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ON FUNCTIONS WITH STRONGLY CLOSED GRAPHS

By
T. NOIRI (Yatsushiro)

1. Introduction

Let X and Y be topological spaces and $f: X \to Y$ be a function of X into Y. The subset $\{(x, f(x)) | x \in X\}$ of the product space $X \times Y$ is called the *graph* of f and is usually denoted by $G(f)$. Properties of functions with closed graphs are considerably known. Quite recently, in [2] L. L. HERRINGTON and P. E. LONG have introduced the concept of strongly closed graphs. By using it they have obtained a necessary and sufficient condition for a Hausdorff space to be H-closed. In the present paper we shall show that if Y is an H-closed space, then for any function $f: X \to Y$ the following a), b) and c) are equivalent:

a) f is almost-continuous and Y is Hausdorff;

b) $G(f)$ is strongly closed;

c) $f^{-1}(K)$ is closed in X whenever K is quasi H-closed relative to Y. Moreover, we shall show that if the assumption "H-closed" on Y is dropt, then a)$\Rightarrow$b)$\Rightarrow$c) holds.

2. Definitions

Let S be a subset of a topological space. By $\mathrm{Cl}\,(S)$ and $\mathrm{Int}\,(S)$ we shall denote the closure of S and the interior of S, respectively.

DEFINITION 1. The graph $G(f)$ is said to be *strongly closed* [2] if for each point $(x, y) \notin G(f)$, there exist open sets $U \subset X$ and $V \subset Y$ containing x and y, respectively, such that $[U \times \mathrm{Cl}\,(V)] \cap G(f) = \emptyset$.

DEFINITION 2. A function $f: X \to Y$ is said to be *almost-continuous* [7] if for each point $x \in X$ and each open set $V \subset Y$ containing $f(x)$, there exists an open set $U \subset X$ containing x such that $f(U) \subset \mathrm{Int}\,(\mathrm{Cl}\,(V))$.

DEFINITION 3. A set $S \subset X$ is said to be *quasi H-closed relative to* X [6] if for every cover $\{U_\alpha | \alpha \in \mathcal{A}\}$ of S by open sets of X, there exists a finite subfamily $\mathcal{A}_0 \subset \mathcal{A}$ such that $S \subset \bigcup \{\mathrm{Cl}\,(U_\alpha) | \alpha \in \mathcal{A}_0\}$. If X is quasi H-closed relative to X, then it is called *quasi H-closed*. When X is Hausdorff, the word "quasi" in these two definitions is dropt.

REMARK 1. Every quasi H-closed subspace of a space X is quasi H-closed relative to X, but the converse does not hold in general [6, p. 161].

DEFINITION 4. A space X is said to be *C-compact* [8] if every closed set of X is quasi H-closed relative to X.

DEFINITION 5. A Hausdorff space X is said to be *locally H-closed* [5] if every point of X hat a neighbourhood which is H-closed.

3. Functions with strongly closed graphs

The following lemma, which follows easily from the definition, is useful in the sequel.

LEMMA. *The graph $G(f)$ is strongly closed if and only if for each point $(x, y) \notin G(f)$, there exist open sets $U \subset X$ and $V \subset Y$ containing x and y, respectively, such that $f(U) \cap \mathrm{Cl}(V) = \emptyset$.*

Every function with the strongly closed graph has the closed graph. The converse is not true [2, Example 3]. It is known that $f: X \to Y$ is almost-continuous and Y is Hausdorff, then $G(f)$ is closed [4, Theorem 9]. The following theorem is an improvement of this result.

THEOREM 1. *If $f: X \to Y$ is almost-continuous and Y is Hausdorff, then $G(f)$ is strongly closed.*

PROOF. Let $(x, y) \notin G(f)$. Then we have $y \neq f(x)$. Since Y is Hausdorff, there exist disjoint open sets V and W in Y containing y and $f(x)$, respectively. Therefore, we have $\mathrm{Cl}(V) \cap \mathrm{Int}(\mathrm{Cl}(W)) = \emptyset$. Since f is almost-continuous, there exists an open set $U \subset X$ containing x such that $f(U) \subset \mathrm{Int}(\mathrm{Cl}(W))$. Therefore, we obtain $f(U) \cap \mathrm{Cl}(V) = \emptyset$. This implies that $G(f)$ is strongly closed.

REMARK 2. The converse to Theorem 1 is not true. In [2, p. 473], it is shown that there exists a function with the strongly closed graph which is not almost-continuous.

In [3] P. KOSTYRKO showed that if $G(f)$ is closed and f is surjective, then the range is a T_1-space, and also showed that there exists a surjective function with the closed graph whose range is not Hausdorff. For a function with the strongly closed graph we have

THEOREM 2. *If $f: X \to Y$ is surjective and $G(f)$ is strongly closed, then Y is Hausdorff.*

PROOF. Let y and z be any distinct points of Y. Then, since f is surjective, there exists a point $x \in X$ such that $f(x) = y$. Therefore, we have $(x, z) \notin G(f)$. Since $G(f)$ is strongly closed, there exist open sets $U \subset X$ and $V \subset Y$ containing x and z, respectively, such that $f(U) \cap \mathrm{Cl}(V) = \emptyset$. Put $W = Y - \mathrm{Cl}(V)$, then W is an open set containing y such that $W \cap V = \emptyset$. This shows that Y is Hausdorff.

It is known that if a function $f: X \to Y$ has the strongly closed graph and Y is H-closed, then f is almost-continuous [2, Theorem 9]. Therefore, by Theorem 2, we obtain the following corollary which shows that the converse to Theorem 1 holds if f is surjective and Y is quasi H-closed.

COROLLARY 1. *Let $f: X \to Y$ be surjective and Y quasi H-closed. If $G(f)$ is strongly closed, then f is almost-continuous and Y is Hausdorff.*

REMARK 3. Under the hypotheses of Corollary 1, f is not necessarily continuous. In [2, Example 2] it is shown that there exists a non-continuous function onto an H-closed space whose graph is strongly closed.

THEOREM 3. *If $G(f)$ is strongly closed, then f has the following property:*
(P) *For every set K quasi H-closed relative to Y, $f^{-1}(K)$ is a closed set of X.*

PROOF. Suppose that $G(f)$ is strongly closed. Assume that there exists a set K quasi H-closed relative to Y such that $f^{-1}(K)$ is not closed in X. Then, there exists a point $x \in \mathrm{Cl}\,(f^{-1}(K))-f^{-1}(K)$. Therefore, for each point $y \in K$, we have $(x, y) \notin G(f)$. Since $G(f)$ is strongly closed, there exist open sets $U_y(x) \subset X$ and $V(y) \subset Y$ containing x and y, respectively, such that $f(U_y(x)) \cap \mathrm{Cl}\,(V(y)) = \varnothing$. Now, the family $\{V(y) | y \in K\}$ is a cover of K by open sets of Y and K is quasi H-closed relative to Y. Therefore, there exist a finite number of points $y_1, y_2, \ldots, y_n$ in K such that $K \subset \bigcup \{\mathrm{Cl}\,(V(y_j)) | 1 \leqq j \leqq n\}$. Put $U = \bigcap \{U_{y_j}(x) | 1 \leqq j \leqq n\}$, then U is an open set of X containing x and $f(U) \cap K \neq \varnothing$ because $x \in \mathrm{Cl}\,(f^{-1}(K))$. Therefore, there exists a positive integer k $(1 \leqq k \leqq n)$ such that

$$f(U_{y_k}(x)) \cap \mathrm{Cl}\,(V(y_k)) \neq \varnothing .$$

This is a contradiction.

REMARK 4. The converse to Theorem 3 is not always true, as the following example due to P. KOSTYRKO [3] shows.

EXAMPLE. Let X and Y be the sets of positive integers. Let X have the discrete topology, Y have the cofinite topology and $f: X \to Y$ be the identity mapping. Then, the closure of any nonempty open set of Y is Y itself. Therefore, f has the property (P). Moreover, although $G(f)$ is closed, it is not strongly closed.

The following corollary shows that the hypothesis „Hausdorff" on the space Y in [2, Theorem 9] is not necessary.

COROLLARY 2. *Let Y be a quasi H-closed space. If $f: X \to Y$ has the strongly closed graph, then it is almost-continuous.*

PROOF. Let F be any regularly closed set of Y. Since Y is quasi H-closed, F is quasi H-closed [6, (2.2), p. 161] and hence quasi H-closed relative to Y. Therefore, by Theorem 3, $f^{-1}(F)$ is closed in X. This shows that f is almost-continuous [7, Theorem 2.2].

The following corollary is an immediate consequence of Theorem 3.

COROLLARY 3 (HERRINGTON and LONG [1]). *Let Y be a C-compact space. If $f: X \to Y$ has the strongly closed graph, then it is continuous.*

As an immediate consequence of Theorem 1 and Theorem 3, we obtain

COROLLARY 4 (LONG and HERRINGTON [4]). *If Y is Hausdorff and $f: X \to Y$ is almost-continuous, then f has the property* (P).

The following theorem shows that the converse to Theorem 3 holds if Y is locally H-closed.

THEOREM 4. *Let Y be a locally H-closed (Hausdorff) space. If $f: X \to Y$ has the following property* (P′), *then $G(f)$ is strongly closed.*

(P′) *For every quasi H-closed set K, $f^{-1}(K)$ is a closed set of X.*

PROOF. Suppose that Y is locally H-closed and $f: X \to Y$ has the property (P′). Let $(x, y) \notin G(f)$, then we have $y \neq f(x)$. Since Y is Hausdorff, there exist disjoint open sets V_x and V_y containing $f(x)$ and y, respectively. Therefore, we have $V_x \cap \mathrm{Cl}\,(V_y) = \varnothing$. Since Y is locally H-closed, there exists a neighbourhood W of

y which is H-closed. Since W is a neighbourhood of y, there exists an open set W_0 such that $y \in W_0 \subset W$. Put $V = V_y \cap W_0$, then V is an open set containing y and $V_x \cap \mathrm{Cl}(V) = \emptyset$. Now, we shall show that $\mathrm{Cl}(V)$ is H-closed. In fact, since W is H-closed and Y is Hausdorff, W is closed in Y and hence $\mathrm{Cl}(V) \subset W$. Since V is open in Y, it is open in W and $\mathrm{Cl}(V)$ is regularly closed in W. Since W is H-closed, so is $\mathrm{Cl}(V)$. Therefore, by (P′), $f^{-1}(\mathrm{Cl}(V))$ is closed in X. Put $U = X - f^{-1}(\mathrm{Cl}(V))$. Then U is an open set containing x such that $f(U) \cap \mathrm{Cl}(V) = \emptyset$ because $V_x \cap \mathrm{Cl}(V) = \emptyset$. This shows that $G(f)$ is strongly closed.

COROLLARY 5. *Let Y be an H-closed space. Then, for any function $f: X \to Y$ the following are equivalent:*

a) *f is almost-continuous,*
b) *$G(f)$ is strongly closed,*
c) *f has the property* (P),
d) *f has the property* (P′).

PROOF. This follows immediately from Theorem 1, Theorem 3, Corollary 2 and Theorem 4.

In conclusion the author wishes to express his thanks to the referee for helpful suggestions.

References

[1] L. L. HERRINGTON and P. E. LONG, Characterizations of C-compact spaces, *Proc. Amer. Math. Soc.*, **52** (1975), 417—426.

[2] L. L. HERRINGTON and P. E. LONG, Characterizations of H-closed spaces, *Proc. Amer. Math. Soc.*, **48** (1975), 469—475.

[3] P. KOSTYRKO, A note on the functions with closed graphs, *Časopis Pěst. Mat.*, **94** (1969), 202—205.

[4] P. E. LONG and L. L. HERRINGTON, Properties of almost-continuous functions, *Boll. Un. Mat. Ital.* **(4) 10** (1974), 336—342.

[5] F. OBREANU, Espaces localement absolument fermés, *Ann. Acad. Repub. Pop. Române, Sect. Sti. Fiz. Chim. Ser. A*, **3** (1950), 375—394.

[6] J. PORTER and J. THOMAS, On H-closed and minimal Hausdorff spaces, *Trans. Amer. Math. Soc.*, **138** (1969), 159—170.

[7] M. K. SINGAL and A. R. SINGAL, Almost-continuous mappings, *Yokohama Math. J.*, **16** (1968), 63—73.

[8] G. VIGLINO, C-compact spaces, *Duke Math. J.*, **36** (1969), 761—764.

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YATSUSHIRO COLLEGE OF TECHNOLOGY
YATSUSHIRO, KUMAMOTO
866 JAPAN

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WEAKLY DISTRIBUTIVE SEMILATTICES

By

W. H. CORNISH and R. C. HICKMAN (Bedford Park)

0. Introduction

A weakly distributive semilattice is a (lower) semilattice such that the infimum distributes over all existent finite suprema, in the sense that if $x_1 \vee \ldots \vee x_n$ exists then $(x \wedge x_1) \vee \ldots \vee (x \wedge x_n)$ exists for any x and equals $x \wedge (x_1 \vee \ldots \vee x_n)$. Such semilattices were first studied by BALBES [4] under the name of *prime semilattices*. Here we have adopted the terminology of a recent paper of VARLET [9]. An important class of weakly distributive semilattices is provided by the *distributive semilattices* of GRÄTZER and SCHMIDT, i.e. those semilattices S in which $a \wedge b \leqq c$ implies $c = a' \wedge b'$ for some $a' \geqq a$ and $b' \geqq b$, or equivalently S is directed-above and has a distributive lattice of filters, c.f. [4; Theorem 4.1] and [9].

In Section 1 of this paper we show that a semilattice is weakly distributive if and only if its lattice of ideals is distributive and that the finitely generated ideals form its free extension in the variety of distributive lattices.

A semilattice is said to have the *upper bound property* if any two elements, which are bounded above, have a supremum. An interesting class of weakly distributive semilattices with the upper bound property is provided by those semilattices in which each principal ideal is a boolean algebra. These semilattices have been studied by ABBOTT [1], [2], [3] under the name of *semiboolean algebras* and mainly from the point of view of Abbot's implication algebras; we show, in Section 2, that the semiboolean algebras are precisely those semilattices with the upper bound property whose finitely generated ideals form a generalized boolean algebra. In Section 2, we shall show that there is an order-isomorphism between the prime filters, in the sense of BALBES [4], of a weakly distributive semilattice, and the prime filters of the lattice of finitely generated ideals, and thus elucidate results in [4] and [9].

In Section 3, we study the congruences on a semilattice which have the substitution property for existent finite suprema. After obtaining a formula for the minimal such congruence identifying a pair (a, b) with $a \leqq b$, we show that, in the presence of the upper bound property, the lattice of such congruences of any weakly distributive semilattice is canonically isomorphic to the lattice of congruences on its free distributive lattice extension. This last result may be regarded as an improvement of a theorem recently proved by FLEISCHER [5, Theorem DL] using very general techniques. In Section 3 we also consider permutability of congruences, the correspondence between ideals and congruences, and the description of the join of congruences.

1. Ideals

Throughout this paper whenever we refer to semilattices we will mean lower semilattices unless otherwise stated. Whenever a semilattice has a least element we will denote it by 0. If $x_1, \dots, x_n$ are elements of a semilattice then when we write $x_1 \vee \dots \vee x_n$ we mean that the supremum of $x_1, \dots, x_n$ exists and $x_1 \vee \dots \vee x_n$ is the symbol denoting this supremum.

A non-empty subset H of a semilattice S is called *hereditary* if *for any* $x \in S$ and $y \in H$, $x \leqq y$ implies $x \in H$. When S does not have a smallest element we also regard the empty set as hereditary. Thus, the set $\mathscr{H}(S)$ of all hereditary subsets of S is a complete distributive lattice when partially ordered by set-inclusion; the meet and join in $\mathscr{H}(S)$ are given by set-theoretic intersection and union, respectively, the largest element is S and the smallest element is $\{0\}$ if $0 \in S$ and the empty set, otherwise. An *ideal* is an hereditary subset which is closed under existent finite suprema. Hence the set $\mathscr{I}(S)$ of all ideals is an algebraic closure system on S, and consequently is an algebraic lattice whose largest element is S and whose smallest element is the smallest hereditary subset. If $J_1, \dots, J_n$ are ideals then $J_1 + \dots + J_n$ denotes their supremum in $\mathscr{I}(S)$. If $A \subseteq S$ then $(A]$ and $\langle A \rangle$ denote the hereditary subset and ideal generated by A, respectively. We note that for any $x \in S$, $\langle x \rangle = (x]$.

THEOREM 1.1. *The following conditions on a semilattice S are equivalent.*

(i) *S is weakly distributive.*

(ii) *For any* $H \in \mathscr{H}(S)$, $\langle H \rangle = \{h_1 \vee \dots \vee h_n : h_1, \dots, h_n \in H\}$.

(iii) *For any* $I, J \in \mathscr{I}(S)$, $I + J = \{a_1 \vee \dots \vee a_n : a_1, \dots, a_n \in I \cup J\}$.

(iv) *$\mathscr{I}(S)$ is a distributive lattice.*

(v) *The map $H \to \langle H \rangle$ is a lattice homomorphism of $\mathscr{H}(S)$ onto $\mathscr{I}(S)$ (which preserves arbitrary suprema).*

PROOF. (i)⇒(ii). The set $J = \{h_1 \vee \dots \vee h_n : h_1, \dots, h_n \in H\}$ is hereditary since S is weakly distributive and is clearly closed under existent finite suprema and so $J = \langle H \rangle$.

(ii)⇒(iii). Since $J + K = \langle J \cup K \rangle$ and $J \cup K$ is hereditary, the result is immediate.

(iii)⇒(iv). Clearly $I \cap J = \{i \wedge j : i \in I, j \in J\}$ and so let $I, J, K \in \mathscr{I}(S)$ and suppose $x \in I \cap (J + K)$. Therefore $x \in I$ and $x = a_1 \vee \dots \vee a_n$, $a_i \in J \cup K$. Since $a_i \leqq x$, $a_i \in I$ and so $a_i \in (I \cap J) \cup (I \cap K)$ showing that $x \in (I \cap J) + (I \cap K)$.

(iv)⇒(i). Let $x, y \in S$ with $y = a_1 \vee \dots \vee a_n$. Now

$$\langle x \wedge y \rangle = \langle x \rangle \cap \langle y \rangle = \langle x \rangle \cap (\langle a_1 \rangle + \dots + \langle a_n \rangle) = \langle x \wedge a_1 \rangle + \dots + \langle x \wedge a_n \rangle,$$

and this implies that $(x \wedge a_1) \vee \dots \vee (x \wedge a_n)$ exists and equals $x \wedge y$.

(i), (ii) and (iii)⇒(v). Let $G, H \in \mathscr{H}(S)$; then by our assumptions $\langle G \cup H \rangle = \langle G \rangle + \langle H \rangle$, and obviously $\langle G \cap H \rangle \subseteq \langle G \rangle \cap \langle H \rangle$. Suppose $x \in \langle G \rangle \cap \langle H \rangle$; that is $x = g_1 \vee \dots \vee g_m = h_1 \vee \dots \vee h_n$ where $g_i \in G$, $h_j \in H$ and so $x = x \wedge x = (g_1 \vee \dots \vee g_m) \wedge (h_1 \vee \dots \vee h_n) = (g_1 \wedge h_1) \vee \dots (g_m \wedge h_n) \in \langle G \cap H \rangle$.

(v)⇒(iv). This is trivial.

Let $\mathscr{I}_f(S)$ denote the set of all finitely generated ideals of a semilattice S. Then $\mathscr{I}_f(S)$ is precisely the set of compact members of $\mathscr{I}(S)$, and in general is only an upper subsemilattice of $\mathscr{I}(S)$. However, when S is weakly distributive and $x_1, \dots, x_m, y_1, \dots, y_n \in S$ it follows from Theorem 1.1 that $\langle x_1, \dots, x_m \rangle \cap \langle y_1, \dots, y_n \rangle =$

$=\sum_{i,j}\langle x_i\wedge y_j\rangle$ and so $\mathscr{I}_f(S)$ is a distributive sublattice of $\mathscr{I}(S)$. Moreover, if $\varepsilon: S\to\mathscr{I}_f(S)$ is defined by $\varepsilon(x)=\langle x\rangle$ for any $x\in S$ then ε is a semilattice monomorphism which preserves existent finite suprema. Hence we have

THEOREM 1.2. *The following conditions on a semilattice S are equivalent.*

(i) *S is weakly distributive.*

(ii) *There is a semilattice monomorphism, preserving existent finite suprema, embedding S into a distributive lattice.*

(iii) *There is a semilattice monomorphism, preserving existent finite suprema, embedding S into a ring of sets.*

COROLLARY. *If $x, y_1, \ldots, y_n$ are elements of S such that $x\vee y_i$ exists for $i=1,\ldots,n$, then $x\vee(y_1\wedge\ldots\wedge y_n)$ exists and*

$$x\vee(y_1\wedge\ldots\wedge y_n)=(x\vee y_1)\wedge\ldots\wedge(x\vee y_n).$$

When S is weakly distributive $\mathscr{I}_f(S)$ is the free extension of S in the variety of distributive lattices in the following sense:

THEOREM 1.3. *Let S be a weakly distributive semilattice, D be a distributive lattice and φ: $S\to D$ a semilattice homomorphism preserving existent finite suprema. Then there is a unique lattice homomorphism $\bar\varphi$: $\mathscr{I}_f(S)\to D$ such that $\bar\varphi\circ\varepsilon=\varphi$. Moreover, $\bar\varphi$ is an injection if and only if φ is an injection.*

PROOF. Let D be a distributive lattice and $\varphi: S\to D$ be as above. Define $\bar\varphi: \mathscr{I}_f(S)\to D$ by $\bar\varphi(I)=\varphi(x_1)\vee\ldots\vee\varphi(x_m)$ when $I=\langle x_1, \ldots, x_m\rangle$ is in $\mathscr{I}_f(S)$. Firstly $\bar\varphi$ is well defined, for suppose $I=\langle x_1, \ldots, x_m\rangle=\langle y_1, \ldots, y_n\rangle$; then each $x_i\in\langle y_1, \ldots, y_n\rangle$ and so $x_i=a_1^i\vee\ldots\vee a_{k(i)}^i$ where $\{a_1^i, \ldots, a_{k(i)}^i\}\subseteq(y_1, \ldots, y_n]$. Hence

$$\varphi(x_1)\vee\ldots\vee\varphi(x_m)=\bigvee_{i=1}^{m}\left(\varphi(a_1^i)\vee\ldots\vee\varphi(a_{k(i)}^i)\right)\leqq\varphi(y_1)\vee\ldots\vee\varphi(y_m),$$

and the reserve inequality is similarly estabished.

Also if $I=\langle x_1, \ldots, x_m\rangle$ and $J=\langle y_1, \ldots, y_n\rangle$ then

$$\bar\varphi(I)\vee\bar\varphi(J)=\varphi(x_1)\vee\ldots\vee\varphi(x_m)\vee\varphi(y_1)\vee\ldots\vee\varphi(y_n)=\bar\varphi(I+J),$$

while

$$\bar\varphi(I)\wedge\bar\varphi(J)=(\varphi(x_1)\vee\ldots\vee\varphi(x_m))\wedge(\varphi(y_1)\vee\ldots\vee\varphi(y_n))=\bigvee_{i,j}\left(\varphi(x_i)\wedge\varphi(y_j)\right)=\bar\varphi(I\cap J)$$

and so $\bar\varphi$ is a lattice homomorphism and $\varphi=\bar\varphi\circ\varepsilon$. It is easily seen that $\bar\varphi$ is the unique homomorphism satisfying $\varphi=\bar\varphi\circ\varepsilon$, and if $\bar\varphi$ is an injection then φ is an injection. Suppose φ is injective and $\bar\varphi(I)=\bar\varphi(J)$ where $I=\langle x_1, \ldots, x_m\rangle$ and $J=\langle y_1, \ldots, y_n\rangle$. Thus, $\varphi(x_1)\vee\ldots\vee\varphi(x_m)=\varphi(y_1)\vee\ldots\vee\varphi(y_n)$ and this implies that $\varphi(x_i)=\varphi(x_i\wedge y_1)\vee\ldots\vee\varphi(x_i\wedge y_n)$ and it follows that $x_i=(x_i\wedge y_1)\vee\ldots\vee(x_i\wedge y_n)$. Hence $x_i\in J$ for all $i=1, \ldots, m$, and so $I\subseteq J$. Similarly $J\subseteq I$ and therefore $I=J$ and we are finished.

COROLLARY. *The category whose objects are weakly distributive semilattices and whose morphisms are semilattice homomorphisms preserving existent finite suprema has the amalgamation property.*

PROOF. Let S, S_1, S_2 be weakly distributive and let $\psi_1: S \to S_1$, $\psi_2: S \to S_2$ be monomorphisms. By Theorem 1.3 there are lattice monomorphisms $\psi_i': \mathcal{I}_f(S) \to \to \mathcal{I}_f(S_i)$ for $i=1, 2$. Since the variety of distributive lattices has the amalgamation property ([5; Corollary 20, p. 148]), there is a distributive lattice D and monomorphisms $\varphi_i: \mathcal{I}_f(S_i) \to D$ for $i=1, 2$ such that $\varphi_1 \circ \psi_1' = \varphi_2 \circ \psi_2'$. Then $\varphi_i | S_i$ is a monomorphism mapping S_i into D for $i=1, 2$ and $\varphi_1 | S_1 \circ \psi_1 = \varphi_2 | S_2 \circ \psi_2$.

2. Prime filters and semiboolean algebras

A *filter* F in a semilattice S is a non-empty subset of S such that if $f_1, f_2 \in F$ and $x \in S$ with $f_1 \leqq x$ then both $f_1 \wedge f_2$ and x are in F. Following BALBES [4] a filter G is called a *prime filter* if $G \neq S$ and at least one of $x_1, \ldots, x_n$ is in G whenever $x_1 \vee \ldots \vee x_n$ exists and is in G. In [4, Theorem 2.2] BALBES showed that a semilattice S is weakly distributive if and only if the prime filters separate the elements of S. Here we show the interrelation between the prime filters of a weakly distributive semilattice S and those of the distributive lattice $\mathcal{I}_f(S)$.

An ideal P of a semilattice S is called a *prime ideal* if $P \neq S$ and $x \wedge y \in P$ implies $x \in P$ or $y \in P$. It is not hard to see that a filter F of a semilattice S is prime if and only if $S \setminus F$ is a prime ideal.

THEOREM 2.1. *Let S be a weakly distributive semilattice and identify S with the subsemilattice $\varepsilon(S)$ of $\mathcal{I}_f(S)$. Then the map $F \to F \cap S$ is an order isomorphism between the partially ordered sets of prime filters of $\mathcal{I}_f(S)$ and S.*

PROOF. Since $\varepsilon: S \to \mathcal{I}_f(S)$ preserves existent finite suprema, $F \cap S$ is a prime filter of S when F is a prime filter of $\mathcal{I}_f(S)$. If F_1 and F_2 are prime filters of $\mathcal{I}_f(S)$ then the primeness of F_1 and F_2 ensures that $F_1 \cap S \subseteq F_2 \cap S$ if and only if $F_1 \subseteq F_2$. It remains to show that the restriction map is a surjection. Let G be a prime filter of S, so that $S \setminus G$ is a prime ideal of S, and thus $X = \{\langle x_1, \ldots, x_n \rangle : x_i \in S \setminus G\}$ is a prime ideal of $\mathcal{I}_f(S)$. Therefore, $\mathcal{I}_f(S) \setminus X$ is a prime filter of $\mathcal{I}_f(S)$ and $S \cap (\mathcal{I}_f(S) \setminus X) = S \setminus (S \cap X) = S \setminus (S \setminus G) = G$.

It is clear that a semilattice has the upper bound property (see Introduction) if and only if each principal ideal is a lattice. Furthermore, if $y_1 \vee \ldots \vee y_n$ exists in a semilattice S with the upper bound property then $y_i \vee y_j$ exists in S for each i and j.

THEOREM 2.2. *A semilattice S is a semiboolean algebra if and only if each of the following conditions is satisfied.*

(i) *S has the upper bound property.*

(ii) *S is weakly distributive.*

(iii) *S has a 0 and for any $x \in S$, $x^* = \{y \in S : y \wedge x = 0\}$ is an ideal and $\langle x \rangle + x^* = S$.*

PROOF. Suppose S is semiboolean. It is immediate that S has a 0 and satisfies the upper bound property. In order to show that S is weakly distributive it is sufficient, in the presence of the upper bound property, to prove that $x \wedge (y_1 \vee y_2) = (x \wedge y_1) \vee \vee (x \wedge y_2)$ when $y_1 \vee y_2$ exists.

Let J be the principal ideal $\langle y_1 \vee y_2 \rangle$. Then $x \wedge y_1$ and $x \wedge y_2$ are both elements of J and so $(x \wedge y_1) \vee (x \wedge y_2)$ exists and is an element of J. If $r_1, r_2 \in J$ are the complements of $x \wedge y_1$ and $x \wedge y_2$ in J, respectively then a routine calculation shows that

$r_1 \wedge r_2$ is the complement in J of both $x \wedge (y_1 \vee y_2)$ and $(x \wedge y_1) \vee (x \wedge y_2)$. Hence, $x \wedge (y_1 \vee y_2) = (x \wedge y_1) \vee (x \wedge y_2)$.

Since S is weakly distributive for any $x \in S$, x^* is an ideal. Let y be an arbitrary element of S. Now $x \wedge y \in \langle y \rangle$ and $\langle y \rangle$ is a Boolean algebra, and so $x \wedge y$ has a complement z in $\langle y \rangle$; that is $z \wedge x \wedge y = 0$ and $z \vee (x \wedge y) = y$. Since $z \in \langle y \rangle$, $z \leqq y$ and so $z \wedge x \wedge y = z \wedge x = 0$, hence $z \in x^*$ and since $x \wedge y \in (x]$ we have $y \in (x] + x^*$. Thus condition (iii) holds.

Now let us assume conditions (i), (ii) and (iii) and let $a, b \in S$ with $a \leqq b$. By assumption $b \in \langle a \rangle + a^*$ and so using the upper bound property we have $b = x \vee y$, where $x \in \langle a \rangle$ and $y \in a^*$. Therefore, $b = b \vee a = a \vee x \vee y = a \vee y$, and since $y \in a^*$, $a \wedge y = 0$, showing that y is the complement of a in $\langle b \rangle$.

THEOREM 2.3. *Let S be a semilattice with* 0 *satisfying the upper bound property. Then the following are equivalent.*

(i) *S is semiboolean.*

(ii) *$\mathscr{I}_f(S)$ is a generalized boolean algebra.*

(iii) *S is weakly distributive and its set of prime filters is unordered by set-inclusion.*

PROOF. (i)$\Rightarrow$(ii). Suppose S is semiboolean. By Theorems 1.3 and 2.2 $\mathscr{I}_f(S)$ is a distributive lattice with 0. Let $\langle 0 \rangle \subseteqq Y \subseteqq X$, where $X = \langle x_1, \ldots, x_n \rangle$, $Y = \langle y_1, \ldots, y_m \rangle \in \mathscr{I}_f(S)$. By Theorem 2.2, for each $i = 1, \ldots, n$, and $j = 1, \ldots, m$ there exist w_{ij} and z_{ij} such that $x_i = w_{ij} \vee z_{ij}$, $w_{ij} \in \langle y_j \rangle$ and $z_{ij} \in y_j^*$. Then $Z = \bigcap_{j=1}^{m} \langle z_{1j}, \ldots, z_{nj} \rangle$ is a finitely generated ideal contained within X. In addition,

$$Y + Z = \bigcap_{j=1}^{m} (\langle y_1, \ldots, y_m \rangle + \langle z_{1j}, \ldots, z_{nj} \rangle) = \bigcap_{j=1}^{m} \langle x_1, \ldots, x_n \rangle = X,$$

and

$$Y \cap Z = (\langle y_1 \rangle \cap Z) + \ldots + (\langle y_n \rangle \cap Z) \subseteqq (\langle y_1 \rangle \cap y_1^*) + \ldots + (\langle y_n \rangle \cap y_n^*) = \langle 0 \rangle.$$

Hence Z is the complement of Y in the interval $[\langle 0 \rangle, X]$ of $\mathscr{I}_f(S)$, and so $\mathscr{I}_f(X)$ is a generalised boolean algebra.

(ii)$\Leftrightarrow$(iii). This is an immediate consequence of Theorem 2.1 and a well known characterization of generalized boolean algebras, c.f. GRÄTZER [6, Theorem 22 p. 76 and exercise 27 p. 79].

(ii)$\Rightarrow$(i). Suppose $\mathscr{I}_f(X)$ is a generalized boolean algebra and let $x, y \in S$ with $0 \leqq x \leqq y$. Because of the upper bound property the complement of $\langle x \rangle$ in the interval $[\langle 0 \rangle, \langle y \rangle]$ is a principal ideal and its generator is the complement of x in $\langle y \rangle$.

3. Congruences

A $\vee$-*congruence* θ of a semilattice S is a congruence of the algebra $(S; \wedge)$ such that if $x_i \equiv y_i(\theta)$ for $i = 1, \ldots, n$ and both $x_1 \vee \ldots \vee x_n$ and $y_1 \vee \ldots \vee y_n$ exist, then $x_1 \vee \ldots \vee x_n \equiv y_1 \vee \ldots \vee y_n(\theta)$. We note that if S satisfies the upper bound property then it is only necessary to state the last condition for $n = 2$.

The set $\mathscr{C}(S)$ of all $\vee$-congruences on S is an algebraic closure system on $S \times S$ and hence, when ordered by set-inclusion, is an algebraic lattice.

THEOREM 3.1. *Let J be an ideal of a weakly distributive semilattice S. Then the equivalence relation $\theta(J)$ defined by $x\equiv y(\theta(J))$ if and only if $(x]+J=(y]+J$ is the smallest $\vee$-congruence having J as a congruence class. Moreover the equivalence relation $R(J)$ defined by $x\equiv y(R(J))$ if and only if, for any $b\in S$, $x\wedge b\in J$ is equivalent to $y\wedge b\in J$ is the largest $\vee$-congruence having J as a congruence class.*

PROOF. Because of Theorem 1.1, $\theta(J)$ has the substitution property for $\wedge$ and it follows that $\theta(J)$ is a $\vee$-congruence having J as a congruence class. Let Φ be any $\vee$-congruence with this last property and let $x\equiv y(\theta(J))$. Without loss of generality, we may assume that $x\leqq y$. Then $y\in\langle x\rangle+J$ and so $y=x_1\vee\ldots\vee x_m\vee j_1\vee\ldots\vee j_n$ for some $x_i\in\langle x\rangle$ and $j_k\in J$. It follows that $y=x\vee j_1\vee\ldots\vee j_n$. Certainly, $x=x\vee(x\wedge j_1)\vee\ldots\vee(x\wedge j_n)$. Because $x\equiv x(\Phi)$ and $j_k\equiv x\wedge j_k(\Phi)$,

$$x\vee j_1\vee\ldots\vee j_n \equiv x\vee(x\wedge j_1)\vee\ldots\vee(x\wedge j_n)(\Phi).$$

That is to say $x\equiv y(\Phi)$. It is well known and easy to show that $R(J)$ is the largest congruence of the semilattice $(S;\ \wedge)$ having the hereditary set J as a congruence class. As S is weakly distributive, $R(J)$ is a $\vee$-congruence and our assertion follows.

Let S be a weakly distributive semilattice and $x\in S$. Then we shall use $\theta(x)$ as an abbreviation for $\theta(\langle x\rangle)$. Also, $\Psi(x)$ denotes the relation defined by $a\equiv b(\Psi(x))$ if and only if $a\wedge x=b\wedge x$; it is easy to see that $\Psi(x)$ is a $\vee$-congruence.

THEOREM 3.2. *Let S be weakly distributive and $a, b\in S$ with $a\leqq b$. Then $T(a, b)$, the smallest $\vee$-congruence identifying a and b is equal to $\Psi(a)\cap\theta(b)$.*

PROOF. Clearly $\Psi(a)\cap\theta(b)$ is a $\vee$-congruence which identifies a and b. Suppose Φ is another such congruence and $x\equiv y(\Psi(a)\cap\theta(b))$ where, without loss of generality, $x\leqq y$. Then $x\wedge a=y\wedge a$ and $y\in\langle x\rangle+\langle b\rangle$. It follows that $y=x\vee b_1\vee\ldots\vee b_n$ for suitable $b_i\in\langle b\rangle$. Since $a\equiv b(\Phi)$ and $x\wedge a=y\wedge a$, $x\wedge b\equiv y\wedge b(\Phi)$ and so $x\equiv x(\Phi)$ and $x\wedge b_i\equiv y\wedge b_i(\Phi)$ for $i=1, \ldots, n$. Therefore

$$y = y\wedge(x\vee b_1\vee\ldots\vee b_n) = x\vee(y\wedge b_1)\vee\ldots\vee(y\wedge b_n) \equiv x\vee(x\wedge b_1)\vee\ldots\vee(x\wedge b_n) = x.$$

That is, $x\equiv y(\Phi)$.

This last theorem shows that for any $a\in S$, a weakly distributive semilattice, $\Psi(a)\cap\theta(a)=\omega$, where ω is the smallest element of $\mathscr{C}(S)$ and is given by $x\equiv y(\omega)$ if and only if $x=y$. Now suppose S is an arbitrary semilattice and let Φ_1 and Φ_2 be two $\vee$-congruences on S. If V denotes the supremum in $\mathscr{C}(S)$ and $z_0, \ldots, z_n\in S$ are such that $z_{i-1}\equiv z_i(\Phi_1$ or $\Phi_2)$ for $i=1, \ldots, n$, then $z_0\equiv z_n(\Phi_1\vee\Phi_2)$. This last remark follows from the inequality $\Phi_1\vee\Phi_2\geqq\Phi_1\bigvee_{\mathscr{E}(S)}\Phi_2$, the supremum in the lattice of equivalence relations, $\mathscr{E}(S)$, on S. If S is weakly distributive we can use this comment to show that for all $a\in S$, $\Psi(a)\vee\theta(a)=\iota=\Psi(a)\bigvee_{\mathscr{E}(S)}\theta(a)$ where ι, the largest $\vee$-congruence on S, is given by $x\equiv y(\iota)$ for all $x, y\in S$: for suppose $x, y\in S$, then $x\equiv x\wedge a(\Psi(a))$, $x\wedge a\equiv y\wedge a(\theta(a))$ and $y\wedge a\equiv y(\Psi(a))$. Given the upper bound property we have more.

LEMMA 3.3. *Let S be a semilattice having the upper bound property and Φ_1, Φ_2 two $\vee$-congruences on S. Then $\Phi_1\vee\Phi_2=\Phi_1\bigvee_{\mathscr{E}(S)}\Phi_2$.*

PROOF. It suffices to show that $\Phi_1 \bigvee_{\mathscr{E}(S)} \Phi_2$ has the substitution property for $\vee$ since general theory tells us that $\Phi_1 \bigvee_{\mathscr{E}(S)} \Phi_2$ is the supremum of Φ_1 and Φ_2 in the lattice of semilattice congruences on S. Suppose $x_i \equiv y_i(\Phi_1 \bigvee_{\mathscr{E}(S)} \Phi_2)$ for $i=1, 2$, and assume both $x_1 \vee x_2$ and $y_1 \vee y_2$ exist. At first assume

$$(1) \qquad x_i \geqq y_i \quad \text{for} \quad i = 1, 2.$$

By assumption there exist elements $z_0^i, z_1^i, \ldots, z_{n(i)}^i$ of S such that $x_i = z_0^i, y_i = z_{n(i)}^i$ and $z_{k-1}^i = z_k^i(\Phi_1 \text{ or } \Phi_2)$ for $k=1, \ldots, n(i)$ and for $i=1, 2$. Set $m = \max\big(n(1)+1, n(2)+1\big)$. For $i=1, 2$ we perform the following alterations to the sequences.

(i) If for some j and k, $z_{k-1}^i \equiv z_k^i(\Phi_j)$ and $z_k^i \equiv z_{k+1}^i(\Phi_j)$ $\big($and so $z_{k-1}^i \equiv z_{k+1}^i(\Phi_j)\big)$ we omit z_k^i from the sequence and renumber the sequence in the natural way.

(ii) Repeat (i) until $z_{k-1}^i \equiv z_k^i(\Phi_j)$ implies $z_k^i \not\equiv z_{k+1}^i(\Phi_j)$.

(iii) If $z_0^i \not\equiv z_1^i(\Phi_1)$ introduce $z_{-1}^i = z_0^i$ and renumber in the natural way.

(iv) Introduce additional terms $z_m^i = z_{m-1}^i = \ldots = z_{n(i)+1}^i = z_{n(i)}^i$.

We now have two sequences $z_0^1, \ldots, z_m^1$ and $z_0^2, \ldots, z_m^2$ such that $x_i = z_0^i$, $y_i = z_m^i$, $z_{k-1}^i \equiv z_k^i(\Phi_1)$ if k is odd and $z_{k-1}^i \equiv z_k^i(\Phi_2)$ if k is even. Define $w_j^i = z_0^i \wedge \ldots \wedge z_j^i$ for $j=1, \ldots, m$ and $i=1, 2$. Therefore $x_i = w_0^i \geqq w_1^i \geqq \ldots \geqq w_m^i$ and also $w_{k-1}^i \equiv w_k^i(\Phi_1)$ if k is odd or $w_{k-1}^i \equiv w_k^i(\Phi_2)$ if k is even. Now since $x_i \geqq w_j^i, y_i$ for $j=1, \ldots, m$ and $i=1, 2$, $w_j^i \vee y_i$ exists and we set $v_j^i = w_j^i \vee y_i$.

Therefore $x_i = v_0^i \geqq v_1^i \geqq \ldots v_m^i = w_m^i \vee y_i = y_i$ and furthermore $v_{k-1}^i \equiv v_k^i(\Phi_1)$ if k is odd or $v_{k-1}^i \equiv v_k^i(\Phi_2)$ if k is even. Since $x_1 \vee x_2$ exists so does $v_j^1 \vee v_j^2$ for all $j=1, \ldots, m$ and hence $x_1 \vee x_2 = v_0^1 \vee v_0^2$, $y_1 \vee y_2 = v_m^1 \vee v_m^2$ and $v_{k-1}^1 \vee v_{k-1}^2 \equiv v_k^1 \vee v_k^2(\Phi_1 \text{ or } \Phi_2)$ proving that $x_1 \vee x_2 \equiv y_1 \vee y_2(\Phi_1 \bigvee_{\mathscr{E}(S)} \Phi_2)$.

If the inequalities in (1) no longer hold, then $x_i \equiv y_i(\Phi_1 \bigvee_{\mathscr{E}(S)} \Phi_2)$ implies $x_i \equiv \equiv x_i \wedge y_i(\Phi_1 \bigvee_{\mathscr{E}(S)} \Phi_2)$ and $y_i \equiv x_i \wedge y_i(\Phi_1 \bigvee_{\mathscr{E}(S)} \Phi_2)$ and so the result follows by an easy argument.

THEOREM 3.4. *Let S be a semilattice with the upper bound property and $\{\Phi_i\colon i \in I\}$ a collection of $\vee$-congruences on S. Then $x \equiv y\big(\bigvee_{i \in I} \Phi_i\big)$ if and only if there exists elements $z_0, \ldots, z_n$ of S such that $x = z_0$, $y = z_n$ and $z_{k-1} \equiv z_k(\Phi_{i(k)})$ for some $i(k) \in I$.*

PROOF. Assume $x \equiv y\big(\bigvee_{i \in I} \Phi_i\big)$ and so $T(x, y) \subseteq \big(\bigvee_{i \in I} \Phi_i\big)$. Since $T(x, y)$ is compact in $\mathscr{C}(S)$, there exist $i_1, \ldots, i_n \in I$ such that $T(x, y) \subseteq \Phi_{i_1} \vee \Phi_{i_2} \vee \ldots \vee \Phi_{i_n}$ and the proof follows from the above lemma. The reverse implication was covered by earlier remarks.

COROLLARY. *For a semilattice S with the upper bound property the $\vee$-congruences form a distributive sublattice of the lattice of (semilattice) congruences on S.*

PROOF. Let $x, y \in S$ with $x \geqq y$, $\Gamma, \Phi_1, \Phi_2 \in \mathscr{C}(S)$ and assume $x \equiv y\big(\Gamma \cap (\Phi_1 \vee \Phi_2)\big)$. It suffices to show that $x \equiv y\big((\Gamma \cap \Phi_1) \vee (\Gamma \cap \Phi_2)\big)$. By Theorem 3.4 there exist $z_0, \ldots, z_n \in S$ with $x = z_0 \geqq z_1 \geqq \ldots \geqq z_n = y$ such that $z_{k-1} \equiv z_k(\Phi_1 \text{ or } \Phi_2)$. Since $x \equiv \equiv y(\Gamma)$, $x \wedge z_i \equiv y \wedge z_i(\Gamma)$ and so $x_i \equiv y(\Gamma)$ for $i=0, 1, \ldots, n$. Therefore $z_{k-1} \equiv z_k(\Gamma)$ and hence $z_{k-1} \equiv z_k(\Gamma \cap \Phi_1 \text{ or } \Gamma \cap \Phi_2)$, giving the desired result.

We note that this last result was stated by GRÄTZER and LAKSER in [7], without proof.

Let S be a weakly distributive semilattice, and denote by $\mathscr{C}(\mathscr{I}_f(S))$ the set of all lattice congruences on $\mathscr{I}_f(S)$. We wish to investigate the relationship between $\mathscr{C}(\mathscr{I}_f(S))$ and $\mathscr{C}(S)$, and to this end we define the restriction map $\varrho\colon \mathscr{C}(\mathscr{I}_f(S))\to\mathscr{C}(S)$ by, for $\theta\in\mathscr{C}(\mathscr{I}_f(S))$, $x\equiv y(\varrho(\theta))$ if and only if $\langle x\rangle\equiv\langle y\rangle(\theta)$. We note the following results without proof.

(i) *If* $\{\theta_i\colon i\in I\}\subseteq\mathscr{C}(\mathscr{I}_f(S))$, *then* $\varrho(\bigcap_{i\in I}\theta_i)=\bigcap_{i\in I}\varrho(\theta_i)$.

(ii) *If* $T(\langle a\rangle,\langle b\rangle)$ *is a principal congruence on* $\mathscr{I}_f(S)$ *with* $\langle a\rangle\subseteq\langle b\rangle$ *then* $\varrho(T(\langle a\rangle,\langle b\rangle))=T(a,b)$.

THEOREM 3.5. *Let S be a weakly distributive semilattice with the upper bound property. Then* $\varrho\colon\mathscr{C}(\mathscr{I}_f(S))\to\mathscr{C}(S)$ *defined as above is a lattice isomorphism.*

PROOF. We first show that ϱ is a homomorphism. It has already been noted that ϱ preserves infima, so now suppose $x\equiv y(\bigvee_{i\in I}\varrho(\theta_i))$. By Theorem 3.4 there exist $z_0,\dots,z_n\in S$ such that $x=z_0$, $y=z_n$ and $z_{k-1}\equiv z_k(\varrho(\theta_{i(k)}))$ for $k=1,\dots,n$ and some $i(k)\in I$. Therefore, $\langle z_{k-1}\rangle\equiv\langle z_k\rangle(\theta_{i(k)})$ and so $\langle x\rangle\equiv\langle y\rangle(\bigvee_{i\in I}\theta_i)$ implying $x\equiv y(\varrho(\bigvee_{i\in I}\theta_i))$. Assume that $x\equiv y(\varrho(\bigvee_{i\in I}\theta_i))$, where, without loss of generality, $x\leqq y$. By the definition of ϱ we have $\langle x\rangle\equiv\langle y\rangle(\bigvee_{i\in I}\theta_i)$ and so there exist finitely generated ideals $J_0,\dots,J_n$ with $\langle x\rangle=J_0\subseteq J_1\subseteq\dots\subseteq J_n=\langle y\rangle$ and $J_{k-1}\equiv J_k(\theta_{i(k)})$ for $k=1,\dots,n$ and some $i(k)\in I$. Since S has the upper bound property, $J_0,\dots,J_n$ are all principal ideals and so there exist $z_0,\dots,z_n$ in S such that $J_k=\langle z_k\rangle$ and hence $z_{k-1}\equiv z_k(\varrho(\theta_{i(k)}))$ showing that $x\equiv y(\bigvee_{i\in I}\varrho(\theta_i))$.

Next we show ϱ is a surjection. Result (ii) above shows that there exists an inverse image for each principal $\vee$-congruence $T(a,b)$, with $a\leqq b$. But for any $\vee$-congruence Φ on S, $\Phi=\bigvee\{T(a,b)\colon a\leqq b, a\equiv b(\Phi)\}$ and so by what we have above $\Phi=\varrho(\theta)$ where $\theta\in\mathscr{C}(\mathscr{I}_f(S))$ defined as $\bigvee\{T(\langle a\rangle,\langle b\rangle)\colon a\leqq b,\ a\equiv b(\Phi)\}$.

Finally ϱ is injective, for suppose $\varrho(\theta)=\varrho(\Phi)$ with $\theta\subseteq\Phi$. Let $I=\langle x_1,\dots,x_m\rangle$, $J=\langle y_1,\dots,y_m\rangle\in\mathscr{I}_f(S)$ with $I\equiv J(\Phi)$. Therefore, $I\cap\langle a\rangle\equiv J\cap\langle a\rangle(\Phi)$ for all $a\in S$. Since S has the upper bound property, both $I\cap\langle a\rangle$ and $J\cap\langle a\rangle$ are principal and will be denoted by $\langle z_0\rangle$ and $\langle z_1\rangle$ respectively. Then, $z_0\equiv z_1(\varrho(\Phi))$ and since $\varrho(\Phi)=\varrho(\theta)$, $\langle z_0\rangle\equiv\langle z_1\rangle(\theta)$. That is, $I\cap\langle a\rangle\equiv J\cap\langle a\rangle(\theta)$ for all $a\in S$. In particular $I\cap\langle y_j\rangle\equiv J\cap\langle y_j\rangle(\theta)$ for $j=1,\dots,n$; that is, $I\cap\langle y_j\rangle\equiv\langle y_j\rangle(\theta)$ and therefore

$$\sum_j (I\cap\langle y_j\rangle)\equiv\Big(\sum_j\langle y_j\rangle\Big)(\theta).$$

Then $I\cap J\equiv J(\theta)$. Since $I\cap J\equiv I(\theta)$ is similar, ϱ is injective.

We now examine a particular example of a weakly distributive semilattice which shows the last result is not true in general. Let Z^- be the negative integers with their natural ordering and let $A=Z^-\cup\{x_1\}\cup\{x_2\}\cup\{a\}$ where we define x_1, x_2 to be less than all the elements of Z^- and set $a=x_1\wedge x_2$. It is clear that given this ordering, A is a weakly distributive semilattice which does not have the upper bound property, since $x_1\vee x_2$ does not exist. Furthermore the only ideal in $\mathscr{I}_f(A)$

which is not a principal ideal is $\langle x_1, x_2\rangle$. We define two congruences Φ_1 and Φ_2 on $\mathcal{J}_f(S)$ by $I\equiv J(\Phi_1)$ if and only if $I\cap\langle x_1, x_2\rangle=J\cap\langle x_1, x_2\rangle$ for all $I, J\in\mathcal{J}_f(S)$, while $I\equiv J(\Phi_2)$ if and only if $I+\langle x_1, x_2\rangle=J+\langle x_1, x_2\rangle$. The theory of congruences on distributive lattices shows that $\Phi_1\vee\Phi_2=\iota$, the largest congruence on $\mathcal{J}_f(S)$. From the very definition of ϱ, we have $\varrho(\Phi_2)=\theta(\langle x_1, x_2\rangle)$, while $a\equiv b(\varrho(\Phi_1))$ if and only if $\langle a\rangle\cap\langle x_1, x_2\rangle=\langle b\rangle\cap\langle x_1, x_2\rangle$ for all $a, b\in A$.

It is now clear that $\varrho(\Phi_1\vee\Phi_2)=\iota$, while $\varrho(\Phi_1)\vee\varrho(\Phi_2)$ has two distinct congruence classes, Z^- and $\{x_1, x_2, a\}$, and so we cannot have $\varrho(\Phi_1\vee\Phi_2)=\varrho(\Phi_1)\vee\varrho(\Phi_2)$.

In the case of semiboolean algebras the lattice of $\vee$-congruences has a special form, as shown by

THEOREM 3.6. *Let S be a weakly distributive semilattice with the upper bound property and assume $0\in S$. Then S is semiboolean if and only if the map $\Phi:\mathcal{J}(S)\to\mathcal{C}(S)$ defined by $\Phi(I)=\theta(I)$, for $I\in\mathcal{J}(S)$, is a lattice isomorphism.*

PROOF. Suppose S is semiboolean. It is clear that ϱ is an injection. Now let $x\equiv y(R(I))$ for some $I\in\mathcal{J}(S)$ and assume that $x\leqq y$. Since S is semiboolean, there exists x' in $\langle y\rangle$ such that $x\wedge x'=0$ and $x\vee x'=y$. Now $x\equiv y(R(I))$ and $x\wedge x'=0\in I$ implies that $y\wedge x'=x'\in I$. Since $y=x\vee x'$, $y\in\langle x\rangle+I$ and hence $\langle y\rangle+I=\langle x\rangle+I$. That is $x\equiv y(\theta(I))$. As $[0]\Phi(=\{x\in S\colon x\equiv 0(\Phi)\})$ is the only ideal which is a congruence class of Φ, for any $\Phi\in\mathcal{C}(S)$, Theorem 3.1 shows that this is sufficient to prove that ϱ is a surjection.

To show that ϱ is an isomorphism, suppose $x\equiv y(\theta(I+J))$ and $x\leqq y$. That is $\langle x\rangle+I+J=\langle y\rangle+I+J$, and so $y\in\langle x\rangle+I+J$ which implies that $y=x\vee i\vee j$ for some $i\in I$, $j\in J$. Since S has the upper bound property $x\vee i$ exists, and so $x\equiv x\vee i(\theta(I))$ while $x\vee i\equiv x\vee i\vee j(\theta(J))$ and this shows that $x\equiv y(\theta(I)\vee\theta(J))$. To prove the reverse implication we note that $a\equiv b(\theta(I)$ or $\theta(J))$ implies that $a\equiv b(\theta(I+J))$ and use Theorem 3.4.

Now assume $x\equiv y(\theta(I)\cap\theta(J))$. Therefore $\langle x\rangle+I=\langle y\rangle+I$ and $\langle x\rangle+J=\langle y\rangle+J$. Using this $\langle x\rangle+(I\cap J)=(\langle x\rangle+I)\cap(\langle x\rangle+J)=(\langle y\rangle+I)\cap(\langle y\rangle+J)=\langle y\rangle+(I\cap J)$, hence $x\equiv y(\theta(I\cap J))$, while the converse is trivial. Thus φ is an isomorphism.

If φ is an isomorphism let $x, y\in S$ with $x\leqq y$ and set $I=[0]\,T(x, y)$. By assumption $\theta(I)=T(x, y)$ and hence $x\equiv y(\theta(I))$; that is $\langle x\rangle+I=\langle y\rangle+I$. Thus $y=x\vee a$ for some $a\in I$, while $x\wedge a\in I$ and so $x\wedge a\equiv 0(T(x, y))$ and it follows that $x\wedge a=0$. Hence a is the complement of x in $\langle y\rangle$.

THEOREM 3.7. *The $\vee$-congruences on a weakly distributive semilattice S with 0 are permutable if and only if it is a generalised boolean algebra.*

PROOF. Let $a, b\in S$ be arbitrary. Since we have already shown that $\theta(a)\cap\Psi(a)=\omega$ and $\theta(a)\underset{\mathcal{E}(S)}{\vee}\Psi(a)=\iota$, the permutability of the $\vee$-congruences implies that S is isomorphic to the product $S/\Psi(a)\times S/\theta(a)$. Also $S/\Psi(a)$ has a largest element $1=[a]\Psi(a)$ and $S/\theta(a)$ has a smallest element $0=[a]\theta(a)$ and a corresponds to the element $(1, 0)$ of the product. If b corresponds to the element (x, y) then it is easy that the element of S corresponding to $(1, y)$ is the supremum of a and b. Hence S is a lattice. The rest follows from well known results.

COROLLARY. *Let S be a distributive semilattice with 0 and 1. Then, S is a boolean algebra if and only if the map $\Psi\colon\mathcal{F}(S)\to\mathcal{C}(S)$, where $\mathcal{F}(S)$ is the lattice of filters* *of S and for* $F\in\mathcal{F}(S)$, $\Psi(F)$ *is the* $\vee$*-congruence* $x\equiv y(\Psi(F))$ $(x, y\in S)$ *iff* $x\wedge f=y\wedge f$ *for some* $f\in F$, *is a lattice isomorphism.*

PROOF. This is an immediate consequence of the theorem since KATRINÁK [8; 3.3, p. 167] has proved that the above congruences associated with filters permute in a distributive semilattice.

We close this section with a discussion of the join in $\mathscr{C}(S)$ of two $\vee$-congruences in the case of a general weakly distributive semilattice. Firstly, we give an example of a weakly distributive semilattice whose $\vee$-congruences are not a sublattice of the lattice of semilattice congruences.

Let $A=\{a, b\}$ be the two element chain defined by $a<b$ and let $\mathcal{N}$ be the natural numbers with the usual order. Let $D=\mathcal{N}\cup A$ be the chain obtained by adjoining A above $\mathcal{N}$; that is $c\in A$ implies $c>n$ for all $n\in\mathcal{N}$. Define C to be the dual lattice of D. We partially order C^3 by saying $(x_1, x_2, x_3)\geqq(y_1, y_2, y_3)$ if and only if $x_i\geqq y_i$ in C for all $i=1, 2, 3$. Let $S'=\bigcup_{i\in C\setminus\{0\}}\{(0, i, 0), (i, i, 0), (0, i, i), (i, i, i)\}$ and induce the partial order of C^3 onto S'.

Define $S=S'\setminus\{(0, a, 0)\}$ and induce the partial order of S' onto S (recall that a is the second smallest element in C). Then we claim that S is a weakly distributive semilattice and that there exist two congruences Γ_1 and Γ_2 such that $\Gamma_1\bigvee_{\mathscr{C}(S)}\Gamma_2\neq\Gamma_1\bigvee_{\mathscr{E}(S)}\Gamma_2$. First we show that S is in fact a semilattice. To do this we need only show that $(0, a, 0)$ cannot be written as $x\wedge y$, $x, y\in S$. Suppose $x\wedge y=(0, a, 0)$. This implies $x\geqq(0, a, 0)$ and $y\geqq(0, a, 0)$. By the construction of S this would imply that x and y are of the form $(0, i, 0)$ and $(0, j, 0)$ for some i and j in C. Hence $x\wedge y=(0, i, 0)\wedge(0, j, 0)=(0, i\wedge j, 0)=(0, a, 0)$ and so $a=i\wedge j$ in C. Since C is a chain this implies $a=i$ or $a=j$ which is a contradiction. Hence S is a semilattice.

We note that the infimum of two elements in S is the same as the infimum of the same two elements in S', and of $y_1\vee\ldots\vee y_n$ exists in S then $y_1\vee\ldots\vee y_n$ exists in S' (since S' is a lattice) and the two suprema are equal. Furthermore it is straightforward to verify that S' is a distributive lattice, for example

$$(i, i, i)\wedge\big((j, j, 0)\vee(0, k, k)\big) = (i, i, i)\wedge(0, j\vee k, 0) = \big(i\wedge(j\vee k), i\wedge(j\vee k), i\wedge(j\vee k)\big),$$

while

$$\big((i, i, i)\wedge(j, j, 0)\big)\vee\big((i, i, i)\wedge(0, k, k)\big) = (i\wedge j, i\wedge j, i\wedge j)\vee(i\wedge k, i\wedge k, i\wedge k) =$$

$$= \big((i\wedge j)\vee(i\wedge k), (i\wedge j)\vee(i\wedge k), (i\wedge j)\vee(i\wedge k)\big) = \big(i\wedge(j\vee k), i\wedge(j\vee k), i\wedge(j\vee k)\big).$$

Hence to show that S is weakly distributive it suffices to show that the existence of $y_1\vee\ldots\vee y_n$ implies the existence of $(x\wedge y_1)\vee\ldots\vee(x\wedge y_n)$ in S for all $x\in S$. Therefore suppose $(x\wedge y_1)\vee\ldots\vee(x\wedge y_n)$ does not exist in S. By our construction this implies $(x\wedge y_1)\vee\ldots\vee(x\wedge y_n)=(0, a, 0)$ in S'. Since S' is distributive this implies $x\wedge(y_1\vee\ldots\vee y_n)=(0, a, 0)$ which leads to a contradiction if both x and $y_1\vee\ldots\vee y_n$ are in S.

Set $\Gamma_1=\theta\big((a, a, a)\big)$ and define Γ_2 as the equivalence relation which has the following equivalence classes (of more than two elements):

$$\{(i, i, 0)\colon\ i\geqq a\},\ \{(0, i, i)\colon\ i\geqq a\},\ \{(i, i, i)\colon\ i\geqq a\}\ \text{ and }\ \{(0, i, 0)\colon\ i>a\}.$$

It is easy to see that Γ_2 is in fact a $\vee$-congruence. Furthermore Γ_1 has as its non-trivial equivalence classes the sets $\{(a, a, 0), (b, b, 0)\}$, $\{(0, a, a), (0, b, b)\}$ and $\{(a, a, a), (b, b, b)\}$. Hence $\Gamma_1 \bigvee_{\mathscr{E}(S)} \Gamma_2$ is defined by the following classes,

$$\{(i, i, 0)\colon i \in C\setminus\{0\}\},\ \{(0, i, i)\colon i\in C\setminus\{0\}\},\ \{(i, i, i)\colon i\in C\setminus\{0\}\},\ \{(0, i, 0)\colon i > a\}$$
$$\text{and}\quad \{(0, b, 0)\}.$$

Hence $(b, b, 0) \equiv (1, 1, 0)\ (\Gamma_1 \bigvee_{\mathscr{E}(S)} \Gamma_2)$ and $(0, b, b) \equiv (0, 1, 1)\ (\Gamma_1 \bigvee_{\mathscr{E}(S)} \Gamma_2)$, both $(1, 1, 0)\vee \vee(1, 1, 0) = (0, 1, 0)$ and $(b, b, 0)\vee(0, b, b) = (0, b, 0)$ exist and yet $(0, 1, 0) \not\equiv \not\equiv (0, b, 0)\ (\Gamma_1 \bigvee_{\mathscr{E}(S)} \Gamma_2)$ showing that $\Gamma_1 \bigvee_{\mathscr{E}(S)} \Gamma_2 \neq \Gamma_1 \bigvee_{\mathscr{C}(S)} \Gamma_2$.

We now give a description of the supremum of $\vee$-congruences in an arbitrary weakly distributive semilattice.

THEOREM 3.8. *Let S be a weakly distributive semilattice and suppose $\{\Gamma_i\colon i\in I\} \subseteq \subseteq \mathscr{C}(S)$. Define $E_0 = \bigvee_{\mathscr{E}(S)} \Gamma_i$, the supremum of $\{\Gamma_i\colon i \in I\}$ in the lattice of equivalence relations on S. Inductively define F_n, E_{n+1} for $n = 0, 1, 2, \dots$ and $x, y\in S$ by $(x, y)\in F_n$ if and only if there exist $x_i, y_i \in S$ for $i = 1, \dots, m$ such that $(x_i, y_i)\in E_n$ and $x = x_1 \vee \dots \dots \vee x_m$, $y = y_1 \vee \dots \vee y_m$; and $(x, y)\in E_{n+1}$ if and only if there exist $z_0, \dots, z_l \in S$ with $x = z_0$, $y = z_l$ and $(z_{i-1}, z_i)\in F_n$ for $i = 1, \dots, l$. Let $T = \bigcup_{n<\omega} E_n$. Then $T = \bigvee_{I\,\mathscr{C}(S)} \Gamma_i$.*

PROOF. We prove by induction that E_k is a semilattice congruence. For $k = 0$ the result is well known. Assume E_{k-1} is a semilattice congruence. Then we claim the following are true of F_{k-1}.

(i) $(x, x)\in F_{k-1}$ for all $x\in S$.

(ii) $(x, y)\in F_{k-1}$ if and only if $(y, x)\in F_{k-1}$ for $x, y\in S$.

(iii) Suppose $(x^i, y^i)\in F_{k-1}$ for $i = 1, \dots, n$ where $x^i, y^i \in S$. Then $(x^1\wedge x^2, y^1\wedge y^2)\in \in F_{k-1}$ and if $x^1\vee \dots \vee x^n$ and $y^1 \vee \dots \vee y^n$ both exist then $(x^1\vee\dots\vee x^n, y^1\vee\dots\vee y^n)\in F_{k-1}$.

The first two of these are trivial. For the third, by definition, there exist elements $x_1^i, \dots, x_{m(i)}^i, y_1^i, \dots, y_{m(i)}^i$ of S such that $(x_j^i, y_j^i)\in E_{k-1}$ and $x^i = x_1^i \vee\dots\vee x_{m(i)}^i$, $y^i = y_1^i\vee\dots\vee y_{m(i)}^i$. Therefore $x^1\wedge x^2 = (x_1^1\wedge x_1^2)\vee(x_1^1\wedge x_2^2)\vee\dots\vee(x_{m(1)}^1\wedge x_{m(2)}^2)$ and $y^1\wedge y^2 = (y_1^1\wedge y_1^2)\vee(y_1^1\wedge y_2^2)\vee\dots\vee(y_{m(1)}^1\wedge y_{m(2)}^2)$. Now $(x_j^i\wedge x_l^k, y_j^i\wedge y_l^k)\in E_{k-1}$ since, by assumption, E_{k-1} is a semilattice congruence, and hence $(x^1\wedge x^2, y^1\wedge y^2)\in F_{k-1}$ from the definition. Also,

$$x^1\vee\dots\vee x^n = x_1^1\vee\dots\vee x_{m(1)}^1\vee x_1^2\vee\dots\vee x_{m(n)}^n$$

and

$$y^1\vee\dots\vee y^n = y_1^1\vee\dots\vee y_{m(1)}^1\vee y_1^2\vee\dots\vee y_{m(n)}^n$$

and thus (iii) is proved.

It is now easy to show that E_k is a semilattice congruence. It is reflexive by (i) and symmetric by (ii) while transitivity follows from the definition. The substitution property for $\wedge$ follows, after an easy manipulation, from (iii).

We note that $E_0\subseteq F_0\subseteq E_1\subseteq F_1\subseteq E_2\subseteq\dots$ and so $T = \bigcup E_n = \bigcup F_n$. T is obviously a semilattice congruence and clearly $T \subseteq \bigvee_{I\,\mathscr{C}(S)} \Gamma_i$. Now suppose $x_i, y_i\in S$ are such that $(x_i, y_i)\in T$ for $i = 1, \dots, n$ and both $x_1\vee\dots\vee x_n$ and $y_1\vee\dots\vee y_n$ exist. By construction there exists an integer k such that $(x_i, y_i)\in F_k$ for all $i = 1, \dots, n$ and so $(x_1\vee\dots\vee x_n, y_1\vee\dots\vee y_n)\in F_k\subseteq T$. It is now clear that $T = \bigvee_{\mathscr{C}(S)} \{\Gamma_i\colon i\in I\}$.

References

[1] J. C. ABBOTT, *Sets, lattices, and Boolean algebras*, Allyn and Bacon (Boston, 1969).
[2] J. C. ABBOTT, Semi-Boolean algebra, *Mat. Vesnik*, **4 (19)** (1967), 177—198.
[3] J. C. ABBOTT, Implication algebras, *Bull. Math. Soc. Sci. Math. Roumanie*, **11** (1967), 3—23.
[4] R. BALBES, A representation theory for prime and implicative semilattices, *Trans. Amer. Math. Soc.*, **136** (1969), 261—267.
[5] I. FLEISCHER, On extending congruences from partial algebras, *Fund. Math.*, **88** (1975), 11—16.
[6] G. GRÄTZER, *Lattice theory. First concepts and distributive lattices*, Freeman (San Francisco, 1971).
[7] G. GRÄTZER and H. LAKSER, Extension theorems on congruences of partial lattices, *Notices Amer. Math. Soc.*, **15** (1968), 732.
[8] T. KATRINÁK, Die Kennzeichnung der distributiven pseudokomplementären Halbverbände, *J. für reine und angew. Math.*, **241** (1970), 160—179.
[9] J. C. VARLET, On separation properties in semilattices, *Semigroup Forum* (to appear).

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SCHOOL OF MATHEMATICAL SCIENCES,
THE FLINDERS UNIVERSITY OF SOUTH AUSTRALIA,
BEDFORD PARK, 5042, S. A.,
AUSTRALIA

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THE RANGE SPACE IN A CLOSED GRAPH THEOREM

By

S. O. IYAHEN (Benin City)

1. A linear map from a Banach space into another is continuous if its graph is closed; it is also open of in addition it is onto. A Hausdorff locally convex space E is barrelled if and only if for each Banach space F, every linear map with a closed graph from E into F is continuous [8]. This note seeks the locally convex range spaces in a closed graph theorem for linear mappings from Banach spaces.

Throughout our linear topological spaces are over a fixed field either the real or complex numbers, and are assumed Hausdorff. A linear map from a linear topological space E to another F is called a closed map if its graph is closed in $E \times F$.

2. V. PTÁK, in [9; 3.6 and 3.8] proved that for a locally convex space E, the following conditions are equivalent:

(i) A continuous linear nearly open one-to-one map from E onto any locally convex space is open.

(ii) A closed linear nearly open one-to-one map from E onto any locally convex space is open.

(iii) A closed linear nearly continuous map from any locally convex space into E is continuous.

Notice that a linear map from a Banach space E into a linear topological space F is nearly continuous, and a linear map from F onto E is nearly open.

Let T be a class of locally convex spaces. As in [3], a locally convex space E is called *$B_r(T)$-space* if for each F in T, any continuous linear one-to-one map from E onto F is open. If "continuous" is replaced by "closed" in this definition, we call E a *$D_r(T)$-space*. We shall denote by $B_r(T)$ and $D_r(T)$ the classes of all $B_r(T)$-spaces and $D_r(T)$-spaces, respectively.

If T is the class of all barrelled spaces, the $D_r(T)$-spaces are the range spaces in the closed graph theorem for linear mappings from members of T [4; Theorem 3.1], and from this it follows that a closed linear subspace of a $D_r(T)$-space is also a $D_r(T)$-space. Every $D_r(T)$-space is a $B_r(T)$-space, but the converse fails [11].

PROPOSITION 1. *Let $(E, \tau) = \times_{\alpha \in \Phi} (E_\alpha, \tau_\alpha)$ be a topological product space of infinitely many non-trivial linear topological spaces. If t is a closed linear map from (E, τ) into a complete metric linear space F, then there is a countable subset Φ_0 of Φ such that*

$$t\Big(\times_{\alpha \in \Phi \setminus \Phi_0} E_\alpha\Big) = 0.$$

If F is locally bounded, then Φ_0 can be chosen to be finite.

PROOF. If η_α is the finest linear topology on E_α, then (E_α, η_α) is ultrabarrelled, and so is

$$(E, \eta) = \times_{\alpha \in \Phi} (E_\alpha, \eta_\alpha),$$

by (1). Since η is finer than τ, the map $t: (E, \eta) \to F$ is closed and therefore continuous by [10; Propositions 13 and 15 (ii)]. If $(V_n)_{n=1}^\infty$ is a fundamental sequence of neighbourhoods in F, then for each n there is a finite subset Ψ_n of Φ such that

$$t(\times_{\alpha \in \Phi \setminus \Psi_n} E_\alpha) \subseteqq V_n,$$

$$t(\times_{\alpha \in \Phi \setminus \bigcup_{n=1}^{\infty} \Psi_n} E_\alpha) \subseteqq \bigcap_{n=1}^{\infty} V_n = 0.$$

If F is locally bounded, V_1 may be taken to be bounded, and $\left(\frac{1}{n} V_1\right)_{n=1}^{\infty}$ is a fundamental sequence of neighbourhoods in F. In the above then,

$$t(\times_{\alpha \in \Phi \setminus \Psi_1} E_\alpha) \subseteqq \bigcap_{n=1}^{\infty} \frac{1}{n} V_1 = 0.$$

(cf. [5; Lemma 3.1]).

We shall denote by B_n and F_r the classes of all Banach and Fréchet spaces, respectively. Clearly $D_r(F_r) \subseteqq D_r(B_n)$.

COROLLARY 1. *Any infinite (uncountably infinite) product of locally convex spaces is a $D_r(B_n)$-space ($D_r(F_r)$-space). In particular any product of Fréchet spaces is a $D_r(F_r)$-space.*

COROLLARY 2. *A locally convex space (E, τ) has the finest locally convex topology if and only if for each $D_r(F_r)$-space F, every closed linear map from (E, τ) into F is continuous. In particular such E is finite dimensional if metrizable.*

PROOF. If η is the finest locally convex topology on E, then (E, η) is a closed linear subspace of some product space F of Banach spaces; F is a $D_r(F_r)$-space. The graph of the identity map $i: (E, \tau) \to (E, \eta)$ is closed in $(E, \tau) \times F$. If i is continuous then $\tau = \eta$.

If in the above argument (E, τ) is a Banach space of infinite dimension, then the closed subspace (E, η) of the $D_r(F_r)$-space F, is not a $D_r(B_n)$-space, and a closed linear map from (E, τ) in B_n into a $D_r(B_n)$-space F need not be continuous.

3. An absolutely convex bounded subset B of a locally convex E is called a *Banach disk* if the linear span of B is complete under the topology with the sequence $\left(\frac{1}{n} B\right)$ as a local base.

A closed sequentially complete absolutely convex bounded subset of a locally convex space is a Banach disk. A Banach disk need not be closed or sequentially complete, since the open unit ball in a Banach space is a Banach disk.

We shall use the notion defined in [2; page 3], of a generalized inductive limit of absolutely convex sets. Let G denote the class of all locally convex spaces which are generalized inductive limits of Banach disks, and G_m the subclass of G consisting of all members of G with the Mackey topology. Let N denote the class of all complete bornological spaces.

PROPOSITION 2. $D_r(G) \subset D_r(G_m) \subset D_r(N) \subseteqq D_r(F_r) \subset D_r(B_n)$.

PROOF. As $B_n \subset F_r \subset N \subset G_m \subset G$, we have that

$$D_r(G) \subseteqq D_r(G_m) \subseteqq D_r(N) \subseteqq D_r(F_r) \subseteqq D_r(B_n).$$

Now $D_r(G) \neq D_m(G_r)$. For if (E, τ) is a reflexive Banach space of infinite dimension, then by the method of proof of Proposition 3.3 of [6], (E, τ) is a $D_r(G_m)$-space. If η is the topology on E of compact convergence, η is strictly coarser than τ, and as (E, η) is in G, (E, τ) is not a $D_r(G)$-space.

$D_r(G_m) \neq D_r(N)$. For if (E, τ) is the strong dual of a Fréchet space and (E, τ) is not bornological (see for example [7; Page 221, Problem G]), then under its associated bornological topology η, E is a $D_r(N)$-space, but not a $D_r(G_m)$-space.

$D_r(F_r) \neq D_r(B_n)$. For, if (E_n, τ_n) is a Banach space of infinite dimension for each positive integer n, and η_n the finest locally convex topology on E_n, then the product space $(E, \eta) = \bigtimes_{n=1}^{\infty} (E_n, \eta_n)$ is a $D_r(B_n)$-space by Proposition 1, but not a $D_r(F_r)$-space, since the identity map from (E, η) onto the Fréchet space $\bigtimes_{n=1}^{\infty} (E_n, \tau_n)$ is continuous but not open.

Notice that with the same examples as above, Proposition 2 holds with $D_r(T)$ replaced throughout by $B_r(T)$.

Let C be the class of all locally convex spaces such that $E \in C$ if and only if either (i) $E \in B_n$ or (ii) E is a linear space under its finest locally convex topology or (iii) E is the topological direct sum of two spaces, one from each of (i) and (ii).

THEOREM 1. *A locally convex space (E, τ) is a $D_r(C)$-space if and only if for each F in C, any closed linear map from F into (E, τ) is continuous. This is true if and only if for each Banach space F, any closed linear map from F into (E, τ) is continuous.*

PROOF. Let t be a closed linear map from some F in C into a $D_r(C)$-space (E, τ). To prove that t is continuous, there is no loss of generality in assuming that F is a Banach space since every member of C is an inductive limit of Banach spaces. For some Hausdorff locally convex topology η, say, on E coarser than τ, $t\colon F \to (E, \eta)$ is continuous, because its graph is closed in $F \times (E, \tau)$. If λ is the finest locally convex topology on E for which t is continuous, then (E, λ) is topologically isomorphic to $(t(F), \Omega) \oplus H$, where Ω is the quotient topology of $F/t^{-1}(0)$, and H is an algebraic supplement of $t(F)$ in E, given the finest locally convex topology [7; Page 148, Problem C (a)]. Thus (E, λ) is in C. Since (E, τ) is a $D_r(C)$-space, and the identity map $(E, \tau) \to (E, \lambda)$ is closed, $\tau \leqq \lambda$. The map $t\colon F \to (E, \tau)$ is therefore continuous.

If $T_1 \subseteqq T_2$ are classes of locally convex spaces, then $D_r(T_2) \subseteqq D_r(T_1)$. Suppose that each member of T_2 is the inductive limit of some members of T_1, and that a closed linear map from any member of T_1 into a $D_r(T_1)$-space F is continuous. Then a closed linear map from a member of T_2 into F is necessarily continuous. This implies that $D_r(T_1) \subseteqq D_r(T_2)$, and thus $D_r(T_1) = D_r(T_2)$. From Theorem 1 we thus have the following

COROLLARY 1. *If B_n^* is the class of all inductive limits of Banach spaces, then* $D_r(C)=D_r(N)=D_r(B_n^*)$.

COROLLARY 2. *A closed linear subspace of a $D_r(N)$-space is a $D_r(N)$-space.*

COROLLARY 3. $D_r(N)\subset D_r(F_r)$.

PROOF. By Proposition 2, $D_r(N)\subseteqq D_r(F_r)$. That $D_r(N)\neq D_r(F_r)$ follows from Corollary 2 above and the remark after Proposition 1 Corollary 2.

If in Theorem 1, C is replaced by G or G_m, then the space $(t(F), \Omega)\oplus H$ is in G or G_m, respectively. The first part of Theorem 1 is thus true with C replaced by either G or G_m. The second part is false by the examples in Proposition 2 showing that $D_r(G)\subset D_r(G_m)\subset D_r(N)$. As can be seen from Propositions 3.2 and 3.3 of (6), a semi-Montel (semi-reflexive) $D_r(N)$-space is a $D_r(G)$-space $(D_r(G_m)$-space$)$. These two latter classes of spaces thus include the distribution spaces ξ, $\mathscr{D}$, $\mathscr{S}$, θ_M, θ_C, and their duals.

References

[1] N. ADASCH, Topologische Produkte gewisser topologischer Vektorraume, *Math. Ann.*, **189** (1970), 280—284.
[2] D. J. H. GARLING, A generalized form of inductive limit topology for vector spaces, *Proc. London Math. Soc.*, **14** (1964), 1—28.
[3] T. HUSAIN, Locally convex spaces with the $B(\tau)$-property, *Math. Ann.*, **164** (1962), 413—422.
[4] S. O. IYAHEN, $D(\tau;\, l)$-spaces and the closed graph theorem, *Proc. Edinburgh Soc.*, **16** (1968), 89—99.
[5] S. O. IYAHEN, Semiconvex spaces, *Glasgow Math. J.*, **9** (1968), 111—118.
[6] S. O. IYAHEN and J. O. POPOOLA, A generalized inductive limit topology for linear spaces, *Glasgow Math. J.*, **14** (1973), 105—110.
[7] J. L. KELLEY and I. NAMIOKA, *Linear topological spaces* (New York, 1963).
[8] N. MAHOWALD, Barrelled spaces and the closed graph theorem, *J. London Math. Soc.*, **36** (1961), 108—110.
[9] V. PTÁK, Completeness and the open mapping theorem, *Bull. Soc. Math. Fr.*, **86** (1958), 41—74.
[10] W. ROBERTSON, Completions of topological vector spaces, *Proc. London Math. Soc.*, **8** (1958), 242—257.
[11] L. J. SULLEY, On $B(\tau)$- and $B_r(\tau)$-locally convex spaces, *Proc. Cambridge Philos. Soc.*, **68** (1970), 95—97.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BENIN
BENIN CITY, NIGERIA

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MEROMORPHE ZWEIGE VON LÖSUNGEN BEI ASYMPTOTISCH LINEAREN OPERATORGLEICHUNGEN

Von

H. HAF (Kassel)

1. Einleitung

Im Zusammenhang mit Fragestellungen der mathematischen Physik treten häufig parameterabhängige Operatorgleichungen der Form

(1.1) $$0 = x + T(x, \lambda), \quad \lambda \in \mathbf{R},$$

mit asymptotisch linearen Operatoren in einem Banachraum X auf. Hierbei heißt $T(\cdot, \lambda)$ für festes λ nach KRASNOSELSKII [7] asymptotisch linear, falls ein linearer Operator L_λ existiert, mit

(1.2) $$\|T(x, \lambda) - L_\lambda x\| = o(\|x\|), \quad \text{für} \quad \|x\| \to \infty.$$

Man nennt L_λ die asymptotische Ableitung von $T(\cdot, \lambda)$. So führt z. B. das nichtlineare Sturm—Liouville-Problem

(1.3) $$-[p(x)y'(x)]' + q(x)y(x) = \lambda y(x) + f(x, y, y', \lambda), \quad x \in (0, \pi)$$

mit den Randbedingungen

(1.4) $$ay(0) + by'(0) = 0, \quad cy(\pi) + dy'(\pi) = 0$$

auf eine Integralgleichung der Form

(1.5) $$0 = y(x) - \int_0^\pi [\lambda y(t) + f(t, y(t), y'(t), \lambda)] G(x, t)\, dt,$$

wobei G die zugehörige Greensche Funktion ist. Für den Fall

(1.6) $$f(\cdot, u, v, \cdot) = o(\sqrt{u^2 + v^2}) \quad \text{für} \quad (u, v) \to \infty$$

ist der Integraloperator in (1.5) asymptotisch linear mit asymptotischer Ableitung

(1.7) $$L_\lambda y(x) = \lambda \int_0^\pi y(t) G(x, t)\, dt.$$

Bei der Behandlung von Problemen dieser Art sind Verzweigungsphänomene von großem Interesse. Ist $T(0, \lambda) = 0$ für alle $\lambda \in \mathbf{R}$, so bildet $\{(0, \lambda) : \lambda \in \mathbf{R}\}$ die Kurve der trivialen Lösungen von (1.1) in $X \times \mathbf{R}$. Falls für ein $\lambda_0 \in \mathbf{R}$ der Punkt $(0, \lambda_0)$ Häufungspunkt von nichttrivialen Lösungen von (1.1) ist, heißt $(0, \lambda_0)$ Verzweigungspunkt. Entsprechend nennt man (∞, λ_0) asymptotischen Verzweigungspunkt, falls es eine

Folge (x_n, λ_n) von Lösungen gibt mit $\lambda_n \to \lambda_0$ und $\|x_n\| \to \infty$. Bekanntlich steht das Lösungsverhalten von (1.1) in einem engen Zusammenhang zum Verhalten der asymptotischen Ableitung L_λ (vgl. z. B. [7]). Mit den üblichen Beweismethoden läßt sich für vollstetige Operatorscharen $T(\cdot, \lambda)$ zeigen (siehe hierzu Lemma 3): Bezeichne $\sigma(L_\lambda)$ das Spektrum von L_λ und gelte für ein $\lambda_0 \in \mathbf{R}$: $-1 \in \sigma(L_{\lambda_0})$ mit Vielfachheit $m < \infty$ ungerade, jedoch $-1 \notin \sigma(L_\lambda)$ in einer Umgebung $U(\lambda_0)/\{\lambda_0\}$ von λ_0. Dann liegt ein asymptotischer Verzweigungspunkt von (1.1) vor.

Dies besagt indessen keineswegs, daß die Lösungsmenge in der Umgebung von (∞, λ_0) eine Kurve enthält, wie sich anhand einfacher Beispiele zeigen läßt.

Die analoge Situation der Verzweigung der trivialen Lösung in $(0, \lambda_0)$, bei in $x=0$ Fréchet-differenzierbaren Abbildungen, ist insbesondere von J. SCHWARTZ [8] und R. BÖHME [1], [2] untersucht worden. Für den Fall komplexer Banachräume und vollstetiger Operatorschar $T(\cdot, \lambda)$, mit $T(x, \lambda)$ analytisch bezüglich λ und x (zum Analytizitätsbegriff vgl. Abschnitt 2) in gewissen Umgebungen von λ_0 bzw. $x=0$, konnte J. Schwartz die Existenz einer Verzweigungslösung in $(0, \lambda_0)$ nachweisen, die sich als Reihenentwicklung nach gebrochenen Potenzen von $(\lambda-\lambda_0)$ darstellen läßt. Das Schwartz'sche Resultat ergibt sich als Anwendung einer Verallgemeinerung eines Satzes von J. CRONIN [3], [4].

Eine systematische Untersuchung unter Einbeziehung reeller Banachräume und nicht notwendig vollstetiger Operatoren wurde von R. Böhme durchgeführt. Böhme legt seinen Betrachtungen das nichtlineare Eigenwertproblem

$$Ax + a_1(x) = (\mu+\lambda)(x + a_2(x)), \quad A \text{ linear,}$$

mit stetigen Operatoren A, a_1, a_2 zugrunde. (Die lineare Parameterabhängigkeit bedeutet keine wesentliche Einschränkung.) Auch hier stellt die Forderung der Analytizität von a_1 und a_2 die entscheidende Voraussetzung für die Existenz von nichttrivialen analytischen Lösungskurven dar. Insbesondere enthält die Arbeit von Böhme Aussagen über die Anzahl der Lösungskurven. Diese Resultate ergeben sich durch Betrachtung topologischer Invarianten und Anwendung funktionentheoretischer Sätze.

In der vorliegenden Arbeit wird in Anlehnung an Beweismethoden von Böhme die Frage untersucht, unter welchen Voraussetzungen an $T(\cdot, \lambda)$ Kurven von Lösungen von (1.1) durch (∞, λ_0) existieren und welche Eigenschaften diese besitzen. Wir beschränken uns hierbei auf reelle Banachräume und leiten durch Inversion

$$x \to \frac{x}{\|x\|^2} \tag{1.8}$$

in Abschnitt 3 eine zu (1.1) äquivalente Gleichung der Form

$$0 = z + \tilde{T}(z, \lambda) \tag{1.9}$$

her, die den Methoden von Böhme zugänglich ist. Es zeigt sich, daß hierbei die asymptotische Ableitung L_λ von $T(\cdot, \lambda)$ in die Fréchetableitung von $\tilde{T}(\cdot, \lambda)$ in $z=0$ übergeht. Ein Vergleich mit den Ergebnissen von Böhme zeigt: Die Eigenschaft der asymptotischen Differenzierbarkeit ist zu schwach, um die Existenz einer Kurve durch (∞, λ_0) zu gewährleisten. Sogar dann, wenn $\tilde{T}$ unendlich oft differenzierbar ist, muß es keine Lösungskurve geben, wie das bekannte Beispiel: $X=\mathbf{R}^2$ und

$$\tilde{T}(x, y; \lambda) = \begin{pmatrix} [-1+(1-\lambda)^3]x-(1-\lambda)y^2 \\ -\lambda y+y^2+\sin\frac{1}{x}\cdot\exp\left(-\frac{1}{x^2}\right) \end{pmatrix} \tag{1.10}$$

zeigt. Dagegen folgt unter stärkeren Voraussetzungen an $T(\cdot, \lambda)$, genauer: an $T(\cdot, \lambda)-L_\lambda$, die Existenz von mindestens zwei meromorphen Lösungskurven, die sich nur in (∞, λ_0) treffen. (Siehe hierzu Satz 1.)

2. Definitionen und Hilfssätze

Wir stellen einige Hilfsmittel zusammen, die wir im folgenden benötigen.

Definition 1. Sei X ein reeller Banachraum, $U\subset X$ offen, und T eine Abbildung von X in sich. T heißt reell analytisch in U, falls es zu jedem $x\in U$ eine η-Umgebung $K_\eta(x)$ gibt, so daß T in $K_\eta(x)$ durch eine gleichmäßig konvergente Taylorreihe der Form

$$T(x+h) = \sum_{n=0}^{\infty} \frac{1}{n!}(D^n T(x))h$$

dargestellt werden kann. Hierbei bedeutet $D^nT(x)$ die n-te Fréchetableitung von T in x (vgl. [5]).

Für analytische Abbildungen gilt die folgende Verschärfung des Satzes über implizite Funktionen:

Lemma 1. *Seien X, Y, Z (reelle) Banachräume, $\Omega\subset X\times Y$ offen und $(0, 0)\in\Omega$. Ferner sei $T\colon \Omega\to Z$ analytisch in Ω mit $T(0, 0)=0$. Die partielle Ableitung $D_2T(0, 0)$ von T im Punkt $(0, 0)$ bezüglich Z sei ein Isomorphismus von Y auf Z. Dann existiert eine Umgebung U von $0\in X$ und eine analytische Abbildung $g\colon U\to X$ mit $g(0)=0$ und $T(x, g(x))=0$ für alle $x\in U$.*

Beweis. Siehe R. Böhme [2].

Schließlich benötigen wir noch

Lemma 2 (curve selection Lemma). *Bezeichne Ω eine offene Umgebung von $0\in\mathbf{R}^p$ und $f_1, \dots, f_m$ sowie $g_1, \dots, g_n$ analytische Abbildungen von Ω in $\mathbf{R}$. Die Mengen M und Q seien durch*

$$M := \{x\in\Omega\colon f_i(x) = 0, \quad i = 1, \dots, m\}$$

$$Q := \{x\in\Omega\colon g_i(x) > 0, \quad i = 1, \dots, n\}$$

erklärt. Falls $0\in\overline{M\cap Q}$[1] gilt, enthält $M\cap Q$ in einer Umgebung von 0 endlich viele analytische Zweige, die sich nur in 0 treffen, genauer, es existieren Intervalle $I_\nu := (-\varepsilon_\nu, \varepsilon_\nu)$ und Abbildungen $\varphi_\nu\colon I_\nu\to\mathbf{R}^p$ mit

1. *φ_ν bildet $(-\varepsilon_\nu, 0)$ und $(0, \varepsilon_\nu)$ je homeomorph auf einen Zweig ab;*

[1] Hierbei ist $\overline{M\cap Q}$ die Abschließung von $M\cap Q$.

2. φ_ν *läßt sich durch*

$$\varphi_\nu(t) = \sum_{i=1}^{\infty} c_{i\nu} t^i, \; c_{i\nu} \in \mathbf{R}^p,$$

darstellen.

BEWEIS. Siehe [2].

Wir beweisen nun in Analogie zu den Ausführungen von BÖHME [2] und in Anlehnung an ein Seminar von K. KIRCHGÄSSNER [6]

LEMMA 3. *Sei X ein (reeller) Banachraum, $\Omega \subset X \times \mathbf{R}$ eine offene und beschränkte Menge, $(0, \lambda_0) \in \Omega$, und $\tilde{T}$ eine vollstetige Abbildung von $\bar{\Omega}$ in X. Ferner sei $\tilde{T}$ analytisch in einer Umgebung von $(0, \lambda_0)$, $\tilde{T}(0, \lambda) = 0$ für alle $(0, \lambda) \in \Omega$ und $D_1 \tilde{T}(0, \lambda) = h(\lambda) \cdot D_1 \tilde{T}(0)$, mit einer in einer Umgebung von λ_0 erklärten (analytischen) streng monotonen reellwertigen Funktion $h(\lambda)$. Gilt dann: $-1 \in \sigma(D_1 \tilde{T}(0, \lambda_0))$ mit ungerader Vielfachheit m und für $\lambda \neq \lambda_0$: $-1 \notin \sigma(D_1 \tilde{T}(0, \lambda))$ in einer Umgebung von λ_0, dann existieren mindestens zwei analytische Zweige von Lösungen der Gleichung*

$$0 = x + \tilde{T}(x, \lambda), \tag{2.1}$$

die sich nur im Punkt $(0, \lambda_0)$ treffen.

BEWEIS. Aufgrund der Voraussetzungen von Lemma 3 ist $(0, \lambda)$ für λ hinreichend nahe bei λ_0 isolierte Lösung von (2.1). Daher existieren λ^-, λ^+ mit $\lambda^- < \lambda_0 < \lambda^+$ und eine Umgebung $K_\varepsilon(0) = \{x : \|x\| < \varepsilon\}$ von $x = 0$, so daß es außer $(0, \lambda^+)$ bzw. $(0, \lambda^-)$ keine weiteren Lösungen auf $K_\varepsilon(0) \times \{\lambda^+\}$ bzw. $K_\varepsilon(0) \times \{\lambda^-\}$ gibt. Da m nach Voraussetzung ungerade ist, folgt unter Verwendung der Homotopie-Invarianz des Abbildungsgrades die Existenz einer Lösung von (2.1) auf dem Zylinder $\dot{K}_\varepsilon(0) \times (\lambda^-, \lambda^+)$, wobei $\dot{K}_\varepsilon(0)$ der Rand von $K_\varepsilon(0)$ ist. D.h. $(0, \lambda_0)$ ist Verzweigungspunkt von (2.1).

Zum Nachweis von analytischen Lösungszweigen läßt sich Problem (2.1) nach der bekannten Methode von Lyapunow—Schmidt auf die Untersuchung einer analytischen Mannigfaltigkeit im $\mathbf{R}^m$ zurückführen. Bezeichne X_1 den Nullraum des vollstetigen Operators $\mathrm{id} + D_1 \tilde{T}(0, \lambda_0)$; dann läßt sich X als direkte Summe $X = X_1 \oplus X_2$ (X_1, X_2 abgeschlossen) darstellen. P_1, P_2 seien die zugehörigen Projektionsoperatoren und $u_1 := P_1 u$, $u_2 := P_2 u$. Ferner verwenden wir die Abkürzung

$$W(x, \lambda - \lambda_0) := \tilde{T}(x, \lambda) - \tilde{T}(0, \lambda_0) - (D\tilde{T}(0, \lambda_0))(x, \lambda - \lambda_0). \tag{2.2}$$

Dann ist (2.1) äquivalent zu

$$\begin{cases} 0 = u_1 + D_1 \tilde{T}(0, \lambda_0) u_1 + (\lambda - \lambda_0) P_1 D_2 \tilde{T}(0, \lambda_0) + P_1 W(u_1 + u_2, \lambda - \lambda_0) \\ 0 = u_2 + D_1 \tilde{T}(0, \lambda_0) u_2 + (\lambda - \lambda_0) P_2 D_2 \tilde{T}(0, \lambda_0) + P_2 W(u_1 + u_2, \lambda - \lambda_0). \end{cases} \tag{2.3}$$

Falls (u_1, λ) in einer hinreichend kleinen Umgebung von $(0, \lambda_0)$ liegt, gibt es wegen Lemma 1 eine bezüglich u_1 und $\lambda - \lambda_0$ analytische Funktion Φ mit $u_2 = \Phi(u_1, \lambda - \lambda_0)$. Damit ist (2.1) äquivalent zu

$$0 = [\mathrm{id} + D_1 \tilde{T}(0, \lambda_0)] u_1 + (\lambda - \lambda_0) P_1 D_2 \tilde{T}(0, \lambda_0) + P_1 W[u_1 + \Phi(u_1, \lambda - \lambda_0), \lambda - \lambda_0]. \tag{2.4}$$

Wegen $\dim X_1 \leqq m < \infty$ läßt sich in X_1 eine Basis einführen und die rechte Seite von (2.4) als analytische Abbildung ψ einer Umgebung von $0 \in \mathbf{R}^{m+1}$ in $\mathbf{R}^m$ auffassen. Damit kann (2.4) in der Form

$$0 = \psi(w, \lambda - \lambda_0) \tag{2.5}$$

geschrieben werden. Setzen wir für $g(x)$ etwa $\|x\|^2$, so folgt mit Lemma 2 die Behauptung.

3. Meromorphe Zweige von Lösungen

Wir untersuchen in diesem Abschnitt die Gleichung

$$0 = x + T(x, \lambda), \tag{3.1}$$

mit $T: X \times \mathbf{R} \to X$ asymptotisch linear für alle $\lambda \in I$, wobei X ein reeller Banachraum[2] und I ein beschränktes offenes Intervall ist. Daher existiert für jedes $\lambda \in I$ ein bezüglich x linearer Operator $L(\cdot, \lambda)$ und eine Abbildung K, $L, K: X \times \mathbf{R} \to X$, so daß

$$K(x, \lambda) = o(\|x\|) \quad \text{für} \quad \|x\| \to \infty, \quad \text{gleichmäßig in } I, \tag{3.2}$$

und

$$T(x, \lambda) = L(x, \lambda) + K(x, \lambda) \tag{3.3}$$

gilt. Damit läßt sich (3.1) in der Form

$$0 = x + T(x, \lambda) = x + L(x, \lambda) + K(x, \lambda) \tag{3.4}$$

darstellen. Ist (x, λ) aus der Lösungsmenge von (3.4) und $\|x\| \neq 0$, dann folgt aus (3.4) durch Inversion

$$x \to \frac{x}{\|x\|^2} =: z \tag{3.5}$$

die Gleichung

$$0 = z + \tilde{T}(z, \lambda) = z + L(z, \lambda) + J(z, \lambda), \tag{3.6}$$

mit

$$J(z, \lambda) := \|z\|^2 K\left(\frac{z}{\|z\|^2}, \lambda\right), \quad \|z\| \neq 0. \tag{3.7}$$

Wir setzen bei stetigem K $J(z, \lambda)$ stetig durch die Festlegung $J(0, \lambda) = 0$ in den Punkt $z = 0$ fort. Der Bedingung (3.2) entspricht im Zusammenhang mit (3.6) die Bedingung

$$J(z, \lambda) = o(\|z\|) \quad \text{für} \quad \|z\| \to 0, \quad \text{gleichmäßig in } I. \tag{3.8}$$

Ist umgekehrt z eine nichttriviale Lösung von (3.6), so genügt

$$x := \frac{z}{\|z\|^2} \tag{3.9}$$

der Ausgangsgleichung (3.1). Aus Lemma 3 ergibt sich dann unter Berücksichtigung von (3.9) der folgende

[2] Im Folgenden betrachten wir solche Banachräume, für die $\|x\|^2$ analytisch ist.

SATZ 1. *Sei X ein reeller Banachraum. Die Abbildungen T, L, K, J: $X\times\mathbf{R}\to X$ seien wie oben erklärt, K stetig in $X\times\mathbf{R}$, L und J vollstetig und analytisch in einer Umgebung von $(0, \lambda_0)$. Für L sei außerdem $L(0, \lambda)=0$, $\lambda\in\mathbf{R}$, erfüllt und die Abhängigkeit vom Parameter λ wie in Lemma 3. Gilt dann: $-1\in\sigma(L(\cdot, \lambda_0))$ mit ungerader Vielfachheit und $-1\notin\sigma(L(\cdot, \lambda))$ für λ hinreichend nahe bei λ_0, so existieren mindestens zwei meromorphe Lösungskurven der Gleichung (3.1), die sich nur im Punkt (∞, λ_0) treffen. Diese genügen für hinreichend kleine $t\in\mathbf{R}$ einer Parameterdarstellung der Form*

$$\varphi(t)=\sum_{i=-k}^{\infty} c_i t^i, \tag{3.10}$$

mit $c_i=(y_i, r_i)\in X\times\mathbf{R}$, $c_k\neq 0$.

BEISPIEL. Ein einfaches Beispiel ist durch $X=\mathbf{R}$ und $0=x+\lambda x+\frac{1}{x}$ gegeben.

Literaturverzeichnis

[1] R. BÖHME, Nichtlineare Störung der isolierten Eigenwerte selbstadjungierter Operatoren, *Math. Z.*, **123** (1971), 61—92.

[2] R. BÖHME, Die Lösung der Verzweigungsgleichungen für nichtlineare Eigenwertprobleme, *Math. Z.*, **127** (1972), 105—126.

[3] J. CRONIN, Analytic functional mappings, *Ann. of. Math.*, **58** (1953), 175—181.

[4] J. CRONIN, Branch points of solutions of equations in Banach spaces II, *Trans. Amer. Math. Soc.*, **76** (1954), 207—222.

[5] E. HILLE, *Functional analysis and semigroups*. Amer. Math. Soc. Colloquium Publications **31** (New York, 1948).

[6] K. KIRCHGÄSSNER, Manuskript zu einem Seminar über „*Nonlinear functional analysis*". (Universität Stuttgart).

[7] M. A. KRASNOSELSKII, *Topological methods in the theory of nonlinear integral equations*. Pergamon Press (Oxford, 1960).

[8] J. SCHWARTZ, Compact analytic mappings of *B*-spaces and a theorem of Jane Cronin, *Comm. Pure Appl. Math.*, **16** (1963), 253—260.

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GESAMTHOCHSCHULE KASSEL
HEINRICH-PLETT-STR. 40
3500 KASSEL
BUNDESREPUBLIK DEUTSCHLAND

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TWO PROBLEMS ON k_R-SPACES

By
J. L. BLASCO* (Valencia)

1. Introduction. A space X is called a k_R-space when every real-valued function with domain X is continuous if its restriction to each compact subset of X is continuous. In this paper we answer the question proposed by K. Bierstedt: If X is a k_R-space so is every open subspace? We give a positive solution when X is a completely regular Hausdorff space and we construct an example of a regular Hausdorff k_R-space which possesses an open subspace that is not a k_R-space.

Associated with each completely regular Hausdorff space X there is a unique completely regular Hausdorff k_R-space, k_RX, having the same underlying set and the same compact subsets as X. The space k_RX is X equipped with the smallest topology making continuous each real-valued function on X whose restriction to compact subsets is continuous. In [1] FROLÍK characterizes the class $\mathfrak{B}$ of the spaces X such that for every pseudocompact space Y the topological product $X \times Y$ is pseudocompact. Let $\mathfrak{B}^*$ be the class of the spaces with the property: Each infinite collection of mutually disjoint, non-empty, open subsets has an infinite subcollection each member of which meets some fixed compact set. According to FROLÍK ([1], T. 3.5, E. 3.7) $\mathfrak{B}^*$ is a proper subclass of $\mathfrak{B}$. Noble has proved that every k-compact space (spaces X for which k_RX is pseudocompact) is in $\mathfrak{B}^*$ ([5], T. 2.1) and leaves as unsolved question if $\mathfrak{B}^*$ is the class of k-compact spaces. A negative answer is provided in Section 3. I am informed by A. Kato that utilizing the Stone—Čech compactification of an uncountable discrete space, he has constructed a space in $\mathfrak{B}^*$ which is not k-compact [3]. Our example is different from the former one.

2. Open subspaces of k_R-spaces. In ([5], 2.3) it is proved that every completely regular Hausdorff space can be embedded as a closed subspace into a pseudocompact completely regular Hausdorff k_R-space, therefore, not every closed subspace of a k_R-space is a k_R-space.

Let X be a topological space and let f be a real-valued function with domain X. If the restriction of f to each of an arbitrary number of open sets, whose union is X, is continuous, then f is continuous. Consequently, we have the following

PROPOSITION. *If every point of a topological space X has a neighbourhood that, provided the topology induced by X, is a k_R-space, then X is a k_R-space.*

If X is a topological space we write $C(X)$ for the ring of all continuous real-valued functions on X and if $f \in C(X)$, $\operatorname{coz} f$ will be the set $\{x \in X : f(x) \neq 0\}$.

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THEOREM. *If X is a completely regular Hausdorff k_R-space, every open subspace of X is a k_R-space.*

PROOF. If $f\in C(X)$ and $A=\operatorname{coz} f$, it will be seen that A is a k_R-space. Since k_RA is completely regular, a basis of neighbourhoods of a point $x\in k_RA$ is the family $\{\operatorname{coz} g: g\in C(k_RA),\ x\in \operatorname{coz} g\}$ and what will be proved is that if $x\in\operatorname{coz} g$, $g\in C(k_RA)$, there exists $h\in C(X)$ such that $x\in\operatorname{coz} h\subset\operatorname{coz} g$, therefore $A=k_RA$ and A will be a k_R-space. Let x be a point of A and let $l\in C(X)$ such that $\overline{\operatorname{coz} l}\cap\{X\sim\operatorname{coz} f\}=\emptyset$, $x\in\operatorname{coz} l$. Given $g\in C(k_RA)$ satisfying $x\in\operatorname{coz} g\subset\operatorname{coz} l$, let $h(y)=g(y)\cdot l(y)$ if $l(y)\neq 0$, $h(y)=0$ if $l(y)=0$. Then $h\in C(k_RX)=C(X)$ and $\operatorname{coz} l\cap\operatorname{coz} h=\operatorname{coz} g$; on the other hand, $\operatorname{coz} l\cap\operatorname{coz} h=\operatorname{coz}(l.\,h)$, $l.\,h\in C(X)$, therefore $\operatorname{coz} g=\operatorname{coz}(l.\,h)$ and A is a k_R-space. Since X is completely regular, every open subspace is a union of sets of the form $\operatorname{coz} f, f\in C(X)$. According to the latter proposition, every open subspace of X is a k_R-space.

The following example shows that the assumption of complete regularity of X is essential.

If β is an ordinal we write $\beta+1$ for the ordinal which follows β and $w(\beta)$ for the set of all ordinals less than β, provided with the order topology. The smallest ordinal of cardinal $\aleph_\alpha$ is denoted by ω_α. If P is the product spacc $w(\omega_1+1)\times\times w(\omega_0\times 1)$, let S be the subspace $P\sim\{(\omega_1,\omega_0)\}$ and let $\{S_n: n\in Z\}$[1] a countable family of copies of S. If A (resp. x) is a subset (resp. a point) of S, we will write A_n (resp. x_n) to denote it in the copy S_n. The topological sum Σ of the spaces S_n, $n\in Z$, is a locally compact Hausdorff space and if we identify in it $(\{\omega_1\}\times w(\omega_0))_{2n}$ with $(\{\omega_1\}\times w(\omega_0))_{2n+1}$ and $(w(\omega_1)\times\{\omega_0\})_{2n-1}$ with $(w(\omega_1)\times\{\omega_0\})_{2n}$, $n\in Z$, the quotient space Σ^* is a locally compact Hausdorff space because the canonical projection φ from Σ onto Σ^* is a perfect map ([4], 5—20). For brevity we will write $\varphi((\alpha,\beta)_n)=(\alpha,\beta)_n^*$ and $\varphi(A_n)=A_n^*$.

Now let $\{\Sigma_p^*: p\in N\}$ be a collection of copies of Σ^* and let Y be its topological sum. Evidently, Y is a locally compact Hausdorff space. We will write $(\alpha,\beta)_{p,n}^*$ for the point $(\alpha,\beta)_n^*\in\Sigma^*$ considered in the copy Σ_p^*. Let X be the set $Y\cup\{\Omega\}$ where Ω is a point which does not belong to Y, and we will define a topology on X. A basis of the neighbourhoods in X of a point of Y will be the family of the neighbourhoods of this point in Y.

Let $\mathcal{U}$ be a free ultrafilter on N.[2] If $p\in N$ and $n\in Z$ we write

$$H(p,n,\alpha_0,\beta_0)=\{(\alpha,\beta)_{p,n}^*\in S_{p,n}^*:\ \alpha_0\leqq\alpha\leqq\omega_1,\ \beta_0\leqq\beta\leqq\omega_0\}$$

where $\alpha_0<\omega_1$ and $\beta_0<\omega_0$. A basis of neighbourhoods of Ω in X will be the family of the sets having the form

$$\{\Omega\}\cup\big[\cup\{H(p,n,\alpha(p,n),\beta(p,n)):\ \beta(p,2n')=\beta(p,2n'+1),\alpha(p,2n'-1)=$$
$$=\alpha(p,2n'),\ p\in F,\ n\leqq -n_0\}\big]\cup\big[\cup\{S_{p,n}^*:\ p\geqq p_0,\ n\geqq n_0\}\big]$$

where $F\in\mathcal{U}$ and $n_0, p_0\in N$. The set X provided with this topology is a regular Hausdorff space.

[1] Z is the set of all integer numbers and $N=\{x\in Z:\ x\geqq 1\}$.
[2] An ultrafilter $\mathcal{F}$ on N is said to be free if $\cap\{F:\ F\in\mathcal{F}\}=\emptyset$.

(I) *If f is a continuous real-valued function on Σ_p^* it can be verified that:*

(a) *For every $n\in Z$, $\lim_\beta f((\omega_1, \beta)_{p,n}^*)=s$.*

(b) *For each $n\in Z$ there exists $\alpha(p, n)<\omega_1$ such that $f((\sigma, \omega_0)_{p,n}^*)=s$ for every $\sigma\geqq\alpha(p, n)$.*

(c) *If ε is a positive number, for each $n\in Z$ there exists a set, $H(p,n,\alpha(p,n),\beta(p, n))$ such that $\beta(p, 2n')=\beta(p, 2n'+1)$, $\alpha(p, 2n'-1)=\alpha(p, 2n')$ and*

$$H(p, n, \alpha(p, n), \beta(p, n)) \subset \{x\in\Sigma_p^*: |f(x)-s| < \varepsilon\}.$$

In fact, if f is a continuous real-valued function on Σ_p^*, its restriction to $S_{p,n}^*$ is continuous, so there exists $\alpha(p, n)<\omega_1$ such that $f((\sigma, \beta)_{p,n}^*)=f(\alpha(p, n), \beta)_{p,n}^*)$ for every $\sigma\geqq\alpha(p, n)$ and $\beta\in w(\omega_0)$ ([2], 8.20). By continuity it follows that $f((\sigma, \omega_0)_{p,n}^*))=f((\alpha(p, n), \omega_0)_{p,n}^*)=s_n$ for every $\sigma\geqq\alpha(p, n)$ and moreover that $\lim_\beta f((\omega_1, \beta)_{p,n}^*)=s_n$. As a consequence of the identifications $\{(\eta, \omega_0)_{p,2n-1}, (\eta, \omega_0)_{p,2n}\}$ and $\{(\omega_1, \beta)_{p,2n}, (\omega_1, \beta)_{p,2n+1}\}$, $n\in Z$, it is concluded that if $n, m\in Z$ then $s_n=s_m$, with which (a) and (b) are proved. Part (c) is a direct consequence of (a) and (b).

(II) *X is a k_R-space.* If T is the subspace of X consisting of

$$\{\Omega\}\cup[\cup\{S_{p,n}^*: p \geqq 1, n \geqq 1\}],$$

then T is a k_R-space because Ω has a countable neighbourhood basis and the remaining points of T have a basis of compact neighbourhoods. If f is a real-valued function which is continuous on each compact subset of X, the restriction of f to T is continuous, and therefore, given a positive number ε there exist two positive integers $p_0, n_0\geqq 2$ such that

$$\cup\{S_{p,n}^*: p\geqq p_0, n\geqq n_0\} \subset \{x\in T: |f(x)-f(\Omega)| < \varepsilon/2\}.$$

Let F be a member of $\mathcal{U}$ satisfying $F\subset\{p\in N: p\geqq p_0\}$; if $p\in F$, from (c), for each $n\leqq -n_0$ there exists a set $H(p, n, \alpha(p, n), \beta(p, n))$ contained in $\{x\in\Sigma_p^*: |f(x)-s|<\varepsilon/2\}$ with $\beta(p, 2n')=\beta(p, 2n'+1)$, $\alpha(p, 2n'-1)=\alpha(p, 2n')$ and $s=\lim_\beta f((\omega_1, \beta)_{p,0}^*)$. From (b) it follows that $|s-f(\Omega)|<\varepsilon/2$ so if $x\in\cup\{H(p, n, \alpha(p, n), \beta(p, n)): n \leqq -n_0\}$, then $|f(x)-f(\Omega)|<\varepsilon$ is concluded. It is verified for the obtained sets $H(p, n, \alpha(p,n), \beta(p, n))$, $p\in F$, $n\leqq -n_0$ the relation $H(p, n, \alpha(p, n), \beta(p, n))\subset\{x\in X: |f(x)-f(\Omega)|<\varepsilon\}$, and, therefore, f is continuous at Ω. Since Y is locally compact we conclude that f is continuous on X, so X is a k_R-space.

Let M be the set $X\sim\cup\{(\alpha, \omega_0)_{p,0}^*: 1\leqq\alpha<\omega_1, p\geqq 1\}$ and put

$$D_p = \cup\{S_{p,-n}^*\cap M: n \geqq 1\}, p\in N.$$

(III) *If $E=\{\Omega\}\cup[\cup\{D_p: p\in N\}]$, a compact subset of E for which Ω is an accumulation point does not exist.* Let $\hat{E}$ be the quotient space which is obtained by identifying the points of each $D_p, p\in N$ in E. Then $\hat{E}$ is homeomorphic to a subspace $N\cup\{\sigma\}$ of βN (Stone—Čech compactification of N) where $\sigma\in\beta N\sim N$. Let us suppose that K is a compact subset of E and that Ω is an accumulation point of K. Evidently K meets infinitely many sets D_p, therefore, the image of K in $\hat{E}$ is an infinite compact subset of $\hat{E}$, which is a contradiction because the infinite compact subsets of βN have cardinality $2^{2^{\aleph_0}}$.

(IV) *There exists an open subspace of X that is not a k_R-space.* The subspace M of X is open and, from (III), the function with domain M whose value in $D_p, p\in N$, is 1 and vanishes otherwise, is continuous on each compact subset of M, but is not continuous in M, so M is not a k_R-space.

3. An example of a space X in $\mathfrak{B}^*$ for wich $k_R X$ is not pseudocompact. Let Z_0 be the subspace of $w(\omega_2+1)$, $Z_0=\{\omega_2\}\cup\{\alpha\in w(\omega_2)\colon \alpha$ has a countable neighbourhood basis$\}$. Let us see that $S=Z_0\sim\{\omega_2\}$ is a sequentially compact space. If $\{x_n\}_{n=1}^\infty$ is a sequence of distinct points in S we can choose a strictly increasing subsequence $\{x_{n_k}\}_{k=1}^\infty$, and since this subsequence is not cofinal in $w(\omega_2)$ ([2], 9K) there exists $\alpha\in w(\omega_2)$ such that $\alpha=\sup\{x_{n_k}\colon k\in N\}$. The sequence $\{x_{n_k}\}_{k=1}^\infty$ converges to α in $w(\omega_2)$ and therefore $\alpha\in S$. Obviously, Z_0 is also sequentially compact.

We shall prove now that ω_2 is not an accumulation point of any compact subset of Z_0. Let us suppose that K is a compact subset of Z_0 such that ω_2 is an accumulation point of K. The set $K\sim\{\omega_2\}$ is cofinal in $w(\omega_2)$. Then, we can define a set $A=\{x_\eta\colon 1\leqq\eta<\omega_1\}\subset K\sim\{\omega_2\}$, where x_η is posterior to $\{x_\beta\colon \beta<\eta\}$ for every $\eta<\omega_1$. Since A has cardinality $\aleph_1$ there exists $\gamma\in w(\omega_2)$ such that $\gamma=\sup\{x_\beta\colon 1\leqq\beta<$ $<\omega_1\}$, then, by our construction γ has no countable neighbourhood basis and hence $\gamma\notin Z_0$. The net $\{x_\eta\colon 1\leqq\eta<\omega_1\}$ does not have convergent subnets in Z_0 because it converges in $w(\omega_2)$ to γ and $w(\omega_2)$ is a Hausdorff space. Therefore $\{x_\eta\colon 1\leqq\eta<\omega_1\}$ is a net in K without cluster points, but this contradicts the fact that K is compact. The function whose value is 1 in ω_2 and vanishes otherwise is not continuous in Z_0, but it is continuous on each compact subset of Z_0, therefore $\{\omega_2\}$ is an open set in $k_R Z_0$.

Let X be the space $(Z_0\times w(\omega_0+1))\sim\{(\omega_2,\omega_0)\}$ and we are to prove that X belongs to $\mathfrak{B}^*$. Since S is sequentially compact, $w(\omega_0+1)$ is compact and the class $\mathfrak{B}^*$ is closed under arbitrary products ([5], T. 3.4), the product space $Y=S\times$ $\times w(\omega_0+1)\in\mathfrak{B}^*$. Since Y is dense in X, it follows that $X\in\mathfrak{B}^*$.

Now let us see that $k_R X$ is not pseudocompact. The set $\{(\omega_2, n)\}$ is open in $k_R X$ for every $n\in N$, so the function whose value is n in (ω_2, n), $n\in N$, and vanishes otherwise is continuous and unbounded in $k_R X$.

References

[1] Z. Frolík, The topological product of two pseudocompact spaces, *Czech. Math. J.*, **10** (1960), 339—48.

[2] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nost. R. Comp. (London, 1960).

[3] A. Kato, A note on pseudocompact spaces and k_R-spaces, *Proc. Amer. Math. Soc.*, **61** (1977), 175—176.

[4] J. Kelley, *General topology*, Van Nost. (Princeton, 1955).

[5] N. Noble, Countably compact and pseudocompact products, *Czech. Math. J.*, **19** (1969), 390—97.

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CATEDRA DE MATEMATICAS II
FACULTAD DE CIENCIAS, PASEO AL MAR 13
VALENCIA 10, ESPAÑA

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ON THE CONSTRUCTIONS OF LOCAL AND ARITHMETIC RINGS

By

LEE SIN-MIN (Paris)

§ 1. Introduction. Following L. FUCHS [3] a ring R is said to be arithmetic if its lattice of ideals $\mathscr{L}(R)$ is a distributive lattice. It is called local if it has a unique maximal ideal. For a commutative local ring with identity, C. U. JENSEN [4] showed that it is arithmetic if and only if its lattice of ideals forms a chain.

In this paper it will be shown that Jensen's result cannot be extended to non-commutative local rings. There methods of construction of local rings are given which show that there exist infinitely many local and arithmetic rings whose lattice of ideals are not totally ordered under set-theoretical inclusion.

§ 2. The first construction of local rings. For each $n \geqq 2$, we denote by $\mathbf{n}$ the linearly ordered set of n elements.

We denote the class of rings R (not necessarily with identity) such that $\mathscr{L}(R)$ is isomorphic to $\mathbf{n}$ by $\mathscr{R}_n$. For example $\mathscr{R}_2$ is the class of all simple rings.

Now let $R \in \mathscr{R}_n$ such that R is commutative with an identity and M be a simple R-module, i.e. M has only (0) and M as its submodules. We construct a ring from the set $\hat{R} = R \times M$ by defining addition and multiplication as follows:

$$(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2), \quad (r_1, m_1) \cdot (r_2, m_2) = (r_1 r_2, r_1 m_2)$$

then $\langle \hat{R}, +, \cdot \rangle$ is a non-commutative ring.

$\hat{R}$ has no identity and also no right identity. However each element in $\{(1, m) \colon m \in M\}$ is a left identity of R. Since the anihilator ideal of M is $J(R)$ thus the left ideals of R are of the forms: $I_k \times 0$ or $I_k \times M$ $(1 \leqq k \leqq n)$ where $0 = I_1 \subset I_2 \subset \dots \dots \subset I_{n-1} \subset I_n = R$ is the chain of ideals of R. Therefore the lattice $\mathscr{L}_1(\hat{R})$ of left ideals of $\hat{R}$ is isomorphic to the lattice product $\mathbf{2} \times \mathbf{n}$.

Each of the left ideal of $\hat{R}$ except $I_n \times 0$ is also a right ideal of $\hat{R}$. For if m is not equal to zero in M then for (r, m) in $\hat{R}$ we have $(1, 0) \cdot (r, m) = (r, m)$ which is not in $I_n \times 0$.

Therefore for each $1 \leqq k < n$ the sets $I_k \times 0$ and $I_k \times M$ are the only proper ideals of $\hat{R}$. The maximal ideal of $\hat{R}$ is $I_{n-1} \times M$.

For a given lattice L we denote the lattice adjoint with a greatest element 1 by L^1. Then we have the following result.

THEOREM 2.1. *For a commutative ring R with identity 1 in $\mathscr{R}_n$ where $n \geqq 2$, the ring $\hat{R}$ constructed as above is a local and arithmetic ring whose lattice of ideals is isomorphic to the lattice* $(\mathbf{2} \times (\mathbf{n} - \mathbf{1}))^1$.

We observe that for $n \geqq 3$, R is a local and arithmetic ring whose lattice of ideals $\mathscr{L}(R)$ do not form a chain.

§ 3. The second construction of local rings. Let $R \in \mathscr{R}_n$ where $n \geqq 2$. Assume R has an identity and its chain of ideals is given by $0 = I_1 \subset I_2 \subset \ldots \subset I_{n-1} \subset I_n = R$.

Let $G_2 = \{x, y\}$ be two element left zero semigroup, i.e. G_2 satisfies the identity $uv = u$. We construct the semigroup ring $R[G_2]$ of G_2 over R. We have

LEMMA 3.2. *For each $1 \leqq i \leqq n$ the set $I_i^* = \{rx - ry : r \in I_i\}$ is an ideal of $R[G_2]$.*

Recall that an ideal I of a ring R is called join-irreducible if $I = J + K$ where J, K are ideals of R implies either $I = J$ or $I = K$. In fact $\{I_i^*\}_{1 \leqq i \leqq n}$ are join-irreducible ideals of $R[G_2]$. Denote $I_i^\bullet = I_i[G_2]$, then $\{I_i^\bullet\}_{1 \leqq i \leqq n}$ are also the join-irreducible ideals of $R[G_2]$.

LEMMA 3.3. *For $1 \leqq i \leqq n$, I_i^* and $I_i^\bullet$ are the only join-irreducible ideals of $R[G_2]$.*

LEMMA 3.4. *For $i \neq j$, we have $I_i^* + I_k^\bullet \neq I_j^* + I_k^\bullet$ where $k \leqq \min \{i, j\}$.*

LEMMA 3.5. *The only ideals of $R[G_2]$ are of the forms:*

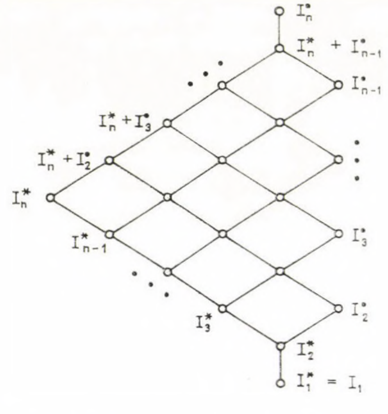
(1) I_i^*, $1 \leqq i \leqq n$

(2) $I_i^\bullet$, $1 \leqq i \leqq n$

(3) $I_i^* + I_j^\bullet$, *where* $2 \leqq j < i \leqq n-1$.

Summarizing the above result we have

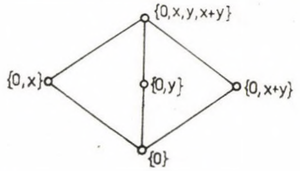
THEOREM 3.6. *For $n \geqq 2$, let $R \in \mathscr{R}_n$ with identity. Then the lattice of ideals of $R[G_2]$ is given by the following lattice:*



Since $\mathscr{L}(R[G_2])$ is a sublattice of $((\mathbf{n}-\mathbf{1}) \times (\mathbf{n}-\mathbf{1}))^{01}$ therefore $\mathscr{L}(R[G_2])$ is distributive.

COROLLARY 3.7. *$R[G_2]$ is a local and arithmetic ring, it is also subdirectly irreducible.*

REMARK 3.8. The above result is not true if R does not have an identity. For example let R be the zero ring $\langle Z/2Z, +, \cdot\rangle$ then $R \in \mathscr{R}_2$. However the lattice of ideals of $R[G_2]$ is the following modular non-distributive lattice $\mathscr{M}_5$:



Clearly $R[G_2]$ is not local.

REMARK 3.9. The above result is also not true if G_2 is replaced by G_m where $m \geqq 3$, i.e. left zero semigroup with m elements.

$R[G_m]$ is still a local ring, however it is not arithmetic for the following ideals: (0), $I_2^*[x, y] = \{rx - ry: r \in I_2\}$, $I_2^*[y, z] = \{ry - rz: r \in I_2\}$, $I_2^*[z, x] = \{rz - rx: r \in I_2\}$ and $I_2^*[x, y, z] = \{ax + by + cz: a, b, c \in I_2 \text{ and } a + b + c = 0\}$ forms a sublattice of $\mathscr{L}(R[G_m])$ which is isomorphic to $\mathscr{M}_5$.

4. The third construction of local rings. The local and arithmetic rings which are constructed by the above methods have no identities. One may think that the condition of existence of an identity in a local and arithmetic ring R may be strong enough to imply that the lattice $\mathscr{L}(R)$ of ideals forms a chain. However this is not the case.

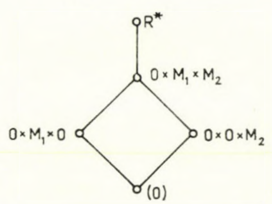
Let R be a simple ring with identity 1 such that there exists two non-isomorphic simple $R-R$-bimodules M_1 and M_2. For example the Weyl algebra $W_1(k)$ on x, y over a commutative field k in characteristic 0 with the defining relation $xy - yx = 1$ is such a simple ring.

Consider $R^* = R \times M_1 \times M_2$ with the component-wise addition and the multiplication is defined as follows:

$$(r, m_1, m_2) \cdot (s, n_1, n_2) = (rs, rn_1 + m_1 s, rn_2 + m_2 s).$$

Then $\langle R^*, +, \cdot\rangle$ is a ring with identity (1, 0, 0).

Its lattice of ideals is isomorphic to $(\mathbf{2} \times \mathbf{2})^1$ and hence R^* is local and arithmetic ring. The lattice $\mathscr{L}(R^*)$ is given as follows:



Finally, we wish to give some conditions for a local and arithmetic ring R whose lattice of ideals $\mathscr{L}(R)$ forms a chain.

THEOREM 4.10. *Let R be a ring with an identity such that*

(1) *R is left (right) local, i.e. it has a unique maximal left (right) ideal and*

(2) *R is left (right) arithmetic ring, i.e. its lattice of left (right) ideals is distributive.*

Then $\mathcal{L}(R)$ *is a chain.*

PROOF. The proof is essentially the same as the commutative case. First of all we show that for any non-zero elements a, b in R, the principal left ideals Ra, Rb are comparable.

Since $Ra = Ra \cap [Rb + R(a-b)] = [Ra \cap Rb] + [Ra \cap R(a-b)]$ therefore $a = t + c(a-b)$ where $t \in Ra \cap Rb$ and $cb \in Ra$. If c is invertible then b is the left multiple of cb and hence $b \in Ra$ which implies $Rb \in Ra$. If c is not invertible then $1-c$ is invertible for R is left local. Since a is the left multiple of $(1-c)a = t - cb \in Rb$ therefore $Ra \subseteq Rb$.

Now we want to show that the lattice $\mathcal{L}_1(R)$ of left ideals of R forms a chain. Suppose I and J are two left ideals of R and $J \not\subseteq I$, then there exists $j \in J$ such that $j \notin I$. Thus the principal left ideal Ri for each $i \in I$ is contained in Rj. Therefore $I \subseteq Rj \subseteq J$.

Since $\mathcal{L}_1(R)$ is a chain therefore $\mathcal{L}(R)$ is a chain.

Finally I wish to thank Professor D. Eisenbud for his helpful discussions.

References

[1] E. A. BEHRENS, *Algebren* (Mannheim, 1965).
[2] P. M. COHN, *Free rings and their relations* (New York, 1971).
[3] L. FUCHS, Über die Ideale arithmetischer Ringe, *Comment. Math. Helv.*, **23** (1949), 330—339.
[4] C. U. JENSEN, Arithmetical rings, *Acta Math. Acad. Sci. Hungar.*, **17** (1966), 113—123.
[5] J. P. LAFON, Anneaux locaux commutatifs sur lesquels module de type fini est somme directe de modules monogenes, *Journal of Algebra*, **17** (1971), 575—591.

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UNIVERSITE DU PARIS—SUD, BÂT. 425
91, ORSAY, FRANCE

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DISTRIBUTIVITY IN SEMILATTICES

By

R. C. HICKMAN (Bedford Park)

0. Introduction

The structure of ideals and congruences in distributive lattice is well known (see GRÄTZER [4]). Our aim here is to construct analogous results for lower semilattices which obey a particular form of distributive law. There are numerous notions of distributivity in semilattices.

A semilattice S is *distributive* if whenever $x, a, b \in S$ are such that $x \geqq a \wedge b$, there exist $a', b' \in S$ with $a' \geqq a$, $b \geqq b'$ and $x = a' \wedge b'$ (see KATRINÁK [5] and GRÄTZER [4, pp. 117—118]).

We call a semilattice S *m-distributive* if and only if it satisfies the equation

$$(\mathrm{D}_m) \qquad y \wedge (x_1 \vee \ldots \vee x_m) = (y \wedge x_1) \vee \ldots \vee (y \wedge x_m)$$

in the sense that whenever the left hand side exists then so does the right hand side and the two sides are equal. This idea was first put forward by SCHEIN [6]. We will denote the class of m-distributive semilattices by $\mathscr{D}_m$ for each $m=2, 3, \ldots$ and the class of semilattices which satisfy D_m for each $m=2, 3, \ldots$ by $\mathscr{D}_\omega$. This last class has been investigated by BALBES [1], VARLET [7] and CORNISH and HICKMAN [2]. If we denote the class of distributive semilattices by $\mathscr{D}$ then the following series of inequalities hold.

$$\mathscr{D}_2 \supseteqq \mathscr{D}_3 \supseteqq \ldots \supseteqq \mathscr{D}_\omega \supseteqq \mathscr{D}.$$

A non-empty subset H of a semilattice S is called *hereditary* if whenever $x \leqq y$ and $y \in H$ then $x \in H$. For any integer $n \geqq 2$ we call a subset I of S an *n-ideal* if I is an hereditary subset of S and whenever $x_1, \ldots, x_n \in I$ are such that $x_1 \vee \ldots \vee x_n$ exists in S then $x_1 \vee \ldots \vee x_n \in I$. A subset J of S is called an *ideal* if J is an n-ideal for each $n=2, 3, \ldots$. In section 1 we show that a semilattice is m-distributive if and only if its m-ideals form a distributive lattice. We also introduce the technical concept of an *m-reducible* join and show that in an m-distributive semilattice the join of two m-ideals I and J is the set of all m-reducible joins of elements from $I \cup J$.

In section 2 we develop a system of congruences on m-distributive semilattices. We consider the class of semilattice congruences that have the following substitution property: suppose $x_1, \ldots, x_m, y_1, \ldots, y_m$ are such that $x_i \equiv y_i(\Phi)$ for $i=1, \ldots, m$ and both $x_1 \vee \ldots \vee x_m$ and $y_1 \vee \ldots \vee y_m$ exist, then $x_1 \vee \ldots \vee x_m \equiv y_1 \vee \ldots \vee y_m(\Phi)$. We call such congruences *m-join partial* and for a given semilattice S denote the collection of all m-join partial congruences by $\mathscr{C}^m(S)$. Our main task in section 2 is to describe the join in the lattice $\mathscr{C}^m(S)$ in the case when S is m-distributive. We conclude by investigating some examples of m-join partial congruences of an m-distributive semilattice; in particular, using the ideas developed in section 1 we are able to determine $T(a, b)$, the smallest m-join partial congruence which identifies a and b for any $a, b \in S$.

1. Ideals

Let S be a semilattice. Recall that for a given integer n, a non-empty subset I of S is called an *n-ideal* if $x \leqq y$ and $y \in I$ implies $x \in I$ and if $x_1, \ldots, x_n \in I$ and $x_1 \vee \ldots \vee x_n$ exists in S then $x_1 \vee \ldots \vee x_n \in I$. Clearly if I is an n-ideal and $x_1, \ldots, x_r$ are elements of I, $r \leqq n$, and $x_1 \vee \ldots \vee x_r$ exists in S then $x_1 \vee \ldots \vee x_r \in I$. We denote the set of all n-ideals of S by $\mathscr{I}^n(S)$.

LEMMA 1. *Let S be a semilattice. If we denote by $\mathscr{I}_0^n(S)$ the set of all n-ideals together with the empty set, then $\mathscr{I}_0^n(S)$ is an algebraic lattice. Furthermore, if H is an hereditary subset of S, then the n-ideal I generated by H is given by the following construction.*

Let $H_0 = H$,

$$(1) \qquad H_k = \{x \in S \colon x \leqq h_1 \vee \ldots \vee h_n \ \ \textit{for some} \ \ h, \ldots, h_n \in H_{k-1}\}$$

for each $k = 1, 2, 3, \ldots$. Then $I = \bigcup_{k<\omega} H_k$.

PROOF. A routine argument shows that $\mathscr{I}_0^n(S)$ is an algebraic lattice. Indeed, since the intersection of two n-ideals is again an n-ideal, $\mathscr{I}^n(S)$ is a lattice and $\mathscr{I}^n(S)$ is an algebraic lattice, provided S possesses a least element.

Let H be hereditary and define H_k, for $k=0, 1, 2, \ldots$, as above. Since each H_k is hereditary, I is hereditary. Suppose $x_1, \ldots, x_n \in I$ are such that $x_1 \vee \ldots \vee x_n$ exists in S. By our construction there exists an integer r such that $x_1, \ldots, x_n \in H_r$. Hence $x_1 \vee \ldots \vee x_n \in H_{r+1} \subseteqq I$, showing that I is an n-ideal. If J is another n-ideal containing H then a simple induction argument shows that $H_k \subseteqq J$ for $k = 0, 1, 2, \ldots$ and so $I \subseteqq J$.

This lemma now gives us a way to describe the join in $\mathscr{I}^n(S)$. Suppose $\{I_\alpha\}_{\alpha \in A}$ is a family of n-ideals of S. Then $\bigcup_{\alpha \in A} I_\alpha$ is an hereditary set and so the join of the I_α in $\mathscr{I}^n(S)$ can be described in terms of the n-ideal generated by $\bigcup_{\alpha \in A} I_\alpha$. For any hereditary set H, we denote the n-ideal generated by H with $\langle H \rangle_n$. Note that in the case when S is n-distributive we may write in Lemma 1

$$(1) \qquad H_k = \{x \in S \colon x = h_1 \vee \ldots \vee h_n \ \ \text{for some} \ \ h_1, \ldots, h_n \in H_{k-1}\}.$$

For any element a in S let $(a]$ denote the set $\{x \in S \colon x \leqq a\}$, that is $(a]$ is the hereditary set generated by a. For simplicity we write $\langle a \rangle_n$ for $\langle (a] \rangle_n$ (in fact $\langle a \rangle_n = (a]$) and $\langle a_1, \ldots, a_r \rangle_n$ for $\langle (a_1] \cup \ldots \cup (a_r] \rangle_n$ where $a_i \in S$ for $i = 1, \ldots, r$.

An n-ideal I is said to be finitely generated if it can be written in the form $\langle a_1, \ldots, a_r \rangle_n$ for some $a_1, \ldots, a_r \in S$. We denote the set of all finitely generated n-ideals by $\mathscr{I}_f^n(S)$. We denote the join in $\mathscr{I}^n(S)$ (and $\mathscr{I}_f^n(S)$, when it is a lattice) by $+^n$.

THEOREM 2. *Let S be a semilattice and $m \geqq 2$ an integer. Then the following are equivalent.*

(i) *$S \in \mathscr{D}_m$ (the class of m-distributive semilattices).*

(ii) *$\mathscr{I}^m(S)$ is a distributive lattice.*

(iii) *$\mathscr{I}_f^m(S)$ is a distributive lattice.*

PROOF. (i)⇒(ii). Let $I, J, K\in\mathcal{I}^m(S)$. We must show that $I\cap(J+^mK)\subseteq (I\cap J)+^m(I\cap K)$, which is the same as $I\cap\langle J\cup K\rangle_m\subseteq\langle(I\cap J)\cup(I\cap K)\rangle_m$.

Let $A_0=J\cup K$, $A_k=\{a_1\vee\ldots\vee a_m\colon a_1, \ldots, a_m\in A_{k-1}\}$ for $k=1, 2, \ldots$. Lemma 1 and the remarks following it show that $\langle J\cup K\rangle_m=\bigcup_{k<\omega} A_k$. Now let $x\in I\cap\langle J\cup K\rangle_m$. Then $x\in I$ and $x\in A_r$ for some integer r. We show, by induction on r, that $x\in\langle(I\cap J)\cup(I\cap K)\rangle_m$. Case $r=0$. Then $x\in I\cap A_0=I\cap(J\cup K)=(I\cap J)\cup(I\cap K)\subseteq\langle(I\cap J)\cup(I\cap K)\rangle_m$.

Assume the result holds for $r=t-1$ and let $x\in I\cap A_t$. Then $x=a_1\vee\ldots\vee a_m$, $a_1, \ldots, a_m\in A_{t-1}$. Now for each $i=1, 2, \ldots, m$ we have $a_i\leqq x\in I$ which implies $a_i\in I\cap A_{t-1}$. Hence by our inductive hypothesis $a_i\in\langle(I\cap J)\cup(I\cap K)\rangle_m$ for each $i=1, 2, \ldots, m$ and so $x\in\langle(I\cap J)\cup(I\cap K)\rangle_m$.

(ii)⇒(iii). We first note that in any semilattice $\langle x\rangle_m\cap\langle y\rangle_m=\langle x\wedge y\rangle_m$ for $x, y\in S$. Now let $\langle x_1, \ldots, x_r\rangle_m$ and $\langle y_1, \ldots, y_s\rangle_m$ be two finitely generated m-ideals. Then,

$$\langle x_1, \ldots, x_r\rangle_m+^m\langle y_1, \ldots, y_s\rangle_m=\langle x_1, \ldots, x_r, y_1, \ldots, y_s\rangle_m \tag{1}$$

and

$$\langle x_1, \ldots, x_r\rangle_m\cap\langle y_1, \ldots, y_s\rangle_m=\langle x_1\wedge y_1, \ldots, x_1\wedge y_s, x_2\wedge y_1, \ldots, x_r\wedge y_s\rangle_m. \tag{2}$$

The second of these results is obtained by a repeated application of (1) and our assumption of (ii). Thus $\mathcal{I}_f^m(S)$ is a sublattice of $\mathcal{I}^m(S)$. Since a sublattice of a distributive lattice is distributive, the result holds.

(iii)⇒(i). Let $x_1, \ldots, x_m\in S$ be such that $x_1\vee\ldots\vee x_m$ exists and let $a\in S$. Then $a\wedge(x_1\vee\ldots\vee x_m)$ is an upper bound for $\{a\wedge x_1, \ldots, a\wedge x_m\}$; let b be any other upper bound. Then $\langle b\rangle_m\supseteq\langle a\wedge x_i\rangle_m$ for each $i=1, \ldots, m$ and so $\langle b\rangle_m\supseteq\langle a\wedge x_1\rangle_m+^m\ldots+^m\langle a\wedge x_m\rangle_m$.

Hence

$$\langle b\rangle_m\supseteq(\langle a\rangle_m\cap\langle x_1\rangle_m)+^m\ldots+^m(\langle a\rangle_m\cap\langle x_m\rangle_m)=\langle a\rangle_m\cap(\langle x_1\rangle_m+^m\ldots+^m\langle x_m\rangle_m)=$$
$$=\langle a\rangle_m\cap\langle x_1, \ldots, x_m\rangle_m=\langle a\rangle_m\cap\langle x_1\vee\ldots\vee x_m\rangle_m=\langle a\wedge(x_1\vee\ldots\vee x_m)\rangle_m,$$

which implies that $b\geqq a\wedge(x_1\vee\ldots\vee x_m)$ and so

$$a\wedge(x_1\vee\ldots\vee x_m)=(a\wedge x_1)\vee\ldots\vee(a\wedge x_m).$$

An n-ideal I of a semilattice S is called *prime* if $x\wedge y\in I$ implies $x\in I$ or $y\in I$. A non-empty subset F of S is called a *filter* if $x\wedge y\in F$ is equivalent to $x\in F$ and $y\in F$. Thus, by using the above theorem, we have the following extension to the Prime Ideal Theorem.

COROLLARY. *Let S be an m-distributive semilattice, I an m-ideal and F a filter such that $I\cap F=\varnothing$. Then there exists a prime m-ideal P such that $I\subseteq P$ and $P\cap F=\varnothing$.*

PROOF. Let $\mathcal{X}$ denote the set of all m-ideals which contain I and are disjoint from F. Clearly $I\in\mathcal{X}$ and if $\mathcal{C}$ is any chain contained in $\mathcal{X}$ then $K=\bigcup\{J\colon J\in\mathcal{C}\}$ is also a member of $\mathcal{X}$. Hence $\mathcal{X}$ satisfies the hypothesis of Zorn's Lemma and so possesses a maximal element P. We claim that P is a prime m-ideal, for suppose $a, b\in S$ and $a\notin P, b\notin P$. Since P is maximal we have $(P+^m\langle a\rangle_m)\cap F\neq\varnothing$ and $(P+^m\langle b\rangle_m)\cap F\neq\varnothing$. Hence there exist $x, y\in F$ such that $x\in P+^m\langle a\rangle_m$ and $y\in P+^m\langle b\rangle_m$, and so

$$x\wedge y\in(P+^m\langle a\rangle_m)\cap(P+^m\langle b\rangle_m)=P+^m\langle a\wedge b\rangle_m$$

which implies that $a\wedge b\notin P$. Thus P is a prime m-ideal which contains I and is disjoint from F.

Following GRÄTZER [4, p. 39] we say that D is the free distributive extension of the partially ordered set P if

(i) D is a distributive lattice.

(ii) $P\subseteq D$, and for $a, b, c\in P$, $\inf\{a, b\}=c$ in P if and only if $a\wedge b=c$ in D, and $\sup\{a, b\}=c$ in P if and only if $a\vee b=c$ in D.

(iii) P generates D as a lattice.

(iv) Let L be a distributive lattice and let $\varphi\colon P\to L$ be an isotone map with the properties that if $a, b, c\in P$, $\inf\{a, b\}=c$ in P, then $a\varphi\wedge b\varphi=c\varphi$ in L, and if $\sup\{a,b\}=c$ in P, then $a\varphi\vee b\varphi=c\varphi$ in L. Then there exists a (lattice) homomorphism $\psi\colon D\to L$ extending φ (that is, $a\varphi=a\psi$ for all $a\in P$).

THEOREM 3. *Let $S\in\mathcal{D}_m$. Then $\mathcal{I}_f^2(S)$ is the free distributive extension of S.*

PROOF. (i) Of course, $\mathcal{D}_2\supseteq\mathcal{D}_m$ for each $m\geqq 2$ and so Theorem 2 shows that $\mathcal{I}_f^2(S)$ is a distributive lattice.

(ii) By identifying $x\in S$ with $\langle x\rangle_2\in\mathcal{I}_f^2(S)$ it is easy to see that both (ii) and (iii) hold.

(iv) Now suppose L and φ are as stated in (iv). Then define $\psi\colon\mathcal{I}_f^2(S)\to L$ by $\langle x_1, \dots, x_r\rangle_2\psi=x_1\varphi\vee\dots\vee x_r\varphi$ for $x_1, \dots, x_r\in S$. First we must show that ψ is well defined. Suppose we have $I=\langle x_1, \dots, x_r\rangle_2=\langle y_1, \dots, y_s\rangle_2$ for some x_i and $y_j\in S$. It is sufficient to show that $x_i\varphi\leqq y_1\varphi\vee\dots\vee y_s\varphi$ for each $i=1, \dots, r$. If we set $H_0=(y_1]\cup\dots\cup(y_s]$, $H_k=\{h_1\vee h_2\colon h_1, h_2\in H_{k-1}\}$ for each $k=1, 2, 3, \dots,$ then by Lemma 1, $I=\bigcup_{k<\omega} H_k$. An easy induction argument shows that $z\in H_k$ implies $z\varphi\leqq\leqq y_1\varphi\vee\dots\vee y_s\varphi$ for each $k=0, 1, 2, \dots$. Now $x_i\in I$, $x_i\in H_{k(i)}$ for some $k(i)$ and so $x_i\varphi\leqq y_1\varphi\vee\dots\vee y_s\varphi$ for $i=1, \dots, r$. Hence ψ is well defined. It is clear that ψ is now in fact a homomorphism and by our construction it is obvious that ψ is an extension of φ.

Grätzer's definition of the free distributive extension is not the only one possible. We could insist that the free extension preserves suprema (and infima) of sets of m elements. That is we change sections (ii) and (iv) of the definition to read

(ii′) $P\subseteq D$, and for $x_1, \dots, x_m, y\in P$, $\inf\{x_1, \dots, x_m\}=y$ in P if and only if $x_1\wedge\dots\wedge x_m=y$ in D, and $\sup\{x_1, \dots, x_m\}=y$ in P if and only if $x_1\vee\dots\vee x_m=y$ in D; and

(iv′) Let L be a distributive lattice and let $\varphi\colon P\to L$ be an isotone map with the properties that if $x_1, \dots, x_m, y\in P$, $\inf\{x_1, \dots, x_m\}=y$ in P, then $x_1\varphi\wedge\dots\wedge x_m\varphi=y\varphi$ in L, and if $\sup\{x_1, \dots, x_m\}=y$ in P, then $x_1\varphi\vee\dots\vee x_m\varphi=y\varphi$ in L. Then there exists a (lattice) homomorphism $\psi\colon D\to L$ extending φ.

If we now call a lattice D which satisfies (i), (ii′), (iii) and (iv′) with respect to P the free distributive m-extension of P we have

THEOREM 3′. *Let $S\in\mathcal{D}_m$. Then $\mathcal{I}_f^m(S)$ is the free distributive m-extension of S.*

In a lattice L it is well known that the distributive equation $a\wedge(x\vee y)=(a\wedge x)\vee\vee(a\wedge y)$ is equivalent to its dual. In the case of m-distributive semilattices we have the following result.

THEOREM 4. *Let S be an m-distributive semilattice. Then S satisfies the equation*

$$(\mathrm{D}'_m) \qquad (y\vee x_1)\wedge(y\vee x_2)\wedge\ldots\wedge(y\vee x_m) = y\vee(x_1\wedge\ldots\wedge x_m)$$

in the sense that if the left hand side exists, then so does the right hand side and the two sides are equal.

PROOF. Suppose $(y\vee x_1)\wedge\ldots\wedge(y\vee x_m)$ exists for $y, x_1, \ldots, x_m\in S$. Certainly $a=(y\vee x_1)\wedge\ldots\wedge(y\vee x_m)$ is an upper bound for $\{y, x_1\wedge\ldots\wedge x_m\}$. Let b be any other upper bound. Then

$$\langle b\rangle_m \supseteq \langle y\rangle_m +^m \langle x_1\wedge\ldots\wedge x_m\rangle_m = \langle y, x_1\rangle_m \cap\ldots\cap \langle y, x_m\rangle_m =$$
$$= \langle y\vee x_1\rangle_m \cap\ldots\cap\langle y\vee x_m\rangle_m = \langle (y\vee x_1)\wedge\ldots\wedge(y\vee x_m)\rangle_m .$$

That is $b\geqq a$ and so $a=y\vee(x_1\wedge\ldots\wedge x_m)$.

Our next task, for $S\in\mathcal{D}_m$, is to find a better description of the join in $\mathcal{I}^m(S)$. For this we need the following definitions.

Let S be a semilattice and $x_1, \ldots, x_r\in S$ be such that $x_1\vee\ldots\vee x_r$ exists in S. A family $\{A_i\}_{i=1}^s$ of subsets of $\{x_1, \ldots, x_r\}$ is called an *n-reducing partition* of $x_1\vee\ldots\vee x_r$ if

(i) $s\leqq n$,

(ii) $\bigcup\limits_{i=1}^{s} A_i = \{x_1, \ldots, x_r\}$,

(iii) for each $i=1, \ldots, s$ the supremum, $\vee A_i$, of all the elements of A_i exists in S.

A join $x_1\vee\ldots\vee x_r$ is called *n-reducible (step 0)* if $r\leqq n$. For $k=1, 2, 3, \ldots$ a join $x_1\vee\ldots\vee x_r$ is called *n-reducible (step k)* if there exists an n-reducing partition $A_1, \ldots, A_s$ of $x_1\vee\ldots\vee x_r$ such that for each $i=1, \ldots, s$ the join $a_i^1\vee\ldots\vee a_i^{j(i)}$ of all the elements of A_i is n-reducible (step $k-1$). A join $x_1\vee\ldots\vee x_r$ is called *n-reducible* if it is n-reducible (step k) for some integer k. Of course, it only makes sense to investigate the case when $n\geqq 2$: a join $x_1\vee\ldots\vee x_r$ is 1-reducible if and only if $r=1$.

LEMMA 5. *Let S be a semilattice and suppose that $x_1^i\vee\ldots\vee x_{r(i)}^i$ are n-reducible joins in S for $i=1, \ldots, s$ with $s\leqq n$, such that $x_i^1\vee\ldots\vee x_{r(1)}^1\vee x_1^2\vee\ldots\vee x_{r(s)}^s$ exists in S. Then, $x_1^1\vee\ldots\vee x_{r(1)}^1\vee x_1^2\vee\ldots\vee x_{r(s)}^s$ is n-reducible.*

PROOF. Suppose $x_1^i\vee\ldots\vee x_{r(i)}^i$ is n-reducible (step $k(i)$) for each i. If we let $k=1+\max\,(k(1), \ldots, k(s))$ it is not very hard to see that $x_1^1\vee\ldots\vee x_{r(1)}^1\vee x_1^2\vee\ldots\vee x_{r(s)}^s$ is n-reducible (step k) since if we let $A_i=\{x_1^i, \ldots, x_{r(i)}^i\}$ then $\{A_i\}_{i=1}^s$ is an n-reducing partition having the necessary properties.

LEMMA 6. *Let $S\in\mathcal{D}_m$, $x_1\vee\ldots\vee x_r$ an m-reducible join in S and $y\in S$. Then $(y\wedge x_1)\vee\ldots\vee(y\wedge x_s)$ exists and equals $y\wedge(x_1\vee\ldots\vee x_r)$.*

PROOF. Suppose $x_1\vee\ldots\vee x_r$ is m-reducible (step k). The proof proceeds by induction on k. The result holds for $k=0$, since $S\in\mathcal{D}_m$ and $r\leqq m$. Suppose the result is true for $k=t-1$, and let $x_1\vee\ldots\vee x_r$ be m-reducible (step t). Then there exists an m-reducing partition $A_1, \ldots, A_s$ of $x_1\vee\ldots\vee x_r$ such that $\vee A_i$ is m-reducible (step $t-1$) for each $i=1, \ldots, s$. If we set $a_i = \vee A_i$, then

$$y\wedge(x_1\vee\ldots\vee x_r) = y\wedge(a_1\vee\ldots\vee a_s) = (y\wedge a_1)\vee\ldots\vee(y\wedge a_s) =$$
$$= (y\wedge(\vee A_1))\vee\ldots\vee(y\wedge(\vee A_s)) = (y\wedge x_1)\vee\ldots\vee(y\wedge x_r)$$

by our inductive hypothesis.

To illustrate the use of this lemma, consider the following example. Let Z^- be the non-positive integers with their natural ordering and let S' be the collection of all couples of the form $(z, -1)$ and $(z, 0)$ where $z\in Z^-$. Under the induced ordering it is not hard to see that S' is in fact a distributive lattice. To S' we adjoin three elements, a, b and c, such that $a\leqq x$ for all $x\in S'$, $b\leqq(z, 0)$ for all $z\in Z^-$, $b\| a$ and $b\|(z, -1)$ for all $z\in Z^-$ and c is the smallest element of $S=S'\cup\{a, b, c\}$. We claim that $S\in\mathscr{D}_\omega$ but $S\notin\mathscr{D}$. Clearly S is a semilattice. Now $(0, -1)\wedge b=c$ and $a\geqq c= =(0, -1)\wedge b$. If $S\in\mathscr{D}$ then there would exist x and y in S with $x\geqq(0, -1)$, $y\geqq b$ such that $a\neq x\wedge y$. Since a is meet irreducible this would imply $a=x$ or $a=y$, both of which would lead to a contradiction. Hence $S\notin\mathscr{D}$. Now we show that $S\in\mathscr{D}_\omega$. We in fact prove that $S\in\mathscr{D}_2$ and that any existent join is 2-reducible, and so, by Lemma 6, we will be finished. To show that $S\in\mathscr{D}_2$ can be done by sheer computation; we just give a few examples.

$$b\wedge((x_1, -1)\vee(x_2, 0)) = b\wedge(\max(x_1, x_2), 0) = b$$
$$(b\wedge(x_1, -1))\vee(b\wedge(x_2, 0)) = c\vee b = b$$
$$b\wedge(a\vee(x, 0)) = b\wedge(x, 0) = b$$
$$(b\wedge a)\vee(b\wedge(x, 0)) = c\vee b = b$$
$$a\wedge(b\vee(x, -1)) = a\wedge(x, 0) = a$$
$$(a\wedge b)\vee(a\wedge(x_1-1)) = c\vee a = a.$$

Now for any integer r and any $x_i\in S$ for $i=1, \ldots, r$, $x_1\vee\ldots\vee x_r$ exists in S unless $\{x_1, \ldots, x_r\}=\{a, b\}$ or $\{x_1, \ldots, x_r\}=\{a, b, c\}$. Using this fact we show, by induction on k, that $x_1\vee\ldots\vee x_k$ is always 2-reducible.

For $k=2$ the statement is trivial. Assume that the result is true for $k=t-1$, and let $x_1\vee\ldots\vee x_t$ exist in S. Without loss of generality we may assume that $x_i\neq x_j$ for $i\neq j$. If for no $i=1, \ldots, t$ is x_i equal to b, then $\{\{x_1, \ldots, x_{t-1}\}, \{x_t\}\}$ is a 2-reducing partition of $x_1\vee\ldots\vee x_t$, and by our inductive hypothesis $x_1\vee\ldots\vee x_{t-1}$ is 2-reducible, giving the result.

If for some $i, x_i=b$, then $\{\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_t\}, \{x_i\}\}$ is a 2-reducing partition of $x_1\vee\ldots\vee x_t$, and again by our inductive hypothesis $x_1\vee\ldots\vee x_{i-1}\vee x_{i+1}\vee\ldots \ldots\vee x_t$ is 2-reducible. Thus $x_1\vee\ldots\vee x_t$ is 2-reducible.

THEOREM 7. *Let $S\in\mathscr{D}_m$ and H an hereditary subset of S. Then*

(i) $\langle H\rangle_m = \{x\in S\colon x = h_1\vee\ldots\vee h_r,\ h_i\in H \text{ for } i = 1, \ldots, r \text{ and } h_1\vee\ldots\vee h_r \text{ } m\text{-reducible}\}$.

PROOF. Since H is an hereditary set, Lemmas 5 and 6 show that the right hand side of (i) is an ideal, and hence must contain $\langle H\rangle_m$. Conversely, if we set $H_0=H$, $H_k=\{h_1\vee\ldots\vee h_m\colon h_i\in H_{k-1}\}$ then we have already seen that $\langle H\rangle_m=\bigcup_{k<\omega} H_k$ and, by using an easy induction argument and Lemma 5, it is possible to show $h_1 \vee \ldots \vee h_r$ is contained in H_k for some k and for each m-reducible join $h_1 \vee \ldots \vee h_r$ of elements of H.

THEOREM 8. *Let $S \in \mathscr{D}_m$, $x_1 \vee \ldots \vee x_r$ and $y_1 \vee \ldots \vee y_s$ be two m-reducible joins in S. Then*

$$(x_1 \wedge y_1) \vee (x_1 \wedge y_2) \vee \ldots \vee (x_1 \wedge y_s) \vee (x_2 \wedge y_1) \vee \ldots \vee (x_r \wedge y_s)$$

is an m-reducible join.

PROOF. A repeated application of Lemma 6 shows that the join $(x_1 \wedge y_1) \vee \vee (x_1 \wedge y_2) \vee \ldots \vee (x_1 \wedge y_s) \vee (x_2 \wedge y_1) \vee \ldots \vee (x_r \wedge y_s)$ actually exists. Suppose $x_1 \vee \ldots \vee x_r$ is m-reducible (step h) and $y_1 \vee \ldots \vee y_s$ is m-reducible (step k). We first do the case when $s=1$. The proof is by induction on h. The case when $h=0$ is obvious. Suppose whenever $z_1 \vee \ldots \vee z_t$ is an m-reducible join (step $u-1$) then so is $(y_1 \wedge z_1) \vee \ldots \ldots \vee (y_1 \wedge z_t)$, and let $x_1 \vee \ldots \vee x_r$ be m-reducible (step u). Now there exists an m-reducing partition $A_1, \ldots, A_v$ of $x_1 \vee \ldots \vee x_r$ such that $\vee A_i$ is m-reducible (step $u-1$) for each $i=1, \ldots, r$. If we set $B_i = \{y_1 \wedge a_i^j : a_j^j \in A_i\}$ our inductive hypothesis shows that $B_1, \ldots, B_v$ is an m-reducing partition of $(y_1 \wedge x_1) \vee \ldots \vee (y_1 \wedge x_r)$ having the necessary properties to ensure our result. We may now return to the case when s is no longer 1. Our proof, again, is by induction on h. The case when $h=0$ is covered by the first part of the proof and Lemma 5. Assume the result is true for $h=u-1$, and suppose $x_1 \vee \ldots \vee x_r$ is m-reducible (step u) where $A_1, \ldots, A_v$ is an m-reducing partition of $x_1 \vee \ldots \vee x_r$ such that $\vee A_i$ is m-reducible (step $u-1$) for each $i=1, \ldots, r$. Now if we set $B_i = \{a_i^j \wedge y_k : a_i^j \in A_i,\ k=1, \ldots, s\}$ then, by our inductive hypothesis, it can be seen that $\{B_i\}_{i=1}^{v}$ is an m-reducing partition of $(x_1 \wedge y_1) \vee \ldots \vee (x_1 \wedge y_s) \vee \vee (x_2 \wedge y_1) \vee \ldots \vee (x_r \wedge y_s)$ such that $\vee B_i$ is m-reducible (step $n(i)$) for some integer $n(i)$ and for each $i=1, \ldots, v$. If we set $n = 1 + \max(n(1), n(2), \ldots, n(v))$ then clearly

$$(x_1 \wedge y_1) \vee \ldots \vee (x_1 \wedge y_s) \vee (x_2 \wedge y_1) \vee \ldots \vee (x_r \wedge y_s)$$

is m-reducible (step n).

We conclude this section by noting that an example, due to Dilworth, of a semilattice which is in $\mathscr{D}_m$ but not $\mathscr{D}_{m+1}$, is presented in CORNISH and HICKMAN [3].

2. Congruences

We are interested in those semilattice congruences Φ, which have the following substitution property for suprema. Let m be a fixed integer and suppose $x_i \equiv y_i(\Phi)$ for $i=1, 2, \ldots, m$ and furthermore both $x_1 \vee \ldots \vee x_m$ and $y_1 \vee \ldots \vee y_m$ exist. Then we require that $x_1 \vee \ldots \vee x_m \equiv y_1 \vee \ldots \vee y_m(\Phi)$. We call such congruences *m-join partial* and denote, for a semilattice S, the set of all m-join partial congruences on S by $\mathscr{C}^m(S)$. This set is easily seen to form an algebraic lattice with smallest element Ω defined by $x \equiv y(\Omega)$ if and only if $x=y$ and largest element Γ defined by $x \equiv y(\Gamma)$ for all x and y. Our first objective is to find a method of describing the join in $\mathscr{C}^m(S)$, in the case where S is m-distributive.

Let S be a semilattice, $x_1, \ldots, x_r, y_1, \ldots, y_r \in S$ be such that both $x_1 \vee \ldots \vee x_r$ and $y_1 \vee \ldots \vee y_r$ exist in S. We call, for a fixed integer $m \geqq 2$, $x_1 \vee \ldots \vee x_r$ and $y_1 \vee \ldots \vee y_r$ *m-comparable* (*step* 0) if $r \leqq m$. If φ is the „map" $\varphi(x_i) = y_i$, we call $x_1 \vee \ldots \vee x_r$ and

$y_1\vee\ldots\vee y_r$ *m-comparable (step k)* if there exist m-reducing partitions $\{A_i\}_{i=1}^t$ and $\{B_i\}_{i=1}^t$ of $x_1\vee\ldots\vee x_r$ and $y_1\vee\ldots\vee y_r$, respectively, such that $\varphi(A_i)=B_i$ for all $i=1, \ldots, t$ and $\vee A_i$ and $\vee B_i$ are m-comparable (step $k-1$) for each $i=1, \ldots, t$. To state it more exactly, for each i we can find integers $j_1, j_2, \ldots, j_{n(i)}$ such that A_i consists of the $j_1^{\text{th}}, j_2^{\text{th}}, \ldots, j_{n(i)}^{\text{th}}$ components of the r-tuple $(x_1, \ldots, x_r)$ and B_i consists of the $j_1^{\text{th}}, j_2^{\text{th}}, \ldots, j_{n(i)}^{\text{th}}$ components of $(y_1, \ldots, y_r)$. We say that $x_1\vee\ldots\vee x_r$ and $y_1\vee\ldots\vee y_r$ are *m-comparable* if they are m-comparable (step k) for some integer k.

LEMMA 1. *Let S be a semilattice and suppose that $x_1\vee\ldots\vee x_r$ and $y_1\vee\ldots\vee y_r$ are m-comparable in S. Then both $x_1\vee\ldots\vee x_r$ and $y_1\vee\ldots\vee y_r$ are m-reducible.*

PROOF. Suppose $x_1\vee\ldots\vee x_r$ and $y_1\vee\ldots\vee y_r$ are m-comparable (step k). The proof proceeds by induction on k. The case $k=0$ is trivial. Suppose that the result holds for $k=t-1$ and let $x_1\vee\ldots\vee x_r$ and $y_1\vee\ldots\vee y_r$ be m-comparable (step t). Then there exist m-reducing partitions $\{A_i\}_{i=1}^s$ and $\{B_i\}_{i=1}^s$ of $x_1\vee\ldots\vee x_r$ and $y_1\vee\ldots\vee y_r$ respectively such that $\vee A_i$ and $\vee B_i$ are m-comparable (step $t-1$). Hence $\vee A_i$, $\vee B_i$ are m-reducible by hypothesis and so the result follows from Lemma 1.5.

LEMMA 2. *Let S be a semilattice and suppose that $x_1^i\vee\ldots\vee x_{r(i)}^i$ and $y_1^i\vee\ldots\vee y_{r(i)}^i$ are m-comparable for each $i=1, \ldots, s$ with $s\leqq m$. If both $x_1^1\vee\ldots\vee x_{r(1)}^1\vee x_1^2\vee\ldots\vee x_{r(S)}^s$ and $y_1^1\vee\ldots\vee y_{r(1)}^1\vee y_1^2\vee\ldots\vee y_{r(S)}^s$ exist in S then they are m-comparable.*

PROOF. An application of the technique used in Lemma 1.5 quickly yields the result.

THEOREM 3. *Let $S\in\mathscr{D}_m$ and suppose that $x_1\vee\ldots\vee x_r$ is m-comparable to $y_1\vee\ldots\vee y_r$, $u_1\vee\ldots\vee u_s$ is m-comparable to $r_1\vee\ldots\vee r_s$. Then $(x_1\wedge u_1)\vee(x_1\wedge u_2)\vee\ldots\vee(x_r\wedge u_s)$ is m-comparable to $(y_1\wedge r_1)\vee(y_1\wedge r_2)\vee\ldots\vee(y_r\wedge r_s)$.*

PROOF. Both these joins exist in S by Lemma 2 and Theorem 1.8. Suppose $x_1\vee\ldots\vee x_r$ and $y_1\vee\ldots\vee y_r$ are m-comparable (step k). The proof proceeds by induction on k. For $k=0$ it suffices, by Lemma 2, to show that $(x_1\wedge u_1)\vee\ldots\vee(x_1\wedge u_s)$ and $(y_1\wedge r_1)\vee\ldots\vee(y_1\wedge r_s)$ are m-comparable. Suppose $u_1\vee\ldots\vee u_s$ and $r_1\vee\ldots\vee r_s$ are m-comparable (step h). We now use induction on h. The case $h=0$ is trivial. Assume that the result holds for $h=t-1$ and $u_1\vee\ldots\vee u_s$ and $r_1\vee\ldots\vee r_s$ are m-comparable (step t). Then, if $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ are the m-reducing partitions of $u_1\vee\ldots\vee u_s$ and $r_1\vee\ldots\vee r_s$ respectively, which satisfy the requirements of the definition, we define $C_i=\{x_1\wedge a_i\colon a_i\in A_i\}$ and $D_i=\{y_1\wedge b_i\colon b_i\in B_i\}$. By an application of the inductive hypothesis we can see that $\{C_i\}_{i=1}^n$ and $\{D_i\}_{i=1}^n$ are m-reducing partitions of $(x_1\wedge u_1)\vee\ldots\vee(x_1\wedge u_s)$ and $(y_1\wedge r_1)\vee\ldots\vee(y_1\wedge r_s)$ satisfying the necessary conditions to ensure that $(x_1\wedge u_1)\vee\ldots\vee(x_1\wedge u_s)$ and $(y_1\wedge r_1)\vee\ldots\vee(y_1\wedge r_s)$ are m-comparable. So we have completed the proof for the case $k=0$. Now assume that the result holds whenever $k=t-1$, and $x_1\vee\ldots\vee x_r$ and $y_1\vee\ldots\vee y_r$ are m-comparable (step t). Let $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ be m-reducing partitions of $x_1\vee\ldots\vee x_r$ and $y_1\vee\ldots\vee y_r$ respectively, which justify the fact that $x_1\vee\ldots\vee x_r$ and $y_1\vee\ldots\vee y_r$ are indeed m-comparable (step t).

If we set $C_i=\{a_i^j\wedge u_k\colon a_i^j\in A_i, k=1, \ldots, s\}$ and $D_i=\{b_i^j\wedge r_k\colon b_i^j\in B_i, k=1, \ldots, s\}$ it is not hard to see that $\{C_i\}_{i=1}^n$ and $\{D_i\}_{i=1}^n$ are m-reducing partitions of $(x_1\wedge u_1)\vee\ldots\vee(x_r\wedge u_s)$ and $(y_1\wedge r_1)\vee\ldots\vee(y_r\wedge r_s)$ respectively. Indeed, it is possible to show that $\{C_i\}_{i=1}^n$ and $\{D_i\}_{i=1}^n$ satisfy the conditions of the definition which ensure that

$(x_1 \wedge u_1) \vee \ldots \vee (x_r \wedge u_s)$ and $(y_1 \wedge r_1) \vee \ldots \vee (y_r \wedge r_s)$ are m-comparable. This has now completed the inductive step.

We are now in a position to describe the join in the lattice of m-join partial congruences on an m-distributive semilattice.

THEOREM 4. *Let S be an m-distributive semilattice and E a semilattice congruence on S. We define $E_0 = E$, and F_0 by $(x, y) \in F_0$ if and only if there exist $x_1, \ldots, x_r$ and $y_1, \ldots, y_r$ in S such that*

(i) $x = x_1 \vee \ldots \vee x_r,\ y = y_1 \vee \ldots \vee y_r$,

(ii) $(x_i, y_i) \in E_0$ *for* $i = 1, \ldots, r$,

(iii) $x_1 \vee \ldots \vee x_r$ *and* $y_1 \vee \ldots \vee y_r$ *are m-comparable.*

We define inductively E_k and F_k for $k = 1, 2, 3, \ldots$ by $(x, y) \in E_k$ if and only if there exist $z_0, \ldots, z_s \in S$ such that $x = z_0$, $y = z_s$ and $(z_{i-1}, z_i) \in F_{k-1}$ for $i = 1, \ldots, s$, while $(x, y) \in F_k$ if and only if there exist $x_1, \ldots, x_r$ and $y_1, \ldots, y_r$ in S such that

(i) $x = x_1 \vee \ldots \vee x_r,\ y = y_1 \vee \ldots \vee y_r$,

(ii) $(x_i, y_i) \in E_k$ *for* $i = 1, \ldots, r$,

(iii) $x_1 \vee \ldots \vee x_r$ *and* $y_1 \vee \ldots \vee y_r$ *are m-comparable.*

If Φ is the m-join partial congruence generated by E, we then have $\Phi = \bigcup_{k<\omega} E_k$.

PROOF. We claim that the following are true for each $k = 0, 1, 2, \ldots$.

(1) E_k is a semilattice congruence,

(2) $(x, x) \in F_k$ for all $x \in S$,

(3) $(x, y) \in F_k$ if and only if $(y, x) \in F_k$,

(4) $(x^i, y^i) \in F_k$ for $i = 1, \ldots, m$ implies $(x^1 \wedge x^2, y^1 \wedge y^2) \in F_k$ and if both $x^1 \vee \ldots \vee x^m$ and $y^1 \vee \ldots \vee y^m$ exist then $(x^1 \vee \ldots \vee x^m, y^1 \vee \ldots \vee y^m) \in F_k$.

The proof proceeds by induction on k. For the case $k = 0$, (1) follows from the definition and (2) and (3) are obvious. To show (4) suppose $(x^i, y^i) \in F_0$ for $i = 1, \ldots, m$. If both $x_1^1 \vee \ldots \vee x^m$ and $y^1 \vee \ldots \vee y^m$ exist then they are certainly m-comparable, in which case the second part holds, while the first part holds by applying the definition and Theorem 3.

Now assume that the result is true for $k = t-1$ and consider the case $k = t$. For (1) it clearly suffices to show that E_t has the substitution property for $\wedge$, so suppose that (x^1, y^1) and $(x^2, y^2) \in E_t$. By definition, there exist $z_0^1, \ldots, z_{r(1)}^1$ and $z_0^2, \ldots, z_{r(2)}^2$ such that $x^i = z_0^i$, $y^i = z_{r(i)}^i$ and $(z_{j-1}^i, z_j^i) \in F_{t-1}$ for $j = 1, \ldots, r(i)$ and $i = 1, 2$. Without loss of generality, we may assume that $r(1) = r(2)$ $(= r)$. Then put $z_j = z_j^1 \wedge z_j^2$ for $j = 0, \ldots, r$ and observe that $x^1 \wedge x^2 = z_0$, $y^1 \wedge y^2 = z_r$ and by our inductive hypothesis $(z_{j-1}, z_j) \in F_{t-1}$ for $j = 1, \ldots, r$. Hence $(x^1 \wedge x^2, y^1 \wedge y^2) \in E_t$ proving (1). Again (2) and (3) are obvious and (4) is shown by the same argument as before.

Let $\Phi = \bigcup_{k<\omega} E_k$. Since $E_0 \subseteq E_1 \subseteq E_2 \subseteq \ldots$, Φ must be a semilattice congruence. Because of our construction it is clearly m-join partial. If Δ is any other m-join partial congruence which contains E then it is easy to show that $E_k \subseteq \Delta$ for each k and so $\Phi \subseteq \Delta$. Thus, Φ is the m-join partial congruence generated by E.

Hence the join of any family of m-join partial congruences may be constructed by first taking their supremum in the lattice of semilattice congruences and then applying the method described above.

Before proceeding with some examples we need an obvious technical lemma.

LEMMA 5. *Let S be an m-distributive semilattice,* $\Phi\in\mathscr{C}^m(S)$ *and* $x_1\vee\ldots\vee x_r$ *and* $y_1\vee\ldots\vee y_r$ *m-comparable. If* $x_i\equiv y_i(\Phi)$ *for* $i=1,\ldots,r$ *then we have*

$$x_1\vee\ldots\vee x_r\equiv y_1\vee\ldots\vee y_r(\Phi).$$

THEOREM 6. *Let S be an m-distributive semilattice and let I be an m-ideal on S. Define the following equivalence relations on S.*

$x\equiv y(\theta(I))$ *if and only if* $\langle x\rangle_m+^m I=\langle y\rangle_m+^m I$, *and*

$x\equiv y(R(I))$ *if and only if* $(x\wedge b\in I\Leftrightarrow y\wedge b\in I,\ \forall b\in S)$.

Then both $\theta(I)$ *and* $R(I)$ *are m-join partial congruences. Furthermore,* $R(I)$ *is the largest m-join partial congruence having I as a congruence class and* $\theta(I)$ *is the smallest m-join partial congruence having I as a congruence class.*

PROOF. By applying Theorem 1.2 it is not hard to show that both $\theta(I)$ and $R(I)$ are m-join partial congruences which have I as a congruence class. Now suppose Δ is another m-join partial congruence which has I as a congruence class. We wish to show that $\theta(I)\subseteq\Delta\subseteq R(I)$. Suppose $x\equiv y(\theta(I))$ and without loss of generality assume $x\geqq y$. Hence $\langle x\rangle_m+^m I=\langle y\rangle_m+^m I$ and so $x\in\langle(y]\cup I\rangle_m$. Thus, x can be expressed as an m-reducible join $h_1\vee\ldots\vee h_r$ where each $h_i\in H=(y]\cup I$, using Theorem 1.7. By Lemma 1.6 $y=y\wedge x=(y\wedge h_1)\vee\ldots\vee(y\wedge h_r)$. Since Δ has I as a congruence class $y\wedge h_i\equiv h_i(\Delta)$ for each $i=1,\ldots,r$. Also $(y\wedge h_1)\vee\ldots\vee(y\wedge h_r)$ and $h_1\vee\ldots\vee h_r$ are clearly m-comparable and so, by Lemma 5, $x\equiv y(\Delta)$.

Now assume that $x\equiv y(\Delta)$ and so $x\wedge b\equiv y\wedge b(\Delta)$ for all $b\in S$ and so $x\wedge b\in I$ if and only if $y\wedge b\in I$, since I is a congruence class of Δ. Hence $x\equiv y(R(I))$.

For a filter F of a semilattice S it is well known that the equivalence relation $\Psi(F)$ defined by $x\equiv y(\Psi(F))$ if and only if $x\wedge f=y\wedge f$ for some $f\in F$, is a semilattice congruence. If S is m-distributive it is easy to show that $\Psi(F)$ is m-join partial. Indeed we have

THEOREM 7. *Let* $S\in\mathscr{D}_m$ *and F a filter in S. Then* $\Psi(F)$ *is an m-join partial congruence. Furthermore if S is directed above, then* $\Psi(F)$ *is the smallest m-join partial congruence which has F as a congruence class.*

For an element a of a semilattice S we let $[a)$ denote the smallest filter which contains a. Clearly $[a)=\{x\in S: x\geqq a\}$.

THEOREM 8. *Let S be an m-distributive semilattice and* $a,b\in S$ *with* $a\leqq b$. *Then* $T(a,b)=\theta(\langle b\rangle_m)\cap\Psi([a))$ *is the smallest m-join partial congruence which identifies a and b.*

PROOF. $T(a,b)$ is clearly an m-join partial congruence which identifies a and b. Suppose Φ is another and let $x\equiv y(T(a,b))$. We may assume that $x\leqq y$. Then $x\wedge a=y\wedge a$ and $\langle x,b\rangle_m=\langle y,b\rangle_m$. Since $a\equiv b(\Phi)$, we have $x\wedge a\equiv x\wedge b(\Phi)$, $y\wedge a\equiv y\wedge \wedge b(\Phi)$ and so $x\wedge b\equiv y\wedge b(\Phi)$.

Now $\langle x,b\rangle_m=\langle y,b\rangle_m$ implies $\langle x\rangle_m\cap\langle x,b\rangle_m=\langle x\rangle_m\cap\langle y,b\rangle_m$. Thus $\langle x\rangle_m= =\langle y,x\wedge b\rangle_m$ and so $x=y\vee(x\wedge b)$. Since $y\equiv y(\Phi)$ and $y\wedge b\equiv x\wedge b(\Phi)$, $y\vee(y\wedge b)\equiv \equiv y\vee(x\wedge b)(\Phi)$.

That is, $y\equiv x(\Phi)$.

During this section we have only considered the case when $S\in\mathcal{D}_m$ for some tengeri m. Similar results for $S\in\mathcal{D}_\omega$ have been given in CORNISH and HICKMAN [2].

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References

[1] R. BALBES, A representation theory for prime and implicative semilattices, *Trans. Amer. Math. Soc.*, **136** (1969), 261—267.
[2] W. H. CORNISH and R. C. HICKMAN, Weakly distributive semilattices, *Acta. Math. Acad. Sci. Hungar.*, **32** (1978), 5—16.
[3] W. H. CORNISH and R. C. HICKMAN, Modular semilattices, manuscript.
[4] G. GRÄTZER, *Lattice theory: First concepts and distributive lattices*, W. H. Freeman (San Francisco, 1971).
[5] T. KATRIŇÁK, Pseudokomplementäre Halbverbände, *Mat. Časopis*, **18** (1968), 121—143.
[6] B. M. SCHEIN, On the definition of distributive semilattices, *Alg. Universalis*, **2** (1972), 1—2.
[7] J. C. VARLET, On separation properties in semilattices, *Semigroup Forum*, **10** (1975), 220—228.

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SCHOOL OF MATHEMATICAL SCIENCES,
THE FLINDERS UNIVERSITY OF SOUTH AUSTRALIA,
BEDFORD PARK, 5042, S.A.,
AUSTRALIA.

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EINE ANALYTISCHE KENNZEICHNUNG EINER KLASSE DISKRETER GRUPPEN UND IHRER RIEMANNSCHEN FLÄCHEN

Von

G. ROSENBERGER (Dortmund)

Einleitung

In der analytischen Theorie der Flächen ist folgende Frage von Bedeutung: Unter welchen Bedingungen sind zwei diskrete, isomorphe Gruppen G_1, G_2 von Automorphismen der oberen Halbebene $\mathfrak{H}$ in der Gruppe $\mathfrak{G}(\mathfrak{H})$ aller Automorphismen von $\mathfrak{H}$ konjugiert zueinander? Sind nämlich zwei solche Gruppen G_1 und G_2 in $\mathfrak{G}(\mathfrak{H})$ konjugiert, so sind die Riemannschen Flächen $\mathfrak{H}/G_1$ und $\mathfrak{H}/G_2$ biholomorph äquivalent.

Seien nun G_1, G_2 zwei diskrete, isomorphe Untergruppen von $\mathfrak{G}(\mathfrak{H})$. Hat G_1 keinen Fundamentalbereich mit endlichem nicht-euklidischen Flächeninhalt, so ist natürlich G_1 in $\mathfrak{G}(\mathfrak{H})$ im allgemeinen nicht konjugiert zu G_2.

Ist $\mathfrak{P}(G_i)$ die Menge der parabolischen Spitzen von G_i $(i=1, 2)$, so bezeichne $\mathfrak{H}^+(G_i)$ die Vereinigungsmenge von $\mathfrak{H}$ und $\mathfrak{P}(G_i)$ für $i=1, 2$.

In [7] wurde gezeigt (vgl. auch [5]): Ist G_i freies Produkt zweier (endlicher oder unendlicher) zyklischer Gruppen und $\mathfrak{H}^+(G_i)/G_i$ kompakte Riemannsche Fläche vom Geschlecht null für $i=1, 2$, so ist G_1 in $\mathfrak{G}(\mathfrak{H})$ konjugiert zu G_2.

Der Fall zweier Erzeugender endlicher Ordnung wurde schon in [2] behandelt. Ist dagegen einer der folgenden Fälle erfüllt, so ist G_1 in $\mathfrak{G}(\mathfrak{H})$ im allgemeinen nicht konjugiert zu G_2:

1. G_i ist freies Produkt zyklischer Gruppen und $\mathfrak{H}^+(G_i)/G_i$ ist kompakte Riemannsche Fläche vom Geschlecht $g \geqq 1$ (vgl. [7], [5] und [10]).

2. G_i hat Rang $\geqq 2$, und $\mathfrak{H}/G_i$ ist kompakte Riemannsche Fläche vom Geschlecht $g \geqq 1$ (vgl. [6] und [10]).

3. G_i hat geometrischen Rang $\geqq 3$, und $\mathfrak{H}/G_i$ ist kompakte Riemannsche Fläche vom Geschlecht null (vgl. [4], [8], [9] und [10]), für $i=1$ und 2.

In dieser Note zeigen wir (Satz 1 und Satz 2): Hat G_i den Rang zwei und den geometrischen Rang zwei, und ist $\mathfrak{H}/G_i$ kompakte Riemannsche Fläche vom Geschlecht null für $i=1, 2$, so sind G_1 und G_2 in der Gruppe aller Automorphismen von $\mathfrak{H}$ konjugiert zueinander und die Riemannschen Flächen $\mathfrak{H}/G_1$ und $\mathfrak{H}/G_2$ biholomorph äquivalent.

Wir liefern einen neuen, elementaren Beweis für teilweise bekannte Ergebnisse (vgl. [3] und [10]).

Vorbemerkungen

Die Arbeit verwendet die Terminologie und Bezeichnungsweise von [3] und [4]. Wir identifizieren die PSL $(2, \mathbf{R})$ mit der Gruppe aller Automorphismen der oberen Halbebene $\mathfrak{G}$. Es ist PSL $(2, \mathbf{R}) =$ SL $(2, \mathbf{R})/\{E, -E\}$, d. h. die PSL $(2, \mathbf{R})$ besteht aus den Paaren $\{W, -W\}$, $W \in$ SL $(2, \mathbf{R})$. Es ruft keine Mißverständnisse hervor, wenn wir kurz W statt $\{W, -W\}$ schreiben. Es bedeute: $\lambda_n = 2\cos\frac{\pi}{n}$, $n \in \mathbf{N}$ und $n \geqq 2$. Sp U die Spur von $U \in \mathrm{GL}(2, \mathbf{R})$.

Sei nun G_1 eine diskrete Untergruppe der PSL $(2, \mathbf{R})$ mit den Eigenschaften:

a) G_1 hat Rang zwei und geometrischen Rang zwei.

b) $\mathfrak{H}/G_1$ ist kompakte Riemannsche Fläche vom Geschlecht null.

Nach [3] und [4] besitzt G_1 eine kombinatorische Beschreibung

$$G = \langle s_1, s_2 | s_1^{\alpha_1} = s_2^{\alpha_2} = (s_1 s_2)^{\alpha_3} = 1) \rangle, \quad 2 \leqq \alpha_1, \alpha_2, \alpha_3; \quad \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} < 1.$$

Man nennt G_1 auch Triangel-Gruppe, und wir werden im folgenden diese Bezeichnung verwenden.

Sei nun G_2 eine diskrete Untergruppe der PSL $(2, \mathbf{R})$, die isomorph zu G_1 ist. Nach Satz 32 von [3] ist G_2 auch eine Triangel-Gruppe und es ist

$$G_2 = \langle u_1, u_2 | u_1^{\alpha_1} = u_2^{\alpha_2} = (u_1 u_2)^{\alpha_3} = 1 \rangle,$$

$\alpha_1, \alpha_2, \alpha_3$ wie bei G_1. Für den Spezialfall $\alpha_1 = 2$, $\alpha_2 = 3$, $\alpha_3 = 7$ wurde Satz 1 und Satz 2 in [3] bewiesen.

Die Resultate

SATZ 1. *Seien $G_1, G_2 \subset$ PSL $(2, \mathbf{R})$ zwei isomorphe Triangelgruppen. Dann ist G_1 in* PSL $(2, \mathbf{R})$ *konjugiert zu G_2, d. h. es gibt eine Transformation $T \in$* PSL $(2, \mathbf{R})$ *mit $G_1 = TG_2T^{-1}$.*

BEWEIS. Sei $G \subset$ PSL $(2, \mathbf{R})$ Triangelgruppe, d. h.

$$G = \langle s_1, s_2 | s_1^{\alpha_1} = s_2^{\alpha_2} = (s_1 s_2)^{\alpha_3} = 1 \rangle \subset \mathrm{PSL}(2, \mathbf{R})$$

mit $2 \leqq \alpha_1, \alpha_2, \alpha_3$; $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} < 1$. Nach Theorem 2.3 von [1] gibt es Erzeugende A und B von G mit

$$\mathrm{Sp}\, A = \lambda_{\alpha_1}, \quad \mathrm{Sp}\, B = \lambda_{\alpha_2} \quad \text{und} \quad \mathrm{Sp}\, AB = -\lambda_{\alpha_3}.$$

BEHAUPTUNG (1.1). *Wir können* $A = \begin{pmatrix} 0 & 1 \\ -1 & \lambda_{\alpha_1} \end{pmatrix}$ *annehmen.*

BEWEIS. Sei etwa $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ PSL $(2, \mathbf{R})$. Wegen $a + d = \lambda_{\alpha_1} < 2$ ist $c \neq 0$. Indem wir eventuell A und B durch A^{-1} und B^{-1} ersetzen, können wir $c < 0$ annehmen. Sei also $c < 0$. Mit

$$N = \frac{1}{\sqrt{-c}} \begin{pmatrix} -c & a \\ 0 & 1 \end{pmatrix} \quad \text{ist} \quad NAN^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & \lambda_{\alpha_1} \end{pmatrix}.$$

Sei also nun

$$A = \begin{pmatrix} 0 & 1 \\ -1 & \lambda_{\alpha_1} \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Sei $M \in$ PSL $(2, \mathbf{R})$ mit $M = xE + yA$, $x, y \in \mathbf{R}$. Dann kommutiert A mit M, d.h.

es ist $MA=AM$. Eine Parameterdarstellung der Zahlen x, y ist gegeben durch

$$x=\cos\vartheta-\cot\frac{\pi}{\alpha_1}\cdot\sin\vartheta,\quad y=\frac{\sin\vartheta}{\sin\frac{\pi}{\alpha_1}}$$

wegen

$$\left(x+\frac{1}{2}\lambda_{\alpha_1}y\right)^2-\left(\frac{1}{4}\lambda_{\alpha_1}^2-1\right)y^2=1.$$

Mit

$$\left(2\sin^2\frac{\pi}{\alpha_1}\right)s:=(b+c)\sin\frac{\pi}{\alpha_1},$$

$$\left(2\sin^2\frac{\pi}{\alpha_1}\right)r:=a-d+(b-c)\cos\frac{\pi}{\alpha_1}\quad\text{und}\quad t:=r-a$$

ist das (1, 1)-Element von MBM^{-1} gerade

$$ax^2+(a\lambda_{\alpha_1}+b+c)xy+(c\lambda_{\alpha_1}+d)y^2=s\sin 2\vartheta+r\cos 2\vartheta-t.$$

Es ist

$$4\sin^2\frac{\pi}{\alpha_1}(s^2+r^2-t^2)=b^2+c^2+2bc+a^2\lambda_{\alpha_1}^2-4ad+2a\lambda_{\alpha_1}(b-c)=$$
$$=(a\lambda_{\alpha_1}+b-c)^2-4=(\mathrm{Sp}\,AB^{-1})^2-4.$$

BEHAUPTUNG (1.2). *Sei* $\alpha_1\geqq\alpha_2\geqq\alpha_3$ *und* $\alpha_2\geqq 4$ *oder* $\alpha_3\geqq 3$. *Dann ist* $\mathrm{Sp}\,AB^{-1}\geqq 2$ *und damit* $(\mathrm{Sp}\,AB^{-1})^2\geqq 4$.

BEWEIS. Es ist $\mathrm{Sp}\,AB^{-1}=\mathrm{Sp}\,A\cdot\mathrm{Sp}\,B-\mathrm{Sp}\,AB=\lambda_{\alpha_1}\cdot\lambda_{\alpha_2}+\lambda_{\alpha_3}$. Ist $\alpha_2\geqq 4$ oder $\alpha_3\geqq 3$, so ist $\lambda_{\alpha_1}\cdot\lambda_{\alpha_2}+\lambda_{\alpha_3}\geqq 2$ wegen $\alpha_1\geqq\alpha_2\geqq\alpha_3$. Damit ist $\mathrm{Sp}\,AB^{-1}\geqq 2$.

Fall (1.3). Es seien zwei der $\alpha_i\geqq 4$ oder alle drei $\alpha_i\geqq 3$. Sei ohne Einschränkung $\alpha_1\geqq\alpha_2\geqq\alpha_3$ und $\alpha_2\geqq 4$ oder $\alpha_3\geqq 3$. Nach (1.2) ist $(\mathrm{Sp}\,AB^{-1})^2\geqq 4$, d. h. $t^2\leqq r^2+s^2$, und es sind r, s nicht beide null. Damit kann eine reelle Zahl ϑ gerade so gewählt werden, daß $s\cdot\sin 2\vartheta+r\cdot\cos 2\vartheta=t$, d.h. das (1, 1)-Element von MBM^{-1} gerade null ist. Wegen $\mathrm{Sp}\,AB^{-1}=\lambda_{\alpha_1}\cdot\lambda_{\alpha_2}+\lambda_{\alpha_3}\geqq 2$ ist das (1, 2)-Element von MBM^{-1} größer als null.

Also ist G in PSL (2, **R**) konjugiert zu einer Gruppe G', die erzeugt wird von

$$\begin{pmatrix}0 & 1\\ -1 & \lambda_{\alpha_1}\end{pmatrix}\quad\text{und}\quad\begin{pmatrix}0 & \varrho\\ -\frac{1}{\varrho} & \lambda_{\alpha_2}\end{pmatrix}$$

mit $\varrho>0$ und $\lambda_{\alpha_1}\lambda_{\alpha_2}+\lambda_{\alpha_3}=\varrho+\frac{1}{\varrho}\geqq 2.$

Sei $\varrho<1$. Die Transformation $X\mapsto PX^{-1}P^{-1}$ mit $P=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ läßt A unverändert und führt $\begin{pmatrix}0 & \varrho\\ -\frac{1}{\varrho} & \lambda_{\alpha_2}\end{pmatrix}$ über in $\begin{pmatrix}0 & \frac{1}{\varrho}\\ -\varrho & \lambda_{\alpha_2}\end{pmatrix}$. Es ist nicht schwer zu zeigen, daß es ein

$R \in \mathrm{GL}(2, \mathbf{R})$ gibt mit $\det R = -1$ und $G'' = RG''R^{-1}$, wobei G'' die von $\begin{pmatrix} 0 & 1 \\ -1 & \lambda_{\alpha_1} \end{pmatrix}$ und $\begin{pmatrix} 0 & \frac{1}{\varrho} \\ -\varrho & \lambda_{\alpha_2} \end{pmatrix}$ erzeugte Gruppe ist.

Also können wir $\varrho \geqq 1$ annehmen, d. h. G ist in $\mathrm{PSL}(2, \mathbf{R})$ konjugiert zu einer Gruppe G', die erzeugt wird von

$$\begin{pmatrix} 0 & 1 \\ -1 & \lambda_{\alpha_1} \end{pmatrix} \quad \text{und} \quad \begin{pmatrix} 0 & \varrho \\ -\frac{1}{\varrho} & \lambda_{\alpha_2} \end{pmatrix}$$

mit $\varrho \geqq 1$ und $\lambda_{\alpha_1}\lambda_{\alpha_2} + \lambda_{\alpha_3} = \varrho + \frac{1}{\varrho} \geqq 2$. Wegen $\varrho \geqq 1$ ist ϱ eindeutig bestimmt, d. h. im Fall (1.3) ist die Behauptung des Satzes gezeigt.

Zu untersuchen ist nun noch der

Fall (1.4). Es sei ein $\alpha_i = 2$ und ein $\alpha_j = 3$. Sei ohne Einschränkung $\alpha_1 = 2$, $\alpha_2 = 3$. Dann ist $s = \frac{b+c}{2}$, $r = \frac{a-d}{2}$ und $t = -\frac{a+d}{2} = -\frac{1}{2}$. Damit ist das (1, 1)-Element von MBM^{-1} gerade

$$\frac{b+c}{2} \sin 2\vartheta + \frac{a-d}{2} \cos 2\vartheta + \frac{1}{2}.$$

Es kann eine reelle Zahl ϑ so gewählt werden, daß gerade

$$(b+c) \sin 2\vartheta + (a-d) \cos 2\vartheta = 0,$$

d. h. das (1, 1)-Element von MBM^{-1} gerade $\frac{1}{2}$ ist. Also ist G in $\mathrm{PSL}(2, \mathbf{R})$ konjugiert zu einer Gruppe G', die erzeugt wird von $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ und $\begin{pmatrix} \frac{1}{2} & g \\ -h & \frac{1}{2} \end{pmatrix}$ mit $g > 0$, $h > 0$, $h + g = \lambda_{\alpha_3}$ und $hg = \frac{3}{4}$. Sei $h < g$. Die Transformation $X \mapsto PX^{-1}P^{-1}$ mit $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ läßt A unverändert und führt $\begin{pmatrix} \frac{1}{2} & g \\ -h & \frac{1}{2} \end{pmatrix}$ über in $\begin{pmatrix} \frac{1}{2} & h \\ -g & \frac{1}{2} \end{pmatrix}$. Analog wie im Fall (1.3) können wir also $h \geqq g$ annehmen, d.h. G ist in $\mathrm{PSL}(2, \mathbf{R})$ konjugiert zu einer Gruppe G', die erzeugt wird von

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{und} \quad \begin{pmatrix} \frac{1}{2} & g \\ -h & \frac{1}{2} \end{pmatrix}$$

mit $h \geqq g > 0$, $h + g = \lambda_{\alpha_3}$ und $hg = \frac{3}{4}$. Wegen $h \geqq g$ ist h eindeutig bestimmt, d. h. auch im Fall (1.4) ist die Behauptung des Satzes gezeigt. Q.e.d.

Als Anwendung erhalten wir

SATZ 2. *Seien $G_1, G_2 \subset \mathrm{PSL}(2, \mathbf{R})$ zwei isomorphe Triangelgruppen. Dann sind die Riemannschen Flächen $\mathfrak{H}/G_1$ und $\mathfrak{H}/G_2$ biholomorph äquivalent.*

BEWEIS. Nach Satz 1 ist $G_1 = TG_2T^{-1}$ für ein $T \in \mathrm{PSL}(2, \mathbf{R})$. Den folgenden Teil des Beweises kann man [3] entnehmen. Wir bringen ihn vollständigkeitshalber. Sind z_1, z_2 Elemente derselben G_2-Bahn, so sind Tz_1, Tz_2 Elemente derselben G_1-Bahn, denn: $z_1 = Az_2$, $A \in G_2$ impliziert $Tz_1 = TAT^{-1}Tz_2$, und es ist $TAT^{-1} \in G_1$. Ist nun Γ eine G_2-Bahn, so sei $T(\Gamma)$ die eindeutig bestimmte G_1-Bahn, die das T-Bild von einem Element aus Γ enthält. Die Zuordnung $\Gamma \mapsto T(\Gamma)$ liefert nun eine biholomorphe Abbildung von $\mathfrak{H}/G_2$ auf $\mathfrak{H}/G_1$. Q.e.d.

Literaturverzeichnis

[1] A. W. KNAPP, Doubly generated Fuchsian groups, *Michigan Math. J.*, **15** (1968), 289—304.
[2] J. LEHNER, M. NEWMAN, Real two dimensional representations of the free product of two cyclic groups, *Cambridge Philos. Soc.*, **62** (1966), 1935—1941.
[3] A. M. MACBEATH, Discontinous groups and birational transformations, *Proc. of the Dundee Summer School* (1961).
[4] N. PECZYNSKI, G. ROSENBERGER, H. ZIESCHANG, Über Erzeugende ebener diskontinuierlicher Gruppen, *Inventiones math.*, **29** (1975), 161—180.
[5] N. PURZITSKY, Two-generator discrete free products, *Math. Z.*, **126** (1972), 209—223.
[6] N. PURZITSKY, G. ROSENBERGER, Two generator Fuchsian groups of genus one, *Math. Z.*, **128** (1972), 245—251. Correction, *Math. Z.*, **132** (1973), 261—262.
[7] G. ROSENBERGER, Fuchssche Gruppen, die freies Produkt zweier zyklischer Gruppen sind, und die Gleichung $x^2+y^2+z^2=xyz$, *Math. Ann.*, **199** (1972), 213—227.
[8] G. ROSENBERGER, R. N. KALIA, Automorphisms of the Fuchsian groups of type $(0; 2, 2, 2, q; 0)$, *Comm. in Algebra*, zu erscheinen.
[9] K. TAKEUCHI, Fuchsian groups contained in $SL_2(\mathbf{Q})$, *J. Math. Soc. Japan*, **23** (1971), 82—94.
[10] H. ZIESCHANG, E. VOGT, H.-D. COLDEWEY, Flächen und ebene diskontinuierliche Gruppen, *Lecture Notes in Mathematics* **122**. Springer (Berlin—Heidelberg—New York; 1970).

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SOME PROPERTIES OF MATRIX OF COEFFICIENTS IN BLOCK'S MULTIPLICATION TABLE OF THE SIMPLE MODULAR LIE ALGEBRAS OF BLOCK TYPE

By

I. B. MOHAMMED (Zaria)[1]

1. Introduction. R. E. BLOCK [1] has constructed simple Lie algebras over a modular field by considering an additive group G of order $p^n>1$, which is a direct sum of a finite number of finite elementary p-groups $G_0, G_1, \dots, G_m$. These groups are then finite dimensional vector spaces over Z_p. He assumed that G_0 can be allowed to be zero, and $p>2$, or $G\neq G_0$. Let $\delta=\delta_1+\delta_2+\dots+\delta_m$ be a fixed element of G, where $0\neq\delta_i\in G_i$ for $i=1, 2, \dots, m$, and let L be a vector space over the modular field Φ whose basis is indexed by the elements of G other than 0 and $-\delta$. Denote by $v(\alpha)$ the basis element of L corresponding to α in G. Hence the basis of L is in one-one correspondence $v(\alpha)\leftrightarrow\alpha$ with elements of G such that $\alpha\neq 0, -\delta$. Furthermore, we assume given, for each i, a nondegenerate and skew-symmetric biadditive function f_i on $G_i\times G_i$ to Φ, such that, for $1\leqq i\leqq m$, there are additive functions g_i, h_i on G_i to Φ, with $g_i(\delta_i)=0$ and $f_i(\alpha,\beta)=g_i(\alpha)h_i(\beta)-g_i(\beta)h_i(\alpha)$ for every α and β in G_i. Multiplication on L is defined by $v(\alpha)\circ v(\beta)=\sum_{i=0}^{m} f_i(\alpha_i,\beta_i)\,v(\infty+\beta-\delta_i)$; where α_i and β_i are the the components of α and β in G_1, respectively; and where δ_0 and $v(0)$ denote zero. With this multiplication L becomes a Lie algebra, which is represented by $L(G,\delta,f)$; where f is the biadditive function on G whose restriction to G_i is f_i.

2. Preliminaries. Let $v(\alpha)$ be denoted by v_α. Block has defined the derivations $D(\gamma_k,-\delta_k)$, $D(\delta,0)$, and $D(\sigma_{ij},0)$ of $L(G,\delta,f)$ by $v_\alpha D(\gamma_k,-\delta_k)=f(\alpha_k,\gamma_k)\,v(\alpha-\delta_k)$, $v_\alpha D(\sigma_{ij},0)=s_{ij}(\alpha)v_\alpha$, and $v_\alpha D(\delta,0)=\left[-1+\sum_{i=1}^{m}s_i(\alpha)\right]v_\alpha$ if $G_0=0$; $D(\delta,0)=0$ if $G_0\neq 0$; where γ_k is any element in G_k for any k; $\sigma_{01},\sigma_{02},\dots,\sigma_{0r_0}$ is the basis of G_0 over Z_p; $\sigma_{i1},\sigma_{i2},\dots,\sigma_{ir_i}$, ϑ_i a basis of G_i over Z_p such that $f(\sigma_{i1},\delta_i)\neq 0$; for $i=1,2,\dots,m$; $s_{ij}(\alpha)$ is the coefficient of σ_{ij} in α for $i=0,1,2,\dots,m$; and $s_i(\alpha)$ is the coefficient of δ_i in α.

If we denote v_α ad v_β for all $\alpha,\beta\neq-\delta$ in G by $v_\alpha R_\beta$, we obtain, using Block's multiplication, a general representation for $v_\alpha R_\beta^p$ as follows:

$$v_\alpha R_\beta^p=(v_\alpha\circ v_\beta)R_\beta^{p-1}=\left[\sum_{i_1=0}^{m} f(\alpha_{i_1},\beta_{i_1})\,v(\infty+\beta-\delta_{i_1})\right]R_\beta^{p-1}=$$

$$=\sum_{i_1=0}^{m} f(\alpha_{i_1},\beta_{i_1})[v(\infty+\beta-\delta_{i_1})\,R_\beta^{p-1}].$$

[1] The paper is based on a doctoral dissertation presented to the University of Oregon in September, 1974.

Continuing we get

$$v_\alpha R_\beta^p =$$

$$= \sum_{i_1, i_2, \ldots, i_p=0}^{m} f(\alpha_{i_1}, \beta_{i_1}) f(\alpha_{i_2}-\delta_{i_1}, \beta_{i_1} \ldots f(\alpha_{i_p}-\delta_{i_1}-\ldots-\delta_{i_{p-1}}, \beta_{i_p}) v(\infty-\delta_{i_1}-\ldots-\delta_{i_p}).$$

Letting the components of α and β in G_i to be α_i and β_i, respectively, we have the following

$$v_\alpha R_{\beta 0}^p = f(\alpha_0, \beta_0)^p v_\alpha, \quad v_\alpha R_{\beta_i}^p = [f(\alpha_i, \beta_i)^p - f(\beta_i, \delta_i)^{p-1} f(\alpha_i, \beta_i)] v,$$

where $i=1, 2, \ldots, m$.

It will be assumed that the algebra $L(G, \delta, f)$ is defined with respect to a group G of order p^n which is a direct sum of elementary p-groups G_i of order p^{n_i}; $i=0, 1, 2, \ldots, m$ and a nondegenerate and skew-symmetric biadditive function f defined on G and taking its values in the ground field Φ such that its restriction to G_i is f_i: $i=0, 1, 2, \ldots, m$. In addition we require that the ground field Φ be algebraically closed of characteristic $p>3$, and $\delta=\delta_1+\delta_2+\ldots+\delta_m$; where $\delta_i \in G_i$. Unless otherwise stated, it is to be assumed that $n \geqq 2$.

The following lemma will be useful.

LEMMA 2.1. *Let Φ be an algebraically closed field of characteristic $p \geqq 3$, G an elementary p-group of order p^n, where $n \geqq 2$, $\{\beta_i | i=1, 2; \ldots\}$ a basis of G, and f a skew-symmetric biadditive function on G with values in Φ. Then f is non-degenerate if and only if the matrix $\big(f(\beta_i, \beta_j)\big)_{1 \leqq i, j \leqq n}$ has no non-trivial linear combination over Z_p of the rows equal to zero.*

REMARK. It needs to be remarked that $f: G \times G \to \Phi$ is a Z_p-bilinear map and not necessarily a bilinear form on G.

3. Matrix of coefficients in Block's multiplication table. THEOREM 3.1. *Let $n \geqq 3$, $G \neq G_0$, and $\{\sigma_{0k}, \sigma_{ij} | j=1, 2, \ldots, n_i;\ i=1, 2, \ldots, m;\ k=1, 2, \ldots, n_0\}$ be some basis of G over Z_p such that $\delta_r = \sigma_{rn_r}$ and $f(\delta_{ri}, \delta_r) \neq 0$; $r=1, 2, \ldots, m$. Then for each $i=0, 1, \ldots, m$ none of the columns of the matrix of the coefficients of $D(\sigma_{ij}, 0)$; $j=1, 2, \ldots, n_i-1$ in $\operatorname{ad} v_{\sigma_{ij}}^p$; $j=1, 2, \ldots, n_i$ is zero.*

PROOF. With the use of $v_\alpha \operatorname{ad} v_\eta^p = \sum_{i=0}^{m} [f(\alpha_i, \eta_i)^p - f(\eta_i, \delta_i)^{p-1} f(\alpha_i, \eta_i)]^p v_\alpha$ we have

$$v_\alpha \operatorname{ad} v_{\eta 0}^p = f_0(\alpha_0, \eta_0)^p v_\alpha$$

and

$$v_\alpha \operatorname{ad} v_{\eta i}^p = [f_i(\alpha_i, \eta_i)^p - f_i(\eta_i, \delta_i)^{p-1} f_i(\alpha_i, \eta_i)] v_\alpha$$

where α_i, η_i are the components of α, η in G_i, respectively; and $i=1, 2, \ldots, m$.

It should be remarked that the nondegeneracy of f_i over Z_p; $i=0, 1, 2, \ldots, m$ has been established by BLOCK in [1].

From the expressions of $v_\alpha \operatorname{ad} v_{\eta 0}^p$; $i=0, 1, \ldots, m$ it follows that to prove the theorem it is sufficient to consider each of the f_i on G_i in turn. In fact for $i=1, 2, \ldots, m$, we only need to consider one of them.

Consider f_0 on G_0. Since $v_\alpha \operatorname{ad} v_{\eta 0}^p = f_0(\alpha_0, \eta_0)^p v_\alpha$ for all $\alpha=(\alpha_0, \ldots, \alpha)$ in G, $\operatorname{ad} v_{\sigma_j}^p = \sum_{i=1}^{n_0} f_0(\sigma_{0i}, \sigma_{0j})^p D(\sigma_{0i}, 0)$; $j=1, 2, \ldots, n_0$. Hence whenever a column of the matrix of the coefficients of $D(\sigma_{0i}, 0)$; $i=1, 2, \ldots, n_0$ in $\operatorname{ad} v_{\sigma_{0j}}^p$; $j=1, 2, \ldots, n_0$

is zero, a column of the matrix of f_0 in the basis σ_{0j}; $j=1, 2, \dots, n_0$ will be zero. This will make f_0 degenerate over Z_p; thereby giving a contradiction. Therefore no column of the matrix of the coefficients of $D(\sigma_{0i}, 0)$; $i=1, 2, \dots, n_0$ in ad $v_{\sigma_{0j}}$; $j=1, 2, \dots, n_0$ will be zero.

We now consider f_1 on G_1. Using

$$v_\alpha \operatorname{ad} v_{\eta_1}^p = [f_1(\alpha_1, \eta_1)^p - f_j(\eta_{1j}\delta_1)^{p-1} f_1(\alpha_1\eta_1)] v_\alpha,$$

we get

$$\operatorname{ad} v_{\sigma_{1j}}^p = \sum_{i=1}^{n_1-1} [f_1(\sigma_{1i}, \sigma_{1j})^p - f_1(\sigma_{1j}, \delta_1)^{p-1} f_1(\sigma_{1i}, \sigma_{1j})] D(\sigma_{1i}, 0); \quad j = 1, 2, \dots, n_1,$$

where δ_1 is defined as σ_{1n_1}.

Suppose one of the columns of the matrix of the coefficients of $D(\sigma_{1i}, 0)$; $i=1, 2, \dots, n_1-1$ in ad $v_{\sigma_{1j}}^p$; $j=1, 2, \dots, n_1$ is zero. Since $f_1(\sigma_{11}, \sigma_{1n_1}) \neq 0$, the first column of the matrix is not zero. Assuming the second column is zero, then $f_1(\sigma_{12}, \sigma_{1n_1})=0$ and

$$f_1(\sigma_{12}, \sigma_{1i})^p - f_1(\sigma_{1i}, \sigma_{1n_1})^{p-1} f_1(\sigma_{12}, \sigma_{1i}) = 0; \quad i = 1, 2, \dots, n_1-1.$$

This says that whenever $f_1(\sigma_{1j}, \sigma_{1n_1})=0$, then $f_1(\sigma_{12}, \sigma_{1j})=0$.

It has been shown in [1] that the Jacobi identity will be satisfied in the algebra $L(G, \delta, f)$ if and only if

$$f(\alpha_i, \beta_i) f(\gamma_i, \delta_i) + f(\beta_i, \gamma_i) f(\alpha_i, \delta_i) + f(\gamma_i, \alpha_i) f(\beta_i, \delta_i) = 0;$$

for $i=1, 2, \dots, m$ and all α, β and γ in G.

Suppose $f_1(\sigma_{12}, \sigma_{1l})=0$, for some l. By the nondegeneracy of f_1 over Z_p, there exists a σ_{1j} such that $f_1(\sigma_{12}, \sigma_{1j}) \neq 0$. Then with the substitution of $\alpha_i=\sigma_{12}$, $\beta_i=\sigma_{1l}$, and $\gamma_i=\sigma_{1j}$ in the alternative Jacobi identity, we get the following:

$$f_1(\sigma_{12}, \sigma_{1l}) f_1(\sigma_{1j}, \sigma_{1n_1}) + f_1(\sigma_{1l}, \sigma_{1j}) f_1(\sigma_{12}, \sigma_{1n_1}) + f_1(\sigma_{1j}, \sigma_{12}) f_1(\sigma_{1l}, \sigma_{1n_1}) = 0.$$

Since $f_1(\sigma_{12}, \sigma_{1l}) = f_1(\sigma_{12}, \sigma_{1n_1}) = 0$ and $f_1(\sigma_{1j}, \sigma_{12}) \neq 0$, the identity reduces to $f_1(\sigma_{1l}, \sigma_{1n_1})=0$. We have thus proved that if $f_1(\sigma_{12}, \sigma_{1l})=0$, then $f_1(\sigma_{1l}, \sigma_{1n_1})=0$.

Combining this with the result that if $f_1(\sigma_{1l}, \sigma_{1n_1})=0$, then $f_1(\sigma_{12}, \sigma_{1l})=0$, we get $f_1(\sigma_{12}, \sigma_{1l})=0$ if and only if $f_1(\sigma_{1l}, \sigma_{1n_1})=0$.

If for some k, $f_1(\sigma_{12}\sigma_{1k}) \neq 0$, then $f_1(\sigma_{1k}\sigma_{1n_1}) \neq 0$, and $\{f_1(\sigma_{12}, \sigma_{1k})[f_1(\sigma_{1k}, \sigma_{1n_1})]^{-1}\}^{p-1}=1$. This implies that $f_1(\sigma_{12}, \sigma_{1k}) = a_k f_1(\sigma_{1k}, \sigma_{1n_1})$ for some nonzero a_k in Z_p.

Suppose for some nonzero a_i, a_j in Z_p, $f_1(\sigma_{12}, \sigma_{1i}) = a_i f_1(\sigma_{1i}, \sigma_{1n_1})$ and $f_1(\sigma_{12}, \sigma_{1j}) = a_j f_1(\sigma_{1j}, \sigma_{1n_1})$. The substituting $\alpha_1=\sigma_{12}$, $\beta_1=\sigma_{1i}$, and $\gamma_1=\sigma_{1j}$ in the condition for the satisfaction of the Jacobi identity, we get

$$f_1(\sigma_{12}, \sigma_{1i}) f_1(\sigma_{1j}, \sigma_{1n_1}) + f_1(\sigma_{1i}, \sigma_{1j}) f_1(\sigma_{12}, \sigma_{1n_1}) + f_1(\sigma_{1j}, \sigma_{12}) f_1(\sigma_{1i}, \sigma_{1n_1}) = 0.$$

Whence

$$a_i f_1(\sigma_{1i}, \sigma_{1n_1}) f_1(\sigma_{1j}, \sigma_{1n_1}) + f_1(\sigma_{1i}, \sigma_{1j}) f_1(\sigma_{12}, \sigma_{1n_1}) - a_j f_1(\sigma_{1j}, \sigma_{1n_1}) f_1(\sigma_{1i}, \sigma_{1n_1}) = 0.$$

Since $f_1(\sigma_{12}, \sigma_{1n_1})=0$, it follows that $(a_i - a_j) f_1(\sigma_{1i}, \sigma_{1n_1}) f_1(\sigma_{1j}, \sigma_{1n_1})=0$. By the assumptions that $f_1(\sigma_{1i}, \sigma_{1n}) \neq 0$ and $f_1(\sigma_{1j}, \sigma_{1n_1}) \neq 0$, we will have $a_i=a_j$.

Hence for each $i=1, 2, \dots, m$, either $f_1(\sigma_{12}, \sigma_{1i})=0$, or $f_1(\sigma_{12}, \sigma_{1i}) = = a f_1(\sigma_{1i}, \sigma_{1n_1})$ for some nonzero $a \in Z$. Since $f_1(\sigma_{12}, \sigma_{1i})=0$ if and only if

$f_1(\sigma_{1i}, \sigma_{1n_1})=0$, we can write $f_1(\sigma_{12}, \sigma_{1i})=af_1(\sigma_{1i}, \sigma_{1n_1})$ for all $i=1, 2, \dots, m$. Inserting the deduction in the matrix of f_1 in the basis σ_{1i}; $i=1, 2, \dots, n_1$, the second row or column of the matrix of f_1 will be a multiple over Z_p of the last row or column, respectively. But Lemma 2.1 says this can only hold if f_1 is degenerate.

Therefore the matrix of the coefficients of $D(\sigma_{1i}, 0)$; $j=1, 2, \dots, n_1-1$ in ad $v^p_{\sigma_{1i}}$; $i=1, 2, \dots, n_1$ cannot have a zero column.

Using an identical argument, we can deduce that for each $i=1, 2, \dots, m$, the matrix of the coefficients of $D(\sigma_{ij}, 0)$; $j=1, 2, \dots, n_1-1$ in ad $v^p_{\sigma_{\cdot j}}$; $\gamma=1, 2, \dots, n_i$ cannot have a zero column.

THEOREM 3.2. *Let $G\neq G_0$, and $n\geqq 3$. If $\{\sigma_{0k}, \sigma_{ij} | j=1, 2, \dots, n_i;\ i=1, 2, \dots, m;\ k=1, 2, \dots, n_0\}$ is a basis of G over Z_p, then for each $i=1, 2, \dots, m$ the columns of matrix of the coefficients of $D(\sigma_{ij}, 0)$; $j=1, 2, \dots, n_i-1$ in* ad $v^p_{\sigma_{ij}}$; *$j=1, 2, \dots, n_i$ are linearly independent over Z_p; and the columns of the matrix of the coefficients of $D(\sigma_{0j}, 0)$; $j=1, 2, \dots, n_0$ in* ad $v^p_{\sigma_{0j}}$; *$j=1, 2, \dots, n_0$ are also linearly independent over Z_p.*

PROOF. Since f_i is the restriction of f to G_i; where $i=0, 1, \dots, m$ it then follows that

$$\operatorname{ad} v^p_{\sigma_{0j}} = \sum_{i=1}^{n_0} f_0(\sigma_{0i}, \sigma_{0j})^p D(\sigma_{0i}, 0);$$

where $j=1, 2, \dots, n_0$ and

$$\operatorname{ad} v^p_{\sigma_{ij}} = \sum_{i=1}^{n_i-1} [f_i(\sigma_{ik}, \sigma_{ij})^p - f_i(\sigma_{ij}, \delta_i)^{p-1} f_i(\sigma_{ik}, \sigma_{ij})] D(\sigma_{ik}, 0)$$

for $j=1, 2, \dots, n_i$; $i=1, 2, \dots, m$.

To prove the theorem we need to consider the matrices of the coefficients of $D(\sigma_{0j}, 0)$; $j=1, 2, \dots, n_0$ in ad $v^p_{\sigma_{0j}}$; $j=1, 2, \dots, n_0$, and $D(\sigma_{ik}, 0)$; $k=1, 2, \dots, n_1-1$ in ad $v^p_{\sigma_{ij}}$; $j=1, 2, \dots, n_i$ for $i=1, 2, \dots, m$.

Suppose the columns of the matrix of the coefficients of $D(\sigma_{0i}, 0)$; $i=1, 2, \dots, n_0$ in ad $v^p_{\sigma_{0j}}$; $j=1, 2, \dots, n_0$ are linearly dependent over Z_p. Then by the skew-symmetry of the matrix the rows will also be linearly dependent over Z_p. Consequently the rows of the matrix of f_0 will be linearly dependent over Z_p; and this contradicts Lemma 2.1. Hence the columns of the matrix of the coefficients of $D(\sigma_{0i}, 0)$; $i=1, 2, \dots, n_0$ in ad $v^p_{\sigma_{ij}}$; $j=1, 2, \dots, n_0$ are linearly independent over Z_p.

We now consider the matrix of the coefficients of $D(\sigma_{1k}, 0)$; $k=1, 2, \dots, n_1-1$ in ad $v^p_{\sigma_{ij}}$; $j=1, 2, \dots, n_1$. Assuming the rows of the matrix are linearly dependent over Z_p; then we can find η_i; $i=1, 2, \dots, n_1-1$ in Z_p not all of which are zero such that they give a dependence relation over Z_p between the columns of the matrix of the coefficients of $D(\beta_{1j}, 0)$; $j=1, 2, \dots, n_1-1$ in ad $v^p_{\beta_{1i}}$; $i=1, 2, \dots, n_1$. If $\gamma=\sum_{i=1}^{n_1-1} \eta_i\beta_{1i}$, then $f_1(\beta_{1n_1}, \gamma)=0$ and

$$f_1(\beta_{1i}, \gamma)^p = f_1(\beta_{1i}, \beta_{1n_1})^{p-1} f_1(\beta_{1i}, \gamma); \quad i = 1, 2, \dots, n_1-1.$$

We note that γ is not a multiple of β_{11} over Z_p as otherwise $f(\beta_{1n_1}, \beta_{11})$ will be zero; which is not possible. Assuming that the coefficient of β_{12} in γ is not zero, then $\beta_{11}, \gamma, \beta_{13}, \dots, \beta_{1n_1}$ is again a basis of G_1 over Z_p.

With this new basis for G_1, $(\operatorname{ad} L(G_1, \delta_1, f_1))^p$ will be spanned by $\operatorname{ad} v_\gamma^p$, $\operatorname{ad} v_{\beta_{1j}}^p$; $j=1, \dots, n_1-1$; $i \neq 2$. The following matrix representation for the expansions of $\operatorname{ad} v_\gamma^p, \operatorname{ad} v_{\beta_{1i}}^p$; $i=1, 2, \dots, n_1-1$; $i \neq 2$ in $D(\beta_{1j}, 0)$, $D(\gamma, 0)$; $j=1, 2, \dots, n_1-1$; $j \neq 2$ will then arise.

$$\begin{bmatrix} \operatorname{ad} v_{\beta_{11}}^p \\ \operatorname{ad} v_\gamma^p \\ \operatorname{ad} v_{13}^p \\ \vdots \\ \operatorname{ad} v_{\beta_{1n_1-1}}^p \\ \operatorname{ad} v_{\beta_1}^p \end{bmatrix} \times$$

$$\times \begin{bmatrix} 0 & 0 & X_{13}^p - X_{n_1 1}^{p-1} X_{13} & \dots & X_{1n_1-1}^p - X_{n_1 1}^{p-1} X_{1n_1-1} \\ -f(\gamma, \beta_{11})^p & 0 & f(\beta_{13}, \gamma)^p & \dots & f(\beta_{1n_1-1}, \gamma)^p \\ -(X_{13}^p - X_{n_{13}}^{p-1} X_{13}) & 0 & 0 & \dots & X_{3n_1-1}^p - X_{n_1 3}^{p-1} X_{3n_1-1} \\ \vdots & \vdots & \vdots & & \vdots \\ -(X_{1n_1-1}^p, X_{n_1 n_1-1}^{p-1} X_{1n_1-1}) & 0 & -(X_{3n_1-1}^p - X_{n_{1n_1-1}}^{p-1} X_{3n_1-1}) & \dots & 0 \\ -X_{1n_1}^p & 0 & -X_{3n_1}^p & \dots & -X_{n_1-1n_1}^p \end{bmatrix} \times$$

$$\times \begin{bmatrix} D(\beta_{11}, 0) \\ D(\gamma, 0) \\ D(\beta_{13}, 0) \\ \vdots \\ D(\beta_{1n_1-1}, 0) \\ \end{bmatrix}$$

Therefore we have found a basis $(\beta_{11}, \gamma, \beta_{13}, \dots, \beta_{1n_1}\}$ of G_1 over Z_p such that the matrix of the coefficients of $D(\gamma, 0)$, $D(\beta_{1j}, 0)$; $j=1, 2, \dots, n_1-1$; $j \neq 2$ in $\operatorname{ad} v_\gamma^p, \operatorname{ad} v_{\beta_{1i}}^p$; $i=1, 2, \dots. n-1$; $i \neq 2$ has a zero column. But according to Theorem 3.1 this is possible only if f_1 is degenerate over Z_p. Since f_1 is nondegenerate over Z_p, we will end up a with a contradiction.

Repeating the same proof for the matrix of the coefficients of $D(\sigma_{ik}, 0)$; $k=1, 2, \dots, n_i-1$ in $\operatorname{ad} v_{\sigma_{ij}}^p$; $j=1, 2, \dots, n_i$, where $i=2, 3, \dots, m$, we conclude that the columns of the matrix of the coefficients of $D(\sigma_{ij}, 0)$; $j=1, 2, \dots, n_i-1$ in $\operatorname{ad} v_{\sigma_{ij}}^p$; $j=1, 2, \dots, n_i$, for $i=1, 2, \dots, m$ are linearly independent over Z_p.

References

[1] R. E. Block, New simple Lie algebras of prime charadteristics, *Trans. A. M. S.*, **89** (1958), 421—449.
[2] R. Godement, *Algebra*, Hermann Publishers (Paris, 1968).
[3] N. Jacobson, *Lie Algebras*, Interscience Publishers (New York, 1962).
[4] I. B. Mohammed, *Isomorphism conditions for the modular Lie algebras of Block type*, Doctoral Dissertation, University of Oregon (1974).
[5] G. B. Seligman, *Modular Lie algebras*, Springer—Verlag (New York, 1967).
[6] D. J. Winter, *Abstract Lie Algebras*, The MIT Press (Cambridge, Massachusetts, 1972).

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AHMADU BELLO UNIVERSITY
ZARIA—NIGERIA

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CLASSICAL ALGEBRAS OF QUOTIENTS AND CENTRAL QUOTIENTS OF ALGEBRAS

By
G. SZETO (Peoria—Chicago)[1]

1. Introduction

L. ROWEN [5] defined the central quotients of an algebra, and answered the question negatively whether the classical algebra of quotients is the central quotient of an algebra. W. D. BLAIR [1] proved that an algebra over a commutative ring finitely generated and projective as a module has a right and left classical algebra of quotients, which is the central quotient of the algebra. The present paper continues the above study. We shall show that any non-zero-divisor of $\mathrm{Hom}_R(M, M)$ is injective if M is finitely generated flat module over a commutative ring R and $R_p \otimes_R \mathrm{Hom}_R(M, M) \cong \mathrm{Hom}_{R_p}(M_p, M_p)$ for each maximal ideal p of R, where R_p is a local ring at p. This generalizes the well known fact when M is finitely generated free or finitely generated and projective over R ([1], Lemmas 1 and 2). Moreover, when the zero divisors $Z(R)$ of R are stable under localizations (that is, whenever $p \subset Z(R)$, $(p)_p \subset Z(R_p)$ for each maximal ideal p), the above non-zero-divisor of $\mathrm{Hom}_R(M, M)$ is also surjective. In this case, $\mathrm{Hom}_R(M, M)$ has a classical algebra of quotients, which is the central quotient of $\mathrm{Hom}_R(M, M)$. Thus this fact is true for an algebra A if $Z(R)$ is stable and if A is finitely generated flat and p-torsion free over R (for p-torsion theory, see [4]). Furthermore, any separable algebra (see [3] for definition) can be shown to have a classical algebra of quotients, which is the central quotient of the algebra.

Throughout, we assume that R is a commutative ring, that all rings have an identity, and that all modules are unitary.

2. Preliminaries

We shall employ the following facts which can be found in [2]. Let $M_n(R)$ be the matrix ring of order n over R. Then we have:

PROPOSITION 2.1. *Let F be a finitely generated free or projective R-module. Then any regular (non-zero-divisor) element of* $\mathrm{Hom}_R(F, F)$ *is injective* ([1], Lemmas 1 and 2).

PROPOSITION 2.2. *If M is a finitely generated R-module, N an R-module, and S a multiplicative closed set excluding 0 of R such that $N \to N_S$ is injective where N_S is the localization of N at S, then $R_S \otimes_R \mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_{R_S}(M_S, N_S)$ under the natural map.*

PROPOSITION 2.3. *If M is a finitely generated and flat R-module over a commutative local ring R, then M is free.*

[1] This work was done during the author's sabbatical leave at the University of Chicago.

PROPOSITION 2.4. *The direct product ΠR_p of local rings R_p at all maximal ideals p of R is faithfully flat over R.*

Let S be the set of non-zero-divisors of R, we denote the total quotient ring of R by R_S, and the set of zero-divisors of R by $Z(R)$. The central quotient of an algebra A is the localization of A at S $(=R_S\otimes_R A)$.

3. Classical algebras of quotients

We now extend Proposition 2.1.

THEOREM 3.1. *Let M be a finitely generated flat R-module such that $R_p\otimes_R \mathrm{Hom}_R(M, M)\cong \mathrm{Hom}_{R_p}(M_p, M_p)$ for each maximal ideal p of R. If f is a regular element of $\mathrm{Hom}_R(M, M)$, f is injective.*

PROOF. Denote the image of f in $\mathrm{Hom}_{R_p}(M_p, M_p)$ by f'. We claim that f' is regular; for otherwise there exists a non-zero element g' in $\mathrm{Hom}_{R_p}(M_p, M_p)$ with g in $\mathrm{Hom}_R(M, M)$ such that $f'g'=0$. $(fg)'=0$, so there exists some s in $(R-p)$ such that $s(fg)=0$, $f(sg)=0$. Since f is regular, $(sg)=0$. Hence $g'=0$, a contradiction. Moreover, $\mathrm{Hom}_{R_p}(M_p, M_p)$ is a matrix ring of finite order over R_p because M_p is finitely generated flat over R_p (Proposition 2.3). Thus f' is injective for each p (Proposition 2.1). Since ΠR_p for all maximal ideals of R is faithfully flat over R (Proposition 2.4), f is also injective.

We note that there exist finitely generated flat R-modules M such that $R_p\otimes \mathrm{Hom}_R(M, M)\cong \mathrm{Hom}_{R_p}(M_p, M_p)$, but M is not projective. For example, let R be the ring given on p. 509 in [6], and M the R-module generated by $f(=(2, 0))$ in R. It is straightforward to verify that $R_p\otimes_R \mathrm{Hom}_R(M, M)\cong$ $\cong \mathrm{Hom}_{R_p}(M_p, M_p)$ for each maximal ideal p or R.

It has been known that some class of algebras have classical algebras of quotients, which are central quotients of the algebras ([1], and [5]). For these algebras, we can assume R is a total quotient ring and claim that each regular element of the algebra is invertible. Next we discuss when the above $\mathrm{Hom}_R(M, M)$ has a classical algebra of quotients.

THEOREM 3.2. *Let A be an R-algebra finitely generated flat as an R-module with $R_p\otimes_R \mathrm{Hom}_R(A, A)\cong \mathrm{Hom}_{R_p}(A_p, A_p)$ for each maximal p contained in $Z(R)$. If A_p is its own classical algebra of quotients, then so is A.*

PROOF. Let x be a regular (non-zero-divisor) element of A. We claim that the image x' of x in A_p is regular in A_p. In fact, suppose y' is a non-zero element in A_p such that $x'y'=0$, $(xy)'=0$. Then there exists an s in $(R-p)$ such that $s(xy)=0$, $x(sy)=0$, so $(sy)=0$; and so $y'=0$ in A_p, a contradiction. By hypothesis on A_p, x' is invertible. On the other hand, the element x induces a left multiplication map f_x of $\mathrm{Hom}_R(A, A)$ such that f_x, is invertible in $\mathrm{Hom}_{R_p}(A_p, A_p)$ for each p. Hence f_x is a bijection in $\mathrm{Hom}_R(A, A)$ because ΠR_p for all maximal ideals p of R is faithfully flat over R (Proposition 2.4). This implies that there exists an y in A such that $f_x(y)=1$, and so $xy=1$. Similarly, there exists a z in A such that $zx=1$, so x is invertible in A.

We observe that there exists a class of commutative rings R with the property that $(p)_p\subset Z(R_p)$ whenever the maximal ideal $p\subset Z(R)$. We call such an R a Z-stable ring.

THEOREM 3.3. (1) *All R of Krull-dimension* 0 *are Z-stable rings.* (2) *All noetherian rings R are Z-stable rings.*

PROOF. (1) Let p be a maximal ideal of R contained in $Z(R)$, and x an element in p such that its image in R_p, $x' \neq 0$. We then have an integer $n > 1$ such that $(x')^n = 0$ since the Krull-dimension of R is 0. Thus x' is a zero divisor of R_p.

(2) Let p be a maximal ideal of R contained in $Z(R)$. Since R is noetherian, p is a maximal associated prime ideal of 0; that is, p = the annihilator of x for some element x. If y is in p such that the image y' of y in R_p is not zero, then x must be in p. Thus $x'y' = 0$. Since p = the annihilator of x, $x' \neq 0$. Thus y' is a zero divisor of R_p.

Now a class of algebras over a Z-stable ring R can be shown to have classical algebras of quotients, which are central quotients of the algebras.

THEOREM 3.4. *Let R be a Z-stable ring, M a finitely generated flat R-module such that* $R_p \otimes_R \mathrm{Hom}_R(M, M) \cong \mathrm{Hom}_{R_p}(M_p, M_p)$ *for each maximal ideal p contained in $Z(R)$, then* $\mathrm{Hom}_R(M, M)$ *has a left and right classical algebra of quotients, which are the central quotients of* $\mathrm{Hom}_R(M, M)$.

PROOF. As given in the remark before Theorem 3.2, R can be assumed to be a total quotient ring. Let f be regular in $\mathrm{Hom}_R(M, M)$. The proof of Theorem 3.1 shows that the image f' of f in $\mathrm{Hom}_{R_p}(M_p, M_p)$ is also regular. Since R is a Z-stable ring, R_p is its own total quotient ring. Then, by Proposition 2.3, f' is invertible since M_p is free over R_p. Hence f' is surjective in $\mathrm{Hom}_{R_p}(M_p, M_p)$ for each p. By noting that R is its own total quotient ring, these p's are all maximal ideals of R, so f is surjective in $\mathrm{Hom}_R(M, M)$ by Proposition 3.4. Thus f is invertible in $\mathrm{Hom}_R(M, M)$.

COROLLARY 3.5. *Let R be a Z-stable ring, and A an R-algebra finitely generated flat as an R-module such that* $R_p \otimes_R \mathrm{Hom}_R(A, A) \cong \mathrm{Hom}_{R_p}(A_p, A_p)$ *for each maximal ideal p contained in $Z(R)$, then A has a left and right classical algebra of quotients, which are the central quotients of A.*

PROOF. Theorem 3.4 implies that $\mathrm{Hom}_R(A, A)$ is its own classical algebra of quotients if R is its own total quotient ring. In this case, since any regular element x in A remains regular in $\mathrm{Hom}_R(A, A)$, where x is considered as a left multiplication map in $\mathrm{Hom}_R(A, A)$, x is invertible in $\mathrm{Hom}_R(A, A)$. Hence x is invertible in A as argued in Theorem 3.2.

Let M be an R-module, p a prime ideal of R. J. Lambek called the submodule N^p the p-torsion of M where

$$N^p = \{x \text{ in } M \mid \text{there exists an } r \text{ not in } p \text{ with } rx = 0\}.$$

If $N^p = 0$, M is called p-torsion free [4]. Clearly, $M \to R_p \otimes_R M$ is injective, so $R_p \otimes_R \mathrm{Hom}_R(M, M) \cong \mathrm{Hom}_{R_p}(M_p, M_p)$ if M is finitely generated over R (Proposition 2.2). Thus we have:

COROLLARY 3.6. *Let R be a Z-stable ring, A an R-algebra finitely generated flat and p-torsion free as an R-module for each maximal ideal p contained in $Z(R)$. Then A has a left and right classical algebra of quotients, which are the central quotients of A.*

Now we give an application for the theorem of BLAIR in [1]. Recall that an R-algebra A is separable if it is projective (left) over $A\otimes_R A^0$, where A^0 is the opposite algebra of A ([3]).

THEOREM 3.7. *Let A be a separable R-algebra. Then A has a left and right classical algebra of quotients, which are the central quotients of A considered as an algebra over its centre C.*

PROOF. Since A is separable over R, it is central separable over its centre C (an Azumaya C-algebra). Hence A is finitely generated and projective over C. Thus the statement is immediate from the theorem in [1].

We conclude the paper with an example of a finitely generated flat module M over a commutative ring R such that:

(1) R is a Z-stable ring;

(2) $R_p\otimes_R \mathrm{Hom}_R(M, M)\cong \mathrm{Hom}_{R_p}(M_p, M_p)$ for all maximal ideals p contained in $Z(R)$; and

(3) M is not projective.

This means that our results provide a new class of algebras with classical algebras of quotients, which are the central quotients of the algebras. The example is due to W. VASCONCELOS ([6], p. 509).

Let A be the ring without identity obtained by taking a non-finite direct sum of copies of $J/2J$ where J is the ring of integers, $A=\oplus\sum_{\alpha}(J/2J)_\alpha$ for α in some index set I and defining addition and multiplication componentwise. Let R be the ring obtained by adding the identity of J to A: $R=J+A$ where addition is defined componentwise and multiplication is given by $(n,a)(n',a')=(nn', na'+n'a+aa')$. Let $f=(2,0)$. Then $M(=Rf)$ is a flat R-module generated by f, but not projective (see [6]). Now let p be a maximal ideal contained in $Z(R)$. Clearly, $(J,0)\subset p$, so f is in p. Hence $M_p=0$. Thus $\mathrm{Hom}_{R_p}(M_p, M_p)=0$. Since $\mathrm{Hom}_R(M,M)\cong M(=Rf)$, $R_p\otimes_R \mathrm{Hom}_R(M,M)=0$. Thus (2) is satisfied. Again, since $(J,0)\subset p$, there exists a component, $J/2J$, not contained in p. Thus $(p)_p=0$. This proves (1).

Finally, we observe that the above R is a Z-stable ring of Krull-dimension 1 although we have proved that all R of Krull-dimension 0 are Z-stable in Theorem 3.3.

References

[1] W. D. BLAIR, Quotient rings of algebras which are module finite and projective, *Acta Sci. Math. Szeged*, **36** (1974), 265—266.

[2] N. BOURBAKI, *Commutative Algebra*, Addison—Wesley Publishing Company (1972).

[3] F. DEMEYER and E. INGRAHAM, *Separable algebras over commutative rings*, Springer-Verlag (Heidelberg—Berlin—New York, 1971).

[4] J. LAMBEK, *Torsion theories, additive semantics, and rings of quotients*, Springer Lecture Notes in Mathematics, **177** (1971).

[5] L. ROWEN, On rings with central polynomials, *J. Algebra*, **31** (1974), 393—426.

[6] W. VASCONCELOS, On finitely generated flat modules, *Trans. Amer. Math. Soc.*, (1969), 495—512.

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BRADLEY UNIVERSITY
PEORIA, ILLINOIS 61 625
AND
THE UNIVERSITY OF CHICAGO
CHICAGO, ILLINOIS 60 637
U.S.A.

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IDEALS IN $Z[x, y]$

By
P. G. TROTTER (Hobart)

1. Introduction

The ideals of $Z[x]$, the ring of polynomials over the integers, have been enumerated by KRONECKER and HENSEL [1]. More recently SZEKERES [4] determined a canonical basis for any ideal of $Z[x]$ and derived from the basis a set of integer invariants that characterize the ideal. RÉDEI has compared the results of these authors in [2], and has generalized them in [3] for ideals of $R[x]$ where R is a commutative principal ideal ring with prime decomposition.

The aim here is to generalize the approach of [4] to obtain a canonical basis for any ideal of $Z[x, y]$ and to characterize the ideal with a set of degree and integer invariants. The invariants of the ideal are chosen in section 5 and are shown to be restricted by certain inequalities between themselves; conversely any set of degrees and integers satisfying the restrictions is a set of invariants of an ideal in $Z[x, y]$.

SZEKERES has enumerated those ideals of $Z[x, y]$ that contain an integer [5], and the homogeneous ideals of $K[x, y, z]$ where K is a field [6]. In these papers a different method is used for determining ideal invariants; in particular, some of the invariants of [6] are chosen as solutions of systems of algebraic, possibly non-linear, equations.

It should be noted that, as with RÉDEI's generalization of [4], the results of this paper can be modified to provide for ideals of $R[x, y]$ where R is a principal ideal domain. Furthermore the methods may be extended for ideals of $Z[x_1, x_2, \ldots, x_n]$. However, because of the additional complication in these cases we consider only ideals in $Z[x, y]$.

The following notation is used throughout. Let

$$\Delta = \{(i, j)\in Z\times Z;\ i, j \geqq 0\} \cup (-\infty, -\infty)$$

be a semigroup under componentwise addition with $(-\infty, -\infty)$ as its zero. Define an order on Δ so that for $(h, k), (i, j)\in\Delta$ then $(i, j)>(h, k)$ if $i+j>h+k$ or $i+j=h+k$ and $j>k$. For $\alpha=(i, j)\in\Delta$ write $X^\alpha=x^iy^j\in Z[x, y]$; if $\alpha=(-\infty, -\infty)$ let $X^\alpha=0$. Call α the *degree* of X^α. For $a\in Z[x, y]$ let $a(\alpha)$ denote the coefficient of the term of degree α of a and let $\bar a\in\Delta$ be largest so that $a(\bar a)\neq 0$. Call $\bar a$ the *degree* of a. Then $a=\sum_{\alpha\leqq\bar a} a(\alpha)X^\alpha$.

For $\alpha\in\Delta$ and $h\in Z$ define $\alpha^h=\alpha+(0, h)$ if $\alpha+(0, h)\in\Delta$ and ${}^h\alpha=\alpha+(h, 0)$ if $\alpha+(h, 0)\in\Delta$ (addition is componentwise). We will consider the following subsets of an ideal Γ of the subsemigroup $\Delta\setminus(-\infty, -\infty)$ of Δ:

$$\Gamma_1 = \{\alpha^1;\ \alpha\in\Gamma\};\quad {}_1\Gamma = \{{}^1\alpha;\ \alpha\in\Gamma\};\quad {}_1\Gamma_1 = \{{}^1\alpha^1;\ \alpha\in\Gamma\};\quad \Phi = \Gamma\setminus(\Gamma_1\cup{}_1\Gamma);$$
$$\Psi = (\Gamma_1\cap{}_1\Gamma)\setminus{}_1\Gamma_1.$$

2. Degree sets of ideals

Let $\mathscr{P}$ be an ideal of $Z[x, y]$ and let $\Gamma=\{\bar{a}; a\in\mathscr{P}\setminus 0\}$ be a set of degrees. Clearly Γ is a semigroup ideal of $\Delta\setminus(-\infty, -\infty)$. We will determine some properties of the subsets Φ and Ψ of Γ.

LEMMA 2.1. *If (h, k) and (p, q) are distinct elements of Φ then either $h>p$ and $k<q$ or $h<p$ and $k>q$. Hence Φ is finite.*

PROOF. Note that $\alpha\in\Phi$ if and only if $\alpha\in\Gamma$ and there exists no $(r, s)\in\Delta$, $(r, s)\neq(0, 0)$, so that ${}^{-r}\alpha^{-s}\in\Gamma$. Since $(h-(h-p), k-(k-q))$ and $(p-(p-h), q-(q-k))$ are in Φ then the first statement follows. Let j be least so that $(i, j)\in\Phi$. If $(i, j)\neq(h, k)\in\Phi$ then $i>h\geqq 0$ so $|\Phi|\leqq i+1$.

We therefore label the elements of Φ by $\alpha_j=(m_j, i_j)$; $j=1, 2, \ldots, |\Phi|$, so that for $|\Phi|\geqq h>k\geqq 1$ then $i_h>i_k$ and $m_h<m_k$. Note that any element of Γ is expressible in the form ${}^h\alpha_j{}^k$ for some $(h, k)\in\Delta$, $\alpha_j\in\Phi$. Hence Φ is a set of generators of the ideal Γ.

LEMMA 2.2. *$\beta\in\Psi$ if and only if $\beta=(m_j, i_{j+1})$ for some j, $1\leqq j<|\Phi|$. Furthermore, $(m_j, i_{j+1})={}^h\alpha_u{}^k$ for some $(h, k)\in\Delta$ only if $u=j$ or $u=j+1$.*

PROOF. If $\beta\in\Psi$ then there exists $\alpha_u, \alpha_j\in\Phi$ and $p, q>0$ so that $(m_u+p, i_u)=\beta=(m_j, i_j+q)$. Then $u>j$. If $u>j+1$ then $i_u>i_{j+1}$ while $m_j>m_{j+1}$ so $\beta=(m_{j+1}, i_{j+1})+(m_j-m_{j+1}, i_u-i_{j+1})\in{}_1\Gamma_1$. But $\beta\in\Psi$ so $u=j+1$ and $\beta=(m_j, i_{j+1})$. Conversely if $(m_j, i_{j+1})\in{}_1\Gamma_1$ then there exists $\alpha_u\in\Phi$ and $p, q>0$ so that $(m_u+p, i_u+q)=(m_j, i_{j+1})$. Then $j+1>u>j$ which is impossible. Clearly $(m_j, i_{j+1})\in\Gamma_1\cap{}_1\Gamma$ so $(m_j, i_{j+1})\in\Psi$. The second statement follows immediately.

We therefore label the elements of Ψ by $\beta_j=(m_j, i_{j+1})$; $j=1, 2, \ldots, |\Phi|-1$.

3. Basis coefficients

An ideal of $Z[x, y]$ is *primitive* if its elements have no non-unit common divisor. In the remainder of the paper $\mathscr{P}$ denotes a primitive ideal of $Z[x, y]$ with degree set $\Gamma\cup(-\infty, -\infty)$. The elements of the subsets Φ and Ψ of Γ will be labelled as in section 2. In this section we describe some bases of $\mathscr{P}$ and from any such basis we determine a set of basis coefficients that uniquely characterize the basis.

By Hilbert's basis theorem there is a least degree $\tau\in\Gamma$ so that the polynomials $a\in\mathscr{P}$, $\bar{a}\leqq\tau$, generate $\mathscr{P}$. For each $\alpha\in\Gamma$ we may choose a polynomial $a_\alpha\in\mathscr{P}$ of degree $\bar{a}_\alpha=\alpha$ so that the leading coefficient $a_\alpha(\alpha)$ is the least positive element of the ideal $\{b(\alpha); b\in\mathscr{P}, \bar{b}\leqq\alpha\}$ of Z. We call the set $\{a_\alpha; \alpha\in\Gamma\}$ an *extended $\mathscr{P}$-basis* and the subset $\{a_\alpha; \alpha\leqq\tau\}$ a *$\mathscr{P}$-basis*. A $\mathscr{P}$-basis is unique if and only if $\Phi=\{\tau\}$. The following is easily proved:

LEMMA 3.1. *If $\{a_\alpha\}$ is an extended $\mathscr{P}$-basis then any $a\in\mathscr{P}$ uniquely determines $R_\alpha\in Z$ so that $a=\sum_{\alpha\leqq\bar{a}} R_\alpha a_\alpha$. Hence a $\mathscr{P}$-basis is a basis of $\mathscr{P}$.*

For an extended $\mathscr{P}$-basis $\{a_\alpha\}$ we have for $\beta\in\Gamma$

(1a) $$a_{{}^{-1}\alpha}x=\sum_{\beta\leqq\alpha} A_{\beta\alpha}a_\beta \quad \text{for} \quad \alpha\in{}_1\Gamma$$

and

(1b) $$a_{\alpha^{-1}}y = \sum_{\beta \leqq \alpha} B_{\beta\alpha} a_\beta \quad \text{for} \quad \alpha \in \Gamma_1,$$

where $A_{\beta\alpha}, B_{\beta\alpha} \in Z$. For convenience write $A_{\beta\alpha}=0$ and $B_{\beta\alpha}=0$ if $\beta \notin \Gamma$. Also let $A_\alpha = A_{\alpha\alpha}$, $B_\alpha = B_{\alpha\alpha}$. Whenever they are defined A_α and B_α are positive integers. $A_{\beta\alpha}$, $B_{\beta\alpha}$ are called *basis coefficients* of $\{a_\alpha\}$.

For $\alpha \in \Gamma_1 \cap {}_1\Gamma$ we get from (1a, b)

(2) $$D_\alpha a_{\alpha^{-1}} y - E_\alpha a_{{}^{-1}\alpha} x = \sum_{\beta < \alpha} F_{\beta\alpha} a_\beta$$

where $C_\alpha D_\alpha = A_\alpha$, $C_\alpha E_\alpha = B_\alpha$, $F_{\beta\alpha} = D_\alpha B_{\beta\alpha} - E_\alpha A_{\beta\alpha}$ and $C_\alpha = (A_\alpha, B_\alpha)$ is the highest common factor of A_α and B_α.

For $\alpha \in {}_1\Gamma_1$ we also have from (1a, b) that

$$a_{{}^{-1}\alpha^{-1}} xy = \sum_{\gamma \leqq \alpha^{-1}} A_{\gamma\alpha^{-1}} a_\gamma y = \sum_{\beta \leqq \gamma \leqq \alpha} A_{(\gamma^{-1})\alpha^{-1}} B_{\beta\gamma} a_\beta.$$

Also $a_{{}^{-1}\alpha^{-1}} yx = \sum_{\beta \leqq \gamma \leqq \alpha} B_{{}^{-1}\gamma({}^{-1}\alpha)} A_{\beta\gamma} a_\beta$. We can equate coefficients by Lemma 3.1 to get for $\alpha \in {}_1\Gamma_1$, $\beta \in \Gamma$

(3) $$A_{\alpha^{-1}} B_{\beta\alpha} - B_{{}^{-1}\alpha} A_{\beta\alpha} = \sum_{\beta \leqq \gamma < \alpha} (B_{{}^{-1}\gamma({}^{-1}\alpha)} A_{\beta\gamma} - A_{(\gamma^{-1})\alpha^{-1}} B_{\beta\gamma}).$$

When $\beta = \alpha$ we get

(4) $$A_{\alpha^{-1}} B_\alpha = B_{{}^{-1}\alpha} A_\alpha \quad \text{for} \quad \alpha \in {}_1\Gamma_1.$$

By comparing (2), (3) and (4) for $\alpha \in {}_1\Gamma_1$ it can be checked that

(5) $$A_{\alpha^{-1}} a_{\alpha^{-1}} y - B_{{}^{-1}\alpha} a_{{}^{-1}\alpha} x = \sum_{\beta < \alpha} G_{\beta\alpha} a_\beta$$

where

$$G_{\beta\alpha} = \sum_{\beta \leqq \gamma < \alpha} (B_{{}^{-1}\gamma({}^{-1}\alpha)} A_{\beta\gamma} - A_{(\gamma^{-1})\alpha^{-1}} B_{\beta\gamma}) = F_{\beta\alpha}(A_{\alpha^{-1}}, B_{{}^{-1}\alpha}).$$

Rewriting equations (1a, b) gives

(6a) $$a_\alpha = \Big(a_{{}^{-1}\alpha} x - \sum_{\beta < \alpha} A_{\beta\alpha} a_\beta\Big)/A_\alpha \quad \text{for} \quad \alpha \in {}_1\Gamma$$

and

(6b) $$a_\alpha = \Big(a_{\alpha^{-1}} y - \sum_{\beta < \alpha} B_{\beta\alpha} a_\beta\Big)/B_\alpha \quad \text{for} \quad \alpha \in \Gamma_1.$$

By using the corresponding equations for degrees less than α and giving precedence to equations of form (6a) we get uniquely

(7) $$a_\alpha = \sum_{k=1}^{n} f_k a_{\alpha_k} \quad \text{where} \quad n = |\Phi|,\ \alpha_k \in \Phi,\ f_k \in Q[x, y];$$

Q denotes the rational number field. Notice that the degree $\overline{f_k a_{\alpha_k}} \leqq \alpha$. In fact, since precedence is given to (6a) in calculating (7) then $\overline{f_k a_{\alpha_k}} = \alpha$ if and only if k is greatest so that $\alpha = {}^p\alpha_k{}^q$ for some $(p, q) \in \Delta$.

THEOREM 3.2. *An extended $\mathcal{P}$-basis $\{a_\alpha\}$ is uniquely determined by its basis coefficients.*

PROOF. Given the polynomials a_{α_k}, $\alpha_k \in \Phi$, an extended $\mathcal{P}$-basis can be uniquely determined from its basis coefficients by equations (1a, b). We must therefore show that the basis coefficients uniquely determine a_{α_k}, $\alpha_k \in \Phi$. If $|\Phi|=1$, $\mathcal{P}$ uniquely gives a_{α_k}. Recall equation (2) for $\alpha=\beta_j\in\Psi$. Using relations (7) to substitute for the terms of (2) (for each j) we get the equations

$$\sum_{k=1}^{n} f_{j,k}a_{\alpha_k}=0 \quad \text{for} \quad j=1,2,\ldots,n-1, \quad f_{j,k}\in Q[x,y].$$

By Lemma 2.2 and the comments following (7) we have

$$\overline{f_{j,k}}+\alpha_k \leqq B_j \quad \text{with equality if and only if} \quad k=j \quad \text{or} \quad k=j+1. \tag{8}$$

Consider the linear equations over the quotient field $Q(x, y)$ of $Z[x, y]$

$$\sum_{k=1}^{n} f_{j,k}z_k=0; \quad j=1,2,\ldots,n-1. \tag{9}$$

We will prove the theorem by showing that there is only one subset of an extended $\mathcal{P}$-basis, with elements of degrees $\alpha_k\in\Phi$, that satisfies (9). We know that $\{a_{\alpha_k};\ \alpha_k\in\Phi\}$ satisfies (9). Let $N=\{1,2,\ldots,n-1\}$ and $N_h=\{1,2,\ldots,h-1,h+1,\ldots,n\}$. Let $\pi\colon N\to N_h$ be a bijection. Then by (8)

$$\overline{\prod_{j\in N} f_{j,\pi(j)}}+\sum_{j\in N}\alpha_{\pi(j)}=\sum_{j\in N}(\overline{f_{j,\pi(j)}}+\alpha_{\pi(j)})\leqq\sum_{j\in N}B_j.$$

Equality is achieved if $\pi(j)=j$ for $j<h$ and $\pi(j)=j+1$ for $j\geqq h$. In fact equality is achieved only for this choice of π; for by (8), since $\pi(h)\neq h$ then $\pi(h)=h+1$, so $\pi(h+1)=h+2$ and so on. Similarly $\pi(h-1)\neq h$ so $\pi(j)=j$ for $j<h$. Hence the determinant $|(f_{j,k})_{k\neq h}|$ has degree

$$\sum_{j\in N}(\beta_j-\alpha_{\pi(j)})=\sum_{j\in N}((m_j,i_{j+1})-(m_{\pi(j)},i_{\pi(j)}))=(m_h-m_n,i_h-i_1),$$

by Lemma 2.2. Hence the polynomial $|(f_{j,k})_{k\neq h}|$ is non-zero. Given some value for z_h the equations (9) are linearly independent over $Q(x,y)$ and hence have a unique solution. Since $\{a_{\alpha_k};\ \alpha_k\in\Phi\}$ satisfies (9) then any solution set of (9) is of the form $\{fa_{\alpha_k};\ \alpha_k\in\Phi\}$ for $f\in Q(x,y)$. But the leading coefficients and degrees of elements of an extended $\mathcal{P}$-basis are uniquely determined by $\mathcal{P}$, so $\{a_{\alpha_k};\ \alpha_k\in\Phi\}$ is the only solution of (9) of the required form.

COROLLARY 3.3. *Let $n=|\Phi|$. Then $m_n=i_1=0$.*

PROOF. By Cramer's rule $\{|(f_{j,k})_{k\neq h}|;\ 1\leqq h\leqq n\}\subseteq Q[x,y]$ is a solution set of (9). By (1a, b) any common divisor of $\{a_{\alpha_k};\ \alpha_k\in\Phi\}$ in $Q[x,y]$ is a common divisor of the elements of $\mathcal{P}$. $\mathcal{P}$ is primitive so we have $fa_{\alpha_h}=|(f_{j,k})_{k\neq h}|$ for some $f\in Q[x,y]$ and $1\leqq h\leqq n$. Then $(m_1+p,i_1+q)=(m_1-m_n,i_1-i_1)$ where $(p,q)=\bar f$. The result follows.

4. The canonical basis

In this section restrictions will be imposed on basis coefficients of extended $\mathcal{P}$-bases so that there is only one extended $\mathcal{P}$-basis with basis coefficients satisfying the restrictions. The underlying $\mathcal{P}$-basis is the canonical basis of $\mathcal{P}$.

Suppose throughout this section that $\{a_\alpha\}$ and $\{b_\alpha\}$ are extended $\mathcal{P}$-bases with basis coefficients $A_{\beta\alpha}$, $B_{\beta\alpha}$ and $A'_{\beta\alpha}$, $B'_{\beta\alpha}$ respectively. Since $a_\alpha(\alpha)=b_\alpha(\alpha)$ then $A_\alpha=A'_\alpha$ if $\alpha\in{}_1\Gamma$ and $B_\alpha=B'_\alpha$ if $\alpha\in\Gamma_1$.

For some $\alpha\in{}_1\Gamma$ assume $a_\beta=b_\beta$ for all $\beta<\alpha$. Combining equations (1a) for the respective extended $\mathcal{P}$-bases gives

$$A_\alpha(a_\alpha-b_\alpha)=\sum_{\beta<\alpha}(A'_{\beta\alpha}-A_{\beta\alpha})a_\beta.$$

By Lemma 3.1 then $A'_{\beta\alpha}\equiv A_{\beta\alpha}\ (\mathrm{mod}\ A_\alpha)$. Conversely, using (1a), we can construct an extended $\mathcal{P}$-basis $\{a_\beta, c_\gamma;\ \beta<\alpha<\gamma\}$ with basis coefficients A_α and $A_{\beta\alpha}+k_{\beta\alpha}A_\alpha$, $k_{\beta\alpha}\in Z$, for degree α. Similar statements apply to $B_{\beta\alpha}$ and $B'_{\beta\alpha}$ when $\alpha\in\Gamma_1$. Let $\alpha_z\in\Phi$ be the least degree in Γ. We can now assume that $\{a_\alpha\}$ has been chosen so that

(10a) $$0\leqq A_{\beta\alpha}<A_\alpha\quad\text{for}\quad \alpha\in{}_1\Gamma\setminus\Psi,\quad\text{or}\quad \alpha=\beta_u\in\Psi,\quad u\geqq z$$

and

(10b) $$0\leqq B_{\beta\alpha}<B_\alpha\quad\text{for}\quad \alpha\in\Gamma_1\setminus({}_1\Gamma\cup\Psi),\quad\text{or}\quad \alpha=\beta_u\in\Psi,\quad u<z.$$

The case $\alpha\in\Psi$ is distinguished for the sake of the next Theorem.

Notice that given the subset $\{a_\alpha;\ \alpha\in\Phi\}$ of an extended $\mathcal{P}$-basis then by (1a, b) there is exactly one extended $\mathcal{P}$-basis with basis coefficients satisfying the restrictions (10a, b). We must therefore restrict the choice of extended $\mathcal{P}$-basis polynomials of degrees $\alpha\in\Phi$.

By Lemma 3.1 it can be seen that

(11) $$b_{\alpha_j}=a_{\alpha_j}+\sum_{\beta<\alpha_j}k_{\beta j}a_\beta\quad\text{for}\quad \alpha_j\in\Phi,\ k_{\beta j}\in Z.$$

Note, since α_z is least in Γ, that $b_{\alpha_z}=a_{\alpha_z}$. For convenience of notation let $\alpha^*=\overline{a_\alpha-b_\alpha}$ and write $b_\alpha\equiv a_\alpha \bmod (a_\beta;\ \beta\leqq\alpha^*)$.

LEMMA 4.1. *Let $\{a_\alpha\}$ and $\{b_\alpha\}$ be extended $\mathcal{P}$-bases satisfying restrictions* (10a, b). *Then* $\alpha^*\leqq\max\{\eta, \max_{\beta<\alpha}\beta^*\}$ *where*

(i) $\eta={}^1(({}^{-1}\alpha)^*)$ *if* $\alpha\in{}_1\Gamma\setminus\Psi$ *or* $\alpha=\beta_u\in\Psi$, $u\geqq z$ *and*

(ii) $\eta=((\alpha^{-1})^*)^1$ *if* $\alpha\in\Gamma_1\setminus({}_1\Gamma\cup\Psi)$ *or* $\alpha=\beta_u\in\Psi$, $u<z$.

PROOF. Consider only case (i), case (ii) is similar. Let $\sigma=\max_{\beta<\alpha}\beta^*$ and $\zeta=\max\{\eta,\sigma\}$. If $({}^{-1}\alpha)^*\in\Gamma$ then there is a non-zero $R\in Z$ so that $b_{{}^{-1}\alpha}\equiv(a_{{}^{-1}\alpha}+Ra_{{}^{-1}\eta}) \bmod (a_\beta;\ \beta<{}^{-1}\eta)$. So by (1a)

(12) $$b_{{}^{-1}\alpha}x\equiv(a_{{}^{-1}\alpha}x+RA_\eta a_\eta) \bmod (a_\beta;\ \beta<\eta).$$

$$b_{{}^{-1}\alpha}x=\sum_{\beta\leqq\alpha}A'_{\beta\alpha}b_\beta\equiv\Big(A_\alpha b_\alpha+\sum_{\sigma<\beta<\alpha}A'_{\beta\alpha}a_\beta\Big) \bmod (a_\beta;\ \beta\leqq\sigma)$$

by (1a). Hence by (12)

$$A_\alpha b_\alpha \equiv \big(a_{-1_\alpha} x - \sum_{\zeta<\beta<\alpha} A'_{\beta\alpha} a_\beta\big) \bmod (a_\beta;\ \beta \leqq \zeta).$$

So by (1a)

$$A_\alpha (b_\alpha - a_\alpha) \equiv \big(\sum_{\zeta<\beta<\alpha} (A_{\beta\alpha} - A'_{\beta\alpha}) a_\beta\big) \bmod (a_\beta;\ \beta \leqq \zeta).$$

By Lemma 3.1, $A_{\beta\alpha} \equiv A'_{\beta\alpha} \pmod{A_\alpha}$ so by (10a) $A_{\beta\alpha} = A'_{\beta\alpha}$ for $\zeta<\beta<\alpha$.

Notice that if $\eta > \max\limits_{\beta<\alpha} \beta^*$ then $A_{\beta\alpha} = A'_{\beta\alpha}$ for $\eta<\beta$, so by (12) and (1a),

$$A_\alpha b_\alpha + A'_{\eta\alpha} a_\eta \equiv \big(A_\alpha a_\alpha + (A_{\eta\alpha} + RA_\eta) a_\eta\big) \bmod (a_\beta;\ \beta < \eta).$$

Then by Lemma 3.1, $A'_{\eta\alpha} \equiv (A_{\eta\alpha} + RA_\eta) \pmod{A_\alpha}$. By (10a) there is a $T \in Z$ so that

(13) $$A'_{\eta\alpha} + TA_\alpha = A_{\eta\alpha} + RA_\eta \quad\text{and}\quad b_\alpha \equiv (a_\alpha + Ta_\eta) \bmod (a_\beta;\ \beta < \eta).$$

In the remainder of the paper the following subsets of Γ will be considered. Recall that α_z is least in Γ, $\Phi = \{\alpha_u;\ 1 \leqq u \leqq |\Phi|\}$ and $\Psi = \{\beta_u;\ 1 \leqq u < |\Phi|\}$. Let

$$\Omega_1 = \{\alpha_u^k \in \Gamma_1 \setminus (\Gamma_1 \cap {}_1\Gamma);\ u < z\};\quad \Omega_2 = \{\alpha_u^k \in \Gamma_1 \setminus (\Gamma_1 \cap {}_1\Gamma);\ u \geqq z\};$$

$${}_1\Omega = \{{}^h\alpha_u \in {}_1\Gamma \setminus (\Gamma_1 \cap {}_1\Gamma);\ u > z\};\quad {}_2\Omega = \{{}^h\alpha_u \in {}_1\Gamma \setminus (\Gamma_1 \cap {}_1\Gamma);\ u \leqq z\};$$

$$\Psi_1 = \{\beta_u \in \Psi;\ u < z\} \quad\text{and}\quad {}_1\Psi = \{\beta_u \in \Psi;\ u \geqq z\}.$$

Define $K_{\beta\alpha} = 1$ if $\alpha^{-1} \in \Phi$, $\beta^{-1} \in \Gamma$ or ${}^{-1}\alpha \in \Phi$, ${}^{-1}\beta \in \Gamma$. Define $K_{\beta\alpha} = 0$ if $\alpha^{-1} \in \Phi$, $\beta^{-1} \notin \Gamma$ or ${}^{-1}\alpha \in \Phi$, ${}^{-1}\beta \notin \Gamma$. Define inductively $K_{(\beta^1)\alpha^1} = K_{\beta\alpha} B_\beta / (K_{\beta\alpha} B_\beta, B_\alpha)$ if $\alpha \in \Omega_1 \cup \Psi_1$, and $K_{{}^1\beta({}^1\alpha)} = K_{\beta\alpha} A_\beta / (K_{\beta\alpha} A_\beta, A_\alpha)$ if $\alpha \in {}_1\Omega \cup {}_1\Psi$.

We will see that there is an extended $\mathscr{P}$-basis with basis coefficients so that

(14a) $$0 \leqq A_{\beta\alpha} < (K_{\beta\alpha} A_\beta, A_\alpha) \quad\text{if}\quad \alpha \in {}_1\Omega \cup {}_1\Psi,$$

(14b) $$0 \leqq B_{\beta\alpha} < (K_{\beta\alpha} B_\beta, B_\alpha) \quad\text{if}\quad \alpha \in \Omega_1 \cup \Psi_1,$$

(15a) $$0 \leqq B_{\beta\alpha} < K_{{}^1\beta({}^1\alpha)} B_\alpha \quad\text{if}\quad \alpha = {}^h\alpha_u \in {}_1\Psi \quad\text{and}\quad {}^{-h}\beta \in \Gamma$$

and

(15b) $$0 \leqq A_{\beta\alpha} < K_{(\beta^1)\alpha^1} A_\alpha \quad\text{if}\quad \alpha = \alpha_u^h \in \Psi_1 \quad\text{and}\quad \beta^{-h} \in \Gamma.$$

THEOREM 4.2. *There is exactly one extended $\mathscr{P}$-basis with basis coefficients satisfying the restrictions* (10a, b), (14a, b) *and* (15a, b).

PROOF. Suppose that the basis coefficients of the extended $\mathscr{P}$-bases $\{a_\alpha\}$ and $\{b_\alpha\}$ are restricted by (10a, b). We have seen that such bases exist and are uniquely determined by their basis coefficients and their elements of degree $\alpha_u \in \Phi$. We will see that with (14a, b) and (15a, b) the elements of degree $\alpha_u \in \Phi$ can also be uniquely determined. Some further information is required.

For $\alpha \in \Gamma$, $\alpha_u \in \Phi$ let $[\alpha - \alpha_u] \in \Delta$ be largest so that $\alpha_u + [\alpha - \alpha_u] \leqq \alpha$. Let $\alpha^+ = \max\limits_{\alpha_u \in \Phi} \{\alpha_u^* + [\alpha - \alpha_u]\}$. We will prove by induction that $\alpha^* \leqq \alpha^+$. Clearly $\alpha_z^* = (-\infty, -\infty) \leqq \alpha_z^+$. Assume $\beta^* \leqq \beta^+$ for all $\beta < \alpha$, $\beta \in \Gamma$. Since $[\beta - \alpha_u] \leqq [\alpha - \alpha_u]$ and $[\alpha^{-1} - \alpha_u]^1 \leqq [\alpha - \alpha_u]$ then $\beta^* \leqq \beta^+ \leqq \alpha^+$ and $((\alpha^{-1})^*)^1 \leqq ((\alpha^{-1})^+)^1 \leqq \alpha^+$. Likewise ${}^1(({}^{-1}\alpha)^*) \leqq \alpha^+$ so by Lemma 4.1 $\alpha^* \leqq \alpha^+$.

Choose $u \neq z$ with the property that $|u-z|$ is least so that $\alpha_v + \alpha_u^* \geqq \alpha_u + \alpha_v^*$ for all $\alpha_v \in \Phi$. Let $\alpha = {}^h\alpha_u \leqq \beta_{u-1}$ if $u > z$ and $\alpha = \alpha_u^h \leqq \beta_u$ if $u < z$. Assume $\alpha_u^* \neq \neq (-\infty, -\infty)$. We will see that $\alpha^+ = \alpha_u^* + [\alpha - \alpha_u] > \beta^+$ for all $\beta < \alpha$. Assume $u > z$; the proof is similar for $u < z$. By Lemma 2.2 we have $[\alpha - \alpha_v] + \alpha_v < \alpha$ if $v \notin \{u, u-1\}$. By the definition of u, if $v = u-1$ then $a_v + a_u^* > \alpha_u + \alpha_v^*$. Hence $\alpha^+ = \alpha_u^* + [\alpha - \alpha_u] > \alpha_v^* + [\alpha - \alpha_v]$ for $v \neq u$ since $\alpha_u + \alpha_v^* + [\alpha - \alpha_v] \leqq \alpha_u^* + \alpha_v + [\alpha - \alpha_v] \leqq \leqq \alpha_u^* + \alpha = \alpha_u^* + \alpha_u + [\alpha - \alpha_u]$. Since $\alpha_v^* + [\beta - \alpha_v] \leqq \alpha_v^* + [\alpha - \alpha_v] < \alpha^+$ and $[\beta - \alpha_u] < < [\alpha - \alpha_u]$ then $\beta^+ < \alpha^+$. Note also that if $\alpha = \beta_{u-1}$ then $[\alpha^{-1} - \alpha_v]^1 \leqq [\alpha - \alpha_v]$ and since $[\alpha - \alpha_u] \notin \Gamma_1$ then $[\alpha^{-1} - \alpha_u]^1 < [\alpha - \alpha_u]$ so $((\alpha^{-1})^+)^1 < \alpha^+$.

The proof of the theorem can now be completed. Choose $u > z$ as above and suppose $a_{\alpha_u} - b_{\alpha_u} \equiv R a_\gamma \bmod (a_\delta; \delta < \gamma)$ for some $R \in Z$ (see (11)). We will see that there is an extended $\mathscr{P}$-basis satisfying (10a, b) and satisfying (14a, b), (15a, b) for $\alpha = {}^h\alpha_u \leqq \beta_{u-1}$ and $\beta = {}^h\gamma$ (then $\alpha \in {}_1\Omega \cup {}_1\Psi$). Furthermore if $\{a_\alpha\}$ and $\{b_\alpha\}$ satisfy these restrictions we will see that $R = 0$. The process is similar if $\alpha \in \Omega_1 \cup \Psi_1$. Let $h = 1$, then $\beta = {}^1(({}^{-1}\alpha)^*) = \alpha_u^* + [\alpha - \alpha_u] = \alpha^+ > \delta^+ \geqq \delta^*$ for all $\delta < \alpha$. Hence as with the derivation of (13) we get $T_1 \in Z$ so that $A'_{\beta\alpha} + T_1 A_\alpha = A_{\beta\alpha} + R A_\beta$ where by (10a), $0 \leqq A_{\beta\alpha}, A'_{\beta\alpha} < A_\alpha$. We may choose $\{b_\alpha\}$ and hence R so that $0 \leqq A'_{\beta\alpha} < (A_\beta, A_\alpha) = = (K_{\beta\alpha} A_\beta, A_\alpha)$. Hence we can assume that the coefficients $A'_{\beta\alpha}$ and $A_{\beta\alpha}$ of $\{b_\alpha\}$ and $\{a_\alpha\}$ respectively satisfy (14a). But then $A'_{\beta\alpha} = A_{\beta\alpha}$ and $T_1 = R A_\beta / A_\alpha = R_1 A_\beta / (A_\beta, A_\alpha)$ where $R_1 = R(A_\beta, A_\alpha)/A_\alpha \in Z$. Depending on the choice of R, R_1 may take any integer value. So $T_1 = R_1 K_{^1\beta(^1\alpha)}$ and by (13) $a_\alpha - b_\alpha \equiv T_1 a_\beta \bmod (a_\delta; \delta < \beta)$. Hence the process may be repeated. For example, suppose $h > 1$ and assume $A'_{^{-1}\beta(^{-1}\alpha)} = = A_{^{-1}\beta(^{-1}\alpha)}$ satisfies (14a), that ${}^{-1}\beta = ({}^{-1}\alpha)^+$ and that $a_{^{-1}\alpha} - b_{^{-1}\alpha} \equiv T_{h-1} a_{^{-1}\beta}$ $\bmod (a_\delta; \delta < {}^{-1}\beta)$ where $T_{h-1} = R_{h-1} K_{\beta\alpha}$ and R_{h-1} can be any integer depending on the choice of R; so that $R = 0$ if and only if $R_{h-1} = 0$. Since ${}^1[{}^{-1}\alpha - \alpha_u] = = [\alpha - \alpha_u]$ then $\beta = \alpha^+$ and as above there exists $T_h \in Z$ so that $A'_{\beta\alpha} + T_h A_\alpha = = A_{\beta\alpha} + T_{h-1} A_\beta$, and by (10a), $0 \leqq A'_{\beta\alpha}, A_{\beta\alpha} < A_\alpha$. We may choose $\{b_\alpha\}$ and thus R, and R_{h-1}, so that $0 \leqq A'_{\beta\alpha} < (K_{\beta\alpha} A_\beta, A_\alpha)$. Therefore assume $A'_{\beta\alpha}$ and $A_{\beta\alpha}$ satisfy (14a). Then $A'_{\beta\alpha} = A_{\beta\alpha}$ and by (13) $a_\alpha - b_\alpha \equiv T_h a_\beta \bmod (a_\delta; \delta < \beta)$ where $T_h = = R_{h-1} K_{\beta\alpha} A_\beta / A_\alpha = R_h K_{^1\beta(^1\alpha)}$ and $R_h = R_{h-1}(K_{\beta\alpha} A_\beta, A_\alpha)/A_\alpha$ takes any integer value, depending on R_{h-1} and hence on R. $R_h = 0$ if and only if $R = 0$.

Now suppose $\alpha = {}^h\alpha_u = \beta_{u-1}$ and $\beta = {}^h\gamma$. By (2) and the above, $D_\alpha a_{\alpha^{-1}} y - - E_\alpha a_{^{-1}\alpha} x = \sum_{\delta < \alpha} F_{\delta\alpha} a_\delta$ and $D_\alpha b_{\alpha^{-1}} y - E_\alpha b_{^{-1}\alpha} x = \sum_{\beta < \delta < \alpha} F_{\delta\alpha} b_\delta + \sum_{\delta \leqq \beta} F'_{\delta\alpha} b_\delta$. Since $\beta = \alpha^+ > \delta^+ \geqq \delta^*$ for all $\delta < \alpha$ and $\alpha^+ > ((\alpha^{-1})^+)^1 \geqq ((\alpha^{-1})^*)^1$ then $\beta = \overline{(a_{^{-1}\alpha} - b_{^{-1}\alpha}) x} > > \overline{(a_{\alpha^{-1}} - b_{\alpha^{-1}}) y}$ and $\beta > \overline{a_\delta - b_\delta}$. So using (12) and comparing terms of degree β gives $F'_{\beta\alpha} = F_{\beta\alpha} + T_{h-1} A_\beta E_\alpha$. By (2)

$$A_\alpha B'_{\beta\alpha} - B_\alpha A'_{\beta\alpha} = A_\alpha B_{\beta\alpha} - B_\alpha A_{\beta\alpha} + T_{h-1} A_\beta B_\alpha.$$

But we have assumed $A'_{\beta\alpha} = A_{\beta\alpha}$ so

$$B'_{\beta\alpha} - B_{\beta\alpha} = T_{h-1} A_\beta B_\alpha / A_\alpha = R_{h-1} K_{\beta\alpha} A_\beta B_\alpha / A_\alpha = R_h K_{^1\beta(^1\alpha)} B_\alpha.$$

We can choose $\{b_\alpha\}$ and hence R and R_h so that $0 \leqq B'_{\beta\alpha} < K_{^1\beta(^1\alpha)} B_\alpha$. So we can assume $B_{\beta\alpha}$ and $B'_{\beta\alpha}$ satisfy (15a). Then $B_{\beta\alpha} = B'_{\beta\alpha}$ so $R_h = R = 0$.

The underlying $\mathscr{P}$-basis of the extended $\mathscr{P}$-basis that satisfies (10a, b), (14a, b) and (15a, b) is the *canonical basis* of $\mathscr{P}$.

5. Invariants

A set of invariants for the primitive ideal $\mathscr{P}$ will now be determined. The set of invariants of $\mathscr{P}$ consists of a set of degrees and a set of non-negative integers satisfying certain restrictions. It will be seen that any set of degrees and integers satisfying the restrictions is the set of invariants of some primitive ideal in $Z[x, y]$.

Let Γ be the degree set of $\mathscr{P}\setminus 0$. Recall the definitions and properties of the following subsets. ${}_1\Gamma=\{{}^1\alpha;\alpha\in\Gamma\}$; $\Gamma_1=\{\alpha^1;\alpha\in\Gamma\}$; ${}_1\Gamma_1=\{{}^1\alpha^1;\alpha\in\Gamma\}$; $\Phi=\{\alpha_u=(m_u, i_u)\in\Gamma;\ 1\leqq u\leqq n$ for $u, n\in Z\}$ where $m_u<m_v$ and $i_v<i_u$ for $1\leqq v<u\leqq n$ and $i_1=m_n=0$; $\Psi=\{\beta_u=(m_u, i_{u+1});\ 1\leqq u<n\}$. For α_z the least degree in Γ,

$$\Omega_1=\{\alpha_u^h\in\Gamma_1\setminus(\Gamma_1\cap{}_1\Gamma);\ u<z\};$$

$${}_1\Omega=\{{}^h\alpha_u\in{}_1\Gamma\setminus(\Gamma_1\cap{}_1\Gamma);\ u>z\};\quad \Omega_2=\Gamma_1\setminus((\Gamma_1\cap{}_1\Gamma)\cup\Omega_1);$$

$${}_2\Omega={}_1\Gamma\setminus((\Gamma_1\cap{}_1\Gamma)\cup{}_1\Omega);\quad \Psi_1=\{\beta_u\in\Psi;\ u<z\}\ \text{ and }\ {}_1\Psi=\{\beta_u\in\Psi;\ u\geqq z\}.$$

Let $\{a_\alpha\}$ be the extended $\mathscr{P}$-basis with basis coefficients satisfying (10a, b), (14a, b) and (15a, b) and let $\{a_\alpha;\alpha\leqq\tau\}$ be the canonical basis. Let $\lambda=\max\{\tau,\beta_u;\beta_u\in\Psi\}$. The set of invariants of $\mathscr{P}$ will contain $\{\lambda\}\cup\Phi$.

We will now choose integer invariants for degrees $\alpha\in\Gamma\setminus\Phi$, $\alpha\leqq\lambda$.

Let Q_α be the highest common factor of the set $\{A_{\beta\gamma}, B_{\beta\gamma};\beta\leqq\gamma\leqq\alpha\}$ and let $Q_\alpha R_\alpha$ be the highest common factor of $\{A_{\beta\gamma}, B_{\beta\gamma};\beta\leqq\gamma<\alpha\}$. Let $A'_{\beta\gamma}Q_\alpha R_\alpha=A_{\beta\gamma}$, $B'_{\beta\gamma}Q_\alpha R_\alpha=B_{\beta\gamma}$ for $\gamma<\alpha$ and let $A'_{\beta\alpha}Q_\alpha=A_{\beta\alpha}$, $B'_{\beta\alpha}Q_\alpha=B_{\beta\alpha}$.

For $\alpha\in{}_1\Gamma_1$ we have by (3) and (5) that $A_{\alpha^{-1}}B_{\beta\alpha}-B_{{}^{-1}\alpha}A_{\beta\alpha}=G_{\beta\alpha}$. Let $G'_{\beta\alpha}(Q_\alpha R_\alpha)^2=G_{\beta\alpha}$ then $A'_{\alpha^{-1}}B'_{\beta\alpha}-B'_{{}^{-1}\alpha}A'_{\beta\alpha}=G'_{\beta\alpha}R_\alpha$. Let $V_{\beta\alpha}$ be the least positive integer so that $A'_{\alpha^{-1}}S-B'_{{}^{-1}\alpha}T=G'_{\beta\alpha}V_{\beta\alpha}$ for some $S, T\in Z$. By (2) and (4), $D_\alpha=A_{\alpha^{-1}}/(A_{\alpha^{-1}}, B_{{}^{-1}\alpha})$ and $E_\alpha=B_{{}^{-1}\alpha}/(A_{\alpha^{-1}}, B_{{}^{-1}\alpha})$. Hence assume $0\leqq T<D_\alpha$. Let V_α be the least common multiple of $\{V_{\beta\alpha};\beta<\alpha\}$, $J_\alpha=R_\alpha/V_\alpha$ and $J_{\beta\alpha}=R_\alpha/V_{\beta\alpha}$. Since $A'_{\alpha^{-1}}SJ_{\beta\alpha}-B'_{{}^{-1}\alpha}TJ_{\beta\alpha}=G'_{\beta\alpha}R_\alpha$ then there is a unique $H_{\beta\alpha}$, $0\leqq H_{\beta\alpha}<A'_\alpha/D_\alpha$ so that $(A'_{\beta\alpha}-TJ_{\beta\alpha})\equiv H_{\beta\alpha}D_\alpha\ (\mathrm{mod}\ A'_\alpha)$. Let $I_\alpha=A'_\alpha/D_\alpha=B'_\alpha/E_\alpha$. The set of invariants of $\mathscr{P}$ will contain $\{H_{\beta\alpha}, I_\alpha, J_\alpha;\beta<\alpha\}$. Note that these are uniquely determined from $\mathscr{P}$ and conversely, given $A_{\beta\gamma}$, $B_{\beta\gamma}$ for $\beta\leqq\gamma<\alpha$, and Q_α then with (10a) we may derive $A_{\beta\alpha}$, $B_{\beta\alpha}$ from these invariants. Further note that the highest common factor U of the invariants is 1; this follows since $I_\alpha=A'_\alpha/D_\alpha=B'_\alpha/E_\alpha$ and $(D_\alpha, E_\alpha)=1$ while for some $W\in Z$, $A'_{\beta\alpha}=TJ_{\beta\alpha}+H_{\beta\alpha}D_\alpha+WA'_\alpha$ and $B'_{\beta\alpha}=SJ_{\beta\alpha}+H_{\beta\alpha}E_\alpha+WB'_\alpha$ so U divides $\{A'_{\beta\alpha}, B'_{\beta\alpha};\beta\leqq\alpha\}$. But J_α divides R_α so U divides $(A_{\beta\gamma}/Q_\alpha, B_{\beta\gamma}/Q_\alpha;\beta\leqq\gamma\leqq\alpha\}$ and hence $U=1$.

For $\alpha\in{}_1\Gamma\setminus(\Gamma_1\cap{}_1\Gamma)$ choose for invariants $H_{\beta\alpha}=A'_{\beta\alpha}$ where $\beta<\alpha$, $I_\alpha=A'_\alpha$ and $J_\alpha=R_\alpha$. By convention let $R_{{}^1\alpha_z}=1$. For $\alpha\in\Gamma_1\setminus(\Gamma_1\cap{}_1\Gamma)$ choose $H_{\beta\alpha}=B'_{\beta\alpha}$ where $\beta<\alpha$, $I_\alpha=B'_\alpha$ and $J_\alpha=R_\alpha$. For $\alpha\in{}_1\Psi$ choose as invariants $D_\alpha, E_\alpha, H_{\beta\alpha}=A'_{\beta\alpha}$ and $L_{\beta\alpha}=B'_{\beta\alpha}$ where $\beta<\alpha$, $I_\alpha=A'_\alpha$ and $J_\alpha=R_\alpha$. For $\alpha\in\Psi_1$ choose as invariants $D_\alpha, E_\alpha, H_{\beta\alpha}=B'_{\beta\alpha}$ and $L_{\beta\alpha}=A'_{\beta\alpha}$ where $\beta<\alpha$, $I_\alpha=B'_\alpha$ and $J_\alpha=R_\alpha$. As in the previous paragraph, in each of these cases the invariants are relatively prime.

Let $Q=Q_\lambda$. Note from the definition of λ that $I_\lambda Q\neq 1$ if $\lambda\notin\Psi$. This follows since $I_\gamma Q=A_\lambda$ or B_λ if $\lambda\notin({}_1\Gamma_1\cup\Psi)$ and $D_\lambda I_\lambda Q=A_\lambda$, $E_\lambda I_\lambda Q=B_\lambda$ if $\lambda\in{}_1\Gamma_1$. We have by (1a, b) the coefficient relations $a_{{}^{-1}\lambda}({}^{-1}\lambda)=A_\lambda a_\lambda(\lambda)$ and $a_{\lambda^{-1}}(\lambda^{-1})=B_\lambda a_\lambda(\lambda)$ if A_λ and B_λ respectively exist. Since $(D_\lambda, E_\lambda)=1$ then in these cases a_λ is generated in $Z[x, y]$ by $\{a_\alpha;\alpha<\lambda\}$ if and only if $I_\lambda Q=1$.

Summarizing, the *$\mathscr{P}$-invariants* of $\mathscr{P}$ are elements of a set of degrees and of a set of non-negative integers. The degree invariants are:

(i) $\alpha_u=(m_u, i_u)$ where $u=1, 2, \dots, n$, $i_1=m_n=0$ and for $u>v$ then $i_u>i_v$, $m_u<m_v$;

(ii) λ where $\lambda \geqq \max\{(m_u, i_{u+1});\ 1\leqq u<n\}$.

Let Γ be the ideal of the semigroup $\Delta\setminus(-\infty, -\infty)$ generated by $\{\alpha_u;\ 1\leqq u\leqq n\}$. Let ${}_1\Gamma, \Gamma_1, \Psi, {}_1\Psi, \Psi_1, {}_1\Omega$ and Ω_1 be subsets as defined earlier in the section and let α_z be least in Γ. The integer invariants are:

(iii) Q where $Q>0$;

(iv) J_α for $\lambda\geqq\alpha\in\Gamma_1\cup{}_1\Gamma$ where $J_\alpha>0$ and $J_{{}^1\alpha_z}=1$;

(v) I_α for $\lambda\geqq\alpha\in\Gamma_1\cup{}_1\Gamma$ where $I_\alpha>0$ and $I_\lambda Q\neq 1$ if $\lambda\notin\Psi$;

(vi) $H_{\beta\alpha}$ for $\lambda\geqq\alpha\in\Gamma_1\cup{}_1\Gamma$ and $\alpha>\beta\in\Gamma$ where $0\leqq H_{\beta\alpha}<I_\alpha$ if $\alpha\notin\Psi\cup{}_1\Omega\cup\Omega_1$, $0\leqq H_{\beta\alpha}<(K_{\beta\alpha}A_\beta/Q_\alpha, I_\alpha)$ if $\alpha\in{}_1\Omega\cup{}_1\Psi$ and $0\leqq H_{\beta\alpha}<(K_{\beta\alpha}B_\beta/Q_\alpha, I_\alpha)$ if $\alpha\in\Omega_1\cup\Psi_1$. Also if $\alpha\notin\Psi$ then $\{I_\alpha, J_\alpha, H_{\beta\alpha};\ \beta<\alpha\}$ is a set of relatively prime numbers;

(vii) D_α and E_α for $\alpha\in\Psi$ where D_α and E_α are relatively prime positive integers and D_α or E_α divides I_α if $\alpha\in{}_1\Psi$ or Ψ_1 respectively;

(viii) $L_{\beta\alpha}$ for $\alpha\in\Psi$ and $\alpha>\beta\in\Gamma$ where $0\leqq L_{\beta\alpha}<K_{{}^1\beta\,({}^1\alpha)}I_\alpha E_\alpha/D_\alpha$ if $\alpha={}^h\alpha_u\in{}_1\Psi$, ${}^{-h}\beta\in\Gamma$ and $0\leqq L_{\beta\alpha}<K_{(\beta^1)\,\alpha^1}I_\alpha D_\alpha/E_\alpha$ if $\alpha=\alpha_u^h\in\Psi_1$, $\beta^{-h}\in\Gamma$. Also $\{I_\alpha, J_\alpha, H_{\beta\alpha}, L_{\beta\alpha};\ \beta<\alpha\}$ is a set of relatively prime numbers.

The restrictions of (vi) and (viii) are derived from (10a, b), (14a, b) and (15a, b).

It will now be shown that the basis coefficients of the $\mathscr{P}$-basis $\{a_\alpha;\ \alpha\leqq\lambda\}$, which contains the canonical basis, can be reconstructed from the $\mathscr{P}$-invariants. Approximate values for the basis coefficients will be calculated successively for degrees in $\Gamma_1\cup{}_1\Gamma$. At the step for degree α the approximations for degree less than α will be modified by a multiplier. At step λ the approximations become exactly the basis coefficient required.

Let $Q_{\beta\alpha}=(\prod_{\beta<\gamma\leqq\alpha} R_\gamma)$. Then $Q_\beta=Q_{\beta\lambda}Q$. The R_γ will be calculated.

Let $A''_\alpha=I_\alpha$ and $A''_{\beta\alpha}=H_{\beta\alpha}$ if $\alpha\in{}_1\Omega\cup{}_2\Omega\cup{}_1\Psi$. Let $B''_\alpha=I_\alpha$ and $B''_{\beta\alpha}=H_{\beta\alpha}$ if $\alpha\in\Omega_1\cup\Omega_2\cup\Psi_1$. For $\alpha\in{}_1\Psi$ let $B''_\alpha=I_\alpha E_\alpha/D_\alpha$ and $B''_{\beta\alpha}=L_{\beta\alpha}$. For $\alpha\in\Psi_1$ let $A''_\alpha=I_\alpha D_\alpha/E_\alpha$ and $A''_{\beta\alpha}=L_{\beta\alpha}$. Notice that from the derivation of the invariants $A_{\beta\alpha}=A''_{\beta\alpha}Q_{\alpha\lambda}Q$ and $B_{\beta\alpha}=B''_{\beta\alpha}Q_{\alpha\lambda}Q$.

Suppose α is least in ${}_1\Gamma_1$, then $R_\gamma=J_\gamma$ for $\gamma<\alpha$. In the terminology used in deriving the invariants, $A'_{\beta\gamma}=A_{\beta\gamma}/Q_\alpha R_\alpha=A''_{\beta\gamma}Q_{\gamma\alpha}/R_\alpha$ and $B'_{\beta\gamma}=B''_{\beta\gamma}Q_{\gamma\alpha}/R_\alpha$ for $\beta\leqq\gamma<\alpha$. So we can calculate, by (5),

$$G'_{\beta\alpha} = \sum_{\beta\leqq\gamma<\alpha} B'_{-1\gamma(-1\alpha)}A'_{\beta\gamma}-A'_{(\gamma-1)\alpha-1}B'_{\beta\gamma}.$$

Let $V_{\beta\alpha}$ be the least positive integer so that $A'_{\alpha^{-1}}S-B'_{-1_\alpha}T=G'_{\beta\alpha}V_{\beta\alpha}$ for some $S, T\in Z$ and let V_α be the least common multiple of $\{V_{\beta\alpha};\ \beta<\alpha\}$. We can compute $D_\alpha=A'_{\alpha^{-1}}/(A'_{\alpha^{-1}}, B'_{-1_\alpha})$, $E_\alpha=B'_{-1_\alpha}/(A'_{\alpha^{-1}}, B'_{-1_\alpha})$ and $J_{\beta\alpha}=J_\alpha V_\alpha/V_{\beta\alpha}$. Define $A''_\alpha=D_\alpha I_\alpha$, $B''_\alpha=E_\alpha I_\alpha$ and $A''_{\beta\alpha}\equiv(TJ_{\beta\alpha}+H_{\beta\alpha}D_\alpha)(\operatorname{mod} A''_\alpha)$ so that $0\leqq A''_{\beta\alpha}<A''_\alpha$ (see (10a)). Let $B''_{\beta\alpha}$ satisfy

$$A'_{\alpha^{-1}}B''_{\beta\alpha}-B'_{-1_\alpha}A''_{\beta\alpha} = G'_{\beta\alpha}J_\alpha V_\alpha.$$

Notice that from the derivation of the invariants $A_{\beta\alpha}=A''_{\beta\alpha}Q_{\alpha\lambda}Q$ and $B_{\beta\alpha}=B''_{\beta\alpha}Q_{\alpha\lambda}Q$. Also note that $R_\alpha=J_\alpha V_\alpha$.

Now assume that $A''_{\delta\gamma}, B''_{\delta\gamma}$ have been calculated for all degress $\gamma<\alpha$. We have $R_\gamma=J_\gamma V_\gamma$ for $\gamma\in{}_1\Gamma_1$ and $R_\gamma=J_\gamma$ otherwise. The process just described may be repeated to calculate $A''_{\beta\alpha}$ and $B''_{\beta\alpha}$ for $\beta\leqq\alpha$, and to determine V_α.

Thus the basis coefficients $A_{\beta\alpha}=A''_{\beta\alpha}Q_{\alpha\lambda}Q$ and $B_{\beta\alpha}=B''_{\beta\alpha}Q_{\alpha\lambda}Q$ of the $\mathscr{P}$-basis $\{a_\alpha;\alpha\leqq\lambda\}$ can be calculated from the invariants of $\mathscr{P}$.

Notice that the numbers $K_{\beta\alpha}$, A_β/Q_α and B_β/Q_α that appear in the definition of $\mathscr{P}$-invariants may be calculated from the invariants for degree less than α. Let $K_{\beta\alpha}=1$ if $\alpha^{-1}\in\Phi$, $\beta^{-1}\in\Gamma$ or ${}^{-1}\alpha\in\Phi$, ${}^{-1}\beta\in\Gamma$. Let $K_{\beta\alpha}=0$ if $\alpha^{-1}\in\Phi$, $\beta^{-1}\notin\Gamma$ or ${}^{-1}\alpha\in\Phi$, ${}^{-1}\beta\notin\Gamma$. Continue inductively. For some $\alpha\in{}_1\Omega\cup{}_1\Psi$ assume $K_{\delta\gamma}$, $A''_{\delta\gamma}$ and $B''_{\delta\gamma}$ have been calculated (where they exist) for $\delta\leqq\gamma<{}^1\alpha$. Then $A_\beta/Q_\alpha=A''_\beta Q_{\beta\alpha}$ and $B_\beta/Q_\alpha=B''_\beta Q_{\beta\alpha}$. Define $K_{{}^1\beta\,({}^1\alpha)}=K_{\beta\alpha}A''_\beta Q_{\beta\alpha}/(K_{\beta\alpha}A''_\beta Q_{\beta\alpha}, A''_\alpha)$. Similarly for $\alpha\in\Omega_1\cup\Psi_1$ define $K_{(\beta^1)\alpha^1}=K_{\beta\alpha}B''_\beta Q_{\beta\alpha}/(K_{\beta\alpha}B''_\beta Q_{\beta\alpha}, B''_\alpha)$.

THEOREM 5.1. *A primitive ideal $\mathscr{P}$ of $Z[x, y]$ uniquely determines a set of $\mathscr{P}$-invariants. Conversely a set of degrees and integers whose elements satisfy the restrictions* (i), ..., (viii) *is a set of $\mathscr{P}$-invariants for some primitive ideal $\mathscr{P}$ of $Z[x, y]$.*

PROOF. The first statement has been proved. Given the set of degrees and integers of the second statement we can determine numbers $A''_{\beta\alpha}, B''_{\beta\alpha}$ for $\lambda\geqq\alpha\in\Gamma_1\cup{}_1\Gamma$ and $\alpha\geqq\beta\in\Gamma$ as in the argument that followed the definition of $\mathscr{P}$-invariants. Likewise $V_\alpha, \alpha\in{}_1\Gamma_1$, can be determined. Let $A_{\beta\alpha}=A''_{\beta\alpha}Q_{\alpha\lambda}Q$ and $B_{\beta\alpha}=B''_{\beta\alpha}Q_{\alpha\lambda}Q$ where $Q_{\alpha\lambda}$ is defined as above for $R_\gamma=J_\gamma$ if $\gamma\in(\Gamma_1\cup{}_1\Gamma)\setminus{}_1\Gamma_1$ and $R_\gamma=J_\gamma V_\gamma$ if $\gamma\in{}_1\Gamma_1$. These numbers satisfy the restrictions (10a, b), (14a, b) and (15a, b). Let $\{d_\alpha;\lambda\geqq\alpha\in\Gamma\}$ be a set of unknowns so that

$$(1a)' \qquad d_{{}^{-1}\alpha}x=\sum_{\beta\leqq\alpha}A_{\beta\alpha}d_\beta \quad \text{for} \quad \alpha\in{}_1\Gamma$$

and

$$(1b)' \qquad d_{\alpha^{-1}}y=\sum_{\beta\leqq\alpha}B_{\beta\alpha}d_\beta \quad \text{for} \quad \alpha\in\Gamma_1.$$

From these equations, equations of form (2), (6a, b) and (7) may be obtained for the unknowns. By their construction from the invariants, the numbers $A_{\beta\alpha}, B_{\beta\alpha}$ satisfy equations (3) and (4) for $\alpha\in{}_1\Gamma_1$. From the equations of form (2) for $\alpha\in\Psi$

$$D_\alpha d_{\alpha^{-1}}y-E_\alpha d_{{}^{-1}\alpha}x-\sum_{\beta<\alpha}F_{\beta\alpha}d_\beta=0$$

we obtain, using equations of form (7), the equations

$$(16) \qquad \sum_{k=1}^{n}f_{j,k}d_{\alpha_k}=0 \quad \text{for} \quad j=1,2,\dots,n-1,\; f_{j,k}\in Q[x,y].$$

As in the proof of Theorem 3.2 equations (16) are satisfied by a set $\{a'_{\alpha_k};k=1,2,\dots,n\}\subseteq Z[x,y]$. Any solution of (16) in $Q[x,y]$ is of the form $\{fa'_{\alpha_k};k=1,2,\dots,n\}$ for some $f\in Q[x,y]$. The equations (1a, b)′ can be solved successively for increasing α to obtain solution set $\{a'_\alpha;\lambda\geqq\alpha\in\Gamma\}$. Multiplying through by a suitable factor $f\in Q[x,y]$ we get uniquely a relatively prime solution set $\{a_\alpha;\lambda\geqq\alpha\in\Gamma\}$ in $Z[x,y]$ for equations (1a, b)′. These polynomials clearly form a $\mathscr{P}$-basis for some primitive ideal $\mathscr{P}$ in $Z[x,y]$ with basis coefficients $A_{\beta\alpha}, B_{\beta\alpha}$ satisfying (10a, b), (14a, b) and (15a, b). Conversely, from the construction, it is clear that the set of degrees and integers with which the proof began are the $\mathscr{P}$-invariants of $\mathscr{P}$.

In conclusion a comment should be made on the complexity of the invariants. The only complication in determining a set of degrees and integers to satisfy (i), ..., (viii) is in the calculation of the numbers $K_{\beta\alpha}$, A_β/Q_α and B_β/Q_α that appear in (vi) and (viii). The method of calculation is described in the comments preceding Theorem 5.1. Although the method involves very elementary manipulations of integers, the number of manipulations involved could be large (depending on the distribution of Φ and Ψ in Γ). Since we have a routine for successively choosing integers to satisfy (iii), ..., (viii) (and hence to construct an ideal with these as invariants), the method described here is less complex than that described in [6] for homogeneous ideals in $K[x, y, z]$. In [6] the invariants include "basis coefficients" that must satisfy systems of algebraic, but not necessarily linear, equations.

References

[1] L. Kronecker and K. Hensel, *Vorlesungen über Zahlentheorie* (Leipzig, 1901).

[2] L. Rédei, Äquivalenz der Sätze von Kronecker—Hensel und von Szekeres, *Acta Sci. Math. Szeged*, **17** (1956), 198—202.

[3] L. Rédei, *Algebra*, Vol. I Pergamon Press (1967).

[4] G. Szekeres, A canonical basis for the ideals of a polynomial domain, *Amer. Math. Monthly*, **59** (1952), 379—386.

[5] G. Szekeres, Metabelian groups with two generators, *Proc. Internat. Conf. Theory of Groups*. (Canberra, 1965), pp. 323—346 (Gordon and Breach, 1967).

[6] G. Szekeres, Homogeneous ideals in $K[x, y, z]$, *Acta Math. Acad. Sci. Hungar.*, **26** (1975), 355—367.

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DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF TASMANIA
BOX 252C, G.P.O.
HOBART, TASMANIA
AUSTRALIA 7001

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COSINE FAMILIES AND ABSTRACT NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

By

C. C. TRAVIS and G. F. WEBB (Nashville)

I. Introduction

Our primary objective is to investigate the abstract semi-linear second order initial value problem

$$d^2/dt^2 w(t) = Aw(t) + f\big(t, w(t), d/dtw(t)\big) \tag{1.1}$$

$$w(t_0) = x \in \mathbf{X}, \quad d/dtw\{t_0) = y \in \mathbf{X}.$$

In (1.1) $\mathbf{X}$ is a Banach space, w is a mapping from $\mathbf{R}$ to $\mathbf{X}$, A is the (possibly unbounded) infinitesimal generator of a strongly continuous cosine family of linear operators in $\mathbf{X}$, and f is a nonlinear mapping from $\mathbf{R}\times\mathbf{X}\times\mathbf{X}$ to $\mathbf{X}$. Our goal will be to give a systematic and general treatment of (1.1) from the standpoint of existence, uniqueness, continuous dependence, and smoothness of solutions. The pioneering work on (1.1) was done by I. SEGAL in [23] and our development follows his approach. Our results extend those of [23] in several respects. First, we allow for a more general linear term A, in that we assume A is the infinitesimal generator of an arbitrary strongly continuous cosine family. Second, we analyze various hypotheses on the nonlinear term f, some of which are more general than found in [23]. Third, we distinguish between cases in which (1.1) can be converted to an equivalent first order system of abstract equations and cases in which it is more advantageous to study the second order equation directly.

As a second objective we will unify and simplify some ideas from the theory of strongly continuous cosine families of linear operators in Banach spaces. The most fundamental and extensive work on cosine families is that of H. FATTORINI in [5, 6]. Important additions to the theory have also been made by M. SOVA [24, 25], G. DAPRATO and E. GIUSTI [3], J. GOLDSTEIN [9, 11], and B. NAGY [20, 21]. We will state some of the main results from these papers and in certain cases give simplified proofs. We will also add some new results concerning inhomogeneous equations, and the equivalence of second order equations and first order systems. The results we present for linear cosine families will be essential to our later treatment of the nonlinear equation (1.1).

II. Strongly continuous cosine families of linear operators

A one parameter family $C(t)$, $t\in\mathbf{R}$ of bounded linear operators mapping the Banach space $\mathbf{X}$ into itself is called a strongly continuous cosine family if and only if

(2.1) $C(s+t)+C(s-t)=2C(s)C(t)$ for all $s, t\in\mathbf{R}$;

(2.2) $C(0)=I$;

(2.3) $C(t)x$ is continuous in t on $\mathbf{R}$ for each fixed $x\in\mathbf{X}$.

If $C(t)$, $t\in\mathbf{R}$ is a strongly continuous cosine family in $\mathbf{X}$, then $S(t)$, $t\in\mathbf{R}$ is the one parameter family of operators in $\mathbf{X}$ defined by

$$S(t)x=\int_0^t C(s)x\,ds,\quad x\in\mathbf{X},\ t\in\mathbf{R}. \tag{2.4}$$

PROPOSITION 2.1. *Let $C(t)$, $t\in\mathbf{R}$ be a strongly continuous cosine family in $\mathbf{X}$. The following are true:*

(2.5) $C(t)=C(-t)$ *for all* $t\in\mathbf{R}$;

(2.6) $C(s)$, $S(s)$, $C(t)$, *and* $S(t)$ *commute for all* $s, t\in\mathbf{R}$;

(2.7) $S(t)x$ *is continuous in t on* $\mathbf{R}$ *for each fixed* $x\in\mathbf{X}$;

(2.8) $S(s+t)+S(s-t)=2S(s)C(t)$ *for all* $s, t\in\mathbf{R}$;

(2.9) $S(s+t)=S(s)C(t)+S(t)C(s)$ *for all* $s, t\in\mathbf{R}$;

(2.10) $S(t)=-S(-t)$ *for all* $t\in\mathbf{R}$;

(2.11) *there exist constants* $K\geqq 1$ *and* $\omega\geqq 0$ *such that* $|C(t)|\leqq Ke^{\omega|t|}$ *for all* $t\in\mathbf{R}$;

$$|S(t)-S(\hat{t})|\leqq K\left|\int_{\hat{t}}^{t} e^{\omega|s|}\,ds\right| \quad \text{for all} \quad t, \hat{t}\in\mathbf{R}. \tag{2.12}$$

PROOF. (2.5) follows from (2.1); (2.6) follows from (2.1) and (2.4); (2.7) follows from (2.4); (2.8) follows by integrating (2.1) with respect to s; (2.9) follows from (2.8); (2.10) follows from (2.8); (2.11) is proved in [5], p. 90, lemma 5.5; and (2.12) follows from (2.11).

The infinitesimal generator of a strongly continuous cosine family $C(t)$, $t\in\mathbf{R}$ is the operator $A\colon \mathbf{X}\to\mathbf{X}$ defined by

$$Ax=d^2/dt^2\,C(0)x \tag{2.13}$$

$D(A)=\{x\in\mathbf{X}: C(t)x$ is a twice continuously differentiable function of $t\}$.

We shall also make use of the set

$E=\{x: C(t)x$ is a once continuously differentiable function of $t\}$.

Proposition 2.2. *Let $C(t)$, $t\in\mathbf{R}$, be a strongly continuous cosine family in $\mathbf{X}$ with infinitesimal generator A. The following are true:*

(2.14) *$D(A)$ is dense in $\mathbf{X}$ and A is a closed operator in $\mathbf{X}$;*

(2.15) *if $x\in\mathbf{X}$ and $r,s\in\mathbf{R}$, then $z\overset{\text{def}}{=}\int_r^s S(u)x\,du\in D(A)$ and $Az=C(s)x-C(r)x$;*

(2.16) *if $x\in\mathbf{X}$ and $r,s\in\mathbf{R}$, then $z\overset{\text{def}}{=}\int_0^s\int_0^r C(u)C(v)x\,du\,dv\in D(A)$ and $Az=$ $=2^{-1}\big(C(s+r)x-C(s-r)x\big)$;*

(2.17) *if $x\in\mathbf{X}$, then $S(t)x\in E$;*

(2.18) *if $x\in E$, then $S(t)x\in D(A)$ and $d/dtC(t)x=AS(t)x$;*

(2.19) *if $x\in D(A)$, then $C(t)x\in D(A)$ and $d^2/dt^2C(t)x=AC(t)x=C(t)Ax$;*

(2.20) *if $x\in E$, then $\lim_{t\to 0}AS(t)x=0$;*

(2.21) *if $x\in E$, then $S(t)x\in D(A)$ and $d^2/dt^2S(t)x=AS(t)x$;*

(2.22) *if $x\in D(A)$, then $S(t)x\in D(A)$ and $AS(t)x=S(t)Ax$;*

(2.23) *$C(t+s)-C(t-s)=2AS(t)S(s)$ for all $s,t\in\mathbf{R}$.*

Proof. The proofs of all but (2.15)—(2.18) and (2.20) can be found in [5]. To prove (2.15), notice that using (2.8) we have

$$C(t)z=\int_r^s C(t)S(u)x\,du=2^{-1}\int_r^s\big(S(u+t)x+S(u-t)x\big)\,du.$$

Thus

$$d/dtC(t)z=2^{-1}\int_r^s\big(C(u+t)x-C(u-t)x\big)\,du=2^{-1}\Big(\int_{r+t}^{s+t}C(u)x\,du-\int_{r-t}^{s-t}C(u)x\,du\Big),$$

and

$$d^2/dt^2C(t)z=2^{-1}\big(C(s+t)x-C(r+t)x+C(s-t)x-C(r-t)x\big),$$

$$d^2/dt^2C(0)z=C(s)x-C(r)x.$$

(2.16) follows from (2.1), (2.5), and

$$C(t)z=\int_0^r\int_0^s C(t)C(u)C(v)x\,du\,dv=\int_0^r\int_0^s 2^{-1}\big(C(t+u)+C(t-u)\big)C(v)x\,du\,dv=$$

$$=2^{-1}\int_0^r\Big(\int_t^{t+s}C(u)C(v)x\,du+\int_{t-s}^{t}C(u)C(v)x\,du\,dv\Big)=2^{-1}\int_0^r\int_{t-s}^{t+s}C(u)C(v)x\,du\,dv,$$

$$d/dtC(t)z=2^{-1}\int_0^r\big(C(t+s)-C(t-s)\big)C(v)x\,dv=$$

$$=2^{-1}\int_0^r 2^{-1}\big(C(t+s+v)+C(t+s-v)-C(t-s+v)-C(t-s-v)\big)x\,dv=$$

$$= 4^{-1}\Big(\int_{t+s}^{t+s+r} C(v)x\,dv + \int_{t+s-r}^{t+s} C(v)x\,dv - \int_{t-s}^{t-s+r} C(v)x\,dv - \int_{t-s-r}^{t-s} C(v)x\,dv\Big) =$$

$$= 4^{-1}\Big(\int_{t+s-r}^{t+s+r} C(v)x\,dv - \int_{t-s-r}^{t-s+r} C(v)x\,dv\Big),$$

$$d^2/dt^2\, C(t)z = 4^{-1}\big(C(t+s+r)x - C(t+s-r)x - C(t-s+r)x + C(t-s-r)x\big),$$

$$d^2/dt^2\, C(0)\, z = 2^{-1}\big(C(s+r)x - C(s-r)x\big).$$

(2.17) follows from (2.1), (2.4), and

$$C(r)\,S(t)x = \int_0^t C(r)\,C(s)x\,ds = 2^{-1}\int_{r-t}^{r+t} C(s)x\,ds,$$

$$d/dr\,C(r)\,S(t)x = 2^{-1}\big(C(r+t)x - C(r-t)x\big).$$

To obtain (2.18) we notice from the previous line that

$$d^2/dr^2\, C(r)\,S(t)x = 2^{-1}\big(d/dr\,C(r+t)x - d/dr\,C(r-t)x\big),$$

$$d^2/dr^2\, C(0)\,S(t)x = d/dt\,C(t)x.$$

To prove (2.20) notice that $x\in E$ implies that $d/dt\,C(t)x$ is a continuous function of t and thus from (2.18),

$$\lim_{t\to 0} AS(t)x = \lim_{t\to 0} d/dt\,C(t)x = d/dt\,C(0)x.$$

Now, let

$$\hat{b} \overset{\text{def}}{=} d/dt\,C(0)x = \lim_{h\to 0}\frac{C(h)x - x}{h}.$$

But, we also have

$$d/dt\,C(0)x = \lim_{t\to 0}\frac{C(-h)x - x}{-h} = \lim_{h\to 0} -\frac{C(h)x - x}{h} = -\hat{b}.$$

Thus $\hat{b}=0$ and the result follows.

It is natural to try to replace definition (2.13) of the infinitesimal generator of a strongly continuous cosine family by the central difference approximation

$$\lim_{t\to 0}\frac{C(-2t)x - 2C(0)x + C(2t)x}{4t^2},$$

of the second derivative of $C(t)$, $t\in\mathbf{X}$, at $t=0$. For arbitrary functions, it is well-known that the central difference definition of the second derivative is not equivalent to the classical definition. However, as the following proposition demonstrates, these two definitions are equivalent for functions satisfying the cosine property (2.1) (see [24], Proposition 2.18).

PROPOSITION 2.3. *Let $C(t)$, $t\in\mathbf{R}$, be a strongly continuous cosine family in $\mathbf{X}$. The operator $\hat{A}$: $\mathbf{X}\to\mathbf{X}$ defined by*

$$\hat{A}x = \lim_{t\to 0}\big(C(2t)x - x\big)/2t^2,$$

with domain those $x \in \mathbf{X}$ for which this limit exists, is the infinitesimal generator of the cosine family $C(t)$, $t \in \mathbf{R}$.

PROOF. Suppose $\lim_{t\to 0}(C(2t)x-x)/2t^2=y$. We would like to show that $x \in D(A)$ and $\lim_{t\to 0}(C(2t)x-x)/2t^2=Ax$. Define

$$x_t = t^{-2}\int_0^t\int_0^t C(s)\,C(r)x\,ds\,dr.$$

Then by (1.16), $x_t \in D(A)$ and

$$Ax_t = \big(C(2t)x-x\big)/2t^2.$$

The result now follows from the fact that A is closed and

$$\lim_{t\to 0} t^{-2}\int_0^t\int_0^t C(s)\,C(r)x\,ds\,dr = x$$

for each $x \in \mathbf{X}$. Thus we have shown that $A=\hat{A}$ on the domain of $\hat{A}$. Now suppose that $x \in D(A)$ and $Ax=y$. Since for $x \in \mathbf{X}$, $C(t)x \to x$ as $t \to 0$, given $\varepsilon>0$ there is a $\delta>0$ such that $0<u<\delta$ implies $\|C(u)Ax-Ax\|<\varepsilon$. Thus if $0<|2t|<\delta$, then

$$\|2t^{-2}(C(2t)x-x)-y\| = \Big\|2t^{-2}\int_0^{2t}\int_0^s \big(C''(u)x-Ax\big)\,du\,ds\Big\| \leqq$$

$$\leqq 2t^{-2}\int_0^{2t}\int_0^s \|C(u)Ax-Ax\|\,du\,ds < \varepsilon,$$

and the proof is complete.

The motivation for the proof of the following proposition concerning the linear inhomogeneous equation corresponding to (1.1) comes from semigroup theory [13], p. 486:

PROPOSITION 2.4. *Let $C(t)$, $t \in \mathbf{R}$ be a strongly continuous cosine family in $\mathbf{X}$ with infinitesimal generator A. If $g\colon \mathbf{R}\to\mathbf{X}$ is continuously differentiable, $x \in D(A)$, $y \in E$, and*

$$w(t) \overset{\text{def}}{=} C(t)x+S(t)y+\int_0^t S(t-s)\,g(s)\,ds, \quad t \in \mathbf{R},$$

then $w(t) \in D(A)$ for $t \in \mathbf{R}$, w is twice continuously differentiable, and w satisfies

$$\text{(2.24)} \qquad d^2/dt^2 w(t) = Aw(t)+g(t), \quad t \in \mathbf{R}, \quad w(0)=x, \quad d/dt\,w(0)=y.$$

Conversely, if $g\colon \mathbf{R}\to\mathbf{X}$ is continuous, $w(t)\colon \mathbf{R}\to\mathbf{X}$ is twice continuously differentiable, $w(t) \in D(A)$ for $t \in \mathbf{R}$, and w satisfies (2.24), then

$$w(t) = C(t)x+S(t)y+\int_0^t S(t-s)\,g(s)\,ds, \quad t \in \mathbf{R}.$$

PROOF. By virtue of (2.19) and (2.21), it suffices to show (2.24) for $x=y=0$. First,

$$\int_0^t S(t-s)g(s)\,ds = \int_0^t S(t-s)\Big(g(0)+\int_0^s g'(u)\,du\Big)ds =$$

$$= \int_0^t S(t-s)\,g(0)\,ds + \int_0^t \int_u^t S(t-s)\,g'(u)\,ds\,du =$$

$$= \int_0^t S(t-s)\,g(0)\,ds + \int_0^t \int_0^{t-u} S(s)\,g'(u)\,ds\,du.$$

By (2.14) and (2.15)

$$A\int_0^t S(t-s)g(s)\,ds = C(t)g(0)-g(0)+\int_0^t \big(C(t-u)-I\big)g'(u)\,du.$$

Then, (2.24) follows from

$$d/dt\,w(t) = \int_0^t C(t-s)\,g(s)\,ds + S(0)\,g(t) = \int_0^t C(s)\,g(t-s)\,ds,$$

$$d^2/dt^2\,w(t) = \int_0^t C(s)\,g'(t-s)\,ds + C(t)\,g(0) = Aw(t)+g(t).$$

To prove the converse statement, observe that

$$d/ds\,S(t-s)\,w'(s) = -C(t-s)\,w'(s)+S(t-s)\,w''(s),$$

$$d/ds\,C(t-s)\,w(s) = -S(t-s)\,Aw(s)+C(t-s)\,w'(s),$$

implies

$$-S(t)\,w'(0) = -\int_0^t C(t-s)w'(s)\,ds + \int_0^t S(t-s)\,w''(s)\,ds,$$

$$w(t)-C(t)\,w(0) = -\int_0^t S(t-s)Aw(s)\,ds + \int_0^t C(t-s)\,w'(s)\,ds.$$

The conclusion follows by adding the two formulas.

The proof of the next proposition, of which we will have later use, can be found in [5].

PROPOSITION 2.5. *Let $C(t)$, $t\in\mathbf{R}$ be a strongly continuous cosine family in $\mathbf{X}$ satisfying $|C(t)|\le Ke^{\omega|t|}$, $t\in\mathbf{R}$, and let A be the infinitesimal generator of $C(t)$, $t\in\mathbf{R}$. Then for $\operatorname{Re}\lambda>\omega$, λ^2 is in the resolvent set of A and*

$$\lambda R(\lambda^2;A)x = \int_0^\infty e^{-\lambda t}C(t)x\,dt \quad \textit{for} \quad x\in\mathbf{X}, \tag{2.25}$$

$$R(\lambda^2;A)x = \int_0^\infty e^{-\lambda t}S(t)x\,dt \quad \textit{for} \quad x\in\mathbf{X}.$$

In [5], p. 95 it is established that if A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t\in\mathbf{R}$, in $\mathbf{X}$ satisfying $\|C(t)\|\leqq Me^{\omega|t|}$, then a translate of A has a square root; that is, if $k\geqq\omega$, there exists an operator B_k in $\mathbf{X}$ such that $B_k^2=A-k^2I$. It is also shown that the domain of B_k is independent of the value of k, and if $k>\omega$ then zero is in the resolvent set of B_k. The following condition, which we call condition (F), is of fundamental importance in the study of strongly continuous cosine families:

Condition (F): If $B^2=A$, where A is the infinitesimal generator of $C(t)$, $t\in\mathbf{R}$, then $S(t)$ maps $\mathbf{X}$ into $D(B)$ for $t\in\mathbf{R}$, $BS(t)$ is bounded in $\mathbf{X}$ for $t\in\mathbf{R}$, and $BS(t)x$ is continuous in t on $\mathbf{R}$ for each fixed $x\in\mathbf{X}$.

In [5] it is also established that A is the infinitesimal generator of a strongly continuous cosine family if and only if $\hat{A}=A-k^2I$, $k\geqq\omega$, is the infinitesimal generator of a strongly continuous cosine family. Thus when dealing with the infinitesimal generator A of a strongly continuous cosine family, we can, without loss of generality, assume that there exists an operator B in $\mathbf{X}$ such that $B^2=A$ and that zero is in the resolvent set of B. In [22], Theorem 3, an example is given of a strongly continuous cosine family that does not satisfy condition (F), even after a suitable translation of the generator.

Some of the ideas in the proposition below originate in [5] and [10].

PROPOSITION 2.6. *Let A and B be linear operators from the Banach space $\mathbf{X}$ to itself, let B commute with every bounded linear operator in $\mathbf{X}$ which commutes with A, let zero be contained in the resolvent set of B, and let $B^2=A$. The following are equivalent:*

(2.26) *A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t\in\mathbf{R}$ in $\mathbf{X}$ satisfying condition* (F);

(2.27) *B is the infinitesimal generator of a strongly continuous group $T(t)$, $t\in\mathbf{R}$ in $\mathbf{X}$;*

(2.28) $\mathscr{B}\overset{\text{def}}{=}\begin{bmatrix}0 & B\\ B & 0\end{bmatrix}$, *with domain $D(B)\times D(B)$, is the infinitesimal generator of a strongly continuous group $U(t)$, $t\in\mathbf{R}$ in $\mathbf{X}\times\mathbf{X}$;*

(2.29) $\mathscr{A}\overset{\text{def}}{=}\begin{bmatrix}0 & I\\ A & 0\end{bmatrix}$ *with domain $D(A)\times D(B)$, is the infinitesimal generator of a strongly continuous group $V(t)$, $t\in\mathbf{R}$ in $Y\overset{\text{def}}{=}[D(B)]\times\mathbf{X}$, where $[D(B)]$ denotes the Banach space $D(B)$ with graph norm $\|x\|_B=\|x\|+\|Bx\|$.*

PROOF. That (2.26) implies (2.27) is proved in [5], p. 97, Theorem 6.6., but we give a more straightforward proof here. Define $T(t)=C(t)+BS(t)$, $t\in\mathbf{R}$. By virtue of the properties of $C(t)$, $S(t)$, and condition (F), it is easily shown that $T(t)$, $t\in\mathbf{R}$ is a strongly continuous group of bounded operators in X. It remains to verify that B is the infinitesimal generator of $T(t)$, $t\in\mathbf{R}$.

Let $x\in D(B)$ and observe

$$t^{-1}\big(T(t)x-x\big)=t^{-1}\big(C(t)x-x\big)+t^{-1}BS(t)x= \tag{2.30}$$

$$=t^{-1}A\int_0^t S(s)x\,ds+t^{-1}S(t)Bx=t^{-1}\int_0^t BS(s)Bx\,ds+t^{-1}\int_0^t C(s)Bx\,ds,$$

where we have used (2.14), (2.19), (2.22) and condition (F). The right-hand side of (2.30) has a limit of Bx as $t\to 0$, and so the infinitesimal generator of $T(t)$, $t\in\mathbf{R}$ and B agree on $D(B)$. Now let x be in the domain of the infinitesimal generator of $T(t)$, $t\in\mathbf{R}$ and observe that

$$t^{-1}\big(T(t)x-x\big)=B\Big(t^{-1}\int_0^t BS(s)x\,ds+t^{-1}\int_0^t C(s)x\,ds\Big), \tag{2.31}$$

where we have used the closedness of B (which follows from the fact that B has a nonempty resolvent set). The expression inside the parentheses of the right-hand side of (2.31) has a limit of x as $t\to 0$, which means $x\in D(B)$, since B is closed.

We shall now demonstrate that (2.27) implies (2.28). For each $t\in\mathbf{R}$, define $C(t)=\big(T(t)+T(-t)\big)/2$, $S(t)=B^{-1}\big(T(t)-T(-t)\big)/2$, and

$$U(t)=\begin{bmatrix} C(t), & BS(t)\\ BS(t), & C(t)\end{bmatrix}. \tag{2.32}$$

Obviously, $U(t)$, $t\in\mathbf{R}$ is a strongly continuous family of bounded operators in $\mathbf{X}\times\mathbf{X}$. A simple calculation shows the group property $U(t)\,U(s)=U(t+s)$. It remains to show that $\mathscr{B}$ generates $U(t), t\in\mathbf{R}$. For $x\in D(B)$, observe that $d/dt\,C(t)x=BS(t)Bx$ and $d/dt\,S(t)x=C(t)x$, and hence for $[x,y]\in D(B)\times D(B)$

$$d/dt\,U(t)[x,y]=[BS(t)Bx+C(t)By,\ C(t)Bx+BS(t)By]. \tag{2.33}$$

Clearly, the right-hand side of (2.33) equals $[By, Bx]$ when t equals zero, which means $\mathscr{B}$ agrees with the infinitesimal generator of $U(t), t\in\mathbf{R}$ on $D(B)\times D(B)$. Now suppose that for $[x,y]$ contained in the domain of the infinitesimal generator of $U(t)$, we have

$$[u,v]=\lim_{t\to 0}t^{-1}[U(t)-I][x,y]. \tag{2.34}$$

Then

$$t^{-1}\big(C(t)x-x\big)+t^{-1}BS(t)y\to u, \tag{2.35}$$

$$t^{-1}\big(BS(t)x\big)+t^{-1}\big(C(t)y-y\big)\to v. \tag{2.36}$$

Replace t by $-t$ in (2.35) and (2.36) and add the result to the equation from which it was obtained, yielding

$$t^{-1}\big(BS(t)y\big)\to u, \tag{2.37}$$

$$t^{-1}\big(BS(t)x\big)\to v. \tag{2.38}$$

It now follows from the closedness of B and the fact that $\lim_{t\to 0}t^{-1}S(t)x=x$ for all $x\in\mathbf{X}$, that $x,y\in D(B)$ and $Bx=v$, $By=u$.

To demonstrate that (2.28) implies (2.29), we define for $t\in\mathbf{R}$, $V(t):[D(B)]\times\times\mathbf{X}\to[D(B)]\times\mathbf{X}$ by

$$V(t)=\begin{bmatrix} B^{-1} & 0\\ 0 & I\end{bmatrix},\quad U(t)=\begin{bmatrix} B & 0\\ 0 & I\end{bmatrix}. \tag{2.39}$$

$V(t)$ is bounded in $[D(B)]\times\mathbf{X}$, since

$$\begin{bmatrix} B^{-1} & 0\\ 0 & I\end{bmatrix}:\ \mathbf{X}\times\mathbf{X}\to[D(B)]\times\mathbf{X},\quad \begin{bmatrix} B & 0\\ 0 & I\end{bmatrix}:\ [D(B)]\times\mathbf{X}\to\mathbf{X}\times\mathbf{X}$$

are both continuous. Obviously, $V(t)$, $t\in\mathbf{R}$ satisfies the group property and is strongly continuous. Moreover, the infinitesimal generator of $V(t)$, $t\in\mathbf{R}$ is

$$\begin{bmatrix} B^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix},$$

with domain $D(A)\times D(B)$.

To demonstrate that (2.29) implies (2.26), define $S(t)y=\pi_1 V(t)[0,y]$ and $C(t)y=\pi_2 V(t)[0,y]$ for $y\in\mathbf{X}$, $t\in\mathbf{R}$, where $\pi_1[x,y]\overset{\text{def}}{=}x$ and $\pi_2[x,y]\overset{\text{def}}{=}y$. Since $V(t)$, $t\in\mathbf{R}$ is a strongly continuous group in $[D(B)]\times\mathbf{X}$, $S(t)$ and $C(t)$ are bounded and strongly continuous in $\mathbf{X}$, $S(t)\mathbf{X}\subset D(B)$, and $BS(t)$ is bounded and strongly continuous in $\mathbf{X}$. If $x\in D(B)$, then $\mathscr{A}V(t)[0,x]=V(t)\mathscr{A}[0,x]$. But $\mathscr{A}V(t)[0,x]=$ $=\mathscr{A}[S(t)x,\ C(t)x]=[C(t)x,\ AS(t)x]$ and $V(t)\mathscr{A}[0,x]=V(t)[x,0]$. Thus $V(t)[x,0]=[C(t)x,\ AS(t)x]$ for $x\in D(B)$. Since for $y\in\mathbf{X}$, $V(t)[0,y]=[S(t)y,\ C(t)y]$, we have that

$$V(t)=\begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}. \tag{2.40}$$

It follows from $V(0)=I$, that $C(0)=I$ and $S(0)=0$. Next observe that if $x\in D(A)$ and $t\in\mathbf{R}$, then

$$Y-d/dt\,V(t)[x,0]=\mathscr{A}[C(t)x,\ AS(t)x]=[AS(t)x,\ AC(t)x], \tag{2.41}$$

where $Y-d/dt$ denotes differentiation with respect to the norm of Y. Equation (2.41) implies that

$$[D(B)]-d/dt\,C(t)x=AS(t)x \quad \text{for} \quad x\in D(A),\quad t\in\mathbf{R}, \tag{2.42}$$

$$X-d/dt\,AS(t)x=AC(t)x \quad \text{for} \quad x\in D(A),\quad t\in\mathbf{R}. \tag{2.43}$$

At this point we would like to demonstrate that

$$C(t)x=C(-t)x \quad \text{and} \quad AS(t)x=-AS(-t)x,\quad t\in\mathbf{R},\quad x\in D(A). \tag{2.44}$$

For $x\in D(A)$, let $v(t)=[C(t)x,\ AS(t)x]$ and $w(t)=[C(-t)x,\ -AS(-t)x]$. Using (2.42) and (2.43), it is easily shown that both v and w satisfy the equation $Y-d/dt\,v(t)=\mathscr{A}v(t)$. Since $v(0)=w(0)$, we can conclude from the uniqueness of solutions to this problem that $C(t)=C(-t)$ and $AS(t)=-AS(-t)$ in $D(A)$. The group property $V(t+s)=V(t)V(s)$, (2.40), and (2.44) now yield that (2.1) holds on $D(A)$. Since $D(A)$ is dense, (2.1) holds in $\mathbf{X}$. Finally, we prove that A is the infinitesimal generator of $C(t)$, $t\in\mathbf{R}$. If $x\in\mathbf{X}$ and $\operatorname{Re}\lambda$ is sufficiently large, then a simple calculation shows that $\pi_2 R(\lambda;\mathscr{A})[0,x]=\lambda R(\lambda^2;A)x$. Using (2.25) and [4, p. 623, lemma 12] we have that the infinitesimal generator $\hat{A}$ of $C(t)$, $t\in\mathbf{R}$ satisfies for $\operatorname{Re}\lambda$ sufficiently large

$$\lambda R(\lambda^2;\hat{A})x=\int_0^\infty e^{-\lambda t}C(t)x\,dt=\int_0^\infty e^{-\lambda t}\pi_2 V(t)[0,x]\,dt=\pi_2 R(\lambda,\mathscr{A})[0,x].$$

But then $\lambda R(\lambda^2;A)=\lambda R(\lambda^2;\hat{A})$ and so $A=\hat{A}$, completing the proof.

Proposition 2.7. *Assume the same hypothesis on A and B is in the previous proposition. A necessary and sufficient condition that A be the infinitesimal generator* *of a strongly continuous cosine family* $C(t)$, $t\in\mathbf{R}$ *in* $\mathbf{X}$ *satisfying condition* (F) *is that there exist constants* $M>0$ *and* $\omega\geqq 0$ *such that*

(2.45) $D(B)$ *is dense in* X;

(2.46) *for real* λ, $\lambda>\omega$, λ^2 *is in the resolvent set* $\varrho(A)$ *of* A;

(2.47) $\lambda(\lambda^2-A)^{-1}$ *and* $B(\lambda^2-A)^{-1}$ *are strongly infinitely differentiable for* $\lambda>\omega$;

(2.48) $$\left\|\frac{(\lambda-\omega)^{N+1}}{N!}(d/d\lambda)^N\lambda(\lambda^2-A)^{-1}\right\|\leqq M \quad \text{for} \quad \lambda>\omega \quad \text{and} \quad N=0,1,2,\ldots;$$

(2.49) $$\left\|\frac{(\lambda-\omega)^{N+1}}{N!}(d/d\lambda)^N B(\lambda^2-A)^{-1}\right\|\leqq M \quad \text{for} \quad \lambda>\omega \quad \text{and} \quad N=0,1,2,\ldots.$$

PROOF. Suppose A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t\in\mathbf{R}$ in $\mathbf{X}$ satisfying condition (F). Then by proposition 2.6, $\mathscr{B}=\begin{bmatrix}0 & B\\ B & 0\end{bmatrix}$, with domain $D(B)\times D(B)$, is the infinitesimal generator of a strongly continuous group $U(t)$, $t\in\mathbf{R}$ in $\mathbf{X}\times\mathbf{X}$ satisfying $\|U(t)\|_{\mathbf{X}\times\mathbf{X}}\leqq Me^{\omega|t|}$, $t\in\mathbf{R}$, for some $M>0$ and $\omega\geqq 0$. It follows from the theory of groups of operators that $D(B)$ is dense in $\mathbf{X}$, and that $(\lambda-\mathscr{B})^{-1}$ exists in $\mathbf{X}\times\mathbf{X}$ as a holomorphic function of λ for $|\lambda|>\omega$, and satisfies $(d/d\lambda)^N(\lambda-\mathscr{B})^{-1}=N!(-1)^N(\lambda-B)^{-N-1}$. However an easy calculation shows that

(2.50) $$(\lambda-\mathscr{B})^{-1}=\begin{bmatrix}\lambda(\lambda^2-A)^{-1} & B(\lambda^2-A)^{-1}\\ B(\lambda^2-A)^{-1} & \lambda(\lambda^2-A)^{-1}\end{bmatrix}.$$

Thus $\lambda(\lambda^2-A)^{-1}$ and $B(\lambda^2-A)^{-1}$ are strongly infinitely differentiable and

$$(\lambda-\mathscr{B})^{-N-1}=\frac{(-1)^N}{N!}(d/d\lambda)^N(\lambda-\mathscr{B})^{-1}=$$

$$=\begin{bmatrix}\frac{(-1)^N}{N!}(d/d\lambda)^N\lambda(\lambda^2-A)^{-1} & \frac{(-1)^N}{N!}(d/d\lambda)^N B(\lambda^2-A)^{-1}\\ \frac{(-1)^N}{N!}(d/d\lambda)^N B(\lambda^2-A)^{-1} & \frac{(-1)^N}{N!}(d/d\lambda)^N\lambda(\lambda^2-A)^{-1}\end{bmatrix}.$$

Since for $|\lambda|>\omega$, the infinitesimal generator $\mathscr{B}$ of the strongly continuous group $U(t)$, $t\in\mathbf{R}$ satisfies

$$\|(\lambda-\mathscr{B})^{-N-1}[x,y]\|_{\mathbf{X}\times\mathbf{X}}\leqq M(|\lambda|-\omega)^{-N-1}(\|x\|+\|y\|), \quad N=0,1,\ldots.$$

We have that for $N=0,1,\ldots$,

$$\|(\lambda-\mathscr{B})^{-N-1}[x,0]\|_{\mathbf{X}\times\mathbf{X}}=$$

$$=\left\|\frac{(-1)^n}{N!}(d/d\lambda)^N\lambda(\lambda^2-A)^{-1}x\right\|+\left\|\frac{(-1)^N}{N!}(d/d\lambda)^N B(\lambda^2-A)^{-1}\right\|\leqq M(|\lambda|-\omega)^{-N-1}\|x\|.$$

Thus (2.48) and (2.49) hold for real $\lambda>\omega$.

Now suppose that conditions (2.45)—(2.49) are satisfied. Then the operator

$$\mathscr{B} = \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix},$$

with domain $D(B)\times D(B)$ is densely defined in $\mathbf{X}\times\mathbf{X}$. It is easily shown that for real λ, $|\lambda|>\omega$, the resolvent of $\mathscr{B}$ exists and is given by (2.50). Conditions (2.48) and (2.49) imply that for $\lambda>\omega$,

$$\|(\pm\lambda-\mathscr{B})^{-N-1}[x, y]\|_{\mathbf{X}\times\mathbf{X}} \leqq$$

$$\leqq \left\|\frac{(-1)^N}{N!}(d/d\lambda)^N \pm\lambda(\lambda^2-A)^{-1}x\right\| + \left\|\frac{(-1)^N}{N!}(d/d\lambda)^N B(\lambda^2-A)^{-1}y\right\| +$$

$$+\left\|\frac{(-1)^N}{N!}(d/d\lambda)^N B(\lambda^2-A)^{-1}x\right\| + \left\|\frac{(-1)^N}{N!}(d/d\lambda)^N \pm\lambda(\lambda^2-A)^{-1}y\right\| \leqq$$

$$\leqq \frac{2M}{(\lambda-\omega)^{N+1}}\|x\| + \frac{2M}{(\lambda-\omega)^{N+1}}\|y\| = \frac{2M}{(\lambda-\omega)^{N+1}}\|[x, y]\|_{\mathbf{X}\times\mathbf{X}}.$$

Thus $\mathscr{B}$ with domain $D(B)\times D(B)$ is the infinitesimal generator of a strongly continuous group $U(t)$, $t\in\mathbf{R}$ in $\mathbf{X}\times\mathbf{X}$ satisfying $\|U(t)\|_{\mathbf{X}\times\mathbf{X}}\leqq 2Me^{\omega|t|}$, $t\in\mathbf{R}$. The proof now follows from the equivalence of (2.26) and (2.28).

If A is the infinitesimal generator of a strongly continuous cosine family in $\mathbf{X}$, then the set E is precisely the set of $y\in\mathbf{X}$ for which the initial value problem

$$u''(t) = Au(t), \quad u(0) = x\in D(A), \quad u'(0) = y,$$

has a twice continuously differentiable solution. Hence the exact determination of this set is of great importance. It is shown in [5], Remark 6.11, that $D(B)\subseteqq E$. The following proposition demonstrates the close relationship between condition (F) and the equality of these two sets.

PROPOSITION 2.8. *Let $C(t)$, $t\in\mathbf{R}$ be a strongly continuous cosine family in* $\mathbf{X}$ *with infinitesimal generator A and let B be an operator in* $\mathbf{X}$ *such that B commutes with every bounded linear operator in* $\mathbf{X}$ *which commutes with A, zero is in the resolvent set of B, and $B^2=A$. A necessary and sufficient condition that $D(B)=E$ is that $C(t)$, $t\in\mathbf{R}$ satisfy condition* (F).

PROOF. Assume that $C(t)$, $t\in\mathbf{R}$ satisfies condition (F). By Proposition 2.6, B is the infinitesimal generator of a strongly continuous group $T(t)$, $t\in\mathbf{R}$ in $\mathbf{X}$ and $C(t)=(T(t)+T(-t))/2$. Assume now that $x\in E$. Then $C(t)x$ is continuously differentiable and hence (see [13], p. 486)

$$u(t) = \int_0^t T(t-s)C(s)x\,ds, \quad t\in\mathbf{R}$$

is the unique continuously differentiable solution to the nonhomogeneous equation

$$u'(t) = Bu(t)+C(t)x, \quad u(0) = 0.$$

Thus $u(t)\in D(B)$, and since $T(-t)$ leaves the $D(B)$ invariant $T(-t)u(t)\in D(B)$. But

$$T(-t)u(t)=\left(tx+\int_0^t T(-2s)x\,ds\right)/2,$$

which implies that $x\in D(B)$, since $\int_0^t T(-2s)x\,ds\in D(B)$.

Now, assume that $D(B)=E$ and we will demonstrate that the cosine family $C(t)$, $t\in\mathbf{R}$ satisfies condition (F). By (2.18), $S(t)\colon \mathbf{X}\to E=D(B)$. Since B is closed and $S(t)$ is bounded, $BS(t)$ is a closed everywhere defined operator, and hence, by the closed graph theorem, is bounded. It remains to show that $BS(t)x$ is continuous in t for fixed $x\in\mathbf{X}$. But for $x\in E=D(B)$, $d/dt\,C(t)x=AS(t)x=BS(t)Bx$ is continuous in t. Thus $BS(t)y$ is continuous in t for y in the range of B. Since B has a bounded inverse, the range of $B=\mathbf{X}$ and the proof is complete.

We conclude this section with some simple examples of cosine families (see [24]).

EXAMPLE 2.1. Let $\mathbf{X}$ be a Banach space and let A be a bounded linear operator in $\mathbf{X}$. Then $C(t)=\sum_{k=0}^{\infty} A^k t^{2k}/(2k)!$ is a strongly continuous cosine family. The corresponding sine family is given by $S(t)=\sum_{k=0}^{\infty} A^k t^{2k+1}/(2k+1)!$. If $\mathbf{X}=\mathbf{R}$, $a>0$, and $A\colon\mathbf{R}\to\mathbf{R}$ is defined by $Ax=ax$, then $C(t)=\cosh\left(t\sqrt{a}\right)$ and $S(t)=\sinh\left(t\sqrt{a}\right)/\sqrt{a}$. If $A\colon\mathbf{R}\to\mathbf{R}$ is defined by $Ax=-ax$, then $C(t)=\cos\left(t\sqrt{a}\right)$ and $S(t)=\sin\left(t\sqrt{a}\right)/\sqrt{a}$.

EXAMPLE 2.2. Let $\mathbf{X}$ be the Banach space of bounded continuous functions on $\mathbf{R}$ with supremum norm. Let $T(t)$, $t\in\mathbf{R}$ be the group of translations on $\mathbf{X}$, that is, $\left(T(t)x\right)(u)=x(u+t)$. Define $C(t)=\left(T(t)+T(-t)\right)/2$. Then

$$(2.55)\qquad \begin{cases} \left(C(t)x\right)(u)=\left(x(u+t)+x(u-t)\right)/2,\\ \left(S(t)x\right)(u)=\left(\int_0^t C(s)x\,ds\right)(u)=2^{-1}\int_{u-t}^{u+t} x(s)\,ds. \end{cases}$$

One easily sees that $C(t)$, $t\in\mathbf{R}$ is a strongly continuous cosine family with infinitesimal generator A given by $Ax=x''$, $D(A)=\{x\in\mathbf{X}\colon x''\in\mathbf{X}\}$. If $B\colon\mathbf{X}\to\mathbf{X}$ is defined by $Bx=x'$, $D(B)=\{x\in X\colon x'\in\mathbf{X}\}$, then $B^2=A$, and it is easily seen that the strongly continuous cosine family defined by (2.55) satisfies condition (F). Since $\left(BS(t)x\right)(u)=\left(x(u+t)-x(u-t)\right)/2$, we note that for $f\in D(A)$, $g\in E=D(B)$,

$$w(x,t)\overset{\text{def}}{=}\left(C(t)f+S(t)g\right)(x)=2^{-1}\left(f(x+t)+f(x-t)\right)+2^{-1}\int_{x-t}^{x+t} g(s)\,ds$$

gives the classical D'Alembert solution of the one-dimensional wave equation

$$\partial^2/\partial t^2\,w(x,t)=\partial^2/\partial x^2\,w(x,t),\quad w(x,0)=f(x),\quad \partial/\partial t\,w(x,0)=g(x),$$

whose abstract formulation is

$$d^2/dt^2\,w(t)=Aw(t),\quad w(0)=f,\quad d/dt\,w(0)=g.$$

III. Abstract nonlinear second order differential equations

We now treat the nonlinear equation (1.1). Henceforth, we suppose that A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t\in\mathbf{R}$ in $\mathbf{X}$. We will seek a "mild" solution of (1.1), that is, a solution of the integral equation

$$w(t) = C(t-t_0)x + S(t-t_0)y + \int_{t_0}^{t} S(t-s)f(s, w(s), d/ds\,w(s))\,ds. \tag{3.1}$$

Equation (3.1) is more general than (1.1) by virtue of Proposition 2.4. We will investigate the existence of solutions to (3.1) under various hypotheses on the cosine family $C(t)$, $t\in\mathbf{R}$, the nonlinear function f, and the initial values x and y.

PROPOSITION 3.1. *Let D be an open subset of $\mathbf{R}\times\mathbf{X}\times\mathbf{X}$ and let $f\colon D\to\mathbf{X}$ be continuous and satisfy*

$$\|f(t, x, y)-f(t, \hat{x}, \hat{y})\| \leqq L(t)(\|x-\hat{x}\|+\|y-\hat{y}\|), \tag{3.2}$$

for (t, x, y), $(t, \hat{x}, \hat{y})\in D$ and some continuous real-valued function L. For each $(t_0, x, y)\in D$ such that $x\in D(A)$ there exists $t_1>0$ and a unique continuously differentiable function $w\colon(t_0-t_1, t_0+t_1)\to\mathbf{X}$ satisfying (3.1). *Furthermore, if $D=\mathbf{R}\times\mathbf{X}\times\mathbf{X}$, then the solution w is defined on $\mathbf{R}$.*

PROOF. Let $(t_0, x, y)\in D$ with $x\in D(A)$, and let N be a neighbourhood of (t_0, x, y) such that $\bar{N}\subset D$. Then there exists $t_1>0$ such that

(i) if $|t-t_0|<t_1$, then $\big(t, C(t-t_0)x+S(t-t_0)y,\ S(t-t_0)Ax+C(t-t_0)y\big)\in N$, and

(ii) if v is a continuously differentiable function from (t_0-t_1, t_0+t_1) to $\mathbf{X}$ such that $\big(s, v(s), v'(s)\big)\in N$ for $|s-t_0|<t_1$, and

$$u(t) \overset{\text{def}}{=} C(t-t_0)x+S(t-t_0)y+\int_{t_0}^{t} S(t-s)f\big(s, v(s), v'(s)\big)\,ds$$

for $|t-t_0|<t_1$, then u is continuously differentiable,

$$u'(t) = S(t-t_0)Ax+C(t-t_0)y+\int_{t_0}^{t} C(t-s)f\big(s, v(s), v'(s)\big)\,ds,$$

and $\big(t, u(t), u'(t)\big)\in N$.

Observe that if $D=\mathbf{R}\times\mathbf{X}\times\mathbf{X}$, we can choose $N=D$ and t_1 can be chosen arbitrarily large. Define $w_0\colon (t_0-t_1, t_0+t_1)\to\mathbf{X}$ by $w_0(t)=C(t-t_0)x+S(t-t_0)y$, and define $w_n\colon (t_0-t_1, t_0+t_1)\to\mathbf{X}$ by

$$w_n(t) = C(t-t_0)x+S(t-t_0)y+\int_{t_0}^{t} S(t-s)f\big(s, w_{n-1}(s), w_{n-1}'(s)\big)\,ds,$$

for $n=1, 2, \ldots$. There exists $M>0$ such that if $|t-t_0|<t_1$, then

$$\Big\|\int_{t_0}^{t} S(t-s)f\big(s, w_0(s), w_0'(s)\big)\,ds\Big\|+\Big\|\int_{t_0}^{t} C(t-s)f\big(s, w_0(s), w_0'(s)\big)\,ds\Big\| \leqq \frac{M}{2}.$$

Thus, for $|t-t_0|<t_1$ and

$$R \stackrel{\text{def}}{=} \sup_{|s|\leqq t_1} (\|S(s)\|, \|C(s)\|) \sup_{|s-t_0|\leqq t_1} L(s)$$

we have that

$$\max\{\|w_1(t)-w_0(t)\|, \|w_1'(t)-w_0'(t)\|\} \leqq \frac{M}{2},$$

$$\max\{\|w_2(t)-w_1(t)\|, \|w_2'(t)-w_1'(t)\|\} \geqq |t-t_0|\, MR,$$

and, in general, for $n=1, 2, \ldots,$

$$\max\{\|w_{n+1}(t)-w_n(t)\|, \quad \|w_{n+1}'(t)-w_n'(t)\|\} \leqq |t-t_0|^n R^n M 2^{n-1}/n!.$$

Then,

$$\lim_{n\to\infty} w_n(t) \stackrel{\text{def}}{=} w(t) \quad \text{and} \quad \lim_{n\to\infty} w_n'(t) \stackrel{\text{def}}{=} v(t)$$

exist uniformly for $|t-t_0|<t_1$, and $w(t)$ is continuously differentiable, $w'(t)=v(t)$, and $(t, w(t), w'(t))\in\bar{N}\subset D$. To establish that w satisfies (3.1), use the estimate

$$\left\|w(t)-C(t-t_0)x-S(t-t_0)y-\int_{t_0}^{t} S(t-s)f(s, w(s), w'(s))\,ds\right\| \leqq$$

$$\leqq \|w(t)-w_{n+1}(t)\| + R\int_{t_0}^{t} (\|w(s)-w_n(s)\| + \|w'(s)-w_n'(s)\|)\,ds.$$

Lastly, the uniqueness claim follows from the following proposition:

PROPOSITION 3.2. *Assume the hypothesis of Proposition* 3.1. *There exist constants* $M=M(t_1)$ *and* $\gamma=\gamma(t_1)$ *such that if* w *and* $\hat{w}$ *satisfy* (3.1) *for* $|t-t_0|<t_1<\infty$ *with* $w(t_0)=x\in D(A)$, $w'(t_0)=y$, $\hat{w}(t_0)=\hat{x}\in D(A)$, *and* $\hat{w}'(t_0)=\hat{y}$, (t_0, x, y), $(t_0, \hat{x}, \hat{y})\in D$, *then for* $|t-t_0|<t_1$

$$\|w(t)-\hat{w}(t)\| + \|w'(t)-\hat{w}'(t)\| \leqq \tag{3.3}$$

$$\leqq M(\|x-\hat{x}\| + \|A(x-\hat{x})\| + \|y-\hat{y}\|)e^{\gamma|t-t_0|}.$$

PROOF. (3.3) follows from Gronwall's Lemma and the estimates

$$\|w(t)-\hat{w}(t)\| \leqq Ke^{\omega|t-t_0|}\|x-\hat{x}\| + K|t-t_0|e^{\omega|t-t_0|}\|y-\hat{y}\| +$$

$$+K|t-t_0|\left|\int_{t_0}^{t} e^{\omega|t-s|}L(s)(\|w(s)-\hat{w}(s)\| + \|w'(s)-\hat{w}'(s)\|)\,ds\right|$$

and

$$\|w'(t)-\hat{w}'(t)\| \leqq K|t-t_0|e^{\omega|t-t_0|}\|A(x-\hat{x})\| + Ke^{\omega|t-t_0|}\|y-\hat{y}\| +$$

$$+K\left|\int_{t_0}^{t} e^{\omega|t-s|}L(s)(\|w(s)-\hat{w}(s)\| + \|w'(s)-\hat{w}'(s)\|)\,ds\right|.$$

PROPOSITION 3.3. *Let the hypothesis of Proposition* 3.1 *hold, let* $(t_0, x, y)\in D$ *where* $x\in D(A)$ *and* $y\in E$, *and also suppose that:*

(3.4) *there exists an open neighbourhood* N *of* (t_0, x, y) *such that* $N\subset D$, f *is continuously Frechet differentiable from* N *to* $\mathbf{X}$, *and there exists a constant* R *such that if* (t, v, w), $(t, v, \hat{w})\in N$, *then*

$$\max\{\|f_1(t, v, w)-f_1(t, v, \hat{w})\|,\ \|f_2(t, v, w)-f_2(t, v, \hat{w})\|,\ \|f_3(t, v, w)-f_3(t, v, \hat{w})\|\}\leqq$$
$$\leqq R\|w-\hat{w}\|,$$

(where f_1, f_2 *and* f_3 *denote the partial derivatives of* f *with respect to its first, second, and third places, respectively).*

Then there exists $t_2\in(0, t_1]$ *such that the solution* w *of* (3.1) *is twice continuously differentiable on* (t_0-t_2, t_0+t_2) *and* w *satisfies* (1.1) *on* (t_0-t_2, t_0+t_2).

PROOF. The method of proof for Proposition 3.1 can be used to show that there exists $t_2\in(0, t_1]$ and a unique continuous function $v:(t_0-t_2, t_0+t_2)\to X$ satisfying

$$v(t) = C(t-t_0)\big(Ax+f(t_0, x, y)\big)+AS(t-t_0)y+ \tag{3.5}$$
$$+\int_{t_0}^{t} C(t-s)\big(f_1(s, w(s), w'(s))+f_2(s, w(s), w'(s))w'(s)+f_3(s, w(s), w'(s))v(s)\big)\,ds.$$

Define $u(t)\overset{\text{def}}{=\!=}y+\int_{t_0}^{t} v(s)ds$ for $|t-t_0|<t_2$. We will show that $u=w'$, which will establish that w is a twice continuously differentiable function. The conclusion of the proposition will then follow from Proposition 2.4. First, by taking the limit of the difference quotient one shows that

$$d/dt\int_{t_0}^{t} C(t-s)f(s, w(s), u(s))\,ds =$$
$$= C(t-t_0)f(t_0, x, y)+\int_{t_0}^{t} C(t-s)\, d/ds f(s, w(s), u(s))\,ds,$$

which implies

$$\int_{t_0}^{t} C(t-s)f(s, w(s), u(s))\,ds = \int_{t_0}^{t} C(s-t_0)f(t_0, x, y)\,ds+ \tag{3.6}$$
$$+\int_{t_0}^{t}\int_{t_0}^{s} C(s-\tau)\, d/d\tau f(\tau, w(\tau), u(\tau))d\tau\, ds.$$

Using (2.19), we obtain

$$u(t) = y+\int_{t_0}^{t} C(s-t_0)\big(Ax+f(t_0, x, y)\big)\,ds+C(t-t_0)y+ \tag{3.7}$$
$$+\int_{t_0}^{t}\int_{t_0}^{s} \big(C(s-\tau)f_1(\tau, w(\tau), w'(\tau))+f_2(\tau, w(\tau), w'(\tau))w'(\tau)+f_3(\tau, w(\tau), w'(\tau))v(\tau)\big)\,d\tau\, ds.$$

From (3.6) and (3.7), we obtain

$$u(t) = C(t-t_0)y + \int_{t_0}^{t} C(s-t_0)Ax\,ds + \tag{3.8}$$

$$+\int_{t_0}^{t}(C(t-s)f(s, w(s)u(s))\,ds - \int_{t_0}^{t}\int_{t_0}^{s} C(s-\tau)\,d/d\tau f(\tau, w(\tau), u(\tau))\,d\tau\,ds +$$

$$+\int_{t_0}^{t}\int_{t_0}^{s}(C(s-\tau)f_1(\tau, w(\tau), w'(\tau)) + f_2(\tau, w(\tau), w'(\tau))w'(\tau) + f_3(\tau, w(\tau), w'(\tau))v(\tau)\,d\tau\,ds.$$

Since for $x \in D(A)$,

$$\int_{t_0}^{t} C(s-t_0)Ax\,ds = S(t-t_0)Ax,$$

(3.8) yields that there exists a positive constant K such that

$$\|u(t)-w'(t)\| \leqq K\int_{t_0}^{t}\|u(s)-w'(s)\|\,ds,$$

which implies, by Gronwall's lemma, that $u(t)=w'(t)$ for $|t-t_0|<t_2$.

COROLLARY 3.4. *Let D be an open subset of $\mathbf{R}\times\mathbf{X}\times\mathbf{X}$ and let $f\colon D\to\mathbf{X}$ be a twice continuously (Frechet) differentiable function on D. For each $(t_0, x, y)\in D$ such that $x\in D(A)$ and $y\in E$, there exists $t_1>0$ and a unique twice continuously differentiable function $w\colon (t_0-t_1, t_0+t_1)\to\mathbf{X}$ satisfying* (1.1).

PROPOSITION 3.5. *Let D be an open subset of $\mathbf{R}\times\mathbf{X}$, let $f\colon D\to\mathbf{X}$ be continuous and satisfy*

$$\|f(t, x)-f(t, \hat{x})\| \leqq L(t)\|x-\hat{x}\|, \tag{3.9}$$

for (t, x), $(t, \hat{x})\in D$ and some continuous real-valued function L. For each $(t_0, x)\in D$ and $y\in\mathbf{X}$, there exists $t_1>0$ and a unique continuous function $w\colon (t_0-t_1, t_0+t_1)\to\mathbf{X}$ satisfying

$$w(t) = C(t-t_0)x + S(t-t_0)y + \int_{t_0}^{t} S(t-s)f(s, w(s))\,ds. \tag{3.10}$$

Further, there exist constants $M=M(t_1)$ and $\gamma=\gamma(t_1)$ such that if (t_0, x), $(t_0, \hat{x})\in D$, y and $\hat{y}\in\mathbf{X}$, w satisfies (3.10) *for $|t-t_0|<t_1$, $\hat{w}$ satisfies* (3.10) *for $|t-t_0|<t_1$ with x and y replaced by $\hat{x}$ and $\hat{y}$, respectively, then for $|t-t_0|<t_1$*

$$\|w(t)-\hat{w}(t)\| \leqq M(\|x-\hat{x}\|+\|y-\hat{y}\|)e^{\gamma|t-t_0|}. \tag{3.11}$$

Finally, if $D=\mathbf{R}\times\mathbf{X}$, then the solution w is defined on all of $\mathbf{R}$.

The proof of Proposition 3.5 is similar to the proof of Propositions 3.1 and 3.2, and hence will be omitted. We next present the analogue of Proposition 3.3 for equation (3.10). Notice that it is not necessary to assume that the Frechet derivative of f is locally Lipschitz.

PROPOSITION 3.6. *Let D be an open subset of $\mathbf{R}\times\mathbf{X}$ and let f: $D\to\mathbf{X}$ be continuously (Frechet) differentiable on D. For each $(t_0, x)\in D$ such that $x\in D(A)$ and for each $y\in E$, there exists a unique twice continuously differentiable function w: $(t_0-t_1, t_0+t_1)\to\mathbf{X}$ satisfying*

$$(3.12)\qquad d^2/dt^2 w(t) = Aw(t)+f(t, w(t)), \quad w(t_0) = x, \quad d/dt w(t_0) = y,$$

for $|t-t_0|<t_1$.

PROOF. Since f is continuously Frechet differentiable, (3.9) is satisfied for some open neighbourhood of (t_0, x), and hence, by Proposition 3.5, equation (3.10) has a unique continuous solution w existing on $|t-t_0|<t_1$, for some positive t_1. Since $x\in D(A)$, it is easily shown that w is continuously differentiable and satisfies

$$w'(t) = AS(t-t_0)x+C(t-t_0)y+\int_{t_0}^{t} C(t-s)f(s, w(s))\,ds.$$

Thus, $f(s, w(s))$ is continuously differentiable for $|s-t_0|<t_1$, so the conclusion follows from Proposition 2.4.

We will now establish local existence of solutions to equation (3.12) under the assumption that f is only continuous. The continuity of f, however, is not sufficient to assure local existence of solutions without further restrictions on the cosine family $C(t)$, $t\in\mathbf{R}$. For our purposes it will suffice to assume that $S(t)$ is compact for each $t\in\mathbf{R}$.

PROPOSITION 3.7. *Let $S(t)$ be compact for each $t\in\mathbf{R}$, let D be an open subset of $\mathbf{R}\times\mathbf{X}$, and let f: $D\to\mathbf{X}$ be continuous. For each $(t_0, x)\in D$ and $y\in\mathbf{X}$, there exist $t_1>0$ and a continuous function $w:(t_0-t_1, t_0+t_1)\to\mathbf{X}$ satisfying* (3.10).

PROOF. Let $(t_0, x)\in D$, $y\in\mathbf{X}$. Let K, ω be constants as in (2.11) and (2.12), and let N be a neighbourhood of $(t_0, x)\in D$ such that $\overline{N}$ is bounded, convex, $\overline{N}\subset D$, and $f(\overline{N})$ is bounded in X. There exists $t_1>0$ such that if u is a continuous function from $[t_0-t_1, t_0+t_1]\to X$ satisfying $(s, u(s))\in\overline{N}$ for $|s-t_0|\leqq t_1$ and if

$$v(t) \overset{\text{def}}{=} C(t-t_0)x+S(t-t_0)y+\int_{t_0}^{t} S(t-s)f(s, u(s))\,ds$$

for $|t-t_0|\leqq t_1$, then $(t, v(t))\in\overline{N}$. Let C be the subset of $C([t_0-t_1, t_0+t_1]; \mathbf{X})$ consisting of all functions u from $[t_0-t_1, t_0+t_1]$ to $\mathbf{X}$ such that $(t, u(t))\in\overline{N}$. Consider the transformation $G: C\to C$ defined by

$$(Gu)(t) = C(t-t_0)x+S(t-t_0)y+\int_{t_0}^{t} S(t-s)f(s, u(s))\,ds.$$

Obviously, G is continuous. To show that G is compact, we will show that if U is a bounded set in C, then $\{(Gu)(t)\colon u\in U\}$ is equicontinuous and precompact in

X for each fixed $t\in[t_0-t_1, t_0+t_1]$. For $u\in U$ and $t_0-t_1\leqq t<\hat{t}\leqq t_0+t_1$,

$$\|(Gu)(t)-(Gu)(\hat{t})\| =$$
$$= \Big\|C(t-t_0)x-C(\hat{t}-t_0)x+S(t-t_0)y-S(\hat{t}-t_0)y+\int_{t_0}^{t} S(t-s)f(s, u(s))ds-$$
$$-\int_{t_0}^{t} S(\hat{t}-s)f(s, u(s))\,ds\Big\| \leqq \|C(t-t_0)x-C(\hat{t}-t_0)x\|+\|S(t-t_0)y-S(\hat{t}-t_0)y\|+$$
$$+\Big\|\int_{t_0}^{t}(S(t-s)-S(\hat{t}-s))\,f(s, u(s))\,ds\Big\|+\Big\|\int_{t_0}^{\hat{t}} S(\hat{t}-s)f(s, u(s))\,ds\Big\|.$$

The equicontinuity now follows from the fact that $C(t-t_0)x$ and $S(t-t_0)y$ are uniformly continuous on $[t_0-t_1, t_0+t_1]$, and from the fact that f bounded on $\overline{N}$ and (2.12) imply

$$\Big\|\int_{t_0}^{t}(S(t-s)-S(\hat{t}-s))f(s, u(s))\,ds\Big\|+\Big\|\int_{t}^{\hat{t}} S(\hat{t}-s)f(s, u(s))\,ds\Big\| \leqq C|\hat{t}-t|,$$

where the constant C does not depend on u. The precompactness of $\{G(u)(t)\colon u\in U\}$ for fixed $t\in[t_0-t_1, t_0+t_1]$ follows from the fact that $S(t)$ is uniformly continuous from $[t_0-t_1, t_0+t_1]$ to $B(\mathbf{X}, \mathbf{X})$ (see (2.12)) and a proof using the compactness of $S(t)$ which is similar to that found in [26], Lemma 2.5.

By Schauder's fixed point theorem, G has a fixed point in C, and the proof of the theorem is complete.

PROPOSITION 3.8. *Let the hypothesis of Proposition* 3.5 *or* 3.7 *hold, and in addition, let f map closed bounded sets in D into bounded sets in* $\mathbf{X}$. *If* $(t_0, x)\in D$, $x\in D(A)$, $y\in\mathbf{X}$, *and w is a solution of* (3.10) *noncontinuable to the right on* $[t_0, b)$, *then either* $b=+\infty$, *or given any closed bounded set U in D, there exists a sequence* $t_k\to b^-$ *such that* $(t_k, w(t_k))\notin U$. *An analogous result holds for a solution noncontinuable to the left.*

PROOF. Assume that $b<\infty$ and the conclusion of the proposition is false. Then there exists a closed bounded set U in D such that $(t, w(t))\in U$ for $t_1\leqq t<b$, where $t_0\leqq t_1<b$. For $t_1<t<\hat{t}<b$

$$\|w(\hat{t})-w(t)\| \leqq \|C(\hat{t}-t_0)\,x-C(t-t_0)x\|+\|S(\hat{t}-t_0)y-S(t-t_0)y\|+$$
$$+\Big\|\int_{t_0}^{t}(S(\hat{t}-s)-S(t-s))f(s, w(s))\,ds\Big\|+\Big\|\int_{t_0}^{\hat{t}} S(\hat{t}-s)f(s, w(s))\,ds\Big\|.$$

Using (2.12) and the fact that f is bounded on U, we have

$$\|w(\hat{t})-w(t)\| \leqq \|C(t-t_0)x-C(\hat{t}-t_0)x\|+\|S(t-t_0)y-S(\hat{t}-t_0)\,y\|+\text{const}\,|\hat{t}-t|.$$

Then, $\lim\limits_{t\to b^-} w(t)\overset{\text{def}}{=\!=}p$ exists, which implies that

$$\lim_{t\to b^-} w'(t) = S(b-t_0)Ax+C(b-t_0)\,y+\int_{t_0}^{b} C(b-s)f(s, w(s))\,ds\overset{\text{def}}{=\!=}q$$

exists. Since $(b, p)\in U\subset D$, one can find a solution of the equation

$$v(t) = C(t-b)\,p+S(t-b)\,q+\int_{b}^{t} S(t-s)f(s, v(s))\,ds$$

for $b \leqq t < t_2$. Extend w to $[t_0, t_2)$ by defining $w(t) = v(t)$ for $b \leqq t < t_2$. Using the identities (2.9) and (2.23) we have for $b \leqq t < t_2$,

$$w(t) = v(t) = C(t-b)\Big(C(b-t_0)x + S(b-t_0)y + \int_{t_0}^{b} S(b-s)f(s, w(s))\,ds\Big) +$$

$$+ S(t-b)\Big(AS(b-t_0)x + C(b-t_0)y + \int_{t_0}^{b} C(b-s)f(s, w(s))\,ds\Big) + \int_{b}^{t} S(t-s)f(s, v(s))\,ds =$$

$$= C(t-t_0)x + S(t-t_0)y + \int_{t_0}^{t} S(t-s)f(s, w(s))\,ds.$$

This contradicts the non-continuability assumption and the proof is complete.

COROLLARY 3.9. *Suppose that the hypothesis of Proposition* 3.7 *holds,* $D = \mathbf{R}\times\mathbf{X}$, *and* f *maps closed bounded sets in* D *into bounded sets in* $\mathbf{X}$. *If* $x \in D(A)$, $y \in \mathbf{X}$, *and* w *is a solution of* (3.10) *noncontinuable to the right on* $[t_0, b)$, *then either* $b = +\infty$ *or* $\overline{\lim}_{t\to b^-} \|w(t)\| = \infty$. *An analogous result holds for a solution of* (3.10) *noncontinuable to the left.*

PROPOSITION 3.10. *Suppose the hypothesis of Proposition* 2.6 *holds and that condition* (F) *is satisfied. Let* D *be an open subset of* $\mathbf{R}\times[D(B)]\times\mathbf{X}$ *and let* $f\colon D\to\mathbf{X}$ *be continuous and satisfy*

$$\|f(t, x, y) - f(t, \hat{x}, \hat{y})\| \leqq L(t)(\|x-\hat{x}\|_B + \|y-\hat{y}\|) \tag{3.13}$$

for $(t, x, y), (t, \hat{x}, \hat{y}) \in D$ *and some continuous real-valued function* L. *For each* $(t_0, x, y) \in D$, *there exists* $t_1 > 0$ *and a unique continuously differentiable function* $w\colon (t_0-t_1, t_0+t_1)\to\mathbf{X}$ *satisfying* (3.1). *Furthermore, if* $D = \mathbf{R}\times[D(B)]\times\mathbf{X}$, *then the solution* w *is defined on* $\mathbf{R}$.

PROOF. We shall make use of the equivalence (2.26) and (2.29) of Proposition 2.6. If we define $Y = [D(B)]\times\mathbf{X}$ and $F(t, [x, y]) = [0, f(t, x, y)]$ for each $(t, x, y) \in D$ then $F\colon \mathbf{R}\times Y\to Y$. It follows from the continuity of f that F is continuous. Furthermore, it is uniformly continuous on finite intervals in its first variable, and Lipschitz continuous in its second variable. The methods of Proposition 3.1 or [23], Theorem 1, can now be used to establish that for each $(t_0, x, y) \in D$ there exists $t_1 > 0$ and a unique continuous function $W\colon (t_0-t_1, t_0+t_1)\to Y$ satisfying

$$W(t) = V(t-t_0)[x, y] + \int_{t_0}^{t} V(t-s)F(s, W(s))\,ds, \tag{3.14}$$

where $V(t)$, $t\in\mathbf{R}$, is the strongly continuous group appearing in (2.29). To establish the claim of this Proposition, we let $\pi_1[x, y] \overset{\text{def}}{=} x$ and $\pi_2[x, y] \overset{\text{def}}{=} y$, and define $w(t) = \pi_1 W(t)$ for $t \in (t_0-t_1, t_0+t_1)$. Then

$$w(t) = C(t-t_0)x + S(t-t_0)y + \int_{t_0}^{t} S(t-s)f(s, w(s), \pi_2 W(s))\,ds,$$

$$d/dt\,w(t) = BS(t-t_0)Bx + C(t-t_0)y + \int_{t_0}^{t} C(t-s)f(s, w(s), \pi_2 W(s))\,ds = \pi_2 W(t).$$

PROPOSITION 3.11. *Let the hypothesis of Proposition* 3.10 *hold. There exist constants* $M=M(t_1)$ *and* $\gamma=\gamma(t_1)$ *such that if* w *and* $\hat{w}$ *satisfy* (3.1) *for* $|t-t_0|<t_1<\infty$, *with* $w(t_0)=x$, $w'(t_0)=y$, $\hat{w}(t_0)=\hat{x}$, $\hat{w}'(t_0)=\hat{y}$, (t_0, x, y), $(t_0, \hat{x}, \hat{y})\in D$, *then for* $|t-t_0|<t_1$

$$\|w(t)-\hat{w}(t)\|_B+\|w'(t)-\hat{w}'(t)\| \leqq M(\|x-\hat{x}\|_B+\|y-\hat{y}\|)e^{\gamma|t-t_0|}. \tag{3.15}$$

The proof is similar to that of Proposition 3.2 and will be omitted.

PROPOSITION 3.12. *Suppose the hypothesis of Proposition* 2.6 *holds and that condition* (F) *is satisfied. Let* D *be an open subset of* $\mathbf{R}\times[D(B)]\times\mathbf{X}$, *and* $f\colon D\to\mathbf{X}$ *be twice continuously (Frechet) differentiable on* D. *For each* $(t_0, x, y)\in D$ *such that* $x\in D(B)$ *and* $y\in D(B)$, *there exists* $t_1>0$ *and a unique twice continuously differentiable function* $w\colon (t_0-t_1, t_0+t_1)\to\mathbf{X}$ *satisfying* (1.1).

The method of proof is similar to that found in [24], Proposition 2.3, and we omit the details.

PROPOSITION 3.13. *Let the hypothesis of Proposition* 3.10 *hold, and in addition, let* f *be uniformly continuous and bounded on bounded sets in* D. *If* $(t_0, x, y)\in D$ *and* w *is a solution of* (3.1) *noncontinuable to the right on* $[t_0, b)$, *then either* $b=+\infty$ *or given any closed bounded set* U *in* D, *there exists a sequence* $t_k\to b^-$ *such that* $(t_k, w(t_k), w'(t_k))\notin U$. *An analogous result holds for a solution noncontinuable to the left.*

PROOF. Assume $b<\infty$ and that the conclusion of the proposition is false. Then there exists a closed bounded set U in D such that $(t, w(t), w'(t))\in U$ for $t_1\leqq t<b$, where $t_0\leqq t_1<b$. For $t_1<t<t+h<b$

$$\|Bw(t+h)-Bw(t)\| \leqq \|C(t+h-t_0)Bx-C(t-t_0)Bx\|+$$
$$+\|BS(t+h-t_0)y-BS(t-t_0)y\|+\Big\|\int_{t_0}^{t_0+h} BS(t+h-s)f(s, w(s), w'(s))\,ds\Big\|+$$
$$+\Big\|\int_{t_0}^{t} BS(t-s)[f(s+h, w(s), w'(s))-f(s, w(s), w'(s))]\,ds\Big\|+$$
$$+\Big\|\int_{t_0}^{t} BS(t-s)[f(s+h, w(s+h), w'(s+h))-f(s+h, w(s), w'(s))]\,ds\Big\|,$$

$$\|w'(t+h)-w'(t)\| \leqq \|BS(t+h-t_0)Bx-BS(t-t_0)Bx\|+$$
$$+\|C(t+h-t_0)y-C(t-t_0)y\|+\Big\|\int_{t_0}^{t_0+h} C(t+h-s)f(s, w(s), w'(s))\,ds\Big\|+$$
$$+\Big\|\int_{t_0}^{t} C(t-s)[f(s+h, w(s), w'(s))-f(s, w(s), w'(s))]\,ds\Big\|+$$
$$+\Big\|\int_{t_0}^{t} C(t-s)f(s+h, w(s+h), w'(s+h))-f(s+h, w(s), w'(s))]\,ds\Big\|.$$

Since F is bounded on bounded sets in D, there exists a constant K such that

$\|f(s, w(s), w'(s))\| \leqq K$ for $s \in [t_0, b)$ (since $s \to (s, w(s), w'(s))$ is continuous for $s \in [t_0, t_1]$ and $(s, w(s), w'(s)) \in U$ for $s \in [t_1, b)$). Also, $|BS(t)|$ is bounded uniformly in finite intervals of t by virtue of the Principle of Uniform Boundedness. For $t_1 < t < t+h < b$, define

$$H(t, h) = \|C(t+h-t_0)Bx - C(t-t_0)Bx\| + \|BS(t+h-t_0)y - BS(t-t_0)y\| +$$
$$+ \|BS(t+h-t_0)Bx - BS(t-t_0)Bx\| + \|C(t+h-t_0)y - C(t-t_0)y\| +$$
$$+ \int_{t_0}^{t_0+h} (|BS(t+h-s)| + |C(t+h-s)|)K\, ds +$$
$$+ \int_{t_0}^{t} (|BS(t-s)| + |C(t-s)|)\|f(s+h, w(s), w'(s)) - f(s, w(s), w'(s))\|\, ds.$$

Using the strong continuity of $C(t)$ and $BS(t)$, the uniform boundedness of $|BS(t)|$ and $|C(t)|$ on finite intervals, and the uniform continuity of f on bounded sets, we see that $\lim_{h \to 0^+} H(t, h) = 0$ uniformly for $t_1 < t < t+h < b$. Define

$$\gamma = \big(\sup_{t_0 \leqq s \leqq b} |BS(s)| + \sup_{t_0 \leqq s \leqq b} |C(s)|\big)\big(\sup_{t_0 \leqq s \leqq b} L(s)\big)$$

where L is as in (3.13) and we see that for $t_1 < t < t+h < b$,

$$\|Bw(t+h) - Bw(t)\| + \|w'(t+h) - w'(t)\| \leqq$$
$$\leqq H(t, h) + \gamma \int_{t_0}^{t} (\|Bw(s+h) - Bw(s)\| + \|w'(s+h) - w'(s)\|)\, ds.$$

Using Gronwall's lemma, we obtain

$$\|Bw(t+h) - Bw(t)\| + \|w'(t+h) - w'(t)\| \leqq H(t, h)\, e^{\gamma(t-t_0)}.$$

Then,

$$\lim_{t \to b^-} Bw(t) \overset{\text{def}}{=} r \quad \text{and} \quad \lim_{t \to b^-} w'(t) \overset{\text{def}}{=} q$$

exist. For $t_1 < t < t+h < b$

$$\|w(t+h) - w(t)\| \leqq \|C(t+h-t_0)x - C(t-t_0)x\| + \|S(t+h-t_0)y - S(t_0-t)y\| +$$
$$+ \Big\|\int_{t_0}^{t} (S(t+h-s) - S(t-s)) f(s, w(s), w'(s))\, ds\Big\| +$$
$$+ \Big\|\int_{t}^{t+h} S(t+h-s) f(s, w(s), w'(s))\, ds\Big\|.$$

Using (2.12) and the fact that f is bounded on U, we have

$$\|w(t+h) - w(t)\| \leqq \|C(t+h-t_0)x - C(t-t_0)x\| + \|S(t+h-t_0)y - S(t-t_0y)\| +$$
$$+ \text{constant times } h.$$

Then, $\lim_{t \to b^-} w(t) \overset{\text{def}}{=} p$ exists, and since B is closed $p \in D(B)$. But then $(b, p, q) \in U \subset D$ and by Proposition 3.10, w can be continued to the right of b as in the proof of Proposition 3.8. This contradicts the noncontinuability assumption on w, completing the proof.

References

[1] V. Barbu, A class of boundary problems for second order abstract differential equations, *J. Fac. Sci. Univ. Tokyo I A Math.*, **19** (1972), 295—319.

[2] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff International (Leyden, The Netherlands, 1975).

[3] G. Da Prato and E. Giusti, Una caratterizzazione dei generatori di funzioni coseno astratte, *Boll. Unione Mat. Italiana*, **22** (1967), 357—362.

[4] N. Dunford and J. T. Schwartz, *Linear Operators, Part* I: *General Theory*, Interscience (New York, 1958).

[5] H. O. Fattorini, Ordinary differential equations in linear topological spaces, I, *J. Differential Equations*, **5** (1968), 72—105.

[6] H. O. Fattorini, Ordinary differential equations in linear topological spaces, II, *J. Differential Equations*, **6** (1969), 50—70.

[7] H. O. Fattorini, Uniformly bounded cosine functions in Hilbert space, *Indiana Univ. Math. J.*, **20** (1970), 411—425.

[8] E. Giusti, Funzioni coseno periodiche, *Boll. Unione Mat. Italiana*, **22** (1967), 478—485.

[9] J. Goldstein, Semi-groups and hyperbolic equations, *J. Functional Analysis*, **4** (1969), 50—70.

[10] J. Goldstein, On a connection between first and second order differential equations in Banach spaces, *J. Math. Anal. Appl.*, **30** (1970), 246—251.

[11] J. Goldstein, On the convergence and approximation of cosine functions, *Aequationes Math.*, **11** (1974), 201—205.

[12] J. Goldstein, *Semigroups of Operators and Abstract Cauchy Problems*, Monograph, Tulane University (New Orleans, 1970).

[13] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag (New York, 1966).

[14] J. Kisyński, On operator-valued solutions of d'Alembert's functional equation, I, *Colloquium Math.*, **23** (1971), 107—114.

[15] Y. Konishi, Cosine functions of operators in locally convex spaces, *J. Fac. Sci. Univ. Tokyo I A Math.*, **18** (1971/72), 443—463.

[16] S. G. Krein, *Linear Differential Equations in Banach Spaces, Amer. Math. Soc. Translations of Math. Monographs*, Vol. **29** (Providence, 1971).

[17] S. Kurepa, On some functional equations in Banach spaces, *Studia Math.*, **19** (1960), 149—158.

[18] S. Kurepa, A cosine funcational equation in Banach algebras, *Acta Sci. Math. Szeged*, **23** (1962), 255—267.

[19] V. Lakshmikantham, *Differential Equations in Abstract Spaces*, Academic Press (New York, 1972).

[20] B. Nagy, On the generators of cosine operator functions, *Publ. Math. Debrecen*, **21** (1974), 151—154.

[21] B. Nagy, On cosine operator functions in Banach spaces, *Acta Sci. Math. Szeged*, **36** (1974), 281—289.

[22] B. Nagy, Cosine operator functions and the abstract Cauchy problem, *Periodica Math. Hungar.*, **7 (3)** (1976), 15—18.

[23] I. Segal, Non-linear semi-groups, *Ann. Math.*, **78** (1963), 339—364.

[24] M. Sova, Cosine operator functions, *Rozprawy Matematiyczne*, **49** (1966), 1—47.

[25] M. Sova, Semigroups and cosine functions of normal operators in Hilbert spaces, *Časopis Pěst. Mat.*, **93** (1968), 437—458.

[26] C. Travis and G. Webb, Existence and stability for partial functional differential equations, *Trans. Amer. Math. Soc.*, **200** (1974), 395—418.

[27] K. Yosida, *Functional Analysis*, Fourth Ed., Springer-Verlag (New York, 1974).

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DEPARTMENT OF MATHEMATICS
VANDERBILT UNIVERSITY
NASHVILLE, TENNESSEE 37 235
USA

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GRAPHS WITH MAXIMAL NUMBER OF ADJACENT PAIRS OF EDGES

By

R. AHLSWEDE[1] (Bielefeld) and G. O. H. KATONA[2] (Budapest)

1. Introduction and statement of the results

Let G_n^N denote an undirected graph (without loops and multiple edges) with n vertices and N edges. $P(G_n^N)$ shall count the number of pairs of different edges which have a common vertex and finally $f(n, N)$ is defined by

$$f(n, N) = \max p(G_n^N), \tag{1.1}$$

where the maximum is taken over all possible graphs G_n^N.

In information theory the problem came up to determine $f(n, N)$ for certain hypergraphs. We give here a solution for graphs and for bipartite graphs. As Vera T. Sós kindly informed us, this problem has been solved by MOSHE KATZ [1] for "nice" N's.

In order to state our results we need the concepts of a quasi-complete graph and of a quasi-star. Suppose the vertices of the graph are denoted by $1, 2, \ldots, n$. We define the *quasi-complete* graph C_n^N with N edges in the following way: i and j are connected for $i, j \leqq a$ $(i \neq j)$ and $a+1$ is connected with $1, 2, \ldots, b$, where a and b are determined by the unique representation

$$N = \binom{a}{2} + b, \quad 0 \leqq b < a. \tag{1.2}$$

A *quasi-star* S_n^N with N edges is defined as follows: use the unique representation

$$\binom{n}{2} - N = \binom{c}{2} + d, \quad 0 \leqq d < c \tag{1.3}$$

and connect the first $n-c-1$ vertices with every other, connect the vertex $n-c$ with the first $n-d$ vertices.

It is easy to see that $S_n^{\binom{n}{2}-N}$ is the complement graph of C_n^N if we change the order of the vertices. We use the abbreviations

$$C(n, N) = p\left(\begin{matrix}N\\ n\end{matrix}\right), \quad S(n, N) = p(S_n^N). \tag{1.4}$$

Let $G_{l,m}^N$ denote an aritrary bipartite graph with N edges and $l+m$ vertices, where l vertices are coloured red and m pink.

[1] Research of this author was supported by the Deutsche Forschungsgemeinschaft.

[2] This work was done while the author visited the Institute of Mathematical Statistics, University of Göttingen.

THEOREM 1. *Suppose* $1\leqq m$, $N=qm+r$, $0\leqq r<m$. *Then* $\max p\,(G^N_{l,m})$ *is assumed for the graph in which q red vertices are all connected with all the pink ones and one more red vertex is connected with r pink ones.*

THEOREM 2. $f(n,N)=\max\,[C(n,N),\ S(n,N)]$.

THEOREM 3.

$$f(n,N)=\begin{cases} S(n,N) & \text{if} \quad 0\leqq N<\frac{1}{2}\binom{n}{2}-\frac{n}{2} \\ C(n,N) & \text{if} \quad \frac{1}{2}\binom{n}{2}+\frac{n}{2}<N\leqq\binom{n}{2}. \end{cases}$$

Moreover, there are infinitely many n's for which $f(n,N)=S(n,N)$ for *all* $N<\frac{1}{2}\binom{n}{2}$ and $f(n,N)=C(n,N)$ for all $N>\frac{1}{2}\binom{n}{2}$. On the other hand there are infinitely many n's for which this is not true. (For further details see Lemma 8.)

We give two different proofs of Theorem 2. The first is more elegant and is based on Theorem 1.

The second proof is more elaborate, however, it uses techniques which we also use in the proof of Theorem 3, and it is worthwhile knowing that both Theorems can be proved by the same approach. The first proof might be more suited for generalizations of Theorem 2.

2. Proof of Theorem 1

The present proof and also the first proof of Theorem 2 are formulated in terms of vertex-vertex incidence matrices.

For the bipartite graph $G^N_{l,m}$ with l red and m pink vertices and with N edges the matrix $J(G^N_{l,m})$ is defined by

$$J(G^N_{l,m})=(a_{ij})^{i=1,\dots,l}_{j=1,\dots,m}$$

where

$$(2.1)\qquad a_{ij}=\begin{cases}1 & \text{if the } i\text{-th red and } j\text{-th pink vertices are connected in } G^N_{l,m}\\ 0 & \text{otherwise.}\end{cases}$$

If $J^N_{l,m}$ is an $l\times m$ $0-1$-matrix with N 1's $q(J^N_{l,m})$ denotes

$$\sum_{i=1}^{l} r_i^2+\sum_{i=1}^{m} s_i^2,$$

where $r_1,\dots,r_l$ and, $s_1,\dots,s_m$ are the numbers of 1's in the rows and columns, respectively. Since

$$(2.2)\qquad q(J(G^N_{l,m}))=2p(G^N_{l,m})+2N,$$

for our purposes it suffices to maximize the quadratic form $q(J^N_{l,m})$. We need

LEMMA 1. $\max q(J^N_{l,m})$ *is assumed for a matrix with the property: if* $a_{ij}=1$ *and* $i'\leqq i$, $j'\leqq j$, *then* $a_{i'j'}=1$.

PROOF. Suppose $J^N_{l,m}$ maximizes and its rows and columns are ordered such that $r_1 \geqq r_2 \geqq \ldots \geqq r_l$ and $s_1 \geqq s_2 \geqq \ldots \geqq s_m$. Suppose that there are entries $a_{ij}=1$, $a_{ij'}=0$ for $j'<j$ (or $a_{i'j}=0$ for $i'<i$). By exchanging this 1 and 0, $q(J^N_{l,m})$ changes by

$$-s_{j'}^2+(s_j-1)^2+(s_{j'}+1)^2-s_{j'}^2 = 2+2(s_{j'}-s_j) > 0,$$

a contradiction and the lemma is proved.

PROOF OF THEOREM 1. We prove it by induction on $n=l+m$. If $l+m=2$ the statement is trivial. Suppose that $l+m>2$ and also that $J^N_{l,m}$ has the properties described in Lemma 1. We can also assume that $r_1 \geqq s_1$, because otherwise we can exchange the role of rows and columns without changing the total number of rows and columns, because $r_1<s_1 \leqq m$, and we can have 1's only in the first r_2 columns. (We write 0's in the undefined places.)

Notice that we have now 1's only in the submatrix J^* determined by the first r_1 columns and first s_1 rows, and by Lemma 1 the first row of this submatrix contains only 1's. Denoting by $J^{N-r_1}_{s_1-1,r_1}$ the $(s_1-1)\times r_1$-matrix, which is derived form J^* by omitting its first row, we can establish the recursion formula

$$q(J^N_{l,m}) = \sum_{i=1}^{m} s_i^2+\sum_{i=1}^{l} r_i^2 = \sum_{i=1}^{r_1} s_i^2+\sum_{i=1}^{s_1} r_i^2 = \tag{2.3}$$

$$= \left(\sum_{i=1}^{r_1}(s_i-1)^2+\sum_{i=2}^{c_1} r_i^2\right)+\sum_{i=1}^{r_1}(2s_i-1)+r_1^2 = q(J^{N-r_1}_{s_1-1,r_1})+2N-r_1+r_1^2.$$

This means that if we want to maximize $q(J^N_{l,m})$ with fixed r_1 and c_1 then we have to maximize $q(J^{N-r_1}_{s_1-1,r_1})$. Here $s_1-1+r_1<l+m$ and we can use the induction hypothesis. Since $c_1-1<r_1$, $\max q(J^{N-r_1}_{s_1-1,r_1})$ is assumed for a matrix in which the first $u-1$ rows are full with 1's and in the u-th row the first r entries are 1's, where

$$N-r_1 = (u-1)r_1+v, \quad 0 \leqq v < r_1. \tag{2.4}$$

All the other entries are 0's.

We have thus proved that it is sufficient to consider the matrices $J^N_{l,m}$ which have 1's in the first r_1 places of the first u rows and in the first v places in the $(u+1)$-th row, where $u \leqq r_1 \leqq m$ and $N=ur_1+v$, $0 \leqq v<r_1$. Denote these matrices by $J(N, r_1)$. We have only to prove that

$$q\big(J(N, r_1+1)\big) \geqq q\big(J(N, r_1)\big). \tag{2.5}$$

For this we use the equation

$$q\big(J(N, r_1)\big) = u_1^2+v^2+v(u+1)^2+(r_1-v)u^2. \tag{2.6}$$

We distinguish two cases:

1. *Case* $v \geqq u$. Then

$$N = u(r_1+1)+v-u, \quad 0 \leqq v-u < r$$

and by (2.6)

$$q\big(j(N, r_1+1)\big) = u(r_1+1)^2+(v-u)^2+(v-u)(u+1)^2+(r_1-v+u+1)u^2. \tag{2.7}$$

An elementary calculation yields

$$q(J(N, r_1+1)) - q(J(N, r_1)) = 2r_1u - 2vu.$$

Since $r_1 > v$ this proves (2.5) in the first case.

2. *Case* $v < u$. Then $N = (u-1)(r_1+1) + (v-u+r_1+1)$, where $0 \leqq v-u+r_1+1 < < r_1+1$ follows from $u \leqq r_1$ and $v < u$. We can again use (2.6) and get

$$(2.8) \quad q(J(N, r_1+1)) = (u-1)(r_1+1)^2 + (v-u+r_1+1)^2 + (v-u+r_1+1)u^2 + + (u-v)(u-1)^2.$$

Again by an elementary calculation

$$q(J(N, r_1+1)) - q(J(N, r_1)) = 2r_1v - 2uv,$$

which is non-negative because $r_1 \geqq u$. We thus have proved (2.5) in both cases and $q(J(N, r_1))$ is maximal if we choose $r_1 = m$. The Theorem is proved.

3. First proof of Theorem 2

Denote the vertex-vertex incidence matrix of G_n^N by

$$I(G_n^N) = (a_{ij})_{i=1,\dots,n}^{j=1,\dots,n};$$

where

$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are connected} \\ 0 & \text{otherwise.} \end{cases}$$

The $n \times n$-matrix $I(G_n^N)$ is symmetric, has $2N$ 1's and 0's in the diagonal. Also every $0-1$-matrix I_n^N is the incidence matrix of a graph G_n^N. Define

$$(3.1) \quad Q(I_n^N) = \sum_{i=1}^{n} c_i^2,$$

where c_i counts the number of 1's in the i-th row or i-th column. Clearly,

$$(3.2) \quad Q(I(G_n^N)) = 2p(I(G_n^N)) + 2N$$

and it suffices to maximize the expression in (3.1). Again we need an auxiliary result

LEMMA 2. $\max Q(I_n^N)$ *is assumed for a matrix with the property: if* $a_{ij}=1$ *and* $i' \leqq i, j' \leqq j, i' \neq j'$, *then* $a_{i'j'} = 1$.

PROOF. Let the rows and columns of an optimal I_n^N be numbered by $1, 2, \dots, n$ such that $c_1 \geqq c_2 \geqq \dots \geqq c_n$. Now suppose that $a_{ij} = 1$, $i < j$, but $a_{i'j} = 0$ for $i' < i$. By exchanging this 1 and 0 (also symmetrically the corresponding term) we get

$$c_{i'}^* = c_{i'} + 1, \quad c_i^* = c_i - 1, \quad c_t^* = c_t \quad \text{for} \quad t \neq i', i,$$

and

$$\sum_{t=1}^{n} c_t^{*2} - \sum_{t=1}^{n} c_t^2 = 2c_i' - 2c_i + 2 > 0,$$

because $c_i' \geqq c_i$. This contradiction proves the lemma.

We begin now the proof of Theorem 2. Suppose I_n^N has the form described in Lemma 2. Denote the largest index i satisfying $a_{i,i-1}=1$ by $w=w(I_n^N)$. Then

$$(3.3)\qquad \begin{cases} a_{w+1,w} = a_{w,w+1} = 0, & a_{ij} = 0, \quad \text{for} \quad w+1 \leqq i, j. \\ a_{ij} = 1 \quad \text{for} \quad 1 \leqq i, j \leqq w & \text{with} \quad i \neq j. \end{cases}$$

In the class of matrices I_n^N maximizing $Q(I_n^N)$, let the minimal and maximal value of $w(I_n^N)$ be denoted by w_1 and w_2, respectively. We distinguish two cases:

1. *Case* $w_2 \geqq \frac{n}{2}$. Cut the matrix I_n^N, with $w_2 = w(I_n^N)$, after the w_2-th row and w_2-th column. It is separated now into four submatrices A, B, C and D. Here A is a $w_2 \times w_2$ matrix with 0's only in the diagonal. B and C are $w_2 \times (n-w_2)$ and $(n-w_2) \times w_2$ matrices (symmetrical images of each other), respectively, D is a 0-matrix.

Consider the expression

$$(3.4)\quad Q(I_n^N) = \sum_{i=j}^{n} c_i^2 = \sum_{i=1}^{w_2} (c_i - (w_2-1))^2 + 2\sum_{i=1}^{w_2} c_i(w_2-1) - \sum_{i=1}^{w_2} (w_2-1)^2 + \sum_{i=w_2+1}^{n} c_i^2 =$$
$$= \sum_{i=1}^{w_2} (c_i - (w_2-1))^2 + \sum_{i=w_2+1}^{n} c_i^2 + 2N(w_2-1) - 2w_2(w_2-1)^2.$$

B and C are incidence matrices of bipartite graphs and

$$(3.5)\qquad q(B) = q(C) = \sum_{i=1}^{w_2} (c_i - (w_2-1))^2 + \sum_{i=w_2+1}^{n} c_i^2.$$

To maximize $Q(I_n^N)$ for fixed w_2 means therefore simply to maximize $q(B)$. Since $w_2 \geqq n - w_2$ it follows from Theorem 1 that the first column of B contains as many 1's as the number of 1's in B permits. This leads to a contradiction if the first column is full, because $a_{w_2, w_2+1} = 1$ contradicts the definition of w_2. It follows that B contains 1's only in the first column and that this number is smaller than w_2.

C is symmetrical to B. The matrix I_n^N, which consists of A, B, C and D is the incidence matrix of a *quasi-complete graph*.

2. *Case* $w_2 < \frac{n}{2}$. In this case also $w_1 < \frac{n}{2}$. Cut now the matrix I_n^N with $w_1 = w(I_n^N)$ after the first $w_1 - 1$ rows and $w_1 - 1$ columns. We obtain four matrices: A, B, C and D. A is a $(w_1-1) \times (w_1-1)$ matrix with 0's only in the diagonal. B is a $(w_1-1) \times (n-w_1+1)$ matrix, C is the symmetrical image of B. D is a 0-matrix since $a_{w_1, w_1+1} = 0$. We can use (3.4) with $w_1 - 1$ instead of w_2. Then $Q(I_n^N)$ can be maximized by maximizing $q(B)$. Since $w_1 - 1 < n - w_1 + 1$ by Theorem 1, $q(B)$ is maximal if we choose B full with 1's in the first rows. However, by definition of w_1, $a_{w_1-1, w_1} = 1$, that is, the first element of the last row of B is 1. It follows that

the first w_1-2 rows of B are full with 1's, because otherwise we could find a B (and I_n^N) which maximizes $Q(I_n^N)$ and has $w(I_n^N)<w_1$.

Thus we have found an optimal matrix which has 1's in the first w_1-2 rows and columns everywhere except the diagonal, furthermore it has 1's at the beginning of the (w_1-1)-th row and column. The other entries are 0. This is the incidence matrix of a *quasi-star graph*. Case 1 and 2 yield

$$f(n, N) = \max\,(C(n, N),\ S(n, N)).$$

Notice that the proof gives also an interesting property of $S(n, N)$:

$$S(n+1, N) \geqq S\{n, N) \quad \text{if} \quad n \geqq 4 \quad \text{and} \quad N \leqq \frac{1}{2}\binom{n}{2}. \tag{3.6}$$

To see this it suffices to check that $w(S_n^N)<\frac{n+1}{2}$. Suppose the contrary, then

$$2N \geqq w(w-1)+2(n+1-w)(w-2) = -w^2+(2n+5)w-4(n+1) \geqq$$

$$\geqq -\left(\frac{n+1}{2}\right)^2+(2n+5)\frac{n+1}{2}-4(n+1) = \frac{(n+1)(3n-7)}{4} > \frac{n(n-1)}{2}.$$

This contradiction for $n \geqq 4$ establishes (3.6).

4. Comparison of $C(n, N)$ and $S(n, N)$, the proof of Theorem 3

At the first moment one might think that it should be easy to compare $C(n, N)$ and $S(n, N)$ for given n and N. However, the functions are given only in an implicit way by number-theoretical-combinatorial expressions. Also around $N=\frac{1}{2}\binom{n}{2}$ they are very close to each other. Of course it is quite easy to make the comparison if $N \ll \frac{1}{2}\binom{n}{2}$ or $N \gg \frac{1}{2}\binom{n}{2}$, but we would like to consider values of N around $\frac{1}{2}\binom{n}{2}$ as well. We shall need several lemmas, which we now state and prove.

LEMMA 3.

a) $S(n, N) = C\left(n, \binom{n}{2}-N\right)-n\binom{n-1}{2}-4N+2nN$

b) $S(n, N)-C(n, N) = C\left(n, \binom{n}{2}-N\right)-S\left(n, \binom{n}{2}-N\right)$.

PROOF. Since the quasi-star with N edges and the quasi-complete graph with $\binom{n}{2}-N$ edges are complementary to each other (if we change the order of the vertices), it suffices to prove the statements for any pair of complementary graphs. If we denote by $c_1, \ldots, c_n$ the valencies of the first graph, then $\sum_{i=1}^{n} c_i=2N$, $\sum_{i=1}^{n}\binom{c_i}{2}$

is the number of adjacencies and $\sum_{i=1}^{n}\binom{n-c_i-1}{2}$ is the corresponding number for the complement graph. Now

$$\sum_{i=1}^{n}\binom{n-c_i-1}{2} = \frac{1}{2}\sum_{i=1}^{n}(n-c_i-1)(n-c_i-2) =$$

$$= \frac{1}{2}\sum_{i=1}^{n}n^2 - \sum_{i=1}^{n}nc_i - \frac{3}{2}\sum_{i=1}^{n}n + 2\sum_{i=1}^{n}c_i + \sum_{i=1}^{n}1 + \frac{1}{2}\sum_{i=1}^{n}(c_i^2-c_i) =$$

$$= \frac{n^3}{2} - 2Nn - \frac{3}{2}n^2 + 4N + n + \sum_{i=1}^{n}\binom{c_i}{2} = \sum_{i=1}^{n}\binom{c_i}{2} + n\binom{n-1}{2} + 4N - 2nN.$$

b) follows easily from a).

After we know now that one of the functions can be expressed in terms of the other one, we express now $C(n, N)$ as partial sum of an infinite sequence. Define β_{ij} by

(4.1) $$\beta_{ij} = i+j, \quad 0 \leqq j \leqq i.$$

The k-th element of the sequence $\beta_{00}, \beta_{10}, \beta_{20}, \beta_{21}, \beta_{22}, \ldots$ is denoted by α_k.

LEMMA 4. $C(n, N) = \sum_{k=1}^{N}\alpha_k$.

PROOF. We proceed by induction on N. The statement clearly holds for $N=1$. Use the expansion

$$N = \binom{a}{2} + b, \quad 0 \leqq b < a.$$

It is easy to see that $\alpha_N = \beta_{a-1,b-1}$. Recall that the quasi-complete graph with N edges is composed out of a complete graph of a vertices and an additional $(a+1)$-st vertex, which is connected with the first b vertices.

Suppose that $b+1<a$, then the quasi-complete graph with $N+1=\binom{a}{2}+b+1$ $(0\leqq b+1<a)$ has one more edge from the $(a+1)$-st vertex. The number of new adjacencies is $a-1+b$. This is, indeed, equal to the new term $\alpha_{N+1}=\beta_{a-1,b}$. If $b+1=a$, then the quasi-complete graph with $N+1=\binom{a+1}{2}$ edges is simply a complete graph with $a+1$ vertices. The number of new adjacencies is $2a-2=$ $=\beta_{a-1,a-1}$ which again equals α_{N+1}. The proof is complete.

LEMMA 5. a) *$C(n, N) \leqq S(n, N)$ is equivalent to*

(4.1) $$\sum_{k=1}^{N}(\alpha_k + \alpha_{\binom{n}{2}-k+1}) \leqq N(2n-4).$$

b) *If $C(n, N) \leqq S(n, N)$ for some $n(\geqq 4)$, $N\left(\leqq \frac{1}{2}\binom{n}{2}\right)$ then it is also true for $n+1, N$.*

PROOF. By Lemma 3a), $C(n, N) \leqq S(n, N)$ is equivalent to

$$C(n, N)+n\binom{n-1}{2}-C\left(n, \binom{n}{2}-N\right) \leqq N(2n-4). \tag{4.2}$$

Since

$$\sum_{i=1}^{n-2} \sum_{j=0}^{i} \beta_{ij} = n\binom{n-1}{2} \tag{4.3}$$

we get from Lemma 4

$$\binom{n-1}{2}-C\left(n, \binom{n}{2}-N\right) = \sum_{i=1}^{n-2} \sum_{j=0}^{i} \beta_{ij} - \sum_{k=1}^{\binom{n}{2}-N} \alpha_k = \sum_{k=1}^{N} \alpha_{\binom{n}{2}-k+1}$$

and that (4.1) is equivalent to

$$\sum_{k=1}^{N} (\alpha_k + \alpha_{\binom{n}{2}-k+1}) \leqq N(2n-4). \tag{4.4}$$

Obviously b) follows from (3.6) and the fact that $C(n+1, N)=C(n, N)$ for $N \leqq \binom{n}{2}$.

We give now a proof which uses only results of the present section. It is far more complicated than the above argument, but it also shows how the new techniques work, which we need later anyhow.

It is clear from a) that it suffices to prove the inequality

$$\sum_{k=1}^{N} (\alpha_k + \alpha_{\binom{n+1}{2}-k+1}) \leqq N(2n-2)$$

or equivalently that

$$\sum_{k=1}^{N} (\alpha_{\binom{n+1}{2}-k+1} - \alpha_{\binom{n}{2}-k+1}) \leqq 2N. \tag{4.5}$$

We prove it first for some special N's. Suppose

$$N = \binom{n}{2} - \binom{l}{2} \quad \left(\left\{\frac{n+1}{2}\right\} \leqq l \leqq n\right)$$

($\{x\}$ is the smallest integer $\geqq x$). Then

$$\sum_{k=1}^{N} \alpha_{\binom{n}{2}-k+1} = \sum_{i=l-1}^{n-2} \sum_{j=0}^{i} \beta_{ij} = \sum_{i=0}^{n-2} \sum_{j=0}^{i} \beta_{ij} - \sum_{i=0}^{l-2} \sum_{j=0}^{i} \beta_{ij} = n\binom{n-1}{2} - l\binom{l-1}{2},$$

and

$$\sum_{k=1}^{N} \alpha_{\binom{n+1}{2}-k+1} = \sum_{i=l+1}^{n-1} \sum_{j=0}^{i} \beta_{ij} + \sum_{j=n-l}^{l} \beta_{l,j} = \sum_{i=0}^{n-1} \sum_{j=0}^{i} \beta_{ij} - \sum_{i=0}^{l} \sum_{j=0}^{i} \beta_{ij} + \sum_{j=n-1}^{l} \beta_{l,j} =$$

$$= (n+1)\binom{n}{2} - (l+2)\binom{l+1}{2} + (2l-n+1)l + (2l-n+1)\frac{n}{2}.$$

Therefore the left hand side of (4.5) is

$$(n+1)\binom{n}{2}-(l+2)\binom{l+1}{2}+(2l-n+1)\left(l+\frac{n}{2}\right)-n\binom{n-1}{2}+l\binom{l-1}{2},$$

which equals

$$2\left(\binom{n}{2}-\binom{l}{2}\right)=2N.$$

We proved that in the case

$$N=\binom{n}{2}-\binom{l}{2}\qquad\left(\left\{\frac{n+1}{2}\right\}\leqq l\leqq n\right)$$

exact equality holds in (4.5).

We shall now verify that the difference of the sides (right-left) of (4.5) is an increasing function of N in the interval

$$\binom{n}{2}-\binom{l}{2}\leqq N\leqq\binom{n+1}{2}-\binom{l+1}{2}\tag{4.6}$$

and it is decreasing in

$$\binom{n+1}{2}-\binom{l+1}{2}\leqq\binom{n}{2}-\binom{l-1}{2}.\tag{4.7}$$

In the last interval (4.6) the last terms of the left hand side of (4.5) are

$$\alpha_{\binom{n+1}{2}-N+1}=\beta_{l,\binom{n+1}{2}-\binom{l+1}{2}-N}$$

and

$$\alpha_{\binom{n}{2}-N+1}=\beta_{l-2,l-1-N+\binom{n}{2}-\binom{l}{2}}\quad\left(\beta_{l-1,0}\ \text{ if }\ N=\binom{n}{2}-\binom{l}{2}\right).$$

Thus for $N+1$, the new terms are

$$\beta_{l,\binom{n+1}{2}-\binom{l+1}{2}-N-1}\quad\text{and}\quad\beta_{l-2,l-2-N+\binom{n}{2}-\binom{l}{2}}.$$

Their difference is

$$l+\binom{n+1}{2}-\binom{l+1}{2}-N-1-l+2-l+2+N-\binom{n}{2}+\binom{l}{2}=n-2l+3.$$

The right hand side of (4.5) is increased by 2, so the change of the difference of the sides is $2l-n-1$, which is non-negative by the supposition $\left\{\frac{n+1}{2}\right\}\leqq l.$ Similarly if we are in the interval (4.7), the last terms in (4.5) are

$$\alpha_{\binom{n+1}{2}-N+1}=\beta_{l-1,l-N+\binom{n+1}{2}-\binom{l+1}{2}}\quad\left(\beta_{l,0}\ \text{ if }\ N=\binom{n+1}{2}-\binom{l+1}{2}\right)$$

and

$$\alpha_{\binom{n}{2}-N+1}=\beta_{l-2,l-1-N+\binom{n}{2}-\binom{l}{2}}.$$

Thus going to $N+1$, the new terms are

$$\beta_{l-1,l-N+\binom{n+1}{2}-\binom{l+1}{2}+1}\quad\text{and}\quad\beta_{l-2,l-N+\binom{n}{2}-\binom{l}{2}}.$$

Their difference is

$$l-1+l-N+\binom{n+1}{2}-\binom{l+1}{2}+1-l+2-l+N-\binom{n}{2}+\binom{l}{2}=n-l+2.$$

The change of the difference of the sides of (4.5) is $l-n$, non-positive because of $l\leqq n$. This proves the statement of our lemma, if

$$N\leqq\binom{n}{2}-\binom{\left\{\frac{n+1}{2}\right\}}{2}$$

but we need it for $N\leqq\frac{1}{2}\binom{n}{2}$. However

$$\frac{1}{2}\binom{n}{2}\leqq\binom{n}{2}-\binom{\left\{\frac{n+1}{2}\right\}}{2}\quad\text{holds, when}\quad 0\leqq\frac{1}{2}\binom{n}{2}-\binom{\frac{n+2}{2}}{2},$$

that is

$$0\leqq 2n^2-2n-n^2-2n=n^2-4n,$$

which is true for $n\geqq 4$. The lemma is proved.

In particular we know that if $C(n,N)\leqq S(n,N)$ holds for $N\leqq\frac{1}{2}\binom{n}{2}-\frac{n}{2}$, then it is also true for $n+1$. However, in order to prove Theorem 3 by induction on n we have to prove the inequality for $N\leqq\frac{1}{2}\binom{n+1}{2}-\frac{n+1}{2}$. Also in some cases it is true up to $\frac{1}{2}\binom{n+1}{2}$ and we want to consider those cases as well. This is done in 3 more lemmas. The last one, Lemma 8, gives the complete solution.

Instead of $C(n,N)\leqq S(n,N)$ we shall use a further modified version of (4.1). Since

$$\sum_{k=1}^{\binom{n}{2}}\alpha_k=n\binom{n-1}{2},$$

the left hand side of (4.1) is equal to

$$n\binom{n-1}{2}-\sum_{k=N+1}^{\binom{n}{2}-N}\alpha_k.$$

Introducing the notation $r=\frac{1}{2}\binom{n}{2}-N$ we obtain from (4.1) necessarily

$$\sum_{k=\frac{1}{2}\binom{n}{2}-r+1}^{r-\frac{1}{2}\binom{n}{2}}\alpha_k\geqq r(2n-4),\tag{4.8}$$

where r is not necessarily an integer, but there are always $2r$ terms on the left hand side.

One more form of (4.8) is

$$\frac{\sum \alpha_k}{r} \geqq 2n-4. \tag{4.9}$$

Let e und f be defined by

$$\frac{1}{2}\binom{n}{2} = \binom{e}{f} + f \quad (0 \leqq f < e). \tag{4.10}$$

(It is easy to see that this is unique).

LEMMA 6. *Suppose* $f < \frac{e}{2}$ *(in (4.10)) if* $2e+2f \geqq 2n-1$. *Then* $C(n, N) \leqq S(n, N)$ *holds when* $r = \frac{1}{2}\binom{n}{2} - N \leqq e+f-1$.

PROOF. *Case A:* $f > \frac{e}{2}+1$. We want to prove (4.8). The sequence of numbers, we are investigating (when f is an integer) is (starting from left to right and from the middle going backward in the next row).

$$\begin{cases} |e-2, \ldots, 2f-5, \; 2f-4, \ldots, 2e-4| \; e-1, \ldots, \; e+f-2 \\ \underbrace{2f-2, \ldots, e+1}_{-e+2f-2} \; |\underbrace{2e, \ldots, 2f,}_{2e-2f+1} \; \underbrace{2f-1, \ldots, e}_{2f-e}| \; \underbrace{2e-2, \ldots, e+f-1}_{e-f}. \end{cases} \tag{4.11}$$

If f is not an integer, the only difference is, that on the right hand side one number, $e+f-\frac{3}{2}$ stands instead of $e+f-2$ and $e+f-1$.

Observe that in one interval in table (4.11) the sum of the numbers standing under each other is constant. The sums are

$$\underbrace{e+2f-4, \ldots,}_{-e+2f-2} \; \underbrace{2e+2f-4, \ldots,}_{2e-2f+1} \; \underbrace{e+2f-2, \ldots,}_{-e+2f} \; \underbrace{2e+2f-3, \ldots}_{e-f}. \tag{4.12}$$

Subcase 1: Prove the statement for $r = \frac{1}{2}\binom{n}{2} - N \leqq e-f$. Using (4.8) we have to prove that the sum of the last $2r$ terms in (4.11) is not smaller than $r(2n-4)$. The sum of the last $2r$ terms of (4.11) is exactly the sum of the last r terms in (4.12). Since $r \leqq e-f$, they are constant. It is enough to prove that

$$2e+2f-3 \geqq 2n-4. \tag{4.13}$$

Later we need the inequality

$$2e+2f-4 \geqq 2n-4 \tag{4.14}$$

as well, thus we make the calculations here together. From the definition (4.10) of e and f we have

$$n(n-1) = 2e^2-2e+4f. \tag{4.15}$$

Suppose, the converse of (4.13) ((4.14)) is true:

$$n > e+f+\frac{1}{2} \quad (n > e+f).$$

Substituting into (4.15) we obtain

$$2e^2-2e+4f > \left(e+f+\frac{1}{2}\right)\left(e+f-\frac{1}{2}\right)$$

$$(2e^2-2e+4f > (e+f)(e+f-1)).$$

Reordering it:

$$e^2-2e-f(2e+f-4)+\frac{1}{4} > 0 \quad (e^2-e-f(2e+f-5) > 0). \tag{4.16}$$

If $e \geqq 2$ then $2e+f-4$ (if $e \geqq 3$ then $2e+f-5$) is non negative, $f(2e+f-4)$ $(f(2e+f-5))$ is an increasing function of f, so (4.16) remains true if we use the inequality $f \geqq \frac{e}{2}$ (supposition of our lemma):

$$\begin{cases} e^2-2e-\frac{e}{2}\left(2e+\frac{e}{2}-4\right)+\frac{1}{4} = -\frac{e^2}{4}+\frac{1}{4} > 0 \\ \left(e^2-e-\frac{e}{2}\left(2e+\frac{e}{2}-5\right) = -\frac{e^2}{4}+\frac{3e}{2} > 0\right). \end{cases} \tag{4.17}$$

However, these inequalities do not hold when $e \geqq 2$ $(e \geqq 6)$. If $n \geqq 3$ then $e \geqq 2$, thus (4.13) is proved for $n \geqq 3$. The inequality (4.14) is proved only for $e \geqq 6$, that is $n \geqq 9$. For the values $n \leqq 3 < 9$ it is easy to check in Table 1 that the supposition $f \geqq \frac{e}{2}$ of the lemma is not satisfied for $n=3, 4, 6, 7$ and for the remaining values $n=5, 8, 9$ (4.14) holds.

n	$\frac{1}{2}\binom{n}{2}$	e	f	n	$\frac{1}{2}\binom{n}{2}$	e	f
3	1,5	2	0,5	23	126,5	16	6,5
4	3	3	0	24	138	17	2
5	5	3	2	25	150	17	14
6	7,5	4	1,5	26	162,5	18	9,5
7	10,5	5	0,5	27	175,5	19	4,5
8	14	5	4	28	189	19	18
9	18	6	3	29	203	20	13
10	22,5	7	1,5	30	217,5	21	7,5
11	27,5	7	6,5	31	232,5	22	1,5
12	33	8	5	32	248	22	17
13	39	9	3	33	264	23	11
14	45,5	10	0,5	34	280,5	24	4,5
15	52,5	10	7,5	35	297,5	24	21,5
16	60	11	5	36	315	25	15
17	68	12	2	37	333	26	8
18	76,5	12	10,5	38	351,5	27	0,5
19	85,5	13	7,5	39	370,5	27	19,5
20	95	14	4	40	390	28	12
21	105	15	0	41	410	29	4
22	115,5	15	10,5	42	430,5	29	24,5

Table 1

Subcase 2: Prove the statement for $e-f<r\leqq f$. The second interval in (4.12) contains the value $e+2f-2$. If $e+2f-2\geqq 2n-4$, we do not have to prove anything. Otherwise the average $\frac{1}{r}\sum \alpha_k$ (see (4.9)) decreases when we use larger values of r. So, it is sufficient to prove (4.8) for $r=f$. The left hand side of (4.8) is

$$(-e+2f)(e+2f-2)+(e-f)(2e+2f-3) = e^2+2f^2-e-f.$$

(4.8) is equivalent to

$$e^2+2f^2-e-f \geqq f(2n-4).$$

Using $e^2-e=\binom{n}{2}-2f$ we obtain

$$\binom{n}{2}+2f^2-3f \geqq f(2n-4)$$

and this is equivalent to

$$2\left(f-\frac{n}{2}\right)\left(f-\frac{n-1}{2}\right) \geqq 0.$$

This inequality always holds, because f is a half of an integer, it cannot satisfy $\frac{n-1}{2}<f<\frac{n}{2}$.

Subcase 3: $f<r\leqq 2e-f+1$. The new term (see (4.12)) in this interval is $2e+$ $+2f-4$. We have proved (4.14), thus the average $\frac{1}{r}\sum \alpha_k$ is increasing in this interval, consequently (4.9) and (4.8) hold.

Subcase 4: $2e-f+1<r\leqq e+f-1$. If the new term $e+2f-4$ is $\geqq 2n-4$, we are done. In the contrary case the average is decreasing, so it is sufficient to prove it for $r=e+f-1$. The statement means that the average of α_k' s for the 4 intervals is $\geqq 2n-4$ (see (4.9)). We have proved it for 2 intervals, so it is sufficient to prove the same for the two new intervals:

$$(-e+2f-2)(e+2f-4)+(2e+2f-4)(2e-2f+1) \geqq (e-1)(2n-4).$$

This is equivalent to

$$3e^2-2f \geqq 2n(e-1). \tag{4.18}$$

Since $f<e$, we can prove (4.18) with $3e$ instead of $2f$:

$$3e \geqq 2n. \tag{4.19}$$

We know $n(n-1)=2e^2-2e+2f$ and using again $f<e$ we have

$$n(n-1) < 2e^2+2e. \tag{4.20}$$

Suppose (4.19) does not hold: $e<\frac{2n}{3}$, and substitute into (4.20):

$$n(n-1) < \frac{8n^2}{9}+\frac{4n}{3}, \quad \text{or} \quad n-21 < 0,$$

which is a contradiction for $n \geqq 21$. For smaller n's it is easy to check from Table 1 that (4.19) holds for $n=3, 4, 6, 7, 9, 10, 12, 13, \dots, 20$. For $n=5, 8$ and 11 (4.19) is not true, but (4.18) holds. Case A is proved.

Case B: If $\frac{e}{2} \leqq f \leqq \frac{e}{2}+1$, then (4.11) and (4.12) have a slightly different form:

$$(4.11') \qquad \begin{cases} |e-2, \dots, 2e-4| \; e-1, \dots, e+f-2 \\ |2e, \dots, e+2f-2, \dots, 2f, \; 2f-1, \dots, e| \; 2e-2, \dots, e+f-1, \end{cases}$$

$$(4.12') \qquad \underbrace{2e+2f-4, \dots}_{e-1}, \; \underbrace{e+2f-2, \dots}_{-e+2f}, \; \underbrace{2e+2f-3, \dots}_{e-f}.$$

So, in this case we consider only 3 intervals. The first two are exactly like in Case A. The length of the third is different, but we did not use it. The lemma is proved.

LEMMA 7. *Suppose* $f < \frac{e}{2}$ *(in (4.10)). If* $2e+2f \geqq 2n-1$, *then* $C(n, N) \leqq$ $\leqq S(n, N)$ *holds when*

$$r = \frac{1}{2}\binom{n}{2} - N \leqq 2e-f+1.$$

If $2e+2f < 2n-1$, *then* $C(n, N) \leqq S(n, N)$ *holds when*

$$\min\left(\frac{n}{2}, e-f\right) \leqq r \leqq \begin{cases} 2e+f-3 & (f<2) \\ 2e-f+1 & (f \geqq 2) \end{cases}$$

and the opposite inequality holds when $0 < r = f$. *The inequality* $C(n, N) \leqq S(n, N)$ *changes its direction only once in this interval.*

PROOF. *Case A:* $f \geqq 2$. The sequence of numbers, we are investigating now, have the following form:

$$(4.21) \begin{cases} \dots, e+2f-7, \dots, 2e-6 \,|\, e-2, \dots, e+2f-3, \dots, 2e-4 \,|\, e-1, \dots, e+f-2 \\ \underbrace{|2e, \dots, e+2f-1}_{e-2f+2}, \; \underbrace{e+2f-2, \dots, e|}_{2f-1} \; \underbrace{2e-2, \dots, e+2f-1}_{e-2f}, \; \underbrace{e+2f-2, \dots, e+f-1}_{f}. \end{cases}$$

Observe that in one interval in table (4.21) the sum of the numbers standing under each other is constant. The sums are

$$(4.22) \qquad \underbrace{3e+2f-7, \dots}_{e-2f+2}, \; \underbrace{2e+2f-4, \dots}_{2f-1}, \; \underbrace{3e+2f-5}_{e-2f}, \; \underbrace{\dots, 2e+2f-3, \dots}_{f}.$$

Subcase 1. If $2e+2f-3 \geqq 2n-4$, then (4.8) holds for $0 < r \leqq f$, otherwise it does not hold. This proves the lemma for the first interval.

Subcase 2. Consider now the interval $f < r \leqq e-f$. If $2e+2f-3 \geqq 2n-4$ is true, $3e+2f-5 \geqq 2n-4$ is also true whenever $e \geqq 2$ (that is, $n \geqq 3$). Then (4.8) holds. If $2e+2f-3 < 2n-4$ holds, then first we prove that (4.8) holds for $r=e-f$. We have to prove

$$(e-2f)(3e+2f-5)+f(2e+2f-3) \geqq (e-f)(2n-4),$$

that is,

(4.23) $$3e^2-2f^2-2ef-e+3f \geqq 2n(e-f).$$

We use $e^2=\binom{n}{2}+e-2f$:

$$3\binom{n}{2}-2f^2-2ef+2e-3f \geqq 2n(e-f).$$

By the suppositions $2e+2f \leqq 2n-2$, so it is enough to prove this last inequality after substituting $-f(2e+2f)$ by $-f(2n-2)$:

$$3\binom{n}{2}-2nf+2f+2e-3f \geqq 2n(e-f)$$

or

$$3\binom{n}{2}+2e-f \geqq 2ne.$$

Since $f<e$, we can write

(4.24) $$3\binom{n}{2}+e \geqq 2ne.$$

We prove it in an indirect way. Suppose the contrary, i.e.

(4.25) $$e > \frac{3\binom{n}{2}}{2n-1}$$

and use $\binom{n}{2}=e^2-e+2f \geqq e^2-e$. As the right hand side of this inequality is increasing in e, it follows

$$\binom{n}{2} > \frac{3\binom{n}{2}}{2n-1}\left(\frac{3\binom{n}{2}}{2n-1}-1\right)$$

by (4.25). Reordering it, we obtain $0>n^2-13n+4$. This is a contradiction for $n \geqq 13$. For smaller n's: (4.26) holds (see Table 1) when $n=5, 6, 8, 9, 10, 11, 12$. For $n=3, 4, 7$ (4.24) is not true, but (4.23) is true.

Subcase 2a. If $\frac{n}{2}<e-f$, we have to prove that the inequality (4.8) changes its direction earlier than $e-f$, at $\frac{n}{2}$. It is enough to prove that it holds for $r=\frac{n}{2}$. In other words:

$$\left(\frac{n}{2}-f\right)(3e+2f-5)+f(2e+2f-3) \geqq (2n-4)\frac{n}{2}$$

has to be proved, or equivalently

(4.26) $$3ne+2nf-2ef+4f \geqq 2n^2+n.$$

It is sufficient to prove $3ne \geqq 2n^2+n$ (we omitted a positive number, as $n \geqq e$), or

(4.27) $$3e \geqq 2n+1.$$

We verify it in an indirect way. Suppose $e<\frac{2n+1}{3}$, and substitute into the inequality

$$n^2-n = 2e^2-2e+4f < 2e^2+2e = 2e(e+1),$$

$$n^2-n < 2\,\frac{2n+1}{3}\cdot\frac{2n+4}{3},$$

or equivalently $n^2-29n-8<0$ which is a contradiction if $n\geqq 30$. For smaller n's: (4.27) holds (see Table 1) when $n=4, 7, 10, 13, 14, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29$. The remaining n's do not belong to this case $\left(f<\frac{e}{2},\ \frac{n}{2}<e-f\right)$.

Subcase 3: $e-f<r\leqq e+f-1$. If $2e+2f-4\geqq 2n-4$, the averages do not decrease in this interval, and we are done. If $2e+2f-4<2n-4$, it is enough to investigate $r=e+f-1$, the average will be the smallest here. We have to prove that the sum in the last 3 small intervals is $\geqq(e+f-1)\ (2n-4)$. The 3^d interval gives

$$(2f-1)(2e+2f-4) = 4ef+4f^2-2e-10f+4.$$

Adding to the sum of the 2 previous ones (see (4.23)):

$$3e^2+2f^2+2ef-7e-3f+4 \geqq (e+f-1)(2n-4)$$

or equivalently

$$3e^2-3e \geqq 2en-2n+f(2n-2f-2e-1). \tag{4.28}$$

It is easy to see, that

$$f(2n-2f-2e-1) = \frac{1}{2}\,2f(2n-2f-2e-1) \leqq \frac{1}{2}\left(\frac{2n-2e-1}{2}\right)^2 =$$

$$= \frac{n^2}{2}+\frac{e^2}{2}+\frac{1}{8}-ne-\frac{n}{2}+\frac{e}{2}.$$

Substitute it into (4.28)

$$6e^2-6e \geqq 4en-4n+n^2+e^2+\frac{1}{4}-2ne-n+e$$

or equivalently

$$-n^2+5e^2-2en-7e+5n-\frac{1}{4} \geqq 0.$$

Use $e^2=\binom{n}{2}+e-2f$:

$$3n^2-4en-4e+5n-20f-\frac{1}{2} \geqq 0. \tag{4.29}$$

Since $2f<e$:

$$3n^2-4en-14e+5n-\frac{1}{2} \geqq 0. \tag{4.30}$$

From $\binom{n}{2}=e^2-e+2f$ we obtain

$$e=\frac{1+\sqrt{1-8f+4\binom{n}{2}}}{2}<\frac{1+\sqrt{2n^2}}{2}=\frac{1+n\sqrt{2}}{2}.$$

Substitute it into

$$(3-2\sqrt{2})n^2-(7\sqrt{2}-3)n-7-\frac{1}{2}\geqq 0. \tag{4.30}$$

This holds for $n\geqq 42$. An ugly compution shows, that for $3\leqq n\leqq 41$, $n\neq 6$, either (4.29) holds or $f\geqq\frac{e}{2}$, that is, not our case. For $n=6$ (4.28) holds.

Subcase 4: $e+f-1<r\leqq 2e-f+1$. If

$$3e+2f-7\geqq 2n-4, \tag{4.31}$$

then the new terms do not decrease the average $\frac{1}{r}\sum\alpha_k$. Use the assumption $f\geqq 2$ of Case A:

$$3e+1\geqq 2n. \tag{4.32}$$

This is weaker than (4.29), which was proved for $n\geqq 21$ and was checked for $n=3, 4, 6, 7, 9, 10, 12, 13, \ldots, 20$. For $n=5, 8$, and 11 (4.32) holds.

Case B: $0<f<2$. (4.21) and (4.22) have a slightly modified form

$$\text{(4.21')}\quad\begin{cases}\ldots|e-3,\ldots,2e-6|\;e-2,\ldots,e+2f-4,\;e+2f-3-4|e-1,\ldots,e+f-2\\ |\underbrace{2e,\ldots,e+2f-1}_{e-2},\;\underbrace{e+2f-2,\ldots,e}_{2f-1}|\;\underbrace{2e-2,\ldots,e+2f-1}_{e-2f},\;\underbrace{e+2f-2,\ldots,e+f-1}_{f}\end{cases}$$

$$\text{(4.22')}\quad\underbrace{3e+2f-7,\ldots}_{e-2},\;\underbrace{2e+2f-4,\ldots}_{2f-1},\;\underbrace{3e+2f-5,\ldots}_{e-2f},\;\underbrace{2e+2f-3,\ldots}_{f}.$$

The only change that the 4th interval is shorter, but we did not use its length. The cases $n=3$ and 7 can be done by an easy computation.

Case C: $f=0$.

$$\text{(4.21'')}\quad\begin{cases}\ldots,2e-8|\;e-3,\ldots,2e-7,\;2e-6|\;e-2,\ldots,2e-4|\\ |2e,\;2e-1,\;2e-2,\;2e-3,\;\underbrace{2e-4,\ldots,e}_{e-3}|\;\underbrace{2e-2}_{1},\;\underbrace{2e-3,\ldots,e-1}_{e-1}\end{cases}$$

$$\text{(4.22'')}\quad\underbrace{3e-7,\ldots}_{e-3},\;\underbrace{4e-8,\ldots}_{1},\;\underbrace{3e-5,\ldots}_{e-1}.$$

We prove that all these numbers are $\geqq 2n-4$. It is enough to prove that $3e-7\geqq$ $\geqq 2n-4$, that is $3e-3\geqq 2n$. We prove it in an indirect way. Suppose $e<\frac{2n+3}{3}$

and use $n(n-1)=2e(e-1)$:

$$n(n-1)<2\frac{2n+3}{3}\cdot\frac{2n}{3},$$

or equivalently $n-21<0$. This is a contradiction if $n\geqq 21$. There is only one case, when $n<21$ and $f=0$ (see Table 1), namely when $n=4$. It is easy to check, that the statement holds for $n=4$.

LEMMA 8. *If* $f\geqq\frac{e}{2}$ *in* (4.10), *then*

$$C(n,N)\leqq S(n,N)\quad \textit{for}\quad 0\leqq N\leqq\frac{1}{2}\binom{n}{2} \tag{4.33}$$

and

$$C(n,N)\geqq S(n,N)\quad \textit{for}\quad \frac{1}{2}\binom{n}{2}\leqq N\leqq\binom{n}{2}. \tag{4.34}$$

If $f<\frac{e}{2}$, *but* $2e+2f\geqq 2n-1$, *then* (4.33) *and* (4.34) *hold, again. If* $f<\frac{e}{2}$, $2e+2f<$ $<2n-1$ *then there is an* R *such that* $f\leqq R\leqq\min\left(\frac{n}{2},\ e-f\right)$, *and*

$$C(n,N)\leqq S(n,N)\quad \textit{for}\quad 0\leqq N\leqq\frac{1}{2}\binom{n}{2}-R$$

$$C(n,N)\geqq S(n,N)\quad \textit{for}\quad \frac{1}{2}\binom{n}{2}-R\leqq N\leqq\frac{1}{2}\binom{n}{2}$$

$$C(n,N)\leqq S(n,N)\quad \textit{for}\quad \frac{1}{2}\binom{n}{2}\leqq N\leqq\frac{1}{2}\binom{n}{2}+R$$

$$C(n,N)\geqq S(n,N)\quad \textit{for}\quad \frac{1}{2}\binom{n}{2}+R\leqq N\leqq\binom{n}{2}.$$

PROOF. We use induction over n. For $n=3$ the statements are true. Let $n>3$. Suppose the statement is true for $n-1$, we want to prove it for n. According to the induction hypothesis $C(n-1,N)\leqq S(n,N)$ if

$$0\leqq N\leqq\frac{1}{2}\binom{n-1}{2}-\frac{n-1}{2}. \tag{4.35}$$

By Lemma 5

$$C(n,N)\leqq S(n,N) \tag{4.36}$$

follows under the condition (4.35). From Lemma 6 we know that (4.36) holds when

$$\frac{1}{2}\binom{n}{2}-z\leqq N\leqq\frac{1}{2}\binom{n}{2}-R, \tag{4.37}$$

where

(4.38)
$$z=\begin{cases} e+f-1 & \text{if } \frac{e}{2}\leqq f \\ 2e-f+1 & \text{if } 2\leqq f<\frac{e}{2} \\ 2e+f-3 & \text{if } 0\leqq f<2 \end{cases}$$

and $R=0$ if $\frac{e}{2}\leqq f$,

$$f\leqq R\leqq \min\left(\frac{n}{2}, e-f\right) \quad \text{if } \frac{e}{2}>f.$$

If we are able to prove that

(4.39)
$$\frac{1}{2}\binom{n}{2}-z\leqq\frac{1}{2}\binom{n-1}{2}-\frac{n-1}{2}$$

(see (4.35) and (4.37)), then (4.36) holds under the condition

$$0\leqq N\leqq\frac{1}{2}\binom{n}{2}-R.$$

(4.39) is equivalent to

(4.40)
$$n-1\leqq z.$$

Case A: $\frac{e}{2}\leqq f$. Then $z=e+f-1$. Assume the opposite of (4.40) holds: $n-1>e+f-1$ or $n>e+f$. Substitute it into the equality $n(n-1)=2e^2-2e+4f$:

$$(e+f)(e+f-1)<2e^2-2e+4f,$$

which is equivalent to

(4.41)
$$e^2-f^2-2ef-e+5f>0.$$

If $f+2e-5\geqq 0$ (if $e\geqq 2$), then

$$f(f+2e-5)\geqq\frac{e}{2}(f+2e-5)\geqq\frac{e}{2}\left(\frac{e}{2}+2e-5\right)=\frac{5e^2}{4}-\frac{5e}{2},$$

(4.41) results in $-e^2+6e>0$. This is a contradiction if $e\geqq 6$, that is, if $n\geqq 9$. For smaller n's either $\frac{e}{2}>f$ or (4.40) holds.

Case B: $2\leqq f<\frac{e}{2}$. We have to prove $n-1\leqq 2e-f+1$ or $2n\leqq 3e+4$. Use

$$\frac{n-1}{2}<\sqrt{\binom{n}{2}}=\sqrt{e^2-e+2f}\leqq e:$$

$$2n\leqq\frac{3n}{\sqrt{2}}-\frac{3}{\sqrt{2}}+4,$$

which holds for $n\geqq 0$.

Case C: $0 \leqq f < 2$. We have to prove $n-1 \leqq 2e+f-3$ or

$$n \leqq 2e-2. \tag{4.42}$$

Use $\frac{n-1}{\sqrt{2}} \leqq e$: $n \leqq \frac{2n-2}{\sqrt{2}} - 2$. This holds for $n \geqq 9$. For $n=4, 6$ and 7 ($n=5$ and 8 do not belong to this case) (4.42) holds. The lemma is proved. Lemma 8 completely proves our Theorem 3.

5. The second proof of Theorem 2

We use the results of Section 3, in particular Lemma 8, and its methods and notation. First we have to prove another inequality:

LEMMA 9. *$C(n, N+n) - C(n, N) \geqq 2N + \binom{n}{2}$ if*

$$\binom{\left[\frac{n}{2}\right]}{2} \leqq N \leqq \binom{n}{2}.$$

PROOF. We start with the case $N = \binom{l}{2}$ $\left(\left[\frac{n}{2}\right] \leqq 1 \leqq n\right)$. Then the statement holds with equality. The difference $C(n, N+n) - C(n, N)$ is the sum of the following terms (see Lemma 4):

$$\beta_{l-1,0}, \ldots, \beta_{l-1,l-1}, \beta_{l,0}, \ldots, \beta_{l,n-l-1}$$

(here we used $1 \leqq n$ and $n \leqq 2l+1$). The sum of these terms is

$$\sum_{i=l-1}^{2l-2} i + \sum_{i=l}^{n-1} i = 3\binom{l}{2} + (n-l)\frac{n+l-1}{2} = 2\binom{l}{2} + \binom{n}{2}.$$

The desired equality is proved. Now we prove that the function

$$C(n, N+n) - C(n, N) - 2N - \binom{n}{2} \tag{5.1}$$

is increasing in the interval

$$\binom{l}{2} \leqq N \leqq \binom{l}{2} + 2l + 1 - n$$

and it is decreasing in

$$\binom{l}{2} + 2l + 1 - n \leqq N \leqq \binom{l+1}{2} \quad \left(\left[\frac{n}{2}\right] \leqq l < n\right).$$

This proves the statement, since (5.1) is 0 for $N = \binom{l}{2}$ and $N = \binom{l+1}{2}$. If

$$\binom{l}{2} \leqq m < \binom{l}{2} + 2l + 1 - n,$$

then the difference $C(n, N+n)-C(n, N)$ is the sum of the terms

$$\beta_{l-1,N-\binom{l}{2}}, \dots, \beta_{l-1,l-1}, \beta_{l,0}, \dots, \beta_{l,n-l-1+N-\binom{l}{2}}.$$

Here

$$0 \leqq n-1+N-\binom{l}{2} \quad \left(\text{because of } l < n, \binom{l}{2} \leqq N\right)$$

and

$$n-l-1+N-\binom{l}{2} < l.$$

If we change N to $N+1$, a new term comes in: $\beta_{l,n-l+N\binom{l}{2}}$, and $\beta_{l-1,N-\binom{l}{2}}$ wil be omitted. The change is

$$n+N-\binom{l}{2}-l+1-N+\binom{l}{2} = n-l+1.$$

However, in (5.1) $2N$ changes to $2(N+1)$. So, the total change is $n-l-1 \geqq 0$, and the function (5.1) is increasing. On the other hand, if

$$\binom{l}{2}+2l+1-n \leqq N < \binom{l+1}{2},$$

then the difference $C(n, N+n)-C(n, N)$ is the sum of the terms

$$\beta_{l-1,N-\binom{l}{2}}, \dots, \beta_{l-1,l-1}, \beta_{l,0}, \dots, \beta_{l,l}, \beta_{l+1,0}, \dots, \beta_{l+1,n+2l+N-\binom{l}{2}},$$

where

$$-1 \leqq n-2l-2+N-\binom{l}{2} < l+1$$

(using $N < \left(\frac{l+1}{2}\right)$ and $n \leqq 2l+1$, if $=-1$, $b_{l,l}$ is the last term). Changing N into $N+1$, the sum changes with

$$\beta_{l+1,n-2l-1+N-\binom{l}{2}}-\beta_{l-1,N-\binom{l}{2}} = n-2l+1,$$

and (5.1) changes with $n-2l-1 \leqq 0$. The function is decreasing in this interval. The lemma is proved.

LEMMA 10.

(5.2) $$f(n, N) = \max\left(f(n-1, N), f(n-1, N-n+1)+\binom{n-1}{2}+2(N-n+1)\right)$$

if $n-1 \leqq N \leqq \binom{n-1}{2}$. *(Otherwise only the defined term is considered on the right hand side.)*

PROOF. Consider an optimal graph. There are two possibilities: a) either each vertex is contained in at least one edge b) or not.

In case b) our graph must be optimal also for $n-1$ vertices, so $f(n, N) = f(n-1, N)$. In Case a) there is an edge containing the last vertex, thus by Lemma 1, the first vertex is connected with every vertex. Consequently, at the first point there are $\binom{n-1}{2}$ adjacencies. The remaining $N-n+1$ edges have one adjacency at their both ends with edges going to the first vertex. This is $2(N-n+1)$. What remained is the number of adjacencies among edges not containing the first vertex. They must form an optimal configuration on $n-1$ points. Thus in case a) the number of adjacencies is

$$f(n-1, N-n+1)+\binom{n-1}{2}+2(N-n+1).$$

The lemma is proved.

PROOF OF THEOREM 3. We prove that $f(n, N)=C(n, N)$ or $S(n, N)$ by induction on n. The exact result is given by Lemma 8. This is true for $n=2, 3$. Suppose it is also true for $n-1$ (with any N) and let us prove it for n.

Case A. Suppose $f(n-1, N)$ and $f(n-1, N-n+1)$ are both assumed for quasi-complete graphs. Since $N \leqq \binom{n}{2}$, also $N-n+1 \leqq \binom{n-1}{2}$.

Subcase 1. If additionally

$$\binom{\left[\frac{n-1}{2}\right]}{2} \leqq N-n+1 \leqq \binom{n-1}{2},$$

then we can use Lemma 9:

$$C(n-1, N)-C(n-1, N-n+1) \geqq \binom{n-1}{2}+2(N-n+1).$$

That is the first term under the max in (5.2) is $\geqq$ than the other one. Consequently $f(n, N)$ is assumed for the quasi-complete graph.

Subcase 2: $N-n+1<\binom{\left[\frac{n-1}{2}\right]}{2}$. In this case we shall prove that $C(n-1, N-n+1) \leqq S(n-1, N-n+1)$. Indeed, it follows from Lemma 8, when

$$\binom{\left[\frac{n-1}{2}\right]}{2}-1 \leqq \frac{1}{2}\binom{n-1}{2}-\frac{n-1}{2},$$

that is always for $n \geqq 4$. It means that in Subcase 2, $f(n-1, N-n+1)$ is assumed by the quasi-star and it belongs to the next case.

Case B: Suppose $f(n-1, N)$ is assumed for the quasi-complete graph and $f(n-1, N-n+1)$ for the quasi-star. It is easy to see that

$$S(n-1, N-n+1)+\binom{n-1}{2}+2(N-n+1)=S(n, N).$$

In (5.2) we have $\max(C(n-1, N), S(n, m))$, we obtain always either a quasi-star or a quasi-complete graph.

Case C: $f(n-1, N)$ is assumed strictly for the quasi-star and $f(n-1, N-n+1)$ strictly for the quasi-complete graph. It means that

$$C(n-1, N) < S(n-1, N) \tag{5.3}$$

and

$$C(n-1, N-n+1) > S(n-1, N-n+1). \tag{5.4}$$

By Lemma 8 (5.3) results in

$$N < \frac{1}{2}\binom{n-1}{2} + \frac{n-1}{2}, \tag{5.5}$$

and (5.4) results in

$$\frac{1}{2}\binom{n-1}{2} - \frac{n-1}{2} < N-n+1. \tag{5.6}$$

But (5.5) and (5.6) contradict each other.

Case D: $f(n-1, N)$ is assumed strictly for the quasi-star and $f(n-1, N-n+1)$ (not necessarily strictly) also for the quasi-star. We have again (5.3) and (5.5). We shall prove that in (5.2) the second term under the maximum is larger, that is, $S(n-1, N) \leqq S(n, N)$. Rewrite this inequality using Lemma 3:

$$C\left(n-1, \binom{n-1}{2}-N\right) - (n-1)\binom{n-2}{2} - 4N + 2(n-1)N \leqq$$

$$\leqq C\left(n, \binom{n}{2}-N\right) - n\binom{n-1}{2} - 4N + 2nN,$$

or

$$C\left(n-1, \binom{n}{2}-N\right) - C\left(n-1, \binom{n-1}{2}-N\right) \geqq n\binom{n-1}{2} - (n-1)\binom{n-2}{2} - 2N =$$

$$= \binom{n-1}{2} + 2\left(\binom{n-1}{2} - N\right).$$

By Lemma 10 this holds when

$$\binom{\left[\frac{n-1}{2}\right]}{2} \leqq \binom{n-1}{2} - m \leqq \binom{n-1}{2}$$

or

$$0 \leqq m \leqq \binom{n-1}{2} - \binom{\left[\frac{n-1}{2}\right]}{2}.$$

However, this follows from (1.2) when

$$\frac{1}{2}\binom{n-1}{2} + \frac{n-1}{2} \leqq \binom{n-1}{2} - \binom{\left[\frac{n-1}{2}\right]}{2} + 1$$

holds, that is always if $n \geqq 4$. The theorem is finally proved.

6. Open questions

1. Some strange number-theoretical combinatorial questions arise. What is the relative density of the numbers n for which $R=0$ (see Lemma 8)? What is the distribution of $\frac{R}{n}$?

2. We started to think about the next problem. N $\left(0 \leqq N \leqq \binom{n}{3}\right)$ different 3-types are given on an n-set. What is the maximal number of pairs of 3-tuples having 2 elements in common. By easy symmetry-arguments it is enough to consider the case $0 \leqq N \leqq \frac{1}{2}\binom{n}{3}$. We conjecture that in this interval the maximum is assumed for a quasi-complete 3-graph, or a quasi-3-star on some $m \leqq n$ elements. Recently Vera T. Sós and M. Simonovits have some results in connection with this problem.

Reference

[1] Moshe Katz, Rearrangements of (0—1) matrices, *Israel J. Math.*, **9** (1971). 53—72.

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST, REÁLTANODA U. 13—15.

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THE ISOMORPHISMS OF THE CATEGORY OF UNIFORM SPACES AND RELATED CATEGORIES

By

E. MAKAI, Jr. (Budapest)

In this paper we determine the isomorphisms of the categories of uniform spaces, proximity spaces and all full embeddings of the categories of several classes of topological spaces (corresponding to usual separation axioms) into themselves. In the lemmas we investigate the relation of the forgetful functors to concrete categories and to functors between concrete categories, in a general setting. To finish we mention several unsolved problems.

Uniform and proximity spaces are not assumed to be separated. Unif denotes the category of uniform spaces, HUnif denotes that of separated uniform spaces.

THEOREM. *Every isomorphism i of* Unif *onto itself is of the following form: for each uniform space X there is an isomorphism $i_X: X \to i(X)$, and for each map $f: X \to Y$ the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} \\ i(X) & \xrightarrow{i(f)} & i(Y) \end{array}$$

(i.e. i is naturally isomorphic to the identity functor).

Before the proof we recall the definitions of the concepts to be used later and remind some facts about them. Let φ be a morphism in a category. If $\varphi: A \to B$, we write $A = D(\varphi)$, $B = R(\varphi)$. φ is called a *constant map* (cf. [3], 14.1.1) if, for every object X, $|\varphi \operatorname{Hom}(X, D(\varphi))| \leqq 1$. φ is called an *extremal monomorphism* (cf. [3], 7.1.1) if φ is a monomorphism, and $\varphi = \psi e$, e is an epimorphism imply e is an isomorphism. φ is called a *pure epimorphism* (cf. [1], p. 7) if φ is an epimorphism, and $\varphi = m\chi$, m is a monomorphism imply χ is an epimorphism. φ is called a *copure monomorphism* (cf. [1], p. 7) if φ is a monomorphism, and $\varphi = \psi e$, e is a pure epimorphism imply e is an isomorphism. In Unif the first of these concepts coincides with constant maps in the usual sense (cf. [3], 14.1.3), in HUnif the following concepts coincide with closed embeddings (by [3], 15.2.3), onto maps (cf. [1], p. 7) and embeddings (cf. [1], p. 7), respectively. For separated uniform spaces p denotes precompact reflection ([2], II); this corresponds to the generated proximity in a one-to-one way. A separated uniform space X is called *proximally fine* (cf. e.g. [9], pp. 141—142) if for each separated uniform space Y the function-sets $\operatorname{Hom}(X, Y)$ and $\operatorname{Hom}(pX, pY)$ coincide. Every metric uniform space is proximally fine, and every proximally fine separated uniform space X has the finest uniformity on the underlying set whose precompact reflection is pX (cf. [2], II. 38 and the remark following it).

First we relate the constant maps in a concrete category to the maps constant in set-theoretic sense, and to the points of the underlying sets.

LEMMA 1. *Let $\mathscr{C}$ be a concrete category, with forgetful functor F into the category of sets. Let*

(1) *for* $\forall C\in\mathscr{C}$ *the maps in* $\mathscr{C}$ *to* C *separate the points of* FC *in the following sense:* $\forall c_1, c_2\in FC,\quad c_1\neq c_2\quad \exists C'\in\mathscr{C},\quad \exists\varphi_1,\varphi_2\colon C'\to C,\quad \exists c'\in FC',\quad (F\varphi_1)(c')=c_1,$ $(F\varphi_2)(c')=c_2$.

Then $F\varphi$ *is constant in set-theoretic sense iff* φ *is a constant map in* $\mathscr{C}$. (In this case $v_F(\varphi)$ denotes the constant value of $F\varphi$ (if $D(F\varphi)\neq\emptyset$).)

Let besides (1) *hold*

(2) $\forall C\in\mathscr{C}\ \forall c\in FC\ \exists C'\in\mathscr{C}, \exists\varphi\colon C'\to C$, *$F\varphi$ is constant on FC' and has value c.*

Then for each $C\in\mathscr{C}$ v_F *is onto from the class of constant maps* φ *with* $R(\varphi)=C$, $D(F\varphi)\neq\emptyset$.

Let (1) *hold. If* φ_1, φ_2 *are constant maps in* $\mathscr{C}$, $D(F\varphi_1), D(F\varphi_2)\neq\emptyset$, *and* $\exists\psi$, $\varphi_1=\varphi_2\psi$, *then* $v_F(\varphi_1)=v_F(\varphi_2)$. *Let* $\sim$ *be the smallest equivalence relation on the class of the above* φ*-s containing the above pairs* (φ_1, φ_2). *Let*

(3) $v_F(\varphi_1) = v_F(\varphi_2) \quad\Rightarrow\quad \varphi_1\sim\varphi_2$.

Then v_F *is one-to-one from the classes of* $\sim$*-equivalence of the above maps with* $R(\varphi)=C$ *to the points of the set* FC.

Proof is evident.

COROLLARY. *Let* $\mathscr{C}$ *be a concrete category, with forgetful functor* F. *If* $\forall C\in\mathscr{C}\ \forall c\in FC\ \exists\varphi\colon C\to C$, *$F\varphi$ is constant and has value c, then the statements of the lemma are valid(and* $\varphi_1\sim\varphi_2$ *iff* $R(\varphi_1)=R(\varphi_2)$ *and* $\exists\psi\colon R(\varphi_1)\to R(\varphi_1)$, ψ *is a constant map,* $\exists\psi_1,\psi_2, \varphi_1=\psi\psi_1,\ \varphi_2=\psi\psi_2$; *especially any such* φ_1 *is of the form* $\psi\psi_1$).

Now we investigate the relation of the functors between concrete categories to the forgetful functors.

LEMMA 2. *Let* $\mathscr{C}$, $\mathscr{D}$ *be concrete categories with forgetful functors* F, G *into the category of sets, both satisfying* (1), $\mathscr{C}$, F *satisfying* (2) *and* (3). *Let further*

(4) $C\in\mathscr{C},\quad FC\neq\emptyset,\quad |\mathrm{Hom}\,(C, C)|=1\Rightarrow\exists C'\in\mathscr{C},\quad C'\neq C,\quad \mathrm{Hom}\,(C', C)\neq\emptyset\vee$ $\vee|\mathrm{Hom}\,(C, C')|>1$, *and similarly for* $\mathscr{D}$, G.

Let H *be a functor from* $\mathscr{C}$ *to* $\mathscr{D}$, *which satisfies*

(5) $\{|\mathrm{Hom}\,(C, C)|>1\vee\exists C'\in\mathscr{C},\quad C'\neq C,\quad [\mathrm{Hom}\,(C', C)\neq\emptyset\vee|\mathrm{Hom}\,(C, C')|>$ $>1]\}\Rightarrow\{|\mathrm{Hom}\,(HC, HC)|>1\vee\exists D'\in\mathscr{D},\quad D'\neq HC,\quad [\mathrm{Hom}\,(D', HC)\neq\emptyset\vee$ $\vee|\mathrm{Hom}\,(HC, D')|>1]\}$ *(e.g. H is an embedding) and*

(6) *φ is a map in $\mathscr{C}$, $H\varphi$ is not a constant map in $\mathscr{D} \Rightarrow \exists \psi_1, \psi_2$ in $\mathscr{C}$, $D(\psi_1)=D(\psi_2)$, $(H\varphi)(H\psi_1) \neq (H\varphi)(H\psi_2)$ (or at least $\varphi\psi_1 \neq \varphi\psi_2$).*

Then there exists a natural transformation $\{h_C\}_{C \in \mathscr{C}}$ from F to GH.
If beside the above conditions (3) *holds for $\mathscr{D}, G$ and*

(7) *H is one-to-one from the $\sim$-equivalence classes in $\mathscr{C}$ with fixed range to those in $\mathscr{D}$,*

then also h_C is one-to-one from FC to GHC.
If beside the above conditions alternatively (2) *holds for $\mathscr{D}, G$ and*

(8) *H is onto from the $\sim$-equivalence classes in $\mathscr{C}$ with fixed range C to those in $\mathscr{D}$ with range HC*

and

(9) *the inverse implication in* (5) *holds,*

then h_C is onto from FC to GHC.

PROOF. (4) uniquely determines that $C \in \mathscr{C}$ and $D \in \mathscr{D}$ for which $FC=\emptyset$ and $GD=\emptyset$ (if there are such objects). By (4) and (5) no $C \in \mathscr{C}$ satisfies $FC \neq \emptyset$, $GHC=\emptyset$. If $FC=\emptyset$, h_C is defined in the unique way.

Let now $C \in \mathscr{C}$, $FC, GHC \neq \emptyset$. Then by (1), (2), (3) the points of FC correspond to the $\sim$-equivalence classes for $\mathscr{C}$ with range C. A $\sim$-equivalence class for $\mathscr{C}$ is mapped by H into a $\sim$-equivalence class for $\mathscr{D}$. In fact by (6) φ is constant $\Rightarrow$ $\Rightarrow$ $H\varphi$ is constant, and according to the above said $D(F\varphi) \neq \emptyset \Rightarrow D(GH\varphi) \neq \emptyset$, and $\varphi_1 = \varphi_2\psi \Rightarrow H\varphi_1 = (H\varphi_2)(H\psi)$. By Lemma 1 for φ_1, φ_2 in $\mathscr{D}$, $\varphi_1 \sim \varphi_2 \Rightarrow v_G(\varphi_1) = = v_G(\varphi_2)$. Hence by the mapping of the $\sim$-equivalence classes, for each $C \in \mathscr{C}$ H induces a map $h_C: FC \to GHC$ (explicitly $h_C[v_F(\varphi)] = v_G(H\varphi)$).

We assert that for any $\varphi: C_1 \to C_2$ in $\mathscr{C}$ the following diagram commutes:

$$\begin{array}{ccc} FC_1 & \xrightarrow{F\varphi} & FC_2 \\ \downarrow{\scriptstyle h_{C_1}} & & \downarrow{\scriptstyle h_{C_2}} \\ GHC_1 & \xrightarrow{GH\varphi} & GHC_2 \end{array}$$

In fact, for $FC_1=\emptyset$ this is evident. If however $FC_1 \neq \emptyset$, then also the other sets in the diagram are non-void. Let now $c_1 \in FC_1$, $c_2 \in FC_2$, $(F\varphi)(c_1)=c_2$. Then by (2) $\exists \varphi_1, \varphi_2$ in $\mathscr{C}$, $F\varphi_1$, $F\varphi_2$ are constant and have values c_1, c_2. By (3), $\varphi\varphi_1 \sim \varphi_2$. Hence, by what has been said above, $H\varphi_1$ and $H\varphi_2$ are constant in $\mathscr{D}$, and $(H\varphi)(H\varphi_1) \sim (H\varphi_2)$. Thus by Lemma 1 $v_G[(H\varphi)(H\varphi_1)] = v_G(H\varphi_2)$, i.e. $(*)$ $(GH\varphi)[v_G(H\varphi_1)] = v_G(H\varphi_2)$. By definition we have $h_{C_1}(c_1) = v_G(H\varphi_1)$, $h_{C_2}(c_2) = = v_G(H\varphi_2)$. Hence $(*)$ can be rewritten as $(GH\varphi)[h_{C_1}(c_1)] = h_{C_2}(c_2)$, i.e. $(GH\varphi)[h_{C_1}(c_1)] = h_{C_2}[(F\varphi)(c_1)]$. Since $c_1 \in FC_1$ was arbitrary, this proves that $\{h_C\}_{C \in \mathscr{C}}$ is a natural transformation.

Let now beside the above conditions (3) for $\mathscr{D}$, G and (7) hold. If φ_1, φ_2 are in $\mathscr{C}$, $R(\varphi_1)=R(\varphi_2)$, $\varphi_1 \nsim \varphi_2$, then (7) assures $H\varphi_1 \nsim H\varphi_2$ and (3) for $\mathscr{D}$, G assures $v_G(H\varphi_1) \neq v_G(H\varphi_2)$. Thus h_C is one-to-one.

Let now beside the above conditions (2) for $\mathscr{D}$, G and (8), (9) hold. Let $FC=\emptyset$. Then by (9) $GHC=\emptyset$, so h_C is onto. Let $FC\neq\emptyset$, so also $GHC\neq\emptyset$. Let $d\in GHC$. Then (2) for $\mathscr{D}$, G assures $\exists\varphi$ in $\mathscr{D}$, $v_G(\varphi)=d$. Further (8) assures $\exists\psi$ in $\mathscr{C}$, $H\psi\sim\varphi$. Thus h_C is onto.

COROLLARY. *Let $\mathscr{C}$, $\mathscr{D}$ be concrete categories with forgetful functors F, G. Let $\forall C\in\mathscr{C}$ $\forall c\in FC$ $\exists\varphi: C\to C$, $F(\varphi)$ is constant, and has value c, and similarly for $\mathscr{D}$. Let $C\in\mathscr{C}$, $|FC|=1\Rightarrow\exists C'\in\mathscr{C}$, $C'\neq C$, $[\mathrm{Hom}\,(C', C)\neq\emptyset\vee|\mathrm{Hom}\,(C, C')|>1]$ and similarly for $\mathscr{D}$. Let H be a functor from $\mathscr{C}$ to $\mathscr{D}$, satisfying $\{\exists C'\in\mathscr{C},\ C'\neq C,$ $[\mathrm{Hom}\,(C', C)\neq\emptyset\vee|\mathrm{Hom}\,(C, C')|>1]\}\Leftrightarrow\{\exists D'\in\mathscr{D},\ D'\neq HC,\ [\mathrm{Hom}\,(D', HC)\neq\emptyset\vee$ $\vee|\mathrm{Hom}\,(HC, D')|>1]\}$, and let H be one-to-one, onto from the constant maps $C\to C$ for any $C\in\mathscr{C}$ to the constant maps $HC\to HC$ (e.g. H is a full embedding, and $\{\exists D'\in\mathscr{D},\ D'\neq HC,\ [\mathrm{Hom}\,(D', HC)\neq\emptyset\vee|\mathrm{Hom}\,(HC, D')|>1]\}\Rightarrow$ there is an HC', $C'\in\mathscr{C}$, instead of D' with the same properties). Then F and GH are naturally isomorphic.*

Lemma 2 is a generalisation of the statement at the beginning of the proof of the Theorem in [5].

PROOF OF THE THEOREM. By Lemma 2, Corollary we have for any $X\in\mathrm{Unif}$ a one-to-one onto map i'_X from the underlying set of X to that of $i(X)$. For objects X and maps φ denote by X' and φ' the corresponding objects and maps in the category of sets. Also by Lemma 2, Corollary for any map $f: X\to Y$ in Unif the following diagram commutes:

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle i'_X}\downarrow & & \downarrow{\scriptstyle i'_Y} \\ i(X)' & \xrightarrow{i(f)'} & i(Y)' \end{array}$$

X is a two-point discrete space iff $|X'|=2$, and $[|Y'|=2$, Y non-isomorphic to $X\Rightarrow|\mathrm{Hom}\,(Y, X)|=2]$. Hence X is a two-point discrete space iff $i(X)$ is. Let $\{0, 1\}$ denote the two-point discrete space.

For any $X\in\mathrm{Unif}$ define the following relation $\sim_X$ on X': for $x_1, x_2\in X'$ $x_1\sim_X x_2\Leftrightarrow\{x_1=x_2\vee[x_1\neq x_2,\ \exists f: X\to X,\ f'(x_1)=x_2,\ f'(x_2)=x_1,\ (x\in X', x\neq x_1, x_2\Rightarrow$ $\Rightarrow f'(x)=x),\ \nexists g_1: X\to\{0, 1\},\ g_1'^{-1}(1)=\{x_1\},\ \nexists g_2: X\to\{0, 1\},\ g_2'^{-1}(1)=\{x_2\}]\}$. Thus $\sim_X$ is carried over by i'_X to $\sim_{i(X)}$. Evidently $x_1\sim_X x_2\Leftrightarrow x_1, x_2$ are not separated by the uniformity of X. Hence i maps the subcategory HUnif onto itself, isomorphically.

i preserves extremal monomorphisms in HUnif, hence f is a closed embedding iff $i(f)$ is. For $X\in\mathrm{HUnif}$ $\{f'(Y'), f: Y\to X$ is an extremal monomorphism in HUnif$\}$ equals the system of all closed sets of X. Hence the map i'_X induces a homeomorphism of the topologies of X and $i(X)$.

Compact T_2 spaces have unique uniformities. Hence if $X\in\mathrm{HUnif}$ is compact (which holds iff $i(X)$ is compact), i'_X induces an isomorphism i_X of X to $i(X)$.

For $X\in\mathrm{HUnif}$ the uniformity weakly generated by all maps from X to compact T_2 spaces of density $\leq|X'|$ is pX. Hence i'_X induces an isomorphism of pX to $pi(X)$.

$X \in \mathrm{HUnif}$ is proximally fine, i.e. $\forall Y \in \mathrm{HUnif}$ $[\mathrm{Hom}(X, Y)]' = [\mathrm{Hom}(pX, pY)]'$ iff $\forall Y \in \mathrm{HUnif}$ $[\mathrm{Hom}(i(X), i(Y))]' = [\mathrm{Hom}(pi(X), pi(Y))]'$, i.e. $\forall Y \in \mathrm{HUnif}$ $[\mathrm{Hom}(i(X), Y)]' = [\mathrm{Hom}(pi(X), pY)]'$, which holds iff $i(X)$ is proximally fine. Further in this case the uniformity of X is the finest one on X' whose precompact reflection is pX. Hence in this case i'_X induces an isomorphism i_X of X to $i(X)$.

Especially for metric uniform spaces X, i'_X induces an isomorphism i_X of X to $i(X)$. Since i preserves products, for any product X of metric uniform spaces i'_X induces an isomorphism i_X of X to $i(X)$.

i preserves copure monomorphisms in HUnif, hence f is an embeddin iff $i(f)$ is. Every $X \in \mathrm{HUnif}$ can be embedded into a product of metric uniform spaces Y. Then $i(X)$ is embedded in $i(Y)$, and i'_X induces an isomorphism i_X of X to $i(X)$.

Now we show that for any $X \in \mathrm{Unif}$, i'_X induces an isomorphism i_X of X to $i(X)$. Denote r_0: Unif$\to$HUnif the T_0-reflection functor. We have $(r_0X)' = X'/\sim_X$. Since $(i'_X \times i'_X)(\sim_X) = \sim_{i(X)}$, i'_X induces a one-to-one onto map j'_X of $(r_0X)'$ onto $(r_0 i(X))'$, and for every $f: X \to Y$ in Unif the following diagram commutes:

$$\begin{array}{ccc} (r_0X)' & \xrightarrow{(r_0f)'} & (r_0Y)' \\ {\scriptstyle j'_X}\downarrow & & \downarrow{\scriptstyle j'_Y} \\ (r_0 i(X))' & \xrightarrow{(r_0 i(f))'} & (r_0 i(Y))' \end{array}$$

Hence by the result already proved for HUnif we have that each j'_X induces an isomorphism of r_0X to $r_0 i(X)$. Thus also i'_X induces an isomorphism i_X of X to $i(X)$. Q.e.d.

REMARK. A related matter is the monadicity of the underlying set functor for categories of algebras with any fixed type and for the category of compact T_2 spaces (cf. [6], VI and also [10]). It might be of interest to construct reasonable forgetful functors for abstractly given categories $\mathscr{C}$ of algebras of given types (possibly ones by which these categories actually become categories of algebras of some types). A possibility is given by $C \to \mathrm{Hom}(C_0, C)$ where C_0 is a free algebra on one generator, provided $C_0 \in \mathscr{C}$ can be recovered from $\mathscr{C}$, cf. [7], 32.21,22 (this is done in a special case at the beginning of the proof of the Theorem in [5]). Also for a concrete category $\mathscr{C}$ of algebras of fixed type, with forgetful functor F, it might be interesting to determine those isomorphisms i of $\mathscr{C}$ onto itself, for which $Fi = F$. Of course it would suffice to consider free algebras and the operations on the generators.

Using only parts of the proof we obtain the following

COROLLARY of the proof. *Every isomorphism of the category of proximity spaces, separated proximity spaces or separated uniform spaces onto itself is of the form given in the theorem.*

REMARKS. 1. It would be desirable to develop the proof of the theorem to a characterisation of the categories in the theorem and the corollary, in the spirit of the characterisations of the category of all topological spaces in [11], Example 4 and [12].

2. Presumably the statements of the theorem and the corollary hold for any full embedding instead of an isomorphism. (These statements for the separated and for the general case are equivalent.)

A free ultraspace ([8], p. 540) is a topological space X, for which $D\subset X\subset \beta D$, $|X\setminus D|=1$ for some discrete space D. A topological space is S_1 if its T_0-reflection is T_1. Utilizing Lemma 2, Corollary and [8], Proposition 6 and Lemmas 5 and 6 (where in fact isomorphic representations of the given category of topological spaces by more general categories are investigated; however the representation is supposed to have the same underlying sets and the same functions as maps) and in the second case constructing T_0-reflections by maps into free ultraspaces in stead of $\{0, 1\}$ we obtain the following

PROPOSITION. *Every full embedding e of any category of topological spaces containing all T_1 spaces or of any category of S_1-spaces containing all free ultraspaces into the category of all topological spaces is of the form given in the theorem (i.e. e is naturally isomorphic to the inclusion functor).*

Utilizing Lemma 2, Corollary, [4], Theorem 4.5 (asserting that for any topological space F whose underlying set is $[0, 1]$, if $\mathrm{Hom}([0, 1], [0, 1])=\mathrm{Hom}(F, F)$ holds, then $F=[0, 1]$, and the fact that any Tychonoff topology (these are not assumed to be T_0) on a given set is determined by its continuous maps to $[0, 1]$, (and noting $\mathrm{Hom}([0, 1], X)=\emptyset \Leftrightarrow X=\emptyset$) we obtain the following

PROPOSITION. *Every full embedding e of any category $\mathscr{C}$ of Tychonoff spaces which contains* $[0, 1]$ *into the category of all Tychonoff spaces is of the form given in the theorem.*

REMARK. The same can be said about the class $R'\mathscr{E}$ of topological spaces, whose T_0-reflections are $\mathscr{E}$-regular (i.e. homeomorphic to subspaces of products of spaces from $\mathscr{E}$, cf. [3], 17.1), where $\mathscr{E}$ is any class of topological spaces (no separation axiom imposed), and any category $\mathscr{C}$ with $\mathscr{E}\subset\mathscr{C}\subset R'\mathscr{E}$, where each $E\in\mathscr{E}$ is $R'\mathscr{E}$-special (i.e. $F\in R'\mathscr{E}$, f one-to-one, onto from the underlying set of E to that of F, $\mathrm{Hom}(E, E)=f^{-1}\,\mathrm{Hom}(F, F)f$ imply f is a homeomorphism, cf. [4], p. 25). Cf. also [8], proof of Proposition 5.

REMARK. Proceeding on the lines of [8] one may ask whether the full embeddings of Unif into the category with objects $(X, \mathscr{U})$, where $\mathscr{U}\subset 2^{X\times X}$, or (X, μ), where μ is a cover of X (f is a map $(X_1, \mathscr{U}_1)\to(X_2, \mathscr{U}_2)$ or $(X_1, \mu_1)\to(X_2, \mu_2)$ if $(f\times f)^{-1}\mathscr{U}_2\subset\mathscr{U}_1$ or $f^{-1}\mu_2\subset\mu_1$) are given by the entourages or by the complements of the entourages, and by the uniform covers, respectively. If only $\mu\subset 2^{2^X}$, then another full embedding is given by $\{\mathscr{U}, \mathscr{U}\subset 2^X, \mathscr{U}$ is a uniform cover on $\bigcup\mathscr{U}\}$. Also if μ is one of the two above mentioned families, $\{2^X\setminus\mathscr{U}, \mathscr{U}\in\mu\}$ gives a full embedding as well. Is there no other full embedding? One can ask the same questions for proximity spaces. Alternatively one can ask, whether all full embeddings of the category of proximity spaces into the categories with objects $(X, \mathscr{V})$, $\mathscr{V}\subset\subset\{(V, W), V, W\subset X\}$ and with maps f for which $f^{-1}\mathscr{V}_2\subset\mathscr{V}_1$, or $f\mathscr{V}_1\subset\mathscr{V}_2$, are given by farness of $\varepsilon_1(V, W)$, $\varepsilon_2(V, W)$ — where $(\varepsilon_1, \varepsilon_2)$ is a one-to-one map defined by a formula containing unions and complements (e.g. $\varepsilon_1(V, W)=X\setminus V$, $\varepsilon_2(V, W)=(V\setminus W)\cup(W\setminus V)$) — or by nearness of V and W, respectively. Analogously one may ask the full embeddings of Unif into the category of (X, μ)-s, with maps f characterised by $f\mu_1 \subset \mu_2$. One such embedding is given by the family of all non-vanishing systems (i.e. systems $\{A_\alpha\} \subset 2^X$, for which for each uniform cover $\mathscr{W}$ $\cap \operatorname{St}(A_\alpha, \mathscr{W}) \neq \emptyset$, cf. [2], p. 86). Is this the only one (for non-empty spaces)?

Remark added in proof (August 8, 1978). L. Márki has drawn my attention to the fact that the question of determining all isomorphisms of a given category onto itself was considered also in the algebra. [14], Ch. 1, Exercises determine all isomorphisms of the categories of sets, small categories, partially ordered sets, abelian groups, groups (although in this case I do not see how he disposed about the "opposite group" functor), while [15] and [16] determine those of the categories of semigroups and R-algebras, commutative or not necessarily commutative, and with 1 or not necessarily whith 1 (where R is a commutative integral domain with 1), respectively. Related matter cf. in [1] too, (Introduction, § 5) about representation of categories by quasiprimitive or primitive classes of algebras and about the diversity of these representations. The case of primitive classes cf. [13] too.

For the last question of the paper another full embedding is given in [17] by the family of all systems consisting of non-empty sets and containing arbitrarily small sets (i.e. systems $\{A_\alpha\} \subset 2^X$, for which for each uniform cover $w \exists A_\alpha$, $\exists W \in w$, $A_\alpha \cup W$). At this question it is natural to restrict ourselves to such μ-s, for which $\mathscr{A} = \{A_\alpha\} \in \mu$ imply $A_\alpha \neq \emptyset$. Namely otherwise $\mu = \mu' \cup \mu''$, where $\mu' = \{\mathscr{A}, \emptyset \notin \mathscr{A} \in \mu\}$, $\mu'' = \{\mathscr{A}, \emptyset \in \mathscr{A} \in \mu\}$ and μ' ahd μ'' give embeddings of Unif into this category; conversely if μ' and μ'' give embeddings, and one of these is full, the μ gives a full embedding. Also we can suppose $\emptyset \notin \mu$.

References

[1] J. R. Isbell, Subobjects, adequacy, completeness and categories of algebras, *Rozprawy Mat.*, **36** (1964).

[2] J. R. Isbell, *Uniform spaces* (Providence, Rhode Island, 1964).

[3] H. Herrlich, *Topologische Reflexionen und Koreflexionen*, Lecture Notes in Mathematics 78, Springer-Verlag (Berlin—Heidelberg—New York, 1968).

[4] J. C. Warndof, Topologies uniquely determined by their continuous self-maps, *Fund. Math.*, **66** (1969/70), 25—43.

[5] M. Sekanina, Embedding of the category of partially ordered sets into the category of topological spaces, *Fund. Math.*, **66** (1969/70), 95—98.

[6] S. MacLane, *Categories for a working mathematician*, Springer-Verlag (New York—Heidelberg—Berlin, 1971).

[7] H. Herrlich—G. E. Strecker, *Category theory*, Allin and Bacon Inc. (Boston, 1973).

[8] J. Rosický—M. Sekanina, Realizations of topologies by set systems, *Topics in Topology*, Proceedings of a Colloquium on Topology (Keszthely, 1972), Colloquia Mathematica Societatis János Bolyai, 8, North Holland Publ. Co., Amsterdam—London; J. Bolyai Math. Soc. Budapest; American Elsevier (New York, 1974), 535—555.

[9] Z. Frolík, Basic refinements of the category of uniform spaces, *Topo 72— General Topology and its Applications*, Second Pittsburgh International Conference 1972, Lecture Notes in Mathematics 378, Springer-Verlag (Berlin—Heidelberg—New York, 1974), 140—158.

[10] Z. Semadeni, A simple topological proof that the underlying set functor for compact spaces is monadic, *ibid.*, 429—435.

[11] M. Hušek, Construction of special functors and its applications, *Comment. Math. Univ. Carolinae*, **8** (1967), 555—566.

[12] D. Schlomiuk, An elementary theory of the category of topological spaces, *Trans. Amer. Math. Soc.*, **149** (1970), 259—278.

[13] F. W. Lawvere, Functorial semantics and algebraic theories, *Proc. Nat. Acad. Sci. USA*, **50** (1963), 869—872.

[14] P. Freyd, *Abelian categories, an introduction to the theory of functors*, Harper's series in modern mathematics, Harper and Row, Publishers (New York, Evanston, London, 1964).

[15] B. P. Brooks, Jr., W. E. Clark, On the categoricity of semigroup-theoretical properties, *Semigroup Forum*, **3** (1971), 259—266.

[16] W. E. Clark, G. M. Bergman, The automorphism class group of the category of rings, *J. of Algebra*, **24** (1973), 80—99.

[17] В. Ю. Сандберг, Новое определение равномерных пространств, *Докл. Акад. Наук. СССР,* **135** (1960), 535—537.

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST, REÁLTANODA U. 13—15.

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ON SOME PROBLEMS OF THE STATISTICAL THEORY OF PARTITIONS WITH APPLICATION TO CHARACTERS OF THE SYMMETRIC GROUP. III

By

M. SZALAY and P. TURÁN, member of the Academy (Budapest)

1. This paper is the third one in a series, the first two papers of which contain five theorems on the distribution of summands in partitions of n (see [1], [2]). We shall give a practical summary of these results in Section 3.

In what follows we are dealing with the statistical properties of the dimensions of the irreducible representations (over the complex field) of S_n, the symmetric group on n letters. It is well-known that the number of pairwise non-equivalent irreducible representations of S_n is equal to the conjugacy class-number of S_n and it is $p(n)$, the number of partitions of n. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_{p(n)}$ be the pairwise non-equivalent irreducible representations of S_n and let us denote by χ_ν the character of Γ_ν and by E the unit class of S_n.

The irreducible representations of S_n are closely connected with the partitions of n by a theorem of FROBENIUS and SCHUR (see [3]). According to this theorem, there is an appropriate one-to-one correspondence between the irreducible representations of S_n and the partitions of n in the following way.

Let

$$\Pi: \begin{cases} n = \lambda_1 + \lambda_2 + \ldots + \lambda_m \\ \lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_m \geqq 1 \end{cases} \tag{1.1}$$

be a generic partition of n. Then for the corresponding irreducible representation Γ_Π

$$\dim \Gamma_\Pi = \chi_\Pi(E) = n! \frac{\prod\limits_{1 \leqq \mu < \nu \leqq m} (\lambda_\mu - \lambda_\nu + \nu - \mu)}{\prod\limits_{\mu=1}^{m} (\lambda_\mu + m - \mu)!}. \tag{1.2}$$

In this paper we prove the following

THEOREM VI. *There are explicitly calculable positive constants c and n_0 such that for $n > n_0$ and at least for $\left(1 - \frac{1}{n}\right) p(n)$ Π's the inequality*

$$\left| \log \chi_\Pi(E) - \log\left(\sqrt{n!}\right) + An \right| < c \cdot n^{7/8} \log^4 n \tag{1.3}$$

holds with the constant A defined by

(1.4)
$$A = -\frac{1}{2} - \log\frac{\pi}{\sqrt{6}} + \frac{6}{\pi^2}\int_0^\infty \frac{y\log y}{\exp(y)-1}\,dy +$$
$$+\frac{6}{\pi^2}\int_0^\infty\int_0^\infty \log\frac{1}{1-\dfrac{\log\dfrac{1}{1-\exp(-x-y)}}{y+\log\dfrac{1}{1-\exp(-x)}}}\,dx\,dy$$

for which

(1.5)
$$A > \frac{6}{\pi^2}0.02.$$

(See also Lemma 8.)

For other observations about the irreducible characters of S_n we refer to [1].

2. In order to estimate the dimensions given by (1.2), we need relatively precise results about the value distribution of the summands in (1.1), at least for the "middle" summands and for "almost all" partitions of n. We shall use the corollary of Theorem II (see [1]). This asserts that if the number of summands $\geqq \Lambda$ in (1.1) is denoted by $S_1(n, \Pi, \Lambda)$ then we have for

(2.1)
$$11\log n \leqq \Lambda \leqq \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n - 3\sqrt{n}\log\log n$$

the *uniform* estimation

(2.2)
$$S_1(n, \Pi, \Lambda) = \left(1 + O\left(\frac{1}{\log n}\right)\right)\frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp\left(-\dfrac{\pi\Lambda}{\sqrt{6n}}\right)}$$

apart from

(2.3)
$$cp(n)n^{-5/4}\log n$$

exceptional Π's at most.

Throughout this paper c stands for explicitly calculable positive constants not necessarily the same in different occurrences. The O and o-signs refer to $n\to\infty$ and the O-constants are explicitly calculable.

For the range (2.1) we shall need also the stronger form of the corollary of Theorem I (see [1]) which follows immediately from Lemmas IV and V of [1]. This asserts for the Λ's in (2.1) the *uniform* estimation

(2.4)
$$S_1(n, \Pi, \Lambda) = \frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp\left(-\dfrac{\pi\Lambda}{\sqrt{6n}}\right)} + O\left(\sqrt{\frac{\sqrt{n}\log n}{\exp\left(\dfrac{\pi\Lambda}{\sqrt{6n}}\right)-1}}\right)$$

for all but $cp(n)n^{-5/4}\log n$ Π's at most.

Owing to the relations

(2.5) $$S_1(n, \Pi, \lambda_\mu) \geqq \mu$$

and

(2.6) $$S_1(n, \Pi, \lambda_\mu + 1) \leqq \mu - 1$$

we can prove similar results for the λ_μ's. Namely, we assert the

LEMMA 1. *For $n > c$, if $\alpha = \alpha(n)$ is restricted by*

(2.7) $$3 \leqq \alpha \leqq \frac{1}{2} \frac{\log n}{\log\log n} - 6$$

then for

(2.8) $$\mu \in \left[\frac{\sqrt{6}}{\pi} \frac{\sqrt{n}}{\log^\alpha n}, \ \alpha \frac{\sqrt{6}}{\pi} \sqrt{n} \log\log n\right] \overset{\text{def}}{=} I_1 \quad (\mu \text{ integer})$$

the relation

(2.9) $$\lambda_\mu = \left(1 + O(n^{-1/4} \log^{(\alpha/2)+2} n)\right) \frac{\sqrt{6}}{\pi} \sqrt{n} \log \frac{1}{1 - \exp\left(-\frac{\pi\mu}{\sqrt{6n}}\right)}$$

holds uniformly in I_1 with the exception of $cp(n)n^{-5/4} \log n$ partitions of n at most.

For the proof of this lemma we mention that owing to (2.7) the relation

(2.10) $$11 \log n \leqq \frac{\sqrt{6}}{\pi} \frac{\sqrt{n}}{\log^{\alpha+1} n} < (\alpha+1) \frac{\sqrt{6}}{\pi} \sqrt{n} \log\log n \leqq$$
$$\leqq \frac{\sqrt{6}}{2\pi} \sqrt{n} \log n - 3\sqrt{n} \log\log n - 1$$

holds for $n > c$. Thus we can apply (2.2) for these expressions, with the exception of $cp(n)n^{-5/4} \log n$ Π's at most, as follows.

$$S_1\left(n, \Pi, \frac{\sqrt{6}}{\pi} \frac{\sqrt{n}}{\log^{\alpha+1} n}\right) = \left(1 + O\left(\frac{1}{\log n}\right)\right) \frac{\sqrt{6}}{\pi} \sqrt{n} \log \frac{1}{1 - \exp(-\log^{-(\alpha+1)} n)} >$$
$$> \left(1 - \frac{c}{\log n}\right) \frac{\sqrt{6}}{\pi} \sqrt{n} \log(\log^{\alpha+1} n) > \left[\alpha \frac{\sqrt{6}}{\pi} \sqrt{n} \log\log n\right] + 1$$

for $n > c$. This gives that

(2.11) $$\lambda_{\left[\alpha \frac{\sqrt{6}}{\pi} \sqrt{n} \log\log n\right] + 1} \geqq \frac{\sqrt{6}}{\pi} \frac{\sqrt{n}}{\log^{\alpha+1} n}.$$

Further,

$$S_1\left(n, \Pi, (\alpha+1) \frac{\sqrt{6}}{\pi} \sqrt{n} \log\log n\right) = \left(1 + O\left(\frac{1}{\log n}\right)\right) \frac{\sqrt{6}}{\pi} \sqrt{n} \log \frac{1}{1 - \log^{-(\alpha+1)} n} <$$
$$< \left(1 + \frac{c}{\log n}\right) \frac{\sqrt{6}}{\pi} \sqrt{n} \log\left(1 + \frac{\log^{-(\alpha+1)} n}{1 - \log^{-(\alpha+1)} n}\right) < c\sqrt{n} \log^{-(\alpha+1)} n < \left[\frac{\sqrt{6}}{\pi} \frac{\sqrt{n}}{\log^\alpha n}\right]$$

for $n>c$. This gives that

$$\lambda_{\left[\frac{\sqrt{6}}{\pi}\frac{\sqrt{n}}{\log^{\alpha} n}\right]} < (\alpha+1)\frac{\sqrt{6}}{\pi}\sqrt{n}\log\log n. \tag{2.12}$$

(2.11) and (2.12) yield that for $\mu\in I_1$ (μ integer) the relation

$$\frac{\sqrt{6}}{\pi}\frac{\sqrt{n}}{\log^{\alpha+1}n} \leqq \lambda_\mu \leqq (\alpha+1)\frac{\sqrt{6}}{\pi}\sqrt{n}\log\log n \tag{2.13}$$

holds uniformly in I_1 for all but $cp(n)n^{-5/4}\log n$ partitions of n at most.

Taking into consideration (2.10) we can apply (2.4) for $\Lambda=\lambda_\mu$ and $\Lambda=\lambda_\mu+1$. With the abbreviation

$$x_0 = \frac{\pi}{\sqrt{6n}} \tag{2.14}$$

we get (using (2.5) and (2.6)) the inequalities

$$\mu \leqq S_1(n, \Pi, \lambda_\mu) \leqq$$

$$\leqq \frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp(-x_0\lambda_\mu)} + c\sqrt{\frac{\sqrt{n}\log n}{\exp(x_0\lambda_\mu)-1}}$$

and

$$\mu \geqq 1+S_1(n, \Pi, \lambda_\mu+1) >$$

$$> \frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp(-x_0(\lambda_\mu+1))} - c\sqrt{\frac{\sqrt{n}\log n}{\exp(x_0(\lambda_\mu+1))-1}} \geqq$$

$$\geqq \frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp(-x_0\lambda_\mu)} - \frac{\sqrt{6}}{\pi}\sqrt{n}\log\left(1+\frac{1-\exp(-x_0)}{\exp(x_0\lambda_\mu)-1}\right) -$$

$$-c\sqrt{\frac{\sqrt{n}\log n}{\exp(x_0\lambda_\mu)-1}} > \frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp(-x_0\lambda_\mu)} -$$

$$-\frac{1}{\exp(x_0\lambda_\mu)-1} - c\sqrt{\frac{\sqrt{n}\log n}{\exp(x_0\lambda_\mu)-1}}.$$

Owing to (2.13) and (2.10) we have

$$\frac{1}{\exp(x_0\lambda_\mu)-1} < c\sqrt{\frac{\sqrt{n}\log n}{\exp(x_0\lambda_\mu)-1}}$$

thus we get the relation

$$\mu = \frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp(-x_0\lambda_\mu)} + O\left(\sqrt{\frac{\sqrt{n}\log n}{\exp(x_0\lambda_\mu)-1}}\right) \tag{2.15}$$

which holds uniformly in I_1 with the exception of $cp(n)n^{-5/4}\log n$ Π's at most. In order to invert this relation we investigate the quotient

$$\frac{\sqrt{\dfrac{\sqrt{n}\log n}{\exp(x_0\lambda_\mu)-1}}}{\sqrt{n}\log\dfrac{1}{1-\exp(-x_0\lambda_\mu)}} \overset{\text{def}}{=\!=} M$$

for the λ_μ's in (2.13).

First let

$$\left(\frac{1}{2x_0}=\right)\frac{\sqrt{6}}{2\pi}\sqrt{n} \leqq \lambda_\mu \leqq (\alpha+1)\frac{\sqrt{6}}{\pi}\sqrt{n}\log\log n.$$

Then

$$M \leqq \sqrt{\frac{\log n}{\sqrt{n}(\exp(x_0\lambda_\mu)-1)}}\cdot\exp(x_0\lambda_\mu) < c\sqrt{\frac{\log n}{\sqrt{n}}\exp(x_0\lambda_\mu)} \leqq$$

$$\leqq c\sqrt{\frac{\log n}{\sqrt{n}}\exp\big((\alpha+1)\log\log n\big)} = cn^{-1/4}\log^{(\alpha/2)+1} n.$$

Next let

$$\frac{\sqrt{6}}{\pi}\frac{\sqrt{n}}{\log^{\alpha+1} n} \leqq \lambda_\mu < \frac{\sqrt{6}}{2\pi}\sqrt{n}\left(=\frac{1}{2x_0}\right).$$

Then

$$M \leqq c\frac{\sqrt{\dfrac{\log n}{\lambda_\mu}}}{\log\dfrac{1}{x_0\lambda_\mu}} < c\sqrt{\frac{\log n}{\lambda_\mu}} \leqq cn^{-1/4}\log^{(\alpha/2)+1} n.$$

Thus we got from (2.15)

$$\mu = \big(1+O(n^{-1/4}\log^{(\alpha/2)+1} n)\big)\frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp(-x_0\lambda_\mu)}.$$

Finally, owing to (2.7) having

$$n^{-1/4}\log^{(\alpha/2)+1} n \leqq \log^{-2} n, \tag{2.16}$$

we got the relation

$$\lambda_\mu = \frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp\{-x_0\mu(1+O(n^{-1/4}\log^{(\alpha/2)+1} n))\}} \tag{2.17}$$

which holds uniformly in I_1 apart from $cp(n)n^{-5/4}\log n$ Π's at most.

Moreover, using (2.16), we have in I_1

$$\mu n^{-3/4}\log^{(\alpha/2)+1} n = O(\log^{-1} n),$$

therefore,

$$\log \frac{1-\exp\{-x_0\mu(1+O(n^{-1/4}\log^{(\alpha/2)+1} n))\}}{1-\exp(-x_0\mu)} =$$

$$= \log\left(1+\frac{1-\exp\{O(\mu n^{-3/4}\log^{(\alpha/2)+1} n)\}}{\exp(x_0\mu)-1}\right) = \log\left(1+\frac{O(\mu n^{-3/4}\log^{(\alpha/2)+1} n)}{\exp(x_0\mu)-1}\right) =$$

$$= \log\left(1+O\left(n^{-1/4}\exp(-x_0\mu)\log^{(\alpha/2)+1} n\right)\frac{x_0\mu}{1-\exp(-x_0\mu)}\right).$$

We investigate the factor

$$\frac{x_0\mu}{1-\exp(-x_0\mu)} \overset{\text{def}}{=} L.$$

For

$$\frac{\sqrt{6}}{\pi}\frac{\sqrt{n}}{\log^\alpha n} \leqq \mu \leqq 2\frac{\sqrt{6}}{\pi}\sqrt{n}\left(=\frac{2}{x_0}\right)$$

we have

$$L = \frac{x_0\mu\exp(x_0\mu)}{\exp(x_0\mu)-1} \leqq \exp(x_0\mu) \leqq e^2$$

and for

$$\left(\frac{2}{x_0}=\right)2\frac{\sqrt{6}}{\pi}\sqrt{n} < \mu \leqq \alpha\sqrt{n}\log\log n$$

we get

$$L \leqq \frac{x_0\mu}{1-\exp(-2)} \leqq c\frac{\mu}{\sqrt{n}} \leqq c\alpha\log\log n \leqq c\log n.$$

These estimations give with (2.17) and (2.16) that

$$\lambda_\mu = \frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp(-x_0\mu)} - \frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1-\exp\{-x_0\mu(1+O(n^{-1/4}\log^{(\alpha/2)+1} n))\}}{1-\exp(-x_0\mu)} =$$

$$= \frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp(-x_0\mu)} - \frac{\sqrt{6}}{\pi}\sqrt{n}\log\left(1+O\left(n^{-1/4}\exp(-x_0\mu)\right)\log^{(\alpha/2)+2} n\right) =$$

$$= \frac{\sqrt{6}}{\pi}\sqrt{n}\left\{\log\frac{1}{1-\exp(-x_0\mu)} + O(n^{-1/4}\log^{(\alpha/2)+2} n)\exp(-x_0\mu)\right\} =$$

$$= \left(1+O(n^{-1/4}\log^{(\alpha/2)+2} n)\right)\frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp(-x_0\mu)}$$

holds uniformly in I_1 apart from $cp(n)n^{-5/4}\log n$ Π's at most. This proves Lemma 1.

In order to make a later estimation easier we mention the

LEMMA 2. *For $n>c$, if $\alpha=\alpha(n)$ is restricted by*

$$3 \leqq \alpha \leqq \frac{1}{4}\frac{\log n}{\log\log n} - 3 \tag{2.18}$$

then for

$$\mu\in\left[\frac{\sqrt{6}}{\pi}\frac{\sqrt{n}}{\log^{\alpha} n}, 2\alpha\frac{\sqrt{6}}{\pi}\sqrt{n}\log\log n\right]\overset{\text{def}}{=} I_2 \quad (\mu \text{ integer}) \tag{2.19}$$

the relation

$$\lambda_\mu = \left(1+O(n^{-1/4}\log^{\alpha+2} n)\right)\frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp\left(-\frac{\pi\mu}{\sqrt{6n}}\right)} \tag{2.20}$$

holds uniformly in I_2 with the exception of $cp(n)n^{-5/4}\log n$ partitions of n at most.

Lemma 2 is an immediate consequence of Lemma 1. Namely, for the α's in (2.18) 2α satisfies (2.7) and Lemma 1 yields (2.20) owing to $I_2(\alpha)\subset I_1(2\alpha)$.

3. Though the applications require good error terms for different choices of $\alpha=\alpha(n)$, we mention the following consequence of Lemma 1 (by maximal α) which is interesting in itself too and can be considered a natural and practical summary of our results in Part I (cf. [1], Theorem II).

COROLLARY 1. *With the restriction*

$$\log^6 n \leqq \mu \leqq \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n - 5\sqrt{n}\log\log n \tag{3.1}$$

the relation

$$\lambda_\mu = \left(1+O\left(\frac{1}{\log n}\right)\right)\frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp\left(-\frac{\pi\mu}{\sqrt{6n}}\right)} \tag{3.2}$$

holds uniformly with the exception of $cp(n)n^{-5/4}\log n$ partitions of n at most.

For the sake of completeness we give a practical summary of our results in Part II (cf. [2], Theorem V). Following the way of Lemma 1 but using also Theorems V and IV (see [2]) one can get easily for the complementary ranges of (3.1) the

COROLLARY 2.

a) *If $\omega(n)\nearrow\infty$ arbitrarily slowly and*

$$\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n - 5\sqrt{n}\log\log n \leqq \mu \leqq \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n - \sqrt{n}\,\omega(n) \tag{3.3}$$

then the relation

$$\lambda_\mu = \exp\left\{\left(\frac{1}{2}\log n - \frac{\pi\mu}{\sqrt{6n}}\right) + O(1)\sqrt{\frac{1}{2}\log n - \frac{\pi\mu}{\sqrt{6n}}}\right\} \tag{3.4}$$

holds for almost all partitions of n, i.e. with the exception of $o(p(n))$ Π's at most.

b) *If $B(n)\nearrow\infty$ arbitrarily slowly and*

$$B(n) \leqq \mu \leqq \log^6 n \tag{3.5}$$

then the relation

$$\lambda_\mu = \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n - \frac{\sqrt{6}}{\pi}\sqrt{n}\log\mu + O(\sqrt{n\log\mu}) \tag{3.6}$$

holds for almost all partitions of n.

c) *If* $\omega(n)\nearrow\infty$ *arbitrarily slowly and*

$$1 \leqq \mu \leqq c \tag{3.7}$$

then the relation

$$\lambda_\mu = \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n + O(\sqrt{n}\,\omega(n)) \tag{3.8}$$

holds for almost all partitions of n.

4. We shall need upper estimations for λ_1 and m (see (1.1)) with the exception of $O(p(n))\exp(-c\log n)$ Π's at most. (Theorem IV gave it for $m=l(\Pi)$ with an exceptional set of measure $O(p(n))\exp(-\omega(n))$ by the condition $\omega(n)=o(\log n)$.) Now we assert the

LEMMA 3. *If* $\beta=\beta(n)$ *is restricted by*

$$0 < \beta < \frac{\pi}{2\sqrt{6}}\cdot\frac{\sqrt{n}}{\log n} - \frac{1}{2} \tag{4.1}$$

then the relations

$$\lambda_1 \leqq (1+2\beta)\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n \tag{4.2}$$

and

$$m \leqq (1+2\beta)\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n \tag{4.3}$$

hold with the exception of $O(p(n)\cdot n^{-\beta})$ Π*'s at most.*

Owing to the associate partitions it is enough to prove (4.2). In order to estimate the number of the exceptional Π's, let

$$F \overset{\text{def}}{=} \left[(1+2\beta)\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n\right]+1.$$

The number of Π's with $\lambda_1=j$, $F\leqq j\leqq n-1$ is at most $p(n-j)$. Hence, the number of the exceptional Π's is

$$\leqq \sum_{j=F}^{n-1} p(n-j)+1 = 1+\sum_{l=1}^{n-F} p(l).$$

Using the partition formula

$$p(n) = (1+o(1))\frac{1}{4n\sqrt{3}}\exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right)$$

of HARDY and RAMANUJAN (see [4]) we get

$$1+\sum_{l=1}^{n-F} p(l) \leqq c+c\sum_{l=1}^{n-F} \frac{1}{l}\exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{l}\right) \leqq$$

$$\leqq c+c\int_{1}^{n-F+1} \frac{1}{x}\exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{x}\right)dx \leqq$$

$$\leqq c+\frac{c}{\sqrt{n-F+1}}\exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n-F+1}\right).$$

Owing to (4.1) we have

$$n-F+1 \geqq \frac{n}{2}$$

thus

$$c+\frac{c}{\sqrt{n-F+1}}\exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n-F+1}\right) \leqq$$

$$\leqq cp(n)\frac{n}{\sqrt{n-F+1}}\exp\left\{-\frac{2\pi}{\sqrt{6}}\left(\sqrt{n}-\sqrt{n-F+1}\right)\right\} \leqq$$

$$\leqq cp(n)\sqrt{n}\exp\left\{-\frac{2\pi}{\sqrt{6}}\cdot\frac{\left[(1+2\beta)\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n\right]}{\sqrt{n}+\sqrt{n-F+1}}\right\} = O\left(p(n)n^{-\beta}\right).$$

Q.e.d.

Choosing $\beta=2$ in Lemma 3 we get the

LEMMA 4. *The inequalities*

(4.4) $$\lambda_1 \leqq 5\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n$$

and

(4.5) $$m \leqq 5\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n$$

hold with the exception of $O\left(p(n)n^{-2}\right)$ Π's at most.

5. The simple observations

$$\prod_{j=1}^{m}\lambda_j!\cdot\prod_{1\leqq\mu<\nu\leqq m}(\lambda_\mu+\nu-\mu) = \prod_{j=1}^{m}\lambda_j!\cdot\prod_{\mu=1}^{m-1}\frac{(\lambda_\mu+m-\mu)!}{\lambda_\mu!} = \prod_{\mu=1}^{m}(\lambda_\mu+m-\mu)!$$

and

$$0<\frac{\lambda_\nu}{\lambda_\mu+\nu-\mu}\leqq\frac{\lambda_\nu}{\lambda_\nu+1}<1 \quad (\text{for} \quad \mu<\nu)$$

give for $\chi_\Pi(E)$ (see (1.2)) the alternative representations

$$\chi_\Pi(E) = \frac{n!}{\lambda_1!\,\lambda_2!\dots\lambda_m!}\prod_{1\le\mu<\nu\le m}\frac{\lambda_\mu-\lambda_\nu+\nu-\mu}{\lambda_\mu+\nu-\mu} = \tag{5.1}$$

$$= \frac{n!}{\lambda_1!\,\lambda_2!\dots\lambda_m!}\exp\left(-\sum_{1\le\mu<\nu\le m}\log\frac{1}{1-\dfrac{\lambda_\nu}{\lambda_\mu+\nu-\mu}}\right).$$

Using Stirling's formula we get

$$\log\frac{n!}{\lambda_1!\,\lambda_2!\dots\lambda_m!} = n\log n - n + \frac{1}{2}\log n + \frac{1}{2}\log 2\pi + O\left(\frac{1}{n}\right) -$$

$$-\left\{\sum_{\mu=1}^{m}\lambda_\mu\log\lambda_\mu - \sum_{\mu=1}^{m}\lambda_\mu + \sum_{\mu=1}^{m}\frac{1}{2}\log\lambda_\mu + \frac{m}{2}\log 2\pi + O\left(\sum_{\mu=1}^{m}\frac{1}{\lambda_\mu}\right)\right\} =$$

$$= n\log n + O(\log n) - \sum_{\mu=1}^{m}\lambda_\mu\log\lambda_\mu + O(m\log\lambda_1) + O(m).$$

Now, Lemma 4 gives that apart from $O\big(p(n)n^{-2}\big)$ Π's at most we have

$$\frac{n!}{\lambda_1!\,\lambda_2!\dots\lambda_m!} = \exp\left\{n\log n - \sum_{\mu=1}^{m}\lambda_\mu\log\lambda_\mu + O\big(\sqrt{n}\log^2 n\big)\right\} =$$

$$= \exp\left\{\frac{1}{2}n\log n + \sum_{\mu=1}^{m}\lambda_\mu\log\sqrt{n} - \sum_{\mu=1}^{m}\lambda_\mu\log\lambda_\mu + O\big(\sqrt{n}\log^2 n\big)\right\} =$$

$$= \sqrt{n!}\exp\left\{\frac{1}{2}n - \sum_{\mu=1}^{m}\lambda_\mu\log\frac{\lambda_\mu}{\sqrt{n}} + O\big(\sqrt{n}\log^2 n\big)\right\} =$$

$$= \sqrt{n!}\exp\left\{\left(\frac{1}{2}+\log\frac{\pi}{\sqrt{6}}\right)n - \sum_{\mu=1}^{m}\lambda_\mu\log\frac{\pi\lambda_\mu}{\sqrt{6n}} + O\big(\sqrt{n}\log^2 n\big)\right\}.$$

This relation gives with (5.1) the fundamental

LEMMA 5. *With the notation* (1.1) *we have*

$$\chi_\Pi(E) = \sqrt{n!}\exp\left\{\left(\frac{1}{2}+\log\frac{\pi}{\sqrt{6}}\right)n - \sum_{\mu=1}^{m}\lambda_\mu\log\frac{\pi\lambda_\mu}{\sqrt{6n}} - \right. \tag{5.2}$$

$$\left. - \sum_{1\le\mu<\nu\le m}\log\frac{1}{1-\dfrac{\lambda_\nu}{\lambda_\mu+\nu-\mu}} + O\big(\sqrt{n}\log^2 n\big)\right\}$$

with the exception of $O\big(p(n)n^{-2}\big)$ *Π's at most.*

6. Next we are going to investigate the sum

$$\sum_{\mu=1}^{m}\lambda_\mu\log\frac{\pi\lambda_\mu}{\sqrt{6n}}. \tag{6.1}$$

Let $\alpha=\alpha(n)$ be restricted by

$$3 \leqq \alpha \leqq \frac{1}{2}\frac{\log n}{\log\log n}-6. \tag{6.2}$$

Then from Lemma 1 we get

$$\lambda_{\left[\alpha\frac{\sqrt{6}}{\pi}\sqrt{n}\log\log n\right]} \leqq c\sqrt{n}\log\frac{1}{1-\exp\left(-\frac{\pi}{\sqrt{6n}}\left[\alpha\frac{\sqrt{6}}{\pi}\sqrt{n}\log\log n\right]\right)} \leqq \tag{6.3}$$

$$\leqq c\sqrt{n}\exp\left(-\frac{\pi}{\sqrt{6n}}\left[\alpha\frac{\sqrt{6}}{\pi}\sqrt{n}\log\log n\right]\right) \leqq c\sqrt{n}\log^{-\alpha} n$$

(6.4): *apart from* $cp(n)n^{-5/4}\log n$ *Π's at most.*

Owing to (6.3) and Lemma 4 we get trivial estimates for two part sums of (6.1) beside (6.4), as follows.

$$\sum_{\substack{\alpha\frac{\sqrt{n}}{\pi}\sqrt{6}\log\log n<\mu\leqq m\\ \mu\text{ integer}}}\lambda_\mu\log\frac{\pi\lambda_\mu}{\sqrt{6n}} = O(m)O\left(\sqrt{n}\log^{-\alpha} n\right)O(\log n) = O(n\log^{2-\alpha} n),$$

$$\sum_{\substack{1\leqq\mu<\frac{\sqrt{6}}{\pi}\sqrt{n}\log^{-\alpha} n\\ \mu\text{ integer}}}\lambda_\mu\log\frac{\pi\lambda_\mu}{\sqrt{6n}} = O\left(\sqrt{n}\log^{-\alpha} n\right)O(\lambda_1)O(\log n) = O(n\log^{2-\alpha} n).$$

Hence, with the notations of Lemma 1, we have beside (6.2) and (6.4)

$$\sum_{\mu=1}^{m}\lambda_\mu\log\frac{\pi\lambda_\mu}{\sqrt{6n}} = \sum_{\substack{\mu\in I_1\\ \mu\text{ integer}}}\lambda_\mu\log\frac{\pi\lambda_\mu}{\sqrt{6n}}+O(n\log^{2-\alpha} n). \tag{6.5}$$

Hereafter, summations over μ refer to integer values of μ. We have to investigate the sum

$$\sum_{\mu\in I_1}\lambda_\mu\log\frac{\pi\lambda_\mu}{\sqrt{6n}}.$$

Using again the abbreviation

$$x_0=\frac{\pi}{\sqrt{6n}} \tag{6.6}$$

Lemma 1 gives for $\mu\in I_1$ beside (6.2) and (6.4)

$$\log\frac{\pi\lambda_\mu}{\sqrt{6n}} = \log\left\{\left(1+O(n^{-1/4}\log^{(\alpha/2)+2} n)\right)\log\frac{1}{1-\exp(-x_0\mu)}\right\} =$$

$$= \log\log\frac{1}{1-\exp(-x_0\mu)}+O\left(n^{-1/4}\log^{(\alpha/2)+2} n\right).$$

From this (using again Lemma 1) we get

$$\sum_{\mu\in I_1}\lambda_\mu\log\frac{\pi\lambda_\mu}{\sqrt{6n}}=\sum_{\mu\in I_1}\lambda_\mu\log\log\frac{1}{1-\exp(-x_0\mu)}+\sum_{\mu\in I_1}\lambda_\mu O(n^{-1/4}\log^{(\alpha/2)+2}n)=$$

$$=\sum_{\mu\in I_1}\bigl(1+O(n^{-1/4}\log^{(\alpha/2)+2}n)\bigr)\frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp(-x_0\mu)}\log\log\frac{1}{1-\exp(-x_0\mu)}+$$

$$+O(n^{3/4}\log^{(\alpha/2)+2}n).\quad \Bigl(\sum_{\mu\in I_1}\lambda_\mu\leqq n.\Bigr)$$

Here

$$O(n^{-1/4}\log^{(\alpha/2)+2}n)\sum_{\mu\in I_1}\frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp(-x_0\mu)}\left|\log\log\frac{1}{1-\exp(-x_0\mu)}\right|=$$

$$=O(n^{-1/4}\log^{(\alpha/2)+2}n)\sum_{\mu\in I_1}O(\lambda_\mu)\left\{\left|\log\frac{\pi\lambda_\mu}{\sqrt{6n}}\right|+O(n^{-1/4}\log^{(\alpha/2)+2}n)\right\}=$$

$$=O(n^{-1/4}\log^{(\alpha/2)+2}n)\sum_{\mu\in I_1}O(\lambda_\mu)O(\log n)=O(n^{3/4}\log^{(\alpha/2)+3}n).$$

Introducing the notations

$$f(t)=\frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1-\exp(-x_0t)}\log\log\frac{1}{1-\exp(-x_0t)}\qquad(t>0)\tag{6.7}$$

and

$$F(\alpha)=\sum_{\mu\in I_1}f(\mu)\tag{6.8}$$

we have beside (6.2) and (6.4)

$$\sum_{\mu\in I_1}\lambda_\mu\log\frac{\pi\lambda_\mu}{\sqrt{6n}}=F(\alpha)+O(n^{3/4}\log^{(\alpha/2)+3}n)$$

i.e. (by (6.5))

$$\sum_{\mu=1}^{m}\lambda_\mu\log\frac{\pi\lambda_\mu}{\sqrt{6n}}=F(\alpha)+O(n^{3/4}\log^{(\alpha/2)+3}n)+O(n\log^{2-\alpha}n).\tag{6.9}$$

Let

$$\begin{cases}t_0=\dfrac{\sqrt{6}}{\pi}\sqrt{n}\log\dfrac{1}{1-\exp(-1)},\\[2mm] t_1=\dfrac{\sqrt{6}}{\pi}\sqrt{n}\log\dfrac{1}{1-\exp(-\exp(-1))};\end{cases}\tag{6.10}$$

$$\begin{cases}t_2=\dfrac{\sqrt{6}}{\pi}\sqrt{n}\log^{-\alpha}n,\\[2mm] t_3=\alpha\dfrac{\sqrt{6}}{\pi}\sqrt{n}\log\log n\end{cases}\qquad(I_1=[t_2,t_3]).\tag{6.11}$$

For $t>0$

$$f'(t)=\frac{1}{\exp(x_0t)-1}\log\frac{\exp(-1)}{\log\dfrac{1}{1-\exp(-x_0t)}}$$

and we have obviously

$$f(t) > 0 \quad \text{if and only if} \quad 0 < t < t_0, \tag{6.12}$$

$$f'(t) > 0 \quad \text{if and only if} \quad t > t_1 (> t_0 > 0) \tag{6.13}$$

and

$$0 < f(t_2) \leqq c\sqrt{n}\log^2 n, \tag{6.14}$$

$$-c\sqrt{n} \leqq f(t_1) < 0. \tag{6.15}$$

Using Euler—Maclaurin's formula and (6.10)—(6.15) it is easy to see that

$$F(\alpha) = \sum_{\mu \in I_1} f(\mu) = \int_{t_2}^{t_3} f(t)\,dt + O\big(\sqrt{n}\log^2 n\big).$$

Substituting

$$y = \log \frac{1}{1-\exp(-x_0 t)}$$

we get

$$F(\alpha) = \frac{6}{\pi^2} n \int_{\log \frac{1}{1-\log^{-\alpha} n}}^{\log \frac{1}{1-\exp(-\log^{-\alpha} n)}} \frac{y \log y}{\exp(y)-1}\,dy + O\big(\sqrt{n}\log^2 n\big). \tag{6.16}$$

But

$$\left| \int_0^{\log \frac{1}{1-\log^{-\alpha} n}} \frac{y\log y}{\exp(y)-1}\,dy \right| = \int_0^{\log \frac{1}{1-\log^{-\alpha} n}} \log\left(\frac{1}{y}\right) \frac{y}{\exp(y)-1}\,dy <$$

$$< \int_0^{2\log^{-\alpha} n} \log \frac{1}{y}\,dy \leqq c \log^{1-\alpha} n$$

and

$$0 < \int_{\log \frac{1}{1-\exp(-\log^{-\alpha} n)} (> \alpha \log\log n)}^{\infty} \frac{y\log y}{\exp(y)-1}\,dy < \int_{\alpha\log\log n}^{\infty} y \log y \frac{\exp(-y)}{1-\exp(-y)}\,dy \leqq$$

$$\leqq c \int_{\alpha \log\log n}^{\infty} y(y-1)\exp(-y)\,dy \leqq c \log^{2-\alpha} n,$$

thus we get from (6.16)

$$F(\alpha) = \frac{6}{\pi^2} n \int_0^{\infty} \frac{y\log y}{\exp(y)-1}\,dy + O(n \log^{2-\alpha} n). \tag{6.17}$$

Finally (6.17) and (6.9) give that the relation

$$\sum_{\mu=1}^{m} \lambda_\mu \log \frac{\pi\lambda_\mu}{\sqrt{6n}} = \frac{6}{\pi^2} n \int_0^{\infty} \frac{y \log y}{\exp(y)-1}\,dy + O(n^{3/4}\log^{(\alpha/2)+3} n) + O(n \log^{2-\alpha} n)$$

holds beside (6.2) and (6.4).

Choosing

$$\alpha = \frac{1}{6}\frac{\log n}{\log\log n} \tag{6.18}$$

the condition (6.2) is fulfilled for $n > c$ and

$$O(n^{3/4}\log^{(\alpha/2)+3} n) + O(n\log^{2-\alpha} n) = O(n^{5/6}\log^3 n).$$

Thus we have proved the

LEMMA 6. *The relation*

$$\sum_{\mu=1}^{m} \lambda_\mu \log \frac{\pi\lambda_\mu}{\sqrt{6n}} = \frac{6}{\pi^2} n \int_0^\infty \frac{y\log y}{\exp(y)-1}\,dy + O(n^{5/6}\log^3 n) \tag{6.19}$$

holds apart from $cp(n)n^{-5/4}\log n$ *Π's at most.*

7. Now we are going to investigate the more intricate sum

$$S = S(\Pi) \overset{\text{def}}{=} \sum_{1\le\mu<\nu\le m} \log \frac{1}{1-\dfrac{\lambda_\nu}{\lambda_\mu+\nu-\mu}} \tag{7.1}$$

in a similar way. We have obviously

$$S = \sum_{h=1}^{m-1}\sum_{d=1}^{m-h} \log \frac{1}{1-\dfrac{\lambda_{h+d}}{\lambda_h+d}}. \tag{7.2}$$

Hereafter, summations over h or d refer to their integer values. Empty sums are to be considered zero.

Let $\alpha = \alpha(n)$ be restricted by

$$3 \le \alpha \le \frac{1}{4}\frac{\log n}{\log\log n} - 3. \tag{7.3}$$

Thus both Lemmas 1 and 2 will be applicable in I_1 and I_2, resp. In order to reduce the investigation to the case $h, d \in I_1$ $(h+d \in I_2)$ we have to estimate the contribution of the "outer" terms.

Let first

$$S_1 \overset{\text{def}}{=} \sum_{h=1}^{m-1} \sum_{\substack{1\le d\le m-h,\\ d<\frac{\sqrt{6}}{\pi}\sqrt{n}\log^{-\alpha} n}} \log \frac{1}{1-\dfrac{\lambda_{h+d}}{\lambda_h+d}}. \tag{7.4}$$

Owing to

$$\frac{\lambda_{h+d}}{\lambda_h+d} \le \frac{\lambda_h}{\lambda_h+1} \le 1 - \frac{1}{\lambda_1+1} \tag{7.5}$$

using also Lemma 4 we get

$$S_1 = O(m)O(\sqrt{n}\log^{-\alpha} n)\log(\lambda_1+1) = O(n\log^{2-\alpha} n)$$

apart from $O(p(n)n^{-2})$ Π's at most.

Next let

$$S_2 \overset{\text{def}}{=} \sum_{1 \leq h < \frac{\sqrt{6}}{\pi}\sqrt{n}\log^{-\alpha} n} \; \sum_{\frac{\sqrt{6}}{\pi}\sqrt{n}\log^{-\alpha} n \leq d \leq m-h} \log \frac{1}{1-\frac{\lambda_{h+d}}{\lambda_h+d}}. \tag{7.6}$$

Using again (7.5) and Lemma 4 we get

$$S_2 = O\big(\sqrt{n}\log^{-\alpha} n\big)O(m)\log(\lambda_1+1) = O(n\log^{2-\alpha} n)$$

apart from $O\big(p(n)n^{-2}\big)$ Π's at most.

Now let

$$S_3 \overset{\text{def}}{=} \sum_{\frac{\sqrt{6}}{\pi}\sqrt{n}\log^{-\alpha} n \leq h \leq m-1} \; \sum_{\alpha\frac{\sqrt{6}}{\pi}\sqrt{n}\log\log n < d \leq m-h} \log \frac{1}{1-\frac{\lambda_{h+d}}{\lambda_h+d}}. \tag{7.7}$$

Owing to $d > \alpha\frac{\sqrt{6}}{\pi}\sqrt{n}\log\log n$ we have

$$\lambda_d \leq \lambda_{\left[\alpha\frac{\sqrt{6}}{\pi}\sqrt{n}\log\log n\right]} \leq c\sqrt{n}\log^{-\alpha} n$$

by Lemma 1

(7.8): *apart from* $cp(n)n^{-5/4}\log n$ *Π's at most.*

Thus we have

$$\frac{\lambda_{h+d}}{\lambda_h+d} < \frac{\lambda_d}{d} < \frac{1}{2} \tag{7.9}$$

for $n > c$ and

$$0 < \log \frac{1}{1-\frac{\lambda_{h+d}}{\lambda_h+d}} < \log\left(1+\frac{\frac{\lambda_d}{d}}{1-\frac{\lambda_d}{d}}\right) < 2\frac{\lambda_d}{d}.$$

Hence, using again Lemma 4

$$S_3 = O(m)\sum_{\alpha\frac{\sqrt{6}}{\pi}\sqrt{n}\log\log n < d \leq m} \frac{\lambda_d}{d} = O(m)O\big(\sqrt{n}\log^{-\alpha} n\big) \sum_{3\frac{\sqrt{6}}{\pi}\sqrt{n}\log\log n < d \leq m} \frac{1}{d} =$$

$$= O(n\log^{2-\alpha} n)$$

beside (7.8).

Defining $\lambda_j = 0$ for $j > m$ we investigate

$$S_4 \overset{\text{def}}{=} \sum_{\alpha\frac{\sqrt{6}}{\pi}\sqrt{n}\log\log n < h \leq m-1} \; \sum_{d \in I_1} \log \frac{1}{1-\frac{\lambda_{h+d}}{\lambda_h+d}}. \tag{7.10}$$

Owing to $h > \alpha\frac{\sqrt{6}}{\pi}\sqrt{n}\log\log n$ we have

$$\lambda_h \leq c\sqrt{n}\log^{-\alpha} n$$

by Lemma 1 beside (7.8), further from

$$0 \leqq \frac{\lambda_{h+d}}{\lambda_h+d} \leqq \frac{\lambda_h}{\lambda_h+d}$$

we get

$$0 \leqq \log \frac{1}{1-\frac{\lambda_{h+d}}{\lambda_h+d}} \leqq$$

$$\leqq \log\left(1+\frac{\lambda_h}{d}\right) < \frac{\lambda_h}{d} = \frac{1}{d} O\left(\sqrt{n}\log^{-\alpha} n\right),$$

therefore,

$$S_4 = O(m) O\left(\sqrt{n}\log^{-\alpha} n\right) \sum_{d\in I_1} \frac{1}{d} = O(n \log^{2-\alpha} n).$$

From the estimations of the sums S_1, S_2, S_3, S_4 we get the fundamental relation

$$S = \sum_{h\in I_1} \sum_{d\in I_1} \log \frac{1}{1-\frac{\lambda_{h+d}}{\lambda_h+d}} + O(n\log^{2-\alpha} n) \tag{7.11}$$

beside (7.3) and (7.8). Here,

$$h, d, h+d \in I_2 \tag{7.12}$$

follows from $h, d \in I_1$ and we can use (2.20) beside (7.3) and (7.8).

We shall apply (2.20) for $\mu=h$ and $\mu=h+d$. ($h, d\in I_1$). Thus we have

$$\lambda_\mu = \left(1+O(n^{-1/4}\log^{\alpha+2} n)\right)\lambda_\mu^* \tag{7.13}$$

where

$$\lambda_\mu^* = \frac{\sqrt{6}}{\pi}\sqrt{n}\log \frac{1}{1-\exp\left(-\frac{\pi\mu}{\sqrt{6n}}\right)} \tag{7.14}$$

(beside (7.3) and (7.8)).

Hereafter we use the restriction

$$3 \leqq \alpha \leqq \frac{1}{8}\frac{\log n}{\log\log n} - 2 \tag{7.15}$$

which is stronger than (7.3).

Thus for $h, d \in I_1$ we get

$$\log \frac{1-\frac{\lambda_{h+d}^*}{\lambda_h^*+d}}{1-\frac{\lambda_{h+d}}{\lambda_h+d}} = \log\left(1+\frac{(\lambda_{h+d}-\lambda_{h+d}^*)-(\lambda_h-\lambda_h^*)\frac{\lambda_{h+d}^*}{\lambda_h^*+d}}{d+\lambda_h-\lambda_{h+d}}\right) =$$

$$= \log\left(1+\frac{1}{d} O(n^{-1/4}\log^{\alpha+2} n)\lambda_{h+d}^*\left(1+\frac{\lambda_h^*}{\lambda_h^*+d}\right)\right) =$$

$$= \log\left(1+\frac{\lambda_{h+d}^*}{d} O(n^{-1/4}\log^{\alpha+2} n)\right) = \frac{\lambda_{h+d}^*}{d} O(n^{-1/4}\log^{\alpha+2} n) = \frac{1}{d} O(n^{1/4}\log^{\alpha+3} n)$$

owing to (7.15), and (for $n>c$)

$$\lambda_{h+d}^* \leqq \alpha \frac{\sqrt{6}}{\pi} \sqrt{n} \log\log n, \quad \frac{\lambda_{h+d}^*}{d} = O(\log^{\alpha+1} n)$$

$\left(\text{from } h+d \geqq 2\frac{\sqrt{6}}{\pi}\sqrt{n}\log^{-\alpha} n \text{ and } d \geqq \frac{\sqrt{6}}{\pi}\sqrt{n}\log^{-\alpha} n\right)$ and

$$\frac{\lambda_{h+d}^*}{d} O(n^{-1/4}\log^{\alpha+2} n) = O(\log^{-1} n).$$

Thus we have

$$\sum_{h\in I_1}\sum_{d\in I_1} \log \frac{1}{1-\frac{\lambda_{h+d}}{\lambda_h+d}} = \sum_{h\in I_1}\sum_{d\in I_1} \log \frac{1}{1-\frac{\lambda_{h+d}^*}{\lambda_h^*+d}} +$$

$$+\sum_{h\in I_1}\sum_{d\in I_1} \frac{1}{d} O(n^{1/4}\log^{\alpha+3} n).$$

Here,

$$\sum_{h\in I_1}\sum_{d\in I_1} \frac{1}{d} = O(\alpha\sqrt{n}\log\log n)O(\log n) = O(\sqrt{n}\log^2 n)$$

and we get from (7.11) the relation

$$S = \sum_{h\in I_1}\sum_{d\in I_1} \log \frac{1}{1-\frac{\lambda_{h+d}^*}{\lambda_h^*+d}} + O(n^{3/4}\log^{\alpha+5} n) + O(n\log^{2-\alpha} n) \tag{7.16}$$

beside (7.15) and (7.8).

8. In order to substitute the sums in (7.16) by integrals let us investigate the sum

$$\sum_{d\in I_1} \log \frac{1}{1-\frac{\lambda_{h+d}^*}{\lambda_h^*+d}}$$

for fixed $h\in I_1$. (The O-constants will be independent of h too.)

Let

$$f_h(d) = \log \frac{1}{1-\frac{\lambda_{h+d}^*}{\lambda_h^*+d}}. \tag{8.1}$$

It follows from (7.14) that $f_h(d)$ is positive and monotonically decreasing in d. Thus we have

$$\sum_{d\in I_1} f_h(d) = \int_{I_1} f_h(y)\,dy + O(f_h(d_0)) \tag{8.2}$$

where $d_0=\frac{\sqrt{6}}{\pi}\sqrt{n}\log^{-\alpha} n$. Using also (7.13) we get (beside (7.15) and (7.8))

$$0<f_h(d_0)\leqq \log\frac{1}{1-\frac{\lambda_h^*}{\lambda_h^*+d_0}}=\log\left(1+\frac{\lambda_h^*}{d_0}\right)=\frac{O(\lambda_h)}{d_0},$$

further,

$$\sum_{h\in I_1} f_h(d_0)=\sum_{h\in I_1}\frac{O(\lambda_h)}{d_0}=O\left(\frac{n}{d_0}\right)=O\left(\sqrt{n}\log^{\alpha} n\right)=O(n^{3/4}\log^{\alpha+5} n).$$

Thus we get from (7.16) and (8.2) the relation

$$S=\int_{I_1}\left(\sum_{h\in I_1}\log\frac{1}{1-\frac{\lambda_{h+y}^*}{\lambda_h^*+y}}\right)dy+O(n^{3/4}\log^{\alpha+5} n)+O(n\log^{2-\alpha} n) \tag{8.3}$$

beside (7.15) and (7.8).

Next we investigate the sum

$$\sum_{h\in I_1}\log\frac{1}{1-\frac{\lambda_{h+y}^*}{\lambda_h^*+y}}$$

for fixed $y\in I_1$. (The O-constants will be independent of y too.)

Let

$$g_y(h)=\log\frac{1}{1-\frac{\lambda_{h+y}^*}{\lambda_h^*+y}}.$$

Using (7.13)—(7.14) and Lemma 4 we get (beside (7.15) and (7.8)) for $h, y\in I_1$

$$0<g_y(h)\leqq\log\frac{1}{1-\frac{\lambda_h^*}{\lambda_h^*+y}}=\log\left(1+\frac{\lambda_h^*}{y}\right)=\frac{1}{y}O(\lambda_h)=\frac{1}{y}O\left(\sqrt{n}\log n\right), \tag{8.4}$$

further for $x, y\in I_1$

$$\frac{d}{dx}g_y(x)=\frac{1}{\lambda_x^*-\lambda_{x+y}^*+y}\left\{\frac{-1}{\exp\left(\frac{\pi(x+y)}{\sqrt{6n}}\right)-1}+\frac{\lambda_{x+y}^*}{\lambda_x^*+y}\cdot\frac{1}{\exp\left(\frac{\pi x}{\sqrt{6n}}\right)-1}\right\},$$

therefore,

$$\left|\frac{d}{dx}g_y(x)\right|<\frac{2}{y}\cdot\frac{1}{\exp\left(\frac{\pi x}{\sqrt{6n}}\right)-1}=O\left(\frac{\sqrt{n}}{y}\cdot\frac{1}{x}\right). \tag{8.5}$$

Using Euler—Maclaurin's formula and (8.4)—(8.5) it is easy to see that

$$\sum_{h\in I_1} g_y(h) = \int_{I_1} g_y(x)\,dx + O\left(\frac{\sqrt{n}\log n}{y}\right) + O\left(\int_{I_1} \frac{\sqrt{n}}{y}\cdot\frac{1}{x}\,dx\right) =$$

$$= \int_{I_1} g_y(x)\,dx + O(\sqrt{n}\log n)\frac{1}{y}.$$

Owing to (8.3) and

$$\int_{I_1} O(\sqrt{n}\log n)\frac{1}{y}\,dy = O(\sqrt{n}\log^2 n) = O(n^{3/4}\log^{\alpha+5} n)$$

we have proved the relation

$$S = \int_{I_1}\int_{I_1} \log\frac{1}{1-\dfrac{\lambda^*_{x+y}}{\lambda^*_x+y}}\,dx\,dy + O(n^{3/4}\log^{\alpha+5} n) + O(n\log^{2-\alpha} n) \tag{8.6}$$

i.e. (by trivial substitutions)

$$S = \frac{6}{\pi^2} n \int_{\log^{-\alpha} n}^{\alpha\log\log n}\int_{\log^{-\alpha} n}^{\alpha\log\log n} \log\frac{1}{1-\dfrac{\log\dfrac{1}{1-\exp(-x-y)}}{y+\log\dfrac{1}{1-\exp(-x)}}}\,dx\,dy + \tag{8.7}$$

$$+O(n^{3/4}\log^{\alpha+5} n) + O(n\log^{2-\alpha} n)$$

beside (7.15) and (7.8).

9. Next we extend the integrals in (8.7). For $x, y > 0$ let

$$f(x, y) \overset{\text{def}}{=} \log\frac{1}{1-\dfrac{\log\dfrac{1}{1-\exp(-x-y)}}{y+\log\dfrac{1}{1-\exp(-x)}}}. \tag{9.1}$$

Owing to

$$f(x, y) < \log\frac{1}{1-\dfrac{\log\dfrac{1}{1-\exp(-x-y)}}{y+\log\dfrac{1}{1-\exp(-x-y)}}}$$

the following inequalities hold obviously for $x, y>0$.

(9.2) $$0<f(x, y)<\log\left(1+\frac{1}{y}\log\frac{1}{1-\exp(-x-y)}\right),$$

(9.3) $$0<f(x, y)<\frac{1}{y}\log\frac{1}{1-\exp(-x-y)},$$

(9.4) $$0<f(x, y)<\frac{1}{y}\log\frac{1}{1-\exp(-x)}.$$

Using (9.4) and the substitution

$$u=\log\frac{1}{1-\exp(-x)}$$

we get

$$0<\int_0^{\log^{-\alpha} n} f(x, y)\,dx \leqq \frac{1}{y}\int_{\log\frac{1}{1-\exp(-\log^{-\alpha} n)}}^{+\infty}\frac{u}{\exp(u)-1}\,du \leqq$$

$$\leqq \frac{1}{y}\int_{\alpha\log\log n}^{+\infty}\frac{u\exp(-u)}{1-\exp(-u)}\,du \leqq \frac{c}{y}\int_{\alpha\log\log n}^{+\infty} u\exp(-u)\,du \leqq \frac{c}{y}\log^{1-\alpha} n$$

and

$$0<\int_{\alpha\log\log n}^{\infty} f(x, y)\,dx \leqq \frac{1}{y}\int_0^{\log\frac{1}{1-\log^{-\alpha} n}}\frac{u}{\exp(u)-1}\,du \leqq$$

$$\leqq \frac{1}{y}\log\frac{1}{1-\log^{-\alpha} n} \leqq \frac{c}{y}\log^{-\alpha} n.$$

Since

$$\int_{\log^{-\alpha} n}^{\alpha\log\log n}\frac{O(\log^{1-\alpha} n)}{y}\,dy = O(\log^{1-\alpha} n)O\big(\log(\log^{\alpha+1} n)\big) = O(\log^{2-\alpha} n)$$

we get from (8.7) the relation

(9.5) $$S=\frac{6}{\pi^2}n\int_{\log^{-\alpha} n}^{\alpha\log\log n}\Big(\int_0^{\infty} f(x, y)\,dx\Big)\,dy+O(n^{3/4}\log^{\alpha+5} n)+O(n\log^{2-\alpha} n)$$

beside (9.1), (7.15) and (7.8).

Using (9.3) and

(9.6) $$\frac{2\cdot\frac{u}{2}}{\exp(u)-1}<2\frac{\exp\left(\frac{u}{2}\right)-1}{\exp(u)-1}<2\exp\left(-\frac{u}{2}\right)$$

we get similarly

$$0<\int_0^\infty f(x,y)\,dx \leqq \frac{1}{y}\int_0^\infty \log\frac{1}{1-\exp(-x-y)}\,dx =$$

$$=\frac{1}{y}\int_y^\infty \log\frac{1}{1-\exp(-x)}\,dx = \frac{1}{y}\int_0^{\log\frac{1}{1-\exp(-y)}} \frac{u}{\exp(u)-1}\,du \leqq \frac{4}{y}\exp(-y).$$

Consequently,

$$0<\int_{\alpha\log\log n}^\infty \left(\int_0^\infty f(x,y)\,dx\right)dy \leqq 4\int_{\alpha\log\log n}^\infty \frac{1}{y}\exp(-y)\,dy \leqq$$

$$\leqq \frac{4}{\alpha\log\log n}\int_{\alpha\log\log n}^\infty \exp(-y)\,dy \leqq c\log^{-\alpha} n.$$

Finally, using (9.2) and (for arbitrary integer $K\geqq 3$ and for $w>0$)

$$\text{(9.7)}\qquad \frac{K\cdot\frac{w}{K}}{1-\exp(-w)} \leqq K\exp(w)\frac{\exp\left(\frac{w}{K}\right)-1}{\exp(w)-1} \leqq K\exp\left(\frac{w}{K}\right)$$

we get

$$0<\int_0^\infty f(x,y)\,dx < \int_0^\infty \log\left(1+\frac{1}{y}\log\frac{1}{1-\exp(-x-y)}\right)dx =$$

$$=\int_y^\infty \log\left(1+\frac{1}{y}\log\frac{1}{1-\exp(-x)}\right)dx =$$

$$=\int_0^{\log\frac{1}{1-\exp(-y)}} \frac{\log\left(1+\frac{u}{y}\right)}{\exp(u)-1}\,du < \int_0^{\log\frac{1}{1-\exp(-y)}} \frac{1}{u}\log\left(1+\frac{u}{y}\right)du =$$

$$=\int_0^{M(y)} \frac{w}{1-\exp(-w)}\,dw \qquad \left(w=\log\left(1+\frac{u}{y}\right)\right)$$

where

$$\text{(9.8)}\qquad M(y)=\log\left(1+\frac{1}{y}\log\frac{1}{1-\exp(-y)}\right).$$

But from (9.7) and (9.8)

$$\int_0^{M(y)} \frac{w}{1-\exp(-w)}\,dw \leqq K\int_0^{M(y)} \exp\left(\frac{w}{K}\right)dw = K^2\left\{\exp\left(\frac{M(y)}{K}\right)-1\right\} <$$

$$< K^2\frac{\exp(M(y))-1}{\exp\left(M(y)\frac{K-1}{K}\right)} < K^2\left(\frac{1}{y}\log\frac{1}{1-\exp(-y)}\right)^{1/K} <$$

$$< K^2\left(\frac{1}{y}\log\left(1+\frac{1}{y}\right)\right)^{1/K} < K^2\left(\frac{1}{y}\right)^{2/K}.$$

Thus we have (for $K \geqq 3$)

$$0 < \int_0^{\log^{-\alpha} n} \left(\int_0^\infty f(x, y)\, dx\right) dy < \int_0^{\log^{-\alpha} n} K^2 \left(\frac{1}{y}\right)^{2/K} dy =$$

$$= \frac{K^2}{1-\frac{2}{K}} (\log^{-\alpha} n)^{1-(2/K)} \leqq 3K^2 (\log^{-\alpha} n) \log^{2\alpha/K} n \leqq 3K^2 (\log^{-\alpha} n) \exp\left(\frac{\log n}{4K}\right)$$

(from (7.15)). Choosing

$$K = [\log n] + 1 \geqq 3 \quad (\text{for} \quad n > c)$$

we get

$$0 < \int_0^{\log^{-\alpha} n} \left(\int_0^\infty f(x, y)\, dx\right) dy \leqq c \log^{2-\alpha} n.$$

In view of (9.5) we have proved the relation

$$S = \frac{6}{\pi^2} n \int_0^\infty \int_0^\infty f(x, y)\, dx\, dy + O(n^{3/4} \log^{\alpha+5} n) + O(n \log^{2-\alpha} n) \tag{9.9}$$

beside (9.1), (7.15) and (7.8).

Choosing

$$\alpha = \frac{1}{8} \frac{\log n}{\log \log n} - 2$$

(7.15) is fulfilled for $n > c$ and we have

$$O(n^{3/4} \log^{\alpha+5} n) + O(n \log^{2-\alpha} n) = O(n^{7/8} \log^4 n).$$

Thus we have proved the

LEMMA 7. *With the notations of* (7.1) *and* (9.1) *the relation*

$$S(\Pi) = \frac{6}{\pi^2} n \int_0^\infty \int_0^\infty f(x, y)\, dx\, dy + O(n^{7/8} \log^4 n) \tag{9.10}$$

holds apart from $cp(n) n^{-5/4} \log n$ *Π's at most.*

10. In view of Lemmas 5, 6, 7 and (for $n > c$)

$$cp(n) n^{-5/4} \log n < \frac{1}{n} p(n)$$

we have proved Theorem VI apart from the estimation (1.5).

Let

$$J \overset{\text{def}}{=} \int_0^\infty \int_0^\infty \log \frac{1}{1 - \dfrac{\log \dfrac{1}{1-\exp(-x-y)}}{y + \log \dfrac{1}{1-\exp(-x)}}}\, dx\, dy. \tag{10.1}$$

We have proved the relation (1.3) with the constant A defined by (1.4). Thus we have

$$(10.2)\qquad A = -\frac{1}{2} - \log\frac{\pi}{\sqrt{6}} + \frac{6}{\pi^2}\int_0^\infty \frac{y\log y}{\exp(y)-1}\,dy + \frac{6}{\pi^2}J$$

and we have to prove the inequality

$$A > \frac{6}{\pi^2}0.02.$$

To determine the integral

$$\int_0^\infty \frac{y\log y}{\exp(y)-1}\,dy$$

we shall need two well-known relations.

Let γ denote Euler's constant. Then we have for $x>0$

$$(10.3)\qquad \frac{\Gamma'(x)}{\Gamma(x)} + \gamma = \int_0^1 \frac{1-t^{x-1}}{1-t}\,dt,$$

in particular,

$$(10.4)\qquad \Gamma'(2) = (1-\gamma)\Gamma(2) = 1-\gamma.$$

Next, we have for $\operatorname{Re} s>1$

$$(10.5)\qquad \zeta(s)\Gamma(s) = \int_0^\infty \frac{u^{s-1}}{\exp(u)-1}\,du,$$

consequently,

$$(10.6)\qquad (\zeta(s)\Gamma(s))' = \int_0^\infty \frac{u^{s-1}\log u}{\exp(u)-1}\,du.$$

Applying (10.6) with $s=2$ and using (10.4) we get

$$(10.7)\qquad \int_0^\infty \frac{y\log y}{\exp(y)-1}\,dy = \zeta'(2)\Gamma(2)+\zeta(2)\Gamma'(2) =$$

$$= -\sum_{\nu=2}^\infty \frac{\log\nu}{\nu^2} + \frac{\pi^2}{6}(1-\gamma)$$

and we have

$$(10.8)\qquad A = \frac{6}{\pi^2}\left\{J - \sum_{\nu=2}^\infty \frac{\log\nu}{\nu^2} + \frac{\pi^2}{6}\left(1-\gamma-\log\frac{\pi}{\sqrt{6}}-\frac{1}{2}\right)\right\}.$$

Using (10.1) it is easy to see that

$$(10.9)\qquad J = \int_0^\infty\int_0^\infty \log\left(1+\frac{\log\dfrac{1}{1-\exp(-x-y)}}{\log\dfrac{\exp(y)-\exp(-x)}{1-\exp(-x)}}\right)dx\,dy.$$

Easy calculations show the validity of the following assertions.

Let for $x, y > 0$

$$(10.10)\qquad \begin{cases} r = \dfrac{\log \dfrac{1}{1-\exp(-x-y)}}{\log \dfrac{\exp(y)-\exp(-x)}{1-\exp(-x)}}, \\ s = \log \dfrac{\exp(y)-\exp(-x)}{1-\exp(-x)}. \end{cases}$$

Then $r, s > 0$. Let for $r, s > 0$

$$(10.11)\qquad \varphi(r, s) \overset{\text{def}}{=} \log \frac{1-\exp(-sr)\{1-\exp(-s)\}}{1-\exp(-sr)},$$

$$(10.12)\qquad \psi(r, s) \overset{\text{def}}{=} \log \frac{1}{1-\exp(-sr)\{1-\exp(-s)\}}.$$

Then

$$(10.13)\qquad \begin{cases} x = \varphi(r, s) \\ y = \psi(r, s) \end{cases}$$

establishes a one-to-one correspondence between the open quarter-planes $\{r, s > 0\}$ and $\{x, y > 0\}$. Further $\varphi(r, s)$ and $\psi(r, s)$ have continuous partial derivatives and

$$\begin{vmatrix} \dfrac{\partial\varphi}{\partial r} & \dfrac{\partial\varphi}{\partial s} \\ \dfrac{\partial\psi}{\partial r} & \dfrac{\partial\psi}{\partial s} \end{vmatrix} = -s \cdot \exp(-sr) \left\{ \frac{1}{1-\exp(-sr)} - \frac{1}{1-\exp(-sr)\{1-\exp(-s)\}} \right\}.$$

Thus we have (from (10.9))

(10.14)

$$J = \int_0^\infty \int_0^\infty \log(1+r) s \cdot \exp(-sr) \left\{ \frac{1}{1-\exp(-sr)} - \frac{1}{1-\exp(-sr)\{1-\exp(-s)\}} \right\} ds\, dr.$$

From (10.14) one can get easily

$$(10.15)\qquad J = \sum_{\nu=2}^\infty \int_0^\infty \log(1+r) \Big(\int_0^\infty s \cdot \exp(-\nu s r) \{1-(1-\exp(-s))^{\nu-1}\}\, ds \Big)\, dr =$$

$$= \sum_{\nu=2}^\infty \sum_{k=0}^{\nu-2} \int_0^\infty \log(1+r) \Big(\int_0^\infty s \cdot \exp(-s(\nu r+1))(1-\exp(-s))^k\, ds \Big)\, dr.$$

We remark that here

$$(10.16)\qquad \int_0^\infty \log(1+r) \Big(\int_0^\infty s \cdot \exp(-s(\nu r+1))(1-\exp(-s))^k\, ds \Big)\, dr \geqq 0.$$

It follows from (10.15) that

$$J=\sum_{\nu=2}^{\infty}\sum_{k=0}^{\nu-2}\left(\sum_{l=0}^{k}\binom{k}{l}(-1)^l\int_0^{\infty}\frac{\log(1+r)}{(\nu r+l+1)^2}\,dr\right)= \tag{10.17}$$

$$=\sum_{\nu=2}^{\infty}\sum_{k=0}^{\nu-2}\frac{1}{\nu}\left(\sum_{l=0}^{k}\binom{k}{l}(-1)^l\frac{\log\nu-\log(l+1)}{\nu-(l+1)}\right)$$

and from (10.16) that here

$$\sum_{l=0}^{k}\binom{k}{l}(-1)^l\frac{\log\nu-\log(l+1)}{\nu-(l+1)}\geqq 0. \tag{10.18}$$

Continuing (10.17),

$$J=\sum_{k=0}^{\infty}\sum_{\nu=k+2}^{\infty}\frac{1}{\nu}\left(\sum_{l=0}^{k}\binom{k}{l}(-1)^l\frac{\log\nu-\log(l+1)}{\nu-(l+1)}\right)=$$

$$=\sum_{\nu=2}^{\infty}\frac{1}{\nu}\frac{\log\nu}{\nu-1}+\sum_{\nu=3}^{\infty}\frac{1}{\nu}\left\{\frac{\log\nu}{\nu-1}-\frac{\log\nu-\log 2}{\nu-2}\right\}+$$

$$+\sum_{\nu=4}^{\infty}\frac{1}{\nu}\left\{\frac{\log\nu}{\nu-1}-\frac{2(\log\nu-\log 2)}{\nu-2}+\frac{\log\nu-\log 3}{\nu-3}\right\}+$$

$$+\sum_{k=3}^{\infty}\sum_{\nu=k+2}^{\infty}\frac{1}{\nu}\left(\sum_{l=0}^{k}\binom{k}{l}(-1)^l\frac{\log\nu-\log(l+1)}{\nu-(l+1)}\right)=$$

$$=\sum_{\nu=2}^{\infty}\frac{\log\nu}{\nu^2}+\frac{1}{72}(141\log 2-52\log 3)+6\sum_{\nu=5}^{\infty}\frac{\log\nu}{\nu^2(\nu-1)(\nu-2)(\nu-3)}+$$

$$+\sum_{k=3}^{\infty}\sum_{\nu=k+2}^{\infty}\frac{1}{\nu}\left(\sum_{l=0}^{k}\binom{k}{l}(-1)^l\frac{\log\nu-\log(l+1)}{\nu-(l+1)}\right).$$

Thus we get from (10.8) the

LEMMA 8. *For the constant A in Theorem* VI *we have*

$$A=\frac{6}{\pi^2}\left\{\frac{1}{72}(141\log 2-52\log 3)-\frac{\pi^2}{6}\left(\gamma-\frac{1}{2}+\frac{1}{2}\log\frac{\pi^2}{6}\right)+\right. \tag{10.19}$$

$$\left.+6\sum_{\nu=5}^{\infty}\frac{\log\nu}{\nu^2(\nu-1)(\nu-2)(\nu-3)}+\sum_{k=3}^{\infty}\sum_{\nu=k+2}^{\infty}\frac{1}{\nu}\left(\sum_{l=0}^{k}\binom{k}{l}(-1)^l\frac{\log\nu-\log(l+1)}{\nu-(l+1)}\right)\right\}.$$

11. It follows from Lemma 8 and (10.18) that

$$A>\frac{6}{\pi^2}\left\{\frac{1}{72}(141\log 2-52\log 3)-\frac{\pi^2}{6}\left(\gamma-\frac{1}{2}+\frac{1}{2}\log\frac{\pi^2}{6}\right)\right\}. \tag{11.1}$$

Using (11.1) and $\gamma=0.577\,215\,664\,9\ldots$ further $\frac{\pi^2}{6}=1.644\,934\,066\,8\ldots$ we get

(11.2) $$A>\frac{6}{\pi^2}0.02.$$

Thus Theorem VI is completely proved.

12. We mentioned the very risky conjecture (2.3) in [1] that the relation

$$\chi_\Pi(E)=\sqrt{n!}\exp\left(-An+A_1\sqrt{n}\log^2 n+A_2\sqrt{n}\log n+O(\sqrt{n}\log\log n)\right)$$

holds with suitable constants A_1 and A_2 for almost all Π's. This relation would be near the best possible since P. Erdős proved (oral communication) that the relation

$$\log\chi_\Pi(E)=g(n)+O\left(\frac{\sqrt{n}}{\log n}\right)$$

cannot be true for *almost all* partitions of n. We sketch his proof here.

From the estimations in Section 11 of [2] it is easy to see that for *arbitrary* fixed positive constant c there exists a positive constant c' such that for $c'p(n)$ partitions of n *at least* the inequality

(12.1) $$S_2(n,\Pi,1)\geqq c\sqrt{n}$$

holds, i.e. these partitions of n contain 1 as summand $[c\sqrt{n}]$-times at least. Omitting $[c\sqrt{n}]$ of the 1's and increasing λ_1 to $\lambda_1+[c\sqrt{n}]$ in the partitions in question we get $c'p(n)$ "new" partitions of n *at least* with the property

(12.2) $$\lambda_1-\lambda_2\geqq[c\sqrt{n}].$$

Now, let us suppose that there exist a function $g(n)$ and a positive constant c_0 such that the relation

(12.3) $$|\log\chi_\Pi(E)-g(n)|\leqq c_0\frac{\sqrt{n}}{\log n}$$

holds for *almost all* partitions of n. Choosing

(12.4) $$c=20\sqrt{c_0},$$

the relations (12.2), (12.3) and

(12.5) $$\lambda_1+m\leqq 5\sqrt{n}\log n\quad(\text{and } m>2)$$

hold simultaneously for $(c'-o(1))p(n)$ partitions of n at least. For *each* partition Π from these ones we construct a partition Π' by

$$\lambda_1'=\lambda_1-\left[\frac{c}{2}\sqrt{n}\right],\quad \lambda_2'=\lambda_2+\left[\frac{c}{2}\sqrt{n}\right],\quad \lambda_3'=\lambda_3,\ldots,\lambda_m'=\lambda_m.$$

(Here the inequality $\lambda_1'\geqq\lambda_2'$ holds owing to (12.2).) It is easy to see that — for

$k=\left[\frac{c}{2}\sqrt{n}\right]$ —

$$\frac{\chi_{\Pi'}(E)}{\chi_{\Pi}(E)} = \left(1-\frac{2k}{\lambda_1-\lambda_2+1}\right)\prod_{\nu=3}^{m}\left(1+\frac{k(\lambda_1-\lambda_2+1-k)}{(\lambda_1-\lambda_\nu+\nu-1)(\lambda_2-\lambda_\nu+\nu-2)}\right)\cdot$$

$$\cdot\prod_{r=1}^{k}\left(1+\frac{\lambda_1-\lambda_2+1-k}{\lambda_2+m-2+r}\right).$$

Using (12.2) and (12.5) we get

$$\frac{\chi_{\Pi'}(E)}{\chi_{\Pi}(E)} \geqq \frac{1}{2k+1}\prod_{r=1}^{k}\left(1+\frac{k+1}{\lambda_1-2k+m-2+r}\right) \geqq$$

$$\geqq \frac{1}{2k+1}\left(1+\frac{k+1}{\lambda_1+m}\right)^k \geqq \frac{1}{2k+1}\left(1+\frac{c}{10\log n}\right)^k.$$

Consequently, for $n>n_0$ we have the inequalities

$$\log\frac{\chi_{\Pi'}(E)}{\chi_{\Pi}(E)} \geqq \frac{kc}{10\log n}-O\left(\frac{\sqrt{n}}{\log^2 n}\right) \geqq \frac{c^2}{100}\frac{\sqrt{n}}{\log n} = 4c_0\frac{\sqrt{n}}{\log n},$$

i.e. (by (12.3))

$$\log\chi_{\Pi'}(E) \geqq g(n)+3c_0\frac{\sqrt{n}}{\log n}.$$

Therefore, for $(c'-o(1))p(n)$ partitions Π' of n *at least* the inequality

$$\log\chi_{\Pi'}(E)-g(n) \geqq 3c_0\frac{\sqrt{n}}{\log n}$$

holds in contradiction with the relation (12.3) of type "almost all".

References

[1] M. Szalay and P. Turán, On some problems of the statistical theory of partitions with application to characters of the symmetric group. I, *Acta Math. Acad. Sci. Hungar.*, **29** (1977), 361—379.

[2] M. Szalay and P. Turán, On some problems of the statistical theory of partitions with application to characters of the symmetric group. II, *Acta Math. Acad. Sci. Hungar.*, **29** (1977), 381—392.

[3] F. D. Murnaghan, *The theory of group representations*, The Johns Hopkins Press (Baltimore, 1949), p. 119.

[4] G. H. Hardy and S. Ramanujan, Asymptotic formulae in combinatory analysis, *Proc. London Math. Soc.*, **17 (2)** (1918), 75—115.

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DEPARTMENT OF ALGEBRA AND NUMBER THEORY
EÖTVÖS LORÁND UNIVERSITY
H-1088 BUDAPEST, MÚZEUM KÖRÚT 6—8.

MATHEMATICAL INSTITUTE OF THE
HUNGARIAN ACADEMY OF SCIENCES
H-1053 BUDAPEST, REÁLTANODA U. 13—15.

Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 32 (1—2), (1978), 157—160.

AN EXTREMAL PROBLEM FOR 3-GRAPHS

By
P. FRANKL (Budapest)

1. Introduction. Let X be a finite set of cardinality n. Let k, l, i be integers, $i<k<n$, $l\geqq 2$. Let $\mathcal{F}$ be a family of k-subsets of X such that there are no l members $F_1, \ldots, F_l$ of $\mathcal{F}$ which form a Δ-system having a kernel of cardinality i, i.e. $F_1\cap F_2= =F_t\cap F_s$ for $1\leqq t<s\leqq l$, and $|F_1\cap F_2|=i$. Let $f(n, l, i, k)$ denote the maximum possible cardinality of $\mathcal{F}$. This function was introduced by Duke and Erdős. In this terminology the ERDŐS—KO—RADO theorem (see [3]) states that for $n\geqq 2k$ $f(n, 2, 0, k)=\binom{n-1}{k-1}$. A recent result of the present author (see [4]) can be restated as $f(n, 2, 1, k)=\binom{n-2}{k-2}$ for $k\geqq 4$, $n>n_0(k)$. DUKE and ERDŐS [2] proved $f(n, l, 1, k)<c(k, l)\binom{n-2}{k-2}$ where $c(k, l)$ is a constant depending on k and l only. In the case $k=3$ they observed $f(n, l, 1, 3)\geqq l(l-1)(n-2l)$ for odd values of l, and $f(n, l, 1, 3)\geqq (l-1)^2(n-2(l-1))$ if l is even. This latter estimation can be improved to $f(n, l, 1, 3)\geqq l\left(l-\frac{3}{2}\right)(n-2l+1)$. To prove this or the lower bound for odd values of l let us make the following construction. Let Y be a subset of X, and let $\mathcal{G}$ be an ordinary 2-graph on Y, which has maximal degree less than l, and contains at most $l-1$ independent edges. Let us define $\mathcal{F}_{\mathcal{G}}=\{E\cup x|E$ is an edge of $\mathcal{G}$, $x\in X-Y\}$. It is not hard to see that $\mathcal{F}_{\mathcal{G}}$ satisfies the assumptions. In the case of even values of l we can take $\mathcal{G}$ to be a graph on $2l-1$ vertices and with degree-sequence $l-1, l-1, \ldots, l-1, l-2$. For odd values of l, let $\mathcal{G}$ be the disjoint union of two copies of the complete graph on l vertices. Duke and Erdős conjectured that $f(n, 3, 1, 3)=6(n-6)+2$. The aim of this paper is to prove the following

THEOREM.

$$f(n, l, 1, 3) < \frac{5}{3}\, l(l-1)n \tag{1}$$

and for $n\geqq 54$

$$f(n, 3, 1, 3) = 6(n-6)+2. \tag{2}$$

2. The proof of the results. Let us define the following family of pairs of sets. $\mathcal{D}=\{(D, x)|x\in X,\ D\subset X,\ |D|=2,\ (D\cup x)\in\mathcal{F}$, there are at least $2l-1$ different members of $\mathcal{F}$ containing $D\}$. Let $\mathcal{F}^*$ be the family of those members of $\mathcal{F}$ which cannot be obtained as a union of the form $D\cup x$, where $(D, x)\in\mathcal{D}$. It is clear from the definitions that the following inequality holds:

$$|\mathcal{F}| \leqq |\mathcal{D}|+|\mathcal{F}^*|. \tag{3}$$

Let us define further for any element x of X:

$$\mathcal{G}_x = \{F-x | x\in\mathcal{F},\ F\in\mathcal{F}^*\},\quad \mathcal{G}'_x = \{D | (D, x)\in\mathcal{D}\}.$$

Then both of these families are ordinary 2-graphs. Let us state some properties of them.

LEMMA 1. *Neither $\mathcal{G}_x$ nor $\mathcal{G}'_x$ contains more than $l-1$ independent edges. The maximum degree of $\mathcal{G}_x$ is at most $2(l-1)$, and that of $\mathcal{G}'_X$ is at most $l-1$.*

PROOF. The first statement is evident. To prove the second let y be a vertex of $\mathcal{G}_x$ of maximal degree, let $z_1, z_2, \dots, z_s$ be the vertices which are adjacent to it. If $s \geqq 2l-1$ then we should have $(\{x, y\}, z_i)\in\mathcal{D}$ for $i=1, \dots, s$ contradicting $\{x, y, z_i\}\in\mathcal{F}^*$. Let now y be a vertex of maximal degree in $\mathcal{G}'_x$, and let $z_1, z_2, \dots, z_t$ be the vertices which are adjacent to y. Suppose that $t \geqq 1$. As for $i=1, \dots, l$ $(\{y, z_i\}, x)\in\mathcal{D}$, and $\left|\left(\bigcup_{j=1}^{l}\{y, z_j\}\right)-\{y, z_i\}\right| = l-1$, we can assign an l-element set to each of the z_i's such that this set is disjoint to $\bigcup_{j=1}^{l}\{y, z_j\}$, and for any element y_i of it $\{y, z_i, y_i\}\in\mathcal{F}$. Consequently, using the mariage principle, we can find elements $x_1, x_2, \dots, x_l$ such that $\{y, z_i, x_i\}\in\mathcal{F}$ for $i=1, 2, \dots, l$, and these sets form a Δ-system with kernel $\{y\}$. This contradiction completes the proof of the lemma.

COROLLARY 1.

$$|\mathcal{G}_x| \leqq \binom{2l-1}{2}, \tag{4}$$

$$|\mathcal{G}'_x| \leqq \begin{cases} l(l-1) & \textit{if } l \textit{ is odd} \\ l\left(l-\dfrac{3}{2}\right) & \textit{if } l \textit{ is even.} \end{cases} \tag{5}$$

For a proof cf. SAUER [5], and ABBOTT—HANSON—SAUER [1], respectively.
Now from (3), (4), and (5) we can deduce

$$|\mathcal{F}| \leqq |\mathcal{D}|+|\mathcal{F}^*| = \frac{1}{3}\sum_{x\in X}|\mathcal{G}_x| + \sum_{x\in X}|\mathcal{G}'_x| < \frac{5}{3}\, l(l-1)\, n$$

proving (1).

Now we restrict ourselves to the case $l=3$. First we give an improvement of Lemma 1 for this case.

LEMMA 2. *Let $D_1, D_2, D_3\in\mathcal{G}_x\cup\mathcal{G}'_x$ for some $x\in X$. Then these sets do not form a Δ-system with empty kernel, moreover, if at least two of them belong to $\mathcal{G}'_x$ then they cannot form a Δ-system with kernel of cardinality 1 either.*

PROOF. The first statement follows trivially from the assumptions of the theorem and $(D_i\cup x)\in\mathcal{F}$ for $i=1, 2, 3$. To prove the second part of the lemma, suppose that $D_2, D_3\in\mathcal{G}'_x$. Then there are at least five different elements of X, say $x_1, x_2, \dots, x_5$, such that $(D_2\cup x_i)\in\mathcal{F}$ for $i=1, 2, \dots, 5$. Hence it is possible to choose one of them which is not an element of $x\cup D_3$. Let x' be this element, and let us set $F_1=D_1\cup x$, $F_2=D_2\cup x'$. As $D_3\in\mathscr{G}'_x$ there are at least five different elements of X, say $z_1,\dots,z_5$ such that for $i=1,\dots,5$, $(D_3\cup z_i)\in\mathscr{F}$. Hence at least one of them, say z_1, is not an element of the 4-set $(F_1\cup F_2)-(D_1\cap D_2)$. Setting $F_3=D_3\cup\cup z_1$ we come to a contradiction as the three sets F_1, F_2, F_3 form a Δ-system with kernel of cardinality 1, establishing the statement of the lemma.

Now we use Lemma 2 to prove the following

LEMMA 3. *For any $x\in X$ either $\mathscr{G}_x$ is empty and $\mathscr{G}'_x$ is the union of two disjoint triangles or we have*

$$\frac{1}{3}|\mathscr{G}_x|+|\mathscr{G}'_x|\leqq\frac{16}{3}. \tag{6}$$

PROOF. We distinguish several cases according to the cardinality of $\mathscr{G}'_x$.

Case (a): $|\mathscr{G}'_x|=6$. By Lemma 1 $\mathscr{G}'_x$ is a 2-graph of maximum degree at most 2. Hence it is the disjoint union of circles and paths. As it contains at most two independent edges in this case the only possibility is that it is the disjoint union of two triangles. Then Lemma 2 implies $\mathscr{G}_x=\emptyset$.

Case (b): $|\mathscr{G}'_x|=5$. Now we have two possibilities for $\mathscr{G}'_x$: either it is a five-circle or the disjoint union of a triangle and a 2-path. In any case Lemma 2 yields $|\mathscr{G}_x|\leqq 1$, implying (6).

Case (c): $|\mathscr{G}'_x|=4$. There are two possibilities again: either $\mathscr{G}'_x$ is the disjoint union of a triangle and an edge or of two 2-paths. In the first case Lemma 2 implies that the edges of $\mathscr{G}_x$ do not meet the triangle and consequently they have pairwise non-empty intersection. Hence by Lemma 1 $|\mathscr{G}_x|\leqq 4$. In the second case a similar argument yields $|\mathscr{G}_x|\leqq 2$. Anyway (6) follows.

Case (d): $|\mathscr{G}'_x|=3$. Now we have three possibilities: either $\mathscr{G}'_x$ is a triangle or a 3-path or the disjoint union of a 2-path and an edge. In the first case the edges of $\mathscr{G}_x$ are disjoint to the triangle and consequently they have pairwise non-empty intersection, yielding $|\mathscr{G}_x|\leqq 4$. In the second case Lemma 2 implies that every edge of $\mathscr{G}_x$ is disjoint to the central edge of the 3-path, whence again by Lemma 2 they have pairwise non-empty intersection. So by Lemma 1 $|\mathscr{G}_x|\leqq 4$. In the third case Lemma 2 implies that any edge of $\mathscr{G}_x$ is either the edge connecting the two endpoints of the 2-path or it intersects the independent edge of $\mathscr{G}'_x$. As $(D, x)\notin\mathscr{D}$ for $D\in\mathscr{G}_x$, the maximum degree in $\mathscr{G}_x\cup\mathscr{G}'_x$ is at most 4, yielding $|\mathscr{G}_x|\leqq 7$. In all three cases (6) follows.

Case (e): $|\mathscr{G}'_x|\leqq 2$. In this case (6) follows at once from $|\mathscr{G}_x|\leqq 10=\binom{5}{2}$.

As by Corollary 1 $|\mathscr{G}'_x|\geqq 7$ is impossible the lemma is proved.

Now we turn to the proof of the theorem. Let Y be the set of those elements of X for which $\mathscr{G}'_x$ is not the union of two triangles. Then (3) and Lemma 3 yield $|\mathscr{F}|\leqq 6n-\frac{2}{3}|Y|$. Assuming that $\mathscr{F}$ is of maximum cardinality we deduce $|Y|\leqq 53$. So for $n\geqq 54$ there exists at least one element of X, say x, such that $\mathscr{G}'_x$ is the disjoint union of two triangles, say $y_1y_2y_3$ and $y_4y_5y_6$. We assert that $\mathscr{G}_{y_i}=\emptyset$ for $i=1, 2, \dots, 6$.

Suppose that it fails for example for $i=3$ and let E be an edge of $\mathcal{G}_{y_3}$. The definition of $\mathcal{F}^*$ implies that $E\cap\{y_1, y_2\}=\emptyset$. Let us set $F_3=E\cup y_3$. As $(\{y_1, y_2\}, x)$ and $(\{y_2, y_3\}, x)$ belong to $\mathcal{D}$ we can proceed in the "usual" way, and find $F_1, F_2\in\mathcal{F}$ such that F_1, F_2, F_3 form a Δ-system with kernel $\{y_3\}$, a contradiction, proving $\mathcal{G}_{y_i}=\emptyset$. A similar argument yields that for $i=1, 2, \ldots, 6$ $|\mathcal{G}'_{y_3}|\leqq 1$ with equality holding if and only if $\mathcal{G}'_{y_3}$ consists of the edge opposite to y_i in the corresponding triangle. However, if this triangle belongs to $\mathcal{F}$ then it is counted three times in (2). So we can deduce

$$|\mathcal{F}|\leqq 2+\sum_{x\in(X-\{y_1,\ldots,y_6\})}|\mathcal{G}'_x|+|\mathcal{G}_x|/3,$$

yielding $|\mathcal{F}|\leqq 6(n-6)+2$ with equality holding only if $Y=\{y_1, \ldots, y_6\}$. But then for every $x'\in X-Y$, $\mathcal{G}'_{x'}$ is the disjoint union of two triangles on the set Y. However if these triangles would not coincide with $y_1y_2y_3$, $y_4y_5y_6$ then we could find three 2-element subsets of Y, forming a Δ-system with a 1-element kernel such that each of the sets is contained in at least 5 different members of $\mathcal{F}$, leading to a contradiction, as so many times before. Hence it follows that the only system — up to isomorphism — for which equality holds in (2) is:

$$\big\{D\cup x\big||D|=2,\ x\notin D\subset\{y_1, y_2, y_3\},\ x\in(X-\{y_4, y_5, y_6\})\big\}\cup$$
$$\cup\big\{D\cup x\big||D|=2,\ x\notin D\subset\{y_4, y_5, y_6\},\ x\in(X-\{y_1, y_2, y_3\})\big\}.$$

Remark. For $n=12$ let X be the union of the two 6-element subsets X_1, X_2, and let us define $\mathcal{H}=\{H\subset X_1\big||H|=3\}\cup\{H\subset X_2\big||H|=3\}$. Then $\mathcal{H}$ does not contain any Δ-system of cardinality 3 and with kernel of cardinality 1. As $|\mathcal{H}|=40$, the theorem certainly does not hold for $n=12$ which proves that the assumption $n\geqq 54$ cannot be omitted. But it can certainly be replaced by a weaker assumption.

References

[1] H. L. Abbott, D. Hanson, N. Sauer, Intersection theorems for systems of sets, *J. Combin. Theory*, **A 12** (1972), 381—389.

[2] R. Duke, P. Erdős, Systems of finite sets having a common intersection, to appear.

[3] P. Erdős, C. Ko, R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford*, **(2) 12** (1961), 313—320.

[4] P. Frankl, On families of finite sets no two of which intersect in a singleton, *Bull. Austral. Math. Soc.*, **17** (1977), 125—134.

[5] N. Sauer, The largest number of edges of graphs such that not more than g intersect in a point or more than n are independent, in: V. N. Welsh, ed., *Combinatorial Mathematics and its Applications*, Academic Press (New York, 1971), 253—257.

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST, REÁLTANODA U. 13—15.

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ON A PROBLEM OF D. NEWMAN

By

D. PETZ (Budapest)

It is a well-known fact that every continuous function on $I=[0, 1]$ can be uniformly approximated by polynomials of a fixed function $f\in C(I)$ if and only if f is strictly monotone. The class of strictly monotone functions is fairly small in the space $C(I)$ so it can be said that in most cases the polynomials of one given function from $C(I)$ do not form a dense set in $C(I)$. As Vera T. Sós informed me, on a joint seminar of Hungarian and American mathematicians held in Budapest, August 1975, Donald Newman raised the question what about taking more than one basic functions and approximating the functions in $C(I)$ by their polynomials. He conjectured that picking three basic functions their polynomials were everywhere dense in most cases.

The aim of this paper is to give one possible topological formulation for Newman's conjecture and to prove it.

Let S^n denote the set of those n-tuples of functions from $C(I)^n$ whose polynomials are everywhere dense in $C(I)$. With this notation we can state the following

THEOREM. *If* $n\leqq 2$ *then* S^n *is nowhere dense in* $C(I)^n$ *and if* $n\geqq 3$ *then* $C(I)^n\setminus S^n$ *is everywhere dense and of first category (i.e. the union of a countable family of nowhere dense sets).*

We say that an n-tuple of functions $(f_1, \dots, f_n)$ is *separate* if for each pair of points $t_1\neq t_2$, $t_1, t_2\in I$, there is a k with $f_k(t_1)\neq f_k(t_2)$. Geometrically this means that the curve produced by the n-tuple in R^n does not intersect itself. Due to Stone—Weierstrass theorem the set of all separate n-tuples is just S^n.

The n-tuple of functions $(f_1, \dots, f_n)$ is said to be *simple* provided that f_k $(1\leqq k\leqq n)$ is piecewise linear and not constant on any subinterval. The range of a simple n-tuple is a polygonal line without any line segment parallel to some coordinate hyperplane. Conversely, such polygonal lines can be observed in this way. Denote by P^n the set of all simple n-tuples of continuous functions. Then

1. P^n is everywhere dense in $C(I)^n$.
2. $C(I)^n\setminus S^n$ is everywhere dense in $C(I)^n$.
3. For $p\in\mathbf{N}$ let

$$F_p^n=\left\{(f_1, \dots, f_n)\in C(I)^n\colon \exists t_1, t_2\in I,\ |t_1-t_2|\geqq \frac{1}{p}, f_k(t_1)=f_k(t_2),\ 1\leqq k\leqq n\right\}.$$

Then F_p^n is closed in $C(I)^n$.

4. $P^3\cap S^3$ is everywhere dense in $C(I)^3$.

Taking $(f_1, f_2, f_3)\in P^3$ and $\varepsilon>0$ we choose the numbers $0=a_0<a_1<\dots$ $\dots<a_n=1$ so that $|f_k(t_1)-f_k(t_2)|<\varepsilon$ for $t_1, t_2\in[a_{i-1}, a_i]$ $(1\leqq k\leqq 3,\ 1\leqq i\leqq n)$. We can make a $(g_1, g_2, g_3)\in P^3$ such that $(g_1, g_2, g_3)([a_{i-1}, a_i])$ does not intersect itself and the curve $(g_1, g_2, g_3)([a_0, a_{i-1}))$ and it is in the sphere $\{x\in R^3: \mathrm{dist}\,(x, Q_i)< <2\varepsilon\}$ where $Q_i=(g_1, g_2, g_3)(a_i)$ $(1\leqq i\leqq n)$. Thus we have $(g_1, g_2, g_3)\in P^3\cap S^3$ and $|g_k-f_k|<4\varepsilon$ $(1\leqq k\leqq 3)$. According to Statement 1 the proof is complete.

5. $C(I)^3\setminus S^3$ is of first category in $C(I)^n$.

It is clear that $C(I)^3\setminus S^3=\cup\{F_p^3: p\in\mathbf{N}\}$. Since F_p^3 is closed and $C(I)^3\setminus F_p^3\supset \supset S^3\cap P^3$ is everywhere dense, F_p^3 is nowhere dense. So $C(I)^3\setminus S^3$ is the union of a countable family of nowhere dense sets.

6. If a simple pair of functions (f_1, f_2) produces a polygonal line having a point of selfintersection different from its endpoints then $(f_1, f_2)\in \mathrm{int}\,(C(I)^2\setminus S^2)$.

7. S^2 is nowhere dense in $C(I)^2$.

This is obtained from Statements 2 and 6.

Now we prove the main theorem. If $n=2$ then by virtue of Statement 7, S^2 is nowhere dense and since $S^1\times C(I)\subset S^2$, S^1 is nowhere dense. If $n=3$ then according to Statement 6, $C(I)^3\setminus S^3$ is of first category. It is obvious that for $n>3$, $C(I)^n\setminus S^n\subset(C(I)^3\setminus S^3)\times C(I)^{n-3}$ and for this reason $C(I)^n\setminus S^n$ is of first category, too. $C(I)^n\setminus S^n$ was stated to be everywhere dense in Statement 2.

Thus the theorem is proved.

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ON THE SZÁSZ—MIRAKIAN OPERATOR

By
T. HERMANN (Budapest)

1. Let S_n be the Szász—Mirakian operator, i.e.

$$S_n[f;\ x] = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) p_{n,k}(x)$$

where

$$p_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}.$$

The operator

$$S_n^*[f;\ x] = \sum_{k=0}^{\infty} n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt \cdot p_{n,k}(x)$$

has been introduced by P. L. BUTZER [1]. It is known [4] that if $f \in C[0, \infty)$ and $f(x) = O(x^{\alpha x})$ then

$$\lim_{n\to\infty} S_n[f;\ x] = f(x) \quad (0 \leqq x < \infty)$$

and if $f \in L[0, \infty)$ then [1]

$$\lim_{n\to\infty} \|S_n^*[f] - f\|_{L^1} = 0.$$

We can consider these operators as analogues of the Bernstein and Kantorovitch polynomials B_n and B_n^* resp. on the interval $[0, \infty)$. It is known [6], [2] that

$$\|B_n[f] - f\|_c \leqq K_c \omega_2\left(f;\ \frac{1}{\sqrt{n}}\right) \quad (f \in C[0, 1])$$

and

$$\|B_n^*[f] - f\|_{L^p} \leqq K_p \omega_2\left(f;\ \frac{1}{\sqrt{n}}\right)_{L^p} \quad (f \in L^p[0, 1],\ \infty > p \geqq 1)$$

where the norms are the usual norms in $C[0, 1]$ and $L^p[0, 1]$ resp. and K_c, K_p are absolute constants.

As a matter of course, the question arises whether similar inequalities hold for the operators S_n and S_n^*:

(1) $$\|S_n[f] - f\|_c \leqq K_c' \omega\left(f;\ \frac{1}{\sqrt{n}}\right)$$

where $f \in C[0, \infty)$ has a finite modulus of continuity $\omega(f; \cdot)$, and

$$\|S_n^*[f]-f\|_{L^1} \leqq K_1' \omega\left(f; \frac{1}{\sqrt{n}}\right)_L \quad (f \in L[0, \infty)); \tag{2}$$

or rather what can we state about the rapidity of the convergence? We try to answer these questions in the following sections.

2. Denote by C_0 the following class of continuous functions on $[0, \infty)$:[1]

$$C_0 = \left\{f \in C[0, \infty) \,\middle|\, \sup_{x \geqq 0} |f(x+\delta)-f(x)| < +\infty \text{ for any } \delta > 0\right\}.$$

Then $f \in C_0$ has modulus of continuity $\omega(f; \delta)$ on $[0, \infty)$ and

$$\sup_{f \in C_0} \frac{\|S_n[f]-f\|_C}{\omega\left(f; \frac{1}{\sqrt{n}}\right)} = +\infty \tag{3}$$

for all n. This is obvious from the following example. Let $f_m(x) = (m-x)_+$ where m is any natural number. Then $\omega(f_m; \delta) = \delta$ and using the Stirling-formula we obtain

$$\|S_n[f_m]-f_m\|_c \geqq S_n[f_m; m]-f_m(m) = S_n[f_m; m] =$$

$$= e^{-nm} \sum_{k=0}^{nm} \left(m-\frac{k}{n}\right) \frac{(nm)^k}{k!} = e^{-nm} \left\{m \sum_{k=0}^{nm} \frac{(nm)^k}{k!} - m \sum_{k=0}^{nm-1} \frac{(nm)^k}{k!}\right\} =$$

$$= me^{-nm} \frac{(nm)^{nm}}{(nm)!} \simeq \sqrt{\frac{m}{2\pi n}}.$$

Hence

$$\lim_{m \to \infty} \frac{\|S_n[f_m]-f_m\|_C}{\omega\left(f_m; \frac{1}{\sqrt{n}}\right)} = +\infty$$

from where (3) follows.

In contrast to this fact we have

THEOREM 1. *Let* $\varrho(x) = \dfrac{1}{1+\sqrt{x}}$. *Then*

$$\sup_{f \in C_0} \frac{\|(S_n[f]-f)\varrho\|}{\omega\left(f; \frac{1}{\sqrt{n}}\right)} \leqq 1$$

for all n.

PROOF. Evidently

$$\sum_{k=0}^{\infty} p_{n,k}(x) = 1, \quad \sum_{k=0}^{\infty} \frac{k}{n} p_{n,k}(x) = x, \quad \sum_{k=0}^{\infty} \left(\frac{k}{n}-x\right)^2 p_{n,k}(x) = \frac{x}{n}. \tag{4}$$

[1] This class of functions is not compact, i.e. $\delta \to 0$ does not necessarily imply $\omega(f; \delta) \to 0$.

Hence

$$|S_n[f; x]-f(x)| = \left|\sum_{k=0}^{\infty} p_{n,k}(x)\left(f\left(\frac{k}{n}\right)-f(x)\right)\right| \leqq$$

$$\leqq \sum_{k=0}^{\infty} p_{n,k}(x)\left|f\left(\frac{k}{n}\right)-f(x)\right| \leqq \sum_{k=0}^{\infty} p_{n,k}(x)\omega\left(f;\ \left|\frac{k}{n}-x\right|\right).$$

Because $\omega(f;\delta\lambda)\leqq(1+[\lambda])\omega(f;\delta)$ so

$$|S_n[f;\ x]-f(x)| \leqq \sum_{k=0}^{\infty} p_{n,k}(x)\left(1+\sqrt{n}\left|\frac{k}{n}-x\right|\right)\omega\left(f;\ \frac{1}{\sqrt{n}}\right) =$$

$$= \omega\left(f;\ \frac{1}{\sqrt{n}}\right)\left[1+\sqrt{n}\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right| p_{n,k}(x)\right].$$

From the Cauchy—Schwarz inequality

$$\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right| p_{n,k}(x) \leqq \left(\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^2 p_{n,k}(x)\right)^{1/2} = \sqrt{\frac{x}{n}}.$$

Hence

$$|S_n[f;\ x]-f(x)| \leqq \omega\left(f;\ \frac{1}{\sqrt{n}}\right)(1+\sqrt{x}).$$

Q.e.d.

Remark. Let $\varepsilon(x)\geqq 0$ any monotone increasing function tending to infinity. Then for the weight function

$$\varrho^*(x) = \frac{\varepsilon(x)}{1+\sqrt{x}}$$

we have

$$\sup_{f\in C_0} \frac{\|(S_n[f]-f)\varrho^*\|}{\omega\left(f;\ \frac{1}{\sqrt{n}}\right)} = +\infty$$

for all n.

Proof. We prove more than what we stated. We give a function $h\in C_0$ such that

$$\sup_{x\geqq 0}\left|\varrho^*(x)\left(S_n[h;\ x]-h(x)\right)\right| = \infty$$

for all n. Let

$$h_m(x) = \begin{cases} 0 & \text{if } 0\leqq x\leqq 2^m \\ x-2^m & \text{if } 2^m\leqq x\leqq 3\cdot 2^{m-1} \\ 2^{m+1}-x & \text{if } 3\cdot 2^{m-1}\leqq x\leqq 2^{m+1} \\ 0 & \text{if } 2^{m+1}\leqq x<\infty \end{cases}$$

and let $h(x)=\sum_{m=1}^{\infty} h_m(x)$. Obviously $\omega(h;\delta)=\delta$ so $h\in C_0$. For the sake of brevity we use the notation $y=2^m$, then

$$\|\varrho^*(S_n[h]-h)\| \geqq \varrho^*(y)|S_n[h;y]-h(y)| = \varrho^*(y)S_n[h;y],$$

and

$$S_n[h;y]\geqq e^{-ny}\sum_{y\leqq\frac{k}{n}\leqq\frac{3}{2}y}\left(\frac{k}{n}-y\right)\frac{(ny)^k}{k!} = e^{-ny}y\left(\sum_{ny\leqq k+1\leqq\frac{3}{2}ny}\frac{(ny)^k}{k!}-\sum_{ny\leqq k\leqq\frac{3}{2}ny}\frac{(ny)^k}{k!}\right)=$$

$$= e^{-ny}y\left(\frac{(ny)^{ny}}{(ny)!}-\frac{(ny)^{ny\,3/2}}{\left(ny\frac{3}{2}\right)!}\right).$$

Applying again the Stirling-formula we obtain the existence of a constant $c_1>0$ so that

$$S_n[h;y]\geqq c_1\sqrt{\frac{y}{n}}\left(1-\left(\sqrt{e\left(\frac{2}{3}\right)^3}\right)^{ny}\right).$$

If $y=2^m$ is large enough then there exists a $c_2>0$ so that

$$S_n[h;y]\geqq c_2\sqrt{\frac{y}{n}}.$$

But

$$\lim_{m\to\infty}\varrho^*(y)|S_n[h;y]-h(y)|\geqq \lim_{y\to\infty}\varepsilon(y)\frac{c_2\sqrt{\frac{y}{n}}}{1+\sqrt{y}}=\infty$$

because $\varepsilon(y)$ is not bounded. Q.e.d.

3. Now let us consider the space $L[0,\infty)$. We prove that the situation is similar to the continuous case. Here the norm is the usual $L[0,\infty)$ norm.

LEMMA 1. *If*

$$f_s(t)=\begin{cases}1 & \text{if} \quad t\leqq s\\ 0 & \text{if} \quad t>s\end{cases}$$

then

$$\|S_n^*[f_s]-f_s\|\sim\sqrt{\frac{s}{n}}.$$

PROOF. It is known [5] that $\|S_n^*\|=1$, thus

$$\|S_n^*[f_{s_1}]-f_{s_1}\|_L\leqq\|S_n^*[f_{s_1}-f_{s_2}]\|_L+\|S_n^*[f_{s_2}]-f_{s_2}\|_L+\|f_{s_2}-f_{s_1}\|\leqq$$

$$\leqq 2\|f_{s_1}-f_{s_2}\|_L+\|S_n^*[f_{s_2}]-f_{s_2}\|_L.$$

Therefore

$$\left|\|S_n^*[f_{s_1}]-f_{s_1}\|_L-\|S_n^*[f_{s_2}]-f_{s_2}\|_L\right|\leqq 2\|f_{s_1}-f_{s_2}\|_L=2|s_1-s_2|.$$

For this reason we may assume without loss of generality that $n \cdot s$ is an integer. Then we obtain

$$\|S_n^*[f_s]-f_s\|_L = \int_0^\infty \left| \sum_{k=0}^\infty p_{n,k}(x) n \int_{k/n}^{\frac{k+1}{n}} f_s(t)\,dt - f_s(x) \right| dx =$$

$$= \int_0^\infty \left| \sum_{k=0}^{ns-1} p_{n,k}(x) - f_s(x) \right| dx = \int_0^s \sum_{k=ns}^\infty p_{n,k}(x)\,dx + \int_s^\infty \sum_{k=s}^{ns-1} p_{n,k}(x)\,dx =$$

$$= \int_0^s \sum_{k=ns}^\infty p_{n,k}(x)\,dx + \sum_{k=0}^{ns-1} \left[\int_0^\infty p_{n,k}(x)\,dx - \int_0^s p_{n,k}(x)\,dx \right] =$$

$$= \int_0^s \sum_{k=ns}^\infty p_{n,k}(x)\,dx + \left(\sum_{k=0}^{ns-1} \frac{1}{n} - \int_0^s \sum_{k=0}^{ns-1} p_{n,k}(x)\,dx \right) =$$

$$= \int_0^s \sum_{k=ns}^\infty p_{n,k}(x)\,dx + \int_0^s \left[1 - \sum_{k=0}^{ns-1} p_{n,k}(x) \right] dx = 2 \int_0^s \sum_{k=ns}^\infty p_{n,k}(x)\,dx =$$

$$= 2 \int_0^s e^{-nx} \sum_{k=ns}^\infty \frac{(nx)^k}{k!}\,dx.$$

Here we have utilized that

$$\sum_{k=0}^\infty p_{n,k}(x) = 1 \quad \text{and} \quad \int_0^\infty e^{-nx} \frac{(nx)^k}{k!}\,dx = \frac{1}{n} \quad (k = 0, 1, \ldots).$$

With a short computation we get:

$$\sum_{k=ns}^\infty \int_0^s e^{-nx} \frac{(nx)^k}{k!}\,dx = \sum_{k=ns}^\infty \left[\frac{e^{-nx}}{-n} \sum_{l=0}^\infty \frac{(nx)^l}{l!} \right]_0^s = \sum_{k=ns}^\infty \frac{e^{-ns}}{n} \sum_{l=k+1}^\infty \frac{(ns)^l}{l!} =$$

$$= \frac{e^{-ns}}{n} \sum_{k=ns+1}^\infty (k-ns) \frac{(ns)^k}{k!} = \frac{e^{-ns}}{n} \left[ns \sum_{k=ns}^\infty \frac{(ns)^k}{k!} - ns \sum_{k=ns+1}^\infty \frac{(ns)^k}{k!} \right] = se^{-ns} \frac{(ns)^{ns}}{(ns)!}.$$

I.e. we obtain with the help of the Stirling formula

$$\|S_n^*[f_s]-f_s\|_L = 2s \frac{\left(\frac{ns}{e}\right)^{ns}}{(ns)!} \simeq \sqrt{\frac{2s}{\pi n}}.$$

Q.e.d.

COROLLARY.

$$\sup_{f \in L(0,\infty)} \frac{\|S_n^*[f]-f\|}{\omega\left(f; \frac{1}{\sqrt{n}}\right)_L} = +\infty.$$

In contrast to this phenomenon, as in the continuous case, the following theorem holds.

THEOREM 2. (i) *Let* $\varrho(x)=\dfrac{1}{1+\sqrt{x}}$. *Then*

$$\sup_{n\in\mathbf{N}}\ \sup_{f\in L(0,\infty)} \frac{\|(S_n^*[f]-f)\varrho\|_L}{\omega\left(f;\frac{1}{\sqrt{n}}\right)_L}<+\infty.$$

(ii) *If* $0\leqq\varepsilon(x)\nearrow\infty$ *and* $\varrho^*(x)=\dfrac{\varepsilon(x)}{1+\sqrt{x}}$ *then*

$$\sup_{f\in L[0,\infty)} \frac{\|(S_n^*f-f)\varrho^*\|_L}{\omega\left(f;\frac{1}{\sqrt{n}}\right)_L}=+\infty.$$

I.e. the weight function obtained is, in a certain sense, the best possible.

PROOF. First we prove (i). Let $g\in L[0,\infty)$ be differentiable and g' integrable, then

$$g(x)=g(0)+\int_0^\infty f_x(t)\,g'(t)\,dt$$

where $f_x(t)$ is the same as in Lemma 1. Thus

$$S_n^*[g;x]-g(x)=\int_0^\infty g'(t)\big[S_n[f_x(t);x]-f_x(t)\big]\,dt.$$

Therefore

$$\|(S_n^*[g;x]-g(x))\varrho(x)\|_L\leqq\int_0^\infty |g'(t)|\cdot\|(S_n^*[f_x(t);x]-f_x(t))\varrho(x)\|_L\,dt.$$

Let us consider now

$$R_n(t)\overset{\text{def}}{=}\|(S_n^*[f_x(t);x]-f_x(t))\varrho(x)\|_L.$$

If we can show that $R_n(t)=O\left(\dfrac{1}{\sqrt{n}}\right)$ independently of t then (i) can be proved as follows:

Let $f_n(x)=\int_0^1 f\left(x+\frac{t}{\sqrt{n}}\right)dt$. Obviously $f_n'(x)=\sqrt{n}\left(f\left(x+\frac{1}{\sqrt{n}}\right)-f(x)\right)$ a.e., and

$$\|\varrho(S_n^*[f]-f)\|_L\leqq\|S_n^*[f-f_n]\varrho\|_L+\|(S_n^*[f_n]-f_n)\varrho\|_L+\|(f_n-f)\varrho\|_L\leqq$$

$$\leqq 2\|f-f_n\|_L+\|(S_n^*[f_n]-f_n)\varrho\|_L.$$

But

$$\|f-f_n\|_L=\int_0^\infty\left|\int_0^1\left(f(x)+f\left(x+\frac{t}{\sqrt{n}}\right)\right)dt\right|dx\leqq\int_0^1\int_0^\infty\left|f\left(x+\frac{t}{\sqrt{n}}\right)-f(x)\right|dx\,dt=$$

$$=\int_0^1\omega\left(f;\frac{t}{\sqrt{n}}\right)_L dt\leqq\omega\left(f;\frac{1}{\sqrt{n}}\right)_L.$$

On the other hand, according to the above considerations

$$\|(S_n^*[f_n]-f_n)\varrho\|_L \leqq \int_0^\infty |f_n'(t)|\,R_n(t)\,dt = O\left(\int_0^\infty \left| f\left(t+\frac{1}{\sqrt{n}}\right)-f(t)\right| dt\right) =$$

$$= O\left(\omega\left(f;\ \frac{1}{\sqrt{n}}\right)_L\right)$$

and from this already (i) follows.

Consider then $R_n(t)$. Except the points $x=t$ we have $f_x(t)+f_t(x)=1$ everywhere, therefore

$$S_n^*[f_x(t);x]-f_x(t) = S_n^*[1-f_t;x]-(1-f_t(x)) = f_t(x)-S_n^*[f_t;x]$$

so

$$R_n(t) = \int_0^\infty \varrho(x)\,|S_n^*[f_t;\ x]-f_t(x)|\,dx.$$

According to Lemma 1, $R_n(t)<c_1\sqrt{\frac{t}{n}}$. So if $t\leqq 2$ then $R_n(t)\leqq\frac{c_2}{\sqrt{n}}$. Assume now that $t>2$. By the definition of $\varrho(x)$ (cf. the proof of Lemma 1)

$$R_n(t) = \int_0^t \frac{e^{-nx}}{1+\sqrt{x}} \sum_{\frac{k}{n}>t} \frac{(nx)^k}{k!}\,dx + \int_0^\infty \frac{e^{-nx}}{1+\sqrt{x}} \sum_{\frac{k}{n}\leqq t} \frac{(nx)^k}{k!}\,dx = I_1+I_2.$$

The function $e^{-nx}\frac{(nx)^k}{k!}$ attains its maximum at $x=\frac{k}{n}$, and in $\left[0,\frac{k}{n}\right]$ and $\left[\frac{k}{n},\infty\right]$ it is monotone increasing and decreasing, resp. By applying the proof of Lemma 1 again, we obtain

$$I_2 \leqq \frac{1}{1+\sqrt{t}} \int_t^\infty e^{-nx} \sum_{\frac{k}{n}\leqq t} \frac{(nx)^k}{k!}\,dx \leqq \frac{1}{1+\sqrt{t}}\,O\left(\sqrt{\frac{t}{n}}\right) = O\left(\frac{1}{\sqrt{n}}\right).$$

As for I_1, we get

$$I_1 = \left(\int_0^{t/2} + \int_{t/2}^t\right) \frac{e^{-nx}}{1+\sqrt{x}} \sum_{\frac{k}{n}>t} \frac{(nx)^k}{k!}\,dx = I_{11}+I_{21}.$$

Evidently (as $t>2$):

$$I_{11} \leqq \frac{t}{2}\,e^{-nt/2}\frac{\left(n\frac{t}{2}\right)^{nt}}{(nt)!}\left[1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\ldots\right] \simeq \sqrt{\frac{t}{2\pi n}}\left(\frac{e}{4}\right)^{nt/2} = O\left(\frac{1}{\sqrt{n}}\right).$$

By using Lemma 1 once more

$$I_{21} \leqq \frac{1}{1+\sqrt{\frac{t}{2}}} \int_0^t e^{-nx} \sum_{\frac{k}{n}>t} \frac{(nx)^h}{k!}\,dx = \frac{\sqrt{2}}{\sqrt{2}+\sqrt{t}}\,O\left(\sqrt{\frac{t}{n}}\right) = O\left(\frac{1}{\sqrt{n}}\right)$$

i.e. $R_n(t) = O\left(\frac{1}{\sqrt{n}}\right)$.

Now we turn to the proof of (ii). Let

$$R_n^*(t) = \|(S_n^*[f_t]-f_t)\,\varrho^*\|_L$$

and let $t>1$. Then by the considerations used before,

$$R_n^*(t) \geqq \int_t^{\infty} \varrho^*(x) \sum_{\frac{k}{n}\leqq t} p_{n,k}(x)\,dx = \int_t^{\infty} \frac{\varepsilon(x)}{1+\sqrt{x}}\, e^{-nx} \sum_{\frac{k}{n}\leqq t} \frac{(nx)^k}{k!}\,dx \geqq$$

$$\geqq \varepsilon(t) \int_t^{\infty} \frac{e^{-nx}}{1+\sqrt{x}} \sum_{\frac{k}{n}\leqq t} \frac{(nx)^k}{k!}\,dx$$

being $\varepsilon(t)$ monotone increasing. Obviously

$$\int_t^{\infty} \frac{e^{-nx}}{1+\sqrt{x}} \sum_{\frac{k}{n}\leqq t} \frac{(nx)^k}{k!}\,dx \geqq \frac{1}{1+\sqrt{2t}} \int_t^{2t} e^{-nx}\, \frac{(nx)^k}{k!}\,dx.$$

For the last integral we obtain

$$\int_0^{2t} e^{-nx} \sum_{\frac{k}{n}\leqq t} \frac{(nx)^k}{k!}\,dx = \sum_{\frac{k}{n}\leqq t} \left[\frac{e^{-nx}}{-n} \sum_{l=0}^{\infty} \frac{(nx)^l}{l!}\right]_t^{2t} =$$

$$= \frac{1}{n} \sum_{k=0}^{nt} \sum_{l=0}^{k} \left[e^{-nx}\frac{(nx)^l}{l!}\right]_{2t}^{t} = \frac{1}{n} \sum_{k=0}^{nt} (nt+1-k) \left[e^{-nx}\frac{(nx)^k}{k!}\right]_{2t}^{t} =$$

$$= \frac{e^{-nt}}{n} \sum_{k=0}^{nt} (nt+1-k)\frac{(nt)^k}{k!} - \frac{e^{-2nt}}{n} \sum_{k=0}^{nt} (nt+1-k)\frac{(2nt)^k}{k!} \overset{\text{def}}{=} J_1 - J_2.$$

Here

$$nJ_1 = e^{-nt}\left\{(nt+1)\sum_{k=0}^{nt}\frac{(nt)^k}{k!} - nt\sum_{k=0}^{nt-1}\frac{(nt)^k}{k!}\right\} = e^{-nt}\left\{(nt+1)\frac{(nt)^{nt}}{(nt)!} + \sum_{k=0}^{nt-1}\frac{(nt)^k}{k!}\right\},$$

$$nJ_2 = e^{-2nt}\left\{(nt+1)\sum_{k=0}^{nt}\frac{(2nt)^k}{k!} - 2nt\sum_{k=0}^{nt-1}\frac{(2nt)^k}{k!}\right\} =$$

$$= e^{-2nt}\left\{(nt+1)\frac{(2nt)^{nt}}{(nt)!} - (nt-1)\sum_{k=0}^{nt-1}\frac{(2nt)^k}{k!}\right\},$$

$$n(J_1-J_2) = e^{-nt}(nt+1)\frac{(nt)^{nt}}{(nt)!}\left[1-\left(\frac{2}{e}\right)^{nt}\right] + e^{-nt}\sum_{k=0}^{nt-1}\frac{(nt)^k}{k!} + e^{-2nt}(nt-1)\sum_{k=0}^{nt-1}\frac{(2nt)^k}{k!}.$$

By omitting the last two terms we obtain

$$J_1 - J_2 \geqq c_3 t\left(\frac{nt}{e}\right)^{nt}\frac{1}{(nt)!} \geqq c_4\sqrt{\frac{t}{n}} \quad (c_3, c_4 > 0).$$

Therefore

$$R_n^*(t) \geqq \varepsilon(t)\frac{c_4\sqrt{\frac{t}{n}}}{1+\sqrt{2t}} \geqq c_5\frac{\varepsilon(t)}{\sqrt{n}} \quad \text{where} \quad c_5 > 0.$$

Thus

$$\frac{\|(S_n^* f_t - f_t)\varrho^*\|_L}{\omega\left(f_t;\, \frac{1}{\sqrt{n}}\right)_L} \geqq c_5\,\varepsilon(t),$$

and here $\varepsilon(t)$ tends to infinity as $t \to \infty$. Q.e.d.

REMARK. It is an open question whether the relations

$$\|S_n[f]-f\|_C \leqq K_c(f)\,\omega\left(f;\, \frac{1}{\sqrt{n}}\right) \quad (f \in C[0, \infty) \text{ and bounded})$$

and

$$\|S_n^*[f]-f\|_L \leqq K_1(f)\,\omega\left(f;\, \frac{1}{\sqrt{n}}\right)_L \quad (f \in L[0, \infty))$$

hold where $K_c(f)$ and $K_1(f)$ are constants depending only on f.

4. In this section we give a Voronovskaya-type theorem for S_n^*:

THEOREM 3. *If* $f \in C^2[0, \infty)$ *and* $f(x) = O(x^{\alpha x})$ $(x \to \infty, \alpha > 0)$ *then*

$$\lim_{n\to\infty} n\left(S_n^*[f;\, x] - f(x)\right) = \frac{1}{2}\left(xf'(x)\right)' \quad (0 \leqq x < +\infty).$$

To prove this theorem we need the following

LEMMA 2. *Let* $f \in C[0, \infty)$ *and* $f(x) = O(x^{\alpha x})$ $(x \to \infty, \alpha > 0)$ *then*

$$\sum_{\left|\frac{k}{n}-x\right| > \delta} p_{n,k}(x)\, n \int_{k/n}^{(k+1)/n} f(t)\, dt = O(e^{-\gamma n}) \quad (\gamma > 0)$$

where γ *depends on* f, x *and* δ.

PROOF OF THE LEMMA.

$$\sum_{\left|\frac{k}{n}-x\right| > \delta} = \sum_{\frac{k}{n}-x > \delta} + \sum_{x-\frac{k}{n} > \delta} = S_1 + S_2.$$

As $f \in C[0, \infty)$ so there exists $M = M(f, x, \delta)$ such that

$$|f(t)| \leqq M \quad \text{if} \quad t \leqq x - \delta.$$

Hence

$$S_2 \leqq M \sum_{x-\frac{k}{n} > \delta} p_{n,k}(x).$$

According to a theorem of HARDY [3, p. 200]

$$S_2 = O(e^{-\gamma_1 n}) \quad (\gamma_1 > 0)$$

because

$$x - \delta \log \frac{x}{x-\delta} < \delta.$$

As $f(x) = O(x^{\alpha x})$, we get

$$S_1 = O\Big(\sum_{\frac{k}{n} > x+\delta} p_{n,k}(x) n \int_{k/n}^{(k+1)/n} t^{\alpha t}\, dt \Big) = O\left(\sum_{\frac{k}{n} > x+\delta} p_{n,k}(x) \left(\frac{k+1}{n} \right)^{\alpha(k+1)/n} \right).$$

Let us introduce the notation

$$a_k = p_{n,k}(x) \left(\frac{k+1}{n} \right)^{\alpha(k+1)/n}.$$

Then by $\frac{k}{n} > x+\delta$ we get

$$\frac{a_k}{a_{k-1}} = \frac{nx}{k} e^{\alpha \left[\frac{k+1}{n} \log \frac{k+1}{n} - \frac{k}{n} \log \frac{k}{n} \right]} = \frac{nx}{k} e^{\frac{\alpha}{n} \left[\log \left(1+\frac{1}{k}\right)^k + \log \frac{k+1}{n} \right]} \leqq \frac{nx}{k} e^{\frac{\alpha}{n} \left[1 + \log \frac{k+1}{n} \right]} =$$

$$= \frac{nx}{k} \left[e \left(\frac{k+1}{n} \right) \right]^{\alpha/n} \leqq \frac{nx}{k} \left(\frac{2ek}{n} \right)^{\alpha/n} = (2e)^{\alpha/n} \frac{x}{\left(\frac{k}{n} \right)^{1-\alpha/n}} \leqq$$

$$\leqq (2e)^{\alpha/n} \frac{x}{(x+\delta)^{1-\alpha/n}} \leqq \frac{x}{x+\frac{\delta}{2}}$$

if n is large enough. Hence

$$S_1 = O\Big(\sum_{k > n(x+\delta)} a_k \Big) = O\left(p_{n, n(x+\delta)}(x) \frac{1}{1 - \frac{x}{x+\frac{\delta}{2}}} \right) = O\big(p_{n, n(x+\delta)}(x)\big).$$

So by the above mentioned theorem of Hardy

$$S_1 = O(e^{-\gamma_2 n}).$$

Q.e.d.

PROOF OF THEOREM 3. By the Taylor formula

$$f(t) - f(x) = (t-x) f'(x) + \frac{(t-x)^2}{2} f''(x) + \frac{(t-x)^2}{2} \eta(f, t, x)$$

where for arbitrary $\varepsilon > 0$, $A > 0$ there exists $\delta > 0$ such that

(5) $$|\eta(f, t, x)| \leqq \varepsilon \quad \text{if} \quad |t-x| \leqq \delta, \quad x \leqq A.$$

With an easy computation we obtain:

(6) $$S_n^*[(t-x);\ x] = \frac{1}{2n} \quad \text{and} \quad S_n^*[(t-x)^2;\ x] = \frac{x}{n}+\frac{1}{3n^2}$$

so

$$S_n^*[f;\ x]-f(x) = \frac{f'(x)}{2n}+\frac{f''(x)}{2}\left(\frac{x}{n}+\frac{1}{3n^2}\right)+$$

$$+\sum_{k=0}^{\infty} p_{n,k}(x)n\int_{k/n}^{(k+1)/n} \eta(f,t,x)\frac{(t-x)^2}{2}\,dt \stackrel{\text{def}}{=} \frac{(xf'(x))'}{2n}+\frac{f''(x)}{6n^2}+\frac{R_n(f)}{n}.$$

We have to prove that $\lim\limits_{n\to\infty} R_n=0$.

$$R_n = \Big(n\sum_{\left|\frac{k}{n}-x\right|\le\delta} + n\sum_{\left|\frac{k}{n}-x\right|>\delta}\Big) = R_{n,1}+R_{n,2}.$$

It follows from (5) and (6)

$$R_{n,1} \le n\varepsilon \sum_{\left|\frac{k}{n}-x\right|\le\delta} p_{n,k}(x)n\int_{k/n}^{(k+1)/n} \frac{(t-x)^2}{2}\,dt \le \varepsilon\left(x+\frac{1}{3n}\right),$$

$$R_{n,2} = n\sum_{\left|\frac{k}{n}-x\right|>\delta} p_{n,k}(x)\,n\int_{k/n}^{(k+1/)n}\left[f(t)-f(x)-(t-x)f'(x)-\frac{(t-x)^2}{2}f''(x)\right]dt.$$

By assumption, the order of the function in square brackets is $O(t^{\alpha t})$ and according to our Lemma 2

$$R_{n,2} = O(e^{-\gamma n})$$

for certain $\gamma>0$. Thus $\lim\limits_{n\to\infty} |R_n|\le\varepsilon x$, but ε was arbitrarily chosen so $\lim\limits_{n\to\infty} R_n=0$. Q.e.d.

References

[1] P. L. Butzer, On the extension of Bernstein polynomials to the infinite interval, *Proc. Amer. Math. Soc.*, **5** (1954), 547—553.

[2] R. DeVore—H. Behrens, Quantitative Korovkin theorems for L_p-spaces, *Proc. of the Symposium on Approximation Theory* (Austin, Texas, 1976).

[3] G. H. Hardy, *Divergent series*. Clarendon Press (Oxford, 1949).

[4] T. Hermann, Approximation of unbounded functions on unbounded intervals, *Acta Math. Acad. Sci. Hungar.*, **29** (1977), 393—398.

[5] T. Hermann, On the Baskakov-type operators, *Acta Math. Acad. Sci. Hungar.*, **31** (1978), 307—316.

[6] G. Freud, On approximation by positive linear methods. II, *Studia Sci. Math. Hung.*, **3** (1971), 347—356.

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COMPUTER AND AUTOMATION INSTITUTE
HUNGARIAN ACADEMY OF SCIENCES
1502 BUDAPEST, KENDE U. 13.

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ON POLYNOMIALS WITH INTEGER COEFFICIENTS AND GIVEN DISCRIMINANT. V
$\mathfrak{p}$-ADIC GENERALIZATIONS

By

K. GYŐRY (Debrecen)

Dedicated with deepest respect to the memory of my teacher Professor P. Turán

1. Introduction

In this paper we continue our investigations on polynomials of given discriminant and give $\mathfrak{p}$-adic generalizations of some of our results obtained in [5], [6], [7], [8] and [9].

Let L be an algebraic number field of degree $n \geqq 1$ with ring of integers $\mathbf{Z}_L$. Denote by h_L and D_L the class number and the absolute value of the discriminant of L, respectively. If $f \in \mathbf{Z}_L[x]$ and $f^*(x) = f(x+a)$ with an $a \in \mathbf{Z}_L$, then for their discriminants $D(f) = D(f^*)$ holds. Such polynomials $f, f^* \in \mathbf{Z}_L[x]$ will be called $\mathbf{Z}_L$-*equivalent*.

As a generalization of some results obtained in [5], [6] and [7] we have proved in [9] that for any monic polynomial $f \in \mathbf{Z}_L[x]$ of degree $k \geqq 3$ with discriminant $D(f) = \delta \neq 0$, $|N_{L/Q}(\delta)| \leqq d$, there is a polynomial f^* $\mathbf{Z}_L$-equivalent to f such that[1]

$$(1) \qquad \overline{|f^*|} < \overline{|\delta|}^{1/(k-1)} \exp\left\{(5nk^3)^{30nk^3}\left((dD_L^k)^{3/2}(\log dD_L)^{nk}\right)^{3(k-1)(k-2)}\right\}.$$

Consequently, there are only finitely many pairwise $\mathbf{Z}_L$ non-equivalent monic polynomials $f \in \mathbf{Z}_L[x]$ with given degree and given discriminant $\delta \neq 0$ and such a system of polynomials is effectively determinable. In the special case $L = \mathbf{Q}$ this was first proved in [5].

In this section we investigate the monic polynomials with integer coefficients in L whose discriminants are divisible only by finitely many fixed prime ideals of L.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be distinct prime ideals of L lying above rational primes not exceeding $P(\geqq 2)$. Throughout this paper, S will denote the multiplicative semigroup of all the integers of L which are not divisible by any prime ideal different from $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$. S obviously contains the group U_L of units of L.

THEOREM 1. *Let L, S be as above and let δ be a non-zero integer in L with $|N_{L/Q}(\delta)| \leqq d$ $(d \geqq 3)$. If $f \in \mathbf{Z}_L[x]$ is a monic polynomial with degree $k \geqq 3$ and discriminant $D(f) \in \delta S$, then it is $\mathbf{Z}_L$-equivalent to a polynomial of the form $\eta^k f^*(\eta^{-1}x)$, where $\eta \in S$, $f^* \in \mathbf{Z}_L[x]$ and*

$$(2) \qquad \overline{|f^*|} < \exp\left\{c_1\left[c_2(s+1)^{c^*}\left((D_L d^{1/k} P^{(s+1)n})^{3/2}(5^{skn}\log D_L dP)^{n+1}\right)^{c_3}\right]^{sc_3+4}\right\}$$

[1] As usual, $\overline{|F|}$ denotes the maximum of the absolute values of the conjugates of the coefficients of a polynomial $F(x)$ with algebraic coefficients.

with effectively computable positive constants $c_3=k(k-1)(k-2)$, $c_1=c_1(n,k)$, $c_2=c_2(n,k)$ *and a positive absolute constant* c^*.[2]

The proof of Theorem 1 easily implies that our theorem is true for $k=2$ with

$$\overline{|f^*|} < \exp\{c_1 3^{sn^2} D_L^2 d P^{s(n+1)} (\log D_L dP)^{3n-2}\} \tag{2'}$$

instead of (2). In what follows we shall state all the consequences only for $k \geqq 3$, but in view of (2′) they remain obviously valid for $k=2$ too with other estimates.

Put $\mathfrak{p}_i^{h_L}=(\pi_i)$ for $i=1, \ldots, s$. As will be shown in the proof, η may be chosen in the form $\varepsilon\pi_1^{w_1}\ldots\pi_s^{w_s}$ with suitable $\varepsilon \in U_L$ and non-negative rational integers $w_1, \ldots, w_s$.

All results of this paper remain valid for $s=0$ (when $S=U_L$), thus they can be regarded as generalizations of some of our theorems obtained in the case $s=0$ (see [5], [6], [7], [9]). However, in case $s=0$ Part IV contains slightly sharper estimates with explicit constants c.

COROLLARY 1. *Let* L, S, δ *be as in Theorem* 1 *and let* $f(x)=x^k+a_1x^{k-1}+\ldots$ $\ldots+a_k \in \mathbf{Z}_L[x]$ *with degree* $k \geqq 3$. *Suppose that for a fixed* i, $2 \leqq i \leqq k$, $a_i=\delta_i\alpha_i$ *with non-zero* $\delta_i, \alpha_i \in \mathbf{Z}_L$, *where* $|N_{L/Q}(\delta_i)| \leqq d_i$, α_i *has at most* t *distinct prime ideal factors with norms* $\leqq P_i$, $P_i \geqq 2$, *and* $f^{(k-i)}(x)$ *has at least two distinct zeros. If* $D(f) \in \delta S$ *and* $N((D(f), \alpha_i)) \leqq M_i$, $M_i \geqq 3$, *then there exists a unit* ε *in* L *such that the polynomial* $f^*(x)=\varepsilon^k f(\varepsilon^{-1}x)$ *satisfies*

$$\overline{|f^*|} < \exp\{c_4(c_5(s+t+1))^{c^*(i^2t+k^3s)}(P_i^n(\log P_i)^t)^{i^2} \log(d_i M_i)(\log\log d_i M_i)^2 \cdot \\ \cdot ((D_L^2 d^{1/k} P^{(s+2)n})^{3/2}(5^{skn}\log D_L d)^{2n})^{c_3(sc_3+4)}\}, \tag{3}$$

where $c_3=k(k-1)(k-2)$, c_4 *and* c_5 *are effectively computable positive constants depending only on* n *and* k *and* c^* *denotes the absolute constant occurring in Theorem* 1.

It is easy to verify that the conditions concerning i and $f^{(k-i)}$ are necessary.

In proving our theorems we employ some recent estimates for linear forms in the logarithms of algebraic numbers [2], [17], [18]. I would like to thank Professors A. Baker, A. J. van der Poorten and R. Tijdeman for kindly permitting me to make use of preprints of their papers [3], [19] and [18].

In Section 2 we give some applications of Theorem 1 and Corollary 1 to algebraic numbers. Further applications will be given in Part VI and in a joint paper with Z. Z. PAPP [10].

2. Applications to algebraic integers

Before stating our theorems on algebraic integers, we establish our notation and recall some standard definitions.

We denote by $\overline{|\alpha|}$ the maximum of the absolute values of the conjugates of

[2] As we shall see, $c^*=\max\{c_2^*, c_4^*\}$ with the effectively computable positive absolute constants c_2^*, c_4^* which occur in Theorems A and B.

Added in proof (August 9, 1978). Replacing Theorems A and B by Theorem 3 of [18] and Theorem 4 of [17], we can get, throughout this paper, slightly better estimates and we may choose $c^*=30$.

an algebraic number α. For an algebraic number α of degree $k \geqq 2$ over L, $D_{L(\alpha)/L}(\alpha)$ and $N_{L(\alpha)/L}(\alpha)$, or, more briefly, $D(\alpha)$ and $N(\alpha)$ will denote the relative discriminant and the relative norm of α with respect to the extension $L(\alpha)/L$. If α is an integer and its minimal polynomial over L is $f \in \mathbf{Z}_L[x]$, then we have $D(\alpha) = D(f)$, $N(\alpha) = \pm f(0)$, $\overline{|f|} \leqq (2\overline{|\alpha|})^k$ and $\overline{|\alpha|} \leqq k\overline{|f|}$.

We say that the algebraic integers α and α^* are *$\mathbf{Z}_L$-equivalent* if $\alpha - \alpha^* \in \mathbf{Z}_L$. In this case their minimal polynomials over L are also $\mathbf{Z}_L$-equivalent. Consequently, Theorem 1 and Corollary 1 easily imply Theorems 2 and 3.

THEOREM 2. *Let L, S, δ be as in Theorem 1 and let α be an algebraic integer with degree $k \geqq 3$ and discriminant $D(\alpha) \in \delta S$ over L. Then α is $\mathbf{Z}_L$-equivalent to an integer of the form $\eta\alpha^*$, where $\eta \in S$ and α^* is an algebraic integer satisfying*

$$\overline{|\alpha^*|} < k \exp\left\{c_1\left[c_2(s+1)^{c^*}\left((D_L d^{1/k} P^{(s+1)n})^{3/2}(5^{skn} \log D_L dP)^{n+1}\right)^{c_3}\right]^{sc_3+4}\right\} \tag{4}$$

with the constants $c_3 = k(k-1)(k-2)$, c_1, c_2 and c^* occurring in Theorem 1.

If in Theorem 2 we restrict ourselves to the integers α of a fixed algebraic number field K of degree $k \geqq 3$ over L, then, writing $D_K = |D_{K/\mathbf{Q}}|$, from (51) we get for α^* that

(4′)
$$\overline{|\alpha^*|} < \exp\left\{c_6 D_L^{1/2} P^{2nc_3}\left[c_7(s+1)^{c^*}(\log P)^2\left(D_K^{3/2k}(\log D_k)^n\right)^{c_3}\right]^{sc_3+4} \log d (\log\log d)^2\right\}$$

with the above c_3 and c^* and with effectively computable $c_6, c_7 > 0$ which depend only on n and k.

In the special case $s=0$ we have obtained in [9] a slightly sharper estimate for $\overline{|\alpha^*|}$ with explicit constants c.

An easy corollary of Theorem 2 is that up to the obvious multiplications by elements of S and the translations by integers of L, there are only finitely many algebraic integers with given degree k and discriminant $D(\alpha) \in \delta S$ over L and they can be effectively determined. We note that this result can be deduced, in an ineffective form, from an ineffective theorem of B. J. BIRCH and J. R. MERRIMAN [4] on binary forms and from the finiteness of the number of solutions of the generalized Thue—Mahler equation ([16], [12], [13]).

Consider now Theorem 2 in the special case when $L = \mathbf{Q}$.

COROLLARY 2. *Let d be a non-zero rational integer and let $p_1, \ldots, p_s$ be rational primes not exceeding P. Let α be an algebraic integer of degree $k \geqq 3$ with discriminant $D(\alpha) = dp_1^{u_1} \ldots p_s^{u_s}$. Then α is $\mathbf{Z}$-equivalent to an integer of the form $p_1^{w_1} \ldots p_s^{w_s}\alpha^*$, where α^* is an algebraic integer such that*

$$\overline{|\alpha^*|} < \exp\left\{c_8\left[c_9(s+1)^{c^*}\left((|d|^{1/k} P^{s+1})^{3/2}(5^{sk} \log 2|d|P)^2\right)^{c_3}\right]^{sc_3+4}\right\} \tag{5}$$

with $c_3 = k(k-1)(k-2)$ and effectively computable positive constants c_8, c_9 depending only on k.

Since the proofs of the present paper were completed, a very recent paper [23] of L. A. TRELINA appeared, in which she proves two interesting theorems similar to the special cases $|d| = |N| = 1$ of our Corollaries 2 and 3. The estimates in [23] are weaker than (5) and (7) in the case $|d| = |N| = 1$, but the exponents of s are explicitly determined. In [21] the proofs depend on a theorem of S. V. KOTOV [13] on the generalized Thue—Mahler equation (see also S. V. KOTOV and V. G. SPRINDŽUK [12]).

An immediate consequence of Corollary 1 is the following

THEOREM 3. *Let L, S, δ be as in Theorem 1 and let α be an algebraic integer with degree $k \geqq 3$ and discriminant $D(\alpha) \in \delta S$ over L. Suppose that the relative norm $N(\alpha)$ of α over L is of the form $N(\alpha) = \vartheta \cdot \sigma$, where $|N_{L/\mathbf{Q}}(\vartheta)| \leqq N$ and σ has at most t distinct prime ideal factors in L with norms $\leqq Q$, $Q \geqq 2$. If $N((D(\alpha), \sigma)) \leqq M$, $NM \geqq 3$, then there exists a unit ε in L such that*

$$(6) \quad \overline{|\alpha\varepsilon|} < k \exp\{c_4(c_5(s+t+1))^{k^2 c^*(ks+t)}(Q^n(\log Q)^t)^{k^2} \log NM(\log\log NM)^2 \cdot \cdot ((D_L^2 d^{1/k} P^{(s+2)n})^{3/2}(5^{skn} \log D_L d)^{2n})^{c_3(sc_3+4)}\}$$

with the constants $c_3 = k(k-1)(k-2)$, c_4, c_5 and c^* occurring in Corollary 1.

In the special case $L = \mathbf{Q}$ Theorem 3 yields

COROLLARY 3. *Let d, N be given non-zero rational integers and let $p_1, \dots, p_s$, $q_1, \dots, q_t$ be fixed rational primes with $\max_i \{p_i\} \leqq P$ and $\max_j \{q_j\} \leqq Q$, $Q \geqq 2$. Let α be an algebraic integer of degree $k \geqq 3$ such that*

$$D(\alpha) = d p_1^{u_1} \dots p_s^{u_s} \quad \text{and} \quad N(\alpha) = N q_1^{v_1} \dots q_t^{v_t}$$

with non-negative rational integers $u_1, \dots, u_s$, $v_1, \dots, v_t$. If $|(D(\alpha), q_1^{v_1} \dots q_t^{v_t})| \leqq M$, $M \geqq 3$, then

$$(7) \quad \overline{|\alpha|} < \exp\{c_{10}(c_{11}(s+t+1))^{k^2 c^*(ks+t)}(Q(\log Q)^t)^{k^2} \log|NM|(\log\log|NM|)^2 \cdot \cdot ((|d|^{1/k} P^{s+2})^{3/2}(5^{sk} \log 2|d|)^2)^{c_3(sc_3+4)}\},$$

where $c_3 = k(k-1)(k-2)$ and c_{10}, c_{11} are effectively computable positive constants depending only on k.

We note that in our theorems and their corollaries $s \leqq n\pi(P) \leqq cnP/\log P$ and $t \leqq n\pi(Q) \leqq cnQ/\log Q$ hold with an effectively computable positive absolute constant c.

The special case $t=0$ of Theorem 3 is of particular interest.

COROLLARY 4. *Let L, S, δ be defined as in Theorem 1 and let α be an algebraic integer with degree $k \geqq 3$ and discriminant $D(\alpha) \in \delta S$ over L. Suppose that $|N_{\mathbf{Q}(\alpha)/\mathbf{Q}}(\alpha)| \leqq N$, $N \geqq 3$. Then there exists a unit ε in L such that*

$$(8) \quad \overline{|\alpha\varepsilon|} < \exp\{c_4'(c_5(s+1))^{c^* k^3 s} \log N(\log\log N)^2((D_L^2 d^{1/k} P^{(s+2)n})^{3/2}(5^{skn} \log D_L d)^{2n})^{c_3(sc_3+4)}\}$$

with the constants $c_3 = k(k-1)(k-2)$, c_5 and c^* occurring in Theorem 3 and with an effectively computable $c_4' = c_4'(n, k)$.

In virtue of $s \leqq cnP/\log P$, Corollary 4 implies

COROLLARY 5. *Let ϱ be an algebraic unit with degree $k \geqq 3$ and discriminant $D(\varrho)$ over L and suppose that $D(\varrho)$ has s distinct prime ideal factors in L. There is a* *unit ε in L such that the greatest $N(\mathfrak{p})$ of the norms of the prime ideals $\mathfrak{p}$ dividing $D(\varrho)$ satisfies*

$$s^2 \log N(\mathfrak{p}) > c_{12} \log\log \overline{|\varrho\varepsilon|}$$

and

$$N(\mathfrak{p}) > c_{13}(\log\log \overline{|\varrho\varepsilon|} \log\log\log \overline{|\varrho\varepsilon|})^{1/2}$$

provided that $\overline{|\varrho\eta|} > c_{14}$ for any unit $\eta \in L$, where $c_{12}, c_{13} > 0$ and c_{14} are effectively computable constants depending only on k, n and D_L.

If ϱ lies in a fixed extension K of L with relative degree $k \geqq 3$, we can derive from (4′) sharper estimates for $s^2 \log N(\mathfrak{p})$ and $N(\mathfrak{p})$ with constants c_{12}, c_{13} and c_{14} depending on k, n, D_L and D_K.

In particular, if ϱ is a unit of degree $k \geqq 3$ over $\mathbf{Q}$, and $D(\varrho)$ has s distinct prime factors the maximum of which is P, then by Corollary 5 we have

$$s^2 \log P > c_{15} \log\log \overline{|\varrho|}$$

and

$$P > c_{16}(\log\log \overline{|\varrho|} \log\log\log \overline{|\varrho|})^{1/2}$$

provided that $\overline{|\varrho|} > c_{17}$, where c_{15}, c_{16}, c_{17} denote effectively computable positive constants depending only on k. In view of $|D(\varrho)| \leqq (2\overline{|\varrho|})^{k(k-1)}$ this allows us to get some information about the arithmetical structure of those rational integers which are discriminants of algebraic units.

COROLLARY 6. *Let D be a rational integer with s distinct prime factors and greatest prime factor P. If there exists an algebraic unit with degree $k \geqq 3$ and discriminant D (over $\mathbf{Q}$), then*

$$s^2 \log P > c_{18} \log\log |D|$$

provided that $|D| \geqq c_{19}$, where c_{18}, c_{19} are effectively computable positive constants which depend only on k.

In the general case Corollary 5 provides a similar consequence.

Consider again a number field L with the above parameters. Take an extension K/L of degree $k \geqq 3$. If $\alpha \in \mathbf{Z}_K$ is a primitive element of K/L, then the principal ideal generated by $D_{K/L}(\alpha)$ may be written in the form

$$(D_{K/L}(\alpha)) = \mathscr{I}^2(\alpha) \cdot D_{K/L}$$

with a suitable non-zero integral ideal $\mathscr{I}(\alpha)$ of L. $\mathscr{I}(\alpha) = \mathscr{I}_{K/L}(\alpha)$ is called the index of α with respect to K/L.

Suppose now that the relative discriminant $D_{K/L}$ is principal and that h_L is odd. Then K/L has a relative integral basis (see e.g. [15]) and $\mathscr{I}_{K/L}(\alpha)$ is also principal, say $\mathscr{I}_{K/L}(\alpha) = (I_{K/L}(\alpha))$ with some $I_{K/L}(\alpha) \in \mathbf{Z}_L$.

Under the above assumptions the theorems and corollaries of this section can be stated for indices of algebraic integers in place of their discriminants. For example, an immediate consequence of Theorem 2 and (4′) is the following

COROLLARY 7. *Let L, S, δ, K be as above, and let α be an integer in K with $I_{K/L}(\alpha)\in\delta S$. Then α is $\mathbf{Z}_L$-equivalent to a number of the form $\eta\alpha^*$, where $\eta\in S$ and α^* is an algebraic integer satisfying*

$$\overline{|\alpha^*|} < \exp\{4c_6 D_L^{1/2} P^{2nc_3}[c_7(s+1)^{c^*}(\log P)^2(D_K^{3/2K}(\log D_K)^n)^{c_3}]^{sc_3+4}\cdot$$
$$\cdot\log dN(D_{K/L})(\log\log dN(D_{K/L}))^2\}$$

with the constants $c_3=k(k-1)(k-2)$, c_6, c_7 and c^* occurring in (4′).

By using Theorem 2 and (4′) one can easily generalize Corollary 7 to any K/L.

3. Auxiliary results

Throughout this section M denotes an algebraic number field of degree m with r_1 real and $2r_2$ complex conjugate fields. Let D_M denote the absolute value of the discriminant of M, R_M its regulator and h_M its class number.

The proofs of our theorems depend on the following powerful results of A. BAKER [2] and A. J. VAN DER POORTEN [17].

Let $\alpha_1, \dots, \alpha_l$ be non-zero elements in M with heights at most $A_1, \dots, A_l$ (all $A_j\geqq 3$), resp. Write

$$\Omega' = \log A_1 \dots \log A_{l-1}, \quad \Omega = \Omega' \log A_l.$$

Further denote by $b_1, \dots, b_l$ rational integers with absolute values at most B $(\geqq 2)$, and write for brevity $\Lambda=\alpha_1^{b_1}\dots\alpha_l^{b_l}-1$.

THEOREM A (A. BAKER [2]). *If $\Lambda\neq 0$, then*

$$|\Lambda| > \exp\{-(c_1^* lm)^{c_2^* l}\,\Omega\log\Omega'\log B\},$$

where c_1^* and c_2^* are effectively computable positive absolute constants.

In fact this theorem is stated in [2] for linear forms in the logarithms of algebraic numbers, but it is easy to see that Theorem A is a consequence of Baker's theorem (see e.g. [19]).

We remark that in [2] the constants corresponding to c_1^* and c_2^* are explicitly determined. Recently, these constants have been further diminished by A. J. VAN DER POORTEN and J. H. LOXTON [18].

Let $\mathfrak{p}$ be a prime ideal in M lying above a prime p.

THEOREM B (A. J. VAN DER POORTEN [17]). *If $\operatorname{ord}_{\mathfrak{p}}\Lambda<\infty$, then*

$$\operatorname{ord}_{\mathfrak{p}}\Lambda < (c_3^* lm)^{c_4^* l}\,\frac{p^{2m}}{\log p}\,\Omega(\log B)^2,$$

where c_3^* and c_4^* are effectively computable positive absolute constants.

As usual, for an $\eta\in M$ we denote by $\eta^{(1)}, \dots, \eta^{(r_1)}$ the real conjugates of η and by $\eta^{(r_1+1)}, \dots, \eta^{(r_1+r_2)}$ the complex conjugates of $\eta^{(r_1+r_2+1)}, \dots, \eta^{(r_1+2r_2)}$, respectively. Put $r=r_1+r_2-1$.

In the proof of Theorem 1 we need the following two lemmas.

LEMMA C. *Suppose* $r \geqq 1$. *There exist independent units* $\eta_1, \dots, \eta_r$ *in* M *such that*

$$\prod_{j=1}^{r} \max(1, \log \overline{|\eta_j|}) < c_{20} R_M$$

and the elements of the inverse matrix of the matrix $(\log |\eta_j^{(i)}|)$, $1 \leqq i, j \leqq r$, *do not exceed* c_{21} *in absolute value, where* c_{20} *and* c_{21} *denote effectively computable positive constants depending only on* m.

This important result follows from the work [20] of C. L. SIEGEL (see H. M. STARK [21]).

We note that by a well known theorem of E. LANDAU [14]

$$R_M h_M < c_{22} D_M^{1/2} (\log D_M)^{m-1} \quad (m \geqq 2) \tag{9}$$

holds with an effectively computable constant c_{22} which depends only on m. An explicit form for c_{22} is given in SIEGEL [20].

Let $\eta_1, \dots, \eta_r$ be units in M with the property given in Lemma C and denote by U the multiplicative group generated by them. If $r=0$, let $U=\{1\}$. Using a well known argument (see BAKER [1]), from Lemma C one can easily deduce the following

LEMMA D. *If* θ *is a non-zero integer in* M, *then there exists an* $\eta \in U$ *such that*

$$\overline{|\theta\eta|} < |N_{M/\mathbf{Q}}(\theta)|^{1/m} \cdot e^{m^2 c_{20} R_M}$$

with the above constant c_{20} $(c_{20}=1$ *if* $r=0)$.

4. Proof of Theorem 1

Put

$$f(x) = (x-\alpha_1)\dots(x-\alpha_k),$$

where $\alpha_1, \dots, \alpha_k$ are distinct algebraic integers. By assumption

$$(D(f)) = \Big(\prod_{1 \leqq i < j \leqq k} (\alpha_j - \alpha_i)^2\Big) = (\delta)\mathfrak{p}_1^{u_1} \dots \mathfrak{p}_s^{u_s} \tag{10}$$

holds with non-negative rational integers $u_1, \dots, u_s$. Write $u_i = h_L v_i + r_i$, $0 \leqq r_i < h_L$ and $\mathfrak{p}_i^{h_L} = (\pi_i)$ with $\pi_i \in \mathbf{Z}_L$ for $i=1, \dots, s$. Then we have

$$\prod_{1 \leqq i < j \leqq k} (\alpha_j - \alpha_i)^2 = \varepsilon \delta' \pi_1^{v_1} \dots \pi_s^{v_s} \tag{11}$$

with a unit ε and an integer δ' of L such that $(\delta') = (\delta)\mathfrak{p}_1^{r_1} \dots \mathfrak{p}_s^{r_s}$ and $|N_{L/Q}(\delta')| \leqq$ $\leqq d P^{sn(h_L - 1)}$. By Lemma D we may suppose that

$$\max_l (\overline{|\pi_l|}) < \exp\{c_{23}(R_L + h_L \log P)\} \tag{12}$$

and

$$\overline{|\delta'|} < \exp\{c_{23}(R_L + \log d + s h_L \log P)\}, \tag{13}$$

where c_{23}, like $c_{24}, c_{25}, \dots$, denote effectively computable positive constants depending only on n and k.

Choose any three from $\alpha_1, \dots, \alpha_k$. Suppose, for convenience, that these are α_1, α_2 and α_3. Let $M=L(\alpha_1, \alpha_2, \alpha_3)$. Then $m=[M\colon \mathbf{Q}]\leqq nk(k-1)(k-2)$. Denote by $\mathfrak{B}_1, \dots, \mathfrak{B}_t$ all the distinct prime ideals of M lying above $\mathfrak{p}_1 \dots \mathfrak{p}_s$. Obviously $t\leqq ms/n$. Applying the unique factorization theorem to (10), we get

$$(14) \qquad (\alpha_j-\alpha_i) = \mathfrak{d}_{ji}\mathfrak{B}_1^{U_{1ji}}\dots\mathfrak{B}_t^{U_{tji}}, \quad 1 \leqq i < j \leqq 3,$$

where $\prod_{1\leqq i<j\leqq 3} \mathfrak{d}_{ji}^2|(\delta)$ and the U_{lji} are non-negative integers. Put $U_{lji}=h_M u_{lji}+r_{lji}$, where h_M is the class number of M/Q and $0\leqq r_{lji}<h_M$ for $l=1, \dots, t$. Write $\mathfrak{B}_l^{h_M}=(\beta_l)$. Then (14) implies

$$(15) \qquad \alpha_j-\alpha_i = \varepsilon_{ji}\delta_{ji}\beta_1^{u_{1ji}}\dots\beta_t^{u_{tji}}, \quad 1 \leqq i < j \leqq 3,$$

where ε_{ji} are units in M and $(\delta_{ji})=\mathfrak{d}_{ji}\mathfrak{B}_1^{r_{1ji}}\dots\mathfrak{B}_t^{r_{tji}}$. In view of Lemma D we may suppose

$$(16) \qquad \max_l (\overline{|\beta_l|}) < \exp\{c_{24}(R_M+h_M\log P)\}$$

and

$$(17) \qquad \max_{1\leqq i<j\leqq 3}(\overline{|\delta_{ji}|}) < \exp\{c_{25}(R_M+\log d+th_M\log P)\},$$

R_M being the regulator of M/Q.

Put $a_l=\min\limits_{1\leqq i<j\leqq 3} u_{lji}$ and $u'_{lji}=u_{lji}-a_l$ for $1\leqq i<j\leqq 3$ and $l=1, \dots, t$. Suppose, for convenience, that $U=\max\limits_{\substack{1\leqq l\leqq t\\ 1\leqq i<j\leqq 3}} u'_{lji}=u'_{121}$ and $u'_{131}=0$. Denote by r the free rank of the group of units in M. If $r>0$, let $\eta_1, \dots, \eta_r$ be units with the property given in Lemma C. By Lemma D we may write

$$\varepsilon_{21}/\varepsilon_{31} = \varepsilon'_{21}\eta_1^{v_{121}}\dots\eta_r^{v_{r21}}, \quad \varepsilon_{32}/\varepsilon_{31} = \varepsilon'_{32}\eta_1^{v_{132}}\dots\eta_r^{v_{r32}},$$

where $v_{1ji}, \dots, v_{rji}$ are rational integers and $\overline{|\varepsilon'_{21}|}$, $\overline{|\varepsilon'_{32}|}$ are bounded above by $\exp\{c_{26}R_M\}$. Put $\varepsilon'_{31}=1$ and $\delta'_{ji}=\varepsilon'_{ji}\cdot\delta_{ji}$, $1\leqq i<j\leqq 3$. Then $\max\limits_{1\leqq i<j\leqq 3}\overline{|\delta'_{ji}|}$ also satisfies (17) with a constant c_{27} in place of c_{25}. Consequently we have

$$(18) \qquad \alpha_j-\alpha_i = \sigma\gamma_{ji}, \quad 1 \leqq i < j \leqq 3,$$

where

$$(19) \qquad \sigma = \varepsilon_{31}\beta_1^{a_1}\dots\beta_t^{a_t} \quad \text{and} \quad \gamma_{ji} = \delta'_{ji}\eta_1^{v_{1ji}}\dots\eta_r^{v_{rji}}\beta_1^{u'_{1ji}}\dots\beta_t^{u'_{tji}}$$

for $1\leqq i<j\leqq 3$ and $v_{l31}=0$ for each $1\leqq l\leqq t$. From

$$(20) \qquad (\alpha_2-\alpha_1)+(\alpha_3-\alpha_2) = (\alpha_3-\alpha_1)$$

it follows that

$$(21) \qquad \gamma_{21}+\gamma_{32} = \gamma_{31},$$

whence

$$(22) \qquad \Lambda = \frac{\gamma_{32}}{\gamma_{31}}-1 = -\frac{\gamma_{21}}{\gamma_{31}} \neq 0.$$

We are going to derive an upper bound for $H=\max(U, V)$, where $V=\max\limits_{l,j,i}(|v_{lji}|)$. We have

$$\text{(23)} \qquad \operatorname{ord}_{\mathfrak{B}_1} \Lambda \geqq h_M U - (h_M + m \log d).$$

Further we obtain from (19) and (22) the equality

$$\Lambda = \frac{\delta'_{32}}{\delta'_{31}} \eta_1^{v_{132}} \dots \eta_r^{v_{r32}} \beta_1^{u'_{132}-u'_{131}} \dots \beta_t^{u'_{t32}-u'_{t31}} - 1.$$

Let us apply now Theorem B with $\mathfrak{B}_1$. Since

$$\Omega < c_{28}^{t+1} R_M (R_M + h_M \log P)^t (R_M + \log d + t h_M \log P) = T_1$$

by (16), (17) and Lemma C, we get by Theorem B that

$$\text{(24)} \qquad \operatorname{ord}_{\mathfrak{B}_1} \Lambda < \big(c_{29}(t+1)\big)^{c_4^*(r+t+1)} \frac{P^{2m}}{\log P} \cdot T_1 (\log H)^2.$$

(23) and (24) imply

(25)

$$U < c_{30}\big(c_{31}(t+1)\big)^{c_4^* t} \frac{P^{2m}}{\log P} T_1 R_M (R_M + h_M \log P)^t (R_M + \log d + t h_M \log P)(\log H)^2 = T_2$$

where c_4^* denotes the effectively computable absolute constant occurring in Theorem B.

We shall prove

$$\text{(26)} \qquad H < c_{32}\big(c_{33}(t+1)\big)^{c^* t} P^{2m} (\log P)(R_M + h_M \log P)^{t+2} \log^2 (R_M + h_M \log P) \cdot$$
$$\cdot \log d (\log\log d)^2 = T_3,$$

where $c^* = \max(c_2^*, c_4^*)$ with the effectively computable positive absolute constant c_2^* appearing in Theorem A. If $V \leqq U$, we obtain immediately (26) from (25). Consequently we may suppose that $U < V = H$. Suppose, for convenience, that $V = |v_{121}|$. (19) implies

$$v_{121} \log |\eta_1^{(a)}| + \dots + v_{r21} \log |\eta_r^{(a)}| = \log |\gamma_{21}^{(a)}| - \log |\delta_{21}'^{(a)}| - \sum_l \mu'_{l21} \log |\beta_l^{(a)}|$$

for any conjugate with $a = 1, \dots, r$, where the conjugates are ordered in the usual manner. Suppose that the right sides attain their maximum in absolute value for $a = J$, $1 \leqq J \leqq r$. By Lemma C we get

$$\text{(27)} \qquad V \leqq c_{34}\Big\{\big|\log |\gamma_{21}^{(J)}|\big| + \big|\log |\delta_{21}'^{(J)}|\big| + \sum_l \mu'_{l21} \big|\log |\beta_l^{(J)}|\big|\Big\}.$$

Thus from (16), (17), (25) and (27) it follows that

$$\big|\log |\gamma_{21}^{(J)}|\big| > c_{35} V - c_{36} T_2 \{R_M + h_M \log P\}.$$

But we have

$$\log |N_{M/\mathbf{Q}}(\gamma_{21})| = \log |N_{M/\mathbf{Q}}(\delta'_{21})| + \sum_l \mu'_{l21} \log |N_{M/\mathbf{Q}}(\beta_l)| \leqq c_{37} h_M T_2 \log P.$$

Hence we obtain for some $1 \leqq g \leqq m$

$$\log |\gamma_{21}^{(g)}| \leqq -c_{38} V + c_{39} T_2 \{R_M + h_M \log P\}$$

with suitable $c_{38}, c_{39}>0$. Further it is easy to see that

$$-\log|\gamma_{31}^{(g)}| \leqq c_{40}\log\overline{|\gamma_{31}|} \leqq c_{41}T_2\{R_M+h_M\log P\}$$

which implies

$$\log|\Lambda^{(g)}| = \log\left|\frac{\gamma_{21}^{(g)}}{\gamma_{31}^{(g)}}\right| < -c_{38}V+c_{42}T_2\{R_M+h_M\log P\}.$$

We may suppose

$$c_{42}T_2(R_M+h_M\log P) < \frac{c_{38}}{2}V, \tag{28}$$

because otherwise we get immediately (26). Consequently

$$\log|\Lambda^{(g)}| < -\frac{c_{38}}{2}H. \tag{29}$$

Applying now Theorem A to

$$\Lambda^{(g)} = \frac{\delta_{32}^{\prime(g)}}{\delta_{31}^{\prime(g)}}\eta_1^{(g)v_{r32}}\dots\eta_r^{(g)v_{r32}}\beta_1^{(g)(u'_{132}-u'_{131})}\dots\beta_t^{(g)(u'_{t32}-u'_{t31})}-1,$$

we obtain

$$\log|\Lambda^{(g)}| > -c_{43}(c_{44}(t+1))^{c_2^* t}T_1\log(R_M+h_M\log P)\log H \tag{30}$$

with the constant c_2^* appearing in Theorem A. From (29) and (30) we get again (26). Thus, using (16), (17), (26) and Lemma C, from (19) we obtain

$$\log\overline{|\gamma_{ji}|} < c_{45}(t+1)(R_M+h_M\log P)T_3 = T_4 \tag{31}$$

for any $1\leqq i<j\leqq 3$.

Using throughout this paper the splitting field $M=L(\alpha_1, \dots, \alpha_k)$ of f instead of $M=L(\alpha_1, \alpha_2, \alpha_3)$, we should obtain an estimate for $\overline{|f^*|}$ in which R_M, h_M and m $(\leqq nk!)$ occur. Since we want to deduce Theorem 2 from Theorem 1, we shall derive an upper bound for $\overline{|f^*|}$ with other parameters. For this reason we need the following notation. Let $M_i=L(\alpha_i)$ with degree $m_i=[M_i:\mathbf{Q}]$ for $i=1, \dots, k$. Denote by D_{M_i} and D_M the absolute values of the discriminants of M_i and M respectively. Put $D=\max_i D_{M_i}$ and let $D=2$ if $L(\alpha_1, \dots, \alpha_k)=\mathbf{Q}$. Then we have (see H. M. STARK [22])

$$D_M|D_{M_1}^{m/m_1}\cdot D_{M_2}^{m/m_2}\cdot D_{M_3}^{m/m_3} \leqq D^{3(k-1)(k-2)}. \tag{32}$$

Thus (9) and (32) imply

$$T_4 < c_{46}P^{2nc_3}\left[c_{47}(s+1)^{c^*}(\log P)^2(D^{3/2k}(\log D)^n)^{c_3}\right]^{sc_3+3}\log d(\log\log d)^2 = T_5 \tag{33}$$

where $c_3=k(k-1)(k-2)$.

If $k>3$, we obtain in a similar manner that for any $3<j\leqq k$

$$\alpha_2-\alpha_1=\sigma_j\gamma_{21j},\quad \alpha_j-\alpha_2=\sigma_j\gamma'_{j2},\quad \alpha_j-\alpha_1=\sigma_j\gamma'_{j1}$$

holds, $\sigma_j, \gamma_{21j}, \gamma'_{j2}, \gamma'_{j1}$ being integers in $L(\alpha_1, \alpha_2, \alpha_j)$ with the property $\overline{|\gamma_{21j}|}, \overline{|\gamma'_{j2}|}, \overline{|\gamma'_{j1}|} < \exp\{c_{48}T_5\}$. Since $\sigma\gamma_{21} = \sigma_j\gamma_{21j}$, we get

$$\alpha_j - \alpha_1 = \sigma\gamma_{j1}$$

for $j=2, \dots, k$ with $\gamma_{j1} = \gamma_{21}\gamma'_{j1}/\gamma_{21j}$ if $j>3$. This implies $\alpha_j - \alpha_i = \sigma\gamma_{ji}$ for any $1 \leqq i < j \leqq k$, where $\gamma_{ji} = \gamma_{j1} - \gamma_{i1}$ for $i \geqq 2$. Since $\alpha_j - \alpha_i$, γ_{21j}, γ_{21i} are integers and $\overline{|\gamma_{21j}|}, \overline{|\gamma_{21i}|} < \exp\{c_{48}T_5\}$, there are rational integers $0 \leqq a'_l \leqq a_l$ such that $\beta_1^{a'_1} \dots \beta_t^{a'_t} \gamma_{ji} = \varrho_{ji}$ are also integers and

$$(34) \qquad \overline{|\varrho_{ji}|} < \exp\{c_{49}(s+1)T_5(\log P)D^{(3/2)(k-1)(k-2)}(\log D)^{nk(k-1)(k-2)-1}\} = T_6$$

for any $1 \leqq i < j \leqq k$. Further, with the notation $b_l = a_l - a'_l (\geqq 0)$, $\varepsilon_{31} = \varepsilon_1$ we have

$$(35) \qquad \alpha_j - \alpha_i = \varepsilon_1 \beta_1^{b_1} \dots \beta_t^{b_t} \varrho_{ji}, \quad 1 \leqq i < j \leqq k.$$

Thus we may write (11) in the form

$$(36) \qquad \varepsilon\delta'\pi_1^{v_1} \dots \pi_s^{v_s} = (\varepsilon_1 \beta_1^{b_1} \dots \beta_t^{b_t})^{k(k-1)} \prod_{1 \leqq i < j \leqq k} \varrho_{ji}^2.$$

Let $1 \leqq l \leqq s$ be a fixed subscript. Let $\mathfrak{B}$ denote an arbitrary prime ideal dividing (π_l) in $G = L(\alpha_1, \dots, \alpha_k)$. $\operatorname{ord}_{\mathfrak{B}} \pi_l = e_l$ does not depend on the choice of $\mathfrak{B}$. $\mathfrak{B}$ divides only one of $\beta_1, \dots, \beta_t$, say $\mathfrak{B}|(\beta_q)$. Let d_l be the greatest rational integer for which

$$(37) \qquad \min\{v_l e_l + \operatorname{ord}_{\mathfrak{B}} \delta' - \operatorname{ord}_{\mathfrak{B}}(\prod_{1 \leqq i < j \leqq k} \varrho_{ji}^2), v_l e_l\} \geqq k(k-1) d_l e_l$$

holds for any $\mathfrak{B}|(\pi_l)$. Since (36) implies

$$(38) \qquad k(k-1) b_q \operatorname{ord}_{\mathfrak{B}} \beta_q = v_l e_l + \operatorname{ord}_{\mathfrak{B}} \delta' - \operatorname{ord}_{\mathfrak{B}}(\prod_{1 \leqq i < j \leqq k} \varrho_{ji}^2),$$

hence $d_l \geqq 0$. By definition of d_l there is a $\mathfrak{B}|(\pi_l)$ such that

$$(39) \qquad k(k-1)(d_l+1)e_l > \min\{v_l e_l + \operatorname{ord}_{\mathfrak{B}} \delta' - \operatorname{ord}_{\mathfrak{B}}(\prod_{1 \leqq i < j \leqq k} \varrho_{ji}^2), v_l e_l\}.$$

But (34) and (13) imply for each $\mathfrak{B}|(\pi_l)$

$$\operatorname{ord}_{\mathfrak{B}}(\prod_{1 \leqq i < j \leqq k} \varrho_{ji}^2) < c_{50} \log T_6$$

and

$$\operatorname{ord}_{\mathfrak{B}} \delta' < c_{51} \log T_6.$$

Therefore from (37) and (39) we get

$$(40) \qquad 0 \leqq v_l e_l - k(k-1) d_l e_l < c_{52} \log T_6$$

and this together with (37) and (38) give

$$(41) \qquad 0 \leqq b_q \operatorname{ord}_{\mathfrak{B}} \beta_q - d_l e_l < c_{53} \log T_6.$$

(40) and (41) are obviously valid for each l, $1 \leqq l \leqq s$, and for every prime ideal $\mathfrak{B}|(\pi_l)$.

Choose now ξ so that $\beta_1^{b_1}\dots\beta_t^{b_t}=\pi_1^{d_1}\dots\pi_s^{d_s}\xi$ holds. By (41) ξ is an integer in the above number field M. There is a unit $\varepsilon_2\in M$ and an integer ξ' such that $\xi=\varepsilon_2\xi'$ and

$$\overline{|\xi'|}<\exp\{c_{54}(s+1)\log P\log T_6\}. \tag{42}$$

In (36) ε can be written in the form $\varepsilon=\eta_1\eta_2^{k(k-1)}$, where η_1 and η_2 are units in L and

$$\overline{|\eta_1|}<\exp\{c_{55}R_L\}. \tag{43}$$

Finally from (36) we obtain that

$$(\varepsilon_1\varepsilon_2\eta_2^{-1})^{k(k-1)}=\eta_1\delta'\left\{\prod_{l=1}^{s}\pi_l^{v_l-k(k-1)d_l}\right\}(\xi')^{-k(k-1)}\Big\{\prod_{1\le i<j\le k}\varrho_{ji}^2\Big\}^{-1},$$

whence, by virtue of (43), (13), (12), (40), (42) and (34), for $\varepsilon_3=\varepsilon_1\varepsilon_2\eta_2^{-1}$ we get

$$\overline{|\varepsilon_3|}<\exp\{c_{56}(s+1)(R_L+h_L\log P)\log T_6\}.$$

This yields with $\varkappa_{ji}=\varepsilon_3\xi'\varrho_{ji}$

$$\alpha_j-\alpha_i=\varepsilon_1\beta_1^{b_1}\dots\beta_t^{b_t}\varrho_{ji}=\eta_2\pi_1^{d_1}\dots\pi_s^{d_s}\varkappa_{ji} \tag{44}$$

for any $1\le i<j\le k$. Writing $\varkappa_{ii}=0$, $\alpha_1+\dots+\alpha_k=a_1$ and $\varkappa_{1i}+\dots+\varkappa_{ki}=-\vartheta_i$, from (44) it follows that

$$k\alpha_i=a_1+\eta_2\pi_1^{d_1}\dots\pi_s^{d_s}\vartheta_i \tag{45}$$

for $i=1,\dots,k$, where $a_1\in\mathbf{Z}_L$ and

$$\overline{|\vartheta_i|}<\exp\{c_{57}(s+1)(R_L+h_L\log P)\log T_6\}=T_7,\quad i=1,\dots,k. \tag{46}$$

From (45) we get

$$\eta_2\pi_1^{d_1}\dots\pi_s^{d_s}\vartheta_i\equiv-a_1\pmod k \tag{47}$$

for $i=1,\dots,k$. If $(k,\pi_1\dots\pi_s)=1$, then there is an $a_2\in\mathbf{Z}_L$ such that

$$\vartheta_i\equiv a_2\pmod k \tag{48}$$

for every i, $1\le i\le k$. Suppose now that, say, $\pi_1,\dots,\pi_l$, $l\le s$, are not relatively prime to k. Consider the ideal decomposition $(k)=\mathfrak{A}_1\cdot\mathfrak{A}_2$ in $G=L(\alpha_1,\dots,\alpha_k)$ such that $(\mathfrak{A}_1,\pi_1\dots\pi_s)=1$ and $\mathfrak{A}_2$ is divisible exactly by those prime ideals which lie above $\pi_1\dots\pi_l$. If for some $1\le j\le l$ and $c_{58}>0$, $d_j\le c_{58}$ holds, then we may take $\vartheta_i'=\pi_j^{d_j}\vartheta_i$ in place of ϑ_i for each i, $1\le i\le k$, where $\overline{|\vartheta_i'|}$ is bounded above by $T_7^{c_{59}}$. In this case we can eliminate π_j from (47). Thus we may assume that in (47) $d_j>c_{60}$ for any j, $1\le j\le l$, and for a suitable $c_{60}>0$. Choose now c_{60} and d_j', $j=1,\dots,l$, with the properties $d_j'\le c_{60}$, $\pi_1^{d_1'}\dots\pi_l^{d_l'}=\tau\equiv0\pmod{\mathfrak{A}_2}$ and $\tau\equiv1\pmod{\mathfrak{A}_1}$. For $\vartheta_i'=\vartheta_i\pi_1^{d_1'}\dots\pi_l^{d_l'}$ we obtain by (47) that

$$\vartheta_i'\equiv a_3\equiv a_3\tau\pmod{\mathfrak{A}_1}$$

and

$$\vartheta_i'\equiv0\equiv a_3\tau\pmod{\mathfrak{A}_2},\quad i=1,\dots,k,$$

with an $a_3 \in \mathbf{Z}_L$. From this we get, similarly to (48)

$$\vartheta_i' \equiv a_3 \tau \pmod{k}$$

for $i=1, \ldots, k$ with $a_3\tau \in \mathbf{Z}_L$.

There exists an integral basis 1, $\omega_2, \ldots, \omega_n$ in L for which $\max_t(|\overline{\omega_t}|) < c_{61} D_L^{(n^2-1)/2}$ holds (see e.g. [6]). Representing $a_3\tau$ in such a basis, it is easy to see that there is an $a_4 \in \mathbf{Z}_L$ congruent to $a_3\tau \pmod{k}$ such that $|\overline{a_4}| < c_{62} D_L^{(n^2-1)/2}$. Write $\vartheta_i' = a_4 + k\gamma_i$ for $i=1, \ldots, k$, $d_j - d_j' = w_j$ for $j=1, \ldots, l$ and $d_j = w_j$ for $j > l$. Then γ_i are integers for any i and

(49)
$$\max_i (|\overline{\gamma_i}|) < T_7^{c_{63}}.$$

Finally, from (45) it follows that for any i

(50)
$$\alpha_i = a + \eta_2 \pi_1^{w_1} \ldots \pi_s^{w_s} \gamma_i$$

with an integer a and a unit η_2 of L.

Take now the polynomial

$$f^*(x) = \prod_{i=1}^{k} (x - \gamma_i) \in \mathbf{Z}_L[x].$$

Putting $\eta_2 \pi_1^{w_1} \ldots \pi_s^{w_s} = \eta$, η belongs obviously to S and $\eta^k f^*(\eta^{-1}x)$ is $\mathbf{Z}_L$-equivalent to f. Further we have

(51)
$$\overline{|f^*|} < T_7^{c_{64}} <$$
$$< \exp\left\{c_{65} D_L^{1/2} P^{2nc_3} \left[c_{66}(s+1)^{c^*} (\log P)^2 (D^{3/2k} (\log D)^n)^{c_3}\right]^{sc_3+4} \log d (\log\log d)^2\right\}$$

by (49), (46), (34) and (33).

In order to prove (2) it suffices now to estimate D from above by P, D_L, d, n, k and s.

If $D = D_L$, that is $M_i = L(\alpha_i) = L$ for each $1 \leqq i \leqq k$, then (2) immediately follows from (51).

Suppose now that $D = D_{M_i}$ for some $M_i = L(\alpha_i) \supsetneqq L$. Denote by $f_i \in \mathbf{Z}_L[x]$ that irreducible monic polynomial for which $f_i(\alpha_i) = 0$. Then $f_i | f$ over $\mathbf{Z}_L$. The relative discriminant $D_{M_i/L}$ lies above $D_{M_i/L}(\alpha_i)$ and by (10)

(52)
$$(D_{M_i/L}(\alpha_i)) = (D(f_i)) \,|\, (D(f)) = (\delta) \mathfrak{p}_1^{u_1} \ldots \mathfrak{p}_s^{u_s}.$$

Further for any j $N(\mathfrak{p}_j) = p_j^{f_j}$ holds with a rational prime p_j. Thus we get

(53)
$$D = D_{M_i} = N_{L/\mathbf{Q}}(D_{M_i/L}) D_L^{[M_i:L]} \,\Big|\, D_L^k N_{L/\mathbf{Q}}(\delta) \prod_{j=1}^{s} p_j^{f_j u_j}.$$

For an arbitrary $p = p_j$, consider the prime ideal decomposition $(p) = \mathfrak{B}_1^{e_1} \ldots \mathfrak{B}_r^{e_r}$ in M_i with $N(\mathfrak{B}_l) = p^{f_l'}$, $l = 1, \ldots, r$. Denote by $p^{v_p(D_{M_i})}$ the highest power of p in D_{M_i} and by $\mathfrak{B}_l^{v_{\mathfrak{B}_l}(\mathscr{D}_{M_i/Q})}$ the highest power of $\mathfrak{B}_l$ in the different $\mathscr{D}_{M_i/\mathbf{Q}}$ for $l = 1, \ldots, r$. By a theorem of K. Hensel [11] (see also W. Narkiewicz [15]) we have

$$v_{\mathfrak{B}_l}(\mathscr{D}_{M_i/\mathbf{Q}}) \leqq e_l + v_{\mathfrak{B}_l}(e_l) - 1.$$

Therefore we get

$$v_p(D_{M_i}) = \sum_l f_l' v_{\mathfrak{B}_l}(\mathcal{D}_{M_i/\mathbf{Q}}) \leqq [M_i:\mathbf{Q}]-1+\sum_l f_l' v_{\mathfrak{B}_l}(e_l) \leqq nk-1+\sum_l f_l' v_{\mathfrak{B}_l}(e_l).$$

But

$$\sum_l f_l' v_{\mathfrak{B}_l}(e_l) \leqq \frac{[M_i:\mathbf{Q}]}{\log p} \sum_l \log e_l \leqq (nk)^2/\log p,$$

consequently

$$\prod_{j=1}^{s} p_j^{v_{p_j}(D_{M_i})} \leqq \left(\prod_{j=1}^{s} p_j\right)^{nk-1} \left(\prod_{j=1}^{s} p_j^{1/\log p_j}\right)^{(nk)^2} \leqq P^{s(nk-1)} \cdot e^{s(nk)^2}.$$

Combining this with (53) we obtain that

$$D \leqq D_L^k d P^{s(nk-1)} e^{s(nk)^2}.$$

Now (51) implies (2) and our theorem is proved.

5. Proof of Corollary 1

By Theorem 1 f may be written in the form $f(x)=\eta^k f^*\big(\eta^{-1}(x+a)\big)$ with $\eta\in S$, $a\in\mathbf{Z}_L, f^*\in\mathbf{Z}_L[x]$ and $\overline{|f^*|}<T_8$, where T_8 denotes the upper bound given in Theorem 1 for $\overline{|f^*|}$. Further, putting $\mathfrak{p}_l^{h_L}=(\pi_l)$ for $l=1, \dots, s$, we may assume that $\eta=\varepsilon\pi_1^{w_1}\dots\pi_s^{w_s}$ with a suitable unit ε of L and non-negative rational integers $w_1, \dots, w_s$ and

$$\max_l (\overline{|\pi_l|}) < \exp\{c_{67}(R_L+h_L\log P)\} = T_9. \tag{54}$$

We have

$$\delta_i\alpha_i = a_i = \frac{\eta^i}{(k-i)!} f^{*(k-i)}(\eta^{-1}a). \tag{55}$$

Let $\theta_1, \dots, \theta_i$ denote the roots of $f^{*(k-i)}(x)$. Then

$$\max_j (\overline{|\theta_j|}) < c_{68}\overline{|f^*|} < c_{69}T_8 \tag{56}$$

holds. By hypothesis $f^{*(k-i)}$ has at least two distinct roots. Suppose, for convenience, that $\theta_1\neq\theta_2$. $\theta_j'=\binom{k}{i}\theta_j$, $j=1, \dots, i$, and $b=\binom{k}{i}a$ are integers and from (55) we obtain that

$$\binom{k}{i}^{i-1} \delta_i\alpha_i = (b-\eta\theta_1')(b-\eta\theta_2') \prod_{j>2} (b-\eta\theta_j'). \tag{57}$$

Put $M=L(\theta_1, \theta_2)$. From (57) we get that

$$(b-\eta\theta_j') = \mathfrak{d}_j \mathfrak{B}_1^{u_{1j}}\dots\mathfrak{B}_r^{u_{rj}}, \quad j=1, 2, \tag{58}$$

where $\mathfrak{d}_j$ denote ideals in M dividing $\binom{k}{i}^{i-1}\delta_i$ and $\mathfrak{B}_1, \dots, \mathfrak{B}_r$ are all the distinct prime ideals in M lying above α_i. Clearly we have $r\leqq ti(i-1)$ and $N(\mathfrak{d}_j)\leqq c_{70}d_i^{i(i-1)}$ for $j=1, 2$. Further $N(\mathfrak{B}_m)\leqq P_i^{i(i-1)}$ for each $m=1, \dots, r$.

Write $u_{mj}=h_M v_{mj}+k_{mj}$ with $0\leqq k_{mj}<h_M$ and $\mathfrak{B}_m^{h_M}=(\beta_m)$ for any m, j, where h_M denotes the class number of $M/\mathbf{Q}$. From (58) it follows that

$$(59)\qquad b-\varepsilon\pi_1^{w_1}\dots\pi_s^{w_s}\theta_j' = \varepsilon_j\beta_1^{v_{1j}}\dots\beta_r^{v_{rj}}\tau_j, \quad j=1,2,$$

with $(\tau_j)=\mathfrak{d}_j\mathfrak{B}_1^{k_{1j}}\dots\mathfrak{B}_r^{k_{rj}}$ for $j=1,2$. We may suppose that

$$(60)\qquad \max_j(|\overline{\tau_j}|) < \exp\{c_{71}(R_M+\log d_i+th_M\log P_i)\} = T_{10}$$

and

$$(61)\qquad \max_m(|\overline{\beta_m}|) < \exp\{c_{72}(R_M+h_M\log P_i)\} = T_{11},$$

R_M being the regulator of $M/\mathbf{Q}$.

In M

$$(\pi_1^{w_1}\dots\pi_s^{w_s}, \beta_1^{v_{1j}}\dots\beta_r^{v_{rj}})\,|\,(D(f), \alpha_i)$$

holds for $j=1,2$. Therefore, if $\mathfrak{B}_m|\pi_l$ for some m and l, then

$$(62)\qquad \min\{v_{mj}, w_l\} < c_{73}\log M_i, \quad j=1,2.$$

Consequently, this allows us to write

$$\pi_1^{w_1}\dots\pi_s^{w_1}(\theta_1'-\theta_2') = \pi_s^{w_1'}\dots\pi_s^{w_s'}\sigma \text{ and } \beta_1^{v_{1j}}\dots\beta_r^{v_{rj}}\tau_j = \beta_1^{v_{1j}'}\dots\beta_r^{v_{rj}'}\sigma_j,$$

where w_l', v_{m_j}' are non-negative rational integers such that $\pi_1^{w_1}\dots\pi_s^{w_s'}$ and $\beta_1^{v_{1j}}\dots\beta_r^{v_{rj}'}$ are relatively prime for $j=1,2$ and by (54), (56), (59), (60), (61) and (62)

$$\log|\overline{\sigma}| < c_{74}\log T_8+c_{75}s\log M_i\log T_9$$

and

$$\log|\overline{\sigma_j}| < c_{76}\log T_{10}+c_{77}t\log M_i\log T_{11}$$

holds for $j=1,2$.

From (59) it follows now that

$$\pi_1^{w_1'}\dots\pi_s^{w_s'}\sigma = (\varepsilon_2/\varepsilon)\cdot\beta_1^{v_{12}'}\dots\beta_r^{v_{r2}'}\sigma_2-(\varepsilon_1/\varepsilon)\cdot\beta_1^{v_{11}'}\dots\beta_r^{v_{r1}'}\sigma_1.$$

Since this equation is of the same type in w_l', v_{mj}' as (21), we can apply the argument used in the proof of Theorem 1 to give an upper bound for $w_1', \dots, w_s', v_{1j}', \dots, v_{rj}'$ and $|\overline{\varepsilon_j/\varepsilon}|$, $j=1,2$, and consequently for $|\overline{\pi_1^{w_1}\dots\pi_s^{w_s}\theta_j'}|$ and $|\overline{a_1}|$, where $a_1=\varepsilon^{-1}a$. After calculating these bounds, we get for the polynomial

$$\bar{f}(x) = (\pi_1^{w_1}\dots\pi_s^{w_s})^k f^*((\pi_1^{w_1}\dots\pi_s^{w_s})^{-1}(x+a_1))$$

that $\bar{f}(x)=\varepsilon^{-k}f(\varepsilon x)$ and

$$\log|\overline{\bar{f}}| < c_{78}(c_{79}(s+t+1))^{c^*(i^2t+k^3s)}(P_i^n(\log P_i)^t)^{i^2}\log d_iM_i(\log\log d_iM_i)^2\cdot$$
$$\cdot((D_L^2d^{1/k}P^{(s+2)n})^{3/2}(5^{skn}\log D_Ld)^{2n})^{c_3(sc_3+4)}$$

with $c_3=k(k-1)(k-2)$. This completes the proof of Corollary 1.

References

[1] A. BAKER, Contributions to the theory of Diophantine equations, *Philos. Trans. Roy. Soc. London, Ser. A.*, **263** (1968), 173—208.

[2] A. BAKER, The theory of linear forms in logarithms, *Proc. Conf. Transcend. Theory* (Cambridge, 1976), to appear.

[3] A BAKER and C. L. STEWART, Further aspects of transcendence theory, *Journées Arithmétiques de Caen, Astérisque*, **41—42** (1977), 153—163.

[4] B. J. BIRCH and J. R. MERRIMAN, Finiteness theorems for binary forms with given discriminant, *Proc. London Math. Soc.*, **25** (1972), 385—394.

[5] K. GYŐRY, Sur les polynômes à coefficients entiers et de discriminant donné, *Acta Arith.*, **23** (1973), 419—426.

[6] K. GYŐRY, Sur les polynômes à coefficients entiers et de discriminant donné, II, *Publ. Math. Debrecen*, **21** (1974), 125—144.

[7] K. GYŐRY, Sur les polynômes à coefficients entiers et de discriminant donné, III, *Publ. Math. Debrecen*, **23** (1976), 141—165.

[8] K. GYŐRY, Polynomials with given discriminant, *Coll. Math. Soc. János Bolyai*, **13** (Debrecen, 1974). Topics in Number Theory (Edited by P. Turán), North-Holland Publ. Comp. (Amsterdam—Oxford—New York, 1976), 65—78.

[9] K. GYŐRY, On polynomials with integer coefficients and given discriminant, IV, *Publ. Math. Debrecen*, **25** (1978), 155—167.

[10] K. GYŐRY and Z. Z. PAPP, On discriminant form and index form equations, to appear.

[11] K. HENSEL, Über die Entwicklung der algebraischen Zahlen in Potenzreihen, *Math. Ann.*, **55** (1902), 305—336.

[12] S. V. KOTOV and V. G. SPRINDŽUK, An effective analysis of the Thue—Mahler equation in relative fields (Russian), *Dokl. Akad. Nauk. BSSR.* **17** (1973), 393—395.

[13] S. V. KOTOV, The Thue—Mahler equation in relative fields (Russian), *Acta Arith.*, **27** (1975), 293—315.

[14] E. LANDAU, Verallgemeinerug eines Pólyaschen Satzes auf algebraische Zahlhörper, *Nachr. Kgl. Ges. Wiss. Göttingen, math.-phys. Kl.* (1918), 478—488.

[15] W. NARKIEWICZ, *Elementary and analytic theory of algebraic numbers*, PWN (Warszawa, 1974).

[16] C. J. PARRY, The $\mathfrak{p}$-adic generalization of the Thue—Siegel theorem, *Acta Math.*, **83** (1950), 1—100.

[17] A. J. VAN DER POORTEN, Linear forms in logarithms in the $\mathfrak{p}$-adic case, *Proc. Conf. Transcend. Theory* (Cambridge, 1976), to appear.

[18] A. J. VAN DER POORTEN and J. H. LOXTON, Multiplicative relations in number fields, *Bull. Austral. Math. Soc.*, **16** (1977), 83—98. Corrigendum and addendum, *ibid.*, **17** (1977), 151—156.

[19] T. N. SHOREY, A. J. VAN DER POORTEN, R. TIJDEMAN and A. SCHINZEL, Applications of the Gel'fond Baker method to Diophantine equations, *Proc. Conf. Transcend. Theory* (Cambridge, 1976), to appear.

[20] C. L. SIEGEL, Abschätzung von Einheiten, *Nachr. Akad. Wiss. Göttingen, math.-phys. Kl.* (1969), 71—86.

[21] H. M. STARK, Effective estimates of solutions of some diophantine equations, *Acta Arith.*, **24** (1973), 251—259.

[22] H. M. STARK, Some effective cases of the Brauer—Siegel theorem, *Inv. Math.*, **23** (1974), 135—152.

[23] L. A. TRELINA, On algebraic integers with discriminants containing fixed prime divisors (Russian), *Mat. Zametki*, **21** (1977), 289—296.

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DEPARTMENT OF MATHEMATICS
KOSSUTH LAJOS UNIVERSITY
4010 DEBRECEN, HUNGARY

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ON THE INTEGRAL OF THE LEBESGUE FUNCTION OF INTERPOLATION

By
P. ERDŐS, member of the Academy and J. SZABADOS (Budapest)

Let

(1) $$-1 \leqq x_{1,n} < x_{2,n} < \ldots < x_{n,n} \leqq 1$$

be the nodes of interpolation (shortly $x_k = x_{k,n}$);

$$l_k(x) = l_{k,n}(x) = \frac{\omega(x)}{\omega'(x_k)(x-x_k)} \quad \left(k = 1, \ldots, n;\ \omega(x) = \prod_{k=1}^{n} (x-x_k)\right)$$

the corresponding fundamental polynomials, and

$$\lambda_n(a, b) = \max_{a \leqq x \leqq b} \sum_{k=1}^{n} |l_k(x)| \quad \text{if} \quad -1 \leqq a < b \leqq 1.$$

The quantity $\lambda_n(-1, 1)$ called Lebesgue constant plays an important role in the theory of Lagrange interpolation; as G. FABER [1] showed [1]

(2) $$\lambda_n(-1, 1) \geqq c_1 \log n$$

for an arbitrary system of nodes (1). Moreover, S. BERNSTEIN [2] proved that

(3) $$\lambda_n(a, b) \geqq c_2 \log n \quad (n \geqq n_1(a, b);\ -1 \leqq a < b \leqq 1)$$

for all systems (1) again.

In this paper we prove a more general result from which (3) will follow as a corollary.

THEOREM. *For an arbitrary system of nodes* (1) *and subinterval* $[a, b] \subseteqq [-1, 1]$ *we have*

(4) $$\int_a^b \sum_{k=1}^{n} |l_k(x)|\, dx \geqq c_3(b-a) \log n \quad (n \geqq n_2(a, b)).$$

In the special case $a=-1$, $b=1$, this result has been announced in [3] (with an indication of a possible method of proof). Our proof is simpler and follows a different pattern.

PROOF. According the growth rate of $\lambda_n(a, b)$ we distinguish two cases.

[1] In what follows, $c_1, c_2, \ldots$ will denote absolute positive constants.

Case 1: $\lambda_n(a, b) \geqq n^3$. Then let $y_n \in [a, b]$ be such that $\lambda_n(a, b) = \sum_{k=1}^{n} |l_k(y_n)|$, say $x_i < y_n < x_{i+1}$. On the interval $[x_i, x_{i+1}]$, $\sum_{k=1}^{n} |l_k(x)|$ is identical with a polynomial of degree less than n, and this polynomial attains its absolute maximum on $[a, b]$ also at y_n. But then by Markov's inequality, the absolute value of this polynomial is $\geqq \frac{1}{2} n^3$ in the interval $\left[y_n - \frac{b-a}{4n^2}, y_n + \frac{b-a}{4n^2}\right]$. Hence

$$\sum_{k=1}^{n} |l_k(x)| \geqq \frac{1}{2} n^3 \quad \text{if} \quad x \in [a, b] \cap \left[y_n - \frac{b-a}{4n^2}, y_n + \frac{b-a}{4n^2}\right],$$

i.e.

$$\int_a^b \sum_{k=1}^{n} |l_k(x)|\, dx \geqq \frac{1}{2} n^3 \frac{b-a}{4n^2} = \frac{b-a}{8} n$$

which is even more than we need.

Case 2: $\lambda_n(a, b) < n^3$. Then, as we shall see from the following lemma, the intervals $[x_k, x_{k+1}] \subseteqq [a, b]$ cannot be "too long".

LEMMA. *We have*

$$\max_{a \leqq x_k < x_{k+1} \leqq b} (x_{k+1} - x_k) \leqq 25 \frac{\log \lambda_n(a, b)}{n} \quad (n \geqq n_3(a, b)) \tag{5}$$

for an arbitrary system of nodes (1).

By a slightly more complicated argument, we could replace x_k by arc cos x_k in this lemma, and then (5) would be a generalization of Theorem IV from [4]. However, the given formulation will be sufficient for our purposes.

PROOF OF THE LEMMA. Assume the contrary; then there exists a subinterval $[c_n, d_n] \subset [a, b]$ of length

$$d_n - c_n = 25 \frac{\log \lambda_n(a, b)}{n}$$

which does not contain any of the nodes x_k, $k = 1, 2, \dots, n$.[2] Let

$$\gamma_n = \frac{3c_n + 2d_n}{5}, \quad \delta_n = \frac{2c_n + 3d_n}{5}$$

and z_k, $k = 1, \dots, n$, the roots of the Chebyshev polynomial $T_n(x)$ of degree n. The polynomial

$$p_n(x) = \prod_{z_k \notin [\gamma_n, \delta_n]} (x - z_k)$$

is of degree less than n. Let $x_0 \in [\gamma_n, \delta_n]$ be a point such that $|T_n(x)|$ attains its local

[2] We may assume that $25 \frac{\log \lambda_n(a, b)}{n} < b - a$; otherwise there is nothing to prove.

maximum at x_0. Such a point exists because by the Bernstein's result (3)

$$\delta_n-\gamma_n=\frac{d_n-c_n}{5}=5\frac{\log\lambda_n(a,b)}{n}>\frac{\pi}{n}>\max_{1\leqq k\leqq n-1}|z_{k+1}-z_k| \quad (n\geqq n_4(a,b))$$

holds. Thus we obtain for $x\in[-1,1]\setminus[c_n,d_n]$

$$|p_n(x)|=\left|\frac{T_n(x)}{\prod\limits_{z_k\in[\gamma_n,\delta_n]}(x-z_k)}\right|=|p_n(x_0)|\cdot\left|\frac{T_n(x)}{T_n(x_0)}\right|\cdot\prod_{z_k\in[\gamma_n,\delta_n]}\left|\frac{x_0-z_k}{x-z_k}\right|\leqq$$

$$\leqq|p_n(x_0)|\cdot\prod_{z_k\in[\gamma_n,\delta_n]}\frac{1}{2}\leqq|p_n(x_0)|\cdot 2^{-\left[\frac{\delta_n-\gamma_n}{\pi}n\right]}<|p_n(x_0)|\lambda_n(a,b)^{-1.1} \quad (n\geqq n_5(a,b)).$$

Hence, by the Lagrange interpolation formula

$$|p_n(x_0)|\leqq\sum_{k=1}^{n}|p_n(x_k)|\cdot|l_k(x_0)|<|p_n(x_0)|\lambda_n(a,b)^{-1.1}\sum_{k=1}^{n}|l_k(x_0)|\leqq|p_n(x_0)|\lambda_n(a,b)^{-0.1},$$

i.e. $\lambda_n(a,b)<1$, a contradiction. The lemma is proved.

Returning to the proof of our theorem, (5) implies that in case $\lambda_n(a,b)<n^3$ we have

$$\max_{a\leqq x_k<x_{k+1}\leqq b}(x_{k+1}-x_k)\leqq 75\frac{\log n}{n} \quad (n\geqq n_3(a,b)). \tag{6}$$

Let

$$(a\leqq)x_i<x_{i+1}<\ldots<x_j(\leqq b)$$

be all the nodes lying in the interval $[a,b]$, then

$$x_i\to a,\quad x_j\to b \quad \text{as} \quad n\to\infty$$

(otherwise even $|l_i(a)|$ or $|l_j(b)|$ would increase at least as a geometric progression). Further

(7)

$$\int_a^b\sum_{k=1}^{n}|l_k(x)|\,dx\geqq\sum_{m=i}^{j-1}\int_{x_m}^{x_{m+1}}\sum_{k=i}^{j}|l_k(x)|\,dx\geqq\frac{1}{2}\sum_{k,m=i}^{j-1}\int_{x_m}^{x_{m+1}}\{|l_k(x)|+|l_{k+1}(x)|\}\,dx>$$

$$>\frac{1}{2}\sum_{m=i}^{j-1}\sum_{k=m}^{j-1}\left\{\int_{x_m}^{x_{m+1}}(|l_k(x)|+|l_{k+1}(x)|)\,dx+\int_{x_k}^{x_{k+1}}(|l_m(x)|+|l_{m+1}(x)|)\,dx\right\}.$$

Let $\Delta x_k=x_{k+1}-x_k$ and

$$y=\frac{\Delta x_k}{\Delta x_m}(x-x_m)+x_k \quad (i\leqq m\leqq k\leqq j-1),$$

then using the inequality

$$l_k(y)+l_{k+1}(y)\geqq 1 \quad (x_k\leqq y\leqq x_{k+1})$$

(cf. [5, Lemma IV]) we get

$$|l_k(x)|+|l_{k+1}(x)| = \left|\frac{\omega(x)}{\omega(y)}\right| \left\{l_k(y)\frac{y-x_k}{x_k-x}+l_{k+1}(y)\frac{x_{k+1}-y}{x_{k+1}-x}\right\} \geqq$$

$$\geqq \left|\frac{\omega(x)}{\omega(y)}\right| \frac{\Delta x_k}{4(x_{k+1}-x_m)}\{l_k(y)+l_{k+1}(y)\} \geqq \left|\frac{\omega(x)}{\omega(y)}\right| \frac{\Delta x_k}{4(x_{k+1}-x_m)}$$

$$\left(x_m+\frac{\Delta x_m}{4} \leqq x \leqq x_{m+1}-\frac{\Delta x_m}{4}\right).$$

Thus

$$\int_{x_m}^{x_{m+1}} \{|l_k(x)|+|l_{k+1}(x)|\}\,dx \geqq \int_{x_m+\frac{\Delta x_m}{4}}^{x_{m+1}-\frac{\Delta x_m}{4}} \left|\frac{\omega(x)}{\omega(y)}\right| dx \cdot \frac{\Delta x_k}{4(x_{k+1}-x_m)} =$$

$$= \frac{\Delta x_m}{4(x_{k+1}-x_m)} \int_{x_k+\frac{\Delta x_k}{4}}^{x_{k+1}-\frac{\Delta x_k}{4}} \left|\frac{\omega(x)}{\omega(y)}\right| dy.$$

Similarly, by changing the roles of k and m, x and y,

$$\int_{x_k}^{x_{k+1}} \{|l_m(x)|+|l_{m+1}(x)|\}\,dx \geqq \frac{\Delta x_m}{4(x_{k+1}-x_m)} \int_{x_k+\frac{\Delta x_k}{4}}^{x_{k+1}-\frac{\Delta x_k}{4}} \left|\frac{\omega(y)}{\omega(x)}\right| dy.$$

Hence and from (7)

$$(8) \qquad \int_a^b \sum_{k=1}^n |l_k(x)|\,dx > \frac{1}{8}\sum_{m=i}^{j-1}\sum_{k=m}^{j-1}\frac{\Delta x_m}{x_{k+1}-x_m}\int_{x_k+\frac{\Delta x_k}{4}}^{x_{k+1}-\frac{\Delta x_k}{4}} \left\{\left|\frac{\omega(x)}{\omega(y)}\right|+\left|\frac{\omega(y)}{\omega(x)}\right|\right\} dy \geqq$$

$$\geqq \frac{1}{8}\sum_{a\leqq x_m\leqq\frac{a+b}{2}} \Delta x_m \sum_{k=m}^{j-1}\frac{\Delta x_k}{x_{k+1}-x_m}.$$

In order to estimate the inner sum, let

$$I_{t,m} = \left[x_m+\frac{75\log n}{n}t,\ x_m+\frac{75\log n}{n}(t+1)\right] \quad (t=0,1,\ldots,s_n)$$

where

$$s_n = \left[\frac{(b-a)n}{150\log n}\right].$$

Then by (6), $I_{t,m} \subset [a, b]$, and each $I_{t,m}$ contains at least one of the nodes x_k. Hence

$$\sum_{k=m}^{j-1} \frac{\Delta x_k}{x_{k+1}-x_m} \geqq \sum_{t=0}^{s_n} \sum_{x_k \in I_{t,m}} \frac{\Delta x_k}{x_{k+1}-x_m} \geqq \sum_{t=0}^{s_n} \frac{n}{75(t+1)\log n} \sum_{x_k \in I_{t,m}} \Delta x_k \geqq$$

$$\geqq \frac{n}{75 \log n} \sum_{t=1}^{\left[\frac{s_n}{3}\right]} \frac{1}{t} \sum_{x_k \in I_{t,m} \cup I_{t+1,m} \cup I_{t+2,m}} \Delta x_k \geqq \sum_{t=1}^{\left[\frac{s_n}{3}\right]} \frac{1}{t} \geqq \frac{1}{2} \log n \quad (n \geqq n_6(a, b)).$$

Thus (8) yields

$$\int_a^b \sum_{k=1}^{n} |l_k(x)|\, dx \geqq \frac{\log n}{16} \sum_{a \leqq x_m \leqq \frac{a+b}{2}} \Delta x_m \geqq \frac{b-a}{40} \log n \quad (n \geqq n_2(a, b)).$$

Q.E.D.

The best constants in (2) and (3) are (roughly speaking) $2/\pi$. Apparently, our c_3 in (4) is far from being best possible, and our method does not seem to be applicable to finding of the largest c_3.

References

[1] G. Faber, Über die interpolatorische Darstellung stetiger Funktionen, *Jahresber. der Deutschen Math. Ver.*, **23** (1914), 190—210.

[2] S. Bernstein, Sur la limitation des valeurs d'un polynome, *Bull. Acad. Sci. de l'URSS*, **8** (1931), 1025—1050.

[3] P. Erdős, Problems and results on the theory of interpolation. II, *Acta Math. Acad. Sci. Hungar.*, **12** (1961), 235—244.

[4] P. Erdős and P. Turán, On interpolation. II, *Annals of Math.*, **39** (1938), 705—724.

[5] P. Erdős and P. Turán, On interpolation. III, *Annals of Math.*, **41** (1940), 510—552.

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[1] G. Szegő, *Orthogonal polynomials,* AMS Coll. Publ. Vol. XXIII (Providence, 1939).
[2] A. Zygmund, Smooth functions, *Duke Math. J.,* **12** (1945), 47—76.

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PERIODIZITÄTSVERHALTEN BOOLE'SCHER DIFFERENZENGLEICHUNGEN

Von
H. KRÖGER (Kiel)

§ 1 Einleitung

Ausgangspunkt ist folgendes System ($t \in \mathbf{R}$, $t \geqq 0$, wobei **R** die Menge der reellen Zahlen bedeutet):

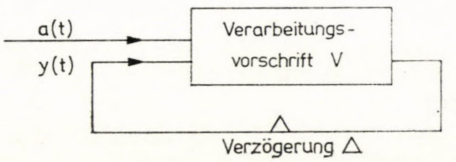

Abb. 1

welches sich als Differenzengleichung schreiben läßt:

$$a(t) \quad \text{für} \quad t \geqq 0 \quad \text{vorgegeben,}$$
$$y(t) := h(t) \quad \text{für} \quad 0 \leqq t < \Delta \quad \text{vorgegeben,}$$
$$y(t+\Delta) := V(a(t), y(t)) \quad \text{für} \quad t \geqq 0.$$

Die Verarbeitungsvorschrift V ist verzögerungsfrei. Die Art der Verarbeitungsvorschrift V sowie die Wertebereiche der Funktionen $a(t)$ und $y(t)$ müssen selbstverständlich miteinander verträglich sein. Der Fall, daß V nicht nur über $a(t)$ und $y(t)$ sondern zusätzlich direkt von t abhängt, soll hier nicht behandelt werden. Die Fragestellung lautet nun: Wann gibt es Zahlen $t_0 \geqq 0$ und $\delta > 0$ derart, daß für $t \geqq t_0$ $y(t+\delta) = y(t)$ gilt? Was kann man über t_0 und δ aussagen?

In Abb. 1 spielt $a(t)$ offenbar die Rolle einer Störfunktion, es handelt sich gewissermaßen um eine inhomogene Differenzengleichung. Falls $a(t)$ periodisch ist, es also ein τ gibt, so daß $a(t+\tau) = a(t)$ gilt, kann man obige inhomogene Differenzengleichung in folgendes homogene Differenzengleichungssystem

$$a(t) \quad \text{für} \quad 0 \leqq t < \tau \quad \text{vorgegeben,}$$
$$y(t) \quad \text{für} \quad 0 \leqq t < \Delta \quad \text{vorgegeben,}$$
$$a(t+\tau) := V_a(a(t), y(t)) \quad \text{für} \quad t \geqq 0 \quad \text{mit} \quad V_a(a(t), y(t)) := a(t),$$
$$y(t+\Delta) := V_y(a(t), y(t)) \quad \text{für} \quad t \geqq 0$$

überführen, dem die Skizze in Abb. 2 entspricht.

Damit liegt es nahe, homogene Differenzengleichungssysteme (mit n Differenzengleichungen, $n \in \mathbf{N}$, $n \geqq 2$, wobei $\mathbf{N}$ die Menge der natürlichen Zahlen bedeutet) zu untersuchen. In dieser Arbeit sollen alle in den Differenzengleichungen auftretenden Funktionen endliche Wertebereiche besitzen und zwar soll stets $\{O, L\}$ — die Menge der Boole'schen Wahrheitswerte *wahr* (L) und *falsch* (O) — als Wertebereich gelten. Die Einschränkung auf endliche Wertebereiche ist wesentlich.

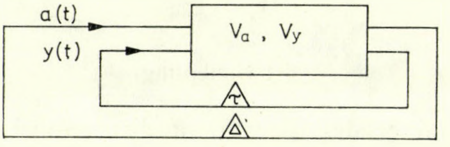

Abb. 2

Die zusätzliche Einschränkung auf den Wertebereich $\{O, L\}$ erlaubt, bekannte Eigenschaften Boole'scher Funktionen auszunutzen. Damit bietet sich eine gewisse Beziehung zur Theorie der Schaltwerke an. Diese Untersuchungen sollen jedoch überwiegend aus der Sicht der Differenzengleichungen durchgeführt werden, denn für von $\{O, L\}$ verschiedene endliche Wertebereiche kann man die allgemeine Periodizitätsaussage von Satz 1 leicht übertragen. Eine weitergehende Spezialisierung wird natürlich sehr stark von dem ausgewählten Wertebereich geprägt.

§ 2 Systeme im rationalen Fall

Unter einem homogenen Boole'schen Differenzengleichungssystem verstehe man folgendes:

Für $1 \leqq i \leqq n$ ($n \in \mathbf{N}$, $n \geqq 2$) seien positive reelle Zahlen Δ_i und Funktionen $h_i(t)$ mit Werten aus $\{O, L\}$ für $0 \leqq t < \Delta_i$ sowie n-stellige Boole'sche Funktionen F_i gegeben und es gelte

$$y_i(t) := h_i(t) \quad \text{für} \quad 0 \leqq t < \Delta_i,$$

$$y_i(t+\Delta_i) := F_i(y_1(t), \ldots, y_n(t)) \quad \text{für} \quad t \geqq 0.$$

Abb. 2 gibt für $n=2$ einen Spezialfall an.

SATZ 1. *Sind in einem homogenen Boole'schen Differenzengleichungssystem die Verhältnisse* $\Delta_1 : \Delta_2, \ldots, \Delta_{n-1} : \Delta_n$ *alle rational, so gilt mit* $\mathfrak{y}(t) := (y_1(t), \ldots, y_n(t))^{\top}$: *Es gibt reelle Zahlen* $t_0 \geqq 0$ *und* $\delta > 0$ *so, daß* $\mathfrak{y}(t+\delta) = \mathfrak{y}(t)$ *für* $t \geqq t_0$.

BEWEIS. Offenbar gibt es ein maximales Δ und $\lambda_i \in \mathbf{N}$ derart, daß $\Delta_i = \lambda_i \cdot \Delta$ gilt, Δ kann als Elementarlänge des Systems bezeichnet werden. Sei

$$\Delta_{\max} := \max(\Delta_1, \ldots, \Delta_n), \quad \lambda_{\max} := \max(\lambda_1, \ldots, \lambda_n).$$

Für festes t' mit $0 \leqq t' < \Delta$ sei für $j \in \mathbf{N} \cup \{0\}$ $\mathfrak{y}_j(t') := \mathfrak{y}(t' + j \cdot \Delta)$. Für $i \in \mathbf{N} \cup \{0\}$ sei

$$S_i(t') := (\mathfrak{y}_i(t'), \mathfrak{y}_{i+1}(t'), \ldots, \mathfrak{y}_{i+\lambda_{\max}-1}(t'))$$

das i-te Stammstück zu t'.

Wegen der Endlichkeit der Wertemenge $\{O, L\}$ der F_i gibt es nur endlich viele verschieden besetzte Stammstücke, nämlich höchstens 2^m mit $m := n \cdot \lambda_{\max}$. Da das i-te Stammstück das $(i+1)$-te Stammstück eindeutig bestimmt, gibt es für jedes t' mit $0 \leqq t' < \Delta$ ganze Zahlen k_0 und k' mit $0 \leqq k_0 < 2^m$, $1 \leqq k' \leqq 2^m$ so, daß für $i \geqq k_0$ $S_{i+k'}(t') = S_i(t')$. Folglich gilt mit $t_0 := 2^m \cdot \Delta \geqq \max \{k_0 : 0 \leqq t' < \Delta\} \cdot \Delta + \Delta$ und $k := kgV\{r : 1 \leqq r \leqq 2^m\}$, $\delta := k \cdot \Delta$ die Aussage $\mathfrak{y}(t+\delta) = \mathfrak{y}(t)$ für $t \geqq t_0$.

Man beachte, daß es sich bei obigen Angaben um Abschätzungen für das gesamte Boole'sche Differenzengleichungssystem — nicht nur für einzelne Komponenten — handelt, daß außerdem keine speziellen Eigenschaften der Funktionen F_i oder der $h_i(t)$ ausgenutzt wurden. Die Bedeutung des Satzes liegt nicht in seiner quantitativen sondern in seiner qualitativen Aussage. Eine Verallgemeinerung für von $\{O, L\}$ verschiedene aber ebenfalls endliche Wertebereiche liegt nahe.

Besonders übersichtliche Verhältnisse findet man, wenn die Funktionen F_i nur i-stellige Funktionen der Argumente $y_1(t), \ldots, y_i(t)$ sind, also der Fall

$$y_i(t+\Delta_i) := F_i(y_1(t), \ldots, y_i(t)).$$

Es handelt sich um ein gestaffeltes System (gewissermaßen in Dreiecksgestalt), das man nicht simultan lösen muß sondern zeilenweise lösen kann. Dies legt es nahe, folgenden Gleichungstyp zu untersuchen.

§ 3 Die allgemeine Gleichung im rationalen Fall

Die Differenzengleichung von Abb. 1 läßt sich offenbar als Spezialfall des Typs

$$a_i(t) \text{ für } t \geqq 0 \text{ und } 1 \leqq i \leqq n \text{ vorgegeben,}$$

$$y(t) := h(t) \text{ für } 0 \leqq t < \Delta \text{ vorgegeben,}$$

$$(*) \qquad y(t+\Delta) := V(a_1(t), \ldots, a_n(t), y(t)) \text{ für } t \geqq 0$$

ansehen. Aber im Fall Boole'scher Funktionen reduziert sich $(*)$ zu

$$y(t+\Delta) := (y(t) \wedge g(t)) \vee (\neg y(t) \wedge f(t)),$$

wobei g und f durch V und $a_1, \ldots, a_n$ eindeutig bestimmt sind, oder anders formuliert

$$y(t+\Delta) := \begin{cases} g(t) & \text{falls } y(t) = L, \\ f(t) & \text{falls } y(t) = O. \end{cases}$$

Nun sollen Boole'sche Differenzengleichungen dieses Typs untersucht werden, bei denen zusätzlich $f(t)$ und $g(t)$ für $t \geqq 0$ eine gemeinsame Periode $\tau > 0$ haben und das Verhältnis $\tau : \Delta$ rational ist. Es gibt also minimale $p, q \in \mathbf{N}$ mit $p \cdot \Delta = q \cdot \tau$. Zunächst soll eine Klassifizierung der Argumente $t \geqq 0$ eingeführt werden.

DEFINITION. Man erkläre:

1) t' hat die Eigenschaft *1*: genau dann, wenn $y(t'+p \cdot \Delta) = y(t')$,

2) t' hat die Eigenschaft *A*: genau dann, wenn es mindestens eine ganze Zahl j mit $0 \leqq j < p$ gibt mit $f(t'+j \cdot \Delta) = g(t'+j \cdot \Delta)$.

Die Eigenschaft 1 vererbt sich, d.h., wenn t' die Eigenschaft 1 hat, so hat auch $t'+i\cdot p\cdot\Delta$ die Eigenschaft 1, $i\in\mathbf{N}\cup\{0\}$.

Beweis durch vollständige Induktion. Es gelte für festes t' und $i\in\mathbf{N}$

$$y(t') = y(t'+p\cdot\Delta) = \ldots = y(t'+i\cdot p\cdot\Delta),$$

dann ist zu zeigen:

$$y(t') = y\big(t'+(i+1)\cdot p\cdot\Delta\big).$$

Wegen $p\cdot\Delta=q\cdot\tau$ gilt für $j\in\mathbf{N}\cup\{0\}$ auch

$$f(t'+j\cdot\Delta)=f(t'+i\cdot p\cdot\Delta+j\cdot\Delta),\quad g(t'+j\cdot\Delta)=g(t'+i\cdot p\cdot\Delta+j\cdot\Delta).$$

Demnach geht die *Berechnungsfolge* von $y(t'+i\cdot p\cdot\Delta)$ bis $y\big(t'+(i+1)\cdot p\cdot\Delta\big)$ aus der von $y(t')$ bis $y(t'+p\cdot\Delta)$ durch Parallelverschiebung um $i\cdot p\cdot\Delta$ hervor und es folgt

$$y(t'+p\cdot\Delta) = y\big(t'+(i+1)\cdot p\cdot\Delta\big).$$

Wegen $p\cdot\Delta=q\cdot\tau$ kann man bei der Definition der Eigenschaft A auf die Bedingung $j<p$ verzichten. Die Eigenschaft A bedeutet offenbar, daß eine *Berechnungsfolge* von $y(t')$ bis $y(t'+p\cdot\Delta)$ sowohl im Fall $y(t')=L$ als auch im Fall $y(t')=O$ dasselbe Endstück hat, das im Extremfall nur aus $y(t'+p\cdot\Delta)$ besteht. Da die Eigenschaft A allein von den Funktionen $f(t)$ und $g(t)$ geprägt wird, ist wieder wegen $p\cdot\Delta=q\cdot\tau$ auch die Eigenschaft A erblich, d.h., wenn t' die Eigenschaft A hat, so hat $t'+i\cdot p\cdot\Delta$ ebenfalls die Eigenschaft A. Hat t' die Eigenschaft A, so gilt dann für $i\in\mathbf{N}$ auch $y(t'+p\cdot\Delta) = y(t'+i\cdot p\cdot\Delta)$.

Als Gegenstück zu A betrachte man die Eigenschaft B, die für t' genau dann gelten soll, wenn die Eigenschaft A für t' nicht gilt, d.h. genau dann, wenn für alle $j\in\mathbf{N}\cup\{0\}$

$$f(t'+j\cdot\Delta) \neq g(t'+j\cdot\Delta)$$

gilt. Ebenso wie A ist auch die Eigenschaft B erblich in dem Sinne, daß mit t' auch $t'+i\cdot p\cdot\Delta$ ($i\in\mathbf{N}$) die Eigenschaft B hat. Wenn t' die Eigenschaft B hat und $y(t')= =W\in\{O, L\}$ für $i\in\mathbf{N}$ die Werte $y(t'+i\cdot\Delta)=W_i\in\{O, L\}$ impliziert, so impliziert $y(t')=\neg W$ für $i\in\mathbf{N}$ die Werte $y(t'+i\cdot\Delta)=\neg W_i$, die Berechnungsfolge wird negiert.

Hat nun t' die Eigenschaft B und nicht die Eigenschaft 1, so gilt zunächst $y(t')=\neg y(t'+p\cdot\Delta)$, wegen der Eigenschaft B muß dann aber für $i\in\mathbf{N}$

$$y\big(t'+(2\cdot i-1)\cdot p\cdot\Delta\big)=\neg\, y(t'),\quad y(t'+2\cdot i\cdot p\cdot\Delta) = y(t')$$

gelten.

Offenbar wird das Periodizitätsverhalten der Lösung $y(t)$ der allgemeinen Boole'schen Differenzengleichung im rationalen Fall durch die Eigenschaften der Argumente im Intervall $0\leqq t<p\cdot\Delta$ festgelegt. Im generellen Fall ist $y(t)$ für $t\geqq p\cdot\Delta$ entweder konstant oder $y(t)$ hat für $t\geqq p\cdot\Delta$ eine kleinste Periode δ und dann gilt für ein geeignetes $r\in\mathbf{N}$ $2\cdot p\cdot\Delta = \delta\cdot r$. Je nach den Eigenschaften der Argumente im Intervall $0\leqq t<p\cdot\Delta$ kann $y(t)$ sogar für $t\geqq 0$ periodisch sein oder für die kleinste Periode δ bei geeignetem $r\in\mathbf{N}$ sogar $p\cdot\Delta = \delta\cdot r$ gelten.

§ 4 Klasseneinteilung 2-stelliger Verknüpfungen

Wählt man in §3 die Funktionen $f(t)$ und $g(t)$ geeignet, so geht der dort behandelte Gleichungstyp über in den Typ

$$\text{(B)}\qquad \begin{cases} a(t) \ \text{ für } \ t \geqq 0 \ \text{ vorgegeben,} \\ y(t) := h(t) \ \text{ für } \ 0 \leqq t < \varDelta \ \text{ vorgegeben,} \\ y(t+\varDelta) := \big(y(t)\varrho a(t)\big) \ \text{ für } \ t \geqq 0, \\ \varrho \in R := R_1 \cup R_2 \cup R_3, \\ R_1 := \{\wedge, \vee, \leftarrow, \nrightarrow\}, \quad R_2 := \{\equiv, \not\equiv\}, \\ R_3 := \{\overline{\wedge}, \overline{\vee}, \nleftarrow, \rightarrow\}. \end{cases}$$

Dabei sei $\wedge =$ und, $\vee =$ oder, $\neg =$ non,

$$x \rightarrow y := \neg x \vee y, \quad x \leftarrow y := y \rightarrow x, \quad x \overline{\wedge} y := \neg (x \wedge y),$$

$$x \overline{\vee} y := \neg (x \vee y), \quad x \nrightarrow y := \neg (x \rightarrow y), \quad x \nleftarrow y := y \nrightarrow x,$$

$$x \equiv y := \big((x \wedge y) \vee (\neg x \wedge \neg y)\big), \qquad x \not\equiv y := \neg (x \equiv y).$$

Jeder Operation $\varrho \in R$ ordne man eine Operation $\tilde{\varrho}$ und $\hat{\varrho}$ mit

$$x \hat{\varrho} y := x \varrho (\neg y) \quad \text{und} \quad x \tilde{\varrho} y := \neg \big((\neg x) \varrho y\big)$$

zu. Damit erhält man folgende Tabelle.

ϱ:	$\wedge$	$\vee$	$\leftarrow$	$\nrightarrow$	$\overline{\wedge}$	$\overline{\vee}$	$\nleftarrow$	$\rightarrow$	$\equiv$	$\not\equiv$
$\hat{\varrho}$:	$\nrightarrow$	$\leftarrow$	$\vee$	$\wedge$	$\rightarrow$	$\nleftarrow$	$\overline{\vee}$	$\overline{\wedge}$	$\not\equiv$	$\equiv$
$\tilde{\varrho}$:	$\leftarrow$	$\nrightarrow$	$\wedge$	$\vee$	$\nleftarrow$	$\rightarrow$	$\overline{\wedge}$	$\overline{\vee}$	$\equiv$	$\not\equiv$
$\hat{\tilde{\varrho}}$:	$\vee$	$\wedge$	$\nrightarrow$	$\leftarrow$	$\overline{\vee}$	$\overline{\wedge}$	$\rightarrow$	$\nleftarrow$	$\not\equiv$	$\equiv$

Offenbar gilt $\tilde{\tilde{\varrho}} = \varrho = \hat{\hat{\varrho}}$, $\hat{\tilde{\varrho}} = \tilde{\hat{\varrho}}$.

Sei nun $a(t)$ mit Periode $\tau > 0$ für $t \geqq 0$ fest vorgegeben, ebenso $\varDelta$ und $h(t)$ für $0 \leqq t < \varDelta$. Die Lösungsfunktion der zugehörigen Differenzengleichung vom Typ (B) werde mit $y(h; \varrho; a)$ bezeichnet. Dann gilt, wie man der Tabelle entnimmt,

$$y(h; \varrho; a) = y(h; \hat{\varrho}; \neg a) = \neg y(\neg h; \tilde{\varrho}; a) = \neg y(\neg h; \hat{\tilde{\varrho}}; \neg a).$$

Sofern man an $a(t)$ und $h(t)$ kein speziellen Bedingungen stellt, kann man den Gleichungstyp (B) auf die Fälle $\varrho \in \{\wedge, \equiv, \rightarrow\}$ reduzieren, für jede der Klassen R_1, R_2, R_3 genügt es, wenn man das Problem für den entsprechenden Repräsentanten lösen kann. Für die Klasse $\{\equiv, \not\equiv\}$ gilt etwa

SATZ 2. *Gegeben sei eine Differenzengleichung vom Typ* (B) *mit* $\varrho \in \{\equiv, \not\equiv\}$, $\tau > 0$ *sei die kleinste Periode von* $a(t)$. *Wenn es* $t_0 \geqq 0$ *und* $\delta > 0$ *so gibt, daß für* $t \geqq t_0$ $y(t+\delta) = y(t)$ *gilt, dann gibt es* $\lambda \in \mathbf{N}$ *so, daß* $\delta = \lambda \cdot \tau$.

BEWEIS. Sei $t' \geqq t_0$. Der Funktionsverlauf von $y(t)$ im Intervall $t' \leqq t < t' + \delta$ ist mit dem im Intervall $t' + \delta \leqq t < t' + 2 \cdot \delta$ identisch. Entsprechendes gilt für $y(t+\varDelta)$. Aus der Wertetafel von $\equiv$ erkennt man sofort, daß in $(x \equiv y) = z$ der

Wert von x durch die Werte von y und z eindeutig bestimmt ist. Daher gilt für $t \geqq t_0$ sofort $a(t+\delta)=a(t)$, aber auch $a(t+\tau)=a(t)$. Da τ als kleinste Periode von $a(t)$ angegeben war, folgt die Behauptung.

Wenn $\tau:\Delta$ rational ist, ist die Existenz von t_0 und δ für Satz 2 durch §3 gesichert. Für die Klasse R_1 erhält man

SATZ 3. *Gegeben sei eine Differenzengleichung vom Typ* (B) *mit* $\varrho \in R_1 = \{\wedge, \vee, \leftarrow, \nrightarrow\}$. *Sei* $\tau:\Delta$ *rational, also* $p \cdot \Delta = q \cdot \tau$ *für geeignete* $p, q \in \mathbf{N}$. *Dann gibt es* $\delta > 0$, $\lambda \in \mathbf{N}$ *und* $t_0 \geqq p \cdot \Delta$ *so, daß für* $t \geqq t_0$ $y(t+\delta) = y(t)$ *gilt, wobei* δ *folgende Eigenschaft hat:* $\lambda \cdot \delta = \Delta$.

BEWEIS. Es genügt, für $\varrho = \wedge$ zu zeigen, daß für $t \geqq t_0$ $y(t+\Delta) = y(t)$ gilt. Für $\varrho = \wedge$ erkennt man $y(t) = O \Rightarrow y(t+\Delta) = O$. Sei nun $t' \geqq t_0$ und $y(t') = L$. Angenommen $y(t'+\Delta) = O$, dann gilt aber als Widerspruch

$$L = y(t') = y(t'+2 \cdot p \cdot \Delta) = y\big(t'+\Delta+(2 \cdot p-1) \cdot \Delta\big) = O.$$

Als offene Frage bleibt, ob es für die Klasse R_3 ähnliche Aussagen wie Satz 2 oder Satz 3 gibt.

§ 5 Weiterführende Beispiele

Bereits bei Differenzengleichungen vom Typ (B) aus §4 mit periodischer Störfunktion $a(t)$ stößt man auf erhebliche Schwierigkeiten, wenn das Verhältnis $\tau:\Delta$ irrational ist. Wegen Satz 2 dürfte der Fall $\varrho \in \{\equiv, \not\equiv\}$ noch am übersichtlichsten sein.

BEISPIEL 1. Sei $\varrho = \equiv$, $\Delta = 2$, $\tau = \sqrt{2}$.

$$a(t) := \begin{cases} L & \text{für} \quad 0 \leqq t < 1, \\ O & \text{für} \quad 1 \leqq t < \sqrt{2}, \end{cases}$$
$$a(t+\tau) := a(t) \quad \text{für} \quad t \geqq 0,$$
$$y(t) := L \quad \text{für} \quad 0 \leqq t < 2,$$
$$y(t+\Delta) := \big(y(t) \equiv a(t)\big) \quad \text{für} \quad t \geqq 0.$$

Man sieht, daß $y(t)$ Sprungstellen für $t = 1+\lambda \cdot \sqrt{2}+2 \cdot \mu$ wobei $\lambda \in \{0\} \cup \mathbf{N}$, $\mu \in \mathbf{N}$, und für $t = \lambda \cdot \sqrt{2}+2 \cdot \mu$ wobei $\lambda, \mu \in \mathbf{N}$, hat.

Besitzt $y(t)$ auf dem Intervall $b \leqq t < b+2$ genau σ Sprungstellen, so besitzt $y(t)$ auf dem Intervall $b+2 \leqq t < b+4$ mindestens $\sigma+2$ Sprungstellen ($b \geqq 0$). Es kann also kein t_0 und kein $\delta > 0$ mit $y(t+\delta) = y(t)$ für $t \geqq t_0$ geben.

BEISPIEL 2. Sei $\varrho = \equiv$, $\Delta = 1$, $\tau = \sqrt{2}$.

$$a(t) := \begin{cases} O & \text{für} \quad 0 \leqq t < 1, \\ L & \text{für} \quad 1 \leqq t < \sqrt{2}, \end{cases}$$
$$a(t+\tau) := a(t) \quad \text{für} \quad t \geqq 0,$$
$$y(t) := L \quad \text{für} \quad 0 \leqq t < 1,$$
$$y(t+\Delta) := \big(y(t) \equiv a(t)\big) \quad \text{für} \quad t \geqq 0.$$

Die Sprungstellen von $y(t)$ sind außer $t=1$ nur noch $t = \lambda \cdot \sqrt{2}+1$ für $\lambda \in \mathbf{N}$. Daher gilt für $t \geqq 0$ mit $\delta := 2 \cdot \sqrt{2}$ $y(t) = y(t+\delta)$.

BEISPIEL 3. Sei $\varrho = \equiv$, $\Delta = 1$, $\tau = \sqrt{2}$.

$$a(t) := \begin{cases} L & \text{für } 0 \leqq t < 1, \\ O & \text{für } 1 \leqq t < \sqrt{2}, \end{cases}$$

$$a(t+\tau) := a(t) \quad \text{für} \quad t \geqq 0,$$

$$y(t) := L \quad \text{für} \quad 0 \leqq t < 1,$$

$$y(t+\Delta) := \big(y(t) \equiv a(t)\big) \quad \text{für} \quad t \geqq 0.$$

Als Sprungstellen von $y(t)$ ermittelt man $t = \lambda$ wobei $\lambda \geqq 2$, $\lambda \in \mathbf{N}$, $t = \mu \cdot \sqrt{2}+1$ wobei $\mu \in \mathbf{N}$. Es gibt kein $t_0 \geqq 0$ und kein $\delta \geqq 0$ derart, daß für $t \geqq t_0$ eine Beziehung $y(t+\delta) = y(t)$ gelten würde. Nach Satz 2 müßte δ nämlich ein Vielfaches von $\sqrt{2}$ sein. Ist aber $\lambda' \in \mathbf{N}$ und $\lambda' \neq 1$, so gibt es kein $\nu \in \mathbf{N}$ derart, daß $\lambda' + \nu \cdot \sqrt{2}$ wieder eine Sprungstelle ist.

Bislang wurden die Differenzengleichungen als exaktes mathematisches Modell behandelt. Man kann sich jedoch auch eine technische Realisierung für Abb. 1 vorstellen, wobei folgendes Problem auftaucht: Aus Trägheitsgründen benötigt der technische Apparat eine gewisse Zeit, um vom Funktionswert O auf den Funktionswert L umzuschalten und umgekehrt. Man kann während der Trägheitspause als Funktionswert entweder einen undefinierten Wert einsetzen — was im mathematischen Modell den Übergang zu einer 3-wertigen Wertemenge oder Logik erfordert — oder man behält den alten Funktionswert solange bei, bis sich der neue Funktionswert durchgesetzt hat, auch das erfordert eine Modifikation des mathematischen Modells. Hier soll an einem Beispiel noch gezeigt werden, wie man in Beispiel 3 bei konsequenter Impulsunterdrückung doch noch zu einer periodischen „Lösung" gelangt.

BEISPIEL 4. Die Ergebnisfunktion $y(t)$ aus Beispiel 3 habe einen O-Impuls bezw. L-Impuls mit den Randpunkten t_1, t_2, wobei $t_2 - t_1 < 1 < t_1 < t_2$. Zu $y(t)$ konstruiere man eine neue Funktion $\dot{y}(t)$ so:

$$\dot{y}(t) := y(t) \quad \text{für} \quad 0 \leqq t < t_1,$$

$$\dot{y}(t) := \neg\, y(t) \quad \text{für} \quad t_1 \leqq t < t_2,$$

$$\dot{y}(t+\Delta) := \big(\dot{y}(t) \equiv a(t)\big) \quad \text{für} \quad t \geqq t_2 - \Delta, \quad \text{mit} \quad \Delta = 1,$$

$a(t)$ wie in Beispiel 3.

Sei

$$t' := \begin{cases} t_2 & \text{falls } t_1 \in \mathbf{N}, \; t_2 \notin \mathbf{N}, \\ t_1 & \text{falls } t_1 \notin \mathbf{N}, \; t_2 \in \mathbf{N}. \end{cases}$$

Offenbar gilt nun

$$\dot{y}(t) = \text{const} \quad \text{für} \quad t' \leqq t < t'+1$$

und

$$\dot{y}(t'+t) = \begin{cases} y(t+1) & \text{wenn} \quad y(t_1) = O \\ \neg y(t+1) & \text{wenn} \quad y(t_1) = L \end{cases} \quad \text{für} \quad t \geqq 0.$$

Der verschluckte Impuls kommt in diesem Beispiel also bei $\dot{y}(t)$ in negierter oder nicht-negierter Form wieder mit den Randpunkten $t_1' := t' + t_1 - 1$ und $t_2' := := t' + t_2 - 1$.

Wiederholt man diesen Prozeß stets, wenn der Impuls negiert oder nicht-negiert wieder auftritt, so erhält man dabei eine Funktion $\overset{*}{y}(t)$ und es gilt für $t \geqq 1$ mit $\delta := t' - 1$ $\overset{*}{y}(t+\delta) = \overset{*}{y}(t)$. Man beachte, daß δ ein Vielfaches von $\sqrt{2}$ ist.

Es bleibt als offene interessante Frage: Wie reagieren andere Beispiele auf einen derartigen Impuls-Unterdrückungsprozeß?

Quellenangaben. Zu diesen Untersuchungen wurde der Verfasser 1970/71 von Prof. Dr. P. Deussen angeregt. Die Resultate wurden zunächst im Bericht Nr. 7107 der Abteilung Mathematik der TU München gesammelt. Für die benutzten Beweismittel sei auf die allgemeine mathematische Literatur verwiesen, besonders auf die Gebiete Verbandstheorie, Boole'sche Algebren, Schaltwerktheorie und Automatentheorie.

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INSTITUT FÜR INFORMATIK UND PRAKTISCHE MATHEMATIK
UNIVERSITÄT KIEL
D—2300 KIEL 1
BUNDESREPUBLIK DEUTSCHLAND

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A LOG LOG LAW FOR ABEL'S SUM

By

R. VASUDEVA (Mysore)

1. Introduction

Let $\{X_n, n \geqq 1\}$ be a sequence of independent random variables (r.v). Write

$$S_n = \sum_{k=1}^{n} X_k, \quad T_n = \sum_{k=1}^{n} (1-k/n)^p X_k \quad \text{and} \quad \sum\nolimits_n^p = \sum_{k=1}^{n} A_{n-k}^p A_n^{-p} X_k,$$

where $p>0$ and

$$A_n^p = \frac{(n+p)(n+p-1)\dots(p+1)}{n!}.$$

For the sequence $\{X_n\}$ with $E(X_n)=0$, $E(X_n^2)=1$ and $|X_n| \leqq M$, where M is a known positive number, GOPOSHKIN [6] proved that $\limsup_{n\to\infty} T_n \left\{\frac{2n}{2p+1} \log\log n\right\}^{-1/2} = 1$ almost surely and $\limsup_{n\to\infty} \sum_n^p \left\{\frac{2n}{2p+1} \log\log n\right\}^{-1/2} = 1$ almost surely. In [1] BASU has established the above results under the weaker conditions (i) $E(X_k)=0$ and $E(X_k^2)=1$ when the X_k's are identically distributed and (ii) $E(X_k)=0$, $E(X_k^2)=1$ and $\sum_{k=1}^{\infty} E(X_k^{2+\delta}) k^{-(1+\delta/2)} < \infty$ when otherwise.

The motive of this paper is to obtain iterated logarithm laws for T_n and $\sum_n^p$ under the following two assumptions on X_n: (i) the X_n's are identically distributed and independent r.v.'s and (ii) for properly selected sequence $\{B_n\}$ of positive constants, the sequence $\{S_n/B_n\}$ converges weakly to a non-normal stable r.v. Our proof rests on the method of GOPOSHKIN [6].

2. Notations and preliminaries

For a real number u let $[u]$ stand for the greatest integer $\leqq u$, let $\varepsilon>0$ and C, C_1, C_2 (>0) stand for absolute constants whose values are not the same at each occurence. $G(\cdot, \alpha, \beta, 0, C)$, $\alpha \in (0, 2)$, $\beta \in [-1, 1]$, $C>0$ stands for the stable distribution function to which $\{S_n/B_n\}$ converges weakly and $L(x)$ stands for a function slowly varying at ∞. Denote $(\log\log n)^{-1}$,

$$(\operatorname{sign} T_n) \left|\frac{T_n}{B_n}\right|^{\theta_n} \quad \text{and} \quad (\operatorname{sign} \textstyle\sum_n^p) \left|\frac{\sum_n^p}{B_n}\right|^{\theta_n} \quad \text{by} \quad \theta_n, U_n \quad \text{and} \quad V_n,$$

respectively. The terms "infinitely often" and "almost surely" are abbreviated as i.o. and a.s., respectively.

We recall two well known results which are useful in proving the main theorem of this paper.

When the X_n's are independent and identically distributed r.v.'s with

$$\lim_{n\to\infty} P(S_n/B_n \leqq x) = G(x, \alpha, \beta, 0, C)$$

then for any $x>0$ (see, for example [4])

$$P(X_1 \leqq -x) \sim C_1 x^{-\alpha} L(x) \quad \text{and} \quad P(X_1 \geqq x) \sim C_2 x^{-\alpha} L(x)$$

whenever $|\beta|<1$, $P(X_1 \geqq x) \sim C_2 x^{-\alpha} L(x)$ when $\beta=1$ and $P(X_1 \leqq -x) \sim \sim C_1 x^{-\alpha} L(x)$ when $\beta=-1$. Further, when $\beta=\pm 1$ the other tail can have any rate but of $o(x^{-\alpha} L(x))$.

Another result is of BEUERMAN [2] which states that the assumptions (i) and (ii) imply

$$\lim_{n\to\infty} P(T_n/B_n \leqq x) = G\left(x; \alpha, \beta, 0, \frac{C}{\alpha p+1}\right)$$

whenever $\alpha \neq 1$. The arguments used in [2] establish the result for $\alpha=1$ also under assumptions (i) and (ii).

3. Results

The main theorems of this paper are presented in this section. A few lemmas are proved first, which are needed in establishing the theorems. In the first three lemmas the sequence (y_n) stands for a sequence of positive numbers with $y_n \to \infty$ as $n \to \infty$.

LEMMA 3.1. *If*

$$\lim_{n\to\infty} P(S_n/B_n \leqq x) = G(x; \alpha, \beta, 0, C)$$

then

$$C_1 y_n^{-(\alpha+\varepsilon)} \leqq P(T_n \geqq y_n B_n) \leqq C_2 y_n^{-(\alpha-\varepsilon)} \tag{1}$$

when $\beta \in (-1, 1]$ *and*

$$C_1 y_n^{-(\alpha+\varepsilon)} \leqq P(T_n \leqq -y_n B_n) \leqq C_2 y_n^{-(\alpha-\varepsilon)} \tag{2}$$

when $\beta \in [-1, 1)$.

PROOF. We give the proof of inequality (1). The proof of (2) is omitted as it could be obtained on similar lines.

Define

$$A_i = \{(1-i/n)^p X_i \geqq (1+\varepsilon) y_n B_n\}$$

and

$$D_i = \left\{\left|\sum_{j=1, j\neq i}^{n} (1-j/n)^p X_j\right| \leqq \varepsilon y_n B_n\right\}, \quad i = 1, 2, \dots, n.$$

Proceeding on the lines of HEYDE [7] one gets

$$P(T_n \geqq y_n B_n) \geqq \sum_{i=1}^{n} P(A_i)\left(P(D_i) - \sum_{j=1}^{n} P(A_j)\right).$$

Since

$$\lim_{n\to\infty} P(T_n/B_n \leqq x) = G\left(x;\ \alpha, \beta, 0, \frac{C}{\alpha p+1}\right),$$

$y_n^{-1} B_n^{-1} T_n \xrightarrow{P} 0$ (in probability) as $n\to\infty$. Hence given a $\delta>0$ with $1-2\delta>0$ we can choose an integer N_1 such that for all $n \geqq N_1$ and for all $i=1, 2, \ldots, n$ $P(D_i) \geqq 1-2\delta$. Also

$$\sum_{j=1}^{n} P(A_j) \leqq \sum_{j=1}^{n} P(|X_j| \geqq y_n B_n) \leqq C_1 n y_n^{-\alpha} B_n^{-\alpha} L(y_n B_n) \leqq C_2 n y_n^{-(\alpha-\varepsilon)} B_n^{-\alpha} L(B_n).$$

Since $nB_n^{-\alpha} L(B_n) \to C$ as $n\to\infty$ (see FELLER [4]) we can find an integer N_2 such that for all $n \geqq N_2$ $\sum_{j=1}^{n} P(A_j) \leqq \delta$. As a consequence

$$P(T_n \geqq y_n B_n) \geqq (1-\delta) \sum_{i=1}^{n} P(A_i). \tag{3}$$

But

$$\sum_{i=1}^{n} P(A_i) \geqq \sum_{i=1}^{[n/2]} P\left(|X_i| \geqq \frac{1}{2^p} y_n B_n\right) \geqq \left[\frac{n}{2}\right] C_1 y_n^{-(\alpha+\varepsilon)} B_n^{-\alpha} L(B_n) \geqq C_2 y_n^{-(\alpha+\varepsilon)}. \tag{4}$$

(3) and (4) together imply that

$$P(T_n \geqq y_n B_n) \geqq C_1 y_n^{-(\alpha+\varepsilon)}.$$

To prove the right half of the inequality define the r.v.'s $X_{k,n}$ as

$$X_{k,n} = \begin{cases} X_k & \text{if } \ |(1-k/n)^p X_k| \leqq Z_n B_n \\ 0 & \text{otherwise,} \end{cases}$$

where $Z_n = y_n^r$, $1/2 < r < 1$, and write

$$T_{nn} = \sum_{k=1}^{n} (1-k/n)^p X_{k,n}.$$

Denote by E_n, F_n and G_n the events

$$E_n = \{(1-k/n)^p X_k \geqq (1-\varepsilon) y_n B_n \quad \text{for at least one} \quad k \leqq n\}.$$

$$F_n = \{|(1-k/n)^p X_k| \geqq Z_n B_n \quad \text{for at least two } k\text{'s}, \quad k \leqq n\}$$

and

$$G_n = \{|T_{nn}| \geqq \varepsilon y_n B_n\}.$$

Then

$$P(T_n \geqq y_n B_n) \leqq P(E_n) + P(F_n) + P(G_n). \tag{5}$$

It is easy to observe that for some $\varepsilon>0$

$$P(E_n) \leqq nP\big(X_1 \geqq (1-\varepsilon) y_n B_n\big) \leqq C_1 y_n^{-(\alpha-\varepsilon)} \tag{6}$$

and

(7) $$P(F_n) \leqq \left(nP(|X_1| \geqq Z_n B_n)\right)^2 \leqq C_1 Z_n^{-2(\alpha-\varepsilon)} \leqq C_2 y_n^{-(\alpha-\varepsilon)}.$$

By Tchebycheff's inequality

$$P(|T_{nn}| \geqq \varepsilon y_n B_n) \leqq \frac{E(T_{nn}^2)}{\varepsilon^2 y_n^2 B_n^2},$$

where

$$E(T_{nn}^2) = \sum_{j=1}^{n} (1-j/n)^{2p} E(X_{j,n}^2) + \sum_{\substack{j,k=1 \\ j\neq k}}^{n} (1-j/n)^p (1-k/n)^p E(X_{j,n}) E(X_{k,n}).$$

From FELLER [4], page 544,

$$E(X_{j,n}^2) \leqq \left(\frac{n}{n-j}\right)^{(2-\alpha)p+\varepsilon} Z_n^{2-\alpha+\varepsilon} B_n^{(2-\alpha)} L(B_n)$$

and hence

(8) $$\sum_{j=1}^{n} \frac{(1-j/n)^{2p}}{\varepsilon^2 B_n^2 y_n^2} E(X_{j,n}^2) \leqq C_1 y_n^{-(\alpha-\varepsilon)}.$$

Easy calculations show that the second term of $E(T_{nn}^2)$ is

$$\leqq \left(\sum_{k=1}^{n} (1-k/n)^p E(|X_{k,n}|)\right)^2.$$

Majorising $E(|X_{k,n}|)$ by $\int_0^{\theta(n,p)} P(|X_1| \geqq x)\,dx$ when $\alpha<1$ and by $\int_{\theta(n,p)}^{\infty} P(|X_1| \geqq x)\,dx$ when $\alpha>1$, where $\theta(n,p)=(n-k)^{-p} n^p Z_n B_n$, and applying Theorem 1 in FELLER [4], page 273, one can obtain the inequality

(9) $$\frac{\left(\sum_{k=1}^{n} (1-k/n)^p E(|X_{k,n}|)\right)^2}{\varepsilon^2 y_n^2 B_n^2} \leqq C_2 y_n^{-(\alpha-\varepsilon)} \quad \text{if} \quad \alpha \neq 1.$$

When $\alpha=1$, the X_n's are essentially symmetric r.v.'s and hence the second term of $E(T_{nn}^2)$ is exactly zero. Hence from (8), (9) and the above argument

(10) $$P(G_n) \leqq (C_1+C_2) y_n^{-(\alpha-\varepsilon)}.$$

The proof of the lemma is complete, once (6), (7) and (10) are substituted in (5).

LEMMA 3.2. $C_1 y_n^{-(\alpha+\varepsilon)} \leqq P(|T_n| \geqq y_n B_n) \leqq C_2 y_n^{-(\alpha-\varepsilon)}$.

Since slight modification in the proof of Lemma 3.1 establishes the above inequality the details are omitted.

LEMMA 3.3.

$$\lim_{n\to\infty} \frac{P(|S_n| \geqq y_n B_n)}{nP(|X_1| \geqq y_n B_n)} = 1.$$

For proof see HEYDE [7], when $\alpha \neq 1$. The arguments given in [7] work in the case of $\alpha=1$ also as the r.v. X_n's reduce to symmetric r.v.'s.

LEMMA 3.4. *For every* $\varepsilon>0$

$$P(|S_n| \geqq B_n(\log n)^{(1+\varepsilon)/\alpha} \text{ i.o.}) = 0.$$

PROOF. Define the sequence $n_r=2^r$, $r=1, 2, \ldots$ and the events

$$A(n) = \{|S_n| \geqq B_n(\log n)^{(1+\varepsilon)/\alpha}\},$$

$$B(r) = \{\sup_{n_r \leqq n < n_{r+1}} |S_n| \geqq B_{n_r}(\log n_r)^{(1+\varepsilon)/\alpha}\},$$

and

$$C(r) = \left\{|S_{n_{r+1}}| \geqq \frac{1}{2} B_{n_r}(\log n_r)^{(1-\varepsilon)/\alpha}\right\}.$$

Observe that

(11) $$(A_n \text{ i.o.}) \subset (B_r \text{ i.o.}).$$

We aim at establishing that for all $r \geqq r_0$ (r_0 is a positive integer)

(12) $$P(B_r) \leqq 2P(C_r).$$

From Lemma 3.21, page 45 [1], (12) holds if we prove that for all n in $n_r \leqq n < n_{r+1}$ and for all $r \geqq r_0$

(13) $$P\left(|S_{n_{r+1}} - S_n| \geqq \frac{1}{2} B_{n_r}(\log n_r)^{(1+\varepsilon)/\alpha}\right) \leqq 1/2.$$

Since

$$P\left\{|S_{n_{r+1}} - S_n| \geqq \frac{1}{2} B_{n_r}(\log n_r)^{(1+\varepsilon)/\alpha}\right\} \leqq P\{|S_{n_{r+1}-n}| \geqq CB_{n_{r+1}-n}(\log n_r)^{(1+\varepsilon)/\alpha}\}$$

from Lemma 3.2 we can find integers N and R such that for all $n_{r+1}-n \geqq N$ and for all $r \geqq R$ (13) holds. When $n_{r+1}-n \leqq N$, as there will be only a finite number of terms in $S_{n_{r+1}}-S_n$, we can always find an r_0 such that (13) holds for all $r \geqq r_0$ and consequently (12) is valid.

Now

$$\sum_{r=1}^{\infty} P(B_r) \leqq \sum_{r=1}^{r_0} P(B_r) + 2 \sum_{r=r_0+1}^{\infty} P(C_r).$$

From Lemma 3.3 one can show that $P(C_r) \leqq Cr^{-(1+\varepsilon)}$ for all $r \geqq r_0$. Hence $\sum_{r=1}^{\infty} P(B_r) < \infty$. By a reference to the Borel—Cantelli lemma and (11), the proof is complete.

THEOREM 3.1. *If*

$$\lim_{n \to \infty} P(S_n/B_n \leqq x) = G(x, \alpha, \beta, 0, C)$$

with $\beta \in (-1, 1]$ *then*

(14) $$P\left(\limsup_{n \to \infty} U_n = e^{1/\alpha}\right) = 1$$

and

(15) $$P\left(\limsup_{n\to\infty} V_n = e^{1/\alpha}\right) = 1.$$

PROOF. To establish (14) it is sufficient if we prove that

(16) $$P\left(T_n \geqq B_n (\log n)^{(1-\varepsilon)/\alpha} \text{ i.o.}\right) = 1$$

and

(17) $$P\left(T_n \geqq B_n (\log n)^{(1+\varepsilon)/\alpha} \text{ i.o.}\right) = 0.$$

Define

$$N_s = s^s, \quad W_s = \sum_{k=N_{s-1}+1}^{N_s} (1-k/N_s)^p X_k$$

and

$$Z_s = \sum_{k=1}^{N_{s-1}} (1-k/N_s)^p X_k, \quad s = 1, 2, \ldots.$$

Then $T_{N_s} = W_s + Z_s$. Hence (16) is obtained by proving

(18) $$P\left(W_s \geqq 2B_{N_s} (\log N_s)^{(1-\varepsilon)/\alpha} \text{ i.o.}\right) = 1$$

and

(19) $$P\left(Z_s \leqq -B_{N_s} (\log N_s)^{(1-\varepsilon)/\alpha} \text{ i.o.}\right) = 0.$$

$$P\left(W_s \geqq 2B_{N_s} (\log N_s)^{(1-\varepsilon)/\alpha}\right) = P\left\{\frac{T_{N_s - N_{s-1}}}{B_{N_s-N_{s-1}}} \geqq 2\frac{B_{N_s}}{B_{N_s-N_{s-1}}}\left(\frac{N_s}{N_s-N_{s-1}}\right)^p (\log N_s)^{(1-\varepsilon)/\alpha}\right\}.$$

Since $\dfrac{N_s}{N_s-N_{s-1}} \to 1$ as $s\to\infty$ $\dfrac{B_{N_s}}{B_{N_s-N_{s-1}}} \to 1$ is immediate and hence

$$P\left(W_s \geqq 2B_{N_s} (\log N_s)^{(1-\varepsilon)/\alpha}\right) \geqq P\left(T_{N_s-N_{s-1}} \geqq CB_{N_s-N_{s-1}} (\log N_s)^{(1-\varepsilon)/\alpha}\right) \geqq$$
$$\geqq C(\log N_s)^{-(1-\varepsilon/2)}.$$

Since W_s, $s=1, 2, \ldots$ are independent r.v.'s an appeal to the Borel—Cantelli lemma yields (18).

Next notice that

$$Z_s B_{N_s}^{-1} (\log N_s)^{-(1-\varepsilon)/\alpha} \xrightarrow{p} 0 \quad \text{as} \quad s\to\infty.$$

Hence by Lemma 3.2 there exists a $\delta>0$ such that

$$P\left(Z_s \leqq -B_{N_s} (\log N_s)^{(1-\varepsilon)/\alpha}\right) \leqq C_1 B_{N_s}^{-1} B_{N_{s-1}}^{\alpha-\delta} (\log N_s)^{-(1-\varepsilon)(\alpha-\delta)/\alpha} \leqq C_2 s^{-(1+\varepsilon)}.$$

Again by Borel—Cantelli lemma (19) is established. We now proceed to claim (17) and thereby complete the proof for (14).

Let $\{n_s\}$ stand for the integer sequence $[\theta^s]$, $s=1, 2, \ldots, \theta>1$. Then from Lemma 3.1

$$P\left(T_{n_s} > B_{n_s} (\log n_s)^{(1+\varepsilon)/\alpha}\right) \leqq Cs^{-(1+\varepsilon)}$$

and hence $P\left(T_{n_s} > B_{n_s} (\log n_s)^{(1+\varepsilon)/\alpha} \text{ i.o.}\right) = 0$. For any n in $n_{s-1} \leqq n < n_s$ write

$$T_n = T_{n_{s-1}} + (T_n - T_{n_{s-1}}).$$

To establish (17) it is enough if we prove that

(20) $$P\left(T_{n_{s-1}} \geqq \frac{1}{2} B_n (\log n)^{(1+\varepsilon)/\alpha} \text{ i.o.}\right) = 0$$

and

(21) $$P\left(T_n - T_{n_{s-1}} \geqq \frac{1}{2} B_n (\log n)^{(1+\varepsilon)/\alpha} \text{ i.o.}\right) = 0.$$

(20) is easily obtained from Lemma 3.1 and the Borel—Cantelli lemma.

To establish (21) we define

$$R_s = \sup_{n_{s-1} \leqq n < n_s} |T_n - T_{n_{s-1}}| B_n^{-1} (\log n)^{-(1+\varepsilon)/\alpha},$$

$$R_s^{(1)} = \sup_{n_{s-1} \leqq n < n_s} \left| \sum_{k=n_{s-1}+1}^{n} (1-k/n)^p X_k \right| B_n^{-1} (\log n)^{-(1+\varepsilon)/\alpha}$$

and

$$R_s^{(2)} = \sup_{n_{s-1} \leqq n < n_s} \left| \sum_{k=1}^{n_{s-1}} \left\{ \left(1 - \frac{k}{n}\right)^p - \left(1 - \frac{k}{n_{s-1}}\right)^p \right\} X_k \right| B_n^{-1} (\log n)^{-(1-\varepsilon)/\alpha}.$$

Proceeding as in [6] one gets

$$R_s^{(1)} \leqq \left(1 - \frac{n_{s-1}}{n_s}\right)^p \sup_{n_{s-1} \leqq n < n_s} |S_n^0| B_n^{-1} (\log n)^{-(1+\varepsilon)/\alpha}$$

where

$$S_n^0 = \sum_{k=n_{s-1}+1}^{n} X_k.$$

From Lemma 3.4 it is easy to conclude that

$$|S_n^0| B_n^{-1} (\log n)^{-(1+\varepsilon)/\alpha} \leqq 1 \quad \text{a.s.}$$

Also from the inequality

$$0 \leqq \left(1 - \frac{n_{s-1}}{n_s}\right)^p \leqq \left(1 - \frac{1}{\theta}\right)^p$$

the right side of which can be made as small as desired by making θ tend to 1, one can easily show that

(22) $$R_s^{(1)} \leqq 1/4 \quad \text{a.s.}$$

Again from [6]

$$R_s^{(2)} \leqq \frac{\sum_{j=1}^{n} \frac{n_s^j - n_{s-1}^j}{n_s^j} |C_p^j| \left| \sum_{k=1}^{n_{s-1}} (k/n_{s-1})^j X_k \right|}{B_{n_{s-1}} (\log n_{s-1})^{(1+\varepsilon)/\alpha}}$$

where

$$C_p^j = \frac{p(p-1)\dots(p-j+1)}{j!}.$$

Clearly

$$P\left(\left|\sum_{k=1}^{n_{s-1}}(k/n_{s-1})^j X_k\right| \geqq B_{n_{s-1}}(\log n_{s-1})^{(1+\varepsilon)/\alpha}\right)=$$

$$=P\left(\left|\sum_{k=1}^{n_{s-1}}(1-k/n_{s-1})^j X_k\right| \geqq B_{n_{s-1}}(\log n_{s-1})^{(1+\varepsilon)/\alpha}\right) \leqq C(s-1)^{-(1+\varepsilon/2)}.$$

Hence

(22) $$\left|\sum_{k=1}^{n_{s-1}}(k/n_{s-1})^j X_k\right| B_{n_{s-1}}^{-1}(\log n_{s-1})^{-(1+\varepsilon)/\alpha} \leqq 1 \quad \text{a.s.}$$

Further since (i) $\frac{n_s^j-n_{s-1}^j}{n_s^j}\sim 1-\frac{1}{\theta^j}$ which can be made as small as desired by taking θ near 1 and (ii) $\sum_{j=1}^{\infty}|C_p^j|<\infty$, (22) implies that $R_s^{(2)}\leqq 1/4$ a.s. Hence $R_s\leqq 1/2$ a.s. and (23) is established. Thus the proof of (14) is complete.

(15) is established below. From the fact that

$$l_{n-k}^p A_n^{-p} = (1-k/n)^p+O(n^{-1})$$

one gets $\sum_n^p=T_n+O(n^{-1})S_n$. Lemma 3.4 implies that $O(n^{-1})\left|\frac{S_n}{B_n}\right|\leqq\varepsilon$ a.s. (15) is equivalent to

(23) $$P\left(\sum_n^p \geqq B_n(\log n)^{(1+\varepsilon)/\alpha} \text{ i.o.}\right)=\begin{cases}0 & \text{if } \varepsilon>0\\ 1 & \text{if } \varepsilon<0.\end{cases}$$

Take $\varepsilon>0$. Then

$$P\left(\sum_n^p \geqq B_n(\log n)^{(1+\varepsilon)/\alpha} \text{ i.o.}\right) \leqq P\left(T_n \geqq B_n(\log n)^{(1+\varepsilon)/\alpha}-\varepsilon \text{ i.o.}\right) \leqq$$

$$\leqq P\left(T_n \geqq B_n(\log n)^{(1+\varepsilon/2)/\alpha} \text{ i.o.}\right)=0,$$

the last step following from (17). Similarly proceeding with $\varepsilon<0$, (23) is obtained.

THEOREM 3.2. *If*

$$\lim_{n\to\infty} P(S_n/B_n \leqq x) = G(x, \alpha, \beta, 0, C)$$

with $\beta\in[-1, 1)$ *then*

(24) $$P\left(\liminf_{n\to\infty} U_n = -e^{1/\alpha}\right)=1$$

and

(25) $$P\left(\liminf_{n\to\infty} V_n = -e^{1/\alpha}\right)=1.$$

Since minor modifications in the arguments of Theorem 3.1 establish (24) and (25), the details of the proof are omitted.

REMARK. Notice that when $|\beta|<1$, the above theorems give both limit superior and limit inferior of the functions U_n and V_n. But when $\beta=1$ ($\beta=-1$), only the limit superior (limit inferior) is given. When the common distribution of the sequence $\{X_n\}$ is $G(\cdot, \alpha, 1, 0, C)$ with $\alpha<1$, MIJNHEER [8] has shown that for a known constant C

(26) $$P\left(\liminf_{n\to\infty} S_n n^{-1/\alpha}(\log\log n)^{(1-\alpha)/\alpha}=C\right)=1.$$

Since $2^{-p}S_{[n/2]}\leqq T_n\leqq S_n$, from (26) we can find constants C_1 and C_2 $(C_1<C_2)$ such that

$$P\big(T_n\leqq C_1 n^{1/\alpha}(\log\log n)^{-(1-\alpha)/\alpha}\ \text{i.o.}\big)=0$$

and

$$P\big(T_n\leqq C_2 n^{1/\alpha}(\log\log n)^{-(1-\alpha)/\alpha}\ \text{i.o.}\big)=1,$$

which imply that

$$P\big(T_n\leqq n^{1/\alpha}(\log n)^{\varepsilon}\ \text{i.o.}\big)=1$$

and

$$P\big(T_n\leqq n^{1/\alpha}(\log n)^{-\varepsilon}\ \text{i.o.}\big)=0$$

and the two together are equivalent to

$$P\big(\liminf_{n\to\infty} U_n=1\big)=1.$$

Now $P(\liminf_{n\to\infty} V_n=1)=1$ can be obtained as an easy consequence.

The following example shows that 1 ceases to be the limit inferior of $U_n(V_n)$ for distributions which are in the domain of attraction of $G(\cdot,\alpha,1,0,C)$, $\alpha<1$.

EXAMPLE. Let $\{X_n\}$ be a sequence of independent r.v.'s with the common distribution F given by

$$F(-x)=x^{-\alpha}(\log x)^{-1/3}$$

and $1-F(x)=x^{-\alpha}$ for $x\geqq e$ and $2/3<\alpha<1$. Then

$$\lim_{n\to\infty} P(S_n/n^{1/\alpha}\leqq x)=G(x;\ \alpha,1,0,C).$$

Taking $y_n=(\log n)^{2(1+\varepsilon)/3\alpha}$ and proceeding as in Lemma 3.4 by giving the value $r=3/4$ in establishing the right half of the inequality, one gets

$$P\big(T_n\leqq -n^{1/\alpha}(\log n)^{2(1+\varepsilon)/3\alpha}\big)\leqq C_1(\log n)^{-(1+\varepsilon)}$$

and

$$P\big(T_n\leqq -n^{1/\alpha}(\log n)^{2(1-\varepsilon)/3\alpha}\big)\geqq C_2(\log n)^{-(1-\varepsilon)}.$$

Repeating the proof that one would give for Theorem 3.2,

(27) $$P\big(T_n\leqq -n^{1/\alpha}(\log n)^{2(1-\varepsilon)/3\alpha}\ \text{i.o.}\big)=1$$

is easily seen.

A slight modification in establishing the other part, which is indicated below, gives

(28) $$P\big(T_n\leqq -n^{1/\alpha}(\log n)^{2(1+\varepsilon)/3\alpha}\ \text{i.o.}\big)=0.$$

The modification is only in proving

$$P\left(T_n-T_{n_{s-1}}\leqq -\frac{1}{2}n^{1/\alpha}(\log n)^{2(1+\varepsilon)/3\alpha}\ \text{i.o.}\right)=0.$$

Define

$$R_s = \inf_{n_{s-1} \leqq n < n_s} (T_n - T_{n_{s-1}}) n^{-1/\alpha} (\log n)^{-2(1+\varepsilon)/3^\alpha},$$

$$R_s^{(1)} = \inf_{n_{s-1} \leqq n < n_s} \left(\sum_{k=n_{s-1}+1}^{n} (1-k/n)^p X_k \right) n^{-1/\alpha} (\log n)^{-2(1+\varepsilon)/3^\alpha}$$

and

$$R_s^{(2)} = \inf_{n_{s-1} \leqq n < n_s} \frac{\sum_{k=1}^{n_{s-1}} (1-k/n)^p X_k - \sum_{k=1}^{n_{s-1}} (1-k/n_{s-1})^p X_k}{n^{-1/\alpha} (\log n)^{2(1+\varepsilon)/3^\alpha}}$$

and the r.v.'s

$$X_k^{-1} = \begin{cases} X_k & \text{if } X_k < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that

$$\left(\sum_{k=1}^{n} X_k^{-1} \right) n^{-1/\alpha} (\log n)^{1/3}$$

converges weakly to a stable r.v. with the same α and with $\beta = -1$. Applying Lemma 3.4 one gets

$$\left| \sum_{k=1}^{n} \frac{X_k^{-1}}{n^{1/\alpha}} \right| \leqq \frac{1}{4} (\log n)^{2(1+\varepsilon)/3^\alpha} \quad \text{a.s.}$$

Therefore

$$\frac{\sum_{k=n_{s-1}+1}^{n} (1-k/n)^p X_k}{n^{1/\alpha} (\log n)^{2(1+\varepsilon)/3^\alpha}} \geqq \frac{\sum_{k=1}^{n} X_k^{-1}}{n^{1/\alpha} (\log n)^{2(1+\varepsilon)/3^\alpha}} \geqq -\frac{1}{4} \quad \text{a.s.}$$

i.e. $R_s^{(1)} \geqq -\frac{1}{4}$ a.s.

Repeating the lines of proof of Theorem 3.2, $R_s^{(2)} \geqq -\frac{1}{4}$ a.s. is easily seen and consequently $P\left(T_n - T_{n_{s-1}} \leqq -\frac{1}{2} n^{1/\alpha} (\log n)^{2(1+\varepsilon)/3^\alpha} \text{ i.o.}\right) = 0$ is established.

(27) and (28) together imply that

$$P\left(\liminf_{n \to \infty} U_n = -e^{2/3^\alpha}\right) = 1.$$

Now

$$P\left(\liminf_{n \to \infty} V_n = -e^{2/3^\alpha}\right) = 1$$

can be easily claimed. Thus we conjecture that the lim inf of $U_n(V_n)$ depends on the rate of convergence of $F(-x)/(1-F(x))$ to zero as $x \to \infty$.

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References

[1] A. K. Basu, A note on Strassen's version of the law of the iterated logarithm, *Proc. Amer. Math. Soc.*, **41** (1973), 596—601.
[2] D. R. Beuerman, Limit distributions for sums of independent and identically distributed random variables, *Canad. Math. Bull.*, **18** (1975), 291—293.
[3] L. Breiman, *Probability*, Addison—Wesley (1968).
[4] W. Feller, *An introduction to probability theory and its applications*, 2, Wiley (1966).
[5] B. V. Gnedenko and A. N. Kolmogorov, Limit distributions for sums of independent random variables, Addison—Wesley (1954).
[6] V. F. Goposhkin, The law of the iterated logarithm for Cesaro's and Abel's methods of summation, *Theor. Prob. applications*, **10** (1965), 411—420.
[7] C. C. Heyde, On large deviation probabilities in case of attraction to non-normal stable law, *Sankhya* A, **30** (1968), 254—258.
[8] J. L. Mijnheer, Sample path properties of stable processes, *Amsterdam Mathematical Centre tract series* **59** (1975).

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DEPARTMENT OF STATISTICS
UNIVERSITY OF MYSORE
MANASAGANGOTRI
MYSORE — 570 006
INDIA

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ON PROPERTIES OF PEANO DERIVATES

By
P. S. CHAKRABARTI (Raniganj)

1. Introduction. Let f be a real function defined in some neighbourhood of the point x_0. If there are numbers $\alpha_1, \alpha_2, \ldots, \alpha_r$ depending on x_0 but not on h such that

$$\lim_{h\to 0}\operatorname{ap}\frac{r!}{h^r}\left\{f(x_0+h)-f(x_0)-\sum_{k=1}^{r}\frac{h^k}{k!}\alpha_k\right\}=0,$$

where lim ap denotes the approximate limit [12, p. 218], then α_r is called the approximate Peano derivative of f at x_0 of order r and is denoted by $f_{r,a}(x_0)$ [2, 4, 9]. From the definition it follows that if $f_{r,a}(x_0)$ exists, then $f_{k,a}(x_0)$ also exists for $1\leqq k\leqq r$. We shall write $f(x_0)=f_{0,a}(x_0)=\alpha_0$.

Let $f_{r,a}(x_0)$ exist for a fixed r. Let

$$\frac{h^{r+1}}{(r+1)!}\Phi_{r+1}(f;\, x_0, h)=f(x_0+h)-\sum_{k=0}^{r}\frac{h^k}{k!}f_{k,a}(x_0).$$

Then writing

$$\bar{f}_{r+1,a}(x_0)=\limsup_{h\to 0}\operatorname{ap}\Phi_{r+1}(f;\, x_0, h),\quad \underline{f}_{r+1,a}(x_0)=\liminf_{h\to 0}\operatorname{ap}\Phi_{r+1}(f;\, x_0, h),$$

$$\bar{f}_{r+1}(x_0)=\limsup_{h\to 0}\Phi_{r+1}(f;\, x_0, h),\quad \underline{f}_{r+1}(x_0)=\liminf_{h\to 0}\Phi_{r+1}(f;\, x_0, h),$$

$\bar{f}_{r+1,a}(x_0)$ and $\underline{f}_{r+1,a}(x_0)$ are called the approximate upper and lower Peano derivates of f at x_0 of order $r+1$, respectively, where lim sup ap and lim inf ap denote the approximate upper and lower limits, respectively [12, p. 218], while $\bar{f}_{r+1}(x_0)$ and $\underline{f}_{r+1}(x_0)$ are called the upper and lower Peano derivates of f at x_0 of order $r+1$, respectively. The definitions of the right hand upper and the approximate right hand upper Peano derivates $\bar{f}^{+}_{r+1}(x_0)$, $\bar{f}^{+}_{r+1,a}(x_0)$ are obtained from $\bar{f}_{r+1}(x_0)$ and $\bar{f}_{r+1,a}(x_0)$, respectively, by restricting h to be positive. The definitions of the other Peano derivates $\bar{f}^{-}_{r+1}(x_0)$, $\bar{f}^{-}_{r+1,a}(x_0)$, $\underline{f}^{+}_{r+1}(x_0)$, $\underline{f}^{+}_{r+1,a}(x_0)$, $\underline{f}^{-}_{r+1}(x_0)$, $\underline{f}^{-}_{r+1,a}(x_0)$ are similar. When all the four derivates $\bar{f}^{+}_{r+1,a}(x_0)$, $\bar{f}^{-}_{r+1,a}(x_0)$, $\underline{f}^{+}_{r+1,a}(x_0)$ and $\underline{f}^{+}_{r+1,a}(x_0)$ are equal, the common value is said to be the approximate Peano derivative (possibly infinite) of f at x_0 of order $r+1$. If in the above definition the approximate limit is replaced by the ordinary limit at every stage, then we get the definition of the Peano derivative f_{r+1} [11].

It is known that a finite approximate Peano derivative belongs to Baire class 1 [4] and it satisfies Darboux property, mean value property, boundedness property, Zahorski's $\mathcal{M}_2$ property and the Denjoy property [9]. It is also known that an ordinary (finite) Peano derivative enjoys Zahorski's property $\mathcal{M}_3$ [13] and that if the approximate derivates are bounded at least on one side then they are the corresponding Dini derivates [6, 5, 7]. These results are extended for the approximate Peano derivates.

2. Terminology and notations. Let f have an approximate Peano derivative of order $n-1$ at each point in the closed interval $[a, b]$. Then f is said to have the mean value property M_n^k with respect to $\bar{f}^+_{n,a}$ in $[a, b]$, $0 \leqq k \leqq n-1$, if for each x and $x+h$ in $[a, b]$ there is x' between x and $x+h$ such that

$$\frac{(n-k)!}{h^{n-k}}\left\{f_{k,a}(x+h)-\sum_{r=k}^{n-1}\frac{h^{r-k}}{(r-k)!}f_{r,a}(x)\right\}=\bar{f}^+_{n,a}(x').$$

If f has the property M_n^k with respect to $\bar{f}^+_{n,a}$, then we shall write $f \in M_n^k(\bar{f}^+_{n,a})$. Set

$$M_n(\bar{f}^+_{n,a}) = \bigcap_{k=0}^{n-1} M_n^k(\bar{f}^+_{n,a}).$$

The mean value property with respect to other derivates, viz., $\underline{f}^+_{n,a}$, $\bar{f}^-_{n,a}$, $\underline{f}^-_{n,a}$, $\bar{f}^+_n$, $\underline{f}^+_n$, $\bar{f}^-_n$ and $\underline{f}^-_n$ are similarly defined. The class of Darboux functions will be denoted by $\mathcal{D}$.

3. Theorem 1. *If $f_{n-1,a}$ $(n \geqq 2)$ exists in $[a, b]$ and if any of $\underline{f}^+_{n,a}$, $\underline{f}^-_{n,a}$, $\bar{f}^+_{n,a}$, $\bar{f}^-_{n,a}$, $\underline{f}^+_n$, $\underline{f}^-_n$, $\bar{f}^+_n$ and $\bar{f}^-_n$ is bounded at least on one side in $[a, b]$, then $f^{(n-1)}$ exists and the Peano derivates of f are equal to the corresponding approximate Peano derivates of f in $[a, b]$.*

Proof. We shall prove that if $\underline{f}^+_{n,a}$ be bounded at least on one side, then $\underline{f}^+_{n,a} = \underline{f}^+_n$. Other cases are similar.

Let us first suppose that $\underline{f}^+_{n,a}$ is bounded below. We may suppose $\underline{f}^+_{n,a} \geqq 0$. Then by Theorem 3 of [9], $f_{n-1,a}$ is non-decreasing and continuous in $[a, b]$ and since $f \in \boldsymbol{B}_{n-1}$, [9], $f^{(n-1)}$ exists in $[a, b]$. If possible let ξ be such that

$$K_1 = \underline{f}^+_n(\xi) < \underline{f}^+_{n,a}(\xi) = K.$$

We may suppose that $0 = \xi = f(\xi) = \ldots = f_{n-1}(\xi)$. Choose $0 < \varepsilon < \dfrac{K-K_1}{2}$. Since $K_1 < K - 2\varepsilon$, there is a sequence $\{h_\nu\}$ such that $h_\nu \to 0+$ as $\nu \to \infty$ and

$$\frac{f(h_\nu)}{h_\nu^n/n!} < K - 2\varepsilon \quad \text{for all} \quad \nu. \tag{1}$$

We may suppose $h_\nu < 1$. Let us fix ν. Then by mean value theorem, there is t_ν, $0 < t_\nu < h_\nu$ such that

$$\frac{f(h_\nu)}{h_\nu^n/n!} = \frac{n}{h_\nu} f_{n-1}(t_\nu).$$

Since f_{n-1} is non-decreasing and $f_{n-1}(0) = 0$, $f_{n-1}(t_\nu) \geqq 0$ i.e. $f(h_\nu) \geqq 0$. Also from (1)

$$\frac{f(h_\nu)}{h_\nu^n/n!} < K - 2\varepsilon < K - \varepsilon.$$

Hence $n!f(h_\nu) < h_\nu^n(K-\varepsilon)$ i.e.

$$\left\{\frac{n!f(h_\nu)}{K-\varepsilon}\right\}^{1/n} < h_\nu .$$

Let $J_\nu=[0, h_\nu]$ and $I_\nu=\left[\left\{\frac{n!f(h_\nu)}{K-\varepsilon}\right\}^{1/n}, h_\nu\right]$. So, from (1)

$$(2) \qquad \frac{m(I_\nu)}{m(J_\nu)} = 1-\left\{\frac{n!f(h_\nu)}{h_\nu^n(K-\varepsilon)}\right\}^{1/n} > 1-\left(\frac{K-2\varepsilon}{K-\varepsilon}\right)^{1/n}.$$

Since $f_{n,a}^{+}(0)=K$ there is N such that the set $E=\left\{x: \frac{f(x)}{x^n/n!}-K>-\varepsilon; x>0\right\}$ satisfies

$$(3) \qquad \frac{m(E\cap J_\nu)}{m(J_\nu)} > \left(\frac{K-2\varepsilon}{K-\varepsilon}\right)^{1/n} \quad \text{for} \quad \nu \geqq N.$$

Since f_{n-1} is non-decreasing in $[0, h_\nu]$ and $f_{n-1}(0)=0$, $f_{n-1}\geqq 0$ in $[0, h_\nu]$. Hence f_{n-2} is non-decreasing in $[0, h_\nu]$. Since $f_{n-2}(0)=0, f_{n-2}\geqq 0$ in $[0, h_\nu]$. Continuing this argument, f is non-decreasing in $[0, h_\nu]$.

Let $x\in I_\nu$. Then $f(x)\leqq f(h_\nu)\leqq \frac{(K-\varepsilon)x^n}{n!}$ i.e. $\frac{f(x)}{x^n/n!}\leqq K-\varepsilon$. So $E\cap I_\nu=0$. Hence from (2)

$$\frac{m(E\cap J_\nu)}{m(J_\nu)} \leqq \frac{m(J_\nu)-m(I_\nu)}{m(J_\nu)} = 1-\frac{m(I_\nu)}{m(J_\nu)} < \left(\frac{K-2\varepsilon}{K-\varepsilon}\right)^{1/n}.$$

This is true for all ν. But this contradicts (3) and the result follows.

Next, let $f_{n,a}^{+}$ be bounded above. We may suppose $f_{n,a}^{+}\leqq 0$. As above, $f^{(n-1)}$ exists and is non-increasing. Suppose $0 = \xi = f(\xi) = \ldots = f_{n-1}(\xi)$. If possible, let ξ be such that $K_1 = f_n^{+}(\xi) < f_{n,a}^{+}(\xi) = K$. Choose $0<\varepsilon<\frac{K-K_1}{2}$. Since $K_1<K-2\varepsilon$, there is a sequence $\{h_\nu\}$ such that $h_\nu\to 0+$ as $\nu\to\infty$ and $0<h_\nu<1$ and

$$(4) \qquad \frac{f(h_\nu)}{h_\nu^n/n!} < K-2\varepsilon \quad \text{for all} \quad \nu.$$

Now, since $K\leqq 0$, $f(h_\nu)<0$ and so $\frac{f(h_\nu)}{K-\varepsilon}>0$. Also from (4) $n!f(h_\nu) < h_\nu^n(K-\varepsilon)$ i.e.

$$\frac{n!f(h_\nu)}{K-\varepsilon} > h_\nu^n.$$

Hence

$$\left\{\frac{n!J(h_\nu)}{K-\varepsilon}\right\}^{1/n} > h_\nu .$$

Let

$$J_\nu = \left[0, \left\{\frac{n!f(h_\nu)}{K-\varepsilon}\right\}^{1/n}\right] \quad \text{and} \quad I_\nu = \left[h_\nu, \left\{\frac{n!f(h_\nu)}{K-\varepsilon}\right\}^{1/n}\right].$$

Then from (4)

$$(5)\qquad \frac{m(I_\nu)}{m(J_\nu)} = 1-\frac{h_\nu}{\left\{\frac{n!f(h_\nu)}{K-\varepsilon}\right\}^{1/n}} = 1-\left\{\frac{h_\nu^n(K-\varepsilon)}{n!f(h_\nu)}\right\}^{1/n} > 1-\left(\frac{K-\varepsilon}{K-2\varepsilon}\right)^{1/n}.$$

Since $\underline{f}_{n,a}^{+}(0)=K$, there is N such that the set

$$E=\left\{x:\ \frac{f(x)}{x^n/n!}-K>-\varepsilon;\ x>0\right\}$$

satisfies

$$(6)\qquad \frac{m(E\cap J_\nu)}{m(J_\nu)} > \left(\frac{K-\varepsilon}{K-2\varepsilon}\right)^{1/n} \quad \text{for all} \quad \nu\geqq N.$$

Since $\underline{f}_{n,a}^{+}\leqq 0$, $f_{n-1,a}$ is non-increasing and since $f_{n-1,a}(0)=0$, $f_{n-1,a}\leqq 0$ in J_ν and as in the first case, f is non-increasing in J_ν. So, if $x\in I_\nu$ then

$$f(x)\leqq f(h_\nu)\leqq \frac{x^n(K-\varepsilon)}{n!}$$

i.e. $\dfrac{f(x)}{x^n/n!}\leqq K-\varepsilon$. So $E\cap I_\nu=0$. Hence from (5)

$$\frac{m(E\cap J_\nu)}{m(J_\nu)}\leqq \frac{m(J_\nu)-m(I_\nu)}{m(J_\nu)} = 1-\frac{m(I_\nu)}{m(J_\nu)} < \left(\frac{K-\varepsilon}{K-2\varepsilon}\right)^{1/n}$$

which contradicts (6) and the result follows.

4. Let $f_{n-1,a}$ exist and let

$$\delta_n(f;x_0,h)=\Phi_n(f;x_0,h)-\Phi_n(f;x_0,-h)$$

where Φ is as defined in Section 1. Then if $\lim_{h\to 0+}\delta_n(f;x_0,h)=0$, f is called smooth of order n at x_0 (see [15, II, p. 62] and [10]). It is clear that if f is smooth of order n at x_0, then $\bar{f}_n^{+}(x_0)=\bar{f}_n^{-}(x_0)$ and similarly if $\lim_{h\to 0+}\text{ap}\ \delta_n(f;x_0,h)=0$ (which may be called the approximate smoothness of f of order n), then $\bar{f}_{n,a}^{+}(x_0)=\bar{f}_{n,a}^{-}(x_0)$. So, the following results, where we assume $\bar{f}_n^{+}=\bar{f}_n^{-}$ or $\bar{f}_{n,a}^{+}=\bar{f}_{n,a}^{-}$, will remain valid if these conditions are replaced by the smoothness or approximate smoothness of f of order n, respectively. We note that if $\bar{f}_{n,a}^{+}=\bar{f}_{n,a}^{-}$ or $\bar{f}_n^{+}=\bar{f}_n^{-}$ then the common value is $\bar{f}_{n,a}$ or $\bar{f}_n$, respectively.

THEOREM 2. *Let $f_{n-1,a}$ $(n\geqq 2)$ exist and let $\bar{f}_{n,a}^{-}=\bar{f}_{n,a}^{+}$ (resp. $\bar{f}_n^{-}=\bar{f}_n^{+}$). If $\bar{f}_{n,a}$ (resp. $\bar{f}_n$) is of Baire class 1, then $\bar{f}_{n,a}$ (resp. $\bar{f}_n$) $\in\mathcal{D}$.*

PROOF. We prove for $\bar{f}_{n,a}$; the proof for $\bar{f}_n$ is similar. Set $g=\bar{f}_{n,a}$ and let α,β, $\alpha<\beta$, be any two real numbers such that $g(\alpha)<0<g(\beta)$. Suppose if possible that there is no $\xi\in(\alpha,\beta)$ such that $g(\xi)=0$. Let

$$E=\{x:\ x\in[\alpha,\beta];\ g(x)>0\},\qquad F=\{x:\ x\in[\alpha,\beta];\ g(x)<0\}.$$

Then $E\cup F=[\alpha,\beta]$.

Let Q be any non-degenerate component of E. Then Q is an interval. Let $c=\inf Q$, $d=\sup Q$. Then by a known result [9] $f_{n-1,a}$ is non-decreasing in (c, d) and since $f_{n-1,a}\in\mathcal{D}$, $f_{n-1,a}$ is non-decreasing and continuous in $[c, d]$. Hence by mean value theorem [9], if $c<c+h<d$, there is h', $0<h'<h$ such that

$$\Phi_n(f;\ c, h) = \frac{n!}{h^n}\left\{f(c+h)-f(c)-\ldots-\frac{h^{n-1}}{(n-1)!}f_{n-1,a}(c)\right\} =$$

$$= \frac{n}{h}\{f_{n-1,a}(c+h')-f_{n-1,a}(c)\} \geqq 0.$$

So $\bar{f}^{+}_{n,a}(c)\geqq 0$. Similarly $\bar{f}^{-}_{n,a}(d)\geqq 0$ i.e. $g(c)\geqq 0$, $g(d)\geqq 0$. Since $g(c)\neq 0\neq g(d)$, $g(c)>0$ and $g(d)>0$. So, $c\in Q$, $d\in Q$. Thus every non-degenerate component of E is a closed interval. Similarly every non-degenerate component of F is a closed interval.

Let $\{Q\}$ and $\{R\}$ be the collections of all non-degenerate components of E and F, respectively, and let $\{S\}=\{Q\}\cup\{R\}$. Then two distinct elements of S are disjoint. Set $P=[\alpha, \beta]-\cup S^0$ where S^0 denotes the interior of S and where the union extends over all S in $\{S\}$. Then P is perfect and g/P has no point of continuity in P, which contradicts the fact that g is in Baire class 1.

THEOREM 3. *Under the hypothesis of Theorem* 2 $f\in M_n(\bar{f}_{n,a})$ (*resp.* $f\in M_n(\bar{f}_n)$).

PROOF. Let $\alpha<\beta$. Set

$$G(t)=f(t)-\frac{f_{n-1,a}(\beta)-f_{n-1,a}(\alpha)}{\beta-\alpha}\cdot\frac{(t-\alpha)^n}{n!}.$$

Then,

$$G_{n-1,a}(t) = f_{n-1,a}(t)-\frac{f_{n-1,a}(\beta)-f_{n-1,a}(\alpha)}{\beta-\alpha}\cdot(t-\alpha)$$

and

$$\bar{G}_{n,a}(t) = \bar{f}_{n,a}(t)-\frac{f_{n-1,a}(\beta)-f_{n-1,a}(\alpha)}{\beta-\alpha}.$$

Now $f_{n-1,a}$ is a Darboux function of Baire class 1 and hence $G_{n-1,a}\in\mathcal{D}$ [1]. Therefore if $\bar{G}_{n,a}(t)>0$ for $t\in(\alpha, \beta)$ then $G_{n-1,a}$ would be nondecreasing in (α, β) and by the property $G_{n-1,a}\in\mathcal{D}$, $G_{n-1,a}$ would be non-decreasing in $[\alpha, \beta]$ and since $G_{n-1,a}(\alpha)=G_{n-1,a}(\beta)$, $G_{n-1,a}$ would be constant in $[\alpha, \beta]$ which is a contradiction to the fact that $\bar{G}_{n,a}(t)>0$ in (α, β). Thus there is $t_1\in(\alpha, \beta)$ such that $\bar{G}_{n,a}(t_1)\leqq 0$. Similarly there is $t_2\in(\alpha, \beta)$ such that $\bar{G}_{n,a}(t_2)\geqq 0$. Since $\bar{G}_{n,a}\in\mathcal{D}$ there is $t_0\in(\alpha, \beta)$ such that $\bar{G}_{n,a}(t_0)=0$. That is,

(1) $$f_{n-1,a}(\beta)-f_{n-1,a}(\alpha) = (\beta-\alpha)\bar{f}_{n,a}(t_0).$$

Next let

$$G(t) = f(t)-\frac{f_{k,a}(\beta)-f_{k,a}(\alpha)-\ldots-\dfrac{(\beta-\alpha)^{n-k-1}}{(n-k-1)!}f_{n-1,a}(\alpha)}{(\beta-\alpha)^{n-k}/(n-k)!}\cdot\frac{(t-\alpha)^n}{n!}.$$

Then

$$G_{k,a}(\beta)-G_{k,a}(\alpha)-\ldots-\frac{(\beta-\alpha)^{n-k-1}}{(n-k-1)!}G_{n-1,a}(\alpha)=0$$

and so by mean value property there is $t_1\in(\alpha,\beta)$ such that

$$G_{n-1,a}(t_1)-G_{n-1,a}(\alpha)=0.$$

Now as above we assert that there are $t_2\in(\alpha,t_1)$ and $t_3\in(\alpha,t_1)$ such that

$$\bar{G}_{n,a}(t_2)\leqq 0 \quad\text{and}\quad \bar{G}_{n,a}(t_3)\geqq 0$$

and since $\bar{G}_{n,a}\in\mathscr{D}$ there is $t_0\in(\alpha,t_1)$ such that $\bar{G}_{n,a}(t_0)=0$ i.e.

$$\bar{f}_{n,a}(t_0)=\frac{f_{k,a}(\beta)-f_{k,a}(\alpha)-\ldots-\dfrac{(\beta-\alpha)^{n-k-1}}{(n-k-1)!}f_{n-1,a}(\alpha)}{(\beta-\alpha)^{n-k}/(n-k)!},$$

completing the proof.

5. We recall that a set $E\in M_2$ if and only if E is an F_σ and every one sided neighbourhood of each point of E intersects E in a set of positive measure and $f\in\mathscr{M}_2$ if and only if for every α and β, the sets $\{x:f(x)>\alpha\}$ and $\{x:f(x)<\beta\}$ belong to the class M_2 [14].

THEOREM 4. *Let*
(i) $f_{n-1,a}$ *exist,*
(ii) $\bar{f}^-_{n,a}=\bar{f}^+_{n,a}$ *(resp.* $\bar{f}^-_n=\bar{f}^+_n$*),*
(iii) $\bar{f}_{n,a}$ *(resp.* $\bar{f}_n$*) belongs to Baire class* 1, *and*
(iv) $\bar{f}_{n,a}$ *(resp.* $\bar{f}_n$*) be finite except on a countable set.*
Then $\bar{f}_{n,a}$ *(resp.* $\bar{f}_n$*)* $\in\mathscr{M}_2$.

PROOF. Since $\bar{f}_{n,a}$ is in Baire class 1, for each α the sets $E_\alpha=\{x:\bar{f}_{n,a}(x)<\alpha\}$ and $E^\alpha=\{x:\bar{f}_{n,a}(x)>\alpha\}$ are F_σ sets. Let $\xi\in E_\alpha$ and $\delta>0$ be arbitrary. Then if $m([\xi,\xi+\delta]\cap E_\alpha)=0$, the function $f_{n-1,a}(x)-\alpha x$ would be non-decreasing in $[\xi,\xi+\delta]$ and hence $\bar{f}_{n,a}(\xi)\geqq\alpha$ which is a contradiction, since $\xi\in E_\alpha$. So $m([\xi,\xi+\delta]\cap E_\alpha)>0$. Similarly if $m([\xi-\delta,\xi]\cap E_\alpha)\neq 0$, then it is of positive measure. Thus $E_\alpha\in M_2$. It can similarly be shown that $E^\alpha\in M_2$.

COROLLARY. *Under the hypothesis of Theorem* 4, $\bar{f}_{n,a}$ *(resp.* $\bar{f}_n$*) possesses Denjoy property* [3], *that is, for any two reals* $\alpha,\beta,\alpha<\beta$, *the set* $\{x:\alpha<\bar{f}_{n,a}$ *(resp.* $\bar{f}_n$*)*$<\beta\}$ *is either void or is of positive measure.*

PROOF. Let $\bar{f}_{n,a}\geqq 0$ a.e. in an interval I. Then $f_{n-1,a}$ is non-decreasing in I and hence $\bar{f}_{n,a}\geqq 0$ in I. So $\bar{f}_{n,a}$ has the property that if $\bar{f}_{n,a}\geqq 0$ a.e. in an interval, then $\bar{f}_{n,a}\geqq 0$ everywhere in that interval. Similarly if $\bar{f}_{n,a}\leqq 0$ a.e. in an interval, then $\bar{f}_{n,a}\leqq 0$ everywhere in that interval. So by a result of [8], $\bar{f}_{n,a}$ possesses Denjoy property.

6. THEOREM 5. *Let* $f_{k-1,a}$ $(k\geqq 2)$ *exist in an interval* $[a,b]$. *If* $\bar{f}_{k,a}\geqq c$ *in* $[a,b]$ *where* $c\geqq 0$ *is fixed, then for each* j, $1\leqq j\leqq k$, *there is a partition of* $[a,b]$ *say*

$a=t_{j,0}<t_{j,1}<\ldots<t_{j,m(j)}=b$ such that $m(j)\leqq 2^j$ and for each i, $1\leqq i\leqq m(j)$, one of the following assertions hold for every x in $[t_{j,i-1}, t_{j,i}]$.

$$1(\mathrm{j})\colon\ f_{k-j,a}(x)-f_{k-j,a}(t_{j,i-1})\geqq \frac{c}{j!}(x^j-t_{j,i-1}^j)\quad \text{and}\quad f_{k-j,a}(t_{j,i-1})\geqq \frac{c}{j!}t_{j,i-1}^j$$

$$2(\mathrm{j})\colon\ f_{k-j,a}(x)-f_{k-j,a}(t_{j,i})\leqq -\frac{c}{j!}(t_{j,i}^j-x^j)\quad \text{and}\quad f_{k-j,a}(t_{j,i})\leqq \frac{c}{j!}t_{j,i}^j$$

$$3(\mathrm{j})\colon\ f_{k-j,a}(x)-f_{k-j,a}(t_{j,i-1})\leqq -\frac{c}{j!}(x^j-t_{j,i-1}^j)\quad \text{and}\quad f_{k-j,a}(t_{j,i-1})\leqq \frac{c}{j!}t_{j,i-1}^j$$

$$4(\mathrm{j})\colon\ f_{k-j,a}(x)-f_{k-j,a}(t_{j,i})\geqq \frac{c}{j!}(t_{j,i}^j-x^j)\quad \text{and}\quad f_{k-j,a}(t_{j,i})\geqq \frac{c}{j!}t_{j,i}^j.$$

PROOF. We shall use the argument of WEIL [13] with essential modifications.

We first prove the theorem for the special case when $c=0$ and then consider the general case. Let $j=1$. Then since $\bar f_{k,a}\geqq 0$, $f_{k-1,a}$ is non-decreasing in $[a,b]$ and by the Darboux property of $f_{k-1,a}$, $f_{k-1,a}$ is continuous in $[a,b]$. So there is a point $t\in[a,b]$ such that

$$(1)\qquad |f_{k-1,a}(t)|\leqq |f_{k-1,a}(x)|,\quad \text{for}\quad x\in[a,b].$$

If $f_{k-1,a}$ is constant in $[a,b]$, we take $t=a$ and if $f_{k-1,a}(a)\geqq 0$, then clearly 1(1) holds for the partition $a=t_{1,0}<t_{1,1}=b$ and if $f_{k-1,a}<0$, then 2(1) holds. So we assume that $f_{k-1,a}$ is non-constant. If $t=a$, then since $f_{k-1,a}$ is non-decreasing in $[a,b]$, $f_{k-1,a}(x)-f_{k-1,a}(a)\geqq 0$. Also $f_{k-1,a}(a)\geqq 0$. For if $f_{k-1,a}(a)<0$, then by (1), $f_{k-1,a}$ would be constant. Hence for the partition $a=t_{1,0}<t_{1,1}=b$, 1(1) holds. If $t=b$ then by the monotonicity of $f_{k-1,a}$, $f_{k-1,a}(x)-f_{k-1,a}(b)\leqq 0$, and as above $f_{k-1,a}(b)\leqq 0$. In this case also the partition is $a=t_{1,0}<t_{1,1}=b$ and 2(1) holds.

Finally, if $t\in(a,b)$, then $f_{k-1,a}(t)=0$. So the partition is $a=t_{1,0}<t_{1,1}=t<t_{1,2}=b$. By the monotonicity of $f_{k-1,a}$, x in $[t_{1,0}, t_{1,1}]$ implies that $f_{k-1,a}(x)-f_{k-1,a}(t_{1,1})\leqq 0$ and so 2(1) is satisfied for any x in $[t_{1,0}, t_{1,1}]$ and x in $[t_{1,1}, t_{1,2}]$ implies that

$$f_{k-1,a}(x)-f_{k-1,a}(t_{1,1})\geqq 0$$

and so 1(1) holds for any x in $[t_{1,1}, t_{1,2}]$. Thus the theorem is true for $j=1$. Let us suppose that the theorem is true for $j=p$ and prove that it is also true for $j=p+1$ where $1\leqq p<k$. So we assume that there is a partition $a=t_{p,0}<t_{p,1}<\ldots<t_{p,m(p)}=b$ such that $m(p)\leqq 2^p$ and for each $i=1,2,\ldots,m(p)$ one of the relations 1(p), 2(p), 3(p) and 4(p) holds for any x in $[t_{p,i-1}, t_{p,i}]$. We shall show that each interval $[t_{p,i-1}, t_{p,i}]$ can be divided into atmost two subintervals on each of which one of the relations $1(p+1)$, $2(p+1)$, $3(p+1)$ and $4(p+1)$ holds for each x in it.

Let $1\leqq i\leqq m(p)$ and let $1(p)$ hold for any x in $[t_{p,i-1}, t_{p,i}]$. Since $f_{k-p,a}(t_{p,i-1})\geqq 0$, $f_{k-p,a}(x)\geqq 0$ for all x in $[t_{p,i-1}, t_{p,i}]$. Hence $f_{k-(p+1),a}$ is non-decreasing and continuous in $[t_{p,i-1}, t_{p,i}]$. So there is a point t in $[t_{p,i-1}, t_{p,i}]$ such that $|f_{k-(p+1),a}(t)|\leqq |f_{k-(p+1),a}(x)|$ for x in $[t_{p,i-1}, t_{p,i}]$. As in the case for $j=1$, we may assume that $f_{k-(p+1),a}$ is non-constant. If $t=t_{p,i-1}$, then since $f_{k-(p+1),a}$ is non-decreasing and continuous in $[t_{p,i-1}, t_{p,i}]$ for each x in $[t_{p,i-1}, t_{p,i}]$,

$$f_{k-(p+1),a}(x)-f_{k-(p+1),a}(t_{p,i-1}) \geqq 0 \quad \text{and} \quad f_{k-(p+1),a}(t_{p,i-1}) \geqq 0$$

then renaming the interval $[t_{p,i-1}, t_{p,i}]$ as $[t_{p+1,i-1}, t_{p+1,i}]$, $1(p+1)$ holds for any x in $[t_{p+1,i-1}, t_{p+1,i}]$ and in this case this interval remains undivided. If $t=t_{p,i}$ then also by the above argument for each x in $[t_{p,i-1}, t_{p,i}]$ we have

$$f_{k-(p+1),a}(x)-f_{k-(p+1),a}(t_{p,i}) \leqq 0 \quad \text{and} \quad f_{k-(p+1),a}(t_{p,i}) \leqq 0$$

then $2(p+1)$ is satisfied for all x in

$$[t_{p+1,i-1}, t_{p+1,i}] = [t_{p,i-1}, t_{p,i}].$$

Finally, assume that $t\in(t_{p,i-1}, t_{p,i})$. Then $f_{k-(p+1),a}(t)=0$. Divide the subinterval $[t_{p,i-1}, t_{p,i}]$ into two subintervals $[t_{p,i-1}, t]$ and $[t, t_{p,i}]$. Now since $f_{k-(p+1),a}$ is non-decreasing in $[t_{p,i-1}, t_{p,i}]$, for each x in $[t_{p,i-1}, t]$

$$f_{k-(p+1),a}(t)-f_{k-(p+1),a}(x) \geqq 0$$

and so $2(p+1)$ holds for all x in $[t_{p,i-1}, t]=[t_{p+1,i-1}, t_{p,i}]$ and for each x in $[t, t_{p,i}]$

$$f_{k-(p+1),a}(x)-f_{k-(p+1),a}(t) \geqq 0$$

and hence $1(p+1)$ is satisfied for all x in

$$[t, t_{p,i}] = [t_{p+1,i}, t_{p+1,i+1}].$$

If 2(p), 3(p) or 4(p) holds for $x\in[t_{p,i-1}, t_{p,i}]$, then it can be proved by similar arguments that one of $1(p+1)$, $2(p+1)$, $3(p+1)$ and $4(p+1)$ also holds for all x in $[t_{p+1,i-1}, t_{p+1,i}]$, $1\leqq i\leqq m(p+1)$. Hence the theorem is also true for $j=p+1$. Thus the proof for the special case when $c=0$ is completed by induction on j.

For the general case, let $c\geqq 0$. Set $g(x)=f(x)-\frac{cx^k}{k!}$. Then $g_{k-1,a}$ exists in $[a, b]$ and $\bar{g}_{k,a}=\bar{f}_{k,a}-c\geqq 0$. So by the special case applied to g, for each j, $1\leqq j\leqq k$, there is a partition of $[a, b]$

$$a = t_{j,0} < t_{j,1} < \ldots < t_{j,m(j)} = b$$

such that $m(j)\leqq 2^j$ and for each i, $1\leqq i\leqq m(j)$ one of the following assertions hold for any x in $[t_{j,i-1}, t_{j,i}]$:

$$1(\mathrm{j})\colon\ f_{k-j,a}(x)-f_{k-j,a}(t_{j,i-1}) \geqq \frac{c}{j!}(x^j-t^j_{j,i-1}) \quad \text{and} \quad f_{k-j,a}(t_{j,i-1}) \geqq \frac{c}{j!}t^j_{j,i-1}$$

$$2(\mathrm{j})\colon\ f_{k-j,a}(x)-f_{k-j,a}(t_{j,i}) \leqq -\frac{c}{j!}(t^j_{j,i}-x^j) \quad \text{and} \quad f_{k-j,a}(t_{j,i}) \leqq \frac{c}{j!}t^j_{j,i}$$

$$3(\mathrm{j})\colon\ f_{k-j,a}(x)-f_{k-j,a}(t_{j,i-1}) \leqq -\frac{c}{j!}(x^j-t^j_{j,i-1}) \quad \text{and} \quad f_{k-j,a}(t_{j,i-1}) \leqq \frac{c}{j!}t^j_{j,i-1}$$

$$4(\mathrm{j})\colon\ f_{k-j,a}(x)-f_{k-j,a}(t_{j,i}) \geqq \frac{c}{j!}(t^j_{j,i}-x^j) \quad \text{and} \quad f_{k-j,a}(t_{j,i}) \geqq \frac{c}{j!}t^j_{j,i},$$

completing the proof.

7. In what follows, by the convergence of a sequence of closed intervals $\{[a_n, b_n]\}$ to a point x, we mean $x \notin \bigcup_{n=1}^{\infty} [a_n, b_n]$ and $\lim_{n\to\infty} a_n = x = \lim_{n\to\infty} b_n$.

LEMMA 1. *Let* $g_{k,a}(0)$ $(k \geqq 2)$ *exist,* $g(0)=g_{1,a}(0)=\ldots=g_{k-1,a}(0)=0$ *and* $g_{k,a}(0)>0$. *Let* $\{I_n=[a_n, b_n]\}$ *be a sequence of closed intervals with positive end-points converging to* 0 *such that for every* n, $x\in I_n$ *implies* $\bar{g}_{k,a}(x)\leqq 0$. *Then*

$$\lim_{n\to\infty} \frac{m(I_n)}{d(0, I_n)} = 0.$$

PROOF. Let $g_{k,a}(0)>\varepsilon>0$ and

$$E = \left\{x:\ g_{k,a}(0)-\varepsilon < \frac{g(x)}{x^k/k!} < g_{k,a}(0)+\varepsilon\right\}.$$

Then 0 is a point of density of the set E. Let n be a fixed positive integer. We shall show that

$$\text{(1)} \qquad \frac{m(E\cap I_n)}{b_n} < \frac{2\varepsilon}{g_{k,a}(0)}.$$

Since $\bar{g}_{k,a}(x)\leqq 0$ for all $x\in I_n$, $g_{k-1,a}(x)$ is non-increasing on I_n and hence $g_{k-1,a}$ is continuous on I_n.

Let $A=E\cap I_n$. If A is empty, then

$$\frac{m(E\cap I_n)}{b_n} = 0 < \frac{2\varepsilon}{g_{k,a}(0)}.$$

If A is non-empty, let $x_2=\sup A$. Since $g_{k-1,a}$ is continuous on I_n, so is $g(x)/(x^n/k!)$ on I_n. Therefore

$$\text{(2)} \qquad \frac{g(x_2)}{x_2^k/k!} \geqq g_{k,a}(0)-\varepsilon > 0.$$

Let

$$\text{(3)} \qquad x_1 = \left[\frac{g_{k,a}(0)-\varepsilon}{g_{k,a}(0)+\varepsilon}\right] x_2.$$

We shall show that $A\subset[x_1, x_2]$. If $x_1\leqq a_n$, then $A\subset[x_1, x_2]$. Let $x_1>a_n$. Since $\bar{g}_{k,a}\leqq 0$ in I_n, $g_{k-1,a}$ exists and $\leqq 0$ in I_n. Now, applying Theorem 5 in the interval $[a_n, x_2]$ for the function $-g$ we see that there is a partition $a_n=t_1<t_2<\ldots$ $\ldots<t_m=x_2$ such that for every $x\in[t_{i-1}, t_i]$ one of the following inequalities holds

$1(k)$: $\quad -g(x)+g(t_{i-1})\geqq 0$ and $-g(t_{i-1})\geqq 0$

$2(k)$: $\quad -g(x)+g(t_i)\leqq 0$ and $-g(t_i)\leqq 0$

$3(k)$: $\quad -g(x)+g(t_{i-1})\leqq 0$ and $-g(t_{i-1})\leqq 0$

$4(k)$: $\quad -g(x)+g(t_i)\geqq 0$ and $-g(t_i)\geqq 0$.

Let $\xi\in[a_n, x_1]$. Then ξ will lie in some subinterval, say $[t_{i_0-1}, t_{i_0}]$, of the partition. Now $g(x_2)>0$ and so from $2(k)$

$$g(x_2) \leqq g(t_{m-1}) \leqq g(t_{m-2}) \leqq \ldots \leqq g(t_{i_0}) \leqq g(\xi).$$

So from (2) and (3)

$$\frac{g(\xi)}{\xi^k/k!} \geqq \frac{g(x_2)}{x_1^k/k!} = \left[\frac{g_{k,a}(0)+\varepsilon}{g_{k,a}(0)-\varepsilon}\right]^k \frac{g(x_2)}{x_2^k/k!} \geqq g_{k,a}(0)+\varepsilon.$$

Hence $\xi\notin E$ and consequently $\xi\notin A$. Thus $A\subset[x_1, x_2]$. Since $E\cap I_n=A$,

$$m(E\cap I_n) \leqq x_2-x_1 = x_2-\frac{g_{k,a}(0)-\varepsilon}{g_{k,a}(0)+\varepsilon}\cdot x_2 = x_2\cdot\frac{2\varepsilon}{g_{k,a}(0)+\varepsilon} < x_2\cdot\frac{2\varepsilon}{g_{k,a}(0)} \leqq$$

$$\leqq b_n\cdot\frac{2\varepsilon}{g_{k,a}(0)},$$

i.e.

$$\frac{m(E\cap I_n)}{b_n} \leqq \frac{2\varepsilon}{g_{k,a}(0)}.$$

Thus (1) is true. Also, since 0 is a point of density of E, we have

$$\lim_{n\to\infty}\frac{m(E\cap[0, b_n])}{b_n} = 1. \tag{4}$$

Now from (1)

$$1 \geqq \frac{a_n}{b_n} \geqq \frac{m(E\cap[0, a_n])}{b_n} = \frac{m(E\cap[0, b_n])}{b_n}-\frac{m(E\cap I_n)}{b_n} > \frac{m(E\cap[0, b_n])}{b_n}-\frac{2\varepsilon}{g_{k,a}(0)}.$$

Hence from (4)

$$1 \geqq \limsup_{n\to\infty}\frac{a_n}{b_n} \geqq \liminf_{n\to\infty}\frac{a_n}{b_n} \geqq 1-\frac{2\varepsilon}{g_{k,a}(0)}.$$

Since $\varepsilon>0$ is arbitrary, $\lim_{n\to\infty}\frac{a_n}{b_n}=1.$

So, $\lim_{n\to\infty}\frac{b_n-a_n}{b_n} = 0.$

THEOREM 6. *Let $f_{k-1,a}$ $(k\geqq 2)$ exist, let $\bar{f}^+_{k,a}=\bar{f}^-_{k,a}$ and let $\bar{f}_{k,a}$ be in Baire class 1. Suppose that $\bar{f}_{k,a}$ is finite except on a countable set. For any two reals $\alpha, \beta, \alpha<\beta$, let*

$$E(\alpha, \beta) = \{x\colon\ \alpha < \bar{f}_{k,a}(x) < \beta\}.$$

If $E(\alpha, \beta)$ contains a point x_0 where $f_{k,a}(x_0)$ exists, then $E(\alpha, \beta)$ has the following property:

For every sequence $\{I_n\}$ of closed intervals converging to x_0 with

$$m[E(\alpha, \beta)\cap I_n] = 0$$

for every n, we have

$$\lim_{n\to\infty} \frac{m(I_n)}{d(x_0, I_n)} = 0$$

where $d(x_0, I_n) = \inf\{|x_0 - y| : y \in I_n\}$.

PROOF. We may assume

$$x_0 = 0 = f(x_0) = f_{1,a}(x_0) = \ldots = f_{k-1,a}(x_0).$$

Now by Corollary of Theorem 4, $m[E(\alpha, \beta)\cap I_n]=0$ implies $E(\alpha, \beta)\cap I_n = \emptyset$ for every n. Thus $x\in I_n$ implies either $\bar{f}_{k,a}(x)\geqq\beta$ or $\bar{f}_{k,a}(x)\leqq\alpha$. Since by Theorem 2, $\bar{f}_{k,a}$ has Darboux property, either $\bar{f}_{k,a}(t)\geqq\beta$ for all $t\in I_n$ or $\bar{f}_{k,a}(t)\leqq\alpha$ for all $t\in I_n$.

Let N be the set of all positive integers. Let

$N_1 = \{n: n\in N; I_n$ has positive end-points and $t\in I_n$ implies $\bar{f}_{k,a}(t)\geqq\beta\}$,

$N_2 = \{n: n\in N; I_n$ has positive end-point and $t\in I_n$ implies $\bar{f}_{k,a}(t)\leqq\alpha\}$,

$N_3 = \{n: n\in N; I_n$ has negative end-points and $t\in I_n$ implies $\bar{f}_{k,a}(t)\geqq\beta\}$,

$N_4 = \{n: n\in N; I_n$ has negative end-points and $t\in I_n$ implies $\bar{f}_{k,a}(t)\leqq\alpha\}$.

Now, the function

$$g(x) = \beta\frac{x^k}{k!} - f(x)$$

is such that $g(0) = g_{1,a}(0) = \ldots = g_{k-1,a}(0) = 0 < g_{k,a}(0)$ and for the intervals I_n, $n\in N_1$, $\bar{g}_{k,a}(t)\leqq 0$ for $t\in I_n$. Hence by Lemma 1

$$\lim_{\substack{n\to\infty\\ n\in N_1}} \frac{m(I_n)}{d(0, I_n)} = 0.$$

Considering $g(x) = f(x) - \alpha\dfrac{x^k}{k!}$,

$$g(x) = \beta\frac{x^k}{k!} - (-1)^k f(-x) \quad \text{and} \quad g(x) = (-1)^k f(-x) - \alpha\frac{x^k}{k!}$$

for the sets N_2, N_3 and N_4 respectively, one gets

$$\lim_{\substack{n\to\infty\\ n\in N_i}} \frac{m(I_n)}{d(0, I_n)} = 0, \quad i = 2, 3, 4$$

and hence

$$\lim_{n\to\infty} \frac{m(I_n)}{d(0, I_n)} = 0.$$

COROLLARY. *Let $f_{k,a}$ exist and be finite. Then $f_{k,a}$ has the following property: For every open interval (α, β), for every x satisfying $\alpha < f_{k,a}(x) < \beta$ and for every sequence $\{I_n\}$ of closed intervals converging to x with $m[I_n\cap\{x: \alpha < f_{k,a}(x) < \beta\}] = 0$ we have*

$$\lim_{n\to\infty} \frac{m(I_n)}{d(x, I_n)} = 0.$$

The class of all functions which satisfy the property of $f_{k,a}$ stated in the above corollary is called by ZAHORSKI [14] the class $\mathcal{M}_3$. Therefore, the above corollary asserts that a finite $f_{k,a} \in \mathcal{M}_3$.

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References

[1] A. M. BRUCKNER, J. G. CEDER and M. WEISS, Uniform limits of Darboux functions, *Colloq. Math.*, (1966), 65—77.
[2] A. DENJOY, Sur l'integration des coefficients differentiels d'order superieur, *Fund. Math.*, **25** (1935), 273—326.
[3] A. DENJOY, Sur une propriete des fonctions derivees exactes, *Enseignement Math.*, **18** (1916), 320—328.
[4] M. J. EVANS, L_p-derivatives and approximate Peano derivatives, *Trans. Amer. Math. Soc.*, **165** (1972), 381—388.
[5] C. GOFFMAN and C. J. NEUGEBAUER, On approximate derivatives, *Proc. Amer. Math. Soc.*, **11** (1960), 962—966.
[6] A. KHINTCHINE, Recherches sur la structure des fonctions mesurables, *Fund. Math.*, **9** (1927), 212—279.
[7] N. C. MANNA and S. N. MUKHOPADHYAY, On approximate derivatives and their properties, *Acta Math. Acad. Sci. Hungar.*, **22** (1971), 309—320.
[8] S. N. MUKHOPADHYAY, On a certain property of the derivative, *Fund. Math.*, **67** (1970), 279—284.
[9] S. N. MUKHOPADHYAY, On the approximate Peano derivatives, *Fund. Math.*, **88** (1975), 133—143.
[10] S. N. MUKHOPADHYAY, On the regularity of the P^n-integral and its application to summable trigonometric series, *Pacific J. Math.*, **55** (1974), 233—247.
[11] H. W. OLIVER, The exact Peano derivative, *Trans. Amer. Math. Soc.*, **76** (1954), 444—456.
[12] S. SAKS, *Theory of the integrals* (Warsaw, 1937).
[13] C. E. WEIL, On properties of derivatives, *Trans. Amer. Math. Soc.*, **114** (1965), 363—376.
[14] Z. ZAHORSKI, Sur la premiere derivee, *Trans. Amer. Math. Soc.*, **69** (1950), 1—54.
[15] A. ZYGMUND, *Trigonometric Series* (Cambridge, 1968).

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DEPARTMENT OF MATHEMATICS
T. D. B. COLLEGE, RANIGANJ
DIST. BURDWAN, WEST BENGAL
INDIA

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TOPOLOGISCHE NULLTEILER UND ENDLICH ERZEUGTE IDEALE IN GEWISSEN ALGEBREN HOLOMORPHER FUNKTIONEN

Von

M. von RENTELN (Giessen)

1. Einleitung und Problemstellung

Die bekanntesten Beispiele von normierten Algebren von Funktionen, die in der offenen Einheitskreisscheibe $D=\{z:|z|<1\}$ holomorph sind, bilden die Algebren $H^\infty=\{f:D\to\mathbf{C}\,|\,f$ holomorph und beschränkt$\}$ und die Disc-Algebra $A(\bar{D})=\{f:\bar{D}\to\mathbf{C}\,|\,f$ stetig auf $\bar{D}$ und holomorph in $D\}$ jeweils unter der Norm $\|f\|=\sup\{|f(z)|:z\in D\}$. Beides sind komplexe Banachalgebren und daher lassen sich viele Resultate mit Hilfe der Gelfandtheorie herleiten. Wir wollen in der vorliegenden Arbeit gewisse normierte Algebren in D holomorpher Funktionen untersuchen, die im allgemeinen weder vollständig noch komplex sind. Während einige Untersuchungen zu reellen Banachalgebren dieser Art vorliegen (vgl. ALLING [1]), scheinen nicht vollständige normierte Algebren über Körpern $\mathbf{K}$ mit $\mathbf{K}\neq\mathbf{C}$ und $\mathbf{K}\neq\mathbf{R}$ nicht untersucht zu sein.

Eines der ungelösten Probleme in der Banachalgebra H^∞ und der Disc-Algebra $A(\bar{D})$ ist die ‚analytische' Charakterisierung der endlich erzeugten Ideale $I=(f_1,\dots,f_n)$. Zur Untersuchung hat sich ein weiteres Ideal

$$W=W(f_1,\dots,f_n)=\left\{f:\ |f(z)|\leqq C\sum_{i=1}^{n}|f_i(z)|\quad\text{in}\quad D\right\}$$

(C ist eine von f abhängige Konstante) als zweckmäßig erwiesen. $I\subseteqq W$ ist klar In H^∞ wurde anfangs vermutet, daß $I=W$ bei jeder Wahl der Funktionen $f_1,\dots,f_n$ gilt (vgl. BIRTEL [2], S. 347, Nr. 12). Für beide Algebren ist dies jedoch falsch. Ebenso ist, im Gegensatz zur Algebra aller im Einheitskreis holomorphen Funktionen, nicht jedes Ideal $I=(f_1,\dots,f_n)$ ein Hauptideal.

Das Ziel unserer Arbeit ist, für gewisse Algebren im Einheitskreis holomorpher Funktionen endlich erzeugte Ideale I anzugeben und zu charakterisieren, für die $I=W$ gilt und die Hauptideale sind. Ein Spezialfall dieses Problems, nämlich $I=W$ und $I=(1)$, wurde für die Algebren H^∞ und $A(\bar{D})$ untersucht und die Lösung im komplizierteren Fall $A=H^\infty$ ist das bekannte Corona-Theorem von L. CARLESON (siehe etwa [3], S. 202) in der folgenden algebraischen Formulierung:

CORONA-THEOREM. *Ist $\mathfrak{G}(A)$ die Gruppe der invertierbaren Elemente der Algebra A, so gilt: $W\cap\mathfrak{G}(A)\neq\emptyset\Leftrightarrow I=W$ und I ist das triviale Hauptideal* (1).

$I=W$ folgt hier schon aus $I=(1)$, wegen der ins Auge gefaßten Verallgemeinerung wählen wir trotzdem obige Formulierung. Der Satz ist auch für $A=A(\bar{D})$ gültig und mittels Gelfandtheorie unschwierig zu beweisen, er wird daher auch als ‚kleines' Corona-Theorem bezeichnet. Das Hauptergebnis unserer Arbeit ist eine Verallgemeinerung des kleinen Corona-Theorems in zweierlei Hinsicht, zum einen werden allgemeinere Algebren A über einem Körper $\mathbf{K} \subseteq \mathbf{C}$ von im Einheitskreis holomorphen Funktionen betrachtet, zum andern wird $\mathfrak{G}(A)$ durch eine geeignete größere Menge ersetzt. Es stellt sich heraus, daß für die gesuchte Charakterisierung diese geeignete Menge gerade das Komplement der topologischen Nullteiler der Algebra A ist.

2. Vorbemerkungen

In diesem Abschnitt möchten wir einige Definitionen, Bemerkungen und eine Proposition bringen.

DEFINITION. Unter einem imaginären Zahlkörper $\mathbf{K}$ verstehen wir einen Unterkörper von $\mathbf{C}$, der mindestens eine nichtreelle Zahl γ enthält mit $|\gamma| \in \mathbf{K}$.

BEISPIEL. Für jedes $m \in \mathbf{N} \setminus \{1, 2\}$ ist $\mathbf{Q}(e^{2\pi i/m})$ ein imaginärer Zahlkörper. Speziell für $m=4$ erhalten wir den Gaußschen Zahlkörper $\mathbf{Q}+i\mathbf{Q}$.

DEFINITION. A sei eine kommutative normierte Algebra über dem Körper $\mathbf{K}$. Ein Element $f \in A$ heißt topologischer Nullteiler von A, falls eine Folge $(g_n)_1^\infty$ von Elementen $g_n \in A$ existiert mit $\|g_n\|=1$ für jedes $n \in \mathbf{N}$ und $fg_n \to 0$ für $n \to \infty$.

BEZEICHUNGEN. Für die Algebra A bezeichne $\mathfrak{G}(A)$ die Menge der in A invertierbaren Elemente und $\mathrm{TNT}(A)$ die Menge der topologischen Nullteiler von A. Fernerhin setzen wir $\mathrm{TNT}^c(A) = A \setminus \mathrm{TNT}(A)$. $\mathbf{K}[z]$ bezeichnet wie üblich die Menge der $\mathbf{K}$-Polynome, d. h. die Koeffizienten der Polynome sind aus dem Körper $\mathbf{K}$. $\mathbf{K}\langle z \rangle$ sei die Menge aller ganzen Funktionen $\sum_{n=0}^{\infty} a_n z^n$ mit $a_n \in \mathbf{K}$.

DEFINITION. Für einen imaginären Zahlkörper $\mathbf{K}$ ist

$$A_{\mathbf{K}}(\bar{D}) := \left\{ f \in A(\bar{D}) \colon f(z) = \sum_{n=0}^{\infty} a_n z^n,\ a_n \in \mathbf{K} \right\}.$$

BEMERKUNGEN. 1. Unter der Sup-Norm ist $A_{\mathbf{K}}(\bar{D})$ eine normierte Algebra über dem Körper $\mathbf{K}$ und nur im Fall $\mathbf{K}=\mathbf{C}$ vollständig.

2. Wie man leicht (unter Beachtung von $\mathbf{Q} \subset \mathbf{K}$ und $\mathbf{K}$ dicht in $\mathbf{C}$) nachrechnet, exisitiert zu jedem $f \in A(\bar{D})$ ein $g \in \mathfrak{G}[A(\bar{D})]$, so daß $fg \in A_{\mathbf{K}}(\bar{D})$. Daraus folgt insbesondere, daß ein Element f genau dann in $A_{\mathbf{K}}(\bar{D})$ invertierbar ist, wenn es keine Nullstelle (in $\bar{D}$) hat.

PROPOSITION. *A sei eine die $\mathbf{K}$-Polynome enthaltende Unteralgebra von $A_{\mathbf{K}}(\bar{D})$ und $f \in A$. Dann gilt: f ist ein topologischer Nullteiler von A genau dann, wenn f eine Nullstelle α auf dem Rand ∂D besitzt.*

BEWEIS. Sei $f \in \mathrm{TNT}(A)$. Dann gibt es eine Funktionenfolge $(g_n)_1^\infty$ mit $g_n \in A$, $\|g_n\|=1$, so daß $g_n f \to 0$ für $n \to \infty$. Nach dem Maximumprinzip existiert eine Punktfolge $(z_n)_1^\infty$ mit $z_n \in \partial D$, so daß $|g_n(z_n)|=1$ für jedes $n \in \mathbf{N}$ gilt. Angenommen f wäre nullstellenfrei auf ∂D, d. h. $|f(z)| \geqq \delta$ auf ∂D für ein $\delta > 0$, so folgt insbesondere $|f(z_n)| \geqq \delta$. Dies impliziert $|(g_n f)(z_n)| \geqq \delta$, das ist aber ein Widerspruch zu $\|g_n f\| \to 0$.

Nun habe umgekehrt $f\in A$ eine Nullstelle $\alpha\in\partial D$. Wir müssen zeigen, daß zu jedem $\varepsilon>0$ ein $n=n(\varepsilon)\in\mathbf{N}$ und $g_n\in A$ existieren, so daß $\|g_n f\|<\varepsilon$ und $\|g_n\|=1$ gilt. Sei $\varepsilon>0$ vorgegeben. Da f stetig und $f(\alpha)=0$ ist, so existiert eine Umgebung $U_\delta(\alpha)=\{z:|z-\alpha|<\delta\}\cap\bar{D}$, in der $|f(z)|<\varepsilon$ ist. Wir bilden die reelle Achse $\mathbf{R}$ durch eine lineare Transformation $L(z)=\dfrac{az+b}{cz+d}$ auf die Einheitskreislinie ab. Da die Einheitskreislinie außer den Punkten -1 und $+1$ noch einen weiteren Punkt (nämlich $\gamma/|\gamma|$) aus $\mathbf{K}$ enthält, kann man die Koeffizienten a, b, c, d sogar aus dem Körper $\mathbf{K}$ wählen. Da $\mathbf{Q}\subset\mathbf{K}$ und $\mathbf{Q}$ dicht in $\mathbf{R}$, so liegt $L(\mathbf{Q})$ dicht in ∂D. Außerdem gilt wegen der Wahl der Koeffizienten $L(\mathbf{Q})\subset\mathbf{K}$. Daher gibt es ein $\beta\in U_\delta(\alpha)\cap\partial D$ mit $\beta\in\mathbf{K}$. Wir setzen $f_\beta(z)=\frac{1}{2}(1+\bar{\beta}z)$. Dann gilt $f_\beta\in A$ wegen $\bar{\beta}=\frac{1}{\beta}\in\mathbf{K}$. $\|f_\beta\|=1$, $f_\beta(\beta)=1$, $|f_\beta(z)|<1$ für jedes $z\in\bar{D}\setminus\{\beta\}$. Insbesondere existiert $\delta_1>0$ mit $|f_\beta(z)|\leqq 1-\delta_1$ in $\bar{D}\setminus U_\delta(\alpha)$. Wir wählen nun n so groß, daß $\|f\|(1-\delta_1)^n<\varepsilon$ gilt. Mit $g_n(z)=f_\beta^n(z)$ folgt $\|g_n\|=1$ und

$$|(g_n f)(z)| = |f_\beta^n(z)||f(z)| \leqq \begin{cases} |f(z)| < \varepsilon & \text{in } U_\delta(\alpha) \\ (1-\delta_1)^n\|f\| < \varepsilon & \text{in } \bar{D}\setminus U_\delta(\alpha). \end{cases}$$

Für jedes $z\in\bar{D}$ gilt also $|(g_n f)(z)|<\varepsilon$, d. h. $\|g_n f\|<\varepsilon$.

3. Das Hauptergebnis

DEFINITION. Wir sagen, daß eine Unteralgebra A von $A_{\mathbf{K}}(\bar{D})$ die schwache Teilereigenschaft besitzt, wenn sie folgende Bedingung erfüllt: Zu jeder Untermenge U einer endlichen Menge $M=\{z_1, \dots, z_m\}$ mit $|z_i|<1$ für jedes $i\in\{1, \dots, m\}$ gibt es ein $f\in A$, welches genau in den z_i einfache Nullstellen hat (sonst keine Nullstellen) und Teiler einer jeden Funktion $g\in A$ ist, die in $z_1, \dots, z_m$ verschwindet.

BEMERKUNGEN. 1. $A_{\mathbf{K}}(\bar{D})$ besitzt für jeden imaginären Zahlkörper $\mathbf{K}$ die schwache Teilereigenschaft. Im Fall der Disc-Algebra nimmt man für f am einfachsten das Polynom $f(z)=(z-z_1)\dots(z-z_m)$. Im Fall $A=A_{\mathbf{K}}(\bar{D})$ wählt man $f(z)=$ $=(z-z_1)\dots(z-z_m)g(z)$ mit $g\in\mathfrak{G}[A(\bar{D})]$, so daß $f\in A$ (vgl. Bemerkung 2 von oben).

2. Im Spezialfall $U=\emptyset$ besagt die Bedingung, daß jedes nullstellenfreie Element in A invertierbar ist.

Für endlich viele Funktionen $f_1, \dots, f_n$ der Unteralgebra A von $A_{\mathbf{K}}(\bar{D})$ bezeichnet $I=I(f_1, \dots, f_n)$ das von den $f_1, \dots, f_n$ (algebraisch) erzeugte Ideal und das Ideal $W=W(f_1, \dots, f_n)$ besteht aus allen Funktionen $f\in A$, die in D einer Abschätzung $|f(z)|\leqq C\sum_{i=1}^{n}|f_i(z)|$ mit einer von f abhängigen Konstanten C genügen.

THEOREM. *A sei eine die $\mathbf{K}$-Polynome enthaltende Unteralgebra von $A_{\mathbf{K}}(\bar{D})$ mit der schwachen Teilereigenschaft. Dann gilt:*

$$W\cap \mathrm{TNT}^c(A) \neq \emptyset \Rightarrow I = W \text{ und } I \text{ ist Hauptideal.}$$

Existiert darüber hinaus zu jedem $f \in A$ mit $f(z_0)=0$ für ein $z_0 \in \partial D$ ein $\varphi \in H^\infty \setminus A$, so daß $f\varphi \in A$, so gilt sogar

$$W \cap \mathrm{TNT}^c(A) \neq \emptyset \Leftrightarrow I = W \text{ und } I \text{ ist Hauptideal.}$$

BEMERKUNG. Im Fall $\mathbf{K}=\mathbf{C}$ gibt es viele bekannte, die Polynome enthaltende, Unteralgebren von $A_{\mathbf{K}}(\bar{D})$, welche die schwache Teilereigenschaft erfüllen, wie z. B. $A^{(n)}$ (KORENBLYUM [6]), A^∞ (TAYLOR and WILLIAMS [7]), A^+ (KAHANE [5], Kap. XI), $A(\bar{D}) \cap \mathrm{Lip}_\alpha(\partial D)$ $(0<\alpha \leqq 1)$ und weitere Algebren (siehe z. B. FAĬVYŠEVSKIĬ [4]).

KOROLLAR. *In jeder Algebra $A=A_{\mathbf{K}}(\bar{D})$ gilt:*

$$W \cap \mathrm{TNT}^c(A) \neq \emptyset \Leftrightarrow I = W \text{ und } I \text{ ist Hauptideal.}$$

BEWEIS. $\mathbf{K}[z] \subset A_{\mathbf{K}}(\bar{D})$ ist klar. Wie weiter oben bemerkt erfüllt $A_{\mathbf{K}}(\bar{D})$ die schwache Teilereigenschaft. Wir haben nur noch zu zeigen, daß $A_{\mathbf{K}}(\bar{D})$ auch die Voraussetzung des zweiten Teils des Satzes erfüllt. Sei also $f \in A$, $f(z_0)=0$ für ein $z_0 \in \partial D$. Wie eine leichte Rechnung zeigt, existiert $f_1 \in \mathfrak{G}[A(\bar{D})]$, so daß $\varphi(z)= :=f_1(z)\exp\left[\frac{z+z_0}{z-z_0}\right]$ eine Entwicklung $\varphi(z)=\sum_{n=0}^{\infty} c_n z^n$ mit $c_n \in \mathbf{K}$ hat. Offensichtlich gilt $\varphi \in H^\infty \setminus A(\bar{D})$, da φ eine singuläre Stelle an z_0 besitzt. Wegen $f(z_0)=0$ ist aber $f\varphi \in A_{\mathbf{K}}(\bar{D})$.

BEWEIS DES SATZES. Zum Beweis des ersten Teiles nehmen wir an, daß W ein Element f enthält, welches kein topologischer Nullteiler (in A) ist. Aus der Proposition folgt, daß f keine Nullstelle auf dem Rand ∂D hat. Also haben die Erzeuger $f_1, \ldots, f_n$ des Ideals I nur endlich viele gemeinsame Nullstellen, von denen keine auf dem Rand liegt. Da A die schwache Teilereigenschaft besitzt, zeigt man leicht, daß auch ein $G \in A$ existiert mit $f_i/G \in A$ und $\sum_{i=1}^{n} |(f_i/G)(z)| \geqq \delta$ mit einem $\delta>0$. Da insbesondere $f_i/G \in A(\bar{D})$ so gibt es nach dem Corona-Theorem für $A(\bar{D})$ Funktionen $G_i \in A(\bar{D})$ mit $1=\sum_{i=1}^{n} G_i(f_i/G)$. Da $\mathbf{K}$ dicht in $\mathbf{C}$ und wegen $\mathbf{K}[z] \subseteqq A$ ist A dicht in $A(\bar{D})$, d. h. es gibt Funktionen $h_i \in A$ mit

$$\|G_i - h_i\| < \frac{1}{2\sum_{i=1}^{n} \|(f_i/G)\|}$$

für jedes $i \in \{1, \ldots, n\}$. Daraus folgt

$$1-\left|\sum_{i=1}^{n} h_i(f_i/G)\right| \leqq \left|1-\sum_{i=1}^{n} h_i(f_i/G)\right| = \left|\sum_{i=1}^{n} (G_i-h_i)(f_i/G)\right| \leqq$$

$$\leqq \sum_{i=1}^{n} \|G_i-h_i\|\,\|f_i/G\| < \frac{1}{2}.$$

Dies impliziert, daß die Funktion $h=\sum_{i=1}^{n} h_i(f_i/G) \in A$ invertierbar ist. Wir erhalten $G=\sum_{i=1}^{n} g_i f_i$ mit $g_i=\frac{h_i}{h} \in A$, d. h. $(G) \subseteqq (f_1, \ldots, f_n)=I$, unter Beachtung von

$f_i/G \in A$, d. h. $(f_i) \subseteq (G)$ für jedes $i \in \{1, \dots, n\}$ folgt $I=(G)$, d. h. I ist ein Hauptideal.

Es bleibt noch zu zeigen $I=W$. $I \subseteq W$ ist klar. Sei also $f \in W$, d. h. $|f(z)| \leqq \leqq C \sum_{i=1}^{n} |f_i(z)|$. Da A die schwache Teilereigenschaft besitzt, ist G auch ein Teiler von f, d. h. $f \in (G) = I$.

Zum Beweis des zweiten Teils ist nur noch zu zeigen, daß aus $W \cap \mathrm{TNT}^c(A) = \emptyset$ folgt $I \neq W$ oder I ist kein Hauptideal. Sei also $W \cap \mathrm{TNT}^c(A) = \emptyset$. Ist I kein Hauptideal, so sind wir fertig. Ist aber I ein Hauptideal, d. h. $I=(d)$ für ein $d \in A$, so müssen wir zeigen $I \neq W$. Nach Voraussetzung besteht W nur aus topologischen Nullteilern, insbesondere muß nach der Proposition $d \in I \subset W$ eine Nullstelle z_0 auf dem Rand ∂D haben. Nach Voraussetzung gibt es zu d eine Funktion $\varphi \in H^\infty \setminus A$, so daß $d\varphi \in A$. Daraus folgt $d\varphi \in W$, aber $d\varphi \notin I$, d. h. $I \neq W$.

Zusatz bei der Korrektur (24. November 1978.) Inzwischen hat mir Herr K. H. Indlekofer einen anderen Beweis der Proposition mitgeteilt, der die Eigenschaft $|\gamma| \in \mathbf{K}$ (bei der Definition des imaginären Zahlkörpers) nicht benötigt.

Literaturverzeichnis

[1] N. L. ALLING, Real Banach algebras and non orientable Klein surfaces I, *J. Reine Angew. Math.*, **241** (1970), 200—208.

[2] F. T. BIRTEL, Function algebras (*Proc. Internat. Sympos. on Function Algebras*, Tulane Univ. 1965). Chicago, 1966.

[3] P. L. DUREN, *Theory of H^p spaces* (New York, 1970).

[4] V. M. FAĬVYŠEVSKIĬ, On the structure of ideals of some algebras of analytic functions, *Dokl. Akad. Nauk SSSR*, **211** (1973), 537—539 [Russ.; engl. Übers.: *Soviet Math. Dkl.*, **14** (1973), 1067—1070].

[5] J. P. KAHANE, *Series de Fourier absolument convergentes* (Berlin—Heidelberg—New York, 1970).

[6] B. I. KORENBLYUM, Closed ideals in the ring A^n, *Funkcional Anal. i Priložen.*, **6** (1972), 38—52 [Russ.; engl. Übers.: *Functional Anal. Appl.*, **6** (1972), 203—214].

[7] B. A. TAYLOR and D. L. WILLIAMS, Ideals in rings of analytic functions with smooth boundary values, *Can. J. Math.*, **22** (1970), 1266—1283.

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MATHEMATISCHES INSTITUT
JUSTUS LIEBIG—UNIVERSITÄT
D—6300 GIESSEN
ARNDTSTR. 2.
BUNDESREPUBLIK DEUTSCHLAND

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E-COMPACTNESS AND CONTINUOUS FUNCTION SPACES

By
W. GOVAERTS[1] (Ghent)

1. Introduction and motivation. For all notions from universal algebra we refer to G. GRÄTZER [12]. In particular, our use of the term "structure" agrees with p. 223—224 in that work. E denotes a Hausdorff space provided with a family of continuous finitary everywhere defined operations and a family of finitary relations with E-compact graph sets. For each topological space X the set $C(X, E)$ of all continuous functions from X into E is provided with pointwisely defined operations and relations so that it is regarded as a structure of type $\tau(E)$.

A character is a homomorphism from $C(X, E)$ into E. In particular, each point $x_0 \in X$ determines a character Π_{x_0} by $\Pi_{x_0}(f)=f(x_0)$ for all $f \in C(X, E)$. Such characters are called evaluations.

In our study of $C(X, E)$ we make use of the pioneering work of S. MRÓWKA ([7], [16]) who introduced the fundamental notions of E-complete regularity and E-compactness (see also [17]).

E is said to be sufficiently complicated iff each character $C(X, E) \to E$ is an evaluation whenever X is E-compact. Essentially, our aim is to investigate which systems E are sufficiently complicated. This problem has already received a considerable attention in literature, as well in the general setting ([4], [6], [17]) as for specific structures E ([19], [14], [18]). Its importance may be seen from the following arguments:

1. For each topological space X there exists an E-compact space Y and a continuous mapping h from X onto a dense subset of Y such that the mapping H from $C(Y, E)$ into $C(X, E)$, obtained by setting $H(f)=f \circ h$ is one-to-one and onto; H is an isomorphism for all operations that are pointwisely definable from continuous operations in E and preserves in both senses all relations whose graphs are E-compact sets (this includes all relations with closed graphs; in some cases, e. g. $E=\mathbf{R}$, all relations have E-compact graphs).

2. If E is sufficiently complicated, each character $K: C(X, E) \to E$ preserves all continuous operations and relations with E-compact graphs, even those not belonging to the structure E. This holds true when X is an arbitrary topological space.

3. If E is sufficiently complicated and X, Y are general topological spaces, then each homomorphism H from $C(X, E)$ into $C(Y, E)$ also preserves all other continuous operations in E and all relations with E-compact graphs; if in addition

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X, Y are both E-compact, there exists a (unique) continuous function $h: Y \to X$ such that $H(f) = f \circ h$ for all $f \in C(X, E)$.

4. If E is sufficiently complicated, two E-compact spaces X, Y are homeomorphic iff the structures $C(X, E)$ and $C(Y, E)$ are isomorphic. For some specific structures E, this basic "determination theorem" implies many others of the same type (e.g. two metric spaces are homeomorphic whenever their rings of continuous real-valued functions are isomorphic; cf. also the nice small paper of M. HENRIKSEN [13]).

We treat the problem in as general a way as possible. In section 2 we obtain a categorical characterization of E-compactness that generalizes some results concerning inverse limits. Section 3 gives applications to the question of representing characters. In section 4 we collect known and original examples.

2. E-compact spaces as inputs of diagrams. For all categorical notions not explained in the text we refer to Ch. III of the work of Z. SEMADENI [19]. In the sequel Top_0 denotes the category of topological T_0-spaces and continuous functions. We also need some full subcategories of Top_0, namely

Top_2 (objects are all Hausdorff spaces)
TFP_E (finite products of E)
TCFP_E (closed subsets of finite products of E)
TCP_E (E-compact spaces)
TCR_E (E-completely regular spaces).

For specific structures E there exist characterizations of E-compact spaces as inverse limits of certain inverse systems (see [5], [21]): however, in general it is more appropriate to use the notion of the input of a diagram in a category (see [19]). Indeed, for some purposes inverse limits are not suitable.

If $\mathscr{D}$ is a small category and $\Gamma: \mathscr{D} \to \mathrm{Top}_0$ (or Top_2) is a diagram, then the input of Γ is easily seen to be the object $I = \{k \in \prod_{D \in \mathscr{D}^0} \Gamma(D) : \Pi_{D_2}(k) = \Gamma(\delta) \circ \Pi_{D_1}(k)$ for each $\mathscr{D}$-morphism $\delta: D_1 \to D_2\}$, together with the projections Π_D from I into $\Gamma(D)$. Hence I is a subset of the product of all $\Gamma(D)$; if each $\Gamma(D)$ is E-compact, I will be E-compact too (the empty set is E-compact for each E).

PROPOSITION 1. *Let $E \in \mathrm{Top}_2$, $X \in \mathrm{TCP}_E$. There is a diagram $\Gamma: \mathscr{D} \to \mathrm{Top}_2$ with $\Gamma(D) \in \mathrm{TCFP}_E$ for all $D \in \mathscr{D}^0$ and there are morphisms $(\sigma_D)_{D \in \mathscr{D}^0}$ from X into $\Gamma(D)$ so that $(X, (\sigma_D)_{D \in \mathscr{D}^0})$ is the input of Γ. Furthermore, we may require*

(a) *the input is an inverse limit*

(b) *for all $f \in C(X, E)$ there exist a $D \in \mathscr{D}^0$ and $f_0 \in C(\Gamma(D), E)$ such that $f = f_0 \circ \sigma_D$.*

PROOF. We set $\mathscr{D}^0 = \bigcup_{n=1}^{\infty} D_n$ where $D_n = C(X, E^n)$. If $f \in D_n$, $g \in D_m$, the set $\langle f, g \rangle$ of all $\mathscr{D}$-morphisms from f into g is the set of all continuous k from $\mathrm{Cl}_{E^n}(f(X))$ into $\mathrm{Cl}_{E^m}(g(X))$ such that $k \circ f = g$. Let $\Gamma: \mathscr{D} \to \mathrm{Top}_2$ be determined by $\Gamma(f) = \mathrm{Cl}_{E^n}(f(X))$ whenever $f \in D_n$ and by $\Gamma(k) = k$ if k is a $\mathscr{D}$-morphism. We also set $\sigma_f = f$. Now the condition (b) trivially holds and it is easily seen that $(\sigma_f)_{f \in \mathscr{D}^0}$ is compatible with Γ.

Let $Y \in \mathrm{Top}_2$, $(\beta_f)_{f \in \mathscr{D}^0}$ a family that is compatible with Γ. We fix $y_0 \in Y$ and show that there is a point $x \in X$ such that $f(x) = \beta_f(y_0)$ for all $f \in D_1$. If not, we introduce $X'=X\cup\{x_0\}$ where $x_0\notin X$. For all $f\in C(X,E)$ we define $f^e:X'\to E$ by $f^e=f$ in all points of X and $f^e(x_0)=\beta_f(y_0)$. We provide X' with the weak topology induced by all $f^e(f\in C(X,E))$. Then X' is an E-completely regular space containing X as a subspace.

We recall from [7] that an E-compact space X is never dense in a proper E-completely regular extension X' such that each continuous $f:X\to E$ may be continuously extended to the whole of X'; so in our situation we arrive to a contradiction by showing that X is dense in X'.

Let $f_1,\dots,f_n\in C(X,E)$ be given, together with open neighbourhoods $G_1,\dots,G_n$ of $\beta_{f_1}(y_0),\dots,\beta_{f_n}(y_0)$ respectively. Let f be the function $(f_1,\dots,f_n)$ from X into E^n. Then $\beta_f(y_0)\in \mathrm{Cl}_{E^n}(f(X))$; for all $i=1,\dots,n$ we have in addition $\pi_i\circ\beta_f(y_0)=\beta_{f_i}(y_0)=f_i^e(y_0)$. This implies that $(G_1\times\dots\times G_n)\cap f(X)\neq\emptyset$.

Hence we may find a point $x\in X$ such that $f(x)\in G_1\times\dots\times G_n$ which means $f_i(x)\in G_i$ for all $i=1,\dots,n$, a contradiction.

We conclude that there is a unique function $\xi:Y\to X$ assigning to each $y\in Y$ a point $x\in X$ such that $\beta_f(y)=f(\xi(y))$ for all $f\in C(X,E)$, i.e. $\beta_f=f\circ\xi$ for each $f\in D_1$. Furthermore, ξ is continuous since each composition $f\circ\xi(f\in C(X,E))$ is continuous.

It remains to show that $\beta_f=f\circ\xi$ whenever $f\in D_n$ with $n>1$. For $i\in\{1,\dots,n\}$ we set $f_i=\pi_i\circ f$, so $f_i\in D_1$. Then $\pi_i\circ\beta_f=\beta_{f_i}=f_i\circ\xi=\pi_i\circ f\circ\xi$; consequently $\beta_f=f\circ\xi$.

Finally, each morphism set $\langle f,g\rangle$ in $\mathscr{D}$ contains at most one point, so that $\mathscr{D}^0$ may be partially ordered by setting $f\leqq g$ whenever $\langle f,g\rangle$ is nonempty; then $\mathscr{D}^0$ is trivially directed downward, which means that the input may be viewed as an inverse limit.

COROLLARY 1. In the case $E=\mathbf{Z}$, spaces of TCFP_E are just finite or countable discrete spaces, so that we find the main result of [5] as a particular case. Furthermore, it is well known ([8], 8.2 p. 115) that all separable metric spaces are realcompact. Proposition 1 thus implies that realcompact spaces are just homeomorphs of inverse limits of separable metric spaces; this agrees with [21]. As another aplication, compact Hausdorff spaces are just inverse limits of finite-dimensional Euclidean compacta; while zerodimensional compact Hausdorff spaces are inverse limits of finite discrete spaces. The latter result implies dually that each Boolean algebra is a direct limit of finite Boolean algebras.

We recall from [11] that a Hausdorff space E is an *ss*-space iff for each closed subset F of each finite product E^n of E and $x\in E^n\setminus F$ there are $f,g\in C(E^n,E)$ such that f and g agree on F but differ in x.

PROPOSITION 2. *Let E be an ss-space, $X\in\mathrm{TCP}_E$. There is a diagram $\Gamma:\mathscr{D}\to\mathrm{Top}_2$ with $\Gamma(D)\in\mathrm{TFP}_E$ for all $D\in\mathscr{D}^0$ and a family $(\sigma_D)_{D\in\mathscr{D}^0}$ of morphisms from X into $\Gamma(D)$ such that $(X,(\sigma_D)_{D\in\mathscr{D}^0})$ is an input of Γ. Furthermore we may require that for each $f\in C(X,E)$ there is a $D\in\mathscr{D}^0$ and $f_0\in C(\Gamma(D),E)$ with $f=f_0\circ\sigma_D$.*

PROOF. Again set $\mathscr{D}^0=\bigcup_{n=1}^{\infty}D_n$ where $D_n=C(X,E^n)$. If $f\in D_n$, $g\in D_m$, $\langle f,g\rangle$ is defined as the set of all $k\in C(E^n,E^m)$ such that $k\circ f=g$. The functor $\Gamma:\mathscr{D}\to\mathrm{Top}_2$ is determined by $\Gamma(f)=E^n$ if $f\in D_n$ and by $\Gamma(k)=k$ whenever k is a $\mathscr{D}$-morphism. Also, $\sigma_f=f$ for each $f\in\mathscr{D}^0$.

The rest of the proof is analogous to that of proposition 1, except for one thing. If $(Y, (\beta_f)_{f\in\mathscr{D}^0})$ is compatible with Γ and if $y_0\in Y$ we need show that $\beta_f(y_0)\in \in \mathrm{Cl}_{E^n}(f(X))$. Suppose this does not hold; then there is $m\geqq 1$ (in fact $m=1$) and continuous $\varphi, \psi: E^n \to E^m$ such that φ, ψ agree on $\mathrm{Cl}_{E^n}(f(X))$ but differ in $\beta_f(y_0)$. Set $h=\varphi\circ f=\psi\circ f$. Then $\beta_h=\varphi\circ\beta_f=\psi\circ\beta_f$, whence $\varphi(\beta_f(y_0))=\psi(\beta_f(y_0))$, a contradiction.

REMARKS. One may wonder whether for each $A\in\mathrm{Top}_2$ and $X\in\mathrm{TCP}_A$ there is a diagram $\Gamma:\mathscr{D}\to\mathrm{Top}_2$ such that each $\Gamma(D)$ is a finite product of A and X is an input of Γ. This does not hold; indeed we know from [10], 3.1 that there exists a Hausdorff space A_0 for which there are A_0-compact spaces that are not A_0-maximal. The negative result now follows from proposition 3, all notions about A-maximal spaces being found in [10]:

PROPOSITION 3. *Let A be a T_0-space, $\Gamma:\mathscr{D}\to\mathrm{Top}_2$ a diagram with $\Gamma(D)$ a finite product of A for each $D\in\mathscr{D}^0$. If X is an input of Γ, together with a family of morphisms $(\sigma_D)_{D\in\mathscr{D}^0}$, then X is A-maximal.*

PROOF. Set $P=\prod_{D\in\mathscr{D}^0}\Gamma(D)$, $\varphi_\delta=\pi_{D_2}$ and $\psi_\delta=\Gamma(\delta)\circ\pi_{D_1}$ for each $\mathscr{D}$-morphism $\delta: D_1\to D_2$. Then P is A-maximal (see [10] 2.5.2)), each φ_δ or ψ_δ is a continuous function into an E-completely regular space and

$$X = \{p\in P: \varphi_\delta(p)=\psi_\delta(p) \text{ for each } \mathscr{D}\text{-morphism } \delta\}.$$

To show that X is A-maximal, let X' be the A-maximal extension of X. It follows from [10] 2.6 that the identity function on X may be extended to a function $f\in C(X', P)$. Then $\varphi_\delta\circ f$ and $\psi_\delta\circ f$ agree on X for each δ, so they agree on the whole of X'; this implies $f(X')\subseteqq X$. The identity on X' now agrees with f on X, so it agrees everywhere and we obtain $X'=X$, i.e. X is A-maximal.

3. Reduction theorems. Let CF_E be the category whose objects are all structures $C(X, E)$ and whose morphisms are structure morphisms in the sense of [12] p. 224. Let $\mathscr{C}_E$ be the contravariant functor from Top_2 into CF_E such that $\mathscr{C}_E(X)=C(X, E)$ for each object X and $[\mathscr{C}_E(\varphi)](f)=f\circ\varphi$ for each continuous $\varphi: X\to Y$ and $f\in C(Y, E)$. It may be seen without difficulty that $\mathscr{C}_E$ is faithful on TCR_E, i.e. whenever $\varphi, \psi\in C(X, Y)$ with $X, Y\in\mathrm{TCR}_E$ and $\mathscr{C}_E(\varphi)=\mathscr{C}_E(\psi)$, then $\varphi=\psi$.

We say that a space $X\in\mathrm{Top}_2$ has the E-evaluation property if each character $C(X, E)\to E$ is an evaluation. If $X_0=\{x_0\}$ is a singleton space in Top_2, this is equivalent to saying that for each morphism $\psi: C(X, E)\to C(X_0, E)$ there is a morphism $\varphi: X_0\to X$ such that $\mathscr{C}_E(\varphi)=\psi$.

PROPOSITION 4. *Let $\Gamma:\mathscr{D}\to\mathrm{Top}_2$ be a diagram; suppose that each $\Gamma(D)$ belongs to TCR_E and has the E-evaluation property. Let $(X, (\sigma_D)_{D\in\mathscr{D}^0})$ be the input of Γ. Suppose that two structure morphisms $\psi_1, \psi_2: C(X, E)\to E$ are equal whenever $\psi_1(f)=\psi_2(f)$ for each $f\in C(X, E)$ that filters through a σ_D. Then X has the E-evaluation property.*

PROOF. For convenience we replace $\mathscr{C}_E$ be a $'$-sign, thus writing X' instead of $C(X, E)$ and φ' instead of $\mathscr{C}_E(\varphi)$. Let $X_0=\{x_0\}$ and suppose $\psi: X'\to X_0'$ is a structure morphism; we are looking for a $\varphi: X_0\to X$ such that $\varphi'=\psi$.

For $D\in\mathscr{D}_0$ set $\psi_D=\psi\circ\sigma'_D$; there is a φ_D such that $\varphi'_D=\psi_D$. To show that $(\varphi_D)_{D\in\mathscr{D}^0}$ is compatible with Γ, we compute for each $\mathscr{D}$-morphism δ

$$(\Gamma(\delta)\circ\varphi_{D_1})'=\varphi'_{D_1}\circ\Gamma(\delta)'=\psi_{D_1}\circ\Gamma(\delta)'=\psi\circ\sigma'_{D_1}\circ\Gamma(\delta)'=$$
$$=\psi\circ(\Gamma(\delta)\circ\sigma_{D_1})'=\psi\circ\sigma'_{D_2}=\psi_{D_2}=\varphi'_{D_2},$$

whence $\Gamma(\delta)\circ\varphi_{D_1}$ indeed equals φ_{D_2}.

Since X is an input of Γ there exists a $\varphi:X_0\to X$ such that $\varphi_D=\sigma_D\circ\varphi$ for all $D\in\mathscr{D}^0$. We conclude that $\psi_D=\varphi'\circ\sigma'_D=\psi\circ\sigma'_D$ for all $D\in\mathscr{D}^0$; this implies that φ' and ψ are structure homomorphisms from $C(X, E)$ into X'_0 that agree in each function which filters through a σ_D; hence they are equal by the hypothesis ad hoc.

We now proceed to the main result of this section, the second part of which is a generalization of [11] 4.2.

PROPOSITION 5. (a) *If each closed subset of each finite product of E has the E-evaluation property, then E is sufficiently complicated.*

(b) *If each finite product of E has the E-evaluation property, and if E is an ss-space, then E is sufficiently complicated.*

PROOFS. (a) follows from propositions 1 and 4 while (b) is a consequence of 2 and 4.

LEMMA 1. *Let $\mathscr{S}$ be a subbase for E, X a closed subset of E^n (n finite). $H:C(X,E)\to E$ arbitrary. Then H is an evaluation iff* (α) *holds:*

(α) *For each finite family $f_1,\dots,f_m\in C(X,E)$ and each family $G_1,\dots,G_m\in\mathscr{S}$ such that $H(f_i)\in G_i$ for all i, there is an $x\in X$ with $f_i(x)\in G_i$ for all i.*

PROOF. The requirement is obviously necessary. Now suppose that it holds; we may assume that $\mathscr{S}$ contains all open sets. Set $x_0=(H(\pi_1),\dots,H(\pi_n))$ where π_i denotes the i^{th} projection of E^n. Whenever $x_0\in G_1\times\dots\times G_n$ with $G_i\in\mathscr{S}$ for all i, there exists an $x\in X$ such that $\pi_i(x)\in G_i$ for all i. So x_0 belongs to the closure of X, i.e. to X itself.

Now let $f\in C(X,E)$ be arbitrary. If $H(f)\neq f(x_0)$, we may choose open disjoint G_0 and G_{n+1} containing $f(x_0)$ and $H(f)$ respectively. By continuity of f, there are $G_1,\dots,G_n\in\mathscr{S}$ such that $x_0\in G_1\times\dots\times G_n$ and $f(G_1\times\dots\times G_n)\subseteq G_0$. Then $H(\pi_i)\in G_i$ for all i and $H(f)\in G_{n+1}$. Suppose $x\in X$ is such that $\pi_i(x)\in G_i$ for all i and $f(x)\in G_{n+1}$. Then $f(x)\in G_0\cap G_{n+1}$, a contradiction.

PROPOSITION 6. *Let $\mathscr{S}$ be a subbase for E. The following conditions are equivalent:*

(i) *E is sufficiently complicated.*

(ii) *Each closed subset of each finite product of E has the E-evaluation property.*

(iii) *Condition* (α) *of lemma* 1 *holds for each E-compact space X and each character H.*

(iv) *Condition* (α) *of lemma* 1 *holds for each closed subset X of each finite product of E and each character H.*

Furthermore, in (ii) *and* (iv) *we may consider only the finite products of E themselves whenever E is an ss-space.*

PROOF. The implications (i)$\Rightarrow$(iii) and (iii)$\Rightarrow$(iv) are obvious. Lemma 1 shows (iv)$\Rightarrow$(ii). Finally, proposition 5 implies (ii)$\Rightarrow$(i).

COROLLARY 2. *E is sufficiently complicated whenever for each E-compact space X, each character H and functions $f_1, \ldots, f_m \in C(X, E)$ there exists an $x \in X$ such that $H(f_i)=f_i(x)$ for all $i=1, \ldots, m$ (cf. [6]).*

A remarkable result concerning topological rings may be deduced from this corollary; since it is known from [4] we state it without proof:

PROPOSITION 7. *Let E be a Hausdorff topological ring, provided with ring operations $+, -, \times$ and all constant unary operations and suppose that the following conditions hold:*

(a) *Whenever $H(f)=0$ for a character $H: C(X, E) \to E$ with $f \in C(X, E)$ and X E-compact, one may find $x \in X$ such that $f(x)=0$.*

(b) *There is a continuous function $x \to x^*$ in E such that $xx^*+yy^*=0$ only when $x=y=0$.*

Then E is sufficiently complicated.

4. Sufficiently complicated structures. In this section we treat old and new examples of sufficiently complicated structures:

1. The real numbers, provided with their usual topology, ring operations and the constant mapping from $\mathbf{R}$ onto 1 ([14]).

2. The two-point discrete lattice $\{0, 1\}$ with lattice operations and constant unary mappings (in this case the class of structures $C(X, E)$ is just the class of Boolean algebras, as follows from a classical representation theorem of M. H. Stone).

3. The ring $\mathbf{Z}$ of integer numbers, provided with a discrete topology, ring operations and constant unary operation onto 1 (see [4]).

4. Field of complex numbers with usual topology, ring operations and constant unary operations (cf. [4]) (analogous result for the quaternions).

5. Subfield of the real numbers with relative topology, ring operations and constant unary operation onto 1 (particular case: rational numbers); cf. [4].

6. Any field with a discrete topology, when provided with ring operations and all constant unary operations ([4]).

7. The interval [0, 1], provided with usual topology, binary operation $(x, y) \to x \cdot y$ and unary operation $x \to 1-x$ (see [9]).

8. The ring of integer real quaternions, provided with a discrete topology, ring operations and constant unary mappings onto i, j (where i, j, k are the imaginary units).

Proof of this example 8. Let X be E-compact, $H: C(X, E) \to E$ a character. Then by hypothesis $H(i)=i$ and $H(j)=j$, so that $H(1)=H(-i \cdot i)= -H(i) \cdot H(i) = -i^2 = 1$ and $H(k)=H(i \cdot j)=i \cdot j=k$. If $x=a+ib+jc+kd$ with a, b, c, d real, then $x-ixi-jxj-kxk=4a$. So $f \in C(X, E)$ is real-valued iff $f-ifi-jfj-kfk=4f$; in that case $H(f)-iH(f)i-jH(f)j-kH(f)k=4H(f)$, whence $H(f)$ is real-valued.

Let H' be the restriction of H to $C(X, Z)$. Then $H'(C(X, Z)) \subseteq Z$ and we conclude from example 3 that there is a point $x_0 \in X$ such that $H(f)=f(x_0)$ for all $f \in C(X, Z)$. Now let f be arbitrary. Then there are $f_1, \ldots, f_4 \in C(X, Z)$ such that $f=f_1+f_2i+f_3j+f_4k$ and we obtain $H(f)=H(f_1)+H(f_2)i+H(f_3)j+$ $+H(f_4)k=f(x_0)$.

9. Field with a Krull valuation, not having characteristic 2, when provided with the valuation topology, ring operations and all constant unary mappings.

Proof of this (as far as we know unpublished) example. We verify the conditions of proposition 7. First remark that in such fields inversion is continuous; so condition (a) clearly holds. Furthermore, the existence of a $*$-function follows from [15] p. 178.

10. Chain, provided with order topology, lattice operations and all constant unary operations. This provides a suitable example whenever the chain is zero-dimensional or when it is connected and separable (see [1], [2], [3]).

11. Let $\mathbf{R}^+=\{r\in\mathbf{R}:r\geqq 0\}$ be provided with usual topology, addition, multiplication and all constant unary operations.

We make use of corollary 2. Let $H:C(X,\mathbf{R}^+)\to\mathbf{R}^+$ be a homomorphism and $H(f_i)=r_i$ for $i=1,\dots,n$. Set $g_i=f_i^2+r_i^2-2f_ir_i$. Then $g_i\in C(X,\mathbf{R}^+)$ and $g_i+2f_ir_i=f_i^2+r_i^2$ whence $H(g_i)=0$. So also $H(g_1+\dots+g_n)=0$, from which we deduce the existence of a point $x_0\in X$ with $g_1(x_0)=\dots=g_n(x_0)=0$; then $f_i(x_0)=$ $=r_i=H(f_i)$ for all i.

12. Let $\mathbf{Z}^+=\{z\in\mathbf{Z}:z\geqq 0\}$ be provided with a discrete topology, addition, multiplication and all constant unary operations. The proof is analogous to the preceding. If this time $H(f)=0$, there is a point $x\in X$ with $f(x)=0$ since otherwise f could be written as $f=1+g$, whence $H(f)=1+H(g)\geqq 1$.

13. Let E be a commutative n-group (see [20]) that is totally ordered so that $e_1+\dots+e_j+\dots+e_n\leqq e_1+\dots+e_j'+\dots+e_n$ whenever $e_j\leqq e_j'$. Let E be provided with the order topology, binary operations $+$ and $\vee$ and all constant unary operations. We verify condition (iii) of proposition 6, taking $\mathcal{S}=\mathcal{S}_1\cup\mathcal{S}_2$ where $\mathcal{S}_1=\{]-\infty,a[:a\in E\}$ and $\mathcal{S}_2=\{]a,+\infty[:a\in E\}$. Let X be E-compact, $H:C(X,E)\to E$ a character, $f_1,\dots,f_m\in C(X,E)$, $H(f_i)=r_i$ and $r_i\in G_i$ with $G_i\in\mathcal{S}$ for $1\leqq i\leqq m$.

First suppose $G_i=]-\infty,e_i[\in\mathcal{S}_1$ for all i. If condition (α) in lemma 1 does not hold, we can find for each $x\in X$ a number $i_x\in\{1,\dots,m\}$ such that $f_{i_x}(x)\geqq e_{i_x}$. We set $g_i=f_i\vee v_{r_i}$ so that $H(g_i)=r_i$ and $g_{i_x}(x)\geqq e_{i_x}$. Choose $k\geqq 1$ so that $1+k(n-1)\geqq m$, then there are $\alpha_1,\dots,\alpha_m$, all $\geqq 1$ such that $\alpha_1+\dots+\alpha_m=$ $=1+k(n-1)$. We set $g=\sum_i\alpha_ig_i$ and $r=\sum_i\alpha_ir_i$. For $1\leqq j\leqq m$ define $e_j'=$ $=\sum_i\alpha_is_{ji}$ where $s_{ji}=r_i$ if $j\neq i$ and $s_{ii}=e_i$. Then $r<e_j'$ for all j; so also $r<e'$ if $e'=\inf\{e_1',\dots,e_m'\}$.

We now have $g(x)\geqq e_{i_x}'\geqq e'$ for all x; consequently $r=\sum_i\alpha_ir_i=\sum_i\alpha_iH(g_i)=$ $=H(\sum_i\alpha_ig_i)=H(g)\geqq H(e')=e'>r$, a contradiction.

Finally, the general case may easily be reduced to this particular situation.

References

[1] P. Brucker, Verbände stetiger Funktionen und kettenwertige Homomorphismen, *Math. Ann.*, **187** (1970), 77—84.
[2] P. Brucker, Eine Charakterisierung K-kompakter topologischer Räume, *Monatshefte für Mathematik*, **75** (1971), 14—25.
[3] P. Brucker, Dualität zwischen Kategorien topologischer Räume und Kategorien von K-Verbänden, *Monatshefte für Mathematik*, **76** (1972), 385—397.
[4] K. P. Chew, Structures of continuous functions IX, Homomorphisms of some function rings, *Bull. Acad. Pol. Sci., Sér. Sci. Math. Astr. Phys.*, **19** (1971), 485—489.
[5] K. P. Chew, N-compact spaces as limits of inverse systems of discrete spaces, *J. Austr. Math. Soc.*, **14** (1972), 467—469.
[6] K. P. Chew and S. Mrówka, Structures of continuous functions XI, Some type of homomorphisms of general structures of continuous functions, *Bull. Acad. Pol. Sci., Sér. Sci. Math. Astr. Phys.*, **19** (1971), 1023—1026.
[7] R. Engelking and S. Mrówka, On E-compact spaces, *ibid.*, **6** (1958), 429—436.
[8] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand (Princeton, 1960).
[9] W. Govaerts, Representation and determination problems: a case study, *Bull. Acad. Pol. Sci. Sér. Sci. Math. Astr. Phys.*, **24** (1976), 57—59.
[10] W. Govaerts, A modified notion of E-compactness, *ibid.*, 61—64.
[11] W. Govaerts, A separation axiom for the study of function space structures, *ibid.*, 65—69.
[12] G. Grätzer, *Universal algebra*, Van Nostrand (Princeton, 1968).
[13] M. Henriksen, On the equivalence of the ring, lattice and semigroup of continuous functions, *Proc. Amer. Math. Soc.*, **7** (1956), 959—960.
[14] E. Hewitt, Rings of real-valued continuous functions I, *Trans. Amer. Math. Soc.*, **64** (1948), 45—99.
[15] I. Kaplansky, Topological rings, *Amer. J. Math.*, **69** (1947), 153—183.
[16] S. Mrówka, Further results on E-compact spaces I, *Acta Math.*, **120** (1968), 161—185.
[17] S. Mrówka, Structures of continuous functions I, *Acta Math. Acad. Sci. Hungar.*, **21** (1970), 239—259.
[18] R. S. Pierce, Rings of integer-valued continuous functions, *Trans. Amer. Math. Soc.*, **100** (1961), 371—394.
[19] Z. Semadeni, *Banach spaces of continuous functions* (vol. I) (Warszawa, 1971).
[20] J. Timm, *Kommutative n-Gruppen*, Diss. (Hamburg, 1967).
[21] P. Zenor, Extending completely regular spaces with inverse limits, *Glasnik Matematički*, Ser III **5** (1970), 157—162.

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SEMINARIE VOOR HOGERE ANALYSE
RIJKSUNIVERSITEIT GENT
KRIJGSLAAN 271
B—9000 GENT
BELGIUM

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PACKING OF SPHERES IN SPACES OF CONSTANT CURVATURE

By
K. BÖRÖCZKY (Budapest)

§ 1. Introduction

The various investigations concerning packings of spheres reach back to LAGRANGE [10] and GAUSS [9] whose results implicitly involve the solution of the problem of densest lattice packing of circles and spheres, respectively. The problem of densest packing of equal circles in the Euclidean plane, without any restriction on the regularity of the arrangement, was solved by THUE [17]. The densest packing of equal spheres in the n-dimensional Euclidean space is not known for $n \geqq 3$. BLICHFELD [1] gave the upper bound $\frac{n+2}{2} \cdot 2^{-n/2}$ for the density of equal spheres in the n-dimensional Euclidean space, which was subsequently improved by RANKIN [12] and ROGERS [14]. Let d_n denote the density of $n+1$ equal spheres mutually touching one another with respect to the simplex spanned by the centres of the spheres. ROGERS [14] proved that d_n is an upper bound for the density of any packing of equal spheres in the n-dimensional Euclidean space. For large values of n, SIDELNIKOV [15] gave a better bound than d_n. Sidelnikov's bound was receently improved by LEVENSTEIN [11].

The analogous problem on the surface of a sphere was raised by TAMMES [16]. For the density of packing equal circles on the sphere, L. FEJES TÓTH [6] gave an upper bound which is exact in some cases. L. FEJES TÓTH [7], [8] and COXETER [4] extended the investigations to spherical and hiperbolic n-spaces. We shall call a spherical, Euclidean or hyperbolic space a space of constant curvature. Without loss of generality we shall assume that the curvature of the space is 1, 0 and -1, respectively.

In an n-dimensional space of constant curvature let $d_n(r)$ be the density of $n+1$ spheres of radius r mutually touching one another with respect to the simplex spanned by the centres of the spheres. FEJES TÓTH [7] and COXETER [4] conjectured that in an n-dimensional space of constant curvature the density of packing spheres of radius r cannot exceed $d_n(r)$. In Euclidean spaces this conjecture has been confirmed by the above mentioned result of ROGERS [14]. The two-dimensional spherical case was formerly settled by FEJES TÓTH [6]. As to the hyperbolic space, I observed (see [3]) that here no reasonable notion of density concerning the entire space can be defined. Thus in the hyperbolic space the formulation of the above conjecture needs a correction.

The main result of our paper is the following

THEOREM 1. *In an n-dimensional space of constant curvature, consider a packing of spheres of radius r. In spherical space suppose that $r<\pi/4$. Then the density of each sphere in its $D-V$ cell cannot exceed the density of $n+1$ spheres of radius r mutually touching one another with respect to the simplex spanned by their centres.*

The *Dirichlet—Voronoi cell,* or in short the *$D-V$ cell* associated with a sphere of a packing consists of those points of the space which lie nearer to the centre of this sphere than to the centre of any other sphere of the packing.

Some years ago I proved the special case of this theorem when $n=3$ (see BÖRÖCZKY—FLORIAN [2]). In spherical spaces the problem of densest packing of equal spheres of radii greater than or equal to $\pi/4$ was previously settled by several authors (see e.g. RANKIN [13]).

§ 2. Orthoschemes and O-simplices

In an n-dimensional space of constant curvature a simplex $A_0A_1\dots A_n$ is called an *orthoscheme* if for $0<i<n$ the i-dimensional subspace $A_0\dots A_i$ is totally orthogonal to the $(n-i)$-dimensional subspace $A_i\dots A_n$ with the additional condition that in spherical space $A_0A_i<\pi/2$ $(0<i\leqq n)$ (Fig. 1). Two subspaces are *totally orthogonal* if together they span the whole space and any two lines from each of them are perpendicular. In what follows we shall write the vertices of an orthoscheme always in accordance with the above definition. Note that by the symmetry of the definition also the opposite order $A_n\dots A_0$ can be used.

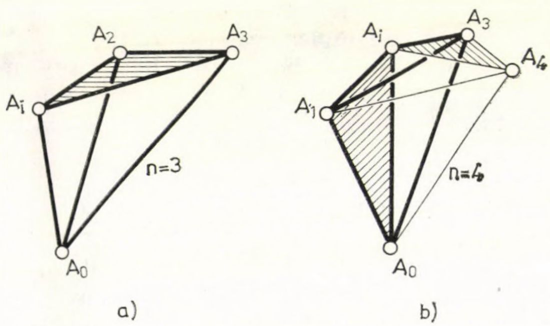

Fig. 1

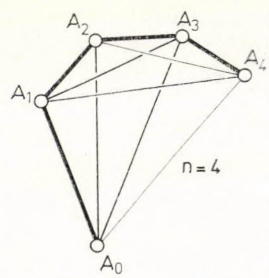

Fig. 2

REMARK 1. *We shall consider a point and a line segment as a* 0- *and* 1-*dimensional orthoscheme, respectively.*

REMARK 2. *The edges A_0A_1, A_1A_2, ..., $A_{n-1}A_n$ of the orthoscheme $A_0A_1\dots A_n$ are pair-wise perpendicular, and conversely, the fact that in a simplex $A_0\dots A_n$ these edges are mutually perpendicular implies that the simplex is an orthoscheme* (Fig. 2).

REMARK 3. *If the point B runs along the polygonal line $A_0A_1\dots A_n$ from A_0 to A_n then the distance A_0B increases and the distance A_nB decreases.*

REMARK 4. *The convex hull of $m\leqq n$ vertices of an n-dimensional orthoscheme forms an $(m-1)$-dimensional orthoscheme which is a cell of the original orthoscheme.*

REMARK 5. *An n-dimensional orthoscheme can be constructed also in the following way: Let $A_0\dots A_i$ be an i-dimensional orthoscheme with $0<i<n$ and consider the*

$(n-i)$-dimensional subspace through A_i totally orthogonal to the subspace $A_0 \dots A_i$. Let $A_i \dots A_n$ be an $(n-i)$-dimensional orthoscheme in this subspace. Then $A_0 A_1 \dots A_n$ is an n-dimensional orthoscheme which is the convex hull of $A_0 \dots A_i$ and $A_i \dots A_n$.

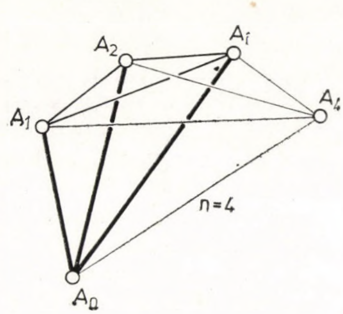

Fig. 3

LEMMA 1. *In an n-dimensional space of constant curvature the simplex $A_0 A_1 \dots A_n$ is an orthoscheme if and only if for any i, $0<i<n$, the orthogonal projection of A_0 onto the $(n-i)$-dimensional subspace $A_i \dots A_n$ is A_i.*

An analogous statement holds for the point A_n and the i-dimensional subspace $A_0 \dots A_i$.

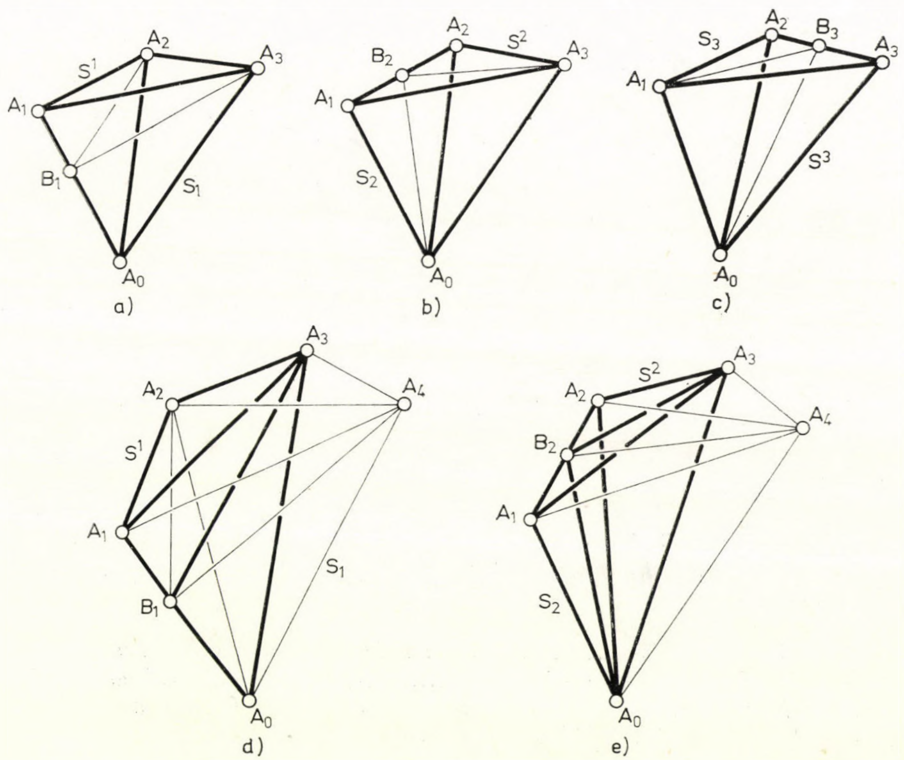

Fig. 4

It is obvious that any orthoscheme fulfils the conditions of the lemma. Suppose that for the simplex $A_0 A_1 \dots A_n$ the vertex A_i is identical with the orthogonal projection of A_0 onto the $(n-i)$-dimensional subspace $A_i \dots A_n$ for any $0<i<n$. If $0<j \leqq i$ then the line $A_0 A_j$ is orthogonal to the $(n-j)$-dimensional subspace $A_j \dots A_i \dots A_n$ thus it is orthogonal also to the subspace $A_i \dots A_n$ (Fig. 3). It follows that the i-dimensional subspace $A_0 \dots A_i$ spanned by the linearly independent

directions $A_0A_1, \ldots, A_0A_i$ is totally orthogonal to the $(n-i)$-dimensional subspace $A_i \ldots A_n$, i.e. $A_0 \ldots A_n$ is an orthoscheme.

In an n-dimensional space of constant curvature let $A_0A_1 \ldots A_n$ be an orthoscheme. Consider an interior point B_i of the edge $A_{i-1}A_i$ $(0<i\leqq n)$. Disect the orthoscheme $A_0A_1 \ldots A_n$ by the $(n-1)$-dimensional hyperplane $A_0 \ldots A_{i-2}B_iA_{i+1} \ldots A_n$ into the simplices $S_i=A_0 \ldots A_{i-1}B_iA_{i+1} \ldots A_n$ and $S^i=A_0 \ldots A_{i-2}B_iA_i \ldots A_n$ (Fig. 4). Here we use the natural convention that $S^1=A_0 \ldots A_{-1}B_1A_1 \ldots A_n=B_1A_1 \ldots A_n$ and $S_n=A_0 \ldots A_{n-1}B_nA_{n+1} \ldots A_n=A_0 \ldots A_{n-1}B_n$. Accordingly $A_0 \ldots A_{-1}$ and $A_{n+1} \ldots A_n$ will denote the empty set. The simplices of type S_i and S^i $(0<i\leqq n)$ obtained by disecting an orthoscheme in two simplices in the above described way will be called *O-simplices.*

REMARK 6. *Any orthoscheme is an O-simplex. The O-simplices $B_1A_1 \ldots A_n$ and $A_0A_1 \ldots A_{n-1}B_n$ are orthoschemes* (Fig. 5).

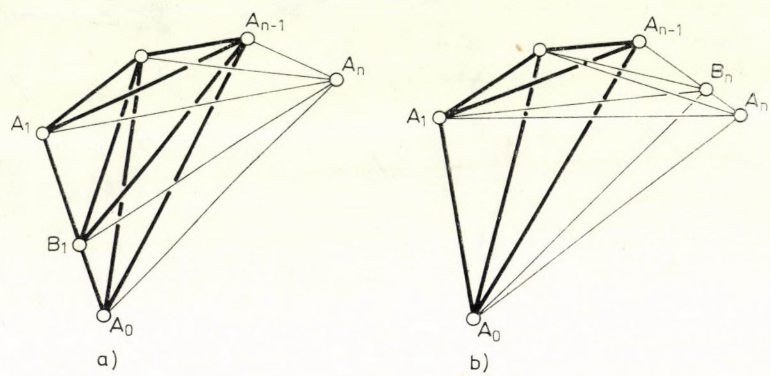

Fig. 5

REMARK 7. *The O-simplices S^i and S_{n-i+1} are of the same type. This can be seen immediately by writing the vertices of the orthoscheme $A_0A_1 \ldots A_n$ in the converse order $A_nA_{n-1} \ldots A_0$.*

LEMMA 2. *Any O-simplex which is not an orthoscheme can be obtained from two different orthoschemes: The O-simplex $A_0 \ldots A_{i-1}B_iA_{i+1} \ldots A_n$ obtained by disecting the orthoscheme $A_0 \ldots A_n$ can be obtained also by disecting an orthoscheme $A_0 \ldots A_{i-1}\underline{A}_iA_{i+1} \ldots A_n$ for which $A_0\underline{A}_i<A_0B_i<A_0A_i$.*

By Remarks 6 and 7, it suffices to consider only O-simplices of type $S_i=A_0 \ldots A_{i-1}B_iA_{i+1} \ldots A_n$ $(0<i<n)$. Since in the right triangle $B_iA_iA_{i+1}$ the angle at B_i is acute, we have $\sphericalangle A_{i-1}B_iA_{i+1}<\pi/2$ (Fig. 6). Let $\underline{A}_i$ be the foot of the perpendicular drawn from the point A_{i-1} to the line $A_{i+1}B_i$. Then we have $A_0\underline{A}_i<A_0B_i<A_0A_i$. Since $A_0 \ldots A_{i-1}B_iA_{i+1} \ldots A_n$ is one of the simplices obtained by disecting the simplex $A_0 \ldots A_{i-1}\underline{A}_iA_{i+1} \ldots A_n$ into two simplices by the hyperplane $A_0 \ldots A_{i-2}B_iA_{i+1} \ldots A_n$, it suffices to show that the simplex $A_0 \ldots A_{i-1}\underline{A}_iA_{i+1} \ldots A_n$ is an orthoscheme (Fig. 7).

$A_{i-1}\underline{A}_iA_{i+1}$ is a right triangle lying in the plane $A_{i-1}A_iA_{i+1}$. Since $A_{i-1}A_i\dots A_n$ is an $(n-i+1)$-dimensional orthoscheme, the plane $A_{i-1}A_iA_{i+1}$ and the $(n-i-1)$-dimensional subspace $A_{i+1}\dots A_n$ are totally orthogonal in the subspace $A_{i-1}A_i\dots A_n$. It follows, by Remark 5, that the convex hull $A_{i-1}\underline{A}_iA_{i+1}\dots A_n$ of the orthoschemes $A_{i-1}\underline{A}_iA_{i+1}$ and $A_{i+1}\dots A_n$ is an orthoscheme.

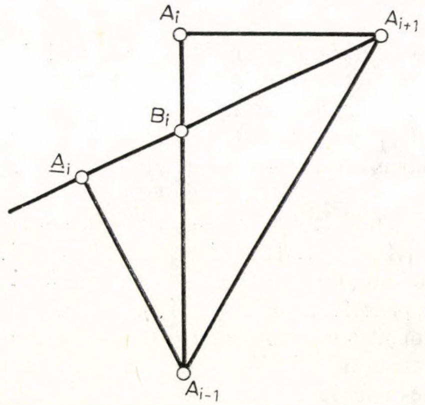

Fig. 6

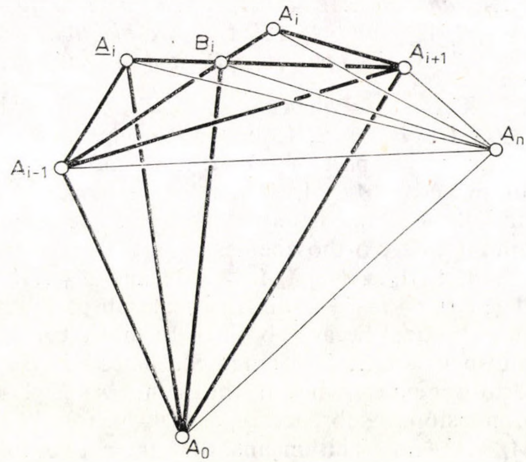

Fig. 7

The orthoschemes $A_{i-1}\underline{A}_iA_{i+1}\dots A_n$ and $A_{i-1}A_i\dots A_n$ lie in the same subspace which is in n-space totally orthogonal to the subspace of the $(i-1)$-dimensional orthoscheme $A_0\dots A_{i-1}$. Thus referring again to Remark 5, we see that $A_0\dots$ $\dots A_{i-1}\underline{A}_iA_{i+1}\dots A_n$ is an orthoscheme.

In what follows the points $\underline{A}_i$ and $A_i=\overline{A}_i$ belonging to the O-simplex $S_i=A_0\dots A_{i-1}B_iA_{i+1}\dots A_n$ will play a distinguished role. We shall call these points *lower and upper orthopoints* of the vertex B_i, respectively. We shall use the symbol $A_0\dots A_{i-1}B_iA_{i+1}\dots A_n$ with $0<i<n$ also for orthoschemes. By convention the lower and upper orthopoints of a vertex of an orthoscheme are identical with the corresponding vertex: $\underline{A}_i=B_i=\overline{A}_i$.

REMARK 8. *If $A_0\dots A_{i-1}B_0A_{i+1}\dots A_n$ is an O-simplex and the point C runs along the polygonal line $A_0\dots A_{i-1}B_iA_{i+1}\dots A_n$ from A_0 to A_n then the distance A_0C increases and the distance A_nC decreases.*

By Remark 3 the statement is true if $A_0\dots A_{i-1}B_iA_{i+1}\dots A_n$ is an orthoscheme. In the opposite case by Lemma 2, the polygonal line $A_0\dots A_{i-1}B_iA_{i+1}\dots A_n$ can be split into two polygonal lines $A_0\dots A_{i-1}B_i$ and $B_iA_{i+1}\dots A_n$, the first belonging to the orthoscheme $A_0A_1\dots A_n$, the second to the orthoscheme $A_0\dots A_{i-1}\underline{A}_iA_{i+1}\dots A_n$. Thus the monotonicity of the distances A_0C and A_nC follows again from Remark 3.

It is natural to call the vertices A_0 and A_n extreme vertices of the O-simplex $A_0\dots A_{i-1}B_iA_{i+1}\dots A_n$.

REMARK 9. *If $A_0\dots A_{i-1}B_iA_{i+1}\dots A_n$ is an O-simplex and the point C runs along the polygonal line $A_0\dots A_{i-1}B_iA_{i+1}\dots A_n$ from A_0 to A_n then the distance A_jC decreases till C reaches A_j and increases afterwards $(0<j<n,\ j\neq i)$.*

The polygonal line $A_0 \dots A_{i-1} B_i A_{i+1} \dots A_n$ can be split into two polygonal lines $A_0 \dots A_j$ and $A_j \dots A_n$ determining two O-simplices with the common extreme vertex A_j. Hence our statement follows from Remark 8.

LEMMA 3. *In an n-dimensional space of constant curvature let* $A_0 \dots A_{i-1} B_i A_{i+1} \dots \dots A_n$ *and* $A_0 \dots A_{i-1} C_i A_{i+1} \dots A_n$ *be two O-simplices. If* $0<i<n$ *then the planes* $A_{i-1} B_i A_{i+1}$ *and* $A_{i-1} C_i A_{i+1}$ *are identical. For* $i=0$ *the lines* $B_0 A_1$ *and* $C_0 A_1$ *and for* $i=n$ *the lines* $A_{n-1} B_n$ *and* $A_{n-1} C_n$ *are identical.*

The cases when $i=0$ or $i=n$ are trivial. Suppose that $0<i<n$, and let $A_0 \dots A_{i-1} B_i A_{i+1} \dots A_n$ be an O-simplex which is not an orthoscheme. Then this simplex is a part of an orthoscheme $A_0 \dots A_i \dots A_n$ such that B_i is an inner point of the edge $A_{i-1} A_i$. Then the planes $A_{i-1} B_i A_{i+1}$ and $A_{i-1} A_i A_{i+1}$ are identical. It follows immediately that we can restrict ourselves to the case when both O-simplices are orthoschemes.

Let $A_0 \dots A_{i-1} A_i A_{i+1} \dots A_n$ and $A_0 \dots A_{i-1} A_i' A_{i+1} \dots A_n$ be two orthoschemes. Then the $(n-i+1)$-dimensional subspaces $A_{i-1} A_i A_{i+1} \dots A_n$ and $A_{i-1} A_i' A_{i+1} \dots A_n$ are identical because both of them are totally orthogonal to the $(i-1)$-dimensional subspace $A_0 \dots A_{i-1}$ (Fig. 8). Since $A_{i-1} A_i A_{i+1} \dots A_n$ and $A_{i-1} A_i' A_{i+1} \dots A_n$ are orthoschemes lying in the same $(n-i+1)$-dimensional subspace, the $(n-i-1)$-dimensional subspace $A_{i+1} \dots A_n$ is totally orthogonal both to $A_{i-1} A_i A_{i+1}$ and $A_{i-1} A_i' A_{i+1}$. This means that these two planes are identical.

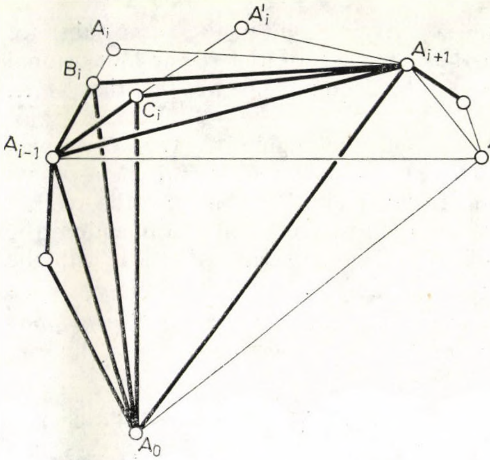

Fig. 8

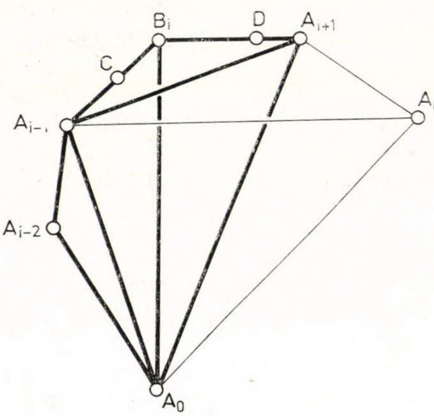

Fig. 9

LEMMA 4. *Let* $A_0 \dots A_{i-1} B_i A_{i+1} \dots A_n$ *be an n-dimensional O-simplex* $(0 \leqq i \leqq n)$. *Let C and D be points of the edges* $A_{i-1} B_i$ *and* $B_i A_{i+1}$ *respectively. Then the* $(n-1)$-*dimensional subspaces* $A_0 \dots A_{i-2} C A_{i+1} \dots A_n$ *and* $A_0 \dots A_{i-1} D A_{i+2} \dots A_n$ *intersect the O-simplex* $A_0 \dots A_{i-1} B_i A_{i+1} \dots A_n$ *in the* $(n-1)$-*dimensional O-simplices* $A_0 \dots \dots A_{i-2} C A_{i+1} \dots A_n$ *and* $A_0 \dots A_{i-1} D A_{i+2} \dots A_n$, *respectively* (Fig. 9).

By Remark 7 it suffices to prove that the $(n-1)$-dimensional subspace $A_0 \dots A_{i-2}CA_{i+1} \dots A_n$ intersects the O-simplex $A_0 \dots A_{i-1}B_iA_{i+1} \dots A_n$ in the $(n-1)$-dimensional O-simplex $A_0 \dots A_{i-2}CA_{i+1} \dots A_n$. Let $\bar{A}_i$ be the upper orthopoint of the vertex B_i. If $i=n$ then $\bar{A}_n=B_n$ and $A_0 \dots A_{i-1}B_iA_{i+1} \dots A_n=A_0 \dots A_{n-1}B_n$ is an orthoscheme (Fig. 10). The $(n-2)$-dimensional subspace $A_0 \dots A_{n-2}$ is totally orthogonal to the plane $A_{n-2}A_{n-1}\bar{A}_n$ and therefore it is totally orthogonal to the segment $A_{n-2}C$ in the $(n-1)$-dimensional subspace $A_0 \dots A_{n-2}C$. Hence Remark 5 implies that $A_0 \dots A_{n-2}C$ is an $(n-1)$-dimensional orthoscheme.

The case when $i=1$ can be settled in a similar way as the case $i=n$. The segment CA_2 is totally orthogonal to the $(n-2)$-dimensional subspace of the orthoscheme $A_2 \dots A_n$ in the $(n-1)$-dimensional subspace $CA_2 \dots A_n$ and therefore, by Remark 5, $CA_2 \dots A_n$ is an $(n-1)$-dimensional orthoscheme (Fig. 10).

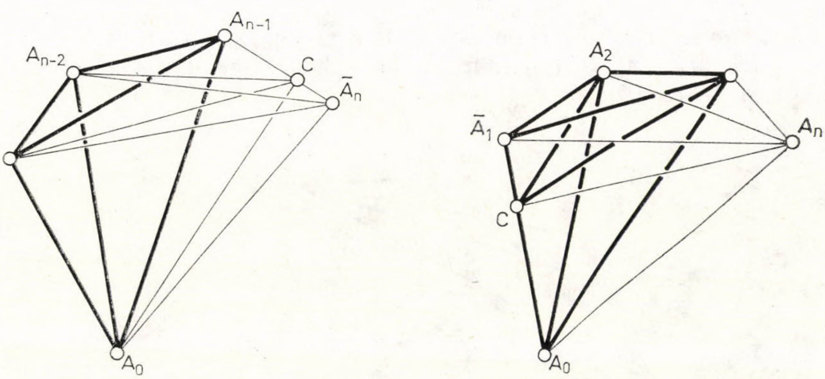

Fig. 10

The Lemma is obvious also in the cases when $1<i<n$ and $C=A_{i-1}$ or $C=\bar{A}_i$. Now the intersections under consideration are the $(n-1)$-dimensional orthoschemes $A_0 \dots A_{i-1}A_{i+1} \dots A_n$ and $A_0 \dots A_{i-2}\bar{A}_i \dots A_n$, respectively.

Consider now the case when $1<i<n$ and C is an inner point of the edge $A_{i-1}\bar{A}_i$. Let $\underline{A}_i$ be the lower orthopoint of the vertex C of the O-simplex

$$A_0 \dots A_{i-1}CA_{i+1} \dots A_n$$

(Fig. 11).

The cell $A_0 \dots A_{i-2}\underline{A}_iA_{i+1} \dots A_n$ of the orthoscheme $A_0 \dots A_{i-1}\underline{A}_iA_{i+1} \dots A_n$ is an $(n-1)$-dimensional orthoscheme which is bisected by the $(n-2)$-dimensional subspace $A_0 \dots A_{i-2}CA_{i+1} \dots A_n$ into two $(n-1)$-dimensional O-simplices. From these two $(n-1)$-dimensional O-simplices the simplex $A_0 \dots$ $\dots A_{i-2}CA_{i+1} \dots A_n$ is nothing else but the intersection of the n-dimensional

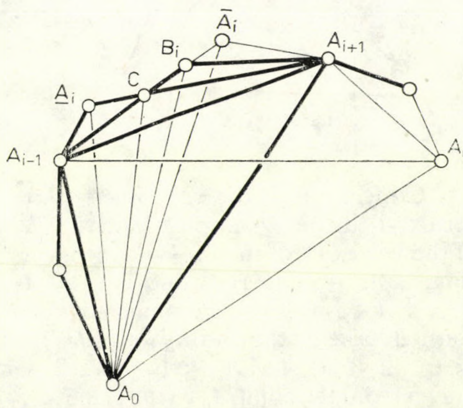

Fig. 11

O-simplex $A_0 \dots A_{i-1} B_i A_{i+1} \dots A_n$ and the $(n-1)$-dimensional subspace $A_0 \dots$ $\dots A_{i-2} C A_{i+1} \dots A_n$.

This completes the proof of Lemma 4.

LEMMA 5. *Let $A_0 \dots A_{i-1} B_i A_{i+1} \dots A_n$ be an n-dimensional O-simplex. For $0<i\leqq n$ let C be a point on the edge $A_{i-1} B_i$ and for $0\leqq i<n$ let D be a point on the edge $B_i A_{i+1}$. Let the points C and D move on the edges $A_{i-1} B_i$ and $B_i A_{i+1}$, resp. so that the distances $A_0 C$ and $A_0 D$ increase. Then the distances between A_0 and the upper and lower orthopoint of the vertex C of the $(n-1)$-dimensional O-simplex $A_0 \dots A_{i-2} C A_{i+1} \dots A_n$ as well as between A_0 and the upper and lower orthopoint of the vertex D of the $(n-1)$-dimensional O-simplex $A_0 \dots A_{i-1} D A_{i+2} \dots A_n$ increase.*

Let C_1 and C_2 be two points on the edge $A_{i-1} B_i$ such that $A_0 C_1 < A_0 C_2$. In view of Remark 3 we have $A_{i-1} C_1 < A_{i-1} C_2$.

We saw in the proof of Lemma 4 that if $i=1$ or $i=n$ then the $(n-1)$-dimensional simplex $A_0 \dots A_{i-2} C A_{i+1} \dots A_n$ is an orthoscheme, thus in this cases the lemma is trivial (Fig. 10).

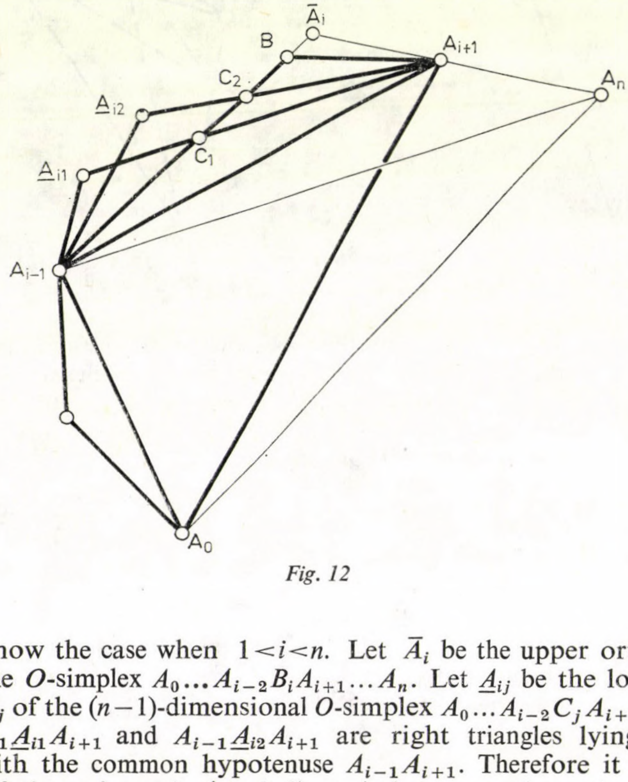

Fig. 12

Consider now the case when $1<i<n$. Let $\bar{A}_i$ be the upper orthopoint of the vertex B_i of the O-simplex $A_0 \dots A_{i-2} B_i A_{i+1} \dots A_n$. Let $\underline{A}_{ij}$ be the lower orthopoint of the vertex C_j of the $(n-1)$-dimensional O-simplex $A_0 \dots A_{i-2} C_j A_{i+1} \dots A_n$ $(j=1, 2)$ (Fig. 12). $A_{i-1} \underline{A}_{i1} A_{i+1}$ and $A_{i-1} \underline{A}_{i2} A_{i+1}$ are right triangles lying in the plane $A_{i-1} \bar{A}_i A_{i+1}$ with the common hypotenuse $A_{i-1} A_{i+1}$. Therefore it follows by the construction of the points $\underline{A}_{ij}$ $(j=1, 2)$ and the assumption $A_{i-1} C_1 < A_{i-1} C_2 <$ $< A_{i-1} \bar{A}_i$ that $A_{i-1} \underline{A}_{i1} < A_{i-1} \underline{A}_{i2}$. Observe that A_{i-1} is the foot of the perpendicular drawn from the point A_0 to the plane $A_{i-1} \bar{A}_i A_{i+1}$. Therefore the inequality $A_{i-1} \underline{A}_{i1} <$ $< A_{i-1} \underline{A}_{i2}$ implies that $A_0 \underline{A}_{i1} < A_0 \underline{A}_{i2}$.

This proves the lemma for the lower orthopoint of the vertex C of $A_0 \ldots A_{i-2} C A_{i+1} \ldots A_n$. The remaining three statements of the lemma can be proved in a similar way. We leave the details to the interested reader.

LEMMA 6. *Let $A_0 \ldots A_{i-1} B_{i1} A_{i+1} \ldots A_n$ and $A_0 \ldots A_{i-1} B_{i2} A_{i+1} \ldots A_n$ be two O-simplices lying on the same side of the hyperplane $A_0 \ldots A_{i-1} A_{i+1} \ldots A_n$ $(0<i<n)$. Suppose that for the lower orthopoints $\underline{A}_{ij}$ and for the upper orthopoints $\bar{A}_{ij}$ of the vertices B_{ij}, $j=1, 2$, we have $A_0 \underline{A}_{i1} < A_0 A_{i2}$ and $A_0 \bar{A}_{i1} < A_0 \bar{A}_{i2}$. Then the segments $A_{i-1} B_{i2}$ and $B_{i1} A_{i+1}$ intersect one another in inner points.*

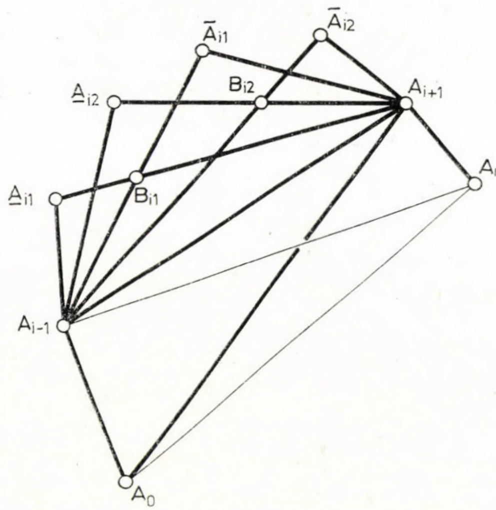

Fig. 13

In view of Lemma 3 the points A_{i-1}, A_{i+1}, $\underline{A}_{i1}$, $\underline{A}_{i2}$, $\bar{A}_{i1}$, $\bar{A}_{i2}$, B_{i1}, B_{i2} lie in the same plane. Further, by the assumptions of the lemma, the points $\underline{A}_{i1}$, $\underline{A}_{i2}$, $\bar{A}_{i1}$, $\bar{A}_{i2}$, B_{i1}, B_{i2} lie on the same side of the line $A_{i-1} A_{i+1}$ (Fig. 13). Since the foot of the perpendicular drawn from the point A_0 to this plane is A_{i-1}, the inequalities $A_0 \underline{A}_{i1} < A_0 \underline{A}_{i2}$ and $A_0 \bar{A}_{i1} < A_0 \bar{A}_{i2}$ imply that $A_{i-1} \underline{A}_{i1} < A_{i-1} \underline{A}_{i2}$ and $A_{i-1} \bar{A}_{i1} < A_{i-1} \bar{A}_{i2}$. Since $A_{i-1} \underline{A}_{i1} A_{i+1}$, $A_{i-1} \underline{A}_{i2} A_{i+1}$, $A_{i-1} \bar{A}_{i1} A_{i+1}$ and $A_{i-1} \bar{A}_{i2} A_{i+1}$ are right triangles with the same hypotenuse, these last two inequalities are equivalent with the following ones:

$$\sphericalangle A_{i-1} A_{i+1} B_{i1} = \sphericalangle A_{i-1} A_{i+1} \underline{A}_{i1} < \sphericalangle A_{i-1} A_{i+1} \underline{A}_{i2} = \sphericalangle A_{i-1} A_{i+1} B_{i2},$$

$$\sphericalangle A_{i+1} A_{i-1} B_{i1} = \sphericalangle A_{i+1} A_{i-1} \bar{A}_{i1} > \sphericalangle A_{i+1} A_{i-1} \bar{A}_{i2} = \sphericalangle A_{i+1} A_{i-1} B_{i2}.$$

It follows that the sides $A_{i-1} B_{i2}$ and $B_{i1} A_{i+1}$ of the triangles $A_{i-1} B_{i2} A_{i+1}$ and $A_{i-1} B_{i1} A_{i+1}$ intersect one another in inner points.

§ 3. Densities of spheres with respect to O-simplices

The density of an n-dimensional sphere S with respect to a body T is defined by $d(T, S) = \frac{S \cap T}{T}$. Here and in what follows we denote a convex body and its n-dimensional volume with the same symbol.

Let $T = A_0 \ldots A_n$ be an orthoscheme and S a sphere which does not intersect the hyperplane $A_1 \ldots A_n$. Our aim is to give an upper bound for the density of S with respect to T in terms of lower bounds of the distances $A_0 A_1, \ldots, A_0 A_n$. One of the main ideas of the proof is to consider the "density" of an n-dimensional sphere S with respect to a k-dimensional simplex $T = P_0 \ldots P_k$ for $k<n$. We consider the k-dimensional simplex T as a degenerated n-dimensional simplex

$P_0 \dots P_k P_{k+1} \dots P_n$ with $P_{k+1} = \dots = P_n = P_k$ and define the *limiting density* $d(T, P_k, S)$ by

$$d(T, P_k, S) = \lim_{\substack{P_i \to P_k \\ k < i \leqq n}} d(P_0 \dots P_k P_{k+1} \dots P_n, S).$$

Let $P_{k+1}, \dots, P_{k+m}$ $(m \geqq n-k)$ be points tending to the point P_k such that the convex hull K of the points $P_0, \dots, P_{k+m}$ is a convex polytope. It is obvious that we have $d(T, P_k, S) = \lim d(K, S)$.

Using this we easily obtain the following alternative definition of the limiting density $d(T, P_k, S)$:

(i) Rotate the simplex T about the subspace $P_0 \dots P_{k-1}$ of the n-dimensional space, obtaining a body B. Then we have $d(T, P_k, S) = d(B, S)$ (Fig. 14).

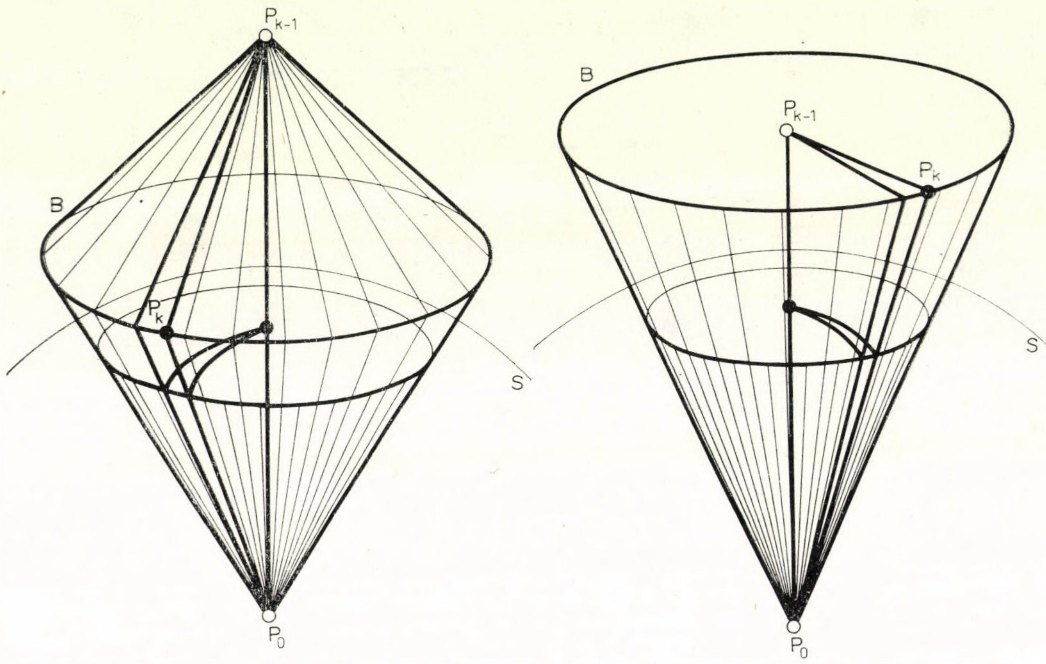

Fig. 14

The following lemma is a simple consequence of the definition of the limiting density (Fig. 15):

LEMMA 7. *In an n-dimensional space of constant curvature let* $T = A_0 \dots A_{i-1} B_i A_{i+1} \dots A_n$ *be an O-simplex of volume V and S a sphere with centre* A_0 *which does not intersect the hyperplane* $A_1 \dots A_{i-1} B_i A_{i+1} \dots A_n$. *Let C and D be the points of the segments* $A_{i-1}B_i$ $(1 < i \leqq n)$ *and* $B_i A_{i+1}$ $(0 < i < n)$, *respectively, such that the simplices* $A_0 \dots A_{i-1} C A_{i+1} \dots A_n$ *and* $A_0 \dots A_{i-1} D A_{i+1} \dots A_n$ *have volume* v $(0 \leqq v \leqq V)$. *Writing* $f(v) = d(A_0 \dots A_{i-1} C A_{i+1} \dots A_n, C, S)$ *and* $g(v) =$

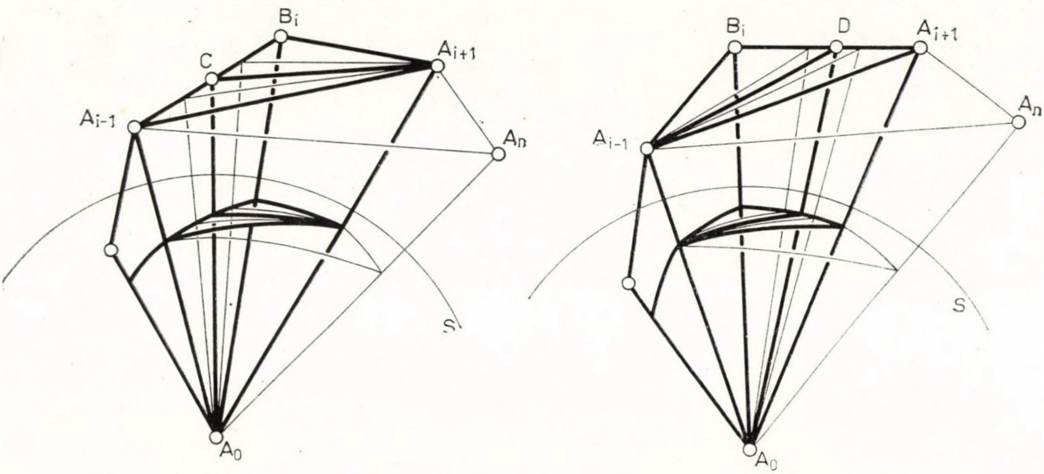

Fig. 15

$=d(A_0 \dots A_{i-1}DA_{i+1} \dots A_n, D, S)$ *we have*

$$d(T, S) = \frac{1}{V} \int_0^V f(v)\, dv = \frac{1}{V} \int_0^V g(v)\, dv.$$

LEMMA 8. *Let* $T=A_0 \dots A_{i-1}B_iA_{i+1} \dots A_k$ *be a k-dimensional O-simplex in an n-dimensional space of constant curvature with* $2 \leqq k<n$ *and S a sphere centred at* A_0 *not intersecting the subspace* $A_1 \dots A_k$. *Let V be the volume of the body obtained by rotating T about the* $(k-1)$*-dimensional subspace* $A_0 \dots A_{i-1}A_{i+1} \dots A_k$. *For* $1<i \leqq k$ *let C be the point of the edge* $A_{i-1}B_i$ *for which the volume of the body obtained by rotating* $A_0 \dots A_{i-1}CA_{i+1} \dots A_k$ *about the subspace* $A_0 \dots A_{i-1}A_{i+1} \dots A_k$ *is equal to v* $(0 \leqq v \leqq V)$. *For* $0<i<k$ *let D be the point of the edge* B_iA_{i+1} *for which the volume of the body obtained by rotating* $A_0 \dots A_{i-1}DA_i \dots A_k$ *about the subspace* $A_0 \dots A_{i-1}A_{i+1} \dots A_k$ *is equal to v* $(0 \leqq v \leqq V)$. *Writing* $f(v)=d(A_0 \dots A_{i-2}CA_{i+1} \dots \dots A_k, C, S)$ *and* $g(v)=d(A_0 \dots A_{i-1}DA_{i+2} \dots A_k, D, S)$ *we have*

$$d(T, B_i, S) = \frac{1}{V} \int_0^V f(v)\, dv = \frac{1}{V} \int_0^V g(v)\, dv.$$

The proof of the lemma is obvious if we use definition (i) of the limiting density (Fig. 16).

In the following lemma it will be convenient to use the symbol $d(P_0 \dots P_n, P_i, S)$ $(0 \leqq i \leqq n)$ to denote the density $d(P_0 \dots P_n, S)$.

LEMMA 9. *If* $T_1=A_0 \dots A_{i-1}B_{i1}A_{i+1} \dots A_k$ *and* $T_2=A_0 \dots A_{i-1}B_{i2}A_{i+1} \dots A_k$ *are two k-dimensional O-simplices* $(1 \leqq k \leqq n,\ 0<i \leqq k)$ *such that for the lower ortho-*

points $\underline{A}_{ij}$ *and for the upper orthopoints* $\bar{A}_{ij}$ *of the vertices* B_{ij}, $j=1,2$, *we have* $A_0\underline{A}_{i1}<A_0\underline{A}_{i2}$ *and* $A_0\bar{A}_{i1}<A_0\bar{A}_{i2}$, *and* S *is an* n*-dimensional sphere centred at* A_0 *which does not intersect the subspaces* $A_1\dots A_{i-1}B_{i1}A_{i+1}\dots A_k$ *and* $A_1\dots$ $\dots A_{i-1}B_{i2}A_{i+1}\dots A_k$ *then*

$$d(T_1, B_{i1}, S) \geqq d(T_2, B_{i2}, S).$$

The proof follows by induction on k. The lemma is obvious for $k=1$. We suppose that it is true for $k<m$ and prove it for $k=m$.

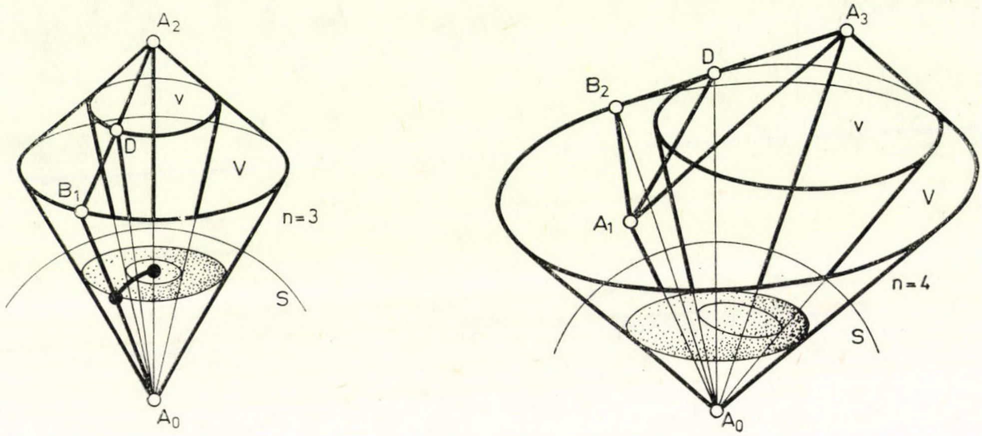

Fig. 16

Let P and Q be points of the segments $B_{i1}A_{i+1}$ $(0<i<m)$ and $A_{i-1}B_{i2}$ $(1<i\leqq m)$, resp. For $m=n$ let $v=v(P)$ denote the volume of $A_0\dots A_{i-1}PA_{i+1}\dots A_m$ and let $w=w(Q)$ denote the volume of $A_0\dots A_{i-1}QA_{i+1}\dots A_m$. If $m<n$ let $v(P)$ and $w(Q)$ be the volumes of the bodies obtained by rotating $A_0\dots A_{i-1}PA_{i+1}\dots A_m$ and $A_0\dots A_{i-1}QA_{i+1}\dots A_m$ about the subspace $A_0\dots A_{i-1}A_{i+1}\dots A_m$, respectively. Write

$$f(v) = d(A_0\dots A_{i-1}PA_{i+2}\dots A_m, P, S)$$

and

$$g(w) = d(A_0\dots A_{i-2}QA_{i+1}\dots A_m, Q, S).$$

If the point P moves on the segment $A_{i+1}B_{i1}$ from A_{i+1} in the direction of B_{i1} then by Remark 8 the distance A_0P decreases. Hence it follows by Lemma 5 that the distance of A_0 from both the upper and lower orthopoints of P of the $(m-1)$-dimensional O-simplex $A_0\dots A_{i-1}PA_{i+2}\dots A_m$, decreases. Therefore, by the induction hypothesis $f(v)=d(A_0\dots A_{i-1}PA_{i+2}\dots A_m, P, S)$ is an increasing function of v. In a similar way we see that $g(w)$ is a decreasing function of w.

We may assume without loss of generality that T_1 and T_2 lie in a common m-dimensional subspace on the same side of the subspace $A_0\dots A_{i-1}A_{i+1}\dots A_m$. If $0<i<m$ then by Lemma 6 the segments $A_{i-1}B_{i2}$ and $B_{i1}A_{i+1}$ intersect one

another in a point C (Fig. 17). First consider the case when $1<i<m$. Then by Lemmas 7 and 8 we have

$$d(A_0 \dots A_{i-1} B_{i1} A_{i+1} \dots A_m, B_{i1}, S) = \frac{1}{v(B_{i1})} \int_0^{v(B_{i1})} f(v)\,dv,$$

$$d(A_0 \dots A_{i-1} B_{i2} A_{i+1} \dots A_m, B_{i2}, S) = \frac{1}{w(B_{i2})} \int_0^{w(B_{i2})} g(w)\,dw$$

and

$$d(A_0 \dots A_{i-1} C A_{i+1} \dots A_m, C, S) = \frac{1}{v(C)} \int_0^{v(C)} f(v)\,dv = \frac{1}{w(C)} \int_0^{w(C)} g(w)\,dw.$$

Since $f(v)$ is an increasing and $g(w)$ a decreasing function, we have

$$d(A_0 \dots A_{i-1} B_{i1} A_{i+1} \dots A_m, B_{i1}, S) \geqq d(A_0 \dots A_{i-1} C A_{i+1} \dots A_m, C, S) \geqq$$
$$\geqq d(A_0 \dots A_{i-1} B_{i2} A_{i+1} \dots A_m, B_{i2}, S).$$

If $i=1$ then the inequalites

$$d(A_0 C A_2 \dots A_m, C, S) \leqq d(A_0 B_{11} \dots A_m, B_{11}, S)$$

can be seen in the same way. Now the relation

$$d(A_0 B_{12} \dots A_m, B_{12}, S) \leqq d(A_0 C A_2 \dots A_m, C, S)$$

does not follow in the above way but it is an obvious consequence of definition (i) of the limiting density (or the definition of the density if $m=n$) (Fig. 18).

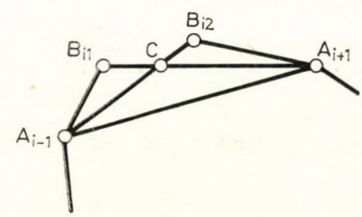

Fig. 17

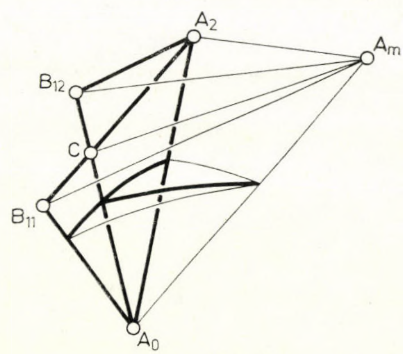

Fig. 18

Finally if $i=m$ then T_1 and T_2 are orthoschemes and the point B_{i1} $(=A_m)$ lies on the edge $A_{i-1}B_{i2}$. Now we have

$$d(A_0 \dots A_{i-1} B_{i1}, B_{i1}, S) = \frac{1}{w(B_{i1})} \int_0^{w(B_{i1})} g(w)\,dw \geqq$$
$$\geqq \frac{1}{w(B_{i2})} \int_0^{w(B_{i2})} g(w)\,dw = d(A_0 \dots A_{i-1} B_{i2}, B_{i2}, S).$$

This completes the proof of Lemma 9.

LEMMA 10. *In an n-dimensional space of constant curvature let* $T=A_0A_1\dots A_n$ *and* $\bar{T}=A_0\bar{A}_1\dots\bar{A}_n$ *be two orthoschemes and* S *a sphere centred at* A_0 *which does not intersect the hyperplanes* $A_1A_2\dots A_n$ *and* $\bar{A}_1\bar{A}_2\dots\bar{A}_n$. *Suppose that* $A_0A_i\geqq A_0\bar{A}_i$ *for* $i=1,\dots,n$. *Then we have* $d(T,S)\leqq d(\bar{T},S)$, *and equality holds only if* $A_0A_i=A_0\bar{A}_i$ *for all* i*'s.*

If $A_0A_i=A_0\bar{A}_i$ for $i=1,2,\dots,n$, then T and $\bar{T}$ are congruent and we have $d(T,S)=d(\bar{T},S)$. If T and $\bar{T}$ are not congruent then consider the least index i for which $A_0A_i>A_0\bar{A}_i$, and construct the orthoscheme $T'=A_0\dots A_{i-1}A_i'A_{i+1}\dots A_n$ for which $A_0A_i'=A_0\bar{A}_i$. By Lemma 9 we have $d(T,S)<d(T',S)$. Repeating this process until we obtain an orthoscheme congruent to $\bar{T}$, the density increases in each step. Thus we have indeed $d(T,S)\leqq d(\bar{T},S)$.

LEMMA 11. *In an n-dimensional space of constant curvature let* $T=A_0A_1\dots A_k$ *be a k-dimensional and* $\bar{T}=A_0\bar{A}_1\dots\bar{A}_n$ *an n-dimensional orthoscheme. Let* S *be a sphere centred at* A_0 *which does not intersect the subspaces* $A_1\dots A_k$ *and* $\bar{A}_1\dots\bar{A}_n$. *Suppose that* $A_0A_i\geqq A_0\bar{A}_i$ *for* $i=1,\dots,k-1$ *and* $A_0A_k\geqq A_0\bar{A}_n$. *Then we have* $d(T,A_k,S)\leqq d(\bar{T},S)$.

We choose a sequence of n-dimensional orthoscheme $O_j=A_0A_1\dots A_kB_{k+1}^j\dots B_n^j$ such that $A_0B_i^j>A_0\bar{A}_i$ for $i=k+1,\dots,n$ and $\lim_{i\to\infty} B_i^j=A_k$ $(i=k+1,\dots,n)$. Then we have, on the one hand, by Lemma 10 $d(O_j,S)\leqq d(\bar{T},S)$ $(j=1,2,\dots)$, and on the other hand, by the definition of the limiting density $d(T,A_k,S)=\lim_{j\to\infty} d(O_j,S)$. This proves the lemma.

§ 4. Further auxiliary results

In an n-dimensional space of constant curvature let $h_n(r)$ denote the distance of the centre of a regular simplex of edgelength $2r$ from the vertices of the simplex. In a 2-dimensional subspace let $h(r)$ denote the distance of the centre of a square of sidelenght $2r$ from its vertices. Note that in an n-dimensional space of constant curvature the distance of the centre of a regular crosspolytope of sidelength $2r$ from its vertices is also $h(r)$, since two pairs of opposite vertices of the crosspolytope form a square.

Let α be the angle spanned by a sphere S with centre O at a point A outside of S. In spherical space let OA be less than $\pi/2$. Then α is a strictly decreasing function of OA. A sphere of radius r spans a wright angle at the distance $h(r)$ and the obtuse angle $2\arccos 1/n$ at the distance $h_n(r)$. Thus we have

$$r=h_1(r)<h_2(r)<\dots<h_n(r)<\dots<h(r).$$

LEMMA 12. *In an n-dimensional space of constant curvature the distance of the radical subspace of* $k+1$ $(2\leqq k\leqq n)$ *disjoint spheres of radius* r *from the centres of the spheres is greater than or equal to* $h_k(r)$, *and equality holds if and only if the spheres mutually touch one another.*

Let U be the subspace spanned by the centres of the spheres and P the intersection of U and the radical subspace of the spheres. Let S be the k-dimensional unit sphere centred at P and lying in U. It is well known that the minimal spherical distance between $k+1$ points lying on the surface of a k-dimensional sphere is at most $2 \arccos 1/k$ with equality if and only if the points form the vertices of a regular simplex. Thus among the projections of the centres of the spheres from P onto S there is a pair with spherical distance at most $2 \arccos 1/k$. It follows that these spheres span an angle at P less than $2 \arccos 1/k$. In view of the above considerations this means that the distance of the centres of the spheres from P is at least $h_k(r)$. The case of the equality is obvious.

LEMMA 13. *If $O, O_1, \ldots, O_{m+2}$ are points in an m-dimensional space of constant curvature such that $\sphericalangle O_{m+2}OO_i \geqq \pi/2$ for $i=1, \ldots, m+1$ then*

$$\min_{\substack{1 \leqq i, j \leqq m+1 \\ i \neq j}} \sphericalangle O_iOO_j \leqq \pi/2.$$

We may assume that

$$\min_{\substack{1 \leqq i, j \leqq m+2 \\ i \neq j}} \sphericalangle O_iOO_j \geqq \pi/2,$$

for otherwise the lemma is trivial. On the other hand, a theorem of DAVENPORT and HAJÓS [5] says that

$$\min_{\substack{1 \leqq i, j \leqq m+2 \\ i \neq j}} \sphericalangle O_iOO_j \leqq \pi/2,$$

and the condition of equality is that $\sphericalangle O_kOO_l=\pi$ for some indices k and l. If one of the indices k and l, say k, is equal to $m+2$ then $\sphericalangle O_iOO_l \leqq \pi/2$ for any i, $1 \leqq i \leqq m+1$. If, on the other hand, $1 \leqq l, k \leqq m+1$, then we have for any point O_i with $1 \leqq i \leqq m+1$, $i \neq k$, $\sphericalangle O_iOO_k + \sphericalangle O_iOO_l = \pi$. Hence we have

$$\min \{\sphericalangle O_iOO_k, \sphericalangle O_iOO_l\} \leqq \pi/2,$$

which completes the proof of the lemma.

LEMMA 14. *In an n-dimensional space of constant curvature let $S_1, \ldots, S_{k+2}$ $(0<k<n)$ be spheres of radius r centred at $O_1, \ldots, O_{k+2}$. Suppose that the radical subspace of the spheres $S_1, \ldots, S_{k+1}$ exists. Let H be the orthogonal projection of the point O_1 on the radical subspace of $S_1, \ldots, S_{k+1}$. If the radical hyperplane of the spheres S_1 and S_{k+2} intersects the segment O_1H, then $O_1H \geqq h(r)$.*

The points $O_1, O_2, \ldots, O_{k+2}$ lie in a $(k+1)$-dimensional subspace U^{k+1} which contains also the point H. Let S_i' be the intersection of the sphere S_i with the subspace U^{k+1} $(i=1, 2, \ldots, k+2)$. Let S' be the $(k+1)$-dimensional sphere in the subspace U^{k+1} with centre H and radius O_1H. The points $O_1, \ldots, O_{k+1}$ lie on the boundary of S'. Since the radical hyperplane of S_1 and S_{k+2} intersects the segment O_1H, we have $O_{k+2}H \leqq O_1H$, showing that the sphere S' contains also the point O_{k+2}. Let U^k be the k-dimensional subspace spanned by the points $O_1, \ldots, O_{k+1}$. Preserving the original notations, we translate the sphere S_{k+2}' in U^{k+1} perpendicularly to the subspace U^k until its centre will lie on the boundary of S'. Since by this process the distance of O_{k+2} from any point of U^k increases, S_{k+2}' in its new position will not intersect any of the spheres $S_1', \ldots, S_{k+1}'$.

Now the points $O_1, \ldots, O_{k+2}$ lie on the boundary of one of the hemispheres of S' bounded by the subspace U^k. Therefore, by Lemma 13 there are two indices, say i and j $(1 \leqq i,\ j \leqq k+2)$, such that $\sphericalangle O_i H O_j \leqq \pi/2$. Since the spheres S_i' and S_j' are disjoint, the angle spanned by these spheres at H is at most $\pi/2$. Thus we have indeed $O_1 H = O_i H \geqq h(r)$.

§ 5. Proof of Theorem 1

In an n-dimensional space of constant curvature, consider a packing of spheres of radius r. We may assume that this packing is saturated. In this case the D$-$V cells of the spheres are convex polytopes. Let S be one of the spheres of the packing, O its centre and Z the D$-$V cell associated with S. Any $(n-k)$-dimensional cell of the polytope Z lies in the radical subspace of $k+1$ spheres of the packing, one of which is S. Thus, by Lemma 12 the distance of an $(n-k)$-dimensional cell of Z from O is at least $h_k(r)$.

Let $\bar{S}$ be the open sphere of radius $h_n(r)$ concentric with S. We show that any cell of Z which intersects $\bar{S}$ contains the orthogonal projection of O onto the subspace of the cell.

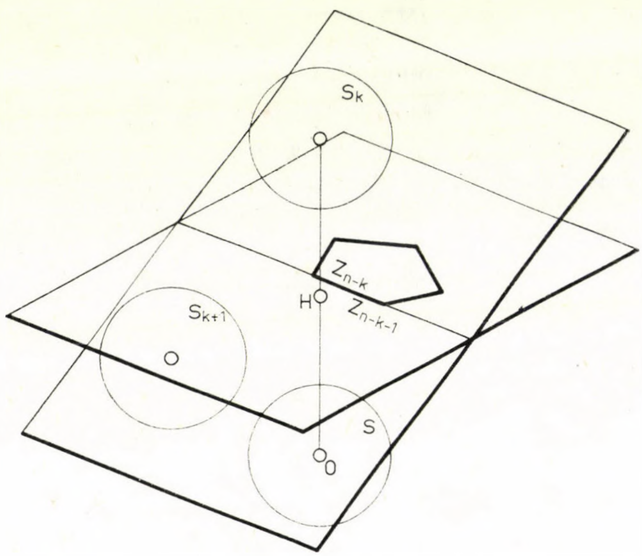

Fig. 19

Let Z_{n-k} be an $(n-k)$-dimensional cell of Z $(0<k<n)$ which does not contain the orthogonal projection H of O onto the subspace of Z_{n-k}. Let the subspace spanned by Z_{n-k} be the radical subspace of the spheres $S, S_1, \ldots, S_k$ of the packing. There is an $(n-k-1)$-dimensional cell Z_{n-k-1} of Z_{n-k} the subspace of which separated H from Z_{n-k} (in the subspace of Z_{n-k}). There is a sphere S_{k+1} of the packings such that the subspace spanned by Z_{n-k-1} is the radical subspace of the spheres $S, S_1, \ldots, S_k, S_{k+1}$ (Fig. 19). The point H and the cell Z are separated by the radical hyperplane of the spheres S and S_{k+1}. Thus this hyperplane inter-

sects the segment OH, and it follows by Lemma 14 that $OH \geqq h(r)$. Since we have $h_n(r) < h(r)$, this means that Z_{n-k} does not intersect $\bar{S}$.

We shall show that the density of S in $Z \cap \bar{S}$ and a fortiori in Z, is less than or equal to the density $d_n(r)$ of $n+1$ spheres of radius r mutually touching one another with respect to the simplex spanned by their centres. To see this we decompose the body $Z \cap \bar{S}$ into smaller bodies and show that the density of S with respect to each of the smaller bodies is at most $d_n(r)$. To describe this decomposition it is convenient to introduce a notation: If $A_0, \dots, A_i$ are points and X is a pointset, we write $A_0 \dots A_i X$ to denote the convex hull of the set formed by adjoining the points $A_0, \dots, A_i$ to the set X. We denote the boundary of $\bar{S}$ by $\bar{S}_0$.

The decomposition of $Z \cap \bar{S}$ follows in n steps which we define by induction: We write $A_0 = O$ and in the first step we divide $Z \cap \bar{S}$ into the body $A_0(Z \cap \bar{S}_0)$ and into the bodies $A_0(Z_{n-1} \cap \bar{S})$, where Z_{n-1} runs over the $(n-1)$-dimensional cells of Z (Fig. 20). Now suppose that in the i^{th} step $(1 \leqq i < n)$ we have decomposed $Z \cap \bar{S}$ into bodies of type $A_0 \dots A_j\,(Z_{n-j} \cap \bar{S}_0)$ for $j = 0, \dots, i-1$, $Z_n = Z$, and into bodies of type $A_0 \dots A_{i-1}(Z_{n-i} \cap \bar{S})$, where Z_k is a k-dimensional cell of Z and A_l is the orthogonal projection of the point $O = A_0$ onto an $(n-l)$-dimensional cell of Z which contains $A_{l+1} \dots A_j Z_{n-j}$ and $A_{l+1} \dots A_{i-1} Z_{n-i}$, respectively. Consider one of the bodies of type $A_0 \dots A_{i-1}(Z_{n-i} \cap \bar{S})$ and denote by A_i the orthogonal projection of A_0 onto the subspace of the cell Z_{n-i}. We have seen that if the set $Z_{n-i} \cap \bar{S}$ is not empty then it contains the point A_i. In the i^{th} step we divide each body of type $A_0 \dots A_{i-1}(Z_{n-i} \cap \bar{S})$ into the body $A_0 \dots A_i(Z_{n-i} \cap \bar{S}_0)$ and into the bodies of the form $A_0 \dots A_i(Z_{n-i-1} \cap \bar{S})$, where Z_{n-i-1} runs over all $(n-i-1)$-dimensional cells of Z_{n-i} (Fig. 21). By Lemma 12 all vertices of Z lie outside of $\bar{S}$, which means that the sets of the form $A_0 \dots A_{n-1}(Z_0 \cap \bar{S})$ are empty. Thus the process ends at the n^{th} step. As a result, we have decomposed $Z \cap \bar{S}$ into bodies of the form $A_0 \dots A_j(Z_{n-j} \cap \bar{S}_0)$ for $j = 0, \dots, n-1$, where Z_k is a k-dimensional cell of Z and A_l is the orthogonal projection of the point $O = A_0$ onto an $(n-l)$-dimensional cell of Z which contains $A_{l+1} \dots A_j Z_{n-j}$. The bodies $A_0 \dots A_{n-1}(Z_1 \cap \bar{S}_0)$ are orthoschemes.

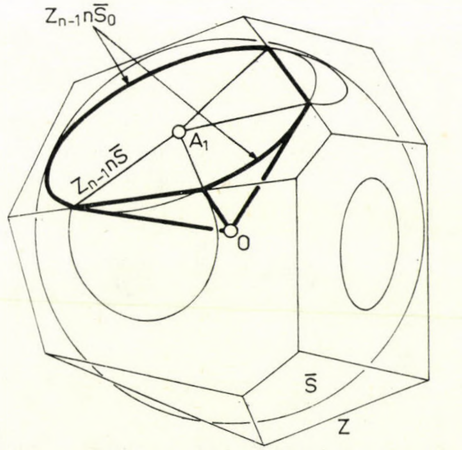

Fig. 20

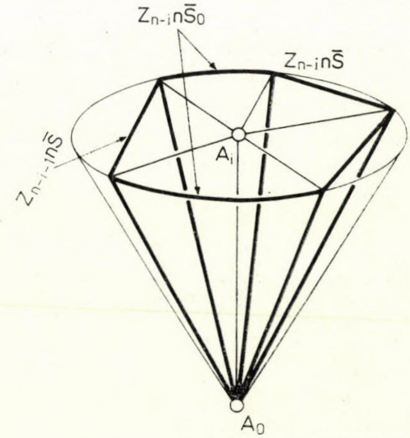

Fig. 21

We continue to show that the density of S with respect to each of the bodies $A_0 \dots A_j(Z_{n-j} \cap \bar{S}_0)$ is at most $d_n(r)$ $(j=0, \dots, n-1)$. Consider $n+1$ spheres of radius r mutually touching one another and let T be the regular simplex spanned by the centres of the spheres. T can be divided into $(n+1)!$ congruent orthoschemes. One of these orthoschemes, say $\bar{A}_0 \bar{A}_1 \dots \bar{A}_n$, is obtained by taking $\bar{A}_0$ to be a vertex of T, and when $\bar{A}_0, \dots, \bar{A}_{i-1}$ have been chosen, taking $\bar{A}_i$ to be the centroid of one of the i-dimensional cells of T that contains $\bar{A}_0, \bar{A}_1, \dots, \bar{A}_{i-1}$. If S' is the sphere of radius r centred at $\bar{A}_0$ then we have

$$d_n(r) = d(\bar{A}_0 \bar{A}_1 \dots \bar{A}_n, S').$$

Consider the body $A_0 \dots A_j(Z_{n-j} \cap \bar{S}_0)$ and let A_{j+1} be an arbitrary point of $Z_{n-j} \cap \bar{S}_0$. Observe that

$$d\big(A_0 \dots A_j(Z_{n-j} \cap \bar{S}_0), S\big) = d(A_0 \dots A_{j+1}, A_{j+1}, S).$$

We have, by Lemma 12, $A_0 A_i \geqq h_i(r)$ for $i=1, \dots, j$ and $A_0 A_{j+1} = h_n(r)$. On the other hand, we have, by definition, $\bar{A}_0 \bar{A}_i = h_i(r)$ for $i=1, \dots, n$. Using the fact that $h_i(r) < h_n(r)$ for $i=1, \dots, n-1$, we get, by Lemma 11,

$$d_n(r) = d(\bar{A}_0 \bar{A}_0 \dots \bar{A}_n, S') \geqq d(A_0 \dots A_{j+1}, A_{j+1}, S) = d\big(A_0 \dots A_j(Z_{n-j} \cap \bar{G}_0), S\big).$$

This completes the proof of Theorem 1.

§ 6. Remarks

In a packing of equal spheres of radius r the density of a sphere in its D–V cell attains the bound $d_n(r)$ only if the D–V cell can be decomposed into orthoschemes congruent with $\bar{A}_0 \dots \bar{A}_n$ $\big(\bar{A}_0 \bar{A}_i = h_i(r)\big)$. This means that the D–V cell is a regular n-dimensional polytope $\{a, 3, \dots, 3\}$ whose i-dimensional cells are at a distance $h_{n-i}(r)$ from the centre of the sphere. Consequently if the density of each sphere with respect to its D–V cell is equal to $d_n(r)$ then the spheres are inspheres of a regular tessellation $\{a, 3, \dots, 3\}$. These tessellations are $\{3, 3, \dots, 3\}$ and $\{4, 3, \dots, 3\}$ in the n-dimensional spherical space, $\{5, 3, 3\}$ in the three-dimensional spherical space and $\{5, 3, 3, 3\}$ in the four-dimensional hyperbolic space. The cell-inspheres of all these tessellations form a densest packing.

In an n-dimensional hyperbolic space let S be horosphere and T a body contained in S. Since the volume of S is infinite the density of T with respect to S cannot be defined in the usual way. But this density can be defined as a limiting value as follows: Let E be the boundary of S. It is known that the hyperbolic metric in E is equivalent to the $(n-1)$-dimensional Euclidean metric. Let A be a subset of E and $C(A)$ the point-set union of the axes of S having a point in common with A. Let O be an arbitrary point of E and $S_{n-1}(R)$ a sphere of radius R centred at O. Now we define the (upper) density of T with respect to S by the value

$$\limsup_{R \to \infty} \frac{C\big(S_{n-1}(R)\big) \cap S}{C\big(S_{n-1}(R)\big) \cap T}.$$

Using the fact that E can be considered as an $(n-1)$-dimensional Euclidean space, it is easily seen that this value does not depend on the choice of the point O.

In the hyperbolic space we can consider, besides packings of spheres, packings of horospheres. In this case we have to extend the investigations of § 2 and § 3 to asymptotic simplices with one ideal vertex A_0. All the definitions and proofs can be transferred without any change to this case. Thus we have the following

THEOREM 4. *In the n-dimensional hyperbolic space, consider a packing of horospheres. Then the density of each horosphere with respect to its $D-V$ cell cannot be greater than the density of $n+1$ horospheres mutually touching one another with respect to the asymptotic simplex spanned by the centres of the horospheres.*

The bound of Theorem 4 is exact for the horospheres inscribed in the cells of the three-dimensional regular tessellation $\{6, 3, 3\}$.

References

[1] H. F. BLICHFELD, The minimum value of quadratic forms and the closest packing of spheres, *Math. Ann.*, **101** (1929), 605—608.

[2] K. BÖRÖCZKY—A. FLORIAN, Über die dichteste Kugelpackung in hyperbolischen Raum, *Acta Math. Acad. Sci. Hungar.*, **15** (1964), 237—245.

[3] K. BÖRÖCZKY, Gömbkitöltés állandó görbületű terekben, *Mat. Lapok*, **25** (1974) (in Hungarian).

[4] H. S. M. COXETER, Arrangements of equal spheres in non-Euclidean spaces, *Acta Math. Acad. Sci. Hungar.*, **4** (1954), 263—274.

[5] H. DAVENPORT—G. HAJÓS, 35. feladat, *Mat. Lapok*, **2** (1951), 68 (in Hungarian).

[6] L. FEJES TÓTH, Über die dichteste Kugellagerung, *Math. Zeitschrift*, **48** (1943), 676—684.

[7] L. FEJES TÓTH, Kreisausfüllungen der hyperbolischen Ebene, *Acta Math. Acad. Sci. Hungar.*, **4** (1953), 103—110.

[8] L. FEJES TÓTH, Über die dichteste Horozyklenlagerung, *Acta Math. Acad. Sci. Hungar.*, **5** (1954), 41—44.

[9] C. F. GAUSS, Untersuchungen über die Eigenschaften der positiven ternären quadratischen Formen von Ludwig August Seeber, *Göttingische gelehrte Anzeigen*, 1831 Juli 9, see *Werke*, Göttingen, 1836 II, 188—196.

[10] J. L. LAGRANGE, Recherches d'arithmétique, *Neuveaux Memoires de l'Académie royal des Sciences et Belles-Lettres de Berlin* (1773), 265—312; Oeuvres, III, 693—758.

[11] V. I. LEVENSTEIN, О максимальной плотности заполнения n-мерного евклидова пространства равными шарами, *Мат. Заметки*, **18** (1975), 301—311.

[12] R. A. RANKIN, On the closest packing of spheres in n-dimensions, *Ann. Math. Princeton II*, **48** (1947), 1062—1081.

[13] R. A. RANKIN, The closest packing of spherical caps in n-dimensions, *Proc. Glasgow Math. Ass.*, **2** (1955), 139—144.

[14] C. A. ROGERS, The packing of the equal spheres, *Proc. London Math. Soc.*, **(3) 8** (1958), 609—620.

[15] V. M. SIDELNIKOV, О плотнейшей укладке шаров на поверхности n-мерной евклидовой сферы и числе векторов двоичного кода с заданным кодовым расстоянием, *Докл. АН СССР*, **213** (1973), 1029—1032.

[16] R. M. L. TAMMES, On the origine of number and arrangement of the places of exit on the surface of pollen grains, *Rec. Trav. Bot. Neerl.*, **27** (1930), 1—84.

[17] A. THUE, Om nogle geometrisk talteoretiske Theoremer, *Fordhl. Skand. Naturforsk.*, **14** (1892), 352—353.

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF GEOMETRY
1088 BUDAPEST, MÚZEUM KRT. 6—8.

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SUR LES GROUPES QUASI-p-PURS-PROJECTIFS

Par

K. BENABDALLAH et R. BRADLEY (Montréal)[1]

Les questions de quasi-projectivité et de quasi-injectivité relatives reçoivent présentement beaucoup d'attention de la part des chercheurs dans ce domaine. Par exemple dans [1] les groupes quasi-purs-injectifs sans torsion de rang fini sont complètement caractérisés (voir également [2]). Dans [3] nous avons caractérisés les groupes quasi-purs-projectifs torsions.

Dans ce travail, nous établissons une caractérisation complète des groupes quasi-p-purs-projectifs. Quelques uns des résultats démontrés dépendent, en tout ou en partie de l'hypothèse généralisée du continu. Ces théorèmes seront marqués d'un astérisque. Les notations utilisées sont de [4] et tous les groupes considérés sont abéliens. De plus, si H est un sous-groupe d'un groupe G, ν_H désignera toujours l'épimorphisme canonique de G sur G/H.

1. Definition et proprietes elementaires

Un groupe G est dit quasi-p-pur-projectif si pour tout sous-groupe p-pur H de G et pour tout homomorphisme $f: G \to G/H$, il existe un endomorphisme φ de G tel que $\nu_H \cdot \varphi = f$. On notera quasi-p-pur-projectif par $q \cdot p^3$ dans ce qui suit.

Il est évident que dans le cas où G est un p-groupe, G est $q \cdot p^3$ si et seulement si G est quasi-pur-projectif. Nous avons précédemment caractérisé les p-groupes quasi-purs-projectifs dans [3]: ce sont soit des sommes directes de groupes cycliques, soit des sommes d'un groupe divisible et d'un groupe borné (en supposant vraie l'hypothèse généralisée du continu). Il est alors logique de se demander si un groupe $q \cdot p^3$ quelconque ne possède pas une structure similaire.

On démontre aisément que:

PROPOSITION 1.1. *Un groupe torsion G est $q \cdot p^3$ si et seulement si G_q est $q \cdot p^3$ pour tout premier q.*

PROPOSITION 1.2. *Tout facteur direct d'un groupe $q \cdot p^3$ est également $q \cdot p^3$.*

Comme tout q-groupe est p-divisible, par la proposition 1.1 et par la discussion qui précède, on obtient:

THÉORÈME* 1.3. *Un groupe torsion G est $q \cdot p^3$ si et seulement si G_q est quasi-projectif pour tout premier q distinct de p et G_p est soit somme directe d'un groupe divisible et d'un groupe borné, soit somme directe de groupes cycliques.*

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THÉORÈME 1.4. *Soit G un groupe p-réduit* $q \cdot p^3$ *et soit B un sous-groupe p-basique de G. Alors* card End $(G/B) \leqq$ card End B.

PREUVE. Soit $\varphi: G/B \to G/B$. Alors il existe un endomorphisme θ de G tel que $\nu_B \cdot \theta = \varphi \cdot \nu_B$. De plus $\theta(B) \leqq B$. Soit f: End $(G/B) \to$ End B l'application définie par $f(\varphi) = \theta|_B = \bar{\theta}$.

Si $\bar{\theta} = \bar{\theta}_1$, $\theta = \theta_1$ car G est p-réduit. Donc $\nu_B \cdot \theta = \nu_B \cdot \theta_1$, $\varphi \cdot \nu_B = \varphi_1 \cdot \nu_B$ et $\varphi = \varphi_1$. Par conséquent f est injective et card End $(G/B) \leqq$ card End B.

Il est facile de démontrer que:

THÉORÈME 1.5. *Si G est* $q \cdot p^3$ *et si H est un sous-groupe p-pur totalement invariant de G alors G/H est également* $q \cdot p^3$.

2. Les groupes $q \cdot p^3$ sans torsion

Nous nous restreignons d'abord aux groupes $q \cdot p^3$ sans torsion. La preuve des lemmes suivants ne présente aucune difficulté.

LEMME 2.1. *Si L est un groupe libre de rang infini, alors il existe un sous-groupe F de L tel que L/F soit divisible sans torsion de même rang que L.*

LEMME 2.2. *Si B est un sous-groupe p-basique d'un groupe G quelconque, alors nB est aussi un sous-groupe p-basique de G et ce pour tout entier n premier avec p.*

Etudions d'abord le cas p-réduit.

THÉORÈME 2.3. *Si G est sans torsion p-réduit* $q \cdot p^3$ *et s'il existe un sous-groupe B p-basique dans G et de rang un, alors* $G \cong \mathbf{Z}$.

PREUVE. Soit m tel que $(m, p) = 1$, m non puissance de premier, alors il existe un endomorphisme idempotent φ de G/mB tel que $\varphi \neq 0, 1$.

G étant $q \cdot p^3$, il existe un endomorphisme θ de G tel que $\nu_{mB} \cdot \theta = \varphi \cdot \nu_{mB}$. Alors $(\theta^2 - \theta)(G) \leqq mB$ et $(\theta^2 - \theta)(B) \leqq mB \leqq B \cong \mathbf{Z}$ d'où $(\theta^2 - \theta)|B = n|B$. Donc $\theta^2 - \theta = n$ car G est p-réduit. (Ici, n désigne un nombre naturel.)

Si $n = 0$, $\theta^2 = \theta$, $\theta \neq 0, 1$ d'où $G = G_1 \oplus G_2$ où $G_1 \neq 0$, $G_2 \neq 0$. Soient B_1 et B_2 des sous-groupes p-basiques de G_1 et G_2 respectivement. Alors $B_1 \neq 0$, $B_2 \neq 0$, $B \cong B_1 \oplus B_2$ ce qui contredit que $B \cong \mathbf{Z}$. Alors $n \neq 0$, $G \cong nG = (\theta^2 - \theta)(G) \leqq B \cong \mathbf{Z}$ d'où $G \cong \mathbf{Z}$.

THÉORÈME 2.4. *Si G est sans torsion p-réduit* $q \cdot p^3$ *et s'il existe un sous-groupe p-basique de rang fini B de G, alors G est libre de même rang que B.*

PREUVE. On démontre ce résultat par induction sur k, le rang de B. Le cas $k = 1$ suit du théorème précédent. On suppose le théorème vrai pour $k < n$. Supposons alors que $k = n$.

Par le procédé utilisé dans la preuve du théorème 2.3, il existe un endomorphisme θ de G tel que $(\theta^2 - \theta)(G) \leqq B$, $\theta \neq 0, 1$. Comme B est libre, ker $(\theta^2 - \theta)$ est facteur direct de G.

Si ker$(\theta^2 - \theta) = 0$, $\theta^2 - \theta$ est un monomorphisme tel que $(\theta^2 - \theta)(G) \leqq B$ libre et donc G est libre. Sinon, on peut écrire $G = G_1 \oplus G_2$ où $G_1 \neq 0$, $G_2 \neq 0$. En effet, si $\ker(\theta^2-\theta)=G$, $G=\theta(G)\oplus\ker\theta$ et si $\ker(\theta^2-\theta)\neq G$, $G=\ker(\theta^2-\theta)\oplus K$, $K\neq 0$. Alors, si B_1 et B_2 sont des sous-groupes p-basiques de G_1 et G_2 respectivement $B_1\oplus B_2\cong B$ et donc rang $B_1<n$, rang $B_2<n$.

Par l'hypothèse d'induction, G_1 et G_2 sont libres d'où G est libre. Donc G est p-basique dans G d'où $G\cong B$.

THÉORÈME* 2.5. *Si G est sans torsion p-réduit $q\cdot p^3$ et si B est un sous-groupe p-basique de G, alors B et G sont de même rang.*

PREUVE. Si le rang de B est fini, le théorème précédent nous donne le résultat. Supposons donc que le rang de B soit infini.

Supposons $\alpha=\text{rang}\, B<\text{rang}\, G=\beta$. Par le lemme 2.1, il existe un sous-groupe B' de B tel que $B/B'\simeq\bigoplus_\alpha \mathbf{Q}$. On vérifie facilement que B' est un sous-groupe p-basique de G. Donc card End $(G/B')\leqq$ card End B' par le théorème 1.4.

Comme $G/B'=B/B'\oplus K/B'$ pour un certain K, on a que card End $(G/B')\geqq$ $\geqq$ card Hom $(G/B', B/B')$. Mais $r_0(B')+r_0(G/B')=r_0(G)=\beta$, $r_0(B')=\alpha$ d'où $r_0(G/B')=\beta$. Donc card Hom $(G/B', B/B')\geqq$ card $(\prod_\beta \mathbf{Q})=2^\beta$.

Donc $\alpha<\beta<2^\beta\leqq$ card End $B'=2^\alpha$ ce qui contredit l'hypothèse généralisée du continu. Par conséquent, B et G sont de même rang.

THÉORÈME* 2.6. *Si G est sans torsion p-réduit, alors G est $q\cdot p^3$ si et seulement si G est libre.*

PREUVE. On suppose d'abord que G est $q\cdot p^3$. Soit B un sous-groupe p-basique de G. Par le théorème précédent, le rang de B et celui de G sont identiques. Si le rang de B est fini, G est libre. On suppose donc que le rang de B est α infini.

Il existe, par le lemme 2.1, un sous-groupe B' de B tel que $B/B'\simeq\bigoplus_\alpha \mathbf{Q}$, $G/B'=B/B'\oplus R/B'$. On définit alors $f\colon G\to G/B'$ par l'inclusion canonique de G dans son enveloppe divisible isomorphe à $B/B'\leqq G/B'$.

Comme f est un homomorphisme et qu'il existe un endomorphisme φ de G tel que $v_B\cdot\varphi=f$, φ est également injectif et $\varphi(G)\leqq B$ libre. Donc G est libre.

L'autre implication est triviale.

Le théorème suivant nous permet de ramener l'étude des groupes $q\cdot p^3$ sans torsion aux sommes directes de p-divisibles $q\cdot p^3$ et de p-réduits $q\cdot p^3$.

THÉORÈME* 2.7. *Si G est sans torsion $q\cdot p^3$, alors G est somme directe d'un groupe p-divisible et d'un groupe p-réduit.*

PREUVE. Soit H le sous-groupe p-divisible maximal de G. Alors H est pur totalement invariant dans G et donc G/H est $q\cdot p^3$ p-réduit par le théorème 1.5.

Alors G/H est libre, $G=H\oplus K$ où $K\cong G/H$ libre et H p-divisible.

On se restreint maintenant au cas p-divisible. On note par $\mathbf{Q}^p$ le sous-groupe de $\mathbf{Q}$ formé des fractions avec dénominateur puissance de p. On vérifie facilement que:

LEMME 2.8. *Un groupe p-divisible réduit G est $q\cdot p^3$ si et seulement si il est quasi-projectif comme $\mathbf{Q}^p$-module.*

Il suffira donc de caractériser les $\mathbf{Q}^p$-modules quasi-projectifs.

Par une preuve analogue à celle du théorème 1.4, on a que:

LEMME 2.9. *Si A est un anneau quelconque, si G est un A-module quasi-projectif et si B est un sous-module de G, alors* card $\mathrm{Hom}_A(G/B, G/B) \leqq$ card $\mathrm{Hom}_A(G, G)$.

On obtient alors:

THÉORÈME 2.10. *Si G est un $\mathbf{Q}^p$-module sans torsion, alors G est quasi-projectif si et seulement si G est libre sur $\mathbf{Q}^p$.*

PREUVE. Supposons d'abord G quasi-projectif. Soit B un sous-module libre maximal inclus dans G. Si le rang de G est fini G/B est torsion tel que card $\mathrm{Hom}_{\mathbf{Q}^p}(G/B, G/B) \leqq$ card $\mathrm{Hom}_{\mathbf{Q}^p}(G, G) \leqq \aleph_0$. Donc G/B est fini et $nG \leqq B$ d'où $G \cong nG$ est libre.

Si le rang de G est infini, il existe B' sous-$\mathbf{Q}^p$-module de B tel que $B/B' \cong \cong \bigoplus_{r(G)} \mathbf{Q}$. Alors $G/B' = B/B' \oplus R/B'$. On définit alors $f\colon G \to G/B'$ comme l'inclusion de G dans son enveloppe divisible $B/B' \leqq G/B'$.

Comme G est quasi-projectif, il existe un endomorphisme φ de G tel que $v_{B'} \cdot \varphi = f$. Donc φ est injectif, $G \cong \varphi(G) \leqq B$ libre. Par conséquent G est libre. L'implication inverse est évidente.

On sait donc que si G est sans torsion $q \cdot p^3$, il peut s'écrire comme somme directe de copies de $\mathbf{Q}^p$ et de copies de $\mathbf{Z}$. Le lemme suivant nous permet de dire plus.

LEMME 2.11. *Si $G = K \oplus L$ où $K \cong \mathbf{Q}^p$ et $L \cong \mathbf{Z}$, alors G n'est pas $q \cdot p^3$.*

PREUVE. On suppose au contraire que G est $q \cdot p^3$. Soit q un nombre premier différent de p et soit $H = K \oplus qL$. Alors H est p-pur dans G et $G/H = \langle \bar{l} \rangle$ où $L = \langle l \rangle$. Soit $x \in K$ tel que $\chi(x) = (\infty, 0, 0, 0, \ldots)$, l'ordre des nombres premiers étant $p, p_1, p_2 \ldots$.

On définit alors $f_0\colon \langle x \rangle \oplus L \to G/H$ par $f(L) = 0$, $f(x) = \bar{l}$. On prolonge f_0 à $f\colon G \to G/H$ de la façon suivante: si $k \in K$ quelconque, il existe un unique entier naturel n et un unique entier a premier avec p tel que $p^n k = ax$. De même, il existe dans G/H un unique élément $\bar{g}$ tel que $p^n \bar{g} = a\bar{l}$. On définit alors $f(k) = \bar{g}$. f est évidemment un homomorphisme bien défini.

Comme G est $q \cdot p^3$, il existe un endomorphisme φ de G tel que $v_H \cdot \varphi = f$. Alors $\varphi(x) + H = \bar{l} \neq 0$. Mais K est totalement invariant dans G et donc $\varphi(x) \in \in K \leqq H$ d'où $\varphi(x) + H = 0$ ce qui est une contradiction. Donc G n'est pas $q \cdot p^3$.

On en déduit la caractérisation suivante:

THÉORÈME* 2.12. *Un groupe sans torsion G est $q \cdot p^3$ si et seulement si G est soit libre, soit somme directe de copies de $\mathbf{Q}^p$.*

COROLLAIRE* 2.13. *Si G est sans torsion $q \cdot p^3$, alors G est q-réduit pour tout premier q différent de p.*

Il est à remarquer qu'on peut démontrer ce résultat indépendamment de l'hypothèse généralisée du continu. Nous pouvons maintenant établir:

THÉORÈME 2.14. *Si G est mixte $q \cdot p^3$, alors G_q est nul pour tout premier q différent de p.*

PREUVE. Supposons au contraire que G_q est non nul pour un nombre premier q distinct de p. Si G_q n'est pas quasi-projectif, ou bien il existe x et y non nuls dans

G_q tels que $G=\langle x\rangle\oplus\langle y\rangle\oplus M$, $O(x)=q^S<q^t=O(y)$, ou bien il existe un facteur direct de G, isomorphe à $\mathbf{Z}(q^\infty)$. De toute façon, ceci contredit le théorème 1.3.

Donc G_q est borné et $G=G_q\oplus K$ avec $K_q=0$. Soit $H=G_q\oplus qK$. Si $qK=K$, $K\cong G/G_q$ est q-divisible et donc G/T est q-divisible $q\cdot p^3$ sans torsion d'où $G=T$ ce qui contredit le fait que G soit mixte.

Donc $qK\neq K$ et $H\neq G$. Evidemment H est p-pur dans G. Soit x un élément non nul de G_q tel que $\langle x\rangle\oplus L=G_q$. Soit $f\colon G\to G/H$ défini par $f(x)=y+H$, un élément non nul quelconque de G/H, et $f(L\oplus K)=0$.

Comme G est $q\cdot p^3$, il existe un endomorphisme φ de G tel que $\nu_H\cdot\varphi=f$ d'où $\varphi(x)+H=y+H\neq 0$. Mais $\varphi(x)\in G_q\leqq H$ d'où $\varphi(x)+H=0$ ce qui est une contradiction.

Donc G_q est nul pour tout q premier distinct de p.

THÉORÈME 2.15. *Un $\mathbf{Q}^p$-module G est quasi-projectif si et seulement si soit $G\cong\bigoplus_{q\neq p}(\oplus\mathbf{Z}(q^{n_q}))$, soit $G\cong\oplus\mathbf{Q}^p$.*

PREUVE. Si G est torsion, $G=\bigoplus_{q\neq p}G_q$ où G_q est quasi-projectif comme $\mathbf{Q}^p$-module. Mais alors G_q est quasi-projectif comme groupe abélien. Donc $G\cong\bigoplus_{q\neq p}(\oplus\mathbf{Z}(q^{n_q}))$. Si G est sans torsion, $G\cong\oplus\mathbf{Q}^p$ par le théorème 2.10. Si G est mixte, $G_p=0$ pour tout premier $q\neq p$. Comme $G_p=0$, G est sans torsion ce qui est une contradiction.

3. Les groupes $q\cdot p^3$ mixtes et la cas general

On se restreint d'abord au cas mixte p-réduit.

THÉORÈME* 3.1. *Si G est mixte p-réduit $q\cdot p^3$ alors G_p est somme direct de groupes cycliques.*

PREUVE. Comme G est p-réduit, G_q est nul pour tout premier q différent de p. Alors G/G_p est $q\cdot p^3$ sans torsion. Si G/G_p est libre $G\cong G_p\oplus G/G_p$ et donc G est somme directe de groupes cycliques.

Si G/G_p est p-divisible, soit B un sous-groupe p-basique de G_p. Alors B est p-basique dans G. Le groupe B ne peut être borné. Sans perte de généralité, on peut supposer que rang $B=$rang final B et rang $G_p=$rang final G_p et rang B infini.

Supposons que rang $B=\alpha<\beta=$rang G_p. Alors rang $G_p/B=\beta$ et G_p/B est divisible d'où $\alpha<\beta<2^\beta=$card End $(G_p/B)\leqq$card End $(G/B)\leqq$card End $B=2^\alpha$ ce qui contredit l'hypothèse généralisée du continu.

Donc, rang $B=$rang G_p. Il existe un sous-groupe pur B' de B tel que B/B' soit divisible et rang $(B/B')=$rang final $B=$rang $B=$rang G_p. On peut alors définir $f_0\colon G_p\to B/B'$ comme l'inclusion canonique de G_p dans son enveloppe divisible. On prolonge f_0 à $f\colon G\to G/B'$ par injectivité de B/B'.

Comme G est $q\cdot p^3$, il existe un endomorphisme φ de G tel que $\nu_{B'}\cdot\varphi=f$. Alors $f|G_p$ est injectif d'où $\varphi|G_p$ est injectif. Donc $G_p\cong\varphi(G_p)\leqq B$.

THÉORÈME 3.2. *Si G est $q\cdot p^3$, si B est un sous-groupe p-pur de G avec $p^\omega B=0$ et si $G/B=H/B\oplus K/B$, alors $p^\omega G=p^\omega H\oplus p^\omega K$.*

PREUVE. Soit f la projection canonique de G/B sur H/B. Alors $f^2=f$. Comme G est $q\cdot p^3$, il existe un endomorphisme φ de G tel que $\nu_B\cdot\varphi=f\cdot\nu_B$. Alors $(\varphi^2-\varphi)(G)\leqq B$ d'où $(\varphi^2-\varphi)(p^\omega G)\leqq p^\omega B=0$.

Par conséquent, si $\bar\varphi=\varphi\,|p^\omega G$, $\bar\varphi=\bar\varphi^2$. Donc $p^\omega G=\bar\varphi(p^\omega G)\oplus\ker\bar\varphi$. Mais $\bar\varphi(p^\omega G)=\varphi(p^\omega G)\leqq p^\omega H$, $\ker\bar\varphi\leqq p^\omega K$ et $p^\omega K\cap p^\omega H=p^\omega B=0$. Donc $p^\omega G=$ $=p^\omega H\oplus p^\omega K$.

THÉORÈME* 3.3. *Si G est p-réduit mixte $q\cdot p^3$, alors G est sans élément de p-hauteur infinie.*

PREUVE. Comme T est somme directe de groupes cycliques, par le théorème 3.1, $p^\omega T=0$. Comme T est pur dans G, $T\cap p^\omega G=0$. Supposons $p^\omega G\neq 0$.

Soit H un $p^\omega G$-haut contenant T. Alors H est p-pur dans G. Si T est borné, T est facteur direct de G et le théorème 2.6 complète la preuve. On suppose donc que T n'est pas borné.

Soit B un sous-groupe pur de T tel que T/B soit divisible. Alors $H/B=T/B\oplus$ $\oplus R/B$, $H/R\cong T/B$ et R est pur dans H et donc p-pur dans G.

Il existe alors h un élément non nul de $H[p]$ tel que $\langle h\rangle\cap R=0$. Soit x un élément non nul de $p^\omega G$. On voit facilement que $(\langle x+h\rangle+R)\cap H=R$ d'où $(\langle x+h\rangle+R)/R\cap H/R=0$. Soit K/R un H/R-haut contenant $(\langle x+h\rangle+R)/R$. Alors $G/R=H/R\oplus K/R$.

Par le théorème précédent, $p^\omega G=p^\omega H\oplus p^\omega K$. Comme $0=H\cap p^\omega G=p^\omega H$, $p^\omega G=p^\omega K$. Donc $x\in K$, $x+h\in K$, $h\in K\cap H=R$ ce qui est une contradiction. Donc $p^\omega G=0$.

THÉORÈME 3.4. *Si $G=G_p\oplus L$ où G_p est une somme directe de groupes cycliques et L est libre, alors G est $q\cdot p^3$.*

PREUVE. Soit H un sous-groupe p-pur de G. Alors $H=H_p\oplus K$ où K est libre car H est somme directe de groupes cycliques. Il est clair que $(G_p+H)/H=(G/H)_p$.

Donc $(G/H)_p=(G_p+H)/H\cong G_p/(G_p\cap H)=G_p/H_p$. Soit θ l'isomorphisme de $(G/H)_p$ à G_p/H_p. Soit $f\colon G\to G/H$ un homomorphisme quelconque. On définit $\bar f\colon G_p\to G_p/H_p$ par $\bar f=\theta\cdot f|_H$.

Comme H_p est pur dans G_p, il existe un endomorphisme φ_1 de G tel que $\nu_{H_p}\cdot\varphi_1=\bar f$. On a alors que $\nu_H\cdot\varphi_1=f\,|H$. De plus, comme L est projectif, il existe un homomorphisme $\varphi_2\colon L\to G$ tel que $\nu_H\cdot\varphi_2=f\,|L$. Soit $\varphi=\varphi_1\oplus\varphi_2$. Alors $\nu_H\cdot\varphi=f$. Donc G est qp^3.

THÉORÈME* 3.5. *Si G est p-réduit mixte, alors G est $q\cdot p^3$ si et seulement si G est somme directe de groupes cycliques.*

PREUVE. Evidemment, $G_q=0$ pour tout q premier différent de p. Alors G/G_p est $q\cdot p^3$ sans torsion. Si G/G_p est libre, $G\cong G_p\oplus G/G_p$ et donc G est somme directe de groupes cycliques.

Supposons donc que G/G_p soit p-divisible. Alors $G/G_p=\bigoplus_{i\in I}\langle\overline{X_{i0}}\rangle_*$ où $\chi(\overline{X_{i0}})=(\infty,0,0,\ldots)$ l'ordre des nombres premiers étant $p,p_1,p_2,\ldots$. Donc, si $\overline{X_{i0}}=p^j\overline{X_{ij}}$, on aura que $\langle\{X_{ij}\,|\,i\in I,\ j\in N\}\rangle+G_p=G$.

On définit l'homomorphisme $f\colon G\to G/G_p$ comme suit: fixons $k\in I$ et soit g

un élément quelconque de G. Alors $g=\sum_{i,j} a_{ij}X_{ij}+t$ où $a_{ij}\in\mathbf{Z}$, $t\in T$, $a_{ij}=0$ sauf pour un nombre fini de couple (i, j). Alors $f(g)=\sum_j a_{kj}\overline{X_{kj+1}}$.

On vérifie d'abord que f est bien défini. Si $g=\sum_{i,j} b_{ij}X_{ij}+t'$ alors $\bar g=\sum_{i,j} a_{ij}\overline{X_{ij}}=\sum_{i,j} b_{ij}\overline{X_{ij}}$. Par indépendance linéaire, $\sum_j a_{kj}\overline{X_{kj}}=\sum_j b_{kj}\overline{X_{kj}}$. Mais $\overline{X_{kj}}=p^{n-j}\overline{X_{kn}}$ si $j\leqq n$ d'où

$$\sum_j a_{kj}\overline{X_{kj}}=\sum_{j=0}^{n} a_{kj}p^{n-j}\overline{X_{kn}},\quad \sum_j b_{kj}\overline{X_{kj}}=\sum_{j=0}^{n} b_{kj}p^{n-j}\overline{X_{kn}}$$

pour un n. Comme G/G_p est sans torsion, $\sum_{j=0}^{n} a_{kj}p^{n-j}=\sum_{j=0}^{n} b_{kj}p^{n-j}$,

$$\sum_{j=0}^{n} a_{kj}p^{n-j}\overline{X_{kn+1}}=\sum_{j=0}^{n} b_{kj}p^{n-j}\overline{X_{kn+1}},\quad \sum_{j=0}^{n} a_{kj}\overline{X_{k,j+1}}=\sum_{j=0}^{n} b_{kj}\overline{X_{k,j+1}}$$

et donc $f(g)$ est uniquement déterminé.

Comme f est évidemment un homomorphisme, il existe un endomorphisme φ de G tel que $v_{G_p}\cdot\varphi=f$. Mais alors $\overline{\varphi(X_{k0})}=\overline{X_{k1}}$, $\overline{\varphi(pX_{k0})}=\overline{X_{k0}}$, $\varphi(pX_{k0})-X_{k0}\in G_p$, $p^n\varphi(pX_{k0})=p^nX_{k0}$, $\varphi(p^{n+1}X_{k0})=p^nX_{k0}$. Par le théorème 3.3, $h_p(p^nX_{k0})=S$ fini car p^nX_{k0} est non nul. Donc $h_p(\varphi(p^{n+1}X_{k0}))\geqq h_p(p^{n+1}X_{k0})\geqq S+1$ ce qui est une contradiction. Donc $I=\emptyset$, $G=G_p$ ce qui contredit le fait que G soit mixte.

On établit maintenant un résultat technique utile.

LEMME 3.6. *Si $G=A\oplus B\oplus C$ où A est un p-groupe divisible, B un p-groupe borné, C soit un groupe libre, soit une somme directe de copies de $\mathbf{Q}^p$ et si H est un sous-groupe p-pur de G, alors $H=A_1\oplus B_1\oplus C_1$, $G=A_2\oplus B_2\oplus C_2$ où $A_2=A$, $B_2\cong B$, $C_2\cong C$, $A_1\leqq A_2$, $B_1\leqq B_2$. Si $C\cong\oplus\mathbf{Q}^p$, $C_1\leqq C_2$.*

PREUVE. Il est évident que H_p est pur dans $A\oplus B$. Alors $H_p=A_1\oplus B_1$ où $A_1\leqq A$ et B_1 est borné. Donc $B_1\cap A=0$ d'où $G_p=A\oplus B=A_2\oplus B_2$ où $A=A_2$, $B_1\leqq B_2$. Si de plus, C est p-divisible, on aura que si $p^nB=0$, $p^nG=A\oplus C$. Comme H_p est facteur direct de H, $H=A_1\oplus B_1\oplus C_1$ où C_1 est sans torsion, $p^nH=A_1\oplus p^nC_1$, avec $p^nC_1\cap A=0$. Soit C_2 un A-haut contenant p^nC_1. Alors p^nC_1 est sous-groupe p-pur de C_2 et $p^nG=A\oplus C_2$. Par conséquent $C_1\cong p^nC_1$ p-divisible, $C_1=p^nC_1$, $C_1\leqq C_2$.

THÉORÈME 3.7. *Si $G=A\oplus B\oplus C$ où A est un p-groupe divisible, B un p-groupe borné et C un groupe libre ou une somme directe de copies de $\mathbf{Q}^p$, alors G est $q\cdot p^3$.*

PREUVE. Soit H un sous-groupe p-pur de G et soit $f\colon G\to G/H$ un homomorphisme quelconque. Nous considérons d'abord le cas ou C est libre. Par le lemme précédent, sans perte de généralité $H=A_1\oplus B_1\oplus C_1$ où $A=A_1\oplus A_2$, $B=B_1\oplus B_2$. Alors $(G/H)_p=\big((A\oplus B)+H\big)/H\cong A\oplus B/\big((A\oplus B)\cap H\big)=A\oplus B/A_1\oplus B_1\cong A_2\oplus B_2$.

Notons par θ l'isomorphisme $\theta\colon(G/H)_p\to A_2\oplus B_2$. Soit $\varphi_1\colon A\oplus B\to A_2\oplus B_2$ défini par $\varphi_1=\theta\cdot f|(A\oplus B)$. Il existe $\varphi_2\colon C\to G$ tel que $v_H\cdot\varphi_2=f$ par la projectivité de C. Si $\varphi=\varphi_1\oplus\varphi_2$, alors $v_H\cdot\varphi=f$ et donc G est $q\cdot p^3$.

Supposons maintenant que C soit p-divisible. Par le lemme 3.6, $H=A_1\oplus\oplus B_1\oplus C_1$ où, sans perte de généralité, $A=A_1\oplus A_2$, $B=B_1\oplus B_2$, $C_1\leqq C$. Alors $G/H\cong A_2\oplus B_2\oplus C/C_1=S$. Soient $\pi_1\colon S\to A_2\oplus B_2$, $\pi_2\colon S\to C/C_1$ et $\theta\colon G/H\to S$ les homomorphismes canoniques.

On définit $\varphi_1\colon A\oplus B\to A_2\oplus B_2$ par $\varphi_1=\theta\cdot f|(A\oplus B)$, $\varphi_2\colon C\to A_2\oplus B_2$ par $\varphi_2=\pi_1\theta\cdot f|C$. De plus, il existe $\varphi_3\colon C\to C$ tel que $\nu_{C_1}\cdot\varphi_3=\pi_2\theta\cdot f|C$. On définit $\varphi=\varphi_1\oplus(\varphi_2+\varphi_3)$. On vérifie facilement que $\nu_H\cdot\varphi=f$. Donc G est $q\cdot p^3$.

THÉORÈME 3.8. *Si $G=A\oplus B$ où A est un p-groupe réduit et B contient $\mathbf{Q}^p$ ou $\mathbf{Z}(p^\infty)$ et si G est $q\cdot p^3$, alors A est borné.*

PREUVE. Sinon, il existe un sous-groupe pur A_1 de A tel que $A/A_1\cong\mathbf{Z}(p^\infty)$. Soit $H=A_1\oplus B$. Alors $G/H\cong\mathbf{Z}(p^\infty)$. Soit x un élément non nul de $\mathbf{Q}^p$ ou $\mathbf{Z}(p^\infty)$ inclus dans B. Soit $f_0\colon\langle x\rangle\to G/H$ défini par $f_0(x)=g+H\neq 0$. On prolonge f_0 à $f\colon G\to G/H$ par l'injectivité de G/H.

Comme G est $q\cdot p^3$, il existe un endomorphisme φ de G tel que $\nu_H\cdot\varphi=f$ d'où $\nu_H\cdot\varphi(x)=f(x)=g+H\neq 0$. Mais $\varphi(x)\in B$ car B contient le sous-groupe p-divisible maximal de G. Donc $\varphi(x)\in H$ et $\nu_H\cdot\varphi(x)=0$ ce qui est une contradiction. Donc A doit être borné.

Le résultat suivant généralise le théorème 2.7.

THÉORÈME* 3.9. *Si G est $q\cdot p^3$, alors $G=A\oplus B$ où A est p-divisible et B p-réduit.*

PREUVE. On écrit d'abord $G=D\oplus R$ où R est réduit et $G\cong\oplus\mathbf{Z}(p^\infty)$ car $\mathbf{Z}(q^\infty)$ et $\mathbf{Q}$ ne sont pas $q\cdot p^3$. Soit M le sous-groupe p-divisible maximal de R. Alors $M\cap R_p=0$. De plus R/M est p-réduit $q\cdot p^3$. Par le théorème 3.1, $R/M=(R/M)_p\oplus\oplus S/M$ où S/M est libre et $(R/M)_p=(R_p\oplus M)/M\cong R_p$ est somme directe de groupes cycliques.

Alors $R=S\oplus R_p$, $S=M\oplus L$ où L libre $G=D\oplus M\oplus L\oplus R_p$. Si on pose $A=D\oplus M$ et $B=L\oplus R_p$ on obtient le résultat.

Nous sommes maintenant en mesure d'établir la caractérisation complète des groupes quasi-p-purs-projectifs:

THÉORÈME* 3.10. *Un groupe G est $q\cdot p^3$ si et seulement si G est d'une des cinq formes suivantes:*

(1) $G=D\oplus B\oplus L$

(2) $G=D\oplus B\oplus S$

(3) $G=A\oplus L$

(4) $G=A\oplus(\bigoplus_{q\neq p}C_q)$

(5) $G=D\oplus B\oplus(\bigoplus_{q\neq p}C_q)$

ou D est un p-groupe divisible, B est un p-groupe borné, L est libre, S est somme directe de copies de $\mathbf{Q}^p$, A est un p-groupe somme directe de groupes cycliques et C_q est un q-groupe quasi-projectif.

PREUVE. Si G est d'une des cinq formes précédentes, G est $q \cdot p^3$ par les théorèmes 1.3, 3.4 et 3.7. Ce résultat est d'ailleurs indépendant de l'hypothèse généralisée du continu.

Supposons maintenant que G est $q \cdot p^3$. Par le théorème précédent, $G = N \oplus M$ où A est p-divisible et B p-réduit. Alors, par les théorèmes 1.3, 2.12, 2.14 et 3.1, $G = D \oplus (\underset{q}{\oplus} C_q) \oplus A \oplus L$ ou $G = D \oplus S \oplus A \oplus L$ selon que N soit torsion ou non, où A, D, C_q, L, S ont la signification donnée dans l'énoncé.

Si $\underset{q}{\oplus} C_q \neq 0$, alors $L = 0$ et G est de la forme (4) ou (5) selon que $D = 0$ ou non. Si $\underset{q}{\oplus} C_q = 0$, $G = D \oplus S \oplus A \oplus L$. Si $D = 0$, $S = 0$ alors G est de la forme (3). Si $S \neq 0$, alors $A = B$ borné et $L = 0$ donc G est de la forme (2). Si $S = 0$, $D \neq 0$, $A = B$ borné et G est de la forme (1).

On remarque ici que plusieurs résultats obtenus en utilisant l'hypothèse généralisée du continu peuvent également être démontré indépendamment comme le corollaire 2.13 et le théorème 3.3 par exemple.

La totalité des résultats restent également valides si on suppose que pour tout groupe $q \cdot p^3$, on peut trouver un sous-groupe p-basique B tel G tel que $r_p(B) = r_p(G)$ et $r_0(B) = r_0(G)$.

Bibliographie

[1] D. M. ARNOLD, B. O'BRIEN and J. D. REID, Torsion free Abelian q.p.i. and q.p.p. groups, à paraître.

[2] D. M. ARNOLD, C. I. VINSONHALER and W. J. WICKLESS, Quasi-pure-projective and injective torsion free Abelian groups of rank 2, *Rocky Mountain Journal of Mathematics*, **6 (1)** (1976), 61—70.

[3] K BENABDALLAH et R. BRADLEY, Sur les groupes quasi-purs-projectifs torsions, *Canad. J. Math.*, **29** (1977), 107—110.

[4] L. FUCHS, *Infinite Abelian Groups*, Volume I, II, Academic Press (New York, 1970, 1973).

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DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ DE MONTRÉAL

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THE EXISTENCE PROBLEM FOR COLOUR CRITICAL LINEAR HYPERGRAPHS

By

H. L. ABBOTT and A. C. LIU (Edmonton)

1. Introduction

One of the most extensively studied concepts of ordinary graph theory is that of chromatic number. We remind the reader that a 2-graph is said to have chromatic number r if r is the least positive integer for which there exists some way of colouring the vertices of the graph in r colours so that no edge joins vertices of the same colour. This notion generalizes in a natural way to hypergraphs.

A *hypergraph* is an ordered pair $(V, \mathcal{F})$ where V is a finite non-empty set and $\mathcal{F}$ is a non-empty collection of subsets of V. We shall always assume that $V = \bigcup \mathcal{F}$, so that we may speak of the hypergraph $\mathcal{F}$ rather than $(V, \mathcal{F})$. The elements of $\bigcup \mathcal{F}$ are called the *vertices* of $\mathcal{F}$ while the elements of $\mathcal{F}$ are called the *edges* of $\mathcal{F}$.

A hypergraph $\mathcal{F}$ is said to be *uniform* if for all $F \in \mathcal{F}$, $|F| = n$ for some integer $n \geqq 2$. Then $\mathcal{F}$ is called an *n-graph*. An n-graph $\mathcal{F}$ is said to be *linear* if $|F \cap F'| \leqq 1$ for all $F, F' \in \mathcal{F}$, $F \neq F'$, and is said to be *regular* of *degree t* if all its vertices belong to exactly t edges for some integer t.

An n-graph $\mathcal{F}$ is said to be *r-colourable* if there exists a function $\varphi: \bigcup \mathcal{F} \to \{1, \ldots, r\}$ such that $|\varphi(F)| \geqq 2$ for all $F \in \mathcal{F}$. We call φ an *r-colouring* of $\mathcal{F}$. $\mathcal{F}$ is *r-chromatic* if it is r-colourable but not $(r-1)$-colourable and we then call r the *chromatic number* of $\mathcal{F}$. $\mathcal{F}$ is *r-critical* if it is r-chromatic and all its proper subgraphs are $(r-1)$-colourable. $\mathcal{F}$ is *critical* if it is r-critical for some r.

Critical 2-graphs were first investigated by Dirac and subsequently by many other authors (see for example [9] and references given there). It is a simple matter to verify that the only 3-critical 2-graphs are the circuits of odd length. It was hoped at one time that a characterization of the 4-critical 2-graphs would be useful in tackling the celebrated Four Colour Problem. However, no such characterization has been found and, in fact, recent results (see, for example, SIMONOVITS [11] and TOFT [12]) indicate that the 4-critical 2-graphs may be quite complicated and that perhaps no simple characterization is possible.

By an *(m, n, r)-graph* we shall mean an r-critical n-graph on m vertices. There are two questions which arise:

(A) Given integers n and r, for which integers m do (m, n, r)-graphs exist?

(B) Given integers n and r, for which integers m do linear (m, n, r)-graphs exist?

For 2-graphs, and here the two problems coincide, $(m, 2, 3)$-graphs exist only when $m \geqq 3$ is odd. This is just a restatement of the fact that the only 3-critical 2-graphs are the odd circuits. For $r \geqq 4$, DIRAC [4] proved that $(m, 2, r)$-graphs exist only when $m = r$ or $m \geqq r + 2$.

For $n \geqq 3$, let

$$M(n, r) = (n-1)(r-1)+1.$$

It is easy to verify that the graph whose edges are the n-element subsets of a set of $M(n, r)$ elements is r-critical. Recently, it has been shown that (m, n, r)-graphs exist for all $m \geqq M(n, r)$. This was done by ABBOTT and HANSON [3] for the case $r=3$ and by TOFT [13] for $r \geqq 4$. Since (m, n, r)-graphs do not exist for $m < M(n, r)$, problem (A) has been solved completely.

The constructions of Toft, Abbott and Hanson do not yield linear n-graphs, so that their work sheds no light on problem (B). *A priori*, it is not obvious whether linear (m, n, r)-graphs exist for any value of m.

The question seems to have been first raised by Gallai. ERDŐS and HAJNAL [5] mention the problem and point out that the Steiner triple system on 7 points is an example of a linear (7, 3, 3)-graph, but give no other examples.

Proofs of the existence of linear (m, n, r)-graphs for each pair of integers n and r, $n \geqq 3$, $r \geqq 3$, and some integer m, were given by several authors at about the same time. A simple proof based on Ramsey's Theorem was given by ABBOTT [1] for the case $r=3$, and his proof easily generalizes to $r \geqq 4$. Other proofs were given by ERDŐS and HAJNAL [6], LOVÁSZ [8] and HALES and JEWETT [7]. In all these papers, the n-graphs constructed are not necessarily critical, but since any r-chromatic n-graph contains an r-critical subgraph, there is no problem.

The papers of Lovász, Erdős and Hajnal referred to above also establish the existence of arbitrarily large integers m for which linear (m, n, r)-graphs exist. A simple proof of this is given by ABBOTT [2], in the case $r=3$, which we shall generalize into one of our Constructions in § 2.

Our main result is the following

THEOREM 1. *For $n \geqq 3$ and $r \geqq 3$, there exists a least integer $M^*(n, r)$ such that for $m \geqq M^*(n, r)$, a linear (m, n, r)-graph exists.*

The determination of the numbers $M^*(n, r)$ seems to be very difficult, and we have succeeded in determining only one value. We state this as

THEOREM 2. $M^*(3, 3)=9$.

Even the next non-trivial values, $M^*(3, 4)$ and $M^*(4, 3)$, seem to be out of reach. The main difficulty lies in finding actual examples of linear (m, n, r)-graphs. We have been able to obtain the following bounds:

THEOREM 3. $M^*(4, 3) \leqq 124$.

THEOREM 4. $M^*(3, 4) \leqq 1399$.

However, we have no reason to believe that these are sharp.

§ 2. Main result

In this section, we establish the existence of linear (m, n, r)-graphs for sufficiently large integers m. We first describe two constructions of such graphs.

CONSTRUCTION 1. *Let $n \geqq 3$ and $r \geqq 3$. Let l be an integer such that a linear (l, n, r)-graph $\mathcal{G}$ exists. For $1 \leqq i \leqq l$, let m_i be an integer such that a linear $(m_i, n, r+1)$-*

graph $\mathscr{F}_i$ *exists. Then there exists a linear* $(m, n, r+1)$*-graph where* $m=1+m_1+\dots$ $\dots+m_l$.

PROOF. Let $\cup\mathscr{G}=\{a_1, \dots, a_l\}$, $V_i=\cup\mathscr{F}_i$ and let the V's be pairwise disjoint. Let F_i be a fixed edge of $\mathscr{F}_i$ and v_i a fixed vertex of F_i. Let v be a vertex which is not in any of the V's. Let $K_i=(F_i-\{v_i\})\cup\{v\}$.

Let $\mathscr{F}$ be the n-graph consisting of all the edges of $\mathscr{F}_i$, $1\leqq i\leqq l$, with F_i replaced by K_i, together with the edges of $\mathscr{G}$ with a_i replaced by v_i, so that $\mathscr{F}$ contains a subgraph isomorphic to $\mathscr{G}$.

It is easily verified that $\mathscr{F}$ is linear and that $|\cup\mathscr{F}|=m$. We need to show that $\mathscr{F}$ is $(r+1)$-critical. We do this in two steps.

Step 1: $\mathscr{F}$ is not r-colourable. Suppose $\mathscr{F}$ has an r-colouring ψ. Then ψ is also an r-colouring of $\mathscr{F}_i-\{F_i\}$ for each i, $1\leqq i\leqq l$. Clearly we must have $\psi(v)\neq\psi(v_i)$. Thus ψ is an $(r-1)$-colouring of the n-graph obtained from $\mathscr{G}$ by replacing a_i by v_i. This is a contradiction since $\mathscr{G}$ is r-chromatic.

Step 2: all subgraphs of $\mathscr{F}$ are r-colourable. We need only consider the subgraphs of the form $\mathscr{F}-\{F\}$ for $F\in\mathscr{F}$. We consider three cases:

(i) $F=\{v_{i_1}, \dots, v_{i_l}\}$ where $G=\{a_{i_1}, \dots, a_{i_l}\}\in\mathscr{G}$. Since $\mathscr{G}$ is r-critical, $\mathscr{G}-\{G\}$ has an $(r-1)$-colouring $\psi_{\mathscr{G}}$. Let ψ_i be an r-colouring of $\mathscr{F}_i-\{F_i\}$ such that $\psi_i(v_i)=\psi_{\mathscr{G}}(a_i)$. Define $\psi:\cup\mathscr{F}\to\{1, \dots, r\}$ by:

$$\psi(x)=\begin{cases}\psi_i(x) & \text{if } x\in V_i\\ r & \text{if } x=v.\end{cases}$$

It is easily verified that ψ is an r-colouring of $\mathscr{F}-\{F\}$.

(ii) $F=K_j$ for some j, $1\leqq j\leqq l$. Since $\mathscr{G}$ is r-critical, $\mathscr{G}$ has an r-colouring $\varphi_{\mathscr{G}}$ such that $\varphi_{\mathscr{G}}(a_j)\neq\varphi_{\mathscr{G}}(x)$ for any $x\in\mathscr{G}$, $x\neq a_j$. Let ψ_i be an r-colouring of $\mathscr{F}_i-\{F_i\}$ such that $\psi_i(v_i)=\varphi_{\mathscr{G}}(a_i)$. Define $\psi:\cup\mathscr{F}\to\{1, \dots, r\}$ by:

$$\psi(x)=\begin{cases}\psi_i(x) & \text{if } x\in V_i\\ \varphi_{\mathscr{G}}(a_j) & \text{if } x=v.\end{cases}$$

(iii) $F\in\mathscr{F}_j-\{F_j\}$ for some j, $1\leqq j\leqq l$. Since $\mathscr{G}$ is r-critical, $\mathscr{G}$ has an r-colouring $\varphi_{\mathscr{G}}$ such that $\varphi_{\mathscr{G}}(a_j)\neq\varphi_{\mathscr{G}}(x)$ for any $x\in\cup\mathscr{G}$, $x\neq a_j$. For $i\neq j$ let ψ_i be an r-colouring of $\mathscr{F}_i-\{F_i\}$ such that $\psi_i(v_i)=\varphi_{\mathscr{G}}(a_i)$. Let ψ_j be an r-colouring of $\mathscr{F}_j-\{F\}$ such that $\psi_j(v_j)=\varphi_{\mathscr{G}}(a_j)$. Define $\psi:\cup\mathscr{F}\to\{1, 2, \dots, r\}$ by:

$$\psi(x)=\begin{cases}\psi_i(x) & \text{if } x\in V_i\\ \varphi_{\mathscr{G}}(a_j) & \text{if } x=v.\end{cases}$$

It is easily verified that ψ is an r-colouring of $\mathscr{F}-\{F\}$.

CONSTRUCTION 1*. *Let* $n\geqq 3$ *and* $(n+1)/2\leqq l\leqq n$. *For* $1\leqq i\leqq l$, *let* m_i *be an integer such that a linear* $(m_i, n, 3)$*-graph* $\mathscr{F}_i$ *exists. Then there exists a linear* $(m, n, 3)$*-graph where* $m=1+m_1+\dots+m_l$.

PROOF. Let $V_i=\cup\mathscr{F}_i$ and let the V's be pairwise disjoint. Let F_i be a fixed edge of $\mathscr{F}_i$ and v_i, v_i' be fixed vertices of F_i. Let v be a vertex which is not in any of the V's. Let $K_i=(F_i-\{v_i\})\cup\{v\}$. Let $\mathscr{F}$ be the n-graph consisting of all the edges of $\mathcal{F}_i$, $1 \leqq i \leqq l$, with F_i replaced by K_i, together with an extra edge containing v_i, $1 \leqq i \leqq l$, and any $n-l$ elements of $\{v_i' \mid 1 \leqq i \leqq l\}$. The proof that $\mathcal{F}$ is a linear $(m, n, 3)$-graph follows closely that of Construction 1, and is omitted.

Construction 2. *Let $n \geqq 3$ and let $k=2$ or 3. Let l be an integer for which there exists a linear $(l, k, 3)$-graph $\mathcal{G}$ which is regular of degree k. For $1 \leqq i \leqq l$, let m_i be an integer for which a linear $(m_i, n, 3)$-graph $\mathcal{F}_i$ exists. Then there exists a linear $(m, n, 3)$-graph where $m = m_1 + \ldots + m_l$.*

Proof. Let $\cup \mathcal{G} = \{a_1, \ldots, a_l\}$. We note first that the number of edges of $\mathcal{G}$ is l and that, by the König—Hall Theorem, one may label the edges $G_1, \ldots, G_l$ so that $a_j \in G_j$. Let $Q = \langle q_{ij} \rangle$ be an $l \times l$ matrix, whose entries are ordered pairs of integers, defined as follows:

$$q_{ij} = \begin{cases} (i, j) & \text{if } a_i \in G_j \\ (0, 0) & \text{otherwise.} \end{cases}$$

Call $(0, 0)$ a zero element and the other entries non-zero elements.

Let $V_i = \cup \mathcal{F}_i$ and let the V's be pairwise disjoint. Let F_i be a fixed edge of $\mathcal{F}_i$ and let H_i be a fixed $(k-1)$-subset of F_i. Let ζ_i be a mapping from F_i into the i-th row of Q which maps $F_i - H_i$ onto the diagonal element (i, i) and maps H_i one-one onto the other $k-1$ non-zero elements of the row. The mappings ζ_i induce a mapping ζ from $F_1 \cup \ldots \cup F_l$ onto the non-zero elements of Q. Let

$$K_j = \{x \in F_1 \cup \ldots \cup F_l \colon \zeta(x) \text{ is in the } j\text{-th column of } Q\}.$$

Clearly each K_j is an n-set. Let $\mathcal{F}$ be the n-graph consisting of all the edges of $\mathcal{F}_i$ with F_i replaced by K_i.

It is easily verified that $\mathcal{F}$ is linear and that $|\cup \mathcal{F}| = m$. We need to show that $\mathcal{F}$ is 3-critical. We do this in two steps.

Step 1: $\mathcal{F}$ is not 2-colourable. Suppose $\mathcal{F}$ has a 2-colouring ψ. Then ψ is also a 2-colouring of $\mathcal{F}_i - \{F_i\}$. Since $\mathcal{F}_i$ is 3-critical, $|\psi(F_i)| = 1$. Define $\psi_{\mathcal{G}} \colon \cup \mathcal{G} \to \{1, 2\}$ by: $\psi_{\mathcal{G}}(a_j) = \psi(F_j)$ for $1 \leqq j \leqq l$. Since $\mathcal{G}$ is 3-chromatic, there exists $G_t \in \mathcal{G}$ such that $|\psi_{\mathcal{G}}(G_t)| = 1$. It follows that $|\psi(K_t)| = 1$. This is a contradiction.

Step 2: all subgraphs of $\mathcal{F}$ are 2-colourable. We need only consider the subgraphs of the form $\mathcal{F} - \{F\}$ for $F \in \mathcal{F}$. We consider two cases:

(i) $F = K_t$ for some t, $1 \leqq t \leqq l$. Since $\mathcal{G}$ is 3-critical, $\mathcal{G} - \{G_t\}$ has a 2-colouring ψ'. Let ψ_i be a 2-colouring of $\mathcal{F}_i - \{F_i\}$ such that $\psi_i(F_i) = \psi'(a_i)$. Clearly the mappings ψ_i induce a 2-colouring of $\mathcal{F} - \{F\}$.

(ii) $F \in \mathcal{F}_t - \{F_t\}$ for some t, $1 \leqq t \leqq l$. We have a 2-colouring ψ_t of $\mathcal{F}_t - \{F\}$. We treat separately the cases $k=2$ and $k=3$.

(a) $k=2$. In this case, $H_t = \{b\}$. for some b. We have a 2-colouring $\psi_{\mathcal{G}}$ of $\mathcal{G} - \{G_t\}$ such that $\psi_{\mathcal{G}}(G_t) = \psi_t(b)$, and a 2-colouring ψ_i of $\mathcal{F}_i - \{F_i\}$, $i \neq t$, such that $\psi_i(F_i) = \psi_{\mathcal{G}}(a_i)$. Clearly the mappings ψ_i induce a 2-colouring of $\mathcal{F} - \{F\}$.

(b) $k=3$. In this case, $H_t = \{b, c\}$ for some b and c. If $\psi_t(b) = \psi_t(c)$, the argument is exactly the same as in case (a). If $\psi_t(b) \neq \psi_t(c)$, we may assume that $\psi_t(b) \neq \psi_t(x)$ for some $x \in F_t - H_t$. Let $\zeta(c)$ belong to the h-th column of Q. Then the argument of case (a) applies with G_t replaced by G_h.

We now prove our main result. We shall need two lemmas, the first one having been mentioned in § 1. The proofs are not difficult and will not be presented.

LEMMA 1. *For* $n \geqq 3$ *and* $r \geqq 3$, *a linear* (m, n, r)*-graph exists for some integer* m.

LEMMA 2. *Let* S *be a set of integers such that* S *contains* a *and* $a+1$ *for some integer* a *and for some integer* $k \geqq 2$, $a_1+\ldots+a_k \in S$ *whenever* $a_1, \ldots, a_k \in S$. *Then* S *contains all integers greater than or equal to* $a(ak-a+1)$.

THEOREM 2.1. *For* $n \geqq 3$ *and* $r \geqq 3$, *there exists a least integer* $M^*(n, r)$ *such that for* $m \geqq M^*(n, r)$, *a linear* (m, n, r)*-graph exists.*

PROOF. We use induction on r. We first establish the existence of $M^*(n, 3)$ for fixed n. Let G be the set of integers m for which linear $(m, n, 3)$-graphs exist. We need to show that S contains all sufficiently large integers. We consider two cases:

(i) *n is odd.* By Lemma 1, there exists an integer $m \in S$. Now an odd circuit of length n is a regular linear $(n, 2, 3)$-graph of degree 2. By Construction 2, $a_1+\ldots+a_n \in S$ if $a_1, \ldots, a_n \in S$. Thus $mn \in S$. By Construction 1*, we have $mn+1 \in S$. By Lemma 2, S contains all sufficiently large integers.

(ii) *n is even.* By Lemma 1, there exists an integer $m \in S$. From the $(7, 3, 3)$-graph we can obtain by Construction 1* a linear $(22, 3, 3)$-graph which is regular of degree 3. By Construction 2, $a_1+\ldots+a_{22} \in S$ if $a_1, \ldots, a_{22} \in S$. Thus $22m \in S$. In Construction 1*, take $l=n$, $m_1=22m$ and $m_i=m$ for $i>1$. It follows that $m(n+21)+1 \in S$. Now an odd circuit of length $21+n$ is a regular linear $(21+n, 2, 3)$-graph of degree 2. By Construction 2, $m(n+21) \in S$. By Lemma 2, S contains all sufficiently large integers.

Suppose $r \geqq 3$ and that $M^*(n, r)$ exists. By Lemma 1, there is an integer m_0 for which there exists a linear $(m_0, n, r+1)$-graph. By Construction 1, there exists a linear $(m_0 M^*(n, r)+1, n, r+1)$-graph. Let

$$m \geqq m_0(m_0+1) M^*(n, r) \tag{1}$$

and write

$$m = qm_0+b, \quad 1 \leqq b \leqq m_0. \tag{2}$$

From (1) and (2), it follows easily that

$$q \geqq m_0 M^*(n, r). \tag{3}$$

Let

$$t = q-(b-1) M^*(n, r) \tag{4}$$

so that by (3) and (4),

$$t \geqq M^*(n, r). \tag{5}$$

By the induction hypothesis, (5) and the fact that $b \geqq 1$, there exists a linear $(t+b-1, n, r)$-graph. In Construction 1, take

$$\begin{cases} m_1 = \ldots = m_t = m_0 \\ m_{t+1} = \ldots = m_{t+b-1} = m_0 M^*(n, r)+1. \end{cases} \tag{6}$$

Now by (2), (4) and (6), $1+m_1+\ldots+m_{t+b-1}=m$ so that, by Construction 1, a linear $(m, n, r+1)$-graph exists. This completes the proof.

§ 3. Some special constructions

In this section, we determine the value of $M^*(3,3)$ and give upper bounds for $M^*(4,3)$ and $M^*(3,4)$. We first describe two other constructions of linear (m, n, r)-graphs.

CONSTRUCTION 3. *Let* $n \geqq 3$ *and let* $n-2 \leqq l \leqq n^2-3n+3$. *For* $1 \leqq i \leqq l$, *let* m_i *be an integer such that a linear* $(m_i, n, 3)$*-graph* $\mathcal{F}_i$ *exists. Then there exists a linear* $(m, n, 3)$*-graph where* $m = n + m_1 + \ldots + m_l$.

PROOF. Let $V_i = \bigcup \mathcal{F}_i$ and let the V's be pairwise disjoint. Let $F_i = \{a_1^i, \ldots, a_n^i\}$ be a fixed edge of $\mathcal{F}_i$. Let $v_1, \ldots, v_n$ be vertices which are not in any of the V's.

Let $\mathcal{F}$ be the n-graph consisting of the following edges:

(1) all edges in $\mathcal{F}_i - \{F_i\}$, $1 \leqq i \leqq l$;

(2) the edges $G_i = \{v_1, a_2^i, \ldots, a_n^i\}$, $1 \leqq i \leqq l$;

(3) the edge $H_1 = \{v_1, \ldots, v_n\}$;

(4) the edges H_j, $2 \leqq j \leqq n$, satisfying the following conditions:

a) $H_j \subset \left(\bigcup_{i=1}^{l} F_i\right) \cup H_1$;

b) $a_1^1 \in H_j$;

c) $H_1 \cap H_j = \{v_j\}$;

d) each of $a_1^2, \ldots, a_1^l$ appears exactly once in $\bigcup_{j=2}^{n} H_j$;

e) each vertex of $\bigcup_{i=1}^{l} (G_i - \{v_1\})$ appears at most once in $\bigcup_{j=2}^{n} H_j$.

The choice of l ensures that these conditions can be met.

It is easily verified that $\mathcal{F}$ is linear and that $|\bigcup \mathcal{F}| = m$. We need to show that $\mathcal{F}$ is 3-critical. We do this in two steps.

Step 1: $\mathcal{F}$ is not 2-colourable. Suppose $\mathcal{F}$ has a 2-colouring ψ. Then ψ is also a 2-colouring of $\mathcal{F}_i - \{F_i\}$ for each i, $1 \leqq i \leqq l$. We may suppose that $\psi(F_1) = 1$. By considering the edge G_1, we conclude that $\psi(v_1) = 2$. It follows that $\psi(F_i) = 1$ also for $2 \leqq i \leqq l$. By considering the edges H_j, we conclude that $\psi(v_j) = 2$ for $2 \leqq j \leqq n$. It follows that $|\psi(H_1)| = 1$. This is a contradiction.

Step 2: all subgraphs of $\mathcal{F}$ are 2-colourable. We need only consider the subgraphs of the form $\mathcal{F} - \{F\}$ for $F \in \mathcal{F}$. We consider six cases:

(i) $F = H_1$. Let ψ_i be a 2-colouring of $\mathcal{F}_i - \{F_i\}$ such that $\psi_i(F_i) = 1$. Define $\psi : \bigcup \mathcal{F} \to \{1, 2\}$ by:

$$\psi(x) = \begin{cases} \psi_i(x) & \text{if } x \in V_i \\ 2 & \text{if } x \in H_1. \end{cases}$$

It is easily verified that ψ is a 2-colouring of $\mathcal{F} - \{F\}$.

(ii) $F=H_j$ *for some* j, $2\leqq j\leqq n$. Let ψ_i be a 2-colouring of $\mathscr{F}_i-\{F_i\}$ such that $\psi_i(F_i)=1$. Define $\psi:\cup\mathscr{F}\to\{1,2\}$ by:

$$\psi(x)=\begin{cases}\psi_i(x) & \text{if } x\in V_i\\ 1 & \text{if } x=v_j\\ 2 & \text{if } x\in H_1-\{v_j\}.\end{cases}$$

Itis (ieasily verified that ψ is a 2-colouring of $\mathscr{F}-\{F\}$.

ii) $F=G_1$. Let ψ_1 be a 2-colouring of $\mathscr{F}_1-\{F_1\}$ such that $\psi_1(F_1)=1$. For $2\leqq i\leqq l$, let ψ_i be a 2-colouring of $\mathscr{F}_i-\{F_i\}$ such that $\psi_i(F_i)=2$. Define $\psi:\cup\mathscr{F}\to\to\{1,2\}$ by

$$\psi(x)=\begin{cases}\psi_i(x) & \text{if } x\in V_i\\ 1 & \text{if } x=v_1\\ 2 & \text{if } x\in H_1-\{v_1\}.\end{cases}$$

It is easily verified that ψ is a 2-colouring of $\mathscr{F}-\{F\}$.

(iv) $F=G_t$ *for some* t, $2\leqq t\leqq l$. Let $a_1^t\in H_j$. Let ψ_t be a 2-colouring of $\mathscr{F}_t-\{F_t\}$ such that $\psi_t(F_t)=1$. For $i\neq t$, let ψ_i be a 2-colouring of $\mathscr{F}_i-\{F_i\}$ such that $\psi_i(F_i)=2$. Define $\psi:\cup\mathscr{F}\to\{1,2\}$ by:

$$\psi(x)=\begin{cases}\psi_i(x) & \text{if } x\in V_i\\ 2 & \text{if } x=v_j\\ 1 & \text{if } x\in H_1-\{v_j\}.\end{cases}$$

It is easily verified that ψ is a 2-colouring of $\mathscr{F}-\{F\}$.

(v) $F\in\mathscr{F}_1-\{F_1\}$. Let ψ_1 be a 2-colouring of $\mathscr{F}_1-\{F\}$ such that $\psi_1(a_1^1)=1$. For $2\leqq i\leqq l$, let ψ_i be a 2-colouring of $\mathscr{F}_i-\{F_i\}$ such that $\psi_i(F_i)=2$. Define $\psi:\cup\mathscr{F}\to\{1,2\}$ by:

$$\psi(x)=\begin{cases}\psi_i(x) & \text{if } x\in V_i\\ 1 & \text{if } x=v_1\\ 2 & \text{if } x\in H_1-\{v_1\}.\end{cases}$$

It is easily verified that ψ is a 2-colouring of $\mathscr{F}-\{F\}$.

(vi) $F\in\mathscr{F}_t-\{F_t\}$ *for some* t, $2\leqq t\leqq l$. Let $a_1^t\in H_j$. Let ψ_t be a 2-colouring of $\mathscr{F}_t-\{F\}$ such that $\psi_t(a_1^t)=1$. For $i\neq t$, let ψ_i be a 2-colouring of $\mathscr{F}_i-\{F_i\}$ such that $\psi_i(F_i)=2$. Define $\psi:\cup\mathscr{F}\to\{1,2\}$ by:

$$\psi(x)=\begin{cases}\psi_i(x) & \text{if } x\in V_i\\ 2 & \text{if } x=v_j\\ 1 & \text{if } x\in H_1-\{v_j\}.\end{cases}$$

It is easily verified that ψ is a 2-colouring of $\mathscr{F}-\{F\}$.

CONSTRUCTION 4. *Let* $r\geqq 2$ *and let* l *be an integer such that a linear* $(l,3,r)$*-graph* $\mathscr{G}$ *exists. For* $1\leqq i\leqq l$, *let* m_i *be an integer such that a linear* $(m_i,3,r+2)$*-graph* $\mathscr{F}_i$ *exists. Then there exists a linear* $(m,3,r+2)$*-graph where* $m=3+$ $+m_1+\ldots+m_l$.

PROOF. Let $\bigcup\mathcal{G}=\{x_1, \dots, x_l\}$. Let $V_i=\bigcup\mathcal{F}_i$ and let the V's be pairwise disjoint. Let $F_i=\{a_1^i, a_2^i, a_3^i\}$ be a fixed edge of $\mathcal{F}_i$. Let v_1, v_2, v_3 be vertices which are not in any of the V's.

Let $\mathcal{F}$ be the n-graph consisting of the following edges:

(1) all edges of $\mathcal{F}_i-\{F_i\}$, $1\leqq i\leqq l$;

(2) the edges $G_1^i=\{v_1, a_2^i, a_3^i\}$, $1\leqq i\leqq l$;

(3) the edges $G_2^i=\{a_1^i, v_2, a_3^i\}$, $1\leqq i\leqq l$;

(4) the edges $G_3^i=\{a_1^i, a_2^i, v_3\}$, $1\leqq i\leqq l$;

(5) the edge $\{v_1, v_2, v_3\}$;

(6) all edges of $\mathcal{G}$, with x_i replaced by a_1^i.

It is easily verified that $\mathcal{F}$ is linear and that $|\bigcup\mathcal{F}|=m$. We need to show that $\mathcal{F}$ is $(r+2)$-critical. We do this in two steps.

Step 1: $\mathcal{F}$ is not $(r+1)$-colourable. Suppose $\mathcal{F}$ has an $(r+1)$-colouring ψ. Then ψ is also an $(r+1)$-colouring of $\mathcal{F}_i-\{F_i\}$ for each i, $1\leqq i\leqq l$. Since $\mathcal{F}_i$ is $(r+2)$-critical, $|\psi(F_i)|=1$. Now $|\psi(\{v_1, v_2, v_3\})|\geqq 2$, and by considering the edges G_1^i, G_2^i and G_3^i, we conclude that $|\psi(\{a_1^i|1\leqq i\leqq l\})|\leqq r-1$. This is a contradiction as $\mathcal{G}$ is r-critical.

Step 2: all subgraphs of $\mathcal{F}$ are $(r+1)$-colourable. We need only to consider the subgraphs of the form $\mathcal{F}-\{F\}$ for $F\in\mathcal{F}$. We consider four cases:

(i) $F=\{v_1, v_2, v_3\}$. Let $\varphi_{\mathcal{G}}$ be an r-colouring of $\mathcal{G}$. Let ψ_i be an $(r+1)$-colouring of $\mathcal{F}_i-\{F_i\}$ such that $\psi_i(a_1^i)=\varphi_{\mathcal{G}}(x_i)$. Define $\psi=\bigcup\mathcal{F}\to\{1, \dots, r+1\}$ by

$$\psi(x)=\begin{cases}\psi_i(x) & \text{if } x\in V_i\\ r+1 & \text{if } x=v_1, v_2, v_3.\end{cases}$$

It is easily verified that ψ is an $(r+1)$-colouring of $\mathcal{F}-\{F\}$.

(ii) $F=G_1^j, G_2^j$ *or* G_3^j *for some* j, $1\leqq j\leqq l$. We may assume that $F=G_1^j$. Since $\mathcal{G}$ is r-critical, $\mathcal{G}$ has an r-colouring $\varphi_{\mathcal{G}}$ such that $\varphi_{\mathcal{G}}(x_j)\neq\varphi_{\mathcal{G}}(x)$ for any $x\in\bigcup\mathcal{G}$, $x\neq x_j$. Let ψ_i be an $(r+1)$-colouring of $\mathcal{F}_i-\{F_i\}$ such that $\psi_i(F_i)=\varphi_{\mathcal{G}}(x_i)$. Define $\psi:\bigcup\mathcal{F}\to\{1, \dots, r+1\}$ by

$$\psi(x)=\begin{cases}\psi_i(x) & \text{if } x\in V_i\\ \varphi_{\mathcal{G}}(x_j) & \text{if } x=v_1\\ r+1 & \text{if } x=v_2, v_3.\end{cases}$$

It is easily verified that ψ is an $(r+1)$-colouring of $\mathcal{F}-\{F\}$.

(iii) $F=\{a_1^{i_1}, a_2^{i_2}, a_1^{i_3}\}$ *where* $G=\{x_{i_1}, x_{i_2}, x_{i_3}\}\in\mathcal{G}$. Since $\mathcal{G}$ is r-critical, $\mathcal{G}-\{G\}$ has an $(r-1)$-colouring $\psi_{\mathcal{G}}$. Let ψ_i be an $(r+1)$-colouring of $\mathcal{F}_i-\{F_i\}$ such that $\psi_i(F_i)=\psi_{\mathcal{G}}(x_i)$. Define $\psi:\bigcup\mathcal{F}\to\{1, \dots, r+1\}$ by

$$\psi(x)=\begin{cases}\psi_i(x) & \text{if } x\in V_i\\ r & \text{if } x=v_1\\ r+1 & \text{if } x=v_2, v_3.\end{cases}$$

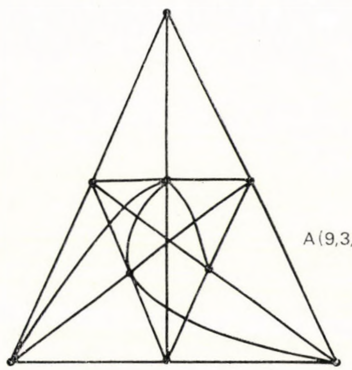
It is easily verified that ψ is an $(r+1)$-colouring of $\mathcal{F}-\{F\}$.

(iv) $F\in\mathcal{F}_j-\{F_j\}$ *for some* j, $1\leqq j\leqq l$. Since $\mathcal{G}$ is r-critical, $\mathcal{G}$ has an r-colouring $\varphi_{\mathcal{G}}$ such that $\varphi_{\mathcal{G}}(x_j)\neq\varphi_{\mathcal{G}}(x)$ for all $x\in\bigcup\mathcal{G}$, $x\neq x_j$. For $i\neq j$, let ψ_i be an $(r+1)$-colouring of $\mathcal{F}_i-\{F_i\}$ such that $\psi_i(F_i)=\varphi_{\mathcal{G}}(x_i)$. Let ψ_j be an $(r+1)$-colouring of $\mathscr{F}_j - \{F\}$. Without loss of generality we may suppose that $\varphi_j(a_1^j) \neq \varphi_j(a_2^j)$. Define $\psi: \cup \mathscr{F} \to \{1, 2, ..., r+1\}$ by

$$\psi(x) = \begin{cases} \psi_i(x) & \text{if } x \in V_i \\ \varphi_{\mathscr{G}}(x_j) & \text{if } x = v_1 \\ r+1 & \text{if } x = v_2 \text{ or } v_3. \end{cases}$$

THEOREM 2. $M^*(3, 3) = 9$,

PROOF. It is not difficult to verify that linear (8, 3, 3)-graphs do not exist. The Steiner triple system on 7 points is a (7, 3, 3)-graph and linear (9, 3, 3) and (11, 3, 3)-graphs are shown in Fig. 1. By Construction 3, with $n=3$ and taking $l=1$, we can obtain a linear $(m+3, 3, 3)$-graph from a linear $(m, 3, 3)$-graph. The Theorem follows immediately.



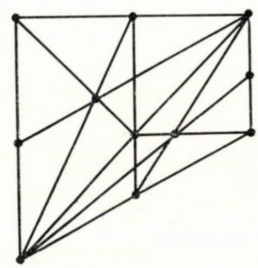


Fig. 1

It is easy to show that linear $(m, 3, 3)$-graphs do not exist for $m \leqq 6$. The existence problem in the case $n=r=3$ is thus solved completely. There exists a linear $(m, 3, 3)$-graph if and only if $m=7$ or $m \geqq 9$.

THEOREM 3. $M^*(4, 3) \leqq 124$.

PROOF. We have verified that there exist block designs with parameters (25, 50, 8, 4, 1) and (28, 63, 9, 4, 1) which contain a linear (25, 4, 3)-graph and a linear (28, 4, 3)-graph, respectively. Using these as building blocks and applying Constructions 1, 1*, 2 and 3, we obtain linear $(m, 4, 3)$-graphs for $124 \leqq m \leqq 149$, a block of 26 consecutive integers. By Construction 1*, with $n=4$ and taking $l=2$, $m_1=m$ and $m_2=25$, we can obtain a linear $(m+26, 4, 3)$-graph from a linear $(m, 4, 3)$-graph. The Theorem follows immediately.

THEOREM 4. $M^*(3, 4) \leqq 1399$.

PROOF. There exists a 4-chromatic linear 3-graph on 31 vertices [10] and we have verified that it contains a linear (31, 3, 4)-graph. Using it as a building block and applying Constructions 1 and 4, we obtain linear $(m, 3, 4)$-graphs for $1399 \leqq m \leqq 1463$, a block of 65 consecutive integers. By Construction 4, with $r=2$ and taking $l=3$ and $m_2=m_3=31$, we can obtain a linear $(m+65, 3, 4)$-graph from a linear $(m, 3, 4)$-graph. The Theorem follows immediately.

References

[1] H. L. Abbott, An application of Ramsey's Theorem to a problem of Erdős and Hajnal, *Can. Math. Bull.*, **8** (1965), 515—517.
[2] H. L. Abbott, On property B of family of sets, *Can. Math. Bull.*, **18** (1975), 133—135.
[3] H. L. Abbott and D. Hanson, On a combinatorial problem of Erdős, *Can. Math. Bull.*, **12** (1969), 823—829.
[4] G. A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, *J. London Math. Soc.*, **27** (1952), 85—92.
[5] P. Erdős and A. Hajnal, On a property of families of sets, *Acta Math. Acad. Sci. Hungar.*, **12** (1961), 87—123.
[6] P. Erdős and A. Hajnal, Chromatic numbers of graphs and set systems, *Acta Math. Acad. Sci. Hungar.*, **17** (1966), 61—99.
[7] A. W. Hales and R. I. Jewett, Regularity and positional games, *Trans. Am. Math. Soc.*, **106** (1963), 222—229.
[8] L. Lovász, On chromatic numbers of finite set systems, *Acta Math. Acad. Sci. Hungar.*, **19** (1968), 59—67.
[9] O. Ore, *The Four Color Problem*, Acad. Press (1967).
[10] A. Rosa, Steiner triple systems and their chromatic number, *Acta Univ. Comen. Math.*, **24** (1970), 159—174.
[11] M. Simonovits, On color-critical graphs, *Stud. Sci. Math. Hungar.*, **7** (1972), 67—81.
[12] B. Toft, Two theorems on critical 4-chromatic graphs, *Stud. Sci. Math. Hungar.*, **7** (1972), 83—89.
[13] B. Toft, On color-critical hypergraphs, *Infinite and Finite Sets*, ed. Hajnal et al., North-Holland Publ. Co. (1975), 1445—1457.

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UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA, CANADA

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MAPPINGS ON METRIC SPACES

By

B. FISHER (Leicester)

A contraction mapping of a metric space (X, ϱ) into itself is a mapping T such that

$$\varrho(Tx, Ty) \leqq c\varrho(x, y),$$

for all x, y in X, where $0 \leqq c < 1$.

In a paper by R. KANNAN [1], he considers a mapping T of a metric space (X, ϱ) into itself such that

$$\varrho(Tx, Ty) \leqq k\{\varrho(x, Tx) + \varrho(y, Ty)\},$$

for all x, y in X, where $0 \leqq k < \frac{1}{2}$. We will call such a mapping a Kannan mapping.

We now prove the following theorem:

THEOREM 1. *Let T be a contraction mapping of the metric space (X, ϱ) into itself. Then T^n is a Kannan mapping for some positive integer n.*

PROOF. Suppose

$$\varrho\{Tx, Ty) \leqq c\varrho(x, y)$$

for all x, y in X, where $0 \leqq c < 1$. Then

$$\varrho(T^n x, T^n y) \leqq c^n \varrho(x, y) \leqq c^n\{\varrho(x, T^n x) + \varrho(T^n x, T^n y) + \varrho(T^n y, y)\}$$

and so

$$\varrho(T^n x, T^n y) \leqq \frac{c^n}{1-c^n}\{\varrho(x, T^n x) + \varrho(y, T^n y)\}$$

for all x, y in X.

Since $c < 1$, we can find an n such that $c^n < 1/3$ and then

$$\frac{c^n}{1-c^n} < 1/2.$$

It follows that for such an n, T^n is a Kannan mapping. This completes the proof of the theorem.

One may now ask the following question. If T is a Kannan mapping of the metric space (X, ϱ) into itself, must T^n be a contraction mapping for some positive integer n?

The answer to this question is in the negative, as can be seen from the following example.

Let X be the set of real numbers x, with $-2<x<2$. Define a metric ϱ on X by

$$\varrho(x, y) = |x-y|.$$

Define a mapping T of X into itself by

$$Tx = \begin{cases} -x/4, & |x| \leqq 1, \\ x/4, & 1 < |x| < 2. \end{cases}$$

Then

$$\varrho(Tx, Ty) \leqq \frac{1}{4}(|x|+|y|), \quad \varrho(x, Tx)+\varrho(y, Ty) \geqq \frac{3}{4}(|x|+|y|).$$

Thus

$$\varrho(Tx, Ty) \leqq \frac{1}{3}\{\varrho(x, Tx)+\varrho(y, Ty)\},$$

for all x, y in X and so T is a Kannan mapping.

However it is obvious that T^n is discontinuous at the point $x=1$ for $n=1, 2, \dots$ and so cannot be a contraction mapping, which is always continuous.

We can however prove the following theorem.

THEOREM 2. *Let T be a Kannan mapping of the metric space (X, ϱ) into itself and suppose that*

$$\varrho(x, Tx)+\varrho(y, Ty) \leqq h\varrho(x, y) \tag{1}$$

for all x, y in X, where $h>0$. Then T^n is a contraction mapping for some positive integer n.

PROOF. Suppose first of all that X is complete and that

$$\varrho(Tx, Ty) \leqq k\{\varrho(x, Tx)+\varrho(y, Ty)\}$$

for all x, y in X, where $0 \leqq k<1/2$. Then

$$\varrho(T^n x, T^{n+1}x) < k\{\varrho(T^{n-1}x, T^n x)+\varrho(T^n x, T^{n+1}x)\}$$

and so

$$\varrho(T^n x, T^{n+1}x) \leqq \frac{k}{1-k}\varrho(T^{n-1}x, T^n x) \leqq \left[\frac{k}{1-k}\right]^n \varrho(x, Tx).$$

Thus

$$\varrho(T^n x, T^{n+r}x) \leqq \varrho(T^n x, T^{n+1}x)+\dots+\varrho(T^{n+r-1}x, T^{n+r}x) \leqq k_n\varrho(x, Tx)$$

where

$$k_n = \left(\frac{k}{1-k}\right)^n \frac{1-k}{1-2k}.$$

It follows that $\{T^n x\}$ is a Cauchy sequence and in fact converges to a unique point z, see [1], with the property that $Tz=z$. On letting r tend to infinity we see that

$$\varrho(T^n x, z) \leqq k_n\varrho(x, Tx).$$

Similarly

$$\varrho(T^n y, z) \leqq k_n\varrho(y, Ty)$$

and so

$$\varrho(T^n x, T^n y) \leqq \varrho(T^n x, z) + \varrho(z, T^n y) \leqq k_n\{\varrho(x, Tx) + \varrho(y, Ty)\} \leqq hk_n \varrho(x, y)$$

for all x, y in X, on using inequality (1). Since $k(1-k)^{-1} < 1$, we can choose an n such that $hk_n < 1$. For this n, T^n is a contraction mapping, completing the proof of the theorem when X is complete.

If X is not complete we note that

$$\varrho(Tx, Ty) \leqq \varrho(x, Tx) + \varrho(x, y) + \varrho(y, Ty) \leqq (h+1)\varrho\{x, y),$$

on using inequality (1). This implies that T is uniformly continuous and so $\tilde{T}$, the completion of T, is a Kannan mapping on $\tilde{X}$, the completion of X and

$$\varrho(x, \tilde{T}x) + \varrho(y, \tilde{T}y) \leqq h\varrho(x, y),$$

for all x, y in $\tilde{X}$. It follows from what we have just proved that $\tilde{T}^n$ is a contraction mapping on $\tilde{X}$ for some n and so for this n, T^n is a contraction mapping on X, completing the proof of the theorem.

Reference

[1] R. KANNAN, Some results on fixed points, *Bull. Calcutta Math. Soc.*, **60** (1968), 71—76.

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DEPARTMENT OF MATHEMATICS
THE UNIVERSITY
LEICESTER LE1 7RH
ENGLAND

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ON A RECURRENT FINSLER MANIFOLD WITH A CONCIRCULAR VECTOR FIELD

By

R. B. MISRA[1] (Turin), F. M. MEHER (Balangir) and N. KISHORE (Allahabad)

Concircular transformation in Riemannian manifolds was introduced and studied in detail by K. YANO in a series of papers [4]. For non-Riemannian manifolds of recurrent curvature K. TAKANO [3] studied affine motions generated by different forms of concircular vector fields. M. OKUMURA [5] continued the study of concircular vector fields in various types of Riemannian manifolds. R. S. SINHA [13], R. B. MISRA and F. M. MEHER [1], [6], [8] to [11] extended the theory of concircular vector fields to Finsler manifolds. In the paper at hand we discuss various forms of concircular vector fields and study the recurrent Finsler manifolds admitting one of them. Notation employed in the paper is based on [1], [6] to [12].

1. Introduction

Let F_n be an n-dimensional Finsler manifold of class at least C^5 endowed with a metric tensor g_{ij}[2] satisfying the requisite conditions [2]. The connection parameters of Berwald are the functions

$$G^i_{jk} \overset{\text{def}}{=} \dot{\partial}_j G^i_k \quad (\dot{\partial}_j \equiv \partial/\partial \dot{x}^j),$$

and are positively homogeneous of degree zero in the directional arguments. Because of homogeneous properties these functions satisfy

$$\text{(1.1)} \qquad \text{a)} \quad G^i_{jk}\dot{x}^j = G^i_k, \qquad \text{b)} \quad G^i_{jkh}\dot{x}^j = 0,$$

where $G^i_{jkh} \overset{\text{def}}{=} \dot{\partial}_j G^i_{kh}$ form a tensor field symmetric in all its lower indices. The Berwald's covariant derivative of a vector $\mathbf{X}$ for these connection parameters is given by

$$\text{(1.2)} \qquad \mathscr{B}_k X^i = \partial_k X^i - (\dot{\partial}_j X^i) G^j_k + X^j G^i_{jk}, \quad (\partial_k \equiv \partial/\partial x^k).$$

In particular, the covariant derivative vanishes for the vector field $\dot{x}^i$.

[1] On leave from the University of Allahabad, Allahabad (India) expresses his gratitude to the Italian National Research Council for offering a Visiting Professorship to him at the Turin University, Turin (Italy).

[2] Unless stated otherwise all the entities are considered as functions of line-element $(x^i, \dot{x}^i)$. The indices $i, j, k, \ldots$ assume positive integral values $1, 2, \ldots, n$.

The operators of partial and covariant differentiation commute according to

$$(1.3)\qquad (\dot{\partial}_j \mathscr{B}_k - \mathscr{B}_k \dot{\partial}_j) X^i = G^i_{jkh} X^h,$$

and

$$(1.4)\qquad 2\mathscr{B}_{[j}\mathscr{B}_{k]} X^i = H^i_{jkh} X^h - H^h_{jk} \dot{\partial}_h X^i,$$ [3]

where the functions H^i_{jkh} are skew-symmetric in j, k and constitute a tensor, **H**, called the curvature tensor of Berwald. The functions H^i_{jk} are connected with the curvature tensor by

$$(1.5)\qquad \text{a)}\ \ H^i_{jkh}\dot{x}^h = H^i_{jk}, \qquad \text{b)}\ \ H^i_{jkh} = \dot{\partial}_h H^i_{jk}.$$

These tensors when transvected with $\dot{x}^k$ or contracted in their indices also give rise to the following relations

$$(1.6)\qquad \text{a)}\ \ H^i_{jk}\dot{x}^k = H^i_j, \qquad \text{b)}\ \ H^i_j \dot{x}^j = 0,$$

$$(1.7)\qquad \begin{cases} \text{a)}\ \ H^i_{ikh} = H_{kh}, & \text{b)}\ \ H^i_{ik} = H_k, & \text{c)}\ \ H^i_{jki} = 2H_{[kj]}, \\ \text{d)}\ \ H_{kh}\dot{x}^h = H_k, & \text{e)}\ \ H_{kh} = \dot{\partial}_h H_k, & \text{f)}\ \ H^i_i = (n-1)H. \end{cases}$$

2. A recurrent manifold and concircular vector field

If there exists a non-null covariant vector field λ_l so that the curvature tensor satisfies

$$(2.1)\qquad \mathscr{B}_l H^i_{jkh} = \lambda_l H^i_{jkh},$$

the manifold is then called recurrent, and is denoted by $HR-F_n$ [7]. It is seen that an $HR-F_n$ also admits

$$(2.2)\qquad \mathscr{B}_l H^i_{jk} = \lambda_l H^i_{jk},$$

and analogous identities satisfied by the entities defined in (1.6) and (1.7).

We consider an infinitesimal point transformation

$$(2.3)\qquad \bar{x}^i = x^i + \varepsilon v^i,$$

where ε is an infinitesimal constant and v^i (independent of the directional arguments) are components of a contravariant vector. As indicated by K. TAKANO [3—V] there exist various possibilities for the vector field v^i:

$$(2.4)\qquad \begin{cases} \text{a)}\ \ \mathscr{B}_j v^i = 0, \\ \text{b)}\ \ \mathscr{B}_j v^i = c\delta^i_j, \quad c \text{ being a non-zero constant}, \\ \text{c)}\ \ \mathscr{B}_j v^i = \varrho(x, \dot{x})\delta^i_j, \quad \varrho \neq 0, \\ \text{d)}\ \ \mathscr{B}_j v^i = \varphi_j(x, \dot{x}) v^i, \quad \varphi_j \text{ being a non-null vector-field}, \\ \text{e)}\ \ \mathscr{B}_j v^i = \varrho(x, \dot{x})\delta^i_j + \varphi_j(x, \dot{x}) v^i, \end{cases}$$

[3] Square brackets denote the skew-symmetric part with respect to the indices enclosed therein.

φ_j being a non-null vector field satisfying

(2.5) $$2\mathscr{B}_{[j}\varphi_{k]} = 0,$$

(2.4) f) $\mathscr{B}_j v^i = \varrho(x, \dot{x})\delta^i_j + \varphi_j(x, \dot{x})v^i$, φ_j being any non-null vector.

Correspondingly, the vector field v^i is called a contravector field, concurrent vector field, special concircular vector field, recurrent vector field, concircular vector field and torse-forming vector field respectively. The forms (2.4b) and (2.4c) of the vector field generating an affine motion in bi-recurrent Finsler manifolds and projectively symmetric Finsler manifolds have been studied in [8] and [11] respectively. Also the form (2.4c) generating a projective motion has been studied in [10]. The form (2.4d) generating an affine motion in symmetric and recurrent Finsler manifolds has been studied in [6] and [9] respectively. In the present paper we discuss the form (2.4e) and study a recurrent manifold admitting an infinitesimal transformation (2.3) of the type (2.4e). Such a manifold will be called a *recurrent Finsler manifold admitting a concircular vector field and will be denoted by* CHR$-$F$_n$.

3. CHR$-$F$_n$

In this section we consider a recurrent Finsler manifold admitting an infinitesimal transformation generated by a concircular vector field v^i of the type (2.4e). Differentiating (2.4e) covariantly with respect to x^k and simplifying by means of (2.4e) itself we derive

(3.1) $$\mathscr{B}_k\mathscr{B}_j v^i = \varrho_k\delta^i_j + \varrho\delta^i_k\varphi_j + \varphi_{kj}v^i,$$

where we have put

(3.2) a) $\varrho_k \stackrel{\text{def}}{=} \mathscr{B}_k\varrho$, b) $\varphi_{kj} \stackrel{\text{def}}{=} \mathscr{B}_k\varphi_j + \varphi_k\varphi_j$.

For (2.5), it may be easily seen that the skew-symmetric part of (3.1), in view of the commutation formula (1.4), yields

(3.3) $$H^i_{jkh}v^h = 2\delta^i_{[j}(\varrho\varphi_{k]} - \varrho_{k]}).$$

Contracting it with respect to i and j, and using (1.7a) we obtain

$$H_{kh}v^h = (n-1)(\varrho\varphi_k - \varrho_k).$$

Substituting from this relation in (3.3), we get

(3.4) $$H^i_{jkh}v^h = \frac{2}{n-1}\,\delta^i_{[j}H_{k]h}v^h.$$

As the contracted curvature tensor is also recurrent in HR$-$F$_n$, the covariant differentiation of (3.4) with respect to x^l, for (2.1), (2.4e), and (3.4) itself, yields

(3.5) $$H^i_{jkl} = \frac{2}{n-1}\,\delta^i_{[j}H_{k]l},$$

for a proper special concircular case, i.e. when neither ϱ nor φ_j are zero.

When contracted with respect to i and l, this relation reduces to

$$H^i_{jki} = \frac{2}{n-1} H_{[kj]},$$

which, in consequence of (1.7c), implies vanishing of H^i_{jki}:

$$H^i_{jki} = 0. \tag{3.6}$$

We, therefore, have the

THEOREM 3.1. *In a* $\mathrm{CHR}-\mathrm{F}_n$ $(n>2)$ *the contracted curvature tensor* H_{kj} *is necessarily symmetric.*

As a consequence of this theorem we may have an analogue of (1.7d):

$$H_{hk}\dot{x}^h = H_k. \tag{3.7}$$

Also, transvection of (3.5) with $\dot{x}^l$, for (1.5a) and (1.7d), determines

$$H^i_{jk} = \frac{2}{n-1}\delta^i_{[j}H_{k]}. \tag{3.8}$$

Employing the relations (3.6), (3.7) and (3.8) it is not difficult to see that the Weyl's projective tensor [2]

$$W^i_{jk} = H^i_{jk} - \frac{\dot{x}^i}{n+1}H^r_{jkr} + \frac{2}{n^2-1}\{nH_{[j} + \dot{x}^r H_{r[j}\}\delta^i_{k]}$$

vanishes in $\mathrm{CHR}-\mathrm{F}_n$. Accordingly, the Weyl's projective curvature tensor

$$W^i_{jkh} \overset{\text{def}}{=} \dot{\partial}_h W^i_{jk},$$

too, vanishes there. This establishes the

COROLLARY 3.1. $\mathrm{CHR}-\mathrm{F}_n$ $(n>2)$ *is projectively flat.*

In the remaining part of this section we deduce a relation between the Berwald's tensor field H^i_{jk} and the recurrence vector field λ_l in $\mathrm{CHR}-\mathrm{F}_n$. Forming the covariant differentiation of (3.8) with respect to x^l and using the recurrence property of H_k we get

$$\mathscr{B}_l H^i_{jk} = \frac{2}{n-1}\lambda_l \delta^i_{[j}H_{k]}.$$

From this relation we obtain two more equations by cyclic interchange of the indices l, j, k and adding the equations so obtained with it we get

$$\lambda_l \delta^i_{[j}H_{k]} + \lambda_j \delta^i_{[k}H_{l]} + \lambda_k \delta^i_{[l}H_{j]} = 0, \tag{3.9}$$

where we have used the Bianchi identity $\mathscr{B}_{[l}H^i_{jk]}=0$. Contracting (3.9) with respect to i and j, we get

$$\lambda_l H_k - \lambda_k H_l = 0, \tag{3.10}$$

for $n>2$. The last relation establishes the fact that $\lambda_l H_k$ is symmetric in l, k. Thus, it seems that there may exist a scalar function $P(x, \dot{x})$ so that

$$H_k = P\lambda_k. \tag{3.11a}$$

As H_k is positively homogeneous of degree 1 and λ_k is independent of $\dot{x}^i$'s, P is homogeneous of degree 1 in $\dot{x}^i$'s. Transvecting (3.11a) with $\dot{x}^k$, we get

$$H_k\dot{x}^k = P\lambda_k\dot{x}^k.$$

From (1.6a), (1.7b) and (1.7f) we notice that $H_k\dot{x}^k=(n-1)H$. Therefore, the above relation determines

$$P = (n-1)(H/\lambda), \tag{3.12}$$

where $\lambda \overset{\text{def}}{=} \lambda_k\dot{x}^k$ is a scalar function supposed to be non-vanishing. From (3.11a) and (3.12) we, therefore, have

$$H_k = (n-1)(H/\lambda)\lambda_k. \tag{3.11b}$$

Substituting from (3.11b) in (3.8) we obtain the desired relation

$$H^i_{jk} = (2H/\lambda)\delta^i_{[j}\lambda_{k]}. \tag{3.13}$$

4. Concircular vector field v^i and the function ϱ

Applying the commutation formula (1.3) for the concircular vector field v^i satisfying (2.4e) we derive

$$(\dot{\partial}_j\varrho)\delta^i_k+(\dot{\partial}_j\varphi_k)v^i = G^i_{jkh}v^h. \tag{4.1}$$

Taking its skew-symmetric part with respect to j, and k, we have

$$2\dot{\partial}_{[j}\varrho\delta^i_{k]}+2v^i\dot{\partial}_{[j}\varphi_{k]} = 0. \tag{4.2}$$

Transvecting it by v^k we get

$$v^i\{\dot{\partial}_j\varrho+2\dot{\partial}_{[j}\varphi_{k]}v^k\}-(v^k\dot{\partial}_k\varrho)\delta^i_j = 0, \tag{4.3}$$

which, on contraction with respect to i and j, yields

$$(1-n)(v^i\dot{\partial}_i\varrho)+2\dot{\partial}_{[i}\varphi_{k]}v^iv^k = 0.$$

The second term being zero this equation simplifies to

$$v^i\dot{\partial}_i\varrho = 0. \tag{4.4}$$

Consequently, (4.3) reduces to

$$\dot{\partial}_j\varrho+2\dot{\partial}_{[j}\varphi_{k]}v^k = 0, \tag{4.5}$$

for arbitrary v^i's. On the other hand, the contraction of (4.2) for i and k yields

$$(n-1)\dot{\partial}_j\varrho+2\dot{\partial}_{[j}\varphi_{k]}v^k = 0. \tag{4.6}$$

Comparing the last two relations we, thus, obtain

$$\dot{\partial}_j\varrho = 0 \tag{4.7}$$

for an F_n $(n>2)$. Also, from (4.2) and (4.7) there follows the identity

(4.8) $$2\dot{\partial}_{[j}\varphi_{k]} = 0.$$

Therefore, we have the

THEOREM 4.1. *If the manifold* F_n $(n>2)$, *admits a concircular vector field* v^i *of the type* (2.4e) *the scalar function* ϱ *is necessarily a point function and the vector field* φ_j *satisfies* (4.8).

References

[1] F. M. MEHER, *Certain investigations in Finsler spaces* (Ph. D. thesis, Berhampur University, India, 1972).

[2] H. RUND, *The differential geometry of Finsler spaces* (Springer-Verlag, 1959).

[3] K. TAKANO, Affine motion in non-Riemannian K^*-spaces, I, II, III (with M. OKUMURA), IV, V, *Tensor (N. S.)*, **11** (1961), 137—143, 161—173, 174—181, 245—253, 270—278.

[4] K. YANO, Concircular geometry, I, II, III, IV, V, *Proc. Imp. Acad. Tokyo*, **16** (1940), 195—200, 354—360, 442—448, 505—511; **18** (1942), 446—451.

[5] M. OKUMURA, On some types of connected spaces with concircular vector fields, *Tensor (N. S.)*, **12** (1962), 33—46.

[6] R. B. MISRA, A symmetric Finsler space, *Tensor (N. S.)*, **24** (1972), 346—350.

[7] R. B. MISRA, On a recurrent Finsler space, *Rev. Roum. Math. Pures Appl.*, **18** (1973), 701—712.

[8] R. B. MISRA, A bi-recurrent Finsler manifold with affine motion, *Indian J. Pure Appl. Math.*, **6** (1975), 1441—1448.

[9] R. B. MISRA—F. M. MEHER, A SHR $-F_n$ admitting an affine motion, *Acta Math. Acad. Sci. Hungar.*, **22** (1971), 423—429.

[10] R. B. MISRA—F. M. MEHER, A Finsler space with special concircular projective motion, *Tensor (N. S.)*, **24** (1972), 288—292.

[11] R. B. MISRA—F. M. MEHER, CA-motion in a PS $-F_n$, *Indian J. Pure Appl. Math.*, **6** (1975), 522—526.

[12] R. B. MISRA—K. S. PANDE, On the Finsler space admitting a holonomy group, *Ann. Mat. Pura Appl.*, **(4) 85** (1970), 327—346.

[13] R. S. SINHA, Affine motion in recurrent Finsler spaces, *Tensor (N. S.)*, **20** (1969), 261—264

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ISTITUTO DI GEOMETRIA
UNIVERSITÀ DI TORINO
VIA PRINCIPE AMEDEO, 8
10 123 TORINO (ITALY)

RAJENDRA COLLEGE
BALANGIR, ORISSA (INDIA)

C. M. P. DEGREE COLLEGE
(UNIVERSITY OF ALLAHABAD)
ALLAHABAD (INDIA).

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A NINE-FOLD PACKING

By

W. J. BLUNDON (St. John's)

Let d_k represent the density of closest k-fold packing of equal circles in the plane when the centres of the circles are at the points of a lattice Λ. It is well known that $d_1=\pi/2\sqrt{3}$, this result being essentially due to Lagrange. HEPPES [4] proved that $d_k/kd_1=1$ for $k=2, 3, 4$ and BLUNDON [1] found the values of d_5 and d_6.

BLUNDON [2] pointed out an inequality in the Heppes paper which leads readily to an estimate for d_k and later [3] he gave a sharper estimate,

$$d_k/kd_1 \geqq f(x/k), \tag{1}$$

where $c=[k\theta]$ and $\theta=\frac{1}{13}(6-\sqrt{10})=0.21828\ldots$, and where

$$f(x)=(1-x^2)/(1-4x^2)^{1/2}.$$

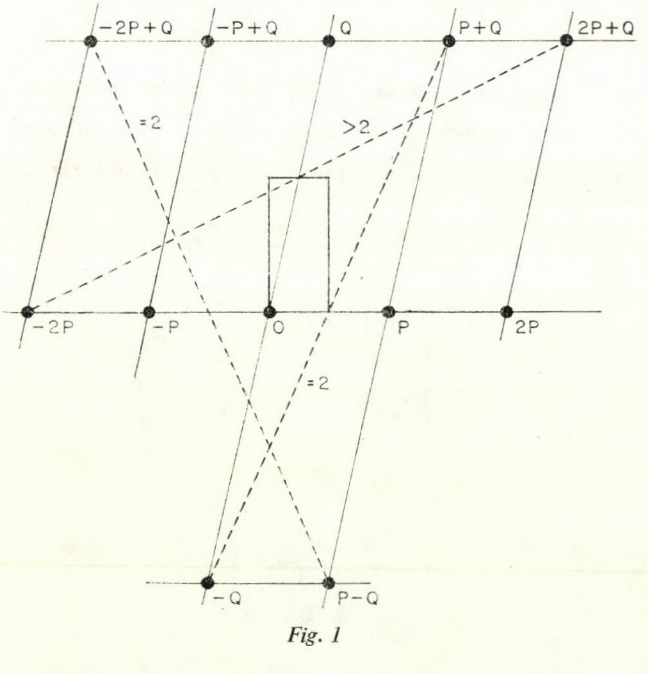

Fig. 1

The values of d_k for $k=1, 2, \ldots, 6$ are all given by equality in (1). Further, (1) gives the best known estimates for d_k for small $k\geqq 7$.

In particular, (1) gives $d_9/9k_1 \geqq 1.01298\ldots$. The purpose of this paper is to give an improved estimate for d_9. We prove the following

THEOREM. $d_9/9d_1 \geqq \frac{25}{63}\sqrt{7} = 1.04990\ldots$, *and the corresponding lattice is that generated by the points* $P(2/5, 0)$ *and* $Q\left(1/5, \frac{1}{5}\sqrt{21}\right)$.

PROOF. Let circles of unit radius be centred at the points of Λ. By symmetry, it is sufficient to prove that no point in the open rectangle with vertices at $O, \frac{1}{2}P, \frac{1}{4}P+\frac{1}{2}Q, -\frac{1}{4}P+\frac{1}{4}Q$ is covered by more than nine circles of Λ. The only circles having points in common with this rectangle are those centred at the twelve lattice points $-2P+Q, -P+Q, Q, P+Q, 2P+Q, -2P, -P, O, P, 2P, -Q, P-Q$.

Since $|P+2Q|=2$, no point of the rectangle can be common to the circles centred at $P+Q$ and $-Q$. Similar considerations apply to the pair $2P+Q$, $-2P$ and also to the pair $P-Q$ and $-2P+Q$. It follows that no point of the rectangle and hence no point of the plane can be covered by more than nine circles.

The estimate for d_9 follows at once from the fact that the determinant of Λ is $\frac{2}{25}\sqrt{21}$ and the determinant of the lattice giving best single packing for circles of unit radius is $2\sqrt{3}$.

References

[1] W. J. BLUNDON, Multiple packing of circles in the plane, *J. London Math. Soc.*, **38** (1968), 176—182.
[2] W. J. BLUNDON, Note on a paper of A. Heppes, *Acta Math. Acad. Sci. Hungar.*, **14** (1963), 317.
[3] W. J. BLUNDON, Some lower bounds for density of multiple packing, *Canad. Math. Bull.*, **7** (1964), 565—572.
[4] A. HEPPES, Mehrfache gitterförmige Kreislagerungen in der Ebene, *Acta Math. Acad. Sci. Hungar.*, **10** (1959), 141—148.

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MEMORIAL UNIVERSITY OF NEWFOUNDLAND,
ST. JOHN'S, NEWFOUNDLAND, CANADA, A1C 5S7.

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THE STRUCTURE OF CRITICAL RAMSEY GRAPHS

By
J. NEŠETŘIL and V. RÖDL (Prague)

The structure of minimal Ramsey graphs

All graphs in this paper are finite and undirected. Let $G=(V, E)$, $G'=(V', E')$. A one-to-one mapping $f: V \to V'$ is said to be an embedding iff $(x, y) \in E \Leftrightarrow \Leftrightarrow (f(x), f(y)) \in E'$. We say that the graph G' is a Ramsey graph for G if for every partition $E'=E_1 \cup E_2$ there exists an embedding $f: G \to G'$ such that $f(E) \subset E_i$ for an $i \in \{1, 2\}$. We abbreviate this by $G \xrightarrow[2]{} G'$. The negation of this statement is denoted by $G \not\xrightarrow[2]{} G'$. Given a graph G, it is an interesting question to characterize all finite graphs G' for which $G \xrightarrow[2]{} G'$. This seems to be very difficult. Presently it is evident that the structure of all graphs G', which are Ramsey graphs for G, is very complex. In this paper we shall go in this direction using the notion of critical Ramsey graph.

DEFINITION. A graph G' is a vertex-critical Ramsey graph for G iff $G \xrightarrow[2]{} G'$, and $G \not\xrightarrow[2]{} G''$ for every vertex deleted subgraph G'' of G. A graph G' is a critical Ramsey graph for G iff $G \xrightarrow[2]{} G'$, and $G \not\xrightarrow[2]{} G''$ for every proper subgraph G'' of G' (i. e. $E(G'') \subsetneqq E(G')$).

Obviously, every critical Ramsey graph for G is a vertex critical Ramsey graph.

The following was conjectured in [6].

CONJECTURE 1. For a graph G, the following three statemens are equivalent:

1) G has an infinite number of nonisomorphic vertex-critical Ramsey graphs

2) $G \not\xrightarrow[2]{} G$

3) G contains at least two edges.

The problem of finding minimal Ramsey graphs was raised by the authors earlier (see [2]).

The statements 1)$\Rightarrow$2), 1)$\Rightarrow$3) and 2)$\Leftrightarrow$3) are obvious. In this paper we shall prove 3)$\Rightarrow$1) for "most frequent graphs" (in the sense specified below). For critical Ramsey graphs the situation seems to be more difficult. It may happen that $G \not\xrightarrow[2]{} G$ and G can have only a finite number of critical Ramsey graphs (an example is provided by k disjoint edges or a star with odd number of edges, see [1]). Nevertheless, we give two constructions of critical Ramsey graphs which correspond to the following theorems:

THEOREM 1. *Let the chromatic number $\chi(G)$ of G be $\geqq 3$. Then G has infinite number of critical Ramsey graphs.*

THEOREM 2. *Let G be 2.5-connected graph. Then G has an infinite number of critical Ramsey graphs.*

(A graph is 2.5-connected if it is 2-connected and the removal of any two points joined by an edge does not disconnect the graph; e.g. every cycle is 2.5-connected.) In fact (in both theorems), we prove more as specified below (e.g. if G does not contain a triangle then there are infinitely many critical Ramsey graphs and there is an infinite number of triangle-free critical Ramsey graphs).

The paper is divided into two parts A, B, according to the two constructions. Part C contains concluding remarks and further results in the direction of the main conjecture. However, we did not achieve the full solution.

Let us finally remark that some special case of the main problem was handled by Burr, Erdős and Lovász [1] who proved that there exists an infinite number of minimal Ramsey graphs for every complete graph. They used a different technique ("signal senders").

A. Nonbipartite graphs

Theorem 1 will be proved in two steps. First we prove

Lemma 1. *Let* K_k, $k \geqq 3$ *be a complete graph,* $a \in \mathbf{N}$. *Then there exists a graph* G *such that*

1) $K_k \xrightarrow[2]{} G$,

2) $\chi(H) \leqq 2k-1$ *for every subgraph* H *of* G *with at most* a *vertices.*

Proof. Put $\langle 1, k\rangle = \{1, 2, \dots, k\}$. Define G recursively as follows. Let G_2 be the isolated edge denoted by $[z_0, z_1]$. Let $G_l = (X_l, E_l)$, $l < 2k-1$ be defined, put $|X_l| = L_l$. Let $(Y_l, \mathcal{M}_l)$ be a set-system such that

1) $M \in \mathcal{M}_l \Rightarrow |M| = L_l$,

2) $\chi(Y_l, \mathcal{M}_l) > 2$,

3) $(Y_l, \mathcal{M}_l)$ does not contain cycles of length $\leqq 2a$ (such a set-system exists by [3] or [4] for every positive integer a).

Let G_{l+1} be defined as follows: For every $M \in \mathcal{M}_l$ let $\iota_M : M \to X_l$ be a fixed bijection. Define $G_{l+1} = (X_{l+1}, E_{l+1})$ by $X_{l+1} = Y_l \cup \{z_l\}$, $[z_l, x] \in E_{l+1}$ for every $x \in Y_l$, where z_l is a new vertex and $[x, y] \in E_{l+1}$, $x, y \in Y_l$ iff $\{x, y\} \subseteqq M \in \mathcal{M}_l$, and $[\iota_M(x), \iota_M(y)] \in E_l$. Put $G = (X, E) = G_{2k-1} = (X_{2k-1}, E_{2k-1})$. Note, that for any pair $x, y \in Y_l$ there is at most one $M \in \mathcal{M}_l$ for which $\{x, y\} \subseteqq M$ because $(Y_l, \mathcal{M}_l)$ contains no 2-cycles. As a consequence, every subgraph of G_{l+1} spanned by $M \in \mathcal{M}_l$ is isomorphic to G_l.

Claim 1: $K_k \to G$.

Let $E = E^1 \cup E^2$. Considering edges $[z_{2k-2}, x]$, $x \in Y_{2k-2} = X_{2k-1} - \{z_{2k-2}\}$ we have an induced partition $Y^1_{2k-2} \cup Y^2_{2k-2}$ of the set Y_{2k-2} defined by $x \in Y^i_{2k-2} \Leftrightarrow$ $\Leftrightarrow [z_{2k-2}, x] \in E^i$. As $\chi(Y_{2k-2}, \mathcal{M}_{2k-2}) > 2$, there exists $M_{2k-2} \in \mathcal{M}_{2k-2}$ such that $M_{2k-2} \subseteqq Y^i_{2k-2}$ for $i=1$ or 2. However, the graph G_{2k-1} restricted to the set M_{2k-2} is isomorphic to G_{2k-2}. Let $\varphi_{2k-2} : X_{2k-2} \to M_{2k-2}$ be an isomorphism. Find $\varphi_{2k-2}(z_{2k-3}) \in M_{2k-2}$ and repeate the above procedure. Consequently, we find a set $\{\varphi_r(z_r);\ r \in \langle 0, 2k-2\rangle\} \subseteqq X$ such that $[\varphi_r(z_r), \varphi_s(z_s)] \subseteqq E_i(s)$ for $r < s$ where $i(s) \in \{1, 2\}$. Hence (by the Dirichlet principle) $K_k \xrightarrow[2]{} G$.

Claim 2: $\chi(H) \leqq l$ for every subgraph H of G_l with at most a vertices.

Proof is by induction on l ($l=2$ is clear). It suffices to prove $\chi(H \cap G_l - \{z_{l-1}\}) \leqq l-1$. If $\mathcal{N}_l = \{N : N \in \mathcal{M}_l, N \cap V(H) \neq \emptyset\}$ then $H = \bigcup_{N \in \mathcal{N}_l} G_N$

where G_N is the subgraph of G_l spanned by N. $\chi(G_N) \leqq l-1$ by the induction hypothesis ($G_N \cong G_{l-1}$). $\chi(H) \leqq l-1$ follows from this using the fact that $\mathcal{N}_l$ does not contain cycles (which can be deduced from condition 3).

COROLLARY. *For K_k, $k \geqq 3$ there exists an infinite number of critical Ramsey graphs.*

PROOF. It suffices to observe that if $K_k \xrightarrow[2]{} G$ then $\chi(G) > 2k-1$ ($k \geqq 3$). Consequently, if $K_k \xrightarrow[2]{} F$, $|V(F)| = a$ and if we take G which has properties of Lemma 1 then every minimal Ramsey graph contained in G has at least $a+1 > |V(F)|$ vertices. This procedure may be repeated.

Now we prove Theorem 1. We prove a stronger theorem (stated below as Theorem 3) which makes use of the following notion:

DEFINITION. A class $\mathcal{G}$ of graphs is said to be ideal if $G \in \mathcal{G} \Rightarrow G \times H \in \mathcal{G}$ for any graph H. A class $\mathcal{G}$ of graphs is said to be Ramsey if for every $G \in \mathcal{G}$ there exists a Ramsey graph $H \in \mathcal{G}$ with $G \xrightarrow[2]{} H$. A class $\mathcal{G}$ of graphs is said to have orderings if for every $G \in \mathcal{G}$ there exists $H \in \mathcal{G}$ such that for every ordering $\leqq_1$ of $V(G)$ and $\leqq_2$ of $V(H)$ there exists a monotone embedding $f: G \to H$. This fact is denoted by $G \xrightarrow{\text{ord}} H$.

We prove the following theorem which implies Theorem 1:

THEOREM 3. *Let $\mathcal{G}$ be an ideal class of graphs which is Ramsey and which has orderings. Then for every graph $G \in \mathcal{G}$, $\chi(G) = k \geqq 3$ there exists an infinite number of Ramsey graphs $H_1, H_2, \dots$ such that $H_i \in \mathcal{G}$ for every $i \in \mathbf{N}$.*

PROOF. Let $G \in \mathcal{G}$, $\chi(G) = k > 2$ be fixed. Let $H_1, H_2, \dots, H_n$ be minimal Ramsey graphs where $|V(H_1)| \leqq |V(H_2)| \leqq \dots \leqq |V(H_n)|$. Let F be a graph which satisfies:

1.a) $K_k \xrightarrow[2]{} F$,

2.a) $\chi(H) < 2k-1$ for every subgraph H of F with at most $|V(H_n)|$ vertices. (The existence of such a graph follows by Lemma 1.)

Let $c_0: G \to [1, k]$ be a fixed colouring. Let $\leqq$ be any ordering of G such that $c_0: (V(G), \leqq) \to [1, k]$ is a monotone mapping. Let $(G', \leqq)$ be an ordered graph (i.e. a graph the vertex set of which is ordered by $\leqq$) which satisfies:

1.b) $G' \in \mathcal{G}$,

2.b) $(G, \leqq) \xrightarrow[r]{} (G', \leqq)$ where $r = 2^l$, $l = |E(F)|$ and $(G, \leqq) \to (G', \leqq)$ means that for every partition $E(G') = E_1 \cup E_2 \cup \dots \cup E_r$ there exists $i \in [1, r]$ and monotone embedding $f: (G, \leqq) \to (G', \leqq)$ such that $f(E(G)) \subseteq E_i$.

The existence of $(G', \leqq)$ follows from the fact that $\mathcal{G}$ is Ramsey which has orderings: Let $G \xrightarrow{\text{ord}} G^*$ and $G^* \xrightarrow[r]{} G'$. Take any ordering $\leqq$ of $V(G')$. One may check that then $(G, \leqq) \xrightarrow[r]{} (G', \leqq)$. Put $H'_{n+1} = G' \times F \in \mathcal{G}$ where $\times$ means the direct product. ($G \times G'$ for $G = (V, E)$, $G' = (V', E')$ means $V(G \times G') = V \times V'$,

$$E(G \times G') = \{[(x, x'), (y, y')];\ (x, y) \in E,\ (x', y') \in E'\}).$$

We prove $G \xrightarrow[2]{} H'_{n+1}$. This follows by a standard argument using the categorial properties of the direct product: Let $c: E(G' \times F) \to \langle 1, 2 \rangle$ be a partition. Assume that $V(G')$ and $V(F)$ are ordered sets (by $\leqq$). Define the induced colouring

$d: E(G') \to 2^{E(F)}$ by

$$d([x', y'])([x, y]) = c([(x', x),\ (y' y)]) \quad \text{for} \quad x < y,\ x' < y'.$$

By 1.b) there exists an embedding $f: G \to G'$ such that $f(E(G))$ belongs to one of the parts of the partition induced by d. But this means that for every edge $[(x', x), (y', y)]$, $[x', y'] \in f(E(G))$ the colour $c([(x', x), (y', y)])$ depends on $[x, y]$ only. Now we use 1.a) and the fact if $f: G \to G'$ is an embedding, $g: G \to K_k$ is a homomorphism and $h: K_k \to F'$ then $(h \circ g) \times f$ is an embedding.

Let H_{n+1} be a critical Ramsey graph contained in H'_{n+1}. Then $|H_{n+1}| > |H_n|$ by Property 2.a). This proves the theorem.

REMARK. Examples of ideal classes which are Ramsey and have orderings:
Class of all graphs [5].
Class of all graphs without triangles [7].
Class of all graphs without K_k [8].
Class of all graphs without short odd cycles [9].
This implies Theorem 1.

B. Bipartite graphs

For bipartite graphs we shall use another construction:

THEOREM. *Let G be a 2.5-connected graph, $|E(G)| > 1$. Then G has an infinite number of critical Ramsey graphs.*

PROOF. Let $G \xrightarrow[2]{} H$, H be critical Ramsey. We shall construct a critical Ramsey graph H^* such that $|H^*| > |H|$. Let us fix $e \in E(H)$, $H = (W, F)$, $e = \{x, y\}$ and put $H' = (W, F - \{e\}) = (W, F')$. We have $G \not\xrightarrow[2]{} H'$ and consequently, there exists a partition $E(H') = E_1 \cup E_2$ such that $G \not\leqq (W, E_i)$. But from $G \xrightarrow[2]{} H'$ and from the 2-connectivity of G follows that

$$(*) \qquad \left|\{z;\ \{x, z\} \in E_i\}\right| \geqq 1 \quad \text{for} \quad i = 1, 2.$$

Put $\{z; \{x, z\} \in F'\} = V'$, $|V'| = r$. Let $(X, \mathcal{M})$ be an r-uniform set system with $\chi(X, \mathcal{M}) > 2$ which does not contain cycles of length $\leqq |W|$ ($r \geqq 2$ by $(*)$). To continue the main construction, assume the following:

1) $\mathcal{M} = \{M_i;\ i \in I\}$.
2) Let $H'_j = (W_j, F'_j)$, $j \in I$ be disjoint copies of $H' = (W, F')$.
3) Let $z \to z_j$, $z \in W$ be an isomorphism $H' \to H'_j$.
4) Put $V'_j = \{z_j, z \in V'\}$.
5) Let $\varepsilon_j: M_j \to V'_j$ be a bijection.

Let $\sim$ be the equivalence on the set $\bigcup_{j \in I} W_j \cup X$ generated by the set $\{(m, \varepsilon_j(m));\ m \in M_j,\ j \in I\}$. For $x \in X \bigcup \cup W_j$ let $[x]$ denote the equivalence class of $\sim$ containing x. Put $H^\vee = (W^\vee, F^\vee)$ where $W^\vee = \{[z];\ z \in X \bigcup \cup W_j\} \cup \{x^*\}$ and $x^* \notin \{[z];\ z \in X \bigcup \cup W_j\}$,

$$F^\vee = \{[[z], [t]],\ [\varepsilon_j[z];\ \varepsilon_j[t]] \in F'\} \cup \{[x, y];\ y \in X\}.$$

We prove $G \xrightarrow[2]{} H^\vee$. Let $F^\vee = F_1 \cup F_2$ be a partition. The restriction of this partition to the set $\{[x^*, y]; y \in X\}$ induces a partition $X = X_1 \cup X_2$ and consequently there exists $M_j \in \mathcal{M}$ such that $\{[[m], x^*], m \in M_j\} \subseteq F_i$. Let $\varepsilon: H' \to H^\vee$ be an isomorphism which satisfies $\varepsilon(x) = x^*$, $\varepsilon(V') = M_j$. Then there exists $G \leqq H'$ such that $\varepsilon(E(G)) \subseteq F_i$ (as every partition $F_1' \cup F_2'$ of $E(H')$ which does not contain H satisfies $(*)$).

Let $H^* = (V^*, F^*)$ be a critical Ramsey graph for G contained in $H^\vee$. Then $x^* \in V^*$ (as G is 2-connected, $(X, \mathcal{M})$ does not contain cycles of length $\leqq |G|$) and $G \leqq H^\vee \Rightarrow G \leqq H_j'$ (as G is 2-edge connected and $(X, \mathcal{M})$ does not contain short cycles).

From this follows that every subgraph $\bar{H}$ of $H^\vee$, $|\bar{H}| \leqq |H|$ fails to satisfy $G \to \bar{H}$ (by the construction of $H^\vee$).

C. Concluding remarks

1) One can prove further partial results for the solution of the original problem.

THEOREM. *For every finite forest T which contains a path of length 3 there exists an infinite number of critical Ramsey graphs.*

PROOF. Let H be a critical Ramsey graph for T. We shall construct a graph H' such that $T \xrightarrow[2]{} H'$ and no subgraph of H with $\leqq |V(H)|$ vertices is a Ramsey graph for T. Let $|V(H)| = r$, $|V(T)| = a$. Let $H' = (V', E')$ be a graph without cycles of length $\leqq r$ and with $\chi(H') > a^2$. Then H' has the following properties:

Let $E(H') = E_1 \cup E_2$ then either $\chi(V', E_1) > a$ or $\chi(V', E_2) > a$. In both cases the graph (V', E_i) does not contain cycles of length $\leqq |T|$ and contains a subgraph $H^\vee$, $\chi(H^\vee) > a$ with each of its vertices of degree $> a$ (e.g. we may take a colour critical subgraph of (V', E_i)). The rest of the statement follows by the fact that every Ramsey graph for a forest T which contains a path of length $\geqq 3$ necessarily contains a cycle.

2) The structure of critical Ramsey graphs could be investigated further. Namely the following could be questioned: Given F and G, does there exist an infinite number of Ramsey critical graphs for F which contain G? This is surely false, as if $F \xrightarrow[2]{} G$ then there exists at most one such a graph (namely G itself — in the case that G is a Ramsey critical graph).

CONJECTURE 2. Let F, G be graphs, $F \underset{2}{\not\to} G$, $F \underset{2}{\not\to} F$. Then there exists an infinite family of vertex critical Ramsey graphs for F which contain G.

3) If we consider partition of vertex-set of graphs (instead of edge-set of graphs) the analogons of both Conjectures 1 and 2 are true. The analogon of Conjecture 1 follows by an easy application of the methods used in [6], and that of Conjecture 2 will be proved in a forthcoming paper by V. Müller, J. Nešetřil and V. Rödl.

4) We could also define the following notion: a graph H is a cocritical Ramsey graph for G if

i) $G \xrightarrow[2]{} H$,

ii) $H \subsetneqq H'$; $V(H) = V(H') \Rightarrow G \underset{2}{\not\to} H'$.

Of course, for every graph there exists a cocritical Ramsey graph. However, the situation seems to be a bit different as indicated by the following: If $G=K_k$ then there exists exactly one cocritical Ramsey graph for G.

5) More generally, one can define an F-Ramsey critical graph for G as a graph H which satisfies

$$\text{i)}\quad G \xrightarrow[2]{F} H, \qquad \text{ii)}\quad G \xrightarrow[2]{F}\!\!\!\!\!\!/\;\; H' \quad \text{for} \quad H' \subsetneqq H.$$

Here $G \xrightarrow[2]{F} H$ (as defined in [5]) means that for every partition of the induced subgraphs of H isomorphic to F into two classes there exists an induced subgraph of H isomorphic to G all whose induced subgraphs isomorphic to F belong to one class.

Most important is the case when F is a complete or discrete graph. One can easily derive the notion of a Ramsey cocritical graph from the F-Ramsey critical graph (using complementation and $F=2$-element discrete graph).

6) All the above questions may be asked also for weak-Ramsey graphs: A graph G' is a weak Ramsey graph for G if for every partition $E(G')=E_1 \cup E_2$ there exists $(\bar{V}, \bar{E}) \simeq G$ such that $\bar{V} \subseteq V(G')$ and $\bar{E} \subseteq E_i$ for $i \in [1, 2]$ (this is the basic concept of the generalized Ramsey theory see [0]). All the above theorems are valid for weak Ramsey graphs as well if we replace 2.5 connectivity by 3-connectivity. This generalizes some results due to S. A. Burr, P. Erdős, R. J. Faudree and R. A. Shelp.

As this is a straightforward translation of the above theorems and proofs, we do not state it explicitly.

References

[0] S. A. Burr, Generalized Ramsey theory for graphs —, a survey, *Graphs and Combinatorics* (R. Bari and F. Harary, eds.) Springer Verlag (Berlin, 1974), 52—75.

[1] S. A. Burr, P. Erdős, L. Lovász, On graphs of Ramsey type (to appear).

[2] P. Erdős, Problems and results on finite and infinite graphs, *Recent advances in graph theory*, Academia (Praha, 1975), 183—192.

[3] P. Erdős, A. Hajnal, On chromatic number of graphs and set systems, *Acta Math. Acad. Sci. Hungar.*, **17** (1966), 61—99.

[4] L. Lovász, On chromatic number of finite set-systems, *Acta Math. Acad. Sci. Hungar.*, **19** (1968), 59—67.

[5] J. Nešetřil, V. Rödl, Type theory of partition problems of graphs, *Recent advances in grap theory*, Academia (Praha, 1975), 405—412.

[6] J. Nešetřil, V. Rödl, Partitions of vertices, *Comment. Math. Univ. Carol.*, **17**, 1 (1976), 85—95.

[7] J. Nešetřil, V. Rödl, A Ramsey graph without triangles exists for any graph without triangles, *Coll. Math. Soc. János Bolyai*, **10**, North-Holland (Amsterdam, 1975), 1127—1132.

[8] J. Nešetřil, V. Rödl, Ramsey property of graphs with forbidden complete subgraphs, *J. Comb. Th.* (B), **20** (1976), 243—249.

[9] J. Nešetřil, V. Rödl, Ramsey property of graphs without short odd cycles (submitted to *Math. Slovaca*).

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KZAA MFF KU
CHARLES UNIVERSITY
SOKOLOVSKÁ 83
186 00 PRAHA 8
ČSSR

FJFI ČVUT
CZECH TECHNICAL UNIVERSITY
HUSOVA 5
110 00 PRAHA 1
ČSSR

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О НЕКОТОРЫХ СВОЙСТВАХ ОБОБЩЕННЫХ РЕШЕНИЙ КВАЗИЛИНЕЙНЫХ ВЫРОЖДАЮЩИХСЯ ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ

Р. КЕРШНЕР (Будапешт)

Введение

0.1. Мы будем рассматривать задачу Коши и первую краевую задачу для уравнения

$$u_t = [a(t)u^{\mu}]_{xx} - [b(t, x, u)]_x - c(t, x, u). \tag{0.1}$$

Условия, которые мы налагаем на функции, образующие уравнение (0.1), сформулированы в п.1.1.1. Эти условия выполнены, например, для модельного уравнения

$$u_t = (u^{\mu})_{xx} - b\cdot(u^{\lambda})_x - cu^{\nu}, \tag{0.2}$$

где $\mu>1$, $\lambda\geqq 1$, $\nu>0$, $c\geqq 0$ и $b\in(-\infty, \infty)$ постоянные.

В случае задачи Коши уравнение (0.1) рассматривается в

$$\mathbf{R}^2_+ = \{(t, x): 0 < t < \infty, x\in\mathbf{R}^1\}$$

с начальным условием

$$u(0, x) = u_0(x). \tag{0.3}$$

В случае первой краевой задачи мы будем рассматривать уравнение (0.1) в $\mathbf{R}^2_{++} = \mathbf{R}^2_+ \cap \{x>0\}$ с начальным и граничным условиями и условием согласования

$$\begin{cases} u(0, x) = u_0(x), \\ u(t, 0) = u_1(t), \\ u_0(0) = u_1(0). \end{cases} \tag{0.4}$$

Уравнение (0.1) является параболическим при $u>0$ и вырождается в уравнение первого порядка при $u=0$. Уравнения вида (0.1) описывают процесс фильтрации жидкости и газа, а также процесс теплопередачи в среде с теплопроводностью, зависящей от температуры. Уравнения такого вида возникают также в теории пограничного слоя, в теории неньютоновских жидкостей и в магнитной гидродинамике. В связи с этим частные решения различных уравнений вида (0.1) строились в целом ряде прикладных работ. Укажем, например, работы [3]–[6], [11], [21], [23], [24] и [33].

В статье [26] О. А. Олейник, А. С. Калашников и Чжоу Юй-линь рассмотрели задачу Коши, первую и вторую краевые задачи для уравнения

$$u_t = [\varphi(u)]_{xx}. \tag{0.5}$$

При соответствующих предположениях относительно данных в [26] доказаны теоремы существования и единственности неотрицательного непрерывного обобщенного решения, ограниченного вместе с обобщенной производной $[\varphi(u)]_x$. Установлено, что в точках, где обобщенное решение положительно, оно удовлетворяет уравнению (0.5) в обычном смысле. В точках, где обобщенное решение обращается в нуль, оно может быть негладким.

Однозначная обобщенная разрешимость задачи Коши для уравнения (0.2) с младшими членами доказана А. С. Калашниковым [17]. Этот вопрос изучался также в статье [9] Гилдингом и Пелетье.

Дифференциальные свойства обобщенных решений уравнений вида (0.1) и (0.2) исследовались Д. Г. Аронсоном [1], С. Н. Кружковым [18], А. С. Калашниковым [14].

В работе Е. С. Сабининой [29] была доказана теорема существования и единственности обобщенного решения задачи Коши для уравнения $u_t = \Delta\varphi(u)$ при предположении, что $\varphi \in C^{2+\alpha}$

В работах Ю. Н. Благовещенского [2], О. А. Олейник [24] и Г. М. Фатеевой [30] рассматривались вопросы существования и единственности решения задачи Коши и краевых задач «в малом» для уравнения

$$u_t = \sum_{i,j=1}^{p} a_{ij}(t, x, u) u_{x_i x_j} + \sum_{i=1}^{p} b_i(t, x, u) u_{x_i} + c(t, x, u) + f(t, x) \tag{0.6}$$

при условии $\sum_{i,j=1}^{p} a_{ij}(t, x, u)\xi_i\xi_j \geqq 0$, $\forall \xi \in \mathbf{R}^p$.

В работе М. И. Фрейдлина [31] указаны условия, обеспечивающие существование и единственность классического решения задачи Коши для уравнения (0.6) «в целом». Заметим, что в работах [2] и [31] применяются вероятностные методы.

А. И. Вольперт и С. И. Худяев [7] изучают задачу Коши для уравнения

$$u_t = [a(t, x, u)u_x]_x - [b(t, x, u)]_x - c(t, x, u),$$

где $a(t, x, u) \geqq 0$, при условиях на функции a, b и c, которые в случае уравнения (0.2) означают, что $\mu \geqq 3$ (или $\mu = 2$), $\lambda \geqq 2$ (или $\lambda = 1$) и $\nu \geqq 1$. В этой работе даются также некоторые обобщения на многомерный случай.

К этому же кругу вопросов относится работа С. Н. Кружкова [19].

В вышеназванных работах можно найти и сведения о других работах по этой теме. См. также статью Ю. А. Дубинского [10] и книгу Ж.—Л. Лионса [22].

0.2. Некоторые характерные черты, свойственные лишь вырождающимся уравнениям, появляются уже в случае самого простого (и наиболее изученного) уравнения вида (0.1)

$$u_t = (u^\mu)_{xx}, \quad \mu > 1. \tag{0.7}$$

В работе [3] Г. И. Баренблатт, рассматривая задачу Коши для этого уравнения при $\mu = 2$ и

$$u_0(x) = \begin{cases} \sigma x^\alpha & \text{при} \quad x > 0 \ (\sigma > 0, 1 < \alpha < 2) \\ 0 & \text{при} \quad x \leqq 0, \end{cases}$$

обнаружил, что при $t > 0$ у производной u_x возникают точки разрыва. Обобщая это, А.С. Калашников в [12] доказал, например, что если $\mu \geqq 2$, $u_0(x)=0$ на отрезке $[a, b]$, $a<b$ и $u_0(x)\not\equiv 0$, то при $t>0$ всегда найдутся точки разрыва функции u_x. Следовательно, задача Коши для уравнения (0.7), вообще говоря, классического решения не имеет. Обобщенное решение определяется через обычное (в таких рассмотрениях) интегральное тождество (см. определения 1.1.1 и 1.4.1).

0.3. Другое важное отличие от невырожденного случая также было впервые открыто для уравнения (0.7): это наличие «конечной скорости распространения возмущений». В работе [11] Я. Б. Зельдовича и А. С. Компанейца было установлено, что при некоторых $u_0(x)\not\equiv 0$, решение $u(t, x)$ финитно по x (для равномерно параболических уравнений это не так).

Пример (см. [11] и [28]). Пусть

$$\lambda(t) = \left[\frac{2\mu(\mu+1)}{\mu-1}(t+1)\right]^{1/(\mu+1)}, \quad t \geqq 0.$$

Тогда функция

$$u(t, x) = \begin{cases} \left\{\dfrac{1}{\lambda(t)}\left[1-\left(\dfrac{x}{\lambda(t)}\right)^2\right]\right\}^{1/(\mu-1)} & \text{при} \quad |x| < \lambda(t) \\ 0 & \text{при} \quad |x| \geqq \lambda(t) \end{cases}$$

является обобщенным решением уравнения (0.7), удовлетворяющим начальному условию

$$u(0, x) = \begin{cases} \left\{\dfrac{1}{\lambda(0)}\left[1-\left(\dfrac{x}{\lambda(0)}\right)^2\right]\right\}^{1/(\mu-1)} & \text{при} \quad |x| < \lambda(0) \\ 0 & \text{при} \quad |x| \geqq \lambda(0). \end{cases}$$

По поводу «конечности скорости распространения возмущений» см. также работы [4], [6] и [26]. В статье [26] доказано в частности, что для конечности скорости распространения возмущений для уравнения (0.5) достаточным является условие

$$\int_0^1 \frac{\varphi'(u)}{u}\, du < \infty. \tag{0.8}$$

Для уравнения (0.7) это эквивалентно условию $\mu>1$. Как доказал А. С. Калашников в [13], условие (0.8) является и необходимым: какова бы ни была функция $u_0\not\equiv 0$, соответствующее обобщенное решение положительно всюду при $t>0$, если интеграл в (0.8) расходится.

Но, как было показано в [12], хотя тепловые возмущения и распространяются с конечной скоростью, они проникают как угодно далеко: для каждой точки x_1 существует такое t_1, что $u(t, x_1)>0$ при $t\geqq t_1$, если только $u_0\not\equiv 0$.

0.4. Наличие младших членов в уравнении может вызвать третий важный эффект, который впервые был обнаружен Л. К. Мартинсоном и К. Б. Павловым [23]. Этот эффект состоит в локализации возмущений: может случится, что даже за бесконечный промежуток времени тепло проникает в среду лишь на конечное расстояние, т.е. существует такое $L>0$, что $u(t, x)=0$ при $|x|\geqq L$ для всех $t>0$.

Пример (ср. [23], [28]). Рассмотрим следующую задачу

$$u_t = (u^\mu)_{xx} - \gamma u \quad \text{при} \quad x \in (-\infty, \infty), \ t > 0, \tag{0.9}$$

$$u(0, x) = \begin{cases} K[1 - x^2 K^2]^{1/(\mu-1)} & \text{при} \quad |x| < K^{-1} \\ 0 & \text{пру} \quad |x| \geqq K^{-1}, \end{cases} \tag{0.10}$$

где

$$K = \left[\frac{2\mu(\mu+1)}{\mu - 1}\right]^{-1/(\mu+1)}.$$

Непосредственно проверяется, что обобщенным решением задачи (0.9), (0.10) является функция

$$u(t, x) = \begin{cases} Ke^{-\gamma t} g(t)[1 - K^2 x^2 g^2(t)]^{1/(\mu-1)} & \text{при} \quad |x| < (Kg)^{-1} \\ 0 & \text{при} \quad |x| \geqq (Kg)^{-1}, \end{cases} \tag{0.11}$$

где

$$g(t) = \left[1 + \frac{1 - e^{-\gamma(\mu-1)t}}{\gamma(\mu-1)}\right]^{-1/(\mu+1)}.$$

(Единственность следует из теоремы 1.2.1.) Из (0.11) видно, что положение фронта (т.е. линии, где $u(t, x)$ обращается в нуль) определяется уравнением

$$x(t) = \pm K^{-1}\left[1 + \frac{1 - e^{-\gamma(\mu-1)t}}{\gamma(\mu-1)}\right]^{1/(\mu+1)},$$

из которого в свою очередь следует, что в данном случае тепло проникает в среду на расстояние

$$L = \sup_t |x(t)| = \lim_{t\to\infty} |x(t)| = K^{-1}\left[1 + \frac{1}{\gamma(\mu-1)}\right]^{1/(\mu+1)}.$$

Условия локализации возмущений в задаче (0.2), (0.3) при произвольной финитной начальной функции изучались в [15]. Аналогичные вопросы были рассмотрены для первой краевой задачи (0.2), (0.4) в [16].

0.5. Для уравнения

$$u_t = (u^\mu)_{xx} - c \cdot u^\nu \tag{0.12}$$

с начальным условием

$$u(0, x) = u_0(x), \tag{0.13}$$

где $\mu > 1$, $0 < \nu < 1$, $c > 0$, $u_0(x)$ — финитная непрерывная функция, изучался поставленный Г. И. Баренблаттом вопрос о направлении движения границы носителя обобщенного решения. Было установлено, что этот носитель может сначала расширяться, а потом сужаться (см. [17]).

Пример. Пусть $\mu + \nu = 2$ и

$$u_0(x) = [(\mu-1)\alpha]^{-1/(\mu-1)}(l^2 - x^2)^{1/(\mu-1)} \quad \text{при} \quad |x| < l \ \text{и} \ u_0(x) = 0 \quad \text{при} \quad |x| \geqq l,$$

где $\alpha > 0$ и $l > 0$ произвольные числа. Можно проверить, что обобщенное решение задачи (0.12), (0.13) дается формулой

$$u(t,x)=\left[\frac{2\mu(\mu+1)}{\mu-1}t+(\mu-1)\alpha\right]^{-1/(\mu-1)}\cdot\left\{\frac{c(\mu-1)^4\alpha^2+4\mu^2l^2}{4\mu^2[(\mu-1)\alpha]^{2/(\mu+1)}}\times\right.$$

$$\left.\times\left[\frac{2\mu(\mu+1)}{\mu-1}t+(\mu-1)\alpha\right]^{2/(\mu+1)}-\frac{c(\mu-1)^2}{4\mu^2}\left[\frac{2\mu(\mu+1)}{\mu-1}t+(\mu-1)\alpha\right]^2-x^2\right\}^{1/(\mu-1)}$$

там, где величина в фигурных скобках неотрицательна, и $u(t,x)=0$ вне этого множества.

Анализ этого решения показывает, что в случае $\alpha<\alpha_0=\dfrac{2l\sqrt{\mu}}{\sqrt{c}(\mu-1)^2}$ при возрастании t носитель $u(t,x)$ сначала расширяется, а затем сужается. В случае $\alpha\geqq\alpha_0$ носитель $u(t,x)$ сужается.

0.6. Настоящая работа посвящена изучению обобщенных решений задач (0.1), (0.3) и (0.1), (0.4). Она состоит из двух глав.

В первом параграфе первой главы при предположениях, выполняющихся, например, в случае уравнения (0.2), доказаны теоремы существования для этих задач.

Обобщенные решения задач (0.1), (0.3) и (0.1), (0.4) строятся как пределы классических решений краевых задач с положительными начальными и граничными функциями в расширяющихся областях.

В определение обобщенного решения включено требование его гельдеровости. Гельдеровость по x доказывается методом вспомогательных функций С. Н. Бернштейна. Согласно работам [18] и [8] из этого следует гельдеровость и по t.

В случае первой краевой задачи оценку нормы Гельдера удается получить таким путем лишь в области вида $x>\delta>0$. Для того, чтобы доказать гельдеровость обобщенного решения вблизи $x=0$, мы устанавливаем, что там аппроксимирующие его классические решения равномерно отделены от нуля.

Во втором параграфе первой главы рассмотрены вопросы единственности обобщенного решения задач (0.1), (0.3) и (0.1), (0.4).

В третьем параграфе первой главы доказаны теоремы сравнения для обобщенных решений. Эти теоремы систематически применяются во второй главе, но имеют и самостоятельный интерес.

Четвертый параграф посвящен многомерному случаю. Оказывается, что примененным в первом параграфе методом существование обобщенных решений задачи Коши и первой краевой задачи для уравнения (0.1) можно доказать при некотором условии близости μ к единице. Доказательства же теорем единственности и сравнения дословно переносятся на многомерный случай.

В первом параграфе второй главы рассмотрен вопрос о локализации возмущений для задачи Коши (0.1), (0.3).

Во втором параграфе второй главы изучаются аналогичные вопросы для первой краевой задачи (0.1), (0.4).

Результаты второй главы иллюстрируются на примере модельного уравнения (0.2).

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I. ТЕОРЕМЫ СУЩЕСТВОВАНИЯ, ЕДИНСТВЕННОСТИ И СРАВНЕНИЯ

§ 1.1. Теоремы существования

1.1.1. *Основные обозначения, предположения и определения.*

$$\mathbf{R}^2_+ = \{(t, x)\colon\ 0 \leqq t < \infty,\ x \in \mathbf{R}^1\}, \quad \mathbf{R}^2_{++} = \mathbf{R}^2_+ \cap \{x \geqq 0\}.$$

Мы будем рассматривать в $\mathbf{R}^2_+$ или в $\mathbf{R}^2_{++}$ уравнение

$$u_t = [a(t)u^\mu]_{xx} - [b(t, x, u)]_x - c(t, x, u). \tag{1}$$

Предполагается выполнение следующих условий.

I. μ — константа, $\mu > 1$.

II. $a(t) \in C^2$ при $0 \leqq t < \infty$ и $0 < \bar{a} \leqq a(t) \leqq \bar{\bar{a}} < \infty$ для всех $t \in [0, \infty]$, где $\bar{a}$ и $\bar{\bar{a}}$ константы.

III. Функции $b(t, x, u)$ и $c(t, x, u)$ определены и непрерывны для значений $u \geqq 0$; при этом $b(t, x, 0) = c(t, x, 0) = 0$, а для $u > 0$ и всех рассматриваемых t, x справедливы соотношения $b(t, x, u) \in C^3$, $c(t, x, u) \in C^2$, $\dfrac{\partial b(t, x, u)}{\partial x} + c(t, x, u) \geqq$ $\geqq 0$ и $c(t, x, u) \geqq 0$.

IV. Существует такое число $\beta \in (\mu - 1, \mu)$, что при $u \in (0, M)$, $\forall M > 0$, и при всех рассматриваемых t, x органичены следующие величины:

$$u^{2\beta+1-\mu}\frac{\partial^2 b(t, x, u)}{\partial u\, \partial x}, \quad u^{\beta+2-\mu}\frac{\partial^2 b(t, x, u)}{\partial u^2}, \quad u^{\beta+1-\mu}\frac{\partial b(t, x, u)}{\partial u},$$

$$u^{3\beta-\mu}\frac{\partial^2 b(t, x, u)}{\partial x^2}, \quad u^{2\beta-\mu}\frac{\partial b(t, x, u)}{\partial x},$$

$$u^{2\beta-\mu}c(t, x, u), \quad u^{2\beta+1-\mu}\frac{\partial c(t, x, u)}{\partial u}, \quad u^{3\beta-\mu}\frac{\partial c(t, x, u)}{\partial x}.$$

Поясним смысл этих условий в случае модельного уравнения:

$$a(t) = 1, \quad b(t, x, u) = b_0 u^\lambda, \quad c(t, x, u) = c_0 u^\nu; \quad c_0, b_0 - \text{const}.$$

Нетрудно проверить, что все сформулированные выше условия выполняются при $c_0 \geqq 0$, $\nu > 0$, $\lambda \geqq 1$ и любом b_0 если

$$\max\left(\mu - 1, \frac{\mu - \nu}{2}\right) < \beta < \mu.$$

Для уравнения (1) будет рассматриваться либо задача Коши в $\mathbf{R}^2_+$ с начальным условием

$$u(0, x) = u_0(x), \quad x \in \mathbf{R}^1, \tag{2}$$

либо первая краевая задача в $\mathbf{R}^2_{++}$ с условиями

$$\begin{cases} u(0, x) = u_0(x), & x \geqq 0 \\ u(t, 0) = u_1(t), & t \geqq 0. \end{cases} \tag{3}$$

Здесь $u_0(x)\geqq 0$ — непрерывная ограниченная функция, причем существует ограниченная обобщенная производная $\frac{du_0^\beta}{dx}$, где β то же, что и выше.

В случае первой краевой задачи дополнительно предполагается следующее

1° $u_0(0)=u_1(0)>0$,

2° Функция $u_1(t)$ ограничена и строго положительна при ограниченных t и имеет обобщенную производную du_1/dt ограниченную при ограниченных t.

Пусть Π — замкнутая подобласть $\mathbf{R}^2_+$ или $\mathbf{R}^2_{++}$, вообще говоря, неограниченная; в частности, может быть $\Pi=\mathbf{R}^2_+$ или $\Pi=\mathbf{R}^2_{++}$.

Определение 1.1.1. Неотрицательная в Π функция $u(t, x)$, удовлетворяющая условию Гельдера и ограниченная при ограниченных t называется обобщенным решением уравнения (1) в Π, если для $u(t, x)$ выполняется интегральное тождество

(4) $$I(u,f;\ t_0,t_1;\ x_0,x_1)=\int\limits_{Ц}\big(uf_t+a(t)u^\mu f_{xx}+b(t,x,u)f_x-$$

$$-c(t,x,u)f\big)\,dx\,dt-\int\limits_{x_0}^{x_1}uf\,dx\,\Big|_{t_0}^{t_1}-\int\limits_{t_0}^{t_1}a(t)u^\mu f_x\,dt\,\Big|_{x_0}^{x_1}=0,$$

каковы бы ни были числа $t_0<t_1$, $x_0<x_1$, такие, что $Ц=[t_0,t_1]\times[x_0,x_1]\subset\Pi$, и функция $f(t,x)\in C^{1,2}_{t,x}(Ц)$, равная нулю при $x=x_0$ и $x=x_1$.

Определение 1.1.2. Обобщенным решением задачи Коши (1), (2) называется ограниченная функция $u(t,x)$ являющаяся обобщенным решением уравнения (1) в $\mathbf{R}^2_+$ и удовлетворяющая условию (2).

Определение 1.1.3. Обобщенным решением первой краевой задачи (1), (3) называется функция $u(t, x)$ являющаяся обобщенным решением уравнения (1) в $\mathbf{R}^2_{++}$ и удовлетворяющая условиям (3).

1.1.2. *Теорема существования для задачи Коши.* Теорема 1.1.1. *Пусть выполнены условия пункта* 1.1.1. *Тогда обобщенное решение задачи Коши* (1), (2) *существует. В тех внутренних точках* $\mathbf{R}^2_+$, *где* $u(t,x)>0$, *функция* $u(t,x)$ *удовлетворяет уравнению* (1) *в обычном смысле.*

Доказательство. Для построения обобщенного решения задачи (1), (2) мы будем следовать схеме, применявшейся в работе [26].

Пусть $v_{0n}(x)$, $n=1, 2, \ldots$ — последовательность ограниченных, положительных, бесконечно дифференцируемых функций, которая, монотонно убывая, равномерно сходится при $n\to\infty$ к функции $v_0(x)=u_0^\beta(x)$ на каждом конечном отрезке; β — то же, что и в п.1.1.1.

Предположим еще, что

$$\sup_{n,x}|dv_{0n}(x)/dx|<\infty \quad \text{и} \quad v_{0n}(x)=\sup_{m,\xi}v_{0m}(\xi)=M \quad \text{для} \quad |x|\geqq n.$$

Обозначим через $v=v_n(t, x)$ решение следующей задачи:

(5) $$v_t = \mu a(t)\left(v^{(\mu-1)/\beta} v_{xx} + \frac{\mu-\beta}{\beta} v^{(\mu-1-\beta)/\beta} v_x^2\right) -$$

$$-\left(\frac{\partial b(t, x, v^{1/\beta})}{\partial u} v_x + v^{(\beta-1)/\beta} \frac{\partial b(t, x, v^{1/\beta})}{\partial x}\right) - \beta v^{(\beta-1)/\beta} c(t, x, v^{1/\beta})$$

в $Q_n = (0, n)\times\{|x| < n\}$,

(6) $$\begin{cases} v(0, x) = v_{0n}(x) \\ v(t, x)\big|_{|x|=n} = M. \end{cases}$$

Из теории невырождающихся параболических уравнений известно (см. [20]), что решение задачи (5), (6) существует и единственно для каждого n. Из принципа максимума следует, что $M \geqq v_n \geqq v_{n+1} > 0$ всюду в Q_n $(n=1, 2, \dots)$, M — то же, что в (6); ясно, что оно не зависит от n. Поэтому в каждой точке $(t, x)\in\mathbf{R}_+^2$ существует $\lim\limits_{n\to\infty} u_n(t, x) = u(t, x)$, где $u_n(t, x) = v_n^{1/\beta}(t, x)$. Мы покажем, что $u(t, x)$ является обобщенным решением задачи (1), (2). Очевидно, что функция $u(t, x)$ неотрицательна, ограничена и удовлетворяет интегральному тождеству (4).

Докажем, что $u(t, x)$ удовлетворяет в $\mathbf{R}_+^2$ условию Гельдера. Обозначим через P_n прямоугольник

$$\{(t, x):\ 0 \leqq t \leqq n,\ |x| \leqq n-1,\ n > 2\}.$$

Мы покажем, что в P_n функция $u_n(t, x)$ удовлетворяет условию Гельдера с показателем и константой, не зависящими от n.

Мы используем метод С. Н. Бернштейна в форме Аронсона [1].

Положим $v_n = f(w_n)$, где $f(w) = \frac{M}{3} w(4-w)$. Тогда $0 < w_n \leqq 1$. Очевидно, что на отрезке [0,1] для функции $f(w)$ выполняются соотношения

(7) $$0 \leqq f \leqq M,\quad \frac{2M}{3} \leqq f' \leqq \frac{4M}{3},\quad f'' = -\frac{2M}{3},\quad \left(\frac{f''}{f'}\right) \leqq -\frac{1}{4}.$$

Функция $w = w_n(t, x)$ удовлетворяет в Q_n уравнению

(8) $$w_t = \mu a(t)\left[f^{(\mu-1)/\beta} w_{xx} + \left(\frac{f''}{f'} f^{(\mu-1)/\beta} + \frac{\mu-\beta}{\beta} f' f^{(\mu-1-\beta)/\beta}\right) w_x^2\right] -$$

$$-\left(\frac{\partial b(t, x, f^{1/\beta})}{\partial u} w_x + \beta \frac{f^{(\beta-1)/\beta}}{f'} \frac{\partial b(t, x, f^{1/\beta})}{\partial x}\right) - \beta \frac{f^{(\beta-1)/\beta}}{f'} c(t, x, f^{1/\beta}).$$

Продифференцируем уравнение (8) по x:

$$(9)\qquad w_{tx}-\mu a(t)f^{(\mu-1)/\beta}w_{xxx}=\mu a(t)\Big\{\frac{\mu-1}{\beta}f'f^{(\mu-1-\beta)/\beta}w_xw_{xx}+$$

$$+2w_xw_{xx}f^{(\mu-1-\beta)/\beta}\left(\frac{f''}{f'}f+\frac{\mu-\beta}{\beta}f'\right)+$$

$$+w_x^3f^{(\mu-1-2\beta)/\beta}\left[\left(\frac{f''}{f'}\right)'f^2+\frac{(\mu-\beta)(\mu-1-\beta)}{\beta^2}f'^2+\frac{2\mu-1-\beta}{\beta}f''f\right]\Big\}-$$

$$-\left[\left(\frac{\partial^2 b(t,x,f^{1/\beta})}{\partial u\,\partial x}+\frac{\partial^2 b(t,x,f^{1/\beta})}{\partial u^2}\frac{1}{\beta}f^{(1-\beta)/\beta}f'w_x\right)w_x+\right.$$

$$+\left(\frac{\partial^2 b(t,x,f^{1/\beta})}{\partial x\,\partial u}\frac{1}{\beta}f^{(1-\beta)/\beta}f'w_x+\frac{\partial^2 b(t,x,f^{1/\beta})}{\partial x^2}\right)\beta\frac{f^{(\beta-1)/\beta}}{f'}+$$

$$\left.+\frac{\partial b(t,x,f^{1/\beta})}{\partial u}w_{xx}+\beta\frac{\partial b(t,x,f^{1/\beta})}{\partial x}\left(\frac{f^{(\beta-1)/\beta}}{f'}\right)'w_x\right]-$$

$$-\beta\frac{\partial c(t,x,f^{1/\beta})}{\partial x}\frac{f^{(\beta-1)/\beta}}{f'}-\frac{\partial c(t,x,f^{1/\beta})}{\partial u}w_x-\beta c(t,x,f^{1/\beta})\left(\frac{f^{(\beta-1)/\beta}}{f'}\right)'w_x.$$

Обозначим через $\zeta=\zeta_n(x)$ гладкую срезающую функцию со следующими свойствами: $\zeta=0$ в окрестности прямых $|x|=n$; $\zeta=1$ при $|x|\leqq n-1$, $0\leqq\zeta\leqq 1$, $|\zeta_x|\leqq M_1$, $|\zeta_{xx}|\leqq M_2$, где M_1 и M_2 не зависят от n.

Рассмотрим в $\overline{Q}_n$ функцию $z(t,x)=\zeta^2(x)w_x^2$. Если $\max z$ достигается при $t=0$, то в точке максимума z выполнено неравенство $z\leqq[(f^{-1})'v_{0nx}]^2\leqq M_3$, где M_3 от n не зависит. Предположим теперь, что $\max z$ достигается в Q_n. Тогда в точке максимума функции $z(t,x)$ выполняются соотношения

$$z_x=0\quad\text{и}\quad z_t-\mu a(t)f^{(\mu-1)/\beta}z_{xx}\geqq 0.$$

Подставляя сюда явное выражение для $z(t,x)$, получаем

$$(10)\qquad \zeta^2w_xw_{xx}=-\zeta_xw_x^2,$$

$$(11)$$

$$\zeta^2w_x\big(w_{tx}-\mu a(t)f^{(\mu-1)/\beta}w_{xxx}\big)\geqq\mu a(t)f^{(\mu-1)/\beta}(\zeta_x^2w_x^2+\zeta_{xx}\zeta w_x^2+4\zeta^2w_xw_{xx}+\zeta^2w_{xx}^2).$$

Величину, стояшую в левой части (11), выразим из уравнения, получающегося умножением обеих частей (9) на ζ^2w_x. Затем возникшее неравенство умножим на $f^{(2\beta+1-\mu)/\beta}$. После некоторой перегруппировки членов полученного неравенства мы приходим к основному неравенству (аргументы у функций a, b

и c не пишем; везде $a(t)=a$, $b(t, x, f^{1/\beta})=b$, $c(t, x, f^{1/\beta})=c$):

$$\mu a\zeta^2\left[-\left(\frac{f''}{f'}\right)'f^2-\frac{(\mu-\beta)(\mu-1-\beta)}{\beta^2}f'^2-\frac{2\mu-1-\beta}{\beta}f''f\right]w_x^4 \leqq \tag{12}$$

$$\leqq \mu a\zeta^2 f^2 w_{xx}^2+\mu a\zeta^2\frac{\mu-1}{\beta}f'fw_{xx}w_x^2+2\mu a\zeta^2\left(\frac{f''}{f'}f^2+\frac{\mu-\beta}{\beta}f'f\right)w_{xx}w_x^2-$$

$$-\mu a f^2(\zeta_x^2 w_x^2+\zeta\zeta_{xx}w_x^2+4\zeta^2 w_{xx}w_x)-$$

$$-\zeta^2\left[\frac{\partial^2 b}{\partial u\,\partial x}f^{(2\beta+1-\mu)/\beta}w_x^2+\frac{\partial^2 b}{\partial u^2}\frac{1}{\beta}f^{(\beta+2-\mu)/\beta}f'w_x^3+\frac{\partial b}{\partial u}f^{(2\beta+1-\mu)/\beta}w_{xx}w_x+\right.$$

$$\left.+\frac{\partial^2 b}{\partial x^2}\beta\frac{f^{(3\beta-\mu)/\beta}}{f'}w_x+\frac{\partial^2 b}{\partial x\,\partial u}f^{(2\beta+1-\mu)/\beta}w_x^2+\frac{\partial b}{\partial x}\beta f^{(2\beta+1-\mu)/\beta}\left(\frac{f^{(\beta-1)/\beta}}{f'}\right)'w_x^2\right]-$$

$$-\zeta^2\left[\frac{\partial c}{\partial x}\beta\frac{f^{(3\beta-\mu)/\beta}}{f'}w_x+\frac{\partial c}{\partial u}f^{(2\beta+1-\mu)/\beta}w_x^2+c\beta f^{(2\beta+1-\mu)/\beta}\left(\frac{f^{(\beta-1)/\beta}}{f'}\right)'w_x^2\right].$$

Величина, стоящая в квадратных скобках левой части (12), в силу выбора β и f (см. (7)) больше фиксированного положительного числа.

Используя для преобразования правой части (12) и равенство (10) и принимая во внимание свойства III и IV п. 1.1.1, мы из (12) получим неравенство

$$\zeta^2 w_x^4 \leqq c_1|w_x|+c_2 w_x^2+c_3|w_x|^3, \tag{13}$$

из которого, в свою очередь, вытекает неравенство $|w_x|\leqq C_4$ в P_n.

Отсюда немедленно следует ограниченность $|v_x|$, а из этого следует, что функция $u_n(t, x)$ удовлетворяет условию Гельдера в P_n с показателем $\min\left(\frac{1}{\beta}, 1\right)$ и константой, не зависящей от n. Как показал С. Н. Кружков [18], из этого следует, что $u_n(t, x)$ удовлетворяет условию Гельдера и по t. Функция $u(t, x)$ обладает этими же свойствами. По построению $u(t, x)$ удовлетворяет начальному условию (2) и, следовательно, является обобщенным решением задачи (1), (2).

Для доказательства последнего утверждения теоремы рассмотрим внутреннюю для $\mathbf{R}_+^2$ точку (t^0, x^0) такую, что $u(t^0, x^0)>0$. Тогда $u^\beta(t, x)\geqq\alpha>0$ в некоторой замкнутой окрестности Ω точки (t^0, x^0). Следовательно $v_n(t, x)\geqq\alpha$ $(n=1, 2, \ldots)$ в Ω. Поэтому уравнение (5) является равномерно относительно n параболическим в Ω. В силу оценок А. Фридмана (см. [32]), последовательность $\{v_n\}$ компактна в $C_{t,x}^{1,2}(\Omega)$. Отсюда вытекает, что функция $u(t, x)$ в Ω удовлетворяет уравнению (1) в обычном смысле. Теорема доказана.

1.1.3. *Теорема существования для первой краевой задачи.* Теорема 1.1.2. *Пусть выполнены условия пункта* 1.1.1. *Тогда обобщенное решение* $u(t, x)$ *первой краевой задачи* (1), (3) *существует. В тех внутренних точках* $\mathbf{R}_{++}^2$, *где* $u(t, x)>0$, *функция* $u(t, x)$ *удовлетворяет уравнению* (1) *в обычном смысле.*

Доказательство в основном аналогично пункту 1.1.2. Достаточно построить обобщенное решение задачи (1), (3) в $\mathbf{R}_{++}^2\cap\{t\leqq T\}$ для произвольного

$T\in(0,\infty)$. Функции $v_n(t,x)$ определим как решения уравнения (5) в $Q_n^+=Q_n\cap\cap\{x\geqq 0\}\cap\{t\leqq T\}$ с условиями

$$\text{(14)}\qquad \begin{cases} v_n(0,x)=v_{0n}(x)\\ v_n|_{x=n}=M\\ v_n(t,0)=v_{1n}(t).\end{cases}$$

Здесь $v_{1n}(t)$, $n=1,2,\dots$ — последовательность положительных бесконечно дифференцируемых функций, монотонно убывая сходящихся к функции $u_1^\beta(t)$. Равномерную по n ограниченность производных v_{nx} удается теперь доказать лишь в прямоугольниках вида $P_{n\delta}=\{(t,x):0\leqq t\leqq T,\ \ 0<\delta\leqq x<n-1\}$ $(\forall\delta>0)$. Отсюда вытекает, что функция $u(t,x)=\lim\limits_{n\to\infty}v_n^{1/\beta}(t,x)$ удовлетворяет условию Гельдера на всяком множестве вида $\mathbf{R}^2_{++}\{x\geqq\delta>0\}$.

Покажем теперь, что в достаточно узкой полосе, примыкающей к прямой $x=0$ решения $v_n(t,x)$ задачи (5), (14) равномерно по n отграничены от нуля.

Лемма 1.1.1. *Пусть выполнены условия пункта* 1.1.1. *Тогда существуют такие положительные постоянные η и ε, не зависящие от n, что при $(t,x)\in P_\eta=\{(t,x):0\leqq t\leqq T,\ 0\leqq x\leqq\eta\}$ имеет место $u_n(t,x)\geqq\varepsilon$.*

Доказательство. По предположению $u_1(t)\geqq\varepsilon_T>0$ при $0\leqq t\leqq T$ и $u_0(x)\geqq\geqq\varepsilon_1>0$ при $0\leqq x\leqq\delta_1$, $\delta_1>0$, по непрерывности функции $u_0(x)$. Рассмотрим в области $D=\{(t,x):0<t<T,\ x>0,\ v=1-x(t+\tau)>0\}$ функцию $y=y(t,x)=\sigma v^\omega$, где τ, σ и ω — вы бираемые ниже положительные постоянные.

Прежде всего, выберем σ и τ таким образом $(\sigma\leqq\sigma_0,\ \tau\geqq\tau_0)$, чтобы мы имели $y(0,x)<u(0,x)$ и $y(t,0)<u(t,0)$.

Тогда $y\leqq u_n$ на параболической границе области $D\cap Q_n^+$ $(n=1,2,\dots)$. Если мы покажем, что при некоторых σ, τ и ω

$$\text{(15)}\qquad \mathscr{L}y=-y_t+[a(t)y^\mu]_{xx}-[b(t,x,y)]_x-c(t,x,y)>0$$

в области D, то из теоремы сравнения А. Фридмана [32] будет следовать, что $y(t,x)\leqq u_n(t,x)$ в $D\cap Q_n^+$. Отсюда предельным переходом получим утверждение леммы. Находим:

$$\mathscr{L}y=\sigma\omega x v^{\omega-1}+a(t)\mu\omega(\omega\mu-1)(t+\tau)^2\sigma^\mu v^{\omega\mu-2}-$$

$$-\frac{\partial b(t,x,y)}{\partial x}+\sigma\omega(t+\tau)\frac{\partial b(t,x,y)}{\partial u}v^{\omega-1}-c(t,x,y).$$

Пусть $\omega>\dfrac{1}{\mu}$. Так как $x>0$ в D, мы имеем

$$\mathscr{L}y>v^{\omega\mu-2}\left[\bar a\omega\mu(\omega\mu-1)\sigma^\mu\tau^2-v^{2-\omega\mu}\left|\frac{\partial b(t,x,y)}{\partial x}\right|-\right.$$

$$\left.-\sigma\omega(T+\tau)\left|\frac{\partial b(t,x,y)}{\partial u}\right|v^{1-\omega(\mu-1)}-v^{2-\omega\mu}c(t,x,y)\right]\equiv v^{\omega\mu-2}\left[\sum\nolimits_1-\sum\nolimits_2-\sum\nolimits_3-\sum\nolimits_4\right].$$

Из условий III п.1.1.1. следует, что для ограниченности $\sum_2$, $\sum_3$ и $\sum_4$ достаточным условием является выполнение следующих неравенств (соответственно):

$$\omega(2\beta-\mu)\leqq 2-\omega\mu,\quad \omega(\beta+1-\mu)\leqq 1-\omega(\mu-1),\quad \omega(2\beta-\mu)\leqq 2-\omega\mu.$$

Все они эквивалентны неравенству $\omega\beta \leqq 1$. Выбор ω из неравенств $\frac{1}{\mu} < \omega \leqq \frac{1}{\beta}$ возможен, так как $\beta < \mu$. Итак, существуют положительные константы C_1, C_2 и C_3 не зависящие от τ, такие, что $\mathscr{L}y > v^{\omega\mu-2}(C_1\tau^2 - C_2\tau - C_3)$. Отсюда видно, что за счет возможного увеличения $\tau \geqq \tau_0$ можно достичь того, чтобы было $\mathscr{L}y > 0$ в D. Лемма доказана.

Согласно лемме, уравнение (5) равномерно по n параболично в P_η. В силу известных оценок из теории невырожденных параболических уравнений (см., напр., [20]) отсюда следует, что функция $u(t, x)$ удовлетворяет условию Гельдера в P_η, а значит, и всюду в $\mathbf{R}^2_{++} \cap \{t \leqq T\}$.

Выполнение интегрального тождества (4) и краевых условий (3) для $u(t, x)$ вытекает из построения. Теорема 1.1.3 доказана.

§ 1.2. Теоремы единственности

1.2.1. *Теорема единственности для задачи Коши.* Теорема 1.2.1. *Обобщенное решение задачи Коши* (1), (2) *единственно.*

Доказательство. Возьмем произвольное $T \in (0, \infty)$. Пусть $u(t, x)$ обобщенное решение задачи (1), (2) в $\mathbf{R}^2_+ \cap \{t \leqq T\}$, построенное в п. 1.1.2. Напомним, что $u(t, x) = \lim_{n\to\infty} u_n(t, x)$, где функции $u_n(t, x) > 0$ удовлетворяют уравнению (1) при $|x| < n$, $t < n$, а следовательно, и тождеству (4), если $t_1 < T < n$ и $[x_0, x_1] \subset \{|x| < n\}$.

Пусть $v(t, x)$ — другое обобщенное решение этой же задачи. Покажем, что $u(t, x) = v(t, x)$ в $\mathbf{R}^2_+ \cap \{t \leqq T\}$. Подставим в тождество (4) вместо u сначала u_n а затем v, вычтем из первого равенства второе. Получим

$$\int_{Ц} (f_t + A_n f_{xx} + B_n f_x - C_n f)(u_n - v)\,dx\,dt - \tag{16}$$

$$- \int_{x_0}^{x_1} f(u_n - v)\,dx \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} a(t) f_x (u_n^\mu - v^\mu)\,dt \Big|_{x_0}^{x_1} = 0,$$

где

$$A_n = \mu a(t) \int_0^1 [\theta u_n + (1-\theta)v]^{\mu-1} d\theta, \quad B_n = \int_0^1 \frac{\partial b\big(t, x, \theta u_n + (1-\theta)v\big)}{\partial u}\,d\theta,$$

$$C_n = \int_0^1 \frac{\partial c\big(t, x, \theta u_n + (1-\theta)v\big)}{\partial u}\,d\theta.$$

В силу предположений относительно функций $a(t)$ и $b(t, x, u)$ и свойств $u_n(t, x)$, доказанных в предыдущем парлграфе, функции $A_n(t, x)$ и $B_n(t, x)$ удовлетворяют условию Гельдера по обоим аргументам, причем

$$A_n \geqq \mu a(t) \inf_{t,x} u_n^{\mu-1} = m_n > 0, \quad n = 1, 2, \ldots.$$

Предположим, что $u \not\equiv v$; тогда найдется круг $E \subset \mathbf{R}^2_+$, в котором разность $u - v$

сохраняет знак. В тождестве (16) положим $t_0=0$, $t_1=T$, $x_0=-R$ и $x_1=R$. Предположим, что числа T и R настолько велики, что $E\subset Ц^{R-1}$ где $Ц^l=[-l, l]\times[0, T]$.

Пусть функция $g(t, x)\in C_0^\infty$ такова, что $g(u-v)>0$ внутри E и $\operatorname{supp} g=\bar{E}$. Обозначим через $\{C_{Rnk}(t, x)\}$ последовательность положительных гладких функций, равномерно сходящуюся к $C_n(t, x)$ при $k\to\infty$ в цилиндре $Ц^R$.

Рассмотрим первую краевую задачу в $Ц^R$ для функции $f(t, x)=f^{Rnk}(t, x)$:

$$(17)\qquad \begin{cases} \mathscr{L}_0 f=f_t+A_n f_{xx}+B_n f_x-C_{Rnk} f=g(t, x),\\ f|_{t=T}=f|_{|x|=R}=0.\end{cases}$$

Известно [32], что задача (17) имеет единственное решение f^{Rnk} для любого $k=1, 2, \ldots$. Нам понадобятся некоторые оценки для f^{Rnk}.

Лемма 1.2.1. *Существуют положительные постоянные α и M, не зависящие от R, n и k такие, что для f в $Ц^R$ выполнено неравенство*

$$|f(t, x)|\leqq M\exp[-\alpha(1+x^2)^{1/2}].$$

Доказательство. Рассмотрим функцию $z_1(t, x)=MT^{-1}(T-t)\exp[-\alpha(1+ +x^2)^{1/2}]$, и положим $w_1=\pm f+z_1$. Очевидно, что $w_1\geqq 0$ при $x=-R$ и $x=R$, и $w_1=0$ при $t=T$. В силу принципа максимума для доказательства леммы достаточно показать, что $\mathscr{L}_0 w_1<0$ в $Ц^R$ при некоторых M и α. Имеем

$$(18)\qquad \mathscr{L}_0 w_1=\pm g+z_1\left[A_n\left(-\frac{\alpha}{(1+x^2)^{1/2}}+\frac{\alpha x^2}{(1+x^2)^{3/2}}+\frac{\alpha^2 x^2}{1+x^2}\right)-B_n\frac{\alpha x}{(1+x^2)^{1/2}}-\right.$$

$$\left.-C_{Rnk}-\frac{1}{T-t}\right]\leqq\pm g+z_1\left[A_n(\alpha+\alpha^2)+|B_n|\alpha-\frac{1}{T}\right].$$

Мы можем предположить, не нарушая общности, что $|g(t, x)|<\frac{1}{2}MT^{-1}\cdot \exp[-(1+x^2)^{1/2}]$ при некотором M. Так как функции A_n и B_n равномерно ограничены, то из неравенства (18) видно, что можно выбрать $\alpha>0$ настолько маленьким, чтобы было $\mathscr{L}_0 w_1<0$ в $Ц^R$. Лемма 1.2.1. доказана.

Лемма 1.2.2. *Существует положительное число $\beta=\beta(n)$, зависящее от n, но не зависящее от R и k, такое, что*

$$\left|\frac{df(t, x)}{dx}\right|_{|x|=R}\leqq M\beta\big(1+(R-1)^2\big)^{1/2}\exp\left[-\frac{\alpha}{2}\big(1+(R-1)^2\big)^{1/2}\right],$$

где α и M то же, что и в лемме 1.2.1.

Доказательство. Воспользуемся методом, применявшемся в статье [27]. Пусть $B=\big(1+(R-1)^2\big)^{1/2}$, $R>2$. Введем функцию $z_2=M\exp\left(-\frac{\alpha}{2}B\right)\cdot \exp[\beta B(x-R)]$ и положим $w_2=\pm f+z_2$.

Имеем

$$w_2|_{x=R}=M\exp\left(-\frac{\alpha}{2}B\right),\quad w_2|_{x=R-1}=\pm f|_{x=R-1}+M\exp\left(-\frac{\alpha}{2}B-\beta B\right).$$

Условие $w_2|_{x=R-1} \leqq w_2|_{x=R}$ выполнено, если (см. лемму 1.2.1) $\exp\left(-\frac{\alpha}{2}B\right)+$ $+\exp(-\beta B) \leqq 1$. Так как $B>1$, то достаточно взять $\beta \geqq \beta_1$, где $\beta_1>0$ определяется равенством $e^{-\beta_1}=1-e^{-\alpha/2}$; очевидно, β_1 не зависит от R, n и k. Далее, в $Ц^R \setminus Ц^{R-1}$ имеем $\mathscr{L}_0 w_2 = z_2(A_n\beta^2B^2+B_n\beta B-C_{Rnk}) \geqq z_2(\alpha_0(n)(R-1)^2\beta^2-\alpha_1(n)R\beta-C_{Rnk})$, где $\alpha_0(n)$ и $\alpha_1(n)$ положительные числа. Поскольку C_{Rnk} равномерно по R и k ограничены, то можно выбрать $\beta_2(n)$ так, чтобы при $\beta \geqq \beta_2(n)$ выполнялось неравенство $\mathscr{L}_0 w_2 > 0$ в $Ц^R \setminus Ц^{R-1}$. Возьмем $\beta \geqq \beta(n) = \max(\beta_1, \beta_2)$. Из доказанного выше следует, что

$$\left.\frac{\partial w_2}{\partial x}\right|_{x=R} \geqq 0, \quad \text{что дает} \quad \left|\frac{\partial f}{\partial x}\right|_{x=R} \leqq M\beta B e^{-\frac{\alpha}{2}B}.$$

Для $x=-R$ доказательство аналогично. Лемма 1.2.2 доказана.

Вернемся к тождеству (16). На основании лемм 1.2.1 и 1.2.2 из него следует, что

$$\int_E g(u_n-v)\,dx\,dt \leqq M_1 \int_{Ц^R} |C_n-C_{Rnk}|\,e^{-\alpha(1+x^2)^{1/2}}\,dx\,dt+$$
$$+M_2\int_{-R}^{R}|u_n(0,x)-u_0(x)|\,e^{-\alpha(1+x^2)^{1/2}}dx+M_3\beta(n)\,Be^{-\frac{\alpha}{2}B}.$$

Устремляя здесь к бесконечности сначала k, затем R и, наконец, n, приходим к неравенству

$$\int_E g(u-v)\,dx\,dt \leqq 0.$$

Это противоречит положительности подинтегральной функции внутри E. Таким образом, $u \equiv v$ всюду при $t \leqq T$, а ввиду произвольности T — всюду в $\mathbf{R}^2_+$. Теорема доказана.

1.2.2. *Теорема единственности для первой краевой задачи.* Теорема 1.2.2. *Обобщенное решение первой краевой задачи* (1), (3) *единственно.*

Доказательство аналогично доказательству предыдущей теоремы. Отличие состоит в том, что в качестве $\{u_n\}$ берется последовательность, построенная в п.1.1.3, $x_0=0$, $x_1=R$.

Вместо задачи (17) рассматривается задача

$$\begin{cases}\mathscr{L}_0 f = g\\ f|_{t=T}=f|_{x=0}=f|_{x=R}=0.\end{cases}$$

§ 1.3. Теоремы сравнения

В этом параграфе будет доказана теорема о монотонной зависимости обобщенного решения уравнения (1) от начальных данных, а также некоторые другие утверждения, играющие важную роль в исследовании свойств обобщенных решений.

1.3.1. Пусть V замкнутая подобласть в $\mathbf{R}^2_+$ или в $\mathbf{R}^2_{++}$.

Определение 1.3.1. Неотрицательная в V функция $v(t, x)$, удовлетворяющая условию Гельдера и ограниченная при ограниченных t называется обобщенным суперрешением уравнения (1) в V, если для $v(t, x)$ выполняется неравенство $I(v, f; t_0, t_1; x_0, x_1) \leqq 0$, каковы бы ни были числа $t_0 < t_1$, $x_0 < x_1$, такие, что $Ц=[x_0, x_1]\times[t_0, t_1]\subset V$, и неотрицательная функция $f(t, x)\in C^{1,2}_{t,x}(Ц)$, равная нулю при $x=x_0$ и $x=x_1$. Здесь $I(v, f; t_0, t_1; x_0, x_1)$ определяется согласно (4).

Замечание 1.3.1. Пусть функция $v(t, x)\geqq 0$, удовлетворяющая в V условию Гельдера и ограниченная при ограниченных t является гладкой вне конечного числа непрерывных кривых вида $x=\xi(t)$ и удовлетворяет там неравенству $\mathscr{L}v\leqq 0$, где

$$\mathscr{L}y = -y_t+[a(t)y^\mu]_{xx}-[b(t, x, y)]_x-c(t, x, y). \tag{19}$$

Пусть, кроме того, производная $\dfrac{dv^\mu}{dx}$ непрерывна при $x=\xi(t)$. Тогда с помощью интегрирования по частям легко убедиться, что $v(t, x)$ является обобщенным суперрешением уравнения (1) в V.

Теорема 1.3.1. *Пусть $u(t, x)$ — обобщенное решение задачи Коши* (1), (2), *а $y(t, x)$ — обобщенное суперрешение уравнения* (1) *в $\mathbf{R}^2_+$. Предположим, что $u_0(x)\leqq y(0, x)$ для $x\in\mathbf{R}^1$. Тогда $u(t, x)\leqq y(t, x)$ всюду в $\mathbf{R}^2_+$.*

Доказательство. Предположим противное. Тогда, так как $u(t, x)$ и $y(t, x)$ непрерывны, существует круг $E\subset\mathbf{R}^2_+\cap\{t\leqq T\}$, в котором $u>y$, и тем более $u_n>y$, дге $\{u_n\}$ построенная в п. 1.1.2 последовательность.

Так как u_n является и обобщенным решением уравнения (1), мы имеем

$$I(y, f;\ t_0, t_1;\ x_0, x_1)-I(u_n, f;\ t_0, t_1;\ x_0, x_1)\leqq 0,$$

то есть

$$\int\limits_{Ц}(f_t+A_nf_{xx}+B_nf_x-C_nf)(y-u_n)\,dx\,dt\leqq \tag{20}$$

$$\leqq\int\limits_{x_0}^{x_1}f(y-u_n)\,dx\Big|_{t_0}^{t_1}+\int\limits_{t_0}^{t_1}a(t)f_x(y^\mu-u_n^\mu)\,dt\Big|_{x_0}^{x_1}.$$

Аналогично тому, как это было сделано в доказательстве теоремы 1.2.1, мы построим последовательность $\{C_{Rnk}(t, x)\}$ с нужными свойствами, где $R>0$ выбирается так же, как там. Полагая в (20) $t_0=0$, $t_1=T$, $Ц=Ц^R$, мы приходим к неравенству

$$\int\limits_{Ц^R}(f_t(t, x)+A_nf_{xx}+B_nf_x-C_{Rnk}f)(y-u_n)\,dx\,dt\leqq \tag{21}$$

$$\leqq\int\limits_{-R}^{R}f(0, x)[u_n(0, x)-y(0, x)]\,dx+\int\limits_{Ц^R}f(C_n-C_{Rnk})(y-u_n)\,dx\,dt+$$

$$+\int\limits_0^T a(t)f_x(y^\mu-u_n^\mu)\,dt\Big|_{-R}^{R}.$$

Пусть $g(t, x)$ — произвольная гладкая финитная функция, $\operatorname{supp} g = \bar{E}$ и $g(t, x) < 0$ в E, а $f(t, x) = f^{Rnk}(t, x)$ — решение задачи

$$\begin{cases} f_t + A_n f_{xx} + B_n f_x - C_{Rnk} f = g \quad \text{в} \quad Ц^R, \\ f(T, x) = f|_{|x|=R} = 0. \end{cases} \tag{22}$$

Оно существует и единственно для любых k, n, R. Из принципа максимума следует, что $f \geqq 0$. Подставим функцию $f = f^{Rnk}$ в неравенство (21), рассуждая подобно тому, как это было сделано в конце доказательства теоремы 1.2.1, мы получаем неравенство $\int_E g(y-u)\,dx\,dt \leqq 0$, и приходим к противоречию. Теорема доказана.

1.3.2. Пусть s — неотрицательное число. Введем множество

$$V_s = \{(t, x) \colon 0 \leqq t < \infty,\ s \leqq x < \infty\}.$$

Теорема 1.3.2. *Пусть $u(t, x)$ обобщенное решение, а $v(t, x)$ обобщенное суперрешение уравнения* (1) *в V_s. Предположим, что $u_0(x) \leqq v(0, x)$ при $s \leqq x < \infty$ и $u(t, s) \leqq v((t, s)$ при $0 \leqq t < \infty$. Тогда $u(t, x) \leqq v(t, x)$ всюду в V_s.*

Доказательство. Рассуждаем аналогично предыдущему пункту. Но вместо решения задачи (22) мы подставляем в неравенство (21) решение следующей задачи:

$$\begin{cases} f_t + A_n f_{xx} + B_n f_x - C_{Rnk} f = g \quad \text{в} \quad Ц = [0, T] \times [s, R], \\ f|_{t=T} = 0, \quad f|_{x=s} = f|_{x=R} = 0, \end{cases}$$

где $R > s$ настолько велико, что $E \subset Ц$.

Из принципа максимума следует, что $f \geqq 0$ в $Ц$, и $\left.\dfrac{df}{dx}\right|_{x=s} \geqq 0$. Аналогично тому, как было получено (21), мы приходим к неравенству

$$\int_E g(v-u_n)\,dx\,dt \leqq \int_s^R f(0, x)[u_n(0, x) - v(0, x)]\,dx +$$
$$+ \int_Ц f(C_n - C_{Rnk})(v - u_n)\,dx\,dt - \int_0^T a(t) f_x(t, s)[v^\mu(t, s) - u_n^\mu(t, s)]\,dt +$$
$$+ \int_0^T a(t) f_x(t, R)[v^\mu(t, R) - u_n^\mu(t, R)]\,dt,$$

и доказательство теоремы можно завершить, как в п.1.3.1.

1.3.3. Обозначим через H область, заданную неравенствами $0 < t < T < \infty$, $\xi_1(t) < x < \xi_2(t)$, где ξ_1 и ξ_2 непрерывные функции. Через ∂H обозначим ее параболическую границу.

Теорема 1.3.3. *Пусть $u(t, x)$ — обобщенное решение задачи* (1), (2), *а $z(t, x) \in C^{1,2}_{t,x}(\bar{H} \setminus \partial H) \cap C(\bar{H})$, причем $u \geqq z$ на ∂H, а в $\bar{H} \setminus \partial H$ выполнены неравенства $z(t, x) > 0$ и $\mathscr{L} z > 0$. Тогда $u(t, x) \geqq z(t, x)$ всюду в H.*

Доказательство. $\mathscr{L} u_n = 0$ в $\bar{H} \setminus \partial H$ и $u_n > z$ на ∂H. По теореме сравнения из [32] следует, что $u_n > z$ в H. Следовательно $u = \lim\limits_{n\to\infty} u_n \geqq z$.

§ 1.4. Некоторые обобщения на многомерный случай

1.4.1. Пусть $\mathbf{R}^p$ — p-мерное евклидово пространство,

$$x = (x_1, \dots, x_p) \in \mathbf{R}^p, \quad x' = (x_2, \dots, x_p).$$

$$\mathbf{R}_+^{p+1} = \{(t, x)\colon\ 0 \leqq t < \infty,\ x \in \mathbf{R}^p\}, \quad \mathbf{R}_{++}^{p+1} = \mathbf{R}_+^{p+1} \cap \{x_1 \geqq 0\}.$$

Мы будем рассматривать в $\mathbf{R}_+^{p+1}$ или $\mathbf{R}_{++}^{p+1}$ уравнение

$$u_t = \sum_{i=1}^{p} [a_i(t) u^\mu]_{x_i x_i} - \sum_{i=1}^{p} [b_i(t, x, u)]_{x_i} - c(t, x, u). \tag{23}$$

Предполагается выполнение следующих условий.

I. $\mu > 1$ — константа.

II. $a_i(t) \in C^2$ при $0 \leqq t < \infty$ и $i = 1, \dots, p$. $0 < \bar{a} \leqq a_i(t) \leqq \bar{\bar{a}} < \infty$ для всех $t \in [0, \infty)$, $i = 1, \dots, p$, где $\bar{a}$ и $\bar{\bar{a}}$ константы.

III. Функции $b_i(t, x, u)$ и $c(t, x, u)$ определены и непрерывны для значений $u \geqq 0$; при этом $b_i(t, x, 0) = c(t, x, 0) = 0$, а для $u > 0$ и всех рассматриваемых t, x справедливы соотношения:

$$b_i(t, x, u) \in C^3, \quad c(t, x, u) \in C^2, \quad c(t, x, u) \geqq 0,$$

$$\sum_{i=1}^{p} \frac{\partial b_i(t, x, u)}{\partial x_i} + c(t, x, u) \geqq 0.$$

IV. Существует такое число $\beta \in (\mu - 1, \mu)$, что при $u \in (0, M)$, $\forall M > 0$ и при всех рассматриваемых t, x ограничены следующие величины

$$u^{2\beta+1-\mu} \frac{\partial^2 b_i(t, x, u)}{\partial u \partial x_i}, \quad u^{\beta+2-\mu} \frac{\partial^2 b_i(t, x, u)}{\partial u^2},$$

$$u^{\beta+1-\mu} \frac{\partial b_i(t, x, u)}{\partial u}, \quad u^{3\beta-\mu} \frac{\partial^2 b_i(t, x, u)}{\partial x_i \partial x_k},$$

$$u^{2\beta-\mu} \frac{\partial b_i(t, x, u)}{\partial x_i}, \quad u^{2\beta-\mu} c(t, x, u), \quad u^{2\beta+1-\mu} \frac{\partial c(t, x, u)}{\partial u},$$

$$u^{3\beta-\mu} \frac{\partial c(t, x, u)}{\partial x_k} \quad (i, k = 1, 2, \dots, p).$$

Эти условия выполнены, например, в случае многомерного модельного уравнения: $a_i(t) = 1$, $b_i(t, x, u) = b_{i0} u^{\lambda_i}$, $c(t, x, u) = c_0 u^\nu$ при условиях $c_0 \geqq 0$, $\nu > 0$, $\lambda_i \geqq 1$ и любых b_{i0}, если $\max\left(\mu - 1, \dfrac{\mu - \nu}{2}\right) < \beta < \mu$; b_{i0} и c_0 — константы.

Для уравнения (2.3) рассмотрим либо задачу Коши в $\mathbf{R}_{+}^{p+1}$ с начальным условием

(24) $$u(0, x) = u_0(x), \quad x \in \mathbf{R}^p,$$

либо первую краевую задачу в $\mathbf{R}_{++}^{p+1}$ с условиями

(25) $$\begin{cases} u(0, x) = u_0(x), & x \in \mathbf{R}^p \cap \{x_1 \geqq 0\} \\ u(t, 0, x') = u_1(t, x'), & t \geqq 0. \end{cases}$$

Здесь $u_0(x) \geqq 0$ — непрерывная ограниченная функция, причем существуют ограниченные обобщенные производные $\partial u_0^\beta(x)/\partial x_i$ $(i=1, \ldots, p)$, где β — то же, что и выше.

В случае первой краевой задачи дополнительно предполагается следующее:

1° Существуют такие $\varepsilon_1 > 0$ и $\delta_1 > 0$, что $u_0(x) \geqq \varepsilon_1$ при $0 \leqq x_1 \leqq \delta_1$;

2° Функция $u_1(t, x')$ непрерывна, ограничена и строго положительна при ограниченных t, имеет обобщенные производные $\partial u_1/\partial t$, $\partial u_1^\beta/\partial x_i$ $(i=2, \ldots, p)$, ограниченные при ограниченных t;

3° Выполняется условие согласования $u_1(0, x') = u_0(0, x')$. Пусть Π замкнутая подобласть $\mathbf{R}_{+}^{p+1}$ или $\mathbf{R}_{++}^{p+1}$.

Определение 1.4.1. Неотрицательная в Π функция $u(t, x)$, удовлетворяющая условию Гельдера и ограниченная при ограниченных t, называется обобщенным решением уравнения (23) в Π, если для $u(t, x)$ выполняется интегральное тождество

$$I(u, f;\, t_0, t_1;\, \Omega) = \int\limits_{Ц} \left(u f_t + \sum_{i=1}^{p} a_i(t) u^\mu f_{x_i x_i} + \sum_{i=1}^{p} b_i(t, x, u) f_{x_i} - c(t, x, u) f \right) dx\, dt -$$

$$- \int\limits_{\Omega} u f\, dx \Big|_{t_0}^{t_1} - \sum_{i=1}^{p} \int\limits_{t_0}^{t} \int\limits_{\partial\Omega} a_i(t) u^\mu f_{x_i} \cos(\nu, x_i)\, ds\, dt = 0,$$

каковы бы ни были числа $t_0 < t_1$ и ограниченная область $\Omega \subset \mathbf{R}^p$ такие, что $Ц = [t_0, t_1] \times \bar{\Omega} \subset \Pi$, и фиункция $f(t, x) \in C_{t, x}^{1, 2}(Ц)$, равная нулю на $[t_0, t_1] \times \partial\Omega$.

Обобщенные решения задач (23), (24) и (23), (25) определяются так же, как в п.1.1.1.

1.4.2. Теорема существования

Теорема 1.4.1. *Пусть вынолнены условия пункта* 1.4.1. *Тогда, если*

$$\mu < 1 + \frac{2}{p} \sqrt{\frac{\bar{a}}{\bar{\bar{a}}} (\mu - \beta)(\beta - \mu + 1)},$$

то обобщенные решения задач (23), (24) *и* (23), (25) *существуют.*

Доказательство. Обобщенное решение строится так же, как в п.1.1.2. Доказывая гельдеровость обобщенного решения, мы приходим к аналогичному (12) неравенству

$$(26)\qquad \sum_{i,k=1}^{p} \mu a_i \zeta^2 \left[-\left(\frac{f''}{f'}\right)' f^2 - \frac{(\mu-\beta)(\mu-1-\beta)}{\beta^2} f'^2 - \frac{2\mu-1-\beta}{\beta} f'' f \right] w_i^2 w_k^2 \leqq$$

$$\leqq -\sum_{i,k=1}^{p} \mu a_i \zeta^2 f^2 w_{ik}^2 + \sum_{i,k=1}^{p} \mu a_i \zeta^2 \frac{\mu-1}{\beta} f' f w_{ii} w_k^2 +$$

$$+\sum_{i,k=1}^{p} 2\mu a_i \zeta^2 \left(\frac{f''}{f'} f^2 + \frac{\mu-\beta}{\beta} f' f \right) w_i w_{ki} w_k - \sum_{i,k=1}^{p} \mu a_i f^2 (\zeta_i^2 w_k^2 + \zeta \zeta_{ii} w_k^2 + 4\zeta^2 w_{ki} w_k) -$$

$$-\sum_{i,k=1}^{p} \zeta^2 \left[\frac{\partial^2 b_i}{\partial u \partial x_n} f^{(2\beta+1-\mu)/\beta} w_i w_k + \frac{\partial^2 b_i}{\partial u^2} \frac{1}{\beta} f^{(\beta+2-\mu)/\beta} f' w_i w_k^2 + \right.$$

$$+\frac{\partial b_i}{\partial u} f^{(2\beta+1-\mu)/\beta} w_{ik} w_k + \frac{\partial^2 b_i}{\partial x_i \partial x_k} \beta \frac{f^{(3\beta-\mu)/\beta}}{f'} w_k + \frac{\partial^2 b_i}{\partial x_i \partial u} f^{(2\beta+1-\mu)/\beta} w_k^2 +$$

$$\left. +\frac{\partial b_i}{\partial x_i} \beta f^{(2\beta+1-\mu)/\beta} \left(\frac{f^{(\beta-1)/\beta}}{f'} \right)' w_k^2 \right] - \sum_{k=1}^{p} \zeta^2 \left[\frac{\partial c}{\partial x_k} \beta \frac{f^{(3\beta-\mu)/\beta}}{f'} w_k + \frac{\partial c}{\partial u} f^{(2\beta+1-\mu)/\beta} w_k^2 + \right.$$

$$\left. + c\beta f^{(2\beta+1-\mu)/\beta} \left(\frac{f^{(\beta-1)/\beta}}{f'} \right)' w_k^2 \right],$$

где $w_i = \dfrac{\partial w}{\partial x_i}$, $w_{ik} = \dfrac{\partial^2 w}{\partial x_i \partial x_k}$. Обозначим через $\sum_j$ j-ый член неравенства (26) так, чтобы оно имело вид

$$\textstyle\sum_1 + \sum_2 + \sum_3 \leqq \sum_4 + \sum_5 + \ldots + \sum_{19}.$$

Имеем $\quad \sum_1 + \sum_2 + \sum_3 \geqq \sum_2 \geqq \displaystyle\sum_{k=1}^{p} 2\mu\zeta^2 \frac{(\mu-\beta)(\beta-\mu+1)}{\beta^2} f'^2 a_k w_k^4 \equiv \sum_{21}$

и

$$\sum_5 \leqq \frac{1}{\varepsilon} \sum_{i,k=1}^{p} \mu a_i \zeta^2 \frac{\mu-1}{\beta} f^2 w_{ii}^2 + \varepsilon \sum_{i,k=1}^{p} \mu a_i \zeta^2 \frac{\mu-1}{\beta} f'^2 w_k^4 \equiv \sum_{s1} + \sum_{s2},$$

где $\varepsilon > 0$ — любое число. Далее, $\qquad \zeta$

$$\sum_4 + \sum_{51} \leqq -\sum_{i=1}^{p} 2\mu a_i \zeta^2 f^2 w_{ii}^2 + \frac{p}{\varepsilon} \sum_{i=1}^{p} \mu a_i \zeta^2 \frac{\mu-1}{\beta} f^2 w_{ii}^2 \leqq 0,$$

если $\varepsilon \geqq \dfrac{p(\mu-1)}{2\beta}$. Фиксируем $\delta \in (0, 1)$ так, чтобы было

$$\mu \leqq 1 + \frac{2}{p} \sqrt{\frac{(1-\delta)\bar{a}}{\bar{\bar{a}}} (\mu-\beta)(\beta-\mu+1)}.$$

Имеем

$$(1-\delta)\sum\nolimits_{21}-\sum\nolimits_{52}=(1-\delta)2\mu\zeta^2\frac{(\mu-\beta)(\beta-\mu+1)}{\beta^2}f'^2\sum_{k=1}^{p}a_k w_k^4-$$

$$-\varepsilon\mu\zeta^2\frac{\mu-1}{\beta}f'^2\left(\sum_{k=1}^{p}a_k\right)\sum_{k=1}^{p}w_k^4\geqq 0,$$

если

$$\varepsilon\leqq 2(1-\delta)\frac{\bar{a}(\mu-\beta)(\beta-\mu+1)}{\bar{\bar{a}}p\beta(\mu-1)}.$$

Ввиду (27), неравенства для ε совместны, поэтому, так же как в п. 1.2.1, мы приходим к неравенству

$$\delta\zeta^2\sum_{k=1}^{p}w_k^{-4}\leqq C_1\sum_{k=1}^{p}|w_k|+C_2\sum_{k=1}^{p}w_k^2+C_3\sum_{k=1}^{p}|w_k|^3,$$

и доказательство теоремы можно закончить так же, как в п.1.2.1.

1.4.3. Теоремы единственности и сравнения в многомерном случае формулируются аналогично §§ 1.2 и 1.3 и доказываются дословно так, как там. При ссылках на эти утверждения мы будем ссылаться на соответствующие утверждения из §§ 1.2 и 1.3.

II. КАЧЕСТВЕННОЕ ИССЛЕДОВАНИЕ ОБОБЩЕННЫХ РЕШЕНИЙ

В этой главе мы будем считать, что выполнены условия, при которых доказаны теоремы существования, единственности и сравнения главы I.

§ 2.1. Локализация возмущений в задаче Коши

Будем предполагать, что функция $u_0(x)$ финитна: $u_0(x)=0$ при $|x|\geqq l$; пусть, кроме того, $u_0(0)>0$.

Определение 2.1.1. Будем говорить, что в задаче (23), (24) происходит локализация возмущений справа по x_k, если существует такое число $l_k>0$, что $u(t,x)=0$ при $x_k\geqq l_k$ для всех $t\geqq 0$, где $u(t,x)$ — обобщенное решение задачи Коши (23), (24). В противном случае мы будем говорить об отсутствии локализации возмущений справа по x_k.

Аналогично определяется локализация возмущений слева по x_k и ее отсутствие.

Если по x_k происходит локализация и справа и слева, то мы будем говорить о двусторонней локализации возмущений по x_k.

Введем обозначение

$$(28)\quad \mathscr{L}y=-y_t+\sum_{i=1}^{p}a_i(t)(y^\mu)_{x_ix_i}-\sum_{i=1}^{p}\left(\frac{\partial b_i(t,x,y)}{\partial u}y_{x_i}+\frac{\partial b_i(t,x,y)}{\partial x_i}\right)-c(t,x,y).$$

Результаты этой главы будут иллюстрироваться на примере одномерного модельного уравнения

$$u_t = (u^\mu)_{xx} - b_0(u^\lambda)_x - c_0 u^\nu, \tag{29}$$

где постоянные, входящие в уравнение, удовлетворяют условиям $\mu > 1$, $\lambda \geqq 1$, $\nu > 0$, $c_0 >$; b_0 — произвольное.

Теорема 2.1.1. *Пусть, кроме условий пункта* 1.4.1, *выполняются следующие предположения: существуют положительные постоянные* $\omega > \mu^{-1}$, C_1, C_2 *и* $k < 1$, *такие, что при* $\theta \in [0, M2^\omega]$, *где* $M = \sup u(t, x)$, *справедливы неравенства*

1° $\theta^\mu \leqq C_1 \theta^{2/\omega} c(t, x, \theta)$,

2° *либо* $\dfrac{\partial b_k(t, x, u)}{\partial u} \leqq 0$, *либо* $\dfrac{\partial b_k(t, x, \theta)}{\partial u}\theta \leqq C_2 \theta^{1/\omega}(t, x, \theta)$,

3° *либо* $\displaystyle\sum_{i=1}^{p} \frac{\partial b_i(t, x, u)}{\partial x_l} \geqq 0$, *либо* $\displaystyle\left|\sum_{i=1}^{p} \frac{\partial b_i(t, x, \theta)}{\partial x_i}\right| \leqq kc(t, x, \theta)$.

Тогда по x_k *происходит локализация возмущений справа.*

Доказательство. Рассмотрим в области $V_k^+ = \{(t, x)\colon 0 < t < \infty,\ 0 < x_k < \infty,\ x \in \mathbf{R}^p\}$ вспомогательную функцию $v_+(t, x) = v_+(x_k)$:

$$v_+(x_k) = \begin{cases} M\left(\dfrac{L - x_k}{L - l}\right)^\omega & \text{при} \quad 0 < x_k < L \\ 0 & \text{при} \quad x_k \geqq L, \end{cases}$$

где $L > 2l$ — число, которое мы выберем позже. Очевидно, что $v_+(0, x) \geqq u_0(x)$, $v_+|_{x_k=0} \geqq u|_{x_k=0}$. Покажем, что $\mathscr{L} v_+ < 0$ при $0 < x_k < L$.

Положим $A = \dfrac{L - x_k}{L - l}$. Имеем

$$\mathscr{L} v_+ = \frac{\omega\mu(\omega\mu - 1) a_k(t)}{(L-l)^2} M^\mu A^{\omega\mu - 2} + \frac{\omega\mu}{L - l} \frac{\partial b_k(t, x, MA^\omega)}{\partial u} MA^{\omega - 1} - \tag{30}$$

$$- \sum_{i=1}^{p} \frac{\partial b_i(t, x, MA^\omega)}{\partial x_i} - c(t, x, MA^\omega).$$

Из условий теоремы следует, что

$$\mathscr{L} v_+ \leqq c(t, x, MA^\omega)\left[\frac{M_1}{(L-l)^2} + \frac{M_2}{L - l} + k - 1\right],$$

где константы M_1 и M_2 от L не зависят. Поэтому мы можем выбрать $L > 2l$ таким большим, чтобы при $0 < x_k < L$ выполнялось неравенство $\mathscr{L} v_+ < 0$. Так как $\mathscr{L} v_+ = 0$ при $x_k \geqq L$ и производная $(v_+^\mu)_x$ непрерывна, то по Замечанию 1.3.1, $v_+(t, x)$ является обобщенным суперрешением уравнения (23) в V_k^+. По теореме 1.3.2 получаем, что $u(t, x) \leqq v_+(t, x)$ всюду в V_k^+, откуда следует утверждение доказываемой теоремы.

Замечание 2.1.1. Как уже отмечалось, для уравнения (29) все условия пункта 1.1.1 (и п.1.4.1) выполнены. Условия теоремы 2.1.1 эквивалентны следующим: существует $\omega > \frac{1}{\mu}$ такое, что при $\theta \in [0, M2^\omega]$

а) $\theta^{\mu-\nu-2/\omega} \leqq C_1$,

б) либо $b_0 \leqq 0$, либо $\theta^{\lambda-\nu-1/\omega} \leqq C_2$.

Отсюда видно, что в качестве ω в случае $b_0 \leqq 0$ можно взять $\frac{2}{\mu-\nu}$, в случае $b_0 > 0$ — число $\max\left(\frac{2}{\mu-\nu}, \frac{1}{\lambda-\nu}\right)$ при условии, что $\mu > \nu$ и $\lambda > \nu$. Теорему 2.1.1 в этом случае можно сформулировать так: пусть $\nu < \min(\mu, \lambda)$, b_0 — любое; тогда по x происходит локализация возмущений справа.

Теорема 2.1.2. *Пусть, кроме условий пункта* 1.4.1, *выполняются следующие предположения: существуют положительные постоянные* $\omega > \mu^{-1}$, C_1, C_2 *и* $k < 1$ *такие, что при* $\theta \in [0, M2^\omega]$ *справедливы неравенства*

1° $\theta^\mu \leqq C_1 \theta^{2/\omega} c(t, x, \theta)$,

2° *либо* $\frac{\partial b_k(t, x, u)}{\partial u} \geqq 0$, *либо* $\left|\frac{\partial b_k(t, x, \theta)}{\partial u}\right| \theta \leqq C_2 \theta^{1/\omega} c(t, x, \theta)$,

3° *либо* $\sum_{i=1}^{p} \frac{\partial b_i(t, x, u)}{\partial x_i} \geqq 0$, *либо* $\left|\sum_{i=1}^{p} \frac{\partial b_i(t, x, \theta)}{\partial x_i}\right| \leqq kc(t, x, \theta)$.

Тогда по x_k *происходит локализация возмущений слева.*

Доказательство. Рассмотрим в области $V_k^- = \{(t, x): 0 < t < \infty, -\infty < x_k < 0, x \in \mathbf{R}^p\}$ вспомогательную функцию

$$v_-(t, x) = \begin{cases} M\left(\frac{L+x_k}{L-l}\right)^\omega & \text{при} \quad -L < x_k < 0 \\ 0 & \text{при} \quad x_k \leqq -L, \end{cases}$$

где число $L > 2l$ будет выбрано ниже.

Очевидно, что $v_-(0, x) \geqq u_0(x)$, $v_-|_{x_k=0} \geqq u|_{x_k=1}$. Покажем, что $\mathscr{L} v_- < 0$ при $-L < x_k < 0$. Положим $A = \frac{L+x_k}{L-l}$. Имеем

$$\mathscr{L} v_- = \frac{\omega\mu(\omega\mu-1)a_k(t)}{(L-l)^2} M^\mu A^{\omega\mu-2} - \frac{\omega\mu}{L-l} \frac{\partial b_k(t, x, MA^\omega)}{\partial u} MA^{\omega-1} - \sum_{i=1}^{p} \frac{\partial b_i(t, x, MA^\omega)}{\partial x_i} - c(t, x, MA^\omega). \tag{31}$$

Из условий теоремы следует, что

$$\mathscr{L} v_- \leqq c(t, x, MA^\omega)\left[\frac{M_1}{(L-l)^2} + \frac{M_2}{L-l} + k - 1\right],$$

где постоянные M_1 и M_2 от L не зависят. Поэтому мы можем выбрать $L>2l$ так, чтобы было $\mathscr{L}v_- <0$ при $-L<x_k<0$. После этого доказательство завершается так же, как в предыдущей теореме.

Замечание 2.1.2. Для модельного уравнения (29) теорема 2.1.2 формулируется так: пусть $\nu<\min(\mu, \lambda)$, b_0 — любое; тогда по x происходит локализация возмущений слева. Принимая во внимание замечание 2.1.1, мы можем сформулировать следующее утверждение: пусть $\nu<\min(\mu, \lambda)$, b_0 — любое. Тогда по x происходит двусторонняя локализация возмущений.

Теорема 2.1.3. *Пусть, кроме условий пункта* 1.4.1, *выполняются следующие предположения: существуют положительные постоянные* $\omega>\mu^{-1}$ *и* C, *такие, что при* $\theta\in[0, M2^{\omega}]$ *справедливы неравенства*

$$1^\circ \quad \frac{\partial b_k(t,x,u)}{\partial u}<0 \quad \textit{при} \quad u>0,$$

$$2^\circ \quad \theta^{\mu}\leqq C\theta^{1+1/\omega}\left|\frac{\partial b_k(t,x,\theta)}{\partial u}\right|,$$

$$3^\circ \quad \sum_{i=1}^{p}\frac{\partial b_i(t,x,u)}{\partial x_i}\geqq 0.$$

Тогда по x_k *происходит локализация возмущений справа.*

Доказательство. Возьмем ту же область и ту же функцию, что и в доказательстве теоремы 2.1.1. Имеем из (30):

$$\mathscr{L}v_+ \leqq \frac{\omega\mu MA^{\omega-1}}{L-l}\left|\frac{\partial b_k(t,x,MA^{\omega})}{\partial u}\right|\left(\frac{M_1}{L-l}-1\right),$$

и опять можем выбрать $L>2l$ таким образом, чтобы было $\mathscr{L}v_+<0$ при $0<x_k<L$. Доказательство завершается так же, как в теореме 2.1.1.

Замечание 2.1.3. Для модельного уравнения (29) условия теоремы 2.1.3 эквивалентны следующим: существуют постоянные $\omega>\mu^{-1}$ и $C>0$ такие, что при $\theta\in[0, M2^{\omega}]$

а) $b_0<0$,

б) $\theta^{\mu-\lambda-1/\omega}\leqq C$.

Отсюда видно, что если $\mu>\lambda$, то можно положить $\omega=\dfrac{1}{\mu-\lambda}$, и теорему сформулировать так: пусть $\lambda<\mu$, $b_0<0$ а $\lambda>0$ — любое. Тогда по x происходит локализация возмущений справа.

Отметим, что при $b_0>0$ и $\min(\mu, \nu)>\lambda=1$ по x справа нет локализации возмущений (см. [17]).

Теорема 2.1.4. *Пусть, кроме условий п.*1.4.1 *выполняются следующие предположения: существуют положительные постоянные* $\omega>\mu^{-1}$ *и* C, *такие, что при* $\theta\in[0, M2^{\omega}]$ *справедливы неравенства*

$$1^\circ \quad \frac{\partial b_k(t,x,u)}{\partial u}>0 \quad \textit{при} \quad u>0,$$

2° $\theta^{\mu} \leqq C\theta^{1+1/\omega} \dfrac{\partial b_k(t, x, \theta)}{\partial u}$,

3° $\sum_{i=1}^{p} \dfrac{\partial b_i(t, x, u)}{\partial x_i} \geqq 0.$

Тогда по x_k происходит локализация возмущений слева.

Доказательство. Рассмотрим ту же область и ту же вспомогательную функцию, что и в доказательстве теоремы 2.1.2. Имеем из (31):

$$\mathscr{L}v_{-} \leqq \frac{\omega\mu M A^{\omega-1}}{L-l} \frac{\partial b_k(t, x, MA^{\omega})}{\partial u} \left(\frac{M_1}{L-l} - 1\right),$$

и мы опять можем завершить доказательство теоремы так же как это было сделано в предыдущих теоремах.

Замечание 2.1.4. В случае модельного уравнения (29) теорема 2.1.4 гласит: пусть $b_0 > 0$ и $\lambda < \mu$. Тогда по x происходит локализация возмущений слева.

Как нетрудно проверить, при $b_0 < 0$ и $\min(\mu, \nu) > \lambda = 1$ имеет место отсутствие локализации возмущений слева по x. (Этот случай сводится к рассмотренному в [17] заменой x на $-x$.)

Теорема 2.1.5. *Предположим, что, кроме условий п.* 1.4.1, *выполнены следующие:*

1° *существуют числа $\lambda_i \geqq \mu$ такие, что*

$$\left|\frac{\partial b_i(t, x, y)}{\partial u}\right| \leqq C_1 y^{\lambda_i - 1} \quad \text{при} \quad y \in [0, 1], \quad i = 1, \ldots, p,$$

2° *для любого $i = 1, \ldots, p$ выполняется либо неравенство $\dfrac{\partial b_i(t, x, u)}{\partial x_i} \leqq 0$, либо неравенство $\dfrac{\partial b_i(t, x, y)}{\partial x_i} \leqq C_2 y^{\lambda_i}$, $y \in [0, 1]$,*

3° *существует число $\nu \geqq \mu$ такое, что $c(t, x, y) \leqq C_3 y^{\nu}$ при $y \in [0, 1]$, где C_1, C_2 и C_3 положительные постоянные.*

Тогда локализация возмущений отсутствует по всем направлениям.

Доказательство. Без ограничения общности мы можем считать, что $u_0(x) \geqq \varepsilon > 0$ при $|x|^2 \leqq \delta$, $\delta > 0$. Положим $H = \{(t, x): 0 < t < \infty, v = \varrho - |x|^2 \ln^{-\gamma}(\ln(t+\tau)) > 0\}$, где положительные постоянные $\tau > \exp(e)$, $\varrho < 1$ и $\gamma < 1$ будут выбраны ниже. Рассмотрим в этой области функцию

$$y(t, x) = (t+\tau)^{-1/(\mu-1)} \ln^{-1/(\mu-1)}(t+\tau)\left[\varrho - |x|^2 \ln^{-\gamma}(\ln(t+\tau))\right]^{1/(\mu-1)}.$$

Положим для сокращения записи $\omega = \dfrac{1}{\mu-1}$, $\theta = t+\tau$; тогда $y(t, x) = \theta^{-\omega} \ln^{-\omega} \theta v^{\omega}$. Если мы выберем τ из неравенства $\tau \ln \tau \geqq \varepsilon^{-1/\omega}$, затем ϱ из неравенства

$$\varrho < \delta \ln^{-1}(\ln \tau), \tag{32}$$

то мы очевидным образом будем иметь: $y(t,x)\leqq u(t,x)$ на параболической границе H, так как $y(t,x)=0$ на боковой границе H. Следовательно, если мы докажем, что $\mathscr{L}y>0$ в H, то наша теорема будет доказана ввиду теоремы 1.3.3. Имеем

$$\mathscr{L}y=\omega\theta^{-\omega-1}\ln^{-\omega}\theta v^{\omega}+\omega\theta^{-\omega-1}\ln^{-\omega-1}\theta v^{\omega}-$$
$$-\sum_{i=1}^{p}\gamma\omega\theta^{-\omega-1}\ln^{-\omega-1}\theta\ln^{-\gamma-1}(\ln\theta)v^{\omega-1}x_i^2+$$
$$+\sum_{i=1}^{p}4a_i(t)\omega\mu(\omega\mu-1)\theta^{-\omega\mu}\ln^{-\omega\mu}\theta\ln^{-2\gamma}(\ln\theta)v^{\omega\mu-2}x_i^2-$$
$$-\sum_{i=1}^{p}2a_i(t)\omega\mu\theta^{-\omega\mu}\ln^{-\omega\mu}\theta\ln^{-\gamma}(\ln\theta)v^{\omega\mu-1}+$$
$$+\sum_{i=1}^{p}\frac{\partial b_i(t,x,y)}{\partial u}\omega\theta^{-\omega}\ln^{-\omega}\theta\ln^{-\gamma}(\ln\theta)v^{\omega-1}\cdot 2x_i-\sum_{i=1}^{p}\frac{\partial b_i(t,x,y)}{\partial x_i}-c(t,x,y). \tag{33}$$

Отметим очевидные соотношения: $2x_i\geqq -1-x_i^2$, $v<\varrho<1$, $x_i^2\ln^{-\gamma}(\ln\theta)<\varrho<1$, $-\omega\mu=-\omega-1$, $\omega\mu-1=\omega$ и $\omega\mu-2=\omega-1$.

Имеем из (33):

$$\mathscr{L}y>\sum_{i=1}^{p}4a_i(t)\omega\mu(\omega\mu-1)\theta^{-\omega-1}\ln^{-\omega-1}\theta\ln^{-2\gamma}(\ln\theta)v^{\omega-1}x_i^2-$$
$$-\sum_{i=1}^{p}\gamma\omega\theta^{-\omega-1}\ln^{-\omega-1}\theta\ln^{-\gamma-1}(\ln\theta)v^{\omega-1}x_i^2+\omega\theta^{-\omega-1}\ln^{-\omega}\theta v^{\omega}-$$
$$-\sum_{i=1}^{p}2a_i(t)\omega\mu\theta^{-\omega-1}\ln^{-\omega-1}\theta\ln^{-\gamma}(\ln\theta)v^{\omega}-\sum_{i=1}^{p}\left|\frac{\partial b_i(t,x,y)}{\partial u}\right|\omega\theta^{-\omega}\ln^{-\omega}\theta v^{\omega-1}-$$
$$-\sum_{i=1}^{p}\left|\frac{\partial b_i(t,x,y)}{\partial u}\right|\omega\theta^{-\omega}\ln^{-\omega}\theta\ln^{-\gamma}(\ln\theta)v^{\omega-1}-\sum_{i=1}^{p}\left|\frac{\partial b_i(t,x,y)}{\partial x_i}\right|-$$
$$-c(t,x,y)\equiv I_1+I_2+\ldots+I_8. \tag{34}$$

Имеем

$$I_1+I_2=\sum_{i=1}^{p}x_i^2\omega\theta^{-\omega-1}\ln^{-\omega-1}\theta\ln^{-2\gamma}(\ln\theta)v^{\omega-1}[4a_i(t)\omega\mu-\gamma\ln^{\gamma-1}(\ln\theta)].$$

Фиксируем $\gamma<1$ и выберем $\tau\geqq\tau_1$ так, чтобы было $4\bar{a}\omega\mu>\ln^{-1+\gamma}(\ln\tau)$. Далее,

$$I_3+I_4+\ldots+I_8\geqq\theta^{-\omega-1}\ln^{-\omega}\theta v^{\omega}\big[\omega-2\bar{\bar{a}}p\omega\mu\ln^{-1}\theta-$$
$$-2C_1\omega\sum_{i=1}^{p}\theta^{-\omega(\lambda_i-\mu)}\ln^{-\omega(\lambda_i-1)}\theta-C_2\sum_{i=1}^{p}\ln^{-\omega(\lambda_i-1)}\theta-C_3\ln^{-\omega(\nu-1)}\theta\big].$$

Напомним, что $\lambda_i\geqq\mu>1$ и $\nu\geqq\mu>1$. Поэтому мы можем выбрать $\tau\geqq\tau_2$ таким образом, чтобы было $I_3+\ldots+I_8>0$ в H. Если мы сначала выберем τ из неравенства $\tau\geqq\max(\exp(e),\varepsilon^{-1/\omega},\tau_1,\tau_2)$, затем $\varrho>0$ из (32), то из неравенства (34) будет следовать, что $\mathscr{L}y>0$ в H. Теорема доказана.

З а м е ч а н и е 2.1.5. В случае модельного уравнения (29) теорему 2.1.5 можно сформулировать так: пусть $\mu\leqq\min(\lambda,\nu)$; тогда по x отсутствует локализация возмущений как слева, так и справа.

Замечания 2.1.1—2.1.5 показывают, что для модельного уравнения (29) при любом соотношении между числами μ, λ и ν и при любом b_0 мы можем сказать, имеет ли место локализация возмущений (слева или справа) или она отсутствует.

§ 2.2. Локализация возмущений в первой краевой задаче

В этом параграфе мы будем предполагать, что $u_0(x)=0$ при $x_1 \geqq l > 0$.

Определение 2.2.1. Будем говорить, что в задаче (23), (25) происходит локализация возмущений по x_1, если существует такое число $l_1>0$, что $u(t, x)=0$ при $x_1 \geqq l_1$ для всех $t \geqq 0$ и $x' \in \mathbf{R}^{p-1}$, где $u(t, x)$ — обобщенное решение задачи (23), (25). В противном случае мы будем говорить об отсутствии локализации возмущений по x_1.

Теорема 2.2.1. *Пусть функция $u_1(t, x')$ ограничена, и пусть, кроме условий п.1.4.1, выполнены следующие условия (мы предполагаем, что u_0 и u_1 ограничены единой константой M): существуют положительные константы $\omega > \mu^{-1}$, C_1, C_2 и $k<1$ такие, что при $G \in [0, M2^{\omega}]$ справедливы неравенства*

1° $\theta^{\mu} \leqq C_1 \theta^{2/\omega} c(t, x, \theta)$,

2° *либо* $\dfrac{\partial b_1(t, x, u)}{\partial u} \leqq 0$, *либо* $\dfrac{\partial b_1(t, x, \theta)}{\partial u} \theta \leqq C_2 \theta^{1/\omega} c(t, x, \theta)$,

3° *для любого $i=1, \ldots, p$ выполняется либо неравенство* $\dfrac{\partial b_i(t, x, u)}{\partial x_i} \geqq 0$, *либо неравенство*

$$\left|\frac{\partial b_i(t, x, \theta)}{\partial x_i}\right| < \frac{k}{p} c(t, x, \theta).$$

Тогда по x_1 происходит локализация возмущений.

Доказательство. Рассмотрим в области $G=\{(t, x): 0<t<\infty,\ 0<x_1<\infty,\ x' \in \mathbf{R}^{p-1}\}$ вспомогательную функцию

$$v(t, x) = \begin{cases} M\left(\dfrac{L-x_1}{L-l}\right)^{\omega} & \text{при} \quad 0<x_1<L \\ 0 & \text{при} \quad x_1 \geqq L, \end{cases}$$

где $L>2l$ — число, которое будет выбрано ниже. Очевидно, что $v(0, x) \geqq u_0(x)$ и $v(t, 0, x') \geqq u_1(t, x')$. Поэтому, как в теореме 2.1.1, достаточно показать, что $\mathscr{L}v<0$ при $0<x_1<L$, где $\mathscr{L}$ определяется согласно (28). Находим, полагая $A=\dfrac{L-x_1}{L-l}$:

$$\mathscr{L}v = \frac{\omega\mu(\omega\mu-1)}{(L-l)^2} a_1(t) M^{\mu} A^{\omega\mu-2} + \frac{\partial b_1(t, x, MA^{\omega})}{\partial u} \frac{\omega\mu}{L-l} MA^{\omega-1} - \sum_{i=1}^{p} \frac{\partial b_i(t, x, MA^{\omega})}{\partial x_i} - c(t, x, MA^{\omega}). \tag{35}$$

Из условий теоремы и из (35) вытекает, что

$$\mathcal{L}v \leqq c(t, x, MA^{\omega})\left[\frac{M_1}{(L-l)^2}+\frac{M_2}{L-l}+k-1\right],$$

где постоянные M_1 и M_2 не зависят от L. Поэтому мы можем выбрать $L>2l$ настолько большим, чтобы при $0<x_1<L$, было $\mathcal{L}v<0$. Теорема доказана.

Замечание 2.2.1. В случае модельного уравнения (29) теорема 2.2.1 утверждает следующее: если функция $u_1(t)$ ограничена и $\nu<\min(\mu, \lambda)$, то при любом b_0 происходит локализация возмущений.

Теорема 2.2.2. *Пусть функция $u_1(t, x')$ ограничена. Пусть, кроме условий п. 1.4.1, выполнены следующие: существуют постоянные $\omega>\mu^{-1}$ и $C>0$ такие, что при $\theta\in[0, M2^{\omega}]$ справедливы соотношения*

1° $\dfrac{\partial b_1(t, x, u)}{\partial u} < 0$ при $u > 0,$

2° $\theta^{\mu} \leqq C\theta^{1+1/\omega}\left|\dfrac{\partial b_i(t, x, \theta)}{\partial u}\right|,$

3° *либо* $b_i(t, x, u) = b_i(t, u),$ *либо* $\dfrac{\partial b_i(t, x, u)}{\partial x_i} \geqq 0, \quad i = 1, \dots, p.$

Тогда по x_1 происходит локализация возмущений.

Доказательство. Сравним обобщенное решение задачи (23), (25) с функцией $v(t, x)$ в области G (см. доказательство теоремы 2.2.1). Из (35) следует, что

$$\mathcal{L}v \leqq \frac{MA^{\omega-1}}{L-l}\left|\frac{\partial b_1(t, x, MA^{\omega})}{\partial u}\right|\left(\frac{M_1}{L-l}-\omega\mu\right) \quad \text{при} \quad 0 < x_1 < L,$$

где M, от L не зависит. Следовательно, мы можем выбрать $L>2l$ так, чтобы выполнялось неравенство $\mathcal{L}v<0$ при $0<x_1<L$. Теорема доказана.

Замечание 2.2.2. Для модельной задачи (29), (3) теорему 2.2.2 можно сформулировать так: пусть $u_1(t)$ ограничена, $b_0<0$ и $\lambda<\mu$. Тогда в задаче (29), (3) происходит локализация возмущений.

Теорема 2.2.3. *Пусть, кроме условий п.* 1.4.1 *выполнены следующие предположения:*

1° $u_1(t, x') \geqq c_1(t+1)^{-1/(\mu-1)},$

2° *либо* $\dfrac{\partial b_1(t, x, u)}{\partial u}\geqq 0,$ *либо существует число* $\lambda_1>\mu,$ *такое, что* $\left|\dfrac{\partial b_1(t, x, y)}{\partial u}\right| \leqq C_2 y^{\lambda_1-1}$ *при* $y\in[0, 1],$

3° *существует число* $\nu\geqq\mu,$ *такое, что* $c(t, x, y)\leqq C_3 y^{\nu}$ *при* $y\in[0, 1],$

4° *либо* $b_i(t, x, u) = b_i(t, u)$, *либо* $\dfrac{\partial b_i(t, x, u)}{\partial x_i} \leqq 0$, *для всех* $i = 1, \ldots, p$, *где* C_1, C_2 *и* C_3 *положительные постоянные. Тогда локализация возмущений по* x_1 *отсутствует.*

Доказательство. Рассмотрим в области

$$G = \{(t, x)\colon 0 < t < \infty, \quad w = \varrho - x_1 \ln^{-\gamma}(\ln(t+\tau)) > 0, \quad x' \in \mathbf{R}^{p-1}\}$$

вспомогательную функцию

$$z(t, x) = (t+\tau)^{-1/(\mu-1)} \ln^{-1/(\mu-1)}(t+\tau)[\varrho - x_1 \ln^{-\gamma} \ln(t+\tau))]^{1/(\mu-1)}$$

где положительные постоянные $\tau > \exp(e) = \tau_0$, $\varrho < 1$ и $\gamma < 1$ будут выбраны ниже. Если мы положим $\omega = (\mu-1)^{-1}$ и $\theta = t+\tau$, то z примет более простой вид: $z(t, x) = \theta^{-\omega} \ln^{-\omega} \theta w^{\omega}$. Сначала мы выберем $\tau \geqq \tau_1$, и $\varrho \leqq \varrho_1 < 1$ так, чтобы мы имели на границе G неравенства $u_1(t, x') \geqq z(t, x')$ и $u_0(x) \geqq z(0, x)$. Это, очевидно, возможно.

Покажем, что при подходящем выборе параметров в G выполняется неравенство $\mathscr{L} z > 0$. Имеем

$$\mathscr{L} z = \omega\theta^{-\omega-1} \ln^{-\omega} \theta w^{\omega} + \omega\theta^{-\omega-1} \ln^{-\omega-1} \theta w^{\omega} -$$
$$-\gamma\omega\theta^{-\omega-1} \ln^{-\omega-1} \theta \ln^{-\gamma-1}(\ln\theta) w^{\omega-1} x_1 +$$
$$+\omega\mu(\omega\mu-1) a_1(t) \theta^{-\omega\mu} \ln^{-\omega\mu} \theta \ln^{-2\gamma}(\ln\theta) w^{\omega\mu-2} +$$
$$+\omega \frac{\partial b_1(t, x, z)}{\partial u} \theta^{-\omega} \ln^{-\omega} \theta \ln^{-\gamma}(\ln\theta) w^{\omega-1} -$$
$$-\sum_{i=1}^{p} \frac{\partial b_i(t, x, z)}{\partial x_i} - c(t, x, z) \equiv I_1 + I_2 + \ldots + I_7.$$

Заметим, что $I_2 \geqq 0$, $I_6 \geqq 0$ и $I_3 > -\varrho\gamma\omega\theta^{-\omega-1} \ln^{-\omega-1} \theta \ln^{-1}(\ln\theta) w^{\omega-1} \equiv I_{31}$, так как $x_1 < \varrho \ln^{\gamma}(\ln\theta)$ в G. Поэтому $\mathscr{L} z > I_1 + I_{31} + I_4 + I_5 + I_7$. Имеем (напомним, что $\omega\mu = \omega + 1$):

$$I_4 + I_{31} + I_5 = \omega\theta^{-\omega\mu} \ln^{-\omega\mu} \theta \ln^{-2\gamma}(\ln\theta) w^{\omega\mu-2} [\mu\omega a_1(t) - \gamma\varrho \ln^{2\gamma-1}(\ln\theta) - - \left|\frac{\partial b_1(t, x, z)}{\partial u}\right| \theta \ln\theta \ln^{\gamma}(\ln\theta)\Big], \tag{36}$$

$$I_1 + I_7 = \theta^{-\omega-1} \ln^{-\omega} \theta w^{\omega} [\omega - \theta^{\omega+1} \ln^{\omega} \theta w^{-\omega} c(t, x, z)]. \tag{37}$$

Обозначим через $\Sigma_1 (\Sigma_2)$ величину в квадратных скобках в (36) (соответственно в (37)). Из условий теоремы следует, что

$$\sum\nolimits_1 \geqq \mu\omega\bar{a} - \ln^{2\gamma-1}(\ln\theta) - C_2 \theta^{-\omega(\lambda_1-\mu)} \ln^{-\omega(\lambda_1-\mu)} \theta \ln^{\gamma}(\ln\theta).$$

Пусть $\gamma < \dfrac{1}{2}$. Так как $\lambda_1 > \mu$, то существует такое число τ_2, что при $\tau \geqq \tau_2$ мы имеем $\Sigma_1 > 0$. Далее, так как $1 < \mu \leqq \nu$ и $w < 1$:

$$\sum\nolimits_2 \geqq \omega - C_2 \theta^{-\omega(\nu-\mu)} \ln^{-\omega(\nu-1)} \theta w^{\omega(\nu-1)} \geqq \omega - C_3 \ln^{-\omega(\nu-1)} \theta.$$

Поэтому существует число τ_3 такое, что при $\tau \geqq \tau_3$ мы имеем $\Sigma_2 > 0$.

Сначала выберем τ из неравенства $\tau \geqq \max(\tau_0, \tau_1, \tau_2, \tau_3)$, потом $\varrho > 0$ из неравенства $\varrho \leqq \varrho_1(\tau)$. При таких значениях τ, γ и ϱ: $\mathscr{L}z > 0$ в G.

Возьмем произвольное $T \in (0, \infty)$. Как показано в п. 1.1.3 (см. также п.1.4.2), функция $u(t, x)$ в $\mathbf{R}^{p+1}_{++} \cap \{t \leqq T\}$ может быть построена как $\lim_{n\to\infty} u_n(t, x)$, где u_n — решения уравнения (23) в расширяющихся областях Q_n^+. Из построения следует, что $u_n > z$ на параболической границе $G \cap \{t \leqq T\}$. Так как $\mathscr{L}z > 0$, то по теореме сравнения из [32] получаем, что $u_n(t, x) \geqq z(t, x)$ в $G \cap \{t \leqq T\}$. Переходя к пределу при $n \to \infty$, ввиду произвольности T получаем, что $u(t, x) \geqq z(t, x)$ всюду в $\mathbf{R}^{p+1}_{++}$, откуда следует утверждение теоремы.

Замечание 2.2.3. Для модельного уравнения (29) при $u_1(t) \geqq C_1(t+1)^{-1/(\mu-1)}$ $C > 0$, теорема 2.2.3 гласит: если $\mu \leqq \nu$ и $\mu < \lambda$, а число b_0 — любое, то локализация возмущений отсутствует; если $b_0 \geqq 0$, то для отсутствия локализации возмущений достаточно, чтобы было $\mu \leqq \nu$.

Литература

[1] D. G. Aronson, Regularity properties of flows through porous media, *SIAM J. Appl .Math.*, **17** (1969).

[2] Ю.Н. Благовещенский, Задача Коши для вырожденных квазилинейных параболических уравнений, *Теория вероятностей и ее применения,* **9** (1964).

[3] Г.И. Баренблатт, Об автомодельных решениях задачи Коши для нелинейного параболического уравнения нестационарной фильтрации газа в пористой среде, *Прикл. матем. и мех.,* **20** (1956).

[4] Г. И. Баренблатт, О некоторых неустановившихся движениях жидкости и газа в пористой среде, *Прикл. матем. и механ.,* **16** (1952).

[5] Г. И. Баренблатт, Об автомодельных движениях сжимаемой жидкости в пористой среде, *Прикл. матем. и механ.,* **18** (1954).

[6] Г. И. Баренблатт, М. И. Вишик, О конечной скорости распространения в задачах нестационарной фильтрации жидкости и газа, *Прикл. матем. и механ.,* **20** (1956).

[7] А. И., Вольперт, С. И. Худяев, О задаче Коши для квазилинейных вырождающихся уравнений второго порядка, *Мат. сб.,* **78 (120)** (1969).

[8] B. H. Gilding, Hölder continuity of solutions of parabolic equations (препринт).

[9] B. H. Gilding, L. A. Peletier, The Cauchy problem for an equation arising in the theory of infiltration, *Archive for Rational Mech. and Anal.* (препринт).

[10] Ю. А. Дубинский, Слабая сходимость в нелинейных эллиптических и параболических уравнениях, *Матем. сб.,* **67** (1965).

[11] Я. Б. Зельдович, А. С. Компанеец, К теории распространения тепла при теплопроводности, зависящей от температуры, «*Сб., посвященный семидесятилетию акад. А. Ф. Иоффе*», Изд. АН СССР (Москва, 1950).

[12] А. С. Калашников, О возникновении особенностей у решений уравнения нестационарной фильтрации, *Ж. вычисл. матем. и матем. физ.,* **7** (1967).

[13] А. С. Калашников, Об уравнениях типа нестационарной фильтрации с бесконечной скоростью распространения возмущений, *Вестник Моск. ун-та, Матем., механ.,* **6** (1972).

[14] А. С. Калашников, О дифференциальных свойствах обобщенных решений уравнений типа нестационарной фильтрации, *Вестник Моск. ун-та, Матем., механ.,* **1** (1974).

[15] А. С. Калашников, О характере распространения возмущений в процессах, описываемых квазилинейными вырождающимися параболическими уравнениями, *Труды семинара имени Г. И. Петровсково,* **1** (1975).

[16] Р. Кершнер, Об условиях локализации тепловых возмущений в полуограниченной движущейся среде при наличии поглощения, *Вестник Моск. ун-та, Матем., механ.,* **4** (1976).

[17] Р. Кершнер, О поведении температурных фронтов в средах с нелинейной теплопроводностью при наличии поглощения, *Вестник Моск. ун-та, Матем., механ.*, (1977) (в печати).

[18] С. Н. Кружков, Результаты о характере непрерывности решений параболических уравнений и некоторые их применения, *Мат. заметки*, **6** (1969).

[19] С. Н. Кружков, О задаче Коши для некоторых классов квазилинейных параболических уравнений, *Мат. заметки*, **6** (1969).

[20] О. А. Ладыженская, В. А. Солонников, Н. Н. Уральцева, *Линейные и квазилинейные уравнения параболического типа*, изд-во «Наука» (Москва, 1967).

[21] Л. С. Лейбензон, *Движение природных жидкостей и газов в пористой среде*, Гостехиздат (Москва, 1947).

[22] Ж.-Л. Лионс, *Некоторые методы решения нелинейных краевых задач*, «Мир» (Москва, 1972).

[23] Л. К. Мартинсон, К. Б. Павлов, К вопросу о пространственной локализации тепловых возмущений в теории нелинейной теплопроводности, *Ж. вычисл. матем. и матем. физ.*, **12** (1972).

[24] Л. К. Мартинсон, К. Б. Павлов, Магнитная гидродинамика неньютоновских жидкостей, *Магнитная гидродинамика*, **1** (1975).

[25] O. A. Oleinik, Alcuni risultati sull equazioni lineari e quasi lineari ellitico-paraboliche a derivate parziali del secondo ordine, *Rend. Accad. Naz. Lincei*, ser. VIII, XL, **5** (1966).

[26] О. А. Олейник, А. С. Калашников, Чжоу Юй-линь, Задача Коши и краевые задачи для уравнений типа нестационарной фильтрации, *Изв. АН СССР, Сер. матем.*, **22** (1958).

[27] О. А. Олейник, Т. Д. Вентцель, Первая краевая задача и задача Коши для квазилинейных уравнений параболического типа, *Матем. сб.*, **41** (1957).

[28] R. E. Prattle, Diffusion from an instantaneous point source with a concentration-dependent coefficient, *Quart. J. Mech. Appl. Math.*, **12** (1959).

[29] Е. С. Сабинина, О задаче Коши для уравнения нестационарной фильтрации газа с многими пространственными переменными, *ДАН СССР*, **136** (1961).

[30] Г. М. Фатеева, О краевых задачах для квазилинейных вырождающихся параболических уравнений, *Мат. сб.*, **76 (118)** (1968).

[31] М. И. Фрейдлин, О существовании «в целом» гладких решений вырождающихся квазилинейных уравнений, *Мат. сб.*, **78 (120)** (1969).

[32] А. Фридман, *Уравнения с частными производными параболического типа*, «Мир» (Москва, 1968).

[33] С. А. Христианович, Движение грунтовых вод, не следующее закон Дарси, *Прикл. матем. и механ.*, **4** (1940).

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ON THE CONTINUITY OF BEST APPROXIMATIONS IN THE SPACE OF INTEGRABLE FUNCTIONS

By

A. KROÓ (Budapest)

Introduction and preliminary results

In the last fifteen years the problems of unicity, strong unicity and continuity of best approximations have been widely investigated. In this paper we shall present several results of this type.

Let M_n be a finite-dimensional subspace of a real Banach space X. Then for each $x \in X$ there exists an element of best approximation to x in M_n. In what follows $p_n(x)$ denotes one of the best approximations to x from M_n and $E_n(x)$ the measure of best approximation.

It is known (see [1]) that if $p_n(x)$ is the unique best approximation to x then it is continuous at the point x, i.e. for any sequence $\{x_m\} \subset X$ such that $\|x - x_m\| \to 0$ we have

$$\sup_{p_n(x_m)} \|p_n(x) - p_n(x_m)\| \to 0.$$

If for a given $x \in X$ the best approximation is unique we define the modulus of continuity of the operator of best approximation at the point x by

$$R_1(x, \varepsilon)_X = \sup_{\substack{p_n(x_1) \\ x_1 \in X,\, \|x - x_1\| \leqq \varepsilon}} \|p_n(x) - p_n(x_1)\|, \quad \varepsilon > 0$$

and the modulus of strong unicity of the operator of best approximation at the point x by

$$R_2(x, \varepsilon)_X = \sup_{\substack{y \in M_n \\ \|x - y\| \leqq E_n(x) + \varepsilon}} \|p_n(x) - y\|, \quad \varepsilon > 0.$$

Then evidently $R_1(x, \varepsilon)_X$ and $R_2(x, \varepsilon)_X$ monotonously tend to zero as $\varepsilon \to +0$. It can be also easily proved [2] that for any $x \in X$ with unique best approximation $p_n(x)$ and $\varepsilon > 0$

(1) $$\varepsilon \leqq R_j(x, \varepsilon)_X \quad (j = 1, 2)$$

and

(2) $$R_1(x, \varepsilon)_X \leqq R_2(x, 2\varepsilon)_X.$$

The orders of $R_1(x, \varepsilon)_X$ and $R_2(x, \varepsilon)_X$ where $\varepsilon \to +0$ are always connected with the specific properties of the space considered and determining of them is, as a role, based on use of the criterion of best approximation in the given space.

As far as I know, the first result in this direction was obtained by G. FREUD [3]. He proved the following

THEOREM A (G. Freud, 1958). *If $X=C[a,b]$, $M_n\subset C[a,b]$ is a Čebysev system on $[a,b]$, then for any $f\in C[a,b]$ and $\varepsilon>0$*

$$\varepsilon \leqq R_1(f,\varepsilon)_C \leqq C_1(f,M_n)\varepsilon. \tag{3}$$

Here and in what follows $C_i(\dots)$ denote constants depending only on quantities specified in the brackets, while C_i denote absolute constants.

D. NEWMAN and H. SHAPIRO [4] proved the more general

THEOREM B (D. Newman, H. Shapiro, 1963). *If $X=C(K)$, $M_n\subset C(K)$ is a Cebysev system on K, where K is a compact set, then for any $f\in C(K)$ and $\varepsilon>0$*

$$\varepsilon \leqq R_2(f,\varepsilon)_C \leqq C_2(f,M_n)\varepsilon. \tag{4}$$

Hence under the conditions of Theorem B $R_1(f,\varepsilon)_C\sim R_2(f,\varepsilon)_C\sim\varepsilon$ for any $f\in C(K)$ and $\varepsilon>0$.

The first result for the space $L_p[-1,1]$ is due to R. HOLMES and B. KRIPKE [5].

THEOREM C. *If $X=L_p[-1,1]$, $p\geqq 2$, $M_n\subset L_p[-1,1]$ is finite-dimensional, then for any $f\in L_p[-1,1]$ and $\varepsilon>0$*

$$\varepsilon \leqq R_1(f,\varepsilon)_{L_p} \leqq C_3(f,M_n)\varepsilon.$$

(Actually they proved a somewhat stronger result.)

In general, in the space $L_p[-1,1]$, R_1 and R_2 are not equivalent. For example, for $p=2$ we evidently have $R_1(f,\varepsilon)_{L_2}\equiv\varepsilon$ for any $f\in L_2$ but $R_2(f,\varepsilon)_{L_2}\sim\varepsilon^{1/2}$ for any $f\in L_2$.

B. O. BJÖRNESTAL [6] studied the case $1\leqq p<2$. He proved the following theorems.

THEOREM D. *If $f\in L_p[-1,1]$, $1<p<2$ and $E_n(f)\leqq 1$ then for any $\varepsilon>0$*

$$R_1(f,\varepsilon)_{L_p} \leqq C_4(p)\varepsilon^{p/2}$$

and in general, this estimation cannot be improved.

THEOREM D′. *If $X=L[-1,1]$, $f\in C[-1,1]$, $p_n(f)\equiv 0$, $M_n\subset C[-1,1]$ is a Čebysev system on $[-1,1]$, then for $0\leqq\varepsilon\leqq C_5(f,M_n)$*

$$R_1(f,\varepsilon)_L \leqq R_2(f,2\varepsilon)_L \leqq S_{\omega_f}^{-1}\big(C_6(f,M_n)\varepsilon\big)$$

where

$$S_{\omega_f}(\varepsilon) = \int_0^{\omega_f^{-1}(\varepsilon)} \big(\varepsilon-\omega_f(t)\big)\,dt, \quad \omega_f(t) = \sup_{\substack{x_1,x_2\in[-1,1]\\ |x_1-x_2|\leqq t}} |f(x_1)-f(x_2)|$$

and in general, $R_1(f,\varepsilon)_L$ is of no smaller order.

Here and in what follows we use the notation

$$f^{-1}(y) = \min\{x\colon f(x)=y\}.$$

Theorem D gives uniform estimation for R_1 when $E_n(f)\leqq 1$. Hence in the case $1<p<2$ the modulus of continuity and uniform modulus of continuity of the operator of best approximation on the set $\{E_n(f)\leqq 1\}$ are equal. In [7] and

[8] the uniform modulus of continuity of the operator of best approximation in the space of continuous real-valued functions was studied. In this case the uniform moduli on certain sets of functions can be determined but their orders are less than ε, i.e. less than the order of the local modulus of continuity of the operator of best approximation.

Theorem D′ gives a local estimation for R_1 and R_2 in the space $L[-1, 1]$. It turned out that this theorem can be generalized in such a way that the constants $C_5(f, M_n)$ and $C_6(f, M_n)$ do not depend on f but only on ω_f or, in other words, the uniform modulus of continuity of the operator of best approximation on the class $H^{\perp}[\omega]=\{f\in C[-1,1]: p_n(f)\equiv 0,\ \omega_f(\delta)\leqq\omega(\delta)\}$ equals to the local modulus which is determined in Theorem D′. This is the main result of the present paper.

At first we give a modification of a known criterion for the best approximation in L, then the main theorem will be proved. We shall also determine a class of "trivial" functions for which $R_1(f,\varepsilon)_L\sim R_2(f,\varepsilon)_L\sim\varepsilon$ and at last some corollaries for algebraic and trigonometric cases will be obtained.

We shall use the notations

$$\|f\|_C=\max_{x\in[-1,1]}|f(x)|,\quad f\in C[-1,1];$$

$$\|g\|_L=\int_{-1}^{1}|g(x)|\,dx,\quad g\in L[-1,1].$$

Characterization of the best approximation in the space $L[-1, 1]$

Let M_n be any finite-dimensional subspace of $L[-1, 1]$. Then $f\in L[-1,1]$ is said to be orthogonal to M_n, written $f\perp M_n$, if and only if $\|f\|_L\leqq\|f-q_n\|_L$ for all $q_n\in M_n$, i.e. 0 is a best approximation of f from M_n. From the general theory of approximation in $L[-1, 1]$ we know two criteria of orthogonality.

Criterion I (Rice [9]). *$f\perp M_n$ if and only if for any $q_n\in M_n$*

$$\left|\int_{-1}^{1} q_n \operatorname{sign} f\,dx\right|\leqq\int_{Z(f)}|q_n|\,dx \tag{5}$$

where $Z(f)=\{x\in[-1,1]: f(x)=0\}$.

Criterion II (Shapiro [10]). *$f\perp M_n$ if and only if there is a function φ on $Z(f)$ such that $|\varphi|\leqq 1$ and for any $q_n\in M_n$*

$$\int_{-1}^{1} q_n \operatorname{sign} f\,dx+\int_{Z(f)}\varphi q_n\,dx=0. \tag{6}$$

If $\mu(Z(f))=0$ then Criteria I and II imply

$$\int_{-1}^{1} q_n \operatorname{sign} f\,dx=0$$

for any $q_n\in M_n$.

Assume now that M_n is a Čebysev system of continuous functions on $[-1, 1]$, $f \perp M_n$ and $\mu(Z(f))>0$. Then the function φ in Criterion II can be determined a.e. on $Z(f)$ and we have the following modification of Criterion II.

THEOREM 1. *Let $f \in L[-1, 1]$ and M_n be a Čebysev system of continuous functions on $[-1, 1]$. Then $f \perp M_n$ in L-norm if and only if there is a function φ on $Z(f)$ such that $|\varphi| \leqq 1$ and for any $q_n \in M_n$ (6) holds.*

Moreover, if $\mu(Z(f))>0$ then $|\varphi| \equiv Q_n(f)$ on $Z(f)$ where

$$Q_n(f) = \sup_{\substack{q_n \in M_n \\ \|q_n\|_C = 1}} \frac{\left| \int_{-1}^{1} q_n \operatorname{sign} f \, dx \right|}{\int_{Z(f)} |q_n| \, dx} \tag{7}$$

for some φ satisfying (6).

PROOF. (5) implies that $0 \leqq Q_n(f) \leqq 1$. If $Q_n(f)=0$ then (7) implies that $\int_{-1}^{1} q_n \operatorname{sign} f \, dx = 0$ for any $q_n \in M_n$ thus we may assume that $\varphi \equiv 0$ in this case. Let $Q_n(f)$ be positive. For some $\bar{q}_n \in M_n$, $\|\bar{q}_n\|_C = 1$

$$\int_{-1}^{1} \bar{q}_n \operatorname{sign} f \, dx = Q_n(f) \int_{Z(f)} |\bar{q}_n| \, dx \tag{8}$$

holds. Let us prove that if for some $q_n^* \in M_n$, $q_n^* \neq \bar{q}_n$, (8) holds then $\operatorname{sign} q_n^* = \operatorname{sign} \bar{q}_n$ a.e. on $Z(f)$. (We may assume of course that $\|q_n^*\|_C = 1$.) Indeed, in this case using (7) we obtain

$$Q_n(f) \int_{Z(f)} |\bar{q}_n| \, dx + Q_n(f) \int_{Z(f)} |q_n^*| \, dx = \int_{-1}^{1} (\bar{q}_n + q_n^*) \operatorname{sign} f \, dx \leqq$$

$$\leqq Q_n(f) \int_{Z(f)} |\bar{q}_n + q_n^*| \, dx \leqq Q_n(f) \int_{Z(f)} \{|\bar{q}_n| + |q_n^*|\} \, dx$$

hence $\operatorname{sign} \bar{q}_n = \operatorname{sign} q_n^*$ a.e. on $Z(f)$.

For any $q_n \in M_n$ define

$$q_n^* = \begin{cases} q_n, & x \in [-1, 1] \setminus Z(f); \\ Q_n(f) q_n, & x \in Z(f). \end{cases} \tag{9}$$

Thus we obtain a finite-dimensional subspace $M_n^* \subset L[-1, 1]$ and (5), (7) and (9) imply that $f \perp M_n^*$. Hence by Criterion II there is a function φ^* on $Z(f)$ such that $|\varphi^*| \leqq 1$ and for any $q_n^* \in M_n^*$

$$\int_{-1}^{1} q_n^* \operatorname{sign} f \, dx + \int_{Z(f)} q_n^* \varphi^* \, dx = 0$$

holds, i.e. for any $q_n \in M_n$

$$\int_{-1}^{1} q_n \operatorname{sign} f \, dx + Q_n(f) \int_{Z(f)} q_n \varphi^* \, dx = 0. \tag{10}$$

Then using (8) and this relation we obtain

$$-\int_{Z(f)} \varphi^* \bar{q}_n \, dx = \int_{Z(f)} |\bar{q}_n| \, dx$$

hence $\varphi^* = -\operatorname{sign} \bar{q}_n$ a.e. on $Z(f)$ and setting $\varphi = Q_n(f)\varphi^*$ in (10) we obtain the statement of the theorem.

REMARK. In what follows we shall always consider φ satisfying Theorem 1.

The main theorem

Let M_n ($n \in \mathbf{N}$) be an n-dimensional Čebysev system of continuous functions on $[-1, 1]$ containing the constant functions, $p_n(f) \in M_n$ and $E_n(f)$ be the polynomial and the measure of best approximation to $f \in C[-1, 1]$ in L-norm, respectively. Further let $I_{\omega_f}(t) = S_{\omega_f}^{-1}(t)$ where $\omega_f(t)$ and $S_{\omega_f}(t)$ are defined in Theorem D′ and $\omega(\delta)$ a uniform modulus of continuity.

Then we have

THEOREM 2. *For any* $f \in C[-1, 1]$, $\omega_f(\delta) \leqq \omega(\delta)$, $f \perp M_n$, $\varepsilon > 0$

$$R_1(f, \varepsilon)_L \leqq R_2(f, 2\varepsilon)_L \leqq C_7(\omega, M_n) I_\omega(\varepsilon) \tag{11}$$

and in general $R_1(f, \varepsilon)_L$ *is of no smaller order.*

Some lemmas

For $q_n \in M_n$ and $0 < \xi < 1$ define the set

$$K(q_n, \xi) = \{x \in [-1, 1]: |q_n(x)| \leqq \xi \|q_n\|_C\}.$$

LEMMA 1. *For any* $q_n \in M_n$, $q_n \not\equiv 0$, $0 < \xi < 1$, $[-1, 1] \setminus K(q_n, \xi)$ *consists of at most* n *disjoint intervals. Moreover, there exists a positive function* $F_n^*(\xi)$ *such that*

$$\mu(K(q_n, \xi)) \leqq F_n^*(\xi) \tag{12}$$

holds where F_n^* *depends only on* M_n *and monotonously tends to zero as* $\xi \to +0$.

PROOF. The first statement follows from the fact that M_n contains the constant functions. Let us prove (12). For any system of points $-1 \leqq x_1 < x_2 < \ldots < x_n \leqq 1$ consider the determinant

$$U_n(x_1, x_2, \ldots, x_n) = \{\varphi_j(x_i)\}_{j=1,2,\ldots,n;}^{i=1,2,\ldots,n;}$$

where $\{\varphi_j(x)\}_{j=1}^n$ is a basis in M_n, and define the function

$$F_n(d) = \inf_{\substack{-1 \leqq x_1 < x_2 < \ldots < x_n \leqq 1 \\ x_{i+1} - x_i \geqq \frac{d}{n},\, i=1,2,\ldots,n-1}} U_n(x_1, x_2, \ldots, x_n), \quad d > 0.$$

Then $F_n(d) > 0$ if $d > 0$, $F_n(d)$ monotonously tends to zero as $d \to +0$ and $F_n(d)$

depends only on M_n. Set $\mu(K(q_n, \xi))=d$, then if $d>0$ we can find a system of points $\{t_i\}_{i=1}^n \in K(q_n, \xi)$ such that

$$t_{j+1}-t_j \geqq \frac{d}{n} \quad (j=1,2,\ldots,n-1),$$

and by Lagrange interpolatory formula

$$\|q_n\|_C \leqq C_8(M_n) \max_{i=1,2,\ldots,n} |q_n(t_i)| \frac{1}{F_n(d)} \leqq C_8(M_n)\xi\|q_n\|_C \frac{1}{F_n(d)}$$

i.e. $F_n(d) \leqq C_8(M_n)\xi$, whence $d \leqq F_n^*(\xi)$, where

$$F_n^*(y) = \max\left\{x\colon \frac{F_n(x)}{C_8(M_n)} = y\right\}.$$

If $d=0$ then (12) is evident. Q.E.D.

LEMMA 2. *For any* $q_n \in M_n$

$$\|q_n\|_C \leqq C_9(M_n)\|q_n\|_L. \tag{13}$$

PROOF. The lemma follows from the equivalence of norms in R^n.

Let $f \in C[-1,1]$ be orthogonal to M_n. Then by Theorem 1 there is a function φ on $Z(f)$ such that relations (6) hold. We shall say that $f \in C[-1,1]$, $f \perp M_n$ is "trivial" if and only if $\mu(Z(f))>0$ and $Q_n(f)<1$.
By Theorem 1 if f is trivial then $|\varphi| \equiv Q_n(f)<1$.

LEMMA 3. *Let f be a trivial function. Then for any* $q_n \in M_n$

$$\|f-q_n\|_L-\|f\|_L \geqq \frac{\xi_0}{2}\left(1-Q_n(f)\right)\mu(Z(f))\|q_n\|_C \tag{14}$$

where $F_n^*(\xi_0)<\mu(Z(f))/2$.

PROOF. Using relation (6) we obtain

$$\|f-q_n\|_L-\|f\|_L = \int_{-1}^{1}(f-q_n)\operatorname{sign}(f-q_n)\,dx - \int_{-1}^{1} f \operatorname{sign} f\,dx = \tag{15}$$

$$= \int_{[-1,1]\setminus Z(f)} (f-q_n)\{\operatorname{sign}(f-q_n)-\operatorname{sign} f\}\,dx + \int_{Z(f)} |q_n|\,dx + \int_{Z(f)} q_n\varphi\,dx =$$

$$= 2\int_{\substack{0<f<q_n \\ q_n<f<0}} |f-q_n|\,dx + \int_{Z(f)} [|q_n|+\varphi q_n]\,dx \geqq \left(1-Q_n(f)\right)\int_{Z(f)} |q_n|\,dx.$$

Let $\xi_0>0$ satisfy $F_n^*(\xi_0)<\mu(Z(f))/2$. Then (15) and (12) imply

$$\|f-q_n\|_L-\|f\|_L \geqq [1-Q_n(f)]\frac{\mu(Z(f))}{2}\xi_0\|q_n\|_C.$$

Q.E.D.

COROLLARY. *Let f be a trivial function. Then for any* $\varepsilon>0$

$$R_1(f, \varepsilon)_L \leqq R_2(f, 2\varepsilon)_L \leqq C_{10}(f, M_n)\varepsilon.$$

Let $f\in C[-1, 1]$, $f\perp M_n$ and assume that φ satisfies (6). Define the following sets:

$$P_1^+(f, q_n) = \{x\in[-1, 1]\colon f \neq 0,\ \operatorname{sign} f = \operatorname{sign} q_n\};$$

$$P_1^-(f, q_n) = \{x\in[-1, 1]\colon f \neq 0,\ \operatorname{sign} f = -\operatorname{sign} q_n\};$$

$$P_2^+(f, q_n) = \{x\in[-1, 1]\colon f = 0,\ \operatorname{sign} \varphi = \operatorname{sign} q_n\};$$

$$P_2^-(f, q_n) = \{x\in[-1, 1]\colon f = 0,\ \operatorname{sign} \varphi = -\operatorname{sign} q_n\};$$

where $q_n\in M_n$.

LEMMA 4. *Let $f\in C[-1, 1]$, $f\perp M_n$ satisfy at least one of the following inequalities:*

$$\mu(Z(f)) < \frac{1}{4C_9(M_n)} \tag{16}$$

(where $C_9(M_n)$ is defined by (13)*); or*

$$Q_n(f) > \frac{1}{2}. \tag{17}$$

Then for any $q_n\in M_n$, $q_n\not\equiv 0$

$$\mu(P_1^-(f, q_n)) + \mu(P_2^-(f, q_n)) \geqq \frac{1}{4C_9(M_n)}. \tag{18}$$

PROOF. By (6) for any $q_n\in M_n$

$$\int\limits_{P_1^+(f,q_n)} |q_n| - \int\limits_{P_1^-(f,q_n)} |q_n| + \int\limits_{Z(f)} \varphi q_n = 0. \tag{19}$$

Therefore, if (16) holds, we get

$$\|q_n\|_L = \int\limits_{P_1^+(f,q_n)} |q_n| + \int\limits_{P_1^-(f,q_n)} |q_n| + \int\limits_{Z(f)} |q_n| =$$

$$= 2\int\limits_{P_1^-(f,q_n)} |q_n| + \int\limits_{Z(f)} \{|q_n| - \varphi q_n\} \leqq 2\|q_n\|_C\{\mu(P_1^-(f, q_n)) + \mu(Z(f))\} \leqq$$

$$\leqq 2C_9(M_n)\|q_n\|_L\left\{\mu(P_1^-(f, q_n)) + \frac{1}{4C_9(M_n)}\right\}$$

i.e.

$$\mu(P_1^-(f, q_n)) \geqq \frac{1}{4C_9(M_n)}.$$

Assume now that f satisfies (17). Then by (19)

$$\int\limits_{P_1^+(f,q_n)} |q_n| + Q_n(f)\int\limits_{P_2^+(f,q_n)} |q_n| = \int\limits_{P_1^-(f,q_n)} |q_n| + Q_n(f)\int\limits_{P_2^-(f,q_n)} |q_n|$$

for any $q_n \in M_n$. Thus

$$\|q_n\|_L \leqq 2\Big\{ \int\limits_{P_1^+(f,q_n)} |q_n| + \int\limits_{P_1^-(f,q_n)} |q_n| + \frac{1}{2} \int\limits_{P_2^+(f,q_n)} |q_n| + \frac{1}{2} \int\limits_{P_2^-(f,q_n)} |q_n| \Big\} \leqq$$

$$\leqq 4\Big\{ \int\limits_{P_1^-(f,q_n)} |q_n| + Q_n(f) \int\limits_{P_2^-(f,q_n)} |q_n| \Big\} \leqq$$

$$\leqq 4C_9(M_n)\|q_n\|_L \{\mu(P_1^-(f,q_n)) + \mu(P_2^-(f,q_n))\}$$

and this proves (18). Q.E.D.

LEMMA 5. *For any uniform modulus of continuity $\omega(t)$, $I_\omega(t)$ is also a uniform modulus of continuity.*

PROOF. It is evident that

$$S_\omega(t) = \int\limits_0^{\omega^{-1}(t)} \{t - \omega(x)\}\, dx$$

is a positive increasing continuous function which converges to zero as $t \to +0$. Let us prove now, that for any $t_1 > 0$, $t_2 > 0$

$$S_\omega(t_1+t_2) > S_\omega(t_1) + S_\omega(t_2). \tag{20}$$

Assume that $t_1 \geqq t_2$. Then $\omega^{-1}(t_1+t_2) \geqq \omega^{-1}(t_1) \geqq \omega^{-1}(t_2)$ and

$$S_\omega(t_1+t_2) = \int\limits_0^{\omega^{-1}(t_1+t_2)} (t_1+t_2-\omega(x))\, dx = \int\limits_0^{\omega^{-1}(t_2)} \{t_1+t_2-\omega(x)\}\, dx +$$

$$+ \int\limits_{\omega^{-1}(t_2)}^{\omega^{-1}(t_1)} (t_1+t_2-\omega(x))\, dx + \int\limits_{\omega^{-1}(t_1)}^{\omega^{-1}(t_1+t_2)} (t_1+t_2-\omega(x))\, dx =$$

$$= \int\limits_0^{\omega^{-1}(t_2)} (t_2-\omega(x))\, dx + \int\limits_0^{\omega^{-1}(t_2)} \omega(x)\, dx + \int\limits_0^{\omega^{-1}(t_2)} (t_1-\omega(x))\, dx +$$

$$+ \int\limits_{\omega^{-1}(t_2)}^{\omega^{-1}(t_1)} (t_1-\omega(x))\, dx + \int\limits_{\omega^{-1}(t_2)}^{\omega^{-1}(t_1)} t_2\, dx + \int\limits_{\omega^{-1}(t_1)}^{\omega^{-1}(t_1+t_2)} (t_1+t_2-\omega(x))\, dx > S_\omega(t_2) + S_\omega(t_1),$$

hence (20) is proved. Furthermore, because of $I_\omega(t) = S_\omega^{-1}(t)$ we have

$$I_\omega(t_1+t_2) < I_\omega(t_1) + I_\omega(t_2)$$

for any $t_1 > 0$, $t_2 > 0$. Being $I_\omega(t)$ also a positive increasing continuous function converging to zero as $t \to +0$, this means that $I_\omega(t)$ is a uniform modulus of continuity, so the lemma is proved.

The proof of the upper estimation in Theorem 2

By (2), it is enough to prove the upper estimation for $R_2(f, \varepsilon)_L$.

Let $f \in C[-1, 1]$, $\omega(f, \delta) \leqq \omega(\delta)$, $f \perp M_n$, $\varepsilon > 0$ and take $q_n \in M_n$, $q_n \not\equiv 0$ such that

(21) $$\|f - q_n\|_L \leqq \|f\|_L + \varepsilon.$$

Obviously, we have to prove the upper estimation only for $0 < \varepsilon < C_{11}(\omega, M_n)$ because if $\varepsilon > C_{11}(\omega, M_n)$ then (21) implies

$$\|q_n\|_L \leqq \|f - q_n\|_L + \|f\|_L \leqq 2\|f\|_L + \varepsilon \leqq 4\|f\|_C + \varepsilon \leqq 4\omega(2) + \varepsilon \leqq C_{12}(\omega, M_n)\varepsilon,$$

hence we obtained an inequality stronger than (11).

Assume now that (17) and (16) do not hold. Then by Lemma 3 and (21) $\varepsilon \geqq C_{13}(M_n)\|q_n\|_L$ and we again obtained a stronger inequality. So we may assume that $0 < \varepsilon < C_{11}(\omega, M_n)$ (this constant will be determined later) and f satisfies at least one of the conditions of Lemma 4. Hence (18) holds for any $q_n \in M_n$, $q_n \not\equiv 0$. Further by (15) and (21)

(22) $$\varepsilon \geqq 2 \int\limits_{\substack{0 < f < q_n \\ q_n < f < 0}} |f - q_n| \, dx + \int\limits_{Z(f)} \{|q_n| + \varphi q_n\} \, dx$$

holds. Set ξ such that

(23) $$F_n^*(\xi) < \frac{1}{48nC_9(M_n)}.$$

Then by (12) we have

(24) $$\mu(K(q_n, \xi)) < \frac{1}{48nC_9(M_n)}.$$

Further

$$K^*(q_n, \xi) = [-1, 1] \setminus K(q_n, \xi) = \bigcup_{i=1}^{k} (a_i, b_i)$$

where $1 \leqq k \leqq n$ and

(25) $$\gamma_i q_n(x) > \xi \|q_n\|_C \quad (x \in (a_i, b_i),\ i = 1, 2, \dots, k)$$

where $\gamma_i = \operatorname{sign} q_n$ on (a_i, b_i) $(i = 1, 2, \dots, k)$.

Let us prove now that there exists an interval (a_j, b_j), $1 \leqq j \leqq k$ on which one of the following cases occur:

Case A: $Z(f) \cap (a_j, b_j) \neq \emptyset$ and

(26) $$\mu(F_j^1) = \mu\{x \in (a_j, b_j) \cap Z(f) \colon \operatorname{sign} \varphi = \gamma_j\} \geqq \frac{1}{32nC_9(M_n)}.$$

Case B: $Z(f) \cap (a_j, b_j) \neq \emptyset$ and

(27) $$\mu(F_j^2) = \mu\{x \in (a_j, b_j) \setminus Z(f) \colon \operatorname{sign} f = \gamma_j\} \geqq \frac{1}{32nC_9(M_n)}.$$

Assume the contrary. Then on each of the intervals (a_i, b_i), $i = 1, 2, \dots, k$ we have the following situation: $Z(f) \cap (a_i, b_i) = \emptyset$, or $Z(f) \cap (a_i, b_i) \neq \emptyset$ but

(28) $$\mu(F_i^1) < \frac{1}{32nC_9(M_n)}$$

and

$$\mu(F_i^2) < \frac{1}{32nC_9(M_n)}. \tag{29}$$

But $k \leqq n$, therefore we can construct a polynomial $\bar{q}_n \in M_n$ such that $\operatorname{sign} \bar{q}_n = \operatorname{sign} f$ on (a_i, b_i) if $Z(f) \cap (a_i, b_i) = \emptyset$ and $\operatorname{sign} \bar{q}_n = -\gamma_i$ on (a_i, b_i) if $Z(f) \cap \cap (a_i, b_i) \neq \emptyset$ $(i = 1, 2, \dots, k)$. Then (24), (28) and (29) imply

$$\mu(P_1^-(f, \bar{q}_n)) + \mu(P_2^-(f, \bar{q}_n)) \leqq \mu(K(q_n, \xi)) +$$

$$+ n \frac{2}{32nC_9(M_n)} \leqq \frac{1}{48C_9(M_n)} + \frac{1}{16C_9(M_n)} = \frac{1}{12C_9(M_n)}$$

and this contradicts (18). Hence for some $1 \leqq j \leqq k$ one of the Cases A or B occurs on (a_j, b_j).

Assume at first that (26) holds. Then (22) and (24) imply

$$\varepsilon \geqq \int_{Z(f)} \{|q_n| + \varphi q_n\} \, dx \geqq \int_{F_j^1} \{|q_n| + \varphi q_n\} \, dx \geqq$$

$$\geqq \int_{F_j^1} |q_n| \, dx \geqq \xi \|q_n\|_C \cdot \frac{1}{96nC_9(M_n)} = C_{14}(M_n) \|q_n\|_C$$

i.e. we obtained an inequality stronger than (11).

Assume now that Case B occurs on (a_j, b_j) and $\gamma_j = 1$ (the case $\gamma_j = -1$ can be settled similarly). Then there is a point $x_1 \in (a_j, b_j)$ such that $f(x_1) = 0$.

Case B′: there is a point $x_2 \in (a_j, b_j)$ such that $f(x_2) = \xi \|q_n\|_C$. Without loss of generality we may assume that $x_1 < x_2$ and $0 < f < \xi \|q_n\|_C$ for $x \in (x_1, x_2)$ hence and by (25) $0 < f < q_n$ for $x \in (x_1, x_2)$. Therefore (22) implies

$$\varepsilon \geqq 2 \int_{\substack{0 < f < q_n \\ q_n < f < 0}} |f - q_n| \, dx \geqq \int_{x_1}^{x_2} (q_n - f) \, dx \geqq \tag{30}$$

$$\geqq \int_0^{\omega^{-1}(\xi \|q_n\|_C)} \{\xi \|q_n\|_C - \omega(x)\} \, dx = S_\omega(\xi \|q_n\|_C).$$

Set $C_{11}(\omega) = \int_0^2 \{\omega(2) - \omega(x)\} dx$, then because of $0 < \varepsilon < C_{11}(\omega)$ (30) implies

$$\xi \|q_n\|_C \leqq I_\omega(\varepsilon). \tag{31}$$

Case B″: $f(x) < \xi \|q_n\|_C$ on (a_j, b_j). Then by (25) $f(x) < q_n(x)$ for $x \in (a_j, b_j)$. Let $P_n(f) = \{x \in (a_j, b_j) : \operatorname{sign} f = 1\}$ then by (27)

$$d = \mu(P_n(f)) \geqq \frac{1}{32nC_9(M_n)} \tag{32}$$

holds. Further $P_n(f) = \bigcup_i (a_j^i, b_j^i)$ (where the number of intervals in the union is finite or countable), $d = \sum_i (b_j^i - a_j^i)$ and we can define a continuous function

$\bar{f}(x)$ on the interval (a_j, a_j+d) which consists of pieces of the function $f(x)$ when $x\in(a_j^i, b_j^i)$. Hence $\bar{f}(a_j)=0$ and $\sup\limits_{\substack{x_1, x_2\in(a_j, a_j+d)\\ |x_1-x_2|\leqq t}} |\bar{f}(x_1)-\bar{f}(x_2)|\leqq\omega(t)$, then (22) yields

$$\varepsilon \geqq \int\limits_{\substack{0<f<q_n\\ q_n<f<0}} |f-q_n|\,dx \geqq \int\limits_{P_n(f)} (q_n-f)\,dx \geqq \tag{33}$$

$$\geqq \sum_i \int\limits_{a_j^i}^{b_j^i} \{\xi\,\|q_n\|_C - f\}\,dx = \int\limits_{a_j}^{a_j+d} \{\xi\,\|q_n\|_C - \bar{f}\}\,dx.$$

Moreover, if $\omega(d)\geqq\xi\,\|q_n\|_C$ then analogously as in Case B′ we obtain (31). If $\frac{\xi}{2}\|q_n\|_C\leqq\omega(d)<\xi\,\|q_n\|_C$ we obtain an estimate twice as large as in (31) and at last if $\omega(d)<\frac{\xi}{2}\|q_n\|_C$ then (33) implies

$$\varepsilon \geqq \int\limits_{a_j}^{a_j+d} \{\xi\,\|q_n\|_C - \omega(x-a_j)\}\,dx \geqq d\,\frac{\xi}{2}\,\|q_n\|_C.$$

Using (32) we obtain again an inequality stronger than needed. This together with (31) completes the proof of the upper estimation in Theorem 2.

The counter-example of Theorem 2

Let us give now an example which shows that in general the upper estimation for $R_1(f, \varepsilon)_L$ is exact. We shall use the method of B. O. Björnestal [6].

Let $\omega(\delta)$, $0\leqq\delta\leqq 2$ be a uniform modulus of continuity and $\{\varphi_i\}_{i=1}^n$ a basis in M_n. By a theorem of Hobby and Rice [11], there exist points $-1=x_0<x_1<x_2<\dots$ $\dots<x_n<x_{n+1}=1$ such that

$$\sum_{i=0}^{n} (-1)^i \int\limits_{x_i}^{x_{i+1}} \varphi_j(x)\,dx = 0 \quad (j=1, 2, \dots, n). \tag{34}$$

Consider the function

$$f(x) = \begin{cases} (-1)^i\,\omega(x-x_i), & x\in\left[x_i, \dfrac{x_i+x_{i+1}}{2}\right]; \\ (-1)^i\,\omega(x_{i+1}-x), & x\in\left[\dfrac{x_i+x_{i+1}}{2}, x_{i+1}\right]; \end{cases} \qquad i=0, 1, \dots, n.$$

By (34), $f\perp M_n$. For $\delta>0$ small enough define the function

$$g_\delta(x) = \begin{cases} (-1)^i\delta, & x\in[x_i, x_i+\omega^{-1}(\delta)]\cup[x_{i+1}-\omega^{-1}(\delta), x_{i+1}]; \\ f(x), & x\in[x_i+\omega^{-1}(\delta), x_{i+1}-\omega^{-1}(\delta)] \end{cases} \qquad (0\leqq i\leqq n).$$

Then

$$\|f-g_\delta\|_L = 2(n+1)\int\limits_0^{\omega^{-1}(\delta)} \{\delta-\omega(x)\}\,dx = 2(n+1)\,S_\omega(\delta).$$

Further $|g_\delta| \geqq \delta$ for any $x \in [-1, 1]$; hence $\operatorname{sign}\left(g_\delta - \frac{\delta}{2}\right) = \operatorname{sign} f$ a.e. on $[-1, 1]$. Because of $f \perp M_n$, $\mu(Z(f)) = 0$ we obtain that $g_\delta - \frac{\delta}{2} \perp M_n$. Thus setting $\delta = I_\omega(\varepsilon)$ we obtain

$$\|f - g_\delta\|_L = 2(n+1)\varepsilon, \quad g_\delta - \frac{I_\omega(\varepsilon)}{2} \perp M_n$$

i.e. the lower estimation in Theorem 2.

REMARK. As we see from the proof, the order of $R_j(f, \varepsilon)_L$ $(j=1, 2)$ depends on the behaviour of the function $f - p_n(f)$ at the neighbourhood of its zeros. That is why further improvement of differential properties of the function f does not improve the order of $R_j(f, \varepsilon)_L$ $(j=1, 2)$.

For example, replacing $f(x)$ by $\bar{f}(x) = \prod_{k=1}^{n} (x - x_k)$ in the counter-example we easily obtain

$$R_1(\bar{f}, \varepsilon)_L \geqq C_{15}(M_n)\varepsilon^{1/2}$$

where $\bar{f}(x)$ is even analytic.

COROLLARY 1. *For any* $f \in \operatorname{Lip} \alpha$, $0 < \alpha \leqq 1$, $f \perp M_n$ *and* $\varepsilon > 0$

$$R_1(f, \varepsilon)_L \leqq R_2(f, 2\varepsilon)_L \leqq C_{16}(\alpha, M_n)\varepsilon^{\alpha/(\alpha+1)}$$

and there is a function $g \in \operatorname{Lip} \alpha$, $g \perp M_n$ *such that*

$$R_1(g, \varepsilon)_L \geqq C_{17}(\alpha, M_n)\varepsilon^{\alpha/(\alpha+1)}.$$

PROOF. Indeed, if $\omega(\delta) = \delta^\alpha$

$$S_\omega(t) = \int_0^{\omega^{-1}(t)} \{t - \omega(x)\}\, dx = \int_0^{t^{1/\alpha}} (t - x^\alpha)\, dx = t^{(\alpha+1)/\alpha} - \frac{1}{\alpha+1} t^{(\alpha+1)/\alpha} = \frac{\alpha}{\alpha+1} t^{(\alpha+1)/\alpha}$$

hence

$$I_\omega(t) = \left(\frac{\alpha+1}{\alpha}\right)^{\alpha/(\alpha+1)} t^{\alpha/(\alpha+1)}.$$

Q.E.D.

This corollary gives an exact uniform estimation for $R_1(f, \varepsilon)_L$ and $R_2(f, \varepsilon)_L$ for functions of $\operatorname{Lip} \alpha$ orthogonal to M_n. With a slight modification of Theorem 1 we can omit the condition of orthogonality of the function f to M_n.

The unit sphere of a finite dimensional space is compact, therefore for any $q_n \in M_n$, $\|q_n\|_C = 1$, $\omega_{q_n}(\delta) \leqq \bar{\omega}(M_n, \delta)$ holds where $\bar{\omega}(M_n, \delta)$ is a fixed modulus of continuity depending only on M_n.

Further setting $f^*(x) = f(x) - f(0)$ for $f \in C[-1, 1]$, $\omega_f(\delta) \leqq \omega(\delta)$ we have

$$\omega(p_n(f), \delta) \equiv \omega(p_n(f^*), \delta) \leqq \|p_n(f^*)\|_C \bar{\omega}(M_n, \delta) \leqq C_9(M_n)\|p_n(f^*)\|_L \bar{\omega}(M_n, \delta) \leqq$$

$$\leqq 2C_9(M_n)\|f^*\|_L \bar{\omega}(M_n, \delta) \leqq 4C_9(M_n)\omega(2)\bar{\omega}(M_n, \delta) = C_{18}(\omega, M_n)\bar{\omega}(M_n, \delta)$$

hence putting

$$\omega^*(\delta) = \max\{\omega(\delta),\ \bar{\omega}(M_n, \delta)\}$$

we obtain

$$\omega(f-p_n(f),\delta) \leqq C_{18}(\omega, M_n)\omega^*(\delta).$$

Using this inequality we obtain

COROLLARY 2. *For any* $f\in C[-1,1]$, $\omega_f(\delta)\leqq\omega(\delta)$ *and* $\varepsilon>0$

$$R_1(f,\varepsilon)_L \leqq R_2(f,2\varepsilon)_L \leqq C_{19}(\omega, M_n)I_{\omega^*}(\varepsilon).$$

The criterion of the local Lip 1 property

By Theorem 2, in general the best possible order of $R_j(f,\varepsilon)_L$ $(j=1,2)$ is $\varepsilon^{1/2}$. But at the same time Lemma 3 shows that if $f-p_n(f)$ is a "trivial" function then

$$R_1(f,\varepsilon)_L \leqq R_2(f,2\varepsilon)_L \leqq C_{10}(f, M_n)\varepsilon.$$

It turns out that the condition of triviality of the function $f-p_n(f)$ is not only sufficient but it is necessary also.

First of all we shall prove that for any $f\in C[-1,1]$, $R_1(f,\varepsilon)_L\sim R_2(f,\varepsilon)_L$ holds.

THEOREM 3. *Let* $f\in C[-1,1]$, $\varepsilon>0$ *and let* $q_n\in M_n$ *satisfy* $\|f-q_n\|_L\leqq \leqq E_n(f)+\varepsilon$. *Then there is a function* $f_1\in L[-1,1]$ *such that* $\|f-f_1\|_L\leqq C_{20}(f)\varepsilon$ *and* $f_1-q_n\perp M_n$.

PROOF. Set $\bar f=f-p_n(f)$, $\bar q_n=q_n-p_n(f)$, then $p_n(\bar f)\equiv 0$ and $\|\bar f-\bar q_n\|_L\leqq \leqq\|\bar f\|_L+\varepsilon$. Thus (15) implies

$$\varepsilon \geqq \|\bar f-\bar q_n\|_L-\|\bar f\|_L = 2\int\limits_{\substack{0<\bar f<\bar q_n\\ \bar q_n<\bar f<0}} |\bar f-\bar q_n|\,dx+\int\limits_{Z(\bar f)}\{|\bar q_n|+\varphi\bar q_n\}\,dx \tag{35}$$

where by Theorem 1, $|\varphi|\equiv Q_n(\bar f)$, $0\leqq Q_n(\bar f)\leqq 1$.

Consider now two cases.

Case A: $Q_n(\bar f)=1$. Then set

$$\bar f_1=\begin{cases}\bar q_n, & \text{if } 0<\bar f\leqq\bar q_n \text{ or } \bar q_n\leqq\bar f<0;\\ |\bar q_n|\varphi+\bar q_n, & \text{if } \bar f=0;\\ \bar f & \text{otherwise.}\end{cases}$$

Then a.e. on $[-1,1]$

$$\operatorname{sign}(\bar f_1-\bar q_n)=\begin{cases}0, & \text{if } 0<\bar f\leqq\bar q_n \text{ or } \bar q_n\leqq\bar f<0;\\ \varphi, & \text{if } \bar f=0;\\ \operatorname{sign}\bar f & \text{otherwise.}\end{cases}$$

Hence setting $\varphi^*=\operatorname{sign}\bar f$ on $\{0<\bar f\leqq\bar q_n$ or $\bar q_n\leqq\bar f<0\}$ we have

$$\int\limits_{-1}^{1} q_n^*\operatorname{sign}(\bar f_1-\bar q_n)\,dx+\int\limits_{Z(\bar f_1-\bar q_n)}\varphi^* q_n^*\,dx=\int\limits_{-1}^{1} q_n^*\operatorname{sign}\bar f\,dx+\int\limits_{Z(\bar f)}\varphi q_n^*\,dx=0$$

for any $q_n^* \in M_n$. This means that $\bar{f}_1 - \bar{q}_n \perp M_n$. On the other hand, from (35) we have

$$\|\bar{f}-\bar{f}_1\|_L = \int\limits_{\substack{0<\bar{f}<\bar{q}_n \\ \bar{q}_n<\bar{f}<0}} |\bar{f}-\bar{q}_n|\,dx + \int\limits_{Z(\bar{f})} \big||\bar{q}_n|\varphi+\bar{q}_n\big|\,dx \leqq \varepsilon,$$

and setting $f_1 = \bar{f}_1 + p_n(f)$ we obtain that $f_1 - q_n \perp M_n$ and $\|f-f_1\|_L = \|\bar{f}-\bar{f}_1\|_L \leqq \varepsilon$.

Case B: $0 \leqq Q_n(\bar{f}) < 1$. Then (35) implies

$$\varepsilon \geqq \{1-Q_n(\bar{f})\} \int\limits_{Z(\bar{f})} |\bar{q}_n|\,dx + 2 \int\limits_{\substack{0<\bar{f}<\bar{q}_n \\ \bar{q}_n<\bar{f}<0}} |\bar{f}-\bar{q}_n|\,dx. \tag{36}$$

Let

$$\bar{f}_1 = \begin{cases} \bar{q}_n, & \text{if } \ 0 \leqq \bar{f} \leqq \bar{q}_n \ \text{ or } \ \bar{q}_n \leqq \bar{f} \leqq 0; \\ \bar{f} & \text{otherwise.} \end{cases}$$

Then

$$\operatorname{sign}(\bar{f}_1 - \bar{q}_n) = \begin{cases} 0, & \text{if } \ 0 \leqq \bar{f} \leqq \bar{q}_n \ \text{ or } \ \bar{q}_n \leqq \bar{f} \leqq 0; \\ \operatorname{sign} \bar{f} & \text{otherwise.} \end{cases}$$

Hence evidently $\bar{f}_1 - \bar{q}_n \perp M_n$. Further (36) implies

$$\|\bar{f}-\bar{f}_1\|_L = \int\limits_{\substack{0\leqq\bar{f}<\bar{q}_n \\ \bar{q}_n<\bar{f}\leqq 0}} |\bar{q}_n-\bar{f}|\,dx = \int\limits_{Z(\bar{f})} |\bar{q}_n|\,dx + \int\limits_{\substack{0<\bar{f}<\bar{q}_n \\ \bar{q}_n<\bar{f}<0}} |\bar{q}_n-\bar{f}|\,dx \leqq$$

$$\leqq \frac{\varepsilon}{1-Q_n(\bar{f})} + \frac{\varepsilon}{2} = C_{20}(f)\,\varepsilon$$

and setting again $f_1 = \bar{f}_1 + p_n(f)$ we obtain $f_1 - q_n \perp M_n$, $\|f-f_1\|_L \leqq C_{20}(f)\varepsilon$. Q.E.D.

Corollary. *For any* $f \in C[-1, 1]$ *and* $\varepsilon > 0$

$$R_1(f, \varepsilon)_L \leqq R_2(f, 2\varepsilon)_L \leqq R_1(f, 2C_{20}(f)\varepsilon)_L.$$

Proof. The proof follows immediately from (1) and Theorem 3.

Now we shall obtain the necessary and sufficient condition for the local Lip 1 property.

Theorem 4. *Let* $f \in C[-1, 1]$. *Then*

$$R_j(f, \varepsilon)_L \leqq C_{21}(f, M_n)\varepsilon \quad (j = 1, 2) \tag{37}$$

for any $\varepsilon > 0$ *if and only if* $f - p_n(f)$ *is trivial.*

Proof. If $f - p_n(f)$ is trivial then (37) follows from the Corollary of Lemma 3. Assume now that for a given $f \in C[-1, 1]$ and arbitrary $\varepsilon > 0$, (37) holds with

$j=1$ or 2. The Corollary of Theorem 3 implies that without loss of generality we may assume

(38) $$R_2(\bar{f}, \varepsilon)_L \equiv R_2(f, \varepsilon)_L \leqq C_{21}(f, M_n)\varepsilon$$

for any $\varepsilon>0$, where $\bar{f}=f-p_n(f)$.

Further for any $q_n \in M_n$, $q_n \not\equiv 0$

(39) $$0<\|\bar{f}-q_n\|_L-\|\bar{f}\|_L \leqq 2 \int_{\substack{0<\bar{f}<q_n\\ q_n<\bar{f}<0}} |\bar{f}-q_n|\,dx+\int_{Z(\bar{f})}|q_n|\,dx-\int_{-1}^{1} q_n \operatorname{sign}\bar{f}\,dx.$$

Assume now that $\bar{f}$ is "nontrivial". Then $\mu(Z(\bar{f}))=0$ or $Q_n(\bar{f})=1$. If $\mu(Z(\bar{f}))=0$ then (39) implies that

$$0<\|\bar{f}-q_n\|_L-\|\bar{f}\|_L=2\int_{\substack{0<\bar{f}<q_n\\ q_n<\bar{f}<0}}|\bar{f}-q_n|\,dx$$

for any $q_n \in M_n$, $q_n \not\equiv 0$.

On the other hand, if $\mu(Z(\bar{f}))>0$ then $Q_n(\bar{f})=1$. Thus for some $\bar{q}_n \in M_n$, $\|\bar{q}_n\|_C=1$

$$\int_{-1}^{1} \bar{q}_n \operatorname{sign} f\,dx=\int_{Z(\bar{f})}|\bar{q}_n|\,dx.$$

Hence and from (39)

(40) $$0<\|\bar{f}-\bar{q}_n\|_L-\|\bar{f}\|_L=2\int_{\substack{0<\bar{f}<\bar{q}_n\\ \bar{q}_n<\bar{f}<0}}|\bar{f}-\bar{q}_n|\,dx.$$

So we may conclude that for some $\bar{q}_n \in M_n$, $\|\bar{q}_n\|_C=1$, (40) holds. Then evidently, for any $\alpha>0$ and $K>0$

(41) $$A_{K,\alpha}=\|K\bar{f}-\alpha\bar{q}_n\|_L-\|K\bar{f}\|_L=2\int_{\substack{0<K\bar{f}<\alpha\bar{q}_n\\ \alpha\bar{q}_n<K\bar{f}<0}}|K\bar{f}-\alpha\bar{q}_n|\,dx.$$

For a given $K>0$, $A_{K,\alpha}$ is a positive monotonously increasing function of $\alpha>0$ and $A_{K,\alpha}\to+0$ as $\alpha\to+0$; $A_{K,\alpha}\to+\infty$ as $\alpha\to+\infty$. Further for $0<\alpha_1<\alpha_2$ we have

$$0<A_{K,\alpha_2}-A_{K,\alpha_1}=\int_{\alpha_2\bar{q}_n<K\bar{f}<0}(K\bar{f}-\alpha_2\bar{q}_n)\,dx+\int_{0<K\bar{f}<\alpha_2\bar{q}_n}(\alpha_2\bar{q}_n-K\bar{f})\,dx-$$

$$-\int_{\alpha_1\bar{q}_n<K\bar{f}<0}(K\bar{f}-\alpha_1\bar{q}_n)\,dx-\int_{0<K\bar{f}<\alpha_1\bar{q}_n}(\alpha_1\bar{q}_n-K\bar{f})\,dx=$$

$$=\int_{\alpha_1\bar{q}_n<K\bar{f}<0}(\alpha_1-\alpha_2)\bar{q}_n\,dx+\int_{\alpha_2\bar{q}_n<K\bar{f}\leqq\alpha_1\bar{q}_n<0}(K\bar{f}-\alpha_2\bar{q}_n)\,dx+\int_{0<K\bar{f}<\alpha_1\bar{q}_n}(\alpha_2-\alpha_1)\bar{q}_n\,dx+$$

$$+\int_{0<\alpha_1\bar{q}_n\leqq K\bar{f}<\alpha_2\bar{q}_n}(\alpha_2\bar{q}_n-K\bar{f})\,dx\leqq 4(\alpha_2-\alpha_1).$$

Hence $A_{K,\alpha}$ is a continuous function of α. Let $K_m>0$ $(m=1,2,\ldots)$, $K_m\to+\infty$ as $m\to\infty$ and take any $\varepsilon_0>0$. Then using the properties of the function $A_{K,\alpha}$ we can choose a positive sequence α_m $(m=1,2,\ldots)$ such that

(42) $$A_{K_m,\alpha_m}=\varepsilon_0 \quad (m=1,2,\ldots).$$

(38) implies that

(43) $$R_2(K_m\bar{f},\varepsilon)_L \leqq C_{21}(f,M_n)\varepsilon \quad (m=1,2,\ldots)$$

for any $\varepsilon>0$. Hence by (41), (42) and (43)

$$\alpha_m\|\bar{q}_n\|_L \leqq C_{21}(f,M_n)\varepsilon_0 \quad (m=1,2,\ldots)$$

thus

(44) $$0<\alpha_m<C_{22}(f,M_n) \quad (m=1,2,\ldots).$$

On the other hand, combining (41), (42) and (44) we obtain

$$0<\varepsilon_0=A_{K_m,\alpha_m}=2\int\limits_{\substack{0<K_m\bar{f}<\alpha_m\bar{q}_n\\ \alpha_m\bar{q}_n<K_m\bar{f}<0}}|K_m\bar{f}-\alpha_m\bar{q}_n|\,dx\leqq$$

$$\leqq 2\alpha_m\|\bar{q}_n\|_L\mu\{x\in[-1,1]\colon 0<K_m\bar{f}<\alpha_m\bar{q}_n \quad\text{or}\quad \alpha_m\bar{q}_n<K_m\bar{f}<0\}\leqq$$

$$\leqq C_{23}(f,M_n)\mu\left\{x\in[-1,1]\colon 0<|\bar{f}|<\frac{C_{24}(f,M_n)}{K_m}\right\}.$$

But

$$\mu\left\{x\in[-1,1]\colon 0<|\bar{f}|<\frac{C_{24}(f,M_n)}{K_m}\right\}\to 0 \quad\text{as}\quad m\to\infty$$

so we obtained a contradiction. Therefore $\bar{f}$ must be "trivial". Q.E.D.

This theorem shows that there is a class of "trivial" functions for which $R_1\sim R_2\sim\varepsilon$ in spite of the fact that, in general, $R_1(\varepsilon)$ and $R_2(\varepsilon)$ do not exceed $\varepsilon^{1/2}$. Now we shall be interested in those "nontrivial" functions for which the order of $R_2(\varepsilon)$ (i.e. $R_1(\varepsilon)$ also) is less than $\varepsilon^{1/2}$. As it was already mentioned, the order of $R_j(f,\varepsilon)_L$ $(j=1,2)$ depends on the behaviour of the function $f-p_n(f)$ near its zeros, and it is natural to expect that if $|f-p_n(f)|$ is of order $t^{1+\gamma}$, $\gamma>0$ near its zeros then we obtain better orders for $R_j(f,\varepsilon)_L$ $(j=1,2)$.

THEOREM 5. *Let $g(t)$, $t>0$ be a positive function monotonously converging to zero as $t\to+0$, and assume that $g^2(t)$ is concave. Then there exists a function $f\in C[-1,1]$ such that for any $0<\varepsilon<C_{25}$*

$$R_j(f,\varepsilon)_L\sim\frac{\varepsilon}{g(\varepsilon)} \quad (j=1,2).$$

PROOF. First of all remark that concavity of $g^2(t)$ implies that $g(t)\geqq C_{26}\sqrt{t}$ hence $\dfrac{\varepsilon}{g(\varepsilon)}\leqq\varepsilon^{1/2}$.

Set $\varphi(t)=\frac{g^{-1}(t)}{t}$ and define

$$f(x)=\begin{cases}(-1)^i\varphi(x-x_i), & x\in\left[x_i, \frac{x_i+x_{i+1}}{2}\right];\\ (-1)^i\varphi(x_{i+1}-x), & x\in\left[\frac{x_i+x_{i+1}}{2}, x_{i+1}\right];\end{cases}\qquad (i=0, 1, \dots, n)$$

where $\{x_i\}_{i=0}^{n+1}$ are as in the counter-example.

Then with the methods of the proof of Theorem 2 it can be easily verified that

$$R_j(f,\varepsilon)_L \sim \Big[\int_0^{\varphi^{-1}(\varepsilon)} \{\varepsilon-\varphi(x)\}\,dx\Big]^{-1} \tag{45}$$

where $[\dots]^{-1}$ denotes the inverse function. Further the concavity of g^2 implies that φ is a convex function, therefore

$$\frac{\varepsilon\varphi^{-1}(\varepsilon)}{2} \leqq \int_0^{\varphi^{-1}(\varepsilon)}\left(\varepsilon-\frac{\varepsilon}{\varphi^{-1}(\varepsilon)}x\right)dx \leqq \int_0^{\varphi^{-1}(\varepsilon)}\{\varepsilon-\varphi(x)\}\,dx \leqq \varepsilon\varphi^{-1}(\varepsilon). \tag{46}$$

On the other hand, the relation $\varphi(t)=\frac{g^{-1}(t)}{t}$ can be transformed in the following way:

$$\varphi(t)=\frac{g^{-1}(t)}{t} \Leftrightarrow \varphi\big(g(t)\big)=\frac{t}{g(t)} \Leftrightarrow g(t)=\varphi^{-1}\left(\frac{t}{g(t)}\right) \Leftrightarrow$$

$$\Leftrightarrow \frac{t}{g(t)}\varphi^{-1}\left(\frac{t}{g(t)}\right)=t \Leftrightarrow \big(t\varphi^{-1}(t)\big)^{-1}=\frac{t}{g(t)}.$$

Combining this inequality with (45) and (46) we obtain the statement of the theorem.

Algebraic case. Let $M_n=P_n$ be the set of algebraic polynomials of order at most $n-1$. Evidently $\bar\omega(P_n,\delta)\leqq C_{26}(n)\delta$ hence by Corollary 2 we have

THEOREM 2a. *If $M_n=P_n$ then for any $f\in C[-1,1]$, $\omega_f(\delta)\leqq\omega(\delta)$ and $\varepsilon>0$*

$$R_1(f,\varepsilon)_L \leqq R_2(f,2\varepsilon)_L \leqq C_{27}(n,\omega)I_\omega(\varepsilon)$$

and this estimate cannot be improved.

COROLLARY 1a. *Let $M_n=P_n$, $0<\alpha\leqq 1$. Then for any $0<\varepsilon<C_{28}(n,\alpha)$*

$$\sup_{f\in \mathrm{Lip}\,\alpha} R_j(f,\varepsilon)_L \sim \varepsilon^{\alpha/(\alpha+1)} \quad (j=1,2)$$

where the constants involved depend only on n and α.

Trigonometric case. For approximation of periodic functions by trigonometric polynomials we have a theorem analogous to Theorem 2a. But for periodic functions the boundedness of $f^{(r)}, r\geqq 1$ implies that $\omega_f(\delta)\leqq C_{29}(r)\delta$, and therefore we can modify Corollary 1a.

Set $W_r \operatorname{Lip} \alpha = \{f | f \text{ periodic}, \omega_{f^{(r)}}(\delta) \leqq \delta^\alpha\}$. Then we have the following

COROLLARY 1b. *In the periodic case*

$$\sup_{f \in W_r \operatorname{Lip} \alpha} R_j(f, \varepsilon)_L \sim \begin{cases} \varepsilon^{\alpha/(\alpha+1)} & \textit{if} \quad r = 0 \\ \varepsilon^{1/2} & \textit{if} \quad r \geqq 1 \end{cases} \quad (j = 1, 2).$$

Let us show that there is no analogous theorem in the algebraic case. Indeed,

$$P_{n+1} \subset W^{n+1} = \{f\colon \|f^{(n+1)}\|_C \leqq 1\},$$

but P_{n+1} is a cone so by a theorem from [2]

$$\sup_{q_{n+1} \in P_{n+1}} R_1(q_{n+1}, \varepsilon)_L \equiv \varepsilon \quad \text{or} \quad \infty.$$

On the other hand, by the Remark made after the proof of Theorem 2

$$R_1(\bar{f}, \varepsilon)_L \geqq C_{15}(M_n)\varepsilon^{1/2}$$

where $\bar{f}(x) = \prod_{k=1}^{n} (x - x_k)$ i.e. $\bar{f} \in P_{n+1}$. Hence

$$\sup_{f \in W^{n+1}} R_1(f, \varepsilon)_L \geqq \sup_{q_{n+1} \in P_{n+1}} R_1(q_{n+1}, \varepsilon)_L \equiv \infty.$$

References

[1] E. W. CHENEY, *Introduction to approximation theory,* McGraw-Hill (New York, 1966).
[2] A. KROÓ, The continuity of best aproximations, *Acta Math. Acad. Sci. Hungar.,* **30** (1977), 175—188.
[3] G. FREUD, Eine Ungleichung für Tshebyscheffsche Approximationspolynome, *Acta Sci. Math. Szeged,* **19** (1958), 162—164.
[4] D. J. NEWMAN, H. S. SHAPIRO, Some theorems on Čebysev approximation, *Duke Math. J.,* **30** (1963), 673—681.
[5] R. HOLMES, B. KRIPKE, Smoothness of approximation, *Mich. Math. J.,* **15** (1968), 225—248.
[6] B. O. BJÖRNESTAL, Continuity of the Metric Projection Operator I—III, *The preprint series of Department of Mathematics,* Royal Institute of Technology, Stockholm, TRITA—MAT, **17** (1974), **20** (1974), **12** (1975).
[7] П. В. Галкин, О модуле непрерывности оператора наилучшего приближения в пространстве непрерывных функций, *Матем. заметки,* **10** (1971), 601—613.
[8] А. В. Кроо, Задача корректности наилучших приближений тригонометрическими полиномами класса $W_0^r H[\omega]_C$, *Матем. заметки,* **22** (1977), 85—103.
[9] J. R. RICE, *The Approximation of Functions I,* Addison—Wesley (1964).
[10] H. S. SHAPIRO, *Topics in Approximation Theory.* Lecture Notes in Math. 187. Springer-Verlag (Berlin—Heidelberg—New York, 1971.)
[11] C. R. HOBBY, J. R. RICE, A moment problem in approximation, *Proc. Amer. Math. Soc.,* **65** (1965), 665—670.

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST, REÁLTANODA U. 13—15.

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HERMITE—FEJÉR TYPE INTERPOLATIONS. I

By
P. VÉRTESI (Budapest)

1. Introduction

1.1. We prove theorems of convergence and divergence type for the Hermite—Fejér and quasi-Hermite—Fejér interpolations based on the Jacobi-roots. Equiconvergence of certain Hermite—Fejér and quasi-Hermite—Fejér processes will be established, too. These statements partially solve a problem raised by P. TURÁN ([11], Problem XXVI).

1.2. Let us denote by

$$(1.1)\qquad (-1\leqq)\, x_{n,n}<x_{n-1,n}<\ldots<x_{1,n}(\leqq 1)\quad (n=1,2,\ldots)$$

n distinct points in $[-1, 1]$ and let $f(x)$ be a continuous function in the same interval. It will be denoted by $f\in C$.

Then, setting

$$(1.2)\qquad \omega_n(x)=c_n(x-x_{1,n})(x-x_{2,n})\ldots(x-x_{n,n})\quad (c_n\neq 0)$$

and

$$(1.3)\qquad l_{k,n}(x)=\frac{\omega_n(x)}{\omega_n(x_{k,n})(x-x_{k,n})}\quad (k=1,2,\ldots,n),$$

the uniquely determined Hermite—Fejér interpolatory polynomials of degree $\leqq 2n-1$ are

$$(1.4)\qquad H_n(f;x)=\sum_{k=1}^{n} f(x_{k,n})h_{k,n}(x)\quad (n=1,2,\ldots)$$

or

$$(1.5)\qquad \bar{H}_n(f;x)=H_n(f;x)+\sum_{k=1}^{n}\beta_{k,n}\mathfrak{H}_{k,n}(x)\quad (n=1,2,\ldots).$$

Here

$$(1.6)\qquad h_{k,n}(x)=v_{k,n}(x)\,l_{k,n}^2(x)\quad (k=1,2,\ldots,n),$$

$$(1.7)\qquad v_{k,n}(x)=1-\frac{\omega_n''(x_{k,n})}{\omega_n'(x_{k,n})}(x-x_{k,n})\quad (k=1,2,\ldots,n),$$

$$(1.8)\qquad \mathfrak{H}_{k,n}(x)=(x-x_{k,n})\,l_{k,n}^2(x)\quad (k=1,2,\ldots,n),$$

the values $\{\beta_{k,n}\}$ are arbitrarily prescribed.

It is well known that $\sum_{k=1}^{n} h_{k,n}(x)=1$, moreover,

$$\text{(1.9)} \qquad \begin{cases} H_n(f;\, x_{k,n}) = \bar{H}_n(f;\, x_{k,n}) = f(x_{k,n}) & (k=1,2,\ldots,n), \\ H_n'(f;\, x_{k,n}) = 0, \quad \bar{H}_n'(f;\, x_{k,n}) = \beta_{k,n} & (k=1,2,\ldots,n) \end{cases}$$

(see, e.g. G. SZEGŐ, [1], 14.1).

1.3. Let us consider for arbitrary fixed complex α and β the n-th Jacobi-polynomial $P_n^{(\alpha,\beta)}(x)$ defined by

$$\text{(1.10)} \qquad P_n^{(\alpha,\beta)}(x) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(n+\alpha+\beta+1)\ldots(n+\alpha+\beta+k)\cdot$$
$$\cdot(\alpha+k+1)\ldots(\alpha+n)\left(\frac{x-1}{2}\right)^k \quad (\alpha,\, \beta\in\mathbf{C})$$

(see [1], (4.21.2) and (4.22 (1); $\gamma\in\mathbf{C}$ means that γ is complex, if γ is real we write $\gamma\in\mathbf{R}$).

If $\alpha, \beta\geqq -1$ then the distinct roots $x_{k,n}^{(\alpha,\beta)}$ $(k=1, 2, \ldots, n)$ of $P_n^{(\alpha,\beta)}(x)$ are in $[-1, 1]$ (see 2.2), so they can serve as fundamental points of the processes H_n or $\bar{H}_n$. For $-1\leqq\alpha, \beta<0$ one can prove that the polynomials $H_n(f; x)$ converge uniformly to $f(x)\in C$ over $[-1, 1]$ (see, e.g., G. GRÜNWALD [2]). Further results can be found in [1], 14.6.

As for the rate of convergence we have

$$\text{(1.11)} \qquad |H_n^{(\alpha,\beta)}(f;\, x)-f(x)| =$$
$$= O(1)\sum_{i=1}^{n}\left[\omega\left(f;\, \frac{i\sin\vartheta}{n}\right)+\omega\left(f;\, \frac{i^2|\cos\vartheta|}{n^2}\right)\right] i^{2\gamma-1} \quad (x\in[-1+\varepsilon, 1],\ \alpha, \beta>-1)$$

where $\gamma=\max(\alpha, -0.5)$, $x=\cos\vartheta$ $(0\leqq\vartheta\leqq\pi)$, $\varepsilon>0$, $\omega(f; t)$ is the modulus of continuity of $f(x)$ and $H_n^{(\alpha,\beta)}$ stands for the process H_n based on the roots of $P_n^{(\alpha,\beta)}(x)$. Similar estimation is valid in $[-1, 1-\varepsilon]$ if γ stands for $\max(\beta, -0.5)$.

In a closed $[a, b]\subset(-1, 1)$ we have

$$\text{(1.12)} \qquad |H_n^{(\alpha,\beta)}(f;\, x)-f(x)| = O(1)\sum_{i=1}^{n}\omega\left(f;\, \frac{i}{n}\right)\frac{1}{i^2} \quad (x\in[a, b],\ \alpha, \beta>-1)$$

(see P. VÉRTESI, [8] and [9]).

1.4. The notion of the quasi-Hermite—Fejér interpolatory polynomials was introduced by P. SZÁSZ [3]. Supposing that $x_{1,n}\equiv 1$, $x_{n,n}\equiv -1$ $(n=1, 2, \ldots)$ he considered the polynomials $Q_n(f; x)$ and $\bar{Q}_n(f; x)$ of degree $\leqq 2n-3$ for which $Q_n(1; x)\equiv 1$ $(n\geqq 2)$ and

$$\text{(1.13)} \qquad \begin{cases} Q_n(f;\, x_{k,n}) = \bar{Q}_n(f;\, x_{k,n}) = f(x_{k,n}) & (k=1,2,\ldots,n), \\ Q_n'(f;\, x_{k,n}) = 0, \quad \bar{Q}_n(f;\, x_{k,n}) = \beta_{k,n} & (k=1,2,\ldots,n-1). \end{cases}$$

It can be established that the uniquely determined Q_n and $\bar{Q}_n$ have the form

$$(1.14)\qquad Q_n(f;\, x) = \sum_{k=1}^{n} f(x_{k,n}) r_{k,n}(x),$$

$$(1.15)\qquad \bar{Q}_n(f;\, x) = Q_n(f;\, x) + \sum_{k=2}^{n-1} \beta_{k,n} s_{k,n}(x),$$

where with $\omega_n(x) = c_n \prod_{k=2}^{n-1} (x - x_{k,n})$

$$(1.16)\qquad \begin{cases} r_{1,n}(x) = \dfrac{1+x}{2\omega_n^2(1)}\,\omega_n^2(x), \quad r_{n,n}(x) = \dfrac{1-x}{2\omega_n^2(-1)}\,\omega_n^2(x), \\[2mm] r_{k,n}(x) = \dfrac{1-x^2}{1-x_{k,n}^2}\, q_{k,n}(x)\, l_{k,n}^2(x) \quad (k = 2, 3, \dots, n-1), \\[2mm] q_{k,n}(x) = 1 + \left[\dfrac{2x_{k,n}}{1-x_{k,n}^2} - \dfrac{\omega_n''(x_{k,n})}{\omega_n'(x_{k,n})}\right](x - x_{k,n}) \quad (k = 2, 3, \dots, n-1), \end{cases}$$

$$(1.17)\qquad s_{k,n}(x) = \frac{1-x^2}{1-x_{k,n}^2}\,(x - x_{k,n})\, l_{k,n}^2(x) \quad (k = 2, 3, \dots, n-1).$$

1.5. If the fundamental points $x_{2n}, x_{3n}, \dots, x_{n-1,n}$ are the roots of $P_{n-2}^{(\alpha,\beta)}(x)$ $(\alpha, \beta > -1)$ we write $Q_n^{(\alpha,\beta)}(f; x)$.

The following result was proved by P. SZÁSZ [3], [13] and J. SÁNTHA [14]. If $0 \leqq \alpha, \beta < 1$ then the polynomials $Q_n^{(\alpha,\beta)}(f; x)$ converge uniformly to $f(x) \in C$ over $[-1, 1]$ (for $\alpha = \beta = 0$ see E. EGERVÁRY, P. TURÁN [4] and T. M. MILLS, A. K. VARMA [15]).

D. L. BERMAN [12] proved that for any fixed $x \in [-1, 1]$ $Q_n^{(\alpha,\alpha)}(f; x) \to f(x)$ $(n \to \infty)$ if $f \in C$ and $\alpha \geqq -0.5$; the convergence is uniform for each $[a, b] \subset (-1, 1)$.

A uniform convergence theorem for $\alpha = \beta = -0.5$ was proved by L. P. POVCHUN and A. A. PRIVALOV in [5].

The rapidity of convergence was established for $\alpha = \beta = 0.5$ by R. B. SAXENA and K. K. MATHUR [6]. They proved that

$$(1.18)\qquad |Q_n^{(0.5,0.5)}(f;\, x)| = O(1) \sum_{i=1}^{n} \left[\omega\left(f;\, \frac{i \sin \vartheta}{n}\right) + \omega\left(f;\, \frac{i^2 |\cos \vartheta|}{n^2}\right)\right] \frac{1}{i^2}, \quad x \in [-1, 1].$$

As we mentioned, the aim of our paper is to prove convergence theorems both for $H_n^{(\alpha,\beta)}$ and $Q_n^{(\alpha,\beta)}$, investigating those values of α and β which were not considered above (Parts 2 and 3). We investigate the rapidity of the convergence, too. Further we prove a certain equiconvergence theorem for $H_n^{(\alpha,\beta)}$ and $Q_n^{(\alpha+1,\beta+1)}$ (Part 3.7).

2. The $H_n^{(\alpha,\beta)}$-process for $\alpha, \beta \geqq -1$

2.1. In this part we prove theorems for the Hermite—Fejér interpolation including the values $\alpha = -1$ and $\beta = -1$. These results will be used in the investigations of the quasi-Hermite—Fejér interpolation, too.

2.2. First of all remark that $x_{k,n}^{(\alpha,\beta)}\in[-1,1]$ $(k=1,2,\dots,n)$ whenever $\alpha,\beta\geqq -1$ and $n\geqq 2$. This is a well-known fact if $\alpha,\beta>-1$ (see, e.g., [1]) and comes from the formula

$$2nP_n^{(-1,\beta)}(x)=(n+\beta)(x-1)P_{n-1}^{(1,\beta)}(x)$$

(see [1], (4.22.2)) if $\min(\alpha,\beta)=-1$.

We often use the fundamental relations

$$P_n^{(\alpha,\beta)}(1)=\binom{n+\alpha}{n}\quad(\alpha\in\mathbf{C}), \tag{2.1}$$

$$P_n^{(\alpha,\beta)}(x)=(-1)^nP_n^{(\beta,\alpha)}(-x)\quad(\alpha\in\mathbf{C}), \tag{2.2}$$

$$\frac{d}{dx}[P_n^{(\alpha,\beta)}(x)]=\frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x)\quad(\alpha\in\mathbf{C}). \tag{2.3}$$

2.3. We now prove

THEOREM 2.1. *Let* $\alpha,\beta\geqq -1$. *Then for any* $f\in C$ *we have*

$$|H_n^{(\alpha,\beta)}(f;x)-f(x)|=O(1)\sum_{i=1}^{n}\omega\left(f;\frac{i}{n}\right)\frac{1}{i^2}\quad \textit{if}\quad x\in[a,b]\subset(-1,1), \tag{2.4}$$

$$\begin{cases} |H_n^{(-1,\beta)}(f;x)-f(x)|= \\ \quad =O(1)\left[(n\vartheta)^4\sum_{i=1}^{n}\omega\left(f;\frac{i^2}{n^2}\right)\frac{1}{i^3}+\omega(f;\sin^2\vartheta)\right] & \textit{if}\quad 0\leqq\vartheta\leqq\frac{c}{n}{}^{1} \\ O(1)\left\{\sum_{i=1}^{n}\left[\omega\left(f;\frac{i\sin\vartheta}{n}\right)+\omega\left(f;\frac{i^2|\cos\vartheta|}{n^2}\right)\right]\frac{1}{i^2}\right\} & \textit{if}\quad \frac{c}{n}\leqq\vartheta\leqq\pi-\varepsilon, \\ O(1)\left\{\sum_{k=1}^{n}\left[\omega\left(f;\frac{i\sin\vartheta}{n}\right)+\omega\left(f;\frac{i^2|\cos\vartheta|}{n^2}\right)\right]i^{2\varphi-1}\right\} & \textit{if}\quad \vartheta\in[\varepsilon,\pi]\ \textit{and}\ \beta>-1. \end{cases} \tag{2.5}$$

Here ε, $c=c(\beta)>0$ *are fixed,* $\varphi=\max(\beta,-0.5)$.

Similar formulae are true for $|H_n^{(\alpha,-1)}(f;x)-f(x)|$ $(\alpha\geqq -1)$.

2.4. COROLLARY 2.1. *If* $-1\leqq\alpha,\beta<0$ *then* $H_n^{(\alpha,\beta)}(f;x)$ *uniformly tends to* $f(x)$ *in* $[-1,1]$, *whenever* $f\in C$.

Indeed, for $\min(\alpha,\beta)>-1$ we obtain Corollary 2.1, e.g. by (1.11). If $\min(\alpha,\beta)=-1$ we can apply (2.5) and (2.6) (see [9], (2.2)).

Using (2.5) we get

COROLLARY 2.2. *Supposing* $0\leqq\vartheta\leqq cn^{-2}$, *we have*

$$|H_n^{(-1,\beta)}(f;x)-f(x)|=O(1)\,\omega(f;\sin^2\vartheta). \tag{2.6}$$

Considering that by the Teliakovski—Gopengauz estimation we can state that for

[1] Here and later $c, c_1, c_2, \dots$ denote fixed positive constants, not necessarily the same in different formulae.

certain $P_n(f; x)$

$$|f(x)-P_n(f;\ x)| = O(1)\omega\left(f;\ \frac{\sin\vartheta}{n}\right) \quad (x\in[-1,1]), \tag{2.7}$$

(2.6) is better than (2.7) if x is "very close" to 1.

Another finer consequence of the estimation (2.5) is

COROLLARY 2.3. *We have for* $f\in \mathrm{Lip}\,\delta$ $(0<\delta\leqq 1)$ *and* $0\leqq \sin\vartheta\leqq cn^{-1}$

$$|H_n^{(-1,-1)}(f;x)-f(x)| = \begin{cases} O(1)\vartheta^2(\vartheta^2 n^2 \ln n+1) & \text{if } \delta=1 \\ O(1)\vartheta^{2\delta}[(n\vartheta)^{4-2\delta}+1] & \text{if } 0<\delta<1. \end{cases} \tag{2.8}$$

Let us remark that by (4.4) and (4.5) we get the well-known formula

$$H_n^{(-1,-1)}(f;x)=f(1)\left[1-\frac{(n-2)(n-1)}{2}(x-1)\right] l_1^2(x)+ \tag{2.9}$$

$$+\sum_{k=2}^{n-1} f(x_k)\, l_k^2(x)+f(-1)\left[1+\frac{(n-2)(n-1)}{2}(x+1)\right] l_n^2(x).$$

2.5. In many cases the estimations (1.11), (1.12) and (2.4)—(2.5) are the best possible. For proving this fact denote $C(\omega)=\{f(x);\ f\in C \text{ and } \omega(f;t)\leqq a(f)\omega(t)\}$ where $\omega(t)$ is a modulus of continuity. (If $a(f)\leqq M$, we write $C_M(\omega)$ instead of $C(\omega)$.) We state

THEOREM 2.2. *Let* $\alpha,\beta\geqq -1$, *moreover* $\lim\limits_{t=0}\omega(t)t^{-1}=\infty$. *Then*

a) *for any fixed* $x^*\in(-1,1)$ *one can choose a sequence* $\{n_i\}$ *and* $f_1\in C(\omega)$ *such that*

$$H_n^{(\alpha,\beta)}(f_1;\ x^*)-f_1(x^*)\geqq \sum_{i=1}^{n}\omega\left(\frac{i}{n}\right)\frac{1}{i^2} \quad (n=n_1,n_2,\ldots);$$

b) *supposing* $0\leqq\xi_n\leqq c_1 n^{-1}$ *for any fixed sequence* $\{z_n=\cos\xi_n\}$ *one can choose a sequence* $\{n_i\}$ *and* $f_2\in C(\omega)$ *such that*

$$H_n^{(-1,\beta)}(f_2;\ z_n)-f_2(z_n)\geqq (n\xi_n)^4\sum_{i=1}^{n}\omega\left(\frac{i^2}{n^2}\right)\frac{1}{i^3}+\omega(\sin^2\xi_n) \quad (n=n_1,n_2,\ldots)$$

(c_1 is a positive fixed number);

c) *supposing* $0\leqq\pi-\xi_n\leqq c_2 n^{-1}$ *moreover* $-1<\beta$, $\beta\neq 0$, *for any fixed sequence* $\{z_n=\cos\xi_n\}$ *one can choose a sequence* $\{n_i\}$ *and* $f_3\in C(\omega)$ *such that*

$$H_n^{(\alpha,\beta)}(f_3;\ z_n)-f_3(z_n)\geqq \sum_{i=1}^{n}\omega\left(\frac{i^2}{n^2}\right) i^{2\beta-1} \quad (n=n_1,n_2,\ldots)$$

(c_2 is a fixed positive number).

2.5.1. Remark that for $\omega(t)=t$ the expressions $\omega(\ldots)$ must be replaced by $\varepsilon_n\omega(\ldots)$ where $\varepsilon_n>0$ and $\varepsilon_n\searrow 0$ arbitrarily slowly (see [10]). Statements analogous to b) and c) can be proved for $H_n^{(\beta,\alpha)}$ in the corresponding intervals.

2.6. If $\alpha\geqq -1$, $\beta>0$, by Theorem 2.2 one can easily prove that for a suitable $f\in C$, $H_n^{(\alpha,\beta)}(f;\ -1)$ does not tend to $f(-1)$. The result can be extended for $\alpha>-1$ and $\beta=0$ (see, e.g., [1] (4.6 (4)).

Further, it is easy to see that for a suitable $g(x)$ $H_n^{(0,-1)}(g;1)$ does not tend to $g(1)$. Indeed, by (4.4) and (4.5) for $\alpha=0$, $\beta=-1$

$$v_1(x)=1-\frac{(n-1)^2}{2}(x-x_1),\quad v_2(x)=\frac{1-x}{1-x_2}\quad (k=2,3,\dots,n),$$

$$v_n(x)=1+\frac{(n-1)^2}{2}(x+1),$$

from where with $g_1(x)=1-x$

$$H_n^{(0,-1)}(g,1)=g(x_n)h_n(1)>h_n(1)>0\neq g(1)=0\quad (n\geqq n_0).$$

Similar result holds for $H_n^{(-1,0)}(f;x)$. I.e., we get

COROLLARY 2.3. *If* $\alpha,\beta\geqq-1$ *and* $\max(\alpha,\beta)\geqq 0$ *then for a suitable* $f\in C$ $\lim_{n\to\infty}\|H_n^{(\alpha,\beta)}(f;x)-f(x)\|>0$. (Here, as usual, $\|g\|_{[a,b]}=\max_{a\leqq x\leqq b}|g(x)|$; if $[a,b]=[-1,1]$ we write $\|g\|$.)

3. The $Q_n^{(\alpha,\beta)}$-process for $\alpha,\beta>-1$

First of all we prove some theorems on the order of convergence.

3.1. THEOREM 3.1. *Supposing* $\alpha,\beta>-1$ *we have the following relations*

$$|Q_n^{(\alpha,\beta)}(f;x)-f(x)|=O(1)\left[\sum_{i=1}^n\omega\left(f;\frac{i}{n}\right)\frac{1}{i^2}+\frac{1}{n^{1+2\alpha}}+\frac{1}{n^{1+2\beta}}\right]\tag{3.1}$$

if $x\in[a,b]\subset(-1,1)$,

$$|Q_n^{(\alpha,\beta)}(f;x)-f(x)|=O(1)\left\{\sum_{i=1}^n\left[\omega\left(f;\frac{i\sin\vartheta}{n}\right)+\right.\right.\tag{3.2}$$

$$\left.\left.+\omega\left(f;\frac{i^2|\cos\vartheta|}{n^2}\right)\right]i^{2\delta-1}+\frac{\omega(f;\sin^2\vartheta)}{(n\vartheta)^{2\alpha+1}}+\frac{\vartheta^{-2\alpha+1}}{n^{2\beta+1}}\right\}\quad \textit{if}\quad \frac{c}{n}\leqq\vartheta\leqq\pi-\varepsilon$$

and $\delta=\max(-0.5,\alpha-1)$.

By (3.2) one can obtain the following

COROLLARY 3.1. *We have*

$$|Q_n^{(\alpha,\beta)}(f;x)-f(x)|=O(1)\left\{\sum_{i=1}^n\left[\omega\left(f;\frac{i\sin\vartheta}{n}\right)+\omega\left(f;\frac{i^2|\cos\vartheta|}{n^2}\right)\right]\frac{1}{i^2}+\right.\tag{3.3}$$

$$\left.+\frac{\vartheta^{-2\alpha+1}}{n^{2\beta+1}}\right\}\quad \textit{if}\quad \frac{c}{n}\leqq\vartheta\leqq\pi-\varepsilon\quad \textit{and}\quad 0\leqq\alpha\leqq 0.5,$$

$$|Q_n^{(\alpha,\beta)}(f;x)-f(x)|=O(1)\left\{\sum_{i=1}^n\left[\omega\left(f;\frac{i\sin\vartheta}{n}\right)+\right.\right.\tag{3.4}$$

$$\left.\left.+\omega\left(f;\frac{i^2|\cos\vartheta|}{n^2}\right)i^{2\alpha-3}\right]+\frac{\vartheta^{-2\alpha+1}}{n^{2\beta+1}}\right\}\quad \textit{if}\quad \frac{c}{n}\leqq\vartheta\leqq\pi-\varepsilon\quad \textit{and}\quad \alpha\geqq 0.5.$$

3.2. Now we investigate the interval $[0, cn^{-1}]$. We state

THEOREM 3.2. *Supposing* $\alpha, \beta > -1$ *we have*

$$|Q_n^{(\alpha,\beta)}(f;x)-f(x)| = O(1)\left[(n\vartheta)^2 \sum_{i=1}^{n} \omega\left(f;\frac{i^2}{n^2}\right) i^{2\alpha-3} + \right. \tag{3.5}$$

$$\left. +\omega(f;\sin^2\vartheta) + \frac{\vartheta^2 \sum_{i=1}^{n} i^{2\beta-1}}{n^{2\beta-2\alpha}}\right] \quad \textit{if} \quad 0 \leqq \vartheta \leqq \frac{c}{n},$$

moreover

$$|Q_n^{(0,0)}(f;x)-f(x)| = O(1)\left[(n\vartheta)^2 \sum_{i=1}^{n} \omega\left(f;\frac{i^2}{n^2}\right)\frac{1}{i^3} + \omega(f;\sin^2\vartheta)\right] \tag{3.6}$$

$$\textit{if} \quad 0 \leqq \vartheta \leqq cn^{-1}.$$

3.3. Using (3.1)—(3.5) and the analogous formulae for $[\varepsilon, \pi]$ we obtain

COROLLARY 3.2. *If* $f \in C$ *then* $\{Q_n^{(\alpha,\beta)}(f;x)\}$ *uniformly tends to* $f(x)$
a) *in* $[a,b] \subset (-1,1)$ *if* $\alpha, \beta > -0.5$;
b) *in* $[-1,1]$ *if* $0 \leqq \alpha, \beta < 1$;
c) *in* $[-1,1]$ *if* $|\alpha|, |\beta| < 0.5$;
d) *in* $[-1,1]$ *if* $\alpha = \beta = -0.5$.

(For the statement d) see [5].)

Remark that by (3.1)—(3.6) we obtain formulae similar to (2.4) and (1.11). Here is an example.

COROLLARY 3.3. *We have*

$$|Q_n^{(\alpha,\beta)}(f;x)-f(x)| = O(1)\sum_{i=1}^{n}\omega\left(f;\frac{i}{n}\right)\frac{1}{i^2} \quad \textit{if} \quad [a,b]\subset(-1,1);\ \alpha,\beta \geqq 0;$$

$$|Q^{(\alpha,\beta)}(f;x)-f(x)| = O(1)\left[\sum_{i=1}^{n}\omega\left(f;\frac{i\sin\vartheta}{n}\right)+\omega\left(f;\frac{i^2|\cos\vartheta|}{n^2}\right)\right] i^{2\delta-1}$$

$$\textit{if} \quad 0 \leqq \vartheta \leqq \pi-\varepsilon \quad \textit{and} \quad \beta \geqq \alpha \geqq 0$$

(*see* (3.1) *and* (3.2)).

3.4. We mention the following estimations of Teliakovski—Gopengauz type for the whole interval $[-1,1]$ if $\alpha=\beta=0.5$ (see [6] and (1.18)).

THEOREM 3.3. *We have for* $-1 \leqq x \leqq 1$

$$|Q_n^{(0.5,0.5)}(f;x)-f(x)| = O(1)\left\{\sin^2(n-1)\vartheta \sum_{i=1}^{n}\left[\omega\left(f;\frac{i\sin\vartheta}{n}\right)+\right.\right. \tag{3.7}$$

$$\left.\left. +\omega\left(f;\frac{i^2|\cos\vartheta|}{n^2}\right)\right]\frac{1}{i^2}+\omega(f;|x-y_j|)\right\}.$$

where y_j is the nearest node to x. By (3.7) we get

COROLLARY 3.4. *We have for* $-1 \leqq x \leqq 1$

$$|Q_n^{(0.5,0.5)}(f;\, x)-f(x)| = O(1)\left\{\sin^2(n-1)\vartheta \sum_{i=1}^{n}\left[\omega\left(f;\, \frac{i\sin\vartheta}{n}\right)+\right.\right.$$

$$\left.\left.+\omega\left(f;\, \frac{i^2|\cos\vartheta|}{n^2}\right)\right]\frac{1}{i^2}+\omega\left(f;\frac{\sin\vartheta}{n}\right)\right\}.$$

3.5. Now we prove that the order of our estimations in many cases is the best possible.

If, e.g., in (3.1) $\sum_{i=1}^{n}\omega\left(\frac{i}{n}\right)\frac{1}{i^2}=O(1)(n^{-1-2\alpha}+n^{-1-2\beta})$ then we say that the sum is negligible. Similar definitions hold for (3.2)—(3.4).

THEOREM 3.4. *Suppose* $\alpha, \beta > -1$, $\lim_{t=0}\omega(t)t^{-1}=\infty$ *and* $e_n \searrow 0$ *arbitrarily slowly. Then*

a) *for any fixed* $x^* \in [-1, 1]$ *one can choose a sequence* $\{n_i\}$ *and a function* $f_1(x) \subset C(\omega)$ *such that*

$$Q_n^{(\alpha,\beta)}(f;\, x^*)-f_1(x^*) \geqq \sum_{i=1}^{n}\omega\left(\frac{i}{n}\right)\frac{1}{i^2}+e_n\left[\frac{1}{n^{1+2\alpha}}+\frac{1}{n^{1+2\beta}}\right] \quad (n = n_1, n_2, \ldots);$$

b) *supposing* $c_1 n^{-1} \leqq \vartheta^* \leqq \pi-\varepsilon$ *(*c_1 *is fixed), for any fixed* $x^*=\cos\vartheta^*$ *one can choose a sequence* $\{n_i\}$ *and a function* $f_2(x) \in C(\omega)$ *such that*

$$Q_n^{(\alpha,\beta)}(f_2;\, x^*)-f_2(x^*) \geqq e_n\left[\frac{\omega(\sin^2\vartheta^*)}{(n\vartheta^*)^{2\alpha+1}}-\frac{(\vartheta^*)^{-2\alpha+1}}{n^{2\beta+1}}\right] \quad (n = n_1, n_2, \ldots)$$

if the sum in the corresponding upper estimation is negligible (see (3.2)—(3.4)*);*

c) *there exists a sequence* $\{z_n=\cos\xi_n\}$ *with* $\xi_n \sim n^{-1}$, *the sequence* $\{n_i\}$ *and* $f_3(x) \in C(\omega)$ *such that*

$$Q_n^{(\alpha,\beta)}(f_3;\, z_n)-f_3(z_n) \geqq \sum_{i=1}^{n}\omega\left(\frac{i^2}{n^2}\right)i^{2\alpha-3}+\frac{e_n}{n^{2(\beta-\alpha+1)}} \quad (n = n_1, n_2, \ldots) \quad \textit{if} \quad \alpha \neq 1;$$

d) *supposing* $0 \leqq \xi_n \leqq c_2 n^{-1}$ *(*c_2 *is fixed), for any fixed sequence* $\{z_n=\cos\xi_n\}$ *one can choose a sequence* $\{n_i\}$ *and a function* $f_4(x) \in C(\omega)$ *such that*

$$Q_n^{(\alpha,\beta)}(f_4;\, z_n)-f_4(z_n) \geqq (n\xi_n)^2 \sum_{i=1}^{n}\omega\left(\frac{i^2}{n^2}\right)i^{2\alpha-3}+$$

$$+\omega(\sin^2\xi_n)+e_n\frac{\xi_n^2\sum_{i=1}^{n}i^{2\beta-1}}{n^{2\beta-2\alpha}} \quad (n = n_1, n_2, \ldots),$$

excluding the cases $\alpha=1$ *and* $\alpha=\beta=0$. *If* $\alpha=\beta=0$, *the last term must be replaced by* $e_n\xi_n^2$.

3.5.1. Remarks analogous to 2.5.1 are valid.

3.6. By Theorem 3.4 we can state the following

COROLLARY 3.5. *One can choose functions* $f \in C$ *such that the corresponding* $\{Q_n^{(\alpha,\beta)}(f; x)\}$ *does not tend uniformly to* $f(x)$

a) *in* $[a, b] \subset (-1, 1)$ *if* $\min(\alpha, \beta) < -0.5$;

b) *in* $[-1, 1]$ *if* $\max(\alpha, \beta) \geqq 1$.

Indeed, we get a) by Theorem 3.4 a). To obtain b), consider that for the function $h(x) = 1 - x$ with $2z_n = 1 + y_2$ we get

$$Q_n^{(1,0)}(h; z_n) \geqq h(-1) r_{n,n}(z_n) \sim n^2 n^{-2} = 1$$

from where $Q_n^{(1,0)}(h; z_n)$ does not tend to $h(z_n) \sim n^{-2}$. If $\alpha = 1, 0 < \beta \leqq 1$ we get by (4.25) and (4.27)

$$Q_n^{(1,\beta)}(h;\ z_n) \geqq \sum_{k=2}^{\left[\frac{n}{2}\right]'} \frac{k^2}{n^2} \frac{k^3}{n^4} \frac{n^4}{k^4} \frac{k^2}{n^2} \sim 1;$$

finally for $\alpha = 1, -1 < \beta < 0$ we obtain

$$\sum_{k=1}^{n} |r_{k,n}(z_n)| \sim n^{-2\beta}$$

i.e. b) is true for $\alpha = 1, \beta \leqq 1$ (or $\beta = 1, \alpha \leqq 1$). Considering Theorem 3.4 c) we obtain the desired result.

3.5. Now we prove an equiconvergence theorem.

THEOREM 3.5. *If* $\alpha, \beta \geqq -1$ *then we have for arbitrary modulus of continuity* $\omega(t)$

$$\frac{\sup\limits_{f \in C_1(\omega)} \|Q_n^{(\alpha+1,\beta+1)}(f;\ x) - f(x)\|}{\sup\limits_{f \in C_1(\omega)} \|H_n^{(\alpha,\beta)}(f;\ x) - f(x)\|} \sim 1. \tag{3.8}$$

3.8. Finally we mention the case $\alpha = \beta = -1$. One can consider $x_{k,n}^{(-1,-1)}$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ as fundamental points of the process $Q_n^{(-1,-1)}$. Moreover, it is easy to prove the statement of Theorem 3.4 a) for $\alpha = \beta = -1$.

The cases $\min(\alpha, \beta) = -1$ can be treated similarly. We omit the details.

4. Proofs

4.1. First of all we mention some relations used later.

$$|P_n^{(\alpha,\beta)}(x)| = \begin{cases} O\left(\vartheta^{-\alpha-\frac{1}{2}} n^{-\frac{1}{2}}\right) & \text{if } \dfrac{c}{n} \leqq \vartheta \leqq \dfrac{\pi}{2}, \quad \alpha, \beta \in \mathbf{R}, \\ O(n^\alpha) & \text{if } 0 \leqq \vartheta \leqq \dfrac{c}{n}, \quad \alpha, \beta \in \mathbf{R}, \\ O(n\vartheta^2) & \text{if } 0 \leqq \vartheta \leqq \dfrac{c}{n} \quad \alpha = -1, \beta \in \mathbf{R}. \end{cases} \tag{4.1}$$

Here the first two relations are from [1], (7.32.5). To prove the third one we can use [1], (4.22.2). (1.7.1(1) and (1.71.10). (Indeed, if $J_\alpha(z)$ stands for the Bessel function of the first kind of order α, we have

$$nP_n^{(-1,\beta)}(\cos\vartheta)=O(1)\frac{n\vartheta}{2}J_{-1}(n\vartheta)=O(1)\left|\frac{n\vartheta}{2}J_1(n\vartheta)\right|=O(1)n^2\vartheta^2$$

uniformly for $\vartheta=O(n^{-1})$, which is the desired estimation.)

(We suppose $n\vartheta\leqq c_1<j_1^{(1)}$ where $j_1^{(\alpha)}$ is the first positive root of $J_\alpha(z)$.)

It is important to remark (4.1) is the best possible in the following sense. Let $z_s=\cos\xi_s$ and $2\xi_s=\vartheta_{s,n}+\vartheta_{s+1,n}\left(1\leqq s<\frac{n}{2}\right)$. Then we have

$$\text{(4.2)}\quad\begin{cases}|P_n^{(\alpha,\beta)}(z_s)|\sim\xi_s^{-\alpha-\frac{1}{2}}n^{-\frac{1}{2}} & \text{if } c_0'\leqq s<\frac{n}{2},\ \alpha,\beta\geqq-1,\\ |P_n^{(\alpha,\beta)}(x)|\sim n^\alpha & \text{if } 2n\vartheta=j_1^{(\alpha)},\quad \alpha,\beta\geqq-1,\\ |P_n^{(-1,\beta)}(x)|\sim n\vartheta^2 & \text{if } 2n\vartheta=j_1^{(1)}\end{cases}$$

(see the arguments of [1], 7.32(3) and (2.3)).

Denote $x_{k,n}=x_{k,n}^{(\alpha,\beta)}=\cos\vartheta_{k,n}^{(\alpha,\beta)}$. Then, sometimes omitting the superfluous notations,

$$\frac{c_1}{n}\leqq\vartheta_{k+1}-\vartheta_k\leqq\frac{c_2}{n}\quad(k=0,1,\dots,n;\ \vartheta_k\neq\vartheta_{k+1};\ \alpha,\beta\geqq-1),$$

where $x_0=1$, $x_{n+1}=-1$. If, e.g., $\alpha=-1$ then $x_0=x_1=1$. Indeed, consider that $P_n^{(\alpha,\beta)}(x)$ satisfy the differential equation

$$\text{(4.3)}\quad(1-x^2)y''+[\beta-\alpha-(\alpha+\beta+2)x]y'+n(n+\alpha+\beta+1)y=0\quad(\alpha,\beta\in\mathbf{C}).$$

([1], (4.21) and 4.2.2). Now by the argument used by G. FREUD, A. SHARMA [7], p. 297 we obtain the desired relation. Another useful relation is

$$\text{(4.4)}\quad\begin{cases}v_k^{(\alpha,\beta)}(x)=1-\dfrac{(\alpha+\beta+2)x_k-\beta+\alpha}{1-x_k^2}(x-x_k)\quad(-1<x_k<1;\ \alpha,\beta\geqq-1),\\ v_1^{(\alpha,\beta)}(x)=O(1)(1+n^2|x-x_1|)\quad(x_1=1;\ \alpha=-1,\beta\geqq-1).\end{cases}$$

The first relation can be obtained by (1.7) and (4.3); further by (2.1)—(2.3)

(4.5)

$$\frac{P_n''^{(\alpha,\beta)}(1)}{P_n'^{(\alpha,\beta)}(1)}=\frac{(n+\alpha+\beta)\binom{n+\alpha}{n-2}}{2\binom{n+\alpha}{n-1}},\quad\frac{P_n''^{(\alpha,\beta)}(-1)}{P_n'^{(\alpha,\beta)}(-1)}=-\frac{(n+\alpha+\beta)\binom{n+\beta}{n-2}}{2\binom{n+\beta}{n-1}}\quad(\alpha,\beta\in\mathbf{C})$$

which can be applied for $v_1(x)$.

If we use [1], Theorem 8.9.1, and 2.2 we can obtain

$$\text{(4.6)}\quad|P_n'^{(\alpha,\beta)}(x_k)|\sim k^{-\alpha-3/2}n^{\alpha+2}\quad(0\leqq\vartheta_k\leqq\pi-\varepsilon;\ \alpha,\beta\geqq-1).$$

We shall use the relations

$$(4.7)\qquad |f(x)-f(x_k)| = O(1)\left[\omega\left(\frac{i\sin\vartheta}{n}\right)+\omega\left(\frac{i^2}{n^2}|\cos\vartheta|\right)\right],$$

where $i=|k-j|$ if $k\neq j$ and $i=1$ if $k=j$ ($x_{j,n}$ is the nearest root to x; see [9], (2.10));

$$(4.8)\qquad \sin\frac{|\vartheta-\vartheta_k|}{2}\sim|\vartheta-\vartheta_k|\sim\frac{i}{n}\quad (k\neq j).$$

4.2. Proof of Theorem 2.1. **4.2.1.** By $\sum_{k=1}^{n} h_k(x)\equiv 1$ we have

$$(4.9)\qquad |H_n(f;x)-f(x)|\leqq\Big(\sum_{k=j}+\sum_{\substack{|x-x_k|\leqq\xi(\alpha,\beta)\\ k\neq j}}+\sum_{x-x_k>\xi(\alpha,\beta)}+\sum_{x_k-x>\xi(\alpha,\beta)}\Big)\cdot$$

$$\cdot|f(x)-f(x_k)|\,|h_k(x)|\equiv\Big(\sum_{\mathrm{I}}+\sum_{\mathrm{II}}+\sum_{\mathrm{III}}+\sum_{\mathrm{IV}}\Big)|f(x)-f(x_k)|\,|h_k(x)|.$$

If we apply the methods used in [8] and the relations obtained above we get for $x\in[a,b]\subset(-1,1)$

$$(4.10)\qquad |h_k(x)|=O(1)i^{-2}\quad\text{if}\quad k\in\mathrm{I}\cup\mathrm{II},\quad \alpha,\beta\geqq-1,$$

$$(4.11)\qquad \sum_{\mathrm{III}}\ldots=O(1)n^{-2\alpha-3}\sum_{l=1}^{n}l^{2\alpha+1},\quad \alpha,\beta\geqq-1,$$

$$(4.12)\qquad \sum_{\mathrm{IV}}\ldots=O(1)n^{-2\beta-3}\sum_{l=1}^{n}l^{2\beta+1},\quad \alpha,\beta\geqq-1.$$

By (4.11) and (4.12) $\sum_{\mathrm{III}}+\sum_{\mathrm{IV}}=O\left(\frac{\ln n}{n}\right)$, i.e. we proved (2.4).

4.2.2. For the whole interval we use

$$|H_n(f;x)-f(x)|=O(1)\Big(\sum_{k=j}+\sum_{\substack{\vartheta_k\leqq\frac{\pi}{2}\\ k\neq j}}+\sum_{\pi-\varepsilon\geqq\vartheta_k>\frac{\pi}{2}}+\sum_{\vartheta_k>\pi-\varepsilon}\Big)|f(x)-f(x_k)|\,|h_k(x)|\equiv$$

$$\equiv\Big(\sum_{\mathrm{I}}+\sum_{\mathrm{II}}+\sum_{\mathrm{III}}+\sum_{\mathrm{IV}}\Big)|f(x)-f(x_k)|\,|h_k(x)|$$

and

$$(4.13)\qquad |h_k(x)|=O(1)\frac{P_n^2(x)}{(1-x_k^2)P_n'^2(x_k)(\vartheta-\vartheta_k)^2}\quad (0\leqq\vartheta\leqq\pi-\varepsilon,\ \vartheta\neq\vartheta_k).$$

By these formulae, as in [9], we obtain as follows.

4.2.2.1. If $0\leqq\vartheta\leqq cn^{-1}$ then we have

$$(4.14)\qquad |h_k(x)|=O(1)\varrho^4 i^{-3}\quad \textit{if}\quad k\in\mathrm{II}\cup\mathrm{III},\quad \alpha=-1,\quad \beta\geqq-1,$$

$$(4.15)\qquad \sum_{\mathrm{IV}}\ldots=O(1)\varrho^4 n^{-2(\beta+2)}\sum_{l=1}^{n}l^{2\beta+1}\quad\text{if}\quad \alpha=-1,\ \beta\geqq-1.$$

By $|h_k(x)|=O(1)$ and elementary estimations we get

$$|f(x)-f(1)|h_1(x) = O(1)\omega(1-x) \quad \textit{if} \quad \alpha=-1, \quad \beta\geqq -1, \tag{4.16}$$

$$\omega(1-x) \leqq \omega(1-x^2) = \omega(\sin^2\vartheta) \quad \text{if} \quad 0\leqq \sin\vartheta \sim \vartheta \leqq cn^{-1}.$$

Using these relations we get the first part of (2.5).

4.2.2.2. If $cn^{-1}\leqq\vartheta\leqq\pi-\varepsilon$ we have the estimations

$$|h_k(x)| = O(1)i^{2\gamma-1} \quad \text{if} \quad k\in\text{II} \quad \text{and} \quad 2\vartheta_k>\vartheta, \quad \alpha,\beta\geqq -1, \tag{4.17}$$

$$|h_k(x)| = O(1)i^{2\gamma-1} \quad \text{if} \quad k\in\text{I}\cup\text{III}; \quad \alpha,\beta\geqq -1, \tag{4.18}$$

$$\sum_{\text{IV}}\ldots = O(1)\vartheta^{-2\alpha-1}n^{-2\beta-3}\sum_{l=1}^{n} l^{2\beta+1} \quad \text{if} \quad \alpha,\beta\geqq -1 \tag{4.19}$$

(see the corresponding parts in [9]; $\gamma=\max(\alpha,\beta,-0.5)$).

For the remaining part of $\sum_{\text{II}}$ (i.e., when $2\vartheta_k<\vartheta$) using (4.4) we get $|v_k(x)|=O(1)$ if $x_k\geqq -1+\varepsilon, k\neq 1, \alpha=-1$ (see further (4.24)). Further, by (4.1) and (4.6) we obtain $|h_1(x)||f(x)-f(1)|=O(1)\omega(\sin^2\vartheta)(n\vartheta)^{-1}$. So the corresponding part of [9] can be estimated as follows:

$$\sum_{\substack{2\vartheta_k<\vartheta\\ k\neq 1}}\ldots = O(1)\sum_{k=1}^{cj}\omega\left(\frac{i}{n}\sin\vartheta\right)\frac{k}{n^2}\frac{\vartheta}{n}\frac{1}{\vartheta^4} = O(1)\omega(\sin^2\vartheta)(n\vartheta)^{-1} \tag{4.20}$$

$$(\alpha=-1, \; \beta\geqq -1).$$

By (4.13)—(4.20) further using that

$$\sin\vartheta\cdot n^{-2\beta-3}\sum l^{2\beta+1} = O(1)\sin\vartheta\cdot n^{-1}\ln n$$

and $\omega(\sin^2\vartheta)(n\vartheta)^{-1}\sim \sum_{k=1}^{cj}\omega((j-k)\sin\vartheta\cdot n^{-1})(j-k)^{-2}$ $(c<1)$ we get the second part of (2.5).

4.2.2.3. To prove the third one, first we investigate the difference

$$[H_n^{(\alpha,-1)}(f;x)-f(x)] \quad \text{in} \quad 0\leqq\vartheta\leqq\pi-\varepsilon \quad (\alpha>-1).$$

By (4.4)

$$v_k^{(\alpha,-1)}(x) = \frac{1-x-\alpha(x-x_k)}{1-x_k} \quad (|x_k|<1) \tag{4.21}$$

from where $|v_k^{(\alpha,-1)}(x)|=O(1)$ if $x_k\leqq 1-\varepsilon$.

By (4.21) and the methods used in [9] we get (2.6) considering (2.2), too. (See the corresponding parts in [9]; the restriction $\alpha>-1$ will be applied at II_1 b.)

4.3. Proof of Theorem 2.2. First we prove b).

4.3.1. We wish to apply the following statement which is a special case of [10], Theorem 3.1.

THEOREM 4.1. *If for the sequence of linear operators* $T_n(f; x)$ *and the functions* $g_n(x)$ $(n=1, 2, \ldots)$ *we have*

(a1) $g_n(x)\in C(\omega)$;

(a2) $T_n(g_n; z_n)\geqq c_4\lambda_n(z_n)$ *for certain* $\{z_n\}\subset[-1, 1]$ *(definition of* $\lambda_n(z_n)$*)*;

(B*) $\bar{f}(x) \overset{\text{def}}{=} \varrho \sum_{i=1}^{\infty} e_{n_i} g_{n_i}(x)\in C(\omega)$ *for certain* $\{n_i\}$ *and* $\{e_{n_i}\}$ $(0<e_n\leqq 1,\ e_{n+1}\leqq e_n)$;

(C*) $c_5\lambda_{n_k}(z_{n_k}) > \sum_{i=k+1}^{\infty} e_{n_i}|T_{n_k}(g_{n_i};\ z_{n_k})| + \sum_{i=k}^{\infty} e_{n_i}|g_{n_i}(z_{n_k})|$ $(k=1, 2, \ldots)$ *with* $c_5 < c_4$,

then there exists an $f\in C(\omega)$ *such that*

$$T_n(f; z_n)-f(z_n) > e_n\lambda_n(z_n) \quad (n = n_1, n_2, \ldots).$$

Let $\{z_n=\cos\xi_n\}$ be a sequence such that $0<\xi_n<c_1 \min(\vartheta_{2,n}^{(-1,\beta)};\ j_1^{(1)})$. By $2\leqq p=p_n(\omega)=o(n)$ and $w_n=x_p-p^2n^{-2}$ define $g_n(x)$ as follows:

$$\begin{cases} g_n(1) = \omega(1-z_n), \\ g_n(x_k) = \omega\left(\dfrac{k^2}{n^2}\right) \quad \text{if} \quad 2 \leqq k \leqq p_n(\omega), \\ g_n(z_n) = g_n(w_n) = g_n(x_k) = 0 \quad (x_k \leqq w_n). \end{cases} \tag{4.22}$$

In the intervals formed by $1, z_n, w_n, x_k$ $(k=2, \ldots, p)$ and -1 let $g_n(x)$ be linear. Obviously $\omega(g_n; t)\leqq c\omega(g_n, t; [1-z_n; 1])=c\omega(t)$ if $0\leqq t\leqq 1-z_n$. (Here $\omega_n(g_n; t; [a, b])$ is the $\omega(g_n; t)$ restricted to $[a, b]$.) I.e.

(a1) $$g_n(x)\in C(\omega).$$

Further we mention that by (4.22)

$$g_N(z_n) = 0 \quad \text{if} \quad N \geqq s(n). \tag{4.23}$$

By (1.7) and (4.5) $v_1(x)=1-c_2n^2|x-x_1|$ i.e., $v_1(z_n)\geqq 0.5$ if c_1 is small enough. Then by (2.1) and (2.3) $h_1(z_n)\sim 1$. Further, using

$$v_k(x) = 1+\frac{\beta+1}{1+x_k}(x-x_k) \quad (k = 2, \ldots, n-1), \tag{4.24}$$

we get $v_k(z_n)>0, v_k(z_n)\sim 1$ $(2\leqq k\leqq p_n(\omega))$.

Using this, (4.22), (4.2) and (4.6) we get

$$H_n^{(-1,\beta)}(g_n;\ z_n) \geqq c_3\left[\omega(\sin^2\xi_n)+\sum_{k=2}^{p} n^2\xi_n^4\frac{n^4}{k^4}\frac{k}{n^2}\omega\left(\frac{k^2}{n^2}\right)\right] \sim$$

$$\sim \left[(n\xi_n)^4 \sum_{i=1}^{p}\omega\left(\frac{i^2}{n^2}\right)\frac{1}{i^3}+\omega(\sin^2\xi_n)\right].$$

So by $\lambda_n(z_n)=[\ldots]$ we obtain

(a2) $$H_n^{(-1,\beta)}(g_n;\ z_n) \geqq c_4\lambda_n(z_n) \quad (n \geqq n_0).$$

Now we prove with $Q>0$

$$\text{(B*)}\qquad \tilde{f}(x)\overset{\text{def}}{=}Q\sum_{i=1}^{\infty} g_{n_i}(x)\in C(\omega)\quad (n_{i+1}>d(n_i)).$$

Indeed, if $1-z_{n_{j+1}}<t\leqq 1-z_{n_j}$ then we get

$$\omega(\tilde{f};\, t)\leqq Q\sum_{i=1}^{\infty}\omega(g_{n_i};\, t)=Q\left[\sum_{i=1}^{j}+\sum_{i=j+1}^{\infty}\right].$$

By (4.22) and $\lim\limits_{t=+0}\omega(t)t^{-1}=\infty$

$$\sum_{i=1}^{j}\omega(g_{n_i};\, t)\leqq ct\sum_{i=1}^{j}\frac{\omega(1-z_{n_i})}{1-z_{n_i}}\leqq 2ct\frac{\omega(1-z_{n_j})}{1-z_{n_j}}\leqq 2c\omega(t)$$

if $\{n_i\}$ is lacunary enough. Further with $0<q<1$

$$\sum_{i=j+1}^{\infty}\omega(g_{n_i};\, t)\leqq \sum_{i=j+1}^{\infty}\omega(g_{n_i};\, 1-w_{n_i})\leqq c\omega(1-w_{n_{j+1}})\sum_{i=0}^{\infty}q^i\leqq \frac{c}{1-q}\omega(t)$$

if $\{n_i\}$ is lacunary such that $\omega(1-w_{n_1})\leqq q$, $\omega(1-w_{n_{i+1}})\leqq q\omega(1-w_{n_i})$. I.e., we proved (B*).

By (4.22) we get $|g_n(x)|\leqq \varrho_n$ where $\lim\limits_{n\to\infty}\varrho_n=0$. Using this and (4.23) we get for a certain $\{n_i\}$

$$\text{(C*)}\qquad \sum_{i=k+1}^{\infty}|H_{n_k}(g_{n_i};\, z_{n_k})|+\sum_{i=k}^{\infty}|g_{n_i}(z_{n_k})|=$$

$$=\left(\sum_{t=1}^{n_k}|h_{t,n_k}(z_{n_k})|\right)\left(\sum_{i=k+1}^{\infty}\varrho_{n_i}\right)<\frac{1}{2}\lambda_{n_k}(z_{n_k}).$$

I.e., by (a1), (a2), (B*), (C*) and $e_n=1$ we get for $f(x)=c_8\tilde{f}(x)\in C(\omega)$

$$\text{(4.25)}\qquad H_n^{(-1,\beta)}(f;\, z_n)-f(z_n)>\lambda_n(z_n)\quad (n=n_1,n_2,\dots)$$

(see Theorem 4.1). To complete our proof, we decide the index $p_n(\omega)$.

First suppose $\omega(t)=O(t^a)$, e.g. $t^{a+\delta}\leqq\omega(t)\leqq ct^{\alpha}$ where $0<a<1,\delta\geqq 0$ is arbitrarily small, t is small enough. We have by a simple computation

$$\sum_{i=1}^{n}\omega\left(\frac{i^2}{n^2}\right)\frac{1}{i^3}=O(1)\sum_{i=1}^{[n^{1-\eta}]}\omega\left(\frac{i^2}{n^2}\right)\frac{1}{i^2}$$

if η is small enough, i.e. one can choose $p_n(\omega)=[n^{1-\eta}]$. For $\omega(t)\neq O(t^{\alpha})$ let $p_n(\omega)=[\sqrt{n}]$, finally if $\omega(t)=t\varphi\left(\frac{1}{t}\right)$ where $\lim\limits_{t=+0}\varphi\left(\frac{1}{t}\right)=\infty$, let $p_n(\omega)=$ $=[n(\ln n)^{-1/2}]$. Using (4.25) we get our assertion.

4.3.2. We sketch the proof of the remaining formulae. To get a) one can choose a sequence $\{n_i\}$ such that $|x_{j,n}-x^*|\sim n^{-1}$ $(n=n_1, n_2, \dots)$ (see, e.g. [1], Theorem 8.9.1). For this subsequence we define $g_n(x)$ as follows:

$$\begin{cases} g_n(x^*) = g_n(x_k) = 0 & \text{if} \quad k \geqq j, \\ g_n(x_k) = \omega\left(\dfrac{j-k}{n}\right) & \text{if} \quad s \equiv j-p_n(\omega) \leqq k < j, \\ g_n(w_n) = g_n(x_k) = 0 & \text{if} \quad x_k \geqq w_n \equiv x_s + p_n n^{-1}. \end{cases}$$

By (4.4) one can see that $v_k(x^*)>0$, $v_k(x^*)\sim 1$ whenever $g_n(x_k)\neq 0$ $(n=n_1, n_2, \dots)$. The remaining parts are similar to 4.3.1.

4.3.3. We prove c) for $z_n=-1$. If $z_n\neq -1$, one can argue similarly. Let $g_n(x)$ $(n=n_1, n_2, \dots)$ define by

$$\begin{cases} g_n(-1) = 0, \\ g_n(x_k) = \omega\left(\dfrac{(n-k+1)^2}{n^2}\right) \operatorname{sign} h_k(-1) \quad \text{if} \quad s \equiv n-p_n \leqq k \leqq n, \\ g_n(w_n) = g_n(x_k) = 0 \quad \text{if} \quad x_k \geqq w_n \equiv x_s + (n-p_n)^2 n^{-2}. \end{cases}$$

By (4.4) $v_k(-1)\sim 1$ and $\operatorname{sign} v_k(-1) = \operatorname{sign}(-\beta)$ if $g_n(x_k)\neq 0$. The further parts are similar to 4.3.1.

4.4. Proof of Theorem 3.1. **4.4.1.** First we state the following simple but important relations:

$$(4.26) \quad \begin{cases} \sin\vartheta\, |P_{n-2}^{(\alpha+1,\beta+1)}(x)| \sim |P_n^{(\alpha,\beta)}(x)| & \text{if} \quad \alpha, \beta \in \mathbf{R}, \ \vartheta \in J_n \\ \sin\vartheta\, |P_{n-2}^{(\alpha+1,\beta+1)}(x)| \sim \varrho |P_n^{(\alpha,\beta)}| & \text{if} \quad \alpha > -1, \ \beta \in \mathbf{R}; \ \vartheta \equiv \dfrac{\varrho}{n} \leqq \dfrac{\vartheta_{1,n}^{(\alpha,\beta)}}{2} \end{cases}$$

where

$$J_n = \left[\frac{c_1}{n}, \pi-\varepsilon\right] \setminus \left\{ \cup \left\{ \left[\vartheta_{k,n}^{(\alpha,\beta)} - \frac{c_2}{n}, \vartheta_{k,n}^{(\alpha,\beta)} + \frac{c_2}{2}\right] \right\} \cup \left\{ \left[\vartheta_{k,n-2}^{(\alpha+1,\psi)} - \frac{c_3}{n}, \vartheta_{k,n-2}^{(\alpha+1,\psi)} + \frac{c_3}{n}\right] \right\} \right\},$$

considering only the real roots of $P_n(x)$'s (see [1], 6.72). Indeed, by the arguments of [1], 7.32 (3) and (2.3) we get

$$(4.27) \qquad |P_n^{(\alpha,\beta)}(x)| \sim \vartheta^{-\alpha-1/2} n^{-1/2} \quad \text{if} \quad \vartheta \in J_n, \ \alpha, \beta \in \mathbf{R}$$

from where by (4.1) we obtain the first relation. The second one can be proved by (2.1) and [1], 6.21 (1). Further, denoting the roots of $P_{n-2}^{(\alpha+1,\beta+1)}(x)$ $(\alpha, \beta > -2)$ by $y_{k,n-2}^{(\alpha+1,\beta+1)} = \cos \eta_{k,n-2}^{(\alpha+1,\beta+1)}$ we get, using (1.16) and (4.3),

$$(4.28) \quad q_{k,n}^{(\alpha+1,\beta+1)}(x) = 1 - \frac{(\alpha+\beta+2)y_k - \beta + \alpha}{1-y_k^2}(x-y_k) \quad (k = 2, 3, \dots, n-1)$$

from where

$$(4.29)\qquad q_{k,n}^{(\alpha+1,\beta+1)}(x) > \min(-\alpha, -\beta) \quad \text{if} \quad -1 \leqq \alpha, \beta \leqq 0, \; k = 2, 3, \dots, n-1$$

(see [1], 14.5(1) and [3]).

4.4.2. If $-1<\alpha, \beta$ and $[a, b]\subset(-1, 1)$ we have by (1.16) and the method applied in 4.21 that

$$|Q_n^{(\alpha,\beta)}(f;\, x)-f(x)| = \sum_{k=1}^{n} [f(y_k)-f(x)]\, r_k(x) = O(1)\left[\sum_{k=2}^{n-1} \dots + \frac{1}{n^{1+2\alpha}} + \frac{1}{n^{1+2\beta}}\right]$$

which proves (3.1).

4.4.3. Now we prove (3.2). By (1.16) we obtain

(4.30)

$$|Q_n^{(\alpha+1,\beta+1)}(f;\, x)-f(x)| = \left|\sum_{k=2}^{n-1} [f(y_k)-f(x)]\left[\frac{\sin\vartheta P_{n-2}^{(\alpha+1,\beta+1)}(x)}{\sin\eta_k P_{n-2}^{\prime(\alpha+1,\beta+1)}(y_k)(x-y_k)}\right]^2 \cdot\right.$$

$$\cdot\, q_k^{(\alpha+1,\beta+1)}(x) + [f(1)-f(x)]\left[\frac{P_{n-2}^{(\alpha+1,\beta+1)}(x)}{P_{n-2}^{(\alpha+1,\beta+1)}(1)}\right]^2 \frac{1+x}{2} +$$

$$\left. +[f(-1)-f(x)]\left[\frac{P_{n-2}^{(\alpha+1,\beta+1)}(x)}{P_{n-1}^{(\alpha+1,\beta+1)}(-1)}\right]^2 \cdot \frac{1-x}{2}\right| = A.$$

First let $\vartheta\in J_n$. One can choose the sequence $\{z_n=\cos\xi_n\}$ such that $|\vartheta-\xi_n|\leqq \leqq cn^{-1}$ moreover $\xi_n\in J_n$. Then by (4.26), (4.28) and former arguments we have in $cn^{-1}\leqq\vartheta\leqq\pi-\varepsilon$

$$(4.31)\qquad A = O(1)\left\{\sum_{k=2}^{n-1} \omega(|z_n-x_k|)\left[\frac{P_n^{(\alpha,\beta)}(z_n)}{P_n^{\prime(\alpha,\beta)}(x_k)(z_n-x_k)}\right]^2 v_k^{(\alpha,\beta)}(z_n) + \right.$$

$$\left. + \frac{\omega(\sin^2\vartheta)}{(n\vartheta)^{2\alpha+3}} + \frac{\vartheta^{-2\alpha-1}}{n^{2\beta+3}}\right\},$$

where $P_n^{\prime(\alpha,\beta)}(x_k) \overset{\text{def}}{=} \sin\eta_k P_{n-2}^{\prime(\alpha+1,\beta+1)}(y_k)$ if $\min(\alpha, \beta)< -1$.

If we use the argument of 4.2.2.2 considering that now $-2<\alpha, \beta$ we get (3.2) substituting α and β by $\alpha+1$ and $\beta+1$, respectively.

Secondly suppose $\vartheta\notin J_n$. Then we can choose ϑ_n^* such that $\vartheta_n^*\in J_n$ and $|\vartheta^*-\vartheta|\leqq \leqq cn^{-1}$. Using

$$(4.32)\qquad \sin\vartheta\, |P_{n-2}^{(\alpha+1,\psi)}(\vartheta)| = O(1)\sin\vartheta_n^*\, |P_{n-2}^{(\alpha+1,\psi)}(\vartheta_n^*)| \quad (0\leqq\vartheta\leqq\pi)$$

we get the previous estimation. This completes the proof of formula (3.2).

4.5. Proof of Theorem 3.2. We have to use (4.30), (4.13), (4.28) and the corresponding estimations.

4.6. PROOF OF THEOREM 3.3. For the sum in (4.30) we obtain

$$\sum_{k=2}^{n-1}[f(y_k)-f(x)]\frac{1-xy_k}{(n-1)^2}\frac{\sin^2(n-1)\vartheta}{(x-y_k)^2}=$$

$$=O(1)\sum_{k=2}^{n-1}[f(y_k)-f(x)]\frac{1-\cos(\vartheta+\eta_k)}{(n-1)^2}\frac{\sin^2(n-1)\vartheta}{4\sin^2\frac{\vartheta-\eta_k}{2}\sin^2\frac{\vartheta+\eta_k}{2}}=$$

$$=O(1)\left[\sin^2(n-1)\vartheta\sum_{k=2}^{n-1}{}'|f(y_k)-f(x)|\frac{1}{(k-j)^2}+\omega(f;\,|x-y_j|)\right]$$

($\sum'$ means that $k\neq j$).

The remaining parts can be incorporated into the above terms.

4.7. PROOF OF THEOREM 3.4. **4.7.1.** For a) first we suppose

$$\max\left(\frac{1}{n^{1+2\alpha}},\frac{1}{n^{1+2\beta}}\right)=O(1)\sum_{i=1}^{n}\omega\left(\frac{i}{n}\right)\frac{1}{i^2}. \tag{4.33}$$

Now, using (4.30), (4.28) and the argument of 4.3.2 we obtain a).

If (4.33) is not valid then let, e.g. $\alpha\leqq\beta$. By (4.28) we can state with a certain $K=K(\alpha,\beta)$

$$\operatorname{sign} q_k(x)=\psi \quad \text{if} \quad x\in[a,b],\quad 2\leqq k\leqq Kn \quad (n\geqq n_0). \tag{4.34}$$

By $s=\max(b+\varepsilon, y_{[Kn]})$ let

$$\begin{cases} g_n(1)=\psi \\ g_n(1-s)=g_n(x)=0 \quad \text{if} \quad x\leqq 1-s \quad (n\geqq n_0). \end{cases}$$

By $\lambda_n(x^*)\overset{\text{def}}{=}n^{-2\alpha-1}$ and $\tilde f(x)=\sum_{i=1}^{\infty}e_{n_i}g_{n_i}(x)$ we get the desired result for a certain $\{n_i\}$ (see Theorem 4.1). The case $\alpha>\beta$ can be treated similarly.

4.7.2. To prove b) let

$$\begin{cases} g_n(y_k)=\operatorname{sign} q_k(x^*)\,\omega(y_k-x^*) \quad \text{if} \quad 1\leqq k\leqq j-1, \\ g_n(x^*)=g_n(y_k)=0 \quad \text{if} \quad k\geqq j \end{cases}$$

and linear in the subintervals. For suitable $\{n_i\}$

$$Q_n^{(\alpha,\beta)}(g_n;\,x^*)\geqq c[\omega(\sin^2\vartheta^*)(n\vartheta^*)^{-2\alpha-1}+(\vartheta^*)^{-2\alpha+1}n^{-2\beta-1}]\overset{\text{def}}{=}\lambda_n(x^*)$$

$$(n=n_1,n_2,\ldots).$$

With $\tilde f(x)=Q\sum_{i=1}^{\infty}e_{n_i}g_{n_i}(x)$ $(e_n\searrow 0)$ we obtain b) by usual argument.

4.7.3. At c) first suppose that for certain $\{e_n\}$ the sum is negligible. By (4.28) we can state with a certain $K_1=K_1(\alpha,\beta)$ and z_n $(1-z_n\sim n^{-2})$, $\operatorname{sign} q_k(z_n)=\psi$ if $K_1n\leqq k\leqq n-1$ $(n\geqq n_0)$. The remaining part is similar to 4.6.1. If the sum is not negligible we can apply (4.26), (4.28) and the ideas 4.3.3.

4.7.4. The statement d) can be proved by the previous arguments.

4.8. PROOF OF THEOREM 3.5. **4.8.1.** First let $-1 \leqq \alpha, \beta \leqq 0$. Define

$$\bar{f}(x;\, x_0) = \begin{cases} f(x) & \text{if } f(x) \geqq f(x_0) \\ 2f(x_0)-f(x) & \text{if } f(x) < f(x_0) \end{cases}$$

and

$$\bar{\bar{f}}(x;\, x_0) = \bar{f}(x;\, x_0) - f(x_0).$$

(See Figure 1.)

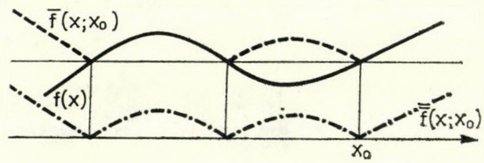

Fig. 1

If $f \in C_1(\omega)$ then $\bar{f} \in C_1(\omega)$ and $\bar{\bar{f}} \in C_1(\omega)$, too. Further by (4.29) $r_{k,n}(x) \geqq 0$ $(k=1, 2, \ldots, n)$, so

$$\|Q_n^{(\alpha+1,\beta+1)}(f;\, x)-f(x)\| = |Q_n^{(\alpha+1,\beta+1)}(f;\, z_n)| \leqq$$

$$\leqq Q_n^{(\alpha+1,\beta+1)}(\bar{f}(\cdot\,;\, z_n);\, z_n) - \bar{f}(z_n;\, z_n) = Q_n^{(\alpha+1,\beta+1)}(\bar{\bar{f}}(\cdot\,;\, z_n);\, z_n) - \bar{\bar{f}}(z_n;\, z_n) =$$

$$= Q_n^{(\alpha+1,\beta+1)}(\bar{\bar{f}}(\cdot\,;\, z_n);\, z_n) \leqq Q_n^{(\alpha+1,\beta+1)}(\omega(\cdot\,;\, z_n);\, z_n),$$

where $\omega(t;\, z_n) = \omega(|t-z_n|)$. I.e. we obtained

$$\sup_{f \in C_1(\omega)} \|Q_n^{(\alpha+1,\beta+1)}(f;\, x)-f(x)\| =$$

$$= \sup_{x \in [-1,1]} Q_n^{(\alpha+1,\beta+1)}(\omega(\cdot\, x);\, x) \stackrel{\text{def}}{=} R_n^{(\alpha+1,\beta+1)}(\omega).$$

By (1.16) it is easy to see that with $|y| \geqq 1-cn^{-2}$

$$Q_n^{(\alpha+1,\beta+1)}(\omega(\cdot\,;\, y);\, y) = O(1) \sup_{-1+\frac{c}{n^2} \leqq x \leqq 1-\frac{c}{n^2}} Q_n^{(\alpha+1,\beta+1)}(\omega(\cdot\,;\, x);\, x),$$

so one can choose the sequence $p_n = \cos \psi_n$ such that for arbitrary fixed $c>0$ $cn^{-1} \leqq \psi_n \leqq \pi - cn^{-1}$, moreover,

$$Q_n^{(\alpha+1,\beta+1)}(\omega(\cdot\,;\, p_n);\, p_n) \sim R_n.$$

By (1.16) we obtain

$$(4.35) \qquad Q_n^{(\alpha+1,\beta+1)}(\omega(\cdot\,;\, p_n);\, p_n) = \sum_{k=1}^{n} \omega(|y_k - p_n|)\, r_k(p_n),$$

where by (4.1) and (4.2) we can suppose

$$(4.36) \qquad |P_{n-2}^{(\alpha+1,\beta+1)}(p_n)| \sim \psi_n^{-\alpha-3/2} n^{-1/2} \quad (p_n \geqq 0).$$

The corresponding estimation holds for $p_n \leqq 0$.

4.8.2. Now we prove

LEMMA 4.1. *We have for* $-1 \leqq \alpha, \beta \leqq 0$

$$Q_n^{(\alpha+1,\beta+1)}(\omega(\cdot\,;\,p_n)) \sim \sum_{k=2}^{n-1} \omega(|y_k - p_n|) r_k(p_n). \tag{4.37}$$

Denote

$$\sum_{k=1}^{n} = \sum_{k=2}^{n-1} + \sum_{k=1} + \sum_{k=n} \equiv I_1 + I_2 + I_3.$$

First we suppose $\alpha, \beta > -1$.

a) If $cn^{-1} \leqq \psi_n \leqq \pi/2$ we get

$$I_1 \geqq c_1 \frac{\psi_n^{-2\alpha-3+2}}{n} \sum_{k=2}^{[n/2]}{}' \frac{\omega(|y_2-p_n|)}{(y_k-p_n)^2} \frac{k^{2\alpha+5-2}}{n^{2\alpha+6-2}} \geqq$$

$$\geqq \frac{\psi_n^{-2\alpha-1}}{n} \sum_{k=2}^{[j/2]} \frac{\omega(|y_k-p_n|)}{|k+j|^2 |k-j|^2} \frac{k^{2\alpha+3}}{n^{2\alpha}} \sim \frac{\omega(\sin^2\psi_n)}{(n\psi_n)^{2\alpha+1}} \frac{j^{2\alpha+4}}{j^4} \sim$$

$$\sim \frac{\omega(\sin^2\psi_n)}{n\psi_n} \geqq c_2 \frac{\omega(\sin^2\psi_n)}{(n\psi_n)^{2\alpha+3}} \sim I_2$$

(see, e.g., (4.30)).

b) By similar computation we have for $\pi/2 \leqq \psi_n \leqq \pi - cn^{-1}$ that $I_1 \geqq c_3 I_3$. (If $j<4$ the argument will be trivial.)

Now we investigate the remaining part.

a) Let $cn^{-1} \leqq \psi_n \leqq \pi/2$. We obtain

$$I_1 \sim \frac{\psi_n^{-2\alpha-1}}{n} \left[\sum_{k=2}^{[3n/4]}{}' \frac{\omega(|y_k-p_n|)}{|y_k-p_n|} \frac{k^{2\alpha+3}}{n^{2\alpha+4}} \left(1+|y_k-p_n|\frac{n^2}{k^2}\right) \frac{1}{|y_k-p_n|} + \sum_{k=2}^{[n/4]} \frac{k^{2\beta+3}}{n^{2\beta+4}} \frac{n^2}{k^2}\right] \geqq$$

$$\geqq c_1 \frac{\psi_n^{-2\alpha+1}}{n^{2\alpha+3}} \left[\sum_{k=1}^{n}{}' \frac{k^{2\alpha+3}}{|k+j||k-j|} + \sum_{k=1}^{n} k^{2\alpha+1} + 1\right] \geqq c_2 \frac{\psi_n^{-2\alpha+1}}{n} \geqq c_2 \frac{\psi_n^{-2\alpha-1}}{n^{2\beta+3}} \sim I_3.$$

b) If $\pi/2 \leqq \psi_n \leqq \pi - cn^{-1}$, by similar estimations $I_1 \geqq c_4 I_2$.

Remarking that similar arguments hold for the remaining α's and β's, we obtain (4.36).

4.8.3. By (4.3.6), (4.3.7) and (4.3.1)

$$R_n \sim Q_n^{(\alpha+1,\beta+1)}(\omega(;\,p_n);\,p_n) \sim \sum_{k=2}^{n-1} \omega(|y_k-p_n|) \left[\frac{P_n^{(\alpha,\beta)}(z_n)}{P_n'^{(\alpha,\beta)}(x_k)(y_k-p_n)}\right]^2 \cdot$$

$$\cdot v_k^{(\alpha,\beta)}(z_n) \sim \sum_{k=2}^{n-1} \omega(|x_k-z_n|) h_k^{(\alpha,\beta)}(z_n) + \omega\left(\frac{1}{n}\right) =$$

$$= O(1) \left[\sup_{f\in C_1(\omega)} \|H_n^{(\alpha,\beta)}(f;\,x) - f(x)\| + \omega\left(\frac{1}{n}\right)\right] =$$

$$= O(1) \sup_{f\in C_1(\omega)} \|H_n^{(\alpha,\beta)}(f;\,x) - f(x)\| \overset{\text{def}}{=} O(1) S_n$$

for a certain $\{z_n = \cos\xi_n\}$ where $|\psi_n - \xi_n| = O(n^{-1})$.

4.8.4. By the corresponding arguments we get $S_n = O(1) R_n$ which gives our theorem for $-1 \leqq \alpha, \beta \leqq 0$.

4.8.5. When $\alpha \geqq \beta$ and $\alpha > 0$ we prove a statement interesting in itself.

LEMMA 4.2. *If* $\alpha \geqq \beta \geqq -1$, $\alpha > 0$ *we have for arbitrary* $\omega(t)$

$$R_n^{(\alpha+1, \beta+1)}(\omega) \sim S_n^{(\alpha, \beta)}(\omega) \sim n^{2\alpha}.$$

To prove this, first we investigate S_n. We prove that $S_n = O(n^{2\alpha})$. Indeed, we have for $f \in C_1(\omega)$ by the previous arguments

$$\|H_n^{(\alpha,\beta)}(f; x) - f(x)\| = |H_n^{(\alpha,\beta)}(f; p_n) - f(p_n)| =$$

$$= O(1) \|P_n^{(\alpha,\beta)}(p_n)\|^2 \left[\sum_{k=1}^{[n/2]}{}' \left(\frac{\omega(|x - x_2|)}{|x - x_k|} \cdot \frac{k^{2\alpha+1}}{n^{2\alpha+2}} + \frac{k^{2\beta+1}}{n^{2\beta+2}} \right) \right].$$

Here

$$[\ldots] \geqq c_1 \sum_{k=1}^{[n/2]} (k^{2\alpha+1} n^{-2\alpha-2} + k^{2\beta+1} n^{-2\beta-2}) \sim 1$$

and

$$[\ldots] \leqq c_2 \left(\sum_{k=1}^{[j/2]} + \sum_{k=[j/2]+1}^{2j} + \sum_{k=2j+1}^{n} \right) \frac{k^{2\alpha+1}}{n^{2\alpha+2} |k+j|\, |k-j|} + c_2 \leqq c_3,$$

so we have

$$\|H_n^{(\alpha,\beta)}(f; x) - f(x)\| = O(1) \|P_n^{(\alpha,\beta)}(p_n)\|^2 = O(n^{2\alpha}) \quad \text{if} \quad \psi_n \sim n^{-1}.$$

Further we get by $f_3(x) = \omega(z_n - x)$ $(x \leqq z_n)$ and $f_3(x) = 0$ $(x > z_n)$, with $\{z_n = \cos \xi_n,\ \xi_n \sim n^{-1}\}$,

$$H_n^{(\alpha,\beta)}(f_3;\ z_n) - f_3(z_n) \sim n^{2\alpha}$$

because for the difference

$$\sum_{i=1}^{n} \omega\left(\frac{i^2}{n^2}\right) i^{2\alpha-1} < \sum_{i=1}^{n} i^{2\alpha-1} \sim n^2$$

and

$$\sum_{i=1}^{n} \omega\left(\frac{i^2}{n^2}\right) i^{2\alpha-1} > c \sum_{i=1}^{n} \frac{i^{2\alpha+1}}{n^2} \sim n^{2\alpha}.$$

So we proved $S_n \sim n^{2\alpha}$. By similar arguments we obtain $R_n \sim n^{2\alpha}$, too. This completes the proof of (3.8).

References

[1] G. SZEGŐ, *Orthogonal polynomials*, AMS Coll. Publ. Vol. XXIII (New York, 1959).
[2] G. GRÜNWALD, On the theory of interpolation, *Acta Math.*, **75** (1942), 219—245.
[3] P. SZÁSZ, On quasi-Hermite—Fejér interpolation, *Acta Math. Acad. Sci. Hungar.*, **10** (1959), 413—439.
[4] E. EGERVÁRY, P. TURÁN, Notes on interpolation. V, *Acta Math. Acad. Sci. Hungar.*, **9** (1958), 259—267.
[5] L. P. POVCHUN, A. A. PRIVALOV, On an interpolatory process defined by Egerváry and Turán, *Izv. Vyss. Ucebn. Zaved. Matematika*, **8 (147)** (1974), 82—87 (Russian).

[6] R. B. SAXENA, K. K. MATHUR, The rapidity of the convergence of quasi-Hermite—Fejér interpolation polynomials, *Acta Math. Acad. Sci. Hungar.*, **29** (1976), 343—347.
[7] G. FREUD, A. SHARMA, Some good sequences of interpolatory polynomials, *Can. J. Math.*, **26 (1)** (1974), 233—246.
[8] P. VÉRTESI, Hermite—Fejér interpolation based on the roots of Jacobi polynomials, *Studia Sci. Math. Hungar.*, **5** (1970), 395—399.
[9] P. VÉRTESI, Notes on the Hermite—Fejér interpolation based on the Jacobi abscissas, *Acta Math. Acad. Sci. Hungar.*, **24** (1973), 233—239.
[10] P. VÉRTESI, On certain linear operators. VIII, *Acta Math. Acad. Sci. Hungar.*, **25** (1974), 171—187 and 449—450.
[11] P. TURÁN, On some unsolved problems in approximation theory, *Mat. Lapok*, **25** (1974), 21—75. (Hungarian).
[12] D. L. BERMAN, On interpolatory processes based for the extended Jacobi abscissas, *Izv. Vyss. Ucebn. Zaved. Matematika*, **10** (137) (1973), 14—22 (Russian).
[13] P. SZÁSZ, On a sum concerning the zeros of the Jacobi polynomials with application to the theory of generalized quasi-step parabolas, *Monatsh. für Math.*, **68** (1964), 167—174.
[14] J. SÁNTHA, Convergence theorems of quasi Hermite—Fejér interpolation, *Publ. Math. Debrecen*, **22** (1975), 23—29.
]15] T. M. MILLS, A. K. VARMA, On a theorem of E. Egerváry and P. Turán on the stability of interpolation (preprint).

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MATHEMATICAL INSTITUTE OF THE
HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST, REÁLTANODA U. 13—15.

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A THEOREM ON RELATIVE UNIVERSAL SPACES WITH GIVEN WEIGHT AND DIMENSION

By

V. V. FILIPPOV (Moscow) and L. ÚRY (Budapest)

Introduction

Let $\mathfrak{F}$ be any class of topological spaces. In this paper let $n\in\{0, 1, \dots\}$ or $n=\infty$, and let $\tau\geqq\omega_0$ be any cardinal. Here ω_0 denotes the first infinite cardinal. "dim" is the usual covering dimension. Let $F\in\mathfrak{F}$ be such that $\dim F\leqq n$, the weight of $F(=wF)\leqq\tau$, $\imath: F\to X$ a closed embedding, i.e. an embedding such that $\imath(F)$ is closed in X.

Definition. $(X, F, \imath)$ is called an (F, τ, n) universal embedding space with respect to $\mathfrak{F}$ iff:

(i) $X\in\mathfrak{F}$, $\dim X\leqq n$, $wX\leqq\tau$,

(ii) for every $Y\in\mathfrak{F}$ and for every closed embedding $\varkappa$ from F into Y such that $\dim Y\leqq n$, $wY\in\tau$ there is an embedding $\gamma: Y\to X$ for which the diagram

$$\begin{array}{ccc} F & \xrightarrow{\ \imath\ } & X \\ & \searrow^{\varkappa} \quad \nearrow_{\gamma} & \\ & Y & \end{array}$$

commutes, i.e. $\imath=\gamma\circ\varkappa$. If $n=\infty$ we simply write (F, τ) instead of (F, τ, ∞).

Denote by $\mathfrak{M}$ and $\mathfrak{C}$ the class of metrizable and the class of compact T_2 topological spaces, resp. Let us suppose, that $\mathfrak{F}=\mathfrak{M}$ or $\mathfrak{C}$. If X and Y are homeomorphic we shall write $X\approx Y$. $|X|$ denotes the cardinality of X. Cardinals are identified with the initial ordinals.

Main theorems

We can summarize our results as follows:

Theorem C. *For every F, n and τ there is an (F, τ, n)-universal embedding space with respect to $\mathfrak{F}$.*

This theorem is obviously a consequence of the following two theorems:

Theorem A. *For every F and τ there is an (F, τ)-universal embedding space with respect to $\mathfrak{F}$.*

Theorem B. *If there is an (F, τ)-universal embedding space with respect to $\mathfrak{F}$, then for every n there is an (F, τ, n)-universal embedding space too.*

Proof of theorem A

(i) *Metric case.* Let F be a metrizable space with weight $\leqq\tau$. Then there is a base B of F such that $B=\bigcup_{i=1}^{\infty} B_i$, where the B_i's are discrete, disjoint families and $|B_i|\leqq\tau$. For every $V\in B$ let $f_V\colon F\to[0,1]$ defined by

$$f_V(x)=\min\left\{\varrho(x, F\setminus V), \frac{1}{2}\right\}$$

where ϱ is any fixed metric on F inducing its topology. For $V\in B_i$ and $k=1,2,\ldots$ let us define a function $t'_{Vk}\colon F\to[0,1]$ such that

$$t'_{Vk}(x)=\frac{1}{2^{k+i}}f_V(x).$$

Let $M_1=\{(V,k)\,|\,V\in B, k\in N\}$, let M_2 be any set such that $M_1\cap M_2=\emptyset$, $|M_2|=\tau$, and let $M=M_1\cup M_2$. For $m\in M_2$ take $t'_m\equiv 0$.

Finally let $\iota\colon F\to R^\tau$ be given by $\iota(x)=\{t'_m(x)\colon m\in M\}$, where R^τ is the τ-dimensional generalized Hilbert-space. Let $L=\{x\in R^\tau\,|\,\forall m\in M_2,\ x_m=0\}$ and take $X(F,\tau)=R^\tau\setminus(L\setminus\iota(F))$.

The map ι is an embedding (see [11] p. 127) and clearly $\iota(F)$ is a closed subspace of $X(F,\tau)$.

Let $\varkappa\colon F\to Y$ be an embedding such that $wY\leqq\tau$, $\varkappa(F)$ is closed in Y. We may suppose, that F is a closed subspace of Y. Let $C=\bigcup_{i=1}^{\infty} C_i$ be a base for Y such that $|C|\leqq\tau$ and the C_i's are disjoint and discrete. Since Y is collectionwise normal, there is an open discrete family $\{G(V)\,|\,V\in B_i\}$ such that $G(V)\supset\overline{V}$. Let $G_k(V)=G(V)\cap U$ if there is a $U\in C_k$ containing $\overline{V}$, and let $G_k(V)=G(V)$ otherwise.

Applying Tietze's theorem for every $V\in B$, $k=1,2,\ldots$ there is a continuous function h_{Vk} such that $h_{Vk}\colon Y\to[0,1]$, $h_{Vk}|_F=f_V$, $h_{Vk}(\operatorname{ext} G_k(V))=0$.

For $V\in B_i$ take

$$t_{Vk}=\frac{1}{2^{k+i}}\cdot h_{Vk}.$$

Obviously the family $\{t_{Vk}^{-1}((0,1])\,|\,V\in B_i,\ k=1,2,\ldots\}$ is σ-discrete, and contains a subfamily being base at the point p in the space Y for every $p\in F$. Let $C^*=\{U\,|\,U\in C, U\cap F=\emptyset\}$. For $U\in C^*$ let $g_U\colon Y\to[0,1]$ be given by $g_U(y)=\min\{\varrho_1(y, Y\setminus U), 1\}$, where ϱ_1 is any metric on Y, inducing the topology of Y. Let $t_U(y)=2^{-i}g_U(y)$ if $U\in C_i\cap C^*$. Since $|C^*|\leqq\tau$ there is an injection $\sigma\colon C^*\to M_2$. Take $t_m=0$ for $m\in M_2\setminus\sigma(C^*)$ and let $t_{\sigma(u)}=t_U$ for $U\in C^*$. Let $\gamma\colon Y\to R^\tau$ be given by $\gamma(y)=\{t_m(y)\colon m\in M\}$. γ is an embedding. Since for every $y\notin F$ there is a $U\in C^*$ such that $t_U(y)\neq 0$ thus if $y\notin F$ then $\gamma(y)\notin L$. Hence $\gamma\colon Y\to X(F,\tau)$ and clearly $\gamma\circ\varkappa=\iota$.

(ii) *Compact case.* Let B be an open base for F such that $|B|\leqq\tau$. Let

$$M_1=\{(U,V,\omega)\,|\,U\supset\overline{V};\ U,V\in B,\ \omega\leqq\tau \text{ any ordinal}\}.$$

For $m=(U, V, \omega)\in M_1$ let $f_m: F\to I$ be a continuous function such that $f(F\setminus U)=0$ and $f(\overline{V})=1$. ($I=[0, 1]$, and F is a compact T_2 space so it is normal.) Let M_2 be any set such that $M_1\cap M_2=\emptyset$, $|M_2|=\tau$ and take $M=M_1\cup M_2$. For $m\in M_2$ let $f_m\equiv 0$. Finally let $\iota: F\to I^{M_1\cup M_2}\approx I^\tau$ be given by $\iota(x)=\{f_m(x)\,|\,m\in M\}$.

Since $\{f_m\}_{m\in M}$ separates the points and the closed sets, ι is an embedding. F is compact, hence $\iota(F)$ is a closed set in I^τ.

Let $\varkappa: F\to Y$ be a closed embedding. We may suppose that F is a closed subspace of Y. Let $C=\{G_\alpha\}_{\alpha\leqq\tau}$ be an open base for Y. For $m=(U, V, \omega)\in M_1$ let $U_m=G_\omega$ if $G_\omega\supset\overline{U}$, otherwise let $U_m=Y$. Let $g_m: Y\setminus U_m\to I$ be given by $g_m(x)=0$. f_m and g_m are compatible and thus by Tietze's theorem there is a $t_m: Y\to X$ such that $t_m|_F=f_m$, $t_m(Y\setminus U_m)=0$. Let

$$C^* = \{(G, H)|G\supset\overline{H};\ G\cap F=\emptyset;\ G, H\in C\}.$$

Since $|C^*|\leqq\tau$, we may suppose that $C^*\subset M_2$. Take $t_m\equiv 0$ for $m\in M_2\setminus C^*$, and let $t_m: Y\to I$ be given by $t_m(Y\setminus G)=0$, $t_m(\overline{H})=1$ if $m\in C^*$.

Now let $\gamma: Y\to I^{M_1\cup M_2}$ be defined by $\gamma(y)=\{t_m(y): m\in M\}$.

Since $\{t_m\}_{m\in M}$ separates the points and the closed sets, γ is an embedding, and clearly $\gamma\circ\varkappa=\iota$.

Two lemmas

Let $\{X_\alpha\}_{\alpha\in I}$ be an almost disjoint family of normal spaces, i.e. $X_\alpha\cap X_\beta=F$ if $\alpha\neq\beta$, and for every $\alpha\in I$ F is a closed subspace of X_α. F is called the kernel of this family. Let $X^0=\bigcup_{\alpha\in I} X_\alpha$ and let $p_\alpha: X_\alpha\to X^0$ be the natural injection. Let the topology of X^0 be induced by $\{p_\alpha\}_{\alpha\in I}$.

The following statements are evident:

(1) U is open (closed) in X^0 iff for every $\alpha\in I$ $p_\alpha^{-1}(U)=X_\alpha\cap U$ is open (closed) in X_α.

(2) p_α is an embedding, X_α is closed in X^0.

(3) $\nu: F\hookrightarrow X^0$ is an embedding.

Lemma 1. *Let H_1, H_2, H be closed subspaces of a normal space Y, $H_1\cap H_2=\emptyset$; U_1, U_2 open sets in H, $U_i\supset H\cap H_i$ $(i=1, 2)$ and $\overline{U}_1\cup\overline{U}_2=\emptyset$. Then there are two open sets V_1, V_2 in Y for which $V_i\supset H_i$, $V_i\cap H=U_i$ $(i=1, 2)$ and $V_1\cap V_2=\emptyset$.*

Proof. For $i=1, 2$, $U_i\cup H_i$ is open in $Z=H_1\cup H_2\cup H$ therefore there are two open sets S_1, S_2 in Y such that $S_i\cap Z=U_i\cup H_i$. Since $(\overline{U}_1\cup H_1)\cap\cap(\overline{U}_2\cup H_2)=\emptyset$, there are open sets G_1, G_2 in Y such that $G_i\supset\overline{U}_i\cup H_i$ and $G_1\cap G_2=\emptyset$. Take $V_i=G_i\cap S_i$ $(i=1, 2)$. It is evident that $V_i\supset H_i$, $V_i\cap H=U_i$ and $V_1\cap V_2=\emptyset$. Q.E.D.

(4) X^0 is normal if the X_α's are normal spaces.

In order to prove this let A_1 and A_2 be disjoint closed subspaces of X^0. Let $F_i=F\cap A_i$ $(i=1, 2)$. F is normal, hence there are U_1, U_2 such that $F_i\subset U_i$, $\overline{U}_1\cap\overline{U}_2=\emptyset$ and the U_i's are open in F. Applying Lemma 1 for the sets $X_\alpha\cap A_1$, $X_\alpha\cap A_2$, F, U_1, U_2 let $S_i^{(\alpha)}=V_i$ $(i=1, 2, \alpha\in I)$. It is clear, that if $G_i=\bigcup_{\alpha\in I} S_i^{(\alpha)}$ then the G_i's are open in X^0, $G_1\cap G_2=\emptyset$ and $G_i\supset A_i$ $(i=1, 2)$.

Lemma 2. *If for every $\alpha\in I$, $\dim X_\alpha\leqq n$ then $\dim X^0\leqq n$.*

PROOF. Let A be any closed subset of X^0 and let $f: A \to S^n$ be any continuous map. The map $f_1 = f|_{A \cap F}$ may be extended to a continuous map $f_2: F \to S^n$, since $A \cap F$ is closed in F and $\dim F \leqq n$ (see [11], p. 270, Theorem 14′). f_2 and f are compatible, and thus there is a continuous map $\tilde{f}: A \cup F \to S^n$ in such a way that $\tilde{f}|_A = f$. Applying Tietze's theorem there is a closed neighbourhood of $A \cup F$ denoted by $\overline{U}$, and a continuous map $\hat{f}: \overline{U} \to S^n$ such that $\hat{f}|_{A \cup F} = \tilde{f}$.

Consider the maps $f_\alpha = \hat{f}|_{X_\alpha \cap \overline{U}}: X_\alpha \cap \overline{U} \to S^n$. Since $X_\alpha \cap \overline{U}$ is closed in X_α, and $\dim X_\alpha \leqq n$ there is a continuous map $g_\alpha: X_\alpha \to S^n$ such that $g_\alpha|_{X_\alpha \cap \overline{U}} = f_\alpha$. It is evident, that $g = \bigcup g_\alpha: X^0 \to S^n$ is continuous and $g|_A = f$. Thus $\dim X^0 \leqq n$. Q.E.D.

Proof of Theorem B

Let $\{X_\alpha\}_{\alpha \in I}$ be an almost disjoint family of normal spaces with the kernel F satisfying the conditions:

(a) $wX_\alpha \leqq \tau$, $\dim X_\alpha \leqq n$ for $\alpha \in I$.

(b) $\{X_\alpha\}_{\alpha \in I}$ is complete in the following sense: For any space Z with $wZ \leqq \tau$, $\dim Z \leqq n$ and for any closed embedding $k: F \to Z$ there exists an $\alpha \in I$ and a homeomorphism $\psi: X_\alpha \to Z$ such that the diagram

$$\begin{array}{ccc} F & \subset & X_\alpha \\ & {\scriptstyle k}\searrow \quad \swarrow{\scriptstyle \psi} & \\ & Z & \end{array}$$

commutes. Obviously, there exists such a family.

Let X^0 be the same as in the Lemmas. For every $\alpha \in I$ let g_α be an embedding such that the diagramm

$$\begin{array}{ccc} F & \subset & X_\alpha \\ & {\scriptstyle \iota}\searrow \quad \swarrow{\scriptstyle g_\alpha} & \\ & X(F, \tau) & \end{array}$$

commutes, where $X(F, \tau)$ with the map ι is an (F, τ)-universal embedding space with respect to $\mathfrak{F}$. Let the map $f^0: X^0 \to X(F, \tau)$ be given by $f^0(x) = g_\alpha(x)$ if $x \in X_\alpha$. Obviously f^0 is well defined, $f^0|_F = \iota$ and it is continuous.

(i) *The metric case.* Applying the factorization theorem (see [11], p. 388) for $f^0: X^0 \to X(F, \tau)$ there are continuous maps $g: X^0 \to \Sigma$, $h: \Sigma \to X(F, \tau)$ in such a way, that $\dim \Sigma \leqq \dim X^0 \leqq n$ and $w\Sigma \leqq wX(F, \tau) \leqq \tau$. Furthermore $X(F, \tau)$ is metrizable hence $\Sigma \in \mathfrak{M}$. Let $\tilde{\iota} = g|_F$. The map g is topological on X_α since $f = h \circ g$ is topological on X_α. Thus Σ with $\tilde{\iota}$ is an (F, τ, n)-universal embedding space with respect to $\mathfrak{M}$ if only $\tilde{\iota}$ is a closed embedding. This proves the following

LEMMA 3. *Let X, Y, Z be Hausdorff spaces and let $f: X \to Y$, $g: Y \to Z$ be continuous maps. If $g \circ f: X \to Z$ is a closed embedding, then so is f, too.*

PROOF. f is obviously an embedding. Suppose that $f(X)$ is not closed in Y, and let $y \in \overline{f(X)} \setminus f(X)$. Then $g(y) \in g\overline{f(X)} \subset \overline{gf(X)} = gf(X)$ and therefore there exists a $y_1 \in f(X)$ such that $g(y_1) = g(y)$. Let G and G_1 be disjoint neighbourhoods of y and y_1, resp. Then $g(G \cap f(X))$ and $g(G_1 \cap f(X))$ are open and disjoint in $gf(X)$. Thus there exists a T open in Z such that $T \cap gf(X) = g(G_1 \cap f(X))$.

Since $g(y) \in T$ there is an open set $G' \subset G$ such that $y \in G'$ and $g(G') \subset T$. Then $g(G' \cap f(X)) \cap g(G_1 \cap f(X)) \neq \emptyset$, while the sets $g(G \cap f(X))$, $g(G_1 \cap f(X))$ are disjoint. This is a contradiction, and thus $f(X)$ is closed in Y.

(ii) *The compact case*. In the compact case let $X' = \beta X^0$, and let f' be any extension of f^0. Applying the factorization theorem (see [11], p. 304) for $f': X' \to X(F, \tau)$ there are continuous maps $g: X' \to \Sigma$, $h: \Sigma \to X(F, \tau)$ such that $f' = h \circ g$. Similarly to the metric case let $\tilde{\imath} = g|_F$. It is evident that $\tilde{\imath}$ is a closed embedding. The map g is topological on X_α since $f' = h \circ g$ is topological on X_α. Thus Σ with $\tilde{\imath}$ is an (F, τ, n)-universal embedding space with respect to $\mathfrak{C}$. Q.E.D.

References

[1] G. Nöbeling, Über eine n-dimensionale Universalmenge im R^{2n+1}, *Math. Ann.*, **104** (1930), 71—80.

[2] А. В. Зарелуа, Универсальный бикомпакт данного веса и данной размерности, *ДАН СССР*, **154** (1964).

[3] J. Nagata, On universal n-dimensional set for metric spaces, *Fund. Math.*, **48** (1959), 1—14.

[4] Б. А. Пасынков, Об универсальных бикомпактах и метрических пространствах данной размерности, *Fund. Math.*, **60** (1967), 285—308.

[5] Б. А. Пасынков, Об универсальных бикомпактах данного веса и данной размерности, *ДАН СССР*, **154** (1964), 1042—1043.

[6] Е. Г. Скляренко, О вдожение нормальных пространств в бикомпакты того же веса и той же размерности, *ДАН СССР*, **123** (1958), 36—39.

[7] Ю. М. Смирнов, Об универсальных пространствах для некоторых классов пространств, *ИАН СССР сер. матем.*, **23** (1959), 185—196.

[8] K. Morita, Normal families and dimension theory in metric spaces, *Math. Ann.*, **128** (1954), 350—362.

[9] W. Hurewicz, Über das Verhältnis separabler Räume zu kompakten Räumen, *Proc. Konink. Akad. Amsterdam*, ser A, **30** (1927), 425—430.

[10] L. Úry, On relative universal embedding spaces, *Acta Math. Acad. Sci. Hungar.*, **28** (1976).

[11] П. С. Александров—Б. А. Пасынков, *Введение в теорию размерности*, Наука (Москва, 1973).

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF GEOMETRY
H—1088 BUDAPEST, MÚZEUM KRT. 6—8.

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ON A SATURATION PROBLEM

By

G. SOMORJAI (Budapest)[1]

Let $f(x)\in C[0,1]$, $\alpha>0$ a real number, and consider the positive linear operators

(1) $$L_n(f,x)=\frac{\sum_{k=0}^{n} f\left(\frac{1}{n}\right)\Big/\left|x-\frac{k}{n}\right|^{\alpha}}{\sum_{k=0}^{n} 1\Big/\left(x-\frac{k}{n}\right)^{\alpha}}\quad (0\leqq x\leqq 1,\ n=1,2,\ldots).$$

Especially, when α is a natural number, (1) is a rational function of degree at most $n\alpha$. In the latter case, these operators were discovered and investigated by J. BALÁZS, and later by J. SZABADOS [2], who also raised the saturation problem (see [1]). In what follows, we give a complete description of the order and class of saturation when $\alpha>2$. The case $0<\alpha\leqq 2$ remains open. $\|\cdot\|$ will always denote the supremum norm in the interval $[0,1]$, and the "o", "O" notations refer to $n\to\infty$.

THEOREM. *If* $\alpha>2$ *is an arbitrary real number then for the operator* (1) *we have the following statements:*

(i) $\|L_n(f)-f\|=o(1/n)$ *iff* $f=$const,

(ii) $\|L_n(f)-f\|=O(1/n)$ *iff* $f\in$ Lip 1.

The direct part of these statements is easily seen. Namely, if $f=$const. then $L_n(f,x)\equiv f(x)$. On the other hand, if $f\in$ Lip 1 then we get from (1)

$$\|f-L_n(f)\|\leqq\frac{\sum_{k=0}^{n}\left|f(x)-f\left(\frac{k}{n}\right)\right|\Big/\left|x-\frac{k}{n}\right|^{\alpha}}{\sum_{k=0}^{n}1\Big/\left|x-\frac{k}{n}\right|^{\alpha}}=\frac{\sum_{k=0}^{n}1\Big/\left|x-\frac{k}{n}\right|^{\alpha-1}}{\sum_{k=0}^{n}1\Big/\left|x-\frac{k}{n}\right|^{\alpha}}=O\left(\frac{1}{n}\right).$$

For the proof of the converse statements we need several lemmas.

LEMMA 1. *If* $\alpha>2$ *then there exists a* $\varrho=\varrho(\alpha)\in(0,1/2)$ *such that for all* $n=1,2,\ldots$, $\left|x-\frac{k}{n}\right|\leqq\varrho/n$ *implies*

(2) $$\sum_{\substack{k'=0\\k'\neq k}}^{n}1\Big/\left|x-\frac{k'}{n}\right|^{\alpha-1}<\frac{1}{20\left|x-\frac{k}{n}\right|^{\alpha-1}}.$$

(Of course, (2) holds with $\alpha-1$ replaced by α, too.)

[1] *Editorial remark.* This paper has been completed by J. Szabados after the premature death of the author on January 15, 1978.

PROOF. The left hand side of (2) is not greater than

$$(2n)^{\alpha-1}+2\sum_{j=1}^{n}\left(\frac{n}{j}\right)^{\alpha-1} \leqq c_1(\alpha)\,n^{\alpha-1},$$

thus it suffices to choose ϱ such that $c_1(\alpha)\,\varrho^{\alpha-1}<1/20$. Q.E.D.

Now let

$$M_f(x) = \limsup_{x'\to x}\left|\frac{f(x')-f(x)}{x'-x}\right| \quad (0 \leqq M_f(x) \leqq +\infty) \tag{3}$$

for any $f\in C[0, 1]$.

LEMMA 2. (i) $M_f(x)\equiv 0$ *iff* $f(x)=\text{const}$,

(ii) $M_f=\sup_{0\leqq x\leqq 1} M_f(x)<+\infty$ *iff* $f\in \text{Lip}\, 1$.

PROOF. (i) and $f\in \text{Lip}\, 1\Rightarrow M_f<\infty$ is well-known. Thus suppose that $M_f<+\infty$, and let $0\leqq x_1<x_2\leqq 1$ be arbitrary. We shall prove that

$$|f(x_2-f(x_1)| \leqq M_f(x_2-x_1). \tag{4}$$

Let $\varepsilon>0$ be arbitrary. To each $x\in[x_1, x_2]$, there exists a $\delta(x)>0$ such that

$$|f(x)-f(y)| \leqq (M_f+\varepsilon)|x-y| \quad \text{if} \quad y\in I(x) = [x-\delta(x), x+\delta(x)]\cap[0, 1] \tag{5}$$

(see (3)). By the Heine—Borel theorem, from the infinite covering $\bigcup_{x\in[x_1, x_2]} I(x)$ of the interval $[x_1, x_2]$, we can select a finite covering $\bigcup_{i=1}^{m} I(y_i)\supseteqq[x_1, x_2]$, such that each point of $[x_1, x_2]$ belongs to at most two $I(y_i)$'s. Let z_i be a common point of $I(y_i)$ and $I(y_{i+1})$ $(i=1, 2, \ldots, m-1)$. Then by (5)

$$|f(x_1)-f(x_2)| \leqq |f(x_1)-f(y_1)|+|f(y_1)-f(z_1)|+|f(z_1)-f(y_2)|+\ldots$$
$$\ldots+|f(y_m)-f(x_2)| \leqq (M_f+\varepsilon)(x_2-x_1).$$

But $\varepsilon>0$ was arbitrary and the proof of (4) is complete.

LEMMA 3. *If* $\alpha>2$ *and* $f\in[0, 1]$ *then*

$$\limsup_{n\to\infty}[n\|L_n(f)-f\|] \geqq \frac{\varrho(\alpha)}{160} M_f.$$

We note that Lemmas 2 and 3 clearly imply the Theorem.

PROOF OF LEMMA 3. We may assume that $M_f>0$, otherwise there is nothing to prove. Let $x_0\in[0, 1]$ be an arbitrary but fixed point such that $M_f(x_0)>0$. If $M_f(x_0)<\infty$ then by (3) there exists a $d_0>0$ such that

$$\left|\frac{f(x)-f(x_0)}{x-x_0}\right| \leqq 2M_f(x_0) \quad \text{if} \quad x\in[x_0-d_0, x_0+d_0]\cap[0, 1]. \tag{6}$$

On the other hand, when $M_f(x_0)=\infty$, (6) clearly holds with $d_0=1$.

Evidently, there exists a sequence $\{x_k\}_{k=1}^{\infty}$ such that

(7) $$x_k \neq x_0 \quad (k=1,2,\ldots), \quad \lim_{k\to\infty} x_k = x_0$$

and

(8) $$\left|\frac{f(x_k)-f(x_0)}{x_k-x_0}\right| \begin{cases} \geqq \dfrac{1}{2} M_f(x_0) & \text{if} \quad M_f(x_0)<\infty \\ \max\limits_{\substack{0\leqq x'\leqq 1 \\ |x'-x_0|\geqq|x_k-x_0|}} \left|\dfrac{f(x')-f(x_0)}{x'-x_0}\right| & \text{if} \quad M_f(x_0)=\infty \end{cases} \quad (k=1,2,\ldots).$$

Thus in both cases

(9) $$\left|\frac{f(x')-f(x_0)}{x'-x_0}\right| \leqq 4\left|\frac{f(x_k)-f(x_0)}{x_k-x_0}\right| \quad \text{if} \quad |x_k-x_0| \leqq |x'-x_0| \leqq d_0 \quad (k=1,2,\ldots).$$

Now if x_0 is irrational then for all $k=1,2,\ldots$ there exist positive integers l_k, m_k such that

(10) $$\left|x_0-\frac{l_k}{m_k}\right| \leqq \frac{2|x_0-x_k|}{\varrho m_k} \quad \text{where} \quad m_k \leqq \frac{\varrho}{2|x_0-x_k|}.$$

(This in an elementary number-theoretic result.) Then by (7) there exist integers n_k which are multiples of m_k and

(11) $$\frac{\varrho}{4n_k} \leqq |x_0-x_k| \leqq \frac{\varrho}{2n_k} \quad (k=1,2,\ldots).$$

Evidently, $\lim\limits_{k\to\infty} m_k=\infty$, otherwise by (7) and (10) the irrational number x_0 could be approximated arbitrarily close by rational numbers l_k/m_k with bounded denominator, which is impossible. But then (10) and (11) imply (with the notation $l_k/m_k = h_k/n_k$, l_k an integer)

(12) $$\left|x_0-\frac{l_k}{n_k}\right| \leqq \frac{\varrho}{4n_k} \quad (k \geqq k_0).$$

This and (11) yield

(13) $$\frac{\varrho}{8n_k} \leqq \left|x_k-\frac{l_k}{n_k}\right| \leqq \frac{\varrho}{n_k} \quad (k \geqq k_0).$$

Now if x_0 is rational, then all the relations (10)—(13) which we will make use in the sequel, trivially hold.

We have by (9) and (11)—(13)

$$\left|\frac{f(x_k)-f\left(\frac{j}{n_k}\right)}{x_k-\frac{j}{n_k}}\right| \leqq \frac{\left|f(x_0)-f\left(\frac{j}{n_k}\right)\right|+|f(x_0)-f(x_k)|}{\left|x_k-\frac{j}{n_k}\right|} = \left|\frac{f(x_0)-f\left(\frac{j}{n_k}\right)}{x_0-\frac{j}{n_k}}\right| \cdot \left|\frac{x_0-\frac{j}{n_k}}{x_k-\frac{j}{n_k}}\right| +$$

$$+\left|\frac{f(x_0)-f(x_k)}{x_0-x_k}\right| \cdot \left|\frac{x_0-x_k}{x_k-\frac{j}{n_k}}\right| \leqq \frac{13}{2}\left|\frac{f(x_k)-f(x_0)}{x_k-x_0}\right| \quad \text{if} \quad \left|x_0-\frac{j}{n_k}\right| \leqq d_0, \ j \neq h_k, \ k \geqq k_0.$$

Thus using (1) and (2) we get

$$(14)\qquad |L_{n_k}(f, x_k)-f(x_k)| \geqq \left\{ \left| \frac{f(x_k)-f\left(\frac{h_k}{n_k}\right)}{x_k-\frac{h_k}{n_k}} \right| \cdot \frac{1}{\left|x_k-\frac{h_k}{n_k}\right|^{\alpha-1}} - \right.$$

$$- \sum_{\substack{j\neq h_k \\ \left|x_0-\frac{j}{n_k}\right|\leqq d_0}} \left| \frac{f(x_k)-f\left(\frac{j}{n_k}\right)}{x_k-\frac{j}{n_k}} \right| \cdot \frac{1}{\left|x_k-\frac{j}{n_k}\right|^{\alpha-1}} -$$

$$\left. \sum_{\substack{j \\ \left|x-\frac{j}{n_k}\right|>d_0}} \left(|f(x_k)|+\left|f\left(\frac{j}{n_k}\right)\right|\right) \frac{1}{\left|x_k-\frac{j}{n_k}\right|^{\alpha}} \right\} \frac{1}{2} \left|x_k-\frac{h_k}{n_k}\right|^{\alpha}.$$

Now if

$$(15)\qquad \left| \frac{f(x_k)-f\left(\frac{h_k}{n_k}\right)}{x_k-\frac{h_k}{n_k}} \right| \geqq \frac{1}{2} \left| \frac{f(x_k)-f(x_0)}{x_k-x_0} \right|$$

then (9), (2) and (13) imply

$$(16)\qquad |L_{n_k}(f, x_k)-f(x_k)| \geqq \left| \frac{f(x_k)-f(x_0)}{x_k-x_0} \right| \left\{ \frac{\varrho}{4n_k} - \frac{\varrho}{10n_k} - O(n_k^{1-\alpha}) \right\} \geqq$$

$$\geqq \frac{\varrho}{10n_k} \left| \frac{f(x_k)-f(x_0)}{x_k-x_0} \right| \quad (k \geqq k_1 \geqq k_0).$$

On the other hand, when (15) does not hold then by (11) and (12)

$$\left| f(x_k)-f\left(\frac{h_k}{n_k}\right) \right| \leqq \frac{1}{2} |f(x_k)-f(x_0)| \cdot \frac{|x_k-x_0|+\left|\frac{h_k}{n_k}-x_0\right|}{|x_k-x_0|} \leqq \frac{3}{4} |f(x_k)-f(x_0)|$$

and thus

$$\left| f(x_0)-f\left(\frac{h_k}{n_k}\right) \right| \geqq \frac{1}{4} |f(x_k)-f(x_0)|.$$

Therefore, similarly as in (14),

$$|L_{n_k}(f, x_0)-f(x_0)| \geqq \left\{\left|\frac{f(x)-f\left(\frac{h_k}{n_k}\right)}{x_0-\frac{h_k}{n_k}}\right| \cdot \frac{1}{\left|x-\frac{h_k}{n_k}\right|^{\alpha-1}} -\right.$$

$$- \sum_{\substack{j\neq h_k \\ \left|x_0-\frac{j}{n_k}\right|\leqq d_0}} \left|\frac{f(x_0)-f\left(\frac{j}{n_k}\right)}{x_0-\frac{j}{n_k}}\right| \cdot \frac{1}{\left|x_0-\frac{j}{n_k}\right|^{\alpha-1}} - \sum_{\substack{j \\ \left|x_0-\frac{j}{n_k}\right|>d_0}} \left.\frac{|f(x_0)|+\left|f\left(\frac{j}{n_k}\right)\right|}{\left|x_0-\frac{j}{n_k}\right|^{\alpha}}\right\} \frac{1}{2}\left|x_0-\frac{h_k}{n_k}\right|^{\alpha} \geqq$$

$$\geqq \frac{1}{8}|f(x_k)-f(x_0)| - \frac{\varrho}{80n_k}\left|\frac{f(x_k)-f(x_0)}{x_k-x_0}\right| - O(n_k^{1-\alpha}) \geqq \frac{3\varrho}{160n_k}\left|\frac{f(x_k)-f(x_0)}{x_k-x_0}\right| - o\left(\frac{1}{n_k}\right).$$

This and (14) imply

$$\|L_{n_k}(f)-f\| \geqq \frac{\varrho}{80n_k}\left|\frac{f(x_k)-f(x_0)}{x_k-x_0}\right| \qquad (k \geqq k_2 \geqq k_1).$$

Hence and from (8)

$$\limsup_{n\to\infty} [\|L_n(f)-f\|\, n] \geqq \frac{\varrho}{160} M_f(x_0).$$

But x_0 was arbitrary and thus Lemma 3 is proved.

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References

[1] *Linear Operators and Approximation*, Proceedings of the Conference held in Oberwolfach, 1977 (Springer Verlag, 1978).

[2] J. SZABADOS, On a problem of R. DeVore, *Acta Math. Acad. Sci. Hungar.*, **27** (1976), 219—223.

MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
1053 BUDAPEST, REÁLTANODA U. 13—15.

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