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GEORGE ALEXITS

(1899 – 1978)

member of the Hungarian Academy of Sciences, editor-in-chief of this journal died on October 14, 1978. He was a leading personality in the Hungarian mathematical life, and a world-wide known scientist in different areas of mathematics such as theory of point sets, orthogonal series, general function series, approximation theory, and metric geometry. Many mathematicians all around the world declare themselves his disciples. Through them and by his own works, his influence is continuously felt on the universal development of our science.

As the reader will observe from the dedications, originally this issue of the Acta Mathematica Hungaricae had been devoted to papers dedicated to his eightieth birthday. Many of the contributors are working in the above mentioned areas of mathematics and were under the influence of his stimulating mathematical activity. The authors of this issue, his friends and pupils, the whole mathematical community feel deep sorrow for this irreplaceable loss which forced the editors to convert the celebration to be expressed in this volume, into mourning.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 33 (1—2), (1979), pp. 3—5.

SOME ABSOLUTELY MONOTONIC FUNCTIONS

By

R. ASKEY (Madison)

To G. Alexits on his 80th birthday

L. J. ROGERS [6] introduced a set of polynomials that generalize the ultraspherical polynomials. The ultraspherical polynomials come from the generating function

$$(1) \qquad (1-2xr+r^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{\lambda}(x)\, r^n .$$

The continuous q-ultraspherical polynomials $C_n(x;\beta\mid q)$ have the generating function

$$(2) \qquad \prod_{k=0}^{\infty} \frac{(1-2xr\beta q^k + r^2\beta^2 q^{2k})}{(1-2xrq^k+r^2q^{2k})} = \sum_{n=0}^{\infty} C_n(x;\beta\mid q)\, r^n, \quad |q|<1 .$$

The orthogonality of these polynomials is demonstrated in [3] and the weight function is given in [1] when $|\beta|<1$, $|q|<1$.

In the case of ultraspherical polynomials, ASKEY and POLLARD [2] proved that

$$(1-r)^{-2|\lambda|}(1-2xr+r^2)^{-\lambda} = \sum_{n=0}^{\infty} g_n^{\lambda}(x)\, r^n$$

with $g_n^{\lambda}(x) \geqq 0$, $-1 \leqq x \leqq 1$, λ real. This was used to obtain a simple proof of Kogbetliantz's theorem on the positivity of the $(C, 2\lambda+1)$ means for ultraspherical series. At present we do not know enough about the continuous q-ultraspherical polynomials to find an analogue of Kogbetliantz's theorem, but it is possible to prove an analogue of the result of Askey and Pollard.

THEOREM 1. *If* $|q|<1$, $|\beta|<1$, *and* $|r|<1$ *then*

$$(3) \qquad \prod_{n=0}^{\infty} \frac{(1-\beta rq^n)^{2|\alpha|}(1-\beta re^{i\theta}q^n)^{\alpha}(1-\beta re^{-i\theta}q^n)^{\alpha}}{(1-rq^n)^{2|\alpha|}(1-re^{i\theta}q^n)^{\alpha}(1-re^{-i\theta}q^n)^{\alpha}} = \sum_{n=0}^{\infty} h_n(\alpha,\beta,q,\theta)\, r^n$$

with $h_n(\alpha,\beta,q,\theta) \geqq 0$ *when* α,β,q,θ *are real.*

PROOF. Assume first that $\alpha \geqq 0$. As in [2], if the left hand side is denoted by $[f(r)]^{\alpha}$ it is sufficient to show that

$$\frac{d}{dr}\log f(r) = \sum_{n=0}^{\infty} k_n r^n$$

with $k_n \geqq 0$. A simple calculation shows that

$$\frac{f'(r)}{f(r)} = 2\sum_{n=0}^{\infty}\left(\frac{1-\beta^{n+1}}{1-q^{n+1}}\right)[1+\cos(n+1)\theta]\,r^n$$

so

$$\log f(r) = 2\sum_{n=0}^{\infty}\left(\frac{1-\beta^{n+1}}{1-q^{n+1}}\right)\frac{[1+\cos(n+1)\theta]\,r^{n+1}}{n+1}.$$

Then

$$f(r) = \sum_{n=0}^{\infty}\frac{\alpha^n[\log f(r)]^n}{n!}.$$

The same proof works for $\alpha < 0$. The only difference is that $1+\cos(n+1)\theta$ is replaced by $1-\cos(n+1)\theta$. The case treated in [2] can be obtained in two ways from Theorem 1. One way is to set $\alpha = 1$, $\beta = q^\lambda$ and let $q \to 1$. The other is to set $\beta = 0$ and let $q \to 0$.

The ultraspherical result can be written as

$$\sum_{k=0}^{n}\frac{(2\lambda)_{n-k}}{(1)_{n-k}}\frac{(2\lambda)_k}{(1)_k}\frac{C_k^\lambda(x)}{C_k^\lambda(1)} \geqq 0, \qquad \lambda > 0.$$

This has been extended to

$$\sum_{k=0}^{n}\frac{(a)_{n-k}}{(1)_{n-k}}\frac{(a)_k}{(1)_k}\frac{C_k^\lambda(x)}{C_k^\lambda(1)} \geqq 0, \qquad 1 \leqq a \leqq 2\lambda.$$

It is unclear how to extend this to the continuous q-ultraspherical polynomials, because $C_n(1;\beta\mid q)$ is not given as a product except in special cases. See [4] and [5] for the current state of knowledge on sums of this type and [1] for more facts about $C_n(x;\beta\mid q)$. One plausible conjecture is that Gasper's result

$$(1-r^2)^{-\lambda}\{(1+2xr+r^2)^{-\lambda}+(1-2xr+r^2)^{-\lambda}\} = \sum_{n=0}^{\infty}c_n^\lambda(x)\,r^n$$

with $c_n^\lambda(x) \geqq 0$ for $\lambda \geqq 1$, can be extended to

$$(4)\qquad \prod_{n=0}^{\infty}\frac{(1-\beta r^2q^n)}{(1-r^2q^n)}\left\{\prod_{n=0}^{\infty}\frac{|1-\beta re^{i\theta}q^n|^2}{|1-re^{i\theta}q^n|^2}+\prod_{n=0}^{\infty}\frac{|1+\beta re^{i\theta}q^n|^2}{|1+re^{i\theta}q^n|^2}\right\} = \sum_{n=0}^{\infty}c_n(\cos\theta;\beta,q)\,r^n$$

with $c_n(\cos\theta,\beta,q) \geqq 0$, $0 \leqq \beta \leqq q < 1$. However before this can be proved (unless it is false) we must learn much more about the continuous q-ultraspherical and Jacobi polynomials.

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A CONSTRUCTION OF PROCESSES OF BEST APPROXIMATION

By

G. BLEIMANN, E. L. STARK and G. WILMES (Aachen)

Dedicated to Professor G. Alexits on the occasion of his 80th birthday on January 5, 1979

1. Motivation. When investigating the approximation of 2π-periodic continuous functions by their Fejér means Professor G. ALEXITS [2] was the first (1941) to observe that the corresponding optimal order of approximation is limited, this being so for a well-determined class of functions (for some historical remarks in this respect see [4, p. 478 f, [6]). This phenomenon which restricts the optimal approximation order of a wide class of general linear approximation processes, is now — under the concept of saturation — an essential part of the constructive theory of functions.

On the other hand, it is well-known that there exist linear approximation processes which approximate with the order of the best approximation, thus do not exhibit the saturation property. In case of periodic functions, for instance, the classical example is given by the famous de la Vallée Poussin sums.

The purpose of this note is to investigate the latter problem further, thus to describe a general method of constructing a family of singular convolution integrals which approximate 2π-periodic functions with the order of the best trigonometric approximation; the sums of de la Vallée Poussin will be contained therein as an extremal example.

2. A certain family of Fourier transform pairs. In order to develop the announced construction just some elementary results from basic Fourier transform theory will be needed. Let $L^p = L^p(\mathbf{R})$, $1 \leqq p \leqq \infty$, denote the spaces of measurable functions on $\mathbf{R}$ for which

$$\|f\|_p := \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(t)|^p\,dt\right\}^{1/p}, \qquad \|f\|_\infty := \operatorname*{ess\,sup}_{x\in\mathbf{R}} |f(x)|,$$

respectively, are finite. The L^1-Fourier transform is defined by

$$f^\wedge(v) := \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,e^{-ivt}\,dt \qquad (f\in L^1;\ v, t \in \mathbf{R});$$

for the corresponding L^2-transform one may consult e.g. [4, p. 174 f]. Moreover, let $\mathscr{E}_{p,v}$ denote the space of all p-th power integrable restrictions to $\mathbf{R}$ of entire functions f on $\mathbf{C}$ which are of exponential type ν, i.e.,

$$|f(z)| \leqq A_\varepsilon e^{(\nu+\varepsilon)|z|} \qquad (\varepsilon, \nu > 0; z \in \mathbf{C});$$

(cf. e.g. [1, p. 161 ff]).

In the sequel two particular functions p_α and ϕ, will be employed. The first one is defined by

(1) $$p_\alpha(v) := \frac{2}{v} \sin \frac{\alpha}{2} v \qquad (\alpha > 0; v \in \mathbf{R});$$

some of its properties are collected in

(2) $$\begin{cases} \text{(i)} \quad p_\alpha(-v) = p_\alpha(v), \quad v \in \mathbf{R}; & \text{(ii)} \quad p_\alpha(0) = \alpha; \\ \text{(iii)} \quad \int_0^\infty p_\alpha(v)\, dv = \pi; & \text{(iv)} \quad \int_0^\infty p_\alpha^2(v)\, dv = \alpha\pi, \quad p_\alpha \in L^2; \\ \text{(v)} \quad p_\alpha(v) = 2\pi \kappa\hat{}_{\left[-\frac{\alpha}{2}, \frac{\alpha}{2}\right]}(v); & \text{(vi)} \quad p_\alpha \in \mathscr{E}_{p, \frac{\alpha}{2}}, \quad 1 < p \leqq \infty; \end{cases}$$

the characteristic function being given by

$$\kappa_{[A,B]} := \begin{cases} 1, A \leqq x \leqq B, \\ 0, \ x \notin [A, B], \end{cases} \qquad (-\infty < A < B < \infty).$$

The second one is assumed to satisfy the conditions

(3) $$\begin{cases} \text{(i)} \quad \phi(-v) = \phi(v), \quad v \in \mathbf{R}; \qquad \text{(ii)} \quad \phi(0) = 1; \\ \text{(iii)} \quad \phi \in \mathscr{E}_{2, \frac{\beta}{2}}, \quad \beta > 0. \end{cases}$$

It is an immediate consequence of the Paley–Wiener theorem (see [1, p. 166]) that

(3, iv) $$(L^2)\, \phi\hat{}(t) \equiv 0, \quad |t| \geqq \beta/2;$$

moreover, in view of (3, ii) there holds

(3, v) $$\int_{-\beta/2}^{\beta/2} \phi\hat{}(t)\, dt = 1.$$

Finally, the function Φ essential in our construction is given by the product

(4) $$\Phi(v) := p_\alpha(v)\, \phi(v)$$

for which

(5) $$\begin{cases} \text{(i)} \quad \Phi(-v) = \Phi(v), \quad v \in \mathbf{R}; \qquad \text{(ii)} \quad \Phi(0) = \alpha; \\ \text{(iii)} \quad \Phi \in L^1; \qquad \text{(iv)} \quad \int_0^\infty \Phi\hat{}(t)\, dt = \frac{\alpha}{2}. \end{cases}$$

It is the Fourier transform of (4) and its properties we are mainly interested in.

THEOREM 1. *Let $\alpha \geqq \beta > 0$. The following representation holds for the Fourier transform of* (4):

$$\text{(6)}\qquad \Phi^{\wedge}(t) = \begin{cases} \text{(i)}\quad 1, & |t| \leqq \dfrac{\alpha-\beta}{2}, \\ \text{(ii)}\quad \displaystyle\int_{-\alpha/2+|t|}^{\beta/2} \phi^{\wedge}(v)\,dv, & \dfrac{\alpha-\beta}{2} \leqq |t| \leqq \dfrac{\alpha+\beta}{2}, \\ \text{(iii)}\quad 0, & |t| \geqq \dfrac{\alpha+\beta}{2}; \end{cases}$$

moreover,

$$\text{(7)}\qquad \Phi^{\wedge}\left(\frac{\alpha}{2}\right) = \frac{1}{2}, \quad \Phi^{\wedge}\left(\frac{\alpha-\beta}{2}+\tau\right) + \Phi^{\wedge}\left(\frac{\alpha+\beta}{2}-\tau\right) \equiv 1, \quad 0 \leqq \tau \leqq \beta.$$

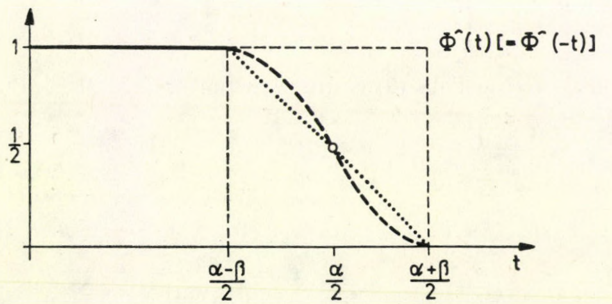
PROOF. By definition, (2, v), and Parseval's formula one has

$$\Phi^{\wedge}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} p_{\alpha}(v)\,\phi(v)\,e^{-ivt}\,dv = \int_{-\infty}^{\infty} \kappa^{\wedge}_{[-\frac{\alpha}{2},\frac{\alpha}{2}]}(v)\,\phi(v)\,e^{-ivt}\,dv =$$

$$= \int_{-\alpha/2}^{\alpha/2} [\phi(\circ)e^{-i\circ t}]^{\wedge}(v)\,dv = \int_{-\alpha/2}^{\alpha/2} \phi^{\wedge}(v+t)\,dv \qquad (t \in \mathbf{R}).$$

Having in mind property (3, iv), the representation (6) follows by straightforward calculations using (3, v). The same applies to (7). Q.E.D.

The latter properties reveal that $\Phi^{\wedge}(t)$ is pointsymmetric in the interval $[(\alpha-\beta)/2, (\alpha+\beta)/2]$ with respect to the point $(\alpha/2, 1/2)$:



(For the particular parameters $(\alpha, \beta) = (3, 1)$ the dotted line refers to Example A, (17), the broken line to Example C below.) The slope of $\Phi^{\wedge}(t)$ in $(\alpha/2, 1/2)$ is given by $\Phi^{\wedge\prime}(\alpha/2) = -\gamma/2\pi$, where $\gamma := \int_{-\infty}^{\infty} \phi(v)dv$ (if this integral exists).

Concerning simple concrete examples the most natural way to construct them is to choose the parameters α and β (in general fractional) in the transform (6) such that both $m:=(\alpha-\beta)/2$ and $n:=(\alpha+\beta)/2$ are (positive) integers.

3. Singular convolution integrals and best trigonometric approximation. As a suitable approximation process for 2π-periodic, e.g. continuous functions, $f\in C_{2\pi}$, Theorem 1 suggests to consider singular convolution integrals of type

$$I_{m,n}(\Phi;f;x):=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\,t_{m,n}(u)\,du \qquad (f\in C_{2\pi}) \tag{8}$$

with kernels $\{t_{m,n}\}$ generated via

$$t_{m,n}(x):=\sum_{k=-\infty}^{\infty}\Phi_{m,n}(x+2k\pi) \qquad (x\in\mathbf{R}), \tag{9}$$

$$\Phi_{m,n}(x):=\frac{\sin(n+m)\dfrac{x}{2}}{x/2}\,\phi\left(\frac{n-m}{2}x\right) \qquad (n,m\in\mathbf{N};\,n>m) \tag{10}$$

where $\phi((n-mx)/2)$ satisfies (3) for $\beta=n-m$. Then (cf. e.g. [4, p. 123]) $t_{m,n}$ is an even integrable 2π-periodic function, $t_{m,n}\in L^1_{2\pi}$, with Fourier coefficients

$$t^{\wedge}_{m,n}(k):=\frac{1}{2\pi}\int_{-\pi}^{\pi} t_{m,n}(u)\,e^{-iku}\,du=\Phi^{\wedge}_{m,n}(k)= \qquad (k\in\mathbf{Z})$$

$$=\begin{cases} 1, & |k|\leqq m,\\ \displaystyle\int_{\frac{2|k|}{n-m}-\frac{n+m}{n-m}}^{1}\phi^{\wedge}(v)\,dv, & m\leqq|k|\leqq n,\\ 0, & |k|\geqq n,\end{cases}$$

the latter in view of (6). Thus (9) is an even normalized trigonometric polynomial of type

$$t_{m,n}(x)=D_m(x)+2\sum_{k=m+1}^{n-1}\lambda_{m,n}(k;\Phi)\cos kx, \tag{11}$$

$$\lambda_{m,n-1}(k;\Phi)\equiv\Phi^{\wedge}_{m,n}(k) \qquad (m+1\leqq k\leqq n-1),$$

$$D_m(x):=1+2\sum_{k=1}^{m}\cos kx \qquad (m\in\mathbf{N}),$$

i.e., involving the Dirichlet kernel.

Finally, let Π_n, $n\in\mathbf{N}$, denote the set of all trigonometric polynomials of degree n and

$$E_n(C_{2\pi};f):=\inf_{g_n\in\Pi_n}\|f(x)-g_n(x)\|_C \qquad (f\in C_{2\pi};\,n\in\mathbf{N}),$$

the best trigonometric approximation (of degree n), where $\|f\|_C := \sup_{x\in R} |f(x)|$. Then one may formulate

THEOREM 2. *For $m, n \in \mathbf{N}$, $n > m$, let the operator $I_{m,n}$ be defined by* (8). *Then there holds*

$$I_{m,n}(\Phi; f; x) \in \Pi_{n-1} \qquad (f \in C_{2\pi}); \tag{12}$$

$$I_{m,n}(\Phi; g_m; x) \equiv g_m(x) \qquad (g_m \in \Pi_m; x \in \mathbf{R}); \tag{13}$$

$$\| I_{m,n} \|_{[C_{2\pi}]} := \sup_{\|f\|_C = 1} \| I_{m,n}(\Phi; f; x) \|_C \leqq \tag{14}$$

$$\leqq c_\phi \left\{ \sqrt{\pi} + \sqrt{2}\left(1 + \log \frac{n+m}{n-m}\right)\right\} = : N_{n,m}; \qquad c_\phi := \frac{2}{\pi} \| \phi \|_2;$$

$$\| I_{m,n}(\Phi; f; x) - f(x) \|_C \leqq N_{n,m} E_m(C_{2\pi}; f). \tag{15}$$

PROOF. Properties (14), (15) are immediate consequences of (11); property (15) follows by a standard argument as given e.g. in [4, p. 105] provided (13) and (14) are fulfilled. Now, concerning (14) note that $\| I_{m,n} \|_{[C_{2\pi}]} \leq \| \Phi_{m,n} \|_1$ (cf. e.g. [4, p. 124]). Moreover, one has by definition

$$\| \Phi_{m,n} \|_1 = \frac{2}{\pi} \int_0^\infty \left| \sin \frac{n+m}{2} t \phi \left(\frac{n-m}{2} t\right)\right| \frac{dt}{t} = \frac{2}{\pi} \int_0^\infty \left| \sin \frac{n+m}{n-m} t \phi(t) \right| \frac{dt}{t} \leqq$$

$$\leqq \frac{2}{\pi} \left\{ \| \phi \|_\infty \int_0^1 \left| \sin \frac{n+m}{n-m} t \right| \frac{dt}{t} + \int_1^\infty | \phi(t) | \frac{dt}{t} \right\} \leqq$$

$$\leqq \frac{2}{\pi} \left\{ \| \phi \|_\infty \left(1 + \log \frac{n+m}{n-m}\right) + \sqrt{\pi} \| \phi \|_2 \right\} \leqq$$

$$\leqq \frac{2}{\pi} \| \phi \|_2 \left\{ \sqrt{2} \left(1 + \log \frac{n+m}{n-m}\right) + \sqrt{\pi} \right\}, \quad c_\phi := \frac{2}{\pi} \| \phi \|_2,$$

since $\| \phi \|_\infty \leqq \sqrt{2} \| \phi \|_2$ by Nikol'skii's inequality (cf. [13, p. 233]), Q.E.D.

REMARKS. (i) For the particular choice $n = m + 1$ the sum in (11) is empty, so that $t_{m,m+1}(x) \equiv D_m(x) \in \Pi_m$, $m \in \mathbf{N}$, reduces to the Dirichlet kernel. Moreover, (15) then implies that

$$\| I_{m,m+1}(\mathfrak{D}; f; x) - f(x) \|_C \leqq c_\phi \{1 + \sqrt{\pi} + \log(2m+1)\} E_m(C_{2\pi}; f),$$

showing that the order of approximation in (15) also cannot be improved in general; cf. e.g. [4, p. 105].

(ii) Concerning the argument of the log-term in (14), the choice $n = 2m$ is *extremal:* in order that $(n+m)/(n-m) = \text{const}$, this choice gives the *smallest* *integer* multiple of m (namely log 3). This leads to the de la Vallée Poussin-type kernels $t_{m,2m} \in \Pi_{2m-1}$, i.e.,

$$t_{2m-1}(x) = D_m^{\sim}(x) + 2 \sum_{k=m+1}^{2m-1} \Phi^{\wedge}_{m,2m}(k) \cos kx \qquad (m \in \mathbf{N}) \tag{16}$$

which only depends upon ϕ of (3), and thus exhibits the symmetry property as given by (7). (It should be noted that for this instant of a constant log-term the proofs of (14) and (15), respectively, may be simplified by another standard argument as reproduced in e.g. [9, p. 150], [8, p. 92 f], [3].)

4. Representative examples. In the *Table* some illustrative examples are collected which are built up from the kernels (on **R**) of (A) *Dirichlet*, (B) *Fejér*, (C) *Rogosinski* (cf. [3], [12, p. 400]), and (D) *Bohman–Zheng Wei-xing* (*Fejér–Korovkin;* cf. [12, p. 400], [16], [17]).

Here some comments are given.

(A) From (10) it follows that

$$\Phi_{m,n}(x) = \frac{\sin (n+m)\dfrac{x}{2} \sin (n-m)\dfrac{x}{2}}{(n-m)(x/2)^2} \qquad (n \geqq m \in \mathbf{N})$$

with Fourier transform (having the well-known trapezoidal graph)

$$\Phi^{\wedge}_{m,n}(v) = \begin{cases} 1, & |v| \leqq m, \\ \dfrac{n-|v|}{n-m}, & m \leqq |v| \leqq n, \\ 0, & |v| \geqq n. \end{cases}$$

(The fact that the graph of $\Phi^{\wedge}_{m,n}(v)$ for $m \leqq |v| \leqq n$ is a straight line exhibits another extremal property of these kernels, this being taken with respect to the symmetry property (7).) The kernel (11) passes into

$$t_{m,n}(x) = D_m(x) + 2 \sum_{k=m+1}^{n-1} \left(1 - \frac{k-m}{n-m}\right) \cos kx =$$

$$= \frac{n}{n-m} F_{n-1}(x) - \frac{m}{n-m} F_{m-1}(x) = \frac{1}{n-m} \sum_{k=m}^{n-1} D_k(x) = V_{m,n}(x),$$

i.e., the kernel of the *general* sums of de la Vallée Poussin; here

$$F_n(x) = \frac{1}{n+1} \left(\frac{\sin (n+1)\dfrac{x}{2}}{\sin \dfrac{x}{2}} \right)^2 = 1 + 2 \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) \cos kx \geqq 0$$

denotes the periodic Fejér kernel.

	$\phi(x)$	$\phi\hat{}(v)$	$\Phi\hat{}_{m,n}(k)$	$\Phi\hat{}_{m,2m}(k)$
A	$\frac{\sin x}{x}$	$1/2, \quad \lvert v \rvert \leqq 1$ $0, \quad \lvert v \rvert < 1$	$1, \quad \lvert k \rvert \leqq m$ $\frac{n-\lvert k \rvert}{n-m}, \quad m \leqq \lvert k \rvert \leqq n$ $0, \quad \lvert k \rvert \geqq n$	$1, \quad \lvert k \rvert \leqq m$ $2-\frac{\lvert k \rvert}{m}, \quad m \leqq \lvert k \rvert \leqq 2m$ $0, \quad \lvert k \rvert \geqq 2m$
B	$\left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2$	$1-\lvert v \rvert, \quad \lvert v \rvert \leqq 1$ $0, \quad \lvert v \rvert \geqq 1$	$1, \quad \lvert k \rvert \leqq m$ $1-2\left(\frac{\lvert k \rvert - m}{n-m}\right)^2, \quad m \leqq \lvert k \rvert \leqq \frac{n+m}{2}$ $2\left(\frac{n-\lvert k \rvert}{n-m}\right)^2, \quad \frac{n+m}{2} \leqq \lvert k \rvert \leqq n$ $0, \quad \lvert k \rvert \geqq n$	$1, \quad \lvert k \rvert \leqq m$ $1-2\left(\frac{\lvert k \rvert}{m}-1\right)^2, \quad m \leqq \lvert k \rvert \leqq \frac{3}{2}m$ $2\left(2-\frac{\lvert k \rvert}{m}\right)^2, \quad \frac{3}{2}m \leqq \lvert k \rvert \leqq 2m$ $0, \quad \lvert k \rvert \geqq 2m$
C	$\pi^2 \frac{\cos x}{\pi^2-4x^2}$	$\frac{\pi}{4}\cos\frac{\pi}{2}v \quad \lvert v \rvert \leqq 1$ $0, \quad \lvert v \rvert \geqq 1$	$1, \quad \lvert k \rvert \leqq m$ $\cos^2\left(\frac{\pi}{2}\frac{\lvert k \rvert - m}{n-m}\right), \quad m \leqq \lvert k \rvert \leqq n$ $0, \quad \lvert k \rvert \geqq n$	$1, \quad \lvert k \rvert \leqq m$ $\sin^2\frac{\pi}{2}\frac{\lvert k \rvert}{m}, \quad m \leqq \lvert k \rvert \leqq 2m$ $0, \quad \lvert k \rvert \geqq 2m$
D	$\pi^4\left(\frac{\cos\frac{x}{2}}{\pi^2-x^2}\right)^2$	$\frac{\pi^2}{8}\left\{(1-\lvert v \rvert)\cos \pi v + \frac{1}{\pi}\sin \pi \lvert v \rvert\right\}, \quad \lvert v \rvert \leqq 1$ $0, \quad \lvert v \rvert \geqq 1$	$1, \quad \lvert k \rvert \leqq m$ $\frac{1}{4}\left\{3-\pi\frac{\lvert k \rvert - m}{n-m}\sin 2\pi\frac{n-\lvert k \rvert}{n-m}+\cos 2\pi\frac{n-\lvert k \rvert}{n-m}\right\}, \quad m \leqq \lvert k \rvert \leqq \frac{n+m}{2}$ $\frac{1}{4}\left\{1-\pi\frac{n-\lvert k \rvert}{n-m}\sin 2\pi\frac{n-\lvert k \rvert}{n-m}-\cos 2\pi\frac{n-\lvert k \rvert}{n-m}\right\}, \quad \frac{n+m}{2} \leqq \lvert k \rvert \leqq n$ $0, \quad \lvert k \rvert \geqq n$	

For the distinguished case $n = 2m$ mentioned above it follows

$$\Phi_{m,2m}(x) = \frac{2}{m}\,\frac{\cos mx - \cos 2mx}{x^2}, \quad \Phi^{\wedge}_{m,2m}(v) = \begin{cases} 1, & |v| \leqq m, \\ 2 - \dfrac{|v|}{m}, & m \leqq |v| \leqq n, \\ 0, & |v| \geqq n, \end{cases}$$

which finally reduces for $m \leqq 1$ to

$$\Phi_{1,2}(x) = \frac{2}{x^2}(\cos x - \cos 2x), \quad \Phi^{\wedge}_{1,2}(v) = \begin{cases} 1, & |v| \leqq 1, \\ 2 - |v|, & 1 \leqq |v| \leqq 2, \\ 0, & |v| \geqq 2 \end{cases}$$

(cf. [1, p. 151]) thus to the particular situation of a kernel of Fejér-type (cf. e.g. [4, p. 202]) for which

$$\lambda_{k,2m-1}(\Phi) = \Phi^{\wedge}_{1,2}\left(\frac{k}{m}\right).$$

For the special de la Vallée Poussin kernel, a particular case of (16), the convergence factors are then given by

$$V^{\wedge}_{2m-1}(k) = \lambda_{k,2m-1}(V) = \begin{cases} 1, & 1 \leqq k \leqq m, \\ 2 - \dfrac{k}{m}, & m \leqq k \leqq 2m, \\ 0, & k \geqq 2m. \end{cases} \tag{17}$$

Recalling that for this kernel one has precisely (see [7, p. 406])

$$\| V_{2m-1} \|_{L^1_{2\pi}} = \frac{1}{3} + \frac{2\sqrt{3}}{\pi} \simeq 1.436 \qquad (m \in \mathbf{N}),$$

i.e., independent of m, this gives one a measure for the quality of the general estimate (14)!

(The choice $m = 1$ corresponds to $\alpha - \beta = 2$, which means $\alpha = 3$, $\beta = 1$ in the minimal integer case such that (1) has the representation

$$p_3(x) = (1 + 2\cos x)\,\frac{\sin \dfrac{x}{2}}{\dfrac{x}{2}};$$

this gives the connection with the construction employed in [3].)

(B) – (D) The analysis of these examples runs along the same lines; all of them establish families of approximation processes which exhibit the structure of the general de la Vallée Poussin sums. Two peculiarities should be noted: the connecting graph of $\Phi^{\wedge}_{m,n}(v)$ for $m \leqq |v| \leqq n$ becomes smoother, i.e., $\Phi^{\wedge}_{m,n} \in C'(\mathbf{R})$, in comparison with example (A); concerning (C), this is even so for a kernel which is not strictly nonnegative; cf. [3]. A disadvantage appears in case (B) and (D), respectively, since $\Phi^{\wedge}_{m,n}(k)$, $m \leqq k \leqq n$, decomposes into two (symmetric) parts; in order to avoid complications in the midpoint $(n+m)/2$ it is practically postulated that $(n+m)/2 \in \mathbf{N}$; this in turn means e.g. for $n = 2m$ that the polynomial degree is raised.

5. Final remarks. (i) There are more general criteria which might be used in order to build up periodic singular integrals achieving the order of the best trigonometric approximation; see for instance [5, p. 559], [14, p. 26], [10, p. 482], [15,p. 50 f], in particular [11] where a rather similar approach is implicitly used, however, as an auxiliary tool for quite different purposes. The intention of this note is to formulate handsome conditions for a straightforward construction of classes of explicit examples, as well as to supplement the direct 2π-periodic approach parallel to Theorem 2 as was given in [3].

(ii) Of course, results analogous to Theorem 2 hold for the approximation by entire functions of exponential type on the real axis if (8) is replaced by

$$I_{m,n}(\Phi; f; x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x-u)\, \Phi_{m,n}(u)\, du$$

where now arbitrary positive real numbers m, n satisfying $m < n$ may be chosen.

(iii) Finally it is remarked that analogous results can be obtained in the N-dimensional case if (1) is replaced by

$$p_\alpha(v) := (\alpha\pi)^{N/2} \frac{J_{N/2}(\alpha \mid v \mid /2)}{\mid v \mid^{N/2}} \qquad (\alpha > 0; v \in \mathbf{R}^N) \tag{18}$$

where $J_{N/2}$ denotes the Bessel function of order $N/2$, $\mathbf{R}^N$ being the Eucledian N-space. Of course, (18) reduces to (1) for $N = 1$. Choosing e. g.

$$\phi(v) := \frac{N\Gamma(N/2)}{2^{1-N/2}} \frac{J_{N/2}(\beta \mid v \mid /2)}{(\beta \mid v \mid /2)^{N/2}} \qquad (\beta > 0; N \in \mathbf{N}; v \in \mathbf{R}^N)$$

which is a radial square integrable function of (radial) exponential type $\beta/2$ satisfying (3, ii, iv, v), it follows that $\phi(v) = \big(\sin(\beta v/2)\big)/(\beta v/2)$, $N = 1$. Thus a multidimensional (radial) counterpart of the de la Vallée Pousin sums (cf. Example (A)) is given by $(\mathbf{Z}^N := \mathbf{Z} \times \ldots \times \mathbf{Z})$

$$I^*_{m,n}(\Phi; f; x) := (2\pi)^{-N} \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} f(x-u)\, t^*_{m,n}(u)\, du\,,$$

$$t^*_{m,n}(x) := \sum_{k \in \mathbf{Z}^N} \Phi^*_{m,n}(x + 2k\pi)\,,$$

$$\Phi^*_{m,n}(x) := \frac{N\Gamma(N/2)}{2^{1-N/2}} \left(2\pi \frac{n+m}{n-m}\right)^{N/2} \frac{J_{N/2}\left((n+m)\frac{\mid x \mid}{2}\right) J_{N/2}\left((n-m)\frac{\mid x \mid}{2}\right)}{\mid x \mid^N}\,.$$

Moreover, if $C_{2\pi}^{(N)}$ denotes the space of continuous functions on $\mathbf{R}^N$ that are 2π-periodic in each variable, then an estimation of the 1-norm of $\Phi_{m,n}^*$ – analogous to that given in [11] – leads to

$$\| I_{m,n}^* \|_{[C_{2\pi}^{(N)}]} \leqq \text{const} \left(\frac{n+m}{n-m} \right)^{(N-1)/2} \qquad (N \geqq 2).$$

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SATURATION CLASSES OF THE CESÀRO AND ABEL—POISSON MEANS OF FOURIER—LEGENDRE SERIES

By

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Dedicated to Professor Georges Alexits on the occasion of his eightieth birthday on 5 January 1979 in high esteem

1. Introduction

As early as 1941 did G. ALEXITS consider the following result. Let us cite it in his own words[1] in his review of [1]:

Bezeichne $\sigma_n^\delta(x)$ das n-te (C,δ)-Mittel der Fourierreihe der 2π periodischen stetigen Funktion $f(x)$, bezeichne ferner $(\sigma_n^\delta)^\sim(x)$ das entsprechende Mittel der konjugierten Reihe. Man setze $\rho_n^\delta = \mathrm{Max}\,|f(x)-\sigma_n^\delta(x)|$ und $(\rho_n^\delta)^\sim = \mathrm{Max}\,|f^\sim(x)-(\sigma_n^\delta)^\sim(x)|$. S. BERNSTEIN [*Mém. Acad. Belg.*, **(2)** 4 (1912), 1—104] *hat bewiesen, daß der Annaherungsgrad $\rho_n^1 = O(1/n^\alpha)$ mit $0<\alpha<1$ notwendig und hinreichend ist, damit $f(x)$ einer Lipschitzbedingung α-ter Ordnung genüge, für $\alpha=1$ gilt dagegen nur $\rho_n^1 = O(\log n/n)$, und es würde nicht einmal aus $\rho_n^1 = O(1/n)$ folgen, daß f einer Lipschitzbedingung erster Ordnung genügt. Verf. beschäftigt sich mit diesem Sprung im Bernsteinschen Satz und beweist folgendes: Genügt $f(x)$ einer Lipschitzbedingung erster Ordnung, so gilt $(\rho_n^\delta)^\sim = O(1/n)$ für alle $\delta>0$. Gilt umgekehrt $(\rho_n^\delta)^\sim = O(1/n)$ für ein $\delta \geqq 1$, so genügt $f(x)$ einer Lipschitzbedingung erster Ordnung.*

By standard methods (e.g. Privalov's theorem) this result can be rewritten in the form $(\delta \geqq 1)$

$$\max_{x\in[-\pi,\pi]} |\sigma_n^\delta(x)-f(x)| = O(1/n) \Leftrightarrow f^\sim(x)\in \mathrm{Lip}(1;C_{2\pi}),$$

where

$$\mathrm{Lip}(1;C_{2\pi}) := \{f\in C_{2\pi};\ \max_{x\in[-\pi,\pi]} |f(x+h)-f(x)| = O(h),\quad h\to 0+\}.$$

This is the difficult part of the saturation theorem for the (C,δ)-means, namely the characterization of the saturation class. The existence of the saturation phenomenon, namely the result that $|\sigma_n^\delta(x)-f(x)| = o(1/n)$ implies $f(x)=$ const. was communicated in case $\delta=1$ by ZYGMUND in a letter to EINAR HILLE already in July 1940 but only published in 1948; see HILLE [15; p. 352]. On the other hand, ZYGMUND himself published the result for any $\delta>0$ in 1945 [26].

[1] Zbl. Math. **8** (1947), 261. See also Alexits' recent paper [3].

In this sense Alexits was the first ever to prove a saturation theorem for a particular summation method of the Fourier series of $f \in C_{2\pi}$. On the other hand, the concept of saturation as such was introduced by JEAN FAVARD in a lecture in 1947; he first recognized its great importance as a basic problem of analysis. This was followed by the dissertation of his doctoral student MARC ZAMANSKY [24] of 1949 (which was preceded by four Comptes Rendus notes on the subject), the first major and comprehensive treatment concerned with saturation.

The chief aim of this paper is to study the saturation behaviour of the (C, δ)-means of the Fourier–Legendre series of $f \in X$, defined by

$$(\sigma_n^\delta f)(x) := (1/A_n^\delta) \sum_{k=0}^{n} A_{n-k}^\delta f^\wedge(k)(2k+1)P_k(x) \qquad (x \in [-1,1];\ n \in \mathbf{P} = \{0,1,2,\dots\})$$

for $\delta > 0$. Here $P_k(x)$ is the Legendre polynomial of degree k,

$$f^\wedge(k) =: \frac{1}{2} \int_{-1}^{1} f(u) P_k(u)\, du \qquad (k \in \mathbf{P})$$

the k-th Fourier–Legendre coefficient (or Legendre transform) of $f \in X$, $A_n^\delta = \binom{n+\delta}{n}$, and X stands either for the space $C[-1,1] \equiv C$ or $L^p(-1,1) \equiv L^p$, $1 \leqq p < \infty$, endowed with the norms

$$\|f\|_C := \sup_{x \in [-1,1]} |f(x)|, \quad \|f\|_p := \left\{\frac{1}{2} \int_{-1}^{1} |f(u)|^p\, du\right\}^{1/p},$$

respectively. Another definition needed is the class

$$\mathrm{Lip}(\alpha; X) := \{f \in X;\ \|f(\cdot + h) - f(\cdot)\|_{X[-1,1-h]} = O(h^\alpha), \quad h \to 0+\}$$

where $\|\cdot\|_{X[a,b]}$ denotes the X-norm with respect to the interval $[a, b]$.

In the case $X = C[-1,1]$ the counterpart of the Alexits result to be established states that $(\delta > 1/2)$

$$(1.1) \qquad \|\sigma_n^\delta f - f\|_C = O(1/n) \Leftrightarrow \sqrt{1-x^2}\, f^\sim(x) \in \mathrm{Lip}(1; C),$$

where $f^\sim$ now denotes the conjugate function in the Legendre frame (see Sec. 3.1 for definition), and that of Zygmund–Hille that

$$\|\sigma_n^\delta f - f\|_C = o(1/n) \Rightarrow f = \text{const}.$$

The factor $\sqrt{1-x^2}$ occuring on the right side of (1.1) is the typical endpoint weight for approximation by algebraic polynomials on the interval $[-1,1]$ ($\sigma_n^\delta f$ is indeed such a polynomial of degree n). This factor takes into account the difficulties at the endpoints in the algebraic instance; these are not present in 2π-periodic approximation.

The saturation question for the Cesàro means will not only be studied in $C[-1,1]$-space but also in $L^p(-1,1), 1 \leq p < \infty$ (see Theorem 3 of Sec. 4). The saturation class is the set $V[X; k]$ (see (3.7)) defined by a relation between Legendre (-Stieltjes) transforms of two functions. Basic will be its characterization in terms of Lipschitz conditions upon the conjugate function (Theorem 1). The latter function is treated in Sec. 3.1, Legendre–Stieltjes transforms in Sec. 2.

The paper concludes with a short discussion on non-optimal approximation of the Cesàro-means, dealt with in Sec. 5.

The problems treated for these means are also discussed for the Abel–Poisson means of the Fourier–Legendre series, their saturation classes being identical.

Let us finally mention that this paper may be regarded as a continuation of [21; 10].

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2. The Legendre–Stieltjes transform

Let $BV[-1,1]$ be the space of all complex valued functions μ, defined on $[-1,1]$, continuous from the left on $[-1,1)$, normalized by $\mu(-1)=0$, and endowed with the norm

$$\|\mu\|_{BV} := (1/2)\int_{-1}^{1} |d\mu(u)| \equiv \frac{1}{2}[\operatorname{Var}\mu]_{-1}^{1}.$$

The Legendre–Stieltjes transform and the (Fourier-) Legendre–Stieltjes series of $\mu \in BV[-1,1]$ are defined by

$$\mu^{\vee}(k) := (1/2)\int_{-1}^{1} P_k(u)\,d\mu(u) \qquad (k \in \mathbf{P}), \tag{2.1}$$

$$(d\mu)(x) \sim \sum_{k=0}^{\infty} \mu^{\vee}(k)(2k+1)P_k(x),$$

respectively, the integral being understood in the Riemann–Stieltjes sense. The Legendre convolution of $f \in X$, $g \in L^1(-1,1)$ is defined by

$$(f * g)(x) := \frac{1}{2}\int_{-1}^{1} (\tau_u f)(x)\,g(u)\,du \qquad (x \in [-1,1]), \tag{2.2}$$

and of $f \in X$, $\mu \in BV[-1,1]$ by

$$(f * d\mu)(x) := \frac{1}{2}\int_{-1}^{1} (\tau_u f)(x)\,d\mu(u) \qquad (x \in [-1,1]), \tag{2.3}$$

the latter integral here being a Lebesgue–Stieltjes integral. Here τ_h denotes the operator of translation in the Legendre frame, defined by

$$(2.4)\qquad (\tau_h f)(x) := \frac{1}{\pi}\int_{-1}^{1} f(xh + y\sqrt{1-x^2}\sqrt{1-h^2})\frac{dy}{\sqrt{1-y^2}} \qquad (x, h \in [-1, 1])$$

having e.g. the properties that $\|\tau_h\|_{[X,X]} = 1, h \in [-1, 1]$, and $\lim_{h\to 1-} \|\tau_h f - f\|_X = 0, f \in X$ (see also [21]).

Elementary calculations show that $f * g$ and $f * d\mu$ exist (a. e.),[2] belong to X, and satisfy the inequalities

$$(2.5)\qquad \|f * g\|_X \leqq \|f\|_X \|g\|_1, \quad \|f * d\mu\|_X \leqq \|f\|_X \|\mu\|_{BV}$$

together with the convolution properties

$$(2.6)\qquad [f * g]^\wedge(k) = f^\wedge(k) g^\wedge(k), \quad [f * d\mu]^\wedge k = f^\wedge(k)\mu^\vee(k) \qquad (k \in \mathbf{P}).$$

Consider the convolution integrals

$$(I_\rho f)(x) := (f * \chi_\rho)(x) \qquad (f \in X;\ \rho \in \mathbf{A};\ x \in [-1, 1]),$$

where $\{\chi_\rho\}_{\rho\in\mathbf{A}}$ is a kernel, i.e., $\chi_\rho \in L^1(-1, 1)$ with $(1/2)\int_{-1}^{1}\chi_\rho(u)\,du = 1, \rho \in \mathbf{A}$, ρ being a parameter ranging over some set $\mathbf{A}$ which is either an interval (a, b) with $0 \leqq a < b \leqq +\infty$, or the set $\mathbf{P}$. The family of integrals $\{I_\rho\}_{\rho\in\mathbf{A}}$ forms an approximation process on X if

$$(2.7)\qquad \lim_{\rho\to\rho_0} \|I_\rho f - f\|_X = 0 \qquad (f \in X),$$

ρ_0 denoting one of the points a, b or $+\infty$.

PROPOSITION 1. *Let $\{\chi_\rho\}_{\rho\in\mathbf{A}}$ be a kernel such that the associated integrals $\{I_\rho\}_{\rho\in\mathbf{A}}$ form an approximation process on* $C[-1, 1]$. *If $\mu \in$* BV$[-1, 1]$, *then for all* $s \in C[-1, 1]$

$$\lim_{\rho\to\rho_0} \int_{-1}^{1} s(x)(\chi_\rho * d\mu)(x)\,dx = \int_{-1}^{1} s(x)\,d\mu(x),$$

i.e., $\chi_\rho * d\mu$ converges in the weak topology of* BV$[-1, 1]$ *to* μ.

PROOF. By Fubini's theorem one has

$$\int_{-1}^{1} s(x)(\chi_\rho * d\mu)(x)\,dx = \int_{-1}^{1}\left\{\frac{1}{2}\int_{-1}^{1}(\tau_u\chi_\rho)(x)s(x)\,dx\right\}d\mu(u) = \int_{-1}^{1}(s * \chi_\rho)(u)\,d\mu(u) \qquad (\rho \in \mathbf{A}).$$

This yields part a) since $s * \chi_\rho$ tends uniformly to s by (2.7). Q.E.D.

[2] "(a. e.)" means that an assertion holds for all $x \in [-1, 1]$ if $X = C[-1, 1]$, and for almost all $x \in [-1, 1]$ if $X = L^p(-1, 1)$, $1 \leqq p < \infty$.

As an example, consider the Abel–Poisson means of the Fourier–Legendre series of $f \in X$ defined by

$$(2.8)\quad f(r;x) := \sum_{k=0}^{\infty} r^k f^{\wedge}(k)(2k+1)P_k(x) = (f * p_r)(x) \qquad (r \in [0,1);\ x \in [-1,1]),$$

where $\{p_r\}_{r\in[0,1)}$ is the Abel–Poisson kernel given by (cf. [20, p. 170])

$$p_r(x) := \sum_{k=0}^{\infty} r^k (2k+1) P_k(x) = \frac{1-r^2}{(1-2rx+r^2)^{3/2}} \qquad (r \in [0,1);\ x \in [-1,1]).$$

Here $p_r(x) \geqq 0$ for all $r \in [0,1)$, $x \in [-1,1]$, and $\lim_{r\to 1-} p_r^{\wedge}(1) = 1$, so that one has by [10, Prop. 2] that

$$(2.9)\qquad \lim_{r\to 1-} \|f(r;\cdot) - f(\cdot)\|_X = 0 \qquad (f \in X).$$

For the Abel–Poisson means of $\mu \in \mathrm{BV}[-1,1]$, namely

$$d\mu(r;x) := \sum_{k=0}^{\infty} r^k \mu^{\vee}(k)(2k+1)P_k(x) = (p_r * d\mu)(x) \qquad (r \in [0,1);\ x \in [-1,1]),$$

one has by Prop. 1 that they are weak* convergent to $\mu \in \mathrm{BV}[-1,1]$. Indeed

COROLLARY 1. *For each* $\mu \in \mathrm{BV}[-1,1]$ *there holds*

$$\lim_{r\to 1-} \int_{-1}^{1} s(x)\, d\mu(r;x)\, dx = \int_{-1}^{1} s(x)\, d\mu(x) \qquad (s \in C[-1,1]).$$

This leads to the uniqueness theorem for the Legendre–Stieltjes transform.

COROLLARY 2. *If* $\mu \in \mathrm{BV}[-1,1]$ *such that* $\mu^{\vee}(k) = 0$ *for all* $k \in \mathbf{P}$, *then* $\mu(x) \equiv 0$.

3. General theorems characterizing two particular function classes

3.1 The conjugate function. The concept of conjugacy in the frame of ultraspherical expansions was introduced by MUCKENHOUPT–STEIN [19]. Let us recall briefly the definitions and results in the Legendre case.

The conjugate series to (2.8) of $f \in X$ is given by

$$(3.1)\qquad f^{\sim}(r;x) := \frac{1}{2}\sum_{k=1}^{\infty} r^k f^{\wedge}(k)(2k+1)\sqrt{1-x^2}\, P_{k-1}^{(1,1)}(x) \qquad (r \in [0,1);\ x \in [-1,1]),$$

$P_n^{(\alpha,\beta)}(x)$ being the Jacobi polynomial. In view of the identity (see [22, (4.7.14)])

$$P_{k-1}^{(1,1)}(x) = \big(2/(k+1)\big) P_k'(x) \qquad (k \in \mathbf{N} = \text{naturals};\ x \in [-1,1])$$

and the Legendre differential equation (see [20, p. 176])

$$\frac{d}{dx}[(1-x^2)P_k'(x)] = -k(k+1)P_k(x) \qquad (k\in \mathbf{P};\ x\in[-1,1]),$$

the series (3.1) may be rewritten as

$$(3.2) \qquad f^{\sim}(r;x) = \sum_{k=1}^{\infty} r^k f^{\wedge}(k)\,[(2k+1)/(k+1)]\sqrt{1-x^2}\,P_k'(x)$$

$$(3.3) \qquad \sqrt{1-x^2}\,f^{\sim}(r;x) = -\int_{-1}^{x}\left\{\sum_{k=1}^{\infty} r^k f^{\wedge}(k)\,k(2k+1)P_k(u)\right\}du$$

$$(r\in[0,1);\ x\in[-1,1]).$$

It is known that for $f\in X$ the pointwise limit of $f^{\sim}(r;x)$ for $r\to 1-$ exists a.e. on $[-1,1]$ and defines the conjugate function of f, namely

$$(3.4) \qquad f^{\sim}(x) := \lim_{r\to 1-} f^{\sim}(r;x).$$

If $f\in L^p(-1,1)$, $1<p<\infty$, one has furthermore

$$(3.5) \qquad \lim_{r\to 1-} \| f^{\sim}(r;\cdot) - f^{\sim}(\cdot)\|_p = 0.$$

For more detailed information concerning the conjugate function, in particular for the generalized Cauchy–Riemann equation which are fundamental for this theory, the reader is referred to [19].

3.2 Two function classes. Two function classes will be needed. If $\{\psi(k)\}_{k\in\mathbf{P}}$ is a sequence of complex numbers, they are

$$(3.6) \qquad W[X;\psi(k)] := \{f\in X;\ \psi(k)f^{\wedge}(k) = g^{\wedge}(k),\ g\in X\}$$

$$(3.7) \quad V[X;\psi(k)] := \begin{cases} \{f\in L^1(-1,1);\ \psi(k)f^{\wedge}(k) = \mu^{\vee}(k),\ \mu\in \mathrm{BV}[-1,1]\} \\ \{f\in L^p(-1,1);\ \psi(k)f^{\wedge}(k) = g^{\wedge}(k),\ g\in L^p(-1,1)\}\ (1<p<\infty) \\ \{f\in C[-1,1];\ \psi(k)f^{\wedge}(k) = g^{\wedge}(k),\ g\in L^{\infty}(-1,1)\}, \end{cases}$$

$L^{\infty}(-1,1)$ denoting the set of all essentially bounded functions on $[-1,1]$ with norm $\|f\|_{\infty} = \operatorname{ess\,sup}_{x\in[-1,1]}|f(x)|$.

Note that the classes $V[L^p;\psi(k)]$ coincide with $W[L^p;\psi(k)]$ for the reflexive spaces $L^p(-1,1)$, $1<p<\infty$. In the cases X being equal to $C[-1,1]$ or $L^1(-1,1)$, one only has $W[X,\psi(k)]\subset V[X;\psi(k)]$.

There is a further, deeper connection between the V- and W-classes. If the classes $W[X;\psi(k)]$ and $V[X;\psi(k)]$ are endowed with the norms

$$(3.8) \qquad \|f\|_{W[X;\psi(k)]} \equiv \|f\|_W := \|f\|_X + \|g\|_X,$$

$$(3.9)\qquad \|f\|_{V[X;\,\psi(k)]} = \|f\|_V := \begin{cases} \|f\|_1 + \|\mu\|_{\mathrm{BV}}, & X = L^1(-1,1) \\ \|f\|_p + \|g\|_p, & X = L^p(-1,1),\ 1<p<\infty, \\ \|f\|_C + \|g\|_\infty, & X = C[-1,1], \end{cases}$$

where μ and g are the functions occurring in the definition of the corresponding classes, then they become normalized Banach subspaces of X (i.e. $\|f\|_X \leqq \|f\|_W$ all $f \in W[X;\psi(k)]$, $\|f\|_X \leqq \|f\|_V$ all $f \in V[X;\psi(k)]$).

If $\mathfrak{X}$ is a Banach space and $\mathfrak{Y}$ a normalized Banach subspace of $\mathfrak{X}$ then the completion of $\mathfrak{Y}$ relative to $\mathfrak{X}$, denoted by $\tilde{\mathfrak{Y}}^{\mathfrak{X}}$, is defined to be the set of those $f \in \mathfrak{X}$ for which there exists a sequence $\{f_n\}_{n\in\mathbf{N}} \subset \mathfrak{Y}$ and a constant $M>0$ such that $\|f_n\|_{\mathfrak{Y}} \leqq M$, all $n \in \mathbf{N}$, and $\lim_{n\to\infty} \|f_n - f\|_{\mathfrak{X}} = 0$. The space $\tilde{\mathfrak{Y}}^{\mathfrak{X}}$ becomes a normalized Banach subspace of $\mathfrak{X}$ under the norm

$$(3.10)\qquad \|f\|_{\tilde{\mathfrak{Y}}^{\mathfrak{X}}} := \inf\Big\{\sup_{n\in\mathbf{N}} \|f\|_{\mathfrak{Y}};\ \{f_n\} \subset \mathfrak{Y},\ \lim_{n\to\infty} \|f_n - f\|_{\mathfrak{X}} = 0\Big\}.$$

Using this concept the following result can be established by modifying the proof of the 2π-periodic counterpart in [8, pp. 373–376].

PROPOSITION 2. *The spaces* $V[X;\psi(k)]$ *and* $\widetilde{W[X;\psi(k)]}^X$ *with norms* (3.9) *and* (3.10), *respectively, are equal with equal norms.*

In [21] it was shown that $W[X;(k(k+1)/2)^r]$, i.e. $\psi(k) = (k(k+1)/2)^r$, $r \in \mathbf{N}$ can be described as the set of functions $f \in X$ for which the strong Legendre derivative $D^r f$ exists and belongs to X. Here the particular case $\psi(k) = k$, $k \in \mathbf{P}$, will be investigated. Indeed, knowledge of the conjugate function will enable us to characterize the particular classes $V[X;k]$, $W[X;k]$; the former will appear as saturation class of specific singular integrals.

3.3 Characterization of the class $V[X;k]$**.** We begin with an elementary lemma which is a particular case of a general result in [4]. Since the proof of the latter is rather complicated we include another one for the special Legendre case.

LEMMA 1. *The function* ξ *defined via*

$$(3.11)\qquad \xi^{\wedge}(k) := \begin{cases} k^{-1}, & k \in \mathbf{N} \\ 1, & k = 0 \end{cases}$$

belongs to $L^p(-1,1)$ *for all* $p \in [1,2)$.

PROOF. Consider the function $\xi_1(x) := 2[2(1-x)]^{-1/2}$, $x \in [-1,1)$. It belongs to $L^p(-1,1)$, all $p \in [1,2)$ and its Legendre coefficients are given by (cf. [11, p. 276])

$$\xi_1^{\wedge}(k) = 2/(2k+1) \qquad (k \in \mathbf{P}).$$

Furthermore, since $\sum_{k=1}^{\infty} (k(2k+1))^{-2}(2k+1) < \infty$, it follows by the Riesz–Fischer theorem for the Legendre system that there exists a $\xi_2 \in L^2(-1,1)$ with coefficients

$$\xi_2^{\wedge}(k) = \begin{cases} (k(2k+1))^{-1}, & k \in \mathbf{N} \\ -1, & k=0. \end{cases}$$

Setting $\xi(x) = \xi_1(x) + \xi_2(x)$ proves the assertion. Q.E.D.

We now come to our first basic result.

THEOREM 1. *Let $f \in X$. The following assertions are equivalent:*

(i) (a) *for* $X = L^1(-1,1)$
(b) *for* $X = L^p(-1,1)$, $1 < p < \infty$
(c) *for* $X = C[-1,1]$
$\Bigg\}$: $f \in V[X;k]$,

(ii) (a) *for* $X = L^1(-1,1)$: $f^{\sim}$ *exists in* L^1*-norm, i.e.,*

$$\lim_{r \to 1-} \| f^{\sim}(r;\cdot) - f(\cdot) \|_1 = 0, \tag{3.12}$$

and there exists some $\mu \in \mathrm{BV}[-1,1]$ *such that* $\sqrt{1-x^2}\, f^{\sim}(x) = -\mu(x)$ *a.e.,*

(b) *for* $X = L^p(-1,1)$, $1 < p < \infty$: *there exists some* $g \in L^p(-1,1)$ *such that*

$$\sqrt{1-x^2}\, f^{\sim}(x) = -\int_{-1}^{x} g(u)\,du \qquad (x \in [-1,1]), \tag{3.13}$$

(c) *for* $X = C[-1,1]$: *there exists some* $g \in L^{\infty}(-1,1)$ *such that* (3.13) *holds,*

(iii) (a) *for* $X = L^1(-1,1)$: $f^{\sim}$ *exists in* L^1*-norm and* $\sqrt{1-x^2}\, f^{\sim}(x) \in \mathrm{Lip}\,(1; L^1)$,
(b) *for* $X = L^p(-1,1)$, $1 < p < \infty$: $\sqrt{1-x^2}\, f^{\sim}(x)$ *is continuous on* $[-1,1]$ *and belongs to* $\mathrm{Lip}\,(1; L^p)$,
(c) *for* $X = C[-1,1]$: $\sqrt{1-x^2}\, f^{\sim}(x) \in \mathrm{Lip}\,(1; C)$.

The function g occurring in (ii)(b) *and* (ii)(c) *is the same as that appearing in the definition of the classes* $V[X;k]$. *The function* μ *in* (ii)(a) *may differ from the corresponding one in* $V[L^1;k]$ *by a constant multiple of the* δ^**-function (see* (3.21) *below).*

PROOF. Since the methods of proof for the different spaces are quite similar, the case $X = L^1(-1,1)$ is considered in detail, the other being sketched.

(i) $\Rightarrow$ (ii): Assuming $f \in V[L^1;k]$, i.e.,

$$kf^{\wedge}(k) = \mu^{\vee}(k) \qquad (k \in \mathbf{P}) \tag{3.14}$$

for some $\mu \in BV[-1, 1]$, one has by (3.3) that

$$(3.15)\quad \sqrt{1-x^2}\, f^{\sim}(r; x) = -\int_{-1}^{x} \left\{ \sum_{k=1}^{\infty} r^k k f^{\wedge}(k)(2k+1) P_k(u) \right\} du = -\int_{-1}^{x} d\mu(r; u)\, du$$
$$(x \in [-1, 1]).$$

Since $f^{\wedge}(k) = \xi^{\wedge}(k)\mu^{\vee}(k) = [\xi * d\mu]^{\wedge}(k)$, $k \in \mathbf{N}$, by (3.11) and (3.14) it follows by the uniqueness theorem for the Legendre transform (cf. [21]) that $f(x) = (\xi * d\mu)(x) + f^{\wedge}(0)$ a.e. So $f \in L^p(-1, 1)$ for all $p \in [1, 2)$ by (2.5), and therefore $f^{\sim}(r; x)$ and the left-hand side of (3.15) converge in L^1-norm to $f^{\sim}(x)$ and $\sqrt{1-x^2}\, f^{\sim}(x)$, respectively. In view of the weak* convergence of $d\mu(r; x)$ to $\mu(x)$ in $BV[-1, 1]$ (Cor. 1), it is easy to show that the right-hand side of (3.15) tends to $-\int_{-1}^{x} d\mu(u) = -\mu(x)$ in the weak topology of $L^1(-1, 1)$ for $r \to 1-$. This implies that it also converges in L^1-norm to $-\mu(x)$. Hence (i) $\Rightarrow$ (ii) if $X = L^1(-1, 1)$.

If $X = L^p(-1, 1)$, $1 < p < \infty$, or $X = C[-1, 1]$, one has instead of (3.15) that

$$(3.16)\qquad \sqrt{1-x^2}\, f^{\sim}(r; x) = -\int_{-1}^{x} g(r; u)\, du \qquad (x \in [-1, 1])$$

for some $g \in L^p(-1, 1)$ or $g \in L^\infty(-1, 1)$. The result then follows since the L^1-convergence of $g(r; x)$ to g implies the uniform convergence of the right-hand side of (3.16) to $-\int_{-1}^{x} g(u)\, du$.

(ii) $\Rightarrow$ (i): In case $X = L^1(-1, 1)$, (ii)(a) yields by integration by parts

$$(3.17)\qquad \mu^{\vee}(k) = \frac{1}{2}\mu(1) + \frac{1}{2}\int_{-1}^{1} \sqrt{1-u^2}\, f^{\sim}(u)\, P_k'(u)\, du \qquad (k \in \mathbf{P}).$$

On the other hand, by (3.12),

$$(3.18)\quad \lim_{r\to 1-} \frac{1}{2}\int_{-1}^{1} \sqrt{1-u^2}\, f^{\sim}(r; u)\, P_k'(u)\, du = \frac{1}{2}\int_{-1}^{1} \sqrt{1-u^2}\, f^{\sim}(u)\, P_k'(u)\, du \quad (k \in \mathbf{P}).$$

Since the series in (3.3) is uniformly convergent with respect to $u \in [-1, 1]$ for each fixed $r \in [0, 1)$, one has

$$(3.19)\qquad \frac{1}{2}\int_{-1}^{1} \sqrt{1-u^2}\, f^{\sim}(r; u)\, P_k'(u)\, du =$$
$$= -\sum_{j=1}^{\infty} r^j f^{\wedge}(j)\, j(2j+1) \frac{1}{2}\int_{-1}^{1} \left(\int_{-1}^{x} P_j(u)\, du \right) P_k'(x)\, dx = r^k k f^{\wedge}(k) \qquad (k \in \mathbf{P})$$

by making use of the orthogonality of $\{P_k\}_{k \in \mathbf{P}}$.

Combining (3.17)–(3.19) yields

$$kf^{\wedge}(k) = \mu^{\vee}(k) - (1/2)\,\mu(1) \qquad (k \in \mathbf{P}). \tag{3.20}$$

Setting now

$$\delta^*(x) := \begin{cases} 2, & x = 1 \\ 0, & -1 \leqq x < 1, \end{cases} \tag{3.21}$$

then $\delta^* \in \mathrm{BV}[-1, 1]$ and $[\delta^*]^{\vee}(k) = 1$. So (3.20) reduces to $kf^{\wedge}(k) = [\mu - (1/2)\,\mu(1)\delta^*]^{\vee}(k)$, $k \in \mathbf{P}$, i.e. $f \in V[L^1; k]$.

If $X = L^p(-1, 1)$ or $X = C[-1, 1]$ the proofs are similar except for the fact that the term corresponding to $(1/2)\,\mu(1)$ in (3.17) vanishes since $f^{\sim}(1) = 0$ by definition, and therefore g must be such that $\int_{-1}^{1} g(u)\,du = 0$.

(ii) ⇔ (iii): In case $X = L^p(-1, 1)$, $1 \leqq p < \infty$, these implication can be deduced from well-known results due to Hardy–Littlewood [14] (see also [**23**, p. 372] for $X = L^1(-1, 1)$). If $X = C[-1, 1]$ the proof in [23] may be easily modified.

Remark. Assertions (ii)(b), (iii)(b) may also be formulated in the form

(ii)(b)′ for $X = L^p(-1, 1)$, $1 < p < \infty$: $\sqrt{1 - x^2}\,f^{\sim}(x)$ is a.e. equal to a function $h \in C[-1, 1]$ having the representation

$$h(x) = -\int_{-1}^{x} g(u)\,du \qquad \big(x \in [-1, 1]\big),$$

where $g \in L^p(-1, 1)$ and $g^{\wedge}(0) = 0$.

(iii)(b)′ for $X = L^p(-1, 1)$, $1 < p < \infty$: $\sqrt{1 - x^2}\,f^{\sim}(x)$ is a.e. equal to a function $h \in C[-1, 1]$ which vanishes at $x = \pm 1$ and belongs to Lip $(1; L^p)$.

For $2 \leqq p < \infty$ the hypothesis $g^{\wedge}(0) = 0$ in (ii)(b)′ or $h(\pm 1) = 0$ in (iii)(b)′ can be dropped. Indeed, $f^{\sim} \in L^p(-1, 1)$ since $f \in L^p(-1, 1)$, and so $h(x)/\sqrt{1 - x^2} \in L^p(-1, 1)$. This implies that $\lim_{x \to 1-} h(x) = 2g^{\wedge}(0) = 0$ and $\lim_{x \to (-1)+} h(x) = 0$.

3.4 The class $W[X; k]$. The result corresponding to Theorem 1 for the classes $W[X; k]$ reads

Theorem 2. *Let $f \in X$. The following assertions are equivalent:*

(i) (a) *for* $X = L^1(-1, 1)$
(c) *for* $X = C[-1, 1]$ $\Big\}$ $: f \in W[X; k]$

(ii) (a) *for $X = L^1(-1, 1)$: $f^{\sim}$ exists in L^1-norm and there exists some $g \in L^1(-1, 1)$ with $g\,(0) = 0$ such that*

$$\sqrt{1 - x^2}\,f^{\sim}(x) \leqq -\int_{-1}^{x} g(u)\,du \qquad a.e. \tag{3.22}$$

(c) *for* $X = C[-1, 1]$: *there exists some* $g \in C[-1, 1]$ *such that* (3.22) *holds for all* $x \in [-1, 1]$.

Assertions of type (b) are omitted since $W[L^p; k] = V[L^p; k]$. The proof is basically the same as that of Theorem 1 (see also [6; 4]).

It would also be possible to characterize $W[X; k]$ by differentiability-properties of $\sqrt{1 - x^2}\, f^{\tilde{}}(x)'$ in a similar way as $V[X; k]$ was characterized by Lipschitz conditions in Theorem 1 (iii).

4. Saturation

4.1 General theory. An approximation process $\{T_\rho\}_{\rho \in \mathbf{A}}$ on X is said to have the saturation property if there exists a positive function ϕ, defined on $\mathbf{A}$ and tending to zero for $\rho \to \rho_0$, such that every $f \in X$ for which

$$\| T_\rho f - f \|_X = o(\phi(\rho)) \qquad (\rho \to \rho_0)$$

is an invariant element of $\{T_\rho\}$, i.e., $T_\rho f = f$ for all $\rho \in \mathbf{A}$, and if the set

$$S[X; T_\rho] := \{ f \in X; \| T_\rho f - f \|_X = O(\phi(\rho)),\ \rho \to \rho_0 \}$$

contains at least one non-invariant element. Then the process is said to be saturated in X with order $O(\phi(\rho))$ and has saturation class $S[X; T_\rho]$.

The following criterion will be useful in deciding whether the saturation property holds for approximation processes defined via singular Legendre convolution integrals $\{I_\rho\}_{\rho \in \mathbf{A}}$.

PROPOSITION 3. *Let* $\{\chi_\rho\}_{\rho \in \mathbf{A}}$ *be the kernel of* $\{I_\rho\}_{\rho \in \mathbf{A}}$ *such that*

$$\| \chi_\rho \|_1 \leqq M \qquad (\rho \in \mathbf{A}), \tag{4.1}$$

$$\lim_{\rho \to \rho_0} \frac{1 - \chi_\rho^\wedge(k)}{\phi(\rho)} = \psi(k) \qquad (k \in \mathbf{P}), \tag{4.2}$$

$\{\psi(k)\}_{k \in \mathbf{P}}$ *being a sequence of complex numbers with* $\psi(0) = 0$, $\psi(k) \neq 0$ *for* $k \in \mathbf{N}$, *and* ϕ *a function on* $\mathbf{A}$ *as above.*

a) *The process* $\{I_\rho\}_{\rho \in \mathbf{A}}$ *is saturated in* X *with order* $O(\phi(\rho))$, $\rho \to \rho_0$, *and the invariant elements are the functions* $f = const.$ *(a.e.).*

b) *The saturation class is contained in* $V[X; \psi(k)]$.

Firstly note that (4.1) and (4.2) imply $\{I_\rho\}$ to form an approximation process (see [10, Prop. 1]). The proof of part a) then follows from [10, Prop. 4], that of part b) along standard lines (cf. [8, Theorem 12.1.4]).

To determine the saturation class completely, i.e. to show that also $V[X; \psi(k)] \subset S[X; I_\rho]$, more information upon the kernel is needed. In the following the matter is confined to two special cases, namely to the (C, δ)-means as well as the Abel–Poisson means of the Fourier–Legendre series of $f \in X$.

4.2 Saturation class of the (C, δ)**-means.** For $\delta > 0$ these (C, δ)-means of $f \in X$ are defined as

$$(4.3)\quad (\sigma_n^\delta f)(x) := (1/A_n^\delta) \sum_{k=0}^{n} A_{n-k}^\delta f^\wedge(k)(2k+1) P_k(x) \qquad (n \in \mathbf{P};\ x \in [-1, 1]),$$

where $A_n^\delta = \binom{n+\delta}{n}$. Setting

$$F_n^\delta(x) := (1/A_n^\delta) \sum_{k=0}^{n} A_{n-k}^\delta (2k+1) P_k(x) \qquad (n \in \mathbf{P}; x \in [-1, 1]),$$

the $\sigma_n^\delta f$ may be rewritten as convolution integrals

$$(\sigma_n^\delta f)(x) = (f * F_n^\delta)(x) \qquad (n \in \mathbf{P}; x \in [-1, 1]).$$

The $(C, 1)$-means are nothing but the Fejér means $\sigma_n f$ treated in [10, Sec. 5]. For any $\delta > 1/2$ one has

$$\| F_n^\delta \|_1 \leqq F^\delta \qquad (n \in \mathbf{P})$$

for some constant $F^\delta \geqq 1$ (see [18]), and $\lim_{n\to\infty} A_{n-k}^\delta / A_n^\delta = 1$, $k \in \mathbf{P}$. This implies by [10, Prop. 1] that $\{\sigma_n^\delta\}$ forms an approximation process on X for each $\delta > 1/2$.

It was shown in [10, Sec. 5] that $\{\sigma_n^\delta\}$ is saturated in X with order $O(n^{-1})$ for $\delta = 1$. It will now be shown that this result extends to any $\delta > 1/2$. This time the saturation class is also determined.

THEOREM 3. *The* (C, δ)*-means,* $\delta > 1/2$, *are saturated in* X *with order* $O(n^{-1})$; *the saturation class is* $V[X; k]$.

PROOF. Since

$$(4.4)\qquad \lim_{n\to\infty} n\big(1 - [F_n^\delta]^\wedge(k)\big) = \lim_{n\to\infty} n\big(1 - (A_{n-k}^\delta / A_n^\delta)\big) = \delta k \qquad (k \in \mathbf{P})$$

(see [7, Section 8]), i.e., $\phi(n) = n^{-1}$, $\psi(k) = \delta k$ in (4.2), the saturation order $O(1/n)$ and that $S[X; \sigma_n^\delta] \subset V(X; k]$ follow from Prop. 3, noting that $V[X; \delta k] = V[X; k]$. Conversely, let $f \in V[X; k]$. If $X = L^1(-1, 1)$ and $\mu \in \mathrm{BV}[-1, 1]$ is such that $kf^\wedge(k) = \mu^\vee(k)$, $k \in \mathbf{P}$, then the following identities are easily verified:

$$\sigma_n^\delta f = \sigma_n^{\delta+1} f + (n+1+\delta)^{-1}(\sigma_n^\delta\, d\mu),$$

$$(4.5)\qquad \sigma_{n+1}^{\delta+1} f - \sigma_n^{\delta+1} f = \frac{(\delta+1)}{(n+1)(n+2+\delta)} (\sigma_{n+1}^\delta\, d\mu),$$

where $(\sigma_n^\delta d\mu)(x) := (F_n^\delta * d\mu)(x)$. This yields

$$(4.6)\qquad \|f-\sigma_n^\delta f\|_1 \leqq \|f-\sigma_n^{\delta+1} f\|_1 + \frac{1}{n+1+\delta}\|\sigma_n^\delta d\mu\|_1 \leqq$$

$$\leqq \sum_{k=n}^{\infty} \|\sigma_{k+1}^{\delta+1} f - \sigma_k^{\delta+1} f\|_1 + F^\delta n^{-1}\|\mu\|_{BV} =$$

$$= \sum_{k=n}^{\infty} \frac{(\delta+1)}{(k+1)(k+2+\delta)} \|\sigma_{k+1}^\delta d\mu\|_1 + F^\delta n^{-1}\|\mu\|_{BV} \leqq$$

$$\leqq M n^{-1}\|\mu\|_{BV} \qquad (n\in \mathbf{N}).$$

This is the desired assertion for $X = L^1(-1, 1)$. For $X = L^p(-1, 1)$, $1 < p < \infty$ or $X = C[-1, 1]$ the same proof works with obvious modifications. Q.E.D.

It is interesting to note that essentially the same method of proof, particularly the identity (4.5) was already used by Alexits [1] in 1941. For further generalizations and applications of this method see [12; 25; 2; 9; 6; 3].

4.3 Saturation class for the Abel–Poisson means. Concerning the Abel–Poisson means $f(r; x) = (f * p_r)(x)$, $f\in X$, $r\in [0, 1)$, $x\in[-1, 1]$, defined in (2.8), condition (4.1) is satisfied with $M = 1$, since $p_r(x) \geqq 0$, $r\in[0, 1)$.

As $p_r^\wedge(k) = r^k$, $k\in \mathbf{P}$, condition (4.2) holds with $\phi(r) = 1-r$ and $\psi(k) = k$. So these means are saturated in X with order $O(1-r)$, $r\to 1-$, and $S[X; f(r; x)] \subset V[X; k]$.

Now let $f\in V[L^1; k]$, i.e., $kf^\wedge(k) = \mu^\vee(k)$, $k\in\mathbf{P}$, $\mu\in BV[-1,1]$. Then for $r\in\mathbf{P}$, $0<r<1$,

$$[f(r;\cdot) - f(\cdot)]^\wedge(k) = (r^k-1)f^\wedge(k) = \int_r^1 t^{k-1}\mu^\vee(k)\,dt =$$

$$= \int_r^1 [d\mu(t;\cdot)]^\vee(k)\,t^{-1}dt = \Big[\int_r^1 d\mu(t;\cdot)\,t^{-1}dt\Big]^\wedge(k),$$

the interchange of the order of integration being valid since $\|d\mu(t;\cdot)\|_1 \leqq \|\mu\|_{BV}$, $0\leqq t<1$. So, for $f\in V[L^1; k]$,

$$\|f(r;\cdot) - f(\cdot)\|_1 = \Big\|\int_r^1 d\mu(t;\cdot)\,t^{-1}dt\Big\|_1 \leqq M(1-r)\|\mu\|_{BV} \quad (r\in[1/2, 1)).$$

Since the corresponding result holds for the other spaces one has

Theorem 4. *The convolution integral $f(r; x)$ of Abel–Poisson is saturated in X with order $O(1-r)$, $r\to 1-$; its saturation class is $V[X, k]$.*

Note that this result could also be deduced using semigroup methods. Such methods for $L^2(-1, 1)$ also give characterizations of $V[X; k]$ in terms of classical derivatives of f, see [16, p. 611].

Applying the characterizations of the class $V[X; k]$ given in Sec. 3, the above results can be formulated as

COROLLARY 3. *The following assertions are equivalent for* $f \in X$:

(i) $\| f - \sigma_n^\delta f \|_X = O(n^{-1}) \quad (n \to \infty;\ \delta > 1/2)$,

(ii) $\| f(\cdot) - f(r; \cdot) \|_X = O(1 - r) \quad (r \to 1-)$,

(iii) (a) *For* $X = L^1(-1, 1)$: $f^\sim$ *exists in* L^1*-norm and* $\sqrt{1-x^2} f^\sim(x) \in \mathrm{Lip}\,(1; L^1)$,

(b) *for* $X = L^p(-1, 1)$, $1 < p < \infty$: $\sqrt{1-x^2} f^\sim(x) \in C[-1, 1]$ *and* $\sqrt{1-x^2} f^\sim(x) \in \mathrm{Lip}\,(1; L^p)$,

(c) *for* $X = C[-1, 1]$: $\sqrt{1-x^2} f^\sim(x) \in \mathrm{Lip}\,(1; C)$.

It should be mentioned that H. BAVINCK [4; 5] also determined the saturation class for the (C, δ) as well as Abel–Poisson means of $f \in X$. However, he expressed this class just by $V[X; k]$.

KALLAEV [17], on the other hand, gave characterizations directly in terms of Lipschitz classes in the particular case $X = C[-1, 1]$ (and not in terms of $V[X; k]$). Let us show that his result coincides with ours. He considered the function

$$G(x) := \int_{-1}^{x} \left(\frac{1-u^2}{x-u} \right)^{1/2} h'(u)\, du, \qquad h(x) := \int_{x}^{1} f(u) \frac{1}{\sqrt{u-x}}\, du,$$

and showed that $S[C; \sigma_n^\delta] = \{ f \in C[-1, 1];\ G \in \mathrm{Lip}\,(1; C) \}$ by using the identity

$$\left[\frac{d}{dx} G(x) \right]^\wedge (k) = -\pi(k + 1/2) f^\wedge(k) \qquad (k \in \mathbf{P}).$$

Now our Theorem 1) (ii) (b) gives

$$\left[\frac{d}{dx} \sqrt{1-x^2} f^\sim(x) \right]^\wedge (k) = k f^\wedge(k) \qquad (k \in \mathbf{P}).$$

This implies that $G(x) = -\pi \{ \sqrt{1-x^2} f^\sim(x) + (1/2) \int_{-1}^{x} f(u)\, du \} + c$. Since $(1/2) \int_{-1}^{x} f(u)\, du + c \in \mathrm{Lip}\ (1; C)$, the Lipschitz conditions for G and $\sqrt{1-x^2} f^\sim(x)$ are equivalent.

5. Non-optimal approximation

Here the results of [10] on the non-optimal approximation of the (C, δ)-means of the Fourier–Legendre series are extended from $\delta = 1$ to arbitrary $\delta > 1/2$.

If the space $W[X; k]$ is endowed with the seminorm $|f|_{W[X;k]} \equiv |f|_W := \| g \|_X$, where $k f^\wedge(x) = g^\wedge(k)$, $k \in \mathbf{P}$, one deduces as in the estimate (4.6) the Jackson-type inequality

$$(5.1) \qquad \| f - \sigma_n^\delta f \|_X \leqq M n^{-1} |f|_{W[X;k]} \qquad (f \in W[X; k];\ n \in \mathbf{N}).$$

By a standard procedure this leads to

$$(5.2) \qquad \| f - \sigma_n^\delta f \|_X \leqq MK(n^{-1}, f; X, W[X; k]) \qquad (f \in X; n \in \mathbf{N}),$$

the K-functional being defined by

$$K(t, f; X, Y) \equiv K(t, f) := \inf_{g \in Y} \{\| f - g \|_X + t \, | g |_Y\} \qquad (t > 0),$$

Y being a subspace of X with seminorm $| \cdot |_Y$.

On the other hand, $p_n \in W[X; k]$ for all $p_n \in \mathcal{P}_n$(= set of all algebraic polynomials of degree $\leqq n$), and it is known (see [13]) that there holds the Bernstein-type inequality

$$(5.3) \qquad | p_n |_{W[X; k]} \leqq Mn \, \| p_n \|_X \qquad (n \in \mathbf{P}),$$

the constant M being independent of n and p_n. This will lead to

PROPOSITION 4. *For any $f \in X$, $0 < \alpha \leqq 1$ and $\delta > 1/2$ one has*

$$(5.4) \qquad \| f - \sigma_n^\delta f \|_X = O(n^{-\alpha}) \qquad (n \to \infty)$$

if and only if

$$(5.5) \qquad K(t, f; X, W[X; k]) = O(t^\alpha) \qquad (t \to 0+).$$

PROOF. The implication (5.5) ⇒ (5.4) follows from (5.2). The converse for $0 < \alpha < 1$ follows along standard lines (see e.g. [8, p. 110]). Indeed, setting $U_2 = \sigma_4^\delta f$, $U_n = \sigma_{2^n}^\delta f - \sigma_{2^{n-1}}^\delta f$ for $n = 3, 4, \dots$ one has for any $m \geqq 2$

$$(5.6) \qquad \sum_{k=2}^{m} U_k = \sigma_{2^m}^\delta f, \quad \| U_k \|_X \leqq M 2^{-k\alpha}.$$

Since the K-functional is subadditive and $U_k \in \mathcal{P}_{2^k}$, there holds

$$K(t, f) \leqq K\left(t, f - \sum_{k=2}^{m} U_k\right) + K\left(t, \sum_{k=2}^{m} U_k\right) \leqq \left\| f - \sum_{k=2}^{m} U_k \right\|_X + t \left| \sum_{k=2}^{m} U_k \right|_W \leqq$$

$$\leqq \| f - \sigma_{2^m}^\delta f \|_X + t \sum_{k=2}^{m} 2^{k(1-\alpha)} \leqq M\{2^{-\alpha m} + t 2^{m(1-\alpha)}\}$$

by (5.3), (5.4) and (5.6). For $0 < t < 1/2$ one can choose $m \geqq 2$ such that $2^{-m} \leqq t < 2^{-m+1}$. This yields $K(t, f) = O(t^\alpha)$, completing the proof for $0 < \alpha < 1$. For $\alpha = 1$ it follows from Theorem 3 that (5.4) implies $f \in V[X; k]$. In view of Prop. 2 one can choose a sequence $\{f_n\}_{n \in \mathbf{N}} \subset W[X; k]$ such that $\| f_n - f \|_X = o(1)$, $n \to \infty$, and $| f_n |_W \leqq M$. This yields

$$K(t, f) \leqq \| f - f_n \|_X + t \, | f_n |_W \leqq \| f - f_n \|_X + tM,$$

giving (5.5) for $n \to \infty$. Q.E.D.

This result shows in particular that the approximation behaviour of the σ_n^δ-means does not depend on $\delta > 1/2$. So one can use the counterpart for the $(C, 1)$-means established in [10] to deduce that (i) $\Leftrightarrow$ (iii) of

THEOREM 5. *The following assertions are equivalent for* $f \in X$, $0 < \alpha < 1$:

(i) $\| f - \sigma_n^\delta f \|_X = O(n^{-\alpha}) \quad (n \to \infty;\ \delta > 1/2)$,

(ii) $\| f(\cdot) - f(r; \cdot) \|_X = O((1-r)^\alpha) \quad (r \to 1-)$,

(iii) $f \in \mathrm{Lip}_1^L(\alpha/2; X)$, *i.e.*, $\| (\tau_h f) - f \|_X = O((1-h)^{\alpha/2}) \quad (h \to 1-)$.

The fact that (ii) $\Leftrightarrow$ (iii) is left to the reader; for the Bernstein-type inequality needed see [13].

It would be possible to characterize the approximation behaviour of the above two means not only in terms of the "global" Lipschitz condition (iii) but also in terms of local Lipschitz classes for $X = C[-1, 1]$; this follows from the results of [21]. Indeed, assertion (iii), and so (i) and (ii) of Theorem 5, are known to be equivalent to

$$\sup_{\substack{x, x+h \in [-1,1] \\ |h| \leq \eta}} (\sqrt{1-x^2})^\alpha \, |\overline{f(x+h)} - f(x)| \leqq M\eta^\alpha \qquad (\eta > 0). \tag{5.7}$$

Note that (5.7) for $\alpha = 1$ is not equivalent to $\sqrt{1-x^2} f(x) \in \mathrm{Lip}(1; C)$, so that, apart from the fact that f must be replaced by $f^{\sim}$, there is a "break" in the smoothness condition between the non-optimal and optimal cases, i.e., when passing from $\alpha < 1$ to $\alpha = 1$. Indeed, the function $f(x) = 1$ satisfies (5.7) for $\alpha = 1$ but $\sqrt{1-x^2} \cdot 1 \notin$ $\notin \mathrm{Lip}(1; C)$.

As a matter of fact, the characterization of the assertions (i) and (ii) in terms of Legendre—Lipschitz classes for the case $\alpha = 1$ in Theorem 5 is an open question. Its solution would probably lead to characterizing Legendre—Lipschitz classes defined by fractional order Legendre differences; see the corresponding open problem for fractional order Legendre derivatives in [21, Sec. 7].

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HOW BIG MUST BE THE INCREMENTS OF A WIENER PROCESS?

By

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Dedicated to Prof. G. Alexits on the occasion of his 80th birthday

1. Introduction

Let $\{W(t), t \geqq 0\}$ be a standard Wiener process and introduce the following notations:

(1.1) $$\mathfrak{J}(t, a_T) = \sup_{s \leqq a_T} |W(t+s) - W(t)|,$$

(1.2) $$I(T) = I(T, a_T) = \sup_{0 \leqq t \leqq T - a_T} \mathfrak{J}(t, a_T),$$

(1.3) $$I_1(T) = \sup_{0 \leqq t \leqq T - a_T} |W(t+a_T) - W(t)|,$$

(1.4) $$\beta(T) = \left(2a_T(\log Ta_T^{-1} + \log\log T)\right)^{-1/2},$$

(1.5) $$\Delta(T) = \frac{[Ta_T^{-1}]}{\log\log T},$$

(1.6) $$\gamma(T) = \left(2a_T \log\left(1 + \frac{\pi^2}{16}\Delta_T\right)\right)^{-1/2},$$

(1.7) $$\delta(T) = (2a_T \log Ta_T^{-1})^{-1/2}.$$

In [3] the following result was proved

THEOREM A. *Let a_T $(T \geqq 0)$ be a non-decreasing function of T for which*
(i) $0 < a_T \leqq T$,
(ii) $a_T T^{-1}$ *is non-increasing.*
Then

(1.8) $$\limsup_{T\to\infty} \beta(T)\, I(T) = 1 \qquad a.s.$$

If we also have
(iii) $\lim\limits_{T\to\infty} \dfrac{\log Ta_T^{-1}}{\log\log T} = \infty$,
then

(1.9) $$\lim_{T\to\infty} \beta(T)\, I(T) = 1 \qquad a.s.$$

The above theorem remains true if $I(T)$ is replaced by $I_1(T)$.

It is natural to ask (and it was asked in [3]) how can the normalizing factors $\gamma^*(T)$ and $\gamma_1^*(T)$ be defined in order to get the relationship

$$\liminf_{T\to\infty} \gamma^*(T)\, I(T) = \liminf_{T\to\infty} \gamma_1^*(T)\, I_1(T) = 1 \qquad \text{a.s.} \tag{1.10}$$

DEO [4] has shown that $\liminf_{T\to\infty} \beta(T)\, I(T)$ differs from $\limsup_{T\to\infty} \beta(T)\, I(T)$ whenever

$$\limsup_{T\to\infty} \frac{\log T a_T^{-1}}{\log\log T} < \infty , \tag{1.11}$$

and the same holds for $I_1(T)$.

Concerning $I_1(T)$ a partial answer to the above question was given by BOOK and SHORE [1], who proved that if

$$\lim_{T\to\infty} \frac{\log (T a_T^{-1})}{\log\log T} = r \tag{1.12}$$

where $0 \leqq r \leqq \infty$, then

$$\liminf_{T\to\infty} \beta(T)\, I_1(T) = \left(\frac{r}{r+1}\right)^{1/2} \qquad \text{a.s.} \tag{1.13}$$

The aim of this paper is to give even a more complete answer concerning the variable $I(T)$. The main result of this paper is the following

THEOREM 1. *Let* a_T $(T \geqq 0)$ *be a nondecreasing function of* T *for which* (i) *and* (ii) *hold true. Then*

$$18^{-1} \leqq \liminf_{T\to\infty} \gamma(T)\, I(T) \leqq 46 \qquad \textit{a.s.} \tag{1.14}$$

If (i) *and* (ii) *and one of the relations*

(iv) $\lim_{T\to\infty} \Delta_T = \infty$

and

(v) $\lim_{T\to\infty} \Delta_T = 0$

hold then we also have

$$\liminf_{T\to\infty} \gamma(T)\, I(T) \geqq 1 \qquad \textit{a.s.} \tag{1.15}$$

Moreover, in case when (i), (ii) *and* (v) *hold then we have*

$$\liminf_{T\to\infty} \gamma(T)\, I(T) \leqq 2\sqrt{2} \qquad \textit{a.s.} \tag{1.16}$$

If we have (i), (ii) *and*

(vi) $$\lim_{T\to\infty} \frac{\log Ta_T^{-1}}{\log\log\log T} = \infty,$$

then

(1.17) $$\liminf_{T\to\infty} \gamma(T)\, I(T) = 1 \qquad a.s.$$

REMARKS 1. The Kolmogorov zero-one law implies that there exists a constant ρ such that

(1.18) $$\liminf_{T\to\infty} \gamma(T)\, I(T) = \rho \qquad \text{a.s.}$$

where the constant ρ may depend on the function a_T. (1.14) states that $18^{-1} \leqq \rho \leqq 46$ for any increasing function a_T satisfying conditions (i), (ii).

2. Since condition (iii) implies (vi), relation (1.17) also follows from Theorem A in the particular case when the condition (vi) is replaced by the stronger condition (iii). (Note that $\beta(T) \approx \gamma(T) \approx \delta(T)$ provided (iii) holds true.)

3. The so called "other law of the iterated logarithm" (see [2] and [5]) states that

(1.19) $$\liminf_{T\to\infty} \sup_{s\leqq T} \left(\frac{8\log\log T}{\pi^2 T}\right)^{1/2} |\, W(s)\,| = 1 \qquad \text{a.s.}$$

This implies that the constant ρ of (1.18) is equal to one in case $a_T = T$. This result does not follow from our Theorem 1.

4. The constants 18^{-1} and 46 in (1.14) are certainly not the best possible ones. It is not hard to get somewhat better constants but we are not in the position to get the exact value of ρ, except in the case when (vi) holds.

5. Our Theorem 1 implies that in the result (1.13) of Book and Shore $I_1(T)$ may be replaced by $I(T)$ and it is not hard to see that $I_1(T)$ in (1.13) may also be replaced either by $\sup_{0\leqq t\leqq T-a_T} \big(W(t+a_T) - W(t)\big)$ or by $\sup_{0\leqq t\leqq T-a_T} \sup_{s\leqq a_T} \big(W(t+s) - W(t)\big)$. It would also be interesting to find the right normalizing factors for these quantities in other cases too. We are able to treat only the variable

(1.20) $$I_1^+(T) = \sup_{0\leqq t\leqq T-a_T} \big(W(t+a_T) - W(t)\big)$$

in the case $a_T/T = \alpha$ (α is a constant). This is given in the next theorem where the curious constant c_α suggests that the constant ρ (see Remark 1) may depend on the function a_T indeed.

THEOREM 2. *For any* $0 < \alpha \leqq 1$ *we have*

(1.21) $$\liminf_{T\to\infty} I_1^+(T)\,(2T\log\log T)^{-1/2} = -c_\alpha \qquad a.s.$$

where

$$c_\alpha = \left(\frac{(2r+1)\alpha - 1}{r(r+1)}\right)^{1/2} \qquad \textit{and} \qquad r = \left[\frac{1}{\alpha}\right]. \tag{1.22}$$

In order to see the meaning of this theorem we mention that STRASSEN's theorem [7] implies that

$$\limsup_{T\to\infty} I_1^+(T)\,(2T\log\log T)^{-1/2} = \alpha^{1/2} \qquad \text{a.s.} \tag{1.23}$$

which can be obtained by considering the function

$$x(s) = \begin{cases} s\alpha^{-1/2} & \text{if} \quad 0 \leqq s \leqq \alpha \\ \alpha^{1/2} & \text{if} \quad \alpha < s \leqq 1 \end{cases} \tag{1.24}$$

in Strassen's class. The fact that $(2T\log\log T)^{-1/2}$ is the right normalizing factor for the lim inf follows also from Strassen's theorem. In case $\alpha = 1$ it is well known that $c_1 = 1$ and this can be obtained by considering the function

$$x(s) = -s, \qquad 0 \leqq s \leqq 1 \tag{1.25}$$

in Strassen's class. It is also immediate from (1.25) that c_α must satisfy $c_\alpha \geqq \alpha$. Theorem 2 says, however, that equality holds (i.e. $c_\alpha = \alpha$) if and only if $1/\alpha$ is an integer, in other cases $c_\alpha > \alpha$.

2. Proof of Theorem 1

We accomplish the proof through several lemmas.

LEMMA 1. *For any $T > e^2$ we have*

$$P\{I(T) < (18\gamma(T))^{-1}\} < (\log T)^{-2}. \tag{2.1}$$

PROOF. We treat the cases when a_T is small and large separately.
Case 1. Suppose that

$$\Delta_T = \frac{[Ta_T^{-1}]}{\log\log T} \leqq \frac{300}{\log(4\pi^{-1})}. \tag{2.2}$$

Then we have

$$P\{I(T) \leqq (18\gamma(T))^{-1}\} = P\{\max_{0\leqq i\leqq [Ta_T^{-1}]-1} \mathfrak{J}(ia_T, a_T) \leqq (18\gamma(T))^{-1}\} =$$

$$= \left(P\{\mathfrak{J}(0, a_T) \leqq (18\gamma(T))^{-1}\}\right)^{[Ta_T^{-1}]} \leqq \left(\frac{4}{\pi}\exp\left(-\frac{\pi^2}{8}(18\gamma(T))^2 a_T\right)\right)^{[Ta_T^{-1}]} \leqq$$

$$\leqq (4\pi^{-1})^{[Ta_T^{-1}]} \exp(-18^2 \log\log T) \leqq (\log T)^{-2}.$$

Here the following inequalities were applied:

$$P\{\sup_{0\leqq t\leqq T} T^{-1/2} \mid W(t)\mid < x\} \leqq \frac{4}{\pi}\exp\left\{-\frac{\pi^2}{8x^2}\right\}, \tag{2.3}$$

$$\log(1+x) \leqq x \qquad \text{for} \qquad x \geqq 0. \tag{2.4}$$

Case 2. Suppose that

$$\Delta_T > \frac{300}{\log(4\pi^{-1})}. \tag{2.5}$$

Then we have

$$P\{I(T) < (2\gamma(T))^{-1}\} \leqq \left(P\{\mathfrak{I}(0, a_T) < (2\gamma(T))^{-1}\}\right)^{[Ta_T^{-1}]} =$$
$$= \left(2\Phi\left((2\gamma(T))^{-1}a_T^{-1/2}\right) - 1\right)^{[Ta_T^{-1}]} \leqq$$
$$\leqq \left(1 - \exp\left(-\frac{1}{2}\log\left(1 + \frac{\pi^2}{16}\Delta_T\right)\right)\right)^{[Ta_T^{-1}]} \leqq \left(1 - \frac{1}{1 + \frac{\pi^2}{32}\Delta_T}\right)^{[Ta_T^{-1}]} \leqq$$
$$\leqq \left(1 - \frac{\log\log T}{\left(\frac{\pi^2}{32} + \frac{\log 4\pi^{-1}}{300}\right)[Ta_T^{-1}]}\right)^{[Ta_T^{-1}]} \leqq (\log T)^{-2}.$$

Here the following inequalities were applied:

$$2\Phi(x) - 1 \leqq 1 - e^{-x^2} \qquad \text{if} \qquad x \geqq 1.8, \tag{2.6}$$

$$(1+x)^{1/2} \leqq 1 + \frac{1}{2}x \qquad \text{if} \qquad x \geqq 0, \tag{2.7}$$

$$1 - x \leqq e^{-x}. \tag{2.8}$$

LEMMA 2. *Suppose that condition* (iv) *holds. Then for any* $\varepsilon > 0$

$$P\{I(T) < ((1+\varepsilon)\gamma(T))^{-1}\} \leqq (\log T)^{-2} \tag{2.9}$$

if T is big enough.

PROOF. We have

$$P\{I(T) < ((1+\varepsilon)\gamma(T))^{-1}\} \leqq \left(2\Phi\left(((1+\varepsilon)\gamma(T))^{-1}a_T^{-1/2}\right) - 1\right)^{[Ta_T^{-1}]} \leqq$$
$$\leqq \left(1 - \exp\left(-(1+2\varepsilon)^{-1}\log\left(1 + \frac{\pi^2}{16}\Delta_T\right)\right)\right)^{[Ta_T^{-1}]} \leqq$$
$$\leqq \left(1 - \frac{1}{\Delta_T^{1-\varepsilon}}\right)^{[Ta_T^{-1}]} \leqq (\log T)^{-2}$$

if T is big enough.

Here the same inequalities were applied as in the proof of Case 2 of Lemma 1.

LEMMA 3. *Suppose that condition* (v) *holds. Then for any* $\varepsilon > 0$ *we have*

$$P\{I(T) < ((1+\varepsilon)\,\gamma(T))^{-1}\} \leqq (\log T)^{-(1+\varepsilon/2)} \tag{2.10}$$

if T is big enough.

PROOF. We have

$$P\{I(T) < ((1+\varepsilon)\,\gamma(T))^{-1}\} \leqq \left(\frac{4}{\pi}\exp\left(-\frac{\pi^2}{8}((1+\varepsilon)\,\gamma(T))^2 a_T\right)\right)^{[Ta_T^{-1}]} \leqq$$
$$\leqq (\log T)^{-(1+\varepsilon/2)}$$

if T is big enough.

LEMMA 4. *Let* a_T $(T \geqq 0)$ *be a non-decreasing function of T for which* (i) *and* (ii) *hold true. Then*

$$\liminf_{T\to\infty} \gamma(T)\,I(T) \geqq 18^{-1} \qquad a.s. \tag{2.11}$$

PROOF. Apply Lemma 1 for the sequence $T_k = \theta^k$ $(k = 1, 2, \dots, \theta > 1)$. Then by the Borel–Cantelli lemma we get

$$\liminf_{k\to\infty} \gamma(T_k)\,I(T_k) \geqq 18^{-1}. \tag{2.12}$$

Now choosing θ close enough to 1, (2.11) follows from the facts that $I(T)$ is non-decreasing in T and

$$\theta^{-1} \leqq \frac{a_T \log\left(1 + \dfrac{\pi^2}{16}\,\dfrac{T}{a_T \log\log T}\right)}{a_{T_k} \log\left(1 + \dfrac{\pi^2}{16}\,\dfrac{T_k}{a_{T_k} \log\log T_k}\right)} \leqq \theta \tag{2.13}$$

for all $T_k \leqq T < T_{k+1}$.

LEMMA 5. *Let* a_T $(T \geqq 0)$ *be a non-decreasing function of T for which* (i), (ii) *and one of* (iv) *and* (v) *hold true. Then*

$$\liminf_{T\to\infty} \gamma(T)\,I(T) \geqq 1 \qquad a.s. \tag{2.14}$$

PROOF. The proof of this Lemma is the same as that of Lemma 4 with the difference that we apply Lemmas 2 and 3 instead of Lemma 1.

LEMMA 6. *For any* $T > e^2$ *we have*

$$P\{\max_{0 \leqq i \leqq [Ta_T^{-1}]-1} \mathfrak{J}(ia_T, a_T) < 15(\gamma(T))^{-1}\} \geqq (\log T)^{-1/2}. \tag{2.15}$$

PROOF. Similarly to the proof of Lemma 1, we consider two cases.
Case 1. Suppose that

$$\Delta_T \leqq 0.03\,. \tag{2.16}$$

Then we have

$$P\{\max_{0\leqq i\leqq [Ta_T^{-1}]-1} \mathfrak{J}(ia_T, a_T) < 10(\gamma(T))^{-1}\} = \left(P\{\mathfrak{J}(0, a_T) < 10(\gamma(T))^{-1}\}\right)^{[Ta_T^{-1}]} \geqq$$

$$\geqq \left(\frac{8}{3\pi}\right) \exp\left(-\frac{\pi^2}{800}(\gamma(T))^2 a_T\right)^{[Ta_T^{-1}]} \geqq \left(\frac{8}{3\pi}\right)^{[Ta_T^{-1}]} \exp\left(-\frac{1}{99}\log\log T\right) \geqq$$

$$\geqq \exp\left(-\left(\frac{1}{99} + 0.03\log\frac{3\pi}{8}\right)\log\log T\right) \geqq (\log T)^{-1/2}.$$

Here the following inequalities were applied

$$P(\sup_{0\leqq t\leqq T} T^{-1/2}|W(t)| < x) \geqq \frac{4}{\pi}\left(e^{-\frac{\pi^2}{8x^2}} - \frac{1}{3}e^{-\frac{9\pi^2}{8x^2}}\right) \geqq \frac{8}{3\pi}e^{-\frac{\pi^2}{8x^2}}, \tag{2.17}$$

$$\log(1+x) \geqq 0.99x \quad \text{if} \quad x \leqq \frac{3\pi^2}{1600}. \tag{2.18}$$

Case 2. Suppose that

$$\Delta_T > 0.03\,. \tag{2.19}$$

Then we have

$$P\{\max_{0\leqq i\leqq [Ta_T^{-1}]-1} \mathfrak{J}(ia_T, a_T) < 15(\gamma(T))^{-1}\} = \left(P\{\mathfrak{J}(0, a_T) < 15(\gamma(T))^{-1}\}\right)^{[Ta_T^{-1}]} \geqq$$

$$\geqq \left(1 - \exp\left(-15^2\log\left(1 + \frac{\pi^2}{16}\Delta_T\right)\right)\right)^{[Ta_T^{-1}]} \geqq$$

$$\geqq \left(1 - \exp\left(-\log\frac{15^1\pi^2}{16}\Delta_T\right)\right)^{[Ta_T^{-1}]} \geqq (\log T)^{-1/2}.$$

Here the following inequalities were applied

$$2\Phi(x) - 1 \geqq 1 - \sqrt{\frac{2}{\pi}}\,\frac{1}{x}e^{-\frac{x^2}{2}} \geqq 1 - e^{-\frac{x^2}{2}} \quad \text{if} \quad x > \sqrt{\frac{2}{\pi}}, \tag{2.20}$$

$$(1+x)^{15^2} \geqq 15^2x^2, \quad 1 - x > e^{-2x} \quad \text{if} \quad 0 < x < 0.6\,. \tag{2.21}$$

LEMMA 7. *Let a_T $(T \geqq 0)$ be a non-decreasing function of T for which* (i) *and* (ii) *hold true. Then*

$$\liminf_{T\to\infty} \gamma(T)\, I(T) \leqq 46 \qquad a.s. \tag{2.22}$$

PROOF. Put $T_k = e^{k \log^2 k}$. Then it follows from Lemma 6 that

$$\liminf_{k\to\infty} \Big(\gamma(T_{k+1}) \max_{\frac{T_k}{a_{T_{k+1}}} \leqq i \leqq \frac{T_{k+1}}{a_{T_{k+1}}} - 1} \mathfrak{I}(i a_{T_{k+1}}, a_{T_{k+1}})\Big) \leqq 15 \qquad a.s. \tag{2.23}$$

It is easily seen that (2.23) implies

$$\liminf_{k\to\infty} \Big(\gamma(T_{k+1}) \sup_{T_k \leqq t \leqq T_{k+1} - a_{T_{k+1}}} \mathfrak{I}(t, a_{T_{k+1}})\Big) \leqq 45 \qquad \text{a.s.} \tag{2.24}$$

We have the following inequality:

$$I(T_{k+1}) \leqq A_k + \sup_{T_k \leqq t \leqq T_{k+1} - a_{T_{k+1}}} \mathfrak{I}(t, a_{T_{k+1}}) \tag{2.25}$$

where

$$A_k = \begin{cases} \sup\limits_{0 \leqq t \leqq T_k} \mathfrak{I}(t, a_{T_{k+1}}) & \text{if } a_{T_{k+1}} \leqq T_k (\log\log T_k)^3 \\ \sup\limits_{0 < u < v \leqq T_k} |W(v) - W(u)| & \text{if } a_{T_{k+1}} < T_k (\log\log T_k)^3. \end{cases} \tag{2.26}$$

We show that

$$\limsup_{k\to\infty} \gamma(T_{k+1})\, A \leqq 1 \qquad \text{a.s.} \tag{2.27}$$

Case 1. $a_{T_{k+1}} \leqq T_k (\log\log T_k)^3$.
Then Theorem A implies that

$$\limsup_{k\to\infty} \frac{\sup\limits_{0 \leqq t \leqq T_k} \mathfrak{I}(t, a_{T_{k+1}})}{\left(2a_{T_{k+1}}\left(\log \dfrac{T_k + a_{T_{k+1}}}{a_{T_{k+1}}} + \log\log (T_k + a_{T_{k+1}})\right)\right)^{1/2}} \leqq 1 \qquad \text{a.s.} \tag{2.28}$$

An easy calculation shows that in this case we have the inequality

$$\gamma(T_{k+1}) \leqq \left[2a_{T_{k+1}}\left(\log \frac{T_k + a_{T_{k+1}}}{a_{T_{k+1}}} + \log\log (T_k + a_{T_{k+1}})\right)\right]^{-1/2} \tag{2.29}$$

for k large enough, hence (2.27) follows.

Case 2. $a_{T_{k+1}} > T_k(\log\log T_k)^3$.
By the law of the iterated logarithm

$$\limsup_{k\to\infty} \frac{\sup_{0<u<v\leqq T_k} |W(v)-W(u)|}{(2T_k \log\log T_k)^{1/2}} = 1 \quad \text{a.s.} \tag{2.30}$$

But in this case

$$\gamma(T_{k+1}) \leqq (2T_k \log\log T_k)^{-1/2} \tag{2.31}$$

for k large enough, proving (2.27) which together with (2.24) and (2.25) implies (2.22).

LEMMA 8. *Let a_T ($T \geqq 0$) be a non-decreasing function of T for which* (i) (ii) *and* $\lim_{T\to\infty} Ta_T^{-1} = \infty$ *hold. Then we have*

$$\liminf_{T\to\infty} \delta(T)\, I(T) \leqq 1 \quad a.s. \tag{2.32}$$

PROOF. On applying Lemma 1* in CSÖRGŐ–RÉVÉSZ [3] we have for any $\varepsilon > 0$

$$P\big(\delta(T)\, I(T) < 1+\varepsilon\big) \leqq C\frac{T}{a_T}\exp\left(-\frac{(1+\varepsilon)^2}{2\delta^2(T)\,(1+\varepsilon)}\right) \leqq C\left(\frac{T}{a_T}\right)^{-\varepsilon} \tag{2.33}$$

i.e.

$$\lim_{T\to\infty} P\big(\delta(T)\, I(T) < 1+\varepsilon\big) = 0. \tag{2.34}$$

A simple calculation (see e.g. the proof of Lemma 2) gives also

$$\lim_{T\to\infty} P\big(\delta(T)\, I(T) < 1-\varepsilon\big) = 0 \tag{2.35}$$

for any $\varepsilon > 0$. That is to say

$$\delta(T)\, I(T) \to 1 \quad \text{as} \quad T\to\infty \quad \text{in probability.} \tag{2.36}$$

It follows that there exists a sequence $T_1 < T_2 < \dots$ such that $\lim_{k\to\infty} T_k = \infty$ and

$$\lim_{k\to\infty} \delta(T_k)\, I(T_k) = 1 \quad \text{a.s.,} \tag{2.37}$$

proving Lemma 8.

Since in the case when (i), (ii) and (vi) hold, we have $\delta(T) \approx \gamma(T)$, and hence also

$$\liminf_{T\to\infty} \gamma(T)\, I(T) \leqq 1 \quad \text{a.s.} \tag{2.38}$$

Now (1.14) follows from Lemmas 4 and 7, (1.15) follows from Lemma 5 and (1.17) follows from Lemmas 5 and 8. It remains to show (1.16).

If $\lim_{T\to\infty} \Delta_T = 0$, then

$$\gamma_T \approx \left(2a_T \frac{\pi^2}{16} \Delta_T\right)^{-1/2} \leqq \left(\frac{16 \log\log T}{\pi^2 T}\right)^{1/2} \tag{2.39}$$

and

$$I(T) \leqq 2 \sup_{0 \leqq t \leqq T} |W(t)|. \tag{2.40}$$

(2.39), (2.40) and Chung's result (1.19) together imply (1.16).

The proof of Theorem 1 is complete.

3. Proof of Theorem 2

It follows from Strassen's result that there exists a constant d_α such that

$$\liminf_{T\to\infty} \frac{\sup_{t \leqq (1-\alpha)T} (W(t+\alpha T) - W(t))}{(2T\log\log T)^{1/2}} = -d_\alpha \quad \text{a.s.} \tag{3.1}$$

We show first that $d_\alpha \geqq c_\alpha$, where c_α is given in Theorem 2. Define the function $x(s)$ as follows: if $1/\alpha = r$ (an integer), then let $x(s) = -s$. If $1/\alpha = r + \tau$, where r is an integer and $0 < \tau < 1$, then split the interval $[0, 1]$ into $2r+1$ parts with the points

$$\begin{aligned} u_{2i} &= i\alpha, & i &= 0, 1, 2, \ldots, r \\ u_{2i+1} &= (i+\tau)\alpha, & i &= 0, 1, 2, \ldots, r. \end{aligned} \tag{3.2}$$

Let $x(s)$ be a continuous piecewise linear function starting from 0 (i.e. $x(0) = 0$) and having slopes

$$x'(s) = \begin{cases} -\dfrac{1}{r+1}\left(\dfrac{1}{\alpha}\dfrac{r(r+1)}{r+1-\tau}\right)^{1/2}, & \text{if } u_{2i} < s < u_{2i+1} \\ -\dfrac{1}{r}\left(\dfrac{1}{\alpha}\dfrac{r(r+1)}{r+1-\tau}\right)^{1/2}, & \text{if } u_{2i-1} < s < u_{2i}. \end{cases} \tag{3.3}$$

It is easily seen that $x(s)$ so defined is in Strassen's class, i.e. $x(0) = 0$, $x(s)$ is absolutely continuous for $0 < s < 1$ and $\int_0^1 x'^2(s)ds = 1$. Since

$$x(s+\alpha) - x(s) = -c_\alpha, \qquad 0 \leqq s \leqq 1-\alpha \tag{3.4}$$

we must have $d_\alpha \geqq c_\alpha$.

Unfortunately we can not accomplish the proof of Theorem 2 by showing that $x(s)$ defined above is extremal within Strassen's class. In order to show that $d_\alpha \leqq c_\alpha$ we give a probabilistic proof.

LEMMA 9. *For $a > 0$ we have*

$$P\left\{\sup_{t \leqq (1-\alpha)T} \frac{W(t+\alpha T) - W(t)}{(\alpha T)^{1/2}} < -a\right\} \leqq e^{-\frac{\alpha a^2}{2c_\alpha^2}}. \tag{3.5}$$

PROOF. It can be seen by scale change that $(\alpha T)^{-1/2} \sup_{t \leqq (1-\alpha)T} \big(W(t+\alpha T) - W(t)\big)$ and $\sup_{t \leqq \frac{1}{\alpha}-1} \big(W(t+1) - W(t)\big)$ have the same distribution. Define the random variables $X_1, \ldots, X_{2r}$ by

$$\begin{aligned} X_{2i-1} &= W(i) - W(i-1), \qquad & i = 1, \ldots, r \\ X_{2i} &= W(i+\tau) - W(i-1+\tau), \qquad & i = 1, \ldots, r. \end{aligned} \tag{3.6}$$

$X_1, \ldots, X_{2r}$ are standard normal variables with covariances

$$\begin{cases} E(X_{2i-1}X_{2i}) = 1 - \tau, \qquad E(X_{2i}X_{2i+1}) = \tau, \\ \qquad E(X_iX_j) = 0 \quad \text{if} \quad |i-j| \geqq 2. \end{cases} \tag{3.7}$$

Moreover

$$P\Big(\sup_{t \leqq \frac{1}{\alpha}-1} \big(W(t+1) - W(t)\big) < -a\Big) \leqq P\big(\max_{1 \leqq i \leqq 2r} X_i < -a\big) =$$

$$= P(X_1 > a, X_2 > a, \ldots, X_{2r} > a).$$

If $\tau = 0$, then $X_{2i-1} = X_{2i}$ and $X_1, X_3, \ldots, X_{2r-1}$ are independent, hence

$$P(X_1 > a, \ldots, X_{2r} > a) = \big(P(X_1 > a)\big)^r \leqq e^{-\frac{a^2 r}{2}} \tag{3.8}$$

by Bernstein's inequality. In this case $\alpha/c_\alpha^2 = r$, thus (3.5) is proved.

If $0 < \tau < 1$, then we apply the multivariate version of Bernstein's inequality:

$$\begin{aligned} P(X_1 > a, \ldots, X_{2r} > a) &\leqq E(e^{z_1(X_1-a)+\ldots+z_{2r}(X_{2r}-a)}) = \\ &= \exp\left\{\frac{1}{2}\sum_{i=1}^{2r}\sum_{j=1}^{2r} \rho_{ij} z_i z_j - a(z_1 + \ldots + z_{2r})\right\} \end{aligned} \tag{3.9}$$

for all $z_1 > 0, \ldots, z_{2r} > 0$ with correlations given by (3.7).

On choosing

$$\begin{cases} z_{2i+1} = \dfrac{a(r-i)}{r+1-\tau}, \qquad i = 0, 1, \ldots, r-1 \\ z_{2i} = \dfrac{ai}{r+1-\tau}, \qquad i = 1, 2, \ldots, r \end{cases} \tag{3.10}$$

we get (3.5) proving Lemma 9.

We remark that the exact distribution of $\sup_{t \leqq T} \big(W(t+1) - W(t)\big)$ is given in SHEPP [6], but his formula does not seem suitable to give an easier proof of Lemma 9.

LEMMA 10. *Let* $T_k = \theta^k$, $k = 1, 2, \ldots; \theta > 1$. *Then*

$$\liminf_{k \to \infty} \frac{\sup\limits_{t \leqq (1-\alpha)T_k} \big(W(t+\alpha T) - W(t)\big)}{(2T_k \log\log T_k)^{1/2}} \geqq -x_\alpha \qquad a.s. \tag{3.11}$$

PROOF. Apply Lemma 9 with $T = T_k$,

$$a = c_\alpha \alpha^{-1/2} (2(1+\varepsilon) \log\log T_k)^{1/2} \tag{3.12}$$

and then appeal to the Borel—Cantelli lemma.

LEMMA 11. *Let* $T_k = \theta^k$, $k = 1, 2, \ldots; \theta > 1$. *Then*

$$\liminf_{k \to \infty} \frac{\inf\limits_{T_k \leqq T < T_{k+1}} \sup\limits_{t \leqq (1-\alpha)T} \big(W(t+\alpha T) - W(t)\big)}{(2T_k \log\log T_k)^{1/2}} \geqq -c_\alpha - \big(\alpha(\theta-1)\big)^{1/2} \quad a.s. \tag{3.13}$$

PROOF. We start from the identity

$$W(t+\alpha T_k) - W(t) = \big(W(t+\alpha T) - W(t)\big) + \big(W(t+\alpha T_k) - W(t+\alpha T)\big). \tag{3.14}$$

Hence for $T_k \leqq T < T_{k+1}$ we have

$$\sup_{t \leqq (1-\alpha)T_k} \big(W(t+\alpha T_k) - W(t)\big) \leqq$$

$$\leqq \sup_{t \leqq (1-\alpha)T} \big(W(t+\alpha T) - W(t)\big) + \sup_{t \leqq (1-\alpha)T_k} \big(W(t+\alpha T_k) - W(t+\alpha T)\big) \leqq$$

$$\leqq \sup_{t \leqq (1-\alpha)T} \big(W(t+\alpha T) - W(t)\big) + \sup_{t \leqq T_k} \sup_{s \leqq \alpha(T_{k+1}-T_k)} |W(t+s) - W(t)|.$$

Therefore

$$\inf_{T_k \leqq T < _{k+1}} \sup_{t \leqq (1-\alpha)T} \big(W(t+\alpha T) - W(t)\big) \geqq$$

$$\geqq \sup_{t \leqq (1-\alpha)T_k} \big(W(t+\alpha T_k) - W(t)\big) - \sup_{t \leqq T_k} \sup_{s \leqq \alpha(T_{k+1}-T_k)} |W(t+s) - W(t)|.$$

The statement of Lemma 11 follows from Lemma 10 and Theorem A.

The inequality $d_\alpha \leqq c_\alpha$ now follows from Lemma 11 since one can choose $\theta - 1$ arbitrarily small.

The proof of Theorem 2 is complete.

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SOME REMARKS ON DISCRETE BAIRE CLASSES

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To Professor G. Alexits on his 80th birthday

0. Introduction. In a recent paper [4], the authors have investigated a modified Baire classification of real-valued functions, by replacing pointwise convergence by other types of convergence, namely by those called "discrete" and "equal" convergence (for the definitions, see 1.2). In particular, it turned out that, under suitable hypotheses on the functional class from which the classification starts, the "equal Baire functions" are the same as the ordinary Baire functions (obtained by pointwise convergence), whereas the "discrete Baire functions" constitute, in general, a proper subset of the class of ordinary Baire functions. Moreover, this subset was characterized by means of a relatively simple condition (see [4], Theorem 7 and Corollary 11).

The main purpose of the present paper is to sharpen the latter condition in order to obtain a characterization not only of discrete Baire functions in general but also of each particular discrete Baire class. This sharpening is based on some results on Borel sets, classical in the case e.g. of metric spaces but not commonly known perhaps in the general setting needed here. Hence section 1 will contain the terminology used in the following, while sections 2 and 3 are devoted to the general theory of Borel classification of sets and Baire classification of functions, respectively; the tools needed later are presented sometimes in a slightly more general form than usually found in the literature (and in a generality that exceeds in some cases the exigencies of the present paper). After this preparation, section 4 contains the characterization of discrete Baire classes.

In section 5, we formulate some consequences of the main result concerning equal Baire classes, and we give another equivalent definition of equal convergence. Finally, section 6 is a supplement to section 4 of [4] and deals with a further analysis of those function classes which constitute the starting point of the Baire classification.

1. Terminology and notation. Let X be a non-empty set. We understand by a *system of sets* a non-empty set whose elements are subsets of X, by a *function* a real-valued function $f: X \to \mathbf{R}$ defined on X, by a *function class* a non-empty set whose elements are functions.

1.1. Let $\mathfrak{M}$ be a system of sets. $\mathfrak{M}$ is said to be a *semi-lattice* if

$$A, B \in \mathfrak{M} \text{ implies } A \cap B \in \mathfrak{M},$$

a *lattice* if

$$A, B \in \mathfrak{M} \text{ implies } A \cup B, A \cap B \in \mathfrak{M},$$

a *ring* if

$$A, B \in \mathfrak{M} \text{ implies } A \cup B, A - B \in \mathfrak{M},$$

a *σ-system* if

$$A_i \in \mathfrak{M} \quad (i \in \mathbf{N}) \text{ implies } \bigcup_1^\infty A_i \in \mathfrak{M},$$

a *δ-system* if

$$A_i \in \mathfrak{M} \quad (i \in \mathbf{N}) \text{ implies } \bigcap_1^\infty A_i \in \mathfrak{M}$$

a *σ-ring* if it is a ring and a σ-system. It is well-known that each ring is a lattice and each σ-ring is a δ-system.

$\mathfrak{M}$ is said to be perfect if, for each $A \in \mathfrak{M}$, there is a representation

$$A = \bigcap_1^\infty B_i$$

where $X - B_i \in \mathfrak{M}$ for each $i \in \mathbf{N}$.

We denote by $\mathfrak{M}^c$ the system of sets composed of the complements (in X) of the elements of $\mathfrak{M}$, by $\mathfrak{M}^\sigma$ and $\mathfrak{M}^\delta$ the smallest σ- and δ-system, respectively, containing $\mathfrak{M}$, composed of the sets having the form

$$\bigcup_1^\infty A_i \quad \text{and} \quad \bigcap_1^\infty A_i \quad \text{where} \quad A_i \in \mathfrak{M},$$

respectively. The smallest σ-ring containing $\mathfrak{M}$ (the existence of which is well-known) s denoted by $\mathfrak{S}(\mathfrak{M})$.

1.2. A sequence (f_n) of functions is said to converge to a function f *discretely* if, for each $x \in X$, there is an index $n_0(x)$ such that $f_n(x) = f(x)$ for $n \geqq n_0(x)$. (f_n) is said to converge *equally* to f if there exists a sequence (ε_n) with $\varepsilon_n > 0$, $\varepsilon_n \to 0$ such that, for each $x \in X$, there is $n_0(x)$ satisfying $|f_n(x) - f(x)| \leqq \varepsilon_n$ for $n \geqq n_0(x)$.

Let Φ be a function class. We denote by Φ^u, Φ^p, Φ^d, Φ^e the set of those functions that are limits of uniformly convergent, pointwise convergent, discretely convergent, and equally convergent sequences taken from Φ. Obviously

$$\left.\begin{matrix} \Phi^u \\ \Phi^d \end{matrix}\right\} \subset \Phi^e \subset \Phi^p.$$

Φ is said to be *complete* if $\Phi = \Phi^u$.

1.3. Let Φ be a function class. Φ is said to be a *lattice* if it *contains all constants* and

$$f, g \in \Phi \text{ implies } \max(f, g) \in \Phi, \ \min(f, g) \in \Phi,$$

a *translation lattice* if it is a lattice and

$$f \in \Phi,\ c \in \mathbf{R} \text{ implies } f + c \in \Phi,$$

a *congruence lattice* if it is a translation lattice and

$$f \in \Phi \text{ implies } -f \in \Phi,$$

a *weakly affine lattice* if it is a congruence lattice and there is a set $C \subset (0, +\infty)$ such that C is not bounded and

$$f \in \Phi,\ c \in C \text{ implies } cf \in \Phi,$$

an *affine lattice* if it is a congruence lattice and

$$f \in \Phi,\ c \in \mathbf{R} \text{ implies } cf \in \Phi,$$

a *subtractive lattice* if it is a lattice and

$$f, g \in \Phi \text{ implies } f - g \in \Phi,$$

an *ordinary class* if it is a subtractive lattice and

$$f, g \in \Phi \text{ implies } fg \in \Phi,$$

$$f \in \Phi,\ f(x) \neq 0 \text{ for } x \in X \text{ implies } \frac{1}{f} \in \Phi.$$

We denote by Φ^v the smallest complete ordinary class containing Φ (i.e. the intersection of all complete ordinary classes containing Φ).

1.4. Let f, g be functions, $c \in \mathbf{R}$. We denote

$$X(f \leqq g) = \{x \in X : f(x) \leqq g(x)\},$$
$$X(f < g) = \{x \in X : f(x) < g(x)\},$$
$$X(f \geqq g) = \{x \in X : f(x) \geqq g(x)\},$$
$$X(f > g) = \{x \in X : f(x) > g(x)\},$$
$$X(f = g) = \{x \in X : f(x) = g(x)\},$$

in particular we use the notation $X(f \leqq c)$ etc. in the case when g is a constant function.

Let Φ be a function class. We denote by $\mathcal{S}(\Phi)$ the system of all sets of the form $X(f \leqq c)$ and $X(f \geqq c)$ for $f \in \Phi$, $c \in \mathbf{R}$. Similarly $\mathcal{Q}(\Phi)$ denotes the system of all sets $X(f < c)$, $X(f > c)$ for $f \in \Phi$, $c \in \mathbf{R}$. Clearly $\mathcal{Q}(\Phi) = \mathcal{S}(\Phi)^c$.

Now let $\mathfrak{M}$ be a system of sets. We denote by $\Phi(\mathfrak{M})$ the function class composed of all functions f such that $X(f \leqq c) \in \mathfrak{M}$ and $X(f \geqq c) \in \mathfrak{M}$ for $c \in \mathbf{R}$.

For a function f and $a, b \in \mathbf{R}$, $a < b$, let us write

$$[f]_a^b = \min\big(b, \max(f, a)\big).$$

2. General Borel sets. Let $\mathcal{S}$ be a perfect semi-lattice with $X \in \mathcal{S}$. We define the *Borel classes belonging to* $\mathcal{S}$ as follows:

$$\mathcal{S}_0 = \mathcal{S}, \quad \mathcal{Q}_0 = \mathcal{S}^c, \tag{1}$$

$$\mathcal{S}_\alpha = \Big(\bigcup_{\xi<\alpha} \mathcal{Q}_\xi\Big)^\delta, \quad \mathcal{Q}_\alpha = \Big(\bigcup_{\xi<\alpha} \mathcal{S}_\xi\Big)^\sigma \qquad (0 < \alpha < \omega_1). \tag{2}$$

The systems of sets $\mathcal{S}_\alpha$ and $\mathcal{Q}_\alpha$ are called *multiplicative* and *additive* classes, belonging to $\mathcal{S}$, respectively. We denote further

$$\mathcal{A}_\alpha = \mathcal{S}_\alpha \cap \mathcal{Q}_\alpha\,;$$

the systems $\mathcal{A}_\alpha$ are the *ambiguous* classes belonging to $\mathcal{S}$.

PROPOSITION 2.1. $\mathcal{S}_\alpha \cup \mathcal{Q}_\alpha \subset \mathcal{A}_\beta$ *for* $0 \leqq \alpha < \beta < \omega_1$.

PROOF. $\mathcal{S}_0 \subset \mathcal{S}_1$ and $\mathcal{Q}_0 \subset \mathcal{Q}_1$ follow from the fact that $\mathcal{S}$ is perfect. $\mathcal{S}_\alpha \subset \mathcal{Q}_{\alpha+1}$, $\mathcal{Q}_\alpha \subset \mathcal{S}_{\alpha+1}$ $(0 \leqq \alpha < \omega_1)$ and $\mathcal{S}_\alpha \subset \mathcal{S}_\beta$, $\mathcal{Q}_\alpha \subset \mathcal{Q}_\beta$ $(0 < \alpha < \beta < \omega_1)$ are obvious by (2).

PROPOSITION 2.2. $\mathcal{Q}_\alpha = \mathcal{S}_\alpha^c$ *for* $0 \leqq \alpha < \omega_1$.

PROOF. Transfinite induction.

PROPOSITION 2.3. $\mathcal{S}_\alpha$ *is a* δ*-lattice*, $\mathcal{Q}_\alpha$ *is a* σ*-lattice for* $0 < \alpha < \omega_1$.

PROOF. $\mathcal{S}_\alpha$ is obviously a δ-system, $\mathcal{Q}_\alpha$ is a σ-system for $0 < \alpha < \omega_1$. $\mathcal{S}_1$ and $\mathcal{Q}_1$ are lattices since $\mathcal{S}$ is a semi-lattice. Then the same is obtained for $\mathcal{S}_\alpha$ and $\mathcal{Q}_\alpha$ $(1 < \alpha < < \omega_1)$ by transfinite induction, taking 2.1 into account.

PROPOSITION 2.4. $\mathcal{A}_\alpha$ *is a ring for* $0 < \alpha < \omega_1$.

PROOF. 2.2 and 2.3.

PROPOSITION 2.5. $\displaystyle\bigcup_{\alpha<\omega_1} \mathcal{S}_\alpha = \bigcup_{\alpha<\omega_1} \mathcal{Q}_\alpha = \mathbb{S}(\mathcal{S})$.

PROOF. The first equality follows from 2.1. This system is a ring by 2.1, 2.3 and 2.2. It is a σ-system since $\bigcup_1^\infty \mathcal{S}_i \in \mathcal{Q}_{\alpha+1}$ if $P_i \in \mathcal{S}_{\alpha_i}$ and $\alpha = \sup\{\alpha_i : i \in \mathbf{N}\} < \omega_1$. Hence this system contains $\mathbb{S}(\mathcal{S})$ and the opposite inclusion is proved by transfinite induction, using the assumption $X \in \mathcal{S}$.

REMARK 2.6. The same construction furnishes, with a slight modification, the system $\mathbb{S}(\mathcal{M})$ for an arbitrary system $\mathcal{M}$ satisfying $X \in \mathcal{M}$. In fact, by defining

$$\mathcal{S} = (\mathcal{M} \cup \mathcal{M}^c)^\delta,$$

it is easily seen that $\mathcal{S}$ is a perfect semi-lattice (moreover a δ-system) and $\mathbb{S}(\mathcal{M}) = \mathbb{S}(\mathcal{S})$.

Based on Propositions 2.1 to 2.5, the proof of the following statements is the same as in the classical theory, e.g. in [8].

PROPOSITION 2.7. *If $Q \in \mathcal{Q}_\alpha$, $0 < \alpha < \omega_1$, then there exists a decomposition*

$$Q = \bigcup_1^\infty A_i, \quad A_i \in \mathcal{A}_\alpha,$$

such that $A_i \cap A_j = \emptyset$ for $i \neq j$.

PROOF. [8], §30, V, Theorem 1 (p. 347).

PROPOSITION 2.8. *If $Q_i \in \mathcal{Q}_\alpha$ ($0 < \alpha < \omega_1$, $i \in I$, $I = \{1, \dots, n\}$ or $I = \mathbf{N}$), then there exist sets $Q_i' \in \mathcal{Q}_\alpha$ such that $Q_i' \subset Q_i$ for $i \in I$, $Q_i' \cap Q_j' = \emptyset$ for $i \neq j$, and*

$$\bigcup_{i \in I} Q_i' = \bigcup_{i \in I} Q_i.$$

If $X = \bigcup_{i \in I} Q_i$ then $Q_i' \in \mathcal{A}_\alpha$.

PROOF. [8], §30, VII, Theorem 1 (p. 350).

PROPOSITION 2.9. *If $A \in \mathcal{A}_{\alpha+1}$ ($0 < \alpha < \omega_1$), then there exist sets $A_i \in \mathcal{A}_{\alpha_i}$ such that $A = \operatorname{Lim} A_i$ and $\alpha_i \leqq \alpha$; if α is a limit ordinal, then we can assume $\alpha_i < \alpha$.*

PROOF. [8], §30, IX, Theorems 1 and 2 (pp. 355 to 357).

3. Baire classes. We begin by a series of easy (and mostly well-known) facts.

PROPOSITION 3.1. *If Φ is a congruence lattice, then $\mathcal{S}(\Phi)$ coincides with the system of all sets $X(f \geqq 0)$ ($f \in \Phi$) and also with the system of all sets $X(f = 0)$ ($f \in \Phi$).*

PROOF.
$$X(f \geqq c) = X(f - c \geqq 0),$$
$$X(f \leqq c) = X(-f + c \geqq 0),$$
$$X(f = 0) = X\big(\max(f, -f) \leqq 0\big),$$
$$X(f \geqq 0) = X\big(\min(f, 0) = 0\big).$$

PROPOSITION 3.2. *If Φ is a congruence lattice, then $\mathcal{S}(\Phi)$ is a perfect lattice and $\emptyset, X \in \mathcal{S}(\Phi)$; if moreover Φ is complete, then $\mathcal{S}(\Phi)$ is a δ-lattice.*

PROOF.
$$\emptyset = X(-1 \geqq 0), \quad X = X(0 \geqq 0),$$
$$X(f \geqq 0) \cap X(g \geqq 0) = X\big(\min(f, g) \geqq 0\big),$$
$$X(f \geqq 0) \cup X(g \geqq 0) = X\big(\max(f, g) \geqq 0\big),$$
$$X(f \geqq 0) = \bigcap_1^\infty X\left(f > -\frac{1}{n}\right).$$

If $P_n = X(f_n = 0)$, $f_n \in \Phi$, then we can assume $0 \leqq f_n \leqq 2^{-n}$; for the functions

$$g_n = \max(f_1, \dots, f_n)$$

we have $g_n \to g$ uniformly, thus $g \in \Phi$, and

$$\bigcap_1^\infty P_n = X(g = 0).$$

PROPOSITION 3.3. *If $\mathcal{S}$ is a δ-lattice, $\emptyset$, $X \in \mathcal{S}$, then $\Phi(\mathcal{S})$ is a complete ordinary class.*

PROOF. [1], 5.6.4 (p. 169), or [6], §41, III (p. 236).

PROPOSITION 3.4. *If Φ is a complete ordinary class and $\mathcal{S} = \mathcal{S}(\Phi)$, then $\Phi = \Phi(\mathcal{S})$.*

PROOF. [1], 5.6.5.1 (p. 171), or [6], §41, VIII (p. 241).

PROPOSITION 3.5. *If Φ is a congruence lattice, $\mathcal{S} = \mathcal{S}(\Phi)$, then $\mathcal{S}(\Phi^v) = \mathcal{S}^\delta$, $\Phi^v_d = \Phi(\mathcal{S}^\delta)$.*

PROOF. $\mathcal{S}^\delta$ is a δ-lattice since $\mathcal{S}$ is a lattice by 3.2, and $\emptyset$, $X \in \mathcal{S}^\delta$. Hence by 3.3 $\Phi(\mathcal{S}^\delta)$ is a complete ordinary class and clearly $\Phi \subset \Phi(\mathcal{S}^\delta)$. Therefore $\Phi^v \subset \Phi(\mathcal{S}^\delta)$, $\mathcal{S}(\Phi^v) \subset \mathcal{S}^\delta$. On the other hand, $\mathcal{S}(\Phi^v)$ is a δ-system by 3.2, it contains $\mathcal{S}$, therefore $\mathcal{S}^\delta \subset \mathcal{S}(\Phi^v)$. The equality $\mathcal{S}^\delta = \mathcal{S}(\Phi^v)$ implies $\Phi^v = \Phi(\mathcal{S}^\delta)$ by 3.4.

PROPOSITION 3.6. *If Φ is a lattice, a translation lattice, a congruence lattice, a weakly affine lattice, an affine lattice, a subtractive lattice, or an ordinary class, then the same holds for Φ^p, Φ^d and Φ^e.*

PROOF. For Φ^p and Φ^d everything is obvious except the condition concerning $\frac{1}{f}$ in the case of an ordinary class. The latter is obtained by considering first an $f > 0$, $f_n \to f$, and putting $g_n = \max\left(f_n, \frac{1}{n}\right)$ so that $g_n > 0$, $g_n \to f$, then using the representation $\frac{1}{f} = f.\ \frac{1}{f^2}$ if $f(x) \neq 0$ for $x \in X$. The nontrivial part of the case of Φ^e is contained in [4], Proposition 3.

PROPOSITION 3.7. *If Φ is a translation lattice, then Φ^p is complete.*

PROOF. Let Φ be a translation lattice. We establish first the following lemma: If $f_n \in \Phi^p$, $f_n \to f$ uniformly and $f_1 \geqq f_2 \geqq f_3 \geqq \ldots$, then $f \in \Phi^p$.
In fact, we can assume

$$f_n \leqq f + \frac{1}{n} \qquad (n \in \mathbf{N}) \tag{1}$$

and

$$f_n = \lim_{k \to \infty} g_{nk} \qquad (g_{nk} \in \Phi)\,.$$

If we replace g_{nk} by $\min(g_{1k}, \ldots, g_{nk})$, we can also assume

$$g_{1k} \geqq g_{2k} \geqq g_{3k} \geqq \ldots \qquad (k \in \mathbf{N})\,. \tag{2}$$

Set

$$g_n = \max\left(g_{1n} - 1,\ g_{2n} - \frac{1}{2},\ \ldots,\ g_{nn} - \frac{1}{n}\right).$$

Then clearly $g_n \in \Phi$ and we show $g_n \to f$.

For a fixed N consider $n > N$, $x \in X$. Then, for n large enough,

$$g_n(x) \geqq g_{Nn}(x) - \frac{1}{N} > f_N(x) - \frac{2}{N} \geqq f(x) - \frac{2}{N}.$$

On the other hand, (2) implies

$$g_n \leqq \max\left(g_{1n} - 1, \dots, g_{Nn} - \frac{1}{N}, g_{Nn} - \frac{1}{N+1}, \dots, g_{Nn} - \frac{1}{n}\right) =$$

$$= \max\left(g_{1n} - 1, \dots, g_{N-1,n} - \frac{1}{N-1}, g_{Nn} - \frac{1}{n}\right),$$

and by denoting the right-hand side by h_n, we have, for $n \to \infty$,

$$h_n \to \max\left(f_1 - 1, \dots, f_{N-1} - \frac{1}{N-1}, f_N\right) = f_N$$

by (1). Hence, for $x \in X$,

$$g_n(x) \leqq h_n(x) < f(x) + \frac{2}{N}$$

for sufficiently large n. Therefore $g_n \to f$ indeed and $f \in \Phi^p$.

A similar argument shows that $f_n \in \Phi^p$, $f_n \to f$ uniformly, $f_1 \leqq f_2 \leqq \dots$ imply $f \in \Phi^p$.

Let us now suppose $f_n \in \Phi^p$, $f_n \to f$ uniformly. We can assume $|f_n - f| < \frac{1}{n}$. Then

$$g_n = \sup\{f_i : i \geqq n\}$$

implies

$$g_1 \geqq g_2 \geqq g_3 \geqq \dots, \qquad |g_n - f| \leqq \frac{1}{n}$$

so that the lemma furnishes $f \in \Phi^p$ as soon as $g_n \in \Phi^p$ is established. Howevei, for n being fixed, we have

$$h_k \to g_n \quad \text{if} \quad h_k = \max(f_n, f_{n+1}, \dots, f_{n+k}),$$

and here

$$h_k \in \Phi^p, \quad h_1 \leqq h_2 \leqq \dots, \quad g_n \geqq h_k \geqq g_n - \frac{2}{n+k}.$$

In order to prove the latter inequality, consider on the one hand that

$$f - \frac{1}{n+k} \leqq f_{n+k} \leqq h_k,$$

and on the other hand that

$$f_m \leqq f + \frac{1}{n+k} \qquad \text{for} \quad m \geqq n+k\,,$$

whence

$$g_n \leqq \max\left(f_n, f_{n+1}, \ldots, f_{n+k}, f + \frac{1}{n+k}\right) =$$

$$= \max\left(h_k, f + \frac{1}{n+k}\right) \leqq h_k + \frac{2}{n+k}\,.$$

Thus the second version of the lemma furnishes $g_n \in \Phi^p$, and the statement obtains.

REMARK 3.8. If Φ is a complete lattice, then Φ^p need not be complete.

For this purpose, consider first the functions

$$t_i(x) = \begin{cases} x + \left(1 - \dfrac{1}{i}\right) & \text{for } x \leqq 0\,, \\ \dfrac{1}{i}x + \left(1 - \dfrac{1}{i}\right) & \text{for } 0 \leqq x \leqq 1\,, \\ x & \text{for } x \geqq 1\,. \end{cases}$$

t_i is strictly increasing, continuous and $t_i(x) \geqq x$ for $x \in \mathbf{R}$, $i \in \mathbf{N}$.

Now let Φ consist of those functions $f\colon [0, 1] \to \mathbf{R}$ for which there is a $g \in C([0, 1])$ such that $f(x) = g(x)$ if x is irrational, $f(x) \leqq t_i(g(x))$ if $x = r_i$, where (r_i) is a sequence composed of all rational numbers in $[0, 1]$.

We show that Φ is a complete lattice. In fact, $C([0, 1]) \subset \Phi$ can be easily seen, hence Φ contains the constants in particular. Moreover, if $f_1, f_2 \in \Phi$ and $g_1, g_2 \in C([0, 1])$ are two functions corresponding to them, then

$$\max\bigl(f_1(x), f_2(x)\bigr) = \max\bigl(g_1(x), g_2(x)\bigr)$$

if x is irrational, and

$$\max\bigl(f_1(x), f_2(x)\bigr) \leqq t_i\bigl(\max\bigl(g_1(x), g_2(x)\bigr)\bigr)$$

if $x = r_i$ so that $\max(f_1, f_2) \in \Phi$. Similarly $\min(f_1, f_2) \in \Phi$. Finally if $f_n \to f$ uniformly, $f_n \in \Phi$, $g_n \in C([0, 1])$ corresponds to f_n, then $g_n \to g \in C([0, 1])$ uniformly and $f(x) = g(x)$ if x is irrational, while $f(r_i) \leqq t_i(g(r_i))$ is easily obtained by the continuity of t_i.

Consider now the functions

$$h_n(x) = \begin{cases} 1 & \text{if} \quad x = r_1, \ldots, r_n\,, \\ 0 & \text{otherwise.} \end{cases}$$

h_n belongs to the first Baire class, hence $h_n = \lim_{k\to\infty} g_{nk}$, $g_{nk} \in C([0, 1])$. Therefore the functions

$$f_{nk}(x) = \begin{cases} g_{nk}(x) & \text{if } x \text{ is irrational}, \\ t_i(g_{nk}(x)) & \text{if } x = r_i \end{cases}$$

belong to Φ and

$$s_n = \lim_{k\to\infty} f_{nk} \in \Phi^p$$

where obviously

$$s_n(x) = \begin{cases} 1 & \text{if } x = r_1, \ldots, r_n, \\ 1 - \dfrac{1}{i} & \text{if } x = r_i, i > n, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Hence $| s_n(x) - D(x) | < \dfrac{1}{n}$ where

$$D(x) = \begin{cases} 0 & \text{if } x \text{ is irrational}, \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

is the Dirichlet function.

Assume $D \in \Phi^p$, i.e. $f_n \to D, f_n \in \Phi$. Then there exists $g_n \in C([0, 1])$ corresponding to f_n, and $g_n(x) \to 0$ if x is irrational. However, there must be an i such that $\underline{\lim}\, g_n(r_i) < 1$ since otherwise the functions $\min(g_n, 1) \in C([0, 1])$ would converge to D which is impossible. Hence, by passing to a suitable subsequence, we can suppose $g_n(r_i) \to c < 1$ so that $1 = \lim f_n(r_i) \leqq \lim t_i(g_n(r_i)) = t_i(c) < 1$. This contradiction shows $D \notin \Phi^p$.

PROPOSITION 3.9. *Let Φ be a weakly affine lattice, $\mathfrak{S} = \mathfrak{S}(\Phi)$. Then Φ^p is a complete ordinary class and*

$$\mathfrak{S}(\Phi^p) = \mathfrak{S}^{c\sigma\delta}, \quad \Phi^p = \Phi(\mathfrak{S}^{c\sigma\delta}).$$

PROOF. $f_n \in \Phi$, $f_n \to f$ implies

$$X(f \geqq c) = \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} X\left(f_n > c - \frac{1}{i}\right) \in \mathfrak{S}^{c\sigma\delta}.$$

Similarly $X(f \leqq c) \in \mathfrak{S}^{c\sigma\delta}$ and $\mathfrak{S}(\Phi^p) \subset \mathfrak{S}^{c\sigma\delta}$.

In order to prove the opposite inclusion, consider first a set

$$Q = \bigcup_{1}^{\infty} Q_i, \quad Q_i \in \mathfrak{S}^c.$$

We show $k_Q \in \Phi^p$ for the characteristic function of Q. In fact, if $Q_i = X(f_i > 0)$, $f_i \in \Phi$, then we can assume $f_i \geqq 0$ and therefore $g_n \to k_Q$ for

$$g_n = \min(1, c_n \max(f_1, \ldots, f_n))$$

where $c_n \in C$, $c_n \to +\infty$ and C is the set of positive numbers figuring in the definition of a weakly affine lattice.

Hence $Q \in \mathfrak{S}(\Phi^p)$, i.e. $\mathfrak{S}^{c\sigma} \subset \mathfrak{S}(\Phi^p)$. By 3.2 and 3.7 $\mathfrak{S}(\Phi^p)$ is a δ-lattice so that $\mathfrak{S}^{c\sigma\delta} \subset \mathfrak{S}(\Phi^p)$ and $\mathfrak{S}^{c\sigma\delta} = \mathfrak{S}(\Phi^p)$. By 3.4 the remaining part of the statement will be proved if we show that Φ^p is a complete ordinary class.

For this purpose, let us denote by Φ_1 the class composed of all bounded functions contained in Φ^u. Clearly Φ_1 is a complete weakly affine lattice so that [2], Theorem 1 furnishes

$$f+g \in \Phi_1, \quad fg \in \Phi_1 \quad \text{for} \quad f, g \in \Phi_1,$$

$$\frac{1}{f} \in \Phi_1 \quad \text{for} \quad f \in \Phi_1, \quad |f| \geqq \delta > 0.$$

If $f, g \in \Phi^p$, then there exist $f_n, g_n \in \Phi$ such that $f_n \to f$, $g_n \to g$. Hence $[f_n]_{-n}^n \in \Phi_1$, $[g_n]_{-n}^n \in \Phi_1$; consequently

$$[f_n]_{-n}^n + [g_n]_{-n}^n \in \Phi_1, \quad [f_n]_{-n}^n \cdot [g_n]_{-n}^n \in \Phi_1,$$

therefore there exist $p_n, q_n \in \Phi$ satisfying

$$|[f_n]_{-n}^n + [g_n]_{-n}^n - p_n| < \frac{1}{n}, \quad |[f_n]_{-n}^n \cdot [g_n]_{-n}^n - q_n| < \frac{1}{n}.$$

Then $p_n \to f+g$, $q_n \to fg$ and $f+g, fg \in \Phi^p$.

Similarly $h \in \Phi^p$, $h > 0$ implies the existence of $h_n \in \Phi$ with $h_n \to h$; then $[h_n]_{1/n}^n \in \Phi_1$, hence

$$\frac{1}{[h_n]_{1/n}^n} \in \Phi_1$$

and there is $s_n \in \Phi$ satisfying

$$\left| \frac{1}{[h_n]_{1/n}^n} - s_n \right| < \frac{1}{n},$$

whence $s_n \to \frac{1}{h}$, $\frac{1}{h} \in \Phi^p$. Finally $h \in \Phi^p$, $h(x) \neq 0$ for $x \in X$ implies $\frac{1}{h} = h \cdot \frac{1}{h^2} \in \Phi^p$, and the weakly affine lattice Φ^p is an ordinary class indeed.

REMARK 3.10. Let $X = [0, 1]$ and Φ consist of those functions f for which

$$|f(x) - f(y)| \leqq |x - y| \qquad (x, y \in [0, 1]).$$

Φ is clearly a complete congruence lattice, moreover $\Phi^p = \Phi$. Now Φ^p is not an ordinary class since $f(x) = x$ implies $f \in \Phi^p$, $2f \notin \Phi^p$. Also $(0, 1] = X(f > 0) \in \mathfrak{S}(\Phi)^c$, but $(0, 1] \notin \mathfrak{S}(\Phi^p)$ since the latter system is composed of closed sets only. Hence $\mathfrak{S}(\Phi^p) \neq \mathfrak{S}(\Phi)^{c\sigma\delta}$ in this case.

3.11. Now we define the ordinary, discrete, and equal Baire classes.

Let Φ be a complete ordinary class. Put

$$\Phi_0^* = \Phi,$$

$$\Phi_\alpha^* = \Big(\bigcup_{\xi<\alpha} \Phi_\xi^*\Big)^p \quad \text{for} \quad 0 < \alpha < \omega_1,$$

then

$$\Phi_0^{(d)} = \Phi,$$

$$\Phi_\alpha^{(d)} = \Big(\bigcup_{\xi<\alpha} \Phi_\xi^{(d)}\Big)^d \quad \text{for} \quad 0 < \alpha < \omega_1,$$

finally

$$\Phi_0^{(e)} = \Phi,$$

$$\Phi_\alpha^{(e)} = \Big(\bigcup_{\xi<\alpha} \Phi_\xi^{(e)}\Big)^e \quad \text{for} \quad 0 < \alpha < \omega_1.$$

Besides the classes Φ_α^* we introduce another, slightly modified construction too:

$$\Phi_0 = \Phi,$$

$$\Phi_{\alpha+1} = \Phi_\alpha^p \qquad (0 \leqq \alpha < \omega_1),$$

$$\Phi_\alpha = \Big(\bigcup_{\xi<\alpha} \Phi_\xi\Big)^v \text{ for a limit ordinal } \alpha < \omega_1.$$

PROPOSITION 3.12. *If Φ is a complete ordinary class, then*

(a) $\Phi_\alpha^* \subset \Phi_\beta^*$, $\Phi_\alpha^{(d)} \subset \Phi_\beta^{(d)}$, $\Phi_\alpha^{(e)} \subset \Phi_\beta^{(e)}$, $\Phi_\alpha \subset \Phi_\beta$ *for* $0 \leqq \alpha < \beta < \omega_1$;

(b) $\Phi_\alpha^{(d)} \subset \Phi_\alpha^{(e)} \subset \Phi_\alpha^*$ *for* $0 \leqq \alpha < \omega_1$;

(c) Φ_α^*, $\Phi_\alpha^{(d)}$, $\Phi_\alpha^{(e)}$ *are ordinary classes for* $0 < \alpha < \omega_1$.

PROOF. The first three inclusions in (a) are obvious, the fourth one, (b) and (c) are obtained by an easy transfinite induction (cf. 3.6).

PROPOSITION 3.13. *The smallest class containing the complete ordinary class Φ and closed with respect to pointwise, discrete, equal convergence is*

$$\Phi^* = \bigcup_{\alpha<\omega_1} \Phi_\alpha^*, \quad \Phi^{(d)} = \bigcup_{\alpha<\omega_1} \Phi_\alpha^{(d)}, \quad \Phi^{(e)} = \bigcup_{\alpha<\omega_1} \Phi_\alpha^{(e)}$$

respectively.

PROOF. If $f_n \to f$, $f_n \in \Phi_{\alpha_n}^*$, $\alpha_n \leqq \alpha < \omega_1$, then $f \in \Phi_{\alpha+1}^*$, and a similar argument holds for the case of discrete or equal convergence.

PROPOSITION 3.14. *Let Φ be a complete ordinary class, $\mathcal{S} = \mathcal{S}(\Phi)$. Then Φ_α is a complete ordinary class for $0 \leqq \alpha < \omega_1$, and $\mathcal{S}_\alpha = \mathcal{S}(\Phi_\alpha)$, $\Phi_\alpha = \Phi(\mathcal{S}_\alpha)$.*

PROOF. By transfinite induction based on 3.6, 3.7, 3.2, 2.1, 2.2, 2.3, 3.9, 3.5, 3.4; cf. [1], 5.6.6.2, p. 171.

PROPOSITION 3.15. *If Φ is a complete ordinary class, then*

(a) $\Phi_\alpha^* = \Phi_\alpha$ *for* $0 \leqq \alpha < \omega$,

(b) $\Phi_\alpha^* = \Phi_{\alpha+1}$ *for* $\omega \leqq \alpha < \omega_1$.

PROOF. By transfinite induction based on 3.14, 2.1, 3.9; cf. [4], (2.1).

4. Characterization of discrete Baire classes. Let Φ be a complete ordinary class, $\mathcal{S} = \mathcal{S}(\Phi)$, and consider the Borel classes $\mathcal{S}_\alpha$, $\mathcal{Q}_\alpha$, $\mathcal{A}_\alpha$ belonging to $\mathcal{S}$; they can be constructed since by 3.2 $\mathcal{S}$ is a perfect δ-lattice containing $\emptyset$ and X.

LEMMA 4.1. *If $f \in \Phi_\alpha^{(d)}$ $(0 \leqq \alpha < \omega_1)$, then there exist a cover $X = \bigcup_1^\infty A_i$ and functions $g_i \in \Phi$ such that $A_i \in \mathcal{Q}_\alpha$ if $0 \leqq \alpha < \omega$, $A_i \in \mathcal{Q}_{\alpha+1}$ if $\omega \leqq \alpha < \omega_1$, and $f \mid A_i = g_i \mid A_i$ for $i \in \mathbf{N}$.*

PROOF. The statement is obvious for $\alpha = 0$. We shall proceed by transfinite induction. Assume first $0 \leqq \alpha < \omega$ and suppose that the statement is valid for α. If now $f \in \Phi_{\alpha+1}^{(d)}$, then we have a sequence (f_n) converging discretely to f and satisfying $f_n \in \Phi_\alpha^{(d)}$ for $n \in \mathbf{N}$. Hence by hypothesis

$$X = \bigcup_{i=1}^\infty A_{ni}, \quad f_n \mid A_{ni} = g_{ni} \mid A_{ni}, \quad g_{ni} \in \Phi \tag{1}$$

and $A_{ni} \in \mathcal{Q}_\alpha$. Define

$$B_n = \bigcap_{k=n}^\infty X(f_k = f_{k+1}). \tag{2}$$

Then clearly $f \mid B_n = f_n \mid B_n$ and $X = \bigcup_1^\infty B_n$, consequently

$$X = \bigcup_{n=1}^\infty \bigcup_{i=1}^\infty (B_n \cap A_{ni}), \; f \mid B_n \cap A_{ni} = g_{ni} \mid B_n \cap A_{ni} \tag{3}$$

and $B_n \in \mathcal{S}_\alpha$ owing to the fact that $X(f_k = f_{k+1}) \in \mathcal{S}_\alpha$ by 3.12(c), 3.12(b), 3.15(a), 3.14, 2.3 and $\mathcal{S}_\alpha$ is a δ-system also for $\alpha = 0$ by 3.2. Therefore $B_n \cap A_{ni} \in \mathcal{Q}_{\alpha+1}$ by 2.1 and 2.3.

For $\omega \leqq \alpha < \omega_1$ the same reasoning can be repeated with the only exception that now $A_{ni} \in \mathcal{Q}_{\alpha+1}$ by the induction hypothesis and 3.15(b) is valid instead of 3.15(a), thus $B_n \in \mathcal{S}_{\alpha+1}$ and $B_n \cap A_{ni} \in \mathcal{Q}_{\alpha+2}$.

Let now $\omega \leqq \alpha < \omega_1$ be a limit ordinal. Then $f \in \Phi_\alpha^{(d)}$ implies that f is the discrete limit of (f_n) with $f_n \in \Phi_{\alpha_n}^{(d)}$, $\alpha_n < \alpha$ and (1), (2), (3) are again valid with $A_{ni} \in \mathcal{Q}_{\alpha_n+1} \subset \mathcal{Q}_\alpha$ by the induction hypothesis and 2.1, further $B_n \in \mathcal{S}_\alpha$ since $f_k - f_{k+1} \in \Phi_{\beta_k}^{(d)} \subset \Phi_{\beta_k}^* \subset \Phi_\alpha$ for $\beta_k = \max(\alpha_k, \alpha_{k+1})$ (cf. 3.12, 3.15) and by 3.14, 3.2, 2.3. Finally $B_n \cap A_{ni} \in \mathcal{Q}_{\alpha+1}$ by 2.1 and 2.3, according to the statement.

LEMMA 4.2. *If $A \in \mathcal{A}_\alpha$ where either $0 \leqq \alpha < \omega$ or $\omega < \alpha = \beta + 1 < \omega_1$, then the characteristic function k_A of A belongs to the class $\Phi_\alpha^{(d)}$ if $\alpha < \omega$ and to $\Phi_\beta^{(d)}$ if $\alpha = \beta + 1 > \omega$.*

REMARK. This is a sharper form of [4], Lemma 6.

PROOF. We proceed by transfinite induction. The statement is true for $\alpha = 0$ by 3.4. If $\alpha = 1$, then

$$A = \bigcup_1^\infty P_n, \quad X - A = \bigcup_1^\infty P'_n$$

where $P_n, P'_n \in \mathcal{S}_0$ and by 3.2 we can assume

$$P_n \subset P_{n+1}, \quad P'_n \subset P'_{n+1} \qquad (n \in \mathbf{N}).$$

By 3.1 there exist functions $g_n, h_n \in \Phi$ such that

$$P_n = X(g_n = 0), \quad P'_n = X(h_n = 0),$$

and $g_n \geqq 0$, $h_n \geqq 0$. Define

$$f_n = \frac{h_n}{g_n + h_n} \in \Phi,$$

then $f_n \to k_A$ discretely so that $k_A \in \Phi_1^{(d)}$.

Assume now $\alpha = \beta + 1$, $0 < \beta < \omega_1$ and suppose that the statement is valid for all non-limit ordinals less than α. By 2.9 and 2.1

$$A = \operatorname{Lim} A_n$$

where $A_n \in \mathcal{A}_\beta$ and, if β is a limit ordinal, then even $A_n \in \mathcal{A}_{\beta_n+1}$, $\beta_n < \beta$. By hypothesis

$$\begin{aligned} &k_{A_n} \in \Phi_\beta^{(d)} && \text{if } \ 0 < \beta < \omega, \\ &k_{A_n} \in \Phi_\gamma^{(d)} && \text{if } \ \omega < \beta = \gamma + 1 < \omega_1, \\ &k_{A_n} \in \Phi_{\beta_n+1}^{(d)} && \text{if } \ \omega \leqq \beta < \omega_1 \text{ is a limit ordinal.} \end{aligned}$$

Since $k_{A_n} \to k_A$ discretely, we obtain

$$\begin{aligned} &k_A \in \Phi_{\beta+1}^{(d)} = \Phi_\alpha^{(d)} && \text{if } \ 1 < \alpha = \beta + 1 < \omega, \\ &k_A \in \Phi_{\gamma+1}^{(d)} = \Phi_\beta^{(d)} && \text{if } \ \omega < \beta = \gamma + 1 < \alpha = \beta + 1 < \omega_1, \\ &k_A \in \Phi_\beta^{(d)} && \text{if } \ \omega \leqq \beta < \alpha = \beta + 1 < \omega_1 \text{ and } \beta \text{ is a limit ordinal.} \end{aligned}$$

LEMMA 4.3. *If*

$$X = \bigcup_1^\infty A_i, \quad f \mid A_i = g_i \mid A_i, \qquad g_i \in \Phi \qquad (i \in \mathbf{N})$$

and either $A_i \in \mathfrak{Q}_\alpha$, $0 \leqq \alpha < \omega$ *or* $A_i \in \mathfrak{Q}_{\alpha+1}$, $\omega \leqq \alpha < \omega_1$, *then* $f \in \Phi_\alpha^{(d)}$.

PROOF. If $\alpha = 0$ then $X(f > c) \in \mathfrak{Q}_0$, $X(f < c) \in \mathfrak{Q}_0$ by 3.2, hence $f \in \Phi$ by 3.4.

If $\alpha = 1$ then we can assume by 2.8 that $A_i \in \mathcal{A}_1$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. Hence

$$A_i = \bigcup_{n=1}^{\infty} P_{in}, \quad P_{in} \in \mathcal{G}_0, \quad P_{in} \subset P_{i,\, n+1}$$

by 3.2. Similarly as in the proof of 4.2 we can construct functions $f_{in} \in \Phi$ such that

$$f_{in}(x) = 1 \quad \text{for} \quad x \in P_{in},$$
$$f_{in}(x) = 0 \quad \text{for} \quad x \in \bigcup_{\substack{j \leq n \\ j \neq i}} P_{jn}.$$

Define

$$f_n = \sum_{i=1}^{n} g_i f_{in} \in \Phi .$$

Then $f_n \to f$ discretely because, for $x \in X$, there is an i such that $x \in A_i$ and then there is an $n_0(x) \geqq i$ such that $x \in P_{in}$ for $n \geqq n_0(x)$ from which $f_n(x) = g_i(x) = f(x)$ for $n \geqq n_0(x)$.

Assume now $1 < \alpha = \beta + 1 < \omega$. Using 2.8 again we can suppose $A_i \in \mathcal{A}_\alpha$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. By 2.9

(1) $$A_i = \operatorname*{Lim}_{n \to \infty} B_{in}$$

where $B_{in} \in \mathcal{A}_\beta$. Define

(2) $$C_{in} = B_{in} - \bigcup_{j=1}^{i-1} B_{jn}$$

and

(3) $$f_n = \sum_{i=1}^{n} g_i k_{C_{in}} .$$

Then $f_n \to f$ discretely. In fact, if $x \in X$, then $x \in A_i$ for some i and $x \in B_{in}$ for n large enough by (1); since $x \notin A_j$ for $j \leqq i-1$, we have also $x \notin B_{jn}$ for $j \leqq i-1$ and sufficiently large n. Hence there is an $n_0(x)$ such that $x \in C_{in}$ for $n \geqq n_0(x)$ and $x \notin C_{jn}$ for the same n and $j \neq i$ because the sets C_{jn} are disjoint for n fixed so that $f_n(x) = g_i(x) = f(x)$ $(n \geqq n_0(x))$.

Now $C_{in} \in \mathcal{A}_\beta$ by 2.4, we have $k_{C_{in}} \in \Phi_\beta^{(d)}$ by 4.2 from which $f_n \in \Phi_\beta^{(d)}$ by 3.6 and $f \in \Phi_\alpha^{(d)}$.

Assume finally $\omega \leqq \alpha < \omega_1$. Then the above reasoning is valid again with the following modifications. The sets A_i belong to $\mathcal{A}_{\alpha+1}$, hence $B_{in} \in \mathcal{A}_\alpha$ or even $B_{in} \in \mathcal{A}_{\alpha_{in}}$, $\alpha_{in} < \alpha$ if α is a limit ordinal; consequently $C_{in} \in \mathcal{A}_\alpha$ or $C_{in} \in \mathcal{A}_{\beta_{in}}$, $\beta_{in} < \alpha$ respectively, thus by 4.2 $k_{C_{in}} \in \Phi_\beta^{(d)}$ for $\omega < \alpha = \beta + 1 < \omega_1$, $k_{C_{in}} \in \Phi_{\beta_{in}}^{(d)}$ for a limit ordinal α, so that $f_n \in \Phi_\beta^{(d)}$ or $f_n \in \Phi_{\gamma_n}^{(d)}$, $\gamma_n < \alpha$ respectively, and in any case $f \in \Phi_\alpha^{(d)}$.

THEOREM 4.4. *$f \in \Phi_\alpha^{(d)}$ if and only if there are a cover $X = \bigcup_1^\infty A_i$ and functions $g_i \in \Phi$ such that $f \mid A_i = g_i \mid A_i$ and*

(1) $$A_i \in \mathfrak{Q}_\alpha \quad \text{for} \quad 0 \leqq \alpha < \omega,$$

(2) $$A_i \in \mathfrak{Q}_{\alpha+1} \quad \text{for} \quad \omega \leqq \alpha < \omega_1.$$

PROOF. 4.1, 4.3.

REMARK 4.5. If $\alpha > 0$, then 4.4(1) and (2) can be replaced, on account of $\mathfrak{Q}_{\alpha+1} = \mathfrak{F}_\alpha^\sigma$, by

(1) $$A_i \in \mathfrak{F}_{\alpha-1} \quad \text{for} \quad 0 < \alpha < \omega,$$

(2) $$A_i \in \mathfrak{F}_\alpha \quad \text{for} \quad \omega \leqq \alpha < \omega_1.$$

This can be found in the special case of $\Phi = C(X)$ in a normal topological space and $\alpha = 1$ in [5], Lemma 3.

COROLLARY 4.6. *$f \in \Phi^{(d)}$ if and only if $f \in \Phi^*$ and there exist a cover $X = \bigcup_1^\infty A_i$ and functions $g_i \in \Phi$ such that $f \mid A_i = g_i \mid A_i$.*

PROOF. The condition is necessary by 3.12 and 4.1. It is sufficient because $f \in \Phi_\alpha^* \subset \Phi_{\alpha+1}$ implies $B_i = X(f = g_i) \in \mathfrak{F}_{\alpha+1} \subset \mathfrak{Q}_{\alpha+2}$ by 3.15, 3.6, 3.14, 2.1, hence $X = \bigcup_1^\infty B_i$ by $A_i \subset B_i$ and $f \mid B_i = g_i \mid B_i$ so that $f \in \Phi_{\alpha+2}^{(d)}$ by 4.3. (See [4], Theorem 7.)

COROLLARY 4.7. *If $f \in \Phi_\alpha \cap \Phi^{(d)}$ then*

(1) $$f \in \Phi_{\alpha+1}^{(d)} \quad \text{for} \quad 0 < \alpha < \omega,$$

(2) $$f \in \Phi_\alpha^{(d)} \quad \text{for} \quad \omega \leqq \alpha < \omega_1.$$

PROOF. We can introduce the same sets B_i as in the proof of 4.6, but now $B_i \in \mathfrak{F}_\alpha$ and 4.5 can be used.

REMARK 4.8. In 4.7(1), $\alpha + 1$ cannot be replaced by α in general. In fact, let $X = \mathbf{R}$, $\Phi = C(\mathbf{R})$ and consider the well-known function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \dfrac{1}{q} & \text{if } x = \dfrac{p}{q},\ p, q \text{ integers, } (p, q) = 1, \quad q > 0. \end{cases}$$

Now clearly $f \in \Phi_1$ but $f \notin \Phi_1^{(d)}$ by [4], Theorem 13.

COROLLARY 4.9. *$f \in \Phi^{(d)}$ if and only if there exist a cover $X = \bigcup_1^\infty A_i$ and functions $g_i \in \Phi$ such that $A_i \in \mathfrak{S}(\mathfrak{F})$ and $f \mid A_i = g_i \mid A_i$ for each i.*

PROOF. The necessity follows from 4.4 and 2.5. The sufficiency is obtained by observing that $A_i \in \mathfrak{F}_{\alpha_i}$ implies $A_i \in \mathfrak{Q}_{\alpha_i+1}$ by 2.1 and $A_i \in \mathfrak{Q}_{\beta_i}$ $(i \in \mathbf{N})$ implies $A_i \in \mathfrak{Q}_\alpha$ for a suitable $\alpha < \omega_1$; then 4.4 can be applied again.

5. Equal convergence. We begin by an equivalent characterization of equal convergence.

THEOREM 5.1. *A sequence* (f_n) *converges equally to* f *if and only if there is a cover* $X = \bigcup_1^\infty A_i$ *such that* $f_n \mid A_i \to f \mid A_i$ *uniformly for every* $i \in \mathbf{N}$.

PROOF. Assume $\varepsilon_n > 0$, $\varepsilon_n \to 0$ and suppose that, for $x \in X$, there is an $n_0(x)$ such that

$$| f_n(x) - f(x) | \leqq \varepsilon_n \quad \text{for} \quad n \geqq n_0(x).$$

Define

$$A_i = \{x \in X: | f_n(x) - f(x) | \leqq \varepsilon_n \quad \text{for} \quad n \geqq i\}.$$

These sets A_i clearly satisfy the condition in the statement.

Conversely suppose

$$X = \bigcup_1^\infty A_i, \quad | f_n(x) - f(x) | \leqq \varepsilon_{in} \quad \text{for} \quad x \in A_i \quad \text{and} \quad n \geqq m(i),$$

where $\varepsilon_{in} \to 0$ for i fixed and $n \to \infty$. Select integers n_k such that $0 < n_1 < n_2 < \dots$ and $\varepsilon_{in} < \frac{1}{k}$ for $i = 1, \dots, k$, $n \geqq n_k$. Define $\varepsilon_n = 1$ for $n < n_2$ and $\varepsilon_n = \frac{1}{k}$ for $n_k \leqq n < n_{k+1}$ $(k = 2, 3, \dots)$. Then $\varepsilon_n \to 0$ and

$$| f_n(x) - f(x) | \leqq \varepsilon_{in} < \varepsilon_n$$

for $x \in A_i$, $n \geqq \max(n_i, m(i))$.

REMARK 5.2. In view of 5.1, the terminology "σ-uniform convergence" could be applied instead of "equal convergence" adopted in [4].

REMARK 5.3. With the help of 5.1, one can easily find an alternative proof of 3.6 for the case of Φ^e, based on results concerning uniform convergence to bounded f or to f with $\frac{1}{f}$ bounded.

Let now Φ be a complete ordinary class as in section 4 and use the notation fixed there. Our next aim is to improve the obvious inclusion $\Phi_\alpha^{(e)} \subset \Phi_\alpha^*$. For this purpose we shall need some lemmas.

LEMMA 5.4. *Let* $\mathfrak{M}$ *be a system of sets and* $\Psi \subset \Phi(\mathfrak{M})$ *a subtractive lattice,* $A \subset X$. *If* (f_n) *is a sequence taken from* Ψ *and* $(f_n \mid A)$ *converges uniformly, then the same holds for* $(f_n \mid B)$ *where* $B \supset A$ *is a suitable set belonging to* $\mathfrak{M}^\delta$.

PROOF. There are a sequence (ε_n) such that $\varepsilon_n > 0$, $\varepsilon_n \to 0$ and an index n_0 such that $| f_n(x) - f_m(x) | \leqq \varepsilon_n$ for $x \in A$ and $n_0 \leqq n \leqq m$. The set

$$B = \bigcap_{n=n_0}^\infty \bigcap_{m=n}^\infty X(| f_n - f_m | \leqq \varepsilon_n)$$

will satisfy our requirements.

LEMMA 5.5. *Suppose* $f_n \in \Phi(\mathfrak{M})$ *and* $f_n \to f$ *uniformly. Then* $f \in \Phi(\mathfrak{M}^\delta)$.

PROOF. We can assume $|f_n - f| \leqq \frac{1}{n}$. Then

$$X(f \geqq c) = \bigcap_1^\infty X\left(f_n \geqq c - \frac{1}{n}\right), \qquad X(f \leqq c) = \bigcap_1^\infty X\left(f_n \leqq c + \frac{1}{n}\right).$$

LEMMA 5.6. *If* $\Psi \subset \Phi(\mathfrak{M})$ *is a subtractive lattice, then* $\Psi^e \subset \Phi(\mathfrak{M}^{\delta\sigma})$.

PROOF. Suppose $f_n \in \Psi$ and $f_n \to f$ equally. Then by 5.1 $X = \bigcup_1^\infty A_i$ and $f_n \mid A_i \to f \mid A_i$ uniformly for every i. By 5.4 we can assume that $A_i \in \mathfrak{M}^\delta$ for each i. Now by 5.5 (applied for A_i instead of X) we get from

$$X(f_n \geqq c) \cap A_i \in \mathfrak{M}^\delta, \qquad X(f_n \leqq c) \cap A_i \in \mathfrak{M}^\delta$$

the relations

$$X(f \geqq c) \cap A_i \in \mathfrak{M}^\delta \quad \text{and} \quad X(f \leqq c) \cap A_i \in \mathfrak{M}^\delta.$$

LEMMA 5.7. *We have for* $0 \leqq \alpha < \omega_1$

$$(\Phi_\alpha)^e \subset \Phi(\mathcal{A}_{\alpha+1}).$$

PROOF. By 3.14 and 5.6

$$(\Phi_\alpha)^e \subset \Phi(\mathcal{S}_\alpha^{\delta\sigma}) = \Phi(\mathcal{Q}_{\alpha+1})$$

in view of 3.2 and 2.3. On the other hand $(\Phi_\alpha)^e \subset (\Phi_\alpha)^p = \Phi_{\alpha+1}$ implies $(\Phi_\alpha)^e \subset \Phi(\mathcal{S}_{\alpha+1})$ by 3.14.

THEOREM 5.8. *We have*

(a) $\Phi_\alpha^{(e)} \subset \Phi(\mathcal{A}_\alpha)$ *for* $0 < \alpha < \omega$,

(b) $\Phi_\alpha^{(e)} \subset \Phi(\mathcal{A}_{\alpha+1})$ *for* $\omega \leqq \alpha < \omega_1$.

PROOF. If $\alpha = \beta + 1$, we have

$$\Phi_\alpha^{(e)} = (\Phi_\beta^{(e)})^e \subset (\Phi_\beta^*)^e$$

whence the statement is obtained by 3.15 and 5.7. If α is a limit ordinal, then

$$\Phi_\alpha^{(e)} = \Big(\bigcup_{\beta<\alpha} \Phi_\beta^{(e)}\Big)^e \subset \Big(\bigcup_{\beta<\alpha} \Phi_\beta^*\Big)^e \subset (\Phi_\alpha)^e$$

by 3.15, and 5.7 can be applied again.

REMARK 5.9. In 5.8 we cannot replace the inclusion by equality. E.g. the function f defined in the proof of [4], Corollary 9 belongs obviously to the class $\Phi(\mathcal{A}_1)$, nevertheless $f \notin \Phi_1^{(d)} = \Phi_1^{(e)}$ for $\Phi = C([0, 1])$ (cf. [4], Corollary 2).

THEOREM 5.10. *If* $0 < \alpha < \omega_1$, *then*

$$\Phi_\alpha^* \subset (\Phi_\alpha^{(d)})^e.$$

PROOF. In the same manner as in the proof of [4], Theorem 10 (which is a weaker form of the present statement), we define, for $f \in \Phi_\alpha^*$, $n \in \mathbf{N}$,

$$A = X(f < -n) \cup X(f > n),$$

$$A_i = X\left(\frac{i-1}{n} < f < \frac{i+1}{n}\right) \qquad (-n^2 \leqq i \leqq n^2),$$

and deduce from 3.15 and 3.14

$$A, A_i \in \mathfrak{Q}_\alpha \quad \text{if} \quad 0 < \alpha < \omega,$$

$$A, A_i \in \mathfrak{Q}_{\alpha+1} \quad \text{if} \quad \omega \leqq \alpha < \omega_1.$$

By 2.8 there are pairwise disjoint sets B, $B_i \in \mathcal{A}_\alpha$ $(0 < \alpha < \omega)$ or B, $B_i \in \mathcal{A}_{\alpha+1}$ $(\omega \leqq \leqq \alpha < \omega_1)$ such that

$$B \subset A, \quad B_i \subset A_i \qquad (-n^2 \leqq i \leqq n^2)$$

and

$$X = \bigcup_{-n^2}^{n^2} B_i \cup B.$$

By defining

$$f_n = \sum_{-n^2}^{n^2} \frac{i}{n} k_{B_i}$$

we have $f_n \to f$ equally and $f_n \in \Phi_\alpha^{(d)}$ because $k_{B_i} \in \Phi_\alpha^{(d)}$ by 4.2 and $\Phi_\alpha^{(d)}$ is an ordinary class by 3.6.

COROLLARY 5.11. *If* $0 \leqq \alpha < \omega_1$, *then*

$$\Phi_\alpha^* \subset \Phi_{\alpha+1}^{(e)} \subset \Phi_{\alpha+1}^*.$$

PROOF. 5.10, 3.12. (This is a sharper form of [4], Corollary 11.)

6. Complete ordinary classes. In [4], Theorem 16 (cf. also [3], Theorem 10), a series of equivalent characterizations of complete ordinary classes was presented, containing partly conditions apparently essentially stronger than those figuring in the definition, partly postulating somewhat less than the definition itself. In particular, it was proved there the following proposition due to J. R. ISBELL [7]: if Φ contains all constants, it is complete, $f, g \in \Phi$ implies $f+g \in \Phi$, $fg \in \Phi$, and $f \in \Phi$, $f(x) \neq 0$ for $x \in X$ implies $\frac{1}{f} \in \Phi$, then Φ is a lattice (and hence a complete ordinary class).

Now we shall prove another statement of a similar character:

THEOREM 6.1. *If Φ is a complete subtractive lattice such that $f \in \Phi, f > 0$ implies $\frac{1}{f} \in \Phi$, then Φ is an ordinary class.*

PROOF. Let Φ_1 denote the class of all bounded functions belonging to Φ. Clearly Φ_1 is a complete subtractive lattice too and then it can be deduced from [2], Theorem 1 that $f, g \in \Phi_1$ implies $fg \in \Phi_1$.

Consider now $f, g \in \Phi$ with $f \geqq 0$, $g \geqq 0$. Then

$$\frac{1}{1+f} \in \Phi_1, \quad \frac{1}{1+g} \in \Phi_1,$$

hence by the above result

$$\frac{1}{1+f} \cdot \frac{1}{1+g} \in \Phi,$$

so that $(1+f)(1+g) \in \Phi$ and

$$fg = (1+f)(1+g) - f - g - 1 \in \Phi.$$

Finally if $f, g \in \Phi$ are arbitrary, we can write

$$f = f_1 - f_2, \qquad g = g_1 - g_2, \qquad f_1, f_2, g_1, g_2 \in \Phi$$

and $f_1 \geqq 0, f_2 \geqq 0, g_1 \geqq 0, g_2 \geqq 0$. Therefore

$$fg = f_1 g_1 - f_2 g_1 - f_1 g_2 + g_1 g_2 \in \Phi.$$

If $f \in \Phi$, $f(x) \neq 0$ for $x \in X$, then $\frac{1}{f} = f \cdot \frac{1}{f^2}$ shows that $\frac{1}{f} \in \Phi$.

REMARK 6.2. The last sentence of the preceding proof shows that the hypotheses of the above quoted theorem of J. R. ISBELL can be slightly weakened by replacing "$f(x) \neq 0$ for $x \in X$" by "$f > 0$".

REMARK 6.3. Theorem 1 of [2] is valid not only for subtractive lattices but also for so-called semi-affine lattices and, in particular, for affine lattices. Therefore it is quite natural to ask whether 6.1 remains true if, in the hypothesis, "subtractive lattice" is replaced by "affine lattice". The following example answers this question by the negative.

Let $X = \mathbf{R}$ and let Φ consist of all continuous functions f for which the limits

$$\lim_{x \to +\infty} f(x) \qquad \text{and} \qquad \lim_{x \to -\infty} f(x)$$

exist (with a finite or infinite value). It is easily seen that Φ is a complete affine lattice. Moreover, if $f \in \Phi$ and $f(x) \neq 0$ for $x \in X$, then either $f > 0$ or $f < 0$ and then it is clear that the limits of $\frac{1}{f(x)}$ do exist for $x \to +\infty$ and $x \to -\infty$.

However, if $f(x) = x$, $g(x) = \sin x - x$ for $x \in X$, then $f, g \in \Phi$ but $f + g \notin \Phi$ so that Φ is not an ordinary class.

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SOME UNCONVENTIONAL PROBLEMS IN NUMBER THEORY

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Dedicated to the 80th birthday of my friend George Alexits

In the paper, we will mostly deal with arithmetic functions, primes, divisors, sieve processes and consecutive integers.

1. Let f be an arithmetic function. The integer n is called a *barrier* for f if

(1) $$m+f(m) \leq n$$

for every $m < n$.

Perhaps I should explain why I considered (1). In the early 1950's, van Wijngaarden told me the following conjecture. Put $\sigma_1(n) = \sigma(n)$, the sum of divisors of n, and $\sigma_k(n) = \sigma_1(\sigma_{k-1}(n))$. Is it true that there is essentially only one sequence $\sigma_k(n)$ $(k = 1, 2, 3, \ldots)$? In other words, if m and n are distinct integers, are there integers k and l such that $\sigma_k(m) = \sigma_l(n)$? Such a conjecture is usually hopeless to prove or disprove. Selfridge and others made some computer experiments and believe that the conjecture is false. I tried to find an airthmetic function for which an analogous conjecture is true and can be proved. Put $f_1(n) = n + \nu(n)$, where $\nu(n)$ is the number of distinct prime factors of n, and $f_k(n) = f_1(f_{k-1}(n))$. Is it true that for any two integers m and n there are integers k and l for which $f_k(m) = f_l(n)$?. This would follow immediately if we could prove that $\nu(n)$ has infinitely many barriers. This problem seems more interesting than my original question. It is easy to find with a pocket computer and a little patience (I do not have either of these) a large number of integers which are barriers for $\nu(n)$, but I am afraid that the question of the existence of infinitely many barriers is hopeless at present. I could not even prove that $\varepsilon\nu(n)$ has infinitely many barriers for some $\varepsilon > 0$. Sieve methods seem the right method of attack, but there are great technical difficulties which I could not overcome.

The following theorem gives a result of this type which can actually be proved.

THEOREM 1. *For* $n = \Pi p_i^{\alpha_i}$ *set* $d_0(n) = \Pi \alpha_i$. *Then* $d_0(n)$ *has infinitely many barriers, that is there are infinitely many* n *such that*

(2) $$m + d_0(m) \leq n \quad \text{for every } m < n.$$

In fact, the density of integers satisfying (2) *is positive.*

I will outline the simple (but slightly messy) proof of Theorem 1 at the end of the paper.

Let me now state a few other difficult problems. Let $\Omega(n)$ denote the total number of prime factors of n, that is $\Omega(n) = \Sigma\alpha_i$ when $n = \Pi p_i^{\alpha_i}$. Probably $\Omega(n)$ has infinitely marry barriers, but this is clearly hopeless at present, since a barrier n would have to satisfy $n-1=p$ and $n-2=2q$ for primes p and q and we are not likely to be able to prove the existence of infinitely many such n in the near future. Selfridge found that 99840 is the largest barrier for $\Omega(n)$ below 10^5. Selfridge and I then investigated whether $d(n)$, the number of divisors of n, has any barriers. Here one has to redefine the barrier a little bit: n is a barrier for $d(n)$ if

$$m + d(m) \leqq n + 2$$

for every $m < n$. This is satisfied by $n = 24$ and we convinced ourselves that if there is any other solution then it is enormously large, far beyond our tables and computers.

Define

$$H_f(n) = \max_{m<n} \big(m + f(m) - n\big).$$

It is quite possible that $H_d(n) \to \infty$ as $n \to \infty$, but these questions are clearly hopeless at the present "state of the art". On the other hand, it would not be very difficult to prove that, for almost all n, $H_\nu(n)/\log\log n(\log\log\log n)^{1/2} \to c(>0)$ as $n \to \infty$. (I have not carried out the details.) The strongest possible conjecture which has a chance of being true is as follows: for every $\varepsilon > 0$, there are infinitely many values of n so that

$$\text{(3)} \qquad \nu(n-k) < (1+\varepsilon)\log k/\log\log k \quad\text{and}\quad \Omega(n-k) < (1+\varepsilon)\log k/\log 2$$

for every k satisfying $k_0(\varepsilon) < k < n$. In may opinion, this has some chance of being true, but there is no chance at all of proving it in the forseeable future. At the present moment, I cannot disprove the following strengthening of (3): there are infinitely many values of n so that

$$\text{(4)} \qquad \nu(n-k) < \frac{\log k}{\log\log k} + C \quad\text{and}\quad \Omega(n-k) < \frac{\log k}{\log 2} + C$$

for every k satisfying $k_0(C) < k < n$. I am convinced that (4) is false for every C and $n > n_0(C)$; perhaps (4) and (3) can be disproved. It seems certain that for every k there are infinitely many values of n for which

$$\max_{n-k<m<n} \big(m + d(m)\big) \leqq n + 2,$$

though this is hopeless with our present methods. It would easily follow from Hypothesis H of Schinzel.

Let $f(n)$ be a non-negative additive or multiplicative function which has a bounded average, that is $\sum_{1\leqq n\leqq x} f(n) < cx$. Then $\liminf_{n\to\infty} H(n) < \infty$. (We suppress the proof since it is very similar to that of Theorem 1.) For $n = \Pi p_i^{\alpha_i}$ define $d_r(n) = \Pi(r+\alpha_i)$. It is not hard to show that if (3) holds then $\liminf_{n\to\infty} H_{d_r}(n) < \infty$.

To conclude this section, we observe that $\sigma(n)$ and $\phi(n)$ increase too fast to have barriers. In fact, it is easy to prove that $\max_{m<n}(m+\phi(m)) = 2n+o(n)$ and if we make plausible (but at present inaccessible) assumptions on the difference of consecutive primes, then it is easy to see that $\max_{m<n}(m+\phi(m)) = 2Q_n - 1$ for all $n > n_0$, where Q_n is the largest prime not exceeding n. Finally a little elementary manipulation with the primes gives $\max_{m<n}(m+\sigma(m)) = \max_{m<n}\sigma(m) + n - o(n)$.

2. Now we discuss some unconventional problems on primes. Denote by $p(m)$ the least and by $P(m)$ the largest of the prime factors of m. Put $F(n) = \max\{m + + p(m): 1 \leqq m < n, m \text{ composite}\}$. Is it true that $F(n) \leqq n$ for infinitely many n? Many related questions occur in a forth-coming triple paper of Eggleton, Selfridge and myself. We conclude that plausible conjectures on primes imply that $F(n) \leqq n$ has only a finite number of solutions. Trivially, $F(n) > n + \sqrt{n}$, but it is quite possible that $F(n) > n + (1-\varepsilon)\sqrt{n}$ for $n > n_0(\varepsilon)$.

Further questions can be posed if we do not want to ignore the primes, as in the definition of $F(n)$, but perhaps it is more natural in this case to consider the numbers $n+i$ instead of $n-i$. Thus, let g be a non-decreasing arithmetic function and let $B(n, g)$ be the smallest i for which $p(n+i) > g(i)$. If such an i does not exist, put $B(n, g) = \infty$. First, take $g(i) = i+1$. It is easy to see that $B(n, i+1)$ is just the smallest prime not dividing $n-1$ and, by the prime number theorem, $B(n, i+1) \leqq \leqq (1+o(1))\log n$. I could not get such a simple estimate for $B(n, g)$ if $g(i) = i + c$, or say $2i+1$. It follows from plausible assumptions on the distribution of primes that $B(n, i^k+1) < \infty$ for $n > n_0(k)$. I wonder if one can prove without any assumptions on the primes that, for every $n > n_0$, there is an i with $p(n+i) > i^2+1$. It follows from Huxley's well-known result on gaps between consecutive primes that, for every $n > n_0(\varepsilon)$, there is an i with $p(n+i) > i^{12/7+\varepsilon}$. It easily follows from well-known results on large gaps between consecutive primes that $p(n+i) < e^{\varepsilon i} + c(\varepsilon)$ $(i = 1, 2, 3, \ldots)$, that is $B(n, e^{\varepsilon i} + c(\varepsilon)) = \infty$ holds for infinitely many n. The additive constant $c(\varepsilon)$ is needed to take care of the very small values of i. In fact, $e^{\varepsilon i}$ can be replaced by $\exp\{ci(\log\log i)^2/\log i \log\log\log i\}$. A well-known conjecture of Cramer states that

$$\limsup_{k\to\infty} (p_{k+1} - p_k)/(\log k)^2 = 1 \tag{5}$$

where $p_1 < p_2 < p_3 < \ldots$ is the sequence of consecutive primes. Let us assume that (5) holds. Then we obtain $B(n, e^{(1-\varepsilon)i^{1/2}}) < \infty$ for every $n > n_0(\varepsilon)$. But I cannot conclude from (5) that $B(n, e^{(1+\varepsilon)i^{1/2}} + c(\varepsilon)) = \infty$ for infinitely many n because, of course, $p(n+i)$ can be very large even if $n+i$ is not a prime. There is clearly not much hope to settle these questions in the near future. Let us therefore be more modest for the moment and try to determine when the integers n satisfying $p(n+i) < < g(i)$ $(i = 1, 2, 3, \ldots)$ have positive density. A more or less routine sieve process shows that a necessary and sufficient condition for the non-decreasing function g to have this property is that

$$\sum_{i=1}^{\infty} \prod_{p<g(i)} \left(1-\frac{1}{p}\right) < \infty .$$

Now let us investigate what can be said about the large values of $p(n+i)$ for $n+i$ composite. First, is it true that for $n > n_0$, there is always an i for which $n+i$ is composite and $p(n+i) > i^2$? This is closely related to questions which we considered with Eggleton and Selfridge. Perhaps it is true that for every k and $n > n_0(k)$, there is an i for which $n+i$ is composite and $p(n+i) > i^k$. Clearly it is hopeless to prove this at present. I thought that for $k > k_0$, there is always an m satisfying $p_k < m < p_{k+1}$ and $p(m) \geqq p_{k+1} - p_k$, with equality say for prime twins. I am now sure that this is not true and I "almost" have a counterexample. Pillai and Szekeres observed that for every $t \leqq 16$, a set of t consecutive integers always contains one which is relatively prime to the others. This is false for $t = 17$, the smallest counterexample being 2184, 2185, . . ., 2200. Consider now the two arithmetic progressions $2183 + d \cdot 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ and $2201 + d \cdot 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$. There certainly will be infinitely many values of d for which the progressions simultaneously represent primes; this follows at once from hypothesis H of Schinzel, but cannot at present be proved. These primes are consecutive and give the required counterexample. I expect that this situation is rather exceptional and that the integers k for which there is no m satisfying $p_k < m < p_{k+1}$ and $p(m) > p_{k+1} - p_k$ have density 0.

Things become much easier if we study $P(m)$. A well-known theorem of Sylvester and Schur states that $P\left(\binom{n}{k}\right) > k$ if $k \leqq \frac{1}{2} n$. In other words, for every k and n with $k \leqq n$, there is an m satisfying $n+1 < m \leqq n+k$ and $P(m) > k$. This is certainly not true for $p(m)$. There are many extensions and sharpenings of the Sylvester–Schur theorem. Although we are very far from being able to prove it, there is no doubt that

$$P\left(\binom{n}{k}\right) > \min\{n-k+1,\ k^{1+c}\} \tag{6}$$

for some absolute constant c. Ramachandra, Shorey and Tijdeman have many results in this direction. It seems certain that (6) actually holds for every c with a finite number of exceptions (depending on c). Cramer's conjecture (5) suggests that perhaps

$$P\left(\binom{n}{k}\right) > \min\{n-k+1,\ e^{(1-\varepsilon)k^{1/2}}\}$$

holds if we disregard a finite number of values of k and n. Let

$$\binom{n}{k} = u_k^{(n)} v_k^{(n)} \quad \text{where} \quad P(u_k^{(n)}) < k,\ p(v_k^{(n)}) \geqq k . \tag{7}$$

In a forthcoming paper, Ecklund, Eggleton, Selfridge and I prove that, for $n \geqq 2k$, we have $v_k^{(n)} > u_k^{(n)}$ except for 12 cases, namely $\binom{8}{3}, \binom{9}{4}, \binom{10}{5}, \binom{12}{5}, \binom{21}{7}, \binom{21}{8}, \binom{30}{7}, \binom{33}{13}, \binom{33}{14}, \binom{36}{13}, \binom{36}{17}$, and $\binom{56}{13}$. If in (7) we modify the definition to $P(u_k^{(n)}) \leqq k, p(v_k^{(n)}) > k$, we can still prove that $v_k^{(n)} > u_k^{(n)}$ for $n \geqq 2k$ for all but a finite number of pairs n and k, but we cannot prove that we have all the exceptional cases. (The unresolved cases correspond to $k = 3$, 5 and 7.) We now give a further result of this type.

THEOREM 2. *Write* $\binom{n}{k} = u_k^{(n)} w_k^{(n)} \pi_k^{(n)}$ *where the prime factors p of* $u_k^{(n)}$, $w_k^{(n)}$ *and* $\pi_k^{(n)}$ *satisfy the respective inequalities* $2 \leqq p \leqq k$, $k < p < n-k+1$ *and* $n-k+1 \leqq p \leqq n$.

(i) *Except for a finite number of cases,* $w_k^{(n)} > 1$ *if* $4 \leqq k < Q$, *where Q is the largest prime not exceeding* $\frac{1}{2}n$.

(ii) *For sufficiently large C and* $n > Ck$, $w_k^{(n)} > \max\{u_k^{(n)}, \pi_k^{(n)}\}$.

The proof is fairly easy since we make no attempt (which would be hopeless in any case) to give all the exceptional k and n. Before we give the proof, let us investigate some of the exceptional cases in (i). For $k = 2$, we have $w_k^{(n)} = 1$ if and only if $n-1$ is a Mersenne prime or n is a Fermat prime. There are probably infinitely many cases with $k = 3$ and $w_k^{(n)} = 1$ arising when $n = 2^\alpha 3^\beta + 1$ and $2^\alpha 3^\beta - 1$ are a prime twin. $\binom{9}{3}$ and $\binom{18}{3}$ are not of this form and give $w_k^{(n)} = 1$, but it is easy to see that there are only a finite number of such exceptional cases and it would be easy to tabulate all of them. Finally, if $k \geqq Q$, then $w_k^{(n)} = 1$ clearly holds.

PROOF OF THEOREM 2. We distinguish several cases.

(a) Assume first that $\frac{n}{20} \leqq k < Q \leqq \frac{n}{2}$ It easily follows from elementary results on primes that $2Q > n-k+1$ for $n > n_0$, as $Q \,\Big|\, \binom{n}{k}$, that is $w_k^{(n)} \geqq Q > 1$.

(b) Assume next that $e^{14} < k < \frac{n}{20}$. It is well-known that if $p^\alpha \,\Big\|\, \binom{n}{k}$, then $p^\alpha \leqq n$. If $w_k^{(n)} = 1$, we therefore have

$$\binom{n}{k} < n^{\pi(k)+\pi(n)-\pi(n-k)} < n^{7k/2\log k},$$

using Montgomery's result $\pi(n) - \pi(n-k) < 2k/\log k$ and the estimate $\pi(k) < 3k/2\log k$. On the other hand, trivially

$$\binom{n}{k} > n^k e^k / k^{k+1}.$$

On combining the last two inequalities and taking a k-th root, we obtain

$$\frac{2n}{k} < \frac{en}{k^{1+1/k}} < n^{7/2 \log k}$$

and this leads to a contradiction for $e^{14} < k < \frac{n}{20}$ and $n > n_0$. This part of the argument could easily be made effective and the n and k with $k < e^{14}$ and $w_k^{(n)} = 1$ could be enumerated. (In fact, I am sure that there are no such values of n and k.) The cases $k \leqq e^{14}$ considered below cannot at present be made effective, but $k \leqq e^{14}$ could be greatly reduced by more careful computations.

(c) Finally assume $4 \leqq k \leqq e^{14}$. Write

$$\prod_{i=1}^{k} (n+i) = \Pi_1 \Pi_2 \quad \text{where} \quad P(\Pi_1) \leqq l,\ p(\Pi_2) > l.$$

A classical theorem of Mahler states that to every $\varepsilon > 0$ there is an $n_0(\varepsilon, k, l)$ so that $\Pi_1 < n^{1+\varepsilon}$ whenever $n > n_0(\varepsilon, k, l)$. Mahler's theorem is not effective and it is a very important open problem to obtain effective bounds. From Mahler's theorem, we obtain

$$\frac{n^k e^k}{k^{k+1}} < \binom{n}{k} = u_k^{(n)} w_k^{(n)} \pi_k^{(n)} < w_k^{(n)} n^{\frac{3}{2} + \pi(n) - \pi(n-k)} \leqq w_k^{(n)} n^{k-\frac{1}{2}}$$

for $n > n_0(k)$, since $\pi(n) - \pi(n-k) \leqq k-2$ for $k \geqq 4$. Thus $w_k^{(n)} > 1$ for $n > n_0$. This completes the proof of (i). We suppress the proof of (ii) since it is similar to that of (i).

We observe that $u_k^{(n)} > \pi_k^{(n)}$ and $\pi_k^{(n)} > u_k^{(n)}$ both hold for infinitely many n for every k. In fact, it is easy to see that for every k, $\pi_k^{(n)} = 1$ for almost all n. If $\pi(n) - \pi(n-k) \geqq 2$, then by Mahler's theorem, $\pi_k^{(n)} > u_k^{(n)}$ for $n > n_0(k)$; perhaps this holds always, or at least with very few exceptions. The reason for this bold and somewhat unmotivated conjecture is that it is not hard to prove $\pi_k^{(n)} > u_k^{(n)}$ for all $n > k^{1+c}$ and $k > k_0$, and I hoped that the first failure of $\pi_k^{(n)} > u_k^{(n)}$ occurs when $\pi(n) = \pi(n-k)$ for the first time. This is certainly false for $k = 4$, since the first failure occurs for $n = 9$. Perhaps it fails for all k. There is not much hope to decide any of these questions in the foreseeable future. It follows easily by elementary methods and a little computation that $\pi_k^{(2k)} > u_k^{(2k)}$ for all k except $k = 5$ and 6. It is also easy to see that if $\pi(n) - \pi(n-k) \geqq 1$, then $\pi_k^{(n)} > u_k^{(n)}$ for all but $o(\pi(x))$ values of $n < x$. Presumably there are infinitely many values of n with $\pi(n) - \pi(n-k) \geqq 1$ and $\pi_k^{(n)} < u_k^{(n)}$, but if true, this will surely be very hard to prove.

It is not difficult to prove that the density $f(c)$ of integers n for which $(u_k^{(n)})^{1/k} > c$ exists and is a continuous strictly decreasing function of c with $f(1) = 1, f(\infty) = 0$. However, the two questions which follow cannot be answered at present because Mahler's theorem is not effective. Denote by $A(n)$ the smallest k for which $u_k^{(n)} > n^2$. By Mahler's theorem, $A(n) \to \infty$ as $n \to \infty$, but we do not know how fast. Perhaps

Baker's results will yield a crude estimate for $A(n)$. Denote by $B(n, k)$ the smallest integer for which

$$\prod_{p^\alpha \| \binom{n}{k},\, p \leqq B(n,k)} p^\alpha > n^2 .$$

Estimate $B(n, k)$ as well as possible.

I investigated if there is a prime $p > k$ so that $p^2 \Big| \binom{n}{k}$. Ordinarily, this does not happen. A simple averaging process shows that, for every $\varepsilon > 0$, there is a $k_0(\varepsilon)$ so that when $k > k_0(\varepsilon)$ the density of integers n for which $p^2 \Big| \binom{n}{k}$ for some $p > k$ is less than ε. Also, for every k, there are infinitely many n for which $\binom{n}{k}$ is squarefree, but the density of these n tends to 0 as $k \to \infty$. The questions connected with $p^2 \Big| \binom{n}{k}$, $p > k$, lead to the following problem which is of independent interest. Is it true that for every $n > n_0$ there is a prime p for which

$$n = up^2 + v, \quad u \geqq 1, \quad 0 \leqq v < p? \tag{8}$$

It easily follows from the sieve of Eratosthenes that (8) is satisfied for almost all n, but it seems likely that (8) has no solution for infinitely many n. More generally, for every $p \leqq \sqrt{n}$, write $n = up^2 + v$ with $0 \leqq v < p^2$ and define $\varepsilon_n = \min\limits_{p \leqq \sqrt{n}} \frac{v}{p}$. Almost certainly $\limsup\limits_{n \to \infty} \varepsilon_n = \infty$ (but $\varepsilon_n \to 0$ as $n \to \infty$ for almost all n). Probably $\varepsilon_n < n^\varepsilon$ for $n > n_0(\varepsilon)$ and every $\varepsilon > 0$.

In a previous paper, I studied the number of prime factors of $\binom{n}{k}$. Trivially,

$$v\left(\binom{n}{k}\right) > \log \binom{n}{k} \Big/ \log n . \tag{9}$$

It is easy to see that if $k > n^{1-o(1)}$, then (9) becomes an asymptotic equality and we have

$$v\left(\binom{n}{k}\right) = (1 + o(1)) \log \binom{n}{k} \Big/ \log n \qquad (k > n^{1-o(1)}).$$

I conjecture that, for "large" k,

$$v\left(\binom{n}{k}\right) = (1 + o(1))k \sum_{k<p<n} \frac{1}{p}.$$

I obtained this conjecture by a simple averaging process. I cannot even prove it if $k > n^\varepsilon$, but perhaps it is true for every $k \geqq (\log n)^c$.

3. I discuss a few miscellaneous problems mostly about consecutive integers. Pomerance and I considered the following problem. Put $A(n, k) = \prod_{1 \leq i \leq k} (n+i)$ and denote by $q(n, k)$ the least prime which does not divide $A(n, k)$. Clearly,

$$q(n, k) < \big(1 + o(1)\big) k \log n . \tag{10}$$

This is clearly very crude. For bounded k and, more generally, for $k = o(\log n)$, the factor $k \log n$ in (10) can perhaps be replaced by $\log n$. An interesting special case is $k = [\log n]$. By choosing n so that it is the product of the primes between $\log n$ and $\big(2 + o(1)\big) \log n$, we see that $q(n, [\log n])$ can be as large as $\big(2 + o(1)\big) \log n$. Is it true that $q(n, [\log n]) < (2 + \varepsilon) \log n$ for $n > n_0(\varepsilon)$? We could not even prove that $q(n, [\log n]) < (1 - \varepsilon)(\log n)^2$. It seems certain that, to every $\varepsilon > 0$, there is a $k(\varepsilon)$ so that the density of integers n for which $P\big(A(n, k(\varepsilon))\big) < n^{1-\varepsilon}$ is less than ε. On probabilistic grounds, one would expect that the density of these integers is asymptotic to

$$\exp\left(-k \sum_{n^{1-\varepsilon} < p < n} \frac{1}{p}\right) = \exp\big(-\big(1 + o(1)\big) k\varepsilon\big)$$

as $n \to \infty$ and $\varepsilon \to 0$, but no sieve method at present applies here. Let $f(c)$ denote the density of integers n for which there is an m with $b < m \leqq n + k$ and $p(m) > e^{ck}$. Using elementary sieve methods, we can prove that $f(c)$ is continuous and strictly decreasing with $f(0) = 1, f(\infty) = 0$. This $f(c)$ could, of course, be determined explicitly. Several times during my long life, I was led to questions of the following type. Estimate, as well as you can, the size of the smallest integer $m_n \geqq n$ for which $\prod_{1 \leqq i \leqq n} (m_n + i)$ has no prime factor p satisfying $n < p < 2n$. I would expect that $m_n > n^k$ for every k if $n > n_0(k)$, but that $m_n < e^{\varepsilon n}$ for every $\varepsilon > 0$ if $n > n_1(\varepsilon)$. However, I could prove nothing non-trivial.

To end this section, I state some older problems. I conjectured more than a year ago that if $m \geqq n + k$, then $[n+1, n+2, \ldots, n+k] \neq [m+1, m+2, \ldots, m+k]$ where the square brackets denote least common multiple. Is it true that $\prod_{1 \leqq i \leqq k} (n+i)$ and $\prod_{1 \leqq i \leqq k} (m+i)$ cannot have the same prime factors for $k > 2$ and $m \geqq n + k$, except for a finite number of values of n, m and k? Put

$$\alpha(m, n, k) = \prod_{i=1}^{k} (m+i) \Big/ \prod_{i=1}^{k} (n+i)$$

and assume $k \geqq 2$ and $m \geqq n + k$. Is it true that $\alpha(m, n, k) = I$ is solvable for every integer $I > 1$? Now let n and k be fixed. Can one say anything about the integers of the form $\alpha(m, n, k)$?

Let me restate an old and very attractive conjecture of Turán and myself on the differences $d_n = p_{n+1} - p_n$ between consecutive primes. We easily proved that $d_{n+1} > d_n$ and $d_{n+1} < d_n$ both have infinitely many solutions. Presumably, $d_n = d_{n+1}$

also holds for infinitely many n but this is well-known to be very difficult. We conjectured that all the $k!$ inequalities of the form $d_{n+i_1} > d_{n+i_2} > \ldots > d_{n+i_k}$ have infinitely many solutions, where $i_1, i_2, \ldots, i_k$ is an arbitrary permutation of $1, 2, \ldots, k$. We certainly could not prove this even for $k = 3$. We could not even prove that there is no n_0 so that $d_{n+1} - d_n$ changes sign when n is replaced by $n+1$ for every $n > n_0$. Perhaps we overlooked a trivial argument; in any case, I offer a hundred dollars for a proof or disproof.

Finally let $B(n)$ (where B stands for Brun) be the smallest integer so that there is a residue a_p for every prime p with $2 \leqq p \leqq B(n)$, and every positive integer $x \leqq n$ satisfies at least one of the congruences $x \equiv a_p \pmod p$. The exact determination of $B(n)$ is probably hopeless, but a good estimate for $B(n)$ would be of the greatest importance for the application of Brun's method. As far as I know, Iwaniec's result $B(n) > c\sqrt{n}$ is the best lower bound known at present. It would be very nice if one could prove that $B(n) > Cn^{1/2}$ for every C and $n > n_0(C)$. It is likely that $B(n) > n^{1-\varepsilon}$ for every $\varepsilon > 0$ and $n > n_1(\varepsilon)$. The method of Rankin (used to give a lower bound on the difference of consecutive primes) gives

$$B(n) < cn(\log\log\log n)^2/\log n \cdot \log\log n \cdot \log\log\log\log n\,.$$

Recently, I considered the following modification of the above problem. Denote by ε_n the smallest number so that there is a residue b_p for every prime p with $n^{\varepsilon_n} < p \leqq n$, and every positive integer $x \leqq n$ satisfies at least one of the congruences $x \equiv b_p \pmod p$. Is it true that $\varepsilon_n \to 0$ as $n \to \infty$? I can prove that $\varepsilon_n > c \log\log\log n/\log\log n$. Are there residues c_p for every prime p with $2 \leqq p \leqq n$ so that every positive integer $x \leqq n$ satisfies at least 2 (or at least r) of the congruences $x \equiv c_p \pmod p$?

4. Proof of Theorem 1. The proof will use a simple averaging process, some of the details of which will be left to the reader. Let $\varepsilon > 0$, k be a sufficiently large integer and A be a multiple of $p_1, p_2, \ldots, p_k$. We shall show that the density of integers n which are barriers for d_0 is greater than $(1-\varepsilon)/A^k$ by considering the integers $n \leqq x$ with $n \equiv 0 \pmod{A^k}$. First, we observe that the density of integers t for which

$$d_0(tA^k - i) > i\,, \tag{11}$$

for some i with $1 \leqq i \leqq k$, is less than $\frac{1}{2}\varepsilon$. Indeed, (11) can only hold if $tA^k - i \equiv$ $\equiv 0 \pmod{p^2}$ for some $p > p_k$ and this easily implies our assertion for $k > k_0(\varepsilon)$. Next, by a simple computation, we obtain

$$\sum_{t=1}^{x} d_0(tA^k - i)^2 > cd_0(i)\,x\,,$$

and from this, the density of integers t satisfying (11) is less than $cd_0(i)/i^2 < c/i^{3/2}$.

Hence, for $k > k_0(\varepsilon)$, the density of integers t for which (11) holds for some $i > k$ is less than

$$\sum_{i>k} \frac{c}{i^{3/2}} < \frac{\varepsilon}{2}.$$

Thus the density of integers t for which tA^k is not a barrier for d_0 is less than ε. This proves Theorem 1.

With a little more trouble, I can prove that the density of integers n for which n is a barrier for $d_0(n)$ exists. More generally let α_i be the density of integers n for which $\max_{m<n} (m + d_0(m)) = n + i$. Then α_i exists for every i and $\sum_{i\geqq 0} \alpha_i = 1$. To end the paper, I state a somewhat special problem. Denote by S_i the set of integers m for which the number of solutions of $n + d_0(n) = m$ is i. I believe that it can be proved that the set S_i has a density $\beta_i \geqq 0$ and $\sum_{i\geqq 0} \beta_i = 1$. (I have not carried out the details and perhaps it is more difficult than I think it is). I am not sure that $\beta_i > 0$ always holds, but $\beta_0 > 0$ seems to hold. I certainly cannot settle the analogous questions for $n + v(n)$, $n + d(n)$, $n + \Phi(n)$, or $n + \sigma(n)$.

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AN EXAMPLE OF A BEST POSSIBLE ZAMANSKY TYPE INEQUALITY WITH EXPONENTIAL ORDER

By

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Dedicated to Professor G. Alexits on the occasion of his 80th birthday

In [1], the fundamental theorems of Jackson, Bernstein, Zamansky, and Stečkin on best approximation in Banach spaces have been extended to exponential orders, i.e. to rates of convergence $O(1/\phi(n))$, $n \to \infty$, where $\phi(n)$ increases faster than any power n^τ, $\tau > 0$. The results there suggested that a certain improvement of Zamansky's inequality should be possible, at least for particular functions and particular pairs of orders.

Indeed, for the trigonometric polynomials of best approximation in $L^2_{2\pi}$ there exists a sharpened version of Zamansky's inequality which is best possible [2; Theorem 2] and is actually attained for a certain class of functions. For general Banach spaces, in particular $C_{2\pi}$, $L^p_{2\pi}$, $1 \leqq p < \infty$, a lower estimate for the best possible order was established in [2; Theorem 4], leaving open the question whether this order is attained. If it were attained, e.g. in $L^1_{2\pi}$, this would imply that here the optimal Zamansky type inequality shows a slower rate of increase than the optimal $L^2_{2\pi}$ version.

The purpose of this note is to exhibit an example of a function in $L^1_{2\pi}$ where the rate of increase of Zamansky's inequality for the polynomials of best approximation can be determined explicitly, and is in fact the same as that given by the lower bound of [2; Theorem 4]. In particular, this rate is smaller than the smallest possible rate in $L^2_{2\pi}$. (Cf. also the comments in Section 3 below.)

1. Preliminaries. Let $L^1_{2\pi}$ be the space of 2π periodic integrable functions with norm $\|f\|_1 = (2\pi)^{-1} \int_{-\pi}^{\pi} |f(x)|\, dx$, $C_{2\pi}$ the space of 2π periodic continuous functions, and Π_n the set of trigonometric polynomials of degree $\leqq n$, where $n \in \mathbf{P} = \{0, 1, 2, \dots\}$. Denoting by $f^\wedge(k) = (2\pi)^{-1} \int_{-\pi}^{\pi} f(x)\, e^{-ikx} dx$, $k \in \mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}$ the Fourier coefficients of a function $f \in L^1_{2\pi}$, we define a subspace Y of $L^1_{2\pi}$ by

$$(1.1) \quad Y = \{f \in L^1_{2\pi};\ \text{there exists } g \in L^1_{2\pi} \text{ such that } e^{|k|} f^\wedge(k) = g^\wedge(k) \quad \forall k \in \mathbf{Z}\}.$$

Then $|f|_Y = \|g\|_1$ is a semi-norm on Y and there are Jackson- and Bernstein type

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inequalities with respect to Y, i.e.

$$E_n[f] \leqq C e^{-n} |f|_Y \qquad (f \in Y,\ n \in \mathbf{P}), \tag{1.2}$$

$$|p_n|_Y \leqq e^n \|p_n\|_1 \qquad (p_n \in \Pi_n,\ n \in \mathbf{P}). \tag{1.3}$$

Here $E_n[f] = \inf\{\|f - p_n\|_1;\ p_n \in \Pi_n\}$ denotes the error of best approximation in $L^1_{2\pi}$ norm and C is a constant, independent of f and n. Inequality (1.2) follows easily, using [4; pp. 80, 81] (setting there $a_k = 2e^{-k}$, cf. also Lemma 1 below), and (1.3) is a particular case of [4; 4.8 (25)].

From the proof of the Favard–Achieser–Krein theorem [4; Section 5.5.1] we collect the following facts.

LEMMA 1. *The function*

$$f(x) = \sum_{k=1}^{\infty} k^{-2} \cos kx \tag{1.4}$$

satisfies

$$E_n[f] = (\pi/4)^2 (n+1)^{-2} \qquad (n \in \mathbf{P}), \tag{1.5}$$

and, denoting by p_n^0 the polynomial of best approximation to f from Π_n with respect to $L^1_{2\pi}$ norm, p_n^0 is given by

$$p_n^0(x) = \sum_{k=0}^{n} b_k^n \cos kx \tag{1.6}$$

where

$$b_0^n = -c_{n+1}^n, \quad b_k^n = k^{-2} - (c_{n+1-k}^n + c_{n+1+k}^n), \qquad 1 \leqq k \leqq n, \tag{1.7}$$

$$c_j^n = \sum_{\nu=0}^{\infty} (-1)^\nu \{(2\nu+1)(n+1) + j\}^{-2} \qquad (j \in \mathbf{P}). \tag{1.8}$$

Indeed, setting $a_0 = 3$, $a_k = k^{-2}$, $k \in \mathbf{N} = \{1, 2, \ldots\}$, the sequence $\{a_k\}_{k \in \mathbf{P}}$ is triply monotonic, i.e. $a_k - a_{k+1} \geqq 0$, $a_k - 2a_{k+1} + a_{k+2} \geqq 0$, and $a_k - 3a_{k+1} + 3a_{k+2} - a_{k+3} \geqq 0$ for each $k \in \mathbf{P}$. Following [4; pp. 80, 81], Markov's theorem yields that the polynomial of best approximation to $f(x) + 3$ in $L^1_{2\pi}$ is given by $\sum_{k=0}^{n} B_k^n \cos kx$ where

$$B_0^n = a_0 - c_{n+1}^n, \quad B_k^n = a_k - (c_{n+1-k}^n + c_{n+1+k}^n), \qquad 1 \leqq k \leqq n,$$

with c_j^n defined by (1.8). Moreover,

$$E_n[f(x)+3] = (2/\pi) \sum_{\nu=0}^{\infty} (-1)^\nu (2\nu+1)^{-1} a_{(2\nu+1)(n+1)} =$$

$$= (2/\pi) \left\{ \sum_{\nu=0}^{\infty} (-1)^\nu (2\nu+1)^{-3} \right\} (n+1)^{-2} = (\pi/4)^2 (n+1)^{-2}$$

(cf. [3; 5.11.26] for the last equality). This implies the above assertions for the function $f(x) = -3 + \sum_{k=0}^{n} B_k^n \cos kx$.

2. An estimate for $|p_n^0|_Y$. In order to show that $|p_n^0|_Y = O(e^n(n+1)^{-3})$, $n \to \infty$ (Lemma 4 below) we first estblish a representation of the coefficients b_k^n (Lemma 2) which admits a study of the properties of the sequence $e^{k/2}b_k^n$ (Lemma 3).

LEMMA 2. *For the coefficients of* (1.6) *the following representation holds:*

$$\text{(2.1)} \qquad b_0^n = -\frac{\pi^2}{48}(n+1)^{-2},$$

$$b_k^n = \frac{\pi^2}{4}(n+1)^{-2} \sin\frac{\pi}{2}\left(1-\frac{k}{n+1}\right)\left\{\cos\frac{\pi}{2}\left(1-\frac{k}{n+1}\right)\right\}^{-2} \quad (1 \leqq k \leqq n).$$

PROOF. By (1.7), (1.8) and, e.g., [3; 5.12.49] one has

$$b_0^n = -\sum_{\nu=0}^{\infty}(-1)^\nu(\nu+1)^{-2}(2n+2)^{-2} = -(2n+2)^{-2}\pi^2/12$$

and, setting $n_\nu = (2\nu+1)(n+1)$ and using [3; 6.1.146] for $1 \leqq k \leqq n$,

$$b_k^n = k^{-2} - \left\{\sum_{\nu=0}^{\infty}(-1)^\nu(n_\nu+n+1+k)^{-2} + \sum_{\nu=0}^{\infty}(-1)^\nu(n_\nu+n+1-k)^{-2}\right\} =$$

$$= k^{-2} + \sum_{\nu=1}^{\infty}(-1)^\nu\big(n_\nu-(n+1-k)\big)^{-2} - \sum_{\nu=0}^{\infty}(-1)^\nu(n_\nu+n+1-k)^{-2} =$$

$$= \sum_{\nu=0}^{\infty}(-1)^\nu\{(n_\nu-(n+1-k))^{-2}-(n_\nu+n+1-k)^{-2}\} =$$

$$= 4(n+1)(n+1-k)\sum_{\nu=0}^{\infty}(-1)^\nu(2\nu+1)\{n_\nu^2-(n+1-k)^2\}^{-2} =$$

$$= 4(n+1)(\pi/4)^2(n+1)^{-3}\sec\frac{\pi}{2}\left(1-\frac{k}{n+1}\right)\tan\frac{\pi}{2}\left(1-\frac{k}{n+1}\right).$$

This establishes (2.1).

LEMMA 3. *Defining $d_k^n = e^{k/2}b_k^n$ for $0 \leqq k \leqq n$, $n \in \mathbf{P}$, there exists $n_0 \in \mathbf{P}$ such that, for $n \geqq n_0$, the sequence $\{d_k^n\}_{k=5}^{n-4}$ is positive, strictly increasing, and strictly convex.*

PROOF. Setting

$$\text{(2.2)} \qquad \delta_n(t) = \pi^2(2n+2)^{-2}\sin\frac{\pi}{2}\left(1-\frac{t}{n+1}\right)\left\{\cos\frac{\pi}{2}\left(1-\frac{t}{n+1}\right)\right\}^{-2}$$

for $0 < t < n+1$, and

$$\text{(2.3)} \qquad \gamma_n(t) = \pi(n+1)^{-1}\left\{3-\cos\pi\left(1-\frac{t}{n+1}\right)\right\}\left\{\sin\pi\left(1-\frac{t}{n+1}\right)\right\}^{-1},$$

one has (cf. (2.1)) $\delta_n(t) > 0$ for $0 < t < n+1$, $\delta_n(k) = d_k^n$ for $1 \leqq k \leqq n$, $k \in \mathbf{N}$, and

$$\lim_{t\to 0+} \delta_n(t) = +\infty, \qquad \lim_{t\to(n+1)-} \delta_n(t) = 0. \tag{2.4}$$

Moreover, for $t \in (0, n+1)$,

$$\delta_n'(t) = (1/2)\,\delta_n(t)\,\{1 - \gamma_n(t)\}, \tag{2.5}$$

$$\delta_n''(t) = (1/2)\,\delta_n(t)\,\{(1-\gamma_n(t))^2 - \gamma_n'(t)\}, \tag{2.6}$$

$$\gamma_n'(t) = -\pi^2(n+1)^{-2}\left\{1 - 3\cos\pi\left(1 - \frac{t}{n+1}\right)\right\}\left\{\sin\pi\left(1-\frac{t}{n+1}\right)\right\}^{-2}, \tag{2.7}$$

$$\gamma_n''(t) = 3\pi^2(n+1)^{-3}\left\{\sin\pi\left(1-\frac{t}{n+1}\right)\right\}^{-3}\left\{\frac{8}{9} + \left[\frac{1}{3} - \cos\pi\left(1-\frac{t}{n+1}\right)\right]^2\right\}, \tag{2.8}$$

whence $\gamma_n''(t) > 0$ for $0 < t < n+1$. Thus, also using (2.3), (2.7), $\gamma_n(t)$ is a positive, strictly convex function, which strictly decreases on $(0, t_0^n)$ and strictly increases on $(t_0^n, n+1)$, where $t_0^n = (n+1)\,(1 - \pi^{-1}\arccos 1/3)$, and which tends to $+\infty$ for $t \to 0+$ and $t \to (n+1)-$.

Choosing $\varepsilon \in (0, 1/2)$, we have $5 - \varepsilon < t_0^n < (n+1) - (5-\varepsilon)$ for n sufficiently large, and, in view of (2.5), the behaviour of $\gamma_n(t)$, and

$$\lim_{n\to\infty}\big(1 - \gamma_n(5-\varepsilon)\big) = \frac{1-\varepsilon}{5-\varepsilon} > 0, \quad \lim_{n\to\infty}\big(1 - \gamma_n((n+1)-(5-\varepsilon))\big) = \frac{3-\varepsilon}{5-\varepsilon} > 0,$$

there is $n_0 \in \mathbf{P}$ such that $\delta_n(t)$ is strictly increasing on $[5-\varepsilon, (n+1)-(5-\varepsilon)]$ for each $n \geqq n_0$. Similarly it follows by (2.6) that $\delta_n''(t) > 0$ for $0 < t \leqq t_0^n$. For $t_0^n < t \leqq$ $\leqq (n+1) - (5-\varepsilon)$ one has

$$\big(1 - \gamma_n(t)\big)^2 - \gamma_n'(t) \geqq 1 - 2\gamma_n\big((n+1)-(5-\varepsilon)\big) - \gamma_n'\big((n+1)-(5-\varepsilon)\big),$$

the right hand side tending to $(3 - 6\varepsilon + \varepsilon^2)\,(5-\varepsilon)^{-2} > 0$ for $n \to \infty$, in view of (2.3) and (2.7). Hence by (2.6), enlarging n_0 if necessary, $\delta_n''(t) > 0$ for each $n \geqq n_0$, $t \in [5-\varepsilon, (n+1)-(5-\varepsilon)]$.

LEMMA 4. *The polynomials* p_n^0, *defined by* (1.6), *satisfy*

$$|\,p_n^0\,|_Y = O\big(e^n(n+1)^{-3}\big), \qquad n \to \infty. \tag{2.9}$$

PROOF. By the definition of $|\cdot|_Y$ one has, with $d_k^n = e^{k/2}\,b_k^n$,

$$|\,p_n^0\,|_Y = \left\| \sum_{k=0}^{n} e^{k/2} d_k^n \cos kx \right\|_1 \leqq \left\| \sum_{k=0}^{4} e^{k/2}(d_k^n - d_5^n)\cos kx \right\|_1 +$$

$$+ \left\| \sum_{k=0}^{4} e^{k/2} d_5^n \cos kx + \sum_{k=5}^{n-4} e^{k/2} d_k^n \cos kx \right\|_1 + \left\| \sum_{k=n-3}^{n} e^k b_k^n \cos kx \right\|_1 = S_1 + S_2 + S_3,$$

say. By (2.1), the coefficients of S_1 tend to

$$\lim_{n\to\infty} e^{k/2}(d_k^n - d_5^n) = \begin{cases} -e^{5/2}5^{-2}; & k=0 \\ e^{k/2}(e^{k/2}k^{-2} - e^{5/2}5^{-2}); & 1 \leqq k \leqq 4, \end{cases}$$

thus $S_1 = O(1)$ as $n \to \infty$. Moreover, $S_3 = O\big(e^n(n+1)^{-3}\big)$, $n \to \infty$, since

$$\lim_{n\to\infty} (n+1)^3 \big(\max_{n-3\leqq k\leqq n} e^k b_k^n\big) \leqq e^n(\pi^2/4) \lim_{n\to\infty} (n+1)\sin\frac{\pi}{2}\left|1 - \frac{n-3}{n+1}\right| = \pi^3 e^n/2.$$

To estimate S_2, we use the convexity of the sequence $\{\kappa_k^n\}_{k=0}^{n-4}$ with $\kappa_k^n = d_5^n$ for $0 \leqq k \leqq 4$, $\kappa_k^n = d_k^n$ for $5 \leqq k \leqq n-4$ (Lemma 3). Thus (cf. e.g. [4; Section 4.8.61])

$$S_2 = \left\| \sum_{k=0}^{n-4} e^{k/2}\kappa_k^n \cos kx \right\|_1 = O\left(\kappa_{n-4}^n \left\| \sum_{k=0}^{n-4} e^{k/2} \cos kx \right\|_1\right), \qquad n \to \infty.$$

The assertion now follows by

$$\lim_{n\to\infty} (n+1)^3 e^{-n/2} \kappa_{n-4}^n = \lim_{n\to\infty} (n+1)^3 b_{n-4}^n < \infty \tag{2.10}$$

and

$$\left\| \sum_{k=0}^{n-4} e^{k/2} \cos kx \right\|_1 \leq C \int_0^{n-3} e^{x/2}\,dx = O(e^{n/2}), \qquad n \to \infty.$$

We further show that the estimate of Lemma 4 cannot be improved.

LEMMA 5. *For the polynomials* (1.6) *one has*

$$\limsup_{n\to\infty} |p_n^0|_Y e^{-n}(n+1)^3 > 0.$$

PROOF. We proceed as in [1; Theorem 2] and [2; Theorem 4], the basic assumptions (W), (E), (M), (S_Y), (J_Y), (B_Y) there being clearly satisfied (cf. (1.2) (1.3)). For the function f given by (1.4) we have

$$E_k[f] \leqq E_{k+1}[f] + E_k[p_{k+1}^0] \qquad (k \in \mathbf{P}),$$

whence, by (W),

$$E_n[f] \leqq \sum_{j=0}^{\infty} E_{n+j}[p_{n+j+1}^0].$$

Now applying the Jackson type inequality (1.2) to each p_{n+j+1}^0 and assuming that

$$|p_n^0|_Y = o\big(e^n(n+1)^{-3}\big), \quad n \to \infty,$$

one obtains

$$E_n[f] \leqq C \sum_{j=0}^{\infty} e^{-(n+j)} |p_{n+j+1}^0|_Y = o\left(\sum_{j=0}^{\infty} (n+j+2)^{-3}\right) = o(n^{-2}), \quad n \to \infty,$$

which contradicts (1.5).

By the same reasoning it can be shown that the order e^{-n} in (1.2) is best possible for the space Y. Indeed, assuming that there exists a null sequence $\{\varepsilon_n\}$ such that $E_n[h] \leqq \varepsilon_n e^{-n} \mid h \mid_Y$ for each $h \in Y$, $n \in \mathbf{P}$, and using (2.9), it follows as above that $E_n[f] = o(n^{-2})$ as $n \to \infty$.

Combining Lemmas 4 and 5, and using the notation $a_n \sim b_n$, $n \to \infty$ for two sequences $\{a_n\}$, $\{b_n\}$ with $0 < \limsup_{n \to \infty} \mid a_n/b_n \mid < \infty$, we have

THEOREM 1. *For the function f of* (1.4) *with polynomials of best approximation p_n^0* (1.6) *and the space Y defined by* (1.1) *the following hold:*

$$E_n[f] = (\pi/4)^2 (n+1)^{-2} \qquad (n \in \mathbf{P}), \tag{2.11}$$

$$\mid p_n^0 \mid_Y \sim e^n n^{-3}, \quad n \to \infty. \tag{2.12}$$

3. Concluding remarks. The factor $e^{|k|}$ has been chosen for the definition (1.1) of Y in order to simplify the computations. In order to compare the present result with those of [1], [2], this factor would have to be replaced, e.g., by $(1 + \mid k \mid)^2 e^{|k|}$ since the function $\phi(x) = e^x$ does not belong to the class Φ as defined in [2] (it does, however, fit into the frame of [1]), whereas $\phi^*(x) = (1 + x)^\alpha e^x$ is a member of Φ for $\alpha > 0$. But it is easily seen that essentially the same results also hold for ϕ^*.

COROLLARY. *Let f, p_n^0 be given by* (1.4), (1.6), *let $\phi^*(x) = (1 + x)^\alpha e^x$ for some $\alpha > 0$, and define the space Z by*

$$Z = \{f \in L_{2\pi}^1;\ \text{there exists } g \in L_{2\pi}^1 \text{ such that } \phi^*(\mid k \mid) f^\wedge(k) = g^\wedge(k) \quad \forall k \in \mathbf{Z}\} \tag{3.1}$$

with semi-norm $\mid f \mid_Z = \| g \|_1$. Then, apart from (2.10), *one has*

$$\mid p_n^0 \mid_Z \sim e^n n^{\alpha-3}, \quad n \to \infty. \tag{3.2}$$

Indeed, leaving Lemmas 1 – 3 unchanged, Lemma 4 is modified as follows. Let

$$\mid p_n^0 \mid_Z = \left\| \sum_{k=0}^{n} (1+k)^\alpha e^{k/2} d_k^n \cos kx \right\|_1 = \sigma_1 + \sigma_2 + \sigma_3,$$

where the σ_i are obtained from the S_i in the proof of Lemma 4 by inserting the factor $(1 + k)^\alpha$ in each term. Then σ_1 and σ_3 may be treated as before, and $\sigma_2 = O\big(e^n (n+1)^{\alpha-3}\big)$, $n \to \infty$ follows by

$$\sigma_2 = \left\| \sum_{k=0}^{n-4} e^{k/2} (1+k)^\alpha \cos kx \right\|_1 = O\left(\kappa_{n-4}^n \left\| \sum_{k=0}^{n-4} e^{k/2} (1+k)^\alpha \cos kx \right\|_1 \right), \quad n \to \infty,$$

using (2.10) and $\sum_{k=0}^{n-4} e^{k/2} (1+k)^\alpha = O\big(e^{n/2} (1+n)^\alpha\big)$, $n \to \infty$.

Since, in general Banach spaces and for general orders ϕ, $\phi^* \in \Phi$, the Zamansky equivalence

$$\text{(i)} \quad E_n[f] = O\big(1/\phi(n)\big), \ n \to \infty \iff \text{(ii)} \quad \mid p_n^0 \mid_Y = O\big(\phi^*(n)/\phi(n)\big), \ n \to \infty$$

is bound up with the condition $\phi \in K_{\phi^*}$ (cf. [1; Theorem 2]), the inverse of Zamansky's theorem (thus the implication (ii) $\Rightarrow$ (i)) may fail to hold in case $\phi \notin K_{\phi^*}$. In fact, it has been shown ([2; Corollary 2]) that then there are functions in $L^2_{2\pi}$ for which (i), (ii) hold as well as functions for which (i) is sharp but

$$(3.3) \qquad | p_n^0 |_Y = O\left(\left(\frac{g'(n)}{g^{*\prime}(n)}\right)^{1/2} \frac{\phi^*(n)}{\phi(n)}\right), \quad n \to \infty,$$

where $g(n) = \log \phi(n)$, $g^*(n) = \log \phi^*(n)$. In general the right hand side of (3.3) increases less rapidly than that of (ii), and (3.3) gives the slowest possible rate of increase for functions $f \in L^2_{2\pi}$ for which (i) is sharp.

The corresponding problem for the spaces $C_{2\pi}$, $L^p_{2\pi}$, $1 \leqq p < \infty$, $p \neq 2$ is unsolved. But for general Banach spaces it was shown in [2; Theorem 4] that, assuming (i), the rate of increase of $| p_n^0 |_Y$ cannot be slower than

$$(3.4) \qquad | p_n^0 |_Y = O\left(\frac{g'(n)}{g^{*\prime}(n)} \frac{\phi^*(n)}{\phi(n)}\right), \quad n \to \infty.$$

The present result now shows that the right hand side of (3.4) is attained, thus it represents a best possible order in Zamansky's inequality as far as all Banach spaces and all pairs of orders ϕ, $\phi^* \in \Phi$ are concerned (and the other hypotheses of [2; Theorem 4] are satisfied). Indeed, setting

$$(3.5) \qquad \phi(x) = (1+x)^2, \quad \phi^*(x) = (1+x)^\alpha e^x, \quad \alpha > 0,$$

the right hand side of (3.2) is just that of (3.4). Nevertheless, since $\lim_{n\to\infty} g^{*\prime}(n) = 1$ here, this does not imply that (3.4) is optimal for general pairs, ϕ, ϕ^*. Moreover, for the spaces $L^p_{2\pi}$, $p \neq 2$, it is expected that the optimal order will depend on p also. Comparing (3.3) and (3.4) for the ϕ, ϕ^* of (3.5), the present result indeed shows that (3.3) is not the best possible version of Zamansky's inequality in general spaces.

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NONSPANNING SETS OF EXPONENTIALS ON CURVES

By

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Dedicated to Professor G. Alexits on his eightieth birthday

1. Uniform Müntz—Szász type approximation on curves

In this paper, $\{p_n\}$ will always be a strictly increasing sequence of positive integers, $\{\rho_n\}$ a sequence of distinct positive real numbers, and $\{\lambda_n\}$ a sequence of distinct complex numbers.

1.1 Introductory remarks. By the theorem of MÜNTZ [13] and SZÁSZ [16], the *powers* 1 *and*

$$x^{\rho_n}, \quad n = 1, 2, \dots, \tag{1.1}$$

where $\rho_n \geqq \delta > 0$, *span the space* $C[0, 1]$ if and only if

$$\Sigma\, 1/\rho_n = \infty\,. \tag{1.2}$$

(Szász also considered complex exponents λ_n: if $\operatorname{Re}\lambda_n \geqq \delta > 0$, the condition is $\Sigma \operatorname{Re}(1/\lambda_n) = \infty$.) Condition (1.2) by itself is necessary and sufficient in order that the *powers* (1.1) *span* $C[a, b]$ when $0 < a < b$ (CLARKSON—ERDŐS [2], SCHWARTZ [15]; for a complex-analysis proof of the necessity, cf. LUXEMBURG—KOREVAAR [9].)

A well-known approximation theorem of WALSH [17] asserts that the *powers* z^n, $n = 0, 1, \dots$ *span* $C(\gamma)$ *for every Jordan arc* γ in the plane. However, there is as yet no nice MÜNTZ—SZÁSZ type result for Jordan arcs. In fact, switching to exponentials

$$e^{\rho_n z}, \quad n = 1, 2, \dots, \tag{1.3}$$

it is known only for the class of *polygonal lines* Γ without vertical chords that condition (1.2) is *both necessary and sufficient* in order that (1.3) be a spanning set for $C(\Gamma)$. (This simple result may be derived from DIXON—KOREVAAR [3].) Condition (1.2) is also *sufficient* in order that (1.3) be a spanning set for $C(\Gamma)$ in the case of all (rectifiable) arcs Γ whose oriented chords make angles with the positive real axis of opening $\leqq \frac{1}{4}\pi$ (KOREVAAR [5]; related results have been obtained by LEONT'EV [8] and MALLIAVIN—SIDDIQI [12]).

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1.2 Recent results on nonspanning sets. We will call a sequence $\{p_n\}$ a *Macintyre sequence* if every nonconstant entire function

$$f(z) = \Sigma a_n z^{p_n}$$

is unbounded on each curve going to infinity. Sufficiently strong nonspanning properties of sets

$$z^{p_n}, \quad n = 1, 2, \dots \tag{1.4}$$

are of interest in connection with the so-called

MACINTYRE CONJECTURE [10]: *$\{p_n\}$ is a Macintyre sequence if and only if*

$$\Sigma \, 1/p_n < \infty. \tag{1.5}$$

The simple condition (1.5) assures that (1.4) fails to span $C(\gamma)$ for all *analytic* Jordan arcs γ (MALLIAVIN–SIDDIQI [11], KOREVAAR [5]). For *arbitrary* arcs γ, the sharpest known result is that (1.4) fails to span $C(\gamma)$ (even in a rather strong sense) whenever $\{p_n\}$ is an *interpolation sequence* as defined in 2.1 below (KOREVAAR–DIXON [6]).

Interpolation sequences $\{p_n\}$ are Macintyry sequences (PAVLOV [14], KOREVAAR–DIXON [6]). Examples of conditions which assure that $\{p_n\}$ be an *interpolation sequence* are:

$$\Sigma \, 1/p_n < \infty, \quad p_n/n \uparrow \tag{1.6}$$

(PAVLOV [14]);

$$p_n \geqq cn \log n \, (\log \log n)^{2+\varepsilon} \qquad (c, \varepsilon > 0) \tag{1.7}$$

(KOREVAAR–DIXON [6]). Condition (1.7) occurs in KŐVÁRI's work [7] as a sufficient condition for a *Macintyre sequence.* It is possible to weaken (1.7) a little (KOREVAAR–DIXON [6]), but there exist sequences $p_n \geqq cn \log n \, (\log \log n)^2$ which are *not* interpolation sequences (BERNDTSSON [1]).

For nonspanning sets (1.4) or (1.3) on arcs, there are *analyticity theorems* for the approximable functions (DIXON–KOREVAAR [3, 6]) somewhat similar to those of CLARKSON–ERDŐS [2] and SCHWARTZ [15] for the case of an interval $[a, b]$.

1.3 Principal result. For arcs γ which satisfy a certain smoothness condition near their endpoints, ERKAMA [4] had shown previous to the recent work of KOREVAAR–DIXON [6] that the condition

$$p_n \geqq cn \, (\log n)^2$$

gives a nonspanning set (1.4) in $C(\gamma)$. Our present main result is as follows.

THEOREM 1. *Let* $0 < \varepsilon < \frac{1}{4}\pi$ *and let* Γ *be a (rectifiable) arc whose oriented chords make angles of opening not exceeding* $\frac{1}{4}\pi - \varepsilon$ *with a fixed direction. Let*

$$|\lambda_n| \geqq nL(n), \quad 0 < L(n)\uparrow, \quad \Sigma\, 1/nL(n) < \infty .$$

Then the (complex) exponentials

$$e^{\lambda_n z}, \quad n = 1, 2, \dots \tag{1.8}$$

fail to span $C(\Gamma)$.

REMARKS. In this theorem one can actually allow arcs whose oriented chords make angles of opening $\leqq \frac{1}{2}\pi - \varepsilon$ with a fixed direction (cf. 3.1). For *analytic* Γ, the sole condition $\Sigma\, 1/|\lambda_n| < \infty$ assures that (1.8) fails to span $C(\Gamma)$ [5, 11, 12].

COROLLARY 1. *For* C^1 *arcs* γ, *the condition*

$$p_n \geqq nL(n), \quad 0 < L(n)\uparrow, \quad \Sigma\, 1/nL(n) < \infty$$

guarantees that the set (1.4) *fails to span* $C(\gamma)$.

2. A nonspanning theorem involving interpolation sequences

As a first step in the proof of Theorem 1 we will derive a strong nonspanning property of integral powers whose exponents form an interpolation sequence. An interpolation problem of the type we need was first considered by PAVLOV [14] in connection with the Macintyre conjecture.

2.1 Interpolation sequences. *A sequence* S *of distinct complex numbers* w_n, $n = 0, 1, \dots$ *is called an interpolation sequence if there is a positive increasing function* $\omega(r) = \omega(r, S)$ *on* $[0, \infty)$ *with the properties*

$$\int_1^\infty r^{-2}\omega(r)\,dr < \infty , \quad r^{-1}\omega(r)\downarrow$$

such that the following is true. For every sequence of complex numbers $\{b_n\}$ *with* $|b_n| \leqq 1$ *there exists an entire function* $\phi(z)$ *for which*

$$\phi(w_n) = b_n, \quad n = 0, 1, \dots; \qquad M(r, \phi) = \max_{|z|=r} |\phi(z)| \leqq e^{\omega(r)}, \quad r \geqq 0 .$$

(Observe that ϕ has to be of exponential type 0.)

It is convenient (and no restriction, cf. [6]) to suppose that in our interpolation sequence, $w_0 = 0$. Considering an interpolating function ϕ as in the definition for which $\phi(0) = 1$, $\phi(w_n) = 0$, $n \geqq 1$, Jensen's formula shows that the enumerative function $N(r)$ for the numbers $w_n \neq 0$ is bounded by $\omega(r)$:

$$N(r) = \int_0^r t^{-1} n(t)\, dt \leqq \log M(r, \phi) \leqq \omega(r).$$

It follows that $n(r) \leqq \omega(er)$ and hence

$$\sum_1^\infty 1/|w_n| < \infty. \tag{2.1}$$

The following lemma extends a result of PAVLOV [14] for positive sequences; a relatively simple proof may be obtained from KOREVAAR–DIXON [6] (see Lemmas 5, 6 and Section 4.3).

LEMMA 1. *Suppose $\Sigma\, 1/p_n < \infty$ and $p_n/n \uparrow$. Then*

$$\ldots, -p_2, -p_1, 0, p_1, p_2, \ldots \tag{2.2}$$

is an interpolation sequence.

2.2 The auxiliary nonspanning theorem. If $\{w_n\}$ is an arbitrary interpolation sequence, (2.1) implies that the exponentials $\exp(w_n z)$ fail to span $C(\Gamma)$ whenever Γ is analytic (cf. 1.3). For interpolation sequences of the form (2.2) and arcs Γ different from a vertical segment, there is a strong nonspanning property which we formulate for powers instead of exponentials. The following result extends our earlier work involving positive interpolation sequences (KOREVAAR–DIXON [6]).

THEOREM 2. *Suppose*

$$\ldots < q_{-2} < q_{-1} < q_0 = 0 < q_1 < q_2 < \ldots$$

is an interpolation sequence consisting of integers with associated function ω. Then for every $a > 1$,

$$\inf_{\gamma=\gamma_a} \inf_c \| 1 - \Sigma' c_n z^{q_n} \|_\gamma \geqq \varepsilon_0(\omega, a) > 0, \tag{2.3}$$

where the inner infimum is taken over all finite sums $\Sigma' c_n z^{q_n}$ with $n \neq 0$, and the outer infimum over all curves γ extending from a point on the circle $C(0, 1/a)$ to a point on $C(0, a)$.

Combining Theorem 2 with Lemma 1 we obtain

COROLLARY 2. *Suppose $\Sigma\, 1/p_n < \infty$ and $p_n/n \uparrow$. Then the exponentials*

$$\exp(\pm \eta p_n z), \quad n = 1, 2, \ldots \tag{2.4}$$

fail to span $C(\Gamma)$ for every arc Γ and every constant $\eta > 0$.

The result with arbitrary $\eta > 0$ is of course equivalent to the result for $\eta = 1$ applied to the curve $\eta\Gamma$.

2.3 Proof of Theorem 2. Although the proof has a good deal in common with the one for the case of positive interpolation sequences ([6], Section 3), we will give all the essential steps here, but we will occasionally refer to [6] for details where they are the same.

(i) *Starting out.* Let $g(z) = g_\varepsilon(z)$ be a polynomial in z and $1/z$ of the form

$$g(z) = \Sigma\, a_n z^{q_n} \text{ with } a_0 = 1/\varepsilon, \quad \varepsilon \text{ small} > 0.$$

In order to prove (2.3), we suppose that for some curve γ extending from $C(0, 1/a)$ to $C(0, a)$ and coefficients a_n (with $a_0 = 1/\varepsilon$), we have

$$\| g(z) \|_\gamma \leqq 1. \tag{2.5}$$

The intention is to show that (2.5) leads to a contradiction if ε is sufficiently small.

We will write $M(r, g) = M(r)$. Since $\log M(r)$ is a convex function of $\log r$, it is either decreasing on $[1/a, 1]$ or increasing on $[1, a]$ (or both). Replacing z by $1/z$ if necessary, we may assume that $M(r)$ is *increasing on* $[1, a]$; we observe that $M(r) \geqq 1/\varepsilon$.

(ii) *A local estimate.* Let $z_r = r \exp(i\beta_r)$, $1 < r < a$, belong to γ. We consider the closed disc $\bar{D} = \bar{D}(z_r, r\delta_r)$, where $0 < \delta_r < 1$ (to be specified later) will be so small that $\bar{D}$ belongs to the annulus $1 \leqq |z| \leqq a$. Since $|g|$ is bounded by $M(re^{\delta_r})$ on $\bar{D}$ and by 1 on γ, a harmonic measure argument of PÓLYA (cf. [6], Lemma 1) shows that

$$|g(z)| \leqq M(re^{\delta_r})^{3/4} \quad \text{for } z \in \bar{D}(z_r, r\delta_r/6). \tag{2.6}$$

(iii) *Estimate for* $M(r)$. We now use interpolation to obtain an estimate for

$$\tilde{g}(r) = \Sigma\, |a_n|\, r^{q_n} = \Sigma\, a_n \exp\{-i(\arg a_n + q_n\beta_r)\}\, z_r^{q_n}$$

which majorizes $M(r)$. We choose an entire function ϕ (of exponential type 0) such that for all n and r

$$\phi(q_n) = \exp\{-i(\arg a_n + q_n\beta_r)\}; \; M(r, \phi) \leqq e^{\omega(r)},$$

with $\omega(r)$ as in 2.1.

An essential role is played by the *Leau–Wigert transform* $\Phi(\zeta)$ of ϕ $(\zeta \neq 1)$:

$$\Phi(\zeta) = \frac{1}{2\pi i} \int_{C_\eta} \frac{B\phi(s)}{1 - \zeta e^s}\, ds = \begin{cases} \sum_0^\infty \phi(k)\, \zeta^k, & |\zeta| < 1, \\ -\sum_{-\infty}^{-1} \phi(k)\, \zeta^k, & |\zeta| > 1. \end{cases}$$

Here $B\phi(s)$ is the Borel transform of ϕ and $C_\eta = C(0, \eta)^+$ is small (depending on $|\zeta - 1|$). Note that Φ is holomorphic on $C^* - \{1\}$.

For $0 < \rho < 1 < \sigma$,

$$\tilde{g}(r) = \Sigma\, a_n \phi(q_n)\, z_r^{q_n} = \frac{1}{2\pi i}\left(\int_{C_\rho} - \int_{C_\sigma}\right) g(z_r/\zeta)\, \Phi(\zeta)\, d\zeta/\zeta =$$

$$= -\frac{1}{2\pi i}\int_{C'} g(z_r/\zeta)\, \Phi(\zeta)\, d\zeta/\zeta\,,$$

where C' is a small circle $C(1, \eta')^+$. We choose $\eta' = \delta_r/7$ so that by (2.6),

$$M(r) \leqq \tilde{g}(r) \leqq \frac{1}{2} M(r e^{\delta_r})^{3/4} \max_{|\zeta-1| = \delta_r/7} |\Phi(\zeta)|\,. \tag{2.7}$$

(iv) *Relation between neighbouring maximum moduli.* From here on the proof is as in [6], Section 3. One has

$$|\Phi(\zeta)| \leqq 2M\left(\frac{1}{2}|\zeta - 1|, B\phi\right) \quad (\zeta \text{ near } 1),$$

$$M(r, B\phi) \leqq \int_0^\infty M(x, \phi)\, e^{-rx}\, dx \leqq H(r) = \int_0^\infty e^{\omega(x) - rx}\, dx\,.$$

Hence by (2.7),

$$M(r) \leqq M(re^{\delta_r})^{3/4}\, H(\delta_r/14)\,. \tag{2.8}$$

The important properties of $H(r)$ are the following: it maps $(0, \infty)$ one-to-one onto $(\infty, 0)$ and if d is given by $H(d) = e$, then

$$\int_0^d \log\log H(r)\, dr < \infty$$

(cf. [6], Lemma 4). We now *define* δ_r by the condition

$$H(\delta_r/14) = M(r)^{1/8} \qquad (\geqq \varepsilon^{-1/8})\,; \tag{2.9}$$

for small ε, δ_r will be small. Thus (2.8) gives

$$M(re^{\delta_r}) \geqq M(r)^{7/6}. \tag{2.10}$$

(v) *Contradiction for small* ε. The final step consists in showing that if ε were less than a certain constant $\varepsilon_0(\omega, a) > 0$, there would be an infinite sequence of points

$$r_0 < r_1 < r_2 < \ldots \text{ inside } (1, a), \text{ with } r_{n+1} = r_n \exp(\delta_{r_n}),$$

to which (2.10) can be applied. Since this is clearly impossible, ε must be $\geqq \varepsilon_0$.

In order to verify that $\Sigma\, \delta_{r_n}$ becomes arbitrarily small as $\varepsilon \downarrow 0$, we remark that by (2.9) and (2.10),

$$\log\log H(\delta_{r_{n+1}}/14) \geqq \log\log H(\delta_{r_n}/14) + \log(7/6)\,.$$

Thus from the graph of $\log\log H(r)$,

$$14 \int_0^{\delta_{r_0}/14} \log\log H(r)\, dr \geqq (\delta_{r_1} + \delta_{r_2} + \ldots) \log(7/6)\,.$$

3. Going to larger exponents; proof of the main theorem

Our proof of Theorem 1 will be based on Corollary 2 in 2.2 and an auxiliary theorem by which we can pass to larger exponents.

3.1 Second auxiliary theorem. The theorem is as follows:

THEOREM 3. *Let* $0 < \varepsilon < \dfrac{1}{4}\pi$ *and let* Γ *be a rectifiable arc whose oriented chords make angles with the positive real axis of opening* $\leqq \dfrac{1}{4}\pi - \varepsilon$. *Let* $\{\rho_n\}$ *be an increasing sequence of positive real numbers such that the exponentials*

$$\exp(\pm \rho_n z)\,, \qquad n = 1, 2, \ldots \tag{3.1}$$

fail to span $C(\Gamma)$. *Then if the* λ_n *are complex numbers such that*

$$|\lambda_n| \geqq (\sin 2\varepsilon)^{-1/2} \varrho_n\,, \qquad n = 1, 2, \ldots, \tag{3.2}$$

the exponentials

$$\exp(\pm \lambda_n z)\,, \qquad n = 1, 2, \ldots \tag{3.3}$$

also fail to span $C(\Gamma)$.

REMARK. In the case $\rho_n = nL(n)$, with $L(n)$ positive and nondecreasing, one can prove a similar result for all arcs Γ whose oriented chords make angles of opening $\leqq \dfrac{1}{2}\pi - \varepsilon$ with the positive real axis.

Idea of the proof. Since the set (3.1) fails to span $C(\Gamma)$, there is a complex Borel measure $\mu \neq 0$ on Γ whose Laplace transform

$$\hat{\mu}(s) = \int_\Gamma e^{sz} \mu(dz) \tag{3.4}$$

vanishes on the sequence $\{\pm \rho_n\}$. We wish to replace μ by a measure $\nu \neq 0$ whose Laplace transform vanishes on the sequence $\{\pm \lambda_n\}$. To this end, we try to convolve

μ on Γ with a function f whose Laplace transform F (taken along the real axis) can be continued analytically to a function with zeros at the points $\pm\lambda_n$ and poles at the points $\pm\rho_n$. After initial modification of μ (see 3.2) it will be possible to take F of the form

$$F(s) = \frac{G(s)}{1 - s^2/\rho_0^2}, \quad G(s) = \prod_1^\infty \frac{1 - s^2/\lambda_n^2}{1 - s^2/\rho_n^2}, \tag{3.5}$$

where $0 < \rho_0 < \rho_1$. Because of (3.2), $F(s)$ will be $O(1/s^2)$ as $s \to \infty$ in the angles

$$V: \quad \frac{1}{4}\pi + \varepsilon \leqq \arg s \leqq \frac{3}{4}\pi - \varepsilon \pmod{\pi}$$

around the imaginary axis. It follows that F is the Laplace transform of a well-behaved function f in the angles

$$U: \quad |\arg z| \leqq \frac{1}{4}\pi - \varepsilon \pmod{\pi}$$

around the real axis. Thanks to the condition on the chords of Γ we can form the convolution

$$v(z) = \int_\Gamma f(z - \zeta)\,\mu(d\zeta), \qquad z \in \Gamma. \tag{3.6}$$

Its Laplace transform on Γ will indeed be given by

$$\hat{v}(s) = F(s)\,\hat{\mu}(s), \tag{3.7}$$

hence $v(z)dz$ will be orthogonal to the set (3.3) on Γ.

3.2 Dividing out zeros of Laplace transforms. Let Γ, μ and $\hat{\mu}$ be as in 3.1

LEMMA 2. *The entire function $\hat{\mu}(s)$ of (3.4) must have infinitely many zeros different from the numbers $\pm\rho_n$.*

PROOF. The support of μ is not just one point, or $\hat{\mu}$ would have no zeros at all. It will follow that for the zeros $s_n \neq 0$ of $\hat{\mu}$,

$$\Sigma\, 1/|s_n| = \infty.$$

Indeed, convergence of this series would imply that $\hat{\mu}(s)$ is of the form $A\,s^m e^{cs} \cdot \Pi(1 - s/s_n)$, and then the indicator function of $\hat{\mu}$ would be the same as that of e^{cs}; the indicator diagram would be a point. However, this would imply that the support of μ reduces to a point. A proof of the last statement may be obtained by the method in DIXON–KOREVAAR [3], where the growth of Laplace transforms on C^1 arcs of limited slope is studied; the saddle-point method of that paper applies also to our arcs Γ whose chords make angles with the real axis of opening not exceeding $\frac{1}{4}\pi - \varepsilon$.

On the other hand we must have $\Sigma\, 1/\rho_n < \infty$, or the set (3.1) would span $C(\Gamma)$ (see 1.1).

LEMMA 3. *Let w be a zero of $\hat{\mu}(s)$. Then*

$$\frac{\hat{\mu}(s)}{w-s} = \hat{\phi}(s), \quad \phi(z) = \int_a^z e^{-w(z-\zeta)}\mu(d\zeta) = \int_b^z \dots, \qquad z \in \Gamma,$$

where a and b are the left-hand and right-hand endpoint of Γ and the integration is along the curve.

PROOF. Letting D denote differentiation along Γ, one solves the boundary-value problem

$$(D+w)\,\phi = \mu \quad \text{on } \Gamma,\ \phi = 0 \text{ at } a \text{ and } b.$$

COROLLARY 3.

$$\frac{s-\alpha}{s-w}\,\hat{\mu}(s) = \hat{\mu}_1(s), \quad \text{where} \quad \mu_1(dz) = \mu(dz) + (\alpha - w)\,\phi(z)\,dz.$$

3.3 Proof of Theorem 3. Let Γ, μ and $\hat{\mu}$ be as in 3.1. By Lemma 2 and Corollary 3 it may be assumed that $\hat{\mu}$ vanishes not only on the sequence $\{\pm\rho_n\}$, but also at real points $\pm\rho_0$, where $0 < \rho_0 < \rho_1$. We now define $F(s)$ by (3.5) and set

$$\text{(3.8)} \quad \begin{cases} F_k(s) = G_k(s)/(1 - s^2/\rho_0^2), \quad G_k(s) = \prod_1^k \dfrac{1 - s^2/\lambda_n^2}{1 - s^2/\rho_n^2}, \\ f(z) = \dfrac{1}{2\pi i}\displaystyle\int_L F(s)\, e^{-zs}\,ds, \quad f_k(z) = \dfrac{1}{2\pi i}\displaystyle\int_L F_k(s)\, e^{-zs}\,ds. \end{cases}$$

Here L is a straight line through 0 and lying in V (see 3.1), traversed upwards.

Because of (3.2), the functions $G(s)$ and $G_k(s)$ of (3.5) and (3.8) will be bounded by 1 for $s \in V$: setting $s = re^{i\theta}$,

$$|1 - s^2/\rho_n^2| = \{1 - 2(r^2/\rho_n^2)\cos 2\theta + r^4/\rho_n^4\}^{1/2} \geqq$$

$$\geqq 1 - (r^2/\rho_n^2)\cos 2\theta \geqq 1 + (r^2/\rho_n^2)\sin 2\varepsilon \geqq 1 + r^2/|\lambda_n|^2 \geqq |1 - s^2/\lambda_n^2|.$$

It follows that the function $f(z)$ is bounded and continuous on U (choose L so that $\operatorname{Re} zs = 0$). Moreover, $f(z)$ is the uniform limit of the functions $f_k(z)$ on U which are equal to exponential polynomials both on U^+ and on U^-. On U^+, $f_k(z)$ is a linear combination of terms $\exp(-\rho_n z)$ and on U^-, terms $\exp(\rho_n z)$. It is clear that $F_k(s)$ is the ordinary Laplace transform of f_k for $|\operatorname{Re} s| < \rho_0$ (the same holds for F and f). Here one may integrate along the real axis or along any reasonable curve in U from $-\infty$ to $+\infty$.

It remains to verify formula (3.7) for the Laplace transform of the convolution ν of f and μ on Γ (3.6). Clearly, ν is the uniform limit, on Γ, of $\nu_k = f_k * \mu$, hence $\hat{\nu} = \lim \hat{\nu}_k$. By Fubini's theorem

$$\hat{\nu}_k(s) = \int_\Gamma \nu_k(z)\, e^{sz} dz = \int_\Gamma e^{sz} dz \int_\Gamma f_k(z-\zeta)\, \mu(d\zeta) = \int_\Gamma e^{s\zeta} \mu(d\zeta) \int_{\Gamma-\zeta} f_k(z)\, e^{sz} dz. \tag{3.9}$$

To handle the last inner integral we extend Γ (endpoints a, b) to a curve Γ_e from $-\infty$ to $+\infty$ by adjoining the horizontal half-lines $\operatorname{Im} z = \operatorname{Im} a$, $\operatorname{Re} z < \operatorname{Re} a$ and $\operatorname{Im} z = \operatorname{Im} b$, $\operatorname{Re} z > \operatorname{Re} b$. Restricting s to the strip $|\operatorname{Re} s| < \rho_0$, we have

$$\int_{\Gamma_e-\zeta} f_k(z)\, e^{sz} dz = \int_{-\infty}^{\infty} f_k(z)\, e^{sz} dz = F_k(s).$$

Now the corresponding integral from $b - \zeta$ to $+\infty$ is equal to a linear combination of terms

$$\int_{b-\zeta}^{+\infty} e^{-\rho_n z + sz} dz = \frac{1}{\rho_n - s} \exp\{(s-\rho_n)(b-\zeta)\},$$

hence since $\hat{\mu}(s)$ vanishes at the points ρ_n,

$$\int_\Gamma e^{s\zeta} \mu(d\zeta) \int_{b-\zeta}^{+\infty} f_k(z)\, e^{sz} dz = 0.$$

One similarly disposes of the integral where the inner integration is from $-\infty$ to $a - \zeta$. The conclusion from (3.9) is that

$$\hat{\nu}_k(s) = \hat{\mu}(s)\, F_k(s)$$

for $|\operatorname{Re} s| < \rho_0$ and hence for all s; (3.7) follows.

3.4 Proof of Theorem 1. With Γ, $\{\lambda_n\}$ and $\{L(n)\}$ as in Theorem 1, we define

$$p_n = n[L(n)/L(1)], \qquad n = 1, 2, \dots .$$

Since $L(n) \uparrow \infty$, the p_n will be positive integers such that $\{p_n/n\}$ is nondecreasing and $\Sigma\, 1/p_n = \infty$. Hence by Corollary 2.2, the exponentials

$$\exp(\pm \eta p_n z), \qquad n = 1, 2, \dots, \ \eta > 0$$

fail to span $C(e^{i\alpha}\Gamma)$ for every real number α.

We choose α so that the oriented chords of $e^{i\alpha}\Gamma$ make angles with the positive real axis of opening $\leqq \frac{1}{4}\pi - \varepsilon$ and we take $\eta = L(1)(\sin 2\varepsilon)^{1/2}$. Since in that case

$$|e^{i\beta}\lambda_n| \geqq nL(n) \geqq L(1)\, p_n = (\sin 2\varepsilon)^{-1/2} \eta p_n, \qquad n = 1, 2, \dots,$$

we may apply Theorem 3 to conclude that the exponentials

$$\exp(e^{i\beta}\lambda_n z), \qquad n = 1, 2, \ldots$$

fail to span $C(e^{i\alpha}\Gamma)$ for every real number β. Taking $\beta = -\alpha$, it follows that the exponentials $\exp(\lambda_n z)$ fail to span $C(\Gamma)$.

REMARKS. The condition on Γ can also be formulated differently. One defines a tangential direction at $z_0 \in \Gamma$ as a limit of directions of oriented chords $[z_1, z_2]$ when $z_1, z_2 \to z_0$ along Γ. Thus if the tangential directions for Γ do not fill up a right angle, the oriented chords make angles with a fixed direction of opening $\leqq \frac{1}{4}\pi - \varepsilon$ for some $\varepsilon > 0$, and conversely.

Suppose, finally, that Γ is a C^1 arc. Then there are subarcs which satisfy the chord condition, hence exponentials (1.8) as in Theorem 1 can not span $C(\Gamma)$. Corollary 1 follows.

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ON THE SUMMABILITY OF FUNCTION SERIES

By

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To Professor G. Alexits on his eightieth birthday

1. Let X be a measurable space with positive measure μ and let $F=\{f_k(x)\}$ $(k=0,1,\dots)$ be a sequence of real-valued, L_μ-integrable functions on a certain measurable set $E\subset X$. Given a matrix $T=(\alpha_{nk})$ $(n,k=0,1,2,\dots)$ we say that the series $\sum_{k=0}^{\infty} c_k f_k(x)$ with real coefficients c_k is T-summable at $x\in E$ if the series $t_n(x)=\sum_{k=0}^{\infty}\alpha_{nk}c_kf_k(x)$ converges for all $n=0,1,2,\dots$ and if $\lim_{n\to\infty} t_n(x)$ exists.

We give sufficient conditions for the matrix T and the function system F, such that $\sum_{k=0}^{\infty} c_kf_k(x)$ is T-summable a.e. on E for all sequences $\{c_k\}$ satisfying

$$\sum_{k=0}^{\infty} c_k^2<\infty\,. \tag{1}$$

We define the Lebesgue functions of the system F with respect to the summation process T by

$$L_n(T,F;x):=\int_E |K_n(T,F;x,t)|\,d\mu(t)\,,$$

where

$$K_n(T,F;x,t):=\sum_{k=0}^{\infty}\alpha_{nk}f_k(x)f_k(t)\,,$$

and we assume that this series converges for all n and $x,t\in E$. Furthermore, for an index set $1\leqq \nu_1<\nu_2<\dots$ define

$$L_{\nu_n}(T,F):=\int_E \max_{1\leqq j\leqq n} L_{\nu_j}(T,F;x)\,d\mu(x)\,.$$

In case $T=I$; i.e. $\alpha_{nk}=1$ for $0\leqq k\leqq n$, $\alpha_{nk}=0$ for $k>n$, such that $t_n(x)=S_n(x)=$ $=\sum_{k=0}^{n}c_kf_k(x)$; we write $L_{\nu_n}(F;x)$ and $L_{\nu_n}(F)$ instead of $L_{\nu_n}(I,F;x)$ and $L_{\nu_n}(I,F)$, resp.

The following results on the influence of the Lebesgue functions on summability are known. ALEXITS and SHARMA [2, Theorem 6, Remark 2] proved that $\sum_{k=0}^{\infty}c_kf_k(x)$ is C_α-summable on E for $\alpha>0$, if (1) holds, $\mu(E)<\infty$, and if $L_n(C_\alpha,F;x)=O(1)$ as $n\to\infty$, uniformly for $x\in E$. The special case $\alpha=1$, F orthonormal is a well-known theorem of KACZMAR [4], SUNOUCHI [7] and LEINDLER [5] have extended Kaczmar's result on orthonormal systems F to Riesz-summability $(R, \lambda, 1)$. Results on general summation processes were obtained by MÓRICZ [6]; and ALEXITS, JOÓ, TANDORI [3] recently proved the following. The series $\sum_{k=0}^{\infty} c_k f_k(x)$ is T-summable a.e. on E for all sequences $\{c_k\}$ with (1) if $\lim_{n\to\infty} \alpha_{nk}$ exists for all k, $L_n(T, F; x) = O_x(1)$ for all $x \in E$, and if

$$(2) \qquad \left| \int_E K_i(T, F; x, t)\, K_j(T, F; y, t)\, d\mu(t) \right| \leqq \sum_{k=0}^{\infty} \beta_k \,|K_k(T, F; x, y)|$$

for $x, y \in E$, $0 \leqq \beta_k = \beta_k(i, j)$ with $\sum_{k=0}^{\infty} \beta_k = O(1)$ uniformly for $i, j = 0, 1, 2, \dots$.

We shall prove a theorem that generalizes the special results on C_α and $(R, \lambda, 1)$ summability to general summation processes. Especially we do not need an assumption like (2).

THEOREM. *Suppose that $E \subset X$ is measurable, and that $T = (\alpha_{nk})$ satisfies*

$$(3) \qquad \begin{cases} 0 \leqq \alpha_{nk} \leqq K; \quad \alpha_{n+1,k} \geqq \alpha_{nk} \text{ for all } k, n = 0, 1, 2, \dots, \text{ and that} \\ T \text{ is row-finite, i.e. } \alpha_{nk} = 0 \text{ for } k > k_n,\ n = 0, 1, \dots. \end{cases}$$

Furthermore, let $F = \{f_k(x)\}$ be a system of L_μ-integrable functions on E satisfying

$$(4) \qquad L_n(T, F) = O(1) \text{ as } n \to \infty .$$

Then $\sum_{k=0}^{\infty} c_k f_k(x)$ is T-summable a.e. on E for all sequences $\{c_k\}$ with (1).

2. The proof of our theorem depends essentially on the following result of ALEXITS [1] on the convergence of function series.

THEOREM A. *Suppose $E \subset X$ is measurable, $F = \{f_k(x)\}$ is a system of L_μ-integrable functions on E, and assume that $L_{\nu_n}(F) = O(1)$ for some index set $1 \leqq \nu_1 < \nu_2 < \dots$. Then* $\lim_{n\to\infty} S_{\nu_n}(x)$ *exists a.e. on E for all $\{c_k\}$ with* (1).

PROOF OF THEOREM. Since $T = (\alpha_{nk})$ is row-finite, i.e. $\alpha_{nk} = 0$ for $k > k_n$, $t_n(x)$ exists for all $x \in E$, $n = 0, 1, \dots$. We will define a function system $\Phi = \{\phi_{nk}(x)\}$, $0 \leqq k \leqq k_n$, $n = 0, 1, \dots$, such that the following hold:

$$\text{(a)} \qquad \phi_{nk}(x) = d_{nk} f_k(x) \quad \text{for } 0 \leqq k \leqq k_n,\ d_{nk} \in \mathbf{R};$$

$$\text{(b)} \qquad t_n(x) = \sum_{r=0}^{n} \sum_{k=0}^{k_r} d_{rk} c_k \phi_{rk}(x) \quad \text{for } x \in E;$$

$$\text{(c)} \qquad K_n(T, F; x, t) = \sum_{r=0}^{n} \sum_{k=0}^{k_r} \phi_{rk}(x)\, \phi_{rk}(t) \quad \text{for } x, t \in E; \text{ and}$$

$$\text{(d)} \qquad \sum_{k=0}^{\infty} c_k^2 < \infty \text{ implies } \sum_{r=0}^{\infty} \sum_{k=0}^{k_r} d_{rk}^2 c_k^2 < \infty .$$

If these conditions are satisfied we can apply Alexits' Theorem A to the function system $\Phi = \{\phi_{nk}(x)\}$.

If $\sum_{k=0}^{\infty} c_k^2 < \infty$, then it follows from (d), (c) and (4) by Theorem A that

$$\lim_{n\to\infty} \sum_{r=0}^{n} \sum_{k=0}^{k_r} d_{rk} c_k \phi_{rk}(x) = \lim_{n\to\infty} t_n(x)$$

(by (b)) exists a.e. on E.

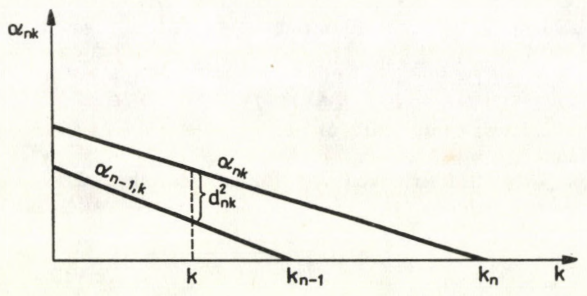
It remains to show the existence of a system Φ satisfying (a) – (d). Because of (a) we have to define numbers d_{nk}, such that (b), (c), (d) hold. It follows from (a) and (c) that the d_{nk} must necessarily satisfy

$$\sum_{k=0}^{\infty} (\alpha_{nk} - \alpha_{n-1,k})\, c_k f_k(x) f_k(t) = \sum_{k=0}^{k_n} d_{nk}^2 f_k(x) f_k(t)$$

for $n = 0, 1, \ldots, x \in E$, where we set $\alpha_{-1,k} = 0$ for all k. Since $\alpha_{nk} \geqq \alpha_{n-1,k}$ by (3) we can define

$$d_{nk} := \sqrt{\alpha_{nk} - \alpha_{n-1,k}} \quad \text{for} \quad 0 \leqq k \leqq k_n, \quad n = 0, 1, \ldots$$

so that (c) is satisfied. This construction of the d_{nk} is illustrated in the following figure.



Clearly, we have

$$(5) \qquad \sum_{r=0}^{n} d_{rk}^2 = \alpha_{nk} \quad \text{for} \quad k, n = 0, 1, 2, \ldots.$$

It follows from this identity that

$$t_n(x) = \sum_{k=0}^{k_n} \alpha_{nk} c_k f_k(x) = \sum_{r=0}^{n} \sum_{k=0}^{k_r} d_{rk}^2 c_k f_k(x) = \sum_{r=0}^{n} \sum_{k=0}^{k_r} d_{rk} c_k \phi_{rk}(x).$$

Now, (d) follows from (5) and (3), since

$$\sum_{r=0}^{n} \sum_{k=0}^{k_r} d_{rk}^2 c_k^2 = \sum_{k=0}^{k_n} \alpha_{nk} c_k^2 \leqq K \cdot \sum_{k=0}^{\infty} c_k^2 \quad \text{for all } n.$$

REMARK. If the measurable space X is σ-finite or if $\mu(E)<\infty$ we may replace condition (4) by the weaker condition $L_n(T, F; x)=O_x(1)$ for all $x \in E$ as in [6]; since we may apply our theorem to each set E_j of a sequence of measurable sets $E_j \subset E$ satisfying.

$$L_n(T, F; x) \leqq j \quad \text{for all } n \text{ and } x \in E_j, \quad \mu(E_j)<\infty$$

and $\mu(E \setminus \bigcup_{j=1}^{\infty} E_j)=0$. This argument was already used in [6] and [8]. Moreover, we mention that we do not need condition (2) (as in [4] and [6]) at all, while our assumptions on the matrix T are somewhat stronger than in the theorems of [4] and [6].

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STRONG APPROXIMATION OF FOURIER SERIES AND STRUCTURAL PROPERTIES OF FUNCTIONS

By

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With admiration to Professor G. Alexits on his eightieth birthday

1. Let $f(x)$ be a continuous and 2π-periodic function and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

be its Fourier series. Denote $s_n(x) = s_n(f, x)$ the n-th partial sum of (1), and denote $\|\cdot\|$ the usual supremum norm.

The problem of the strong approximation of Fourier series is due to Professor G. Alexits.

Generalizing one of the results of Alexits and Králik [1], in [5], among others, we proved the following

Theorem A. *If $f \in \operatorname{Lip} \alpha$, $0 < \alpha \leqq 1$, $p > 0$, then*

$$H_n(f, p) = \left\| \left\{ \frac{1}{n+1} \sum_{\nu=0}^{n} |s_\nu - f|^p \right\}^{1/p} \right\| = \begin{cases} O(n^{-\alpha}) & \text{if } \alpha < \dfrac{1}{p}, \\ O\left(\left(\dfrac{\log n}{n}\right)^{\alpha}\right) & \text{if } \alpha = \dfrac{1}{p}, \end{cases} \tag{2}$$

and these estimations are best possible.

This theorem can be interpreted such that the estimation

$$H_n(f, p) = O\left(\left(\frac{\log n}{n}\right)^{1/p}\right)$$

cannot be improved for the whole class $\operatorname{Lip} \dfrac{1}{p}$, $p \geqq 1$.

The question of the following type: whether an estimation like

$$H_n(f, p) = o\left(\left(\frac{\log n}{n}\right)^{1/p}\right),$$

what properties of the function involves, or what subclass of $\operatorname{Lip} 1/p$ characterizes, was first raised and solved by Freud [3] in the following special case:

[1] This research was made while the author worked in the Department of Mathematics of Justus-Liebig University, Giessen, as a visiting professor by a grant of the "Deutsche Forschungsgemeinschaft".

If $p>1$ *and*

$$H_n(f,p)=O(n^{-1/p}),$$

or equivalently, if

$$\left\|\sum_{k=1}^{\infty}|s_k-f|^p\right\|<\infty \qquad (p>1), \tag{3}$$

then

$$\lim_{h\to 0} h^{-1/p}\,(f(x+h)-f(x))=0 \tag{4}$$

holds almost everywhere.

He also raised, *whether* (4) *holds for all* x?

In [7] we verified *that* (3) *does not imply* (4) *everywhere.* Our counterexample is the function

$$f_0(x)=\sum_{n=1}^{\infty}\frac{\sin nx}{n^{1+1/p}}$$

which satisfies (3) but at the point $x=0$ it does not fulfil (4).

In [8] we generalized these results to such strong summability means which are determined by a general triangular matrix and we also investigated the case $p=1$.

A collected form of the theorems published in [8] reads:

THEOREM B. *Let* $p\geqq 1$. *Suppose that the triangular matrix* $\|\alpha_{nk}\|$ $\left(\alpha_{nk}=\dfrac{\lambda_k}{\Lambda_n},\ \lambda_k>0;\ n=0,1,\dots;\ k=0,1,\dots,n \text{ and } \Lambda_n=\sum_{k=0}^{n}\lambda_k\right)$ *has the following properties:* $n\Lambda_n^{-1/p}$ *is increasing,*

$$\sum_{k=0}^{n}\Lambda_k^{-1/p}\leqq Kn\Lambda_n^{-1/p}, \tag{5}$$

$$\sum_{k=n}^{\infty}k^{-1}\Lambda_k^{-1/p}\leqq K\Lambda_n^{-1/p}, \tag{6}$$

$$\left\{\sum_{k=n}^{2n}\lambda_k^{1/(1-p)}\right\}^{p-1}\leqq Kn^p\Lambda_n^{-1}, \qquad \text{if} \quad p>1, \tag{7}$$

and

$$\Lambda_n\leqq Kn\lambda_n \qquad \text{if} \quad p=1. \tag{8}$$

Then

$$\left\|\sum_{n=0}^{\infty}\lambda_n|s_n-f|^p\right\|<\infty \tag{9}$$

implies

$$|f(x+h)-f(x)|\leqq K\Lambda^{-1/p}\left(\frac{1}{h}\right) \qquad \text{for all} \quad x; \tag{10}$$

and

(11) $$\lim_{h\to 0} \Lambda^{1/p}\left(\frac{1}{h}\right)\big(f(x+h)-f(x)\big)=0 \qquad \textit{for almost every } x;$$

furthermore there exists a function $f_0(x)$ *satisfying* (9) *but*

(12) $$f_0\left(\frac{\pi}{2^n}\right)-f_0(0)>\frac{1}{4}\Lambda^{-1/p}\left(\frac{2^n}{\pi}\right),$$

where $\Lambda(x)$ *denotes the increasing function being linear between n and* $n+1$, *moreover* $\Lambda(n)=\Lambda_n$.

It is clear that in the case $\lambda_k=1$ and $p>1$ all of the conditions of Theorem B are satisfied therefore Theorem B includes the result of Freud.

It is also easy to verify that many of the Riesz-means, $(C, 1)$-means satisfy the conditions of Theorem B, for example, if

$$\lambda_k=k^{\beta-1},\quad 0<\beta<p;\quad \text{or}\quad \lambda_k=\frac{\log k}{k^\beta},\quad 0<\beta<1.$$

But we remark that if $p=1$ and $\lambda_k=1$ then condition (5) is not fulfilled, therefore this theorem says nothing about what condition (9) with $p=1$ and $\lambda_k=1$ implies.

This problem was investigated in a joint paper of LEINDLER and NIKISIN [13] proving

THEOREM C. *If*

(13) $$\left\|\sum_{n=0}^{\infty}|s_n-f|\right\|<\infty$$

then

(14) $$|f(x+h)-f(x)|\leqq Kh\log\frac{1}{h} \qquad \textit{for all } x,$$

and

(15) $$|f(x+h)-f(x)|=O_x(h) \qquad \textit{for almost every } x.$$

Furthermore there exists a function $f_0(x)$ *satisfying* (13) *but*

(16) $$f_0\left(\frac{\pi}{2^n}\right)-f_0(0)>\frac{1}{8}\,\frac{\pi}{2^n}\log\frac{2^n}{\pi} \qquad \textit{for all } n\geqq 6.$$

This result was extended to the r-th derivative of f in [9].

Inequality (16) shows that condition (13) does not imply that $f\in \text{Lip }1$. In connection with this fact I raised (see [9], [10]) the following problem: *Does the condition*

(17) $$\left\|\sum_{n=0}^{\infty}|s_n-f|^p\right\|<\infty \qquad \textit{with } 0<p<1$$

imply $f\in\text{Lip }1$?

The answer was given in an affirmative form recently by OSKOLKOV [14] and SZABADOS [15] independently. They both proved the following stronger statement:

THEOREM D. *If*

$$\left\| \sum_{n=1}^{\infty} \Omega(|s_n - f|) \right\| < \infty \tag{18}$$

and

$$\int_0^1 \frac{dx}{\Omega(x)} < \infty, \tag{19}$$

then $f \in \text{Lip } 1$, *where* $\Omega(\delta)$ *denotes an arbitrary modulus of continuity.*

Under a certain restriction on $\Omega(\delta)$ they also proved the necessity of condition (19).

Setting $\Omega(x) = x^p$ with $0 < p < 1$, (19) is fulfilled, (18) reduces to (17), thus Theorem D answers our problem, too.

Szabados also proved that *if* $0 < p < 1$, $\frac{1}{p} = r + \alpha$, *where* $r = \left[\frac{1}{p}\right]$ *then condition* (17) *implies that* $f^{(r-1)}(x)$ *is continuous and*

$$\omega(f^{(r-1)}; \delta) = \begin{cases} O(\delta(\log 1/\delta)^{1/p}) & \text{if } \alpha = 0, \\ O(\delta) & \text{if } \alpha > 0, \end{cases}$$

where $\omega(f; \delta)$ denotes the modulus of continuity of f.

This result was generalized by us ([11], [12]) and one of the generalizations can be read here as Corollary 2.2.

Very recently, in a joint paper of KROTOV and LEINDLER [4], a condition of different type from (18) is given, but in connection with the strong approximation, it also implies the inclusion $f \in \text{Lip } 1$, and in a certain sense it is best possible. A special case of our result is

THEOREM E. *If* $0 < p < \infty$ *and* $\{\lambda_n\}$ *is a monotone sequence such that* $\{n^\theta \lambda_n\}$ *with a certain* $0 < \theta < 1$ *increases then the condition*

$$\sum_{n=1}^{\infty} \frac{1}{(n\lambda_n)^{1/p}} < \infty \tag{20}$$

is necessary and sufficient that

$$\left\{ f : \left\| \sum_{n=1}^{\infty} \lambda_n |s_n - f|^p \right\| < \infty \right\} \subset \text{Lip } 1. \tag{21}$$

Theorem E with $0 < p < 1$ and $\lambda_n = 1$ also gives an affirmative answer to the problem raised at (17).

We can also observe that Theorem B does not include Theorem E, namely (10) with $\Lambda_n = n^p$ would give (21) but then condition (5) is not satisfied. This fact, and as we have also seen at Theorem C, condition (5), is the most critical point in the usage of Theorem B, and if we investigate the proof of Theorem B more carefully we can observe that it is just for the sake of simplicity. Therefore we try to modify condition (5) and by this to extend the use of Theorem B. Moreover we intend to generalize Theorem B for any positive p which seems to be the most important part of our result.

2. The main aim of the present work is to prove Theorems 1 and 2, but we shall also give some further results.

THEOREM 1. *Let $p>0$ and let $\rho(x)$ be a nonincreasing function. Suppose that the triangular matrix (α_{nk}) $\left(\alpha_{nk}=\frac{\lambda_k}{\Lambda_n},\ \lambda_k>0;\ n=0,1,\dots;\ k=0,1,\dots,n;\ \text{and } \Lambda_n=\sum_{k=1}^{n}\lambda_k\right)$ has the following properties:*

$$\frac{1}{n}\sum_{k=1}^{n}\Lambda_k^{-1/p}+\sum_{k=n}^{\infty}k^{-1}\Lambda_k^{-1/p}\leqq K\rho(n)\ [2] \tag{2.1}$$

$$\left\{\sum_{k=n}^{2n}\lambda_k^{1/(1-p)}\right\}^{p-1}\leqq Kn^p\Lambda_n^{-1},\quad \text{if } p>1, \tag{2.2.1}$$

$$\Lambda_n\leqq K\lambda_n\cdot n,\quad \text{if } p=1, \tag{2.2.2}$$

$$\left\{\sum_{k=n}^{2n}\lambda_k^{p/(p-1)}\right\}^{1-p}\leqq Kn\Lambda_n^{-p},\quad \text{if } p<1. \tag{2.2.3}$$

Then

$$\left\|\sum_{n=0}^{\infty}\lambda_n\,|\,s_n-f\,|^p\right\|<\infty \tag{2.3}$$

implies for all x

$$|f(x+h)-f(x)|\leqq K\rho\left(\frac{1}{h}\right),\quad \text{i.e. } \omega(f;h)=O\left(\rho\left(\frac{1}{h}\right)\right). \tag{2.4}$$

Moreover, suppose the function $\rho(x)$ satisfies the following additional conditions: for any $\varepsilon>0$ there exists a number $N(\varepsilon)$ such that if $N\geqq N(\varepsilon)$ then

$$\rho(x)\leqq N\varepsilon\rho(Nx)\leqq N\varepsilon^2\rho(x) \tag{2.5}$$

hold for any $x>x_0(\varepsilon)$. Then (2.3) also implies

$$\lim_{h\to 0}\rho^{-1}\left(\frac{1}{h}\right)\left(f(x+h)-f(x)\right)=0 \tag{2.6}$$

for almost every x.

[2] We use $K, K_1, K_2, \dots$ to denote various positive numbers not necessarily the same at each occurence.

THEOREM 2. *Suppose that the matrix* (α_{nk}) *and the function* $\rho(x)$ *satisfy conditions* (2.1) *and* (2.2.*i*). *If r is a nonnegative integer and*

$$C_m = \left\| \sum_{n=m}^{\infty} \lambda_n (n^r \mid s_n - f \mid)^p \right\|,$$

then the condition $C_0 < \infty$ *implies*

$$\omega(f^{(r)}, h) = O\left(\rho\left(\frac{1}{h}\right)\right), \tag{2.7}$$

and if $C_m \to 0$ *and* $\rho(x) \leqq N\varepsilon\rho(Nx)$ *for any* $N \geqq N(\varepsilon)$ *and* $x > x_0(\varepsilon)$, *then*

$$\lim_{h\to 0} \rho^{-1}\left(\frac{1}{h}\right)(f^{(r)}(x+h) - f^{(r)}(x)) = 0 \tag{2.8}$$

also holds everywhere.

The generality of Theorems 1 and 2 makes the danger that one cannot see their efficacy, therefore it seems to be worthy to present some simple and interesting collor-aries of these theorems.

First we list some corollaries of Theorem 1.

COROLLARY 1.1. *Condition* (17) *implies* $f \in \text{Lip } 1$.

COROLLARY 1.2. *If* $p > 0$ *and the sequence* $\{\lambda_k\}$ *is monotone and for a certain* $0 < \theta < 1$ *the sequence* $\{n^\theta \lambda_n\}$ *is nondecreasing, then the condition*

$$\left\| \sum_{n=1}^{\infty} \lambda_n \mid s_n - f \mid^p \right\| < \infty$$

under (20) *implies* $f \in \text{Lip } 1$.

It is clear that Corollary 1.2 with $\lambda_n = 1$ and $0 < p < 1$ includes Corollary 1.1.

COROLLARY 1.3. *If* $0 < p < \infty$ *and*

$$\left\| \sum_{n=1}^{\infty} \frac{1}{n} (n \mid s_n - f \mid)^p \right\| < \infty$$

then

$$\omega(f; h) \leqq Kh \log \frac{1}{h}.$$

We observe that these corollaries can be deduced from Theorem 1 of [4], too.

COROLLARY 1.4. *If* $0 < \alpha < 1$, $p > 0$ *and*

$$\left\| \sum_{n=1}^{\infty} \frac{1}{n} (n^\alpha \mid s_n - f \mid)^p \right\| < \infty$$

then $f \in \operatorname{Lip} \alpha$; *furthermore*

$$\lim_{h\to 0} h^{-\alpha}\big(f(x+h)-f(x)\big)=0$$

holds almost everywhere.

We can show that *Theorem* B *is also a corollary of Theorem* 1. Namely it is evident that if $\rho(x)=\Lambda(x)^{-1/p}$ then the conditions of Theorem B are the same as those of Theorem 1 for $p \geqq 1$, thus (2.4) and (2.6) imply (10) and (11), respectively. The only crux is to verify that conditions (2.5) hold, but we prove it as follows:

Let $\varepsilon > 0$ be an arbitrary fixed number. Let us choose a positive integer μ such that $\mu^{-1} < \varepsilon$. Let $N = 2^\mu$. Then using conditions of Theorem B, we obtain that

$$\rho(x)=\Lambda(x)^{-1/p}=\frac{1}{x}\,x\,\Lambda(x)^{-1/p}\leqq\frac{1}{x}\left(\frac{1}{\mu}\sum_{k=0}^{\mu}2^k x\,\Lambda(2^k x)^{-1/p}\right)\leqq$$

$$\leqq\frac{K}{\mu x}2^\mu x\,\Lambda(2^\mu x)^{-1/p}\leqq K\varepsilon N\Lambda(Nx)^{-1/p}=K\varepsilon N\rho(Nx)$$

and

$$\rho(Nx)=\Lambda(2^\mu x)^{-1/p}\leqq\frac{1}{\mu}\sum_{n=0}^{\mu}\Lambda(2^n x)^{-1/p}\leqq\frac{K}{\mu}\Lambda(x)^{-1/p}\leqq K\varepsilon\rho(x)$$

whence (2.5) follows.

It is plain that Corollaries 1.1, 1.3 and 1.4 immediately follow from Theorem 1 by choosing $\lambda_k = 1$, $\rho(x)=\frac{1}{x}$; $\lambda_k = k^{p-1}$, $\rho(x)=\frac{1}{x}\log x$ and $\lambda_k = k^{\alpha p-1}$, $\rho(x) = x^{-\alpha}$, respectively; namely in these cases conditions (2.1) and (2.2.*i*) are obviously satisfied.

To prove Corollary 1.2, i.e. that (20) implies (2.1) and (2.2.*i*), it also just needs some elementary calculations. Now we verify only that (20) $\Rightarrow$ (2.1) for $p \geqq 1$, namely all of the cases would run similarly. Since λ_n must be increasing thus $\frac{n}{2}\lambda_{n/2}\leqq\Lambda_n\leqq n\lambda_n$, whence by (20)

$$\frac{1}{n}\sum_{k=1}^{n}\Lambda_k^{-1/p}\leqq K\frac{1}{n}\sum_{k=1}^{\infty}(k\lambda_k)^{-1/p}\leqq\frac{K}{n}$$

and

$$\sum_{k=n}^{\infty}k^{-1}\Lambda_k^{-1/p}\leqq K\sum_{k\geqq n/2}^{\infty}k^{-1}(k\lambda_k)^{-1/p}\leqq\frac{K}{n}\sum_{k=1}^{\infty}(k\lambda_k)^{-1/p}\leqq\frac{K}{n},$$

therefore we can choose $\rho(x)=\frac{1}{x}$, and then (2.4) ensures $f\in\operatorname{Lip} 1$.

Before presenting corollaries of Theorem 2 we would like to emphasize that, unfortunately, Theorem 1 does not give statement (15) of Theorem C, but it includes most of the results cited before.

Corollaries of Theorem 2:

First of all we can state in a very succinct form that if *the n-th term of the series appearing in the conditions of the previous corollaries is multiplied by* n^{rp} *and the supremum norm of these new series is finite, then in the statements of the corollaries* f *can be replaced by* $f^{(r)}$, *except the second statement of Corollary* 1.4.

E.g. we have as a generalization of Corollary 1.1 the following

COROLLARY 2.1. *If* $0 < p < 1$ *and* r *is a nonnegative integer, then*

$$\left\| \sum_{n=1}^{\infty} n^{rp} \,|\, s_n - f \,|^p \right\| < \infty$$

implies $f^{(r)} \in \mathrm{Lip}\ 1$.

COROLLARY 2.2. *If* $0 < p < 1$ *and* $\frac{1}{p} = r + \alpha$, *where* r *is a positive integer and* $0 < \alpha \leqq 1$, *then* (17) *implies*

$$\omega(f^{(r)}; \delta) = \begin{cases} O\left(\delta \log \frac{1}{\delta}\right), & \text{if } \alpha = 1, \\ O(\delta^{\alpha}), & \text{if } 0 < \alpha < 1. \end{cases}$$

This corollary has been proved in [12].

To deduce Corollary 2.2 we only have to set $\lambda_n = n^{-rp}$ and to verify that the conditions of Theorem 2 are satisfied with

$$\rho(x) = \frac{1}{x} \log x \quad \text{if} \quad \alpha = 1; \quad \text{and} \quad \rho(x) = x^{\alpha} \quad \text{if} \quad 0 < \alpha < 1.$$

But this can be done by a straightforward calculation therefore we omit it.

COROLLARY 2.3. *If* $0 < \alpha < 1$, $p > 0$ *and*

$$\left\| \sum_{n=m}^{\infty} n^{(\alpha+r)p-1} \,|\, s_n - f \,|^p \right\| \to 0 \quad \text{as} \quad m \to \infty, \tag{2.9}$$

then

$$\lim_{h \to 0} h^{-\alpha}\big(f^{(r)}(x+h) - f^{(r)}(x)\big) = 0 \tag{2.10}$$

holds everywhere.

It is evident that Corollary 2.3 cannot be extended to the case $\alpha = 1$, namely (2.9) does not imply that $f^{(r)}$ should be constant.

Moreover, the following Theorem 3 (or Corollary 3.1) shows that (2.9) is, in general, necessary that (2.10) should hold.

Theorem 3 will also show that Theorems 1 and 2 cannot be strengthened generally, that is, the the given estimations are best possible.

THEOREM 3. *Let* $p > 0$ *and let* r *be a nonnegative integer. Suppose that the monotone* $\{\lambda_k\}$ *satisfies the conditions:* $\{n^{rp-1}\Lambda_n\}$ *is monotone, for a certain* $\eta < 1 - pr$

$\{n^{\eta+rp-1}\Lambda_n\}$ is nondecreasing, and $n\lambda_n \leqq K\Lambda_n$. Then there exists a function $F(x) = F(\lambda, r, p; x)$ such that

$$\left\| \sum_{n=1}^{\infty} \lambda_n (n^r \mid s_n(F) - F \mid)^p \right\| < \infty \tag{2.11}$$

but

$$\omega\left(F^{(r)}; \frac{1}{n}\right) \geqq C \frac{1}{n^2} \sum_{k=1}^{n} k\Lambda_k^{-1/p}\,; \tag{2.12}$$

and if r is even, then

$$\omega\left(F^{(r)}; \frac{1}{n}\right) \geqq C \frac{1}{n} \sum_{k=1}^{n} \Lambda_k^{-1/p}, \tag{2.13}$$

if r is odd, then

$$\omega\left(\tilde{F}^{(r)}; \frac{1}{n}\right) \geqq C \frac{1}{n} \sum_{k=1}^{n} \Lambda_k^{-1/p}, \tag{2.14}$$

where C is a positive absolute constant.

It is obvious that (2.13) and (2.14) are stronger statements than (2.12). We also observe that (2.13) cannot be extended to an odd integer (or (2.14) to an even one), generally, namely if $p = 1$, $\lambda_n = 1$ and r is odd, then by one of our results ([9], Theorem 2) (2.11) implies $F^{(r)} \in \text{Lip } 1$, but in this case (2.13) would reduce to

$$\omega\left(F^{(r)}; \frac{1}{n}\right) \geqq C \frac{1}{n} \log n\,,$$

in contradiction to $F^{(r)} \in \text{Lip } 1$. The cited result of [9] also gives, in the same special case, that if r is even, then (2.11) implies $\tilde{F}^{(r)} \in \text{Lip } 1$, and this shows that (2.14) does not hold for even r.

One more observation; if

$$\rho(n) := \frac{1}{n} \sum_{k=1}^{n} \Lambda_k^{-1/p},$$

then Theorem 3 shows that the estimations (2.4) and (2.7) cannot be strengthened, as we have stated before. Namely, it is easy to verify that the conditions (2.1) and (2.2.i) are fulfilled.

We shall see later on that the special case $\lambda_n = n^{\alpha p - 1}$ of Theorem 3 is worthy for formulating as a

COROLLARY 3.1. *Let $p > 0$, $0 < \alpha \leqq 1$ and r be a nonnegative integer. Then there exists a function $F(x) = F(r, \alpha; x)$ such that*

$$\left\| \sum_{n=1}^{\infty} n^{(r+\alpha)p-1} \mid s_n(F) - F \mid^p \right\| < \infty\,, \tag{2.15}$$

but

$$\omega(F^{(r)};\delta) \geqq C\delta^{\alpha} \quad \textit{if} \quad 0<\alpha\leqq 1\,; \tag{2.16}$$

moreover if $\alpha = 1$, *then*

$$\omega(F^{(r)};\delta) \geqq C\delta \log\frac{1}{\delta} \qquad \textit{for even } r,$$

and

$$\omega(\tilde{F}^{(r)};\delta) \geqq C\delta \log\frac{1}{\delta} \qquad \textit{for odd } r,$$

where $C > 0$.

Since (2.5) holds with $\rho(x) = x^{-\alpha}$ for $0<\alpha<1$, *thus* (2.16) *verifies that* (2.6) *cannot be extended to every* x; *that* (2.7) *is best possible*; *moreover* (2.16) *shows that the condition* $C_m \to 0$ *is necessary to* (2.8).

Summing up our remarks we can state that *the statements of Theorems* 1 *and* 2, *in general, are best possible.*

In order to show that under the assumptions of Corollary 3.1 the conditions of Theorem 3 are fulfilled we have just to mention that η can be chosen such that $1-pr-\alpha p<\eta<1-pr$ holds.

Finally we mention one more result.

Corollary 3.1 with $r=0$, $p=\alpha=1$ shows that the condition

$$\left\| \sum_{n=1}^{\infty} |s_n(f)-f| \right\| < \infty \tag{2.17}$$

does not imply $f\in \mathrm{Lip}\, 1$, but the condition

$$\sum_{n=1}^{\infty} E_n < \infty \tag{2.18}$$

where $E_n = E_n(f)$ denotes the best approximation of f by trigonometric polynomials of order at most n, by the following wellknown inequality of Steckin (see [2], p. 534)

$$\omega\left(f;\frac{1}{n}\right) \leqq \frac{1}{n}\sum_{k=1}^{n} E_k$$

does imply $f\in \mathrm{Lip}\, 1$; hence it is obvious that these conditions are not equivalent. But if we strengthen condition (2.17) just a little bit, namely if we claim

$$\sum_{m=0}^{\infty} \left\| \sum_{n=2^m+1}^{2^{m+1}} |s_n(f)-f| \right\| < \infty,$$

this will already be equivalent to (2.18), namely we have

THEOREM 4. *Let $p > 0$ and let $\{\mu_n\}$ denote a monotone sequence with the property $0 < k \leqq \mu_{2^{n+1}}/\mu_{2^n} \leqq K < \infty$ for all n. Then the conditions*

$$\sum_{m=0}^{\infty} \left\| \sum_{n=2^m+1}^{2^{m+1}} \mu_n \,|\, s_n - f \,|^p \right\| < \infty \tag{2.19}$$

and

$$\sum_{n=0}^{\infty} \mu_n E_n^p < \infty \tag{2.20}$$

are equivalent.

3. We require some lemmas to prove our theorems.

LEMMA 1 ([6], Theorem 5). *We have for any positive p and natural number n*

$$\left\| \left\{ \frac{1}{n} \sum_{k=n}^{2n} |\, s_k - f \,|^p \right\}^{1/p} \right\| = O(E_n).$$

LEMMA 2 ([12], Lemma 2). *If $0 < p \leqq 1$ then*

$$E_n \left(\frac{E_{2n}}{E_n} \right)^{1/p^2} = O\left(\left\| \left\{ \frac{1}{n} \sum_{k=n+1}^{2n} |\, s_k - f \,|^p \right\}^{1/p} \right\| \right).$$

LEMMA 3. *Let $p > 0$ and r be a nonnegative integer. Suppose that the matrix (α_{nk}) satisfies conditions (2.2.i) ($1 \leqq i \leqq 3$). Then, with a certain μ,*

$$C_\mu = \left\| \sum_{n=\mu+1}^{\infty} \lambda_n (n^r \,|\, s_n - f \,|)^p \right\| < \infty \tag{3.1}$$

implies

$$\sum_{n=m+1}^{\infty} 2^{nr} E_{2^n} \leqq K \left(\sum_{n=m}^{\infty} C_{2^n} \Lambda_{2^n}^{-1/p} + 2^{-m} \sum_{n=0}^{m} 2^n C_{2^n} \Lambda_{2^n}^{-1/p} \right)^3 \tag{3.2}$$

and

$$\sum_{n=0}^{m+1} 2^{n(r+1)} E_{2^n} \leqq K \sum_{n=0}^{m} 2^n C_{2^n} \Lambda_{2^n}^{-1/p}. \tag{3.3}$$

PROOF. It is obvious that if $V_n(x) = \frac{1}{n} \sum_{\nu=n+1}^{2n} s_\nu(x)$ then

$$E_{2n} \leqq \|\, f(x) - V_n(x) \,\|. \tag{3.4}$$

[3] The second sum on the right-hand side of (3.2) for $p \geqq 1$ can be omitted.

If $p > 1$ then, by (2.2.1), (3.1) and using Hölder's inequality, we obtain that

$$|f(x) - V_n(x)| \leqq \frac{1}{n} \sum_{k=n+1}^{2n} |s_k(x) - f(x)| \leqq \tag{3.5}$$

$$\leqq \frac{1}{n} \left\{ \sum_{k=n+1}^{2n} \lambda_k |s_k(x) - f(x)|^p \right\}^{1/p} \left\{ \sum_{k=n+1}^{2n} \lambda_k^{1/(1-p)} \right\}^{(p-1)/p} \leqq$$

$$\leqq \frac{K}{n^{r+1}} \left\{ \sum_{k=n+1}^{2n} \lambda_k (k^r |s_k(x) - f(x)|)^p \right\}^{1/p} n \Lambda_n^{-1/p} \leqq K n^{-r} \Lambda_n^{-1/p};$$

and if $p = 1$ then by (2.2.2)

$$|f(x) - V_n(x)| \leqq \frac{K}{n^{r+1}} \sum_{k=n+1}^{2n} \lambda_k k^r |s_k(x) - f(x)| n\Lambda_n^{-1} \leqq K n^{-r} \Lambda_n^{-1}. \tag{3.6}$$

Hence, by (3.4), (3.2) and (3.3) follow for $p \geqq 1$ (see footnote[3]).

The proof of these inequalities for $0 < p < 1$ requires a longer calculation. First we use Lemma 2 with p^2. This gives

$$E_n \left(\frac{E_{2n}}{E_n} \right)^{1/p^4} \leqq K \left\| \left\{ \frac{1}{n} \sum_{k=n+1}^{2n} |s_k - f|^{p^2} \right\}^{1/p^2} \right\|.$$

Hence, by (2.2.3) and (3.1), we get

$$E_n \left(\frac{E_{2n}}{E_n} \right)^{1/p^4} \leqq K \left\| \left\{ \frac{1}{n} \left(\sum_{k=n+1}^{2n} \lambda_k |s_k - f|^p \right)^p \left(\sum_{k=n+1}^{2n} \lambda_k^{p/(p-1)} \right)^{1-p} \right\}^{1/p^2} \right\| \leqq \tag{3.7}$$

$$\leqq K n^{-r} \Lambda_n^{-1/p} \left\| \left(\sum_{k=n+1}^{2n} \lambda_k (k^r |s_k - f|)^p \right)^{1/p} \right\| \leqq K C_n n^{-r} \Lambda_n^{-1/p},$$

which implies

$$\sum_{n=m}^{\infty} 2^{nr} E_{2^n} \left(\frac{E_{2^{n+1}}}{E_{2^n}} \right)^{1/p^4} \leqq K \sum_{n=0}^{m} C_{2^n} \Lambda_{2^n}^{-1/p} \tag{3.8}$$

and

$$\sum_{n=0}^{m} 2^{n(r+1)} E_{2^n} \left(\frac{E_{2^{n+1}}}{E_{2^n}} \right)^{1/p^4} \leqq K \sum_{n=0}^{m} 2^n C_{2^n} \Lambda_{2^n}^{-1/p}. \tag{3.9}$$

It seems to be almost evident that (3.8) $\Rightarrow$ (3.2) and (3.9) $\Rightarrow$ (3.3), but the correct proofs of these statements take quite a long calculation.

First we divide the natural numbers $n (\in N)$ into two sets N_1 and N_2 according that

$$\frac{E_{2^{n+1}}}{E_{2^n}} \leqq \frac{1}{4^{r+1}} \tag{3.10}$$

holds or not. If for a certain n, $E_{2^n}=0$ we consider n to be in N_1. It is clear that (3.8) and (3.9) imply

$$\sum_{m\leqq n\in N_1} 2^{nr} E_{2^n} \leqq K_1 \sum_{n=m}^{\infty} C_{2^n} \Lambda_{2^n}^{-1/p} \tag{3.11}$$

and

$$\sum_{\substack{n\in N_1\\ n\leqq m}} 2^{n(r+1)} E_{2^n} \leqq K_1 \sum_{n=0}^{m} 2^n C_{2^n} \Lambda_{2^n}^{-1/p}. \tag{3.12}$$

The crux of the proof now comes to verify (3.2) and (3.3) for the natural numbers n belonging to N_2. If N_2 is not empty then we can give indices m_i and M_i such that

$$N_2 = \bigcup_i \{n : m_i < n \leqq M_i\}.$$

It can happen, of course, that $m_1=-1$ and for a certain i, $M_i=\infty$.

For a given m let M_{i_0} denote the smallest M_i with $m\leqq M_{i_0}$. Then if $(0\leqq)\ m\leqq \leqq m_{i_0}$, we have the following estimation:

$$\sum_{m\leqq n\in N_2} 2^{nr} E_{2^n} = \sum_{i\geqq i_0} \sum_{k=m_i+1}^{M_i} 2^{kr} E_{2^k}; \tag{3.13}$$

and if $m>m_{i_0}$ then

$$\sum_{m\leqq n\in N_2} 2^{nr} E_{2^n} = \sum_{k=m}^{M_{i_0}} 2^{kr} E_{2^k} + \sum_{i\geqq i_0+1} \sum_{k=m_i+1}^{M_i} 2^{kr} E_{2^k}. \tag{3.14}$$

In view of the converse of (3.10) an easy calculation gives that for any i

$$\sum_{k=m_i+1}^{M_i} 2^{nr} E_{2^n} \leqq K\, 2^{m_i r} E_{2^{m_i}}. \tag{3.15}$$

Thus, if $m\leqq m_{i_0}$, we have proved (3.2), namely for $i\geqq i_0$, $m\leqq m_i\in N_1$, consequently these indices appeared in (3.11), i.e. by (3.13) and (3.15)

$$\sum_{m\leqq n\in N_2} 2^{rn} E_{2^n} \leqq \sum_{m\leqq n\in N_1} 2^{nr} E_{2^n},$$

whence by (3.11) we obtain (3.2).

If $m>m_{i_0}$, we can estimate the second sum of (3.14) as before, and the first one can also be estimated by its first term, that is,

$$\sum_{k=m}^{M_{i_0}} 2^{kr} E_{2^k} \leqq K\, 2^{mr} E_{2^m}. \tag{3.16}$$

But now the index m does not belong to N_1, namely it is between m_{i_0} and M_{i_0}, thus we cannot use (3.11) in estimating (3.14). In this case, using (3.10), we have the following estimations: If $m_{i_0}>-1$ then

$$E_{2^m} \leqq \frac{1}{(4^{r+1})^{m-m_{i_0}}} E_{2^{m_{i_0}}}, \tag{3.17}$$

and since $m_{i_0} \in N_1$, by (3.7), (3.10) and (3.17),

$$(3.18)\qquad 2^{mr} E_{2^m} \leqq K\, 2^{mr} \frac{2^{(r+2)m_{i_0}}}{2^{(2r+2)m}} C_{2^{m_{i_0}}} \Lambda_{2^{m_{i_0}}}^{-1/p} \leqq K\, 2^{-m} \sum_{n=0}^{m} 2^n C_{2^n} \Lambda_{2^n}^{-1/p}.$$

In the case $m_{i_0} = -1$ we can also give this upper estimation but by another reason. Then we have by (3.10)

$$(3.19)\qquad 2^{mr} E_{2^m} \leqq \frac{2^{mr}}{(4^{r+1})^{m-1}} E_2 \leqq \frac{K}{2^m} \leqq K\, 2^{-m} \sum_{n=0}^{m} 2^n C_{2^n} \Lambda_{2^n}^{-1/p}.$$

Now collecting our estimations, by (3.12), (3.14), (3.15), (3.16), (3.18) and (3.19), we obtain (3.2).

The proof of (3.3) runs similarly. Using the same notations as before it is clear that we have to detail only the inequality

$$(3.20)\qquad \sum_{\substack{n \in N_2 \\ n \leqq m}} 2^{n(r+1)} E_{2^n} \leqq K \sum_{n=0}^{m} 2^n C_{2^n} \Lambda_{2^n}^{-1/p}.$$

If $(0 \leqq)\ m < m_{i_0}$ then

$$(3.21)\qquad \sum_{\substack{n \in N_2 \\ n \leqq m}} 2^{n(r+1)} E_{2^n} = \sum_{i=1}^{i_0-1} \sum_{k=m_i+1}^{M_i} 2^{k(r+1)} E_{2^k}.$$

By (3.10) we also have

$$(3.22)\qquad \sum_{k=m_i+1}^{M_i} 2^{k(r+1)} E_{2^k} \leqq K\, 2^{m_i(r+1)} E_{2^{m_i}},$$

whence (3.20) follows by (3.12) and (3.21).

If $m > m_{i_0}$ then in respect to (3.21) and (3 .22)

$$\sum_{\substack{n \in N_2 \\ n \leqq m}} 2^{n(r+1)} E_{2^n} \leqq K \sum_{i=1}^{i_0-1} 2^{m_i(r+1)} E_{2^{m_i}} + \sum_{k=m_{i_0}+1}^{m} 2^{k(r+1)} E_{2^k} \leqq K \sum_{i=1}^{i_0} 2^{m_i(r+1)} E_{2^{m_i}} \leqq$$
$$\leqq K \sum_{\substack{n \in N_1 \\ n \leqq m}} 2^{n(r+1)} E_{2^n},$$

which proves (3.20) by (3.12), and hereby (3.3) is also verified.

The proof of Lemma 3 is complete.

LEMMA 4 ([4], Lemma). *Let $\{\rho_n\}$ be a nonincreasing sequence of positive numbers such that $\sum_{n=1}^{\infty} \rho_n n^{-1} < \infty$ and let*

$$\rho(x) = \sum_{n=1}^{\infty} \rho_n n^{-1} \sin nx.$$

Then for any $m > 2^9$

$$(3.23)\qquad \rho\left(\frac{\pi}{m}\right) > \frac{1}{2} \frac{1}{m} \sum_{n=1}^{m} \rho_n.$$

LEMMA 5 ([4], Lemma 4). *Let $p > 0$ and let $\gamma = \{\gamma_k\}$ be a monotone sequence such that for a certain $\theta < 1$ the sequence $\{k^\theta \gamma_k\}$ is nondecreasing. Denote*

$$F(x) = F(\gamma, p;\ x) = \sum_{n=1}^{\infty} (n\,\gamma_n)^{-1/p}\, n^{-1} \sin nx\,,$$

then

$$\left\| \sum_{n=1}^{\infty} \gamma_n\, |s_n(F) - F|^p \right\| < \infty\,.$$

4. PROOF OF THEOREM 1. First we prove (2.4). Let $U_n(x) = V_{2^n}(x) - V_{2^{n-1}}(x)$, where $V_{2^{-1}}(x) \equiv 0$ and $V_n(x) = \frac{1}{n} \sum_{k=n+1}^{2n} s_k(x)$, $n \geqq 0$. It is clear that

$$f(x) = \sum_{n=0}^{\infty} U_n(x)\,.$$

Thus, if $2^{-m} < h \leqq 2^{-m+1}$, we have

$$|f(x+h) - f(x)| \leqq \sum_{n=0}^{m} |U_n(x+h) - U_n(x)| + 2 \sum_{n=m+1}^{\infty} \|U_n\|\,.$$

Since

$$|U_n(x)| \leqq |V_{2^n}(x) - f(x)| + |f(x) - V_{2^{n-1}}(x)|\,,$$

e.g. by Lemma 1

$$\|U_n\| \leqq K E_{2^{n-1}}\,. \tag{4.1}$$

Using the wellknown Bernstein's inequality we obtain

$$|f(x+h) - f(x)| \leqq Kh \sum_{n=0}^{m-1} 2^n E_{2^n} + K \sum_{n=m}^{\infty} E_{2^n}\,.$$

By (2.3) we can use Lemma 3 with $r = 0$, thus the previous inequality, by (2.1) and (3.2)–(3.3), implies that

$$|f(x+h) - f(x)| \leqq K\rho\,(2^m) \leqq K\rho\left(\frac{1}{h}\right),$$

which gives (2.4).

The proof of (2.6) is more difficult. Let η be an arbitrary positive number. By Jegorov's theorem and by (2.3) there exists a perfect set $H_\eta \subset [0, 2\pi]$ such that $\mu(H_\eta) > 2\pi - \eta$ [4] and on the set H_η the series

$$\sum_{n=0}^{\infty} \lambda_n\, |s_n(x) - f(x)|^p \tag{4.2}$$

converges uniformly. By a known theorem of Lebesgue, there exists a subset H_η^* of H_η such that $\mu(H_\eta^*) = \mu(H_\eta)$ and the points of H_η^* are of density 1.

[4] $\mu(H)$ denotes the Lebesgue measure of H.

Let x be an arbitrary fixed point of H_η^* and let ε_1 be an arbitrary fixed positive number. Furthermore let $N = N(\varepsilon_1)$ denote the number satisfying (2.5) with ε_1, and denote $\varepsilon = N^{-1}\,(\leqq \varepsilon_1)$ and $\beta = p^2 + p^{-4}$. By $x \in H_\eta^*$ there exists a positive $\delta = \delta(\varepsilon)$ such that if $0 < h \leqq \delta$ then

$$\mu\big([x-h, x] \cap H_\eta^*\big) > (1-\varepsilon)\,h \quad \text{and} \quad \mu\big([x, x+h] \cap H_\eta^*\big) > (1-\varepsilon)\,h.$$

Since the series (4.2) converges uniformly on H_η, there exists an integer $\mu^*(\geqq 4)$ such that for all $t \in H_\eta$

$$\sum_{n=\mu}^{\infty} \lambda_n\, |s_n(t) - f(t)|^p \leqq \varepsilon^\beta . \tag{4.3}$$

Hence, similarly as in (3.5) and (3.6), we obtain for $p \geqq 1$ and $2^n \geqq \mu^*$ the estimation

$$|f(t) - V_{2^n}(t)| \leqq K\, \varepsilon^{\beta/p}\, \Lambda_{2^n}^{-1/p} \tag{4.4}$$

at any $t \in H_\eta$.

Next we prove a similar estimation for $0 < p < 1$. Using Hölder's inequality twice, Lemma 1 and (2.2.3) we obtain

$$n\, |f(t) - V_n(t)| = \sum_{k=n+1}^{2n} |s_k(t) - f(t)|^{p^4}\, |s_k(t) - f(t)|^{1-p^4} \leqq$$

$$\leqq \left\{ \sum_{k=n+1}^{2n} |s_k(t) - f(t)|^{p^2} \right\}^{p^2} \left\{ \sum_{k=n+1}^{2n} |s_k(t) - f(t)|^{1+p^2} \right\}^{1-p^2} \leqq$$

$$\leqq \left\{ \left(\sum_{k=n+1}^{2n} \lambda_k\, |s_k(t) - f(t)|^p \right)^p \left(\sum_{k=n-1}^{2n} \lambda_k^{p/(p-1)} \right)^{1-p} \right\}^{p^2} n^{1-p^2}\, E_n^{1-p^4} \leqq$$

$$\leqq K n\, E_n^{1-p^4}\, \Lambda_n^{-p^3} \left\{ \sum_{k=n+1}^{2n} \lambda_k\, |s_k(t) - f(t)|^p \right\}^{p^3} .$$

Hence, by (4.3), assuming $t \in H_\eta$ and $2^n \geqq \mu^*$, we get

$$|f(t) - V_{2^n}(t)| \leqq K\, \varepsilon^{\beta p^3}\, E_{2^n} (E_{2^n} \Lambda_{2^n}^{1/p})^{-p^4} \leqq K\, \varepsilon^{\beta p^3} (E_{2^n} + \Lambda_{2^n}^{-1/p}), \tag{4.5}$$

which requires some explanation only when

$$E_{2^n}\, \Lambda_{2^n}^{1/p} < 1 ,$$

but then, by $p < 1$,

$$E_{2^n}^{1-p^4}\, \Lambda_{2^n}^{-p^3} < (\Lambda_{2^n}^{-1/p})^{1-p^4}\, \Lambda_{2^n}^{-p^3} = \Lambda_{2^n}^{-1/p} ,$$

whence (4.5) obviously follows.

Let us choose ξ such that $|x - \xi| < \min \left(\delta, \dfrac{\varepsilon}{\mu^*}, \dfrac{1}{x_0(\varepsilon_1)} \right)$. Let $\nu = \nu(\xi)$ be the smallest natural number with $\varepsilon \leqq 2^\nu\, |x - \xi| < 2\varepsilon$. It is clear that $2^\nu > \mu^*$. Since

$|x-\xi|<\delta$, there exists a point $\xi_1 \in H_\eta$ lying between x and ξ such that $|\xi-\xi_1| \leqq \varepsilon|x-\xi|$ and $\xi_1 \neq \xi$.

By (2.4), (4.4) and (4.5) we obtain

$$(4.6)\quad |f(\xi)-f(x)| \leqq |f(\xi)-f(\xi_1)| + |f(\xi_1)-V_{2^\nu}(\xi_1)| + |V_{2^\nu}(\xi_1)-V_{2^\nu}(x)| + \\ + |V_{2^\nu}(x)-f(x)| \leqq K\rho\left(\frac{1}{|\xi-\xi_1|}\right) + K\varepsilon^{\beta'}(E_{2^\nu}+\Lambda_{2^\nu}^{-1/p}) + |\xi_1-x|\,\|V'_{2^\nu}\|,$$

where $\beta' = \min(\beta p^3, \beta/p) > 1$ for any $p>0$.

Here the first term on the right-hand side, by (2.5), can be estimated easily:

$$\rho\left(\frac{1}{|\xi-\xi_1|}\right) \leqq \rho\left(\frac{1}{\varepsilon|x-\xi|}\right) = \rho\left(\frac{N}{|x-\xi|}\right) \leqq \varepsilon_1\rho\left(\frac{1}{|x-\xi|}\right).$$

Since, by (2.1), (2.3) and (3.5) (with $r=0$), we have

$$E_{2^\nu} + \Lambda_{2^\nu}^{-1/p} \leqq K\rho(2^\nu),$$

the second term of (4.6), also by (2.5), does not exceed

$$K\varepsilon\rho(2^\nu) \leqq K\varepsilon\rho\left(\frac{\varepsilon}{|x-\xi|}\right) = K\frac{1}{N}\rho\left(\frac{1}{N|x-\xi|}\right) \leqq K\varepsilon_1\rho\left(\frac{1}{|x-\xi|}\right).$$

In order to estimate the third term in (4.6) we set

$$V'_{2^\nu}(x) = \sum_{n=0}^{\nu}\left(V'_{2^\nu}(x) - V'_{2^{n-1}}(x)\right),$$

whence, using Bernstein's inequality, conditions (2.1), (2.3), (3.3) (with $r=0$) and (4.1) give that

$$\|V'_{2^\nu}(x)\| \leqq \sum_{n=0}^{\nu} 2^n \|U_n\| \leqq K\sum_{n=0}^{\nu} 2^n E_{2^n} \leqq K\,2^\nu\rho(2^\nu)$$

From this it follows, as before, that

$$|\xi_1-x|\,\|V'_{2^\nu}\| \leqq K|x-\xi|\,2^\nu\rho(2^\nu) \leqq K\,2\,\varepsilon\,\rho(2^\nu) \leqq K_1\varepsilon_1\rho\left(\frac{1}{|x-\xi|}\right).$$

Summing up, we obtain

$$(4.7)\qquad |f(\xi)-f(x)| \leqq K\varepsilon_1\rho\left(\frac{1}{|x-\xi|}\right).$$

Since ε_1 was arbitrary, (4.7) implies (2.6) for all $x\in H_\eta^*$. Let $G(f)$ denote the subset of $[0, 2\pi]$ where (2.6) does not hold. It is obvious that $G(f) \subset [0, 2\pi]\setminus H_\eta^*$, thus the exterior measure of $G(f)$ is less than η. Since η was also arbitrary, the measure of $G(f)$ is zero, that is, (2.6) is proved for almost every x.

We have completed the proof of Theorem 1.

PROOF OF THEOREM 2. First we show that

$$f^{(r)}(x) = \sum_{n=0}^{\infty} U_n^{(r)}(x). \tag{4.8}$$

To prove this it is enough to show that $\Sigma\, U_n^{(r)}(x)$ has a convergent numerical majorant series. By (4.1), using Bernstein's inequality, we have

$$\| U_n^{(r)} \| \leqq K\, 2^{nr} E_{2^n-1}, \tag{4.9}$$

whence, by (2.1), (3.2) and $C_0 < \infty$,

$$\sum_{n=0}^{\infty} \| U_n^{(r)} \| < \infty$$

follows, which implies (4.8).

Now we prove (2.7). Let $2^{-m} < h \leqq 2^{-m+1}$. Then, by (2.1), (3.2), (3.3), (4.9) and $C_0 < \infty$, we have that

$$|f^{(r)}(x+h) - f^{(r)}(x)| \leqq \sum_{n=0}^{m+1} |U_n^{(r)}(x+h) - U_n^{(r)}(x)| + 2 \sum_{n=m+2}^{\infty} \| U_n^{(r)} \| \leqq$$

$$\leqq h \sum_{n=0}^{m+1} \| U_n^{(r+1)} \| + K \sum_{k=m+1}^{\infty} 2^{kr} E_{2^k} \leqq$$

$$\leqq K\, 2^{-m} \sum_{n=0}^{m+1} 2^{n(r+1)} E_{2^n} + K \sum_{n=m+1}^{\infty} 2^{nr} E_{2^n} \leqq K \rho(2^m) \leqq K \rho\left(\frac{1}{h}\right),$$

which proves (2.7).

In order to prove (2.8) we first remark that the condition $\rho(x) \leqq N\varepsilon\rho(Nx)$ implies

$$x\rho(x) \to \infty \quad \text{as} \quad x \to \infty. \tag{4.10}$$

Now let ε_1 be an arbitrary fixed positive number.

By $C_m \to 0$ and (4.10) there exist indices $m_0 = m_0(\varepsilon_1)$ and $\nu_0 = \nu_0(\varepsilon_1)$ such that

$$C_m < \varepsilon_1 \quad \text{for} \quad m \geqq m_0 \quad \text{and} \quad \varepsilon_1^{-1}\, 2^{m_0} \rho(2^{m_0}) < \rho\,(2^\nu) 2^\nu \quad \text{for} \quad \nu \geqq \nu_0. \tag{4.11}$$

If $m > \max(m_0, \nu_0)$ then, using (4.11), (2.1), (3.2) and (3.3), we obtain

$$\sum_{n=m+1}^{\infty} 2^{nr} E_{2^n} \leqq K \left(\varepsilon_1 \sum_{n=m}^{\infty} \Lambda_{2^n}^{-1/p} + 2^{-m} \left(\sum_{n=0}^{m_0} + \sum_{n=m_0+1}^{m} \right) 2^n C_{2^n} \Lambda_{2^n}^{-1/p} \right) + \tag{4.12}$$

$$\leqq \left(K(\varepsilon_1 \rho(2^m) + 2^{-m} 2^{m_0} \rho(2^{m_0}) + \varepsilon_1 2^{-m} \sum_{n=m_0+1}^{m} 2^n \Lambda_{2^n}^{-1/p} \right) \leqq K \varepsilon_1 \rho(2^m)$$

and

$$\sum_{n=0}^{m} 2^{n(r+1)} E_{2^n} \leqq \left(\sum_{n+1}^{m_0} + \sum_{n=m_0+1}^{m} \right) 2^{n(r+1)} E_{2^n} \leqq K \sum_{n=0}^{m_0} 2^{nr} \Lambda_{2^n}^{-1/p} + \tag{4.13}$$

$$+ \varepsilon_1 K \sum_{n=m_0+1}^{m} 2^n \Lambda_{2^n}^{-1/p} \leqq K\big(2^{m_0} \rho(2^{m_0}) + \varepsilon_1 2^m \rho(2^m)\big) \leqq 2 K \varepsilon_1 2^m \rho(2^m).$$

Henceforth the proof runs similarly to that of (2.7). Let $2^{-m} < h \leqq 2^{-m+1}$. Then, by (4.9), (4.12) and (4.13), we obtain

$$|f^{(r)}(x+h) - f^{(r)}(x)| \leqq \sum_{n=0}^{m+1} |U_n^{(r)}(x+h) - U_n^{(r)}(x)| + 2 \sum_{n=m+2}^{\infty} ||U_n^{(r)}|| \leqq$$

$$\leqq h \sum_{n=0}^{m+1} ||U_n^{(r+1)}|| + K\varepsilon_1 \rho(2^m) \leqq K_1 \varepsilon_1 \rho(2^m),$$

which proves (2.8).

We have completed the proof of Theorem 2.

Proof of Theorem 3. Let

$$F(x) = F(\lambda, r, p;\, x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^{r+1} \Lambda_n^{1/p}}.$$

First we verify (2.11). Using Lemma 5 with $\gamma_n = n^{rp-1} \Lambda_n$ we immediately obtain that

$$\left\| \sum_{n=1}^{\infty} \Lambda_n n^{-1} (n^r |s_n(F) - F|)^p \right\| < \infty,$$

whence, by $n\lambda_n = K\Lambda_n$, (2.11) follows.

In order to prove (2.12), (2.13) and (2.14) we use Lemma 4.

If r is an even integer it is fairly obvious that (3.23) gives (2.13) (in respect to $F(0) = 0$), whence (2.12) also follows.

If r is odd then (3.23) also clearly gives (2.14) but this does not imply (2.12). Then (i.e. for an odd integer r) we have to verify (2.12) by a straightforward calculation. Being

$$F^{(r)}(x) = \pm \sum_{n=0}^{\infty} \frac{\cos nx}{n \Lambda_n^{1/p}},$$

thus, in view of $\sum_{n=1}^{\infty} n^{-1} \Lambda_n^{-1/p} < \infty$ (which follows by the conditions $\eta < 1 - pr$ and $n^{\eta+rp-1} \Lambda_n \uparrow$) we have

$$\left| F^{(r)}\left(\frac{\pi}{m}\right) - F^{(r)}(0) \right| = \sum_{n=1}^{\infty} \frac{2}{n \Lambda_n^{1/p}} \left(\sin n \frac{\pi}{2m}\right)^2 \geqq \frac{1}{m^2} \sum_{n=1}^{m} n \Lambda_n^{-1/p},$$

which proves (2.12).

The proof is complete.

Proof of Theorem 4. By Lemma 1 the implication (2.20) $\Rightarrow$ (2.19) immediately follows.

To prove the inverse implication we distinguish two cases according $p \geqq 1$ or $0 < p < 1$.

If $p \geqq 1$ then

$$E_n \leqq \left\| \frac{1}{n} \sum_{k=n+1}^{2n} |s_k - f| \right\| \leqq \left\| \left\{ \frac{1}{n} \sum_{k=n+1}^{2n} |s_k - f|^p \right\}^{1/p} \right\|$$

whence

$$2^n E_{2^n}^p \leqq \left\| \sum_{k=2^n+1}^{2^{n+1}} |s_k - f|^p \right\|$$

holds, which makes clear that (2.19) implies (2.20) in view of the restriction on $\{\mu_n\}$.

If $0 < p < 1$ then, by Lemma 2, we have

$$2^n E_{2^n}^p \left(\frac{E_{2^{n+1}}}{E_{2^n}} \right)^{1/p} = O\left(\left\| \sum_{h=2^n+1}^{2^{n+1}} |s_k - f|^p \right\| \right),$$

which by (2.19) implies that

$$\sum_{n=1}^{\infty} \mu_{2^n} 2^n E_{2^n}^p \left(\frac{E_{2^{n+1}}}{E_{2^n}} \right)^{1/p} < \infty . \tag{4.14}$$

If we now divide the natural numbers n into two sets N_1 and N_2 according that

$$\frac{E_{2^{n+1}}}{E_{2^n}} \geqq \frac{1}{4K}$$

holds or not; and use the same technique as in the proof of Lemma 3, we obtain that (4.14) implies

$$\sum_{n=1}^{\infty} \mu_{2^n} 2^n E_{2^n}^p < \infty ,$$

which proves (2.20); and Theorem 4 is proved.

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PROBABILISTIC APPROACH TO SCHOENBERG'S PROBLEM IN BIRKHOFF INTERPOLATION

By

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To Professor G. Alexits on the occasion of his eightieth birthday

1. Introduction. Let $E = (e_{ik})_{i=1}^{m}\,_{k=0}^{n-1}$ be an $m \times n$ interpolation matrix, with elements e_{ik} that are zero or one, with exactly n ones. With E we associate a *Birkhoff interpolation problem* for polynomials P of degree at most $n-1$,

$$P^{(k)}(x_i) = c_{ik}\,, \tag{1.1}$$

the equations corresponding to pairs i, k with $e_{ik} = 1$. A matrix E is *regular* if equations (1.1) have a solution for each selection of the knots $x_1 < \ldots < x_m$ and of constants c_{ik}. Otherwise E is *singular*. In a famous paper, SCHOENBERG [8] proposed to find all regular matrices. This problem — at least at present — seems to be hopelessly difficult. One is perhaps justified in trying to simplify the problem. We ask whether *most* interpolation matrices are singular, and give a positive answer to this question if m is not small compared with n, more exactly if m satisfies condition (3.2).

We do not exclude the possibility that some rows of E consist of zeros. (In fact, if $m > n$, such rows must be necessarily present.) Of importance for interpolation matrices E is the Pólya condition

$$M_k \geqq k+1\,, \qquad k = 0, 1, \ldots, n-1 \tag{1.2}$$

and the stronger Birkhoff condition

$$M_k \geqq k+2\,, \qquad k = 0, 1, \ldots, n-2\,; \tag{1.3}$$

here M_k is the number of ones located in the columns $0, 1, \ldots, k$ of E. By $M(m, n)$, $P(m, n)$, $B(m, n)$ we denote the numbers of *all $m \times n$ interpolation matrices*, of *all Pólya matrices*, of *all Birkhoff matrices*, respectively. First we have to determine, at least asymptotically, these numbers. Obviously $M(m, n) = \binom{mn}{n}$, and by Stirling's formula,

$$M(m, n) = \binom{mn}{n} \approx \frac{1}{\sqrt{2\pi n}}\, m^n \left(1 + \frac{1}{m-1}\right)^{(m-1)n} \leqq \text{Const}\, \frac{1}{\sqrt{n}}\, m^n e^n\,. \tag{1.4}$$

We have $B(m, n) \leqq P(m, n) \leqq M(m, n)$. We shall prove that $P(m, n) = o(M(m, n))$ for $n \to \infty$, and that $B(m, n) \geqq \rho P(m, n)$ for some constant $\rho > 0$. The first relation

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already solves our singularity problem, for all non-Pólya matrices are singular. Even more, by the theorem of Birkhoff, Ferguson and Nemeth, for such matrices equations (1.1) are not solvable for *any selection of knots* $x_1 < \ldots < x_m$ and for some c_{ik}. But the real problem lies deeper. We shall prove that most Birkhoff matrices, and that most Pólya matrices are singular.

A row i of E will be called a *singleton*, if it contains exactly one one, $e_{ik} = 1$. This singleton is *supported*, if E contains ones $e_{i_1k_1} = e_{i_2k_2} = 1$ for which $i_1 < i < i_2$, $k_1, k_2 < k$. We shall use the following result of LORENTZ and ZELLER [6]: *A Birkhoff matrix is singular if it contains a supported singleton.* There exist stronger theorems of singularity (LORENTZ [4]), but it is not clear how to use them probabilistically.

The theorem quoted does not hold for Pólya matrices. Therefore with ATKINSON and SHARMA [1] we consider vertical decompositions of a Pólya matrix into Birkhoff (or one column) matrices. There exists a maximal decomposition (*canonical decomposition*) of this kind:

$$E = E_1 \oplus \ldots \oplus E_\lambda, \qquad n_1 + \ldots + n_\lambda = n. \tag{1.5}$$

Here each E_l, an $m \times n_l$ matrix with exactly n_l ones — is a Birkhoff matrix if $n_l > 1$, or a one column matrix; E is singular if and only if one of the E_l is singular. (See [1] or LORENTZ [5], where an exposition of Birkhoff interpolation theory is given.)

Here and everywhere else in the paper "singularity" can be replaced by "strong singularity", which means that the determinant of the system (1.1) changes sign, not merely vanishes.

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2. Count of Birkhoff and of Pólya matrices. For an $m \times n$ interpolation matrix E, let m_k, $k = 0, \ldots, n-1$ denote the number of ones in its k-th column. If E is a Birkhoff matrix, it follows from (1.3) that $m_{n-1} = 0$. We can as well omit the last column and study the $m \times (n-1)$ submatrix E' of E; this matrix E' has also exactly n ones. Its column numbers m_k form an $n-1$ tuple $\overline{m} = (m_0, m_1, \ldots, m_{n-2})$ with $\sum_0^{n-2} m_l = n$.

We shall study cyclic permutations of columns of a matrix E' — an idea that in probability goes back at least to SPITZER [9]. Results very close to our Theorems 1 and 2 can be found in [7]. We supply a complete proof of Theorem 1, since the equivalence classes discussed there are very important for the sequel.

The cyclic permutations of columns of E' will produce again matrices of the same class. With E', its $(n-1)$ tuple $\overline{m}$ will also undergo cyclic permutations, producing

$$\sigma^i \overline{m} = (m_i, \ldots, m_{n-2}, m_0, \ldots, m_{i-1}), \qquad i = 0, \ldots, n-2. \tag{2.1}$$

Two matrices E' of our type will be called *equivalent*, if one of them can be obtained from the other by a cyclic permutation of its columns. Obviously, this is an equivalence relation among $m \times (n-1)$ matrices E' with n ones.

We shall need the functional

(2.2) $$A(\bar{m}) = (n-2)\,m_0 + (n-3)m_1 + \ldots + 1m_{n-3} + 0m_{n-2}.$$

(This $A(\bar{m})$ can be interpreted as the area under an obvious graph connected with the $(n-1)$ tuple $\bar{m}$.)

THEOREM 1. *Each equivalence class of matrices E' consists of $(n-1)$ different matrices. Exactly one matrix in each class produces a Birkhoff $m \times n$ matrix E, if a last column of zeros is added to it. All $m \times n$ Birkhoff matrices are obtained in this way. In particular,*

(2.3) $$B(m, n) = \frac{1}{n-1}\binom{m(n-1)}{n}, \qquad n \geqq 2.$$

PROOF. We have

(2.4) $$A(\sigma^i\bar{m}) - A(\sigma^{i+1}\bar{m}) = (n-1)\,m_i - n, \qquad i = 0, \ldots, n-3,$$

hence for $j > i$,

(2.5) $$A(\sigma^i\bar{m}) - A(\sigma^j\bar{m}) = (n-1)\sum_{l=i}^{j-1} m_l - (j-i)\,n, \qquad j > i.$$

To prove that all matrices in an equivalence class are different, it is sufficient to show that all numbers $A(\sigma^i\bar{m})$, $i = 0, \ldots, n-2$ are different. If the difference (2.5) is zero, then $\sum_i^{j-1} m_l$ is congruent zero modulo n. This means that this sum is either n or 0. In the first case, (2.5) yields $n(j-i) = n(n-1)$, which is impossible, in the second case we have $j - i = 0$, as required.

It remains to show that exactly one of the permutations corresponds to a Birkhoff matrix. This is the permutation with the *largest value* of $A(\sigma^i\bar{m})$. Let, for example, the largest value correspond to $i = 0$. This is equivalent to $A(\bar{m}) - A(\sigma^j\bar{m}) > 0$, $j = 1, \ldots, n-2$, or to

$$A(\bar{m}) - A(\sigma^j\bar{m}) = (n-1)\sum_{l=0}^{j-1} m_l - jn > 0,$$

or to $\sum_0^{j-1} m_l > jn/(n-1)$, that is to $\sum_0^{j-1} m_l \geqq j+1$, $j = 1, \ldots, n-2$ and this is identical with the assumption that the extended matrix E satisfies (1.3).

Similar arguments allow the counting of all $m \times n$ Pólya matrices E (with n ones). It is necessary here to consider $m \times (n+1)$ matrices E' with n ones, and cyclic permutations of their columns. To a matrix E' corresponds a counting function $\bar{m} = (m_0, \ldots, m_n)$ with $\sum_0^n m_l = n$, and with its permutations $\sigma^i\bar{m}$, $i = 0, \ldots, n$. Some of the E' have $m_n = 0$, then they reduce to $m \times n$ matrices E.

THEOREM 2. *Each equivalence class of matrices E' consists of $(n+1)$ different matrices. Exactly one of them produces an $m \times n$ Pólya matrix, if the last column of E' is omitted. All $m \times n$ Pólya matrices are obtained in this way. Consequently*

$$P(m, n) = \frac{1}{n+1}\binom{m(n+1)}{n}. \tag{2.6}$$

The proof is similar to that of Theorem 1, and uses the functional

$$A(\bar{m}) = nm_0 + (n-1)\, m_1 + \ldots + 0m_n. \tag{2.7}$$

Formulas (2.3), (2.6) and Stirling's formula allow one to get asymptotic expressions for $P(m, n)$, $B(m, n)$. For example ,we have the strong equivalences

$$P(n, n) \approx \sqrt{\frac{e}{2\pi}}\, n^{n-3/2} e^n, \qquad B(n, n) \approx \sqrt{\frac{1}{2\pi e^3}}\, n^{n-3/2} e^n.$$

PROPOSITION 1. *For $m \geqq 3$, $n \geqq 2$ one has the following inequalities:*

$$\frac{B(m, n)}{P(m, n)} \geqq \rho \quad \textit{for some constant} \quad \rho > 0, \tag{2.8}$$

$$\frac{B(m+1, n+1)}{B(m, n)} \leqq (m+1)\left(1 + \frac{2}{n-1} + \frac{1}{m-1}\right)^n. \tag{2.9}$$

PROOF. One can take $\rho = e^{-5}$. For the first inequality, one has

$$\frac{P(m, n)}{B(m, n)} \leqq \frac{\binom{mn+m}{n}}{\binom{mn-m}{n}} = \prod_{k=0}^{n-1} \frac{mn+m-k}{mn-m-k} \leqq \left(\frac{mn+m-n+1}{mn-m-n+1}\right)^n =$$

$$= \left(1 + \frac{2m}{(m-1)(n-1)}\right)^n \leqq \left(1 + \frac{3}{n-1}\right)^n \leqq \frac{1}{\rho}.$$

On the other hand,

$$\frac{B(m+1, n+1)}{B(m, n)} \leqq \frac{\binom{mn+n}{n+1}}{\binom{mn-m}{n}} = \frac{(m+1)\, n}{n+1} \prod_{k=0}^{n-1} \frac{mn+n-k-1}{mn-m-k} \leqq$$

$$\leqq (m+1)\left(\frac{mn}{mn-m-n+1}\right)^n = (m+1)\left(1 + \frac{m+n-1}{(m-1)(n-1)}\right)^n \leqq$$

$$\leqq (m+1)\left(1 + \frac{2}{n-1} + \frac{1}{m-1}\right)^n.$$

3. Probabilistic considerations. We wish to prove that most Birkhoff and most Pólya matrices have *many* singletons. By $B(m, n)$, $B^p(m, n)$, $B_p(m, n)$ we denote the classes of all Birkhoff $m \times n$ matrices, or those of them that have at least p, or have exactly p singletons, respectively. Also the numbers of these matrices will be denoted in this way. The number of ones in the matrices will be exactly n. However, $\bar{B}(m, n)$, $\bar{B}^p(m, n)$, $\bar{B}_p(m, n)$ will denote the corresponding classes and numbers for matrices with exactly $n+1$ ones, which satisfy the Birkhoff condition (1.3) (for $k = 0, \ldots, n-1$, and, trivially, for $k = n$). For example, consider the class $\bar{B}(m+1, n+1)$. An $(m+1) \times (n+1)$ matrix belongs to this class exactly if it has an empty last column, and if its first n columns form an $(m+1) \times n$ matrix with $(n+1)$ ones which satisfies the Birkhoff condition. For the corresponding numbers we have

$$B(m+1, n+1) = \bar{B}(m+1, n). \tag{3.1}$$

THEOREM 3. (i) *Let m satisfy*

$$(1+\delta)\frac{n}{\log n} \leqq m, \qquad \delta > 0 \quad \textit{constant}. \tag{3.2}$$

Then for each p and all large n, almost all $m \times n$ Birkhoff matrices have at least p singletons:

$$\lim_{n\to\infty} \frac{B^p(m, n)}{B(m, n)} = 1. \tag{3.3}$$

(ii) *The same statement is true for Pólya matrices:*

$$\lim_{n\to\infty} \frac{P^p(m, n)}{P(m, n)} = 1. \tag{3.4}$$

We shall prove only (i), which is needed later. The proof of (ii) is similar but simpler.

Let $E \in B_{k-1}(m, n)$ be an $m \times n$ Birkhoff matrix with exactly $k-1$ singletons, let F be an $(m+1) \times n$ matrix with exactly one one. There are $(m+1)n$ such F. Combining E and F, we construct an $(m+1) \times n$ matrix $E' = E * F$ by placing the m rows of E, in their natural order, into the m empty rows of F. Clearly $E' \in \bar{B}_k(m+1, n)$. In this way we will get some matrices E' of the class $\bar{B}_k(m+1, n)$, and when so, then at most k times, for we can use at most k singleton rows of E' for F. Therefore

$$k\bar{B}_k(m+1, n) \geqq (m+1)nB_{k-1}(m, n).$$

Summation for $k = 1, 2, \ldots, p$ yields

$$\bar{B}_1(m+1, n) + \ldots + \bar{B}_p(m+1, n) \geqq \frac{(m+1)n}{p}\,[B(m, n) - B^p(m, n)],$$

hence

$$B(m, n) - B^p(m, n) \leqq \frac{p}{(m+1)n}\,\bar{B}(m+1, n), \tag{3.5}$$

and using (3.1) and (2.9) we obtain

$$1-\frac{B^p(m,n)}{B(m,n)}\leqq\frac{p}{n}\left(1+\frac{2}{n-1}+\frac{1}{m-1}\right)^n. \tag{3.6}$$

Because of the condition (3.2), $\log\left(1+\frac{2}{n-1}+\frac{1}{m-1}\right)^n\leqq(1-\delta')\log n$ for some $\delta'>0$ and all large n. Thus, the right hand side of (3.6) does not exceed $pn^{-\delta'}\to 0$.

Theorem 3(ii), and without doubt 3(i) as well, could be also derived from a theorem of BÉKÉSSY [2] (quoted in the book [3, p. 336]), which studies the asymptotic behaviour of certain probabilities and uses the saddle-point method. This approach requires again condition (3.2); our method of proof is simpler.

We have now to prove that most functions on integers are strongly non-monotone. We consider functions $f(i)$, $i=1,2,\ldots,p$, with values $k=1,\ldots,n$. There are altogether n^p such functions. We say that f has at most four monotone branches, if there exist four (perhaps degenerate) intervals $p_j\leqq k<p_{j+1}$, $p_0=1$, $p_4=p$, on each of which f is monotone.

PROPOSITION 2. (i) *There are* $\binom{n+p-1}{p}$ *monotone increasing functions* f.

(ii) *The number of functions with at most four monotone branches is*

$$\leqq\frac{2^{3p+4}}{p\,!}n^p\leqq\frac{1}{p^\sigma}n^p, \tag{3.7}$$

for each $\sigma>0$ *and sufficiently large* p, $p\leqq n$.

PROOF. Monotone increasing functions f are in one-to-one correspondence with paths which connect points (1,1) and (p,n), move at each step one unit upward or one unit horizontally, and also mark a special point (p,n'), $n'\leqq n$ (for the value $n'=f(p)$). On the other hand these paths can be described as follows. On the horizontal axis we mark $(p-1)+(n-1)+1=p+n-1$ points. We select some $(p-1)$ of them, different from the last point — for the left end points of the horizontal steps of the graph — and one additional point for the point (p,n') of the graph. This selection determines the graph uniquely. Hence, there are $\binom{n+p-1}{p}$ graphs and increasing functions.

A function f with at most four branches is determined by the selection of four intervals of lengths l_j, $\sum_1^4 l_j=p$, and of the sense of increase or decrease of f on each of them; the range of f on each interval is at most $k=1,\ldots,n$. An upper bound for the number of functions is therefore

$$2^4\sum_{l_1+\ldots+l_4=p}\prod_{j=1}^4\binom{n+l_j-1}{l_j}\leqq 2^4(n+p)^p\sum_{l_1+\ldots+l_4=p}\frac{1}{l_1\,!\ldots l_4\,!}=2^4(n+p)^p\frac{4^p}{p\,!}$$

and (3.7) follows if $p\leqq n$.

If we treat a function f as a $p \times n$ matrix, we can generate new functions f' by cyclic permutations of the columns of the matrix. For a given f, all n functions f' obtained in this way are different. We call them equivalent. The following fact is essential: If f has at most two monotone branches, then all equivalent functions f' have at most four branches.

Consider all n^{p-1} equivalence classes of functions f. Let α be the number of classes which contain functions with at most two branches. Since each such class consists of functions with at most four, we have by (3.7), $\alpha n \leqq p^{-\sigma} n^p$, that is $\alpha \leqq \leqq p^{-\sigma} n^{p-1}$. On the other hand, a function f with not less than three monotone branches must contain the configuration of Figure 1, that is, f must contain a supported singleton. We have proved:

Fig. 1

PROPOSITION 3. *Among all n^{p-1} equivalence classes of functions f, all but $p^{-\sigma} n^{p-1}$ of them consist solely of functions with a supported singleton.*

4. Main theorems. THEOREM 4. *Let $\varepsilon > 0$ be given, let m satisfy* (3.2), *then for all large n, $n \geqq n_0$, all but $\varepsilon B(m, n)$ of the $B(m, n)$ Birkhoff $m \times n$ matrices are singular; even more, all but $\varepsilon B(m, n)$ of them have supported singletons.*

PROOF. Because of (3.3), it is sufficient to prove, for some p, that all but $\varepsilon B^p(m, n)$ matrices of the class $B^p(m, n)$ are singular (more exactly, that they contain supported singletons). We take p so large that $p^{-\sigma} < \varepsilon$. In proving our statement, we can replace $B^p(m, n)$ by a set $B_p(m, n)$ and even by a set $B' \subset B_p(m, n)$ given by a subset $A \subset \subset (1, \ldots, m)$ of p elements, on which matrices $E \in B'$ have singletons, because $B^p(m, n)$ is a disjoint union of the sets B' (with not necessarily the same values of p).

As usual, we identify the set B' with equivalence classes of a certain set of $m \times \times (n-1)$ matrices M'. Matrices $E \in M'$ can be thought of as pairs $E = (f, \bar{E})$, where f is a function on A, with values $1, \ldots, n-1$, and $\bar{E} \in \bar{M}$ is an arbitrary $(m-p) \times (n-1)$ matrix with exactly $n-p$ ones. Let β be the number of matrices in $\bar{M}$.

We split B' into even smaller disjoint sets B''. Each $B'' = B''_C$ consists of pairs $(f, \bar{E})$, where f belongs to a fixed permutation class C of functions and $\bar{E} \in \bar{M}$ is arbitrary. Since C and $\bar{M}$ are invariant under permutations, B''_C is a union of equivalence classes of M'. Each set B''_C contains $(n-1)\beta$ matrices, hence consists of the same number β of equivalence classes. Since $B' = \bigcup_C B''_C$, the set B' consists of $(n-1)^{p-1}\beta$ equivalence classes. By Proposition 3, except for at most $p^{-\sigma}(n-1)^{p-1} < < \varepsilon(n-1)^{p-1}$ sets B''_C, each of them contains only matrices with supported singletons, hence generates a singular Birkhoff matrix. Hence all but at most $\varepsilon(n-1)^{p-1}\beta = \varepsilon B'$ Birkhoff matrices of B' are singular .This completes the proof.

THEOREM 5. *Let $\varepsilon > 0$ be given, let m satisfy* (3.2). *Then for all large n, all but $\varepsilon P(m, n)$ of the $P(m, n)$ Pólya matrices are singular.*

PROOF. We consider the canonical decompositions

$$E = E_1 \oplus \dots \oplus E_\lambda, \qquad n = n_1 + \dots + n_\lambda \tag{4.1}$$

of the $m \times n$ Pólya matrices E into $m \times n_l$ matrices E_l that are either Birkhoff matrices, or have one column. We prove that for most such E, there are among the E_l Birkhoff matrices with supported singletons.

For given $\varepsilon > 0$, we select a sufficiently large integer p_0. The first requirement is that $p_0 \geqq n_0$, where n_0 is given by Theorem 4. The second requirement for p_0 will be given later.

We first consider matrices (4.1) with fixed λ, $n_1, \dots, n_\lambda$, with the property that $n_l \geqq p_0$ for some l. The probability of E_l to be regular is then $< \varepsilon$, the probability for E to be regular is even less. Representations (4.1) with different $n_1. \dots, n_\lambda$ give rise to disjoint sets of matrices E, hence our conclusion is valid for the set of all such E.

Next we consider matrices E which satisfy $n_l \leqq p_0$ for all $l = 1, \dots, \lambda$ in (4.1). It is easier to handle matrices of approximately equal but not too small length. By combining some of the matrices E_l together, we can obtain another decomposition of E,

$$E = F_1 \oplus \dots \oplus F_\mu, \qquad p_1 + \dots + p_\mu = n, \tag{4.2}$$

where each F_l, $l = 1, \dots, \mu$ is a Pólya $m \times p_l$ matrix with

$$p_0 \leqq p_l \leqq 2p_0. \tag{4.3}$$

The number of matrices E of this kind does not exceed the number γ of all possible sums in (4.2).

In order to evaluate γ, we start with an estimate of $P(m, p)$ from above.

Using (2.6) and (1.4) we have

$$P(m,p) = \frac{1}{p+1}\binom{m(p+1)}{p} \leqq \frac{1}{(m-1)p}\binom{m(p+1)}{p+1} \leqq$$

$$\leqq C_1 \frac{1}{p^{3/2}} m^n \left(1 + \frac{1}{m-1}\right)^{(m-1)(p+1)} \leqq C_2 \frac{1}{p^{3/2}} m^n \left(1 + \frac{1}{m-1}\right)^{(m-1)p},$$

where C_1 and $C_2 = C_1 e$ are absolute constants. For $p = p_l$ we have $p_l^{-3/2} \leqq p_0^{-3/2}$.

If μ, $p_1, \dots, p_\mu$ are fixed, the number of representations (4.2) does not exceed

$$\prod_{l=1}^{\mu} P(m, p_l) \leqq C_2^\mu \frac{1}{p_0^{3\mu/2}} \prod_{l=1}^{\mu} m^{p_l} \left(1 + \frac{1}{m-1}\right)^{(m-1)p_l} = C_2^\mu \frac{1}{p_0^{3\mu/2}} m^n \left(1 + \frac{1}{m-1}\right)^{(m-1)n}.$$

For given μ, there are at most $p_0+1 \leqq 2p_0$ choices of p_1 (in the interval $[p_0, 2p_0]$), after that at most $2p_0$ choices of p_2, and so on, altogether not more than $(2p_0)^\mu$ choices of $p_1, \dots, p_\mu$. And there are at most n choices of μ. We see that the number γ of representations (4.2) satisfies

$$\gamma \leqq n(2p_0)^\mu \prod P(m, p_l) \leqq \left(\frac{2C_2}{\sqrt{p_0}}\right)^\mu nm^n \left(1+\frac{1}{m-1}\right)^{(m-1)n}.$$

We now fix p_0 so large that $C_2 \leqq \frac{1}{4}\sqrt{p_0}$. Then because $\mu \geqq n/(2p_0)$,

$$\gamma \leqq 2^{-n/(2p_0)} nm^n \left(1+\frac{1}{m-1}\right)^{(m-1)n}. \tag{4.4}$$

It is easy to estimate $P(m, n)$ from below by means of (2.6) and (1.4):

$$P(m, n) \geqq \frac{1}{n+1}\binom{mn}{n} \geqq C_3 n^{-3/2} m^n \left(1+\frac{1}{m-1}\right)^{(m-1)n}. \tag{4.5}$$

From (4.4) and (4.5) it follows that $\gamma = o(P(m, n))$. This completes the proof.

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ÜBER CLUSTER SETS ANALYTISCHER FUNKTIONEN

Von

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Herrn Professor Georg Alexits zum 80. Geburtstag gewidmet

1. Einleitung

Es sei $G\subset \mathbf{C}$ ein Gebiet, es sei ζ ein Randpunkt von G, und die Funktion f sei analytisch in G. Als cluster set $S(f, G, \zeta)$ von f in G bezüglich ζ wird bekanntlich die Menge aller Punkte $\omega\in\hat{\mathbf{C}}$ bezeichnet, für welche eine Punktfolge $\{\zeta_n\}_{n=1}^{\infty}$ mit

$$\zeta_n\in G \ (n=1,2,\dots);\quad \zeta_n\to\zeta,\quad f(\zeta_n)\to\omega\quad (n\to\infty)$$

existiert. Eine Zusammenstellung der klassischen Ergebnisse über cluster sets findet man bei NOSHIRO [14]. Ist z. B. $\zeta\in\partial G$ eine wesentliche Singularität der im Gebiet G analytischen Funktion f, so gilt $S(f, G, \zeta)=\hat{\mathbf{C}}$; in diesem Fall ist also $S(f, G, \zeta)$ die größtmögliche Punktmenge. Wir werden jedoch hier zeigen, daß es noch "größere" cluster sets gibt, sofern diese anders erklärt werden.

Zur Motivierung der neu zu definierenden cluster sets betrachten wir ein interessantes Ergebnis über das Verhalten der Riemannschen Zetafunktion $Z(z)$ im Teilgebiet

$$H:=\left\{z:\frac{1}{2}<\operatorname{Re} z<1\right\}$$

ihres kritischen Streifens. VORONIN [21] hat gezeigt, daß zu jeder Kreisscheibe $B=\{z:|z|\leqq r\}$ mit $r<\frac{1}{4}$ und jeder auf B analytischen, nullstellenfreien Funktion g eine Folge $\{y_n\}_{n=1}^{\infty}$ reeller Zahlen mit $|y_n|\to\infty$ existiert, für welche

$$Z\left(z+\frac{3}{4}+iy_n\right)\underset{B}{\rightrightarrows} g(z)\qquad (n\to\infty)$$

gilt. Ein entsprechendes Ergebnis für allgemeine Eulerprodukte wurde kürzlich von REICH [16] bewiesen.

Das Resultat von Voronin läßt es wünschenswert erscheinen, einen neuen Typ von cluster sets wie folgt zu erklären. Es sei $G\subset\mathbf{C}$ ein Gebiet, es sei ζ ein Randpunkt von G, und die Funktion f sei analytisch in G. Ferner sei $B\subset\mathbf{C}$ eine kompakte Menge. Als modifiziertes cluster set $S(f, G, \zeta, B)$ von f in G bezüglich ζ und B bezeichnen wir die Menge aller Funktionen g, für welche Folgen $\{a_n\}_{n=1}^{\infty}$ und $\{b_n\}_{n=1}^{\infty}$ existieren mit

$$a_n z+b_n\in G\quad\text{für alle}\quad z\in B\quad\text{und alle}\quad n\in\mathbf{N},$$
$$a_n z+b_n\to\zeta\quad\text{für alle}\quad z\in B\quad\text{und}\quad n\to\infty$$

derart, daß gilt

$$f(a_n z + (b_n) \underset{B}{\Longrightarrow} g(z) \qquad (n \to \infty).$$

In dieser Terminologie besagt das Ergebnis von Voronin, daß für die Zetafunktion $Z(z)$ und jedes Kompaktum $B = \{z : |z| \leqq r\}$ mit $r < \frac{1}{4}$ die Funktionenmenge $S(Z, H, \infty, B)$ alle auf B nullstellenfreien, analytischen Funktionen enthält.

Bezeichnen wir für ein Kompaktum B wie üblich mit $A(B)$ den Banach-Raum der auf B stetigen und im Inneren von B analytischen Funktionen, so ist klar, daß $S(f, G, \zeta, B)$ stets eine (evtl. leere) Teilmenge von $A(B)$ ist. Unser Ziel ist hier die Konstruktion von sog. universellen Funktionen f, die ein einfach zusammenhängenden Gebieten G analytisch sind und für welche $S(f, G, \zeta, B)$ maximal große Funktionenmengen sind.

Bei den von uns konstruierten universellen Funktionen ergeben sich interessante Aspekte im Zusammenhang mit den sog. Universalreihen, die zunächst im Reellen bei trigonometrischen Reihen von MENCHOFF [12] und später bei allgemeinen Orthogonalreihen von MARCINKIEWICZ [11] und TALALYAN [20] behandelt wurden. Siehe hierzu die gute Darstellung bei ALEXITS [1, S. 143 ff.]. Universalreihen im Komplexen sind konstruiert worden von CHUI—PARNES [3] und LUH [7, 8, 9].

2. Funktionen mit maximalen cluster sets

Wir bezeichnen mit M die Familie aller kompakten Teilmengen $B \subset \mathbf{C}$, deren Komplement zusammenhängend ist.

Durch Anwendung von Ergebnissen der Approximationstheorie im Komplexen können wir in einfach zusammenhängenden Gebieten G analytische Funktionen f konstruieren, für welche $S(f, G, \zeta, B)$ maximale Funktionenmengen sind.

Zunächst betrachten wir einfach zusammenhängende Gebiete $G \subset \mathbf{C}$ mit $G \neq \mathbf{C}$.

SATZ 1. *Es sei $G \subset \mathbf{C}$, $G \neq \mathbf{C}$ ein einfach zusammenhängendes Gebiet. Dann existiert eine in G analytische Funktion f mit der Eigenschaft: Für jedes $\zeta \in \partial G$ und jedes $B \in M$ gilt $S(f, G, \zeta, B) = A(B)$.*

BEWEIS. Ohne Beschränkung der Allgemeinheit können wir annehmen, daß der Einheitskreis $\mathbf{D} := \{z : |z| < 1\}$ in G enthalten ist. Wir betrachten eine konforme Abbildung ϕ von G auf $\mathbf{D}$ und setzen für $n \in \mathbf{N}$:

$$G_n := \left\{z : z \in G,\ |\phi(z)| < 1 - \frac{1}{2n}\right\}.$$

1. Es sei $\{\zeta^{(k)}\}_{k=1}^{\infty}$ eine Folge mit $\zeta^{(k)} \in \partial G$ für $k = 1, 2, \ldots$, welche in ∂G dicht liegt. Zu jedem $k \in \mathbf{N}$ wählen wir eine Folge $\{z_n^{(k)}\}_{n=1}^{\infty}$ mit

$$z_n^{(k)} \in G \setminus \overline{G_n} \quad (n = 1, 2, \ldots), \qquad z_n^{(k)} \to \zeta^{(k)} \quad (n \to \infty).$$

Nun wird in induktiver Weise eine Folge $\{n_\nu\}_{\nu=1}^{\infty}$ natürlicher Zahlen n_ν konstruiert. Es sei $n_1 = 1$ und für ein $\nu \geqq 1$ sei n_ν bekannt. Wir wählen $n_{\nu+1} > n_\nu$ so groß, daß gilt

$$z_{n_\nu}^{(1)}, z_{n_\nu}^{(2)}, \dots, z_{n_\nu}^{(n_\nu)} \in G_{n_{\nu+1}} \setminus \overline{G_{n_\nu}}.$$

Ferner sei $\{r_\nu\}_{\nu=1}^{\infty}$ eine Folge reeller Zahlen, wobei $r_\nu \in \left(0, \frac{1}{\nu}\right)$ so klein gewählt wird, daß die Kreisscheiben

$$|z - z_{n_\nu}^{(\mu)}| \leqq r_\nu \qquad (\mu = 1, \dots, n_\nu)$$

paarweise disjunkt sind (sofern die Mittelpunkte $z_{n_\nu}^{(\mu)}$ verschieden sind) und alle in $G_{n_{\nu+1}} \setminus \overline{G_{n_\nu}}$ enthalten sind.

2. Es bezeichne $\{Q_n\}_{n=0}^{\infty}$ die Folge der Polynome, deren Koeffizienten rationalen Real- und Imaginärteil haben. Wir konstruieren eine Folge $\{P_\nu\}_{\nu=0}^{\infty}$ von Polynomen. Es sei $P_0(w) = 0$ und für ein $\nu \geqq 1$ seien die Polynome $P_0, \dots, P_{\nu-1}$ schon bekannt. Nach dem Approximationssatz von Runge gibt es ein Polynom P_ν, welches folgende Eigenschaften gleichzeitig erfüllt:

$$\max_{\overline{G_{n_\nu}}} |P_\nu(w) - P_{\nu-1}(w)| < \frac{1}{\nu^2}, \tag{1}$$

$$\max_{|w - z_{n_\nu}^{(\mu)}| \leqq r_\nu} \left| P_\nu(w) - Q_\nu\left(\frac{\nu}{(\nu+1) r_\nu} \cdot (w - z_{n_\nu}^{(\mu)})\right) \right| < \frac{1}{\nu} \qquad (\mu = 1, \dots, n_\nu). \tag{2}$$

3. Die Funktion f werde definiert durch die Polynomreihe

$$f(w) := \sum_{\nu=1}^{\infty} \{P_\nu(w) - P_{\nu-1}(w)\}.$$

Wegen (1) ist f eine in G analytische Funktion. Sie genügt wegen (2) für $\mu = 1, \dots, n_\nu$; $\nu = 1, 2, \dots$ der Beziehung

$$\max_{|w - z_{n_\nu}^{(\mu)}| \leqq r_\nu} \left| f(w) - Q_\nu\left(\frac{\nu}{(\nu+1) r_\nu} \cdot (w - z_{n_\nu}^{(\mu)})\right) \right| \leqq \max_{|w - z_{n_\nu}^{(\mu)}| \leqq r_\nu} |f(w) - P_\nu(w)| +$$

$$\max_{|w - z_{n_\nu}^{(\mu)}| \leqq r_\nu} |P_\nu(w) - Q_\nu(\dots)| < \max_{|w - z_{n_\nu}^{(\mu)}| \leqq r_\nu} \left| \sum_{\lambda=\nu+1}^{\infty} \{P_\lambda(w) - P_{\lambda-1}(w)\} \right| + \frac{1}{\nu} \leqq$$

$$\leqq \sum_{\lambda=\nu+1}^{\infty} \max_{\overline{G_{n_\lambda}}} |P_\lambda(w) - P_{\lambda-1}(w)| + \frac{1}{\nu} < \sum_{\lambda=\nu+1}^{\infty} \frac{1}{\lambda^2} + \frac{1}{\nu} =: \varepsilon_\nu.$$

Hierbei gilt $\varepsilon_\nu \to 0$ für $\nu \to \infty$. Setzen wir $z := \frac{\nu}{(\nu+1) r_\nu} \cdot (w - z_{n_\nu}^{(\mu)})$, so erhalten wir also für $\mu = 1, \dots, n_\nu$; $\nu = 1, 2, \dots$

$$\max_{|z| \leqq \frac{\nu}{\nu+1}} \left| f\left(\frac{\nu+1}{\nu} r_\nu z + z_{n_\nu}^{(\mu)}\right) - Q_\nu(z) \right| < \varepsilon_\nu. \tag{3}$$

4. Es sei nun ein $\zeta \in \partial G$, eine Menge $B \in M$ und eine Funktion $g \in A(B)$ gegeben. Wir bestimmen ein $R \geqq 1$ so, daß die Menge

$$B_R := \left\{ z : z = \frac{z^*}{R},\ z^* \in B \right\}$$

in $\mathbf{D}$ enthalten ist. Die Funktion g_R mit

$$g_R(z) := g(Rz) \qquad (z \in B_R)$$

gehört dann zur Klasse $A(B_R)$. Nach dem Approximationssatz von MERGELYAN [13] gibt es eine Folge $\{\nu_k\}_{k=1}^{\infty}$ mit $\nu_k \to \infty$ für $k \to \infty$ und

$$(4) \qquad \max_{B_R} |g_R(z) - Q_{\nu_k}(z)| < \frac{1}{k} \qquad (k = 1, 2, \ldots).$$

Für genügend große k ist B_R enthalten in $|z| \leqq \dfrac{\nu_k}{\nu_k + 1}$, und es folgt mit (3) und (4) für $\mu = 1, \ldots, n_{\nu_k}$ und alle genügend großen k:

$$(5) \qquad \max_{B_R} \left| f\left(\frac{\nu_k + 1}{\nu_k} r_{\nu_k} z + z_{n_{\nu_k}}^{(\mu)} \right) - g_R(z) \right| < \varepsilon_{\nu_k} + \frac{1}{k}.$$

Gemäß ihrer Konstruktion hat die Menge der Punkte

$$z_{n_{\nu_k}}^{(\mu)},\ \mu = 1, \ldots, n_{\nu_k}; \quad k = 1, 2, \ldots$$

jeden Randpunkt von G als Häufungspunkt. Es existieren daher Folgen $\{k_s\}$ und $\{\mu_s\}$ mit

$$1 \leqq \mu_s \leqq n_{\nu_{k_s}} \quad \text{und} \quad z_{n_{\nu_{k_s}}}^{(\mu_s)} \to \zeta \qquad (s \to \infty).$$

Setzen wir nun

$$a_s := \frac{\nu_{k_s} + 1}{\nu_{k_s}} \cdot r_{\nu_{k_s}} \cdot \frac{1}{R}, \quad b_s := z_{n_{\nu_{k_s}}}^{(\mu_s)},$$

so gilt $a_s \to 0$, $b_s \to \zeta$ und mit (5) folgt für $s \to \infty$

$$\max_{B_R} |f(a_s Rz + b_s) - g_R(z)| \to 0.$$

Hieraus ergibt sich für $s \to \infty$

$$\max_{B} |f(a_s z + b_s) - g(z)| \to 0,$$

woraus die Behauptung folgt.

Für den Fall $G = \mathbf{C}$ erhalten wir das folgende Ergebnis.

SATZ 2. *Es existiert eine ganze Funktion f mit der Eigenschaft: Für jedes $B \in M$ gilt $S(f, \mathbf{C}, \infty, B) = A(B)$.*

BEWEIS. In [9] haben wir zu einer Folge $\{\lambda_n\}$ komplexer Zahlen mit dem Häufungspunkt ∞ eine ganze Funktion f konstruiert, welche folgende Eigenschaft hat: Zu jeder Menge $B \in M$ und jeder Funktion $g \in A(B)$ gibt es eine Folge $\{n_k\}$ natürlicher Zahlen mit $\lambda_{n_k} \to \infty$ und

$$f(z + \lambda_{n_k}) \underset{B}{\Longrightarrow} g(z) \qquad (k \to \infty).$$

Hieraus folgt unmittelbar die Behauptung.

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DISCRETE POLYNOMIAL SPLINES ON THE CIRCLE

By

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Dedicated to Professor G. Alexits on his 80th birthday

1. Introduction. In 1971, MANGASARIAN and SCHUMAKER ([3], [4]) introduced the notion of "discrete splines" and gave its application to best summation formulae and mathematical programming. Later TOM LYCHE [2] gave many results for discrete splines which are analogues of results for polynomial splines. SCHUMAKER [8] has recently further extended the class of discrete splines to what he calls "generalized spline functions".

The object of this note is to define a class of discrete splines on the unit circle and to consider the corresponding interpolation problem. In the limiting case we get the results of SCHOENBERG [5] for interpolation by polynomial splines on the circle. In the literature, complex splines have received careful attention from AHLBERG, NILSON and WALSH [1]. However we prefer to follow the simpler and novel approach of Schoenberg.

In Section 2, we state the problem of interpolation and our main result. Section 3 deals with the B-splines $M_{n,\alpha}(z)$ for this class of splines and their Fourier series. In Section 4, we obtain the finite Fourier series for the periodic sequence of the values of $M_{n,\alpha}(z)$ at the nodes of interpolation. We also obtain two lemmas to evaluate the finite Fourier coefficients. The proof of Theorem 1 is given in Section 5. In Section 6, we obtain discrete monosplines of least L_2-norm. In Section 7, we find the fundamental functions of the interpolation problem. We also obtain interpolatory splines of degree n, which interpolate a given polynomial.

2. The interpolation problem. Let U denote the circle $|z|=1$, n, k integers, $k>n$. Let ω denote the k-th root of unity and let α be a complex number with $|\alpha|=1$ and $0 \leqq \arg\alpha < \frac{2\pi}{nk}$. If A_ν denotes the arc from ω^ν to $\omega^{\nu+1}$, we shall denote by $\mathfrak{S}_{n,k}^{(\alpha)}$ the class of complex valued functions $S(x)$ defined on U by the following three properties:

(1) $S(z) \in C(U)$

(2) On each arc A_ν, $S(z)$ coincides with a polynomial $P_\nu(z)$ of degree n $(\nu = 0, 1, \ldots, k-1)$.

(3) If n is odd $(= 2m+1)$, then for $\nu = 0, 1, 2, \ldots, k-1$,

(3a) $P_\nu(\omega^\nu\alpha^j) = P_{\nu-1}(\omega^\nu\alpha^j)$, $\quad j = 0, \pm 1, \pm 2, \ldots, \pm m$.

If n is even $(= 2m)$; then for $\nu = 0, 1, \ldots, k-1$,

(3b) $P_\nu(\omega^\nu \alpha^j) = P_{\nu-1}(\omega^\nu \alpha^j), \quad j = 0, \pm 1, \ldots, \pm(m-1), m.$

We shall call the functions $S_n(z)$ discrete symmetric splines on U. Following Schoenberg, we propose the following problem:

PROBLEM I. *Given* $\psi = e^{i\gamma}$, $\alpha = e^{i\beta}$, $\gamma \neq 0$, $\varepsilon = 0$ *or* 1, *for what choices of* n, k, γ *and* α *does the interpolation problem*

$$S_n(\psi^\varepsilon \omega^j) = w_j, \qquad j = 0, 1, \ldots, k-1$$

admit a unique solution, where $S_n(z) \in \mathfrak{S}_{n,k}^{(\alpha)}$?

For $\alpha = 1$, $\psi = e^{\frac{i\pi}{k}}$, the problem has been completely solved by SCHOENBERG [5] in an elegant way (see also [1]).

A similar class of discrete splines can be defined if we replace (3a) and (3b) by (3c):

(3c) $P_\nu(\omega^\nu \alpha^j) = P_{\nu-1}(\omega^\nu \alpha^j), \quad j = 0, 1, 2, \ldots, n-1.$

We shall denote this class by $\tilde{\mathfrak{S}}_{n,k}^{(\alpha)}$ and shall call it the discrete forward spline. It is possible to propose the same problem as Problem I for $S_n(z) \in \tilde{\mathfrak{S}}_{n,k}^{(\alpha)}$. However, we shall not dwell upon this problem here, but we shall refer to it as Problem II. We shall make two natural assumptions on α and γ. More precisely we require that

(a) $0 < \beta = \arg \alpha < \dfrac{2\pi}{nk}$

(b) $\gamma = \arg \psi$ is such that either $0 \leqq k\gamma \leqq m\beta$ or $\pi \leqq k\gamma \leqq 2\pi$.

Our principal result is

THEOREM 1. *If* β *and* γ *satisfy the assumptions* (a) *and* (b), *then the interpolation problem* I *has a unique solution except in the following cases:*

Case 1. $\varepsilon = 0, n = 2m, \beta = 0, k = 2h.$

Case 2. $\varepsilon = 1, n = 2m+1, \gamma = \dfrac{\pi}{k}, k = 2h+1.$

REMARK 1. It follows exactly as in [5] that in either of the two exceptional cases, there is exactly one spline $S_0(z) \in \mathfrak{S}_{n,k}^{(\alpha)}$, which vanishes at the k points $\{\psi^\varepsilon \omega^\nu\}_0^{k-1}$. In fact

$$S_0(z) = \sum_{j=0}^{k-1} \omega^{(h+m)j} M_{n,\alpha}(z\omega^{-j})$$

where $M_{n,\alpha}(z)$ is given by (3.2).

REMARK 2. We do not know what happens when n is even and the assumptions (a) and (b) on β and γ are omitted. More precisely, when $m\beta < k\gamma < \pi$.

We shall need the identity

$$(2.1) \qquad (x-1)(x-\alpha)\ldots(x-\alpha^n) = \sum_{\nu=0}^{n+1} (-1)^\nu \begin{bmatrix} n+1 \\ \nu \end{bmatrix}_{(\alpha)} \alpha^{\frac{\nu(\nu-1)}{2}} x^{n-\nu}$$

where $\left[\begin{matrix} n+1 \\ \nu \end{matrix}\right]_{(\alpha)}$ denotes the Gaussian coefficients:

$$\left[\begin{matrix} n+1 \\ \nu \end{matrix}\right]_{(\alpha)} = \frac{(\alpha^{n+1}-1)(\alpha^n-1)\dots(\alpha^{n-\nu+2}-1)}{(\alpha-1)(\alpha^2-1)\dots(\alpha^\nu-1)}.$$

Taking $n = 2m+1$ in (2.1) and putting $x = z\alpha^m$, we get

$$\prod_{j=-m}^{m}(z-\alpha^j) = \sum_{\nu=0}^{2m+1}(-1)^\nu \left[\begin{matrix} 2m+1 \\ \nu \end{matrix}\right]_{(\alpha)} \alpha^{\frac{\nu(\nu-2m-1)}{2}} z^{2m+1-\nu}. \tag{2.2}$$

Set

$$\phi_{n,\nu}(z;\alpha) = \begin{cases} \prod_{j=-m}^{m}(z-\omega^\nu\alpha^j), & n = 2m+1 \\ \prod_{j=-m+1}^{m}(z-\omega^\nu\alpha^j), & n = 2m. \end{cases} \tag{2.3}$$

Then it follows from (2.2) that if n is odd ($= 2m+1$),

$$\phi_{n,\nu}(z;\alpha) = \sum_{j=0}^{2m+1}(-1)^\nu \left[\begin{matrix} 2m+1 \\ j \end{matrix}\right]_{(\alpha)} \alpha^{\frac{j}{2}(j-2m-1)} z^{2m+1-j}\omega^{\nu j}. \tag{2.4}$$

If $n = 2m$, then

$$\phi_{n,\nu}(z;\alpha) = (z-\omega^\nu\alpha^m)\,\phi_{n-1,\nu}(z;\alpha). \tag{2.5}$$

3. The B-spline $M_{n,\alpha}(z)$ and its Fourier series. Since $S_n(z) \in \mathfrak{S}_{n,k}^{(\alpha)}$ coincides on the arc A_ν with the polynomial $P_\nu(z)$, it follows from requirements (3a) and (3b) that

$$P_\nu(z) = P_{\nu-1}(z) + C_\nu\phi_{n,\nu}(z,\alpha).$$

Progressing round the circle, we get the identity

$$\sum_{\nu=0}^{k-1} C_\nu\phi_{n,\nu}(z;\alpha) = 0. \tag{3.1}$$

Using (2.4) or (2.5), we see that (3.1) is equivalent to the condition that $\sum_{j=0}^{k-1} C_j x^j$ vanishes if $x = 1, \omega, \omega^2, \dots, \omega^n$.

The B-splines are obtained by assuming that $\sum_{j=0}^{k-1} C_j x^j$ is of least degree. It follows as in [5] that the B-spline $M_{n,\alpha}(z)$ is given by

$$M_{n,\alpha}(z) = \frac{1}{n!}\omega^{-\frac{n}{2}(n+1)} \sum_{\nu=0}^{n+1}(-1)^\nu \left[\begin{matrix} n+1 \\ \nu \end{matrix}\right]_{(\alpha)} \omega^{\frac{(n-\nu)(n-\nu+1)}{2}} \phi^+_{n,\nu}(z,\alpha) \tag{3.2}$$

where

$$\phi^+_{n,\nu}(z;\alpha) = \begin{cases} 0, & z \in A_{\nu-1} \\ \phi_{n,\nu}(z;\alpha), & z \in A_j, \quad j \geqq \nu. \end{cases}$$

It can be proved exactly as in [7] that every $S_n(z) \in \mathfrak{S}^{(\alpha)}_{n,k}$ can be uniquely represented in the form

$$S_n(z) = \sum_{j=0}^{k-1} C'_j M_{n,\alpha}(z, \omega^{-j}). \tag{3.3}$$

Since $z - \omega^\mu \alpha^j = \alpha^j [z - \omega^\mu - (1 - \alpha^{-j}) z]$, it follows that

$$\prod_{j=0}^{m} (z - \omega^\mu \alpha^j) = \sum_{l=0}^{m} (-1)^l (z - \omega^\mu)^{m-l+1} \sigma_l z^l, \tag{3.4}$$

where $\sigma_0 = 1$ and $\sigma_1, \sigma_2, \ldots, \sigma_m$ are the elementary symmetric functions of $1 - \alpha^{-1}$, $1 - \alpha^{-2}, \ldots, 1 - \alpha^{-m}$. Hence we have for $n = 2m + 1$,

$$\begin{aligned} \phi_{n,\mu}(z; \alpha) &= \frac{1}{(z - \omega^\mu)} \prod_{j=0}^{m} (z - \omega^\mu \alpha^j) \prod_{j=0}^{m} (z - \omega^\mu \alpha^{-j}) = \\ &= \sum_{\nu=0}^{2m} (-1)^\nu s_{\nu, 2m+1} z^\nu (z - \omega^\mu)^{2m+1-\nu}, \end{aligned} \tag{3.5}$$

where

$$s_{0, 2m+1} = 1, \quad s_{1, m2+1} = \sigma_1 + \bar{\sigma}_1, \quad s_{2, 2m+1} = \sigma_2 + \bar{\sigma}_2 + \sigma_1 \bar{\sigma}_1 \ldots.$$

For $n = 2m$, we use (2.5) and get

$$\phi_{n,\mu}(z; \alpha) = \sum_{\nu=0}^{2m-1} (-1)^\nu s_{\nu, 2m} z^\nu (z - \omega^\mu)^{2m-\nu} \tag{3.6}$$

where

$$s_{\nu, 2m} = \alpha^m \{ s_{\nu, 2\bar{m}-1} + (1 - \alpha^{-m}) s_{\nu-1, 2m-1} \}. \tag{3.7}$$

Using (3.5) or (3.6) according as n is odd or even, we have from (3.2) after interchange of summation

$$M_{n,\alpha}(z) = \sum_{\nu=0}^{n+1} \frac{(-1)^\nu s_{\nu, n} z^\nu M_n^{(\nu)}(z)}{n(n-1) \ldots (n - \nu + 1)}, \tag{3.8}$$

where $M_n(z)$ denotes the B-spline of degree n as given by SCHOENBERG [5]. (Here we have slightly changed the notation of Schoenberg to fit into this scheme.) Indeed from [5], we have

$$M_n(z) = \frac{1}{n!} \omega^{-\frac{n(n+1)}{2}} \sum_{\nu=0}^{n+1} (-1)^\nu \begin{bmatrix} n+1 \\ \nu \end{bmatrix}_{(\omega)} \omega^{\frac{(n-\mu)(n-\mu+1)}{2}} (z - \omega^\mu)_+^{n-\nu}.$$

The Fourier series for $M_n(z)$ is given by SCHOENBERG [5]:

$$M_n(z) = \frac{1}{2\pi i} \sum_{-\infty}^{\infty} b_\nu b_{\nu-1} \ldots b_{\nu-n} z^\nu, \qquad 0 \leq n < k - 1 \tag{3.9}$$

where

$$b_\nu = \frac{1-\omega^{-\nu}}{\nu}, \quad \nu = 1, 2, \ldots, \quad b_0 = \frac{2\pi i}{k}. \tag{3.9a}$$

Combining (3.8) and (3.9), we get the Fourier series for $M_{n,\alpha}(z)$:

$$M_{n,\alpha}(z) = \frac{1}{2\pi i} \sum_{\nu=-\infty}^{\infty} b_\nu b_{\nu-1} \ldots b_{\nu-n} \beta(\nu, n) z^\nu \tag{3.10}$$

where

$$\beta(\nu, n) = \sum_{j=0}^{n-1} (-1)^j s_{j,n} \frac{\nu(\nu-1)\ldots(\nu-j+1)}{n(n-1)\ldots(n-j+1)}. \tag{3.11}$$

4. Finite Fourier series for $\{M_{n,\alpha}(\psi^\varepsilon \omega^j)\}$. From the unique representation of every $s(x) \in \mathfrak{S}_{n,k}^{(\alpha)}$ as a linear combination of $M_{n,\alpha}(z\omega^{-j})$, $j = 0, 1, \ldots, k-1$, it follows that solution to Problem I depends on the singularity or otherwise on the matrix $\| M_{n,\alpha}(\psi^\varepsilon \omega^{\nu-j}) \|$. Since $\{M_{n,\alpha}(\psi^\varepsilon \omega^j)\}$ is a periodic sequence with period k, we can write

$$M_{n,\alpha}(\psi^\varepsilon \omega^j) = \sum_{\nu=0}^{k-1} \zeta_{\nu,\varepsilon} \omega^{\nu j}, \quad j = 0, 1, \ldots, k-1 \quad (\varepsilon = 0 \text{ or } 1), \tag{4.1}$$

which is called its finite Fourier series. Following SCHOENBERG [5] it follows that

$$\operatorname{Det} \| M_{n,\alpha}(\psi^\varepsilon \omega^{\nu-j}) \| = k^k \sum_{\nu=0}^{k-1} \zeta_{\nu,\varepsilon}.$$

We shall prove

LEMMA 1. *For* $0 \leqq \nu \leqq n$, $\zeta_{\nu,\varepsilon} \neq 0$.

PROOF. From (4.1), we have

$$\zeta_{\nu,\varepsilon} = \frac{1}{k} \sum_{j=0}^{k-1} M_{n,\alpha}(\psi^\varepsilon \omega^j)\, \omega^{-\nu j} \tag{4.2}$$

so that using (3.10), we easily get

$$\zeta_{\nu,\varepsilon} = \frac{\psi^{\nu\varepsilon}}{2\pi i} \sum_{s=-\infty}^{\infty} b_{ks+\nu} b_{ks+\nu-1} \ldots b_{ks+\nu-n} \beta(ks+\nu, n) \psi^{\varepsilon ks}. \tag{4.3}$$

Since $0 \leqq \nu \leqq n$, one of the numbers $\nu, \nu-1, \ldots, \nu-n$ will vanish and since $b_{ks} = 0$ unless $s = 0$, we have

$$\zeta_{\nu,\varepsilon} = \frac{\psi^{\nu\varepsilon}}{2\pi i} b_\nu b_{\nu-1} \ldots b_{\nu-n} \beta(\nu, n).$$

Also from (3.11), we have

$$\beta(\nu, n) = \frac{\nu!}{n!} \rho_n^{(n-\nu)}(1) \tag{4.4}$$

where

$$(4.5)\qquad \rho_n(x) = \sum_{\nu=0}^{n-1} (-1)^\nu s_{\nu,n} x^{n-\nu} = \begin{cases} \prod_{j=-m}^{m} (x-1+\alpha^{-j}), & n = 2m+1 \\ \prod_{j=-m+1}^{m} (x-1+\alpha^{-j}), & n = 2m. \end{cases}$$

In particular $\beta(0, n) = 1$, $\beta(1, 2m+1) = 1$, $\beta(1, 2m) = \alpha^{-m}$. From Gauss–Lucas theorem it follows that $\rho_n^{(n-\nu)}(1) \neq 0$, $\nu = 0, 1, 2, \ldots, n$. Thus $\beta(\nu, n) \neq 0$. Since $b_\nu, b_{\nu-1}, \ldots, b_{\nu-n}$ are all non-vanishing for $0 \leqq \nu \leqq n$ from (3.9a), it follows that $\zeta_{\nu,\varepsilon} \neq 0$. This proves the lemma.

LEMMA 2. *If $n+1 \leqq \nu \leqq k-1$, and if $r = \nu - n$, we have*

$$(4.6)\qquad \zeta_{\nu,\varepsilon} = \begin{cases} \dfrac{c}{n!} \displaystyle\int_0^1 \frac{\rho_n(x)(1-x)^{r-1}}{|1-(1-x)^k \psi^{k\varepsilon}|^2} K(x)\,dx, & n = 2m+1 \\ \dfrac{c}{n!} \displaystyle\int_0^1 \rho_{n-1}(x) \frac{(1-x)^{r-1}}{|1-(1-x)^k \psi^{k\varepsilon}|^2} K_1(x)\,dx, & n = 2m \end{cases}$$

where

$$(4.7)\qquad c = \frac{\psi^{\nu\varepsilon}}{2\pi i} \prod_{l=r}^{n+r} (1-\omega^{-l}),$$

$$(4.8)\qquad K(x) = 1-(1-x)^{2k-2r-n} + \psi^{-k\varepsilon}(1-x)^{k-2r-n}[1-(1-x)^{2r+n}]$$

and

$$(4.9)\qquad K_1(x) = (x-1+\alpha^{-m})\{1-\psi^{-k\varepsilon}(1-x)^k\} + \\ + (-1)^{n+1}\alpha^{-m}\psi^{-k\varepsilon}(1-x)^{k-2r-n}(x-1+\alpha^m)\{1-(1-x)^k\psi^{k\varepsilon}\}.$$

PROOF. From (4.3), we see that $\zeta_{\nu,\varepsilon} = cN(\alpha)$, where c is given by (4.7) and

$$(4.10)\quad N(\alpha) = \sum_{s=0}^{\infty} + \sum_{s=-\infty}^{-1} \frac{\psi^{k\varepsilon s}\beta(ks+r+n, n)}{(ks+r)(ks+r+1)\ldots(ks+r+n)} \equiv N_1(\alpha) + N_2(\alpha).$$

We can rewrite $N_2(\alpha)$ by a change of variable, so that

$$(4.11)\quad N_2(\alpha) = (-1)^{n+1}\psi^{-k\varepsilon} \sum_{s=0}^{\infty} \frac{\psi^{-k\varepsilon s}\beta(-ks-k+r+n; n)}{(ks+k-r)(ks+k-r-1)\ldots(ks+k-r-n)}.$$

From (3.8) and (4.5), it is easy to verify that

$$(4.12)\qquad \frac{\beta(ks+r+n, n)}{(ks+r)\ldots(ks+r+n)} = \frac{1}{n!}\int_0^1 \rho_n(x)(1-x)^{ks+r-1}\,dx.$$

Similarly, we have

$$(4.13)\quad \frac{\beta(-ks-k+n+r,n)}{(ks+k-r)\dots(ks+k-r-n)} = \frac{(-1)^n}{n!}\int_0^1 \rho_n\left(\frac{-x}{1-x}\right)(1-x)^{ks+k-r-1}\,dx\,.$$

From the definition of $\rho_n(x)$ in (4.5), we see that

$$(4.14)\quad \rho_n\left(-\frac{x}{1-x}\right) = \begin{cases} (-1)^n \dfrac{\rho_n(x)}{(1-x)^n}, & n = 2m+1 \\ \dfrac{(-1)^n\alpha^{-m}(x-1+\alpha^m)\,\rho_{n-1}(x)}{(1-x)^n}, & n = 2m\,. \end{cases}$$

Combining (4.10) and (4.12), we have

$$(4.15)\quad N_1(\alpha) = \frac{1}{n!}\int_0^1 \rho_n(x)\frac{(1-x)^{r-1}}{1-(1-x)^k\psi^{k\varepsilon}}\,dx \text{ for all } n.$$

From (4.11), (4.13) and (4.14), we have

$$(4.16)\quad N_2(\alpha) = \begin{cases} \dfrac{1}{n!}\displaystyle\int_0^1 \frac{(-1)^{n+1}\rho_n(x)\,(1-x)^{k-r-n-1}\psi^{-k\varepsilon}}{1-(1-x)^k\psi^{-k\varepsilon}}\,dx \quad \text{for} \quad n = 2m+1 \\ \dfrac{1}{n!}\displaystyle\int_0^1 (-1)^{n+1}\frac{\rho_{n-1}(x)\left(1+\alpha^{-m}(x-1)\right)(1-x)^{k-r-n-1}\psi^{-k\varepsilon}}{1-(1-x)^k\psi^{-k\varepsilon}}\,dx, \\ \qquad n = 2m\,. \end{cases}$$

From (4.15), (4.16) and (4.10), we get $N(\alpha)$ which easily gives (4.6).

5. Proof of Theorem 1. In order to prove Theorem 1, it is enough to show that none of the $\zeta_{\nu,\varepsilon}$'s vanishes. For $0 \leq \nu \leq n$ this follows from Lemma 1.

For $n+1 \leq \nu \leq k-1$, we use (4.6) in Lemma 2 and observe that if $n = 2m+1$, we have

$$\rho_n(x) = x\prod_{j=1}^{m} |x-1+\alpha^{-j}| \geq 0 \quad \text{in} \quad [0,1]\,.$$

We shall now consider two cases.

Case I (n is odd). If $\varepsilon = 0$, (4.8) shows that $K(x) > 0$ in $(0, 1)$. Therefore, $\zeta_{\nu,0} \neq 0$.

If $\varepsilon = 1$ and $k-2r-n = 0$, we have

$$K(x) = (1+\psi^{-k})\left(1-(1-x)^k\right)$$

which shows that $K(x) = 0$ if and only if $\psi^{-k} = -1$.

If $\varepsilon = 1$ and $k - 2r - n \neq 0$, we see that for $0 < x < 1$

$$\operatorname{Im}(K(x)) = -\sin k\gamma \cdot (1-x)^{k-2r-n}\,[1-(1-x)^{2r+n}] < 0$$

so that

$$\operatorname{Im} \zeta_{\nu,1} \neq 0 .$$

This proves the theorem, when n is odd.

Case II (n is even). In this case $\rho_{n-1}(x) \geqq 0$. If $\varepsilon = 0$ and $\alpha = 1$, we have from (4.9)

$$K_1(x) = x\{1-(1-x)^k\}\,\{1-(1-x)^{k-2r-n}\}$$

which vanishes if and only if $k - 2r - n = 0$.

If $\varepsilon = 0$ and $\alpha \neq 1$ (i.e., $\beta \neq 0$), we have

$$\operatorname{Im} K_1(x) = -\sin m\beta\,\{1-(1-x)^k\}\,\{1+(1-x)^{k-2r-n+1}\} \leqq 0$$

so that $\zeta_{\nu,0} \neq 0$, when $\alpha \neq 1$.

If $\varepsilon = 1$, since $\alpha = e^{i\beta}$, $\psi = e^{i\gamma}$, we have after some simplification

$$\operatorname{Im}\left[K_1(x)\,\alpha^{\overline{\frac{m}{2}}}\,\psi^{\overline{\frac{k}{2}}}\right] = -[1+(1-x)^{k-2r-n}] \times$$

$$\times\left[\sin\frac{m\beta-k\gamma}{2}\{1-(1-x)^{k+1}\} + \sin\frac{m\beta+k\gamma}{2}\{(1-x)-(1-x)^k\}\right].$$

If $0 \leqq k\gamma \leqq m\beta$, both $\sin\dfrac{m\beta-k\gamma}{2}$ and $\sin\dfrac{m\beta+k\gamma}{2}$ are non-negative, since $n\beta < \dfrac{2\pi}{k}$ i.e. $m\beta < \dfrac{\pi}{k}$.

Hence we have

$$\operatorname{Im}\left[\alpha^{\overline{\frac{m}{2}}}\,\psi^{\overline{\frac{k}{2}}}\,K_1(x)\right] \quad \text{in} \quad (0, 1). \tag{5.1}$$

Similarly, if $0 \leqq m\beta < k\gamma$ and $\pi < k\gamma < 2\pi$, we have $\dfrac{\pi}{2} \leqq \dfrac{m\beta+k\gamma}{2} < \pi + \dfrac{\pi}{2k}$ so that $\sin\dfrac{k\gamma-m\beta}{2} > \sin\dfrac{k\gamma+m\beta}{2}$. Since $1-(1-x)^{k+1} > (1-x)-(1-x)^k$, we see that in this case

$$\operatorname{Im}\left[\alpha^{\overline{\frac{m}{2}}}\,\psi^{\overline{\frac{k}{2}}}\,K_1(x)\right] > 0 \quad \text{in} \quad (0, 1). \tag{5.2}$$

Combining (4.6), (5.1) and (5.2), we see that

$$\operatorname{Im}\left(\alpha^{\overline{\frac{m}{2}}}\,\psi^{\overline{\frac{k}{2}}}\,\zeta_{\nu,1}\right) \neq 0,$$

so that $\zeta_{\nu,1} \neq 0$. This completes the proof of the theorem.

6. Discrete monospline of least L_2-norm. If $S(z) \in \mathfrak{S}^{(\alpha)}_{n,k}$, we shall call $K(z)$ a discrete monospline of degree $n+1$, if

$$K(z) = \frac{z^{n+1}}{(n+1)!} - S(z). \tag{6.1}$$

We want to determine the discrete monospline of least L_2-norm. The most general element $S(z) \in \mathfrak{S}^{(\alpha)}_{n,k}$ is given by

$$S(z) = \frac{1}{2\pi i} \sum_{-\infty}^{\infty} b_\nu b_{\nu-1} \dots b_{\nu-n} \beta(\nu, n) z^\nu \eta_\nu \tag{6.2}$$

where $\{\eta_\nu\}$ is an arbitrary periodic sequence of numbers of period k. From (6.1) and (6.2), we have on using Parseval's formula:

$$\frac{1}{2\pi i} \int_0^{2\pi} |K(z)|^2 d\theta = \left| \frac{1}{(n+1)!} - \frac{1}{2\pi i} b_{n+1} \dots b_1 \beta(n+1, n) \eta_{n+1} \right|^2 +$$

$$+ \frac{|\eta_{n+1}|^2}{4\pi^2} \sum_{\substack{\nu \equiv n+1(k) \\ \nu \neq n+1}} |b_\nu b_{\nu-1} \dots b_{\nu-n}|^2 |\beta(\nu, n)|^2 +$$

$$+ \frac{1}{4\pi^2} \sum_{\substack{l=0 \\ l \neq n+1}} \sum_{\nu \equiv l(k)} |b_\nu b_{\nu-1} \dots b_{\nu-n}|^2 |\beta(\nu, n)|^2 .$$

Since $\beta(\nu, n) \neq 0$ for all integers ν (from (4.4), (4.12) and (4.13)), we have

$$\sum_{\nu \equiv l(k)} |b_\nu b_{\nu-1} \dots b_{\nu-n}|^2 |\beta(\nu, n)|^2 > 0$$

so that to minimize $\| K \|_2$, we must have $\eta_l = 0$, $l = 0, 1, \dots, n, n+2, \dots, k-1$. Hence

$$K(z) = \frac{z^{n+1}}{(n+1)!} - \frac{\eta_{n+1}}{2\pi i} \sum_{\nu \equiv n+1(k)} b_\nu b_{\nu-1} \dots b_{\nu-n} \beta(\nu, n) z^\nu .$$

Since $K(\omega z) = \omega^{n+1} K(z)$, we can rewrite

$$K(z) = \frac{z^{n+1}}{(n+1)!} - \lambda \sum_{s=-\infty}^{\infty} \frac{\beta(ks+n+1, n) z^{ks+n+1}}{(ks+1) \dots (ks+n+1) \beta(n+1, n)} .$$

Repeating the reasoning of SCHOENBERG in [6], we now have

THEOREM 2. *The unique monospline of the form* (6.1) *of least L_2-norm is given by*

$$K_*(z) = \frac{z^{n+1}}{(n+1)!} - \lambda_* S_n(z) \tag{6.3}$$

where

$$S_n(z) = \sum_{-\infty}^{\infty} \frac{\beta(ks+n+1, n) z^{ks+n+1}}{\beta(n+1, n)(ks+1)(ks+2) \dots (ks+n+1)} \tag{6.4}$$

and

$$\lambda_* = \sum_{-\infty}^{\infty} \left| \frac{\beta(ks+n+1, n)}{\beta(n+1, n)} \right|^2 \frac{((n+1)!)^2}{[(ks+1)\dots(ks+n+1)]^2}. \tag{6.5}$$

7. Fundamental functions $L_{n,\varepsilon}^{(\alpha)}(z)$. The element $L_{n,\varepsilon}^{(\alpha)}(z) \in \mathfrak{S}_{n,k}^{(\alpha)}$ satisfying the relation

$$\begin{cases} L_{n,\varepsilon}^{(\alpha)}(\psi^\varepsilon) = 1 \\ L_{n,\varepsilon}(\psi^\varepsilon \omega^j) = 0, \qquad j = 1, 2, \dots, k-1 \end{cases} \tag{7.1}$$

is called the fundamental function of the interpolation problem. It is easy to express the function $L_{n,\varepsilon}^{(\alpha)}(z)$ in terms of the B-splines and the numbers $\zeta_{\nu,\varepsilon}$ of (4.1). Indeed we have

$$L_{n,\varepsilon}^{(\alpha)}(z) = \sum_{j=0}^{k-1} a_j M_{n,\alpha}(z\omega^{-j}) \tag{7.2}$$

where

$$a_j = k^{-2} \sum_{\nu=0}^{k-1} \zeta_{\nu,\varepsilon}^{-1} \omega^{\nu j}. \tag{7.3}$$

This can be verified from (4.1) and (7.1).

From (7.2) and (7.3), we can now easily show that the unique spline $S(z) \in \mathfrak{S}_{n,k}^{(\alpha)}$ interpolating the polynomial z^r, $0 \leqq r \leqq k-1$, is given by

$$S(z) = \begin{cases} z^r, & 0 \leqq r \leqq n \\ \dfrac{1}{k} \zeta_{r,\varepsilon}^{-1} \displaystyle\sum_{\nu=0}^{k-1} \omega^{\nu r} M_{n,\alpha}(z\omega)^{-\nu}, & n+1 \leqq r \leqq k-1. \end{cases} \tag{7.4}$$

From (3.10), we can obtain the Fourier series for $S(z)$, which interpolates z^r at the $\omega^\nu \psi^\varepsilon$ ($\nu = 0, 1, \dots, k-1$):

$$S(z) = C \sum_{-\infty}^{\infty} \frac{\beta(ks+r, n)}{(ks+r)(ks+r-1)\dots(ks+r-n)} z^{ks+r} \tag{7.5}$$

where

$$C = \frac{\zeta_{r,\varepsilon}^{-1}}{2\pi i} \prod_{j=r-n}^{r} (1 - \omega^{-j}).$$

Since $S(z)$ satisfies the fundamental equation $S(\omega z) = \omega^r S(z)$ we may call $S(z)$, an (r, α)-flower in the footsteps of Schoenberg [6].

REMARK 3. Recently in a forthcoming paper, J. TZIMBALARIO [9] has given a unified treatment of cardinal polynomial spline interpolation, trigonometric interpolation and interpolation by splines on the circle. It would be interesting to see how far his method can be adapted to resolve our problem.

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APPROXIMATION BY QUADRATIC SPLINES

By

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Dedicated to Professor G. Alexits on his 80th birthday

1. This note presents an approximation method to continuous functions by a sequence of quadratic spline functions with equidistant nodes. Features which make the procedure noteworthy are:

(i) For any given function $f(x)$, the corresponding spline approximants $\Phi_n(f; x)$ are given by simple formulae requiring only the values of f at equidistant points.

(ii) The $\Phi_n(f; x)$ are continuously differentiable, while we do not have to assume the same concerning $f(x)$.

(iii) For convex functions $f(x)$, we have $\Phi_n(f; x) \geqq f(x)$ for all n and x.

For definitions and earlier results we refer the reader to [1], [4] and [5] and for related results to [2], [3] and [6].

2. We formulate our result for functions f defined on [0, 1]. For convenience we shall use the notation $x_k = \dfrac{k-1}{n}$, $\xi_k = \dfrac{2k-1}{2n}$, $k = 0, 1, \dots, n+1$ and extend the definition of f outside [0, 1] by $f(\xi) = f(0)$ for $\xi < 0$ and $f(\xi) = f(1)$ for $\xi > 1$.

THEOREM. *Suppose $f \in C[0, 1]$ and that f is the integral of $f_1 \in \mathrm{BV}[0, 1]$. For $n = 1, 2, \dots$ let the spline approximants $\Phi_n(f; x)$ be defined by*

$$\Phi_n(f;\, x) = f(0) + \sum_{k=0}^{n} \lambda_k(f)\,(x - x_k)_+^2 \tag{2.1}$$

with

$$\lambda_k(f) = \frac{n^2}{2}\,[f(\xi_{k+1}) - 3f(\xi_k) + 3f(\xi_{k-1}) - f(\xi_{k-2})]\,. \tag{2.2}$$

Then we have

$$\|\, \Phi_n(f) - f \,\|_{L^1} \leqq \frac{V_1}{6n^2}\,, \tag{2.3}$$

$$\|\, \Phi_n(f) - f \,\|_{\infty} \leqq \frac{V_1}{4n} \tag{2.4}$$

where V_1 is the total variation of f_1.

3. We shall need the following

LEMMA. *For fixed* ξ, $\xi_k \leqq \xi \leqq \xi_{k+1}$ $(k = 0, 1, \dots, n)$ *let*

$$\Gamma_k(x;\xi) = a_k(\xi)(x - x_k)_+^2 + b_k(\xi)(x - x_{k+1})_+^2 + c_k(\xi)(x - x_{k+2})_+^2$$

with

$$(3.1)\quad a_k(\xi) = \frac{n}{4}(2k+1-2n\xi), \quad b_k(\xi) = n(n\xi - k), \quad c_k(\xi) = -\frac{n}{4}(2n\xi - 2k + 1).$$

Then

$$(3.2)\qquad \Gamma_k(x;\xi) = (x-\xi)_+ \quad \textit{for} \quad x_k > x \quad \textit{or} \quad x > x_{k+2},$$

$$(3.3)\qquad \Gamma_k(x;\xi) \geqq (x-\xi)_+ \quad \textit{for all} \quad x,$$

and

$$(3.4)\qquad \int_0^1 [\Gamma_k(x;\xi) - (x-\xi)_+]\,dx \leqq \frac{1}{6n^2}$$

for $\xi \in [0, 1]$, $k = 0, 1, \dots, n$.

PROOF. If we solve the indetity

$$a_k(x - x_k)^2 + b_k(x - x_{k+1})^2 + c_k(x - x_{k+2})^2 \equiv x - \xi$$

valid for $x > x_{k+2}$, we obtain for a_k, b_k and c_k the formulae given by (3.1). This proves (3.2), (3.3) and (3.4) follow by straightforward computations.

PROOF OF THE THEOREM. We define $f_1(\xi) = 0$ for $\xi < 0$ and $\xi > 1$ and set

$$(3.5)\qquad \Phi_n(f;x) = f(0) + \sum_{k=0}^{n} \int_{\xi_k}^{\xi_{k+1}} \Gamma_k(x;\xi)\,df_1(\xi)$$

with $\Gamma_k(x;\xi)$ as given in the Lemma. Evaluating the various integrals yields, after combining suitable terms, that $\Phi_n(f;x)$ is given by (2.1) with $\lambda_k(f)$ given by (2.2). Note that all those terms obtained by integration which involve values of f_1, vanish through summation; hence the coefficients $\lambda_k(f)$ involve values of f only.

In order to prove (2.3) we observe that for $x \in [0, 1]$ we may write

$$(3.6)\qquad f(x) = f(0) + \int_{\xi_0}^{\xi_{n+1}} (x-\xi)_+\,df_1(\xi)$$

since $\xi_0 < 0$, $\xi_{n+1} > 1$ and $f_1(\xi) = 0$ for $\xi < 0$ or $\xi > 1$. From (3.5) and (3.6)

$$\int_0^1 |\Phi_n(f;x) - f(x)|\,dx \leqq \sum_{k=0}^{n} \int_{\xi_k}^{\xi_{k+1}} |df_1(\xi)| \int_0^1 [\Gamma_k(x;\xi) - (x-\xi)_+]\,dx$$

which,on account of (3.4) $\leqq \frac{V_1}{6n^2}$. The inequality (2.4) can be established by similar considerations.

REMARKS. From (3.3), (3.5) and (3.6) it follows that $\Phi_n(f; x) \geqq f(x)$ for all $x \in [0, 1]$, if f is a convex function. Clearly the opposite inequality holds if f is concave.

It follows from (2.2) that if f has a bounded third derivative in $[0, 1]$, then for the coefficients $\lambda_k(f)$ we have $\lambda_k(f) = O\left(\frac{1}{n}\right) \|f'''\|$ for $3 \leqq k \leqq n-1$.

EXAMPLES. If $f(x) = x$, then $\lambda_k(f) = 0$ for $k \geqq 3$, hence $\Phi_n(f; x)$ reduces to $\frac{n}{4}\left[\left(x+\frac{1}{n}\right)^2 - \left(x-\frac{1}{n}\right)_+^2\right]$, so that $\Phi_n(f; x) = f(x)$ for $\frac{1}{n} \leqq x \leqq 1$ in this case.

If $f(x) = x^2$, then $\lambda_k(f) = 0$ for $k \geqq 3$, so $\Phi_n(f; x) = \frac{1}{8}\left(x+\frac{1}{n}\right)^2 + \frac{3}{4}x^2 + \frac{1}{8}\left(x-\frac{1}{n}\right)_+^2$. Hence, if $\frac{1}{n} \leqq x \leqq 1$, $\Phi_n(f; x) = f(x) + \frac{1}{4n^2}$. Since now $V_1 = 2$, we have in this case $\|\Phi_n(f) - f\|_{L^1} \geqq \frac{V_1}{8n^2}$. This shows that our estimate (2.3) is not far from being best possible.

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EXPONENTIAL ESTIMATES FOR THE MAXIMUM OF PARTIAL SUMS. I

(SEQUENCES OF RV'S)

By

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Dedicated to Professor G. Alexits on his 80th birthday

§ 1. Introduction

Let $\{\xi_k\}$ be a sequence of random variables (in abbreviation: rv's). Set

$$S(b, m) = \sum_{k=b+1}^{b+m} \xi_k$$

and

$$M(b, m) = \max_{1 \leqq k \leqq m} |S(b, k)| \quad (S(b, 0) = M(b, 0) = 0),$$

where $b \geqq 0$ and $m \geqq 1$ are integers. It is not assumed that the ξ_k are independent or that they are identically distributed. The only restriction on the dependence will be that imposed by the assumed bounds of exponential type on $P\{|S(b, m)| \geqq \lambda\}$ for $0 < \lambda < \Lambda$. These bounds are guaranteed under a suitable independence assumption (e.g., mutual independence, martingale differences, strong mixing, weak multiplicativity, or the like).

In the following, $g(b, m)$ will denote a non-negative function depending on the joint df of $\xi_{b+1}, \xi_{b+2}, \ldots, \xi_{b+m}$, and having the property

$$g(b, h) + g(b+h, m-h) \leqq g(b, m) \tag{1}$$

for all $b \geqq 0$ and $1 \leqq h < m$. In other words, condition (1) means that $g(b, m)$, as a function of the interval $[b+1, b+m]$, is "superadditive". Examples are $g(b, m) = m^\alpha$ with an $\alpha \geqq 1$ or $g(b, m) = \sum_{k=b+1}^{b+m} \sigma_k^2$, in the latter case assuming the existence of the finite variances σ_k^2 of the rv's ξ_k.

The upper bound for $P\{|S(b, m)| \geqq \lambda\}$ will be considered in the form of a rather general condition:

$$P\{|S(b, m)| \geqq \lambda\} \leqq C \exp\left(-\frac{\lambda^2}{g(b, m)}\right), \tag{2}$$

where $\lambda \in (0, \Lambda)$, $0 < \Lambda \leqq \infty$, C is a constant, and $g(b, m)$ satisfies (1). Throughout the paper, Λ denotes a fixed positive number or ∞, and C_1, C_2, C_γ, and C_δ denote positive constants.

In case $g(b, m) = 0$ for a certain $b \geqq 0$ and $m \geqq 1$, the right-hand side of (2) is equal to zero for all $\lambda > 0$ as an agreement. That is $P\{|S(b, m)| = 0\} = 1$ which means $S(b, m) = 0$ almost surely (in abbreviation : a.s.).

We note that condition (2) is satisfied, among others, with $C = 2$, $\Lambda = \infty$, and a function $g(b, m)$ proportional to $\sum_{k=b+1}^{b+m} \sigma_k^2$ if $\{\xi_k\}$ is a sequence of uniformly bounded

(i) independent rv's, or
(ii) martingale differences, or
(iii) multiplicative rv's.

The notion of multiplicativity is due to ALEXITS [1, pp. 186–187] and means that for each choice $1 \leqq k_1 < k_2 < \ldots < k_r$, $r \geqq 1$, we have

$$E(\xi_{k_1} \xi_{k_2} \ldots \xi_{k_r}) = 0. \tag{3}$$

§ 2. Main result

The following theorem, which is our main result, will certainly have a broad scope of applications.

THEOREM 1. *Suppose that there exists a non-negative function $g(b, m)$ satisfying (1) such that (2) holds for all $\lambda \in (0, \Lambda)$, $b \geqq 0$, and $m \geqq 1$. Then, for any $0 < \varepsilon < 1$, there exists a constant $C_1 = C_1(\varepsilon)$ such that*

$$P\{M(b, m) \geqq \lambda\} \leqq C_1 \exp\left(-\frac{(1-\varepsilon)\lambda^2}{g(b, m)}\right)$$

holds for all $\lambda \in (0, \Lambda)$, $b \geqq 0$, and $m \geqq 1$.

PROOF. The proof is based on the "bisection" technique (see, for example, BILLINGSLEY [2, pp. 87–103]), which goes back to Rademacher and Menšov.

We are to find two positive constants C_1 and C_2, the latter as close to 1 as wanted, such that

$$P\{M(b, k) \geqq \lambda\} \leqq C_1 \exp\left(-\frac{C_2\lambda^2}{g(b, k)}\right) \tag{4}$$

holds true for all $\lambda \in (0, \Lambda)$, $b \geqq 0$, and $k \geqq 1$.

The proof goes by induction on k. For $k = 1$, inequality (4) is an obvious consequence of (2) provided $C_1 \geqq C$ and $0 < C_2 \leqq 1$. Assume now, as induction hypothesis, that (4) holds for all $k < m$ (and for all $b \geqq 0$), and prove it for $k = m$ (and for all $b \geqq 0$). For a fixed $b \geqq 0$ two cases are possible: $g(b, m) = 0$ or $g(b, m) \neq 0$.

Case 1. If

$$g(b, m) = 0 \tag{5}$$

for some b, then by (1) we also have $g(b, k) = 0$ and, consequently, $S(b, k) = 0$

a.s. for $k = 1, 2, \ldots, m$. Hence $M(b, m) = 0$ a.s. or, equivalently,

$$P\{M(b, m) \geqq \lambda\} = 0 \qquad \text{for all} \qquad \lambda > 0 .$$

This means that (4) obviously holds.

Case 2. Let α be a real number that will be determined later. For the moment we only assume that $0 < \alpha < 1$. Since now $g(b, m) \neq 0$, there exists an integer $h = h(\alpha)$, $1 \leqq h \leqq m$, such that

$$g(b, h-1) \leqq \alpha\, g(b, m) < g(b, h) , \tag{6}$$

where $g(b, h-1)$ on the left is 0 if $h = 1$. Then (1) implies

$$g(b+h, m-h) \leqq g(b, m) - g(b, h) < (1-\alpha)\, g(b, m) . \tag{7}$$

It is clear that, for $1 \leqq k < h$, we have

$$|S(b, k)| \leqq M(b, h-1) ,$$

and, for $h \leqq k \leqq m$,

$$|S(b, k)| \leqq |S(b, h)| + M(b+h, m-h) .$$

Hence, for all $\lambda > 0$, we have

$$P\{M(b, m) \geqq \lambda\} \leqq P\{M(b, h-1) \geqq \lambda\} + P\{|S(b, h)| + M(b+h, m-h) \geqq \lambda\} .$$

If λ_1 and λ_2 are positive numbers and $\lambda_1 + \lambda_2 = \lambda$, then

$$\begin{aligned} P\{ M(b, m) \geqq \lambda\} \leqq {} & P\{M(b, h-1) \geqq \lambda\} + \\ & + P\{|S(b, h)| \geqq \lambda_1\} + P\{M(b+h, m-h) \geqq \lambda_2\} . \end{aligned} \tag{8}$$

Applying the induction hypothesis to $h - 1 < m$, we have

$$P\{M(b, h-1) \geqq \lambda\} \leqq C_1 \exp\left(-\frac{C_2 \lambda^2}{g(b, h-1)}\right),$$

and by (6)

$$P\{M(b, h-1) \geqq \lambda\} \leqq C_1 \exp\left(-\frac{C_2 \lambda^2}{\alpha\, g(b, m)}\right) \tag{9}$$

for all $\lambda \in (0, \Lambda)$. Applying the induction hypothesis now to $m - h < m$ and using (7), we find that

$$P\{M(b+h, m-h) \geqq \lambda_2\} \leqq C_1 \exp\left(-\frac{C_2 \lambda_2^2}{(1-\alpha)\, g(b, m)}\right) \tag{10}$$

for all $\lambda_2 \in (0, \Lambda)$. Finally, by (2)

$$P\{|S(b, h)| \geqq \lambda_1 \} \leqq C \exp\left(-\frac{\lambda_1^2}{g(b, h)}\right) \leqq C \exp\left(-\frac{\lambda_1^2}{g(b, m)}\right) \tag{11}$$

for all $\lambda_1 \in (0, \Lambda)$. Putting the inequalities (9)–(11) into the right-hand side of (8), we arrive at

$$P\{M(b, m) \geqq \lambda\} \leqq C_1 \exp\left(-\frac{C_2 \lambda^2}{\alpha g(b, m)}\right) + \tag{12}$$

$$+ C \exp\left(-\frac{\lambda_1^2}{g(b, m)}\right) + C_1 \exp\left(-\frac{C_2 \lambda_2^2}{(1-\alpha) g(b, m)}\right).$$

Let us choose α, λ_1 and λ_2 in such a way that the three exponents on the right-hand side of (12) coincide (do not forget that $0 < \alpha < 1$ and $\lambda_1 + \lambda_2 = \lambda$!). This is the case if

$$\frac{\lambda_2^2}{\lambda^2} = \frac{1-\alpha}{\alpha} \tag{13}$$

and

$$\frac{\lambda_1^2}{\lambda^2} = \frac{C_2}{\alpha}. \tag{14}$$

From (13) it follows that we necessarily have $1/2 < \alpha < 1$. For the moment assume that such a choice of α, λ_1 and λ_2 is possible.

Then inequality (12) may be rewritten as follows:

$$P\{M(b, m) \geqq \lambda\} \leqq 3\, C_1 \exp\left(-\frac{C_2 \lambda^2}{\alpha g(b, m)}\right).$$

Here the right-hand side does not exceed

$$C_1 \exp\left(-\frac{C_2 \lambda^2}{g(b, m)}\right)$$

provided

$$\ln 3 - \frac{C_2 \lambda^2}{\alpha g(b, m)} \leqq -\frac{C_2 \lambda^2}{g(b, m)},$$

which is so, when $g(b, m)$ is not too large:

$$g(b, m) \leqq \frac{(1-\alpha)\, C_2 \lambda^2}{\alpha \ln 3}. \tag{15}$$

Thus we obtain the wanted (4) under the assumption (15).

On the other hand, if (15) is not satisfied, i.e., if

$$g(b, m) > \frac{(1-\alpha)\, C_2 \lambda^2}{\alpha \ln 3}, \tag{16}$$

then by choosing

$$C_1 \geqq 3^{\alpha/(1-\alpha)}, \tag{17}$$

we have

$$C_1 \exp\left(-\frac{C_2\lambda^2}{g(b,m)}\right) > C_1 \exp\left(-\frac{\alpha \ln 3}{1-\alpha}\right) \geqq 1 .$$

Consequently, (4) holds again, since in any case $P\{M(b,m) \geqq \lambda\} \leqq 1$.

The cases (5), (15), and (16) together cover all possible values of $g(b,m)$. This completes the induction step.

Finally, we have to check that α, λ_1 and λ_2 can be chosen in such a way that (13) and (14) be fulfilled. If α is sufficiently close to 1 (from below), then by (13) the ratio λ_2^2/λ^2 can be made as small as required. Then $\lambda_1+\lambda_2=\lambda$ shows that λ_1^2/λ^2 can approach 1 (from below) within any prescribed accuracy. By (14), we have $C_2=\alpha\lambda_1^2/\lambda^2$. Summing up the above reasoning, we can fix an $\alpha=\alpha(\varepsilon)$ so that $C_2 \geqq 1-\varepsilon$. By (17), the constant

$$C_1 = \max\,(C, 3^{\alpha/(1-\alpha)})$$

will be appropriate for us.

Thus Theorem 1 is completely proved.

We note that since $\alpha(\varepsilon) \to 1-0$ as $\varepsilon \to +0$, therefore $C_1(\varepsilon) \to +\infty$ as $\varepsilon \to +0$.

§ 3. A few remarks and supplements

In most cases it is enough to use the following less sharp result instead of Theorem 1.

THEOREM 1*. *Suppose that there exists a non-negative function* $g(b,m)$ *satisfying* (1) *such that* (2) *holds for all* $\lambda \in (0, \Lambda)$, $b \geqq 0$, *and* $m \geqq 1$. *Then*

$$P\{M(b,m) \geqq \lambda\} \leqq \max\,(81, C) \exp\left(-\frac{\lambda^2}{5g(b,m)}\right).$$

Indeed, let $\alpha = 4/5$. Then, by (13), $\lambda_2 = \lambda/2$. Hence $\lambda_1 = \lambda/2$ and, by (14), $C_2 = 1/5$. Finally, by (17), $C_1 \geqq 81$.

The method of proving Theorem 1 applies in treating the more general case, when λ^2 in the exponent of the right-hand side of (2) is substituted by a function $\phi(\lambda)$, defined on $[0, \Lambda)$, subject to the following weak assumptions:

(i) $\phi(0) = 0$,

(ii) $\phi(\lambda)$ is strictly increasing on $[0, \Lambda)$, and

(iii) $\phi(\lambda)$ is continuous on $[0, \Lambda)$.

THEOREM 2. *Suppose that there exist a function* $\phi(\lambda)$ *with the properties* (i)–(iii), *and a non-negative function* $g(b,m)$ *satisfying* (1) *such that*

$$P\{|S(b,m)| \geqq \lambda\} \leqq C \exp\left(-\frac{\phi(\lambda)}{g(b,m)}\right)$$

holds for all $\lambda \in (0, \Lambda)$, $b \geqq 0$, *and* $m \geqq 1$. *Then*

$$P\{M(b,m) \geqq \lambda\} \leqq C_1 \exp\left(-\frac{C_2 \phi(\lambda)}{g(b, m)}\right),$$

and even C_2 *can be made, by increasing* C_1, *as close to* 1 *as required.*

To obtain upper bounds of exponential type for $P\{M(b, m) \geqq \lambda\}$ another approach was used in [4]. We recall a theorem, proved there.

THEOREM A. *Suppose that there exists a non-negative function* $g(b, m)$ *satisfying* (1) *such that*

$$E\{\exp(\lambda\, |S(b, m)|)\} \leqq C \exp(\lambda^2 g(b, m)) \tag{18}$$

holds for all $\lambda > 0$, $b \geqq 0$, *and* $m \geqq 1$. *Then*

$$E\{\exp(\lambda M(b, m))\} \leqq 8C \exp(12\lambda^2 g(b, m)).$$

In comparison with Theorem 1*, an essential difference is that the fulfilment of (18) for all $\lambda > 0$ is required, while (2) need hold only in a (right-hand side) neighbourhood of $\lambda = 0$. Unfortunately, we are not able to prove Theorem A when (18) is satisfied merely for $\lambda \in (0, \Lambda)$ with finite Λ.

On the other hand, the proof of Theorem A given in [4] makes use of no full power of a probability space. Theorem A is true on any measurable space $(X, \mathcal{A}, \mu)$, taking integrals over X with respect to μ in place of the expectations. The above proof of Theorem 1 (and so that of Theorem 1*), on the contrary, heavily uses the finiteness of the underlying measure space. Namely, the argument in the case of (16) breaks down if $\mu(X) = \infty$.

§ 4. Applications: LIL, convergence rates in the LIL and SLLN, complete convergence of normed M_n

It turns out that the "$\leqq$" part of the LIL is a consequence of the inequality (2) of exponential type. The results below can be obtained by adaptation of more or less standard arguments (see, e.g. [6]) by making use of Theorem 1*. Therefore we omit their proofs (as to the details we refer to [4] where the proofs are performed in the special case

$$g(b, m) = C \sum_{k=b+1}^{b+m} \sigma_k^2).$$

In case $b = 0$ we shall use the abbreviated notations

$$S(0, m) = S(m), \quad M(0, m) = M(m) \quad \text{and} \quad g(0, m) = g(m);$$

furthermore, set

$$d(m) = g(m) - g(m-1) \quad \text{for} \quad m \geqq 1 \quad (g(0) = 0).$$

The following theorem is the "$\leqq$" part of the LIL under fairly general conditions.

THEOREM 3. *Suppose that there exists a non-negative function* $g(b, m)$ *satisfying* (1) *such that* (2) *holds for all* $\lambda > 0$, $b \geqq 0$, *and* $m \geqq 1$. *If* $g(m) \to \infty$ *as* $m \to \infty$, *then*

$$P\left\{\limsup_{m\to\infty} \frac{|S(m)|}{\big(g(m)\ln\ln g(m)\big)^{1/2}} \leqq 1\right\} = 1\,. \tag{19}$$

We remark that the conclusion of Theorem 3 in the special case $g(b, m) = 2B^2m$ was proved by SERFLING [8, Theorem 4.1] for uniformly bounded rv's, $|\xi_k| \leqq B$ a.s. ($k = 1, 2, \ldots$), having the following two properties:

a) for every $\gamma > 2$

$$E\,|S(b, m)|^\gamma \leqq C_\gamma\, m^{\gamma/2} \qquad (b \geqq 0, m \geqq 1)\,,$$

b) for every $\lambda > 0$

$$P\{|S(b, m)| \geqq \lambda\} \leqq 2\exp\left(-\frac{\lambda^2}{2B^2 m}\right) \qquad (b \geqq 0, m \geqq 1)\,.$$

Due to the maximal inequality of exponential type in Theorem 1*, condition a) is superfluous.

Turning to the convergence rate in (19), we can state

THEOREM 4. *Suppose that there exists a non-negative function* $g(b, m)$ *satisfying* (1) *such that* (2) *holds for all* $\lambda > 0$, $b \geqq 0$, *and* $m \geqq 1$. *If*

$$g(m) \to \infty \quad \textit{and} \quad d(m) = o\big(g(m)\big) \quad \textit{as} \quad m \to \infty\,,$$

then, for any $\theta > 1$, *we have*

$$\sum_m \frac{d(m)}{g(m)\ln g(m)}\, P\left\{\sup_{k\geqq m} \frac{|S(k)|}{\big(\theta\, g(k)\ln\ln g(k)\big)^{1/2}} \geqq 1\right\} < \infty\,. \tag{20}$$

If the factor $\big(\theta \ln\ln g(k)\big)^{1/2}$ in the expression (20) is replaced by a less sharp factor $\big(\ln\ln g(k)\big)^{\alpha/2}$ or by an even rougher factor $\big(\ln g(k)\big)^{\alpha/2}$ with an $\alpha > 1$ in both cases, then an essentially better rate of convergence, not depending on α, can be achieved.

THEOREM 5. *Under the conditions of Theorem* 4, *for any* $\alpha > 1$ *and* $\beta > 0$, *we have*

$$\sum_m \frac{d(m)\big(\ln g(m)\big)^\beta}{g(m)}\, P\left\{\sup_{k\geqq m} \frac{|S(k)|}{\big(g(k)\big(\ln\ln g(k)\big)^\alpha\big)^{1/2}} \geqq 1\right\} < \infty$$

and

$$\sum_m d(m)\, g^\beta(m)\, P\left\{\sup_{k\geqq m} \frac{|S(k)|}{\big(g(k)\big(\ln g(k)\big)^\alpha\big)^{1/2}} \geqq 1\right\} < \infty.$$

This theorem essentially improves a result of SERFLING [8, Theorem 5.3].

A trivial consequence of the LIL is that $S(m)/g(m) \to 0$ a.s. as $m \to \infty$. It is of some interest to obtain information on the rate of convergence in this SLLN.

THEOREM 6. *Suppose that there exists a non-negative function $g(b, m)$ satisfying (1) such that (2) holds for all $\lambda > 0$, $b \geqq 0$, and $m \geqq 1$. Furthermore, suppose that with a $\beta > 0$ we have*

$$m^\beta = O\big(g(m)\big) \quad \text{and} \quad d(m) = o\big(g(m)\big) \quad \text{as} \quad m \to \infty .$$

Then, for any positive numbers ε and $\tau < \exp(\varepsilon^2)$, we have

$$\sum_m \tau^{g(m)} P\left\{\sup_{k \geqq m} \frac{|S(k)|}{g(k)} \geqq \varepsilon\right\} < \infty .$$

Now we come to the question of norming $M(m)$ suitably (say, into the form $M(m)/c(m)$, $c(m)$ is a sequence of numbers) in order that it converge completely to zero in the sense of HSU and ROBBINS [3]. The following result is a simple consequence of Theorem 1*.

THEOREM 7. *Under the conditions of Theorem 6, for any $\varepsilon > 0$, we have*

$$\sum_m P\left\{\frac{M(m)}{\big(v(m)\, g(m) \ln g(m)\big)^{1/2}} \geqq \varepsilon\right\} < \infty ,$$

provided $v(m) \to \infty$ as $m \to \infty$.

We note that in the special case when the ξ_k are independent and $g(b, m) = 2 \sum_{k=b+1}^{b+m} \sigma_k^2$, the statements of Theorems 3–7 are well-known. For multiplicative rv's various extensions of the LIL have been made by many authors (see, e.g., [7] or [5], the latter containing a historical review). Our Theorem 3 covers almost all of these results as far as the "$\leqq$" part of the LIL is concerned.

To be more specific, let $\{\phi_k\}$ be a sequence of rv's with the following properties:

(i) $|\phi_k| \leqq B$ a.s. $(k = 1, 2, \ldots)$,

(ii) for each even number $r \geqq 2$

$$W_r = \Big(\sum_{1 \leqq k_1 < k_2 < \ldots < k_r} E^2(\phi_{k_1} \phi_{k_2} \ldots \phi_{k_r})\Big)^{1/2} < \infty ,$$

(iii) $W = \limsup_{r \to \infty} W_r^{1/r} < \infty$,

and let $\{a_k\}$ be a sequence of numbers. Then inequality (2) is satisfied for the sequence $\{\xi_k = a_k \phi_k\}$ with

$$g(b, m) = 2(B^2 + W^2 + \delta) \sum_{k=b+1}^{b+m} a_k^2$$

and $C = C_\delta$ for every $\delta > 0$ (cf. [5, Lemma 2]). Consequently Theorems 3–7 are valid for such sequences $\{\xi_k\}$ of rv's if we substitute $2(B^2 + W^2) \sum_{k=b+1}^{b+m} a_k^2$ for $g(b, m)$ in them.

The above conditions (ii) and (iii) express a certain kind of weak multiplicativity of the sequence $\{\phi_k\}$ of rv's. In particular, if $\{\xi_k = \phi_k\}$ $(a_k \equiv 1)$ is a multiplicative sequence of rv's (i.e. satisfying (3)), then (ii) and (iii) obviously hold with $W = W_r = 0$.

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NIKOLSKII-TYPE INEQUALITIES IN CONNECTION WITH REGULAR SPECTRAL MEASURES

By

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Dedicated to Professor George Alexits on the occasion of his eightieth birthday on January 5, 1979, in high esteem

1. Introduction. If $L_{2\pi}^p$, $0 < p \leqq \infty$, is the space of 2π-periodic functions, pth power integrable over $(-\pi, \pi]$, Hölder's inequality states that for any $f \in L_{2\pi}^p$ and $0 < q \leqq p \leqq \infty$

$$\|f\|_{q,2\pi} \leqq \|f\|_{p,2\pi}. \tag{1.1}$$

In the case of functions with compact spectra, thus for trigonometric polynomials $t_n(x) := \sum_{k=-n}^{n} c_k \exp\{ikx\}$, NIKOLSKII [10] proved a converse result, namely for $0 < p \leqq q \leqq \infty$

$$\|t_n\|_{q,2\pi} \leqq C_{p,q}\, n^{1/p-1/q} \|t_n\|_{p,2\pi}. \tag{1.2}$$

There are counterparts for polynomials in several variables as well as for entire functions of exponential type, also connected with the names of D. Jackson, M. Plancherel, G. Polya, G. Szegö, A. Zygmund; for all the details and comprehensive bibliographical comments see [8, 15, 16].

The aim of the present note is to construct a framework which enables one to consider inequalities of type (1.2) within a general class of orthogonal expansions in Banach spaces.

To this end, Section 2 develops a multiplier concept for Banach spaces X which are admissible with respect to a regular spectral measure E for some appropriate Hilbert space H. This continues and extends slightly our approach given in [4] which already turned out to be quite useful in connection with other generalizations (cf. [16] and the literature cited there). Let us mention that the classical situation (1.2) is covered for $1 \leqq p \leqq \infty$ with $H = L_{2\pi}^2$ and $X = L_{2\pi}^p$, $1 \leqq p < \infty$, or $X = L_{2\pi}^\infty = (L_{2\pi}^1)^*$, respectively. Section 3 defines polynomials in this general frame and sets up de la Vallée Poussin (or delayed) means (see Theorem 2), a basic tool in the treatment of problems (such as (1.2)) concerned with functions having compact spectra. Section 4 first deals with interpolation of admissible Banach spaces. Here we only discuss the real method of Lions—Peetre but other constructions such as the complex method of Calderon may be considered as well (see [9, 18] for details). This is then used in Theorem 3 to derive general Nikolskii-type inequalities. Finally, Section 5 is concerned with some applications to illustrate the wide applicability of the general results obtained. While Section 5.1 considers trigonometric polynomials in several variables for different scales of admissible Banach spaces, Section 5.2 treats the (continuous)

Fourier spectral measure in order to derive Nikolskii-type inequalities for entire functions of exponential type in Besov spaces. Such inequalities may be used to deduce some sharp estimates of ZYGMUND [20] concerning best approximation by entire functions of exponential type in L^p-spaces, $1 < p < \infty$. Let us emphasize that one may give many further applications to other specific, discrete or continuous orthogonal expansions such as to those into Hermite or Laguerre functions, Jacobi polynomials or Hankel transforms in various weight spaces. For all the details, however, we refer to [9, 18]; see also [7].

Though our approach is rather general, let us also point out that it does not give any contribution to the problem of best constants in (1.2) (or in related inequalities) — still an open problem — nor does it enable one to recapture the classical results for the quasi-Banach spaces $L^p_{2\pi}$, $0 < p < 1$.

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2. Multipliers for admissible Banach spaces. Let **C**, **R**, **Z**, **P**, **N** be the sets of all complex, real, integral, non-negative integral, and natural numbers, respectively. For any complex Banach space X let $[X, X] = [X]$ be the space of all bounded linear operators of X into itself, $X^* = [X, \mathbf{C}]$ denoting the dual space of bounded linear functionals on X.

For a complex Hilbert space H with inner product $(\cdot,\cdot)$ let E be a (countably additive, self adjoint, bounded, linear) spectral measure in $\mathbf{R}^N$, the Euclidean N-space with inner product $uv := \sum_{j=1}^{N} u_j v_j$ and norm $|u| := \sqrt{uu}$, i.e., E maps the family Σ of all Borel measurable sets in $\mathbf{R}^N$ into the set of all self adjoint projections in $[H]$ such that ($\emptyset$ being the void set, I the identity mapping)

(i) $E(\sigma_1 \cap \sigma_2) = E(\sigma_1)\, E(\sigma_2)$ for all $\sigma_1, \sigma_2 \in \Sigma$,

(ii) $E(\emptyset) = 0$, $E(\mathbf{R}^N) = I$,

(iii) $E\left(\bigcup_{j=1}^{\infty} \sigma_j\right) = \sum_{j=1}^{\infty} E(\sigma_j)$, where $\sigma_j \in \Sigma$, $\sigma_i \cap \sigma_j = \emptyset$ for $i \neq j$.

Then for any $\tau \in L^\infty(\mathbf{R}^N, E)$, the space of complex-valued, E-essentially bounded functions, the integral

$$T^\tau := \int_{\mathbf{R}^N} \tau(u)\, dE(u) \tag{2.1}$$

is well-defined as an element of $[H]$. Moreover, the map $\tau \to T^\tau$ has the properties that $\big(\tau, \tau_1, \tau_2 \in L^\infty(\mathbf{R}^N, E)\big)$

$$\int_{\mathbf{R}^N} \tau_1(u)\, \tau_2(u)\, dE(u) = \Big[\int_{\mathbf{R}^N} \tau_1(u)\, dE(u)\Big]\Big[\int_{\mathbf{R}^N} \tau_2(u)\, dE(u)\Big], \tag{2.2}$$

$$\| T^\tau \|_{[H]} = \| \tau \|_{\infty, E} := E - \operatorname*{ess\,sup}_{u \in \mathbf{R}^N} |\tau(u)|, \tag{2.3}$$

$$(T^\tau f, g) = (f, T^{\bar{\tau}} g), \tag{2.4}$$

the bar denoting the complex conjugate number. In other words, the map $\tau \to T^\tau$ is an isometric homomorphism of the algebra $L^\infty(\mathbf{R}^N, E)$ onto a commutative subalgebra $[H]_M$ of $[H]$. For these basic facts compare [5, p. 900].

Now let X be a Banach space such that

$$\overline{X \cap H}^{\|\cdot\|_X} = X, \quad \overline{X \cap H}^{\|\cdot\|_H} = H, \tag{2.5}$$

i.e., $X \cap H$ is dense in X and H. Then, according to [4] (compare also [12, pp. 3, 121]) $\tau \in L^\infty(\mathbf{R}^N, E)$ is called a multiplier on X if for each $f \in H \cap X$

$$T^\tau f := f^\tau := \int_{\mathbf{R}^N} \tau(u)\, dE(u) f \in H \cap X, \quad \|f^\tau\|_X \leqq A \|f\|_X. \tag{2.6}$$

In view of (2.5), the closure of T^τ (it shall be denoted by the same symbol) belongs to $[X]$. The set of all multipliers τ on X is denoted by $M = M(X, E, H)$, the corresponding set of multiplier operators T^τ by $[X]_M$. Setting

$$\|\tau\|_M := \|T^\tau\|_{[X]} := \sup\{\|f^\tau\|_X ; f \in H \cap X, \|f\|_X \leqq 1\}, \tag{2.7}$$

M is a commutative Banach algebra with respect to the natural vector operations and pointwise multiplication, isometrically isomorphic to the subspace $[X]_M \subset [X]$.

If, however, the closure of $H \cap X$ in X is a proper, nontrivial subspace of X, the previous procedure does not lead to multiplier operators defined on all of X. In this situation we assume that X can be identified with the dual Y^* of a Banach space Y satisfying (2.5). This identification is supposed to be given by a bijective isometry J_X which satisfies

$$J_X : X \to Y^*, \quad J_X(\alpha f_1 + \beta f_2) = \bar{\alpha} J_X(f_1) + \bar{\beta} J_X(f_2) \tag{2.8}$$

and coincides with the natural isometry $J_H : H \to H^*$ given via the theorem of F. Riesz (on the duals of Hilbert spaces) when restricted to $H \cap X$, i.e.,

$$J_X f(g) = (g, f) \qquad (f \in H \cap X, g \in H \cap Y). \tag{2.9}$$

Then $\tau \in L^\infty(\mathbf{R}^N, E)$ is called a multiplier on X if $\bar{\tau} \in M(Y)$ according to the previous definition, the corresponding multiplier operator T^τ being given via

$$T^\tau f := J_X^{-1}(T^{\bar{\tau}})^* J_X f \qquad (f \in X), \tag{2.10}$$

where $(T^{\bar{\tau}})^*$ denotes the dual of $T^{\bar{\tau}} \in [Y]_M$ defined by

$$(T^{\bar{\tau}})^* f^*(g) = f^*(T^{\bar{\tau}} g) \qquad (f^* \in Y^*, g \in Y).$$

Indeed, we have $T^\tau \in [X]$ and $\|T^\tau\|_{[X]} = \|T^{\bar{\tau}}\|_{[Y]}$. Moreover, (2.10) yields for any $f \in H \cap X$, $g \in H \cap Y$ (see (2.4), (2.9))

$$J_X T^\tau f(g) = (T^{\bar{\tau}})^* J_X f(g) = J_X f(T^{\bar{\tau}} g) = \Big(\int_{\mathbf{R}^N} \overline{\tau(u)}\, dE(u)\, g, f\Big) = \Big(g, \int_{\mathbf{R}^N} \tau(u)\, dE(u) f\Big).$$

Therefore for any $f \in H \cap X$

$$T^{\tau} f = \int_{\mathbf{R}^N} \tau(u)\, dE(u)\, f,$$

justifying the terminology (for further details see [9]).

We conclude with a criterion for radial multipliers with respect to Riesz bounded spectral measures (see [4]). For a Banach space X satisfying (2.5) a spectral measure E is called regular (cf. [11]) if it is (R, α)-bounded for some $\alpha \geqq 0$, i.e.,

$$(2.11) \qquad r_\alpha(u/\rho) : = (1 - |u|/\rho)_+^\alpha : = [\max \{(1 - |u|/\rho), 0\}]^\alpha$$

belongs to M and, uniformly for all $\rho > 0$,

$$(2.12) \qquad \| r_\alpha(u/\rho) \|_M \leqq C_\alpha .$$

THEOREM 1. *Let E be a regular spectral measure. If $\alpha = j \in \mathbf{P}$ is such that* (2.11–12) *holds, and if $\tau_\rho(u) := \lambda(|u|/\rho)$ for a (sufficiently smooth) function λ satisfying*

$$(2.13) \qquad \|\lambda\|_{\mathrm{BV}_{j+1}} : = \frac{1}{j!} \int_0^\infty t^j |\lambda^{(j+1)}(t)|\, dt + \lim_{t\to\infty} |\lambda(t)| < \infty ,$$

then uniformly for all $\rho > 0$

$$(2.14) \qquad \|\tau_\rho\|_M \leqq C_j \|\lambda\|_{\mathrm{BV}_{j+1}} .$$

In the following we shall call a Banach space X admissible (with respect to H, E) if either one of the following conditions holds true:

(2.15) (i) X satisfies (2.5), and E is regular for X.

(ii) J_X maps (cf. (2.8)) X onto the dual Y^* of a Banach space Y satisfying (i).

3. Generalized polynomials and de la Vallée Poussin means. Let H, E be as in Section 2. We call $f \in H$ a polynomial (with respect to E) if there exists a compact set $\sigma \subset \mathbf{R}^N$ such that $E(\sigma)g = f$ for some $g \in H$. The set of all polynomials in H is denoted by

$$(3.1) \qquad \Pi^H : = \bigcup_{\sigma \text{ compact}} E(\sigma)\, H .$$

Since $E(\sigma)$ is a projection, $E(\sigma)g = f$ implies $E(\sigma)f = f$. The elements of

$$(3.2) \qquad \Pi_\rho^H : = \{ f \in \Pi^H;\ E(\{x \in \mathbf{R}^N;\ |x| \leqq \rho\}) f = f \}$$

are called polynomials (in H) of (radial) degree $\rho > 0$.

In order to extend this definition to admissible Banach spaces, let us introduce a family $\{\lambda_\varepsilon\}_{\varepsilon>0}$ of infinitely differentiable, real-valued functions on $[0, \infty)$ satisfying

$$0 \leqq \lambda_\varepsilon(t) \leqq 1; \quad \lambda_\varepsilon(t) = \begin{cases} 1, & 0 \leqq t \leqq t + \varepsilon/2 \\ 0, & t \geqq 1 + \varepsilon . \end{cases} \tag{3.3}$$

Setting

$$\tau_{\varepsilon,\rho}(u) := \lambda_\varepsilon(|u|/\rho), \quad T_{\varepsilon,\rho} := T^{\tau_{\varepsilon,\rho}} \qquad (\varepsilon, \rho > 0), \tag{3.4}$$

it follows from Theorem 1 (and (2.10)) that the family $\{T_{\varepsilon,\rho}\}_{\varepsilon>0}$ is well-defined on admissible Banach spaces X. Thus we can define the set of all polynomials (of radial degree $\rho > 0$) in X by

$$\begin{cases} \Pi_\rho^X := \{f \in X;\ T_{\varepsilon,\rho} f = f \ \text{ for all } \varepsilon > 0\}, \\ \Pi^X := \bigcup_{\rho>0} \Pi_\rho^X . \end{cases} \tag{3.5}$$

In case $H = X$ this is equivalent to (3.1 – 2).

The basic tool concerning a treatment of Nikolskii-type inequalities for these (generalized) polynomials will be a family of de la Vallée Poussin (or delayed) means (cf. [15, p. 522] for the one-dimensional trigonometric system, see also [6, 7, 11] for other orthogonal expansions). On H these operators are defined by

$$V_\rho f := \int_{|u| \leqq 2\rho} e(u/\rho)\, dE(u) f \qquad (\rho > 0), \tag{3.6}$$

where the corresponding multiplier e is given via

$$e(u) := \lambda(|u|), \quad \lambda(t) := \begin{cases} 1, & 0 \leqq t \leqq 1 \\ 0, & t \geqq 2, \end{cases} \tag{3.7}$$

$0 \leqq \lambda(t) \leqq 1$ denoting an infinitely differentiable function on $[0, \infty)$. It follows from Theorem 1 that the operators V_ρ are well-defined on each admissible space X.

THEOREM 2. *Let X be an admissible Banach space.*

(i) *For any $\rho > 0$ one has $V_\rho \in [X]$, and $\|V_\rho\|_{[X]}$ is a continuous, bounded function of ρ. In particular, for some $j \in \mathbf{P}$ (cf. (2.14))*

$$\|V_\rho\|_{[X]} \leqq C_j \|\lambda\|_{BV_{j+1}} =: M. \tag{3.8}$$

(ii) *For any $f \in X$, $\rho > 0$ one has $V_\rho f \in \Pi_{2\rho}^X$.*

(iii) *For any $P_\rho \in \Pi_\rho^X$ there holds the projection property*

$$V_\rho P_\rho = P_\rho \qquad (\rho > 0). \tag{3.9}$$

PROOF. Since X is admissible, (3.8) is an immediate consequence of Theorem 1 (and (2.10)). Moreover, for each fixed $\rho > 0$ and $h \in \mathbf{R}$ with $|h| < \rho/2$

$$e_{\rho,h}(u) := e_\rho(u) - e_{\rho+h}(u) := \lambda(|u|/\rho) - \lambda(|u|/(\rho + h))$$

is the multiplier corresponding to $V_\rho - V_{\rho+h}$. Thus by Theorem 1

$$\|V_\rho - V_{\rho+h}\|_{[X]} \leqq C_j \left\| \lambda\left(\frac{\cdot}{\rho}\right) - \lambda\left(\frac{\cdot}{\rho+h}\right)\right\|_{\mathrm{BV}_{j+1}} \leqq$$

$$\leqq C_j \int_0^\infty t^j \left| \lambda^{(j+1)}(t) - \lambda^{(j+1)}\left(\frac{t\rho}{\rho+h}\right)\right| dt +$$

$$+ C_j \int_0^\infty t^j \left| \lambda^{(j+1)}\left(\frac{t\rho}{\rho+h}\right) - \left(\frac{\rho}{\rho+h}\right)^{j+1} \lambda^{(j+1)}\left(\frac{t\rho}{\rho+h}\right)\right| dt = : I_{1,\rho}(h) + I_{2,\rho}(h).$$

Since $\lambda(t) = 0$ for $t \geqq 2$ and $|h| < \rho/2$, thus $2/3 < \rho/(\rho+h) < 2$, one obviously has $(h \to 0)$

$$I_{1,\rho}(h) = C_j \int_0^3 t^j \left| \lambda^{(j+1)}(t) - \lambda^{(j+1)}\left(\frac{t\rho}{\rho+h}\right)\right| dt = o(1),$$

$$I_{2,\rho}(h) = \left| \left(\frac{\rho+h}{\rho}\right)^{j+1} - 1 \right| C_j \int_0^3 t^j |\lambda^{(j+1)}(t)| \, dt = o(1).$$

Thus $\|V_\rho\|_{[X]}$ is continuous in $\rho > 0$. Moreover, since

$$\tau_{\varepsilon,\rho}(u)\, e(u/(\rho+\rho\varepsilon)) = \tau_{\varepsilon,\rho}(u) \qquad (u \in \mathbf{R}^N;\ \varepsilon, \rho > 0),$$

we have $T_{\varepsilon,\rho} = V_{\rho+\rho\varepsilon} T_{\varepsilon,\rho}$, and consequently by (i) for any $P_\rho \in \Pi_\rho^X$

$$\|P_\rho - V_\rho P_\rho\|_X \leqq \|P_\rho - V_{\rho+\rho\varepsilon} T_{\varepsilon,\rho} P_\rho\|_X + \|V_{\rho+\rho\varepsilon} P_\rho - V_\rho P_\rho\|_X \leqq$$

$$\leqq \|V_{\rho+\rho\varepsilon} - V_\rho\|_{[X]} \|P_\rho\|_X = o(1) \qquad (\varepsilon \to 0+).$$

This completes the proof.

4. Inequalities of Nikolskii-type. Throughout this section we deal with interpolation couples $\tilde{X} := (X_0, X_1)$ of admissible (with respect to a given pair E, H) Banach spaces, i.e., we assume X_0, X_1 to be embedded in a topological Hausdorff space $\mathfrak{X}$ such that their sum and intersection

$$\Sigma(\tilde{X}) := \{f \in \mathfrak{X};\ f = f_0 + f_1,\ f_0 \in X_0,\ f_1 \in X_1\},$$

$$\|f\|_{\Sigma(\tilde{X})} := \inf_{f=f_0+f_1} \{\|f_0\|_{X_0} + \|f_1\|_{X_1}\},$$

$$\Delta(\tilde{X}) := X_0 \cap X_1,\ \|f\|_{\Delta(\tilde{X})} := \max\{\|f\|_{X_0}, \|f\|_{X_1}\}$$

are well-defined Banach spaces. Then two Banach spaces X, Y are called (exact) interpolation spaces (of type $\theta, 0 \leqq \theta \leqq 1$) with respect to two given couples $\tilde{X}$, $\tilde{Y}$ if

$$\Delta(\tilde{X}) \subset X \subset \Sigma(\tilde{X}), \quad \Delta(\tilde{Y}) \subset Y \subset \Sigma(\tilde{Y}),$$

and if the assumption

(4.1) $$T \in [X_0, Y_0] \cap [X_1, Y_1], \quad \|T\|_{[X_j, Y_j]} \leqq M_j \quad (j = 0, 1)$$

implies

(4.2) $$T \in [X, Y], \quad \|T\|_{[X, Y]} \leqq M_0^{1-\theta} M_1^{\theta}.$$

For $0 < \theta < 1$, $1 \leqq q < \infty$ and $0 \leqq \theta \leqq 1$, $q = \infty$ denote by $(X_0, X_1)_{\theta, q} = \tilde{X}_{\theta,q}$ the interpolation spaces, generated by the real K-interpolation method of Lions–Peetre, i.e.,

$$\tilde{X}_{\Theta,q} := \{f \in \Sigma(\tilde{X});\ \|f\|_{\theta,q} < \infty\},$$

$$\|f\|_{\theta,q} := \begin{cases} \left(\int\limits_0^\infty [t^{-\theta} K(t,f)]^q \dfrac{dt}{t} \right)^{1/q}, & 1 \leqq q < \infty \\ \sup\limits_{t>0} t^{-\theta} K(t,f), & q = \infty, \end{cases}$$

(4.3) $$K(t, f, \tilde{X}) := \inf_{f = f_0 + f_1} \{\|f_0\|_{X_0} + t\|f_1\|_{X_1}\} \quad (t > 0).$$

An interpolation couple $\tilde{X} := (X_0, X_1)$ of admissible Banach spaces is said to have the "Nikolskii property" (cf. [17]) if

(4.4) $$\Pi^{X_0} \subset \Pi^{X_1}, \quad \|P_\rho\|_{X_1} \leqq \phi(\rho)\|P_\rho\|_{X_0} \quad (P_\rho \in \Pi_\rho^{X_0},\ \rho > 0)$$

for some function $\phi(\rho)$, strictly positive and monotonically increasing on $(0, \infty)$.

THEOREM 3. *Let $\tilde{X} := (X_0, X_1)$ be an interpolation couple of admissible Banach spaces, having the Nikolskii property* (4.4). *Then for $0 < \theta < \theta' < 1$, $1 \leqq q, q' \leqq \infty$ there hold the following Nikolskii-type inequalities for $P_\rho \in \Pi_\rho^{\tilde{X}_{\theta',q'}}$, $P_\rho \in \Pi_\rho^{\tilde{X}_{\theta,q}}$, and $P_\rho \in \Pi_\rho^{X_0}$, respectively:*

$$\|P_\rho\|_{1,\infty} \leqq \|P_\rho\|_{X_1} \leqq K(\theta', q')\,[\phi(2\rho)]^{1-\theta'}\,\|P_\rho\|_{\theta',q'},$$

$$\|P_\rho\|_{\theta',q'} \leqq K(\theta, \theta', q, q')\,[\phi(2\rho)]^{\theta'-\theta}\,\|P_\rho\|_{\theta,q},$$

$$\|P_\rho\|_{\theta,q} \leqq K(\theta, q)\,[\phi(2\rho)]^{\theta}\,\|P_\rho\|_{X_0}.$$

PROOF. First of all, it follows by (4.1–2) that the operators $T_{\varepsilon,\rho}$ as well as the de la Vallée Poussin operators V_ρ are well-defined on the interpolation spaces $\tilde{X}_{\theta,q}$. Thus (3.5) makes sense with X replaced by $\tilde{X}_{\theta,q}$ so that $\Pi_\rho^{\tilde{X}_{\theta,q}}$ is well-defined for $0 < \theta < 1$, $1 \leqq q < \infty$ and $0 \leqq \theta \leqq 1$, $q = \infty$. Moreover,

$$\|V_{\rho+\varepsilon} - V_\rho\|_{[\tilde{X}_{\theta,q}]} \leqq \|V_{\rho+\varepsilon} - V_\rho\|_{[X_0]}^{1-\theta}\,\|V_{\rho+\varepsilon} - V_\rho\|_{[X_1]}^{\theta}.$$

Thus the argument used in the proof of Theorem 2 (iii) shows that the projection property (3.9) holds for all $P_\rho \in \tilde{X}_{\theta,q}$.

According to (3.8) one has

$$\max_{j=0,1} \sup_{\rho>0} \| V_\rho \|_{[X_j]} \leqq M < \infty . \tag{4.5}$$

Hence Theorem 2 (ii) and the Nikolskii property (4.4) yield

$$\| V_\rho \|_{[X_0, X_1]} \leqq M\phi(2\rho) \qquad (\rho > 0) . \tag{4.6}$$

In view of $(X_i, X_i)_{\theta,q} = X_i$ (cf. [1, p. 46]),

$$[\tilde{X}_{\theta',q'}, X_1] = [(X_0, X_1)_{\theta',q'}, (X_1, X_1)_{\theta',q'}], \tag{4.7}$$

$$[X_0, \tilde{X}_{\theta,q}] = [(X_0, X_0)_{\theta,q}, (X_0, X_1)_{\theta,q}], \tag{4.8}$$

whereas the reiteration theorem (cf. [1, p. 50]) yields $(\eta := \theta/\theta')$

$$[\tilde{X}_{\theta,q}, \tilde{X}_{\theta',q'}] = [(X_0, \tilde{X}_{\theta',q'})_{\eta,q}, (\tilde{X}_{\theta',q'}, \tilde{X}_{\theta',q'})_{\eta,q}] \tag{4.9}$$

with equivalent norms. Thus (4.1−2) in connection with (4.5−8) implies

$$\| V_\rho \|_{[\tilde{X}_{\theta',q'}, X_1]} \leqq K(\theta', q') [\phi(2\rho)]^{1-\theta'}, \tag{4.10}$$

$$\| V_\rho \|_{[X_0, \tilde{X}_{\theta,q}]} \leqq K(\theta, q) [\phi(2\rho)]^{\theta} . \tag{4.11}$$

Moreover, this yields by (4.9) that

$$\| V_\rho \|_{[\tilde{X}_{\theta,q}, \tilde{X}_{\theta',q'}]} \leqq K(\theta, \theta', q, q') [\phi(2\rho)]^{\theta'-\theta} . \tag{4.12}$$

Since in any case $\|f\|_{\tilde{X}_{j,\infty}} \leqq \|f\|_{X_j}, j = 0, 1$ (cf. [3, p. 168]), the desired inequalities now follow from (4.10−12) and the projection property (3.9).

5. Applications. In order to illustrate the wide applicability of the previous results, let us consider certain concrete instances of spaces H, X, and spectral measures E. Rather than to give a complete list of possible applications, our aim is to show that the procedure of Sections 2−4 may serve as a unifying approach to classical and new Nikolskii-type inequalities. Further examples are worked out in [7, 9, 18].

5.1 Trigonometric polynomials of several variables. Let $\mathbf{Q}^N \subset \mathbf{R}^N$ be the cube $\{x \in \mathbf{R}^N;\ -\pi \leqq x_j < \pi, 1 \leqq j \leqq N\}$, and $L^p_{2\pi}$, $1 \leqq p \leqq \infty$, the space of measurable functions with period 2π in each variable for which

$$\|f\|_{p,2\pi} := \Big\{(2\pi)^{-N} \int_{\mathbf{Q}^N} |f(u)|^p\, du\Big\}^{1/p}, \quad \|f\|_{\infty,2\pi} := \operatorname*{ess\,sup}_{u\in\mathbf{Q}^N} |f(u)| ,$$

respectively, is finite. For $f \in L^p_{2\pi}$ and $k := (k_1, \ldots, k_N) \in \mathbf{Z}^N$, the N-fold Cartesian product of $\mathbf{Z}$, the k-th Fourier coefficient is given by

$$f^\wedge(k) := (2\pi)^{-N} \int_{\mathbf{Q}^N} f(u) \exp\{-iku\}\, du .$$

If one sets

$$E(\sigma) := \sum_{k \in \sigma \cap \mathbf{Z}^N} f^\wedge(k)\, e^{ikx} \qquad (\sigma \in \Sigma, f \in L^2_{2\pi}), \tag{5.1}$$

then E is a spectral measure for the Hilbert space $H = L^2_{2\pi}$ with inner product

$$(f, g) := (2\pi)^{-N} \int_{\mathbf{Q}^N} f(u)\, \overline{g(u)}\, du\,.$$

Moreover, E is (R, j)-bounded for $L^p_{2\pi}$, $1 \leqq p \leqq \infty$, if e.g. $j > (N-1)/2$ (cf. [14, p. 271]). Thus the spaces $X = L^p_{2\pi}$, $1 \leqq p \leqq \infty$, are admissible Banach spaces with respect to H, E since $L^p_{2\pi} \cap L^2_{2\pi}$ is dense in $L^2_{2\pi}$ and $L^p_{2\pi}$, $1 \leqq p < \infty$, whereas $L^\infty_{2\pi}$ may be identified with $(L^1_{2\pi})^*$. Obviously, $\| \tau \|_{\infty, E} = \sup_{k \in \mathbf{Z}^N} | \tau(k) |$ and $L^\infty(\mathbf{R}^N, E) = l^\infty(\mathbf{Z}^N)$, the set of all bounded sequences $\{\tau_k\}_{k \in \mathbf{Z}^N} \subset \mathbf{C}$, so that (2.1) reads for $f \in L^2_{2\pi}$ (cf. [5, p. 899])

$$T^\tau f := \int_{\mathbf{R}^N} \tau(u)\, dE(u) f = \sum_{k \in \mathbf{Z}^N} \tau(k) f^\wedge(k)\, e^{ikx}\,.$$

Moreover, Π^X_ρ coincides with the set of all trigonometric polynomials of radial degree ρ, i.e.,

$$\Pi^X_\rho := \{t_\rho;\ t_\rho(x) = \sum_{|k| \leqq \rho} c_k\, e^{ikx},\ c_k := t^\wedge_\rho(k) \in \mathbf{C}\}$$

(for more details cf. [9, 18]).

Let us consider the interpolation couple

$$\tilde{X} := (X_0, X_1) := (L^1_{2\pi}, L^\infty_{2\pi})\,.$$

Since any trigonometric polynomial of (radial) degree ρ has at most $(2\rho+1)^N$ nonzero coefficients $| t^\wedge_\rho(k) | \leqq \| t_\rho \|_{1, 2\pi}$, the couple $\tilde{X}$ obviously has the Nikolskii property (4.4) with $\phi(\rho) := (2\rho+1)^N$. Moreover, the interpolation spaces $\tilde{X}_{\theta, q}$, $0 < \theta < 1$, $1 \leqq q < \infty$ and $0 \leqq \theta \leqq 1$, $q = \infty$ are equal to the Lorentz spaces $L(p, q)$, $\theta = 1 - 1/p$ (cf. [3, p. 186]). Thus, as an application of Theorem 3, we obtain

COROLLARY 1. *Let* $1 < p_0 < p_1 < \infty$, $1 \leqq q_0, q_1 \leqq \infty$, *or* $p_0 = 1, q_0 = \infty$, *or* $p_1 = q_1 = \infty$. *Then for any trigonometric polynomial* t_ρ *of radial degree* ρ *there holds* $(i = 0, 1)$

$$\| t_\rho \|_{L(p_1, q_1)} \leqq C_{p_i q_i}\, \rho^{N(1/p_0 - 1/p_1)}\, \| t_\rho \|_{L(p_0, q_0)}\,.$$

Since $L(p, p) = L^p_{2\pi}$, $1 < p < \infty$, $L(1, \infty) = L^1_{2\pi}$, and $L(\infty, \infty) = L^\infty_{2\pi}$ (cf. [3, p. 186]), Corollary 1 indeed contains the classical Nikolskii inequality (1.2) for $1 \leqq p \leqq \infty$.

Another couple of admissible spaces is given by

$$\tilde{Y} := (Y_0, Y_1) := (C_{2\pi}, C^r_{2\pi}) \qquad (r \in \mathbf{P}),$$

where $C^r_{2\pi} \subset L^\infty_{2\pi}$ denotes the space of functions having continuous partial derivatives up to the order r (in particular $C_{2\pi} := C_{2\pi}^0$) with norm

$$\|f\|_{C_{2\pi}^r} := \|f\|_{\infty,2\pi} + \max_{k_1+\ldots+k_N=r} \left\| \frac{\partial^{k_1+\ldots+k_N} f}{\partial x_1^{k_1} \ldots \partial x_N^{k_N}} \right\|_{\infty,2\pi}.$$

Here the interpolation spaces $\tilde{Y}_{\theta,\infty}$, $0 \leqq \theta \leqq 1$, are equal to the Lipschitz spaces Lip $(\alpha, r, C_{2\pi})$, $\alpha = r\theta$, equipped with norm

$$\|f\|_{\mathrm{Lip}\,\alpha} := \|f\|_{C_{2\pi}} + \sup_{t>0} t^{-\alpha} \omega_r(t, f) \qquad (0 \leqq \alpha \leqq r),$$

ω_r denoting the modulus of continuity of order $r \in \mathbf{P}$ (cf. (5.7)). It follows from Bernstein's inequality

$$\left\| \frac{\partial^{k_1+\ldots+k_N} t_\rho}{\partial x_1^{k_1} \ldots \partial x_N^{k_N}} \right\|_{\infty,2\pi} \leqq \rho^{k_1+\ldots+k_N} \|t_\rho\|_{\infty,2\pi}$$

that $\tilde{Y}$ has the Nikolskii property (4.4) with $\phi(\rho) := 1 + \rho^r$. Thus we may again apply Theorem 3 to deduce

COROLLARY 2. *For any trigonometric polynomial of radial degree $\rho \geqq 1$ there hold the inequalities*

$$\|t_\rho\|_{\mathrm{Lip}\,\alpha} \leqq C_{\alpha,\beta}\, \rho^{\alpha-\beta} \|t_\rho\|_{\mathrm{Lip}\,\beta} \qquad (0 \leqq \beta \leqq \alpha \leqq r),$$

$$\|t_\rho\|_{C_{2\pi}^r} \leqq C_\alpha\, \rho^{r-\alpha} \|t_\rho\|_{\mathrm{Lip}\,\alpha} \qquad (0 \leqq \alpha \leqq r).$$

5.2 Fourier spectral measure. Let $L^p := L^p(\mathbf{R}^N)$ be the space of Lebesgue measurable functions on $\mathbf{R}^N$ for which the norm

$$\|f\|_p := \begin{cases} \{(2\pi)^{-N/2} \int_{\mathbf{R}^N} |f(u)|^p\, du\}^{1/p}, & 1 \leqq p < \infty \\ \operatorname*{ess\,sup}_{u \in \mathbf{R}^N} |f(u)|, & p = \infty, \end{cases}$$

respectively, is finite. Let $\mathfrak{S} := \mathfrak{S}(\mathbf{R}^N)$ denote the Schwartzian space of infinitely differentiable functions, rapidly decreasing at infinity, and $\mathfrak{S}'$ be the corresponding dual space of tempered distributions. The Fourier transform of $f \in \mathfrak{S}'$ is given by $(\phi \in \mathfrak{S})$

$$\mathfrak{F}f(\phi) := f(\phi\hat{\ }), \quad \phi\hat{\ }(v) := (2\pi)^{-N/2} \int_{\mathbf{R}^N} \phi(x) \exp\{-ivx\}\, dx.$$

Let $\mathfrak{F}^{-1}$ be its inverse and $\mathfrak{P}_\sigma$, $\sigma \in \Sigma$, the multiplication projection

$$\mathfrak{P}_\sigma f(u) := \chi_\sigma(u) f(u), \quad \chi_\sigma(u) := \begin{cases} 1, & u \in \sigma \\ 0, & u \notin \sigma. \end{cases}$$

Then (cf. [5, p. 1989])

$$E(\sigma) := \mathfrak{F}^{-1} \mathfrak{P}_\sigma \mathfrak{F} \qquad (\sigma \in \Sigma) \tag{5.2}$$

is a spectral measure for $H = L^2(\mathbf{R}^N)$ with inner product

$$(f, g) := (2\pi)^{-N/2} \int_{\mathbf{R}^N} f(u)\, \overline{g(u)}\, du .$$

Now $L^\infty(\mathbf{R}^N, E) = L^\infty$, and for any $\tau \in L^\infty$ the integral (2.1) admits the representation

$$T^\tau f := \int_{\mathbf{R}^N} \tau(u)\, dE(u) f = \mathcal{F}^{-1}[\tau(u) f^\wedge(u)] .$$

Furthermore, for any compact set $\sigma \subset \mathbf{R}^N$

$$E(\sigma) f(x) = (2\pi)^{-N/2} \int_\sigma f^\wedge(v)\, e^{ivx}\, dv \qquad (f \in L^2) .$$

It is an immediate consequence of the Paley—Wiener theorem (cf. [14, p. 112]) that Π_ρ^H is the set $\mathcal{E}_{2,\rho}$, where $\mathcal{E}_{p,\rho}$, $1 \leqq p \leqq \infty$, denotes the space of all p-th power integrable restrictions to $\mathbf{R}^N$ of entire functions f on $\mathbf{C}^N$ which are of (radial) exponential type ρ, i.e.,

$$|f(z)| \leqq A_\varepsilon \exp\{(\rho + \varepsilon)\, |z|\} \qquad (\varepsilon > 0, z \in \mathbf{C}^N) .$$

Note that E is (R, j)-bounded for $X = L^p$ if e.g. $j > (N - 1)\, |\, 1/p - 1/2\, |$ (cf. [13, p. 114]). Thus, all the spaces L^p, $1 \leqq p \leqq \infty$, are admissible, and it follows that the multiplier definition of Section 2 coincides with the classical one of Fourier multipliers (cf. [13, p. 94]). Moreover, the theorem of Paley—Wiener—Schwartz implies $\Pi_\rho^X = \mathcal{E}_{p,\rho}$ for $X = L^p$, $1 \leqq p \leqq \infty$. Instead of formulating counterparts of Corollaries 1—2 for entire functions of exponential type, let us derive Nikolskii-type inequalities in the Bessel potential- and Besov spaces

$$L_\alpha^p := \{f \in \mathcal{S}' ;\ \|f\|_\alpha^p < \infty\} \qquad (1 \leqq p \leqq \infty, \alpha \in \mathbf{R}),$$

$$B_{p,q}^s := \{f \in \mathcal{S}' ;\ \|f\|_{pq}^s < \infty\} \qquad (1 \leqq p, q \leqq \infty, s \in \mathbf{R}),$$

the norms being given by (with the obvious modifications for $q \doteq \infty$)

$$\|f\|_p^\alpha := \|I^\alpha f\|_p := \|\mathcal{F}^{-1}[(1 + v^2)^{\alpha/2} f^\wedge(v)]\|_p , \tag{5.3}$$

$$\|f\|_{pq}^s := \|\mathcal{F}^{-1}(\psi f^\wedge)\|_p + \left\{\sum_{k=1}^\infty (2^{sk}\, \|\mathcal{F}^{-1} \phi_k f^\wedge\|_p)^q\right\}^{1/q} , \tag{5.4}$$

respectively. Here ψ, $\phi_k \in \mathcal{S}$, $k \in \mathbf{Z}$, satisfy

$$\phi_k(v) := \phi(2^{-k} v); \quad \phi(v) \begin{cases} > 0, & 1/2 < |v| < 2 \\ = 0, & \text{otherwise,} \end{cases}$$

$$\sum_{k=-\infty}^\infty \phi_k(v) = 1; \quad \psi(v) := 1 - \sum_{k=1}^\infty \phi_k(v) .$$

It follows (cf. [1, pp. 142, 146]) that the spaces L_α^p, $1 \leqq p \leqq \infty$, $\alpha \in \mathbf{R}$, are admissible whereas $B^s_{p,q}$ is equal to the interpolation space

$$(L^p_\alpha, L^p_\beta)_{\theta,q}, \ 1 \leqq p, q \leqq \infty, \ 0 < \theta < 1, \ s = (1-\theta)\alpha + \theta\beta .$$

Let us now consider, for example, the interpolation couple

$$\tilde{X} := (B^s_{p,\infty}, B^s_{p,1})$$

for some fixed $1 \leqq p \leqq \infty$, $s \in \mathbf{R}$. Then (cf. [1, p. 153])

$$\tilde{X}_{\theta,q} = B^s_{p,q} \qquad (0 < \theta < 1, q = 1/\theta) .$$

Moreover, for any $P_\rho \in \mathscr{E}_{p,\rho}$ it follows from the theorem of Paley–Wiener–Schwartz that the sum in (5.4) contains at most $k_0 \in \mathbf{N}$ terms for some $k_0 \leqq 1 + (\log \rho)/(\log 2) < k_0 + 1$. Thus $\tilde{X}$ has the Nikolskii property (4.4) with

$$\phi(\rho) := c(1 + \log^+ \rho) := c \max \{1, 1 + \log \rho\} .$$

An application of Theorem 3 then leads to

COROLLARY 3. Let $s \in \mathbf{R}$, $1 \leqq p \leqq \infty$, $1 \leqq q_0 < q_1 \leqq \infty$. *Then one has for any* $P_\rho \in \mathscr{E}_{p,\rho}$

$$\|P_\rho\|^s_{pq_0} \leqq C_{q_0,q_1} (1 + \log^+ \rho)^{1/q_0 - 1/q_1} \|P_\rho\|^s_{pq_1} .$$

Taking into account the embedding theorem for Besov spaces (cf. [1, p. 152]) one in particular has

$$\|P_\rho\|^s_p \leqq C_{p,q} (1 + \log^+ \rho)^{1/p - 1/q} \|P_\rho\|^s_{pq} \qquad (1 < p \leqq 2, p < q), \tag{5.5}$$

$$\|P_\rho\|^s_p \leqq C_{p,q} (1 + \log^+ \rho)^{1/2 - 1/q} \|P_\rho\|^s_{pq} \qquad (2 \leqq p < \infty, 2 < q). \tag{5.6}$$

By means of these inequalities we would like to derive an inverse theorem for the best approximation

$$E_{p,\rho}(f) := \inf_{P_\rho \in \mathscr{E}_{p,\rho}} \|f - P_\rho\|_p$$

by entire functions of exponential type. To this end, let the radial modulus of continuity of (fractional) order $\alpha > 0$ be defined by

$$\omega_{\alpha,p}(t,f) := \sup_{|h| \leqq t} \left\| \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x + kh) \right\|_p . \tag{5.7}$$

COROLLARY 4. *If* $f \in L^p$, $1 < p < \infty$, *satisfies for some* $\alpha > 0$

$$E_{p,\rho}(f) = O(\rho^{-\alpha}) \qquad (\rho \to \infty), \tag{5.8}$$

then $(\tilde{p} := \min\{p, 2\})$

$$\omega_{\alpha,p}(t,f) = O(t^\alpha |\log t|^{1/\tilde{p}}) . \tag{5.9}$$

PROOF. Note that (5.8) holds if and only if $f \in B^{\alpha}_{p,\infty}$ (cf. [1, p. 181]), whereas it follows from [19] that (5.9) is equivalent to (cf. (4.3))

$$K(t^{\alpha}, f, L^p, L^p_{\alpha}) = O(t^{\alpha} \,|\log t\,|^{1/\tilde{p}}) \qquad (t \to 0+).$$

Moreover, if $P_{\rho}(f)$ denotes the element of best approximation to $f \in B^{\alpha}_{p,\infty}$, i.e., $E_{p,\rho}(f) = \|f - P_{\rho}(f)\|_p$, then (5.5–6) imply ($t := \rho^{-1}$)

$$K(t^{\alpha}, f, L^p, L^p_{\alpha}) \leqq \|f - P_{1/t}(f)\|_p + t^{\alpha}\, \|P_{1/t}(f)\|^{\alpha}_p \leqq$$

$$\leqq Ct^{\alpha} + C' t^{\alpha}\big(1 + \log^{+}(1/t)\big)^{1/\tilde{p}}\, \|f\|^{\alpha}_{p\infty},$$

which completes the proof.

We conclude with the remark that Corollaries 3,4 apply mutatis mutandito 2π-periodic functions of several variables. Thus, in particular, Corollary 4 extends a result of A. ZYGMUND [20], who proved an analogous estimate for $\alpha = 1$ conscerning the approximation by one-dimensional trigonometric polynomials (for a different approach see [2]).

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ON A GENERALIZATION OF THE HAAR SYSTEM

By

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To Professor G. Alexits on his 80th birthday

1. Introduction

Let (X, A, P) be a probability space, $\mathbf{N} = \{0, 1, 2, \dots\}$, $L^p = L^p(X, A, P)$ $(1 \leqq p \leqq \infty)$, and $\|f\|_p$ $(1 \leqq p \leqq \infty)$ denote the L^p-norm of the function $f \in L^p$. Recently, as a generalization of the property of lacunary series, G. ALEXITS [1], [2], [3] has introduced the notion of strongly multiplicative, multiplicative and weakly multiplicative systems. The system $\Phi = \{\phi_n : n \in \mathbf{N}\} \subset L^\infty$ is called strongly multiplicative, multiplicative, and weakly multiplicative if the functions ψ_n $(n \in \mathbf{N})$ of the product system Ψ of Φ are orthogonal,

$$(1) \qquad \begin{cases} \text{(i)} \int\limits_X \psi_n dP = 0 \qquad \left(n \in \mathbf{N}^* = \mathbf{N} \setminus \{0\}\right). \\ \text{(ii)} \ A = \sum\limits_{n=0}^{\infty} \left| \int\limits_X \psi_n dP \right| < \infty, \end{cases}$$

respectively. The product system Ψ is defined by

$$(2) \qquad \psi_n = \prod_{j=0}^{\infty} \phi_j^{n_j} \qquad \left(n = \sum_{j=0}^{\infty} n_j 2^j \in \mathbf{N},\ n_j \in \{0, 1\}\right).$$

The Rademacher system $R = \{r_n : n \in \mathbf{N}\}$ is a strongly multiplicative system and the product system of R is the Walsh–Paley system $W = \{w_n : n \in \mathbf{N}\}$. The following relation between the Haar system $\{\chi_n : n \in \mathbf{N}\}$ and the Walsh–Paley system is well-known [1]

$$(3) \qquad \begin{cases} \text{(i)} \ w_{2^n+i} = \sum\limits_{0 \leqq j < 2^n} \alpha_{ij}^n \chi_{2^n+j} \\ \text{(ii)} \ \chi_{2^n+i} = \sum\limits_{0 \leqq j < 2^n} \alpha_{ij}^n w_{2^n+j} \end{cases} \qquad (i \in \{0, 1, 2, \dots, 2^n - 1\},\quad n \in \mathbf{N}),$$

where $\alpha_{ij}^n = 2^{-n/2}\ w_i(j 2^{-n})$.

The following generalization of the Haar system was introduced by G. ALEXITS [2]. Let Ψ be the product system of Φ, then the system

$$(4) \qquad h_0 = 1, \quad h_{2^n+i} = \sum_{0 \leqq j < 2^n} \alpha_{ij}^n \psi_{2^n+j} \qquad (i \in \{0, 1, \dots, 2^n - 1\},\quad n \in \mathbf{N})$$

is called the H-system generated by the system Φ. The H-system generated by R is the Haar system.

ALEXITS [2] showed the following

THEOREM A. *Let Φ be a strongly multiplicative system with $\|\phi_n\|_\infty \leqq 1$ and $\|\psi_n\|_2 = 1$ $(n \in \mathbf{N})$. Then the Fourier series with respect to the H-system generated by Φ of every function $f \in L^2$ is a.e. $(c, \alpha > 0)$ summable.*

In this paper, among others, we give a generalization of this theorem.

THEOREM 1. *Let Φ be a weakly multiplicative system with $\|\phi_n\|_\infty \leqq 1$ $(n \in \mathbf{N})$ and let H be the H-system generated by Φ. Then H is a convergence system.*

2. A relation between function systems

Let (X, A, P) and (Y, B, Q) be two probability spaces and for $K \in L^\infty(X \times Y, A \times B, P \times Q)$

$$\begin{aligned} \|K\|_{(p,\infty)} &= \operatorname*{vrai\,max}_{y \in Y} \Big(\int_X |K(x,y)|^p \, dP\Big)^{1/p}, \\ \|K\|_{(\infty,q)} &= \operatorname*{vrai\,max}_{x \in X} \Big(\int_Y |K(x,y)|^q \, dQ\Big)^{1/q} \end{aligned} \qquad (1 \leqq p, q < \infty).$$

We shall use integral operators of the form

$$(Kg)(x) = \int_Y g(y) K(x,y) \, dQ \qquad \big(g \in L^2(Y, B, Q)\big),$$

and introduce the following notion. The function system $G = \{g_n : n \in \mathbf{N}\} \subset L^\infty(Y, B, Q)$ is called a better system than $F = \{f_n : n \in \mathbf{N}\} \subset L^\infty(X, A, P)$ (in notation $G < F$) if there exist a sequence $K_n \in L^\infty(X \times Y, A \times B, P \times Q)$ $(n \in \mathbf{N})$ of functions and an index sequence $(m_n, n \in \mathbf{N})$ such that

$$\text{(5)} \qquad \begin{cases} \text{(i)} \ f_k = K_n g_k \qquad (0 \leqq k < m_n, \ n \in \mathbf{N}), \\ \text{(ii)} \ \sup\limits_n \|K_n\|_{(1,\infty)} \leqq M < \infty, \qquad \sup\limits_n \|K_n\|_{(\infty,1)} \leqq M < \infty. \end{cases}$$

It is obvious that the relation $<$ is transitive.

We shall prove the following

THEOREM 2. *If $G < F$, then for every sequence $(a_n, n \in \mathbf{N})$ and for all $1 \leqq p \leqq \infty$ we have*

$$\text{(6)} \qquad \begin{cases} \Big\| \sum\limits_{k=0}^n a_k f_k \Big\|_p \leqq M \Big\| \sum\limits_{k=0}^n a_k g_k \Big\|_p \qquad (n \in \mathbf{N}), \\ \Big\| \sup\limits_n \Big| \sum\limits_{k=0}^n a_k f_k \Big| \Big\|_p \leqq M \Big\| \sup\limits_n \Big| \sum\limits_{k=0}^n a_k g_k \Big| \Big\|_p . \end{cases}$$

Let Ψ be the product system of the weakly multiplicative system $\Phi \subset L^\infty(X, A, P)$. Then $\Psi > W$.

Indeed, let

$$K_n(x, y) = \sum_{0 \leqq k < 2^n} \psi_k(x)\, w_k(y) = \prod_{i=0}^{n-1} \left(1 + \phi_i(x)\, r_i(y)\right) \geqq 0, \quad m_n = 2^n \qquad (n \in \mathbf{N}).$$

Then by the orthogonality of the system W (5)(i) holds. Furthermore by (1) we have

$$\int_0^1 |\, K_n(x, y) \,|\, dy = \sum_{k=0}^{2^n-1} \psi_k(x) \int_0^1 w_k(y)\, dy = 1 \qquad (x \in X, n \in \mathbf{N}),$$

$$\int_X |\, K_n(x, y) \,|\, dP = \sum_{k=0}^{2^n-1} w_k(y) \int_X \psi_k \, dP \leqq A \qquad (y \in [0, 1], \; n \in \mathbf{N}).$$

From this follows (see also [8])

THEOREM 3. *If Ψ is the product system of a weakly multiplicative system, then for every sequence $(c_n, n \in \mathbf{N})$ we have*

$$\left\| \sup_n \left| \sum_{k=0}^{n} c_k \psi_k \right| \right\|_2 \leqq C \left(\Sigma \,|\, c_n \,|^2\right)^{1/2},$$

where the constant C depends only on A in (1).

3. W-systems

Starting from (3) (i) we introduce the following notion. Let $H = \{H_n : n \in \mathbf{N}\} \subset$ $\subset L^\infty(X, A, P)$ a function system. Then the system

$$W_0 = H_0, \quad W_{2^n+i} = \sum_{j=0}^{2^n-1} \alpha_{ij}^n H_{2^n+j} \qquad (i \in \{0, 1, \ldots, 2^n - 1\}, \quad n \in \mathbf{N})$$

is called the W-system generated by H.

Further we investigate systems H for which there exists a constant K such that

$$(*) \qquad \begin{cases} \text{(i)} \quad \left\| \sum_{0 \leqq j < 2^n} H_{2^n+j} \right\|_\infty \leqq K \cdot 2^{n/2}, \\ \text{(ii)} \quad \|\, H_{2^n+i} \,\|_1 \leqq K \cdot 2^{-n/2} \qquad (i \in \{0, 1, \ldots, 2^n - 1\}, \quad n \in \mathbf{N}). \end{cases}$$

It is obvious that the Haar-system has these properties. Z. CIESIELSKI [7] has proved that for the Franklin system (*) also holds.

We show that for all H-systems, generated by weakly multiplicative systems $\{\phi_n : n \in \mathbf{N}\}$ with $\|\, \phi_n \,\|_\infty \leqq 1$ (*) is true.

Indeed, according to definition

$$h_{2^n+i} = 2^{-n/2} \sum_{j=0}^{2^n-1} W_j(i 2^{-n})\, \psi_{2^n+j} = 2^{-n/2} \phi_n \prod_{k=0}^{n-1} \left(1 + r_k(i 2^{-n})\, \phi_k\right)$$
$$(i \in \{0, 1, \ldots, 2^n - 1\}, \quad n \in \mathbf{N}).$$

Thus, by $\| \phi_n \|_\infty \leqq 1$ we get

(7) $$\| h_{2^n+i} \|_1 \leqq 2^{-n/2} \int_X \prod_{k=0}^{n-1} (1 + r_k(i2^{-n}) \phi_k)\, dP \leqq 2^{-n/2} \sum_{j=0}^{\infty} \left| \int_X \psi_j \, dP \right| = A2^{-n/2},$$

$$\sum_{j=0}^{2^n-1} | h_{2^n+j}(x) | \leqq \left| \sum_{j=0}^{2^n-1} 2^{-n/2} \sum_{i=0}^{2^n-1} W_i(j2^{-n}) \psi_{2^n+i}(x) \right| \leqq 2^{n/2}.$$

It is easy to show that every W-system generated by H systems having the property (*) are uniformly bounded. The W-system of the Franklin system was introduced by Z. CIESIELSKI [6].

For a sequence $a = (a_n, n \in \mathbf{N}) \in l_2$ we define the sequence $\tilde{a} = (\tilde{a}_n, n \in \mathbf{N}) \in l_2$ as follows:

$$\tilde{a}_0 = a_0, \quad \tilde{a}_{2^n+i} = \sum_{j=0}^{2^n-1} \alpha_{ij}^n a_{2^n+j} \qquad (i \in \{0, 1, 2, \ldots, 2^n - 1\}, \ n \in \mathbf{N})$$

then the map $l_2 \in a \to \tilde{a} \in l_2$ is a linear isomorphism and

(8) $$\sum_{j=0}^{2^n-1} a_{2^n+j} W_{2^n+j} = \sum_{j=0}^{2^n-1} \tilde{a}_{2^n+j} H_{2^n+j} \qquad (n \in \mathbf{N}).$$

Applying Theorem 2 we get (see [9])

THEOREM 4. *Let $H = \{H_n : n \in \mathbf{N}\}$ be a system with the property (*) and let $\{W_n : n \in \mathbf{N}\}$ the W-system generated by H. Then for every sequence $(a_n, n \in \mathbf{N})$ we have*

$$\left\| \sup_n \left| \sum_{k=0}^{n} a_k W_k \right| \right\|_2 \leqq C \left(\left\| \sup_n \left| \sum_{k=0}^{2^n-1} \tilde{a}_k H_k \right| \right\|_2 + \left(\sum_{k=1}^{\infty} a_k^2 \right)^{1/2} \right),$$

where the constant C depends only on K in (*).

4. Proof of the theorems

PROOF OF THEOREM 1. Denote $L_n(G)$ the n^{th} Lebesgue function of the system $G = \{G_n : n \in \mathbf{N}\} \subset L^1(X, A, P)$, i.e. let

$$L_n(G : x) = \int_X \left| \sum_{k=0}^{n-1} G_n(x) G_n(y) \right| dP(y) \qquad (x \in X, n \in \mathbf{N}^*).$$

We shall show that if H is the H-system generated by the weakly multiplicative system Φ, then

(9) $$\| L_n(H) \|_\infty \leqq 2A \qquad (n \in \mathbf{N}),$$

where A denotes the constant in (1)(ii).

Indeed, by the orthogonality of the matrix $(\alpha_{ij}^n, i, j = 0, 1, \ldots, 2^n - 1)$

$$\sum_{k=0}^{2^n-1} h_{2^n+k}(x)\, h_{2^n+k}(y) = \sum_{k=0}^{2^n-1} \psi_{2^n+k}(x)\, \psi_{2^n+k}(y), \qquad (n \in \mathbf{N}),$$

thus

$$L_{2^n}(H; x) = L_{2^n}(\Psi; x) = \int_X \left| \prod_{i=0}^{n-1} \left(1 + \phi_i(x)\, \phi_i(y)\right) \right| dP(y) \leqq$$

$$\leqq \sum_{k=0}^{\infty} \left| \int_X \psi_k(y)\, dP(y) \right| = A \qquad (n \in \mathbf{N}).$$

By the property (*) of the system $\{h_n : n \in \mathbf{N}\}$ (see [7]) we have

$$\max_{0 \leqq m < 2^n} \int_X \left| \sum_{k=2^n}^{2^n+m} h_k(x)\, h_k(y) \right| dP(y) \leqq \max_{2^n \leqq m < 2^{n+1}} \| h_m \|_1 \sum_{k=2^n}^{2^{n+1}-1} | h_k(x) | \leqq A \qquad (n \in \mathbf{N}),$$

and (9) is proved.

Applying a theorem of G. ALEXITS [4] we get Theorem 1.

PROOF OF THEOREM 2. To prove (6) we need the following

LEMMA. *If for the function* $K \in L^1(X \times Y, A \times B, P \times Q)$, $\| K \|_{(1, \infty)} < M$ *and* $\| K \|_{(\infty, 1)} < M$, *then for every* $1 \leqq p \leqq \infty$ *the operator* $K : L^p(Y, B, Q) \to L^p(X, A, P)$ *is bounded, and for the* L^p*-norm we have* $\| K \|_{L^p} \leqq M$. (See e.g. [7], pp. 518.)

By the definition of the relation $<$ we have

$$\sum_{k=0}^{m} a_k f_k = K_n \left(\sum_{k=0}^{m} a_k g_k \right) \qquad (m < m_n),$$

$$\sup_{0 \leqq k < m_n} \left| \sum_{i=0}^{k} a_i f_i(x) \right| \leqq \int_Y \sup_{0 \leqq k < m_n} \left| \sum_{i=0}^{k} a_i g_i(y) \right| | K_n(x, y) |\, dQ.$$

Applying the lemma, we get Theorem 2.

PROOF OF THEOREM 4. It is obwious, that by (8) we have

$$\left\| \sup_n \left| \sum_{k=0}^{n} a_k W_k \right| \right\|_2^2 \leqq 2 \left\| \sup_n \left| \sum_{k=0}^{2^n-1} a_k W_k \right| \right\|_2^2 + 2 \sum_{n=0}^{\infty} \left\| \sup_{2^n \leqq m < 2^{n+1}} \left| \sum_{k=2^n}^{m} a_k W_k \right| \right\|_2^2. \tag{10}$$

Let

$$K_n(x, t) = \sum_{j=0}^{2^n-1} H_{2^n+j}(x)\, \chi_{2^n+j}(t) \qquad (x \in X,\ t \in [0, 1),\ n \in \mathbf{N}).$$

Then

$$K_n(w_{2^n+k}) = W_{2^n+k} \qquad (k \in \{0, 1, \ldots, 2^n - 1\},\ n \in \mathbf{N}),$$

$$\| K_n \|_{(1, \infty)} \leqq K, \quad \| K_n \|_{(\infty, 1)} \leqq K \qquad (n \in \mathbf{N}).$$

Indeed, by the orthogonality of the Haar-system we have

$$K_n(w_{2^n+k})(x) = \int_0^1 \sum_{j=0}^{2^n-1} H_{2^n+j}(x)\,\chi_{2^n+j}(t) \sum_{i=0}^{2^n-1} \alpha_{ki}^n \chi_{2^n+i}(t)\,dt =$$

$$= \sum_{j=0}^{2^n-1} \alpha_{kj}^n H_{2^n+j}(x) = W_{2^n+k}(x),$$

$$\| K_n \|_{(\infty,1)} \leqq 2^{-n/2} \left\| \sum_{j=0}^{2^n-1} H_{2^n+j} \right\|_\infty \leqq K,$$

$$\| K_n \|_{(1,\infty)} \leqq 2^{n/2} \max_{0 \leqq j < 2^n} \| H_{2^n+j} \|_1 \leqq K \qquad (n \in \mathbf{N}).$$

From Theorem 2 and from a Theorem of P. SJÖLIN [10] we get

$$\left\| \sup_{0 \leqq m < 2^n} \left| \sum_{k=0}^{m} a_{2^n+k} W_{2^n+k} \right| \right\|_2^2 \leqq C \left\| \sup_{0 \leqq m < 2^n} \left| \sum_{k=0}^{m} a_{2^n+k} W_{2^n+k} \right| \right\|_2^2 \leqq \tilde{C} \left(\sum_{k=2^n}^{2^{n+1}-1} a_k^2 \right),$$

where $\tilde{C}$ depends only on K. This and (9) imply Theorem 4.

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ON THE CONVERGENCE OF VILENKIN—FOURIER SERIES

By

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To the eightieth birthday of Professor G. Alexits

Introduction

In this paper we are concerned with the almost everywhere and L^1-convergence of Vilenkin–Fourier series. First we give an estimation for the partial sums of Vilekin-series. This estimation implies a sufficient condition for functions, belonging to the L^1 space, that their Vilenkin–Fourier series converges in L^1-norm. This condition is similar to the well-known Dini–Lipschitz condition. We formulate a corollary for Vilenkin groups with a certain boundedness property and prove that in this corollary the assumption made concerning the L^1-modulus of continuity is not sufficient for the L^1-convergence for the Vilenkin-groups without this property. We remark that a similar result was proved by C. W. ONNEWEER [1] for Walsh–Fourier series. Finally we prove the analogue of a theorem of F. SCHIPP [4] for Vilenkin–Fourier series. Namely, several sufficient conditions are formulated for the a.e. convergence of Vilenkin series.

§. 1

In this section we introduce some notations and definitions. Let

$$m = (m_0, m_1, \dots, m_k, \dots) \qquad \left(2 \leqq m_k, m_k \in \mathbf{N}, k \in \mathbf{N} := \{0, 1, \dots\}\right)$$

be a sequence of natural numbers and denote by Z_{m_k} the m_k^{th} discrete cyclic group, i.e.

$$Z_{m_k} := \{0, 1, \dots, m_k - 1\} \qquad (k \in \mathbf{N}).$$

Furthermore, if we define the group G_m as the direct product of the groups Z_{m_k}, then G_m is a compact Abelian group. Thus the elements of G_m are of the form $x = (x_0, x_1, \dots, x_k, \dots)$ with $0 \leqq x_k < m_k$ $(k \in \mathbf{N})$ and for x, y in G_m their sum $x \dotplus y$ is obtained by adding the n^{th} coordinates of x and y modulo m_n. We define the sets $I_n(x)$ of G_m as follows:

$$I_n(x) := \{y \in G_m : y = (x_0, \dots, x_{n-1}, y_n, \dots)\} \qquad (x \in G_m, n \in \mathbf{N}).$$

Then the $I_n(0)$'s $(n \in \mathbf{N})$ form a basis for the neighbourhoods of $0 \in G_m$ in G_m and these sets completely determine the topology of G_m.

Next, let $\hat{G}_m = \{\psi_n : n \in \mathbf{N}\}$ (the so-called Vilenkin system) denote the character group of G_m. We enumerate the elements of $\hat{G}_m$ as follows. For $k \in \mathbf{N}$ and $x \in G_m$

let r_k be the function defined by

$$r_k(x) := \exp \frac{2\pi i x_k}{m_k} \qquad (x \in G,\, i := \sqrt{-1}).$$

If we define the sequence $(M_k, k \in \mathbf{N})$ by $M_0 := 1$, $M_{k+1} := m_k M_k$ $(k \in \mathbf{N})$, then each $n \in \mathbf{N}$ has a unique representation of the form

$$n = \sum_{k=0}^{\infty} n_k M_k,$$

where $0 \leqq n_k < m_k$, $n_k \in \mathbf{N}$. For such $n \in \mathbf{N}$ we define the function ψ_n by

$$\psi_n := \prod_{k=0}^{\infty} (r_k)^{n_k}.$$

We remark that $\hat{G}_m$ is a complete orthonormal system with respect to the normalized Haar measure dx on G_m [7]. Furthermore, if $m_k = 2$ $(k \in \mathbf{N})$ then G_m is the so-called dyadic group and the elements of the character group $\hat{G}_m$ are the Walsh–Paley functions.

For $f \in L^1(G_m)$ we define its partial sums by

$$\hat{f}(n) := \int_{G_m} f(t)\, \overline{\psi_n(t)}\, dt, \quad S_n(f) := \sum_{k=0}^{n-1} \hat{f}(k)\, \psi_k \qquad (n \in \mathbf{N}).$$

Then we have the formula

$$S_n(f)(x) = \int_{G_m} f(t)\, D_n(x \dot{-} t)\, dt =: f * D_n(x) \qquad (x \in G_m, n \in \mathbf{N}), \tag{1}$$

where for the so-called Dirichlet-kernels D_n $(n \in \mathbf{N})$ it was proved the following formula [5]:

$$D_n = \psi_n \sum_{k=0}^{\infty} \left(\sum_{j=m_k-n_k}^{m_k-1} r_k^j \right) D_{M_k} \qquad (n \in \mathbf{N}). \tag{2}$$

The integral modulus of continuity is defined by

$$\omega_1(f, \delta) := \sup_{\lambda(y) \leqq \delta} \int_{G_m} |f(x) - f(x \dot{-} y)|\, dx \qquad (\delta > 0, f \in L^1(G_m)),$$

where

$$\lambda(y) := \sum_{k=0}^{\infty} \frac{y_k}{M_{k+1}} \qquad (y \in G_m).$$

§. 2

It is known that for all $1 < p < \infty$ the Vilenkin–Fourier series of a function $f \in L^p(G_m)$ converges in L^p-norm and the analoguous statement does not hold in the case $p = 1, \infty$ [6], [3], [8], [5]. We prove that a certain assumption for $\omega_1(f, \cdot)$

($f \in L^1$) implies the L^1-convergence of the Fourier-series of f with respect to the system $\hat{G}_m$.

THEOREM 1. *Let f be a function in $L^1(G_m)$ for which the following condition holds:*

$$\omega_1(f, 1/M_k) = o(m_k \log M_k)^{-1} \quad (k \to \infty). \tag{3}$$

Then the sequence of the partial sums $(S_n(f), n \in \mathbf{N})$ converges in L^1-norm to f.

From this theorem we obtain immediately the

COROLLARY. *If the group G_m has the boundedness property*

$$\overline{\lim}\,(m_k, k \in \mathbf{N}) < \infty, \tag{4}$$

then the condition

$$\omega_1(f, 1/M_k) = o(\log M_k)^{-1} \qquad (k \to \infty, f \in L^1(G_m)) \tag{5}$$

implies the L^1-convergence of the Vilenkin—Fourier series of f.

The above corollary is an analogue of the known Dini—Lipschitz test foruni form convergence of Vilenkin-series [2]. We prove that the condition (4) is essential in the corollary. This follows from the next

THEOREM 2. *There exists a group G_m and a function $f \in L^1(G_m)$ for which the assumption* (5) *holds, however the sequence $(S_n(f), n \in \mathbf{N})$ is not L^1-convergent.*

We remark that in the conditions (3) and (5) the "o" cannot be replaced by "O" [1]. Furthermore, by the corollary it is obvious that the group G_m in Theorem 2 cannot have the boundedness property (4).

In the following theorem we are concerned with the a.e. convergence of Vilenkin-series.

THEOREM 3. *Let $f \in L^1(G_m)$ be an integrable function. Then each of the following three conditions implies the a.e. convergence of $(S_n(f), n \in \mathbf{N})$:*

(i) $$\sum_{k=0}^{\infty} m_k \int_{G_m} \int_{G_m} |f(x \dot{-} u) - f(x)|\, D_{M_k}(u)\, du\, dx < \infty,$$

(ii) $$\sum_{k=0}^{\infty} m_k\, \omega_1(f, 1/M_k) < \infty,$$

(iii) *there exists a positive number q, for which*

$$\sum_{k=1}^{\infty} \frac{m_k}{k^{1+q}} < \infty, \quad \omega_1(f, \delta) = O(\log 1/\delta)^{-1-q} \qquad (\delta \to 0)$$

holds.

We remark that similar results for a.e. convergence of Walsh—Fourier series were proved by F. SCHIPP [4]. In the proof of Theorem 3 we follow the method of his paper.

§. 3. Proof of the theorems

PROOF OF THEOREM 1. The statement of Theorem 1 is an immediate consequence of the following estimation

$$\text{(6)} \qquad \| S_n(f) - f \|_1 \leqq C \cdot m_k \cdot \log M_k \cdot \omega_1(f, 1/M_k)$$
$$\big(M_k \leqq n < M_{k+1},\ n, k \in \mathbf{N},\ f \in L^1(G)\big),$$

where $C > 0$ is an absolute constant.

To prove (6) let us write

$$n = j \cdot M_k + \sum_{l=0}^{k-1} n_l M_l \qquad (1 \leqq j < m_k)$$

for the natural number n, then from (2) we obtain the following decomposition of $S_n(f)$:

$$S_n(f) = S_{j \cdot M_k}(f) + f * (r_k^j D_{n^*}),$$

where $n^* := \sum_{l=0}^{k-1} n_l M_l$. From this follows that

$$\text{(7)} \qquad \| S_n(f) - f \|_1 \leqq \| S_{jM_k}(f) - f \|_1 + \| f * (r_k^j D_{n^*}) \|_1 .$$

By (2) and $\int_{G_m} D_s = 1$ $(s = 1, 2, \ldots)$ we have

$$\| S_{jM_k}(f) - f \|_1 = \int_{G_m} \Big| \int_{G_m} (f(x) - f(x \dot- t)) D_{jM_k}(t)\, dt \Big|\, dx \leqq$$
$$\leqq m_k \int_{G_m} \int_{G_m} | f(x) - f(x \dot- t) |\, | D_{M_k}(t) |\, dt\, dx .$$

It is known [7] that

$$\text{(8)} \qquad D_{M_k}(t) = \begin{cases} M_k & (t \in I_k(0)) \\ 0 & (t \notin I_k(0)) \end{cases} \qquad (k \in \mathbf{N}),$$

thus applying Fubini's theorem we see that

$$\text{(9)} \quad \| S_{jM_k}(f) - f \|_1 \leqq m_k M_k \int_{I_k(0)} \Big(\int_{G_m} | f(x) - f(x \dot- t) |\, dx \Big)\, dt \leqq m_k \cdot \omega_1(f, 1/M_k).$$

For the second term in (7) we can establish an estimation as follows. Let us denote by e_k $(k \in \mathbf{N})$ the following element of G_m

$$e_k := (0, 0, \ldots, 0, \underbrace{1}_{k+1}, 0, \ldots).$$

Then

$$r_k(e_k)^{-j} \big(f * (r_k^j D_{n^*}) \big)(x) = \int_{G_m} f(t)\, r_k^j(x \dot- t \dot- e_k)\, D_{n^*}(x \dot- t \dot- e_k)\, dt =$$
$$= \int_{G_m} f(t \dot- e_k)\, r_k^j(x \dot- t)\, D_{n^*}(x \dot- t)\, dt \qquad (x \in G_m).$$

Indeed $D_{n*}(y)$ depends only on the first k coordinates of the element $y \in G_m$. From this equation we obtain that

$$|1 - r_k(e_k)^{-j}| \, \| f * (r_k^j D_{n*}) \|_1 \leqq \int_{G_m} \int_{G_m} | f(t) \dot{-} f(t - e_k) | \, | D_{n*}(x \dot{-} t) | \, dt \, dx .$$

Applying Fubini's theorem and noticing, that $\int_{G_m} | D_{n*}(x \dot{-} t) | \, dx = \| D_{n*} \|_1 \; (t \in G_m)$, we have

$$|1 - r_k(e_k)^{-j}| \cdot \| f * (r_k^j D_{n*}) \|_1 \leqq \| D_{n*} \|_1 \, \omega_1 (f, 1/M_{k+1}) .$$

On the other hand the following estimation is valid:

$$|1 - r_k(e_k)^{-j}| = \left| 1 - \exp \frac{-2\pi i j}{m_k} \right| = 2 \sin \frac{\pi j}{m_k} \geqq 2 \cdot \sin \frac{\pi}{m_k} \geqq \frac{4}{m_k},$$

thus

$$\| f * (r_k^j D_{n*}) \|_1 \leqq \frac{1}{4} m_k \, \| D_{n*} \|_1 \, \omega_1 (f, M_{k+1}) .$$

Since $\| D_{n*} \|_1 = O(\log n^*) = O(\log M_k)$ [7] and evidently $\omega_1(f, 1/M_{k+1}) \leqq \omega_1(f, 1/M_k)$, we get

$$\| f * (r_k^j D_{n*}) \|_1 = O\big(m_k \cdot \log M_k \cdot \omega_1(f, 1/M_k)\big) . \tag{10}$$

From (7), (9) and (10), (6) follows immediately.

This completes the proof of Theorem 1.

Proof of Theorem 2. We define a sequence m by

$$m_0 := 2, \quad m_n := M_n^{2^n} \qquad (n = 1, 2, \ldots) .$$

A simple consequence of (8) is $\| D_{M_k} \|_1 = 1$ $(k \in \mathbf{N})$, thus for all $\alpha_k > 0$ $(k \in \mathbf{N})$, $\sum_{k=0}^{\infty} \alpha_k < \infty$ the series

$$\sum_{k=0}^{\infty} \alpha_k (D_{M_{k+1}} - D_{M_k})$$

is L^1-convergent. We denote its limit by $f \in L^1(G_m)$. If we define the k_j's $(j \in \mathbf{N})$ as

$$k_j := \frac{m_j}{2} + 1 ,$$

then

$$\| S_{k_j M_j}(f) - S_{M_j}(f) \|_1 = \alpha_j \left\| \sum_{s=m_j}^{k_j M_j - 1} \psi_s \right\|_1 = \alpha_j \left\| \sum_{s=1}^{k_j - 1} \sum_{l = s M_j}^{(s+1) M_j - 1} \psi_l \right\|_1 =$$

$$= \alpha_j \left\| \left(\sum_{s=1}^{k_j - 1} \psi_{s M_j} \right) D_{M_j} \right\|_1 \qquad (j \in \mathbf{N}) .$$

Hence by the definition of the functions ψ_n $(n \in \mathbf{N})$ and from (8) we have

$$\| S_{k_j M_j}(f) - S_{M_j}(f) \|_1 = \alpha_j M_j \int_{I_j(0)} \left| \sum_{s=1}^{k_j-1} \psi_{sM_j} \right| =$$

$$= \frac{\alpha_j}{m_j} \sum_{s=0}^{m_j-1} \left| \sum_{l=1}^{k_j-1} \exp \frac{2\pi ils}{m_j} \right| > \frac{\alpha_j}{m_j} \sum_{s=1}^{m_j-1} \frac{\left| 1 - \exp \dfrac{2\pi is(k_j-1)}{m_j} \right|}{\left| 1 - \exp \dfrac{2\pi is}{m_j} \right|} =$$

$$= \frac{\alpha_j}{m_j} \sum_{s=1}^{m_j-1} \frac{\left| \sin \dfrac{\pi s}{2} \right|}{\sin \dfrac{\pi s}{m_j}} \geqq \frac{\alpha_j}{m_j} \cdot \frac{m_j}{\pi} \sum_{\substack{s=1 \\ s \equiv 1(2)}}^{m_j-1} \frac{1}{s} \geqq C \cdot \alpha_j \cdot \log m_j ,$$

where $C > 0$ is an absolute constant $(j \in \mathbf{N})$. It is obvious that for

$$\alpha_j := \frac{1}{\log m_j} \qquad (j \in \mathbf{N})$$

all the conditions are satisfied, thus

$$\| S_{k_j M_j}(f) - S_{M_j}(f) \|_1 \geqq C > 0 \qquad (j \in \mathbf{N}).$$

Hence the sequence $(S_n(f),\ n \in \mathbf{N})$ is not L^1-convergent.

We can easily prove that for the function f the relation (5) holds. Indeed, let n be a fixed natural number, then for all $y \in I_n(0)$ we have

$$\int_{G_m} | f(x) - f(x \dot{-} y) | \, dx = \int_{G_m} \left| \sum_{k=n}^{\infty} \alpha_k (R_k(x) - R_k(x \dot{-} y) \right| dx$$

(where $R_k := D_{M_{k+1}} - D_{M_k}$ $(k \in \mathbf{N})$), i.e. by (8)

$$\int_{G_m} | f(x) - f(x \dot{-} y) | \, dx \leqq 4 \sum_{k=n}^{\infty} \alpha_k .$$

From this follows by reason of the definition of ω_1 that

$$\omega_1(f, 1/M_n) \leqq 4 \sum_{k=n}^{\infty} \alpha_k ,$$

thus

$$\omega_1(f, 1/M_n) \cdot \log M_n \leqq 4 \cdot \sum_{k=n}^{\infty} \frac{\log M_n}{2^k \cdot \log M_n} \leqq 4 \cdot \sum_{k=n}^{\infty} \frac{1}{2^k} = o(1) \quad (n \to \infty).$$

This proves Theorem 2.

Proof of Theorem 3. We suppose first that condition (i) holds. Then by the theorem of Beppo-Levi we have

$$\sum_{k=0}^{\infty} m_k \int_{G_m} | f(x \dot{-} u) - f(x) | \, D_{M_k}(u) \, du < \infty \qquad (\text{a.e. } x \in G).$$

From (1) and (2) follows for all $k_0 \in \mathbf{N}$ that

$$|S_n(f)(x) - f(x)| = |\int_{G_m} (f(x \dot{-} u) - f(x)) D_n(u)\, du| =$$

$$= \left| \int_{G_m} (f(x \dot{-} u) - f(x)) \psi_n(u) \sum_{k=0}^{\infty} \sum_{j=m_k-n_k}^{m_k-1} (r_k(u))^j D_{M_k}(u)\, du \right| \leqq$$

$$\leqq \sum_{k=0}^{k_0} \left| \int_{G_m} (f(x \dot{-} u) - f(x)) \sum_{j=m_k-n_k}^{m_k-1} (r_k(u))^j D_{M_k}(u) \psi_n(u)\, du \right| +$$

$$+ \sum_{k=k_0+1}^{\infty} \int_{G_m} |f(x \dot{-} u) - f(x)| \left| \sum_{j=m_k-n_k}^{m_k-1} (r_k(u))^j \right| D_{M_k}(u)\, du \leqq$$

$$\leqq \sum_{k=0}^{k_0} |\int_{G_m} H_{k,x,n}(u) \psi_n(u)\, du| + \sum_{k=k_0+1}^{\infty} m_k \int_{G_m} |f(x \dot{-} u) - f(x)| D_{M_k}(u)\, du,$$

where

$$H_{k,x,n}(u) := (f(x \dot{-} u) - f(x)) \sum_{j=m_k-n_k}^{m_k-1} (r_k(u))^j D_{M_k}(u) \qquad (u, x \in G_m, n, k \in \mathbf{N}).$$

Since for a fixed $k_0 \in \mathbf{N}$ we have, by definitions, at most M_{k_0+1} distinct functions among the functions $H_{k,x,n}$ ($k = 0, \dots, k_0$; $x \in G_m$, $n \in \mathbf{N}$) and for a $g \in L^1(G_m)$ the relation $\int_{G_m} g \cdot \bar{\psi}_n = o(1)$ ($n \to \infty$) holds, thus we can easily see that

$$\sum_{k=0}^{k_0} |\int_{G_m} H_{k,x,n}(u) \psi_n(u)\, du| = o(1) \qquad (u \to \infty).$$

From this by reason of condition (i) and by the preceding decomposition of $S_n(f)(x) - f(x)$ follows the a.e. convergence of $(S_n(f), n \in \mathbf{N})$.

We prove now that condition (ii) implies (i). Indeed, for a fixed $k \in \mathbf{N}$ we have

$$\int_{G_m} \int_{G_m} |f(x \dot{-} u) - f(x)| D_{M_k}(u)\, du\, dx = \int_{G_m} D_{M_k}(u) \int_{G_m} |f(x \dot{-} u) - f(x)|\, dx\, du =$$

$$= M_k \int_{I_k(0)} \left(\int_{G_m} |f(x \dot{-} u) - f(x)|\, dx \right) du \leqq \omega_1(f, 1/M_k).$$

Thus condition (ii) is also sufficient for the a.e. convergence.

Finally, when (iii) holds, then

$$\omega_1(f, 1/M_k) = O(\log M_k)^{-1-q} = O(\log 2^k)^{-1-q} = O(k^{-1-q}),$$

hence from (iii) condition (ii) follows and thus Theorem 3 is proved.

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ON SOME PROBLEMS CONNECTED WITH POLYNOMIALS ORTHOGONAL ON THE COMPLEX UNIT CIRCLE

By

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Dedicated to Professor G. Alexits on his eightieth birthday

1. Introduction, notations

Let $f(t)$ be a nonnegative, 2π-periodic Lebesgue integrable function on the real line such that $\int_0^{2\pi} f(t)\,dt > 0$. Then there exists a unique sequence of polynomials $\{\Phi_n(z)\}_{n=0}^{\infty}$ of the complex variable z with the following properties:

(i) $\Phi_n(z)$ is of degree exactly n, with positive leading coefficient;

(ii) $\frac{1}{2\pi}\int_0^{2\pi} f(t)\,\Phi_n(e^{it})\,\overline{\Phi_N(e^{it})}\,dt = \delta_{nN} \qquad (n, N = 0, 1, \dots)$

(cf. SZEGÖ [1], pp. 287—288).

$\{\Phi_n(z)\}_{n=0}^{\infty}$ is called the system of orthonormal polynomials with respect to the weight-function $f(t)$.

The theory of these polynomials, from certain point of view, is rather well developed (see e.g. SZEGÖ [1], Chapters X—XIII; FREUD [2], Chapter V). Thus, for example, the asymptotic behaviour of $\Phi_n(z)$ when $n\to\infty$, is well-known for $|z|\geqq 1$ under certain, rather general conditions for the weight-function $f(t)$. Nevertheless, we know much less about the properties of $\Phi_n(z)$ *inside* the unit circle, while this problem is equally important. Namely, all the roots of $\Phi_n(z)$ lie in $|z|<1$ (cf. e.g. SZEGÖ [1], Theorem 11.4.1), but beyond this fact, not much is known on the finer distribution of the roots. Special examples show that in this respect the most various cases may occur. Thus, if $f(t)\equiv 1$ then $\Phi_n(z)\equiv z^n$, i.e. all the roots are 0, while if $f(t)=\cos^2\frac{t}{2}$ then

$$\Phi_n(z) = 2\,\frac{1+2z+\dots+(n+1)\,z^n}{\sqrt{(n+1)\,(3n+2)}}$$

and here the roots are uniformly close to $|z|=1$ and asymptotically uniformly distributed in any angular region $\alpha\leqq \arg z\leqq\beta$. More precisely, if $Z(f)$ denotes the set of cluster-points of the roots of the polynomials $\Phi_n(z)$ $(n=1, 2, \dots)$ then

$$Z(f) = 0 \quad\text{and}\quad Z(f) = \{z\colon |z| = 1\}$$

in the above mentioned cases, respectively (cf. TURÁN [3], p. 60).

The purpose of the present paper is to get asymptotic formulae for $\Phi_n(z)$ and to characterize the set $Z(f)$, when $f(t)$ is a nonnegative trigonometric polynomial of precise degree m. Perhaps our approach can be applied for more general cases, e.g. when $f(t)$ is continuous. At the end we give a partial answer for a problem raised by P. TURÁN [3].

Thus, in the sequel, $f(t) = f_m(t)$ will always denote a nonnegative trigonometric polynomial of fixed degree m. (The trivial case $m = 0$ will be excluded; see the above example.) Such a polynomial always can be written uniquely in the form

(1) $$f_m(t) = | h_m(e^{it}) |^2$$

where

(2) $$h_m(z) = c_m \prod_{k=1}^{s} (z - z_k)^{m_k} \qquad \left(\sum_{k=1}^{s} m_k = m, \quad c_m \neq 0 \right)$$

is a polynomial of degree m of the complex variable z. Here the z_k's are pairwise different and satisfy

$$| z_k | \geqq 1 \qquad (k = 1, \ldots, s)$$

(cf. SZEGŐ [1], Theorem 1.2.2). Introducing the polynomial

(3) $$h_m^*(z) = \bar{c}_m \prod_{k=1}^{s} (1 - z\bar{z}_k)^{m_k}$$

of degree m, we get from (1) the representation

$$f_m(t) = e^{-itm} h_m(e^{it})\, h_m^*(e^{it}).$$

We shall denote by

$$z_k^* = \bar{z}_k^{-1} \qquad (| z_k^* | \leqq 1; \quad k = 1, \ldots, s)$$

the different roots of $h_m^*(z)$.

Using the previous notations, the quantities

$$\rho_m = \max_{1 \leqq k \leqq s} | z_k^* | \qquad (0 < \rho_m \leqq 1)$$

and

$$\kappa_m = \prod_{k=1}^{s} (- z_k)^{m_k}$$

will play an important role. We shall also use the notation

$$M = \max_{| z_k^* | = \rho_m} m_k \qquad (1 \leqq M \leqq m).$$

2. Asymptotic properties of $\Phi_n(z)$

At first we deal with positive weight-functions ($\rho_m < 1$):

THEOREM 1. *If* $\rho_m < 1$ *then*

$$\Phi_n(z) = \begin{cases} -\dfrac{\bar{c}_m \mid \kappa_m \mid M n^{M-1}}{c_m \kappa_m h_m(z)} \sum\limits_{\substack{|z_k^*|=\rho_m \\ m_k = M}} \dfrac{z_k^{*n+m+1-M} h_m(z_k^*)}{h_m^{*(M)}(z_k^*)(z-z_k^*)} + o(n^{M-1}\rho_m^n) \ \textit{if} \ |z| < \rho_m; \\ \dfrac{\bar{c}_m \mid \kappa_m \mid z^{n+m}}{c_m \kappa_m h_m^*(z)} + o(|z|^n) \ \textit{if} \ |z| > \rho_m. \end{cases}$$

Now consider the case when the weight-function has zeros ($\rho_m = 1$):

THEOREM 2. *If* $\rho_m = 1$ *then*

$$\Phi_n(z) = \begin{cases} -\dfrac{\bar{c}_m \mid \kappa_m \mid}{c_m \kappa_m n h_m(z)} \sum\limits_{|z_k^*|=1} \dfrac{(-1)^{m_k} m_k z_k^{*n+m+1} h_m^{(m_k)}(z_k^*)}{h_m^{*(m_k)}(z_k^*)(z-z_k^*)} + o\left(\dfrac{1}{n}\right) \quad \textit{if} \quad |z| < 1; \\ \dfrac{(-1)^{m_k} n^{m_k} z_k^{*n+m-m_k} \bar{c}_m \mid \kappa_m \mid}{c_m \kappa_m \dbinom{2m_k}{m_k} h_m^{*(m_k)}(z_k^*)} + o(n^{m_k}) \quad \textit{if} \quad z = z_k^* \quad \textit{with} \quad |z_k^*| = 1; \\ \dfrac{\bar{c}_m \mid \kappa_m \mid z^{n+m}}{c_m \kappa_m h_m^*(z)} + o(|z|^n) \quad \textit{if} \quad |z| \geqq 1 \quad \textit{but} \quad z \neq z_k^*. \end{cases}$$

REMARK. In case $|z| \geqq 1$ and $|z| > 1$ the last statements of Theorems 1 and 2 follow from Theorems 12.1.1 and 12.1.3 of SZEGÖ [1], respectively. In Theorem 1, we could have stated asymptotic formulas for $|z| = \rho_m$, too, but the situation is rather complicated, depending on the multiplicities of the roots $|z_k^*| = \rho_m$.

PROOF. The proofs of Theorems 1 and 2 run parallel and consist of several steps. At first we prove that sufficiently large n's, $\Phi_n(z)$ can be written in the form

$$(5) \qquad \Phi_n(z) =$$

$$= \frac{\gamma_n z^{n+m}}{h_m(z)} \left\{ 1 - \sum_{k=1}^{s} \sum_{l=1}^{m_k} a_{k,l,n} \left[\left(\frac{z}{z-z_k^*} \right)^l - \left(\frac{z_k^*}{z} \right)^{n+m} \sum_{j=0}^{l-1} \binom{n+m+l}{j} \left(\frac{z_k^*}{z-z_k^*} \right)^{l-j} \right] \right\}$$

with suitable complex numbers γ_n and $a_{k,l,n}$. In order to show this, observe that here with the notation

$$(6) \qquad r_{k,l,n}(z) = \left(\frac{z}{z-z_k^*} \right)^l - \left(\frac{z_k^*}{z} \right)^{n+m} \sum_{j=0}^{l-1} \binom{n+m+l}{j} \left(\frac{z_k^*}{z-z_k^*} \right)^{l-j}$$

the expressions

$$(7)\qquad z^{n+m} r_{k,l,n}(z) = (z - z_k^*)^{n+m}\left[\left(1 + \frac{z_k^*}{z - z_k^*}\right)^{n+m+l} - \right.$$

$$\left. - \sum_{j=0}^{l-1} \binom{n+m+l}{j}\left(\frac{z_k^*}{z - z_k^*}\right)^{n+m+l-j}\right] = z_k^{*n+m} \sum_{j=l}^{n+m+l} \binom{n+m+l}{j}\left(\frac{z_k^*}{z - z_k^*}\right)^{l-j} =$$

$$= \sum_{j=0}^{n+m} \binom{n+m+l}{j+l} (z - z_k^*)^j \, z_k^{*n+m-j}$$

are polynomials of degree $n + m$. Thus, if we prescribe the conditions

$$(8)\qquad \sum_{k=1}^{s} \sum_{l=1}^{m_k} a_{k,l,n} r_{k,l,n}^{(\nu)}(z_\mu) = \delta_{\nu 0} \qquad (\nu = 0, 1, \ldots, m_\mu - 1;\ \mu = 1, \ldots, s)$$

then the expression on the right hand side of (5) will be, indeed, a polynomial of degree at most n (in fact, it will turn out that it is of degree *exactly* n). Hence we have to prove that, for sufficiently large n's, the system of m linear equations (8) for the unknowns $a_{k,l,n}$ is nonsingular. To this end, let us calculate the asymptotic values of $r_{k,l,n}^{(\nu)}(z_\mu)$ from (6):

$$r_{k,l,n}^{(\nu)}(z_\mu) = \left[\left(\frac{z}{z - z_k^*}\right)^l\right]_{z=z_\mu}^{(\nu)} + O\left(n^{\nu+l-1}\left|\frac{z_k^*}{z_\mu}\right|^n\right) \qquad (\text{if } \ z_\mu \neq z_k^*, \ \ \nu = 0, 1, \ldots);$$

and from (7):

$$(9)\qquad r_{\mu,l,n}^{(\nu)}(z_\mu) = z_\mu^{n+m}\left[z^{-n-m} \sum_{j=0}^{n+m} \binom{n+m+l}{j+l} (z - z_\mu)^j z_\mu^{-j}\right]_{z=z_\mu}^{(\nu)} =$$

$$= z_\mu^{n+m} \sum_{\lambda=0}^{\nu} \binom{\nu}{\lambda} (-n)^\lambda z_\mu^{-n-m-\lambda} \frac{n^{\nu-\lambda+l}(\nu-\lambda)!}{(\nu-\lambda+l)!} z_\mu^{\lambda-\nu} + O(n^{\nu+l-1}) =$$

$$= z_\mu^{-\nu} n^{\nu+l} \sum_{\lambda=0}^{\nu} \left(\frac{(-1)^\lambda \nu!}{\lambda!\,(\nu-\lambda+l)!}\right) + O(n^{\nu+l-1}) = \frac{z_\mu^{-\nu} n^{\nu+l} \nu!}{(\nu+l)!} \sum_{\lambda=0}^{\nu} (-1)^\lambda \binom{\nu+l}{\lambda} +$$

$$+ O(n^{\nu+l-1}) = \frac{(-1)^\nu z_\mu^{-\nu} n^{\nu+l}}{(\nu+l)(l-1)!} + O(n^{\nu+l-1}) \qquad (\nu = 0, 1, \ldots, m;\ \ |z_\mu| = 1).$$

Hence, introducing the notation

$$(10)\qquad b_{k,l,n} = \begin{cases} a_{k,l,n} & \text{if } \ |z_k^*| < 1 \\ \dfrac{n^l}{(l-1)!} a_{k,l,n} & \text{if } \ |z_k^*| = 1, \end{cases}$$

(8) takes the form

$$(11)\qquad \sum_{|z_k^*|<1} \sum_{l=1}^{m_k} b_{k,l,n}\left\{\left[\left(\frac{z}{z - z_k^*}\right)^l\right]_{z=z_\mu}^{(\nu)} + O\left(n^{\nu+l-1}\left|\frac{z_k^*}{z_\mu}\right|^n\right)\right\} +$$

$$+ \sum_{|z_k^*|=1} \sum_{l=1}^{m_k} O(n^{-1})\, b_{k,l,n} = \delta_{\nu o} \qquad (\nu = 0, 1, \ldots, m_\mu - 1;\ \ |z_\mu| 1),$$

and

$$(12)\qquad \sum_{|z_k^*|<1}\sum_{l=1}^{m_k} b_{k,l,n}\left\{\left[\left(\frac{z}{z-z_k^*}\right)^l\right]^{(\nu)}_{z=z_\mu} + O(n^{\nu+l-1}\,|\,z_k^*\,|^n)\right\} +$$

$$+\sum_{\substack{|z_k^*|=1\\ k\neq\mu}}\sum_{l=1}^{m_k} O(n^{\nu-1})\,b_{k,l,n} + \sum_{l=1}^{m_\mu} b_{k,l,n}\left\{\frac{(-1)^\nu z_\mu^{-\nu} n^\nu}{\nu+l} + O(n^{\nu-1})\right\} = \delta_{\nu G}$$

$$(\nu = 0, 1, \ldots, m_\mu - 1;\quad |\,z_\mu\,| = 1).$$

Now let n tend to infinity in (11) and (12) (in the latter case after dividing by n^ν). Then we obtain a system of equations

$$(13)\qquad \sum_{|z_k^*|<1}\sum_{l=1}^{m_k} b_{k,l}\left[\left(\frac{z}{z-z_k^*}\right)^l\right]^{(\nu)}_{z=z_\mu} = \delta_{\nu 0}\quad (\nu = 0, 1, \ldots, m_\mu - 1;\quad |\,z_\mu\,|>1),$$

$$(14)\qquad \delta_{\nu 0}\sum_{|z_k^*|<1}\sum_{l=1}^{m_k} b_{k,l}\left(\frac{z_\mu}{z_\mu - z_k^*}\right)^l + \sum_{l=1}^{m_\mu}\frac{b_{\mu,l}}{\nu+l} = \delta_{\nu 0}$$

$$(\nu = 0, 1, \ldots, m_\mu - 1;\quad |\,z_\mu\,| = 1)$$

for the unknowns $b_{k,l}$. We show that this system is nonsingular by solving it explicitly. If we succeed in determining complex numbers $b_{k,l}$ such that

$$(15)\qquad \sum_{|z_k^*|<1}\sum_{l=1}^{m_k} b_{k,l}\left(\frac{z}{z-z_k^*}\right)^l = 1 - \frac{\bar c_m h_m(z)}{c_m\kappa_m h_m^*(z)}$$

then these numbers evidently satisfy (13). But easy to check that

$$(16)\qquad b_{k,l} = -\frac{\bar c_m}{c_m\kappa_m(m_k - l)!}\left[\frac{(1-z_k^* z)^{m_k} h_m\left(\frac{1}{z}\right)}{h_m^*\left(\frac{1}{z}\right)}\right]^{(m_k - l)}_{z=\frac{1}{z_k^*}}$$

$$(l = 1, \ldots, m_k;\quad |\,z_k^*\,| < 1)$$

will serve this purpose. Now (15) yields

$$\sum_{|z_k^*|<1}\sum_{l=1}^{m_k} b_{k,l}\left(\frac{z_\mu}{z_\mu - z_k^*}\right)^l = 1 - \frac{\bar c_m h_m^{(m_\mu)}(z_\mu)}{c_m\kappa_m h_m^{*(m_\mu)}(z_\mu)}\qquad (|\,z_\mu\,| = 1),$$

and thus (14) takes the form

$$\sum_{l=1}^{m_\mu}\frac{b_{\mu,l}}{\nu+l} = \delta_{\nu 0}\,\frac{\bar c_m h_m^{(m_\mu)}(z_\mu)}{c_m\kappa_m h_m^{*(m_\mu)}(z_\mu)}\qquad (\nu = 0, 1, \ldots, m_\mu - 1;\quad |\,z_\mu\,| = 1).$$

For each fixed μ, this system has again a unique solution

$$(17)\qquad b_{\mu,l} = -\,l\binom{m_\mu}{l}\binom{-m_\mu}{l}\frac{\bar c_m h_m^{(m_\mu)}(z_\mu)}{c_m\kappa_m h_m^{*(m_\mu)}(z_\mu)}\qquad (l = 1, \ldots, m_\mu;\quad |\,z_\mu\,| = 1).$$

Collecting our results, we get that the system (13)–(14) has a unique solution (16)–(17), which implies that for sufficiently large n's the system (11)–(12) is also solvable and

$$\lim_{n\to\infty} b_{k,l,n} = b_{k,l} \qquad (l=1,\dots,m_k;\quad k=1,\dots,s). \tag{18}$$

Hence (5) is, indeed, a polynomial of degree at most n provided that $b_{k,l,n}$ satisfy (11)–(12) and $a_{k,l,n}$ are defined through (10).

We still have to show that (5) is orthogonal to all z^p $(p=0,1,\dots,n-1)$ with respect to the weight-function (4). We have from (5)

$$\int_0^{2\pi} f(t)\,\Phi_n(e^{it})\,e^{-ipt}\,dt = \gamma_n \int_0^{2\pi} e^{i(n-p)t}\Bigg\{h_m^*(e^{it}) -$$

$$-\sum_{k=1}^{s}\sum_{l=1}^{m_k} a_{k,l,n}\frac{h_m^*(e^{it})}{(e^{it}-z_k^*)^l}\left[e^{ilt}-\left(\frac{z_k^*}{e^{it}}\right)^{n+m}\sum_{j=0}^{l-1}\binom{n+m+l}{j}(e^{it}-z_k^*)^j z_k^{*l-j}\right]\Bigg\}\,dt = 0$$

$$(0 \leqq p \leqq n-1).$$

Now determine γ_n such that the system $\{\Phi_n(z)\}_{n=0}^{\infty}$ be ortho*normal*. We have from (2) and (5)

$$\Phi_n(z) = \frac{\gamma_n}{c_m}\left(1-\sum_{k=1}^{s}\sum_{l=1}^{m_k} a_{k,l,n}\right) z^n + \dots$$

and thus by (5)

$$\frac{1}{2\pi}\int_0^{2\pi} f(t)\,|\,\Phi_n(e^{it})\,|^2\,dt = \frac{|\,\gamma_n\,|^2}{2\pi\,\bar c_m}\left(1-\sum_{k=1}^{s}\sum_{l=1}^{m_k}\bar a_{k,l,n}\right)\int_0^{2\pi}\Bigg\{h_m^*(e^{it}) - \tag{19}$$

$$-\sum_{k=1}^{s}\sum_{l=1}^{m_k} a_{k,l,n}\frac{h_m^*(e^{it})}{(e^{it}-z_k^*)^l}\left[e^{itl}-\left(\frac{z_k^*}{e^{it}}\right)^{n+m}\sum_{j=0}^{l-1}\binom{n+m+l}{j}(e^{it}-z_k^*)^j z_k^{*l-j}\right]\Bigg\}\,dt =$$

$$= |\,\gamma_n\,|^2\left(1-\sum_{k=1}^{s}\sum_{l=1}^{m_k}\bar a_{k,l,n}\right).$$

Here the left hand side is real and positive, hence the same is true for the right hand side. Therefore we may define

$$\gamma_n = \left(1-\sum_{k=1}^{s}\sum_{l=1}^{m_k}\bar a_{k,l,n}\right)^{-1/2},$$

thus ensuring the orthonormality of the system $\{\Phi_n(z)\}_{n=0}^{\infty}$. By (10), (18), (15), (2) and (3) we have

$$\lim_{n\to\infty}\sum_{k=1}^{s}\sum_{l=1}^{m_k} a_{k,l,n} = \lim_{n\to\infty}\sum_{|z_k^*|<1}\sum_{l=1}^{m_k} b_{k,l,n} = \sum_{|z_k^*|<1}\sum_{l=1}^{m_k} b_{k,l} = 1-\frac{\bar c_m}{c_m\kappa_m}\lim_{z\to\infty}\frac{h_m^*(z)}{h_m(z)} =$$

$$= 1-\frac{\bar c_m}{c_m\kappa_m}\cdot\frac{c_m}{\bar c_m\prod_{k=1}^{s}(-\bar z_k)^{m_k}} = 1-\frac{1}{\prod_{k=1}^{s}|\,z_k\,|^{2m_k}} = 1-\frac{1}{|\,\kappa_m\,|^2} \tag{20}$$

i.e.

$$\lim_{n\to\infty} \gamma_n = |\kappa_m|. \tag{21}$$

Now the previous considerations enable us to prove Theorems 1 and 2. Easy to see from (6) and (7) that

$$z^{n+m} r_{k,l,n}(z) = \begin{cases} -\dfrac{z_k^{*n+m+1} n^{l-1}}{(l-1)!\,(z-z_k^*)} + O(n^{l-2}\,|z_k^*|^n) & \text{if } |z|<|z_k^*| \text{ or} \\ & |z|=|z_k^*| \text{ but } l \geqq 2 \text{ and } z\neq z_k^*; \\ \dfrac{z^{n+m+1}-z_k^{*n+m+1}}{z-z_k^*} & \text{if } |z|=|z_k^*|,\ l=1,\ z\neq z_k^*; \\ \dfrac{z_k^{*n+m}\,n^l}{l!} + O(n^{l-1}\,|z_k^*|^n) & \text{if } z=z_k^*; \\ \dfrac{z^{n+m+l}}{(z-z_k^*)^l} + O(n^{l-1}\,|z_k^*|^n) & \text{if } |z|>|z_k^*|. \end{cases} \tag{22}$$

Using also the relation

$$a_{k,m_k\cdot n} = -\frac{\bar c_m m_k!\, h_m(z_k^*)}{c_m \kappa_m z^{*m_k} h_m^{*(m_k)}(z_k^*)} + O(1) \qquad (|z_k^*|<1) \tag{23}$$

(which follows from (2), (3), (10) and (16)), we may select the greatest terms in (5), and obtain Theorem 1.

To get the first relation in Theorem 2, we use (5), (6), (21), (22) and (17):

$$\begin{aligned} \Phi_n(z) &= \frac{\gamma_n}{h_m(z)} \sum_{|z_k^*|=1} \sum_{l=1}^{m_k} a_{k,l,n} \left\{ \frac{z_k^{*\,n+m+1}\, n^{l-1}}{(l-1)!\,(z-z_k^*)} + O(n^{l-2}) \right\} = \\ &= \frac{\gamma_n}{nh_m(z)} \sum_{|z_k^*|=1} \frac{z_k^{*n+m+1}}{z-z_k^*} \sum_{l=1}^{m_k} b_{k,l,n} + O(n^{-2}) = \\ &= \frac{|\kappa_m|}{nh_m(z)} \sum_{|z_k^*|=1} \frac{z_k^{*n+m+1}}{z-z_k^*} \sum_{l=1}^{m_k} b_{k,l} + o(n^{-1}) = \\ &= -\frac{\bar c_m |\kappa_m|}{c_m \kappa_m n h_m(z)} \sum_{|z_k^*|=1} \frac{(-1)^{m_k} m_k h_m^{(m_k)}(z_k^*)\, z_k^{*n+m+1}}{h_m^{*(m_k)}(z_k^*)\,(z-z_k^*)} + o(n^{-1}) \quad (|z|<1). \end{aligned}$$

As for the second relation, the l'Hospital rule yields from (5), (6), (9) and (17)

$$\Phi_n(z_k^*) = \gamma_n z_k^{*n+m} \left. \frac{1 - \sum\limits_{p=1}^{s} \sum\limits_{l=1}^{m_p} a_{p,l,n} r_{p,l,n}(z)}{h_m(z)} \right|_{z=z_k^*} =$$

$$= -\gamma_n z_k^{*n+m} \frac{\sum_{p=1}^{s}\sum_{l=1}^{m_p} a_{p,l,n} r_{p,l,n}^{(m_k)}(z_k^*)}{h_m^{(m_k)}(z_k^*)} = \frac{-\gamma_n z_k^{*n+m}}{h_m^{(m_k)}(z_k^*)} \left\{ \sum_{\substack{p=1\\p\neq k}}^{s} \sum_{l=1}^{m_p} a_{p,l,n} \left\{ \left[\left(\frac{z}{z-z_p^*} \right)^l \right]_{z=z_k^*}^{(m_k)} + \right.\right.$$

$$\left.\left. + O(n^{m_k+l-1} \mid z_p^* \mid^n) \right\} + \sum_{l=1}^{m_k} a_{k,l,n} \left[\frac{(-1)^{m_k} z_k^{*-m_k} n^{m_k+l}}{(m_k+l)(l-1)!} + O(n^{m_k+l-1}) \right] \right\} =$$

$$= -\gamma_n \frac{z_k^{*n+m}}{h_m^{(m_k)}(z_k^*)} \left[(-1)^{m_k} z_k^{*-m_k} n^{m_k} \sum_{l=1}^{m_k} \frac{b_{k,l,n}}{m_k+l} + O(n^{m_k-1}) \right] =$$

$$= -\frac{\mid \kappa_m \mid z_k^{*n+m-m_k} n^{m_k}(-1)^{m_k}}{h_m^{(m_k)}(z_k^*)} \sum_{l=1}^{m_k} \frac{b_{k,l}}{m_k+l} + o(n^{m_k}) =$$

$$= \frac{\bar{c}_m \mid \kappa_m \mid (-1)^{m_k} z_k^{*n+m-m_k} n^{m_k}}{c_m \kappa_m \binom{2m_k}{m_k} h_m^{*(m_k)}(z_k^*)} + O(n^{m_k}).$$

The last relation in Theorem 2 follows again from (5) by (10) and (15):

$$\Phi_n(z) = \frac{\gamma_n z^{n+m}}{h_m(z)} \left[1 - \sum_{|z_k^*|<1} \sum_{l=1}^{m_k} a_{k,\cdot,n} \left(\frac{z}{z-z_k^*} \right)^l + O\left(\frac{1}{n} \right) \right] = \frac{\bar{c}_m \mid \kappa_m \mid z^{n+m}}{c_m \kappa_m h_m^*(z)} + o(\mid z \mid^n).$$

3. The distribution of roots of $\Phi_n(z)$

Now we are in the position as to characterize the asymptotic distribution of the roots of $\Phi_n(z)$. Using the notation $Z(f_m)$ for the set of cluster-points of the roots of the $\Phi_n(z)$'s, we state

THEOREM 3. $Z(f_m) \cap \{z: \mid z \mid < \rho_m\} = \emptyset$.

PROOF. Theorems 1 and 2 imply that

$$\frac{c_m \kappa_m}{\bar{c}_m \mid \kappa_m \mid} z^{-n} \Phi_n(z) \Rightarrow \frac{z^m}{h_m^*(z)} \quad \text{as } n \to \infty$$

in every closed subdomain of $\mid z \mid > \rho_m$. The limit function $z^m/h_m^*(z)$ does not have zeros in $\mid z \mid > \rho_m + \varepsilon$ ($\varepsilon > 0$ arbitrary) thus by Hurwitz's theorem (cf. e.g. MARDEN [4], p. 5) so does $z^{-n}\Phi_n(z)$ for large n's, which proves the statement. Note that in case $\rho_m = 1$ Theorem 3 trivially holds.

The next step is to characterize the circle $\mid z \mid = \rho_m$.

THEOREM 4. $\{z: \mid z \mid = \rho_m\} \subseteqq Z(f_m)$.

PROOF. Let $z_1^*, \ldots, z_t^*$ be those roots of $h_m^*(z)$ for which $\mid z_k^* \mid = \rho_m$, $m_k = M$ ($k = 1, \ldots, t$), and assume that $\rho_m < 1$. (The proof in case $\rho_m = 1$ runs along the same lines (it is even simpler), therefore we omit it.) At first we prove that there exists an integer p, $0 \leqq p < t$ such that

$$\sum_{k=1}^{t} z_k^{*p} \frac{h_m(z_k^*)}{h_m^{*(M)}(z_k^*)} \neq 0 . \tag{24}$$

Namely, if the left hand side of (24) were zero for all $p = 0, 1, \ldots, t-1$ then this would mean that the corresponding homogeneous linear system for the "unknowns" $\frac{h_m(z_k^*)}{h_m^{*(M)}(z_k^*)}$ ($\neq 0$) would have a non-trivial solution, i.e. it would be singular. This is impossible, because the determinant of this system is the Vandermonde-determinant of the pairwise different elements $z_1^*, \ldots, z_t^*$, thus different from 0.

The next step is to prove the existence of a sequence of integers $n_1 < n_2 < \ldots$ such that

$$\lim_{j\to\infty} \left(\frac{z_k^*}{\rho_m}\right)^{n_j} = \left(\frac{z_k^*}{\rho_m}\right)^{p-m+M} \qquad (k = 1, \ldots, t). \tag{25}$$

The selection of this sequence $\{n_j\}$ can be made in the following manner. Divide the z_k^*'s ($k = 1, \ldots, t$) into two classes:

$$R_0 = \left\{ z_k^* : \ \frac{1}{\pi} \arg z_k^* \quad \text{rational} \right\},$$

$$R_1 = \left\{ z_k^* : \ \frac{1}{\pi} \arg z_k^* \quad \text{irrational} \right\}$$

(one of these sets may be empty). In R_1 (if $R_1 \neq \emptyset$), there exists a so-called *basis**

$$z_k^* = \rho_m e^{2\pi\omega_k i} \in R_1 \qquad (\omega_k \text{ irrational}; \ k = 1, \ldots, u; \ u \leqq t)$$

with the following properties:

(a) If with certain rational numbers $\lambda, \lambda_1, \ldots, \lambda_u$, $\sum_{k=1}^{u} \lambda_k \omega_k = \lambda$ holds then $\lambda = \lambda_1 = \ldots = \lambda_u = 0$.

(b) If $z_k^* = \rho_m e^{2\pi\chi_k i} \in R_1$ then there exist rational numbers $r_{k1}, \ldots, r_{ku}, r_k$ such that

$$\chi_k = \sum_{j=1}^{u} r_{kj}\omega_j + r_k . \tag{26}$$

These properties are clearly independent of the particular choice of the ω_j's mod 1.

Evidently, there exist positive integers q and n_0 such that

$$\left(\frac{z_k^*}{\rho_m}\right)^{nq} = 1 \qquad (z_k^* \in R_0, n \geqq n_0)$$

* We may assume that among the z_k^*'s, the first u from this basis.

and

$$e^{2\pi r_k n q i} = 1 \qquad (n \geqq n_0) \tag{27}$$

where r_k are the rational numbers defined by property (b) above.

Now, according to a classical result of H. WEYL [5, Satz 4, p. 491], by property (a) there exists a subsequence $\{p_j\}_{j=1}^{\infty} \subset \{1, 2, \ldots\}$ such that

$$\operatorname*{ilm}_{j\to\infty} \left(\frac{\rho_m}{z_k^*}\right)^{p_j q} = 1 \qquad (k = 1, \ldots, u).$$

But then by (26), (27) and property (b) we have

$$\lim_{j\to\infty} \left(\frac{z_k^*}{\rho_m}\right)^{p_j q} = \lim_{j\to\infty} e^{2\pi\chi_k p_j i q} = \prod_{v=1}^{u} \lim_{j\to\infty} e^{2\pi r_{kv}\omega_v p_j i q} = 1 \qquad (z_k^* \in R_1)$$

and thus (25) is completely proved with $n_j = p_j q + p - m + M$.

Now let $\varepsilon > 0$ be arbitrarily small. Then by Theorem 1 and (25) we have

$$\frac{c_m \varkappa_m \rho_m^{m+1-M} h_m(z) \Phi_{n_j}(z)}{\bar{c}_m \,|\, \varkappa_m \,|\, M n_j^{M-1} \rho_m^{n_j}} \Rightarrow - \sum_{k=1}^{t} \frac{z_k^{*p+1} h_m(z_k^*)}{h_m^{*(M)}(z_k^*)\,(z - z_k^*)} \quad \text{as } j \to \infty \quad (|z| \leqq \rho_m - \varepsilon).$$

By Hurwitz's theorem this means that for sufficiently large j's, $j \geqq j_1(\varepsilon)$, the polynomial $\Phi_{n_j}(z)$ has at most $t-1$ zeros in $|z| \leqq \rho_m - \varepsilon$. On the other hand, by Theorem 3, for sufficiently large j's, $j \geqq j_2(\varepsilon)$, $\Phi_{n_j}(z)$ does not have any root in $1 \geqq |z| \geqq \geqq \rho_m + \varepsilon$. Hence $\Phi_{n_j}(z)$ has at least $n_j - t + 1$ roots in $\rho_m - \varepsilon < |z| < \rho_m + \varepsilon$ provided that $j \geqq j_3(\varepsilon) = \max\big(j_1(\varepsilon), j_2(\varepsilon)\big)$.

Thus we have proved that "most" of the roots of $\Phi_{n_j}(z)$ lie "near" $|z| = \rho_m$. The next step will be to prove that these roots are asymptotically uniformly distributed in angles with vertex at 0. To this end we use a deep and general theorem of ERDŐS and TURÁN [6, Theorem I] which states that if $\zeta_1, \ldots, \zeta_N$ are the roots of the polynomial $a_0 + a_1 z + \ldots + a_N z^N$ $(a_0, a_N \neq 0)$ then for any $0 = \alpha \leqq \beta \leqq 2\pi$ we have

$$\left| \sum_{\alpha \leqq \arg \zeta_k \leqq \beta} 1 - \frac{\beta - \alpha}{2\pi} N \right| \leqq 16 \sqrt{N \log \frac{|a_0| + \ldots + |a_N|}{\sqrt{|a_0 a_N|}}}. \tag{28}$$

We get from (5)

$$\gamma_{n_j}^{-1} h_m(z)\, \Phi_{n_j}(z) = z^{n_j+m} - \sum_{k=1}^{s} \sum_{l=1}^{m_k} a_{k,l,n_j} \left[\frac{z^{n_j+m+l}}{(z - z_k^*)^l} - \sum_{u=0}^{l-1} \binom{n_j + m + l}{u} \frac{z_k^{*n_j+m+l-u}}{(z - z_k^*)^{l-u}} \right]$$

i.e.

$$\bar{c}_m^{-1} \bar{\varkappa}_m^{-1} \gamma_{n_j}^{-1} h_m(z)\, h_m^*(z)\, \Phi_{n_j}(z) = \left(1 - \sum_{k=1}^{s} \sum_{l=1}^{m_k} a_{k,l,n_j}\right) z^{n_j+2m} + \sum_{t=0}^{m-1} c_{t,j} z^{n_j+m+t} + \tag{29}$$

$$+ \sum_{t=1}^{m-1} d_{t,j} z^t + \bar{\varkappa}_m^{-1} \sum_{k=1}^{s} \sum_{l=1}^{m_k} a_{k,l,n_j} z_k^{*n_j+m} \sum_{u=0}^{l-1} (-1)^{l-u} \binom{n_j + m + l}{u}$$

where by (10) and (18)

$$c_{t,j} = O(1),\ d_{t,j} = O(n_j^{m-1}\rho_m^{n_j}) \qquad (j \to \infty). \tag{30}$$

Further, the leading coefficient in (29) tends to $|\kappa_m|^{-2}$ as $j \to \infty$ (see (20)), and the constant term is

$$-\bar{\varkappa}_m^{-1} \sum_{k=1}^{s} \sum_{l=1}^{m_k} a_{k,l,n_j} z_k^{*n_j+m} \left(\frac{n_j^{l-1}}{(l-1)!} + O(n_j^{l-2}) \right) = -\frac{n_j^{M-1}\bar{\varkappa}_m^{-1}}{(M-1)!} \sum_{\substack{|z_k^*|=\rho_m \\ m_k=M}} a_{k,M,n_j} z_k^{*n_j+1} +$$

$$+ O(n_j^{M-2}\rho_m^{n_j}) = \frac{Mn_j^{M-1}\bar{c}_m}{c_m|\kappa_m|^2} \sum_{k=1}^{t} \frac{z_k^{*n_j+m-M} h_m(z_k^*)}{h_m^{*(M)}(z_k^*)} + o(n_j^{M-1}\rho_m^{n_j}) =$$

$$= \frac{Mn_j^{M-1}\rho_m^{n_j}\bar{c}_m}{c_m|\kappa_m|^2} \sum_{k=1}^{t} \frac{z_k^{*p} h_m(z_k^*)}{h_m^{*(M)}(z_k^*)} + o(n_j^{M-1}\rho_m^{n_j})$$

where we made use of the relations (23) and (25). Hence, introducing a new variable in (29) by $z = w\rho_m$, we get

$$c_m\bar{c}_m^{-1}|\kappa_m|^2 n_j^{1-M}\rho_m^{-n_j}\gamma_{n_j}^{-1} h_m(w\rho_m)\, h_m^*(w\rho_m)\, \Phi_{n_j}(w\rho_m) = \sum_{t=0}^{n_j+2m} \beta_{t,j} w^t$$

where by (30) and (24)

$$\beta_{0,j} \to M \sum_{k=1}^{t} \frac{z_k^{*p} h_m(z_k^*)}{h_m^{*(M)}(z_k^*)} \neq 0 \qquad (j \to \infty),$$

$$\beta_{t,j} = O(n_j^{m-M}) \qquad (1 \leqq t \leqq n_j + 2m,\ j \to \infty)$$

and

$$\varliminf_{j\to\infty} n_j^{M-1} |\beta_{n_j+2m,j}| > 0.$$

Hence, applying (28) we have that if ξ is an arbitrary point on $|z| = \rho_m$ then the polynomial (31) has

$$\frac{\varepsilon(n_j+2m)}{\pi} + O(\sqrt{n_j \log n_j}) \geq \frac{\varepsilon}{4} n_j \qquad (j \geqq j_5)$$

roots in the angular region $|\arg z - \arg \xi| \leqq \varepsilon$. But the polynomials $h_m(z)$ and $h_m^*(z)$ have altogether $2m$ roots, thus $\Phi_{n_j}(z)$ has at least $\frac{\varepsilon}{5} n_j$ roots in $|\arg z\text{-}\arg \xi| \leqq \varepsilon$ $(j \geqq i_6)$. On the other hand, we have seen that at most $t-1$ of the roots of $\Phi_{n_j}(z)$ may lie outside of $\rho_m - \varepsilon < |z| < \rho_m + \varepsilon$ $(j \geqq j_3)$. Therefore the "ε-neighbourhood"

$$\{z\colon \rho_m - \varepsilon < |z| < \rho_m + \varepsilon,\ |\arg z - \arg \xi| \leqq \varepsilon\}$$

of the point ξ contains at least $\frac{\varepsilon}{6} n_j$ roots of $\Phi_{n_j}(z)$, $j \geqq j_7 = \max(j_3, j_6)$, which means (being $\varepsilon > 0$ arbitrary) that $\xi \in Z(f_m)$. Q.E.D.

Finally, we would like to examine which points in $|z| < \rho_m$ belong to $Z(f)$. The situation strongly depends on whether the z_k^*'s are in R_0 or R_1. At first we turn to the case $\rho_m < 1$. As before, let z_k^* $(k = 1, \ldots, t)$ be those roots of $h_m^*(z)$ for which $|z_k^*| = \rho_m$, $m_k = M$ $(k = 1, \ldots, t)$ holds. These can be written in the form (see (26))

$$z_k^* = e^{2i\pi\left(\sum_{j=1}^{u} r_{kj}\omega_j + \frac{p_k}{q_k}\right)} \qquad (k = 1, \ldots, t)$$

where p_k, q_k are integers, $0 \leqq p_k < q_k$. This is clear when $z_k^* \in R_1$, and in case $z_k^* \in R_0$ we can put $r_{k1} = \ldots = r_{ku} = 0$.

THEOREM 5. *Let $\rho_m < 1$. Then all the points $z \in Z(f_m)$ in $|z| < \rho_m$ satisfy*

$$\sum_{k=1}^{t} e^{2i\pi\left(\sum_{j=1}^{u} r_{kj}\alpha_j + \frac{s_k}{q_k}\right)} \frac{h_m(z_k^*)}{h_m^{*(M)}(z_k^*)(z - z_k^*)} = 0 \tag{32}$$

where s_k, $0 \leqq s_k < q_k$ $(k = 1, \ldots, t)$ are integers and α_j $(j = 1, \ldots, u)$ are arbitrary real numbers.

REMARKS. 1. If $R_1 = \emptyset$ then the number of cluster points in $|z| < \rho_m$ is finite, namely at most $(t-1)q_1 \ldots q_t$. In particular, when $t = 1$, then $Z(f_m) = \{z : |z| = \rho_m\}$ (even if $R_1 \neq \emptyset$).

2. If $u = 1$, i.e. the basis in R_1 consists of one element, then for fixed r_{k1}, s_k, q_k, the condition (32) prescribes algebraic curves of degree at most t for $z \in Z(f_m)$, as α_1 varies. In particular, when $t = 2$, parts of these curves lying in $|z| < \rho_m$ are circular arcs or diameters of $|z| = \rho_m$.

3. If $u \geqq 2$, i.e. the basis in R_1 consists of at least two elements, then (32) prescribes two-dimensional domains bounded by algebraic curves, for $z \in Z(f_m)$.

PROOF OF THEOREM 5. Given α_j $(j = 1, \ldots, u)$ arbitrary, $0 \leqq s_k < q_k$ $(k = 1, \ldots, t)$, by the quoted theorem of Weyl, there exists a sequence of integers $n_1 < n_2 < \ldots$ such that

$$\lim_{l \to \infty} \left(\frac{z_k^*}{\rho_m}\right)^{n_l} = e^{2\pi i\left(\sum_{j=1}^{u} r_{kj}\alpha_j + \frac{s_k}{q_k}\right)} \qquad (k = 1, \ldots, t).$$

Thus our statement follows from Theorem 1 and Hurwitz's theorem.

The case $\rho_m = 1$ can be handled similarly, and therefore we omit the proof of the next theorem. As for notation now $z_1^*, \ldots, z_t^*$ denote *all* those roots of $h_m^*(z)$ for which $|z_k^*| = 1$. All other notations remain unchanged.

THEOREM 6. *Let $\rho_m = 1$. Then all the points $t \in Z(f_m)$ in $|z| < 1$ satisfy*

$$\sum_{k=1}^{t} e^{2i\pi\left(\sum_{j=1}^{u} r_{kj}\alpha_j + \frac{s_k}{q_k}\right)} \frac{(-1)^{m_k} m_k h_m^{(m_k)}(z_k^*)}{h_m^{*(m_k)}(z_k^*)(z - z_k^*)} = 0$$

where s_k, $0 \leqq s_k < q_k$ $(k = 1, \ldots, t)$ are integers and α_j $(j = 1, \ldots, u)$ are arbitrary real numbers.

The same remarks apply to this theorem as to the previous one.

4. A problem of Turán

From the preceding considerations a natural question arises: does there exist a weight-function $f(t)$ such that $Z(f) = \{z : |z| \leqq 1\}$? This problem was raised by P. TURÁN [3, Problem 67]. In what follows, we give a partial answer in this connection.

THEOREM 7. *Given an arbitrary* $\varepsilon > 0$, *there exists a weight-function* $f(t, \varepsilon)$ *such that the two-dimensional Lebesgue-measure of the set* $Z(f)$ *is greater than* $\pi - \varepsilon$.

The full answer for the above question remains open.

PROOF OF THEOREM 7. Let $m = 3$, $z_1 = z_1^* = e^{2\pi\omega_1 i}$, $z_2 = z_2^* = e^{2\pi\omega_2 i}$ $\left(0 < \omega_1 < < \omega_2 < \frac{1}{6\pi}\sqrt{\frac{\varepsilon}{\pi}}\right)$, $z_3 = z_3^* = 1$ $(t = m = 3)$ where z_1 and z_2 form a basis in the previously defined sense (i.e. $u = 2$). Consider the corresponding weight-function $f_3(t) = |h_3(e^{it})|^2$ where

$$h_3(z) = (z - 1)(z - z_1)(z - z_2).$$

Theorem 6 yields that all the points $z \in Z(f)$, $|z| < 1$ satisfy

$$\frac{1}{z-1} + \frac{e^{2\pi\alpha_1 i}}{z - z_1} + \frac{e^{2\pi\alpha_2 i}}{z - z_2} = 0 \tag{33}$$

where α_1 and α_2 are arbitrary real numbers.

Now let z be such that $|z| < 1$ and $|z - 1| > \sqrt{\frac{\varepsilon}{\pi}}$. Then by $|z_j - 1| < \frac{1}{3}\sqrt{\frac{\varepsilon}{\pi}}$ $(j = 1, 2)$ easy to see that the quantities

$$\frac{1}{|z-1|}, \quad \frac{1}{|z-z_1|}, \quad \frac{1}{|z-z_2|}$$

satisfy the triangle-inequalities, which implies that a proper choice of α_1, α_2 yields relation (33), i.e. $z \in Z(f)$. The restrictions $|z| < 1$, $|z - 1| > \sqrt{\frac{\varepsilon}{\pi}}$ mean that a circle with centre 1 and radius $\sqrt{\frac{\varepsilon}{\pi}}$ (i.e. of area ε) must be excluded from $|z| \leqq 1$. Q.E.D.

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BEMERKUNG ZUR KONVERGENZ DER FUNKTIONENREIHEN

Von

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Herrn Professor G. Alexits zum 80. Geburtstag gewidmet

1. Es sei $\lambda=\{\lambda_n\}_1^\infty$ eine nichtabnehmende Folge von positiven Zahlen. Mit $\Omega(\lambda)$, bzw. mit $\Omega^*(\lambda)$ bezeichnen wir die Klasse der im Intervall (0, 1) orthonormierten Funktionensysteme $\phi=\{\phi_n(x)\}_1^\infty$, für die

$$L_n(\phi;x)=\int_0^1\left|\sum_{k=1}^n \phi_k(x)\,\phi_k(t)\right|dt\leqq\lambda_n\qquad\big(x\in(0,1);\ n=1,2,\ldots\big),$$

bzw.

$$\int_0^1\sup_n\frac{L_n(\phi;x)}{\lambda_n}\,dx\leqq 1$$

erfüllt wird. Für eine Folge $a=\{a_n\}_1^\infty$ setzen wir

$$\|a;\Omega(\lambda)\|=\sup_{\phi\in\Omega(\lambda)}\int_0^1\sup_{1\leqq i\leqq j<\infty}|a_i\,\phi_i(x)+\ldots+a_j\,\phi_j(x)|\,dx,$$

$$\|a;\Omega^*(\lambda)\|=\sup_{\phi\in\Omega^*(\lambda)}\int_0^1\sup_{1\leqq i\leqq j<\infty}|a_i\,\phi_i(x)+\ldots+a_j\,\phi_j(x)|\,dx.$$

Es sei endlich

$$M(\lambda)=\{a:\|a;\Omega(\lambda)\|<\infty\},\quad M^*(\lambda)=\{a:\|a;\Omega^*(\lambda)\|<\infty\}.$$

Für eine Folge a und für natürliche Zahlen M,N ($M\leqq N$), setzen wir

$$a(N,\infty)=\{0,\ldots,0,a_N,a_{N+1},\ldots\},$$
$$a(M,N)=\{0,\ldots,0,a_M,\ldots,a_N,0,\ldots\}.$$

Auf Grund der Definitionen der Normen $\|\cdot\,;\Omega(\lambda)\|$, $\|\cdot\,;\Omega^*(\lambda)\|$ ist es klar, daß für jede Folge a

$$\|a(1,N);\Omega(\lambda)\|\leqq\|a(1,N+1);\Omega(\lambda)\|,\quad\|a(1,N);\Omega^*(\lambda)\|\leqq\|a(1,N+1);\Omega^*(\lambda)\|,$$
$$\|a(N,\infty);\Omega(\lambda)\|\geqq\|a(N+1,\infty);\Omega(\lambda)\|,$$
$$\|a(N,\infty);\Omega^*(\lambda)\|\geqq\|a(N+1,\infty);\Omega^*(\lambda)\|\qquad(N=1,2,\ldots)$$

erfüllt sind; weiterhin gelten

$$\|a(1,N);\,\Omega(\lambda)\| \nearrow \|a;\,\Omega(\lambda)\|,\quad \|a(1,N);\,\Omega^*(\lambda)\| \nearrow \|a;\,\Omega^*(\lambda)\| \qquad (N\nearrow\infty).$$

In der Arbeit [3] haben wir die folgende Behauptung bewiesen.

SATZ A. *Gilt* $\|a(N,\infty);\,\Omega(\lambda)\|\to 0$ $(N\to\infty)$, *bzw.* $\|a(N,\infty);\,\Omega^*(\lambda)\|\to 0$ $(N\to\infty)$, *so ist die Reihe*

$$\sum_{n=1}^{\infty} a_n\,\phi_n(x)$$

für jedes System $\phi\in\Omega(\lambda)$, *bzw. für jedes System* $\phi\in\Omega^*(\lambda)$ *im Intervall* (0,1) *fast überall konvergent.*

2. In dieser Note werden wir einen einfachen Beweis für die folgende Behauptung geben:

SATZ I. *Im Falle* $a\in M^*(\lambda)$ *gilt* $\|a(N,\infty);\,\Omega^*(\lambda)\|\to 0$ $(N\to\infty)$.

Aus dem Satz A folgt also, daß im Falle $a\in M^*(\lambda)$ die Reihe $\Sigma\, a_n\phi_n(x)$ für jedes System $\phi\in\Omega^*(\lambda)$ fast überall konvergiert.

In der Arbeit [1] haben wir diese Behauptung mit einer komplizierten Methode bewiesen.

3. BEWEIS DES SATZES I. Wir bringen zunächst einen

HILFSSATZ. *Für jede Folge a und für jede endliche Folge von natürlichen Zahlen* $n_0,\dots,n_N$ $(n_0<\dots<n_N)$ *gilt*

$$\sum_{i=1}^{N}\|a(n_{i-1}+1,n_i);\,\Omega^*(\lambda)\|^2 \leqq 2\,\|a(n_0+1,n_N);\,\Omega^*(\lambda)\|^2.$$

BEWEIS DES HILFSSATZES. Auf Grund der Definition der Norm $\|\cdot\,;\Omega^*(\lambda)\|$ gibt es für eine beliebige positive Zahl ε ein System $\phi^{(i)}=\{\phi_n^{(i)}(x)\}_1^\infty\in\Omega^*(\lambda)$ mit

$$(1)\qquad J_i=\int_0^1 \sup_{n_{i-1}<k\leqq l\leqq n_i}\left|\sum_{n=k}^{l} a_n\,\phi_n^{(i)}(x)\right| dx \geqq \|a(n_{i-1}+1,n_i);\,\Omega^*(\lambda)\|-\varepsilon .$$

Es sei $0=\alpha_0<\dots<\alpha_N=1$, und setzen wir

$$\phi_n(x)=\begin{cases}\dfrac{1}{\sqrt{\alpha_1-\alpha_0}}\,\phi_n^{(1)}\!\left(\dfrac{x-\alpha_0}{\alpha_1-\alpha_0}\right), & x\in(\alpha_0,\alpha_1),\quad n=1,\dots,n_1,\\[2ex] \dfrac{1}{\sqrt{\alpha_i-\alpha_{i-1}}}\,\phi_n^{(i)}\!\left(\dfrac{x-\alpha_{i-1}}{\alpha_i-\alpha_{i-1}}\right), & x\in(\alpha_{i-1},\alpha_i),\quad n=n_{i-1}+1,\dots,n_i;\\ & i=2,\dots,N-1,\\[2ex] \dfrac{1}{\sqrt{\alpha_N-\alpha_{N-1}}}\,\phi_n^{(N)}\!\left(\dfrac{x-\alpha_{N-1}}{\alpha_N-\alpha_{N-1}}\right), & x\in(\alpha_{N-1},\alpha_N),\quad n=n_{N-1}+1,\\ & n_{N-1}+2,\dots .\end{cases}$$

Es ist klar, daß $\phi = \{\phi_n(x)\}_1^\infty$ in (0, 1) ein orthonormiertes System ist. Weiterhin gelten die folgenden Gleichungen:

$$(2)\qquad L_n(\phi, x) = \begin{cases} L_n\left(\phi^{(1)}; \dfrac{x-\alpha_0}{\alpha_1-\alpha_0}\right), & x \in (\alpha_0, \alpha_1);\quad n = 1, \ldots, n_1, \\ L_{n_1}\left(\phi^{(1)}; \dfrac{x-\alpha_0}{\alpha_1-\alpha_0}\right), & x \in (\alpha_0, \alpha_1);\quad n = n_1+1,\ n_1+2, \ldots, \end{cases}$$

$$(3)\qquad L_n(\phi; x) =$$

$$= \begin{cases} 0, & x \in (\alpha_{i-1}, \alpha_i);\quad n = 1, \ldots, n_1, \\ \displaystyle\int_0^1 \left| \sum_{k=n_{i-1}+1}^{n} \phi_k^{(i)}\left(\frac{x-\alpha_{i-1}}{\alpha_i-\alpha_{i-1}}\right)\phi_k^{(i)}(t)\right| dt, & x \in (\alpha_{i-1}, \alpha_i);\quad n = n_{i-1}+1, \ldots, n_i, \\ \displaystyle\int_0^1 \left| \sum_{k=n_{i-1}+1}^{n_i} \phi_k^{(i)}\left(\frac{x-\alpha_{i-1}}{\alpha_i-\alpha_{i-1}}\right)\phi_k^{(i)}(t)\right| dt, & x \in (\alpha_{i-1}, \alpha_i);\quad n = n_i+1, \ldots \end{cases}$$

$$(i = 2, \ldots, N-1),$$

$$(4)\qquad L_n(\phi; x) =$$

$$= \begin{cases} 0, & x \notin (\alpha_{N-1}, \alpha_N), \\ \displaystyle\int_0^1 \left| \sum_{k=n_{N-1}+1}^{n} \phi_k^{(N)}\left(\frac{x-\alpha_{N-1}}{\alpha_N-\alpha_{N-1}}\right)\phi_k^{(N)}(t)\right| dt, & x \in (\alpha_{N-1}, \alpha_N);\quad n = n_{N-1}+1, \ldots. \end{cases}$$

Da im Falle $n > n_{i-1}$ $(i = 2, \ldots, N-1)$

$$\int_0^1 \left| \sum_{k=n_{i-1}+1}^{n} \phi_k^{(i)}\left(\frac{x-\alpha_{i-1}}{\alpha_i-\alpha_{i-1}}\right)\phi_k^{(i)}(t)\right| dt \leqq$$

$$\leqq L_{\min(n, n_i)}\left(\phi^{(i)}; \left(\frac{x-\alpha_{i-1}}{\alpha_i-\alpha_{i-1}}\right) + L_{n_{i-1}}\left(\phi^{(i)}; \frac{x-\alpha_{i-1}}{\alpha_i-\alpha_{i-1}}\right)\right)$$

und im Falle $n > n_{N-1}$

$$\int_0^1 \left| \sum_{k=n_{N-1}+1}^{n} \phi_k^{(N)}\left(\frac{x-\alpha_{N-1}}{\alpha_N-\alpha_{N-1}}\right)\phi_k^{(N)}(t)\right| dt \leqq$$

$$\leqq L_n\left(\phi^{(N)}; \frac{x-\alpha_{N-1}}{\alpha_N-\alpha_{N-1}}\right) + L_{n_{N-1}}\left(\phi^{(N)}; \frac{x-\alpha_{N-1}}{\alpha_N-\alpha_{N-1}}\right)$$

gilt, erhalten wir aus (2), (3) und (4)

$$\int_0^1 \sup_n \frac{L_n(\phi; x)}{\lambda_n}\, dx = \sum_{i=1}^{N} \int_{\alpha_{i-1}}^{\alpha_i} \sup_n \frac{L_n(\phi; x)}{\lambda_n}\, dx =$$

$$= \int_{\alpha_0}^{\alpha_1} \sup_{n \leqq n_1} \frac{L_n\left(\phi^{(1)}; \dfrac{x-\alpha_0}{\alpha_1-\alpha_0}\right)}{\lambda_n}\, dx +$$

$$+2\sum_{i=2}^{N-1} \int_{\alpha_{i-1}}^{\alpha_i} \sup_{n_{i-1}<n\leqq n_i} \frac{L_n\left(\phi^{(i)}; \dfrac{x-\alpha_{i-1}}{\alpha_i-\alpha_{i-1}}\right)}{\lambda_n}\, dx + 2\int_{\alpha_{N-1}}^{\alpha_N} \sup_{n_{N-1}<n} \frac{L_n\left(\phi^{(N)}; \dfrac{x-\alpha_{N-1}}{\alpha_N-\alpha_{N-1}}\right)}{\lambda_n}\, dx \leqq$$

$$\leqq (\alpha_1-\alpha_0)\int_0^1 \sup_n \frac{L_n(\phi^{(1)}; x)}{\lambda_n}\, dx + 2\sum_{i=2}^{N} (\alpha_i-\alpha_{i-1}) \int_0^1 \sup_n \frac{L_n(\phi^{(i)}; x)}{\lambda_n}\, dx \leqq$$

$$\leqq 2\sum_{i=1}^{N} (\alpha_i-\alpha_{i-1}) = 2\,,$$

also gilt $\phi \in \Omega^*(2\lambda)$, wobei $2\lambda = \{2\lambda_n\}_1^\infty$ ist.

Weiterhin folgt aus (1):

$$\|a(n_0+1, n_N); \Omega^*(\lambda)\| \geqq \int_0^1 \sup_{n_0<i\leqq j\leqq n_N} \left|\sum_{k=i}^{j} a_k \phi_k(x)\right| dx \geqq$$

$$\geqq \sum_{i=1}^{N} \frac{1}{\sqrt{\alpha_i-\alpha_{i-1}}} \int_{\alpha_{i-1}}^{\alpha_i} \sup_{n_{i-1}<k\leqq l\leqq n_i} \left|\sum_{n=k}^{l} a_n\, \phi_n^{(i)}\left(\frac{x-\alpha_{i-1}}{\alpha_i-\alpha_{i-1}}\right)\right| dx =$$

$$= \sum_{i=1}^{N} \sqrt{\alpha_i-\alpha_{i-1}}\, J_i \geqq \sum_{i=1}^{N} \sqrt{\alpha_i-\alpha_{i-1}}\, \|a(n_{i-1}+1, n_i); \Omega^*(\lambda)\| - N\varepsilon\,.$$

Da diese Ungleichung für jede Folge $0=\alpha_0<\ldots<\alpha_N=1$ besteht, erhalten wir daraus

$$\|a(n_0+1, n_N); \Omega^*(2\lambda)\| \geqq \sqrt{\sum_{i=1}^{N} \|a(n_{i-1}+1, n_i)\ \Omega^*(\lambda)\|^2} - N\varepsilon\,.$$

Da $\varepsilon>0$ beliebig ist, gilt auch

$$\sum_{i=1}^{N} \|a(n_{i-1}+1, n_i); \Omega^*(\lambda)\|^2 \leqq \|a(n_0+1, n_N); \Omega^*(2\lambda)\|^2\,. \tag{5}$$

In der Arbeit [2] haben wir gezeigt, daß für jede Folge a

$$\|a; \Omega^*(K\lambda)\| \leqq \sqrt{K}\, \|a; \Omega^*(\lambda)\|$$

ist. Daraus und aus (5) bekommen wir die Behauptung des Hilfssatzes.

Den Satz I können wir jetzt leicht beweisen. Gilt nämlich, für eine Folge a,

$$\|a(N, \infty); \Omega^*(\lambda)\| \to 0 \quad (N \to \infty),$$

so gibt es eine Indexfolge $(0 =) n_0 < \ldots < n_k < \ldots$ und eine positive Zahl ρ mit

$$\|a(n_{k-1}+1, n_k); \Omega^*(\lambda)\| > \sqrt{\rho} \qquad (k = 1, 2, \ldots).$$

Auf Grund des Hilfssatzes ist aber

$$N\rho \leqq \sum_{k=1}^{N} \|a(n_{k-1}+1, n_k); \Omega^*(\lambda)\|^2 \leqq 2\,\|a(1, n_N); \Omega^*(\lambda)\|^2$$

für $N = 1, 2, \ldots$, woraus sich $\|a; \Omega^*(\lambda)\| = \infty$, d. h. $a \notin M^*(\lambda)$ ergibt.

4. Mit dieser Methode kann man auch den folgenden Satz beweisen.

SATZ II. *Im Falle* $a \in M(2\lambda)$ *gilt* $\|a(N, \infty); \Omega(\lambda)\| \searrow 0$ $(N \nearrow \infty)$.

Ein schwächere Behauptung haben wir in [3] bewiesen.

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ON INTEGRABILITY OF LACUNARY TRIGONOMETRIC SERIES WITH WEIGHT

By

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Dedicated to Professor G. Alexits on his 80th birthday

Let $n_1, n_2, \ldots$ be an arbitrary sequence of positive numbers which satisfies the gap condition of Hadamard

$$\frac{n_{k+1}}{n_k} \geqq q > 1, \quad k = 1, 2, \ldots. \tag{1}$$

If for real numbers a_k, b_k the condition $\Sigma(a_k^2 + b_k^2) < \infty$ holds then it is well-known (see [1, ch. 2, § 1, no. 1,3]) that the series

$$a_0 + \sum_{k=1}^{\infty} (a_k \cos n_k x + b_k \sin n_k x) \tag{2}$$

converges a.e. and its sum denoted by $F(x)$ in what follows is integrable over each finite interval in the p-th power, for all $p > 0$. In the case of pure sine series (2), i.e. if $a_k \equiv 0$, M. WEISS ([2]; [3, Theorem 5.27]) found necessary and sufficient conditions for integrability of the function $(1/x)\, F(x)$ over $[0, \pi]$. Putting forward this investigation of hers, L. A. BALAŠOV and the author obtained a criterion in [4] for integrability of the function

$$\phi(x)\, F(x) \tag{3}$$

over $[0, \pi]$, where the weight function $\phi(x)$ increases infinitely as $x \to 0$ and the condition

$$\phi(x)\, x \log^{\alpha} \frac{1}{x} \downarrow 0 \tag{4}$$

holds for some α for sufficiently small x. Let Φ stand, as in [4], for the class of functions ϕ with the just mentioned requirements.

Here we generalize these results by consideration of integrability problem for the function $\phi(x)\,|F(x)|^p$ where $0 < p < \infty$. Moreover, the class of functions ϕ will be widened somewhat for which integrability conditions of (3) are investigated. The author's attention to the possibility and desirability of weakening the restrictions on the function ϕ was driven by L. Leindler and S. B. Stečkin when discussing the results of [4].

We say that a function ϕ is in the class Φ_p $(0 < p < \infty)$ if it is defined for $x > 0$, positive, increases infinitely as $x \to 0$ and the conditions

$$x\phi(x) = o\left(\int_x^1 \phi(t)\,dt\right) \tag{5}$$

and

$$\int_0^x t^\rho \phi(t)\,dt = O\big(x^{1+\rho}\,\phi(x)\big) \tag{6}$$

hold, where $\rho = \min(p, 1)$. It may be easily seen from (6) that

$$\phi(x) = O\big(\phi(2x)\big) \qquad (x \to 0). \tag{7}$$

It is not hard to verify also that (4) implies (5) and (6) and thus $\Phi \subset \Phi_p$ for all $p > 0$. Note that for $0 < \alpha \leqq 1$ the function $\phi(x) = 1/x^\alpha$ is in the classes Φ_p while for $\alpha > 1$ this is no longer true.

In what follows A stands for positive numbers which may be different at different occurrences, and depend on q, p and the function ϕ, and we keep in mind that the corresponding estimates are true if A is sufficiently large.

THEOREM. *Let the sequence of numbers* $\{n_k\}$ *satisfy* (1), *the series* $\Sigma(a_k^2 + b_k^2)$ *be convergent and* $\phi \in \Phi_p$ $(0 < p < \infty)$. *Then the integral*

$$\int_{+0} |F(x)|^p \phi(x)\,dx \tag{8}$$

is finite if and only if the series

$$\sum_{s=0}^{\infty} \phi_s \left\{ \left| \sum_{k=0}^{s} a_k \right|^p + \left[\sum_{k=s}^{\infty} (a_k^2 + b_k^2) \right]^{p/2} \right\}, \tag{9}$$

is convergent, where

$$\phi_s = \int_{1/n_{s+1}}^{1/n_s} \phi(x)\,dx. \tag{10}$$

If the series (9) *converges then the estimate*

$$\int_0^{1/n_1} |F(x)|^p \phi(x)\,dx \leqq AS \tag{11}$$

holds, where S *denotes the sum of* (9).

If $p = 1$ then the above assertion implies Theorem 1 of [4], while if $p = 1$, $\phi(x) = \dfrac{1}{x}$ and $a_k \equiv 0$ it coincides with the result of WEISS [2].

In the proof we use the following well-known estimates.

LEMMA 1. *Let* $q > 1$ *and* $0 < p < \infty$. *Then there exist positive numbers* Λ *and* Q *depending only on* q *and* p *such that if* $0 \leqq \beta \leqq 1/q$, $\Sigma(c_k^2 + d_k^2) < \infty$ *and* $\lambda_1, \lambda_2, \ldots$

are arbitrary real numbers satisfying conditions $\lambda_1 \geqq \Lambda$ *and* $\lambda_{k+1}/\lambda_k \geqq q > 1$ $(k = 1, 2, \ldots)$ *then*

$$\frac{1}{Q}\left\{|c_0|^p + \left[\sum_{k=1}^{\infty}(c_k^2+d_k^2)\right]^{p/2}\right\} \leqq \tag{12}$$

$$\leqq \int_{\beta}^{1}\left|c_0 + \sum_{k=1}^{\infty}(c_k\cos\lambda_k x + d_k\sin\lambda_k x)\right|^p dx \leqq Q\left\{|c_0|^p + \left[\sum_{k=1}^{\infty}(c_k^2+d_k^2)\right]^{p/2}\right\}.$$

If λ_k are integers, the above lemma is a consequence of the estimates (6.6) and (8.20) in Ch. 5 of [5]. Generalization to the case of nonintegral λ_k is discussed in [1, Ch. 2, § 1, no. 3]. Further note that for $p \leqq 4$ (12) may be easily deduced from lemmas contained in [6, § 11].

LEMMA 2. *Let* $0 < p \leqq 2$ *and* ψ_s *be positive. Then*

$$\sum_{s=1}^{N}\psi_s\left(\sum_{k=1}^{s}\psi_k/\psi_s\right)^{p/2}|a_s|^p \leqq 9\sum_{s=1}^{N}\psi_s\left(\sum_{k=s}^{N}a_k^2\right)^{p/2} \tag{13}$$

for arbitrary real numbers a_s.

(13) is a generalization of Hardy–Littlewood inequality [7, Theorem 346]. This estimate is due to LEINDLER [8].

PROOF OF THE THEOREM. Without any loss of generality we may assume that the inequalities

$$1 < q \leqq \frac{n_{k+1}}{n_k} \leqq q^2, \quad k = 1, 2, \ldots \tag{14}$$

hold for the n_k's. Indeed, the latter conditions will be obviously satisfied if we insert the terms into the series (2) with zero coefficients a_k, b_k and meanwhile the series (9) remains unchanged. On account of (14) and (7)

$$\frac{1}{A}\frac{1}{n_s}\phi\left(\frac{1}{n_s}\right) \leqq \phi_s \leqq A\frac{1}{n_s}\phi\left(\frac{1}{n_s}\right). \tag{15}$$

Introduce the notation $E_s = \left[\frac{1}{n_{s+1}}, \frac{1}{n_s}\right]$ and estimate the integral

$$\int_{1/n_{M+1}}^{1/n_1}|F(x)|^p\phi(x)dx = \sum_{s=1}^{M}\int_{E_s}|F(x)|^p\phi(x)\,dx. \tag{16}$$

Let r be the least integer such that

$$q^r > \Lambda, \tag{17}$$

where Λ is the number from Lemma 1. For $x \in E_s$ we represent $F(x)$ as

$$F(x) = \sum_{k=1}^{s+r-1}[a_k(\cos n_k x - 1) + b_k\sin n_k x] + \tag{18}$$

$$+\sum_{k=0}^{s+r-1}a_k + \sum_{k=s+r}^{\infty}(a_k\cos n_k x + b_k\sin n_k x).$$

We start with estimate of integrals of the first sum in the right hand side of (18). We have

$$\sigma \equiv \sum_{s=1}^{M} \int_{E_s} \left| \sum_{k=1}^{s+r-1} [a_k(\cos n_k x - 1) + b_k \sin n_k x] \right|^p \phi(x)\,dx \leqq$$

$$\leqq \sum_{s=1}^{M} \left[\sum_{k=1}^{s+r-1} (|a_k| + |b_k|)\, n_k \right]^p \int_{E_s} x^p \phi(x)\,dx .$$

This implies by Jensen inequality [7, Theorem 19] for $0 < p \leqq 1$

$$\sigma \leqq \sum_{s=1}^{M} \sum_{k=1}^{s+r-1} (|a_k|^p + |b_k|^p)\, n_k^p \int_{E_s} x^p \phi(x)\,dx .$$

On the other hand, if $p > 1$ then using Hölder inequality and the right estimate (14) we get

$$\sigma \leqq A \sum_{s=1}^{M} \sum_{k=1}^{s+r-1} (|a_k|^p + |b_k|^p)\, n_k\, n_{s+r-1}^{p-1} \int_{E_s} x^p \phi(x)\,dx \leqq$$

$$\leqq A \sum_{s=1}^{M} \sum_{k=1}^{s+r-1} (|a_k|^p + |b_k|^p)\, n_k \int_{E_s} x\, \phi(x)\,dx .$$

Thus for all $p > 0$

$$\sigma \leqq A \sum_{s=1}^{M} \sum_{k=1}^{s+r-1} (|a_k|^p + |b_k|^p)\, n_k^\rho \int_{E_s} x^\rho \phi(x)\,dx \leqq$$

$$\leqq A \sum_{k=1}^{r-1} (|a_k|^p + |b_k|^p)\, n_k^\rho \int_0^{1/n_1} x^\rho \phi(x)\,dx +$$

$$+ A \sum_{k=r}^{M+r-1} (|a_k|^p + |b_k|^p)\, n_k^\rho \int_0^{1/n_{k-r-1}} x^\rho \phi(x)\,dx ,$$

whence by (6), (14) and (7) we find

$$\sigma \leqq A \sum_{k=1}^{M+r-1} (|a_k|^p + |b_k|^p)\, \phi_k . \tag{19}$$

Now we estimate integrals of the remaining sums in the right hand side of (18). According to (17), the sequence $\lambda_k = n_{k+s+r-1}/n_s$ $(k = 1, 2, \ldots)$ satisfies the requirements of Lemma 1 for each s. Consequently the following estimates from above hold:

$$\sigma_s \equiv \int_{E_s} \left| \sum_{k=0}^{s+r-1} a_k + \sum_{k=s+r}^{\infty} (a_k \cos n_k x + b_k \sin n_k x) \right|^p \phi(x)\,dx \leqq \tag{20}$$

$$\leqq \frac{1}{n_s} \phi\left(\frac{1}{n_{s+1}}\right) \int_{n_s/n_{s+1}}^{1} \left| \sum_{k=0}^{s+r-1} a_k + \sum_{k=s+r}^{\infty} \left(a_k \cos \frac{n_k}{n_s} x + b_k \sin \frac{n_k}{n_s} x \right) \right|^p dx \leqq$$

$$\leqq \frac{A}{n_s} \phi\left(\frac{1}{n_{s+1}}\right) \left\{ \left| \sum_{k=0}^{s+r-1} a_k \right|^p + \left[\sum_{k=s+r}^{\infty} (a_k^2 + b_k^2) \right]^{p/2} \right\}$$

and, from below,

$$(21)\qquad \sigma_s \geqq \frac{1}{n_s}\phi\left(\frac{1}{n_s}\right)\int\limits_{n_s/n_{s+1}}^{1}\left|\sum_{k=0}^{s+r-1} a_k + \sum_{k=s+r}^{\infty}\left(a_k\cos\frac{n_k}{n_s}x + b_k\sin\frac{n_k}{n_s}x\right)\right|^p dx \geqq$$

$$\geqq \frac{1}{An_s}\phi\left(\frac{1}{n_s}\right)\left\{\left|\sum_{k=0}^{s+r-1} a_k\right|^p + \left[\sum_{k=s+r}^{\infty}(a_k^2+b_k^2)\right]^{p/2}\right\}.$$

In view of (7) and (14) we can replace $\frac{1}{n_s}\phi\left(\frac{1}{n_{s+1}}\right)$, $\frac{1}{n_s}\phi\left(\frac{1}{n_s}\right)$ by ϕ_{s+r} in (20), (21), and the sum $\sum_{k=0}^{s+r-1} a_k$ by $\sum_{k=0}^{s+r} a_k$.

Then we get from (16), (18), (19), (20)

$$(22)\qquad \int\limits_{1/n_{M+1}}^{1/n_1} |F(x)|^p\phi(x)\,dx \leqq A\sum_{k=1}^{M+r-1}(|a_k|^p + |b_k|^p)\,\phi_k +$$

$$+ A\sum_{s=1}^{M}\phi_{s+r}\left\{\left|\sum_{k=0}^{s+r} a_k\right|^p + \left[\sum_{k=s+r}^{\infty}(a_k^2+b_k^2)\right]^{p/2}\right\} \leqq$$

$$\leqq A\sum_{s=1}^{M+r}\phi_s\left\{\left|\sum_{k=0}^{s} a_k\right|^p + \left[\sum_{k=s}^{\infty}(a_k^2+b_k^2)\right]^{p/2}\right\}.$$

(22) implies the sufficiency of the conditions of our theorem for the finiteness of the integral (8).

Similarly, (16), (18), (19) and (21) yield the following estimate from below:

$$(23)\qquad \int\limits_{1/n_{M+1}}^{1/n_1} |F(x)|^p\phi(x)\,dx \geqq \frac{1}{A}\sum_{s=r+1}^{M+r}\phi_s\left\{\left|\sum_{k=0}^{s} a_k\right|^p + \left[\sum_{k=s}^{\infty}(a_k^2+b_k^2)\right]^{p/2}\right\} -$$

$$- A\sum_{s=1}^{M+r-1}\phi_s(|a_s|^p + |b_s|^p).$$

(23) implies the necessity of the conditions of our theorem. The latter statement is obvious if the series

$$(24)\qquad \sum_{s=1}^{\infty}\phi_s(|a_s|^p + |b_s|^p)$$

converges. On the other hand, if the series (24) is divergent then we use the following consideration. Since by (10) and (15)

$$\sum_{k=1}^{s}\frac{\phi_k}{\phi_s} \geqq \frac{1}{A\frac{1}{n_s}\phi\left(\frac{1}{n_s}\right)}\int\limits_{1/n_s}^{1/n_1}\phi(x)\,dx,$$

(5) implies that

$$\sum_{k=1}^{s} \frac{\phi_k}{\phi_s} \to \infty \qquad (s \to \infty).$$

Thus if the series (24) diverges then

$$\sum_{s=1}^{N} \phi_s(|a_s|^p + |b_s|^p) = o\left(\sum_{s=1}^{N} \phi_s \left(\sum_{k=1}^{s} \frac{\phi_k}{\phi_s}\right)^{p/2} (|a_s|^p + |b_s|^p)\right)$$

as $N \to \infty$, and consequently by Lemma 2

$$\sum_{s=1}^{N} \phi_s(|a_s|^p + |b_s|^p) = o\left(\sum_{s=1}^{N} \phi_s \left[\sum_{k=s}^{N} (a_k^2 + b_k^2)\right]^{p/2}\right). \tag{25}$$

From (23) and (25) we obtain the necessity of the conditions of the theorem in the case of the divergent series (24). Q.E.D.

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CONVERGENT INTERPOLATORY PROCESSES FOR ARBITRARY SYSTEMS OF NODES

By

P. VÉRTESI (Budapest)

To Professor G. Alexits on his 80th birthday

1. Introduction and preliminary results

1.1. Considering any matrix $X=\{x_{k,n}=\cos\xi_{k,n}\}$ $(k=1,2,\ldots,n;\,n=1,2,\ldots)$ in $[-1,1]$ with

$$(1.1)\qquad -1\leqq x_{n,n}<x_{n-1,n}<\ldots<x_{2,n}<x_{1,n}\leqq 1\qquad (n=1,2,\ldots)$$

we know that the Lagrange interpolatory procedure based on the nodes (1.1) cannot be uniformly convergent for all $f\in C$ (C is the set of functions continuous on $[-1,1]$).

1.2. However, if we do not restrict ourselves to the Lagrange interpolation we can state positive results.

For example S. BERNSTEIN [1] proved as follows:

THEOREM A. *If $f\in C$ then for every fixed $c>0$ there exist polynomials $P_n(f;x)$ of degree $\leqq n-1$ interpolating f at least at $n(1-c)$ roots of $T_n(x)$ and $\|P_n(f;x)-f(x)\|\to 0$ $(n\to\infty)$.*

Here and later $T_n(x)=\cos n\vartheta$ $(x=\cos\vartheta)$ are the Chebyshev polynomials having the roots $T=\{t_{k,n}\}$ where

$$(1.2)\quad t_{k,n}=\cos\vartheta_{k,n}=\cos\frac{2k-1}{2n}\pi\qquad (k=1,2,\ldots,n;\quad n=1,2,\ldots);$$

further $\|g(x)\|=\max\limits_{-1\leqq x\leqq 1}|g(x)|$ for $g\in C$.

Later G. GRÜNWALD [16] proved his important results on the Hermite—Fejér step-parabolas.

THEOREM B. *If the nodes form a strongly normal matrix then for every $f\in C$ there exists a sequence of polynomials $H_n(f;x)$ of degree $\leqq 2n-1$ such that $H_n(f;x_{i,n})=f(x_{i,n})$ $(i=1,2,\ldots,n)$ further $\|H_n(f;x)-f(x)\|\to 0$ $(n\to\infty)$.*

E.g., the roots of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ form a strongly normal matrix if $-1<\alpha,\beta<0$ (see [2]).

I wish to remark that the processes mentioned above are linear operators.

1.3. In 1943 P. ERDŐS [3] stated the following

THEOREM C. *Let X be defined by* (1.1). *A necessary and sufficient condition that to every fixed $c > 0$ there should exist a sequence of operators $\{L_n(f; c; x) = L_n(f; x)\}$ defined for all $f \in C$ so that*

(i) $L_n(f; x)$ *is a polynomial of degree* $\leqq n(1 + c)$ $(n = 1, 2, \dots)$,
(ii) $L_n(f; x_{\kappa,n}) = f(x_{k,n})$ $\quad (k = 1, 2, \dots, n)$,
(iii) $\lim\limits_{n\to\infty} \| L_n(f; x) - f(x) \| = 0$ *for every* $f \in C$,

is that

$$\text{(E)} \qquad \begin{cases} \textit{if} \;\; n(\beta_n - \alpha_n) \to \infty, \;\; 0 \leqq \alpha_n < \beta_n \leqq \pi \;\; \textit{then} \\ \overline{\lim\limits_{n\to\infty}} \dfrac{N_n(\alpha_n, \beta_n)}{n(\beta_n - \alpha_n)} \leqq \dfrac{1}{\pi} \quad \textit{and} \\ \underline{\lim\limits_{n\to\infty}} (\xi_{i+1,n} - \xi_{i,n})\, n > 0 \quad (i \textit{ arbitrary}). \end{cases}$$

Here $N_n(X; \alpha, \beta) = N_n(\alpha, \beta)$ stands for the number of $\xi_{k,n}$'s in $[\alpha, \beta]$.

As Erdős remarked, the first part of (E) means that every interval (in ϑ) which is large compared to $1/n$ contains asymptotically at most as many x_i's as $T_n(x)$. The classical orthogonal polynomials, as is well known, satisfy (E).

We will show that the $L_n(f; x)$'s can be *linear* operators, too. Namely we prove

THEOREM 1.1. *Considering an arbitrarily fixed matrix X, the necessary and sufficient condition that to every fixed $c > 0$ there should exist a sequence of linear operators $L_n(f; x)$ on C having* (i), (ii) *and* (iii) *is that* (E) *should be satisfied.*

1.4. In his paper [4] G. FREUD investigated the rate of the convergence. He called the point system $\{x_{k,n}\}$ "approximating" if for every fixed $c > 0$ there is a sequence $L_n(f; x)$ of *linear* operators on C so that (i), (ii) and (iii) are true for L_n; if, moreover

$$\text{(iv)} \qquad \| L_n(f; x) - f(x) \| \leqq K(c)\, E_n(f) \qquad (n = 1, 2, \dots)$$

then X is called "well approximating". (Here $E_n(f)$ is the best approximation of $f(x)$ by polynomials of degree $\leqq n$.)

G. Freud proved

THEOREM D. *A point system is well approximating if and only if it is approximating.*

1.5. It is an interesting problem to state positive theorems for *arbitrary* systems of nodes. This was recently investigated by J. SZABADOS [5]. To formulate his result, let

$$\text{(1.3)} \qquad d_n = d_n(X) = \min_{1 \leqq k \leqq n-1} (\xi_{k+1,n} - \xi_{k,n}) \qquad (n = 1, 2, \dots).$$

Now the following statement holds.

THEOREM E. *For any fixed X and $f \in C$ there exist polynomials $p_n(f; x)$ $(n = 1, 2, \dots)$ of degree $m \leqq c/d_n$ for which* (ii) *is valid; further*

$$\| p_n(f; x) - f(x) \| = O(1)\, E_m(f) \qquad (n = 1, 2, \dots). \tag{1.5}$$

Here and later the "O" sign does not depend on f, n and (at the pointwise estimations) x.

In his proof the constant c must be chosen greater than $\pi/2 + 2\pi^3/\sqrt{3} \approx 37.5$; moreover, generally the polynomials $p_n(f; x)$ are not linear operators.

1.6. Our aim is now to define a *linear* operator sequence where, as we shall see, the acting constant, in a certain sense, will be the best possible. Further we deal with pointwise estimations, too.

2. New results

2.1. Uniform approximation for arbitrary X. **2.11.** Using the above notations we can prove

THEOREM 2.1. *Let us consider an arbitrary matrix X in $[-1, 1]$. Then for every fixed $c > 0$ and $\varepsilon > 0$ there exists a sequence of linear operators $L_n(f, c, \varepsilon; x) = L_n(f; x)$ defined on C so that*

(a) *$L_n(f; x)$ is a polynomial of degree* $N \leqq \dfrac{\pi}{d_n}(1 + c)$ $(n = 1, 2, \dots)$,

(b) $L_n(f; x_{k,n}) = f(x_{k,n})$ $(k = 1, 2, \dots, n)$,

(c) $\| L_n(f; x) - f(x) \| = O(1)\, E_{[N(1-\varepsilon)]}(f)$ $(n = 1, 2, \dots;\ f \in C)$.

2.12. Considering the proof, especially condition (E), observe that d_n can be replaced by

$$D_n = \min_{\substack{1 \leqq k \leqq n-1 \\ k \neq i_1, i_2, \dots, i_L}} (\xi_{k+1,n} - \xi_{k,n}) \qquad (n = 1, 2, \dots) \tag{1.4}$$

(where $1 \leqq i_1 < \dots i_L \leqq n - 1$ and L does not depend on n) supposing that $D_n = O(d_n)$. Obviously $D_n \geqq d_n$ which means that for some matrices X the degree of $L_n(f; x)$ can be reduced.

Generally, if we can find a matrix $Y = \{y_{k,m_n}\}$ $(k = 1, 2, \dots, m_n;\ n \geqq n_0)$ such that $x_{i,n} \in \{y_{k,m_n}\}_{k=1}^{m_n}$ $(i = 1, 2, \dots, n;\ n \geqq n_0)$ and Y satisfies (E) (with m_n) then we can prove Theorem 2.1 for $N = m_n(1 + c)$.

2.13. On the other hand, generally the constant π cannot be changed to $\pi - \rho$ $(\rho > 0)$.

Indeed, if $d_n = \pi/n$ then

$$N \leqq n(1 - \rho/\pi)(1 + c) < n - 1 \qquad (n \geqq n_0),$$

whenever c is small enough, contradicting (b). But if we suppose $\pi/d_n \geqq \alpha n$ where $\alpha > 1$, one can ask whether we can decrease the degree by requiring only

(d) $$\lim_{n\to\infty} \| L_n(f; x) - f(x) \| = 0 \qquad (n = 1, 2, \ldots; f \in C)$$

instead of (c). The following statement can be proved like Theorem D.

THEOREM 2.2. *If for a sequence of linear operators $L_n(f; c, x)$* (a), (b) *and* (d) *hold then one can construct another sequence of linear operators $B_n(f; c, \varepsilon, x)$ so that or $\{B_n\}$* (a), (b) *and* (c) *are valid.*

2.2 Pointwise approximation for arbitrary X. **2.21.** We quote the following well-known estimations.

THEOREM F. *If $f^{(r)} \in C$ ($r \geqq 0$ fixed integer) then there are sequences of linear polynomial operators $J_{n,r}(f; x)$ and $G_{n,r}(f; x)$ of degree $\leqq n$ such that for any $x \in [-1, 1]$*

(2.1) $$|f(x) - J_{n,r}(f; x)| = O(1)\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^r \omega\left(f^{(r)}; \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right) \quad (n > r)$$

and

(2.2) $$|f(x) - G_{n,r}(f; x)| = O(1)\left(\frac{\sqrt{1-x^2}}{n}\right)^r \omega\left(f^{(r)}; \frac{\sqrt{1-x^2}}{n}\right) \qquad (n > r).$$

(*Here $\omega(g; t)$ is the modulus of continuity of $g(x)$ in* $[-1, 1]$.)

(See e.g. A. F. TIMAN [7], 5.2 for (2.1) and I. E. GOPENGAUZ [8] and S. A. TELYAKOVSKI [9] for (2.2).)

For $r = 0$ several papers were devoted to prove (2.1) or (2.2) by polynomials $P_n(f; x)$ of degree $\approx 4n$ interpolating to $f(x)$ in specially chosen $x_{k,n}$. (See e.g. [10]—[12]; for a more detailed reference see [13].) A recent result proved by G. FREUD and A. SHARMA [13] states as follows.

THEOREM G. *Let $\{x_{k,n}\}$ ($k = 1, 2, \ldots, n$) denote the zeros of $P_n^{(\alpha,\beta)}(x)$ ($\alpha, \beta > -1$ arbitrary but fixed). If $f \in C$, then for every fixed $c > 0$ there exist linear polynomial operators $A_n^{(\alpha,\beta)}(f; x) = A_n(f; x)$ of degree $\leqq n(1 + c)$ so that*

(2.3) $$\begin{cases} A_n(f; x_{k,n}) = f(x_{k,n}) \qquad (k = 1, 2, \ldots, n;\ \ n \geqq n_0), \\ | A_n(f; x) - f(x) | = O(1)\, \omega\left(f; \dfrac{\sqrt{1-x^2}}{n} + \dfrac{1}{n^2}\right) \qquad (n \geqq n_0). \end{cases}$$

2.22. Now we give a theorem dealing with arbitrary X and $r \geqq 0$.

THEOREM 2.3. *Let the matrix of nodes $X \subset [-1, 1]$ be arbitrary, $r \geqq 0$ fixed integer. Then for every fixed $c > 0$ there exists a sequence of linear operators $K_n(f; c; x) = K_n(f; x)$ defined for every $f^{(r)} \in C$ for which* (a), (b) *and*

(e) $$| K_n(f; x) - f(x) | = O(1)\left(\frac{\sqrt{1-x^2}}{N} + \frac{1}{N^2}\right)^r \omega\left(f^{(r)}; \frac{\sqrt{1-x^2}}{N} + \frac{1}{N^2}\right) (n \geqq n_0)$$

are valid.

As for the bound $\pi(1 + c)/d_n$, see 2.12 and 2.13.

2.3. A problem of G. Freud. In connection with Theorem G in 1972 G. FREUD raised the following problem ([14], Problem 10). Find necessary and sufficient criteria concerning $X=\{x_{k,n}\}$ so that for every $c>0$ there exists a sequence $\{A_n\}$ of linear polynomial operators defined on C for which the degree of $A_n(f; x)$ is $\leqq n(1+c)$, further (2.3) is true (see further [15], pp. 274).

Now we solve a generalized form of this question.

THEOREM 2.4. *Let $X\subset[-1, 1]$ and $r\geqq 0$ fixed. Then the necessary and sufficient condition that for every $c>0$ there should exist a sequence $\{A_n\}$ of linear polynomial operators defined for $f^{(r)}\in C$ such that*

$$(2.4)\quad \begin{cases} \deg A_n(f; x)\leqq n(1+c), \\ A_n(f; x_{k,n})=f(x_{k,n}) \quad (k=1, 2, \ldots, n), \\ |A_n(f; x)-f(x)|=O(1)\left(\dfrac{\sqrt{1-x^2}}{n}+\dfrac{1}{n^2}\right)^r \omega\left(f^{(r)}; \dfrac{\sqrt{1-x^2}}{n}+\dfrac{1}{n^2}\right) (n\geqq n_0), \end{cases}$$

is that (E) *should be satisfied.*

2.4. Estimation of Gopengauz – Telyakovski-type. Now we prove two theorems concerning the pointwise estimation (2.2). For the sake of definiteness suppose $X\subset(-1,1)$ and define

$$(2.5)\quad \delta_n=\delta_n(X)=\min_{0\leqq k\leqq n}(\xi_{k+1,n}-\xi_{k,n}) \qquad (n=1, 2, \ldots)$$

where $x_{0,n}=\cos\xi_{0,n}=1$ and $x_{n+1,n}=\cos\xi_{n+1,n}=-1$ (compare with (1.3)). We prove

THEOREM 2.5. *Let $X\subset(-1, 1)$ and $r\geqq 0$ be fixed. Then for $c>0$ there exists a sequence of linear polynomial operators $M_n(f; x)$ defined for $f^{(r)}\in C$ such that*

$$(2.6)\quad \begin{cases} \deg M_n(f; x)=N\leqq\left[\dfrac{\pi}{\delta_n}(1+c)\right], \\ M_n(f; x_{k,n})=f(x_{k,n}) \qquad (k=1, 2, \ldots, n), \\ |M_n(f; x)-f(x)|=O(1)\left(\dfrac{\sqrt{1-x^2}}{N}\right)^r \omega\left(f^{(r)}; \dfrac{\sqrt{1-x^2}}{N}\right) \qquad (n\geqq n_0). \end{cases}$$

Similarly, we can prove

THEOREM 2.6. *Let $X\subset(-1, 1)$ and $r\geqq 0$ be fixed. The necessary and sufficient condition that for every $c>0$ there should exist a sequence $\{M_n\}$ of linear polynomial operators for $f^{(r)}\in C$ satisfying* (2.6) *with $N\leqq n(1+c)$ is that* (E) *should hold for $\tilde{X}=\{x_{k,n}\}$ $(k=0, 1, \ldots, n+1; n=1, 2, \ldots)$.*

Remarks analogous to 2.12 and 2.13 hold.

2.5. REMARK. We can state corresponding theorems for the trigonometric case, too.

3. Proofs

3.1. PROOF OF THEOREM 1.1. First we quote an important statement found by P. ERDŐS [4] in 1967.

THEOREM H. *Let* $\{x_{k,n}\}_{k=1}^{n}$ *be an arbitrary system of nodes in* $[-1, 1]$. *Then the necessary and sufficient condition that to every* $c_1 > 0$ *there should exist an* $A(c_1)$ *so that to every* $f_{k,n}$, $|f_{k,n}| \leqq 1$, $1 \leqq k \leqq n$, $n = 1, 2, \ldots$, *there should exist a polynomial* $P_q(x)$ *of degree* $q \leqq n(1 + c_1)$ *satisfying*

$$(3.1) \qquad P_q(x_{k,n}) = f_{k,n}, \quad \| P_q(x) \| \leqq A(c_1) \qquad (1 \leqq k \leqq n; \quad n = 1, 2, \ldots),$$

is that (E) *should hold.*

By this theorem we can choose a sequence of polynomials $Q(x_{k,n}; x)$ such that

$$(3.2) \qquad \begin{cases} Q(x_{k,n}; x_{j,n}) = \delta_{k,j}, \quad \| Q(x_{k,n}; x) \| \leqq A(c_1), \\ \deg Q(x_{k,n}; x) \leqq n\left(1 + \dfrac{c}{2}\right) \qquad (1 \leqq j, k \leqq n, n \geqq n_0). \end{cases}$$

3.11. Now let

$$(3.3) \qquad s = \left[\frac{cn}{4}\right] \qquad (n \geqq n_0).$$

Denote $\min\limits_{1 \leqq i \leqq s} |\xi_{k,n} - \vartheta_{i,s}| = |\xi_{k,n} - \vartheta_{j_k,s}|$ $(k = 1, 2, \ldots, n)$. (Whenever there exist two such ϑ_i we can choose any of them.) It may occur that $\vartheta_{j_k,s} = \vartheta_{j_{k+1},s} = \ldots = \vartheta_{j_{k+l},s}$ but then $l \leqq M(c)$.

By a simple computation we get

$$(3.4) \qquad a_{j_k,s}(x_{k,n}) \geqq \alpha > 0 \qquad (k = 1, 2, \ldots, n; n \geqq n_0)$$

where $a_{i,s}(x)$ are the fundamental polynomials of the Lagrange interpolation based on the roots of $T_s(x)$, i.e.

$$(3.5) \qquad a_{i,s}(x) = \frac{(-1)^{i+1} T_s(x) \sin \vartheta_{i,s}}{s(x - t_{i,s})} \qquad (i = 1, 2, \ldots, s).$$

We shall use the fact

$$(3.6) \qquad \sum_{i=1}^{s} a_{i,s}^2(x) \leqq 2 \qquad (x \in [-1, 1], \quad s = 1, 2, \ldots),$$

too (see e.g. [6]).

Define

$$(3.7) \qquad h_{k,s}(x) = \frac{a_{j_k,s}^2(x)}{a_{j_k,s}^2(x_{k,n})} \qquad (k = 1, 2, \ldots, n; \quad n \geqq n_0).$$

3.12. For the positive integer $N \geqq N_0$ and $\varepsilon > 0$ let

$$(3.8) \qquad V_N(f; \varepsilon; x) = \frac{1}{N - [N(1-\varepsilon)] + 1} \sum_{k=[N(1-\varepsilon)]}^{N} s_k(f; x)$$

where $s_k(f; \cos \vartheta)$ is the k-th partial sum of the Fourier series of $f(\cos \vartheta)$. Using de la Vallée Poussin's result we have for the polynomial V_n of degree $\leqq N$

$$(3.9) \qquad \| f(x) - V_N(f; \varepsilon; x) \| \leqq K(\varepsilon) E_{[N(1-\varepsilon)]}(f).$$

3.13. Now we shall prove (i), (ii) and (iii). Let

$$(3.10) \qquad L_n(f; x) = V_N(f; \varepsilon; x) + \sum_{k=1}^{n} [f(x_{k,n}) - V_N(f; \varepsilon, x_{k,n})] \, h_{k,s}(x) \, Q(x_{k,n}; x)$$

with $N = [n(1+c)]$.

By (3,2), (3.5), (3.7) and (3.8) $L_n(f; x)$ is a linear polynomial operator of degree $\leqq N$. Using (3.2) and (3.7)

$$L_n(f; x_{k,n}) = f(x_{k,n}) \qquad (k = 1, 2, \ldots, n),$$

moreover, by (3.2), (3.4) and (3.6) for $n \geqq n_0$

$$| L_n(f; x) - f(x) | \leqq | V_N(f; \varepsilon; x) - f(x) | +$$
$$+ 2M \alpha^{-2} \cdot A(c_1) \, k(\varepsilon) \cdot E_{[N(1-\varepsilon)]}(f) = O(1) \, E_{[N(1-\varepsilon)]}(f),$$

which is actually (iv) (see 1.4).

3.2. Proofs of Theorems 2.1 and 2.2. For convenience first we deal with Theorem 2.2.

3.21. The proof is essentially due to Freud [5]. We shall sketch it since the original is not easily accessible. Let $L_n(f; x)$ be an operator having properties (a), (b) and (d). By (d) we have, using the Banach–Steinhaus theorem,

$$\| L_n(g; x) \| \leqq M(c) \, \| g(x) \| \quad \text{for any } g \in C.$$

Using this and (3.9) we can easily get that the operator

$$(3.11) \qquad B_n(f; x) = V_N(f; \varepsilon, x) + L_n[f - V_N(f; \varepsilon); x]$$

fulfills the conditions (a), (b) and (c).

3.22. Consider now Theorem 2.1. Let us choose $c_2 > 0$ such that

$$(3.12) \qquad 1 + \frac{c}{2} > (1 + c_2)^2 \qquad (0 < c_2 < c).$$

Let

$$m = \left[\frac{\pi}{d_n}(1 + c_2)\right] \qquad (n \geqq n_0) \tag{3.13}$$

then $Y = \{y_{i,m} = \cos \eta_{i,m}\}_{i=1}^{p}$ is a system of nodes containing each $x_{k,n}$ $(k = 1, 2, \ldots, n)$ further those $t_{j,m}$'s for which $\min\limits_{1 \leqq k \leqq n} | \xi_{k,n} - \vartheta_{j,m} | > \frac{\pi}{2m}$ or $\min\limits_{1 \leqq k \leqq n} (\xi_{k,n} - \vartheta_{j,m}) = \frac{\pi}{2m}$. If p would be $m + 1$ we omit the first $\vartheta_{j,m} \in \{y_{i,m}\}$. By this we get that $p(n) = m$ for each n. Moreover, the nodes $\{y_{i,m}\}$ $(i = 1, 2, \ldots, m;\ n \geqq n_0)$ satisfy condition (E) (for m). Indeed, $m(\eta_{i+1,m} - \eta_{i,m}) \geqq \frac{\pi}{2}$, further, by our construction, $N_m(T; \alpha_m, \beta_m) + 2 \geqq N_m(Y; \alpha_m, \beta_m)$, from where we obtain the second requirement.

Now, using Theorem 1.1, further Theorem 2.2 for the matrix Y, we obtain our statement.

3.3. Proof of Theorem 2.3. **3.31.** Using the notations of Section 3.1, and (3.2), let now

$$s = \left[\frac{\pi c}{2id_n}\right] \qquad (n \geqq n_0, \quad i = r + 3) \tag{3.14}$$

and

$$h_{k,s}(x) = \left[\frac{a_{j_{k},s}(x)}{a_{j_{k},s}(x_{k,n})}\right]^i \qquad (k = 1, 2, \ldots, n; \quad n \geqq n_0). \tag{3.15}$$

Let further with $N = [\pi(1 + c)/d_n]$

$$K_n(f; x) = J_{N,r}(f; x) + \sum_{k=1}^{n} [f(x_{k,n}) - J_{N,r}(x_{k,n})]\, h_{k,s}(x)\, Q(x_{k,n}; x), \tag{3.16}$$

where the Q's are defined for Y with $\deg Q \leqq \pi\left(1 + \frac{c}{2}\right)\Big/ d_n$ (see 3.22). Obviously we have (a) and (b) for $K_n(f; x)$ (see (3.14), (3.15) and (3.2)). So we have to prove only the requirement (e).

3.32. For this aim we state

$$\left| \sum_{k=1}^{n} \left(\frac{\sin \xi_{k,n}}{N} + \frac{1}{N^2}\right)^r \omega\left(f^{(r)}; \frac{\sin \xi_{k,n}}{N} + \frac{1}{N^2}\right) \left[\frac{T_s(x) \sin \vartheta_{j_k,s}}{s(x - t_{j_k,s})}\right]^i \right| = \tag{3.17}$$

$$= O(1) \left(\frac{\sin \vartheta}{N} + \frac{1}{N^2}\right)^r \omega\left(f^{(r)}; \frac{\sin \vartheta}{N} + \frac{1}{N^2}\right) \qquad (x \in [-1, 1]).$$

As it can be proved with $|\vartheta_{j_l}-\vartheta|\leqq\pi/2s$

$$\text{(3.18)}\qquad\begin{cases}\sin\vartheta_k\sim\dfrac{k}{s}\quad(k\leqq s/2)\,,\quad \sin\vartheta_k\sim\dfrac{n-k+1}{s}\quad(k\geqq s/2)\,,\\[2ex] \dfrac{\sin\vartheta_k}{|\cos\vartheta-\cos\vartheta_k|}\leqq\dfrac{1}{\sin\dfrac{|\vartheta-\vartheta_k|}{2}}\,,\\[2ex] \dfrac{\sin\vartheta}{N}+\dfrac{1}{N^2}\sim\dfrac{\sin\vartheta_{j_l}}{N}\,.\end{cases}$$

So using (3.18), $s\sim N$, $|a_k(x)|=O(1)$ and the notation

$$\Delta_n(\vartheta)=\frac{\sin\vartheta}{n}+\frac{1}{n^2}\,,$$

we have

$$\sum_{\substack{k\\ j_k=j_l}}\Delta_N^r(\xi_k)\,\omega\big(\Delta_N(\xi_k)\big)=O(1)\,\Delta_N^r(\vartheta)\,\omega\big(\Delta_N(\vartheta)\big)\,.$$

Further, by (3.18)

$$\Bigg|\sum_{\substack{k=1\\ j_k\neq j_l}}^{n}\cdots\Bigg|=O(1)\,\Delta_N^r(\vartheta)\,\omega\big(\Delta_N(\vartheta)\big)\sum_{j_k\neq j_l}\frac{\Delta_N^r(\xi_k)}{\Delta_N^r(\vartheta)}\left(\frac{\Delta_N(\xi_k)}{\Delta_N(\vartheta)}+1\right)\left|\frac{T_s(x)\sin\vartheta_{j_k}}{s(x-t_{j_k})}\right|^i=$$

$$=O(1)\,\Delta_N^r(\vartheta)\,\omega\big(\Delta_N(\vartheta)\big)\max_{t=r,\;r+1}\sum_{j_k\neq j_l}\left(\frac{j_k}{j_l}\right)^t\frac{1}{|j_k-j_l|^i}=O(1)$$

because by $i=r+3$

$$\sum_{\substack{k=1\\ k\neq l}}^{n}\left(\frac{k}{l}\right)^t\frac{1}{|k-l|^i}=\sum_{k<\frac{l}{2}}+\sum_{k>2l}+\sum_{\substack{\frac{l}{2}\leqq k\leqq 2l\\ k\neq l}}=O(1)\sum_{k=1}^{n}\frac{1}{k^{i-t}}=O(1)\,.$$

3.33. Using (3.17) and (2.1) we get from (3.16) the relation (e) as in 3.13.

3.4. PROOF OF THEOREM 2.4. The necessity of (E) is obvious by Theorem C. So we have to prove the sufficiency. We shall use the ideas and notations of 3.1 and 3.2. Let $m=n$, $\{y_{k,n}\}=\{x_{k,n}\}$,

$$\deg Q(x_{k,n};x)\leqq n\left(1+\frac{c}{2}\right)\quad\text{and}\quad s=\left[\frac{nc}{2(r+3)}\right].$$

We state with $\vartheta_{0,n}=0$ and $\vartheta_{n+1,n}=\pi$

$$\text{(3.19)}\qquad N_n(X;\vartheta_{k,n},\vartheta_{k+1,n})\leqq M\qquad(k=1,2,\dots,n;\quad n=1,2,\dots)\,.$$

Indeed, in the contrary case for a certain sequence $\{n_i\}$ and $\{\phi_{n_i}\}$ $(i = 1, 2, \ldots;$ $\lim\limits_{i\to\infty} \phi_{n_i} = \infty;\ \phi_{n_i} \leqq n_i)$ we have

$$N_n(X; \vartheta_{j(n),n}, \vartheta_{j(n)+1,n}) \geqq \phi_n \qquad (n = n_1, n_2, \ldots).$$

Considering the interval $[\alpha_n, \beta_n] = [\vartheta_{j,n}; \vartheta_{j+[\sqrt{\phi_n}],n}]$ (or $[\vartheta_{j-[\sqrt{\phi_n}],n}, \vartheta_{j+1,n}]$ if $j + \sqrt{\phi_n} > n$) we have $n(\beta_n - \alpha_n) \sim \sqrt{\phi_n}$, so

$$\overline{\lim_{n\to\infty}} \frac{N_n(X; \alpha_n, \beta_n)}{n(\beta_n - \alpha_n)} \geqq K \lim_{n_i\to\infty} \frac{\phi_{n_i}}{\sqrt{\phi_{n_i}}} = \infty,$$

which contradicts (E).

Suppose

$$\vartheta_{j_k,s} = \vartheta_{j_{k+1},s} = \ldots = \vartheta_{j_{k+l},s} \qquad (n \geqq n_0).$$

Then by (3.19) we can state $l \leqq M_1$ uniformly in n and k. The remaining part is similar to 3.3.

3.5. Proof of Theorem 2.5. **3.51.** Let now

$$s = \left[\frac{\pi c}{2i\,\delta_n}\right] \qquad (n \geqq n_0;\quad i \geqq r + 3 \quad \text{and even}) \tag{3.20}$$

and

$$h_{k,s}(x) = \left[\frac{(1 - x^2)\, a^2_{j_k,s}(x)}{(1 - x^2_{k,n})\, a^2_{j_k,s}(x_{k,n})}\right]^{\frac{i}{2}} \qquad (k = 1, 2, \ldots, n). \tag{3.21}$$

We form

$$M_n(f; x) = G_{N,r}(f; x) + \sum_{k=1}^{n} [f(x_{k,n}) - G_{N,r}(f; x_{k,n})]\, h_{k,s}(x)\, Q(x_{k,n}; x) \tag{3.22}$$

where $N = [\pi(1 + c)/\delta_n]$ and the polynomials Q are defined for the matrix Y constructed to the nodes $\tilde{X} = \{x_{k,n}\}$ $(k = 0, 1, \ldots, n + 1; n = 1, 2, \ldots)$ by the methods applied in 3.22. So we can suppose

$$\deg Q \leqq \left[\pi\left(1 + \frac{c}{2}\right)\Big/\delta_n\right].$$

Obviously we have the first two relations of (2.6). So we have to prove the third one.

3.52. Clearly we may suppose $x \neq x_{k,n}$ $(k = 0, 1, \dots, n+1)$. First let $\vartheta < \vartheta_{1,s}/2$ We have by $\vartheta_{j_k} \sim \vartheta_{j_k} \pm \vartheta$ and $\sin\vartheta = O(N^{-1})$ that

$$I = \left| \sum_{k=1}^{n} [f(x_k) - G_N(x_k)]\, h_{k,s}(x)\, Q(x_{k,n}\, x) \right| =$$

$$= O(1) \sum_{k=1}^{n} \left(\frac{\sin\xi_k}{N}\right)^r \omega\left(\frac{\sin\xi_k}{N}\right) \left[\frac{\sin\vartheta}{\sin\xi_k} \cdot \frac{\sin\vartheta_{j_k}}{N\sin\frac{\vartheta+\vartheta_{j_k}}{2}\sin\frac{\vartheta-\vartheta_{j_n}}{2}}\right]^i =$$

$$= O(1) \left(\frac{\sin\vartheta}{N}\right)^r \omega\left(\frac{\sin\vartheta}{N}\right) \sum_{k=1}^{n} \left(\frac{\sin\xi_k}{\sin\vartheta} + 1\right) \left(\frac{\sin\vartheta}{\sin\xi_k}\right)^{i-r} \frac{1}{(N\sin\vartheta_{j_k})^i} =$$

$$= O(1) \left(\frac{\sin\vartheta}{N}\right)^r \omega\left(\frac{\sin\vartheta}{N}\right) \sum_{k=1}^{n} \frac{1}{N^{i-r}} \frac{N^{i-r}}{k^2} = O(1) \left(\frac{\sin\vartheta}{N}\right)^r \omega\left(\frac{\sin\vartheta}{N}\right).$$

We get the same result for $\vartheta_{s,s} + \frac{\pi}{4s} < \vartheta < \pi$. For the remaining interval, applying

$$(3.23) \qquad \begin{cases} \dfrac{\sin\xi_k}{N} = O(1)\left(\dfrac{\sin\xi_k}{N} + \dfrac{1}{N^2}\right), \\ \sin\xi_{k,n} \sim \sin\vartheta_{j_k,s} \qquad (k = 1, 2, \dots, n), \\ \sin\vartheta_{j_l,s} \sim \sin\vartheta \qquad (|\vartheta - \vartheta_{j_l}| \leqq \pi/2s), \\ \sin\vartheta \leqq \sin\vartheta + \sin\vartheta_k \leqq 2\sin\dfrac{\vartheta+\vartheta_k}{2}, \end{cases}$$

we get the desired estimation by the method used in 3.32, considering $\sin\vartheta + N^{-1} = O(\sin\vartheta)$ for our ϑ's.

3.6. Proof of Theorem 2.6. The necessity of (E) for $\tilde{X}$ is obvious by $M_n(f; x_{k,n}) = f(x_{k,n})$ $(k = 0, 1, \dots, n+1)$ and Theorem C. We can prove the sufficiency as in 3.4, using $\tilde{X}$ instead of X.

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DECOMPOSITION OF UNIVERSAL FUNCTION SPACES

By
W. GOVAERTS[1] (Gent)

1. Introductory concepts. We continue the study of decomposition theorems by KAPLANSKY, BLAIR, BURRILL and SHORE in [1], [5], [6] in a generalized setting. For all notions from universal algebra we refer to [4]. Slightly modifying a definition in [2] we say that a universal algebra E is a (β)-algebra in an n-ary polynomial symbol $*$ of its type $(n\geqq 2)$, iff

(a) there exist mappings $\varphi_2, \dots \varphi_n$: $E\to E$ such that $a*\varphi_2(b)*\dots*\varphi_n(b)=b$ for all $a, b\in E$;

(b) there exist mappings $\psi_2, \dots \psi_n$: $E^2\to E$ such that $a*\psi_2(a, b)*\dots*\psi_n(a, b)=a$ and $b*\psi_2(a, b)*\dots*\psi_n(a, b)=b$ for all $a, b\in E$.

Furthermore, if E is a topological universal algebra, then it is said to be a topological (β)-algebra if in addition the mappings φ_k and ψ_k are all continuous.

EXAMPLE 1 ([2]). Consider a lattice $L, \vee, \wedge$; set $n=3$, $a*b*c=(a\vee b)\wedge c$, $\varphi_2(a)=\varphi_3(a)=a$, $\psi_2(a, b)=a\wedge b$, $\psi_3(a, b)=a\vee b$ for all $a, b, c\in L$.

EXAMPLE 2 ([2]). Let $R, +, \cdot$ be a ring with identity; set $n=3$, $a*b*c=a\cdot b+c$, $\varphi_2(a)=0$, $\varphi_3(a)=a$; $\psi_2(a, b)=1$, $\psi_3(a, b)=0$.

EXAMPLE 3. Let J be the fundamental interval [0, 1], provided with binary operation $(x, y)\to x\cdot y$ and unary operation $x\to 1-x$ (cf. [3]). Set $a*b*c=$ $=(1-(1-a)b)c$, $\varphi_2(a)=0$, $\varphi_3(a)=a$, $\psi_2(a, b)=\psi_3(a, b)=1$.

Example 2 is closely related to the so-called ternary rings. It may be generalized to the case where R has no identity if there exists a continuous mapping ξ: $R\to R$ such that $r\cdot\xi(r)=r$ for all r; one may define $a*b*c=a\cdot b+c$, $\varphi_2(a)=0$, $\varphi_3(a)=a$, $\psi_2(a, b)=\xi(a-b)$, $\psi_3(a, b)=a-a\cdot\xi(a-b)$. However, we know of no such examples for non-discrete rings R.

Two properties of (β)-algebras are essential, one of which is a generalization of the lemma in [6]:

2. Proposition. (a) *Let A, B be universal algebras of the same type and suppose that A is a (β)-algebra in the n-ary polynomial symbol $*$. If there exists a homomorphism from A onto B, then B is a (β)-algebra in the same polynomial symbol.* (b) *Let S, T be nonempty universal algebras of the same type, each being a (β)-algebra in the*

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polynomial symbol $*$, *with functions* $\varphi_k^{(1)}, \psi_k^{(1)}$ *and* $\varphi_k^{(2)}, \psi_k^{(2)}$ *respectively* $(2 \leqq k \leqq n)$. *Then* $S \times T$ *is a* (β)*-algebra in* $*$ *with functions* φ_k, ψ_k $(2 \leqq k \leqq n)$ *determined by*

$$\varphi_k(s, t) = (\varphi_k^{(1)}(s), \varphi_k^{(2)}(t)), \quad \psi_k((s, t), (s', t')) = (\psi_k^{(1)}(s, s'), \psi_k^{(2)}(t, t')).$$

Furthermore, for each congruence relation $\sim$ *in* $S \times T$ *there are congruence relations* $\sim_1$ *in* S *and* $\sim_2$ *in* T *such that*

(i) $s_1 \sim_1 s_2$ *iff there are* t, t' *in* T *such that* $(s_1, t) \sim (s_2, t')$;

(ii) $s_1 \sim_1 s_2$ *iff* $(s_1, t) \sim (s_2, t)$ *for all* $t \in T$;

(iii), (iv): *analogously for* $\sim_2$;

(v) $(S \times T)/\sim$ *is isomorphic to* $S/\sim_1 \times T/\sim_2$.

PROOF. (a) The easy demonstration may be found in [2], prop. 2.3.

(b) Clearly, $S \times T$ is a (β)-algebra in $*$ with functions φ_k and ψ_k as mentioned. Given a congruence relation $\sim$ in $S \times T$ we use (i) and (iii) to define $\sim_1$ and $\sim_2$. The "if"-part in (ii) is then immediate; now suppose that $(s_1, t) \sim (s_2, t')$ and let $t_0 \in T$ be arbitrary. Then

$$(s_1, t) * (\psi_2^{(1)}(s_1, s_2), \varphi_2^{(2)}(t_0)) * \dots * (\psi_n^{(1)}(s_1, s_2), \varphi_n^{(2)}(t_0)) \sim$$

$$\sim (s_2, t') * (\psi_2^{(1)}(s_1, s_2), \varphi_2^{(2)}(t_0)) * \dots * (\psi_n^{(1)}(s_1, s_2), \varphi_n^{(2)}(t_0))$$

or $(s_1, t_0) \sim (s_2, t_0)$. This establishes the "only if" part of (ii). From (i) and (ii) we conclude that $\sim_1$ is a congruence relation in S; (iii) and (iv) are obtained similarly.

A mapping θ from $(S \times T)/\sim$ onto $S/\sim_1 \times T/\sim_2$ is defined by $\theta((s, t)/\sim) = (s/\sim_1, t/\sim_2)$; the definition is made possible by (i) and (iii). If $(s/\sim_1, t/\sim_2) = (s'/\sim_1, t'/\sim_2)$, we conclude from (ii) and (iv) that $(s, t) \sim (s', t)$ and $(s', t) \sim (s', t')$, so that $(s, t) \sim (s', t')$; this proves that θ is one-to-one. Now θ is clearly an isomorphism.

3. Definitions. Let E be a topological (β)-algebra, X a topological space, $L \subseteqq C(X, E)$. For each $x \in X$ the spectrum of x relative to L is defined as $\mathrm{Sp}_L(x) = \{f(x) : f \in L\}$. L is said to be adequate (cf. [1] and [6]) if it is a subalgebra of $C(X, E)$ that is closed under composition with functions φ_k, ψ_k $(2 \leqq k \leqq n)$ and if each spectrum contains at least two points. Clearly an adequate subset is itself a (β)-algebra. A mapping Φ: $E_1 \times E_2 \to L$ is said to determine a decomposition of L if E_1, E_2 have the same type as E and if Φ is an isomorphism. By $f|_A$ we denote the restriction of a function f to a subset A of its domain of definition.

4. Theorem. *Let* L *be an adequate subset of* $C(X, E)$ *where* E *is a Hausdorff* (β)*-algebra; suppose that each spectrum is indecomposable (i.e. is not isomorphic to some direct product with factors each containing more than one point). If a mapping* $\Phi: E_1 \times E_2 \to L$ *determines a decomposition of* L, *then* X *may be written as* $X = X_1 \cup X_2$ *with* X_1, X_2 *open-and-closed disjoint subsets so that* E_i *is isomorphic to* $\{f|_{X_i} : f \in L\}$ $(i = 1, 2)$.

PROOF. Let Π_1, Π_2 denote the natural mappings from $E_1 \times E_2$ onto E_1 and E_2 respectively. Set

$$X_1 = \bigcap \{Z(f, g)\colon \Pi_1 \Phi^{-1}(f) = \Pi_1 \Phi^{-1}(g)\},\ X_2 = \bigcap \{Z(f, g)\colon \Pi_2 \Phi^{-1}(f) = \Pi_2 \Phi^{-1}(g)\}$$

where $Z(f, g) = \{x \in X\colon f(x) = g(x)\}$.

Since E is Hausdorff, X_1 and X_2 are closed in X. Suppose $x \in X_1 \cap X_2$. There are $f, g \in L$ such that $f(x) \neq g(x)$. Let $\Phi^{-1}(f) = (e_1, e_2)$, $\Phi^{-1}(g) = (e_1', e_2')$, $h = \Phi(e_1, e_2')$. Then $x \in Z(f, h) \cap Z(g, h)$, a contradiction.

To prove that $X_1 \cup X_2 = X$, choose $x_0 \in X$ arbitrary and let $\Pi_{x_0}\colon L \to E$ be defined by $\Pi_{x_0}(f) = f(x_0)$. If E is a topological (β)-algebra in $*$, it follows from part (a) of proposition 2 that E_1 and E_2 are (β)-algebras in $*$. Now let $\sim$ be induced in $E_1 \times E_2$ by the homomorphism $\Pi_{x_0} \circ \Phi$. There are congruence relations $\sim_1$ in E_1 and $\sim_2$ in E_2 that satisfy the conditions (i)—(iv) of proposition 2 (b); hence $E_1/\sim_1 \times E_2/\sim_2$ is isomorphic to $\mathrm{Sp}_L(x_0)$ from which we conclude that one of $\sim_1$, $\sim_2$ is the universal relation; let $\sim_2$ be so. If $f, g \in L$ with $\Pi_1 \Phi^{-1}(f) = \Pi_1 \Phi^{-1}(g)$, then $\Phi^{-1}(f) \sim \Phi^{-1}(g)$, whence $f(x_0) = g(x_0)$. This implies $x_0 \in Z(f, g)$ so that $x_0 \in X_1$.

Finally, a mapping Φ_1 may be defined from E_1 onto $\{f|_{X_1}\colon f \in L\}$ by assigning to each $e_1 \in E_1$ the restriction to X_1 of any $f \in L$ such that $\Pi_1 \Phi^{-1}(f) = e_1$. To prove that Φ_1 is one-to-one, suppose $\Phi_1(e_1) = \Phi_1(e_1')$ and let $e_2 \in E_2$ be arbitrary, $h = \Phi(e_1, e_2)$, $k = \Phi(e_1', e_2)$. Since h and k have the same restrictions to both X_1 and X_2, they coincide, whence $e_1 = e_1'$. A routine proof shows that Φ_1 is an isomorphism.

5. Corollaries. *Let E be a Hausdorff (β)-algebra that is algebraically indecomposable, X an arbitrary topological space. Then there is a one-to-one correspondence between all decompositions of $C(X, E)$ as a direct product and all decompositions of X as a union of two disjoint open-and-closed subsets. In particular, $C(X, E)$ is algebraically indecomposable iff X is connected.*

6. Final remarks. (a) When E is the topological algebraic product $E_1 \times E_2$ of two topological universal algebras E_1 and E_2, then each function space $C(X, E)$ has a decomposition into $C(X, E_1)$ and $C(X, E_2)$. Hence some indecomposability assumption about E seems likely.

(b) Some intrinsic algebraic conditions seem also necessary. As for an example, let E be the set $\mathbf{R}$ of all real numbers, considered as a vector space over itself (hence provided with addition, subtraction and all unary multiplications $x \to cx$ for $c \in \mathbf{R}$). Set $J = [0, 1]$, then $C(J, \mathbf{R})$ is a real vector space having many decompositions (which may be obtained theoretically by partitioning a Hamel basis of $C(J, \mathbf{R})$).

(c) As we remarked in [2], a universal algebra E is indecomposable if there is a binary operation $\circ$ such that one of the following conditions holds:

(1) $a \circ b = c$ implies $a = c$ or $b = c$ (lattice derived from a chain)

(2) there exists $e \in E$ such that $a \circ b = e$ if and only if $a = e$ or $b = e$ (ring without zero divisors; Example 3 of this paper).

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THE MAXIMAL k-FREE DIVISOR OF m WHICH IS PRIME TO n. II

By

D. SURYANARAYANA and P. SUBRAHMANYAM (Waltair)

§ 1. Introduction. Let k be a fixed integer $\geqq 2$. We recall that a positive integer m is called k-free if it is not divisible by the k-th power of any prime. Let Q_k denote the set of k-free integers. Let n denote a fixed positive integer and let $\gamma_k(m; n)$ denote the maximal k-free divisor of m which is prime to n. Let $\gamma_k(m)$ and $\delta_k(m)$ denote the maximal k-free divisor of m and the maximal odd k-free divisor of m, respectively. It is clear that $\gamma_k(m; 1)=\gamma_k(m)$ and $\gamma_k(m; 2)=\delta_k(m)$. In particular, when $k=2$, $\gamma_2(m; n)=\gamma(m; n)$, the maximal square-free divisor of m which is prime to n. Also, $\gamma_2(m)=\gamma(m; 1)=\gamma(m)$, the maximal square-free divisor (or the core) of m and $\delta_2(m)=\gamma(m; 2)=\delta(m)$, the maximal odd square-free divisor of m. We essentially follow the notation adopted in our earlier paper [16]. As stated at the end of [16], we establish in the present paper the asymptotic formulae for

$$\sum_{\substack{m \leqq n \\ (m,n)=1}} \frac{q_k(m)\varphi(m)}{m^2} \quad \text{and} \quad \sum_{m \leqq x} \frac{\gamma_k(m;\, n)}{m^2}$$

with uniform O-estimates for error terms (see § 4), where $\varphi(m)$ is the Euler totient function and $q_k(m)=1$ or 0 according as $m \in Q_k$ or $m \notin Q_k$. Also, we improve the O-estimates of the error terms on the assumption of the Riemann hypothesis. As particular cases of these asymptotic formulae, we deduce asymptotic formulae for

$$\sum_{m \leqq x} \frac{\gamma_k(m)}{m^2}, \quad \sum_{m \leqq x} \frac{\delta_k(m)}{m^2}.$$

We also discuss the case $k=2$ and make some remarks about the earlier work (if any) on the orders of the error terms. In fact, the orders of the error terms obtained in this paper are improved ones over those existing in the literature.

In § 2 we prepare the necessary background and prove some lemmas. In § 3 we prove some more lemmas which are needed in establishing the asymptotic formulae of § 4. Especially, Lemmas 3.1, 3.2 and 3.3 are general in nature which can be used in establishing some identities, which in turn are useful in deriving asymptotic formulae for a particular type of summatory functions. For example, Lemma 3.1 can be used to derive the identity (3.7) of our earlier paper [16], which has been used in deriving an asymptotic formula for the sum $\sum_{mn \leqq x} \varphi(mn)$, where the summation is extended over all positive integers m such that $mn \leqq x$. Lemma 3.2 can be used to derive Lemma 5.3 of [15] (stated below as Lemma 2.11), which in turn can be used to derive formulae (3) of [1] as was pointed out at the end of [15]. Lemma 3.2 can also be used to derive formula (2.5) given below. Also, Lemma 3.3 can be used to derive formula (2.6) given below and an interesting identity will be derived in Remark 3.3 below from which the identities derived by APOSTOL at the end of his paper [1] will follow.

§ 2. Preliminaries. Let $\mu(n)$ denote the Möbius function. It is well known that

$$q_k(n) = \sum_{d^k \delta = n} \mu(d). \tag{2.1}$$

Let $\psi(n)$ denote the Dedekind ψ-function (cf. [8], p. 123 or cf. [6]) and $J_k(n)$ denote the Jordan totient function (cf. [8], p. 147 or cf. [4]). The functions $\varphi(n)$, $\psi(n)$ and $J_k(n)$ have the following arithmetical forms:

$$\varphi(n) = \sum_{d\delta = n} \mu(d)\delta = n \prod_{p|n} \left(1 - \frac{1}{p}\right), \tag{2.2}$$

$$\psi(n) = \sum_{d\delta = n} \mu^2(d)\delta = n \prod_{p|n} \left(1 + \frac{1}{p}\right), \tag{2.3}$$

$$J_k(n) = \sum_{d\delta = n} \mu(d)\delta^k = n^k \prod_{p|n} \left(1 - \frac{1}{p^k}\right). \tag{2.4}$$

REMARK 2.1. It is clear that $\varphi(n) \leqq n$, $\psi(n) \geqq n$ and $\frac{1}{J_k(n)} = O\left(\frac{1}{n^k}\right)$, since

$$J_k(n) > n^k \prod_p \left(1 - \frac{1}{p^k}\right) = \frac{n^k}{\zeta(k)}$$

(cf. [9], Theorem 280), where $\zeta(s)$ is the Riemann Zeta function defined by $\zeta(s) = \sum_{m=1}^{\infty} m^{-s}$ for $s > 1$.

LEMMA 2.1 (cf. [5], Lemma 5.1 or cf. [11], Lemma 2.3, $s=2$).

$$\sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \frac{\mu(m)}{m^2} = \frac{n^2}{\zeta(2) J_2(n)} = \frac{n^2}{\zeta(2)\varphi(n)\psi(n)}. \tag{2.5}$$

LEMMA 2.2 (cf. [11], Lemma 2.5, $s=2$).

$$\sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \frac{\mu(m) \log m}{m^2} = \frac{n^2}{\zeta(2)\varphi(n)\psi(n)} \left\{\beta(n) + \frac{\zeta'(2)}{\zeta(2)}\right\}, \tag{2.6}$$

where

$$\beta(n) \equiv -\frac{n^2}{J_2(n)} \sum_{d|n} \frac{\mu(d)\log d}{d^2} = \begin{cases} \sum_{p|n} \frac{\log p}{p^2 - 1}, & \text{if } n > 1 \\ 0, & \text{if } n = 1. \end{cases} \tag{2.7}$$

Throughout the following x denotes real variable $\geqq 1$, unless otherwise stated.

LEMMA 2.3 (cf. [2], p. 71). *For* $x \geqq 2$,

$$\sum_{m \leqq x} \frac{\varphi(m)}{m^2} = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2}\left(\log\frac{x}{m}+\gamma\right)+O\left(\frac{\log x}{x}\right) = \tag{2.8}$$

$$= \frac{1}{\zeta(2)}\left(\log x+\gamma-\frac{\zeta'(2)}{\zeta(2)}\right)+O\left(\frac{\log x}{x}\right),$$

where γ *is Euler's constant and* $\zeta'(s)$ *is the derivative of the Riemann Zeta function* $\zeta(s)$.

LEMMA 2.4. *For* $x \geqq 3$,

$$\sum_{m \leqq x} \frac{\varphi(m)}{m^2} = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2}\left(\log\frac{x}{m}+\gamma\right)+O\left(\frac{\lambda(x)}{x}\right) = \tag{2.9}$$

$$= \frac{1}{\zeta(2)}\left(\log x+\gamma-\frac{\zeta'(2)}{\zeta(2)}\right)+O\left(\frac{\lambda(x)}{x}\right),$$

where $\lambda(x)$ *is the function defined by*

$$\lambda(x) = \begin{cases} \log^{2/3} x\,(\log\log x)^{4/3} & \text{for} \quad x \geqq 3 \\ 1 \quad \text{for} \quad 0 < x < 3. \end{cases} \tag{2.10}$$

REMARK 2.2. We note that $\lambda(x)$ is monotonic increasing and $\dfrac{\lambda(x)}{x^{\varepsilon}}$ is monotonic decreasing for every $\varepsilon > 0$.

PROOF. It is known (cf. [18], Satz. 1, p. 144) that

$$\Phi(x) = \sum_{m \leqq x} \varphi(m) = \frac{x^2}{2\zeta(2)}+O(x\lambda(x)). \tag{2.11}$$

Hence by partial summation (cf. [9], Theorem 421), we get

$$\sum_{m \leqq x} \frac{\varphi(m)}{m^2} = \frac{\Phi(x)}{x^2}+2\int_1^x \frac{\Phi(t)}{t^3}\,dt = \tag{2.12}$$

$$= \frac{1}{2\zeta(2)}+O\left(\frac{\lambda(x)}{x}\right)+2\int_1^x \left\{\frac{1}{2\zeta(2)}\cdot\frac{1}{t}+E(t)\right\}dt,$$

where $E(t)=O\left(\dfrac{\lambda(t)}{t^2}\right)$.

By Remark 2.2, it follows that

$$\int_x^{\infty} E(t)\,dt = O\left(\int_x^{\infty} \frac{\lambda(t)}{x^{\varepsilon}}\cdot\frac{1}{t^{2-\varepsilon}}\,dt\right) = O\left(\frac{\lambda(x)}{x^{\varepsilon}}\int_x^{\infty} \frac{dt}{t^{2-\varepsilon}}\right) = O\left(\frac{\lambda(x)}{x}\right).$$

Hence by (2.12), we get

$$\sum_{m\leqq x}\frac{\varphi(m)}{m^2}=\frac{1}{\zeta(2)}(\log x+C)+O\left(\frac{\lambda(x)}{x}\right), \tag{2.13}$$

where $C=\frac{1}{2}+2\zeta(2)\int\limits_1^\infty E(t)\,dt$.

Now, comparing (2.13) with (2.8), we find that $C=\gamma-\frac{\zeta'(2)}{\zeta(2)}$. Substituting this value of C in (2.13), we get (2.9). Hence Lemma 2.4 follows.

LEMMA 2.5. (cf. [13], Lemma 3).

$$\sum_{d|n}\frac{\mu(d)}{\psi(d)}=\frac{n}{\psi(n)}.$$

LEMMA 2.6.

$$A(n)\equiv-\frac{\psi(n)}{n}\sum_{d|n}\frac{\mu(d)\log d}{\psi(d)}=\begin{cases}\sum\limits_{p|n}\frac{\log p}{p}, & \textit{if}\quad n>1\\ 0, & \textit{if}\quad n=1.\end{cases}$$

LEMMA 2.7.

$$B(n)\equiv-\frac{\psi(n)}{n}\sum_{d|n}\frac{\mu(d)\beta(d)}{\psi(d)}=\begin{cases}\sum\limits_{p|n}\frac{\log p}{p(p^2-1)}, & \textit{if}\quad n>1\\ 0, & \textit{if}\quad n=1.\end{cases}$$

REMARK 2.3. Proofs of Lemmas 2.6 and 2.7 can be given in the same way as the proof of the following which was given originally in (cf. [7], pp. 293—294) and then in [3]:

$$\alpha(n)\equiv-\frac{n}{\varphi(n)}\sum_{d|n}\frac{\mu(d)\log d}{d}=\begin{cases}\sum\limits_{p|n}\frac{\log p}{p-1}, & \text{if}\quad n>1\\ 0, & \text{if}\quad n=1.\end{cases} \tag{2.14}$$

In fact, H. DAVENPORT (cf. [7], p. 293) used the notation $w(n)$ for $\alpha(n)$ defined above.

Let $H_k(n)$ be the arithmetical function defined by $H_k(1)=1$ and

$$H_k(n)=n^k\prod_{p|n}\left(1-\frac{1}{p^{k-1}(p+1)}\right)\quad\text{for}\quad n>1. \tag{2.15}$$

REMARK 2.4. It is clear that $H_k(n)>\alpha_k n^k$, where α_k is the constant defined by

$$\alpha_k=\prod_p\left(1-\frac{1}{p^{k-1}(p+1)}\right), \tag{2.16}$$

the product being extended over all primes p.

Let $A_k(n)$ and $B_k(n)$ be the arithmetical functions defined by

$$A_k(n)=\begin{cases}\sum\limits_{p|n}\frac{\log p}{\{p^{k-1}(p+1)-1\}}, & \text{if}\quad n>1\\ 0, & \text{if}\quad n=1,\end{cases} \tag{2.17}$$

and

$$(2.18)\qquad B_k(n) = \begin{cases} \sum_{p|n} \dfrac{\log p}{(p^2-1)\{p^{k-1}(p+1)-1\}}, & \text{if } n>1 \\ 0, & \text{if } n=1. \end{cases}$$

Let s be any positive real number and let $H(s, n)$, $A(s, n)$ and $B(s, n)$ be the arithmetical functions given by

$$(2.19)\qquad H(s, n) \equiv n^s \sum_{d|n} \frac{\mu(d)}{\psi(d) d^{s-1}} = n^s \prod_{p|n} \left(1-\frac{1}{p^{s-1}(p+1)}\right).$$

$$(2.20)\quad A(s, n) \equiv -\frac{n^s}{H(s, n)} \sum_{d|n} \frac{\mu(d)\log d}{\psi(d)\, d^{s-1}} = \begin{cases} \sum_{p|n} \dfrac{\log p}{\{p^{s-1}(p+1)-1\}}, & \text{if } n>1 \\ 0, & \text{if } n=1. \end{cases}$$

$$(2.21)\quad B(s, n) \equiv -\frac{n^s}{H(s, n)} \sum_{d|n} \frac{\mu(d)\beta(d)}{\psi(d) d^{s-1}} = \begin{cases} \sum_{p|n} \dfrac{\log p}{(p^2-1)\{p^{s-1}(p+1)-1\}}, & \text{if } n>1 \\ 0, & \text{if } n=1. \end{cases}$$

Taking logarithms of both sides of (2.19) and then differentiating with respect to s, we get

$$(2.22)\qquad H'(s, n) = H(s, n)\{\log n + A(s, n)\}.$$

It is clear that

$$(2.23)\qquad H(1, n) = \frac{n^2}{\psi(n)} \quad \text{and} \quad H(k, n) = H_k(n),$$

$$(2.24)\qquad A(1, n) = A(n) \quad \text{and} \quad A(k, n) = A_k(n),$$

$$(2.25)\qquad B(1, n) = B(n) \quad \text{and} \quad B(k, n) = B_k(n).$$

REMARK 2.5. Proofs of (2.20) and (2.21) can be given in the same way as it was given for (2.14) in [3].

LEMMA 2.8. *For* $s>1$ *and* $n \geqq 1$,

$$(2.26)\qquad \sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \frac{\mu(m)}{\psi(m) m^{s-1}} = \prod_p \left(1-\frac{1}{p^{s-1}(p+1)}\right) \frac{n^s}{H(s, n)}.$$

PROOF. The sery is absolutely convergent for $s>1$, since $\psi(m)>m$ and the general term of the series is a multiplicative function of m. Hence by expanding the series into an infinite product of Euler type (cf. [9], Theorem 286), we get (2.26), by making use of (2.19).

LEMMA 2.9. *For* $s>1$ *and* $n \geqq 1$,

(2.27)

$$\sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \frac{\mu(m)\log m}{\psi(m) m^{s-1}} = \prod_p \left(1-\frac{1}{p^{s-1}(p+1)}\right) \frac{n^s}{H(s, n)} \left\{A(s, n) - \sum_p \frac{\log p}{p^{s-1}(p+1)-1}\right\}.$$

PROOF. The series is uniformly convergent for $s \geqq 1+\varepsilon > 1$ and so by termwise differentiation of the series in (2.26) with respect to s, we get (2.27). In finding out the derivative of the right side expression of (2.26), formula (2.22) is made use of. This completes the proof of (2.27).

Let $\sigma_t^*(n)$ denote the sum of the t-th powers of the square-free divisors of n and let $\theta(n)$ denote the number of square-free divisors of n. It is clear that $\sigma_0^*(n)=\theta(n)=2^{\nu(n)}$, where $\nu(n)$ is the number of distinct prime factors of $n>1$, $\nu(1)=0$.

Throughout the following ε denotes any pre-assigned positive real number. All the O-estimates that appear in this paper are independent of x and n (and of k also, if it appears in the formula), but might depend on ε (in case a term involving ε appears in the O-estimate). We describe this situation by mentioning the word "uniformly" at the end of each asymptotic formula.

LEMMA 2.10 (cf. [14], Lemma 3.5). *For* $x \geqq 3$ *and* $n \geqq 1$,

$$M_n(x) = \sum_{\substack{m \leqq x \\ (m,n)=1}} \mu(m) = O\big(\sigma^*_{-1+\varepsilon}(n)\, x\delta(x)\big), \tag{2.28}$$

uniformly, where $\delta(x)$ *is the function defined by*

$$\delta(x) = \begin{cases} \exp\{-A \log^{3/5} x\, (\log\log x)^{-1/5}\} & \text{for } x \geqq 3 \\ 1 \quad \text{for} \quad 0 < x < 3; \end{cases} \tag{2.29}$$

A being an absolute positive constant.

REMARK 2.6. If h is a positive constant, then it is clear that $\delta(x)\log^h x = O\big(\exp\{-A' \log^{3/5} x\, (\log\log x)^{-1/5}\}\big)$, where A' is an absolute constant such that $0 < A' < A$. So, we may replace $\delta(x)\log^h x$ appearing in any O-term by $\delta(x)$ itself. Also, we note that $\delta(x)$ is monotonic decreasing and $x^\varepsilon \delta(x)$ is monotonic increasing for every $\varepsilon > 0$.

REMARK 2.7. Sometimes it is convenient to replace $\sigma^*_{-1+\varepsilon}(n)$ appearing in any O-term by $\theta(n)$ or $\tau(n)$. Clearly, $\sigma^*_{-1+\varepsilon}(n) \leqq \theta(n) \leqq \tau(n)$, where $\tau(n)$ is the number of all divisors of n. It is well known that $\tau(n) = O(n^\varepsilon)$ for every $\varepsilon > 0$ (cf. [9], Theorem 315).

LEMMA 2.11 (cf. [15], Lemma 5.3). $\displaystyle\sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \frac{\mu(m)}{m} = 0$ *for all* $n \geqq 1$.

LEMMA 2.12. *For* $x \geqq 3$ *and* $n \geqq 1$,

$$L_n(n) \equiv \sum_{\substack{m \leqq x \\ (m,n)=1}} \frac{\mu(m)}{m} = O\big(\sigma^*_{-1+\varepsilon}(n)\delta(x)\log x\big), \tag{2.30}$$

uniformly.

PROOF. It follows by Lemma 2.11 that

$$\sum_{\substack{m \leqq x \\ (m,n)=1}} \frac{\mu(m)}{m} = -\sum_{\substack{m > x \\ (m,n)=1}} \frac{\mu(m)}{m}. \tag{2.31}$$

Now, writing $f(m)=\frac{1}{m}$, we see that $f(m+1)-f(m)=O\left(\frac{1}{m^2}\right)$. By partial summation and Lemma 2.10, we have

$$(2.32)\quad L_n(x)=\sum_{\substack{m>x\\(m,n)=1}}\frac{\mu(m)}{m}=M_n(x)f([x]+1)+\sum_{m>x}M_n(m)\{f(m+1)-f(m)\}=$$
$$=O\big(\sigma^*_{-1+\varepsilon}(n)\delta(x)\big)+O\left(\sigma^*_{-1+\varepsilon}(n)\sum_{m>x}\frac{\delta(m)}{m}\right).$$

It is easy to see that $\delta(m)\log^2 m$ is monotonic decreasing for sufficiently large m, so that

$$\sum_{m>x}\frac{\delta(m)}{m}=\sum_{m>x}\frac{\delta(m)\log^2 m}{m\log^2 m}\leqq\delta(x)\log^2 x\sum_{m>x}\frac{1}{m\log^2 m}=O\big(\delta(x)\log x\big).$$

Hence by (2.32) and the above, Lemma 2.12 follows.

LEMMA 2.13. *For* $x\geqq 3$ *and* $n\geqq 1$,

$$(2.33)\quad S_n(x)=\sum_{\substack{m\leqq x\\(m,n)=1}}\frac{\mu(m)}{\psi(m)}=O\big(\sigma^*_{-1+\varepsilon}(n)\delta(x)\log x\big),$$

uniformly.

PROOF. By Lemma 2.5 and Lemma 2.2, we have

$$\sum_{\substack{m\leqq x\\(m,n)=1}}\frac{\mu(m)}{\psi(m)}=\sum_{\substack{m\leqq x\\(m,n)=1}}\frac{\mu(m)}{m}\sum_{d\delta=m}\frac{\mu(d)}{\psi(d)}=\sum_{\substack{d\delta\leqq x\\(d\delta,n)=1}}\frac{\mu(d\delta)\mu(d)}{\psi(d)\,d\delta}=$$
$$=\sum_{\substack{d\delta\leqq x\\(d,\delta)=1\\(d,n)=(d,n)=1}}\frac{\mu^2(d)\mu(d)\mu(\delta)}{\psi(d)\,d\delta}=\sum_{\substack{d\leqq x\\(d,n)=1}}\frac{\mu(d)}{\psi(d)d}\sum_{\substack{\delta\leqq x/d\\(\delta,dn)=1}}\frac{\mu(\delta)}{\delta}=\sum_{\substack{d\leqq x\\(d,n)=1}}\frac{\mu(d)}{\psi(d)d}L_{dn}\left(\frac{x}{d}\right)=$$
$$=O\left(\sum_{\substack{d\leqq x\\(d,n)=1}}\frac{\mu^2(d)}{\psi(d)d}\sigma^*_{-1+\varepsilon}(dn)\delta\left(\frac{x}{d}\right)\log\left(\frac{x}{d}\right)\right)=O\left(\sigma^*_{-1+\varepsilon}(n)\log x\sum_{d\leqq x}\frac{\sigma^*_{-1+\varepsilon}(d)}{\psi(d)\,d}\delta\left(\frac{x}{d}\right)\right).$$

Now, by Remarks 2.1, 2.6 and 2.7, we obtain

$$\sum_{d\leqq x}\frac{\sigma^*_{-1+\varepsilon}(d)}{\psi(d)d}\delta\left(\frac{x}{d}\right)=\frac{1}{x^\varepsilon}\sum_{d\leqq x}\frac{\tau(d)}{d^{2-\varepsilon}}\left(\frac{x}{d}\right)^\varepsilon\delta\left(\frac{x}{d}\right)\leqq\frac{1}{x^\varepsilon}x^\varepsilon\delta(x)\sum_{d\leqq x}\frac{\tau(d)}{d^{2-\varepsilon}}=O\big(\delta(x)\big).$$

Hence Lemma 2.13 follows.

LEMMA 2.14. *For* $s>1$, $x\geqq 3$ *and* $n\geqq 1$,

$$(2.34)\quad \sum_{\substack{m>x\\(m,n)=1}}\frac{\mu(m)}{\psi(m)m^{s-1}}=O\left(\frac{\sigma^*_{-1+\varepsilon}(n)\delta(x)\log x}{x^{s-1}}\right),$$

uniformly.

PROOF. Writing $f(m)=\frac{1}{m^{s-1}}$, we see that $f(m+1)-f(m)=O\left(\frac{1}{m^s}\right)$. By partial summation, Lemma 2.13 and Remark 2.6, we obtain

$$\sum_{\substack{m>x\\(m,n)=1}}\frac{\mu(m)}{\psi(m)m^{s-1}}=-S_n(x)f([x]+1)-\sum_{m>x}S_n(x)\{f(m+1)-f(m)\}=$$

$$=O\left(\frac{\sigma^*_{-1+\varepsilon}(n)\delta(x)\log x}{x^{s-1}}\right)+O\left(\sigma^*_{-1+\varepsilon}(n)\sum_{m>x}\frac{\delta(m)\log m}{m^s}\right)=$$

$$=O\left(\frac{\sigma^*_{-1+\varepsilon}(n)\delta(x)\log x}{x^{s-1}}\right)+O\left(\frac{\sigma^*_{-1+\varepsilon}(n)\delta(x)\log x}{x^{s-1}}\right).$$

Hence Lemma 2.14 follows.

LEMMA 2.15. *For* $s>1$, $x\geqq 3$ *and* $n\geqq 1$,

$$\sum_{\substack{m>x\\(m,n)=1}}\frac{\mu(m)\log m}{\psi(m)m^{s-1}}=O\left(\frac{\sigma^*_{-1+\varepsilon}(n)\delta(x)\log^2 x}{x^{s-1}}\right), \tag{2.35}$$

uniformly.

PROOF. Writing $g(m)=\frac{\log m}{m^{s-1}}$, we see that $g(m+1)-g(m)=O\left(\frac{\log m}{m^s}\right)$. Using partial summation, Lemma 2.13 and Remark 2.6, we get this Lemma following the same procedure adopted in Lemma 2.14.

LEMMA 2.16. *For* $x\geqq 3$ *and* $n\geqq 1$,

$$\sum_{\substack{m\leqq x\\(m,n)=1}}\frac{\mu(m)}{\psi(m)m^{k-1}}=\frac{\alpha_k n^k}{H_k(n)}+O\left(\frac{\sigma^*_{-1+\varepsilon}(n)\delta(x)}{x^{k-1}}\right), \tag{2.36}$$

$$\sum_{\substack{m\leqq x\\(m,n)=1}}\frac{\mu(m)\log m}{\psi(m)m^{k-1}}=\frac{\alpha_k n^k}{H_k(n)}\left\{A_k(n)-\sum_p\frac{\log p}{p^{k-1}(p+1)-1}\right\}+O\left\{\frac{\sigma^*_{-1+\varepsilon}(n)\delta(x)}{x^{k-1}}\right\}, \tag{2.37}$$

uniformly, where α_k *is the constant given by* (2.16).

PROOF. (2.36) follows from (2.26) and (2.34) for $s=k$, (2.23) and Remark 2.6 Also, (2.37) follows from (2.27) and (2.35) for $s=k$; (2.23), (2.24) and Remark 2.6.

REMARK 2.8. An alternative proof of (2.36) has been given in our earlier paper (cf [16], Lemma 2.5).

LEMMA 2.17. *For* $x\geqq 3$ *and* $n\geqq 1$,

(2.38)

$$\sum_{\substack{m\leqq x\\(m,n)=1}}\frac{\mu(m)\beta(m)}{\psi(m)m^{k-1}}=\frac{\alpha_k n^k}{H_k(n)}\left\{B_k(n)-\sum_p\frac{\log p}{(p^2-1)\{p^{k-1}(p+1)-1\}}\right\}+O\left(\frac{\sigma^*_{-1+\varepsilon}(n)\delta(x)}{x^{k-1}}\right),$$

uniformly, where $B_k(n)$ *is given by* (2.18).

PROOF. We have by (2.7) and (2.36),

(2.39)

$$\sum_{\substack{m\leqq x\\(m,n)=1}}\frac{\mu(m)\beta(m)}{\psi(m)m^{k-1}}=\sum_{\substack{m\leqq x\\(m,n)=1}}\frac{\mu(m)}{\psi(m)\,m^{k-1}}\sum_{p\delta=m}\frac{\log p}{p^2-1}=\sum_{\substack{p\delta\leqq x\\(p\delta,n)=1}}\frac{\mu(p\delta)\log p}{\psi(p\delta)\,p^{k-1}\delta^{k-1}(p^2-1)}=$$

$$=\sum_{\substack{p\delta\leqq x\\(p,\delta)=1\\(p,n)=(\delta,n)=1}}\frac{\mu(p)\mu(\delta)\log p}{\psi(p)\psi(\delta)\,p^{k-1}\delta^{k-1}(p^2-1)}=$$

$$=-\sum_{\substack{p\leqq x\\(p,n)=1}}\frac{\log p}{(p+1)\,p^{k-1}(p^2-1)}\sum_{\substack{\delta\leqq x/p\\(\delta,pn)=1}}\frac{\mu(\delta)}{\psi(\delta)\delta^{k-1}}=$$

$$=-\sum_{\substack{p\leqq x\\(p,n)=1}}\frac{\log p}{(p+1)\,p^{k-1}(p^2-1)}\left\{\frac{\alpha_k p^k n^k}{H_k(p)\,H_k(n)}+O\left(\frac{\sigma^*_{-1+\varepsilon}(pn)\delta\left(\frac{x}{p}\right)}{(x/p)^{k-1}}\right)\right\}=$$

$$=-\frac{\alpha_k n^k}{H_k(n)}\sum_{\substack{p\leqq x\\(p,n)=1}}\frac{p\log p}{H_k(p)(p+1)(p^2-1)}+$$

$$+O\left(\sigma^*_{-1+\varepsilon}(n)\sum_{p\leqq x}\frac{\sigma^*_{-1+\varepsilon}(p)(p/x)^{k-1}\delta(x/p)\log p}{(p+1)\,p^{k-1}(p^2-1)}\right).$$

Since $(p+1)p^{k-1}(p^2-1)>p^{k+2}$, we get by Remarks 2.6 and 2.7 that for $0<\varepsilon<1$, the O-term in the above is

$$O\left(\sigma^*_{-1+\varepsilon}(n)\sum_{p\leqq x}\frac{\sigma^*_{-1+\varepsilon}(p)(p/x)^{k-1+\varepsilon}(x/p)^{\varepsilon}\delta(x/p)\log p}{p^{k+2}}\right)=$$

$$=O\left(\frac{\sigma^*_{-1+\varepsilon}(n)x^{\varepsilon}\delta(x)}{x^{k-1+\varepsilon}}\sum_{p\leqq x}\frac{\tau(p)\log p}{p^{3-\varepsilon}}\right)=O\left(\frac{\sigma^*_{-1+\varepsilon}(n)\delta(x)}{x^{k-1}}\right).$$

Also, by (2.15) and Remark 2.4 the first term in (2.39) is

(2.40)
$$=-\frac{\alpha_k n^k}{H_k(n)}\sum_{\substack{p\leqq x\\(p,n)=1}}\frac{\log p}{(p^2-1)\{p^{k-1}(p+1)-1\}}=$$

$$=-\frac{\alpha_k n^k}{H_k(n)}\sum_{\substack{p\\(p,n)=1}}\frac{\log p}{(p^2-1)\{p^{k-1}(p+1)-1\}}+O\left(\sum_{p>x}\frac{\log p}{p^{k+2}}\right)=$$

$$=\frac{\alpha_k n^k}{H_k(n)}\left\{B_k(n)-\sum_{p}\frac{\log p}{(p^2-1)\{p^{k-1}(p+1)-1\}}\right\}+O\left(\frac{\log x}{x^{k+1}}\right).$$

Hence by (2.39) and (2.40), Lemma 2.17 follows.

§ 3. Auxiliary results. In this section we prove some more lemmas which are needed in our present discussion. We first prove the following:

LEMMA 3.1. *Let $f(m)$ be a completely multiplicative arithmetical function and $g(m)$ be any arithmetical function. Let*

$$F(m) = f(m) \prod_{p|m} g(p), \tag{3.1}$$

where the product is extended over all prime divisors p of n. Also let

$$G_n(x) = \sum_{mn \leqq x} F(mn), \tag{3.2}$$

where the summation is extended over all positive integers m such that $mn \leqq x$, n being a fixed positive integer. Then for any prime p such that $(p, n)=1$ and any integer $\alpha \geqq 1$, we have

$$G_{np^\alpha}(x) = F(p^\alpha) G_n\left(\frac{x}{p^\alpha}\right) + F(p)\big(f(p^\alpha) - F(p^\alpha)\big) \sum_{r=0}^{c-1} (f(p) - F(p))^r G_n\left(\frac{x}{p^{r+\alpha+1}}\right), \tag{3.3}$$

where $c = [\log_p x]$.

PROOF. We have by (3.2)

$$G_{np^\alpha}(x) = \sum_{mnp^\alpha \leqq x} F(mnp^\alpha) = \sum_{\substack{mnp^\alpha \leqq x \\ (p,m)=1}} F(mnp^\alpha) + \sum_{\substack{mnp^\alpha \leqq x \\ p|m}} F(mnp^\alpha). \tag{3.4}$$

Since $f(n)$ is completely multiplicative and $\prod_{p|n} g(p)$ is multiplicative, it follows by (3.1) that $F(n)$ is multiplicative. Hence for $(p, m)=1$, we have $F(mnp^\alpha) = F(mn) F(p^\alpha)$. Also, for $p|m$, we have

$$F(mnp^\alpha) = f(mnp^\alpha) \prod_{q|mnp^\alpha} g(q) = f(mn) f(p^\alpha) \prod_{q|mn} g(q) =$$

$$= f(mn) f(p^\alpha) F(mn)/f(mn) = f(p^\alpha) F(mn).$$

Hence by (3.4), we get

$$G_{np^\alpha}(x) = F(p^\alpha) \sum_{\substack{mnp^\alpha \leqq x \\ (p,m)=1}} F(mn) + f(p^\alpha) \sum_{\substack{mnp^\alpha \leqq x \\ p|m}} F(mn) = \tag{3.5}$$

$$= F(p^\alpha) \Big\{ \sum_{mnp^\alpha \leqq x} F(mn) - \sum_{\substack{mnp^\alpha \leqq x \\ p|m}} F(mn) \Big\} + f(p^\alpha) \sum_{\substack{mnp^\alpha \leqq x \\ p|m}} F(mn) =$$

$$= F(p^\alpha) \sum_{mn \leqq \frac{x}{p^\alpha}} F(mn) + \big(f(p^\alpha) - F(p^\alpha)\big) \sum_{tpn \leqq \frac{x}{p^\alpha}} F(tpn) =$$

$$= F(p^\alpha) G_n\left(\frac{x}{p^\alpha}\right) + \big(f(p^\alpha) - F(p^\alpha)\big) G_{np}\left(\frac{x}{p^\alpha}\right).$$

Putting $\alpha = 1$ in (3.5), we get

$$G_{np}(x) = F(p) G_n\left(\frac{x}{p}\right) + \big(f(p) - F(p)\big) G_{np}\left(\frac{x}{p}\right). \tag{3.6}$$

Now, substituting $\frac{x}{p}, \frac{x}{p^2}, \dots, \frac{x}{p^{c-1}}$ in (3.7) for x, where $c=[\log_p x]$ and then simplifying, we get

(3.7)
$$G_{np}(x) = F(p)G_n\left(\frac{x}{p}\right)+F(p)\sum_{r=1}^{c-1}(f(p)-F(p))^r G_n\left(\frac{x}{p^{r+1}}\right)+(f(p)-F(p))^c G_n\left(\frac{x}{p^c}\right) =$$
$$= F(p)\sum_{r=0}^{c-1}(f(p)-F(p))^r G_n\left(\frac{x}{p^{r+1}}\right),$$

since
$$G_{np}\left(\frac{x}{p^c}\right) = \sum_{mnp\le \frac{x}{p^c}} F(mnp) = \sum_{mn\le \frac{x}{p^{c+1}}} F(mnp) = 0$$

for
$$(x/p^{c+1}) < 1,\ c = [\log_p x].$$

Substituting $\frac{x}{p^\alpha}$ for x in (3.7), we get

$$G_{np}\left(\frac{x}{p^\alpha}\right) = F(p)\sum_{r=0}^{c-1}(f(p)-F(p))^r G_n\left(\frac{x}{p^{r+\alpha+1}}\right). \tag{3.8}$$

Now, (3.3) follows by (3.5) and (3.8).

REMARK 3.1. Since $G_n\left(\frac{x}{p^{r+\alpha+1}}\right)=0$ for $r\ge c$, we can rewrite (3.3) as

$$G_{np^\alpha}(x) = F(p^\alpha)G_n\left(\frac{x}{p^\alpha}\right)+F(p)(f(p^\alpha)-F(p^\alpha))\sum_{r=0}^{\infty}(f(p)-F(p))^r G_n\left(\frac{x}{p^{r+\alpha+1}}\right). \tag{3.9}$$

LEMMA 3.2. *Let $h(m)$ be a multiplicative arithmetical function and let*

$$U_n = \sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \mu(m)h(m). \tag{3.10}$$

Then for any prime p such that $(p,n)=1$ and any integer $\alpha\ge 1$, we have

$$(1-h(p))U_{np^\alpha} = U_n. \tag{3.11}$$

PROOF. We have

$$\sum_{\substack{m=1 \\ (m,np^\alpha)=1}}^{\infty} \mu(m)h(m) = \sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \mu(m)h(m) = \sum_{\substack{m=1 \\ (m,n)=1 \\ p|m}}^{\infty} \mu(m)h(m) = \tag{3.12}$$
$$= U_n - \sum_{\substack{t=1 \\ (t,n)=1}}^{\infty} \mu(pt)\mu(pt) = U_n + h(p)\sum_{\substack{t=1 \\ (t,np^\alpha)=1}}^{\infty} \mu(t)h(t).$$

Hence (3.11) follows by (3.10) and (3.12).

LEMMA 3.3. *Let $h(m)$ be multiplicative and let*

$$V_n = \sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \mu(m)h(m)\log m. \tag{3.13}$$

Then for any prime p such that $(p,n)=1$ and any integer $\alpha \geqq 1$, we have

$$(1-h(p))V_{np^\alpha} = V_n + h(p)\log p\, U_{np^\alpha}. \tag{3.14}$$

PROOF. We have

$$\sum_{\substack{m=1 \\ (m,np^\alpha)=1}}^{\infty} \mu(m)h(m)\log m = \sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \mu(m)h(m)\log m - \sum_{\substack{m=1 \\ (m,n)=1 \\ p|m}}^{\infty} \mu(m)h(m)\log m = \tag{3.15}$$

$$= V_n - \sum_{\substack{t=1 \\ (t,n)=1}}^{\infty} \mu(pt)h(pt)\log pt = V_n + h(p) \sum_{\substack{t=1 \\ (t,np^\alpha)=1}}^{\infty} \mu(t)h(t)\{\log p + \log t\}.$$

Hence (3.14) follows by (3.10), (3.13) and (3.15).

REMARK 3.2. By (3.11) and (3.14), we obtain

$$V_{np^\alpha} = \frac{1}{1-h(p)} V_n + \frac{h(p)\log p}{(1-h(p))^2} U_n. \tag{3.16}$$

REMARK 3.3. It is well known (cf. [10], § 156—§ 159) that $\sum_{m=1}^{\infty} \frac{\mu(m)}{m} = 0$ and $\sum_{m=1}^{\infty} \frac{\mu(m)\log m}{m} = -1$. Hence taking $h(m) = \frac{1}{m}$ in Lemmas 3.2 and 3.3, we see from the above known results that $U_1=0$ and $V_1=-1$. Using induction on n, we can easily prove that $U_n=0$ for all n, by making use of the identity (3.11). Now, using $U_n=0$ and (3.16), we can easily prove the following identity, by induction on n:

$$\sum_{\substack{m=1 \\ (m,n)=1}}^{n} \frac{\mu(m)\log m}{m} = -\frac{n}{\varphi(n)}. \tag{3.17}$$

By taking $n=p$ (a prime) in (3.17), we get that $\sum_{\substack{m=1 \\ p\nmid m}}^{\infty} \frac{\mu(m)\log m}{m} = \frac{p}{1-p}$.

From this and $\sum_{m=1}^{\infty} \frac{\mu(m)\log m}{m} = -1$, it follows that $\sum_{\substack{m=1 \\ p|m}}^{\infty} \frac{\mu(m)\log m}{m} = \frac{1}{p-1}$.

LEMMA 3.4. *For $x \geqq 3$, $n \geqq 1$*

$$\Phi_n''(x) \equiv \sum_{mn \leqq x} \frac{\varphi(mn)}{m^2 n^2} = \frac{\varphi(n)}{n^2} \sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \frac{\mu(m)}{m^2}\left(\log\frac{x}{mn} + \gamma\right) + O\left(\frac{\lambda(x/n)}{x}\right), \tag{3.18}$$

where the O-estimate is uniform in x and n and $\lambda(x)$ is given by (2.10).

PROOF. Taking $f(m)=\frac{1}{m}$ and $g(m)=1-\frac{1}{m}$, we see by (3.1) and (2.2) that $F(m)=\frac{\varphi(m)}{m^2}$ and hence by Lemma 3.1 and Remark 3.1, we have for $(p,n)=1$,

$$(3.19)\quad \Phi''_{np^\alpha}(x)=\frac{\varphi(p^\alpha)}{p^{2\alpha}}\Phi''_n\left(\frac{x}{p^\alpha}\right)+\frac{\varphi(p)}{p^2}\left(\frac{1}{p^\alpha}-\frac{\varphi(p^\alpha)}{p^{2\alpha}}\right)\sum_{r=0}^{\infty}\left(\frac{1}{p}-\frac{\varphi(p)}{p^2}\right)^r\Phi''_n\left(\frac{x}{p^{r+\alpha+1}}\right)=$$

$$=\frac{\varphi(p^\alpha)}{p^{2\alpha}}\Phi''_n\left(\frac{x}{p^\alpha}\right)-\frac{p-1}{p^{\alpha+3}}\sum_{r=0}^{\infty}\frac{1}{p^{2r}}\Phi''_n\left(\frac{x}{p^{r+\alpha+1}}\right)=$$

$$=\frac{\varphi(p^\alpha)}{p^{2\alpha}}\Phi''_n\left(\frac{x}{p^\alpha}\right)-\frac{\varphi(p^\alpha)}{p^{2\alpha}}\sum_{r=0}^{\infty}\frac{1}{p^{2r+2}}\Phi''_n\left(\frac{x}{p^{r+\alpha+1}}\right)=\frac{\varphi(p^\alpha)}{p^{2\alpha}}\sum_{r=0}^{\infty}\frac{1}{p^{2r}}\Phi''_n\left(\frac{x}{p^{r+\alpha}}\right).$$

We prove (3.18) by induction on n. From (2.9), it is clear that (3.18) is true for $n=1$. Let us assume that (3.18) is true for $1, 2, 3, \dots, n-1$, where $n>1$ and prove it for n.

Since $n>1$, there is a prime p such that $p|n$. Let $n=Np^\alpha$, where $(p,N)=1$. Clearly $1\leqq N\leqq n-1$. Hence by our induction assumption, we have

$$(3.20)\qquad \Phi''_N(x)=\frac{\varphi(N)}{N^2}\sum_{\substack{m=1\\(m,N)=1}}^{\infty}\frac{\mu(m)}{m^2}\left(\log\frac{x}{mN}+\gamma\right)+O\left(\frac{\lambda(x/N)}{x}\right),$$

where the O-estimate is uniform in x and N.

Now, by (3.19) and (3.20), we have

$$(3.21)\qquad \Phi''_{Np^\alpha}(x)=\frac{\varphi(p^\alpha)}{p^{2\alpha}}\sum_{r=0}^{\infty}\frac{1}{p^{2r}}\Phi''_N\left(\frac{x}{p^{r+\alpha}}\right)=$$

$$=\frac{\varphi(p^\alpha)}{p^{2\alpha}}\sum_{r=0}^{\infty}\frac{1}{p^{2r}}\left\{\frac{\varphi(N)}{N^2}\sum_{\substack{m=1\\(m,N)=1}}^{\infty}\frac{\mu(m)}{m^2}\left(\log\frac{x}{mNp^{r+\alpha}}+\gamma\right)+O\left(\frac{\lambda(x/Np^{r+\alpha})}{x/p^{r+\alpha}}\right)\right\}=$$

$$=\frac{\varphi(Np^\alpha)}{(Np^\alpha)^2}\sum_{r=0}^{\infty}\frac{1}{p^{2r}}\sum_{\substack{m=1\\(m,N)=1}}^{\infty}\frac{\mu(m)}{m^2}\left(\log\frac{x}{mNp^\alpha}+\gamma\right)-\frac{\varphi(Np^\alpha)}{(Np^\alpha)^2}\sum_{r=0}^{\infty}\frac{1}{p^{2r}}\sum_{\substack{m=1\\(m,N)=1}}^{\infty}\frac{\mu(m)}{m^2}+$$

$$+O\left(\frac{\lambda(x/Np^\alpha)}{x}\frac{\varphi(p^\alpha)}{p^\alpha}\sum_{r=0}^{\infty}\frac{1}{p^r}\right)=\frac{\varphi(Np^\alpha)}{(Np^\alpha)^2}\frac{p^2}{p^2-1}\sum_{\substack{m=1\\(m,N)=1}}^{\infty}\frac{\mu(m)}{m^2}\left(\log\frac{x}{mNp^\alpha}+\gamma\right)-$$

$$-\frac{\varphi(Np^\alpha)}{(Np^\alpha)^2}\frac{p^2\log p}{(p^2-1)^2}\sum_{\substack{m=1\\(m,N)=1}}^{\infty}\frac{\mu(m)}{m^2}+O\left(\frac{\lambda(x/Np^\alpha)}{x}\frac{\varphi(p^\alpha)}{p^\alpha}\frac{p}{p-1}\right)=$$

$$=\frac{\varphi(Np^\alpha)}{(Np^\alpha)^2}\frac{p^2}{p^2-1}\sum_{\substack{m=1\\(m,N)=1}}^{\infty}\frac{\mu(m)}{m^2}\left(\log\frac{x}{mNp^\alpha}+\gamma\right)-\frac{\varphi(Np^\alpha)}{(Np^\alpha)^2}\frac{p^2\log p}{(p^2-1)^2}\sum_{\substack{m=1\\(m,N)=1}}^{\infty}\frac{\mu(m)}{m^2}+$$

$$+O\left(\frac{\lambda(x/Np^\alpha)}{x}\right)=\frac{\varphi(Np^\alpha)}{(Np^\alpha)^2}\frac{p^2}{p^2-1}\left(\log\frac{x}{Np^\alpha}+\gamma\right)\sum_{\substack{m=1\\(m,N)=1}}^{\infty}\frac{\mu(m)}{m^2}-$$

$$-\frac{\varphi(Np^\alpha)}{(Np^\alpha)^2}\frac{p^2}{p^2-1}\left\{\sum_{\substack{m=1\\(m,N)=1}}^{\infty}\frac{\mu(m)\log m}{m^2}+\frac{\log p}{p^2-1}\sum_{\substack{m=1\\(m,N)=1}}^{\infty}\frac{\mu(m)}{m^2}\right\}+O\left(\frac{\lambda(x/Np^\alpha)}{x}\right).$$

From (3.16), taking $h(m)=\frac{1}{m^2}$, we get that

$$(3.22)\qquad \sum_{\substack{m=1\\(m,Np^\alpha)=1}}^{\infty}\frac{\mu(m)\log m}{m^2}=\frac{p^2}{p^2-1}\left\{\sum_{\substack{m=1\\(m,N)=1}}^{\infty}\frac{\mu(m)\log m}{m^2}+\frac{\log p}{p^2-1}\sum_{\substack{m=1\\(m,N)=1}}^{\infty}\frac{\mu(m)}{m^2}\right\}.$$

From (3.21), (3.22) and (3.11) with $h(m)=\frac{1}{m^2}$, we get

$$\Phi''_{Np^\alpha}(x)=\frac{\varphi(Np^\alpha)}{(Np^\alpha)^2}\left(\log\frac{x}{Np^\alpha}+\gamma\right)\sum_{\substack{m=1\\(m,Np^\alpha)=1}}^{\infty}\frac{\mu(m)}{m^2}-\frac{\varphi(Np^\alpha)}{(Np^\alpha)^2}\sum_{\substack{m=1\\(m,Np^\alpha)=1}}^{\infty}\frac{\mu(m)\log m}{m^2}+$$

$$+O\left(\frac{\lambda(x/Np^\alpha)}{x}\right)=\frac{\varphi(n)}{n^2}\sum_{\substack{m=1\\(m,n)=1}}^{\infty}\frac{\mu(m)}{m^2}\left(\log\frac{x}{mn}+\gamma\right)+O\left(\frac{\lambda(x/n)}{x}\right).$$

Hence Lemma 3.4 follows.

REMARK 3.4. Using (2.5) and (2.6), we can rewrite (3.18) as follows:

$$(3.23)\qquad \Phi''_n(x)=\frac{1}{\zeta(2)\psi(n)}\left(\log\frac{x}{n}+\gamma-\beta(n)-\frac{\zeta'(2)}{\zeta(2)}\right)+O\left(\frac{\lambda(x/n)}{x}\right),$$

where the O-estimate is uniform in x and n.

LEMMA 3.5. *For $x\geqq 3$, $n\geqq 1$ and $u\geqq 1$ such that $(n,u)=1$,*

$$(3.24)\qquad \sum_{\substack{mn\leqq x\\(m,u)=1}}\frac{\varphi(mn)}{m^2n^2}=$$

$$=\frac{u}{\zeta(2)\psi(nu)}\left(\log\frac{x}{n}+\gamma-\beta(n)-\frac{\zeta'(2)}{\zeta(2)}+A(u)+B(u)\right)+O\left(\frac{\theta(u)\lambda(x/n)}{x}\right),$$

where the O-estimate is uniform in x, n and u.

PROOF. We have by (3.23),

$$\sum_{\substack{mn\leqq x\\(m,u)=1}}\frac{\varphi(mn)}{m^2n^2}=\sum_{mn\leqq x}\frac{\varphi(mn)}{m^2n^2}\sum_{\substack{d\delta=m\\d|u}}\mu(d)=\sum_{d|u}\mu(d)\sum_{\delta dn\leqq x}\frac{\varphi(\delta dn)}{\delta^2d^2n^2}=\sum_{d|n}\mu(d)\Phi''_{dn}(x)=$$

$$=\sum_{d|u}\mu(d)\left\{\frac{1}{\zeta(2)\psi(dn)}\left(\log\frac{x}{dn}+\gamma-\beta(dn)-\frac{\zeta'(2)}{\zeta(2)}\right)+O\left(\frac{\lambda(x/dn)}{x}\right)\right\}.$$

Since $(u, n)=1$, we have $(d, n)=1$ for $d|u$. Hence by Lemmas 2.5, 2.6 and 2.7,

$$\sum_{\substack{mn\leqq x\\(m,u)=1}} \frac{\varphi(mn)}{m^2 n^2} = \frac{1}{\zeta(2)\psi(n)}\left(\log\frac{x}{n}+\gamma-\beta(n)-\frac{\zeta'(2)}{\zeta(2)}\right)\sum_{d|u}\frac{\mu(d)}{\psi(d)}-$$

$$-\frac{1}{\zeta(2)\psi(n)}\sum_{d|u}\frac{\mu(d)\log d}{\psi(d)}-\frac{1}{\zeta(2)\psi(n)}\sum_{d|u}\frac{\mu(d)\beta(d)}{\psi(d)}+O\left(\frac{\lambda(x/n)}{x}\sum_{d|u}\mu^2(d)\right)=$$

$$=\frac{u}{\zeta(2)\psi(nu)}\left(\log\frac{x}{n}+\gamma-\beta(n)-\frac{\zeta'(2)}{\zeta(2)}+A(u)+B(u)\right)+O\left(\frac{\theta(u)\lambda(x/n)}{x}\right).$$

Hence Lemma 3.5 follows.

§ 4. Main results. First we prove following:

THEOREM 4.1. *For* $x\geqq 3$ *and* $n\geqq 1$,

$$\sum_{\substack{m\leqq x\\(m,n)=1}}\frac{q_k(m)\varphi(m)}{m^2}=\frac{\alpha_k n^{k+1}}{\zeta(2)\psi(n)H_k(n)}\left[\log x+\gamma-\frac{\zeta'(2)}{\zeta(2)}+A(n)-A_k(n)+B(n)-\right.\tag{4.1}$$

$$\left.-B_k(n)+\sum_p\frac{\{k(p^2-1)+1\}\log p}{(p^2-1)\{p^{k-1}(p+1)-1\}}\right]+O\left(\theta(n)x^{-1+1/k}\delta(x)\right)$$

uniformly, where $A(n)$, $B(n)$, $H_k(n)$, $A_k(n)$, $B_k(n)$ *are given by Lemma* 2.6, *Lemma* 2.7, (2.15), (2.17), (2.18), *respectively*, α_k *is the constant given by* (2.16) *and* $\delta(x)$, $\lambda(x)$ *are given by* (2.29) *and* (2.10), *respectively*.

PROOF. We have by (2.1),

$$\sum_{\substack{m\leqq x\\(m,n)=1}}\frac{q_k(m)\varphi(m)}{m^2}=\sum_{\substack{m\leqq x\\(m,n)=1}}\frac{\varphi(m)}{m^2}\sum_{d^k\delta=m}\mu(d)=\sum_{\substack{d^k\delta\leqq x\\(d,n)=(\delta,n)=1}}\frac{\mu(d)\varphi(d^k\delta)}{d^{2k}\delta^2},$$

where the summation is taken over all ordered pairs (d,δ) such that $d^k\delta\leqq x$ and $(d,n)=(\delta,n)=1$. Hence we have by Lemma 3.5,

$$\sum_{\substack{m\leqq x\\(m,n)=1}}\frac{q_k(m)\varphi(m)}{m^2}=\sum_{\substack{d\leqq\sqrt[k]{x}\\(d,n)=1}}\mu(d)\sum_{\substack{\delta\leqq\frac{x}{d^k}\\(\delta,n)=1}}\frac{\varphi(\delta d^k)}{\delta^2 d^{2k}}=\tag{4.2}$$

$$=\sum_{\substack{d\leqq\sqrt[k]{x}\\(d,n)=1}}\mu(d)\left\{\frac{n}{\zeta(2)\psi(d^k n)}\left(\log\frac{x}{d^k}+\gamma-\beta(d^k)-\frac{\zeta'(2)}{\zeta(2)}+A(n)+B(n)\right)+\right.$$

$$\left.+O\left(\frac{\theta(n)\lambda(x/d^k)}{x}\right)\right\}=\frac{n}{\zeta(2)\psi(n)}\left[\left(\log x+\gamma-\frac{\zeta'(2)}{\zeta(2)}+A(n)+B(n)\right)\sum_{\substack{d\leqq\sqrt[k]{x}\\(d,n)=1}}\frac{\mu(d)}{\psi(d)d^{k-1}}-\right.$$

$$\left.-k\sum_{\substack{d\leqq\sqrt[k]{x}\\(d,n)=1}}\frac{\mu(d)\log d}{\psi(d)d^{k-1}}-\sum_{\substack{d\leqq\sqrt[k]{x}\\(d,n)=1}}\frac{\mu(d)\beta(d)}{\psi(d)d^{k-1}}\right]+O\left(\frac{\theta(n)\lambda(x)}{x}\sum_{\substack{d\leqq\sqrt[k]{x}\\(d,n)=1}}\mu^2(d)\right).$$

Since

$$\sum_{\substack{d \leqq \sqrt[k]{x} \\ (d,n)=1}} \mu^2(d) = \sum_{d \leqq \sqrt[k]{x}} 1 = O\left(x^{\frac{1}{k}}\right),$$

uniformly; we have that the O-term in (4.2) $O\left(\theta(n)\frac{\lambda(x)}{x}x^{\frac{1}{k}}\right)$.

Hence by (2.36), (2.37) and (2.38), we obtain from (4.2),

$$(4.3) \quad \sum_{\substack{m \leqq x \\ (m,n)=1}} \frac{q_k(m)\varphi(m)}{m^2} = \frac{n}{\zeta(2)\psi(n)}\left[\left(\log x + \gamma - \frac{\zeta'(2)}{\zeta(2)} + A(n) + B(n)\right)\times\right.$$

$$\times\left\{\frac{\alpha_k n^k}{H_k(n)} + O\left(\frac{\sigma^*_{-1+\varepsilon}(n)\delta(\sqrt[k]{x})}{x^{1-1/k}}\right)\right\} - \frac{k\alpha_k n^k}{H_k(n)}\left\{A_k(n) - \sum_p \frac{\log p}{p^{k-1}(p+1)-1}\right\} +$$

$$+ O\left(\frac{\sigma^*_{-1+\varepsilon}(n)\delta(\sqrt[k]{x})}{x^{1-1/k}}\right) - \frac{\alpha_k n^k}{H_k(n)}\left\{B_k(n) - \sum_p \frac{\log p}{(p^2-1)\{p^{k-1}(p+1)-1\}}\right\} +$$

$$\left. + O\left(\frac{\sigma^*_{-1+\varepsilon}(n)\delta(\sqrt[k]{x})}{x^{1-1/k}}\right)\right] + O(\theta(n)\lambda(x)x^{-1+1/k}) = \frac{\alpha_k n^{k+1}}{\zeta(2)\psi(n)H_k(n)}\times$$

$$\times\left[\log x + \gamma - \frac{\zeta'(2)}{\zeta(2)} + A(n) - kA_k(n) + B(n) - B_k(n) + \sum_p \frac{\{k(p^2-1)+1\}\log p}{(p^2-1)\{p^{k-1}(p+1)-1\}}\right] +$$

$$+ O(\theta(n)\lambda(x)x^{-1+1/k}).$$

It has been proved in our earlier paper (cf. [16], Theorem 4.1) that

$$(4.4) \quad Q_k(x;\, n, \varphi) \equiv \sum_{\substack{m \leqq x \\ (m,n)=1}} q_k(m)\varphi(m) =$$

$$= \frac{\alpha_k n^{k+1} x^2}{2\zeta(2)\psi(n)H_k(n)} + O\left(\theta(n)x^{1+1/k}\delta(x)\right),$$

uniformly.

Hence by partial summation (cf. [9], Theorem 421), and (4.4), we get

$$(4.5) \quad \sum_{\substack{m \leqq x \\ (m,n)=1}} \frac{q_k(m)\varphi(m)}{m^2} = \frac{Q_k(x;\, n, \varphi)}{x^2} + 2\int_1^x \frac{Q(t;\, n, \varphi)}{t^3}\,dt =$$

$$= \frac{\alpha_k n^{k+1}}{2\zeta(2)\psi(n)H_k(n)} + O\left(\theta(n)x^{1+1/k}\delta(x)\right) +$$

$$+ 2\int_1^x \left\{\frac{\alpha_k n^{k+1}}{2\zeta(2)\psi(n)H_k(n)}\,\frac{1}{t} + E_k(t;\, n, \varphi)\right\}dt,$$

where

$$E_k(t;\, n, \varphi) = O\left(\theta(n)t^{-2+1/k}\delta(t)\right).$$

By Remark 2.6, it follows that

$$\int_x^\infty E_k(t;\, n, \varphi)\, dt = O\left(\theta(n)\delta(x)\int_x^\infty t^{-2+1/k}\, dt\right) = O\big(\theta(n)x^{-1+1/k}\delta(x)\big).$$

Hence by (4.5), we get

$$\sum_{\substack{m\leqq x\\ (m,n)=1}} \frac{q_k(m)\varphi(m)}{m^2} = \frac{\alpha_k n^{k+1}}{\zeta(2)\psi(n)H_k(n)}[\log x + C_k(n)] + O\big(\theta(n)x^{-1+1/k}\delta(x)\big), \tag{4.6}$$

where

$$C_k(n) = \frac{1}{2} + \frac{2\zeta(2)\psi(n)H_k(n)}{\alpha_k n^{k+1}}\int_1^\infty E_k(t;\, n, \varphi)\, dt.$$

Now, comparing (4.6) with (4.3), we find that

$$C_k(n) = \gamma - \frac{\zeta'(2)}{\zeta(2)} + A(n) - kA_k(n) + B(n) - B_k(n) + \sum_p \frac{\{k(p^2-1)+1\}\log p}{(p^2-1)\{p^{k-1}(p+1)-1\}}.$$

Substituting this value of $C_k(n)$ in (4.6), we obtain (4.1).
Thus Theorem 4.1 is proved.

COROLLARY 4.1.1 ($n=1$). *For* $x\geqq 3$ *and* $k\geqq 2$,

$$\sum_{m\leqq x} \frac{q_k(m)\varphi(m)}{m^2} = \frac{\alpha_k}{\zeta(2)}\left[\log x + \gamma - \frac{\zeta'(2)}{\zeta(2)} + \sum_p \frac{\{k(p^2-1)+1\}\log p}{(p^2-1)\{p^{k-1}(p+1)-1\}}\right] + O\big(x^{-1+1/k}\delta(x)\big). \tag{4.7}$$

COROLLARY 4.1.2 ($n=1$, $k=2$). *For* $x\geqq 3$,

(4.8)

$$\sum_{m\leqq x} \frac{\mu^2(m)\varphi(m)}{m^2} = \frac{\alpha}{\zeta(2)}\left[\log x + \gamma - \frac{\zeta'(2)}{\zeta(2)} + \sum_p \frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)}\right] + O\big(x^{-1/2}\delta(x)\big),$$

where α *is the constant given by*

$$\alpha = \prod_p \left(1 - \frac{1}{p(p+1)}\right). \tag{4.9}$$

REMARK 4.1. In 1971, S. UCHIYAMA (cf. [17], (6)) obtained the following asymptotic formula: For $x\geqq 2$,

$$\sum_{m\leqq x} \frac{\mu^2(m)\varphi(m)}{m^2} = \frac{\alpha}{\zeta(2)}\log x + c_3 + O(x^{-1/2}\log x), \tag{4.10}$$

where c_3 is a constant, the value of which has not been given explicitly. It may be noted that our asymptotic formula given above in (4.8) not only gives the explicit value of c_3, but also gives an improvement in the O-estimate of the error term in (4.10).

COROLLARY 4.1.3 ($k=2$). *For* $x \geqq 3$ *and* $n \geqq 1$

$$(4.11) \quad \sum_{\substack{m \leqq x \\ (m,n)=1}} \frac{\mu^2(m)\varphi(m)}{m^2} = \frac{\alpha n^3}{\zeta(2)\psi(n)H_2(n)}\left[\log x + \gamma - \frac{\zeta'(2)}{\zeta(2)} + A(n) - 2A_2(n) + \right.$$

$$\left. + B(n) - B_2(n) + \sum_p \frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)}\right] + O\big(\theta(n)x^{-1/2}\delta(x)\big),$$

uniformly.

THEOREM 4.2. *If the Riemann hypothesis is true, then for* $x \geqq 3$ *and* $n \geqq 1$, *the error term in* (4.1) *can be replaced by* $O\big(\theta(n)x^{-1+2/(2k+1)}\omega(x)\big)$, *where* $\omega(x)$ *is the function defined by*

$$(4.12) \qquad \omega(x) = \begin{cases} \exp\{A \log x (\log\log x)^{-1}\} & \text{for} \quad x \geqq 3 \\ 1 \quad \text{for} \quad 0 < x < 3; \end{cases}$$

A being a positive absolute constant.

PROOF. It has been shown in our earlier paper (cf. [16], Theorem 4.2) that under the assumption of the Riemann hypothesis the error term in (4.4) can be replaced by $O\big(\theta(n)x^{-1+2/(2k+1)}\omega(x)\big)$. Now, using (4.4) with this O-estimate for the error term and using partial summation as in the proof of Theorem 4.1, we get Theorem 4.2.

COROLLARY 4.2.1 ($n=1$). *If the Riemann hypothesis is true, then for* $x \geqq 3$, $k \geqq 2$, *the error term in* (4.7) *can be replaced by* $O\big(x^{-1+2/(2k+1)}\omega(x)\big)$.

COROLLARY 4.2.2 ($n=1, k=2$). *If the Riemann hypothesis is true, then for* $x \geqq 3$, *the error term in* (4.8) *can be replaced by* $O\big(x^{-3/5}\omega(x)\big)$.

COROLLARY 4.2.3 ($k=2$). *If the Riemann hypothesis is true, then for* $x \geqq 3$ *and* $n \geqq 1$, *the error term in* (4.11) *can be replaced by* $O\big(\theta(n)x^{-3/5}\omega(x)\big)$.

THEOREM 4.3. *For* $x \geqq 3$ *and* $n \geqq 1$,

$$(4.13) \quad \sum_{m \leqq x} \frac{\gamma_k(m;n)}{m^2} = \frac{\alpha_k n^{k+1}}{\psi(n)H_k(n)}\left[\log x + \gamma - A(n) - kA_k(n) + B(n) - B_k(n) + \right.$$

$$\left. + \sum_p \frac{\{k(p^2-1)+1\}\log p}{(p^2-1)\{p^{k-1}(p+1)-1\}}\right] + O\big(\theta(n)x^{-1+1/k}\delta(x)\big),$$

uniformly.

PROOF. We have

$$\gamma_k(m;n) = \sum_{d \mid \gamma_k(m;n)} \varphi(d) = \sum_{\substack{d \mid m \\ d \in Q_k \\ (d,n)=1}} \varphi(d) = \sum_{\substack{d\delta = m \\ (d,n)=1}} q_k(d)\varphi(d).$$

Hence we have

$$(4.14)\qquad \sum_{m\leqq x}\frac{\gamma_k(m;\,n)}{m^2}=\sum_{\substack{d\delta\leqq x\\(d,n)=1}}\frac{q_k(d)\varphi(d)}{d^2\delta^2}=\sum_{\delta\leqq x}\frac{1}{\delta^2}\sum_{\substack{d\leqq\frac{x}{\delta}\\(d,n)=1}}\frac{q_k(d)\varphi(d)}{d^2}=$$

$$=\sum_{\delta\leqq x}\frac{1}{\delta^2}\left[\frac{\alpha_k n^{k+1}}{\zeta(2)\psi(n)H_k(n)}\left(\log\frac{x}{\delta}+\gamma-\frac{\zeta'(2)}{\zeta(2)}+A(n)-kA_k(n)+B(n)-B_k(n)+\right.\right.$$

$$\left.\left.+\sum_p\frac{\{k(p^2-1)+1\}\log p}{(p^2-1)\{p^{k-1}(p+1)-1\}}\right)+O\left(\theta(n)\left(\frac{x}{\delta}\right)^{-1+1/k}\delta\left(\frac{x}{\delta}\right)\right)\right]=$$

$$=\frac{\alpha_k n^{k+1}}{\zeta(2)\psi(2)H_k(n)}\left[\log x+\gamma-\frac{\zeta'(2)}{\zeta(2)}+A(n)-kA_k(n)+B(n)-B_k(n)+\right.$$

$$\left.+\sum_p\frac{\{k(p^2-1)+1\}\log p}{(p^2-1)\{p^{k-1}(p+1)-1\}}\right]\left\{\zeta(2)+O\left(\frac{1}{x}\right)\right\}-$$

$$-\frac{\alpha_k n^{k+1}}{\zeta(2)\psi(n)H_k(n)}\left\{-\zeta'(2)+O\left(\frac{\log x}{x}\right)\right\}+O\left(\theta(n)\sum_{m\leqq x}\frac{1}{m^2}\left(\frac{x}{m}\right)^{-1+1/k}\delta\left(\frac{x}{m}\right)\right).$$

By Remark 2.6, it follows that for $0<\varepsilon<1/k$, the last O-term in (4.14) is

$$O\left(\theta(n)\sum_{m\leqq x}\frac{1}{m^2}\left(\frac{x}{m}\right)^{-1+1/k-\varepsilon}\left(\frac{x}{m}\right)^{\varepsilon}\delta\left(\frac{x}{m}\right)\right)=$$

$$=O\big(\theta(n)x^{\varepsilon}\delta(x)x^{-1-1/k+\varepsilon}\sum_{m\leqq x}m^{-1+1/k-\varepsilon}\big)=O\big(\theta(n)x^{-1+1/k}\delta(x)\big).$$

By Remarks 2.1 and 2.4, it is clear that $\dfrac{\alpha_k n^{k+1}}{\psi(n)H_k(n)}<1$. Also, by Lemmas 2.6, 2.7, (2.17) and (2.18), it is easy to see that each of $kA_k(n)$, $B(n)$, $B_k(n)$ is less than $A(n)$, so that the term contained in square brackets of (4.14) is $O(A(n)\log x)$. However,

$$A(n)=\sum_{p|n}\frac{\log p}{p}<\sum_{p|n}1=\nu(n)<\theta(n).$$

Hence the O-term that arises from the first term of (4.14) is

$$O\left(\theta(n)\frac{\log x}{x}\right)=O\big(\theta(n)x^{-1+1/k}\delta(x)\big).$$

Also, the O-term that arises from the second term of (4.14) is

$$O\left(\frac{\log x}{x}\right)=O\big(\theta(n)x^{-1+1/k}\delta(x)\big).$$

Now, Theorem 4.3 follows from the above discussion and (4.14).

COROLLARY 4.3.1 ($n=1$). *For* $x \geqq 3$ *and* $k \geqq 2$,

$$\sum_{m \leqq x} \frac{\gamma_k(m)}{m^2} = \alpha_k \left[\log x + \gamma + \sum_p \frac{\{k(p^2-1)+1\} \log p}{(p^2-1)\{p^{k-1}(p+1)-1\}}\right] + O\left(x^{-1+1/k}\delta(x)\right). \tag{4.15}$$

COROLLARY 4.3.2 ($n=2$). *For* $x \geqq 3$ *and* $k \geqq 2$,

$$\sum_{m \leqq x} \frac{\delta_k(m)}{m^2} = \frac{2^k \alpha_k}{(2^k+2^{k-1}-1)} \left[\log x + \gamma + \frac{(2^k-k-1)\log 2}{(2^k+2^{k-1}-1)} + \right. \tag{4.16}$$

$$\left. + \sum_p \frac{\{k(p^2-1)+1\} \log p}{(p^2-1)\{p^{k-1}(p+1)-1\}}\right] + O\left(x^{-1+1/k}\delta(x)\right).$$

COROLLARY 4.3.3 ($k=2$). *For* $x \geqq 3$ *and* $n \geqq 1$,

$$\sum_{m \leqq x} \frac{\gamma(m; n)}{m^2} = \frac{\alpha n^3}{\psi(n) H_2(n)} \left[\log x + \gamma + A(n) - 2A_2(n) + B(n) - B_2(n) + \right. \tag{4.17}$$

$$\left. + \sum_p \frac{\{2(p^2-1)+1\} \log p}{(p^2-1)(p^2+p-1)}\right] + O\left(\theta(n) x^{-1/2} \delta(x)\right),$$

uniformly, where α *is the constant given by* (4.9).

COROLLARY 4.3.4 ($k=2, n=1$). *For* $x \geqq 3$,

$$\sum_{m \leqq x} \frac{\gamma(m)}{m^2} = \alpha \left[\log x + \gamma + \sum_p \frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)}\right] + O\left(x^{-1/2}\delta(x)\right). \tag{4.18}$$

REMARK 4.2. In 1971, S. UCHIYAMA [17] obtained the following asymptotic formula in an attempt to solve a problem posed by the first named author (cf. [12], Problem 17(2): For $x \geqq 2$,

$$\sum_{m \leqq x} \frac{\gamma(m)}{m^2} = \alpha \log x + \beta + O(x^{-1/2} \log x), \tag{4.19}$$

where α is the constant given by (4.9) and β is the constant given by $\beta = \zeta(2) c_3 + \alpha \frac{\zeta'(2)}{\zeta(2)}$, c_3 being the constant mentioned in the asymptotic formula (4.10).) In fact, he also gave the following concise presentation of the constant β: Let $F(s)$ be the function defined by

$$F(s) = \prod_p \left(\frac{p-1}{p^{s+1}(p^{s+2}-1)}\right),$$

which is analytic for $\operatorname{Re}(s) > -\frac{1}{2}$. Then $\alpha = F(0)$ and $\beta = \gamma F(0) + F'(0)$, γ being Euler's constant.

It may be noted that our formula given in (4.18) gives an improvement in the O-estimate of the error term in (4.19) and it can be easily verified that the value of the constant β given by S. Uchiyama coincides with the one that can be obtained from our formula (4.18).

COROLLARY 4.3.5 ($k=2, n=2$). *For* $x \geqq 3$,

$$\sum_{m \leqq x} \frac{\delta(m)}{m^2} = \frac{4\alpha}{5}\left[\log x + \gamma + \frac{1}{5}\log 2 + \sum_p \frac{(2p^2-1)\log p}{(p^2-1)(p^2+p-1)}\right] + O\big(x^{-1/2}\delta(x)\big). \tag{4.20}$$

THEOREM 4.4. *If the Riemann hypothesis is true, then for* $x \geqq 3$ *and* $n \geqq 1$, *the error term in* (4.13) *can be replaced by* $O\big(\theta(n)\, x^{-1+2/(2k+1)}\omega(x)\big)$, *where* $\omega(x)$ *is given by* (4.12).

PROOF. It has been shown in our earlier paper (cf. [16], Theorem 4.4) that under the assumption of the Riemann hypothesis, we have

$$\sum_{m \leqq x} \gamma_k(m; n) = \frac{\alpha_k n^{k+1} x^2}{2\psi(n) H_k(n)} + O\big(\theta(n) x^{1+2/(2k+1)}\omega(x)\big). \tag{4.21}$$

By partial summation (cf. [9], Theorem 421) and (4.21), we get

$$\sum_{m \leqq x} \frac{\gamma_k(m;\, n)}{m^2} = \frac{\alpha_k n^{k+1}}{\psi(n) H_k(n)}\big(\log x + C_k'(n)\big) + O\big(\theta(n) x^{-1+2/(2k+1)}\omega(x)\big), \tag{4.22}$$

where $C_k'(n)$ is a number depending on k and n.

Now, comparing (4.22) with (4.13), we find that

$$C_k'(n) = \gamma + A(n) - kA_k(n) + B(n) - B_k(n) + \sum_p \frac{\{k(p^2-1)+1\}\log p}{(p^2-1)\{p^{k-1}(p+1)-1\}}.$$

Substituting this value of $C_k'(n)$ in (4.22), we obtain (4.13) with the O-term replaced by $O\big(\theta(n) x^{-1+2(2k+1)}\omega(x)\big)$. Hence Theorem 4.4 follows.

COROLLARY 4.4.1. ($n=1$ or $n=2$). *If the Riemann hypothesis is true, then for* $x \geqq 3$ *and* $k \geqq 2$, *the error terms in* (4.15) *and* (4.16) *can be replaced by* $O\big(x^{-1+2/(2k+1)}\omega(n)\big)$.

COROLLARY 4.4.2. ($k=2$). *If the Riemann hypothesis is true, then for* $x \geqq 3$ *and* $n \geqq 1$, *the error term in* (4.17) *can be replaced by* $O\big(\theta(n) x^{3/5}\omega(x)\big)$.

COROLLARY 4.4.3 ($k=2$; $n=1$ or 2). *If the Riemann hypothesis is true, then for* $x \geqq 3$, *the error terms in* (4.18) *and* (4.20) *can be replaced by* $O\big(x^{-3/5}\omega(x)\big)$.

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A NOTE ON SEMI-PRIMARY GROUP RINGS

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A ring R is semi-primary if its Jacobson radical $J(R)$ is nilpotent and $R/J(R)$ is artinian. Let R be a ring, G a group and RG denote the group ring of G over R. In this note, we prove the following:

THEOREM. *RG is semi-primary iff R is semi-primary and G is finite.*

To prove the theorem, we shall need the following results on group rings:

THEOREM 1. (RENAULT [2], WOODS [3].) *RG is perfect iff R is perfect and G is finite.*

LEMMA 2. (CONNEL [1], p. 665.) *If G is locally finite, then* $J(RG)\cap R=J(R)$.

PROOF OF THE THEOREM. Assume that RG is semi-primary. Then it is perfect. Therefore R is perfect and G is finite by Theorem 1. Hence $R/J(R)$ is artinian. As G is finite, $J(R)=J(RG)\cap R$ by Lemma 2. It follows that $J(R)$ is nilpotent.

Conversely, assume that R is semi-primary and G is finite. Then RG is perfect and so $RG/J(RG)$ is artinian. We are left to show that $J(RG)$ is nilpotent. We note that $J(R)G\subseteqq J(RG)$. Also, $RG/J(R)G\cong[R/J(R)]G$ canonically. Since $R/J(R)$ is artinian and G is finite, $[R/J(R)]G$ is artinian. It follows that its Jacobson radical J_1 is nilpotent. Now define $f\colon RG\to RG/J(R)G$ by $f(x)=x+J(R)G$ for $x\in RG$. Then

$$f(J(RG)) = J(RG)/J(R)G \subseteqq J_1.$$

Since $J(R)$ is nilpotent, so is $J(R)G$. It follows readily that $J(RG)$ is nilpotent. This shows that RG is semi-primary.

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TRIGONOMETRISCHE REIHEN ÜBER MULTIPLIKATIVEN ZAHLENMENGEN. II

Von

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Theodor Schneider zum 65. Geburtstag gewidmet

Einleitung. Im Zusammenhang mit der Charakterisierung der Γ-Funktion als eindeutiger Lösung gewisser Funktionalgleichungen (s. ARTIN [1], § 6; LUCHT [3], [4]) stellte sich folgende Frage: Man finde eine möglichst umfassende Menge T von Primzahlen $p \in \mathbf{P}$, für die die Reihe

$$\sum_{\substack{n \\ p|n \Rightarrow p \in T}} \frac{\sin(2\pi n\alpha)}{n}$$

gleichmäßig in α konvergiert. In der ersten Arbeit zu diesem Thema [5] gaben wir zu jedem $\varepsilon > 0$ solche Mengen T an, die innerhalb von $\mathbf{P}$ eine Relativdichte $> 1-\varepsilon$ besitzen. Es lag nahe zu vermuten, daß sich Mengen T finden lassen für die $\sum_{p \notin T} \frac{1}{p}$ „gerade noch divergiert". Dies soll hier gezeigt werden.

HAUPTSATZ. *Ez sei $g_1 \colon \mathbf{R}^+ \to \mathbf{R}^+$ monoton wachsend und $g_1(\infty) = \infty$. Dann existiert eine Primzahlmenge T mit den Eigenschaften*

a) $g(x) = \sum_{p \leqq x,\, p \notin T} \frac{1}{p} \leqq g_1(x)$ *für alle* $x \geqq x_0$,

b) $g(\infty) = \infty$,

c) $\sum_{\substack{n \\ p|n \Rightarrow p \in T}} \frac{\sin(2\pi n\alpha)}{n}$ *gleichmaßig konvergent in* $\alpha \in \mathbf{R}$.

Bezüglich der Γ-Funktion besagt dies (s. die Bemerkungen am Beginn von [5]): Die Funktionalgleichungen

$$\begin{cases} F(\alpha+1) = \alpha F(\alpha), \\ F\left(\frac{\alpha}{m}\right) F\left(\frac{\alpha+1}{m}\right) \dots F\left(\frac{\alpha+m-1}{m}\right) = \frac{(2\pi)^{m-1/2}}{m^{\alpha-1/2}} F(\alpha)(p|m \Rightarrow p \in T) \end{cases}$$

besitzen außer der Γ-Funktion eine weitere, für positive α positivwertige Lösung F.

Unter Berücksichtigung von [4] ist damit die vom ersten Autor aufgeworfene Frage im wesentlichen gelöst.

Der Hauptsatz wird in ähnlicher Weise hergeleitet wie die Ergebnisse von [5]; der Beweisgang ist allerdings erheblich verwickelter. In den Teilen I und II wird

— noch unter recht allgemeinen Voraussetzungen — die zu betrachtende Reihe überführt in eine, die über die „dünne" Zahlenmenge $L=\{d: p|d \Rightarrow p \notin T\}$ erstreckt ist (Satz 2). Dies hat den Vorteil, daß lange Abschnitte trivial abgeschätzt werden können (Teil IV). In Teil III wird $S=\mathbf{P}-T$ definiert. Wie in [5] wird unterschieden, ob α durch rationale Zahlen mit „kleinem" (Teil V) oder „großem" (Teil VI) Nenner approximiert werden kann. Hier benutzen wir erneut Ergebnisse über L-Reihen bzw. Exponentialsummen.

Eine einfache Charakterisierung der Mengen T mit gleichmäßig konvergenter Sinusreihe zu geben, scheint schwierig zu sein. Das hier beschriebene Beispiel ist zu speziell, um Aufschluß über die allgemeine Situation zu geben.

Herrn Prof. Paul Erdős, der uns mit wertvollen Ratschlägen zu dieser Arbeit ermunterte, danken wir herzlich.

Bezeichnungen. N, P, Z, Q, R bezeichnen die Menge der natürlichen Zahlen, der Primzahlen, der ganzen, der rationalen bzw. der reellen Zahlen.

Der Buchstabe p, evtl. indiziert, bezeichnet stets Primzahlen. $M(x)$ bezeichnet die Anzahl der Elemente $\leqq x$ aus der Menge M, desgleichen für Zahlenmengen S usw. $\Pi(x)$ bedeutet die Anzahl der Primzahlen $\leqq x$.

$s=\sigma+it$ ist eine komplexe Variable mit dem Realteil σ und dem Imaginärteil t.

C ist eine genügend groß gewählte Konstante; alle $\ll$-, O- und o-Abschätzungen sind, soweit nicht ausdrücklich geregelt, höchstens von C und der Menge S abhängig, in keinem Fall von α.

$c, c_1, c_2, \ldots$ sind positive Konstanten, deren genauer Wert nicht interessiert. $\log_r$ bezeichnet den r-fach iterierten Logarithmus ($r \in \mathbf{N}$), also $\log_1=\log$, $\log_{r+1}=\log\circ\log_r$.

μ ist die Möbius-Funktion, ω die zahlentheoretische Funktion, die dem Argument die Anzahl seiner (paarweise verschiedenen) Primteiler zordnet. $e(x)=\exp(2\pi ix)=e^{2\pi ix}$.

$[x]$, $\{x\}$ bedeuten den ganzen bzw. den gebrochenen Teil von $x\in\mathbf{R}$, also $x=[x]+\{x\}$ mit $0\leqq\{x\}<1$, $[x]\in\mathbf{Z}$. $\|x\|=\min(\{x\}, 1-\{x\})$ ist der Abstand von $x\in\mathbf{R}$ zur nächstgelegenen ganzen Zahl.

sgn x steht für das Signum (Vorzeichen) von $x\in\mathbf{R}$,

$$\operatorname{sgn} x = \begin{cases} 1 & \text{für} \quad x>0, \\ 0 & \text{für} \quad x=0, \\ -1 & \text{für} \quad x<0. \end{cases}$$

1. Es sei S eine Menge von Primzahlen,

$$g(x)=\sum_{\substack{p\leqq x\\ p\in S}}\frac{1}{p}, \quad h(x)=\frac{1}{x}S(x), \quad D(x)=\sum_{\substack{d\leqq x\\ d\in L}}\mu^2(d),$$

wobei L die von S multiplikativ erzeugte Halbgruppe bezeichnet. Der folgende Satz enthält Abschätzungen einiger über L erstreckter Summen unter schwachen Voraussetzungen über die Funktionen g und h. Bekanntlich ist stets $g(x)\ll\log_2 x$, $h(x)\log x\ll 1$.

SATZ 1. *Mit den obigen Bezeichnungen sei*

$$\frac{h(y)}{h(x)} \ll 1 \quad \textit{für} \quad \sqrt{x} \leqq y \leqq x \, (x \geqq x_0). \tag{1}$$

Dann gilt mit einer absoluten Konstanten $c>0$

a) $D(x) \ll S(x) \exp\big(cg(x)\big)$,

b) $g(x) = o(\log_2 x) \Rightarrow$

$$\sum_{\substack{A<d\leqq B\\ d\in L}} \frac{\mu^2(d)}{d} \ll \log^{-1/2} A + \big(g(B)-g(A)\big)\exp\big(cg(B)\big) \quad (B > A \to \infty),$$

c) $h(x) = o\big((\log x \log_2 x)^{-1}\big) \Rightarrow \sum\limits_{\substack{d\leqq x\\ d\in L\\ (d,d_0)=1}} \dfrac{\mu(d)}{d} = \prod\limits_{\substack{p\leqq x\\ p\in S\\ p\nmid d_0}} \left(1-\dfrac{1}{p}\right) + o(1)$

gleichmäßig für $d_0 \in \mathbf{N}$ $(x \to \infty)$.

BEWEIS. a) Wegen (1) existiert eine absolute Konstante $c>0$ mit

$$\sum_{\substack{p\leqq\sqrt{x}\\ p\in S}} S\left(\frac{x}{p}\right) = S(x) \sum_{\substack{p\leqq\sqrt{x}\\ p\in S}} \frac{1}{p} \frac{h\left(\frac{x}{p}\right)}{h(x)} \leqq cS(x)g(x). \tag{2}$$

Setzt man für $k=1, 2, \ldots$

$$S_k(x) = \sum_{\substack{d\leqq x,\, d\in L\\ \omega(d)=k}} \mu^2(d),$$

so folgt wie bei HARDY—RAMANUJAN [2], p. 79, die rekursive Ungleichung

$$S_{k+1}(x) \leqq \frac{1}{k} \sum_{\substack{p\leqq\sqrt{x}\\ p\in S}} S_k\left(\frac{x}{p}\right).$$

Unter Verwendung von (2) liefert sie durch Induktion rasch

$$S_k(x) \leqq \frac{1}{(k-1)!} S(x)\big(cg(x)\big)^{k-1},$$

was, eingesetzt in

$$D(x) = 1 + \sum_{k\geqq 1} S_k(x),$$

die Behauptung a) ergibt. Wir bemerken, daß sich unter stärkeren Schwankungsbedingungen für die Funktion h anstelle von (1) die Konstante c in a) durch 1 ersetzen läßt, was sich auch auf b) überträgt.

b) Partielle Summation liefert

$$g(x) = \int_1^x \frac{dS(t)}{t} = \frac{S(x)}{x} + \int_1^x \frac{S(t)}{t^2}\, dt, \tag{3}$$

also

$$\int_A^B \frac{S(t)}{t^2}\,dt = g(B)-g(A)+O\left(\frac{1}{\log A}\right).$$

Entsprechend folgt mit a)

$$\sum_{\substack{A<d\leq B\\ d\in L}} \frac{\mu^2(d)}{d} = \int_A^B \frac{dD(t)}{t} \ll \frac{D(A)}{A} + \left(g(B)-g(A)+O\left(\frac{1}{\log A}\right)\right)\exp\left(cg(B)\right).$$

Darin ist nach Voraussetzung $g(A)=o(\log_2 A)$, also wegen a)

$$\frac{D(A)}{A} \ll \frac{S(A)}{A}\exp\left(cg(A)\right) \ll (\log A)^{-1+o(1)} \ll \log^{-1/2} A.$$

Ist $B>A$ so klein, daß $g(B)-g(A)\ll\dfrac{1}{\log A}$ ausfällt, so wird

$$\left(g(B)-g(A)+O\left(\frac{1}{\log A}\right)\right)\exp\left(cg(B)\right) \ll \frac{1}{\log A}\exp\left(cg(A)\right) \ll \log^{-1/2} A;$$

für größere B ist ohne weiteres

$$\left(g(B)-g(A)+O\left(\frac{1}{\log A}\right)\right)\exp\left(cg(B)\right) \ll \left(g(B)-g(A)\right)\exp\left(cg(B)\right).$$

Insgesamt ergibt sich b).

c) Einsetzen der Voraussetzung $h(x)=o\left((\log x \log_2 x)^{-1}\right)$ in (3) liefert

(4) $$g(x) = o(\log_3 x)$$

sowie für $y>x\to\infty$ noch

(5) $$g(y)-g(x) = o\left(\frac{1}{\log x}\right)+o\left(\log\left(\frac{\log_2 y}{\log_2 x}\right)\right).$$

Wir haben

$$\prod_{\substack{p\leq x\\ p\in S\\ p\nmid d_0}}\left(1-\frac{1}{p}\right) - \sum_{\substack{d\leq x\\ d\in L\\ (d,d_0)=1}} \frac{\mu(d)}{d} \ll \sum_{\substack{d>x\\ d\in L\\ p|d\Rightarrow p\leq x}} \frac{\mu^2(d)}{d}.$$

Für $g(x)\ll 1$ ist die rechte Seite ein $o(1)$; andernfalls wählen wir

$$y = x^{3g(x)}, \quad \eta = \frac{1}{\log x},$$

und zerlegen nach einer Idee von RANKIN [7] (vgl. WOLKE [11])

$$\sum_{\substack{d>x\\ d\in L\\ p|d\Rightarrow p\leq x}} \frac{\mu^2(d)}{d} \ll \sum_{\substack{x<d\leq y\\ d\in L}} \frac{\mu^2(d)}{d} + \sum_{\substack{d>y\\ d\in L\\ p|d\Rightarrow p\leq x}} \frac{\mu^2(d)}{d}\left(\frac{d}{y}\right)^{\eta} = \sum\nolimits_1 + \sum\nolimits_2.$$

Zur Abschätzung von $\sum_1$ können wir wegen (4) die Abschätzung aus b) mit $A=x$, $B=y$ benutzen; indem wir auch (5) beachten, wird

$$\sum\nolimits_1 \ll \log^{-1/2} x + \log\left(\frac{\log_2 y}{\log_2 x}\right) \exp\left(o(\log_3 y)\right) \ll$$

$$\ll \frac{\log g(x)}{\log_2 x} (\log_2 x^{3g(x)})^{o(1)} \ll \frac{\log g(x)}{\log_2 x} (\log_2 x)^{1/2} = o(1).$$

Weiter ist

$$\sum\nolimits_2 \ll y^{-\eta} \sum_{\substack{d>y \\ d\in L \\ p|d \Rightarrow p \leqq x}} \frac{\mu^2(d)}{d^{1-\eta}} \ll y^{-\eta} \prod_{\substack{p\leqq x \\ p\in S}} \left(1+\frac{1}{p^{1-\eta}}\right) \ll y^{-\eta} \exp\left(\sum_{\substack{p\leqq x \\ p\in S}} \frac{1}{p^{1-\eta}}\right) \ll$$

$$\ll \exp\left(-\eta \log y + x^{\eta} g(x)\right) = o(1),$$

denn der Exponent ist gleich

$$-\frac{\log y}{\log x} + e g(x) = -(3-e) g(x)$$

und strebt für $x \to \infty$ gegen $-\infty$.

II. Aus I übernehmen wir die Bezeichnungen S, g, h, D, L und erklären M als die von der zu S komplementären Primzahlmenge $T = \mathbf{P} - S$ multiplikativ erzeugte Halbgruppe. Ferner setzen wir

$$(6) \qquad H(\zeta) = \sum_{n=1}^{\infty} \frac{\sin(2\pi n\zeta)}{n} = \begin{cases} \pi\left(\frac{1}{2}-\zeta\right) & \text{für } 0<\zeta<1, \\ 0 & \text{für } \zeta = 0, \\ \text{1-periodisch fortgesetzt.} \end{cases}$$

Dann besteht der folgende

Satz 2. *Ist neben* $(x \to \infty)$

$$(1) \qquad \frac{h(y)}{h(x)} \ll 1 \quad \textit{für} \quad \sqrt{x} \leqq y \leqq x,$$

$$(7) \qquad h(x) = o\left((\log x \log_2 x)^{-1}\right)$$

noch

$$(8) \qquad g(\infty) = \infty$$

erfüllt, so gilt gleichmäßig für alle $\alpha \in \mathbf{R}$

$$(9) \qquad \sum_{\substack{n\leqq x \\ n\in M}} \frac{\sin(2\pi n\alpha)}{n} = \sum_{\substack{d\leqq x \\ d\in L}} \frac{\mu(d)}{d} H(d\alpha) + o(1).$$

Wir bemerken, daß unter den in Satz 2 aufgeführten Bedingungen, wie schon in I festgestellt,

$$(4) \qquad g(x) = o(\log_3 x)$$

sowie alle in Satz 1 angegebenen Abschätzungen gültig sind. Der Satz 2 übersetzt das Konvergenzverhalten der in (9) links stehenden Summe mit stetigen Summanden, die über die ziemlich dichte Menge M erstreckt ist, auf die rechts stehende Summe mit unstetigen Summanden, die gemäß Satz 1 a) über die ziemlich dünne Menge der quadratfreien Zahlen aus L läuft. Nach einem Satz von WIRSING ([10], Satz 1) gilt nämlich für die Anzahlfunktion $M(x)$ der Menge M eine asymptotische Beziehung der Art

$$M(x) \sim c_1 x \exp(-g(x)).$$

Zum Beweis von Satz 2 haben wir wegen $L \cap M = \{1\}$ und der multiplikativen Struktur der Mengen L, M zunächst die Darstellung

$$\sum_{\substack{d|n \\ d \in L}} \mu(d) = \begin{cases} 1 & \text{für} \quad n \in M, \\ 0 & \text{für} \quad n \notin M \end{cases}$$

der charakteristischen Funktion von M. Durch Umsummierung ergibt sich damit

$$\sum_{\substack{n \leqq x \\ n \in M}} \frac{\sin(2\pi n\alpha)}{n} = \sum_{n \leqq x} \frac{\sin(2\pi n\alpha)}{n} \sum_{\substack{d|n \\ d \in L}} \mu(d) =$$

$$= \sum_{\substack{d \leqq x \\ d \in L}} \frac{\mu(d)}{d} \sum_{n \leqq \frac{x}{d}} \frac{\sin(2\pi nd\alpha)}{n} = \sum_{\substack{d \leqq x \\ d \in L}} \frac{\mu(d)}{d} H(d\alpha) - \sum_{\substack{d \leqq x \\ d \in L}} \frac{\mu(d)}{d} \sum_{n > \frac{x}{d}} \frac{\sin(2\pi nd\alpha)}{n},$$

so daß zu zeigen bleibt

$$\sum_{\substack{d \leqq x \\ d \in L}} \frac{\mu(d)}{d} \sum_{n > \frac{x}{d}} \frac{\sin(2\pi nd\alpha)}{n} = o(1) \quad (x \to \infty). \tag{10}$$

Nach TITCHMARSH [9], p. 42, gilt für $0 \leqq \beta < 1$, $N \in \mathbf{N}$ die Integraldarstellung

$$\sum_{n \leqq N} \frac{\sin(2\pi n\beta)}{n} =$$

$$= \int_0^{N+1/2} \frac{\sin(2\pi\beta t)}{t} dt + 2\pi \int_0^{\beta} \left(\frac{1}{2 \sin \pi t} - \frac{1}{2\pi t} \right) \sin\left(\left(N + \frac{1}{2} \right) 2\pi t \right) dt - \pi\beta.$$

Für $0 \leqq \beta \leqq \frac{1}{2}$ ist das zweite Integral durch partielle Integration $\ll \frac{1}{N}$ und somit

$$\sum_{n > N} \frac{\sin(2\pi n\beta)}{n} = \operatorname{sgn} H(\beta) \int_{N+1/2}^{\infty} \frac{\sin(2\pi)\|\beta\| t)}{t} dt + O\left(\frac{1}{N}\right).$$

Für die innere Summe in (10) erhält man also

$$\sum_{n>\frac{x}{d}} \frac{\sin(2\pi n d\alpha)}{n} = \operatorname{sgn} H(d\alpha) \int_{\left[\frac{x}{d}\right]+\frac{1}{2}}^{\infty} \frac{\sin(2\pi\|d\alpha\| t)}{t}\, dt + O\left(\frac{d}{x}\right). \tag{11}$$

Insbesondere ist wegen $\int_T^\infty \frac{\sin t}{t}\, dt \ll \frac{1}{T}$ für alle x gleichmäßig in $\alpha\in\mathbf{R}$, $d\leqq x$

$$\sum_{n>\frac{x}{d}} \frac{\sin(2\pi n d\alpha)}{n} \ll \min\left(1, \frac{d}{x\|d\alpha\|}\right). \tag{12}$$

Aus Satz 1 b) entnehmen wir

$$\sum_{\substack{x^{1/3}<d\leqq x\\ d\in L}} \frac{\mu^2(d)}{d} \ll \log^{-1/2} x + \left(g(x)-g(x^{1/3})\right)\exp\left(cg(x)\right),$$

und dies ist wegen $g(x)=o(\log_3 x)$ und

$$g(x)-g(x^{1/3}) \ll \int_{x^{1/3}}^{x} \frac{h(t)}{t}\, dt \ll \log_3 x - \log_3 x^{1/3} \ll \frac{1}{\log_2 x}$$

ein $o(1)$.

Zusammen mit (12) folgt

$$\sum_{\substack{x^{1/3}<d\leqq x\\ d\in L}} \frac{\mu(d)}{d} \sum_{n>\frac{x}{d}} \frac{\sin(2\pi n d\alpha)}{n} \ll \sum_{\substack{x^{1/3}<d\leqq x\\ d\in L}} \frac{\mu^2(d)}{d} = o(1).$$

Von der Summe in (10) bleibt nach Einsetzen von (11)

$$\sum_{\substack{d\leqq x^{1/3}\\ d\in L}} \frac{\mu(d)}{d} \operatorname{sgn} H(d\alpha) \int_{\left[\frac{x}{d}\right]+\frac{1}{2}}^{\infty} \frac{\sin(2\pi\|d\alpha\| t)}{t}\, dt + O\left(\frac{1}{x} \sum_{\substack{d\leqq x^{1/3}\\ d\in L}} \mu^2(d)\right)$$

übrig, wobei das Restglied ein $o(1)$ ist.

Anstelle von (10) ist also noch

$$\sum_{\substack{d\leqq x^{1/3}\\ d\leqq L}} \frac{\mu(d)}{d} \operatorname{sgn} H(d\alpha) \int_{\left[\frac{x}{d}\right]+\frac{1}{2}}^{\infty} \frac{\sin(2\pi\|d\alpha\| t)}{t}\, dt = o(1) \tag{13}$$

nachzuweisen.

Wir zerlegen die Summe in zwei Bestandteile, je nachdem ob $\|d\alpha\|<x^{-2/3}$ oder $\geqq x^{-2/3}$ ist.

Mit (12) ergibt sich für den zweiten Teil die Abschätzung

$$\sum_{\substack{d \leqq x^{1/3} \\ d \in L \\ \|d\alpha\| \geqq x^{-2/3}}} \ll \sum_{\substack{d \leqq x^{1/3} \\ d \in L \\ \|d\alpha\| \geqq x^{-2/3}}} \frac{\mu^2(d)}{d} \frac{d}{x\,\|d\alpha\|} \ll x^{-1/3} D(x^{1/3}) = o(1),$$

letzteres wegen Satz 1 a).

Bezeichnet

$$L^* = L^*(\alpha, x) = \{d \in L \colon d \leqq x^{1/3}, \|d\alpha\| < x^{-2/3}\} \quad (x \geqq 3),$$

so bleibt

$$\sum_{d \in L^*} \frac{\mu(d)}{d} \operatorname{sgn} H(d\alpha) \int_{\left[\frac{x}{d}\right]+\frac{1}{2}}^{\infty} \frac{\sin(2\pi \|d\alpha\|\, t)}{t}\, dt \tag{14}$$

abzuschätzen.

Wir benutzen das nachstehende Lemma, das wir später beweisen.

Lemma 1. *Ist L^* nicht leer und $d_0 \in L^*$ minimal gewählt, so gilt für alle $d \in L^*$*

$$d_0 | d, \quad \|d\alpha\| = \frac{d}{d_0} \|d_0\alpha\|, \quad \operatorname{sgn} H(d\alpha) = \operatorname{sgn} H(d_0\alpha), \tag{15}$$

und es ist

$$L^* = \left\{d \in L \colon d_0 | d,\ d \leqq \min\left(x^{1/3}, \frac{d_0}{\|d_0\alpha\|} x^{-2/3}\right)\right\}. \tag{16}$$

Es sei $L^* \neq \emptyset$, anderenfalls ist nichts zu zeigen. Als Folge von Lemma 1 ergibt sich die Darstellung von (14) als

$$\operatorname{sgn} H(d_0\alpha) \sum_{d \in L^*} \frac{\mu(d)}{d} \int_x^{\infty} \frac{\sin\left(2\pi \frac{\|d_0\alpha\|}{d_0} t\right)}{t}\, dt \tag{17}$$

bis auf ein Restglied

$$\ll \sum_{d \in L^*} \frac{\mu^2(d)}{d} \left| \int_x^{\left(\left[\frac{x}{d}\right]+\frac{1}{2}\right)d} \frac{dt}{t} \right| \ll \frac{1}{x} \sum_{d \in L^*} \mu^2(d) = o(1).$$

Zur Abschätzung von (17) setzen wir

$$z = \min\left(x^{1/3}, \frac{d_0}{\|d_0\alpha\|} x^{-2/3}\right)$$

und wählen eine monoton und unbeschränkt wachsende Funktion f etwa mit $f(x) \ll \log x$, so daß

$$\sum_{\substack{f(x) < p \leqq x \\ p \leqq S}} \frac{1}{p} = g(x) - g(f(x))$$

für $x\to\infty$ ebenfalls monoton und unbeschränkt wächst (Die Existenz ist klar). Schließlich bemerken wir, daß die in (17) auftretenden Faktoren alle gleichmäßig beschränkt sind. Für den mittleren Faktor folgt dies unter Verwendung von Satz 1c) und Lemma 1 aus

$$\sum_{d\in L^*}\frac{\mu(d)}{d}=\frac{\mu(d_0)}{d_0}\sum_{\substack{d\leqq\frac{z}{d_0}\\ d\in L\\ (d,d_0)=1}}\frac{\mu(d)}{d}=\frac{\mu(d_0)}{d_0}\prod_{\substack{p\leqq\frac{z}{d_0}\\ p\in S\\ p\nmid d_0}}\left(1-\frac{1}{p}\right)+o(1). \tag{18}$$

Indem wir mehrere Fälle unterscheiden, zeigen wir, daß stets einer der Faktoren in (17) ein $o(1)$ ist.

1. Fall. Es sei $z=x^{1/3}$ und $d_0>f(x)$. Wegen (18) ist dann offenbar

$$\sum_{d\in L^*}\frac{\mu(d)}{d}=o(1). \tag{19}$$

2. Fall. Es sei $z=x^{1/3}$ und $d_0\leqq f(x)$, dann folgt aus

$$\prod_{\substack{p\leqq\frac{z}{d_0}\\ p\nmid d_0\\ p\in S}}\left(1-\frac{1}{p}\right)\leqq\prod_{d_0<p\leqq\frac{z}{d_0}}\left(1-\frac{1}{p}\right)\ll\exp\Bigg(-\sum_{\substack{f(x)<p\leqq x\\ p\in S}}\frac{1}{p}\Bigg)=o(1)$$

ebenfalls (19).

3. Fall. Es sei

$$z=\frac{d_0}{\|d_0\alpha\|}x^{-2/3}\quad\left(\text{also }\frac{\|d_0\alpha\|}{d_0}\geqq\frac{1}{x}\right)\quad\text{und}\quad\frac{\|d_0\alpha\|}{d_0}\leqq\frac{\log x}{x}.$$

Dann ist $\dfrac{z}{d_0}\geqq\dfrac{x^{1/3}}{\log x}$, und je nach Größe von d_0 kommt wie in den vorigen Fällen (19).

4. Fall. Es sei

$$z=\frac{d_0}{\|d_0\alpha\|}x^{-2/3}\quad\text{und}\quad\frac{\|d_0\alpha\|}{d_0}>\frac{\log x}{x}.$$

Diesmal betrachten wir das Integral aus (17).

$$\int_x^\infty\frac{\sin\left(2\pi\frac{\|d_0\alpha\|}{d_0}t\right)}{t}\,dt=\int_{2\pi\frac{\|d_0\alpha\|}{d_0}x}^\infty\frac{\sin t}{t}\,dt\ll\frac{d_0}{x\|d_0\alpha\|}\ll\frac{1}{\log x}=o(1).$$

Insgesamt ist damit gezeigt, daß der Ausdruck in (17) ein $o(1)$ gleichmäßig für $\alpha\in\mathbf{R}$ ist.

Zum Abschluß des Beweises von Satz 2 bleibt noch Lemma 1 zu erledigen. Zu jedem $d\in L^*$ existiert ein ganzes a mit

$$\left|\alpha-\frac{a}{d}\right| < \frac{1}{d}x^{-2/3}.$$

Dieses a ist eindeutig bestimmt, denn sonst $(a\neq a')$ wäre

$$\frac{1}{d} \leqq \left|\frac{a}{d}-\frac{a'}{d}\right| \leqq \left|\alpha-\frac{a}{d}\right|+\left|\alpha-\frac{a'}{d}\right| < \frac{2}{d}x^{-2/3}$$

entgegen $x\geqq 3$. Sei $d_0\in L^*$ minimal, $d>d_0$, $d\in L^*$. Dann gilt $\frac{a}{d}=\frac{a_0}{d_0}$, denn andernfalls wäre

$$\frac{1}{dd_0} \leqq \left|\frac{a}{d}-\frac{a_0}{d_0}\right| \leqq \left|\alpha-\frac{a}{d}\right|+\left|\alpha-\frac{a_0}{d_0}\right| < x^{-2/3}\left(\frac{1}{d}+\frac{1}{d_0}\right)$$

entgegen $d, d_0\leqq x^{1/3}$. Daraus folgt $d_0|d$ (denn auf Grund der Minimalität von d_0 ist $(d_0, a_0)=1$), $\|d\alpha\|=\frac{d}{d_0}\|d_0\alpha\|$ für alle $d\in L^*$ sowie (16). In (15) bleibt $\operatorname{sgn} H(d\alpha)=$ $=\operatorname{sgn} H(d_0\alpha)$ zu zeigen übrig oder gleichwertig, daß $\{d\alpha\}$ bzw. $\{d_0\alpha\}$ stets entweder beide in $\left[0,\frac{1}{2}\right)$ oder beide in $\left[\frac{1}{2},1\right)$ liegen. Es sei zuerst $\{d_0\alpha\}=\|d_0\alpha\|$. Nach dem bereits bewiesenen Teil von (15) folgt

$$\{d\alpha\} \leqq \frac{d}{d_0}\{d_0\alpha\} = \frac{d}{d_0}\|d_0\alpha\| = \|d\alpha\|, \quad \text{also} \quad \{d\alpha\} = \|d\alpha\|.$$

Ist umgekehrt $\{d_0\alpha\}=1-\|d_0\alpha\|$, so setzen wir $\beta=1-\alpha$ und erhalten $\{d_0\beta\}=$ $=1-\{d_0\alpha\}=\|d_0\alpha\|=\|d_0\beta\|$, also wie im ersten Fall $\{d\beta\}=\|d\beta\|$. Es folgt $\{d\alpha\}=$ $=1-\{d\beta\}=1-\|d\beta\|=1-\|d\alpha\|$. Damit ist Lemma 1 vollständig bewiesen.

III. Der Satz 2 ermöglicht leicht die Angabe von Primzahlmengen T der Relativdichte 1 in $\mathbf{P}$, so daß die trigonometrische Reihe

$$\sum_{\substack{n=1\\ n\in M}}^{\infty} \frac{\sin(2\pi n\alpha)}{n} \quad (\alpha\in\mathbf{R}), \tag{20}$$

erstreckt über die von T multiplikativ erzeugte Halbgruppe M, stellenweise divergiert. Etwa enthalte T alle Primzahlen mit Ausnahme einer (dünnen) Menge S von Primzahlen $\equiv 3 \bmod 4$, für die die Voraussetzungen von Satz 2 erfüllt sind. Bei $\alpha=\frac{1}{4}$ gilt dann

$$\sum_{\substack{d\leqq x\\ d\in L}} \frac{\mu(d)}{d} H\left(\frac{d}{4}\right) = H\left(\frac{1}{4}\right)\Bigg\{ \sum_{\substack{d\leqq x\\ d\in L\\ d\equiv 1(4)}} \frac{\mu(d)}{d} - \sum_{\substack{d\leqq x\\ d\in L\\ d\equiv 3(4)}} \frac{\mu(d)}{d}\Bigg\} = \frac{\pi}{4} \sum_{\substack{d\leqq x\\ d\in L}} \frac{\mu^2(d)}{d},$$

und Satz 2 liefert die Divergenz der Reihe (20) an der Stelle $\alpha=\frac{1}{4}$.

Im weiteren konstruieren wir spezielle Primzahlmengen S der Relativdichte 0 in $\mathbf{P}$ (durch deren Angabe die Mengen T, L, M wie früher festgelegt sind), für die sich die Reihe (20) als gleichmäßig konvergent erweisen wird. Es sei A_1 genügend groß (etwa $\log_3 A_1 \geqq e$) und für $k \in \mathbf{N}$

$$\log_2 A_{k+1} = \log_2 A_k \log_3 A_k. \tag{21}$$

Induktion zeigt

$$k \ll \log_3 A_k < \log_3 A_{k+1} \ll k \log k \quad (k \to \infty). \tag{22}$$

Es sei ferner ϑ: $\mathbf{N} \to \mathbf{R}$ eine Funktion mit

$$k \leqq \vartheta(k) < \vartheta(k+1) = \bigl(1+o(1)\bigr)\vartheta(k) \quad (k \to \infty), \tag{23}$$

$$\sum_{k=1}^{\infty} \frac{1}{\vartheta(k)} = \infty, \tag{24}$$

$$\sum_{\varkappa=1}^{k} \frac{1}{\vartheta(\varkappa)} = o\bigl(\log \vartheta(k)\bigr) \quad (k \to \infty). \tag{25}$$

Solche Funktionen existieren, wie das Beispiel $\vartheta(k) = k \log (ek)$ zeigt. Schließlich sei $C > 0$ eine hinreichend große Konstante und p_k die kleinste Primzahl mit

$$p_k \geqq (\log A_k)^{1/C}. \tag{26}$$

Mit den Bezeichnungen

$$r_k = \left[\frac{p_k}{\vartheta(k) \log_2 A_k \log_3 A_k}\right], \quad v_k = \left[\frac{\log_4 A_{k+1}}{\log 2}\right]$$

und $R_k(v) = \left\{1, 2, \ldots, \left[\frac{r_k}{2^v}\right]\right\}$ setzen wir

$$S = \bigcup_{k \in \mathbf{N}} \bigcup_{v=0}^{v_k} \bigcup_{a \in R_k(v)} \{p \in \mathbf{P}\colon A_k < p \leqq A_{k+1}^{2^v},\ p \equiv a \bmod p_k\}. \tag{27}$$

Diese Menge S ist zusammengesetzt aus den in den Abschnitten $(A_k, A_{k+1}]$ enthaltenen Primzahlen, die in r_k festen Restklassen mod p_k liegen, wobei jeweils in kurzen Teilabschnitten $(A_k, A_k^*]$ mit

$$A_k^* = A_k^{\log_3 A_k} \tag{28}$$

weitere Primzahlen hinzugefügt werden. Letzteres dient lediglich der Glättung von S im Hinblick auf die Bedingung (1) in den Sätzen 1 und 2. Indem wir den Primzahlsatz von Page—Siegel—Walfisz (s. etwa PRACHAR [6]) verwenden, folgt für $A_k^{2^v} < x \leqq A_k^{2^{v+1}}$, $0 \leqq v \leqq v_{k-1} - 1$

$$\frac{r_{k-1}}{2^v \varphi(p_{k-1})} \frac{x}{\log x} \ll S(x) \ll \left(\frac{r_{k-1}}{2^{v-1} \varphi(p_{k-1})} + \frac{r_k}{\varphi(p_k)}\right) \frac{x}{\log x}$$

und für $A_k^* < x \leqq A_{k+1}$

$$\frac{r_k}{\varphi(p_k)} \frac{x}{\log x} \ll S(x) \ll \left(\frac{r_{k-1}}{2^{v_{k-1}} \varphi(p_{k-1})} + \frac{r_k}{\varphi(p_k)}\right) \frac{x}{\log x}.$$

Wegen (21), der Wahl von v_k und r_k sind die oberen und unteren Schranken in diesen Abschätzungen jeweils von derselben Größenordnung und liefern überdies (1). Speziell ergibt sich noch die globale Abschätzung

$$\frac{x}{\vartheta(k) \log x \log_2 x \log_3 x} \ll S(x) \ll \frac{x}{\vartheta(k) \log x \log_2 x} \quad (A_k < x \leqq A_{k+1}), \tag{29}$$

die die Gültigkeit von (7) zeigt. Schließlich haben wir

$$g(A_{k+1}) - g(A_k) = \sum_{\substack{A_k < p \leqq A_k^* \\ p \in S}} \frac{1}{p} + \sum_{\substack{A_k^* < p \leqq A_{k+1} \\ p \in S}} \frac{1}{p},$$

worin die erste Summe

$$\ll \left(\frac{r_{k-1}}{\varphi(p_{k-1})} + \frac{r_k}{\varphi(p_k)}\right) \log\left(\frac{\log A_k^*}{\log A_k}\right) \ll \frac{\log_4 A_k}{\vartheta(k) \log_2 A_k}$$

ist, während die zweite Summe asymptotisch gleich

$$\frac{r_k}{\varphi(p_k)} (\log_2 A_{k+1} - \log_2 A_k^*) \sim \frac{1}{\vartheta(k)}$$

ist. Insgesamt folgt

$$g(A_{k+1}) - g(A_k) \sim \frac{1}{\vartheta(k)} \tag{29}$$

und damit weiter

$$\sum_{\varkappa \leqq k} \frac{1}{\vartheta(\varkappa)} \ll g(x) \ll \sum_{\varkappa \leqq k} \frac{1}{\vartheta(\varkappa)} \quad (A_k < x \leqq A_{k+1}). \tag{30}$$

Wegen (24) und (30) ist insbesondere auch (8) erfüllt.

Für spätere Abschätzungen sei noch mit dem C aus (26)

$$A_k' = \exp(\log A_k)^C. \tag{31}$$

Dann gilt $A_k < A_k^* < A_k' < A_{k+1}$ für alle in Abhängigkeit von C hinreichend großen k, und es ergibt sich analog mit (22)

$$g(A_k') - g(A_k) \ll \frac{1}{\vartheta(k) \log_3 A_k} \ll \frac{1}{k\vartheta(k)}. \tag{32}$$

Nach diesen Vorbereitungen formulieren wir den

SATZ 3. *Es sei S durch* (27) *erklärt und M die von $T = \mathbf{P} - S$ multiplikativ erzeugte Halbgruppe. Dann konvergiert die trigonometrische Reihe* (20) *gleichmäßig in $\alpha \in \mathbf{R}$.*

Offensichtlich beinhaltet Satz 3 den in der Einleitung genannten Hauptsatz, denn durch geeignete Wahl der Funktion ϑ kann ein beliebig langsames Wachstum von g erreicht werden.

Gemäß Satz 2 ist zum Beweis von Satz 3 zu zeigen, daß für $B>A\to\infty$

$$\sum_{\substack{A<d\leqq B\\ d\in L}}\frac{\mu(d)}{d}H(d\alpha)=o(1) \tag{33}$$

gleichmäßig für $\alpha\in\mathbf{R}$ gilt. Dem Nachweis von (33) sind die folgenden Abschnitte gewidmet.

IV. In Abhängigkeit von den Approximationseigenschaften der Zahl $\alpha\in\mathbf{R}$ unterscheiden wir vier Arten von Intervallen $I\subset[N_0(C),\infty)$, wobei $N_0(C)$ genügend groß in Abhängigkeit von der Konstanten C aus (26) gewählt sei.

Def. I heißt ein α-Intervall von 1. Art $\Leftrightarrow$

$$\bigvee_{\substack{a,q\\(a,q)=1}}\bigwedge_{N\in I}\left(1\leqq q\leqq \log^{2C}N\wedge\frac{1}{N\log^C N}<\left|\alpha-\frac{a}{q}\right|\leqq\frac{\log^C N}{qN}\right),$$

von 2. Art $\Leftrightarrow$

$$\bigvee_{\substack{a,q\\(a,q)=1}}\bigwedge_{N\in I}\left(\exp\left(\frac{\log_2 N}{C\log_3 N}\right)<q\leqq \log^{2C}N\wedge\left|\alpha-\frac{a}{q}\right|\leqq\frac{1}{N\log^C N}\right),$$

von 3. Art $\Leftrightarrow$

$$\bigvee_{\substack{a,q\\(a,q)=1}}\bigwedge_{N\in I}\left(1\leqq q\leqq \exp\left(\frac{\log_2 N}{C\log_3 N}\right)\wedge\left|\alpha-\frac{a}{q}\right|\leqq\frac{1}{N\log^C N}\right),$$

von 4. Art $\Leftrightarrow$

$$\bigvee_{\substack{a,q\\(a,q)=1}}\bigwedge_{N\in I}\left(\log^{2C}N<q\leqq \frac{N}{\log^C N}\wedge\left|\alpha-\frac{a}{q}\right|\leqq\frac{\log^C N}{qN}\right).$$

Nach dem Dirichletschen Approximationssatz

$$\bigwedge_{N\geqq N_0(C)}\bigvee_{q}\left(1\leqq q\leqq\frac{N}{\log^C N}\wedge\|q\alpha\|\leqq\frac{\log^C N}{N}\right)$$

liegt jedes $N\geqq N_0(C)$ in einem dieser α-Intervalle. Die α-Intervalle 1., 2. und 3. Art sind untereinander disjunkt (s. die Beweise zu den folgenden Lemmata), bei denen 4. Art ist dies im allgemeinen nicht der Fall. Der Bereich $[N_0,\infty)$ werde ausgeschöpft durch die maximal gewählten α-Intervalle 1., 2. und 3. Art sowie durch Intervalle 4. Art. Die Beiträge der α-Intervalle verschiedener Art zur Summe in (33) schätzen wir getrennt ab. Wir zeigen zunächst, daß die Abschnitte 1. oder 2. Art „kurz" sind und „weit auseinander" liegen.

LEMMA 2. *Es sein $(N_\nu, N'_\nu]$ und $(N_{\nu+1}, N'_{\nu+1}]$ zwei α-Intervalle* 1. *Art mit maximaler Länge und $N'_\nu<N'_{\nu+1}$. Dann gilt*

$$N'_\nu\leqq N_\nu\log^{3C}N_\nu,\ \log_2 N_{\nu+1}\gg\log N_\nu. \tag{34}$$

Die zweite Ungleichung ist auch für α-Intervalle 3. *Art richtig.*

BEWEIS. Es seien $\frac{a_\nu}{q_\nu}, \frac{a_{\nu+1}}{q_{\nu+1}}$ die zu den genannten Intervallen gehörenden rationalen Approximationen von α. Dann folgt die erste Ungleichung in (34) aus

$$\frac{1}{N_\nu \log^C N_\nu} \leqq \left|\alpha - \frac{a_\nu}{q_\nu}\right| \leqq \frac{\log^C N_\nu'}{N_\nu'}.$$

Zum Nachweis der zweiten Ungleichung in (34) beachten wir $q_\nu < q_{\nu+1}$ sowie

$$\frac{1}{\log^{4C} N_{\nu+1}'} \leqq \frac{1}{q_\nu q_{\nu+1}} \leqq \left|\frac{a_\nu}{q_\nu} - \frac{a_{\nu+1}}{q_{\nu+1}}\right| \leqq \left|\alpha - \frac{a_\nu}{q_\nu}\right| + \left|\alpha - \frac{a_{\nu+1}}{q_{\nu+1}}\right| \leqq 2\frac{\log^C N_\nu'}{N_\nu'},$$

woraus nach leichten Umformungen die Behauptung folgt.

LEMMA 3. *Es seien $(N_\nu, N_\nu']$ und $(N_{\nu+1}, N_{\nu+1}']$ zwei α-Intervalle 2. Art mit maximaler Länge und $N_\nu' < N_{\nu+1}'$. Dann gilt*

(35) $$\log_2 N_\nu' \ll \log_2 N_\nu \log_3 N_\nu, \quad \log_2 N_{\nu+1} \log_3 N_{\nu+1} \gg \log N_\nu.$$

BEWEIS. Die erste Ungleichung ergibt sich aus

$$\exp\left(\frac{\log_2 N_\nu'}{C \log_3 N_\nu'}\right) \leqq q_\nu \leqq \log^{2C} N_\nu,$$

die zweite aus $q_\nu < q_{\nu+1}$ und

$$\frac{1}{\log^{4C} N_{\nu+1}'} \leqq \frac{1}{q_\nu q_{\nu+1}} \leqq \left|\frac{a_\nu}{q_\nu} - \frac{a_{\nu+1}}{q_{\nu+1}}\right| \leqq \left|\alpha - \frac{a_\nu}{q_\nu}\right| + \left|\alpha - \frac{a_{\nu+1}}{q_{\nu+1}}\right| \leqq \frac{2}{N_\nu' \log^C N_\nu'}$$

wie in Lemma 2.

Ein Vergleich von (21) mit (34) bzw. (35) zeigt, daß höchstens $\ll 1$ Intervalle $(A_k, A_{k+1}]$ ein α-Intervall 1. oder 2. Art überdecken. Ist $A_k \leqq N_\nu$, wobei N_ν den Anfang eines α-Intervalls 1. oder 2. Art bezeichnet, so liefern (21), (34), (35) nach kurzer Rechnung (wir erinnern an $N_\nu \geqq N_0(C)$) weiter

(36) $$A_{k^2} \leqq N_{\nu+1}.$$

Triviale Abschätzung des Beitrags eines α-Intervalls $(N_\nu, N_\nu']$ von 1. oder 2. Art mit $A_k \leqq N_\nu$ zur Summe in (33) gibt bei Verwendung von Satz 1b)

$$\sum_{\substack{N_\nu < d \leqq N_\nu' \\ d \in L}} \frac{\mu(d)}{d} H(d\alpha) \ll \sum_{\substack{N_\nu < d \leqq N_\nu' \\ d \in L}} \frac{\mu^2(d)}{d} \ll \frac{1}{\vartheta(k)} \exp\left(cg(A_k)\right).$$

Wegen (25) und $\vartheta(k) \gg k$ ist dies $\ll k^{-1/2}$. Bezeichnet l die größte Zahl mit $A_l \leqq A$, so erweist sich infolge (36) der Beitrag aller α-Intervalle 1. oder 2. Art zur Summe in (33) als

$$\ll \sum_{\nu \geqq 0} (l^{2^\nu})^{-1/2} \ll l^{-1/2}.$$

Für $A \to \infty$ ist dies ein $o(1)$, unabhängig von $\alpha \in \mathbf{R}$.

V. Wir schätzen den Beitrag der α-Intervalle 3. Art zur Summe

$$\sum(A, B, \alpha) = \sum_{\substack{A<d\leqq B \\ d\in L}} \frac{\mu(d)}{d} H(d\alpha)$$

ab, zunächst für rationale α.

LEMMA 4. *Für* $A_k \leqq N < N' \leqq A_{k+1}$, $1 \leqq q \leqq \exp\left(\frac{\log_2 N'}{C \log_3 N'}\right)$, $(a, q)=1$ gilt

$$\sum\left(N, N', \frac{a}{q}\right) \ll k^{-3/2}. \tag{37}$$

BEWEIS. Wir dürfen k und damit A_k in Abhängigkeit von C als genügend groß annehmen. Für $A_k \leqq N' \leqq A'_k$ liefert Satz 1 b) bei trivialer Abschätzung

$$\sum\left(N, N', \frac{a}{q}\right) \ll \sum_{\substack{A_k<d\leqq A'_k \\ d\in L}} \frac{\mu^2(d)}{d} \ll \left(g(A'_k)-g(A_k)\right)\exp\left(cg(A'_k)\right) \ll k^{-3/2},$$

letzteres wegen (23), (25), (30) und (32). Für $A'_k \leqq N < N' \leqq A_{k+1}$ haben wir die Zerlegung

$$\sum\left(N, N', \frac{a}{q}\right) = \sum_{1\leqq \nu \leqq \frac{q-1}{2}} H\left(\frac{\nu a}{q}\right)\left\{\sum_{\substack{N<d\leqq N' \\ d\in L \\ d\equiv \nu(q)}} \frac{\mu(d)}{d} - \sum_{\substack{N<d\leqq N' \\ d\in L \\ d\equiv -\nu(q)}} \frac{\mu(d)}{d}\right\}. \tag{38}$$

Der Inhalt der geschweiften Klammer läßt sich mit Hilfe von Charaktersummen darstellen. Mit den Bezeichnungen $(\nu, q)=d'$, $q'=\frac{q}{d'}$, $\nu'=\frac{\nu}{d'}$ ist für $d' \in L$

$$\sum_{\substack{N<d\leqq N' \\ d\in L \\ d\equiv \nu(q)}} \frac{\mu(d)}{d} = \frac{\mu(d')}{d'\varphi(q')} \sum_{\chi \bmod q'} \bar{\chi}(\nu') \sum_{\substack{N<dd'\leqq N' \\ d\in L \\ (d,d')=1}} \frac{\mu(d)\chi(d)}{d},$$

für $d' \notin L$ verschwindet die linke Seite. Einsetzen in (38) zeigt, daß die Beiträge der jeweiligen Hauptcharaktere $\chi_0 \bmod q'$ verschwinden, und liefert die Abschätzung

$$\sum\left(N, N', \frac{a}{q}\right) \ll q \max_{q'|q} \max_{\chi\neq\chi_0 \bmod q'} \left|\sum_{\substack{N<dd'\leqq N' \\ d\in L \\ (d,d')=1}} \frac{\mu(d)\chi(d)}{d}\right|. \tag{39}$$

Wegen $N' \leqq A_{k+1}$ ist bei genügend großem k

$$q \leqq \exp\left(\frac{\log_2 A_{k+1}}{C \log_3 A_{k+1}}\right) < (\log A_k)^{1/C} \leqq p_k, \tag{40}$$

erst recht $q', d' \leqq q < A_k'^{1/2} \leqq N^{1/2}$. Besteht für $A_k'^{1/2} < x \leqq A_{k+1}$, $\chi \neq \chi_0 \bmod q'$ die Abschätzung

$$\sum_{\substack{d\leqq x \\ d\in L \\ (d,d')=1}} \chi(d)\mu(d) \ll x(\log x)^{-2}, \tag{41}$$

so ergibt sich mittels partieller Summation und (31)

$$\sum_{\substack{N<dd'\leqq N' \\ d\in L \\ (d,d')=1}} \frac{\mu(d)\chi(d)}{d} \ll \int_{A_k'^{1/2}}^{N'} \frac{dx}{x\log^2 x} \ll (\log A_k)^{-C}.$$

Aus (39), (40) und (22) folgt daraus (bei genügend großem C) die Behauptung (37) von Lemma 4.

Es bleibt (41) zu zeigen übrig. Als erzeugende Funktion für die Summe linker Hand kann nach Definition der Mengen S, L für $A_k'^{1/2}<x\leqq A_{k+1}$ der folgende Ausdruck $F(s,\chi)$ gewählt werden (dabei sei $R_k=R_k(0)$):

$$F(s,\chi) = \prod_{\substack{p\leqq A_k^* \\ p\in S \\ p\nmid d'}} \left(1-\frac{\chi(p)}{p^s}\right) \prod_{a\in R_k} \prod_{\substack{p>A_k^* \\ p\equiv a(p_k)}} \left(1-\frac{\chi(p)}{p^s}\right) = f_1(s) \prod_{a\in R_k} \prod_{p\equiv a(p_k)} \left(1-\frac{\chi(p)}{p^s}\right).$$

Darin ist $f_1(s)$ eine für $\sigma\geqq\frac{3}{4}$ analytische Funktion mit

$$f_1(s) \ll \prod_{p\leqq A_k^*} \left(1+\frac{1}{p^\sigma}\right) \ll \exp\left(\sum_{p\leqq A_k^*} \frac{1}{p^\sigma}\right).$$

Das Produkt über die $a\in R_k$, $p\equiv a(p_k)$ stellen wir durch Dirichletsche L-Reihen dar. Es gilt

$$\prod_{a\in R_k} \prod_{p\equiv a(p_k)} \left(1-\frac{\chi(p)}{p^s}\right) = f_2(s) \prod_{\psi \bmod p_k} L(s,\chi\psi)^{-\frac{1}{\varphi(p_k)}\sum_{a\in R_k}\bar\psi(a)},$$

worin mit einer absoluten Konstanten $c_1>0$ (wir erinnern an $|R_k|=r_k$) $f_2(s)\ll c_1^{r_k}$ eine ebenfalls für $\sigma\geqq\frac{3}{4}$ analytische Funktion bedeutet. Insgesamt ist also

$$F(s,\chi) = f_3(s) \prod_{\psi \bmod p_k} L(s,\chi\psi)^{-\frac{1}{\varphi(p_k)}\sum_{a\in R_k}\bar\psi(a)} \tag{42}$$

und darin

$$f_3(s) \ll c_1^{r_k} \exp\left(\sum_{p\leqq A_k^*} \frac{1}{p^\sigma}\right) \tag{43}$$

analytisch für $\sigma\geqq\frac{3}{4}$. Zwischen der Summe in (41) und der erzeugenden Funktion $F(s,\chi)$ besteht für $A_k'^{1/2}<x\leqq A_{k+1}$, $x\notin \mathbf{N}$, der Zusammenhang (vgl. etwa PRACHAR [6], Satz A 3.1)

$$\sum_{\substack{d\leqq x \\ d\in L \\ (d,d')=1}} \mu(d)\chi(d) = \frac{1}{2\pi i} \int_{a-iT}^{a+iT} F(s,\chi)\frac{x^s}{s}\,ds + O\left(\frac{x\log x}{T}\right), \tag{44}$$

wobei zur Abkürzung $a=1+\frac{1}{\log x}$, $T=\exp(\log^{1/20} x)$ gesetzt ist. Zur günstigen Auswertung des Integrals verschieben wir den Integrationsweg innerhalb des Analytizitätsgebietes von $F(s,\chi)$ möglichst weit nach links über $\sigma=1$ hinaus. Zur Durchführung sind drei Tatsachen wesentlich: Erstens besteht gemäß (41) die Ungleichung $q'<p_k$, zweitens ist p_k Primzahl, drittens treten nur Charaktere $\chi\neq\chi_0 \bmod q'$ auf. Demnach ist keiner der in (42) vorkommenden Charaktere $\chi\psi$ der Hauptcharakter $\bmod q'p_k$. Die zu diesen Charakteren gehörenden L-Reihen sind also analytisch und nullstellenfrei in dem Bereich

$$1-\frac{1}{\log^{10} T} \leqq \sigma \leqq a, \quad |t| \leqq T \tag{45}$$

und genügen dort der Abschätzung

$$(\log T)^{-10} \ll |L(s, \chi\psi)| \ll (\log T)^2$$

(vgl. etwa SCHWARZ [8], S. 165, Satz 7.1).

In dem durch (45) gegebenen Bereich hat dies die Analytizität von $F(s, \chi)$ sowie gemäß (43) die Abschätzung

$$F(s, \chi) \ll (\log T)^{8r_k} \exp\Big(\sum_{p\leqq A_k^*} \frac{1}{p^\sigma}\Big)$$

zur Folge. Kurze Rechnung nach Einbringen der Bedeutungen von T, r_k und der Ungleichung $\sigma\geqq 1-(\log T)^{-10}$ liefert

$$F(s, \chi) \ll \exp\big((\log A_k)^{1/C}\big). \tag{46}$$

Indem wir den Integrationsweg in (44) durch den Rechtwinkelzug mit den Eckpunkten $a-iT$, $1-\frac{1}{\log^{10} T}-iT$, $1-\frac{1}{\log^{10} T}+iT$, $a+iT$ ersetzen, ergibt sich zusammen mit (46) für die Summe in (41) die Schranke

$$x\exp\big((\log A_k)^{1/C}\big)\Big(\log T \exp\Big(-\frac{\log x}{\log^{10} T}\Big)+\frac{1}{T}\Big) \ll x(\log x)^{-2}.$$

Das war in (41) behauptet, womit Lemma 4 vollständig bewiesen ist.

Es sei $(N_\nu, N_\nu']$ mit $N_\nu\geqq N_0(C)$ ein α-Intervall 3. Art von maximaler Länge und a_ν/q_ν die zugehörige rationale Approximation von α, $\beta_\nu=\alpha-a_\nu/q_\nu$ mit $|\beta_\nu|\leqq \leqq (N_\nu' \log^C N_\nu')^{-1}$. Für die $d\in L$, $N_\nu<d\leqq N_\nu'$ ist dann

$$H(d\alpha)=\begin{cases} H\Big(d\frac{a_\nu}{q_\nu}\Big)-\pi d\beta_\nu+\frac{\pi}{2}\operatorname{sgn}\beta_\nu & \text{für } q_\nu | d,\\ H\Big(d\frac{a_\nu}{q_\nu}\Big)-\pi d\beta_\nu & \text{für } q_\nu \nmid d,\end{cases}$$

letzteres wegen

$$|d\beta_\nu|<\frac{1}{q_\nu} \quad\text{und}\quad \frac{1}{q_\nu}\leqq \Big\{d\frac{a_\nu}{q_\nu}\Big\}\leqq \frac{q_\nu-1}{q_\nu}.$$

Es folgt

$$\sum_{\substack{N_\nu<d\leqq N_\nu'\\ d\in L}} \frac{\mu(d)}{d} H(d\alpha) = \sum_{\substack{N_\nu<d\leqq N_\nu'\\ d\in L}} \frac{\mu(d)}{d} H\Big(d\frac{a_\nu}{q_\nu}\Big)-\pi\beta_\nu \sum_{\substack{N_\nu<d\leqq N_\nu'\\ d\in L}} \mu(d)+ \tag{47}$$

$$+\frac{\pi}{2}\operatorname{sgn}\beta_\nu \sum_{\substack{N_\nu<d\leqq N_\nu'\\ d\in L\\ q_\nu|d}} \frac{\mu(d)}{d}.$$

Für zwei derartige α-Intervalle 3. Art $(N_\nu, N'_\nu]$, $(N_{\nu+1}, N'_{\nu+1}]$ mit $N'_\nu < N'_{\nu+1}$ gilt $q_\nu < q_{\nu+1}$. Die Intervalle sind disjunkt (s. Lemma 2), genauer erhält man aus

$$(\log N_{\nu+1})^{-1} \leqq \frac{1}{q_{\nu+1}^2} \leqq \left|\frac{a_{\nu+1}}{q_{\nu+1}} - \frac{a_\nu}{q_\nu}\right| \leqq \frac{2}{N'_\nu \log^C N'_\nu} \leqq \frac{1}{N'_\nu}$$

die Ungleichungen

$$N_{\nu+1} > e^{N'_\nu}, \tag{48}$$

sowie

$$q_{\nu+1} > N'^{1/2}_\nu > e^{q_\nu}. \tag{49}$$

Ist nun $l \in \mathbf{N}$ durch $A_l < A \leqq A_{l+1}$ bestimmt und ν_0 mit $A < N'_{\nu_0}$ minimal gewählt, so tragen wegen Lemma 4 die beiden ersten Summen rechter Hand in (47), summiert über alle α-Intervalle 3. Art aus $(A, B]$, zur Summe in (33) nur $\ll l^{-1/2}$ bei, was für $A \to \infty$, also $l \to \infty$, gleichmäßig gegen Null geht. Es bleibt noch der vom letzten Summanden in (47) stammende Anteil

$$\sum_{\nu \geqq \nu_0} \left| \sum_{\substack{\max(A, N_\nu) < d \leqq N'_\nu \\ d \in L \\ d_\nu | d}} \frac{\mu(d)}{d} \right| \tag{50}$$

abzuschätzen. Vom zweiten Summanden an $(\nu > \nu_0)$ ist $A < N_\nu$ und

$$\sum_{\substack{N_\nu < d \leqq N'_\nu \\ d \in L \\ q_\nu | d}} \frac{\mu(d)}{d} = \frac{\mu(q_\nu)}{q_\nu} \sum_{\substack{\frac{N_\nu}{q_\nu} < d \leqq \frac{N'_\nu}{q_\nu} \\ (q_\nu, d) = 1}} \frac{\mu(d)}{d} \quad \text{für} \quad q_\nu \in L$$

bzw. Null, falls $q_\nu \notin L$. Nach Satz 1 c) ist darin die Summe rechts gleichmäßig beschränkt, also der Beitrag der Summanden für $\nu > \nu_0$ zu (50)

$$\ll \sum_{\nu > \nu_0} \frac{1}{q_\nu} \ll A^{-1/2},$$

letzteres wegen $q_{\nu_0+1} > A^{1/2}$ und (49). Für $A \to \infty$ geht dies gleichmäßig gegen Null. Für den ersten Summanden mit $\nu = \nu_0$ erhalten wir

$$\left| \sum_{\substack{\max(A, N_{\nu_0}) < d \leqq N'_{\nu_0} \\ d \in L \\ d_{\nu_0} | d}} \frac{\mu(d)}{d} \right| \ll \frac{1}{q_{\nu_0}} \left| \sum_{\substack{x < d \leqq y \\ d \in L \\ q_{\nu_0} | d}} \frac{\mu(d)}{d} \right|,$$

wobei $x = \frac{1}{q_{\nu_0}} \max(A, N_{\nu_0})$, $y = \frac{1}{q_{\nu_0}} N'_{\nu_0}$ gesetzt ist. Die rechte Seite läßt sich, abgesehen von einer gleichmäßigen Nullfolge für $x \to \infty$, bei Verwendung von Satz 1 c) abschätzen durch

$$\frac{1}{q_{\nu_0}} \prod_{\substack{p \leqq x \\ p \in S \\ p \nmid q_{\nu_0}}} \left(1 - \frac{1}{p}\right) \ll \frac{1}{q_{\nu_0}} \exp(-g(x) + g(q_{\nu_0})). \tag{51}$$

Gemäß (24) gibt es eine monoton und unbeschränkt wachsende Funktion f mit $g(f(x))=o(g(x))$. Indem wir die Fälle $q_{\nu_0} \leqq f(x)$ und $q_{\nu_0} > f(x)$ unterscheiden, erweist sich auch der Betrag von (51) zur Summe in (50) als ein gleichmäßiges $o(1)$ für $x \to \infty$ oder gleichwertig $A \to \infty$.

Insgesamt ist damit gezeigt, daß der Beitrag aller α-Intervalle 3. Art zur Summe in (33) für $A \to \infty$ gleichmäßig gegen Null geht.

VI. Zur Abschätzung des Beitrages der α-Intervalle 4. Art zu $\sum (A, B, \alpha)$ sei mit ganzen, teilerfremden Zahlen a, q

$$\log^{2C} N < q \leqq Q = \frac{N}{\log^C N}, \qquad \left|\alpha - \frac{a}{q}\right| \leqq \frac{1}{qQ}$$

und

$$I_{a/q} = \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right].$$

Ist $H(d\alpha)$ für ein festes $d \leqq N$ stetig in $\alpha \in I_{a/q}$, so gilt

$$H(d\alpha) - H\left(d\frac{a}{q}\right) \ll \frac{d}{qQ} \ll \log^{-C} N. \tag{52}$$

Hat dagegen $H(d\alpha)$ für ein festes $d \leqq N$ eine Sprungstelle in $I_{a/q}$, so genau eine und zwar bei $\frac{b}{d}$ mit einem gewissen $b \in \mathbf{Z}$. Es gilt dann

$$|ad - bq| = qd\left|\frac{a}{q} - \frac{b}{d}\right| \leqq \frac{d}{Q} \leqq \log^C N.$$

Die Anzahl ϱ aller Sprungstellen von $H(d\alpha)$ mit $d \leqq N$, $\alpha \in I_{a/q}$ ist daher

$$\leqq \sum_{|l| \leqq \log^C N} \sum_{\substack{d \leqq N \\ \bigvee_{b \in \mathbf{Z}} ad - bq = l}} 1 = \sum_{|l| \leqq \log^C N} \sum_{\substack{d \leqq N \\ ad \equiv l(q)}} 1.$$

Wegen $(a, q) = 1$ bestimmt die Kongruenz $ad \equiv l(q)$ genau eine Restklasse $d \bmod q$, und es folgt weiter

$$\varrho \leqq \sum_{|l| \leqq \log^C N} \left(\frac{N}{q} + 1\right) \ll \frac{N}{\log^C N}. \tag{53}$$

Aus (52) und (53) erhalten wir für $\alpha \in I_{a/q}$

$$\sum_{\substack{d \leqq N \\ d \in L}} \mu(d) H(d\alpha) = \sum_{\substack{d \leqq N \\ d \in L}} \mu(d) H\left(d\frac{a}{q}\right) + O\left(\frac{N}{\log^C N}\right). \tag{54}$$

Wegen (6) ist

$$H\left(d\frac{a}{q}\right) = \sum_{n \leqq N} \frac{\sin\left(2\pi nd\frac{a}{q}\right)}{n} + \sum_{n > N} \frac{\sin\left(2\pi nd\frac{a}{q}\right)}{n}.$$

Indem wir für den zweiten Anteil die Darstellung (12) mit $\frac{a}{q}$, N an Stelle von α bzw. $\frac{x}{d}$ übernehmen und $\left\|d\frac{a}{q}\right\| \geqq \frac{1}{q}$ im Fall $q \nmid d$ beachten, folgt

$$\sum_{n>N} \frac{\sin\left(2\pi n d\,\frac{a}{q}\right)}{n} \ll \log^{-C} N.$$

Für $q \mid d$ gilt dies sowieso. Einsetzen in (54) liefert

$$\sum_{\substack{d \leqq N \\ d \in L}} \mu(d) H(d\alpha) = \sum_{n \leqq N} \frac{1}{n} \sum_{\substack{d \leqq N \\ d \in L}} \mu(d) \sin\left(2\pi n d\,\frac{a}{q}\right) + O\left(\frac{N}{\log^C N}\right). \tag{55}$$

Wir behaupten, daß für alle N mit $\log^{2C} N < q \leqq \frac{N}{\log^C N}$ die Abschätzung

$$\sum_{n \leqq N} \frac{1}{n} \sum_{\substack{d \leqq N \\ d \in L}} \mu(d) e\left(nd\,\frac{a}{q}\right) \ll \frac{N}{\log^2 N} \tag{56}$$

besteht. Nehmen wir (56) als richtig an, so ergibt sich für den Beitrag der α-Intervalle 4. Art unserer in IV vorgenommenen Intervalleinteilung zu $\sum(A, B, \alpha)$ ein $o(1)$: Die Abschätzung (56), die auch für den Imaginärteil der Summe linker Hand besteht, liefert, in (55) eingesetzt, für alle N aus einem α-Intervall 4. Art zunächst

$$\sum_{\substack{d \leqq N \\ d \in L}} \mu(d) H(d\alpha) \ll \frac{N}{\log^2 N}. \tag{57}$$

Es sei $(N_\nu, N'_\nu]$ ein (evtl. unendliches) Intervall maximaler Länge mit

$$\left|\alpha - \frac{a_\nu}{q_\nu}\right| \leqq \frac{\log^C N}{q_\nu N}, \quad 1 \leqq q_\nu \leqq \log^{2C} N \quad \text{für alle} \quad N \in (N_\nu, N'_\nu].$$

Nach den Erörterungen aus IV ist $(N_\nu, N'_\nu]$ zusammengesetzt aus maximalen α-Intervallen 1., 2. oder 3. Art. Wie in IV erkennt man, daß für das nächste derartige Intervall $(N_{\nu+1}, N'_{\nu+1}]$ mit $N'_\nu < N'_{\nu+1}$, falls vorhanden, die Abschätzung $N'_\nu \ll \log^{4C} N_{\nu+1}$ oder erst recht

$$\log^2 N_\nu \leqq \log^2 N'_\nu \leqq \log N_{\nu+1} \leqq \log N'_{\nu+1} \quad (N'_\nu \geqq N_0(C)) \tag{58}$$

besteht. Die Lücke zwischen diesen Intervallen ist aus α-Intervallen 4. Art zusammengesetzt. Deren Beitrag zu $\sum(A, B, \alpha)$ ist wegen (57) mit $N^*_\nu = \max(A, N'_\nu)$ bei partieller Summation

$$\ll \int_{N^*_\nu}^{N_{\nu+1}} \frac{dt}{t \log^2 t} \ll \frac{1}{\log N^*_\nu}.$$

Die sämtlichen α-Intervalle 4. Art der in IV verabredeten Intervalleinteilung tragen zu $\sum(A, B, \alpha)$ daher höchstens

$$\ll \sum_{\substack{\nu \\ N'_\nu \geqq A}} \frac{1}{\log N^*_\nu} \ll \frac{1}{\log A}$$

bei, letzteres wegen (58). Damit ist unter Annahme von (56) der Satz 3 vollständig nachgewiesen. Es bleibt noch (56) zu zeigen.

Wir reduzieren die Behauptung (56) weiter und zerlegen dazu die äußere Summe in zwei Bestandteile, je nachdem in der gekürzten Darstellung $\frac{an'}{q'} = \frac{an}{q}$, $(an', q')=1$, für den Nenner q' die Ungleichung $q' \leqq \log^C N$ oder $\log^C N < q' \leqq \frac{N}{\log^C N}$ besteht. Für den ersten Anteil $(q' \leqq \log^C N)$ erhalten wir die Abschätzung

$$\ll \sum_{\substack{d \leqq N \\ d \in L}} \mu^2(d) \sum_{\substack{n \leqq N, q'n = qn' \\ (n', q')=1, q' \leqq \log^C N}} \frac{1}{n} \ll N \log N \sum_{\substack{q' \leqq \log^C N \\ q' | q}} \frac{q'}{q} \ll$$

$$\ll N(\log N)^{C+1} \frac{\tau(q)}{q} \ll N(\log N)^{1-C/2},$$

letzteres wegen $\tau(q) \ll q^{1/4}$. Für genügend großes C, etwa $C \geqq 6$, ist die rechte Seite $\ll N(\log N)^{-2}$, wie verlangt.

Für den zweiten Anteil gilt mit $a' = an'$ die Abschätzung

$$\ll \log N \max_{(a', q')=1} \left| \sum_{\substack{d \leqq N \\ d \in L}} \mu(d) e\left(\frac{a'}{q'} d\right) \right|.$$

Es genügt also zum Nachweis von (56), für $(a, q)=1$, $\log^C N < q \leqq N(\log N)^{-C}$ zu zeigen (wir haben die Striche fortgelassen)

(59) $$\sum_{\substack{d \leqq N \\ d \in L}} \mu(d) e\left(\frac{a}{q} d\right) \ll \frac{N}{\log^3 N}.$$ [1]

Wir setzen $\nu_0 = [(\log_2 N)^2]$ und klassifizieren nach der Anzahl $\omega(d)$ der Prim teiler von d,

(60) $$\sum_{\substack{d \leqq N \\ d \in L}} \mu(d) e\left(\frac{a}{q} d\right) = e\left(\frac{a}{q}\right) + \sum_{\substack{d \leqq N \\ d \in L \\ \omega(d) \geqq \nu_0}} \mu(d) e\left(\frac{a}{q} d\right) - \sum_{\substack{d \leqq N \\ p \in S}} e\left(\frac{a}{q} p\right) +$$

$$+ \sum_{2 \leqq \nu < \nu_0} (-1)^\nu \sum_{\substack{d \leqq N \\ d \in L \\ \omega(d) = \nu}} \mu^2(d) e\left(\frac{a}{q} d\right),$$

[1] Vgl. hierzu H. DAVENPORT, On some infinite series involving arithmetic functions (II), *Quarterly J. of Math.*, **8** (1937), 313—320.

und werden für jeden Anteil rechter Hand die in (59) geforderte Abschätzung einzeln nachweisen.

Für den ersten Term $e\left(\frac{a}{q}\right)$ ist dies offensichtlich. Zweitens haben wir

$$\sum_{\substack{d \leqq N \\ d \in L \\ \omega(d) \geqq \nu_0}} \mu(d) e\left(\frac{a}{q} d\right) \ll \sum_{\substack{n \leqq N \\ \omega(n) \geqq \nu_0}} \mu^2(d) = \sum_{\nu \geqq \nu_0} \Pi_\nu(N),$$

wobei $\Pi_\nu(N)$ die Anzahl der quadratfreien Zahlen $\leqq N$ mit genau ν Primfaktoren bezeichnet. Nach HARDY—RAMANUJAN [2], Lemma A, existieren absolute Konstanten $c_1, c_2 > 0$ mit

$$\Pi_\nu(N) < c_1 \frac{N}{\log N} \frac{(\log_2 N + c_2)^{\nu-1}}{(\nu-1)!} \qquad (N \geqq 2,\ \nu \in \mathbf{N}).$$

Nach kurzer Rechnung erweist sich damitder zweite Term in der Zerlegung (60) als $\ll N(\log N)^{-3}$, wie gewünscht.

Die Abschätzung des dritten Terms, der über die Primzahlen aus S erstreckten Epxonentialsumme, besorgt das folgende

LEMMA 5. *Es sei S die in* III *erklärte Primzahlmenge. Für* $C \geqq 8$, $R > 1$, $\log^R N < q \leqq N(\log N)^{-R}$, $(a, q) = 1$ *gilt*

$$\sum_{\substack{p \leqq N \\ p \in S}} e\left(\frac{a}{q} p\right) \ll N(\log N)^{5-R/2}. \tag{61}$$

BEWEIS. Mit den Bezeichnungen aus III gelte $A_k < N \leqq A_{k+1}$, und es sei q_k eine der Zahlen $p_{k-1}, p_k, p_{k-1}p_k$. Für $A_k \geqq N_0(C)$ gilt dann $q_k \leqq (\log N)^{2/C}$. Ferner ist für $(\nu, q_k) = 1$

$$\sum_{\substack{p \leqq N \\ p \equiv \nu(q_k)}} e\left(\frac{a}{q} p\right) \ll \max_{\psi \bmod q_k} \left| \sum_{p \leqq N} \psi(p) e\left(\frac{a}{q} p\right) \right|. \tag{62}$$

Indem wir in der Summe rechts den Charakter $\psi \bmod q_k$ durch seinen zugehörigen primitiven Charakter $\psi^* \bmod q_k^*$ mit $q_k^* | q_k$ ersetzen, machen wir einen Fehler $\ll 1$. Bezeichnet

$$G(\bar{\psi}^*) = \sum_{\substack{1 \leqq \lambda \leqq q_k^* \\ (\lambda, q_k^*) = 1}} \bar{\psi}^*(\lambda) e\left(\frac{\lambda}{q_k^*}\right)$$

die zu $\bar{\psi}^* \bmod q_k^*$ gehörige Gaußsche Summe, so ist $|G(\bar{\psi}^*)| = \sqrt{q_k^*}$ und (s. etwa PRACHAR [6], Lemma VII. 1.1)

$$\psi^*(p) = \frac{1}{G(\bar{\psi}^*)} \sum_{\substack{1 \leqq \lambda \leqq q_k^* \\ (\lambda, q_k^*) = 1}} \bar{\psi}^*(\lambda) e\left(\frac{\lambda p}{q_k^*}\right).$$

Einsetzen in (62) mit ψ^* an Stelle von ψ liefert rechts

$$\left|\sum_{p\le N}\psi^*(p)e\left(\frac{a}{q}p\right)\right| = \frac{1}{\sqrt{q_k^*}}\left|\sum_{\substack{1\le\lambda\le q_k^*\\(\lambda,q_k^*)=1}}\bar{\psi}^*(\lambda)\sum_{p\le N}e\left(\left(\frac{a}{q}+\frac{\lambda}{q_k^*}\right)p\right)\right| \le$$

$$\le \sqrt{q_k^*}\max_{\substack{1\le\lambda\le q_k^*\\(\lambda,q_k^*)=1}}\left|\sum_{p\le N}e\left(\left(\frac{a}{q}+\frac{\lambda}{q_k^*}\right)p\right)\right|.$$

Darin genügt der Nenner des gekürzten Bruches $\frac{a''}{q''}=\frac{a}{q}+\frac{\lambda}{q_k^*}$ der Ungleichung

$$(\log N)^{R-2/C} < \frac{q}{q_k^*} \le q'' \le qq_k^* \le N(\log N)^{-(R-2/C)}.$$

Der Satz von Vinogradov (s. etwa SCHWARZ [8], VII, Satz 2.1) liefert

$$\sum_{p\le N}e\left(\frac{a''}{q''}p\right) \ll N(\log N)^{9/2-R/2+1/C}$$

und, mit dem Faktor $\sqrt{q_k^*}\le(\log N)^{1/C}$ versehen,

$$\sum_{\substack{p\le N\\p\equiv\nu(q_k)}}e\left(\frac{a}{q}p\right) \ll N(\log N)^{9/2-R/2+2/C}.$$

Nach der Definition der aus Restklassen abschnittsweise zusammengesetzten Primzahlmenge S folgt für $A_k<N\le A_{k+1}$

$$\sum_{\substack{p\le N\\p\in S}}e\left(\frac{a}{q}p\right) \ll p_{k-1}p_kN(\log N)^{9/2-R/2+2/C}+A_{k-1}^* \ll N(\log N)^{5-R/2},$$

wie in (61) behauptet.

Mit $R=C\ge 16$ ergibt sich aus Lemma 5 für den dritten Term auf der rechten Seite von (60) ebenfalls die gewünschte obere Schranke $N(\log N)^{-3}$.

Es bleibt der letzte Anteil in (60) geeignet abzuschätzen. Offenbar genügt es,

$$\sum_{\substack{\sqrt{N}<d\le N\\d\in L,\,\omega(d)=\nu}}\mu^2(d)\,e\left(\frac{a}{q}d\right) \ll N(\log N)^{-4} \quad (2\le\nu\le\nu_0) \tag{63}$$

nachzuweisen. Als wesentliches Hilfsmittel benötigen wir

LEMMA 6. *Es seien $x\ge x_0$, $R>1$, $(\log x)^R<q\le x(\log x)^{-R}$, $(a,q)=1$, $R'>R/4$ und (a_i), (b_j) zwei streng monoton wachsende Folgen von natürlichen Zahlen. (Die Folgen dürfen von x abhängen oder endlich sein.) Dann gilt*

$$\sum_{\substack{a_i,b_j>(\log x)^{R'}\\a_ib_j\le x}}e\left(\frac{a}{q}a_ib_j\right) \ll x(\log x)^{4-R/8}. \tag{64}$$

Zum Nachweis von (63) nehmen wir zunächst die Richtigkeit von Lemma 6 an und setzen $E=C/4$. Jedes $d\in L$ besitzt eine eindeutige Darstellung $d=d_1d_2$, wobei $p\leqq\log^E N$ für alle Primteiler p von d_1 und $p>\log^E N$ für alle Primteiler p von d_2 gilt. Aus

$$d_1 \leqq (\log^E N)^{\nu_0} < N^{1/2} \quad (N\geqq N_0(C))$$

erhalten wir $d_2\neq 1$, also bereits

$$d_2 > \log^E N \tag{65}$$

ür alle in (63) gezählten $d\in L$.

Wir spalten die Summe in (63) auf,

$$\sum\nolimits_1+\sum\nolimits_2 = \Big(\sum_{\substack{\sqrt{N}<d_1d_2\leqq N\\ d_1>\log^E N}}^{*} + \sum_{\substack{\sqrt{N}<d_1d_2\leqq N\\ d_1\leqq\log^E N}}^{*}\Big)\mu^2(d_1d_2)e\left(\frac{a}{q}d_1d_2\right),$$

wobei rechts die weiteren Summationsbedingungen $\omega(d_1d_2)=\nu$ und $d_1, d_2\in L$ durch $*$ angedeutet sind.

Auf $\sum_1$ kann wegen (65) das Lemma 6 mit $R=C$, $R'=E$ und den Folgen (d_1), (d_2) von quadratfreien Zahlen aus L mit den obigen Primteilerbedingungen angewendet werden. Es folgt

$$\sum\nolimits_1 \ll \sqrt{N}+N(\log N)^{4-C/8} \ll N(\log N)^{-4},$$

letzteres für $C\geqq 64$.

Für $\sum_2$ erhalten wir die Darstellung $(2\leqq\nu<\nu_0)$

$$\sum\nolimits_2 = \sum_{\substack{d_1\leqq\log^E N\\ d_1\in L}}\mu^2(d_1)\sum_{\varkappa=1}^{\nu-\omega(d_1)}\sum_{\substack{\frac{\sqrt{N}}{d_1}<d_2\leqq\frac{N}{d_1}\\ d_2\in L,\ \omega(d_2)=\varkappa}}\mu^2(d_2)e\left(\frac{a}{q}d_1d_2\right). \tag{66}$$

Wir erinnern daran, daß sämtliche Primteiler von d_2 größer als $\log^E N$ sind, und bezeichnen die innere, über d_2 erstreckte Summe in (66) mit $L_\varkappa(\sqrt{N}/d_1, N/d_1)$. Eine Auszählung liefert für $\varkappa\geqq 2$ den Zusammenhang

$$L_\varkappa\left(\frac{\sqrt{N}}{d_1},\frac{N}{d_1}\right) = \frac{1}{\varkappa}\sum_{\substack{\log^E N<p\leqq\frac{N}{d_1}\\ p\in S}} L_{\varkappa-1}\left(\frac{\sqrt{N}}{pd_1},\frac{N}{pd_1}\right)+O\Big(\sum_{\substack{\log^E N<p\leqq\frac{N}{d_1}\\ p\in S}}\ \sum_{\substack{d_2'\leqq\frac{N}{d_1p^2}\\ d_2'\in L}}1\Big), \tag{67}$$

wobei das Restglied die nicht quadratfreien d_2 berücksichtigt, über die im Hauptglied summiert wird. Beachten wir noch

$$\sum_{p>\log^E N}\frac{1}{p^2} \ll (\log N)^{-E},$$

so ergibt sich für das Fehlerglied in (67) die Abschätzung $O\left(\frac{N}{d_1\log^E N}\right)$. Einsetzen von (67) in (66) liefert

(68)

$$\sum\nolimits_2 = \sum_{\substack{d_1 \leqq \log^E N \\ d_1 \in L}} \mu^2(d_1) L_1\left(\frac{N}{d_1}\right) + \sum_{\substack{d_1 \leqq \log^E N \\ d_1 \in L}} \mu^2(d_1) \sum_{\varkappa=2}^{\nu-\omega(d_1)} \frac{1}{\varkappa} \sum_{\substack{\log^E N < p \leqq \frac{N}{d_1} \\ p \in S}} L_{\varkappa-1}\left(\frac{\sqrt{N}}{pd_1}, \frac{N}{pd_1}\right) +$$

$$+ O\big(N (\log N)^{-E} (\log_2 N)^3\big).$$

Hierin ist das Restglied $\ll N (\log N)^{-4}$ für $C \geqq 20$. Auf

$$L_1\left(\frac{N}{d_1}\right) = \sum_{\frac{\sqrt{N}}{d_1} < p \leqq \frac{N}{d_1}} e\left(\frac{a}{q} d_1 p\right)$$

kann (für jedes $d_1 \leqq \log^E N$) das Lemma 5 mit N/d_1 an Stelle von N und $R = 2E$ angewendet werden. Es folgt

$$L_1\left(\frac{N}{d_1}\right) \ll \frac{\sqrt{N}}{d_1} + \frac{N}{d_1} (\log N)^{5-E} \ll \frac{N}{d_1} (\log N)^{-5},$$

letzteres für $C \geqq 40$, und damit

$$\sum_{\substack{d_1 \leqq \log^E N \\ d_1 \in L}} \mu^2(d_1) L_1\left(\frac{N}{d_1}\right) \ll N (\log N)^{-4}.$$

Schließlich kann für $2 \leqq \varkappa < \nu_0$ auf

$$\sum_{\substack{\log^E N \leqq p \leqq \frac{N}{d_1} \\ p \in S}} L_{\varkappa-1}\left(\frac{\sqrt{N}}{pd_1}, \frac{N}{pd_1}\right) = \sum_{\substack{p \in S, d_2 \in L \\ pd_2 \leqq \frac{N}{d_1};\, p, d_2 \geqq \log^E N \\ \omega(d_2) = \varkappa - 1}} \mu^2(d_2) e\left(\frac{a}{q} d_1 p d_2\right) + O(\varkappa \sqrt{N})$$

das Lemma 6 mit $R = 2E$, $R' = E$ und N/d_1 an Stelle von N (für jedes $d_1 \leqq \log^E N$) angewendet werden. Für die Folgenglieder a_i, b_j sind die $p \in S$ bzw. die quadratfreien $d_2 \in L$ mit $\omega(d_2) = \varkappa - 1$ und Primfaktoren $\geqq \log^E N$ zu wählen. Es folgt

$$\sum_{\substack{\log^E N \leqq p \leqq \frac{N}{d_1} \\ p \in S}} L_{\varkappa-1}\left(\frac{N}{pd_1}\right) \ll \frac{N}{d_1} (\log N)^{4-E/4} \ll \frac{N}{d_1} (\log N)^{-5},$$

letzteres für $C \geqq 144$. Damit wird auch der mittlere Anteil auf der rechten Seite in

(68) $\ll N (\log N)^{-5} (\log_2 N)^3 \ll N (\log N)^{-4}$.

Bei Wahl eines festen $C \geqq 144$ trifft daher (63) zu, und der Satz 3 ist unter Annahme von Lemma 6 vollständig bewiesen.

BEWEIS VON LEMMA 6. Mit $R''=\frac{R}{8}$ gilt

$$\sum_{\substack{a_i, b_j \geqq (\log)^{R'} \\ a_i b_j \leqq x (\log x)^{-R''}}} e\left(\frac{a}{q} a_i b_j\right) \ll \sum_{n \leqq x (\log x)^{-R''}} \tau(n) \ll x (\log x)^{1-R/8}.$$

Die Summe über die Produkte $a_i b_j$ mit $x(\log x)^{-R''} < a_i b_j \leqq x$ spalten wir auf in $\ll \log x$ Teile $\sum_\nu$, in denen über Intervalle $T_\nu < a_i b_j \leqq T'_\nu$, $T'_\nu \leqq 2T_\nu$, summiert wird. Jedes $\sum_\nu$ spalten wir weiter auf in $\ll \log x$ Summen $\sum_{\nu\varkappa}$ mit den zusätzlichen Bedingungen $W_\varkappa < a_i \leqq W'_\varkappa$, $W'_\varkappa \leqq 2W_\varkappa$, wobei also $(\log x)^{R'} < W_\varkappa \leqq$ $\leqq x(\log x)^{-R'}$ gilt.

Auf jedes $\sum_{\nu\varkappa}$ kann dann der Vinogradovsche Drei-Folgen-Satz (s. etwa PRACHAR [6], VI, Lemma 6.4) angewendet werden. In der dortigen Bezeichnungsweise seien die Folgen von Zahlen u_1, u_2, v durch $u_1=1$, $u_{2i}=a_i$, $v_j=b_j$ bestimmt. Wir erhalten

$$\sum\nolimits_{\nu\varkappa} \ll T_\nu \log^2 T_\nu \left(\frac{1}{q}+\frac{q}{T_\nu}+\frac{1}{W_\varkappa}+\frac{W_\varkappa}{T_\nu}\right)^{1/2} \ll x \log^2 x \left((\log x)^{-R}+(\log x)\right)^{-R')^{1/2}} \ll$$

$$\ll x \log^2 x (\log x)^{-R/8}.$$

Da nur $\ll \log^2 x$ Summen $\sum_{\nu\varkappa}$ auftreten, folgt die Behauptung von Lemma 6.

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SOME RESULTS ON FIXED POINTS

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We first of all prove the following theorem:

THEOREM 1. *If S is a continuous mapping and T is a mapping of the complete metric space X into itself satisfying the inequality*

$$d(STx, TSy) \leqq c\{d(x, Sy)+d(y, Tx)\} \tag{1}$$

for all x, y in X, where $0<c<\frac{1}{2}$, *then S and T have a unique common fixed point.*

PROOF. Let x be an arbitrary point in X. Then

$$\begin{aligned} d((ST)^n x, T(ST)^n x) &\leqq cd((ST)^{n-1}x, (ST)^n x) \leqq \\ &\leqq c\{d((ST)^{n-1}x, T(ST)^{n-1}x)+d(T(ST)^{n-1}x, (ST)^n x)\} = \\ &= a_1 d((ST)^{n-1}x, T(ST)^{n-1}x)+b_1 d(T(ST)^{n-1}x, (ST)^n x) \leqq \\ &\leqq a_1 d((ST)^{n-1}x, T(ST)^{n-1}x)+cb_1 d(T(ST)^{n-2}x, T(ST)^{n-1}x) \leqq \\ &\leqq (a_1+cb_1) d((ST)^{n-1}x, T(ST)^{n-1}x)+cb_1 d(T(ST)^{n-2}x, (ST)^{n-1}x) = \\ &= a_2 d(T(ST)^{n-2}x, (ST)^{n-1}x)+b_2 d((ST)^{n-1}x, T(ST)^{n-1}x) \end{aligned}$$

where

$$a_1 = b_1 = c, \quad a_2 = cb_1, \quad b_2 = a_1+cb_1.$$

Further use of inequality (1) and the triangular inequality gives us

$$\begin{aligned} &d((ST)^n x, T(ST)^n x) \leqq \\ &\leqq a_{2r} d(T(ST)^{n-r-1}x, (ST)^{n-r}x)+b_{2r} d((ST)^{n-r}x, T(ST)^{n-r}x) \leqq \\ &\leqq a_{2r} d(T(ST)^{n-r-1}x, (ST)^{n-r}x) + \\ &+cb_{2r}\{d((ST)^{n-r-1}x, T(ST)^{n-r-1}x)+d(T(ST)^{n-r-1}x, (ST)^{n-r}x)\} = \\ &= a_{2r+1} d((ST)^{n-r-1}x, T(ST)^{n-r-1}x)+b_{2r+1} d(T(ST)^{n-r-1}x, (ST)^{n-r}x) \leqq \\ &\leqq a_{2r+1} d((ST)^{n-r-1}x, T(ST)^{n-r-1}x)+ \\ &+cb_{2r+1}\{d(T(ST)^{n-r-2}x, (ST)^{n-r-1}x)+d((ST)^{n-r-1}x, T(ST)^{n-r-1}x)\} = \\ &= a_{2r+2} d(T(ST)^{n-r-2}x, (ST)^{n-r-1}x)+b_{2r+2} d((ST)^{n-r-1}x, T(ST)^{n-r-1}x) \leqq \\ &\leqq a_{2n-1} d(x, Tx)+b_{2n-1} d(Tx, STx), \end{aligned}$$

where

$$a_{r+1} = cb_r, \quad b_{r+1} = a_r + cb_r$$

for $r=1, 2, \ldots, 2n-1$. It follows that

$$a_{r+2} - ca_{r+1} - ca_r = 0 = b_{r+2} - cb_{r+1} - cb_r$$

so that a_r and b_r are of the form

$$a_r = A\alpha^r + B\beta^r, \quad b_r = C\alpha^r + D\beta^r,$$

where

$$\alpha = \frac{1}{2}\{c-(c^2+4c)^{1/2}\}, \quad \beta = \frac{1}{2}\{c+(c^2+4c)^{1/2}\}$$

and $0<\alpha<\beta<1$, since $0<c<1/2$. The coefficients A, B, C and D can of course be found on using the initial conditions but it is sufficient to notice that there must exist a fixed $k>0$ such that

$$d\big((ST)^n x, T(ST)^n x\big) \leqq k\beta^{2n-1}$$

for $n=1, 2, \ldots$.

Similarly, there must exist a fixed $k'>0$ such that

$$d\big(T(ST)^n x, (ST)^{n+1} x\big) \leqq k'\beta^{2n-1}$$

for $n=1, 2, \ldots$ and since $\beta<1$, it follows that the sequence

$$\{x, Tx, STx, \ldots, (ST)^n x, T(ST)^n x, \ldots\}$$

is a Cauchy sequence in the complete metric space X and so has a limit z in X. Thus

$$\lim_{n\to\infty} (ST)^n x = \lim_{n\to\infty} T(ST)^n x = z = \lim_{n\to\infty} S\big(T(ST)^n\big)x = Sz,$$

since S is continuous. Hence z is a fixed point of S.

We will now show that z is also a fixed point of T. We have

$$d(z, Tz) = \lim_{n\to\infty} d\big((ST)^n x, TSz\big) \leqq c \lim_{n\to\infty} \big\{d\big((ST)^{n-1}x, Sz\big) + d\big(z, T(ST)^{n-1}x\big)\big\} = 0.$$

It follows that z is a common fixed point of S and T.

Now suppose that S and T have a second common fixed point w. Then

$$d(z, w) = d(STz, TSw) \leqq c\{d(z, Sw) + d(w, Tz)\} = 2cd(z, w).$$

Since $2c<1$, it follows that the common fixed point z must be unique. This completes the proof of the theorem.

We note that although the common fixed point z is unique it is possible for S or T to have other fixed points. This is easily seen by considering any complete metric space X having at least two points. Choose a point z in X and define mappings S and T on X by

$$Sx = x, \quad Tx = z$$

for all x in X. S is continuous and

$$d(STx, TSy) = d(z, z) = 0$$

for all x, y in X. Inequality (1) is therefore satisfied with $c=1/4$, but every point in X is a fixed point of S.

The condition that S be continuous is also necessary. To show this, let

$$X = \left\{0, 2, \frac{1}{2}, \left(\frac{1}{2}\right)^2, \ldots, \left(\frac{1}{2}\right)^n, \ldots\right\}$$

with the usual metric for real numbers and define discontinuous mappings S and T on X by

$$S(0) = T(0) = 2, \quad S(2) = T(2) = \frac{1}{2},$$

$$S\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{n+1}, \quad T\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{n+2},$$

for $n=1, 2, \ldots$. Inequality (1) can be shown to be satisfied with $c=1/3$, but S and T have no fixed points.

If we now put $S=T$ in Theorem 1, we have the following

THEOREM 2. *If T is a continuous mapping of the complete metric space X into itself satisfying the inequality*

$$d(T^2x, T^2y) \leqq c\{d(x, Ty)+d(y, Tx)\}$$

for all x, y in X, where $0<c<\frac{1}{2}$, then T has a unique fixed point.

We finally prove two theorems for compact metric spaces.

THEOREM 3. *If S and T are continuous mappings of the compact metric space X into itself satisfying either the inequality*

$$d(STx, TSy) < \frac{1}{2}\{d(x, Sy)+d(y, Tx)\}, \quad \text{if} \quad d(x, Sy)+d(y, Tx) \neq 0$$

or the equality

$$d(STx, TSy) = 0, \quad \text{if} \quad d(x, Sy)+d(x, Tx) = 0$$

for all x, y in X, then S and T each have a fixed point. Further, if S and T have a common fixed point, then it is unique.

PROOF. If there exists $c<1/2$ such that

$$d(STx, TSy) \leqq c\{d(x, Sy)+d(y, Tx)\}$$

for all x, y in X, then the result follows from Theorem 1 with S and T then having a common fixed point.

Let us suppose then that no such c exists. Then there exists a sequence of positive real numbers $\{c_n\}$ converging to zero and sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$d(STx_n, TSy_n) > \left(\frac{1}{2}-c_n\right)\{d(x_n, Sy_n)+d(y_n, Tx_n)\}$$

for $n=1, 2, \ldots$. Since X is compact, we can find convergent subsequences $\{x_{n(r)}\}$ and $\{y_{n(r)}\}$ of $\{x_n\}$ and $\{y_n\}$ converging to x and y respectively. Thus

$$d(STx_{n(r)}, TSy_{n(r)}) > \left(\frac{1}{2}-c_{n(r)}\right)\{d(x_{n(r)}, Sy_{n(r)})+d(y_{n(r)}, Tx_{n(r)})\}$$

and on letting r tend to infinity we see that since S and T are continuous

$$d(STx, TSy) \geqq \frac{1}{2}\{d(x, Sy)+d(y, Tx)\},$$

giving a contradiction unless we have

$$STx = TSy, \quad Sy = x, \quad Tx = y.$$

It follows that

$$Tx = y = TSy = STx$$

and so Tx is a fixed point of S.

Similarly, Sy is a fixed point of T.

Now suppose that z and w are distinct common fixed points of S and T. Then

$$d(z, w) = d(STz, STw) = 0,$$

giving a contradiction. The common fixed point must therefore be unique if one exists. This completes the proof of the theorem.

The proof of the last theorem is immediate.

THEOREM 4. *If T is a continuous mapping of the compact metric space X into itself satisfying either the inequality*

$$d(T^2x, T^2y) < \frac{1}{2}\{d(x, Ty)+d(y, Tx)\}, \quad \text{if} \quad d(x, Ty)+d(y, Tx) \neq 0$$

or the equality

$$d(T^2x, T^2y) = 0, \quad \text{if} \quad d(x, Ty)+d(y, Tx) = 0$$

for all x, y in X, then T has a unique fixed point.

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A NOTE ON p-COMPATIBLE AC-LATTICES

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Dedicated to the memory of Professor Pál Turán

1. In his paper [3] M. F. JANOWITZ introduced the notion of a p-compatible AC-lattice and developed a structure theory for a certain class of these lattices. The importance of the concept of p-compatibility is also seen from the other results of [3]. The class of p-compatible AC-lattices includes all finite-statisch AC-lattices. Finite-statisch AC-lattices were introduced by M. F. JANOWITZ [1] in order to generalize the notion of statisch AC-lattices as defined by R. WILLE [8] (in fact, the notions of statisch, finite-statisch and p-compatible are defined in the more general setting of atomistic lattices).

In this note it is our aim to prove a result on p-compatible AC-lattices (Theorem 4) similar to a result on finite-statisch AC-lattices in [2]. Restricting slightly the notion of p-compatibility we are able to prove a certain converse to Theorem 4 (Theorem 5).

2. An element b of an atomistic lattice L is called finite if it is either 0 or the join of a finite number of atoms; by $F(L)$ we denote the set of the finite elements of L. An AC-lattice is an atomistic lattice with the covering property:

$$p \text{ an atom, } p \not\leqq a \text{ implies } a \vee p \text{ covers } a.$$

In the sequel, $a \prec b$ means that b covers a. For the theory of AC-lattices we refer to the book [4] by F. MAEDA and S. MAEDA.

An atomistic lattice is called finite-statisch if $p \leqq q \vee a$ with p, q atoms implies $p \leqq q \vee a_1$ for some finite element $a_1 \leqq a$.

Two elements a, b of a lattice with 0 are called perspective (denoted by $a \sim b$) in case there is an element x such that $a \vee x = b \vee x$ and $a \wedge x = b \wedge x = 0$.

An atomistic lattice L is p-compatible if $p \sim q$ in L (p, q atoms) implies $p \sim q$ in $F(L)$.

LEMMA 1 (cf. [3, Lemma 3]). *Every finite-statisch AC-lattice is a p-compatible AC-lattice.*

On the other hand, there exists a p-compatible AC-lattice that is not finite-statisch (cf. [3, Example 1]).

REMARK 2. Let $p \sim q$ for a finite-statisch AC-lattice L or for the lattice L of [3, Example 1], that is, $p \vee x = q \vee x$ and $p \wedge x = q \wedge x = 0$ for some $x \in L$. Then it is always possible to choose an $x_1 \leqq x$ with $x_1 \in F(L)$ such that $p \vee x_1 = q \vee x_1$ and $p \wedge x_1 = q \wedge x_1 = 0$.

We shall also need

LEMMA 3 (cf. [2, Lemma 2] or [4, Lemma 8.18, p. 39]). *Let* $a<b$ *in an AC-lattice L. Then*

(i) *The principal dual ideal* $[a)$ *is an AC-lattice.*

(ii) *An element* $c\in L$ *is an atom of* $[a)$ *if and only if there exists an atom* $p\in L$ *such that* $c=a\vee p$ *and* $p\not\leqq a$.

(iii) *An element* $c\in L$ *is a finite element of* $[a)$ *if and only if* $c=a\vee d$ *for some finite element* $d\in L$. *In particular, if* $a\in F(L)$ *then* $c\in L$ *is a finite element of* $[a)$ *if and only if* $c\geqq a$ *and* $c\in F(L)$.

3. Now we are ready to prove a result which is similar to [2, Theorem 3].

THEOREM 4. *Let L be an AC-lattice such that every infinite element dominates a finite element* $a\in F(L)$ *having the property that* $[a)$ *is p-compatible. Then L is p-compatible.*

PROOF. Let p, q be atoms of L with $p\sim q$. Without loss of generality we may assume that $p\neq q$. Then there exists an $x\in L$ with

$$x\vee p = x\vee q \quad \text{and} \quad x\wedge p = x\wedge q = 0. \tag{1}$$

Suppose that $x\notin F(L)$. By assumption there exists an $a\leqq x$ such that $a\in F(L)$ and $[a)$ is p-compatible. We have $p, q\not\leqq a$; otherwise we had $p, q\leqq a\leqq x$ in contradiction to (1). Thus

$$a \prec a\vee p \quad \text{and} \quad a\prec a\vee q \tag{2}$$

by Lemma 3.

If $a\vee p=a\vee q$ then by $a\wedge p=0=a\wedge q$ we have $p\sim q$ in $F(L)$.

Let now $a\vee p\neq a\vee q$. Then

$$x\vee(a\vee p) = (x\vee a)\vee p = x\vee p \tag{3}$$

and

$$x\vee(a\vee q) = (x\vee a)\vee q = x\vee q. \tag{4}$$

Because of (1) it follows from (3) and (4) that

$$x\vee(a\vee p) = x\vee(a\vee q). \tag{5}$$

We have furthermore

$$a\vee p, a\vee q \not\leqq x. \tag{6}$$

Namely, if e.g. $a\vee p\leqq x$ then $x\vee(a\vee p)=x\vee p\leqq x$ which contradicts (1). We note that $a\vee p$, $a\vee q>x$ is not possible. Namely, if e.g. $a\leqq x<a\vee p$, then it follows from (2) that $x=a$ which contradicts our assumption $x\notin F(L)$. Thus (2) and (6) together yield

$$x\wedge(a\vee p) = a = x\wedge(a\vee q). \tag{7}$$

From (5) and (7) we obtain $(a\vee p)\sim(a\vee q)$ in $[a)$. By the p-compatibility of $[a)$ there exists an element x_1 which is finite in $[a)$ such that

$$x_1\vee(a\vee p) = x_1\vee p = x_1\vee q = x_1\vee(a\vee q) \tag{8}$$

by (3), (4) and (5), and

$$x_1 \wedge (a \vee p) = a = x_1 \wedge (a \vee q) \tag{9}$$

by (7).

Moreover, we have $p, q \not\leqq x_1$. For, if e.g. $p \leqq x_1$ then $p \vee a \leqq x_1$ in contradiction to (9) and (2). Furthermore $0 \leqq x_1 < p$ yields $x_1 = 0$ which implies $p = q$ by (8), a contradiction to our assumption $p \neq q$. Thus $x_1 \neq 0$ and

$$p \wedge x_1 = 0 = q \wedge x_1. \tag{10}$$

From (8) and (10) we obtain that $p \sim q$ holds in $F(L)$. It follows that L is p-compatible which was to be proved.

We now slightly restrict the notion of p-compatibility in the following manner:

An AC-lattice L is called strongly p-compatible if whenever $p \vee x = q \vee x$ and $p \wedge x = q \wedge x = 0$ (p, q atoms) in L then there exists an $x_1 \leqq x$ with $x_1 \in F(L)$ such that $p \vee x_1 = q \vee x_1$ and $p \wedge x_1 = p \wedge x_1 = 0$.

Note that in the definition of p-compatibility it is only assumed that $x_1 \in F(L)$ but not that $x_1 \leqq x$. On the other hand, we see from Remark 2 that every finite-statisch AC-lattice as well as the lattice of [3, Example 1] are strongly p-compatible.

For strongly p-compatible AC-lattices we are able to prove a certain converse of Theorem 4.

THEOREM 5. *Let L be a strongly p-compatible AC-lattice. Then $[a)$ is a strongly p-compatible AC-lattice for every $a \in L$.*

PROOF. Let

$$p' \vee x = q' \vee x \quad \text{and} \quad p' \wedge x = q' \wedge x = a \tag{11}$$

in $[a)$ with $p' \neq q'$ atoms of $[a)$. By Lemma 2 there exist atoms p, q of L such that

$$p' = a \vee p \quad \text{and} \quad q' = a \vee q. \tag{12}$$

Then

$$p, q \not\leqq x. \tag{13}$$

For, if e.g. $p \leqq x$ then $p' = a \vee p \leqq x$ in contradiction to (11) since $x = a$ is not possible because of $p' \succ a$. Furthermore $p, q > x$ cannot hold. Namely, e.g. $0 \leqq x < p$ yields $x = 0$ which implies $p' = q'$ by (11), a contradiction to our assumption $p' \neq q'$. Thus from (13) and (12) it follows that

$$x \prec x \vee p \leqq x \vee p' = x \vee (a \vee p) = x \vee p$$

and

$$x \prec x \vee q \leqq x \vee q' = x \vee (a \vee q) = x \vee q.$$

This implies $x \vee p = x \vee p'$ and $x \vee q = x \vee q'$ and thus

$$x \vee p = x \vee q \tag{14}$$

because of (11). Since L is strongly p-compatible, it follows from (13) and (14) that there exists an $x_1 \in F(L)$ with $x_1 \leqq x$ such that

$$p \vee x_1 = q \vee x_1 \quad \text{and} \quad p \wedge x_1 = q \wedge x_1 = 0. \tag{15}$$

Then $a\vee(p\vee x_1)=a\vee(q\vee x_1)$ and thus

$$(a\vee p)\vee(a\vee x_1)=(a\vee q)\vee(a\vee x_1)$$

that is

$$p'\vee(a\vee x_1)=q'\vee(a\vee x_1) \tag{16}$$

by (12). We have

$$x_1 \not\leqq a. \tag{17}$$

For, if $x_1\leqq a$ then $a\vee x_1=a$ and it follows by (12) and (15) that

$$p'=a\vee p=a\vee x_1\vee p=a\vee x_1\vee q=a\vee q=q'$$

which contradicts our assumption $p'\neq q'$. Moreover, we have

$$p'\wedge(a\vee x_1)=a=q'\wedge(a\vee x_1) \tag{18}$$

which follows from $p',q'\succ a$, from $a\vee x_1\succ a$ (cf. (17)) and from $p',q'\not\leqq a\vee x_1$. The latter relation holds true because if e.g. $p'\leqq a\vee x_1$ then $a\leqq x$ (cf. (11)) and $x_1\leqq x$ (since L is strongly p-compatible) imply $p'\leqq x$ which contradicts (11). Since $x_1\in F(L)$, it follows from Lemma 2 that $a\vee x_1$ is finite in the AC-lattice $[a)$. Moreover we have $a\vee x_1\leqq x$ and therefore (16) and (18) yield that $[a)$ is strongly p-compatible.

4. It is not difficult to prove a result similar to Theorem 5 for finite statisch AC-lattices, resp. for statisch AC-lattices.

LEMMA 6. *Let L be a finite-statisch AC-lattice. The $[a)$ is a finite-statisch AC-lattice for every $a\in L$. If L is a statisch AC-lattice, then $[a)$ is a statisch AC-lattice for every $a\in L$.*

Before giving some corollaries we clarify some notions. Similar to the notion of incidence geometry of grade n as introduced by R. WILLE [9], one might give the following definition: AC-lattices in which the principal dual ideal $[b)$ is a finite-modular (resp. modular) AC-lattice for every element b of height n are called finite-modular (resp. modular) AC-lattices of grade n (cf. [6], [7]). For the notions of height, finite-modular, strongly planar and weakly modular we refer to [4].

COROLLARY 7. *Let L be a finite-modular AC-lattice of grade n. If L is statisch, then L is a modular AC-lattice of grade n.*

PROOF. Let $a\in L$ be an arbitrary element of height n. By assumption $[a)$ is a finite-modular AC-lattice and hence $F_{[a)}(L)$ (the ideal of the finite elements of $[a)$) is a standard ideal of $[a)$ by [1, Theorem 4.6]. The dual ideal $[a)$ is a statisch AC-lattice by Lemma 6. Hence $F_{[a)}(L)$ is even a neutral ideal of $[a)$ by [5, Theorem 3.2]. By [5, Theorem 3.3] it follows that $[a)$ is modular since $F_{[a)}(L)$ is a modular sublattice of $[a)$.

COROLLARY 8. *A finite-modular AC-lattice of grade n is statisch if and only if it is a modular AC-lattice of grade n.*

PROOF. If a finite-modular AC-lattice of grade n is statisch, then it is a modular AC-lattice of grade n by Corollary 7.

Conversely, let L be a modular AC-lattice of grade n. Then $[b)$ is statisch for every element $b \in L$ with height n by [8, Satz 3.11]. By [2, Theorem 4] it follows that L is statisch.

We conclude this note by

COROLLARY 9 ([1, Theorem 4.10]). *A statisch AC-lattice is finite-modular if and only if it is modular, it is strongly planar if and only if it is weakly modular.*

PROOF. Apply Corollary 5 for $n=0, 1$, respectively.

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ON THE BEHAVIOUR OF A CERTAIN CLASS OF APPROXIMATION OPERATORS FOR DISCONTINUOUS FUNCTIONS

By

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§ 0. Introduction

In this paper we study the behaviour of a wide class of approximation operators when the approximated functions are bounded and discontinuous. Our theorems apply to the Bernstein polynomials, the Szász operator, an operator of Weierstrass, the power series of Meyer—König and Zeller and to many other operators as well.

In order to analyze the properties of that many operators in a compact way we first represent them using a convenient probablistic form. Then we use probablistic tools such as the Chebishev inequality and the central limit theorem to estimate the difference between the values of the functions and their approximating operators at discontinuity points. Our methods simplify proofs, unify them, and enable one to get easily quite general results.

In § 1 we summarize our main results. In § 2 we show how one can apply our general results to several well known approximation operators, and in § 3 the proofs to the theorems are provided.

§ 1. The main results

Let $X_n (n \geqq 1)$ be independent and identically distributed (abbreviated i.i.d.) random variables (r.v.'s) with (common) expected value x and finite variance. Then $P_n(f, x)$ defined by

$$P_n(f, x) = \mathscr{E} f\left(\frac{X_1 + \ldots + X_n}{n}\right)^{1} \tag{1.1}$$

has important approximation properties for continuous functions. In fact many well known approximation operators are obtained from this general operator by choosing a particular distribution for the X_n's.

The approximation properties of $P_n(f, x)$ for continuous and bounded functions are discussed briefly by W. FELLER (in [5], Chapter VII, Section 1) and in great detail by D. STANCU [11]. S. KARLIN shows that the boundedness assumption on f can be replaced by an appropriate growth condition provided the r.v.'s are of Pólya type. (See [7], Chapter 7, § 6.) In the same book (Chapter 6, § 3) Karlin discusses monotonicity properties of the operator for convex functions.

We proceed now to state the main results of this paper on the approximation properties of $P_n(f, x)$ for (measurable) bounded discontinuous functions.

[1] $\mathscr{E}$ stands for expectation.

THEOREM 1. *Let X_n be i.i.d.r.v.'s taking values in some interval J. Assume*

$$\mathcal{E}X_n = x_0;\ x_0 \quad \text{is an internal point of } J. \tag{1.2}$$

$$0 < \operatorname{Var} X_n < \infty. \tag{1.3}$$

Then for any bounded (measurable) function f on J

$$\frac{1}{2}(l^+ + l^-) \leqq \varliminf_{n\to\infty} P_n(f, x_0) \leqq \varlimsup_{n\to\infty} P_n(f, x_0) \leqq \frac{1}{2}(L^+ + L^-) \tag{1.4}$$

where l^-, l^+, L^-, L^+ are the left and right lower limits and the left and right upper limits of f at x_0, respectively.

In particular if x_0 is a discontinuity point of the first kind for f then

$$\lim_{n\to\infty} P_n(f, x_0) = \frac{1}{2}\big(f(x_0+) + f(x_0-)\big). \tag{1.5}$$

REMARKS. (i) If the X_n's are one sided Pólya r.v.s. we can relax the boundedness assumption. Instead we have to impose that f is finite for every finite x and satisfies the following growth condition:

$$\lim_{x\to\infty} \frac{f(x)}{e^{\gamma x}} = 0 \quad \text{for every} \quad \gamma > 0. \tag{1.6}$$

(ii) Certain approximation operators may be constructed by a similar procedure as follows. Let

$$\begin{cases} \mathcal{E}X_n = h(x_0),\ h(x_0) \text{ is an internal point of } J \text{ and } h \text{ is a continuous strictly} \\ \text{monotone function from } J^* \text{ onto } J. \end{cases} \tag{1.2a}$$

If X_n are i.i.d.r.v.'s with finite variances, $S_n = \sum_{k=1}^{n} X_k$, then $h^{-1}\left(\frac{S_n}{n}\right)$ converges in probability to x_0; so one is led to construct

$$P_n^*(f, x_0) = \mathcal{E}f\left(h^{-1}\left(\frac{S_n}{n}\right)\right). \tag{1.1a}$$

THEOREM 1a. *Assume all the conditions of Theorem 1 are in force except for* (1.2) *which is replaced by* (1.2a). *Then for every (measurable) bounded function f on J^**

$$\frac{1}{2}(l^+ + l^-) \leqq \varliminf_{n\to\infty} P_n^*(f, x_0) \leqq \varlimsup_{n\to\infty} P_n^*(f, x_0) \leqq \frac{1}{2}(L^+ + L^-). \tag{1.4a}$$

For small perturbations we prove:

THEOREM 2. *Let the r.v's $X_{n,k}$ $(n \geqq 1, k = 1, 2, \ldots, r(n))$ take values on some interval J. For each integer n assume that $X_{n,1}, X_{n,2}, \ldots, X_{n,r(n)}$ are i.i.d. with finite expectations x_n and positive finite variances b_n^2 satisfying*

$$0 < m \leqq b_n \leqq M < \infty \quad (n \geqq 1) \tag{1.7}$$

and

(1.8) $$(x_0-x_n)\sqrt{n}\,b_n^{-1}\to\beta \quad (n\to\infty).$$

Define

(1.9) $$Q_n(f,x_n)=\mathscr{E}f\left(\frac{X_{n,1}+\ldots+X_{n,r(n)}}{r(n)}\right).$$

Then for any bounded (measurable) function f on J

(1.10) $$\Phi(\beta)l^-+(1-\Phi(\beta))l^+\leqq\varliminf_{n\to\infty}Q_n(f,x_n)\leqq$$
$$\leqq\varlimsup_{n\to\infty}Q_n(f,x_n)\leqq\Phi(\beta)L^-+(1-\Phi(\beta))L^+$$

where l^+, l^-, L^+, L^- are as in Theorem 1 *and $\Phi(z)$ is the standard normal distribution, i.e.*

(1.11) $$\Phi(z)=\int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-(1/2)y^2}\,dy.$$

So if x_0 is a discontinuity point of the first kind, then

(1.12) $$\lim_{n\to\infty}Q_n(f,x_0)=\Phi(\beta)f(x_0-)+(1-\Phi(\beta))f(x_0+).$$

REMARK. One can find an analogue of Theorem la to this case.

§ 2. Examples

We may use now the results of the previous section to indicate how several operators approximate bounded discontinuous functions. First we show how these operators may be obtained from the general operator $P_n(f,x)$ (or $Q_n(f,x)$) by choosing appropriately the distribution of the X_n's (or $X_{n,k}$'s). Then we apply Theorem 1 (or Theorem 2).

A) *The Bernstein polynomials.* Let X_n be i.i.d.r.v.'s having the Bernoulli distribution $B(1,x)$ i.e.

$$\Pr\{X_n=i\}=x^i(1-x)^{1-i}\quad (i=0,1;\ 0\leqq x\leqq 1).$$

So

$$P_n(f,x)=B_n(f,x)=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}\quad (0\leqq x\leqq 1).$$

These are the classic Bernstein polynomials. Using Theorem 1 we deduce

$$\frac{1}{2}(l^++l^-)\leqq\varliminf_{n\to\infty}B_n(f,x)\leqq\varlimsup_{n\to\infty}B_n(f,x)\leqq\frac{1}{2}(L^++L^-).$$

This result was first proved by I. CHLODOVSKI [4] and later by F. HERZOG and J. D. HILL [6].

If for each fixed n, $X_{n,k}$ $(1\leqq k\leqq n)$ are independent and distributed according to

$$\Pr\{X_{n,k}=i\}=x_n^i(1-x_n)^{1-i}\quad (i=0,1)$$

then

$$Q_n(f,x_n)=B_n(f,x_n)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x_n^k(1-x_n)^{n-k}.$$

So if $(x_0(1-x_0))^{-1/2}(x_0-x_n)\sqrt{n}\to\beta$ $(n\to\infty)$ then

$$\Phi(\beta)l^-+(1-\Phi(\beta))l^+\leqq \varliminf_{n\to\infty} B_n(f,x_n)\leqq \varlimsup_{n\to\infty} B_n(f,x_n)\leqq \Phi(\beta)L^-+(1-\Phi(\beta))L^+.$$

B) *The Szász operator*. Assume X_n are i.i.d.r.v.'s having the Poisson law with mean x i.e.

$$\Pr(X_n=k)=e^{-x}\frac{x^k}{k!}\quad (k=0,1,2,\ldots).$$

Then

$$P_n(f,x)=S_n(f,x)=\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)e^{-nx}\frac{(nx)^k}{k!}.$$

This operator was introduced by O. SZÁSZ (see [12]).

To get the limit of $S_n(f,x_n)$ when x_n approaches a discontinuity point choose $X_{n,k}$ to be independent Poisson r.v.'s with mean x_n. Then $Q_n(f,x_n)$ reduces to $S_n(f,x_n)$ and Theorem 2 reads: If $(x_0-x_n)\sqrt{nx_0^{-1}}\to\beta$ $(n\to\infty)$ then for any finite function on $[0,\infty)$ satisfying (1.6), (1.10) holds.

C) *The Weierstrass operator*. Let X_n $(n\geqq 1)$ be independent and identically normally distributed r.v.'s with mean x and variance σ^2 i.e. their densities are given by

$$f_{x_n}(z)=\frac{1}{\sqrt{2\pi\sigma}}\exp\left\{-\frac{1}{2}\left(\frac{z-x}{\sigma}\right)^2\right\}\quad (-\infty<x<\infty,\ \infty<z<\infty,\ \sigma>0).$$

Then

$$P_n(f,x)=W_n(f,x)=\sqrt{\frac{n}{2\pi}}\frac{1}{\sigma}\int_{-\infty}^{\infty} f(z)\exp\left(-\frac{n}{2}\left(\frac{z-x}{\sigma}\right)^2\right)dz.$$

If $\sigma=\frac{1}{\sqrt{2}}$ this is the classical Weirstrass operator.

For any finite function f satisfying

$$\lim_{x\to\infty}\frac{f(x)}{e^{\gamma x}}=0\quad (\gamma>0);\quad \lim_{x\to-\infty}\frac{f(x)}{e^{\gamma x}}=0\quad (\gamma<0)$$

we have

$$\frac{1}{2}(l^++l^-)\leqq \varliminf_{n\to\infty} W_n(f,x_0)\leqq \varlimsup_{n\to\infty} W_n(f,x_0)\leqq \frac{1}{2}(L^++L^-).$$

D) *The operator of Cheney and Sharma.* Our methods apply also to the following operator of E. W. CHENEY and A. SHARMA [3]:

$$K_n(f,t,x) = \sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right)(1-x)^{n+1}\exp\left(\frac{tx}{1-x}\right)\mathcal{L}_k^n(t)x^k.$$

$\mathcal{L}_k^\alpha(t)$ are the Laugerre polynomials of order α and degree k. Using the generating function of these polynomials one can show that $K_n(f,t,x)$ can be written probablisticly as $K_n(f,t,x)=\mathcal{E}f(Z_n/(Z_n+n))$, where $Z_n=X_1+\ldots+X_n+Y, X_1, \ldots, X_n, Y$ are independent r.v.'s; each X_j is distributed according to

$$\Pr(X=k) = x^k(1-x) \quad (k \geqq 0;\ 0 \leqq x \leqq 1),$$

and Y according to

$$\Pr(Y=k) = \exp\left(\frac{tx}{1-x}\right)\mathcal{L}_k(t)x^k(1-x), \quad k \geqq 0; \quad t > 0; \quad 0 \leqq x \leqq 1.$$

(It seems that this is the first time a probablistic interpretation is given to this operator).

As Y/n is asymptotically negligible with respect to $(X_1+\ldots+X_n)/n$ we deduce that

$$\Phi(\beta)l^- + (1-\Phi(\beta))l^+ \leqq \varliminf_{n\to\infty} K_n(f,t,x_n) \leqq \varlimsup_{n\to\infty} K_n(f,t,x_n) \leqq$$

$$\leqq \Phi(\beta)L^- + (1-\Phi(\beta))L^+$$

where $\beta = \lim_{n\to\infty}(x_0-x_n)\sqrt{n}(1-x_0)/\sqrt{x_0}$.

In particular this result holds for the operator of MEYER—KÖNIG and ZELLER (see [8]).

Our methods apply to many other operators such as those of V. A. BASKAKOV [1], M. MÜLLER [9] and D. STANCU [11].

§ 3. Proofs of the theorems

First let us show that Theorem 1 is an easy consequence of Theorem 2. Suppose $X_1, \ldots, X_n$ are i.i.d.r.v's with mean x and finite positive variance.

Denote $X_{n,k}=X_k$ $(n\geqq 1, k=1, \ldots, r(n))$. Then $X_{n,k}$ $(n\geqq 1, k=1, \ldots, r(n))$ satisfy the conditions of Theorem 2 with $x_n\equiv x$ (for all n) And $\beta=0$. Thus (1.10) and (1.12) hold for $Q_n(f,x_n)$ with $\Phi(\beta)=\Phi(0)=\frac{1}{2}$.

But in this case $P_n(f,x_n)\equiv Q_n(f,x_n)$ so (1.4) and (1.5) hold. Thus Theorem 1 follows immediately from Theorem 2. So we have only to prove the latter. For this end we need the following

LEMMA 1. *Suppose for each n, $X_{n,1}, X_{n,2}, \dots X_{n,r(n)}$ are i.i.d.r.v.'s with means x_n and variances b_n^2 satisfying* (1.7) *and* (1.8). *Then*

$$\lim_{n\to\infty} \Pr\left\{\left(\frac{S_n}{r(n)}-x_n\right)\sqrt{r(n)}\, b_n^{-1} \leqq z\right\} = \Phi(z) \tag{3.1}$$

uniformly for z in R $\left(S_n = \sum_{k=1}^{r(n)} X_{n,k}\right)$.

PROOF. The result of this lemma will follow once we show that the Lindeberg condition

$$\frac{1}{\operatorname{Var} S_n} \sum_{k=1}^{r(n)} \int\limits_{|z-x_n|>\varepsilon \operatorname{Var} S_n} (z-x_n)^2\, dF_{n,k}(z) \to 0 \quad (n\to\infty) \tag{3.2}$$

holds. ($F_{n,k}(z)$ is the distribution function of $X_{n,k}$).

But for any fixed n, $X_{n,k}$ $(1 \leqq k \leqq r(n))$, are i.i.d.r.v.'s with variances b_n^2; hence

$$0 \leqq \frac{1}{\operatorname{Var} S_n} \sum_{k=1}^{r(n)} \int\limits_{|z-x_n|>\varepsilon \operatorname{Var} S_n} (z-x_n)^2\, dF_{n,k} =$$

$$= \frac{1}{b_n^2} \int\limits_{|z-x_n|>\varepsilon r(n) b_n^2} (z-x_n)^2\, dF_{n,k} \leqq \frac{1}{m} \int\limits_{|z-x_n|>\varepsilon r(n) m} (z-x_n)^2\, dF_{n,k} \to 0 \quad (n\to\infty)$$

(as $M \geqq b_n^2 \geqq m$; $r(n) \to \infty$).

Thus the Lindeberg condition (3.2) is satisfied, hence (3.1) follows. The fact that the convergence in (3.1) is uniform follows from Theorem 4.3.3 in RÉNYI [10].

PROOF OF THEOREM 2. Assume, without loss of generality that $r(n)=n$, so

$$Q_n(f, x_n) = \mathscr{E} f\left(\frac{X_{n,1}+\dots+X_{n,n}}{n}\right) = \int f(z)\, dF_n(z)$$

$\left(\text{where } F_n \text{ is the distribution function of } \sum_{k=0}^{n} X_{n,k}/n\right)$.

As f is a bounded function, the upper and lower limits at any point exist. Given $\varepsilon>0$ we can find $\delta \equiv \delta(\varepsilon)>0$ such that

$$l^- - \varepsilon \leqq f(z) \leqq L^- + \varepsilon \quad \text{for} \quad z \in J \quad \text{and} \quad x_0 - \delta \leqq z < x_0$$

$$l^+ - \varepsilon \leqq f(z) \leqq L^+ + \varepsilon \quad \text{for} \quad z \in J \quad \text{and} \quad x_0 < z \leqq x_0 + \delta.$$

Let

$$J_1 = J \cap \{z\colon |z-x_0| > \delta\}; \quad J_2 = J \cap \{z\colon x_0 - \delta \leqq z < x_0\};$$

$$J_3 \leqq J \cap \{z\colon x_0 < z \leqq x_0 + \delta\}; \quad J_4 = \{x_0\}.$$

Write $Q_n(f, x_n)$ as

$$Q_n(f, x_n) = \left\{\int_{J_1} + \int_{J_2} + \int_{J_3} + \int_{J_4}\right\} f(z)\, dF_n \equiv I_1(n) + I_2(n) + I_3(n) + I_4(n). \tag{3.3}$$

First let us estimate $I_1(n)$. As f is bounded, say $|f(z)| \leqq M$ for all real z, we have

$$|I_1(n)| = \int_{J_1} f(z)\, dF_n \leqq M \int_{J_1} dF_n = M \Pr\left\{\left|\frac{S_n}{n} - x_0\right| > \delta\right\}.$$

But by the Chebyshev inequality and the boundedness of the b_n's

$$\Pr\left\{\left|\frac{S_n}{n} - x_0\right| > \delta\right\} \leqq \Pr\left\{\left|\frac{S_n}{n} - x_n\right| > \frac{\delta}{2}\right\} \leqq \frac{4b_n^2}{\delta^2 n} \to 0 \quad (n \to \infty). \tag{3.4}$$

Hence $I_1(n) \to 0$ $(n \to \infty)$.

Let us now estimate $I_2(n)$.

$$I_2(n) \geqq (l^- - \varepsilon) \int_{J_2} dF_n = (l^- - \varepsilon)\left[\Pr\left\{\frac{S_n}{n} < x_0\right\} - \Pr\left\{\frac{S_n}{n} < x_0 - \delta\right\}\right] =$$

$$= (l^- - \varepsilon)\left[\Pr\left\{\left(\frac{S_n}{n} - x_n\right)\sqrt{n}\, b_n^{-1} < (x_0 - x_n)\sqrt{n}\, b_n^{-1}\right\} - \Pr\left\{\frac{S_n}{n} < x_0 - \delta\right\}\right].$$

Now by (3.1) of Lemma 1 and (1.8)

$$\lim_{n\to\infty} \Pr\left\{\left(\frac{S_n}{n} - x_n\right)\sqrt{n}\, b_n^{-1} \leqq (x_0 - x_n)\sqrt{n}\, b_n^{-1}\right\} = \Phi(\beta)$$

and by (3.4)

$$\lim_{n\to\infty} \Pr\left\{\frac{S_n}{n} < x_0 - \delta\right\} = 0.$$

Hence $\varliminf_{n\to\infty} I_2(n) \geqq \Phi(\beta)\, l^-$, similarly $\varlimsup_{n\to\infty} I_2(n) \leqq \Phi(\beta)\, L^-$ and

$$\big(1 - \Phi(\beta)\big)\, l^+ \leqq \varliminf_{n\to\infty} I_3(n) \leqq \varlimsup_{n\to\infty} I_3(n) \leqq \big(1 - \Phi(\beta)\big)\, L^+. \tag{3.5}$$

The fact that $I_4(n) \to 0$ follows immediately from the continuity of the limiting (normal) distribution.

Combining our estimates yields the theorem. Q.E.D.

The proof of Theorem 1a is similar, hence it would not be given here.

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ON FUNCTIONAL CENTRAL LIMIT THEOREM FOR STATIONARY MARTINGALE RANDOM FIELDS

By

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0. Introduction

The term random field is often used to denote a collection of random variables with a parameter space which is a subset of the q-dimensional Euclidean space R^q. Stationary random fields are of great practical importance and hence also of theoretical interest. Examples of random fields occur in biological investigations concerning the distribution of plants or animals over a given area, when $q=2$ and $\mathbf{t}=(t_1, t_2)$ is a point of the area. In problems involving propagation of electromagnetic waves through random media the natural parameter space is a subset of R^4, representing space and time. Further important examples occur in the theory of turbulance where, for example, one may consider the case $q=4$ and $\mathbf{t}$ is a point in space-time, while $\xi_1(\mathbf{t}), \xi_2(\mathbf{t}), \xi_3(\mathbf{t})$ are the velocity components of a turbulent fluid at the point $\mathbf{t}$. Multiparameter stochastic process (the so-called random field) plays a prominent role in weak convergence of empirical process to Kiefer process (a two-dimensional Brownian bridge), Brownian sheets, and sample spacings. In this paper we extend the concept of martingale to random fields and obtain a functional central limit theorem for such random fields. An important example of martingales with a partially ordered parameter set is the following generalization of Wiener process. Let $\mathscr{A}^q$ be the family of all Borel sets in R^q having finite Lebesgue measure. Let $\{X_A, A\in\mathscr{A}^q\}$ be a real Gaussian additive random set function with $E(X_A)=0$, $E(X_A X_B)=m(A\cap B)$ where m denotes the Lebesgue measure. Intuitively, X_A can be thought of as the integral over A of a Gaussian White noise. Such integral of Gaussian White noise has extensively been used by Physicists and engineers.

1. Martingale random fields

Martingales with a partially ordered parameter have been considered by Cairoli [4], Wong and Zakai [10], and recently by Shorack and Smythe [8].

Cairoli's definition of martingale is through product type of probability spaces which is very complicated and of limited scope. Wong and Zakai's approach through increasing path is not suitable for weak convergence. Shorack and Smythe's [8] definition is stronger than our definition if $q>1$. The definition of martingale field given in § 3 is natural for stationary processes which is our primary interest.

2. Notation

Let Z^q denote the set of all q-tuples of integers ($q \geqq 1$, a positive integer). The points in Z^q will be denoted by $\mathbf{m}, \mathbf{n}$, etc., or sometime, when necessary, more explicitly by $(m_1, m_2, \ldots, m_q)$, $(n_1, n_2, \ldots, n_q)$, etc. Z^q is partially ordered by stipulating $\mathbf{m} \leqq \mathbf{n}$ iff $m_i \leqq n_i$ for each $i, 1 \leqq i \leqq q$. We write 0 and 1 for points $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$ in Z^q, respectively.

Let $\{\xi_{\mathbf{n}}: \mathbf{n} \in Z^q\}$ be a random field, i.e., a collection of random variables indexed by time-set Z^q. The random field is said to be stationary if for each finite subset S of Z^q, and each $\mathbf{m} \in Z^q$, the joint distribution of $\{\xi_{\mathbf{n}+\mathbf{m}}: \mathbf{n} \in S\}$ is the same as that of $\{\xi_{\mathbf{n}}: \mathbf{n} \in S\}$. Here $\mathbf{n}+\mathbf{m}$ is the usual coordinatewise sum.

For each $\mathbf{n}(\mathbf{n} \geqq \mathbf{1})$, let $F_{\mathbf{n}}$ be the σ-field generated by $\{\xi_{\mathbf{p}}: p_j \geqq 1$ for $j=1, \ldots, q$ and for at least one i $1 \leqq p_i \leqq n_i\}$. Note that $F_{\mathbf{n}}$ is the σ-field generated by $\{\xi_{\mathbf{p}}: \mathbf{p} \ngtr \mathbf{n}\}$. For $\mathbf{n} \geqq \mathbf{1}$, define the partial sum $S_{\mathbf{n}} = \sum_{\mathbf{1} \leqq \mathbf{j} \leqq \mathbf{n}} \xi_{\mathbf{j}}$. Whenever convenient we will extend the domain of $S_{\mathbf{n}}$ to include indices some of whose coordinates may be zero. In such cases we define $S_{\mathbf{n}}$ to be zero.

Let T be the closed unit interval [0, 1] and T^q the q-fold Cartesian product of T. Let C^q be the set of all continuous functions on T^q with the uniform metric and, as in BICKEL and WICHURA [1], let us denote by D^q the Skorohod function space on T^q. All the properties of D^q that we need can be found in BICKEL and WICHURA [1].

If $\mathbf{n}=(n_1, n_2, \ldots, n_q)$, let $|\mathbf{n}|$ stand for the product $n_1 n_2 \ldots n_q$. Define $|\mathbf{t}|$ similarly for $\mathbf{t} \in T^q$. In this paper the limit $\mathbf{n} \to \infty$ will mean $\min_{1 \leqq i \leqq q} n_i \to \infty$. On T^q as well as Z^q we use the maximum norm, i.e., if $\mathbf{t} \in T^q$ or $\mathbf{n} \in Z^q$, then $\|\mathbf{t}\| = \max_{1 \leqq j \leqq q} |t_j|$ and $\|\mathbf{n}\| = \max_{1 \leqq j \leqq q} |n_j|$.

If $E\{\xi_{\mathbf{n}}^2\} = \sigma^2 > 0$ for all $\mathbf{n} \geqq \mathbf{1}$, we define for $\mathbf{t} \in T^q$ and $\mathbf{n} \geqq \mathbf{1}$,

$$X_{\mathbf{n}}(t) = (\sigma^2 |\mathbf{n}|)^{-1/2} \times S_{[n_1 t_1], \ldots, [n_q t_q]},$$

where $[\cdot]$ is the usual greatest integer function. The stochastic process $X_{\mathbf{n}}$ has sample paths in D^q.

3. The central limit and related theorems

The main theorem of this paper is

THEOREM 1. *Let* $\{\xi_{\mathbf{n}}: \mathbf{n} \geqq \mathbf{1}\}$ *be a stationary, ergodic random field for which*

(1) $E(\xi_{\mathbf{n}} \| F_{\mathbf{m}}) = 0$ *whenever* $\mathbf{m} < \mathbf{n}$ *with probability* 1

and for which $E\{\xi_{\mathbf{n}}^2\} = \sigma^2$ *is positive and finite. Then the net* $\{X_{\mathbf{n}}: \mathbf{n} \geqq \mathbf{1}\}$ *of stochastic processes converges weakly, in* D^q, *to the q-parameter Wiener process.*

REMARK. There is no loss of generality in working with the random field $\{\xi_{\mathbf{n}}: \mathbf{n} \in Z^q\}$ since given a random field with "one-sided" time set, we can construct a new random field with time set all of Z^q and with the same finite-dimensional distributions (cf. M. ROSENBLATT [7] in the case $q=2$). Now if we let $F_{\mathbf{n}}^*$ be the

σ-field generated by $\{\xi_{\mathbf{p}}$: for at least one $i, p_i \leqq n_i\}$, (1) becomes by stationarity

(1′) $$E\{\xi_{\mathbf{p}} \| F^*_{\mathbf{p}-1}\} = 0 \quad \text{with probability 1.}$$

The techniques used in the proof of this theorem are, with some exceptions, essentially those used by BILLINGSLEY [2] to prove Theorem 23.1 there and further exploited by C. M. DEO [5]. The proof will be carried out in the following series of theorems and lemmas.

THEOREM 2 (A characterization of the q-parameter Wiener process). *Let X be a random element of D^q with $P(X \in C^q) = 1$ and $P(X(0)) = 1$. Moreover X satisfies*

Condition 1. *For $\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(k)}, \mathbf{t} \in T^q$ and G the σ-field generated by $\{X(\mathbf{t}^{(1)}), \dots$ $\dots, X(\mathbf{t}^{(k)}), X(\mathbf{t})\}$. If $0 \leqq t_j^{(1)}, \dots, t_j^{(k)} \leqq t_j < 1$ then*

(2) $$\lim_{h_j \downarrow 0} \frac{1}{h_j} E\{|E\{X(t_1, \dots, t_j + h_j, \dots, t_q) - X(t_1, \dots, t_q)| |G\}|\} = 0$$

for all $j = 1, 2, \dots, q$.

(3) $$\lim_{h_j \downarrow 0} \frac{1}{h_j} E\{|E\{(X(t_1, \dots, t_j + h_j, \dots, t_q) - X(t_1, \dots, t_q))^2| |G\} - $$
$$- h_j \prod_{i \neq j} t_i|\} = 0 \quad \textit{for all} \quad j = 1, 2, \dots, q.$$

Condition 2. *Let $\mathbf{t} \in T^q$. If $t_j < 1$ then*

(4) $$\lim_{\alpha \to \infty} \limsup_{h_j \downarrow 0} \frac{1}{h_j} \int\limits_{\{(X(t_1, \dots, t_j + h_j, \dots, t_q) - X(t_1, \dots, t_q))^2 \geqq \alpha h_j\}} (X(t_1, \dots, t_j + h_j, \dots, t_q) -$$
$$- X(t_1, \dots, t_q))^2 \, dP = 0$$

Then X is the Wiener process on D^q.

PROOF. The proof, with few modifications, is similar to the proof of Theorem 19.3 of BILLINGSLEY [2]. We will prove for the case $q = 2$ (for general q the proof is analogous).

Fix points $\mathbf{t}^{(j)} \in T^2$ $(j = 1, \dots, k)$ with $0 \leqq t_1^{(1)}, \dots, t_1^{(k)} < 1$. Fix real numbers $u_1, \dots, u_k$. Let $\mathbf{t} \in T^2$ with t_2 fixed and let t_1 and u vary over the strip $\max(t_1^{(1)}, \dots, t_1^{(k)}) \leqq$ $\leqq t_1 < 1$, $-\infty < u < \infty$. Put

(5) $$\psi(t_1, t_2, u) = E\left\{\exp\left(\sum_{j=1}^{k} iu_j X(\mathbf{t}^{(j)})\right) e^{iuX(\mathbf{t})}\right\}.$$

Proceeding as in BILLINGSLEY [2] one obtains for arbitrary v and $0 \leqq t_1^{(1)}, \dots, t_1^{(k)} \leqq$ $\leqq t_1 \leqq s_1 \leqq 1$

(6) $$E\left\{\exp\left(\sum_{j=1}^{k} iu_j X(\mathbf{t}^{(j)})\right) e^{iuX(\mathbf{t})} e^{iv[X(s_1, t_2) - X(t_1, t_2)]}\right\} =$$
$$= E\left\{\exp\left(\sum_{j=1}^{k} iu_j X(\mathbf{t}^{(j)})\right) e^{iuX(\mathbf{t})}\right\} e^{-1/2 v^2 t_2 (s_1 - t_1)}.$$

Thus we have shown: given $\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(k)}, \mathbf{t} \in T^2$, if $0 \leqq t_1^{(1)}, \dots, t_1^{(k)} \leqq t_1 \leqq s_1 \leqq 1$ then $X(s_1, t_2) - X(t_1, t_2)$ has distribution $N(0, t_2(s_1 - t_1))$ and is independent of any linear combination of the random variables $X(\mathbf{t}^{(1)}), \dots, X(\mathbf{t}^{(k)}), X(\mathbf{t})$.

Doing exactly the same for the second coordinate we have: if $0 \leqq t_2^{(1)}, \dots, t_2^{(k)} \leqq \leqq t_2 \leqq s_2 \leqq 1$ then $X(t_1, s_2) - X(t_1, t_2)$ has distribution $N(0, t_1(s_2 - t_2))$ and is independent of any linear combination of the random variables $X(\mathbf{t}^{(1)}), \dots, X(\mathbf{t}^{(k)}), X(\mathbf{t})$. From above it follows that $X(\mathbf{t})$ is $N(0, |\mathbf{t}|)$ and

$$E\{X(\mathbf{t})X(\mathbf{s})\} = \min(t_1, s_1) \times \min(t_2, s_2).$$

By Cramér—Wold device (see BILLINGSLEY [2], p. 48) it remains to show that all finite-dimensional distributions are normal. It suffices to show that for arbitrary points $\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(n)}$ of T^2 any linear combination of the r.v. $X(\mathbf{t}^{(1)}), \dots, X(\mathbf{t}^{(n)})$ is univariate normal (with zero mean). The proof will be done by induction. If $n=1$ we have shown that $X(\mathbf{t})$ is $N(0, |\mathbf{t}|)$. Suppose true for $n=k-1$ $(k \geqq 2)$. Now for $n=k$, given k points in T^2 we can always order them with respect to the first coordinate and relabel them so that we have $\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(k)} \in T^2$ with $0 \leqq t_1^{(1)} \leqq t_1^{(2)} \leqq \dots \dots \leqq t_1^{(k)}$. Given real numbers $a_1, \dots, a_k$ let $Z_1 = \sum_{j=1}^{k} a_j X(\mathbf{t}^{(j)})$. Then

$$E\{e^{iuZ_1}\} = e^{-1/2\, u^2 \alpha_1} E\{e^{iuZ_2}\}, \tag{7}$$

where

$$\alpha_1 = a_k^2 t_2^{(k)} (t_1^{(k)} - t_1^{(k-1)}) \geqq 0 \quad \text{and} \quad Z_2 = \sum_{j=1}^{k-1} a_j X(\mathbf{t}^{(j)}) + a_k X(\mathbf{t}^*)$$

with $t^* = (t_1^{(k-1)}, t_2^{(k)})$. Notice Z_2 is a linear combination of $X(\mathbf{t}^{(1)}), \dots, X(\mathbf{t}^{(k-1)})$, $X(\mathbf{t}^*)$ and we have one match in the first coordinate, namely, $t_1^{(k-1)} = t_1^*$. Now order $\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(k-1)}, \mathbf{t}^*$ with respect to the second coordinate and relabelling them we have $Z_2 = \sum_{j=1}^{k} b_j X(\mathbf{s}^{(j)})$ (for conveniently chosen b_j's) where $s_2^{(1)} \leqq s_2^{(2)} \leqq \dots \leqq s_2^{(k)}$, then

$$E\{e^{iuZ_1}\} = e^{-1/2\, u^2 \alpha_1} e^{-1/2\, u^2 \alpha_2} E\{e^{iuZ_3}\}, \tag{8}$$

where

$$\alpha_2 = b_k^2 s_1^{(k)} (s_2^{(k)} - s_2^{(k-1)}) \geqq 0 \quad \text{and} \quad Z_3 = \sum_{j=1}^{k-1} b_j X(\mathbf{s}^{(j)}) + b_k X(\mathbf{s}^*)$$

with $\mathbf{s}^* = (s_1^{(k)}, s_2^{(k-1)})$. Notice Z_3 is a linear combination of $X(\mathbf{s}^{(1)}), \dots, X(\mathbf{s}^{(k-1)})$, $X(\mathbf{s}^*)$ where we have one match in the first coordinate and one match in the second coordinate. After k times this procedure, we have

$$E\{e^{iuZ_1}\} = e^{-1/2\, u^2 (\alpha_1 + \dots + \alpha_k)} E\{e^{iuZ_{k+1}}\}, \tag{9}$$

where Z_{k+1} is a linear combination of $\{X(\mathbf{r}^{(j)}), j=1, \dots, k\}$ and we have k matches among the coordinates of $\{\mathbf{r}^{(j)}, j=1, 2, \dots, k\}$. This implies that Z_{k+1} is a linear combination of $\{X(\mathbf{v}^{(j)}), j=1, \dots, k-1\}$ (relabelling the r_j's) and by induction assumption Z_{k+1} is univariate normal with zero mean, thus Z_1 is univariate normal and this completes the proof.

For $X \in D^q$ and $0<\delta<1$, define the modulus $w(X;\delta)$ by

(10) $$w(X;\delta) = \sup\{|X(\mathbf{t})-X(\mathbf{s})|:\ \|\mathbf{t}-\mathbf{s}\| \leqq \delta\}.$$

THEOREM 3. *Suppose that* $\{X_{\mathbf{n}}^2(\mathbf{t}): \mathbf{n}\in Z^q, \mathbf{t}\in T^q, \mathbf{n}\geqq \mathbf{1}\}$ *is uniformly integrable for each* $\mathbf{t}\in T^q$, *that* $X_{\mathbf{n}}(0) \xrightarrow{p} 0$, *and that for each* $\varepsilon>0, \eta>0$ *we can find a* $\delta>0$ *such that*

(11) $$P\big(w(X_{\mathbf{n}};\delta) \geqq \varepsilon\big) < \eta$$

for all sufficiently large $\mathbf{n}$. *Moreover* $X_{\mathbf{n}}$ *satisfies*

Condition 1. *For* $\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(k)}$, $\mathbf{t}\in T^q$ *and* $G_{\mathbf{n}}$ *the* σ-*field generated by* $\{X_{\mathbf{n}}(\mathbf{t}^{(1)}), \dots, X_{\mathbf{n}}(\mathbf{t}^{(k)}), X(\mathbf{t})\}$. *If* $0\leqq t_j^{(1)}, \dots, t_j^{(k)} \leqq t_j<1$ *then*

(12) $$\lim_{h_j\downarrow 0} \limsup_{\mathbf{n}\to\infty} \frac{1}{h_j} E\big\{|E\{X_{\mathbf{n}}(t_1, \dots, t_j+h_j, \dots, t_q)-X(t_1, \dots, t_q)|\,|G_{\mathbf{n}}\}|\big\} = 0$$

for all $j=1, 2, \dots, q$.

(13)
$$\lim_{h_j\downarrow 0} \limsup_{\mathbf{n}\to\infty} \frac{1}{h_j} E\big\{\big|E\big\{(X_{\mathbf{n}}(t_1, \dots, t_j+h_j, \dots, t_q)-X(t_1, \dots, t_q))^2\big|\,G_{\mathbf{n}}\big\}-h_j \prod_{i\neq j} t_i\big|\big\} = 0$$

for all $j=1, 2, \dots, q$.

Condition 2. *Let* $\mathbf{t}\in T^q$. *If* $t_j<1$ *then*

(14)
$$\lim_{\alpha\to\infty} \limsup_{h_j\downarrow 0} \limsup_{\mathbf{n}\to\infty} \frac{1}{h_j} \int\limits_{\{(X_{\mathbf{n}}(t_1, \dots, t_j+h_j, \dots, t_q)-X(t_1, \dots, t_q))^2 \geqq \alpha h_j\}} (X_{\mathbf{n}}(t_1, \dots, t_j+h_j, \dots, t_q)- -X(t_1, \dots, t_q))^2\, dP = 0$$

for all $j=1, 2, \dots, q$.

Then X_n *converges weakly to the q-parameter Wiener process* $X(\mathbf{t})$.

PROOF. Using Theorem 2 of this paper, the proof is similar to that of Theorem 19.4 of BILLINGSLEY [2].

The next theorem is a multiparameter version of the ergodic theorem 6.21 of BREIMAN [3].

On (R_∞^q, B_∞^q) define the transformation $T=T^{1,\dots,1}: R_\infty^q \to R_\infty^q$, $T(x_{1,1,\dots,1}, x_{1,1,\dots,2}, \dots)=(x_{2,2,\dots,2}, x_{2,2,\dots,3}, \dots)$. Then we have q commutating shift transformations (again, cf. ROSENBLATT [7], p. 555) $\tau_1=T^{1,0,\dots,0}$, $\tau_2=T^{0,1,0,\dots,0}, \dots, \tau_q=T^{0,0,\dots,0,1}$ with $\tau_1 X_{\mathbf{n}}=X_{n_1+1,n_2,\dots,n_q}$, $\tau_2 X_{\mathbf{n}}=X_{n_1,n_2+1,\dots,n_q}$, $\dots$ $\tau_q X_{\mathbf{n}}=X_{n_1,n_2,\dots,n_q+1}$. Stationarity of process implies invariance of the probability measure P under q shifts i.e. for any measurable set A

$$P(\tau_1 A) = P(\tau_2 A) = \dots = P(\tau_q A) = P(A).$$

By ergodicity of process we mean that

$$\lim_{\mathbf{n}\to\infty} \frac{1}{|\mathbf{n}|} \sum_{\substack{k_i=1\\ 1\leqq i\leqq q}}^{n_i} P(\tau_1^{k_1}\tau_2^{k_2}\dots\tau_q^{k_q} A) = P(A)$$

for each measurable set A. A set A is said to be invariant if $\tau_1 A=\tau_2 A=\ldots=\tau_q A$ up to an exceptional set of probability zero. If $\{\xi_{\mathbf{n}}\}$ is stationary then T is measure-preserving.

THEOREM 4. *Let T be measure-preserving on (Ω, F, P). Then for X any random variable such that $E|X|<\infty$:*

(a) *for each j $(j=1, \ldots, q)$ and for fixed n_i $(i\neq j)$,*

$$\lim_{n_j\to\infty}\frac{1}{|\mathbf{n}|}\sum_{\substack{k_i=0\\1\leq i\leq q}}^{n_i-1} X(T^{k_1,\ldots,k_q}\omega)=E(X|J)\quad \textit{a.s. and uniformly in}\quad \{n_i,\, i\neq j\}.$$

(b) $$\lim_{\mathbf{n}\to\infty}\frac{1}{|\mathbf{n}|}\sum_{\substack{k_i=0\\1\leq i\leq q}}^{n_i-1} X(T^{k_1,\ldots,k_q}\omega)=E(X|J)\quad \text{a.s.},$$

where J is the class of invariant sets with respect to T.

PROOF. It can be done as in BREIMAN [3] with few modifications as follows: for each j $(j=1, \ldots, q)$ and for fixed n_i $(i\neq j)$ one can show that

$$\int\limits_{\{M_{\mathbf{n}}^j>0\}} Y_{n_j}\, dP \geq 0, \tag{15}$$

(For example if $q=2$, $Y_{n_2}(\omega)=X(\omega)+X(T^{0,1}\omega)+\ldots+X(T^{0,n_2-1}\omega)$, $N_{n_1}(\omega)=$ $=Y_{n_2}(\omega)+Y_{n_2}(T^{1,0}\omega)+\ldots+Y_{n_2}(T^{n_1-1,0}\omega)$.) where

$$Y_{n_j}=\sum_{\substack{k_i=0\\(i\neq j,\, k_j=0)}}^{n_i-1} X(T^{k_1,\ldots,k_q}\omega)\quad\text{and}\quad M_{\mathbf{n}}^j=\max\{0, N_1, \ldots, N_{n_j}\},$$

where

$$N_{n_l}=\sum_{k_j=0}^{l-1}\sum_{\substack{k_i=0\\i\neq j}}^{n_i-1} X(T^{k_1,\ldots,k_q}\omega).$$

Then (a) follows and due to the uniform convergence (b) also follows.

COROLLARY 1. *Under the conditions of Theorem 4*
(a) *for each j $(j=1, \ldots, q)$ and fixed n_i $(i\neq j)$*

$$\lim_{n_j\to\infty} E\left|\frac{1}{|\mathbf{n}|}\sum_{\substack{k_i=0\\1\leq i\leq q}}^{n_i-1} X(T^{k_1,\ldots,k_q}\omega)-E(X|J)\right|=0\quad \textit{uniformly in}\quad \{n_i,\, i\neq j\}.$$

(b) $$\lim_{\mathbf{n}\to\infty} E\left|\frac{1}{|\mathbf{n}|}\sum_{\substack{k_i=0\\1\leq i\leq q}}^{n_i-1} X(T^{k_1,\ldots,k_q}\omega)-E(X|J)\right|=0.$$

PROOF. Using Theorem 4, the proof is similar to that of Corollary 6.25 of BREIMAN [3].

LEMMA 1. *Under the hypotheses of Theorem* 1 *and assuming that* $|\xi_0| \leqq l$ *with probability* 1, *we have*

$$ES^4_{n_1 \dots n_q} \leqq 6^q |\mathbf{n}|^2 l^4. \tag{16}$$

PROOF. First notice that from (1′) it follows that the r.v's $\xi_{\mathbf{p}}$ are orthogonal and by stationarity it follows that

$$ES^2_{n_1 \dots n_q} = n_1 \dots n_q \sigma^2. \tag{17}$$

The proof will be done by induction, the case $q=1$ has been shown by BILLINGSLEY [2]. Assume it is true for $q=m-1$, then for $q=m$, $ES^4_{\mathbf{n}} = \sum E\{\xi_{\mathbf{i}}\xi_{\mathbf{j}}\xi_{\mathbf{k}}\xi_{\mathbf{l}}\}$ with the indices $1 \leqq i_s, j_s, k_s, l_s \leqq n_s$ for $s=1, \dots, m$. Take the first coordinate of the indices, i.e., i_1, j_1, k_1, l_1, if the largest of them is not matched by any other then by (1′) the term vanishes; if we let $S(i_1, j_1, k_1, l_1) = \sum \xi_{\mathbf{i}}\xi_{\mathbf{j}}\xi_{\mathbf{k}}\xi_{\mathbf{l}}$ with the indices $1 \leqq i_s, j_s, k_s, l_s \leqq n_s$ for $s=2, \dots, m$ and i_1, j_1, k_1, l_1 fixed, then we have

$$E\{S^4_{\mathbf{n}}\} = \sum_{k_1} ES(k_1, \dots, k_1) + 4 \sum_{i_1 < k_1} ES(i_1, k_1, k_1, k_1) + 6 \sum_{i_1, j_1 < k_1} ES(i_1, j_1, k_1, k_1).$$

By using the induction hypotheses the first two terms on the right contribute at most $3n_1^2(6^{m-1}n_2^2 \dots n_m^2 l^4)$ in all, and the last sum is by (1′)

$$6 \sum_{k_1 = l_1} \sum_{\substack{k_s, l_s \\ s=2, \dots, m}} E\{S^2_{k_1 - 1, n_2, \dots, n_m} \xi_{\mathbf{k}} \xi_{\mathbf{l}}\} = 6 \sum_{\substack{k_s \\ 1 \leqq s \leqq m}} E\{S^2_{k_1 - 1, n_2, \dots, n_m} \xi^2_{\mathbf{k}}\}$$

and by (18) it cannot exceed $3|\mathbf{n}|^2 l^4$.

Our next lemma is a multiparameter version of a submartingale inequality of DOOB [6].

LEMMA 2. *Under the conditions of Theorem* 1, *and for* $\gamma > 1$

$$E\{\max_{\mathbf{k} < \mathbf{n}} |S_{\mathbf{k}}|^\gamma\} \leqq \left(\frac{\gamma}{\gamma - 1}\right)^{q\gamma} E\{|S_{\mathbf{n}}|^\gamma\}. \tag{18}$$

PROOF. The proof is similar to the one given by WICHURA [9] and will be done by induction. For $q=1$ (see DOOB [6], p. 317), suppose true for $q=p-1$, we will show it also holds for $q=p$. Given $\mathbf{n}=(n_1, \dots, n_p)$, let $1 \leqq m \leqq n_p$ and

$$S_{k_1, \dots, k_{p-1}}(m) = \sum_{\substack{1 \leqq j_i \leqq k_i \\ 1 \leqq i < p}} \xi_{j_1, \dots, j_{p-1}, m},$$

$$U_m = (S_{k_1, \dots, k_{p-1}}(m))_{1 \leqq k_i \leqq n_i, 1 \leqq i < p}$$

and put

$$V_m = (S_{k_1, \dots, k_{p-1}, m})_{1 \leqq k_i \leqq n_i, 1 \leqq i < p}, \quad \|V_m\| = \max_{\substack{1 \leqq k_i \leqq n_i \\ 1 \leqq i < p}} |S_{k_1, \dots, k_{p-1}, m}|.$$

Now by (1′)

$$E(V_{m+1} - V_m \| V_1, \dots, V_m) = E(U_{m+1} \| U_1, \dots, U_m) = 0.$$

Thus $(V_m)_{1 \leqq m \leqq n_p}$ is a martingale in $R^{n_1 + \dots + n_{p-1}}$ and the proof can be completed as in WICHURA [9]. Professor D. J. Scott drew our attention that this lemma could probably be proved by Lemma 1 of SHORACK and SMYTHE [8].

We now proceed to the proof of Theorem 1. Enough to show that the hypotheses of Theorem 3 are satisfied. Given

$$\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(k)}, \mathbf{t} \in T^q \quad \text{with} \quad 0 \leqq t_j^{(1)}, \ldots, t_j^{(k)} \leqq t_j < 1$$

for some j $(j=1, \ldots, q)$ let

$$Z_{\mathbf{n}}^j = X_{\mathbf{n}}(t_1, \ldots, t_j+h_j, \ldots, t_q) - X_{\mathbf{n}}(t_1, \ldots, t_q)$$

and $G_{\mathbf{n}}$ the σ-field generated by $\{X_{\mathbf{n}}(\mathbf{t}^{(1)}), \ldots, X_{\mathbf{n}}(\mathbf{t}^{(k)}), X_{\mathbf{n}}(\mathbf{t})\}$. Now

$$Z_{\mathbf{n}}^j = (\sigma^2 |\mathbf{n}|)^{-1/2} \sum_{\mathbf{p} \in A} \xi_{\mathbf{p}},$$

where $A = \{\mathbf{p}\colon 1 \leqq p_i \leqq [n_i t_i],\ i \neq j,\ [n_j t_j] < p_j \leqq [n_j(t_j+h_j)]\}$. Using the fact that $G_{\mathbf{n}} \subseteqq F^*_{\mathbf{p}-1}$ for $\mathbf{p} \in A$, (13) will follow from (1′). The remaining of the proof can be done as in BILLINGSLEY [2].

4. Integrals in place of sums

Theorem 1 has a natural formulation with $\xi_{\mathbf{n}}$ replaced by a process in continuous multidimensional time. Let $\{v_{\mathbf{t}}(\omega)\colon -\infty < t_i < \infty;\ 1 \leqq i \leqq q\}$ be a stationary random field satisfying $E(v_0^2) < \infty$ defined on a completed probability space (Ω, F, P), i.e., for every finite subset $\{\mathbf{t}^{(1)}, \mathbf{t}^{(2)}, \ldots, \mathbf{t}^{(q)}\}$ of R^q and each $\mathbf{s} \in R^q$, the joint distribution of $\{v_{\mathbf{t}^{(1)}+\mathbf{s}}, \ldots, v_{\mathbf{t}^{(q)}+\mathbf{s}}\}$ is the same as $\{v_{\mathbf{t}^{(1)}}, \ldots, v_{\mathbf{t}^{(q)}}\}$. Here $\mathbf{t}+\mathbf{s}$ is the usual coordinatewise sum. Suppose that $\{v_t\}$ is measurable with respect to $\sigma(F \times B^q)$, B_q is the Borel field of R^q, (DOOB [6]), so that the Lebesgue integrals

$$\int_{s_1}^{t_1} \ldots \int_{s_q}^{t_q} v_{\mathbf{u}}(\omega)\, du_1 \ldots du_q$$

are well defined and finite with probability 1. Let $Y_{\mathbf{n}}$ be the random field of T^q defined by

$$Y_{\mathbf{n}}(\mathbf{t}, \omega) = (\sigma^2 |\mathbf{n}|)^{-1/2} \int_0^{n_1 t_1} \ldots \int_0^{n_q t_q} v_{\mathbf{s}}(\omega)\, ds_1 \ldots ds_q$$

where $\sigma^2 = E(v_{\mathbf{n}}^2) > 0$ for all $\mathbf{n} \geqq \mathbf{1}$. (Note $Y_{\mathbf{n}}$ lies in C^q.) A subset B of T^q is called a block if it is of the form $\prod_{j=1}^{q} (s_j, t_j]$, $(s_j, t_j]$'s being half-closed subintervals of $[0, 1]$. If $V = \{v_{\mathbf{t}}\colon \mathbf{t} \in T^q\}$ is a stochastic process, then the increment $V(B)$ of V around a block $B = \prod_{j=1}^{q} (s_j, t_j]$ is given by

$$V(B) = \sum_{\varepsilon_1=0,1} \sum_{\varepsilon_2=0,1} \cdots \sum_{\varepsilon_q=0,1} (-1)^{q-\Sigma \varepsilon_j} v_{(s_1+\varepsilon_1(t_1-s_1),\, s_2+\varepsilon_2(t_2-s_2), \ldots,\, s_q+\varepsilon_q(t_q-s_q))}.$$

For every finite subset $\mathbf{t}^{(1)} < \mathbf{t}^{(2)} < \ldots < \mathbf{t}^{(n)}$ of T^q $E(v_{\mathbf{t}^{(n)}} \| F_{t^{(n-1)}}) = 0$ where $F_{t^{(n)}}$ is the σ-field generated by $\{v_{\mathbf{t}^{(p)}}\colon t_j^{(p)} \geqq 1$ for $j = 1, 2, \ldots, q$ and for at least one j, $1 \leqq t_j^{(p)} \leqq t_j^{(n)}\}$.

THEOREM 5. *The net* $\{Y_{\mathbf{n}}: \mathbf{n}\geqq 1\}$ *of stochastic processes converges in* C^q *to the q-parameter Wiener process* W.

PROOF. Although we could imitate the arguments used before for discrete time, it is simpler to reduce the present case to the previous one. Following BILLINGSLEY [2], p. 178, let

$$\xi_{\mathbf{n}} = \int_{n_1-1}^{n_1} \dots \int_{n_q-1}^{n_q} v_{\mathbf{s}}\, ds_1 \dots ds_q,$$

then by applications of Fubini's theorem

$$E\{\xi_{\mathbf{0}}^2\} < \infty. \tag{19}$$

Now if the blocks $B_1, B_2, \dots, B_k$ are disjoint in T^q the $Y_{\mathbf{n}}(B_i)$'s are orthogonal (DOOB [6], p. 514). Since the process $\{v_{\mathbf{t}}\}$ is stationary, by DOOB [6], p. 514, the process $\{v_{\mathbf{t}}\}$ is metrically transitive (ergodic) and hence $\{\xi_{\mathbf{n}}, \mathbf{n}\geqq 1\}$ is a stationary ergodic process satisfying (1) or (1′). By Theorem 1, $X_{\mathbf{n}}\Rightarrow W$, the q-parameter Wiener process (BILLINGSLEY [2], p. 23). Now

$$\delta_{\mathbf{n}} = \sup_{\mathbf{t}\in T^q} |Y_{\mathbf{n}}(\mathbf{t}) - X_{\mathbf{n}}(\mathbf{t})| \leqq (\sigma^2|\mathbf{n}|)^{-1/2} \times \max_{\substack{1\leqq p_i\leqq n_i \\ 1\leqq i\leqq q}} \int_{p_1-1}^{p_1} \int_{p_2-1}^{p_2} \dots \int_{p_q-1}^{p_q} |v_{\mathbf{t}}|\, dt_1\, dt_2 \dots dt_q$$

so that

$$P\{\delta_{\mathbf{n}} \geqq \varepsilon\} \leqq \left(\frac{1}{\varepsilon^2\sigma^2}\right)^q \int_{\{|\xi|\geqq \varepsilon\sigma|\mathbf{n}|^{-1/2}\}} \zeta^2\, dP,$$

where

$$\zeta = \int_0^1 \dots \int_0^1 |v_{\mathbf{s}}|\, ds_1 \dots ds_q.$$

By (20) $\delta_{\mathbf{n}} \xrightarrow{P} 0$ and theorem follows by Theorem 4.1 of BILLINGSLEY [2], since $X_{\mathbf{n}}\Rightarrow W$.

REMARKS. The proof of our Theorem 1 could have been shortened considerably if we could prove Theorem 1 using Lemma 3 of DEO [5]. The work on page 7 (cf. also the proof of Billingsley's theorem 19.2) suggests that condition (iii) of Deo's lemma may hold in the situation under consideration. This amounts to prove that the conditions of our Theorem 1 imply that the random variables X_n are asymptotically independent. But this is a very difficult problem which we are unable to solve at present.

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A TURÁN'S PROBLEM ON THE 0—2 INTERPOLATION BASED ON ZEROS OF JACOBI-POLYNOMIALS

By

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Let $\{x_j\}$, $j=1,2,\ldots,n$ be a sequence of points satisfying

$$(1) \qquad 1 \geqq x_1 > x_2 > \ldots > x_n \geqq -1.$$

J. SURÁNYI and P. TURÁN [1] called a sequence $\{x_j\}$ with the property (1) a 0—2 interpolation sequence if for arbitrarily prescribed numbers y_j, z_j, there is a unique polynomial $\Pi_{2n-1}(x)$ of degree $\leqq 2n-1$ so that

$$(2) \qquad \Pi_{2n-1}(x_j) = y_j, \quad \Pi''_{2n-1}(x_j) = z_j, \quad j = 1, 2, \ldots, n.$$

Let $\{x_j\}$ be the zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. For the ultraspherical polynomials $P_n^{(\alpha,\alpha)}(x)$, SURÁNYI and TURÁN [1, Theorem III] has determined all values n and α for which the sequence of zeros x_j of $P_n^{(\alpha,\alpha)}(x)$ is a 0—2 interpolation sequence. In [3, Problem 9], TURÁN asks: is it true for $\alpha \neq \beta$ that the zeros of all $P_n^{(\alpha,\beta)}(x)$ is a 0—2 interpolation sequence? If not, then determine the "bad" cases when (2) is not fulfilled.

In our present note, we shall answer this question by proving the following:

THEOREM. *Let $\{x_j\}$, $j=1,2,\ldots,n$ be the zeros of $P_n^{(\alpha,\beta)}(x)$ and let the sum $x=\alpha+\beta$ be not an even integer and satisfy*

$$(3) \quad x^3+(n+5+\varepsilon)x^2+[3\varepsilon(n-1)-2(2n^2-9n+3)]x+n[\varepsilon(2n-3)-4(n-1)(n-3)] = 0.$$

Then for all sufficiently small $\varepsilon>0$ and $0<|\alpha-\beta|<\varepsilon$, the sequence $\{x_j\}$ is not a 0—2 interpolation sequence.

PROOF. It is sufficient to prove that if there is a polynomial $\Pi_{2n-1}(x)$ satisfying (2), then there are infinitely many polynomials satisfying (2). This, however, is equivalent to the existence of a polynomial $K_{2n-1}(x)\not\equiv 0$ for which (2) holds for the particular value $y_j=z_j=0$, $j=1,2,\ldots,n$, because the family of polynomials $\Pi_{2n-1}(x)+cK_{2n-1}(x)$ will again satisfy (2).

Since $y_j=0$, $K_{2n-1}(x)$ can be written as

$$K_{2n-1}(x) = P_n^{(\alpha,\beta)}(x) R_{n-1}(x).$$

For simplicity, we replace $P_n^{(\alpha,\beta)}(x)$ by $P_n(x)$, then

$$(4) \qquad K_{2n-1}(x) = P_n(x) R_{n-1}(x).$$

By the conditions $z_j=0$, we also have for $j=1,2,\ldots,n$,

$$(5) \qquad P''_n(x_j) R_{n-1}(x_j) + 2P'_n(x_j) R'_{n-1}(x_j) = 0.$$

* I am indebted to Professor Turán for his guidance on this problem when both of us visited the University of Montreal.

The Jacobi polynomial $P_n(x)$ satisfies the following differential equation (see SZEGŐ [2, p. 59]):

$$(1-x^2)y''+[\beta-\alpha-(\alpha+\beta+2)x]y'+n(n+\alpha+\beta+1)y = 0.$$

It follows that

$$(1-x_j^2)P_n''(x_j)+[\beta-\alpha-(\alpha+\beta+2)x_j]P_n'(x_j) = 0. \tag{6}$$

Since each x_j is a simple zero of $P_n(x)$, $P_n'(x_j)\neq 0$, (5) and (6) yield that for each $j=1, 2, \dots, n$,

$$(1-x_j^2)R_{n-1}'(x_j)-1/2[\beta-\alpha-(\alpha+\beta+2)x_j]R_{n-1}(x_j) = 0.$$

We then obtain the following identity:

$$(1-x^2)R_{n-1}'(x)-1/2[\beta-\alpha-(\alpha+\beta+2)x]R_{n-1}(x) \equiv cP_n(x). \tag{7}$$

To compute $R_{n-1}(x)$, set

$$R_{n-1}(x) = \sum_{\nu=0}^{n-1} c_\nu P_\nu(x). \tag{8}$$

(7) and (8) give

$$(1-x^2)\sum_{\nu=0}^{n-1} c_\nu P_\nu'(x)-1/2[\beta-\alpha-(\alpha+\beta+2)x]\sum_{\nu=0}^{n-1} c_\nu P_\nu(x) \equiv cP_n(x). \tag{9}$$

To compare both sides of (9), we need two recurrence formulas of SZEGŐ [2, p. 70—71]:

$$(1-x^2)P_\nu'(x) = A_\nu P_{\nu-1}(x)+B_\nu P_\nu(x)+C_\nu P_{\nu+1}(x), \tag{10}$$

$$\begin{cases} A_\nu = \dfrac{2(\nu+\alpha)(\nu+\beta)(\nu+\alpha+\beta+1)}{(2\nu+\alpha+\beta)(2\nu+\alpha+\beta+1)}, \\ B_\nu = \dfrac{2\nu(\alpha-\beta)(\nu+\alpha+\beta+1)}{(2\nu+\alpha+\beta)(2\nu+\alpha+\beta+2)}, \\ C_\nu = -\dfrac{2\nu(\nu+1)(\nu+\alpha+\beta+1)}{(2\nu+\alpha+\beta+1)(2\nu+\alpha+\beta+2)}, \end{cases} \tag{11}$$

$$xP_\nu(x) = D_\nu P_\nu(x)+E_\nu P_{\nu+1}(x)+F_\nu P_{\nu-1}(x), \tag{12}$$

$$\begin{cases} D_\nu = \dfrac{\beta^2-\alpha^2}{(2\nu+\alpha+\beta)(2\nu+\alpha+\beta+2)}, \\ E_\nu = \dfrac{2(\nu+1)(2+\alpha+\beta+1)}{(2\nu+\alpha+\beta+1)(2\nu+\alpha+\beta+2)}, \\ F_\nu = \dfrac{2(\nu+\alpha)(\nu+\beta)}{(2\nu+\alpha+\beta)(2\nu+\alpha+\beta+1)}, \end{cases} \tag{13}$$

where $\alpha>-1$, $\beta>-1$, $\nu=1, 2, \dots, n-1$.

Combining (9), (10), (11), (12) and (13) we obtain

$$(14)\qquad cP_n(x) = P_0(x)\{c_1[A_1+1/2(\alpha+\beta+2)F_1]\}+$$
$$+P_1(x)\{c_2[A_2+1/2(\alpha+\beta+2)F_2]+c_1[B_1-1/2(\beta-\alpha)+1/2(\alpha+\beta+2)D_1]+c_0\}+\dots$$
$$+P_{n-2}(x)\{c_{n-1}[A_{n-1}+1/2(\alpha+\beta+2)F_{n-1}]+$$
$$+c_{n-2}[B_{n-2}-1/2(\beta-\alpha)+1/2(\alpha+\beta+2)D_{n-2}]+c_{n-3}[C_{n-3}+1/2(\alpha+\beta+2)E_{n-3}]\}+$$
$$+P_{n-1}(x)\{c_{n-1}[B_{n-1}-1/2(\beta-\alpha)+1/2(\alpha+\beta+2)D_{n-1}]+$$
$$+c_{n-2}[C_{n-2}+1/2(\alpha+\beta+2)E_{n-2}]\}+P_n(x)\{c_{n-1}[C_{n-1}+1/2(\alpha+\beta+2)E_{n-1}]\}.$$

By the orthogonality of $P_\nu(x)$, we find from (14) that all coefficients of $P_\nu(x)$, $\nu=0,1,2,\dots,n-1$ must be zero. This yields the following identities:

$$(15)\qquad c_{\nu+1}[A_{\nu+1}+1/2(\alpha+\beta+2)F_{\nu+1}]+c_\nu[B_\nu-1/2(\beta-\alpha)+1/2(\alpha+\beta+2)D_\nu]+$$
$$+c_{\nu-1}[C_{\nu-1}+1/2(\alpha+\beta+2)E_{\nu-1}]=0,$$

where $\nu=1,2,\dots,n-1$.

In view of (11), (13) and the condition $\alpha+\beta\neq$even, we obtain

$$(16)\qquad A_{\nu+1}+1/2(\alpha+\beta+2)F_{\nu+1}=$$
$$=\frac{2(\nu+1+\alpha)(\nu+1+\beta)[(\nu+\alpha+\beta+2)+1/2(\alpha+\beta+2)]}{(2\nu+\alpha+\beta+2)(2\nu+\alpha+\beta+3)}>0,$$

$$(17)\qquad B_\nu-1/2(\beta-\alpha)+1/2(\alpha+\beta+2)D_\nu=$$
$$=\frac{(\alpha-\beta)[2\nu(\nu+\alpha+\beta+1)+1/2(2\nu+\alpha+\beta)(2\nu+\alpha+\beta+2)-1/2(\alpha+\beta+2)(\alpha+\beta)]}{(2\nu+\alpha+\beta)(2\nu+\alpha+\beta+2)}\neq 0,$$

$$(18)\qquad C_{\nu-1}+1/2(\alpha+\beta+2)E_{\nu-1}=-\frac{2\nu(\nu+\alpha+\beta)[(\nu-1)-1/2(\alpha+\beta+2)]}{(2\nu+\alpha+\beta-1)(2\nu+\alpha+\beta)}\quad 0.$$

By choosing $c_0\neq 0$, and applying (15), (16), (17) and (18), we can compute all c_j in terms of c_0, $j=1,2,\dots,n-1$. The only thing we have to worry is the coefficients of $P_{n-2}(x)$ and $P_{n-1}(x)$. The numbers c_{n-1} and c_{n-2} which have been computed in the coefficient of $P_{n-2}(x)$, must satisfy the coefficient of $P_{n-1}(x)$, namely,

$$(19)\qquad c_{n-1}[B_{n-1}-1/2(\beta-\alpha)+1/2(\alpha+\beta+2)D_{n-1}]+$$
$$+c_{n-2}[C_{n-2}+1/2(\alpha+\beta+2)E_{n-2}]=0,$$

where $c_{n-1}\neq 0$, $c_{n-2}\neq 0$.

Let $\alpha-\beta=\frac{c_{n-2}}{c_{n-1}}\varepsilon$ and $\alpha+\beta=x$, then by (17), (18) and (19), we find the equation (3). Clearly, for any $\varepsilon<1$ and $n\geqq 4$, we have

$$f(0,\varepsilon)=n[\varepsilon(2n-3)-4(n-1)(n-3)]<0,$$

and

$$f(2n,\varepsilon)=n[\varepsilon(12n-9)+72n-24]>0.$$

It follows that the equation (3) has a positive solution s between 0 and $2n$. This gives

$$\begin{cases} \alpha-\beta = \dfrac{c_{n-2}}{c_{n-1}}\,\varepsilon = \delta \\ \alpha+\beta = s, \end{cases}$$

and therefore we obtain the solution

$$\alpha = \frac{s+\delta}{2} \quad \text{and} \quad \beta = \frac{s-\delta}{2}.$$

By choosing sufficiently small $\varepsilon>0$, we have $\alpha>-1$, and $\beta>-1$. This implies the existence of all coefficients c_ν, $\nu=0, 1, \dots n-1$, and the polynomials $R_{n-1}(x)$ and $K_{2n-1}(x)$. Hence $\{x_j\}$ is not a 0—2 interpolation sequence. The proof is complete.

REMARK. In the above proof we start from the coefficients of $P_0(x)$, $P_1(x)$, ..., $P_n(x)$. Similarly, we can reverse the process. Starting from the last one we can compute c_{n-1} in terms of c, then $c_{n-2}, c_{n-3}, \dots$. We come to the coefficients of $P_3(x)$ and $P_2(x)$. In $P_3(x)$, the numbers c_3 and c_2 are determined in terms of c. Therefore these two numbers must satisfy the coefficient of $P_2(x)$,

(20) $$c_3[A_3+1/2(\alpha+\beta+2)F_3]+c_2[B_2-1/2(\beta-\alpha)+1/2(\alpha+\beta+2)D_2] = 0,$$

where $c_2\neq 0$, $c_3\neq 0$. Again, let $\alpha-\beta=\dfrac{c_3}{c_2}\varepsilon=\delta$ and $\alpha+\beta=x$, then by (16), (17), (20) and the equality $\alpha\beta=(x^2-\delta^2)/4$, we get

(21) $$g(x, \varepsilon, \delta) = 3x^4+58x^3+(148-3\delta^2+32\varepsilon)x^2+$$
$$+(1272-22\delta^2+320\varepsilon)x+8(180-5\delta^2+84\varepsilon) = 0.$$

For sufficiently small ε, we have

$$g(0, \varepsilon, \delta) > 0 \quad \text{and} \quad g(-2, \varepsilon, \delta) < 0.$$

Therefore equation (21) has a solution $-2<s<0$ such that $\alpha>-1$ and $\beta>-1$. This again gives the existence of $P_n^{(\alpha,\beta)}(x)$ such that its zeros are not a 0—2 interpolation sequence. In both cases, $\varepsilon>0$ can be taken through a small interval. It follows that there are continuum many such polynomials $P_n^{(\alpha,\beta)}(x)$, $\alpha\neq\beta$, whose zeros are not a 0—2 interpolation sequence.

In the above theorem, we require that the sum $s=\alpha+\beta$ be not an even integer and satisfy (3). It is clear that the condition (3) is necessary. Moreover, the first condition is also necessary. If s is an even integer, then the theorem is no longer true due to the following

EXAMPLE. Let $\alpha=1+\sqrt{2/3}$ and $\beta=1-\sqrt{2/3}$. Then by (18), we have $C_2+1/2(\alpha+\beta+2)E_2=0$. For convenience, we denote the numbers in (16), (17) and (18) by $X_{\nu+1}$, Y_ν, and $Z_{\nu-1}$ respectively. We then have

$$c_1 = (\beta-\alpha)c_0/2X_1, \quad c_2 = -Y_1(\beta-\alpha)c_0/2X_1X_2,$$

and

$$c_3 = (Y_1 Y_2 - X_2 Z_1)(\beta - \alpha) c_0 / 2X_1 X_2 X_3.$$

The numbers of α and β are the solutions of $Y_1 Y_2 - X_2 Z_1 = 0$ with $\alpha + \beta = 2$. Hence we have $c_3 = 0$. It follows from (15) and (16) that $c_4 = 0$. By repeating (15), we obtain $c_3 = c_4 = c_5 = \ldots = c_{n-1} = 0$. Since $c_1 \neq 0$, $c_2 \neq 0$, the solution of $R_{n-1}(x)$ does not exist. Hence the theorem is not true.

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A GENERALIZATION OF A THEOREM OF A. KERTÉSZ

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The purpose of this note is to prove the following

THEOREM. *Let A be a subring of a ring R. If the additive group $(R, +)$ of R is the sum of $(A, +)$ and a subgroup $(B, +)$: $(R, +)=(A, +)+(B, +)$, $AB=BA=0$, and for each $x\in R$ there exists $e\in R$ such that $x-ex\in A$, then R is the ringtheoretical direct sum of A and B^2.*

Needless to say, our theorem includes that of A. KERTÉSZ [2] as well as DINH VAN HUYNH [1]. As for notations used here, we follow [1].

In advance of proving our theorem, we state the next

LEMMA. *Under the hypothesis of our theorem, if $x_1, \ldots, x_n$ are in R then there exists $e\in R$ such that $x_i-ex_i\in A$ for all i. Especially, in case $x_1, \ldots, x_n$ are in B, such an e can be chosen from B.*

PROOF. Choose an element $e_n\in R$ such that $x_n-e_nx_n\in A$, and set $y_i=x_i-ex_i$ $(i=1, \ldots, n-1)$. By induction method, there exists $e'\in R$ such that $y_i-e'y_i\in A$ $(i=1, \ldots, n-1)$. Let $e=e'+e_n-e'e_n$. Then we have $x_i-ex_i=y_i-e'y_i\in A$ $(i=1, \ldots, n-1)$. Noting here that A is an ideal of R, it follows $x_n-ex_n= =(x_n-e_nx_n)-e'(x_n-e_nx_n)\in A$. The latter assertion will be almost evident.

PROOF OF THE THEOREM. Since $B^2B^2\subseteqq B(A+B)=B^2$, the additive group B^2 forms a subring of R and $AB^2=B^2A=0$. Now, let $r=a+b$ $(a\in A, b\in B)$ be an arbitrary element of R, and choose an element $e\in B$ with $b-eb=a'\in A$ (cf. the lemma). Then $r=(a+a')+eb\in A+B^2$. Finally, we shall show that $A\cap B^2=0$, which will complete the proof. Let $a=x_1y_1+\ldots+x_ny_n$ be an arbitrary element of $A\cap B^2$ $(x_i, y_i\in B)$. Again by the lemma, there exists $e'\in B$ with $x_i-e'x_i\in A$ for all i. Then $a=(x_1-e'x_1)y_1+\ldots+(x_n-e'x_n)y_n+e'a=0$.

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POLYNOMIAL INEQUALITIES AND MARKOV'S INEQUALITY IN WEIGHTED L^p-SPACES

By

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Notations and main results

Let $\|\cdot\|_p$ and $\|\cdot\|_p^*$ be the usual norm for functions in $L^p(-1,+1)$ and for 2π-periodic functions whose restriction to $[0, 2\pi]$ belongs to $L^p(0, 2\pi)$, respectively. Let Π_n and Π_n^* be the set of algebraic and 2π-periodic trigonometric polynomials of degree at most n, respectively.

1. *Inequalities for trigonometric polynomials.* Let $a_1, a_2, \ldots, a_s, b_1, b_2, \ldots, b_s$, be real and positive, $\theta_1, \theta_2, \ldots, \theta_s$ satisfying $\theta_1 < \theta_2 < \ldots < \theta_s < \theta_1 + 2\pi$, and w, v, 2π-periodic functions such that for $\theta \in [(\theta_1+\theta_s)/2-\pi,\ (\theta_1+\theta_s)/2+\pi]$,

$$w(\theta) = \prod_1^s |\theta-\theta_i|^{a_i} \varphi(\theta), \quad v(\theta) = \prod_1^s |\theta-\theta_i|^{b_i} \psi(\theta),$$

where φ and ψ are two measurable functions bounded from 0 and infinity. We set $r = \max(b_1, b_2, \ldots, b_s)$.

THEOREM 1. *With the previous notations, there exists a constant $C(v, w)$ such that for any $p \in [1, +\infty]$ and arbitrary $T \in \Pi_n^* (n \geqq 1)$ the estimate*

$$\|Tw\|_p^* \leqq C(v, w) n^r \|Twv\|_p^*$$

holds true. Furthermore r is the best possible exponent.

2. *Inequalities for algebraic polynomials.* Let $\alpha_1, \alpha_2, \beta_1, \beta_2, c_1, c_2, \ldots, c_s, d_1, d_2, \ldots, d_s$ be real and positive, $x_1, x_2, \ldots, x_s \in]-1, +1[$. We define u_1 and u_2 by

$$\left.\begin{aligned} u_1(x) &= (1-x)^{\alpha_1}(1+x)^{\beta_1} \prod_1^s |x-x_i|^{c_i} f_1(x) \\ u_2(x) &= (1-x)^{\alpha_2}(1+x)^{\beta_2} \prod_1^s |x-x_i|^{d_i} f_2(x) \end{aligned}\right\} (x \in [-1, +1])$$

where f_1 and f_2 are measurable functions bounded from 0 and infinity. Set $d = \max(2\alpha_2, 2\beta_2, d_1, d_2, \ldots, d_s)$.

THEOREM 2. *There exists a constant $C(u_1, u_2)$ such that for any $p \in [1, +\infty]$ and arbitrary $P \in \Pi_n$ $(n \geqq 1)$ the estimate*

$$\|Pu_1\|_p \leqq C(u_1, u_2) n^d \|Pu_1 u_2\|_p$$

holds true. Furthermore d is the best possible exponent.

3. *Markov's inequality in weighted L^p-spaces.*

THEOREM 3. *Let u_1 as in Theorem 2. There exists a constant $C(u_1)$ such that for any $p\in[1,+\infty]$ and arbitrary $P\in\Pi_n$ the estimate*

$$\|P'u_1\|_p \leqq C(u_1)n^2\|Pu_1\|_p$$

holds true. Furthermore the exponent 2 is optimal.

These results contain as particular cases some theorems of [3] and [5], specify some results of [2], and improve [1] dealing with the case $p=2$.

Proof of Theorem 1

A fundamental tool is the Bernstein's inequality:

For any $T\in\Pi_n^*$ the estimate $\|T'\|_\infty^*\leqq n\|T\|_\infty^*$ holds true. It follows that if $\|T\|_\infty^*=|T(\theta_0)|$ then, for $|\theta-\theta_0|\leqq(2n)^{-1}$ we have: $|T(\theta)-T(\theta_0)|\leqq|\theta-\theta_0|n\|T\|_\infty^*\leqq \leqq\frac{1}{2}\|T\|_\infty^*$, therefore

$$|T(\theta)|\geqq\frac{1}{2}\|T\|_\infty^* \quad \text{if} \quad |\theta-\theta_0|\leqq(2n)^{-1}. \tag{1}$$

To establish Theorem 1, we need a lemma. Let $\delta_1,\delta_2,\dots,\delta_l$ be real and positive, $\varphi_1,\varphi_2,\dots,\varphi_l$ satisfying $\varphi_1<\varphi_2<\dots<\varphi_l<\varphi_1+2\pi$, u a 2π-periodic function such that for $\varphi\in J=[(\varphi_1+\varphi_l)/2-\pi,\ (\varphi_1+\varphi_l)/2+\pi]$

$$u(\varphi)=\prod_1^l|\varphi-\varphi_i|^{\delta_i}u_0(\varphi),$$

where u_0 is a measurable function bounded from 0. Set $\delta=\max(\delta_1,\delta_2,\dots,\delta_l)$.

LEMMA 1. *There exists a constant $C_1>0$ such that for any $p\in[1,+\infty[$, $n\geqq1$ and arbitrary $\varphi_0\in\mathbf{R}$, the estimate*

$$C_1^p n^{-\delta p-1}\leqq\int_{\varphi_0-(2n)^{-1}}^{\varphi_0+(2n)^{-}}|u(\varphi)|^p\,d\varphi$$

holds true.

PROOF. It is sufficient to prove the lemma when $\varphi_0\in J$. Let N be an integer such that $\varphi_{k+1}-\varphi_k\geqq2N^{-1}$ $(k=1,\dots,l-1)$ and $\pi-(\varphi_l-\varphi_1)/2\geqq2N^{-1}$. We set $I_k=[\varphi_k-(2N)^{-1},\varphi_k+(2N)^{-1}]$ $(k=1,\dots,l)$, $I_0=[\varphi_0-(2n)^{-1},\varphi_0+(2n)^{-1}]$. There exists a constant $C_2>0$ such that for any $\varphi\in J\setminus\bigcup_{1\leqq k\leqq l}I_k$ $|u(\varphi)|\geqq C_2$. There exists a constant $C_{3,k}>0$ such that for any $\varphi\in I_k$, $|u(\varphi)|\geqq C_{3,k}|\varphi-\varphi_k|^\delta$. We set $C_3=\inf_k C_{3,k}$. If $n\geqq N$ one can find a subinterval U of I_0 of length $(2n)^{-1}$ included either in some I_k or in $J\setminus\bigcup_{1\leqq k\leqq l}I_k$. Then

$$\int_{I_0}|u(\varphi)|^p\,d\varphi\geqq\int_U|u(\varphi)|^p\,d\varphi$$

and the right side of the last estimate is greater than either $2C_3^p(\delta p+1)^{-1}(4n)^{-\delta p-1}$ or $C_2^p(2n)^{-1}$ therefore greater than

$$C_4^p(\delta p+1)^{-1}(4n)^{-\delta p-1} \quad \text{with} \quad C_4 = \inf(C_2, C_3).$$

If $n<N$, obviously:

$$\int_{I_0} |u(\varphi)|^p\, d\varphi \geqq C_4^p(4n)^{-\delta p-1}(\delta p+1)^{-1} \geqq C_4^p(4N)^{-\delta p-1}(\delta p+1)^{-1} n^{-\delta p-1},$$

and in any case

$$\int_{I_0} |u(\varphi)|^p\, d\varphi \geqq [C_4(4N)^{-\delta-1}(\delta+1)^{-1}e^{-1/e}]^p n^{-\delta p-1}. \quad \text{Q.E.D.}$$

PROOF OF THEOREM 1. It is sufficient to prove Theorem 1 when p is finite (since $\|h\|_p^* \xrightarrow[p\to\infty]{} \|h\|_\infty^*$) and for $n\geqq N_0$ (N_0 given) (because a linear mapping of a finite dimensional space in an other is continuous). Set $I_{k,n}=[\theta_k-n^{-1}, \theta_k+n^{-1}]$ and choose N_0 such that for $n\geqq N_0$, the intervals $I_{k,n}$ be disjoint. Define $E_n= = \bigcup_{1\leqq k\leqq s} I_{k,n}$. Let $n\geqq N_0$, $T\in \Pi_n^*$, $p\geqq 1$. We have

$$\int_{I_{k,n}} |T(\theta)\, w(\theta)|^p\, d\theta \leqq C_5^p n^{-a_k p} \int_{I_{k,n}} \Big|T(\theta) \prod_{i\neq k} |\theta-\theta_i|^{a_i}\Big|^p d\theta.$$

We define a trigonometric polynomial R_k by

$$R_k(\theta) = \prod_{i\neq k} (1-\cos(\theta-\theta_i))^{[a_i]+1}$$

($[a_i]$ denotes the integer part of a_i). R_k is positive and bounded from 0 on $I_{k,n}$ and thus

$$\prod_{i\neq k} |\theta-\theta_i|^{a_i} \leqq C_6 R_k(\theta) \quad \text{if} \quad \theta\in I_{k,n}.$$

Therefore

$$\int_{I_{k,n}} |T(\theta)\, w(\theta)|^p\, d\theta \leqq C_7^p n^{-a_k p} \int_{I_{k,n}} |T(\theta)\, R_k(\theta)|^p\, d\theta \leqq$$

$$\leqq (2C_7)^p n^{-a_k p-1} \|TR_k\|_\infty^{*p}.$$

Let θ_0 be such that $\|TR_k\|_\infty^* = |T(\theta_0)\, R_k(\theta_0)|$ and

$$J_n = [\theta_0-(2n)^{-1}, \theta_0+(2n)^{-1}].$$

Lemma 1 and (1) applied to $|\theta-\theta_k|^{a_k} v(\theta)$ yield

$$\int_{J_n} \big|T(\theta)\, R_k(\theta)\, |\theta-\theta_k|^{a_k} v(\theta)\big|^p\, d\theta \geqq C_8^p \|TR_k\|_\infty^{*p} n^{-(r+a_k)p-1}.$$

Therefore

$$\int_{I_{k,n}} |T(\theta)\, w(\theta)|^p\, d\theta \leqq C_9^p n^{rp} \big\| |\theta-\theta_k|^{a_k} TR_k v \big\|_p^{*p}.$$

But we have $R_k(\theta)\, |\theta-\theta_k|^{a_k} \leqq C_{10} |w(\theta)|$, hence

$$\int_{E_n} |T(\theta)\, w(\theta)|^p\, d\theta \leqq C_{11}^p n^{rp} \|Twv\|_p^{*p}.$$

On the other hand, there exists a constant C_{12} such that if $\theta\in J\setminus E_n$ $|v(\theta)|\geqq C_{12}n^{-r}$ holds, whence

$$\int_{J\setminus E_n} |T(\theta)w(\theta)|^p\,d\theta \leqq C_{12}^p n^{rp} \int_{J\setminus E_n} |T(\theta)w(\theta)v(\theta)|^p\,d\theta \leqq C_{12}^p n^{rp}\|Twv\|_p^{*p}.$$

Then it is sufficient to take $C(v, w)=C_{11}+C_{12}$.

Now we shall prove that the exponent r is sharp by showing that for each n one can find $T\in\Pi_p^*$ such that

$$\|Tw\|_p^* \geqq C_{20}n^r\|Twv\|_p^*.$$

Obviously we can assume $b_1=r, \theta_1=0$. We have

$$\|Twv\|_p^* \leqq C_{13}\int_{-\pi}^{+\pi} \big||\theta|^{r+a_1}T(\theta)\big|^p\,d\theta$$

for some constant C_{13}. Let $\gamma>r+a_1+1$. Using the notations of [6], Ch. **IV for** Jacobi polynomials, we define T by

$$T(\theta) = P_n^{(\gamma,0)}(\cos\theta).$$

It is well known that

$$\|T\|_\infty^* = |T(0)| = \binom{n+\gamma}{n} \geqq C_{14}n^\gamma$$

([6], p. 168) then using (1), if $n\geqq N_0$

$$\int_{-(2n)^{-1}}^{(2n)^{-1}} |T(\theta)w(\theta)|^p\,d\theta \geqq C_{15}^p \int_{-(2n)^{-1}}^{(2n)^{-1}} \big|T(\theta)\,|\theta|^{a_1}\big|^p\,d\theta \geqq C_{16}^p n^{\gamma p-a_1p-1}.$$

On the other hand, using [6], p. 168—169,

$$P_n^{(\gamma,0)}(\cos\theta) \leqq \begin{cases} C_{17}n^{-1/2}\theta^{-\gamma-1/2} & \text{if } n^{-1}\leqq\theta\leqq\pi/2,\\ C_{18}n^\gamma & \text{if } 0\leqq\theta\leqq n^{-1},\\ 1 & \text{if } \pi/2\leqq\theta\leqq\pi.\end{cases}$$

Therefore

$$\int_{-\pi}^{+\pi} \big||\theta|^{r+a_1}T(\theta)\big|^p\,d\theta \leqq 2C_{18}^p n^{\gamma p}\int_0^{n^{-1}} \theta^{(r+a_1)p}\,d\theta +$$

$$+2C_{17}^p n^{-p/2}\int_{n^{-1}}^{\pi/2} \theta^{p(r+a_1-\gamma-1/2)}\,d\theta+\frac{\pi}{2} \leqq C_{19}^p n^{p(\gamma-r-a_1)-1},$$

whence $\|Tw\|_p^*\geqq C_{20}n^r\|Twv\|_p^*$.

PROOF OF THEOREM 2. We prove immediately the inequality by putting $x=\cos\theta$ and using Theorem 1.

To prove the sharpness of the exponent d we carry out the same proof as in Theorem 1, with the polynomials

$$P_n^{(\gamma,0)}(x) \quad \text{if} \quad d = 2\alpha_2,$$

$$P_n^{(0,\gamma)}(x) \quad \text{if} \quad d = 2\beta_2,$$

$$P_n^{(0,\gamma)}\left(2\left(\frac{x-x_k}{1+\varepsilon x_k}\right)^2 - 1\right) \quad \text{if} \quad d = d_k \quad \text{and} \quad \varepsilon = \operatorname{sgn}(x_k).$$

Application to orthonormal polynomials and approximation

Let u_1 be defined as in Theorem 2. Let us assume that for some positive constants L and λ, $f(\theta)=|u_1(\cos\theta)\sin\theta|$ $(\theta\in[0,\pi])$ satisfies $|f(\theta+\delta)-f(\theta)|\leqq \leqq L\,|\log\delta|^{1-\lambda}$.

Then the orthonormal polynomials $\{p_n\}$ with respect to the weightfunction $|u_1|$ $\left(\text{i.e. such that } \int_{-1}^{+1} p_i(x)p_j(x)\,|u_1(x)|\,dx=\delta_{i,j}\right)$ satisfy

$$\left|(1-x^2)^{1/4}\,|u_1(x)|^{1/2}\,p_n(x)\right| \leqq K_0$$

for some constant K_0. (See [6], p. 297.) Therefore, using Theorem 2:

$$\|p_n\|_\infty \leqq C_{21}n^{d'} \quad (n\geqq 1) \tag{2}$$

where $d'=\frac{1}{2}\max(c_1, c_2, \dots, c_s, 2\alpha_1+1, 2\beta_1+1)$.

For a measurable function f such that $f\sqrt{|u_1|}\in L^2(I)$ we put

$$E_{n,w}(f) = \inf_{P\in\pi_n} \left\|(f-P)\sqrt{|u_1|}\right\|_2$$

and

$$E_n(f) = \inf_{P\in\pi_n} \|f-P\|_\infty.$$

Proposition 1. *Let f be a function such that $f\sqrt{|u_1|}\in L^2(I)$, and for a $q>d'-1/2$, $E_{n,w}(f)\leqq Mn^{-q}$. Then there exists C and g such that $E_n(g)\leqq Cn^{d'+1/2-q}$, and $g=f$ almost everywhere.*

Lemma 2. *Let q and k be such that $q>k+1/2$ and (b_i) a sequence of complex numbers. If there exists $M>0$ such that for each n,*

$$\left(\sum_{i=n+1}^{\infty} |b_i|^2\right)^{1/2} \leqq Mn^{-q},$$

then

$$\sum_{i=n+1}^{\infty} i^k|b_i| \leqq Kn^{k+1/2-q}.$$

Proof. From Cauchy's inequality we get

$$\sum_{i=2^m+1}^{2^{m+1}} |b_i| \leqq 2^{1/2m}\left(\sum_{i=2^m+1}^{2^{m+1}} |b_i|^2\right)^{1/2} \leqq M2^{m(1/2-q)}.$$

Therefore

$$\sum_{n=2^m+1}^{2^{m+1}} n^k |b_n| \leqq 2^{k(m+1)+m(1/2-q)} M = 2^k M 2^{m(k-q+1/2)}$$

and if $q>k+1/2$, $\sum_{n=1}^{\infty} n^k |b_n|$ converges. Furthermore,

$$\sum_{n=2^m+1}^{\infty} n^k |b_n| \leqq 2^k M \sum_{n=m}^{\infty} 2^{n(k+1/2-q)} = \frac{2^k M 2^{m(k+1/2-q)}}{1-2^{k+1/2-q}}$$

from where the lemma results immediately.

PROOF OF THE PROPOSITION. Let $\sum_0^{\infty} b_i p_i$ be the orthonormal expansion of f with respect to the system $\{p_n\}$. Then by assumption $\left(\sum_{n+1}^{\infty} |b_i|^2\right)^{1/2} \leqq Mn^{-q}$, and Lemma 2 yields:

$$\sum_{i=n+1}^{\infty} (i+1)^{d'} |b_i| \leqq 2Kn^{d'+1/2-q}.$$

Thus from (2) $\sum_0^{\infty} b_i p_i$ converges uniformly to a function g equal to f almost everywhere and

$$\left\| g - \sum_0^n b_i p_i \right\|_{\infty} \leqq Cn^{d'+1/2-q}.$$

PROOF OF THEOREM 3.

PROPOSITION 2. *Let* $a \geqq 0$. *There exists a constant* $C(a)$ *such that for any* $p \in [1, +\infty]$ *and arbitrary* $P \in \Pi_n$ *the estimate*

$$\left\| |x|^a P' \right\|_p \leqq C(a) n^2 \left\| |x|^a P \right\|_p$$

holds true. Furthermore 2 *is the best possible exponent.*

PROOF. First, we recall a result of [4], p. 412: if $P \in \Pi_n$ then

$$\|(1-x^2)^{1/2} P'\|_p \leqq C_{22} n \|P\|_p.$$

Whence in particular:

$$\|P'\|_{L^p[-1/\sqrt{2}, 1/\sqrt{2}]} \leqq C_{22} \sqrt{2} n \|P\|_p.$$

Let us assume $a \in [0, 1[$. Then from Theorem 2

$$\left\| |x|^a P' \right\|_{L^p[-1/\sqrt{2}, 1/\sqrt{2}]} \leqq \|P'\|_{L^p[-1/\sqrt{2}, 1/\sqrt{2}]} \leqq$$

$$\leqq C_{22} \sqrt{2} n \|P\|_p \leqq C_{23} n^{1+a} \left\| |x|^a P \right\|_p \leqq C_{23} n^2 \left\| |x|^a P \right\|_p,$$

and from [4]

$$\left\| |x|^a P' \right\|_{L^p[1/\sqrt{2}, 1]} \leqq \|P'\|_{L^p[1/\sqrt{2}, 1]} \leqq C_{24} n^2 \|P\|_{L^p[1/\sqrt{2}, 1]} \leqq$$

$$\leqq C_{24} n^2 2^{a/2} \left\| |x|^a P \right\|_{L^p[1/\sqrt{2}, 1]} \leqq C_{24} n^2 \sqrt{2} \left\| |x|^a P \right\|_p.$$

Analogous estimate holds for $\left\| |x|^a P \right\|_{L^p[-1, -1/\sqrt{2}]}$. Thus Proposition 1 is proved when $a<1$.

If $a\geqq 1$ we put $a=[a]+\alpha$ with $\alpha\in[0,1[$, $[a]$ integer. Then

$$|x|^a|P'(x)| = |x|^\alpha|x^{[a]}P'(x)| = |x|^\alpha\big|(x^{[a]}P(x))'-|a|x^{[a]-1}P(x)\big| \leqq$$
$$\leqq |x|^\alpha\big|(x^{[a]}P(x))'\big|+[a]|x|^\alpha|x^{[a]-1}P(x)| = A+B.$$

From the first part of the proof we get

$$\|A\|_p \leqq C_{25}(n+a)^2\big\||x|^aP\big\|_p,$$

and from Theorem 2

$$\|B\|_p \leqq C_{26}n\big\||x|^aP\big\|_p.$$

Q.E.D.

To establish the sharpness of the exponent 2, one can use polynomials $P_n^{(\gamma,0)}(2x^2-1)$.

PROPOSITION 3. *Let $\alpha\geqq 0$; there exists a constant $C(\alpha)$ such that for any $P\in\Pi_n$ and $p\in[1,+\infty]$ the estimate*

$$\|(1\pm x)^\alpha P'(x)\|_p \leqq C(\alpha)n^2\|(1\pm x)^\alpha P(x)\|_p$$

holds true. Furthermore the exponent 2 is the best possible.

PROOF. (The proof is made with $(1-x)^\alpha$) From [5], p. 473

$$\|(1-x)^\alpha P'(x)\|_{L^p[0,1]} \leqq \|(1-x^2)^\alpha P'(x)\|_{L^p[0,1]} \leqq$$
$$\leqq \|(1-x^2)^\alpha P'(x)\|_p \leqq C_{27}n^2\|(1-x^2)^\alpha P(x)\|_p \leqq C_{27}2^\alpha n^2\|(1-x)^\alpha P\|_p$$

since for $|x|\leqq 1$, $1-x^2\leqq 2(1-x)$.

On the other hand, it is obvious that

$$\|(1-x)^\alpha P'(x)\|_{L^p[-1,0]} \leqq 2^\alpha\|P'\|_{L^p[-1,0]} \leqq C_{28}n^2\|P\|_{L^p[-1,0]} \leqq$$
$$\leqq C_{28}n^2\|(1-x)^\alpha P(x)\|_{L^p[-1,0]} \leqq C_{28}n^2\|(1-x)^\alpha P(x)\|_p.$$

The sharpness of exponent 2 is easily seen by the polynomials $P_n^{(\gamma,0)}(x)$.

Now Theorem 3 is an immediate corollary of Propositions 2 and 3.

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HERMITE—FEJÉR TYPE INTERPOLATIONS. II

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1. Introduction

We investigate general convergence criteria for the quasi-Hermite—Fejér interpolation process. Some problems of P. Turán will be solved. Further, we discuss the convergence of Hermite—Fejér and quasi-Hermite—Fejér processes based on the roots of orthogonal polynomials generated by the weight $(1-x^2)^\beta |x|^{2\alpha+1} (-1<\alpha, \beta)$.

2. A general convergence theorem. Application

2.1. For an arbitrary $f \in C$ ($=f$ is continuous on $[-1, 1]$) we construct the uniquely defined Hermite—Fejér interpolatory polynomials $H_n(f; X; x)$ of degree $2n-1$ satisfying

(2.1) $$H_n(f; X; x_{k,n}) = f(x_{k,n}), \quad H_n'(f; X; x_{k,n}) = 0 \quad (k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$$

where X stands for the matrix

(2.2) $$-1 < x_{n,n} < x_{n,n-1} < \ldots < x_{2,n} < x_{1,n} < 1 \quad (n = 1, 2, \ldots).$$

Similarly, one can define the uniquely determined quasi-Hermite—Fejér interpolating polynomials $Q_n(f; X; x)$ of degree $2n+1$ by

(2.3) $$\begin{cases} Q_n(f; X; 1) = f(1), \quad Q_n(f; X; -1) = f(-1), \quad Q_n(f; X; x_{k,n}) = f(x_{k,n}); \\ Q_n'(f; X; x_{k,n}) = 0 \quad (k = 1, 2, \ldots, n;\ n = 1, 2, \ldots). \end{cases}$$

2.2. Let $\omega_n(X; x) = c_n \prod_{k=1}^{n} (x - x_{k,n})$. For $\omega_n(X; x) = P_n^{(\alpha,\beta)}(x)$ (where $\{P_n^{(\alpha,\beta)}(x)\}$ is the orthogonal polynomial system in $[-1, 1]$ belonging to the weight $(1-x)^\alpha (1+x)^\beta$, $-1<\alpha, \beta$), the investigation of the convergence of $\{H_n(f; X; x)\}$ to $f(x)$ can be found in G. Szegő [1]. Similar questions for $\{Q_n\}$ were investigated, e.g., by P. Szász [2] and P. Vértesi [10].

In his paper [3], P. Turán raised the following problems ([3], XXVI and XXVII).

1) *Define the class of weight functions $p(x)$ such that for the corresponding matrices P*

$$\lim_{n\to\infty} Q_n(f; P; x) = f(x) \quad \textit{uniformly in } [-1, 1] \textit{ for } f \in C.$$

2) *For which* $p(x)$ *and* P *is*

$$\lim_{n\to\infty} \int_{-1}^{1} [f(x)-Q_n(f;\ P;\ x)]^2 p(x)\,dx = 0 \quad (f\in C)$$

true?

(The matrix P corresponds to the weight $p(x)$ if its n-th row contains the n roots of the n-th orthogonal polynomial belonging to $p(x)$. As usual, we suppose $p(x)\geqq 0$ and $p(x)\in L$.)

2.3. We provide answers for the questions raised above. Omitting the superfluous notations let

$$d_n(f;\ X;\ x) \overset{\text{def}}{=} \frac{\omega_n^2(x)}{2}\cdot \tag{2.4}$$

$$\cdot\left\{\left[\frac{f(1)-H_n(f;\ 1)}{\omega_n^2(1)}\right](1+x)+\left[\frac{f(-1)-H_n(f;\ -1)}{\omega_n^2(-1)}\right](1-x)\right\} \quad (f\in C).$$

We state the following

THEOREM 2.1. *For any fixed* X *and* $f\in C$

$$Q_n(f;\ X;\ x)-H_n(f;\ X;\ x) = d_n(f;\ X;\ x). \tag{2.5}$$

Using (2.5) and the simple relation

$$Q_n(f;\ x)-H_n(f;\ x) = Q_n(f;\ x)-f(x)+f(x)-H_n(f;\ x) \tag{2.6}$$

we obtain the following statements:

COROLLARY 2.2. *If for a fixed* $x\in[-1,1]$ *and* $f\in C$

$$\begin{cases} \lim_{n\to\infty} d_n(f;\ x) = 0 \quad \textit{and} \quad \lim_{n\to\infty} H_n(f;\ x) = f(x) \\ \textit{then} \quad \lim_{n\to\infty} Q_n(f;\ x) = f(x); \end{cases} \tag{2.7}$$

or

$$\begin{cases} \lim_{n\to\infty} d_n(f;\ x) = 0 \quad \textit{and} \quad \lim_{n\to\infty} Q_n(f;\ x) = f(x) \\ \textit{then} \quad \lim_{n\to\infty} H_n(f;\ x) = f(x). \end{cases} \tag{2.8}$$

If the premises of (2.7) (*or* (2.8)) *are uniformly valid in* $[a,b]\subseteqq[-1,1]$ *then the same holds for the conclusion, too.*

As for the second problem of 2.2. we obtain as follows

COROLLARY 2.3. *If for a fixed matrix P corresponding to $p(x)$ and $f\in C$*

$$(2.9)\qquad \begin{cases} \lim_{n\to\infty} \|d_n(f;x)\| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \int_{-1}^{1} [f(x)-H_n(f;x)]^2 p(x)\,dx = 0 \\ \textit{then} \quad \lim_{n\to\infty} \int_{-1}^{1} [f(x)-Q_n(f;x)]^2 p(x)\,dx = 0; \end{cases}$$

$$(2.10)\qquad \begin{cases} \lim_{n\to\infty} \|d_n(f;x)\| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \int_{-1}^{1} [f(x)-Q_n(f;x)]^2 p(x)\,dx = 0 \\ \textit{then} \quad \lim_{n\to\infty} \int_{-1}^{1} [f(x)-H_n(f;x)]^2 p(x)\,dx = 0. \end{cases}$$

(Here, as usual, $\|g\|_{[a,b]} = \max_{a\leqq x\leqq b} |g(x)|$; $\|g\| = \|g\|_{[-1,1]}$.) By (2.4) and Corollaries 2.2 we get

COROLLARY 2.4. *If* $\lim_{n\to\infty} \|H_n(f;x)-f(x)\| = 0$ *and*

$$\max\left(\frac{\|\omega_n(x)\|}{|\omega_n(1)|}, \frac{\|\omega_n(x)\|}{|\omega_n(-1)|}\right) \leqq K \quad (n = 1, 2, \ldots)$$

then

$$\lim_{n\to\infty} \|Q_n(f;x)-f(x)\| = 0 \quad (f\in C).$$

Analogous statement holds for the convergence of $H_n(f;x)$ to $f(x)$.

2.3.1. Let us see two simple applications to the above statements. By Corollary 2.4 we get as follows.

a) *For* $f\in C$

$$\lim_{n\to\infty} \|Q_n^{(\alpha,\alpha)}(f;x)-f(x)\| = 0 \quad \textit{if} \quad -0.5 \leqq \alpha < 0.$$

(Here and later $Q_n^{(\alpha,\beta)}$ (or $H_n^{(\alpha,\beta)}$) stands for the process Q_n (or H_n) based on the roots of $P_n^{(\alpha,\beta)}(x)$).

Indeed, it is well known that $\lim_{n\to\infty} \|H_n^{(\alpha,\alpha)}(f;x)-f(x)\| = 0$ if $-1<\alpha<0$. Further $\|P_n^{(\alpha,\alpha)}(x)\| = |P_n^{(\alpha,\alpha)}(\pm 1)| \sim n^\alpha$ $(\alpha \geqq -1/2)$ (see e.g. [1]).

This statement was proved in [10], too. Using finer arguments we can get results which have not been settled. By Corollary 2.2 we obtain:

b) *If* $f\in C$ *then*

$$\lim_{n\to\infty} \|Q_n^{(\alpha,\beta)}(f;x)-f(x)\| = 0 \quad \textit{if} \quad -0.5 \leqq \alpha, \beta < 0.$$

Indeed, by [1], 7.32.5 we obtain with $x = \cos\vartheta$

$$[P_n^{(\alpha,\beta)}(x)]^2(1+x) = \begin{cases} O(1)n^{2\alpha} & \text{if} \quad x \geqq 0, \\ O(1)\dfrac{\sin^2\vartheta}{n(\sin\vartheta)^{2\beta+1}} = O(1)n^{-1} & \text{if} \quad x \leqq 0 \end{cases}$$

and

$$[P_n^{(\alpha,\beta)}(x)]^2(1-x) = \begin{cases} O(1)\,n^{2\beta} & \text{if} \quad x \leqq 0, \\ O(1)\dfrac{\sin^2 \vartheta}{n(\sin\vartheta)^{2\alpha+1}} = O(1)\,n^{-1} & \text{if} \quad x \geqq 0. \end{cases}$$

Using these, $|P_n^{(\alpha,\beta)}(1)| \sim n^\alpha$, $|P_n^{(\alpha,\beta)}(-1)| \sim n^\beta$ and Corollary 2.2 we obtain b).

2.4. By (2.4)—(2.6) we can get divergence theorems, too. Here is an example:

COROLLARY 2.5. *If* $\lim_{n\to\infty} \|H_n(f; x) - f(x)\| = 0$ *and*

$$\overline{\lim_{n\to\infty}} \|d_n(f; x)\| \geqq c > 0 \quad \textit{then} \quad \overline{\lim_{n\to\infty}} \|Q_n(f; x) - f(x)\| > 0 \quad (f \in C).$$

2.5. REMARKS.

2.5.1. By similar methods the statement 2.3.1. a) for $\alpha = \beta = -0.5$ was proved by D. L. BERMAN [9].

2.5.2. Using (2.4) and (2.6) we can get estimations for the order of the convergence, too. We omit the details.

3. Another application for two questions raised by P. Turán

3.1. For the matrix (2.2) it is well-known that

$$H_n(f; X, x) = \sum_{k=1}^{n} f(x_{k,n})\, v_{k,n}(X, x)\, l_{k,n}^2(X, x) \tag{3.1}$$

and

$$Q_n(f; X, x) = \sum_{k=1}^{n} f(x_{k,n}) \frac{1-x^2}{1-x_{k,n}^2}\, q_{k,n}(X; x)\, l_{k,n}^2(X, x) + \tag{3.2}$$

$$+ f(1)\frac{1+x}{2}\frac{\omega_n^2(X; x)}{\omega_n^2(X; 1)} + f(-1)\frac{1-x}{2}\frac{\omega_n^2(X; x)}{\omega_n^2(X; -1)},$$

where

$$v_{k,n}(X, x) = 1 - \frac{\omega_n''(x_k)}{\omega_n'(x_k)}(x - x_k) \quad (k = 1, 2, \dots, n), \tag{3.3}$$

$$l_{k,n}(X, x) = \frac{\omega_n(x)}{\omega_n'(x_{k,n})(x - x_{k,n})} \quad (k = 1, 2, \dots, n), \tag{3.4}$$

$$q_{k,n}(X, x) = 1 + \left[\frac{2x_{k,n}}{1-x_{k,n}^2} - \frac{\omega_n''(x_{k,n})}{\omega_n'(x_{k,n})}\right](x - x_{k,n}) \quad (k = 1, 2, \dots, n) \tag{3.5}$$

(see, e.g., [2] and [5]).

If for any $x \in [-1, 1]$

$$v_{k,n}(X, x) \geqq \varrho > 0 \quad (k = 1, 2, \dots, n;\ n = 1, 2, \dots) \tag{3.6}$$

then the matrix X is called ϱ-normal, similarly if

$$q_{k,n}(X; x) \geqq \varrho > 0 \tag{3.7}$$

then X is ϱ-quasi-normal. These properties are very important considering the convergence behaviour of our processes. Namely we have

THEOREM 3.1. *If X is ϱ-normal then $\{H_n(f; X, x)\}$ uniformly tends to $f(x)$ in $[-1, 1]$ whenever $f \in C$; the same holds for $\{Q_n(f; X; x)\}$ supposing X is ϱ-quasi-normal.*

(The first part of this statement was proved by G. GRÜNWALD [5], the second one by J. SÁNTHA [6].)

3.2. If X is ϱ-normal then the root $s_{k,n}$ of $v_{k,n}(x)$ is *outside of* $[-1, 1]$; if X is ϱ-quasi-normal then the root $t_{k,n}$ of $q_{k,n}(x)$ is *outside of* $[-1, 1]$ $(k=1, 2, \ldots, n;$ $n=1, 2, \ldots)$. The points $\{s_{k,n}\}$ are the *conjugate points*; the roots $\{t_{k,n}\}$ form the *quasi-conjugate points*. By these definitions we can formulate two problems raised by P. TURÁN.

1) *Define a matrix X for which the set $\{s_{k,n}\}$ is dense in $[-1, 1]$ and* $\lim\limits_{n\to\infty} \|H_n(f; X; x) - f(x)\| = 0$ *if $f \in C$.*

2) *Define a weight-function $p(x)$ vanishing and continuous for a certain $x_0 \in (-1, 1)$ such that for the corresponding matrix P*

$$\lim_{n\to\infty} \|H_n(f; P; x) - f(x)\| = 0 \quad \text{if} \quad f \in C.$$

3.3. These questions were solved by J. BALÁZS [4]. Our aim is twofold. First we generalize his results and by these new statements we settle the analogous problems for the process $\{Q_n\}$ using the results proved in Part 2.

3.4. Let us consider the weight function

$$p(\alpha, \beta; x) = |x|^{2\alpha+1}(1-x^2)^\beta \quad (\alpha, \beta > -1,\ x \in [-1, 1]) \tag{3.8}$$

and denote the corresponding orthogonal system by $\{R_n^{(\alpha,\beta)}(x)\}$ (see LASCENOV [7]). One can prove

$$\begin{cases} R_{2n}^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(1-2x^2), \\ R_{2n+1}^{(\alpha,\beta)}(x) = xP_n^{(\alpha+1,\beta)}(1-2x^2) \end{cases} \quad (n = 0, 1), \tag{3.9}$$

from where for the roots $\{y_{k,n}^{(\alpha,\beta)}\}_{k,n} \overset{\text{def}}{=} Y(\alpha, \beta)$ of $\{R_{2n(+1)}^{(\alpha,\beta)}(x)\}$

$$y_{k,n}^{(\alpha,\beta)} = -y_{-k,n}^{(\alpha,\beta)} = \begin{cases} \sin \dfrac{\vartheta_{k,n}^{(\alpha,\beta)}}{2} & (n \text{ is even};\ k = 1, 2, \ldots, n), \\ \sin \dfrac{\vartheta_{k,n}^{(\alpha+1,\beta)}}{2} & (n \text{ is odd};\ k = 0, 1, 2, \ldots, n) \end{cases} \tag{3.10}$$

where $\{x_{k,n}^{(\alpha,\beta)} = \cos \vartheta_{k,n}^{(\alpha,\beta)}\}$ $(k=1, 2, \ldots, n)$ are the roots of $P_n^{(\alpha,\beta)}(x)$ and $\vartheta_{0,n}^{(\alpha,\beta)} = 0$ (see, e.g., VÉRTESI [11]).

First we state

THEOREM 3.2. *The necessary and sufficient condition that for any* $f\in C$

$$\lim_{n\to\infty} \|H_n(f; Y(\alpha,\beta); x)-f(x)\| = 0 \tag{3.11}$$

is that the relations

$$-1<\alpha,\quad \beta<0 \quad and \quad \beta-\alpha \leqq 0.5 \tag{A}$$

should hold.

By Theorem 3.2 we get the following special cases:

COROLLARY 3.3. *Supposing that*

(a) $-1<\beta\leqq\alpha<0$, *or*

(b) $-1<\beta\leqq-0.5$ *and* $-1<\alpha<0$, *or*

(c) $-1<\beta<0$ *and* $-0.5\leqq\alpha<0$,

then

$$\lim_{n\to\infty} \left\|H_n(f; Y(\alpha,\beta); x)-f(x)\right\| = 0 \quad for \quad f\in C.$$

For the conjugate points of Y we state

THEOREM 3.4. *If* $\alpha\neq-1/2$ *then the conjugate points* $\{s_{k,n}\}$ *of* $Y(\alpha,\beta)$ *are dense in* $[-1,1]$. *Moreover, if* $\alpha>-\frac{1}{2}$ *they are dense on the real line, too.*

3.5. By Corollary 3.3 and Theorem 3.4 we get

COROLLARY 3.5. *Supposing* $-0.5<\alpha,\beta<0$, *the matrix* $Y(\alpha,\beta)$ *and the weight* $p(\alpha,\beta;x)$ *solve the problems of* 3.2.

3.6. Using Corollaries 2.4 and 3.3 we obtain

COROLLARY 3.6. *If* $-0.5\leqq\alpha\leqq\beta<0$ *then*

$$\lim_{n\to\infty} \|Q_n(f; Y(\alpha,\beta), x)-f(x)\| = 0 \quad for \quad f\in C. \tag{3.12}$$

Corollary 3.6 solves the second problem of 3.2 for Q_n if $-0.5<\alpha<0$.

3.7. Let us consider the quasi-conjugate points of Y. We state

THEOREM 3.7. *Let* $-1<\beta<0$. *Then, supposing* $\alpha\neq-\frac{1}{2}$, *the quasi-conjugate points* $\{t_{k,n}\}$ *of* $Y(\alpha,\beta)$ *are dense in* $[-1,1]$. *Moreover, if* $\alpha<-\frac{1}{2}$ *they are dense on the real line. If* $\beta>0$ *a statement analogous to Theorem* 3.4 *holds.*

3.8. By Corollary 3.6 and Theorem 3.7 we get

COROLLARY 3.8. *Supposing* $-0.5<\alpha\leqq\beta<0$, *the matrix* $Y(\alpha,\beta)$ *and the weight* $p(\alpha,\beta;x)$ *solve the problems of* 3.2 *for the quasi-Hermite—Fejér interpolation.*

3.9. REMARKS. **3.9.1.** For the special cases $-0.5<\alpha<0$, and $\beta=-0.5$ the relation (3.11) was settled by J. BALÁZS [4]. Similarly, using a method different from our one, he proved, that the conjugate points are dense in $[-1, 1]$ whenever $-0.5<\alpha<0$ and $\beta=-0.5$.

3.9.2. If $\alpha=-0.5$ then by (3.8) the orthogonal system will be the ultraspherical Jacobi polynomials which are settled in [1] (see Theorems 3.2 and 3.4.) Using [10], (4.28) and [8], one can obtain the position of the conjugate and quasi-conjugate points for $\alpha=-0.5$ (See Theorems 3.4. and 3.7.)

3.9.3. In their paper [12], K. K. MATHUR and R. B. SAXENA investigated the problems raised in 3.2 for $Q_n(f; Y)$. Unfortunately, their proof is erroneous, (Namely they use $P_m^{(p,q-1/2)}(2x^2-1)$ and $P_m^{(p,q+1/2)}(2x^2-1)$ instead of $P_m^{(p,(q-1)/2)}(2x^2-1)$ and $P_m^{(p,(q+1)/2)}(2x^2-1)$, respectively (see [12], (4.1)). Further [12], Lemma 6.1 states $(1-x^2)p_n^2(x)=O(n^{-1})$ $(-1\leqq x\leqq 1)$ which is not valid, e.g. for $x=0$.)

4. Proofs

4.1. PROOF OF THEOREM 2.1. By (2.1) and (2.3)

$$Q_n(f; X; x)-H_n(f; X; x) = \omega_n^2(X; x)(a_n x+b_n)$$

because $S_n(x)=Q_n(x)-H_n(x)$ is a polynomial of degree $\leqq 2n+1$ for which $S_n(x_k)=S_n'(x_k)=0$ $(k=1, 2, \dots, n)$. Moreover, for $x=\pm 1$ we have

$$\begin{cases} f(1)-H_n(f; 1) = \omega_n^2(1)(b_n+a_n), \\ f(-1)-H_n(f; -1) = \omega_n^2(-1)(b_n-a_n). \end{cases}$$

Solving this system for the unknowns a_n and b_n, we get

$$a_n = \frac{1}{2}\left[\frac{f(1)-H_n(f; 1)}{\omega_n^2(1)} - \frac{f(-1)-H_n(f; -1)}{\omega_n^2(-1)}\right],$$

$$b_n = \frac{1}{2}\left[\frac{f(1)-H_n(f; 1)}{\omega_n^2(1)} + \frac{f(-1)-H_n(f; -1)}{\omega_n^2(-1)}\right],$$

which give (2 5).

4.2. PROOF OF THEOREM 3.2. **4.2.1.** Suppose (A) holds. First we quote a lemma which was essentially proved by G. GRÜNWALD (see the proof of Theorem 2 in [5]).

LEMMA 4.1. *If* $\sum_{k=1}^{n} |h_k(x)|=O(1)$ *and* $\lim_{n\to\infty} \sum_{k=1}^{n} |\eta_{k,n}(x)|=0$ *uniformly in* $x\in[-1, 1]$, *then* $\lim_{n\to\infty} \|H_n(f; x)-f(x)\|=0$ *for* $f\in C$. (Here, as usual, $h_{k,n}(X; x)= =h_k(x)=v_k(x)\,l_k^2(x)$ and $\eta_{k,n}(X; x)=\eta_{k,n}(x)=(x-x_k)\,l_k^2(x)$.)

4.2.2. We apply this lemma for Y. By [11], 4.1 and a simple computation we have

$$(4.1)\quad l_{k,n}(Y;x)=\begin{cases}-\dfrac{P_n(1-2x^2)}{4y_kP_n'(x_{|k|})(x-y_k)} & (n \text{ is even, } k=\pm1,\pm2,\dots,\pm n),\\ -\dfrac{xP_n(1-2x^2)}{4y_k^2P_n'(x_{|k|})(x-y_k)} & (n \text{ is odd, } k=\pm1,\pm2,\dots,\pm n),\\ \dfrac{P_n(1-2x^2)}{P_n(1)} & (n \text{ is odd, } k=0).\end{cases}$$

(Here and later on, if n is even then the corresponding parameters of P_n, P_n' and $x_{k,n}$ are (α,β); for odd n they are $(\alpha+1,\beta)$.) Further

$$(4.2)\quad v_{k,n}(Y,x)=\begin{cases}1+\dfrac{1}{y_k}\left[2\dfrac{\alpha-\beta+(\alpha+\beta+2)x_{|k|}}{1+x_{|k|}}-1\right](x-y_k), & (n=1,2,\dots,\ k=\pm1,\pm2,\dots,\pm n)\\ 1, \quad n \text{ is odd}, \quad k=0.\end{cases}$$

4.2.3. First let n be even. Then, as in [11], (4.1) we get $\sum_{|k|=1}^{n}|h_{k,n}(Y;y)|=O(1)$ supposing that (A) holds. (More exactly we obtain for *arbitrary* $-1<\alpha,\beta$

$$\sum_{|k|=1}^{n}\left|h_{k,n}(Y(\alpha,\beta);x)\right|=O(1)\left(1+n^{2\alpha}+n^{2\beta}+\frac{n^{2\beta}}{n^{2\alpha+1}}\sum_{k=1}^{n}k^{2\alpha}\right)$$

from where by (A) we get (3.11).)

Further, by a rather long and not so trivial calculation similar to [11], 4.1, we can prove with $\gamma=\max(\alpha,\beta,-0.5)$

$$(4.3)\qquad \sum_{|k|=1}^{n}|\eta_{k,n}(Y;x)|=\begin{cases}\dfrac{\ln n}{n} & \text{if } \gamma\leqq-0.5,\\ n^{2\gamma} & \text{if } \gamma>-0.5.\end{cases}$$

By these and using the lemma we obtain (3.11).

4.2.4. For odd n, using (4.1) and the corresponding formulae for the Jacobi polynomials and roots, it is easy to see that

$$|l_{k,2n+1}(Y;x)|=\left|\frac{xP_n^{(\alpha+1,\beta)}(1-2x^2)}{4y_{k,2n+1}P_n'^{(\alpha+1,\beta)}(x_{|k|,n}^{(\alpha+1,\beta)})(x-y_{k,2n+1})}\right|=$$

$$=O(1)\left|\frac{P_n^{(\alpha,\beta)}(1-2x^2)}{4y_{k,2n}P_n'^{(\alpha,\beta)}(x_{|k|,n}^{(\alpha,\beta)})(x-y_{k,2n})}\right|=O(1)|l_{k,2n}(Y;x)| \quad k\neq0$$

and $l_{0,2n+1}(Y,x)=O(1)$. So this case can be calculated as the above one. This proves (3.11) for each n.

4.2.5. Now we prove that condition (A) is necessary, too. First let $-1<\alpha, \beta<0$ and $\beta-\alpha>0.5$ i.e. $\alpha<-0.5$. By (4.2) we get for a suitable $0<c_1<1$

$$|v_k(1)| = \left|1+\frac{1}{y_k}\left\{\frac{2}{2+(x_k-1)}[2\alpha+2+(\alpha+\beta+2)(x_k-1)]-1\right\}(1-y_k)\right| \sim$$

$$\sim \left|1+\frac{1}{y_k}\left\{\frac{2}{2-\delta}[2\alpha+2+(\alpha+\beta+2)\delta]-1\right\}(1-y_k)\right| \sim \frac{n}{k}$$

if $0<k\leqq c_1 n$, i.e. $|x_k-1|\leqq\delta$. Now by (4.1)

$$\sum_{0<k\leqq c_1 n} |h_k(1)| \sim n^{2\beta} \sum_{k=1}^{n} \frac{n}{k}\frac{n^2}{k^2}\frac{k^{2\alpha+3}}{n^{2\alpha+4}} \sim \frac{n^{2\beta}}{n^{2\alpha+1}} \sum_{k=1}^{n} k^{2\alpha} \neq O(1),$$

i.e. (3.11) does not hold for a certain $f\in C$.

If $\alpha\geqq 0$ then, as we proved in [11], $H_n(f_1;0)\sim n^{2\alpha}$ for $n=2,4,\dots$ and $f_1=x^2$. For $\beta\geqq 0$ consider $f_2(x)=1-x$. We have, as above, supposing $\alpha\neq-0.5$

$$|H_n(f_2;1)-f_2(1)| = |H_n(f_2;1)| \geqq \Big|\sum_{0<k<c_1 n} f_2(y_2)h_k(1)\Big| \sim$$

$$\sim \frac{n^{2\beta}}{n^{2\alpha+1}}\sum_{k=1}^{n} k^{2\alpha} \geqq c_2\cdot n^{2\beta}.$$

Finally if $\alpha=-0.5$, $\beta\geqq 0$ then by (3.8) $R_n^{(-0.5,\beta)}(x)=c(n,\beta)P_n^{(\beta,\beta)}(x)$, for which we know the statement. So we completely proved Theorem 3.2.

4.3. PROOF OF THEOREM 3.4. **4.3.1.** By (4.2) we obtain

$$v_{k,n}(0) = 2(\beta+1)\frac{1-x_{|k|}}{1+x_{|k|}}-2\alpha \quad (k\neq 0) \tag{4.4}$$

and by definition

$$v_k(y_k) = 1. \tag{4.5}$$

Using (4.2)

$$s_{k,n} = y_k+\frac{y_k(1+x_{|k|})}{2(\beta+1)(1-x_{|k|})-(2\alpha+1)(1+x_{|k|})} \quad (k\neq 0). \tag{4.6}$$

4.3.2. First let $-1<\alpha<-0.5$. Define $0<a<b<1$, $b-a=\delta$. We shall prove that for certain n and k, $s_{n,k}\in[a,b]$. Indeed, by (4.4) and (4.5) one can verify that $v_{k,n}(0)>-2\alpha>1+\eta$ $(k=\pm1, \pm2, \dots \pm n; \eta>0)$ from where, using (4.4)—(4.6), we can say that there exist $0<c_1<c_2<1$ such that

$$s_{[c_1 n],n}\leqq a \quad\text{and}\quad s_{[c_2 n],n}\geqq b \quad (n\geqq n_0). \tag{4.7}$$

Moreover, by (4.6)

$$|s_{k+1,n}-s_{k,n}|\leqq\delta \quad\text{if}\quad [c_1 n]\leqq k\leqq[c_2 n], \quad n\geqq n_0. \tag{4.8}$$

By (4.7) and (4.8) we get the desired relation. The case $[a,b]\subset(-1,0)$ can be treated similarly.

4.3.3. Let $-0.5<\alpha$. Using the above notations one can choose a constant $c_3>0$ such that

$$-2\alpha < v_{k,n}(0) < 1-\eta \quad (-c_3 n \geqq k < 0). \tag{4.9}$$

By (4.9) and $v_{k,n}(0)<v_{k+1,n}(0)$ $(k<0)$ we can state that for a certain $c_4>0$ $s_{-[c_4 n],n}<a$. To go further let us denote the nearest root (of $P_{2n}(x)$) to $[2(\beta+1)-(2\alpha+1)][2(\beta+1)+(2\alpha+1)]^{-1} \overset{\text{def}}{=} A$ by $x_{j(n),n}$. Obviously $|A|<1$ so $jn^{-1}\to c_5$ where $0<c_5<1$. Further by (4.6) $\lim_{n\to\infty} s_{-j,n}=\infty$. I.e., for suitable $0<c_7<c_6<c_5$

$$b \leqq s_{-[c_7 n],n} < s_{-[c_6 n],n} \leqq 2b. \tag{4.10}$$

The remaining parts are analogous to 4.3.2. The case $a<-1$ or $a>1$ can be treated similarly.

4.4. Proof of Theorem 3.7. We use ideas analogous to 4.3. We sketch the proof. First we state

$$q_{k,n}(0) = 2\beta\frac{y_k^2}{1-y_k^2}-2\alpha = 2\beta\frac{1-x_{|k|}}{1+x_{|k|}}-2\alpha \quad (k\neq 0). \tag{4.11}$$

Indeed, by [2], 1. §, (5)

$$q_k(x) = v_k(x)+\frac{2y_k}{1-y_k^2}(x-y_k) \tag{4.12}$$

from where, using (3.10) and (4.4), we obtain (4.11). By simple computation

$$t_{k,n} = -\frac{y_k\left(2\beta\frac{y_k^2}{1-y_k^2}-2\alpha\right)}{1-\left(2\beta\frac{y_k^2}{1-y_k^2}-2\alpha\right)} \quad (k\neq 0). \tag{4.13}$$

4.4.1. Let $\beta<0$ and $\alpha<-0.5$. By (4.11) $\lim_{n\to\infty} t_{1,n}=0$ and $t_{k,n}>1$ if $q_{k,n}(0)\approx 1$, i.e. $y_k^2(1-y_k^2)^{-1}\approx(2\alpha+1)(2\beta)^{-1}$ $(k>0)$. By these we can get the desired result.

4.4.2. Let $\beta<0$ and $-0.5<\alpha<0$. Now $q_{k,n}(0)\approx 0$ if $y_k^2(1-y_k^2)^{-1}\approx\alpha\beta^{-1}$. For this $k>0$ $t_{k,n}\approx 0$. Further $\lim_{n\to\infty} t_{n,n}=1$; these give the corresponding statement. If $\alpha\geqq 0$, k can be equal to 1.

4.4.3. The case $\beta>0$, by (4.4) and (4.11), can be considered as 4.3. because of $q_k(Y(\alpha,\beta);0)\approx v_k(Y(\alpha,\beta-1),0)$.

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ON THE GROWTH OF SOME ADDITIVE FUNCTIONS ON SMALL INTERVALS

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1. The letters $c, c_1, c_2, \ldots$ denote suitable, $\varepsilon, \varepsilon_1, \varepsilon_2, \ldots, \delta$ small positive constants. $\varepsilon_1, \varepsilon_2, \ldots$ will depend on ε. p_n denotes the n^{th} prime number, $p, q, q_1, q_2, \ldots$ are primes. $\sum\limits_p$ denotes a summation over primes indicated. $\pi(x) = \sum\limits_{p \leqq x} 1$. $\omega(n)$ denotes the number of distinct prime factors of n. (a, b) and $[a, b]$ denote the greatest common divisor and the least common multiple of a and b, resp. $[x]$ denotes the integer part of x. For the sake of brevity we shall write $x_{i+1} = \log x_i$ $(i=0, 1, 2)$, $x_0 = x$.

Let

$$O_k(n) = \max_{j=1,\ldots,k} \omega(n+j), \quad o_k(n) = \min_{j=1,\ldots,k} \omega(n+j). \tag{1.1}$$

One of us (see [1]) proved the following assertions. For every $\varepsilon > 0$, apart from a set of n's having zero density, the inequalities

$$O_k(n) \leqq (1+\varepsilon)\,\varrho\left(\frac{\log k}{\log\log n}\right)\log\log n, \quad o_k(n) \geqq (1-\varepsilon)\,\bar{\varrho}\left(\frac{\log k}{\log\log n}\right)\log\log n$$

hold for every $k=1, 2, \ldots$. Here $\varrho(u)$ $(u \geqq 0)$ is defined as the inverse function of $\psi(r) = r\log\frac{z}{e} + 1$ defined in $z \geqq 1$, and $\bar{\varrho}(n)$ $(n \geqq 0)$ is the inverse function of the same $\psi(r)$ defined in $0 < z \leqq 1$. In the same paper it was conjectured that

$$O_k(n) \geqq (1-\varepsilon)\,\varrho\left(\frac{\log k}{\log\log n}\right)\log\log n, \tag{1.2}$$

and

$$o_k(n) \leqq (1+\varepsilon)\,\bar{\varrho}\left(\frac{\log k}{\log\log n}\right)\log\log n,$$

for every $k \geqq 1$ and for almost all n. The last conjecture is false, since for $k = \log n$, $o_k(n) = 0$ would follow, which is impossible. Instead of it we state

$$o_k(n) \leqq \left\{\bar{\varrho}\left(\frac{\log k}{\log\log n}\right) + \varepsilon\right\}\log\log n, \tag{1.3}$$

where $\bar{\varrho}(u) = 0$ or $u \geqq 1$.

We shall prove

THEOREM 1. *For every $\varepsilon > 0$ the inequalities* (1.2), (1.3) *hold for every $k \geqq 1$, apart from a set of n's having zero density.*

Let $g(n)$ be a non-negative strongly additive function, i.e. $g(p^\alpha)=g(p)$ for every prime p. Let

$$f_k(n) = \max_{j=1,\dots,k} g(n+j). \tag{1.4}$$

It is obvious that $f_k(n) \geqq f_k(0)$. We are interested in the conditions which imply that

$$f_k(n) \leqq (1+\varepsilon) f_k(0) \tag{1.5}$$

holds for every $k > k_0$, apart from a set of n's having upper density at most $\delta(\varepsilon, k_0)$, where $\delta(\varepsilon, k_0) \to 0$ as $k_0 \to \infty$.

This question was considered for some special functions in [2].

Let

$$g^+(p) = \begin{cases} g(p), & \text{if } g(p) \leqq 1, \\ 1, & \text{if } g(p) > 1, \end{cases}$$

and $g^+(n)$ is defined as a strongly additive function generated by the values $g^+(p)$. By using the wellknown Turán—Kubilius inequality

$$\sum_{n \leqq x} (g^+(n) - A_x)^2 \leqq c \times B_x \quad (\leqq c \times A_x)$$

$$A_x = \sum_{p \leqq x} \frac{g^+(p)}{p}, \quad B_x = \sum_{p \leqq x} \frac{g^{+2}(p)}{p} \quad (\leqq A_x),$$

and that $g(n) \geqq g^+(n)$, we immediately have that the convergence of

$$\sum \frac{g^+(p)}{p}$$

is a necessary condition for the truth of (1.5).

We are unable to decide if

$$\sum \frac{g(p)}{p} < \infty \tag{1.6}$$

is necessary for (1.5).*

Assume that $g(p)$ tends to zero monotonically as $p \to \infty$. We shall prove that (1.6) is not sufficient for (1.5). This disproves the conjecture stated in [2], namely that from the convergence of the series $\sum \frac{g^+(p)}{p}$, $\sum_{g(p)>1} \frac{1}{p} > 1$ (1.5) would follow. Finally, assuming some regularity conditions on

$$A(y) = \sum_{p \leqq y} g(p)$$

we shall show that (1.5) holds.

Let $t(x)$ be a real valued monotonically decreasing function defined for $x \geqq 1$. Let

$$A(y) = \sum_{p \leqq y} t(p), \tag{1.7}$$

* Remark. We decided this question affirmatively. We shall publish this in a forthcoming paper in this journal.

and suppose that

$$\sum_p \frac{t(p)}{p} < \infty, \tag{1.8}$$

and that for every positive constant δ

$$\lim_{y\to\infty} \frac{A(y)}{yt(\exp(\exp(y^\delta)))} = \infty.$$

Let $g(n)$ be the strongly additive function defined for primes as $g(p)=t(p)$.

THEOREM 2. *Assume that the conditions* (1.7), (1.8) *hold. Let* ε *be an arbitrary positive constant. Then for every integer* k_0 *the inequality*

$$f_k(n) < (1+\varepsilon) f_k(0)$$

holds for every $k \geqq k_0$ *and for all but* $\delta(k_0, \varepsilon)x$ *integers* n *in* $[1, x]$. *Here* $\delta(k_0, \varepsilon) \to 0$ $(k_0 \to \infty)$.

We shall prove these assertions in the following sections.

Now we make the following remark. In [3], IVÁNYI and KÁTAI proved the existence of a completely additive $f(n)$ not identically zero for which $f(n)=A_j$, $n \in [N_j, N_j+\tau(N_j)]$ on a suitable set $N_1<N_2<\dots$ of integers, where $\tau(N)= =\exp\left(c\sqrt{(\log N)(\log\log\log N)}\right)$, A_j are arbitrary complex or real values.

Now we prove the following

THEOREM 3. *Let* $\varepsilon>0$ *and* $x>x_0(\varepsilon)$. *Then there exists a completely additive function* $f(n)$ *for which*

$$f(n) = 0 \quad in \quad [N+1, N+\lambda(x)],$$

where $\frac{x}{2} \leqq N \leqq x$ *and*

$$\lambda(x) = \left[\exp\left(\left(\frac{1}{2}-\varepsilon\right)\frac{(\log x)(\log\log\log x)}{\log\log x}\right)\right],$$

and which takes on a non-zero value in $[1, \sqrt{x}]$.

REMARK. Unfortunately we can not prove that there is an $f(n)$ with infinitely many such intervals.

PROOF. Denote by $N(x, y)$ the number of integers $n \leqq x$ all prime factors of which are not greater than y. By a theorem of RANKIN [4]

$$N(x, y) < x\exp\left(-\frac{\log\log\log y}{\log y}\log x + \log\log y + O\left(\frac{\log\log y}{\log\log\log y}\right)\right). \tag{1.9}$$

Let $k=\lambda(x)$, x large. (1.9) implies

$$N(x, k) < \left[\frac{x}{2k}\right]\pi(k).$$

Thus it is easy to see that there is an interval $[N+1, N+k]$ in $\frac{x}{2} \leqq N < N+k \leqq x$, for which the number of integers all prime factors of which do not exceed k is smaller than $\pi(k)$. Let $n=A(n)B(n)$, where $A(n)$ is composed of the prime factors $\leqq k$ of n. Let $n+l_i$ $(i=1, \dots, h)$, $h<\pi(k)$ be the n's in $[N+1, N+k]$ for which $B(n+l_i)=1$.

The additivity leads to the following linear system of equations:

$$f(A(n+l_j))=0 \quad (j=1, \dots, h), \tag{1.10}$$

$$f(B(n+r))=-f(A(n+r)) \quad (r\neq l_j (j=1, \dots, h)), \tag{1.11}$$

where the indeterminates are the values $f(p)$ for primes p contained in $(N+1), \dots$ $\dots, (N+k)$. (1.9) is a homogeneous system, the number h of equations is smaller than $\pi(k)$, therefore we can choose values $f(p_1), \dots, f(p_{\pi(k)})$ non-trivially such that (1.10) hold. This holds in the case $h=0$, too. To finish the proof we need to take into account only that $B(n+r)$ $(r\neq l_j,\ j=1, \dots, k)$ are mutually coprime, so we can solve (1.11). This completes the proof of Theorem 3.

2. Lemmas. Let k be an integer, $\mathscr{P}$ be a finite set of primes greater than k. Let $\mathscr{T}_r$ denote the set of integers of the form $t_r=q_1q_2\cdots q_r$, $q_i\in\mathscr{P}$, $q_i\neq q_j$ $(i\neq j)$,

$$P=\sum_{p\in\mathscr{P}} 1/p, \quad T_r=\sum_{t_r\in\mathscr{T}_r} 1/t_r, \tag{2.1}$$

$$a=\sum_{p\in\mathscr{P}}\frac{1}{p^2}. \tag{2.2}$$

Let Π_r be the number of elements of $\mathscr{T}_r$.

LEMMA 1. *For every $r\geqq 2$ we have*

$$\frac{P^r}{r!}-\frac{a}{2}\frac{P^{r-2}}{(r-2)!}\leqq T_r\leqq\frac{P^r}{r!}. \tag{2.3}$$

PROOF. The right hand side of (2.3) is obvious. We prove the left hand side by using induction. The assertion holds for $r=2$, since

$$T_2=\frac{1}{2}(P^2-a).$$

Observing that

$$T_rP\leqq T_{r+1}(r+1)+\sum_{p\in\mathscr{P}}\frac{1}{p^2}\Big\{\sum_{(t_{r-1},p)=1}\frac{1}{t_{r-1}}\Big\}\leqq T_{r+1}(r+1)+aT_{r-1},$$

we get

$$T_{r+1}\geqq\frac{T_rP}{r+1}-\frac{a}{r+1}T_{r-1},$$

and by the induction hypothesis

$$T_{r+1}\geqq\Big\{\frac{P^r}{r!}-\frac{a}{2}\frac{P^{r-2}}{(r-2)!}\Big\}\frac{P}{r+1}-\frac{a}{r+1}\frac{P^{r-1}}{(r-1)!}=\frac{P^{r+1}}{(r+1)!}-\frac{a}{2}\frac{P^{r-1}}{(r-1)!}.$$

By this Lemma 1 is proved.

We shall use Brun's sieve in the form of Theorem 2.5 in [5], or in the simpler form of [6], Theorem 6.2. Namely we shall use the following result, which we state now as

LEMMA 2. *Let $a_1, a_2, \dots$ be positive integers, $\mathcal{R}$ a finite set of primes, all of them smaller than z. Let*

$$\eta(y,d) = \left| \sum_{\substack{a_\nu \equiv o(d) \\ a_\nu \leqq y}} 1 - \frac{\gamma(d)}{d} y \right|,$$

where $\gamma(d)$ is a multiplicative function on the set of square free numbers all prime factors of which are in $\mathcal{R}$. Suppose that $\eta(y,d) \leqq \gamma(d)$ for all such d, and $\gamma(p) = O(1)$, $\gamma(p) \leqq p-1$ for all $p \in \mathcal{R}$. Putting $R = \prod_{p \in \mathcal{P}} p$, for $y \geqq r$ we get

$$\sum_{\substack{a_\nu \leqq y \\ (a_\nu, R)=1}} 1 = y \prod_{p \in \mathcal{R}} \left(1 - \frac{\gamma(p)}{p}\right) \left\{1 + O\left(\exp\left(-\frac{1}{2}\frac{\log y}{\log z}\right)\right)\right\}. \tag{2.4}$$

Let now $\mathcal{P}$ be the set of all primes in (k, r), where $z < x^{1/4r}$. Let $\mathcal{A}$ be the set of integers $n = t_r b$, where $t_r \in \mathcal{T}_r$, $(b, \prod_{p \in \mathcal{P}} p) = 1$. Let

$$V(n) = \begin{cases} 1, & \text{if } n \in \mathcal{A}, \\ 0, & \text{if } n \notin \mathcal{A}, \end{cases}$$

and put

$$\sum\nolimits^{(0)} = \sum_{n \leqq x} V(n), \quad \sum\nolimits^{(h)} = \sum_{n+h \leqq x} V(n)V(n+h) \quad (h = 1, \dots, k). \tag{2.5}$$

Let

$$\Gamma_1 = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right), \quad \Gamma_2 = \prod_{p \in \mathcal{P}} \left(1 - \frac{2}{p}\right), \tag{2.6}$$

and $\lambda(n)$ a multiplicative function on the square free integers defined for primes p by $\lambda(p) = \left(1 - \frac{1}{p}\right)\left(1 - \frac{2}{p}\right)^{-1}$.

For the computation of $\sum^{(0)}, \sum^{(h)}$ we shall use the previous lemma. Let $N(y|\mathcal{P})$ be the number of $b \leqq y$, which have no prime factors in $\mathcal{P}$. By (2.4),

$$N(y|\mathcal{P}) = y\Gamma_1 \left\{1 + O\left(\exp\left(-\frac{2r \log y}{\log x}\right)\right)\right\},$$

since $x/t_r \geqq x^{3/4}$. Consequently

$$\sum\nolimits^{(0)} = \sum_{t_r \in \mathcal{T}_r} N\left(\frac{x}{t_r} | \mathcal{P}\right) = T_r \Gamma_1 \times (1 + O(e^{-r})). \tag{2.7}$$

Consider now $\sum^{(h)}$ $(h \geqq 1)$. First we count the integers n, $n = t_r^{(1)} b_1$, $n + h = t_r^{(2)} b_2 \leqq x$ with fixed $t_r^{(1)}, t_r^{(2)} \in \mathcal{T}_r$. There is a solution only if $(t_r^{(1)}, t_r^{(2)}) = 1$. The solutions b_1, b_2 of $t_r^{(2)} b_2 - t_r^{(1)} b_1 = h$ are in the progressions $b_2 = b_2^{(0)} + s t_r^{(1)}$,

$b_1=b_1^{(0)}+st_r^{(1)}$ $(s=0, 1, 2, \dots)$. Sieving those elements $b_1 b_2$ which have prime factors in $\mathscr{P}$, we get that $\gamma(p)=2$ if $p\nmid t_r^{(1)} t_r^{(2)}$, and $\gamma(p)=1$, if $p| t_r^{(1)} t_r^{(2)}$. Thus by Lemma 2,

$$\sum\nolimits^{(h)} = x\Gamma_2\big(1+O(\bar{e}^r)\big)A, \tag{2.8}$$

$$A = \sum_{(t_r^{(1)},t_r^{(2)})=1} \frac{\lambda(t_r^{(1)}t_r^{(2)})}{t_r^{(1)}t_r^{(2)}}.$$

Since $t_r^{(1)}t_r^{(2)}=t_{2r}$ has $\binom{2r}{r}$ solutions for fixed t_{2r} we have

$$A = \binom{2r}{r}\sum \frac{\lambda(t_{2r})}{t_{2r}}.$$

Let $h(d)$ be the Moebius transform of $\lambda(d)$. Then $h(p)=\frac{1}{p-2}$, $h(d)$ is multiplicative, and we have

$$T_{2r} \leqq \sum \frac{\lambda(t_{2r})}{t_{2r}} \leqq T_{2r}+\sum_{\nu=1}^{2r}\Big\{\sum_{\delta\in\mathscr{T}_\nu}\frac{h(\delta)}{\delta}\Big\}T_{2r-\nu}.$$

Taking into account that

$$\sum_{\delta\in\mathscr{T}_\nu}\frac{h(\delta)}{\delta} \leqq \frac{1}{\nu!}\Big\{\sum_{p>k}\frac{1}{p(p-2)}\Big\}^\nu \leqq \frac{1}{\nu!}\Big(\frac{c}{k\log k}\Big)^\nu,$$

from Lemma 1 we get

$$\sum \frac{\lambda(t_{2r})}{t_{2r}} \leqq \frac{p^{2r}}{(2r)!}\exp\Big(\frac{2rc}{Pk\log k}\Big).$$

Furthermore Lemma 1 implies that

$$T_{2r} \geqq \Big(1-\frac{4ar}{P}\Big)\frac{P^{2r}}{(2r)!},$$

and so

$$A = \Big(\frac{P^r}{r!}\Big)^2\Big(1+O\Big(\frac{r}{Pk\log k}\Big)\Big),$$

if

$$\frac{r}{Pk\log k}=O(1). \tag{2.9}$$

We have

$$\log\Gamma_2 = 2\log\Gamma_1+O(a),\quad \log\Gamma_1 = -P+O(a),$$

whence

$$\Gamma_1 = e^{-p}\big(1+O(a)\big),\quad \Gamma_2 = e^{-2p}\big(1+O(a)\big).$$

Consequently

$$\sum\nolimits^{(0)} = xe^{-p}\frac{P^r}{r!^2}\Big(1+O(e^{-r})+O\Big(\Big(\frac{r}{P}+1\Big)\frac{1}{k\log k}\Big)\Big), \tag{2.10}$$

$$\sum\nolimits^{(h)} = xe^{-2p}\frac{P^{2r}}{r!}\Big(1+O(e^{-r})+O\Big(\Big(\frac{r}{P}+1\Big)\frac{1}{k\log k}\Big)\Big), \tag{2.11}$$

if (2.5) holds.

Let

$$F_k(n) = \sum_{i=1}^{k} V(n+i), \quad \Lambda = ke^{-p}\frac{P^r}{r!}, \tag{2.12}$$

$$E = \sum_{n \leqq x} (F_k(n) - \Lambda)^2. \tag{2.13}$$

We have

$$E = \sum_{n \leqq x} F_k^2(n) - 2\Lambda \sum_{n \leqq x} F_k(n) + \Lambda^2 x,$$

and observe that

$$\sum_{n \leqq x} F_k(n) = k \sum{}^{(0)} + O(k^2),$$

$$\sum_{n \leqq x} F_k^2(n) = k \sum{}^{(0)} + \sum_{h=1}^{k} 2(k-h) \sum{}^{(h)} + O(k^3).$$

Collecting our results we get

LEMMA 3. *If* (2.9) *holds, then*

$$E = O\left(x(\Lambda^2+\Lambda)\left(e^{-r} + \frac{r+P}{Pk\log k}\right) + k^3 + k^2\Lambda\right). \tag{2.14}$$

Let now $\mathscr{P}$ be an arbitrary set of primes, $P = \sum_{p \in \mathscr{P}} 1/p$,

$$\omega(n|\mathscr{P}) = \sum_{\substack{p|n \\ p \in \mathscr{P}}} 1, \tag{2.15}$$

$$O_k(n) = \max_{j=1,\dots,k} \omega(n+j|\mathscr{P}), \quad o_k(n) = \min_{j=1,\dots,k} \omega(n+j|\mathscr{P}). \tag{2.16}$$

Let $D_k(x, L|\mathscr{P})$ be the number of $n \leqq x$ for which $O_k(n|\mathscr{P}) \geqq L$. It is obvious that

$$D_k(x, L|\mathscr{P}) \leqq z^{-L} \sum_{n \leqq x} z^{O_k(n|\mathscr{P})} \leqq z^{-L} k \sum_{n \leqq x+k} z^{\omega(n|\mathscr{P})},$$

for $z \geqq 1$. Observing that

$$\sum_{n \leqq x+k} z^{\omega(n|\mathscr{P})} \leqq (x+k) \prod_{p \in \mathscr{P}} \left(1 + \frac{z-1}{p}\right) < (x+k)\exp(zP),$$

by substituting $z = L/p$, we get immediately

LEMMA 4. *If* $1 \leqq k \leqq x$, $L \geqq P$, *then*

$$D_k(x, L|\mathscr{P}) \leqq 2x \exp\left(\log k - L \log \frac{L}{Pe}\right). \tag{2.17}$$

3. Proof of Theorem 1. First we prove (1.2). Let B be a suitable large constant depending on ε. First we shall prove (1.2) for

$$k \geqq \exp\left((\log\log n)^B\right). \tag{3.1}$$

Indeed, if we define t_k to be the largest integer l so that the product of the first l primes is smaller than k, then we get $O_k(n) \geqq O_k(0) = t_k$. From the prime number theorem we get

$$\log k \sim \sum_{j=1}^{t_k} \log p_j \sim p_{t_k} \sim t_k \log t_k,$$

whence

$$t_k \sim \frac{\log k}{\log\log k} \quad (k \to \infty).$$

Furthermore, as it is easy to show, $\varrho(u) \sim \frac{u}{\log u}$ $(u \to \infty)$, whence

$$\varrho\left(\frac{\log k}{\log\log n}\right)\log\log n \geqq \left(1-\frac{\varepsilon}{2}\right)\frac{\log k}{\log\log k},$$

if B is large enough. Thus (1.2) holds if (3.1) satisfies.

Let B be fixed, x large, and put

$$\alpha = \frac{\log k}{x_2}. \tag{3.2}$$

Observing that $\varrho(\lambda) \sim 1+\sqrt{2\lambda}$ $(\lambda \sim 0)$, therefore by choosing ε_1 to satisfy $(1+2\sqrt{\varepsilon_1})\left(1-\frac{\varepsilon}{2}\right)<1$, we get $\left(1-\frac{\varepsilon}{2}\right)\varrho(\varepsilon_1)<1$. We can choose $\varepsilon_1 = \frac{\varepsilon^2}{16}$. By using Hardy—Ramanujan's wellknown theorem that $\omega(n) \sim \log\log n$ for almost all n, we get (1.2) in $0 \leqq \alpha \leqq \varepsilon_1$.

Assume that

$$\varepsilon_1 x_2 \leqq \log k \leqq x_2^B. \tag{3.3}$$

Let r be an integer for which

$$r = \Delta x_2 + O(1), \quad \Delta = (1-\varepsilon_2)\varrho(\alpha), \tag{3.4}$$

ε_2 being a small positive constant.

Let $\mathscr{P}$ be the set of primes in $(k, x^{1/4r})$ and $N_{k,r}(x)$ denote the number of $n \leqq x$ for which $O_k(n)<r$. For these numbers $F_k(n)=0$, and by Lemma 4

$$N_{k,r}(x) \leqq \frac{E}{\Lambda^2} \leqq O\left(x\left(1+\frac{1}{\Lambda}\right)\left(e^{-r}+\frac{r+P}{Pk\log k}\right)+\frac{k^3+k^2\Lambda}{\Lambda^2}\right). \tag{3.5}$$

From (3.3), (3.4) we have

$$\alpha \leqq x_2^{B-1}, \quad \Delta \leqq c x_2^{B-1}, \quad \log r = O(x_3),$$

$$P = x_2 + O(x_3),$$

$$\frac{r+P}{Pk\log k} \ll \frac{(\Delta+1)x_2}{x_2 e^{\alpha x_2}\alpha x_2} = O(x_1^{-\alpha/2}).$$

By using Stirling formula,

$$\log \Lambda = \log k - P - r\log\frac{r}{Pe} + O(\log r) = \big(\alpha - \psi(\Delta)\big)x_2 + O(x_3).$$

Since

$$\psi(\Delta) = (1-\varepsilon_2)\psi(\varrho) + \varepsilon_2 + (1-\varepsilon_2)\varrho\log(1-\varepsilon_2)$$

and $\psi(\varrho)=\alpha$, therefore by using that $\varrho(\lambda)\sim 1+\sqrt{2\lambda}$ $(\lambda\sim 0)$, we get $\alpha-\psi(\Delta)\geqq \geqq \varepsilon_2^2/2$, if $\alpha\geqq 4\varepsilon_2^2$, ε_2 being small. Choosing $\varepsilon_2\leqq\sqrt{2\varepsilon_1}$, we get that $\Lambda\geqq 1$ for all large x and for all α in (3.3).

Since $e^{-r}\ll e^{-\Delta x_2}$, we obtain that

$$N_{k,r}(x)\leqq c_2 x\{e^{-\Delta x_2}+e^{-\alpha x_2/2}\}+O(x^{1/2}). \tag{3.6}$$

Let now $\alpha_j=j\varepsilon_1$, $k_j=[e^{\alpha_j x_2}]$, $j=1,\dots,T$, and $T-1$ is the largest integer for which $\alpha_{T-1}\leqq x_2^{B-1}$. Thus $T=O\left(\frac{1}{\varepsilon_1}x_2^{B-1}\right)$, and from (3.6)

$$\sum_{i=1}^{T} N_{k_i,r}(x)\ll xe^{-\frac{\varepsilon_1}{3}x_2}. \tag{3.7}$$

Hence it follows that for all but $O\big(xx_1^{-\frac{\varepsilon_1}{3}}\big)$ integers n in $\left[\frac{x}{2},x\right]$

$$O_{k_i}(n)>\left(1-\frac{\varepsilon}{2}\right)\varrho\left(\frac{\log k_i}{x_2}\right)x_2 \quad (i=1,\dots,T). \tag{3.8}$$

Let $k\in[k_i,k_{i+1})$ and suppose that (3.8) holds for an n. Since $O_k(n)\geqq O_{k_i}(n)$ and $\varrho(\alpha)<(1+c_3\varepsilon_1)\varrho(\alpha_i)$, therefore

$$O_k(n)>\left(1-\frac{2\varepsilon}{3}\right)\varrho(\alpha)\log\log n.$$

Since $\log\log n$ increases very slowly therefore

$$O_k(n)>(1-\varepsilon)\varrho\left(\frac{\log k}{\log\log n}\right)\log\log n$$

holds for all but $O\big(xx_1^{-\frac{\varepsilon_1}{3}}\big)$ integers $n\in\left[\frac{x}{2},x\right]$. This assertion holds for $x\geqq X_0$. Choosing now $x=2^\nu X_0$ $(\nu=0,1,\dots)$ and using our result, we obtain (1.2).

The proof of (1.3) is very similar. Since $\bar\varrho(\lambda)\sim 1-\sqrt{2\lambda}$ $(\lambda\sim 0)$, therefore (1.3) is obvious if $\alpha\leqq\frac{\varepsilon^2}{3}$.

Let $\mathscr{P}$ be the set of primes in $(k,x^{1/4r})$,

$$\alpha=\frac{\log k}{x_2},\quad \frac{\varepsilon^2}{3}\leqq\alpha\leqq 1,$$

r be an integer for which $r=Hx_2+O(1)$, $H=\bar\varrho(\alpha)+\varepsilon_3$.

Let $B_{k,r}(x)$ be the number of $n\leqq x$, for which $o_k(n|\mathscr{P})>r$. For these n's $F_k(n)=0$, and by Lemma 4 we get

$$B_{k,r}(x)\leqq c_3x\left(1+\frac{1}{\Lambda}\right)\left(e^{-r}+\frac{1}{k\log k}\right)+c_4\frac{k^3+k^2\Lambda}{\Lambda^2}. \tag{3.9}$$

From Stirling formula

$$\log \Lambda = \log\left(ke^{-p}\frac{P^r}{r!}\right) = \left(\alpha-1-H\log\frac{H}{e}\right)x_2+O(x_3) = (\alpha-\psi(H))x_2+O(x_3).$$

Since $-\psi'(z)=-\log z$ is decreasing,

$$\psi(\bar\varrho)-\psi(H) = \int_{\bar\varrho}^{H} -\log z\,dz \geqq (H-\bar\varrho)\log\frac{1}{H} = \varepsilon_3\log\frac{1}{H},$$

consequently

$$\alpha-\psi(H) = \psi(\bar\varrho)-\psi(H) \geqq \varepsilon_3^2 \quad \text{in} \quad \alpha\in\left[-\frac{\varepsilon^2}{3}, 1\right],$$

if ε_3 is sufficiently small.

Thus $\Lambda\geqq 1$, and

$$B_{k,r}(x) \leqq c_5x(e^{-r}+k^{-1}). \tag{3.10}$$

Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be the set of primes in the intervals $[1, k]$, $[x^{1/4r}, x]$, respectively, and

$$P_1 = \sum_{p<k} 1/p = \log\log k+O(1), \quad P_2 = \sum_{x^{1/4r}<p\leqq x} 1/p\log 4r+O(1).$$

Applying Lemma 4 by

$$(L=)\,L_1 = \frac{4\log k}{\log\log k},$$

we get

$$B_k(x, L_1|\mathscr{P}_1) \leqq x/k^3. \tag{3.11}$$

Observing that $\log k=\alpha x_2\geqq\frac{\varepsilon^2}{3}x_2$, and $P_2=O(x_3)$, by choosing $L=L_1$, we get

$$B_k(x, L_1|\mathscr{P}_2) \leqq c(\varepsilon)\frac{x}{k^3}. \tag{3.12}$$

Since

$$o_k(n) \leqq o_k(n|\mathscr{P})+O_k(n|\mathscr{P}_1)+O_k(n|\mathscr{P}_2),$$

from (3.10), (3.11), (3.12) we have that for large x

$$o_k(n) \leqq r+2L_1 \leqq (\bar\varrho(\alpha)+2\varepsilon_3)x_2, \tag{3.13}$$

apart from at most

$$c_1(\varepsilon)x\{e^{-(\bar\varrho(\alpha)+\varepsilon_3)x_2}+e^{-\alpha x_2/2}\} \tag{3.14}$$

n in $[1, x]$.

Let $\alpha_t=t\frac{\varepsilon^2}{12}$ $(t=1, \dots, T)$, $T=\left[\frac{12}{\varepsilon^2}\right]+1$, $k_t=[x_1^{\alpha_t}]$. From (3.13) and (3.14) we deduce that

$$o_{k_j}(n) \leqq (\bar\varrho(\alpha_j)+2\varepsilon_3)x_2 \quad (j=1, \dots, T) \tag{3.15}$$

holds for all but $c_2(\varepsilon)xe^{-\varepsilon_3x_2}$ n in $[1, x]$, assuming that ε_3 is sufficiently small.

(3.15) easily implies that

$$o_k(n) \leqq \left(\bar{\varrho}(\alpha_j) + \frac{3\varepsilon}{4}\right) x_2 \tag{3.16}$$

for every $k \in [k_1, k_T]$. This is an immediate consequence of the fact that $0 \leqq \bar{\varrho}(\alpha_j) - \bar{\varrho}(\alpha_{j+1}) < \frac{\varepsilon}{4}$. Indeed, since $\psi'(2) = \log z$, $-\bar{\varrho}'$ is increasing, we get

$$\bar{\varrho}(\alpha_j) - \bar{\varrho}(\alpha_{j+1}) \leqq -\bar{\varrho}'(\alpha_1)\frac{\varepsilon^2}{12} = -\frac{1}{\log \bar{\varrho}(\alpha_1)}\frac{\varepsilon^2}{12} \sim -\frac{1}{\log(1-\sqrt{2\alpha_1})}\frac{\varepsilon^2}{12} < \frac{\varepsilon}{4}.$$

Putting $\log\log n$ instead of x_2 in (3.16), we get that

$$o_k(n) \leqq \left\{\bar{\varrho}\left(\frac{\log k}{\log\log n}\right) + \varepsilon\right\} \log\log n \tag{3.17}$$

holds for all but $c_2(\varepsilon) x e^{-\varepsilon_3 x_2}$ n in $\left[\frac{x}{2}, x\right]$.

Choosing a large X_0 and putting $x = 2^\nu X_0$ $(\nu = 0, 1, \dots)$ we get (1.3) immediately. Theorem 1 is proved.

4. A counter example. Now we give a non-negative strongly additive $g(n)$ for which $g(p)$ is monotonic, $\sum \frac{g(p)}{p} < \infty$, and (1.5) does not hold.

Let $R_1 = 1$, $R_{s+1} = \exp(\exp(R_s))$, $J_s = [R_s, R_{s+1})$. We define g for primes p as follows:

$$g(p) = \frac{1}{R_s s^2} \quad (p \in J_s),\ s = 1, 2, \dots.$$

Sincc

$$\sum_{A<p<B} \frac{1}{p} = \log \frac{\log B}{\log A} + O\left(\frac{1}{\log A}\right),$$

therefore

$$\sum_p \frac{g(p)}{p} = \sum_{s=1}^{\infty} \frac{1}{R_s s^2}\left\{\sum_{p\in J_s} \frac{1}{p}\right\} \ll \sum \frac{1}{s^2} = O(1).$$

Let μ be a large integer, $\mathcal{P}$ be the set of all primes in $(k, R_{\mu+2}]$. Let

$$r = 2R_{\mu+1}^2, \quad \log k = (2+\tau) R_{\mu+1}^2 \log R_{\mu+1}, \quad \frac{1}{4} \leqq \tau \leqq \frac{1}{2}.$$

Let $x \geqq R_{\mu+5}$.

Now we use Lemma 3. Its conditions are fulfilled. By an easy computation we get

$$\sum_{\substack{n \leqq x \\ F_k(n)=0}} 1 \leqq x e^{-R_{\mu+1}^2} \tag{4.1}$$

for large μ.

Let δ be small, μ be so large that $\delta > e^{-R_{\mu+1}^2}$. Then for all but δx n in $[1, x]$ $F_k(n) \neq 0$. For such an n for at least one j, $1 \leqq j \leqq k$, $n+j$ has at least r prime factors in $[1, R_{\mu+2})$, and so

$$g(n+j) \geqq \frac{r}{R_{\mu+1}(\mu+1)^2}.$$

Consequently

$$f_k(n) \geqq \frac{r}{R_{\mu+1}(\mu+1)^2} = \frac{2R_{\mu+1}}{(\mu+1)^2}. \tag{4.2}$$

Consider now $f_k(0)$. Let t_k be defined as above, i.e. $p_1 \dots p_{t_k} \leqq k \leqq p_1 \dots p_{t_k} p_{t_k+1}$. It is obvious that $f_k(0) = g(t_k)$. From the prime number theorem we get

$$\log k \sim p_{t_k} \sim t_k \log t_k \quad (\mu \to \infty).$$

Let

$$A_s = \prod_{p \in J_s} p \quad (s = 1, \dots, \mu), \quad B = \prod_{R_{\mu+1} \leqq p \leqq p_{t_k}} p.$$

Then

$$g(A_s) = \frac{1}{R_s s^2} \{\pi(R_{s+1}) - \pi(R_s)\},$$

and so

$$\sum_{s=1}^{\mu} g(A_s) \leqq 2 \sum_{s=1}^{\mu} \frac{R_{s+1}}{R_s s^2} \leqq \frac{3R_{\mu+1}}{R_\mu \mu^2}.$$

Furthermore, for an arbitrary but fixed $\varepsilon > 0$

$$g(B) = \frac{1}{R_{\mu+1}(\mu+1)^2} \{\pi(p_{t_k}) - \pi(R_{\mu+1})\} \leqq \frac{t_k}{R_{\mu+1}(\mu+1)^2} \leqq$$

$$\leqq (1+\varepsilon) \frac{\log k}{(\log\log k) R_{\mu+1}(\mu+1)^2} \leqq (1+\varepsilon)\left(1+\frac{\tau}{2}\right) \frac{R_{\mu+1}}{(\mu+1)^2},$$

if μ is sufficiently large. Consequently for large μ

$$f_k(0) < 1{,}6 \frac{R_{\mu+1}}{(\mu+1)^2}, \quad \text{and} \quad f_k(m) > 2 \frac{R_{\mu+1}}{(\mu+1)^2}$$

for all but δx of n's in $[1, x]$.

5. Proof of Theorem 2. Suppose that the conditions (1.7), (1.8) are fulfilled. If $A(y)$ is bounded then the assertion is almost obvious. Indeed, if $A(\infty) = B$, then $\sup g(n) = B$, i.e. $f_k(n) \leqq B$. Furthermore $f_k(0) \to B$, and so $f_k(n) - f_k(0) < \varepsilon f_k(0)$ for every n, if k is large enough.

Suppose now that $A(y) \to \infty$ $(y \to \infty)$. Observe that the prime number theorem easily implies

$$f_k(0) = \big(1 + o(1)\big) A(\log k) \quad k(\to \infty). \tag{5.1}$$

Furthermore from $t(y) \to 0$ $(y \to \infty)$ we obtain

$$f_{2k}(0) = f_k(0) + o(1) = \big(1 + o(1)\big) f_k(0). \tag{5.2}$$

Hence

$$f_k(0) \leqq f_k(n) \leqq f_{k+x}(0) \leqq f_{2k}(0) \leqq (1+\varepsilon) f_k(0), \tag{5.3}$$

if $k > x, n \leqq x, k$ is large.

Now we assume that $k \leqq x$. Let δ be small,

$$H = \exp\left(\exp\left((\log k)^\delta\right)\right),$$

and

$$g_1(p) = \begin{cases} g(p), & \text{if } p \leqq H, \\ 0, & \text{if } p > H; \end{cases}$$

$$g_2(p) = \begin{cases} 0, & \text{if } p \leqq H, \\ g(p), & \text{if } p > H, \end{cases}$$

and $g_1(n)$, $g_2(n)$ are the corresponding additive functions. Let

$$f_k^{(i)}(n) = \max_{j=1,\dots,k} g_i(n+j) \quad (i = 1, 2).$$

It is obvious that

$$f_k(n) \leqq f_k^{(1)}(n) + f_k^{(2)}(n).$$

Let $\theta = 1 + 2\delta$,

$$r = \left[\theta \frac{\log k}{\log \log k}\right].$$

Let $C_r(x)$ be the number of those $n \leqq x$ that have at least r prime divisors in $[1, H]$. It is obvious that

$$C_r(x) \leqq \sum_{t_r} \left[\frac{x}{t_r}\right] \leqq \frac{xP^r}{r!}, \quad P = \sum_{p \leqq H} \frac{1}{p}.$$

We have

$$kC_r(x) \leqq x \exp\left(\log k - r \log \frac{r}{Pe} + O(\log r)\right),$$

and by

$$P = (\log k)^\delta + O(1)$$

we get

$$\log k - r \log \frac{r}{Pe} + O(\log r) \leqq -\frac{\delta}{4} \log k,$$

i.e.

$$kC_r(x) \leqq \frac{x}{k^{\delta/4}}. \tag{5.4}$$

If the integers $n+j$ $(j=1, \dots, r)$ have no r distinct prime factors from $[1, H]$, then

$$f_k^{(1)}(n) \leqq g(p_1 \dots p_{r-1}) \leqq (1+3\delta) A(\log k).$$

Thus we proved that

$$f_k^{(1)}(n) < (1+3\delta) A(\log k)$$

for all but $x/k^{\delta/4}$ integers $n \in [1, x]$.

Let now η be a small positive constant, $\Delta=\eta A(\log k)$. We put $z=e^u$ $(u\geqq 0)$,

$$D(x, z)=\sum_{n\leqq x} z^{g_2(n)}.$$

The function $z^{g_2(n)}$ is multiplicative, and its Moebius transform $l(n)$ is defined for prime powers as

$$l(p)=\begin{cases} e^{ug(p)}-1, & p>H, \\ 0, & p<H, \end{cases}$$

$l(p^\alpha)=0\ (\alpha\geqq 2)$.

Consequently

$$D(x, z)=\sum_{d\leqq x} l(d)\left[\frac{x}{d}\right]\leqq x \prod_{H<p\leqq x}\left(1+\frac{e^{ug(p)}-1}{p}\right).$$

Let $u=\dfrac{1}{2t(H)}$. Then from $e^{ug(p)}-1<2ug(p)$ it follows that

$$D(x, z)\leqq x\exp\left(2u\sum_{H<p<x}\frac{t(p)}{p}\right).$$

Let $B(x,\eta,k)$ denote the number of those $n\leqq x$, for which $f_2(n)\geqq\Delta$. We obtain

$$B(x,\eta,k)\leqq k\sum_{n\leqq x} z^{g_2(n)-\Delta u}\leqq x\exp\left(-\Delta u+2u\sum_{H<p<x}\frac{t(p)}{p}+\log k\right).$$

From (1.9) we have

$$-\Delta u+2u\sum_{H<p<x}\frac{t(p)}{p}+\log k<-3\log k$$

for large k, i.e.

$$B(x,\eta,k)\leqq\frac{x}{k^3}.$$

Consequently

$$f_k(n)<(1+3\delta+\eta)A(\log k)$$

for all but $\left(\dfrac{1}{k^{\delta/4}}+\dfrac{1}{k^3}\right)x$ integers n in $[1, x]$, for every large k. Let $3\delta+\eta<\dfrac{\varepsilon}{4}$. From (5.1) we get

$$f_k(n)<\left(1+\frac{\varepsilon}{2}\right)f_k(0), \tag{5.6}$$

if $k\geqq c(\varepsilon)$.

We choose $(k=)k_\nu=2^\nu k_0$ $(\nu=0,1,2,\ldots)$. Then

$$f_{k_\nu}(n)<\left(1+\frac{\varepsilon}{2}\right)f_k(0)\quad(\nu=0,1,2,\ldots), \tag{5.7}$$

allowing at most

$$2x\sum_{\nu=1}^{\infty}k_\nu^{-\delta/4}\leqq\frac{cx}{k_0^{\delta/4}}$$

integers n in $[1, x]$. Suppose that (5.7) holds for an n. If $k \geqq k_0$ $k \in [k_\nu, k_{\nu+1})$, then from

$$f_k(n) \leqq f_{k_{\nu+1}}(n) \leqq \left(1+\frac{\varepsilon}{2}\right) f_{k_{\nu+1}}(0) < \left(1+\frac{\varepsilon}{2}\right)\left(1+\frac{\varepsilon}{4}\right) f_k(0),$$

the inequality

$$f_k(n) < (1+\varepsilon) f_k(0)$$

follows for every $k \geqq k_0$, which completes the proof of Theorem 2.

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ON THE STRUCTURE OF SPACES WHICH ARE PARACOMPACT p-SPACES HEREDITARILY

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Introduction

Paracompact p-spaces (or paracompact M-spaces) have been proved to be exactly the regular T_1 spaces which have a perfect map onto a metrizable space (see [1]) or, equivalently, the spaces which are homeomorphic to a closed subspace of the topological product of a metrizable and a compact T_2 space (see [11]). Clearly, subspaces of paracompact p-spaces are not in general paracompact p-spaces. (Take, for example, any T_2 compactification of a non-paracompact, completely regular space.) Let us call a space which is hereditarily a paracompact p-space an F_{pp}-space. (This terminology is adapted from A. V. ARHANGEL'SKIĬ [3].) The one-point compactification of any uncountable discrete space is an example of a non-metrizable F_{pp}-space.

In the present paper we prove a general decomposition theorem concerning perfect maps. It follows then that every F_{pp}-space is the union of at most ω_1 of its metrizable subspaces. With the aid of this result we are able to give affirmative answers to the following two problems of A. V. ARHANGEL'SKIĬ [3] (Theorems 3.2 and 3.4):

1. Is the Souslin number (the supremum of cardinalities of pairwise disjoint families $+\omega$) of an F_{pp}-space equal to its weight?

2. Does every F_{pp}-space have a dense metrizable subspace?[1]

Earlier these problems were affirmatively answered under GCH and CH, respectively. (See [7], [9], [4] and [5].)

Concerning our terminology and notation we remark that $\varkappa$ always denotes an infinite cardinal. Cardinals are identified with initial ordinals. Given a topological space (X, τ) (or briefly X) and a subset Y of X, $\tau|Y$ denotes the subspace topology on Y relative to τ, $\mathrm{cl}_\tau Y$ (or briefly $\mathrm{cl}\, Y$) denotes the closure of Y in (X, τ). The notation concerning perfect maps and relations is explaned in §1.

§ 1. Preliminaries

We shall use the following notation.

Let (X, τ) be a topological space and R an equivalence relation on X. For any point x in X, let $[x]_R$ denote the equivalence class of x. If $A\subset X$ then let $[A]_R = \bigcup\{[x]_R : x\in A\}$. Let τ_R denote the topology on X consisting of those τ-open sub-

[1] By using a method different from ours, an affirmative answer to the second problem has independently been obtained by Arhangel'skiĭ (*Vestnik Moskov. Univ. Ser. I Mat. Meh.* **5** (1977), 30—36), too. The author was revising the present paper when Arhangel'skiĭ's paper appeared.

sets which can be represented as unions of some equivalence classes of R. If R' is an equivalence relation on a non-void subset X' of X then instead of $(\tau|X')_{R'}$ we write briefly $\tau_{R'}$. R is said to be *closed* if for every closed subset A in (X, τ) $[A]_R$ is also a closed subset in (X, τ). Let us say that R is *perfect* if R is closed and all equivalence classes of R are compact subsets of (X, τ). A continuous map $f\colon (X, \tau)\to(Z, \tau_Z)$ is called perfect if it is closed and for each point z in Z, $f^{-1}(z)$ is compact in (X, τ).

Let us now recall some properties of perfect maps and perfect relations. In most of this paper it will be more convenient to use perfect equivalence relations instead of perfect maps; for their equivalency see [6], p. 71.

PROPOSITION 1.1. *Let (X, τ) be a topological space and R an equivalence relation on X. Then the following assertions are equivalent.*

(i) *R is perfect.*

(ii) *Each equivalence class $[x]_R$ of R is compact in τ and the family of all τ_R-open sets containing $[x]_R$ forms a neighbourhood base of $[x]_R$ in τ, too.*

PROOF. See [6], p. 75.

LEMMA 1.2. *Let (X, τ) be a T_2 space, X' and X_i $(i\in I)$ non-void subsets of X such that $X'\subset X_i$ for each i in I. Suppose that R' and R_i $(i\in I)$ are perfect equivalence relations on $(X', \tau|X')$ and $(X_i, \tau|X_i)$ $(i\in I)$, respectively. Define an equivalence relation R on X' by putting*

$$x \sim_R y \text{ iff } x \sim_{R'} y \text{ and } x \sim_{R_i} y \quad \text{for each } i \text{ in } I.$$

Then

(a) *R is a perfect equivalence relation on $(X', \tau|X')$;*

(b) *$\tau_R=\sup\{\tau_{R'}, \tau_{R_i}|X' : i\in I\}$.*

PROOF. Let $\tau_R=\bar\tau$ and $\sup\{\tau_{R'}, \tau_{R_i}|X'\colon i\in I\}=\tau^*$. In order to prove our assertion, we shall apply Proposition 1.1. So let us choose an arbitrary point x in X', and let $\mathfrak{B}'$ and $\mathfrak{B}_i$ denote an open neighbourhood base of $[x]_{R'}$ and $[x]_{R_i}$ in $\tau_{R'}$ and τ_{R_i} (for all i in I), respectively. Then, introducing the notation $\mathfrak{B}_i'=\{V\cap X'\colon V\in\mathfrak{B}_i\}$ we shall prove that

$(*)$ $\mathfrak{B}=\{$finite intersections of members in $\mathfrak{B}'\cup\bigcup_{i\in I}\mathfrak{B}_i'\}$ is a neighbourhood base for $[x]_R=[x]_{R'}\cap\bigcap_{i\in I}[x]_{R_i}$ in $\tau|X'$.

To prove $(*)$, suppose indirectly that there is a $\tau|X'$-open set $U\supset[x]_R$ such that no member of $\mathfrak{B}$ is contained in U, i.e. $\mathfrak{N}=\mathfrak{B}\cup\{X'-U\}$ has the finite intersection property. Since by Proposition 1.1 $\mathfrak{B}'$ is a neighbourhood base for the compact set $[x]_{R'}$ in $\tau|X'$ and $\mathfrak{N}\supset\mathfrak{B}'$ we infer that $M=\bigcap\{\mathrm{cl}_{\tau|X'}N\colon N\in\mathfrak{N}\}\neq\emptyset$. On the other hand, making use of the definition of $\mathfrak{B}$, $M\subset[x]_{R'}\cap\bigcap_{i\in I}[x]_{R_i}\cap(X'-U)=\emptyset$, a contradiction.

Clearly $\mathfrak{B}$ consists of $\bar\tau$-open sets; hence (a) follows from $(*)$ and Proposition 1.1. To prove (b), it is enough to note that obviously $\tau^*\subset\bar\tau$ and $\bar\tau\subset\tau^*$ follows from $(*)$.

§ 2. A decomposition theorem

DEFINITION 2.1. We say that a topological property P is *$\varkappa$-complete* if given any family $\{\tau_i : i\in I\}$ of topologies possessing property P on the same non-void set X, $|I|<\varkappa$ implies that $\sup\{\tau_i : i\in I\}$ also possesses P.

A topological property P is said to be *hereditary* if every subspace of a topological space with property P also possesses P.

PROPOSITION 2.2. *Having a σ-locally finite base is an ω_1-complete and hereditary property.*

PROOF. Obvious.

REMARK. The reader might easily find other ω_1-complete and hereditary properties of interest such as having a development, having a σ-point finite base, having a point-countable base etc.

LEMMA 2.3. *Let P be a $\varkappa$-complete and hereditary topological property. Suppose that (X, τ) is a T_2 space satisfying the following conditions:*

(i) *for every subspace $(Y, \tau|Y)$ of (X, τ) there is a perfect equivalence relation R_Y on $(Y, \tau|Y)$ such that τ_{R_Y} possesses property P;*

(ii) *in addition, R_X is such that for each x in X, $[x]_{R_X}$ contains no strictly decreasing sequence $\{C_\alpha : \alpha<\varkappa\}$ of length $\varkappa$ of its τ-compact subsets.*

Then (X, τ) is the union of at most $\varkappa$ of its subspaces possessing property P.

PROOF. We construct a transfinite sequence $\{(T_\alpha, R_\alpha, \tau_\alpha, \{x(C_\alpha)\}, X_\alpha) : \alpha<\varkappa\}$ as follows.

Let $T_0=X$, $R_0=R_X$ the perfect equivalence relation of (ii), and $\tau_0=\tau_{R_0}$. For every equivalence class C_0 of R_0, let us choose a point $x(C_0)$ arbitrarily, and let $X_0=\{x(C_0) : C_0$ is an equivalence class of $R_0\}$.

Suppose now that $0<\alpha<\varkappa$ and for every $\beta<\alpha$ $T_\beta, R_\beta, \tau_\beta, \{x(C_\beta)\}, X_\beta$ are already defined. Then let

$$T_\alpha = X - \bigcup_{\beta<\alpha} X_\beta .$$

Let R_{T_α} be a perfect equivalence relation on $(T_\alpha, \tau|T_\alpha)$ as required in (i), and let R_α be the equivalence relation on $(T_\alpha, \tau|T_\alpha)$ defined by

$$x \sim_{R_\alpha} y \quad \text{iff} \quad x \sim_{R_{T_\alpha}} y \quad \text{and} \quad x \sim_{R_\beta} y \quad \text{for every} \quad \beta<\alpha.$$

Define τ_α by putting $\tau_\alpha=\tau_{R_\alpha}$.

Finally, let us choose a point $x(C_\alpha)\in C_\alpha$ arbitrarily for every equivalence class C_α of R_α, and let $X_\alpha=\{x(C_\alpha) : C_\alpha$ is an equivalence class of $R_\alpha\}$.

We shall prove our lemma by showing that

(a) R_α is perfect and τ_α possesses property P for each $\alpha<\varkappa$;

(b) $\tau|X_\alpha$ possesses P for each $\alpha<\varkappa$;

(c) $X=\bigcup_{\alpha<\varkappa} X_\alpha$.

We shall verify (a) by transfinite induction. It is clearly satisfied for $\alpha=0$. Suppose now that $0<\alpha<\varkappa$ and that we have proved (a) for all $\beta<\alpha$. By virtue of Lemma 1.2 and by the definition of R_α we infer then that R_α is a perfect equivalence relation and that $\tau_\alpha=\tau_{R_\alpha}=\sup\{\tau_{R_{T_\alpha}}, \tau_\beta|T_\alpha\colon \beta<\alpha\}$. Since P is $\varkappa$-complete and hereditary it follows that τ_α possesses P.

To prove (b), let R'_α be a perfect equivalence relation on $(X_\alpha, \tau|X_\alpha)$ as required in (i), and let $\tau'_\alpha=\tau_{R'_\alpha}$. Denote by R the equivalence relation on X_α defined by

$$x\sim_R y \quad \text{iff} \quad x\sim_{R_\alpha} y \quad \text{and} \quad x\sim_{R'_\alpha} y.$$

By the definition of X_α, R is the relation $x=y$ on X_α; applying Lemma 1.2 we conclude that

$$\tau|X_\alpha=\tau_R=\sup\{\tau'_\alpha, \tau_\alpha|X_\alpha\}$$

has property P, since P is hereditary and $\varkappa$-complete.

Suppose now indirectly that (c) is not true, i.e. there is a point $x\in X-\bigcup_{\alpha<\varkappa} X_\alpha$. Then $\{[x]_{R_\alpha}\colon \alpha<\varkappa\}$ is a strictly decreasing sequence of τ-compact subsets of $[x]_{R_X}$ in contradiction with (ii).

THEOREM 2.4. *Let $\varkappa$ be a regular cardinal, and let P be a $\varkappa$-complete and hereditary topological property. Suppose that (X, τ) is a T_2 space satisfying the following conditions:*

(1) *for every subspace $(Y, \tau|Y)$ of (X, τ), there is a perfect equivalence relation R_Y on $(Y, \tau|Y)$ such that τ_{R_Y} possesses property P;*

(2) *if $(Y, \tau|Y)$ is a subspace with density $<\varkappa$ then there is no strictly decreasing sequence $\{C_\alpha\colon\alpha<\varkappa\}$ of length $\varkappa$ of its compact subsets;*

(3) *for every point x in X and every subset A of X with $x\in\mathrm{cl}_\tau A$, there is a subset $A'\subset A$ with $|A'|<\varkappa$, $x\in\mathrm{cl}_\tau A'$.*

Then (X, τ) is the union of at most $\varkappa$ of its subspaces possessing property P.

PROOF. We shall construct a transfinite sequence

$$\{(T_\alpha, R_\alpha, \tau_\alpha, \{K(C_\alpha)\}, \{X(C_\alpha)\}, K_\alpha, X_\alpha)\colon \alpha<\varkappa\}$$

as follows.

Let $T_0=X$, R_0 be a perfect equivalence relation on (X, τ) as required in (1), and $\tau_0=\tau_{R_0}$. For every equivalence class C_0 of R_0, let us choose as $\{X(C_0)\}$ an arbitrary subset of C_0 consisting of a single point and put $K(C_0)=\emptyset$. Finally, let $K_0=\emptyset$ and $X_0=\bigcup\{X(C_0)\colon C_0$ is an equivalence class of $R_0\}$.

Assume now that $0<\alpha<\varkappa$ and that we have already defined $T_\beta, R_\beta, \tau_\beta$, $\{K(C_\beta)\}$, $\{X(C_\beta)\}$, K_β, X_β for every $\beta<\alpha$. Then let

$$T_\alpha=X-\bigcup_{\beta<\alpha} X_\alpha-\bigcup_{\beta<\alpha} K_\beta.$$

Let R_{T_α} be a perfect equivalence relation on $(T_\alpha, \tau|T_\alpha)$ as required in (1), and let R_α be the equivalence relation on $(T_\alpha, \tau|T_\alpha)$ defined by

$$x\sim_{R_\alpha} y \quad \text{iff} \quad x\sim_{R_{T_\alpha}} y \text{ and } x\sim_{R_\beta} y \quad \text{for every} \quad \beta<\alpha.$$

Let us put $\tau_\alpha=\tau_{R_\alpha}$.

Now, for every equivalence class $C_\alpha=[x]_{R_\alpha}$ of R_α, let

$$K(C_\alpha)=\mathrm{cl}\,\tau\big(\bigcup_{\beta<\alpha} X(C_\beta)\big)\cap C_\alpha,$$

where C_β denotes the equivalence class $[x]_{R_\beta}$ of R_β. For every equivalence class C_α of R_α, $X(C_\alpha)$ is defined by

$$X(C_\alpha)=\begin{cases}\text{an arbitrarily chosen set consisting of a}\\ \text{single point in } C_\alpha-K(C_\alpha), \text{ if } C_\alpha-K(C_\alpha)\neq\emptyset;\\ \emptyset, \text{ otherwise.}\end{cases}$$

Finally, let

$$K_\alpha=\bigcup\{K(C_\alpha)\colon C_\alpha \text{ is an equivalence class of } R_\alpha\}$$

and

$$X_\alpha=\bigcup\{X(C_\alpha)\colon C_\alpha \text{ is an equivalence class of } R_\alpha\}.$$

In order to prove our theorem, it is enough to show that the following assertions are valid:

(a) R_α is perfect and τ_α possesses property P for each $\alpha<\varkappa$;

(b) $\tau|X_\alpha$ possesses P for each $\alpha<\varkappa$;

(c) for each $\alpha<\varkappa$ $(K_\alpha, \tau|K_\alpha)$ is the union of at most $\varkappa$ of its subspaces possessing P;

(d) $X=\bigcup\limits_{\alpha<\varkappa} X_\alpha \cup \bigcup\limits_{\alpha<\varkappa} K_\alpha$.

(a) and (b) can be shown by the same argument as we did in the proof of Lemma 2.3.

We shall obtain (c) by proving that $(K_\alpha, \tau|K_\alpha)$ satisfies the conditions of Lemma 2.3 for each $\alpha<\varkappa$. Only (ii) needs proof. To prove it, let R'_α be a perfect equivalence relation on $(K_\alpha, \tau|K_\alpha)$ as required in (1). Let R be the equivalence relation on $(K_\alpha, \tau|K_\alpha)$ defined by

$$x\sim_R y \quad\text{iff}\quad x\sim_{R_\alpha} y \quad\text{and}\quad x\sim_{R'_\alpha} y.$$

By virtue of Lemma 1.2, and since P is a $\varkappa$-complete and hereditary property, it follows that R is such a relation as required in (i). Moreover, since by the definition of $K(C_\alpha)$ each equivalence class of R is contained in a subspace of (X, τ) with density $<\varkappa$, we have (ii) by (2).

Finally, let us suppose indirectly that (d) is not true, i.e. there is a point

$$x\in X-\bigcup_{\alpha<\varkappa} X_\alpha-\bigcup_{\alpha<\varkappa} K_\alpha.$$

Let C_α denote the equivalence class $[x]_{R_\alpha}$ of R_α. By its definition $X(C_\alpha)$ consists of exactly one point x_α for every $\alpha<\varkappa$. We shall prove that $\{x_\alpha\colon\alpha<\varkappa\}$ is a free sequence in (X, τ). To prove this, let α_0 be an arbitrary ordinal with $0<\alpha_0<\varkappa$. Then

$$\mathrm{cl}_\tau\{x_\alpha\colon \alpha<\alpha_0\}\cap \mathrm{cl}_\tau\{x_\alpha\colon \alpha\geqq\alpha_0\}\subset \mathrm{cl}_\tau\{x_\alpha\colon \alpha<\alpha_0\}\cap(\{x_{\alpha_0}\}\cup C_{\alpha_0+1})=$$

$$\mathrm{cl}_\tau\{x_\alpha\colon \alpha<\alpha_0\}\cap C_{\alpha_0+1}=\mathrm{cl}_\tau\{x_\alpha\colon \alpha<\alpha_0\}\cap(C_{\alpha_0}\cap C_{\alpha_0+1})=$$

$$K(C_{\alpha_0})\cap C_{\alpha_0+1}\subset K_{\alpha_0}\cap T_{\alpha_0+1}=\emptyset \quad \text{q.e.d.}$$

We have proved that $\{x_\alpha\colon\alpha<\varkappa\}$ is a free sequence of length $\varkappa$ in the compact T_2 space $(C_0, \tau|C_0)$. By virtue of (3) and the regularity of $\varkappa$, this is a contradiction.

REMARKS. 1. If $\varkappa=\lambda^+$ with $\lambda\geqq\omega$ then (3) means that (X, τ) has tightness $\leqq\lambda$.

2. (2) and (3) are rather weak additional conditions; it turns out in § 3 that e.g. for $\varkappa=\omega_1$ and $P=$(having a σ-locally finite base), (1) implies both of them.

3. It is enough to require (2) and (3) in the equivalence classes of R_0 only.

§ 3. On two problems of A.V. Arhangel'skiĭ

Let us first recall that the *Souslin number* of a topological space X, denoted $c(X)$, is the smallest infinite cardinal number $\varkappa$ such that every family of pairwise disjoint open subsets of X has cardinality $\leqq\varkappa$. X is called $\varkappa$-Lindelöf if each open cover of X has a subcover of cardinality $\leqq\varkappa$. X is said to have countable *tightness* if for every point $x\in X$ and every subset $A\subset X$ with $x\in\operatorname{cl} A$ there is a countable subset $A'\subset A$ with $x\in\operatorname{cl} A'$. A space X is said to be of pointwise countable type if it can be covered by its compact subsets having countable character. (Evidently, F_{pp}-spaces are such hereditarily.)

THEOREM 3.1. *Every F_{pp}-space X is the union of at most ω_1 of its metrizable subspaces.*

PROOF. We shall show that X satisfies the conditions of Theorem 2.4 with $\varkappa=\omega_1$ and $P=$(having a σ-locally finite base). By Proposition 2.2 P is ω_1-complete and hereditary. Recall that the inverse image of the topology of Z by a perfect map $f:(X^*,\tau^*)\to Z$ coincides with τ_R^*, where R is the equivalence relation $f(x)=$ $=f(y)$ on X^*. Therefore condition (1) follows from the Nagata—Smirnov metrization theorem and the definition of an F_{pp}-space. Since X is hereditarily paracompact we infer that every separable subspace of X is hereditarily Lindelöf, hence (2) is also fulfilled. Since every F_{pp}-space is hereditarily a space of pointwise countable type (and thus a very-k-space) we infer that X has countable tightness (see [3], [2]). Therefore (3) is also satisfied.

REMARK. As we remarked after Proposition 2.2, there are many ω_1-complete and hereditary properties of interest; so Theorem 2.4 might be used to obtain other results of type Theorem 3.1. On the other hand, it seems to be a complicated problem whether every F_{pp}-space is the union of only countably many of its metrizable subspaces.

THEOREM 3.2. *The weight of an F_{pp}-space X is equal to its Souslin number.*

PROOF. Let $c(X)=\varkappa$. If $\varkappa=\omega$ then our theorem is proved in [3] of Arhangel'skiĭ; hence we may assume $\varkappa\geqq\omega_1$. Obviously, X has weight $\geqq\varkappa$. Since X is hereditarily paracompact, $c(X)=\varkappa$ implies that X is hereditarily $\varkappa$-Lindelöf (see e.g. [7]). Since a $\varkappa$-Lindelöf metrizable space has weight $\leqq\varkappa$, we conclude by Theorem 3.1 that X is the union of at most $\omega_1\leqq\varkappa$ of its subspaces each of which has weight $\leqq\varkappa$. Since the weight is additive for p-spaces (see [1]) it follows that X has weight $\leqq\varkappa$.

LEMMA 3.3. *Suppose that a first countable F_{pp}-space X^* is the union of a family $\{A_\alpha:\alpha<\omega_1\}$ of its discrete subspaces such that $\bigcup_{\beta<\alpha} A_\beta$ is closed for each $\alpha<\omega_1$. Then X^* is metrizable.*

PROOF. We shall make use of a well-known result of J. NAGATA [10] that a paracompact p-space is metrizable iff it has a point-countable separating open cover. In order to show that our space X^* has such a cover, let us define for each $\alpha<\omega_1$ a family $\mathfrak{G}_\alpha$ of open subsets of X^* as follows.

For every point x in A_α let $\mathfrak{V}_{x\alpha}=\{V^n_{x\alpha}: n<\omega\}$ be an open neighbourhood base of x in X^* such that $V^n_{x\alpha}\cap A_\alpha=\{x\}$ for every $n<\omega$. Since X^* is hereditarily paracompact we infer that $\mathfrak{V}^n_\alpha=\{V^n_{x\alpha}: x\in A_\alpha\}$ has a σ-point finite open refinement $\mathfrak{G}^n_\alpha$ with $\bigcup\mathfrak{V}^n_\alpha=\bigcup\mathfrak{G}^n_\alpha$. Let $\mathfrak{G}'_\alpha=\bigcup_{n<\omega}\mathfrak{G}^n_\alpha$ and

$$\mathfrak{G}_\alpha=\{G'-\bigcup_{\beta<\alpha}A_\beta: G'\in\mathfrak{G}'_\alpha\}.$$

Finally, let

$$\mathfrak{G}=\bigcup_{\alpha<\omega_1}\mathfrak{G}_\alpha.$$

Clearly $\mathfrak{G}$ is a point-countable open cover of X^*. We shall prove that $\mathfrak{G}$ is also point separating. To prove this, let $x, y\in X^*$, $x\neq y$, and let α be the smallest ordinal with $x\in A'_\alpha$. Then there is a $V^n_{x\alpha}\in\mathfrak{V}_{x\alpha}$ with $y\notin V^n_{x\alpha}$. Since by definition $V^n_{x\alpha}$ is the only member of $\mathfrak{V}^n_\alpha$ which contains x we infer that there is a G' in $\mathfrak{G}^n_\alpha$ with $x\in G'\subset V^n_{x\alpha}$. Let $G=G'-\bigcup_{\beta<\alpha}A_\beta$, then $G\in\mathfrak{G}$, $x\in G$ and $y\notin G$.

THEOREM 3.4. *Every F_{pp}-space has a dense metrizable subspace.*

PROOF. Since by Bing's metrization theorem every metrizable space has a dense subspace which is the union of countably many discrete subspaces we infer by Theorem 3.1 that every F_{pp}-space has a dense subspace X which is the union of a family $\{D_\alpha: \alpha<\omega_1\}$ of its discrete subspaces. By the main result in M. ISMAIL [8] every space which is hereditarily of pointwise countable type (a fortiori every F_{pp}-space) has an open dense first countable subspace. Hence X may be assumed to be first countable. To prove our theorem, it is enough to show that such an F_{pp}-space X has a dense metrizable subspace. To show this ,let us define $\{A_\alpha: \alpha<\omega_1\}$ by

$$A_\alpha=D_\alpha-\operatorname{cl}\big(\bigcup_{\beta<\alpha}A_\beta\big),$$

and let $X^*=\bigcup_{\alpha<\omega_1}A_\alpha$. Clearly, X^* satisfies the conditions of Lemma 3.3; hence X^* is metrizable. We shall also prove that X^* is dense in X. To prove this, suppose indirectly that there is a point $x\in X-\operatorname{cl}X^*$, and let $\alpha<\omega_1$ be an ordinal with $x\in D_\alpha$. Then, by virtue of the definition of A_α, $x\notin\operatorname{cl}X^*$ and $x\in D_\alpha$ implies $x\in A_\alpha\subset X^*$, a contradiction.

REMARK. As we have already indicated Theorem 3.2 and Theorem 3.4 give affirmative answers to Problem 2 and Problem 4 in ARHANGEL'SKIĬ [3], respectively.

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ÜBER NICHT-HOLONOME ŌTSUKISCHE RÄUME

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§ 1. Einleitung

Die Theorie der nicht-holonomen Räume entwickelte sich aus der Theorie der Riemannschen Räume und kann im wesentlichen als eine Verallgemeinerung der Theorie der Unterräume betrachtet werden. Im folgenden wollen wir solche — *Bahnkurven* oder kurz *Bahnen* genannte — Kurven $x^i = x^i(s)$ eines n-dimensionalen Punktraumes V_n^0 bestimmen, in dem eine Ōtsukische Übertragungstheorie [2] existiert, die Übertragungsparameter $''\Gamma_j{}^i{}_k$ sind aus einem metrischen Grundtensor bestimmt, und die Kurven $x^i(s)$ $(i=1, 2, \dots, n)$ — der Parameter „s" bedeutet jetzt und im folgenden immer die Bogenlänge — genügen dem System der folgenden Differentialgleichungen (vgl. die Theorie der nicht-holonomen Räume in [1], [3] und in erster Reihe in [4]):

$$A_{(\varrho)i}(x)\frac{dx^i}{ds} = 0 \quad (\varrho = 1, 2, \dots, m;\ i = 1, 2, \dots, n), \tag{1.1}$$

$$g_{ik}\frac{Dx'^i}{ds} = \sum_{\varrho=1}^{m} \pi_{(\varrho)}(x, x')A_{(\varrho)k}(x), \quad x'^i \overset{\text{def}}{=} \frac{dx^i}{ds}, \tag{1.2}$$

wo die $A_{\varrho(k)}$ m kovariante Vektorfelder des n-dimensionalen Raumes, die $\pi_{(\varrho)}(x, x')$ aber m Skalare bedeuten. Bezüglich der Bezeichnungen wollen wir festlegen, daß die *griechischen Indizes* immer die Zahlen $1, 2, \dots, m$, *die lateinischen Indizes* aber die Zahlen $1, 2, \dots, n$ bedeuten werden, und ferner die *Einsteinsche Summationskonvention* bezüglich der doppelt vorkommenden Indizes nur *auf die lateinischen Indizes bedingt wird.* Selbstverständich gilt immer: $m<n$. Die griechischen Indizes bestimmen keinen tensoriellen Charakter, die lateinischen Indizes bezeichnen aber immer die Komponenten der Tensoren.

Bilden die Vektoren $A_{(\varrho)i}$ die Normalenvektoren eines $(n-m)$-dimensionalen Unterraumes $\mathfrak{U}_{(n-m)}$, so drückt (1.1) aus, daß die Kurve $x^i(s)$ im Unterraum $\mathfrak{U}_{(n-m)}$ liegt, und nach (1.2) ist ihr Hauptnormalenvektor Dx'^i/ds ein Vektor des normalen Unterraumes $\mathfrak{U}_m^\perp$ von $\mathfrak{U}_{(n-m)}$.

Die Systeme (1.1) und (1.2) der Differentialgleichungen können in einem System zweiter Ordnung (vgl. Gleichung (3.9) und das Korollar von Satz 2) vereinigt werden, wo dann nur noch die Anfangsbedingungen $A_{(\varrho)i}(x(s_0))x'^i(s_0)=0$ befriedigt werden müssen.

Von den Vektoren wollen wir im folgenden noch annehmen, daß sie paarweise aufeinander senkrecht stehen, d. h. es gilt:

$$A_{(\mu)t} A_{(\nu)}{}^t \equiv g^{it} A_{(\mu)i} A_{(\nu)t} = \delta_{\mu\nu}, \tag{1.3}$$

wo (hier und im folgenden) $\delta_{\mu\nu}$ bzw. δ_μ^ν und auch $\delta^{\mu\nu}$ alle das Kronecker-δ bedeuten werden, obwohl die griechischen Indizes keinen tensoriellen Charakter bestimmen. $\delta_{\mu\nu}$ bedeutet hiernach $1/2m(m+1)$ Konstanten.

§ 2. Grundformeln der Ōtsukischen Räume

In diesem Paragraphen stellen wir kurz diejenigen Formeln der Ōtsukischen Räume [2] zusammen, die wir im folgenden benützen werden. Wir beschränken uns auf den lokalen Teil der Theorie der Ōtsukischen Übertragung (vgl. in erster Reihe § 3 und § 4 von [2]), doch wollen wir noch annehmen, daß im Raum ein metrischer Tensor $g_{ij}(x)$ existiert, von dem die affinen Übertragungsparameter mit Hilfe des Grundtensors $P_j^i(x)$, der für die Ōtsukische Übertragung kennzeichnend ist, bestimmt werden.

Es seien also $''\Gamma_i{}^j{}_k(x)$ die aus g_{ij} bestimmten Übertragungsparameter (Christoffelsche Symbole zweiter Art), d. h.

$$''\Gamma_i{}^j{}_k \overset{\text{def}}{=} \frac{1}{2} g^{jr}(\partial_k g_{ir} + \partial_i g_{rk} - \partial_r g_{ik}), \tag{2.1}$$

wo g^{jr} — wie gewöhnlich — den inversen Tensor von g_{ij} bedeutet. Bedeutet nun Q_t^i den inversen Tensor von P_j^i, d. h. gelten neben $\mathrm{Det}\,(P_j^i) \neq 0$ die Relationen:

(2.2a) $\quad P_r^i Q_t^r = \delta_t^i, \qquad$ (2.2b) $\quad P_i^t Q_t^r = \delta_i^r$

(δ_k^i bezeichnet das Kronecker-δ des n-dimensionalen Raumes), so sind neben $''\Gamma_i{}^j{}_k$ durch die Formeln

$$\partial_k P_j^i + ''\Gamma_r{}^i{}_k P_j^r - P_r^i\, '\Gamma_j{}^r{}_k = 0 \tag{2.3}$$

(vgl. [2], (3.13)) auch die Übertragungsparameter $'\Gamma_j{}^r{}_k$ eindeutig festgelegt, wie das durch eine Kontraktion mit Q_i^h nach (2.2b) leicht bestätigt werden kann.

In den Ōtsukischen Räumen existieren nun die folgenden kovarianten bzw. invarianten Ableitungen:

$$V^{i_1\dots i_p}_{j_1\dots j_q|k} \overset{\text{def}}{=} \partial_k V^{i_1\dots i_p}_{j_1\dots j_q} + \sum_{s=1}^{p} {}'\Gamma_h{}^{i_s}{}_k V^{i_1\dots i_{s-1} h i_{s+1}\dots i_p}_{j_1 j_2 \dots\dots\dots j_q} - \sum_{s=1}^{q} {}''\Gamma_{j_s}{}^h{}_k V^{i_1 i_2 \dots\dots\dots i_p}_{j_1\dots j_{s-1} h j_{s+1}\dots j_q}, \tag{2.4}$$

$$\nabla_k V^{i_1\dots i_p}_{j_1\dots j_q} \overset{\text{def}}{=} P^{i_1}_{r_1}\dots P^{i_p}_{r_p} V^{r_1\dots r_p}_{s_1\dots s_q|k} P^{s_1}_{j_1}\dots P^{s_q}_{j_q}, \tag{2.5}$$

$$\frac{\bar{D}}{dt} V^{i_1\dots i_p}_{j_1\dots j_q} \overset{\text{def}}{=} V^{i_1\dots i_p}_{j_1\dots j_q|k} \frac{dx^k}{dt} \tag{2.6}$$

und nach (2.4), (2.5) und (2.6):

$$(2.7) \qquad \frac{D}{dt} V^{i_1 \dots i_p}_{j_1 \dots j_q} \overset{\text{def}}{=\!=} (\nabla_k V^{i_1 \dots i_p}_{j_1 \dots j_q}) \frac{dx^k}{dt} \equiv P^{i_1}_{r_1} \dots P^{i_p}_{r_p} \left(\frac{\bar{D}}{dt} V^{r_1 \dots r_p}_{s_1 \dots s_q} \right) P^{s_1}_{j_1} \dots P^{s_q}_{j_q} \equiv$$

$$\equiv P^{i_1}_{r_1} \dots P^{i_p}_{r_p} \left\{ \frac{d}{dt} V^{r_1 \dots r_p}_{s_1 \dots s_q} + \left(\sum_{t=1}^{p} {}'\Gamma_h{}^{r_t}{}_k V^{r_1 \dots r_{t-1} h r_{t+1} \dots r_p}_{s_1 s_2 \; \cdot \; \cdot \; \cdot \; s_q} - \right.\right.$$

$$\left.\left. - \sum_{t=1}^{q} {}''\Gamma_{s_t}{}^h{}_k V^{r_1 r_2 \; \cdot \; \cdot \; \cdot \; r_p}_{s_1 \dots s_{t-1} h s_{t+1} \dots s_q} \right) \frac{dx^k}{dt} \right\} P^{s_1}_{j_1} \dots P^{s_q}_{j_q}.$$

Im wesentlichen definiert (2.7) die Ōtsukische Übertragung, in der es charakteristisch ist, daß die kovarianten Ableitungen der kontra- bzw. kovarianten Tensoren mit verschiedenen Übertragungsparametern gebildet und die Tensoren P^i_j vorhanden sind. Ist $P^i_j = \delta^i_j$, so geht die Ōtsukische Theorie in die gewöhnliche Theorie der affinzusammenhängenden Punkträume über, wie das auf Grund von (2.3)—(2.7) unmittelbar bestätigt werden kann.

Wir wollen im folgenden immer die Bogenlänge

$$(2.8) \qquad s = \int_{t_0}^{t} \sqrt{g_{ij}(x(t)) \dot{x}^i \dot{x}^j} \, dt, \quad \dot{x}^i \overset{\text{def}}{=\!=} \frac{dx^i}{dt}$$

als Parameter benützen, und — wie üblich — die Ableitung nach s, d. h. die Operation d/ds durch einen Strich bezeichnen; nur die Ōtsukischen Übertragungsparameter $'\Gamma$ und $''\Gamma$ bilden eine Ausnahme von dieser Vereinbarung. Diese Parameter sind im allgemeinen von dem affinen Parameter von Ōtsuki (vgl. [2], Formel (4.7)) verschieden, es gilt aber der

SATZ 1. *Dann und nur dann, wenn längs einer, mit dem Parameter s angegebenen Extremalkurve $x^i(s)$ von (2.8) die Relation*

$$(2.9) \qquad {}'\Gamma_r{}^i{}_k x'^r x'^k = {}''\Gamma_r{}^i{}_k x'^r x'^k + \psi(s) x'^i, \quad x'^j \overset{\text{def}}{=\!=} \frac{dx^j}{ds}$$

besteht, wo $\psi(s)$ einen Skalar bedeutet, ist $x^i(s)$ gleichzeitig eine affine Bahn des Ōtsukischen Raumes, und nur im Falle $\psi(s) \equiv 0$ ist die Bogenlänge gleichzeitig ein affiner Parameter im Ōtsukischen Sinn.

BEWEIS. Bekanntlich sind die Extremalkurven des durch (2.8) bestimmten Riemannschen Raumes nach (2.1) durch

$$(2.10) \qquad \frac{d^2 x^i}{ds^2} + {}''\Gamma_j{}^i{}_k(x) x'^j x'^k = 0, \quad x'^j \overset{\text{def}}{=\!=} \frac{dx^j}{ds}$$

festgelegt. Die Gleichung einer Bahn des Ōtsukischen Raumes ist nach (4.4) von [2]:

$$\frac{D}{ds}\left(\frac{dx^i}{ds}\right) = \psi(s) P^i_j \frac{dx^j}{ds},$$

wo aber jetzt selbstverständlich s nicht den affinen Parameter, sondern die Bogenlänge als Parameter bedeutet. Da die Gleichung der Bahnen in der Form

$$(2.11)\qquad Q^i_j \frac{D}{ds}\left(\frac{dx^j}{ds}\right) \equiv \frac{d^2x^i}{ds^2} + {'\Gamma_j}^i{}_k(x)x'^j x'^k = \psi(s)\frac{dx^i}{ds}$$

geschrieben werden kann, folgt aus (2.10) und (2.11), daß diese Gleichungen dann und nur dann übereinstimmen, wenn (2.9) besteht. Im Falle $\psi(s)=0$ ist s nach den genannten Formeln offenbar ein affiner Parameter, w. z. b. w.

BEMERKUNG. ${''\Gamma_i}^j{}_k$ ist nach (2.1) in den unteren Indizes immer symmetrisch, hingegen gilt das für ${'\Gamma_i}^j{}_k$ im allgemeinen nicht. In (2.9) und (2.11) kommt aber nur der in (i, k) symmetrische Teil von ${'\Gamma_i}^j{}_k$ vor.

Die Relation (2.9) ist für $P^i_r=\delta^i_r$ mit $\psi\equiv 0$ gültig; das gilt aber auch für $P^i_r=\lambda\delta^i_r$, falls $\lambda=Konst.$ besteht. In diesem Falle ist nämlich nach (2.2a): $Q^i_r=\lambda^{-1}\delta^i_r$ und nach (2.3) wird ${'\Gamma_i}^j{}_k(x)\equiv{''\Gamma_i}^j{}_k(x)$.

§ 3. Bahnen der allgemeinen nicht-holonomen Ōtsukischen Übertragung

Die Bahnen einer nicht-holonomen Ōtsukischen Übertragung sind durch (1.1) und (1.2) festgelegt. Mit einer Methode, die in den nicht-holonomen Theorien im allgemeinen üblich ist, bestimmen wir die $\pi_{(\varrho)}$ $(\varrho=1, \dots, m)$ in (1.2) ausgedrückt mit den $A_{(\varrho)i}$ bzw. mit den $DA_{(\varrho)i}$. Ziehen wir in (1.2) den Index k hinauf, so wird nach einer Kontraktion mit Q^j_k und nach Vertauschungen der Summationsindizes im Hinblick auf (2.2b) und (2.7):

$$(3.1)\qquad \frac{\bar{D}x'^j}{dt} = \sum_{\nu=1}^{m} \pi_{(\nu)}(x, x')Q^j_t A_{(\nu)}{}^t.$$

Eine Kontraktion dieser Gleichung mit $A_{\mu(j)}$ ergibt

$$(3.2)\qquad A_{(\mu)j}\frac{\bar{D}x'^j}{ds} = \sum_{\nu=1}^{m} \pi_{(\nu)}\lambda_{\mu\nu};$$

$$(3.3)\qquad \lambda_{\mu\nu} \overset{\text{def}}{=\!=} A_{(\mu)j}Q^j_t A_{(\nu)}{}^t.$$

Wir stellen nun die *Forderung,* daß *die* $\lambda_{\mu\nu}$ *eine inverse Größe haben,* d. h.

$$(3.4)\qquad \sum_{\mu} \lambda_{\nu\mu}\lambda^{\varrho\mu} \equiv \sum_{\mu} \lambda_{\mu\nu}\lambda^{\mu\varrho} = \delta^{\varrho}_{\nu}$$

nach $\lambda^{\mu\varrho}$ eindeutig lösbar ist. Die Formel (3.2) ergibt nun nach (3.4):

$$(3.5)\qquad \pi_{(\varrho)}(x, x') = \sum_{\mu=1}^{m} \lambda^{\mu\varrho}A_{(\mu)j}\frac{\bar{D}x'^j}{ds}.$$

BEMERKUNG. Da die $\pi_{(\varrho)}$, $\lambda^{\mu\varrho}$, $\lambda_{\mu\nu}$ skalare Funktionen bedeuten, ist es offenbar gleichgültig, ob diese Indizes oben, oder unten stehen.

Wir formen jetzt (1.1) um. Die Ableitung nach dem Parameter s ergibt:

$$\frac{dA_{(\varrho)i}}{ds}\frac{dx^i}{ds}+A_{(\varrho)i}\frac{d^2x^i}{ds^2}=0.$$

Auf Grund der Definition der Operation $\bar{D}/ds$ (vgl. unsere Formeln (2.6) und (2.4)) kann diese Formel in der Form:

$$\frac{\bar{D}A_{(\mu)k}}{ds}\frac{dx^k}{ds}+A_{(\mu)k}\frac{\bar{D}x'^k}{ds}-({'\Gamma}_j{}^t{}_k-{''\Gamma}_j{}^t{}_k)A_{(\mu)t}\frac{dx^j}{ds}\frac{dx^k}{ds}=0 \tag{3.6}$$

geschrieben werden. Beachten wir nun, daß nach (2.4), (2.6) und (2.7)

$$\frac{D\delta^a_b}{ds}=P^a_hP^r_b({'\Gamma}_r{}^h{}_k-{''\Gamma}_r{}^h{}_k)\frac{dx^k}{ds} \tag{3.7}$$

besteht, so wird nach einer Kontraktion mit $Q^t_aQ^b_j$ aus (3.7)

$$({'\Gamma}_j{}^t{}_k-{''\Gamma}_j{}^t{}_k)\frac{dx^k}{ds}=Q^t_aQ^b_j\frac{D\delta^a_b}{ds}. \tag{3.7a}$$

Substituieren wir das in (3.6), so wird:

$$\frac{\bar{D}A_{(\mu)j}}{ds}\frac{dx^j}{ds}+A_{(\mu)j}\frac{\bar{D}x'^j}{ds}-Q^t_aQ^b_j\frac{D\delta^a_b}{ds}A_{(\mu)t}\frac{dx^j}{ds}=0. \tag{3.8}$$

Mit Hilfe von (3.8) kann jetzt aus (3.5) die Größe $A_{\mu(j)}\bar{D}x'^j$ durch

$$A_{(\mu)j}\frac{\bar{D}x'^j}{ds}=-Q^b_j\frac{DA_{(\mu)b}}{ds}\frac{dx^j}{ds}+Q^t_aQ^b_j\frac{D\delta^a_b}{ds}A_{(\mu)t}\frac{dx^j}{ds}$$

ausgedrückt werden. Substituieren wir die in dieser Weise erhaltenen Größen $\pi_{(\varrho)}$ von (3.5) in (1.2), ziehen wir noch in (1.2) den Index k herauf, so wird (nach einigen Vertauschungen der Indizes):

$$\frac{Dx'^i}{ds}+\sum_{\mu,\varrho}\lambda^{\mu\varrho}Q^t_j\frac{dx^j}{ds}A_{(\varrho)}{}^i\left(\frac{DA_{(\mu)t}}{ds}-Q^k_r\frac{D\delta^r_t}{ds}A_{(\mu)k}\right)=0. \tag{3.9}$$

Die Gleichung (3.9) *bestimmt die nicht-holonomen Bahnkurven des Raumes.*

Wir beweisen den

SATZ 2. *Es gilt für die Lösungskurven* $x^i(s)$ *von* (3.9):

$$A_{(\varrho)i}(x(s))\frac{dx^i}{ds}=konst. \tag{3.10}$$

BEWEIS. Wir nehmen an, daß $x^i(s)$ eine Lösungskurve des Differentialgleichungssystems (3.9) ist. Es gilt ferner die Formel:

$$\frac{d}{ds}(A_{(\sigma)i}x'^i)\equiv Q^h_i\left(\frac{DA_{(\sigma)h}}{ds}\frac{dx^i}{ds}+A_{(\sigma)h}\frac{Dx'^i}{ds}\right)-Q^j_rQ^t_i\frac{D\delta^r_t}{ds}\frac{dx^i}{ds}A_{(\sigma)j}, \tag{3.11}$$

wie das auf Grund der Definitionsformel (2.7) des invarianten Differentials — in diesem Falle auf Vektoren angewendet — leicht bestätigt werden kann. Substituieren wir nun in die Formel (3.11) den Wert von Dx'^i aus (3.9), und beachten dann, daß nach (3.3) und (3.4):

$$\sum_{\varrho} Q_i^h A_{(\sigma)h} A_{(\varrho)}{}^i \lambda^{\mu\varrho} = \sum_{\varrho} \lambda_{\sigma\varrho} \lambda^{\mu\varrho} = \delta_\sigma^\mu$$

besteht, so folgt aus (3.11):

$$\frac{d}{ds}(A_{(\sigma)i} x'^i) = 0;$$

das beweist eben (3.10).

Aus dem Satz 2 folgt das

KOROLLAR ZU SATZ 2. *Gelten längs einer Lösungskurve $x^i(s)$ von (3.9) die Anfangsbedingungen*

$$\left[A_{(\sigma)i}(x(s)) \frac{dx^i}{ds}\right]_{s=s_0} = 0, \tag{3.12}$$

so ist $A_{(\sigma)i} x'^i = 0$ längs $x^i(s)$ immer gültig, die Kurve $x^i(s)$ genügt also der Bedingung (1.1).

Die Bahnen des nicht-holonomen Ōtsukischen Raumes können also durch (3.9) mit den Anfangsbedingungen (3.12) charakterisiert werden.

Bezüglich der durch (3.3) bestimmten Skalare $\lambda_{\mu\nu}$ beweisen wir den folgenden

SATZ 3. *Ist $P_{ij} \equiv P_i^r g_{rj}$ in (i, j) symmetrisch, so ist auch $\lambda_{\mu\nu}$ in (μ, ν) symmetrisch.*

BEWEIS. Vor allem zeigen wir, daß aus der Symmetrie von P_{ij} in (i, j) auch die von $Q^{ab} \overset{\text{def}}{=} g^{bm} Q_m^a$ in (a, b) folgt. Nach einer Kontraktion von

$$g_{ir} P_j^r = g_{jr} P_i^r \tag{3.13}$$

mit $g^{jm} g^{ik} Q_m^a Q_k^b$ erhält man im Hinblick auf (2.2b):

$$g^{bm} Q_m^a = g^{ak} Q_k^b,$$

womit aus (3.3) unmittelbar folgt:

$$\lambda_{\mu\nu} = A_{(\mu)j} Q_t^j g^{ti} A_{(\nu)i} = A_{(\mu)j} Q_t^i g^{tj} A_{(\nu)i} = \lambda_{\nu\mu}, \tag{3.14}$$

w. z. b. w.

Ist $\lambda_{\mu\nu}$ symmetrisch, so besteht $\lambda_{\mu\nu}$ nur aus $1/2m(m+1)$ Komponenten; im nicht-symmetrischen Fall hat es aber m^2 Komponenten.

§ 4. Die nicht-holonomen Übertragungsparameter

Auf Grund der Formel (3.9) können wir eine nicht-holonome Ōtsukische affine Übertragung ableiten. Wir stellen die folgende

FORDERUNG. *Es seien die nicht-holonomen Ōtsukischen Übertragungsparameter $\tilde{Q}_j{}^i{}_k$ für die Übertragung der kontravarianten Vektoren so bestimmt, daß für x'^i die Parallelverschiebung eben durch* (3.9) *angegeben sei.*

Da die Parallelverschiebung durch $\tilde{D}x'^i=0$ definiert ist, wo

$$\frac{\tilde{D}x'^i}{ds} = P^i_t\left(\frac{d^2x^t}{ds^2}+\tilde{Q}_j{}^t{}_k(x)x'^jx'^k\right) \tag{4.1}$$

bedeutet, müssen wir (3.9) unformen. Eine kurze Rechnung auf Grund von (2.7) und (2.2b) bzw. (3.7) gibt die folgende Identität:

$$\frac{DA_{(\mu)t}}{ds}-Q^k_r\frac{D\delta^r_t}{ds}A_{(\mu)k}\equiv P^m_t(\partial_hA_{(\mu)m}-'\Gamma_m{}^k{}_hA_{(\mu)k})\frac{dx^h}{ds}.$$

Substituieren wir das in (3.9), so wird wieder im Hinblick auf (2.2a):

$$\frac{Dx'^i}{ds}+\sum_{\mu,\varrho}\lambda^{\mu\varrho}A_{(\varrho)}{}^i(\partial_hA_{(\mu)j}-'\Gamma_j{}^k{}_hA_{(\mu)k})\frac{dx^j}{ds}\frac{dx^h}{ds}=0,$$

was noch in der Form:

$$P^i_t\left\{\frac{d^2x^t}{ds^2}+\left('\Gamma_h{}^t{}_j+\sum_{\mu,\varrho}\lambda^{\mu\varrho}Q^t_mA_{(\varrho)}{}^m(\partial_hA_{(\mu)j}-'\Gamma_j{}^k{}_hA_{(\mu)k})\right)\frac{dx^j}{ds}\frac{dx^h}{ds}\right\}=0 \tag{4.2}$$

geschrieben werden kann.

Ein Vergleich von (4.1) und (4.2) zeigt sofort, daß für $\tilde{Q}_j{}^t{}_h$ die Formel

$$\tilde{Q}_j{}^t{}_h\overset{\text{def}}{=}{}'\Gamma_{(h}{}^t{}_{j)}+\sum_{\mu,\varrho}\lambda^{\mu\varrho}Q^t_mA_{(\varrho)}{}^m(\partial_{(h}A_{(\mu)j)}-'\Gamma_{(j}{}^k{}_{h)}A_{(\mu)k}) \tag{4.3}$$

geeignet ist; die Klammern bei den lateinischen Indizes bedeuten nach der Schoutenschen Symbolik den symmetrischen Teil der entsprechenden Größen, da in (4.2), wegen der Symmetrie von $x'^jx'^h$ in (h, j), in dem mit $x'^jx'^h$ multiplizierten Glied offenbar nur die in (h, j) symmetrischen Teile vorhanden sein können.

Ist nun $\tilde{Q}_j{}^t{}_h$ bestimmt, so können die Übertragungsparameter $\tilde{\tilde{Q}}_j{}^t{}_h$ für die Übertragung der kovarianten Vektoren nach der Theorie der Ōtsukischen Übertragung durch die Formel

$$\partial_kP^i_j+\tilde{\tilde{Q}}_r{}^i{}_kP^r_j-P^i_r\tilde{Q}_j{}^r{}_k=0 \tag{4.4}$$

bestimmt werden, was zur Formel (2.3) analog ist.

Aus (4.2) könnten auch andere Übertragungsparameter bestimmt werden, wenn nämlich in (4.3) für $\tilde{Q}_j{}^t{}_h$ nicht der in (h, j) symmetrischen Teil $'\Gamma_{(h}{}^t{}_{j)}$ genommen wäre.

§ 5. Der Fall der Eigenvektoren

Der Begriff des kontravarianten Eigenvektors ist in [2] § 5 angegeben, wir geben aber hier eine im wesentlichen analoge Definition für *kovariante Eigenvektoren.*

DEFINITION. Ein kovarianter Eigenvektor, der längs einer Kurve $x^i(s)$ definiert ist, ist ein Vektor, für das eine Relation von der Form:

$$P^j_t(x)V_j(x)=\tau(x)V_t(x),\quad x^i=x^i(s) \tag{5.1}$$

besteht, wo $\tau(x)$ ein Skalarfeld bezeichnet.

Aus (5.1) bestimmen wir eine wichtige Formel für die kovarianten Eigenvektoren, deren Analogon für den kontravarianten Fall in [2], Formel (5.8) angegeben ist. Bilden wir die Operation $\bar{D}/ds$ von beiden Seiten von (5.1), so wird:

$$\frac{\bar{D}V_j}{ds} P_t^j + V_j \left(\frac{dP_t^j}{ds} + {}''\Gamma_m{}^j{}_k P_t^m - {}''\Gamma_t{}^m{}_k P_m^j \right) \frac{dx^k}{ds} = \frac{d\tau}{ds} V_t + \tau \frac{\bar{D}V_t}{ds}.$$

Das erste Glied ist nach (2.7) DV_t/ds; das Glied dP_t^j/ds können wir mit Hilfe von (2.3) eliminieren, wenn (2.3) mit dx^k/ds kontrahiert wird. Kontrahieren wir die erhaltene Gleichung mit P_i^t, so wird im Hinblick auf (5.1):

$$P_i^t \frac{DV_t}{ds} + V_j \frac{D\delta_i^j}{ds} = \tau \left(\frac{DV_i}{ds} + \frac{d\tau}{ds} V_i \right). \tag{5.2}$$

Im folgenden wollen wir den Fall untersuchen, in dem die Vektoren $A_{(\varrho)i}$ *Eigenvektorfelder* sind, d. h.

$$P_t^j(x) A_{(\varrho)j}(x) = \tau_{(\varrho)}(x) A_{(\varrho)t}(x), \quad x^i = x^i(s). \tag{5.3}$$

Statt dieser Bedingung wollen wir eine stärkere Forderung stellen, daß nämlich (5.3) längs jeder Kurve $x^i(s)$ bestehe. Offenbar folgt daraus, daß die Vektoren $A_{(\varrho)i}(x)$ *längs jeder Kurve* Eigenvektoren sind. Es wird somit (5.2) für die $A_{(\varrho)i}$ längs aller Kurven des Raumes bestehen.

Wir stellen also die folgende

FORDERUNG. *Die kovarianten Vektoren $A_{(\mu)i}$ seien Eigenvektorfelder, d. h. es gelten die Relationen:*

$$P_r^j(x) A_{(\mu)j}(x) = \tau_{(\mu)}(x) A_{(\mu)r}(x), \tag{5.4}$$

$$P_i^r \frac{DA_{(\mu)r}}{ds} + A_{(\mu)r} \frac{D\delta_i^r}{ds} = \tau_{(\mu)} \left(\frac{DA_{(\mu)i}}{ds} + \frac{d\tau_{(\mu)}}{ds} A_{(\mu)i} \right). \tag{5.5}$$

(In (5.4) und in (5.5) soll selbstverständlich — wie vereinbart — nach μ nicht summiert werden.)

Vor allem berechnen wir die Größen $\lambda_{\mu\nu}$. Da nach (5.4) und (2.2a)

$$\tau_{(\mu)}^{-1} A_{(\mu)i} = Q_i^r A_{(\mu)r} \tag{5.6}$$

ist, bekommt man aus (3.3) im Hinblick auf die Orthogonalitätsrelation (1.3) die Formel:

$$\lambda_{\mu\nu} = \tau_{(\mu)}^{-1} A_{(\mu)t} A_{(\nu)}{}^t = \tau_{(\mu)}^{-1} \delta_{\mu\nu}. \tag{5.7a}$$

Auf Grund der Definitionsformel (3.4) erhält man für die Skalare $\lambda^{\mu\nu}$ unmittelbar

$$\lambda^{\mu\nu} = \tau_{(\mu)} \delta_{\mu\nu}. \tag{5.7b}$$

Beachten wir jetzt die Formeln (5.6) und (5.7b), so wird aus (3.9):

$$\frac{Dx'^i}{ds} + \sum_{\varrho=1}^{m} \tau_{(\varrho)} Q_j^t \frac{dx^j}{ds} A_{(\varrho)}{}^i \left(\frac{DA_{(\varrho)t}}{ds} - \tau_{(\varrho)}^{-1} A_{(\varrho)r} \frac{D\delta_t^r}{ds} \right) = 0. \tag{5.8}$$

Die Formel (5.8) entspricht im Fall, wo die $A_{(\mu)i}$ Eigenvektoren sind, der Formel (3.9) der Bahnen, doch kann jetzt (5.8) mittels der Formel (5.5) noch weiter vereinfacht werden. Schreiben wir (5.5) in der Form:

$$A_{(\varrho)r}\frac{D\delta^r_t}{ds} = -P^r_t\frac{DA_{(\varrho)r}}{ds}+\tau_{(\varrho)}\left(\frac{DA_{(\varrho)t}}{ds}+\frac{d\tau_{(\varrho)}}{ds}A_{(\varrho)t}\right),$$

so geht (5.8) nach Herunterziehen des Index i in

$$g_{ij}\frac{Dx'^j}{ds}+\sum_{\varrho=1}^{m}\tau_{(\varrho)}Q^t_j x'^j A_{(\varrho)i}\left(\tau^{-1}_{(\varrho)}P^r_t\frac{DA_{(\varrho)r}}{ds}-\frac{d\tau_{(\varrho)}}{ds}A_{(\varrho)t}\right)=0$$

über. Es ist aber nach (5.6) und nach (1.1)

$$Q^t_j A_{(\varrho)t}x'^j = \tau^{-1}_{(\varrho)}A_{(\varrho)j}x'^j = 0,$$

und somit wird im Hinblick auf (2.2a)

$$g_{ij}\frac{Dx'^j}{ds}+\sum_{\varrho=1}^{m}A_{(\varrho)i}\frac{DA_{(\varrho)j}}{ds}x'^j = 0, \tag{5.9}$$

was mit der Gleichung der Bahnen der nicht-holonomen Räume vollständig übereinstimmt. Es gilt daher der

SATZ 4. *Die Gleichung* (3.9) *der Bahnen der nicht-holonomen Ōtsukischen Räume geht im Falle, wo die* $A_{(\varrho)i}$ *Eigenvektoren sind, in* (5.9) *über, und diese Gleichung stimmt formal mit der Gleichung der Bahnen der gewöhnlichen nicht-holonomen Räume überein* (vgl. [1], Formel (2.18), die zwar für Linienelementräume angegeben ist, aber offenbar auch für Punkträume gilt).

Die Übertragungsparameter $\tilde{Q}_j{}^t{}_k$ können aus der Formel (4.3) mittels (5.7b) und (5.6) bestimmt werden. Es wird, wenn die Symmetrie von Q^{tm} in (t, m) bedingt wird:

$$\tilde{Q}_j{}^t{}_h = {}'\Gamma_{(h}{}^t{}_{j)}+\sum_{\varrho=1}^{m}A_{(\varrho)}{}^t(\partial_{(h}A_{(\varrho)j)}-{}'\Gamma_{(j}{}^k{}_{h)}A_{(\varrho)k}). \tag{5.10}$$

Die Übertragungsparameter $\tilde{\tilde{Q}}_j{}^t{}_h$ können wieder mittels (4.4) bestimmt werden.

Wir wollen noch denjenigen Fall untersuchen, der durch

$$P^i_j(x) = \tau(x)\delta^i_j \tag{5.11}$$

gekennzeichnet ist. In diesem Fall sind offenbar alle Vektoren Eigenvektoren, sogar Eigenvektorfelder, wie das nach (5.1) unmittelbar verifiziert werden kann. Die Relation (5.3) gilt jetzt mit der Bedingung, daß $\tau_{(1)}=\tau_{(2)}=\ldots=\tau_{(m)}=\tau(x)$ besteht.

Die Übertragungsparameter ${}'\Gamma_j{}^i{}_k$ und ${}''\Gamma_j{}^i{}_k$ stimmen nach (2.3) nicht überein, aber D/ds und $\bar{D}/ds$ unterscheiden sich nur um einen skalaren Faktor. Ist $V^{\cdots}_{\cdots}$

ein Tensor r-ter Stufe, so ist nach (2.7) und (5.11) $DV^{\dots}_{\dots}=\tau^r \bar{D}V^{\dots}_{\dots}$. Aus der Formel (5.9) wird somit die Gleichung der Bahnen:

$$g_{ij}\frac{\bar{D}x'^j}{ds}+\sum_{\varrho=1}^{m}A_{(\varrho)i}\frac{\bar{D}A_{(\varrho)j}}{ds}x'^j=0. \tag{5.12}$$

Nehmen wir endlich noch an, daß neben (5.4) das Kronecker-δ einer der Ralationen

$$\frac{D\delta^r_t}{ds}=\tau\delta^r_t \tag{5.13}$$

$$\frac{D\delta^r_t}{ds}=\tau P^r_t \tag{5.14}$$

genügt (vgl. [2], (5.9) und (5.10)). Es gilt:

SATZ 5. *Ist eine der Formeln* (5.13) *bzw.* (5.14) *gültig, so vereinfacht sich die Gleichung* (5.8) *der Bahnen auf:*

$$\frac{Dx'^i}{ds}+\sum_{\varrho=1}^{m}\tau_{(\varrho)}Q^t_j\frac{dx^j}{ds}A_{(\varrho)}{}^i\frac{DA_{(\varrho)t}}{ds}=0. \tag{5.15}$$

BEWEIS. Das Glied, welches $D\delta^r_t$ enthält, wird nach (5.13) aus der Gleichung (5.8) herausfallen, da nach (5.6) und (1.1):

$$Q^t_j\frac{dx^j}{ds}A_{(\varrho)r}\frac{D\delta^r_t}{ds}=\tau Q^t_j\frac{dx^j}{ds}A_{(\varrho)t}=\tau\tau^{-1}_{(\varrho)}A_{(\varrho)j}\frac{dx^j}{ds}=0,$$

folglich (5.15) gültig ist.

Aus (5.14) und (2.2a) wird wieder

$$Q^t_j\frac{dx^j}{ds}A_{(\varrho)r}\frac{D\delta^r_t}{ds}=\tau\frac{dx^j}{ds}A_{(\varrho)j}=0,$$

somit geht (5.8) auch in diesem Fall in (5.15) über, w. z. b. w.

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ON AN OPERATIONAL CALCULUS FOR MEROMORPHIC FUNCTIONS

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1. Introduction and notations

In [2] H. A. GINDLER has extended the Dunford—Taylor operational calculus to a certain class of meromorphic functions, and proved a mapping theorem concerning the fine structure of the spectra. The aim of this paper is to continue and complete his investigations. An independent definition of the operator $f(T)$ is given, which is shown to be equivalent to the original definition of $f_b(T)$ for each b in the (nonvoid) resolvent set of the operator T. We complete the proof of the closedness of the operator $f(T)$ and show that the mapping $f \to f(T)$ has, in an appropriate sense, the properties of an operational calculus.

Let C denote the complex plane and $\bar{C}$ its compactification. Let X be a complex Banach space and $B(X)$ the Banach algebra of all bounded linear operators from X into X. Let T denote a closed linear operator with domain $D(T)$ and range $R(T)$ in X, with nonvoid resolvent set $r(T)$ and resolvent operator $R(z, T) = (z-T)^{-1} \in B(X)$ for $z \in r(T)$, and with spectrum $s(T)$. The extended spectrum of T, $s_e(T)$, is defined as $s(T)$ if $T \in B(X)$, and $s(T) \cup \{\infty\}$ if $T \notin B(X)$, and it is a nonvoid compact subset of $\bar{C}$. We will say that the complex function f belongs to the class $M(T)$, or $f \in M(T)$, if

(i) f is meromorphic on $s_e(T)$,

(ii) for each pole $p \in s(T)$ of f, p does not belong to $Ps(T)$, the point spectrum of T.

In particular, we say that $f \in A(T) \subset M(T)$, if f has no poles on $s_e(T)$ or, equivalently, f is holomorphic on $s_e(T)$.

If $f \in M(T)$, then f has (at most) a finite number of poles on $s_e(T)$, say $p_0 = \infty$, $p_1, \dots, p_k$ with orders $n_i \geqq 0$ $(i = 0, 1, \dots, k)$, respectively. Define the polynomial

$$P(z) = \prod_{i=1}^{k} (p_i - z)^{n_i}, \tag{1}$$

put $m = n_1 + \dots + n_k$, $n = m + n_0$, and suppose $b \in r(T)$. Define (with the usual conventions) the function

$$F_b(z) = f(z) P(z) (b-z)^{-n}, \tag{2}$$

then $F_b \in A(T)$. Hence the Dunford—Taylor calculus yields $F_b(T) \in B(X)$. By (ii), the operator $P(T)(b-T)^{-n}$ has the inverse $(b-T)^n P(T)^{-1}$, which is a closed operator. Now GINDLER has given ([2; p. 33]) the following

DEFINITION G. The operator $f_b(T)$ is defined by

$$f_b(T) = F_b(T)(b-T)^n P(T)^{-1}. \tag{3}$$

(Here we have stressed the possible dependence on $b \in r(T)$.) In Section 2 we will give another definition of $f(T)$, which will turn out to be equivalent to $f_b(T)$ for every $b \in r(T)$, thus proving the independence of b. Further, the properties of the mapping $f \to f(T)$ will be investigated.

2. The operational calculus

The following lemma is of purely algebraic character.

LEMMA 1. *Let P and Q be polynomials and let S denote their least common multiple. Assume that T is a linear operator from $D(T) \subset X$ into X and that the null spaces of both $P(T)$ and $Q(T)$ are $\{0\}$. Then*

$$R(P(T)) \cap R(Q(T)) = R(S(T)). \tag{4}$$

PROOF. It is clearly sufficient to show that the left side of (4) is contained in the right one. To avoid trivialities we may assume that P and Q are at least of degree 1 and that they have at least one pair of different factors. Suppose now that they are of degree 1 with roots c and d, respectively, and that

$$x = (c-T)y = (d-T)v.$$

Then $(c-T)y = (c-T)v + (d-c)v$, hence $v = (c-T)w$ for some $w \in D(T)$, and $x = (d-T)(c-T)w$.

Assume now that (4) is true if the given polynomials have degree at most n, and that $x = P(T)y = Q(T)v$, where P and Q have degree at most $n+1$. If they have a common factor, say $c-T$, then

$$(c-T)^{-1}x = P^*(T)y = Q^*(T)v,$$

where P^* and Q^* have degree at most n. By assumption and with obvious notation, then $(c-T)^{-1}x \in R(S^*(T))$, hence $x \in R(S(T))$. If P and Q have no common factors, then we have, say,

$$x = (c-T)P'(T)y = (d-T)Q'(T)v,$$

where P' and Q' have degree at most n. As above, we obtain $Q'(T)v = (c-T)w = z$ and, since c is not a zero of Q', the inductive assumption yields that

$$z = (c-T)Q'(T)v' \quad \text{for some} \quad v' \in D(Q(T)).$$

By a similar reasoning we obtain

$$x = (c-T)(d-T)P'(T)y' = (c-T)(d-T)Q'(T)v',$$

or

$$(c-T)^{-1}(d-T)^{-1}x = P'(T)y' = Q'(T)v',$$

and another application of the inductive hypothesis completes the proof.

Suppose from now on that T is a closed linear operator in X with nonvoid resolvent set and $f \in M(T)$. Using the notations of Section 1, it is well-known that

$$f(z) = H(z) + Q_0(z) + \sum_{i=1}^{k} \sum_{j=1}^{n_i} a_{ij}(p_i - z)^{-j}, \tag{5}$$

where $H(z)$ is holomorphic on $s_e(T)$, and the other members on the right side of (5) are the principal parts of the Laurent expansion of f at the points $p_0=\infty$, $p_1, \dots, p_k$, in this order (cf. [5; p. 146]). Thus Q_0 is a polynomial of degree n_0, $H\in A(T)$, and we can give

DEFINITION 1. The operator $f(T)$ is defined by

$$f(T) = H(T)+Q_0(T)+\sum_{i=1}^{k}\sum_{j=1}^{n_i} a_{ij}(p_i-T)^{-j}. \tag{6}$$

Using Definition G and Lemma 1, we prove now

THEOREM 1. *For every $b\in r(T)$ we have $f_b(T)=f(T)$. Further, $D(f(T))=$ $=D(T^{n_0})\cap R(P(T))$.*

PROOF. With the notations

$$P_i(z) = P(z)(p_i-z)^{-n_i} \quad (i = 1, 2, \dots, k),$$

we obtain from (2) and (5)

$$F_b(z) = H(z)P(z)(b-z)^{-n}+Q_0(z)P(z)(b-z)^{-n}+$$

$$+\sum_{i=1}^{k}\sum_{j=1}^{n_i} a_{ij}(p_i-z)^{n_i-j}P_i(z)(b-z)^{-n}.$$

By the Dunford—Taylor calculus, in particular by [6; 5.6—D]

$$F_b(T) = H(T)P(T)(b-T)^{-n}+Q_0(T)P(T)(b-T)^{-n}+$$

$$+\sum_{i=1}^{k}\sum_{j=1}^{n_i} a_{ij}(p_i-T)^{n_i-j}P_i(T)(b-T)^{-n}.$$

If $x\in D(f_b(T))$, then we obtain from this equality $f_b(T)x=f(T)x$, for every $b\in r(T)$. From (6) we have

$$D(f(T)) = D(T^{n_0})\cap\bigcap_{i=1}^{k} R((p_i-T)^{n_i}),$$

thus Lemma 1 yields the last assertion of the theorem. Since $P(T)^{-1}$ maps every $x\in D(T^{n_0})\cap R(P(T))$ into $D(T^{n_0+m})=D(T^n)$, thus $x\in D(f(T))$ implies $x\in D(f_b(T))$, hence $f_b(T)=f(T)$ for every $b\in r(T)$.

Thus we may and will suppress the index b in $f_b(T)$.

COROLLARY. *If $f\in M(T)$ and $c\neq 0$ is a complex number, then $cf\in M(T)$ and $(cf)(T)=cf(T)$. If, in additon, $g\in M(T)$, then $f+g\in M(T)$ and $f(T)+g(T)\subset$ $\subset(f+g)(T)$. Further,*

$$D(f(T)+g(T)) = D((f+g)(T))\cap D(g(T)).$$

PROOF. The first four statements follow from Definition 1 and the preceding discussion. The last one can be proved as the similar assertion for analytic functions of a spectral operator (cf. [1; p. 2236]).

REMARK. This corollary generalizes [2; Lemma 4], and its proof is made trivial by Theorem 1.

The following lemma will be fundamental in proving the closedness of the operator $f(T)$.

LEMMA 2. *Suppose* $F\in A(T)$, $F(\infty)=c\neq 0$, *n is a positive integer and* $F(T)x\in D(T^n)$, *then* $x\in D(T^n)$.

PROOF. We may and will suppose that $T\notin B(X)$. If $x\in X$ and D is a suitable Cauchy domain with positively oriented boundary $+B(D)$ (cf. [6; pp. 292—293]), then

$$F(T)x = cx+\frac{1}{2\pi i}\int_{+B(D)} F(z)R(z,T)x\,dz. \tag{7}$$

If $b\in r(T)$, then by the resolvent equation

$$R(z,T) = R(b,T)\big(I+(b-z)R(z,T)\big);$$

hence

$$\int_{+B(D)} F(z)R(z,T)x\,dz = R(b,T)\int_{+B(D)} F(z)\big(I+(b-z)R(z,T)\big)x\,dz\in D(T). \tag{8}$$

From this we obtain the statement for $n=1$. Assume now that it is true for $n=r$, and that $F(T)x\in D(T^{r+1})$. Then $x\in D(T^r)$, hence $x=R(b,T)^r y$ for some $y\in X$ and, by (8),

$$\int_{+B(D)} F(z)R(z,T)x\,dz = R(b,T)^r\int_{+B(D)} F(z)R(z,T)y\,dz\in D(T^{r+1}).$$

From (7) we obtain that the statement is true for $n=r+1$, thus the proof is complete.

THEOREM 2. *If* $f\in M(T)$ *and* $b\in r(T)$, *then*

$$f(T) = (b-T)^n P(T)^{-1}F_b(T), \tag{9}$$

hence the operator $f(T)$ *is closed.*

PROOF. It is shown in the proof of [2; Lemma 3] that

$$f(T) = F(T)(b-T)^n P(T)^{-1}\subset (b-T)^n P(T)^{-1}F(T).$$

(In what follows we will always suppress b in $F_b(T)$ if no misunderstanding can arise.) Suppose now $x_n\to x$ and $f(T)x_n\to y$. Then $F(T)x_n\to F(T)x$ and, since $(b-T)^n P(T)^{-1}$ is closed, $y=(b-T)^n P(T)^{-1}F(T)x$. Thus, to prove that $f(T)$ is closed, we shall show that (9) is true, which will also have independent importance.

We have to show that $F(T)x\in D\big(f(T)\big)$ implies $x\in D\big(f(T)\big)$. By Theorem 1, this amounts to proving that $F(T)x\in D(T^{n_0})\cap R\big(P(T)\big)$ implies $x\in D(T^{n_0})\cap R\big(P(T)\big)$. If $n_0>0$ then, by the definition of $F(z)$, $F(\infty)=c\neq 0$, hence Lemma 2 is applicable. Thus, for all possible values of n_0, $F(T)x\in D(T^{n_0})$ implies $x\in D(T^{n_0})$, and it suffices to show that

$$F(T)x\in R\big(P(T)\big) \quad\text{implies}\quad x\in R\big(P(T)\big). \tag{10}$$

For this it is enough to prove that for every root p_i of $P(z)$

(11) $$F(T)x\in R(T-p_i) \quad \text{implies} \quad x\in R(T-p_i).$$

Indeed, if (11) is true, the set $\{q_i;\ i=1,2,\dots,m\}$ consists exactly of the roots p_i according to their multiplicities, and $F(T)x=\prod_{i=1}^{m}(T-q_i)u$ for some $u\in X$, then $x=(T-q_1)w$ for some $w\in X$, by (11). Hence, by [6; 5.6—F],

$$(T-q_1)F(T)w=F(T)x=\prod_{i=1}^{m}(T-q_i)u.$$

Since q_i $(i=1,2,\dots,m)$ is not in $Ps(T)$, we obtain that $F(T)w=\prod_{i=2}^{m}(T-q_i)u$ and, by repeated applications of (11), that $x\in R(P(T))$, i.e. (10) is valid.

If we put $F_i(z)=F(z+p_i)$, then it is easy to see that $F_i\in A(T-p_i)$ and $F_i(T-p_i)=F(T)$. Hence we may suppose $p_i=0$ in (11), more exactly, it suffices to prove that if there exists T^{-1}, $F(0)=c\neq 0$ and $F(T)x\in R(T)$, then $x\in R(T)$.

By [4; III. 6. 15], under these conditions $s_e(T^{-1})=(s_e(T))^{-1}$, where the mapping $z\to z^{-1}$ is considered on $\bar{C}$. In particular, $r(T^{-1})$ is nonvoid, and for the function $G(z)=F(z^{-1})$ we have $G\in A(T^{-1})$ and $G(\infty)=F(0)=c\neq 0$. We show that $G(T^{-1})=F(T)$.

Suppose V is a suitable Cauchy domain containing $s_e(T)$, with positively oriented boundary $+B(V)$. Then $U=V^{-1}$ is also a suitable Cauchy domain containing $s_e(T^{-1})$, and its boundary, $B(U)=\dfrac{1}{+B(V)}$ has positive orientation. Further,

$$R(z,T^{-1})=z^{-1}I-z^{-2}R(z^{-1},T) \quad \text{for} \quad z\neq 0,\quad z\in r(T^{-1}),$$

hence

$$G(T^{-1})=G(\infty)I+\frac{1}{2\pi i}\int_{+B(U)}G(z)R(z,T^{-1})\,dz=$$

$$=\big(G(\infty)+G(0)-G(\infty)\big)I-\frac{1}{2\pi i}\int_{+B(U)}F(z^{-1})z^{-2}R(z^{-1},T)\,dz=$$

$$=F(\infty)I+\frac{1}{2\pi i}\int_{+B(V)}F(w)R(w,T)\,dw=F(T).$$

Now if $F(T)x\in R(T)$, i.e. $G(T^{-1})x\in D(T^{-1})$, then Lemma 2 yields $x\in D(T^{-1})=R(T)$, and the proof is complete.

REMARK. The proof of (9) is missing in the proof of [2; Lemma 3].

THEOREM 3. *If $f,g\in M(T)$, then $fg\in M(T)$ and $f(T)g(T)\subset(fg)(T)$. Moreover,*

$$D\big(f(T)g(T)\big)=D\big((fg)(T)\big)\cap D\big(g(T)\big).$$

PROOF. For the purpose of this proof we will suitably modify the standard notations used so far. It is clear that $fg\in M(T)$, and if $\{p_1,p_2,\dots,p_k\}$ denotes the set of all poles of either f or g on $s(T)$, then this set contains every pole of

fg on $s(T)$. Denoting fg by h, the orders of the poles are $n_i(f), n_i(g), n_i(h) \geqq 0$, respectively, where

$$n_i(h)+d_i = n_i(f)+n_i(g) \quad (d_i \geqq 0;\ i = 1, 2, \dots, k).$$

Similarly, at infinity we have

$$n_0(h)+c = n_0(f)+n_0(g) \quad (c \geqq 0).$$

Denoting the polynomials in Definition G by P_f, P_g, P_h, respectively, and $\prod_{i=1}^{k} (p_i-z)^{d_i}$ by $P(z)$, we have

$$(12) \qquad P_h(z)P(z) = P_f(z)P_g(z),$$

where P is of degree $d=d_1+\dots+d_k \geqq 0$ and, with obvious notations,

$$(13) \qquad m(h)+d = m(f)+m(g).$$

Similarly,

$$(14) \qquad n(h)+q = n(f)+n(g), \quad \text{where} \quad q = d+c \geqq 0.$$

If $b \in r(T)$, and F, G, H denote the respective analytical functions in Definition G (the index b is omitted again), then

$$F(z)G(z) = f(z)P_f(z)(b-z)^{-n(f)}g(z)P_g(z)(b-z)^{-n(g)} =$$
$$= H(z)P(z)(b-z)^{-q},$$

by (12) and (14). Since $d \leqq q$, [6; 5.6—D] yields

$$(15) \qquad F(T)G(T) = P(T)R(b, T)^q H(T).$$

According to Theorem 2,

$$(16) \qquad f(T)g(T) = (b-T)^{n(f)}P_f(T)^{-1}F(T)(b-T)^{n(g)}P_g(T)^{-1}G(T).$$

By [6; 5.6—F], each polynomial $Q(T)$ commutes with $F(T)$, i.e.

$$F(T)Q(T) \subset Q(T)F(T).$$

Hence, by [4; III. 5. 37], if there exists $Q(T)^{-1}$, then

$$F(T)Q(T)^{-1} \subset Q(T)^{-1}F(T).$$

Further, since $b \in r(T)$, we have for each $n=0, 1, 2, \dots$

$$(17) \qquad Q(T)^{-1}(b-T)^n \subset (b-T)^n Q(T)^{-1}.$$

Indeed, if $Q(T)^{-1}(b-T)^n x = y$, then $x = R(b, T)^n Q(T)y = Q(T)R(b, T)^n y$, again by [6; 5.6—F], and hence $(b-T)^n Q(T)^{-1}x = y$. Using these results and (16), we obtain

$$f(T)g(T) \subset (b-T)^{n(f)+n(g)}P_f(T)^{-1}P_g(T)^{-1}F(T)G(T).$$

From (12), (14) and (15) we get

$$f(T)g(T) \subset (b-T)^{n(h)+q}P_h(T)^{-1}P(T)^{-1}P(T)R(b, T)^q H(T).$$

Now $P_h(T)^{-1}R(b,T)^q x=y$ implies $x=(b-T)^q P_h(T)y=P_h(T)(b-T)^q y$, hence $y=R(b,T)^q P_h(T)^{-1}x$, and conversely, thus

$$f(T)g(T)\subset(b-T)^{n(h)+q}R(b,T)^q P_h(T)^{-1}H(T)=(fg)(T).$$

Suppose now that $x\in D(g(T))\cap D(h(T))$, then there exists

$$u=F(T)(b-T)^{n(g)}P_g(T)^{-1}G(T)x.$$

By our previous remarks and (15),

$$u=(b-T)^{n(g)}P_g(T)^{-1}P(T)R(b,T)^q H(T)x. \tag{18}$$

Since $x\in D(h(T))$, there exists $P_h(T)^{-1}H(T)x\in D(T^{n(h)})$, and

$$\begin{aligned}P_h(T)^{-1}R(b,T)^q H(T)x=\\ =R(b,T)^q P_h(T)^{-1}H(T)x\in D(T^{n(h)+q})\subset D(T^{m(f)+m(g)}),\end{aligned} \tag{19}$$

by (13) and (14). For every $y\in D(T^{m(f)+m(g)})$ we have $P(T)P_h(T)y=P_g(T)P_f(T)y$ or, equivalently, $P_g(T)^{-1}P(T)P_h(T)y=P_f(T)y$. Hence if $P_h(T)^{-1}w=$ $=y\in D(T^{m(f)+m(g)})$, then

$$P_g(T)^{-1}P(T)w=P_f(T)P_h(T)^{-1}w.$$

From this, (18) and (19) we obtain

$$\begin{aligned}u&=(b-T)^{n(g)}P_f(T)P_h(T)^{-1}R(b,T)^q H(T)x=\\&=P_f(T)(b-T)^{n(g)}R(b,T)^q P_h(T)^{-1}H(T)x.\end{aligned}$$

But then there exists

$$\begin{aligned}f(T)g(T)x&=(b-T)^{n(f)}P_f(T)^{-1}u=\\&=(b-T)^{n(h)+q}R(b,T)^q P_h(T)^{-1}H(T)x=h(T)x,\end{aligned}$$

and the proof is complete.

COROLLARY. *Suppose $f\in M(T)$, f is not identically 0 on each component of its domain, and that $f(z)=0$ implies $z\notin Ps(T)$. Then $\frac{1}{f}\in M(T)$ and there exists $f(T)^{-1}=\frac{1}{f}(T)$.*

PROOF. Let z_i denote the roots of f on $s_e(T)$. By assumption, there are only a finite number of them ($i=1,2,\dots,k$), with finite orders n_i. Then, clearly, $g=\frac{1}{f}$ is meromorphic on $s_e(T)$, z_i are all the poles of g on $s_e(T)$ with orders n_i, and the finite poles are not in $Ps(T)$. Hence $g\in M(T)$ and, by Theorem 3,

$$f(T)g(T)x=x \quad\text{for}\quad x\in D(g(T)),$$
$$g(T)f(T)x=x \quad\text{for}\quad x\in D(f(T)).$$

Hence $g(T)=f(T)^{-1}$, by the same reasoning as in [1; pp. 2235—2236].

Before we prove the spectral mapping theorem for meromorphic functions, recall that we adopt the usual convention that $f(z)=\infty$ if z is a pole of f.

THEOREM 4. *If $f\in M(T)$, then $s_e(f(T))=f(s_e(T))$.*

PROOF. If $a\in s(T)$ and a is not a pole of f, then $f(a)\in s(f(T))$, by [2; Theorem 5]. If $a\in s_e(T)$ and a is a pole of f, then $f(a)=\infty$. According to a remark of GINDLER [2; pp. 33—34], $f(T)\in B(X)$ if and only if $f\in A(T)$, hence in our case $f(a)=\infty\in s_e(f(T))$. Suppose now that $\infty\in s_e(T)$ and ∞ is not a pole of f, then $T\notin B(X)$ and $f(\infty)=c$ is finite. If there exists $z\in s(T)$ such that $f(z)=c$, then $c\in s(f(T))$, by a previous remark. Supposing the contrary, the function $g(z)=$ $=c-f(z)$ is 0 at ∞ but has no zeros on $s(T)$, and $g\in M(T)$. By Corollary to Theorem 1, $g(T)=c-f(T)$, and $c\in s(f(T))$ if and only if $g(T)^{-1}\notin B(X)$.

Returning to the notations of Section 1, we have

$$G(z) = g(z)P(z)(b-z)^{-m} \quad \text{and} \quad g(T) = G(T)(b-T)^m P(T)^{-1}. \tag{20}$$

Further, G has no zeros on $s(T)$, but $G(\infty)=0$, hence either G has a zero of finite order q at infinity, or it vanishes in a neighbourhood of ∞. In the first case $R(G(T))=D(T^q)\neq X$, by [6; 5.6—H], hence $R(g(T))\subset R(G(T))\neq X$, by (20). In the second case $s(T)$ must be bounded, and [6; 5.7—C] yields that $G(T)x=0$ for some $x\neq 0$. But then $g(T)x=0$, by Theorem 2, hence $g(T)$ has no inverse. Thus we have proved that $f(s_e(T))\subset s_e(f(T))$.

To prove the converse relation, assume that $c\in s_e(f(T))$. If $c=\infty$, then $f(T)\notin B(X)$, hence $f\in M(T)\setminus A(T)$ and there is a pole $p\in s_e(T)$ of f, thus $c\in f(s_e(T))$. If $c\in s(f(T))$ and $f(z)\equiv c$ on some such component K of the domain of f that $s_e(T)\cap K$ is nonvoid, then $c\in f(s_e(T))$ trivially, thus from now on we assume the contrary: $f(z)\not\equiv c$ on each such component. If, in addition, $c\neq f(\infty)$, then [2; Theorem 5] yields that $c\in f(s(T))$, and that the same is true if $c=f(\infty)$ and $T\in B(X)$. Finally, if $c=f(\infty)$ and $T\notin B(X)$, then clearly $c\in f(s_e(T))$, thus $s_e(f(T))\subset f(s_e(T))$.

Our next aim is to supplement [2; Theorem 5]. To avoid any uncertainty of terminology, which occurs in [3; pp. 204—206], we emphasize that a linear operator T will be called bounded, if it is bounded on $D(T)$, and unbounded otherwise. Recall that T has the spectral properties P_1, P_2, P_3 if and only if T is not 1—1, $R(T)$ is not dense in X, T has no bounded inverse, respectively. We note that if $f\in M(T)$ and $f(z)\equiv c$ on some component K of the domain of f such that $s_e(T)\cap K$ is nonvoid, then $c-f(T)$ has properties P_1, P_2 and P_3. The proof requires only a slight change in Remark on p. 206 of [3]. Therefore, investigating the properties of $c-f(T)$, we will assume that $f(z)\not\equiv c$ on each such component. Now we recall the following result of Hille and Phillips in a slightly supplemented form.

THEOREM H. *Suppose $T\notin B(X)$, $f\in A(T)$ and $f(\infty)=c$. If $D(T)$ is nondense in X, then $c-f(T)$ has property P_2, while if T is unbounded, then $c-f(T)$ has P_3. On the other hand, if $c-f(T)$ has*

(i) P_1, *or*

(ii) P_2 *and $D(T)$ is dense in X, or*

(iii) P_3 *and T is bounded,*

then for some $a\in s(T)$, $f(a)=c$, *the operator* $a-T$ *has* (i) P_1, (ii) P_2, (iii) P_3, *respectively.*

PROOF. Essentially all is proved in [3; p. 206], except (iii). To prove this, assume that $b\in r(T)$, the set $\{a_1, a_2, \dots, a_n\}$ consists of all roots of the equation $f(a)=c$ in $s(T)$, according to their respective multiplicities, and $q_k(T)=(a_k-T)R(b, T)$ $(k=1, 2, \dots, n)$. Then $q_k(T)\in B(X)$, and there exists $B\in B(X)$ such that

$$R(b,T)^m \prod_{k=1}^{n} q_k(T) = B(c-f(T)),$$

where m denotes the order of ∞ as a zero of $g(z)=c-f(z)$ (cf. [3; p. 206]). Now if $c-f(T)$ has property P_3, then there is a sequence $\{x_r\}\subset X$ such that $|x_r|=1$ and

$$y_r = R(b, T)^m \prod_{k=1}^{n} q_k(T)x_r \to 0.$$

By assumption, $(b-T)^m$ is bounded. Should each $q_k(T)$ have a bounded inverse, then

$$x_r = q_n(T)^{-1} \dots q_1(T)^{-1}(b-T)^m y_r \to 0,$$

a contradiction. Thus the set $\{a_1, a_2, \dots, a_n\}$ is nonvoid, and for some k the operator $q_k(T)$ has property P_3. By a similar reasoning, then a_k-T also has P_3, and the proof is complete.

Now we will prove

THEOREM 5. *In the first sentence of Theorem* H *replace* $A(T)$ *by* $M(T)$ *and assume that* c *is finite. Under these more general conditions all assertions in Theorem* H *are valid.*

PROOF. Put $g(z)=c-f(z)$, then $g\in M(T)$, $g(\infty)=0$ and $g(T)=c-f(T)$. Further, by Section 1,

$$G(z) = (c-f(z))P(z)(b-z)^{-n}, \tag{21}$$

where the polynomial P and n are determined by the poles of f (or, equivalently, of g), and

$$g(T) = G(T)(b-T)^n P(T)^{-1}. \tag{22}$$

Here $G\in A(T), G(\infty)=0$ and $G(z)\not\equiv 0$ on each component of its domain that has nonvoid intersection with $s_e(T)$, thus Theorem H applies to G. Therefore, if $D(T)$ is nondense in X, then $R(G(T))$, hence $R(c-f(T))$ are also nondense. If T is unbounded, then $G(T)$ has property P_3. Now if $G(T)x=0$ for some $x\neq 0$, then $g(T)x=0$, by Theorem 2. On the other hand, if $G(T)^{-1}$ exists, so does $g(T)^{-1}$, by (22). Further, since $G(T)$ has P_3, $R(G(T))$ cannot be closed, by [4; IV. 5. 2]. Since $P(z)$ is of degree n, $R(g(T))=R(G(T))$, hence $g(T)$ also has P_3, again by [4; IV. 5. 2].

If $a\in s(T)$ then, according to (21), $f(a)=c$ if and only if $G(a)=0$. Hence, to prove the last three statements it suffices to show that under the given conditions $G(T)$ has properties P_1, P_2 or P_3, respectively. Indeed, if $g(T)$ has P_2, then

$R(G(T))=R(g(T))$ implies that $G(T)$ also has P_2. Finally, Theorem 3 yields that $G(T)\supset P(T)R(b, T)^n g(T)$, where $P(T)R(b, T)^n$ belongs to $B(X)$. This proves the similar statements for properties P_1 and P_3, which completes the proof of the theorem.

Before proving the composite function theorem, we note that $g(f(z))$ will also be denoted by $(g\circ f)(z)$ and the domain of a function f by $\operatorname{dom}(f)$. Recall that if f, meromorphic at a point z_0, assumes the value v_0 p-tuply at this point, and the function g, meromorphic at the point v_0, assumes the value w_0 k-tuply at this point, then the function $g\circ f$ assumes the value w_0 kp-tuply at z_0 (cf. [5; p. 151]).

THEOREM 6. *Suppose $f\in M(T)$, $r(f(T))$ is nonvoid, $g\in M(f(T))$, $f(z)$ is not identically equal to q on each component of* $\operatorname{dom}(f)$ *if q is a pole in $s(f(T))$ of g, and $h=g\circ f$. Then $h\in M(T)$ and $h(T)=g(f(T))$.*

PROOF. By the preceding remark, from the assumptions clearly follows that h is meromorphic on $s_e(T)$. Further, we may suppose that $\operatorname{dom}(g)$ contains only those poles of g that are in $s_e(f(T))$, and that f assumes these values or has poles only at points in $s_e(T)$. By [6; pp. 289—293], there are Cauchy domains D and D_1 such that

$$f(s_e(T)) = s_e(f(T)) \subset D \subset \bar{D} \subset \operatorname{dom}(g),$$

$$s_e(T) \subset D_1 \subset \bar{D}_1 \subset f^{-1}(D) \subset \operatorname{dom}(f),$$

hence $f(\bar{D}_1)\subset D$ (here $\bar{D}$ and $f^{-1}(D)$ denote the closure and inverse image of D, respectively).

Note first that $v\in r(f(T))$ implies that the function $r_v(z)=(v-f(z))^{-1}$ belongs to $A(T)$, by Theorem 4. Further, $r_v(z)(v-f(z))\equiv 1$ implies $r_v(T)=R(v, f(T))$ in view of the calculus developed so far. Thus

$$R(v,f(T)) = r_v(\infty)I+\frac{1}{2\pi i}\int_{+B(D_1)}(v-f(z))^{-1}R(z,T)\,dz. \tag{23}$$

Consider now the particular case $g\in A(f(T))$. Then $h\in A(T)$, and

$$g(f(T)) = g(\infty)I+$$

$$+\frac{1}{2\pi i}\int_{+B(D)} g(v)\left[r_v(\infty)I+\frac{1}{2\pi i}\int_{+B(D_1)}(v-f(z))^{-1}R(z,T)\,dz\right]dv =$$

$$= g(\infty)I+\frac{1}{2\pi i}\int_{+B(D)} g(v)(v-f(\infty))^{-1}\,dv\,I+$$

$$+\frac{1}{2\pi i}\int_{+B(D_1)}(h(z)-g(\infty))R(z,T)\,dz,$$

by (23) and Fubini's theorem. Here the second term is $(h(\infty)-g(\infty))I$, while the last term yields $h(T)-h(\infty)I$, by the Dunford—Taylor calculus. Thus $g(f(T))=$ $=h(T)$, hence the statement is proved in this particular case.

Next suppose that the function $g(z)$ is $P(z)=a_0+a_1z+\ldots+a_nz^n$, a polynomial of degree n. If f has the poles p_i with orders n_i $(i=0, 1, \ldots, k)$, then $h=P\circ f$ has the same poles with orders n_in, hence $h\in M(T)$. By our previous results,

$$P(f(T)) = a_0I+a_1f(T)+\ldots+a_nf(T)^n \subset$$
$$\subset (a_0+a_1f+\ldots+a_nf^n)(T) = h(T).$$

Further, with obvious notations in accordance with Section 1,

$$F(z) = f(z)P_f(z)(b-z)^{-n(f)} \quad \text{and} \quad H(z) = h(z)P_f(z)^n(b-z)^{-n(f)n}.$$

Since

$$f(T) = F(T)(b-T)^{n(f)}P_f(T)^{-1},$$

by Theorem 2 we obtain that

$$f(T)^n = F(T)^n((b-T)^{n(f)}P_f(T)^{-1})^n.$$

By (17) we have

$$(24) \qquad ((b-T)^{n(f)}P_f(T)^{-1})^n \subset (b-T)^{n(f)n}P_f(T)^{-n},$$

where both sides are invertible operators. The left side has the inverse $(P_f(T)R(b, T)^{n(f)})^n\in B(X)$, thus it can have no proper invertible extension, and we have equality in (24). But this implies $P(f(T))=(P\circ f)(T)$.

Consider now the general case when $g\in M(f(T))$ and g has the poles z_i of orders $n_i>0$ $(i=1, 2, \ldots, k)$ on $s_e(f(T))$. If f assumes the values z_i at the points p_i^r (the index r runs over the values $1, 2, \ldots, r(i)$; $r(i)$ is finite by assumption) q_i^r-tuply on $s_e(T)$, then p_i^r are all the poles of $h=g\circ f$ on $s_e(T)$ of orders $n_iq_i^r$. Further, if $f(p_i^r)=\infty$, then $p_i^r\notin Ps(T)$, by the assumption $f\in M(T)$. On the other hand, if $f(p_i^r)=z_i\neq\infty$, then $z_i\notin Ps(f(T))$ and [2; Theorem 5] imply that $p_i^r\notin Ps(T)$, hence $h\in M(T)$.

Choose now $b\in r(T)$, $c\in r(f(T))$ and define

$$H_b(z) = h(z)P_h(z)(b-z)^{-n(h)}, \quad G_c(v) = g(v)P_g(v)(c-v)^{-n(g)},$$

in accordance with the notations of Section 1. Then

$$(G_c\circ f)(z)P_h(z)(b-z)^{-n(h)} = H_b(z)(P_g\circ f)(z)(c-f(z))^{-n(g)}.$$

If f has the poles p_i of orders j_i and P_g has degree n, then $P_g\circ f$ has the poles p_i of orders nj_i, while at p_i the function $(c-f(z))^{-n(g)}$ has a zero of order $n(g)j_i\geqq \geqq nj_i$. Hence the function

$$k(z) = (P_g\circ f)(z)(c-f(z))^{-n(g)}$$

belongs to $A(T)$, and so does clearly $(c-f(z))^{-n(g)}$. By our previous results then

$$k(T) = P_g(f(T))R(c, f(T))^{n(g)}.$$

Again by a previous result we obtain $(G_c\circ f)(T)=G_c(f(T))$ and, with the notation $V=f(T)$,

$$G_c(V)P_h(T)R(b, T)^{n(h)} = H_b(T)P_g(V)R(c, V)^{n(g)}.$$

Here, on both sides, the first factors commute with the rests. Suppose now $x \in D(g(V))$, then

$$P_h(T)R(b,T)^{n(h)}G_c(V)(c-V)^{n(g)}P_g(V)^{-1}x = H_b(T)x,$$

hence $g(V)x = h(T)x$, by Theorem 2. The converse relation $h(T) \subset g(V)$ can be proved similarly, thus the proof is complete.

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BEMERKUNG ZU EINEM SATZ VON G. ALEXITS UND A. SHARMA

Von

K. TANDORI (Szeged), Mitglied der Akademie

1. Für ein Funktionensystem $\varphi=\{\varphi_n(x)\}_1^\infty$, $\varphi_n(x)\in L(0,1)$ $(n=1,2,\dots)$ setzen wir

$$L_N(\varphi;x)=\int_0^1\left|\sum_{n=1}^N \varphi_n(x)\varphi_n(t)\right|dt,$$

$$L_N^*(\varphi;x)=\int_0^1 \max_{1\leqq k\leqq N}\left|\sum_{n=1}^k \varphi_n(x)\varphi_n(t)\right|dt$$

$(N=1,2,\dots)$. Es gilt

(1) $$L_N(\varphi;x)\leqq L_N^*(\varphi;x)\quad (x\in(0,1);\ N=1,2,\dots).$$

G. ALEXITS und A. SHARMA [1] haben den folgenden Satz bewiesen.

SATZ A. *Gilt*

(2) $$L_N(\varphi;x)=O(1)\quad (x\in(0,1);\ N=1,2,\dots),$$

dann ist die Reihe

(3) $$\sum_{n=1}^\infty c_n\varphi_n(x)$$

im Falle $c=\{c_n\}_1^\infty\in l^2$ *im Intervall* (0, 1) *fast überall konvergent.*

In der Arbeit [3] haben wir folgendes gezeigt.

SATZ B. *Ist* $c\notin l^2$, *dann gibt es ein im Intervall* (0, 1) *orthonormiertes System* φ *derart, daß* (2) *besteht, und die Reihe* (3) *in* (0, 1) *fast überall divergiert.*

In dieser Note werden wir zuerst die folgende Behauptung zeigen.

SATZ I. *Ist* $c\notin l^2$, *dann gibt es ein Funktionensystem* φ *mit* $\varphi_n(x)\in L^2(0,1)$ $(n=1,2,\dots)$ *derart, daß*

(4) $$L_N^*(\varphi;x)=O(1)\quad (x\in 0,1);\ N=1,2,\dots)$$

besteht, und die Reihe (3) *in einer Menge mit positivem Maß divergiert.*

Ob diese Behauptung auch für ein orthonormiertes System φ und für fast überall Divergenz gilt, ist noch nicht bekannt.

Es soll bemerkt werden, daß im Falle (4) für jede Folge $c \in l^2$

$$\sup_k \left| \sum_{n=1}^{k} c_n \varphi_n(x) \right| \in L^2(0, 1)$$

gilt. Diese Relation folgt aus der folgenden allgemeinen Behauptung.

SATZ II. *Es sei* $\lambda = \{\lambda_n\}_1^\infty$ *eine nichtabnehmende Folge von positiven Zahlen. Gilt* $\varphi_n(x) \in L^2(0, 1)$ $(n = 1, 2, \ldots)$ *und*

$$L_N^*(\varphi; x) = O(\lambda_N) \quad (x \in (0, 1);\ N = 1, 2, \ldots),$$

dann ist im Falle $c \in l^2$

$$\sup_k \frac{1}{\sqrt{\lambda_k}} \left| \sum_{n=1}^{k} c_n \varphi_n(x) \right| \in L^2(0, 1).$$

Diese Behauptung ist eine Verallgemeinerung eines Satzes von L. CSERNYÁK [2] für nicht orthonormierte Funktionensysteme.

2. BEWEIS DES SATZES I. Es sei c eine Folge mit $\sum_{n=1}^{\infty} c_n^2 = \infty$. Dann gibt es eine positive Konstante C und Indizes $(0=)n_0 < \ldots < n_i < \ldots$ mit

$$C(i+1)^{1+\varepsilon} \leqq \sum_{n=n_i+1}^{n_{i+1}} c_n^2 \quad (\varepsilon > 0;\ i = 0, 1, \ldots). \tag{5}$$

Es sei $I_n(i)$ $(n = n_i + 1, \ldots, n_{i+1})$ eine Einteilung des Intervalls $(0, 1)$ in paarweise disjunkte Intervalle mit

$$\operatorname{mes}(I_n(i)) = \frac{c_n^2}{(i+1)^{1+\varepsilon}} \Bigg/ \sum_{m=n_i+1}^{n_{i+1}} \frac{c_m^2}{(i+1)^{1+\varepsilon}} \quad (n = n_i + 1, \ldots, n_{i+1}).$$

Wir setzen

$$\psi_n(x) = \begin{cases} \dfrac{\sqrt{(i+1)^{1+\varepsilon}}}{c_n} & (x \in I_n(i)), \\ 0 \quad \text{sonst} \end{cases} \quad (n = n_i + 1, \ldots, n_{i+1}),$$

Dann ist

$$\sum_{n=n_i+1}^{n_{i+1}} c_n \psi_n(x) = \sqrt{(i+1)^{1+\varepsilon}} \quad (x \in (0, 1);\ i = 0, 1, \ldots). \tag{6}$$

Es sei endlich

$$\varphi_n(x) = \psi_n(x) / \sqrt{(i+1)^{1+\varepsilon}} \quad (n = 1, 2, \ldots);$$

es ist klar, daß $\varphi_n(x) \in L^2(0, 1)$ $(n = 1, 2, \ldots)$.

Es sei i eine nichtnegative Zahl und $x \in (0, 1)$. Dann gibt es einen Index n_0 $(n_i < n_0 \leqq n_{i+1})$ mit $x \in I_{n_0}(i)$ und gilt

$$\int_0^1 \max_{n_i < k \leqq n_{i+1}} \left| \sum_{n=n_i+1}^{k} \varphi_n(x)\varphi_n(t) \right| dt = \frac{1}{(i+1)^{1+\varepsilon}} \int_0^1 \max_{n_i < k \leqq n_{i+1}} \left| \sum_{n=n_i+1}^{k} \psi_n(x)\psi_n(t) \right| dt =$$

$$= \frac{1}{(i+1)^{1+\varepsilon}} \frac{\sqrt{(i+1)^{1+\varepsilon}}}{c_{n_0}} \int_{I_{n_0}(i)} |\psi_{n_0}(t)|\, dt = \frac{1}{c_{n_0}^2} \operatorname{mes}\left(I_{n_0}(i)\right) \leqq 1/C(i+1)^{1+\varepsilon},$$

auf Grund von (5).

Daraus folgt

$$L_N^*(\varphi;\, x) \leqq \sum_{i=0}^{\infty} \int_0^1 \max_{n_i < k \leqq n_{i+1}} \left| \sum_{n=n_i+1}^{k} \varphi_n(x)\varphi_n(t) \right| dt \leqq$$

$$\leqq \frac{1}{C} \sum_{i=0}^{\infty} \frac{1}{(i+1)^{1+\varepsilon}} < \infty \quad (x \in (0, 1);\ N = 1, 2, \ldots).$$

Weiterhin folgt aus (6)

$$\sum_{n=n_i+1}^{n_{i+1}} c_n \varphi_n(x) = 1 \quad (x \in (0, 1);\ i = 0, 1, \ldots),$$

somit divergiert die Reihe (3) in (0, 1) überall

Beweis des Satzes II. Der Beweis geht mit einer bekannten Methode (s. [1], [2]).

Die n-te Partialsumme der Reihe (3) bezeichnen wir mit $s_n(x)$. Es sei N eine natürliche Zahl und $N(x)$ die kleinste natürliche Zahl, für die

$$\frac{s_{N(x)}(x)}{\sqrt{\lambda_{N(x)}}} = \max_{1 \leqq k \leqq N} \frac{s_k(x)}{\sqrt{\lambda_k}}$$

besteht. Es sei $\{\psi_n(x)\}_1^\infty$ ein orthonormiertes System in (0, 1) und sei $g(x) \in L^2(0, 1)$. Dann ist

$$I = \int_0^1 \frac{s_{N(x)}(x)\, g(x)}{\sqrt{\lambda_{N(x)}}}\, dx = \int_0^1 \left(\frac{g(x)}{\sqrt{\lambda_{N(x)}}} \int_0^1 \left(\sum_{n=1}^{N} c_n \psi_n(t) \right) \left(\sum_{n=1}^{N(x)} \varphi_n(x)\psi_n(t) \right) dt \right) dx.$$

Durch Anwendung der Schwarzschen Ungleichung erhalten wir

$$I = \left(\int_0^1 \left(\sum_{n=1}^{N} c_n \psi_n(t) \right)^2 dt \right)^{1/2} \times$$

$$\times \left(\int_0^1 \int_0^1 \int_0^1 \left(\frac{g(x)}{\sqrt{\lambda_{N(x)}}} \frac{g(y)}{\sqrt{\lambda_{N(y)}}} \right) \left(\sum_{n=1}^{N(x)} \varphi_n(x)\psi_n(t) \right) \left(\sum_{n=1}^{N(y)} \varphi_n(y)\psi_n(t) \right) dx\, dy\, dt \right)^{1/2} \leqq$$

$$\leqq \left(\sum_{n=1}^{\infty} c_n^2 \right)^{1/2} \left(\int_0^1 \int_0^1 \frac{g(x)\, g(y)}{\sqrt{\lambda_{N(x)} \lambda_{N(y)}}} \left(\sum_{n=1}^{N(x,y)} \varphi_n(x)\, \varphi_n(y) \right) dx\, dy \right)^{1/2},$$

wobei $N(x, y) = \min(N(x), N(y))$. Daraus folgt

$$I \leqq O(1)\left(\int_0^1 \frac{g^2(x)}{\lambda_{N(x)}} \int_0^1 \left|\sum_{n=1}^{N(x,y)} \varphi_n(x)\varphi_n(y)\right| dx\,dy + \int_0^1 \frac{g^2(y)}{\lambda_{N(y)}} \int_0^1 \left|\sum_{n=1}^{N(x,y)} \varphi_n(x)\varphi_n(y)\right| dx\,dy\right).$$

Auf Grund der Bedingung über $L_N^*(\varphi, x)$ bekommen wir

$$I \leqq O(1)\int_0^1 g^2(x)\,dx,$$

und im Falle

$$g(x) = \frac{s_{N(x)}(x)}{\sqrt{\lambda_{N(x)}}} \Big/ \left\{\int_0^1 \left(\frac{s_{N(x)}(x)}{\sqrt{\lambda_{N(x)}}}\right)^2 dx\right\}^{1/2}$$

ergibt sich

$$I = \left\{\int_0^1 \frac{s_{N(x)}^2(x)}{\sqrt{\lambda_{N(x)}}}\,dx\right\}^{1/2} = O(1).$$

Da $\left\{\frac{s_{N(x)}(x)}{\sqrt{\lambda_{N(x)}}}\right\}_{N=1}^{\infty}$ eine nichtabnehmende Folge ist, erhalten wir

$$\sup_k \frac{s_k(x)}{\sqrt{\lambda_k}} \in L^2(0, 1).$$

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APPROXIMATELY CONTINUOUS FUNCTIONS WHICH ARE CONTINUOUS ALMOST EVERYWHERE

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1. Introduction

In this paper we are concerned with continuing a study, begun in [4], of several subclasses of approximately continuous functions f: $[0, 1] \to R$. In [4] a topology r was generated which was the coarsest topology relative to which all approximately differentiable functions are continuous. It was also shown that the set of r-continuous functions would contain every approximately continuous function which was continuous almost everywhere. Such a function was labeled a.e. continuous. However, it was pointed out that there was a topology, strictly coarser than r, relative to which the a.e. continuous functions become continuous. This coarser topology was defined and labeled the a.e. topology. At that time, it was an open question whether the a.e. topology was the coarsest topology. In this paper that question is answered affirmatively.

After establishing that fact we consider the separation properties of certain subclasses of the a.e. continuous functions. In that part of the work an analogue of the Lusin—Menchoff Theorem is needed and proved. It is used to establish the main separation property of the a.e. topology. Throughout the paper we will adopt the notation and definitions developed in a very interesting paper of LACZKOVICH [3].

2. Preliminary notations, definitions and theorems

All functions will be defined on [0, 1].

A = The class of approximately continuous functions.
D_1 = The class of functions whose points of discontinuity form a set of at most 1 point.
D_f = The class of functions whose points of discontinuity form a finite set.
D_z = The class of functions whose points of discontinuity form a set of measure zero.
AD_j = $A \cap D_j$, $j=1$, f or z.
H^0 = The Euclidean interior of H.
$\overline{H}$ = The Euclidean closure of H.
(In other topologies t, the prefix $t-H^0$ ($t-\overline{H}$) denotes the interior (closure) of H relative to the topology t.)
H^{-1} = The complement of H.
$\lambda(H)$ = The (outer) Lebesgue measure of H.

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$d(x, H)$ = The upper density of H at x, that is

$$d(x, H) = \limsup_{\substack{x\in I \\ \lambda(I)\to 0}} \frac{\lambda(H\cap I)}{\lambda(I)}.$$

The following definitions and theorems can be found in [1], [4].

DEFINITION 2.1. A set U is said to be d-open if U is measurable and $d(x, U^{-1})=0$ for all x in U. That is, U has density 1 at all of its points.

THEOREM 2.1. *The collection of d-open sets forms a topology d which is the coarsest topology making every function from A continuous.*

DEFINITION 2.2. A set U is said to be almost open if U is d-open and $\lambda(U)=\lambda(U^0)$.

THEOREM 2.2. *The collection of almost open sets forms a topology, a.e., relative to which every function in AD_z is continuous.*

DEFINITION 2.3. Let B be the family of all sets which are F_σ and G_δ and d-open. Then let r be the collection of all sets which are the union of some subfamily of B.

THEOREM 2.3. *The collection r forms a topology which is the coarsest topology making each approximately differentiable function continuous.*

THEOREM 2.4. *The a.e. topology is strictly coarser than the r-topology.*

In the next section we will need several trivial facts about the a.e. topology. We present these propositions here:

PROPOSITION 2.1. *Let X be an a.e. closed set. Then* $\lambda(X)=\lambda(\overline{X})$.

PROOF. Consider $X^{-1}=U$. This is an almost open set, so we have $\lambda(U)=\lambda(U^0)$. It is easy to see that $U^0=(\overline{X})^{-1}$. Hence $\lambda(X)=1-\lambda(U)=1-\lambda(U^0)=\lambda(\overline{X})$.

PROPOSITION 2.2. *Let U be almost open and have density* 1 *at a point* x_0. *Then* $U\cup\{x_0\}$ *is almost open.*

PROOF. Obvious.

PROPOSITION 2.3. *Let X be an a.e. closed set. Let* $X\subset Y\subset\overline{X}$. *Then Y is a.e. closed.*

PROOF. Consider $X^{-1}=U$ and U^0. Let x belong to Y^{-1}. Then U has density 1 at x. Since $\lambda(U^0)=\lambda(U)$, by Prop. 2.2, $U^0\cup\{x\}$ is almost open, and $U^0\cup\{x\}\subset Y^{-1}$.

PROPOSITION 2.4. *Let X be a.e. closed and* $\lambda(X)=0$. *Let* $Y\subset X$. *Then Y is a.e. closed.*

PROOF. Noting that $\lambda(X^{-1})=1$ the proof is the same as that in Proposition 2.3.

We are now ready to proceed to the main section of the paper.

3. Main theorems and separation properties

THEOREM 3.1. *The a.e. topology is the coarsest topology making each function in AD_1 continuous.*

PROOF. We have already shown that each function in AD_1 is a.e. continuous since $AD_1 \subset AD_z$. It is easy to see that the correct topology for AD_1 will have as subbasis the following family of sets:

$$Q = \{E\colon E = \{x\colon f(x) > 0\} \text{ for some } f \text{ in } AD_1\}.$$

Therefore, let W be any almost open set and x_0 any fixed point of W. If we show that there is a function f in AD_1 such that $f(x_0) > 0$ and $\{x: f(x) > 0\} \subset W$, then we are finished.

The point x_0 is either in W^0 or in $W \setminus W^0$. By the normality of the Euclidean topology it is clear that we need only consider the case where x_0 belongs to $W \setminus W^0$. We may also assume that $x_0 = 0$. By Definition 2.2 we are guaranteed that W^0 has density 1 at x_0. Therefore, we may select from W^0 a sequence of closed intervals $[a_n, b_n]$ such that $\bigcup_{n=1}^{\infty} [a_n, b_n]$ has density 1 at 0, and $\lim_{n\to+\infty} a_n = 0 = \lim_{n\to+\infty} b_n$. Further, inside each $[a_n, b_n]$ we may select a closed subinterval with $a_n < c_n < d_n < b_n$ and $\bigcup_{n=1}^{\infty} [c_n, d_n]$ having density 1 at 0.

We define a function f as follows:

$$f = \begin{cases} 1 & \text{if } \; x \in \bigcup_{n=1}^{\infty} [c_n, d_n] \cup \{0\}, \\ 0 & \text{if } \; x \in \left(\bigcup_{n=1}^{\infty} (a_n, b_n) \cup \{0\} \right)^{-1}. \end{cases}$$

Next define f to be continuous and linear over $[a_n, c_n]$ and $[d_n, b_n]$ for each n. Then f is continuous on $(0, 1]$ and approximately continuous at 0. Further, $\{x: f(x) > 0\} = \bigcup_{n=1}^{\infty} (a_n, b_n) \cup \{0\} \subset W$, which completes the proof.

COROLLARY 3.1. *The a.e. topology is the coarsest topology making each function in AD_z (AD_f) continuous.*

PROOF. This is obvious from Theorem 2.2 and Theorem 3.1, since $AD_1 \subset AD_f \subset AD_z$.

THEOREM 3.2. *The collection S of functions which are a.e. continuous is precisely AD_z.*

PROOF. We have $AD_z \subset S$. Now let f be an a.e. continuous function. Since the a.e. topology is coarser than the d-topology f is approximately continuous everywhere, and we need only show that f is continuous almost everywhere. Let $r < s$ be any two fixed rationals, and let $U_{r,s} = \{x: r < f(x) < s\}$. The set $U_{r,s}$ is almost open; hence $U_{r,s} = U_{r,s}^0 \cup Z_{r,s}$ where $Z_{r,s}$ has measure zero. Let $Z = \bigcup Z_{r,s}$ the union being taken over all pairs of rational numbers r and s with $r < s$. The set Z has $\lambda(Z)=0$. Let x_0 belong to Z^{-1}. Let $r_0<s_0$ be any two rationals satisfying $r_0<f(x_0)<s_0$. Then $x_0\in U^0_{r_0,s_0}$, and so f is continuous at x_0.

In [3], LACZKOVICH presented a set of three separation properties. We will need a little discussion here to establish a framework in which to examine these properties for AD_1, AD_f and AD_z.

For a class of functions F we say that H is an F-zero set if there is a function f from F with $H=\{x:f(x)=0\}$. For each $X\subset[0,1]$, let $T_F(X)$ be the intersection of all F-zero sets containing X. We say that the class F is an ordinary system if F contains the constants and if for any two f and g in F, $f+g$, fg, and fg^{-1} belong to F (provided $g^{-1}\neq 0$). Laczkovich proves:

If F is an ordinary system and if F has the additional property that $|f|$ belongs to F for every f in F, then $T_F(X)$ is a closure operator for the coarsest topology in which all the elements of F are continuous.

Now it is clear that AD_1 is not an ordinary system, but AD_f and AD_z satisfy the conditions of the above theorem of Laczkovich. Therefore, the topologies generated by T_{AD_f} and T_{AD_z} are the same, namely the a.e. topology. Further, $T_{AD_z}(X)=\text{a.e.}-\bar{X}$.

DEFINITION 3.1. A class F has the first separation property S_1 if for every pair of disjoint F-zero sets X and Y there is a function in F with $f(x)=0$ for x in X and $f(x)=1$ for x in Y.

DEFINITION 3.2. A class F has the second separation property S_2 if for every pair of disjoint G_δ sets X and Y with $T_F(X)=X$ and $T_F(Y)=Y$ there is a function f in F with $f(x)=0$ for x in X and $f(x)=1$ for x in Y.

DEFINITION 3.3. A class F has the third separation property S_3 if for every pair of disjoint G_δ sets X and Y with $T_F(X)\cap T_F(Y)=\varnothing$ there is a function f in F with $f(x)=0$ for x in X and $f(x)=1$ for x in Y.

We now prove:

i) AD_z has S_2,

ii) AD_f has S_1 but not S_2, and

iii) AD_1 does not have S_1.

Whether or not AD_z has S_3 is an open question. To show that AD_z has S_2 we will need an analogue of the Lusin—Menchoff theorem [1].

THEOREM 3.3. *Let X be a closed set and U an almost open set containing X. Then there is a closed set P such that*

$$X\subset \text{a.e.}-P^0\subset P\subset U.$$

PROOF. Let $V=U^0$ and let $Y=X\cap V^{-1}$. This Y is a closed set. Further $\lambda(Y)=0$ because $Y\subset U\setminus U^0$. We note that V has density 1 at every point of Y. Let

$$R_n=\left\{y:\frac{1}{n+1}\leqq \operatorname{dist}(y,Y)\leqq\frac{1}{n}\right\},\quad n=1,2,\ldots.$$

We set $X_n=X\cap R_n$ and $V_n=V\cap R_n$. If for each n we select a closed set $P_n\subset V_n$ with $\lambda(P_n)>\lambda(V_n)-\frac{1}{2^n}$ and set $P=\bigcup_{n=1}^{\infty}P_n\cup Y$, then P will have density 1 at every point of Y. The proof of this fact can be found on page 500 of [1]. We, however, must take some care in our selection of P_n in order to obtain the additional conditions on P.

First we examine the structure of R_n. It is easy to see that each R_n consists of a finite number of closed intervals. However, some of these intervals may be degenerate, that is, consist of a single point. If this happens at a point x, then $\operatorname{dist}(x, Y)=1/(n+1)$, and x will belong to a nondegenerated interval of R_{n+1}. For this reason we consider R_n as only consisting of closed nondegenerate intervals. Next, each X_n is a compact subset of V_n, which is open relative to R_n. We select $P_n \subset V_n$ so that

1) P_n consists of a finite number of closed intervals,
2) P_n has X_n in its interior, relative to R_n, and
3) $\lambda(P_n) > \lambda(V_n) - 2^{-n}$.

Now let x be an endpoint of an interval of R_n and also belong to X. Then both P_n and P_{n+1} will have an interval, with x as endpoint, but from opposite sides. Therefore x will be in the interior of $\left(\bigcup_{n=1}^{\infty} P_n\right) \cup Y = P$. This means that $X \setminus Y \subset P^0$. We have that $\lambda(P^0)=\lambda(P)$, and P has density 1 at every point of Y. Thus by Proposition 2.2, $P^0 \cup Y$ is almost open. Therefore, $X \subset \text{a.e.} - P^0$, which completes the proof.

COROLLARY 3.3. *Let U be an F_σ almost open set. Then U can be expressed as the union of a sequence of closed sets E_n such that $E_n \subset \text{a.e.} - E_{n+1}^0 \subset E_{n+1}$.*

PROOF. (The proof is a paraphrase of one in [4] but is short enough to be given here.)

Since U is an F_σ set it can be expressed as the union of closed sets F_n. By Theorem 3.3 for F_1 there is a closed set P such that $F_1 \subset \text{a.e.} - P^0 \subset P \subset U$. Let $E_1 = F_1$ and $E_2 = P$. Assume that E_{n+1} has been chosen so that

$$E_1 \subset \text{a.e.} - E_2^0 \subset E_2 \subset \ldots \subset \text{a.e.} - E_{n+1}^0 \subset E_{n+1} \subset U.$$

Consider the set $E_{n+1} \cup F_{n+1} = H_{n+1}$. Then H_{n+1} is a closed set. Again an application of Theorem 3.3 shows that there is a closed set P having $H_{n+1} \subset \text{a.e.} - P^0 \subset P \subset U$. Let $E_{n+2} = P$. This completes the proof.

The following lemma, also from [4], is necessary. The proof is not given here.

LEMMA 3.1. *Let X and Y be disjoint, non-empty, closed subsets of* $[0, 1]$. *Then there is a differentiable function g satisfying*

1) $g(x)=1$ *for all x in X,*
2) $g(x)=0$ *for all x in Y,*
3) $0 < g(x) < 1$ *for all x in $(X \cup Y)^{-1}$, and*
4) $g'(x)=0$ *for all x in $X \cup Y$.*

THEOREM 3.4. *Let U be an $F_\sigma - G_\delta$ almost open set and X_0 any closed subset of U. There is a function g satisfying*

1) *g is a.e. continuous,*
2) *g is uppersemicontinuous,*
3) *g is differentiable almost everywhere in* $[0, 1]$,

4) g *is approximately differentiable for all* x *in* $[0, 1]$,
5) $g(x)=1$ *for all* x *in* X_0,
6) $0<g(x)<1$ *for all* x *in* $U\setminus X_0$, *and*
7) $g(x)=0$ *for all* x in U^{-1}.

PROOF. (The construction of g follows the same lines as that of Theorem 3.1 of [4].) The set U^{-1} is an F_σ set. We express U^{-1} as the union of a sequence of closed sets Z_n with $Z_n \subset Z_{n+1}$. The set U is an F_σ almost open set. Using the above corollary we express U as the union of closed sets E_n with $E_1=X_0$ and $E_n \subset$ a.e.$-$ $-E_{n+1}^0$ for all n. For each n, let f_n be a differentiable function satisfying the 4 properties of Lemma 3.1 for $E_n=X$ and $Y=Z_n$. Let g_n be the product of f_1 through f_n. For each x the sequence $g_n(x)$ is a non-increasing sequence of non-negative numbers. Hence the pointwise limit of g_n exists and is uppersemicontinuous. We label it $g(x)$.

As mentioned above, a perusal of the proof of Theorem 3.1 of [4] will show that g has 4), 5), 6) and 7). Also, it will show that g is differentiable, with $g'(x)=0$, for all x in U^{-1}. We note that property 1) will follow from 3) and 4). Therefore, we need only show that g is differentiable almost everywhere in U.

We have expressed U as the union of closed sets E_n with $E_n \subset$ a.e.$-E_{n+1}^0$ for each n. For each n set $E_{n+1}^0 = V_{n+1}$. Then $\lambda(V_{n+1}) = \lambda(\text{a.e.}-E_{n+1}^0) \geqq \lambda(E_n)$. Let $V=\bigcup_{n+1}^{\infty} V_{n+1}$. Then $\lambda(V)=\lambda(U)$. Note that $V_j \subset V_{j+1}$ for all j. We claim that g is differentiable at every point of V. Let j be fixed. Consider the functions g_j and $f_{j+1}, \ldots$. These are all differentiable functions on V_j. We have $f_{j+k}=1$ over V_j, $k=1, 2, \ldots$. Hence $g_k(x)=g_j(x)$ for all $n>j$ and x in V_j. So that $g(x)=g_j(x)$ over V_j. Hence g is differentiable over V.

THEOREM 3.5. *Let X and Y be disjoint G_δ sets having $T_F(X)=X$ and $T_F(Y)=Y$. Then there is a function h such that*

1) $h(x)=1$ *for all* x *in* X,
2) $h(x)=0$ *for all* x *in* Y, *and*
3) $h(x)$ *is a.e. continuous.*

Hence AD_z has property S_2.

PROOF. As mentioned before $T_F(X)=\text{a.e.}-\overline{X}$, so we are dealing with a pair of disjoint G_δ a.e. closed sets. By KURATOWSKI [2], there is a set W, which is both F_σ and G_δ, such that $X\subset W$ and $W\cap Y=\emptyset$. Consider $\overline{X}$ and $\overline{Y}$. By Proposition 2.1, $\lambda(X)=\lambda(\overline{X})$ and $\lambda(Y)=\lambda(\overline{Y})$. Let $X_1=\overline{X}\cap W$ and $Y_1=\overline{Y}\cap W^{-1}$. Then X_1 and Y_1 are disjoint, and both are $F_\sigma-G_\delta$ sets. Since $X\subset X_1\subset \overline{X}$ and $Y\subset Y_1\subset \overline{Y}$, by Proposition 2.3, X_1 and Y_1 are a.e. closed sets. Pick a point u from X_1^{-1} and a point v from Y_1^{-1}. By Theorem 3.4 there are two functions f and g such that

a) f and g are a.e. continuous,
b) $f(u)=1=g(v)$,
c) $f(x)=0$ for all x in X_1,
d) $g(x)=0$ for all x in Y_1,
e) $0<f(x)\leqq 1$ for x in $X_1^{-1}\supset Y_1\supset Y$, and
f) $0<g(x)\leqq 1$ for all x in $Y_1^{-1}\supset X_1\supset X$.

Consider the function $h(x)=\frac{g(x)}{f(x)+g(x)}$. This function h has properties 1), 2) and 3).

COROLLARY 3.5. *The a.e. topology is completely regular.*

PROOF. Obvious from Theorem 3.5.

However, we also have:

THEOREM 3.6. *The a.e. topology is not normal.*

PROOF. Let C be the Cantor set. Let X and Y be two disjoint dense subsets of C. The set C is closed and hence a.e. closed. Further $\lambda(C)=0$. Therefore, Proposition 2.4 guarantees that X and Y are a.e. closed. If there were an a.e. continuous function such that $f(x)=1$ for all x in X and $f(x)=0$ for all x in Y then f could not be Baire 1. However, every a.e. continuous function is approximately continuous and thus must be Baire 1.

We now look at the separation properties of AD_f and AD_1. As mentioned before, the family AD_f forms an ordinary system. Hence Theorem 1.2.3 of Laczkovich shows that AD_f has S_1. Therefore, we need only prove that AD_f does not possess S_2. To this end we must provide an example of two disjoint G_δ a.e. closed sets, X and Y, which cannot be separated by a function from AD_f. We need the following simple lemma.

LEMMA 3.2. *Let f belong to AD_f and α be fixed. Let $X=\{x:f(x)=\alpha\}$. Then $\overline{X}\setminus X$ is contained in the set of discontinuities of f and hence is finite.*

PROOF. Let x_0 belong to $\overline{X}\setminus X$. Then there is a sequence of points of x_n converging to x_0 with x_n in X for all n. Since x_0 does not belong to X, $f(x_0)\neq\alpha$, but $\lim_{n\to+\infty} f(x_n)=\alpha$. Therefore, f is discontinuous at x_0.

THEOREM 3.7. *The class AD_f does not have S_2.*

PROOF. Let C be the Cantor set and $U=C^{-1}$. Let the components of U be arranged in a sequence $I_n=(a_n, b_n)$. Let

$$X=\left\{x_n\colon x_n=a_n+\frac{1}{3}(b_n-a_n),\quad n=1,2,\ldots\right\},$$

and

$$Y=\left\{y_n\colon y_n=b_n-\frac{1}{3}(b_n-a_n),\quad n=1,2,\ldots\right\}.$$

The sets X and Y are disjoint, and both are G_δ sets. Since $\overline{X}=C\cup X$ and $\overline{Y}=C\cup Y$ we have by Proposition 2.4 that X and Y are a.e. closed. If f is any function from AD_f with $f(x)=0$ for all x in X, then Lemma 3.2 says that $C\setminus\{x:f(x)=0\}$ is a finite set. Similarly, if $f(x)=1$ for all x in Y then $C\setminus\{x:f(x)=1\}$ is a finite set. Therefore, no function f from AD_f can have both $f(x)=0$ for all x in X and $f(x)=1$ for all x in Y.

Finally we consider the class AD_1.

THEOREM 3.8. *The class AD_1 does not have S_1.*

PROOF. Let $[a_n, b_n]$ and $[c_n, d_n]$ be two sequences of intervals satisfying:

i) $b_{n+1} < a_n < c_n < d_n < b_n < 1/2$, for all n, and

ii) $\bigcup_{n=1}^{\infty} [c_n, d_n]$ has density 1 at 0.

Let f be defined as follows:

$$f = \begin{cases} 1, & x \in \bigcup_{n=1}^{\infty} [c_n, d_n] \cup \{0\} \cup \left\{\frac{1}{2}\right\} = A, \\ 0, & x \in \bigcup_{n=1}^{\infty} [b_{n+1}, a_n] \cup \{1\} = B. \end{cases}$$

Next define f to be continuous and linear on each component of $(A \cup B)^{-1}$. Then f is continuous on $(0, 1]$ and approximately continuous at $x=0$. Hence f belongs to AD_1. The set $E=\{x: f(x)=0\}=B$. Let $g(x)=f(1-x)$. Then this function is also in AD_1. Let $D=\{x: g(x)=0\}$. The two sets B and D are disjoint AD_1-zero sets. If h is a function such that $h(x)=0$ on B and $h(x)=1$ on D then clearly $h(1)=0$ and $h(0)=1$. However, $h=0$ on $[b_{n+1}, a_n]$ so that h is discontinuous at 0. Similarly, h is discontinuous at 1. Therefore, h cannot belong to AD_1.

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A NOTE ON APPROXIMATELY CONTINUOUS AND A.E. CONTINUOUS FUNCTIONS

By

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Let A denote the class of approximately continuous functions in [0, 1] and let AD_z denote the class of functions which are approximately continuous everywhere and continuous almost everywhere in [0, 1]. In this note we prove that the classes A and AD_z do not have the separation property S_3. (The definition of property S_3 can be found in [1], 1.2 or [3], Definition 3.3.)

Consider first the class A. We shall construct two disjoint G_δ sets, G_1 and G_2 such that $T_A(G_1)\cap T_A(G_2)=\emptyset$ and G_1 and G_2 are not separable by any function from A ($T_A(X)$ is defined as the intersection of the A-zero sets containing X, see [1], 1.1).

Let C denote the Cantor ternary set and let the components of $(0, 1)\setminus C$ be arranged in a sequence (a_n, b_n). We put

$$I_k^n = \left(a_n+\frac{b_n-a_n}{n}\left(1-\frac{1}{4n}\right)\frac{1}{2^k},\ a_n+\frac{b_n-a_n}{n}\cdot\frac{1}{2^k}\right) \quad (n=1,2,\dots;\ k=1,2,\dots)$$

and $H_n=\bigcup_{k=1}^{\infty} I_k^n$ for every $n=1,2,\dots$. Then obviously

$$H_n\subset\left(a_n,\ a_n+\frac{b_n-a_n}{2n}\right). \tag{1}$$

We show that

(2) for every interval I containing either a_n or b_n, we have $\lambda(I\cap H_n)\leqq\frac{1}{n}\lambda(I)$, and

$$\bar{d}(a_n, H_n)>0, \tag{3}$$

where $\bar{d}(a, X)$ denotes the upper density of the set X at the point a. Indeed, if the interval I contains b_n then either $\lambda(I\cap H_n)=0$, or, by (1) we have

$$\lambda(I\cap H_n)/\lambda(I)\leqq\frac{b_n-a_n}{2n}\bigg/(b_n-a_n)\left(1-\frac{1}{2n}\right)\leqq\frac{1}{n}.$$

If $I=(c, d)$ contains a_n and

$$a_n+\frac{(b_n-a_n)}{n}\frac{1}{2^{i+1}}<d\leqq a_n+\frac{(b_n-a_n)}{n}\frac{1}{2^i} \quad (i=1,2,\dots)$$

then $\lambda(I\cap H_n)\leqq\frac{(b_n-a_n)}{n}\frac{1}{4n}\sum_{k=i}^{\infty}\frac{1}{2^k}=\frac{(b_n-a_n)}{4n^2}\frac{1}{2^{i-1}}$ and $\lambda(I)\geqq\frac{(b_n-a_n)}{n}\frac{1}{2^{i+1}}$ whence we get (2). In addition, denoting the interval $\left(a_n, a_n+\frac{(b_n-a_n)}{n}\frac{1}{2^k}\right)$ by J_k we have $d(a_n, H_n)\geqq\limsup_{k\to\infty}\lambda(H_n\cap J_k)/\lambda(J_k)\geqq 1/4n$ which proves (3).

Now we put $G_1=\bigcup_{n=1}^{\infty} H_n$ and $G_2=C\setminus\{a_n\}_{n=1}^{\infty}$, then G_1 and G_2 are disjoint G_δ sets. Next we prove

$$T_A(G_2)=G_2 \tag{4}$$

and

$$T_A(G_1)=\bigcup_{n=1}^{\infty}\overline{H_n}. \tag{5}$$

We remark that the set $T_A(X)$ can be obtained as the d-closure of X, that is

$$T_A(X)=X\cup\{x;\, d(x,X)>0\} \quad \text{holds for any} \quad X\subset[0,1]. \tag{6}$$

(See [1], 2.2.1, p. 408.) Hence (4) follows from (6) and $\lambda(G_2)=0$. Now the definition of H_n, (6) and (3) easily imply $T_A(G_1)\supset\bigcup_{n=1}^{\infty}\overline{H_n}$. In order to prove $T_A(G_1)\subset\bigcup_{n=1}^{\infty}\overline{H_n}$ we have to show that $x\notin\bigcup_{n=1}^{\infty}\overline{H_n}$ implies $d(x, G_1)=0$. This is obvious for $x\notin\overline{G_1}=C\cup\bigcup_{n=1}^{\infty}\overline{H_n}$ so that we can suppose $x\in\overline{G_1}\setminus\bigcup_{n=1}^{\infty}\overline{H_n}=C\setminus\{a_n\}_{n=1}^{\infty}$. Then for every $N\geqq 1$ there exists a $\delta>0$ such that

(i) if $x\notin\{b_n\}_{n=1}^{\infty}$ then $(x-\delta, x+\delta)\cap\bigcup_{n=1}^{N}(a_n, b_n)=\emptyset$;

(ii) if $x=b_k$ then $(x-\delta, x)\cap H_k=\emptyset$ and

$(x, x+\delta)\cap\bigcup_{n=1}^{N}(a_n, b_n)=\emptyset$.

Let I be an arbitrary interval with $x\in I\subset(x-\delta, x+\delta)$. Then $I\cap H_n=\emptyset$ for $n\leqq N$ and $I\cap[a_n, b_n]\neq\emptyset$ implies either $a_n\in I$, or $b_n\in I$. Hence, applying (2) for the intervals $I\cap[a_n, b_n]$ we have

$$\lambda(I\cap G_1)=\sum_{n=1}^{\infty}\lambda(I\cap H_n)=\sum_{n=N+1}^{\infty}\lambda(I\cap H_n)=$$

$$=\sum_{n=N+1}^{\infty}\lambda(I\cap[a_n, b_n]\cap H_n)\leqq\sum_{n=N+1}^{\infty}\frac{1}{N+1}\lambda(I\cap[a_n, b_n])<\frac{1}{N}\lambda(I).$$

Thus we get $d(x, G_1)=0$ which proves (5).

Hence, by (4) and (5) we have $T_A(G_1)\cap T_A(G_2)=\emptyset$. Suppose that there is an $f\in A$ such that $f(x)=0$ for all x in G_1 and $f(x)=1$ for all x in G_2. Then $f(a_n)=0$ holds for every n since f is approximately continuous at a_n and $d(a_n, G_1)>0$ by (3). That is $f(x)=\begin{cases}0 & \text{if } x\in\{a_n\},\\ 1 & \text{if } x\in C\setminus\{a_n\}\end{cases}$. Since the sets $\{a_n\}$ and

$C\setminus\{a_n\}$ are dense in C this implies that the restricted function $f|_C$ has no continuity points which contradicts the fact that f is a Baire 1 function. This contradiction proves that the class A does not have property S_3.

Now we consider the class AD_z. We prove that

$$T_{AD_z}(G_1) = T_A(G_1) \quad \text{and} \quad T_{AD_z}(G_2) = G_2. \tag{7}$$

Since $AD_z \subset A$, G_1 and G_2 can not be separated by any function from AD_z. In addition, (7) and (5) prove $T_{AD_z}(G_1) \cap T_{AD_z}(G_2) = \emptyset$ showing that AD_z does not have property S_3, either. O'Malley proved that $T_{AD_z}(X)$ can be obtained as the closure of X in the a.e. topology (see [3]; the definition of the a.e. topology is given in [2], Theorem 3.11 or [3], Theorem 2.2). $T_A(G_1)$ and G_2 are a.e. closed sets since they are d-closed and satisfy

$$\lambda(\overline{T_A(G_1)}) = \lambda\left(C \cup \bigcup_{n=1}^{\infty} \bar{H}_n\right) = \lambda(T_A(G_1))$$

and $\lambda(\bar{G}_2) = \lambda(C) = 0 = \lambda(G_2)$. Thus we have $T_{AD_z}(G_1) \subset T_A(G_1)$ and $T_{AD_z}(G_2) \subset G_2$. Since the opposite relations are obvious, we get (7) and the proof is complete.

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THE CATEGORY OF UNARY ALGEBRAS, CONTAINING A GIVEN SUBALGEBRA I.

By

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I. Introduction

Let A, B be categories. A functor $\mathbf{F}: A \to B$ which is one-to-one on every $\mathrm{Mor}_A(a, a')$ is called *faithful*. If, in addition, $\mathbf{F}$ is also one-to-one on the class of objects of A we call it an *embedding*. $\mathbf{F}$ is full if every morphism $\delta: \mathbf{F}(a) \to \mathbf{F}(a)(a')$ of B has the form $\delta = \mathbf{F}(\gamma)$ for some $\gamma \cdot a \to a'$. A *full embedding* thus defines an isomorphism between A and a *full subcategory* of B.

A *concrete category* is a pair $(A, \square)$ where A is a category and $\square$ is a faithful functor $\square: A \to SETS$. A category of universal algebras will always be considered as a concrete category, and $\square$ denotes the usual underlying-set functor.

Let $(A_1, \square_1)$, $(A_2, \square_2)$ be concrete categories. A full embedding $\mathbf{F}: A_1 \to A_2$ is called a *strong embedding* if $\mathbf{H} \cdot \square_1 = \square_2 \cdot \mathbf{F}$ for some faithful functor $\mathbf{H}: SETS \to SETS$ ([12]).

A category is *binding* (or *universal*) if every category of universal algebras has a full embedding into A. A concrete category $(A, \square)$ is called *strongly binding* (or *strongly universal*) if every category of universal algebras can be strongly embedded into $(A, \square)$. Hence, every strongly binding category is binding.

$a \in \mathrm{Ob}\, A$ is *rigid*, if $\mathrm{End}\, a = \mathrm{Mor}\,(a, a) = \{1_a\}$. A family B of objects of A is called *mutually rigid if each* $a \in B$ is rigid, and $\mathrm{Mor}\,(a, b) = \emptyset$ whenever $a, b \in B$, $a \neq b$. Note, that if A is binding, then for each cardinal $\varkappa$, there exists a rigid family $B \subset \mathrm{Ob}\, A$ with $\mathrm{card}\, B = \varkappa$.

HEDRLIN and PULTR [11] have shown, using [17], that the *category of graphs and the category of* 2-*unary algebras are binding*. These results have been used by several authors to prove that various categories of universal algebras are binding such as the categories of commutative groupoids (SICHLER [14]), semigroups (SICHLER [15], TRNKOVA [16]), integral domains with unity (FRIED—SICHLER [7]), (0, 1)-lattices (GRÄTZER—SICHLER [8]), certain primitive classes of 2-unary algebras (PULTR—SICHLER [13]).

Later on the following question was proposed by HEDRLIN and MENDELSOHN [10]. Let A be a strongly binding category, and $a \in \mathrm{Ob}\, A$. Let $A(a)$ denote the full subcategory of A, defined by $\mathrm{Ob}\, A(a) = \{b \in \mathrm{Ob}\, A,\ \mathrm{Mono}\,(a, b) \neq \emptyset\}$. Under what conditions on a, will $A(a)$ also be binding (or strongly binding).

In [10], this problem was solved for the category of directed graphs, and later for undirected graphs (BABAI [2], BABAI—NEŠETŘIL [3]). The answer was obtained recently for integral domains (FRIED [5]), and for (0; 1)-lattices (ADAMS—SICHLER [1]). In each case there are only obvious restrictions on a, if any.

The aim of this paper is to study the above question for the category of $\varkappa$-unary algebras $A_\varkappa$ ($\varkappa$ denotes an arbitrary cardinal). It is known that $A_\varkappa$ is strongly binding iff $\varkappa \geqq 2$ (PULTR [12]). Let $\mathfrak{A} \in \mathrm{Ob}\, A_\varkappa$. Let $A_\varkappa(\mathfrak{A})$ denote the category of $\varkappa$-unary algebras having a subalgebra isomorphic to $\mathfrak{A}$. We shall prove the following

THEOREM 1.1. *If* $\varkappa \geqq 2$, card $\mathfrak{A} < 2^{\varkappa\omega}$ *and* $\mathfrak{A}$ *has no one-element subalgebra, then* $A_\varkappa(\mathfrak{A})$ *is strongly binding.*

On the other hand, in a sense this is best possible:

THEOREM 1.2. *Let* $\varkappa \geqq 2$. *Then there exists a* $\varkappa$*-unary algebra* $\mathfrak{A}$, *such that* card $\mathfrak{A} = 2^{\varkappa\omega}$, $\mathfrak{A}$ *has no one-element subalgebra and any* $\varkappa$*-unary algebra* $\mathfrak{C}$, *containing no one-element subalgebra has a homomorphism into* $\mathfrak{A}$. *In particular,* $\mathfrak{A}$ *is not contained in any rigid* $\varkappa$*-unary algebra. Hence* $A_\varkappa(\mathfrak{A})$ *is not binding.*

E. FRIED [6] raised the question whether $A_\varkappa(\mathfrak{A})$ would be binding for rigid $\mathfrak{A}$. We shall prove the following stronger statement:

THEOREM 1.3. *Let* $\varkappa \geqq 2$ *and* $\mathfrak{A}$ *a* $\varkappa$*-unary algebra with* card $\mathfrak{A} > 1$. *The following statements are equivalent:*

(1) $A_\varkappa(\mathfrak{A})$ *is strongly binding.*

(2) $A_\varkappa(\mathfrak{A})$ *is binding.*

(3) *There exists a rigid* $\mathfrak{B} \in A_\varkappa(\mathfrak{A})$.

(4) *There exists a* $\mathfrak{B} \in A_\varkappa(\mathfrak{A})$ *such that* card $\mathfrak{B} > 2^{\varkappa\omega}$ *and* End $\mathfrak{B} = $ Aut $\mathfrak{B}$.

We shall prove further, that in (4) the assumption card $\mathfrak{B} > 2^{\varkappa\omega}$ cannot be omitted, at least for finite $\varkappa$.

As a byproduct of our investigations we shall obtain that, roughly speaking, every small lattice can be embedded into Con $(\mathfrak{A})$, for any sufficiently large $\varkappa$-unary algebra $\mathfrak{A}$. (Corollary 3.5). This in turn implies

COROLLARY 3.6. *Let* $\mathfrak{A}$ *be a* $\varkappa$*-unary algebra, and assume that* card $\mathfrak{A} \geqq 2$. *If* L *denotes the class of all lattices, then* $\mathbf{S}(\mathrm{Con}\,(\mathbf{P}\mathfrak{A})) = L$.

II. Strong embedding into $A_\varkappa(\mathfrak{A})$ for small $\mathfrak{A}$

Let us call an algebra *cyclic* if it is generated by one element.

LEMMA 2.1. *A cyclic* $\varkappa$*-unary algebra has at most* $\varkappa\omega$ *elements. There are at most* $2^{\varkappa\omega}$ *non isomorphic cyclic* $\varkappa$*-unary algebras.*

PROOF. Clear.

LEMMA 2.2. *Let* $2 \leqq \varkappa$. *Then there exists a mutually rigid family of cyclic* $\varkappa$*-unary algebras* $B = \{\mathfrak{B}_\alpha : \alpha < 2^{\varkappa\omega}\}$, *such that if* φ *is a homomorphism of* $\mathfrak{B}_\alpha$ *into an algebra* $\mathfrak{C}$ *having no one-element subalgebra, then* φ *is an injective mapping.*

PROOF. First assume $\varkappa < \omega$. Let $f_1, \dots, f_\varkappa$ be the operations. Let $\mathbf{N} = \{0, 1, 2, \dots\}$ and $\mathbf{M} = \mathbf{N} - \{n! : n \in \mathbf{N}\}$. To each $\alpha < 2^\omega$ there corresponds a function $g_\alpha : \mathbf{M} \to \{3, 4\}$, $g_\alpha = g_\beta$ iff $\alpha = \beta$. Let us define $\mathfrak{B}_\alpha$ by

(i) $\square \mathfrak{B}_\alpha = \mathbf{N}_\alpha = \{n_\alpha : n \in \mathbf{N}\}$;

(ii) $f_k(n) = n + 1$ for each $k \geqq 2$;

(iii) $f_1(0)=1$, $f_1((2n)!)=(2n)!+2$ $(n\geqq 1)$, $f_1((2n+1)!)=(2n+1)!+1$;

(iv) $f_1(m)=m+g_\alpha(m)$ $(m\in\mathbf{M})$.

It is easy to see that $f_1f_1(k)=f_2f_2(k)$ iff $k=0$, hence if $\chi\in\operatorname{End}\mathfrak{B}_\alpha$, then $\chi(0_\alpha)=0_\alpha$, hence because of (ii) $\chi=1_{\mathfrak{B}_\alpha}$. For the same reason, if $\chi\in\operatorname{Mor}(\mathfrak{B}_\alpha,\mathfrak{B}_\beta)$, then $(0_\alpha)=0_\beta$ hence $\chi(k_\alpha)=k_\beta$, but this is impossible because of (iv).

Now assume that θ is a congruence of $\mathfrak{B}_\alpha$, and $\mathfrak{B}_\alpha/\theta$ has no one-element subalgebra. Let $k\equiv k+d$ (θ) $(d>0)$. Then by (ii) $k+l_1d+m\equiv k+l_2d+m$ (θ). Hence for a sufficiently large n, if n is divisible by d then $(2n+1)!\equiv(2n)!$ (θ) hence by (iii) $(2n+1)!+1\equiv(2n)!+2$ (θ) and by (ii) $(2n+1)!+1\equiv(2n)!+1$ (θ) hence $m_1\equiv m_2$ (θ) if $m_1,m_2\geqq(2n)!+1$. But then $\mathfrak{B}_\alpha/\theta$ has a one-element subalgebra, therefore θ is trivial.

Now assume that $\varkappa\geqq\omega$. Let f_α $(1\leqq\alpha\leqq\varkappa)$ be the operations. To each $\beta<2^\varkappa$ we associate a function $g_\beta:\varkappa-\{0\}\to\{1,2\}$. Now we define $\mathfrak{B}_\alpha$ by

(v) $\square\mathfrak{B}_\beta=\varkappa$;

(vi) $f_\alpha(0)=\alpha$;

(vii) $f_\alpha(\gamma)=\gamma$ if $\alpha\neq\gamma$; $f_\alpha(\alpha)=\alpha+g_\beta(\alpha)$.

It is obvious that $B=\{\mathfrak{B}_\alpha:\alpha<2^\varkappa\}$ is rigid. Let θ be a congruence of $\mathfrak{B}_\alpha$. Assume $\gamma_1\equiv\gamma_2$ (θ) for some $\gamma_1\neq\gamma_2$. If $\gamma_1,\gamma_2>0$ then by (vii) $\mathfrak{B}_\alpha/\theta$ has a one element subalgebra. If $\gamma_1=0$, $\gamma_2>0$ then, setting $\gamma_1'=\gamma_2+g_\beta(\gamma_2)$, we have $\gamma_2\equiv\gamma_1'(\theta)$ by (vi), whence the previous case again.

Proof of Theorem 1.1. Let $\mathfrak{A}$ be a $\varkappa$-unary algebra without one-element subalgebras, and $\operatorname{card}\mathfrak{A}<2^{\varkappa\omega}$. Let B be the family of cyclic $\varkappa$-unary algebras constructed in Lemma 2.2. As $\mathfrak{A}$ has less than $2^{\varkappa\omega}$ cyclic subalgebras, B has a subset $C=\{\mathfrak{C}_\alpha:\alpha<2^{\varkappa\omega}\}$ such that $\operatorname{Mor}(\mathfrak{C}_\alpha,\mathfrak{A})=\emptyset$ for each α. We may assume that $\square\mathfrak{A}=\{\beta:\beta<\tau\}$ $(\tau<2^{\varkappa\omega})$. Let us define an algebra $\overline{\mathfrak{A}}=\langle\square\overline{\mathfrak{A}},\bar f_\gamma\rangle$ by

(ix) $\square\overline{\mathfrak{A}}=((\square\mathfrak{A})\times\{0,1\})\cup(\bigcup_{\alpha<\tau}\square\mathfrak{C}_\alpha)$ (we assume this union to be disjoint);

(x) $\bar f_\gamma(a)=f_\gamma(a)$ if $a\in(\square\mathfrak{A})\times\{0\}$ or $a\in\bigcup_{\alpha<\tau}\square\mathfrak{C}_\alpha$;

(xi) $\bar f_\gamma(\alpha,1)=(\alpha,0)$ if $\gamma\geqq 2$, $\bar f_1(\alpha,1)=0_\alpha$ (the element 0 of $\mathfrak{C}_\alpha$).

Hence we obtain a $\varkappa$-unary algebra, and $\overline{\mathfrak{A}}|(\square\mathfrak{A})\times\{0\}\cong\mathfrak{A}$. We assert that $\overline{\mathfrak{A}}$ is rigid. For let $\chi\in\operatorname{End}\mathfrak{A}$. 0_α generates a subalgebra isomorphic to $\mathfrak{C}_\alpha$. Hence $\chi(0_\alpha)$ has the same property, because of Lemma 2.2. ($\overline{\mathfrak{A}}$ has no one-element subalgebra). Hence $\chi(0_\alpha)\notin(\square\mathfrak{A})\times\{0\}$ and $\chi(0_\alpha)\notin\square\mathfrak{C}_\beta$ for any $\beta\neq\alpha$. But $\chi(0_\alpha)\notin(\square\mathfrak{A})\times\{1\}$ for if $\varkappa<\omega$ then $f_1(0_\alpha)=f_2(0_\alpha)$ and for $a\in(\square\mathfrak{A})\times\{1\}$, $\bar f_1(a)\neq\bar f_2(a)$, and if $\varkappa\geqq\omega$ then $f_1f_1(0_\alpha)=f_1(0_\alpha)$ and for $a\in(\square\mathfrak{A})\times\{1\}$, $f_1f_1(0_\alpha)=1_\alpha\neq f_1(0_\alpha)$. Hence $\chi(0_\alpha)\in\square\mathfrak{C}_\alpha$. By the rigidity of $\mathfrak{C}_\alpha$ this implies $\chi|\mathfrak{C}_\alpha=\mathrm{id}_{\mathfrak{C}_\alpha}$ for each $\alpha<\tau$.

Moreover, $f_1(\alpha,1)=0_\alpha$, hence $f_1(\chi(\alpha,1))=0_\alpha$ and therefore $\chi(\alpha,1)=(\alpha,1)$, and $\chi(\alpha,0)=\chi(f_2(\alpha,1))=f_2(\chi(\alpha,1))=(\alpha,0)$, proving that $\overline{\mathfrak{A}}$ is rigid.

We can now apply Theorem 1.3 ((3)→(1)) to obtain the desired strong embedding into $A_\varkappa(\mathfrak{A})$. The proof of Theorem 1.3 is postponed to Section 4.

III. Construction of the counterexample

Let $\mathfrak{A}=\langle\Box\mathfrak{A}, f_\alpha:\alpha<\varkappa\rangle$ be a $\varkappa$-unary algebra, θ a congruence of $\mathfrak{A}$. Let $[a]_\theta$ denote the image of a under the natural homomorphism $\varphi_\theta:\mathfrak{A}\to\mathfrak{A}/\theta$. We shall say that θ *preserves loops* if for any $a\in\Box\mathfrak{A}, \alpha<\varkappa,\ f_\alpha(a)=a$ iff $f_\alpha([a]_\theta)=[a]_\theta$.

LEMMA 3.1. *Any $\varkappa$-unary algebra $\mathfrak{A}$ has a maximal loop preserving congruence θ.*

PROOF. Let $\theta_0\leqq\theta_1\leqq\dots$ be a chain of loop preserving congruences. It is obvious that $\vee\theta_i$ also preserves loops hence by Zorn's lemma we have a maximal such θ.

We shall need some results of combinatorial set theory. Let $P=\{\mathbf{P}_\alpha\}$ be a system of sets. We shall say (cf. [4]) that P is strong Δ-system, if $\mathbf{P}_{\alpha_1}\cap\mathbf{P}_{\alpha_2}=\cap P$ for all $\alpha_1\neq\alpha_2$. The symbol $(\beta,\gamma)\to\mathrm{st}\,\Delta(\delta)$ means, that if we have an arbitrary set system $Q=\{\mathbf{Q}_\alpha\}$ with $\mathrm{card}\,Q=\beta$, $\mathrm{card}\,\mathbf{Q}_\alpha\leqq\gamma$, then it contains a strong Δ-system of cardinality δ. cf β denotes the cofinality of β, β^+ is the successor cardinal of β. Using this notations there holds

LEMMA 3.2 (ERDŐS—RADO [4]). *If $\varkappa\geqq\omega$ then $((2^\varkappa)^+,\varkappa)\to\mathrm{st}\,\Delta((2^\varkappa)^+)$.*

Now let $\mathfrak{A}$ be a universal algebra, $C=\{\mathfrak{C}_\alpha\}$ a system of subalgebras of $\mathfrak{A}$. We shall say that $\mathfrak{C}$ is a *strong algebraic Δ-system* if $\Box C=\{\Box\mathfrak{C}_\alpha\}$ is a strong Δ-system, and $\mathfrak{C}_{\alpha_1}$ and $\mathfrak{C}_{\alpha_2}$ are isomorphic over $\mathbf{X}=\cap\Box C$ for all α_1,α_2 (i.e. there exists an isomorphism $\gamma:\mathfrak{C}_{\alpha_1}\to\mathfrak{C}_{\alpha_2}$ with $\gamma|\mathbf{X}=\mathrm{id}_\mathbf{X}$).

Let $\mathfrak{A}$ be a universal algebra of type $\Omega=\{\varkappa_\alpha:\alpha<\gamma\}$. Let $\sum\Omega=\sum_{\alpha<\gamma}\varkappa_\alpha$. We define the symbol $(\beta,\varkappa)\to\mathrm{sta}\,\Delta(\delta)$ to mean that given an algebra of type Ω and $\varkappa=\sum\Omega$ and $\mathrm{card}\,\mathfrak{A}=\beta$ then $\mathfrak{A}$ has a set of cyclic subalgebras $C=\{\mathfrak{C}_\alpha:\alpha<\delta\}$, which is a strong algebraic Δ-system.

LEMMA 3.3. *Let $\beta=2^{\varkappa\omega}$. Then $(\beta^+,\varkappa)\to\mathrm{sta}\,\Delta(\beta^+)$.*

PROOF. Let $\mathfrak{B}$ be an algebra of type Ω, where $\sum\Omega=\varkappa$ and $\mathrm{card}\,\mathfrak{B}=\beta^+$. Clearly, there is a cyclic algebra $\mathfrak{C}$ of type Ω isomorphic to β^+ different subalgebras $\mathfrak{C}_\gamma(\gamma<\beta^+)$ of $\mathfrak{B}$. Let $\chi_\gamma:\mathfrak{C}_\gamma\to\mathfrak{C}$ be an isomorphism for each $\gamma<\beta^+$. According to Lemma 3.2, there exists a subset $\mathbf{H}$ of β^+ such that $\mathrm{card}\,\mathbf{H}=\beta^+$ and a set $\mathbf{X}\subset\Box\mathfrak{B}$ such that $\Box\mathfrak{C}_{\delta_1}\cap\Box\mathfrak{C}_{\delta_2}=\mathbf{X}$ for all $\delta_1,\delta_2\in\mathbf{H}$, $\delta_1\neq\delta_2$. $\chi_\gamma|\mathbf{X}$ is a mapping from $\mathbf{X}$ into $\mathfrak{C}$. But there are only $2^{\varkappa\omega}$ different mappings from $\mathbf{X}$ into $\mathfrak{C}$, hence $\mathbf{H}$ has a subset $\mathbf{G}$ of power β^+ such that $\chi_{\delta_1}|\mathbf{X}=\chi_{\delta_2}|\mathbf{X}$ for any $\delta_1,\delta_2\in\mathbf{G}$. $G=\{\mathfrak{C}_\alpha:\alpha\in\mathbf{G}\}$ is obviously a strong algebraic Δ-system.

LEMMA 3.4. *Let $\mathfrak{A}$ and θ be as in Lemma* 3.1. *Then* $\mathrm{card}\,(\mathfrak{A}/\theta)\leqq 2^{\varkappa\omega}$.

PROOF. Assume that $\mathrm{card}\,(\mathfrak{A}/\theta)>2^{\varkappa\omega}$. By the maximality of θ, $\mathfrak{A}/\theta$ has no loop preserving congruence. Hence it suffices to prove, that if $\mathfrak{B}$ is a $\varkappa$-unary algebra and $\mathrm{card}\,(\mathfrak{B})>2^{\varkappa\omega}$ then $\mathfrak{B}$ has a non trivial loop preserving congruence. Applying Lemma 3.3 to $\mathfrak{B}$, we obtain in particular, two subalgebras $\mathfrak{C}_0$ and $\mathfrak{C}_1$ of $\mathfrak{B}$ which are isomorphic over their intersection. Let $\chi:\mathfrak{C}_0\cup\mathfrak{C}_1\to\mathfrak{C}_0$ denote a mapping such that $\chi|\mathfrak{C}_i$ is an isomorphism of $\mathfrak{C}_i$ into $\mathfrak{C}_0$ $(i=0,1)$.

Now we define the congruence Φ on $\mathfrak{B}$ by setting $a\equiv b(\Phi)$ iff either $a=b$ or $a,b\in\mathfrak{C}_0\cup\mathfrak{C}_1$ and $\chi(a)=\chi(b)$. Φ is obviously an equivalence and every equivalence class has one or two elements. Clearly Φ is a loop preserving congruence.

From Lemma 3.3 we deduce three corollaries. For $\mathbf{X}$ a set, let $\mathrm{Eq}(\mathbf{X})$ denote the lattice of all equivalence relations on $\mathbf{X}$.

COROLLARY 3.5. *Let $\mathfrak{A}$ be a $\varkappa$-unary algebra. Assume that* $\mathrm{card}\,(\mathfrak{A})>2^{\nu}$ *where* $\nu\geqq\varkappa\omega$. *Then* $\mathrm{Con}\,(\mathfrak{A})$ *contains a complete sublattice isomorphic to* $Eq\,((2^{\nu})^{+})$ *such that* $0\in\mathrm{Eq}\,((2^{\nu})^{+})$ *corresponds to* $\omega\in\mathrm{Con}\,(\mathfrak{A})$.

PROOF. Let $\lambda=2^{\nu}$. By Lemma 3.3 we have a system of subsets of $\mathfrak{A}$, $C=\{\mathfrak{C}_{\alpha}:\alpha<\lambda^{+}\}$ being a strong algebraic Δ-system, and we have the isomorphisms $\chi_{\alpha}:\mathfrak{C}_{\alpha}\to\mathfrak{C}$. Using this system we shall embed $\mathrm{Eq}\,(\lambda^{+})$ into $\mathrm{Con}\,(\mathfrak{A})$. Let $\theta\in\mathrm{Eq}\,(\lambda^{+})$. We define $\bar\theta\in\mathrm{Con}\,(\mathfrak{A})$ by $a\equiv b\;(\bar\theta)$ iff $a=b$ or there exist $\alpha_1,\alpha_2<\lambda^{+}$ such that $\alpha_1\equiv\alpha_2\;(\theta)$ and $\chi_{\alpha_1}(a)=\chi_{\alpha_2}(b)$. We remark that if a is not in $\mathbf{Z}=\bigcup\{\square\mathfrak{C}_{\alpha}:\alpha<\lambda^{+}\}-\mathbf{X}$ then $a\equiv b\;(\bar\theta)$ iff $a=b$ and if $a\in\mathbf{Z}$ then $\chi_{\alpha}(a)$ is defined for precisely one α. So, it is obvious that $\bar\theta$ is a congruence relation, and the mapping $\theta\to\bar\theta$ is a complete embedding of $\mathrm{Eq}\,(\lambda^{+})$ into $\mathrm{Con}\,(\mathfrak{A})$.

COROLLARY 3.6. *Let $\mathfrak{A}$ be a non-trivial $\varkappa$-unary algebra, and let L denote the class of all lattices. Then* $\mathbf{S}\,(\mathrm{Con}\,(\mathbf{P}\mathfrak{A}))=L$.

PROOF. By Whitman's theorem [18], every lattice can be embedded into a partition lattice, hence Corollary 3.5 implies Corollary 3.6.

COROLLARY 3.7. *Let $\mathfrak{A}$ be a subdirect irreducible $\varkappa$-unary algebra. Then* $\mathrm{card}\,(\mathfrak{A})\leqq 2^{\varkappa\omega}$.

PROOF. 0 is meet reducible in $\mathrm{Eq}\,(3)$, and if $\mathrm{card}\,(\mathfrak{A})>2^{\varkappa\omega}$ then by Corollary 3.5, ω is not meet irreducible in $\mathrm{Con}\,(\mathfrak{A})$. (We have used, in fact, only that $\mathfrak{A}$ is finitely subdirect irreducible.)

PROOF OF THEOREM 1.2. Let $\mathfrak{A}$ be a $\varkappa$-unary algebra without one-element subalgebras. Let θ be as in Lemma 3.1. Then $\mathrm{card}\,(\mathfrak{A}/\theta)\leqq 2^{\varkappa\omega}$, and since θ preserves loops, $\mathfrak{A}/\theta$ has no one-element subalgebra either. Let $\mathfrak{D}_1$ denote the disjoint union of all $\varkappa$-unary algebras $\mathfrak{A}$ of cardinality $\leqq 2^{\varkappa\omega}$ having no one-element subalgebra, taking one algebra from every isomorphy class. Then $\mathfrak{D}_1$ has no one element subalgebra, and if $\mathfrak{C}$ has neither, then by Lemma 3.4, there exists a homomorphism $\varphi:\mathfrak{C}\to\mathfrak{D}_1$. Again by Lemma 3.4, $\mathfrak{D}_1$ has a loop preserving congruence Φ, such that $\mathrm{card}\,(\mathfrak{D}_1/\Phi)\leqq 2^{\varkappa\omega}$. Let $\mathfrak{D}_2=\mathfrak{D}_1/\Phi$. We state that $\mathfrak{A}=\mathfrak{D}_2$ satisfies the conditions of Theorem 1.2. As in the proof of Lemma 5.3 it is easy to see that $\mathfrak{A}$ itself is not rigid, moreover it has a non-injective endomorphism. Let $\mathfrak{C}$ be a $\varkappa$-unary algebra without one-element subalgebras. Then $\mathfrak{C}$ has a homomorphism into $\mathfrak{D}_1$, hence $\mathfrak{C}$ has a homomorphism into $\mathfrak{A}$. Let now $\mathfrak{B}\supset\mathfrak{A}$. If $\mathfrak{B}$ has a one-element subalgebra then $\mathfrak{B}$ is not rigid. Otherwise, by the above, there exists a homomorphism of $\mathfrak{B}$ into $\mathfrak{A}$, hence again $\mathfrak{B}$ is not rigid. By Theorem 1.1, $\mathrm{card}\,(\mathfrak{A})\geqq 2^{\varkappa\omega}$, and this completes the proof.

We have proved essentially the following stronger

COROLLARY 3.8. *Let $\mathfrak{A}$ as constructed above, and $\mathfrak{B}\supseteqq\mathfrak{A}$. Then $\mathfrak{B}$ has a non injective endomorphism.*

COROLLARY 3.9. *Let $\mathbf{X}$ be a set, and $\mathfrak{S}$ a transformation semigroup on $\mathbf{X}$, and* $\mathrm{card}\,(\mathfrak{S})\leqq\varkappa$, *and assume that $\mathfrak{S}$ has no common fixed point. Then there exists an*

$\mathfrak{S}$-invariant partition of $\mathbf{X}$ *into at most* $2^{\varkappa\omega}$ *classes such that* $\mathfrak{S}$ *has no common fixed class.*

PROOF. One can naturally interpret $(\mathbf{X}, \mathfrak{S})$ as a $\varkappa$-unary algebra. The $\mathfrak{S}$-invariant partitions are exactly the congruences, hence the corollary is straightforward by Lemma 3.4.

COROLLARY 3.10. *There exists a* $\varkappa$*-unary algebra* $\mathfrak{B}$ *with* $\operatorname{card}\mathfrak{B}=2^{\varkappa\omega}$ *such that if* $\mathfrak{A}$ *is a* $\varkappa$*-unary algebra and* $\operatorname{card}(\mathfrak{A})>1$, *then there exists a homomorphism* $\varphi:\mathfrak{A}\to\mathfrak{B}$ *satisfying* $\operatorname{card}(\varphi\mathfrak{A})>1$.

PROOF. Let $\mathfrak{C}$ denote the disjoint union of all $\varkappa$-unary algebras of cardinality not exceeding $2^{\varkappa\omega}$, taking one algebra from each isomorphy class. Let θ be as in Lemma 3.1. Set $\mathfrak{B}=\mathfrak{C}/\theta\cup\mathfrak{T}$, where $\mathfrak{T}$ denotes the one element algebra. Let $\operatorname{card}(\mathfrak{A})>1$. If there exists f and a $\mathfrak{A}$ such that $f_\alpha(a)\neq a$, then there exists a $\varphi:\mathfrak{A}\to\mathfrak{C}/\theta$ with $\operatorname{card}(\varphi\mathfrak{A})>1$. If for all f_α and a we have $f_\alpha(a)=a$, then the existence of such φ is trivial.

IV. $A_\varkappa(\mathfrak{A})$ in the case of rigid $\mathfrak{A}$

In this section we shall prove Theorem 1.3. The implications $(1)\to(2)\to(3)$, $(2)\to(4)$ are trivial.

4.1. First we prove that $(3)\to(1)$. Let $\mathfrak{B}\in A_\varkappa(\mathfrak{A})$ be a rigid algebra. It suffices to prove that $A_\varkappa(\mathfrak{B})$ is strongly binding. To preserve the original notation we put $\mathfrak{A}=\mathfrak{B}$ and we shall prove that $A_\varkappa(\mathfrak{A})$ is binding.

We shall say that $\mathfrak{C}\subset\square\mathfrak{A}$ has the *common coimage property,* if for any function $b:\varkappa\to\square\mathfrak{C}$ there exists a $c\in\square\mathfrak{A}$ with $f_i(c)=b(i)$.

$\mathfrak{A}$ has the common coimage property if $\square\mathfrak{A}$ has. We shall distinguish three cases.

Case (a): $\mathfrak{A}$ does not have the common coimage property. Let $\mathbf{D}$ be the set of functions violating the common coimage property. Let $\mathfrak{A}'$ be defined by $\square\mathfrak{A}'=\square\mathfrak{A}\cup\mathbf{D}$ (disjoint union); $f_i'(a)=f_i(a)$ if $a\in\square\mathfrak{A}$, $f_i'(b)=b(i)$ where $b\in\mathbf{D}$. Let us fix some $c\in\mathbf{D}$. We construct a strong embedding $\mathbf{F}:A_2\to A_\varkappa(\mathfrak{A})$ by $\mathbf{F}:\mathfrak{B}=\langle\square\mathfrak{B},h_i\rangle\to\langle\square\mathbf{F}(\mathfrak{B}),\bar f_i\rangle$ by

$$\square\mathbf{F}(\mathfrak{B})=\square\mathfrak{B}\times(\square\mathfrak{A}'\cup\{1,2,3,4,5,6,7\})$$

and

$$\begin{aligned}
&\bar f_i(b,a)=\big(b,f_i(a)\big), \quad b\in\square\mathfrak{B},\quad a\in\square\mathfrak{A}',\\
&\bar f_0(b,1)=(b,1),\quad \bar f_i(b,1)=(b,c) \qquad (i\geqq 1),\\
&\bar f_i(b,2)=(b,2),\quad \bar f_0(b,2)=(b,c) \qquad (i\geqq 1),\\
&\bar f_i(b,5)=(b,c),\\
&\bar f_0(b,3)=(b,3),\quad \bar f_i(b,3)=(b,5) \qquad (i\geqq 1),\\
&\bar f_i(b,4)=(b,4),\quad \bar f_0(b,4)=(b,5) \qquad (i\geqq 1),\\
&\bar f_0(b,6)=(b,1),\quad \bar f_i(b,6)=\big(h_2(b),2\big) \quad (i\geqq 1),\\
&\bar f_0(b,7)=(b,3),\quad \bar f_i(b,7)=\big(h_1(b),4\big) \quad (i\geqq 1).
\end{aligned}$$

If $\varphi:\mathfrak{B}_1\to\mathfrak{B}_2$ is a homomorphism then $\mathbf{F}(\varphi)$ can be defined in the natural way.

We remark that from the construction it is easy to see that if $\mathbf{H}$ is a subalgebra of $\mathbf{F}(\mathfrak{B})$ and it has the common coimage property then there exists a $b\in\Box\mathfrak{B}$ with $\Box\mathbf{H}\subset\{b\}\times\Box\mathfrak{A}$. Now let $\chi:\mathbf{F}(\mathfrak{B}_1)\to\mathbf{F}(\mathfrak{B}_2)$ be a homomorphism. Hence by the above remark $\chi(\{b_1\}\times\Box\mathfrak{A})=\{b_2\}\times\Box\mathfrak{A}$ and we have a mapping $\bar{\chi}:\mathfrak{B}_1\to\mathfrak{B}_2$ defined by $\bar{\chi}(b_1)=b_2$ with the above b_1 and b_2.

Now it is a routine matter to check that $\chi(b,d)=(\bar{\chi}(b),d)$ for each $(b,d)\in\Box\mathbf{F}(\mathfrak{B})$. We shall prove that $\bar{\chi}$ is a homomorphism. Indeed,

$$(\bar{\chi}(h_1(b)),4)=\chi(h_1(b),4)=\chi(\bar{f}_2(b,7))=\bar{f}_2(\chi(b,7))=\bar{f}_2(\bar{\chi}(b),7)=(h_1(\bar{\chi}(b)),4)$$

hence $\bar{\chi}(h_1(b))=h_1(\bar{\chi}(b))$.

The same holds for h_2 hence $\bar{\chi}$ is a homomorphism. But obviously $\mathbf{F}(\bar{\chi})=\chi$ hence $\mathbf{F}$ is full. From the construction it it obvious that it is also strong.

Case (b): $\mathfrak{A}$ statisfies the common coimage property and has no subalgebra isomorphic to $\mathfrak{B}(2)$, where $\Box\mathfrak{B}(2)=\{1,2\}$, $f_i(1)=2$, $f_i(2)=1$.

Let $\mathfrak{A}'$ be defined by $\Box\mathfrak{A}'=\Box\mathfrak{A}\cup\Box\mathfrak{B}(2)\cup\{c\}$ (disjoint union), $f_i'(a)=f_i(a)$ if $a\in\Box\mathfrak{A}\cup\Box\mathfrak{B}(2)$, $f_0'(c)=a_0$ for some $a_0\in\Box\mathfrak{A}$, $f_0'(c)=1$ for $i\geqq 1$. It is easy to see that $\mathfrak{A}'$ is also rigid, and $\mathfrak{A}'$ does not have the common coimage property, hence we arrived at the previous case.

Case (c): $\mathfrak{A}$ satisfies the common coimage property and has $\mathfrak{B}(2)$ as a subalgebra.

Let $2\mathfrak{A}$ denote the disjoint union of two copies of $\mathfrak{A}$, $\Box 2\mathfrak{A}=\Box\mathfrak{A}\times\{0,1\}$. Let θ be the following congruence on $2\mathfrak{A}$: $(1,0)\equiv(2,1)$ (θ), $(2,0)\equiv(1,1)$ (θ) and all the other classes have one element. This is obviously a congruence. Let $\overline{\mathfrak{A}}=2\mathfrak{A}/\theta$, $a\in\Box\mathfrak{A}$, $a\neq 1,2$ and b a new symbol. We define $\mathfrak{A}'$ by

$$\Box\mathfrak{A}'=\Box\overline{\mathfrak{A}}\cup\{b\},\quad f_i'(c)=f(c),\quad c\in\Box\overline{\mathfrak{A}},\quad f_0'(b)=(a,0),\quad f_i'(b)=(a,1)$$

$$\text{for}\quad i\geqq 1.$$

We shall call $[\Box\mathfrak{A}\times\{0\}]_\theta=\mathbf{L}$ and $[\Box\mathfrak{A}\times\{1\}]_\theta=\mathbf{R}$ the left and right side of $\mathfrak{A}'$, respectively. Note that $\mathbf{L}\cap\mathbf{R}=\Box\mathfrak{B}(2)$.

It is clear from the construction that both the left and right sides are subalgebras of $\mathfrak{A}'$ isomorphic to $\mathfrak{A}$. Moreover, a two element subset of $\Box\mathfrak{A}'$ has the common coimage property iff both elements belong to the same side. Hence endomorphisms of $\mathfrak{A}$ map the sides into sides. If $\chi\in\operatorname{End}\mathfrak{A}'$ and both $\mathbf{L}$ and $\mathbf{R}$ are mapped into $\mathbf{L}$ then $\chi|\mathbf{L}$ and $\chi|\mathbf{R}$ induce two endomorphism of $\mathfrak{A}$, and they are different as their action on $\mathfrak{B}(2)$ differs by the definition of θ. Hence either $\chi(\mathbf{L})\leqq\mathbf{L}$, $\chi(\mathbf{R})\leqq\mathbf{R}$ or conversely. But b makes the converse impossible. Therefore and by the assumption that $\mathfrak{A}$ was rigid we conclude that $\mathfrak{A}'$ is also rigid. But clearly $\mathfrak{A}'$ does not have the common coimage property. Therefore again Case (a) applies, completing the proof.

4.2. Now we prove (4)$\to$(3). We shall distinguish two cases.

Case (d): $\mathfrak{A}$ does not satisfy the common coimage property. Let $\mathfrak{A}'$ be constructed as in Case (a) and $\varkappa\in\operatorname{End}\mathfrak{A}'$. Then $\varkappa|\Box\mathfrak{A}\in\operatorname{End}\mathfrak{A}$ hence $\varkappa|\Box\mathfrak{A}$ is bijective, which implies that $\varkappa|\Box\mathfrak{A}'$ is one-to-one. Let $b\in\mathbf{D}$. Then $(\chi^{-1}(b_i): i<\varkappa)$ does not have the common coimage property since $(b_i: i<\varkappa)$ did not have it, hence there exists $\chi^{-1}(b)$. It follows that $\chi|\Box\mathfrak{A}'$ is onto. This implies $\operatorname{End}(\mathfrak{A}')=\operatorname{Aut}(\mathfrak{A}')$. Now applying the construction of Case (a) again we obtain $\mathfrak{A}''$, $\Box\mathfrak{A}''=\Box\mathfrak{A}\cup\mathbf{D}\cup\mathbf{D}'$ and (*): to each $a_0\in\Box\mathfrak{A}$ there exists $(a_i\in\mathbf{D}: 0\leqq i<\varkappa)$ and $b\in\mathbf{D}'$ with $f_i(b)=a_i$.

Let $\operatorname{card}\mathbf{D}'=\tau$ and let $(\mathfrak{C}_\alpha:\alpha<\tau)$ be a class of mutually rigid $\varkappa$-unary algebras, such that if $c\in\Box\mathfrak{C}_\alpha$ then $f_1(c)\neq f_2(c)$. This means that no subset of $\mathfrak{C}_\alpha$ has the common coimage property. The existence of such a class follows from [13]. Let some $c_\alpha\in\Box\mathfrak{C}_\alpha$ be specified for each $\alpha<\tau$, and let $\mathbf{D}'=\{b_\alpha:\alpha<\tau\}$. Now we define an algebra $\mathfrak{D}$ by $\Box\mathfrak{D}=\Box\mathfrak{A}''\cup(\bigcup_{\alpha<\tau}\mathfrak{C})\cup\tau$ (disjoint union), $\bar{f}_i(x)=f_i(x)$ if $x\notin\tau$, $\bar{f}_0(\alpha)=b_\alpha$, $\bar{f}_i(\alpha)=c_\alpha$ for $i\geqq 1$. Let $\chi\in\operatorname{End}\mathfrak{D}$. By the choice of the $\mathfrak{C}_\alpha$-s, $\chi(\Box\mathfrak{A}'')\subseteqq\Box\mathfrak{A}''$, hence $\chi(b_\alpha)=b_{\bar{\chi}(\alpha)}$ for some $\bar{\chi}:\tau\to\tau$ and $\bar{\chi}$ is a bijection. But α is the only element with $\bar{f}_0(\alpha)=b_\alpha$, hence $\chi(\alpha)=\bar{\chi}(\alpha)$ and therefore $\chi(\Box\mathfrak{C}_\alpha)=\Box\mathfrak{C}_{\bar{\chi}(\alpha)}$. This is possible iff $\bar{\chi}=\mathrm{id}_\tau$, hence $\chi(b_\alpha)=b_\alpha$. Now let $a\in\Box\mathfrak{A}$. By the observation (*) we have a b_α with $\bar{f}_0(b_\alpha)=a$, therefore $\chi(a)=a$ implying that $\chi|\Box\mathfrak{A}''=\mathrm{id}$, but $\chi|\Box\mathfrak{C}_\alpha=\mathrm{id}$ since $\mathfrak{C}_\alpha$ is rigid, proving that $\mathfrak{D}$ itself is also rigid.

Case (e): $\mathfrak{A}$ satisfies the common coimage property. Since $\operatorname{card}\mathfrak{A}>2^{\varkappa\omega}$, by Lemma 3.3 we have a strong algebraic Δ-system in $\mathfrak{A}$, say $C=\{\mathfrak{A}_\alpha:\alpha<\tau\}$. Let $\mathbf{X}=\bigcap_{\alpha<\tau}\Box\mathfrak{C}_\alpha$. We may assume that C is a maximal sta Δ-system. We have $\tau\geqq\omega$. Now let $B=\{\mathfrak{B}_\alpha:\alpha<\tau+1\}$ be a rearrangement of C of order type $\tau+1$. Let $\mathfrak{A}'$ be a disjoint copy of $\mathfrak{A}$. Let $\mathbf{X}'$ and $\mathfrak{B}'_\alpha$ denote the subalgebras of $\mathfrak{A}'$ corresponding to $\mathbf{X}$ and $\mathfrak{B}_\alpha$, resp. Let $\varphi_\alpha:\mathfrak{C}_\alpha\to\mathfrak{B}_\alpha$ $(\alpha<\tau)$ be a family of isomorphisms agreeing on $\mathbf{X}$ and satisfying $\varphi_\alpha(\mathbf{X})=\mathbf{X}'$.

We define a congruence on $\mathfrak{A}\cup\mathfrak{A}'$ by $a\equiv b(\theta)$ iff $a=b$ or there exists an $\alpha<\tau$ with $\varphi_\alpha(a)=b$ or $\varphi_\alpha(b)=a$. Let $\overline{\mathfrak{A}}=(\mathfrak{A}\cup\mathfrak{A}')/\theta$.

We shall call $\mathbf{L}=[\Box\mathfrak{A}]_\theta$ and $\mathbf{R}=[\Box\mathfrak{A}']_\theta$ the left and right side of $\overline{\mathfrak{A}}$, resp. Both sides are subalgebras isomorphic to $\mathfrak{A}$. Obviously $[\mathfrak{C}_\alpha]_\theta=[\mathfrak{B}_\alpha]_\theta$ $(\alpha<\tau)$ and $\mathbf{L}\cap\mathbf{R}=\bigcup\{\Box[\mathfrak{C}_\alpha]_\theta:\alpha<\tau\}$. $\Box\mathfrak{A}-\bigcup_{\alpha<\tau}\Box\mathfrak{C}_\alpha\neq\emptyset$ since otherwise we could construct a non bijective endomorphism of $\mathfrak{A}$. Therefore both $\mathbf{L}-\mathbf{R}$ and $\mathbf{R}-\mathbf{L}$ are non-void, implying that $\overline{\mathfrak{A}}$ does not have the common coimage property.

Let $\chi\in\operatorname{End}(\overline{\mathfrak{A}})$. As in the Case (c) we again have that $\chi(\mathbf{L})\leqq\mathbf{L}$ or $\chi(\mathbf{L})\leqq\mathbf{R}$, and the same for $\mathbf{R}$. Assume that $\chi(\mathbf{L}), \chi(\mathbf{R})\leqq\mathbf{L}$. From $\chi(\mathbf{L})\leqq\mathbf{L}$ we infer that $\chi(\mathbf{L}\cap\mathbf{R})$ is the union of a Δ-system which is maximal in $\mathbf{L}$ since $\chi|\mathbf{L}$ is an automorphism of $\mathbf{L}$. But by $\chi(\mathbf{R})\leqq\mathbf{L}$, $\chi([\mathfrak{B}_\tau]_\theta)$ can be added to this Δ-system, a contradiction. Hence using again that $\operatorname{End}(\mathfrak{A})=\operatorname{Aut}(\mathfrak{A})$ we get $\operatorname{End}(\overline{\mathfrak{A}})=\operatorname{Aut}(\overline{\mathfrak{A}})$. Thus we obtained the previous case, completing the proof of Theorem 1.3.

V. The case $\varkappa<\omega$

In this section we shall investigate the counterexample given in Section 3 in more detail. We are mainly concerned with the proof of the following

THEOREM 5.1. *Let* $2\leqq\varkappa<\omega$.

(i) *There exists a $\varkappa$-unary algebra $\mathfrak{A}$ such that*

  (α) *$\mathfrak{A}$ has no one-element subalgebra,*

  (β) *if $\mathfrak{B}$ has no one-element subalgebra then* $\operatorname{Hom}(\mathfrak{B},\mathfrak{A})\neq\emptyset$,

  (γ) *if $\mathfrak{A}'$ satisfies (α) and (β) then $\mathfrak{A}'$ contains a subalgebra isomorphic to $\mathfrak{A}$.*

(ii) *$\mathfrak{A}$ is unique (up to isomorphism).*

(iii) card $(\mathfrak{A})=2^{\omega}$.

(iv) $\operatorname{End}(\mathfrak{A})=\operatorname{Aut}(\mathfrak{A})\cong\prod_{p} C_p^{\frac{p^{\varkappa}-1}{p-1}}$ *(The product is taken over all prime numbers p).*

PROBLEM. *Does there exists an algebra satisfying* (i) *for* $\varkappa\geqq\omega$?

Henceforth we assume $2\leqq\varkappa<\omega$.

In Section 3 we have constructed a $\varkappa$-unary algebra $\mathfrak{D}_2$ such that it had no one-element subalgebra and if neither $\mathfrak{B}$ had one then $\operatorname{Hom}(\mathfrak{B},\mathfrak{D}_2)\neq\emptyset$. One can feel that $\mathfrak{D}_2$ is not the "smallest possible" such algebra, for there exists a homomorphism $\varphi:\mathfrak{B}\to\mathfrak{D}_2$ with $\operatorname{Ker}\varphi$ being a loop preserving congruence, and this is a rather strong condition. To construct a "smaller" algebra we need the following

LEMMA 5.2. *Assume that* $\theta_0\leqq\theta_1\leqq\ldots$ *are congruences of* $\mathfrak{B}$ *and* $\mathfrak{B}/\theta_i$ *has no one-element subalgebra. Then neither has* $\mathfrak{B}/\bigvee_i\theta_i$.

PROOF. Obvious.

Now let $\mathfrak{D}_2$ be the counterexample as in Section 3. By Lemma 5.2 and by Zorn's Lemma we have a maximal θ such that $\mathfrak{A}=\mathfrak{D}_2/\theta$ has no one-element subalgebra and therefore each proper homomorphic image of $\mathfrak{A}$ has a one-element subalgebra. Henceforth $\mathfrak{A}$ will denote always this algebra. We assert that we do obtain a new algebra this way:

LEMMA 5.3. $\mathfrak{A}\not\cong\mathfrak{D}_2$.

PROOF. Let $\mathfrak{B}$ denote the algebra $\square\mathfrak{B}=\{1,2,3\}$; $f_1(1)=f_1(2)=f_1(3)=2$, $f_i(1)=f_i(2)=f_i(3)=3$ $(i\geqq 2)$.

It is easy to see that $\mathfrak{B}$ has no proper loop preserving congruence hence $\mathfrak{D}_2$ has a subalgebra isomorphic to $\mathfrak{B}$. On the other hand, $\mathfrak{B}$ has a two element homomorphic image containing no one-element subalgebra, hence $\mathfrak{A}$ has no subalgebra isomorphic to $\mathfrak{B}$.

$\mathfrak{A}$ satisfies (i) of Theorem 5.1. This is clear for (α) and (β). (γ) follows from the fact that $\operatorname{Hom}(\mathfrak{A},\mathfrak{A}')\neq\emptyset$ but since $\mathfrak{A}'$ has no one-element subalgebra, all homomorphisms from $\mathfrak{A}$ into $\mathfrak{A}'$ are one-to-one. Obviously, card $(\mathfrak{A})\leqq$ card $(\mathfrak{D}_2)=2^{\omega}$ (Theorem 1.2). The converse inequality follows from Theorem 1.1 proving (iii). Moreover, (iv) and (i) (γ) imply (ii), hence it suffices to prove (iv).

We shall say that a $\varkappa$-unary algebra is *circulant* if it has no one-element subalgebra and its automorphism group has a cyclic subgroup acting transitively on the underlying set of the algebra.

LEMMA 5.4. *Each circulant $\varkappa$-unary algebra has a homomorphic image which is circulant of prime order. If p is a fixed prime then there are* $\dfrac{p^{\varkappa}-1}{p-1}$ *non-isomorphic circulant algebras of order p. All these are simple and their automorphism group is* $\mathbf{C}_p$.

PROOF. The first part of the lemma is obvious. If we distinguish between the elements of the underlying set then there are $p^{\varkappa}-1$ circulant algebras on p elements. Clearly we obtained each algebra $(p-1)$ times hence the second statement is also true while the third is trivial.

We shall denote these algebras by $\mathfrak{C}_p(i)$ $\left(i=1, \ldots, \frac{p^{\varkappa}-1}{p-1}\right)$. Clearly, $\mathfrak{A}$ has exactly one subalgebra isomorphic to $\mathfrak{C}_p(i)$ for all p and i. We shall prove that the endomorphisms of $\mathfrak{A}$ are fully determined by their action on these subalgebras.

LEMMA 5.5. *Let $\varphi \in \operatorname{End} \mathfrak{A}$ and assume that φ acts identically on each $\mathfrak{C}_p(i)$. Then $\varphi = 1_{\mathfrak{A}}$.*

PROOF. Assume to the contrary that $\varphi \neq 1_{\mathfrak{A}}$. Obviously φ is injective. Now we define a congruence on $\mathfrak{A}$ by $a \equiv b\ (\theta)$ iff there exists a $d \in \square\mathfrak{A}$, $n, m \geqq 0$ satisfying $\varphi^n(d)=a$, $\varphi^m(d)=b$. Using that φ is one-to-one, it is easy to check that θ is a congruence, and the following stronger form of the transitivity also holds: if $a_i \equiv a_j\ (\theta)$ for all $1 \leqq i, j \leqq n$ then there exists a $d \in \square\mathfrak{A}$ and $n_i \geqq 0$ satisfying $\varphi^{n_i}(d)=a_i$.

θ is a proper congruence of $\mathfrak{A}$ hence $\mathfrak{A}/\theta$ contains a one-element subalgebra. This means that there exist $a_1 \equiv a_i\ (\theta)\ (1 \leqq i \leqq n)$, $a_i \equiv f_i(a_i)\ (\theta)\ (1 \leqq i \leqq n)$. By the above remark this means that there exists a $d \in \square\mathfrak{A}$ and natural numbers s_i, t_i for $1 \leqq i \leqq n$ such that $a_i = \varphi^{t_i}(d)$, $f_i(a_i) = \varphi^{s_i}(d)$. But φ is injective and therefore from $\varphi^{t_i}(f_i(d)) = f_i(a_i) = \varphi^{s_i}(d)$ it follows that $f_i(d) = \varphi^{n_i}(d)$ $(n_i = s_i - t_i)$. (It may well happen that some n_i's are negative, but the right side exists and is unique.)

If there exist i_1, i_2 with $n_{i_1} < 0, n_{i_2} > 0$ then setting $\mathfrak{D} = \{\varphi^n(d) : n \in \mathbf{Z}\}$ we see that $\mathfrak{D}$ is a subalgebra and the restriction $\varphi|\mathfrak{D}$ generates a group acting transitively on $\mathfrak{D}$. Using Lemma 5.3 we get a contradiction.

Assume now $n_i \geqq 0$ for $i=1, \ldots, \varkappa$. If all the elements $\varphi^n(d)$ $(n=0, 1, \ldots)$ are distinct then it is easy to see that $\mathfrak{D} = \{\varphi^n(d) : n \geqq 0\}$ has a homomorphic image isomorphic to $\mathfrak{C}_p(i)$ for suitable p, i. In the remaining case $\varphi^n(d)=d$ for an $n>0$ hence $\mathfrak{D} = \{\varphi^n(d) : n \geqq 0\}$ is a circulant subalgebra, again a contradiction.

The case when each $n_i \leqq 0$ goes similarly.

Now we are in the position to prove (iv). First we prove that any automorphism of $\mathfrak{C} = \bigcup_i \bigcup_i \mathfrak{C}_p(i)$ extends to an endomorphism of $\mathfrak{A}$.

Observe that $\Phi = \operatorname{End}(\mathfrak{C}) = \operatorname{Aut}(\mathfrak{C}) \cong \prod_i \prod_i \operatorname{Aut}(\mathfrak{C}_p(i))$. We define a $\varkappa$-unary algebra $\mathfrak{F}$ by setting $\mathfrak{F} = \mathfrak{A} \times \Phi$, $f_i(a, \Psi) = (f_i(a), \Psi)$. By definition $1_\Psi : a \to (a, \Psi)$ is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{F}|(\square\mathfrak{A} \times \{\Psi\})$. We define a congruence on $\mathfrak{F}$ by $(a, \varphi) \equiv (b, \Psi)\ (\theta)$ iff either $(a, \varphi) = (b, \Psi)$ or there exists a pair (p, i) with $a, b \in \square\mathfrak{C}_\nu(i)$ and if for $\varphi = (\varphi_{p,i})$, $\Psi = (\Psi_{p,i})\, \varphi_{p,i}^{-1}(a) = \Psi_{p,i}^{-1}(b)$. It is a routine computation to see that θ is a congruence. Let $\overline{\mathfrak{F}} = \mathfrak{F}/\theta$. Hence we obtain homomorphisms $\bar{1}_\Psi : \mathfrak{A} \to \overline{\mathfrak{F}}$ defined by $\bar{1}_\Psi : a \to [(a, \Psi)]_\theta$. Then clearly $\bar{1}_\varphi(\mathfrak{C}) = \bar{1}_\Psi(\mathfrak{C}) \cong \mathfrak{C}$ for any φ, Ψ. Let $\overline{\mathfrak{C}}$ denote this subalgebra of $\overline{\mathfrak{F}}$. Moreover each isomorphism $\mathfrak{C} \to \overline{\mathfrak{C}}$ is obtained as $\bar{1}_\varphi$ for exactly one $\varphi \in \operatorname{Aut}(\mathfrak{C})$.

Now any homomorphism $\chi : \overline{\mathfrak{F}} \to \mathfrak{A}$ maps $\overline{\mathfrak{C}}$ onto $\mathfrak{C}$ hence $\{\chi \circ \bar{1}_\varphi | \mathfrak{C} : \varphi \in \operatorname{Aut}(\mathfrak{C})\} = \operatorname{Aut}(\mathfrak{C})$. As $\chi \circ \bar{1}_\varphi \in \operatorname{End}(\mathfrak{A})$, this proves the assertion we have started with.

Hence by restriction we obtained a surjection $\operatorname{End}(\mathfrak{A}) \to \operatorname{Aut}(\mathfrak{C})$ which is, by Lemma 5.4, also an injection (using the fact that every endomorphism of $\mathfrak{A}$ is a monomorphism). Hence $\operatorname{End}(\mathfrak{A}) = \operatorname{Aut}(\mathfrak{A})$ is a group. As

$$\operatorname{Aut}(\mathfrak{C}) \cong \Pi \mathbf{C}_p^{p^{\varkappa}-1/p-1}$$

we conclude that

$$\operatorname{End}(\mathfrak{A}) = \operatorname{Aut}(\mathfrak{A}) \cong \Pi \mathbf{C}_p^{p^{\varkappa}-1/p-1}.$$

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