Denote by $\sim$ the least inverse semigroup congruence on S. If $s\in S$, let us denote by $r(s)$ and $l(s)$ the element $\alpha\in Y$ with the property that E_α is the left unit and the right unit, respectively, of the $\sim$-class containing s. Clearly, for every inverse s' of s, we have $r(s')=l(s)$ and $l(s')=r(s)$, and, moreover, $ss'\in E_{r(s)}$ and $s's\in E_{l(s)}$. Denote by τ_s the mapping of $r(s)Y$ onto $l(s)Y$ which assigns to every element $\alpha(\leqq r(s))$ in Y the element β with $s'E_\alpha s\subseteqq E_\beta$. It is well known that τ_s is an isomorphism. In particular, if e is idempotent then τ_e is the identity automorphism of $r(e)Y$. Furthermore, if $s, \bar{s}\in S$ then $\tau_s\tau_{\bar{s}}=\tau_{s\bar{s}}$ and $l(s\bar{s})=(l(s)r(\bar{s}))\tau_{\bar{s}}$. If s and s' are inverses of each other in S then τ_s and $\tau_{s'}$ are also inverses of each other. In the sequel we apply these facts without any comment.

Let M be a band E of monoids $M_e (e\in E)$ with identities denoted by 1_e. If $\{1_e\colon e\in E\}$ is a subband in M then M is called a *proper band E of monoids* $M_e(e\in E)$. Then, clearly, the equality $1_e 1_f=1_{ef}$ holds for every $e, f\in E$. Hence $\{1_e\colon e\in E\}$ is isomorphic to E.

Let Σ_α be an orthodox monoid for every $\alpha\in Y$. Denote by ε_α the identity of Σ_α and by $U(\Sigma_\alpha)$ the group of units in Σ_α. Suppose Σ is a proper semilattice Y of the orthodox monoids $\Sigma_\alpha (\alpha\in Y)$. It is well known that in this case there exist monoid homomorphisms $\varphi_{\alpha,\beta}\colon \Sigma_\alpha\to\Sigma_\beta$ for every $\alpha, \beta\in Y$ with $\alpha\geqq\beta$ such that (i) $\varphi_{\alpha,\alpha}$ is the identity automorphism of Σ_α for every $\alpha\in Y$, (ii) $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma}=\varphi_{\alpha,\gamma}$ provided $\alpha\geqq\beta\geqq\gamma$ and (iii) $\varrho\cdot\sigma=\varrho\varphi_{\alpha,\alpha\beta}\cdot\sigma\varphi_{\beta,\alpha\beta}$ holds in Σ for every $\varrho\in\Sigma_\alpha$ and $\sigma\in\Sigma_\beta$. Property (iii) implies that Σ is orthodox as Σ_α is orthodox for each $\alpha\in Y$.

For every $s\in S$, let h_s be an endomorphism of Σ with the property that $\varepsilon_\alpha h_s=\varepsilon_{(\alpha r(s))\tau_s}$ for every $\alpha\in Y$. Moreover, let $\chi_{s,\bar{s}}\in U(\Sigma_{l(s\bar{s})})$ be given for every pair of elements $s, \bar{s}\in S$. Assume that the mappings $h\colon S\to \mathrm{End}\,\Sigma$, $s\mapsto h_s$ and $\chi\colon S\times S\to\Sigma$, $(s,\bar{s})\mapsto\chi_{s,\bar{s}}$ satisfy the following conditions:

(A1) if $s\sim s_1$, $\bar{s}\sim\bar{s}_1$ then $h_s=h_{s_1}$ and $\chi_{s,\bar{s}}=\chi_{s_1,\bar{s}_1}$;

(A2) $\chi_{s,\bar{s}}h_t=\chi_{s\bar{s},t}^{-1}\cdot\chi_{s,\bar{s}t}\cdot\chi_{\bar{s},t}$ for every $s, \bar{s}, t\in S$;

(A3) $\sigma h_s h_{\bar{s}}=\chi_{s,\bar{s}}^{-1}\cdot\sigma h_{s\bar{s}}\cdot\chi_{s,\bar{s}}$ for every $s, \bar{s}\in S$ and $\sigma\in\Sigma$;

(A4) if $e \in E$ and $\sigma \in \Sigma_\alpha$ then $\sigma h_e = \sigma \cdot \varepsilon_{l(e)} (= \sigma \varphi_{\alpha, \alpha l(e)})$;

(A5) if $s \in S$ and $e, f \in E$ with $e \mathscr{R} s \mathscr{L} f$ then $\chi_{e,s} = \chi_{s,f} = \varepsilon_{l(s)}$;

(A6) $\chi_{e,f} = \varepsilon_{l(ef)}$ provided $e, f \in E$.

Here χ^{-1} is used to denote the inverse of $\chi \in U(\Sigma_\alpha)$ in the group $U(\Sigma_\alpha)$.

A pair of mappings h, χ fulfilling these conditions is called an (S, Σ)*-pair*.

LEMMA 1.1. *If h, χ form an (S, Σ)-pair then $\Sigma_\alpha h_s \subseteq \Sigma_{(\alpha r(s))\tau_s}$ for every $\alpha \in Y$.*

PROOF. Suppose $\varrho \in \Sigma_\alpha$. Then

$$\varrho h_s = (\varrho \cdot \varepsilon_\alpha) h_s = \varrho h_s \cdot \varepsilon_{(\alpha r(s))\tau_s}, \tag{1}$$

for ε_α is an identity in Σ_α, h_s is an endomorphism and $\varepsilon_\alpha h_s = \varepsilon_{(\alpha r(s))\tau_s}$. However, $(\alpha r(s))\tau_s \leqq r(s)\tau_s$ implies $\varepsilon_{(\alpha r(s))\tau_s} \cdot \varepsilon_{r(s)\tau_s} = \varepsilon_{(\alpha r(s))\tau_s}$ whence we have $\varrho h_s = \varrho h_s \cdot \varepsilon_{r(s)\tau_s} = \varrho h_s \cdot \varepsilon_{r(s)} h_s = (\varrho \cdot \varepsilon_{r(s)}) h_s$. Here $\varrho \cdot \varepsilon_{r(s)} \in \Sigma_{\alpha r(s)}$. Thus the assertion to be proved is reduced to the following one: $\Sigma_\alpha h_s \subseteq \Sigma_{\alpha\tau_s}$ provided $\alpha \in Y$ with $\alpha \leqq r(s)$. Now let $\sigma \in \Sigma_\alpha$ where $\alpha \leqq r(s)$. Then, by (1), we have $\sigma h_s \in \Sigma_\beta$ where $\beta \leqq \alpha\tau_s$. If s' is an inverse of s then, according to (A3) and (A4), we infer $\sigma h_s h_{s'} = \chi_{s,s'}^{-1} \cdot \sigma h_{ss'} \cdot \chi_{s,s'} = \chi_{s,s'}^{-1} \cdot \sigma \cdot \varepsilon_{l(ss')} \cdot \chi_{s,s'}$. Here $l(ss') = l(s') = r(s)$ whence $\varepsilon_\alpha \cdot \varepsilon_{r(s)} = \varepsilon_\alpha$ implies

$$\sigma h_s h_{s'} = \chi_{s,s'}^{-1} \cdot \sigma \cdot \chi_{s,s'} \in \Sigma_\alpha .$$

Hence, by (1), the inequality $\alpha \leqq (\beta r(s'))\tau_{s'}$ follows. Since $\beta \leqq \alpha\tau_s \leqq l(s) = r(s')$ this means that $\alpha \leqq \beta\tau_s^{-1}$. Thus $\beta \leqq \alpha\tau_s \leqq \beta\tau_s^{-1}\tau_s = \beta$, that is, we have $\beta = \alpha\tau_s$ which was to be proved.

Let us define a multiplication on the set

$$\mathbf{S} = \{(s, \sigma) \colon s \in S,\ \sigma \in \Sigma_{l(s)}\}$$

as follows: if $s, \bar{s} \in S$ and $\sigma \in \Sigma_{l(s)}, \bar{\sigma} \in \Sigma_{l(\bar{s})}$ then

$$(s, \sigma)(\bar{s}, \bar{\sigma}) = (s\bar{s}, \chi_{s,\bar{s}} \cdot \sigma h_{\bar{s}} \cdot \bar{\sigma}). \tag{2}$$

The product in the second component is taken in Σ. By definition, $\chi_{s,\bar{s}} \in \Sigma_{l(s\bar{s})}$ and, by Lemma 1.1, $\sigma h_{\bar{s}} \in \Sigma_{(l(s)r(\bar{s}))\tau_{\bar{s}}}$. Since $l(s\bar{s}) = (l(s)r(\bar{s}))\tau_{\bar{s}} \leqq r(\bar{s})\tau_{\bar{s}} = l(\bar{s})$ we can see that $\chi_{s,\bar{s}} \cdot \sigma h_{\bar{s}} \cdot \bar{\sigma} \in \Sigma_{l(s\bar{s})}$. Thus $\mathbf{S}$ is closed under the multiplication defined in (2).

LEMMA 1.2. *The multiplication defined in* (2) *is associative.*

PROOF. Let $s, \bar{s}, \bar{\bar{s}} \in S$ and $\sigma \in \Sigma_{l(s)}$, $\bar{\sigma} \in \Sigma_{l(\bar{s})}$ and $\bar{\bar{\sigma}} \in \Sigma_{l(\bar{\bar{s}})}$. Then, by (2), we have $((s, \sigma)(\bar{s}, \bar{\sigma}))(\bar{\bar{s}}, \bar{\bar{\sigma}}) = ((s\bar{s})\bar{\bar{s}}, \sigma_1)$ and $(s, \sigma)((\bar{s}, \bar{\sigma})(\bar{\bar{s}}, \bar{\bar{\sigma}})) = (s(\bar{s}\bar{\bar{s}}), \sigma_2)$ where $\sigma_1 = \chi_{s\bar{s},\bar{\bar{s}}} \cdot (\chi_{s,\bar{s}} \cdot \sigma h_{\bar{s}} \cdot \bar{\sigma}) h_{\bar{\bar{s}}} \cdot \bar{\bar{\sigma}}$ and $\sigma_2 = \chi_{s,\bar{s}\bar{\bar{s}}} \cdot \sigma h_{\bar{s}\bar{\bar{s}}} \cdot (\chi_{\dot{s},\bar{\bar{s}}} \cdot \bar{\sigma} h_{\bar{\bar{s}}} \cdot \bar{\bar{\sigma}})$. Since S is associative all we have to show is that $\sigma_1 = \sigma_2$. Utilizing the fact that h_s is an endomorphism, (A2) and (A3) imply that

$$\sigma_1 = \chi_{s\bar{s},\bar{\bar{s}}} \cdot \chi_{s,\bar{s}} h_{\bar{\bar{s}}} \cdot \sigma h_{\bar{s}} h_{\bar{\bar{s}}} \cdot \bar{\sigma} h_{\bar{\bar{s}}} \cdot \bar{\bar{\sigma}} = \chi_{s\bar{s},\bar{\bar{s}}} \cdot \chi^{-1}_{s\bar{s},\bar{\bar{s}}} \cdot \chi_{s,\bar{s}\bar{\bar{s}}} \cdot \chi_{\bar{s},\bar{\bar{s}}} \cdot \chi^{-1}_{\bar{s},\bar{\bar{s}}} \cdot \sigma h_{\bar{s}\bar{\bar{s}}} \cdot \chi_{\bar{s},\bar{\bar{s}}} \cdot \bar{\sigma} h_{\bar{\bar{s}}} \cdot \bar{\bar{\sigma}} =$$

$$= \varepsilon_{l(s\bar{s}\bar{\bar{s}})} \cdot \chi_{s,\bar{s}\bar{\bar{s}}} \cdot \varepsilon_{l(\bar{s}\bar{\bar{s}})} \cdot \sigma h_{\bar{s}\bar{\bar{s}}} \cdot \chi_{\bar{s},\bar{\bar{s}}} \cdot \bar{\sigma} h_{\bar{\bar{s}}} \cdot \bar{\bar{\sigma}}$$

in the semigroup Σ. Here $\chi_{s,\bar{s}\bar{\bar{s}}}\in\Sigma_{l(s\bar{s}\bar{\bar{s}})}$. Thus $\chi_{s,\bar{s}\bar{\bar{s}}}=\varepsilon_{l(s\bar{s}\bar{\bar{s}})}\cdot\chi_{s,\bar{s}\bar{\bar{s}}}=\chi_{s,\bar{s}\bar{\bar{s}}}\cdot\varepsilon_{l(s\bar{s}\bar{\bar{s}})}$. Moreover, the equality $\varepsilon_{l(s\bar{s}\bar{\bar{s}})}\cdot\varepsilon_{l(\bar{s}\bar{\bar{s}})}=\varepsilon_{l(s\bar{s}\bar{\bar{s}})}$ follows from the inequality $l(s\bar{s}\bar{\bar{s}})\leqq l(\bar{s}\bar{\bar{s}})$. Hence $\sigma_1=\chi_{s,\bar{s}\bar{\bar{s}}}\cdot\sigma h_{\bar{s}\bar{\bar{s}}}\cdot\chi_{\bar{s},\bar{\bar{s}}}\cdot\bar{\sigma}h_{\bar{\bar{s}}}\cdot\bar{\bar{\sigma}}=\sigma_2$ which completes the proof.

Lemma 1.2 shows that the set $\mathbf{S}$ forms a semigroup with respect to the multiplication defined in (2). In what follows this semigroup will be denoted by $\mathscr{S}(S,\Sigma;h,\chi)$.

LEMMA 1.3. *The semigroup* $\mathbf{S}=\mathscr{S}(S,\Sigma;h,\chi)$ *is orthodox and its band of idempotents is*

$$\mathbf{B}=\{(e,\iota)\colon e\in S,\ \iota\in\Sigma_{l(e)}\ \textit{are idempotents}\}.$$

The product of two elements (e,ι) *and* $(\bar{e},\bar{\iota})$ *in* $\mathbf{B}$ *equals* $(e\bar{e},\iota\bar{\iota})$.

PROOF. First we prove that $\mathbf{S}$ is regular. Let s and s' be inverses of each other in S and let $\sigma,\sigma^*\in\Sigma_{l(s)}$. Then, by Lemma 1.1, $\sigma^*h_{s'}\in\Sigma_{l(s')}$. Making use of the definition (2) we obtain that $(s,\sigma)(s',\sigma^*h_{s'})(s,\sigma)=(ss's,\sigma_1)=(s,\sigma_1)$ where $\sigma_1=\chi_{s,s's}\cdot\sigma h_{s's}\cdot\chi_{s',s}\cdot\sigma^*h_{s'}h_s\cdot\sigma$. Here $s's\in E_{l(s)}$. Thus (A4) and (A5) ensure that $\chi_{s,s's}=\varepsilon_{l(s)}$ and $\sigma h_{s's}=\sigma\cdot\varepsilon_{l(s)}$, $\sigma^*h_{s's}=\sigma^*\cdot\varepsilon_{l(s)}$. Moreover, $\chi_{s',s}\cdot\chi_{s',s}^{-1}=\varepsilon_{l(s)}$. Since $\varepsilon_{l(s)}$ is an identity in $\Sigma_{l(s)}$, (A3) implies that

$$\sigma_1=\varepsilon_{l(s)}\cdot\sigma\cdot\varepsilon_{l(s)}\cdot\chi_{s',s}\cdot\chi_{s',s}^{-1}\cdot\sigma^*h_{s's}\cdot\chi_{s's}\cdot\sigma=$$

$$=\sigma\cdot\varepsilon_{l(s)}\cdot\sigma^*\cdot\varepsilon_{l(s)}\cdot\chi_{s',s}\cdot\sigma=\sigma\cdot(\sigma^*\cdot\chi_{s',s})\cdot\sigma$$

where $\sigma^*\cdot\chi_{s',s}\in\Sigma_{l(s)}$. Since S and Σ are regular every $s\in S$ and $\sigma\in\Sigma$ has an inverse s' and σ', respectively. If $\sigma\in\Sigma_{l(s)}$ then σ' is also contained in $\Sigma_{l(s)}$. Since $\chi_{s',s}\in U(\Sigma_{l(s)})$ the equality $\sigma'=\sigma'\cdot\varepsilon_{l(s)}=\sigma^*\cdot\chi_{s',s}$ holds for the element $\sigma^*=\sigma'\cdot\chi_{s',s}^{-1}$. Thus the foregoing argument shows that $(s,\sigma)(s',(\sigma'\cdot\chi_{s',s}^{-1})h_{s'})(s,\sigma)=$ $=(s,\sigma)$. Thus we have proved $\mathbf{S}$ to be regular.

Assume that (e,ι) is an idempotent element in $\mathbf{S}$. Then, by (2), e is idempotent and $\chi_{e,e}\cdot\iota h_e\cdot\iota=\iota$. (A5) implies that $\chi_{e,e}=\varepsilon_{l(e)}$ and (A4) that $\iota h_e=\iota\cdot\varepsilon_{l(e)}$. Since $\iota\in\Sigma_{l(e)}$ we obtain that $\iota=\varepsilon_{l(e)}\cdot\iota\cdot\varepsilon_{l(e)}\cdot\iota=\iota^2$, that is, ι is idempotent. Conversely, suppose $e\in E$ and $\iota\in\Sigma_{l(e)}$ are idempotents. Then (A5) and (A4) ensure $\chi_{e,e}=\varepsilon_{l(e)}$ and $\iota h_e=\iota$, respectively. Thus $\iota=\iota^2=\chi_{e,e}\cdot\iota h_e\cdot\iota$ whence it follows by definition, that (e,ι) is idempotent. We have verified that the set $\mathbf{B}$ defined in (3) is the set of idempotents of $\mathbf{S}$. Finally, we have to prove that $\mathbf{B}$ is closed under the multiplication (2). Let (e,ι) and $(\bar{e},\bar{\iota})\in\mathbf{B}$. By (2), we have $(e,\iota)(\bar{e},\bar{\iota})=(e\bar{e},\iota_1)$ where $\iota_1=\chi_{e,\bar{e}}\cdot\iota h_{\bar{e}}\cdot\bar{\iota}$. Since S is orthodox $e\bar{e}$ is idempotent. On the other hand, owing to (A4) and (A6), we can see that $\iota_1=\varepsilon_{l(e)l(\bar{e})}\cdot\iota\cdot\varepsilon_{l(\bar{e})}\cdot\bar{\iota}$. Since $\iota_1\in\Sigma_{l(e\bar{e})}$ and $l(e\bar{e})=l(e)l(\bar{e})\leqq l(\bar{e})$ we obtain that $\iota_1=\iota\cdot\bar{\iota}$. Σ is orthodox whence if follows that ι_1 is idempotent. The proof is complete.

Let us define a relation $\mathscr{C}$ on the semigroup $\mathbf{S}$ in the following way:

$$(4)\qquad (s,\sigma)\mathscr{C}(\bar{s},\bar{\sigma})\quad\text{if and only if}\quad s=\bar{s}.$$

It is immediate that $\mathscr{C}$ is a congruence relation on $\mathbf{S}$ and $\mathbf{S}/\mathscr{C}$ is isomorphic to S.

Observe that

$$\overline{\mathbf{B}}=\{(e,\varepsilon_{l(e)})\colon e\in E\}$$

is a subband of $\mathbf{B}$ since $l(e)l(\bar{e})=l(e\bar{e})$. Moreover,

$$F_e=\{(e,\iota)\colon \iota\in\Sigma_{l(e)}\text{ is idempotent}\}$$

is a subband of $\mathbf{B}$ for every $e \in E$. The element $(e, \varepsilon_{l(e)})$ is just the identity in F_e. Thus the congruence $\mathscr{C}$ is $\overline{\mathbf{B}}$-separating.

LEMMA 1.4. *The relation* $\mathscr{C}$ *defined in* (4) *is a strong* $\overline{\mathbf{B}}$*-separating congruence on* $\mathbf{S}$.

PROOF. We have seen that $\mathscr{C}$ is a $\overline{\mathbf{B}}$-separating congruence. All we have to show is that every $\mathscr{C}$-class contains an element which is $\mathscr{L}$- and $\mathscr{R}$-equivalent to some idempotents in $\overline{\mathbf{B}}$. We intend to prove that $(s, \varepsilon_{l(s)})$ has this property for every $s \in S$. Let e be an idempotent of S with $e\mathscr{R}s$. Then we have $(e, \varepsilon_{l(e)})(s, \varepsilon_{l(s)}) = (es, \chi_{e,s} \cdot \varepsilon_{l(e)}h_s \cdot \varepsilon_{l(s)})$ by (2). Here $es = s$ and, according to (A5), $\chi_{e,s} = \varepsilon_{l(s)}$. Furthermore, $\varepsilon_{l(e)}h_s = \varepsilon_{(l(e)r(s))\tau_s} = \varepsilon_{l(s)}$ as $(l(e)r(s))\tau_s = r(es)\tau_s = r(s)\tau_s = l(s)$. Thus we have

$$(e, \varepsilon_{l(e)})(s, \varepsilon_{l(s)}) = (s, \varepsilon_{l(s)}). \tag{5}$$

Consider an inverse s' of s with $s'\mathscr{L}e$. In fact, such an inverse always exists. Then $ss' = e$. Clearly, we have $l(s') = l(e)$ and $\chi_{s,s'} \in \Sigma_{l(ss')} = \Sigma_{l(s')}$. Thus $(s', \chi_{s,s'}^{-1}) \in \mathbf{S}$ and, by (2), $(s, \varepsilon_{l(s)})(s', \chi_{s,s'}^{-1}) = (e, \sigma)$ where $\sigma = \chi_{s,s'} \cdot \varepsilon_{l(s)}h_{s'} \cdot \chi_{s,s'}^{-1}$. Since $\varepsilon_{l(s)}h_{s'} = \varepsilon_{r(s')}h_{s'} = \varepsilon_{r(s')\tau_{s'}} = \varepsilon_{l(s')} = \varepsilon_{l(e)}$ and $\chi_{s,s'}$ and $\chi_{s,s'}^{-1}$ are inverses of each other in $U(\Sigma_{l(e)})$, we obtain that $\sigma = \varepsilon_{l(e)}$, that is, $(s, \varepsilon_{l(s)})(s', \chi_{s,s'}^{-1}) = (e, \varepsilon_{l(e)})$.

Together with the equality (5) this means that $(s, \varepsilon_{l(s)})\mathscr{R}(e, \varepsilon_{l(e)})$ where $(e, \varepsilon_{l(e)}) \in \overline{\mathbf{B}}$. Dually, we can prove that $(s, \varepsilon_{l(s)})\mathscr{L}(f, \varepsilon_{l(f)})$ provided f is an idempotent in S with $s\mathscr{L}f$. This completes the proof of the lemma.

The following theorem sums up the results proved in Lemmas 1.2, 1.3 and 1.4.

THEOREM 1.5. *Let* S *be an orthodox semigroup whose band of idempotents is a semilattice* Y *of rectangular bands. For every* $\alpha \in Y$, *let* Σ_α *be an orthodox monoid with identity* ε_α. *Suppose* Σ *is a proper semilattice* Y *of monoids* $\Sigma_\alpha (\alpha \in Y)$. *If* h, χ *form an* (S, Σ)*-pair then* $\mathbf{S} = \mathscr{S}(S, \Sigma; h, \chi)$ *is an orthodox semigroup, the relation* $\mathscr{C}$ *defined in* (4) *is a strong subband-separating congruence on* $\mathbf{S}$ *and* $\mathbf{S}/\mathscr{C}$ *is isomorphic to* S.

Our purpose is to prove the converse of this theorem. We shall prove that if T is an orthodox semigroup, $\varkappa$ is a strong subband-separating congruence on T and $T/\varkappa = S$ then T is isomorphic to a semigroup $\mathscr{S}(S, \Sigma; h, \chi)$ where Σ, h and χ have the properties required in Theorem 1.5.

2. The kernel of a subband-separating congruence

Now we describe the kernel of subband-separating congruences. More precisely, we describe the structure of the union of the idempotent congruence classes in the case of subband-separating congruences.

We need the concept of the spined product of two groupoids.

Let A be a groupoid and Y a semilattice. Term A a semilattice Y of groupoids $A_\alpha (\alpha \in Y)$ if the following are satisfied: (i) $A = \bigcup \{A_\alpha : \alpha \in Y\}$, (ii) $A_\alpha \cap A_\beta = \square$ provided $\alpha, \beta \in Y$ with $\alpha \neq \beta$, and (iii) $A_\alpha A_\beta \subseteq A_{\alpha\beta}$ for every $\alpha, \beta \in Y$.

Let Y be a semilattice. Suppose A and B are groupoids which are semilattices Y of groupoids $A_\alpha (\alpha \in Y)$ and $B_\alpha (\alpha \in Y)$, respectively. By the *spined product of*

A and B over Y we mean the subgroupoid $A\times_Y B=\bigcup\{A_\alpha\times B_\alpha\colon \alpha\in Y\}$ of the direct product of A and B. Clearly, the spined product $A\times_Y B$ is a semigroup if and only if both of the groupoids A and B are semigroups.

The following theorem characterizes proper bands of monoids. The structure of proper bands of monoids was described by B. M. SCHEIN [10] in another way. Our description is a generalization of M. YAMADA's result [20] concerning the orthodox bands of groups.

THEOREM 2.1. *Let E be a band which is a semilattice Y of rectangular bands $E_\alpha(\alpha\in Y)$. For every $\alpha\in Y$, suppose Σ_α to be a monoid with identity ε_α. Let Σ be a semigroup which is a proper semilattice Y of monoids $\Sigma_\alpha(\alpha\in Y)$. Then the spined product $\tilde\Sigma=E\times_Y\Sigma$ is a semigroup which is a proper band E of its submonoids $\tilde\Sigma_e=\{(e,\sigma)\colon\sigma\in\Sigma_\alpha\}$ $(\alpha\in Y,\ e\in E_\alpha)$ with identities (e,ε_α). Moreover, Σ is isomorphic to $\tilde\Sigma/\eta$ where η is the least congruence on $\tilde\Sigma$ for which $\eta|\{(e,\varepsilon_\alpha)\colon e\in E_\alpha,\alpha\in Y\}=\mathcal{D}$.*

Conversely, if $\tilde\Sigma$ is a proper band E of its submonoids $\tilde\Sigma_e(e\in E)$ with identities $\tilde\varepsilon_e$ then $\tilde\Sigma$ is isomorphic to $E\times_Y(\tilde\Sigma/\eta)$ where η is the least congruence on $\tilde\Sigma$ for which $\eta|\{\tilde\varepsilon_e\colon e\in E\}=\mathcal{D}$. Here $\tilde\Sigma/\eta$ is a proper semilattice Y of submonoids.

PROOF. The first statement in the direct part of the theorem follows immediately from the definition of the spined product $\tilde\Sigma$. Define an equivalence relation η on $\tilde\Sigma$ by putting $(e,\sigma)\eta(f,\tau)$ if and only if $\sigma=\tau$. Clearly, η is a congruence on $\tilde\Sigma$, $\eta|\{(e,\varepsilon_\alpha)\colon e\in E_\alpha,\alpha\in Y\}=\mathcal{D}$ and $\tilde\Sigma/\eta$ is isomorphic to Σ. Assume that ν is a congruence on $\tilde\Sigma$ with $\nu|\{(e,\varepsilon_\alpha)\colon e\in E_\alpha,\alpha\in Y\}=\mathcal{D}$. Then $(e,\varepsilon_\alpha)\nu(f,\varepsilon_\alpha)$ holds provided $e,f\in E_\alpha$. This implies that

$$(e,\sigma)=(e,\varepsilon_\alpha)(e,\sigma)(e,\varepsilon_\alpha)\,\nu\,(f,\varepsilon_\alpha)(e,\sigma)(f,\varepsilon_\alpha)=(f,\sigma)$$

for every $\sigma\in\Sigma_\alpha$. Hence we see that $\eta\subseteq\nu$, that is, η is the least congruence on $\tilde\Sigma$ with $\eta|\{(e,\varepsilon_\alpha)\colon e\in E_\alpha,\alpha\in Y\}=\mathcal{D}$.

In order to prove the converse part assume that $\tilde\Sigma$ is a semigroup which is a proper band E of its submonoids $\tilde\Sigma_e(e\in E)$ with identities denoted by $\tilde\varepsilon_e$. Then we have $\tilde\varepsilon_e\tilde\varepsilon_{\bar e}=\tilde\varepsilon_{e\bar e}$ for every $e,\bar e\in E$. For every $\alpha\in Y$, let us select and fix an element e_α in E_α. Denote $\tilde\Sigma_{e_\alpha}$ by Σ_α and, similarly, $\tilde\varepsilon_{e_\alpha}$ by ε_α. We define a multiplication denoted by "$\circ$" on the set $\Sigma=\bigcup\{\Sigma_\alpha\colon\alpha\in Y\}$ such that $\varrho\circ\sigma=\varepsilon_{\alpha\beta}\varrho\sigma\varepsilon_{\alpha\beta}$ for every $\varrho\in\Sigma_\alpha$, $\sigma\in\Sigma_\beta$. Since $\varrho\sigma\in\tilde\Sigma_{e_\alpha e_\beta}$ and $e_\alpha e_\beta\in E_{\alpha\beta}$ it is easily seen that $\varrho\circ\sigma\in\Sigma_{\alpha\beta}$. Thus Σ is a groupoid which is a semilattice Y of semigroups Σ_α $(\alpha\in Y)$.

Let us define a mapping $\Phi\colon\tilde\Sigma\to E\times_Y\Sigma$ by $\sigma\Phi=(e,\varepsilon_\alpha\sigma\varepsilon_\alpha)$ for every $\sigma\in\tilde\Sigma_e$ where $e\in E_\alpha$. Clearly, $\varepsilon_\alpha\sigma\varepsilon_\alpha\in\Sigma_\alpha$ as $e_\alpha ee_\alpha=e_\alpha$. Thus $\sigma\Phi$ is, in fact, in $E\times_Y\Sigma$. Denote by Φ' the mapping of $E\times_Y\Sigma$ into $\tilde\Sigma$ which is defined by $(e,\sigma)\Phi'=\tilde\varepsilon_e\sigma\tilde\varepsilon_e$ for every $e\in E_\alpha$ and $\sigma\in\Sigma_\alpha$. Since $ee_\alpha e=e$ we have $\tilde\varepsilon_e\sigma\tilde\varepsilon_e\in\tilde\Sigma_e$. The mappings Φ and Φ' are inverses of each other. For if $\sigma\in\tilde\Sigma_e$ with $e\in E_\alpha$ then $\sigma\Phi\Phi'=$ $=(e,\varepsilon_\alpha\sigma\varepsilon_\alpha)\Phi'=\tilde\varepsilon_e\varepsilon_\alpha\sigma\varepsilon_\alpha\tilde\varepsilon_e=\sigma$ and, conversely, if $e\in E_\alpha$ and $\sigma\in\Sigma_\alpha$ then $(e,\sigma)\Phi'\Phi=$ $=(\tilde\varepsilon_e\sigma\tilde\varepsilon_e)\Phi=(e,\varepsilon_\alpha\tilde\varepsilon_e\sigma\tilde\varepsilon_e\varepsilon_\alpha)=(e,\sigma)$. This implies that Φ is one-to-one and onto. Now we prove that Φ is a homomorphism. Let $\varrho\in\tilde\Sigma_e$, $\sigma\in\tilde\Sigma_f$ where $e\in E_\alpha$ and $f\in E_\beta$. Then $\varrho\sigma\in\tilde\Sigma_{ef}$ and $ef\in E_{\alpha\beta}$. By definition, $\varrho\Phi\cdot\sigma\Phi=(e,\varepsilon_\alpha\varrho\varepsilon_\alpha)(f,\varepsilon_\beta\sigma\varepsilon_\beta)=(ef,\sigma_1)$ where

$$\sigma_1=\varepsilon_{\alpha\beta}\varepsilon_\alpha\varrho\varepsilon_\alpha\varepsilon_\beta\sigma\varepsilon_\beta\varepsilon_{\alpha\beta}. \tag{6}$$

On the other hand, $(\varrho\sigma)\Phi=(ef,\varepsilon_{\alpha\beta}\varrho\sigma\varepsilon_{\alpha\beta})$. We have to show that $\sigma_1=\varepsilon_{\alpha\beta}\varrho\sigma\varepsilon_{\alpha\beta}$. Since $e_\alpha e_\beta\in E_{\alpha\beta}$ and $\alpha\beta\leqq\alpha$ we have $e_{\alpha\beta}e_\alpha(ee_\alpha e_\beta)=e_{\alpha\beta}(ee_\alpha e_\beta)$. Hence $\varepsilon_{\alpha\beta}\varepsilon_\alpha\tilde\varepsilon_{ee_\alpha e_\beta}=$

$=\varepsilon_{\alpha\beta}\tilde{\varepsilon}_{ee_\alpha e_\beta}$. Since $\varrho\varepsilon_\alpha\varepsilon_\beta\in\tilde{\Sigma}_{ee_\alpha e_\beta}$ this implies the equality $\varepsilon_{\alpha\beta}\varepsilon_\alpha\varrho\varepsilon_\alpha\varepsilon_\beta=\varepsilon_{\alpha\beta}\varrho\varepsilon_\alpha\varepsilon_\beta$. Similarly, we can see that $\varepsilon_\alpha\varepsilon_\beta\sigma\varepsilon_\beta\varepsilon_{\alpha\beta}=\varepsilon_\alpha\varepsilon_\beta\sigma\varepsilon_{\alpha\beta}$. Thus we obtain from (6) that

$$(6)' \qquad \sigma_1 = \varepsilon_{\alpha\beta}\,\varrho\varepsilon_\alpha\varepsilon_\beta\sigma\varepsilon_{\alpha\beta}.$$

Here $\varepsilon_{\alpha\beta}\varrho\in\tilde{\Sigma}_{e_{\alpha\beta}e}, \sigma\varepsilon_{\alpha\beta}\in\tilde{\Sigma}_{fe_{\alpha\beta}}$ and $e_{\alpha\beta}e, fe_{\alpha\beta}\in E_{\alpha\beta}$. Therefore $(e_{\alpha\beta}e)(e_\alpha e_\beta)(fe_{\alpha\beta})=$ $=(e_{\alpha\beta}e)(fe_{\alpha\beta})$ whence $\tilde{\varepsilon}_{e_{\alpha\beta}e}\varepsilon_\alpha\varepsilon_\beta\tilde{\varepsilon}_{fe_{\alpha\beta}}=\tilde{\varepsilon}_{e_{\alpha\beta}e}\tilde{\varepsilon}_{fe_{\alpha\beta}}$. By (6′), this implies $\sigma_1=\varepsilon_{\alpha\beta}\varrho\sigma\varepsilon_{\alpha\beta}$ which was to be proved. Thus it is shown that $\tilde{\Sigma}$ and $E\times_Y\Sigma$ are isomorphic to each other. Since $\tilde{\Sigma}$ is associative we infer that the same holds for Σ. Furthermore, Σ is a proper semilattice Y of submonoids $\Sigma_\alpha(\alpha\in Y)$ since $\varepsilon_\alpha\circ\varepsilon_\beta=\varepsilon_{\alpha\beta}$ for every $\alpha,\beta\in Y$. The direct part of the theorem ensures that Σ is isomorphic to $\tilde{\Sigma}/\eta$ where η is the least congruence on $\tilde{\Sigma}$ with $\eta|\{\tilde{\varepsilon}_e\colon e\in E\}=\mathscr{D}$. The proof is complete.

Let T be an orthodox semigroup with band of idempotents B and $\bar{B}$ a subband of B. Let $\varkappa$ be a $\bar{B}$-separating congruence on T. Denote by $\ker\varkappa$ and $\bigcup\ker\varkappa$ the set of the idempotent $\varkappa$-classes and the union of the idempotent $\varkappa$-classes, respectively. It is proved in [11] that $\bigcup\ker\varkappa$ is a proper band $\bar{B}$ of orthodox monoids. Thus we obtain the following corollaries.

COROLLARY 2.2. *Suppose T is an orthodox semigroup with band of idempotents B and $\bar{B}$ is a subband in B. Let $\bar{B}$ be a semilattice Y of rectangular bands. Let $\varkappa$ be a $\bar{B}$-separating congruence on T. Then $\bigcup\ker\varkappa$ is isomorphic to $\bar{B}\times_Y\Sigma$ with $\Sigma=(\bigcup\ker\varkappa)/\eta$ where η is the least $(\bar{B},\mathscr{D})$-parcelling congruence on the orthodox semigroup $\bigcup\ker\varkappa$.*

COROLLARY 2.3. *Let S and $\bar{S}$ be orthodox semigroups with bands of idempotents E and $\bar{E}$, respectively, which are semilattices Y and $\bar{Y}$ of rectangular bands, respectively. Let Σ and $\bar{\Sigma}$ be proper semilattices Y and $\bar{Y}$ of orthodox monoids Σ_α and $\bar{\Sigma}_{\bar{\alpha}}$ $(\alpha\in Y,\bar{\alpha}\in\bar{Y})$, respectively. Suppose there exists an onto isomorphism $\varphi\colon\mathscr{S}(S,\Sigma;h,\chi)\to\mathscr{S}(\bar{S},\bar{\Sigma};\bar{h},\bar{\chi})$ such that $(s,\sigma)\mathscr{C}(s_1,\sigma_1)$ if and only if $(s,\sigma)\varphi\mathscr{C}(s_1,\sigma_1)\varphi$. Then S is isomorphic to $\bar{S}$ and Σ is isomorphic to $\bar{\Sigma}$.*

PROOF. Since every $\mathscr{C}$-class [$\bar{\mathscr{C}}$-class] corresponds to a unique element s of S [$\bar{s}$ of $\bar{S}$] we can define a one-to-one mapping φ_1 of S onto $\bar{S}$ such that $(s,\sigma)\varphi=$ $=(s\varphi_1,\bar{\sigma})$ for some $\bar{\sigma}\in\bar{\Sigma}$. Since φ is an onto isomorphism, φ_1 is also an onto isomorphism. On the other hand, $\bigcup\ker\mathscr{C}=E\times_Y\Sigma$ and $\bigcup\ker\bar{\mathscr{C}}=\bar{E}\times_Y\Sigma$ by definition. The isomorphism φ maps $\tilde{\Sigma}_e=\{(e,\sigma)\colon\sigma\in\Sigma_\alpha\}$ onto $\tilde{\bar{\Sigma}}_{e\varphi_1}=\{(e\varphi_1,\bar{\sigma})\colon$ $\bar{\sigma}\in\bar{\Sigma}_{\bar{\alpha}}\}$ for every $e\in E_\alpha$ where $e\varphi_1\in\bar{E}_{\bar{\alpha}}$. Identifying $\bar{E}$ with E under φ_1 and $\bigcup\ker\bar{\mathscr{C}}$ with $\bigcup\ker\mathscr{C}$ under φ we conclude by Theorem 2.1 that Σ is isomorphic to Σ.

3. The main theorem

Before stating the main theorem describing all the strong subband-separating extensions of orthodox semigroups we prove a lemma.

LEMMA 3.1. *Let T be an orthodox semigroup with band of idempotents B. Suppose $\varkappa$ to be a $\bar{B}$-separating congruence on T. Denote $T/\varkappa$ by S. Then there exists a cross-section $\{u_s\colon s\in S, u_s\varkappa=s\}$ of the $\varkappa$-classes with the properties that*

(i) $u_s \in S_B$ *for every* $s \in S$ *and* (ii) $u_s u_{s'} = u_{ss'}$ *provided* s *and* s' *are inverses of each other in* S. *In particular, if* e *is an idempotent in* S *then* u_e *is the greatest idempotent in the* $\varkappa$*-class* e.

Proof. Assume that the band of idempotents in S is $E = \bigcup_{\alpha \in Y} E_\alpha$. Select and fix an element e_α in E_α for every $\alpha \in Y$. The $\varkappa$-class $e \in E$ contains a unique idempotent belonging to $\bar{B}$, the greatest idempotent in the $\varkappa$-class e, which will be denoted by i_e^*. If a cross-section $\{u_s \colon s \in S,\ u_s\varkappa = s\}$ of $\varkappa$-classes satisfies the conditions (i) and (ii) then $u_e \in \bar{B}$ for every $e \in E$. Thus $u_e = i_e^*$ necessarily holds.

Now we define a cross-section with the required properties. Let $u_e = i_e^*$ for every $e \in E$. It follows from the results in [11] that if s and s' are inverses of each other such that $ss' = e_\alpha$ and $s's = e_\beta$ for some $\alpha, \beta \in Y$ then there exist elements $t, t' \in S_B$ such that they are inverses of each other and $t\varkappa = s$, $t'\varkappa = s'$. Then $tt' = i_{e_\alpha}^*$ and $t't = i_{e_\beta}^*$. Let us define u_s to be t and $u_{s'}$ to be t'. Now let $\tilde{s}$ and $\tilde{s}'$ be arbitrary elements in S such that they are inverses of each other. Suppose $\tilde{s}\tilde{s}' = e$ and $\tilde{s}'\tilde{s} = f$ where $e \in E_\alpha$ and $f \in E_\beta$. Then $s = e_\alpha \tilde{s} e_\beta$ and $s' = e_\beta \tilde{s}' e_\alpha$ are inverses of each other in S with the property that $ss' = e_\alpha$ and $s's = e_\beta$. Hence u_s and $u_{s'}$ are defined. Let us define $u_{\tilde{s}}$ to be $i_e^* u_s i_f^*$ and $u_{\tilde{s}'}$ to be $i_f^* u_{s'} i_e^*$. This definition is independent of the choice of e and f. Furthermore, $u_{\tilde{s}}\varkappa = \tilde{s}$, $u_{\tilde{s}'}\varkappa = \tilde{s}'$ and $u_{\tilde{s}} u_{\tilde{s}'} = i_e^*$, $u_{\tilde{s}'} u_{\tilde{s}} = i_f^*$ clearly hold. Thus we have defined a cross-section of $\varkappa$-classes satisfying the required conditions.

The following theorem describes all the strong subband-separating extensions of orthodox semigroups.

Theorem 3.2. *Let* T *be an orthodox semigroup and* $\varkappa$ *a strong subband-separating congruence on* T. *Denote by* S *the factor semigroup* $T/\varkappa$. *Suppose the band of idempotents of* S *is a semilattice* Y *of rectangular bands. Then there exists a unique (up to isomorphism) orthodox semigroup* Σ *which is a proper semilattice* Y *of orthodox monoids* $\Sigma_\alpha (\alpha \in Y)$ *such that* T *is isomorphic to* $\mathcal{S}(S, \Sigma; h, \chi)$ *for some* (S, Σ)*-pair* h, χ.

Proof. We make use of the notations $l(s), r(s), \tau_s$ on S introduced at the beginning of Section 1. Suppose the band of idempotents of T is B and $\varkappa$ separates its subband $\bar{B}$. By definition, every idempotent $\varkappa$-class e contains a unique element i_e^* from $\bar{B}$ which is the greatest idempotent in that $\varkappa$-class. Lemma 3.1 ensures the existence of a cross-section $\{u_s \colon s \in S,\ u_s\varkappa = s\} \subseteq S_B$ of $\varkappa$-classes such that $u_s u_{s'} = u_{ss'}$ provided s and s' are inverses of each other in S. If s and s' are inverses of each other in S then, on the one hand, we have $u_{ss'} = i_{ss'}^*$. On the other hand, u_s and $u_{s'}$ are inverses of each other in T. For if u_s' is an inverse of u_s then $u_s' \in S_B$ (cf. [11]) and $(u_s u_s')\varkappa = s \cdot u_s'\varkappa \mathcal{R} ss' = i_{ss'}^*\varkappa$. Here $u_s u_s'$ and $i_{ss'}^*$ are elements of $\bar{B}$ whence it follows that $i_{ss'}^* \mathcal{R} u_s$. Thus, $u_s u_{s'} u_s = i_{ss'}^* u_s = u_s$ and, similarly, $u_{s'} u_s u_{s'} = u_{s'}$. The idempotent $\varkappa$-classes are orthodox monoids as it has been proved in [11]. Denote by $\tilde{\Sigma}_e$ the $\varkappa$-class $e \in E$ as a subsemigroup of T. Obviously, the identity of $\tilde{\Sigma}_e$ is i_e^*. The union $\tilde{\Sigma}$ of the idempotent $\varkappa$-classes $\tilde{\Sigma}_e (e \in E)$ is a proper band E of the orthodox monoids $\tilde{\Sigma}_e (e \in E)$. For every $\alpha \in Y$, choose and fix an element e_α in E_α. For brevity, denote the semigroup $\tilde{\Sigma}_{e_\alpha}$ by Σ_α and its identity $i_{e_\alpha}^*$ by i_α. Put $\Sigma = \bigcup \{\Sigma_\alpha \colon \alpha \in Y\}$. Define a multiplication on Σ in such a way that $\varrho \circ \sigma = i_{\alpha\beta} \varrho\sigma i_{\alpha\beta}$ provided $\varrho \in \Sigma_\alpha$ and $\sigma \in \Sigma_\beta$. According to the proof of Theorem 2.1, Σ is a proper semilattice Y of the orthodox monoids

Σ_α ($\alpha\in Y$). Furthermore, the mapping $\Phi\colon \tilde{\Sigma}\to E\times_Y\Sigma$ assigning $(e, i_\alpha\sigma i_\alpha)$ to the element $\sigma\in\tilde{\Sigma}_e$ where $e\in E_\alpha$ is an onto isomorphism.

Now we turn to proving Theorem 3.2. We divide the proof into several steps.

1° *If $t\in T$ and $x\in S_B$ with $t\varkappa=x\varkappa=s$ then t is uniquely expressed in the form $x\sigma i_e^*$ where $e\in E$, $s\mathcal{L}e\mathcal{R}e_{l(s)}$ and $\sigma\in\Sigma_{l(s)}$.* Let $f\in E$ with $s\mathcal{R}f$. Denote by s' the inverse of s for which $e\mathcal{R}s'\mathcal{L}f$. Then $ss'=f$ and $s's=e$. Since $e\mathcal{R}e_{l(s)}$ we have $i_e^*\mathcal{R}i_{l(s)}$. By a result in [11], there exists an inverse x' of x such that $x'\varkappa=s'$. For such an x', we have $xx'=i_f^*$ and $x'x=i_e^*$. Suppose b is an idempotent in T with $b\mathcal{R}t$. Let $i_{\tilde{f}}^*$ be the greatest idempotent in the $\varkappa$-class $b\varkappa$. Since $i_{\tilde{f}}^*\varkappa=b\varkappa\mathcal{R}t\varkappa=s$ we have $\tilde{f}\mathcal{R}s$. Thus $f\mathcal{R}\tilde{f}$ which implies $i_f^*\mathcal{R}i_{\tilde{f}}^*$. This yields the equality $t=bt=i_{\tilde{f}}^*bt=i_{\tilde{f}}^*t=i_f^*i_{\tilde{f}}^*t=i_f^*t$. One can show similarly that $t=ti_e^*$. Applying these equalities we obtain that $t=xx'ti_e^*=x(x'ti_{l(s)})i_e^*$. Here $(x'ti_{l(s)})\varkappa=s'se_{l(s)}=ee_{l(s)}=e_{l(s)}$ whence $\sigma=x'ti_{l(s)}\in\Sigma_{l(s)}$. Thus t is expressed in the required form. If t were expressed in two different ways then $t=x\sigma i_e^*=x\tilde{\sigma}i_e^*$ for the conditions $s=t\varkappa$ and $s\mathcal{L}e\mathcal{R}e_{l(s)}$ uniquely determine e. Multiplying the equality $x\sigma i_e^*=x\tilde{\sigma}i_e^*$ by x' on the left and by $i_{l(s)}$ on the right we obtain that $i_e^*\sigma i_e^*i_{l(s)}=i_e^*\tilde{\sigma}i_e^*i_{l(s)}$. Hence $\sigma=\tilde{\sigma}$ follows as $\sigma,\tilde{\sigma}\in\Sigma_{l(s)}$ which implies $\sigma=i_{l(s)}\sigma$, $\tilde{\sigma}=i_{l(s)}\tilde{\sigma}$ and $i_e^*i_{l(s)}=i_{l(s)}$. Note that it turns out from this argument that $\sigma=x'ti_{l(s)}$ holds for every inverse x' of x with $i_{l(s)}\mathcal{R}x'$ provided $t=x\sigma i_e^*$ where $x\in S_B$, $t\varkappa=x\varkappa=s$, $\sigma\in\Sigma_{l(s)}$ and $e\in E$ with $s\mathcal{L}e\mathcal{R}e_{l(s)}$.

2° *The dual of* 1°. *If $t\in T$ and $x\in S_B$ with $t\varkappa=x\varkappa=s$ then t is uniquely expressed in the form $i_f^*\sigma x$ where $\sigma\in\Sigma_{r(s)}$ and $s\mathcal{R}f\mathcal{L}e_{r(s)}$.* Moreover, $\sigma=i_{r(s)}tx'$ is valid for arbitrary inverses x' of x with $i_{r(s)}\mathcal{L}x'$.

In particular, every $t\in T$ is expressed in the form $t=u_s\sigma i_e^*$ where $s=t\varkappa$, $\sigma\in\Sigma_{l(s)}$ and $s\mathcal{L}e\mathcal{R}e_{l(s)}$. Therefore we can define a one-to-one mapping $\psi\colon T\to\{(s,\sigma)\colon s\in S,\ \sigma\in\Sigma_{l(s)}\}$ such that if $t=u_s\sigma i_e^*$ where $s=t\varkappa$, $\sigma\in\Sigma_{l(s)}$ and $s\mathcal{L}e\mathcal{R}e_{l(s)}$ then $t\psi=(s,\sigma)$. In the sequel we define an (S,Σ)-pair h,χ such that ψ becomes an isomorphism of T onto $\mathcal{S}(S,\Sigma;h,\chi)$.

3° *Definition of h which satisfies* (A1). Let $\sigma\in\Sigma_\alpha$, $e\in E$ with $e\mathcal{R}e_\alpha$ and $s\in S$. Then $(i_e^*u_s)\varkappa=es$ and $i_e^*u_s\in S_B$. Moreover, we have $(\sigma i_e^*u_s)\varkappa=e_\alpha es=es$. Thus, by 1° the element $\sigma i_e^*u_s$ is of the form $i_e^*u_s\tilde{\sigma}i_f^*$ where $\tilde{\sigma}\in\Sigma_{l(es)}$ and $es\mathcal{L}f\mathcal{R}e_{l(es)}$. Here $\tilde{\sigma}=i_{l(es)}(i_e^*u_s)'(\sigma i_e^*u_s)i_{l(es)}$ where $(i_e^*u_s)'$ denotes an arbitrary inverse of $i_e^*u_s$. In particular, if s' and e' are inverses of s and e, respectively, then $u_{s'}i_{e'}^*$ is an inverse of $i_e^*u_s$ whence

$$\tilde{\sigma}=i_{l(es)}u_{s'}i_{e'}^*\sigma i_e^*u_si_{l(es)}. \tag{7}$$

Let s' and e' be inverses of s and e, respectively. Suppose $e'\mathcal{L}e_\alpha$. Dually to the foregoing argument one obtains that $u_{s'}i_{e'}^*\sigma$ is uniquely expressed in the form $i_g^*\tilde{\tilde{\sigma}}u_{s'}i_{e'}^*$ where $\tilde{\tilde{\sigma}}\in\Sigma_{r(s'e')}$ and $s'e'\mathcal{R}g\mathcal{L}e_{r(s'e')}$. Furthermore,

$$\tilde{\tilde{\sigma}}=i_{r(s'e')}u_{s'}i_{e'}^*\sigma i_{(e')'}^*u_{(s')'}i_{r(s'e')} \tag{8}$$

holds for arbitrary inverses $(e')'$ and $(s')'$ of e' and s', respectively. Clearly, we have $(\alpha r(s))\tau_s=l(es)=r(s'e')$. Thus, choosing e' in (7) in such a way that $e'\mathcal{L}e_\alpha$ and choosing $(s')'$ and $(e')'$ in (8) to be s and e, respectively, the equality $\tilde{\sigma}=\tilde{\tilde{\sigma}}$ is easily seen. If $e'=(e')'=e_\alpha$ in (8) then $\tilde{\tilde{\sigma}}=i_{(\alpha r(s))\tau_s}u_{s'}\sigma u_{(s')'}i_{(\alpha r(s))\tau_s}$. Hence

$$\tilde{\sigma}=i_{(\alpha r(s))\tau_s}u_{s'}\sigma u_{(s')'}i_{(\alpha r(s))\tau_s} \tag{9}$$

is the unique element in $\Sigma_{(\alpha r(s))\tau_s}$ such that $\sigma i_e^* u_s = i_e^* u_s \tilde{\sigma} i_f^*$ holds where $es\mathcal{L}f\mathcal{R}e_{(\alpha r(s))\tau_s}$. This implies on the one hand, that $\tilde{\sigma}$ does not depend on e. On the other hand, if s_1 and s_2 have a common inverse s' in S, that is, $s_1 \sim s_2$ then by choosing s_1' to be s' and $(s')'$ to be s_2 one can see that the respective $\tilde{\sigma}$-s are equal. Let us define a transformation h_s of Σ for each $s \in S$ in such a way that if $\sigma \in \Sigma_\alpha$ then σh_s equals $\tilde{\sigma}$ given in (9). Clearly, $\sigma h_s \in \Sigma_{(\alpha r(s))\tau_s}$ and σh_s is the unique element in $\Sigma_{(\alpha r(s))\tau_s}$ such that if $\sigma \in \Sigma_\alpha$ and $e\mathcal{R}e_\alpha$ then

$$\sigma i_e^* u_s = i_e^* u_s (\sigma h_s) i_f^* \tag{10}$$

where $es\mathcal{L}f\mathcal{R}e_{(\alpha r(s))\tau_s}$. The mapping h is shown to have property (A1).

4° *For every* $s \in S$, h_s *is an endomorphism with* $i_\alpha h_s = i_{(\alpha r(s))\tau_s}$. Let $\varrho \in \Sigma_\alpha$ and $\sigma \in \Sigma_\beta$. Then $\varrho h_s \in \Sigma_{(\alpha r(s))\tau_s}$ and $\sigma h_s \in \Sigma_{(\beta r(s))\tau_s}$. This implies $\varrho h_s \circ \sigma h_s \in \Sigma_\gamma$ where $\gamma = (\alpha\beta r(s))\tau_s$. Thus, choosing $(s')'$ to be s, we obtain by (9) that

$$\varrho h_s \circ \sigma h_s = i_\gamma i_{(\alpha r(s))\tau_s} u_{s'} \varrho u_s i_{(\alpha r(s))\tau_s} i_{(\beta r(s))\tau_s} u_{s'} \sigma u_s i_{(\beta r(s))\tau_s} i_\gamma .$$

Here $(u_{s'} \varrho u_s)\Phi = (s' e_\alpha s, \hat{\varrho})$ where $\hat{\varrho} = i_{(\alpha r(s))\tau_s} u_{s'} \varrho u_s i_{(\alpha r(s))\tau_s}$ since $s' e_\alpha s \in E_{(\alpha r(s))\tau_s}$. Similarly, $(u_{s'} \sigma u_s)\Phi = (s' e_\beta s, \hat{\sigma})$ where $\hat{\sigma} = i_{(\beta r(s))\tau_s} u_{s'} \sigma u_s i_{(\beta r(s))\tau_s}$. Since Φ is an isomorphism of $\bar{\Sigma}$ onto $E \times_Y \Sigma$ this implies that

$$(\varrho h_s \circ \sigma h_s)\Phi = (e_\gamma, i_\gamma)(e_{(\alpha r(s))\tau_s}, i_{(\alpha r(s))\tau_s})(s' e_\alpha s, \hat{\varrho})(e_{(\alpha r(s))\tau_s}, i_{(\alpha r(s))\tau_s}) \cdot$$

$$\cdot (e_{(\beta r(s))\tau_s}, i_{(\beta r(s))\tau_s})(s' e_\beta s, \hat{\sigma})(e_{(\beta r(s))\tau_s}, i_{(\beta r(s))\tau_s})(e_\gamma, i_\gamma) =$$

$$= (e_\gamma, i_\gamma \circ \hat{\varrho} \circ \hat{\sigma} \circ i_\gamma) = (e_\gamma, i_\gamma)(s' e_\alpha s, \hat{\varrho})(s' e_\beta s, \hat{\sigma})(e_\gamma, i_\gamma) = (i_\gamma (u_{s'} \varrho u_s)(u_{s'} \sigma u_s) i_\gamma \Phi)$$

and therefore $\varrho h_s \circ \sigma h_s = i_\gamma u_{s'} \varrho u_s u_{s'} \sigma u_s i_\gamma$. Since u_s and $u_{s'}$ are inverses of each other in T the elements $i_\gamma u_{s'}$ and $u_s i_\gamma$ are also inverses of each other. Thus

$$\varrho h_s \circ \sigma h_s = i_\gamma u_{s'} (u_s i_\gamma u_{s'}) \varrho u_s u_{s'} \sigma (u_s i_\gamma u_{s'}) u_s i_\gamma . \tag{11}$$

Here $u_s i_\gamma u_{s'} = i^*_{se_\gamma s'}$ where $se_\gamma s' \in E_{\gamma\tau_s^{-1}} = E_{\alpha\beta r(s)}$ as $u_s i_\gamma u_{s'} \in \bar{B}$ and $(u_s i_\gamma u_{s'})\varkappa = se_\gamma s'$. Moreover, we have $u_s u_{s'} = i^*_{ss'}$. Since Φ is an isomorphism, utilizing the equalities $\varrho \circ i_{r(s)} = i_{r(s)} \circ \varrho$ and $i_{r(s)} \circ i_{\alpha\beta r(s)} = i_{\alpha\beta r(s)}$ we obtain that

$$((u_s i_\gamma u_{s'}) \varrho u_s u_{s'} \sigma (u_s i_\gamma u_{s'}))\Phi = (se_\gamma s', i_{\alpha\beta r(s)})(e_\alpha, \varrho)(e_{r(s)}, i_{r(s)})(e_\beta, \sigma)(se_\gamma s', i_{\alpha\beta r(s)}) =$$

$$= (se_\gamma s', i_{\alpha\beta r(s)} \circ \varrho \circ \sigma \circ i_{\alpha\beta r(s)}) = (se_\gamma s', i_{\alpha\beta r(s)})(e_{\alpha\beta}, \varrho \circ \sigma)(se_\gamma s', i_{\alpha\beta r(s)}) =$$

$$= ((u_s i_\gamma u_{s'})(\varrho \circ \sigma)(u_s i_\gamma u_{s'}))\Phi .$$

Φ is one-to-one whence it follows that $(u_s i_\gamma u_{s'}) \varrho u_s u_{s'} \sigma (u_s i_\gamma u_{s'}) = (u_s i_\gamma u_{s'})(\varrho \circ \sigma) \cdot$ $\cdot (u_s i_\gamma u_{s'})$. Applying (11) one infers that $\varrho h_s \circ \sigma h_s = i_\gamma u_{s'} (\varrho \circ \sigma) u_s i_\gamma$. The right hand side of this equality is just $(\varrho \circ \sigma) h_s$ since $\varrho \circ \sigma \in \Sigma_{\alpha\beta}$. Thus h_s is shown to be an endomorphism of Σ. (9) immediately implies that $i_\alpha h_s = i_{(\alpha r(s))\tau_s}$ for $(u_{s'} i_\alpha u_s)\varkappa = s' e_\alpha s$ where $s' e_\alpha s \in E_{(\alpha r(s))\tau_s}$. On the other hand, we have $u_{s'} i_\alpha u_s \in \bar{B}$. Hence $u_{s'} i_\alpha u_s = i^*_{s' e_\alpha s}$. Therefore it is clear that $i_{(\alpha r(s))\tau_s} i^*_{s' e_\alpha s} i_{(\alpha r(s))\tau_s} = i_{(\alpha r(s))\tau_s}$.

5° *Property* (A4). Suppose $e \in E_\alpha$ and $\sigma \in \Sigma_\beta$. In consequence of (9) we have $\sigma h_e = i_{\alpha\beta} i_e^* \sigma i_e^* i_{\alpha\beta}$ as $r(e)=\alpha$ and τ_e is the identity automorphism of the semilattice αY. Since $i_\alpha \circ \sigma = \sigma \circ i_\alpha \in \Sigma_{\alpha\beta}$ we have

$$(i_{\alpha\beta} i_e^* \sigma i_e^* i_{\alpha\beta})\Phi = (e_{\alpha\beta}, i_{\alpha\beta})(e, i_\alpha)(e_\beta, \sigma)(e, i_\alpha)(e_{\alpha\beta}, i_{\alpha\beta}) =$$
$$= (e_{\alpha\beta}, i_{\alpha\beta} \circ i_\alpha \circ \sigma \circ i_\alpha \circ i_{\alpha\beta}) = (e_{\alpha\beta}, \sigma \circ i_\alpha) = (\sigma \circ i_\alpha)\Phi.$$

Thus we conclude $\sigma h_e = \sigma \circ i_\alpha$ for Φ is one-to-one. This was to be proved.

6° *Definition of* χ *which satisfies* (A1). Let s and $\bar{s}$ be arbitrary elements in S. Since $(u_s u_{\bar{s}})\varkappa = s\bar{s}$ the element $u_s u_{\bar{s}}$ is uniquely expressed in the form $u_{s\bar{s}} \sigma i_e^*$ where $\sigma \in \Sigma_{l(s\bar{s})}$ and $s\bar{s} \mathcal{L} e \mathcal{R} e_{l(s\bar{s})}$. The uniquely determined element σ will be denoted by $\chi_{s,\bar{s}}$. As we know

$$\chi_{s,\bar{s}} = i_{l(s\bar{s})} u_{(s\bar{s})'} u_s u_{\bar{s}} i_{l(s\bar{s})} \tag{12}$$

holds for any inverse $(s\bar{s})'$ of $s\bar{s}$ since $u_{(s\bar{s})'}$ is an inverse of $u_{s\bar{s}}$ in S_B provided $(s\bar{s})'$ is an inverse of $s\bar{s}$ in S. By (12), one can see that $\chi_{s,\bar{s}} \in S_B$ and, consequently, $\chi_{s,\bar{s}} \in U(\Sigma_{l(s\bar{s})})$. The dual argument shows that if s' and $\bar{s}'$ are inverses of s and $\bar{s}$, respectively, then there exists a unique element $\tilde{\sigma} \in \Sigma_{r(\bar{s}'s')}$ with $u_{\bar{s}'} u_{s'} = i_f^* \tilde{\sigma} u_{\bar{s}'s'}$ where $\bar{s}'s' \mathcal{R} f \mathcal{L} e_{r(\bar{s}'s')}$. Denoting this $\tilde{\sigma}$ by $\tilde{\chi}_{\bar{s}',s'}$ we obtain that

$$\tilde{\chi}_{\bar{s}',s'} = i_{r(\bar{s}'s')} u_{\bar{s}'} u_{s'} u_{(\bar{s}'s')'} i_{r(\bar{s}'s')} \tag{13}$$

holds for every inverse $(\bar{s}'s')'$ of $\bar{s}'s'$ and that $\tilde{\chi}_{\bar{s}',s'} \in U(\Sigma_{r(\bar{s}'s')})$. Obviously, $l(s\bar{s}) = r(\bar{s}'s')$. By choosing $(\bar{s}'s')'$ to be $s\bar{s}$ in (13) it immediately follows from (12) and (13) that $\chi_{s,\bar{s}}$ and $\tilde{\chi}_{\bar{s}',s'}$ are inverses of each other in T and, since they are contained in the same group $U(\Sigma_{l(s\bar{s})})$, we have $\tilde{\chi}_{\bar{s}',s'} = \chi_{s,\bar{s}}^{-1}$. If $s_1 \sim s_2$ and $\bar{s}_1 \sim \bar{s}_2$ then putting $s_1' = s_2'$ to be s' and $\bar{s}_1' = \bar{s}_2'$ to be $\bar{s}'$ where s' and $\bar{s}'$ are common inverses of s_1, s_2 and of $\bar{s}_1, \bar{s}_2$, respectively, (12) and (13) imply $\tilde{\chi}_{\bar{s}',s'} = \chi_{s_1,\bar{s}_1}^{-1} = \chi_{s_2,\bar{s}_2}^{-1}$. Therefore $\chi_{s_1,\bar{s}_1} = \chi_{s_2,\bar{s}_2}$, that is, property (A1) is valid for χ.

7° *Properties* (A5) *and* (A6). If $s \in S$ and $e, f \in E$ with $e \mathcal{R} s \mathcal{L} f$ then $es = sf = s$ and $i_e^* u_s = u_s i_f^* = u_s$. Hence, by (12), we have $\chi_{e,s} = \chi_{s,f} = i_{l(s)}$ since $u_{s'} u_s = i_{s's}^*$ where $s's \in E_{l(s)}$. Now let e and f be two idempotents in S. Then $\chi_{e,f} = i_{l(ef)} i_{ef}^* i_e^* i_f^* i_{l(ef)} = i_{l(ef)}$ according to (12), for $i_{ef}^* i_e^* i_f^* = i_{ef}^*$ and $ef \in E_{l(ef)}$.

8° *Property* (A2). Let $s, \bar{s}$ and $t \in S$. Applying the definition of the operation "$\circ$" on Σ we obtain by (12) and (13) that

$$\chi_{s\bar{s},t}^{-1} \circ \chi_{s,\bar{s}t} \circ \chi_{\bar{s},t} = i_{l(s\bar{s}t)} u_{t'} u_{(s\bar{s})'} u_{s\bar{s}t} i_{l(s\bar{s}t)} \cdot \tag{14}$$
$$\cdot\, i_{l(s\bar{s}t)} u_{(s\bar{s}t)'} u_s u_{\bar{s}t} i_{l(s\bar{s}t)} \cdot i_{l(\bar{s}t)} u_{(\bar{s}t)'} u_{\bar{s}} u_t i_{l(\bar{s}t)} i_{l(s\bar{s}t)}.$$

Introduce the following notations: let $\sigma_1 = u_{\bar{s}t} i_{l(s\bar{s}t)} i_{l(\bar{s}t)} u_{(\bar{s}t)'}$ and $\sigma_2 = u_{\bar{s}} u_t i_{l(\bar{s}t)} i_{l(s\bar{s}t)} u_{t'} u_{\bar{s}'}$. Since $l(s\bar{s}t)\tau_{\bar{s}t}^{-1} = l(s) r(\bar{s}t)$ we have $\sigma_1 \in \tilde{\Sigma}_{e_1}$ and $\sigma_2 \in \tilde{\Sigma}_{e_2}$ where $e_1, e_2 \in E_{l(s)r(\bar{s}t)}$. Utilizing the inequality $l(s)r(\bar{s}t) \leqq l(s)$ one can see that

$$(\sigma_1\sigma_2)\Phi = \sigma_1\Phi \cdot \sigma_2\Phi = (e_1, i_{l(s)r(\bar{s}t)})(e_2, i_{l(s)r(\bar{s}t)}) =$$
$$= (e_1 e_2, i_{l(s)r(\bar{s}t)}) = (e_1 s' s e_2, i_{l(s)r(\bar{s}t)} \circ i_{l(s)} \circ i_{l(s)r(\bar{s}t)}) =$$
$$= (e_1, i_{l(s)r(\bar{s}t)})(s's, i_{l(s)})(e_2, i_{l(s)r(\bar{s}t)}) = \sigma_1\Phi \cdot (u_{s'} u_s)\Phi \cdot \sigma_2\Phi = (\sigma_1 u_{s'} u_s \sigma_2)\Phi.$$

Since Φ is one-to-one this implies $\sigma_1\sigma_2=\sigma_1 u_{s'} u_s \sigma_2$. Again by the definition of Φ, we have $(u_{s\bar{s}} u_t i_{l(s\bar{s}t)} u_{t'} u_{(s\bar{s})'})\Phi=(s\bar{s}te_{l(s\bar{s}t)}t'(s\bar{s})', i_{r(s\bar{s}t)})$, $(u_{s\bar{s}t} i_{l(s\bar{s}t)} u_{(s\bar{s}t)'})\Phi=(s\bar{s}te_{l(s\bar{s}t)}(s\bar{s}t)', i_{r(s\bar{s}t)})$, $(u_s u_{\bar{s}t} i_{l(s\bar{s}t)} i_{l(\bar{s}t)} u_{(\bar{s}t)'} u_{s'})\Phi=(s\bar{s}te_{l(s\bar{s}t)} e_{l(\bar{s}t)}(\bar{s}t)' s', i_{r(s\bar{s}t)})$ and $(u_s u_{\bar{s}} u_t i_{l(\bar{s}t)} i_{l(s\bar{s}t)} u_{t'} u_{\bar{s}'} u_{s'})$ $\Phi=(s\bar{s}te_{l(\bar{s}t)} e_{l(s\bar{s}t)} t'\bar{s}'s', i_{r(s\bar{s}t)})$. Here the first components are all in $E_{r(s\bar{s}t)}$. Hence

$$[(u_{s\bar{s}} u_t i_{l(s\bar{s}t)} u_{t'} u_{(s\bar{s})'})(u_{s\bar{s}t} i_{l(s\bar{s}t)} u_{(s\bar{s}t)'})(u_s u_{\bar{s}t} i_{l(s\bar{s}t)} i_{l(\bar{s}t)} u_{(\bar{s}t)'} u_{s'})\cdot$$
$$\cdot(u_s u_{\bar{s}} u_t i_{l(\bar{s}t)} i_{l(s\bar{s}t)} u_{t'} u_{\bar{s}'} u_{s'})]\Phi = (s\bar{s}te_{l(s\bar{s}t)} t'(s\bar{s})' s\bar{s}te_{l(\bar{s}t)} e_{l(s\bar{s}t)} t'\bar{s}'s', i_{r(s\bar{s}t)}) =$$
$$= [(u_{s\bar{s}} u_t i_{l(s\bar{s}t)} u_{t'} u_{(s\bar{s})'})(u_s u_{\bar{s}} u_t i_{l(\bar{s}t)} i_{l(s\bar{s}t)} u_{t'} u_{\bar{s}'} u_{s'})]\Phi.$$

If the images of two elements under Φ are equal then the elements themselves are equal. Since $u_{s\bar{s}} u_t i_{l(s\bar{s}t)}$ and $i_{l(s\bar{s}t)} u_{t'} u_{(s\bar{s})'}$, and, similarly, $u_s u_{\bar{s}} u_t i_{l(\bar{s}t)} i_{l(s\bar{s}t)}$ and $i_{l(s\bar{s}t)} u_{t'} u_{\bar{s}'} u_{s'}$ are inverses of each other (14) implies

$$\chi^{-1}_{s\bar{s},t}\circ\chi_{s,\bar{s}t}\circ\chi_{\bar{s},t} = i_{l(s\bar{s}t)} u_{t'} u_{(s\bar{s})'} u_s u_{\bar{s}} u_t i_{l(\bar{s}t)} i_{l(s\bar{s}t)}.$$

Here $u_{t'} u_{(s\bar{s})'} u_s u_{\bar{s}} u_t \in \tilde{\Sigma}_{t'(s\bar{s})' s\bar{s}t}$ and $t'(s\bar{s})' s\bar{s}t \in E_{l(s\bar{s}t)}$. Since $l(\bar{s}t) \geqq l(s\bar{s}t)$ we have $i^*_{t'(s\bar{s})' s\bar{s}t} i_{l(\bar{s}t)} i_{l(s\bar{s}t)} = i^*_{t'(s\bar{s})' s\bar{s}t} i_{l(s\bar{s}t)}$. Thus

$$\chi^{-1}_{s\bar{s},t}\circ\chi_{s,\bar{s}t}\circ\chi_{\bar{s},t} = i_{l(s\bar{s}t)} u_{t'} u_{(s\bar{s})'} u_s u_{\bar{s}} u_t i_{l(s\bar{s}t)}. \tag{15}$$

Observe that $(u_t i_{l(s\bar{s}t)} u_{t'})\Phi=(te_{l(s\bar{s}t)} t', i_{l(s\bar{s})r(t)})$ and $(u_{(s\bar{s})'} u_s u_{\bar{s}})\Phi=((s\bar{s})' s\bar{s}, \varrho)$ where $\varrho\in\Sigma_{l(s\bar{s})}$. Therefore

$$[(u_t i_{l(s\bar{s}t)} u_{t'})(u_{(s\bar{s})'} u_s u_{\bar{s}})(u_t i_{l(s\bar{s}t)} u_{t'})]\Phi =$$
$$= ((te_{l(s\bar{s}t)} t')((s\bar{s})' s\bar{s})(te_{l(s\bar{s}t)} t'), i_{l(s\bar{s}) r(t)}\circ\varrho\circ i_{l(s\bar{s}) r(t)}) =$$
$$= ((te_{l(s\bar{s}t)} t') e_{l(s\bar{s})}((s\bar{s})' s\bar{s}) e_{l(s\bar{s})}(te_{l(s\bar{s}t)} t'), i_{l(s\bar{s}) r(t)}\circ i_{l(s\bar{s})}\circ\varrho\circ i_{l(s\bar{s})}\circ i_{l(s\bar{s}) r(t)}) =$$
$$= [(u_t i_{l(s\bar{s}t)} u_{t'}) i_{l(s\bar{s})}(u_{(s\bar{s})'} u_s u_{\bar{s}}) i_{l(s\bar{s})}(u_t i_{l(s\bar{s}t)} u_{t'})]\Phi.$$

If the images of two elements under Φ are equal then the elements themselves are equal. Utilizing the fact that $i_{l(s\bar{s}t)} u_{t'}$ and $u_t i_{l(s\bar{s}t)}$ are inverses of each other, we obtain by (15) that

$$\chi^{-1}_{s\bar{s},t}\circ\chi_{s,\bar{s}t}\circ\chi_{\bar{s},t} = i_{l(s\bar{s}t)} u_t (i_{l(s\bar{s})} u_{(s\bar{s})'} u_s u_{\bar{s}} i_{l(s\bar{s})}) u_{t'} i_{l(s\bar{s}t)}.$$

According to (12) and (9), the right hand side of this equality is just $\chi_{s,\bar{s}} h_t$. Thus (A2) is shown.

9° *The equality*

$$i^*_k u_s u_{\bar{s}}\sigma = i^*_k u_{s\bar{s}}(\chi_{s,\bar{s}}\circ\sigma) \tag{16}$$

holds provided $k\in E_\alpha$, $s, \bar{s}\in S$ *and* $\sigma\in\Sigma_\beta$ *where* $\beta=(\alpha r(s\bar{s}))\tau_{s\bar{s}}$. By the definition of $\chi_{s,\bar{s}}$, we have $u_s u_{\bar{s}}=u_{s\bar{s}}\chi_{s,\bar{s}} i^*_e$ where $s\bar{s}\mathcal{L}e\mathcal{R}e_{l(s\bar{s})}$. Therefore

$$i^*_k u_s u_{\bar{s}}\sigma = i^*_k u_{s\bar{s}}\chi_{s,\bar{s}} i^*_e\sigma = i^*_k u_{s\bar{s}}(i_\beta u_{(s\bar{s})'} i^*_k u_{s\bar{s}})\chi_{s,\bar{s}} i^*_e\sigma. \tag{17}$$

The latter equality holds because $i_\beta u_{(s\bar{s})'} i^*_k$ is an inverse of $i^*_k u_{s\bar{s}}$. Applying the isomorphism Φ we can see that

$$[(i_\beta u_{(s\bar{s})'} i^*_k u_{s\bar{s}})\chi_{s,\bar{s}} i^*_e\sigma]\Phi = (e_\beta, i_\beta)((s\bar{s})' ks\bar{s}, i_\beta)(e_{l(s\bar{s})}, \chi_{s,\bar{s}})(e, i_{l(s\bar{s})})(e_\beta, \sigma) =$$
$$= (e_\beta, i_\beta\circ\chi_{s,\bar{s}}\circ\sigma) = (e_\beta, \chi_{s,\bar{s}}\circ\sigma) = (\chi_{s,\bar{s}}\circ\sigma)\Phi.$$

Here the facts are used that $(s\bar{s})'ks\bar{s}\in E_\beta$ and $e\in E_{l(s\bar{s})}\geqq E_\beta$. Hence the equality (16) immediately follows by (17) since Φ is one-to-one.

10° *We have*

$$\sigma i_f^* \varrho i_e^* = (\sigma \circ \varrho) i_f^* \tag{18}$$

provided $\varrho\in\Sigma_\alpha$, $\sigma\in\Sigma_\beta$ *and* $e\mathcal{R}e_\alpha$, $f\mathcal{R}e_\beta$ *with* $fe=f$. Clearly, $\beta\leqq\alpha$ by the equality $fe=f$. Applying the isomorphism Φ we have

$$(\sigma i_f^* \varrho i_e^*)\Phi = (e_\beta, \sigma)(f, i_\beta)(e_\alpha, \varrho)(e, i_\alpha) = (e_\beta f e_\alpha e, \sigma \circ i_\beta \circ \varrho \circ i_\alpha) =$$
$$= (f, \sigma\circ\varrho) = (e_\beta f, \sigma\circ\varrho\circ i_\beta) = (e_\beta, \sigma\circ\varrho)(f, i_\beta) = ((\sigma\circ\varrho)i_f^*)\Phi.$$

This implies the equality (18) since Φ is one-to-one.

11° *Property* (A3). Now let $s, \bar{s}\in S$ and $\sigma\in\Sigma_\alpha$. By (10), we have

$$\sigma u_s u_{\bar{s}} = (\sigma i_\alpha u_s) u_{\bar{s}} = \big(i_\alpha u_s (\sigma h_s) i_f^*\big) u_{\bar{s}} = i_\alpha u_s \big((\sigma h_s) i_f^*\big) u_{\bar{s}} = i_\alpha u_s i_f^* u_{\bar{s}} (\sigma h_s h_{\bar{s}}) i_g^* \tag{19}$$

where $e_\alpha s\mathcal{L}f\mathcal{R}e_{(\alpha r(s))\tau_s}$ and $f\bar{s}\mathcal{L}g\mathcal{R}e_{((\alpha r(s))\tau_s r(\bar{s}))\tau_{\bar{s}}}$. Here $\big((\alpha r(s))\tau_s r(\bar{s})\big)\tau_{\bar{s}}=(\alpha r(s\bar{s}))\tau_{s\bar{s}}$. Hence $\sigma h_s h_{\bar{s}}\in\Sigma_{(\alpha r(s\bar{s}))\tau_{s\bar{s}}}$. Since $e_\alpha sf=e_\alpha s$ and $u_{s'} i_\alpha u_s = i^*_{s' e_\alpha s} = i^*_{s' e_\alpha s} i_f^*$ we have $i_\alpha u_s = i_\alpha u_s (u_{s'} i_\alpha u_s) = i_\alpha u_s (u_{s'} i_\alpha u_s) i_f^* = i_\alpha u_s i_f^*$. Furthermore, $e_\alpha s\bar{s}\mathcal{L}f\bar{s}$ for $e_\alpha s\mathcal{L}f$. Therefore (19) implies $\sigma u_s u_{\bar{s}} = i_\alpha u_s u_{\bar{s}} (\sigma h_s h_{\bar{s}}) i_g^*$ where $e_\alpha s\bar{s}\mathcal{L}g\mathcal{R}e_{(\alpha r(s\bar{s}))\tau_{s\bar{s}}}$. Hence we obtain by (16) that

$$\sigma u_s u_{\bar{s}} = i_\alpha u_{s\bar{s}} (\chi_{s,\bar{s}} \circ \sigma h_s h_{\bar{s}}) i_g^*. \tag{20}$$

On the other hand, we have $u_s u_{\bar{s}} = u_{s\bar{s}} \chi_{s,\bar{s}} i_e^*$ where $s\bar{s}\mathcal{L}e\mathcal{R}e_{l(s\bar{s})}$ whence it follows by (10) that

$$\sigma u_s u_{\bar{s}} = \sigma u_{s\bar{s}} \chi_{s,\bar{s}} i_e^* = \sigma (i_\alpha u_{s\bar{s}}) \chi_{s,\bar{s}} i_e^* = i_\alpha u_{s\bar{s}} (\sigma h_{s\bar{s}}) i_g^* \chi_{s,\bar{s}} i_e^*.$$

Since $e_\alpha s\bar{s}\mathcal{L}g$ and $s\bar{s}\mathcal{L}e$ we infer $ge=g$. This ensures by (18) that $\sigma u_s u_{\bar{s}} = i_\alpha u_{s\bar{s}} (\sigma h_{s\bar{s}} \circ \chi_{s,\bar{s}}) i_g^*$.

Since $\sigma u_s u_{\bar{s}}$ is uniquely expressed in the form $i_\alpha u_{s\bar{s}} \varrho i_g^*$ where $\varrho\in\Sigma_\beta$ the equality $\chi_{s,\bar{s}}\circ\sigma h_s h_{\bar{s}} = \sigma h_{s\bar{s}}\circ\chi_{s,\bar{s}}$ holds in Σ by (20). Multiplying this equality by $\chi_{s,\bar{s}}^{-1}$ the equality in (A3) yields because $\chi_{s,\bar{s}}^{-1}\circ\chi_{s,\bar{s}} = i_{l(s\bar{s})} \geqq i_{(\alpha r(s\bar{s}))\tau_{s\bar{s}}}$.

Thus h, χ is proved to be an (S, Σ)-pair.

12° ψ *is an isomorphism of* T *onto* $\mathcal{S}(S, \Sigma; h, \chi)$. The mapping ψ defined at the beginning of the proof is one-to-one. All we have to prove is that, for every $s, \bar{s}\in S$, $\sigma\in\Sigma_{l(s)}$ and $\bar{\sigma}\in\Sigma_{l(\bar{s})}$, the equality

$$(u_s \sigma i_e^*)(u_{\bar{s}} \bar{\sigma} i_{\bar{e}}^*) = u_{s\bar{s}} (\chi_{s,\bar{s}} \circ \sigma h_{\bar{s}} \circ \bar{\sigma}) i_f^* \tag{21}$$

is valid where e, $\bar{e}$ and f are idempotents with $s\mathcal{L}e\mathcal{R}e_{l(s)}$, $\bar{s}\mathcal{L}\bar{e}\mathcal{R}e_{l(\bar{s})}$ and $s\bar{s}\mathcal{L}f\mathcal{R}e_{l(s\bar{s})}$. By the definition of $h_{\bar{s}}$, (9) implies $\sigma i_e^* u_{\bar{s}} = i_e^* u_{\bar{s}} (\sigma h_{\bar{s}}) i_f^*$ since $e\bar{s}\mathcal{L}s\bar{s}\mathcal{L}f$. Since $u_s i_e^* = u_s$ we have $(u_s \sigma i_e^*)(u_{\bar{s}} \bar{\sigma} i_{\bar{e}}^*) = u_s u_{\bar{s}} (\sigma h_{\bar{s}}) i_f^* \bar{\sigma} i_{\bar{e}}^*$.

Applying the equality (16) with $k=ss'$ we conclude $i_{ss'}^* u_s u_{\bar{s}} \sigma h_{\bar{s}} = i_{ss'}^* u_{s\bar{s}} (\chi_{s,\bar{s}}\circ\sigma h_{\bar{s}})$. Here $i_{ss'}^* u_s = u_s$ and $i_{ss'}^* u_{s\bar{s}} = u_{s\bar{s}}$. Hence $(u_s \sigma i_e^*)(u_{\bar{s}} \bar{\sigma} i_{\bar{e}}^*) = u_{s\bar{s}} (\chi_{s,\bar{s}}\circ\sigma h_{\bar{s}}) i_f^* \bar{\sigma} i_{\bar{e}}^*$ where $s\bar{s}\mathcal{L}f$ and $\bar{s}\mathcal{L}\bar{e}$ and therefore $f\bar{e}=f$. Thus, the equality (21) follows from (18) whence we infer that T is isomorphic to $\mathcal{S}(S, \Sigma; h, \chi)$. Corollary 2.3 ensures that Σ is unique up to isomorphism which completes the proof of the theorem.

REMARK. The idempotent separating congruences on an orthodox semigroup T with band of idempotents B are just the B-separating congruences which are trivially strong. Thus Theorems 1.5 and 3.2 give a description of all idempotent separating extensions of orthodox semigroups if we require Σ to be a semilattice of groups.

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NEGATIVE BEANTWORTUNG EINER APPROXIMATIONSFRAGE

Von

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Es sei S_n der Operator von Szász—Mirakyan, das heißt

$$S_n(f;\, x) = e^{-nx} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!}.$$

Bezeichne C^* die Menge der im Intervall $[0, \infty)$ gleichmäßig stetigen und beschränkten Funktionen. Setzen wir

$$\|f\|_{C^*} = \sup_{0 \leqq x < \infty} |f(x)| \quad (f \in C^*),$$

$$\omega(f;\, \delta) = \sup_{0 \leqq \delta_1 \leqq \delta} \|f(x+\delta_1) - f(x)\|_{C^*} \quad (f \in C^*,\, \delta > 0),$$

T. Hermann bewies [1], daß es keine Konstante K gibt, so daß für alle $f \in C^*$ die Ungleichung

$$\|S_n(f) - f\|_{C^*} \leqq K\omega\left(f;\, \frac{1}{\sqrt{n}}\right)$$

gilt. In Zusammenhang damit stellte er die Frage, ob zu jeder Funktion $f \in C^*$ eine (nur von f abhängende) Konstante $K(f)$ existiert, derart, daß die Abschätzung

$$\|S_n(f) - f\|_{C^*} \leqq K(f)\omega\left(f;\, \frac{1}{\sqrt{n}}\right)$$

zutrifft. Nun, diese Frage kann man negativ beantworten, das geht aus folgender Behauptung hervor:

Satz. *Es existiert eine, im Intervall* $[0, \infty)$ *beschränkte, differenzierbare Funktion* $f \in C^*$ *mit beschränkter Ableitung, so daß für hinreichend große* n *die Ungleichung*

$$\|S_n(f) - f\|_{C^*} > \varepsilon$$

gilt, wobei ε *eine geeignete positive Zahl ist.*

Folgerung. Es existiert eine Funktion $f \in C^*$, für welche die folgende Limesbeziehung gilt:

$$\lim_{n \to \infty} \frac{\|S_n(f) - f\|_{C^*}}{\omega\left(f;\, \frac{1}{\sqrt{n}}\right)} = \infty$$

$\Big($Aus $|f'(x)|\leqq M$ $(0\leqq x<\infty)$ folgt nämlich die Ungleichung $\omega(f;\delta)\leqq M\delta$, und daraus $\lim\limits_{n\to\infty}\omega\left(f:\dfrac{1}{\sqrt{n}}\right)=0.\Big)$

Beweis. Betrachten wir die nach der Gleichung $f(x)=\sin\left(\dfrac{\pi}{6}x\right)$ $(0\leqq x<\infty)$ definierte Funktion f. Es sei $n>0$ eine natürliche Zahl. Es existiert zu jeder $t\geqq 0$ eine Stelle ξ zwischen den Stellen t und $12n+3$, so daß

$$f(t)=1-\frac{\dfrac{\pi^2}{36}\sin\left(\dfrac{\pi}{6}\xi\right)}{2!}[t-(12n+3)]^2$$

ist. (Taylor-Formel.) Daraus folgt, daß (unabhängig von t und n) $\sin\left(\dfrac{\pi}{6}\xi\right)\geqq 0$ gilt. Also gelten die Gleichungen

$$f\left(\frac{k}{n}\right)=1-\frac{\dfrac{\pi^2}{36}\sin\left(\dfrac{\pi}{6}\xi_{n,k}\right)}{2!}\left[\frac{k}{n}-(12n+3)\right]^2\quad(k=0,1,2,\ldots)$$

mit $\sin\left(\dfrac{\pi}{6}\xi_{n,k}\right)\geqq 0$, wobei sich die Stelle $\xi_{n,k}$ zwischen den Stellen k/n und $12n+3$ befindet $(k=0,1,2,\ldots)$. So erhalten wir

$$S_n(f;12n+3)=1-\frac{\pi^2}{72}e^{-12n^2-3n}\sum_{k=1}^{\infty}\sin\left(\frac{\pi}{6}\xi_{n,k}\right)\left[\frac{k}{n}-(12n+3)\right]^2\frac{(12n^2+3n)^k}{k!}.$$

Da $f(12n+3)=1$ ist, ergibt sich

$$\Delta_n\overset{\mathrm{Def}}{=\!=}|f(12n+3)-S_n(f;12n+3)|>$$

$$>\frac{\pi^2}{72}e^{-12n^2-3n}\sum_{k=12n^2+4n}^{12n^2+5n}\sin\left(\frac{\pi}{6}\xi_{n,k}\right)\left[\frac{k}{n}-(12n+3)\right]^2\frac{(12n^2+3n)^k}{k!}.$$

Aus $12n+4\leqq\dfrac{k}{n}\leqq 12n+5$ folgen die Ungleichungen $12n+3<\xi_{n,k}<12n+5$ und daraus $\sin\left(\dfrac{\pi}{6}\xi_{n,k}\right)>\dfrac{1}{2}$. Ferner ist es evident, daß $\dfrac{k}{n}-(12n+3)\geqq 1$ ist, so gilt die Abschätzung

$$\Delta_n>\frac{\pi^2}{144}e^{-12n^2-3n}\sum_{j=0}^{n}\frac{(12n^2+3n)^{12n^2+4n+j}}{(12n^2+4n+j)!}.$$

Da das letzte Glied das kleinste der Glieder der Summe ist, bekommt man

$$\Delta_n>\frac{\pi^2}{144}e^{-12n^2-3n}(n+1)\frac{(12n^2+3n)^{12n^2+5n}}{(12n^2+5n)!}\overset{\mathrm{Def}}{=\!=}\frac{\pi^2}{144}B_n.$$

Aufgrund der Stirling-Formel ergibt sich

$$B_n=e^{-12n^2-3n}(n+1)\frac{(12n^2+3n)^{12n^2+5n}}{\omega_n\sqrt{2\pi(12n^2+5n)}\,e^{-12n^2-5n}(12n^2+5n)^{12n^2+5n}}$$

mit $\lim_{n\to\infty} \omega_n = 1$. So erhalten wir

$$\lim_{n\to\infty} B_n = \frac{1}{2\sqrt{6\pi}} \lim_{n\to\infty} \frac{e^{2n}}{\left(1+\frac{2}{12n+3}\right)^{12n^2+5n}}.$$

Die Funktion g definieren wir nach der Gleichung

$$g(x) = \frac{e^{2x}}{\left(1+\frac{2}{12x+3}\right)^{12x^2+5x}} \quad (x>0).$$

Verwenden wir die Bezeichnung $y = \frac{1}{12x+3}$. Dann ist $x = \frac{1}{12y} - \frac{1}{4}$, und

$$g(x) = \exp\left[\frac{1}{6y} - \frac{1}{2} - \left(\frac{1}{12y^2} - \frac{1}{12y} - \frac{1}{2}\right)\log(1+2y)\right] =$$

$$= \exp\left[\frac{-1+\log(1+2y)}{2} + \frac{\log(1+2y)}{12y} + \frac{2y-\log(1+2y)}{12y^2}\right].$$

Unter Anwendung der Regel von l'Hospital erhalten wir

$$\lim_{y\to 0+} [\ldots] = -\frac{1}{2} + \frac{1}{6} + \frac{1}{6} = -\frac{1}{6},$$

das heißt $g(x) \to e^{-1/6}$ $(x\to\infty)$. Deshalb ist

$$\lim_{n\to\infty} B_n = \frac{1}{2\sqrt{6\pi}\sqrt[6]{e}}.$$

Dann existiert eine solche Zahl n_0, daß für $n > n_0$ $B_n > \frac{1}{2e\sqrt{6\pi}}$ its. Mit $f(x) = \sin\left(\frac{\pi}{6}x\right)$ $(x \geqq 0)$ hat sich also die Gültigkeit der Abschätzung

$$\|S_n(f) - f\|_{C^*} > \frac{\pi^2}{288e\sqrt{6\pi}} \quad (n > n_0)$$

bewiesen.

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WHICH DISTRIBUTIVE LATTICES HAVE 2-DISTRIBUTIVE SUBLATTICE LATTICES?

By

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I. Introduction

The concept of *n-distributivity* was introduced by HUHN (cf. [4] and [5]). A lattice is said to be n-distributive $(n \geqq 1)$ if it satisfies the identity

$$x \wedge \bigvee_{i=0}^{n} y_i \leqq \bigvee_{j=0}^{n} \Big(x \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n} y_i \Big). \tag{1}$$

The n-distributivity of subalgebra lattices (or congruence lattices) of universal algebras proved to be an important property in several cases. E.g., as it was proved by HUHN ([3] and [4]), the subgroup lattice of an abelian group A is n-distributive iff every finitely generated subgroup of A can be generated by at most n elements.

Sublattice lattices were investigated by FILIPPOV [1]. He gave necessary and sufficient conditions for having isomorphism between sublattice lattices of two given lattices. Lattices having modular and (upper) semi-modular sublattice lattices were characterized by KOH [6] and LAKSER [7], respectively.

Our aim is to characterize distributive lattices having n-distributive sublattice lattices in case $n \leqq 2$.

II. Preliminaries

For an *idempotent* algebra A let $\mathrm{Su}(A)$ denote the lattice of subalgebras of A. (It contains the empty set as a subalgebra.) Let us recall a non-published result of A. P. HUHN:

LEMMA 1. *For an arbitrary idempotent algebra A and $n \geqq 1$, $\mathrm{Su}(A)$ is n-distributive iff for any subset H of A we have*

$$[H] = \bigcup \{[G] \colon G \subseteqq H \quad \text{and} \quad |G| \leqq n\}. \tag{2}$$

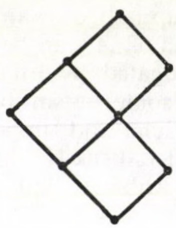
(Here, $[H]$ means the subalgebra generated by H and $\bigcup$ stands for the set-theoretical union.)

PROOF. Suppose $\mathrm{Su}(A)$ is n-distributive and $H = \{h_0, h_1, \ldots, h_m\} \subseteqq A$ for some $m \geqq n$. n-distributivity implies m-distributivity for all $m \geqq n$ (HUHN [5]), so for an arbitrary $a \in [H]$, we have

$$a \in \{a\} \wedge \bigvee_{i=0}^{m} \{h_i\} \subseteqq \bigvee_{j=0}^{m} \Big(\{a\} \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{m} \{h_i\} \Big).$$

Hence $a \in [\{h_0, \ldots, h_{j-1}, h_{j+1}, \ldots, h_m\}]$ for some j. This proves (2) for any *finite* H whence (2) holds for *any* subset H. Conversely, suppose that (2) holds. Then for any $Y_i \in \mathrm{Su}(A)$, we have $\bigvee_{i=0}^{n} Y_i = \bigcup_{j=0}^{n} \bigvee_{\substack{i=0 \\ i \neq j}}^{n} Y_i$ whence the n-distributivity of $\mathrm{Su}(A)$ follows easily. Q.e.d.

Now we define the concept of the *special sum* of lattices. Let $(I, \leqq)$ be a chain and for every $i \in I$ let L_i be a lattice. Let $\sum_{i \in I} L_i$ denote the *ordinal sum* of lattices in the usual sense. (I.e., consider the disjoint union of the L_i-s and let $x \leqq y$ mean that $x \in L_i$, $y \in L_j$ and $i < j$, or $x, y \in L_i$ and $x \leqq y$.) For $a, b \in \sum_{i \in I} L_i$, let $a \vartheta b$ denote that "a is the greatest element of L_i, b is the least element of L_j and $i \prec j$, for some $i, j \in I$". Let Θ be the *equivalence* relation on $\sum_{i \in I} L_i$ generated by the binary relation ϑ. Then, as it can be seen easily, Θ is a congruence relation. Now, denoting by $\sum'_{i \in I} L_i$, the definition of the special sum is the following: $\sum'_{i \in I} L_i = \sum_{i \in I} L_i / \Theta$. Let us agree that we write $\sum'_{i \in I} L_i = \sum_{i \in I} L_i$ iff Θ is the equality relation. Denoting the lattice



by K we can state our main

THEOREM. *For any distributive lattice L the following three conditions are equivalent:*

(i) $\mathrm{Su}(L)$ *is 2-distributive*

(ii) L *contains neither a sublattice isomorphic to K nor a three-element antichain (antichain means a set of pairwise incomparable elements)*

(iii) L *is isomorphic to a special sum $\sum'_{i \in I} L_i$, where for each $i \in I$, L_i is a chain or $L_i = \underset{\sim}{2} \times C$ for some chain C. ($\underset{\sim}{2}$ denotes the two-element chain.)*

REMARK. As for 1-distributivity, which is the usual distributivity, it is very easy to show that an arbitrary lattice L has distributive sublattice lattice iff L is a chain.

In what follows lattices isomorphic to $\underset{\sim}{2} \times C$ for some chain C will be referred to as *ladders*. The following sections deal with the proof of the theorem, namely the implications (iii)→(i), (i)→(ii) and (ii)→(iii) are proved.

III. The first part of the proof

In this section the implications (iii)$\to$(i) and (i)$\to$(ii) will be verified.

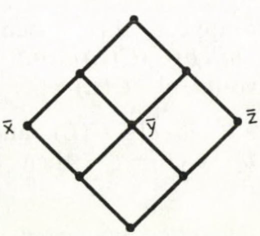
PROPOSITION 1. *If a lattice* L *is a chain or a ladder then* $\mathrm{Su}(L)$ *is 2-distributive. If* L_2 *is a homomorphic image of a lattice* L_1 *and* $\mathrm{Su}(L_1)$ *is 2-distributive then so is* $\mathrm{Su}(L_2)$.

The proof is straightforward by making use of Lemma 1, so it will be omitted. It can be easily shown that

$$\mathrm{Su}\,(\sum_{i\in I} L_i) \cong \prod_{i\in I} \mathrm{Su}\,(L_i) \tag{3}$$

for an arbitrary ordinal sum $\sum_{i\in I} L_i$ (cf. also FILIPPOV [1, Lemma 1.2]). Now Proposition 1 and (3) yield the proof of (iii)$\to$(i).

To prove (i)$\to$(ii) suppose L is a distributive lattice and $\mathrm{Su}(L)$ is 2-distributive. Since neither $\mathrm{Su}(K)$ nor $\mathrm{Su}(\underset{\sim}{2}^3)$ are 2-distributive by Lemma 1, L does not contain any sublattice isomorphic to K or $\underset{\sim}{2}^3$. Suppose $\{a, b, c\}$ is a three-element antichain in L. Then $\{a\vee b, a\vee c, b\vee c\}$ cannot be a three-element antichain since otherwise it would generate a sublattice isomorphic to $\underset{\sim}{2}^3$ (cf. GRÄTZER [2, p. 45]). Hence $a\vee b\vee c\in\{a\vee b, a\vee c, b\vee c\}$ and, by the lattice theoretical Duality Principle, $a\wedge b\wedge c\in\{a\wedge b, a\wedge c, b\wedge c\}$. If we had $a\vee b\vee c=a\vee b$ and $a\wedge b\wedge c=b\wedge c$ then $c=(a\vee b\vee c)\wedge c=(a\vee b)\wedge c\leqq(a\vee b)\wedge(a\vee c)=a\vee(b\wedge c)=a\vee(a\wedge b\wedge c)=a$ would contradict $a\| c$. So $a\vee c=a\vee b\vee c$ and $a\wedge c=a\wedge b\wedge c$ can be assumed. Now $[a, b, c]=[\{a, b, c\}]$ is a homomorphic image of $\mathrm{FD}(3)/\Theta$ where $\mathrm{FD}(3)$ denotes the free distributive lattice freely generated by $\{x, y, z\}$ and Θ denotes the smallest congruence for which $a\wedge c\Theta a\wedge b\wedge c$ and $a\vee c\Theta a\vee b\vee c$. Since the structure of $\mathrm{FD}(3)$ is well-known (c.f. GRÄTZER [2, p. 46]), it is easy to check that $\mathrm{FD}(3)/\Theta$ is the following lattice:



An arbitrary homomorphism $\varphi\colon \mathrm{FD}(3)/\Theta\to[a, b, c]$, $\bar{x}\mapsto a$, $\bar{y}\mapsto b$, $\bar{z}\mapsto c$ must be injective because a, b, c are pairwise incomparable. Hence $\mathrm{FD}(3)/\Theta$ can be embedded into L. So can K, which is a contradiction. Thus the proof of (i)$\to$(ii) is complete.

IV. A decomposition of lattices

The (decomposability) lemma given below will be an important tool to prove (ii)$\to$(iii). First, for a *partially ordered set* L, we set

$$C(L) = \{x \in L: x \nparallel y \text{ for all } y \in L\}$$

and

$$C'(L) = \{x \in C(L): x \neq 0_L \text{ and } x \neq 1_L\}.$$

(Note that L is not necessarily bounded and the above two sets may coincide.)

LEMMA 2. *An arbitrary lattice L is isomorphic to a special sum $\sum_{i\in I}' L_i$ where, for all $i \in I$, L_i is a chain or $C'(L_i)=\varnothing$.*

Before proving this lemma we need some preliminaries. Define the binary relation $\varrho = \varrho_L$ on L in the following way: set $a\varrho b$ iff one of the conditions

— $a \parallel b$

— $a \nparallel b$ and $[a, b] \cap C(L) = \varnothing$

— $a \nparallel b$ and $[a, b] \subseteqq C(L)$

holds, where $[a, b] = \{x \in L: a \leqq x \leqq b \text{ or } b \leqq x \leqq a\}$.

PROPOSITION 2. *For an arbitrary lattice L, $\varrho = \varrho_L$ is an equivalence relation.*

PROOF. Suppose we have $a\varrho b$ and $b\varrho c$ for some $a, b, c \in L$ and let us show that $a\varrho c$. Evidently ϱ is reflexive and symmetric so, by the Duality Principle, we have to deal only with the following four cases.

Case 1. $a\parallel b$ and $b \parallel c$. If $a \nparallel c$, say $a<c$, then $[a, c]\cap C(L)=\varnothing$ because $x \parallel b$ for all $x\in[a, c]$.

Case 2. $a \parallel b$ and $b<c$. Suppose $a\nparallel c$, then $a<c$. Now $[a, c]\cap C(L)= =([a\vee b, c]\cap C(L))\cup(([a, a\vee b]\setminus\{a\vee b\})\cap C(L))$. But, for all $x\in[a, a\vee b]\setminus\{a\vee b\}$, $x\parallel b$ and $[a\vee b, c]\subseteqq[b, c]$ so we have $[a, c]\cap C(L)=\varnothing$.

Case 3. $a<b$ and $b<c$. If $[a, b]\subseteqq C(L)$ and $[b, c]\subseteqq C(L)$ then $[a, c]= =[a, b]\cup[b, c]\subseteqq C(L)$. If $[a, b]\cap C(L)=\varnothing$ then $[a, c]\cap C(L)=([a, b]\cap C(L))\cup \cup([b, c]\cap C(L))=\varnothing$.

Case 4. $a<b$ and $b>c$. If $a\nparallel c$, say $a<c$, then $[a, c]\subseteqq[a, b]$ and either $[a, c]\subseteqq[a, b]\subseteqq C(L)$ or $[a, c]\cap C(L)\subseteqq[a, b]\cap C(L)=\varnothing$. This completes the proof of Proposition 2.

PROPOSITION 3. *Let L be a lattice and M a ϱ_L-class in L. Then either $M\subseteqq C(L)$ or $M\cap C(L)=\varnothing$. If $M\subseteqq C(L)$ then M is a chain. If $M\cap C(L)=\varnothing$ then M has neither greatest element nor least element, and exactly one of the following four possibilities*

$$[M]=M,\quad [M]=M\cup\{1_M\},\quad [M]=M\cup\{0_M\},\quad [M]=M\cup\{0_M, 1_M\}$$

holds where $0_M, 1_M\in C(L)\setminus M$ and they are the zero and unit of $[M]$, respectively.

PROOF. It is sufficient to prove the last statement. Let $M\cap C(L)=\emptyset$. Then $C(M)=\emptyset$ whence M has neither greatest nor least element. Suppose M is not a join sub-semilattice of L, say $a, b\in M$ but $a\vee b\notin M$, and let us show that $a\vee b\in C(L)$ and $M\cup\{a\vee b\}$ is a join-semilattice. Since $a\bar{\varrho}a\vee b$ (i.e., $a\varrho a\vee b$ does not hold), there is an $x\in[a, a\vee b]\cap C(L)$. Now $a\|b$ implies $x\ngeqq b$ and $b\in M$ implies $x\nparallel b$ and so $a\vee b\leqq x\in[a, a\vee b]$ implies $a\vee b=x\in C(L)$. Now x is the unit of $M\cup\{x\}$ because otherwise $x<c$ for some $c\in M$ and $x\in[a, c]\cap C(L)=\emptyset$, a contradiction. If, for $d, e\in M, y=d\vee e\notin M$, then $y\leqq x$. But the role of x and y can be interchanged so $x\leqq y$ as well. I.e., $M\cup\{a\vee b\}$ is a join-semilattice indeed and the proof is complete by the Duality Principle.

PROOF OF LEMMA 2. Let $\varrho=\varrho_L$ and for $D_1, D_2\in L/\varrho$ we define $D_1\leqq D_2$ by the formula $(\exists X_1\in D_1)(\exists X_2\in D_2)(X_1\leqq X_2)$. An easy calculation shows that $(L/\varrho, \leqq)$ is a chain. We assert $L\cong \sum\limits_{D\in L/\varrho}' [D]$. Let $\varphi\colon \sum\limits_{D\in L/\varrho} [D]\to L$ be the map for which $\varphi|_{[D]}$ is the natural embedding of D into L. It follows easily that φ is a homomorphism onto L. It is also seen that if $D, E\in L/\varrho$, $D<E$ and $x\in[E]\cap[D]$ then $x=1_D=0_E$, and $D\prec E$ or $D\prec\{x\}\prec E$. Therefore $L\cong \sum\limits_{D\in L/\varrho} [D]/\mathrm{Ker}\,\varphi = \sum\limits_{D\in L/\varrho}' [D]$, indeed. If $D\in L/\varrho$ is not a chain then $C'([D])=\emptyset$ follows from Proposition 3. Q.e.d.

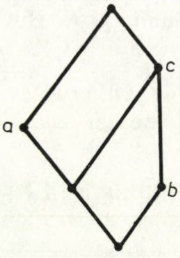
V. The second part of the proof

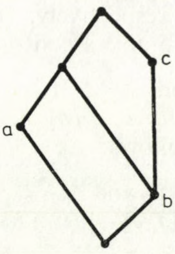
Having Lemma 2, the proof of (ii)$\to$(iii) will be complete if we prove the following

LEMMA 3. *Let L be a distributive lattice which satisfies* (ii), $C'(L)=\emptyset$ *and* $|L|\geqq 3$. *Then L is of the form $\sum\limits_{i\in I} L_i=\sum\limits_{i\in I}' L_i$ where $(I, \leqq)$ is a chain and for all $i\in I$, L_i is a ladder or consists of a single element.*

The proof of this lemma requires several preliminary statements.

PROPOSITION 4. *Let a, b, c be elements of a distributive lattice satisfying* (ii). *If $a\|b$, $a\|c$ and $b<c$ then $[a, b, c]$ is isomorphic to one of the following two lattices*





PROOF. It is an immediate consequence of the well-known fact (cf. GRÄTZER [2, p. 14]) that the free distributive lattice generated by the partially ordered set $(\{a, b, c\}, b\leqq c)$ is isomorphic to K.

Let us agree that any use of Proposition 4 in case of $[x, y, z]$ will also mean $x\|y$, $x\|z$ and $y\leqq z$.

An element x in L is called *join-reducible* if $x=a\vee b\notin\{a,b\}$ for some $a,b\in L$. An element x is said to be *reducible (doubly reducible)* if it is either (both) join-reducible or (and) meet-reducible.

PROPOSITION 5. *The lattice L from Lemma* 3 (*i.e., L distributive,* (ii) *and* $C'(L)=\emptyset$) *does not contain any doubly reducible element.*

PROOF. Suppose $x\in L$ is doubly reducible. Then $x=e\vee b=f\wedge c$, $e\|b$, $f\|c$ and $a\|x$ for some $a,b,c,e,f\in L$. Now $\{a,b,e\}$ and $\{a,c,f\}$ are not antichains so $a\|c$, $a<f$, $a\|b$, $a>e$ can be assumed. Consider $[a,b,c]$. Then we have $a\vee b\leqq f$, $a\vee c\not\leqq f$, $a\wedge c\geqq e$ and $a\wedge b\not\geqq e$, which contradicts Proposition 4. Q.e.d.

For a lattice L define the binary relation ψ_L on L as follows:

Set $a\psi_L b$ iff for any $e,f\in[a\wedge b,a\vee b]$, where e is join-reducible (in L) and f is meet-reducible (in L), $e\not\leqq f$ holds.

LEMMA 4. *Let L be a distributive lattice satisfying* (ii) *and* $C'(L)=\emptyset$. *Then $\psi=\psi_L$ is a congruence, L/ψ is a chain and $a\|b$ implies $a\psi b$ for any $a,b\in L$.*

PROOF. Throughout the proof, let e and f stand for join-reducible and meet-reducible elements of L, respectively. Suppose $a\|b$ but $a\psi b$ does not hold. So $e,f\in[a\wedge b,a\vee b]$ and $e\leqq f$ for some e,f. First, exactly one of $a\|e$ and $b\|e$ holds because otherwise $e=a\wedge b$ or $e=f=a\vee b$ would contradict Proposition 5. Suppose $a\|e$ and $b\not\| e$. If we had $e<b$, then e would be doubly reducible by Proposition 4 (considering $[a,e,b]$). If $b<e$, then $a\|f$ (otherwise $f=a\vee b$ is doubly reducible) and, considering $[a,e,f]$, e or f is doubly reducible. Thus $e=b$. Similarly, $f\in\{a,b\}$. Since $f\leqq e$, $f=b=e$ which contradicts Proposition 5. Now we have shown:

$$a\|b \quad \text{implies} \quad a\psi b. \tag{4}$$

Suppose we have $a\psi b$ and $b\psi c$. Since ψ is reflexive by Proposition 5 and symmetric, to prove transitivity only the following four cases have to be considered.

Case 1. $a<b$ and $b<c$ but $a\overline{\psi}c$. Then there are $e,f\in[a,c]$, $e<f$. Both $b\|e$ and $b\|f$ do not hold by Propositions 4 and 5. Say $b\not\| f$, and so, from $\{e,f\}\not\subseteq[a,b]$, $b<f$. Then $e<b$ because otherwise $\{b\vee e,f\}\subseteq[b,c]$. Choose an $x\in L$ such that $x\|b$. Both $e\leqq x$ and $x\leqq f$ lead to a contradiction: $a\leqq e\leqq x\wedge b\leqq b$ or $b\leqq x\vee b\leqq f\leqq c$, respectively. Therefore $x\|e$ and $x\|f$. But regarding $[x,e,f]$ Probositions 4 and 5 give a contradiction.

Case 2. $a\|b$ and $b<c$. If $a\|c$ then $a\psi c$ by (4). Otherwise $a<c$. Set $b'=a\vee b$, then from $[a,b']\subseteq[a\wedge b,a\vee b]$ and $[b',c]\subseteq[b,c]$ we get $a\psi b'$ and $b'\psi c$ whence, by Case 1, $a\psi c$ follows.

Case 3. $a\|b$, $b\|c$ and $a<c$. Now, by Proposition 4, we have either $[a,c]\subseteq\subseteq[a\wedge b,a\vee b]$ or $[a,c]\subseteq[b\wedge c,b\vee c]$, and so $a\psi c$.

Case 4. $a<b$ and $c<b$ and $a<c$. Then $a\psi c$ follows from $[a,c]\subseteq[a,b]$.

Now we have that ψ is an equivalence relation. Let us have $a,b,c\in L$, $a\psi b$. If $a\psi c$ then $a\wedge c\psi a\psi b\psi b\wedge c$, while in case of $a\overline{\psi}c$ $a\wedge c\psi b\wedge c$ follows from (4). Therefore, by the Duality Principle, ψ is a congruence. Finally, (4) implies that L/ψ is a chain. Q.e.d.

COROLLARY 1. *Let* L *be a distributive lattice satisfying* (ii) *and* $C'(L)=\varnothing$. *Then* $L=\sum'_{D\in L/\psi} D=\sum_{D\in L/\psi} D$, *where* ψ *denotes* ψ_L, *for all* $D\in L/\psi$, $C'(D)=\varnothing$ *and* $\psi_D=D^2$.

LEMMA 5. *Let* L *be a distributive lattice for which* $\psi_L=L^2$, $C'(L)=\varnothing$, $|L|\geqq 5$ *and* (ii) *hold. Suppose* L *contains neither a join-irreducible unit nor a meet-irreducible zero. Then* L *is a ladder.*

PROOF. The proof consists of several steps. Let E and F denote the set of join-reducible and meet-reducible elements of L, respectively. Then $e\not\leqq f$ for any $e\in E$ and $f\in F$ and therefore $E\cap F=\varnothing$.

STEP 1. F is an ideal and E is a dual ideal of L and both are chains.

PROOF. First we show that F is a chain. Suppose $a, b\in F$, $a\| b$. Let $a=x\wedge y$ and $b=u\wedge v$ where $x\| y$ and $u\| v$. Since $\{a, u, v\}$ is not an antichain, $a<u$ and $a\| v$ can be assumed. Similarly, $b<y$ and $b\| x$ can be also assumed. So $a=x\wedge(y\wedge u)$ and $b=v\wedge(y\wedge u)$. Hence $x\| y\wedge u$ and $v\| y\wedge u$ and $x\| v$ (from $a\| b$), which is a contradiction. I.e., F is a chain and so is E. Now let $f\in F$, $x\in L\setminus F$ and $x<f$. Then x is not the zero of L and so $x\| y$ for some $y\in L$. Here $y\| f$ since otherwise $f\geqq x\vee y\in E$. Considering $[y, x, f]$, Proposition 4 leads to a contradiction because of $f\notin E$. I.e., F is an ideal and E is a dual ideal by the Duality Principle. Q.e.d.

STEP 2. Both E and F have at least two elements.

PROOF. Suppose $|F|<2$ and let a, b be incomparable elements in L. Then, by Proposition 4, $x\| a$ implies $x=b$ and $x\| b$ implies $x=a$ for any $x\in L$. Thus $L=\{a, b\}\cup(a\wedge b]\cup[a\vee b)$, which is a contradiction. The proof is complete by the Duality Principle.

STEP 3. If $a, b\in L\setminus(E\cup F)$, $f\in F$, $a\leqq b$ and $f<a$ then $a=b$.

PROOF. Suppose $a<b$, then $a\| c$ for some $c\in L$. In case $c\| b$, by Proposition 4, $\{a, b\}\cap(E\cup F)\neq\varnothing$. Thus $c<b$. Hence $b\geqq a\vee c\in E$, which contradicts Step 1.

STEP 4. If $f\in F$, $a, b\in L\setminus(E\cup F)$, $a>f$, $b>f$ and f is not the zero element of L, then $a=b$.

PROOF. If $a\neq b$ then $a\| b$ by Step 3. Choose an element q in L so that $f\| q$. Now $q\notin F$ by Step 1. $\{a, b, q\}$ is not an antichain, so $q\in E$ contradicts Step 1 and $q\notin E$ contradicts Step 3.

STEP 5. If $f\in F$, f is not the zero of L and $[f)\cap(L\setminus(E\cup F))=\varnothing$, then F is a prime ideal.

PROOF. Suppose $a\wedge b\in F$ but $a\notin F$ and $b\notin F$. $\{a, b\}\not\subseteq E$ by Step 1 so, e.g., $a\notin E$. Consequently, by Step 1 and Proposition 4, we have $a\| f$, $f<b$ and $a\vee f\| b$. Hence, by Step 1, $b\notin E$, which is a contradiction. Q.e.d.

Now F' is defined as follows: If F is a prime ideal then set $F'=F$. Otherwise set $F'=F\cup\{f'\}$ where $f'\in L\setminus(E\cup F)$ and, for some $f\in F$, $0_L\neq f<f'$. E' is defined dually. The previous steps yield the correctness of this definition.

STEP 6. F' is a prime ideal and a chain, and in case $F' \neq F, f'$ is the greatest element of F'. The dual statement is valid for E'.

PROOF. It is enough to consider the case $F' \neq F$. First we show that $g < f'$ for any $g \in F$. If $g \nless f'$ for some $g \in F$, then $g \| f'$ by Step 1. Since $g \wedge f' \neq 0_L$ by Step 1, $b \| f' \wedge g$ for some $b \in L \setminus F$. We have $b \| g$ by Step 1 and, $\{f', g, b\}$ being not an antichain, $b < f'$. Applying Proposition 4 to $[g, b, f']$ we get $f' \in E$ which is a contradiction. Therefore f' is the greatest element of F and, by Step 1, F' is a chain. Suppose $x \in L$ and $x < f'$ but $x \notin F$. For *any* $f \in F$ $x \| f$ by Steps 1 and 3, however $x \not\| x \wedge f \in F$ is a contradiction. Therefore F' is an ideal. Suppose $a, b \notin F'$, but $a \wedge b \in F'$. Since $a \| b, b \notin E$ can be assumed by Step 1. From Step 3 we have $f' \| b$. Since $\{b, f', a\}$ is not an antichain, $f' < a$. Considering $[b, f', a]$, Proposition 4 yields $f' \in F$ or $a \wedge b \| f'$, which is a contradiction. Q.E.D.

STEP 7. $E' \cap F' = \emptyset$.

PROOF. Suppose $E' \cap F' \neq \emptyset$. Since $E \cap F = \emptyset$, we have $e' = f'$. Consider the set $H = \{x \in L: x \| f'\}$. $H \neq \emptyset$. Suppose H consists of a single element x. Let $f_1, f_2 \in F, f_1 < f_2$. Since $f_2 \notin C'(L), f_2 \| x$ and, considering $[x, f_2, f']$, Proposition 4 yields a contradiction. Suppose $x, y \in H, x \not\equiv y$. Then $x > y$ and considering $[f', y, x]$, Proposition 4 yields a contradiction again. Q.e.d.

STEP 8. $E' \cup F' = L$.

PROOF. Suppose $x \in L \setminus (E \cup F)$. Let $0_L \neq f \in F$ and $1_L \neq e \in E$. Since $x \not\equiv f$ and $e \not\equiv x$ by Step 1, we have $f < x$ or $x < e$ by Step 1 and Proposition 4. Therefore Step 4 (or its dual statement) implies $x = f'$ or $x = e'$. Q.e.d.

Now we define a map $\tau: E' \to F'$ as follows:

$$e\tau = \begin{cases} f \wedge e, & \text{if } f \| e \quad \text{for some} \quad f \in F' \\ f', & \text{if } f < e \quad \text{for all} \quad f \in F'. \end{cases}$$

STEP 9. The definition of τ is correct.

PROOF. Suppose $f_1 < f_2, f_1 \| e$ and $f_2 \| e$ for some $f_1, f_2 \in F', e \in E'$. Considering $[e, f_1, f_2]$, Proposition 4 and Step 6 imply $e \wedge f_1 = e \wedge f_2$. Suppose we have an $e \in E'$ such that $e \not\| f$ for all $f \in F'$. Steps 6, 8 and $C'(L) = \emptyset$ imply $e = 1 = 1_L$.

We have $a, b \in L$ such that $a \| b$ and $a \vee b = 1$. By Steps 6 and 8 $a \in F'$ and $b \in E'$ can be assumed. If $a \in F$ then, for some $c, d \in L$, $a = c \wedge d$, $c \| d$ and, from $1 = c \vee b = d \vee b$, $\{b, c, d\}$ is an antichain. Therefore $a = f' \notin F$ and $e\tau = 1\tau$ is defined. Q.e.d.

STEP 10. The map τ is a bijective lattice homomorphism.

PROOF. First we show that $e_1 < e_2$ $(e_1, e_2 \in E')$ implies $e_1\tau < e_2\tau$. As we have already seen in the proof of Step 9, $e_2\tau = f'$ implies $e_2 = 1$. Therefore $e_2 \neq 1$ can be assumed. Let $e_i\tau = e_i \wedge f_i$ $(i = 1, 2, f_i \in F')$. Then $f_2 \| e_1, e_i\tau = f_2 \wedge e_i$ $(i = 1, 2)$ and $f_2 \vee e_1 \not\| e_2$. Considering $[f_2, e_1, e_2]$, Proposition 4 implies $e_1\tau < e_2\tau$. Now, if $f' \in F' \setminus F$ exists then $a \wedge b \in F$ and $a \| b$ for some $a, b \in L \setminus F$. Hence, by Step 6, $f' \in \{a, b\}$ and $f' = (a \vee b)\tau$. If $f \in F$ then, by Step 6, $f = c \wedge d$ and $c \| d$ for some $c \in F'$ and $d \in E'$. So $f = d\tau$. I.e., τ is surjective. Q.e.d.

Now let $\underset{\sim}{2}=\{0, 1\}$ be the two-element chain and let us define a map $\eta\colon \underset{\sim}{2}\times E'\to L$ by $(1, e)\mapsto e$ and $(0, e)\mapsto e\tau$. Our previous steps imply that η is a (required) isomorphism between $\underset{\sim}{2}\times E'$ and L. The proof of Lemma 5 is complete.

Finally, Lemmas 4 and 5 and Corollary 1 imply Lemma 3.

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A RECURRENT FINSLER MANIFOLD WITH A CONCIRCULAR VECTOR FIELD

By

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Recently R. B. MISRA, F. M. MEHER and N. KISHORE [1] studied a recurrent Finsler manifold $F_n\,(n>2)$ admitting an infinitesimal transformation $\bar{x}^i = x^i + \varepsilon v^i(x^j)$; where ε is an infinitesimal constant and v^i are components of a contravariant vector satisfying

$$\text{(A)} \qquad \text{i)}\ \mathscr{B}_k v^i = \varrho(x, \dot{x})\delta^i_k + \varphi_k(x, \dot{x})v^i \quad \text{ii)}\ \mathscr{B}_j \varphi_k = \mathscr{B}_k \varphi_j,$$

$\mathscr{B}_k$ being Berwald's covariant differential operator. Such a manifold is called, by the above authors, a recurrent Finsler manifold admitting a concircular vector field and is denoted by $\mathrm{CHR}-F_n\,(n>2)$. In this note we prove that such a $\mathrm{CHR}-F_n\,(n>2)$ does not exist. The notation employed here is based on the above paper [1].

Let F_n be an n-dimensional Finsler manifold of class at least C^5 equipped with a metric function F[1] satisfying the required conditions [3]. Let the components of the metric tensor, Berwald's connection, and curvature tensor of Berwald be denoted by g_{ij}, G^i_{jk} and H^i_{jkh} respectively; H^i_{jkh} being skew-symmetric in j and k. Transvections of H^i_{jkh} by $\dot{x}$'s and contractions of indices yield the following:

$$\text{(1)} \qquad \text{a)}\ H^i_{jk} = H^i_{jkh}\dot{x}^h, \quad \text{b)}\ H_{kh} = H^i_{ikh}, \quad \text{c)}\ H_k = H^i_{ik}.$$

These are also connected by

$$\text{(2)} \qquad \text{a)}\ H_{kh}\dot{x}^h = H_k, \quad \text{b)}\ H_k\dot{x}^k = (n-1)H.$$

The author [2] proved that H^i_{jk} satisfy

$$\text{(3)} \qquad y_i H^i_{jk} = 0,$$

where $y_i = g_{ij}\dot{x}^j$. It is well known that y_i satisfy

$$\text{(4)} \qquad \mathscr{B}_m y_i = 0.$$

A Finsler manifold F_n is called recurrent if there exists a non-null covariant vector field λ_m satisfying

$$\text{(5)} \qquad \mathscr{B}_m H^i_{jkh} = \lambda_m H^i_{jkh}.$$

Transvecting (5) by $\dot{x}^h$, and using (1a) we get

$$\text{(6)} \qquad \mathscr{B}_m H^i_{jk} = \lambda_m H^i_{jk}.$$

Now, we propose

[1] Unless stated otherwise all the entities are considered as functions of line-elements $(x^i, \dot{x}^i)$. The indices $i, j, k, \ldots$ assume positive integral values 1 to n.

THEOREM 1. *A* CHR$-F_n$ $(n>2)$ *does not exist.*

PROOF. We claim that a CHR$-F_n$ does not exist, if not, we assume the contrary. We know that the curvature tensor of a CHR$-F_n$ $(n>2)$ is expressible in the form [1]

$$H^i_{jkh} = \frac{2}{n-1}\,\delta^i_{[j}\,H_{k]h}. \tag{7}$$

Transvecting (7) by $\dot{x}^h$, and using (1a) and (2a) we have

$$H^i_{jk} = \frac{2}{n-1}\,\delta^i_{[j}\,H_{k]}. \tag{8}$$

Further transvection of (8) by y_i, in view of (3), yields

$$y_j H_k = y_k H_j. \tag{9}$$

Transvecting (9) by $\dot{x}^j$, using $y_j\dot{x}^j = F^2$ and (2b) we get $H_k = (n-1)\dfrac{H}{F^2}\,y_k$; in view of which (8) reduces to

$$H^i_{jk} = 2R\delta^i_{[j}\,y_{k]} \tag{10}$$

where $R \overset{\text{def}}{=} H/F^2$. Utilizing BERWALD's theorem [3, pp. 133], we conclude that R is a constant. Differentiating (10) covariantly with respect to x^m, and using (4) and (6) we get $\lambda_m = 0$, a contradiction.

If we put $\varphi_k = 0$ in (A), the above infinitesimal transformation is called a special concircular transformation. If we approach in the similar way of [1] we may easily prove that the curvature tensor of a recurrent Finsler manifold admitting a special concircular transformation is expressible in the form (7), which will lead to a contradiction. Thus, we have

THEOREM 2. *A recurrent Finsler manifold* F_n $(n>2)$ *admitting a special concircular transformation does not exist.*

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INDEX

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[1] G. Szegő, *Orthogonal polynomials,* AMS Coll. Publ. Vol. XXIII (Providence, 1939).
[2] A. Zygmund, Smooth functions, *Duke Math. J.,* **12** (1945), 47—76.

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