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О ЕДИНСТВЕННОСТИ РЯДОВ ПО ЦЕНТРИРОВАННЫМ *H*-СИСТЕМАМ

Г. Г. ГЕВОРКЯН (Ереван)

Пусть $(X, \mathscr{F}, \mu)$ — вероятностное пространство и $\{\mathscr{F}_n\}_{n=1}^{\infty}$ — последовательность σ-алгебр, входящих в $\mathscr{F}$ и обладающих свойствами

1) $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \ldots \subset \mathscr{F}_n \subset \ldots$, $X \in \mathscr{F}_n$;

2) $\mathscr{F}_1 = \{X, \varnothing\}$ и каждая σ-алгебр $\mathscr{F}_n$ порождена ровно n атомами $\Delta_1^{(n)}$, $\Delta_2^{(n)}, \ldots, \Delta_n^{(n)}$, $\Delta_i^{(n)} \cap \Delta_j^{(n)} = \varnothing$, $i \neq j$ и $\bigcup\limits_{i=1}^{n} \Delta_i^{(n)} = X$.

Ясно, что при вышеуказанных предположениях, совокупность атомов, образующих $\mathscr{F}_{n+1}$, получается расщеплением одного из атомов $\Delta_i^{(n)}$ на два атома.

Через $\mathscr{F}_\infty$ обозначается минимальная σ-алгебра, содержащая все $\mathscr{F}_n$, $n=1, 2, \ldots$.

Определение 1. Система $\{\varphi_n(x)\}_{n=1}^{\infty}$ ортонормированных в пространстве $(X, \mathscr{F}, \mu)$ функций, называется *H*-системой, если для некоторой последовательности $\mathscr{F}_n$, удовлетворяющей условиям 1), 2), минимальная σ-алгебра относительно которой измеримы все функции $\{\varphi_i\}_{i=1}^{n}$, совпадает с $\mathscr{F}_n$, для всех $n=1, 2, \ldots$.

Определение *H*-системы было введено в работе GUNDY [4]. Из определения видно, что $\varphi_1(x) \equiv 1$, $x \in X$, каждая функция φ_n, $n > 1$ обращается в нуль вне некоторого атома $\Delta_i^{(n-1)}$ из σ-алгербы $\mathscr{F}_{n-1}$ и принимает постоянные значения (разных знаков) на каждом из двух атомов σ-алгебры $\mathscr{F}_n$, полученных расщеплением атома $\Delta_i^{(n-1)}$.

Из определения немедленно следует также, что

$$\int_X \varphi_n \, d\mu = \int_{\Delta_i^{(n-1)}} \varphi_n \, d\mu = 0, \quad n > 1.$$

В том частном случае; когда $(X, \mathscr{F}, \mu)$ совпадает с пространством Лебега на отрезке $[0, 1)$ и $\mathscr{F}_{2^n}$ порождается из атомов $\Delta_i^{(2^n)} = \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)$, $1 \leqq i \leqq 2$, система $\{\varphi_n(x)\}_{n=1}^{\infty}$ совпадает, с точностью до значений в двоично рациональных точках с известной системой Хаара $\{\chi_n(x)\}$. В класс *H*-систем входят также системы типа Хаара (см. [5]).

В работе [2] Ф. Г. Арутюняном и А. А. Талаляном для рядов по системам Хаара и Уолша был установлен аналог известной теоремы Валле-Пус-

* Независимо аналогичный результат был получен Критенденом и Шапиро [3].

сена о единственности тригонометрических рядов.* Для системы Хаара эта теорема формулируется следующим образом.

Теорема А (Ф. Г. Арутюнян, А. А. Талалян). *Пусть ряд*

$$(*)\qquad \sum_{n=0}^{\infty} a_n \chi_n(x),$$

где $\{\chi_n(x)\}$ *— система Хаара, обладает свойствами:*

α) *некоторая последовательность* $\{S_{N_j}(x)\}$ *частных сумм ряда* $(*)$ *сходится к суммируемой функции* $f(x)$ *всюду на отрезке* $[0, 1]$, *кроме, быть может, счетного множества точек;*

β) *для любой точки* $x_0 \in [0, 1]$ $\lim\limits_{k\to\infty} \dfrac{a_{n_k}}{\chi_{n_k}(x_0)} = 0$, *где* $n_1 < n_2 < \ldots < n_k < \ldots$ *суть все те номера* n, *для которых* $\chi_{n_k}(x_0) \neq 0$.

Тогда ряд $(*)$ *является рядом Фурье функции* $f(x)$ *по системе Хаара, т.е.*

$$a_n = \int_0^1 f(x)\chi_n(x)\,dx.$$

В дальнейшем появилось много работ (см., например, [1], [8], [9]) в которых для системы Хаара получены разные обобщения и усиления теоремы А.

В настоящей работе вышеуказанная теорема распространяется на произвольные H-системы, рассмотренные в пространстве $(X, \mathscr{F}, \mu)$.

При этом для доказательства соответствующей теоремы приходится ввести некоторую модификацию понятия точки пространства $(X, \mathscr{F}, \mu)$.

Определение 2. Скажем, что $\xi = \{\delta_n\}_{n=1}^{\infty}$ является точкой пространства $(X, \mathscr{F}_\infty, \mu)$ (относительно $\{\mathscr{F}_n\}_{n=1}^{\infty}$), если $\delta_1 \supset \delta_2 \supset \ldots \supset \delta_n \supset \ldots$ и δ_n атом из σ-алгебры $\mathscr{F}_n$, $n = 1, 2, \ldots$.

Заметим, что в последовательности $\{\delta_n\}_{n=1}^{\infty}$ некоторые атомы могут повторяться, так как они не всегда расщепляются на две части.

Скажем, что точка $\xi = \{\delta_n\}$ имеет меру нуль, если $\lim\limits_{n\to\infty} \mu(\delta_n) = 0$.

Если $\{\varphi_n(x)\}_{n=1}^{\infty}$ является H-системой относительно последовательности σ-алгебр $\{\mathscr{F}_n\}$ и $\xi = \{\delta_n\}_{n=1}^{\infty}$ — точка (относительно $\{\mathscr{F}_n\}$) пространства $(X, \mathscr{F}_\infty, \mu)$, то значение принимаемое этой функцией на атоме δ_n, $n = 1, 2, \ldots$.

Если задан ряд $\sum\limits_{k=1}^{\infty} a_k \varphi_k(x)$, то значением $S_n(\xi)$ частичной суммы $S_n(x) = \sum\limits_{k=1}^{n} a_k \varphi_k(x)$ в точке $\xi = \{\delta_n\}_{n=1}^{\infty}$ считается величина $S_n(\xi) = \sum\limits_{k=1}^{n} a_k \varphi_k(\xi)$.

В дальнейшем через ξ обозначаются модифицированные точки пространства $(X, \mathscr{F}, \mu)$.

Имеет место следующая

Теорема 1. *Пусть ряд*

$$(1)\qquad \sum_{n=1}^{\infty} a_n \varphi_n(x),$$

где $\{\varphi_n(x)\}_{n=1}^{\infty}$ *H-система, обладает следующими свойствами:*

A) *некоторая фиксированная последовательность частичных сумм* $\{S_{N_i}(x)\}$ *ряда* (1) *сходится во всех, кроме, быть может, счетного множества точках* $\xi=\{\delta_n\}$, *т. е. существует конечный предел* $\lim\limits_{i\to\infty} S_{N_i}(\xi)$ *для всех* $\xi=\{\delta_n\}_{n=1}^{\infty}\notin A$, *где* A *не более чем счетное множество;*

B) *ряд* (1) *по мере сходится к интегрируемой функции* $f(x)$;

C) *для любой точки* ξ *имеет место* $\lim\limits_{k\to\infty}\frac{a_{n_k}}{\varphi_{n_k}(\xi)}=0$, *где* $n_1<n_2<\ldots n_k<\ldots$ *суть все те номера* n, *для которых* $\varphi_n(\xi)\neq 0$. *Тогда ряд* (1) *является рядом Фурье функции* $f(x)$ *по системе* $\{\varphi_n(x)\}_{n=1}^{\infty}$, *т. е.*

$$a_n = \int\limits_X f(x)\varphi_n(x)\,d\mu.$$

Легко видеть, что теорема 1 содержит в себе теорему А.

Доказательство. При доказательстве мы в основном пользуемся схемой предложенной в работе [2]. Из условия B) следует, что $f(x)$ является $\mathscr{F}_\infty$-измеримой, поэтому разложение функции $f(x)$ по системе $\{\varphi_n(x)\}_{n=1}^{\infty}$

$$(2) \qquad \sum_{n=1}^{\infty} c_n\varphi_n(x)$$

сходится к $f(x)$ почти всюду и в метрике $L_1(X, \mathscr{F}, \mu)$ (см. [4], [6]), т. е.

$$(3) \qquad \lim_{n\to\infty}\sigma_n(x) = f(x)\ \text{п.в.на}\ X$$

и

$$(4) \qquad \lim_{n\to\infty}\int\limits_X |\sigma_n(x)-f(x)|\,d\mu = 0.$$

Пусть $\{\xi_k\}$ те точки в которых ряд (1) расходится. Предполагая, что ряды (1) и (2) не совпадают, найдем точку ξ, отличную от всех точек ξ_k $(k=1, 2, \ldots)$, где частные суммы $S_{N_i}(\xi)$ ряда (1) расходятся. Тем самым придем к противоречию.

Введем некоторые обозначения. Через $\Delta_k^{(1)}$ и $\Delta_k^{(2)}$ обозначим те атомы, на которых функции $\varphi_k(x)$ принимает положительное и, соответственно, отрицательное значение. Через $\varphi_{k,n}^{(1)}$ и $\varphi_{k,n}^{(2)}$ $(n=1, 2, \ldots)$ обозначим те функции из $\{\varphi_n(x)\}_{n=1}^{\infty}$, которые равны нулю, соответственно, вне атомов $\Delta_k^{(1)}$ и $\Delta_k^{(2)}$.

Далее, через $a_{k,n}^{(i)}$ и $c_{k,n}^{(i)}$ $(i=1, 2;\ n=1, 2, \ldots)$, обозначим коэффициенты функций $\varphi_{k,n}^{(i)}$, соответственно в рядах (1) и (2).

Пусть ряды (1) и (2) не совпадают и k_1 наименьший номер для которого $a_n\neq c_n$. Тогда на атоме $\Delta_{k_1}^{(i_1)}$ $(i_1=1$ или $2)$, частные суммы $S_{k_1}(x)$ и $\sigma_{k_1}(x)$ рядов (1) и (2) принимают отличные друг от друга постоянные значения (см., Определение 1).

Для доказательства теоремы нам достаточно доказать следующую лемму.

Лемма. *Пусть* ξ_0 *произвольная точка пространства* X *(относительно* $\{\mathscr{F}_n\}_{n=1}^{\infty}$*), имеющая меру нуль и* k_0*-произвольное натуральное число, для которого выполнено следующее условие* *α) на некотором атоме вида $\Delta_{k_0}^{(i_0)}$ ($i_0=1$ или 2) частичные суммы $S_{k_0}(x)$ и $\sigma_{k_0}(x)$ рядов (1) и (2) принимают отличные друг от друга постоянные значения.*

Тогда для любого $M>0$ и натурального N можно определить число N_j, принадлежащее последовательности $\{N_j\}$, натуральное число p и атом вида $\Delta_p^{(i_p)}$, $i_p=1$ или 2, которые обладают следующими свойствами:

1°) $N_j>N$;

2°) $\Delta_p^{(i_p)}\subset\Delta_{k_0}^{(i_0)}$ и атом $\Delta_p^{(i_p)}$ не содержат точку ξ_0, т. е. если $\xi_0=\{\delta_n\}_{n=1}^{\infty}$, то существует атом δ_{n_0}, такое, что $\delta_{n_0}\cap\Delta_p^{(i_p)}=\emptyset$;

3° частичная сумма $S_{N_j}(x)$ ряда (1) внутри атома $\Delta_p^{(i_p)}$ постоянна и по абсолютной величине больше M;

4° для числа p и атома $\Delta_p^{(i_p)}$ выполнено условие α), в котором вместо k_0 взято p.

Доказательство леммы. Сначала докажем существование числа k_0' и атома вида $\Delta_{k_0'}^{(i_0')}$, $i_0'=1$ или 2, которые удовлетворяют условию α), где вместо k_0 взято k_0', причем $\Delta_{k_0'}^{(i_0')}\subset\Delta_{k_0}^{(i_0)}$ и $\xi_0\notin\Delta_{k_0'}^{(i_0')}$. В случае, когда $\xi_0\notin\Delta_{k_0}^{(i_0)}$, это утверждение верно. Предположим, что $\xi_0\in\Delta_{k_0}^{(i_0)}$ (т. е., если $\xi_0=\{\delta_n\}_{n=1}^{\infty}$, то существует атом δ_{n_0}, такой что, $\delta_{n_0}\subset\Delta_{k_0}^{(i_0)}$). Из условия α) следует:

(5) $$S_{k_0}(x)-\sigma_{k_0}(x)=d\neq 0 \quad \text{на атоме} \quad \Delta_{k_0}^{(i_0)}.$$

Рассмотрим ряд

(6) $$d+\sum_{n=1}^{\infty}(a_{k_0,n}^{(i_0)}-c_{k_0,n}^{(i_0)})\,\varphi_{k_0,n}^{(i_0)}(x).$$

Положим

(7) $$d_n=a_{k_0,n}^{(i_0)}-c_{k_0,n}^{(i_0)} \quad (n=1,2,\ldots).$$

Из точки $\xi_0=\{\delta_n\}_{n=1}^{\infty}$ выберем такую подпоследовательность $\{\delta_{n_k}\}$, члены которой отличаются друг от друга и в эту подпоследовательность входят все отличные друг от друга атомы последовательности $\{\delta_n\}_{n=1}^{\infty}$. Подпоследовательность $\{\delta_{n_k}\}$ бесконечна и $\lim\limits_{k\to\infty}\mu(\delta_{n_k})=0$, так как мера точки ξ_0 равна нулю.

Существует такой номер $n_{k'}$, что $\delta_{n_{k'}}=\Delta_{k_0}^{(i_0)}$.

Далее, обозначим через $\Delta_{k_0,n}^{(1)}$ и $\Delta_{k_0,n}^{(2)}$, $n=1,2,\ldots$, те атомы, на которых функция $\varphi_{k_0,n}^{(i_0)}(x)$ принимает, соответственно, положительные и отрицательные значения.

Из атомов $\Delta_{k_0,n}^{(1)}$ и $\Delta_{k_0,n}^{(2)}$ выберем последовательность атомов $\Delta_{k_0,m_k}^{(i_k)}$, $i_k=1$ или 2, $\Delta_{k_0,m_1}^{(i_1)}\supset\Delta_{k_0,m_2}^{(i_2)}\supset\ldots\supset\Delta_{k_0,m_k}^{(i_k)}\supset\ldots$ таких что

(8) $$\Delta_{k_0,m_k}^{(i_k)}=\delta_{n_{k'+k}}.$$

Рассмотрим также последовательность атомов $\Delta_{k_0,m_k}^{(i_k')}$ где $i_k'\neq i_k$, $i_k'=1$ или 2. Возможны только два случая:

I. Частные суммы ряда (6) с номерами m_k, $k=1, 2, \ldots$, обращаются в нуль на атомах $\Delta^{(i_0')}_{k_0, m_k}$, для всех k, т. е. (см. (7))

$$d+\sum_{n=1}^{m_k} d_n \varphi^{(i_0)}_{k_0, n} = 0 \quad \text{на атоме} \quad \Delta^{(i_k')}_{k_0, m_k} \quad \text{для всех} \quad k. \tag{9}$$

II. Равенство (9) имеет место не для всех $k=1, 2, \ldots$.

Легко убедиться (по индукции), что когда имеет место первое утверждение, то

$$d_{m_k} = -\frac{d}{\beta_k}\left(1-\frac{\alpha_1}{\beta_1}\right)\left(1-\frac{\alpha_2}{\beta_2}\right)\ldots\left(1-\frac{\alpha_{k-1}}{\beta_{k-1}}\right) \tag{10}$$

где α_i и β_i, соответственно, те значения которые принимает функция $\varphi^{(i_0)}_{k_0, m_i}$ на множествах $\delta_{n_{k'+i}}$ и $\delta_{n_{k'+i-1}} \setminus \delta_{n_{k'+i}}$ т. е.

$$\varphi^{(i_0)}_{k_0, m_i}(x) = \begin{cases} \alpha_i & \text{на} \quad \delta_{n_{k'+1}} \\ \beta_i & \text{на} \quad \delta_{n_{k'+i-1}} \setminus \delta_{n_{k'+i}}. \end{cases} \tag{11}$$

Из определения 1 следует, что числа α_i и β_i удовлетворяют следующим условиям

$$|\alpha_i|\mu(\delta_{n_{k'+i}}) = |\beta_i|\mu(\delta_{n_{k'+i-1}} \setminus \delta_{n_{k'+i}}), \tag{12}$$

$$\alpha_i^2 \mu(\delta_{n_{k'+i}}) + \beta_i^2 \mu(\delta_{n_{k'+i-1}} \setminus \delta_{n_{k'+1}}) = 1, \tag{13}$$

$$\alpha_i \beta_i < 0. \tag{14}$$

Отсюда и из (10) следует

$$\frac{d_{m_k}}{\alpha_k} = -\frac{d}{\alpha_k \beta_k}\left(1-\frac{\alpha_1}{\beta_1}\right)\ldots\left(1-\frac{\alpha_{k-1}}{\beta_{k-1}}\right) = \frac{d}{|\alpha_k \beta_k|}\times \tag{15}$$

$$\times\left[1+\frac{\mu(\delta_{n_{k'}} \setminus \delta_{n_{k'+1}})}{\mu(\delta_{n_{k'+1}})}\right]\ldots\left[1+\frac{\mu(\delta_{n_{k'+k-2}} \setminus \delta_{n_{k'+k-1}})}{\mu(\delta_{n_{k'+k-1}})}\right] =$$

$$= \frac{d}{|\alpha_k \beta_k|}\frac{\mu(\delta_{n_{k'}})}{\mu(\delta_{n_{k'+1}})}\ldots\frac{\mu(\delta_{n_{k'+k-2}})}{\mu(\delta_{n_{k'+k-1}})} = \frac{d\mu(\delta_{n_{k'}})}{|\alpha_k \beta_k|\mu(\delta_{n_{k'+k-1}})}.$$

Используя (12) и (13), легко убедиться, что

$$\frac{d_{m_k}}{\alpha_k} = \frac{d\mu(\delta_{n_{k'}})}{|\alpha_k \beta_k|\,\mu(\delta_{n_{k'+k-1}})} = d\mu(\delta_{n_{k'}}). \tag{16}$$

Из (16), а также из (7) и условия C) теоремы следует:

$$\lim_{k\to\infty} \frac{c^{(i_0)}_{k_0, m_k}}{\alpha_k} = -d\mu(\delta_{n_{k'}}). \tag{17}$$

Докажем, что (17) противоречит тому, что ряд (2) сходится в метрике L_1. Без ограничения общности можно считать, что $-d\mu(\delta_{n_{k'}})=1$. Тогда из (17) следует существование такого p что для произвольного $s>p$, $\frac{c_{k_0,m_s}^{(i_0)}}{\alpha_s}>\frac{1}{2}$. Легко видеть, что

$$\int\limits_{\delta_{n_{k'}}}\left|\sum_{n=n_s}^{n_q} c_{k_0,n}^{(i_0)}\varphi_{k_0,n}^{(i_0)}(x)\right|d\mu > \int\limits_{\delta_{n_{k'+q}}}\sum_{k=s}^{q} c_{k_0,m_k}^{(i_0)}\varphi_{k_0,m_k}^{(i_0)}(x)\,d\mu > \frac{1}{2}\int\limits_{\delta_{n_{k'+q}}}\sum_{k=s}^{q}\alpha_k^2\,d\mu. \tag{18}$$

Из (12) и (13) находим $\alpha_k^2=\frac{1}{\mu(\delta_{n_{k'+k}})}-\frac{1}{\mu(\delta_{n_{k'+k-1}})}$. Возьмем произвольное $s>p$ и q такое, что $\frac{\mu(\delta_{n_{k'+q}})}{\mu(\delta_{n_{k'+s-1}})}<\frac{1}{2}$. Тогда имеем:

$$\left\|\sum_{n=n_s}^{n_q} c_{k_0,n}^{(i_0)}\varphi_{k_0,n}^{(i_0)}(x)\right\|_1 > \frac{1}{2}\int\limits_{\delta_{n_{k'+q}}}\sum_{k=m}^{q}\alpha_k^2\,d\mu = \tag{19}$$

$$=\frac{1}{2}\mu(\delta_{n_{k'+q}})\sum_{k=s}^{q}\left(\frac{1}{\mu(\delta_{n_{k'+k}})}-\frac{1}{\mu(\delta_{n_{k'+k-1}})}\right)=\frac{1}{2}\mu(\delta_{n_{k'+q}})\left[\frac{1}{\mu(\delta_{n_{k'+q}})}-\frac{1}{\mu(\delta_{n_{k'+m-1}})}\right]=$$

$$=\frac{1}{2}-\frac{1}{2}\frac{\mu(\delta_{n_{k'+q}})}{\mu(\delta_{n_{k'+m-1}})}>\frac{1}{4}.$$

Но это противоречит тому, что ряд (2) сходится в метрике L_1. Таким образом, предположение, что имеет место случай 1, приводит к противоречию.

Итак, мы доказали, что существует число k_0' и атом вида $\Delta_{k_0'}^{(i_0')}$, $i_0'=1$ или 2, которые удовлетворяют условию α), где вместо k_0 взято k_0', причем

$$\Delta_{k_0'}^{(i_0')}\subset\Delta_{k_0}^{(i_0)} \quad \text{и} \quad \xi_0\notin\Delta_{k_0'}^{(i_0')}, \tag{20}$$

$$S_{k_0'}(x)-\sigma_{k_0'}(x)=c\neq 0 \quad \text{на} \quad \Delta_{k_0'}^{(i_0')}. \tag{21}$$

Рассмотрим ряд

$$c+\sum_{m=1}^{\infty}\left(a_{k_0',m}^{(i_0')}-c_{k_0',m}^{(i_0')}\right)\varphi_{k_0',m}^{(i_0')}(x). \tag{22}$$

Пусть $\varphi_{k_0',1}^{(i_0')}$ в H-системе $\{\varphi_k\}_{n=1}^{\infty}$ имеет номер q_1, т.е.

$$\varphi_{k_0',1}^{(i_0')}(x)=\varphi_{q_1}(x) \tag{23}$$

и N_j — наименьшее число из последовательности $\{N_j\}$ такое, что

$$N_{j_1}\geqq q_1. \tag{24}$$

Для данного $j\geqq j_1$ обозначим через m_j наибольшее натуральное число, для которого функция $\varphi_{k_0',m_j}^{(i_0')}(x)$ в системе $\{\varphi_n(x)\}_{n=1}^{\infty}$ имеет номер, не превосходящий числа N_j.

Из выборов функций $\varphi^{(i_0')}_{k_0',m}$ и чисел m_j следует, что на атоме $\Delta^{(i_0')}_{k_0'}$ выполняются равенства:

$$S_{k_0'}(x)+\sum_{m=1}^{m_j} a^{(i_0')}_{k_0',m}\varphi^{(i_0')}_{k_0',m}(x)=\sum_{n=1}^{N_j} a_n\varphi_n(x),\quad j\geqq j_1, \tag{25}$$

$$\sigma_{k_0'}(x)+\sum_{m=1}^{m_j} c^{(i_0')}_{k_0',m}\varphi^{(i_0')}_{k_0',m}(x)=\sum_{n=1}^{N_j} c_n\varphi_n(x),\quad j\geqq j_1. \tag{26}$$

Отсюда, в силу (3), (21) и условия B) теоремы, вытекает:

$$\lim_{j\to\infty}\left[c+\sum_{m=1}^{m_j}\left(a^{(i_0')}_{k_0',m}-c^{(i_0')}_{k_0',m}\right)\varphi^{(i_0')}_{k_0',m}(x)\right]=0 \quad \text{по мере на} \quad \Delta^{(i_0')}_{k_0'}. \tag{27}$$

Из (4) вытекает

$$\lim_{j\to\infty}\int\limits_{\Delta^{(i_0')}_{k_0'}}\left|\sigma_{k_0'}(x)+\sum_{m=1}^{mj} c^{(i_0')}_{k_0',m}\varphi^{(i_0')}_{k_0',m}(x)-f(x)\right|d\mu=0. \tag{28}$$

Обозначим, для краткости,

$$\Phi_j(x)=\sigma_{k_0'}(x)+\sum_{m=1}^{m_j} c^{(i_0')}_{k_0',m}\varphi^{(i_0')}_{k_0',m}(x)\quad \text{на}\quad \Delta^{(i_0')}_{k_0'}, \tag{29}$$

$$\Psi_j(x)=S_{k_0'}(x)+\sum_{m=1}^{m_j} a^{(i_0')}_{k_0',m}\varphi^{(i_0')}_{k_0',m}(x)\quad \text{на}\quad \Delta^{(i_0')}_{k_0'}. \tag{30}$$

Пусть $M>0$ и натуральное N — числа, фигурирующие в формулировке леммы.

Возьмем j_0 такое, что

$$j_0>j_1 \quad \text{и} \quad N_{j_0}>N. \tag{31}$$

Покажем, что неравенства

$$|\Psi_j(x)|\leqq M+|\Phi_j(x)|,\quad j\geqq j_0 \tag{32}$$

не могут выполняться всюду на атоме $\Delta^{(i_0')}_{k_0'}$. В самом деле, из (28) следует, что функции $\Phi_j(x)$ имеют равностепенно абсолютно непрерывные интегралы на $\Delta^{(i_0')}_{k_0'}$ и если бы каждое из неравенств (32) выполнялось всюду на $\Delta^{(i_0')}_{k_0'}$, то функции $\Psi_j(x)$, $j\geqq j_1$ тоже имели бы равностепенно абсолютно непрерывные интегралы на $\Delta^{(i_0')}_{k_0'}$. Тогда в силу теоремы Витали о переходе к пределу под знаком интеграла (см. [7])., мы имеем (см. (27) и (21))

$$\lim_{j\to\infty}\int\limits_{\Delta^{(i_0')}_{k_0'}}(\Psi_j(x)-\Phi_j(x)\,d\mu=0. \tag{33}$$

Но из (21) и из определения функций $\varphi^{(i_0')}_{k_0',m}(x)$ вытекает

$$\int\limits_{\Delta^{(i_0')}_{k_0'}} \left(\Psi_j(x)-\Phi_j(x)\right) d\mu = c\mu\left(\Delta^{(i_0')}_{k_0'}\right) \neq 0, \quad j \geqq j_1, \tag{34}$$

которое противоречит равенству (33).

Пусть $j \geqq j_0$ наименьшее число, для которого не выполнено неравенство (32), т. е. неравенство

$$|\Psi_j(x)| > M+|\Phi_j(x)|, \quad j \geqq j_0 \tag{35}$$

имеет место на некоторых атомах вида $\Delta^{(i_m)}_{k_0',m}$, $i_m=1$ или 2, (см., Определение 1), представляющих атомы постоянства функций $\varphi^{(i_0')}_{k_0',m}$, $m<m_j$.

Пусть m' — наибольшее число среди указанных чисел $m<m_j$ и атом $\Delta^{(i_{m'})}_{k_0',m}$, $i_{m'}=1$ или 2, один из атомов постоянства функции $\varphi^{(i_0')}_{k_0',m}$, где выполнено неравенство (35). Ясно, что на атоме $\Delta^{(i_{m'})}_{k_0',m}$, функции $\Phi_j(x)$ и $\Psi_j(x)$ постоянны. Если p тот номер, для которого $\varphi_p(x)=\varphi^{(i_0')}_{k_0',m'}(x)$, то атом $\Delta^{(i_{m'})}_{k_0',m'}$ совпадает с атомом $\Delta^{(i_p)}_p$, где $i_p=i_{m'}$. Легко видеть, что найденные числа N_j, p и атом $\Delta^{(i_p)}_p$ удовлетворяют всем требованиям леммы.

Доказательство теоремы непосредственно следует из доказанной леммы. Прежде всего, отметим, что из условия B) следует, что точки расходимости ξ_k ряда (1) имеют меру нуль. Последовательным применением леммы, в формулировке которой вместо ξ_0 берутся точки $\xi_1, \xi_2, \ldots, \xi_k, \ldots$, можно определить подпоследовательность N_{j_k} последовательности N_j и атомы $\Delta^{(i_{p_k})}_{p_k}$, $k=1, 2, \ldots$, $i_{p_k}=1$ или 2, которые обладают следующими свойствами

$$\xi_k \notin \Delta^{(i_{p_k})}_{p_k} \quad (k=1, 2, \ldots), \tag{36}$$

$$\Delta^{(i_1)}_{k_1} \supset \Delta^{(i_{p_1})}_{p_1}, \quad \Delta^{(i_{p_{k+1}})}_{p_{k+1}} \subset \Delta^{(i_{p_k})}_{p_k} \quad (k=1, 2, \ldots), \tag{37}$$

$$|S_{N_{j_k}}(x)|>K \quad \text{на} \quad \Delta^{(i_{p_k})}_{p_k}. \tag{38}$$

Последовательность $\left\{\Delta^{(i_{p_k})}_{p_k}\right\}_{k=1}^{\infty}$ является подпоследовательностью некоторой точки ξ (однозначно определяющейся), в которой последовательность частичных сумм $S_{N_j}(x)$ расходится, чего не может быть так как $\xi \neq \xi_n$, $n=1, 2, \ldots$ (см. (37)).

Тем самым теорема доказана.

Замечание 1. Из доказательства Теоремы 1 (см. доказательство леммы) видно, что верна следующая

Теорема 2. *Пусть ряд*

$$\sum_{n=1}^{\infty} a_n \varphi_n(x), \tag{39}$$

где $\{\varphi_n(x)\}_{n=1}^{\infty}$ *H-система, обладает следующими свойствами:*

1) *некоторая последовательность частичных сумм* $\{S_{N_j}(x)\}$ *ряда* (39) *сходится во всех точках;*

2) *ряд* (39) *по мере сходится к интегрируемой функции* $f(x)$.

Тогда ряд (39) *является рядом Фурье функции* $f(x)$ *по системе* $\{\varphi_n(x)\}_{n=1}^{\infty}$.

Заметим также, что в Теореме 2 на коэффициенты рядов никаких ограничений не налагаются. Для мартингалов Теорема 2 имеет следующую формулировку.

Теорема 3. *Пусть* $\{\mathscr{F}_n\}_{n=1}^{\infty}$ *последовательность* σ*-алгебр удовлетворяющая условиям* 1), 2) *и* $\{f_n, \mathscr{F}_n\}$ *мартингал относительно* $\{\mathscr{F}_n\}$, *которая обладает следующими свойствами:*

1) *некоторая подпоследовательность* f_{n_k} *во всех точках сходится (т. е. для любой точки* $\xi=\{\delta_n\}_{n=1}^{\infty}$ *последовательность* $f_{n_k}(\xi)=f_{n_k}(\delta_{n_k})$ *сходится к конечному пределу);*

2) *последовательность* $f_n(x)$ *по мере сходится к интегрируемой функции* $f(x)$.

Тогда $f_n=E_n f$, *где* E_n *оператор условного математического ожидания относительно* $\mathscr{F}_n$ *(см.* [6]*).*

Замечание 2. В формулировке Теоремы 1 условие C) на коэффициенты необходимо. В самом деле пусть $\xi=\{\delta_n\}_{n=1}^{\infty}$, такая точка, что $\mu(\delta_n)\to 0$, $n\to\infty\dots$ Возьмем $\{\delta_{n_k}\}$ подпоследовательность различных между собой атомов последовательности $\{\delta_n\}$ и $\{\Psi_k\}_{k=1}^{\infty}$, те функции из системы $\{\varphi_n\}_{n=1}^{\infty}$, носители которых равны δ_{n_k}, т. е.

$$\Psi_k=\begin{cases}\alpha_k & \text{на} \quad \delta_{n_{k+1}}\\ \beta_k & \text{на} \quad \delta_{n_k}\setminus\delta_{n_{k+1}}\end{cases}.$$

Легко видеть, что ряд $\sum\limits_{k=1}^{\infty}\alpha_k\Psi_k(x)$ во всех точках, за исключением ξ, сходится к нулю, но не является рядом Фурье функции $f(x)\equiv 0$. Условие C) нарушено, так как $\dfrac{\alpha_k}{\Psi_k(\xi)}=1$. Приведенный пример ряда является аналогом примера Фабера, приведенного им для рядов Хаара [10].

Замечание 3. Существует *H*-система $\{\varphi_k(x)\}$, ряд Фурье по которой некоторой интегрируемой $\mathscr{F}_\infty$-измеримой функции сходится во всех точках, но коэффициенты которого не удовлетворяют условию C). В самом деле, пусть $X=[0, 1]$ с Лебеговой мерой и

$$\varphi_k(x)=\begin{cases}\sqrt{\dfrac{2^k}{(2^k+1)(2^{k-1}+1)}} & \text{если} \quad x\in\left[0;\ \dfrac{1}{2}+\dfrac{1}{2^{k+1}}\right),\\[2ex] -\sqrt{\dfrac{(2^k+1)\,2^k}{2^{k-1}+1}} & \text{если} \quad x\in\left[\dfrac{1}{2}+\dfrac{1}{2^{k+1}},\ \dfrac{1}{2}+\dfrac{1}{2^k}\right),\end{cases}$$

когда $k>1$, а $\varphi_1(x)\equiv 1$. Тогда $\sum_{k=1}^{\infty}\varphi_k(x)$ во всех точках и в метрике $L_1[0, 1]$ сходится. Следовательно, $\sum_{k=1}^{\infty}\varphi_k(x)$ является рядом Фурье некоторой $\mathcal{F}_\infty$-измеримой интегрируемой функции, но в «точке» $\left\{\left[0, \frac{1}{2}+\frac{1}{2^k}\right)\right\}_{k=1}^{\infty}$ условие С) не выполнено. Приведенный пример показывает, что Теорема 2 не содержится в Теореме 1.

Замечание 4. Утверждение Теоремы 1 не будет верным, если заменить условие С), поставленное в каждой модифицированной точке, таким же условием поставленной в каждой обычной точке. Если рассмотреть систему Хаара, $\chi_0^{(0)}$, $\chi_0^{(1)}$, $\{\chi_n^{(k)}\}$, $n=1, 2, \ldots$, $1\leqq k\leqq 2^n$ на полуоткрытом интервале (0, 1], считая значения этих функций в точках разрыва равными их левосторонним пределам, то она будет H-системой на (0, 1] и ряд

$$\chi_0^{(0)}(x)+\chi_0^{(1)}(x)+\sum_{n=1}^{\infty}\sqrt{2^n}\,\chi_n^{(1)}(x)$$

будет удовлетворять условиям А), В) и условию С) в обычном смысле, но не будет рядом Фурье своей суммы.

В заключении выражаю благодарность профессору А. А. Талаляну, под руководством которого выполнена настоящая работа.

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ARTINIAN RINGS IN WHICH ONE SIDED IDEALS ARE QUASI-PROJECTIVE

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Introduction

For a ring R, an R-module M is said to be quasi-injective in case the natural homomorphism $\mathrm{Hom}_R(M, M) \to \mathrm{Hom}_R(K, M)$ is epic for all submodules K of M. Dually M is said to be quasi-projective in case the natural homomorphism $\mathrm{Hom}_R(M, M) \to \mathrm{Hom}_R(M, N)$ is epic for all factor modules N of M. Rings whose left ideals are quasi-injective have been studied by a number of authors ([3], [5], [7]), and for suitable conditions on the ring a number of structure theorems have been obtained ([5], [7]). The main object of this paper is to investigate artinian rings whose left ideals are quasi-projective. These rings include artinian hereditary rings, but many examples exist which are not hereditary. (See Section 4.)

The first three sections are devoted to characterizing artinian rings whose left ideals are quasi-projective. The main theorem (Theorem 3.5) appears in Section 3. There, these rings are characterized in terms of their primitive idempotents and two sided ideals i.e., given a basic set of primitive idempotents and the set of ideals of an artinian ring R, it is possible to determine if R has all left ideals quasi-projective by considering each left R-module $J^\alpha e$ where α is a positive integer, J is the Jacobson radical and e is a primitive idempotent. It will be shown that $J^\alpha e$ must have a certain decomposition for rings with left ideals quasi-projective, and that with the addition of a suitable hypothesis, this decomposition completely determines such rings.

The final section is devoted to a number of examples to show that the conditions of the structure theorems in 3 are necessary and the best possible.

We shall use the following notation. The ring R is associative with unity. The letter J denotes the Jacobson radical and ${}_RM(M_R)$ signifies that M is a left (right) R-module. The socle of a module M, which is the largest semi-simple submodule of M, will be denoted by $S(M)$. When R is semi-local (i.e., R/J is artinian semi-simple), the semi-simple module M/JM, called the top of M, will be denoted by $T(M)$. Also the notation $M^{(A)}$ means $\oplus \Sigma M_\alpha$ where $M_\alpha \cong M$.

Preliminaries

A number of concepts will be needed in the development of the results which follow. We begin with the following

DEFINITION. Let P be a projective R-module. Then P is said to be *hereditary* in case every submodule of P is projective.

Clearly any submodule of a hereditary module is again hereditary. Also observe that for a given set $\{P_\alpha\}_{\alpha \in A}$ of hereditary modules, the direct sum $\oplus \Sigma P_\alpha$ is always hereditary ([6], Proposition 7, page 85). Although many rings with left (right) ideals quasi-projective are not hereditary, a common feature of many of these rings is that they possess a 'large' hereditary left ideal.

We also will need the following lemmas which allow us to simplify some of the proofs in much of the subsequent work. Recall that a module Q is said to be *projective relative* to M if for all factor modules N of M the natural homomorphism $\mathrm{Hom}_R(Q, M) \to \mathrm{Hom}_R(Q, N)$ is epic. The class of modules to which Q is projective is closed under taking submodules, factors, and finite direct sums [8]. From this it is easily seen that $M_1 \oplus M_2$ is quasi-projective if and only if M_i is projective relative to M_j for $i, j = 1, 2$.

1.1. Lemma. *Let R be a ring. Suppose the module $Re \oplus Re/Ie$ is quasi-projective where e is a primitive idempotent and Ie is a left ideal. Then $Ie = 0$.*

Proof. By ([8], Proposition 1.2), Re/Ie is projective relative to Re. Thus the map $Re \to Re/Ie$ splits. Since e is primitive, this forces $Ie = 0$.

1.2. Lemma. *Let R be a ring with every left ideal quasi-projective. Let f be a primitive idempotent and I a left ideal such that $I \cap Rf = 0$. Suppose $fI \neq 0$. Then there exists a monomorphism $\varphi: Rf \to I$, given by right multiplication of an element $x \in I$.*

Proof. Since $fI \neq 0$, there exists an $x \in I$ such that $fx \neq 0$. Let φ be the map given by right multiplication of x. Then $Rf/Kf \cong \mathrm{Im}(\varphi) \subseteq I$. As $I \cap Rf = 0$, $Rf/Kf \oplus Rf$ is isomorphic to a left ideal of R. Hence by 1.1, $Kf = 0$. This shows that φ is monic.

2. The Loewy series decomposition

For a left R-module M, the Loewy series is the sequence of left R-modules

$$M \supset JM \supset \ldots \supset J^k M \supset \ldots .$$

The k-th Loewy factor is the module $J^{k-1}M/J^k M$. One defines the Loewy series, for right modules in a similar way. The Loewy series will be used to obtain a decomposition for artinian rings whose left ideals are quasi-projective. In light of this, we make the following

Definition. Let R be left artinian and e a primitive idempotent. Let

$$Re \supset Je \supset \ldots \supset J^n e \supset 0,$$

be the Loewy series for Re. For each α such $1 \leqq \alpha \leqq n$, $J^\alpha e$ may be decomposed into a direct sum of indecomposables say $J^\alpha e = \oplus \sum_{i_\alpha = 1}^{k_\alpha} I_{i_\alpha}$. Then we may express the Loewy series as,

$$Re \supset \oplus \sum^{k_1} I_{i_1} \supset \ldots \supset \oplus \sum^{k_n} I_{i_n} \supset 0.$$

The above expression will be called a *Loewy series decomposition* for the module Re.

It will be shown that rings with every left ideal quasi-projective have a particularly nice Loewy series decomposition for each of their principal indecomposable projective modules. This Loewy series decomposition will be used to characterize left artinian rings with every left ideal quasi-projective in terms of the primitive idempotents and two sided ideals of the ring.

REMARK. Note that for each $\alpha > 0$, $J^\alpha e$ has a unique decomposition using the Krull—Schmidt theorem for artinian rings.

Thus the Loewy series decomposition for each principal indecomposable projective is unique up to isomorphism.

The remainder of this section will be devoted to obtaining the Loewy series decomposition for each Re, where e is a primitive idempotent and R is an artinian ring with every left ideal quasi-projective.

2.1 LEMMA. *Let R be a left artinian ring with every left ideal quasi-projective. Let f be any primitive idempotent and $L \subseteq Rf$ a left ideal. Then L admits a decomposition $L = P \oplus K$ such that:*

(1) *P is projective and $fP = 0$.*

(2) *$K \cong (Rf/If)^{(n)}$ for some two sided ideal I.*

Here either P or K may be 0.

PROOF. The left ideal L is quasi-projective, so by ([4], Theorem 1.10),

$$L \cong (Re_1/Ie_1)^{(n_1)} \oplus \ldots \oplus (Re_k/Ie_k)^{(n_k)}$$

where $\{e_j\}_{j=1}^k$ are a set of primitive orthogonal idempotents and $Re_i \not\cong Re_j$ when $i \neq j$, and I is a 2-sided ideal in R. As $Re_j \not\cong Rf$ for all j with at most one possible exception, let $P \cong \oplus \Sigma (Re_j/Ie_j)^{(n_j)}$ where $1 \leq j \leq k$ and $Re_j \not\cong Rf$. Then there exists for each j, a left ideal isomorphic to $Re_j \oplus Re_j/Ie_j$, where $Re_j \not\cong Rf$. By 1.1 $Ie_j = 0$. Hence $P \cong \oplus \Sigma (Re_j)^{(n_j)}$. Now suppose $fP \neq 0$. Using 1.2 there is an isomorphic copy of Rf contained in $P \subseteq L$ contradicting R left artinian. Thus $f \cdot P = 0$, and $L \cong P$ or $L \cong P \oplus (Rf/If)^{(n)}$ depending on whether there exists $Re_j \cong Rf$ for some $j \leq k$.

2.2 LEMMA. *Let R be left artinian with every left ideal quasiprojective, and let P and f be as in Lemma 2.1. Then P is hereditary.*

PROOF. Consider $K \subseteq P$. Then

$$K \cong (Rf_1/If_1)^{(n_1)} \oplus \ldots \oplus (Rf_m/If_m)^{(n_m)}$$

where each f_j is a primitive idempotent and I is a two sided ideal. By 2.1 $fK = 0$ which implies that each $Rf_j \not\cong Rf$, $(1 \leq j \leq m)$. Hence, there exists a left ideal isomorphic to $Rf_j \oplus Rf_j/If_j$. So by 1.1. $If_j = 0$ for each j. This shows that K is projective, so P is hereditary.

2.3 LEMMA. *Let R be left artinian with every left ideal quasi-projective. Let K be as in Lemma 2.1. Suppose $K \cong (Rf/If)^{(n)}$ where $n > 1$ and f a primitive idempotent, I a two sided ideal. Then*

(1) *$f \cdot JK = 0$ i.e., JK has no composition factor isomorphic to $T(Rf)$.*

(2) *JK is hereditary.*

PROOF. Suppose that $f \cdot JK \neq 0$. Then $f \cdot Jf/If \neq 0$. This induces a homomorphism of Rf into Jf/If. Hence there is a factor module of Rf, $N \subseteq Jf/If$. Since K is a direct sum of at least two copies of Rf/If, there is a submodule of K isomorphic to $N \oplus Rf/If$.

Thus N is projective relative to Rf/If. Using this and that $T(N)\cong T(Rf)$, we have the following diagram,

$$\begin{tikzcd} & N \arrow[dl, "\varphi"'] \arrow[d, "\pi"] & \\ Rf/If \arrow[r] & T(Rf) \arrow[r] & 0 \end{tikzcd}$$

in which the map π can be extended to a map $\varphi\colon N\to Rf/If$. But φ is epic since Jf/If is superfluous in Rf/If. Thus Rf/If is isomorphic to an epimorphic image of N. This contradicts R being left artinian.

To prove (2), we need only note that as JK is quasi-projective $JK\cong\oplus\Sigma Re_\alpha/Ie_\alpha$ where each $Re_\alpha\not\cong Rf$. Thus $Ie_\alpha=0$ for each e_α follows from 1.2. This shows that JK is projective. The proof that JK is hereditary is similar to the proof of 2.2.

Lemma's 2.1, 2.2, 2.3 provide the motivation for the following

Definition. We will say that a left artinian ring R has a *Loewy series decomposition of type qp* if the following conditions hold: For each primitive idempotent f, $J^\alpha f=K_\alpha\oplus P_\alpha$ where P_α is hereditary, and $K_\alpha\cong(Rf/I_\alpha f)^{(n_\alpha)}$ where I_α is some two sided ideal.

The K_α, P_α satisfy,

1. $K_1\supset K_2\supset\ldots\supset K_n=0$ and $JK_\alpha=K_{\alpha+1}\oplus Q_{\alpha+1}$, $Q_{\alpha+1}\subset P_{\alpha+1}$.
2. If $K_1\cong(Rf/I_1f)^{(n_1)}$, $n_1>1$, then $K_\alpha=0$, $\alpha>1$.
3. If $K_1\cong Rf/I_1f$ then for $\alpha>1$ where $K_\alpha\neq 0$, $K_\alpha\cong Rf/I_\alpha f$.

2.4 Proposition. *Let R be a left artinian ring with every left ideal quasi-projective. Then R has a decomposition of type qp.*

Proof. Let f be any primitive idempotent. Then by 2.1 and 2.2, $J^\alpha f=K_\alpha\oplus P_\alpha$ where $K_\alpha\cong(Rf/I_\alpha f)^{(n_\alpha)}$, P_α is hereditary, and I_α is a two sided ideal. To show (1) we use induction to construct $K_{\alpha+1}$ and $P_{\alpha+1}$ from K_α and P_α as follows: Let $J^{\alpha+1}f=J(K_\alpha\oplus P_\alpha)=JK_\alpha\oplus JP_\alpha$. By 2.1 and 2.2, $JK_\alpha=K_{\alpha+1}\oplus Q_{\alpha+1}$ where $Q_{\alpha+1}$ is hereditary and $K_{\alpha+1}\cong(Rf/I_{\alpha+1}f)^{n_{\alpha+1}}$. Clearly $K_{\alpha+1}\subset K_\alpha$. Now $J^{\alpha+1}f=K_{\alpha+1}\oplus Q_{\alpha+1}\oplus JP_\alpha$. Let $P_{\alpha+1}=Q_{\alpha+1}\oplus JP_\alpha$. Then $P_{\alpha+1}$ is hereditary and $Q_{\alpha+1}$ is a direct summand of $P_{\alpha+1}$.

For statement (2), we note that it follows easily from 2.3. For (3) let $K_1\cong Rf/I_1f\subseteq Jf$. Suppose $K_\alpha\cong Rf/I_\alpha f$ and $K_{\alpha+1}\cong(Rf/I_{\alpha+1})^{(n_{\alpha+1})}$ where $n_{\alpha+1}\geqq 1$. Then $Rf\to K_\alpha\to 0$ whence $Jf\to JK_\alpha\to 0$. Now using that Jf/J^2f has one isomorphic copy of $T(Rf)$ we have $T(K_{\alpha+1})\cong T(Rf)$. So $n_{\alpha+1}=1$.

Remark. In the future the terminalogy $K_{\text{subscript}}$, $P_{\text{subscript}}$ will be used to stand for the modules K_α, P_α when $J^\alpha f=K_\alpha\oplus P_\alpha$ whenever R has a decomposition of type *qp* and f is a primitive idempotent.

3. Left artinian rings whose left ideals are quasi-projective

The Loewy series decomposition of type qp will now be used to characterize the rings of this section. An additional property must be satisfied by the above decomposition in order to completely determine the structure of these rings. This is indicated by the following

3.1 LEMMA. *Let R be as in Lemma 2.1. Suppose $J^{\alpha}f=K_{\alpha}\oplus P_{\alpha}$ where $K_{\alpha}\cong(Rf/If)^{(n)}$. Then for any indecomposable projective left ideal P, $P\cong Re$, e a primitive idempotent, such that $P\cap K_{\alpha}=0$, we have $e\cdot K_{\alpha}\neq 0$ if and only if there exists an isomorphic copy of P contained in K_{α}.*

PROOF. If $e\cdot K_{\alpha}\neq 0$, then by 1.2 K_{α} contains a copy of Re whenever $Re\not\cong Rf$. Otherwise 1.1 applies and K_{α} contains a copy of Rf, a contradiction to R artinian.

We now examine the left ideals of rings possessing a decomposition of type qp.

3.2 LEMMA. *Let R be a left artinian ring with a decomposition of type qp. Then for any left ideal $L\subseteq Rf$, f a primitive idempotent, $L\cong M_{\alpha}\oplus N$, where N is hereditary, and $M_{\alpha}\cong(Rf/I_{\alpha}f)^{(n_{\alpha})}$ where M_{α} is a direct summand of K_{α} and $J^{\alpha}f=K_{\alpha}\oplus P_{\alpha}$.*

PROOF. Since $L\subseteq Rf$, there exists α_1 such that $L\subseteq J^{\alpha_1}f$, $L\nsubseteq J^{\alpha_1+1}f$. Since $J^{\alpha_1}f=K_{\alpha_1}\oplus P_{\alpha_1}$ the restriction to L of the canonical projection of $J^{\alpha_1}f$ onto P_{α_1} maps L onto a submodule L_{α_1} of P_{α_1}. As P_{α_1} is hereditary, L_{α_1} is projective, hence $L\cong L_{\alpha_1}\oplus M_{\alpha_1}$ where $L_{\alpha_1}\subseteq P_{\alpha_1}$, $M_{\alpha_1}\subseteq K_{\alpha_1}$.

Now we consider two cases:

Case 1: $\alpha_1=1$, $K_1\cong(Rf/I_1f)^{(n_1)}$, $n_1>1$. Consider the restriction to M_1 of the canonical projection π of K_1 onto each of the indecomposable summands $I\cong Rf/I_1f$ of K_1. If the restriction is epic for one of the indecomposable direct summands I, $M_1\subseteq K_1$ and I quasi-projective imply that $M_1\cong I\oplus M_2$ where $M_2\subseteq K_1$. Now apply the same argument to M_2 as was done to M_1 in case one of the projections onto an indecomposable direct summand of K_1 is epic when restricted to M_2. Since K_1 is a finite direct sum of indecomposable quasi-projective modules, continue the process until

$$M_1\cong(Rf/I_1f)^{(s)}\oplus M_{s+1},\quad s\leqq n_1$$

and $M_{s+1}\subseteq K_1$ has the property that for each $\pi\colon K_1\to I$, when restricted to M_{s+1} is not epic. This means that $\pi(M_{s+1})\subseteq JI$ for all indecomposable I in the direct sum decomposition of K_1. Therefore,

$$M_{s+1}\subseteqq JK_1\subseteqq J(K_1\oplus P_1)=J^2f.$$

But by property 2 of the Loewy series decomposition of type qp, $J^2f=P_2$, P_2 hereditary. Hence M_{s+1} is hereditary. Setting $N=M_{s+1}\oplus L_1$, we have $L\cong N\oplus M_s$ where $M_s\cong(Rf/I_1f)^{(s)}$. Thus the conditions of the lemma are satisfied.

Case 2: $K_{\alpha_1}\cong Rf/I_{\alpha_1}f$. If $M_{\alpha_1}=K_{\alpha_1}$ there is nothing to prove. Otherwise

$$M_{\alpha_1}\subseteqq JK_{\alpha_1}\subseteqq P_{\alpha_2}\oplus K_{\alpha_2}=J^{\alpha_2}f$$

where $\alpha_2=\alpha_1+1$. The projection $\pi\colon J^{\alpha_2}f\to P_{\alpha_2}$ maps M_{α_1} onto a hereditary submodule L_{α_2} of P_{α_2}. Hence $M_{\alpha_1}\cong L_{\alpha_2}\oplus M_{\alpha_2}$, $M_{\alpha_2}\subseteqq K_{\alpha_2}$. If $K_{\alpha_2}=0$, we are through.

Otherwise, property 3 of a Loewy series of type qp implies that $K_{\alpha_2}\cong Rf/I_{\alpha_2}f$. If $K_{\alpha_2}=M_{\alpha_2}$ there is nothing more to prove. Otherwise, using that R is artinian, we can continue the process s number of times until we obtain M_{α_s} such that M_{α_s} is hereditary or $M_{\alpha_s}\cong L_{\alpha_{s+1}}\oplus M_{\alpha_{s+1}}$, $L_{\alpha_{s+1}}$ hereditary, $M_{\alpha_{s+1}}\cong Rf/I_{\alpha_{s+1}}f$. Let $N=\oplus\sum_{i=1}^{s+1} L_{\alpha_i}$. Then $L\cong N$ or $L\cong N\oplus M_{\alpha_{s+1}}$. In either case the lemma is satisfied. This completes the proof.

3.3. Lemma. *Suppose R has a decomposition of type qp and satisfies the conclusion of* 3.1. *Then K_α is projective relative to P where P is a projective left ideal such that $K_\alpha\cap P=0$. Thus $K_\alpha\oplus P$ is quasi-projective.*

Proof. Clearly the last statement follows from the first and the remark before 1.1. Recall that $K_\alpha\cong(Rf/If)^{(n)}$ where K_α is a direct summand of $J^\alpha f$, and f is a primitive idempotent. Now $P=\oplus\sum^{k}P_i$, $P_i\cong Re_i$, where each $e_i\in R$ is a primitive idempotent. We show that K_α is projective relative to P by first showing that it is projective relative to each P_i. So let g be a map $g\colon K_\alpha\to P_i/K_i$, P_i/K_i a factor module of P_i, and π a map $\pi\colon P_i\to P_i/K_i$ which is epic. Now consider the module

$$H=\{x\in P_i\colon\ \pi(x)\in\mathrm{Im}\,(g)\}.$$

By 3.2,

$$H=H_1\oplus\ldots\oplus H_{t-1}\oplus(H_t)^{(s)}\tag{1}$$

where $H_j\cong Rf_j$, $1\le j\le t-1$, $f_j\in R$ a primitive idempotent, and $H_t=Re_i/Ie_i$, H_t quasi-projective. In the following discussion set $e_i=f_t$.

Let $H=M_1\oplus M_2$ where M_2 is the direct sum of all the indecomposable modules in (1) contained in the ker (π). Hence for each indecomposable module $H_j\subseteq M_1$, K_α has a composition factor isomorphic to $T(Rf_j)$. This implies that $f_jK_\alpha\ne0$ for each $H_j\subseteq M_1$. Clearly each $H_j\cap K_\alpha=0$ for $1\le j\le t-1$. Thus for each $H_j\subseteq M_1$, $(1\le j\le t-1)$ there is an isomorphic copy of H_j contained in K_α since R satisfies the conclusion of 3.1. By the same argument, if $H_t\subseteq M_1$, $P_i\cap K_\alpha=0$ implies that K_α contains an isomorphic copy of P_i. These two statements imply that K_α is projective relative to M_1. Thus it is possible to extend g to M_1 (and hence to H). So K_α is projective relative to Re_i for each i, and is therefore projective relative to P.

3.4 Lemma. *Let R be a left artinian ring. Suppose R has a decomposition of type qp and satisfies the conclusion of* 3.1. *If $1=\sum^{k}e_i$, where $\{e_i\}$ is a set of primitive orthogonal idempotents, then for any left ideal $L\subseteq R$, L is quasi-projective and $L\cong$ $\cong\oplus\sum^{k}(M_{\alpha_i}\oplus N_i)$ where $M_{\alpha_i}\oplus N_i\subseteq Re_i$ and N_i is hereditary, M_{α_i} a direct summand of K_{α_i}, K_{α_i} as in* 3.2.

Proof. We first show that any left ideal of the form $L=L_1e_1\oplus\ldots\oplus L_ke_k$ is quasi-projective where L_i, $i=1,\ldots,k$, are left ideals.

By 3.2, $L_ie_i=M_{\alpha_i}\oplus N_i$ where $M_{\alpha_i}\cong(Re_i/Ie_i)^{(n_{\alpha_i})}$ and N_i is hereditary. Thus $L=\oplus\Sigma M_{\alpha_i}\oplus N_i$. Using 3.3, M_{α_i} is projective relative to N_j for all $1\le j\le k$. Since $M_{\alpha_i}\subseteq Re_i$, and $Re_i\cap Re_j=0$ for $j\ne i$, 3.3 implies that M_{α_i} is projective relative to Re_j. Since M_{α_i} is a direct sum of factor modules of Re_j, M_{α_i} is projective relative to

M_{α_j} $(i \neq j)$. Now using the remark before 1.1 and the quasi-projectivity of each M_{α_i}, it is easily seen that $\oplus \Sigma L_i e_i \cong \oplus \Sigma M_{\alpha_i} \oplus N_i$ is quasi-projective.

We need only show that $L = \oplus \Sigma L_i e_i$ for suitably chosen left ideals L_i, $i=1, ..., k$. As $L \subseteqq \oplus \Sigma L e_i$, and $\oplus \Sigma L e_i$ is quasi-projective by the previous remarks, Le_1 is projective relative to L by the remark before 1.1. Thus, the canonical epimorphism $\pi_1 : L \to Le_1$ given by right multiplication by e_1 splits. Hence $L \cong Le_1 \oplus L_2$ where $L_2 \subseteqq L$, and $L_2 e_1 = 0$. Now there exists a canonical epimorphism π_2 of L_2 onto $L_2 e_2$. Using that $L_2 \subseteqq \oplus \Sigma L_2 e_i$ quasi-projective, we can apply the same argument on L_2 as on L. Thus $L_2 \cong L_2 e_2 \oplus L_3$ where $L_3 \subseteqq L_2$ and $L_3 e_2 = 0$. By the application of this argument for at most k times, L can be expressed as

$$L \cong Le_1 \oplus L_2 e_2 \oplus \ldots \oplus L_k e_k \oplus L_{k+1}$$

where $L_i \cong L_i e_i \oplus L_{i+1}$, $L_{i+1} \subseteqq L_i$, and $L_{i+1} e_i = 0$. Since $L \supseteqq L_2 \supseteqq \ldots \supseteqq L_k \supseteqq L_{k+1}$ and $L_{i+1} e_i = 0$, we have $L_{k+1} e_i = 0$ $(1 \leqq 1 \leqq k)$. So $L_{k+1} = 0$. Therefore $L \cong Le_1 \oplus \oplus L_2 e_2 \oplus \ldots \oplus L_k e_k$. By the remarks at the beginning of the proof, L is quasi-projective,

Now the following theorem can be proved which completely characterizes the left artinian rings whose left ideals are quasi-projective.

3.5 Theorem. *Let R be a left artinian ring. Then R has every left ideal quasi-projective if and only if R satisfies the following conditions:*

(1) *For each primitive idempotent f, Rf has a decomposition of type qp.*

(2) *For each K_α such that $J^\alpha f = K_\alpha \oplus N_\alpha \subseteqq Rf$, and indecomposable projective left ideal P, $P \cong Re$, e a primitive idempotent, such that $P \cap K_\alpha = 0$, either $e \cdot K_\alpha = 0$ or K_α contains an isomoprhic copy of P.*

Proof. $\Rightarrow$ follows from 2.4 and 3.1. $\Leftarrow$ is a consequence of 3.4.

4. Examples

This section presents a number of examples of rings which serve to illustrate the main features of the decomposition used to characterize rings with every left ideal quasi-projective. The first two examples show that such rings cannot be completely characterized by their Loewy decomposition for each principal indecomposable module — we really need to know the two sided ideals of the ring. The following notation will be used. The 2-sided ideal $l_R(M) = \{x \in R : xM = 0\}$ is the left annihilator of the module M. It is known that for left artinian rings M is quasi-projective if and only if M is projective over $R/l_R(M)$ [2].

1. Let F be a field and R the ring of matrices of the form,

$$R = \left\{ \begin{vmatrix} \alpha & \lambda_4 & \lambda_3 & \lambda_2 \\ 0 & \gamma & 0 & \lambda_1 \\ 0 & 0 & \gamma & \lambda_1 \\ 0 & 0 & 0 & \gamma \end{vmatrix} \right\} \quad (\lambda_i \in F,\ i = 1, \ldots, 4,\ \alpha, \gamma \in F)$$

with primitive idempotents

$$e_1 = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Then $Je_2 \cong T(Re_1) \oplus T(Re_1) \oplus K$ where K is a uniserial left ideal with $T(K) \cong$ $\cong T(Re_2)$, $S(K) \cong T(Re_1)$. So the Loewy series decomposition for Re_2 is

$$\begin{array}{r|c|c|} \multicolumn{1}{r}{} & \multicolumn{1}{|c}{\cong T(Re_2)} & \multicolumn{1}{c}{} \\ \cline{2-3} \begin{matrix} T(Re_2) \\ T(Re_1) \end{matrix} & T(Re_1) & T(Re_1) \end{array}$$

However, the decomposition for Re_2 is not of type qP. For R does not have every left ideal quasi-projective as the uniserial left ideal

$$K = \left\{ \begin{bmatrix} 0 & 0 & 0 & \lambda_2 \\ & 0 & 0 & \lambda_1 \\ & & 0 & \lambda_1 \\ & & & 0 \end{bmatrix} \right\}$$

is not quasi-projective, since K is not projective over $R/l_R(K) = R/K$.

The next example gives a ring with every left ideal quasi-projective and with a Loewy series decomposition the same as the ring in 1.

2. Let F be a field and R the ring of matrices of the form,

$$R = \left\{ \begin{bmatrix} \alpha & \lambda_3 & \lambda_2 & & & \\ & \gamma & \lambda_1 & & 0 & \\ & & \gamma & & & \\ & & & \alpha & \lambda_3 & \lambda_4 \\ & 0 & & & \gamma & \lambda_1 \\ & & & & & \gamma \end{bmatrix} \right\}$$

with primitive idempotents

$$e_1 = \begin{bmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}.$$

It is easily checked that

$$K = \left\{ \begin{bmatrix} 0 & 0 & \lambda & & & \\ & 0 & \lambda_1 & & 0 & \\ & & 0 & & & \\ & & & 0 & 0 & \lambda \\ & 0 & & & 0 & \lambda_1 \\ & & & & & 0 \end{bmatrix} \right\}$$

where K is generated by the element

$$x = \begin{bmatrix} 0 & 0 & 0 & & & \\ & 0 & 1 & & 0 & \\ & & 0 & & & \\ & & & 0 & 0 & 0 \\ & 0 & & & 0 & 1 \\ & & & & & 0 \end{bmatrix}.$$

Thus $Je_2 \cong T(Re_1) \oplus T(Re_1) \oplus K$ where $K = Rx$ is a uniserial module such $T(K) \cong$ $\cong T(Re_2)$, $S(K) \cong T(Re_1)$.

So the Loewy series decomposition is of form,

$$\begin{array}{c|c|c|} & T(Re_2) & \\ \hline \begin{matrix} T(Re_2) \\ T(Re_1) \end{matrix} & T(Re_1) & T(Re_1) \end{array}.$$

It is easily checked that K is projective over $R/l(K)$ and in fact that every left ideal is quasi-projective.

3. This example gives a ring with every left ideal quasi-projective, with a Loewy series decomposition for a principle indecomposable Re_2 such that Je_2/J^2e_2 has more than one copy of $T(Re_2)$. Let K be a field, and R the set of matrices of the form.

$$R = \left\{ \begin{bmatrix} \alpha & \lambda_3 & \lambda_2 & & & \\ & \gamma & \lambda_1 & & 0 & \\ & & \gamma & & & \\ & & & \alpha & \lambda_3 & \lambda_5 \\ & 0 & & & \gamma & \lambda_4 \\ & & & & & \gamma \end{bmatrix} \right\} \quad (\alpha, \gamma \in K,\ \lambda_i \in K,\ i = 1, \ldots, 5).$$

with primitive idempotents

$$e_2 = \begin{bmatrix} 0 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}.$$

It is easily checked that R has every left ideal quasi-projective and that $Je_2 \cong T(Re_1) \oplus \oplus K_1 \oplus K_2$ where $K_1 \cong K_2$ and $T(K_1) \cong T(Re_2)$, $S(K_1) \cong T(Re_1)$.

So the Loewy series decomposition for Re_2 is

$$\begin{array}{r|c|c|} \multicolumn{2}{c|}{} & T(Re_2) \\ \cline{2-3} \begin{array}{r} T(Re_2) \\ T(Re_1) \end{array} & \begin{array}{c} T(Re_2) \\ T(Re_1) \end{array} & T(Re_1) \end{array}$$

4. This example shows that condition (2) of 3.5 is necessary by exhibiting a ring with a Loewy series of type qp without having all left ideals quasi-projective.

Let S be any local uniserial ring with a composition series of length 2, so that $JS \cong T(S)$. Define R to be the matrix ring $M_n(S)$, n an integer such that $n > 1$. Then for any primitive idempotent $e \in R$, $T(Re) \cong S(Re)$. So it is easily seen that R has a Loewy series decomposition of type qp. But for f any primitive idempotent such that $Rf \cap Re = 0$, we must have $fJe \neq 0$. Thus condition 2 does not hold. Clearly R does not have every left ideal quasi-projective since $Re \oplus T(Re)$ is not quasi-projective.

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NEW ESTIMATION FOR THE LEBESGUE FUNCTION OF LAGRANGE INTERPOLATION

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To Professor P. Erdős on his 70th birthday

1. Introduction

Let $Z=\{x_{kn}\}$, $n=1,2,\ldots;\ k=1,2,\ldots,n$, be any triangular matrix with

$$-1 \equiv x_{n+1,n} \leqq x_{nn} < \ldots < x_{1n} \leqq x_{0n} \equiv 1 \quad (n=1,2,\ldots). \tag{1.1}$$

Putting, sometimes omitting the superfluous notations,

$$\omega(x) = \omega_n(Z,x) = \prod_{k=1}^{n} (x-x_k), \tag{1.2}$$

$$l_k(x) = l_{kn}(Z,x) = \frac{\omega(x)}{\omega'(x_k)(x-x_k)} \quad (k=1,2,\ldots,n) \tag{1.3}$$

are the corresponding fundamental polynomials of degree $n-1$ of the Lagrange interpolation. It is well known that the so called Lebesgue function and Lebesgue constant

$$\lambda_n(x) = \lambda_n(Z,x) = \sum_{k=1}^{n} |l_k(x)|, \quad \lambda_n = \lambda_n(Z) = \max_{-1\leqq x\leqq 1} \lambda_n(x) \tag{1.4}$$

play a fundamental role in the study of the convergence and divergence properties of the Lagrange interpolatory polynomials. Here we quote two results which, in certain sense, generalize the previous statements of G. FABER [1] and S. BERNSTEIN [2].

In 1958, P. ERDŐS [4] proved as follows.

THEOREM 1.1. *Let ε and A be any given positive numbers. Then, considering arbitrary matrix Z, the measure of the set in $x\,(-\infty<x<\infty)$ for which*

$$\lambda_n(x) \leqq A \quad \text{if} \quad n \geqq n_0(A,\varepsilon), \tag{1.5}$$

is less than ε.

The following result, proved recently by P. ERDŐS and P. VÉRTESI [11], gives the best possible order.

THEOREM 1.2. *Let $\varepsilon>0$ be any given number. Then for arbitrary matrix Z there exist sets $H_n=H_n(\varepsilon,Z)$, $|H_n|\leqq\varepsilon$, and $\eta=\eta(\varepsilon)$, $\eta>0$ such that*

$$\lambda_n(x) > \eta \ln n \quad \text{if} \quad x\in[-1,1]\setminus H_n \quad \text{and} \quad n \geqq n_0(\varepsilon). \tag{1.6}$$

Here, as it comes from the proof, $\eta=c\varepsilon^3$. A natural question is whether this estimation can not be improved. Or more exactly: Prove the relation $\eta=c\varepsilon$ which, considering the Chebyshev nodes, would be the best possible order, (see ERDŐS [4], especially Theorem 2).

2. Results

2.1. In this part we answer the above problem in the following general form.

THEOREM 2.1. *There exists a positive constant c such that if* $\varepsilon=\{\varepsilon_n\}$ *is any sequence of positive numbers, then for an arbitrary matrix Z, there exist sets* $H_n=H_n(\varepsilon, Z)$, $|H_n|\leqq\varepsilon_n$, *for which*

$$\lambda_n(x) > c\varepsilon_n \ln n \tag{2.1}$$

if $x\in[-1, 1]\setminus H_n$ *and* $n = 1, 2, \ldots$.

2.2. Choosing $\varepsilon_n=A/(c\ln n)$, or $\varepsilon_n=\varepsilon$, we get improvements of Theorems 1.1 and 1.2, respectively.

2.3. Further, it is easy to gain the following

COROLLARY 2.2. *If* $S_n\subseteqq[-1, 1]$ *are arbitrary measurable sets then, using the above notations, for any Z,*

$$\int_{S_n} \lambda_n(x)\, dx > (|S_n|-\varepsilon_n)\, c\varepsilon_n \ln n \quad \text{if} \quad n = 1, 2, \ldots. \tag{2.2}$$

The case $S_n=S=[a, b]$ was treated by P. ERDŐS and J. SZABADOS [6].

2.4. Further remarkable results on $\lambda_n(x)$ including extremal problems can be found in the papers [3], [5], [8], [9] and [10].

3. Proof

If $\varepsilon_n\leqq(c\ln n)^{-1}$, then (2.1) obviously holds if $x\neq x_{kn}$ $(k=1, 2, \ldots, n)$ because $\lambda_n(x)>1$ whenever e.g., $n\geqq6$, if we exclude the nodes (see [12], Lemma IV for $x_n<x<x_1$; if, e.g. $x_1<x\leqq1$ then clearly $l_1(x)>1$). So *from now on we shall suppose*

$$\varepsilon_n > \frac{1}{c\ln n} \quad \text{and} \quad n \geqq 6. \tag{3.1}$$

3.1. In what follows let $J_{kn} = [x_{k+1,n}, x_{kn}]$ $(k = 0, 1, \ldots, n;\ n = 1, 2, \ldots)$. If $|J_{kn}|\leqq\delta_n\overset{\text{def}}{=}n^{-1/6}$ we say that the interval is *short*; the others are the *long* ones.

3.2. First we settle the long intervals: As in our paper [7], Lemma 4.4 we can prove

LEMMA 3.1. *Let* $|J_{kn}|>\delta_n$ *(k is fixed,* $0\leqq k\leqq n$*). Then for any* $(\ln n)^{-2}\leqq s_n\leqq1/4$ *we can define the index* $t=t(k, n)$ *and the set* $h_{kn}\subset J_{kn}$ *so that* $|h_{kn}|\leqq4s_n|J_{kn}|$, *moreover*

$$|l_{tn}(x)| \geqq 3^{\sqrt{n}} \quad \text{if} \quad x\in J_{kn}\setminus h_{kn} \quad \text{and} \quad n \geqq n_1. \tag{3.2}$$

(n_1 is an absolute positive constant.)

Now if $s_n=1/\ln^2 n$, *we obtain* (2.1) *for the long intervals apart from the set* H_{1n} *of measure* $\leqq8/\ln^2 n$.

3.3. To settle the short intervals we introduce the following notations.

$$(3.3)\qquad \begin{cases} J_k(q_k) = J_{kn}(q_{J_{kn}}) = [x_{k+1}+q_k|J_k|,\ x_k-q_k|J_k|], \\ \overline{J_k} = \overline{J_k(q_k)} = \overline{J_{kn}(q_{J_{kn}})} = J_k\setminus J_k(q_k), \quad 0 \leqq k \leqq n, \end{cases}$$

where $0<q_k\leqq 1/2$. Let $z_k=z_k(q_k)$ be defined by

$$(3.4)\qquad |\omega_n(z_k)| = \min_{x\in J_k(q_k)} |\omega_n(x)|, \quad k=0,1,\dots,n,$$

finally, let

$$(3.5)\qquad |J_i, J_k| = \max(|x_{i+1}-x_k|, |x_{k+1}-x_i|), \quad 0 \leqq i, k \leqq n,$$

$$(3.6)\qquad \varrho(J_i, J_k) = \min(|x_{i+1}-x_k|, |x_{k+1}-x_i|), \quad 0 \leqq i, k \leqq n.$$

LEMMA 3.2. *If* $1\leqq k, r\leqq n$, *then*

$$(3.7)\qquad |l_k(x)|+|l_{k+1}(x)| > \frac{1}{4}\frac{|\omega_n(z_r)|}{|\omega_n(z_k)|}\frac{|\overline{J_k}|}{|J_r, J_k|}, \quad n \geqq 6,$$

if $x\in J_r(q_r)$, $\varrho(J_r, J_k)\geqq\delta_n$ *and* $|J_r|\leqq\delta_n$.

The proof is similar to [7], 4.1.1. First we verify

$$(3.8)\qquad |l_s(x)| = \left|\frac{\omega(x)}{\omega'(x_s)(x-x_s)}\right| = \frac{|\omega(x)|}{|\omega(z_r)|}\frac{|z_r-x_s|}{|x-x_s|}|l_s(z_r)| \geqq \frac{1}{2}|l_s(z_r)|$$

if $s=k, k+1$ and $x\in J_r(q_r)$. Indeed,

$$\frac{|z_r-x_s|}{|x-x_s|} \geqq \frac{|z_r-x_s|+\delta_n-\delta_n}{|z_r-x_s|+\delta_n} \geqq 1-\frac{\delta_n}{2\delta_n} = 1/2,$$

which gives (3.8). So we can write if, e.g. $r<k$

$$|l_k(x)|+|l_{k+1}(x)| \geqq \frac{1}{2}[|l_k(z_r)|+|l_{k+1}(z_r)|] =$$

$$= \frac{1}{2}\frac{|\omega(z_r)|}{|\omega(z_k)|}\left[|l_k(z_k)|\frac{x_k-z_k}{z_r-x_k}+|l_{k+1}(z_k)|\frac{z_k-x_{k+1}}{z_r-x_{k+1}}\right] \geqq$$

$$\geqq \frac{1}{2}\frac{|\omega(z_r)|}{|\omega(z_k)|}\frac{q_k|J_k|}{|J_r, J_k|}[|l_k(z_k)|+|l_{k+1}(z_k)|] \quad (x\in J_r(q_r)),$$

which is (3.7), considering that $2q_k|J_k|=|\overline{J_k}|$ and $[\dots]>1$ ([12], Lemma IV).

3.4. Using mutatis mutandis the notations of 3.3 we state

LEMMA 3.3. *Let* $I_k=[a_k, b_k]$, $1\leqq k\leqq t$, $t\geqq 2$, *be any* t *intervals in* $[-1, 1]$ *with* $|I_k\cap I_j| = 0$ $(k\neq j)$, $|I_k| \leqq \delta$ $(1 \leqq k \leqq t)$, $\sum_{k=1}^{t}|\overline{I_k}| = \mu$. *Let* $\xi\geqq\delta$ *be fixed. Supposing that for certain integer* $R\geqq 4$ *we have* $\mu\geqq 2^R\xi$, *there exists the index* s, $1\leqq s\leqq t$, *such that*

$$F=\sum_{\substack{k=1\\ \varrho(I_s,I_k)\geqq\xi}}^{t}\frac{|\overline{I_k}|}{|I_s,I_k|}\geqq\frac{R}{8}\mu-\frac{3}{2}. \tag{3.9}$$

I_s *will be called accumulation interval* of $\{I_k\}_{k=1}^t$.

Note that we do not require $b_k\leqq a_{k+1}$.

Let us remark that considering an arbitrary fixed interval [a, b] instead of [−1, 1], we obtain by analogous argument

$$F\geqq\frac{1}{b-a}\left(\frac{R}{4}\mu-3\right). \tag{3.10}$$

The lemma and its proof correspond to [7], 4.1.3. and [11], 3.4.

Indeed, dropping the interval I_j containing the middle point of $[-1,1]$, and bisecting the same interval $[-1,1]$, we have (say) in $[0,1]$ a set of measure $\geqq(\mu-|\overline{I_j}|)/2\geqq$ $\geqq(\mu-\delta)/2$ consisting of certain $\overline{I_k}$. Doing the same, after the l-th bisection we obtain that interval of length 2^{1-l} which contains certain $\overline{I_k}$ of aggregate measure $>2^{-l}\mu-\delta\geqq2^{-l-1}\mu\geqq\xi$ for $1\leqq l\leqq p\overset{\text{def}}{=}R-1$.

Consider these intervals $L_1^*, L_1^*, \ldots, L_1^*$ (Fig. 1).

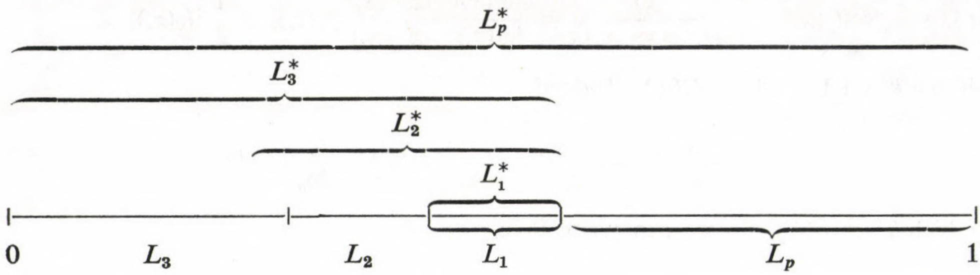

Fig. 1,

Obviously $|L_l^*|=2^{l-p}(\geqq2\xi)$ contains at least 2^{l-1} sets $\overline{I_k}$ because

$$\sum_{\substack{k\\ \overline{I_k}\subset L_l^*}}|\overline{I_k}|\geqq2^{l-p-2}\mu\quad(1\leqq l\leqq p). \tag{3.11}$$

Let $L_1=L_1^*$ further $L_l=L_l^*\setminus L_{l-1}^*$ $(2\leqq l\leqq p)$ (see Figure 1). If s is an index for which $I_s\subset L_1$, we can write

$$F\geqq{\sum_{l=1}^{p}}'\sum_{\substack{k\\ I_k\subset L_l}}\frac{|\overline{I_k}|}{|I_s,I_k|}\overset{\text{def}}{=}B,$$

where the dash means that we exclude l whenever $\varrho(I_s,L_l)<\xi$. To estimate B, let

$$\sum_{\substack{k\\ I_k\subset L_l}}|\overline{I_k}|\overset{\text{def}}{=}\alpha_l\mu,\quad 1\leqq l\leqq p. \tag{3.12}$$

By (3.11) and construction we can write

$$\mu\sum_{l=1}^{i}\alpha_l \geqq 2^{i-p-2}\mu \quad (1\leqq i\leqq p), \tag{3.13}$$

$$|I_s, I_i| \leqq 2^{l-p} \quad \text{if} \quad I_i\subset L_l \quad (1\leqq l\leqq p). \tag{3.14}$$

We shall use the relation

$$\alpha_l \leqq 2^{l-2}\alpha_1 \quad (2\leqq l\leqq p). \tag{3.15}$$

(Indeed, by construction $\alpha_2\leqq\alpha_1$, $\alpha_l\leqq\sum_{i=1}^{l-1}\alpha_i\leqq 2\sum_{i=1}^{l-2}\alpha_i$, $3\leqq l\leqq p$, from where we get (3.15)). Now by (3.14), (3.12), (3.13), the Abel transformation, finally by (3.15) we obtain

$$B\geqq \mu 2^p\sum_{l=1}^{p}{}' 2^{-l}\alpha_l \geqq \mu 2^p\left(\sum_{l=1}^{p}2^{-l}\alpha_l-3\max_{1\leqq l\leqq p}2^{-l}\alpha_l\right)$$

$$\geqq \mu 2^p\left[\sum_{l=1}^{p-1}2^{-l-1}\left(\sum_{i=1}^{l}\alpha_i\right)+2^{-p}\sum_{i=1}^{p}\alpha_i-3\cdot 2^{-2}\alpha_1\right]\geqq$$

$$\geqq \mu 2^p\left[\sum_{l=1}^{p-1}2^{l-p-2-l-1}+2^{-p-2}-3(2^{p+1}\mu)^{-1}\right]=\frac{p+1}{8}\mu-\frac{3}{2},$$

as it was stated.

3.5. Now we decide $q_{J_{kn}}$ for the short intervals. For this aim we define the index set K_n' and the set D_n' by $|J_{kn}|\leqq\delta_n$ if $k\in K_n'$, $|J_{kn}|>\delta_n$ if $k\notin K_n'$, $D_n'=\bigcup_{K\in K_n'}J_{kn}$. If y_k denotes the middle point of J_k, let $k\in K_n\overset{\text{def}}{=}\{K_n'\}\setminus\{0,n\}$, further

$$\beta_{kn} = \max\{y\colon x_{k+1}\leqq y\leqq y_k \text{ and (2.1) does not hold for } y\},$$
$$\gamma_{kn} = \min\{y\colon y_k\leqq y\leqq x_k \text{ and (2.1) does not hold for } y\},$$
$$d_{kn} = \max[(x_k-\gamma_k),(\beta_k-x_{k+1})],$$

$$q_k = q_{J_{kn}} = d_k/|I_k|, \quad k\in K_n. \tag{3.16}$$

Using that $\lambda_n(x_i)=1$ $(1\leqq i\leqq n)$, we obtain that $0<q_k<1/2$.

It is important to remark that (2.1) *holds whenever x is from the interior of* $J_k(q_k)$ $(k\in K_n)$.

For the remaining "bad" sets $\bar{J}_k$ *we shall prove*

$$\sum_{k\in K_n}|\bar{J}_k| \overset{\text{def}}{=} \mu_n \leqq \frac{\varepsilon_n}{2} \quad \text{if} \quad n\geqq n_2. \tag{3.17}$$

(n_2 is an absolute constant.)

To prove (3.17) it is enough to consider those indices $\{n_i\}_{i=1}^{\infty}=N$ for which $\mu_{n_i}\geqq\varepsilon_{n_i}/3$. We can use Lemma 3.3 for $D_n=\bigcup J_k$, $k\in K_n$, with $\mu=\mu_n$, $\xi=\delta=\delta_n$ and $R=[{}^2\log n^{1/7}]$ if $n\in N$ and $n\geqq n_2$ (shortly $n\in N_1$). Denote by $M_1=M_{1n}$ the accumulation interval. Dropping M_1, we apply Lemma 3.3 again for the remaining intervals of D_n with $\mu=\mu_n-|\bar{M}_1|>\mu_n/2$ and the above ξ, δ and R, supposing $\mu_n\geqq\xi\,2^{R+1}$ if $n\in N_1$. We get the accumulation interval M_2. At the i-th step $(2\leqq i\leqq\psi_n)$

we drop $M_1, M_2, \ldots, M_{i-1}$ and apply Lemma 3.3 for the remaining intervals of D_n with $\mu=\mu_n-\sum_{j=1}^{i-1}|\overline{M}_i|$ using the same ξ, δ and R; ψ_n denotes the first index for which

$$\sum_{i=1}^{\psi_n-1}|\overline{M}_i| \leqq \frac{\mu_n}{2} \quad \text{but} \quad \sum_{i=1}^{\psi_n}|\overline{M}_i| > \frac{\mu_n}{2} \quad (n\in N_1). \tag{3.18}$$

Denoting by $M_{\psi_n+1}, M_{\psi_n+2}, \ldots, M_{\varphi_n}$ the further (i.e. not accumulation) intervals of D_n, by (3.9) we get for $n\in N_1$

$$\sum_{k=r}^{\varphi_n}{}' \frac{|\overline{M}_k|}{|M_r, M_k|} \geqq \frac{\mu_n \ln n}{112}-\frac{3}{2} \geqq \frac{\mu_n \ln n}{2\cdot 112}, \quad 1\leqq r\leqq \psi_n, \tag{3.19}$$

if $\varepsilon_n \geqq 9\cdot 112/\ln n$. We shall see that $1/c>9\cdot 112$, i.e. this condition can be satisfied (see (3.1)). (Here and later the dash indicates that we omit those indices k for which $\varrho(M_r, M_k)<\varrho_n$).

3.6. By the definition (3.16) of q_k we can choose points $u_{in}\in M_{in}(q_{in}/2)$ such that (2.1) does not hold $(1\leqq i\leqq \varphi_n, n\in N_1)$.

If for a fixed $n\in N_1$ there exists an index t $(1\leqq t\leqq \varphi_n)$ such that

$$\lambda_n(u_{tn}) \geqq 2c\mu_n \ln n, \tag{3.20}$$

(where $c>0$ will be determined later), by $c\varepsilon_n \ln n \geqq \lambda_n(u_{tn})$ we obtain (3.17) for this n. *We shall verify* (3.20) *for arbitrary* $n\in N_1$ *with suitable* $t=t(n)$. Indeed, let us suppose that for a certain $m\in N_1$

$$\lambda_m(u_{rm}) < 2c\mu_m \ln m \quad \text{where} \quad u_{rm}\in M_{rm}(q_{rm}/2), \quad 1\leqq r\leqq \varphi_m. \tag{3.21}$$

Then by (3.21) we have

$$\sum_{r=1}^{\varphi_m}|\overline{M}_{rm}|\,\lambda_m(u_{rm}) < 2c\mu_m^2 \ln m \quad \text{where} \quad m\in N_1. \tag{3.22}$$

On the other hand, for an arbitrary $n\in N_1$ we can write by (3.7), with $\bar{z}_k$ corresponding to (3.4),

$$|\overline{M}_r| \sum_{k=1}^{n}|l_k(u_r)| \geqq \frac{1}{2}|\overline{M}_r| \sum_{k\in K_n}{}' [|l_k(u_r)|+|l_{k+1}(u_r)|] >$$

$$> \frac{1}{8}|\overline{M}_r| \sum_{k=1}^{\varphi_n}{}' \frac{|\omega(\bar{z}_r)|}{|\omega(\bar{z}_k)|}\frac{|\overline{M}_k|}{|M_r, M_k|} \quad (1\leqq r\leqq \varphi_n),$$

so by (3.18) and (3.19)

$$\sum_{r=1}^{\varphi_n}|\overline{M}_r|\lambda_n(u_r) > \frac{1}{8}\sum_{r=1}^{\varphi_n}\sum_{k=1}^{\varphi_n}{}' \frac{|\omega(\bar{z}_r)|}{|\omega(\bar{z}_k)|}\frac{|\overline{M}_r||\overline{M}_k|}{|M_r, M_k|} =$$

$$= \frac{1}{8}\sum_{r=1}^{\varphi_n}\sum_{k=r}^{\varphi_n}{}' \left[\frac{|\omega(\bar{z}_r)|}{|\omega(\bar{z}_k)|}+\frac{|\omega(\bar{z}_k)|}{|\omega(\bar{z}_r)|}\right]\frac{|\overline{M}_r||\overline{M}_k|}{|M_r, M_k|} \geqq$$

$$\geqq \frac{1}{8}\sum_{r=1}^{\psi_n}|\overline{M}_r| \sum_{k=1}^{\varphi_n}{}' \frac{|\overline{M}_k|}{|M_r, M_k|} > \frac{\mu_u^2 \ln n}{8\cdot 2\cdot 2\cdot 112} = 2c\mu_n^2 \ln n$$

if $c=1/(64\cdot 112)=1/7168$. This contradicts (3.22), i.e. (3.20) is true for any $n\in N_1$ with suitable $t=t(n)$. This proves (3.17).

3.7. Now we estimate $|H_n|$. If J_{0n} is short it should belong to H_n. The same should be done with J_{nn}. So by (3.1), 3.2 and (3.17), with $n_0=\max(6, n_1, n_2)$

$$|H_n| \leqq |H_{1n}|+\mu_n+2\delta_n \leqq \varepsilon_n \quad \text{if} \quad n \geqq n_0,$$

which completes the proof if $n\geqq n_0$.

3.8. Obviously, we can suppose $\varepsilon_n\leqq 2$, $n=1, 2, \ldots$. Using this and $\lambda_n(x)\geqq 1$ we obtain the theorem with another $c>0$.

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GENERALISED SMOOTH FUNCTIONS

By

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Introduction. Let f be a real function defined in some neighbourhood of the point x_0 on the real line R, and let $f(x_0)=\alpha_0$. If there exist real numbers $\alpha_2, \alpha_4, \dots, \alpha_{2k}$ depending on x_0 but not on h such that

$$\frac{1}{2}\{f(x_0+h)+f(x_0-h)\} = \sum_{r=0}^{k} \frac{h^{2r}}{(2r)!}\alpha_{2r}+o(h^{2k}),$$

then α_{2k} is called the summetric de la Vallée Poussin (d.l.V.P.) derivative of f at x_0 of order $2k$ and is denoted by $D^{2k}f(x_0)$. (It follows from the definition that if $D^{2k}f(x_0)$ exists, then $D^{2r}f(x_0)$ also exist for all r, $0\leqq r\leqq k$, where $D^0f(x_0)$ is $f(x_0)$.) Similarly if there are numbers $\beta_1, \beta_3, \dots, \beta_{2k+1}$ depending on x_0 but not on h such that

$$\frac{1}{2}\{f(x_0+h)-f(x_0-h)\} = \sum_{r=0}^{k} \frac{h^{2r+1}}{(2r+1)!}\beta_{2r+1}+o(h^{2k+1}),$$

then β_{2k+1} is called the symmetric d.l.V.P. derivative of f at x_0 of order $2k+1$ and is denoted by $D^{2k+1}f(x_0)$.

Suppose that $D^{2m-2}f(x_0)$ exists, $m\geqq 1$, and write

$$\theta_{2m}(f;\, x_0, h) = \frac{(2m)!}{h^{2m}}\left[\frac{1}{2}\{f(x_0+h)+f(x_0-h)\} - \sum_{r=0}^{m-1}\frac{h^{2r}}{(2r)!}D^{2r}f(x_0)\right].$$

Then f is said to be *smooth at* x_0 *of order* $2m$, [3] (or $2m$-smooth) if $\lim_{h\to 0} h\theta_{2m}(f;\, x_0, h) = 0$.

Smoothness of order $2m+1$ is defined similarly. It is clear from the definition that if f is smooth of order r then it is smooth of order $r-2$ and that smoothness of order 2 is the usual smoothness. It can be verified that Zygmund's definition of generalised smoothness of order r [9; II; p. 62] is equivalent to the present definition of smoothness of order $r+1$ (see [5]).

With the same assumption on f if there are numbers $v_0, v_1, v_2, \dots, v_m$ depending on x_0 but not on h such that

$$f(x_0+h) = \sum_{r=0}^{m}\frac{h^r}{r!}v_r+o(h^m),$$

then v_m is called the unsymmetric d.l.V.P. derivative (also called the Peano-derivative) of f at x_0 of order m and is denoted by $f_{(m)}(x_0)$. It is clear from the above definition that if $f_{(m)}(x_0)$ exists then $D^m f(x_0)$ also exists with equal value.

A function f is said to satisfy the property $\mathscr{R}$ in an interval I if for every perfect set $P\subset I$ there is a portion of P in which f restricted to P is continuous. The property $\mathscr{D}$ will mean Darboux property and $|E|$ will denote the (Lebesgue) measure of the measurable set E. It is known that the Peano derivative $f_{(m)}$ possesses the property $\mathscr{D}$, the mean value property, and the property that it becomes the ordinary derivative $f^{(m)}$ whenever $f_{(m)}$ is bounded at least on one side [4, 7]. This fact will be used frequently.

It is known that a continuous (usual) smooth function on an interval I is differentiable on a set which is of the power of the continuum in any subinterval of I and the derivative possesses Darboux property on the set of its existence [8]. In the present paper we have proved these results in our more general setting. Also some other interesting properties of generalized smooth functions are investigated.

LEMMA 1 (Auerbach). *If $\Sigma\Phi_n(x)$ is a series of continuous functions in $[a, b]$ and Σa_n is a convergent series of positive constant terms such that for each $x\in[a, b]$ there is a positive number $N(x)$ with the property that $|\Phi_n(x)|\leqq a_n$ whenever $n>N(x)$ then there is a subinterval of $[a, b]$ where $\Sigma\Phi_n(x)$ converges uniformly.*

For the proof see [1].

LEMMA 2. *Let f be continuous in (a, b) and let $D^{2k-1}f$, $k=1, 2, \ldots, m$, exist and be continuous in (a, b). Then the ordinary derivative $f^{(2m-1)}$ exists and is continuous in (a, b).*

PROOF. If $m=1$ the result follows from Corollary 2 of Theorem 4 of [2]. So we suppose that the result is true for $m=n$ and prove it for $m=n+1$. The proof will then follow by induction. Since by hypothesis $D^{2k-1}f$, $k=1, 2, \ldots, n+1$, exist and are continuous in (a, b) and since the result is true for $m=n, f^{(2n-1)}$ exists and is continuous in (a, b). Let $[\alpha, \beta]\subset(a, b)$. For each $x\in(a, b)$ and each h with $x\pm h\in$ $\in(a, b)$, there is, by mean value theorem, a θ, $0<\theta<1$, such that

$$\frac{(2n+1)!}{h^{2n+1}}\left[\frac{1}{2}\{f(x+h)-f(x-h)\}-\sum_{k=1}^{n}\frac{h^{2k-1}}{(2k-1)!}f^{(2k-1)}(x)\right]=$$

$$=\frac{f^{(2n-1)}(x+\theta h)+f^{(2n-1)}(x-\theta h)-2f^{(2n-1)}(x)}{(\theta h)^2}.$$

Hence, writing $\underline{D}^2\Phi(x)=\liminf_{h\to 0}\theta_2(\Phi; x, h)$ etc., we have $D^2f^{(2n-1)}(x)\leqq D^{2n+1}f(x)\leqq$ $\underline{D}^2 f^{(2n-1)}(x)$ for all $x\in(a, b)$. Since $D^{2n+1}f$ is continuous in $[\alpha, \beta]$, the function

$$f^{(2n-1)}(x)-\int_{\alpha}^{x}dt\int_{\alpha}^{t}D^{2n+1}f(u)\,du$$

is linear in $[\alpha, \beta]$ [9, I; p. 327] and hence $f^{(2n+1)}=D^{2n+1}f$ in (α, β). Since $[\alpha, \beta]\subset(a, b)$ is arbitrary $f^{(2n+1)}(x)=D^{2n+1}f(x)$ for all x in (a, b).

LEMMA 3. *Let f be continuous in (a, b) and let $D^{2k}f$, $k=1, 2, \ldots, m$, exist and be continuous in (a, b). Then $f^{(2m)}$ exists and is continuous in (a, b).*

If $m=1$ the result is true. For, if $[\alpha, \beta]\subset(a, b)$ then by [9, I; p. 327], the function

$$f(x)-\int_{\alpha}^{x} dt \int_{\alpha}^{t} D^2 f(u)\, du$$

is linear in $[\alpha, \beta]$ and hence as above $f^{(2)}=D^2 f$ in (a, b). Supposing the result to be true for $m=n$ it can be proved as in Lemma 2, that it is also true for $m=n+1$ and the proof follows by induction.

THEOREM 1. *Let f be continuous and $D^{m-2} f$ exist in (a, b). If f is m-smooth in (a, b), then $f^{(m-2)}$ exists and is continuous on a dense open set in (a, b).*

PROOF. We prove the theorem when m is even. Let $m=2k$. The theorem is obviously true for $k=1$. Now suppose that the theorem is true for $k=r$. We show that it is also true for $k=r+1$. Let $[a', b']\subset(a, b)$. Choose a sequence $\{h_n\}$ such that Σh_n is convergent and $h_n>h_{n+1}>0$, $a'-h_n>a$, $b'+h_n<b$ for all n. Since, f is $2r+2$ smooth it is $2r$ smooth. So, by our supposition there is an interval $[c, d]\subset \subset(a', b')$ such that $f^{(2r-2)}$ exists and continuous in $[c, d]$. Set $\Psi_n^{2r}(x)=\theta_{2r}(f; x, h_n)$, $n=1, 2, \ldots$. Then Ψ_n^{2r} is continuous in $[c, d]$ for all n. Since f is $2r+2$ smooth, we have for $x\in[c, d]$ $\lim_{h\to 0} h\theta_{2r+2}(f; x, h) = 0$. Therefore,

$$\lim_{h\to 0} \frac{1}{h}[\theta_{2r}(f; x, h)-D^{2r} f(x)] = 0$$

i.e.,

$$\lim_{n\to\infty} \frac{1}{h_n}[\Psi_n^{2r}(x)-D^{2r} f(x)] = 0.$$

So, there is a positive integer $N(x)$ such that $|\Psi_n^{2r}(x)-D^{2r} f(x)| < h_n$ whenever $n>N(x)$. Therefore, for $x\in[c, d]$ and $n>N(x)$

$$|\Psi_{n+1}^{2r}(x)-\Psi_n^{2r}(x)| \leqq |\Psi_{n+1}^{2r}(x)-D^{2r} f(x)|+|\Psi_n^{2r}(x)-D^{2r} f(x)| < h_{n+1}+h_n < 2h_n.$$

So, the series $\sum_{n=1}^{\infty} (\Psi_{n+1}^{2r}-\Psi_n^{2r})$ satisfies all the conditions of Lemma 1 in $[c, d]$. So there is a subinterval say (a'', b'') of $[c, d]$ where $\sum_{n=1}^{\infty} (\Psi_{n+1}^{2r}-\Psi_n^{2r})$ converges uniformly and hence $\Psi_1^{2r}+\sum_{n=1}^{\infty} (\Psi_{n+1}^{2r}-\Psi_n^{2r})$ i.e. $\{\Psi_n^{2r}\}$ converges uniformly. So the limit function $D^{2r} f$ is continuous in $(a'', b'')\subset(a', b')$. Hence by Lemma 3, $f^{(2r)}$ exists and is continuous in (a'', b''). Thus the theorem is true for $k=r+1$. Hence by induction the theorem is true for all k.

Now if m be odd, say $m=2k+1$, the proof is similar. For $k=1$, it can be proved by considering $\Psi_n^1(x)=\frac{1}{2h_n}[f(x+h_n)-f(x-h_n)]$ and using Lemmas 3 and 2. Then supposing the theorem to be true for $k=r$, it can be shown, as above, by considering $\Psi_n^{2r+1}(x)=\theta_{2r+1}(f; x, h_n)$ and using Lemmas 1 and 2 that the theorem is true for $k=r+1$. The proof then follows by induction.

Lemma 4. *Let $f_{(m-2)}$ exist in (a, b) and let f be smooth of order m in (a, b) where $m \geqq 2$. If $f_{(m-2)}$ attains a local maximum or minimum at $x_0 \in (a, b)$ then $f_{(m-1)}(x_0)$ exists and equals zero.*

Proof. If m is odd then

$$h\theta_m(f; x_0, h) = \frac{m!}{2h^{m-1}}\left[f(x_0+h)-f(x_0-h)-2\sum_{k=0}^{(m-3)/2}\frac{h^{2k+1}}{(2k+1)!}D^{2k+1}f(x_0)\right] =$$

$$= \frac{m!}{2h^{m-1}}\left[f(x_0+h)-\sum_{k=0}^{m-2}\frac{h^k}{k!}f_{(k)}(x_0)-f(x_0-h)+\sum_{k=0}^{m-2}\frac{(-h)^k}{k!}f_{(k)}(x_0)\right] =$$

$$= \frac{m!}{2h^{m-1}}\left[f(x_0+h)-\sum_{k=0}^{m-2}\frac{h^k}{k!}f_{(k)}(x_0)\right] +$$

$$+\left[-\frac{m!}{2(-h)^{m-1}}\left(f(x_0-h)-\sum_{k=0}^{m-2}\frac{(-h)^k}{k!}f_{(k)}(x_0)\right)\right]$$

and if m is even then,

$$h\theta_m(f; x_0, h) = \frac{m!}{2h^{m-1}}\left[f(x_0+h)+f(x_0-h)-2\sum_{k=0}^{(m-2)/2}\frac{h^{2k}}{(2k)!}D^{2k}f(x_0)\right] =$$

$$= \frac{m!}{2h^{m-1}}\left[f(x_0+h)-\sum_{k=0}^{m-2}\frac{h^k}{k!}f_{(k)}(x_0)+f(x_0-h)-\sum_{k=0}^{m-2}\frac{(-h)^k}{k!}f_{(k)}(x_0)\right] =$$

$$= \frac{m!}{2h^{m-1}}\left[f(x_0+h)-\sum_{k=0}^{m-2}\frac{h^k}{k!}f_{(k)}(x_0)\right] +$$

$$+\left[-\frac{m!}{2(-h)^{m-1}}\left(f(x_0-h)-\sum_{k=0}^{m-2}\frac{(-h)^k}{k!}f_{(k)}(x_0)\right)\right].$$

Thus in any case

$$h\theta_m(f; x_0, h) = \frac{m!}{2h^{m-1}}\left[f(x_0+h)-\sum_{k=0}^{m-2}\frac{h^k}{k!}f_{(k)}(x_0)\right] + \tag{1}$$

$$+\left[-\frac{m!}{2(-h)^{m-1}}\left(f(x_0-h)-\sum_{k=0}^{m-2}\frac{(-h)^k}{k!}f_{(k)}(x_0)\right)\right].$$

Choose $h(\neq 0)$ such that $x_0 \pm h \in (a, b)$. Then by the mean value property of the Peano derivative there are $\delta_1, \delta_2, 0<\delta_1<1, 0<\delta_2<1$ such that

$$\frac{m!}{2h^{m-1}}\left[f(x_0+h)-\sum_{k=0}^{m-2}\frac{h^k}{k!}f_{(k)}(x_0)\right] = \tag{2}$$

$$= \frac{m!}{2h^{m-1}}\left[\frac{h^{m-2}}{(m-2)!}f_{(m-2)}(x_0+\delta_1 h)-\frac{h^{m-2}}{(m-2)!}f_{(m-2)}(x_0)\right] =$$

$$= \frac{m(m-1)}{2h}[f_{(m-2)}(x_0+\delta_1 h)-f_{(m-2)}(x_0)]$$

and

$$(3)\qquad -\frac{m!}{2(-h)^{m-1}}\left[f(x_0-h)-\sum_{k=0}^{m-2}\frac{(-h)^k}{k!}f_{(k)}(x_0)\right]=$$

$$=-\frac{m!}{2(-h)^{m-1}}\left[\frac{(-h)^{m-2}}{(m-2)!}f_{(m-2)}(x_0-\delta_2 h)-\frac{(-h)^{m-2}}{(m-2)!}f_{(m-2)}(x_0)\right]=$$

$$=\frac{m(m-1)}{2h}[f_{(m-2)}(x_0-\delta_2 h)-f_{(m-2)}(x_0)].$$

Since $f_{(m-2)}$ attains a maximum or minimum at x_0, from (2) and (3) both terms in the right of (1) are of the same sign for small $|h|$. Since f is smooth of order m at x_0, from (1), it follows that $f_{(m-1)}(x_0)$ exists and equals zero.

THEOREM 2. *Let f be continuous and $f_{(m-2)}$ exist in (a, b) and let f be smooth of order m in (a, b). Then the set E of points x in (a, b) where $f_{(m-1)}(x)$ exists and is finite, is of the power of the continuum in every subinterval of (a, b).*

PROOF. *Let* (a', b') be any subinterval of (a, b). We shall show that $(a', b')\cap E$ is of the power of the continuum. By Theorem 1, there is a subinterval say $[\alpha, \beta]$ of (a', b') where $f^{(m-2)}$ exists and is continuous. If $f^{(m-2)}$ is linear in some closed interval $[\alpha, \beta']$, $\alpha<\beta'<\beta$, then $f^{(m-1)}$ exists everywhere in (α, β') and the result follows. So we suppose that $f^{(m-2)}$ is not linear in $[\alpha, \beta']$ for all β', $\alpha<\beta'<\beta$. Let β', $\alpha<\beta'<\beta$ be fixed. Set

$$g(x)=f(x)-\frac{1}{(m-2)!}f_{(m-2)}(\alpha)x^{m-2}-$$

$$-\frac{1}{(\beta'-\alpha)(m-1)!}\{f_{(m-2)}(\beta')-f_{(m-2)}(\alpha)\}(x-\alpha)^{m-1}.$$

Then $g^{(m-2)}$ is continuous in $[\alpha, \beta']$. Also $g^{(m-2)}(\alpha)=0=g^{(m-2)}(\beta')$. Since $f^{(m-2)}$ is not linear in $[\alpha, \beta']$, $g^{(m-2)}$ is not constant in $[\alpha, \beta']$. Hence there is $\xi\in(\alpha, \beta')$ where $g^{(m-2)}$ attains a maximum or minimum. So by Lemma 4, $g_{(m-1)}(\xi)$ exists and equals zero, i.e.,

$$(1)\qquad f_{(m-1)}(\xi)=\frac{1}{\beta'-\alpha}\{f_{(m-2)}(\beta')-f_{(m-2)}(\alpha)\}.$$

Thus for each β', $\alpha<\beta'<\beta$ there is ξ, $\alpha<\xi<\beta'$ such that (1) holds. But since $f^{(m-2)}$ is continuous and not linear in $[\alpha, \beta']$ for all β', $\alpha<\beta'<\beta$ the set of values $\frac{1}{\beta'-\alpha}\{f_{(m-2)}(\beta')-f_{(m-2)})(\alpha)\}$ is of the power of the continuum as β' varies over (α, β). Thus the set of points ξ in (α, β) for which $f_{(m-1)}(\xi)$ exists is of cardinality continuum. This proves the theorem.

From Theorems 1 and 2 we have

THEOREM 3. *Let f be continuous and $D^{m-2}f$ exist in (a, b). If f is smooth of order m in (a, b), then the set E of points x in (a, b) where $f_{(m-1)}(x)$ exists and is finite, is of the power of the continuum in any subinterval of (a, b).*

LEMMA 5. *Let f be continuous in (a, b) and $D^{m-2}f$ exist and possess property*

$\mathscr{R}$ and $\mathscr{D}$ in (a, b). Also let $D^{m-2}f$ do not attain a local maximum or minimum in (a, b). Then the ordinary derivative $f^{(m-2)}$ exists and is continuous and monotone in (a, b).

PROOF. Let $[\alpha, \beta]\subset(a, b)$. Let O be the set of all point x in $[\alpha, \beta]$ such that there is a neighbourhood of x (relative to $[\alpha, \beta]$) in which $D^{m-2}f$ is continuous. Clearly O is open (relative to $[\alpha, \beta]$) and hence the set $P=[\alpha, \beta]-O$ is closed.

Now for any interval $J\subset O$ since $D^{m-2}f$ is continuous and has no local maximum or minimum in J, it is monotone in J and since $D^{m-2}f$ possesses Darboux property, it is continuous and monotone in $\bar{J}$ (the closure of J). Thus $f_{(m-2)}$ is continuous and monotone in $\bar{J}$ for every interval $J\subset O$. This fact will be used in the following argument.

If $\xi\in(\alpha, \beta)$ is an isolated point of P, then there are $\delta_1>0, \delta_2>0$ such that $(\xi-\delta_1, \xi)\cup(\xi, \xi+\delta_2)\subset O$ and hence $D^{m-2}f$ is continuous in $[\xi-\delta_1, \xi]$ and in $[\xi, \xi+\delta_2]$ which shows that $D^{m-2}f$ is continuous in $(\xi-\delta_1, \xi+\delta_2)$ and this is a contradiction since $\xi\notin O$. So, P has no isolated point in (α, β). By similar argument it can be shown that if α (or β) belongs to P then α (or β) is not an isolated point of P. Thus P is perfect.

If possible let P be non-void. Since $D^{m-2}f$ has the property $\mathscr{R}$ in (a, b) there is a portion of P, say $(p, q)\cap P$ where $D^{m-2}f$ restricted to P is continuous. Let $\xi\in(p, q)\cap P$ and let $\{\xi_n\}$ be any sequence such that $\xi_n\to\xi$ as $n\to\infty$. We shall show that $D^{m-2}f(\xi_n)\to D^{m-2}f(\xi)$ as $n\to\infty$. If $\xi_n\in P$ for all n then $D^{m-2}f(\xi_n)\to D^{m-2}f(\xi)$ as $n\to\infty$. So we suppose that $\xi\in O$ for all n. If ξ is an isolated point of P from one side, say from the left, then there is $\delta>0$ such that $(\xi-\delta, \xi)\subset O$ and so $D^{m-2}f$ is continuous at ξ from the left. Hence, if ξ is an isolated point of P from one side and if $\xi_n\to\xi$ from that side then $D^{m-2}f(\xi_n)\to D^{m-2}f(\xi)$. So we suppose that for each n there is a component interval $(s_n, t_n)\subset O$ such that $s_n<\xi_n<t_n$ and $s_n\to\xi$, $t_n\to\xi$. Since $D^{m-2}f$ is monotone in $[s_n, t_n]$, $D^{m-2}f(\xi_n)$ lies in the closed interval with end points $D^{m-2}f(s_n)$ and $D^{m-2}f(t_n)$ which tend to $D^{m-2}f(\xi)$ and so $D^{m-2}f(\xi_n)\to$ $\to D^{m-2}f(\xi)$. Therefore $D^{m-2}f$ is continuous in $(p, q)\cap P$ and hence continuous in (p, q). But this is a contradiction since (p, q) contains points of P. So, P is void. Hence $[\alpha, \beta]\subset O$. Therefore $D^{m-2}f$ is continuous in $[\alpha, \beta]$ and since it has no local maximum or minimum in $[\alpha, \beta]$ it is monotone in $[\alpha, \beta]$. Since $[\alpha, \beta]\subset(a, b)$ is arbitrary, $D^{m-2}f$ is continuous and monotone in (a, b). So by Lemma 2 or 3, $f^{(m-2)}$ exists and is continuous and monotone in (a, b).

THEOREM 4. *Let f be continuous and $f_{(m-2)}$ exist in (a, b). Also let $f_{(m-2)}$ possess the property $\mathscr{R}$ in (a, b) and let f be smooth of order m in (a, b). Let $E=\{x\colon x\in(a, b)$, $f_{(m-1)}(x)$ exists$\}$. If $f_{(m-1)}(x)\geqq 0$ for all $x\in E$ then $f_{(m-2)}$ is continuous and nondecreasing in (a, b).*

PROOF. Let us first suppose that $f_{(m-1)}>0$ in E. Clearly $f_{(m-2)}$ does not attain a local maximum or minimum in (a, b). For, if $f_{(m-2)}$ attains a local maximum or minimum in (a, b) at $x_0\in(a, b)$ then by Lemma 4, $x_0\in E$ and $f_{(m-1)}(x_0)=0$ a contradiction. Since $f_{(m-2)}$ possesses Darboux property, by Lemma 5, $f_{(m-2)}$ is continuous and monotone in (a, b). If $f_{(m-2)}$ is nonincreasing in (a, b) then for any point $x_0\in E$ and $x_0+h\in(a, b)$ there is, by mean value theorem, γ, $0<\gamma<1$ such that

$$\frac{(m-1)!}{h^{m-1}}\left[f(x_0+h)-\sum_{k=0}^{m-2}\frac{h^k}{k!}f_{(k)}(x_0)\right]=\frac{m-1}{h}\left[f_{(m-2)}(x_0+\gamma h)-f_{(m-2)}(x_0)\right]\leqq 0,$$

and hence $f_{(m-1)}(x_0)\leqq 0$ which is a contradiction. Hence $f_{(m-2)}$ is non-decreasing and continuous in (a, b).

To complete the proof consider the function $g(x)=f(x)+\varepsilon \dfrac{x^{m-1}}{(m-1)!}$ where $\varepsilon>0$ is arbitrary. Then $g_{(m-1)}(x)=f_{(m-1)}(x)+\varepsilon>0$ for all $x\in E$. Since g satisfies other hypotheses of the theorem, from the first part $g_{(m-2)}$ is non-decreasing and continuous in (a, b) and since $\varepsilon>0$ is arbitrary, $f_{(m-2)}$ is continuous and non-decreasing in (a, b).

THEOREM 5. *Let f be continuous and $f_{(m-2)}$ exist in (a, b). Also let $f_{(m-2)}$ possess the property $\mathcal{R}$ and f be smooth of order m in (a, b). Let $E=\{x\colon x\in(a, b), f_{(m-1)}(x)$ exists$\}$. Then $f_{(m-1)}$ has the property $\mathcal{D}$ on E.*

PROOF. Let α, β, $\alpha<\beta$ be any two points in E and let $f_{(m-1)}(\alpha)<f_{(m-1)}(\beta)$. Let $f_{(m-1)}(\alpha)<c<f_{(m-1)}(\beta)$. We show that there is a point $\gamma\in(\alpha, \beta)$ such that $f_{(m-1)}(\gamma)=c$. Set $g(x)=f(x)-\dfrac{c}{(m-1)!}\, x^{m-1}$. Then $g_{(m-1)}(\alpha)=f_{(m-1)}(\alpha)-c<0$ and $g_{(m-1)}(\beta)=f_{(m-1)}(\beta)-c>0$.

If $g_{(m-2)}$ attains a local maximum or minimum at some point $\gamma\in(\alpha, \beta)$, then by Lemma 4, $g_{(m-1)}(\gamma)=0$ i.e., $f_{(m-1)}(\gamma)=c$ and so the theorem is proved. Thus if we prove that $g_{(m-2)}$ attains a local maximum or minimum in (α, β) the proof will be complete.

If possible suppose that $g_{(m-2)}$ does not attain a local maximum or minimum in (α, β). So by Lemma 5, $g_{(m-2)}$ is continuous and monotone in (α, β) and by Darboux property of $g_{(m-2)}$ it is continuous and monotone in $[\alpha, \beta]$. If $g_{(m-2)}$ is nondecreasing in $[\alpha, \beta]$, by mean value theorem, there is δ, $0<\delta<1$ such that

$$\frac{(m-1)!}{h^{m-1}}\left[g(\alpha+h)-\sum_{k=0}^{m-2}\frac{h^k}{k!}\,g_{(k)}(\alpha)\right]=\frac{m-1}{h}\,[g_{(m-2)}(\alpha+\delta h)-g_{(m-2)}(\alpha)]\geqq 0$$

and hence $g_{(m-1)}(\alpha)\geqq 0$ which is a contradiction. If $g_{(m-2)}$ is nonincreasing in $[\alpha, \beta]$ by similar argument $g_{(m-1)}(\beta)\leqq 0$ which is also a contradiction. Thus $g_{(m-2)}$ must attain a local maximum or minimum in (α, β). This completes the proof.

THEOREM 6. *Let f be continuous and $f_{(m-2)}$ exist in (a, b). Also let $f_{(m-2)}$ possess the property $\mathcal{R}$ and f be smooth of order m in (a, b). Let $E=\{x\colon x\in(a, b), f_{(m-1)}(x)$ exists$\}$. Then for any k, $0\leqq k\leqq m-2$ and for each x and $x+h$ in (a, b) there is $x'\in E$ between x and $x+h$ such that*

$$\frac{(m-k-1)!}{h^{m-k-1}}\left\{f_{(k)}(x+h)-\sum_{r=k}^{m-2}\frac{h^{r-k}}{(r-k)!}f_{(r)}(x)\right\}=f_{(m-1)}(x').$$

PROOF. First we prove the theorem for $k=m-2$. We assume $h>0$, the case $h<0$ is similar. It is sufficient to suppose that $f_{(m-2)}(x+h)=f_{(m-2)}(x)$ and prove that there is $x'\in(x, x+h)\cap E$ such that $f_{(m-1)}(x')=0$.

If $f_{(m-1)}(t)\geqq 0$ for all $t\in(x, x+h)\cap E$ then by Theorem 4, $f_{(m-2)}$ is nondecreasing in $(x, x+h)$ and by Darboux property of $f_{(m-2)}$, it is nondecreasing in $[x, x+h]$. Since $f_{(m-2)}(x+h)=f_{(m-2)}(x)$, $f_{(m-2)}$ is constant in $[x, x+h]$ and hence $f_{(m-1)}(t)=0$ for all $t\in(x, x+h)\cap E$. Also if $f_{(m-1)}(t)\leqq 0$ in $(x, x+h)\cap E$, by similar argument there is $x' \in (x, x+h) \cap E$ such that $f_{(m-1)}(x')=0$. Finally, if there are $\xi, \eta \in \in (x, x+h) \cap E$ such that $f_{(m-1)}(\xi)<0$ and $f_{(m-1)}(\eta)>0$ then by Theorem 5, there is $x' \in (x, x+h) \cap E$ such that $f_{(m-1)}(x')=0$. This completes the proof for $k=m-2$.

Let now $0 \leqq k \leqq m-3$. Set

$$g(t) = f(t) - \frac{(m-k-1)!}{h^{m-k-1}} \left\{ f_{(k)}(x+h) - \sum_{r=k}^{n-2} \frac{h^{r-k}}{(r-k)!} f_{(r)}(x) \right\} \frac{(t-x)^{m-1}}{(m-1)!}.$$

Then

$$g_{(k)}(x+h) - \sum_{r=k}^{m-2} \frac{h^{r-k}}{(r-k)!} g_{(r)}(x) = 0. \tag{1}$$

Since $g_{(m-2)}$ exists in (a, b) by mean value theorem there is x_0, $x<x_0<x+h$ such that

$$g_{(k)}(x+h) - \sum_{r=k}^{m-3} \frac{h^{r-k}}{(r-k)!} g_{(r)}(x) = \frac{h^{m-k-2}}{(m-k-2)!} g_{(m-2)}(x_0). \tag{2}$$

So by (1) and (2) $g_{(m-2)}(x_0) - g_{(m-2)}(x)=0$. Therefore from the first part of the proof there is x', $x<x'<x_0$ such that $g_{(m-1)}(x')=0$ i.e.,

$$\frac{(m-k-1)!}{h^{m-k-1}} \left\{ f_{(k)}(x+h) - \sum_{r=k}^{m-2} \frac{h^{r-k}}{(r-k)!} f_{(r)}(x) \right\} = f_{(m-1)}(x').$$

This completes the proof.

THEOREM 7. *Let f be continuous and $D^{m-2}f$ exist in (a, b), and f be smooth of order m in (a, b). Also let $E=\{x: x\in(a, b), f_{(m-1)}(x)$ exists$\}$ and $|E \cap I|<|I|$ for every interval $I \subset (a, b)$. Then $f_{(m-1)}$ has the property $\mathcal{D}$ on E.*

PROOF. By Theorem 3, $E \cap I$ has the power of the continuum for every interval I. Let α, β, $\alpha<\beta$, be any two points in E and let $f_{(m-1)}(\alpha)<f_{(m-1)}(\beta)$. Let $f_{(m-1)}(\alpha)< <c<f_{(m-1)}(\beta)$. Suppose, if possible, that $f_{(m-1)}(x) \neq c$ for all $x \in (\alpha, \beta) \cap E$. We may clearly assume that $c=0$. Let

$$E_+ = \{x:\ x\in(\alpha, \beta) \cap E,\ f_{(m-1)}(x) > 0\}, \quad E_- = \{x:\ x\in(\alpha, \beta) \cap E,\ f_{(m-1)}(x) < 0\}.$$

Then $(\alpha, \beta) \cap E = E_+ \cup E_-$. Now by Theorem 1, let $f^{(m-2)}$ exist and be continuous in an open interval $I \subset (\alpha, \beta)$. Then, by Theorem 5, $f_{(m-1)}$ has the property $\mathcal{D}$ on $I \cap E$ and hence either $I \cap E \subset E_+$ or $I \cap E \subset E_-$. Let $I \cap E \subset E_+$. Then by Theorem 4, $f^{(m-2)}$ is nondecreasing in I and hence $f^{(m-1)}$ exists almost everywhere in I. So $|E \cap I|=|I|$, a contradiction. If $I \cap E \subset E_-$ we would get a similar contradiction.

THEOREM 8. *Let f be continuous and $f_{(m-2)}$ exist in (a, b). Also let f be smooth of order m in (a, b) and let $E=\{x: x\in(a, b), f_{(m-1)}(x)$ exists$\}$ and $|E \cap I|<|I|$ for every interval $I \subset (a, b)$. Then for any k, $0 \leqq k \leqq m-2$ and for each x and $x+h$ in (a, b) there is $x' \in E$ between x and $x+h$ such that*

$$\frac{(m-k-1)!}{h^{m-k-1}} \left\{ f_{(k)}(x+h) - \sum_{r=k}^{m-2} \frac{h^{r-k}}{(r-k)!} f_{(r)}(x) \right\} = f_{(m-1)}(x').$$

PROOF. We assume $h>0$, the case $h<0$ is similar. For $k=m-2$ as in Theorem 6, it is sufficient to suppose that $f_{(m-2)}(x+h)=f_{(m-2)}(x)$ and prove that there is $x'\in(x, x+h)\cap E$ such that $f_{(m-1)}(x')=0$. If $f_{(m-1)}(t)\neq 0$ for all $t\in(x, x+h)\cap E$ then proceeding as in Theorem 7, we arrive at a contradiction. This proves the theorem for $k=m-2$. For $0\leqq k\leqq m-3$ the proof is similar to that of the second part of Theorem 6.

REMARKS. If $m>2$ then the existence of $f_{(m-2)}$ implies the continuity of f. Hence if $m>2$ the continuity condition in Theorems 2, 4, 5, 6 and 8 are superfluous. On the other hand, if $m=2$ then from Theorems 5 and 6 it follows that the condition $|I\cap E|<|I|$ in Theorems 7 and 8 are redundant; this condition will be needed in this case if the continuity of f is replaced by the measurabilty [6]. Also if $m=2$, then the supposition that $f_{(m-2)}$ or $D^{m-2}f$ exists is superfluous.

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SELECTIVE DIFFERENTIATION: REDEFINING SELECTIONS

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The concept of selective differentiation was introduced and developed in [4]. Here we present selective differentiation from a different viewpoint or, rather, in a new framework. This framework takes some effort to erect but is profitable in terms of the results obtained. It originates from several very simple observations. First, any closed nondegenerate interval $[a, b]$ can be considered as a point (a, b) in the upper half-plane $H=\{(x, y); x<y\}$. Second, most notions of a sequence of intervals $I_n=[a_n, b_n]$ converging to a point x_0 can be translated into an equivalent notion of the sequence (a_n, b_n) in H converging to the point (x_0, x_0) of the boundary D. For example, $[a_n, b_n]$ w-converges to x_0, using the definition in [5], if and only if (a_n, b_n) converges to (x_0, x_0) inside some Stolz angle. Third, for any function $f\colon R\to R$ let $G(x, y)=\frac{f(y)-f(x)}{y-x}$, $G\colon H\to R$. Then the study of various differentiability properties of f can be accomplished through study of the boundary behavior of G at D. See for example [2] or [1, pp. 68—70]. These three facts form the foundation of the framework.

A selection S consists of choosing a point from the interior of each nondegenerate closed interval $[a, b]$ and labeling the point $P[a, b]$. Within the above framework, it is clear that a selection S is identifiable with a function $s\colon H\to R$ satisfying $x<s(x, y)<y$. Throughout this paper, we will use this equivalence. For example, the original definition of selective derivative at x_0 becomes:

DEFINITION 1. *Let $f\colon R\to R$ be fixed and let $s(x, y)$ be a selection on H. Then f is said to have a selective derivative at x_0 if there is a number α such that*

$$\underset{(x,y)\to(x_0,x_0)}{\text{hv-lim}} \frac{f(s(x, y))-f(x_0)}{s(x, y)-x_0} = \alpha,$$

where $\underset{(x,y)\to(x_0,x_0)}{\text{hv-lim}}$ is the notation to represent that (x, y) approaches (x_0, x_0) along the horizontal and vertical line segments in H, ending at (x_0, x_0).

We will have numerous instances of evaluating limits with different methods of approach to the point in question. In each case a prefix such as hv, hopefully self-explanatory, will be adjoined to the notation. Further, where no confusion will arise we will delete the $(x, y)\to(x_0, y_0)$.

DEFINITION 2. For each fixed real number a a subset r of H is called a right approach set for a if every (x, y) in r satisfies $a\leqq x$ and (a, a) is the only limit point of r in D. A left approach set l is defined similarly. When necessary we use the notation $r(a)$ and $l(a)$ to show that l and r depend on a.

Note that if we consider the horizontal and vertical line segments at (x_0, x_0) as left and right approach sets at x_0, then we may think of selective differentiation as depending jointly on the selection and the approach set. We explore this possibility.

DEFINITION 3. Suppose that for each a in R a right and a left approach set, dependent on a, have been chosen. The collection C of these sets is said to have the intersection property if for each fixed a there is a $\delta>0$ such that if $a-\delta<a_1<a<a_2<$ $<a+\delta$ then $r(a_1)\cap l(a)\neq\emptyset$ and $r(a)\cap l(a_2)\neq\emptyset$.

Note that if for each a the right and left approach set consists of a line in H then the collection thus obtained would have the intersection property.

Our first theorem deals with our ability to change selections and is somewhat deceptively simple.

THEOREM 1. *Let $s\colon H\to R$ be any selection and let C be any collection of right and left approach sets with the intersection property. Then there is a new selection t such that for any function $G\colon R\setminus\{x_0\}\to R$ and any x_0*

$$\operatorname*{lr\text{-}lim\,inf}_{(x,y)\to(x_0,x_0)} G(s(x,y)) \leqq \operatorname*{hv\text{-}lim\,inf}_{(x,y)\to(x_0,x_0)} G(t(x,y)),$$

where lr *is the notation for approach to (x_0, x_0) along the $r(x_0)\cup l(x_0)$ in C. Further, for the same t the relation*

$$\text{lr-lim sup}\, G(s(x,y)) \geqq \text{hv-lim sup}\, G(t(x,y))$$

exists.

PROOF. Let (a,b) belong to H. If $r(a)\cap l(b)=\emptyset$, let $t(a,b)=s(a,b)$. If $r(a)\cap l(b)\neq\emptyset$, let (c,d) be any point in this intersection. Then $a<c<s(c,d)<d<b$. Let $t(a,b)=s(c,d)$. This defines t.

It will now suffice to show that for any sequence $h_n\to 0$, $h_n>0$, and G and x_0 fixed we have

$$\text{lr-lim inf}\, G(s(x,y)) \leqq \liminf_{n\to+\infty} G(t(x_0, x_0+h_n)).$$

To see this, we note that there is an N such that for $n>N$; $x_0<x_0+h_n<x_0+\delta(x_0)$ so that $r(x_0)\cap l(x_0+h_n)\neq\emptyset$. Then $t(x_0, x_0+h_n)=s(c_n, d_n)$ where (c_n, d_n) belongs to $r(x_0)\cap l(x_0+h_n)$. So $x_0<c_n<d_n<x_0+h_n$. This implies that (c_n, d_n) approaches (x_0, x_0) through $l(x_0)\cup r(x_0)$ and

$$\liminf_{n\to+\infty} G(t(x_0, x_0+h_n)) = \liminf_{n\to+\infty} G(s(c_n, d_n)) \geqq \text{lr-lim inf}\, G(s(x,y)).$$

COROLLARY 1. *Let s and C be as in Theorem 1. Let $f\colon R\to R$ be such that*

$$\text{lr-lim inf}\, \frac{f(s(x,y))-f(x_0)}{s(x,y)-x_0} > 0$$

for each x_0. Then f is increasing.

PROOF. The one t selection defined in Theorem 1 applied repeatedly for each x_0 and

$$G(s(x,y)) = \frac{f(s(x,y))-f(x_0)}{s(x,y)-x_0}$$

gives

$$\text{hv-lim inf}\,\frac{f\big(t(x, y)\big)-f(x_0)}{t(x, y)-x_0} > 0.$$

In the terminology of paper [4] this says that ${}_tf'(x_0)>0$ for all x_0, and Theorem 1 of [4] yields the result.

COROLLARY 2. *Let s and C be as in Theorem 1. Let $f\colon R\to R$ and $g\colon R\to R$ be such that*

$$\text{lr-lim}\,\frac{f\big(s(x, y)\big)-f(x_0)}{s(x, y)-x_0} = g(x_0)$$

for all x_0. Then there is another selection t such that g is the selective derivative of f relative to t.

PROOF. This is clear from Theorem 1 and Definition 1. Here we must apply Theorem 1 and t for both lr-lim sup and lr-lim inf. It follows that g will have all the properties of selective derivatives. These include being Baire class 2 [3] but not necessarily Baire class 1 [4] and having the Denjoy-Clarkson property [6].

COROLLARY 3. *Let s, C, f and g be as in Corollary 2. Suppose in addition that f has a selective derivative relative to s, sf', in the sense of Definition 1. Then $\{x\colon g(x)\neq sf'(x)\}$ is countable.*

PROOF. This follows from Corollary 2 above and Theorem 9 of [4].

These three corollaries should suffice to illustrate that virtually all of the results of [4] have analogues in terms of collections of right and left approach sets with the intersection property. These ideas should dovetail nicely with the very interesting results in [3].

We now examine what improvements can be made if we require that f has a selective derivative along more than just a right and left approach set. This will require that we place a more restrictive condition on our approach sets.

DEFINITION 4. *Let* $0<\alpha<1<\beta$ be two fixed numbers. Let $C(\alpha, \beta)$ be the collection of left and right approach sets with the property that for each a, $l(a)$ is the line segment with slope α ending at (a, a) and $r(a)$ is the line segment with slope β ending at (a, a).

THEOREM 2. *Let $s\colon H\to R$ be a selection and $0<\alpha<1<\beta$ be fixed. Suppose $f\colon R\to R$ and $g\colon R\to R$ are such that*

$$\alpha\text{hv}\beta\text{-lim}\,\frac{f\big(s(x, y)\big)-f(x_0)}{s(x, y)-x_0} = g(x_0)$$

for all x_0. Then g is Baire Class 1.

PROOF. It will suffice to show that for every a $\{x\colon g(x)\geqq a\}$ is a G_δ set. Let a be fixed and let $A=\{x\colon g(x)\geqq a\}$. Further, let m be any fixed integer. For every x_0 in A there is a $1>\delta(x_0)>0$ such that for all (x, y) on $l(x_0)\cup h(x_0)\cup v(x_0)\cup r(x_0)$

within distance δ of (x_0, x_0) we have

$$\frac{f(s(x,y))-f(x_0)}{s(x,y)-x_0} \geqq a-\frac{1}{m}.$$

The set of points within δ distance of (x_0, x_0) on $l(x_0)\cup h(x_0)\cup v(x_0)\cup r(x_0)$ consists of four line segments, labeled l_1, l_2, l_3 and l_4 as we proceed from left to right. The endpoints of these segments in H have coordinates

$$p_1=\left(x_0-\frac{\delta}{\sqrt{1+\alpha^2}},\, x_0-\frac{\alpha\delta}{\sqrt{1+\alpha^2}}\right), \quad p_2=(x_0-\delta, x_0),$$

$$p_3=(x_0, x_0+\delta), \quad p_4=\left(x_0+\frac{\delta}{\sqrt{1+\beta^2}},\, x_0+\frac{\beta\delta}{\sqrt{1+\beta^2}}\right).$$

We project these points onto D to determine an interval I dependant on x_0 and δ in the following way:

Project p_1 parallel to l_2 to $\left(x_0-\frac{\alpha\delta}{\sqrt{1+\alpha^2}},\, x_0-\frac{\alpha\delta}{\sqrt{1+\alpha^2}}\right)$,

Project p_2 parallel to l_1 to $\left(x_0+\frac{\alpha\delta}{1-\alpha},\, x_0+\frac{\alpha\delta}{1-\alpha}\right)$,

Project p_3 parallel to l_4 to $\left(x_0-\frac{\delta}{\beta-1},\, x_0-\frac{\delta}{\beta-1}\right)$,

Project p_4 parallel to l_3 to $\left(x_0+\frac{\delta}{\sqrt{1+\beta^2}},\, x_0+\frac{\delta}{\sqrt{1+\beta^2}}\right)$.

Next we let

$$c=\max\left(x_0-\frac{\alpha\delta}{\sqrt{1+\alpha^2}},\, x_0-\frac{\delta}{\beta-1}\right),$$

and

$$d=\min\left(x_0+\frac{\delta}{\sqrt{1+\beta^2}},\, x_0+\frac{\alpha\delta}{1-\alpha}\right).$$

Finally, let $I(x_0, \delta)=(c, d)$ and let $G_n=\bigcup_{x\in A} I\left(x, \frac{\delta(x)}{n}\right)$. G_n is open for every n, and $A\subset G_n$. Let $B_m=\bigcap_{n=1}^{\infty} G_n$. We claim that for every $x_0\in B_m$, $g(x)\geqq a-1/m$.

This is clear if x_0 belongs to A. Suppose x_0 belongs to $B_m\setminus A$ and $g(x_0)<a-1/m$. Then there is a $\delta(x_0)>0$ such that if (x, y) belongs to $l(x_0)\cup h(x_0)\cup v(x_0)\cup r(x_0)$ and is within $\delta(x_0)$ distance of (x_0, x_0) then

$$\frac{f(s(x,y))-f(x_0)}{s(x,y)-x_0} < a-\frac{1}{m}.$$

Since x_0 belongs to $B_m \setminus A$ there exists x_n belonging to A such that x_0 belongs to $I\left(x_n, \frac{\delta(x_n)}{n}\right) = I_n$ and $x_n \to x_0$. We may assume without loss of generality that $x_n > x_0$ for every n. By the way I_n was defined, $l(x_n)$ intersects line $y = x_0$ at a point p_n which is within distance $\frac{\delta(x_n)}{n}$ of (x_n, x_n). Further, the line $r(x_0)$ intersects the line $x = x_n$ at q_n which is within distance $\frac{\delta(x_n)}{n}$ of (x_n, x_n). As $n \to +\infty$, p_n and $q_n \to (x_0, x_0)$. Therefore, there is an N such that for $n > N$, p_n and q_n are within distance $\delta(x_0)$ of (x_0, x_0), and these points lie on $h(x_0) \cup r(x_0)$. Select one such n. Then:

i) $$s(p_n) < x_0 < x_n < s(q_n),$$

ii) $$\frac{f(s(p_n)) - f(x_n)}{s(p_n) - x_n} \geqq a - \frac{1}{m},$$

iii) $$\frac{f(s(q_n)) - f(x_n)}{s(q_n) - x_n} \geqq a - \frac{1}{m},$$

iv) $$\frac{f(s(p_n)) - f(x_0)}{s(p_n) - x_0} < a - \frac{1}{m},$$

and

v) $$\frac{f(s(q_n)) - f(x_0)}{s(q_n) - x_0} < a - \frac{1}{m}.$$

Then ii) and iii) imply that

$$\frac{f(s(q_n)) - f(s(p_n))}{s(q_n) - s(p_n)} \geqq a - \frac{1}{m}$$

while iv) and v) imply that

$$\frac{f(s(q_n)) - f(s(p_n))}{s(q_n) - s(p_n)} < a - \frac{1}{m}.$$

This contradiction implies that $f(x) \geqq a - 1/m$ for all $x \in B_m \supset A$. This implies that $\bigcap_{m=1}^{\infty} B_m = A$, so that A is a G_δ set.

We end the paper by giving a demonstration of a situation where the hypothesis of Theorem 2 is satisfied.

THEOREM 3. *Let $f\colon R \to R$ have an approximate derivative g for all x. Then there is a selection t such that*

$$\frac{1}{2}\,\text{hv 2-lim}\frac{f(t(x, y)) - f(x_0)}{t(x, y) - x_0} = g(x_0)$$

for all x_0.

PROOF. For each x in R there is a measurable set $P(x)$ having density 1 at x such that

$$P(x)-\lim_{y\to x}\frac{f(y)-f(x)}{y-x}=g(x).$$

Let (a, b) be fixed and $x_i=a+i(b-a)$, $i=-1, 0, 1, 2$. Set

$$Q_i=\begin{cases} P(x_i)\cap(a, b) & \text{if } m\big(P(x_i)\cap(a, b)\big)>\frac{4}{5}(b-a)\\ (a, b) & \text{otherwise.}\end{cases}$$

Then $\bigcap_{i=-1}^{2} Q_i\neq\emptyset$, and we pick a point from this intersection to be $t(a, b)$. It is a simple matter to check that for any x_0 and any sequence (x_n, y_n) approaching (x_0, x_0) along one of the four lines there is an N such that for all $n>N$, $t(x_n, y_n)$ was chosen from $P(x_0)\cap(x_n, y_n)$. This completes the proof and the paper.

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ON THE PRODUCT OF TWO b_R-SPACES AND THE CLASS $\mathfrak{B}$ OF FROLÍK

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Introduction. All spaces considered here are Tychonoff. A subset B of a space X is called bounded (in X) if each continuous real-valued function on X is bounded on B. We call a space X a b_R-space (resp. p_R-space, k_R-space) if a real-valued function on X is continuous whenever its restriction to each bounded (resp. pseudocompact, compact) subset of X is continuous. Clearly k_R-spaces and locally pseudocompact spaces are p_R-spaces and therefore b_R-spaces.

NOBLE [8] introduced the notion of b_R-space as an aid in studying conditions under which a projection on a product is z-closed. In [2] and [10] this notion has been used in order to give conditions under which $\theta(X\times Y)=\theta X\times\theta Y$, where θX is the topological completion of X. In [1] the author gives several conditions sufficient to insure that a product of two b_R-spaces be a b_R-space. The present paper is concerned with the following questions raised in [1]:

(1) Suppose that X is locally pseudocompact and that Y is a b_R-space. Is $X\times Y$ a b_R-space?

(2) Is the product of two pseudocompact spaces a b_R-space?

Following FROLÍK [4] let $\mathfrak{B}$ be the class of spaces X such that for every pseudocompact space Y the topological product $X\times Y$ is pseudocompact. We answer negatively the above questions by showing that if X is a pseudocompact space which is not in $\mathfrak{B}$, then there exists a pseudocompact space Z such that $X\times Z$ is not a b_R-space. We apply our results to give the following characterization of the class $\mathfrak{B}$: A pseudocompact space X belongs to $\mathfrak{B}$ if and only if the product $X\times Y$ is a p_R-space whenever Y is. From this result, it is natural to raise the question: Under what conditions on a space X is the product $X\times Y$ a p_R-space for every p_R-space Y? The class of all such spaces is characterized in the last section.

Notations and preliminaries. Throughout this paper we adopt the notation and terminology of [4] and [5]. βX denotes the Stone-Čech compactification of X. A zero-set is the set of zeros of a real-valued continuous function. $Z(X)$ denotes the family of all zero-sets in X. N is the discrete space of positive integers. Recall that a subset A of X is regular closed in case A is the closure of an open set. For later use, the following fundamental facts are needed.

(F.1) ([9], Proposition 2.3) A subset B of a space X is bounded if and only if for each locally finite family $\mathcal{U}$ of mutually disjoint, non-empty open sets in X, only finitely many members of $\mathcal{U}$ meet B.

(F.2) [6] A space is pseudocompact if and only if every locally finite family of its open subsets is finite.

(F.3) ([4], Theorem 3.6) A space X is in $\mathfrak{B}$ if and only if each infinite family of mutually disjoint, non-empty open subsets of X contains an infinite subfamily $\{U_n: n=1, 2, \ldots\}$ which satisfies the following condition: For each filter $\mathscr{F}$ of infinite subsets of N we have

$$\bigcap_{F\in\mathscr{F}} \mathrm{cl}_X\big(\bigcup_{n\in F} U_n\big) \neq \emptyset.$$

The topological product of two b_R-spaces. The following theorem is the main result.

THEOREM 1. *Let V be a non-empty regular closed set of a space X. If V is pseudocompact and does not belong to $\mathfrak{B}$, then there is a pseudocompact subspace Z of βN such that $X\times Z$ is not a b_R-space.*

PROOF. Since $V\notin\mathfrak{B}$, according to F.3 there exists a sequence $\{U'_n: n=1, 2, \ldots\}$ of mutually disjoint, non-empty open subsets of V which satisfies $(*)$, where $(*)$ is the condition: If M is any infinite subset of N, then

$$\bigcap_{F\in\mathscr{F}(M)} \mathrm{cl}_V\big(\bigcup_{n\in F} U'_n\big) = \emptyset$$

for some filter $\mathscr{F}(M)$ of infinite subsets of M. Let $U_n=U'_n\cap\mathrm{int}\, V$, $n=1, 2, \ldots$. Since $V=\mathrm{cl}\,(\mathrm{int}\, V)$, $\{U_n: n=1, 2, \ldots\}$ is a sequence of non-empty open subsets of X which satisfies $(*)$ (with U_n instead of U'_n).

For each infinite subset M of N, we choose a filter $\mathscr{F}(M)$ on M such that

$$\bigcap_{F\in\mathscr{F}(M)} \mathrm{cl}_V\big(\bigcup_{n\in F} U_n\big) = \emptyset$$

and let $\mathscr{U}_M$ be an ultrafilter on N containing $\mathscr{F}(M)$. Consider the following subspace of βN: $Y=N\cup\{p(M)\in\beta N-N: \mathscr{U}_M$ converges to $p(M)$, $M\subset N$, M infinite$\}$. The space Y is pseudocompact. In fact, given any sequence $\{A_k: k=1, 2, \ldots\}$ of mutually disjoint, non-empty open subsets of Y, let M be the set $\{n_k\in N: n_k\in N\cap A_k, k=1, 2, \ldots\}$. By our assumption there is an ultrafilter $\mathscr{U}_M$ on N which converges to $p(M)\in Y$ and contains M. Consequently, $p(M)\in\mathrm{cl}_{\beta N} M$ and therefore $p(M)$ is a cluster point of $\{A_k: k=1, 2, \ldots\}$. From F.2, Y is pseudocompact.

Since X is completely regular and U_n is open in X, we can choose zero-sets Z_n and Z'_n in $Z(X)$ such that $\mathrm{int}_X Z_n\neq\emptyset$ and $Z_n\subset X-Z'_n\subset U_n$. The set V is pseudocompact, so there is a point $\alpha\in V$ such that each neighborhood of α meets infinitely many of the sets Z_n. Let $\mathscr{E}(\alpha)$ be the family of all neighborhoods of α. For each $E\in\mathscr{E}(\alpha)$ put $T(E)=\{n\in N: E\cap Z_n\neq\emptyset\}$. As the family $\{\mathrm{cl}_{\beta N}T(E): E\in\mathscr{E}(\alpha)\}$ has the finite intersection property, it follows that $K=\bigcap\{\mathrm{cl}_{\beta N}T(E): E\in\mathscr{E}(\alpha)\}$ is a non-empty compact subset of βN. Let η be a point of K. Let us see that $\eta\notin Y$. Suppose there is a point $p(M)\in Y$ such that $\eta=p(M)$. Since

$$\bigcap_{F\in\mathscr{F}(M)} \mathrm{cl}_V\big(\bigcup_{n\in F} U_n\big) = \emptyset$$

there exists a neighborhood E_0 of α and $F_0 \in \mathcal{F}(M)$ such that $E_0 \cap (\bigcup_{n \in F_0} U_n) = \emptyset$. On the other hand, $p(M) \in \mathrm{cl}_{\beta N} T(E_0)$, which is to say that $T(E_0) \in \mathcal{U}_M$. Furthermore $\mathcal{F}(M) \subset \mathcal{U}_M$, therefore $F_0 \in \mathcal{U}_M$ and consequently $T(E_0) \cap F_0 \neq \emptyset$. From the definition of $T(E_0)$, for some $n_0 \in F_0$ we have $E_0 \cap Z_{n_0} \neq \emptyset$, which is a contradiction. Therefore $\eta \notin Y - N$.

Let us see that $\eta \notin N$. Since α is a cluster point of the sequence $\{Z_n : n=1, 2, \dots\}$ and each set U_n is open, we have that $\alpha \notin U_n$ for every $n=1, 2, \dots$. Thus, for each n there exists $E_n \in \mathcal{E}(\alpha)$ such that $n \notin T(E_n)$. Therefore $n \notin K$ for every $n \in N$. So $\eta \notin Y$.

Evidently, the space $Z = Y \cup \{\eta\}$ is pseudocompact. In order to see that $X \times Z$ is not a b_R-space, we need the following fact: *If B is a bounded subset of $X \times Z$, then the set $\{n \in N : B \cap (U_n \times \{n\}) \neq \emptyset\}$ is finite.* Suppose that B is a subset of $X \times Z$ meeting $U_n \times \{n\}$ for every $n \in M' \subset N$, M' infinite. Choose an infinite subset M of M' such that $\eta \notin \mathrm{cl}_{\beta N} M$. We can write $M = \{n_k : k=1, 2, \dots\}$, $n_k < n_{k+1}$. It will be seen that B is not bounded by showing that the family $\{U_{n_k} \times \{n_k\} : k=1, 2, \dots\}$, is discrete in $X \times Z$. Let $x \in V$ and $p(N_0) \in Y - N$. Since

$$\bigcap_{F \in \mathcal{U}_{N_0}} \mathrm{cl}_V (\bigcup_{n \in F} U_n) = \emptyset,$$

there exists a neighborhood W of x and $F_0 \in \mathcal{U}_{N_0}$ such that

$$W \cap (\bigcup_{n \in F_0} U_n) = \emptyset.$$

Therefore $W \times (Z \cap \mathrm{cl}_{\beta N} F_0)$ is a neighborhood of $(x, p(N_0))$ meeting no set $U_n \times \{n\}$. Now consider a point (x, η), $x \in X$. Since $\eta \notin \mathrm{cl}_{\beta N} M$ and $\mathrm{cl}_{\beta N} M$ is clopen in βN, it follows that $X \times (Z \cap \mathrm{cl}_{\beta N}(N - M))$ is a neighborhood of (x, η) meeting no set $U_{n_k} \times \{n_k\}$. For the points $(x, p(N_0)) \in (X - V) \times (Y - N)$ and $(x, n) \in X \times N$ the conclusion holds clearly. From F.1 the set B is not bounded in $X \times Z$.

Finally, let us see that $X \times Z$ is not a b_R-space. For each $n=1, 2, \dots$ let f_n be a real-valued continuous function on X such that $f_n(Z_n) = \{1\}$ and $f_n(Z_n') = \{0\}$. Now consider the function f defined in $X \times Z$ $f(x, n) = f_n(x)$, $(x, n) \in X \times N$ and vanishes otherwise. The function f is not continuous on $X \times Z$ because (α, η) is a cluster point of the sequence $\{Z_n \times \{n\} : n=1, 2, \dots\}$ and f is 0 in (α, η) and 1 on each set $Z_n \times \{n\}$. If B is any bounded subset of $X \times Z$, then B meets at most finitely many sets $U_n \times \{n\}$. Consequently, the restriction of f to B is continuous. This concludes the proof.

COROLLARY 2. *If X is a pseudocompact space which is not in $\mathfrak{B}$, there exists a pseudocompact space Z such that $X \times Z$ is not a b_R-space.*

COROLLARY 3. *Let X be a locally pseudocompact space. If for every b_R-space (resp. p_R-space) Y the product $X \times Y$ is a b_R-space (resp. p_R-space), then each point of X has a neighborhood in $\mathfrak{B}$.*

PROOF. Suppose that the point $x \in X$ has no neighborhood in $\mathfrak{B}$, and let W be a pseudocompact neighborhood of x. According to ([3], Proposition 4.2) the set

$V=\mathrm{cl}(\mathrm{int}\, W)$ is a pseudocompact neighborhood of x, which is a regular closed set. Since $V \notin \mathfrak{B}$, from Theorem 1 the conclusion follows.

p_R-spaces and the class $\mathfrak{B}$. We use $\mathfrak{S}$ to denote the class of all spaces X such that $X \times Y$ is a p_R-space whenever Y is. $\mathfrak{S}^*$ is the subset of $\mathfrak{S}$, consisting of all pseudocompact spaces in $\mathfrak{S}$. $C^*(X)$ will denote the Banach space of all bounded, real-valued, continuous functions over the space X, with the supremum-norm.

The following result is a consequence of 3.10(b) in [5].

LEMMA 4. *Let g be a function from a p_R-space X into a (completely regular) space Y. If the restriction of g to each pseudocompact subset of X is continuous, then g is continuous on X.*

THEOREM 5. $\mathfrak{B}=\mathfrak{S}^*$.

PROOF. From Corollary 2 we have $\mathfrak{S}^* \subset \mathfrak{B}$. Now let $X \in \mathfrak{B}$ and let Y be a p_R-space. Suppose that f is a real-valued function whose restriction to each pseudocompact subset of $X \times Y$ is continuous. If S is a pseudocompact subset of Y, then $X \times S$ is pseudocompact. Therefore $f \in C^*(X \times S)$. Let ψ be the function from Y into $C^*(X)$ defined $\psi(p)=f(\cdot, p)$ $p \in Y$. According to ([4], Theorem 2.2) the restriction of ψ to S is continuous and from Lemma 4, ψ is continuous on Y. Now it is easy to see that f is continuous on $X \times Y$. Thus $X \times Y$ is a p_R-space and $X \in \mathfrak{S}^*$. Therefore $\mathfrak{B}=\mathfrak{S}^*$.

HUŠEK [7] proved the following: If X is not locally bounded, there are a paracompact sequential (hence a p_R-space) Y and a real-valued function f which is not continuous on $X \times Y$ but it is continuous on each compact subset of $X \times Y$. Actually, the function f is continuous on each bounded subset of $X \times Y$, as an easy check shows. Therefore every space belonging to $\mathfrak{S}$ is locally bounded.

THEOREM 6. *A space X belongs to $\mathfrak{S}$ if and only if each point of X has a neighborhood in $\mathfrak{B}$.*

PROOF. *Necessity*. From ([3], Proposition 4.2) a bounded regular closed subset (of a completely regular space) is pseudocompact. Then, from the former observation, every space of the class $\mathfrak{S}$ is locally pseudocompact. The necessity now follows from Corollary 3.

Sufficiency. Suppose that each point $x \in X$ has a neighborhood $V(x) \in \mathfrak{B}$ and let Y be a p_R-space. By Theorem 5, $V(x) \times Y$ is a p_R-space. Let f be a real-valued function whose restriction to each pseudocompact subset of $X \times Y$ is continuous. Since f is continuous on each member of the family $\{\mathrm{int}\, V(x) \times Y : x \in X\}$, it follows that f is continuous on $X \times Y$. Therefore $X \times Y$ is a p_R-space and the proof is concluded.

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ON THE STABILITY OF SYSTEMS OF NEUTRAL TYPE

By
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1. Introduction

Liapunov's second method for ordinary differential equations can be generalized to deal with systems of differential equations of neutral type without trouble using scalar functions. But, one of the problems of practical nature is that to ensure the definite sign of the derivative $V'_{(1)}$ along all the solutions of (1) for every initial function, one must impose fairly strong conditions on the functional "g". RAZUMIKHIN noticed in [4] that it is unnecessary to study the sign of $V'_{(1)}$ for every initial function. Indeed, if a solution of (1) starts inside a ball and leaves it at a certain instant t, then $|x(t+s)| \leqq |x(t)|$ for every $s \in [-r, 0]$. Therefore, it is enough to consider initial functions satisfying this last condition.

The purpose of this work is to show that for a system of differential equations of neutral type, Razumikhin's condition ensures only the stability of the solutions in the sense of Liapunov. In this respect we give an example showing that Razumikhin's condition is not sufficient to guarantee the asymptotic stability of the solutions. Moreover, we show that substituting Razumikhin's condition for Krasovskiy's, asymptotic stability holds under certain additional conditions.

2. Preliminaries

Here we consider a system of differential equations of neutral type

$$x'(t) = g(t, x_t, x'_t), \quad x_{t_0} = \varphi; \quad t_0 \geqq \tau \tag{1}$$

where $g \in C^1[J \times C \times C, R^n]$, $\varphi \in C_1 = C^1[[-r, 0], R^n]$, $g(t, 0, 0) = 0$ for every $t \in J = [\tau, +\infty)$; $x_t, x'_t : [-r, 0] \to R^n$ are functions defined by $x_t(s) = x(t+s)$, $x'_t(s) = x'(t+s)$.

The solution of (1) through (t_0, φ) is denoted by $x(t_0, \varphi)$. Furthermore, the following notations will be used: $C_i = C_i[[-r, 0], R^n]$ is the space of functions with norm $\|\cdot\|_i$, $i = 0, 1$ respectively and

$$\|\varphi\|_0 = \max_{s \in [-r, 0]} |\varphi(s)|, \quad \|\varphi\|_1 = \max\{\|\varphi\|_0, \|\dot{\varphi}\|_0\}.$$

DEFINITION 1. The zero solution of (1) is uniformly stable in C_i $(i=0, 1)$, if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any $t_0 \geqq \tau$ if $\|\varphi\|_i < \delta$, we have $\|x_t(t_0, \varphi)\|_i < \varepsilon$ for all $t \geqq t_0$. If in addition there exists a $\Delta > 0$ such that $\|\varphi\|_i < 0$ implies $\lim_{t \to \infty} \|x_t(t_0, \varphi)\|_i = 0$, then we shall say that the trivial solution of (1) is uniformly asymptotically stable in C_i $(i = 0, 1)$.

DEFINITION 2. The zero solution of the functional equation

$$Z(t)-g(t, x_t, z_t)=0 \tag{2}$$

is said to be functionally stable, if for any $\varepsilon>0$, $t_0 \geqq \tau$ there are $\delta_1>0$ and $\delta_2>0$ such that if $\|Z_{t_0}\|_0<\delta_1$ and $\|x_t\|_0<\delta_2$ for every $t \geqq t_0$, we get $\|Z_t\|_0<\varepsilon$ for all $t \geqq t_0$. If in addition $\lim_{t\to\infty} \|Z_t\|_0=0$, then the trivial solution of (2) is said to be functionally asymptotically stable.

If $V: R \times R^n \to R$ is a continuous function, then $\dot V(t, \varphi(0))$, the derivative of V along the solutions of (1) is defined to be

$$\dot V(t, \varphi(0)) = \lim_{h\to 0^+} \frac{1}{h}\left[V\big(t+h, x(t, \varphi)(t+h)\big)-V\big(t, \varphi(0)\big)\right],$$

where $x(t, \varphi)$ is the solution of (1) through (t, φ).

3. Results

Consider the equation

$$x'(t)=f\big(t, x(t), x(t-r), x'(t-r)\big), \tag{3}$$

in which f is given by the following relation:

$$f(t, x, y, z)=\begin{cases} -x, & \text{if } \; x^2 \geqq y^2 \\ -a(t)x-x\dfrac{a(t)}{1+z^2}, & \text{if } \; x^2 \exp\left[2\displaystyle\int_{t-n}^{t} a(s)ds\right] \leqq y^2 \\ -x+\dfrac{(1-a(t))(y^2-x^2)}{x\left(\exp\left(2\displaystyle\int_{t-n}^{t} a(s)\,ds\right)-1\right)}-\dfrac{a(t)(1+z^2)^{-1}(y^2-x^2)}{\left[\exp\left(2\displaystyle\int_{t-n}^{t} a(s)\,ds\right)-1\right]x}, & \\ \quad \text{if } \; 0<x^2<y^2<x^2\exp\left(2\displaystyle\int_{t-n}^{t} a(s)\,ds\right), & \end{cases}$$

where $0<a(t)<1$ for all $t \geqq -r$, a is a continuous function such that

$$\int_0^\infty a(t)\,dt=\alpha<+\infty.$$

We shall prove that there exists a positive definite continuously differentiable function $V: R \to R^+$, such that

$$\dot V_{(3)}\big(x(t)\big)=\frac{\partial V\big(x(t)\big)}{\partial x} f\big(t, x(t), x(t-r), x'(t-r)\big)$$

is negative definite along the solutions of (3) that verify the Razumikhin's condition

$$V\big(x(s)\big) \leqq V\big(x(t)\big) \quad \text{for every} \quad t \geqq t_0 \quad \text{and} \quad s \leqq t, \tag{4}$$

however the zero solution of (3) is not asymptotically stable in C_0.

In fact, if we set $V(x)=x^2$, then

$$\dot{V}_{(1)} = -2V[x(t)], \tag{5}$$

if $V[x(t)] \geqq V[x(t-r)]$;

$$\dot{V}_{(1)} = -2a(t)V[x(t)] - 2a(t)V[x(t)][1+V[x'(t-r)]]^{-1}, \tag{6}$$

if $V[x(t)] \exp\left(2 \int_{t-n}^{t} a(s)\,ds\right) \leqq V[x(t-r)]$;

$$\dot{V}_{(1)} = -2V[x(t)] + \frac{2(1-a(t))(V[x(t-r)]-V[x(t)]}{\exp\left(2 \int_{t-n}^{t} a(s)\,ds\right)-1} = \tag{7}$$

$$= -2\frac{a(t)(1+V[x'(t-r)])^{-1}(V[x(t-r)]-V[x(t)])}{\exp\left(2 \int_{t-n}^{t} a(s)\,ds\right)-1},$$

if $V[x(t)] \leqq V[x(t-r)] \leqq V[x(t)] \exp\left(2 \int_{t-n}^{t} a(s)\,ds\right)$.

Therefore, from the properties of the function a, we obtain $\frac{dV(x(t))}{dt} \leqq$ $\leqq -2a(t)V(x(t))$ for $t \geqq 0$. Setting $w(t)=V[x(t)]$ and integrating the last inequality, we get $w(s) \geqq w(t)\exp\left(-2\int_t^s a(u)\,du\right)$ for $s \in [0, t]$. Hence, if we put $s=t-r$, then $w(t-r) \geqq w(t)\exp\left(2\int_{t-n}^{t} a(s)\,ds\right)$ for all $t \geqq r$. But this together with (6) imply that $w'(t) = -2a(t)w(t) - \frac{2a(t)w(t)}{1+V[x'(t-r)]}$ for all $t \geqq r$. Since V is positive definite, it follows that, $w'(t) \geqq -4a(t)w(t)$ for all $t \geqq r$. Finally, integrating the last inequality, we have

$$w(t) \geqq w(r)\exp\left(-4\int_n^t a(s)\,ds\right) \geqq w(r)e^{-4\alpha}, \quad t \geqq r.$$

Consequently, the trivial solution of (3) cannot be asymptotically stable in C_0, since $|x(t)| \geqq |x(r)| \exp(-4\alpha) > 0$ for all $t \geqq r$.

On the other hand, from (5) it is clear that the condition (4) holds, since if $V(x(t-r)) \leqq V(x(t))$, then $V'_{(3)}(x(t)) = -2V(x(t))$.

A similar example was published by Z. MIKOLAJSKA in [2], related to differential equations with time lag.

Now, we give sufficient conditions for the asymptotic stability of the solution $x=0$ of equation (1).

THEOREM 1. *Suppose* $a, b, w: R^+ \to R^+$ *are continuous, nondecreasing functions* $a(s), b(s), w(s)$ *positive for* $s>0$ $a(0)=b(0)=w(0)=0$. *If there is a continuous func-*

tion $V: R\times R^n \to R^+$ *such that*

$$a(|x|) \leqq V(t, x) \leqq b(|x|), \quad t\in \mathbf{R}, \quad x\in R^n,$$

and there is a continuous nondecreasing function $p(s)>s$ *for* $s>0$ *such that*

$$\dot V(t, \varphi(0)) \leqq -w(|\varphi(0)|), \quad if \quad V(t+s, \varphi(s)) < p(V(t, \varphi(0)), \quad s\in[-r, 0],$$

then the trivial solution of (1) *is uniform-asymptotically stable in* C_0. *If, in addition, the trivial solution of the functional equation* (2) *is functional-asymptotically stable, then the trivial solution of* (1) *is asymptotically stable in* C_1.

The proof of the uniform asymptotic stability in C_0 runs exactly as the proof of Theorem 7 in [1]. Now, from the uniform asymptotic stability in C_0 of $x=0$, we get that for any $\varepsilon_1>0$ there exist $\delta(\varepsilon_1)>0$ and $\Delta>0$ such that $\|\varphi\|_0<\delta$ implies $\|x_t(t_0, \varphi)\|_0<\varepsilon_1$ for all $t\geqq t_0$, and further $\lim_{t\to\infty} \|x_t(t_0, \varphi)\|_0=0$, if $\|\varphi\|_0<\Delta$. Therefore, by functional-asymptotic stability of the trivial solution of (2), for any $\varepsilon>0$ we obtain that there is $\delta_1(\varepsilon)>0$ such that $\|\dot x\|_0<\varepsilon$, for all $t\geqq t_0$ if $\|\dot\varphi\|_0< <\delta_1(\varepsilon)$, and $\lim_{t\to\infty} \|\dot x_t\|_0=0$.

The last result follows immediately setting $\delta_2=\varepsilon_1$ in Definition 2. Finally, if we choose $\varepsilon_2=\max(\varepsilon, \varepsilon_1)$, the uniform asymptotic stability of the $x=0$ holds.

Misnik and Nosov in [3] tried to prove a similar result, but they did not notice that Razumikhin's condition is not sufficient for asymptotic stability.

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DOUBLY ORDERED LINEAR RANK STATISTICS

By

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1. Introduction

Let $N=m+n$, where m and n are positive integers. Let the real numbers $x_1, \ldots, x_N$ be pairwise different from each other. Let Rank x_k and rank x_k denote the rank of x_k within the sequences $x_1, \ldots, x_N$, and $x_1, \ldots, x_n$, respectively. In other words, if the rearrangement according to size $z_1 < \ldots < z_N$ of those numbers $x_k = z_{r_k}$, and if the rearrangement according to size $y_1 < \ldots < y_n$ of the numbers $x_1, \ldots, x_n$, $x_k = y_{i_k}$, then we say that x_k has rank r_k and i_k with respect to these orders, and we write Rank $x_k = r_k$, and rank $x_k = i_k$ $(k=1, \ldots, m)$, respectively.

Denote by $\Pi_m^{(N)}$ the set of all $(r_1, \ldots, r_m)$ chosen without repetition from the elements $1, \ldots, N$, and by P_m the set of all permutations of the elements $1, \ldots, m$ without repetition.

Let the distribution functions of the real random variables $X_1, \ldots, X_N$ be continuous. Then $P(X_j = X_k) = 0, j \neq k$.

Denote by $\{r_1, \ldots, r_m\}$ and $[i_1, \ldots, i_m]$ the vectors with the components $r_1, \ldots, r_m$ and $i_1, \ldots, i_m$, where $(r_1, \ldots, r_m) \in \Pi_m^{(N)}$, $(i_1, \ldots, i_m) \in P_m$, respectively.

DEFINITION 1.2. The vector $\{r_1, \ldots, r_m\}$ is said to be the outer-rank of the random variables $X_1, \ldots, X_m$ with respect to the random variables $X_1, \ldots, X_N$, if

$$\{\text{Rank}\, X_1, \ldots, \text{Rank}\, X_m\} = \{r_1, \ldots, r_m\}.$$

DEFINITION 1.1. The vector $[i_1, \ldots, i_m]$ is said to be the inner-rank of the random variables $Y_1, \ldots, Y_m$, if

$$[\text{rank}\, Y_1, \ldots, \text{rank}\, Y_m] = [i_1, \ldots, i_m].$$

Obviously, the random events $\{r_1, \ldots, r_m\}$ and $[i_1, \ldots, i_m]$ are independent if the random variables $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_m$ are independent, i.e. in this case

$$P(\{r_1, \ldots, r_m\}, [i_1, \ldots, i_m]) = P(\{r_1, \ldots, r_m\}) P([i_1, \ldots, i_m]).$$

By the help of this formula we get the following theorem on the basis of [3] (p. 369, Satz 10).

THEOREM 1.1. *Let the random variables $Z_1 = (X_1, Y_1), \ldots, Z_m = (X_m, Y_m)$ and the random variables $X_{m+1}, \ldots, X_N$ be given. If the random vector variables $(X_1, \ldots, X_N)$ and $(Y_1, \ldots, Y_m)$ are independent and if the joint distribution functions of the random variables $X_1, \ldots, X_N$ and the random variables $Y_1, \ldots, Y_m$ are symmetric functions of* *their variables, and they are continuous in each of the variables, then*

$$P(\{r_1, \ldots, r_m\}, [i_1, \ldots, i_m]) = \frac{1}{m!(n+1)\ldots(n+m)},$$

$$(r_1, \ldots, r_m) \in \Pi_m^{(N)}, \quad (i_1, \ldots, i_m) \in P_m.$$

The conditions of Theorem 1.1 will be satisfied if $X_1, \ldots, X_N$ and if $Y_1, \ldots, Y_m$ are samples with continuous distribution functions, and these random variables are independent.

Let the matrices

$$A_j = \begin{pmatrix} a_{11}^{(j)} \ldots a_{1N}^{(j)} \\ \cdot \quad \ldots \quad \cdot \\ a_{m1}^{(j)} \ldots a_{mN}^{(j)} \end{pmatrix} \quad (j = 1, \ldots, m) \tag{1.1}$$

with real elements be given, and let $A = A_1 \ldots A_m$ be the $m \times mN$ matrix with blocks (1.1).

On the basis of Theorem 1.1 we give the following definition.

DEFINITION 1.3. The random variable $X_{m,n}^{(N)}$ is said to be a doubly ordered linear rank statistics generated by the matrix A if

$$P(X_{m,n}^{(N)} = a_{i_1 r_1}^{(1)} + \ldots + a_{i_m r_m}^{(m)}) = \frac{1}{m!(n+1)\ldots(n+m)}, \tag{1.2}$$

where $(r_1, \ldots, r_m)$ and $(i_1, \ldots, i_m)$ run over the sets $\Pi_m^{(N)}$ and P_m, respectively.

Let the random vector variables $Z_1 = (X_1, Y_1), \ldots, Z_m = (X_m, Y_m)$ and the random variables $X_{m+1}, \ldots, X_N$ be given. Suppose that the random variables $X_1, \ldots, X_N$, $Y_1, \ldots, Y_m$ are independent with continuous distribution functions. Suppose that $X_1, \ldots, X_m$ and $X_{m+1}, \ldots, X_N$ are samples with distribution functions $F(x)$ and $G(x)$, respectively. Then the doubly ordered linear rank statistics $X_{m,n}^{(N)}$ defined by (1.2) give us the possibility to decide on the acceptance or the rejection of the joint hypothesis

a) the second components of the random vector variables $Z_1, \ldots, Z_m$ have a common distribution function;

b) $F(x) = G(x)$, $x \in R_1$.

If all rows of the matrix A are equal, then $X_{m,n}^{(N)}$ give us the possibility to take a decision on the acceptance or rejection of the hypothesis b).

Denote by $\Psi_{m,n}^{(N)}(t)$ the characteristic function of the random variable $X_{m,n}^{(N)}$ defined by (1.2).

The aim of this paper is to investigate the characteristic function $\Psi_{m,n}^{(N)}(t)$. Beside the Introduction the paper contains two sections. In Section 2 the characteristic function $\Psi_{m,n}^{(N)}(t)$ will be approximated by the permanents of simpler characteristic functions. On the basis of this approximation theorem we give an asymptotic formula for $\Psi_{m,n}^{(N)}(t)$. The theorems of Section 3 are dealing with the construction of doubly ordered linear rank statistics with given limit distribution. To do this it is necessary to extend te well-known Koksma's inequality for arbitrary distributions.

2. The characteristic function of the doubly ordered linear rank statistics

First of all we prove the following theorem.

THEOREM 2.1. *Let $\Psi_{m,n}^{(N)}(t)$ be the characteristic function of the doubly ordered linear rank statistics defined by* (1.2). *Let*

$$\varphi_{jk}^{(N)}(t) = \frac{1}{N}\sum_{n=1}^{N} e^{it\,a_{jn}^{(k)}} \quad (j, k = 1, \ldots, m)$$

and

$$\Phi_m^{(N)}(t) = \begin{pmatrix} \varphi_{11}^{(N)}(t) \ldots \varphi_{1m}^{(N)}(t) \\ \cdot \quad \ldots \quad \cdot \\ \varphi_{m1}^{(N)}(t) \ldots \varphi_{mm}^{(N)}(t) \end{pmatrix}. \tag{2.1}$$

Then

$$\left|\Psi_{m,n}^{(N)}(t) - \frac{N^m}{(n+1)\ldots(n+m)}\frac{1}{m!}\operatorname{Per}\Phi_m^{(N)}(t)\right| \le \frac{N^m}{(n+1)\ldots(n+m)} - 1$$

for $t \in R_1$.

PROOF. If $M=(a_{jk})$ is a square matrix of order n with complex numbers as its elements, then the permanent of M, denoted by $\operatorname{Per} M$, is defined as follows:

$$\operatorname{Per} M = \sum_{(i_1, \ldots, i_m)} a_{1i_1}, \ldots a_{mi_m},$$

where $(i_1, \ldots, i_m)$ runs over the full symmetric group.

On the basis of (1.2) we get that

$$\Psi_{m,n}^{(N)}(t) = \frac{1}{m!(n+1)\ldots(n+m)} \sum_{(r_1, \ldots, r_m)\in \Pi_m^{(N)}}$$

$$\sum_{(i_1, \ldots, i_m)\in P_m} \exp\{it(a_{i_1 r_1}^{(1)} + \ldots + a_{i_m r_m}^{(m)})\} = \frac{1}{m!(n+1)\ldots(n+m)}\cdot$$

$$\sum_{(r_1, \ldots, r_m)\in \Pi_m^{(N)}} \operatorname{Per} \begin{pmatrix} e^{it\,a_{1r_1}^{(1)}} \ldots e^{it\,a_{1r_m}^{(m)}} \\ \cdot \quad \ldots \quad \cdot \\ e^{it\,a_{mr_1}^{(1)}} \ldots e^{it\,a_{mr_m}^{(m)}} \end{pmatrix}.$$

Let $B=B_1 \ldots B_m$ the $m\times mN$ matrix with blocks $B_1, \ldots, B_m$, where the B_j matrix is defined as follows. The elements of the j-th row are equal to one, while the remaining elements are equal to zero. Therefore the permanent of the $m\times m$ submatrix M of B is different from zero if and only if M has one column from each of the matrices $B_1, \ldots; B_m$. The number of such submatrices of B is N^m and the permanent of these is equal to one.

Let

$$(2.2)\qquad A_j(t)=\begin{pmatrix} e^{ita_{11}^{(j)}}\dots e^{ita_{1N}^{(j)}}\\ \cdot \quad \dots \quad \cdot\\ e^{ita_{m1}^{(j)}}\dots e^{ita_{mN}^{(j)}}\end{pmatrix}\quad (j=1,\dots,m),$$

and let $A(t)=A_1(t)\dots A_m(t)$ be the $m\times m^N$ matrix with blocks (2.2).

We have

$$(2.3)\qquad \operatorname{Per}\big(A(t)B^*\big)=N^m \operatorname{Per}\Phi_m^{(N)}(t),$$

where $\Phi_m^N(t)$ is defined by (2.1). On the other hand, using the Cauchy-Binet expansion theorem,

$$(2.4)\qquad \operatorname{Per}\big(A(t)B^*\big)=m!(n+1)\dots(n+m)\Psi_{m,n}^{(N)}(t)+H(t),$$

where $H(t)$ is equal to the sum of the permanents of those $m\times m$ submatrices of $A(t)$, which have one and only one column from each of the matrices (2.2), but at least two columns have the same column index. The number of such matrices is equal to $N^m-(n+1)\dots(n+m)$. Since the moduli of the elements of these matrices are equal to one, the moduli of the permanents of these matrices are less than or equal to $m!$ Thus on the basis of (2.3) and (2.4) we get the statement of Theorem 2.1.

By the help of Theorem 2.1 we obtain easily the following theorem.

THEOREM 2.2. *Let the sequence* $\{X_{m,n}^{(N)}\}_{n=1}^{\infty}$ *of doubly ordered linear rank statistics be given, where* $X_{m,n}^{(N)}$ *is generated by the* $m\times mN$ *matrix* $A^{(N)}=A_1^{(N)}\dots A_m^{(N)}$ *with blocks*

$$A_j^{(N)}=\begin{pmatrix} a_{11}^{(j)}(N)\dots a_{1N}^{(j)}(N)\\ \cdot \quad \dots \quad \cdot\\ a_{m1}^{(j)}(N)\dots a_{mN}^{(j)}(N)\end{pmatrix}\quad (j=1,\dots,m).$$

Then uniformly in $t\in R_1$ *we have*

$$\lim_{n\to\infty}\left[\Psi_{m,n}^{(N)}(t)-\frac{N^m}{(n+1)\dots(n+m)}\frac{1}{m!}\operatorname{Per}\Phi_m^{(N)}(t)\right]=0,$$

where $\Psi_{m,n}^{(N)}(t)$ *is the characteristic function of* $X_{m,n}^{(N)}$.

3. Doubly ordered linear rank statistics with given limit distribution

This section consists of two parts. In the first one we extend the well-known Koksma's inequality for arbitrary distribution. In the second part we use this inequality to construct doubly ordered linear rank statistics with given limit distribution.

a) Let $H(a,b)$ be the set of the strictly monoton increasing continuous distribution functions $F(x)$, for which $a=\sup\{x\in R_1|F(x)=0\}$, $b=\inf\{x\in R_1|F(x)=1\}$, where $a<b$ are real numbers.

Let the sequence

$$(3.1)\qquad \omega=\{x_n\}_{n=1}^{\infty},\quad x_n\in[a,b)$$

be given. For a positive integer N and a subset E of $[a, b)$ let the counting function $A(E; N)$ be defined as the number of terms x_n, $1 \leqq n \leqq N$, for which $x_n \in E$.

DEFINITION 3.1. Let $F(x) \in H(a, b)$. The sequence (3.1) is said to be $F(x)$-distributed, if for every pair α, β of real numbers with $a \leqq \alpha < \beta \leqq b$ we have

$$\lim_{n\to\infty} \frac{1}{N} A\big([\alpha, \beta); N\big) = F(\beta) - F(\alpha).$$

As we said we shall use the $F(x)$-distributed sequences in the second part of this chapter to construct doubly ordered linear rank statistics with given asymptotic. Therefore in this first part of this chapter we compile the necessary definitions and theorems, which will be used in the above mentioned constructions. Definition 3.1 can be found in [2] (p. 54), but the following definitions and theorems only in the case of uniformly distributed sequences. The proofs of the following theorems are almost the same as in the case of uniformly distributed sequences. Therefore we shall refer only to the corresponding pages of the book [2].

DEFINITION 3.2. For a finite sequence $x_1, \ldots, x_N$ of real numbers, $x_n \in [a, b)$

$$D_N(F) = D_N(x_1, \ldots, x_N | F) = \sup_{a < \alpha \leqq b} \left| \frac{1}{N} A\big([a, \alpha); N\big) - F(\alpha) \right|$$

is said to be the discrepancy of the given sequence with respect to $F(x) \in H(a, b)$.

THEOREM 3.1. *Let $x_1 \leqq x_2 \leqq \ldots \leqq x_N$ be N numbers in $[a, b)$. Then their discrepancy with respect to $F(x) \in H(a, b)$ is given by*

$$D_N(F) = \max_{j=1, \ldots, N} \max\left(\left| F(x_j) - \frac{j}{N} \right|, \ \left| F(x_j) - \frac{j-1}{N} \right| \right) =$$

$$= \frac{1}{2N} + \max_{j=1, \ldots, N} \left| F(x_j) - \frac{2j-1}{2N} \right|.$$

PROOF. The proof is the same as in the case of uniformly distributed sequences ([2], p. 91, Theorem 1.4), but it is necessary to use that $F(x) \in H(a, b)$.

For a finite sequence $x_1, \ldots, x_N$ of real numbers, $x_n \in [a, b)$, let

$$D_N^*(F) = D_N^*(x_1, \ldots, x_N | F) = \sup_{a \leqq \alpha < \beta \leqq b} \left| \frac{1}{N} A\big([\alpha, \beta); N\big) - [F(\beta) - F(\alpha)] \right|.$$

LEMMA 3.1. *The sequence* (3.1) *is $F(x)$-distributed if and only if* $\lim_{N\to\infty} D_N^*(\omega | F) = 0$.

PROOF. The proof is the same as in the case of uniformly distributed sequences ([2], p. 89, Theorem 1.1) if we take into consideration that $F(x) \in H(a, b)$.

LEMMA 3.2. *The quantities $D_N(F)$ and $D_N^*(F)$ are related by the inequality*

$$D_N(F) \leqq D_N^*(F) \leqq 2D_N(F).$$

PROOF. The proof is the same as in the case of uniformly distributed sequences ([2], p. 91, Theorem 1.3).

As an immediate consequence of Lemmata 1 and 2, we get the following theorem.

THEOREM 3.2. *The sequence* (3.1) *is* $F(x)$*-distributed if and only if*

$$\lim_{N\to\infty} D_N(\omega|F) = 0.$$

LEMMA 3.3. *Let* $x_1 \leqq \ldots \leqq x_N$ *be given* N *points in* $[a, b)$, *and let* g *be a function of bounded variation on* $[a, b]$. *Then with* $x_0 = a$, $x_{N+1} = b$, *we have the identity*

$$\frac{1}{N}\sum_{n=1}^{N} g(x_n) - \int_a^b g(t)\,dF(t) = \sum_{n=0}^{N}\int_{x_n}^{x_{n+1}} \left(F(t) - \frac{n}{N}\right) dg(t),$$

where $F(x) \in H(a, b)$.

PROOF. (See [2], p. 143, Lemma 5.1.) Using integration by parts and Abel's summation formula, we get

$$\sum_{n=0}^{N}\int_{x_n}^{x_{n+1}} \left(F(t) - \frac{n}{N}\right) dg(t) = \int_a^b F(t)\,dg(t) - \sum_{n=0}^{N}\frac{n}{N}[g(x_{n+1}) - g(x_n)] =$$

$$= [F(t)g(t)]_a^b - \int_a^b g(t)\,dF(t) + \frac{1}{N}\sum_{n=0}^{N-1} g(x_{n+1}) - g(b) =$$

$$= \frac{1}{N}\sum_{n=1}^{N} g(x_n) - \int_a^b g(t)\,dF(t),$$

because $F(b)g(b) = g(b)$, $F(a)g(a) = 0$.

In the following we prove an extension of the Koksma's inequality ([2], p. 143, Theorem 5.1).

THEOREM 3.3. *Let* $F(x) \in H(a, b)$. *Let* g *be a function of bounded variation* $V(g)$ *on* $[a, b]$, *and suppose we are given* N *points* $x_1, \ldots, x_N$ *on* $[a, b)$ *with discrepancy* $D_N(F)$. *Then*

$$\left|\frac{1}{N}\sum_{m=1}^{N} g(x_n) - \int_a^b g(t)\,dF(t)\right| \leqq V(g)\,D_N(F).$$

PROOF. Without loss of generality, we may assume that $x_1 \leqq \ldots \leqq x_N$. Thus, we can apply Lemma 3.3. For fixed n with $0 \leqq n \leqq N$, because $F(x) \in H(a, b)$, we have

$$\left|F(t) - \frac{n}{N}\right| \leqq \max\left(\left|F(x_n) - \frac{n}{N}\right|, \left|F(x_{n+1}) - \frac{n}{N}\right|\right) \leqq D_N(F)$$

for $x_n \leqq t \leqq x_{n+1}$ by Theorem 3.1, and the desired inequality follows immediately.

COROLLARY 3.1. *Let* $F(x)\in H(a,b)$. *Let* g *be a function of bounded variation* $V(g)>0$ *on* $[a,b]$, *and suppose we are given the sequence* (3.1). *Then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} g(x_n)=\int_a^b g(t)\,dF(t)$$

holds if and only if the sequence (3.1) *is* $F(x)$*-distributed.*

PROOF. Using Theorems 3.2 and 3.3, the statement follows immediately.

b) In this second part of the section we give a procedure to construct doubly ordered linear rank statistics with given limit distribution. To this construction we need some lemmata.

LEMMA 3.4. *If* f *is continuous function of bounded variation* $V(f)$ *on* $[a,b]$, *then* $V(e^{itf})\leqq|t|V(f)$, $t\in R_1$.

PROOF. See [1], Lemma 3.1.

Using Theorem 3.3 and Lemma 3.4, we get the following lemma.

LEMMA 3.5. *Let* f *be a continuous function of bounded variation* $V(f)$ *on* $[a,b]$, *and suppose we are given* N *points* $x_1,\dots,x_N$ *in* $[a,b)$ *with discrepancy* $D_N(F)$ *with respect to the distribution function* $F(x)\in H(a,b)$. *If*

$$\varphi^{(N)}(t)=\frac{1}{N}\sum_{n=1}^{N} e^{itf(x_n)},$$

then

$$\left|\varphi^{(N)}(t)-\int_a^b e^{itf(x)}\,dF(x)\right|\leqq |t|V(f)\,D_N(F)$$

for $t\in R_1$.

LEMMA 3.6. *Let* f_j *be a continuous function of bounded variation* $V(f_j)$ *on* $[a,b]$, *and suppose we are given* N *points* $x_1^{(j)},\dots,x_N^{(j)}$ *in* $[a,b)$ *with discrepancy* $D_N^{(j)}(F_j)$ *with respect to the distribution function* $F_j(x)\in H(a,b)$, *where* $j=1,\dots,m$. *Moreover we define the matrices*

$$\Phi_m(t)=\begin{pmatrix}\varphi_{11}(t)\dots\varphi_{1m}(t)\\ \cdot\quad\dots\quad\cdot\\ \varphi_{m1}(t)\dots\varphi_{mm}(t)\end{pmatrix}, \tag{3.2}$$

and

$$\Phi_m^{(N)}(t)=\begin{pmatrix}\varphi_{11}^{(N)}(t)\dots\varphi_{1m}^{(N)}(t)\\ \cdot\quad\dots\quad\cdot\\ \varphi_{m1}^{(N)}(t)\dots\varphi_{mm}^{(N)}(t)\end{pmatrix}$$

with the elements

$$\varphi_{jk}(t)=\int_a^b e^{itf_j(x)}\,dF_k(x)\quad (j,k=1,\dots,m) \tag{3.3}$$

and

$$\varphi_{jk}^{(N)}(t)=\frac{1}{N}\sum_{n=1}^{N} e^{itf_j(x_n^{(k)})}\quad (j,k=1,\dots,m), \tag{3.4}$$

respectively. Then

$$\frac{1}{m!}\,|\mathrm{Per}\,\Phi_m^{(N)}(t)-\mathrm{Per}\,\Phi(t)| \leqq \frac{|t|}{m}\left[\sum_{j=1}^{m} V(f_j)\right]\left[\sum_{j=1}^{m} D_N^{(j)}(F_j)\right]$$

for $t\in R_1$.

PROOF. Let us introduce the notation

$$\Delta_j=\frac{1}{m!}\,\mathrm{Per}\begin{pmatrix}\varphi_{11}(t)\dots\varphi_{1j-1}(t) & \varphi_{1j}^{(N)}(t)-\varphi_{1j}(t) & \varphi_{1j+1}^{(N)}(t) & \dots\varphi_{1m}^{(N)}(t)\\ \cdot\ \ \dots\ \ \cdot & \cdot & \cdot & \dots\ \ \cdot\\ \varphi_{m1}(t)\dots\varphi_{mj-1}(t) & \varphi_{mj}^{(N)}(t)-\varphi_{mj}(t) & \varphi_{mj+1}^{(N)}(t)\dots & \varphi_{mm}^{(N)}(t)\end{pmatrix}$$

$$(j=1,\dots,m),$$

then we can easily verify that

$$\frac{1}{m!}\,[\mathrm{Per}\,\Phi_m^{(N)}(t)-\mathrm{Per}\,\Phi_m(t)]=\Delta_1+\dots+\Delta_m. \tag{3.5}$$

Since the moduli of the characteristic functions (3.3) and (3.4) are less than or equal to one, on the basis of Lemma 3.5 we get

$$|\Delta_j|\leqq\frac{(m-1)!}{m!}\sum_{k=1}^{m}|\varphi_{kj}^{(N)}(t)-\varphi_{kj}(t)|\leqq\frac{|t|}{m}V(f_j)\sum_{k=1}^{m}D_N^{(k)}(F_k)\quad (j=1,\dots,m).$$

Utilizing this inequality in (3.5), we obtain the statement of our lemma.

Using Lemma 3.6 we get the following theorem as a consequence of Theorem 2.1.

THEOREM 3.4. *Let f_j be a continuous function of bounded variation $V(f_j)$ on $[a,b]$, and suppose we are given N points $x_1^{(j)},\dots,x_N^{(j)}$ in $[a,b)$ with discrepancy $D_N^{(j)}(F_j)$ with respect to the distribution function $F_j(x)\in H(a,b)$, where $j=1,\dots,m$. Let*

$$A_j^{(N)}=\begin{pmatrix}f_j(x_1^{(1)})\dots f_j(x_N^{(1)})\\ \cdot\ \ \dots\ \ \cdot\\ f_j(x_1^{(m)})\dots f_j(x_N^{(m)})\end{pmatrix}\quad (j=1,\dots,m). \tag{3.6}$$

Let us denote by $\Psi_{m,n}^{(N)}(t)$ the characteristic function of the doubly ordered linear rank statistics $X_{m,n}^{(N)}$ generated by the matrix $A=A_1^{(N)}\dots A_m^{(N)}$. Then we get

$$\left|\Psi_{m,n}^{(N)}(t)-\frac{N^m}{(n+1)\dots(n+m)}\frac{1}{m!}\,\mathrm{Per}\,\Phi_m(t)\right|\leqq \tag{3.7}$$

$$\leqq\frac{N^m}{(n+1)\dots(n+m)}\left[1+\frac{|t|}{m}\sum_{j=1}^{m}V(f_j)\cdot\sum_{j=1}^{m}D_N^{(j)}(F_j)\right]-1$$

for $t\in R_1$, where the matrix $\Phi_m(t)$ is defined by (3.2).

THEOREM 3.5. *Let f_j be a continuous function of bounded variation $V(f_j)$ on $[a,b]$, and suppose we are given the sequence $\omega_j=\{x_n^{(j)}\}_{n=1}^{\infty}$, $x_n^{(j)}\in[a,b)$ with discrepancy $D_N^{(j)}(F_j)$ of the points $x_1^{(j)},\dots,x_m^{(j)}$ with respect to the distribution function $F_j(x)\in\in H(a,b)$ for $j=1,\dots,m$. Let $V(f_1)+\dots+V(f_m)>0$. Denote by $\Psi_{m,n}^{(N)}(t)$ the charac-*

teristic function of the doubly ordered linear rank statistics $X_{m,n}^{(N)}$ *generated by the matrix* $A^{(N)}=A_1^{(N)}\dots A_m^{(N)}$, *where* $A_j^{(N)}$ *is defined by* (3.6). *Then*

$$\lim_{n\to\infty} \Psi_{m,n}^{(N)}(t) = \frac{1}{m!} \operatorname{Per} \Phi_m(t)$$

holds uniformly in any finite interval $t\in[-T, T]$ *if and only if the sequence* ω_j *is* $F_j(x)$*-distributed for* $j=1, \dots, m$, *where* $\Phi_m(t)$ *is defined by* (3.2).

PROOF. Since

$$\lim_{n\to\infty} \frac{N^m}{(n+1)\dots(n+m)} = 1,$$

the right hand side of (3.7) has the limit zero uniformly in any finite interval $t\in[-T, T]$ if and only if $\lim_{N\to\infty} D_N^{(j)}(F_j) = 0 \quad (j = 1, \dots, m)$. Thus we get the statement of our theorem using Theorem 3.2.

Denote by Y_{jk} the random variable with characteristic function (3.3). Then Theorem 3.5 can be expressed in the following alternative.

Under the conditions of Theorem 3.5 the sequence of the random variables $\{X_{m,n}^{(N)}\}_{n=1}^{\infty}$ converges weakly to the mixture of the random variables $Y_{1i_1}+\dots+Y_{mi_m}$, $(i_1, \dots, i_m)\in P_m$ with weights $\dfrac{1}{m!}$ if and only if, the sequence ω_j is $F_j(x)$-distributed for $j=1, \dots, m$. (The random variables $Y_{1i_1}, \dots, Y_{mi_m}$ are independent.)

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REMARKS ON POSITIONAL GAMES. I

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1. Introduction and results

We start with some terminology. A *hypergraph* is a collection of sets. The sets in the hypergraph are called *edges*, and the elements of these edges are called *vertices*. We deal with finite hypergraphs only. $|A|$ denotes the number of elements of the set A.

Let p and q be positive integers and $\mathcal{H}$ be a hypergraph. A $(p, q, \mathcal{H})$-game is a game in which two players select previously unselected vertices of $\mathcal{H}$. The first player selects p vertices per move and the second player selects q vertices per move. The first player wins whenever he selects all the vertices of some eddge of $\mathcal{H}$, other wise the second player wins. In the case $p=q=1$, ERDŐS and SELFRIDGE [3] have found a sufficient condition for the second player's win: If

$$\sum_{A\in\mathcal{H}} 2^{-|A|} < 1/2,$$

then the second player has a winning strategy for the $(1, 1, \mathcal{H})$-game.

This theorem is sharp in a strong sense, see [3]. Our first aim is to prove a generalization of this result.

THEOREM 1. *If*

$$\sum_{A\in\mathcal{H}} (1+q)^{-|A|/p} < \frac{1}{1+q},$$

then the second player has a winning strategy for the $(p, q, \mathcal{H})$-game.

A weaker result was proved by CSIRMAZ [2].

Theorem 1 is also sharp in the following sense. For each p, q there are infinitely many hypergraphs $\mathcal{H}$ such that in the formula above equality holds, and the first player wins the game $(p, q, \mathcal{H})$. The form of the extremal graph is a tree of height h in which every node has exactly $q+1$ immediate successors. Put q points in place of each node, and an edge of the extremal hypergraph is the union of points along a full branch. Obviously this hypergraph has $(1+q)^{h-1}$ edges, every edge contains hp points, and is a win for the first player.

Secondly, we give a sufficient condition for the first player's win.

THEOREM 2. *Let $v(\mathcal{H})$ denote the number of vertices of $\mathcal{H}$ and denote by $d_2(\mathcal{H})$ the maximum number of edges of $\mathcal{H}$ containing two given vertices. If*

$$\sum_{A\in\mathcal{H}} \left(1+\frac{q}{p}\right)^{-|A|} > p^2q^2(p+q)^{-3}\, d_2(\mathcal{H})\, v(\mathcal{H}),$$

then the first player has a winning strategy for the $(p, q, \mathcal{H})$-game.

Let $W(R, r)$ be the family of all possible arithmetic progressions of r terms in the interval $\{1, 2, ..., R\}$. By a way of illustration of the two theorems above consider the $(p, q, W(R, r))$-game, i.e., the first player wants an arithmetic progression of r terms. It is easy to see that $R^2/4(r-1) < |W(R, r)| < R^2/(r-1)$. Since at most $\binom{r}{2}$ arithmetic progressions of r terms can contain two given integers, we have $d_2(W(R, r)) \leqq \binom{r}{2}$. By the application of Theorems 1 and 2 we obtain that if $r < c_1 \log R$ then the first, if $r > c_2 \log R$ then the second player has a winning strategy for the $(p, q, W(R, r))$-game. Throughout this paper log stands for base e logarithms, and the constants $c_1 = c_1(p, q)$ and $c_2 = c_2(p, q)$ depend only on p and q.

Let us consider two further applications of Theorem 1. CHVÁTAL and ERDŐS [1] introduced the following graph-game. Two players, Maker and Breaker, with Breaker going first, play a game on a complete graph of n vertices. By $T(n, b)$, we shall denote the game where on each move, Breaker claims b previously unclaimed edges and Maker claims one previously unclaimed edge. Maker wins if he claims all the edges of some spanning tree of the complete graph of n vertices; otherwise Breaker wins. CHVÁTAL and ERDŐS [1] raised the question: What is the largest $f(n)$ such that Maker has a winning strategy for $T(n, f(n))$? They proved that

$$\frac{n}{(4+\varepsilon)\log n} < f(n) < \frac{(1+\varepsilon)n}{\log n}. \tag{1}$$

As a direct application of Theorem 1 we shall prove a slight improvement on the left-hand side of (1).

THEOREM 3. $\dfrac{(\log 2-\varepsilon)n}{\log n} < f(n)$ *for* $n > n_0(\varepsilon)$.

Following CHVÁTAL and ERDŐS [1] we shall denote by $H(n, b)$ the game which differs from $T(n, b)$ in only one respect: Maker's aim is to claim all the edges of some Hamiltonian cycle. What is the largest $g(n)$ such that Maker has a winning strategy for $H(n, g(n))$? CHVÁTAL and ERDŐS [1] proved that $g(n) \geqq 1$ for all sufficiently large n. They suggested that $g(n) \to \infty$ as $n \to \infty$. In Part II of this paper we shall prove this conjecture in the following stronger form.

THEOREM 4. $g(n) > n^{1/2-\varepsilon}$ *for* $n > n_1(\varepsilon)$.

2. Proof of Theorem 1

Given a hypergraph $\mathcal{G}$, disjoint subsets X and Y of the vertex-set $V(\mathcal{G})$ of $\mathcal{G}$ denote $\varphi(X, Y, \mathcal{G}) = \sum_A{}' (1+\mu)^{-|A \setminus X|}$ where the summation Σ' is extended over those $A \in \mathcal{G}$ for which $A \cap Y = \emptyset$. The parameter $\mu > 0$ will be fixed later.

Given $z \in V(\mathcal{G})$, let $\varphi(X, Y, \mathcal{G}, z) = \sum_A{}'' (1+\mu)^{-|A \setminus X|}$ where the summation Σ'' is taken over those $A \in \mathcal{G}$ for which $z \in A$ and $A \cap Y = \emptyset$.

We start with two observations:

$$\varphi(X, Y\cup\{y_1\}, \mathcal{G}, y_2) \leqq \varphi(X, Y, \mathcal{G}, y_2), \tag{2}$$

$$\varphi(X\cup\{x_1\}, Y, \mathcal{G}, x_2) \leqq (1+\mu)\varphi(X, Y, \mathcal{G}, x_2). \tag{3}$$

Now consider a play according to the rules. Let $x_i^{(1)}, \ldots, x_i^{(p)}$ and $y_i^{(1)}, \ldots, y_i^{(q)}$ denote the vertices chosen by the first and the second player at the i-th move, respectively. Let

$$X_i = \{x_1^{(1)}, \ldots, x_1^{(p)}, \ldots, x_i^{(1)}, \ldots, x_i^{(p)}\},$$

$$Y_i = \{y_1^{(1)}, \ldots, y_1^{(q)}, \ldots, y_i^{(1)}, \ldots, y_i^{(q)}\}.$$

Moreover, let

$$X_{i,j} = X_i\cup\{x_{i+1}^{(1)}, \ldots, x_{i+1}^{(j)}\}, \quad Y_{i,k} = Y_i\cup\{y_{i+1}^{(1)}, \ldots, y_{i+1}^{(k)}\}.$$

Now we define the hypergraph $\mathcal{H}_i$ as follows: Throw away the edges of $\mathcal{H}$ blocked by some element of Y_{i-1} and from the remaining edges subtract X_i, i.e.

$$\mathcal{H}_i = \{A\setminus X_i\colon A\in\mathcal{H} \quad\text{and}\quad A\cap Y_{i-1} = \emptyset\}.$$

Let $\psi(\mathcal{H}_i) = \sum_{B\in\mathcal{H}_i} (1+\mu)^{-|B|}$, that is $\psi(\mathcal{H}_i) = \varphi(X_i, Y_{i-1}, \mathcal{H})$.

The first player wins if and only if some of the $\mathcal{H}_i$'s contains the empty set. Since the cardinality of the empty set is zero, in this case $\psi(\mathcal{H}_i)\geqq(1+\mu)^0=1$. It follows that if $\psi(\mathcal{H}_i)<1$ for every $i\geqq 1$ then the second player wins.

Here is the second player's winning strategy: at his i-th move he computes the values $\varphi(X_i, Y_{i-1,k-1}, G, Y)$ for each vertex $y\in V(\mathcal{H})\setminus(X_i\cup Y_{i-1,k-1})$ and then picks $y_i^{(k)}$ for which the maximum is attained, $1\leqq k\leqq q$.

Let $\mu=(1+q)^{1/p}-1$. We claim that making this choice,

$$\psi(\mathcal{H}_{i+1}) \leqq \psi(\mathcal{H}_i) \tag{4}$$

ındependently of the first player's move. It we prove (4), we are ready. Indeed, by the hypothesis of the theorem

$$\psi(\mathcal{H}_1) = \sum_{A\in\mathcal{H}} (1+\mu)^{-|A\setminus X_1|} \leqq \sum_{A\in\mathcal{H}} (1+\mu)^{-|A|+p} = \sum_{A\in\mathcal{H}} (1+q)^{1-(|A|/p)} < 1$$

so $\psi(\mathcal{H}_i)\leqq\psi(\mathcal{H}_1)<1$ for all $i\geqq 1$.

In order to show (4) observe that on his i-th move the second player subtracts

$$\sum_{k=1}^{q} \varphi(X_i, Y_{i-1,k-1}, \mathcal{H}, y_i^{(k)})$$

from $\psi(\mathcal{H}_i)$. After the $(i+1)$-st move of the first player

$$\psi(\mathcal{H}_{i+1}) = \psi(\mathcal{H}_i) - \sum_{k=1}^{q} \varphi(X_i, Y_{i-1,k-1}, \mathcal{H}, y_i^{(k)}) + \mu \sum_{j=1}^{p} \varphi(X_{i,j-1}, Y_i, \mathcal{H}, x_{i+1}^{(j)}). \tag{5}$$

By (2), we have

$$(6) \qquad \varphi(X_i, Y_{i-1,k}, \mathscr{H}, y_i^{(k+1)}) \leqq \varphi(X_i, Y_{i-1,k-1}, \mathscr{H}, y_i^{(k+1)}), \quad 1 \leqq k \leqq q-1.$$

On the other hand, using the maximum property of $y_i^{(k)}$

$$(7) \qquad \varphi(X_i, Y_{i-1,k-1}, \mathscr{H}, y_i^{(k+1)}) \leqq \varphi(X_i, Y_{i-1,k-1}, \mathscr{H}, y_i^{(k)}), \quad 1 \leqq k \leqq q-1.$$

Combining (6) and (7) we obtain

$$(8) \qquad \varphi(X_i, Y_{i-1,k}, \mathscr{H}, y_i^{(k+1)}) \leqq \varphi(X_i, Y_{i-1,k-1}, \mathscr{H}, y_i^{(k)}), \quad 1 \leqq k \leqq q-1.$$

We get similarly

$$(9) \qquad \varphi(X_i, Y_i, \mathscr{H}, x_{i+1}^{(j+1)}) \leqq \varphi(X_i, Y_{i-1,q-1}, \mathscr{H}, y_i^{(q)}),$$

for each $0 \leqq j \leqq p-1$, and by (3)

$$(10) \qquad \varphi(X_{i,j}, Y_i, \mathscr{H}, z) \leqq (1+\mu)\varphi(X_{i,j-1}, Y_i, \mathscr{H}, z), \quad 1 \leqq j \leqq p.$$

By repeated application of (8) we obtain

$$(11) \qquad \varphi(X_i, Y_{i-1,q-1}, \mathscr{H}, y_i^{(q)}) \leqq \varphi(X_i, Y_{i-1,j-1}, \mathscr{H}, y_i^{(j)}), \quad 1 \leqq j \leqq q.$$

By (10) and (9)

$$(12) \qquad \varphi(X_{i,j}, Y_i, \mathscr{H}, x_{i+1}^{(j+1)}) \leqq (1+\mu)^j \varphi(X_i, Y_i, \mathscr{H}, x_{i+1}^{(j+1)}) \leqq$$
$$\leqq (1+\mu)^j \varphi(X_i, Y_{i-1,q-1}, \mathscr{H}, y_i^{(q)})$$

for each $0 \leqq j \leqq p-1$.

Returning now to (5), by (11) and (12) we conclude

$$\psi(\mathscr{H}_{i+1}) \leqq \psi(\mathscr{H}_i) - \left\{ q - \sum_{j=0}^{p-1} \mu(1+\mu)^j \right\} \varphi(X_i, Y_{i-1,q-1}, \mathscr{H}, y_i^{(q)}) = \psi(\mathscr{H}_i),$$

since $q = \sum_{j=0}^{p-1} \mu(1+\mu)^j$ where $\mu = (1+q)^{1/p} - 1$. This completes the proof of (4), and thereby the proof of Theorem 1.

3. Proof of Theorem 2

Let $\varphi, x_i^{(j)}, y_i^{(k)}, X_i, Y_i$ be as in the proof of Theorem 1, and let $X_0 = Y_0 = \varnothing$. Here is the strategy for the first player: at his $(i+1)$-st move $(i \geqq 0)$ he computes the values $\varphi(X_i, Y_i, \mathscr{H}, x)$ for each vertex $x \in V(\mathscr{H}) \setminus (X_i \cup Y_i)$ and for some $\mu > 0$ determined later, and then picks $x_{i+1}^{(1)}, \ldots, x_{i+1}^{(p)}$ which are of the largest value.

Now let $\psi_i = (X_i, Y_i, \mathscr{H})$. We want to give a lower bound for the difference $\psi_{i+1} - \psi_i$, which is equal to

$$(13) \qquad \sum_{A: A \cap Y_{i+1} = \varnothing} \{(1+\mu)^{-|A \setminus X_{i+1}|} - (1+\mu)^{-|A \setminus X_i|}\} - \sum_A{}^* (1+\mu)^{-|A \setminus X_i|}$$

where the summation Σ^* is taken over those $A\in\mathscr{H}$ for which $A\cap Y_{i+1}\neq\emptyset$ but $A\cap Y_i=\emptyset$. The first sum in (13) is

$$F=\sum_{j=1}^{p}\{(1+\mu)^j-1\}\sum_A{}^{**}(1+\mu)^{-|A\setminus X_i|}$$

where the summation Σ^{**} is taken over those $A\in\mathscr{H}$ for which $A\cap Y_{i+1}=\emptyset$ and $|A\cap(X_{i+1}\setminus X_i)|=j$. Now using the fact that $(1+\mu)^j\geqq 1+\mu j$, we get

$$F\geqq\sum_{j=1}^{p}\mu j\sum_A{}^{**}(1+\mu)^{-|A\setminus X_i|}=\mu\sum_{j=1}^{p}(X_i, Y_{i+1}, \mathscr{H}, x_{i+1}^{(j)}). \tag{14}$$

Let

$$\mathscr{H}_i^j=\{A\in\mathscr{H}: x_{i+1}^{(j)}\in A,\ A\cap Y_i=\emptyset,\quad\text{but}\quad A\cap(Y_{i+1}\setminus Y_i)\neq\emptyset\}.$$

Obviously, $|\mathscr{H}_i^j|\leqq q\,d_2(\mathscr{H})$, and

$$\varphi(X_i, Y_{i+1}, \mathscr{H}, x_{i+1}^{(j)})=\varphi(X_i, Y_i, \mathscr{H}, x_{i+1}^{(j)})-\sum_{A\in\mathscr{H}_i^j}(1+\mu)^{-|A\setminus X_i|}\geqq$$

$$\geqq\varphi(X_i, Y_i, \mathscr{H}, x_{i+1}^{(j)})-q\,d_2(\mathscr{H})(1+\mu)^{-2}.$$

Therefore, by (14)

$$F\geqq\mu\sum_{j-1}^{p}\varphi(X_i, Y_i, \mathscr{H}, x_{i+1}^{(j)})-pq\,d_2(\mathscr{H})\mu(1+\mu)^{-2}. \tag{15}$$

On the other hand, the second sum in (13) does not exceed $\sum_{k=1}^{q}\varphi(X_i, Y_i, \mathscr{H}, y_{i+1}^{(k)})$, thus, by (15)

$$\psi_{i+1}-\psi_i\geqq\mu\sum_{j=1}^{p}\varphi(X_i, Y_i, \mathscr{H}, x_{i+1}^{(j)})-\sum_{k=0}^{q}\varphi(X_i, Y_i, \mathscr{H}, y_{i+1}^{(k)})-pq\,d_2(\mathscr{H})\mu(1+\mu)^{-2}.$$

By the choice of $x_{i+1}^{(j)}$ we have $\varphi(X_i, Y_i, \mathscr{H}, x_{i+1}^{(j)})\geqq\varphi(X_i, Y_i, \mathscr{H}, y_{i+1}^{(k)})$ for each $1\leqq k\leqq q$. It follows that

$$\frac{q}{p}\sum_{k=1}^{p}\varphi(X_i, Y_i, \mathscr{H}, x_{i+1}^{(j)})\geqq\sum_{k=1}^{q}\varphi(X_i, Y_i, \mathscr{H}, y_{i+1}^{(k)}),$$

so choosing $\mu=q/p$ we obtain

$$\psi_{i+1}-\psi_i\geqq-\left(\frac{pq}{p+q}\right)^2 d_2(\mathscr{H}).$$

By repeated application of this lower bound we get the desired lower estimate for ψ_i:

$$\psi_i\geqq\psi_0-i\left(\frac{pq}{p+q}\right)^2 d_2(\mathscr{H}).$$

Since $\psi_0 = \sum_{A\in\mathscr{H}} (1+\mu)^{-|A|}$, the hypothesis of the theorem implies $\psi_i > 0$ for each $i \leqq v(\mathscr{H})/(p+q)$, that is, the first player cannot lose the play until this move. But there is no more move, so we are done.

4. Proof of Theorem 3

Let K_n denote a complete graph of n vertices. Let $\mathscr{H}_n$ denote the set of all complete $t\times(n-t)$, $1\leqq t\leqq n-1$ bipartite subgraphs of K_n.

A graph $G\subseteq K_n$ contains a spanning tree of K_n if and only if it is not representable as $G_1\cup G_2$, where G_1 and G_2 are vertex-disjoint subgraphs of K_n. From this follows that Maker has a winning strategy for the game $T(n, b)$ if and only if the second player has a winning strategy for the $(b, 1, \mathscr{H}_n)$-game.

We claim

$$\sum_{A\in\mathscr{H}_n} 2^{-|A|/b} = \sum_{t=1}^{n-1}\binom{n}{t} 2^{-t(n-t)/b} < 1/2 \tag{16}$$

with $b = \lfloor(\log 2-\varepsilon)n/\log n\rfloor$ ($\lfloor x\rfloor$ denotes the greatest integer not exceeding x) and $n > n_0(\varepsilon)$.

Using the well-known fact $\binom{n}{t} \leqq \left(\frac{en}{t}\right)^t$ we have

$$\sum_{t=1}^{n}\binom{n}{t} 2^{-t(n-t)/b} = 2\sum_{t=1}^{n/2}\binom{n}{t} 2^{-t(n-t)/b} \leqq 2\sum_{t=1}^{n/2}\left\{\frac{en}{t} 2^{-(n-t)/b}\right\}^t. \tag{17}$$

At first, if $1\leqq t\leqq n^{1/2}$ and $n > n_2(\varepsilon)$ then

$$\frac{n-t}{b} \geqq (n-n^{1/2})/\lfloor(\log 2-\varepsilon)n(\log)^{-1}\rfloor \geqq (1+\varepsilon)\log n/\log 2,$$

and

$$\frac{en}{t} 2^{-(n-t)/b} \leqq en2^{-(1+\varepsilon)\log n/\log 2} = en^{-\varepsilon} \leqq 1/9$$

for $n > n_4(\varepsilon)$. Therefore, if $n > \max\{n_2(\varepsilon), n_3(\varepsilon)\}$ then

$$\sum_{1\leqq t\leqq n^{1/2}}\left\{\frac{en}{t} 2^{-(n-t)/b}\right\}^t < \sum_{t=1}^{\infty} 9^{-t} = 1/8. \tag{18}$$

Secondly, if $n^{1/2}\leqq t\leqq n/2$ then

$$(n-t)/b \geqq n/2b \geqq (1+\varepsilon)\log n/(2\log 2),$$

and

$$\frac{en}{t} 2^{-(n-t)/b} \leqq \frac{en}{n^{1/2}} 2^{-(1+\varepsilon)\log n/2\log 2} = en^{-\varepsilon/2} \leqq 1/9$$

for $n > n_4(\varepsilon)$. Hence

$$\sum_{n^{1/2} \leq t \leq n/2} \left\{ \frac{en}{t} 2^{-(n-t)/b} \right\}^t < \sum_{t=1}^{\infty} 9^{-t} = 1/8. \tag{19}$$

By (17), (18) and (19) we obtain (16).

Applying Theorem 1 we get that the second player can force a win in the $(b, 1, \mathcal{H}_n)$-game, and Theorem 3 follows.

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CHARACTERIZING AND CONSTRUCTING SPECIAL RADICALS

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1. Introduction and preliminaries

The theory of special radicals was developed in the fundamental papers [3] and [4] of ANDRUNAKIEVIČ. Though major contributions to the theory have subsequently been made by ANDRUNAKIEVIČ and RJABUHIN [5], HEYMAN and ROOS [10], JAEGERMANN and SANDS [11], LEAVITT and WATTERS [12], NICHOLSON and WATTERS [13] and RJABUHIN [16], no internal characterization of special radicals has up till now been given. For the semisimple classes of special radicals, on the other hand, intrinsic characterization can be found in a recent paper [17] of RJABUHIN and WIEGANDT.

The main purpose of the present paper is to characterize special radical classes by closure properties and to investigate the smallest special radical class containing a given class. In Section 2 we give characterizations of supernilpotent, special and special dual radicals, and also of their semisimple classes. Section 3 is devoted to the "lower special radical" construction, in the obvious sense. It is shown in this section, as an example, that the lower radical defined by a variety containing all zerorings is special. The results of this section lead us to introduce the concept of a *saturated class*, which is discussed in Section 4. This notion is utilized in the final two sections of the paper in which we obtain new descriptions of the standard examples of special radicals.

Throughout the paper, all rings under consideration are associative. We note, however, that the results in Section 2 are valid for alternative rings. All classes of rings considered are assumed to be closed under isomorphisms and to contain all one-element rings. The minor ambiguities occasionally resulting from the latter convention should not cause any confusion. Classes will always be denoted by bold face capitals; $\mathbf{Z}$ will denote the class of zerorings, $\mathbf{O}$ the class of one-element rings, $\mathbf{P}$ the class of prime rings, $\mathbf{H}$ the class of semiprime rings, $\mathbf{Q}$ the class of subdirectly irreducible prime rings, $\mathbf{I}$ the class of simple rings with identity, $\mathbf{B}$ the Baer lower radical class. Ideals will be indicated by the symbol $\lhd$. $H(A)$ will denote the heart of a subdirectly irreducible ring A. If a ring A has a homomorphic image B in a class $\mathbf{X}$, B will be called an $\mathbf{X}$-*factor* of A. $\mathbf{X}$-*ideals* are defined analogously. A class $\mathbf{X}$ is *hereditary* if it closed under ideals and *regular* if it has the weaker property that every non-zero ideal of a ring in $\mathbf{X}$ has a non-zero $\mathbf{X}$-factor. Finally, a class is *essentially closed* if $B\in\mathbf{X}$ implies $A\in\mathbf{X}$ whenever B is an essential ideal of A. The *essential closure* of a class is the smallest essentially closed class containing it.

We shall make use of the following propositions. The first is an easy consequence of Andrunakievič's Lemma.

PROPOSITION 1. *If* $C\lhd B\lhd A$ *and* B/C *is semiprime, then* $C\lhd A$.

PROPOSITION 2. (cf. [2], Proposition 1). *If* $B \lhd A$, *then either* B *is an essential ideal of* A *or* $(B+C)/C$ *is an essential ideal of* A/C, *where* C *is an ideal of* A *such that* $B \cap C = 0$ *and* C *is maximal with respect to this property.*

Radical and semisimple classes are meant in the sense of Kuroš and Amitsur. For details see [6], [8], [19], [20]. For a class $\mathbf{M}$ of rings, we define $\mathcal{U}\mathbf{M} = \{A|A$ has no non-zero homomorphic images in $\mathbf{M}\}$ and $\mathcal{S}\mathbf{M} = \{A|A$ has no non-zero ideals in $\mathbf{M}\}$. If $\mathbf{M}$ is regular, then $\mathcal{U}\mathbf{M}$ is the upper radical class defined by $\mathbf{M}$. If $\mathbf{R}$ is a radical class, the $\mathbf{R}$-radical of a ring A will be denoted by $\mathbf{R}(A)$.

PROPOSITION 3 (cf. [7]). *A radical class* $\mathbf{R}$ *is hereditary if and only if the corresponding semisimple class* $\mathcal{S}\mathbf{R}$ *is essentially closed.*

A hereditary radical class $\mathbf{R}$ is *supernilpotent if* $\mathbf{Z} \subseteq \mathbf{R}$. In view of [10] a *weakly special class* can be defined as an essentially closed hereditary class of semiprime rings. The upper radical of a weakly special class is supernilpotent, and the semisimple class of a supernilpotent radical is weakly special (cf. [16]). We say that a radical $\mathbf{R}$ has the *intersection property relative to the class* $\mathbf{M}$ if

$$\mathbf{R}(A) = \cap \{I \lhd A | A/I \in \mathbf{M}\}$$

for every ring A. $\mathbf{R}$ is then, of course, the upper radical defined by $\mathbf{M}$. A weakly special class $\mathbf{W}$ of prime rings is called a *special class,* and its upper radical $\mathcal{U}\mathbf{W}$ is called a *special radical.* Every special radical $\mathbf{R}$ is supernilpotent and has the intersection property relative to $P \cap \mathcal{S}\mathbf{R}$ (see [3]).

2. Radical and semisimple classes

In what follows, $\mathbf{C}$ will always denote a *weakly special class,* i.e. an *essentially closed, hereditary class of semiprime rings.* Examples of such classes include $\mathbf{H}$, $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{I}$ and the semisimple class $\mathcal{S}\mathbf{R}$ of any supernilpotent radical $\mathbf{R}$.

THEOREM 1. *A class* $\mathbf{R}$ *is a supernilpotent radical class with the intersection property relative to* $\mathbf{C} \cap \mathcal{S}\mathbf{R}$ *if and only if* $\mathbf{R}$ *is homomorphically closed and hereditary and satisfies*

(R) *If every non-zero* $\mathbf{C}$*-factor of a ring* A *has a non-zero* $\mathbf{R}$*-ideal, then* $A \in \mathbf{R}$. *Furthermore* $\mathbf{R}$ *is then the upper radical class defined by* $\mathbf{C} \cap \mathcal{S}\mathbf{R}$.

PROOF. Let $\mathbf{R}$ be a supernilpotent radical class with the intersection property relative to $C \cap \mathcal{S}\mathbf{R}$. Then $\mathbf{M} = \mathbf{C} \cap \mathcal{S}\mathbf{R}$ is a weakly special class in view of Proposition 3. Let A be a ring of which every non-zero $\mathbf{C}$-factor has a non-zero $\mathbf{R}$-ideal. By the intersection property, $A/\mathbf{R}(A)$ is a subdirect product of $\mathbf{M}$-rings C_λ, $\lambda \in \Lambda$. Since each C_λ is a $\mathbf{C}$-factor of A without $\mathbf{R}$-ideals, we have $C_\lambda = 0$ for each λ, whence $A = \mathbf{R}(A) \in \mathbf{R}$. Thus condition (R) is satisfied. As well, $\mathbf{R}$ is obviously homomorphically closed and hereditary.

Conversely, if $\mathbf{R}$ is homomorphically closed and hereditary and satisfies (R), then if every non-zero homomorphic image of a ring A has a non-zero $\mathbf{R}$-ideal, condition (R) implies that $A \in \mathbf{R}$. This (since $\mathbf{R}$ is homomorphically closed) means that $\mathbf{R}$ is a radical class. Since zerorings have no $\mathbf{C}$-factors, $\mathbf{R}$ is supernilpotent. By Proposition 3, $\mathbf{M}=\mathbf{C}\cap\mathscr{S}\mathbf{R}$ is weakly special; moreover, $\mathbf{R}\subseteq\mathscr{U}\mathbf{M}$. If K is a ring in $\mathscr{U}\mathbf{M}$, then $K/\mathbf{R}(K)\in\mathscr{U}\mathbf{M}\cap\mathscr{S}\mathbf{R}$. Hence every non-zero $\mathbf{C}$-factor of $K/\mathbf{R}(K)$ is not in $\mathbf{M}$ and consequently not in $\mathscr{S}\mathbf{R}$. Thus every non-zero $\mathbf{C}$-factor of $K/\mathbf{R}(K)$ has a non-zero $\mathbf{R}$-ideal, whence by (R) it follows that $K/\mathbf{R}(K)\in\mathbf{R}$, i.e. $K=\mathbf{R}(K)\in\mathbf{R}$. This proves that $\mathbf{R}=\mathscr{U}\mathbf{M}$.

The last assertion is straightforward.

COROLLARY 1. *Let* $\mathbf{C}=\mathbf{H}, \mathbf{P}$. *Then Theorem 1 gives characterizations of supernilpotent and special radicals, respectively.*

COROLLARY 2. $\mathbf{R}$ *is the upper radical class defined by a class of simple rings with identity if and only if* $\mathbf{R}$ *is homomorphically closed and hereditary and satisfies*

$(\mathrm{R_I})$ *If every* $\mathbf{I}$*-factor of* A *is in* $\mathbf{R}$, *then* $A\in\mathbf{R}$.

PROOF. Put $\mathbf{C}=\mathbf{I}$ in Theorem 1.

Before applying Theorem 1 to the class $\mathbf{Q}$, we reformulate Theorem 7 of ANDRUNAKIEVIČ [3].

Let $\mathbf{E}$ denote the class of simple prime rings.

PROPOSITION 4. *Let* $\mathbf{F}$ *be a subclass of* $\mathbf{E}$. *Then*

$$\mathscr{Q}\mathbf{F}=\{A\in\mathbf{Q}\mid H(A)\in\mathbf{F}\}$$

is the essential closure of $\mathbf{F}$, *and* $\mathscr{Q}\mathbf{F}$ *is a special class. Conversely, every special class* $\mathbf{M}$ *of subdirectly irreducible rings has the form* $\mathscr{Q}\mathbf{F}$ *where* $\mathbf{F}=\mathbf{E}\cap\mathbf{M}$.

A *special dual radical* is the upper radical of a special class of subdirectly irreducible (prime) rings (for the terminology we refer to [3]).

COROLLARY 3. $\mathbf{D}$ *is a special dual radical if and only if* $\mathbf{D}$ *is homomorphically closed and hereditary and satisfies*

$(\mathrm{R_Q})$ *If* $H(K)\in\mathbf{D}$ *for every subdirectly irreducible prime factor* K *of* A, *then* $A\in\mathbf{D}$.

PROOF. Put $\mathbf{C}=\mathbf{Q}$ in Theorem 1 and observe that a subdirectly irreducible ring has a non-zero $\mathbf{D}$-ideal if and only if its heart is in $\mathbf{D}$ (since $\mathbf{D}$ is hereditary).

The next theorem characterizes the special dual radicals and generalizes SULIŃSKI's theorem [18] asserting that the upper radical of a class $\mathbf{F}$ of simple rings is hereditary and has the intersection property relative to $\mathbf{F}$ if and only if each ring in $\mathbf{F}$ has an identity.

THEOREM 2. *Let* $\mathbf{M}$ *be a regular class of subdirectly irreducible prime rings. Then* $\mathscr{U}\mathbf{M}$ *is hereditary and has the intersection property relative to* $\mathbf{M}$ *if and only if* $\mathbf{M}$ *is a special class. If* $\mathbf{D}$ *is any special dual radical, then* $\mathbf{M}=\mathbf{Q}\cap\mathscr{S}\mathbf{D}$ *is the only special class of subdirectly irreducible prime rings such that* $\mathbf{D}=\mathscr{U}\mathbf{M}$ *and* $\mathbf{D}$ *has the intersection property relative to* $\mathbf{M}$.

PROOF. Assume that $\mathscr{U}\mathbf{M}$ is hereditary and has the intersection propety relative to $\mathbf{M}$. Then since $\mathscr{U}\mathbf{M}$ is hereditary, $\mathbf{Q}\cap\mathscr{S}\mathscr{U}\mathbf{M}$ is essentially closed and hence a special class. If $A\in\mathbf{Q}\cap\mathscr{S}\mathscr{U}\mathbf{M}$, but $A\notin\mathbf{M}$, then $\bigcap\{I\lhd A\mid A/I\in\mathbf{M}\}=0$ while $\bigcap\{I\lhd A\mid A/I\in\mathbf{M}\}\supseteq H(A)\neq 0$. We conclude that $\mathbf{Q}\cap\mathscr{S}\mathscr{U}\mathbf{M}\subseteq\mathbf{M}\subseteq\mathbf{Q}\cap\mathscr{S}\mathscr{U}\mathbf{M}$, whence $\mathbf{M}$ is special.

The converse is straightforward.

If $\mathbf{D}$ is a special dual radical, then $\mathbf{D}=\mathscr{U}(\mathbf{Q}\cap\mathscr{S}\mathbf{D})$ and the reasoning used above shows that $\mathbf{M}=\mathbf{Q}\cap\mathscr{S}\mathbf{D}$ is the only special class of subdirectly irreducible rings such that $\mathbf{D}=\mathscr{U}\mathbf{M}$ and $\mathbf{D}$ has the intersection property relative to $\mathbf{M}$.

The next theorem is a companion piece to Theorem 1 inasmuch as it characterizes the semisimple classes of supernilpotent radicals with a specified intersection property.

THEOREM 3. *The following conditions are equivalent for a class* $\mathbf{S}$.

(i) $\mathbf{S}$ *is the semisimple class of a supernilpotent radical which has the intersection property relative to* $\mathbf{C}\cap\mathbf{S}$.

(ii) $\mathbf{S}$ *is an essentially closed and subdirectly closed class of rings satisfying*

(S) *If* $A\in\mathbf{S}$, *then every non-zero ideal of A has a non-zero* $\mathbf{C}$*-factor in* $\mathbf{S}$ *and the upper radical of* $\mathbf{S}$ *has the intersection property relative to* $\mathbf{C}\cap\mathbf{S}$.

(iii) $\mathbf{S}$ *is an essentially closed, subdirectly closed, regular class of rings satisfying*

(T) *Every* $\mathbf{S}$*-ring is a subdirect product of* $\mathbf{C}\cap\mathbf{S}$*-rings.*

PROOF. (i)⇒(ii): Since $\mathbf{S}$ is the semisimple class of a hereditary radical, $\mathbf{S}$ is essentially closed as well as subdirectly closed. If $0\neq I\lhd A\in\mathbf{S}$, then $I\in\mathbf{S}$, so by the intersection property, I is a subdirect product of $\mathbf{C}\cap\mathbf{S}$-rings; in particular, I has a non-zero $\mathbf{C}\cap\mathbf{S}$-factor. Thus (S) is satisfied.

(ii)⇒(iii): It follows readily from (S) that $\mathbf{S}$ is regular. Let A be an $\mathbf{S}$-ring. Let $\mathbf{T}=\mathbf{C}\cap\mathbf{S}$. Then $\mathscr{U}\mathbf{T}(A)$ has no non-zero $\mathbf{T}$-factors, so (S) implies that $\mathscr{U}\mathbf{T}(A)=0$. Thus A is a subdirect product of $\mathbf{T}$-rings and (T) has been established.

(iii)⇒(i): By Corollary 2 of [1], an essentially closed, subdirectly closed regular class $\mathbf{S}$ of rings is the semisimple class of the hereditary radical class $\mathscr{U}\mathbf{S}$. Since $\mathbf{C}$ consists of semiprime rings, by condition (T), $\mathscr{U}\mathbf{S}$ is supernilpotent. Moreover, (T) also says that $\mathscr{U}\mathbf{S}$ has the intersection property relative to $\mathbf{C}\cap\mathbf{S}$.

COROLLARY 4. *Let* $\mathbf{C}=\mathbf{H},\mathbf{P},\mathbf{Q}$ *or* $\mathbf{I}$. *Then Theorem 3 gives, respectively, characterizations of the semisimple classes of supernilpontent, special and special dual radicals and the upper radicals defined by classes of simple rings with identity.*

Let us note that for $\mathbf{C}=\mathbf{P}$, the equivalence of (i) and (iii) in Theorem 3 is Theorem 4 of [17].

REMARK. All the results of this section are valid also for alternative rings.

3. Radical constructions

Again, let $\mathbf{C}$ be a weakly special class. Further, let $\mathbf{K}$ be a class of rings such that

(i) *if* $I\lhd A\in\mathbf{K}$ *and* $I\in\mathbf{C}$, *then* $I\in\mathbf{K}$ and

(ii) *if* $I\lhd A\in\mathbf{K}$ *and* $A/I\in\mathbf{C}$, *then* $A/I\in\mathbf{K}$.

We define the class $\mathbf{L}=\mathscr{L}(\mathbf{C},\mathbf{K})$ as follows:

$\mathbf{L}=\{A\mid$every non-zero $\mathbf{C}$-factor of A has a non-zero $\mathbf{K}$-ideal$\}$.

THEOREM 4. $\mathbf{L}=\mathscr{L}(\mathbf{C},\mathbf{K})$ *is a supernilpotent radical class which contains* $\mathbf{K}$ *and has the intersection property relative to* $\mathbf{C}\cap\mathscr{S}\mathbf{L}$ *and is the smallest such radical class.*

PROOF. Clearly $\mathbf{L}$ is homomorphically closed. It follows readily from (ii) that $\mathbf{K}\subseteq\mathbf{L}$.

We next show that $\mathbf{L}$ is hereditary. Let $B\lhd A\in\mathbf{L}$ and let B/C be a non-zero $\mathbf{C}$-factor of B. Since $\mathbf{C}$ consists of semiprime rings, by Proposition 1 we have $C\lhd A$ and thus $B/C\lhd A/C$. If B/C is an essential ideal of A/C, then $A/C\in\mathbf{C}$, as $\mathbf{C}$ is essentially closed. Since A/C is in $\mathbf{L}$, it has a non-zero $\mathbf{K}$-ideal I/C. Since B/C is an essential ideal of A/C, we then have $(B/C)\cap(I/C)=D/C\neq 0$. Moreover, $D/C\lhd A/C\in\mathbf{C}$, so $D/C\in\mathbf{C}$. But $D/C\lhd I/C\in\mathbf{K}$, so by (i), $D/C\in\mathbf{K}$. If B/C is not an essential ideal of A/C, then by Proposition 2 there is an ideal K of A such that $B\cap K=C$ and $(B+K)/K\cong B/C$ is an essential ideal of A/K. Arguing as above, we see that $(B+K)/K$ has a non-zero $\mathbf{K}$-ideal. Hence B/C does. Thus in either case, B/C has a non-zero $\mathbf{K}$-ideal. It follows that B is in $\mathbf{L}$ so the latter is hereditary.

We now establish condition (R) of Theorem 1 for $\mathbf{L}$. Let A be a ring such that every non-zero $\mathbf{C}$-factor of A has a non-zero $\mathbf{L}$-ideal. Let B be such a factor, C a non-zero $\mathbf{L}$-ideal. By definition of $\mathbf{L}$ (since $\mathbf{C}$ is hereditary and thus C is in $\mathbf{C}$) C has a non-zero $\mathbf{K}$-ideal D. Let $\bar{D}$ denote the ideal of B generated by D. By Andrunakievič's Lemma, $\bar{D}^3\subseteq D$, and since $\bar{D}\lhd B\in\mathbf{C}$, and $\mathbf{C}$ is a hereditary class of semiprime rings, $\bar{D}^3\neq 0$. Since $\bar{D}^3\lhd B\in\mathbf{C}$ and $\bar{D}^3\lhd D\in\mathbf{K}$, (i) implies that $\bar{D}^3\in K$. Thus B has a non-zero $\mathbf{K}$-ideal (viz. $\bar{D}^3$). It follows that A is in $\mathbf{L}$, i.e. (R) is satisfied.

Applying Theorem 1, we see that $\mathbf{L}$ is a supernilpotent radical with the intersection property relative to $\mathbf{C}\cap\mathscr{S}\mathbf{L}$.

Finally, let $\mathbf{J}$ be any supernilpotent radical class such that $\mathbf{K}\subseteq\mathbf{J}$ and $\mathbf{J}$ has the intersection property relative to $\mathbf{C}\cap\mathscr{S}\mathbf{J}$. Let A be a ring in $\mathbf{L}$. Then every non-zero $\mathbf{C}$-factor of A has a non-zero $\mathbf{K}$-ideal, and so a non-zero $\mathbf{J}$-ideal. Hence by Theorem 1, A is in $\mathbf{J}$. Thus $\mathbf{L}\subseteq\mathbf{J}$.

REMARK. In the case of alternative rings, the smallest supernilpotent radical $\mathbf{L}$ containing $\mathbf{K}$ and with the intersection property relative to $\mathbf{C}\cap\mathscr{S}\mathbf{L}$ can be obtained by a transfinite iteration of the construction defined above. It is unclear how many steps are necessary.

In the following proposition we note some examples of the situation just described.

PROPOSITION 5.

(a) *If* $\mathbf{C}\subseteq\mathbf{K}$, *then* $\mathscr{L}(\mathbf{C},\mathbf{K})$ *is the class of all rings.*

(b) *If* $\mathbf{C}\cap\mathbf{K}=\mathbf{O}$, *then* $\mathscr{L}(\mathbf{C},\mathbf{K})=\mathscr{U}\mathbf{C}$.

(c) *If* $\mathbf{Z}\subseteq\mathbf{K}$, *then* $\mathscr{L}(\mathbf{H},\mathbf{K})$ *is the lower radical class* $\mathscr{L}\mathbf{K}$ *defined by* $\mathbf{K}$.

(d) $\mathscr{L}(\mathbf{P},\mathbf{K})$ *is the smallest special radical class containing* $\mathbf{K}$.

PROOF. (a), (b) and (d) are straightforward. To prove (c), we note that $\mathscr{L}\mathbf{K}=\subseteq\mathscr{L}(\mathbf{H},\mathbf{K})$, while if $A\in\mathscr{L}(\mathbf{H},\mathbf{K})$ and B is a non-zero homomorphic image of A, then either B has a non-zero ideal in $\mathbf{Z}\subseteq\mathbf{K}$, or B is in $\mathbf{H}$, in which case again B has a non-zero $\mathbf{K}$-ideal, whence it follows that A is in $\mathscr{L}\mathbf{K}$.

The following result is stated by OSBORN ([14], p. 309).

PROPOSITION 6. *If a non-zero ideal B of a prime ring A satisfies a polynomial identity f, then A also satisfies f.*

Theorem 4 and Proposition 6 enable us to construct special classes from varieties. Before going into details, we recall some notation. If $\mathbf{X}$ and $\mathbf{Y}$ are classes of rings, then

$$\mathbf{X}\circ\mathbf{Y} = \{A \mid \text{there exists an } I \lhd A \text{ with } I\in\mathbf{X}, A/I\in\mathbf{Y}\}.$$

PROPOSITION 7. *Let* $\mathbf{V}$ *be a variety of rings. Then* $\mathbf{Z}\circ\mathbf{V}$ *is a variety and*

$$\mathbf{B}\circ\mathbf{V} = \{A \mid A/\mathbf{B}(A)\in\mathbf{V}\} = \mathbf{B}\circ(\mathbf{Z}\circ\mathbf{V}).$$

PROOF. The first assertion is well known, and the first equality is clear since $\mathbf{V}$ is homomorphically closed. Clearly also $\mathbf{B}\circ\mathbf{V}\subseteq\mathbf{B}\circ(\mathbf{Z}\circ\mathbf{V})$. If A is in $\mathbf{B}\circ(\mathbf{Z}\circ\mathbf{V})$, then A has an ideal B in $\mathbf{B}$ and an ideal C such that $B\subseteq C$, $C/B\in\mathbf{Z}$ and $A/C\cong(A/B)/(C/B)\in\mathbf{V}$. But then C is in $\mathbf{B}$, so A is in $\mathbf{B}\circ\mathbf{V}$.

THEOREM 5. *Let* $\mathbf{J}$ *be a special radical,* $\mathbf{V}$ *a variety. Then* $\mathbf{J}\circ\mathbf{V}$ *is a special radical; in fact*

$$\mathcal{L}(\mathbf{J}\cup\mathbf{V}) = \mathcal{L}(\mathbf{P}, (\mathbf{P}\cap\mathbf{J})\cup\mathbf{V}) = \mathbf{J}\circ\mathbf{V}.$$

PROOF. Let A be in $\mathcal{L}(\mathbf{J}\cup\mathbf{V})$. Then since $\mathbf{J}\cup\mathbf{V}$ is hereditary and $\mathbf{Z}\subseteq\mathbf{J}\cup\mathbf{V}$, every non-zero homomorphic image of A has a non-zero ideal in $\mathbf{J}\cup\mathbf{V}$. Let B be a prime homomorphic image of A. Then B has a non-zero ideal C in $\mathbf{J}\cup\mathbf{V}$. If $C\notin\mathbf{V}$, then $C\in\mathbf{J}$ and $C\lhd B\in\mathbf{P}$, $C\in\mathbf{P}\cap\mathbf{J}$. Thus $\mathcal{L}(\mathbf{J}\cup\mathbf{V})\subseteq\mathcal{L}(\mathbf{P}, (\mathbf{P}\cap\mathbf{J})\cup\mathbf{V})$.

If now D is a ring in $\mathcal{L}(\mathbf{P}, (\mathbf{P}\cap\mathbf{J})\cup\mathbf{V})$, then $\bar{D}=D/\mathbf{J}(D)$ is also in $\mathcal{L}(\mathbf{P}, (\mathbf{P}\cap\mathbf{J})\cup\mathbf{V})$. Since $\mathbf{J}(\bar{D})=0$, $\bar{D}$ is a subdirect product of a set $\{D_\lambda \mid \lambda\in\Lambda\}$ of $\mathbf{J}$-semisimple prime rings. But each D_λ is also in $\mathcal{L}(\mathbf{P}, (\mathbf{P}\cap\mathbf{J})\cup\mathbf{V})$ so it has a non-zero ideal in $(\mathbf{P}\cap\mathbf{J})\cup\mathbf{V}$, and hence in $\mathbf{V}$. Then Proposition 6 says that D_λ is in $\mathbf{V}$. Thus $\bar{D}=D/\mathbf{J}(D)$, as a subdirect product of $\mathbf{V}$-rings, is in $\mathbf{V}$, and $\mathcal{L}(\mathbf{P}, (\mathbf{P}\cap\mathbf{J})\cup\cup\mathbf{V})\subseteq\mathbf{J}\circ\mathbf{V}$.

Since $\mathcal{L}(\mathbf{J}\cup\mathbf{V})$ is closed under extensions, we have $\mathbf{J}\circ\mathbf{V}\subseteq\mathcal{L}(\mathbf{J}\cup\mathbf{V})$. This completes the proof.

COROLLARY 5. *For any variety* $\mathbf{V}$, $\mathbf{B}\circ\mathbf{V}$ *is a special radical; in fact*

$$\mathcal{L}(\mathbf{Z}\cup\mathbf{V}) = \mathcal{L}(\mathbf{P}, \mathbf{V}) = \mathbf{B}\circ\mathbf{V}.$$

PROOF. Clearly $\mathcal{L}(\mathbf{Z}\cup\mathbf{V}) = \mathcal{L}(\mathbf{B}\cup\mathbf{V})$.

COROLLARY 6. *Let* $\mathbf{V}$ *be a variety such that* $\mathbf{Z}\subseteq\mathbf{V}$. *Then* $\mathcal{L}\mathbf{V}$ *is special, and* $\mathcal{L}\mathbf{V}=\mathbf{B}\circ\mathbf{V}$.

It has previously been shown by the first author ([9], Theorem 3.6) that $\mathcal{L}\mathbf{G}_n=\mathbf{B}\circ\mathbf{G}_n$ where $\mathbf{G}_n$ is the variety defined by the standard identity of degree n. Also P. N. Stewart has shown that $\mathbf{U}\circ\mathbf{G}_2$ is a special radical where $\mathbf{U}$ is any special radical containing the generalized nil radical and $\mathbf{G}_2$ is the variety of commutative rings (private communication).

4. Saturated subclasses

We say that a subclass $\mathbf{N}$ of a weakly special class $\mathbf{C}$ is a *saturated subclass* of $\mathbf{C}$ if $\mathbf{N}$ satisfies conditions (i) and (ii) of Section 3 and also

(iii) *If every non-zero* $\mathbf{C}$*-factor of a ring* $A\in\mathbf{C}$ *has a non-zero* $\mathbf{N}$*-ideal, then* $A\in\mathbf{N}$.

Condition (iii) is clearly equivalent to $\mathcal{L}(\mathbf{C}, \mathbf{N})\cap\mathbf{C}=\mathbf{N}$.

Since $\mathbf{N} \subseteqq \mathbf{C}$, $\mathbf{N}$ is hereditary by (i). We shall consider the class

$$\mathscr{K}(\mathbf{C}, \mathbf{N}) = \{A \mid \text{every } \mathbf{C}\text{-factor of } A \text{ is in } \mathbf{N}\} = \mathscr{U}(\mathbf{C} \setminus \mathbf{N}).$$

THEOREM 6. $\mathscr{K}(\mathbf{C}, \mathbf{N}) = \mathscr{L}(\mathbf{C}, \mathbf{N})$ *for every saturated subclass* $\mathbf{N}$ *of* $\mathbf{C}$.

PROOF. Obviously $\mathscr{K}(\mathbf{C}, \mathbf{N}) \subseteqq \mathscr{L}(\mathbf{C}, \mathbf{N})$. Let A be in $\mathscr{L}(\mathbf{C}, \mathbf{N})$, B a $\mathbf{C}$-factor of A. Then B is also in $\mathscr{L}(\mathbf{C}, \mathbf{N})$, so every non-zero $\mathbf{C}$-factor of B has a non-zero $\mathbf{N}$-ideal. By (iii), B is in $\mathbf{N}$ and thus A is in $\mathscr{K}(\mathbf{C}, \mathbf{N})$.

A subclass $\mathbf{M}$ of $\mathbf{C}$ will be called a *cosaturated subclass* if $\mathbf{M}$ is weakly special and satisfies the condition

(M) *If* $A \in \mathbf{C}$ *and every non-zero ideal of* A *has a non-zero* $\mathbf{M}$*-factor, then* $A \in \mathbf{M}$.

Though (M) is dual to (iii), the notion of a cosaturated subclass is not categorically dual to that of a saturated subclass. Such classes, however, determine each other uniquely as we shall see in the next Theorem, which justifies the terminology.

THEOREM 7. *Let* $\mathbf{R}$ *be a supernilpotent radical with the intersection property relative to* $\mathbf{C} \cap \mathscr{S}\mathbf{R}$. *Then* $\mathbf{R}$ *contains a unique saturated subclass* $\mathbf{N}$ *of* $\mathbf{C}$ *such that* $\mathbf{R} = \mathscr{K}(\mathbf{C}, \mathbf{N})$ *(namely,* $\mathbf{N} = \mathbf{C} \cap \mathbf{R}$*). Also* $\mathbf{R}$ *determines a unique cosaturated subclass* $\mathbf{M}$ *of* $\mathbf{C}$ *such that* $\mathbf{R} = \mathscr{U}\mathbf{M}$ *(namely,* $\mathbf{M} = \mathbf{C} \cap \mathscr{S}\mathbf{R}$*). There is a bijective correspondence between the saturated subclasses* $\mathbf{N}$ *and the cosaturated subclasses* $\mathbf{M}$ *of* $\mathbf{C}$, *given by*

$$\mathbf{N} \mapsto \mathbf{C} \cap \mathscr{S}\mathscr{K}(\mathbf{C}, \mathbf{N}); \quad \mathbf{M} \mapsto \mathbf{C} \cap \mathscr{U}\mathbf{M}.$$

If $\mathbf{N}$ *and* $\mathbf{M}$ *are corresponding classes, then* $\mathscr{K}(\mathbf{C}, \mathbf{N}) = \mathscr{U}\mathbf{M}$.

PROOF. By Theorem 1 (R) it is clear that $\mathbf{N}$ is a saturated subclass of $\mathbf{C}$. Moreover, $\mathbf{N}$ is the largest saturated subclass of $\mathbf{C}$ contained in $\mathbf{R}$. Let $\mathbf{N}_1$ be another saturated subclass of $\mathbf{C}$ such that $\mathbf{R} = \mathscr{K}(\mathbf{C}, \mathbf{N}_1)$. If A is in $\mathbf{N}$, then, since $\mathbf{N} \subseteqq \mathbf{R} = \mathscr{K}(\mathbf{C}, \mathbf{N}_1)$, every $\mathbf{C}$-factor of A, and hence in particular A itself, is in $\mathbf{N}_1$. Thus $\mathbf{N} \subseteqq \mathbf{N}_1$, so $\mathbf{N} = \mathbf{N}_1$. On the other hand, $\mathbf{R} = \mathscr{K}(\mathbf{C}, \mathbf{N})$, by a standard argument.

Clearly $\mathbf{M} = \mathbf{C} \cap \mathscr{S}\mathbf{R}$ is weakly special. If $A \in \mathbf{C}$ and every non-zero ideal of A has a non-zero $\mathbf{M}$-factor, then A is in the semisimple class $\mathscr{S}\mathbf{R}$ and thus in $\mathbf{M}$. Hence $\mathbf{M}$ is a cosaturated subclass of $\mathbf{C}$, clearly the largest such contained in $\mathscr{S}\mathbf{R}$. Also, $\mathbf{R} = \mathscr{U}\mathbf{M}$. Let $\mathbf{M}_1$ be any cosaturated subclass of $\mathbf{C}$ for which $\mathbf{R} = \mathscr{U}\mathbf{M}_1$. If $A \in \mathbf{M}$ then since $\mathbf{M} \subseteqq \mathscr{S}\mathbf{R} \subseteqq \mathscr{S}\mathscr{U}\mathbf{M}_1$, every non-zero ideal of A has a non-zero $\mathbf{M}_1$-factor, and so by (M), A is in $\mathbf{M}_1$. Thus $\mathbf{M}_1 = \mathbf{M}$.

The other assertions are straightforward.

THEOREM 8. *Let* $\mathbf{N}$ *be a saturated subclass of* $\mathbf{C}$. *Then* $\mathbf{R} = \mathscr{K}(\mathbf{C}, \mathbf{N})$ *if and only if* $\mathbf{R}$ *is homomorphically closed and hereditary and satisfies the following conditions.*

(N1) $\mathbf{N} = \mathbf{C} \cap \mathbf{R}$.

(N2) *If every* $\mathbf{C}$*-factor of a ring* A *is in* $\mathbf{N}$, *then* $A \in \mathbf{R}$.

PROOF. By Theorems 1, 4, 6 and 7, $\mathscr{K}(\mathbf{C}, \mathbf{N})$ satisfies (N1) and (N2).

Conversely, if (N1) and (N2) are satisfied, then by Theorem 1, $\mathbf{R}$ is a supernilpotent radical with the intersection property relative to $\mathbf{M}_1 = \mathbf{C} \cap \mathscr{S}\mathbf{R}$. Hence by Theorem 7, $\mathbf{M}_1$ is the cosaturated subclass of $\mathbf{C}$ such that $\mathbf{R} = \mathscr{U}\mathbf{M}_1$. The corresponding saturated subclass of $\mathbf{C}$ is $\mathbf{N}_1 = \mathbf{C} \cap \mathscr{U}\mathbf{M}_1 = \mathbf{C} \cap \mathbf{R} = \mathbf{N}$ by condition (N1). On the other hand, the cosaturated subclass of $\mathbf{C}$ corresponding to $\mathbf{N}$ is $\mathbf{M} = \mathbf{C} \cap \mathscr{S}\mathscr{K}(\mathbf{C}, \mathbf{N})$

and since $\mathbf{N}_1=\mathbf{N}$, Theorem 7 implies that $\mathbf{M}_1=\mathbf{M}$. Therefore, again by Theorem 7, we have $\mathbf{R}=\mathscr{U}\mathbf{M}_1=\mathscr{U}\mathbf{M}=\mathscr{K}(\mathbf{C}, \mathbf{N})$.

We end this section with the consideration of saturated and cosaturated classes associated with varieties and simple rings with identity.

PROPOSITION 8. *If* $\mathbf{U}\subseteqq\mathbf{I}$, *then* $\mathbf{U}$ *is a saturated subclass of* $\mathbf{P}$, $\mathbf{Q}$ *and* $\mathbf{I}$; *the corresponding cosaturated subclasses are* $\mathbf{P}\setminus\mathbf{U}$, $\mathbf{Q}\setminus\mathbf{U}$ *and* $\mathbf{I}\setminus\mathbf{U}$ *respectively.*

PROOF. That $\mathbf{U}$ is a saturated subclass in each case follows readily from the observation that if A is in $\mathbf{U}$ and $A\lhd B\in\mathbf{P}$, then A is a direct summand of B and thus $A=B$. Let $\mathbf{A}=\mathscr{L}(\mathbf{P}, \mathbf{U})=\mathscr{K}(\mathbf{P}, \mathbf{U})$. Clearly $\mathbf{P}\cap\mathscr{S}\mathbf{A}\subseteqq\mathbf{P}\setminus\mathbf{U}$. Let K be a non-zero ring in $\mathbf{P}\setminus\mathbf{U}$. Then $\mathbf{A}(K)\in\mathbf{P}$ and every prime factor of $\mathbf{A}(K)$ is in $\mathbf{U}$. But $\mathbf{A}(K)\lhd\lhd\mathbf{K}\in\mathbf{P}$, so $\mathbf{A}(K)$ is prime, and thus $\mathbf{A}(K)$ is in $\mathbf{U}$ or $\mathbf{A}(K)=0$. But if $\mathbf{A}(K)\in\mathbf{U}$, then as above, $K=\mathbf{A}(K)\in\mathbf{U}$. Consequently $\mathbf{A}(K)=0$, so that $\mathbf{P}\setminus\mathbf{U}=\mathbf{P}\cap\mathscr{S}\mathbf{A}$.

For $\mathbf{Q}$ and $\mathbf{I}$, the proof is the same.

We next look at varieties.

PROPOSITION 9. *Let* $\mathbf{X}$ *be a variety. Then* $\mathbf{P}\cap\mathbf{X}$ *is a saturated subclass of* $\mathbf{P}$; *the corresponding cosaturated subclass is* $\mathbf{P}\setminus\mathbf{X}$.

PROOF. By Theorem 5, $\mathbf{B}\circ\mathbf{X}=\mathscr{L}(\mathbf{Z}\cup\mathbf{X})$ is a special radical. Hence by Theorem 7, the class $\mathbf{P}\cap(\mathbf{B}\circ\mathbf{X})$ is a saturated subclass of $\mathbf{P}$. We shall show that $\mathbf{P}\cap\mathbf{X}=\mathbf{P}\cap(\mathbf{B}\circ\mathbf{X})$. If A is in $\mathbf{P}\cap(\mathbf{B}\circ\mathbf{X})$, then $\mathbf{B}(A)=0$, so $A\in\mathbf{X}$. Hence $\mathbf{P}\cap(\mathbf{B}\circ\mathbf{X})\subseteqq\mathbf{P}\cap\mathbf{X}$. The reverse inclusion is obvious.

Again by Theorem 7, to complete the proof we have to show that $\mathbf{P}\setminus\mathbf{X}=\mathbf{P}\cap\mathscr{S}(\mathbf{B}\circ\mathbf{X})$. Clearly $\mathbf{P}\cap\mathscr{S}(\mathbf{B}\circ\mathbf{X})\subseteqq\mathbf{P}\setminus\mathbf{X}$. Let A be in $\mathbf{P}\setminus\mathbf{X}$ and let $B=(\mathbf{B}\circ\mathbf{X})(A)$. If $B\neq0$, then since $\mathbf{B}\circ\mathbf{X}=\mathscr{L}(\mathbf{Z}\cup\mathbf{X})$, B has a non-zero ideal in $\mathbf{Z}\cup\mathbf{X}$ (since the lower radical construction over a hereditary homomorphically closed class containing all zerorings terminates at the second step). Since $B\lhd A\in\mathbf{P}$, B has an ideal $C\in\mathbf{X}$. From Proposition 6, it then follows that B, and then A, is in $\mathbf{X}$, contradicting our specification of A. Thus $B=0$ and A is in $\mathbf{P}\cap\mathscr{S}(\mathbf{B}\circ\mathbf{X})$, so $\mathbf{P}\setminus\mathbf{X}=\mathbf{P}\cap\mathscr{S}(\mathbf{B}\circ\mathbf{X})$.

PROPOSITION 10. *Let* $\mathbf{X}$ *be a variety. Then* $\mathbf{P}\cap\mathbf{X}$ *is a cosaturated subclass of* $\mathbf{P}$.

PROOF. Take $A\in\mathbf{P}$ and I to be the unique smallest ideal of A such that $A/I\in\mathbf{X}$. If $I\neq0$ then $0\neq I/J\in\mathbf{P}\cap\mathbf{X}$ implies $J\lhd A$ and $A/J\in\mathbf{X}$ by Proposition 6. Thus $I=0$ and $A\in\mathbf{X}$. That $\mathbf{P}\cap\mathbf{X}$ is weakly special is straightforward using Proposition 6.

Thus for any variety $\mathbf{X}$, $\mathbf{P}\cap\mathbf{X}$ is a "saturated-cosaturated" subclass of $\mathbf{P}$. Any subclass of $\mathbf{I}$ has the analogous property with respect to $\mathbf{I}$.

5. Concrete radicals

Making use of Theorems 7 and 8, we easily obtain the following result, which we can use to describe a number of well known special radicals.

THEOREM 9. *For a class* $\mathbf{U}$ *of rings, the following conditions are equivalent.*

(1) $\mathbf{U}$ *is a supernilpotent radical with the intersection property relative to* $\mathbf{C}\cap\mathscr{S}\mathbf{U}$.

(2) $\mathbf{U}=\mathscr{K}(\mathbf{C}, \mathbf{N})$ *where* $\mathbf{N}=\mathbf{C}\cap\mathbf{U}$ *is a saturated subclass of* $\mathbf{C}$.

(3) $\mathbf{U}$ *is homomorphically closed and hereditary, and* $\mathbf{N}=\mathbf{C}\cap\mathbf{U}$ *is a saturated subclass of* $\mathbf{C}$ *such that if every non-zero* $\mathbf{C}$*-factor of a ring* A *is in* $\mathbf{N}$, *then* A *is in* $\mathbf{U}$.

COROLLARY 7. *In the notation of Theorem 9,*

(1) $\mathbf{U}=\mathbf{B}$ *if* $\mathbf{C}=\mathbf{P}$ *and* $\mathbf{N}=\mathbf{O}$.

(2) $\mathbf{U}$ *is the Levitzki (locally nilpotent) radical if* $\mathbf{C}=\mathbf{P}$ *and* $\mathbf{N}=\{A\in\mathbf{P}|A$ *is locally nilpotent*$\}$,

(3) $\mathbf{U}$ *is the nil radical if* $\mathbf{C}=\mathbf{P}$ *and* $\mathbf{N}=\{A\in\mathbf{P}|A$ *is nil*$\}$,

(4) $\mathbf{U}$ *is the Jacobson radical if* $\mathbf{C}=\mathbf{P}$ *and* $\mathbf{N}=\{A\in\mathbf{P}|A$ *is quasi-regular*$\}$,

(5) $\mathbf{U}$ *is the Brown-McCoy radical if* $\mathbf{C}=\mathbf{P}$ *or* $\mathbf{Q}$ *and* $\mathbf{N}=\{A\in\mathbf{C}|A$ *is G-regular*$\}$,

(6) $\mathbf{U}$ *is Thierrin's corpoidal radical if either*

(a) $\mathbf{C}=\mathbf{P}$ *or* $\mathbf{Q}$ *and* $\mathbf{N}=\{A\in\mathbf{C}|A$ *has no homomorphic image which is a field*$\}$ *or*

(b) $\mathbf{C}=\mathbf{I}$ *and* $\mathbf{N}=\{A\in\mathbf{I}|A$ *is not a field*$\}$,

(7) $\mathbf{U}$ *is the Behrens radical if* $\mathbf{C}=\mathbf{P}$ *or* $\mathbf{Q}$ *and* $\mathbf{N}=\{A\in\mathbf{C}|A$ *has no homomorphic image with non-zero idempotents*$\}$,

(8) $\mathbf{U}$ *is the generalized nil radical if* $\mathbf{C}=\mathbf{P}$ *and* $\mathbf{N}=\{A\in\mathbf{P}|$*every non-zero homomorphic image of* A *has a zero divisor*$\}$,

(9) $\mathbf{U}$ *is the antisimple radical if* $\mathbf{C}=\mathbf{P}$ *and* $\mathbf{N}=\{A\in\mathbf{P}|A$ *has no (non-zero) subdirectly irreducible prime factors*$\}$.

PROOF. (1), (2), (3) and (4) follow readily from Theorem 9.

(5) Since the Brown-McCoy radical $\mathbf{G}$ satisfies

$$\mathbf{G}=\mathscr{U}\mathscr{S}\mathbf{G}=\mathscr{U}(\mathbf{P}\cap\mathscr{S}\mathbf{G})\subseteqq\mathscr{U}(\mathbf{Q}\cap\mathscr{S}\mathbf{G})\subseteqq\mathscr{U}\mathbf{I}=\mathbf{G},$$

it has the intersection property relative to $\mathbf{P}\cap\mathscr{S}\mathbf{G}$ and $\mathbf{Q}\cap\mathscr{S}\mathbf{G}$. Thus Theorem 9 is applicable. The same applies to (6).

(7) Propes [15] has shown that the Behrens radical $\mathbf{T}$ is the class of rings A, every homomorphic image of which contains no non-zero idempotents. Since $\mathbf{T}$ is a special radical, we have

$$\mathbf{T}=\mathscr{U}\mathscr{S}\mathbf{T}=\mathscr{U}(\mathbf{P}\cap\mathscr{S}\mathbf{T})\subseteqq\mathscr{U}(\mathbf{Q}\cap\mathscr{S}\mathbf{T})=\mathbf{T}.$$

Theorem 9 now yields the required assertion.

(8) The generalized nil radical is the upper radical defined by the class of rings without zero-divisors.

(9) The antisimple radical $\mathbf{A}$ is $\mathscr{U}\mathbf{Q}$. $\mathbf{P}\cap\mathbf{A}$ is the saturated subclass of $\mathbf{P}$ corresponding to the cosaturated subclass $\mathbf{P}\cap\mathscr{S}\mathbf{A}$. Clearly $\mathbf{P}\cap\mathbf{A}=\{A\in\mathbf{P}|A$ has no non-zero $\mathbf{Q}$-factors$\}$. Now Theorem 9 characterizes the special radical $\mathbf{J}=\mathscr{K}(\mathbf{P},\mathbf{P}\cap\mathbf{A})$ with the intersection property relative to $\mathbf{P}\cap\mathscr{S}\mathbf{J}$. By Theorems 4 and 6 we have $\mathbf{J}\subseteqq\mathbf{A}$. Let A be in$\mathbf{A}$ be and let B be a non-zero $\mathbf{P}$-factor of A. We have $B\in\mathbf{P}\cap\mathbf{A}$. Thus $A\in\mathscr{K}(\mathbf{P},\mathbf{P}\cap\mathbf{A})=\mathbf{J}$. We conclude that $\mathbf{A}=\mathbf{J}=\mathscr{K}(\mathbf{P},\mathbf{P}\cap\mathbf{A})$.

6. The structure of special radical rings

Let **W** be any special class, $\mathbf{R}=\mathcal{U}\mathbf{W}$. By Theorem 6, $\mathbf{N}=\mathbf{P}\cap\mathbf{R}$ is a saturated subclass of **P** and $\mathbf{R}=\mathcal{K}(\mathbf{P},\mathbf{N})$. In this last section we shall discuss the structure of **R**-rings in terms of **N** and **B**. Using the fact that a semiprime ring has a minimal prime ideal one can easily prove the following.

THEOREM 10. *Let A be a ring, $B=A/\mathbf{B}(A)$. Then $A\in\mathcal{K}(\mathbf{P},\mathbf{N})$ if and only if $B/J\in\mathbf{N}$ for every minimal prime ideal J of B. In addition, if $A\in\mathcal{K}(\mathbf{P},\mathbf{N})$, there is a chain*

$$B\supset J_1\supset J_2\supset\ldots\supset J_\beta\supset\ldots\supset 0$$

of ideals of B such that $J_{\beta+1}$ is a minimal prime ideal of J_β and $J_\beta/J_{\beta+1}\in\mathbf{N}$, for each ordinal β.

When $\mathbf{N}\subseteqq\mathbf{I}$, we can obtain a slightly better description of the rings in $\mathcal{K}(\mathbf{L},\mathbf{N})$ $\mathcal{K}(\mathbf{Q},\mathbf{N})$ and $\mathcal{K}(\mathbf{I},\mathbf{N})$. For brevity, we do this only for $\mathcal{K}(\mathbf{P},\mathbf{I})$.

COROLLARY 8. *$A\in\mathcal{K}(\mathbf{P},\mathbf{I})$ if and only if $B=A/\mathbf{B}(A)$ is Brown-McCoy semisimple and every prime factor of B is in* **I**.

COROLLARY 9. *$A\in\mathcal{S}\mathcal{K}(\mathbf{P},\mathbf{I})$ if and only if A is a subdirect product of rings in* $\mathbf{P}\setminus\mathbf{I}$.

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TENSOR PRODUCT OF MODULES OVER DEDEKIND DOMAINS

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1. Introduction

In this paper we study the structural properties of the tensor product of two modules over Dedeking domains. We not only generalize some results known for tensor products of abelian groups (see [1]), but also obtain some new ones. The main step in this direction is to generalize Fuchs's p-basic subgroup (see [1]). An attempt in this direction was first made by YOUSUFZAI who introduced P-basic submodules in [7], but he did not prove the existence of a P-basic submodule in every module over a Dedekind domain. This we achieve in Section 2. In Section 3 we use P-basic submodules in the discussion of tensor products. In particular, we prove that if B_1, B_2 are P-basic submodules of modules M_1, M_2 over a Dedekind domain, then $B_1 \otimes B_2$ is isomoprhic to a P-basic submodule of $M_1 \otimes M_2$.

Let N be a submodule of a module M over a Dedekind domain R, and P a prime ideal of R. Then we define N to be *P-pure* in M if $N \cap P^m M = P^m N$ for all positive integers m, and call N *pure* in M if N is P-pure in M for all prime ideals P of R. For modules over Dedekind domains this definition of purity is equivalent to all other well-known definitions of purity (see [1], [3], [4], [5], [6]). We also call a module M *P-divisible* if $PM = M$.

Throughout this paper R denotes a Dedekind domain and K its field of fractions, P a prime ideal of R, M an R-module, and M_t the torsion submodule of M. M is called a P-module, if the order (i.e. the annihilator of each element of M) is a power of P. Also we call an R-module simply a module.

2. P-basic submodules

In this section we show that every module M contains a P-basic submodule and that any two P-basic submodule of M are isomorphic. We recall that a submodule B of M is a *P-basic submodule* if the following conditions hold:

(i) B is a direct sum of cyclic P-modules and/or cyclic modules of order 0.

(ii) B is P-pure in M.

(iii) M/B is P-divisible.

First, a definition and a couple of lemmas.

DEFINITION 2.1. An element $x \in M$ is of *height m at P* if $x \in P^m M \setminus P^{m+1} M$, and x is of *infinite height at P* if $x \in P^m M$ for all positive integers m.

LEMMA 2.2. *Let $\lambda \in R$ and $x \in M$, where M is a torsion-free module, then the height of λx at P is the sum of heights at P of λ and x.*

PROOF. The result is trivial if λ or x is of infinite height. Suppose now that λ and x are of heights m and n respectively and $\lambda x \in P^k M$, where $k > m+n$. Let $I = \{\mu : \mu \in R,\ \mu x \in P^k M\}$. Clearly I is an ideal of R and $I \supseteqq P^{k-n}$. Hence $I = P^r$, $r \leqq m$, for $\lambda \notin P^{m+1}$. Thus $P^r x \subseteqq P^k M$, $r \leqq m$. Since M can be embedded in a vector space over K, we can multiply both the sides by P^{-1}, whence it follows that $x \in P^{k-r} M$. But $k - r > n$, which contradicts the fact that x is of height n.

LEMMA 2.3. *Let M be a torsion-free module. Then $M/PM \cong P^m M/P^{m+1} M$, where m is any positive integer.*

PROOF. Let $\lambda \in P \setminus P^2$, then $\lambda^m \in P^m \setminus P^{m+1}$. Define $\theta : M/PM \to P^m M/P^{m+1} M$ by setting $\theta(x+PM) = \lambda^m x + P^{m+1} M$. It is clear that θ is a homomorphism. If $\theta(x+PM) = 0$, then $\lambda^m x \in P^{m+1} M$, and so $x \in PM$ by Lemma 2.2. Hence θ is injective. Suppose now that $y + P^{m+1} M$ is an element of $P^m M/P^{m+1} M$. Since $P^m = R\lambda^m + P^{m+1}$, it follows that $P^m M = \lambda^m M + P^{m+1} M$, and so $y = \lambda^m x + z$ for some $x \in M$ and $z \in P^{m+1} M$. Hence $y + P^{m+1} M = \lambda^m x + P^{m+1} M = \theta(x+PM)$. Hence θ is surjective, and so it is an isomorphism.

THEOREM 2.4. *Let M be a torsion-free module. Then M contains a P-basic submodule, and any two P-basic submodules of M are isomorphic.*

PROOF. Let $\{a_i : i \in I\}$ be a basis of the vector space M/PM over the field R/P. Lift it to a set $\{b_i : i \in I\}$ in M, and let B be the submodule of M generated by the set $\{b_i : i \in I\}$. We claim that B is a P-basic submodule of M. Since $P(M/B) = (PM+B)/B = M/B$, it follows that M/B is P-divisible. Now let $\sum_{j \in J} \lambda_j b_j = 0$, where $\lambda_j \in R$ and J is a finite subset of I. Hence $\Sigma \lambda_j a_j = 0$, and so each $\lambda_j \in P$. We prove by induction that each $\lambda_j \in P^n$ for all positive integers n, and so each $\lambda_j = 0$. Suppose that each $\lambda_j \in P^m$, $m \geqq 1$. Since $P^m = R\lambda^m + P^{m+1}$, $\lambda \in P/P^2$, we can write $\lambda_j = \mu_j \lambda^m + v_j$ for some $\mu_j \in R$ and $v_j \in P^{m+1}$. Hence the relation $\Sigma \lambda_j b_j = 0$ implies that $\Sigma(\mu_j \lambda^m b_j + v_j b_j) = 0$, whence it follows that $\Sigma \mu_j(\lambda^m b_j + P^{m+1} M) = 0$. But $\{\lambda^m b_i + P^{m+1} M\}$ is a basis of the vector space $P^m M/P^{m+1} M$ over R/P by Lemma 2.3. Hence $\mu_j \in P$, so $\lambda_j \in P^{m+1}$. Thus each $\lambda_j \in P^n$ for all positive integers n. Hence $\{b_i : i \in I\}$ is R-independent. We finally prove that B is P-pure in M. Let $x \in B \cap P^m M$, then $x = \sum_{j \in J_1} \lambda_j b_j$ for some finite subset J_1 of I, where $\Sigma \lambda_j b_j \in P^m M$. We shall show that each $\lambda_j \in P^m$, and so $x \in P^m B$. We first note that if $\Sigma \lambda_j b_j \in PM$, then $\Sigma \lambda_j a_j = 0$, and so each $\lambda_j \in P$. The proof can then be completed by induction as above. Hence B is a P-basic submodule of M. Moreover, it can be shown that if $B = \bigoplus_{i \in I} Rb_i$ is a P-basic submodule of M, then $\{b_i + PM : i \in I\}$ is a basis of the vector space P/PM over R/P. Hence any two P-basic submodules of M are isomorphic.

REMARK 2.5. If M is a P-module, then M can be regarded as a module over the local ring R_P, which is a discrete valuation ring. Hence M contains a basic submodule and any two basic submodules of M are isomorphic by [Lemma 21; 4].

THEOREM 2.6. *Any module M contains a P-basic submodule and any two P-basic submodules of M are isomorphic.*

PROOF. Let B_1 be a P-basic submodule of the torsion-free module M/M_t (see Theorem 2.4). The exact sequence $0 \to M_t \xrightarrow{\lambda} M \xrightarrow{\mu} M/M_t \to 0$ induces the exact sequence $0 \to M_t \to M_1 \to B_1 \to 0$, where $M_1 = \mu^{-1}(B_1)$. The latter sequence splits, for B_1 is free. Hence $M_1 = M_t \oplus B_2$, where $B_2 \cong B_1$. Let A be the P-component of M_t, and let B_3 be a basic submodule of A. We claim that $\bar{B} = B_3 \oplus B_2$ is a P-basic submodule of M. First, we show that $\bar{B}$ is P-pure in M. Since B_3 is pure in A and A is pure in M_t, it follows that B_3 is pure in M_t. Hence $\bar{B} = B_3 \oplus B_2$ is certainly P-pure in $M_1 = M_t \oplus B_2$. Also $M_1/M_t = B_1$ is P-pure in M/M_t, so M_1 is P-pure in M, for M_t is pure in M. Hence $\bar{B}$ is P-pure in M. Next, we prove that $M/\bar{B}$ is P-divisible. Consider the exact sequence $0 \to M_1/\bar{B} \to M/\bar{B} \to M/M_1 \to 0$. Since $M_1/\bar{B} \cong M_t/B_3 \cong A/B_3 \oplus M_t/A$, it is P-divisible, for A/B_3 and M_t/A are both P-divisible. Also M/M_1, being isomorphic to $M/M_t/B_1$ is P-divisible, for B_1 is a P-basic submodule of M/M_t. Hence $M/\bar{B}$ is also P-divisible. Thus $\bar{B}$ is a P-basic submodule of M. Also it follows from Theorem 2.4 and Remark 2.5 that any two P-basic submodules of M are isomorphic.

DEFINITION 2.7. We call the number of summands of a P-basic submodule of a module M the *P-rank* of M.

3. Tensor product

First, we state some results, which can be easily verified.

(i) If M is a P-module and N a Q-module, where P, Q are distinct prime ideals of R, then $M \otimes N = 0$.

(ii) If M, N are two cyclic P-modules of orders P^m and P^n respectively, then $M \otimes N$ is a cyclic P-module of order P^r, where $r = \min(m, n)$.

(iii) If M is a P-module and N is P-divisible, then $M \otimes N = 0$.

(iv) If M, N are two P-modules, then $M \otimes N \cong B_1 \otimes B_2$, where B_1, B_2 are basic submodules of M, N respectively. Hence $M \otimes N$ is a direct sum of cyclic P-modules.

(v) The tensor product of two torsion modules is a direct sum of cyclic modules, being the direct sum of the tensor products of the corresponding P-components.

(vi) The tensor product of two torsion-free modules is again torsion-free.

(vii) A module is flat if and only if it is torsion-free.

THEOREM 3.1. *Let B be a P-basic module of a module M and let N be a P-module then $M \otimes N \cong B \otimes N$.*

PROOF. The P-pure exact sequence $0 \to B \to M \to M/B \to 0$ gives rise to the exact sequence

$$0 \to B \otimes N \to M \otimes N \to (M/B) \otimes N \to 0.$$

Since M/B is P-divisible $(M/B) \otimes N = 0$, whence the result follows.

COROLLARY 3.2. *If M is a torsion-free module and N is a P-module, then $M \otimes N$ is isomorphic to a direct sum of copies of N.*

REMARK 3.3. The above corollary gives the structure of the tensor product of a torsion-free module and a torsion module, for a torsion module is a direct sum of its P-components.

THEOREM 3.4. *Let* $0 \to M' \xrightarrow{\lambda} M \xrightarrow{\mu} M'' \to 0$ *be a pure exact sequence, and let* N *be a torsion module. Then the exact sequence*

$$0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0 \tag{3.5}$$

splits.

PROOF. It is enough to prove the theorem, taking N to be a P-module. Let B be a P-basic submodule of M''. Consider the pure exact sequence $0 \to M' \to M_1 \to B \to 0$, where $M_1 = \mu^{-1}(B)$. The sequence splits, for B is pure-projective (see [Theorem 3; 3]). This gives rise to the splitting sequence

$$0 \to M' \otimes N \to M_1 \otimes N \to B \otimes N \to 0. \tag{3.6}$$

Since $M'' \otimes N \cong B \otimes N$ by Theorem 3.1, the sequences (3.5) and (3.6) are isomorphic by 5-lemma. Hence the sequences (3.5) splits.

COROLLARY 3.7. *Let* $0 \to M' \to M \to M'' \to 0$ *be a P-pure exact sequence and* N *a P-module. Then the exact sequence* $0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$ *splits.*

COROLLARY 3.8. *Let* $0 \to M' \to M \to M'' \to 0$ *be a P-pure exact sequence, and let* B', B, B'' *be P-basic submodules of* M', M, M'', *respectively. Then* $B \cong B' \oplus B''$.

PROOF. This follows from Theorem 3.1 and Corollary 3.7.

COROLLARY 3.9. *Let* $0 \to M' \to M \to M'' \to 0$ *be a pure exact sequence and let* N *be any module, and let* B', B, B'' *be P-basic submodules of* $M' \otimes N, M \otimes N, M'' \otimes N$, *respectively. Then* $B \cong B' \oplus B''$.

PROOF. Note that the sequence $0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$ is also pure exact (see [2] or [6]).

COROLLARY 3.10. *Let* M, N *be any modules, then*

$$(M \otimes N)_t \cong M_t \otimes N_t \oplus M_t \otimes (N/N_t) \oplus (M/M_t) \otimes N_t.$$

PROOF. Since $(M \otimes N)/(M_t \otimes N + M \otimes N_t) \cong (M/M_t) \otimes (N/N_t)$ is torsion-free, it follows that $(M_t \otimes N + M \otimes N_t) = (M \otimes N)_t$. The result then follows from Theorem 3.4.

REMARK 3.11. From (iv), (v), Corollary 3.2 and Corollary 3.10 the structure of $(M \otimes N)_t$ is completely determined.

LEMMA 3.12. *Let* $0 \to M' \to M \to M'' \to 0$ *be a P-pure exact sequence and let* N *be a torsion-free module, then the exact sequence* $0 \to N \otimes M' \to N \otimes M \to N \otimes M'' \to 0$ *is also P-pure exact.*

PROOF. Let A be any arbitrary P-module. Tensoring the given sequence by A we get the exact sequence

$$0 \to A \otimes M' \to A \otimes M \to A \otimes M'' \to 0,$$

which gives rise to the exact sequence

$$0 \to N \otimes (A \otimes M') \to N \otimes (A \otimes M) \to N \otimes (A \otimes M'') \to 0.$$

Hence the isomorphic sequence

$$0 \to A\otimes(N\otimes M') \to A\otimes(N\otimes M) \to A\otimes(N\otimes M'') \to 0$$

is also exact. This implies that the sequence

$$0 \to N\otimes M' \to N\otimes M \to N\otimes M'' \to 0$$

is P-pure exact.

THEOREM 3.13. *Let $0\to M'\to M\to M''\to 0$ be a P-pure exact sequence, and let N be any module such that its torsion submodule is a P-module. Then the sequence*

$$0 \to N\otimes M' \to N\otimes M \to N\otimes M'' \to 0$$

is P-pure exact.

PROOF. Let us express N as the direct limit of the family $\{N_\lambda : \lambda\in\Lambda\}$ of its finitely generated submodules. The sequences $0\to N_\lambda\otimes M'\to N_\lambda\otimes M\to N_\lambda\otimes M''\to 0$ are P-pure exact by Corollary 3.7 and Lemma 3.12, for N_λ is a direct sum of a P-module and a torsion-free module. Taking the direct limit of this direct system of P-pure exact sequences we get the result.

THEOREM 3.14. *Let M_1, M_2 be two modules such that the torsion submodule of one of them is a P-module, and let B_1, B_2 be P-basic submodules of M_1, M_2, respectively. Then $B_1\otimes B_2$ is a P-basic submodule of $M_1\otimes M_2$.*

PROOF. Let the torsion submodule of M_1 be a P-module. The sequences

$$0 \to B_1\otimes B_2 \to M_1\otimes B_2 \to (M_1/B_1)\otimes B_2 \to 0,$$

$$0 \to M_1\otimes B_2 \to M_1\otimes M_2 \to M_1\otimes(M_2/B_2) \to 0$$

are P-pure exact by Theorem 3.13. Hence $B_1\otimes B_2$ is P-pure in $M_1\otimes M_2$. Also we have the exact sequence

$$0 \to \frac{M_1\otimes B_2}{B_1\otimes B_2} \to \frac{M_1\otimes M_2}{B_1\otimes B_2} \to \frac{M_1\otimes M_2}{M_1\otimes B_2} \to 0.$$

Since

$$\frac{M_1\otimes B_2}{B_1\otimes B_2} \cong (M_1/B_1)\otimes B_2, \quad \frac{M_1\otimes M_2}{M_1\otimes B_2} \cong M_1\otimes(M_2/B_2)$$

are both P-divisible, it follows that $\dfrac{M_1\otimes M_2}{B_1\otimes B_2}$ is also P-divisible. Clearly $B_1\otimes B_2$ is a direct sum of cyclic P-modules and/or cyclic modules of order 0, so it is a P-basic submodule of $M_1\otimes M_2$.

Finally, we strengthen our theorem and prove that if B_1, B_2 are P-basic submodules of any two modules M_1, M_2 respectively, then $B_1\otimes B_2$ is isomorphic to a P-basic submodule of $M_1\otimes M_2$.

First, a definition and a couple of lemmas.

DEFINITION 3.15. The *P'-component* of a module M is the direct sum of all the Q-components of M, where Q is a prime ideal of R different from P. We can also define a *P'-module* and a *P'-torsion-free* module in an obvious way.

NOTATION 3.16. In the rest of this section we shall denote the P'-component of M by M' and M/M' by M''.

LEMMA 3.17. *Let M_1, M_2 be two modules. Then*

$$(M_1 \otimes M_2)' \cong M_1' \otimes M_2' \oplus M_1' \otimes M_2'' \oplus M_1'' \otimes M_2',$$

and $(M_1 \otimes M_2)'' \cong M_1'' \otimes M_2''$.

PROOF. Since M_1', M_2' are pure submodules of M_1, M_2 respectively, we have the following exact commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M_1' \otimes M_2' & \to & M_1' \otimes M_2 & \to & M_1' \otimes M_2'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M_1 \otimes M_2' & \to & M_1 \otimes M_2 & \to & M_1 \otimes M_2'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M_1'' \otimes M_2' & \to & M_1'' \otimes M_2 & \to & M_1'' \otimes M_2'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

From the above diagram we can easily show that $M_1 \otimes M_2/(M_1' \otimes M_2 + M_1 \otimes M_2') \cong$ $\cong M_1'' \otimes M_2''$. Since $(M_1' \otimes M_2 + M_1 \otimes M_2')$ is a P'-module and $M_1'' \otimes M_2''$ is P'-torsion-free, it follows that $(M_1' \otimes M_2 + M_1 \otimes M_2') = (M_1 \otimes M_2)'$ and $(M_1 \otimes M_2)'' \cong$ $\cong M_1'' \otimes M_2''$. Since M_1', M_2' are torsion modules the first row and the first column of the above diagram split by Theorem 3.4. Hence

$$M_1' \otimes M_2 + M_1 \otimes M_2' \cong M_1' \otimes M_2' \oplus M_1' \otimes M_2'' \oplus M_1'' \otimes M_2'.$$

LEMMA 3.18. *A P-basic submodule of M is isomorphic to a P-basic submodule of M''.*

PROOF. Let B be a P-basic submodule of M. Then $(M'+B)/M' \cong B/M' \cap B \cong B$, for $M' \cap B = 0$. We claim that $(M'+B)/M'$ is a P-basic submodule of M''. Since

$$M''/(M'+B)/M' \cong M/(M'+B) \cong M/B/(M'+B)/B,$$

it follows that $M''/(M'+B)/M'$ is P-divisible for M/B is P-divisible. Now $(M'+B)/B \cong M'$ is P-pure in M/B, for M' is P-divisible, and B is P-pure in M, so $(M'+B)$ is P-pure in M. Hence $(M'+B)/M'$ is P-pure in $M/M' = M''$.

THEOREM 3.19. *Let B_1, B_2 be P-basic submodules of any two modules M_1, M_2 respectively. Then $B_1 \otimes B_2$ is isomorphic to a P-basic submodule of $M_1 \otimes M_2$.*

PROOF. Let $B, \tilde{B}_1, \tilde{B}_2$ be P-basic submodules of $M_1 \otimes M_2, M_1'', M_2''$ respectively. By the above lemma B is isomorphic to a P-basic submodule of $(M_1 \otimes M_2)'' \cong$

$\cong M_1'' \otimes M_2''$ (by Lemma 3.17). But a P-basic submodule of $M_1'' \otimes M_2''$ is isomorphic to $\tilde{B}_1 \otimes \tilde{B}_2$ by Theorem 3.14. Since $\tilde{B}_1 \cong B_1$ and $\tilde{B}_2 \cong B_2$ by Lemma 3.18, the result follows.

COROLLARY 3.20. *The P-rank of* $M_1 \otimes M_2$ *is the product of the P-ranks of* M_1 *and* M_2.

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SOME ADDITIONAL RESULTS ON THE STRONG APPROXIMATION OF ORTHOGONAL SERIES

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Introduction

Let $\{\varphi_n(x)\}$ be an orthonormal system on the finite interval (a, b). We consider the orthogonal series

(1) $$\sum_{n=0}^{\infty} c_n \varphi_n(x) \quad \text{with} \quad \sum_{n=0}^{\infty} c_n^2 < \infty.$$

By the Riesz-Fischer theorem the series (1) converges in L^2 to a square-integrable function $f(x)$. Let us denote the partial sums of the series (1) by $s_n(x)$.

One of the first results in connection with the strong approximation of orthogonal series is due to G. SUNOUCHI [8] who generalized an ordinary approximation theorem of the author [1] as follows:

THEOREM A. *If* $0<\gamma<1$ *and*

(2) $$\sum_{n=1}^{\infty} c_n^2 n^{2\gamma} < \infty$$

then

(3) $$\left\{\frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |s_\nu(x)-f(x)|^p\right\}^{1/p} = o_x(n^{-\gamma})$$

holds almost everywhere (a.e.) *in* (a, b) *for any* $\alpha>0$ *and* $0<p<\gamma^{-1}$.

This result was generalized into several directions. For example the partial sums $s_\nu(x)$ in (3) were replaced by $s_{k_\nu}(x)$ or $s_{\mu_\nu}(x)$, where $\{k_\nu\}$ and $\{\mu_\nu\}$ denote an increasing and a "mixed" sequence of natural numbers, respectively. There are such generalizations where the partial sums are replaced by (C, β)-means of negative β, or the (C, α)-summation method is exchanged by another one (see [2], [3], [5]).

Very recently in [5] we proved

THEOREM B. *Suppose that* $\gamma>0$, $0<p<\gamma^{-1}$ *and* $p\leqq 2$, *that* $\alpha>p \max\left(\frac{1}{2}, \gamma\right)$ *or if* $p=2$ *then* $\alpha\geqq 1$; *moreover that* (2) *holds.*
Then

(4) $$C_n(f, \alpha, p, \{\mu_\nu\}; x) := \left\{\frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |s_{\mu_\nu}(x)-f(x)|^p\right\}^{1/p} = o_x(n^{-\gamma})$$

holds a.e. in (a, b) *for any (not necessarily monotone) sequence* $\{\mu_\nu\}$ *of distinct positive integers.*

From the results of [6] and [7] we can unify the following

THEOREM C. *Suppose that* $\gamma>0$, $0<p\gamma<\beta$ *and that* (2) *holds. Moreover if*
(i) $\beta\leqq 2$ *or* $\beta>2$ *but at least either* $\gamma<1$ *or* $p\leqq 2$,
(ii) $p\geqq 2$ *and* $\sum_{n=1}^{\infty} c_n^2 n^{2\gamma+1-\frac{2}{p}}<\infty$,
then

$$(5)\qquad h_n(f,\beta;\, p, \{k_\nu\};\, x):=\left\{\frac{1}{(n+1)^\beta}\sum_{\nu=0}^{n}(\nu+1)^{\beta-1}|s_{k_\nu}(x)-f(x)|^p\right\}^{1/p}=o_x(n^{-\gamma})$$

holds a.e. in (a,b) *for any increasing sequence* $\{k_\nu\}$.

Later on we shall use the following consequence of Theorem B.

PROPOSITION A. *Suppose that* $\gamma>0$, $0<p<\gamma^{-1}$ *and* $p\leqq 2$, *that* $\beta>p\max\left(\frac{1}{2},\gamma\right)$ *or if* $p=2$ *then* $\beta\geqq 1$, *moreover that* (2) *holds.*
Then

$$(6)\qquad h_n(f,\beta,p,\{\mu_\nu\};\,x)=o_x(n^{-\gamma})$$

holds a.e. in (a,b) *for any sequence* $\{\mu_\nu\}$ *of distinct positive integers.*

Comparing the restrictions on the parameters in the previous results the assumptions

$$(7)\qquad \alpha>\frac{p}{2}\quad\text{and}\quad \beta>\frac{p}{2}\ \left(\text{if } \gamma<\frac{1}{2}\right)$$

seem to be artificial. Analysing the proof of (4) it turns out that the conditions (7) spring from the fact that (4) was first proved for $\alpha=1$, and from this result it can be extended just to $\alpha>\frac{p}{2}$. But a more careful investigation shows that without the restriction $\alpha>\frac{p}{2}$ we cannot prove (4), indeed, but the assumption $\beta>\frac{p}{2}$ is superfluous and we can omit it. This will be proved in Theorem 1.

If we assume that $\alpha=\frac{p}{2}$ then the approximation-order in (4) will increase with the factor $(\log n)^{\frac{1}{p}-\frac{1}{2}}$ $(p\leqq 2)$; and if $\alpha<\frac{p}{2}$ then we require a sharper condition instead of (2) in order to have (4), see Theorem 2.

Comparing the assumptions of Theorem B and C we see that the restriction $p\gamma<1$ appears only in connection with the means $C_n(f,\alpha,\ldots)$. This has raised the next question: *Can we omit the restriction* $p\gamma<1$ *among the assumptions of Theorem B if* $\alpha>1$? As the example to be given soon shows *the answer for this question is negative*, generally. Moreover the assumption $\gamma p<1$ is required not just for the so-called extra strong approximation, i.e. if the sequence $\{\mu_\nu\}$ in the means given under (4) is mixed, but for the simplest case $\mu_\nu\equiv\nu$, too. Namely if e.g. $\alpha>\gamma p\geqq 1$ and $f(x)\neq s_0(x)$ a.e. then

$$(8)\qquad n^{p\gamma}C_n^p(f,\alpha,p,\{\nu\};\,x)\geqq Kn^{p\gamma-\alpha}n^{\alpha-1}|s_0(x)-f(x)|^p,^*$$

* $K, K_1, K_2, \ldots$ denote positive constants not necessarily the same at each occurrence.

whence it can be seen that (4) does not hold, and not only for a mixed sequence $\{\mu_\nu\}$, but neither for the sequence of the natural numbers.

We would like to call the attention, once more, to the fact that in the case of the means $h_n(f,\beta,\ldots)$ with $\beta>1$ the condition $\gamma p<1$ is not required to the estimation (5) for the strong and very strong approximation, i.e. if the sequence $\{k_\nu\}$ in (5) is either the sequence of the natural numbers $(k_\nu\equiv\nu)$ or an arbitrary increasing subsequence of the natural numbers. But we mention that it can also be proved — not so easily as in (8) for the means $C_n(f,\alpha,\ldots)$ — that the proof of (6) also requires the assumption $p\gamma<1$ for $\beta>1$ and an arbitrary mixed sequence $\{\mu_\nu\}$.

These remarks and the results show that the means $h_n(f,\beta,\ldots)$ and $C_n(f,\alpha,\ldots)$ in connection with the extra strong approximation behave similarly only if $\alpha,\beta>\frac{p}{2}$, but if $\alpha,\beta\leqq\frac{p}{2}$ then the means $h_n(f,\beta,\ldots)$ are more effective than the means $C_n(f,\alpha,\ldots)$ are. Moreover this phenomenon appears also for $\alpha,\beta>1$ if we consider the strong and very strong approximation (compare Theorem 3 and Theorem C).

Continuing the investigation of similarity of the means $C_n(f,\alpha,\ldots)$ and $h_n(f,\beta,\ldots)$ for $\alpha,\beta\leqq1$ we shall show that if the investigations are confined just for the very strong approximation then the assumption $p\leqq2$ can be omitted among the conditions of Theorem B and the remaining assumptions imply that

$$C_n(f,\alpha,p,\{k_\nu\};x)=o_x(n^{-\gamma}) \tag{9}$$

holds a.e. in (a,b) for any increasing sequence $\{k_\nu\}$, which is the perfect analogue of (5) for $\alpha\leqq1$ (see Theorem 3).

Our next problem is also connected by the restriction $p\leqq2$, but for $\alpha,\beta>1$. We do believe that the assumption $p\leqq2$ can be omitted without more change, but we are not able to prove this. In order to have the order $o_x(n^{-\gamma})$ we have to claim a more powerful condition instead of (2). Such extensions are given by Theorem 4, 5 and 6.

Finally we investigate the approximation order of the strong means in the limit cases $\gamma p=\alpha=\beta\leqq1$ and show that the approximation order does not exceed $O_x(n^{-\gamma})$ a.e., but for $p<2$ and $\alpha<1$ we can ensure this order only under conditions claiming a little more than (2) with $\gamma=\frac{\alpha}{p}$ does. Namely if we claimed only (2) then the order of approximation in the case $p\gamma=\alpha=\beta\leqq2$ would be only $o_x\big(n^{-\gamma}(\log n)^{\frac{1}{p}-\frac{1}{2}}\big)$.

More precisely we prove the following theorems:

THEOREM 1. *Suppose that* $\beta,\gamma>0, 0<p\leqq1$ *and* $p\gamma<\beta\leqq\frac{p}{2}$, *moreover that* (2) *holds. Then* (6) *holds a.e. in* (a,b) *for any sequence* $\{\mu_\nu\}$ *of distinct positive integers.*

We mention that this theorem is a significant strengthening of Theorem 2 of [4].

THEOREM 2. *Suppose that* $\alpha,\gamma>0, 0<p\leqq2$ *and* $p\gamma<\alpha\leqq\frac{p}{2}$. *Then* (2) *and* $\alpha=\frac{p}{2}$ *imply*

$$C_n(f,\alpha,p,\{\mu_\nu\};x)=o_x\big(n^{-\alpha}(\log n)^{\frac{1}{p}-\frac{1}{2}}\big); \tag{10}$$

furthermore $\alpha<\frac{p}{2}$ *and*

$$\sum_{n=1}^{\infty} c_n^2 n^{2\gamma-\frac{2\alpha}{p}+1}<\infty \tag{11}$$

ensure that (4) *holds a.e. in* (a, b) *for any sequence* $\{\mu_\nu\}$ *of distinct positive integers.*

THEOREM 3. *If* $\gamma>0$ *and* $0<p\gamma<\alpha\leqq 1$ *then the condition* (2) *implies that* (9) *holds a.e. in* (a, b) *for any increasing sequence* $\{k_\nu\}$ *of natural numbers.*

Theorem 3 is a remarkable generalization of Theorem A given in [5].

THEOREM 4. *If* $\gamma>0, p\geqq 2$ *and* $p\gamma<\min(\alpha, 1)$ *then*

$$\sum_{n=1}^{\infty} c_n^2 n^{2\gamma+1-\frac{2}{p}}<\infty \tag{12}$$

implies (9) *a.e. in* (a, b) *for any increasing sequence* $\{k_\nu\}$ *of the natural numbers.*

THEOREM 5. *If* $\gamma>0, p\geqq 2$ *and* $p\gamma<\min(\beta, 1)$ *then* (12) *implies* (6) *a.e. in* (a, b) *for any sequence* $\{\mu_\nu\}$ *of distinct positive integers.*

THEOREM 6. *Suppose that* $\gamma>0, p\geqq 2$ *and that* $p\gamma<\min(\alpha, 1)$. *Then either* $\alpha\geqq 1$ *and* (12) *or* $\alpha<1$ *and* (11) *imply that* (4) *holds a.e. in* (a, b) *for any sequence* $\{\mu_\nu\}$ *of distinct positive integers.*

THEOREM 7. *Let* $0<\alpha\leqq 1$ *and* $p>0$. *Then each of the following pairs of condition:*

$$\alpha<\frac{p}{2}\leqq 1 \quad \text{and} \quad \sum_{n=1}^{\infty} n^{\frac{p}{2}-1}\left(\sum_{k=n}^{\infty} c_k^2\right)^{p/2}<\infty, \tag{13}$$

$$\alpha=\frac{p}{2} \quad \text{and} \quad \sum_{n=4}^{\infty}\left(\frac{n}{\log n}\right)^{\frac{p}{2}-1}\left(\sum_{k=n}^{\infty} c_k^2\right)^{p/2}<\infty, \tag{14}$$

$$\alpha>\frac{p}{2} \quad \text{and} \quad \sum_{n=1}^{\infty} n^{\alpha-1}\left(\sum_{k=n}^{\infty} c_k^2\right)^{p/2}<\infty, \tag{15}$$

imply that

$$C_n(f, \alpha, p, \{\mu_\nu\}; x)=O_x\left(n^{-\frac{\alpha}{p}}\right) \tag{16}$$

holds a.e. in (a, b) *for any sequence* $\{\mu_\nu\}$ *of distinct positive integers.*

If $p\geqq 2$ *and*

$$\sum_{n=1}^{\infty} n c_n^2<\infty \tag{17}$$

then we also have that (16) *holds for any* $0<\alpha\leqq 1$.

THEOREM 8. *Suppose that* $0<\beta\leqq 1$ *and* $p>0$. *Then*

$$p \leqq 2 \quad \text{and} \quad \sum_{n=1}^{\infty} n^{\beta-1}\left(\sum_{k=n}^{\infty} c_k^2\right)^{p/2} < \infty \tag{18}$$

$$p \geqq 2 \quad \text{and} \quad \sum_{n=1}^{\infty} c_n^2 n^{2\frac{\beta}{p}-\frac{2}{p}+1} < \infty \tag{19}$$

imply that

$$h_n(f, \beta, p, \{\mu_\nu\}; x) = O_x(n^{-\frac{\beta}{p}}) \tag{20}$$

holds a.e. in (a, b) *for any sequence* $\{\mu_\nu\}$ *of distinct positive integers.*

We mention that (20) generalizes the statement (15) of [7] in the case $\beta\leqq 1$; and the case $p\geqq 2$, but only for increasing $\{k_n\}$, is also treated in [7].

§ 1. Lemmas

In order to prove these theorems we require some lemmas.

LEMMA 1 ([7], Lemma 3). *Let* $\varkappa>0$ *and* $\{\lambda_n\}$ *be an arbitrary sequence of positive numbers. Assuming that the condition*

$$\sum_{n=1}^{\infty} \lambda_n \left\{\sum_{k=n}^{\infty} c_k^2\right\}^{\varkappa} < \infty \tag{1.1}$$

implies a "certain property $T=T(\{s_n(x)\})$" *of the partial sums* $s_n(x)$ *of* (1) *for any orthonormal system, then* (1.1) *implies that the partial sums* $s_{k_n}(x)$ *of* (1) *also have the same proparty* T *for any increasing sequence* $\{k_n\}$ *i.e. if*

$$(1.1) \Rightarrow T(\{s_n(x)\}) \quad \text{then} \quad (1.1) \Rightarrow T(\{s_{k_n}(x)\})$$

for any increasing sequence $\{k_n\}$.

LEMMA 2 ([2], Lemma 5). *Let* $\{\lambda_n\}$ *be a monotone sequence of positive numbers such that*

$$\sum_{n=1}^{m} \lambda_{2^n}^2 \leqq K\lambda_{2^m}^2.$$

Then the condition

$$\sum_{n=0}^{\infty} c_n^2 \lambda_n^2 < \infty$$

implies that

$$s_{2^n}(x) - f(x) = o_x(\lambda_{2^n}^{-1})$$

holds a.e. in (a, b).

LEMMA 3 ([7], Lemma 1). *If* $0<p, q\leqq 2$ *then*

$$\int_a^b \left(\sum_{\nu=n}^{2n-1} |s_\nu(x)-f(x)|^q\right)^{p/q} dx \leqq Kn^{p/q} E_n^p,$$

where

$$E_n = \left\{ \sum_{k=n+1}^{\infty} c_k^2 \right\}^{1/2}.$$

LEMMA 4. *Suppose that* $0<\gamma<1/2$ *and* (2) *holds. Then for any sequence* $\{\mu_\nu\}$ *of distinct positive integers and the sequence* $\{m_\nu\}$ *defined by* $m_\nu=2^m$ *if* $2^m \leqq \mu_\nu < 2^{m+1}$ *we have that*

$$\int_a^b \left\{ \sum_{\nu=0}^{\infty} (\nu+1)^{2\gamma-1} |s_{\mu_\nu}(x) - s_{m_\nu}(x)|^2 \right\} dx \leqq K \sum_{n=1}^{\infty} c_n^2 n^{2\gamma}.$$

The statement of Lemma 4 can be found in the proof of Lemma 3 in [5], implicitly (see (1.7)).

LEMMA 5. *Suppose that* $\beta, \gamma>0, 0<p\leqq 2, p\gamma<\beta\leqq\frac{p}{2}$ *and that* (2) *holds. Then*

$$\int_a^b \left\{ \sup_{0\leqq n<\infty} \left(\frac{n^{p\gamma}}{(n+1)^\beta} \sum_{\nu=0}^{n} (\nu+1)^{\beta-1} |s_{\mu_\nu}(x) - s_{m_\nu}(x)|^p \right)^{1/p} \right\}^2 dx \leqq K \sum_{n=1}^{\infty} c_n^2 n^{2\gamma} \tag{1.2}$$

holds for any sequence $\{\mu_\nu\}$ *of distinct positive integers and* $\{m_\nu\}$ *defined in Lemma* 4.

PROOF. First we show that

$$\sup_{0\leqq n<\infty} \frac{n^{p\gamma}}{(n+1)^\beta} \sum_{\nu=0}^{n} (\nu+1)^{\beta-1} |s_{\mu_\nu}(x) - s_{m_\nu}(x)|^p \leqq \tag{1.3}$$

$$\leqq K \left\{ \sum_{\nu=0}^{\infty} (\nu+1)^{2\gamma-1} |s_{\mu_\nu}(x) - s_{m_\nu}(x)|^2 \right\}^{p/2}.$$

If $p=2$ then (1.3) is an obvious consequence of the following elementary inequality

$$\frac{n^{2\gamma}}{(n+1)^\beta} (\nu+1)^{\beta-1} \leqq (\nu+1)^{2\gamma-1} \quad (\nu\leqq n).$$

If $p<2$ then we use the Hölder inequality with $\frac{2}{p}$ and $\frac{2}{2-p}$ and obtain in view of $\beta>p\gamma$ that

$$\sum_{\nu=0}^{n} (\nu+1)^{\beta-1} |s_{\mu_\nu}(x) - s_{m_\nu}(x)|^p \leqq$$

$$\leqq \left\{ \sum_{\nu=0}^{n} (\nu+1)^{2\gamma-1} |s_{\mu_\nu}(x) - s_{m_\nu}(x)|^2 \right\}^{p/2} \left\{ \sum_{\nu=0}^{n} (\nu+1)^{\frac{2(\beta-1)}{2-p}+\frac{p}{2-p}(1-2\gamma)} \right\}^{1-\frac{p}{2}} \leqq$$

$$\leqq K \left\{ \sum_{\nu=0}^{n} (\nu+1)^{2\gamma-1} |s_{\mu_\nu}(x) - s_{m_\nu}(x)|^2 \right\}^{p/2} (n+1)^{\beta-p\gamma},$$

whence (1.3) obviously follows.

Finally, by (1.3) and Lemma 4, we get (1.2).

LEMMA 6. *Suppose that* $\alpha, \gamma>0, 0<p\leqq 2$ *and* $p\gamma<\alpha\leqq\frac{p}{2}$. *Then* (2) *and* $\alpha=\frac{p}{2}$ *imply that*

$$(1.4)\qquad \int_a^b\left\{\sup_{0\leqq n<\infty}\left(\frac{n^{p\gamma}(\log n)^{\frac{p}{2}-1}}{A_n^\alpha}\sum_{\nu=0}^n A_{n-\nu}^{\alpha-1}|s_{\mu_\nu}(x)-s_{m_\nu}(x)|^p\right)^{1/p}\right\}^2 dx\leqq K\sum_{n=1}^\infty c_n^2 n^{2\gamma};$$

furthermore $\alpha<p/2$ *and*

$$(1.5)\qquad \sum_{n=2}^\infty c_n^2 n^{2\gamma-\frac{2\alpha}{p}+1}<\infty$$

imply that

$$(1.6)\qquad \int_a^b\left\{\sup_{0\leqq n<\infty}\left(\frac{n^{p\gamma}}{A_n^\alpha}\sum_{\nu=0}^n A_{n-\nu}^{\alpha-1}|s_{\mu_\nu}(x)-s_{m_\nu}(x)|^p\right)^{1/p}\right\}^2 dx\leqq K\sum_{n=1}^\infty c_n^2 n^{2\gamma-\frac{2\alpha}{p}+1}$$

holds for any sequence $\{\mu_\nu\}$ *and* $\{m_\nu\}$ *determined in Lemma* 4.

PROOF. The assumptions imply that $\gamma<1/2$. First we show that if $\alpha=\frac{p}{2}$ then

$$(1.7)\qquad \sup_{0\leqq n<\infty}\frac{n^{p\gamma}(\log n)^{\frac{p}{2}-1}}{A_n^\alpha}\sum_{\nu=0}^n A_{n-\nu}^{\alpha-1}|s_{\mu_\nu}(x)-s_{m_\nu}(x)|^p\leqq$$

$$\leqq K\left\{\sum_{\nu=0}^\infty(\nu+1)^{2\gamma-1}|s_{\mu_\nu}(x)-s_{m_\nu}(x)|^2\right\}^{p/2}.$$

If $p=2$ then this follows on account of $\alpha\leqq 1$ and $p\gamma<\alpha$.
If $p<2$ then we use the Hölder inequality and obtain that

$$(1.8)\qquad \sum_{\nu=0}^n A_{n-\nu}^{\alpha-1}|s_{\mu_\nu}(x)-s_{m_\nu}(x)|^p\leqq$$

$$\leqq\left\{\sum_{\nu=0}^n(\nu+1)^{2\gamma-1}|s_{\mu_\nu}(x)-s_{m_\nu}(x)|^2\right\}^{p/2}\left\{\sum_{\nu=0}^n\left((\nu+1)^{\frac{p}{2}(1-2\gamma)}A_{n-\nu}^{\alpha-1}\right)^{\frac{2}{2-p}}\right\}^{1-\frac{p}{2}}.$$

Since $2\gamma<1$, $\alpha=\frac{p}{2}$ and $p\gamma<\alpha$ we have that

$$\sum_{\nu=0}^n(\nu+1)^{\frac{p}{2-p}(1-2\gamma)}(A_{n-\nu}^{\alpha-1})^{\frac{2}{2-p}}\leqq Kn^{\frac{2(\alpha-1)}{2-p}}\sum_{\nu=0}^{n/2}(\nu+1)^{\frac{p}{2-p}(1-2\gamma)}+$$

$$+Kn^{\frac{p}{2-p}(1-2\gamma)}\sum_{\nu=n/2}^n(A_{n-\nu}^{\alpha-1})^{\frac{2}{2-p}}\leqq Kn^{\frac{2(\alpha-1)}{2-p}+\frac{p}{2-p}(1-2\gamma)+1}+$$

$$+Kn^{\frac{p}{2-p}(1-2\gamma)}\log n\leqq K_1 n^{\frac{p}{2-p}(1-2\gamma)}\log n\leqq K_2 n^{\frac{2}{2-p}(\alpha-p\gamma)}\log n,$$

whence, by (1.8), (1.7) obviously follows.

(1.7) and Lemma 4 prove (1.4).

Next we prove (1.6). In order to prove (1.6) we first verify the following inequality

$$(1.9)\qquad \sup_{0\le n<\infty} \frac{n^{p\gamma}}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\gamma-1} |s_{\mu_\nu}(x)-s_{m_\nu}(x)|^p \le$$

$$\le K\left\{\sum_{\nu=0}^{\infty} (\nu+1)^{2\left(\gamma-\frac{\alpha}{p}\right)} |s_{\mu_\nu}(x)-s_{m_\nu}(x)|^2\right\}^{p/2}.$$

If $p=2$ then (1.9) follows by the following facts:

$$p\gamma = 2\gamma < \alpha < 1,\quad A_{n-\nu}^{\alpha-n} \le K,\quad \frac{n^{2\gamma}}{A_n^\alpha} \le K\frac{(\nu+1)^{2\gamma}}{(\nu+1)^\alpha}\quad (\nu\le n).$$

In the case $p<2$ we use again the Hölder inequality which gives that

$$(1.10)\qquad \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |s_{\mu_\nu}(x)-s_{m_\nu}(x)|^p \le$$

$$\le \left\{\sum_{\nu=0}^{n} (\nu+1)^{2\gamma-\frac{2\alpha}{p}} |s_{\mu_\nu}(x)-s_{m_\nu}(x)|^2\right\}^{\frac{p}{2}} \left\{\sum_{\nu=0}^{n} \left((\nu+1)^{\alpha-p\gamma} A_{n-\nu}^{\alpha-1}\right)^{\frac{2}{2-p}}\right\}^{1-\frac{p}{2}}.$$

By $p\gamma<\alpha<\frac{p}{2}<1$ we have that

$$(1.11)\qquad \sum_{\nu=0}^{n} (\nu+1)^{\frac{2}{2-p}(\alpha-p\gamma)} (A_{n-\nu}^{\alpha-1})^{\frac{2}{2-p}} \le$$

$$\le K\left\{n^{\frac{2(\alpha-1)}{2-p}} \sum_{\nu=0}^{n} (\nu+1)^{\frac{2}{2-p}(\alpha-p\gamma)} + n^{\frac{2}{2-p}(\alpha-p\gamma)} \sum_{\nu=1}^{n} \nu^{(\alpha-1)\frac{2}{2-p}}\right\} \le$$

$$\le K_1\left\{n^{\frac{2(\alpha-1)}{2-p}+\frac{2}{2-p}(\alpha-p\gamma)+1} + n^{\frac{2}{2-p}(\alpha-p\gamma)}\right\} =$$

$$= K_1\left\{n^{\frac{2}{2-p}\left(\alpha+\alpha-p\gamma-\frac{p}{2}\right)} + n^{\frac{2}{2-p}(\alpha-p\gamma)}\right\} \le K_2 n^{\frac{2}{2-p}(\alpha-p\gamma)}.$$

This and (1.10) painly verify (1.9).

To prove (1.6) we show that

$$(1.12)\qquad \int_a^b \left\{\sum_{\nu=0}^{\infty} (\nu+1)^{2\left(\gamma-\frac{\alpha}{p}\right)} |s_{\mu_\nu}(x)-s_{m_\nu}(x)|^2\right\} dx \le K\sum_{n=1}^{\infty} c_n^2 n^{2\gamma-\frac{2\alpha}{p}+1}.$$

An elementary calculation shows that

$$\sum_{\nu=0}^{\infty} (\nu+1)^{2\left(\gamma-\frac{\alpha}{p}\right)} \int_a^b |s_{\mu_\nu}(x)-s_{m_\nu}(x)|^2\,dx = \sum_{\nu=0}^{\infty} (\nu+1)^{2\left(\gamma-\frac{\alpha}{p}\right)} \sum_{n=m_\nu+1}^{\mu_\nu} c_n^2 =$$

$$= \sum_{m=0}^{\infty} \sum_{2^m\le\mu_\nu<2^{m+1}} (\nu+1)^{2\left(\gamma-\frac{\alpha}{p}\right)} \sum_{n=m_\nu+1}^{\mu_\nu} c_n^2 \le K\sum_{m=0}^{\infty} \left(\sum_{\nu=1}^{2^m} \nu^{2\left(\gamma-\frac{\alpha}{p}\right)} \sum_{n=2^m+1}^{12^{m+}} c_n^2\right) \le$$

$$\le K_1 \sum_{m=0}^{\infty} 2^{m\left(2\left(\gamma-\frac{\alpha}{p}\right)+1\right)} \sum_{n=2^m+1}^{2^{m+1}} c_n^2 \le K_2 \sum_{n=1}^{\infty} c_n^2 n^{2\gamma-\frac{2\alpha}{p}+1},$$

which proves (1.12). Furthermore (1.10), (1.11) and (1.12) imply (1.6) plainly, and so the proof is completed.

§ 2. Proof of the theorems

PROOF OF THEOREM 1. By (2) Lemma 2 gives tha

$$|s_{2m}(x)-f(x)| = o_x(2^{-m\gamma}) \tag{2.1}$$

holds a.e. Hence, using the notation $m_\nu=2^m$ if $2^m \leqq \mu_\nu < 2^{m+1}$, we obtain that

$$h_n^p(f, \beta, p, \{\mu_\nu\}; x) \leqq K\frac{1}{n^\beta}\sum_{\nu=0}^{n}(\nu+1)^{\beta-1}(|s_{\mu_\nu}(x)-s_{m_\nu}(x)|^p+|s_{m_\nu}(x)-f(x)|^p). \tag{2.2}$$

Now let $N_n(m)$ denote the number of μ_ν lying in the interval $[2^m, 2^{m+1})$ and $\nu \leqq n$. It is obvious that

$$N_n(m) \leqq \min(n+1, 2^m) \quad \text{and} \quad \sum_{m=0}^{\infty} N_n(m) = n+1. \tag{2.3}$$

If $2^{l-1}<n\leqq 2^l$ then, by (2.1), (2.3) and $0<\gamma p<\beta\leqq 1$, we have that

$$\frac{1}{n^\beta}\sum_{\nu=0}^{n}(\nu+1)^{\beta-1}|s_{m_\nu}(x)-f(x)|^p \leqq \frac{1}{n^\beta}\sum_{m=0}^{\infty} N_n(m)^\beta o_x(2^{-m\gamma p}) \leqq \tag{2.4}$$

$$\leqq \frac{1}{n^\beta}\left\{\sum_{m=0}^{l} 2^{m\beta} o_x(2^{-m\gamma p}) + \sum_{m=l+1}^{\infty} n^\beta o_x(2^{-m\gamma p})\right\} = o_x(n^{-\gamma p}).$$

This shows that if we can prove that

$$\frac{1}{n^\beta}\sum_{\nu=0}^{n}(\nu+1)^{\beta-1}|s_{\mu_\nu}(x)-s_{m_\nu}(x)|^p = o_x(n^{-\gamma p}) \tag{2.5}$$

also holds a.e. then by (2.2) the statement (6) is also proved.

So the rest of the proof is to verify (2.5). This can be proved by Lemma 5. First we divide the series (1) into two series as follows. For any fixed positive ε we choose $\varkappa$ such that

$$\sum_{n=\varkappa}^{\infty} c_n^2 n^{2\gamma} < \varepsilon^3. \tag{2.6}$$

Now let

$$\sum_{n=0}^{\infty} a_n\varphi_n(x) \quad \text{with} \quad a_n = \begin{cases} c_n & \text{for} \quad n \leqq \varkappa \\ 0 & \text{for} \quad n > \varkappa, \end{cases} \tag{2.7}$$

and

$$\sum_{n=0}^{\infty} b_n\varphi_n(x) \quad \text{with} \quad b_n = \begin{cases} 0 & \text{for} \quad n \leqq \varkappa, \\ c_n & \text{for} \quad n > \varkappa. \end{cases} \tag{2.8}$$

Denote $s_n(a; x)$ and $s_n(b; x)$ the n-th partial sums of the series (2.7) and (2.8), respectively.

By the definitions it is clear that $s_n(x)=s_n(a; x)+s_n(b; x)$.

Since the number of the coefficients $a_n\neq 0$ is finite and $p\gamma<\beta$ we can apply Lemma 5 for the series (2.7) and a parameter γ' instead of γ which satisfies the conditions $\gamma'>\gamma$ and $p\gamma'<\beta$. Then we obtain that

$$\sup_{0\leqq n<\infty}\left(\frac{n^{p\gamma'}}{(n+1)^{\beta}}\sum_{\nu=0}^{n}(\nu+1)^{\beta-1}|s_{\mu_\nu}(a;x)-s_{m_\nu}(a;x)|^p\right)^{1/l}$$

is finite a.e., whence

$$\frac{n^{p\gamma}}{(n+1)^{\beta}}\sum_{\nu=0}^{n}(\nu+1)^{\beta-1}|s_{\mu_\nu}(a;x)-s_{m_\nu}(a;x)|^p=o_x(1) \tag{2.9}$$

holds a.e.

On the other hand Lemma 5 applying to (2.8), by (2.6), gives that

$$\int_a^b\left\{\sup_{0\leqq n<\infty}\left(\frac{n^{p\gamma}}{(n+1)^{\beta}}\sum_{\nu=0}^{n}(\nu+1)^{\beta-1}|s_{\mu_\nu}(b;x)-s_{m_\nu}(b;x)|^p\right)^{1/p}\right\}^2dx\leqq K\varepsilon^3.$$

This gives that

$$\text{meas}\left\{x\,|\,\limsup\left(\frac{n^{p\gamma}}{(n+1)^{\beta}}\sum_{\nu=0}^{n}(\nu+1)^{\beta-1}|s_{\mu_\nu}(b;x)-s_{m_\nu}(b;x)|^p\right)^{1/p}>\varepsilon\right\}\leqq K\varepsilon.$$

whence, by (2.9), the statement (2.5) follows a.e.

As we have stated this completes the proof.

PROOF OF THEOREM 2. The proof is similar to that of Theorem 1 and we shall use the notations given there. Here we have that

$$C_n(f,\alpha,p,\{\mu_\nu\};x)\leqq\frac{K}{A_n^{\alpha}}\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha-1}\left(|s_{\mu_\nu}(x)-s_{m_\nu}(x)|^p+|s_{m_\nu}(x)-f(x)|^p\right). \tag{2.10}$$

The second sum in (2.10) can be estimated by (2.1), (2.3) and $p\gamma<\alpha\leqq 1$ as follows

$$\frac{1}{A_n^{\alpha}}\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha-1}|s_{m_\nu}(x)-f(x)|^p\leqq\frac{K}{n^{\alpha}}\sum_{m=0}^{\infty}N_n(m)^{\alpha}o_x(2^{-m\gamma p})\leqq \tag{2.11}$$

$$\leqq\frac{K}{n^{\alpha}}\left\{\sum_{m=0}^{l}2^{m\alpha}o_x(2^{-m\gamma p})+\sum_{m=l+1}^{\infty}n^{\alpha}o_x(2^{-m\gamma p})\right\}=o_x(n^{-\gamma p}),$$

and so we only have to prove that

$$\frac{1}{A_n^{\alpha}}\sum_{\nu=0}^{n}A_{n-\gamma}^{\alpha-1}|s_{\mu_\nu}(x)-s_{m_\nu}(x)|^p=o_x\left(n^{-\gamma p}(\log n)^{\frac{p}{2}-1}\right). \tag{2.12}$$

The proof of (2.12) follows the same line as that of (2.5), the only difference is that we use Lemma 6 (more precisely (1.4)) instead of Lemma 5, therefore we omit the details.

The statement (10) ensues from (2.10), (2.11) and (2.12) obviously.

Next we prove (4) under the assumptions $\alpha<\frac{p}{2}$ and (11). Then Lemma 2, by (11), gives the following estimation

$$|s_{2^m}(x)-f(x)|=o_x\left(2^{-m\left(\gamma-\frac{\alpha}{p}+\frac{1}{2}\right)}\right)$$

instead of (2.1), but this is sharper on account of $\alpha<\frac{p}{2}$ than (2.1) is, so we can use the estimation (2.1) during the proof. Consequently (2.11) holds. In order to prove (4) it is sufficient to verify that

$$\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|s_{\mu_\nu}(x)-s_{m_\nu}(x)|^p = o_x(n^{-\gamma p}) \tag{2.13}$$

also holds a.e. in (a, b). But (2.13) follows from (1.6) repeating the same considerations as we have made in the proof of (2.5).

Summing up our estimations we obtain (4).

Thus the proof is complete.

Proof of Theorem 3. First of all it is clear that if $p\geqq 2$ then the condition $p\gamma<\alpha(\leqq 1)$ implies that $\gamma<1/2$; thus we can apply Theorem A and obtain the estimation (3), whence, by Lemma 1, (9) follows. If $p<2$ then in view of Theorem B we have only to prove (9) for such parameters α to be varying in the interval $\gamma p<\alpha\leqq\frac{p}{2}$. But then the parameter γ is also less than 1/2, so we can conclude to (9) as before, and this ends the proof.

Proof of Theorem 4. We prove (9) by means of Theorem 3 and Theorem B. Namely we show that if $\alpha\leqq 1$, moreover

$$\alpha' = \frac{2}{p}(\alpha-1)+1 \quad\text{and}\quad \gamma' = \gamma+\frac{1}{2}-\frac{1}{p} \tag{2.14}$$

then the assumptions of Theorem 3 with $p'=2$ and α', γ' instead of p, α and γ are satisfied. Indeed, it is clear by $\alpha\leqq 1$ and $p\geqq 2$ that $0<\alpha'\leqq 1$, and by $p\gamma<\alpha$

$$2\gamma'=2\gamma+1-\frac{2}{p}<\frac{2}{p}\alpha+1-\frac{2}{p}=\alpha',$$

moreover on account of (12)

$$\sum_{n=1}^{\infty} c_n^2 n^{2\gamma'} = \sum_{n=1}^{\infty} c_n^2 n^{2\gamma+1-\frac{2}{p}} < \infty.$$

If $\alpha>1$ then the assumptions of Theorem B are satisfied with $p'=2$ and α', γ' given under (2.14). Namely by $p\gamma<1$ $2\gamma'=2\gamma+1-\frac{2}{p}<1$, and $\alpha'>1$ plainly.

Next we use the following well-known inequality

$$\sum_{n=0}^{\infty} a_n^r \leqq \left(\sum_{n=0}^{\infty} a_n\right)^r \quad\text{for}\quad r\geqq 1 \quad\text{and}\quad a_n\geqq 0; \tag{2.15}$$

and the following properties of the binomial-coefficients

$$0<Kn^\alpha\leqq A_n^\alpha\leqq K_1 n^\alpha \quad\text{for}\quad \alpha>-1. \tag{2.16}$$

These estimations ensure that

$$\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|s_{k_\nu}(x)-f(x)|^p \leqq \left(\sum_{\nu=0}^{n} (A_{n-\nu}^{\alpha-1})^{\frac{2}{p}} |s_{k_\nu}(x)-f(x)|^2\right)^{p/2} \leqq$$

$$\leqq K\left(\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha'-1}|s_{k_\nu}(x)-f(x)|^2\right)^{\frac{p}{2}}.$$

Hence we obtain that

$$(2.17)\qquad C_n(f, \alpha, p, \{k_\nu\}; x) \leqq K\frac{1}{(A_n^\alpha)^{1/p}}\left(\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha'-1}|s_{k_\nu}(x)-f(x)|^2\right)^{1/2} \leqq$$

$$\leqq K_1 n^{\frac{\alpha'}{2}-\frac{\alpha}{p}}\left(\frac{1}{A_n^{\alpha'}}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha'-1}|s_{k_\nu}(x)-f(x)|^2\right)^{1/2}.$$

Now either Theorem 3 or Theorem B gives that $C_n(f, \alpha', 2, \{k_\nu\}; x)=o_x(n^{-\gamma'})$, and thus by (2.17)

$$C_n(f, \alpha, p, \{k_\nu\}; x) = o_x\left(n^{\frac{\alpha'}{2}-\frac{\alpha}{p}-\gamma'}\right) = o_x(n^{-\gamma})$$

holds a.e. in (a, b) in accordance with our statement.

Proof of Theorem 5. The proof runs similarly to that of Theorem 4, but now we shall use the results of Theorem 1 and Proposition A with $p'=2$, $\beta'=\frac{2}{p}(\beta-1)+1$ and $\gamma'=\gamma+\frac{1}{2}-\frac{1}{p}$. An easy calculation shows that $\beta', \gamma'>0$, and if $\beta\leqq 1$ then $2\gamma'=2\gamma+1-\frac{2}{p}<\beta'\leqq 1$, and for $\beta>1$, by $p\gamma<1$, $2\gamma'<1<\beta$.

So we have verified that with these parameters β', γ' and $p'=2$ the assumptions required to the estimation (6) are satisfied, thus we have the following estimation

$$(2.18)\qquad h_n(f, \beta', 2, \{\mu_\nu\}; x) = o_x(n^{-\gamma'})$$

a.e. in (a, b). Consequently, using inequalities (2.15) and (2.18), we obtain that

$$\sum_{\nu=0}^{n} (\nu+1)^{\beta-1}|s_{\mu_\nu}(x)-f(x)|^p \leqq \left(\sum_{\nu=0}^{n} (\nu+1)^{\frac{2}{p}(\beta-1)}|s_{\mu_\nu}(x)-f(x)|^2\right)^{p/2} \leqq$$

$$\leqq Kh_n^p(f, \beta', 2, \{\mu_{\nu_\nu}\}; x)\, n^{\beta'\frac{p}{2}} = o_x(n^{\beta-\gamma p})$$

holds a.e., whence (6) obviously follows.

We have completed our proof.

Proof of Theorem 6. Now Theorem 2 and Theorem B give the kernel of the proof. If α' and γ' have the same meaning as in (2.14) and $p'=2$, then an elementary calculation shows that for $\alpha<1$ the assumptions of Theorem 2 are fulfilled; and if $\alpha\geqq 1$ then the conditions of Theorem B hold with the parameters α', γ' and p'. Thus we always have the estimation $C_n(f, \alpha', 2, \{\mu_\nu\}; x) = o_x(n^{-\gamma'})$. On the other hand by (2.17)

$$C_n(f, \alpha, p; \{k_\nu\}: x) \leqq Kn^{\frac{\alpha'}{2}-\frac{\alpha}{p}} C_n(f, \alpha', 2, \{\mu_\nu\}; x).$$

These estimations together prove the statement of Theorem 6.

PROOF OF THEOREM 7. First we prove that if $p \leqq 2$ then (13), (14) and (15) imply (16). It is clear that if we can show that

$$\Sigma_n^{\alpha, p} := \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |s_{\mu_\nu}(x) - f(x)|^p = O_x(1) \tag{2.19}$$

then (16) follows.

Denote

$$[A_{n-\nu}^{\alpha-1}]^+ = \begin{cases} A_{n-\nu}^{\alpha-1} & \text{for} \quad \nu \leqq n, \\ 0 & \text{for} \quad \nu > n. \end{cases}$$

Using this notation and the Hölder inequality we obtain that

$$\Sigma_n^{\alpha, p}(x) = \sum_{m=0}^{\infty} \sum_{2^m \leqq \mu_\nu < 2^{m+1}} [A_{n-\nu}^{\alpha-1}]^+ |s_{\mu_\nu}(x) - f(x)|^p \leqq \tag{2.20}$$

$$\leqq \sum_{m=0}^{\infty} \Big(\sum_{2^m \leqq \mu_\nu < 2^{m+1}} |s_{\mu_\nu}(x) - f(x)|^2 \Big)^{p/2} \Big(\sum_{2^m \leqq \mu_\nu < 2^{m+1}} ([A_{n-\nu}^{\alpha-1}]^+)^{\frac{2}{2-p}} \Big)^{1-\frac{p}{2}}.$$

For $\alpha \leqq 1$ an elementary calculation gives that for any n

$$\sum_{2^m \leqq \mu_\nu < 2^{m+1}} ([A_{n-\nu}^{\alpha-2}]^+)^{\frac{2}{2-p}} \leqq K \sum_{\nu=1}^{2^m} \nu^{(\alpha-1)\frac{2}{2-p}} \equiv P_m^{\alpha, p}. \tag{2.21}$$

If (13) holds then $P_m^{\alpha, p} \leqq K$ and so by (2.20) and (2.21)

$$\Sigma_n^{\alpha, p}(x) = \sum_{m=0}^{\infty} \Big(\sum_{k=2^m}^{2^{m+1}} |s_k(x) - f(x)|^2 \Big)^{p/2} \equiv \Sigma_p(x). \tag{2.22}$$

Next we show that under the condition (13) the series $\Sigma_p(x)$ converges a.e. in (a, b). Namely, by Lemma 3 and (13), the integral of $\Sigma_p(x)$ is finite, since

$$\sum_{m=0}^{\infty} \int_a^b \Big(\sum_{k=2^m}^{2^{m+1}-1} |s_k(x) - f(x)|^2 \Big)^{p/2} \leqq K \sum_{m=0}^{\infty} 2^{m\frac{p}{2}} E_{2^m}^p \leqq K \sum_{n=1}^{\infty} n^{\frac{p}{2}-1} E_n^p,$$

and thus the Beppo Levi theorem ensures the convergence of the series $\Sigma_p(x)$ a.e. This and (2.22) imply (2.19).

Under assumption (14) $P_m^{\alpha, p} \leqq Km$, and then (2.20) and (2.21) give that

$$\Sigma_n^{\alpha, p}(x) \leqq \sum_{m=0}^{\infty} \Big(\sum_{k=2^m}^{2^{m+1}-1} |s_k(x) - f(x)|^2 \Big)^{p/2} m^{1-\frac{p}{2}} \equiv \Sigma_p^*(x).$$

If we integrate term by term the last series we obtain by (14) and Lemma 3, that

$$\int_a^b \sum\nolimits_p^*(x)\,dx \leqq K \sum_{m=0}^{\infty} 2^{m-\frac{2}{p}} E_{2^m}^p m^{1-\frac{p}{2}} \leqq K \sum_{n=1}^{\infty} n^{\frac{p}{2}-1} (\log n)^{1-\frac{p}{2}} E_n^p < \infty.$$

and hence we can get (2.19) as before.

If (15) holds then $P_m^{\alpha,p} \leqq K\, 2^{m\left(\alpha-\frac{p}{2}\right)}$ and thus we have to show that

$$I \equiv \int_a^b \left\{ \sum_{m=0}^{\infty} \left(\sum_{k=2^m}^{2^{m+1}-1} |s_k(x)-f(x)|^2 \right)^{p/2} 2^{m\left(\alpha-\frac{p}{2}\right)} \right\} dx < \infty.$$

But Lemma 3 and (15) imply that

$$I \leqq K \sum_{m=0}^{\infty} 2^{m\alpha} E_{2^m}^p \leqq K_1 \sum_{n=1}^{\infty} n^{\alpha-1} E_n^p < \infty$$

and this gives the way to conclude to (2.19).

Finally we prove the implication (17)⇒(16), or what is the same we prove that (2.19) follows from (17).

By (2.15) and (2.16) we obtain that

$$\sum\nolimits_n^{\alpha,p}(x) \leqq \left(\sum_{\nu=0}^{n} (A_{n-\nu}^{\alpha-1})^{\frac{2}{p}} |s_{\mu_\nu}(x)-f(x)|^2 \right)^{\frac{p}{2}} \leqq \tag{2.23}$$

$$\leqq K \left(\sum_{\nu=0}^{n} A_{\nu-n}^{\alpha'-1} |s_{\mu_\nu}(x)-f(x)|^2 \right)^{\frac{p}{2}} \equiv K\left(\sum\nolimits_n^{\alpha',2}(x)\right)^{\frac{p}{2}},$$

where $\alpha' = \dfrac{2}{p}(\alpha-1)+1$.

It is clear that $0<\alpha'\leqq 1$ and thus by (17) with $\alpha=\alpha'$ and $p=2$ the conditions under (13) are satisfied, therefore $\sum_n^{\alpha',2}(x)=O_x(1)$ a.e. in (a,b). This and (2.23) prove the implication (17)⇒(16) obviously.

Thus Theorem 7 is proved.

Proof of Theorem 8. In order to prove (20) we show that

$$\sum\nolimits^{\beta,p}(x) = \sum_{\nu=0}^{\infty} (\nu+1)^{\beta-1} |s_{\mu_\nu}(x)-f(x)|^p = O_x(1) \quad \text{a.e.} \tag{2.24}$$

In view of $\beta\leqq 1$ and (18), in the case $p=2$ (2.24) can be proved easily, namely if we integrate the series $\Sigma^{\beta,p}(x)$ term by term then

$$\int_a^b \sum\nolimits^{\beta,p}(x)\,dx \leqq \sum_{\nu=0}^{\infty} (\nu+1)^{\beta-1} E_{\mu_\nu}^2 \leqq \sum_{\nu=0}^{\infty} (\nu+1)^{\beta-1} E_\nu^2 < \infty,$$

and thus Beppo Levi theorem proves (2.24).

If $p<2$ then first we use the Hölder's inequality as follows

$$\int_a^b \sum^{\beta,p}(x)\,dx = \sum_{\nu=0}^{\infty}(\nu+1)^{\beta-1}\int_a^b |s_{\mu_\nu}(x)-f(x)|^p\,dx \leqq$$

$$\leqq K\sum_{\nu=0}^{\infty}(\nu+1)^{\beta-1}\Big(\int_a^b |s_{\mu_\nu}(x)-f(x)|^2\,dx\Big)^{p/2} \leqq$$

$$\leqq K\sum_{\nu=0}^{\infty}(\nu+1)^{\beta-1}E_{\mu_\nu}^p \leqq K\sum_{\nu=0}^{\infty}(\nu+1)^{\beta-1}E_\nu^p < \infty,$$

and this also implices (2.24) and (20).

If $p\geqq 2$ then first we define β' by $\beta'=(\beta-1)\dfrac{2}{p}+1$. Then by (2.15) we have obviously that

$$\text{(2.25)} \qquad \sum^{\beta,p}(x) \leqq \Big(\sum_{\nu=0}^{\infty}(\nu+1)^{(\beta-1)\frac{2}{p}}|s_{\mu_\nu}(x)-f(x)|^2\Big)^{\frac{p}{2}} \leqq$$

$$\leqq \Big(\sum_{\nu=0}^{\infty}(\nu+1)^{\beta'-1}|s_{\mu_\nu}(x)-f(x)|^2\Big)^{\frac{p}{2}} = \big(\sum^{\beta',2}(x)\big)^{\frac{p}{2}}.$$

The assumptions $0<\beta\leqq 1$ and $p\geqq 2$ imply that $0<\beta'\leqq 1$ and thus the conditions given in (18) with β' and $p'=2$ are satisfied, whence $\sum^{\beta',2}(x)=O_x(1)$ holds a.e. in (a, b) and this, by (2.25), proves (2.24) for $p\geqq 2$, too. The proof is complete.

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SOME EXAMPLES OF SUPERNILPOTENT NON-SPECIAL RADICALS

By

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For a given associative ring A, we will denote by $\bar{A}$ any nonzero homomorphic image of A, and write $I \lhd A$ when I is a nonzero ideal of A. For a given class $\mathbf{M}$ we have as usual the dual definitions $U\mathbf{M}=\{A/\text{all } \bar{A} \notin \mathbf{M}\}$ and $S\mathbf{M}=\{A/\text{if } I \lhd A \text{ then } I \notin \mathbf{M}\}$. Denote $\mathbf{R}(A)=\bigcap\{I/I \lhd A, A/I \in S\mathbf{R}\}$ for a given radical class $\mathbf{R}$ and a ring A. For the basic notions and results of the radical theory we refer to [2] and [10].

The first example of supernilpotent non-special radical was given by J. M. RJABUHIN [7]. Later several authors were interested in problems of construction "sufficiently many" supernilpotent non-special radicals which satisfy some additional requirement. For example L. C. A. VAN LEEUWEN and T. L. JENKINS [4] have given countably many supernilpotent non-special radicals such that their semisimple classes contain all fields and J. M. RJABUHIN (see [8] or [2] p. 252, Theorem 5) has shown that every supernilpotent radical different from lower nil radical is the union of supernilpotent non-special radicals.

F. SZÁSZ posed the following problem ([10], Problem 17): is it possible to construct a countable infinite set of supernilpotent non-special radical classes $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_n, \ldots$ such that $S\mathbf{R}_m \cap S\mathbf{R}_n=\{0\}$ for $m \neq n$? Note that F. SZÁSZ [9] has constructed a countable infinite set of special radical classes $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_n, \ldots$ such that $S\mathbf{R}_m \cap \cap S\mathbf{R}_n=\{0\}$ for $n \neq m$.

In the present note we give an affirmate answer for this question of F. Szász.

Let K be a subset of a ring A. We denote $\mathrm{Ann}_A K=\{a \in A/aK=0=Ka\}$. A class $\mathbf{M}$ of rings is called a weakly special class of rings if it satisfies the following three conditions:

(a) Every ring in the class $\mathbf{M}$ is a semiprime ring.

(b) Every nonzero ideal of a ring in $\mathbf{M}$ is itself a ring in $\mathbf{M}$.

(c) If I is a ring in the class $\mathbf{M}$, I is an ideal of a ring A and $\mathrm{Ann}_A I=0$ then $A \in \mathbf{M}$ (see [10] p. 67, or [2] p. 171).

A radical class $\mathbf{R}$ is called supernilpotent if it is hereditary and contains all nilpotent rings.

THEOREM 1 (see [10] p. 67, or [2] p. 172). *The radical class* $\mathbf{R}$ *is supernilpotent iff* $S\mathbf{R}$ *is a weakly special class. If* $\mathbf{M}$ *is a weakly special class and* $\mathbf{R}=U\mathbf{M}$ *then* $\mathbf{R}(A)= =\bigcap\{I/I$ *is an ideal of the ring* A *and* $A/I \in \mathbf{M}\}$ *for all rings* A.

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THEOREM 2 (see [10] p. 72 or [2] p. 232). *A supernilpotent radical class* **R** *is special iff every* **R**-*semisimple nonzero ring has an* **R**-*semisimple nonzero prime homomorphic image.*

COROLLARY 3. *A supernilpotent radical class* **R** *is not special iff there exists an* **R**-*semisimple nonzero ring every homomorphic prime image of which is not* **R**-*semisimple.*

Let A be a ring and $\mathrm{St}_n(X)$ be the standard identity of degree n (see [3], p. 328). Denote $d(A)=\min\{n/\mathrm{St}_{2n}(X)$ is an identity of the ring $A\}$. Let $\mathbf{N}_n$ be the class of all rings such that:

(i) Every ring in the class $\mathbf{N}_n$ is a semiprime ring.

(ii) If $A\in\mathbf{N}_n$ and $I\lhd A$ then $d(I)=n$.

(iii) If $A\in\mathbf{N}_n$ and $I\lhd A$ then the ring I is not a prime ring.

LEMMA 4 (see [6] Theorem A or [5] Corollary 1). *Let A be a prime ring, and let $C(A)$ be the centre of the ring A. If* $\mathrm{St}_n(X)$ *is a polynomial identity of the ring A, then $C(A)\neq 0$.*

LEMMA 5. *Let A be a semiprime ring, $I\lhd A$,* $\mathrm{Ann}_A I=0$ *and $d(I)=n$. Then $d(A)=n$.*

PROOF. 1) Suppose that A is a prime ring and $\mathrm{St}_{2n}(X)$ is a polynomial identity of the ring I. We shall show that $\mathrm{St}_{2n}(X)$ is a polynomial identity of the ring A. Indeed, by Lemma 4, $C(I)\neq 0$. Let $S=C(I)\setminus\{0\}$. If $s,t\in S$ and $a\in A$, then $ta\in I$ and

$$t(sa-as)=tsa-(ta)s=tsa-s(ta)=tsa-tsa=0.$$

But a prime ring has no nonzero central zero-divisors. Hence $sa=as$ for all $s\in S$, $a\in A$ and $S\subseteq C(A)$. Let $S^{-1}A$ be a ring of fractions. It is clear that $A\subseteq S^{-1}A$. Further, let $S_i^{-1}a_i\in S^{-1}A$, where $s_i\in S$, $a_i\in A$ for all $i=1,2,\dots,2n$. Then

$$\mathrm{St}_{2n}(s_1^{-1}a_1,\dots,s_{2n}^{-1}a_{2n})=(s_1^2\dots s_{2n}^2)^{-1}St_{2n}(s_1a_1,\dots,s_{2n}a_{2n})=0$$

because $s_ia_i\in I$ for all $i=1,2,\dots,2n$. Hence $\mathrm{St}_{2n}(X)$ is an identity for rings $S^{-1}A$ and A.

2) Let **R** be the lower radical class generated by the class of all nilpotent rings. It is well-known that **R** is a hereditary radical class. Further, let $\mathscr{P}=\{P\lhd A/P$ is a prime ideal$\}$, $\mathscr{P}_1=\{P\in\mathscr{P}/P\not\supseteq I\}$ and $L=\cap\{P/P\in\mathscr{P}_1\}$. It is well-known, that

$$\mathbf{R}(A)=\bigcap_{P\in\mathscr{P}}P,\quad \mathbf{R}(I)=\mathbf{R}(A)\cap I=(\bigcap_{P\in\mathscr{P}_1}P)\cap I=L\cap I.$$

Since A is a semiprime ring, $\mathbf{R}(A)=\mathbf{R}(I)=0$. Hence $L\cap I=0$. But $\mathrm{Ann}_A I=0$. Therefore $L=0$. We have $d(I)=n$. Hence $\mathrm{St}_{2n}(X)$ is a polynomial identity of the ring I. Let $P\in\mathscr{P}_1$, $\bar{A}=A/P$, $\bar{I}=I/I\cap P$. It is clear that $\bar{I}\lhd\bar{A}$ and $\mathrm{St}_{2n}(X)$ is a polynomial identity of the ring $\bar{I}$. By (1) $\mathrm{St}_{2n}(X)$ is an identity of the ring $\bar{A}$. Since $\cap\{P/P\in\mathscr{P}_1\}=L=0$, the ring A is a subdirect product of rings which satisfy the polynomial identity $\mathrm{St}_{2n}(X)$. Hence $\mathrm{St}_{2n}(X)$ is a polynomial identity of the ring A and $d(A)\leqq n$. On the other hand $d(A)\geqq d(I)=n$. Thus $d(A)=n$.

THEOREM 6. *The class* $\mathbf{N}_n$ *satisfies the following conditions:*

1) *The class* $\mathbf{N}_n$ *is nonzero for all* $n=1, 2, 3, \ldots$.

2) *The class* $\mathbf{N}_n$ *is a weakly special class.*

3) *The radical class* $\mathbf{R}_n=U\mathbf{N}_n$ *determined by the weakly special class* $\mathbf{N}_n$ *is a supernilpotent non-special radical class.*

4) *If* $A\in S\mathbf{R}_n$ *and* $A\neq 0$ *then* $d(A)=n$.

5) $S\mathbf{R}_m\cap S\mathbf{R}_n$ *for all* $m\neq n$.

PROOF. 1) Let B be a Boolean ring without ideals with two elements (such rings were used by Rjabuhin in the construction of a supernilpotent non-special radical (see [7]) and $A=M_n(B)$ is the ring of $n\times n$ matrices over B. Since B is a commutative ring, $d(A)=n$ (see [1]). Clearly, A is a semiprime ring. Let I be an ideal of the ring A. Then $I=M_n(L)$, where L is an ideal of B. Thus $d(I)=d(M_n(L))=n$. If I is a prime ring, then L is also a prime ring. Since B is a Boolean ring without ideals with two elements $L=0$ and $I=0$. Remark that $\mathbf{N}_n\neq\{0\}$ because $A\in\mathbf{N}_n$.

2) Clearly, the class $\mathbf{N}_n$ has the property (a) from the definition of the weakly special class. Let I be a nozero ideal of the ring A and $A\in\mathbf{N}_n$. We shall show that $I\in\mathbf{N}_n$. Let L be a nonzero ideal of the ring I and let $\bar{L}$ be an ideal of the ring A generated by L. It follows that $\bar{L}^3\subseteq L$ (see [10] p. 35). Since $A\in\mathbf{N}_n$, A is a semiprime ring and so $\bar{L}^3\neq 0$, $d(I)=n=d(\bar{L}^3)$. Consider the inclusion $\bar{L}^3\subseteq L\subseteq I$. It follows that $n=d(\bar{L}^3)\leqq d(L)\leqq d(I)=n$ and the condition (ii) is satisfied.

If L is a prime ring then so is $\bar{L}^3$. But $\bar{L}^3$ is an ideal of the ring A and this contradicts $A\in\mathbf{N}_n$. Hence condition (iii) is satisfied and $I\in\mathbf{N}_n$. Therefore condition (b) is also satisfied for the class $\mathbf{N}_n$.

Let I be a nonzero ideal of the ring A, $\mathrm{Ann}_A I=0$ and $I\in\mathbf{N}_n$. We shall prove that $A\in\mathbf{N}_n$ and condition (c) holds. Clearly, A is a semiprime ring. So condition (i) holds. Let $L\lhd A$. Since $I\in\mathbf{N}_n$, $d(I)=n$ and $d(L\cap I)=n$. By Lemma 5, $d(A)=n$. We have $L\cap I\subseteq L\subseteq A$. Hence $n=d(L\cap I)\leqq d(L)\leqq d(A)=n$ and $d(L)=n$. Thus (ii) holds.

If L is a prime ring then so is $L\cap I$. But $L\cap I$ is an ideal of the ring I and $I\in\mathbf{N}_n$. Therefore $L\cap I=0$, $LI=0$ and $L\subseteq\mathrm{Ann}_A I=0$. This contradicts $L\neq 0$. Hence the ring A satisfies condition (iii) and $A\in\mathbf{N}_n$. Then $\mathbf{N}_n$ is a weakly special class becasue conditions (a), (b), (c) hold.

3) Let A be the ring which was constructed in step 1 of our proof. We have proved that $A\in\mathbf{N}_n$. Hence $A\in S\mathbf{R}_n$. If P is a prime ideal of the ring A, then $P=$ $=M_n(L)$, where L is a prime ideal of the ring B. Clearly, $A/P=M_n(B/L)$ and B/L is a two element field. Therefore A/P is a simple ring. If $A/P\in S\mathbf{R}_n$, then A/P is a subdirect product of rings from the class $\mathbf{N}_n$. Then $A/P\in\mathbf{N}_n$, because A/P is a simple ring. But the class $\mathbf{N}_n$ has no prime rings. This is a contradiction. Hence $A/P\notin S\mathbf{R}_n$. By Corollary 3, $\mathbf{R}$ is not a special radical class.

4) Let A be a ring and $A\in S\mathbf{R}_n$. By Theorem 1, A is a subdirect product of rings from the class $\mathbf{N}_n$. Hence $d(A)=n$.

5) It is clear now that $S\mathbf{R}_m\cap S\mathbf{R}_n=\{0\}$, $m\neq n$.

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A REMARK ON A PROBLEM OF A. RÉNYI

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Introduction

In 1952, A. Rényi raised the following question: Let $f\in C[a, b]$ $(a, b\in \mathbf{R}, a<b)$ be strictly positive, further let

$$a = x_{0,n} < x_{1,n} < \ldots < x_{K(n),n} = b$$

be a system of partitions of the interval $[a, b]$ into $K(n)$ subintervals, with the property

$$\lim_{n\to\infty} \max_i (x_{i,n} - x_{i-1,n}) = 0.$$

Introducing the quantities

$$s_{i,n} = \frac{\int\limits_{x_{i-1,n}}^{x_{i,n}} xf(x)\,dx}{\int\limits_{x_{i-1,n}}^{x_{i,n}} f(x)\,dx} \qquad (i = 1, 2, \ldots, K(n),\ n = 1, 2, \ldots)$$

the question is whether these numbers $s_{i,n}$ $(i=1, 2, \ldots, K(n);\ n=1, 2, \ldots)$ uniquely determine the function f up to a constant factor.

This question was partially solved by I. VINCZE [1], who proved that the answer is affirmative in the case f is differentiable in $[a, b]$.

In the present paper we give a positive answer for a special class of partitions and arbitrary f. On the other hand, a counterexample will be given which shows that the answer is in general negative.

Results

THEOREM. *Let*

$$a = x_{0,n} < x_{1,n} < \ldots < x_{K(n),n} = b$$

be a system of partitions of the finite interval $[a, b]$, *with the following properties:*

A) $\lim\limits_{n\to\infty} \max\limits_i (x_{i,n} - x_{i-1,n}) = 0,$

B) $x_{i,n}$ *is an element of the set* $\{x_{0,n+1}, x_{1,n+1}, \ldots, x_{K(n+1),n+1}\}$ *for all* $n\in\mathbf{N}$, $0\leqq i\leqq K(n)$, *and in all open intervals* $(x_{i,n}, x_{i+1,n})$ *we have at most one element of* $\{x_{0,n+1}, x_{1,n+1}, \ldots, x_{K(n+1),n+1}\}$.

C) $K(1)=1$.

Then the values

$$s_{i,n} = \frac{\int_{x_{i-1,n}}^{x_{i,n}} xf(x)\,dx}{\int_{x_{i-1,n}}^{x_{i,n}} f(x)\,dx} \qquad i = 1, \ldots, K(n),\ n = 1, 2, \ldots$$

uniquely determine the strictly positive function $f \in C[a, b]$ with the additional assumption

$$1 = \int_a^b f(x)\,dx.$$

PROOF. For the proof, it is enough to show that the values $s_{i,n}$ uniquely determine the integrals

$$H_{i,n} \equiv \int_{x_{i-1,n}}^{x_{i,1}} f(x)\,dx \quad (1 \leqq i \leqq K(n),\ n = 1, 2, \ldots).$$

This can be proved by induction for n. For $n=1$,

$$H_{1,1} = \int_{x_{0,1}}^{x_{1,1}} f(x)\,dx = \int_a^b f(x)\,dx = 1.$$

In the inductional step we distinguish two cases.

Case I. $x_{i-1,n+1} = x_{j-1,n}$ and $x_{i,n+1} = x_{j,n}$ for some $1 \leqq j \leqq K(n)$. Then $H_{i,n+1} = H_{j,n}$.

Case II. $x_{i-1,n+1} = x_{j-1,n}$ and $x_{i+1,n+1} = x_{j,n}$ for some $1 \leqq j \leqq K(n)$. In this case we have the following equations:

$$H_{i,n+1} + H_{i+1,n+1} = H_{j,n},$$

$$s_{i,n+1}H_{i,n+1} + s_{i+1,n+1}H_{i+1,n+1} = s_{j,n}H_{j,n}.$$

Due to the elementary fact $s_{i,n+1} \neq s_{i+1,n+1}$ this linear system uniquely determines the values $H_{i,n+1}$ and $H_{i+1,n+1}$, hence the theorem is proved.

COUNTEREXAMPLE. *There exists a continuous non-constant function $f>0$ defined on the interval* $[0, 1]$ *and a partition of the interval satisfying Rényi's requirements such that for all intervals* $(x_{i-1,n}, x_{i,n})$,

$$s_{i,n} = \frac{\int_{x_{i-1,n}}^{x_{i,n}} xf(x)\,dx}{\int_{x_{i-1,n}}^{x_{i,n}} f(x)\,dx} = \frac{1}{2}(x_{i-1n} + x_{in}) \quad (1 \leqq i \leqq K(n),\ n = 1, 2, \ldots)$$

i.e. f has the same mass centers on the intervals of the system as the constant function

PROOF. Let the numbers $x_{i,n}$ be defined in the following way:

$$x_{0,1}=0,\quad x_{1,1}=\frac{1}{100},\quad x_{2,1}=\frac{1}{10},\quad x_{3,1}=\frac{9}{10},\ x_{4,1}=\frac{99}{100},\ x_{5,1}=1.$$

For $n>1,\ 0\leqq i\leqq 5$,

$$x_{i,n}=x_{\left[\frac{i}{5}\right],n-1}\quad \text{if}\quad i\equiv 0 (\operatorname{mod} 5),$$

$$x_{i,n}=x_{\left[\frac{i}{5}\right],n-1}+\frac{1}{100}\left(x_{\left[\frac{i}{5}+1\right],n-1}-x_{\left[\frac{i}{5}\right],n-1}\right)\quad \text{if}\quad i\equiv 1 (\operatorname{mod} 5),$$

$$x_{i,n}=x_{\left[\frac{i}{5}\right],n-1}+\frac{1}{10}\left(x_{\left[\frac{i}{5}+1\right],n-1}-x_{\left[\frac{i}{5}\right],n-1}\right)\quad \text{if}\quad i\equiv 2 (\operatorname{mod} 5),$$

$$x_{i,n}=x_{\left[\frac{i}{5}\right],n-1}+\frac{9}{10}\left(x_{\left[\frac{i}{5}+1\right],n-1}-x_{\left[\frac{i}{5}\right],n-1}\right)\quad \text{if}\quad i\equiv 3 (\operatorname{mod} 5),$$

$$x_{i,n}=x_{\left[\frac{i}{5}\right],n-1}+\frac{99}{100}\left(x_{\left[\frac{i}{5}+1\right],n-1}-x_{\left[\frac{i}{5}\right],n-1}\right)\quad \text{if}\quad i\equiv 4 (\operatorname{mod} 5).$$

We shall construct positive step functions $h_1, h_2, \ldots$ such that

$$\text{(i)}\qquad \frac{\int\limits_{x_{i-1,K}}^{x_{i,K}} x h_n(x)\,dx}{\int\limits_{x_{i-1,K}}^{x_{i,K}} h_n(x)\,dx}=\frac{1}{2}(x_{i-1,K}+x_{i,K})$$

for all $1\leqq K\leqq n,\ 0<i\leqq 5^K$.

(ii) All h_n are constants on the intervals $(x_{i-1,n}, x_{i,n})$ for $0<i\leqq 5^n$.

$$\text{(iii)}\qquad \lim_{n\to\infty}\max_{1\leqq i\leqq 5^n-1}|h_n(x_{i,n}-0)-h_n(x_{i,n}+0)|=0.$$

$$\text{(iv)}\qquad \lim_{n,m\to\infty}\max_{x\in[0,1]}|h_n(x)-h_m(x)|=0.$$

$$\text{(v)}\qquad h_n(0)=10,\quad h_n(1)=11\quad \text{for all}\quad n\in\mathbf{N}.$$

$$\text{(vi)}\qquad h_n(x)\geqq 1\quad \text{for all}\quad n\in\mathbf{N},\quad x\in[0,1].$$

(i)–(vi) imply (using well-known results of the classical analysis) that h_n converges uniformly to a function f, which has the desired properties.

$$h_1(x) = \begin{cases} 10 & \text{if } x \in \left[0, \frac{1}{100}\right] \\ 10.5 + \frac{495}{8010} & \text{if } x \in \left(\frac{1}{100}, \frac{1}{10}\right] \\ 10.5 & \text{if } x \in \left(\frac{1}{10}, \frac{9}{10}\right] \\ 10.5 - \frac{495}{8010} & \text{if } x \in \left(\frac{9}{10}, \frac{99}{100}\right] \\ 11 & \text{if } x \in \left(\frac{99}{100}, 1\right] \end{cases}$$

$$h_{n+1}(x) = \begin{cases} 10 & \text{if } x \in [0, x_{1,n+1}] \\ 11 & \text{if } x \in (x_{5^{n+1}-1,n+1}, 1] \\ c_1 & \text{if } x \in (x_{i-1,n+1}, x_{i,n+1}] (i \equiv 1 (\mathrm{mod}\ 5), i > 1) \\ c_2 & \text{if } x \in (x_{i-1,n+1}, x_{i,n+1}] (i \equiv 2 (\mathrm{mod}\ 5)) \\ c_3 & \text{if } x \in (x_{i-1,n+1}, x_{i,n+1}] (i \equiv 3 (\mathrm{mod}\ 5)) \\ c_4 & \text{if } x \in (x_{i-1,n+1}, x_{i,n+1}] (i \equiv 4 (\mathrm{mod}\ 5)) \\ c_5 & \text{if } x \in (x_{i-1,n+1}, x_{i,n+1}] (i \equiv 0 (\mathrm{mod}\ 5),\ i < 5^{n+1}) \end{cases}$$

where

$$c_1 = \frac{2}{3} h_n(x_{i-1,n+1}+0) + \frac{1}{3} h_n(x_{i-1,n+1}-0),$$

$$c_5 = \frac{2}{3} h_n(x_{i,n+1}-0) + \frac{1}{3} h_n(x_{i,n+1}+0),$$

$$c_2 = h_n(x_{i,n+1}) - \frac{495}{8010}(c_1 - c_5),$$

$$c_4 = h(x_{i,n+1}) + \frac{495}{8010}(c_1 - c_5),$$

$$c_3 = h_n(x_{i,n+1}) + \frac{1}{80}\big(h_n(x_{i-2,n+1}+0) - c_1\big) + \frac{1}{80}\big(h_n(x_{i+2,n+1}-0) - c_5\big).$$

It is easy to show that all h_n satisfy (ii).

(i) follows from (ii) and

$$\int_{x_{i-1,K}}^{x_{i,K}} x h_l(x)\,dx = \int_{x_{i-1,K}}^{x_{i,K}} x h_{l+1}(x)\,dx, \quad \int_{x_{i-1,K}}^{x_{i,K}} h_l(x)\,dx = \int_{x_{i-1,K}}^{x_{i,K}} h_{l+1}(x)\,dx$$

for all $l \geqq k$, which can be checked easily.

Also, an elementary computation shows that

$$\max_{1\leqq i\leqq 5^{n+1}-1}|h_{n+1}(x_{i,n+1}-0)-h_{n+1}(x_{i,n+1}+0)|\leqq \tag{1}$$

$$\leqq\frac{9}{10}\max_{1\leqq i\leqq 5^{n}-1}|h_{n}(x_{i,n}-0)-h_{n}(x_{i,n}+0)|.$$

This clearly implies (iii). (v) is straightforward.

The following relations can be derived by means of the definitions of $c_1, c_2, c_3, c_4, c_5, h_n, h_{n+1}$.

$$|c_1-h_n(x_{i,n+1}+0)|=\frac{1}{3}|h_n(x_{i,n+1}+0)-h_n(x_{i,n+1}-0)|\quad (i\equiv 1 (\mathrm{mod}\ 5)), \tag{2}$$

$$|c_5-h_n(x_{i,n+1}-0)|=\frac{1}{3}|h_n(x_{i,n+1}+0)-h_n(x_{i,n+1}-0)|\quad (i\equiv 0 (\mathrm{mod}\ 5)), \tag{3}$$

$$|c_2-h_n(x_{i,n+1})|=\frac{495}{8010}\cdot\frac{1}{3}|h_n(x_{i-1,n+1}-0)-h_n(x_{i+3,n+1}+0)|\quad (i\equiv 2 (\mathrm{mod}\ 5)),$$

$$|c_4-h_n(x_{i,n+1})|=\frac{495}{8010}\cdot\frac{1}{3}|h_n(x_{i-3,n+1}-0)-h_n(x_{i+1,n+1}+0)|\quad (i\equiv 4 (\mathrm{mod}\ 5)),$$

$$|c_3-h_n(x_{i,n+1})|\leqq\frac{1}{80}\cdot\frac{1}{3}|h_n(x_{i-2,n+1}-0)-h_n(x_{i-2,n+1}+0)|+$$

$$+\frac{1}{80}\cdot\frac{1}{3}|h_n(x_{i+2,n+1}-0)-h_n(x_{i+2,n+1}+0)|\quad (i\equiv 3 (\mathrm{mod}\ 5))$$

(the last formula follows from (2) and (3)).

According to the definition of h_{n+1},

$$\max_{x\in[0,1]}|h_n(x)-h_{n+1}(x)|\leqq\max_{1\leqq i\leqq 5^n-1}|h_n(x_{i,n}+0)-h_n(x_{i,n}-0)|.$$

By (1), this implies (iv), and also (vi). Being $f(0)=10$, and $f(1)=11$, f cannot be constant. Q.E.D.

In consequence of the result by Vincze we have the following.

COROLLARY. *The function given in the counterexample is non-differentiable on an everywhere dense set in* [0, 1].

It is still an open question whether f is non-differentiable almost everywhere in [0, 1].

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ON COMMON TERMS OF LINEAR RECURRENCES

By

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Let $G=\{G_n\}_{n=0}^{\infty}$ be a linear recurrence defined by rational integer constants $k(>1)$, $G_0, G_1, \ldots, G_{k-1}, A_1, A_2, \ldots, A_k$ and by the recursion

$$G_n = A_1 G_{n-1}+A_2 G_{n-2}+\ldots+A_k G_{n-k} \quad (n \geqq k).$$

We suppose that not all the G_i's are zero and $A_k \neq 0$. Denote the distinct roots of the characteristic polynomial

$$g(x) = x^k - A_1 x^{k-1} - \ldots - A_{k-1}x - A_k$$

by $\alpha=\alpha_1, \alpha_2, \ldots, \alpha_u$, where α_i has multiplicity m_i. We suppose that $u>1$ and $|\alpha|>|\alpha_i|$ for $i=2, 3, \ldots, u$. It is known that

$$G_n = a\alpha^n + P_2(n)\alpha_2^n + \ldots + P_u(n)\alpha_u^n$$

for $n \geqq 0$, where $P_i(n)$ is a polynomial of degree at most m_i-1; furthermore a and the coefficients of $P_i(n)$ are algebraic numbers from the field $\mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_u)$. In the following we assume $a \neq 0$.

Let $H=\{H_n\}_{n=0}^{\infty}$ be another linear recurrence with the characteristic polynomial

$$h(x) = x^r - B_1 x^{r-1} - \ldots - B_{r-1}x - B_r$$

and with explicit form

$$H_n = b\beta^n + F_2(n)\beta_2^n + \ldots + F_v(n)\beta_v^n,$$

where $\beta=\beta_1, \beta_2, \ldots, \beta_v$ are the distinct roots of $h(x)$. We suppose that $v>1$, $b \neq 0$ and $|\beta|>|\beta_i|$ for $i=2, 3, \ldots, v$.

Let $p_1, p_2, \ldots, p_s$ be rational primes and denote the set of rational integers which have only these primes as prime factors by S.

In [6], with K. GYŐRY and A. SCHINZEL we showed that if G is Lucas or Lehmer sequence ($k=2$), then

$$G_x \in S \tag{1}$$

holds only for finitely many sequences G and for finitely many integers x. K. GYŐRY [5] improved this result giving explicit upper bound for x and for the constants of the sequences which satisfy (1). J. H. LOXTON and A. J. VAN DER POORTEN [8] proved that if $u \geqq 2$ and G is a non-degenerate sequence (i.e. neither α_i nor α_i/α_j are roots of unity for $i, j=1, 2, \ldots, u$ and $i \neq j$) then the set of integers x satifying (1) has density zero.

The diophantine equation

$$G_x = wy^q \tag{2}$$

was also studied by several authors. O. WYLER [15] and J. H. E. COHN [4] proved that if G is the Fibonacci sequence, $w=1$ or 2 and $q=2$, then (2) has only a finite number of integer solutions x, y. T. N. SHOREY and C. L. STEWART [14] showed that if G is a non-degenerate recurrence, $y>1$ and $q>1$, then (2) implies the inequality $q<C$, where C is an effectively computable constant in terms of w and the parameters of the sequence G. They proved that x and y are also bounded for second order recurrences. A. PETHŐ [11, 12] proved similar results for second order recurrences supposing $(A_1, A_2)=1$ and $w\in S$, furthermore he gave an upper bound for w, too. If G is the Fibonacci sequence, A. PETHŐ [13] gave all the solutions of the equations $G_x=py^3$ and $G_x=p^2y^3$, where p is a prime with some restrictions.

M. MIGNOTTE [9, 10] studied the common terms of two sequences. He proved that if α and β are multiplicatively independent, then the equation

$$G_x = H_y \tag{3}$$

has only finitely many integer solutions x, y. He showed that if (3) has infinitely many solutions then α and β are multiplicatively dependent and the set of solutions is the union of a finite set and a finite number of arithmetical progressions.

If the sequence H is equivalent to G then (3) has the form

$$G_x = G_y. \tag{4}$$

Denote the number of solutions of (4) with fixed x and $G_x=t$ by $m(t)$. The maximum value of $m(t)$ is called the multiplicity of sequence G. For non-degenerate second order sequences K. K. KUBOTA [7] prowed that $m(t)\leqq 4$. F. BEUKERS [3] improved this result by showing $m(t)+m(-t)\leqq 3$, with finitely many exceptions.

The purpose of this paper is to study the generalizations of relations (1), (2) and (4). We prove the following theorems and consequences.

THEOREM 1. *Let $G=\{G_n\}_{n=0}^{\infty}$ be a linear recurrence with $|\alpha|=|\alpha_1|>|\alpha_i|$ for $i=2, 3, ..., u$. Suppose that $G_i\neq a\alpha^i$ for $i>n_0$, where n_0 is a constant integer. Let S be the set of integers which can be written in the form $\pm p_1^{e_1}...p_s^{e_s}$, where $p_1, ..., p_s$ are fixed primes and $e_i\geqq 0$ $(i=1, 2, ..., s)$. If $G_x\in S$ then $x<n_1$, where n_1 is an explicitly computable number depending only on the set S, the parameters of the sequence G and on n_0.*

THEOREM 2. *Let G and H be linear recurrences with conditions $|\alpha|=|\alpha_1|>|\alpha_i|$ and $|\beta|=|\beta_1|>|\beta_j|$ $(i=2, 3, ..., u$ and $j=2, 3, ..., v)$ and let S be the set defined in Theorem 1. Suppose that $G_i\neq a\alpha^i$, $H_j\neq b\beta^j$ and $s_1a\alpha^i\neq s_2b\beta^j$ for $i, j>n_0$ and for any integer $s_1, s_2\in S$. If*

$$s_1 G_x = s_2 H_y \tag{5}$$

with $s_1, s_2\in S$, then $\max(x, y)<n_2$, where n_2 is an explicitly computable number depending only on the set S, the parameters of sequences G and H and on n_0.

COROLLARY 1. *Let G and S be defined as in Theorem 1. Suppose that $\alpha^i\notin S$ for any integer i. If $s_1G_x=s_2G_y$ with $s_1, s_2\in S$ and $x\neq y$, then $\max(x, y)<n_3$, where n_3 is an explicitly computable number depending only on the set S, the parameters of sequence G and on n_0.*

COROLLARY 2. *Let G be a linear recurrence defined as in Theorem 1. Then the multiplicity of the sequence G is finite.*

We note that the theorems do not hold generally without conditions. For example, let G be a second order recurrence defined by parameters $G_0=1$, $G_1=3$, $A_1=5$ and $A_2=-6$. In this case $\alpha=\alpha_1=3$, $\alpha_2=2$ and $G_n=3^n$ for $n\geqq 0$. Thus if $3\in S$ and $s_1G_x=s_2H_y$ has a solution then it has infinitely many solutions.

For the proofs of theorems we need a result due to A. Baker.

LEMMA. *Let*

$$\Lambda=\gamma_0+\gamma_1\log\omega_1+\dots+\gamma_n\log\omega_n,$$

where the γ's and ω's denote algebraic numbers ($\omega_i\neq 0$ or 1). We assume that not all the γ's are 0, and that the logarithms mean their principal values. Suppose that ω_i and γ_i have heights at most M_i ($\geqq 4$) and B ($\geqq 4$), respectively, and that the field K generated by the ω's and γ's over the rational numbers has degree at most d. If $\Lambda\neq 0$ then

$$|\Lambda|>(B\Omega)^{-C\Omega\log\Omega'}$$

where

$$\Omega=\log M_1\cdot\log M_2\cdot\ldots\cdot\log M_n,\quad \Omega'=\Omega/\log M_n$$

and $C=(16\,nd)^{200n}$.

If $\gamma_0=0$ and $\gamma_1,\dots,\gamma_n$ are rational integers then $|\Lambda|>B^{-C\Omega\log\Omega'}$.

(See A. BAKER [1] or A. BAKER and C. L. STEWART [2].)

Theorem 1 follows from Theorem 2 since we can choose a sequence H such that the assumptions of Theorem 2 are satisfied but 1 is contained in H as a term. Thus we have to prove only Theorem 2.

PROOF OF THEOREM 2. Without loss of generality we may assume that $x\geqq y\geqq 0$ and $(s_1,s_2)=1$ if equation (5) holds for some integers x, y, s_1 and s_2. In what follows $c_1, c_2, \dots, n_4, n_5, \dots$ denote positive numbers which are explicitly computable and depend only on the set S and the parameters of sequences G and H.

Since

$$G_x=a\alpha^x\left[1+P_2(x)\left(\frac{\alpha_2}{\alpha}\right)^x+\dots+P_u(x)\left(\frac{\alpha_u}{\alpha}\right)^x\right], \tag{6}$$

$|\alpha_i/\alpha|<1$ $(i=2,3,\dots,u)$ and $|\alpha|>1$,

$$|G_x|<e^{c_1x}. \tag{7}$$

In a similar manner we get

$$|H_y|<e^{c_22^y} \tag{8}$$

for $y>0$. If (5) holds for integers x, y, s_1, s_2 then by $s_1, s_2\in S$ we have $|s_1|=\prod_{i=1}^{s}p_i^{e_i'}$ $(e_i'\geqq 0)$ and $|s_2|=\prod_{i=1}^{s}p_i^{e_i}$ $(e_i\geqq 0)$, therefore $p_i^{e_i}|G$ and $p_i^{e_i'}|H_y$, by (7) and (8), imply the inequalities $0\leqq e_i<\dfrac{c_1x}{\log p_i}<c_3x$ and $0\leqq e_i'<\dfrac{c_2y}{\log p_i}<c_4y$

for $i=1, 2, \ldots, s$. Using (6), equation (5) can be written in the form

$$\frac{s_1 a\alpha^x}{s_2 H_y} = (1+\varepsilon_1)^{-1}, \tag{9}$$

where, by the inequalities $|\alpha_i/\alpha|<1$ $(i=2, 3, \ldots, u)$,

$$|\varepsilon_1| = \left|\sum_{i=2}^{u} P_i(x)\left(\frac{\alpha_i}{\alpha}\right)^x\right| < e^{-c_5 x}$$

for $x>n_4$. We may assume that $H_y\neq 0$ since $G_x\neq 0$ for x large enough, further $\varepsilon_1\neq 0$ by the conditions of the theorem. Thus by (9) we get

$$\big|\log|s_1|-\log|s_2|+\log|a|+x\log|\alpha|-\log|H_y|\big| = |\Lambda| = \log|1+\varepsilon_1| < e^{-c_6 x} \tag{10}$$

and $0<|\Lambda|$ for $x>n_5$. We shall use Baker's result detailed in the Lemma. Now $\log|s_1|=e_1'\log p_1+\ldots+e_s'\log p_s$ and $\log|s_2|=e_1\log p_1+\ldots+e_s\log p_s$, where $e_i<c_3x$ and $e_i'<c_4y$, thus in our case $n\leqq s+3$ (since $(s_1, s_2)=1$), $M_i<c_7$ $(c_7\geqq 4$ and $i<n)$, $M_n=|H_y|=|b\beta^y(1+\varepsilon_2)|<e^{c_8 y}$ (for $y>0$), $d\leqq c_9$, $\Omega\leqq c_{10}y$ and $\Omega'\leqq c_{11}$. By the Lemma we have

$$|\Lambda| > B^{-c_{12}y} > (c_{13}x)^{-c_{12}y} > e^{-c_{14}y\log x} \tag{11}$$

since $B<c_{13}x$ by the condition $x\geqq y$. Comparing inequalities (10) and (11), we get

$$y > c_{15}\frac{x}{\log x} \tag{12}$$

for $x>n_5$.

If $y=0$ then $M_n=|H_0|$ and so $\Omega=c_{16}$. This implies the inequality $|\Lambda|>e^{-c_{17}\log x}$. But this contradicts (10) if $x>n_6$.

Equation (5) can be written in the form

$$\frac{s_1 a\alpha^x}{s_2 b\beta^y} = \frac{1+\varepsilon_2}{1+\varepsilon_1}, \tag{13}$$

too, where

$$|\varepsilon_2| = \left|\sum_{i=2}^{v} F_i(y)\left(\frac{\beta_i}{\beta}\right)^y\right| < e^{-c_{18}y}$$

for $y>n_7$. We may suppose that $y>n_7$ if $x>n_8$, by (12). Using the assumption $x\geqq y$ and the conditions of the theorem, (13) implies the inequality

$$\left|\log|s_1|-\log|s_2|+\log\left|\frac{a}{b}\right|+x\log|\alpha|-y\log|\beta|\right| = \tag{14}$$

$$= |\Lambda'| = \big|\log|1+\varepsilon_2|-\log|1+\varepsilon_1|\big| \leqq 2(|\varepsilon_1|+|\varepsilon_2|) < e^{-c_{19}y}$$

for $x>n_9$, furthermore $|\Lambda'|>0$. We again use the Lemma. In this case $n\leqq s+3$, $B<c_{20}x$, $M_i<c_{21}$ $(\geqq 4$ and $i=1, 2, \ldots, n)$, $d\leqq c_{22}$ and $\Omega\log\Omega'<c_{23}$, so

$$|\Lambda'| > (c_{20}x)^{-c_{24}} > e^{-c_{25}\log x}. \tag{15}$$

But (14) and (15) imply $y<c_{26}\log x$ which contradicts (12) if $x>n_{10}$, thus Theorem 2 is true with $n_2=\max(n_0, n_4, n_5, \ldots, n_{10})$.

In the proof of Theorem 2 we did not use the restriction $G\neq H$ therefore Corollary 1 is true since the sequences G and $H=G$ satisfy the conditions. Corollary 2 is also true as an obvious consequence of Corollary 1.

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TEMPERED GENERALIZED FUNCTIONS AND THEIR FOURIER TRANSFORMS

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0. Introduction. This paper forms a continuation of our study of quotient multipliers and generalized functions [12]—[22]. However, the reader is assumed to be well acquainted only with the first section of [13], where the multiplier extension of admissible vector modules was described to have a general algebraic framework for an abstract theory of generalized functions.

The paper [13] needs some corrigendum. In Definition 1.1, we forgot to stress that $\mathscr{B}\neq\{0\}$. Definition 1.6 contains two misprints, however the reader can easily correct them. The addition in $\mathfrak{M}(\mathscr{A},\mathscr{B})$, according to our original notation [12], should be denoted by a boldface plus. Moreover, we meantime observed that our construction of the multiplier extensions of admissible vector modules greatly resembles that of quotient modules defined by Gabriel topologies [7].

In the present paper, we are mainly concerned with the multiplier extension $\mathfrak{M}(\mathscr{S})=\mathfrak{M}(\mathscr{S},\mathscr{S})$ of the convolution algebra $\mathscr{S}$ of rapidly decreasing functions and its Fourier transform. However, as some helpful tools, the multiplier extensions $\mathfrak{M}(\mathscr{D})$, $\mathfrak{M}(\mathscr{D},\mathscr{S})$, $\mathfrak{M}(\mathscr{D},\mathscr{O})$, $\mathfrak{M}(\mathscr{S},\mathscr{O})$ and $\mathfrak{M}(\mathscr{D},\mathscr{E})$, where $\mathscr{O}$ stands for either $\mathscr{O}_C$ or $\mathscr{O}_M$ [3], are also used.

In Section 1, we study the non-zero divisor subsets [13] of the above admissible convolution vector modules of test functions. For example, we show that a subset D of $\mathscr{S}$ is not a divisor of zero in $\mathscr{S}$ if and only if the set

$$Z(D)=\bigcap_{\varphi\in D}\{t\in\mathbf{R}^k:\hat{\varphi}(t)=0\}$$

has empty interior in $\mathbf{R}^k$.

In Section 2, using a reasonable concept of identification of generalized functions, we show that

$$\mathfrak{M}(\mathscr{D})=\mathfrak{N}(\mathscr{D},\mathscr{S})\subset\mathfrak{M}(\mathscr{D},\mathscr{S})\subset\mathfrak{Q}(\mathscr{S})\subset\mathfrak{M}(\mathscr{S}),$$

and

$$\mathfrak{M}(\mathscr{D},\mathscr{O})\subset\mathfrak{M}(\mathscr{S},\mathscr{O})\quad\text{and}\quad\mathfrak{M}(\mathscr{S},\mathscr{O}_C)=\mathfrak{M}(\mathscr{S},\mathscr{O}_M),$$

where $\mathfrak{Q}(\mathscr{S})$ denotes the classical quotient algebra of $\mathscr{S}$.

In Section 3, identifying Schwartz distributions as convolutors (convolution operators), we prove that

$$\mathscr{D}'\subset\mathfrak{M}(\mathscr{D},\mathscr{E}),\quad\mathscr{S}'\subset\mathfrak{M}(\mathscr{S},\mathscr{O}_M),\quad\mathscr{O}_C'\subset\mathfrak{M}(\mathscr{S}),\quad\mathscr{E}'\subset\mathfrak{M}(\mathscr{D})$$

such that the corresponding distributions are the only total (resp. continuous) elements of the corresponding multiplier extensions.

In Section 4, we define the Fourier transform of $\mathfrak{M}(\mathcal{S})$ as a natural extension of the classical Fourier transform of $\mathcal{S}$ such that it turns out to be an algebraic and topological isomorphism of $\mathfrak{M}(\mathcal{S})$ onto $\mathfrak{M}(\hat{\mathcal{S}})$, where $\hat{\mathcal{S}}$ denotes $\mathcal{S}$ as a function algebra, and the multiplier extensions $\mathfrak{M}(\mathcal{S})$ and $\mathfrak{M}(\hat{\mathcal{S}})$ are considered to be equipped with the locally convex topologies described in [19]. After embedding some distributions into $\mathfrak{M}(\hat{\mathcal{S}})$, we can show that the Fourier transform of $\mathfrak{M}(\mathcal{S})$ also extends a larger restriction of the distributional one, and we have $(\mathcal{S}' \cap \mathfrak{M}(\mathcal{S}))\hat{} = \mathcal{S}' \cap \mathfrak{M}(\hat{\mathcal{S}})$, and $(\mathfrak{M}(\mathcal{S}) \cap \mathfrak{M}(\mathcal{S}, \mathcal{O}_M))\hat{} \subset \mathcal{E}$ and $(\mathfrak{M}(\mathcal{S}) \cap \mathfrak{N}(\mathcal{D}, \mathcal{O}_M))\hat{} \subset \mathcal{A}$, where $\mathcal{A}$ denotes the subspace of $\mathcal{E}$ consisting of all analytic functions.

Finally, we indicate that using the Fourier transform of $\mathfrak{M}(\hat{\mathcal{S}})$, it is also possible to define the Fourier transforms of the multiplier extensions $\mathfrak{M}(\mathcal{S}, \mathcal{O}_M)$ and $\mathfrak{M}(\mathcal{D}, \mathcal{E})$ (and also the Mikusiński operator field $\mathfrak{M}(\mathcal{E}_R)$, where $\mathcal{E}_R$ denotes the convolution algebra of functions in $\mathcal{E}$ with supports in various right-sided orthants [10]) in a natural way within the framework of our theory.

1. Test functions. The spaces $\mathcal{D} \subset \mathcal{S} \subset \mathcal{O}_C \subset \mathcal{O}_M \subset \mathcal{E}$ of infinitely differentiable functions from $\mathbf{R}^k$ into $\mathbf{C}$ are the most important test function spaces for Schwartz distributions [3], [23].

Under convolution

$$(f * g)(x) = \int_{\mathbf{R}^k} f(x-y)\, g(y)\, dm_k(y),$$

where $dm_k(y) = (2\pi)^{-k/2} dy$, they become, as called in [13], admissible vector modules.

For instance, $\mathcal{D}$ and $\mathcal{S}$ are admissible algebras, $\mathcal{O}_C$ and $\mathcal{O}_M$ are admissible $\mathcal{S}$-vector modules, and $\mathcal{E}$ is an admissible $\mathcal{D}$-vector module.

Before considering their multiplier extensions, it is suitable to study their non-zero divisor subsets. For this, we need the Fourier transform

$$\hat{f}(t) = \int_{\mathbf{R}^k} f(y) e^{-ity}\, dm_k(y)$$

which is a topological vector space isomorphism of $\mathcal{S}$ onto $\mathcal{S}$ such that $(f*g)\hat{} = \hat{f}\hat{g}$ for all $f, g \in \mathcal{S}$ [6].

DEFINITION 1.1. For $D \subset \mathcal{S}$, define

$$Z(D) = \bigcap_{\varphi \in D} \{t \in \mathbf{R}^k : \hat{\varphi}(t) = 0\}.$$

THEOREM 1.2. *Let $D \subset \mathcal{S}$. Then D is not a divisor of zero in $\mathcal{S}$ if and only if $Z(D)^0 = \emptyset$.*

PROOF. Suppose first that D is not a divisor of zero in $\mathcal{S}$. To prove that $Z(D)^0 = \emptyset$, assume to the contrary that there exists an open set $\emptyset \neq U \subset \mathbf{R}^k$ such that $U \subset Z(D)$. Then, since $\mathcal{D} \subset \mathcal{S}$ and $\hat{\mathcal{S}} = \mathcal{S}$, there exists $0 \neq f \in \mathcal{S}$ such that $\operatorname{supp} \hat{f} \subset U$. Thus, we have $(f * \varphi)\hat{} = \hat{f}\hat{g} = 0$, and hence $f * \varphi = 0$ for all $\varphi \in D$, a contradiction.

Now suppose that $Z(D)^0 = \emptyset$. To prove that D is not a divisor of zero in $\mathcal{S}$, assume that $f \in \mathcal{S}$ such that $f * \varphi = 0$ for all $\varphi \in D$. Then, we have $\hat{f}\hat{g} = (f * \varphi)\hat{} = 0$

for all $\varphi \in D$. Hence, since $Z(D)^0 = \varnothing$ and $\hat{f}$ is continuous, we can infer that $\hat{f}=0$, which implies $f=0$.

COROLLARY 1.3. *The admissible $\mathscr{D}$-vector module $\mathscr{S}$ has no proper divisors of zero.*

PROOF. If $0 \neq \varphi \in \mathscr{D}$, then $\hat{\varphi}$ is a non-zero analytic function on $\mathbf{R}^k$ [6, 7.22 Theorem] which may have only isolated zeros, and thus by Theorem 1.2, φ is not a divisor of zero in $\mathscr{S}$.

THEOREM 1.4. *Let $D \subset \mathscr{S}$. Then D is not a divisor of zero in $\mathscr{O}_C$ if and only if D is not a divisor of zero in $\mathscr{O}_M$.*

PROOF. To prove the nontrivial part, suppose that D is not a divisor of zero in $\mathscr{O}_C$. Let $f \in \mathscr{O}_M$ such that $f * \varphi = 0$ for all $\varphi \in D$. Then, we also have $(f*\sigma)*\varphi = (f*\varphi)*\sigma = 0$ for all $\varphi \in D$ and $\sigma \in \mathscr{D}$. Hence, since $f*\sigma \in \mathscr{O}_C$ for all $\sigma \in \mathscr{D}$ [3, Proposition 4.11.7], by the assumption, we can infer that $f*\sigma=0$ for all $\sigma \in \mathscr{D}$, and this implies $f=0$.

THEOREM 1.5. *Let $D \subset \mathscr{S}$, and suppose that D is not a divisor of zero in $\mathscr{O}_M$. Then $Z(D) = \varnothing$.*

PROOF. Let $t \in \mathbf{R}^k$, and define the function e_t on $\mathbf{R}^k$ by $e_t(x) = \exp itx$. Then $e_t \in \mathscr{O}_M$, and thus by the assumption, there exists $\varphi \in D$ such that $e_t * \varphi \neq 0$. Hence, since $e_t * \varphi = \hat{\varphi}(t) e_t$, it follows that $\hat{\varphi}(t) \neq 0$. This shows that $t \notin Z(D)$.

PROBLEMS 1.6. (i) Is the converse of Theorem 1.5 also true? (Our conjecture is that this is the case, and we think that an analogue of [6, 9.3. Theorem] can be used to prove it.) (ii) Do $D \subset \mathscr{D}$ and $Z(D) = \varnothing$ imply that D is not a divisor of zero in $\mathscr{E}$?

2. Identification of generalized functions. After some natural identifications, for the duals of test function spaces we have $\mathscr{E}' \subset \mathscr{O}'_M \subset \mathscr{O}'_C \subset \mathscr{S}' \subset \mathscr{D}'$. For the multiplier extensions of the admissible convolution vector modules of test functions there are no such straightforward inclusions. However, we still have the following obvious theorems.

THEOREM 2.1. *The mapping defined on $\mathfrak{M}(\mathscr{D})$ by $F \to \bar{F}$, where $\bar{F}$ denotes the maximal extension of F in the $\mathscr{D}$-vector module $\mathscr{S}$* [13, *Definition* 1.4], *is an algebra isomorphism of $\mathfrak{M}(\mathscr{D})$ onto $\mathfrak{N}(\mathscr{D}, \mathscr{S})$ which also preserves $\mathscr{D}$ functions.*

PROOF. This is quite obvious by Corollary 1.3. (Note that by [13, Definition 1.14], we have $\mathscr{D} \subset \mathfrak{M}(\mathscr{D})$ and $\mathscr{D} \subset \mathfrak{N}(\mathscr{D}, \mathscr{S})$.)

DEFINITION 2.2. Let S be the family of all elements of $\mathscr{S}$ which are not divisors of zero in $\mathscr{S}$, and define

$$\mathfrak{Q}(\mathscr{S}) = \{F \in \mathfrak{M}(\mathscr{S}) : D_F \cap S \neq \varnothing\}.$$

Note that $\mathfrak{Q}(\mathscr{S})$ consists of all elements F of $\mathfrak{M}(\mathscr{S})$ which can be written in the form $F = F(\varphi)/\varphi$ [13. Definition 1.19], and thus $\mathfrak{Q}(\mathscr{S})$ may be viewed as the classical quotient algebra of $\mathscr{S}$.

THEOREM 2.3. *The mapping defined on* $\mathfrak{M}(\mathcal{D}, \mathcal{S})$ *by* $F \to \bar{F}$, *where* $\bar{F}$ *denotes the maximal extension of* F *in* $\mathcal{S}$, *is a vector space isomorphism of* $\mathfrak{M}(\mathcal{D}, \mathcal{S})$ *into* $\mathfrak{Q}(\mathcal{S})$ *such that* $\overline{F * \Phi} = \bar{F} * \bar{\Phi}$ *for all* $F \in \mathfrak{M}(\mathcal{D}, \mathcal{S})$ *and* $\Phi \in \mathfrak{N}(\mathcal{D}, \mathcal{S})$. *Moreover, this mapping also preserves* $\mathcal{S}$ *functions.*

PROOF. This is also quite obvious by Corollary 1.3.

THEOREM 2.4. *Let* $\mathcal{O}$ *stand for either* $\mathcal{O}_C$ *or* $\mathcal{O}_M$. *Then the mapping defined on* $\mathfrak{M}(\mathcal{D}, \mathcal{O})$ *by* $F \to \bar{F}$, *where* $\bar{F}$ *denotes the maximal extension of* F *in the* $\mathcal{S}$*-vector module* $\mathcal{O}$, *is a vector space isomorphism of* $\mathfrak{M}(\mathcal{D}, \mathcal{O})$ *into* $\mathfrak{M}(\mathcal{S}, \mathcal{O})$ *which takes* $\mathfrak{N}(\mathcal{D}, \mathcal{O})$ *into* $\mathfrak{N}(\mathcal{S}, \mathcal{O})$ *such that* $\overline{F * \Phi} = \bar{F} * \bar{\Phi}$ *for all* $F \in \mathfrak{M}(\mathcal{D}, \mathcal{O})$ *and* $\Phi \in \mathfrak{N}(\mathcal{D}, \mathcal{O})$. *Moreover. this mapping also preserves* $\mathcal{O}$ *functions.*

PROOF. Obvious.

THEOREM 2.5. *The mapping defined on* $\mathfrak{M}(\mathcal{S}, \mathcal{O}_C)$ *by* $F \to \bar{F}$, *where* $\bar{F}$ *denotes the maximal extension of* F *in the* $\mathcal{S}$*-vector module* $\mathcal{O}_M$, *is a vector space isomorphism of* $\mathfrak{M}(\mathcal{S}, \mathcal{O}_C)$ *onto* $\mathfrak{M}(\mathcal{S}, \mathcal{O}_M)$ *which takes* $\mathfrak{N}(\mathcal{S}, \mathcal{O}_C)$ *onto* $\mathfrak{N}(\mathcal{S}, \mathcal{O}_M)$ *such that* $\overline{F * \Phi} = \bar{F} * \bar{\Phi}$ *for all* $F \in \mathfrak{M}(\mathcal{S}, \mathcal{O}_C)$ *and* $\Phi \in \mathfrak{N}(\mathcal{S}, \mathcal{O}_C)$. *Moreover, this mapping also preserves* $\mathcal{O}_C$ *functions.*

PROOF. This is quite obvious by Theorem 1.4. (To prove that the above mapping is onto $\mathfrak{M}(\mathcal{S}, \mathcal{O}_M)$, note that if $F \in \mathfrak{M}(\mathcal{S}, \mathcal{O}_M)$, then $F(\varphi * \psi) = F(\varphi) * \psi \in \mathcal{O}_C$ for all $\varphi \in D_F$ and $\psi \in \mathcal{S}$, and thus the domain D_{F_0} of $F_0 = F \cap (\mathcal{S} \times \mathcal{O}_C)$ is not a divisor of zero in $\mathcal{O}_M$ since $D_F * \mathcal{S} \subset D_{F_0}$. Consequently, $F_0 \in \mathfrak{M}(\mathcal{S}, \mathcal{O}_C)$ and $\bar{F}_0 = F$.)

REMARK 2.6. By the above theorems, we may write

$$\mathfrak{M}(\mathcal{D}) = \mathfrak{N}(\mathcal{D}, \mathcal{S}) \subset \mathfrak{M}(\mathcal{D}, \mathcal{S}) \subset \mathfrak{Q}(\mathcal{S}) \subset \mathfrak{M}(\mathcal{S}),$$

and

$$\mathfrak{M}(\mathcal{D}, \mathcal{O}) \subset \mathfrak{M}(\mathcal{S}, \mathcal{O}) \quad \text{and} \quad \mathfrak{N}(\mathcal{D}, \mathcal{O}) = \mathfrak{N}(\mathcal{S}, \mathcal{O}),$$

where $\mathcal{O} = \mathcal{O}_C$ or $\mathcal{O}_M$; and $\mathfrak{M}(\mathcal{S}, \mathcal{O}_C) = \mathfrak{M}(\mathcal{S}, \mathcal{O}_M)$ and $\mathfrak{N}(\mathcal{S}, \mathcal{O}_C) = \mathfrak{N}(\mathcal{S}, \mathcal{O}_M)$.

EXAMPLE 2.7. We have $\mathfrak{N}(\mathcal{D}, \mathcal{S}) \neq \mathfrak{M}(\mathcal{D}, \mathcal{S})$.

To see this, pick $0 \neq f \in \mathcal{S}$ such that $\hat{f} \in \mathcal{D}$. Then $f \in \mathfrak{M}(\mathcal{D}, \mathcal{S})$, but since $(f * \varphi)^{\wedge} = \hat{f}\hat{g} \in \mathcal{D} \setminus \{0\}$, and hence $f * \varphi \in \mathcal{S} \setminus \mathcal{D}$ for all $0 \neq \varphi \in \mathcal{D}$, we have $f \notin \mathfrak{N}(\mathcal{D}, \mathcal{S})$.

PROBLEMS 2.8. Are the other inclusions in Remark 2.6 also proper?

REMARK 2.9. The importance of the embeddings made complete in Remark 2.6 lies in that they show that it is sufficient to consider only the multiplier extensions $\mathfrak{M}(\mathcal{S})$, $\mathfrak{M}(\mathcal{S}, \mathcal{O}_M)$ and $\mathfrak{M}(\mathcal{D}, \mathcal{E})$.

The investigation of the relationship of the above multiplier extensions needs a general concept of identification of generalized functions, and also the solution of Problems 1.6.

The following definition seems to be quite natural. For $i = 1, 2$, let $\mathcal{D} \subset \mathcal{A}_i \subset \subset \mathcal{B}_i \subset \mathcal{E}$ be appropriate subspaces of $\mathcal{E}$ such that under convolution $\mathcal{B}_i$ forms an $\mathcal{A}_i$-vector module. The generalized functions $F_i \in \mathfrak{M}(\mathcal{A}_i, \mathcal{B}_i)$ $(i = 1, 2)$ are to be identified if the domain of $F_1 \cap F_2$ is not a divisor of zero in both $\mathcal{B}_1$ and $\mathcal{B}_2$. (Note that in this case we have $F=F_1\cap F_2\in\mathfrak{M}(\mathscr{A}_1\cap\mathscr{A}_2, \mathscr{B}_1\cap\mathscr{B}_2)$, $\bar{F}^1=F_1$ and $\bar{F}^2=F_2$ where the bars denote the corresponding maximal extensions.)

However, we do not know whether or not this definition may lead to confusions or contradictions. In spite of this, it seems reasonable to use the notations $\mathfrak{M}(\mathscr{S})\cap\cap\mathfrak{M}(\mathscr{S}, \mathscr{O}_M)$, $\mathfrak{M}(\mathscr{S})\cap\mathfrak{M}(\mathscr{D}, \mathscr{E})$, $\mathfrak{M}(\mathscr{S}, \mathscr{O}_M)\cap\mathfrak{M}(\mathscr{D}, \mathscr{E})$ etc. in the above sense. As an illustration, the obvious inclusion $\mathfrak{N}(\mathscr{D}, \mathscr{E})\subset\mathfrak{M}(\mathscr{S})\cap\mathfrak{M}(\mathscr{S}, \mathscr{O}_M)$ can be mentioned.

3. Embedding of distributions. For embedding of $\mathscr{D}'$ into $\mathfrak{M}(\mathscr{D}, \mathscr{E})$, in [12] and [21], we have proved the following theorem which has its origin in [8].

THEOREM 3.1. *For $\Lambda\in\mathscr{D}'$, let F_Λ be the function defined on $\mathscr{D}$ by $F_\Lambda(\varphi)=\Lambda*\varphi$. Then the mapping defined on $\mathscr{D}'$ by $\Lambda\to F_\Lambda$ is a vector space isomorphism of $\mathscr{D}'$ into $\mathfrak{M}(\mathscr{D}, \mathscr{E})$ which takes $\mathscr{D}'$ onto* $\mathrm{Hom}_{\mathscr{D}}(\mathscr{D}, \mathscr{E})$ *and $\mathscr{E}'$ onto* $\mathrm{Hom}_{\mathscr{D}}(\mathscr{D}, \mathscr{D})$ *such that $F_{\Lambda_1*\Lambda_2}=F_{\Lambda_1}*F_{\Lambda_2}$ for all $\Lambda_1\in\mathscr{D}'$ and $\Lambda_2\in\mathscr{E}'$. Moreover, this mapping also preserves $\mathscr{E}$ functions.*

For embedding of $\mathscr{S}'$ into $\mathfrak{M}(\mathscr{S}, \mathscr{O}_M)$, we can prove here a similar theorem.

THEOREM 3.2. *For $\Lambda\in\mathscr{S}'$, let F_Λ be the function defined on $\mathscr{S}$ by $F_\Lambda(\varphi)=\Lambda*\varphi$. Then the mapping defined on $\mathscr{S}'$ by $\Lambda\to F_\Lambda$ is a vector space isomrphism of $\mathscr{S}'$ into $\mathfrak{M}(\mathscr{S}, \mathscr{O}_M)$ which takes $\mathscr{S}'$ onto* $\mathrm{Hom}_{\mathscr{S}}(\mathscr{S}, \mathscr{O}_M)$ *and $\mathscr{O}_C'$ onto* $\mathrm{Hom}_{\mathscr{S}}(\mathscr{S}, \mathscr{S})$ *such that $F_{\Lambda_1*\Lambda_2}=F_{\Lambda_1}*F_{\Lambda_2}$ for all $\Lambda_1\in\mathscr{S}'$ and $\Lambda_2\in\mathscr{O}_C'$. Moreover, this mapping also preserves $\mathscr{O}_M$ functions.*

PROOF. The only part which needs proof is that the above mapping takes $\mathscr{S}'$ onto $\mathrm{Hom}_{\mathscr{S}}(\mathscr{S}, \mathscr{O}_M)$ and $\mathscr{O}_C'$ onto $\mathrm{Hom}_{\mathscr{S}}(\mathscr{S}, \mathscr{S})$. We shall prove only the latter assertion, since the proof of the former one is similar, but simpler.

If $\Lambda\in\mathscr{O}_C'$, then by [3, Theorem 4.11.3 and Proposition 4.11.5], $F_\Lambda(\varphi)\hat{}=(\Lambda*\varphi)\hat{}=\hat{\varphi}\hat{\Lambda}\in\mathscr{S}$, and hence $F_\Lambda(\varphi)\in\mathscr{S}$ for all $\varphi\in\mathscr{S}$, which shows that $F_\Lambda\in\mathrm{Hom}_{\mathscr{S}}(\mathscr{S},\mathscr{S})$. (For another proof, one may turn to [22, Theorem 30.1].)

Now suppose that $F\in\mathrm{Hom}_{\mathscr{S}}(\mathscr{S}, \mathscr{S})$, and define the functional Λ on $\mathscr{S}$ by $\Lambda(\varphi)=F(\check{\varphi})(0)$. Then $\Lambda\in\mathscr{S}'$, since by the closed graph theorem F is a continuons mapping of $\mathscr{S}$ into $\mathscr{S}$ [22]. Furthermore, a similar computation as in the proof of [12, Theorem 3.5] shows that $F(\varphi)=\Lambda*\varphi=F_\Lambda(\varphi)$ for all $\varphi\in\mathscr{S}$. Thus, it remains only to show that $\Lambda\in\mathscr{O}_C'$. We have $\hat{\varphi}\hat{\Lambda}=(\Lambda*\varphi)\hat{}=F(\varphi)\hat{}\in\mathscr{S}$ for all $\varphi\in\mathscr{S}$. Hence, choosing $\varphi_n\in\mathscr{S}$ such that $\hat{\varphi}_n(t)=1$ if $|t|\leqq n$, we can infer that $\hat{\Lambda}\in\mathscr{E}$. Thus, by [3, Proposition 4.11.5], we also have $\hat{\Lambda}\in\mathscr{O}_M$. Hence, by [3, Theorem 4.11.3], it is clear that $\Lambda\in\mathscr{O}_C'$.

REMARK 3.3. Since by [3, Proposition 4.11.7], $\Lambda*\varphi\in\mathscr{O}_C$ for all $\Lambda\in\mathscr{S}'$ and $\varphi\in\mathscr{S}$, one may replace $\mathscr{O}_M$ by $\mathscr{O}_C$ in the above theorem. However, the space $\mathscr{O}_M$ seems to be more convenient for all purposes.

REMARK 3.4. By Theorems 3.1 and 3.2, we may write $\mathscr{D}'\subset\mathfrak{M}(\mathscr{D}, \mathscr{E})$ and $\mathscr{S}'\subset\mathfrak{M}(\mathscr{S}, \mathscr{O}_M)$. After these embeddings, we also have $\mathscr{E}'\subset\mathfrak{M}(\mathscr{D})$ and $\mathscr{O}_C'\subset\mathfrak{M}(\mathscr{S})$. Moreover, it is noteworthy that the corresponding distributions are the only total (resp. continuous) elements of the corresponding multiplier extensions.

REMARK 3.5. The above embeddings of distributions are also consistent with the identification of generalized functions considered in Remark 2.9.

Namely, if $\Lambda_1 \in \mathscr{D}'$ and $\Lambda_2 \in \mathscr{S}'$ such that the domain $D_{F_{\Lambda_1} \cap F_{\Lambda_2}}$ of $F_{\Lambda_1} \cap F_{\Lambda_2}$ is not a divisor of zero in $\mathscr{E}$, then we have

$$(\Lambda_1 * \varphi - \Lambda_2 * \varphi) * \psi = (\Lambda_1 * \psi - \Lambda_2 * \psi)\varphi = (F_{\Lambda_1}(\psi) - F_{\Lambda_2}(\psi)) * \varphi = 0$$

for all $\varphi \in \mathscr{D}$ and $\psi \in \mathscr{D}_{F_{\Lambda_1} \cap F_{\Lambda_2}}$, and hence $\Lambda_1 * \varphi = \Lambda_2 * \varphi$ for all $\varphi \in \mathscr{D}$, which implies that $\Lambda_1 = \Lambda_2$.

Problems 3.6. In the sense of Remark 2.9, we may ask: Are the inclusions

$$\mathscr{S}' \subset \mathscr{D}' \cap \mathfrak{M}(\mathscr{S}, \mathscr{O}_M) \subset \mathscr{D}' \quad \text{and} \quad \mathscr{O}'_C \subset \mathscr{S}' \cap \mathfrak{M}(\mathscr{S}) \subset \mathscr{S}'$$

where $\mathscr{S}'$ is now considered as a subspace of $\mathfrak{M}(\mathscr{S}, \mathscr{O}_M)$, proper?

4. Fourier transforms. Though $\hat{\mathscr{S}} = \mathscr{S}$, we still need the notation $\hat{\mathscr{S}}$ to indicate that $\mathscr{S}$ is considered as an algebra under pointwise multiplication. Since the Fourier transform of $\mathscr{S}$ is an isomorphism of the convolution algebra $\mathscr{S}$ onto the function algebra $\hat{\mathscr{S}}$, it seems quite natural to have

Definition 4.1. For $F \in \mathfrak{M}(\mathscr{S})$, the function $\hat{F}$ defined on $\hat{D}_F$ by $\hat{F}(\hat{\varphi}) = F(\varphi)\hat{}$ is called the Fourier transform of F.

Theorem 4.2. *The mapping defined on $\mathfrak{M}(\mathscr{S})$ by $F \to \hat{F}$ is the unique algebra isomorphism of $\mathfrak{M}(\mathscr{S})$ onto $\mathfrak{M}(\hat{\mathscr{S}})$ which extends the classical Fourier transform of $\mathscr{S}$.*

Proof. Everything stated here is clear. However, for a better understanding of Definition 4.1, we show the uniqueness of the Fourier transform of $\mathfrak{M}(\mathscr{S})$. For this, suppose that $F \to \tilde{F}$ is a mapping of $\mathfrak{M}(\mathscr{S})$ into $\mathfrak{M}(\hat{\mathscr{S}})$ such that $\tilde{f} = \hat{f}$ and $(F * f)\tilde{} = \tilde{F}\tilde{f}$ for all $F \in \mathfrak{M}(\mathscr{S})$ and $f \in \mathscr{S}$. If $F \in \mathfrak{M}(\mathscr{S})$, then using [13, Proposition 1.16], we have

$$\hat{F}(\hat{\varphi}) = F(\varphi)\hat{} = F(\varphi)\tilde{} = (F * \varphi)\tilde{} = \tilde{F}\tilde{\varphi} = \tilde{F}\hat{\varphi}$$

and hence $\tilde{F}(\hat{\varphi}) = \hat{F}(\hat{\varphi})$ for all $\varphi \in D_F$. Since $\hat{D}_F$ is not a divisor of zero in $\hat{\mathscr{S}}$, this implies that $\tilde{F} = \hat{F}$.

To show that the Fourier transform of $\mathfrak{M}(\mathscr{S})$ also extends some larger restriction of distributional one, we have to consider some tempered distributions to be embedded in $\mathfrak{M}(\hat{\mathscr{S}})$.

Theorem 4.3. *Let $\mathscr{N}$ be the family of all distributions $\Lambda \in \mathscr{S}'$ for which the set $\hat{E}_\Lambda$, where $E_\Lambda = \{\varphi \in \mathscr{S} : \varphi\Lambda \in \mathscr{S}\}$, is not a divisor of zero in $\mathscr{O}_M$; and for $\Lambda \in \mathscr{N}$, let M_Λ be the function defined on E_Λ by $M_\Lambda(\varphi) = \varphi\Lambda$. Then $\mathscr{N}$ is a subspace of $\mathscr{S}'$ such that $\mathscr{O}_M \subset \mathscr{N}$, and the mapping defined on $\mathscr{N}$ by $\Lambda \to M_\Lambda$ is a vector space isomorphism of $\mathscr{N}$ into $\mathfrak{M}(\hat{\mathscr{S}})$ which takes $\mathscr{O}_M$ onto* $\mathrm{Hom}_{\hat{\mathscr{S}}}(\hat{\mathscr{S}}, \hat{\mathscr{S}})$ *such that $M_{f\Lambda} = M_f M_\Lambda$ for all $f \in \mathscr{O}_M$ and $\Lambda \in \mathscr{N}$.*

Proof. It is clear that $M_\Lambda(\varphi)\psi = \varphi M_\Lambda(\psi)$ for all $\varphi, \psi \in E_\Lambda$. Moreover, if $(\varphi, f) \in \mathscr{S} \times \mathscr{S}$ such that $f\sigma = \varphi M_\Lambda(\sigma)$ for all $\sigma \in E_\Lambda$, then we have $f\sigma = \sigma(\varphi\Lambda)$, and hence $\hat{f} * \hat{\sigma} = (\hat{\Lambda} * \hat{\varphi}) * \hat{\sigma}$, i.e., $(\hat{f} - \hat{\Lambda} * \hat{\varphi}) * \hat{\sigma} = 0$ for all $\sigma \in E_\Lambda$. Hence, since $\hat{\Lambda} * \hat{\varphi} \in \mathscr{O}_M$, and $\hat{E}_\Lambda$ is not a divisor of zero in $\mathscr{O}_M$, we can infer that $\hat{f} - \hat{\Lambda} * \hat{\varphi} = 0$, i.e., $f = \varphi\Lambda = M_\Lambda(\varphi)$. This shows that $M_\Lambda \in \mathfrak{M}(\hat{\mathscr{S}})$ for all $\Lambda \in \mathscr{N}$.

If $\Lambda_1, \Lambda_2 \in \mathcal{N}$ such that $M_{\Lambda_1} = M_{\Lambda_2}$, then we have $\varphi\Lambda_1 = \varphi\Lambda_2$, and hence $\hat{\Lambda}_1 * \hat{\varphi} = \hat{\Lambda}_2 * \hat{\varphi}$ for all $\varphi \in E = E_{\Lambda_1} = E_{\Lambda_2}$. Hence, since $\hat{E}$ is not a divisor of zero in $\mathcal{O}_M$, using a similar argument as in Remark 3.5, we can infer that $\hat{\Lambda}_1 * \varphi = \hat{\Lambda}_2 * \varphi$ for all $\varphi \in \mathcal{S}$. This implies that $\hat{\Lambda}_1 = \hat{\Lambda}_2$, i.e., $\Lambda_1 = \Lambda_2$. Thus, we have shown that the mapping $\Lambda \to M_\Lambda$ is injective.

The above arguments clarify the definition of $\mathcal{N}$, and the remaining part may now be omitted. (To prove that the mapping $\Lambda \to M_\Lambda$ takes $\mathcal{O}_M$ onto $\mathrm{Hom}_{\hat{\mathcal{S}}}(\hat{\mathcal{S}}, \hat{\mathcal{S}})$ use Theorem 3.2.)

REMARK 4.4. By Theorem 4.3, $\mathcal{N}$ can be embedded in $\mathfrak{M}(\hat{\mathcal{S}})$, and after this embedding, we may write $\mathcal{S}' \cap \mathfrak{M}(\hat{\mathcal{S}})$ instead of $\mathcal{N}$.

THEOREM 4.5. *The Fourier transform of $\mathfrak{M}(\mathcal{S})$ extends the distributional Fourier transform of $\mathcal{S}' \cap \mathfrak{M}(\mathcal{S})$, where $\mathcal{S}'$ is considered as a subspace of $\mathfrak{M}(\mathcal{S}, \mathcal{O}_M)$, and moreover, we have*

$$(\mathcal{S}' \cap \mathfrak{M}(\mathcal{S}))\hat{} = \mathcal{S}' \cap \mathfrak{M}(\hat{\mathcal{S}}).$$

PROOF. Suppose that $\Lambda \in \mathcal{S}' \cup \mathfrak{M}(\mathcal{S})$. Then, by Remarks 3.4 and 2.9, $\Lambda \in \mathcal{S}'$ such that $D_\Lambda = \{\varphi \in \mathcal{S} : \Lambda * \varphi \in \mathcal{S}\}$ is not a divisor of zero in $\mathcal{O}_M$, and Λ is identified with the element F_Λ of $\mathfrak{M}(\mathcal{S})$ defined on D_Λ by $F_\Lambda(\varphi) = \Lambda * \varphi$. By Definition 4.1, we have $\hat{F}_\Lambda(\hat{\varphi}) = (\Lambda * \varphi)\hat{} = \hat{\varphi}\hat{\Lambda}$ for all $\varphi \in D_\Lambda$. Hence, since $\check{D}_\Lambda$ is also not a divisor of zero in $\mathcal{O}_M$, it is clear that $\hat{\Lambda} \in \mathcal{N}$. Moreover, since $\hat{D}_\Lambda$ is not a divisor of zero in $\hat{\mathcal{S}}$, we also have $\hat{F}_\Lambda = M_{\hat{\Lambda}}$.

To prove the converse inclusion, suppose now that $\Lambda \in \mathcal{S}' \cap \mathfrak{M}(\hat{\mathcal{S}})$. Then $\Lambda \in \mathcal{S}'$ such that $\hat{E}_\Lambda$ is not a divisor of zero in $\mathcal{O}_M$, and Λ is identified with the element M_Λ of $\mathfrak{M}(\hat{\mathcal{S}})$. Since $(\mathcal{S}')\hat{} = \mathcal{S}'$, there exists $\Lambda_0 \in \mathcal{S}'$ such that $\hat{\Lambda}_0 = \Lambda$. Moreover, since $(\Lambda_0 * \varphi)\hat{} = \hat{\varphi}\hat{\Lambda}_0 = \hat{\varphi}\Lambda = M_\Lambda(\hat{\varphi})$ for all $\varphi \in (\hat{E}_\Lambda)\check{}$ and $(\hat{E}_\Lambda)\check{}$ is also not a divisor of zero in $\mathcal{O}_M$, it is clear that $\Lambda_0 \in \mathcal{S}' \cap \mathfrak{M}(\mathcal{S})$.

To show that the Fourier transforms of some elements of $\mathfrak{M}(\mathcal{S})$ are analytic functions, we have to consider $\mathcal{E}$ to be embedded in $\mathfrak{M}(\hat{\mathcal{S}})$.

THEOREM 4.6. *For $f \in \mathcal{E}$, let M_f be the function defined on $E_f = \{\varphi \in \mathcal{S} : f\varphi \in \mathcal{S}\}$ by $M_f(\varphi) = f\varphi$. Then the mapping defined on $\mathcal{E}$ by $f \to M_f$ is a vector space isomorphism of $\mathcal{E}$ into $\mathfrak{M}(\hat{\mathcal{S}})$ such that $M_{fg} = M_f M_g$ for all $f, g \in \mathcal{E}$.*

PROOF. Simple computation. (Note that $\mathcal{D} \subset E_f$ for all $f \in \mathcal{E}$.)

REMARK 4.7. By this theorem, we may consider $\mathcal{E}$ to be embedded in $\mathfrak{M}(\hat{\mathcal{S}})$. Note that this embedding is also consistent with the former ones. (Recall that $\mathcal{S}$ was embedded in $\mathfrak{M}(\hat{\mathcal{S}})$ in [13, Definition 1.12], and $\mathcal{O}_M$ was embedded in $\mathfrak{M}(\hat{\mathcal{S}})$ together with $\mathcal{N}$ in Remark 4.4.)

THEOREM 4.8. *We have $(\mathfrak{M}(\mathcal{S}) \cap \mathfrak{M}(\mathcal{S}, \mathcal{O}_M))\hat{} \subset \mathcal{E}$.*

PROOF. Let $F \in \mathfrak{M}(\mathcal{S}) \cap \mathfrak{M}(\mathcal{S}, \mathcal{O}_M)$. Then, by Remark 2.9, $F \in \mathfrak{M}(\mathcal{S})$ such that D_F is not a divisor of zero in $\mathcal{O}_M$. Thus, by Theorem 1.5, $Z(D_F) = \varnothing$, i.e., for each $t \in \mathbf{R}^k$, there exists $\varphi_t \in D_F$ such that $\hat{\varphi}_t(t) \neq 0$. Define the function f on $\mathbf{R}^k$ by

$f(t)=F(\varphi_t)^{\wedge}(t)/\hat{\varphi}_t(t)$. Then, since $\hat{F}(\hat{\varphi})\hat{\varphi}_t=\hat{\varphi}\hat{F}(\hat{\varphi}_t)=\hat{\varphi}F(\varphi_t)^{\wedge}$, we have $\hat{F}(\hat{\varphi})==f\hat{\varphi}$ for all $\varphi\in D_F$. Hence, it is clear that $\hat{F}=f\in\mathscr{E}$.

THEOREM 4.9. *Let $\mathscr{A}=\mathscr{A}(\mathbf{R}^k)$ be the space of all analytic functions from $\mathbf{R}^k$ into $\mathbf{C}$. Then*

$$(\mathfrak{M}(\mathscr{S})\cap\mathfrak{N}(\mathscr{D},\mathscr{O}_M))^{\wedge}\subset\mathscr{A}.$$

PROOF. Let $F\in\mathfrak{M}(\mathscr{S})\cap\mathfrak{N}(\mathscr{D},\mathscr{O}_M)$. In the sense of Remark 2.9, this means that $F\in\mathfrak{M}(\mathscr{S})$ such that there exists $G\in\mathfrak{N}(\mathscr{D},\mathscr{O}_M)$ such that $D_{F\cap G}$ is not a divisor of zero in $\mathscr{O}_M$. Moreover, since $G\in\mathfrak{N}(\mathscr{D},\mathscr{O}_M)$, $G^{-1}(\mathscr{D})$ is not a divisor of zero in $\mathscr{O}_M$ [13. Definition 1.6]. Thus, $D=G^{-1}(\mathscr{D})\cap D_{F\cap G}$ is also not a divisor of zero in $\mathscr{O}_M$. On the other hand, it is clear that $F(\varphi)=G(\varphi)\in\mathscr{D}$ for all $\varphi\in D$.

By [13, Proposition 1.16], we have $F(\varphi)^{\wedge}=\hat{F}(\hat{\varphi})=\hat{F}\hat{\varphi}$ for all $\varphi\in D_F$. Moreover, since we also have $F\in\mathfrak{M}(\mathscr{S},\mathscr{O}_M)$, Theorem 4.8 shows that $\hat{F}\in\mathscr{E}$.

Let $s\in\mathbf{R}^k$. Then, by Theorem 1.5, there exists $\varphi\in D$ such that $\hat{\varphi}(s)\neq 0$. Since $\hat{\varphi}$ is continuous, there exists an open neighbourhood V of s such that $\hat{\varphi}(t)\neq 0$ for all $t\in V$. Thus, we have $\hat{F}(t)=F(\varphi)^{\wedge}(t)/\hat{\varphi}(t)$ for all $t\in V$. Hence, since $\hat{\varphi}$ and $F(\varphi)^{\wedge}$ are analytic functions on $\mathbf{R}^k$, it is clear that $\hat{F}$ is analytic in V.

Finally, to study the continuity properties of the Fourier transform of $\mathfrak{M}(\mathscr{S})$, we have to consider some topologies on $\mathfrak{M}(\mathscr{S})$ and $\mathfrak{M}(\hat{\mathscr{S}})$. The most natural topologies for the multiplier extensions of admissible locally convex vector modules seem to be the locally convex inductive limit topologies described in [19]. (See also [15, Remark 3.9].) Therefore, in the following, we shall consider $\mathfrak{M}(\mathscr{S})$ and $\mathfrak{M}(\hat{\mathscr{S}})$ to be equipped with those locally convex topologies.

THEOREM 4.10. *The Fourier transform of $\mathfrak{M}(\mathscr{S})$ is also a topological isomorphism of $\mathfrak{M}(\mathscr{S})$ onto $\mathfrak{M}(\hat{\mathscr{S}})$.*

PROOF. By a well-known property of locally convex inductive limit topologies [3, Proposition 2.12.1], it is enough to show that for any net (F_ν) in $\mathfrak{M}(\mathscr{S})$ and any $F\in\mathfrak{M}(\mathscr{S})$, the conditions $F\in\lim\limits_{\nu}{}_{\mathfrak{M}}F_\nu$ and $\hat{F}\in\lim\limits_{\nu}{}_{\hat{\mathfrak{M}}}\hat{F}_\nu$, where $\lim_{\mathfrak{M}}$ and $\lim_{\hat{\mathfrak{M}}}$ are the Mikusiński-type convergences [15] in $\mathfrak{M}=\mathfrak{M}(\mathscr{S})$ and $\hat{\mathfrak{M}}=\mathfrak{M}(\hat{\mathscr{S}})$, respectively, are equivalent. However, this is quite obvious, since for any $\varphi\in\mathscr{S}$, we have $F(\varphi)=\lim\limits_{\nu}F_\nu(\varphi)$ in $\mathscr{S}$ if and only if $\hat{F}(\hat{\varphi})=F(\varphi)^{\wedge}=\lim\limits_{\nu}F_\nu(\varphi)^{\wedge}=\lim\limits_{\nu}\hat{F}_\nu(\hat{\varphi})$ in $\hat{\mathscr{S}}$, and moreover, since for any $D\subset\mathscr{S}$, D is not a divisor of zero in $\mathscr{S}$ if and only if $\hat{D}$ is not a divisor of zero in $\hat{\mathscr{S}}$.

REMARK 4.11. A similar argument as above shows that this theorem also remains valid if we consider $\mathfrak{M}(\mathscr{S})$ and $\mathfrak{M}(\hat{\mathscr{S}})$ to be equipped with either the T_1-topologies described in [15], or the Hausdorff topologies described in [20]. However, these topologies seem now to be very unnatural.

For the distributional Fourier transform, we have $\hat{\hat{\ }}=\vee$ [6, p. 177]. To have this identity also in the present theory, we have to define the Fourier transform of $\mathfrak{M}(\hat{\mathscr{S}})$ too. The transform $\vee$ for $\mathfrak{M}(\mathscr{S})$ or $\mathfrak{M}(\hat{\mathscr{S}})$ is considered to be defined according to Section 2 of [16].

DEFINITION 4.12. For $F\in\mathfrak{M}(\hat{\mathcal{S}})$, the function $\hat{F}$ defined on $\hat{D}_F$ by $\hat{F}(\hat{\varphi})=F(\varphi)\hat{}$ is called the Fourier transform of F.

THEOREM 4.13. *The mapping defined on* $\mathfrak{M}(\hat{\mathcal{S}})$ *by* $F\to\hat{F}$ *is an algebraic and topological isomorphism of* $\mathfrak{M}(\hat{\mathcal{S}})$ *onto* $\mathfrak{M}(\mathcal{S})$ *which extends the distributional Fourier transform of* $\mathcal{S}'\cap\mathfrak{M}(\hat{\mathcal{S}})$. *Moreover, we have* $(\hat{F})\hat{}=\check{F}$ *for all* $F\in\mathfrak{M}(\mathcal{S})\cup\mathfrak{M}(\hat{\mathcal{S}})$.

PROOF. If $F\in\mathfrak{M}(\hat{\mathcal{S}})$, then a simple computation shows that $\hat{F}\in\mathfrak{M}(\mathcal{S})$. Moreover, we have

$$(\hat{F})\hat{}\,(\varphi)=(\hat{F})\hat{}\big((\check{\varphi})\hat{}\big)=\big(\hat{F}((\check{\varphi})\hat{})\big)\hat{}=\big(F(\check{\varphi})\big)\hat{}=F(\check{\varphi})\check{}=\check{F}(\varphi)$$

for all $\varphi\in\check{D}_F$, which shows that $(\hat{F})\hat{}=\check{F}$, i.e. $\hat{F}=(\hat{F})^{\wedge^{-1}}$ where $\wedge^{-1}$ denotes the inverse of the Fourier transform of $\mathfrak{M}(\mathcal{S})$. Hence, the properties of the Fourier transform of $\mathfrak{M}(\hat{\mathcal{S}})$ can easily be derived.

REMARK 4.14. Since $\mathcal{E}\subset\mathfrak{M}(\hat{\mathcal{S}})$, by considering the Fourier transform of $\mathfrak{M}(\hat{\mathcal{S}})$ restricted to $\mathcal{E}$, we get a Fourier transform for $\mathcal{E}$. In particular, by Theorem 4.13, we have $\hat{f}*\hat{\varphi}=(f\varphi)\hat{}\in\hat{\mathcal{D}}$, and hence also $\hat{f}(\hat{\varphi})\in\hat{\hat{\mathcal{D}}}$ for all $\varphi\in\mathcal{D}$.

Using the above Fourier transform of $\mathcal{E}$, it is also possible to define the Fourier transforms of the multiplier extensions $\mathfrak{M}(\mathcal{S},\mathcal{O}_M)$ and $\mathfrak{M}(\mathcal{D},\mathcal{E})$ (and also the Mikusiński operator field $\mathfrak{M}(\mathcal{E}_R)$, where $\mathcal{E}_R$ is the convolution algebra of functions in $\mathcal{E}$ with supports in various right-sided orthants $\mathbf{R}^k_\lambda=\{x\in\mathbf{R}^k: x\geqq\lambda\}$) within the framework of our theory. However, this may only be the subject of same forthcoming papers.

Moreover, we also plan to show that our Fourier transforms are also compatible with a slight modification of the one given by EHRENPREIS [2]. Note that, using [2], we can also consider $\mathcal{D}'\cap\mathfrak{M}(\hat{\mathcal{S}})$. (For some ideas in this respect, see Theorems 4.3 and 4.6.)

For all these purposes, it seems convenient to use that, after some natural identifications, we also have $\mathfrak{M}(\mathcal{S})=\mathfrak{M}(\mathcal{Z})$ and $\mathfrak{M}(\hat{\mathcal{S}})=\mathfrak{M}(\hat{\mathcal{Z}})$, where $\mathcal{Z}$ denotes $\hat{\mathcal{D}}$ as a convolution algebra and $\hat{\mathcal{Z}}$ denotes $\mathcal{D}$ as a function algebra.

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DISTRIBUTION OF THE VALUES OF ω IN SHORT INTERVALS

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Introduction. Let $\omega(m)$ denote the number of prime factors of m, $1 \leqq b(n) \leqq n$ be a sequence of integers and let

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy.$$

In [1], it is shown that

$$(1) \quad \frac{1}{b(n)} \text{ card } \{(n, n+b(n))] \cap (m: \omega(m) - \log\log m < x(\log\log m)^{1/2})\} \to \Phi(x)$$

as $n \to \infty$, provided $b(n) \geqq Tn^{\alpha}$ for some $T > 1$ and $0 < \alpha \leqq 1$. Similar results for general additive arithmetic functions are also proved in [1]. In this connection P. Erdős raised the following question. How small can one let $b(n)$ to be, so that (1) still holds? We have the following result in this direction.

THEOREM. *Let* $1 \leqq a(n) \leqq (\log\log n)^{1/2}$ *be a sequence of real numbers tending to infinity. Then* (1) *holds if* $b(n) \geqq n^{a(n)(\log\log n)^{-1/2}}$.

NOTATIONS. Let Q denote the set of all primes and for any set E of integers let

$$\nu_n(E) = \frac{1}{b(n)} \text{ card } \{E \cap (n, n+b(n)]\}.$$

We require the following lemma (for a proof see [1]).

LEMMA (Lemma 1 of [1]). *Let* $\delta_p(m) = 1 - \dfrac{1}{p}$ *or* $-\dfrac{1}{p}$ *according as* $p|m$ *or not. Let* $\{a_p\}$ *be a sequence of real numbers and let* $r, k \leqq s$ *be integers. Then*

$$\Sigma' \big(\Sigma'' a_p \delta_p(m)\big)^2 \leqq (2s + 8k^2)\, \Sigma'' \frac{1}{p} a_p^2,$$

where Σ' *denotes the sum over all integers* $m \in (r, r+s]$ *and* Σ'' *denotes the sum over all primes* $p < k$.

Proof of the theorem. To avoid repetitions of arguments of [1] and [2], we give only the main steps of the proof. Let $k = k_n = 1 + [(b(n))^{1/4}]$ and for any $t \geqq 1$,

* Part of this work was done while the author was visiting the Mathematical Institute of the Hungarian Academy of Sciences, Budapest, in November 1980.

let $\omega_t(m)=\sum_{p|m,\, p<t} 1$. For $m\in(n, n+b(n)]$,

$$|\omega(m)-\omega_k(m)| \leqq \sum_{p|m,\, k<p\leqq 2n} 1 \leqq 1+((\log 2n)/\log k) \ll \frac{(\log\log n)^{1/2}}{a(n)} \tag{2}$$

and

$$0 \leqq \log\log 2n-\log\log m \leqq \log\log 2n - \log\log n \to 0 \tag{3}$$

as $n\to\infty$. Since

$$\sum_{k<p\leqq n}\frac{1}{p} = \log\log n-\log\log k+o(1) \ll \log\log\log n,$$

by (2) and (3) it is enough to prove

$$\nu_n\left\{m:\omega_k(m)-\sum_{p\leqq k}\frac{1}{p} < x\left(\sum_{p<k}\frac{1}{p}\right)^{1/2}\right\} \to \Phi(x). \tag{4}$$

Now let $r=r_n=\exp\left((\log n)(\log\log n)^{-1/2}\right)$. Clearly

$$\left(\sum_{p\leqq r}\frac{1}{p}\right)\left(\sum_{p\leqq k}\frac{1}{p}\right)^{-1} \to 1 \tag{5}$$

as $n\to\infty$. So by the lemma, it follows that for every $\varepsilon>0$,

$$\nu_n\left\{m:\left|\omega_r(m)-\omega_k(m)+\sum_{r<p\leqq k}\frac{1}{p}\right| > \varepsilon\left(\sum_{p\leqq k}\frac{1}{p}\right)^{1/2}\right\} \to 0 \tag{6}$$

as $n\to\infty$. In view of (5) and (6) it is enough to show that, as $n\to\infty$,

$$F_n(x)=\nu_n\left\{m:\omega_r(m)-\sum_{p<r}\frac{1}{p} < x\left(\sum_{p<r}\frac{1}{p}\right)^{1/2}\right\} \to \Phi(x). \tag{7}$$

To prove (7), we introduce a sequence $\{\xi_p : p\in Q\}$ of independent random variables with

$$P\left(\xi_p = 1-\frac{1}{p}\right)=\frac{1}{p} \quad\text{and}\quad P\left(\xi_p=-\frac{1}{p}\right)=1-\frac{1}{p}.$$

Put $\zeta_n=\left(\sum_{p\leqq r}\frac{1}{p}\right)^{-1/2}\sum_{p\leqq r}\xi_p$. It follows that

$$\zeta_n \text{ converges weakly to } \Phi. \tag{8}$$

We shall now show that the distribution of ζ_n does not differ much from F_n. By a proof similar to the proof of lemma 3.1 of [2], we have for any integer $t\geqq 1$

$$\left|E(\zeta_n^t)-\frac{1}{b(n)}\sum_{n<m\leqq n+b(n)}\left(\omega_r(m)-\sum_{p\leqq r}\frac{1}{p}\right)^k\right| \leqq \frac{2^t r^t}{b(n)} \tag{9}$$

and by lemma 3.2 of [2] we have

$$|E(\zeta_n^t)| \leqq t!\, e^t. \tag{10}$$

Since $r_n^t/b(n)\to 0$ as $n\to\infty$ for any $t\geqq 1$, (7) follows from (8), (9), (10) above and Theorem 11.2 of [2]. This completes the proof.

REMARKS. If $c(m)\to\infty$, then, except possibly for $o(b(n))$ integers $m\in(n, n+b(n)]$, we have $|\omega(m)-\log\log m|<c(m)(\log\log m)^{1/2}$. In particular, by taking $b(n)=\exp\{\log n)(\log\log\log n/\log\log n)^{1/2}\}$ we have

$$|\omega(m)-\log\log m| < (\log\log\log m)(\log\log m)^{1/2}$$

for all but $o(b(n))$ many $m\in(n, n+b(n)]$.

As in [1] and [2], by going over to the Brownian motion we obtain, as $n\to\infty$, that

$$\frac{1}{b(n)}\operatorname{card}\left\{n < m \leqq n+b(n): \max_{t\leqq n}(\omega_t(m)-\log\log t) < x(\log\log m)^{1/2}\right\} \to$$
$$\to \sqrt{\frac{2}{\pi}}\int_0^x e^{-\frac{1}{2}y^2}\,dy.$$

As a consequence, in particular, it follows that

$$\left|\max_{t\leqq n}(\omega_t(m)-\log\log t)\right| < (\log\log\log m)(\log\log m)^{1/2}$$

for all but $o(b(n))$ many integers $m\in(n, n+b(n)]$.

One can show that many results of [1] still hold for general arithmetic functions, when the restriction $b(n)\geqq Tn^{\alpha}$ is weakened to $b(n)\geqq Tn^{\alpha(n)}$, where $\alpha(n)\to 0$ at an appropriate rate.

Some open problems. In this connection P. Erdős and I. Z. Ruzsa raised the following questions.

(a) What is the largest value of $a(n)$ such that, if $b(n)<a(n)$ for all n, then (1) fails to hold?

(b) Does (1) hold if $b(n)=n^{(\log\log n)^{-1/2}}$?

The author wishes to thank P. Erdős and I. Z. Ruzsa for many valuable discussions.

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A HIERARCHY OF REGULAR OPEN SETS OF THE CANTOR SPACE

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The present paper is a revised version of my unpublished paper entitled "The Boolean algebra of recursively regular open sets of the Cantor space." In this paper we endeavor to present a few new results together with neater constructions and corrections to old proofs.

The main concepts of this paper were defined in [1] where our preoccupation was with the concept of "constructive Boolean algebraic extensions" of the atomless Boolean algebra, say $\mathscr{B}_a$. For convenience we viewed $\mathscr{B}_a$ as the clopen sets of the Cantor space, say $\mathfrak{C}$. For the clopen sets the Boolean operations $\wedge$, $\vee$ and $\perp$ are taken as the ordinary intersection, union and complement. For an extension $\mathscr{B}$ of $\mathscr{B}_a$ consisting of regular open sets of $\mathfrak{C}$, the operations $\wedge$, $\vee$ and $\perp$ are taken as intersection, interior of the closure of union and complement of closure.

In the present paper we study various subclasses of open sets of $\mathfrak{C}$ whose elements can be "constructed" in a sense that is made precise using recursion theory. Just as the concept of constructive extension of $\mathscr{B}_a$ depends on the computability of $\mathscr{B}_a$ "constructive open sets" depend on a fixed indexing φ of $\mathscr{B}_a$ arising from the computability of $\mathscr{B}_a$. Although $\mathscr{B}_a$ has infinitely many indices (uncountably many, even), any two presentations of $\mathscr{B}_a$ corresponding to different indices are recursively isomorphic.

Basic concepts

Let U be an open set of $\mathfrak{C}$. To express that U is *regular open* we write $U^{\top\top}=U$ (i.e., if a clopen set $\mathcal{O}\subseteqq U^-$ then already $\mathcal{O}\subseteqq U$). Of course, it is well-known that any open set of $\mathfrak{C}$ is some countable union of elements of $\mathscr{B}_a$. With an admissible indexing φ of $\mathscr{B}_a$ presumed fixed we have a fixed enumeration $\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_n, \dots$ of the clopen sets of $\mathfrak{C}$ (where $\mathcal{O}_0=\varnothing$ and $\mathcal{O}_1=\mathfrak{C}$). An open set U is *r.e. open* (relative to φ) in case there is a recursive function f such that $U=\bigcup_{n\in\omega}\mathcal{O}_{f(n)}$. U is *recursive open* (relative to φ) in case U and $U^\perp$ are both r.e. open. U is *recursively regular open* (relative to φ) in case there exist recursive functions f and g such that $U=(\bigcup_{n\in\omega}\mathcal{O}_{g(n)})^{\perp\perp}$ and $U^\perp=(\bigcup_{n\in\omega}\mathcal{O}_{g(n)})^{\perp\perp}$ (i.e., U and $U^\perp$ are suprema of recursive sequences of clopen sets).

* The author thanks G. Nelson for many stimulating conversations about the constructions in this paper.

Throughout this paper, when we use the terms r.e. open, recursive open, and recursively regular open we shall omit the phrase "relative to φ" with the clear understanding that a set U which is r.e. open (rel. to φ) may in fact not be r.e. open relative to some admissible indexing ψ. However, when we give a procedure for constructing an r.e. open set U (relative to φ) this same procedure will produce an r.e. open set V relative to ψ. Our approach will be to construct open sets U which are, say, r.e. open relative to φ rather than choosing an open set U having some a priori existence and worrying about whether or not U is r.e. open relative to some φ.

The following figure indicates those subclasses of regular open sets which interest us. Their relative "sizes" are indicated in a tree diagram. The inclusions indicated in Figure 1 are more or less obvious:

$$(1) \subseteqq (2) \subseteqq (3) \subseteqq (4) \subseteqq (5) \subseteqq (7) \subseteqq (9),$$

$$(4) \subseteqq (6) \subseteqq (8) \subseteqq (9) \quad \text{and} \quad (5) \subseteqq (8).$$

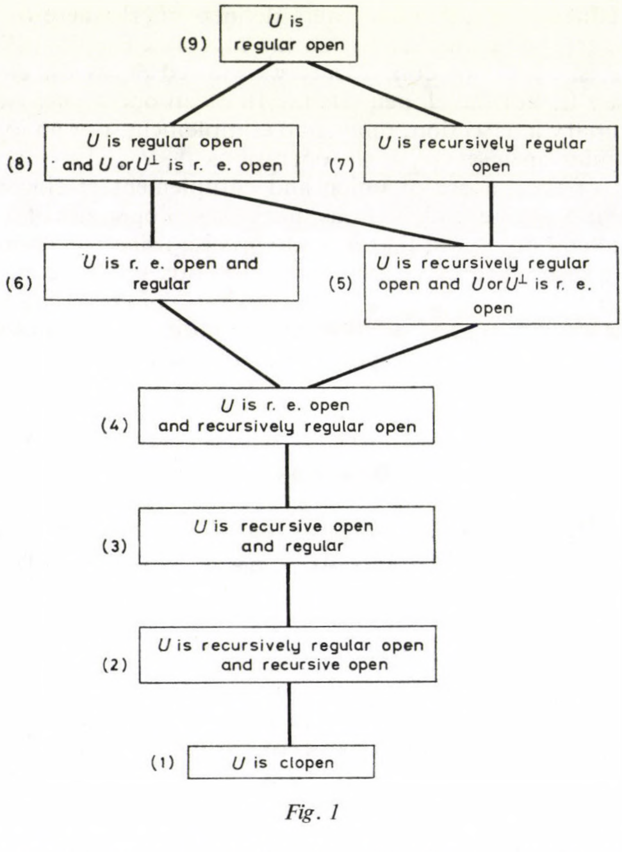

Fig. 1

The main results of this paper determine that all inclusions except $(2) \subseteqq (3)$ are proper. Some of the proper inclusions are immediate consequences of results established in [1]. For example, Theorem 7 in [1] proved the existence of recursive regular open sets U_a and U_b such that $(U_a \cup U_b)^{\perp\perp}$ is not r.e. open (but $U_a^{\perp} \cap U_b^{\perp}$ is r.e. open). By letting $U = U_a^{\perp} \cap U_b^{\perp}$, we establish that (3)$\subset$(4). By letting $U = (U_a \cup U_b)^{\perp\perp}$, we establish that (4)$\subset$(5) and (6)$\subset$(8). For (1)$\subset$(2), choose $U = U_a$.

Before proving other proper inclusions, it is instructive to give a nicer proof of Theorem 7 in [1]. In the proof of Lemma 1 below, we introduce a simpler version of the construction given in the proof of Theorem 7 in [1]. The key to the simplicity is that there is no need to worry about "regularity" of sets being constructed. Regularity is obtained by using Lemma 2 (splitting lemma) below.

LEMMA 1. *There are disjoint r.e. open sets V and W such that V is recursive open and not regular while W is regular and not recursive open such that*
(i) $V^{\perp} = W$, *and*
(ii) $V^{\perp\perp}$ *is recursively regular open and not r.e. open.*

LEMMA 2 (Splitting Lemma). *Every recursive open set can be split (nontrivially) into two disjoint sets which are both recursive open and regular.*

PROOF OF LEMMA 2. This lemma is just Theorem 16 (iii) of [1].

PROPOSITION 3 (Theorem 7 in [1]). *There exist two sets U_a and U_b such that both are recursive open and regular and $(U_a \cup U_b)^{\perp\perp}$ is not r.e. open.*

PROOF. Use Lemma 1 to build V and W. Then use Lemma 2 to split V into U_a and U_b.

Before beginning the crucial part of the argument (i.e., the proof of Lemma 1), we recall the function $\alpha(e)$ from [1]. Let $\{\mathcal{O}_{\alpha(e)}\}_{e\in\omega}$ be as in [1]. In particular,
(i) $\alpha(e)$ is a recursive function,
(ii) the $\mathcal{O}_{\alpha(e)}$'s are pairwise disjoint,
(iii) $(\bigcup_{e\in\omega} \mathcal{O}_{\alpha(e)}) \subset \mathfrak{C}$ and $(\bigcup_{e\in\omega} \mathcal{O}_{\alpha(e)})^{-} = \mathfrak{C}$,
(iv) for any e, either $\mathcal{O}_e \subset \bigcup_{k\leqq e} \mathcal{O}_{\alpha(k)}$ or $(\bigcup_{k\leqq e} \mathcal{O}_{\alpha(e)})' \subset \mathcal{O}_e$,
(v) Bd $(\bigcup_{e\in\omega} \mathcal{O}_{\alpha(e)})$ (i.e., boundary of $\bigcup_{e\in\omega} \mathcal{O}_{\alpha(e)})$ is just a single point, say x_0.
(Otherwise, we can determine an $\mathcal{O}_e$ which denies (iv).)

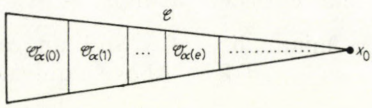

Fig. 2

PROOF OF LEMMA 1. Let $j: \omega\times\omega \to \omega$ be the pairing function which enumerates the pairs in the order (0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), ... At stage $t = j(e, s)$, we use $V^{(t)}$ and $W^{(t)}$ to denote clopen sets which will "approximate" V and W, respectively. We shall define recursive functions $a(t)$ and $b(t)$ such that

$$V = \bigcup_{t\in\omega} V^{(t)} = \bigcup_{t\in\omega} \mathcal{O}_{a(t)}$$

and

$$W = \bigcup_{t\in\omega} W^{(t)} = \bigcup_{t\in\omega} \mathcal{O}_{b(t)}.$$

Assume that $V^{(t-1)}$ and $W^{(t-1)}$ have already been constructed. Let $t=j(e,s)$. In the enumeration of $\varphi_0(n), \dots, \varphi_e(n), \dots$ corresponding to a universal Turing machine, compute each of $\varphi_t(0), \varphi_e(1), \dots, \varphi_e(s)$ for $s+1$-steps.

Case 1. $\mathcal{O}_{\alpha(e)} \subseteqq \bigcup \{\mathcal{O}_{\varphi_e(n)} : n \leqq s+1$ and $\varphi_e(n)\downarrow$ within $s+1$ steps$\}$. Set $a(t)=0$ and $b(t)=\mu_j(\mathcal{O}_j = \mathcal{O}_{\alpha(e)} - V^{(t-1)})$.

Case 2. Otherwise. Define $A(t)$, the index of the clopen set "attacked" at stage t by $A(t)=\mu k(\mathcal{O}_k \subseteqq \mathcal{O}_{\alpha(e)}$ and $((\mathcal{O}_k - V^{(t-1)}) \neq \emptyset$ and $(\forall t' < t)(A(t') \neq k))$, set $a(t)=\mu_j(\emptyset \neq \mathcal{O}_j \subset \mathcal{O}_{A(t)} - V^{(t-1)})$ and $b(t)=0$.

This completes the construction.

Easy arguments give rise to the following consequences:

(1) $V \cap W = \emptyset$.

(2) If $\mathcal{O}_e \cap V = \emptyset$ or $\mathcal{O}_e \cap W = \emptyset$ then $\mathcal{O}_e \subseteqq \bigcup_{k \leqq e} \mathcal{O}_{\alpha(k)}$.

(3) $A(t)$ is one-to-one on its domain.

(4) For any given e, either $\mathcal{O}_{\alpha(e)} \subseteqq V^-$ and $\mathcal{O}_{\alpha(e)} \nsubseteqq V$, or $\mathcal{O}_{\alpha(e)} \subseteqq W$ or $\mathcal{O}_{\alpha(e)} \subseteqq V \cup W$. Moreover, if $\mathcal{O}_{\alpha(e)} \subseteqq W^-$ then already $\mathcal{O}_{\alpha(e)} \subseteqq W$.

Claim (i). $V^\perp = W$.

Clearly, $W \subseteqq V^\perp$. Suppose that the inclusion is proper. Then there is a clopen set $\mathcal{O}_e$ such that $\mathcal{O}_e \subseteqq V^\perp$ and $\mathcal{O}_e \nsubseteqq W$. Thus, $\mathcal{O}_e \cap V^{\perp\perp} = \emptyset$; whence $\mathcal{O}_e \cap V = \emptyset$. Therefore $\mathcal{O}_e \subseteqq \bigcup_{k \leqq e} \mathcal{O}_{\alpha(k)}$ by (2). So, for some k, $\mathcal{O}_e \cap \mathcal{O}_{\alpha(k)} \nsubseteqq W$. Now, suppose that Case 1 holds at stage $t=j(k,s)$, for some $s \in w$. Then $\mathcal{O}_{\alpha(k)} - V^{(t-1)} \subseteqq W$. But $\mathcal{O}_{\alpha(k)} \cap \mathcal{O}_e \subseteqq \mathcal{O}_{\alpha(k)} - V^{(t-1)}$. Hence $\mathcal{O}_{\alpha(k)} \cap \mathcal{O}_e \subseteqq W$; a contradiction follows.

Claim (ii). $V^{\perp\perp}$ is not r.e. open.

Suppose, on the contrary, that $V^{\perp\perp} = \bigcup_{n \in \omega} \mathcal{O}_{\varphi_e(n)}$, for some e. So, $V^{\perp\perp -} = (\bigcup_{n \in \omega} \mathcal{O}_{\varphi_e(n)})^-$. Thus, $V^- = (\bigcup_{n \in \omega} \mathcal{O}_{\varphi_e(n)})^-$. Thus, $\mathcal{O}_k \subseteqq V^-$ iff $\mathcal{O}_k \subseteqq (\bigcup_{n \in \omega} \mathcal{O}_{\varphi_e(n)})^-$ iff $\mathcal{O}_k \subseteqq \bigcup_{n \in \omega} \mathcal{O}_{\varphi_e(n)}$, by regularity. It suffices to refute this equivalence.

Case 1. $\mathcal{O}_{\alpha(e)} \subseteqq \bigcup_{n \in \omega} \mathcal{O}_{\varphi_e(n)}$. Let s be the least such natural number such that $\mathcal{O}_{\alpha(e)} \subseteqq \bigcup \{\mathcal{O}_{\varphi_e(n)} : n \leqq s+1$ and $\varphi_e(n)\downarrow$ in at most $(s+1)$ steps$\}$. By compactness, s exists. Let $t=j(e,s)$. The minimality of s and an easy induction imply that $\mathcal{O}_{\alpha(e)} - V^{(t)} \neq \emptyset$. Hence, $\emptyset \neq \mathcal{O}_{\alpha(e)} \nsubseteqq V^-$. Above equivalence is refuted by using $\mathcal{O}_{\alpha(e)}$ for $\mathcal{O}_k$.

Case 2. At stage $t=j(e,s)$, Case 1 does not hold. Suppose that $\mathcal{O}_{\alpha(e)} \nsubseteqq V^-$. Then, the nonempty open set $(\mathcal{O}_{\alpha(e)} - V)$ contains a clopen set $\mathcal{O}_m$. Hence there is a number t such that $A(t)=m$. Hence $\mathcal{O}_m \cap V \neq \emptyset$. This contradicts the fact that $\mathcal{O}_m \subseteqq (\mathcal{O}_{\alpha(e)} - V^-)$. Thus, $\mathcal{O}_{\alpha(e)} \subseteqq V^-$.

Note that $V^{\perp\perp} \supset V$, since V is r.e. open and $V^{\perp\perp}$ is not. Hence, unlike W, V is not regular. But V is recursive open, since V and $V^\perp = W$ are both r.e. open. Clearly W is regular, since "$\perp$" applied to any open set yields a regular open set. But W is not recursive open, since $W^\perp = V^{\perp\perp}$ is not r.e. open.

Moreover, $V^{\perp\perp}$ is recursively regular open, since $V^\perp = W^{\perp\perp}$ and both V and W are r.e. open.

COROLLARY 4. *There exists a recursively regular open set U such that U is not r.e. open and $U^{\perp}$ is r.e. open.*

PROOF. Choose V of Lemma 1. Let $U=V^{\perp\perp}$.

We can give a slight modification of the above construction to produce a result which is related to Corollary 4.

THEOREM 5. *There exists a recursively regular open set V such that neither V nor $V^{\perp}$ is r.e. open.*

The construction. Let the sequence $\{\mathcal{O}_{\alpha(e)}\}_{e\in\omega}$ be as in the proof of Lemma1. Now construct W_1 and W_2 in a stepwise fashion. At stage $t=j(e,s)$, $W_1^{(t)}$ and $W_2^{(t)}$ will be clopen sets which "approximate" W_1 and W_2, respectively. As before, we want to define recursive functions $a(t)$ and $b(t)$ in such a way that $W_1=\bigcup_{t\in\omega} W_1^{(t)}=$ $=\bigcup_{t\in\omega}\mathcal{O}_{a(t)}$ and $W_2=\bigcup_{t\in\omega} W_2^{(t)}=\bigcup_{t\in\omega}\mathcal{O}_{b(t)}$.

Consider stage $t=j(e,s)$.

Case 1. $\mathcal{O}_{\alpha(f)}\subseteq\bigcup\{\mathcal{O}_{\varphi_f(n)}: n\leqq s+1$ and $\varphi_s(n)\downarrow$ within $s+1$ steps$\}$, where $e=2f+1$ or $e=2f$.

For $e=2f+1$ set $a(t)=\mu_j(\mathcal{O}_j\subseteq\mathcal{O}_{\alpha(f)}\wedge(\mathcal{O}_j-W_2^{(t-1)}\neq\emptyset))$; set $b(t)=\mu_j[\mathcal{O}_j=$ $=(\mathcal{O}_{\alpha(f)}-\mathcal{O}_{a(t)})]$.

For $e=2f$, set $b(t)=\mu_j(\mathcal{O}_j\subseteq\mathcal{O}_{\alpha(f)}\wedge(\mathcal{O}_j-W_1^{(t-1)}\neq\emptyset))$; set $a(t)=\mu_j[\mathcal{O}_j=$ $=(\mathcal{O}_{\alpha(f)}-\mathcal{O}_{b(t)})]$.

Case 2. $\mathcal{O}_{\alpha(f)}\nsubseteq\bigcup\{\mathcal{O}_{\varphi_f(n)}: n\leqq s+1$ and $\varphi_s(n)\downarrow$ within $s+1$ steps$\}$, where $e=2f+1$ or $e=2f$.

Define $A(t)$, the index of the clopen set "attacked" at stage t, by

$$A(t)=\mu_k(\mathcal{O}_k\subseteq\mathcal{O}_{\alpha(f)}\wedge\mathcal{O}_m-(W_1^{(t-1)}UW_2^{(t-1)}))\neq\emptyset.$$

For $e=2f+1$, set $a(t)=\mu_j[\emptyset\neq\mathcal{O}_j\subset\mathcal{O}_{A(t)}]$ and $b(t)=0$.

For $e=2f$, set $b(t)=\mu_j[\emptyset\neq\mathcal{O}_j\subset\mathcal{O}_{A(t)}]$ and $a(t)=0$.

This ends the construction. Intuitively, as $\mathcal{O}_{\alpha(f)}$ arises each time we alternate back and forth putting pieces of it into W_1 and W_2 until such time as it is decided that $\mathcal{O}_{\alpha(f)}$ is to be contained in $W_1\cup W_2$. If this decision is not reached then undoubtedly $\mathcal{O}_{\alpha(f)}$ intersects both $(W_1^- - W_1)$ and $(W_2^- - W_2)$.

Easy consequences of the construction are:

(0) The functions $a(t)$ and $b(t)$ are recursive and $A(t)$ is partial recursive.

(1) $W_1\cap W_2=\emptyset$.

(2) $\mathcal{O}_n\cap W_1=\emptyset$ or $\mathcal{O}_e\cap W_2=\emptyset$ implies $\mathcal{O}_e\subseteq\bigcup_{k\leqq e}\mathcal{O}_{\alpha(k)}$.

Claim: $W_1^{\perp}=W_2^{\perp\perp}$. (Hence $W_2^{\perp}=W_1^{\perp\perp}$.) Clearly, $W_2^{\perp\perp}\subseteq W_1^{\perp}$.

Suppose that the inclusion is proper. Then $W_2^-\subset W_1^{\perp}$. Otherwise, we are done. So $W_1^{\perp\perp}\subset W_2^{\perp}$ and $W_2^{\perp\perp}\subset W_1^{\perp}$. Also, $W_1^{\perp\perp -}\subset W_2^{\perp}$ and $W_2^{\perp\perp -}\subset W_1^{\perp}$, lest we are done. Thus, $W_1^{\perp}\cap W_2^{\perp}\neq\emptyset$. Let $\mathcal{O}_m\subseteq W_1^{\perp}\cap W_2^{\perp}$. Then $\mathcal{O}_m\cap W_1=\emptyset$, $\mathcal{O}_m\cap W_2=\emptyset$. Hence, by (2), $\mathcal{O}_m\subseteq\bigcup_{k\leqq m}\mathcal{O}_{\alpha(k)}$. For some $f\leqq m$, $\mathcal{O}_{\alpha(f)}\cap\mathcal{O}_m\neq\emptyset$. Let $\mathcal{O}_d=\mathcal{O}_{\alpha(f)}\cap\mathcal{O}_m$. Now $\mathcal{O}_d\subseteq\mathcal{O}_{\alpha(f)}$ and $\mathcal{O}_d-(W_1^{(t-1)}\cap W_2^{(t-1)})\neq\emptyset$, for t. Thus at some stage $t=j(2f+1,s)$ or $t=j(2f,s)$, $\mathcal{O}_d$ is "attacked". A contradiction follows.

Another consequence is:

(3) $\mathcal{O}_{\alpha(f)} \cap W_1^{\perp\perp} \subseteq W_1$ iff $\mathcal{O}_{\alpha(f)} \cap W_2^{\perp\perp} \subseteq W_2$. Also, $\mathcal{O}_{\alpha(f)} \cap W_1 \neq \emptyset$ and $\mathcal{O}_{\alpha(f)} \cap W_2 \neq \emptyset$, for all f. Moreover, $W_1^- \cap W_2^- \neq \emptyset$, since easily $x_0 \in W_1^- \cap W_2^-$.

Claim: $W_1^{\perp\perp}$ is not r.e. open. (Hence $W_2^{\perp\perp}$ is not r.e. open.) As before, suppose $W_1^{\perp\perp} = \bigcup_{n\in\omega} \mathcal{O}_{\varphi_f(n)}$, i.e., $W_1^- = \big(\bigcup_{n\in\omega} \mathcal{O}_{\varphi_e(n)}\big)^-$. Then, by regularity,

$$(*) \qquad \mathcal{O}_k \subseteq W_1^- \quad \text{iff} \quad \mathcal{O}_k \subseteq \bigcup_{n\in\omega} \mathcal{O}_{\varphi_f(n)}.$$

By choosing $\mathcal{O}_k$ appropriately, we refute this equivalence.

Case 1. $\mathcal{O}_{\alpha(f)} \subseteq \big(\bigcup_{n\in\omega} \mathcal{O}_{\varphi_f(n)}\big)$.

Since $\mathcal{O}_{\alpha(f)} \nsubseteq W_1^-$, we refute $(*)$ under this case by choosing $\mathcal{O}_k = \mathcal{O}_{\alpha(k)}$.

Case 2. $\mathcal{O}_{\alpha(f)} \nsubseteq \big(\bigcup_{n\in\omega} \mathcal{O}_{\varphi_f(n)}\big)$.

Consider the open set $S = \mathcal{O}_{\alpha(f)} - W_2^-$; $S = \mathcal{O}_{\alpha(f)} \cap W_2^{\perp} = \mathcal{O}_{\alpha(e)} \cap W_1^{\perp\perp}$. Whence, $S \subseteq W_1^{\perp\perp}$.

Subcase 2a. $S \subseteq W_1$. Then $\mathcal{O}_{\alpha(f)} \cap W_1^{\perp} \subseteq W_2$. In fact, $\mathcal{O}_{\alpha(f)} \subseteq W_1 \cap W_2$. This implies $\mathcal{O}_{\alpha(f)} \subseteq \bigcup_{n\in\omega} \mathcal{O}_{\varphi_f(n)}$, a contradiction.

Subcase 2b. $S \nsubseteq W_1$. Choose $\mathcal{O}_k$ such that $\mathcal{O}_k \subseteq S$ and $\mathcal{O}_k \nsubseteq W_1$. In particular, $\mathcal{O}_k \subseteq \mathcal{O}_{\alpha(f)}$. Also $\mathcal{O}_k \nsubseteq \big(\bigcup_{n\in\omega} \mathcal{O}_{\varphi_f(n)}\big)$, lest it be attacked at some stages $t = j(2f, s)$ and $t' = j(2f+1, s)$. A contradiction follows.

The proof is completely symmetric in W_1 and W_2; hence, we can show similarly that $W_2^{\perp\perp}$ is not r.e. open.

Referring again to Figure 1, we have as a consequence of Theorem 5 that $(7) \supset (5)$ and $(9) \supset (8)$. Of course, by cardinality considerations, we already knew that $(9) \supset (8)$, but the present proof avoids a cardinality argument.

The following theorem yields the remaining proper conclusions: $(4) \subset (6)$, $(5) \subset (8)$ and $(7) \subset (9)$. Of course, $(7) \subset (9)$ is known by cardinality considerations.

THEOREM 6. *There exists an r.e. open and regular set U such that U is not recursively regular open.*

We first give remarks. Let $\{\mathcal{O}_{\alpha(e)}\}_{e\in\omega}$ be as before. If $A \subset \omega$ then $U = \bigcup_{e\in A} \mathcal{O}_{\alpha(e)}$ is regular open. Easily, if A is finite then U is clopen. Otherwise, $U^- = U \cup \{x_0\}$, which follows from the relationship between $\mathcal{O}_e$ and $\mathcal{O}_{\alpha(e)}$. It suffices to show that, if $\mathcal{O}_e \subseteq U^-$ then $\mathcal{O}_e \subseteq U$.

If $\mathcal{O}_e \subseteq U^-$ then either $x_0 \in \mathcal{O}_e$ or $x_0 \notin \mathcal{O}_e$. If $x_0 \notin \mathcal{O}_e$ then already $\mathcal{O}_e \subseteq U$. Suppose that $x_0 \in \mathcal{O}_e$. Then by definition of $\alpha(e)$, $\big(\bigcup_{k<e} \mathcal{O}_{\alpha(e)}\big)' \subseteq \mathcal{O}_e$. In particular, $\mathcal{O}_{\alpha(j)} \subseteq \mathcal{O}_e$, for $j \geqq e$. Hence, $\mathcal{O}_e \cap U^{\perp} \neq \emptyset$. Therefore, $\mathcal{O}_e \subseteq U^-$, a contradiction.

Also, if A is a recursive subset of ω then U is recursive open. This, of course, gives us a simple way to construct recursive open and regular sets which are not clopen. For example, if S denotes odd integers then $U=\bigcup_{e\in S}\mathcal{O}_{\alpha(e)}$ is recursive open and regular, but not clopen.

PROOF OF THEOREM 6. Let K be the complete set. Let $U=\bigcup_{e\in K}\mathcal{O}_{\alpha(e)}$. Then, from the above remarks, U is r.e. open and regular. It suffices to show that U is not recursively regular open. Since $U=(\bigcup_{j\in\omega}\mathcal{O}_{h(j)})^{\perp\perp}$, we must show that there is no recursive function g such that $U^{\perp}=(\bigcup_{n}\mathcal{O}_{g(n)})^{\perp\perp}$. Suppose the contrary. For any $e\in\omega$, either $\mathcal{O}_{\alpha(e)}\subseteqq U$ or $\mathcal{O}_{\alpha(e)}\subseteqq U^{\perp}$. Thus for some s, either $\mathcal{O}_{\alpha(e)}\subseteqq\bigcup_{j\leqq s}\mathcal{O}_{h(j)}$ or $\mathcal{O}_{\alpha(e)}\cap\cap\mathcal{O}_{g(s)}\neq\emptyset$. Whichever happens first determines whether $e\in K$ or $e\in K'$, a contradiction.

COROLLARY 7. *If $A\subseteqq\omega$ is a set such that neither A nor A' is r.e. then $U=\bigcup_{e\in A}\mathcal{O}_{\alpha(e)}$ is regular open and neither U nor $U^{\perp}$ is r.e. open. Also, U is not recursively regular open.*

PROOF. Assume otherwise. Then there exists recursive functions f and g such that

$$\bigcup_{e\in A}\mathcal{O}_{\alpha(e)}=\Big(\bigcup_{e\in\omega}\mathcal{O}_{f(e)}\Big)^{\perp\perp}\quad\text{and}\quad\Big(\bigcup_{e\in A'}\mathcal{O}_{\alpha(e)}\Big)=\Big(\bigcup_{e\in\omega}\mathcal{O}_{g(e)}\Big)^{\perp\perp}.$$

Thus,

$$\Big(\bigcup_{e\in A}\mathcal{O}_{\alpha(e)}\Big)^{\perp}=\Big(\bigcup_{e\in\omega}\mathcal{O}_{g(e)}\Big)^{\perp}\quad\text{and}\quad\Big(\bigcup_{e\in A'}\mathcal{O}_{\alpha(e)}\Big)^{\perp}=\Big(\bigcup_{e\in\omega}\mathcal{O}_{g(e)}\Big)^{\perp},$$

i.e.,

$$\Big(\bigcup_{e\in A}\mathcal{O}_{\alpha(e)}\Big)^{-}=\Big(\bigcup_{e\in\omega}\mathcal{O}_{f(e)}\Big)^{-}\quad\text{and}\quad\Big(\bigcup_{e\in A'}\mathcal{O}_{\alpha(e)}\Big)^{-}=\Big(\bigcup_{e\in\omega}\mathcal{O}_{g(e)}\Big)^{-}.$$

Choose $e\in\omega$. By regularity, either $\mathcal{O}_{\alpha(e)}\subseteqq\bigcup_{k\in\omega}\mathcal{O}_{f(k)}$ or $\mathcal{O}_{\alpha(e)}\subseteqq\bigcup_{k\in\omega}\mathcal{O}_{g(k)}$. Then, alternately checking

$$\mathcal{O}_{\alpha(e)}\subseteqq\mathcal{O}_{f(0)}\cup\mathcal{O}_{f(1)}\cup\ldots\cup\mathcal{O}_{f(s)}\quad\text{and}\quad\mathcal{O}_{\alpha(e)}\subseteqq\mathcal{O}_{g(0)}\cup\mathcal{O}_{g(1)}\cup\ldots\cup\mathcal{O}_{g(t)},$$

we determine whether or not $e\in A$ or $e\in A'$, a contradiction.

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POSITIVE DEFINITE GENERALIZED MEASURES ON THE DYADIC FIELD

By

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1. Introduction. In 1974, L. ARGABRIGHT and J. G. DE LAMADRID [1] defined a generalized Fourier transform of a measure on a locally compact abelian group and extended Bochner's theorem on Fourier transforms of positive definite functions.

In this paper, we consider a generalized Fourier transform on a special locally compact abelian group called the dyadic field. The dyadic field was introduced by N. J. FINE [2]. He defined the Fourier transform of an integrable function and discussed some of its properties. On the dyadic field we can define a generalized measure called dyadic measure or quasi-measure (see [3]) and its generalized Fourier transform. Bochner's theorem is extended on the dyadic field.

2. Dyadic field, dyadic measure and its Walsh—Fourier transform. The dyadic field is the set of all 0, 1 sequences, $x=(\dots, x_i, \dots)$ such that $\lim_{i\to-\infty} x_i=0$, in which the group operation $\dotplus$ is addition such that

$$(\dots, x_i, \dots)\dotplus(\dots, y_i, \dots) = (\dots, z_i, \dots)$$

where $z_i=x_i+y_i$ (mod 2) and the group operation $\cdot$ is a product such that

$$(\dots, x_i, \dots)\cdot(\dots, y_i, \dots) = (\dots, z_k, \dots)$$

where $z_k=\sum_{i+j=k} x_i y_j$ (mod 2).

Let λ be a function which maps the dyadic field onto $[0, \infty)$ such that

$$\lambda(x) = \sum_{n=-\infty}^{\infty} x_n/2^n.$$

λ does not have a single valued inverse on the dyadic rationals. When $x=(\dots, x_n, \dots)$, it is convenient to write $x=\lambda(x)$ if $\lim_{n\to\infty} x_n\neq 1$ and $x=\lambda(x)^-$ if $\lim_{n\to\infty} x_n=1$; for example $(\dots, 0, \overset{0}{1}, 0, \dots)=1$ and $(\dots, \overset{0}{0}, 1, 1, \dots)=1^-$. When x is a dyadic rational, x^- is called the conjugate dyadyc rational of x. It is easy to see that $(1/2^n)^-=1/2^n\cdot 1^-$. Therefore it is natural to write $(1/2^n)^-=1^-/2^n$. Moreover we have $((p+1)/2^n)^-=(p\dotplus 1^-)/2^n$.

We introduce an order $<$ in the dyadic field. When $\lambda(x)<\lambda(y)$, we write $x<y$ and when x is a dyadic rational, we write $x^-<x$. $[y, x]$ is the set of all z such that $x\leqq z\leqq y$ and (x, y) is the set of all z such that $x<z<y$. Then $[0, \infty)$ is the dyadic field and $[0, 1^-]$ is the dyadic group. Especially, for each $n=0, \pm 1, \pm 2, \dots$ and

$p=0, 1, \ldots,$ $I=[p/2^n, ((p+1)/2^n)^-]$ is called a dyadic interval of rank n and $I_n(x)$ is the dyadic interval of rank n which contains x.

Let m be a set function which is defined on the family of dyadic intervals. m is called a dyadic measure or a quasi-measure on the dyadic field, if it satisfies

$$m([p/2^n, ((p+1)/2^n)^-]) = m([2p/2^{n+1}, ((2p+1)/2^{n+1})^-]) + \\ + m([(2p+1)/2^{n+1}, ((2p+2)/2^{n+1})^-])$$

for each $n=0, \pm 1, \pm 2, \ldots$ and $p=0, 1, \ldots$. If a dyadic measure m satisfies

$$\sup_n \sum_{p=0}^{\infty} |m([p/2^n, ((p+1)/2^n)^-])| < \infty,$$

then m is a measure on the dyadic field. When f is an integrable function, a set function m_f defined by

$$\int_I f(x)\,dx = m_f(I),$$

where dx is the Haar measure and I is a dyadic interval, is a dyadic measure. When $m(I) \geqq 0$ for each dyadic interval I, m is called a positive dyadic measure, and if

$$\sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j m(I_n(x_i \dotplus x_j)) \geqq 0$$

for arbitrary real numbers $a_1, a_2, \ldots, a_N$ and arbitrary elements $x_1, x_2, \ldots, x_N$, m is called a positive definite dyadic measure.

Two dyadic measures m_1 and m_2 are said to be identical if and only if $m_1(I) = m_2(I)$ for each dyadic interval I.

Let $w_y(x)$ be the Walsh function defined by

$$w_y(x) = (-1)^{\sum_{i+j=1} x_i y_j}$$

where $x=(\ldots, x_i, \ldots)$ and $y=(\ldots, y_j, \ldots)$.

N. J. Fine [2] defined the Walsh—Fourier transform of an integrable function f as

$$\hat{f}(x) = \int_0^{\infty} f(y) w_y(x)\,dy.$$

$\hat{f}(x)$ is a continuous function on the dyadic field. Therefore we can define a dyadic measure $m_{\hat{f}}$ such that

$$m_{\hat{f}}(I) = \int_I \hat{f}(y)\,dy.$$

We can easily prove that

$$m_{\hat{f}}([p/2^n, ((p+1)/2^n)^-]) = 1/2^n \int_0^{(2^n)^-} f(y) w_y(p/2^n)\,dy = 1/2^n \int_0^{(2^n)^-} w_y(p/2^n) m_f(dy).$$

$m_{\hat{f}}$ is a dyadic measure and we shall write $m_{\hat{f}} = \hat{m}_f$. When m is a dyadic measure and f is constant on each closed interval $[z_i, z_{i+1}^-]$ where z_i is a dyadic rational, it is convenient to write

$$\sum_{i=1}^{n-1} f(z_i)\, m([z_i, z_{i+1}^-]) = \int_{z_1}^{z_n^-} f(y)\, m(dy).$$

Therefore when m is a dyadic measure, we can give the Walsh—Fourier transform of m by the following dyadic measure $\hat{m}$:

$$\hat{m}\big([p/2^n, ((p+1)/2^n)^-]\big) = 1/2^n \int_0^{(2^n)^-} w_y(p/2^n)\, m(dy)$$

and generally

$$\hat{m}\big(I_n(x)\big) = 1/2^n \int_0^{(2^n)^-} w_y(x)\, m(dy).$$

Any dyadic measure m has its Walsh Fourier transform $\hat{m}$ and the relation $\hat{\hat{m}}=m$ holds.

3. Positive definite dyadic mesures. Bochner's theorem is extended for dyadic measures as follows:

THEOREM 1. *m is a positive dyadic measure if and only if $\hat{m}$ is a positive definite dyadic measure.*

PROOF. For any $a_1, \dots, a_N, x_1, \dots, x_N$ and sufficiently large n,

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j \hat{m}\big(I_n(x_i \dotplus x_j)\big) = \sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j 1/2^n \int_0^{(2^n)^-} w_y(x_i \dotplus x_j)\, m(dy) =$$

$$= 1/2^n \int_0^{(2^n)^-} \sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j w_y(x_i) w_y(x_j)\, m(dy) = 1/2^n \int_0^{(2^n)^-} \left(\sum_{i=1}^{N} a_i w_y(x_i)\right)^2 m(dy).$$

If m is a positive dyadic measure, then $\hat{m}$ is a positive definite dyadic measure.

Conversely, for a sub-dyadic interval I, of $[0, 2^{n-}]$, let $h(y)$ be the characteristic function of $\bigcup_{k=0}^{\infty} (k2^n \dotplus I)$. There exist $x_1, \dots, x_N$ and $a_1, \dots, a_N$ satisfying

$$\sum_{i=1}^{N} a_i w_{x_i}(y) = h(y).$$

Therefore if $\hat{m}$ is a positive definite dyadic measure, then we get

$$1/2^n m(I) = 1/2^n \int_0^{(2^n)^-} [h(y)]^2 m(dy) = 1/2^n \int_0^{(2^n)^-} \left(\sum_{i=1}^{N} a_i w_{x_i}(y)\right)^2 m(dy) =$$

$$= 1/2^n \sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j \int_0^{(2^n)^-} w_y(x_i) w_y(x_j)\, m(dy) = \sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j m\big(I_n(x_i \dotplus x_j)\big) \geqq 0.$$

Hence we have $m(I) \geqq 0$.

THEOREM 2. *If a positive dyadic measure m satisfies* $m([0,\infty))<\infty$, *then there is a continuous function* $\hat{f}(x)$ *such that* $\hat{m}=m_{\hat{f}}$ *and*

$$(*)\qquad \hat{f}(x)=\int_0^\infty w_y(x)m(dy).$$

PROOF. For each $x\in[2^N,(2^{N+1})^-]$, we have

$$2^n\hat{m}(I_n(x))=\int_0^{(2n)^-} w_y(x)m(dy)=$$

$$=\sum_{k=0}^{2^{N+n}-1}\Delta m([k/2^N,((k+1)/2^N)^-])\,w_{k/2^N}(x)=\sum_{k=0}^{2^{N+n}-1}\Delta m([k/2^N,((k+1)/2^N)^-])w_k(x/2^N)$$

where

$$\Delta m([k/2^N,((k+1)/2^N)^-])=$$
$$=m([2k/2^{N+1},((2k+1)/2^{N+1})^-])-m([(2k+1)/2^{N+1},((2k+2)/2^{N+1})^-]).$$

The last series is absolutely convergent by the hypothesis. Let f be the limit function of this series. It is continuous on each dyadic interval $[2^N,2^{N+1}]$ for $N=0,\pm1,\pm2,\ldots$ and evidently

$$\hat{m}(I)=\int_I \hat{f}(x)\,dx=m_{\hat{f}}(I)$$

for each dyadic interval I and (*).

COROLLARY. *If a positive dyadic measure m satisfies* $m([0,\infty))=\infty$ *and*

$$\sum_{k=0}^{\infty}\left|\Delta m([k/2^N,((k+1)/2^N)^-])\right|^2<\infty$$

for each $N=0,\pm1,\pm2,\ldots$, *then there exists a function f such that* $\hat{f}\in L([2^N,(2^{N+1})^-])$ *for* $N=0,\pm1,\pm2,\ldots$ *and* $\hat{m}=m_{\hat{f}}$. *Moreover we have*

$$\lim_{n\to\infty}\int_0^{(2n)^-} w_y(x)m(dy)=\hat{f}(x)\quad a.e.$$

For a dyadic measure m, set

$$\Delta_x^* m(I_n(x))=m(I_{n+1}(x))-m(I_n(x)\setminus I_{n+1}(x)).$$

THEOREM 3. *m is a dyadic measure which satisfies* $m(I)\geqq0$ *for each dyadic interval which does not contain* 0, *if and only if* $\hat{m}$ *satisfies*

$$\sum_{i=1}^N\sum_{j=1}^N a_ia_j\Delta^*_{x_i\dot{+}x_j}\hat{m}(I_n(x_i\dot{+}x_j))\geqq0$$

for any $a_1,\ldots,a_N,x_1,\ldots,x_N$ *and sufficiently large* n.

PROOF. At first we have

$$\Delta_x^* \hat{m}(I_n(x)) = 1/2^{n+1} \int_0^{(2^n+1)-} w_y(x)m(dy) - 1/2^{n+1} \int_0^{(2^n+1)-} w_y(x \dot{+} (1/2^{n+1}))m(dy) =$$

$$= 1/2^{n+1} \int_0^{(2^n+1)-} w_y(x)(1-w_y(1/2^{n+1}))m(dy) = 1/2^n \int_{2^n}^{(2^n+1)-} w_y(x)m(dy).$$

Therefore the following formula gives Theorem 3:

$$\sum_{i=1}^N \sum_{j=1}^N a_i a_j \Delta_{x_i \dot{+} x_j}^* \hat{m}(I_n(x_i \dot{+} x_j)) = 1/2^n \int_{2^n}^{(2^n+1)-} \left(\sum_{i=1}^N a_i w_y(x_i)\right)^2 m(dy) \geqq 0.$$

At last we give two examples of positive dyadic measures and their Walsh-Fourier transforms:

(i) If $f(x)=1$ for $x\in[0, 1^-]$ and $=0$ for $x\in[1, \infty)$, then we have $\hat{m}_f=m_f$.

(ii) If $m(I)=1$ for a sufficiently small dyadic interval I satisfying $I\cap Z\neq\emptyset$ and $m(I)=0$ for I satisfying $I\cap Z=\emptyset$ where $Z=\{0, 1, \ldots\}$, then we have $\hat{m}=m$.

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Proof. At first, we have

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GENERALIZED SLLN UNDER WEAK MULTIPLICATIVE DEPENDENCE RESTRICTIONS

By

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1. Introduction. For random variables $X_1, X_2, \dots$ and constants $a_1, a_2, \dots$, put $S_n = \sum_1^n a_k X_k$. The main result of this paper is a strong law of large numbers for $\{S_n\}$ with respect to a sequence of constants $\{b_n\}$, where the dependence restrictions on the sequence $\{X_j\}$ will be of the weak multiplicative type. A rate of convergence will be provided for the strong law. These results will broaden the theorems found in MÓRICZ [3].

2. Dependence restrictions and previous results. The term "weak multiplicative" refers to any form of restriction on the product moments $E\{X_{j_1} X_{j_2} \dots X_{j_\nu}\}$ of order ν. Three different but related conditions were formulated in LONGNECKER and SERFLING [2]. They will be stated in this section for completeness. The first two conditions are orthogonality related dependence restrictions.

DEFINITION. A sequence of random variables $\{X_j\}$ satisfies Condition A with respect to an even integer ν, a sequence of constants $\{a_j\}$, and a symmetric function g of $\nu-1$ arguments if

$$|E\{X_{j_1} X_{j_2} \dots X_{j_\nu}\}| \leqq g(j_2-j_1, j_3-j_2, \dots, j_\nu-j_{\nu-1}) a_{j_1} a_{j_2} \dots a_{j_\nu} \tag{2.1a}$$

for all $1 \leqq j_1 < \dots < j_\nu$, and if

$$\sum_{k=1}^{\infty} \sum_{j_1=1}^{k} \dots \sum_{j_{\nu-2}=1}^{k} g(j_1, \dots, j_{\nu-2}, k) < \infty. \tag{2.1b}$$

DEFINITION. A sequence of random variables $\{X_j\}$ satisfies Condition B with respect to an even integer ν, a sequence of constants $\{a_j\}$, and a symmetric function g of $\nu/2$ arguments if

$$|E\{X_{j_1} \dots X_{j_\nu}\}| \leqq g(j_2-j_1, j_4-j_3, \dots, j_\nu-j_{\nu-1}) a_{j_1} \dots a_{j_\nu} \tag{2.2a}$$

for all $1 \leqq j_1 < \dots < j_\nu$, and if

$$\sum_{k=1}^{\infty} \sum_{j_1=1}^{k} \dots \sum_{j_{\nu/2-1}=1}^{k} g(j_1, \dots, j_{\nu/2-1}, k) < \infty. \tag{2.2b}$$

A third dependence restriction which is related to Gaussian time series is

DEFINITION. A sequence of random variables $\{X_j\}$ satisfies Condition C with respect to an even integer ν, constants $\{a_j\}$, a function $f(j)$, and a function g of

$\nu/2-1$ arguments if

(2.3a)

$$|E\{X_{j_1}\dots X_{j_\nu}\}| \leqq \min\{f(j_2-j_1), f(j_\nu-j_{\nu-1})\}\, g(j_3-j_2, j_5-j_4, \dots, j_\nu-j_{\nu-1})\, a_{j_1}\dots a_{j_\nu},$$

for all $1 \leqq j_1 < \dots < j_\nu$, if

(2.3b)
$$\sum_{j=1}^{\infty} f(j) < \infty,$$

and if

(2.3c)
$$\sum_{l=1}^{\nu/2-1} \sum_{j_l=1}^{\infty} \sum_{j_1=1}^{j_l} \dots \sum_{j_{l-1}=1}^{j_l} \sum_{j_{l+1}=1}^{j_l} \dots \sum_{j_{\nu/2-1}=1}^{j_l} g(j_1, \dots, j_{\nu/2-1}) < \infty.$$

A discussion of the interrelationship of the above three dependence restrictions with other well-known dependence restrictions is contained in LONGNECKER and SERFLING [2].

A general maximal inequality is derived in LONGNECKER and SERFLING [1]. It is directly applicable to the partial sum of random variables satisfying either Condition A, Condition B, or Condition C.

THEOREM 2.1. *Let the sequence $\{X_j\}$ satisfy, for an even integer $\nu > 2$, either Condition* A, *Condition* B, *or Condition* C, *and $b_j = E(\{X_j^\nu\}) < \infty$ (all j). Then for all positive λ,*

(2.4)
$$P\left\{\max_{1 \leqq j \leqq n} \left|\sum_{k=1}^{j} a_k X_k\right| \geqq \lambda\right\} \leqq C_\nu \left[\sum_{k=1}^{n} b_k^2 a_k^2\right]^{\nu/2} \lambda^{-\nu},$$

where C_ν is a constant depending on only ν.

The above inequality will be used to extend the following result of MÓRICZ [3].

THEOREM 2.2. *Let ν be an even integer, $\nu \geqq 4$. Let $\{X_j\}$ be a system of random variables for which*

(2.5)
$$E\{X_j^\nu\} \leqq K < \infty \quad (j = 1, 2, \dots)$$

and

(2.6)
$$\|B_\nu\|^2 = \sum_{1 \leqq j_1 < \dots j_\nu} (E\{X_{j_1} \dots X_{j_\nu}\})^2 < \infty.$$

Let $\{a_k\}$ be a sequence of numbers such that

(2.7)
$$A_n = \left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \to \infty, \quad as \quad n \to \infty.$$

Then the relation

(2.8)
$$P\left[\lim_{n\to\infty} \frac{|S_n|}{A_n (\log A_n)^{1/\nu} (\log\log A_n)^{(1+\varepsilon)/\nu}} = 0\right] = 1$$

holds true for any positive ε.

In particular, with probability 1 *we have*

(2.9)
$$\sum_{k=1}^{n} X_k = o\{n^{1/2} (\log n)^{1/\nu} (\log\log n)^{(1+\varepsilon)/\nu}\}, \quad as \quad n \to \infty.$$

By letting the moment restriction ν tend to infinity, the following strengthening of (2.8) is obtained.

COROLLARY 2.1. *Suppose that a system* $\{X_j\}$ *of random variables satisfies* (2.5) *and* (2.6) *for infinitely many even integers* ν. *Then, under condition* (2.7) *the relation*

$$P\left[\lim_{n\to\infty}\frac{|S_n|}{A_n(\log A_n)^{\varepsilon}}=0\right]=1 \tag{2.10}$$

holds true for any positive ε.

Móricz also establishes rates of convergence for the strong laws given in (2.8) and (2.10).

Results similar to Theorem 2.2 and Corollary 2.1 will be derived in the next section for sequences of random variables satisfying any of the conditions A, B, or C. Móricz' results will be extended by relaxing two of his conditions. First, the uniform boundedness of the νth moments will be removed by requiring $\sum_{k=1}^{n} b_k^2 a_k^2 \to \infty$, where $b_k = E\{X_k^\nu\}$. Also, with

$$g(j_1, j_2, \dots, j_{\nu-1}) = \sup_i |E\{X_i X_{i+j_1} \cdots X_{i+j_1+\dots+j_{\nu-1}}\}|,$$

it is seen that

$$\sum_{k=1}^{\infty}\sum_{j_1=1}^{k}\cdots\sum_{j_{\nu-2}=1}^{k} g^2(j_1, \dots, j_{\nu-2}, k) \leqq \|B_\nu\|^2.$$

Thus, the generalized strong law for sequences satisfying Condition A will be applicable to a larger class of sequences $\{X_i\}$ then covered by Móricz' result.

3. Generalized strong law. The main result of this paper is the following generalized strong law for sequences of random variables satisfying one of the three types of weak multiplicative dependence restrictions given in Section 2. A rate of convergence for the strong law will be provided which is of the same order as obtained in MÓRICZ [3]. The method of proof in the following results are similar in nature to the proofs in MÓRICZ [3].

THEOREM 3.1. *Let the sequence* $\{X_i\}$ *satisfy, for an even integer* $\nu > 2$, *either Condition* A, *Condition* B, *or Condition* C, *and* $b_k = E\{X_k^\nu\} < \infty$ *(all* k*). Let* $\{a_k\}$ *be a sequence of constants such that*

$$A_n = \left(\sum_{k=1}^{n} b_k^2 a_k^2\right)^{1/2} \to \infty, \quad as \quad n \to \infty. \tag{3.1}$$

Then the relation

$$P\left[\lim_{n\to\infty}\frac{|S_n|}{A_n(\log A_n)^{1/\nu}(\log\log A_n)^{(1+\varepsilon)/\nu}}=0\right]=1 \tag{3.2}$$

holds true for any $\varepsilon > 0$.

PROOF. Let $M(n) = \max_{1\leqq k\leqq n} |S_k|$ and

$$B(n) = A_n(\log A_n^2)^{1/\nu}(\log\log A_n^2)^{(1+\varepsilon)/\nu} \quad (n = 1, 2, \dots).$$

Then by Theorem 2.1,

$$P[M(n) \geqq B(n)] \leqq \frac{C_\nu}{(\log A_n^2)(\log\log A_n^2)^{1+\varepsilon}} \quad (n = 1, 2, \ldots). \tag{3.3}$$

Since $A_n^2 \to \infty$, as $n \to \infty$, there exists a non-decreasing sequence of positive integers $\{n_k\}$ such that

$$A_{n_k-1}^2 \leqq e^k < A_{n_k}^2 \quad (k = 1, 2, \ldots, A_0 = 0). \tag{3.4}$$

Expressions (3.3) and (3.4) yield

$$\Sigma_{(k)} P[M(n_k) \geqq B(n_k)] \leqq \Sigma_{(k)} \frac{C_\nu}{(\log A_{n_k}^2)(\log\log A_{n_k}^2)^{1+\varepsilon}} \leqq \tag{3.5}$$

$$\leqq C_\nu \sum_{k=2}^{\infty} \frac{1}{k(\log k)^{1+\varepsilon}} < \infty,$$

(where $\Sigma_{(k)}$ is the summation over distinct n_k's).

Therefore, the Borel—Cantelli Lemma implies that the inequality

$$M(n_k) < B(n_k) \tag{3.6}$$

holds with probability one for sufficiently large k.

By similar arguments it is seen that

$$M(n_k - 1) < B(n_k - 1) \tag{3.7}$$

holds with probability one for sufficiently large k.

Suppose $n_n \leqq n < n_{k+1}$ then $B(n) \geqq B(n_k)$ and $|S_n| \leqq M_{n_{k+1}-1}$. Thus with probability one, for sufficiently large k

$$\frac{|S_n|}{B(n)} \leqq \frac{M_{n_{k+1}-1}}{B(n_k)} \leqq \frac{B(n_{k+1}-1)}{B(n_k)}. \tag{3.8}$$

But, $B(n_{k+1}-1)/B(n_k)$ is bounded by $De^{\frac{1}{2}}$ for sufficiently large k. Thus,

$$P[|S_n| = O(A_n \log A_n^2)^{1/\nu}(\log\log A_n^2)^{(1+\varepsilon)/\nu}] = 1, \tag{3.9}$$

for all $\varepsilon > 0$. Since ε is an arbitrary positive number, (3.9) implies (3.2).

The following stronger result is obtained by restricting the sequence of random variables to those satisfying the dependence restrictions for an infinite number of even integers.

COROLLARY 3.1. *Let the sequence of random variables $\{X_k\}$ satisfy the conditions of Theorem* 3.1 *for an infinite number of even integers* ν. *Then, for all* $\varepsilon > 0$,

$$P\big[|S_n| = o\big(A_n(\log A_n^2)^\varepsilon\big)\big] = 1. \tag{3.10}$$

The next theorem will provide the rate of convergence in the strong law of Theorem 3.1.

THEOREM 3.2. *Let $\{X_k\}$ be a sequence of random variables which satisfies the conditions of Theorem* 3.1 *for an even integer* $\nu > 2$. *Let $\{a_k\}$ be a sequence of constants* *satisfying*

(3.11) $$A_n = \left(\sum_{k=1}^{n} a_k^2 b_k^2\right)^{1/2} \to \infty, \quad as \quad n \to \infty,$$

and

(3.12) $$a_n^2 b_n^2 < (1-e^{-1}) A_n^2 \quad (for\ all\ n \geqq n_0).$$

Then, for any choice of α *and* β *satisfying*

(3.13) $$0 < \beta < \alpha\nu - 1,$$

(3.14) $$\sum_{n=1}^{\infty} \frac{a_n^2}{A_n^2 (\log A_n^2)^{1-\beta}} P\left[\sup_{k \geqq n} \frac{|S_k|}{A_k (\log A_k^2)^{\alpha}} > 1\right] < \infty.$$

PROOF. Conditions (3.11) and (3.12) imply the existence of a strictly increasing sequence of positive integers $\{n_j\}$ such that

(3.15) $$e^j \leqq A_{n_j}^2 < e^{j+1} \quad \text{(for all } j \text{ suffciently large).}$$

For each n define the integer $j_0 = j_0(n)$ by

(3.16) $$n_{j_0} \leqq n \leqq n_{j_0+1}.$$

By Theorem 2.1 and (3.15),

$$P\left[\sup_{k \geqq n} \frac{|S_k|}{A_k (\log A_k^2)^{\alpha}} > 1\right] \leqq \sum_{j=j_0}^{\infty} P\left[\max_{n_j \leqq k \leqq n_{j+1}} \frac{|S_k|}{A_k (\log A_k^2)^{\alpha}} > 1\right] \leqq$$

$$\leqq \sum_{j=j_0}^{\infty} C_\nu A_{n_j}^{-\nu} (\log A_{n_j}^2)^{-\alpha\nu} A_{n_{j+1}}^{\nu} \leqq C_{\nu,\alpha} (\log A_n^2)^{1-\alpha\nu},$$

where $C_{\nu,\alpha}$ is suitably defined constant involving just ν and α. Hence

$$\sum_{n=1}^{\infty} \frac{a_n^2}{A_n^2 (\log A_{n1}^2)^{1-\beta}} P\left[\sup_{k \geqq n} \frac{|S_k|}{A_k (\log A_k)^{\alpha}} > 1\right] \leqq C_{\nu,\alpha} \sum_{n=1}^{\infty} \frac{a_n^2}{A_n^2 (\log A_n^2)^{\alpha\nu-\beta}}.$$

Thus, since $\alpha\nu - \beta - 1 > 0$, we have the above series converges.

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MINIMAX THEOREMS FOR UPPER SEMICONTINUOUS FUNCTIONS

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1. The various generalizations of von Neumann's classical minimax theorem [1] constitute an important chapter of the modern analysis. In the economic applications it might have some interest to prove minimax theorems for vector-valued functions, e.g. for functions mapping into $\mathbf{R}^n$, endowed with the lexicographic order. As the first theorem of this paper shows, without any further conditions Neumann's result does not remain true for such functions.

In a recent publication [5], C.-W. HA generalized Neumann's minimax theorem (see also in [3]) for upper semicontinuous functions. Our second theorem establishes a slightly more general form of this result, which contains also Theorem 1 in [4]. Our proof is based on the considerations, developed by I. Joó in [2] and [3]; thus we can eliminate the application of Brouwer's fixed point theorem, essentially used in [5]. Theorem 2 is formulated for functions mapping into a linearly ordered space. Thus we obtain a positive answer for the minimax problem of vector-valued functions.

The third theorem of this paper asserts that in case if one of the underlying spaces is a convex subset of some topological vector space, the continuity conditions of Theorem 2 can be weakened.

The author is grateful to I. Joó for proposing the minimax problem of vector-valued functions.

2. THEOREM 1. *There exists a continuous function* $f\colon [0,1]\times[-1,1]\to\to[-1,1]\times[-1,1]$ *such that*

(1^x) *the subfunctions* $f(\cdot, y)$ *are concave for any fixed* $y\in[-1,1]$,

(1^y) *the subfunctions* $f(x, \cdot)$ *are convex for any fixed* $x\in[0,1]$;

nevertheless

$$\max_x \min_y f(x,y) = (0,-1) \neq (0,0) = \min_y \max_x f(x,y) \tag{2}$$

($[-1,1]\times[-1,1]$ *is equipped with the lexicographic order*).

PROOF. Consider the continuous function

$$f\colon [0,1]\times[-1,1]\to[-1,1]\times[-1,1],\quad f(x,y)=(xy,-y).$$

It is easy to see that

$$\min_y f(x,y) = \begin{cases} (0,-1) & \text{if } x=0,\\ (-x,1) & \text{if } 0<x\leqq 1,\end{cases}$$

$$\max_x f(x,y) = \begin{cases} (0,-y) & \text{if } -1\leqq y\leqq 0,\\ (y,-y) & \text{if } 0<y\leqq 1.\end{cases}$$

From these relations we obtain (2) at once.

To prove (1^x) and (1^y), we have to show that

$$f(tx_1+(1-t)x_2, y) \geqq tf(x_1, y)+(1-t)f(x_2, y)$$

for any $x_1, x_2 \in [0, 1]$, $y \in [-1, 1]$, $t \in [0, 1]$, and

$$f(x, ty_1+(1-t)y_2) \leqq tf(x, y_1)+(1-t)f(x, y_2)$$

for any $x \in [0, 1]$, $y_1, y_2 \in [-1, 1]$, $t \in [0, 1]$. But these conditions are obviously satisfied; moreover, we have not only inequality but also equality in both cases:

$$(tx_1y+(1-t)x_2y, -y) = t(x_1y, -y)+(1-t)(x_2y, -y)$$

and

$$(xty_1+x(1-t)y_2, -ty_1-(1-t)y_2) = t(xy_1, -y_1)+(1-t)(xy_2, -y_2).$$

The theorem is proved.

3. We recall that by an *interval space* (see [4]) we mean a topological space X endowed with a mapping $[\cdot, \cdot]: X \times X \to \{\text{connected subsets of } X\}$ such that $x_1, x_2 \in [x_1, x_2]=[x_2, x_1]$ for all $x_1, x_2 \in X$. A subset K of an interval space is *convex* if for every $x_1, x_2 \in K$ we have $[x_1, x_2] \subset K$. Any convex subset of a real topological vector space is an interval space with its natural interval structure.

A *linearly ordered space* (see [6]) is called *complete* if every subset has a least upper bound. Such spaces are the extended real line $\overline{\mathbf{R}}$, the extended euclidean n-space $\overline{\mathbf{R}}^n$ or any compact (in the euclidean topology) subset of $\mathbf{R}^n$ with respect to the lexicographic order.

Let X be an interval space and Z a complete linearly ordered space. A function $f: X \to Z$ is called *quasiconvex* (resp. *quasiconcave*) if the sets

$$\{x \in X: f(x) \leqq z\} \quad (\text{resp. } \{x \in X: f(x) \geqq z\})$$

are convex for all $z \in Z$. Furthermore, f is called *upper semicontinuous* if all the sets

$$\{x \in X: f(x) \geqq z\}, \quad z \in Z,$$

are closed in X.

If X is compact and $f: X \to Z$ is upper semicontinuous, then there exists an $x_0 \in X$ such that $f(x_0)=\sup_{x \in X} f(x)$. Given a family $(f_i)_{i \in I}$ of upper semicontinuous functions from X into Z, the map $\inf_{i \in I} f_i$ is also upper semicontinuous. These statements are proved in the same way as in case $Z=\overline{\mathbf{R}}$.

THEOREM 2. *Let X be a compact interval space, Y an arbitrary interval space, Z a complete linearly ordered space and $f: X \times Y \to Z$ an upper semicontinuous function such that*

(3^x) *the subfunctions $f(\cdot, y)$ are quasiconcave on X for any fixed $y \in Y$,*

(3^y) *the subfunctions $f(x, \cdot)$ are quasiconvex on Y for any fixed $x \in X$.*

Then

$$\max_x \inf_y f(x, y) = \inf_y \max_x f(x, y). \tag{4}$$

PROOF. The expressions in (4) make sense by the two statements mentioned just before this theorem. Being the relation $\max_x \inf_y f(x, y) \leqq \inf_y \max_x f(x, y)$ obvious, it is enough to show that the family of sets

$$\{K(y) \equiv \{x \in X : f(x, y) \geqq \inf_y \max_x f(x, y)\} : y \in Y\}$$

has a non-empty intersection.

For any $y \in Y$, the set $K(y)$ is convex by (3^x) and non-empty by the definition of $\inf_y \max_x f(x, y) \equiv z^*$. Moreover, $K(y)$ is compact because X is compact and f is upper semicontinuous.

It follows from (3^y) that for any $y_1, y_2 \in Y$ and $y \in [y_1, y_2]$, $K(y) \subset K(y_1) \cup K(y_2)$. Finally, if $\lim_{i \in I} x_i = x$, $\lim_{i \in I} y_i = y$ and $x_i \in K(y_i)$ for all $i \in I$, then $x \in K(y)$. Indeed, we have $f(x_i, y_i) \geqq z^*$ for all $i \in I$ and $\lim_{i \in I} (x_i, y_i) = (x, y)$. Hence, by the upper semicontinuity of f, $f(x, y) \geqq z^*$, i.e. $x \in K(y)$.

On the basis of these properties, our theorem follows from the fixed point theorem of I. Joó [2], which can be proved by simple tools (the present formulation is due to L. L. Stachó [4]):

Let X, Y be interval spaces and $K(\cdot)$ a mapping of Y into the family of compact convex subsets of X, such that

(i) $K(y) \neq \emptyset$ for all $y \in Y$,

(ii) $K(y) \subset K(y_1) \cup K(y_2)$ whenever $y \in [y_1, y_2]$ and $y_1, y_2 \in Y$,

(iii) $x \in K(y)$ whenever $y = \lim_{i \in I} y_i$, $x = \lim_{i \in I} x_i$ and $x_i \in K(y_i)$ for all $i \in I$.

Then we have

$$\bigcap_{y \in Y} K(y) \neq \emptyset.$$

4. Theorem 3. *Let X be a compact interval space, Y a convex subset of some real topological vector space, Z a complete linearly ordered space and $f: X \times Y \to Z$ a function, having the properties*

(5^x) *the subfunctions $f(\cdot, y)$ are quasiconcave on X and upper semicontinuous on X for all fixed $y \in Y$,*

(5^y) *the subfunctions $f(x, \cdot)$ are quasiconvex on Y and upper semicontinuous on any interval of Y for all fixed $x \in X$.*

Then

$$\max_x \inf_y f(x, y) = \inf_y \max_x f(x, y).$$

Remark. As Theorem 2 in [4] shows, this assertion is true if we require in (5^y) lower semicontinuity instead of upper semicontinuity.

Proof. It suffices again to prove that the family of sets

$$\mathscr{F} \equiv \{K(y) \equiv \{x \in X : f(x, y) \geqq \inf_y \max_x f(x, y)\} : y \in Y\}$$

has a non-empty intersection. Being the elements of $\mathscr{F}$ compact (because of (5^x) and the compactness of X), it suffices to show that $\mathscr{F}$ has the finite intersection property. The definition of $\inf_y \max_x f(x, y) \equiv z^*$ ensures that $K(y) \neq \emptyset$ for all $y \in Y$.

Assume now that $\bigcap_{i=1}^{n} K(y_i) \neq \emptyset$ for every choice of $y_1, \ldots, y_n \in Y$, but

$\bigcap_{i=1}^{n+1} K(y_i^*) = \varnothing$ for some $y_1^*, \ldots, y_{n+1}^* \in Y$. To complete the proof, we show that this is impossible. Set $K^*(y) \equiv \bigcap_{i=3}^{n+1} K(y_i^*) \cap K(y)$ for all $y \in Y$, then

(6) $$K^*(y_1^*) \cap K^*(y_2^*) = \varnothing.$$

It follows from the inductive hypothesis and (5^x) that

(7) $K^*(y)$ is non-empty, convex and compact for all $y \in Y$.

(5^y) implies that

(8) $K^*(y) \subset K^*(y_1) \cup K^*(y_2)$ whenever $y_1, y_2 \in Y$ and $y \in [y_1, y_2]$.

Furthermore,

(9) either $K^*(y) \subset K^*(y_1^*)$ or $K^*(y) \subset K^*(y_2^*)$ for any $y \in [y_1^*, y_2^*]$.

Indeed, if there were points x_1, x_2 such that $x_i \in K^*(y) \cap K^*(y_i^*)$ $(i=1, 2)$ for some $y \in [y_1^*, y_2^*]$, then — using (6), (7) and (8) — the connected set $[x_1, x_2]$ could be represented as the union of two closed, non-empty and disjoint subsets:

$$[x_1, x_2] = \bigcup_{i=1}^{2} [x_1, x_2] \cap K^*(y) \cap K^*(y_i^*),$$

which is impossible.

For brevity, we write henceforth $[y_1, y_2) = (y_2, y_1]$ instead of $[y_1, y_2] \setminus \{y_2\}$. It follows from (6)—(9) that the sets

$$\{y \in [y_1^*, y_2^*] : K^*(y) \subset K^*(y_i^*)\}, \quad i = 1, 2$$

are disjoint convex sets and their union is $[y_1^*, y_2^*]$. Therefore there exists a point $y_0 \in [y_1^*, y_2^*]$ such that

(10) $$K^*(y) \subset K^*(y_i^*) \quad \text{for all} \quad y \in [y_i^*, y_0), \quad i = 1, 2.$$

Suppose

(11) $$K^*(y_0) \subset K^*(y_1^*)$$

(the case $K^*(y_0) \subset K^*(y_2^*)$ is similar). Then $\bigcap_{y \in [y_2^*, y_0)} K^*(y) \neq \varnothing$. Indeed, being the sets $K^*(y)$ compact, it is enough to show that for any $y_1 \in (y_0, y_2]$, $y_2 \in (y_0, y_2^*]$: $K^*(y_1) \subset K^*(y_2)$. But this is true: the application of (8), (11), (10) and (6) gives

$$K^*(y_1) \subset \big(K^*(y_0) \cup K^*(y_2)\big) \cap K^*(y_2^*) \subset \big(K^*(y_1^*) \cup K^*(y_2)\big) \cap K^*(y_2^*) =$$

$$= \big(K^*(y_1^*) \cap K^*(y_2^*)\big) \cup \big(K^*(y_2) \cap K^*(y_2^*)\big) = \varnothing \cup K^*(y_2) = K^*(y_2).$$

Choosing an arbitrary $x_0 \in \bigcap_{y \in [y_2^*, y_0)} K^*(y)$, we have by definition $f(x_0, y) \geqq z^*$ for all $y\in[y_2^*, y_0)$; and taking the limit $y\to y_0$, we obtain by (5^y)

$$f(x_0, y_0) \geqq z^*. \tag{12}$$

On the other hand; $x_0\in K^*(y_2^*)$, (6) and (11) imply $x_0\neq K^*(y_0)$ i.e. $f(x_0, y_0)<z^*$, contradicting (12). This contradiction proves the theorem.

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p-NILPOTENCE AND FACTOR GROUPS OF p-SUBGROUPS

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Let G be a finite group and P a Sylow p-subgroup of G. Our aim is to get statements of the following type: "If P belongs to a certain class of finite p-groups and $N_G(P)$ is p-nilpotent then so is G itself."

If P is chosen to be abelian we have the classical result of Burnside. The assertion is true also for regular P as a corollary to Wielandt's theorem (see [1]). At last we mention the case when P is of maximal class and it is not isomorphic to $Z_p \wr Z_p$ (the wreath product of two cyclic groups of order p). The fact, that for odd p it presents a true statement easily follows from any of the results in [6].

One can see that in all cases listed above no factor group of P is isomorphic to $Z_p \wr Z_p$ (as for the last case consult [4], p. 372).

EXAMPLE. For any prime p let q be a prime dividing $(p^p-1)/(p-1)$ and $V=\{v_\alpha|\alpha\in F\}$ an elementary abelian group of order p^p indexed by the elements of $F=GF(p^p)$ such that $v_\alpha\cdot v_\beta=v_{\alpha+\beta}$ $(\alpha,\beta\in F)$. For a fixed element γ of multiplicative order q in F we denote by g that automorphism of V for which $v_\alpha^g=v_{\alpha\gamma}$ holds. We may define an other automorphism h of V by $v_\alpha^h=v_{\alpha^p}$. Let G be the semidirect product of V and $\langle g,h\rangle$ then $h^{-1}gh=g^p$, $\langle V,h\rangle=N_G(\langle V,h\rangle)$ is a Sylow p-subgroup of G. G is not p-nilpotent, as $\langle g\rangle$ is its nonnormal Sylow q-subgroup. Denoting by $\{\alpha,\alpha^p,\alpha^{p^2},\dots,\alpha^{p^{p-1}}\}=\{\alpha_1,\alpha_2,\dots,\alpha_p\}$ a normal basis in F, $\langle V,h\rangle=\bigtimes_{1\leq i\leq p}(\langle v_{\alpha_i}\rangle)\langle h\rangle$ is isomorphic to $Z_p\wr Z_p$.

LEMMA. *Let P be the semidirect product of an elementary abelian p-group V by a cyclic p-group $\langle b\rangle$. Then either the class of P is at most $p-1$ or $P/N\cong Z_p\wr Z_p$ for a suitable normal subgroup N.*

PROOF. According to the Jordan-form of b, $V=\bigtimes_{1\leq i\leq k} V_i$ with $V_i=\bigtimes_{1\leq j\leq n_i}\langle a_{j,i}\rangle$ and

$$b^{-1}a_{j,i}b=\begin{cases} a_{j,i}\cdot a_{j+1,i}, & \text{if } j<n_i\\ a_{j,i}, & \text{if } j=n_i.\end{cases}$$

The class of P is max $\{n_i|1\leq i\leq k\}$, as all the V_i-s are $\langle b\rangle$-invariant and abelian. Assume that $n=n_1\geq p$. Let $N=\langle a_{p+1,1},\dots,a_{n,1}\rangle\times V_2\times\dots\times V_k\langle b^p\rangle$ then $P/N\cong Z_p\wr Z_p$.

DEFINITION. Let $K\leq H\leq G$. K is called strongly closed in H with respect to G, if for any element $g\in G$, $K^g\cap H\leq K$.

THEOREM. *Let G be a finite group, p a prime, $P\in \mathrm{Syl}_p(G)$. Suppose P contains a* *subgroup T, strongly closed in P with respect to G and T does not possess any factor group isomorphic to $Z_p \wr Z_p$. If $N_G(T)$ is p-nilpotent and*

(1) $p>2$, or

(2) $p=2$ and either G is 2-solvable (i.e. solvable) or G does not involve S_4 (the symmetric group on 4 letters);

then G is p-nilpotent.

REMARK. Taking the special case $T=P$, for $p>2$ the theorem is of the desired type outlined in the introduction, and it generalizes (for odd primes) all the statements listed there. We will deal with some applications of this theorem in a forthcoming paper.

PROOF. Suppose the theorem were false, and let G be a minimal counterexample. All the conditions remain valid in $G_1=G/0_{p'}(G)$. In fact, for any element $g\in G$ there exists an element h in $0_{p'}(G)$ such that $T^g\cap P\cdot 0_{p'}(G)=T^g\cap P^h=(T^{gh^{-1}}\cap P)^h\leqq \leqq T^h$, hence $T\cdot 0_{p'}(G)/0_{p'}(G)$ is strongly closed in $P\cdot 0_{p'}(G)/0_{p'}(G)$ with respect to G_1. Clearly $N_{G_1}(T\cdot 0_{p'}(G)/0_{p'}(G))=N_G(T\cdot 0_{p'}(G))/0_{p'}(G)=N_G(T)\cdot 0_{p'}(G)/0_{p'}(G)$ is p-nilpotent, thus $0_{p'}(G)=1$ by the minimality of G. Using the theorems of THOMPSON [3] and that of GLAUBERMAN [5] we can deduce that $0_p(G)>1$. Let R be a minimal normal p-subgroup of G. All the conditions of the theorem remain valid in G/R, thus G/R is p-nilpotent, hence $R\not\leqq\Phi(G)$ and R is the unique minimal normal p-subgroup. Let M be a maximal subgroup in G with $R\not\leqq M$ then $G=R\cdot M$ and $R\cap M=1$, as R is (elementary) abelian. $P=P\cap G=P\cap RM=R(P\cap M)= =R\cdot P_1$ where $P_1=P\cap M$ is a Sylow p-subgroup of M and $M\cong G/R$ is p-nilpotent.

Let $K=0_{p'}(M)$ and $1\neq Q\in \mathrm{Syl}_q(K)$. Suppose that $Q\neq K$. We may assume by $M=K\cdot N_M(Q)$ that $P_1\leqq N_M(Q)$, so $P_1\cdot Q<M$ and $P\cdot Q=R\cdot P_1\cdot Q<G$ yields $P\leqq N_G(Q)$. Repeating this for all primes dividing the order fo K we should get $P\leqq N_G(K)$ and so $K\lhd G$, a contradiction to $0_{p'}(G)=1$; thus $Q=K$. Let $Q_0<Q$, Q_0 be a characteristic subgroup of Q then $Q_0\cdot R\lhd G$ by $Q_0\cdot R/R$ char $Q\cdot R/R\lhd \lhd G/R$, so $P\cdot Q_0=P_1\cdot R\cdot Q_0<G$, $P\leqq N_G(Q_0)$, $Q_0\leqq 0_{p'}(G)=1$, so Q is elementary abelian.

Let $L=C_G(R)$ and $Q_1=L\cap Q$ then $Q_1\lhd M$. As $R\leqq C_G(Q_1)$, $Q_1\leqq 0_q(G)=1$, so L is a p-group. Suppose that $L>R$; $L\cap M\lhd M$ and $R\leqq C_G(L\cap M)$ gives $L\cap M\lhd G$, hence $L\cap M=1$ by $(L\cap M)\cap R=1$ and the uniqueness of R. Thus $C_G(R)=L=L\cap P=L\cap R\cdot P_1=R(L\cap P_1)=R(L\cap M)=R$. As $Z(P)\leqq C_G(R)=R$, for any $g\in G$ $0_p(G)=0_p(G)^g\leqq C_G(Z(P)^g)$, so $0_p(G)\leqq C_G(\langle Z(P)^g|g\in G\rangle)=C_G(R)=R$. It simply implies that $C_G(Q)=Q$.

$T\lhd P$, thus $T\cap R>1$. If $h\in G$, then $(T\cap R)^h=T^h\cap R=(T^h\cap P)\cap R\leqq T\cap R$, hence $T\cap R\lhd G$, and so $T\geqq R$ by the minimality of R. $T=T\cap P=T\cap R\cdot P_1= R(T\cap P_1)>R$ by the conditions, so $T\cdot Q=R((T\cap P_1)Q)$ is not p-nilpotent, consequently $TQ=G$ by the minimality of G, thus $T=P$.

R is an irreducible M-module, so by Clifford's theorem there is a decomposition $R=R_1\times R_2\times\ldots\times R_t$ with $Q\leqq N_G(R_i)$, $R_i^y=R_{i(y)}$ for all i and $y\in P_1$, and for any i R_i is the direct product of minimal Q-invariant subgroups which are Q-isomorphic; therefore $Q/C_Q(R_i)$ is cyclic $(i=1, 2, \ldots, t)$. Suppose $t>1$. Let S be maximal among those normal subgroups of P with a corresponding factor group of the type $\bar{P}=U\cdot\bar{P}_1$ such that $U=\bigtimes_{1\leqq i\leqq s} U_i$, $U_i^b=U_{i(b)}$ for any $b\in\bar{P}_1$, where the action of

$\bar{P}_1$ is nontrivial and transitive on the index-set and $U\cap\bar{P}_1=1$. Assume that $|U_r|>p$ for some r. Let $1\neq y=y_1\cdot y_2\cdot\ldots\cdot y_s\in\Omega_1(Z(\bar{P})\cap U)$ then $\bigtimes_i\langle y_i\rangle\lhd\bar{P}$ and $\bar{P}/\bigtimes_i\langle y_i\rangle$ remains of the same type, thus $|U_1|=\ldots=|U_s|=p$. Clearly $N=\{y\in\bar{P}_1|U_i^y=U_i\ (i=1,2,\ldots,s)\}=C_{\bar{P}_1}(U)\lhd\bar{P}$ and produces a factor group of the same type, so $N=1$. Let $1\neq z\in\Omega_1(Z(\bar{P}_1))$ then z induces a fixpoint-free permutation on the index-set, so it may be assumed that $U=V_1\times\ldots\times V_e$ with $V_i=\bigtimes_{1\leq j\leq p}U_{(i-1)p+j}$ and $z\in N(V_i)$ for all i. Obviously $V_i\langle z\rangle\cong Z_p\wr Z_p$; if $e=1$, then $\bar{P}_1=C_{\bar{P}_1}(\langle z\rangle)=\langle z\rangle$ and $\bar{P}\cong Z_p\wr Z_p$, a contradiction, thus $e>1$. Denoting by W_i the derived group of $V_i\langle z\rangle$, $|V_i\colon W_i|=p$ and $\bigtimes_{1\leq i\leq e}W_i\lhd\bar{P}$. Its factor group is of the same type, too. We get by this contradiction $t=1$, hence $Q=Q/C_Q(R)=Q/C_Q(R_1)$ is cyclic. Thus $P_1\cong M/Q=N_G(Q)/C_G(Q)$ is also cyclic, and P satisfies the conditions of the lemma. It follows immediately that $p\neq 2\neq q$, so we can conclude by the HALL—HIGMAN Theorem B [2] that the class of P is at least p, which yields the final contradiction by our lemma.

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SECTIONWISE PROPERTIES AND MEASURABILITY OF FUNCTIONS OF TWO VARIABLES

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1. Let $\mathcal{F}$ and $\mathcal{G}$ be function classes on $[0, 1]$. We denote by $\mathcal{F}\times\mathcal{G}$ the class of functions f defined on $Q=[0, 1]\times[0, 1]$ with the property $f_x\in\mathcal{G}$ and $f^y\in\mathcal{F}$ for every $x, y\in[0, 1]$, that is, all the horizontal sections $f^y(x)=f(x, y)$ belong to $\mathcal{F}$ and all the vertical sections $f_x(y)=f(x, y)$ belong to $\mathcal{G}$.

In this paper first we summarize the measurability properties of $\mathcal{F}\times\mathcal{G}$ where $\mathcal{F}$ and $\mathcal{G}$ run through the following classes defined on $[0, 1]$:

$C = C[0, 1] = \{f; f \text{ is continuous}\}$,

$\mathcal{A} = \{f; f \text{ is approximately continuous}\}$,

$b_1\Delta = \{f; |f|\leqq 1 \text{ and } f \text{ is a derivative}\}$,

$\Delta = \{f; f \text{ is a derivative}\}$,

$\mathcal{DB}_1 = \{f; f \text{ is Darboux Baire 1}\}$,

$\mathcal{B}_\alpha$: the α'th class of Baire, $\alpha=1, 2, \dots$.

We also use the notation $\mathcal{B}_\alpha$ for the Baire classes of functions defined on Q. $\mathcal{M}$ denotes the class of Lebesgue measurable functions on Q.

The following chart makes it easy to look through the measurability results.

	C	$\mathcal{A}$	$b_1\Delta$	Δ	$\mathcal{DB}_1$	$\mathcal{B}_1$	$\mathcal{B}_2$
C	$\mathcal{B}_1$	$\mathcal{B}_1$	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_2$	$\mathcal{B}_2$	$\mathcal{B}_3$
$\mathcal{A}$		$\mathcal{B}_2$	$\mathcal{B}_2$	$\mathcal{B}_2$	$\mathcal{B}_2$	$\mathcal{B}_2$	*
$b_1\Delta$			$\mathcal{B}_2$	$\mathcal{B}_2$	$\mathcal{B}_2$	$\mathcal{B}_2$	*
Δ				$\mathcal{M}$	$\mathcal{M}$	$\mathcal{M}$	*
$\mathcal{DB}_1$					*	*	*
$\mathcal{B}_1$						*	*
$\mathcal{B}_2$							*

* indicates that the corresponding classes $\mathcal{F}\times\mathcal{G}$ contain non-measurable functions.

The reader should realize that our classes are listed in increasing order apart from the independent classes $\mathcal{A}$ and $b_1\Delta$. It is obvious that a measurability property of $\mathcal{F}\times\mathcal{G}$ also holds for $\mathcal{F}'\times\mathcal{G}'$ whenever $\mathcal{F}'\subset\mathcal{F}$, $\mathcal{G}'\subset\mathcal{G}$.

The known results:

(i) $C\times C\subset\mathscr{B}_1$ and, in general, $C\times\mathscr{B}_\alpha\subset\mathscr{B}_{\alpha+1}$ is a piece of classic due to Lebesgue ([7], §27, V. Théorème 2, p. 285).

(ii) $C\times b_1\Delta\subset\mathscr{B}_1$ must be known. We could not find a reference and hence a simple proof is provided as follows. Let $f\in C\times b_1\Delta$ and put

$$F(x, y)=\int_0^y f(x, t)\,dt \quad ((x, y)\in Q).$$

It is immediate by Lebesque's convergence theorem that the sections F^y are continuous. On the other hand, the sections F_x are uniformly Lipschitz 1 functions and these imply the continuity of F on Q. Hence

$$f(x, y)=\lim_{n\to\infty} n\left(F\left(x, y+\frac{1}{n}\right)-F(x, y)\right)$$

is Baire 1 on Q.

(iii) If $f\in C\times\mathscr{A}$, then $\frac{2}{\pi}\operatorname{arctg} f\in C\times b_1\Delta\subset\mathscr{B}_1$ and hence $f\in\mathscr{B}_1$. (We used here the fact that bounded approximately continuous functions are derivatives, [1] p. 21.)

(iv) $C\times\Delta\subset C\times\mathscr{D}\mathscr{B}_1\subset C\times\mathscr{B}_1\subset\mathscr{B}_2$ and $C\times\mathscr{B}_2\subset\mathscr{B}_3$

follows by (i) and hence the assertions in the first row are verified.

(v) $\mathscr{A}\times\mathscr{A}\subset\mathscr{B}_2$ was proved by R. O. Davies [2]. $\mathscr{A}\times\mathscr{B}_1\subset\mathscr{B}_2$ is proved in [9] and this implies everything in the second row apart from $*$.

(vi) If $2^{\aleph_0}=\aleph_1$ then there exists a non-measurable function in $\mathscr{A}\times\mathscr{B}_2$. A construction can be found in [3], Theorem 11 and [4], Théorème 3. The authors of these papers claim only the Lebesgue measurability of the sections f_x, but their constructions actually give Baire 2 functions. We do not know whether the continuum hypothesis is necessary for $\mathscr{A}\times\mathscr{B}_2\not\subset\mathscr{M}$.

(vii) $b_1\Delta\times b_1\Delta\subset\mathscr{B}_2$ is due to Z. Grande ([6], Théorème 3). The stronger assertion $b_1\Delta\times\mathscr{B}_1\subset\mathscr{B}_2$ is contained in [9].

(viii) All the stars in the last column follow from (vi). In fact, if $f\in(\mathscr{A}\times\mathscr{B}_2)-\mathscr{M}$ then $\frac{2}{\pi}\operatorname{arctg} f\in(b_1\Delta\times\mathscr{B}_2)-\mathscr{M}\subset\ldots\subset(\mathscr{B}_2\times\mathscr{B}_2)-\mathscr{M}$. All these relations rely upon $2^{\aleph_0}=\aleph_1$.

(ix) $\Delta\times\mathscr{B}_1\subset\mathscr{M}$ was proved by M. Laczkovich in [8]. We do not know whether or not stronger measurability properties (e.g. $\Delta\times\Delta\subset\mathscr{B}_4$) hold in the fourth row.

(x) $\mathscr{D}\mathscr{B}_1\times\mathscr{D}\mathscr{B}_1\not\subset\mathscr{M}$ is a theorem of J. S. Lipiński [10]. This implies all the remaining stars in the chart. We remark that the first result in this topic is a theorem of Sierpiński stating $\mathscr{B}_1\times\mathscr{B}_1\not\subset\mathscr{M}$ ([11], p. 147). It is remarkable that Lipiński's counterexample is sectionwise approximately continuous with at most one exceptional point for each section. The sharp contrast between this fact and $\mathscr{A}\times\mathscr{A}\subset\mathscr{B}_2$ shows that the two dimensional measurability very delicately depends on those of the sections.

(xi) As we mentioned above, it is not known whether in the fourth row we have sharp results. On the other hand all the positive results in the first three rows are sharp. $\mathscr{A}\times\mathscr{A}\not\subset\mathscr{B}_1$ was proved by Davies ([2], Theorem 2). He constructed a bounded function, thus he also proved $\mathcal{A}\times b_1\Delta \not\subset \mathcal{B}_1$ and $b_1\Delta\times b_1\Delta\not\subset\mathcal{B}_1$. Hence the second and the third row cannot be improved.

(xii) $C\times\mathcal{D}\mathcal{B}_1\not\subset\mathcal{B}_1$ was shown by Z. GRANDE ([5], Théorème 3). The only gap remained to fill up is to show $C\times\Delta\not\subset\mathcal{B}_1$. In the next section of our paper we are going to prove this relation.

2. Our result is stated in the following

THEOREM. *There exists a function $f(x, y)$ defined on the unit square Q such that the section f^y is continuous for every $y\in[0, 1]$, the section f_x is a derivative for every $x\in[0, 1]$ and f does not belong to the first class of Baire.*

PROOF. We represent out function f as a sum $f=g+h$ where g and h satisfy the following properties.

$(1)_g$ g^y is continuous for every $y\in[0, 1]$;

$(2)_g$ the function

$$g_x^*(y)=\begin{cases} g_x(y) & \text{if} \quad y\neq x\\ 0 & \text{if} \quad y=x\end{cases}$$

is a derivative for every $x\in[0, 1]$;

$(3)_g$

$$g(x, x)=\begin{cases} 1 & \text{if} \quad x=s_k \quad (k=1, 2, \ldots),\\ 0 & \text{otherwise};\end{cases}$$

where $\{s_k\}_{k=1}^{\infty}$ is a suitable sequence everywhere dense in $[0, 1]$;

$(1)_h$ h^y is continuous for every $y\in[0, 1]$;

$(2)_h$ If $x\in[0, 1]-\{s_k\}_{k=1}^{\infty}$ then the section h_x is a derivative. Furthermore, the function

$$\begin{cases} h_{s_k}(y) & \text{if} \quad y\neq s_k\\ 1 & \text{if} \quad y=s_k\end{cases}$$

is a derivative for every $k=1, 2, \ldots$;

$(3)_h$ $$h(x, x)=0 \quad \text{for every} \quad x\in[0, 1].$$

Having these properties above, the function $f\overset{\text{def}}{=}g+h$ obviously has the required continuous and derivative sections, respectively. Since

$$f(x, x)=\begin{cases} 1 & \text{if} \quad x=s_k \quad (k=1, 2, \ldots),\\ 0 & \text{otherwise}\end{cases}$$

and $\{s_k\}_{k=1}^{\infty}$ is everywhere dense in $[0, 1]$, f cannot be a Baire 1 function.

Let the rational numbers of $(0, 1)$ be enumerated in a sequence $\{r_k\}_{k=1}^{\infty}$. Let $s_1=r_1$, $\delta_1=\min(s_1, 1-s_1)$ and let P_1 be a nowhere dense perfect set in $(s_1-\delta_1, s_1+\delta_1)$ such that the points s_1 and $s_1\pm\frac{1}{n}$ are all density points of P_1 for every $n>\frac{1}{\delta_1}$. Suppose the points $s_1, \ldots, s_{k-1}$ and the nowhere dense perfect sets

$P_1, \ldots, P_{k-1}$ have been defined such that $s_i \in P_i$ $(i=1, 2, \ldots, k-1)$. Now we choose a point

$$s_k \in (0, 1) - \bigcup_{i=1}^{k-1} P_i$$

such that $|s_k - r_k| < \frac{1}{k}$ and put

$$\delta_k = \operatorname{dist}\left(s_k, \bigcup_{i=1}^{k-1} P_i \cup \{0, 1\}\right).$$

Let P_k be a nowhere dense perfect subset of $(s_k - \delta_k, s_k + \delta_k)$ such that the points s_k and $s_k \pm \frac{1}{n}$ $\left(n > \frac{1}{\delta_k}\right)$ are density points of P_k. Thus we have defined the sequences $\{s_k\}_{k=1}^{\infty}$ and $\{P_k\}_{k=1}^{\infty}$ by induction.

LEMMA 1. *Let $P \subset (0, 1)$ be measurable, $a \in P$ be a density point of P and let $0 < \eta \leqq \varepsilon \leqq 1$ be given. Then there exists a function g defined on Q such that*

(i) $$0 \leqq g \leqq 1;$$

(ii) $$g(x, x) = \begin{cases} 1 & \text{if} \quad x = a, \\ 0 & \text{otherwise}; \end{cases}$$

(iii) $g(x, y) = 0$ *if* $x = a$, $y \neq a$ *or* $|x-a| \geqq \varepsilon$ *or* $|y-a| \geqq \varepsilon$ *or* $y \notin P$;
(iv) g_x *is approximately continuous for every* $x \in [0, 1]$, $x \neq a$;
(v) g^y *is continuous for every* $y \in [0, 1]$;

(vi) $$\int_0^1 g_x \, dy \leqq \eta |x-a| \quad \text{for every} \quad x \in [0, 1].$$

PROOF. Consider the set

$$D = \left\{(x, y);\ |x-a| \leqq \varepsilon,\ |y-a| \leqq \frac{\eta}{2} |x-a|\right\}.$$

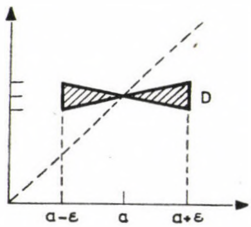

Fig. 1

Let

$$\chi(x, y) = \begin{cases} 0 & \text{if} \quad (x, y) \notin D, \\ 1 & \text{if} \quad (x, y) = (a, a), \\ \left(1 - \frac{|x-a|}{\varepsilon}\right)\left(1 - \frac{2}{\eta}\left|\frac{y-a}{x-a}\right|\right) & \text{if} \quad (x, y) \in D - \{(a, a)\}. \end{cases}$$

Then $0\leqq\chi\leqq 1$, χ is continuous except in (a, a) and $\chi(x, a)$ is continuous in $[0, 1]$. Applying a lemma of ZAHORSKI ([12], Lemme 12, p. 29, or [1], p. 28) it is easy to find an approximately continuous function $p(y)$ defined on $[0, 1]$ such that $0\leqq p\leqq 1$, $p(a)=1$ and $p(y)=0$ for $y\notin P$. We put $g(x, y)\overset{\text{def}}{=}\chi(x, y)p(y)$. The easy verification of (i)—(vi) is left to the reader.

Apply Lemma 1 with $P=P_k$, $a=s_k$, $\varepsilon=\frac{1}{k}$, $\eta=\frac{1}{k^2}$ and let D_k and g_k denote the corresponding domain and function ($k=1, 2, \ldots$). We put $g=\sum_{k=1}^{\infty} g_k$. Observe that, for every $y\in[0, 1]$, $g_k^y\equiv 0$ apart from at most one exceptional k and hence the definition of g makes sense. We have to verify $(1)_g$, $(2)_g$ and $(3)_g$. $(1)_g$ and $(3)_g$ are obvious from the definition. In order to prove $(2)_g$ first we observe

(4) $0\leqq g\leqq 1$ and that

(5) g_x is approximately continuous everywhere except at the point $y=x$, for every $x\in[0, 1]$.

Indeed, if $y\neq x$ that is the point (x, y) is off the diagonal then, there is a neighbourhood of (x, y) which meets only a finite number of the rectangles

$$\left[s_k-\frac{1}{k}, s_k+\frac{1}{k}\right]\times\left[s_k-\frac{1}{k}, s_k+\frac{1}{k}\right]$$

and hence, by Lemma 1 (iii) and (iv), g_x is approximately continuous at y. Let $x\in[0, 1]$ be fixed and put

$$G(y)=\int_x^y g(x, t)\,dt \quad (y\in[0, 1]).$$

By (4) and (5), we have $G'(y)=g_x(y)$ for every $y\neq x$. It is enough to prove

$$\lim_{y\to x}\frac{G(y)}{y-x}=0. \tag{6}$$

Suppose first $x\in[0, 1]-\{s_k\}_{k=1}^{\infty}$ and let N be fixed. Choose δ so small that $|y-x|<\delta$ implies $(\{x\}\times[x, y])\cap\left(\bigcup_{k=1}^{N} D_k\right)=\emptyset$. Observe that $(\{x\}\times[x, y])\cap D_k\neq\emptyset$ implies $|x-s_k|\leqq 2|x-y|$ ($k=1, 2, \ldots$). Hence, by Beppo Levi's theorem, we have

$$G(y)=\int_x^y g(x, t)\,dt=\sum_{k=1}^{\infty}\int_x^y g_k(x, t)\,dt=\sum{}'\int_x^y g_k(x, t)\,dt\leqq$$

$$\leqq\sum{}'\int_0^1 g_k(x, t)\,dt\leqq\sum{}'\frac{1}{k^2}|x-s_k|\leqq\sum_{k=N+1}^{\infty}\frac{2}{k^2}|x-y|=o(|x-y|)$$

where in $\sum'$ the summation is extended to the indices k with $(\{x\}\times[x, y])\cap D_k\neq\emptyset$. This proves (6). The same estimation holds for $x=s_j$ as well since, by Lemma 1 (iii), $g_j(s_j, y)=0$ if $y\neq s_j$.

Now we turn to the construction of h.

LEMMA 2. *Let M be a measurable set in* $[0, 1]$ *such that*

$$\lambda(M\cap(1-\delta, 1)) > 0 \quad \textit{for every} \quad \delta > 0. \tag{7}$$

Then there exists a function $u\colon Q\to \mathbf{R}$ *such that*

(i) u_x *is bounded and approximately continuous for every* $x\in[0, 1]$;

(ii) u^y *is continuous for every* $y\in[0, 1]$;

(iii) $u(x, y)=0$ *if* $x=0$ *or* $y=0$ *or* $y=1$ *or* $y\notin M$;

(iv) $$\int_0^1 u_x\,dy = \begin{cases} 1 & \textit{if} \quad x = 1 \\ 0 & \textit{if} \quad x < 1; \end{cases}$$

(v) $$\left|\int_0^t u_x dy\right| \leqq 1 \quad \textit{for every} \quad x,\ t\in[0, 1].$$

PROOF. By our assumption (7) we can find a sequence of density points of M $0 < y_1 < y_2 < \dots,\quad y_n \to 1-0$.

Let H denote the set of density points of M and let V be an F_σ set,

$$\{y_k\}_{k=1}^\infty \subset V \subset H, \quad \lambda(V) = \lambda(H) = \lambda(M).$$

Then we can apply Zahorski's lemma and obtain an approximately continuous function $0\leqq p\leqq 1$ which separates the $\mathcal{G}_\delta$ sets $([0, 1]-V)\cup\{0, 1\}$ and $\{y_k\}_{k=1}^\infty$ both closed in the density topology. That is

$$p(y_k) = 1 \quad (k = 1, 2, \dots), \quad p(y) = 0 \quad \big(y\in([0, 1]-V)\cup\{0, 1\}\big).$$

We define

$$q(y) = \frac{1}{\int_0^1 p(t)\,dt}\, p(y)$$

and observe that

$$\int_{1-\delta}^1 q(y)\,dy > 0 \quad (\delta>0) \tag{8}$$

and

$$\int_0^1 q(y)\,dy = 1.$$

We put

$$\alpha(x) = \int_0^x \frac{x-t}{1-t}\, q(t)\,dt, \quad \beta(x) = \int_x^1 (t-x)\, q(t)\,dt \quad (0 \leqq x \leqq 1).$$

By (8) we have $\beta(x)>0$ $(x<1)$. Define

$$u(0, y) = 0, \quad u(1, y) = q(y)$$

and

$$u(x, y) = \begin{cases} -\dfrac{\alpha(x)}{\beta(x)}(y-x)\,q(y) & \text{if } \ 0 < x < y \leqq 1, \\ \dfrac{x-y}{1-y}\,q(y) & \text{if } \ 0 \leqq y \leqq x < 1. \end{cases}$$

Now we have to verify properties (i)–(v). (i) follows immediately from the approximate continuity of q. (ii) and (iii) are easy by the definitions. (iv) follows by direct calculations. The estimation under (v) is trivial for $x=0$ and $x=1$. If $0<x<1$ then the integrand u_x has one and only one change of sign at $y=x$ where it turns from positive to negative. Hence

$$\Big|\int_0^t u_x\,dy\Big| \leqq \Big|\int_0^x u_x\,dy\Big| = \int_0^x \frac{x-y}{1-y}\,q(y)\,dy \leqq \int_0^x q(y)\,dy \leqq \int_0^1 q(y)\,dy = 1.$$

LEMMA 3. *Let T denote the rectangle $[a-\delta, a+\delta]\times[b, b+\delta]$, let $P\subset[b, b+\delta]$ be a measurable subset such that $b+\delta$ is a left hand side density point of P and let a positive number c be given. Then there exists a function $\varphi\colon T\to\mathbf{R}$ such that*

(i) *$\varphi(x, y)=0$ if (x, y) is a boundary point of T or $y\notin P$;*

(ii) *φ_x is a bounded approximately continuous function for every $x\in[a-\delta, a+\delta]$;*

(iii) *φ^y is continuous for every $y\in[b, b+\delta]$;*

(iv)
$$\int_b^{b+\delta} \varphi_x\,dy = \begin{cases} c & \text{if } \ x = a, \\ 0 & \text{if } \ 0 < |x-a| \leqq \delta; \end{cases}$$

(v)
$$\Big|\int_b^t \varphi_x\,dy\Big| \leqq c \quad \text{for every} \quad (x, t)\in T.$$

PROOF. Let u denote the function constructed in Lemma 2 with $M=\{y;\ \delta y+b\in P\}$. We define the function u_1 by

$$u_1(x, y) = \begin{cases} u(x+1, y) & \text{if } \ (x, y)\in[-1, 0]\times[0, 1], \\ u(1-x, y) & \text{if } \ (x, y)\in[0, 1]\times[0, 1] \end{cases}$$

and

$$\varphi(x, y) = c u_1\left(\frac{x-a}{\delta}, \frac{y-b}{\delta}\right) \quad \text{if} \quad (x, y)\in T.$$

Properties (i)—(v) follow immediately from Lemma 2.

Now consider the rectangles

$$T_{k,n} = \left[s_k - \frac{1}{n(n+1)},\ s_k + \frac{1}{n(n+1)}\right]\times\left[s_k + \frac{1}{n+1},\ s_k + \frac{1}{n}\right]$$

and

$$T'_{k,n} = \left[s_k - \frac{1}{n(n+1)},\ s_k + \frac{1}{n(n+1)}\right]\times\left[s_k - \frac{1}{n},\ s_k - \frac{1}{n+1}\right]$$

$$(k = 1, 2, \ldots;\ n = 2, 3, \ldots).$$

Since

$$(s_k, s_k) \notin \bigcup_{j=1}^{k-1} \bigcup_{m=2}^{\infty} (T_{j,m} \cup T'_{j,m}),$$

there exists a natural number $n_k > \frac{1}{\delta_k}$ such that

$$T_{k,n} \cap T_{j,m} = T_{k,n} \cap T'_{j,m} = T'_{k,n} \cap T_{j,m} = T'_{k,n} \cap T'_{j,m} = \emptyset \tag{9}$$

if $n \geqq n_k$, $j < k$ and $m \geqq 2$.

For a fixed pair (k, n) we apply Lemma 3 for each of the rectangles $T_{k,n}$, $T'_{k,n}$ with $c = \frac{1}{n(n+1)}$, $P = P_k \cap \left[s_k + \frac{1}{n+1},\ s_k + \frac{1}{n}\right]$ or $P = P_k \cap \left[s_k - \frac{1}{n}, s_k - \frac{1}{n+1}\right]$ and obtain the functions $\varphi_{k,n}$ and $\psi_{k,n}$, respectively. We put

$$h(x, y) = \begin{cases} \varphi_{k,n}(x, y) & \text{if } (x, y) \in T_{k,n} \quad (k = 1, 2, \ldots;\ n \geqq n_k), \\ \psi_{k,n}(x, y) & \text{if } (x, y) \in T'_{k,n} \quad (k = 1, 2, \ldots;\ n \geqq n_k), \\ 0 \text{ otherwise.} \end{cases}$$

By Lemma 3 (i), the definition of h is unambiguous. We have to prove $(1)_h$, $(2)_h$ and $(3)_h$. $(3)_h$ is obvious. If $y \notin \bigcup_{k=1}^{\infty} P_k$ then $h^y \equiv 0$ by Lemma 3 (i). If $y \in P_k$ then $h(x, y)$ can take non-zero values only in one of the rectangles $T_{k,n}$, $T'_{k,m}$ and hence h^y is continuous by Lemma 3 (i) and (iii) and hence $(1)_h$ follows.

In order to prove $(2)_h$ we observe that for every $x \in [0, 1]$ the section h_x is locally bounded and approximately continuous everywhere except at the point $y = x$. Indeed, if $x \neq y$ then the point (x, y) has a neighbourhood meeting only a finite number (at most two) of the rectangles $T_{k,n}$ and $T'_{k,n}$. Thus we can refer to Lemma 3 (i) and (ii).

Let now $x_0 \in [0, 1] - \{s_k\}_{k=1}^{\infty}$ be fixed and let

$$H(y) = \begin{cases} 0 & \text{if } y = x_0, \\ -\int_y^1 h_{x_0}(t)\, dt & \text{if } x_0 < y \leqq 1, \\ \int_0^y h_{x_0}(t)\, dt & \text{if } 0 \leqq y < x_0. \end{cases}$$

We prove that H is a primitive of h_{x_0}. $H'(y) = h_{x_0}(y)$ is obvious for $y \neq x_0$ by the remark above. We have to prove

$$\lim_{y \to x_0} \frac{H(y)}{y - x_0} = 0 \quad (= h_{x_0}(x_0)). \tag{10}$$

If $(x_0, y) \notin \bigcup_{k=1}^{\infty} \bigcup_{n=n_k}^{\infty} (T_{k,n} \cup T'_{k,n})$ then $H(y) = 0$ by Lemma 3 (iv). Suppose

$$y > x_0, \quad (x_0, y) \in T_{k,n} \tag{11}$$

$(n \geqq n_k)$. Then

$$H(y) = -\int_y^1 h_{x_0}(t)\,dt = \int_{s_k+\frac{1}{n+1}}^{y} \varphi_k(x_0, t)\,dt$$

and thus, by Lemma 3 (v),

$$|H(y)| \leqq \frac{1}{n(n+1)}.$$

On the other hand, (11) implies

$$s_k - \frac{1}{n(n+1)} \leqq x_0 \leqq s_k + \frac{1}{n(n+1)}, \quad s_k + \frac{1}{n+1} \leqq y \leqq s_k + \frac{1}{n}$$

from which $y - x_0 \geqq \frac{1}{n+1} - \frac{1}{n(n+1)} \geqq \frac{1}{2(n+1)}$ if $n \geqq 2$. Therefore $\left|\frac{H(y)}{y-x_0}\right| \leqq \frac{2}{n}$. $y \to x_0$ implies $n \to \infty$, proving (10). (The proof is similar for $y < x_0$.)

Now let $x_0 = s_k$ and put

$$H(y) = -\int_y^1 h(s_k, t)\,dt, \quad s_k < y \leqq 1.$$

Since h_{s_k} is locally bounded and approximately continuous if $y > s_k$, we have $H'(y) = h_{s_k}(y)$ $(s_k < y \leqq 1)$. Furthermore, if $s_k + \frac{1}{n+1} \leqq y \leqq s_k + \frac{1}{n}$ and $n > n_k$, then

$$H(y) = -\int_y^1 h(s_k, t)\,dt = -\sum_{j=n_k}^{n} \int_{s_k+\frac{1}{j+1}}^{s_k+\frac{1}{j}} \varphi_{k,j}(s_k, t)\,dt + \int_{s_k+\frac{1}{n+1}}^{y} \varphi_{k,n}(s_k, t)\,dt =$$

$$= -\sum_{j=n_k}^{n} \frac{1}{j(j+1)} + \int_{s_k+\frac{1}{n+1}}^{y} \varphi_{k,n}(s_k, t)\,dt = \frac{1}{n+1} - \frac{1}{n_k} + O\left(\frac{1}{n(n+1)}\right)$$

using Lemma 3 (iv) and (v).

This implies

$$\lim_{y \to s_k+0} H(y) = -\frac{1}{n_k}$$

and

$$\lim_{y \to s_k+0} \frac{H(y) + \frac{1}{n_k}}{y - s_k} = 1.$$

That is, if we define $H(s_k) = -\frac{1}{n_k}$, then we have $H'_+(s_k) = 1$. Similarly, if we define

$$H(y) = \int_0^y h(s_k, t)\,dt - \frac{2}{n_k} \quad (0 \leqq y < s_k),$$

then we have again

$$\lim_{y \to s_k - 0} H(y) = -\frac{1}{n_k} \quad \text{and} \quad \lim_{y \to s_k - 0} \frac{H(y) + \dfrac{1}{n_k}}{y - s_k} = 1.$$

Thus we have verified $(2)_h$ and hence the proof of our theorem is complete.

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SOME THEOREMS ON UNICITY OF MULTIVARIATE L_1-APPROXIMATION

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Introduction

Let X be a normed linear space and consider a finite dimensional subspace $U\subset X$. We say that $p\in U$ is a best approximation of $x\in X$ if $\|x-p\|=\inf\{\|x-q\|: q\in U\}$. Since U is finite dimensional each $x\in X$ possesses a best approximation; moreover, if the norm in X is strictly convex, this best approximation is unique. But a number of important norms (Chebyshev norm, L_1-norm) are not strictly convex, therefore the study of uniqueness of best approximation in these cases needs deeper considerations.

In the present note we shall study the unicity of L_1-approximation. Let $K\subset\mathbf{R}^m$ be a compact convex set in the real Euclidian space $\mathbf{R}^m$ $(m\geqq 1)$ having nonempty interior. Consider the space $X=C_1(K)$ of real valued continuous functions on K with norm $\|f\|=\int_K|f|d\mu_m$. (μ_m denotes the Lebesgue measure on K.) Let U_n be an n-dimensional subspace of $C_1(K)$. U_n is called a unicity subspace of $C_1(K)$ if each $f\in C_1(K)$ possesses a unique best approximation out of U_n. Furthermore, we say that U_n is a Haar subspace if zero is the only function in U_n vanishing more than $n-1$ times on K.

The classical theorem of Jackson and Krein (see [11], p. 236) states that if $m=1$ and $K=I=[0, 1]$ then any Haar subspace of $C_1(K)$ is a unicity subspace. Some generalizations of this result for complex and vector valued functions can be found in [5] and [6].

We shall study the unicity of L_1-approximation of functions of more than one variable. By a wellknown result of MAIRHUBER [8] there are no Haar subspaces of dimension greater than 1 in $C_1(K)$, when $m>1$. This fact is an essential difficulty in the extension of the theory of Chebyshev approximation to functions of several variables, because the Haar property is a necessary and sufficient condition for the uniqueness of Chebyshev approximation. On the other hand it is known that different families of spline functions are unicity subspaces of $C_1(I)$ where $I=[0,1]$, $m=1$ (see [1], [3], [12]). Thus it turnes out that in contrast to Chebyshev approximation, the Haar property does not characterize the unicity subspaces of $C_1(K)$. This fact gives a hope that in spite of the absence of Haar subspaces in $C_1(K)$, when $m>1$, there may exist unicity subspaces in $C_1(K)$.

In the present note we shall give several results on uniqueness of best L_1-approximation of continuous functions of more than one variable.

In the first section we consider continuous functions of m variables and prove the unicity of L_1-approximation by linear functions. We also give a general uniqueness theorem for separating functions.

In the second section we study functions of two variables. The main result of this section is the uniqueness theorem for L_1-approximation by algebraic polynomials which are linear in one variable and of arbitrary degree with respect to the other variable.

§. 1

Let $G\subset C_1(K)$ be a linear subspace of $C_1(K)$ and let $U\subset G$ be a finite dimensional subspace of G. Then U is called a unicity subspace of G if each $f\in G$ possesses a unique best approximation out of U. In what follows we shall denote by $\mu_m(A)$ the Lebesgue measure of $A\subset\mathbf{R}^m$ $(m\geqq 1)$. The subspace $U\subset C_1(K)$ is called a Z_m subspace if for any $p\in U\setminus\{0\}$, $\mu_m(Z(p))=0$, where $Z(p)=\{x\in K: p(x)=0\}$.

The following lemma gives a sufficient condition for a Z_m subspace to be a unicity subspace.

LEMMA 1. *Let $m\in\mathbf{N}$ and let G be a linear subspace of $C_1(K)$. Moreover, assume that U is a finite dimensional Z_m subspace of G, which is not a unicity subspace of G. Then there exist $f\in G$ and $p\in U\setminus\{0\}$ such that $Z(f)\subset Z(p)$ and*

$$\int_K q\operatorname{sgn} f\,d\mu_m = 0 \tag{1}$$

for any $q\in U$.

PROOF. Since U is not a unicity subspace of G, some $f^*\in G$ possesses two distinct best approximations $p_1, p_2\in U$. Then $(p_1+p_2)/2\in U$ is also a best approximation. Therefore

$$2\int_K |f^*-(p_1+p_2)/2|\,d\mu_m = \int_K |f^*-p_1|\,d\mu_m+\int_K |f^*-p_2|\,d\mu_m.$$

This yields

$$2|f^*-(p_1+p_2)/2| = |f^*-p_1|+|f^*-p_2| \tag{2}$$

μ_m-a.e. on K. By continuity of the functions involved, (2) holds for any interior point of K. Gut K is convex and compact, hence K is equal to the closure of its interior. Thus, finally, we obtain that (2) holds for any $x\in K$. Set $f=f^*-(p_1+p_2)/2$, $p=p_1-p_2$. Then $f\in G, p\in U\setminus\{0\}$ and (2) implies that $Z(f)\subset Z(p)$. Moreover by definition of f, 0 is a best approximation of f. Then by a wellknown characterization theorem for best L_1-approximation (see [11], p. 46)

$$\Big|\int_{K\setminus Z(f)} q\operatorname{sgn} f\,d\mu_m\Big| \leqq \int_{Z(f)} |q|\,d\mu_m \tag{3}$$

for any $q\in U$. But using that $Z(f)\subset Z(p)$ and U is a Z_m subspace we have $\mu_m(Z(f))=0$. Thus (1) follows immediately from (3). The lemma is proved.

Let L_{m+1} be the subspace of linear functions on K, i.e. $L_{m+1}=\{f\in C_1(K)$: $f(x)=f(x_1, x_2, \dots, x_m)=\sum_{i=1}^m a_ix_i+a_{m+1},\ a_i\in\mathbf{R},\ 1\leqq i\leqq m+1\}$. Evidently, L_{m+1} is an $m+1$ dimensional Z_m subspace of $C_1(K)$.

THEOREM 1. *Let K be a compact convex subset of $\mathbf{R}^m$ with nonempty interior $(m\geqq 1)$. Then L_{m+1} is a unicity subspace of $C_1(K)$.*

PROOF. Assume the contrary. Then by Lemma 1 there exist $f\in C_1(K)$ and $p\in L_{m+1}\setminus\{0\}$ such that $Z(f)\subset Z(p)$ and (1) holds for any $q\in L_{m+1}$. Set $S_i=\{x\in\mathbf{R}^m: (-1)^i p(x)>0\}$, $K_i=K\cap S_i$ $(i=1,2)$. It can be easily shown that S_i is convex, thus K_i is also convex $(i=1,2)$. Since $Z(f)\subset Z(p)$, f does not vanish on the convex set K_i. Therefore f has constant sign on K_i, i.e. $\operatorname{sgn} f=\gamma_i$ on K_i $(|\gamma_i|=1, i=1,2)$. If $\gamma_1=\gamma_2$, then $\operatorname{sgn} f=\gamma_1$ μ_m-a.e. on K, a contradiction to (1). On the other hand, if $\gamma_1=-\gamma_2$, then setting $\bar p=\gamma_2 p$ we obtain that $\operatorname{sgn}\bar p=\operatorname{sgn} f$ μ_m-a.e. on K. This again contradicts (1). The theorem is proved.

Set now $m>1$; $I=[0,1]$, $K=I^m=\{x=(x_1,x_2,\dots,x_m)\in\mathbf{R}^m: x_i\in I, 1\le i\le m\}$. For a given $f\in C_1(I^m)$ we put

$$f_i^*(x_i)=\int_{I^{m-1}}\operatorname{sgn} f(x_1,\dots,x_m)\,dx_1\dots dx_{i-1}\,dx_{i+1}\dots dx_m \quad (1\le i\le m). \tag{4}$$

LEMMA 2. *If $f\in C_1(I^m)$ $(m\ge 1)$ and for given $x'\in I$*

$$\mu_{m-1}\{(x_1,\dots,x_{i-1},x_{i+1},\dots,x_m)\in I^{m-1}: f(x_1,\dots,x_{i-1},x',x_{i+1},\dots,x_m)=0\}=0 \tag{5}$$

then f_i^* is continuous at x' $(1\le i\le m)$.

PROOF. For arbitrary $\varepsilon>0$ we set $B_\varepsilon=\{(x_1,\dots,x_{i-1},x_{i+1},\dots,x_m)\in I^{m-1}: |f(x_1,\dots,x_{i-1},x',x_{i+1},\dots,x_m)|\le\varepsilon\}$. It follows from (5) that

$$\mu_{m-1}(B_\varepsilon)\to 0 \quad (\varepsilon\to 0). \tag{6}$$

Since f is continuous on I^m, $|f(x)-f(x^*)|\le\varepsilon$ for any $x,x^*\in I^m$ such that $\varrho(x,x^*)\le\delta(\varepsilon)$. ($\varrho(.,.)$ denotes the Euclidean distance.) Therefore we have for $|h|\le\delta(\varepsilon)$

$$|f_i^*(x')-f_i^*(x'+h)|\le\int_{I^{m-1}}|\operatorname{sgn} f(x_1,\dots,x_{i-1},x',x_{i+1},\dots,x_m)-$$

$$-\operatorname{sgn} f(x_1,\dots,x_{i-1},x'+h,x_{i+1},\dots,x_m)|\,dx_1\dots dx_{i-1}\,dx_{i+1}\dots dx_m=$$

$$=\int_{I^{m-1}\setminus B_\varepsilon}+\int_{B_\varepsilon}=\int_{B_\varepsilon}\le 2\mu_{m-1}(B_\varepsilon).$$

This and (6) imply the statement of the lemma.

Now we shall consider the special case of functions separating the variables. Set $C_1^*(I^m)=\{f\in C_1(I^m): f(x)=f(x_1,\dots,x_m)=\sum_{i=1}^m f_i(x_i)\}$. This is the subspace of separating functions in $C_1(I^m)$. Let $M_{n_i}\subset C_1(I)$ be n_i-dimensional Haar subspaces of $C_1(I)$, $n_i\ge 1$ $(1\le i\le m)$. We assume that each M_{n_i} contains the constant functions. Set $M_N=\{\sum_{i=1}^m q_i(x_i): q_i\in M_{n_i}, 1\le i\le m\}$. Then M_N is an $N=\sum_{i=1}^m n_i-m+1$ dimensional subspace of $C_1^*(I^m)$.

LEMMA 3. *M_N is a Z_m subspace of $C_1^*(I^m)$.*

PROOF. Assume the contrary. Then for some $q(x_1,\dots,x_m)=\sum_{i=1}^m q_i(x_i)\in M_N\setminus\{0\}$, $\mu_m(Z(q))>0$. Since $q\in M_N\setminus\{0\}$ there exists $1\le j\le m$ such that q_j is not a constant function. Let χ be the characteristic function of $Z(q)$. Then

$$(7) \qquad 0<\mu_m(Z(q))=\int_{I^m}\chi\,d\mu_m=\int_{I^{m-1}}\Big(\int_I \chi\,dx_j\Big)\,dx_1\dots dx_{j-1}\,dx_{j+1}\dots dx_m.$$

For a given $(\bar{x}_1,\dots,\bar{x}_{j-1},\bar{x}_{j+1},\dots,\bar{x}_m)\in I^{m-1}$, $\chi(\bar{x}_1,\dots,\bar{x}_{j-1},x,\bar{x}_{j+1},\dots,\bar{x}_m)=1$ if and only if $q_j(x)=-\sum_{\substack{i=1\\ i\neq j}}^{m} q_i(\bar{x}_i)$. Using that $q_j\in M_{n_j}$ is not a constant function we obtain that the latter relation holds for at most n_j-1 values of x. Hence $\int_I \chi dx_j=0$ for any $(\bar{x}_1,\dots,\bar{x}_{j-1},\bar{x}_{j+1},\dots,\bar{x}_m)\in I^{m-1}$. This evidently contradicts (7). The lemma is proved.

COROLLARY 1. *If $q\in M_N$ is not a constant function, then for any $a\in\mathbf{R}$, $\mu_m\{x\in I^m: q(x)=a\}=0$.*

The following property of Haar subspaces will play an important role in the present paper. Its proof can be obtained from a more general result proved in [7], p. 41.

LEMMA 4. *Let U_n be an n-dimensional Haar subspace of $C_1(I)$. Then for any choice of points $0=t_0<t_1<t_2<\dots<t_k<t_{k+1}=1$ $(k\leqq n-1)$ and signs γ_i $(|\gamma_i|=1,\ 0\leqq i\leqq k)$ there exists a $q\in U_n\setminus\{0\}$ such that* $\operatorname{sgn} q(x)=\gamma_i$ *for* $t_i<x<t_{i+1}$ $(0\leqq i\leqq k)$.

The subspace M_N defined above is a natural "Haar type" subspace of separating functions. It turnes out that M_N satisfies the unicity property.

THEOREM 2. *M_N is a unicity subspace of $C_1^*(I^m)$.*

PROOF. Assume the contrary. By Lemma 3 M_N is a Z_m subspace, hence we may apply Lemma 1 for $G=C_1^*(I^m)$ and $U=M_N$. Thus for some $f\in C_1^*(I^m)$ and $p\in M_N\setminus\{0\}$ we have $Z(f)\subset Z(p)$ and $\int_{I^m} q\operatorname{sgn} f\,d\mu_m=0$ for any $q\in M_N$. By this and definition (4) we have

$$(8) \qquad \int_I q_i f_i^*\,dx_i=0$$

for any $q_i\in M_{n_i}$ $(1\leqq i\leqq m)$. Since $p\in M_N\setminus\{0\}$, $p(x_1,\dots,x_m)=\sum_{i=1}^m p_i(x_i)$, where $p_i\in M_{n_i}$ $(1\leqq i\leqq m)$. We shall consider two cases.

Case 1: for some $1\leqq k,\ j\leqq m$, $k\neq j$, p_j and p_k are not constant functions. Then $\sum_{\substack{i=1\\ i\neq j}}^{m} p_i$ is not a constant function and applying Corollary 1 (for smaller dimension) we easily obtain that

$$\mu_{m-1}\{(x_1,\dots,x_{j-1},x_{j+1},\dots,x_m)\in I^{m-1}: p(x_1,\dots,x_{j-1},x',x_{j+1},\dots,x_m)=0\}=0$$

for any $x'\in I$. Since $Z(f)\subset Z(p)$ this implies that

$$\mu_{m-1}\{(x_1,\dots,x_{j-1},x_{j+1},\dots,x_m)\in I^{m-1}: f(x_1,\dots,x_{j-1},x',x_{j+1},\dots,x_m)=0\}=0$$

for any $x'\in I$. Thus by Lemma 2, f_j^* is continuous on I. Moreover it follows from (8) that f_j^* has at least n_j zeros $0<\bar{x}_1<\bar{x}_2<\ldots<\bar{x}_{n_j}<1$ inside I. (Otherwise by Lemma 4 for some $q^*\in M_{n_j}\setminus\{0\}$, sgn $q^*=\operatorname{sgn} f_j^*$ μ_1-a.e. on I, a contradiction to (8).) Therefore using that $f(x_1, \ldots, x_m)=\sum_{i=1}^m f_i(x_i)$ we have

(9)

$$0=f_j^*(\bar{x}_s)=\int_{I^{m-1}} \operatorname{sgn} f(x_1, \ldots, x_{j-1}, \bar{x}_s, x_{j+1}, \ldots, x_m)\, dx_1 \ldots dx_{j-1}\, dx_{j+1} \ldots dx_m =$$

$$=\mu_{m-1}\{(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m)\in I^{m-1}: \sum_{\substack{i=1\\ i\neq j}}^m f_i(x_i)>-f_j(\bar{x}_s)\}-$$

$$-\mu_{m-1}\{(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m)\in I^{m-1}: \sum_{\substack{i=1\\ i\neq j}}^m f_i(x_i)<-f_j(\bar{x}_s)\},$$

where $s=1, 2, \ldots, n_j$. Set $\bar{f}_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m)=\sum_{\substack{i=1\\ i\neq j}}^m f_i(x_i)$, $\bar{f}_j\in C_1^*(I^{m-1})$.

It is easy to see that the equation

$$\mu_{m-1}\{x\in I^{m-1}: \bar{f}_j(x)>a\}=\mu_{m-1}\{x\in I^{m-1}: \bar{f}_j(x)<a\}$$

has at most one solution $a=a_0$, where

$$\min_{x\in I^{m-1}} \bar{f}_j(x)\leqq a_0\leqq \max_{x\in I^{m-1}} \bar{f}_j(x). \tag{10}$$

Therefore by (9) we obtain

$$f_j(\bar{x}_s)=-a_0, \quad s=1, 2, \ldots, n_j. \tag{11}$$

Furthermore (10) implies that there exists $x^*\in I^{m-1}$ such that $\bar{f}_j(x^*)=a_0$, i.e. $\sum_{\substack{i=1\\ i\neq j}}^m f_i(x_i^*)=a_0$, where $x_i^*\in I$ $(1\leqq i\leqq m, i\neq j)$. Hence and by (11)

$$f(x_1^*, \ldots, x_{j-1}^*, \bar{x}_s, x_{j+1}^*, \ldots, x_m^*)=0 \quad (1\leqq s\leqq n_j).$$

Since $Z(f)\subset Z(p)$ we get

$$0=p(x_1^*, \ldots, x_{j-1}^*, \bar{x}_s, x_{j+1}^*, \ldots, x_m^*)=\sum_{\substack{i=1\\ i\neq j}}^m p_i(x_i^*)+p_j(\bar{x}_s) \quad (1\leqq s\leqq n_j).$$

This relations and the Haar property yield that p_j is a constant function, but this contradicts our assumption.

Case 2: at most one of the p_i-s is not a constant function. Then $p(x_1, \ldots, x_m)=$ $=\bar{p}(x_k)\in M_{n_k}\setminus\{0\}$ $(1\leqq k\leqq m)$. Let $0\leqq x_1'<x_2'<\ldots<x_r'\leqq 1$ be the all zeros of $\bar{p}$, $r\leqq n_k-1$. Since $Z(f)\subset Z(p)$, f does not vanish on the rectangles

$$B_j=\{(x_1, \ldots, x_m)\in I^m: 0<x_i<1;\quad x_j'<x_k<x_{j+1}',\ 1\leqq i\leqq m,\ i\neq k\},$$

where $j=0, 1, \ldots, r$; $x_0'=0$, $x_{r+1}'=1$. (If $x_1'=0$ or $x_r'=1$, then the first or, respectively, the last rectangle is empty.) Therefore $\operatorname{sgn} f=\gamma_j$ for $x\in B_j$ $(|\gamma_j|=1, 0\leqq j\leqq r)$.

By Lemma 4 there exists a $q\in M_{n_k}\setminus\{0\}$ such that $\operatorname{sgn} q(x)=\gamma_j$ while $x'_j<x<x'_{j+1}$ $(0\leqq j\leqq r)$. Setting $q^*(x_1, \ldots, x_m)=q(x_k)\in M_N\setminus\{0\}$ we obtain $\operatorname{sgn} f=\operatorname{sgn} q^*$ μ_m-a.e. on I^m, i.e. $\int_{I^m} q^* \operatorname{sgn} f\, d\mu_m>0$. We again arrived at a contradiction.

The proof of Theorem 2 is complete.

Let P_n denote the set of algebraic polynomials on I of degree at most n. For a given $\bar{n}=(n_1, n_2, \ldots, n_m)\in Z_+^m$ we set $P_{\bar{n}}^*=\{\sum_{i=1}^{m} p_{n_i}(x_i)\colon p_{n_i}\in P_{n_i},\ 1\leqq i\leqq m\}$. Then by Theorem 2 we immediately obtain the following

COROLLARY 2. *$P_{\bar{n}}^*$ is a unicity subspace of $C_1^*(I^m)$.*

REMARK. The Chebyshev approximation of separating functions was studied by D. NEWMAN and H. SHAPIRO [9]. They proved that a best Chebyshev approximation of a continuous separating function of two variables can be given by the sum of Chebyshev approximants of its component functions. Later in [4] it was shown that this is the unique best Chebyshev approximation of a separating function. In connection with Theorem 2 a natural question arises: is the best L_1-approximation of a continuous separating function equal to the sum of the best L_1-approximants of its component functions? The following example shows that in general the answer is no.

EXAMPLE. Set $m=2$ and consider the function $f(x_1, x_2)=f_1(x_1)+f_2(x_2)\in C_1^*(I^2)$, where

$$f_1(x_1)=\begin{cases} 0, & x_1\in\left[0, \dfrac{1}{2}\right], \\ x_1-\dfrac{1}{2}, & x_1\in\left[\dfrac{1}{2}, 1\right]; \end{cases} \qquad f_2(x_2)=x_2-\frac{1}{2},\ x_2\in[0, 1].$$

It can be easily verified that 0 is the best L_1-approximation of both f_1 and f_2 on I by constant functions. Assume that the best L_1-approximation of $f(x_1, x_2)$ by constants is also zero. Since $\mu_2(Z(f))=0$,

$$\int_{I^2} \operatorname{sgn} f(x_1, x_2)\, dx_1\, dx_2 = 0,$$

i.e.

$$\int_I f_1^*(x_1)\, dx_1 = 0, \tag{12}$$

where $f_1^*(x_1)=\int_I \operatorname{sgn} f(x_1, x_2) dx_2$. It can be derived by simple calculations that $f_1^*(x_1)=2f_1(x_1)$. But this contradicts (12).

§. 2

In this section we shall study the L_1-approximation of functions of two variables. Thus we set $m=2$ and consider the space $C_1(I^2)$, where $I=[0, 1]$. Theorem 1 in the previous section states that the subspace of linear polynomials $L_3=\{ax+by+c\colon a, b, c\in \mathbf{R}\}$ is a unicity subspace of $C_1(I^2)$. This is another illustration of the fact that the Haar property is not necessary for the uniqueness of best L_1-approximation. A theorem of this type can not hold for Chebyshev approximation. L. COLLATZ [2] investigated the unicity of restricted Chebyshev approximation problem on the plane. He verified the uniqueness of Chebyshev approximation of differentiable functions by linear algebraic polynomials of two variables. But later T. J. RIVLIN and H. S. SHAPIRO [10] proved that linearity of the approximating polynomials is essential in this case. They showed that if the approximating algebraic polynomials are quadratic with respect to at least one of the variables, then there exists an infinitely differentiable function possessing more than one best Chebyshev approximation.

In this section it will be shown that linear polynomials are not the only unicity subspaces of $C_1(I^2)$. In particular, we shall verify the uniqueness of L_1-approximation when the approximating polynomials are linear with respect to only one of the variables.

Let $k, n\in \mathbf{N}$, $k\leqq n$ and let U_k^* and U_n^* be Haar subspaces of $C_1(I)$ of dimension k and n, respectively. Moreover, we assume that $U_k^*\subset U_n^*$. Consider an arbitrary continuous strictly increasing function φ on I and set

$$\overline{U}_{n+k} = \{q(x, y) = \varphi(y)\, q_k(x) + p_n(x) : q_k\in U_k^*,\ p_n\in U_n^*\}.$$

This is an $n+k$-dimensional subspace of $C_1(I^2)$.

REMARK. If $U_k^*=U_n^*$, then $\overline{U}_{n+k}=\overline{U}_{2n}$ can be considered as the tensor product of U_n^* and the linear span of 1 and φ. The linear span of 1 and φ is a 2-dimensional Haar subspace, hence in this case $\overline{U}_{2n}$ is a product of two Haar subspaces.

THEOREM 3. *$\overline{U}_{n+k}$ is a unicity subspace of $C_1(I^2)$.*

PROOF. First of all we shall verify that $\overline{U}_{n+k}$ is a Z_2 subspace. Take an arbitrary $q(x,y)=\varphi(y)q_k(x)+p_n(x)\in\overline{U}_{n+k}\setminus\{0\}$. Let us prove that $\mu_2(Z(q))=0$. Set $E(\bar{x})=\{(x,y)\in I^2\colon x=\bar{x},\ y\in I\}$ $(\bar{x}\in I)$ and let $x_i\in I$ $(1\leqq i\leqq s)$ be the all common zeros of q_k and p_n, $s\leqq n-1$. Then evidently $E(x_i)\subset Z(q)$. Further, if $\bar{x}\neq x_i$ for each $1\leqq i\leqq s$ and $E(\bar{x})\cap Z(q)\neq\emptyset$ then q_k can not vanish at $\bar{x}$. Therefore in this case $E(\bar{x})\cap Z(q)$ consists of a single point $\{x=\bar{x};\ y=\varphi^{-1}(-p_n(\bar{x})/q_k(\bar{x}))\}$, where φ^{-1} denotes the inverse function of φ. Thus $\mu_1(E(\bar{x})\cap Z(q))=0$ for almost all $\bar{x}\in I$, hence $\mu_2(Z(q))=0$. This implies that $\overline{U}_{n+k}$ is a Z_2 subspace.

Assume now that the statement of the theorem is false. Then by Lemma 1 there exist $f\in C_1(I^2)$ and $q^*(x,y)=\varphi(y)q_k^*(x)+p_n^*(x)\in\overline{U}_{n+k}\setminus\{0\}$ such that $Z(f)\subset Z(q^*)$ and

$$\int_{I^2} q\,\operatorname{sgn} f\,d\mu_2 = \int_I dx\Big(\int_I q\,\operatorname{sgn} f\,dy\Big) = 0 \tag{13}$$

for any $q\in\overline{U}_{n+k}$.

We shall consider several cases.

Case 1: q_k^* is the zero function. Then $q^*(x,y)=p_n^*(x)$, where $p_n^*\in U_n^*\setminus\{0\}$. Therefore $Z(q^*)=\bigcup_{i=1}^{r} E(x_i)$, where $0\leqq x_1<\ldots<x_r\leqq 1$ are the all zeros of p_n^*, $r\leqq n-1$. Since $Z(f)\subset Z(q^*)$, f does not vanish on the rectangles $B_i=\{x_{i-1}<x<x_i,\ 0<y<1\}$. Thus $\operatorname{sgn} f=\gamma_i$ on B_i, where $|\gamma_i|=1$ $(1\leqq i\leqq r+1,\ x_0=0,\ x_{r+1}=1)$. Since $r\leqq n-1$ by Lemma 4 there exists a $\bar{p}_n\in U_n^*\setminus\{0\}$ such that $\operatorname{sgn}\bar{p}_n=\gamma_i$ while

$x_{i-1}<x<x_i$ $(1\le i\le r+1)$. Thus considering $\bar{p}_n$ as an element of $\bar{U}_{n+k}$ we have $\operatorname{sgn} f=\operatorname{sgn}\bar{p}_n$ on B_i $(1\le i\le r+1)$, i.e. $\int_{I^2}\bar{p}_n \operatorname{sgn} f\,d\mu_2>0$. This contradicts (13).

Case 2: $q_k^*\in U_k^*\setminus\{0\}$, $\varphi\left(\frac{1}{2}\right)q_k^*(x)+p_n^*(x)$ is the zero function. Then $q^*(x,y)=$ $=q_k^*(x)\left\{\varphi(y)-\varphi\left(\frac{1}{2}\right)\right\}$. Thus $Z(q^*)=\left(\bigcup_{i=1}^{s}E(x_i)\right)\cup\left\{x\in I;\ y=\frac{1}{2}\right\}$, where $0\le x_1<$ $<x_2<\ldots<x_s\le 1$ are the all zeros of q_k^*, $s\le k-1$. Set

$$\left.\begin{aligned} A_i &= \left\{x_{i-1}<x<x_i;\quad 0<y<\frac{1}{2}\right\},\\ B_i &= \left\{x_{i-1}<x<x_i;\quad \frac{1}{2}<y<1\right\}\end{aligned}\right.\qquad (1\le i\le s+1;\quad x_0=0;\quad x_{s+1}=1).$$

Since $Z(f)\subset Z(q^*)$, f preserves sign on each A_i and B_i. Hence

$$\operatorname{sgn} f=\begin{cases}\gamma_i & \text{on}\quad A_i\\ \varepsilon_i & \text{on}\quad B_i,\end{cases}\tag{14}$$

where $|\gamma_i|=|\varepsilon_i|=1$, $1\le i\le s+1$. Furthermore (13) implies that

$$\int_I p_n(x)f_1^*(x)\,dx=0\quad (p_n\in U_n^*),\tag{15}$$

where $f_1^*(x)=\int_I \operatorname{sgn} f(x,y)dy$. From (14) we easily derive that $f_1^*(x)=\frac{1}{2}(\gamma_i+\varepsilon_i)=\beta_i$ for $x_{i-1}<x<x_i$ $(1\le i\le s+1)$. Evidently β_i equals 1, -1 or 0 $(1\le i\le s+1)$. Since $s\le k-1$, Lemma 4 implies that there exists a $\bar{q}_k\in U_k^*\setminus\{0\}$ such that $f_1^*\bar{q}_k\ge 0$ μ_1-a.e. on I. But $\bar{q}_k\in U_k^*\subset U_n^*$, hence it follows from (15) that $f_1^*\bar{q}_k=0$ μ_1-a.e. on I, i.e. $f_1^*=0$ μ_1-a.e. on I. Thus $\beta_i=0$ and therefore $\gamma_i=-\varepsilon_i$ for each $1\le i\le s+1$. Using again Lemma 4, consider a polynomial $q_k\in U_k^*\setminus\{0\}$ such that $\operatorname{sgn} q_k=\varepsilon_i$ for $x_{i-1}<x<x_i$ $(1\le i\le s+1)$. Furthermore, set $\bar{q}(x,y)=\varphi(y)q_k(x)-\varphi\left(\frac{1}{2}\right)q_k(x)=$ $=q_k(x)\left\{\varphi(y)-\varphi\left(\frac{1}{2}\right)\right\}\in\bar{U}_{n+k}\setminus\{0\}$. Then

$$\operatorname{sgn}\bar{q}(x,y)=\operatorname{sgn} q_k(x)\operatorname{sgn}\left\{\varphi(y)-\varphi\left(\frac{1}{2}\right)\right\}=\begin{cases}-\varepsilon_i & \text{on}\quad A_i\\ \ \ \varepsilon_i & \text{on}\quad B_i\end{cases}=\begin{cases}\gamma_i & \text{on}\quad A_i\\ \varepsilon_i & \text{on}\quad B_i.\end{cases}$$

This and (14) imply that $\int_{I^2}\bar{q}\operatorname{sgn} f\,d\mu_2>0$. Thus we again obtained a contradiction to (13).

Case 3: $q_k^*\in U_k^*\setminus\{0\}$, $\varphi\left(\frac{1}{2}\right)q_k^*(x)+p_n^*(x)\in U_n^*\setminus\{0\}$. Let $0\le x_1'<x_2'<\ldots<x_r'\le 1$ be the all common zeros of q_k^* and p_n^*, $r\le k-1$. As it was shown above if $x\ne x_i'$ for each $1\le i\le r$, then $E(x)\cap Z(q^*)$ contains at most one point. Since $Z(f)\subset Z(q^*)$, $E(x)\cap Z(f)$ also contains at most one point in this case. Thus applying Lemma 2 we obtain that $f_1^*(x)$ is continuous while $x_{i-1}'<x<x_i'$ $(1\le i\le r+1;\ x_0'=0;\ x_{r+1}'=1)$. Furthermore (15) and Lemma 4 imply that f_1^* has at least $n-r$ distinct zeros on

$Q=\bigcup_{i=1}^{r+1}(x'_{i-1}, x'_i)$. (Otherwise f_1^* changes sign at at most $n-1$ points and this, in view of Lemma 4, contradicts (15).) Let $\bar{x}_i$, $1\leqq i\leqq n-r$, be any distinct zero of f_1^* on Q. Since $\bar{x}_i\in Q$, $\bar{x}_i$ is not a common zero of q_k^* and p_n^*, hence $E(\bar{x}_i)$ contains at most one zero of f. On the other hand $E(\bar{x}_i)$ should contain a zero of f, because otherwise $f_1^*(\bar{x}_i)$ equals 1 or -1. Thus $E(\bar{x}_i)\cap Z(f)=\{x=\bar{x}_i;\ y=\bar{y}_i\}$, where, as it was shown above,

$$\bar{y}_i=\varphi^{-1}(-p_n^*(\bar{x}_i)/q_k^*)\bar{x}_i)),\quad 1\leqq i\leqq n-r. \tag{16}$$

Moreover f must change sign on $E(\bar{x}_i)$, hence $\operatorname{sgn} f(\bar{x}_i, y)=\beta_i$ while $0<y<\bar{y}_i$ and $\operatorname{sgn} f(\bar{x}_i, y)=-\beta_i$ while $\bar{y}_i<y<1$ ($|\beta_i|=1$; $1\leqq i\leqq n-r$). Thus

$$0=f_1^*(\bar{x}_i)=\int_I \operatorname{sgn} f(\bar{x}_i, y)\,dy=\beta_i\bar{y}_i-\beta_i(1-\bar{y}_i)=\beta_i(2\bar{y}_i-1).$$

Therefore we obtain $\bar{y}_i=\frac{1}{2}$ $(1\leqq i\leqq n-r)$. From this and (16) we easily derive

$$\varphi\left(\frac{1}{2}\right)q_k^*(\bar{x}_i)+p_n^*(\bar{x}_i)=0,\quad 1\leqq i\leqq n-r. \tag{17}$$

Furthermore $q_k^*(x'_i)=p_n^*(x'_i)=0$ $(1\leqq i\leqq r)$, where $x'_i\neq\bar{x}_j$ for all i and j. This together with (17) imply that the polynomial $\varphi\left(\frac{1}{2}\right)q_k^*(x)+p_n^*(x)\in U_n^*\setminus\{0\}$ has n distinct zeros on I. Thus we arrived at a contradiction to the Haar property.

The proof of the theorem is complete.

Consider the set of algebraic polynomials on I^2 of degree m with respect to x and k with respect to y

$$P_{m,k}=\left\{\sum_{i=0}^{k}\sum_{r=0}^{m}a_{i,r}\,x^r y^i : a_{i,r}\in\mathbf{R}\right\}\quad (k, m\in\mathbf{Z}_+).$$

It can be easily seen that $P_{0,k}$ and $P_{m,0}$ are unicity subspaces of $C_1(I^2)$. This can be proved by the arguments used in Case 1 of the Theorem 3. (Note that $P_{0,k}=P_k$, $P_{m,0}=P_m$.) Moreover from Theorem 3 we can derive the following

COROLLARY 3. *$P_{m,1}$ and $P_{1,k}$ are unicity subspaces of $C_1(I^2)$.*

The question whether $P_{m,k}$ is a unicity subspace of $C_1(I^2)$ if $k, m\geqq 2$, remains open. We have a feeling that the answer to this question is affirmative. Even the attempts to settle the case when k (or m) equals 2 were unsuccessful. We are able to prove only a weaker result. Set

$$P_{m,2}^*=\{p_m(x)+q_2(y),\quad \text{where}\quad p_m\in P_m,\ q_2\in P_2\}.$$

Then we have the following

THEOREM 4. *For any $m\in\mathbf{N}$, $P_{m,2}^*$ is a unicity subspace of $C_1(I^2)$.*

PROOF. Evidently $P_{m,2}^*$ is a Z_2 subspace. If the statement of the theorem is false, then by Lemma 1 there exist an $f\in C_1(I^2)$ and $p^*(x, y)=p_m^*(x)+q_2^*(y)\in P_{m,2}^*\setminus\{0\}$

such that $Z(f)\subset Z(p^*)$ and

$$\int_{I^2} p(x, y)\,\mathrm{sgn}\, f(x, y)\, dy\, dx = 0 \quad (p\in P^*_{m,2}). \tag{18}$$

In particular

$$\int_I p_m(x) f_1^*(x)\, dx = 0 \quad (p_m\in P_m), \tag{19}$$

where $f_1^*(x)=\int_I \mathrm{sgn}\, f(x, y) dy$.

If $\deg q_2^*<2$, then p^* can be written in the form $p^*(x, y)=a_1 y+\tilde{p}_m(x)$ $(\tilde{p}_m\in P_m)$. In this case we can arrive at a contradiction analogously as in the proof of Theorem 3. Therefore we may assume, that $\deg q_2^*=2$. Then the solution of the equation $p^*(x, y)=0$ is given by two curves

$$\gamma_1:\begin{cases} y = y_0+\sqrt{\bar{p}_m(x)}, \\ x\in I', \end{cases} \qquad \gamma_2:\begin{cases} y = y_0-\sqrt{\bar{p}_m(x)}, \\ x\in I', \end{cases} \tag{20}$$

where $\bar{p}_m\in P_m$, $y_0\in \mathbf{R}$, $I'=\{x\in I\colon \bar{p}_m(x)\geqq 0\}$. Since $Z(f)\subset Z(p^*)$, f can vanish only on γ_1 and γ_2.

Case 1: $\bar{p}_m$ is a constant function. Then f can change sign only on the line segments $\{x\in I;\, y=y_1\}$ and $\{x\in I;\, y=y_2\}$, where $y_1=y_0+\sqrt{\bar{p}_m}$, $y_2=y_0-\sqrt{\bar{p}_m}$ and $y_1, y_2\in\mathbf{R}$. Hence one of the polynomials β, $\varepsilon(y-y_1)$, $\theta(y-y_2)$, $\delta(y-y_1)(y-y_2)\in \in P^*_{m,2}$ $(\beta, \varepsilon, \theta, \delta=\pm 1)$ has the same sign as f μ_2-a.e. on I^2. But this contradicts (18).

Case 2: $\bar{p}_m$ is not a constant function. It follows from (20), that for any $x\in I$, $E(x)$ contains at most two zeros of f. Thus by Lemma 2 f_1^* is continuous on I. Moreover, it is easy to see that for any $x\in I$, $f_1^*(x)$ can take only one of the following values: ± 1; $\pm(1-2y_0-2\sqrt{\bar{p}_m(x)})$; $\pm(1-2y_0+2\sqrt{\bar{p}_m(x)})$; $\pm(1-4\sqrt{\bar{p}_m(x)})$. Since $\bar{p}_m\in P_m$ is not a constant function we obtain that f_1^* has at most $3m$ zeros on I. Furthermore, by continuity of f_1^* and (19) we have that f_1^* has at least $m+1$ zeros on I.

Let $0<x_1<x_2<\ldots<x_k<1$ be the all zeros of $\bar{p}_m'$ inside I, $k\leqq m-1$. Then $\bar{p}_m$ is strictly monotone on each interval $[x_{i-1}, x_i]$ $(1\leqq i\leqq k+1,\ x_0=0,\ x_{k+1}=1)$. Let us prove that f_1^* is also monotone on $[x_{i-1}, x_i]$, $1\leqq i\leqq k+1$. Assume, e.g., that $\bar{p}_m$ is increasing on $[x_{i-1}, x_i]$. (The case when $\bar{p}_m$ is decreasing on this interval can be considered analogously.) If $\bar{p}_m(x_i)\leqq 0$, then the curves γ_1 and γ_2 do not intersect the rectangle $A_i=\{x_{i-1}<x<x_i;\ 0<y<1\}$. Hence in this case f_1^* is a constant $(-1$ or $1)$ on $[x_{i-1}, x_i]$ i.e. it is monotone on this interval. Thus we may assume that $\bar{p}_m(x_i)>0$. Then $\bar{p}_m>0$ for $x^*_{i-1}<x<x_i$, where $x_{i-1}\leqq x^*_{i-1}<x_i$ and $x^*_{i-1}= =x_{i-1}$ if $\bar{p}_m(x_{i-1})\geqq 0$ and $\bar{p}_m(x^*_{i-1})=0$ if $\bar{p}_m(x_{i-1})<0$. Since $\bar{p}_m$ is positive and increasing for $x^*_{i-1}<x<x_i$, γ_1 is strictly increasing, γ_2 is strictly decreasing and $\gamma_1>\gamma_2$ on this interval. Set $\bar{\gamma}_i(x)=\max\{0, \min\{1, \gamma_i(x)\}\}$, $i=0, 1$. Clearly, $\bar{\gamma}_1$ $(\bar{\gamma}_2)$ is continuous and increasing (decreasing) and $0\leqq\bar{\gamma}_2\leqq\bar{\gamma}_1\leqq 1$ for $x^*_{i-1}<x<x_i$. Furthermore, f can vanish only on $\bar{\gamma}_1$ and $\bar{\gamma}_2$. The curves $\bar{\gamma}_1$ and $\bar{\gamma}_2$ divide the rectangle $\}x^*_{i-1}<x<x_i;\ 0<y<1\}$ to three connected parts:

$$B_1 = \{x^*_{i-1} < x < x_i;\quad \bar{\gamma}_1(x) < y < 1\};$$
$$B_2 = \{x^*_{i-1} < x < x_i;\quad \bar{\gamma}_2(x) < y < \bar{\gamma}_1(x)\};$$
$$B_3 = \{x^*_{i-1} < x < x_i;\quad 0 < y < \bar{\gamma}_2(x)\}.$$

(Some of these regions may be empty.) Since f does not vanish on B_i, f preserves sign there, i.e. $\operatorname{sgn} f=\beta_i$ on B_i ($|\beta_i|=1$; $1\leqq i\leqq 3$). Therefore we have for $x_{i-1}^*<x<x_i$

$$f_1^*(x)=\int_I \operatorname{sgn} f(x,y)\,dy=\int_{\bar\gamma_1(x)}^{1}\operatorname{sgn} f(x,y)\,dy+\int_{\bar\gamma_2(x)}^{\bar\gamma_1(x)}\operatorname{sgn} f(x,y)\,dy+$$
$$+\int_0^{\bar\gamma_2(x)}\operatorname{sgn} f(x,y)\,dy=\beta_1\big(1-\bar\gamma_1(x)\big)+\beta_2\big(\bar\gamma_1(x)-\bar\gamma_2(x)\big)+\beta_3\bar\gamma_2(x)=$$
$$=\begin{cases}\beta_1, & \text{if}\quad \beta_1=\beta_2=\beta_3;\\ \beta_1\big(1-2\bar\gamma_2(x)\big), & \text{if}\quad \beta_1=\beta_2=-\beta_3;\\ \beta_1\big(1-2\bar\gamma_1(x)+2\bar\gamma_2(x)\big), & \text{if}\quad \beta_1=-\beta_2=\beta_3;\\ \beta_1\big(1-2\bar\gamma_1(x)\big), & \text{if}\quad \beta_1=-\beta_2=-\beta_3.\end{cases}$$

Since $\bar\gamma_1$ ($\bar\gamma_2$) is increasing (decreasing) for $x_{i-1}^*<x<x_i$ this implies that f_1^* is monotone on this interval. If $x_{i-1}^*=x_{i-1}$, then we are ready. On the other hand if $x_{i-1}<x_{i-1}^*$, then $\bar p_m<0$ for $x_{i-1}<x<x_{i-1}^*$. Therefore the curves γ_1 and γ_2 do not intersect the rectangle $\{x_{i-1}<x<x_{i-1}^*,\ 0<y<1\}$ and hence $f_1^*(x)=\xi$ for any $x_{i-1}<x<x_{i-1}^*$, ($|\xi|=1$). This and the continuity of f_1^* imply that f_1^* is monotone for $x_{i-1}<x<x_i$ ($1\leqq i\leqq k+1$). Furthermore, it was shown above that f_1^* has at least $m+1$ zeros on I. Since $k+1\leqq m$, there exists an interval $[x_{j-1},x_j]$ ($1\leqq j\leqq k+1$) containing at least two distinct zeros of f_1^*. But f_1^* is monotone on $[x_{j-1},x_j]$, hence f_1^* vanishes on this nondegenerate interval. This is a contradiction, since we have proved above that f_1^* has at most $3m$ zeros on I. The theorem is proved.

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[1] G. Szegő, *Orthogonal polynomials*, AMS Coll. Publ. Vol. XXIII (Providence, 1939).
[2] A. Zygmund, Smooth functions, *Duke Math. J.*, **12** (1945), 47—76.

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ON A PROBLEM OF TURÁN ABOUT THE OPTIMAL (0,2)-INTERPOLATION

J. S. HWANG (Taipei)

1. Introduction. The "optimal (0,2)-interpolation" has been systematically investigated by P. Turán with J. Surányi [6] and J. Balázs [1, 2, 3]. Let $\{x_j\}$, $j=1, 2, \ldots, n$ be a sequence of n points satisfying

$$(1) \qquad 1 \geqq x_1 > x_2 > \ldots > x_n \geqq -1.$$

Suppose the points (1) are such that for arbitrarily prescribed numbers y_j, z_j, $j=1, 2, \ldots, n$, there is a unique polynomial $\Pi_{2n-1}(x)$ of degree $\leqq 2n-1$ so that

$$(2) \qquad \Pi_{2n-1}(x_j) = y_j \quad \text{and} \quad \Pi''_{2n-1}(x_j) = z_j, \quad j = 1, 2, \ldots, n.$$

We then can associate to each function $f(x)$ which is continuous on $[-1, 1]$, a unique polynomial $R_n(x;f)$ of degree $\leqq 2n-1$ so that

$$(3) \qquad R_n(x_j;f) = f(x_j) \quad \text{and} \quad R''_n(x_j;f) = z_j, \quad j = 1, 2, \ldots, n.$$

In order to write (3) into one equality, we may use the following fundamental functions $r_j(x)$ and $s_j(x)$ of first and second kind, respectively, with degree $\leqq 2n-1$,

$$(4) \qquad r_j(x_k) = \begin{cases} 1, & k = j \\ 0, & k \neq j, \end{cases} \quad \text{and} \quad r''_j(x_k) = 0, \quad k = 1, 2, \ldots, n$$

$$(5) \qquad s''_j(x_k) = \begin{cases} 1, & k = j \\ 0, & k \neq j, \end{cases} \quad \text{and} \quad s_j(x_k) = 0, \quad k = 1, 2, \ldots, n.$$

With the help of (4) and (5), we can write (3) as

$$(6) \qquad R_n(x; f) = \sum_{j=1}^{n} f(x_j) r_j(x) + \sum_{j=1}^{n} z_j s_j(x).$$

Surányi and Turán called a sequence $\{x_j\}$ with the property (1) a (0,2)-interpolation sequence if for any prescribed numbers y_j, z_j, there is a unique polynomial $\Pi_{2n-1}(x)$ satisfying (2). In [6, Problem 9], Turán asks: is it true for $\alpha \neq \beta$ that the zeros of all Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is a (0,2)-interpolation sequence? Recently we have answered this question in the negative sense [4].

In the same paper [6, Problem 8], Turán asks: whether there exists n points with the property (1) for which all fundamental functions $s_j(x)$ of second kind exist, but not all fundamental functions $r_j(x)$ of first kind? In this paper, we shall answer this question in the affirmative sense as follows.

THEOREM 1. *For $n=4$, there are four points $\{x_j\}$, $j=1, 2, 3, 4$, satisfying* (1) *such that all $s_j(x)$ exist, but not all $r_j(x)$.*

The proof of the above Theorem 1 depends mainly on the suitable representations of those fundamental functions which are interesting in themselves.

2. Representation of $s_j(x)$. We set

$$(7)\qquad \begin{aligned} P_j(x) &= (x-x_1)\dots(x-x_{j-1})(x-x_{j+1})\dots(x-x_n) = \\ &= ((x-x_j)+(x_j-x_1))\dots((x-x_j)+(x_j-x_{j-1}))((x-x_j)+ \\ &\quad +(x_j-x_{j+1}))\dots((x-x_j)+(x_j-x_n)) = \\ &= (x-x_j)^{n-1}+C_1(x-x_j)^{n-2}+\dots+C_{n-2}(x-x_j)+C_{n-1}. \end{aligned}$$

Then by virtue of equation (5), $s_j''(x)$ can be written as

$$(8)\quad \begin{aligned} s_j''(x) &= P_j(x)\{a_0(x-x_j)^{n-2}+a_1(x-x_j)^{n-3}+\dots+a_{n-3}(x-x_j)+a_{n-2}\} = \\ &= a_0P_j(x)(x-x_j)^{n-2}+a_1P_j(x)(x-x_j)^{n-3}+\dots+a_{n-3}P_j(x-x_j)+a_{n-2}P_j(x). \end{aligned}$$

Integrating both sides of (8) twice from x_j to x, and adding two more parameters a_{n-1} and a_n, we obtain

$$(9)\qquad s_j(x)=a_0Q_{j,0}(x)+a_1Q_{j,1}(x)+\dots+a_{n-2}Q_{j,n-2}(x)+a_{n-1}(x-x_j)+a_n,$$

where

$$Q_{j,k}(x) = \int_{x_j}^{x}\int_{x_j}^{y} P_j(t)(t-x_j)^{n-(k+2)}\,dt\,dy,\quad k=0,1,\dots,n-2.$$

Hence

$$\begin{aligned} s_j(x) &= \int_{x_j}^{x}\int_{x_j}^{y} \{(t-x_j)^{2n-(k+3)}+C_1(t-x_j)^{2n-(k+4)}+\dots+C_{n-1}(t-x_j)^{n-(k+2)}\}\,d\,tdy = \\ &= \int_{x_j}^{x}\left\{\frac{(y-x_j)^{2n-(k+2)}}{2n-(k+2)}+\frac{C_1(y-x_j)^{2n-(k+3)}}{2n-(k+3)}+\dots+\frac{C_{n-1}(y-x_j)^{n-(k+1)}}{n-(k+1)}\right\}dy = \\ &= \frac{(x-x_j)^{2n-(k+1)}}{(2n-(k+1))(2n-(k+2))}+\frac{C_1(x-x_j)^{2n-(k+2)}}{(2n-(k+2))(2n-(k+3))}+\dots \\ &\quad \dots+\frac{C_{n-1}(x-x_j)^{n-k}}{(n-k)(n-(k+1))}. \end{aligned}$$

Since $s_j(x_j)=0$ and $Q_{j,k}(x_j)=0$ for $k=0,1,\dots,n-2$, $j=1,2,\dots,n$, it follows that $a_n=0$. Combining with (5), (7), (8), and (9), we have

(10)

$$\begin{cases} s_j''(x_j) = a_0\cdot 0 + a_1\cdot 0 + \dots + a_{n-2}\cdot C_{n-1} + a_{n-1}\cdot 0 = 1 \\ s_j(x_1) = a_0Q_{j,0}(x_1)+a_1Q_{j,1}(x_1)+\dots+a_{n-2}Q_{j,n-2}(x_1)+a_{n-1}(x_1-x_j) = 0 \\ \vdots \\ s_j(x_{j-1}) = a_0Q_{j,0}(x_{j-1})+a_1Q_{j,1}(x_{j-1})+\dots+a_{n-2}Q_{j,n-2}(x_{j-1})+a_{n-1}(x_{j-1}-x_j) = 0 \\ s_j(x_{j+1}) = a_0Q_{j,0}(x_{j+1})+a_1Q_{j,1}(x_{j+1})+\dots+a_{n-2}Q_{j,n-2}(x_{j+1})+a_{n-1}(x_{j+1}-x_j) = 0 \\ \vdots \\ s_j(x_n) = a_0Q_{j,0}(x_n)+a_1Q_{j,1}(x_n)+\dots+a_{n-2}Q_{j,n-2}(x_n)+a_{n-1}(x_n-x_j) = 0. \end{cases}$$

Let Δ_j be the determinant of the coefficients of (10) with respect to $a_0, a_1, \dots, a_{n-1}$, we then have

$$(11)\qquad \Delta_j = (-1)^{n-1} C_{n-1} \begin{vmatrix} Q_{j,0}(x_1) & Q_{j,1}(x_1) & \dots Q_{j,n-3}(x_1) & (x_1-x_j) \\ \vdots & & & \\ Q_{j,0}(x_{j-1}) & Q_{j,1}(x_{j-1}) & \dots Q_{j,n-3}(x_{j-1}) & (x_{j-1}-x_j) \\ Q_{j,0}(x_{j+1}) & Q_{j,1}(x_{j+1}) & \dots Q_{j,n-3}(x_{j+1}) & (x_{j+1}-x_j) \\ \vdots & & & \\ Q_{j,0}(x_n) & Q_{j,1}(x_n) & \dots Q_{j,n-3}(x_n) & (x_n-x_j) \end{vmatrix}$$

where $C_{n-1} = P_j(x_j) \neq 0$.

It is clear that the system $(a_0, a_1, \dots, a_{n-1})$ has a unique solution if and only if $\Delta_j \neq 0$. In such a case, the function $s_j(x)$ can be formulated as

$$(12)\qquad s_j(x) = \frac{1}{\Delta_j} \begin{vmatrix} Q_{j,0}(x) & Q_{j,1}(x) & \dots Q_{j,n-2}(x) & (x-x_j) \\ Q_{j,0}(x_1) & Q_{j,1}(x_1) & \dots Q_{j,n-2}(x_1) & (x_1-x_j) \\ \vdots & & & \\ Q_{j,0}(x_{j-1}) & Q_{j,1}(x_{j-1}) & \dots Q_{j,n-2}(x_{j-1}) & (x_{j-1}-x_j) \\ Q_{j,0}(x_{j+1}) & Q_{j,1}(x_{j+1}) & \dots Q_{j,n-2}(x_{j+1}) & (x_{j+1}-x_j) \\ \vdots & & & \\ Q_{j,0}(x_n) & Q_{j,1}(x_n) & \dots Q_{j,n-2}(x_n) & (x_n-x_j) \end{vmatrix}.$$

This yields the following

THEOREM 2. *The fundamental functions $s_j(x)$ of second kind can be represented by* (12) *provided $\Delta_j \neq 0$, where Δ_j and $Q_{j,k}(x)$ are defined by* (11) *and* (9), *respectively.*

3. Representation of $r_j(x)$. According to (4) and (7), $r_j''(x)$ can be written as

$$(13)\qquad r_j''(x) = P_j(x)(x-x_j)\{b_0(x-x_j)^{n-3} + b_1(x-x_j)^{n-4} + \dots + b_{n-4}(x-x_j) + b_{n-3}\} =$$
$$= b_0 P_j(x)(x-x_j)^{n-2} + b_1 P_j(x)(x-x_j)^{n-3} + \dots$$
$$\dots + b_{n-4} P_j(x)(x-x_j)^2 + b_{n-3} P_j(x)(x-x_j).$$

Integrating both sides of (13) twice from x_j to x, and adding two more parameters b_{n-2} and b_{n-1}, we obtain

$$(14)\qquad r_j(x) = b_0 Q_{j,0}(x) + b_1 Q_{j,1}(x) + \dots + b_{n-3} Q_{j,n-3}(x) + b_{n-2}(x-x_j) + b_{n-1},$$

where $Q_{j,k}(x)$ are the same as that of (9).

By virtue of (4) and (14), we have the following simultaneous equations:

(15)

$$\begin{cases} r_j(x_j) = \\ b_0\cdot 0 \quad +b_1\cdot 0 \quad +\ldots+b_{n-3}\cdot 0 \quad +b_{n-2}\cdot 0 \quad +b_{n-1}=1 \\ r_j(x_1) = \\ b_0Q_{j,0}(x_1) \quad +b_1Q_{j,1}(x_1) \quad +\ldots+b_{n-3}Q_{j,n-3}(x_1) \quad +b_{n-2}(x_1-x_j) \quad +b_{n-1}=0 \\ \vdots \\ r_j(x_{j-1}) = \\ b_0Q_{j,0}(x_{j-1})+b_1Q_{j,1}(x_{j-1})+\ldots+b_{n-3}Q_{j,n-3}(x_{j-1})+b_{n-2}(x_{j-1}-x_j)+b_{n-1}=0 \\ r_j(x_{j+1}) = \\ b_0Q_{j,0}(x_{j+1})+b_1Q_{j,1}(x_{j+1})+\ldots+b_{n-3}Q_{j,n-3}(x_{j+1})+b_{n-2}(x_{j+1}-x_j)+b_{n-1}=0 \\ \vdots \\ r_j(x_n) = \\ b_0Q_{j,0}(x_n) \quad +b_1Q_{j,1}(x_n) \quad +\ldots+b_{n-3}Q_{j,n-3}(x_n) \quad +b_{n-2}(x_n-x_j) \quad +b_{n-1}=0. \end{cases}$$

Let ∇_j be the determinant of the coefficients of (15), we then have

$$\nabla_j=(-1)^n\begin{vmatrix} Q_{j,0}(x_1) & Q_{j,1}(x_1) & \ldots Q_{j,n-3}(x_1) & (x_1-x_j) \\ \vdots & & & \\ Q_{j,0}(x_{j-1}) & Q_{j,1}(x_{j-1}) & \ldots Q_{j,n-3}(x_{j-1}) & (x_{j-1}-x_j) \\ Q_{j,0}(x_{j+1}) & Q_{j,1}(x_{j+1}) & \ldots Q_{j,n-3}(x_{j+1}) & (x_{j+1}-x_j) \\ \vdots & & & \\ Q_{j,0}(x_n) & Q_{j,1}(x_n) & \ldots Q_{j,n-3}(x_n) & (x_n-x_j) \end{vmatrix}. \tag{16}$$

Clearly the system $(b_0, b_1, \ldots, b_{n-2})$ has a unique solution if and only if $\nabla_j\neq 0$ by which $r_j(x)$ can be represented as

$$r_j(x)=\frac{1}{\nabla_j}\begin{vmatrix} Q_{j,0}(x) & Q_{j,1}(x) & \ldots Q_{j,n-3}(x) & (x-x_j) & 1 \\ Q_{j,0}(x_1) & Q_{j,1}(x_1) & \ldots Q_{j,n-3}(x_1) & (x_1-x_j) & 1 \\ \vdots & & & & \\ Q_{j,0}(x_{j-1}) & Q_{j,1}(x_{j-1}) & \ldots Q_{j,n-3}(x_{j-1}) & (x_{j-1}-x_j) & 1 \\ Q_{j,0}(x_{j+1}) & Q_{j,1}(x_{j+1}) & \ldots Q_{j,n-3}(x_{j+1}) & (x_{j+1}-x_j) & 1 \\ \vdots & & & & \\ Q_{j,0}(x_n) & Q_{j,1}(x_n) & \ldots Q_{j,n-3}(x_n) & (x_n-x_j) & 1 \end{vmatrix}. \tag{17}$$

This gives the following

THEOREM 3. *The fundamental functions $r_j(x)$ of first kind can be represented by* (17) *provided $\nabla_j\neq 0$, where ∇_j and $Q_{j,k}(x)$ are defined by* (16) *and* (9), *respectively.*

As a consequence of Theorems 2 and 3, i.e. (11) and (16), we find that $\Delta_j\neq 0$ if and only if $\nabla_j\neq 0$. This yields the following existence relation between $s_j(x)$ and $r_j(x)$.

COROLLARY. *For each j, the function $s_j(x)$ exists uniquely if and only if $r_j(x)$ does.*

4. Proof of Theorem 1. For simplicity let $x_1=0$. We are going to find the solutions of x_2, x_3, and x_4 such that all $s_j(x)$ exist and in particular, $s_1(x)$ exists infinitely many while $r_1(x)$ does not exist.

According to (5), (7), (8), (9), (10), and (15), we can easily obtain

$$(18)\qquad \begin{cases} s_1(x_2) = a_0Q_{1,0}(x_2)+a_1Q_{1,1}(x_2)-Q_{1,2}(x_2)/x_2x_3x_4+a_3x_2 = 0 \\ s_1(x_3) = a_0Q_{1,0}(x_3)+a_1Q_{1,1}(x_3)-Q_{1,2}(x_3)/x_2x_3x_4+a_3x_3 = 0 \\ s_1(x_4) = a_0Q_{1,0}(x_4)+a_1Q_{1,1}(x_4)-Q_{1,2}(x_4)/x_2x_3x_4+a_3x_4 = 0 \end{cases}$$

$$(19)\qquad \begin{cases} r_1(x_2) = b_0Q_{1,0}(x_2)+b_1Q_{1,1}(x_2)+b_2x_2+1 = 0 \\ r_1(x_3) = b_0Q_{1,0}(x_3)+b_1Q_{1,1}(x_3)+b_2x_3+1 = 0 \\ r_1(x_4) = b_0Q_{1,0}(x_4)+b_1Q_{1,1}(x_4)+b_2x_4+1 = 0 \end{cases}$$

where

$$Q_{1,0}(x) = x^7/42-(x_2+x_3+x_4)x^6/30+(x_2x_3+x_3x_4+x_4x_2)x^5/20-x_2x_3x_4x^4/12$$

$$Q_{1,1}(x) = x^6/30-(x_2+x_3+x_4)x^5/20+(x_2x_3+x_3x_4+x_4x_2)x^4/12-x_2x_3x_4x^3/6$$

$$Q_{1,2}(x) = x^5/20-(x_2+x_3+x_4)x^4/12+(x_2x_3+x_3x_4+x_4x_2)x^3/6 \ -x_2x_3x_4x^2/2.$$

We set

$$(20)\qquad \frac{Q_{1,0}(x_2)}{Q_{1,0}(x_3)} = \frac{Q_{1,1}(x_2)}{Q_{1,1}(x_3)} = \frac{Q_{1,2}(x_2)}{Q_{1,2}(x_3)} = \frac{x_2}{x_3} = t.$$

If we can find a solution x_2, x_3, and x_4 which satisfies (20) then the determinant of the coefficients of (18) and (19) will be zero. This means that the solutions of a_0, a_1, and a_3 exist infinitely many while that of b_0, b_1, and b_2 do not exist. It follows that $s_1(x)$ exists infinitely many and $r_1(x)$ does not exist.

Substitute (20) into (18) and observe that $x_2x_3(x_2-x_3)\neq 0$, we obtain

$$(21)\qquad x_3((1+t)(2+t)(1+2t))-x_4(5(1-3t+t^2)) = 10t(1+t)$$

$$(22)\qquad x_3((1+t)^4+2t^2)-x^4((1+t)(2-t)(1-2t)) = 5t(1+t+t^2)$$

$$(23)\qquad x_3(4(1+t^5)+18t(1+t+t^2+t^3))-x_4(7(1-t-t^2-t^3+4t^4)) = \\ = 21t(1+t)(1+t^2).$$

Solving (21) and (22) for x_3 and x_4 in terms of t, we get

$$(24)\qquad x_3(t) = N_3(t)/D(t) \quad \text{and} \quad x_4(t) = N_4(t)/D(t),$$

where $N_3(t)=5t(1-4t-13t^2-4t^3+t^4)$, $N_4(t)=5t^2(1+t)(1-7t-t^2)$ and $D(t)=1-3t-2t^2-46t^3-t^4-3t^5+t^6$.

Substitute (24) into (23) and observe that $t(1+t)\neq 0$, we have

$$(25)\qquad F(t) = 1-18t+219t^2+146t^3+962t^4+391t^5-425t^6-223t^7+21t^8 = 0.$$

It is easy to see that the function $F(t)$ has two real roots t_1 and t_2 such that $1<t_1<2$ and $10<t_2<11$. Substituting t_1 or t_2 into (24), we can see the solutions of the syssem $(x_1(t), x_2(t), x_3(t), x_4(t))=(0, tx_3(t), x_3(t), x_4(t))$ exist. These solutions do not tatisty the condition (1) and therefore we have to shrink them. To do this, we let

$$x_j^*(t) = cx_j(t), \quad j = 1, 2, 3, 4, \quad \text{and} \quad c < 1. \tag{26}$$

From (19), we can see that $Q_{1,k}(x_j)$ are homogeneous, where $k=0, 1, 2$, and $j=2, 3$. Let $Q_{1,k}^*(x)$ be the polynomials induced from $Q_{1,k}(x)$ by replacing x_2, x_3, and x_4 by x_2^*, x_3^*, and x^*. Then by virtue of (20), we have

$$\frac{Q_{1,0}^*(x_2^*)}{Q_{1,0}^*(x_3^*)} = \frac{Q_{1,1}^*(x_2^*)}{Q_{1,1}^*(x_3^*)} = \frac{Q_{1,2}^*(x_2^*)}{Q_{1,2}^*(x_3^*)} = \frac{x_2^*}{x_3^*} = t. \tag{20}^*$$

This shows that the system $\{x_j\}$ can be replaced by the system $\{cx_j\}$. Therefore by choosing $c<1/\max(|x_j|)$, $j=2, 3, 4$, we obtain a system $\{cx_j\}$ of four points such that $s_1(x)$ exists infinitely many, but $r_1(x)$ does not exist.

Now, for this system $\{cx_j\}$, if needed c can be chosen so that the determinants of the coefficients of $s_2(x)$, $s_3(x)$, and $s_4(x)$ are all different from zero. By virtue of Corollary, we find that all $s_j(x)$ and $r_j(x)$ exist uniquely, where $j=2, 3, 4$. This completes the proof of Theorem 1.

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ON NON-AVERAGING SETS OF INTEGERS

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§ 1. Introduction. A set S of integers is said to be non-averaging if the arithmetic mean of two or more members of S is not in S. If A is a set of integers, we denote by $\|A\|$ the size of a largest non-averaging subset of A. The study of non-averaging sets was initiated by Straus [5] who proved that if $f(n)=\|\{1, 2, \ldots, n\}\|$ then, for some positive constant c and all sufficiently large n, $f(n)>\exp(c\sqrt{\log n})$. Later, Erdős and Straus [3] proved that $f(n)<cn^{2/3}$ and conjectured that $f(n)<\exp(c\sqrt{\log n})$. This conjecture was shown to be false by the author [1] who showed that

$$f(n) > cn^{1/10}. \tag{1}$$

Put $g(n)=\min \|A\|$ where the minimum is taken over all sets of integers A of size n. Erdős asked whether there exists a constant $\beta>0$ such that $g(n)>n^{\beta}$. Some partial results in this direction were obtained in [2]. For example, it was shown that if P denotes the set of the first n primes then $\|P\|>n^{\beta}$ for any $\beta<\frac{1}{10}$. It was also shown that for any $\beta<\frac{1}{20}$ almost all sets A of size n satisfy $\|A\|>n^{\beta}$.

In the present work we answer Erdős' question in the affirmative by proving the following result:

Theorem 1. *For any* $\beta<\frac{1}{13}$, $g(n)>n^{\beta}$ *for all sufficiently large* n.

We shall, in our proof of the theorem, make heavy use of methods developed in the paper of Komlós, Sulyok and Szemerédi [4] whose principal result we now formulate. Let

$$\sum_{j=1}^{l} a_{ij}x_j = 0, \quad i = 1, 2, \ldots, m \tag{ϱ}$$

be a system of equations with integer coefficients. Suppose that for each i, $\sum a_{ij}=0$ so that the solutions of (ϱ) are translation invariant. Let

$$\alpha = \max_{1\le i\le m} \sum_{j=1}^{l} |a_{ij}|.$$

Denote by $f_{\varrho}(n)$ the size of a largest subset of $\{1, 2, \ldots, n\}$ which does not contain any solution of (ϱ) in integers $x_1, \ldots, x_l$. (It is understood, of course, that

$x_1=x_2=\ldots=x_l$ is not considered as a solution.) Denote by $g_\varrho(n)$ the largest integer such that every set of n integers contains a subset of size $g_\varrho(n)$ which does not contain a solution of (ϱ). It is proved in [4] that

$$g_\varrho(n) \geqq \frac{1}{8\alpha^6} f_\varrho(n). \tag{2}$$

It is clear that our problem on non-averaging sets can be essentially formulated in this context since $g(n)$ is the largest integer such that every set A of n integers contains a subset of size $g(n)$ which does not contain a solution to any of the equations

$$x_1+x_2+\ldots+x_l = lx_{l+1}, \quad l = 2, 3, \ldots, g(n-1) \tag{3}$$

in distinct integers $x_1, x_2, \ldots, x_{i+1}$. However, (3) differs from (ϱ) in that the number of variables, and hence the size of α, grows with n, and thus (2) loses its significance. Professor Erdős suggested to the author that, nevertheless, the arguments used in [4], suitably adapted, may yield some information about $g(n)$. We have carried out this suggestion and have found that the methods of [4] together with the techniques used in [2] lead to a proof of Theorem 1.

We modify the problem as follows. Let H be an integer, depending on n, to be specified later. Let A be a set of integers and denote by $\|A\|_H$ the size of a largest subset of A not containing a solution of any of the equations

$$x_1+x_2+\ldots+x_l = lx_{l+1}, \quad l = 2, 3, \ldots, H \tag{4}$$

in distinct integers $x_1, x_2, \ldots, x_{l+1}$. Let $g^H(n)=\min \|A\|_H$ where the minimum is taken over all sets A of size n. Then, clearly,

$$g(n) \geqq \max_H \min \{H+1, g^H(n)\}. \tag{5}$$

Thus it will suffice to work with $g^H(n)$ and subsequently show that a suitable choice for H will yield the theorem.

We formulate in § 2 a number of lemmas. The first five of these are adaptations of similar results established in [4]. We do not present proofs of these lemmas here since the arguments, although differing in detail from those in [4], do not involve any new ideas. The sixth lemma is a special case of a result proved in [2]. In § 3 we give the proof of the theorem.

§ 2. Some lemmas. LEMMA 1. *Let* $A=\{a_1, a_2, \ldots, a_n\}$ *be a set of positive integers and suppose there exists a positive integer* q *such that* a_i *can be written as* $a_i=h_iq+r_i$ *where* $|r_i|<q/2H$. *Then, if* $R=\{r_1, \ldots, r_n\}$, $\|A\|_H \geqq \|R\|_H$.

LEMMA 2. *Given integers* $0<a_1<a_2<\ldots<a_n$, $a_n \geqq 3^{(2H)^n}$, *there exists a number* $q<a_n$ *such that* $a_i=k_iq+r_i$ *where* $|r_i|<q/2H$ *and* $r_i \neq r_j$ *unless* $i=j$. *Furthermore* $\|a_1, a_2, \ldots, a_n\|_H \geqq \|r_1, \ldots, r_n\|_H$.

LEMMA 3. *Given a set of* n *integers* $a_1<a_2<\ldots<a_n$ *and a prime* $q \geqq 2H$ *which does not divide any of the numbers* q_i *or* a_i-a_j *then there exists an integer* t, $1 \leqq t \leqq q-1$,

such that for $m \geqq [n/2H]$ *of the numbers* a_i *we have* $ta_i = h_i q + r_i$, $0 < r_i \leqq q/H$ *and* $r_i \neq r_j$ *if* $i \neq j$. *Furthermore, if the distinct remainders are denoted by* $b_1, b_2, \dots, b_m$, *we have* $\|a_1, a_2, \dots, a_n\|_H \geqq \|b_1, b_2, \dots, b_m\|_H$.

LEMMA 4. *Let* $0 < a_1 < a_2 < \dots < a_n \leqq n^{\Delta}$, $\Delta > 1$. *Then there exist* $0 < b_1 < b_2 < \dots < b_m \leqq 2n(\log n)^2/H$ *such that* $n \geqq [n/4H]$ *and* $\|a_1, a_2, \dots, a_n\|_H \geqq \|b_1, b_2, \dots, b_m\|_H$.

LEMMA 5. *Let* $0 < a_1 < a_2 < \dots < a_n$. *Then there exist integers* $0 < b_1 < b_2 < \dots < b_m \leqq \frac{2n^2}{H} \log^2 a_n$ *where* $m \geqq \left[\frac{n}{2H}\right]$ *and* $\|a_1, a_2, \dots, a_n\|_H \geqq \|b_1, b_2, \dots, b_m\|_H$.

LEMMA 6. *Let* B, s *and* N *be positive integers satisfying* $(H, B) = 1$, $s = 4H^4(B-1)^2$, $N = B^4 - 1$. *Then there exists a partition of* $\{1, 2, \dots, N\}$ *into* s *sets* $A_1, A_2, \dots, A_s$ *such that for each* l, $2 \leqq l \leqq H$, *and each* i, $1 \leqq i \leqq s$, *no* l *members of* A *have arithmetic mean in* A_i.

§ 3. Proof of Theorem 1. Let $A = \{a_1, a_2, \dots, a_n\}$, $0 < a_1 < a_2 < \dots < a_n$, be a set of n integers. By Lemma 2, we may suppose that $a_n \leqq 3^{(2H)^n}$. By Lemma 5, there exist integers $0 < b_1 < b_2 < \dots < b_m$, $m = [n/2H]$, such that $b_m \leqq 2n^2 \log^2 a_n < 2^{2n+2}$ and such that $\|a_1, a_2, \dots, a_n\|_H \geqq \|b_1, b_2, \dots, b_m\|_H$. By Lemma 5, there exist integers $0 < c_1 < c_2 < \dots < c_k$, $k = \left[\frac{m}{2H}\right]$, such that $c_k \leqq 2m^2 \log^2 b_m < \frac{3n^2}{H^2} \log^2 n$. If we choose $H < n^{1/3}$ we find that $c_k \leqq k^3$. By Lemma 4, there exist integers $0 < d_1 < d_2 < \dots < d_r$, $r = \left[\frac{k}{4H}\right]$, $d_r \leqq 6\frac{k}{H}(\log k)^2$ and such that $\|c_1, c_2, \dots, c_k\|_H \geqq \|d_1, d_2, \dots, d_r\|_H$. One finds that the numbers $d_1, d_2, \dots, d_r$ are in the interval $\left[1, \frac{3}{2}\frac{n}{H}(\log n)^2\right]$. Choose B in Lemma 6 to be the least prime exceeding $\left(3n(\log n^2)/2H\right)^{1/4} + 1$ and let $N = B^4 - 1$. Then the numbers $d_1, d_2, \dots, d_r$ lie in the interval $[1, N]$. Our eventual choice for H will be such that $H < B$ and hence that $(H, B) = 1$. One of the sets $A_1, A_2, \dots, A_s$ obtained via Lemma 6 contains a subset B of at least $\left[\frac{r}{s}\right]$ of the numbers $d_1, d_2, \dots, d_r$, so that $\|A\|_H \geqq \|B\|_H \geqq \left[\frac{r}{s}\right]$. Choose $H = \left[\frac{1}{2} n^{1/13}/(\log n)^{2/13}\right]$. One then finds, after some routine calculations, that $\left[\frac{r}{s}\right] \geqq H + 1$. The desired conclusion now follows from (5). Actually, we have proved the slightly stronger result that $g(n) > \frac{1}{2} n^{1/13}/(\log n)^{2/13}$ if n is sufficiently large.

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$3K_2$-DECOMPOSITION OF A GRAPH

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1. Introduction

Graphs in this paper are finite, have no multiple edges and loops.

DEFINITION 1. A graph $G=G(V, E)$ is said to have an H-decomposition if it is the union of edge-disjoint isomorphic copies of the graph H.

Necessary and sufficient conditions for H-decomposition have been determined mostly for the complete graph Kn, see [1, 2], but also for complete bipartite [2] and complete multipartite graphs [2, 5]. However only for particular graphs H. Recently Y. Caro and J. Schönheim considered H-decomposition of a general graph where H is $2K_2$ or P_2 (two-bars or a path of length 2). This problem was completely solved [3, 4]. This paper determines the graph G which have $3K_2$-decomposition. It is proved that the necessary conditions are also sufficient excluding a list of 26 graphs.

2. Preliminary results

The following two conditions for $G=G(V, E)$ to have a $3K_2$-decomposition are obviously necessary:

(1) $E(G)=3k$,

(2) $\deg(v) \leqq k$, for all $v \in V(G)$.

By a simple computation one could easily see that (1) and (2) imply $|V| \geqq 6$.

In the course of this paper we shall deal with the sufficiency problem.

DEFINITION 2. If U is a subset of vertices of the graph $G(V, E)$ then $\deg_U(v)$ will denote the degree of v in the graph induced by $U \cup \{v\}$.

DEFINITION 3. Let $G=G(V, E)$ satisfy (1) and (2) for a certain k. Denote $V_1=\{v|v \in V(G), \deg(v)=k\}$, and its cardinality by α.

DEFINITION 4. Let $G=G(V, E)$ be a graph and H a subgraph of G. Denote by $G \setminus H$ the graph whose set of vertices is the same as that of G and its set of edges is the set $E(G) \setminus E(H)$.

DEFINITION 5. Let $G=G(V, E)$ satisfy (1) and (2) for a certain k. Define $X=\sum_{v \in V_1} \deg_{V_1}(v)$, $Y=\sum_{v \in V_1} \deg_{V \setminus V_1}(v)$, where V_1 is as in Definition 3.

THEOREM 1. *Let* $G=G(V,E)$ *satisfy* (1) *and* (2) *for* k. *Then,*

(a) $$X \geqq 2(\alpha-3)k$$

(b) $$\alpha \leqq 6, \quad \textit{for all } k$$

(c) $$\alpha \leqq 3, \quad \textit{for all } k \geqq 7.$$

PROOF. Summing degrees in G implies

$$X+Y = \alpha k. \tag{3}$$

Counting edges implies

$$\frac{X}{2}+Y \leqq 3k, \tag{4}$$

or

$$X+2Y \leqq 6k. \tag{4'}$$

Subtracting (4) from (3) implies (a). Subtracting (3) from (4′) implies

$$Y \leqq (6-\alpha)k. \tag{5}$$

Since Y is a non-negative integer, $\alpha \leqq 6$ for all k, and (b) is proved. By definition of X,

$$X \leqq \alpha(\alpha-1). \tag{6}$$

Substituting (5) and (6) in (3) we obtain:

$$(6-\alpha)k+\alpha(\alpha-1) \geqq \alpha k, \tag{7}$$

or

$$\alpha^2-\alpha-2\alpha k+6k \geqq 0. \tag{7'}$$

(8) Substituting $\alpha = 6$ in (7′) implies $k \leqq 5$.

(9) Substituting $\alpha = 5$ in (7′) implies $k \leqq 5$.

(10) Substituting $\alpha = 4$ in (7′) implies $k \leqq 6$.

Hence $k \geqq 7$ implies $\alpha \leqq 3$. Thus (c) is proved.

3. Main theorem

DEFINITION 6. Ex (k) will denote the set of graphs satisfying (1) and (2) for k, but having no $3K_2$-decomposition.

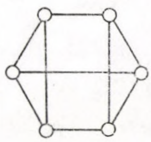
DEFINITION 7. $\overline{\text{Ex}\,(k)}$ will denote the set of graphs which satisfy (1) and (2) for $k+1$ and whose edges are the disjoint union of $3K_2$ and some element of Ex (k).

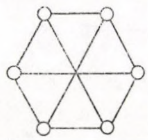
DEFINITION 8. Exx (4) will denote the set of graphs which satisfy (1) and (2) for $k=4$ and contain K_5 as a subgraph.

THEOREM 2. (a) *For* $k=2$ *and* $k>3$, $\mathrm{Ex}\,(k+1)\subset\overline{\mathrm{Ex}\,(k)}$,
(b) $\mathrm{Ex}\,(4)\subset\overline{\mathrm{Ex}\,(3)}\cup\mathrm{Exx}\,(4)$.

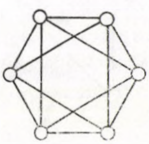
PROOF. Let $G=G(V,E)\in\mathrm{Ex}\,(k+1)$. By Theorem 1 (b), $0\leqq\alpha\leqq 6$. We shall deal with the various cases of α.

Case 1. $\alpha=6$. Theorem 1 (a) and (4′) imply $X=6k$ and $V=V_1$. Moreover $G\subseteqq K_6$. We shall introduce here the only graphs which satisfy (1) and (2) for various values of $k+1$. For $k+1=3$ we obtain the following two graphs:





For $k+1=4$ we obtain the following graph:



and for $k+1=5$ we obtain K_6.

One can easily see that none of the listed graphs belongs to Ex (3), Ex (4), Ex (5), respectively. Hence, the case $\alpha=6$ is settled.

Case 2. $\alpha=5$. Let $V_1=\{v_1, v_2, v_3, v_4, v_5\}$. We shall deal with the various cases of $k+1$.

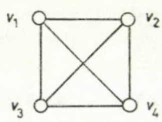
Let $k+1=3$. Since $\dfrac{3\times 5}{2}$ is not an integer, we may assume without loss of generality that there is an edge (v_1, x) where $v_1\in V_1$ and $x\notin V_1$. By Theorem 1 (a) there are at least six edges in the graph induced by V_1. Since $\deg_{V_1}(v_1)\leqq 2$ there are at least four edges in the graph induced by $V_1\setminus\{v_1\}$ and one can find there $2K_2$. Adding the edge (v_1, x) to $2K_2$ we have found $3K_2$ as a subgraph of G. Consider now the graph $G\setminus 3K_2$ which has six edges and the degree of its vertices does not exceed 2. Since $G\in\mathrm{Ex}\,(k+1)$, $G\setminus 3K_2\in\mathrm{Ex}\,(k)$. Hence $G\in\overline{\mathrm{Ex}\,(k)}$.

Let $k+1=4$. If $Y=0$ then $K_5\subset G$ and we are through. Otherwise, we apply the same arguments as for $k+1=3$. Without loss of generality, there is an edge (v_1, x), $x\notin V_1$. By Theorem 1 (a) there are at least 8 edges in the graph induced by V_1. Since $\deg_{V_1}(v_1)\leqq 3$, there are at least 5 edges in the graph induced by $V_1\setminus\{v_1\}$. One can see that G contains $3K_2$ obtained by taking (v_1, x) and $2K_2$ from the graph induced by $V_1 \setminus \{v_1\}$. $G \setminus 3K_2$ has 9 edges and the degree of its vertices does not exceed 3. Since $G \in \mathrm{Ex}(k+1)$, $G \setminus 3K_2 \in \mathrm{Ex}(k)$. Hence $G \in \overline{\mathrm{Ex}(k)}$.

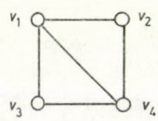
Let $k+1=5$. By Theorem 1 (a), $X \geqq 20$. Hence the graph induced by V_1 is K_5. Take $v_1 \in V_1$ then there exists $x \notin V_1$ such that (v_1, x) is an edge of G. Now taking (v_1, x) and (v_2, v_3) and (v_4, v_5) we certainly obtain $3K_2$. $G \setminus 3K_2$ has 12 edges and the degree of each vertex does not exceed 4. Since $G \in \mathrm{Ex}(k+1)$, $G \setminus 3K_2 \in \mathrm{Ex}(k)$. Hence, $G \in \overline{\mathrm{Ex}(k)}$.

By (9), $k+1<6$. Thus Case 2 is settled.

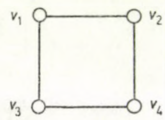
Case 3. $\alpha=4$. Let $V_1=\{v_1, v_2, v_3, v_4\}$. Following (10), obviously we have $3 \leqq k+1 \leqq 6$. Then, the graph induced by V_1 has 3, 4, 5 or 6 edges [Theorem 1 (a)], and it is one of the following:



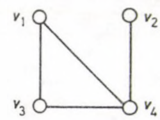
H_1



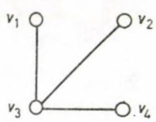
H_2



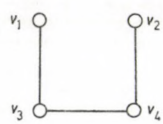
H_3



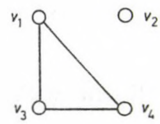
H_4



H_5



H_6



H_7

If $k+1=6$ the only possible graph is H_1.

If $k+1=5$ the possible graphs are H_1 and H_2.

If $k+1=4$ the possible graphs are H_1, H_2, H_3, and H_4.

In any of the above cases we choose $\{(v_1, x), (v_2, y), (v_3, v_4)\}$, where $x, y \notin V_1$, to be the set of edges of $3K_2$. An easy computation shows that such a choice is always possible.

If $k+1=3$ the possible graphs are H_1, H_2, H_3, H_4, H_5, H_6 or H_7.

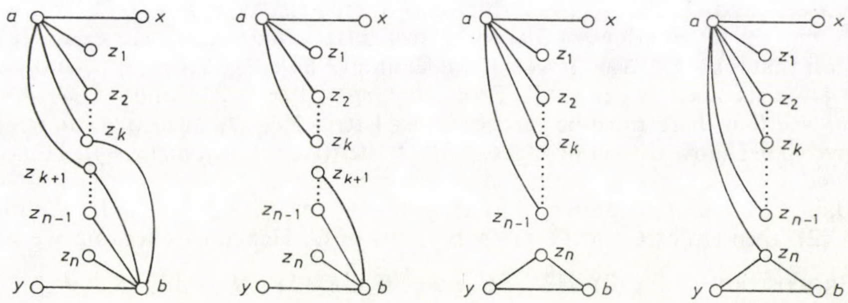
If the graph induced by V_1 is H_1, H_2 or H_3 we choose $\{(v_1, v_2), (v_3, v_4), (x, y)\}$, where $x, y \notin V_1$, to be the set of edges of $3K_2$.

If the graph induced by V_1 is H_4, H_5, H_6 or H_7 we choose $\{(v_1, x), (v_2, y), (v_3, v_4)\}$ where $x, y \notin V_1$, to be the set of edges of $3K_2$. Again, such a choice is always possible. Since $G \in \mathrm{Ex}(k+1)$ then $G \setminus 3K_2 \in \mathrm{Ex}(k)$. Hence $G \in \overline{\mathrm{Ex}(k)}$. Thus the case $\alpha=4$ is settled.

Case 4. $\alpha \leqq 3$. In order to prove this case, we shall prove the following statement:

Let $G=G(V, E)$ be a graph satisfying (1) *and* (2) *for $k+1 \geqq 3$. Then we can find $3K_2$ in G. Moreover, if there exist $a, b \in V(G)$ such that* $\deg(a) \geqq 3$ *or* $\deg(b) \geqq 3$ *then $a, b \in V(3K_2)$.*

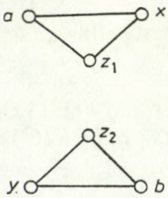
PROOF. The only graph that does not contain $2K_2$ are K_3 and a star. But K_3 and a star do not satisfy (1) and (2) for any k. Hence our graph contains $2K_2$. If there are $a, b \in V(G)$ such that $\deg(a) \geqq 3$ or $\deg(b) \geqq 3$ then there exist vertices $x, y \neq a, b$ such that $(x, a), (y, b) \in E(G)$. In any case we shall take $2K_2$ as (x, a) and (y, b). Let $z_1, \ldots, z_n$ $(n \geqq 2)$ be the vertices left in G. If there is any edge (z_i, z_j), we are through. Otherwise all the vertices z_i, $i \neq 1, \ldots, n$ are adjacent only to $\{a, b, x, y\}$. Let $G_2=G_2(V_2, E_2)$ be a graph such that $V_2=\{a, b, x, y\}$ and $E_2=\{(s, t) \in G \mid s, t \in V_2, (s, t) \neq (x, a), (y, b)\}$. Denote $\beta=|E(G_2)|$. Then obviously $0 \leqq \beta \leqq 4$. Denote by $\gamma(a)$ and $\gamma(b)$ the degrees of a and b in G_2, respectively. Then $0 \leqq \gamma(a)$, $\gamma(b) \leqq 2$. If there exist edges $(z_l, a), (z_m, x)$ for $z_l \neq a_m$ [or $(z_l, b), (z_m, y)$ where $z_l \neq z_m$] then we shall choose the set of edges of $3K_2$ to be $\{(z_l, a), (z_m, x), (b, y)\}$ [or $\{(z_l, b), (z_m, y), (a, x)\}$]. Otherwise, without loss of generality the possible $G \setminus G_2$ graphs are:



A B C D

Then, in cases A and B either the vertex a or the vertex b has degree at least $\dfrac{3(k+1)-\beta}{2}+\gamma$ where γ is either $\gamma(a)$ or $\gamma(b)$. Hence if G satisfies (2) then $\dfrac{3(k+1)-\beta}{2}+\gamma \leqq k+1$ which implies $k+1 \leqq \beta-2\gamma$. Using the definitions of β and γ one can easily see that the last inequality implies $k+1<3$ which is impossible. Thus, n oneof these graphs satisfy (2) which contradicts our demands in the statement. In the cases C and D we have $k+1 \geqq \deg(a) \geqq 3(k+1)-3-\beta+\gamma$ where $\gamma=\gamma(a)$ and G satisfying (2). The last inequality implies $2(k+1) \leqq \beta+3-\gamma$, or $k+1 \leqq \dfrac{\beta-\gamma+3}{2}$. Again using definitions of β and γ we obtain that $k+1<3$. Thus, these graphs do not satisfy (2).

We note that in case $|V(G)|=6$ we can also have that $G \setminus G_2$ is of the form



Calculations for a or b as above show that $k+1<3$. Hence our statement is proved in all cases.

Now, returning to the case $\alpha \leqq 3$, the statement guarantees the existence of $3K_2$ in our graph. In order to complete the proof of the theorem, we have to point out, in each case of α, which edges to choose for $3K_2$.

For $\alpha=0$ the statement ensures us that there exist $3K_2$ in G. For $\alpha=1$ again we have $3K_2$ in G, but since there is a vertex t_1 such that $\deg(t_1) \geqq 3$ we choose the $3K_2$ such that $t_1 \in V(3K_2)$.

For $\alpha=2$, there exist t_1 and t_2 vertices in G such that $\deg(t_1) \geqq 3$, and $\deg(t_2) \geqq 3$. We take $3K_2$ such that $t_1, t_2 \in V(3K_2)$. If $G \in \mathrm{Ex}(k+1)$ then $G \setminus 3K_2 \in \mathrm{Ex}(k)$, implying $G \in \overline{\mathrm{Ex}(k)}$ and we are through if $\alpha=0, 1$, or 2.

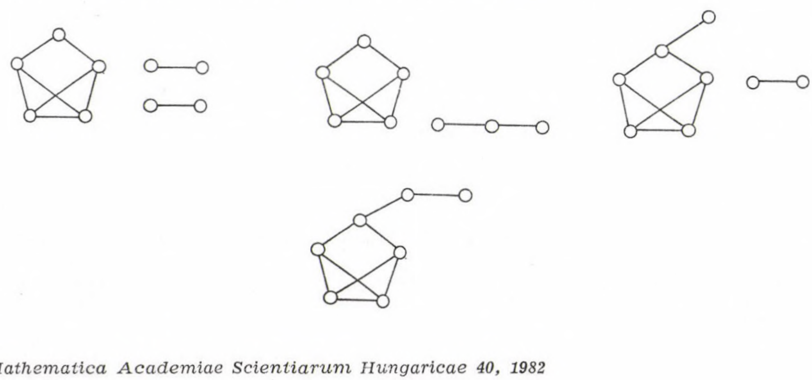
Let $\alpha=3$. As before we define $V_1=\{v_1, v_2, v_3\}$. We shall deal with the two cases $X=0$ and $X \geqq 1$. In the former case there exist $x, y, z \in V(G) \setminus V_1$, $x \neq y \neq z$. We choose the set of edges of $3K_2$ to be $\{(v_1, x), (v_2, y), (v_3, z)\}$. Hence we can find $3K_2$ such that $V_1 \subset V(3K_2)$. If $X \geqq 1$ we shall take an edge, say (v_1, v_2), and an edge (v_3, t) [t exists since $\deg(v_3) \geqq 3$]. From $\deg(v_i) \geqq 3$, $i=1, 2, 3$ and $|E(G)| \geqq 9$ it is easy to see that there must be vertices $u, w \in V(G) \setminus V_1 \cup \{t\}$ such that $(u, w) \in E(G)$. Then we shall choose the set of edges of $3K_2$ to be $\{(v_1, v_2), (v_3, t), (u, w)\}$. Thus case 4 is proved and the proof of the theorem is complete.

Using Definition 6 above, it is easy to see that $\mathrm{Ex}(1)=\emptyset$. In addition, if $G \in \mathrm{Ex}(2)$ then either C_3 or C_5 are subgraphs of G. Hence by checking we obtain:

PROPOSITION 1. $\mathrm{Ex}(2)=\{2K_3, C_5 \cup K_2, K_3 \cup K_2 \cup P_2, K_3 \cup 3K_2, K_3 \cup P_3\}$.

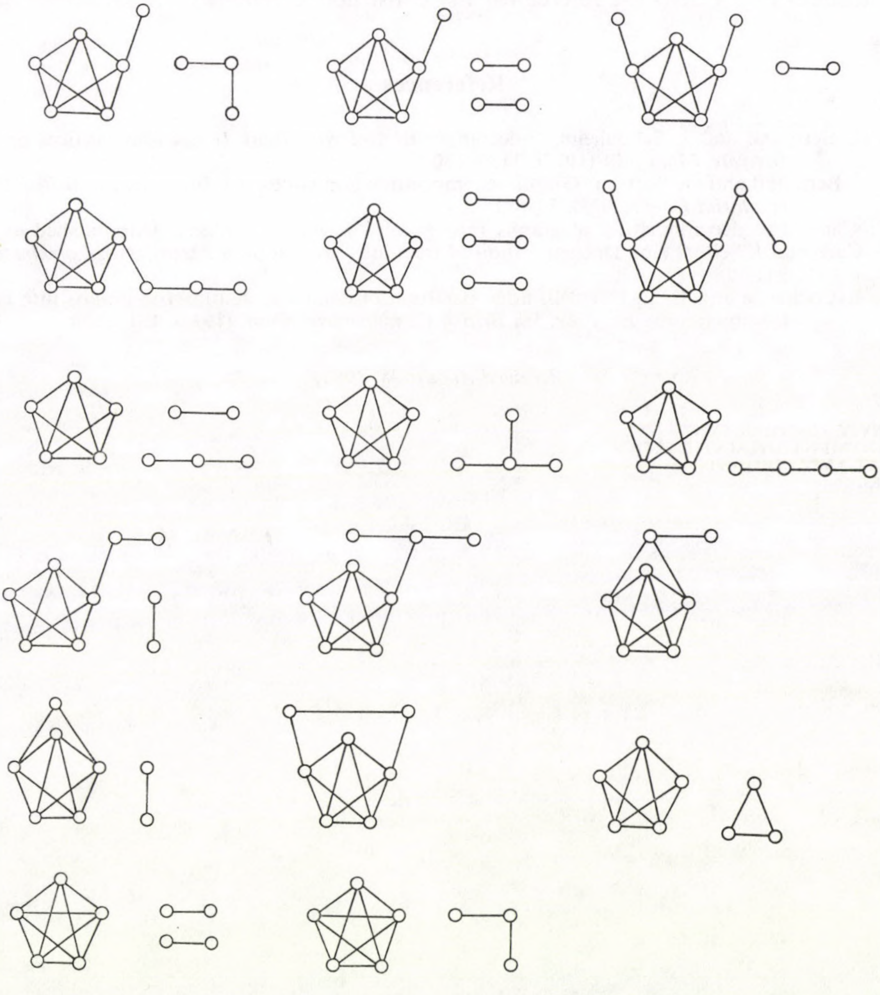
Using Theorem 2 and Proposition 1 we obtain:

PROPOSITION 2. Ex (3) *consists of the following graphs:*



Using Theorem 2 and Proposition 2 we obtain:

PROPOSITION 3. Ex (4) *consists of the following graphs:*



Using Theorem 2 and Proporsition 3 we obtain:

PROPOSITION 4. Ex $(5)=\emptyset$.

THEOREM 3 (Main Theorem). *Let* $G=G(V, E)$ *be a graph such that G is none of the listed graphs in Propositions* 1, 2, 3. *Then* (1) *and* (2) *are necessary and sufficient for* G *to have* $3K_2$*-decomposition.*

PROOF. The necessity is obvious. By Theorem 2 and Proposition 4 it follows that Ex $(k)=\emptyset$ for all $k\geqq 5$. Hence (1) and (2) are sufficient.

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ON ORBIT SPACES OF COMPACT GROUP ACTIONS

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1. Introduction. If a compact group G acts continuously on a given topological space X then many local as well as global properties of X are known to pass on to the orbit space [4]. The quotient map $\nu\colon X\to X/G$ is a special case of a continuous closed as well as open surjection $f\colon X\to Y$ in which inverse image of each $y\in Y$ is a compact space i.e., f is an open proper map. Therefore, any topological or algebraic topological property which is preserved by such a map f will obviously be passed on to the orbit space X/G from the space X. For example, topological properties like paracompactness, normality, local compactness, local connectedness etc. are passed on to the orbit space X/G from X. Observe that the orbit map $\nu\colon X\to X/G$ is closed simply because the group G is compact and therefore if the group is non-compact, none of the above mentioned properties may be passed on from X to the orbit space X/G. Contrary to this, even if the group G is assumed to be compact the algebraic topological properties may not be passed on from the space X to the orbit space X/G. As a matter of fact, some reasonable restrictions have to be imposed not only on the group G but also on the transformation group (G, X) for the validity of such results. For example suppose, X is of finite cohomological dimension over $K=\mathbf{Z}$, $\mathbf{Z}_p$ or Q and G is a compact Lie group acting continuously on X with finitely many orbit types. Then X is acyclic over K implies the orbit space X/G are also acyclic over K [10]. Under similar conditions if X is locally compact separable metric and AR (resp. ANR) then so is the orbit space X/G. These are some of the deep results which are especially true for the actions of compact Lie groups. The objective of this paper is to study the cohomology eigenvalues of a given equivariant map. Suppose X is a G-space and $f\colon X\to X$ is an equivariant map. Let $\hat{f}\colon X/G\to X/G$ be the induced self map of the orbit space. If $H_c^*(X, K)$ denotes the Alexander—Spanier cohomology with compact supports then for any field K, $f^*\colon H_c^*(X, K)\to H_c^*(X, K)$ and $\hat{f}^*\colon H_c^*(X/G, K)\to H_c^*(X/G, K)$ are two operators. With this notation our main result is.

THEOREM 1. *Let X be a locally compact space and G be a compact finite dimensional group acting continuously on X. If $f\colon X\to X$ is an equivariant map, then any eigen value of $\hat{f}^*\colon \tilde{H}_c^*(X/G, K_0)\to\tilde{H}_c^*(X/G, K_0)$ is also an eigen value of $f^*\colon \tilde{H}_c^*(X, K_0)\to \tilde{H}_c^*(X, K_0)$ where K_0 is a field of characteristic zero and $\tilde{H}_c^*$ denotes the reduced groups.*

The first result of above kind was proved by Skjelbred [12] when the group G is a compact Lie group and the space X is a paracompact Hausdorff space. Skjelbred's result is a far reaching generalization of Oliver's proof of the Conner conjecture [10]. The above result of ours is interesting for two reasons. First of all, in the theory of topological transformation groups the class of locally compact spaces is not only as good as that of paracompact spaces but is also an independent class [9]. Secondly, for such a class of spaces the result is proved not only for compact Lie groups but also for any finite dimensional compact group — a more general situation. The final section is devoted to the comparison of the orbit map for a compact group and a general open proper map.

The authors are thankful to the referee for his kind suggestions which improved the exposition of this paper.

2. Preliminaries. The functor $H_c^*(\,.\,, K_0)$ stands for the sheaf cohomology functor of the constant sheaf K_0 with compact supports. Here K_0 is a field of characteristic zero. Since the compact subsets of a locally compact space X form a paracompactifying family of supports, the groups $H_c^*(X, K_0)$ are isomorphic to the Alexander—Spanier cohomology groups. Since the cohomological dimension is a local property $\dim_{K_0}(X)$ with respect to H^* is the same as with respect to H_c^*. *Unless otherwise stated, our space X is assumed to be of finite cohomological dimension over the ground ring under consideration.* We also recall that if a compact group G acts on a space X and N is a closed normal subgroup of G then G/N acts on X/N in an obvious manner and it is easily checked that if G acts on X with finitely many orbit types then so does G/N on X/N.

Suppose X is a G-space. Then the set $M(G, X)$ of all equivariant maps from X to itself is a monoid and this monoid acts on $H^*(X, K_0)$ or $H_c^*(X, K_0)$ on the right as follows: For $\alpha\in H^*(X, K_0)$ and $f\in M(G, X)$, let $\alpha f=f^*(\alpha)$. Similarly if $\hat{f}$ is the self map on X/G induced by f then $M(G, X)$ also acts on $H^*(X/G, K_0)$ or $H_c^*(X/G, K_0)$ as follows: For $\alpha\in H^*(X/G, K_0)$ and $f\in M(G, X)$ let $\alpha f=\hat{f}^*(\alpha)$. As a matter of fact if $\mathcal{J}$ is any monoid and there is a homomorphism from $\mathcal{J}$ to $M(G, X)$ then both $H^*(X, K_0)$ as well as $H^*(X/G, K_0)$ can be regarded as $\mathcal{J}$-modules in an obvious way. Suppose M is a $\mathcal{J}$-module. Then by a simple subquotient of M we mean a simple $\mathcal{J}$-module isomorphic to M_2/M_1 where M_1 and M_2 are $\mathcal{J}$-submodules of M.

3. A more general form of the main theorem. First of all we show that Theorem 1 is a special case of the following.

THEOREM 3.1. *Let X be a locally compact G-space of finite cohomological dimension over K_0 where G is a finite dimensional compact group acting continuously on X and $\mathcal{J}$ be a monoid of equivariant self maps of X. Then every simple subquotient of the $\mathcal{J}$-module $H_c^*(X/G, K_0)$ is a simple subquotient of the $\mathcal{J}$-module $H_c^*(X, K_0)$.*

To see how Theorem 1 follows from Theorem 3.1 let us consider the monoid $\mathcal{J}=\{1, f, f^2, \dots, f^n, \dots\}$ of equivariant self-maps of X generated by f. Now if $\alpha\in K$ is an eigenvalue of $\hat{f}^*$: $H_c^*(X/G, K_0)\to H_c^*(X/G, K_0)$ then there is a one-dimensional subspace, say $K(x)$ of $H_c^*(X/G, K_0)$ generated by an eigenvector x of $\hat{f}^*$. This subspace is clearly invariant under f and so it is a $\mathcal{J}$-module. Clearly it is a simple subquotient of $H_c^*(X/G, K_0)$. Hence by Theorem 3.1, there exists $\mathcal{J}$-submodules, M_1 and M of $H_c^*(X, K_0)$ such that $M/M_1=K_0(x)$. This means there is a vector $x'\in H_c^*(X, K_0)$ such that $K_0(x')$ is invariant under $\mathcal{J}$, i.e., invariant under f^*. Now let g: $K_0(x)\to K_0(x')$ be a $\mathcal{J}$-isomorphism then $f^*(x)=\alpha x$ means $g(\hat{f}^*(x))=g(\alpha(x))=$ $=\alpha g(x)$. But since g is a $\mathcal{J}$-isomorphism $g\hat{f}^*=f^*g$ and we have $f^*(g(x))=\alpha(g(x))$ i.e., $f^*|K_0(x')$ is multiplication by α. Hence α is an eigenvalue of f^* also.

4. Proof of theorem 3.1. For convenience the proof of Theorem 3.1 will be completed in four steps depending upon the nature of the group G. The four cases are as follows:

(i) when G is compact totally disconnected;
(ii) when $G=S^1$ the circle group acting semifreely;
(iii) when G is connected simple and Lie
(iv) when G is any compact finite dimensional group — the general case.

General case. First of all we show that the general case follows from (i), (ii) and (iii). As a matter of fact (i) implies that the theorem is true for any finite group G. Now we remark that if N is a closed normal subgroup of a compact group G such that Theorem 3.1 is true for N as well as G/N then the theorem is true for G itself. Next, suppose G is a compact Lie group and G_0 is the component of identity. Then G/G_0 is finite and therefore the theorem will be true for G if it is proved for any compact connected Lie group. Further, if G is abelian then it is a torus, say T^n and by our remark it suffices to prove the theorem for S^1. Now because there are only finite number of isotropy subgroups, we can assume that there is a finite group N of S^1 which contains all of the isotropy subgroups. Since $S^1/N \sim S^1$ acts semifreely on X the theorem is true for S^1/N by (ii) and therefore by our remark the theorem is true for S^1. On the other hand, if G is non-abelian then the theorem follows by induction on dim G, our remark and (iii). Thus the theorem is proved for any compact Lie group G. Finally, suppose G is any compact finite dimensional group acting continuously on a locally compact space X. Then there exists a compact totally disconnected normal subgroup H of G such that G/H is a Lie group [9]. Therefore, the proof of the theorem at once follows from (i) and our remark.

Proof when G is compact totally disconnected. It is a standard fact that when G is compact totally disconnected then given any neighbourhood of the identity e of G, there exists a closed normal subgroup G_i of G contained in the neighbourhood such that G/G_i is a finite group. Since the intersection of all open neighbourhoods of e is $\{e\}$ we find that intersection of all such closed normal subgroups G_i of G is also $\{e\}$. If $G_i \subset G_j$ we let $\pi_{ij}\colon G/G_i \to G/G_j$ be the homomorphism induced by identity map of G. Then we have an inverse system of finite groups G/G_i and homomorphisms π_{ij} whose inverse limit is the group G with π_i as the canonical map. Similarly, we have an inverse system of locally compact spaces X/G_i with bonding maps $f_{ij}\colon X/G_i \to K/G_j$ if $G_i \subset G_j$ whose inverse limit is the space X with obvious orbit maps $X \to X/G_i$ as the canonical maps. In fact, it is easily seen that these two inverse systems give rise to an inverse system of transformation groups $(G/G_i, X/G_i)$ with obvious bonding equivariant maps whose inverse limit is the transformation group (G, X). Now, if we consider the orbit spaces we find an inverse system in which the induced bonding maps are homeomorphisms so are the canonical maps $X/G \to (X/G_i)/(G/G_i)$ making various triangles commutative, which shows that $X/G \approx \varprojlim (X/G_i)/(G/G_i)$. If we apply any cohomology functor H^p to this inverse system with inverse limit X/G then we find that

$$\text{(i)} \qquad H^p(X/G) \approx \varinjlim H^p\big((X/G_i)/(G/G_i)\big).$$

If H^p stands for Alexander—Spanier cohomology and K_0 is a field of characteristic zero then we know that there is a canonical isomorphism induced by orbit map

$$H_c^p((X/G_i)/(G/G_i), K_0) \approx [H_c^p(X/G_i, K_0)]^{G/G_i},$$

where $[H_c^p(X/G_i, K_0)]^{G/G_i}$ means the fixed point set of the group G/G_i; the action of the group G/G_i on the group $H_c^p(X/G_i, K_0)$ is the one induced by the action of the group G/G_i on the space X/G_i. But one can verify that

$$\text{(ii)} \qquad \varinjlim [H_c^p(X/G_i, K_0)]^{G/G_i} \approx [H_c^p(X, K_0)]^G.$$

Then, by combining (i) and (ii) we get

$$\text{(iii)} \qquad H_c^p(X/G, K_0) \approx [H_c^p(X, K_0)]^G.$$

Also, observe that

$$\text{(iv)} \qquad [H_c^p(X, K_0)]^G \subset H_c^p(X, K_0).$$

Then, by (iii) and (iv) we have the following

PROPOSITION 4.1. *Let X be a locally compact space and G be a compact totally disconnected group acting continuously on X. Let $\mathcal{J}$ be a monoid of equivariant self-maps of X. Then every simple subquotient of the $\mathcal{J}$-module $\tilde{H}_c^*(X/G, K_0)$ is a simple subquotient of the $\mathcal{J}$-module $H_c^*(X, K_0)$.*

Proof of Theorem 3.1 *when* $G=S^1$. Let X_G be the Borel space of the G-action. Then in the fiber bundle $X \to X_G \to B_G$, X_G is the total space and B_G is the classifying space of the principal G-bundle. For further details we refer to [1, 5, 11]. Because G is acting with finite number of orbit types, by our remark, we can assume that G is acting semifreely. First let us see the following.

PROPOSITION 4.2. *Let G be a compact Lie group acting semifreely on a locally compact space X with fixed point set F. Then there is a long exact Mayer—Vietoris sequence of the form*

$$\dots \xrightarrow{\delta} H_c^p(X/G) \to H_c^p(F) \oplus H_c^p(X_G) \to H_c^p(F_G) \xrightarrow{\delta} \dots .$$

Also, when F is nonempty, there is a reduced Mayer—Vietoris sequence

$$\dots \xrightarrow{\delta} \tilde{H}_c^*(X/G) \to \tilde{H}_c^*(F) \oplus \tilde{H}_c^*(X_G) \to \tilde{H}_c^*(F_G) \xrightarrow{\delta} \dots .$$

PROOF. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{\delta} & H_c^*(X_G, F_G) & \to & H_c^*(X_G) & \to & H_c^*(F_G) & \xrightarrow{\delta} & \dots \\
 & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} & & \\
\dots & \xrightarrow{\delta} & H_c^*(X/G, F) & \to & H_c^*(X/G) & \to & H_c^*(F) & \xrightarrow{\delta} & \dots
\end{array}$$

and note that $\pi\colon X_G \to X/G$ induces isomorphism $\pi^*\colon H_c^*(X/G, F) \to H_c^*(X_G, F_G)$ because the action is semifree and the supports are in the family of all compact subsets of X. Now the Mayer—Vietoris sequence as well as the reduced Mayer—Vietoris sequence can be deduced from above diagram by the standard arguments of generalized homology-cohomology theories.

Now we prove

LEMMA 4.3. *Let* $G=S^1$ *be acting semifreely on* X *with nonempty fixed point set* F. *Let* $\mathscr{J}$ *be a monoid of equivariant self-maps of* X. *Then every simple subquotient of any of the three* $\mathscr{J}$*-modules* $\tilde{H}_c^*(X_G, K_0)$, $\tilde{H}_c^*(F_G, K_0)$ *and* $\tilde{H}_c^*(F, K_0)$ *is a subquotient of the* $\mathscr{J}$*-module* $\tilde{H}_c^*(X, K_0)$.

PROOF. Every simple subquotient of the $\mathscr{J}$-modules $\tilde{H}_c^*(F, K_0)$ and $\tilde{H}_c^*(F_G, K_0)$ is a subquotient of $\tilde{H}_c^*(X_G, K_0)$ follows from $\tilde{H}_c^*(F_G, K_0)=\tilde{H}_c^*(F, K_0)\otimes H_c^*(B_G, K_0)$ and the restriction homomorphism $H_c^*(X_G, K_0)\to H_c^*(F_G, K_0)$ is onto in high degrees.

The fibre bundle $X\to X_G\to B_G$ gives a spectral sequence converging to $\tilde{H}_c^*(X_G, K_0)$ with $E_1=\mathscr{C}_{\mathrm{cell}}^*(B_G, \tilde{H}_c^*(X, K_0))$. Hence every simple subquotient of the $\mathscr{J}$-module $\tilde{H}_c^*(X_G, K_0)$ is a simple subquotient of the $\mathscr{J}$-module $\tilde{H}_c^*(X, K_0)$.

COROLLARY 4.4. *If* $F\neq\emptyset$ *then Theorem* 3.1 *holds for* $G=S^1$.

PROOF. The reduced Mayer—Vietoris sequence shows that every simple subquotient of $\tilde{H}_c^*(X/G, K_0)$ is a subquotient of $\tilde{H}_c^*(F_G, K_0)\oplus\tilde{H}_c^*(X_G, K_0)\oplus\tilde{H}_c^*(F, K_0)$. By Lemma 4.3, it is a subquotient of the $\mathscr{J}$-module $\tilde{H}_c^*(X, K_0)$.

COROLLARY 4.5. *If* $F=\emptyset$ *then Theorem* 3.1 *holds when* $G=S^1$.

PROOF. When $F=\emptyset$, $G=S^1$ is acting freely and there is an isomorphism $H_c^*(X/G, K_0)\approx H_c^*(X_G, K_0)$. The spectral sequence of the fibering $X_G\to B_G$ with $E_1=\mathscr{C}_{\mathrm{cell}}^*(B_G, H_c^*(X, K_0))$, $E_2^{ab}=H_c^a(\mathbf{C}P^\infty)\otimes H_c^b(X, K_0)$ for $G=S^1$ converges to $H_c^*(X/G, K_0)$. Now to complete the proof it is enough to show that every simple subquotient of the $\mathscr{J}$-module E_∞/K_0 (where $K_0\subset E_\infty^{00}$ is the field of coefficients) is a subquotient of $H_c^*(X, K_0)$. Clearly for $r\geqq 1$, $b>0$, every simple subquotient of E^{ab} is a subquotient of $H_c^b(X, K_0)$. Hence, for $r\geqq 2$, every simple subquotient of $d_r(E_r)$ is a subquotient of $H_c^+(X, K_0)=\sum_{b>0} H_c^b(X, K_0)$. For $a>c$, $c=$ the cohomology dimension of X over K_0, $E_\infty^{00}=0$. It follows that for $a>c$, each simple subquotient of E_2^{a0} is a subquotient of $H_c^+(X, K_0)$. E_2^{a0} and E_2^{a+20} as $\mathscr{J}$-modules are isomorphic for $a>0$ and so the last statement is valid for $a>0$. It remains only the module E_∞^{00}/K_0 wich is contained in $\tilde{H}_c^0(X, K_0)$ and the proof is complete.

Proof of theorem 3.1 *when G is connected, simple and Lie.* In this case we shall use the fact [12] that for such a group G there is a compact G-space Z which has the property that if H is any closed subgroup of G then the orbit map $Z\to Z/H$ induces an isomorphism $H^*(Z/H, \mathbf{Z})\approx H^*(Z, \mathbf{Z})$. Furthermore, Z is a compact $G-CW$ complex, by a result of Matumoto [8] and G has no fixed points in Z. The $G-CW$ structure on Z defines a finite cell complex structure on Z/G [7]. For each cell c of Z/G, choose $x\in Z$ such that $G(x)$ is in the interior of c, and also set $G_c=G_x$. The cellular system (G_c) will be used in the Borel construction. Let X and Z be two G-spaces and let $X\times Z$ be a G-space with diagonal action. Then pr_1: $(Z\times X)/G\to Z/G$, pr_2: $(Z\times X)/G\to X/G$ are projections of orbit spaces. The fiber for pr_1 is X/G_z and the fiber for pr_2 is Z/G_x for $x\in X$ and $z\in Z$. Now apply the Leray spectral sequence to the proper mappings pr_2 and p_2 of the following commutative diagram (the vertical arrows are induced by π where π: $Z\to S^1$ is a projection map on the second factor and $S^1=[0,1]/\{0,1\}$):

$$\begin{array}{ccccc} Z/G_x & \to & (Z\times X)/G & \xrightarrow{pr_2} & X/G \\ \downarrow{\scriptstyle \pi} & & & & \downarrow{\scriptstyle 1} \\ S^1 & \to & S^1\times(X/G) & \xrightarrow{p_2} & X/G. \end{array}$$

Since π induces cohomology isomorphisms of the fibers, we have $H^*(S^1)\otimes H_c^*(X/G)\approx\approx H_c^*((Z\times X)/G)$ for any coefficient ring. Now it is clear that this is an isomorphism of $\mathscr{J}$-modules. For the mapping pr_1: $(Z\times X)/G\to Z/G$ we obtain a spectral sequence defined by the skeleton filtration of the cell complex Z/G with $E_1=\mathscr{C}^*_{\text{cell}}(Z/G, H_c^*(X/G_c))$ and converging to $H_c^*((Z\times X)/G)\approx H^*(S^1)\otimes H_c^*(X/G)$. For reduced cohomology, the spectral sequence $\tilde{E}$ with $\tilde{E}_1=\mathscr{C}^*_{\text{cell}}(Z/G, H_c^*(X/G_c, K_0)$ converges to $H^*(S^1)\otimes\tilde{H}_c^*(X/G, K_0)$. This is a spectral sequence of $\mathscr{J}$-modules. A simple subquotient of the $\mathscr{J}$-module $\tilde{H}_c^*(X/G, K_0)$ must be a simple subquotient of $\tilde{E}_1$ and hence of some $\tilde{H}_c^*(X/G_c, K_0)$. For each c, $G_c<G$ and Z is without fixed points and hence by induction on $\dim G$ it follows that Theorem 3.1 is valid for actions of G_c. Hence each simple subquotient of $\tilde{H}_c^*(X/G_c, K_0)$ is a subquotient of $\tilde{H}_c^*(X, K_0)$ and this completes the proof of Theorem 3.1 for the case when G is connected simple.

5. Open proper map and orbit map. As pointed out in the introduction the orbit map v: $X\to X/G$ by a compact group is a continuous closed as well as open surjection such that the inverse image of each point is compact. In other words the orbit map v is an open proper map. The converse can not be obviously true. A general question which can be asked is the following: *What are those topological properties of X which are passed on to the orbit space X/G but are not in general, preserved by an open proper map f: $X\to Y$.*

We do not know the answer to this question. However, the very fact that the inverse image of each point in the case of the orbit map v: $X\to X/G$ is not only a homogeneous space but also a coset space makes us believe that there should be a number of topological properties which should be preserved by the orbit map and not by an open proper map. Here we wish to point out that if X is completely regular and f: $X\to Y$ is an open proper map then so is Y [6]. Therefore, if X is completely regular and G is compact then the orbit space X/G is also completely regular. But the proof of the latter fact given by Palais [1] using Haar integral is comparatively extremely elegant.

Recall that a space X is said to be functionally Hausdorff if given any two points $x, y\in X$ there exists a continuous real valued function f: $X\to\mathbf{R}$ such that $f(x)=0$ and $f(y)=1$. Using Haar integral of the compact group we prove the following.

THEOREM 5.1. *Let X be a functionally Hausdorff (completely Hausdorff) space and G be a compact group acting continuously on X. Then the orbit space X/G is also functionally Hausdorff.*

PROOF. Suppose $G(x)$ and $G(y)$ are two distinct orbits of X representing two distinct points of the orbit space X/G. Fix a point $x_0\in G(x)$ and for all $y\in G(y)$ find a continuous function f_y: $X\to I$ such that $f_y(x_0)=0$ and $f(V_y)>3/4$ for each point of a neighbourhood V_y of y. Select a finite number of neighbourhoods. $V_{y_1}, \dots, V_{y_n}$ which cover the compact orbit $G(y)$. Then the function $\hat{f}$: $X\to\mathbf{R}$ defined by $\hat{f}=\sup\{f_{y_1},\dots,f_{y_n}\}$ is such that $\hat{f}(x_0)=0$ and $\hat{f}(G(y))>3/4$. Now varying $x\in G(x)$ we can find a function $\hat{\hat{f}}: X\to\mathbf{R}$ such that $\hat{\hat{f}}(G(x))<1/4$ and $\hat{\hat{f}}(G(y))>3/4$. Now define a function $F: X\to\mathbf{R}$ such that $F(x)=\int_G \hat{\hat{f}}(gx)\,dg$. Then F is continuous and constant on orbits of X. Therefore F induces a continuous function $\hat{F}: X/G\to R$ such that

$$\hat{F}(G(x)) = \int_G \hat{\hat{f}}(gx)\,dg < \int_G \hat{\hat{f}}(gy)\,dy = \hat{F}(G(y)).$$

Now it is easy to verify that X/G is functionally Hausdorff.

We do not know whether or not functionally Hausdorff is preserved by an open proper map.

There is another crucial difference between the orbit map by a compact group and an open proper map. We observe that, in general, the restriction of a closed map to any subset A of X, unless of course A itself is closed, is not closed and therefore under a proper map hereditary properties like complete normality may not be preserved. But, if A is an invariant subset of X then $v: A\to A/G$ is a closed-open surjection and therefore we have the following

THEOREM 5.2. *Suppose P is a topological property which is invariant under a closed or open continuous surjection. If G is a compact group acting continuously on a space X such that each subspace of X has P then each subspace of the orbit space X/G also has P.*

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ON SUMS OF LEBESGUE FUNCTION TYPE

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1. Introduction

1.1. Let $X=\{x_{kn}\}$, $n=1, 2, \ldots$; $1 \leqq k \leqq n$, be a triangular matrix where

$$(1.1) \qquad -1 \leqq x_{nn} < x_{n-1,n} < \ldots < x_{1n} \leqq 1 \quad (n = 1, 2, \ldots).$$

If, sometimes omitting the superfluous notation

$$(1.2) \qquad \omega(x) = \omega_n(X, x) = \prod_{k=1}^{n} (x-x_k) \quad (n = 1, 2, \ldots)$$

then

$$(1.3) \qquad l_k(x) = l_{kn}(X, x) = \frac{\omega(x)}{\omega'(x_k)(x-x_k)} \quad (k = 1, 2, \ldots, n)$$

are the corresponding fundamental functions of the Lagrange interpolation. The definitions of the Lebesgue constant and Lebesgue function are

$$(1.4) \qquad \lambda_n(x) = \lambda_n(X, x) = \prod_{k=1}^{n} |l_k(x)|, \quad \lambda_n = \lambda_n(X) = \max_{-1 \leqq x \leqq 1} \lambda_n(x),$$

respectively. As it is well known, they play a fundamental role in the study of the convergence and divergence behaviour of the Lagrange interpolation.

Here we quote three results which, in certain sense, generalize previous statements of G. Faber [1] and S. Bernstein [2].

In 1958, P. Erdős [3] proved as follows:

THEOREM 1.1. *Let ε and A be any positive numbers Then, considering an arbitrary matrix X, the measure of the set in x $(-\infty < x < \infty)$ for which*

$$(1.5) \qquad \lambda_n(x) \leqq A \quad \text{if} \quad n \geqq n_0(A, \varepsilon)$$

is less than ε.

The following result, proved recently by P. Erdős and P. Vértesi [4], states more.

THEOREM 1.2. *Let $\varepsilon > 0$ be any given number. Then for an arbitrary matrix X there exist sets $H_n = H_n(\varepsilon, X)$ of measure $\leqq \varepsilon$ and $\eta = \eta(\varepsilon), \eta > 0$ such that*

$$(1.6) \qquad \lambda_n(x) > \eta \ln n \quad \text{if} \quad x \in [-1, 1] \setminus H_n \quad \text{and} \quad n \geqq n_0(\varepsilon).$$

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In a very recent paper [5] I proved that $\eta=c\varepsilon$. More precisely

THEOREM 1.3. *There exists a positive constant c such that if* $\varepsilon=\{\varepsilon_n\}$ *is any sequence of positive numbers then for an arbitrary matrix X, there exist sets* $H_n=H_n(\varepsilon, X)$ *of measure* $\leqq\varepsilon_n$ *for which*

$$(1.7)\qquad \lambda_n(x) > c\varepsilon_n \ln n \quad \textit{if} \quad x\in[-1,1]\setminus H_n \quad \textit{and} \quad n=1,2,\ldots.$$

As it is easy to see the order of (1.7) is the best possible (take the Chebyshev nodes $\cos[(2k-1)\pi(2n)^{-1}]$, $k=1,2,\ldots,n$).

1.2. Let us consider now the fundamental functions of Hermite—Fejèr interpolation having the form

$$(1.8)\qquad \mathfrak{h}_{kn}(X,x)=(x-x_k)\, l_k^2(X,x) \quad (k=1,2,\ldots,n).$$

In their paper [6], P. Erdős and P. Turán proved the following deep result.

THEOREM 1.4. *By whatever choice of the matrix X we have the inequality*

$$(1.9)\qquad \max_{-1\leqq x\leqq 1}\sum_{k=1}^{n}|\mathfrak{h}_{kn}(X,x)| \geqq \frac{2}{\pi n}(\ln n - c_1 \ln\ln n).$$

Here and later on $c, c_1, c_2, \ldots$ denote positive numerical constants. In the same paper they conjectured:

"Probably also the inequality

$$(1.10)\qquad \int_{-1}^{1}\left\{\sum_{k=1}^{n}|\mathfrak{h}_{kn}(X,x)|\right\}dx > c_2\frac{\ln n}{n}$$

holds or even the inequality

$$(1.11)\qquad \sum_{k=1}^{n}|\mathfrak{h}_{kn}(X,x)| > c_3\frac{\ln n}{n}$$

in $[-1,1]$ with the exception of a set with measure tending to zero with $1/n$..." (see [6], p. 224).

1.3. In this paper I am going to give a result which generalizes Theorem 1.3, moreover proves (1.11), actually with the best possible order. I will investigate the complex and trigonometric cases, too.

2. Results

2.1. Let us denote by $H_{ns}(x)=H_{ns}(X,x)$ the uniquely determined Hermite interpolatory polynomial of degree $\leqq ns-1$ with

$$(2.1)\qquad H_{ns}^{(i)}(x_{kn})=d_{kni} \quad (1\leqq k\leqq n,\ 0\leqq i\leqq s-1).$$

Here $\{d_{kni}\}$ are fixed real numbers, s is a given positive integer. This polynomial can be written as follows

$$H_{ns}(x) = \sum_{i=0}^{s-1} \sum_{k=1}^{n} d_{kni} l_{kni}(X, x) \tag{2.2}$$

where the fundamental polynomials l_{kni} of degree exactly $ns-1$ fulfill the conditions

$$l_{kni}^{(j)}(x_{tn}) = \begin{cases} 0 & \text{if } j \neq i, \quad 0 \leqq j \leqq s-1, \quad 1 \leqq t \leqq n, \\ \delta_{tk} & \text{if } j = i, \quad 1 \leqq t \leqq n. \end{cases} \tag{2.3}$$

It is easy to see that by (1.3)

$$l_{k,n,s-1}(x) = \frac{(x-x_k)^{s-1} l_k^s(x)}{(s-1)!} \quad (1 \leqq k \leqq n). \tag{2.4}$$

2.2. Using this motivation we investigate the behaviour of

$$\lambda_n(s)(x) = \lambda_n(s)(X, x) \stackrel{\text{def}}{=} \sum_{k=1}^{n} |x-x_k|^{s-1} |l_k^s(x)| \quad (n = 1, 2, \ldots) \tag{2.5}$$

for a fixed positive integer s. If $s=1$, $\lambda_n(1)(x)=\lambda_n(x)$ is the Lebesgue function (see 1.6); for $s=2$ we obtain that $\lambda_n(2)(x)=\sum_{k=1}^{n} |\mathfrak{h}_{kn}(x)|$ (see (1.8)).

The first statement is

THEOREM 2.1. *If $s \geqq 1$ is a fixed integer, there exists a constant $c=c(s)>0$ such that if $\varepsilon=\{\varepsilon_n\}$ is any sequence of positive numbers then for arbitrary matrix X, there exist sets $H_n=H_n(\varepsilon, X, s)$ of measure $\leqq \varepsilon_n$ such that*

$$\lambda_n(s)(x) > c\varepsilon_n^s \frac{\ln n}{n^{s-1}} \tag{2.6}$$

if $x \in [-1, 1] \setminus H_n$ and $n=1, 2, \ldots$.

2.3.a) Considering the Chebyshev nodes it is easy to see that the order of (2.6) is the best possible.

2.3.b) If $s=2$ and $\varepsilon_n=\eta$ $(n=1, 2, \ldots)$ we obtain the answer for (1.11). Generally it is easy to gain the following result.

COROLLARY 2.2. *If $P_n \subseteqq [-1, 1]$ are arbitrary measurable sets then, using the above notations, for any X*

$$\int_{P_n} \lambda_n(s)(x)\, dx > (|P_n| - \varepsilon_n) c\varepsilon_n^s \frac{\ln n}{n^{s-1}} \quad \text{if} \quad n = 1, 2, \ldots. \tag{2.7}$$

The case $P_n=P=[a, b]$ and $s=1$ was treated by P. Erdős and J. Szabados [7]. If $s=2$ and $\varepsilon_n=\eta$, we obtain (1.10).

2.4. Let us see now the corresponding theorem for the complex case. Let $Z=\{z_{kn}=\exp(i\vartheta_{kn})\}$, $k=1, 2, \ldots, n$, $n=1, 2, \ldots$ with

$$0 \leqq \vartheta_{1n} < \vartheta_{2n} < \ldots < \vartheta_{nn} < 2\pi, \quad n = 1, 2, \ldots \tag{2.8}$$

be a triangular matrix on the unit circle line $\Gamma=\{z;\ z=\exp(i\theta),\ 0\leqq\theta<2\pi\}$. Let

$$(2.9)\qquad l_k(z)=l_{kn}(Z,z)=\prod_{j\neq k}\frac{z-z_{jn}}{z_{kn}-z_{jn}},\quad k=1,2,\ldots,n$$

be the fundamental polynomials of the corresponding Lagrange interpolation. Using the previous motivation we investigate

$$(2.10)\qquad \lambda_n(s)(z)=\lambda_n(s)(Z,z)\stackrel{\text{def}}{=}\sum_{k=1}^{n}|z-z_k|^{s-1}|l_k^s(z)|,\quad n=1,2,\ldots$$

for a fixed positive integer s.

For $s=1$ S. Y. Alper [9] proved that for any matrix Z

$$(2.11)\qquad \lambda_n>\frac{\ln n}{8\sqrt{\pi}},\quad n\geqq n_0$$

where $\lambda_n=\max_{z\in\Gamma}\lambda_n(1)(Z,z)$. A far-reaching generalization is the following

THEOREM 2.3. *If $s\geqq1$ is a fixed integer, there exists a certain constant $c_1=c_1(s)>0$ such that if $\varepsilon=\{\varepsilon_n\}$ is any sequence of positive number then for an arbitrary matrix Z, there exist sets $H_n=H_n(\varepsilon,Z,s)\subset\Gamma$ of measure $\leqq\varepsilon_n$ such that*

$$(2.12)\qquad \lambda_n(s)(z)>c_1\varepsilon_n^s\frac{\ln n}{n^{s-1}}\quad \textit{if}\quad z\in\Gamma\setminus H_n\quad\textit{and}\quad n=1,2,\ldots.$$

Considering the nodes $z_{kn}=\exp(i2k\pi/n)$ we can say that the order of (2.12) is the best possible.

2.4. Finally let us see the trigonometric case. By (2.8) for the matrix $\Theta=\{\theta_{kn}\}$, $k=1,2,\ldots,n$; $n=1,2,\ldots,$ we define the functions

$$(2.13)\qquad l_k(\theta)=l_{kn}(\Theta,\theta)=\prod_{j\neq k}\frac{\sin\dfrac{\theta-\theta_j}{2}}{\sin\dfrac{\theta_k-\theta_j}{2}},\quad k=1,2,\ldots,n$$

(which are the fundamental trigonometric interpolatory polynomials of degree $(n-1)/2$ whenever n is odd). Let

$$(2.14)\qquad \lambda_n(s)(\theta)=\lambda_n(\Theta,\theta)\stackrel{\text{def}}{=}\sum_{k=1}^{n}\left|\sin\frac{\theta-\theta_k}{2}\right|^{s-1}|l_k^s(\theta)|,\quad n=1,2,\ldots.$$

We state

THEOREM 2.4. *If $s\geqq1$ is a fixed integer, there exists a certain constant $c_1=c_1(s)>0$ such that if $\varepsilon=\{\varepsilon_n\}$ is any sequence of positive numbers then for arbitrary matrix Θ there exist sets $H_n=H_n(\varepsilon,\Theta,s)\subseteqq[0,2\pi)$ of measure $\leqq\varepsilon_n$ such that*

$$(2.14)\qquad \lambda_n(s)(\theta)>c_1\varepsilon_n^s\frac{\ln n}{n^{s-1}}\quad\textit{if}\quad \theta\in[0,2\pi)\setminus H_n\quad\textit{and}\quad n=1,2,\ldots.$$

If $\theta_{kn}=2k\pi/n$, $k=0, 1, \dots, n-1$, we obtain that the order is the best possible.

REMARK. If n and s are odd, then

$$l_{k,n,s-1}(\theta) = \frac{1}{(s-1)!}\left(2\sin\frac{\theta-\theta_k}{2}\right)^{s-1} l_{kn}^s(\theta) \quad (1 \leqq k \leqq n)$$

are trigonometric polynomials of degree $(ns-1)/2$ having analogous properties to those of (2.4).

3. Proofs

3.1. PROOF OF TEOREM 2.1. We shall use many ideas from [4], [5], and [8]. In what follows let $J_{kn} = [x_{k+1\ n}, x_{k,n}]$ $(k = 0, 1, \dots, n;\ n = 1, 2, \dots)$ with $x_0 \equiv 1$ and $x_{n+1} \equiv -1$. If $|J_k| \leqq \delta_n \overset{\text{def}}{=} n^{-1/6}$ we say that the interval is *short*; the others are the *long* ones.

3.2. First we settle the long intervals. As in our paper [8], Lemma 4.4., we can prove

LEMMA 3.1. *Let* $|J_{kn}|>\delta_n$ *for a certain* k $(0<k<n)$. *Then for any* $(\ln n)^{-2} \leqq s_n \leqq \leqq 1/4$ *we can define the index* $t=t(k, n)$ *and the set* $h_{kn} \subset J_{kn}$ *so that* $|h_{kn}| \leqq 4s_n|J_{kn}|$, *moreover*

$$|l_{tn}(x)| \geqq 3^{\sqrt{n}} \quad \text{if} \quad x \in J_{kn} \setminus h_{kn} \quad \text{and} \quad n \geqq n_1. \tag{3.1}$$

Here n_1 *is an absolute constant, if* $x \in J_{kn} \setminus h_{kn}$ *then* $\min(|x-x_k|, |x-x_{k+1}|) \geqq \geqq s_n|J_{kn}|$.

Now let $s_n = 1/\ln^2 n$. If J_{kn} is long then for $x \in J_{kn} \setminus h_{kn}$,

$$\lambda_n(s)(x) \geqq |x-x_t|^{s-1}|l_t(x)|^s \geqq (s_n|J_k|)^{s-1} 3^{s\sqrt{n}} > \ln^2 n,$$

i.e. (2.6) is true. So we obtain that the estimation (2.6) is true for the long intervals not considering a set H_{1n} of measure $\leqq 8/\ln^2 n$.

3.3. To settle the short intervals we introduce the following notations:

$$\begin{cases} J_k(q_k) = J_{kn}(q(J_{kn})) = [x_{k+1}+q_k|J_k|, x_k-q_k|J_k|], \\ \overline{J_k} = \overline{J_k(q_k)} = \overline{J_{kn}(q(J_{kn}))} = J_k \setminus J_k(q_k), \quad 0 \leqq k \leqq n, \end{cases} \tag{3.2}$$

where $0 \leqq q_k \leqq 1/2$. Let $z_k = z_k(q_k)$ be defined by

$$|\omega_n(z_k)| = \min_{x \in J_k(q_k)} |\omega_n(x)|, \quad k = 0, 1, \dots, n, \tag{3.3}$$

finally let

$$|J_i, J_k| = \max(|x_{i+1}-x_k|, |x_{k+1}-x_i|), \quad 0 \leqq i, k \leqq n, \tag{3.4}$$

$$\varrho(J_i, J_k) = \min(|x_{i+1}-x_k|, |x_{k+1}-x_i|), \quad 0 \leqq i, k \leqq n. \tag{3.5}$$

We state (see [5], Lemma 3.2)

LEMMA 3.2. *If* $1 \leqq k, r \leqq n$ *and* $q_k > 0$ *then*

$$|L_k(x)| + |L_{k+1}(x)| > \frac{1}{2 \cdot 4^s} \left| \frac{\omega_n(z_r)}{\omega_n(z_k)} \right|^s \frac{|\bar{J}_k|^s}{|J_r, J_k|}, \quad n \geqq 6 \tag{3.6}$$

if $x \in J_r(q_r)$, $\varrho(J_r, J_k) \geqq \delta_n$ *and* $|J_r| \leqq \delta_n$, *where we used the notation* $L_{kn}(s)(x) = L_k(x) = (x - x_k)^{s-1} l_k^s(x)$.

First we remark that

$$|L_i(x)| \geqq \frac{1}{2} |L_i(z_r)| \quad \text{if} \quad x \in J_r(q_r), \quad i = k, k+1. \tag{3.7}$$

Indeed, by definition

$$|L_i(x)| = \left| \frac{\omega(x)}{\omega(z_r)} \right|^s \left| \frac{z_r - x_i}{x - x_i} \right| |L_i(z_r)|.$$

Considering that $|\omega(x)| \geqq |\omega(z_r)|$ and

$$\left| \frac{z_r - x_i}{x - x_i} \right| \geqq \frac{|z_r - x_i| + \delta_n - \delta_n}{|x_r - x_i| + \delta_n} \geqq 1 - \frac{\delta_n}{2\delta_n} = 1/2,$$

we obtain (3.7). So

$$|L_k(x)| + |L_{k+1}(x)| \geqq \frac{1}{2} [|L_k(z_r)| + |L_{k+1}(z_r)|] =$$

$$= \frac{1}{2} \left| \frac{\omega(z_r)}{\omega(z_k)} \right|^s \left[|L_k(z_k)| \left| \frac{x_k - z_k}{z_r - x_k} \right| + |L_{k+1}(z_k)| \left| \frac{x_{k+1} - z_k}{z_r - x_{k+1}} \right| \right] =$$

$$= \frac{1}{2} \left| \frac{\omega(z_r)}{\omega(z_k)} \right|^s \left[\frac{|x_k - z_k|^s}{|z_r - x_k|} |l_k^s(z_k)| + \frac{|x_{k+1} - z_k|^s}{|z_r - x_{k+1}|} |l_{k+1}^s(z_k)| \right] \geqq$$

$$\geqq \frac{q_k^s}{2} \left| \frac{\omega(z_r)}{\omega(z_k)} \right|^s \frac{|J_k|^s}{|J_r, J_k|} [|l_k^s(z_k)| + |l_{k-1}^s(z_k)|,$$

from where we obtain (3.6) using that $[\ldots] > 2^{-s}$ because of $l_k(x) + l_{k+1}(x) > 1$ $\big(x \in J_k(q_k)\big)$.

3.4. Now we define $q(J_{kn})$ for the short intervals. For this aim let us determine the index set K_n' and the set D_n' by $|J_{kn}| \leqq \delta_n$ if $k \in K_n'$, $|J_{kn}| > \delta_n$ if $k \notin K_n'$, $D_n' = \bigcup_{k \in K_n'} J_{kn}$. If y_k denotes the middle point of J_k, let $k \in K_n \overset{\text{def}}{=} K_n' \setminus \{0, n\}$, further

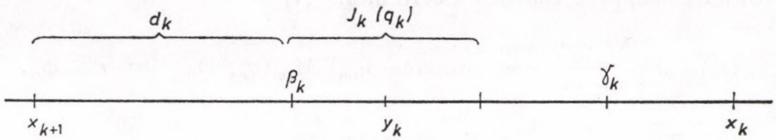
$$\beta_{kn} = \max\{y\colon\ x_{k+1} \leqq y \leqq y_k \text{ and (2.6) does not hold for } y\},$$

$$\gamma_{kn} = \min\{y\colon\ y_k \leqq y \leqq x_k \text{ and (2.6) does not hold for } y\},$$

$$d_{kn} = \max(x_k - \gamma_k,\ \beta_k - x_{k+1}),$$

$$q_k = q(J_{kn}) = d_k/|J_k|. \tag{3.8}$$

Let us remark that (2.6) holds whenever x belongs to the interior of $J_k(q_k)$.



Supposing that

$$\varepsilon_n > \frac{1008}{\ln n} \quad \text{and} \quad n \geqq 6, \tag{3.9}$$

we get that $q_k > 0$ (because $\lambda_n(s)(x_k)=0$ if $s \geqq 2$ and $\lambda_n(1)(x_k)=1$, $1 \leqq k \leqq n$). By (3.9) we intend to estimate the measure of the "bad" set $\bigcup_{k\in K_n} |\overline{J_{kn}}|$ (i.e. where (2.6) does not hold). Let us denote this measure by ν_n. Omitting those short intervals for which $|\bar{J}_k| \leqq \frac{\nu_n}{4n}$ we shall prove for the remaining ones

$$\sum_{\substack{k\in K_n \\ |J_k| > \frac{\nu_n}{4n}}} |\bar{J}_{kn}| \overset{\text{def}}{=} \mu_n \leqq \frac{\varepsilon_n}{2} \quad \text{if} \quad n \geqq n_2. \tag{3.10}$$

Obviously $3\nu_n/4 \leqq \mu_n \leqq \nu_n$ so if (3.10) holds, then

$$\sum_{k\in K_n} |\bar{J}_k| \leqq \frac{2}{3}\varepsilon_n \quad \text{if} \quad n \geqq n_2. \tag{3.11}$$

3.5. Denoting by φ_n the number of the terms in (3.10), we can reorder the corresponding intervals J_k such that for the reordered intervals M_k

$$\sum_{k=r}^{\varphi_n}{}' \frac{|\overline{M}_k|}{|M_r, M_k|} \geqq \frac{\mu_n \ln n}{224}, \quad 1 \leqq r \leqq \psi_n, \quad n \in N_1 \tag{3.12}$$

if $\varepsilon_n > 1008/\ln n$ which we supposed. Here ψ_n is defined by

$$\sum_{i=1}^{\psi_n - 1} |\overline{M}_i| \leqq \frac{\mu_n}{2} \quad \text{but} \quad \sum_{i=1}^{\psi_n} |\overline{M}_i| > \frac{\mu_n}{2} \quad (n \in N_1), \tag{3.13}$$

N_1 is a suitable sequence; the dash indicates that we omit those indices k for which $\varrho(M_r, M_k) < \delta_n$ (see [5] 3.4 and 3.5, especially (3.19)).

3.6. By the definition of q_k we can choose points $u_{in} \in M_{in}(q_{in}/2)$ for which (2.6) does not hold ($1 \leqq t \leqq \varphi_n$, $n \in N_1$).

If for a fixed $n \in N_1$ there exists an index t ($1 \leqq t \leqq \varphi_n$) such that

$$\lambda_n(s)(u_{tn}) \geqq \frac{2^s c \mu_n^s \ln n}{n^{s-1}} \tag{3.14}$$

(where $c > 0$ will be determined later), by $c\varepsilon_n^s \ln n/n^{s-1} \geqq \lambda_n(s)(u_{tn})$ we obtain (3.10) for this n. *We prove* (3.14) *for arbitrary* $n \in N_1$ *with suitable* $t = t(n)$.

Indeed, let us suppose that for a certain $m \in N_1$

$$\lambda_m(s)(u_{tm}) < \frac{2^s c\mu_m^s \ln m}{m^{s-1}} \quad \text{where} \quad u_{rm} \in M_{rm}(q_{rm}/2), \quad 1 \leqq r \leqq \varphi_m. \tag{3.15}$$

Then

$$\sum_{r=1}^{\varphi_m} |M_{rm}| \lambda_m(s)(u_{rm}) < \frac{2^s c\mu_m^{s+1} \ln m}{m^{s-1}} \quad \text{where} \quad m \in N_1. \tag{3.16}$$

On the other hand, for arbitrary $n \in N_1$ we can write by (3.6) with $\bar{z}_k$ corresponding to (3.3)

$$|\overline{M}_r| \sum_{k=1}^{n} |L_k(u_r)| \geqq \frac{1}{2} |\overline{M}_r| \sum_{\substack{k \in K_n \\ |J_k| > \frac{v_n}{4n}}}' [|L_k(u_r)| + |L_{k+1}(u_r)|] =$$

$$= \frac{|\overline{M}_r|}{4^{s+1}} \sum_{k=1}^{\varphi_n}{}' \left| \frac{\omega(\bar{z}_r)}{\omega(\bar{z}_k)} \right|^s \frac{|\overline{M}_k|^s}{|M_r, M_k|} > \frac{\mu_n^{s-1}}{16^s n^{s-1}} \sum_{k=1}^{\varphi_n}{}' \left| \frac{\omega(\bar{z}_r)}{\omega(\bar{z}_k)} \right|^s \frac{|\overline{M}_r| |\overline{M}_k|}{|M_r, M_k|},$$

using that $v_n \geqq \mu_n$. So by (3.12) and (3.13)

$$\sum_{r=1}^{\varphi_n} |\overline{M}_r| \lambda_n(s)(u_r) > \frac{\mu_n^{s-1}}{16^s n^{s-1}} \sum_{r=1}^{\varphi_n} \sum_{k=1}^{\varphi_n}{}' \left| \frac{\omega(\bar{z}_r)}{\omega(\bar{z}_k)} \right|^s \frac{|\overline{M}_r| |\overline{M}_k|}{|M_r, M_k|} = \tag{3.17}$$

$$= \frac{\mu_n^{s-1}}{16^s n^{s-1}} \sum_{r=1}^{\varphi_n} \sum_{k=r}^{\varphi_n}{}' \left[\left| \frac{\omega(\bar{z}_r)}{\omega(\bar{z}_k)} \right|^s + \left| \frac{\omega(\bar{z}_k)}{\omega(\bar{z}_r)} \right|^s \right] \frac{|\overline{M}_r| |\overline{M}_k|}{|M_r, M_k|} \geqq$$

$$\geqq \frac{\mu_n^{s-1}}{16^s n^{s-1}} \sum_{r=1}^{\psi_n} |\overline{M}_r| \sum_{k=r}^{\varphi_n}{}' \frac{|\overline{M}_k|}{|M_r, M_k|} > \frac{\mu_n^{s+1} \ln n}{2 \cdot 224 \cdot 16^s n^{s-1}} = \frac{2^s c\mu_n^{s+1} \ln n}{n^{s-1}}, \quad n \in N_1$$

if $c = (448 \cdot 32^s)^{-1}$. This contradicts (3.16) i.e. (3.14) is true for any $n \in N_1$ with suitable $t = t(n)$. So we proved (3.10) and (3.11).

3.7. To estimate $|H_n|$ we write by (3.7), (3.2) and (3.11)

$$|H_n| \leqq |H_{1n}| + \frac{2}{3} \varepsilon_n + 2\delta_n \leqq \varepsilon_n \quad \text{if} \quad n \geqq n_0 \overset{\text{def}}{=} \max(6, n_1, n_2)$$

where $2\delta_n$ stands for the measure of J_0 and J_n whenever they are short.

3.8. Now let

$$0 < \varepsilon_n < \frac{1008}{\ln n}. \tag{3.18}$$

Let $q_{kn} = q_n = \varepsilon_n [2(n+1)]^{-1}$, $0 \leqq k \leqq n$, and let us omit the set $H_n \overset{\text{def}}{=} \bigcup_{k=0}^{n} \bar{J}_k$ of measure ε_n. If $x \in [-1, 1] \setminus H_n$ and $x \in [x_{j+1}, x_j]$ $(j = 1, 2, \ldots, n-1)$, by (3.18) we can write

$$\lambda_n(s)(x) > |x - x_j|^{s-1} |l_j^s(x)| + |x - x_{j+1}|^{s-1} |l_{j+1}^s(x)| \geqq$$

$$\geqq \left[\frac{\varepsilon_n}{2(n+1)} \right]^{s-1} [|l_j^s(x)| + |l_{j+1}^s(x)|] \geqq \frac{\varepsilon_n^{s-1}}{4^s n^{s-1}} > \frac{c\varepsilon_n^s \ln n}{n^{s-1}}, \quad \text{if} \quad n \geqq 6.$$

If e.g. $|J_0|>0$ and $x\in J_0(q_n)$, we use that $|l_j(x)|>1$. So the proof is complete whenever $n\geqq n_0$.

Now let $n\leqq n_0$. Using the same q_n and H_n as above we can write by $\lambda_n(x)\geqq 1$ $(n=1, 2, \ldots)$ and $0<\varepsilon_n\leqq 2$

$$\lambda_n(s)(x) = \sum_{k=1}^{n} |x-x_k|^{s-1}|l_k(x)|^s \geqq \left[\frac{\varepsilon_n}{2(n+1)}\right]^{s-1}\frac{1}{n^s} \geqq \frac{\varepsilon_n^s \ln n}{n^{s-1}}\,\frac{1}{2n_0[2(n_0+1)]^{s-1}\ln n_0}$$

if $1\leqq n\leqq n_0$, $x\in[-1, 1]\setminus H_n$; from where we obtain the theorem with perhaps another constant c.

3.9. Proof of teorem 2.3. Not only the theorem but also the proof is analogous to the real case. Let now

(3.19) $J_{kn} = [\theta_{kn}, \theta_{k+1\, n}]$, $(k = 0, 1, \ldots, n;\ n = 1, 2, \ldots)$ with $\theta_0 = \theta_{n+1} = 0$.

Again, if $|J_k|\leqq\delta_n \overset{\text{def}}{=} n^{-1/6}$ we say that J_k is short; the others are the long ones.

3.10. First we state

Lemma 3.3. *Let* $|J_{kn}|>\delta_n$ *for a certain* k $(0<k<n)$. *Then for any* $(\ln n)^{-2}\leqq$ $\leqq s_n\leqq 1/4$ *we can define the index* $t=t(k,n)$ *and the set* $h_{kn}\subset J_{kn}$ *so that* $|h_{kn}|\leqq$ $\leqq 4s_n|J_{kn}|$, *moreover*

$$|l_{tn}(e^{i\theta})| \geqq 3^{\sqrt{n}} \quad \textit{if} \quad \theta\in J_{kn}\setminus h_{kn} \quad \textit{and} \quad n\geqq n_1. \tag{3.20}$$

Here n_1 is an absolute constant; if $\theta\in J_{kn}/h_{kn}$, *then* $\min(|\theta-\theta_k|^*, |\theta-\theta_{k+1}|^*)\geqq s_n|J_{kn}|$. *Here and later* $|\alpha|^*=\min(|\alpha|, 2\pi-|\alpha|)$, $0\leqq|\alpha|\leqq 2\pi$.

This lemma can be proved by the method of [10], 4.3.

If $s_n=1/\ln^2 n$, the argument is similar to 3.2. We obtain that (2.12) holds for the long intervals not considering the sets H_{1n} of measure $\leqq 8\pi/1n^2 n$.

3.11. Let

$$\omega_n(\theta) \overset{\text{def}}{=} c_n \prod_{k=1}^{n} \sin\frac{\theta-\theta_k}{2}, \quad c_n\neq 0. \tag{3.21}$$

Then

$$l_{kn}(e^{i\theta}) = \exp\left[i(n-1)\frac{\theta-\theta_k}{2}\right]\frac{\omega(\theta)}{2\omega'(\theta_k)\sin\dfrac{\theta-\theta_k}{2}}, \quad 0\leqq\theta\leqq 2\pi. \tag{3.22}$$

(see e.g. [10], (4.5)). Now using mutatis mutandis the notations (3.2), we state by

$$|\omega_n(\xi_k)| = \min_{\theta\in J_k(q_k)} |\omega_n(\theta)|, \quad k = 0, 1, \ldots, n, \tag{3.23}$$

and

$$|J_i, J_k| = \max(|\theta_{i+1}-\theta_k|^*, \quad |\theta_{k+1}-\theta_i|^*), \quad 0\leqq i, \quad k\leqq n, \tag{3.24}$$

$$\varrho(J_i, J_k) = \min(|\theta_{i+1}-\theta_k|^*, \ |\theta_{k+1}-\theta_i|^*), \quad 0\leqq i, \quad k\leqq n, \tag{3.25}$$

as follows.

LEMMA 3.4. *If* $1 \leqq k, r \leqq n$ *and* $q_k > 0$ *then*

$$(3.25)\qquad |L_k(e^{i\theta})| + |L_{k+1}(e^{i\theta})| > \frac{1}{2(4\pi)^s} \left| \frac{\omega_n(\xi_r)}{\omega_n(\xi_k)} \right|^s \frac{|\bar{J}_k|}{|J_r, J_k|}, \quad n \geqq n_0,$$

whenever

$$\theta \in J_r(q_r), \quad \varrho(J_r, J_k) \geqq \delta_n \quad and \quad |J_r| \leqq \delta_n,$$

where we used the notation $L_{kn}(s)(z) = L_k(z) = (z - z_k)^{s-1} l_k^s(z)$.

Indeed, using the argument of [5], Lemma 3.2, we can write

$$(3.26)\qquad |L_t(e^{i\theta})| \geqq \frac{1}{4} |L_t(e^{i\xi_r})| \quad \text{if} \quad t = k, \quad k+1; \; \theta \in J_r(q_r) \quad \text{and} \quad n \geqq n_0.$$

(Indeed, $|L_t(e^{i\theta})| = \left| \dfrac{\omega(\theta)}{\omega(\xi_r)} \right|^s \left| \dfrac{\sin \frac{\xi_r - \theta_t}{2}}{\sin \frac{\theta - \theta_t}{2}} \right| |L_t(e^{i\xi_r})|$, where

$$\frac{\sin \left| \frac{\xi_r - \theta_t}{2} \right|}{\sin \frac{|\theta - \theta_t|}{2}} = \frac{\sin \frac{|\xi_r - \theta_t|^*}{2}}{\sin \frac{|\theta - \theta_t|^*}{2}} > \frac{\sin \left(\frac{|\xi_r - \theta_t|^*}{2} + \frac{\delta_n}{2} - \frac{\delta_n}{2} \right)}{2 \sin \left(\frac{|\xi_r - \theta_t|^*}{2} + \frac{\delta_n}{2} \right)} \geqq \frac{1}{4}$$

if $n \geqq n_0$, from where we get (3.26).)

The remaining parts are analogous to **3.3**, considering that $|l_k(e^{i\theta})| + |l_{k+1}(e^{i\theta})| \geqq 1$ if $\theta \in J_k$ (see [10], Lemma 4.2.1).

3.12. The remaining parts are analogous to **3.4—3.8**. We mention that instead of (3.12) we obtain $\mu_n \ln n (224\pi)^{-1}$ on the right hand-side. We omit further details.

3.13. PROOF OF THEOREM 2.4. The proof is based on the relation

$$l_{kn}(\Theta, \theta) = \frac{\omega(\theta)}{2\omega'(\theta_{kn}) \sin \frac{\theta - \theta_{kn}}{2}}$$

where $\omega(\theta)$ is defined by (3.2). The remaining parts are analogous to the complex case. We omit the details.

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ON AN IMPROVED RATE OF CONVERGENCE TO NORMALITY FOR SUMS OF DEPENDENT RANDOM VARIABLES WITH APPLICATIONS TO STOCHASTIC APPROXIMATION

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1. Introduction

The error term in the central limit theorem for sums of martingale differences has been considered by Basu [1], Erickson, Quine and Weber [3], and Heyde and Brown [4] In each case the only moment assumptions were that the sum of the conditional variances converges to 1 (or some other constant >0) in probability, almost surely or in L_1, and that the sum of the absolute $(2+\delta)$-order moments converges for some $\delta>0$. In each case the error rate, when specialized to i.i.d. random variables with third moments, becomes $O(n^{-1/8})$ [Although this rate is only $O(n^{-1/12})$ according to Theorem 2 of Basu [1] a minor change in the choice of the upper limit of the integral near the end of the proof makes it $O(n^{-1/8})$]. Of course, because of the methods used, this specialization to the i.i.d. case only amounts to the assumption of identical and almost surely constant conditional second and third absolute moments of the martingale differences. The aim of this paper is to investigate by how much (if any) the rate of convergence can be improved by assuming what Dvoretzky [2] calls "near constancy" assumption about the conditionnal variances. Theorem (2.1) shows that even under the weakest assumptions (see [3]) an improved rate can be found implicitly as an easy consequence of Lemma 3.2 of Dvoretzky [2]. However, it is not obvious how to use the theorem to get a better convergence rate than that obtainable by using Theorem 2.3 of [3]. We illustrate this with some applications to stochastic approximation. It is shown that a convergence rate of $O(n^{-1/4})$ can be obtained in some cases including the case of i.i.d. random variables with third moments. We have contented ourselves with order of magnitude calculations only since it is almost impossible to give one single expression which is the best (or nearly so) explicit bound in all cases.

2. Notation and results

Our procedure and assumptions are similar to those of Erickson et al [3] who have considered the problem under the weakest assumptions.

Let $X=\{X_{nk},\ k=1, \dots, n;\ n=1, 2, \dots\}$ be an array of random variables. Put $S_{nk}=\sum_{j=1}^{k} X_{nj}$, $S_n=S_{nn}$. Let $F=\{F_{nk},\ k=0, 1, \dots, n;\ n=1, 2, \dots\}$ be an array of σ-fields, where F_{n0} is the trivial σ-field. Assume X_{nk} is F_{nk} measurable and $F_{nk}\subset F_{n,k+1}$. Define the conditional expectation operators $E_{nk}(\cdot)=E(\cdot\,|F_{nk})$ and write

$$\mu_{nk} = E_{n,k-1}X_{nk},\ \sigma^2_{nk} = E_{n,k-1}X^2_{nk}-\mu^2_{nk},\ s^2_{nk} = \sum_{j=1}^{k}\sigma^2_{nj},\ \sigma^2_n = s^2_{nn},\ \gamma^{2+\delta}_{nk} = = E_{n,k-1}|X_{nk}-\mu_{nk}|^{2+\delta}$$

for $\delta>0$.

Let N be a $N(0, 1)$ random variable. Let $\Delta(X, Y)$ denote the Kolmogorov (supnorm) distance between the distribution functions of the random variables X and Y.

Throughout we use C, with or without subscripts, as generic positive constants, independent of "time variables" k, n, etc. Thus we may have $2C \leqq C$. Equalities (inequalities) among random variables are almost sure equalities (inequalities).

(2.1) THEOREM. *Given (X, F) above, if $\mu_{nk}=0$ for all n, k, then for $\delta \in (0, 1]$ and $T>0$*

$$\Delta(S_n, N) \leqq C_1 \int_0^T \sum_{k=1}^n t^2 E[e^{-(t^2/2)(1-s_{nk}^2)\wedge 0} \gamma_{nk}^{2+\delta}]\, dt + C_2 T^{-1} +$$

$$+ C_3 \{E|\sigma_n^2 - 1|^{1+\delta/2} + \sum_{k=1}^n E[\gamma_{nk}^{2+\delta} I(s_{nk}^2 > 1)]\}^{1/(3+\delta)}.$$

REMARKS. One minimizes the right hand side with respect to T to obtain the best convergence rate. The possible improvement to Theorem 2.3 of [3] comes from the usage of the exponential expressions in the first term and of the indicator functions in the last. Majorizing all these expressions by 1 yields Theorem 2.3 of [3].

PROOF OF THEOREM (2.1). We first state and prove a lemma which gives an improved bound for the difference of the characteristic functions of S_n and N.

LEMMA. *If the assumption $\sigma_n^2=1$ is added to the conditions of Theorem (2.1) then*

$$|Ee^{itS_n} - e^{-t^2/2}| \leqq Ct^2 \sum_{k=1}^n E[e^{-(t^2/2)(1-s_{nk}^2)} \gamma_{nk}^{2+\delta}]$$

for all real t.

PROOF OF LEMMA. Let $Y_{n1}, \dots, Y_{nn}$ be completely independent standard normal random variables and independent of F_{nn}. Let $Z_{nk} = S_{n,k-1} + R_{n,k+1}$ for $1 \leqq k \leqq n$, where $R_{nk} = \sum_{j=k}^n \sigma_{nj} Y_{nj}$ and $S_{n0} = R_{n,n+1} = 0$. Then

$$S_{nk} = \sum_{k=1}^n (Z_{nk} + X_{nk}) - \sum_{k=1}^n (Z_{nk} + \sigma_{nk} Y_{nk}).$$

Using Lemma 3.2 of Dvoretzky [2]

$$E_{nk} e^{itR_{n,k+1}} = e^{-(t^2/2)} \sum_{j=k+1}^n \sigma_{nj}^2 = e^{-(t^2/2)(1-s_{nk}^2)},$$

which is $F_{n,k-1}$ measurable. Thus

$$|Ee^{itS_n} - e^{-t^2/2}| = \left| \sum_{k=1}^n E\{e^{itS_{n,k-1}} E_{n,k-1}[(e^{itX_{nk}} - e^{it\sigma_{nk} Y_{nk}}) E_{nk} e^{itR_{n,k+1}}]\} \right| =$$

$$= \left| \sum_{k=1}^n E\{e^{itS_{n,k-1}} e^{-(t^2/2)(1-s_{nk}^2)} E_{n,k-1}(e^{itX_{nk}} - e^{it\sigma_{nk} Y_{nk}})\} \right| \leqq Ct^2 \sum_{k=1}^n E[e^{-(t^2/2)(1-s_{nk}^2)} \gamma_{nk}^{2+\delta}],$$

the proof of the last step being the same as in [3].

The rest of the proof Theorem (2.1) is identical to that of Theorem (2.3) of [3] with the following two modified estimates:

1) Defining $\tau_n = \max\{0 \leqq k \leqq n: s_{nk}^2 \leqq 1\}$, $\alpha_n^2 = S_{n,\tau_n}^2$ and N_n as a standard normal random variable independent of all other random variables, we use the lemma above and Esseen's theorem (see e.g. Loéve [6], p. 285) to get

$$\Delta(S_{n,\tau_n} + (1-\alpha_n^2)^{1/2} N_n, N) \leqq C_1 \int_0^T t^2 E\left[\sum_{k=1}^{\tau_n} e^{-(t^2/2)(1-s_{nk}^2)} \gamma_{nk}^{2+\delta}\right] dt + C_2 T^{-1} \leqq$$

$$\leqq C_1 \int_0^T t^2 \sum_{k=1}^{n} E[e^{-(t^2/2)(1-s_{nk}^2)\wedge 0} \gamma_{nk}^{2+\delta}]\, dt + C_2 T^{-1}$$

for all $T > 0$. (Our S_{n,τ_n} is the same as $\sum_{k=1}^{n} W_{nk}$ in [3].) It should be clear from the proof of our lemma that the additional term $t^2 E(1-\alpha_n^2)^{1+\delta/2}$ in the integrand used in [3] is not necessary.

2) We use $E[(1-\alpha_n^2)^{1+\delta/2} I(\tau_n < n)] \leqq \sum_{k=1}^{n} E[\gamma_{nk}^{2+\delta} I(s_{nk}^2 > 1)]$ instead of the corresponding bound in [3] without the indicator functions.

It should be pointed out that there are several minor mistakes and misprints in the proof of Theorem 2.3 in [3]. The assumption $\sigma_n^2 = 1$ should be added to the first part of the proof; different symbols should be used for the (almost surely) bounded random variable $|\theta|$ and its bound; in the string of inequalities (3.6) although the third term is less than or equal to the first, it is not necessarily less than or equal to the second; the exponent $1+1/2\delta$ should be read as $1+\delta/2$.

3. Applications

A large and interesting area of application of Theorem (2.1) is where $\{Z_k\}$ is a martingale difference sequence adapted to the σ-fields $\{F_{nk} = F_k\}$ where F_k is the σ-field generated by $\{Z_1, \ldots, Z_k\}$, $\{b_{nk}\}$ is a fixed or an adaptive sequence, and $X_{nk} = b_{nk} Z_k$. Thus S_n represents a weighted average of the Z_k's with possibly random adaptive weights. The area of stochastic approximation offers a large variety of such cases. It is an active with a vast literature. To illustrate how Theorem (2.1) can be used we consider a Robbins—Monro process for which asymptotic normality has been proven, among others, by Sacks [7]. One should consult this reference for the proofs of results cited.

Robbins—Monro process. Suppose F_x is a distribution function for each real x with mean $m(x)$ which is finite, variance $\sigma^2(x)$ and absolute $(2+\delta)$-order central moment $\gamma^{2+\delta}(x)$ for $\delta > 0$. The Robbins—Monro procedure finds the root of the regression function $m(\cdot)$ sequentially by unbiased sampling from $\{F_x\}$ at appropriate (known) x's. Assume

(3.1) $m(x) = x - \theta, \quad -\infty < x < \infty,$

(3.2) $\sigma^2(x) \leqq \sigma^2$ and $\gamma^{2+}(x) \leqq \gamma^{2+\delta}$ for all x with finite σ^2 and $\gamma^{2+\delta}$ for some $\delta \in (0, 1]$, (3.3) $\sigma^2(\cdot)$ and $\gamma^{2+\delta}(\cdot)$ are Borel measurable functions with δ as in (3.2), and

(3.4) $\sigma^2(x)=\sigma^2(\theta)[1+O(|x-\theta|^d)]$ as $x\to\theta$ for some $d\in(0, 2]$, and $\sigma^2(\theta)>0$.

Let $X_1=\theta$ and define recursively $X_{n+1}=X_n-a_n Y_n$, $n>1$, where $a_n=a/n$ for some $a>1/2$, and the conditional distribution function of Y_n given $\{X_1, \dots, X_n; Y_1, \dots, Y_{n-1}\}$ is F_{X_n}. For this procedure it is known [7] that under assumptions (3.1)—(3.4)

(3.5) $X_n\to\theta$ a.s.,

(3.6) $E(X_n-\theta)^2=O(n^{-1})$, and

(3.7) $\sqrt{n}(X_{n+1}-\theta)\to N(0, a^2\sigma^2(\theta)/(2a-1))$ in distribution.

For notational simplicity we assume $\theta=0$ and $\sigma^2(\theta)=(2a-1)/a^2$. Thus the variance of the limiting distribution in (3.7) is 1.

Define $\varepsilon=a-1/2$.

(3.8) THEOREM. *Under the assumptions* (3.1)—(3.4) *with* $\delta=1$ *and* $d=2$

(i)
$$\Delta(\sqrt{n}\, X_{n+1}, N) = \begin{cases} O(n^{-\varepsilon/2}) & \text{if } \varepsilon < 1/2, \\ O((n/\log n)^{-1/4}) & \text{if } \varepsilon = 1/2, \quad \text{and} \\ O(n^{-1/4}) & \text{if } \varepsilon > 1/2. \end{cases}$$

(ii) *Under the additional assumption* $\sigma^2(x)=\sigma^2(0)$ *for all* x *and* $\varepsilon=1/2$

$$\Delta(\sqrt{n}\, X_{n+1}, N) = O(n^{1/4}).$$

REMARKS. 1) Sacks [7] proved asymptotic normality of $\sqrt{n}X_{n+1}$ under the assumption (among others) $\sigma^2(x)\to\sigma^2(0)>0$ as $x\to0$. We have needed (3.4) to get a rate of convergence.

2) By starting the process at 0 and assuming strict linearity of $m(\cdot)$ we have removed the bias term in $\sqrt{n}X_{n+1}$ due to the initial bias and non-linearity, respectively [7]. This way we have isolated that part of the convergence rate problem which is in the realm of Theorem (2.1). We have also assumed $\delta=1$ in (3.2) and $d=2$ in (3.4) in order to reduce the number of parameters (one has to consider the convergence rates in more than 3^3 subsets of the $\varepsilon-\delta-d$ parameter space in general) and for ease of comparisons.

Using our methods it is possible to compute the convergence rates for a more general process where one may consider $m(\cdot)$ to be quasi-linear with some finite positive derivative α at 0, arbitrary $\delta\in(0, 1]$ and $d\in(0, 2]$, X_1 to be a random variable with finite second moment, $a_n=A_n/n^\lambda$ for some $1/2<\lambda\leqq1$ and A_n an adaptive random variable converging almost surely to $1/\alpha$ where $\alpha=m'(0)$ (see e.g. Venter [8]), etc. However, the computations are exceedingly long and tedious.

PROOF OF THEOREM (3.8). For $k\geqq1$ define $Z_k=Y_k-m(X_k)=Y_k-X_k$. Then the $\{Z_k\}$ form a martingale difference sequence adapted to $\{F_{nk}=F_k=\sigma(Z_1, \dots, Z_k)\}$. Iterating $X_{n+1}=X_n-a_n(X_n+Z_n)$ we have $X_{n+1}=\sum_{k=1}^{n} a_k\beta_{nk}Z_k$, where $\beta_{nk}=\prod_{j=k+1}^{n}(1-a_j)$ for $0\leqq k<$n and $\beta_{nn}=1$. If we identify $n^{1/2}a_k\beta_{nk}Z_k$ with X_{nk} and $\sigma(Z_1, \dots, Z_k)$ with F_{nk} then Theorem (2.1) applies to this (X, F) with $\delta=1$.

We note that

$$\sum_{k=1}^{n} E[e^{-(t^2/2)(1-s_{nk}^2)\wedge 0}\gamma_{nk}^3] \leqq \sum_{k<n-\sqrt{n}} e^{-(t^2/2)[1-(k/n)^\varepsilon]}E(\gamma_{nk}^3)+$$

$$+\sum_{k<n-\sqrt{n}} E[\gamma_{nk}^3 I(s_{nk}^2 > (k/n)^\varepsilon)]+\sum_{k\geqq n-\sqrt{n}} E(\gamma_{nk}^3).$$

(For other problems instead of $\sqrt{n}$ some other $o(n)$ — term may be appropriate.) Using this in Theorem (2.1) we have

(3.9) $$\Delta(\sqrt{n}X_{n+1}, N)\leqq$$

(E1) $$C_1 \sum_{k<n-\sqrt{n}} \int_0^T t^2 e^{-(t^2/2)[1-(k/n)^\varepsilon]}E(\gamma_{nk}^3)\,dt+$$

(E2) $$C_2 T^3 \sum_{k<n-\sqrt{n}} E[\gamma_{nk}^2 I(s_{nk}^2 > (k/n)^\varepsilon)]+$$

(E3) $$C_3 T^3 \sum_{k\geqq n-\sqrt{n}} E(\gamma_{nk}^3)+$$

(E4) $$C_4 T^{-1}+$$

(E5) $$C_5\{E|\sigma_n^2-1|^{3/2}\}^{1/4}+$$

(E6) $$C_6\left\{\sum_{k=1}^{n} E[\gamma_{nk}^3 I(s_{nk}^2 > 1)]\right\}^{1/4},$$

where we have used $|a+b|^r\leqq|a|^r+|b|^r$ for $r\leqq 1$ to separate the last two terms. We will estimate the expressions (E1)—(E6) for various ranges of the parameter ε.

From the description of the process and assumption (3.2)

$$\sigma_{nk}^2 = na_k^2\beta_{nk}^2\sigma^2(X_k) = na_k^2\beta_{nk}^2\sigma^2(0)[1+O(X_k^2)]$$

and $\gamma_{nk}^3\leqq n^{3/2}a_k^3|\beta_{nk}|^3\gamma^3$. (Here $O(X_k^2)$ is a random variable such that $O(X_k^2)/X_k^2$ is almost surely bounded uniformly in k on $\{X_k\neq 0\}$. Sequences of the form $O(k^{-p})$ in "time variables" are always meant to be $O(k^{-p})$ as $k\to\infty$ uniformly in all other "time variables".) Using the well-known estimate $\beta_{nk}=(k/n)^a[1+O(k^{-1})]$ (as can be seen by logarithmic expansion of β_{nk}) and (3.6) we have

$$s_{nk}^2 = n\sum_{j=1}^{k} a_j^2\beta_{nj}^2\sigma^2(X_j) = n^{-2\varepsilon}\sum_{j=1}^{k} a^2 j^{2a-2}[1+O(j^{-1})]\sigma^2(0)[1+O(X_j^2)] =$$

$$= (k/n)^{2\varepsilon}[1+O(k^{-(2\varepsilon\wedge 1)})]+n^{-2\varepsilon}\sum_{j=1}^{k} j^{2\varepsilon-1}[O(j^{-1})+O(X_j^2)];$$

(3.10) $$E|s_{nk}^2-(k/n)^{2\varepsilon}| = (k/n)^{2\varepsilon}O(k)^{-(2\varepsilon\wedge 1)}+n^{-2\varepsilon}\sum_{j=1}^{k} j^{2\varepsilon-1}O(j^{-1}) =$$

$$= \begin{cases} O(n^{-2\varepsilon}) & \text{if } \varepsilon < 1/2,\\ O(n^{-1}\log k) & \text{if } \varepsilon = 1/2,\\ O((k/n)^{2\varepsilon}k^{-1}) & \text{if } \varepsilon > 1/2;\end{cases}$$

(3.11) $$P\{s_{nk}^2 > (k/n)^\varepsilon\} \leqq P\{|s_{nk}^2-(k/n)^{2\varepsilon}| > (k/n)^\varepsilon[1-(k/n)^\varepsilon]\} \leqq$$

$$\leqq E|s_{nk}^2-(k/n)^{2\varepsilon}|/(k/n)^\varepsilon[1-(k/n)^\varepsilon],$$

and

$$\gamma_{nk}^3 \leqq Cn^{3/2}k^3(k/n)^{3a} = Cn^{-3\varepsilon}k^{3\varepsilon-3/2}. \tag{3.12}$$

In (E2)—E(4) we will choose $T \leqq n^{1/4}$ for all values of ε. Thus, using (3.12) and $T \leqq n^{1/4}$,

$$\text{(E1)} \qquad \leqq Cn^{-3\varepsilon} \sum_{k<n-\sqrt{n}} k^{3\varepsilon-3/2} \int_0^{n^{1/4}} t^2 e^{-(t^2/2)[1-(k/n)^\varepsilon]}\, dt =$$

$$= Cn^{-3\varepsilon} \sum_{k<n-\sqrt{n}} k^{3\varepsilon-3/2}[1-(k/n)^\varepsilon]^{-3/2} \int_0^{n^{1/4}[1-(k/n)^\varepsilon]^{1/2}} z^2 e^{-z^2/2}\, dz \leqq$$

$$\leqq Cn^{-3/2} \sum_{k<n-\sqrt{n}} (k/n)^{3\varepsilon-3/2}[1-(k/n)^\varepsilon]^{-3/2} = O\left(n^{-1/2} \int_{1/n}^{1-1\sqrt{n}} x^{3\varepsilon-3/2}(1-x^\varepsilon)^{-3/2}\, dx\right) =$$

$$= O\left(n^{-1/2} \int_{1/n}^{1/2} x^{3\varepsilon-3/2}\, dx\right) + O\left(n^{-1/2} \int_{1/2}^{1-1\sqrt{n}} (1-x^\varepsilon)^{-3/2}\, dx\right).$$

The first term in the expression above is

$$\begin{cases} O(n^{-3\varepsilon}) & \text{if} \quad \varepsilon < 1/6, \\ O(n^{-1/2}\log n) & \text{if} \quad \varepsilon = 1/6, \\ O(n^{-1/2}) & \text{if} \quad \varepsilon > 1/6. \end{cases}$$

By substituting $y = 1-x^\varepsilon$ in the second term it is easily seen to be $O(n^{-1/4})$. Thus

$$(E1) = O(n^{-(3\varepsilon \wedge 1/4)}). \tag{3.13}$$

From (3.10) and (3.12)

$$\sum_{k<n-\sqrt{n}} E[\gamma_{nk}^3 I(s_{nk}^2 > (k/n)^\varepsilon)] \leqq Cn^{-3\varepsilon} \sum_{k<n-\sqrt{n}} k^{3\varepsilon-3/2} E\,|s_{nk}^2-(k/n)^{2\varepsilon}|/(k/n)^\varepsilon[1-(k/n)^\varepsilon].$$

Using (3.11) and making integral comparisons as before we have

$$\sum_{k<n-\sqrt{n}} E[\gamma_{nk}^3 I(s_{nk}^2 > (k/n)^\varepsilon)] = \begin{cases} O(n^{-1/2-2\varepsilon}\log n) & \text{if} \quad \varepsilon < 1/4, \\ O(n^{-1}\log n) & \text{if} \quad \varepsilon = 1/4, \\ O(n^{-1}) & \text{if} \quad \varepsilon > 1/4. \end{cases} \tag{3.14}$$

We have not used more refined estimates for $\varepsilon > 1/4$ because (E2) will be dominated by other expressions in this range of ε as we shall see.

From (3.12)

(3.15)

$$\sum_{k<n-\sqrt{n}} E(\gamma_{nk}^3) \leqq Cn^{-3\varepsilon} \sum_{k\geqq n-\sqrt{n}} k^{3\varepsilon-3/2} \leqq Cn^{-3\varepsilon}\sqrt{n}\,[(n-\sqrt{n})^{-3\varepsilon-3/2} \vee n^{3\varepsilon-3/2}] = O(n^{-1}).$$

Since $\{X_k\colon 1 \leqq k \leqq n\}$ does not have a simple probability structure it is difficult to bound $E|\sigma_n^2-1|^{3/2}$ by anything $o(E|\sigma_n^2-1|)$. However, σ_n^2 is uniformly bounded by assumption (3.2). Hence

$$E|\sigma_n^2-1|^{3/2} \leqq CE|\sigma_n^2-1| \tag{3.16}$$

whose bounds are given by (3.10) with $k=n$.

$$\sum_{k=1}^{n} E[\gamma_{nk}^3 I(s_{nk}^2 > 1)] \leqq \sum_{k<n-\sqrt{n}} E[\gamma_{nk}^3 I(s_{nk}^2 > 1)] + \sum_{k\geqq n-\sqrt{n}} E(\gamma_{nk}^3).$$

Thus (E6) is majorized by $\min_{T>0} \{(\mathrm{E2})+(\mathrm{E3})+(\mathrm{E4})\}$ and hence (E6) can be dropped for order of magnitude calculations.

Now choose

$$T = \begin{cases} n^{\varepsilon/2} & \text{if } \varepsilon < 1/2, \\ (n/\log n)^{1/4} & \text{if } \varepsilon = 1/2, \\ n^{1/4} & \text{if } \varepsilon > 1/2. \end{cases}$$

Part (i) of the theorem now follows from (3.9) and (3.13)—(3.16). If $\varepsilon=1/2$ $(a=1)$ then $a_k=1/k$, $\beta_{nk}=k/n$, and $X_{nk}=Z_k/\sqrt{n}$, $1\leqq k\leqq n$. If, in addition, $\sigma^2(x)=\sigma^2(0)$ for all x then $s_{nk}^2=k/n$, $1\leqq k\leqq n$. Thus

$$\Delta(\sqrt{n}X_{n+1}, N) \leqq C_1 \sum_{k<n-\sqrt{n}} \int_0^T t^2 e^{-(t^2/2)[1-(k/n)]} E(\gamma_{nk}^3)\, dt + C_2 T^3 \sum_{k\geqq n-\sqrt{n}} E(\gamma_{nk}^3) + C_3 T^{-1}$$

for all $T>0$ from Theorem (2.1). Part (ii) of Theorem (3.8) now follows from (3.13) and (3.15) by choosing $T=n^{1/4}$.

REMARKS. (1) From part (ii) of Theorem (3.8) we get a convergence rate of $O(n^{-1/4})$ for the i.i.d. case with third moments.

(2) If instead of just moment assumptions we assume that the Z_k's become "almost i.i.d." much better convergence rates can be obtained. If we add the assumptions (a) $\varepsilon=1/2$, (b) $F_x\to F_y$ weakly as $x\to y$ for all real y, and (c) $\int_0^1 |F_x^{-1}(u)-F_0^{-1}(u)|^2 du = O(|x|^2)$ as $x\to 0$, where $F_x^{-1}(u)=\sup\{t\colon F_x(t)\leqq u\}$, to the conditions of part (i) of Theorem (3.8), then from a slight modification of Kersting's [5] proof of his Theorem 1 it follows that

$$\sqrt{n}X_{n+1}+n^{-1/2}\sum_{k=1}^{n} V_k = O(n^{-1/2}) \text{ a.s.},$$

where $\{V_k\}$ is an i.i.d. sequence, independent of F_{nn}, with distribution function F_0 on some common probability space. From Berry—Essen theorem

$$\Delta\left(n^{-1/2}\sum_{k=1}^{n} V_k, N\right) = o(n^{-1/2}).$$

From a well-known relation between Lévy and Kolmogorov distances between random variables when one of them is a normal (see e.g. equation (3.5) in [3]) it then follows that $\Delta(\sqrt{n}X_{n+1}, N)=O(n^{-1/2})$.

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LAGUERRE TYPE POLYNOMIALS UNDER AN INDEFINITE INNER PRODUCT

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Introduction. Recently [2] it was shown that the Laguerre-type polynomials

$$r_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(k+1)!} \binom{n}{k} [k(R+n+1)+R]x^k$$

satisfy a fourth order differential equation

$$e^{-x} ly = e^{-x} \lambda_n y, \quad n = 0, 1, \dots,$$

where

$$e^{-x} ly = (x^2 e^{-x} y'')'' - (([2R+2]x+2)e^{-x} y')'$$

and

$$\lambda_n = (2R+2)n + n(n-1),$$

and are orthogonal with respect to the Stieltjes measure ψ given by

$$\psi(x) = \begin{cases} 0, & -\infty < x < 0 \\ \dfrac{R+1}{R} - e^{-x}, & 0 \leqq x < \infty, \end{cases}$$

when $R>0$. Further it was shown that $\int_{0-}^{\infty} r_n(x)^2 d\psi = (R+n+1)(R+n)$; that the set $\{r_n\}_{n=0}^{\infty}$ spans the Hilbert space H generated by ψ, and that in H l gives rise to a self-adjoint differential operator.

The purpose of this article is to note that when $R<0$, but is not a negative integer, then ψ generates an indefinite inner product space K which is a Pontrjagin space [1], the polynomials $\{r_n\}_{n=0}^{\infty}$ span K, and, again, l gives rise to a self-adjoint operator A in K.

While the extension may seem superficially easy, the subtleties of indefinite inner product spaces assure that it is not. For instance, when R is a negative number $-n$, then the polynomial r_n is of degree $n-1$, and the space K becomes a degenerate indefinite inner product space. Exactly the right path must be followed in order for everything to work.

The polynomials. We assume that for some integer $N \geqq 0$, $-(N+1) < R < -N$. With R so constrained we then note that r_n is a polynomial exactly of degree n and that the formulas $e^{-x}lr_n=e^{-x}\lambda_n r_n$,

$$\langle r_n, r_n\rangle = \int_0^\infty r_n(x)^2\, d\psi = (R+n+1)(R+n)$$

still hold [2]. Since $(R+n+1)>0$ when $n\geqq N$, and $(R+n)>0$ when $n\geqq N+1$, we find that

$$\langle r_j, r_j\rangle \begin{cases} >0 & \text{if } j=0, \dots, N-1, \\ <0 & \text{if } j=N, \\ >0 & \text{if } j=N+1, \dots . \end{cases}$$

Let K be the indefinite inner product space generated by $\langle \cdot, \cdot\rangle$. Then K admits a Hilbert majorant given by

$$[f, g] = \frac{1}{|R|} f(0)\overline{g(0)} + \int_0^\infty f(x)\overline{g(x)}e^{-x}\,dx.$$

According to [1; p. 89] $K=K^+\oplus K^0\oplus K^-$, where K^+ is a positive definite subspace, K^0 is neutral and K^- is negative definite. By using the Fourier—Stieltjes transform [2] it is possible to show that $K^0=\{0\}$. Further, if $R^+=\text{span}\,\{r_n: \langle r_n, r_n\rangle>0\}$, and $R^-=\text{span}\,\{r_n: \langle r_n, r_n\rangle<0\}$, the same argument shows that the orthocomplement of $R^+\oplus R^-$ is $\{0\}$ and so $K=R^+\oplus R^-$.

THEOREM 1. *The indefinite inner product space K, generated by $\langle \cdot, \cdot\rangle$, is a Pontrjagin space spanned by $\{r_n\}_{n=0}^\infty$, $K=R^+\oplus R^-$. The subspace R^- is one dimensional.*

$$f=\sum_{n=0}^\infty c_n r_n, \quad \textit{where } c_n = \langle f, r_n\rangle/\langle r_n, r_n\rangle.$$

COROLLARY. *If f is in K, then $f=\sum_{n=0}^\infty c_n r_n$, where $c_n=\langle f, r_n\rangle/\langle r_n, r_n\rangle$.*

The differential operator. While the differential expression $e^{-x}l$ is formally selfadjoint, the boundary value problem

$$e^{-x}ly = \lambda e^{-x}y, \quad -2Ry'(0) = \lambda y(0),$$

which is required to show symmetry in Green's formula, involves a λ-dependent boundary condition. As a result a slightly different formulation is convient in order to fully exhibit the role played by 0.

We denote by $\mathcal{K}$ the indefinite inner product space $L^2(0, \infty; e^{-x})\times C$, where for $F=(f(x), f_0)^T$ and $G=(g(x), g_0)^T$ in $\mathcal{K}$,

$$\langle F, G\rangle_{\mathcal{K}} = \int_0^\infty f(x)\overline{g(x)}e^{-x}\,dx + (1/R)f_0\bar{g}_0.$$

It is evident that K and $\mathcal{K}$ are isomorphic. The operator A is defined as follows: Let D_A denote those elements $Y=(y(x), y_0)^T$ in $\mathcal{K}$ satisfying:

1. y is in $L^2(0, \infty; e^{-x})$.
2. y', y'', y''' exist and are absolutely continuous.

3. $e^{-x}ly$ exist a.e. and is in $L^2(0, \infty)$.
4. $y_0=y(0)$.

We then define A by setting

$$AY = \begin{pmatrix} ly \\ (ly)(0) \end{pmatrix} = \begin{pmatrix} ly \\ -2Ry'(0) \end{pmatrix}.$$

Green's formula establishes that A is symmetric. An argument similar to that in [2] shows

THEOREM 2. *A is self-adjoint in $\mathcal{K}$.*

THEOREM 3. *The spectrum of A is real and discrete. It consists only of eigenvalues $\sigma_p(A)=\{\lambda_n\}_{n=0}^{\infty}$.*

Here the argument is similar to that found in [3] and is omitted.

THEOREM 4. *Let $R_n=(r_n, R)^T$, $n=0, 1, \ldots$. For all F in $\mathcal{K}$, $F=\sum_{n=0}^{\infty} C_n R_n$, where $C_n = \langle F, R_n\rangle_{\mathcal{K}}/(R+n+1)(R+n)$. For all Y in D_A, $AY=\sum_{n=0}^{\infty} \lambda_n C_n R_n$, where $C_n = \langle Y, R_n\rangle_{\mathcal{K}}/(R+n+1)(R+n)$. Further Y is in D_A if and only if $\sum_{n=0}^{\infty} n^2\lambda_n^2|C_n|^2<\infty$.*

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A NOTE ON A PAPER OF AYOUB

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Let R be a ring, R^+ the additive group of R, and R_t the torsion part of R. If R_t is a ring direct summand of R, then R is said to be a fissible ring. A result of Ayoub [1, Corollary to Theorem 7], may be restated as follows:

PROPOSITION. *Let $G=T\oplus D$, T a torsion group which is not reduced, and D a divisible torsion free group. Then there exists an associative ring R, with $R^+=G$, such that R is not fissible.*

The purpose of this is to generalize the above Proposition by replacing D with an arbitrary torsion free group, and to prove the converse of the Proposition.

THEOREM 1. *Let $G=T\oplus H$, T a torsion group which is not reduced, and $H\neq 0$, a torsion free group. Then there exists a ring R with $R^+=G$ such that R is not fissible.*

PROOF. There exists a prime p such that $T=Z(p^\infty)\oplus A$. Clearly there exists a non-zero homomorphism $f\colon H\otimes H\to Z(p^\infty)$. Let $g_i=c_i+a_i+h_i$, $c_i\in Z(p^\infty)$, $a_i\in A$, $h_i\in H$, $i=1,2$. Define $g_1\cdot g_2=f(h_1\otimes h_2)$. This multiplication induces a ring structure R on G. Since $R^3=0$, R is clearly associative. Suppose that $R=R_t\oplus S$ is a ring direct sum. Then $0\neq R^2=S^2\subseteqq R_t$, and so $S^2\cap R_t\neq 0$, a contradiction.

THEOREM 2. *Let $G=T\oplus D$, T a torsion group, D a torsion free divisible group. The following are equivalent:*

1) *Every ring R with $R^+=G$ is fissible.*
2) *Every associative ring R with $R^+=G$ is fissible.*
3) *T is reduced.*

PROOF. Clearly 1)$\Rightarrow$2), and the implication 2)$\Rightarrow$3) is the contrapositive of the above Proposition.

3)$\Rightarrow$1). Let $G=T\oplus D$, T a reduced torsion group, D a divisible torsion free group. Let R be a ring with $R^+=G$. Clearly T is an ideal in R. Let $a\in T$, $a\neq 0$, and let $x\in D$. There exists $y\in D$ such that $x=|a|y$. Hence $ax=a(|a|y)=(|a|a)y=0$, and similarly $xa=0$. Therefore $TD=DT=0$. To show that R is fissible, it suffices to prove that $D^2\subseteqq D$. The map $\mu\colon D\times D\to G$ defined by $\mu[(a,b)]=a\cdot b$ is bilinear. Hence there exists a homomorphism $f\colon D\otimes D\to G$ satisfying $f(a\otimes b)=a\cdot b$ for all $a,b\in D$. Therefore $D^2=f(D\otimes D)$ is a homomorphic image of a divisible group, and hence divisible. Since T is reduced, $D^2\subseteqq D$.

The converse of Theorem 2 is not true.

EXAMPLE. Let T be a p-reduced p-group, p a fixed prime and let H be a p-divisible torsion free group which is not divisible, e.g., the subgroup of the rationals generated by $\left\{\frac{1}{p^n}; n=1,2,\ldots\right\}$. For any ring R with $R^+=T\oplus H$, clearly (1) $T^2\subseteqq T$. The map: $T\otimes H\to R^+$ from the cartesian product of T and H into R^+ defined by $[(x,y)]=x\cdot y$, the product on the right being multiplication in R, is bilinear and so factors through $T\otimes H$. However it is readily seen that $T\otimes H=0$ and so (2) $TH=0$, and similarly (3) $HT=0$. The same argument applied to $H\otimes H$ yields that H^2 is contained in $\varphi(H\otimes H)$, $\varphi\in\operatorname{Hom}(H\otimes H, R^+)$. Let π_T be the natural projection of R^+ onto T. Since $H\otimes H$ is p-divisible, $\pi_T\varphi(H\otimes H)$ is p-divisible. Since T is p-reduced, this implies that $\pi_T\varphi(H\otimes H)=0$ which in turn yields that $\pi_T(H^2)=0$, i.e., (4) $H^2\subseteqq H$. Clearly (1)—(4) imply that R is a ring direct sum $R=H\oplus T$, and so every ring with additive group $T\oplus H$ is fissible.

QUESTION. Let G be a mixed group satisfying $p(G/G_t)\neq G/G_t$ for every prime p, G_t=the torsion part of G. Must there exist a non-fissible (associative) ring R with $R^+=G$?

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CHARACTERIZATION OF QUASIDEVIATION MEANS

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In the theory of means the concept of *quasiarithmetic means* plays important role (see Hardy—Littlewood—Pólya [9]). In 1930, Kolmogorov [10] raised the following question: How can quasiarithmetic means be characterized? The answer was given independently by Kolmogorov [10], Nagumo [12] and de Finetti [8]. (See Theorem 4.0.)

During the development of the theory of means, a number of papers dealt with different generalizations of the quasiarithmetic means. One of the most important generalizations was given by Daróczy [4], who defined the *deviation functions* and *deviation means*. There are numerous papers (e.g. Daróczy [4,5], Losonczi [11], Páles [13], Daróczy—Páles [6, 7]) investigating inequalities and equations concerning these mean values.

In this paper we generalize the concepts of deviation functions and deviation means by defining *quasideviation functions* and *quasideviation means*. It turns out that the results obtained for deviation means by Daróczy [4, 5], Páles [13] and Daróczy—Páles [6, 7] remain valid for quasideviation means too.

The aim of the present paper is the investigation of the question similar to Kolmogorov's one: How can quasideviation means be characterized?

To answer this question we need some new concepts which have not been defined. These are the *strongly intern mean*, the *infinitesimal mean* and the *inequality of bisymmetry* (see Definitions 1.2, 1.4 and 1.6, respectively). In addition to these definitions, in § 1 we also list some known concepts which will be used later. § 2 deals with the definition of quasideviation functions, discrete and weighted quasideviation means.

The §§ 3 and 4 contain the main results of the paper. In § 3 a characterization theorem is given for the weighted quasideviation means. Using this result in § 4 we characterize the discrete quasideviation means.

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§ 1. Discrete and weighted symmetric means

Let $\mathbf{R}$, $\mathbf{R}_+$, $\mathbf{Q}$, $\mathbf{Z}$ and $\mathbf{N}$ denote the set of real numbers, positive real numbers, rational numbers, integer numbers and natural numbers, respectively.

For $x=(x_1, \ldots, x_n)\in\mathbf{R}^n$ let

$$\langle x\rangle = \langle x_1, \ldots, x_n\rangle = \left\{\sum_{i=1}^{n} \lambda_i x_i \,\middle|\, \lambda_i > 0,\ \sum_{i=1}^{n} \lambda_i = 1\right\}.$$

It is easy to see that

$$\langle x\rangle = \{x_1\}$$

if $x_1=\ldots=x_n$ and

$$\langle x\rangle = \left]\min_{1\le i\le n} x_i,\ \max_{1\le i\le n} x_i\right[$$

otherwise.

If $x=(x_1, \ldots, x_n)\in\mathbf{R}^n$ and $y=(y_1, \ldots, y_m)\in\mathbf{R}^m$ then let (x, y) mean the $(n+m)$-tuple $(x_1, \ldots, x_n, y_1, \ldots, y_m)\in\mathbf{R}^{n+m}$. Let $\mathscr{D}(I)=\bigcup_{n=1}^{\infty} I^n$ where $I\subseteqq\mathbf{R}$ is an arbitrary interval.

Definition 1.1. The function $M\colon \mathscr{D}(I)\to I$ is said to be a *discrete symmetric mean* on the interval I, if it has the following two properties:

a) M is *intern*, i.e.

$$M(x)\in\langle x\rangle \tag{1.1}$$

for all $x\in\mathscr{D}(I)$.

b) M is *symmetric*, i.e. for all $n\in\mathbf{N}$

$$M_n = M|_{I^n} \tag{1.2}$$

is a symmetric function.

The class of discrete symmetric means on I is denoted by $\mathscr{M}(I)$.

Definition 1.2. A discrete symmetric mean $M\in\mathscr{M}(I)$ is said to be *strongly intern* if for every $x_1, \ldots, x_n\in\mathscr{D}(I)$ $n\in\mathbf{N}$ we have

$$M(x_1, \ldots, x_n)\in\langle M(x_1), \ldots, M(x_n)\rangle. \tag{1.3}$$

Definition 1.3. We say that the discrete mean $M\in\mathscr{M}(I)$ is *associative* if the identity

$$M(x_1, \ldots, x_n) = M\big(\underbrace{M(x_1), \ldots, M(x_1)}_{k_1}, \ldots, \underbrace{M(x_n), \ldots, M(x_n)}_{k_n}\big) \tag{1.4}$$

holds for all $x_1\in I^{k_1}, \ldots, x_n\in I^{k_n}, k_1, \ldots, k_n, n\in\mathbf{N}$.

Remark. Using the intern property, it is obvious that every associative mean is strongly intern.

Definition 1.4. Let $M\in\mathscr{M}(I)$ and

$$\Omega^k_{x,y}(M) = \max_{1\le l\le k}\big|M(\underbrace{x, \ldots, x}_{l}, \underbrace{y, \ldots, y}_{k-l}) - M(\underbrace{x, \ldots, x}_{l-1}, \underbrace{y, \ldots, y}_{k-l+1})\big| \tag{1.5}$$

for $x, y\in I$, $n\in\mathbf{N}$. The mean M is said to be *infinitesimal* if for all $x, y\in I$

$$\lim_{k\to\infty} \Omega^k_{x,y}(M) = 0. \tag{1.6}$$

Let $\Delta = (\mathbf{R}_+\cup\{0\})^2\setminus(0, 0)$ and $\tilde{\mathscr{D}}(I) = I^2\times\Delta$.

Definition 1.5. The function $\tilde{M}\colon \tilde{\mathscr{D}}(I)\to I$ is called *weighted symmetric mean* on I if it has the following properties:

a) $\tilde{M}$ is *symmetric*, i.e.

$$\tilde{M}(x, y; \lambda, \mu) = \tilde{M}(y, x; \mu, \lambda) \tag{1.7}$$

for all $x, y \in I$, $(\lambda, \mu) \in \Delta$.

b) $\tilde{M}$ is *reflexive*, i.e.

$$\tilde{M}(x, x; \lambda, \mu) = x \tag{1.8}$$

for $x \in I$, $(\lambda, \mu) \in \Delta$.

c) $\tilde{M}$ is *intern*, i.e.

$$x = \tilde{M}(x, y; 1, 0) < \tilde{M}(x, y; \lambda, \mu) < \tilde{M}(x, y; 0, 1) = y \tag{1.9}$$

if $x, y \in I$, $x < y$, $\lambda, \mu \in \mathbf{R}_+$.

d) $\tilde{M}$ is *nullhomogeneous in the weights*, i.e.

$$\tilde{M}(x, y; t\lambda, t\mu) = \tilde{M}(x, y; \lambda, \mu) \tag{1.10}$$

for all $x, y \in I$, $(\lambda, \mu) \in \Delta$, $t \in \mathbf{R}_+$.

The class of weighted symmetric means on I is denoted by $\tilde{\mathcal{M}}(I)$.

DEFINITION 1.6. Let $\tilde{M} \in \tilde{\mathcal{M}}(I)$. We say that $\tilde{M}$ *satisfies the inequality of bisymmetry* if

(1.11)

$$\min \{\tilde{M}(x, y; \lambda_x, \lambda_y),\ \tilde{M}(u, v; \lambda_u, \lambda_v)\} \leqq \max \{\tilde{M}(x, u; \lambda_x, \lambda_u),\ \tilde{M}(y, v; \lambda_y, \lambda_v)\}$$

for all $x, y, u, v \in I$, $\lambda_x, \lambda_y, \lambda_u, \lambda_v \in \mathbf{R}_+$.

DEFINITION 1.7. The mean $\tilde{M} \in \tilde{\mathcal{M}}(I)$ is said to be *bisymmetric* if it satisfies the *equation of bisymmetry* i.e.

$$\begin{aligned} &\tilde{M}\big(\tilde{M}(x, y; \lambda_x, \lambda_y),\ \tilde{M}(u, v; \lambda_u, \lambda_v);\ \lambda_x+\lambda_y,\ \lambda_u+\lambda_v\big) = \\ &= \tilde{M}\big(\tilde{M}(x, u; \lambda_x, \lambda_u),\ \tilde{M}(y, v; \lambda_y, \lambda_v);\ \lambda_x+\lambda_u,\ \lambda_y+\lambda_v\big) \end{aligned} \tag{1.12}$$

for all $x, y, u, v \in I$, $\lambda_x, \lambda_y, \lambda_u, \lambda_v \in \mathbf{R}_+$.

REMARK. It is easy to see that if $\tilde{M}$ fulfils the equation of bisymmetry then $\tilde{M}$ also satisfies the inequality of bisymmetry.

DEFINITION 1.8. We say that $\tilde{M} \in \tilde{\mathcal{M}}(I)$ is *regular* if for $x, y \in I$, $x < y$ the function

$$\lambda \to \tilde{M}(x, y; \lambda, 1-\lambda), \quad \lambda \in [0, 1] \tag{1.13}$$

is continuous and strictly monoton decreasing.

DEFINITION 1.9. Let $M \in \mathcal{M}(I)$ and $\tilde{M} \in \tilde{\mathcal{M}}(I)$. The means M, $\tilde{M}$ are said to be *associated* if

$$M(\underbrace{x, \ldots, x}_{k}, \underbrace{y, \ldots, y}_{l}) = \tilde{M}(x, y; k, l) \tag{1.14}$$

for all $x, y \in I$, $k, l \in \mathbf{N}$.

§ 2. Quasideviations and quasideviation means

DEFINITION 2.1. Let $I\subseteqq\mathbf{R}$ be an interval. The function $E:I^2\to\mathbf{R}$ is said to be a *quasideviation* on I if

(E1) for all $x,\ t\in I$

$$\operatorname{sgn} E(x, t) = \operatorname{sgn}(x-t), \tag{2.1}$$

(E2) the function

$$t \to E(x, t), \quad t\in I \tag{2.2}$$

is continuous for all $x\in I$,

(E3) the function

$$p_{x,y}^{(E)}(t) = -\frac{E(y, t)}{E(x, t)}, \quad t\in]x, y[\tag{2.3}$$

is strictly monotone decreasing for $x, y\in I$, $x<y$.

The class of quasideviations on I is denoted by $\mathscr{E}(I)$.

REMARK. The concept of *deviation* has been introduced by Daróczy [4]. The function $E:I^2\to\mathbf{R}$ is called a deviation if it has the properties (E1), (E2) and

(E3)* The function (2.2) is strictly decreasing for $x\in I$.

It can be easily shown that every deviation is a quasideviation, but not conversely.

The following result is needed to explain the concept of discrete quasideviation mean.

THEOREM 2.1. *Let* $E\in\mathscr{E}(I)$, $n\in\mathbf{N}$, $x_1, \dots, x_n\in I$. *Define for* $t\in I$ *the function* e *by*

$$e(t) = \sum_{i=1}^{n} E(x_i, t). \tag{2.4}$$

Then there exists a value $t_0\in I$ *such that*

$$\operatorname{sgn} e(t) = \operatorname{sgn}(t_0-t) \tag{2.5}$$

for $t\in I$ *and*

$$t_0\in\langle x_1, \dots, x_n\rangle. \tag{2.6}$$

PROOF. Without loss of generality we can assume that $x_1\leqq\dots\leqq x_n$. If $x_1=x_n$, then $t_0=x_1=x_n$ and (2.5) follows from (E1). Thus we may assume that $x_1<x_n$.

Using (E1) it is easy to see that

$$e(x_1) = e\big(\min_{1\leqq i\leqq n} x_i\big) > 0, \quad e(x_n) = e\big(\max_{1\leqq i\leqq n} x_i\big) < 0. \tag{2.7}$$

By (E2) e is a continuous function, hence there exists a value $t_0\in]x_1, x_n[=$ $=\langle x_1, \dots, x_n\rangle$ such that $e(t_0)=0$. Thus (2.6) is satisfied. To prove (2.5) let $t<t_0$ be an arbitrary element of I. If $t\leqq x_1$ then the inequality $e(t)>0$ (which is equivalent to (2.5)) follows from (E1). Otherwise we may assume that

$$x_1 \leqq\dots\leqq x_k < t \leqq x_{k+1} \leqq\dots\leqq x_l \leqq t_0 < x_{l+1} \leqq\dots\leqq x_n. \tag{2.8}$$

Let $1 \leqq i \leqq k$ and $l+1 \leqq j \leqq n$. By (E3) the function

$$t \to \frac{E(x_j, t)}{E(x_i, t)}, \quad t \in]x_i, x_j[$$

is strictly monotone increasing, hence

$$\frac{E(x_j, t)}{E(x_i, t)} < \frac{E(x_j, t_0)}{E(x_i, t_0)}$$

because $x_i < t < t_0 < x_j$. Rearranging the above inequality we get

$$E(x_i, t_0)E(x_j, t) < E(x_j, t_0)E(x_i, t). \tag{2.9}$$

Using (E1) it can also be verified that (2.9) is valid if $k+1 \leqq i \leqq l$ and $l+1 \leqq j \leqq n$. Adding the inequalities obtained and applying the equation $e(t_0)=0$ we have

$$\sum_{i=1}^{l} E(x_i, t_0) \sum_{j=l+1}^{n} E(x_j, t) < \sum_{j=l+1}^{n} E(x_j, t_0) \sum_{i=1}^{l} E(x_i, t) =$$

$$= \sum_{i=l+1}^{n} E(x_i, t_0) \sum_{j=1}^{l} E(x_j, t) = \left(-\sum_{i=1}^{l} E(x_i, t_0)\right) \sum_{j=1}^{l} E(x_j, t).$$

Thus

$$\sum_{i=1}^{l} E(x_i, t_0) \sum_{j=1}^{n} E(x_j, t) < 0. \tag{2.10}$$

By (E1) and (2.8) $\sum_{i=1}^{l} E(x_i, t_0) < 0$ thus by (2.10) we get $e(t) > 0$.

When $t_0 < t$ it can analogously be seen that $e(t) < 0$. This completes the proof.

REMARK. The analogue of Theorem 2.1 for deviations was proved by Daróczy [4].

DEFINITION 2.2. Let $E \in \mathscr{E}(I)$ and $x=(x_1, \ldots, x_n) \in I^n \subset \mathscr{D}(I)$. The unique solution $t=t_0$ of the equation

$$\sum_{i=1}^{n} E(x_i, t) = 0 \tag{2.11}$$

is called the *quasideviation mean of x generated by E* and is denoted by $\mathfrak{M}_E(x)$.

By Theorem 2.1 this definition is correct and $\mathfrak{M}_E$ is a discrete symmetric mean on I.

THEOREM 2.2. *Let $E \in \mathscr{E}(I)$, $x, y \in I$, and $(\lambda, \mu) \in \Delta$. Further let*

$$e(t) = \lambda E(x, t) + \mu E(y, t), \quad t \in I. \tag{2.12}$$

Then there exists a $t_0 \in I$ such that

$$\operatorname{sgn} e(t) = \operatorname{sgn}(t_0 - t) \tag{2.13}$$

for $t \in I$. If $x \neq y$, $\lambda, \mu > 0$ then

$$\min\{x, y\} < t_0 < \max\{x, y\}. \tag{2.14}$$

PROOF. If $x=y$ or $\lambda=0$ or $\mu=0$ then (2.13) follows from (E1). Thus (by symmetry reasons) we may assume that $x<y$, $\lambda>0$ and $\mu>0$. By (E3) the function $p_{x,y}^{(E)}$ defined in (2.3) is strictly monotone decreasing and maps the interval $]x, y[$ onto $\mathbf{R}_+$. Thus there exists a unique value $t_0\in]x, y[$ such that $p_{x,y}^{(E)}(t_0)=\dfrac{\lambda}{\mu}$ i.e. $e(t_0)=0$. If $t\leqq x$ and $y\leqq t$ then the inequalities $e(t)>0$ and $e(t)<0$ follow from (E1), respectively. If $t\in]x, y[$ then by (E3)

$$\operatorname{sgn}\left(p_{x,y}^{(E)}(t)-\frac{\lambda}{\mu}\right)=\operatorname{sgn}(t_0-t). \tag{2.15}$$

From (2.15) we obtain (2.13).

REMARK. Theorem 2.2 and Definition 2.3 below for deviations have been formed by Daróczy—Páles [6].

DEFINITION 2.3. Let $E\in\mathscr{E}(I)$, $x, y\in I$ and $(\lambda, \mu)\in\Delta$. The unique solution $t=t_0$ of the equation

$$\lambda E(x, t)+\mu E(y, t)=0 \tag{2.16}$$

is called the *weighted symmetric quasideviation mean of* x, y *with weights* λ, μ *generated by* E and is denoted by $\tilde{\mathfrak{M}}_E(x, y; \lambda, \mu)$.

REMARK. If $E\in\mathscr{E}(I)$ then the means $\mathfrak{M}_E$ and $\tilde{\mathfrak{M}}_E$ are associated.

Denote by $\Omega(I)$ the set of real valued functions which are *continuous* and *strictly monotone increasing* on I. Let further $\mathscr{P}(I)$ be the class of *positive* real valued functions on I. If $\varphi\in\Omega(I), f\in\mathscr{P}(I)$ then the function

$$E(x, t)=E_{\varphi,f}(x, t)=f(x)\big(\varphi(x)-\varphi(t)\big)\quad (x, t\in I) \tag{2.17}$$

is a deviation and also a quasideviation on I.

For this deviation (2.17) the unique solutions of (2.11) and (2.16) have the form

$$t_0=\mathfrak{M}_{E_{\varphi,f}}(x)=M_{\varphi,f}(x)=\varphi^{-1}\left(\sum_{i=1}^{n} f(x_i)\varphi(x_i)\Big/\sum_{i=1}^{n} f(x_i)\right) \tag{2.18}$$

and

$$t_0=\tilde{\mathfrak{M}}_{E_{\varphi,f}}(x, y;\ \lambda, \mu)=\tilde{M}_{\varphi,f}(x, y;\ \lambda, \mu)=\varphi^{-1}\left(\frac{\lambda f(x)\varphi(x)+\mu f(y)\varphi(y)}{\lambda f(x)+\mu f(y)}\right) \tag{2.19}$$

respectively. (φ^{-1} denotes the inverse function of φ.)

The means $M_{\varphi,f}$ and $\tilde{M}_{\varphi,f}$ defined by (2.18) and (2.19) will be called discrete and weighted quasiarithmetic mean with weightfunction, respectively (see Bajraktarevic [3] and Aczél—Daróczy [2]).

If $f(x)=p$ is a positive constant in (2.18) and (2.19), we obtain the well-known quasiarithmetic means. The theory of these mean values can be found in the book of Hardy—Littlewood—Pólya [9].

§ 3. Properties and characterization of weighted quasideviation means

The weighted quasiarithmetic means are characterized by the following theorem.

THEOREM 3.0. *Let $I \subseteqq \mathbf{R}$ be an interval. A weighted symmetric mean on I is quasiarithmetic if and only if it is regular and fulfils the equation of bisymmetry.*

The proof of Theorem 3.0 can be found in the book of Aczél [1].

The most important results of this section are summarized in the following

CHARACTERIZATION THEOREM 1. *Let $I \subseteqq \mathbf{R}$ be an open interval. A weighted symmetric mean on I is generated by a quasideviation if and only if it is regular and satisfies the inequality of bisymmetry.*

First we show the necessity of the conditions.

THEOREM 3.1. *Let $I \subseteqq \mathbf{R}$ be an interval and let $E \in \mathscr{E}(I)$. Then the mean $\tilde{\mathfrak{M}}_E$ is regular.*

PROOF. Let $x, y \in I$, $x<y$ be fixed values, and let

$$f_{x,y}(t) = \frac{E(y, t)}{E(y, t)-E(x, t)} \tag{3.1}$$

for $t \in [x, y]$. By (E1) and (E2), $f_{x,y}\colon [x, y] \to [0, 1]$ is a continuous function, $f_{x,y}(x)=1$, $f_{x,y}(y)=0$ and by the notation (2.3) we may write

$$f_{x,y}(t) = \frac{1}{1-\dfrac{E(x, t)}{E(y, t)}} = \frac{1}{1+\dfrac{1}{p^{(E)}_{x,y}(t)}}$$

for $t \in]x, y[$. Using (E3) we obtain that $f_{x,y}$ is strictly monotone decreasing. This implies that $f_{x,y}$ is invertible and its inverse is continuous and strictly monotone decreasing, too. To prove the regularity of $\tilde{\mathfrak{M}}_E$ it is enough to show that for $\lambda \in [0, 1]$ the equation

$$f^{-1}(\lambda) = \tilde{\mathfrak{M}}_E(x, y;\ \lambda, 1-\lambda) \tag{3.2}$$

holds.

Let $t_\lambda = \tilde{\mathfrak{M}}_E(x, y;\ \lambda, 1-\lambda)$, for $\lambda \in [0, 1]$ then $\lambda E(x, t_\lambda)+(1-\lambda)E(y, t_\lambda) = 0$ hence $f_{x,y}(t_\lambda)=\lambda$. Thus $f_{x,y}^{-1}(\lambda) = t_\lambda = \tilde{\mathfrak{M}}_E(x, y;\ \lambda, 1-\lambda)$.

REMARK. Theorem 3.1 was proved by Daróczy—Páles [6] for weighted deviation means.

THEOREM 3.2. *Let $I \subseteqq \mathbf{R}$ be an interval and let $E \in \mathscr{E}(I)$. Then the mean $\tilde{\mathfrak{M}}_E$ satisfies the inequality of bisymmetry.*

PROOF. Suppose on the contrary that,

$$\begin{aligned} \min\{\tilde{\mathfrak{M}}_E(x, y;\ \lambda_x, \lambda_y),\ \ \tilde{\mathfrak{M}}_E(u, v;\ \lambda_u, \lambda_v)\} &> \\ > \max\{\tilde{\mathfrak{M}}_E(x, u;\ \lambda_x, \lambda_u),\ \ \tilde{\mathfrak{M}}_E(y, v;\ \lambda_y, \lambda_v)\}& \end{aligned} \tag{3.3}$$

for some values $x, y, u, v \in I$, $\lambda_x, \lambda_y, \lambda_u, \lambda_v \in \mathbf{R}_+$. Then we can find a $t \in I$ such that

$$\tilde{\mathfrak{M}}_E(x, y;\ \lambda_x, \lambda_y) > t, \quad \tilde{\mathfrak{M}}_E(u, v;\ \lambda_u, \lambda_v) > t, \tag{3.4}$$

and

$$\tilde{\mathfrak{M}}_E(x, u;\ \lambda_x, \lambda_u) < t, \quad \tilde{\mathfrak{M}}_E(y, v;\ \lambda_y, \lambda_v) < t. \tag{3.5}$$

Using Theorem 2.2 we obtain from (3.4) that

$$\lambda_x E(x, t) + \lambda_y E(y, t) > 0, \quad \lambda_u E(u, t) + \lambda_v E(v, t) > 0.$$

Hence

$$\lambda_x E(x, t) + \lambda_y E(y, t) + \lambda_u E(u, t) + \lambda_v E(v, t) > 0. \tag{3.6}$$

Starting with (3.5) we obtain, in a similar way, the inequality reversed to (3.6). This contradiction proves our theorem.

Theorems 3.1 and 3.2 show that the conditions in our Characterization Theorem 1 are necessary. To prove the sufficiency part we construct a suitable quasideviation. Our construction consist of two steps.

THEOREM 3.3. *Let* $I \subseteq \mathbf{R}$ *be an interval and let* $\tilde{M} \in \tilde{\mathcal{M}}(I)$ *be a regular weighted mean satisfying the inequality of bisymmetry. Then the function* $p_{x,y}\colon]x, y[\to \mathbf{R}_+$ *where* $x, y \in I$, $x < y$, *defined by*

$$t = \tilde{M}(x, y;\ p_{x,y}(t), 1), \quad t \in]x\, y[\tag{3.7}$$

is strictly decreasing, continuous and satisfies the relations

$$\lim_{t \to x} p_{x,y}(t) = \infty, \quad \lim_{t \to y} p_{x,y}(t) = 0. \tag{3.8}$$

If $x, y, u, v \in I$, $x, v < t < u, y$, *then*

$$p_{x,y}(t)\, p_{v,u}(t) = p_{x,u}(t)\, p_{v,y}(t). \tag{3.9}$$

PROOF. If $x, y \in I$ and $x < y$ then by the regularity and nullhomogeneity of $\tilde{M}$ the function $\lambda \mapsto \tilde{M}(x, y;\ \lambda, 1)$, $\lambda \in \mathbf{R}_+$ is continuous and strictly monotone decreasing. Thus its inverse $p_{x,y}$ defined by (3.7), also has these properties.

From the relations

$$\lim_{\lambda \to \infty} \tilde{M}(x, y;\ \lambda, 1) = \lim_{\lambda \to \infty} \tilde{M}\left(x, y;\ \frac{\lambda}{\lambda+1}, \frac{1}{\lambda+1}\right) = \lim_{\mu \to 1} \tilde{M}(x, y;\ \mu, 1-\mu) = x$$

and

$$\lim_{\lambda \to 0} \tilde{M}(x, y;\ \lambda, 1) = \tilde{M}(x, y;\ 0, 1) = y,$$

(3.8) follows.

Now we prove (3.9). Let $\varepsilon > 0$ and let

$$\lambda_x = p_{x,y}(t), \quad \lambda_y = 1+\varepsilon, \quad \lambda_u = \frac{p_{x,y}(t)}{p_{x,u}(t)}, \quad \lambda_v = (1+\varepsilon)\, p_{v,y}(t).$$

Then we have

$$\tilde{M}(x, y;\ \lambda_x, \lambda_y) = \tilde{M}(x, y;\ p_{x,y}(t), 1+\varepsilon) = \tag{3.10}$$

$$= \tilde{M}\left(x, y;\ \frac{p_{x,y}(t)}{1+\varepsilon}, 1\right) > \tilde{M}(x, y;\ p_{x,y}(t), 1) = t.$$

Further

$$(3.11)\quad \tilde{M}(x, u;\ \lambda_x, \lambda_u) = \tilde{M}\left(x, u;\ p_{x,y}(t), \frac{p_{x,y}(t)}{p_{x,u}(t)}\right) = \tilde{M}(x, u;\ p_{x,u}(t), 1) = t,$$

and

$$(3.12)\quad \tilde{M}(y, v;\ \lambda_y, \lambda_v) = \tilde{M}(y, v;\ 1+\varepsilon, (1+\varepsilon)p_{v,y}(t)) = \tilde{M}(v, y;\ p_{v,y}(t), 1) = t.$$

Using the inequality of bisymmetry (1.11), we get from (3.10), (3.11) and (3.12) that

$$(3.13)\qquad \tilde{M}(u, v;\ \lambda_u, \lambda_v) \leqq t$$

i.e.

$$(3.14)\qquad \tilde{M}\left(v, u;\ (1+\varepsilon)p_{v,y}(t), \frac{p_{x,y}(t)}{p_{x,u}(t)}\right) \leqq t = \tilde{M}(v, u;\ p_{v,u}(t), 1).$$

Since the function $\lambda \to \tilde{M}(v, u;\ \lambda, 1)$, $\lambda \in \mathbf{R}_+$ is strictly monotone decreasing, it follows from (3.14) that

$$\frac{(1+\varepsilon)p_{v,y}(t)p_{x,u}(t)}{p_{x,y}(t)} \geqq p_{v,u}(t).$$

Letting $\varepsilon \to 0$ we get

$$(3.15)\qquad p_{x,y}(t)p_{v,u}(t) \leqq p_{v,y}(t)p_{x,u}(t).$$

Exchanging the roles of u and y in the above proof, we obtain (3.15) with reversed inequality sign. Thus (3.9) is fulfilled.

THEOREM 3.4. *Let $I \subseteq \mathbf{R}$ be an open interval and let $\tilde{M} \in \tilde{\mathcal{M}}(I)$ be a regular mean satisfying the inequality of bisymmetry. Then there exists a quasideviation $E \in \mathcal{E}(I)$ such that*

$$(3.16)\qquad \tilde{M} = \tilde{\mathfrak{M}}_E.$$

PROOF. Let $p_{x,y}$ be the function defined by (3.7) where $x, y \in I$, $x<y$.

The basic idea of our proof is to construct a quasideviation $E \in \mathcal{E}(I)$ such that

$$(3.17)\qquad p_{x,y}(t)E(x, t)+E(y, t) = 0$$

holds for $x, y \in I$, $x<t<y$.

Let $a_1, a_2 \in I$, $a_1<A<a_2$ be fixed values, $I_1 = \{t \in I \mid t \geqq A\}$, $I_2 = \{t \in I \mid t \leqq A\}$, and choose $c_1 \in]-\infty\ 0[$, and $c_2 \in]0, \infty[$ such that

$$(3.18)\qquad p_{a_1,a_2}(A)c_1+c_2 = 0.$$

Let

$$E_1(x, t) = \begin{cases} -p_{a_1,x}(t)c_1, & t < x \\ 0, & t = x \\ \dfrac{p_{a_1,z_1}(t)}{p_{x,z_1}(t)}c_1, & t > x(z_1 \in I_1,\ z_1 > t) \end{cases}$$

for $(x, t) \in I \times I_1$. If $z_1' > t$, $z_1' \in I_1$ then by (3.9)

$$\frac{p_{a_1,z_1}(t)}{p_{x,z_1}(t)}c_1 = \frac{p_{a_1,z_1'}(t)}{p_{x,z_1'}(t)}c_1.$$

Therefore the definition of E_1 is correct. (At the definition of E_1 it is essential that I is open: for every $t\in I_1$ there exists a $z_1\in I_1$ with $z_1>t$.)

The choice of c_1 implies that, for $(x, t)\in I\times I_1$, $\operatorname{sgn} E_1(x, t)=\operatorname{sgn}(x-t)$. Hence E_1 has the property (E1). Now we show that the function

$$t \to E_1(x, t), \quad t\in I_1 \tag{3.19}$$

is continuous for each $x\in I$, i.e. E_1 has property (E2), too.

Let $(x, t_0)\in I\times I_1$ and apply Theorem 3.3. If $t_0<x$ then

$$\lim_{\substack{t\to t_0\\ A\leqq t}} E_1(x, t) = \lim_{\substack{t\to t_0\\ A\leqq t<x}} E_1(x, t) = \lim_{\substack{t\to t_0\\ A\leqq t<x}} -p_{a_1,x}(t)c_1 = -p_{a_1,x}(t_0)c_1 = E_1(x, t_0).$$

Thus (3.19) is continuous at the point $t_0\in I$, $t_0<x$.

If $t_0=x$ and $x\neq A$ then

$$\lim_{\substack{t\to t_0-0\\ A\leqq t}} E_1(x, t) = \lim_{\substack{t\to t_0\\ A\leqq t<x}} E_1(x, t) = \lim_{\substack{t\to t_0\\ A\leqq t<x}} -p_{a_1,x}(t)c_1 = 0 = E_1(x, t_0).$$

If $t_0=x$ and $z_1>x$, $z_1\in I$ then

$$\lim_{t\to t_0+0} E_1(x, t) = \lim_{\substack{t\to t_0\\ x<t<z_1}} E_1(x, t) = \lim_{\substack{t\to x\\ x<t<z_1}} \frac{p_{a_1,z_1}(t)}{p_{x,z_1}(t)}c_1 = 0 = E_1(x, t_0).$$

Hence (3.19) is continuous at the point $t_0\in I$, $t_0=x$.

Finally if $t_0>x$, $z_1>t_0$ and $z_1\in I_1$ then

$$\lim_{\substack{t\to t_0\\ A\leqq t}} E_1(x, t) = \lim_{\substack{t\to t_0\\ A\leqq t\leqq z_1}} E_1(x, t) = \lim_{\substack{t\to t_0\\ A\leqq t\leqq z_1}} \frac{p_{a_1,z_1}(t)}{p_{x,z_1}(t)}c_1 = \frac{p_{a_1,z_1}(t_0)}{p_{x,z_1}(t_0)}c_1 = E_1(x, t_0).$$

Thus (3.19) is also continuous at $t_0\in I$, if $t_0>x$.

It can be seen that (3.17) remains valid for $x<t<y$, $x, y\in I$, $t\in I_1$ if we write E_1 instead of E. For $(x, t)\in I\times I_2$ let

$$E_2(x, t) = \begin{cases} \dfrac{p_{z_2,x}(t)}{p_{z_2,a_2}(t)}c_2, & t<x \quad (z_2\in I,\ z_2<t)\\ 0, & t=x\\ -\dfrac{1}{p_{x,a_2}(t)}c_2, & t>x. \end{cases}$$

It can analogously be seen that the definition of E_2 is correct and E_2 has the properties (E1) and (E2), and (3.17) remains valid for $x<t<y$, $x, y\in I$, $t\in I$ if we write E_2 instead of E.

Let finally

$$E(x, t) = \begin{cases} E_1(x, t), & t\geqq A,\\ E_2(x, t), & t<A, \end{cases} \qquad x, t\in I.$$

We prove that E is a quasideviation on I. (E1) is obviously satisfied. To prove (E2) we have to verify the equation

$$E_1(x, A) = E_2(x, A), \quad x\in I. \tag{3.20}$$

But (3.20) directly follows from (3.18) and (3.9). To prove (E3) we have to notice that (3.17) is satisfied for $x<t<y$, $x, y\in I$. Rearranging (3.17) we obtain

$$p_{x,y}(t) = -\frac{E(y,t)}{E(x,t)}, \quad x<t<y, \quad x, y\in I.$$

Using the properties of $p_{x,y}$ we can see that (E3) is satisfied.

Finally we prove (3.16). It is sufficient to show that

$$\lambda E(x, \tilde{M}(x,y;\lambda,\mu)) + \mu E(y, \tilde{M}(x,y;\lambda,\mu)) = 0 \tag{3.21}$$

for all $x, y\in I$, $(\lambda,\mu)\in\Delta$.

If $\lambda=0$ or $\mu=0$ or $x=y$ then (3.21) is obvious. Suppose that $\lambda>0$, $\mu>0$ and $x<y$ (the case $x>y$ is similar). Then using (3.7) and (3.17) we have

$$E(y, \tilde{M}(x,y;\lambda,\mu)) = -p_{x,y}(\tilde{M}(x,y;\lambda,\mu))E(x, \tilde{M}(x,y;\lambda,\mu)) =$$

$$= -p_{x,y}\left(\tilde{M}\left(x,y;\frac{\lambda}{\mu},1\right)\right)E(x, \tilde{M}(x,y;\lambda,\mu)) = -\frac{\lambda}{\mu}E(x, \tilde{M}(x,y;\lambda,\mu)).$$

Thus the proof is complete.

§ 4. Properties and characterization of discrete quasideviation means

The discrete quasiarithmetic means were characterized by Kolmogorov [10]:

THEOREM 4.0. *Let $I\subseteq\mathbf{R}$ be an interval. The mean $M\in\mathcal{M}(I)$ is quasiarithmetic if and only if M is associative and, for $n\in\mathbf{N}$, M_n is a continuous and strictly monoton increasing function of its variables.*

The following theorem contains our most important result on characterization of discrete quasideviation means.

CHARACTERIZATION THEOREM 2. *Let $I\subseteq\mathbf{R}$ be an open interval. A discrete symmetric mean on I is generated by a quasideviation if and only if it is strongly intern and infinitesimal.*

First we show that the conditions are necessary.

THEOREM 4.1. *Let $I\subseteq\mathbf{R}$ be an interval, and let $E\in\mathcal{E}(I)$. Then the mean $\mathfrak{M}_E$ is infinitesimal.*

PROOF. Let $x, y\in I$. If $x=y$ then $\Omega^k_{x,y}(\mathfrak{M}_E)=0$ and (1.6) is obviously satisfied. Thus without loss of generality we may assume that $x<y$. Let $f_{x,y}$ be the function defined in (3.1). In the proof of Theorem 3.1 we have shown that $f_{x,y}$ is invertible, continuous and strictly decreasing. Let

$$\omega_{x,y}(h) = \sup_{\substack{|\lambda_1-\lambda_2|\le h\\ 0\le\lambda_1,\lambda_2\le 1}} |f^{-1}_{x,y}(\lambda_1) - f^{-1}_{x,y}(\lambda_2)|, \quad h>0. \tag{4.1}$$

$f_{x,y}^{-1}$ is uniformly continuous on [0, 1] hence

$$\lim_{h\to 0}\omega_{x,y}(h) = 0. \tag{4.2}$$

Let $0\leq l\leq k$, $k>0$ be integer numbers. Using (3.2) and the fact that $\mathfrak{M}_E$ and $\tilde{\mathfrak{M}}_E$ are associated we get

$$\mathfrak{M}_E(\underbrace{x, \ldots, x}_{l}, \underbrace{y, \ldots, y}_{k-l}) = \tilde{\mathfrak{M}}_E(x, y;\, l, k-l) = \tilde{\mathfrak{M}}_E\left(x, y;\, \frac{l}{k},\, 1-\frac{l}{k}\right) = f_{x,y}^{-1}\left(\frac{l}{k}\right).$$

Then

$$\Omega_{x,y}^k(\mathfrak{M}_E) = \max_{1\leq l\leq k}|\mathfrak{M}_E(\underbrace{x, \ldots, x}_{l}, \underbrace{y, \ldots, y}_{k-l}) - \mathfrak{M}_E(\underbrace{x, \ldots, x}_{l-1}, \underbrace{y, \ldots, y}_{k-l+1})| =$$

$$= \max_{1\leq l\leq k}\left|f_{x,y}^{-1}\left(\frac{l}{k}\right) - f_{x,y}^{-1}\left(\frac{l-1}{k}\right)\right| \leq \omega_{x,y}\left(\frac{1}{k}\right).$$

Hence by (4.2) we obtain (1.6).

THEOREM 4.2. *Let $I\subseteq\mathbf{R}$ be an interval and let $E\in\mathscr{E}(I)$. Then the mean $\mathfrak{M}_E$ is strongly intern.*

PROOF. Let $x_i = (x_{i1}, \ldots, x_{ik_i})\in I^{k_i}$, $i = 1, \ldots, n$; $k_1, \ldots, k_n\in\mathbf{N}$, and let

$$t = \min_{1\leq i\leq n}\mathfrak{M}_E(x_i), \quad s = \max_{1\leq i\leq n}\mathfrak{M}_E(x_i).$$

We have to show that

$$\mathfrak{M}_E(x_1, \ldots, x_n)\in\langle\mathfrak{M}_E(x_1), \ldots, \mathfrak{M}_E(x_n)\rangle = \langle t, s\rangle. \tag{4.3}$$

Using Theorem 2.1 we have

$$\sum_{j=1}^{k_i} E(x_{ij}, t) \geq 0, \quad i = 1, \ldots, n. \tag{4.4}$$

Thus

$$\sum_{i=1}^{n}\sum_{j=1}^{k_i} E(x_{ij}, t) \geq 0. \tag{4.5}$$

Applying Theorem 2.1 again we get

$$t \leq \mathfrak{M}_E(x_1, \ldots, x_n). \tag{4.6}$$

In a similar way we can also obtain the inquality

$$\mathfrak{M}_E(x_1, \ldots, x_n) \leq s. \tag{4.7}$$

If $t=s$ then (4.6) and (4.7) prove (4.3). If $t<s$ then we show that (4.6) and (4.7) hold with strict inequality sign. Suppose, on the contrary, that (4.6) holds with equality sign. Then (4.5) and (4.4) also hold with this sign. Thus $\mathfrak{M}_E(x_i)=t$ for $i=1, \ldots, n$. Hence $t=s$. This means that (4.3) is valid.

Theorems 4.1 and 4.2 show that the conditions of Characterization Theorem 2 are necessary. Now we prove the sufficiency. Our proof consists of two steps.

THEOREM 4.3. *Let $I \subseteqq \mathbf{R}$ be an interval and let $M \in \mathcal{M}(I)$ be an infinitesimal and strongly intern discrete mean. Then there exists a weighted mean $\tilde{M} \in \tilde{\mathcal{M}}(I)$ associated to M which is regular and satisfies the inequality of bisymmetry.*

PROOF. Let

(4.8) $$\Delta_{\mathbf{Q}} = \{(\lambda, \mu) \in \Delta \mid \lambda/(\lambda+\mu) \in \mathbf{Q}\}.$$

Let further $x, y \in I$, $x<y$ be fixed values.

For $(\lambda, \mu) \in \Delta_{\mathbf{Q}}$, $\lambda l = \mu k$, $(k, l \in \mathbf{N} \cup \{0\},\ k+l>0)$ let

(4.9) $$m_{x,y}(\lambda, \mu) = M(\underbrace{x, \dots, x}_{k}, \underbrace{y, \dots, y}_{l}).$$

The value $m_{x,y}(\lambda, \mu)$ does not depend on the choice of the integers k, l because M is strongly intern. It is obvious from (4.9) that $m_{x,y}$ is a nullhomogeneous and intern function, i.e.

(4.10) $$m_{x,y}(\lambda, \mu) = m_{x,y}(t\lambda, t\mu)$$

for $(\lambda, \mu) \in \Delta_{\mathbf{Q}}$, $t \in \mathbf{R}_+$, and

(4.11) $$x < m_{x,y}(\lambda, \mu) < y,$$

if $(\lambda, \mu) \in \Delta_{\mathbf{Q}}$, $\lambda, \mu > 0$.

Let us consider the function

(4.12) $$\lambda \mapsto m_{x,y}(\lambda, 1-\lambda), \quad \lambda \in [0, 1] \cap \mathbf{Q}.$$

We need the following propositions:

PROPOSITION A. *The function* (4.12) *is strictly monotone decreasing on* $[0, 1] \cap \mathbf{Q}$.

PROPOSITION B. *The function* (4.12) *is uniformly continuous on* $[0, 1] \cap \mathbf{Q}$.

PROOF OF PROPOSITION A. Let $\lambda_1, \lambda_2, \in \mathbf{Q}$ and $0 \leqq \lambda_1 < \lambda_2 \leqq 1$. If $\lambda_1 = 0$ or $\lambda_2 = 1$ then

$$m_{x\ y}(\lambda_1, 1-\lambda_1) > m_{x,y}(\lambda_2, 1-\lambda_2)$$

obviously follows from (4.11). Hence we may assume that $0 < \lambda_1 < \lambda_2 < 1$. Let

$$\lambda_i = \frac{l_i}{k_i}, \quad 0 < l_i < k_i, \quad l_i, k_i \in \mathbf{N}$$

for $i=1, 2$. Now $\lambda_1 < \lambda_2$ means that

(4.13) $$l_1 k_2 < l_2 k_1.$$

Applying (4.9), (4.13) and the strongly intern property of M we have

$$m_{x,y}(\lambda_2, 1-\lambda_2) = M(\underbrace{x, \dots, x}_{l_1 l_2}, \underbrace{y, \dots, y}_{l_1(k_2-l_2)}) = \min \{M(\underbrace{x, \dots, x}_{l_1 l_2}, \underbrace{y, \dots, y}_{l_1(k_2-l_2)}), \underbrace{y, \dots, y}_{l_2 k_1 - l_1 k_2}\} <$$

$$< M(\underbrace{x, \dots, x}_{l_1 l_2}, \underbrace{y, \dots, y}_{(k_1-l_1) l_2}) = m_{x,y}(\lambda_1, 1-\lambda_1).$$

PROOF OF PROPOSITION B. Let

$$\omega_{x,y}(h) = \sup_{\substack{|\lambda_1-\lambda_2|\leqq h \\ \lambda_1,\lambda_2\in[0,1]\cap\mathbf{Q}}} |m_{x,y}(\lambda_1, 1-\lambda_1)-m_{x,y}(\lambda_2, 1-\lambda_2)|$$

for $h>0$. We have to show that

$$\lim_{h\to 0}\omega_{x,y}(h) = 0. \tag{4.14}$$

Let $h\in[0, 1]$ be fixed and $k=\left[\frac{1}{h}\right]$. Further let $\lambda_1, \lambda_2\in[0, 1]\cap\mathbf{Q}$, $\lambda_1<\lambda_2<\lambda_1+h$, $l_1=[k\lambda_1]$ and

$$l_2 = \begin{cases} [k\lambda_2]+1, & \text{if } \lambda_2<1, \\ k, & \text{if } \lambda_2 = 1. \end{cases}$$

It is clear that

$$0 \leqq \frac{l_1}{k} \leqq \lambda_1, \quad \lambda_2 \leqq \frac{l_2}{k} \leqq 1.$$

By Proposition A we have

$$\begin{cases} m_{x,y}\left(\frac{l_1}{k}, 1-\frac{l_1}{k}\right) \geqq m_{x,y}(\lambda_1, 1-\lambda_1), \\ m_{x,y}\left(\frac{l_2}{k}, 1-\frac{l_2}{k}\right) \leqq m_{x,y}(\lambda_2, 1-\lambda_2). \end{cases} \tag{4.15}$$

Further $k\lambda_2\leqq k\lambda_1+kh\leqq k\lambda_1+1$, that is $[k\lambda_2]\leqq[k\lambda_1]+1$. Thus

$$0 \leqq l_2-l_1 \leqq [k\lambda_2]+1-[k\lambda_1] \leqq 2. \tag{4.16}$$

Applying (4.15) and (4.16) we obtain

$$|m_{x,y}(\lambda_1, 1-\lambda_1)-m_{x,y}(\lambda_2, 1-\lambda_2)| \leqq$$

$$\leqq \left|m_{x,y}\left(\frac{l_1}{k}, 1-\frac{l_1}{k}\right)-m_{x,y}\left(\frac{l_2}{k}, 1-\frac{l_2}{k}\right)\right| =$$

$$= |M(\underbrace{x, \dots, x}_{l_1}, \underbrace{y, \dots, y}_{k-l_1})-M(\underbrace{x, \dots, x}_{l_2}, \underbrace{y, \dots, y}_{k-l_2})| \leqq 2\Omega^k_{x,y}(M).$$

Consequently

$$\omega_{x,y}(h) \leqq 2\Omega^{\left[\frac{1}{h}\right]}_{x,y}(M) \tag{4.17}$$

for $h\in]0, 1]$. Since M is infinitesimal, (4.17) gives (4.14).

Let us continue the proof of Theorem 4.3.

Using Propositions A and B, it can be seen that there exists a continuous and strictly monotone decreasing extension of the function (4.12). Thus we can define a function $m^*_{x,y}\colon \Delta\to\mathbf{R}$ having the following properties:

a) $m^*_{x,y}$ is an extension of $m_{x,y}$, i.e.

$$m^*_{x,y}(\lambda, \mu) = M(\underbrace{x, \ldots, x}_{k}, \underbrace{y, \ldots, y}_{l}) \tag{4.18}$$

if $(\lambda, \mu) \in \Delta_{\mathbf{Q}}$, $\lambda l = \mu k$.

b) $m^*_{x,y}$ is nullhomogeneous, i.e.

$$m^*_{x,y}(\lambda, \mu) = m^*_{x,y}(t\lambda, t\mu) \tag{4.19}$$

for $(\lambda, \mu) \in \Delta$, $t \in \mathbf{R}_+$.

c) $m^*_{x,y}$ is intern, i.e.

$$x < m^*_{x,y}(\lambda, \mu) < y \tag{4.20}$$

for $\lambda, \mu > 0$.

d) $m^*_{x,y}$ is regular, i.e. the function

$$\lambda \to m^*_{x,y}(\lambda, 1-\lambda), \quad \lambda \in [0, 1] \tag{4.21}$$

is strictly monotone decreasing and continuous on $[0, 1]$.

Now we define the function $\tilde{M}\colon \tilde{\mathcal{D}}(I) \to \mathbf{R}$ by

$$\tilde{M}(x, y;\ \lambda, \mu) = \begin{cases} m^*_{x,y}(\lambda, \mu), & x < y, \\ x, & x = y, \\ m^*_{y,x}(\mu, \lambda), & x > y \end{cases} \quad (x, y \in I, (\lambda, \mu) \in \Delta). \tag{4.22}$$

Using (4.19), (4.20), (4.21) and (4.22) we can verify that $\tilde{M}$ is a weighted regular symmetric mean. By (4.18) it is obvious that M and $\tilde{M}$ are associated. We have to show that $\tilde{M}$ satisfies the inequality of bisymmetry.

Let $x, y, u, v \in I$, $\lambda_x, \lambda_y, \lambda_u, \lambda_v \in \mathbf{R}_+$. By the regularity of $\tilde{M}$, it is sufficient to show that (1.11) holds for rational values of $\lambda_x, \lambda_y, \lambda_u, \lambda_v$.

Suppose that

$$\lambda_x = \frac{l_x}{k}, \quad \lambda_y = \frac{l_y}{k}, \quad \lambda_u = \frac{l_u}{k}, \quad \lambda_v = \frac{l_v}{k} \quad (l_x, l_y, l_u, l_v, k \in \mathbf{N}).$$

M is strongly intern therefore

$$\min \{M(\underbrace{x, \ldots, x}_{l_x}, \underbrace{y, \ldots, y}_{l_y}), M(\underbrace{u, \ldots, u}_{l_u}, \underbrace{v, \ldots, v}_{l_v})\} \leqq$$

$$\leqq M(\underbrace{x, \ldots, x}_{l_x}, \underbrace{y, \ldots, y}_{l_y}, \underbrace{u, \ldots, u}_{l_u}, \underbrace{v, \ldots, v}_{l_v}) \leqq$$

$$\leqq \max \{M(\underbrace{x, \ldots, x}_{l_x}, \underbrace{u, \ldots, u}_{l_u}), M(\underbrace{y, \ldots, y}_{l_y}, \underbrace{v, \ldots, v}_{l_v})\}.$$

M and $\tilde{M}$ are associated, thus

$$\begin{aligned} &\min \{\tilde{M}(x, y;\ l_x, l_y), \tilde{M}(u, v;\ l_u, l_v)\} \leqq \\ &\leqq \max \{\tilde{M}(x, u;\ l_x, l_u), \tilde{M}(y, v;\ l_y, l_v)\}. \end{aligned} \tag{4.23}$$

Using the nullhomogeneity of $\tilde{M}$ for the value $t = \frac{1}{k}$ we get (1.11) from (4.23).

THEOREM 4.4. *Let $I \subseteqq \mathbf{R}$ be an open interval and let $M \in \mathcal{M}(I)$ be an infinitesimal and strongly intern mean. Then there exists a quasideviation $E \in \mathcal{E}(I)$ such that M is generated by E, i.e. $M = \mathfrak{M}_E$.*

PROOF. Using Theorems 4.3 and 3.4 we can see that there exists a quasideviation $E \in \mathcal{E}(I)$ such that M and $\tilde{\mathfrak{M}}_E$ are associated. $\mathfrak{M}_E$ and $\tilde{\mathfrak{M}}_E$ are also associated, thus

$$M(\underbrace{x, \ldots, x}_{k}, \underbrace{y, \ldots, y}_{l}) = \mathfrak{M}_E(\underbrace{x, \ldots, x}_{k}, \underbrace{y, \ldots, y}_{l}) \tag{4.24}$$

for all $x, y \in I$ and $k, l \in \mathbf{N} \cup \{0\}$, $k+l>0$. We are going to show that

$$M(x) = \mathfrak{M}_E(x) \tag{4.25}$$

for $x \in \mathcal{D}(I)$, too.

Let $x \in I^n$, $n \in \mathbf{N}$, $x = (x_1, \ldots, x_n)$. Without loss of generality we may assume that $x_1 \leqq \ldots \leqq x_n$, $x_1 < x_n$.

Let $t_0 = \mathfrak{M}_E(x)$. Instead of (4.25) we show that

$$\operatorname{sgn}(t - t_0) = \operatorname{sgn}\big(t - M(x)\big) \tag{4.26}$$

for $t \in I \setminus \{t_0\}$. We prove (4.26) in the case $t < t_0$. In the case $t_0 < t$ the proof is similar.

Let $I \ni t < t_0$. If $t \leqq x_1$ then (4.26) is obvious. Suppose that $x_1 < t$. Then there exist integers $1 \leqq k < l \leqq n$ such that

$$x_1 \leqq \ldots \leqq x_k < t \leqq x_{k+1} \leqq \ldots \leqq x_l \leqq t_0 < x_{l+1} \leqq \ldots \leqq x_n. \tag{4.27}$$

We need the following

PROPOSITION C. *There exist integers $u_1, \ldots, u_n$ such that*

$$\operatorname{sgn} u_i = \operatorname{sgn} E(x_i, t_0), \quad i = 1, \ldots, n, \tag{4.28}$$

$$\sum_{i=1}^{n} u_i = 0, \tag{4.29}$$

and

$$u_j E(x_i, t) - u_i E(x_j, t) > 0 \tag{4.30}$$

for all $(i, j) \in \{1, \ldots, l\} \times \{l+1, \ldots, n\}$.

PROOF. For $i = 1, \ldots, n$, let $\lambda_i^{(m)}$, $m = 1, 2, \ldots$ be a sequence of rational numbers such that

$$\lambda_i^{(m)} = 0 \quad \text{if} \quad x_i = t_0, \quad (m = 1, 2, \ldots), \tag{4.31}$$

$$\lim_{m \to \infty} \lambda_i^{(m)} = E(x_i, t_0), \quad (i = 1, \ldots, n), \tag{4.32}$$

and

$$\sum_{i=1}^{n} \lambda_i^{(m)} = 0, \quad (m = 1, 2, \ldots). \tag{4.33}$$

As we have seen in the proof of Theorem 2.1, it follows from (4.27) that

$$E(x_i, t)E(x_j, t_0) > E(x_i, t_0)E(x_j, t)$$

for $(i,j)\in\{1, \dots, l\}\times\{l+1, \dots, n\}$.

By (4.31) and (4.32) we can choose an index m_0 such that,

$$\lambda_j^{(m_0)} E(x_i, t) - \lambda_i^{m_0} E(x_j, t) > 0 \tag{4.34}$$

for $(i,j)\in\{1, \dots, l\}\times\{l+1, \dots, n\}$, and

$$\operatorname{sgn} \lambda_i^{(m_0)} = \operatorname{sgn} E(x_i, t_0) \tag{4.35}$$

for $i=1, \dots, n$.

Since $\lambda_1^{(m_0)}, \dots, \lambda_n^{(m_0)}$ are rational numbers, there exist a natural number u^* and integers $u_1, \dots, u_n$ such that $\lambda_i^{(m_0)} = \dfrac{u_i}{u^*}$.

Using (4.34), (4.35) and (4.33) we see that $u_1, \dots, u_n$ satisfy (4.28), (4.29) and (4.30). Thus Proposition C is proved.

Let us continue the proof of Theorem 4.4. Applying Theorem 2.1 and (4.30) we obtain

$$t < \mathfrak{M}_E(\underbrace{x_i, \dots, x_i}_{u_j}, \underbrace{x_j, \dots, x_j}_{-u_i}).$$

Using (4.24) we have

$$t < M(\underbrace{x_i, \dots, x_i}_{u_j}, \underbrace{x_j, \dots, x_j}_{-u_i}) \tag{4.36}$$

for $(i,j)\in\{1, \dots, l\}\times\{l+1, \dots, n\}$.

Since M is strongly intern we have

$$t < \min_{\substack{1\leq i\leq l\\ l+1\leq j\leq n}} M(\underbrace{x_i, \dots, x_i}_{u_j}, \underbrace{x_j, \dots, x_j}_{-u_i}) \leqq$$

$$\leqq M(\dots, \underbrace{x_i, \dots, x_i}_{u_{l+1}+\dots+u_n}, \dots, \underbrace{x_j, \dots, x_j}_{-u_1-u_2-\dots-u_l}, \dots) = M(x_1, \dots, x_n) = M(x).$$

(Here we used the equality (4.24) in the form $u_{l+1}+\dots+u_n = -u_1-u_2-\dots-u_l$.)

Thus (4.26) is proved.

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ON RADICALS OF SEMIGROUP RINGS

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In [4] Krempa studied radicals of semigroup rings. Among others, for any semigroup P with unity and a radical S he defined the class PS of rings by $R \in PS$ iff the semigroup ring $R[P]$ is in S. This notion was introduced earlier by Ortiz [6] for associative rings and by Gardner [2] for associative rings and the infinite cyclic semigroup P. In the above quoted papers it was proved that the class PS is radical and many properties of it were investigated. Some generalizations were made in [3].

In [4] Krempa proved (Corollary 1) that for any radical S and any ring R, $(S(R[P]) \cap R)[P] \subseteq S(R[P])$ and he called S to be P-normal if for any ring R, $(S(R[P]) \cap R)[P] = S(R[P])$. In particular, for any radical S and any ring R, $(PS)(R) \subseteq S(R[P]) \cap R$. If $(S(R[P]) \cap R)[P]$ is an S ideal of $R[P]$ then $(PS)(R) = S(R[P]) \cap R$. Krempa asked ([4], p. 61) if there exists a semigroup P such that for some radical S in the class of all rings and some ring R, $(PS)(R) \neq S(R[P]) \cap R$. In this paper we answer this question affirmatively by showing that the infinite cyclic semigroup (Section 1) and non-trivial finite groups (Section 2) have this property. Actually, it seems to be probable that all non-trivial semigroups have this property. We obtain a partial result in this direction (Corollary 1) by showing that for any non-trivial semigroup P there exists a radical S which is not P-normal. The situation is quite different with radicals in the class of all associative algebras over a field F. It is known [5] that for some F there exist non-trivial semigroups P such that any radical S of this class is P-normal, so for any F-algebra A, $S(A[P]) \cap A \in PS$. In Section 3 we show that for some F there exist semigroups P such that $S(A[P]) \cap A \in PS$ for all A and S although not all radicals are P-normal.

At first, let us remark that using Tangeman—Kreiling construction [7] of the lower radical we obtain immediately

PROPOSIPION 1. *Let S be a radical in the class $\mathscr{A}$ of all associative rings and let $\bar{S}$ be the lower radical in the class of all rings determined by S. Then $\bar{S} \cap \mathscr{A} = S$.*

Proposition 1 shows that to answer Krempa's cited question it suffices to consider associative rings and radicals in the class of al associative rings only. So, unless stated otherwise, in the sequel all rings will be associative and all radicals will be in the class of all associative rings.

For undefined terms and used facts of radicals we refer to [8].

1. If P is the infinite cyclic semigroup with unity then for any ring R, $R[P]$ is isomorphic to the polynomial ring $R[x]$ of indeterminate x. In that case for any radical S we will write $P_1 S$ instead of PS. Before constructing a radical S and a ring R such that $P_1 S(R) \neq S(R[x]) \cap R$ we prove two lemmas.

LEMMA 1. *Let K be a field and S the locally finite radical in the class of K-algebras (i.e. $A \in S$ iff any finitely generated subalgebra of A is finite dimensional). Then for any K-algebra R, $S(R[x]) = L(R)[x]$, where $L(R)$ is the locally nilpotent radical of R.*

PROOF. The inclusion $L(R)[x] \subseteq S(R[x])$ issues from the fact that any finitely generated nilpotent K-algebra is finite dimensional. Now since any subalgebra of an S-algebra is in S, by [1], $S(R[x]) = I[x]$ for some ideal I of R. By the foregoing $I \supseteq L(R)$. We will prove that I is locally nilpotent. If not then I contains a subalgebra B generated by elements $r_1, \ldots, r_k$ which is not nilpotent. Thus for any natural number n there exists $0 \neq a_n \in B^n$. But then $a_n x^n$ are lineary independent elements of the subalgebra C of $I[x]$ generated by B and $r_1 x, \ldots, r_k x$. Of course the subalgebra C is finitely generated and infinite dimensional. This contradiction proves that $I \subseteq L(R)$.

LEMMA 2. *If A is a simple ring without non-zero central elements then no ideal of $A[x]$ contains non-zero central elements.*

PROOF. Let J be a non-zero ideal of $A[x]$. For $m=0, 1, \ldots$ define

$$J_m = \{a \in A \mid ax^m + a_{m+1}x^{m+1} + \ldots + a_k x^k \in J \text{ for some } a_{m+1}, \ldots, a_k \in A\}.$$

Of course J_m are ideals in A, so for any m, $J_m = 0$ or $J_m = A$. Let $a = a_n x^n + \ldots + a_k x^k$ be a central element of J with $a_n \neq 0$. If $b = b_n x^n + \ldots + b_r x^r \in J$ then $ab = ba$ and, in consequence, $a_n b_n = b_n a_n$. This and the fact that $J_n = A$ implies that a_n is a central element of A. This contradiction proves the lemma.

Now let K be the field of p elements for some prime p and let A be a simple locally finite K-algebra without non-zero central elements (for example the algebra of infinite matrices with finite number of non-zero entries from K). Let A^* be the natural extension of A to a K-algebra with unity such that $A^*/A \approx K$ and let I be the ideal of $A^*[x]$ generated by A and $x^p - x$. Then we have

THEOREM 1. *If S is the lower radical determined by I then $(P_1 S)(A^*) \neq A = S(A^*[x]) \cap A^*$.*

PROOF. To prove that $(P_1 S)(A^*) \neq A$ it suffices to show that $S(A[x]) = 0$. If $S(A[x]) \neq 0$ then $A[x]$ contains a non-zero accessible subring R which is a homomorphic image of I by a homomorphism f. Since $A[x]$ is a semiprime ring, for any natural number n, $R^n \neq 0$. But for some n, R^n is an ideal of $A[x]$. Since $(x^p - x)^n$ is a central element of I^n and $f(I^n) = R^n$, $f\big((x^p - x)^n\big) = 0$. This shows that R is a homomorphic image of $I/\big((x^p - x)^n A^*[x]\big)$. But $I/\big((x^p - x)^n A^*[x]\big)$ is a locally finite K-algebra, so the locally finite radical of $A[x]$ is not zero. Thus Lemma 1 implies that A is locally nilpotent. Since A is simple, $A^2 = 0$. This contradicts the assumption that A does not contain non-zero central elements.

Now we will prove that $S(A^*[x]) \cap A^* = A$. Clearly $S(A^*[x]) \cap A^* \supseteq I \cap A^* = A$, so, since $A^*/A \approx K$, it is enough to prove that $K \notin S$. If $K \in S$ then K is a homomorphic image of I by a homomorphism $f\colon I \to K$. Since K is a ring with unity and I is an ideal of $A^*[x]$, we can extend f to a homomorphism $\bar{f}\colon A^*[x] \to K$. Now $\bar{f}(x^p - x) = \big(\bar{f}(x)\big)^p - \bar{f}(x) = 0$. Also $\bar{f}(A) = 0$ as K does not contain non-trivial subrings and A is a simple ring which is not isomorphic with K. Hence $f(I) = \bar{f}(I) = 0$ as I is generated by A and $x^p - x$. This contradiction ends the proof.

2. LEMMA 3. *Let $G\neq 1$ be a group such that for any $g\in G$, $g^2=1$, Z the ring of integers and $2Z$ the ring of even integers. If S is the lower radical determined by the ideal I of $Z[G]$ generated by $2Z$ and $\{1-g|g\in G\}$ then*

a) $S(Z[G])\cap Z=2Z$;

b) $S((2Z)[G])=0$.

PROOF. Certainly $S(Z[G])\cap Z\supseteqq 2Z$. If $S(Z[G])\cap Z\neq 2Z$ then $S(Z[G])\cap Z=Z$ so $Z[G]\in S$. Thus $Z_2=Z/2Z\in S$. This means that Z_2 is a homomorphic image of I by a homomorphism $f\colon I\to Z_2$. Since Z_2 is a ring with unity and I is an ideal of $Z[G]$, we can extend f to a homomorphism $\bar{f}\colon Z[G]\to Z_2$. But then $\bar{f}(1-g)=0$ for any $g\in G$, so $\operatorname{Ker} f\subseteqq J=\{\sum a_i g_i|a_i\in 2Z,\ g_i\in G,\ \sum a_i=0\}$. In particular Z_2 is a homomorphic image of $(2Z)[G]/J\approx 2Z$. This contradiction shows a).

If $S((2Z)[G])\neq 0$ then $(2Z)[G]$ contains a non-zero accessible subring A which is a homomorphic image of I by a homomorphism $f\colon I\to A$. Let us observe that $f(2)+f(2)=f(4)=f(2)f(2)$ and $(f(1-g))^2=f((1-g)^2)=f(2(1-g))=2f(1-g)$ for $g\in G$. Thus $f(2)$ and $f(1-g)$ are elements of $(2Z)[G]$ which satisfy the equality $x^2=2x$. If $a=2a_1+\sum_{1\neq g\in G}(2a_g)g\in(2Z)[G]$ and $a^2=2a$ then, since $g^2=1$ for $g\in G$, we obtain $a_1^2+\sum_{g\neq 1}a_g^2=a_1$. Thus $a_g=0$ for $g\neq 1$ and $a\in 2Z$. Hence $A\subseteqq 2Z$. This is impossible as A is a non-zero accessible subring of $(2Z)[G]$.

COROLLARY 1. *For any semigroup $P\neq 1$ there exists a radical which is not P-normal.*

PROOF. Let K be the field of three elements and S the upper radical determined by K. If S is P-normal then $S(K[P])=0$. In particular $K[P]$ is a subdirect sum of copies of K, so any element of $K[P]$ satisfies the equality $x^3-x=0$. Thus P is a group such that $p^2=1$ for $p\in P$. Now the rest follows from Lemma 3.

LEMMA 4. *Let $A=xK[[x]]$, where $K[[x]]$ is the power series ring over a prime finite field K of indeterminate x. Let S be the lower radical determined by $\{A, A^*\otimes_K F_1, \dots, A^*\otimes_K F_n\}$ where F_i are finite simple K-algebras with unity non-isomorphic with K. Then*

a) $K\notin S$;

b) $A\otimes_K F_i\notin S$ for $i=1, \dots, n$.

PROOF. If $K\in S$ then K is a homomorphic image of A or $A^*\otimes_K F_i$ for some $1\leqq i\leqq n$. The first case is impossible as any proper homomorphic image of A is a nilpotent ring. If $f\colon A^*\otimes_K F_i\to K$ is a homomorphism then, since $F_i\not\approx K$, $f(F_i)=0$. Thus $f=0$ and the second case is impossible too. This proves a).

Now if $S(A\otimes_K F_i)\neq 0$ then $A\otimes_K F_i$ contains a non-zero accessible subring R which is a homomorphic image of A or $A^*\otimes_K F_j$ for some $1\leqq j\leqq n$. But $A\otimes_K F_i$ is isomorphic to a subring of $xF_i[[x]]$, so $A\otimes_K F_i$ does not contain non-zero idempotents. This eliminates the second possibility. Since F_i is a ring of matrices over a finite field F, the ring $A\otimes_K F_i$ is semiprime. Thus if R is a homomorphic image of A then $R\approx A$ as any proper homomorphic image of A is nilpotent. But R is an accessible subring of $A^*\otimes_K F_i$. Thus for some m, R^m is an ideal of $A^*\otimes_K F_i$. Clearly

$R^m \approx A^m$ and $R^m \neq 0$. This is impossible if F_i is non-commutative as then R^m is non-commutative. If F_i is commutative then R^m is an F_i-algebra and A^m, as it is easy to check, is not one. This contradiction ends the proof.

THEOREM 2. *For any finite group* $G \neq 1$ *there exist a radical* S *and a ring* R *such that* $(GS)(R) \neq S(R[G]) \cap R$.

PROOF. If for any $g \in G$, $g^2=1$ then the result follows from Lemma 3. So we can assume that G contains an element of order $n>2$. Since any arithmetical progression $kn+r$ of integers with n, r relatively prime contains infinitely many primes, there exists a prime p greater than the order of G such that n does not divide $p-1$. Let K be the field of p elements. Then $K[G]$ is a semiprime finite dimensional K-algebra, so $K[G]=R_1 \oplus \ldots \oplus R_k$, where R_i are simple K-algebras with unity. By the choice of p it follows that some R_i are not isomorphic to K. But some of them are, as K is, a homomorphic image of $K[G]$. Now if S and A are those of Lemma 4 then $S(A^*[G]) \cap A=A$ and $S(A[G]) \neq A[G]$, so $(GS)(A^*) \neq S(A^*[G]) \cap A$.

3. Let Z_2 be the field of two elements and C_2 the cyclic group of order 2. Then not all radicals in the class of all Z_2-algebras are C_2-normal. For example $Z_2[C_2]$ is neither Jacobson radical nor semiprimitive. But we have

PROPOSITION 2. *For any radical* S *in the class of all* Z_2*-algebras and any* Z_2*-algebra* A, $(C_2S)(A)=S(A[C_2]) \cap A$.

PROOF. If $C_2=\{1, g\}$ then for any Z_2-algebra A, $\omega(A[C_2])$ will denote the ideal $\{a1+bg \mid a, b \in A\}$. Of course $(\omega(A[C_2]))^2=0$ and $\omega(A[C_2])$ is the kernel of the natural homomorphism $\pi\colon A[C_2] \to A$ sending $a1+bg$ on $a+b$. Now if for some Z_2-algebra A, $S(A[C_2]) \neq 0$ and $S(A[C_2]) \cap A=0$ then any zero Z_2-algebra is in S. Indeed, let $\pi\big(S(A[C_2])\big)=I$. Then I is an ideal of A and $\varphi(S(A[C_2])) \subseteq \omega((A/I)[C_2])$, where φ is the natural homomorphism of $A[C_2]$ onto $(A/I)[C_2]$. So if $\varphi\big(S(A[C_2])\big) \neq 0$ then $S(A[C_2])$ can be homomorphically mapped onto the zero Z_2-algebra Z_2^0 on the additive group of Z_2. This implies that any zero Z_2-algebra is in S. If $\varphi\big(S(A[C_2])\big)=0$ then $S(A[C_2]) \subseteq I[C_2]$. But for every $a+bg \in S(A[C_2])$, $(a+b)+(a+b)g=(a+bg)(1+g) \in S(A[C_2])$, so $\omega(I[C_2]) \subseteq S(A[C_2])$. Thus $S(A[C_2])=I[C_2]$, a contradiction.

Now let B be a Z_2-algebra and let $C=S(B[C_2]) \cap B$. Then $C[C_2] \subseteq S(B[C_2])$. If $C[C_2]=S(B[C_2])$ then the result follows. So, let $C[C_2] \subsetneqq S(B[C_2])$. Then $S\big((B/C)[C_2]\big) \neq 0$ and $S\big((B/C)[C_2]\big) \cap (B/C)=0$. Hence, by the preceding paragraph, any zero Z_2-algebra is in S. If $I=\pi\big(S(B[C_2])\big)$, where π is the natural homomorphism of $B[C_2]$ onto B, then $C \subseteq I \in S$ and by foregoing $\omega(I[C_2]) \in S$. But $I[C_2]/\omega(I[C_2]) \approx I$, so $I[C_2] \in S$. This shows that $C[C_2]=I[C_2] \in S$ and ends the proof.

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ON NASH EQUILIBRIUM. I

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The notion of "Nash equilibrium" (see [1]) was developed and generalized in different directions (see [4], [5]). The purpose of this paper is to present one generalization of this concept which is connected with optimality over cones. The existence of equilibrium will be established and Arrow—Debreu's model will be discussed with its help.

1. Notations and definitions

Let us consider a multiobjective model consisting of n subsystems denoted by $\mathcal{S}_1, \dots, \mathcal{S}_n$. Each $\mathcal{S}_i$ is given by the set of its possible actions Z_i and the multivalued mapping φ_i restricting the domain of actions, from the product $Z=Z_1\times \dots \times Z_n$ into Z_i. Denote S the set of states of this model, i.e. S consists of all points $z=(z_1, \dots, z_n)\in Z$ with $z_i\in\varphi_i(z)$ $(i=1, \dots, n)$. Let u_i be an objective function of $\mathcal{S}_i$ defined on Z with values in a Euclidean space R^{m_i}.

DEFINITION 1. Suppose that M is a closed convex cone in R^m, K is a subset of S and f is a function from S into R^m. A state $z\in K$ is called to be M-optimal of K if there is no $z'\in K$ so that $f(z')\neq f(z)$ and $f(z')-f(z)\in M$. We write $z\in MO(f|K)$. Suppose $M_1, \dots, M_n$ are closed convex cones containing no lines in $R^{m_1}, \dots, R^{m_n}$, respectively.

DEFINITION 2. A state $z\in S$ is called to be a Nash-M equilibrium of the model if z belongs to $M_iO(u_i|S_i(z))$ for all $i=1, \dots, n$, where

$$S_i(z) = \{z' = (z'_1, \dots, z'_n)\in Z\colon\ z'_j = z_j,\ j\neq i \text{ and } z'_i\in\varphi_i(z)\}.$$

REMARK. If $M_i=R^{m_i}_+$ is the non-negative orthant in R^{m_i} then M_i-optimality is Pareto optimality and Nash-M equilibrium is Nash-Pareto equilibrium defined in [2]. In particular, if $m_i=1$ for all $i=1, \dots, n$ and $M_i=\{x\in R,\ x\geqq 0\}$, then we have the well-known Nash-equilibrium (see [1]). The results in this paper generalize those in [2] and of course they generalize the theorems of § 17 and § 18 in [1].

2. Existence of equilibria

Before proving the existence of equilibrium of the model, we recall that a function f is called to be M-concave if

$$f(\lambda x+(1-\lambda) y)-\lambda f(x)-(1-\lambda)f(y)\in M$$

for all $\lambda\in[0, 1]$. Here we suppose that S is a subset in a locally convex space. The lower semi-continuity and upper semi-continuity of multi-valued mappings can be found in [3].

LEMMA 1. *For a closed convex cone M containing no lines, there is $p\in M$ so that $\langle p, x\rangle > 0$ for every $x \neq 0$, $x\in M$.*

PROOF. We prove the lemma by induction on $\dim M$ (by definition, $\dim M$ is the dimension of the smallest space containing M). The assertion is trivial for $\dim M=0$. Suppose the lemma is proved for M with $\dim M<m$. We have to show it for M with $\dim M=m$. The convex cone M contains no lines, hence there is a hyperplane $H=\{x:\langle x, b\rangle=0$ for some fixed $b\in M\}$ which separates M and $(-M)$. If $\langle x, b\rangle>0$ for every $x\neq 0, x\in M$ then the proof is finished. Otherwise $M\cap H$ is a closed convex cone with $0<\dim M\cap H<m$. By induction there is $q\in M\cap H$ so that $\langle q, y\rangle>0$ for every $y\neq 0$, $y\in M\cap H$. We claim that there exists a positive λ such that $\langle \lambda b+q, x\rangle>0$ for every $x\neq 0, x\in M$. It is sufficient to prove that the latter inequality holds for $x\in M$ and $\|x\|=1$. Suppose the opposite: for $\lambda=1, 2, \ldots$ there are $x_1, x_2, \ldots$ in M with $\|x_i\|=1$, and

$$\langle ib+q, x_i\rangle \leqq 0. \tag{1}$$

In view of the compactness of the set $K=\{x\in M, \|x\|=1\}$ we may assume that $\{x_i\}$ converges to $x_0\in M$. If $x_0\in H$ one can find some positive ε so that $\langle q, x_0\rangle>\varepsilon$. Hence, for sufficiently large N

$$\langle q, x_i\rangle > \frac{\varepsilon}{2}, \quad i \geqq N. \tag{2}$$

This contradicts (1) because of $\langle b, x_i\rangle\geqq 0$. In the other case $x_0\notin H$, i.e. $\langle b, x_0\rangle>\delta$ for some positive δ. Consequently $\langle b, x_i\rangle>\dfrac{\delta}{2}$ if i is larger thansome integer N'. The function $\langle q, \cdot\rangle$ is bounded below on K, therefore

$$\langle ib+q, x_i\rangle = i\langle b, x_i\rangle+\langle q, x_i\rangle > i\,\frac{\delta}{2}+\min_{x\in K}\langle q, x\rangle > 0 \tag{3}$$

for i sufficiently large. (3) contradicts (1), which completes the proof.

LEMMA 2. *If $z^*\in K$ satisfies $\langle p, f(z^*)\rangle\geqq\langle p, f(z)\rangle$ for each $z\in K$, where p is defined as in Lemma 1, then z^* is M-optimal on K.*

PROOF. If z^* were not M-optimal on K, we would find some $y\in K$ such that $f(y)\neq f(z^*)$ and $f(y)-f(z^*)\in M$. Hence $\langle p, f(y)-f(z^*)\rangle>0$ that contradicts the condition of Lemma 2.

THEOREM 1. *Suppose that the model satisfies the following conditions:*

i) *$Z_1, \ldots, Z_n$ are non-empty convex compacta in a locally convex space;*

ii) *$u_1, \ldots, u_n$ are continuous functions, u_i is M_i-concave on z_i, $i=1, \ldots, n$, respectively;*

iii) *$\varphi_1, \ldots, \varphi_n$ are Hausdorff continuous and the image of any point is a non-empty convex compactum.*

Then the model possesses Nash-M equilibrium.

PROOF. First we note that by Lemma 1, there are $p_1 \in M_1, \dots, p_n \in M_n$ with the property in Lemma 1. Let $\bar{u}_i$ be a composition of p_i and u_i, i.e. $\bar{u}_i(z) = [p_i \circ u_i](z) = \langle p_i, u_i(z) \rangle$ from Z into R. Define a multivalued mapping F from Z into itself by $F(z) = F_1(z) \times \dots \times F_n(z)$, where

$$F_i(z) = \{z' \in S_i(z),\ \bar{u}_i(z') = \max [\bar{u}_i(y) | y \in S_i(z)]\}.$$

Suppose that there exists $z^* \in Z$ such that z^* is a fixed point of F, i.e. $z^* \in F(z^*)$. Observe that z^* is Nash-M equilibrium. Indeed, z^* satisfies

$$\bar{u}_i(z^*) = \max [\bar{u}_i(y) | y \in S_i(z^*)] \quad (i = 1, \dots, n),$$

i.e. by Lemma 2, z^* belongs to $M_i O(u_i | S_i(z^*))$ for every i. By definition z^* is Nash-M equilibrium.

To finish the proof we have to show that F has a fixed point on Z. In view of KyFan's Fixed Point Theorem (see [3]) it suffices to show

a) $F(z)$ is nonempty convex for every $z \in Z$;
b) F is upper semi-continuous from the convex compact Z into itself.

This is equivalent to showing that F_i satisfies a) and b) for each i. For a), by assumption on φ_i we see that $S_i(z)$ is nonempty convex, so is $F_i(z)$. Let z^1 and z^2 be two points in $F_i(z)$ and λ be a number in $[0, 1]$. By the M_i-concavity of u_i, $u_i(\lambda z^1 + (1-\lambda) z^2) - \lambda u_i(z^1) - (1-\lambda) u_i(z^2) \in M_i$, hence $\langle p_i, u_i(\lambda z^1 + (1-\lambda) z^2) \rangle - \langle p_i, \lambda u_i(z^1) + (1-\lambda) u_i(z^2) \geqq 0$. It follows

$$\bar{u}_i(\lambda z^1 + (1-\lambda) z^2) \geqq \lambda \bar{u}_i(z^1) + (1-\lambda) \bar{u}_i(z^2) = \max [\bar{u}_i(y) | y \in S_i(z)],$$

hence $\lambda z^1 + (1-\lambda) z^2 \in F_i(z)$. The convexity of $F_i(z)$ is proved. For b), suppose the contrary that there are a sequence $\{z^k\}$, $z^k \in Z$ converging to $z^0 \in Z$ and a sequence $\{y^k\}$, $y^k \in F_i(z^k)$ so that

$$\varrho(y^k, F_i(z^0)) \geqq \varepsilon_0 \quad \text{for some positive } \varepsilon_0. \tag{4}$$

Without loss of generality we may assume that $\{y^k\}$ converges to y^0. The mapping φ_i is continuous (lower and upper semi-continuous) therefore $y^0 \in S_i(z^0)$. From (4) we have $\varrho(y^0, F_i(z^0)) \geqq \varepsilon_0$ or

$$\max [\bar{u}_i(y) | y \in S_i(z^0)] - \bar{u}_i(y^0) \geqq \varepsilon_1 \tag{5}$$

for some positive ε_1. By the continuity of φ_i, the compactness of S and the continuity of $\bar{u}_i$ we have

$$\max [\bar{u}_i(y) | y \in S_i(z^0)] - \max [\bar{u}_i(y) | y \in S_i(z^k)] < \frac{1}{4} \varepsilon_1 \tag{6}$$

for k larger than some integer N. The convergence of $\{y^k\}$ to y^0 and (6) yield

$$(7)\quad \max[\bar{u}_i(y)|y\in S_i(z^0)]-\bar{u}_i(y^0) = \max[\bar{u}_i(y)|y\in S_i(z^0)]-\max[\bar{u}_i(y)|y\in S_i(z^k)]+$$

$$+\max[\bar{u}_i(y)|y\in S_i(z^k)]-\bar{u}_i(y^0) \leqq \frac{1}{4}\varepsilon_1+|\bar{u}_i(y^k)-\bar{u}_i(y^0)| < \frac{\varepsilon_1}{2}$$

for k sufficiently large. (7) contradicts (5) which shows the upper semicontinuity of F_i. The theorem is proved.

3. Arrow—Debreu's model

We first describe the generalized Arrow—Debreu's model. Suppose that our economy has l kinds of commodities and there are m agents, n consumers and a price system. These agents have their sets of production possibilities $X_1, \dots, X_m$ in R^l, where every $x_i\in X_i$ characterizes a production process, for which positive coordinates correspond to product-output and negative ones to product-expediture. For the i-th consumer there is a utility function f_i defined on R^l_+ with values in R^{n_i}. Assume that a closed convex cone M_i, containing no lines is given in R^{n_i}. A price system is a set of vectors $p\in R^l_+$. We shall deal with the normed price

$$p\in P = \left\{p = (p^1, \dots, p^l)\in R^l_+,\ \sum_{k=1}^{l} p^k = 1\right\}.$$

A balance relation of production with consumption is characterized by the multi-valued mapping

$$\Theta\colon Q = X_1\times\dots\times X_m\times(R^l_+)^n\times P \to R^n$$

satisfying

i) If $\sum_{i=1}^{n} p(x_i)\geqq 0$ then $\Theta(q)\cap R^n_+\neq\emptyset$, where $q=(x, y, p)\in Q$.

ii) If $x=(x_1, \dots, x_m)$, $x_i\in X_i$, $y=(y_1, \dots, y_n)$, $y_j\in R^l_+$ and $p\in P$, $q=(x, y, p)\in Q$ then for any $z\in\Theta(q)$, $\sum_{k=1}^{n} z^k \leqq \sum_{i=1}^{n} p(x_i)$ where $z = (z^1, \dots, z^n)\in R^n$ and

$$p(x_i) = \sum_{k=1}^{l} p^k x_i^k,\quad x_i = (x_i^1, \dots, x_i^l)\in R^l.$$

Definition 3. A state $\bar{q}=(\bar{x}_1, \dots, \bar{x}_m, \bar{y}_1, \dots, \bar{y}_n, \bar{p})\in Q$ is called to be equilibrium iff

i) $$\sum_{i=1}^{m} \bar{x}_i \geqq \sum_{j=1}^{n} \bar{y}_j,$$

ii) $$\max[\bar{p}(x_i)|x_i\in X_i] = \bar{p}(\bar{x}_i)\quad (i = 1, \dots, m),$$

iii) $$\bar{y}_j\in M_jO\big(f_j|\{y_j\in R^l_+ : \bar{p}(y_j) \leqq z^j \text{ for some } (z^1, \dots, z^n)\in\Theta(\bar{q})\}\big).$$

REMARK. In particular, if Θ is single-valued, $\Theta=(\Theta_1, \ldots, \Theta_n)$ with $\Theta_j=\sum_{i=1}^{m} \alpha_{ij} p(x_j)$ where $\alpha_{ij} \geqq 0$, $\sum_{j=1}^{m} \alpha_{ij}=1$, f_j has its values in R^1 and $M_j=R^1_+$ then above-described model gives Arrow—Debreu's model in the classical sense (see [1]).

THEOREM 2. *Suppose that the model satisfies the following conditions:*

i) *X_i is convex compact containing* 0.
ii) *f_j is continuous, M_j-concave.*
iii) *Θ is continuous with a convex image of every point.*

Then the model possesses its equilibrium.

PROOF. We shall reduce this model to the one described in Section 2. First note that a state equilibrium must verify hypothesis i) in Definition 3. Therefore we can regard that the domain of change of y is not R^l_+, but it is Y, consisting of $y \in R^l_+$ with $y \leqq \bar{y}$ where $\bar{y} = (\bar{y}^1, \ldots, \bar{y}^l)$,

$$\bar{y}^k = \sum_{i=1}^{n} \max [x_i^k | x_i \in X_i] \quad (k = 1, \ldots, l).$$

Let us consider the following multiobjective model with $n+m+1$ subsystems: The sets of actions are

$$Z_i = X_i \quad (i = 1, \ldots, m), \quad Z_{m+j} = Y \quad (j = 1, \ldots, n), \quad Z_{m+n+1} = P.$$

The mappings restricting the domain of actions are:

$$\varphi_i(z) = X_i \quad (i = 1, \ldots, m); \quad z = (x_1, \ldots, x_m, y_1, \ldots, y_n, p);$$

$\varphi_{m+j}(z)=\{y \in Y | p(y) \leqq t^j$ for some $(t^1, \ldots, t^n) \in \Theta(z)\}$ in the case $Pr_j \Theta(z) \cap \{y \in Y: p(y)=0\} \neq \emptyset$ where $Pr_j \Theta(z)$ stands for the projection of $\Theta(z)$ on j-coordinate, $\varphi_{m+j}(z)=\{y \in Y: p(y)=0\}$ otherwise $(j=1, \ldots, n)$, $\varphi_{m+n+1}(z)=P$. The objective functions are:

$$u_i(z) = p(x_i) \quad \text{from } Z \text{ into } R^1, \quad i = 1, \ldots, m,$$

$$u_{m+j}(z) = f_j(y_j) \quad \text{from } Z \text{ into } R^{n_j}, \quad j = 1, \ldots, n,$$

$$u_{m+n+1}(z) = p\Big(\sum_j y_j - \sum_i x_i\Big) \quad \text{from } Z \text{ into } R^1.$$

In order to apply Theorem 1 we have to show that conditions i), ii) and iii) in this theorem are satisfied. For, $u_1, \ldots, u_m$ and u_{m+n+1} are concave (or R^1_+-concave), u_{m+j} is M_j-concave on $z_{m+j}=y_j$, $j=1, \ldots, n$ (by assumption on f_j). It is clear that $\varphi_1, \ldots, \varphi_m$ and φ_{m+n+1} satisfy hypothesis of Theorem 1.

For φ_{m+j} $(j=1, \ldots, m)$ we have to show that $\varphi_{m+j}(z)$ is non-empty convex compact for every $z \in Z$ and φ_{m+j} is continuous. The continuity of φ_{m+j} follows from Θ and p. Now we verify the convexity of $\varphi_{m+j}(z)$. It is sufficient to verify it in the case $Pr_j \Theta(z) \cap \{y \in Y: p(y)=0\} \neq \emptyset$, otherwise it is trivial. Let y and y' be two elements of $\varphi_{m+j}(z)$. For all $\lambda \in [0, 1]$, $\lambda y+(1-\lambda)y' \in Y$ as Y is convex. By definition, there are t and t' in $\Theta(z)$ such that $p(y)$ is smaller than t^j (the j-component of t) and

$p(y')$ is smaller that t'^j. Hence $p(\lambda y+(1-\lambda)y')=\lambda p(y)+(1-\lambda)p(y')\leqq\lambda t^j+(1-\lambda)t'^j$. Take $\bar{t}=\lambda t+(1-\lambda)t'\in\Theta(z)$, then $p(\lambda y+(1-\lambda)y')$ is smaller than $\bar{t}^j$. This shows that $\lambda y+(1-\lambda)y'\in\varphi_{m+j}(z)$. Obviously $\varphi_{m+j}(z)$ is non-empty. Its compactness follows from the compactness of Y and continuity of φ_{m+j}. Using Theorem 1 we obtain Nash-M equilibrium $\bar{z}=(\bar{x}_1, \ldots, \bar{x}_m, \bar{y}_1, \ldots, \bar{y}_n, \bar{p})$ satisfying conditions ii) and iii) in Definition 3. Moreover, $O\in X_i$ follows $p(\bar{x}_i)\geqq 0$. Thus $\Theta(\bar{z})\cap R^n_+\neq\emptyset$. For condition i), suppose the contrary that there are r kinds of commodities $k_1, \ldots, k_r$ such that

$$(8) \qquad \sum_j \bar{y}_j^{k_s}-\sum_i \bar{x}_i^{k_s}>0\geqq\sum_j \bar{y}_j^k-\sum_i \bar{x}_i^k \quad (s=1, \ldots, r;\ k\in\{1, \ldots, l\}\setminus\{k_1, \ldots, k_r\}).$$

Since u_{m+n+1} maximizes $p(\sum \bar{y}_j-\sum \bar{x}_i)$, we have

$$(9) \qquad \sum_{s=1}^r \bar{p}^{k_s}=1 \quad \text{and} \quad \bar{p}^k=0 \quad \text{for} \quad k\in\{1, \ldots, l\}\setminus\{k_1, \ldots, k_r\}.$$

From the assumption on Θ we have $\sum \bar{t}^j\leqq\sum \bar{p}(\bar{x}_i)$ hence

$$(10) \qquad \sum \bar{p}(\bar{y}_j)\leqq\sum \bar{p}(\bar{x}_i).$$

(8) and (9) imply

$$(11) \qquad \sum_j \bar{p}(\bar{y}_j)>\sum_i \bar{p}(\bar{x}_i).$$

The contradiction between (10) and (11) shows that $\sum \bar{x}_i\geqq\sum \bar{y}_j$. Theorem 2 is proved.

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ABSOLUTE CONVERGENCE OF FOURIER SERIES OF FUNCTIONS OF $\Lambda BV^{(p)}$ AND $\varphi\Lambda BV$

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1. Let f be a real valued function defined on an interval I of R^1. For $I_n=[a_n, b_n]\subset I$, set $f(I_n)=f(b_n)-f(a_n)$. The intervals I_n, $n=1, 2, \ldots$ shall be assumed to be nonoverlapping. If $\Lambda=\{\lambda_n\}$ is a nondecreasing sequence of positive real numbers such that $\sum 1/\lambda_n=\infty$, we say that f is of *Λ-bounded variation* (ΛBV) if, for every $\{I_n\}$,

$$\sum |f(I_n)|/\lambda_n<\infty.$$

This is equivalent to requiring that the sums be uniformly bounded (see [4]). We shall suppose that $I=[0, 2\pi]$.

If φ is a nonnegative convex function defined for $0\leqq x<\infty$ such that $\varphi(x)/x\to 0$ as $x\to 0$, φ is said to have property Δ_2 (or to "be Δ_2") if there is a constant $d(d\geqq 2)$ so that $\varphi(2x)\leqq d\varphi(x)$ for all $x\geqq 0$. If φ is Δ_2, we say that f is of *$\varphi\Lambda$-bounded variation* ($\varphi\Lambda$BV) if, for every $\{I_n\}$,

$$\sum \varphi(|f(I_n)|)/\lambda_n<\infty. \tag{1}$$

When $\varphi(x)=x^p$, $p>1$, this class is called $\Lambda BV^{(p)}$. For $p\geqq 1$, the integral modulus of continuity of order p of f is

$$\omega_p(f; \delta) = \sup_{0<t\leqq\delta}\Big(\int_I |f(x+t)-f(x)|^p\,dx\Big)^{1/p},$$

where f has been extended periodically to R^1.

M. Shiba [2] has shown the following

THEOREM. *If* $f\in\Lambda BV^{(p)}$, $1\leqq p<2r$, $1<r<\infty$, *and*

$$\sum_{n=1}^{\infty}\lambda_n^{1/2r}\big(\omega_{p+(2-p)s}(f; \pi/n)\big)^{1-p/2r}/n^{1-1/2s}<\infty,$$

where $1/r+1/s=1$, *then the Fourier series of f converges absolutely.*

We note that there is a misprint in the theorem as it is stated in [2].

In this paper we first improve this theorem by refining the method of Shiba and then prove a similar result for functions in the class $\varphi\Lambda$BV. The result for $\varphi\Lambda$BV, Theorem 2, is more general in that it is more widely applicable, but unfortunately it does not contain Theorem 1 as a special case. We have encountered this phenomenon previously (see [1], Theorem 1).

We will make use of the following

LEMMA. *If* $c_1 \geqq c_2 \geqq \ldots \geqq c_n > 0$, $\sum_{k=1}^{n} c_k = 1$, *and* $a_1 \geqq a_2 \geqq \ldots \geqq a_n$, *then*

$$(1/n) \sum_{k=1}^{n} a_k \leqq \sum_{k=1}^{n} c_k a_k. \tag{2}$$

We now state our results.

THEOREM 1. *If* $f \in \Lambda BV^{(p)}$, $1 \leqq p < 2r$, $1 \leqq r < \infty$, *and*

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} 1/\lambda_k \right)^{-1/2r} (\omega_{p+(2-p)s}(f; \pi/n))^{1-p/2r}/n^{1/2} < \infty,$$

where $1/r + 1/s = 1$, *then the Fourier series of f converges absolutely.*

When $r=1$, $s=\infty$, we take $\omega_\infty(f; \pi/n) = \omega(f; \pi/n)$, the ordinary modulus of continuity of f.

Since $n/\lambda_n \leqq \sum_{k=1}^{n} 1/\lambda_k$, Theorem 1 is an improvement on the theorem of Shiba. In fact it is possible that $n/\lambda_n = o\left(\sum_{k=1}^{n} 1/\lambda_k \right)$, for example, we have the functions of harmonic bounded variation (HBV), for which $\lambda_n \equiv n$, so that $n/\lambda_n \equiv 1$ and $\sum_{k=1}^{n} 1/\lambda_n \sim \log n$.

The case $p=1$, $r=1$, $s=\infty$ has been given by S. Wang [3].

THEOREM 2. *If* φ *is* Δ_2, $f \in \varphi \Lambda BV$, $1 \leqq p < 2r$, $1 \leqq r < \infty$, *and*

$$\sum_{n=1}^{\infty} \left[\varphi^{-1} \left(\left(\sum_{k=1}^{n} 1/\lambda_k \right)^{-1} \omega_{p+(2-p)s}^{2r-p}(f; \pi/n) \right) \right]^{1/2r} \Big/ n^{1/2} < \infty,$$

where $1/r + 1/s = 1$, *then the Fourier series of f converges absolutely.*

In what follows we shall use C to denote constants, which are not necessarily the same at each occurrence.

2. PROOF OF THE LEMMA. Apply summation by parts to the difference of the two sides of (2) to obtain

$$\sum_{k=1}^{n} (c_k - 1/n) a_k = \sum_{k=1}^{n-1} (a_k - a_{k+1}) \left(\sum_{i=1}^{k} (c_i - 1/n) \right) \geqq 0$$

since $\sum_{i=1}^{k} (c_i - 1/n) \geqq 0$ and $a_k - a_{k+1} \geqq 0$ for $k = 1, \ldots, n-1$.

3. PROOF OF THEOREM 1. Suppose that $r > 1$, and note that $2 = ((2-p)s + p)/s + p/r$. We set $I_k(x) = [x + (k-1)\pi/N, x + k\pi/N]$, and obtain by Hölder's inequality

$$\int_0^{2\pi} |f(I_k(x))|^2 dx \leqq \left(\int_0^{2\pi} |f(I_k(x))|^{(2-p)s+p} dx \right)^{1/s} \left(\int_0^{2\pi} |f(I_k(x))|^p dx \right)^{1/r} \leqq$$

$$\leqq \Omega_N^{1/r} \left(\int_0^{2\pi} |f(I_k(x))|^p dx \right)^{1/r},$$

where $\Omega_N=(\omega_{(2-p)s+p}(f;\pi/N))^{2r-p}$. Thus

$$\sum_{k=1}^{N}\Big(\int_0^{2\pi}|f(I_k(x))|^2\,dx\Big)^r=\Omega_N\int_0^{2\pi}\sum_{k=1}^{N}|f(I_k(x))|^p\,dx.$$

If $k_j=k_j(x)$, $j=1,\dots,N$, denotes a rearrangement of $k=1,\dots,N$, such that $\{|f(I_{k_j}(x))|\}$ is nondecreasing, then, by the lemma, the above is not greater than

$$\Omega_N\Big(N/\sum_{k=1}^{N}1/\lambda_k\Big)\int_0^{2\pi}\sum_{j=1}^{N}|f(I_{k_j}(x))|^p/\lambda_{k_j}\,dx\leqq C\Omega_N\Big(N/\sum_{k=1}^{N}1/\lambda_k\Big)\,(V_\Lambda^{(p)}(f))^p,$$

where $(V_\Lambda^{(p)}(f))^p$ is the supremum of sums of the form (1) with $\varphi(x)=x^p$. By hypothesis, $V_\Lambda^{(p)}(f)<\infty$, so the above is not greater than $C\Omega_N\Big(N/\sum_{k=1}^{N}1/\lambda_k\Big)$. If a_n and b_n are the Fourier cosine and sine coefficients of f respectively and $\varrho_n^2=a_n^2+b_n^2$, by Parseval's relation,

$$\frac{1}{\pi}\int_0^{2\pi}|f(x+h)-f(x-h)|^2\,dx\geqq 4\sum_{n=1}^{N}\varrho_n^2\sin^2 nh.$$

Thus

$$N\Big(\sum_{n=1}^{N}\varrho_n^2\sin^2(n\pi/2N)\Big)^r\leqq C\Omega_N N/\sum_{k=1}^{N}1/\lambda_k,\quad \sum_{n=1}^{N}\varrho_n^2\sin^2(n\pi/2N)\leqq C\Omega_N^{1/r}\Big(\sum_{k=1}^{N}1/\lambda_k\Big)^{-1/r},$$

and since $n/N\leqq\sin(n\pi/2N)$ for $n=1,2,\dots,N$,

$$\sum_{n=1}^{N}n^2\varrho_n^2\leqq CN^2\Omega_N^{1/r}\Big(\sum_{k=1}^{N}1/\lambda_k\Big)^{-1/r}. \tag{3}$$

Let $\psi_N=\sum_{n=1}^{N}n\varrho_n$, then

$$\psi_N\leqq N^{1/2}\Big(\sum_{n=1}^{N}n^2\varrho_n^2\Big)^{1/2}\leqq CN^{3/2}\Omega_N^{1/2r}\Big(\sum_{k=1}^{N}1/\lambda_k\Big)^{-1/2r}.$$

Now

$$\sum_{n=1}^{N}\varrho_n=\sum_{n=1}^{N-1}\psi_n\big(1/n-1/(n+1)\big)+\psi_N/N\leqq\sum_{n=1}^{N-1}\psi_n/n^2+\psi_N/N\leqq$$

$$\leqq C\sum_{n=1}^{N-1}n^{-1/2}\Omega_N^{1/2r}\Big(\sum_{k=1}^{n}1/\lambda_k\Big)^{-1/2r}+CN^{1/2}\Omega_N^{1/2r}\Big(\sum_{k=1}^{N}1/\lambda_k\Big)^{-1/2r}=O(1)\quad\text{as}\quad N\to\infty,$$

and the theorem is proved.

For the case $r=1$, $s=\infty$, simply note that

$$|f(I_k(x))|^2=|f(x+(k-1)\pi/N)-f(x+k\pi/N)|^{2-p}|f(I_k(x))|^p\leqq$$
$$\leqq\omega(f;\pi/N)^{2-p}|f(I_k(x))|^p$$

and proceed as above.

4. PROOF OF THEOREM 2. Since multiplying f by a positive constant alters $\omega_p(f;\delta)$ by the same constant, and φ is Δ_2, we may assume that $|f(x)|\leqq\frac{1}{2}$ for all x. As above, we obtain, for $r>1$,

$$\int_0^{2\pi}|f(I_k(x))|^2\,dx \leqq \Omega_N^{1/r}\Big(\int_0^{2\pi}|f(I_k(x))|^p\,dx\Big)^{1/r} \leqq \Omega_N^{1/r}\Big(\int_0^{2\pi}|f(I_k(x))|\,dx\Big)^{1/r},$$

and

$$\Big(\sum_{n=1}^{N} n^2\varrho_n^2/N^2\Big)^r \leqq C\Omega_N\int_0^{2\pi}|f(I_k(x))|\,dx.$$

Since $\varphi(2x)\leqq d\varphi(x)$, we have $\varphi(ax)\leqq d^{\log_2 a}\,\varphi(x)$, so

$$\varphi\Big((1/2\pi)\Big(\sum_{n=1}^{N} n^2\varrho_n^2/N^2\Big)^r\Big) \leqq d^{\log_2 C\Omega_N}\varphi\Big((1/2\pi)\int_0^{2\pi}|f(I_k(x))|\,dx\Big) =$$

$$= C\Omega_N^{\log_2 d}\varphi\Big((1/2\pi)\int_0^{2\pi}|f(I_k(x))|\,dx\Big) = C\Omega_N^{-1+\log_2 d}\,\Omega_N\,\varphi\Big((1/2\pi)\int_0^{2\pi}|f(I_k(x))|\,dx\Big),$$

and by Jensen's inequality this is not greater than $C\Omega_N\int_0^{2\pi}\varphi(|f(I_k(x))|)\,dx.$ Since the left side of the above inequality is independent of k, on averaging both sides we obtain

$$\varphi\Big(\Big(\sum_{n=1}^{N} n^2\varrho_n^2/N^2\Big)^r\Big) \leqq C\Omega_N\Big(\int_0^{2\pi}\sum_{k=1}^{N}\varphi(|f(I_k(x))|)/\lambda_k\,dx\Big)\Big/\sum_{k=1}^{N}1/\lambda_k \leqq$$

$$\leqq C\Omega_N V_{\varphi\Lambda}(f)\Big/\sum_{k=1}^{N}1/\lambda_k \leqq C\Omega_N\Big/\sum_{k=1}^{N}1/\lambda_k,$$

where $V_{\varphi\Lambda}(f)$ is the supremum of sums of the form (1). We have obtained

$$\sum_{n=1}^{N} n^2\varrho_n^2 \leqq CN^2\Big(\varphi^{-1}\Big(\Omega_N\Big/\sum_{k=1}^{N}1/\lambda_k\Big)\Big)^{1/r},$$

and the proof, along with the case $r=1$, $s=\infty$, is completed as in the proof of Theorem 1, beginning at (3).

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ТЕОРЕМЫ О РАВНОСХОДИМОСТИ

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Цель настоящей работы состоит в доказательстве нескольких теорем типа равносходимости, которые дополняют результаты главы IX книги Г. Сеге [4]. Исследование проводится методом В. А. Ильина (см. например [1], [2]), путем надлежащего развития идей работы [3]. Перейдем к точной формулировке результатов.

Пусть $G=(a, b)$ — конечный или бесконечный интервал, $K=[a', b']$ — произвольный, но фиксированный (конечный) отрезок в интервале G, $0<\delta_0<\varrho(\partial K, \partial G)$ фиксированное число (здесь $\varrho(\partial K, \partial G)$-расстояние между границами отрезка K и интервала G), $q\in L^2_{\mathrm{loc}}(G)$, т.е. $q(x)$ интегрируема с квадратом в смысле Лебега на любом компакте интервала G.

Рассмотрим призвольное неотрицательное самосопряженное расширение оператора $Lu=-u''+qu$ типа Шредингера с точечным спектром (хотя это условие в действительности излишне, наша первая теорема справедлива также в случае общего самосопряженного расширения). Обозначим через $\{U_n(x)\}$ полную ортонормированную систему собственных функций этого расширения, а через $\{\lambda_n\}$-соответствующую систему неотрицательных собственных значений, у которой наличие точек сгущения не исключается.

Для любой функции $f(x)\in L^2(G)$ составим частичную сумму

$$\sigma_\mu(f, x) = \sum_{\sqrt{\lambda_n}<\mu} (f, U_n)U_n(x), \quad (\mu>0) \tag{1}$$

(Можно показать, что эта сумма абсолютно сходится при фиксированных $x\in G$ и $\mu>0$; таким образом, результат не зависит от порядка слагаемых.)

Частичные суммы (1) будут сравниваться с модифицированными преобразованием Фурье той же самой функции $f(x)$:

$$S_\mu(f, x) = \frac{1}{\pi}\int_{x-\delta_0}^{x+\delta_0} f(t)\frac{\sin\mu(x-t)}{x-t}\,dt; \quad x\in K. \tag{2}$$

1. Основным результатом этого пункта является следующая теорема, обобщающая теоремы 9.1.2, 9.1.5 и 9.1.6 книги Г. Сегё [4].

Теорема 1. *При сделанных выше предположениях, имеет место оценка*

$$S_\mu(f, x)-\sigma_\mu(f, x) = o(1), \quad (\mu\to\infty) \tag{3}$$

равномерно относительно $x\in K$.

Доказательство. Будем исходить из равенства (34) работы [3]: при любых фиксированных $x \in K$ и $\mu > 0$ в метрике $L^2(G)$ по переменной y имеет место равенство

$$S_{\delta_0} V_\delta(|x-y|, \mu) - \sum_{\sqrt{\lambda_n} < \mu} U_n(x) U_n(y) = \frac{1}{2} \sum_{\sqrt{\lambda_n} = \mu} U_n(x) U_n(y) - \tag{4}$$

$$- \sum_{n=1}^{\infty} U_n(x) U_n(y) S_\delta [I^{\mu}_{\sqrt{\lambda_n}}(\delta)] - \frac{2}{\pi} \sum_{n=1}^{\infty} S_{\delta_0} \Big[\int_0^\delta \frac{\sin \mu t}{t} h_n(x, t)\, dt \Big] U_n(y),$$

где

$$V_\delta(t, \mu) = \begin{cases} \dfrac{1}{\pi} \dfrac{\sin \mu t}{t}, & \text{при} \quad 0 \leqq t \leqq \delta \\ 0, & \text{при} \quad t > \delta, \end{cases} \tag{5}$$

$$I^{\mu}_{\sqrt{\lambda_n}}(\delta) = \frac{2}{\pi} \int_\delta^\infty \frac{\sin \mu t}{t} \cos \sqrt{\lambda_n}\, t\, dt, \tag{6}$$

$$h_n(x, t) = \frac{1}{2\sqrt{\lambda_n}} \int_{x-t}^{x+t} q(\xi) U_n(\xi) \sin \sqrt{\lambda_n} (|x-\xi| - t)\, d\xi, \tag{7}$$

$$S_{\delta_0}\big(g(\delta)\big) = \frac{2}{\delta_0} \int_{\delta_0/2}^{\delta_0} g(\delta)\, d\delta. \tag{8}$$

В работе [1] В. А. Ильин доказал, что при предположениях теоремы 1 имеет место оценка

$$\sum_{|\sqrt{\lambda_n} - \mu| \leqq 1} |U_n(x)|^2 \leqq C \quad (x \in K,\ \mu > 0); \tag{9}$$

далее, интегрируя по частям, легко доказать оценку

$$|S_{\delta_0}[I^{\mu}_{\sqrt{\lambda_n}}(\delta)]| \leqq \frac{C(\delta_0)}{1 + |\sqrt{\lambda_n} - \mu|^2}, \quad (\mu > 0,\ \sqrt{\lambda_n} \geqq 0). \tag{10}$$

Используя эти оценки, можно доказать, что ряды, находящиеся в правой части (4), сходятся абсолютно при любых фиксированных $x \in K$, $\mu > 0$, $y \in G$. Например, для второй («главной») суммы это доказывается следующим образом (применяя неравенство Коши-Буняковского «по пачкам» и оценки (9) и (10)):

$$\sum_{n=1}^{\infty} |U_n(x) U_n(y) S_{\delta_0}[I^{\mu}_{\sqrt{\lambda_n}}(\delta)]| \leqq$$

$$\leqq \sum_{k=1}^{\infty} \Big(\sum_{k \leqq \sqrt{\lambda_n} \leqq k+1} |U_n(x)|^2 \Big)^{1/2} \Big(\sum_{k \leqq \sqrt{\lambda_n} \leqq k+1} |U_n(y)|^2 \Big)^{1/2} \frac{c(\delta_0)}{1 + |\mu - k|^2} \leqq C.$$

Умножая обе части (4) на $f(y)$ и почленно интегрируя по y на G (это законно в силу теоремы Беппо-Леви [5]), получим

$$(11)\qquad \int_a^b \Big[S_{\delta_0}V_\delta(|x-y|,\mu)-\sum_{\sqrt{\lambda_n}<\mu} U_n(x)U_n(y)\Big]f(y)\,dy =$$

$$= \frac{1}{2}\sum_{\sqrt{\lambda_n}=\mu} U_n(x)\int_a^b U_n(y)f(y)\,dy-\sum_{n=1}^{\infty} U_n(x)S_{\delta_0}[I^{\mu}_{\sqrt{\lambda_n}}(\delta)]\int_a^b U_n(y)f(y)\,dy-$$

$$-\frac{2}{\pi}\sum_{n=1}^{\infty} S_{\delta_0}\Big[\int_a^\delta \frac{\sin\mu t}{t}h_n(x,t)\,dt\Big]\int_a^b U_n(y)f(y)\,dy.$$

Так как при выполнении условий теоремы

$$\{d_k\}_{k=0}^{\infty} = \Big\{\Big(\sum_{k\le\sqrt{\lambda_n}\le k+1}\Big|\int_a^b U_n(y)f(y)\,dy\Big|^2\Big)^{1/2}\Big\}_{k=0}^{\infty}\in l_2,$$

то, используя неравенство Коши—Буняковского «по пачкам», оценки (9) и (10), для «главный суммы» правой части равенства (11) получим оценку

$$\sum_{k-1}^{\infty}\Big(\sum_{k\le\sqrt{\lambda_n}\le k+1}|U_n(x)|^2\Big)^{1/2}\Big(\sum_{k\le\sqrt{\lambda_n}\le k+1}\Big|\int_a^b U_n(y)f(y)\,dy\Big|^2\Big)^{1/2}\frac{C(\delta_0)}{1+|\mu-k|^2} = o(1);$$

$$\mu\to\infty.$$

Первая сумма правой части (11) оценивается тривиально, а третяя сумма оценивается так же как аналогичная сумма из работы [3] (стр. 1183—1184).

Таким образом, доказана следующая оценка:

$$\int_a^b S_{\delta_0}V_\delta(|x-y|,\mu)f(y)\,dy-\sigma_\mu(f,x) = o(1),\quad \mu\to\infty,\quad x\in K.$$

Для завершения доказательства теоремы 1 достаточно доказать оценку

$$\int_a^b S_{\delta_0}V_\delta(|x-y|,\mu)f(y)\,dy-\int_a^b V_{\delta_0}(|x-y|,\mu)f(y)\,dy = o(1),\quad \mu\to\infty,\quad x\in K.$$

Для этого выпишем явный вид функции $S_{\delta_0}v_\delta(|x-y|,\mu)$:

$$(12)\qquad S_{\delta_0}V_\delta(|x-y|,\mu) = \begin{cases} V_{\delta_0}(|x-y|,\mu), & \text{при}\quad |x-y|\le\dfrac{\delta_0}{2}\\ \dfrac{2(\delta_0-|x-y|)}{\delta_0}V_{\delta_0}(|x-y|,\mu), & \text{при}\quad \dfrac{\delta_0}{2}\le|x-y|\le\delta_0\\ 0, & \text{при}\quad |x-y|>\delta_0. \end{cases}$$

Воспользовавшись (12), будем иметь

$$\int_a^b [S_{\delta_0} V_\delta(|x-y|, \mu) - V_{\delta_0}(|x-y|, \mu)] f(y)\, dy =$$

$$= \frac{1}{\pi} \int_{x+\frac{\delta_0}{2}}^{x+\delta_0} \frac{\delta_0 - 2|x-y|}{\delta_0} \frac{\sin \mu(x-y)}{x-y} f(y)\, dy +$$

$$+ \frac{1}{\pi} \int_{x-\delta_0}^{x+\frac{\delta_0}{2}} \frac{\delta_0 - 2|x-y|}{\delta_0} \frac{\sin \mu(x-y)}{x-y} f(y)\, dy.$$

Интегралы в правой части последнего равенства в силу теремы Римана—Лебега стремятся к нулю при $\mu \to \infty$ $(x \in K)$. Теорема 1 доказана.

2. В этом пункте рассмотрим вопрос об оценке близости частичных сумм разложений по системе функций Якоби и тригонометрической системе. При этом, как и выше, будем следовать методу работы [3].

Пусть $G = (0, \pi)$; при вещественных $\alpha, \beta > -1$ обозначим

$$q(x) = q^{(\alpha,\beta)}(x) = -\left\{\frac{\frac{1}{4} - \alpha^2}{4\sin^2 \frac{x}{2}} + \frac{\frac{1}{4} - \beta^2}{4\cos^2 \frac{x}{2}}\right\}, \quad \lambda_n = \lambda_n^{(\alpha,\beta)} = \left(n + \frac{\alpha+\beta+1}{2}\right)^2,$$

$$C_n^{(\alpha,\beta)} = (2n+\alpha+\beta+1) \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)},$$

$$U_n(x) = U_n^{(\alpha,\beta)}(x) = C_n^{(\alpha,\beta)} \left(\sin \frac{x}{2}\right)^{\alpha+\frac{1}{2}} \left(\cos \frac{x}{2}\right)^{\beta+\frac{1}{2}} P_n^{(\alpha,\beta)}(\cos x);$$

Здесь $P_n^{(\alpha,\beta)}(x)$ обозначает полином Якоби степени n с нормировкой $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$.

Отметим, что

$$\int_0^\pi |U_n(x)|^2\, dx = 1,$$

и при $n \to \infty$: $C_n^{(\alpha,\beta)} = O(1) \sim n$.

В работе [3] для частного случая $\alpha = \beta = 0$ доказано, что для абсолютно непрерывной на $(0, \pi)$ функции $f(x)$ (класс таким функций обозначим через $W_1^1(0, \pi)$) имеет место оценка

$$S_\mu(f, x) - \sigma_\mu(f, x) = O\left(\frac{1}{\mu}\right), \quad (\mu \geqq 1)$$

равномерно относительно x из компакта K.

Здесь докажем эту оценку при произвольных $\alpha>-1$ и $\beta>-1$. Именно, справедлива следующая

Теорема 2. *Пусть* $\alpha, \beta>-1$, $f\in W_1^1(0, \pi)$. *Тогда, равномерно относительно* x *из компакта* K, *имеет место оценка*

$$S_\mu(f, x)-\sigma_\mu^{(\alpha,\beta)}(f, x) = O\left(\frac{1}{\mu}\right), \quad (\mu \geqq 1); \tag{13}$$

где

$$\sigma_\mu^{(\alpha,\beta)}(f, x) = \sum_{\sqrt{\lambda_n^{(\alpha,\beta)}}<\mu} (f, U_n^{(\alpha,\beta)})U_n^{(\alpha,\beta)}(x).$$

Замечание. Из результатов работы [3] следует, что оценка в этой теореме неулучшаема.

Доказательство соотношения (13) проводится по схеме доказательства теоремы 1, которая, как уже отмечалось, является развитием метода работы [3]. При этом существенную роль играет следующая

Лемма 1. *Если* $f\in W_1^1(0, \pi)$, *то при* $\alpha, \beta>=1$ *имеет место оценка*

$$\int_0^\pi f(x)U_n^{(\alpha,\beta)}(x)\,dx = O\left(\frac{1}{n}\right), \quad (n \geqq 1). \tag{14}$$

Доказательство Леммы 1. Представим интеграл из (14) в следующем виде

$$\int_0^\pi = \int_0^{\pi/2} + \int_{\pi/2}^\pi f(x)U_n^{(\alpha,\beta)}(x)\,dx = I_1+I_2.$$

Так как оценка интегралов I_1 и I_2 проводится аналогично, то ограничимся оценкой I_2.

Интегруруя по частям, получим

$$\int_0^{\pi/2} f(t)U_n^{(\alpha,\beta)}(t)\,dt = -\left[f(t)\int_t^{\pi/2} U_n^{(\alpha,\beta)}(\theta)\,d\theta\right]_t^{t=\frac{\pi}{2}}+\int_0^{\pi/2} f'(t)\left(\int_0^{\pi/2} U_n^{(\alpha,\beta)}(\theta)\,d\theta\right)dt.$$

Отсюда видно, что для получения требуемой оценки для I_1, достаточно доказать неравенство

$$\left|\int_t^{\pi/2} U_n^{(\alpha,\beta)}(\theta)\,d\theta\right| \leqq \frac{c}{n}; \tag{15}$$

где C не зависит от t. Для этой цели нам нужно следующее представление для $U_n^{(\alpha,\beta)}(e)$, которое следует из формулы (8.21.17) книги Г. Сеге [4] (стр. 205):

$$U_n^{(\alpha,\beta)}(\theta) = C_n^{(\alpha,\beta)}N^{-\alpha}\frac{\Gamma(n+\alpha+1)}{n!}\sqrt{\theta}\,J_\alpha(N\theta)+ \tag{16}$$

$$+\begin{cases}\sqrt{\theta}O(n^{-1}), & \text{при} \quad cn^{-1} \leqq \theta \leqq \pi/2\\ \theta^{\alpha+2}O\left(n^{\alpha-\frac{1}{2}}\right), & \text{при} \quad 0<\theta<cn^{-1},\end{cases}$$

где $J_\alpha(x)$ — функция Бесселя первого рода порядка α, C — фиксированное положительное число, $N=n+\dfrac{\alpha+\beta+1}{2}$.

Интегрирование обеих частей равенства (16), с учетом оценок для $C_n^{(\alpha,\beta)}$ и $\Gamma(n+\alpha+1)$, даст

$$\int_t^{\pi/2} U_n^{(\alpha,\beta)}(\theta)\,d\theta = O(1)\sqrt{n}\int_t^{\pi/2}\sqrt{\theta}J_\alpha(N\theta)\,d\theta+O\left(\frac{1}{n}\right). \tag{17}$$

Теперь исследуем интеграл в правой части последнего равенства. Для этого нам нужны следующие оценки функции Бесселя (см. [4], стр. 30, формулы (1.71.10) и (1.71.8)):

$$J_\alpha(z)\sim z^d,\quad z\to+0,\quad \alpha>-1,$$

$$\left|J_\alpha(z)-\sqrt{\frac{2}{\pi z}}\left[\cos\left(z-\frac{\alpha\pi}{2}-\frac{\pi}{4}\right)\left(1+\frac{a_1}{z^2}\right)+\right.\right. \tag{17'}$$

$$\left.\left.+\sin\left(z-\frac{\alpha\pi}{2}-\frac{\pi}{4}\right)\left(\frac{b_0}{z}+\frac{b_1}{z^3}\right)\right]\right|\leqq\frac{c(\delta,\alpha)}{z^{9/2}},\quad z\geqq\delta>0,\quad \alpha>-1;$$

где a_1, b_0, b_1 — некоторые постоянные, зависящие лишь от α. Итак

$$\int_t^{\pi/2}\sqrt{\theta}\,J_\alpha(N\theta)\,d\theta = N^{-3/2}\int_{Nt}^{\pi N/2}\sqrt{z}\,J_\alpha(z)\,dz = N^{-2/3}\left\{\int_{Nt}^{\delta}+\int_{\delta}^{\pi N/2}\sqrt{z}\,J_\alpha(z)\,dz\right\} = \tag{18}$$

$$=O(N^{-3/2})\int_0^{\delta}z^{1/2+\alpha}\,dz+O(N^{-3/2})\left\{\int_\delta^{\pi N/2}\cos\left(z-\frac{\alpha\pi}{2}-\frac{\pi}{4}\right)\left(1+\frac{a_1}{z^2}\right)dz+\right.$$

$$\left.+\int_\delta^{\pi N/2}\sin\left(z-\frac{\alpha\pi}{2}-\frac{\pi}{4}\right)\left(\frac{b_0}{z}+\frac{b_1}{z^3}\right)dz+\int_\delta^{\pi N}\frac{dz}{z^4}\right\}=O\left(\frac{1}{n^{3/2}}\right).$$

Здесь для третьего интеграла правой части использована оценка

$$\left|\int_{\delta_1}^{\delta_2}\frac{\sin(\theta-\sigma)}{\theta}\,d\theta\right|\leqq c(\delta_1),\quad (0<\delta_1\leqq\delta_2,\ -\infty<\sigma<\infty);$$

остальные интегралы оцениваются тривиально.

Подставив оценку (18) в равенство (17), получим неравенство (15), что, как отмечалось выше, приводит к утверждению леммы.

Обозначим через $H_p^a(0,\pi)$ $(1\leqq p<\infty,\ 0<a\leqq 1)$ класс функций введенный на странице 47 книги Г. Х. Харди и В. В. Рогозинского [7].

Теперь установим для функций из класса $H_p^a(0,\pi)$ теорему аналогичную теореме 2.

Теорема 3. *Пусть* $\alpha,\beta>=1$ *и функция* $f(x)\in H_2^a(0,\pi)$; $a<\dfrac{1}{2}$ *удовлетворяет условию*

$$\int_0^{\pi}\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}|f(\theta)|\,d\theta<\infty. \tag{19}$$

Тогда имеет место соотношение

$$\sigma_\mu^{(\alpha,\beta)}(f,x)-\sigma_\mu(f,x)=O\left(\frac{1}{\mu^a}\right)\quad(\mu\geqq 1), \tag{20}$$

(*где* $\sigma_\mu(f,x)$ *обозначает* μ*-частичную сумму ряда фурье* (*по косинусам*) *функции* f) *равномерно по* x *на каждом компакте интервала* $(0,\pi)$. *Оценка является точной по порядку в том смысле, что* $O\left(\frac{1}{\mu a}\right)$ *нельзя заменить на* $o\left(\frac{1}{\mu a}\right)$.

Доказательство. Будем исходить из равенства (11). При этом также, как в теореме 2, ограничимся оценкой коэффициента Фурье

$$\int_0^\pi f(x)U_n^{(\alpha,\beta)}(x)\,dx; \tag{21}$$

потому что остальная часть доказательства проводится по той же схеме, которая использовалась при доказательстве теоремы 1.

Также как в лемме 1 ограничимся оценкой интеграла

$$\int_0^{\pi/2} f(t)U_n^{(\alpha,\beta)}(t)\,dt=\int_0^{1/n}+\int_{1/n}^{\pi/2} f(t)U_n^{(\alpha,\beta)}(t)\,dt=A_1+A_2.$$

Обращаясь ко второй части формулы (16), находим (используя (17))

$$A_1=O(1)\sqrt{n}\int_0^{1/n}\sqrt{\theta}\,J_\alpha(N\theta)f(\theta)\,d\theta+O(n^{\alpha+1/2})\int_0^{1/n}f(\theta)\theta^{\alpha+2}\,d\theta=$$

$$=O(1)\sqrt{n}\int_0^{1/n}\theta^{1/2}(N\theta)^\alpha|f(\theta)|\,d\theta+O(n^{-3/2})=O(1)\int_0^{1/n}|f(\theta)|\,d\theta+O(n^{-3/2}).$$

Отсюда, используя для оценки последнего интеграла неравенство Коши—Буняковского, получим $A_1=O\left(\frac{1}{\sqrt{n}}\right)$.

Для оценивания A_2 используем первую часть (16) и следующую асимптотическую формулу для бесселевых функций, которая является грубой формой соотношения (17):

$$J_\alpha(x)=\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\alpha\pi}{2}-\frac{\pi}{4}\right)+O\left(\frac{1}{x^{3/2}}\right),\quad x\to\infty.$$

Имеем

$$A_2=O(\sqrt{n})\int_{1/n}^{\pi/2}\sqrt{\theta}\sqrt{\frac{2}{\pi\theta N}}\cos\left(N\theta-\frac{\alpha\pi}{2}-\frac{\pi}{4}\right)f(\theta)\,d\theta+ \tag{22}$$

$$+O(\sqrt{n})\int_{1/n}^{\pi/2}\sqrt{\theta}\frac{1}{(n\theta)^{3/2}}|f(\theta)|\,d\theta=O(1)\int_{1/n}^{\pi/2}f(\theta)\cos\left(N\theta-\frac{\alpha\pi}{2}-\frac{\pi}{4}\right)d\theta+$$

$$+O\left(\frac{1}{n}\right)\int_{1/n}^{\pi/2}\frac{|f(\theta)|}{\theta}\,d\theta=O\left(\frac{1}{n^a}\right)+O\left(\frac{1}{n^a}\right)\int_{1/n}^{\pi/2}\frac{|f(\theta)|}{\theta^a}\,d\theta.$$

Здесь была использована теорема о порядке коэффициентов Фурье по тригонометрической системе для функций из класса $H_p^a(0, \pi)$ (см. [7], стр. 47, теорема 36), а также теорема Бонне [5].

Далее, применяя к последнему интегралу в равенстве (22) неравенство Коши—Буняковского, получим $A_2 = O\left(\frac{1}{n^a}\right)$.

Лемма доказана.

4. Пусть на интервале $G=(0, \pi)$ заданы операторы $Lu=-u''+q(x)u$ и $\hat{L}u=-u''+\hat{q}(x)u$, с потенциалами $q(x)$ и $\hat{q}(x)$, удовлетворяющими условию: $q(x)$, $\hat{q}(x)\in L^p(G)$ при некотором $p>1$.

Обозначим через $\{U_n(x)\}$ и $\{\hat{U}_n(x)\}$ полные ортонормированные системы собственных функций операторов L и $\hat{L}$ соответственно (мы предполагаем, что потенциалы q и $\hat{q}$ допускают существование таких систем), а через $\{\lambda_n\}$ и $\{\hat{\lambda}_n\}$ — соответсвующие системы неотрицательных собственных значений.

Для функции $f\in H_1^a(G)$, $0<a<1$ обозначим через $\sigma_\mu(f, x)$ и $\hat{\sigma}_\mu(f, x)$ μ-ые частичные суммы разложений, отвечающих системам $\{U_n(x)\}$ и $\{\hat{U}_n(x)\}$ соответственно.

Теорема 4. *При сделанных выше соглашениях, для любой функции $f\in H_1^a(G)$, $0<a<1$ имеет место точная по порядку оценка*

$$\sigma_\mu(f, x)-\hat{\sigma}_\mu(f, x) = O\left(\frac{1}{\mu^a}\right), \quad (\mu \geqq 1),$$

равномерная по x на любом компакте интервала G.

Эта теорема усиливает результат Н. Лажетича [9]. Её доказательство основано на следующих фактах (а именно: на неравенстве (24)).

Пусть $f(x)\in H_1^a(G)$, $0<a<1$, тогда, воспользовавшись теоремой Бонне [5] и теоремой 36 книги [7] (стр. 47), будем иметь

$$\int_0^\pi f(x)e^{i\lambda x}\,dx = \int_0^\pi f(x)\cos([\lambda]+\{\lambda\})x\,dx+ \tag{23}$$

$$+i\int_0^\pi f(x)\sin([\lambda]+\{\lambda\})x\,dx = O\left(\frac{1}{1+|\lambda|^a}\right) \quad (-\infty<\lambda<\infty).$$

Из (23), в силу односторонней формулы Э. Ч. Титчмарша [8] (см. стр. 26) для $U_n(x)$:

$$U_n(x) = U_n(0)\cos\sqrt{\lambda_n}\,x+\frac{U_n'(0)}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}\,x+$$

$$+\frac{1}{\sqrt{\lambda_n}}\int_0^\pi q(\xi)U_n(\xi)\sin\sqrt{\lambda_n}\,(x-\xi)\,d\xi,$$

и оценок (см. [10])

$$|U_n(x)|\leqq c, \quad |U_n'(x)|\leqq c\sqrt{\lambda_n}, \quad (0\leqq x\leqq \pi), \quad \sum_{|\sqrt{\lambda_n}-\lambda|\leqq 1} 1\leqq c \quad (\mu\geqq 1),$$

вытекает неравенство

$$(24)\qquad \Big(\sum_{|\sqrt{\lambda_n}-\mu|\le 1}\Big|\int_0^\pi f(x)U_n(x)\,dx\Big)^{1/2} \leqq \frac{c}{(1+\sqrt{\lambda_n})^a}, \quad n=1,2,\ldots.$$

Замечание. Теорема 4 позволяет высказать предложение о справедливости оценки (20) для каждой функции f из класса $H_1^a(G)$, $0<a<1$.

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THE ASYMPTOTIC BEHAVIOUR OF THE EIGENFUNCTIONS OF HIGHER ORDER OF THE SCHRÖDINGER OPERATOR

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In the spectral theory of non-selfadjoint differential operators several properties of the eigenfunctions have been investigated (see [2], [3], [4], [5]). In [2] and [3] V. A. Il'in gave a necessary and sufficient condition to settle whether a given system of eigenfunctions of arbitrary order form a basis or not. The authors of [4] and [5] proved that — in case of the Schrödinger operator — the proportion of the different L^p norms of such a function is essentially the same as for the exponential functions. The aim of this paper is to examine the asymptotic behavior of the eigenfunctions of arbitrary order of the Schrödinger operator. Our result — which extends an old theorem of G. D. Birkhoff — shows the connection between the eigenfunctions of higher order and the exponential functions from another point of view.

Let $G=[0, 1]$, $q\in L^1(G)$ an arbitrary complex function and consider the formal differential operator

$$Lu = -u''+qu.$$

As usual, a function u_i: $G\to\mathbf{C}$, $u_i\not\equiv 0$ $(i=0, 1, \dots)$ is said to be an eigenfunction of order i (of the operator L) with the eigenvalue $\lambda\in\mathbf{C}$, if u_i is absolute continuous on G together with its derivative and if for almost all $x\in G$,

$$-u_i''(x)+q(x)u_i(x) = \lambda u_i(x)-u_{i-1}(x); \tag{1}$$

here $u_{i-1}\equiv 0$ for $i=0$ and u_{i-1} is an eigenfunction of order $i-1$, with the eigenvalue λ for $i\geqq 1$.

Introducing the notation

$$T_i \equiv \left\{z\in\mathbf{C}\colon \operatorname{Re} z \geqq 0, |z| > \max\left\{1, (4i+4)\int_G |q(x)|\,dx\right\}\right\},$$

the main result of this paper is the following:

THEOREM. *There exist functions* u_{ij}: $G\times T_i\to\mathbf{C}$ $(i=0, 1, \dots, j=1, 2)$ *such that*

(i) *for any fixed* i, j *and* ϱ, $u_{ij}(\cdot, \varrho)$: $G\to\mathbf{C}$ *is an eigenfunction of order* i *with the eigenvalue* $-\varrho^2$;

(ii) *for any fixed* i, j *and* x, $u_{ij}(x, \cdot)$: $T_i\to\mathbf{C}$ *is holomorphic in* $\operatorname{int}T_i$ *and continuous in* T_i;

(iii) *for any* $i\geqq 0$, *uniformly in* $x\in G$

$$u_{i1}(x, \varrho) = \exp(\varrho x)[x^i+O(1/\varrho)], \quad D_1 u_{i1}(x, \varrho) = \varrho\exp(\varrho x)[x^i+O(1/\varrho)],$$

$$u_{i2}(x, \varrho) = \exp(-\varrho x)[x^i+O(1/\varrho)], \quad D_1 u_{i2}(x, \varrho) = -\varrho\exp(-\varrho x)[x^i+O(1/\varrho)],$$

if $|\varrho|\to\infty$.

It follows easily from these properties the existence of constants $C(k)$, $k\geqq 0$ such that for any fixed k, the functions $u_{ij}(\cdot, \varrho)$, $i=0, 1, \dots, k, j=1, 2$, form a basis in the linear space of the eigenfunctions of order $\leqq k$ with the eigenvalue $-\varrho^2$ if $|\varrho|>C(k)$.

COROLLARY. *For any fixed* $k\geqq 0$ *and* $\varepsilon>0$ *there exists a constant* $C'(k, \varepsilon)$ *such that for* $|\varrho|>C'(k, \varepsilon)$ *all the eigenfunctions* u_k *of order* k *with the eigenvalue* $-\varrho^2$, *have the form*

$$u_k(x) = \sum_{i=0}^{k} A_{i1} f_{i1}(x) \exp(\varrho x) + A_{i2} f_{i2}(x) \exp(-\varrho x),$$

where the numbers A_{ij} *are constants, the functions* f_{ij} *are continuous and for all* $i=0, 1, \dots, k, j=1, 2.$

$$\sup\{|x^i - f_{ij}(x)|:\ 0\leqq x\leqq 1\} < \varepsilon.$$

REMARK. The case $i=0$ of the theorem is well-known [1], even for the case of an arbitrary linear differenctial operator. In a forthcoming publication we shall extend, the results of this paper to the case of an arbitrary linear differential operator, too.

The theorem will follow from several lemmas.

LEMMA 1. *Let* u_k *be an eigenfunction of order* $k\geqq 0$ *with the eigenvalue* $\lambda=-\varrho^2\neq 0$ *and put* $u_{k-1}\equiv\lambda u_k - Lu_k$. *Then for all* $x\in G$,

$$u_k(x) = \left(\frac{u_k(0)}{2}+\frac{u_k'(0)}{2\varrho}\right)\exp(\varrho x) + \left(\frac{u_k(0)}{2}-\frac{u_k'(0)}{2\varrho}\right)\exp(-\varrho x) + \int_0^x \frac{\operatorname{sh}\varrho(x-t)}{\varrho}[q(t)u_k(t)+u_{k-1}(t)]\,dt. \tag{2}$$

PROOF. We have by (1)

$$\int_0^x \frac{\operatorname{sh}\varrho(x-t)}{\varrho}[q(t)u_k(t)+u_{k-1}(t)]\,dt =$$

$$= \int_0^x \frac{\operatorname{sh}\varrho(x-t)}{\varrho}\lambda u_k(t)\,dt + \int_0^x \frac{\operatorname{sh}\varrho(x-t)}{\varrho}u_k''(t)\,dt.$$

Integrating by parts the second integral on the right side twice, we obtain (2).

COROLLARY. *If* u_{k-1} *is an arbitrary eigenfunction of order* $k-1\geqq -1$ $(u_{-1}\equiv 0)$ *with the eigenvalue* $\lambda=-\varrho^2\neq 0$ *and* a_k, b_k *are arbitrary complex numbers, then there exists a unique eigenfunction* u_k *of order* k *with this same eigenvalue such that* $u_{k-1}=\lambda u_k - Lu_k$ *and for all* $x\in G$,

$$u_k(x) = a_k\exp(\varrho x)+b_k\exp(-\varrho x)+\int_0^x \frac{\operatorname{sh}\varrho(x-t)}{\varrho}[q(t)u_k(t)+u_{k-1}(t)]\,dt. \tag{3}$$

Indeed, this follows from the existence theorem of linear differential equations.

One can easily show by induction that there exist polynomials P_j, Q_j of degree j $(j=0, 1, \dots)$ such that for any $j \geqq 0$ and $t \in \mathbf{R}$,

$$\int_0^t \operatorname{sh}(t-t_j) \int_0^{t_j} \dots \int_0^{t_1} \operatorname{sh}(t_1-t_0) \exp(t_0)\, dt_0 \dots dt_1 = P_{j+1}(t) \exp(t) + Q_j(t) \exp(-t).$$

We shall also use the notations $P_0 \equiv 1$, $Q_{-1} \equiv 0$.

LEMMA 2. *Given any complex numbers a_k, b_k, $k=0, 1, \dots, i$ and $\varrho \neq 0$, there exists a unique eigenfunction u_i of order i with the eigenvalue $\lambda = -\varrho^2$ such that — introducing the functions $u_{k-1} \equiv \lambda u_k - L u_k$, $k = i, i-1, \dots, 1$, for any $0 \leqq k \leqq i$ and for all $x \in G$,*

$$u_k(x) = \sum_{j=0}^{k} \varrho^{-2j} a_{k-j} [P_j(\varrho x) \exp(\varrho x) + Q_{j-1}(\varrho x) \exp(-\varrho x)] + \tag{4}$$
$$+ \sum_{j=0}^{k} \varrho^{-2j} b_{k-j} [P_j(-\varrho x) \exp(-\varrho x) + Q_{j-1}(-\varrho x) \exp(\varrho x)] +$$
$$+ \sum_{j=0}^{k} \varrho^{-j-1} \int_0^x \operatorname{sh} \varrho(x-x_j) \int_0^{x_j} \dots \int_0^{x_1} \operatorname{sh} \varrho(x_1-x_0)\, q(x_0)\, u_{k-j}(x_0)\, dx_0 \dots dx_j.$$

PROOF. Let us define $u_0, u_1, \dots, u_i$ recursively so as to satisfy (3) $(u_{-1} \equiv 0)$. Then, by the repeated application of (3) we obtain for any $0 \leqq k \leqq i$ and $x \in G$,

$$u_k(x) = a_k \exp(\varrho x) + b_k \exp(-\varrho x) +$$
$$+ \sum_{j=1}^{k} \varrho^{-j} a_{k-j} \int_0^x \operatorname{sh} \varrho(x-x_j) \int_0^{x_j} \dots \int_0^{x_2} \operatorname{sh} \varrho(x_2-x_1) \exp(\varrho x_1)\, dx_1 \dots dx_j +$$
$$+ \sum_{j=1}^{k} \varrho^{-j} b_{k-j} \int_0^x \operatorname{sh} \varrho(x-x_j) \int_0^{x_j} \dots \int_0^{x_2} \operatorname{sh} \varrho(x_2-x_1) \exp(-\varrho x_1)\, dx_1 \dots dx_j +$$
$$+ \sum_{j=0}^{k} \varrho^{-j-1} \int_0^x \operatorname{sh} \varrho(x-x_j) \int_0^{x_j} \dots \int_0^{x_1} \operatorname{sh} \varrho(x_1-x_0)\, q(x_0) u_{k-j}(x_0)\, dx_0 \dots dx_j.$$

In view of the definition of the polynomials P_j, Q_j, hence (4) follows after the substitution $t_j = \varrho x_j$.

LEMMA 3. *For any fixed $i \geqq 0$, the numbers a_k, b_k, $k=0, \dots, i$, can be chosen uniquely so that the first two sums on the right side of (4) for $k=i$ reduce to $x^i \exp(\varrho x)$. Moreover, in this case*

$$a_k = A_k \varrho^{i-2k} \quad b_k = B_k \varrho^{i-2k} \quad k = 0, 1, \dots, i, \tag{5}$$

where the numbers A_k, B_k do not depend on ϱ.

PROOF. We must choose the numbers a_k, b_k so as to satisfy the identity

$$x^i \exp(\varrho x) = \sum_{j=0}^{i} \varrho^{-2j} a_{i-j} [P_j(\varrho x) \exp(\varrho x) + Q_{j-1}(\varrho x) \exp(-\varrho x)] +$$
$$+ \sum_{j=0}^{i} \varrho^{-2j} b_{i-j} [P_j(-\varrho x) \exp(-\varrho x) + Q_{j-1}(-\varrho x) \exp(\varrho x)].$$

Let us write the polynomials P_j, Q_j in explicit form:

$$P_j(t) = \sum_{k=0}^{j} p_{jk} t^k, \quad p_{jj} \neq 0, \quad Q_j(t) = \sum_{k=0}^{j} q_{jk} t^k,$$

then the required identity takes the following form:

$$x^i \exp(\varrho x) = \sum_{k=0}^{i} x^k \exp(\varrho x) \left[\sum_{j=k}^{i} a_{i-j} p_{jk} \varrho^{k-2j} + \sum_{j=k+1}^{i} b_{i-j} q_{j-1,k} (-\varrho)^{k-2j} \right] +$$

$$+ \sum_{k=0}^{i} x^k \exp(-\varrho x) \left[\sum_{j=k+1}^{i} a_{i-j} q_{j-1,k} \varrho^{k-2j} + \sum_{j=k}^{i} b_{i-j} p_{jk} (-\varrho)^{k-2j} \right].$$

This is equivalent to the following system of linear equations:

$$\sum_{j=k}^{i} a_{i-j} p_{jk} \varrho^{k-2j} + \sum_{j=k+1}^{i} b_{i-j} q_{j-1,k} (-\varrho)^{k-2j} = \delta_{ik},$$

$$\sum_{j=k+1}^{i} a_{i-j} q_{j-1,k} \varrho^{k-2j} + \sum_{j=k}^{i} b_{i-j} p_{jk} (-\varrho)^{k-2j} = 0,$$

$k=0, 1, \ldots, i$. Considering these equations for $k=i$;

$$a_0 p_{ii} \varrho^{-i} = 1, \quad b_0 p_{ii} (-\varrho)^{-i} = 0,$$

we obtain $(p_{ii} \neq 0!)$

$$a_0 = p_{ii}^{-1} \varrho^i \equiv A_0 \varrho^i, \quad b_0 = 0 \equiv 0 \cdot \varrho^i.$$

Suppose now that $a_0, b_0, \ldots, a_m, b_m$ are determined uniquely from the equations with $k=i, i-1, \ldots, i-m$ $(m<i)$ and

$$a_k = A_k \varrho^{i-2k}, \quad b_k = B_k \varrho^{i-2k}, \quad k = 0, \ldots, m,$$

where the numbers A_k, B_k are independent of ϱ. Then the equations for $k=i-m-1$ can be written in the following form:

$$a_{m+1} p_{i-m-1, i-m-1} \varrho^{m+1-i} + \sum_{j=i-m}^{i} A_{i-j} p_{j, i-m-1} \varrho^{-m-1} +$$

$$+ \sum_{j=i-m}^{i} B_{i-j} q_{j-1, i-m-1} (-\varrho)^{-m-1} = 0,$$

$$\sum_{j=i-m}^{i} A_{i-j} q_{j-1, i-m-1} \varrho^{-m-1} + \sum_{j=i-m}^{i} B_{i-j} p_{j, i-m-1} (-\varrho)^{-m-1} +$$

$$+ b_{m+1} p_{i-m-1, i-m-1} (-\varrho)^{m+1-i} = 0.$$

Hence a_{m+1}, b_{m+1} are determined uniquely and

$$a_{m+1} = A_{m+1} \varrho^{i-2m-2}, \quad b_{m+1} = B_{m+1} \varrho^{i-2m-2},$$

where the numbers A_{m+1}, B_{m+1} do not depend on ϱ. Continuing this procedure, we obtain the statement of the lemma.

Let us now define for any $i \geqq 0$ the function $u_i \equiv u_{i1}\colon G \times T_i \to \mathbf{C}$ so that for any $\varrho \in T_i$ let the subfunction $u_{i1}(\cdot, \varrho)$ be equal to the eigenfunction described in Lemmas 2 and 3. Introducing the functions

$$u_{k-1}\colon G \times T_i \to \mathbf{C}, \quad u_{k-1} = -\varrho^2 u_k + D_{11} u_k - q u_k, \quad 1 \leqq k \leqq i,$$

we have by (4) and (5) for any $0 \leqq k \leqq i$, $x \in G$ and $\varrho \in T_i$,

$$\text{(6)} \qquad u_k(x, \varrho) = \sum_{j=0}^{k} \varrho^{i-2k+j} x^j [A_{kj} \exp(\varrho x) + B_{kj} \exp(-\varrho x)] +$$

$$+ \sum_{j=0}^{k} \varrho^{-j-1} \int_0^x \operatorname{sh} \varrho(x - x_j) \int_0^{x_j} \dots \int_0^{x_1} \operatorname{sh} \varrho(x_1 - x_0)\, q(x_0)\, u_{k-j}(x_0, \varrho)\, dx_0 \dots dx_j,$$

where the numbers A_{kj}, B_{kj} $(0 \leqq j \leqq k \leqq i)$ do not depend on ϱ and $A_{ij} = \delta_{ij}$, $B_{ij} = 0$, $0 \leqq j \leqq i$. Taking the derivative of both sides, we obtain

$$\text{(7)} \qquad D_1 u_k(x, \varrho) = \sum_{j=0}^{k} \varrho^{i-2k+j+1} x^j [A_{kj} \exp(\varrho x) - B_{kj} \exp(-\varrho x)] +$$

$$+ \sum_{j=1}^{k} \varrho^{i-2k+j} j x^{j-1} [A_{kj} \exp(\varrho x) + B_{kj} \exp(-\varrho x)] +$$

$$+ \sum_{j=0}^{k} \varrho^{-j} \int_0^x \operatorname{ch} \varrho(x - x_j) \int_0^{x_j} \dots \int_0^{x_1} \operatorname{sh} \varrho(x_1 - x_0)\, q(x_0)\, u_{k-j}(x_0, \varrho)\, dx_0 \dots dx_j.$$

Let us now introduce the following notations:

$$y_k(x, \varrho) = \varrho^{k-i} \exp(-\varrho x)\, u_k(x, \varrho),$$

$$z_k(x, \varrho) = \varrho^{k-i-1} \exp(-\varrho x)\, D_1 u_k(x, \varrho),$$

$$f_k(x, \varrho) = \sum_{j=0}^{k} \varrho^{j-k} x^j [A_{kj} + B_{kj} \exp(-2\varrho x)],$$

$$g_k(x, \varrho) = \sum_{j=0}^{k} \varrho^{j-k-1} [\varrho x^j + j x^{j-1}] [A_{kj} + B_{kj} \exp(-2\varrho x)],$$

$$F_{kj}(x, t, \varrho) = \begin{cases} 0 \quad \text{if} \quad t \geqq x, \\ \displaystyle\int_t^x \dots \int_t^{x_2} \frac{1 - \exp(-2\varrho(x - x_j))}{2} \cdots \frac{1 - \exp(-2\varrho(x_1 - t))}{2} \frac{q(t)}{\varrho}\, dx_1 \dots dx_j \\ \hfill \text{if} \quad t < x, \end{cases}$$

$$G_{kj}(x, t, \varrho) = \begin{cases} 0 \quad \text{if} \quad t \geqq x, \\ \displaystyle\int_t^x \dots \int_t^{x_2} \frac{1 + \exp(-2\varrho(x - x_j))}{2} \cdots \frac{1 - \exp(-2\varrho(x_1 - t))}{2} \frac{q(t)}{\varrho}\, dx_1 \dots dx_j \\ \hfill \text{if} \quad t < x, \end{cases}$$

then the above system of integral equations (6)—(7) can be written in the following form:

$$(8)\qquad \begin{cases} y_k(x,\varrho) = f_k(x,\varrho) + \sum_{j=0}^{k} \int_0^1 F_{kj}(x,t,\varrho)\, y_{k-j}(t,\varrho)\,dt, \\ z_k(x,\varrho) = g_k(x,\varrho) + \sum_{j=0}^{k} \int_0^1 G_{kj}(x,t,\varrho)\, y_{k-j}(t,\varrho)\,dt, \end{cases} \qquad k = 0, 1, \dots, i.$$

Now we prove that the function u_i has the properties (i)—(iii) in the theorem, required from u_{i1}. (i) follows at once by the construction of u_i. (ii) and (iii) follow by the following proposition, which can be applied to the system (8) with

$$r = 2i+2, \quad C_1 = \sum_{0 \le j \le k \le i} (1+i)(|A_{kj}| + |B_{kj}|), \quad C_2 = \int_0^1 |q(x)|\,dx.$$

(We do not prove this proposition because a slightly weaker form of it can be found in [2], p. 44, and the same proof works in our case, too.)

LEMMA 4. *Consider the system of integral equations*

$$y_k(x,\varrho) = f_k(x,\varrho) + \sum_{j=1}^{r} \int_G F_{kj}(x,t,\varrho)\, y_j(t,\varrho)\,dt, \quad k = 1, \dots, r,$$

where G is a bounded interval, T is a subset of the complex plane and the measurable functions $f_k\colon G \times T \to \mathbf{C}$, $F_{kj}\colon G \times G \times T \to \mathbf{C}$ have the following properties:

a) *There exists a constant C_1 such that for all k, $x \in G$ and $\varrho \in T$, $|f_k(x,\varrho)| \le C_1$;*
b) *For all k, j and $\varrho \in T$, $F_{kj}(\cdot, \cdot, \varrho) \in L^1(G \times G)$;*
c) *There exists a constant C_2 such that for all k, j, $x \in G$ and $\varrho \in T$,*

$$\int_G |F_{kj}(x,t,\varrho)|\,dt \le C_2 |\varrho|^{-1};$$

d) *For all k, j and $x, y \in G$, the functions $f_k(x, \cdot)$ and $F_{kj}(x, y, \cdot)$ are continuous in T and holomorphic in* $\operatorname{int} T$.

Then for any fixed $\varrho \in T' \equiv \{\varrho \in T\colon |\varrho| \ge 2rC_2\}$ the above system has a unique bounded, measurable solution. Moreover,

e) *For any k and $x \in G$, $y_k(x, \cdot)$ is continuous in T' and holomorphic in* $\operatorname{int} T'$,
f) *For any k, $x \in G$ and $\varrho \in T'$,*

$$|y_k(x,\varrho) - f_k(x,\varrho)| \le 2rC_1 C_2 |\varrho|^{-1}.$$

The construction of the functions u_{i2} is analogous to the above construction of the functions u_{i1} and the theorem is proved.

The Corollary is an obvious consequence of the theorem.

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ON SZÉP'S DECOMPOSITIONS OF IDEMPOTENT-GENERATED SEMIGROUPS

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Introduction

In [5], Szép proved that every semigroup S, without nonzero annihilators, has the following disjoint decompositions:

(1) $$S = \bigcup_{i=0}^{5} S_i,$$

$$S_0 = \{a \in S \mid aS \subset S,\ \exists x \in S\colon\ x \neq 0 \text{ and } ax = 0\},$$

$$S_1 = \{a \in S \mid aS = S,\ \exists x \in S\colon\ x \neq 0 \text{ and } ax = 0\},$$

$$S_2 = \{a \in S \setminus (S_0 \cup S_1) \mid aS \subset S,\ \exists x_1, x_2 \in S\colon\ x_1 \neq x_2 \text{ and } ax_1 = ax_2\},$$

$$S_3 = \{a \in S \setminus (S_0 \cup S_1) \mid aS = S,\ \exists x_1, x_2 \in S\colon\ x_1 \neq x_2 \text{ and } ax_1 = ax_2\},$$

$$S_4 = \left\{a \in S \setminus \bigcup_{i=0}^{3} S_i / aS \subset S\right\},$$

$$S_5 = \left\{a \in S \setminus \bigcup_{i=0}^{3} S_i / aS = S\right\},$$

(2) $$S = \bigcup_{i=0}^{5} T_i,$$

$$T_0 = \{a \in S \mid Sa \subset S,\ \exists y \in S\colon\ y \neq 0 \text{ and } ya = 0\},$$

$$T_1 = \{a \in S \mid Sa = S,\ \exists y \in S\colon\ y \neq 0 \text{ and } ya = 0\},$$

$$T_2 = \{a \in S \setminus (T_0 \cup T_1) \mid Sa \subset S,\ \exists y_1, y_2 \in S\colon\ y_1 \neq y_2 \text{ and } y_1 a = y_2 a\},$$

$$T_3 = \{a \in S \setminus (T_0 \cup T_1) \mid Sa = S,\ \exists y_1, y_2 \in S\colon\ y_1 \neq y_2 \text{ and } y_1 a = y_2 a\},$$

$$T_4 = \left\{a \in S \setminus \bigcup_{i=0}^{3} T_i / Sa \subset S\right\},$$

$$T_5 = \left\{a \in S \setminus \bigcup_{i=0}^{3} T_i / Sa = S\right\}.$$

It is easy to see that the components S_i and T_i $(i=0, \dots, 5)$ are semigroups, and

* This work was performed in the sphere of G.N.S.A.G.A. of C.N.R.

many relations hold among them. The following ones will be useful for us (see bibliography for the proof):

$$S_i \cap S_j = \emptyset, \quad T_i \cap T_j = \emptyset \quad (i \neq j), \quad S_5 S_i \subseteqq S_i; \quad S_i S_5 \subseteqq S_i \quad (0 \leqq i \leqq 5)$$

$$T_i T_5 \subseteqq T_i; \quad T_5 T_i \subseteqq T_i \quad (0 \leqq i \leqq 5), \quad S_0 S_1 \subseteqq S_0; \quad T_1 T_0 \subseteqq T_0.$$

From now on, decomposition (1) will be named *left decomposition* of S, or $D_L(S)$, and decomposition (2) will be named *right decomposition* of S, or $D_R(S)$.

In the present paper, idempotent-generated semigroups are studied, using such decompositions. Hereafter we suppose that S is a semigroup, without nonzero annihilators. This is not a restriction, since every semigroup can be reduced to that case.

Besides, all our theorems can be transformed into their dual theorems, changing right operations into left operations, every subsemigroup S_i into corresponding T_i, and so on (and, of course, vice versa).

In § 1, a characterization will be given for idempotent-generated semigroups, using decomposition $D_L(S)$.

In § 2, idempotent semigroups will be studied, in the same way.

In § 3, some properties will be given for subsemigroups, appearing in decompositions $D_L(S)$ and $D_R(S)$, particularly when S is idempotent-generated.

Finally, in § 4, a characterization will be given for idempotent-generated semigroups U, which can be extended to an idempotent-generated semigroup S, such that U is a particular subsemigroup of S.

§ 1.

Let S be an idempotent-generated semigroup, and let E be a set of idempotent generators for S. In this section we will suppose that E is minimal, i.e. for every subset $E' \subset E$, E' is not a set of generator of S. At last, let I be the set of all idempotent elements of S.

LEMMA 1.1. *If $b \in S$ and $bS = S$, then $b \in I$ and b is a left unit of S. So b may appear only as the last term in every decomposition of elements of S in idempotent elements.*

PROOF. Let $b = e_1 e_2 \dots e_n$: $bS = S$; $e_i e \in E$, $i = 1, \dots, n$. Then, e_1 is a left unit of S, since $e_1 b = b$. So, if $b' = e_2 \dots e_n$, we get $b = e_1 b' = b'$. In the same way, one can prove that $b = e_n$, so that b is idempotent, and (since $bS = S$), b is a left unit for S. Then, if b appears in an irreducible decomposition by idempotents of an elements of S, it must be the last term. Q. e. d.

THEOREM 1.2. *S_5 is rectangular*[1], *and every element of S_5 is idempotent and a left unit for S. Conversely, every left unit belongs to S_5.*

PROOF. If b is a left unit of S, we get $bS = S$, and certainly $b \notin S_1 \cup S_3$. So $b \in S_5$. Conversely, Lemma 1.1 implies that for every $b \in S_5$, b is a left unit for S. At last, if $a, b \in S_5$, we get $aba = ba = a$, and S_5 is rectangular. Q. e. d.

[1] A semigroup is called *rectangular* when it is a rectangular band; a semigroup is *idempotent* when it is a band.

THEOREM 1.3. *$S_1 = S_3 = \emptyset$, so that S has no magnifying element.*

PROOF. If $bS = S$, then b is a left unit of S, and therefore it belongs to S_5. Q.e.d.

THEOREM 1.4. *If $S_0 \neq \emptyset$ then at least one generator of S is in S_0.*

PROOF. Let $b \in S_0$. Then there is $x \neq 0$ such that $bx = 0$. Let $b = e_1 e_2 \dots e_n$ $(e_1, \dots, e_n \in E)$. If $(e_2 \dots e_n)x \neq 0$, we get $e_1 \in S_0$. Otherwise let $b' = e_2 \dots e_n$; $b' \in S_0$. In such a way, one proves that one of the following cases holds: there exists i: $(e_{i+1} \dots e_n)x \neq 0$, or $e_n x = 0$. In the first case, $e_i \in S_0$; in the second case $e_n \in S_0$. Q. e. d.

LEMMA 1.5. *Let $b = e_1 \dots e_n$ $(e_1, \dots, e_n \in E)$. If one of the following conditions holds, then $b \in S_0 \cup S_2$:*

i) $e_n \in S_0 \cup S_2$,
ii) $e_{n-1} \in S_0 \cup S_2$ *and* $e_n \in S_5$.

PROOF. If $e_n \in S_0$, then there exists $x \neq 0$: $e_n x = 0$; then $bx = (e_1 \dots e_{n-1}) e_n x = 0$. The same holds if $e_{n-1} \in S_0$ and $e_n \in S_5$.

If $e_n \in S_2$ or $e_{n-1} \in S_2$ and $e_n \in S_5$ then, in a similar way, $b \in S - S_0$ yields $b \in S_2$. Q. e. d.

LEMMA 1.6. *If $b \in S_4$, then the following properties hold:*

i) $e_n \in S_4 \cup S_5$,
ii) *if* $e_n \in S_5$, *then* $e_{n-1} \in S_4$.

PROOF. It is a simple corollary of the former lemma.

THEOREM 1.7. $S_4 = \emptyset$.

PROOF. Suppose $S_4 \neq \emptyset$, and let $b = e_1 e_2 \dots e_n \in S_4$. Lemma 1.6 implies that $e_n \in S_4 \cup S_5$.

If $e_n \notin S_5$ then there exists at least one lement x of S such that $e_n x \neq x$; so that $bx = (e_1 \dots e_n)x = e_1 \dots e_n x = e_1 \dots e_n e_n x = b(e_n x)$. Then $b \notin S_0$ implies $b \in S_2$. Since $b \in S_4$ we get that $e_n \in S_5$.

In the same way as above, it can be shown that $e_1 \dots e_{n-1} \in S_0 \cup S_2$; but $S_0 S_5 \subseteq S_0$ and $S_2 S_5 \subseteq S_2$; this yields that $b \in S_0 \cup S_2$ and $S_4 = \emptyset$. Q. e. d.

THEOREM 1.8. *$\bar{S} = S_0 \cup S_2$ is an idempotent-generated subsemigroup of S.*

PROOF. It is well-known (see [3]) that $\bar{S}$ is a subsemigroup; let us prove that it is idempotent-generated.

Let $\bar{I}$ be the set of idempotent elements of $\bar{S}$. For every $x \in \bar{S}$, we have $x = e_1 e_2 \dots e_n e$, $e_1, \dots, e_n \in \bar{I}$ and $e \in \bar{I} \cup S_5$ (by Lemma 1.1). If $e \in S_5$, $(e_n e)(e_n e) = e_n(e e_n)e = e_n e_n e = e_n e$, i.e. $e_n e$ is idempotent. Besides, $e_n e \in \bar{S} S_5 \subseteq \bar{S}$ and so $e_n e \in \bar{I}$. Q. e. d.

Now, we can prove the following theorem:

THEOREM 1.9. *A semigroup S is idempotent-generated if and only if it has the following left decomposition, $D_L(S)$:*

1) $S_1=S_3=S_4=\emptyset$;
2) *S_5 is rectangular and its elements are the left units of S;*
3) *$\bar{S}=S_0\cup S_2$ is idempotent-generated.*

PROOF. From the former theorems we get that the condition is necessary.

Conversely, as every element in S_5 is idempotent, it follows immediately that the condition is sufficient. Q. e. d.

THEOREM 1.10. $\bar{S}_0=S_0$.

PROOF. If $x\in\bar{S}_0$ then $x\bar{S}\subset\bar{S}$; so that $x\notin S_5$ and there exists $y=0$, $y\in\bar{S}$: $xy=0$. This yields $x\in S_0$ and $\bar{S}_0\subseteqq S_0$.

Conversely, let $x\in S_0$ $(\subseteqq\bar{S})$, $x\neq 0$. If $x\bar{S}=\bar{S}$, x would be idempotent, a left unit for $\bar{S}$, and for every $y\neq 0$: $xy=0$, it would be $y\in S_5$; so that $0=0x=(xy)x=$ $=x(yx)=x^2=x$, and $x=0$.

We get $x\bar{S}\subset\bar{S}$ for every $x\in S_0$. Now, suppose that for every $y\in S$, $y\neq 0$: $xy=0$, y belongs to S_5. Then $x^2=(xy)x=0$ so there exists y $(=x)$, $y\in S$, $y\neq 0$, $xy=0$. So $x\in\bar{S}_0$. Q. e. d.

The following corollary follows immediately:

COROLLARY 1.11. *If S is an idempotent-generated semigroup, then $S_2=\bar{S}_2\cup\bar{S}_5$.*

COROLLARY 1.12. *Let S be an idempotent-generated semigroup. Using the notations $\bar{S}=\bar{S}^1$, $S_5=S_5^1$, $(\overline{\bar{S}^i})=\bar{S}^{i+1}$, $(S_5^i)=S_5^{i+1}$, we get the following succession:*

$$S = \bar{S}^1\cup S_5^1$$

$$\bar{S}^1 = \bar{S}^2\cup S_5^2$$

$$\vdots$$

$$\bar{S}^i = \bar{S}^{i+1}\cup S_5^{i+1}$$

$$\vdots$$

where the following properties hold:

a) *subsemigroups $\bar{S}^k$ are idempotent-generated; subsemigroups S_5^k are rectangular and their elements are the left units of $\bar{S}^{k-1}$;*

b) *if $k\geqq j$ then $s'\in S_5^j$, $s''\in S_5^k$ or $s''\in\bar{S}^k$ implies $s's''=s''$;*

c) *if $k>j$ then $S_5^kS_5^j\subseteqq S_5^k$, $\bar{S}^kS_5^j\subseteqq\bar{S}^k$.*

PROOF. Assertion a) is a corollary of Theorem 1.9.

Assertion b) follows immediately from Theorem 1.2, since $\bar{S}^k\subseteqq\bar{S}^j$, if $j\leqq k$.

Assertion c) can be proved by induction on k. Without any loss of generality, we may suppose $j=1$, writing $\bar{S}$, instead of $\bar{S}^1$, and S_5 instead of S_5^1.

It follows immediately that $S_5S_5=S_5$ and $\bar{S}S_5\subseteqq\bar{S}$. Now, suppose that we have proved that $S_5^kS_5\subseteqq S_5^k$ and $\bar{S}^kS_5\subseteqq\bar{S}^k$, if $k\geqq 1$.

Let $t\in S_5^{k+1}$ and $s\in S_5$. As $S_5^{k+1}\subseteqq\bar{S}^k$, we get $ts\in\bar{S}^k$. But, if $y\in\bar{S}^k$, $(ts)y=$ $=t(sy)=ty=y$ and ts is a left unit for $\bar{S}^k$; so that $S_5^{k+1}S_5\subseteqq S_5^{k+1}$.

Again, let $t\in\bar{S}^{k+1}$ and $s\in S_5$; since $\bar{S}^{k+1}\subseteqq\bar{S}^k$, $ts\in\bar{S}^k$ and $ts\bar{S}^k=t\bar{S}^k\subseteqq\bar{S}^k$; $\bar{S}^{k+1}S_5\subseteqq\bar{S}^{k+1}$. Q. e. d.

Corollary 1.12 implies

COROLLARY 1.13. *An idempotent-generated semigroup has one of the following four types of decomposition:*

i) $S=((((\dots)\cup S_5^4)\cup S_5^3)\cup S_5^2)\cup S_5^1$, *with infinitely many rectangular subsemigroups; then S is idempotent and* $S_0=\emptyset$;

ii) $S=S'\cup((((\dots)\cup S_5^3)\cup S_5^2)\cup S_5^1)$, *with infinitely many rectangular subsemigroups, and an idempotent-generated one,* $S'=S_0\cup S_2'$, *without any left identity;*

iii) $S=((((S_5^n\cup\dots)\cup S_5^3)\cup S_5^2)\cup S_5^1)$, *with n rectangular subsemigroups; then S is idempotent and* $S_0=\emptyset$;

iv) $S=((((\bar{S}^m\cup S_5^m)\dots)\cup S_5^3)\cup S_5^2)\cup S_5^1$, *with m rectangular subsemigroups and an idempotent-generated one,* $\bar{S}^m=S_0\cup\bar{S}_2^m$, *without any left identity.*

Decompositions i) and iii) correspond to idempotent semigroups. In the following we will see that idempotent (and even rectangular) semigroups may have different decompositions from the above ones.

§ 2.

Now, let us consider Green's relation $\mathscr{L}$ on the semigroup S, defined in the well-known manner: for every $a,b\in S$, $a\mathscr{L}b$ iff $S^1a=S^1b$ $(S^1=S\cup\{1\})$, i.e. iff there exist x and y in S^1 such that $b=xa$ and $a=yb$. If $a\in S$, we call L_a its $\mathscr{L}$-class of equivalence in S.

THEOREM 2.1. *If S is an idempotent-generated semigroup, then $a\in S_i$ implies that* $L_a\subseteqq S_i$ $(i=0,2,5)$.

PROOF. Let $a\in S_0$ and $b\in S$, with $a\mathscr{L}b$. Then there exists $x\in S$: $b=xa$ and if $y\neq 0$, $ay=0$, then $by=xay=0$ and $b\in S_0$.

In the same way, by the symmetric property of $\mathscr{L}$, we get the proof for S_2 and S_5. Q. e. d.

COROLLARY 2.2. *If the semigroup S is idempotent-generated, then for every a in S_5 we have* $L_a=\{a\}$.

PROOF. Suppose that $e_1,e_2\in S_5$, $e_1\neq e_2$. Then there exists $x\in S$: $e_1=xe_2$ if $e_1\mathscr{L}e_2$. But by Lemma 1.1, $e_1\in S_5$ can appear only as the last element in any decomposition of elements of S.

If $e_1=xe_2=(e_{i1},\dots,e_{in})e_2$, e_1 would not appear, even as the last element, and so it could not belong to any set of generators of S, in contradiction to Theorem 1.2 Q. e. d.

Now, we want to study Szép's decompositions of an idempotent semigroup. Let us prove the following lemma:

LEMMA 2.3. *The Green's $\mathscr{L}$-classes of an idempotent semigroup S are rectangular subsemigroups of S. A similar property holds for $\mathscr{R}$-classes, etc.*

PROOF. If $a,b\in S$, $a\mathscr{L}b$, then $S^1(ab)=S^1ab=(S^1a)b=(S^1b)b=S^1b$ and $ab\mathscr{L}b\mathscr{L}a$. Then L_a is a subsemigroup of S. Besides, since $a\mathscr{L}b$, there exists $x\in S$: $a=xb$. Then, $aba=xbbxb=xbxb=xb=a$, and L_a is rectangular. Q. e. d.

THEOREM 2.4. *A semigroup S is idempotent if and only if it has the following decomposition $D_L(S)$:*

1) $S_1=S_3=S_4=\emptyset$;

2) S_5 *is rectangular, and its elements are the left units of S;*

3) $S_i=\bigcup_{a\in S_i} L_a$ $(i=0,2)$ *and every class L_a is a rectangular subsemigroup of S.*

PROOF. Such a semigroup is idempotent, since it is the union of rectangular semigroups.

Conversely, if S is idempotent, Lemma 2.3 and Theorem 1.9 imply the theorem. Q. e. d.

COROLLARY 2.5. *If S is a rectangular semigroup and $S\neq\emptyset$, then $S=S_2$* or $S=S_5$.

PROOF. If S is rectangular, not trivial, then $0\notin S$ (in fact, otherwise, for every $x\in S$, $x=x0x=x0=0$). So $S=S_2\cup S_5$. Suppose that $S_5\neq\emptyset$ and $e\in S_5$. For every $x, y\in S$ we can say $xy=(ex)(ey)=(exe)y=ey=y$. It implies that $S=S_5$, if $S_5\neq\emptyset$.

Anyhow, since S is isomorphic to the cartesian product $I\times J$, with the operation $(i_1, j_1)(i_2, j_2)=(i_1, j_2)$, it is easy to show that the case $S=S_5$ corresponds to $|I|=1$, and the case $S=S_2$ corresponds to $|I|>1$. Q. e. d.

§ 3.

Hereafter, we will suppose that S does not contain any unit element. We state the dual of Theorem 1.9.

THEOREM 3.1. *A semigroup S is idempotent-generated if and only if it has a decomposition $D_R(S)$ with the following properties:*

1) $T_1=T_3=T_4=\emptyset$;
2) T_5 *is rectangular, and every element of T_5 is a right unit for S;*
3) $S=T_0\cup T_2$ *is an idempotent-generated subsemigroup of S.*

Now, we want to study some properties of subsemigroups S_i, T_i $(i=0, \dots, 5)$ in the general case, or particularly for idempotent-generated semigroups.

THEOREM 3.2. *If S is idempotent-generated, $T_5\neq\emptyset$ implies $S_5=\emptyset$, and $S_5\neq\emptyset$ implies $T_5=\emptyset$.*

PROOF. Suppose that there exist $e_1\in T_5$ and $e_2\in S_5$. Then, $e_2e_1=e_1$ (since e_2 is a left unit for S), $e_2e_1=e_2$ (since e_1 is a right for S), and so $e_1=e_2$, and $T_5=S_5=\{e_1\}$, where e_1 is a unit element for S; and we have excluded this case. Q. e. d.

Anyway, both T_5 and S_5 can be empty.

EXAMPLE 3.3. Consider the semigroup $S=\{\langle a, b,\rangle\ a^2=a,\ b^2=b,\ (ab)^2=ba\}=\{a, b, ab, ba\}$ where $ba=0$.

We get $S_0=\{b, ab, ba\}$, $S_2=\{a\}$, $T_0=\{a, ab, ba\}$, $T_2=\{b\}$ and $S_5=T_5=\varnothing$. Q. e. d.

COROLLARY 3.4. *If S is an idempotent-generated semigroup, then $S_5 \subsetneqq T_2$ implies $S=T_0$; dually, $T_5 \subsetneqq S_2$ implies $S=S_0$.*

PROOF. It is enough to prove that if $S_5 \subsetneqq T_2$, then $S \subseteqq T_0$.

If $S_5 \subsetneqq T_2$, then there exists $e \in S_5$, $e \notin T_2$; so that $e \in T_0$ (by the former lemma) and so there exists $x \in S$: $x \neq 0$, and $xe=0$. Then for every $y \in S$, $xy=x(ey)= =(xe)y=0y=0$ and $y \in T_0$. Q. e. d.

THEOREM 3.5. *For every semigroup S, $T_0 \neq \{0\}$ iff $S_0 \neq \{0\}$.*

PROOF. By the dual property, we have only to prove that $T_0 \neq \{0\}$ implies that $S_0 \neq \{0\}$. But this is trivial, since for every $x \in T_0$ there exists $y \in S, y \neq 0$: $yx=0$ and if $x \neq 0$ then $y \in S_0$. Q. e. d.

Recall the following definitions:

$$C(a, x) = \{y \in S:\ ax = ay\},\quad D(a, x) = \{y \in S:\ xa = ya\},$$

$$C(x) = \bigcap_{a \in S} C(a, x) = \{y \in S:\ \forall a \in S,\ ax = ay\},$$

$$D(x) = \bigcap_{a \in S} D(a, x) = \{y \in S:\ \forall a \in S,\ xa = ya\}.$$

THEOREM 3.6. *For every semigroup S, if $a \in S_2$ and $x \in S_0$, then $C(a, x) \subseteqq S_0$ (dually, if $a \in T_2$, $x \in T_0$, then $D(a, x) \subseteqq T_0$).*

PROOF. If $a \in S_2$, then there exist $x_1, x_2 \in S_2$, $x_1 \neq x_2$: $ax_1=ax_2$. If $x_1 \in S_0$, then there exists $y \in S$, $y \neq 0$: $x_1 y=0$. Then $x_2 y=0$; in fact, if $x_2 y \neq 0$ we get $a(x_2 y)=(ax_2)y=(ax_1)y=a(x_1 y)=0$ and $a \in S_0$. Since $a \in S_2$, this yields $x_2 \in S_0$. Q. e. d.

COROLLARY 3.7. *Let S be an idempotent-generated semigroup and let e belong to S_5. Then:*

i) $S_5 \subsetneqq T_2$ *implies* $D(e, x) \subseteqq T_0$; $S_5 \subseteqq T_2$ *implies that if* $D(e, x) \subsetneqq T_0$, *then* $D(e, x) \subseteqq T_2$;

ii) *if x is idempotent, then $D(e, x)$ is a semigroup.*

PROOF. i) The first part of assertion i) follows from Corollary 3.4.

For the second part, if $S_5 \subsetneqq T_0$, $e \in S_5$ then $S_5 \neq \varnothing$, $T_5=\varnothing$ and, by Theorem 3.6. $D(e, x) \subseteqq T_2$, since there exists $y \in D(e, x)$: $y \notin T_0$.

ii) Let x be idempotent and $t_1, t_2 \in D(e, x)$; since $xe=t_1 e=t_2 e$, one can say $xe=x^2 e=x(ex)e=(xe)(xe)=(t_1 e)(t_2 e)=t_1(et_2)e=t_1 t_2 e$ and $t_1 t_2 \in D(e, x)$. Q. e. d.

COROLLARY 3.8. *If S is idempotent-generated, then $x \in S_5$ implies that $C(a, x)$ is a subsemigroup for every $a \in S$; dually $x \in T_5$ implies that $D(a, x)$ is a subsemigroup for every $a \in S$.*

PROOF. Let $t_1, t_2 \in C(a, x)$, i.e. $ax=at_1$, $ax=at_2$. Then $a(t_1 t_2)=(at_1)t_2= =(ax)t_2=a(xt_2)=at_2=ax$ and $t_1 t_2 \in C(a, x)$. Q. e. d.

THEOREM 3.9. *If S is a semigroup and x is idempotent then*
i) *$D(x)$ is a subsemigroup and for every $y\in D(x)$, $D(y)=D(x)$;*
ii) *every element of $D(x)$ has index $\leqq 2$, and period$=1$;*
iii) *if $e\in D(x)$, e idempotent, then $e\mathcal{R}x$.*

PROOF. i) Let $t_1, t_2\in D(x)$, and $a\in S$, $(t_1t_2)a=t_1(t_2a)=x(t_2a)=x(xa)=x^2a=xa$ and $t_1t_2\in D(x)$. If $y\in D(x)$, then $ya=xa$ for every $a\in S$, and so $D(x)=D(y)$.

ii) If $t\in D(x)$, then $ta=xa$ for every $a\in S$. Since $t\in S$, $t^2=tt=xt$. But suppose $t^{n-1}=xt$; we get $t^n=tt^{n-1}=t(xt)=x(xt)=x^2t=xt$. Therefore: $xt=t^2=t^3=\ldots=t^n$, $n\geqq 2$.

iii) from assertion ii), we get $xe=e^2=e$. But $ex=xx=x$ and this yields $e\mathcal{R}x$. Q. e. d.

THEOREM 3.10. *If S is an idempotent-generated semigroup then*
i) *if e is idempotent, then $D(e)$ is an idempotent subsemigroup;*
ii) *if $x\in S_i$ $(i=0, 2, 5)$, then $D(x)\subseteqq S_i$, for every x;*
iii) *if e is idempotent, $e\in T_i$ $(i=0, 2, 5)$, then $D(e)\subseteqq T_i$;*
iv) *for every $x\in T_5$, $D(x)=\{x\}$;*
v) *for every $x\in S_5$, $D(x)=S_5$.*

PROOF. i) Let e be idempotent; if $S(e)=\{e\}$, the assertion is trivial. Suppose that there exists $x\neq e$: $x\in D(e)$, $x=e_1\ldots e_n$ $(e_1, \ldots, e_n\in E)$. Then for every $a\in S$, $ea=xa=e_1\ldots e_na=xe_na=ee_na$ and $ee_n\in D(e)$. But $ee_n=e(een)=(ee_n)(ee_n)=(ee_n)$ and ee_n is idempotent. But $ee_n=xe_n=e_1\ldots e_ne_n=x$ and so x is idempotent.

ii) Suppose $x\in S_0$. There exists $y\in S$, $y\neq 0$: $xy=0$; if $t\in D(x)$ then $ty=xy=0$ and so $t\in S_0$. If $x\in S_5$ then $xa=a$ for every $a\in S$. This yields that every $t\in D(x)$ must be a left unit for S, and conversely, every left unit for S must belong to $D(x)$. Therefore if $x\in S_5$, $D(x)=S_5$. This proves assertions ii) and v).

iii) By assertion i) and by the former theorem, every element $e'\in D(e)$ has the property $e'\mathcal{R}e$ when e is idempotent. But for every $x\in T_i$ $(i=0, 2, 5)$, we get $R_x\subseteqq T_i$, by the dual property of Lemma 2.1.

iv) Suppose $e\in T_5$ and $y\in D(e)$. Then $ye=ee=e$. But $e\in T_5$ implies $ye=y$ and so $y=e$. Q. e. d.

Remark that the dual theorems of 3.9 and 3.10 hold, for $C(x)$.

§ 4.

By using some properties which have been shown in Section 3, in this part we want to characterize some particular idempotent-generated semigroups. Namely, we want to give an answer to the following question: if S is an idempotent-generated semigroup, which conditions are necessary and sufficient to get an extension H of S such that H is an idempotent-generated semigroup, and S is isomorphic to $\bar{H}(=H_0\cup H_2)$?

As an answer, we shall prove the following theorem:

THEOREM 4.1. *Let S be an idempotent-generated semigroup, and let I be the set of idempotents of S. Let T be a semigroup such that every element in it is a left unit for T.*

The following conditions are equivalent:

a) i) *There exists* $\varphi\colon I\to I$ $(\varphi\neq id.)$: $\varphi(e)e'=ee'$, $\forall e, e'\in I$.

ii) *Let* $\Phi(I)$ *be the set of such applications from* I *to itself; there exists an homomorphism* $\alpha\colon T\to\Phi(I)$ *which is not trivial.*

b) i) *There exists* $\varphi'\colon S\to S$ $(\varphi'\neq id.)$: $\varphi'(x)y=xy$, $\forall x, y\in S$.

ii) *Let* $\Phi(S)$ *be the set of such applications from* S *to itself; there exists an homomorphism* $\alpha'\colon T\to\Phi(S)$, *which is not trivial.*

c) *There exists an idempotent-generated semigroup* H, *such that* $H_0\cup H_2\simeq S$ *and* $H_5\simeq T$.

PROOF. a)$\Rightarrow$b). Consider $\Phi(I)$ with the composition operator $(\varphi_1\circ\varphi_2)(e)=$ $=\varphi_2(\varphi_1(e))$. It is easy to show that $\Phi(I)$ is a semigroup (e.g. $\varphi_1\circ\varphi_2\in\Phi(I)$, in fact $\varphi_2(\varphi_1(e))e'=\varphi_1(e)e'=ee'$).

Since α is a nontrivial homomorphism, there exists in $\Phi(I)$ a subsemigroup such that every element in it is a left unit and such that it has at least one element $\bar\varphi\neq id$.

Let $\varphi'\colon S\to S$ be an application such that for every $x=e_1e_2\dots e_n\in S$, $\varphi'(x)=$ $=e_1\dots e_{n-1}\bar\varphi(e_n)$. Then for every $x\in S$ and for every $e\in I$,

$$\varphi'(x)e = e_1\dots e_{n-1}\bar\varphi(e_n)e = e_1\dots e_{n-1}e_ne = xe;$$

and for every $x, y\in S$,

$$\varphi'(x)y = \varphi'(x)e_1'\dots e_m' = xe_1'\dots e_m' = xy.$$

Therefore there exists $\varphi'\colon S\to S(\varphi'\neq id.)$: $\varphi'(x)y=xy$, $\forall x, y\in S$.

Let $\Phi(S)$ be the semigroup of all these applications $\varphi'\colon S\to S$ which can be obtained as above from the elements of $\alpha(T)$, and let $\alpha'\colon T\to\Phi(S)$ be the following application: $\forall t\in T$, $\alpha'(t)=\varphi_t\colon S\to S$ such that for every $x=e_1\dots e_n$, $\varphi_t(x)=$ $=e_1\dots e_{n-1}(\alpha(t)(e_n))$.

Since α is an homomorphism, α' is an homomorphism, too. In fact:

$$\begin{aligned}\varphi_{t_1\circ t_2}(x) &= e_1\dots e_{n-1}(\alpha(t_1t_2)(e_n)) = e_1\dots e_{n-1}((\alpha(t_1)\alpha(t_2))(e_n)) =\\ &= e_1\dots e_{n-1}(\alpha(t_2)(\alpha(t_1)(e_n))).\\ (\varphi_{t_1}\circ\varphi_{t_2})(x) &= \varphi_{t_2}(\varphi_{t_1}(x)) = \varphi_{t_2}(e_1\dots e_{n-1}(\alpha(t_1)(e_n))) =\\ &= e_1\dots e_{n-1}(\alpha(t_2)(\alpha(t_1)(e_n))).\end{aligned}$$

Therefore, $\varphi_{t_1\circ t_2}=\varphi_{t_1}\circ\varphi_{t_2}$, and $\alpha'(t_1t_2)=\alpha'(t_1)\alpha'(t_2)$. Since there exists a nontrivial α, we can guarantee that there exists a nontrivial α' such that every element in $\alpha'(T)$ is a left unit for $\alpha'(T)$.

b)$\Rightarrow$c). Consider now $H=S\cup T$ and define the following operation $+$ on H:

1) $a, b\in S$ $\quad a+b = ab\in S$,

2) $a, b\in T$ $\quad a+b = ab = b\in T$,

3) $a\in T$, $b\in S$ $\quad a+b = b\in S$,

4) $a\in S$, $b\in T$ $\quad a+b = \varphi_b(a)\in S$.

$(H, +)$ is a semigroup. In fact, the associative property holds in it. The proof is trivial in almost all cases except the following ones:

i) $a\in S$, $b\in S$, $c\in T$, $a+(b+c)=a\varphi_c(b)$, $(a+b)+c=\varphi_c(ab)$, and if $b=e_1\dots e_n$, we get $\varphi_c(ab)=\varphi_c(ae_1\dots e_n)=ae_1\dots e_{n-1}\varphi_c(e_n)=a\varphi_c(b)$.

ii) $a\in S,\quad b\in T,\quad c\in S,\quad a+(b+c)=a+c=ac,\quad (a+b)+c=\varphi_b(a)c=ac.$

iii) $a\in S,\quad b\in T,\quad c\in T,\quad a+(b+c)=\psi_{bc}(a)=\varphi_c(a),\quad (a+b)+c=$

$$=\varphi_c(\varphi_b(a))=\varphi_b\circ\varphi_c(a)=\varphi_{bc}(a)=\varphi_c(a).$$

If T° is the isomorphic image of T in H, the elements in T° are left units for H (by 2) and 3)). So $T^\circ\subseteqq H_5$.

If, for an $h\in H$, $h+x=x$, $\forall x\in H$ then $h\in S^\circ=H-T^\circ$ is impossible. In fact, $h+t=\varphi_t(h)$, $\forall t\in T$ and $h+t=t$ cannot hold. So $T^\circ\supseteqq H_5$ and $T\simeq T^\circ=H_5$, and $S\simeq S^\circ=H-T^\circ=H_0\cup H_2=\bar{H}$.

c)$\Rightarrow$a). From now on, we will identify $\bar{H}$ and S, and H_5 and T. Call $I_{\bar{H}}$ the set of idempotent elements of $\bar{H}$.

If $t\in H_5$, let φ_t be the application $\varphi_t\colon I_{\bar{H}}\to H$, $\forall e\in I_{\bar{H}}$, $\varphi_t(e)=e+t$. Then $\varphi_t(e)\in I_{\bar{H}}$. In fact $2(e+t)=(e+t)+(e+t)=e+(t+e)+t=e+e+t=e+t$. Besides $(e+t)\in\bar{H}$, by Theorem 1.8.

If it were $\varphi_t(e)=e$ for every $e\in I_{\bar{H}}$, then t would be a right unit for $\bar{H}$ and this would be in contrast to Lemma 1.1. Therefore $\varphi_t(e)\neq e$ for at least one $e\in I_{\bar{H}}$ and $\varphi_t\neq id$. $\varphi_t(e)e'=(e+t)+e'=e+(t+e')=e+e'$ for every $e, e'\in I_{\bar{H}}$. The application $\alpha\colon H_5\to\Phi(I_{\bar{H}})$, such that $\alpha(t)=\varphi_t$, $\forall t\in H_5$, is a not trivial homomorphism. In fact,

$$\alpha(t_1+t_2)(e)=\varphi_{t_1+t_2}(e)=e+(t_1+t_2),$$

$$\big(\alpha(t_1)\alpha(t_2)\big)(e)=\alpha(t_2)(e+t_1)=(e+t_1)+t_2=e+(t_1+t_2).$$

Q. e. d.

REMARK. Condition a) ii) implies some restrictions about the type of application of a) i). In fact it will be necessary that φ is idempotent, and that it is a left unit of $\Phi(I)$. This makes much minor the number of chances, as it is shown by the following example:

EXAMPLE 4.2. Let $S=\{\langle a, b\rangle,\ a^2=a,\ b^2=b,\ aba=a,\ bab=b\}=\{a, b, ab, ba\}$ and let $T=\{\langle u, v\rangle,\ u^2=u,\ v^2=v,\ uv=v,\ vu=u\}=\{u, v\}$.

Condition a) i) of Theorem 4.1 is satisfied by every application which can be obtained by using one or more of the following correspondences, and by taking the other elements as fixed points: $a\to ab$, $b\to ba$, $ab\to a$, $ba\to b$.

But by condition a) ii) we must only refer to the subsemigroup of $\Phi(I)$ which contains only the following two applications from I to itself:

$$\varphi_1\begin{cases} a\to ab\\ b\to b\\ ab\to ab\\ ba\to b\end{cases}\qquad \varphi_2\begin{cases} a\to a\\ b\to ba\\ ab\to a\\ ba\to ba.\end{cases}$$

So, we get the following possible extensions of S by T:

$$H' = \{a, b, ab, ba, u, v;\ /\varphi_u = \varphi_1 = \varphi_v\}$$

$$H'' = \{a, b, ab, ba, u, v;\ /\varphi_u = \varphi_2 = \varphi_v\}$$

$$H''' = \{a, b, ab, ba, u, v;\ /\varphi_u = \varphi_1,\ \varphi_v = \varphi_2\}$$

$$H'''' = \{a, b, ab, ba, u, v;\ /\varphi_u = \varphi_2,\ \varphi_v = \varphi_1\}.$$

Remark that in the former example $S=S_2$ and therefore $S_5=\emptyset$.

So to get that a semigroup S can be extended, $S_5\neq\emptyset$ is not necessary. But such a condition is sufficient, as it is shown by the following theorem:

THEOREM 4.3. *If the idempotent-generated semigroup S has a non empty subsemigroup S_5, then condition a) i) of Theorem* 4.1 *is always satisfied.*

PROOF. Let $k\in S_5$, and suppose that k is not a unit for S. So there must exist $e\in I$: $ek\neq e$. Then $ek=ekk$, and we have $ek\in D(k, e)$; so for every $t\in S$ we get $(ek)t=e(kt)=et$, and we may say $\varphi_k(e)=ek$. Q. e. d.

Now consider the following equivalence relation on the elements of I: $\forall e_1, e_2\in I$, $e_1\sim e_2$ iff $D(e_1)=D(e_2)$. By Theorem 3.10, we get

COROLLARY 4.4. *For every idempotent-generated semigroup S*

a) *S_5 is an equivalence class for $\sim$;*

b) *all classes of elements in T_5 are singletons;*

c) *every class is included in one and only one of the subsemigroups S_i $(i=0, 2, 5)$ and T_i $(i=0, 2, 5)$.*

Finally, it can be easily proved that

COROLLARY 4.5. *An idempotent-generated semigroup S satisfies condition a) i) of Theorem* 4.1 *if and only if there exists at least one class in $I/\sim$ which is different from a singleton.*

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SOME RESULTS ON RECORD VALUES FROM THE EXPONENTIAL AND WEIBULL LAW

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1. Introduction

Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with a common continuous distribution function $F(x)$. Define the *record times* $L(n)$ $(n \geqq 1)$ by $L(0)=1$ and $L(n)=\min\{j\colon X_j > X_{L(n-1)}\}$ $(n>1)$. Set $Y_n = X_{L(n)}$ $(n \geqq 0)$ and call Y_n $(n \geqq 0)$ the *sequence of upper records*. Because $F(x)$ is continuous we note that both $L(n)$ and Y_n $(n \geqq 0)$ are well defined.

The *joint probability element* of $Y_0, \ldots, Y_s$ is given by

$$dG(y_0, \ldots, y_s) = \frac{dF(y_0) \ldots dF(y_s)}{p(y_0) \cdots p(y_{s-1})}, \quad y_0 < \ldots < y_s, \tag{1.1}$$

where $p(x) = 1 - F(x)$. Integrating out $y_0, \ldots, y_{s-1}$ we get the *probability element* of Y_s $(s \geqq 0)$, which is given by

$$dG_s(y_s) = (s!)^{-1}\bigl(-\log p(y_s)\bigr)^s\, dF(y_s). \tag{1.2}$$

In an analogous way we can define the *sequence of lower records*. If $u=t(x)$ is a strictly increasing function then it can be easily shown that $t(Y_n)$ $(n \geqq 0)$ is a sequence of upper records resulting from a series of i.i.d. random variables each with distribution function $F\bigl(t^{-1}(u)\bigr)$. If $t(x)$ is strictly decreasing then $t(Y_n)$ $(n \geqq 0)$ is a sequence of lower records from $p\bigl(t^{-1}(u)\bigr)$. Therefore we limit ourselves to the case of upper records calling them simply *records*.

Basic in the study of records is the function $R(x) = -\log\bigl(1-F(x)\bigr)$. This function has the property of transforming records from any continuous strictly increasing distribution $F(x)$ to records from the *exponential distribution* $E(0, 1)$, where

$$E(a, b) = \begin{cases} 1-\exp\bigl(-b(x-a)\bigr) & \text{if} \quad x \geqq a, \quad b > 0 \\ 0 & \text{if} \quad x < a. \end{cases}$$

In this note we indicate that using this property some results on records can be obtained in a similar way to that introduced for the study of *order statistics* by A. Rényi [1]. The results are of similar nature to those given by G. Hajós and A. Rényi. [2]. Next, we give a characterization of the *Weibull law*, related to the results obtained

2. The exponential law

Let $Y_0, Y_1, \ldots$ be record values generated by a strictly increasing distribution function $F(x)$. We may confine ourselves to the case $F(x)=E(0,1)$. Then the *joint density* of $Y_0, \ldots, Y_s$ is given by

$$f(y_0, \ldots, y_s) = \exp(-y_s), \quad 0 < y_0 < \ldots < y_s. \tag{2.1}$$

Considering now the case where $Y_k=c_k, \ldots, Y_s=c_s$ $(1 \leqq k \leqq s)$ are fixed, we can easily see that the conditional density of the remaining variables is given by

$$f(y_0, \ldots, y_{k-1}|c_k, \ldots, c_s) = k!/c_k^k, \quad 0 < y < \ldots < y_{k-1} < c_k. \tag{2.2}$$

That is they have the same distribution as the order statistics of a random sample of size k from the *uniform distribution* on the interval $(0, c_k)$.

In a similar way, when $Y_0=c_0, \ldots, Y_k=c_k$ $(0 \leqq k \leqq s-1)$ are fixed, we get

$$f(y_{k+1}, \ldots, y_s|c_0, \ldots, c_k) = \exp\{-(y_s-c_k)\} \tag{2.3}$$

where $c_k<y_{k+1}<\ldots<y_s$. Comparing (2.3) with (2.1) we deduce that the conditional distribution of the remaining $s-k$ records is the same as the unconditional one of the first $s-k$ records from $E(c_k, 1)$.

Since (2.2) and (2.3) depend only on c_k, the previous statements hold also true when only $Y_k=c_k$ $(1 \leqq k \leqq s-1)$ is fixed. Using analogous arguments one can prove that the sets of variables $(Y_0, \ldots, Y_{k-1})$ and $(Y_{k+1}, \ldots, Y_s)$ are conditionally independent when $Y_k=c_k$ $(1 \leqq k \leqq s-1)$ is fixed. That is the sequence Y_i $(i \geqq 0)$ forms a *Markov chain.* In the case of an arbitrary continuous distribution $F(x)$, using arguments similar to the ones given in § 1 of [1], one can prove that Y_i $(i \geqq 0)$ forms a Markov chain.

Now we state the following theorem.

THEOREM 2.1. *Let $Y_0, Y_1, \ldots$ be record values from $E(0,1)$. Then Y_i/Y_k, where $0 \leqq i < k$, has the same distribution as that of the $(i+1)^{st}$ order statistic of a random sample of size k from the uniform distribution on the interval* $(0, 1)$. *Also Y_i/Y_k and Y_k are independent. In addition, the random variables $(Y_{i-1}/Y_i)^i$ $(1 \leqq i \leqq s)$ are i.i.d. each having as distribution, the uniform one on the interval* $(0, 1)$.

The proof of the above theorem follows exactly the same steps as the proof of corresponding statements for order statistics that is given in Section 6 of [2]. Therefore we omit it here. A direct but rather complicated proof of the above theorem was given by M. L. Aggarwal and A. Nagabushanam in [3]. Corresponding statements for $E(a, b)$ can be made, provided that some obvious modifications are made.

3. The Weibull law

A random variable X with probability density function given by

$$f(x) = \begin{cases} \gamma x^{\gamma-1}\exp(-\lambda x^\gamma) & \text{if} \quad x>0, \quad \lambda>0, \quad \gamma>0 \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

is said to have the Weibull law with parameters λ and γ.

Consider now the records $Y_0, Y_1, \dots, Y_k$ generated by the distribution (3.1). Making the appropriate substitutions and transformations in (1.1) we conclude that $Y_0/Y_1, Y_1/Y_2, \dots, Y_{k-1}/Y_k$ and Y_k are mutually independent. From this we deduce that Y_i/Y_k and Y_k, where $0 \leqq i < k$, are independent. We will prove below that, under some distributional assumptions, this is a characteristic property of the Weibull law.

Assume that the records Y_i and Y_k, $Y_i < Y_k$ are generated by a continuous distribution function $F(x)$. Before giving the characterization we establish some properties of $F(x)$ which follow as immediate consequences of the independence of Y_i/Y_k and Y_k.

PROPOSITION 3.1. *The independence of Y_i/Y_k and Y_k implies either $F(0)=0$ $(X \geqq 0)$ or $F(0)=1$ $(X \leqq 0)$.*

PROOF. Suppose $P(X<0)>0$ and $P(X>0)>0$, that is X takes both negative and positive values with positive probability. Then $Y_k \leqq 0$ implies $Y_i/Y_k > 0$. Now if $Y_k > 0$ then Y_i/Y_k can take both negative and positive values. But this is a contradiction to the assumption of independence. Hence the proof of the proposition.

This proposition suggests the examination of two separate cases. So we consider first the case $X \geqq 0$.

PROPOSITION 3.2. *Assume $F(x)$ is continuous, $F(0)=0$ and Y_i/Y_k is independent of Y_k. Then $0 \leqq x_1 < x_2$ implies $F(x_1) < F(x_2)$.*

PROOF. Let $x_1 < x_2$ and $F(x_1) = F(x_2) < 1$. Denote by b the smallest *point of increase* of $F(x)$ such that $x_2 \leqq b$. Then $F(b) = F(x_2)$. Suppose now $Y_k = c_k$ with $c_k > b$. Using the Markov property of record values, the conditional probability element of Y_i when $Y_k = c_k$, is given by the expression

$$\frac{k!}{i!(k-i-1)!} \frac{[-\log p(y_i)]^i}{[-\log p(c_k)]^k} \left[-\log \frac{p(c_k)}{p(y_i)}\right]^{k-i-1} \frac{dF(y_i)}{p(y_i)},$$

where $0 \leqq y < c_k$. Therefore the conditional distribution $F_i(x)$ of Y_i/Y_k given $Y_k = c_k$ is given by

$$F_i(x) = \int_0^{xc_k} \frac{k!}{i!(k-i-1)!} \left[\frac{\log p(y_i)}{\log p(c_k)}\right]^i \left[1 - \frac{\log p(y_i)}{\log p(c_k)}\right]^{k-i-1} \frac{d \log p(y_i)}{\log p(c_k)}$$

with $0 \leqq x < 1$. Next we transform the integral on the right hand side introducing the variable $t = \log p(y_i)/\log p(c_k)$ to find

$$F_i(x) = \int_0^{\log p(c_k x)/\log p(c_k)} (k-i)\binom{k}{i} z^i (1-z)^{k-i-1}\, dz. \tag{3.2}$$

From the above expression we conclude that $F_i(x_1/c_k) = F_i(x_2/c_k) = F_i(b/c_k)$ and that b/c_k is a point of increase of $F_i(x)$. Now as $F(x)$ is continuous we can find another point of increase of $F(x)$, say h, such that $c_k < h$ and $x_1/c_k < b/h$. Assume now $Y_k = h$. Then, as the conditional distribution $F_i(x)$ remains the same because of the assumed independence, we have that b/h is a point of increase of $F_i(x)$. This is a contradiction. Hence $F(x_1) < F(x_2)$. This proves the proposition.

Now we state and prove the characterization theorem.

THEOREM 3.1. *Let* $Y_i < Y_k$ *with* $0 \leqq i < k$ *be the corresponding i-th and k-th records from a continuous distribution* $F(x)$ *with* $F(0)=0$. *Then the independence of* Y_i/Y_k *and* Y_k *implies that X is distributed according to the Weibull law* (3.1).

PROOF. Proposition 3.2 suggests that the *support* of $F(x)$ must be of the form $[0, a)$ where $0 < a \leqq \infty$. From (3.2) and the assumption of independence we have that $p(x)$ must satisfy the *functional equation*

$$\log p(c_k x) = A(x) \log p(c_k), \quad 0 < c_k \leqq a, \quad 0 < x < 1. \tag{3.3}$$

If we set $A(1)=1$ then (3.3) holds for $0 < x \leqq 1$. Setting $x = \exp(-y)$, $c_k = \exp(-y_k)$ and $a(x) = \log p(e^{-x})$ we are led to the equation

$$a(y+y_k) = b(y)a(y_k), \quad 0 \leqq y, \quad \log a^{-1} < y_k.$$

This is a *Cauchy type functional equation* and its solution is $a(y) = B\exp(\theta y)$, $y \geqq \log a^{-1}$, $B<0$. Hence $p(x) = \exp(Bx^{-\theta})$, $0 \leqq x \leqq a$, $B<0$. As $p(x)$ must be decreasing and continuous we must have $\theta < 0$ and $a = +\infty$. This implies that X has the Weibull law (3.1) with $\lambda = -B$ and $\gamma = -\theta$. This concludes the proof of theorem.

Suppose now $X \leqq 0$. The analysis can be carried out in a similar fashion. As in Proposition 3.2 we can prove that X is not bounded from below and that $x_1 < x_2$ implies $F(x_1) < F(x_2)$. Therefore the support of X is of the form $(-\infty, a]$. The conditional distribution of Y_i/Y_k when $Y_k = c_k$ is given by

$$F_i(x) = \int_{\log p(c_k x)/\log p(c_k)}^{1} \frac{k!}{(k-i-1)!} t^i (1-t)^{n-i-1}\, dt, \quad x > 1, \quad c_k < a \leqq 0.$$

The independence leads to the equation $\log p(c_k x) = B(x) \log p(c_k)$, $x > 1$, $c_k < a$ and the general solution is given by $p(y) = \exp(\beta y^{\lambda})$ with $y < 0$, $\lambda < 0$, $\beta > 0$ where $a = 0$ because of continuity. This is a Weibull type distribution on the negative axis.

Instead of the assumption of independence, we may impose the stronger assumption that Y_i/Y_k given Y_k has the same distribution as the $i+1$ order statistics of a random sample of size k from the uniform distribution on $(0, 1)$. Then after some calculation we can show that this distribution is $E(0, b)$. Note also that using strictly monotonous transforms and the remarks given in the introduction, several distributions can be characterized. For this purpose we may use the independence of suitable functions of records, upper or lower ones, which result when the transforms are applied on Y_i/Y_k and Y_k.

The *monotone failure rate distributions, increasing failure rate (IFR)* and *decreasing failure rate (DFR)*, play an important role in *reliability* studies. A result connected with these distributions is quoted and proven as a corollary of Theorem 3.1.

COROLLARY 3.1. *Let* $F(0)=0$ *and* $F(y)$ *be IFR (DFR) and continuous. Then the independence of* Y_i/Y_k *and* Y_k *implies that* Y_i/Y_k *is stochastically greater (smaller) than the* $(i+1)^{st}$ *order statistic of a random sample of size k from the uniform distribution on* $[0, 1]$.

PROOF. As $F(y)$ is IFR (DFR), then $-\log p(y)$ is *convex (concave)*, for any y in $\{y : F(y) < 1, y \geqq 0\}$ (see e.g. Theorem 4.1 on page 25 of [4]). Then for any $0 < x < 1$ and any $c_k > 0$, we get, $\log p(c_k x)/\log p(c_k) \leqq x (\geqq x)$ as $\log p(0) = 0$

and $-\log p(x)$ is convex (concave). Then from (3.2) we have

$$F_i(x) \leqq (\geqq) \int_0^x \frac{k!}{i!(k-i-1)!} z^i(1-z)^{k-i-1}\,dz.$$

This proves the corollary, as the integral is the distribution function of the $(i+1)^{st}$ order statistic.

4. Remarks

It is easy to see that Y_k/Y_m and Y_n/Y_r, where $0<k<m<n<r$, are independent when $F(x)$ is the Weibull distribution. Now consider the *Pareto distribution* whose density is given by $f(x)=\frac{v}{a}\left(\frac{a}{x}\right)^{v-1}$, $0<a<x$, $0<v$. Let $H_0, H_1, \ldots$ be a sequence of records from this distribution. Then a direct computation shows that H_0, $H_0/H_1, \ldots, H_{s-1}/H_s$ are independent. This implies that the independence of any pair Y_k/Y_m and Y_n/Y_r where $0\leqq k<m\leqq n<r$ does not lead to a characterization of the Weibull or Pareto distribution.

Analogous remarks can be made for the differences between records. One can easily show that $Y_1-Y_0, \ldots, Y_s-Y_{s-1}$ are independent and that Y_i-Y_{i-1} has an exponential distribution when $Y_0, Y_1, \ldots$ come from $E(a, b)$. Consider now a random variable with distribution function $1-\exp(-\exp x)$, $-\infty<x<\infty$. Then using the definitions, after some calculations, we can see that the differences Y_i-Y_{i-1} $(1\leqq i\leqq s)$ are independent. The conclusion is that the independence of Y_m-Y_k and Y_r-Y_n where $0\leqq k<m\leqq n<r$ does not lead to a characterization of $E(a, b)$.

Both these results can be contrasted with the case of the order statistics, where the independence of the previously mentioned quotients, correspondingly differences, characterizes the Pareto distribution, correspondingly $E(a, b)$. These results have been proved by H. J. Rossberg [5].

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ÜBER EIN INTERPOLATIONSVERFAHREN VON S. N. BERNSTEIN

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Einleitung und Ergebnisse

Gegeben seien die Knoten $t_v=\tau+v\pi/n$, $v=1, 2, 3, \ldots, 2n$, und

$$(1) \qquad S_n[g](t) = \sum_{i=1}^{2n} d_n(t-t_i)g(t_i)$$

sei das trigonometrische Interpolationspolynom der Ordnung n der 2π-periodischen Funktion g. Hierbei ist

$$(2) \qquad d_n(t) = \frac{1}{2n}\sin nt \operatorname{ctg}\frac{t}{2} = \frac{1}{2n}+\frac{1}{n}\sum_{j=1}^{n-1}\cos jt+\frac{1}{2n}\cos nt.$$

Für τ wird häufig $\tau=0$ oder $\tau=-\pi/(2n)$ gewählt, doch ist für Fehlerabschätzungen die Wahl von τ unerheblich, da sie als eine Translation von g und $S_n[g]$ interpretiert werden kann.

Eine Verallgemeinerung von (1) sind die Ausdrücke

$$(3) \qquad S_{nk}[g](t) = \sum_{i=1}^{2n} d_n\left(t-\frac{2i-1}{2n}\pi\right)\cdot\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}g\left(\frac{2i-2j+k-1}{2n}\pi\right),$$

vgl. Kis [5, (7)]. Sie lassen sich auch durch S_n darstellen, so gilt zum Beispiel $S_{n0}=S^u$ mit $\tau=-\pi/(2n)$,

$$S_{n1}[g](t) = \frac{1}{2}\left(S_n[g]\left(t-\frac{\pi}{2n}\right)+S_n[g]\left(t+\frac{\pi}{2n}\right)\right)$$

mit $\tau=0$, und

$$S_{n2}[g](t) = \frac{1}{4}\left(S_n[g]\left(t-\frac{\pi}{n}\right)+2S_n[g](t)+S_n[g]\left(t+\frac{\pi}{n}\right)\right)$$

mit $\tau=-\pi/(2n)$. Ausdrücke dieser Art wurden zuerst von S. N. Bernstein [1] untersucht. Sie lassen sich auch schreiben in der Form

$$(4) \qquad \begin{cases} S_{nk}[g](t) = \displaystyle\sum_{i=1}^{2n} g\left(\frac{i\pi}{n}\right)s_i(t), & k \text{ ungerade,} \\ S_{nk}[g](t) = \displaystyle\sum_{i=1}^{2n} g\left(\frac{2i-1}{2n}\pi\right)s_i(t), & k \text{ gerade,} \end{cases}$$

mit

$$(5)\qquad \begin{cases} s_i(t) = \dfrac{1}{2^k}\displaystyle\sum_{j=0}^{k}\binom{k}{j} d_n\left(t-\dfrac{2i+2j-k}{2n}\pi\right), & k, \text{ ungerade} \\ s_i(t) = \dfrac{1}{2^k}\displaystyle\sum_{j=0}^{k}\binom{k}{j} d_n\left(t-\dfrac{2i+2j-k-1}{2n}\pi\right), & k, \text{ gerade} \end{cases}$$

vgl. Kis [5, Lemma 1].

Wie üblich sei mit H_ω die Klasse der Funktionen bezeichnet, deren Stetigkeitsmodul nicht größer als ein vorgegebener Stetigkeitsmodul ω ist. In [8] beweist Kis u. a. die Abschätzung

$$(6)\qquad |S_{nk}[g](t)-g(t)| \leqq \omega\left(\frac{\pi}{2n}\right)+(C_{nk}-1)\omega\left(\frac{\pi}{n}\right), \quad \omega \text{ konkav.}$$

Hierbei wurde von Kis [6] gezeigt, daß $C_{n2}=5/4$, $C_{n1}=\left(1+\frac{1}{\pi}\right)+O(1/n^2)$, $C_{n1}\leqq\left(1+\frac{1}{\pi}\right)$ für $n=1, 3, 5, \ldots$.

In dieser Arbeit sollen Fehlerabschätzungen für die Funktionenklasse $\mathrm{Lip}_M 1$ bewiesen werden, also für alle g mit $|g(x)-g(y)|\leqq M|x-y|$. Für $k=0$ ist bereits bekannt [3]

$$|S_n[g](t)-g(t)| \leqq \|S_n\|\frac{M\pi}{2n},$$

wobei $\|S_n\|$ die Norm des Interpolationsoperators S_n bezeichnet. Asymptotische Ausdrücke hierfür sind in [4] angegeben. Die bestmögliche Approximation durch trigonometrische Polynome n-ten Grades ist gegeben durch

$$E_n(g) \leqq \frac{M\pi}{2(n+1)}, \quad g\in \mathrm{Lip}_M 1.$$

Wir betrachten nun für $k\geqq 1$ Fehlerabschätzungen der Form

$$(7)\qquad |S_{nk}[g](t)-g(t)| \leqq c_{nk}\frac{M\pi}{2n}, \quad g\in \mathrm{Lip}_M 1.$$

Die Abschätzung (6) liefert mit $\omega(\delta)=M\delta$

$$(8)\qquad c_{n1} \leqq \left(1+\frac{2}{\pi}\right), \quad n = 1, 3, 5, \ldots, \qquad c_{n2} \leqq \frac{3}{2}, \quad n = 1, 2, 3, \ldots.$$

Als Hauptziel dieser Arbeit soll gezeigt werden, daß c_{nk} am kleinsten ist für $k=1$. Herbei sei stets $k<2n$, falls nicht anders vermerkt. Die Abschätzung in (8) für c_{n1} läßt sich wesentlich verbessern. Zunächst zeigen wir für $n=1, 2, 3, \ldots$

SATZ 1.

$$c_{n,2v} \geqq \frac{2v}{2^{2v}}\binom{2v}{v}, \quad c_{n,2v-1} \geqq \frac{2v}{2^{2v}}\binom{2v}{v}, \quad v = 1, 2, 3, \ldots.$$

Die auf der rechten Seite der Abschätzung stehenden Größen sind als Funktion von v streng monoton wachsend, wie man etwa durch vollständige Induktion beweisen kann. Wir erhalten somit

KOROLLAR 1.

$$c_{nk} \geqq \frac{3}{2}, \quad k = 3, 4, 5, \ldots.$$

Für $k=1$ und $k=2$ liefert Satz 1 mit $v=1$ nur $c_{n2} \geqq 1$ und $c_{n1} \geqq 1$. Wir zeigen daher als nächstes

SATZ 2.

$$c_{n2} > 1{,}36 - \frac{1}{2n^2}.$$

Am schwierigsten is der Fall $k=1$, denn hier ist eine möglichst genaue Abschätzung von c_{n1} nach oben und nach unten sinnvoll. Wir beweisen

SATZ 3.

$$c_{n1} = c + r_n \quad \textit{mit} \quad c = 1{,}1696\ldots \quad \textit{und} \quad -\frac{3}{4n^2} < r_n < \frac{1}{2n^2}.$$

Als Folgerung erhält man zunächst für $n>2$

KOROLLAR 2. $c_{nk} > c_{n1}$, $k=2, 3, 4, \ldots$.

Dieses Korollar ist auch richtig für $n=2$, denn hierfür errechnet man leicht durch Diskussion der Beweise für diesen konkreten Fall $c_{22} > 1{,}29$, aber $c_{21} = 1{,}1914\ldots$. Für $n=1$ ergibt sich dagegen $c_{12} = c_{11} = 1$.

Beweis der Sätze 1 und 2

Es gilt $S_{nk}[g]\left(\frac{\pi}{2n}\right) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} g\left(\frac{k+1-2j}{2n}\pi\right)$, vgl. [5, (1)], und damit

$$\left|S_{nk}[g]\left(\frac{\pi}{2n}\right) - g\left(\frac{\pi}{2n}\right)\right| \leqq \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\left|g\left(\frac{k+1-2j}{2n}\pi\right) - g\left(\frac{\pi}{2n}\right)\right| \leqq$$

$$\leqq \frac{M\pi}{2n}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\left|\frac{k}{2} - j\right|.$$

Es ist leicht, für $k<2n$ eine Funktion $g \in \mathrm{Lip}_M 1$ zu finden, für die in dieser Ungleichungskette das Gleichheitszeichen gilt. Die rechts erhaltene Summe wertet man durch Fallunterscheidung k gerade bzw. k ungerade aus, wie dies bereits in [5] beim Beweis von (22) bzw. (24) durchgeführt wurde. Damit ist Satz 1 bewiesen.

Zum Beweis von Satz 2 gehen wir aus von der Darstellung (4). Durch partielle Summation ergibt sich zunächst

$$(9)\qquad |S_{n2}[g](t)-g(t)| = \sum_{i=1-n}^{-1}\left(g\left(\frac{2i-1}{2n}\pi\right)-g\left(\frac{2i+1}{2n}\pi\right)\right)\sigma_i(t)+$$

$$+\left(g\left(-\frac{\pi}{2n}\right)-g(t)\right)\sigma_0(t)+\left(g\left(\frac{\pi}{2n}\right)-g(t)\right)\sigma_1(t)+$$

$$+\sum_{i=2}^{n}\left(g\left(\frac{2i-1}{2n}\pi\right)-g\left(\frac{2i-3}{2n}\pi\right)\right)\sigma_i(t).$$

Hierbei wurde wie bei Kis [6] für bebliebiges k zur Abkürzung eingeführt

$$(10)\qquad \sigma_i(t) = \sum_{j=1-n}^{i} s_j(t),\quad 1-n \leqq i \leqq 0;\quad \sigma_i(t) = \sum_{j=i}^{n} s_j(t),\quad 1 \leqq i \leqq n.$$

Von großer Bedeutung sind die beiden folgenden Lemmata, vgl. [5, Lemmata 3—6], [6, Lemma 2]

LEMMA 1. *Sei* $0\leqq t\leqq\frac{\pi}{2n}$. *Wenn* $2n\leqq k$, *dann ist* $s_i(t)\geqq 0$, $-n<i\leqq n$. *Wenn* $2n>k$, *dann gilt für* k *gerade*

$$(-1)^{i+\frac{k}{2}}s_i(t)\geqq 0,\quad -n<i\leqq\frac{k}{2},\quad s_i(t)\geqq 0,\quad -\frac{k}{2}<i\leqq\frac{k}{2},$$

$$(-1)^{i+\frac{k}{2}+1}s_i(t)\geqq 0,\quad \frac{k}{2}<i\leqq n,$$

für k *ungerade gilt*

$$(-1)^{i+\frac{k+1}{2}}s_i(t)\geqq 0,\quad -n<i\leqq-\frac{k+1}{2},\quad \frac{k+1}{2}\leqq i\leqq n,$$

$$s_i(t)\geqq 0,\quad -\frac{k-1}{2}\leqq i\leqq\frac{k-1}{2}.$$

Der Beweis in [5] kann wesentlich vereinfacht werden. Sei z.B. $2n>k$ und k gerade. Wie bei Kis geht man aus von (5) und (2)

$$2ns_i(t) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sin n\left(t-\frac{2i+2j-k-1}{2n}\pi\right)\operatorname{ctg}\left(\frac{t}{2}-\frac{2i+2j-k-1}{4n}\pi\right) =$$

$$=\frac{1}{2^k}\sin n\left(t+\frac{\pi}{2n}\right)(-1)^{i-\frac{k}{2}}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\operatorname{ctg}\left(\frac{t}{2}-\frac{2i+2j-k-1}{4n}\pi\right).$$

Es bleibt nur das Vorzeichen der letzten Summe zu bestimmen. Für $-n<i\leqq-k/2$ ist z. B. das Argument von $f(x)=\operatorname{ctg} x$ aus dem Intervall $(0,\pi)$. Die betreffende Summe kann nun aber als k-te Differenz von f mit der Schrittweite $h=\pi/(2n)$ inter-

pretiert werden,

$$\Delta_h^k f(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(x+kh-jh), \quad x = \frac{t}{2} - \frac{2i+k-1}{4n}\pi.$$

Hat nun $f^{(k)}$ festes Vorzeichen, so gilt bekanntlich $\operatorname{sgn} \Delta_h^k f(x) = \operatorname{sgn} f^{(k)}(x)$. In vorliegendem Falle gilt $\operatorname{sgn} f^{(k)} = (-1)^k$. Letzteres läßt sich z. B. beweisen, indem man die Beziehung $f' = -1 - f^2$ nach der Produktregel differenziert, womit man

$$f^{(k+1)}(x) = -\sum_{i=0}^{k} \binom{k}{j} f^{(j)}(x) f^{(k-j)}(x) \quad (k > 0),$$

erhält, und hiermit vollständige Induktion anwendet.

Lemma 1 benutzt Kis [6] zum Beweis von

LEMMA 2. *In Lemma* 1 *kann* s_i *ersetzt werden durch* σ_i.

Zum weiteren Beweis von Satz 2 wählen wir nun in (9) als $g(t)$, $0 \leqq t \leqq \pi$, eine Funktion g mit $g(0) = M\pi/(2n)$ und

$$g\left(\frac{2i-1}{2n}\pi\right) = \begin{cases} M\pi/n, & i = 1, 3, 5, \dots \\ 2M\pi/n, & i = 2, 4, 6, \dots, \end{cases}$$

dazwischen sei g linear. Mit $g(-t) = g(t)$ ist g auf $[-\pi, \pi]$ definiert. Aus (9) und Lemma 2 ergibt sich

$$S_{n2}[g](0) - g(0) = \frac{M\pi}{2n}\left(1 + 2\sum_{\substack{i=1-n \\ i \neq 0,1}}^{n} |\sigma_i(0)|\right). \tag{11}$$

Wie in [6, (42)] gezeigt und wegen $\sigma_0 + \sigma_1 = 1$ gilt

$$\sum_{\substack{i=1-n \\ i \neq 0,1}}^{n} |\sigma_i(t)| = \frac{1}{2} - \frac{1}{4} l_1(\cos t) - \frac{1}{4}(-1)^{n+1} l_n(\cos t)$$

mit

$$l_i(\cos t) = \frac{1}{n}(-1)^{i+1} \sin\frac{2i-1}{2n}\pi \cos nt \bigg/ \left(\cos t - \cos\frac{2i-1}{2n}\pi\right).$$

Damit errechnet sich

$$l_1(\cos 0) = \frac{1}{n}\sin\frac{\pi}{2n} \bigg/ \left(1 - \cos\frac{\pi}{2n}\right) = \frac{1}{n}\operatorname{ctg}\frac{\pi}{4n} < \frac{4}{\pi},$$

$$(-1)^{n+1} l_n(\cos 0) = \frac{1}{n}\sin\frac{\pi}{2n} \bigg/ \left(1 + \cos\frac{\pi}{2n}\right) = \frac{1}{n}\tan\frac{\pi}{4n} < \frac{1}{n^2}.$$

Somit folgt schließlich

$$\sum_{\substack{i=1-n \\ i \neq 0,1}}^{n} |\sigma_i(0)| > \left(\frac{1}{2} - \frac{1}{\pi}\right) - \frac{1}{4n^2}.$$

Dies eingesetzt in (11) ergibt die Aussage von Satz 2.

Beweis von Satz 3

Zunächst ist es nützlich zu zeigen, daß

$$(12)\qquad r_n(t) = \sup_{g \in \mathrm{Lip}_M 1} |S_{n1}[g](t) - g(t)|$$

nur für $0 \leqq t \leqq \pi/(2n)$ diskutiert werdßen muß, denn es gilt

LEMMA 3.

$$r_n(t) = r_n(-t), \qquad t \in \left[0, \frac{\pi}{2n}\right],$$

$$r_n(t) = r_n\left(t + \frac{v\pi}{n}\right), \quad t \in \left[0, \frac{\pi}{n}\right], \quad v = \pm 1, \pm 2, \ldots.$$

BEWEIS. Daß das Supremum in (12) existiert für $0 \leqq t \leqq \pi/(2n)$ folgt aus den Rechnungen zu Lemma 4. Es ist

$$r_n(-t) = \sup_{f \in \mathrm{Lip}_M 1} |S_{n1}[f](-t) - f(-t)|.$$

Sei $g(t) = f(-t)$, $t \in \mathbf{R}$. Es gilt $g \in \mathrm{Lip}_M 1$ genau dann, wenn $f \in \mathrm{Lip}_M 1$; durchläuft f alle Funktionen von $\mathrm{Lip}_M 1$, dann auch g und umgekehrt. Da $S_{n1}[f](-t) = S_{n1}[g](t)$ ist die erste Beziehung bewiesen. Analog folgt die zweite Aussage mit $g(t) = f(t + v\pi/n)$ und $S_{n1}[f](t + v\pi/n) = S_{n1}[g](t)$.

Für die folgenden Beweise werde zur Abkürzung gesetzt

$$\varepsilon(t) = \frac{1}{2}[\sigma_0(t) - \sigma_1(t)],$$

wobei die σ_i wie in (10) definiert sind. Wir beweisen nun

LEMMA 4. $|S_{n1}[g](t) - g(t)| \leqq \left(1 + \frac{4}{\pi} nt\varepsilon(t)\right)\frac{M\pi}{2n}, \ 0 \leqq t \leqq \frac{\pi}{2n}.$

BEWEIS. Wir gehen aus von

$$(13)\qquad \sum_{i=1-n}^{n} s_i(t) = 1,$$

vgl. [6, (24)]. Dies kann auch aus (5) mit Hilfe von

$$\sum_{i=1-n}^{n} d_n(t - t_i) = 1$$

gefolgert werden. Damit ist wegen (4)

$$S_{n1}[g](t) - g(t) = \sum_{i=1-n}^{n} \left(g\left(\frac{i\pi}{n}\right) - g(t)\right) s_i(t).$$

Durch partielle Summation ergibt sich mit (10)

$$S_{n1}[g](t)-g(t) = \sum_{i=1-n}^{-1}\left(g\left(\frac{i\pi}{n}\right)-g\left(\frac{i+1}{n}\pi\right)\right)\sigma_i(t)+(g(0)-g(t))\sigma_0(t)+ \tag{14}$$

$$+\left(g\left(\frac{\pi}{n}\right)-g(t)\right)\sigma_1(t)+\sum_{i=2}^{n}\left(g\left(\frac{i\pi}{n}\right)-g\left(\frac{i-1}{n}\pi\right)\right)\sigma_i(t).$$

Für $g\in \mathrm{Lip}_M 1$ folgt hieraus

$$|S_{n1}[g](t)-g(t)| \leqq \frac{M\pi}{n}\sum_{\substack{i=1-n\\ i\neq 0,1}}^{n}|\sigma_i(t)|+Mt|\sigma_0(t)|+M\left(\frac{\pi}{n}-t\right)|\sigma_1(t)|. \tag{15}$$

Zur weiteren Auswertung beachte man zunächst, daß aus (10) und (13) folgt $\sigma_0(t)+\sigma_1(t)=1$. Daher gilt

$$\sigma_0(t) = \frac{1}{2}+\varepsilon(t), \quad \sigma_1(t) = \frac{1}{2}-\varepsilon(t).$$

Wie in Lemma 2 angegeben, ist in dem zu Grunde liegenden Intervall $\sigma_0(t)\geqq 0$, $\sigma_1(t)\geqq 0$. In [6, (62)] wurde gezeigt

$$\sum_{\substack{i=1-n\\ i\neq 0}}^{n}|\sigma_i(t)| = \frac{1}{2}.$$

Hiermit folgt nun unmittelbar

$$\sum_{\substack{i=1-n\\ i\neq 0,1}}^{n}|\sigma_i(t)| = \sum_{\substack{i=1-n\\ i\neq 0}}^{n}|\sigma_i(t)|-|\sigma_1(t)| = \varepsilon(t). \tag{16}$$

Auf ähnliche Weise zeigt man

$$t\cdot|\sigma_0(t)|+\left(\frac{\pi}{n}-t\right)|\sigma_1(t)| = \frac{\pi}{2n}+\varepsilon(t)\left(2t-\frac{\pi}{n}\right).$$

Hiermit und mit (16) läßt sich nun (15) umformen zu der in Lemma 4 angegebenen Form.

Aus Lemma 3 und Lemma 4 kann nun gefolgert werden

$$c_{n1} \leqq 1+\frac{4}{\pi}\sup_{0\leqq t\leqq\frac{\pi}{2n}}(nt\varepsilon(t)). \tag{17}$$

In anderer Richtung gilt

LEMMA 5.

$$c_{n1} \geqq 1+\frac{4}{\pi}\sup_{0\leqq t\leqq\frac{\pi}{2n}}(nt\varepsilon(t)), \qquad n \text{ ungerade},$$

$$c_{n1} \geqq 1+\frac{4}{\pi}\sup_{0\leqq t\leqq\frac{\pi}{2n}}(nt\varepsilon(t))-\frac{1}{4n^2}, \quad n \text{ gerade}.$$

BEWEIS. Für $k=1$ ergibt sich in Lemma 2 speziel $\sigma_0(t)\geqq 0$, $\sigma_1(t)\geqq 0$ und

$$\operatorname{sgn}\sigma_i(t) = (-1)^{i+1}, \quad -n < i \leqq -1, \quad 2 \leqq i \leqq n. \tag{18}$$

Für jedes feste $t'\in[0, \pi/(2n)]$, n ungerade, betrachten wir folgende Funktion γ:

$$\gamma\left(\frac{i\pi}{n}\right) = \begin{cases} \dfrac{M\pi}{n} & \text{für} \quad i = 0, -2, -4, \dots \\ \dfrac{2M\pi}{n} & \text{für} \quad i = -1, -3, -5, \dots \end{cases}, \quad -\pi < \frac{i\pi}{n} \leqq 0,$$

$$\gamma(t') = M\left(\frac{\pi}{n} - t'\right)$$

$$\gamma\left(\frac{i\pi}{n}\right) = \begin{cases} M\left(\dfrac{2\pi}{n} - 2t'\right) & \text{für} \quad i = 1, 3, 5, \dots \\ M\left(\dfrac{\pi}{n} - 2t'\right) & \text{für} \quad i = 2, 4, 6, \dots \end{cases}, \quad 0 < \frac{i\pi}{n} \leqq \pi.$$

Im übrigen soll γ linear verlaufen. Man prüft leicht nach, daß $\gamma\in\operatorname{Lip}_M 1$ und

$$\gamma\left(\frac{i\pi}{n}\right) - \gamma\left(\frac{i+1}{n}\pi\right) = \frac{M\pi}{n}(-1)^{i+1}, \quad -n < i \leqq -1,$$

$$\gamma\left(\frac{i\pi}{n}\right) - \gamma\left(\frac{i-1}{n}\pi\right) = \frac{M\pi}{n}(-1)^{i+1}, \quad 2 \leqq i \leqq n,$$

$$\gamma(0) - \gamma(t') = Mt', \quad \gamma\left(\frac{\pi}{n}\right) - \gamma(t') = M\left(\frac{\pi}{n} - t'\right).$$

Dies eingesetzt in (14) ergibt auf Grund von (18)

$$S_{n1}[\gamma](t') - \gamma(t') = \frac{M\pi}{n} \sum_{\substack{i=1-n \\ i\neq 0,1}}^{n} |\sigma_i(t')| + Mt'|\sigma_0(t')| + M\left(\frac{\pi}{n} - t'\right)|\sigma_1(t')|.$$

Die weiteren Umformungen wie bei (15) ergeben

$$S_{n1}[\gamma](t') - \gamma(t') = \frac{M\pi}{2n}\left[1 + \frac{4}{\pi} nt'\varepsilon(t')\right].$$

Für t', $0\leqq t'\leqq \pi/(2n)$, wähle man denjenigen Wert, für den der Ausdruck $nt\cdot\varepsilon(t)$ maximal wird. Damit ist Lemma 5 für n ungerade bewiesen.

Für n gerade sei γ wie oben mit Ausnahme von $\gamma(\pi)$. Hierfür wählen wir $\gamma(\pi) = \dfrac{M\pi}{n}$. Hiermit ist γ 2π-periodisch und $\gamma\in\operatorname{Lip}_M 1$. Es gilt aber

$$\left[\gamma(\pi) - \gamma\left(\frac{n-1}{n}\pi\right)\right]\sigma_n(t') = \left[\frac{M\pi}{n} - M2t'\right]|\sigma_n(t')|.$$

Nach weiterer Rechnung ergibt sich hiermit

$$S_{n1}[\gamma](t')-\gamma(t') = \frac{M\pi}{2n}\left[1+\frac{4}{\pi}\,nt'\varepsilon(t)-\frac{4}{\pi}\,nt'|\sigma_n(t')|\right].$$

Beachtet man $\sigma_n=s_n$, so erhält man

$$|\sigma_n(t)| = \frac{1}{4n}\cos nt\left[\tan\left(\frac{t}{2}+\frac{\pi}{4n}\right)+\tan\left(\frac{t}{2}-\frac{\pi}{4n}\right)\right] < \frac{nt\cos nt}{\pi n^2}.$$

Hieraus errechnet sich mit der Substitution $nt=z$, $0\leqq t\leqq\pi/(2n)$,

$$\frac{4}{\pi}\,nt'|\sigma_n(t')| < \frac{1}{4n^2},$$

womit Lemma 5 vollständig bewiesen ist.

LEMMA 6.

$$\frac{4}{\pi}\sup_{0\leqq t\leqq\frac{\pi}{2n}}(nt\varepsilon(t)) = 0{,}1696\,\ldots+r_n^*,\quad |r_n^*| < \frac{1}{2n^2}\quad (n>2).$$

BEWEIS. Wir gehen aus von $\sigma_0(t)=1/2+\varepsilon(t)$ und der Darstellung von σ_0 in [6, S. 187], woraus folgt

$$\varepsilon(t) = \frac{1}{n}\sum_{j=1}^{[n/2]}\sin\left[(2j-1)\left(\frac{\pi}{2n}-t\right)\right]\operatorname{ctg}\frac{2j-1}{2n}\,\pi.$$

Aus der Potenzreihendarstellung des Sinus $\sin x=\sum_{v=1}^{\infty}a_v x^v$ ergibt sich somit nach einigen Umformungen

$$\varepsilon(t) = \sum_{v=1}^{\infty}a_v\left(\frac{\pi}{2}-nt\right)^v\left[\frac{2^v}{\pi^{v+1}}\sum_{j=1}^{[n/2]}\frac{\pi}{n}\left(\frac{2j-1}{2n}\,\pi\right)^v\operatorname{ctg}\frac{2j-1}{2n}\,\pi\right]. \tag{19}$$

Die Summe in eckigen Klammern kann als Mittelpunktformel bezüglich $f_v(x)=x^v\operatorname{ctg}x$ mit der Schrittweite $h=\pi/n$ interpretiert werden. Zunächst sei der Fall n gerade betrachtet. Es ist

$$\frac{2^v}{\pi^{v+1}}\sum_{j=1}^{[n/2]}\frac{\pi}{n}\left(\frac{2j-1}{2n}\,\pi\right)^v\operatorname{ctg}\left(\frac{2j-1}{2n}\,\pi\right) = \frac{2^v}{\pi^{v+1}}\int_0^{\pi/2}f_v(x)\,dx+r_{nv}. \tag{20}$$

Für die Mittelpunktformel ist bekannt [2, S. 43]

$$\left|\sum_{j=1}^{m}hf\left(a+\frac{2j-1}{2}\,h\right)-\int_a^b f(x)\,dx\right| \leqq (b-a)\frac{h^2}{24}\sup_{a\leqq x\leqq b}|f''(x)|.$$

Hier gilt

$$\sup_{0 \leq x \leq \frac{\pi}{2}} |f_v''(x)| = \left|f_v''\left(\frac{\pi}{2}\right)\right| = 2v\left(\frac{\pi}{2}\right)^{v-1}.$$

Im Falle n gerade ist $a=0$, $b=\pi/2$ und $h=\pi/n$ zu setzen. Im Falle n ungerade erhalten wir $a=0$, $b=\pi/2-\pi/(2n)$. Wir korrigieren in (20) daher mit dem Term

$$0 < \frac{2^v}{\pi^{v+1}} \int_b^{\pi/2} f_v(x)\,dx < \frac{\pi}{8n^2}, \quad (v \geq 1),$$

und erhalten somit für beliebiges n

$$|r_{n1}| \leq \frac{\pi}{n^2} \cdot \frac{1}{8}, \quad |r_{nv}| \leq \frac{\pi}{n^2}\left(\frac{2v+3}{24}\right), \quad v > 1.$$

Im Falle $v=1$ wurde berücksichtigt, daß f konkav ist und die Mittelpunktformel daher einen größeren Wert liefert als das Integral. Die Integrale in (20) sind für kleine v leicht auszuwerten, für große v beachte man die einfache Abschätzung $\operatorname{ctg} x < 1/x$. Die erhaltenen Werte sind in (19) einzusetzen. Nach Substitution von $nt=z$ und sorgfältiger Auswertung der entstandenen Ausdrücke erhält man das in Lemma 6 zitierte Resultat.

Aus (17), Lemma 5 und Lemma 6 folgt nun die Aussage von Satz 3.

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BORSUK'S THEOREM AND THE NUMBER OF FACETS OF CENTRALLY SYMMETRIC POLYTOPES

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1. Introduction

Let $C^n=\{x\in\mathbf{R}^n\colon\ |x_i|\leqq 1\ \ i=1,\dots,n\}$ be the n-dimensional cube and A be a d-dimensional subspace of $\mathbf{R}^n$ having no point in common with the $(n-d-1)$-dimensional faces of C^n. We want to find a lower bound on the number of vertices of the polytope $A\cap C^n$. More generally, given an n-dimensional centrally symmetric polytope K (whose center is at the origin) and a d-dimensional subspace $A\subset\mathbf{R}^n$, find lower bound on the number of vertices of $A\cap K$. We are going to prove two theorems concerning this question. These theorems have several interesting corollaries, for instance the following "lower bound"-type one. Every d-dimensional, centrally symmetric simplicial polytope has at least 2^d facets. (In fact this theorem is equivalent to our main result when $K=C^n$.)

This question was motivated by the following problem of Erdős [2]. Given $a_1,\dots,a_n\in\mathbf{R}^d$ vectors of at most unit length, at least how many of the 2^n vectors $\sum_{i=1}^{n}\varepsilon_i a_i$ $(\varepsilon_i=+1$ or $-1)$ lie in the ball $\sqrt{d}\,B^d$, where B^d is the euclidean unit ball of $\mathbf{R}^d$. Erdős conjectured that this number is at least $c(d)2^n n^{-\frac{d}{2}}$ for some positive constant $c(d)$ depending only on d. This conjecture has been proved very recently by J. Beck [1]. In this paper we do not contribute to this problem because our results would imply only that the number in question is at least $2^{n-d}\Big/\binom{n}{d}$.

In the proofs we shall need Borsuk's theorem on antipodal maps. A continuous map $\varphi\colon S^n\to\mathbf{R}^m$ is said to be *antipodal* if $\varphi(-x)=-\varphi(x)$ for every $x\in S^n$.

BORSUK'S THEOREM. *If $m<n$, then there is no antipodal map $\varphi\colon S^n\to S^m$.*

This theorem is equivalent to the following.

If $\varphi\colon S^n\to\mathbf{R}^n$ is an antipodal map, then there exists an $x\in S^n$ with $\varphi(x)=0$.

We shall prove the following extension of Borsuk's theorem.

If $\varphi\colon S^n\to S^m$ is antipodal, then the n-dimensional measure of $\varphi(S^n)$ is not less than the (n-dimensional) measure of S^n.

2. Notation and results

Let K be a convex polytope in $\mathbf{R}^n$. The *support* of $x\in K$ is defined as the minimal face of K containing x. A face is understood to be closed. If x lies in ∂K, the boundary of K, then $t(x)=t(x,K)$ denotes the set of outer normals of unit length to K at x. It is clear that $t(x)\subset S^{n-1}$ is nonempty. The set $t(x)$ consists of one point if the boundary of K is smooth at x. The d-dimensional outer angle of K at x $(d=1, 2, \dots, n)$ is defined as

$$\alpha_d(x, K) = \frac{\lambda_{d-1}(t(x))}{\lambda_{d-1}(S^{d-1})},$$

where λ_{d-1} is the $(d-1)$-dimensional Lebesgue measure in $\mathbf{R}^n$ and S^{d-1} is supposed to be isometrically imbedded into $\mathbf{R}^n$. Obviously,

$$\alpha_d(x, K) = \begin{cases} 0, & \text{if the support of } x \text{ is more than } (n-d)\text{-dimensional,} \\ \infty, & \text{if the support of } x \text{ is less than } (n-d)\text{-dimensional.} \end{cases}$$

Let $\mathscr{A}^{(d)}$ denote the set of d-dimensional subspaces of $\mathbf{R}^n$. We shall consider sections of type $A\cap K$ where $K\subset\mathbf{R}^n$ is a centrally symmetric n-dimensional polytope (with center at the origin) and $A\in\mathscr{A}^{(d)}$. A section $A\cap K$ is called *regular* if A has no point in common with the $(n-d-1)$-dimensional faces of K.

THEOREM 1. *Let K be a centrally symmetric, n-dimensional polytope and $A\in\mathscr{A}^{(d)}$. Then*

$$\sum_{x\in \operatorname{vert}(A\cap K)} \alpha_d(x, K) \geqq 1 \tag{1}$$

where $\operatorname{vert}(A\cap K)$ *is the set of vertices of* $A\cap K$.

COROLLARY 1. *If $A\cap K$ is a regular section, then*

$$|\operatorname{vert}(A\cap K)| \geqq \frac{1}{\alpha_d(K)},$$

where $\alpha_d(K)=\max\{\alpha_d(x, K)$: *the support of x is $(n-d)$-dimensional*$\}$.

COROLLARY 2. *Any regular, d-dimensional section of C^n has at least 2^d vertices.*

COROLLARY 3. *Any d-dimensional, centrally symmetric, simplicial polytope has at least 2^d facets.*

COROLLARY 4. [cf. Erdős, Beck]. *If $a_1, \dots, a_n\in B^d$, then at least $2^{n-d}\Big/\binom{n}{d}$ vectors out of the 2^n vectors $\sum_{i=1}^{n} \varepsilon_i a_i$ $(\varepsilon_i=+1$ or $-1)$ lie in the ball $\sqrt{d}\,B^d$.*

Let $\mathscr{L}^{(n-d)}=\mathscr{L}^{(n-d)}(K)$ be the set of all $(n-d)$-dimensional faces of K. To present our next theorem we define a map $\varphi\colon S^{n-d}\to \operatorname{skel}_{n-d} K$ to be *special* if

(i) φ is antipodal

(ii) for each $L\in\mathscr{L}^{(n-d)}$ either $L\subset\varphi(S^{n-d})$ *or* $\operatorname{int} L\cap\varphi(S^{n-d})=\varnothing$.

Here $\operatorname{int} L$ denotes the relative interior of the face L.

We mention that some projections $\pi\colon \mathbf{R}^n\to A$ (where $A\in\mathscr{A}^{(n-d+1)}$) induce a special map $\varphi_\pi\colon S^{n-d}\to \operatorname{skel}_{n-d} K$ in a natural way. Suppose that π is a projection such that the image of every $L\in\mathscr{L}^{(n-d+1)}$ is $(n-d+1)$-dimensional. Then π, restricted to K is one-to-one on every face $L\in\mathscr{L}^{(n-d+1)}$. On the other hand, $\pi(K)$ is a convex polytope whose boundary is the "same" as S^{n-d}, and π has an inverse on this boundary. Denoting this inverse by φ_π we have the induced special map $\varphi_\pi\colon S^{n-d}\to \operatorname{skel}_{n-d} K$.

Our next theorem gives a lower bound on the number of vertices of a regular section of K through the following discrete linear program.

$$(2)\qquad \begin{cases} \text{minimize} & \sum\limits_{L\in\mathscr{L}^{(n-d)}} x(L) \\ \text{subject to} & x(L)=0 \text{ or } 1 \qquad (\forall L), \\ & x(L)=x(-L) \qquad (\forall L), \\ & \sum\limits_{\substack{L\in\mathscr{L}^{(n-d)}\\ L\subset\varphi(S^{n-d})}} x(L)\geqq 2 \quad (\forall\varphi \text{ special}). \end{cases}$$

Denote the minimum of this problem by M. In other words, M is the minimum size of a centrally symmetric set of $(n-d)$-faces of K meeting all special images of S^{n-d}.

THEOREM 2. *Every regular section of a centrally symmetric n-dimensional polytope K has at least M vertices.*

Corollaries 2, 3 and 4 follow from this theorem as well. Moreover we can sharpen Corollary 2 (and, similarly Corollary 3):

COROLLARY 2′. *Any regular d-dimensional section of* C^n *has at least* 2^d *vertices. Equality holds if and only if the section is a d-dimensional parallelepiped.*

Further we have

COROLLARY 5. *Every d-dimensional regular section of the d-dimensional octahedron has exactly* $2\binom{n}{d-1}$ *vertices.*

COROLLARY 6. *Every 2-dimensional regular section of the dodecahedron (icosahedron) has at least* 6 *(resp.* 10*) vertices.*

The proof of Theorem 1 will be based on the following extension of Borsuk's theorem.

THEOREM 3. *If* $\varphi\colon S^k\to S^n$ *is an antipodal map, then* $\lambda_k(\varphi(S^k))\geqq\lambda_k(S^k)$. *Here* λ_k *is the k-dimensional Lebesgue measure (both in* $\mathbf{R}^{k+1}$ *and* $\mathbf{R}^{n+1}$*) normalized so that* $\lambda_k(S^k)$ *equals the k-dimensional mesaure of any copy of* S^k *isometrically imbedded into* S^k.

Let us mention two open problems: The first one arises from an attempt to find an alternative proof of Theorem 3. Let $K\subset\mathbf{R}^n$ be a symmetric convex polytope and $\varphi\colon \operatorname{vert} K\to\mathbf{R}^m-\{0\}$ such that for every vertex v, if $v_1, \dots, v_r$ are the neighbours of v then there exist coefficients $\lambda_1, \dots, \lambda_r>0$ such that

$$\varphi(v)=\lambda_1\varphi(v_1)+\dots+\lambda_r\varphi(v_r).$$

Then we conjecture that $\varphi(\operatorname{vert} K)$ lies in an n-dimensional subspace of $\mathbf{R}^m$. This conjecture would imply Theorem 3.

To present the second problem write $f_k(P)$ for the number of k-dimensional faces of the polytope P. Suppose P is symmetric, simple and d-dimensional with

$2n$ facets. The lower bound theorem would say that $f(P)$ is not less than a function of d, n and k. An obvious guess for that function is

$$f_0(P) \geqq 2^d+2(n-d)(d-1),$$
$$f_k(P) \geqq 2^{d-k}\binom{d}{k}+2(n-d)\binom{d}{k+1} \quad \text{for} \quad 1 \leqq k \leqq d-1.$$

This is supported by a kind communication of P. McMullen [4]. If the guess is correct, the minimal polytopes would be obtained from the cube by successive centrally symmetric truncations of vertices.

3. Proofs

PROOF OF THEOREM 1. Let us choose an $\varepsilon>0$ such that if L is a face of K and $A\cap L=\emptyset$, then $A\cap(L+\varepsilon B^n)=\emptyset$. Such an ε exists because each face of K is compact.

Put now $K_\varepsilon=K+\varepsilon B^n$ and let S^{d-1} be the unit sphere of the subspace A. The map $\pi\colon A\cap\partial K\to S^{d-1}$ defined by $\pi(y)=\dfrac{y}{\|y\|}$ is one-to-one and antipodal. We define a map $\varphi\colon S^{d-1}\to S^{n-1}$ by $\varphi(z)=t(\pi^{-1}(z), K_\varepsilon)$. Since K_ε is smooth at every point of its boundary, φ is well defined, continuous and antipodal. Theorem 3 then implies

$$\lambda_{d-1}(S^{d-1}) \leqq \lambda_{d-1}(\varphi(S^{d-1})) = \lambda_{d-1}(t(A\cap\partial K_\varepsilon, K_\varepsilon)).$$

Claim. $t(A\cap\partial K_\varepsilon, K_\varepsilon)\subseteqq \bigcup t(\operatorname{int} L, K)$, where the union is taken over all faces L of K with $L\cap A\neq\emptyset$.

Suppose $z\in t(y, K_\varepsilon)$ for some $y\in A\cap\partial K_\varepsilon$. Then $y=x+\varepsilon z$ where $x\in\partial K$ and $z\in t(x,K)$, as one can check easily. Write L for the support of x (in K), then $x\in\operatorname{int} L$ and $z\in t(\operatorname{int} L, K)$. All we have to show is that $L\cap A\neq\emptyset$. Suppose that $L\cap A=\emptyset$, then by the choice of ε, $A\cap(L+\varepsilon B^n)=\emptyset$, too. But $y\in A$ and $y=x+\varepsilon z\in L+\varepsilon B^n$, a contradiction.

From this we have

$$\lambda_{d-1}(S^{d-1}) \leqq \lambda_{d-1}(t(A\cap\partial K_\varepsilon, K_\varepsilon)) \leqq \sum_{L\cap A\neq\emptyset} \lambda_{d-1}(t(\operatorname{int} L, K)).$$

Clearly λ_{d-1} $(t(\operatorname{int} L, K))=0$ if $\dim L>n-d$. Suppose $A\cap K$ a regular section, then $L\cap A=\emptyset$ for every face L with $\dim L<n-d$. Thus

$$1 \leqq \sum_{\substack{L\cap A\neq\emptyset\\ L\in\mathcal{L}^{(n-d)}}} \frac{\lambda_{d-1}(t(\operatorname{int} L, K))}{\lambda_{d-1}(S^{d-1})} = \sum_{x\in \operatorname{vert} A\cap K} \alpha_d(x, L),$$

because $t(\operatorname{int} L, K)$ coincides with $t(x, K)$ for every $x\in\operatorname{int} L$ and $L\cap A=\emptyset$ for some $L\in\mathcal{L}^{(n-d)}$ if and only if $A\cap L$ is a vertex of $A\cap K$.

Finally, if $A\cap K$ is not a regular section, then some member of the left hand side of (1) equals $+\infty$.

Corollary 1 is an immediate consequence.

PROOF OF COROLLARY 2. It is easy to see that $\alpha_d(x, C^n)=2^{-d}$ if the support of x is $(n-d)$-dimensional. Using Corollary 1 this fact implies the result.

PROOF OF COROLLARY 3. It is easy to check and actually well known [3] that every d-dimensional, centrally symmetric and simple polytope is a regular section of C^n for some n. So Corollary 2 says that every d-dimensional, centrally symmetric and simple polytope has at least 2^d vertices. Dualizing this statement we get Corollary 3.

Here we mention that Corollary 2 does not hold for non-regular sections. This follows from the fact that every d-dimensional, symmetric polytope with $2n$ facets is a section of C^n. For instance, the d-dimensional octahedron is a (non-regular) section of C^{2d-1} and it has only $2d$ vertices.

PROOF OF COROLLARY 4. We may clearly suppose that the vectors $a_1, \dots, a_n \in B^d$ are in general position, say their entries are algebraically independent over the rationals. Put

$$A = \left\{x \in \mathbf{R}\colon \sum_{i=1}^{n} x_i a_i = 0\right\} \in \mathscr{A}^{(n-d)}.$$

$P = A \cap C^n$ is a regular section because the points $a_1, \dots, a_n$ are in general position. By Corollary 2, $|\mathrm{vert}\, P| \geqq 2^{n-d}$. To each vertex x^0 of P there corresponds a sign sequence $\varepsilon_1, \dots, \varepsilon_n$ such that $\varepsilon_i = x_i^0$ if $|x_i^0| = 1$ and $\left\|\sum_1^n \varepsilon_i a_i\right\| \leqq \sqrt{d}$. This is a simple geometric fact the proof of which is left to the reader. On the other hand any sign sequence can correspond to at most $\binom{n}{d}$ vertices of P. (One can slightly improve this bound, but it would not influence the order of magnitude. It is easy to construct an example where a sign sequence corresponds to $\binom{n-1}{d}$ vertices of P.) This shows that at least $2^{n-d}\Big/\binom{n}{d}$ vectors out of the 2^n vectors $\sum_{i=1}^{n} \varepsilon_i a_i$ $(\varepsilon_i = \pm 1)$ lie in the ball $\sqrt{d}\, B^d$.

PROOF OF THEOREM 2. Suppose that $A \in \mathscr{A}^d$ and that the section $A \cap K$ is regular. For $L \in \mathscr{L}^{(n-d)}$ put

$$x_A(L) = \begin{cases} 1 & \text{if } A \cap L \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\sum_{L \in \mathscr{L}^{(n-d)}} x_A(L) = |\mathrm{vert}\, A \cap K|$. We show $x_A(L)$ satisfies the conditions of the discrete linear program (2). All we have to check is the condition

$$\sum_{\substack{L \subseteq \varphi(S^{n-d}) \\ L \in \mathscr{L}^{(n-d)}}} x_A(L) \geqq 2 \tag{3}$$

for each special map $\varphi\colon S^{n-d} \to s\,\mathrm{kel}_{n-d} K$. Now let φ be a special map, then, for $L \subseteq \varphi(S^{n-d})$ $x_A(L) = 1$ iff $L \cap A \neq \emptyset$. So (3) holds iff $A \cap \varphi(S^{n-d})$ consists of at least two pints. Consider the orthogonal complement, $A^\perp$, of A and let $\pi\colon \mathbf{R}^n \to A^\perp$ be the orthogonal projection. Since φ is antipodal, $A \cap \varphi(S^{n-d})$ contains two points iff $0 \in \pi \circ \varphi(S^{n-d})$. But $\pi \circ \varphi\colon S^{n-d} \to A^\perp$ $(\cong \mathbf{R}^{n-d})$, so by Borsuk's theorem there exists a $z \in S^{n-d}$ with $\pi \circ \varphi(z) = 0$.

Corollary 2 follows from Theorem 2 as well. In order to see this take the special map $\varphi\colon S^{n-d}\to \operatorname{skel}_{n-d} C^n$ which is induced by some projection and consider the set of special maps $\{g\circ\varphi\colon g\in G\}$ where G is the group generated by the reflections of C^n. Clearly $L\subseteqq g\circ\varphi(S^{n-d})$ for exactly 2^{n-d+1} elements $g\in G$ (for each fixed $L\in\mathscr{L}^{(n-d)}$) and $|G|=2^n$. So summing up the inequalities

$$\sum_{L\subseteqq g\circ\varphi(S^{n-d})} x_A(L)\geqq 2$$

for every $g\in G$ we get $\sum_L x_A(L)\geqq 2^d$. This implies $M\geqq 2^d$. The same method gives Corollary 2′ as well. Indeed, if the set $\{L\in\mathscr{L}^{(n-d)}\colon L\cap A\neq\emptyset\}$ contains two faces, L_1 and L_2 that are not parallel, then one can find a special map φ (induced by same projection) so that both $L_1, L_2\subseteqq\varphi(S^{n-d})$. Consequently

$$\sum_{L\subseteqq\varphi(S^{n-d})} x_A(L)\geqq 4>2.$$

This implies $M>2^d$.

To see that Corollary 5 holds we use the method of proof of Theorem 2. The $(n-d+1)$-dimensional subspace $x_{i_1}=\ldots=x_{i_{d-1}}=0$ $(1\leqq i_1<i_{d-1}\leqq n)$ intersects the octahedron

$$O^n=\left\{x\in\mathbf{R}^n\colon \sum_{i=1}^n |x_i|\leqq 1\right\}$$

in an $(n-d+1)$-dimensional octahedron $O^{n-d+1}_{i_1,\ldots,i_{d-1}}$ whose boundary is clearly the image of a special map $\varphi\colon S^{n-d}\to\operatorname{skel}_{n-d} O^n$. Since the section $A\cap O^n$ is regular and $O^{n-d+1}_{i_1,\ldots,i_{d-1}}$ lies in a subspace,

$$\sum_{L\in\varphi(S^{n-d})} x_A(L)=2.$$

Summing up these equalities for each such φ we get

$$|\operatorname{vert} A\cap O^n|=\sum_{L\in Z^{(n-d)}} x_A(L)=2\binom{n}{d-1},$$

because every $L\in Z^{(n-d)}$ lies on the boundary of exactly one octahedron $O^{n-d+1}_{i_1,\ldots,i_{d-1}}$.

We mention that Corollary 1 does not imply Corollary 5 (for $n\geqq 4$ and $d=2$ for instance). And in general, Theorem 2 seems to be stronger than Theorem 1.

Corollary 6 can be proven using a suitable set of special maps.

Proof of Theorem 3. We can suppose that $n\geqq k$. We are going to use the following formula which is a consequence of the Fubini theorem. If $X\subseteqq S^n$ is λ_k measurable, then

$$\lambda_k(X)=\int_{\mathscr{A}} |X\cap A|\,d\mu \tag{4}$$

where μ is the invariant measure on the set $\mathscr{A}$ of all $(n+1-k)$-dimensional subspaces of $\mathbf{R}^{n+1}$, normalized suitably. Applying this formula to $X=\varphi(S^k)$,

$$\lambda_k(\varphi(S^k))=\int |\varphi(S^k)\cap A|\,d\mu\geqq\int 2d\mu,$$

because $|\varphi(S^k)\cap A|\geqq 2$ for every $A\in\mathscr{A}$ as we have seen in the proof of Theorem 2. Let φ_0: $S^k\to S^n$ be an isometric imbedding of S^k into S^n. Then $|\varphi_0(S^k)\cap A|=2$ for μ-almost every $A\in\mathscr{A}$. Applying (4) again with $X=\varphi_0(S^k)$

$$\lambda_k(\varphi_0(S^k)) = \int 2d\mu,$$

and this proves the theorem.

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APPROXIMATION BY BERNSTEIN TYPE RATIONAL FUNCTIONS. II

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In [1], the first of us introduced and considered some approximation properties of the Bernstein type discrete linear operator

$$R_n(f, x) = \frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} f\left(\frac{k}{b_n}\right)\binom{n}{k}(a_n x)^k.$$

Among others, it was proved that if $f(x)$ is continuous in $[0, \infty)$, $|f(x)| = O(e^{\alpha x})$ $(x \to \infty)$ with some α then in any interval $[0, A]$ $(A > 0)$ the estimate

$$(1) \qquad \Delta_n(f, x) \overset{\text{def}}{=} |f(x) - R_n(f, x)| \leqq c_0 \omega_{2A}(n^{-1/3}) \quad (0 \leqq x \leqq A)$$

holds for sufficiently large n's provided $a_n = n^{-1/3}$, $b_n = n^{2/3}$. Here c_0 depends only on A and α, and $\omega_{2A}(.)$ is the modulus of continuity of $f(x)$ on the interval $[0, 2A]$. As it was noted in [1], the convergence of $R_n(f, x)$ holds under the more general conditions $a_n = b_n/n \to 0$, $b_n \to \infty$ $(n \to \infty)$ as well.

The aim of the present paper is to improve the estimate (1) by an appropriate choice of a_n and b_n in the case when $f(x)$ satisfies some more restrictive conditions. Furthermore we shall show that these results can be applied to approximate certain improper integrals by quadrature sums of positive coefficients based on finite number of equidistant nodes.

First we assume that $f(x)$ is uniformly continuous in $[0, \infty)$; then the modulus of continuity $\omega_f(.)$ of $f(x)$ exists on the entire positive half-axis.

THEOREM 1. *If $f(x)$ is uniformly continuous in $[0, \infty)$ and*

$$(2) \qquad a_n = n^{\beta-1}, \quad b_n = n^{\beta} \quad (0 < \beta \leqq 2/3)$$

then

$$(3) \qquad \Delta_n(f, x) \leqq 2\left(1 + x^{\frac{1}{2(1-\beta)}}\right) \omega_f\left(\sqrt{\frac{x}{n^{\beta}}}\right) \quad (0 \leqq x < \infty).$$

This estimate has several advantages compared to (1). First of all, $\Delta_n(f, 0) = 0$, i.e. (3) reflects the interpolation property $R_n(f, 0) = f(0)$ of the operator $R_n(f, x)$; even the rate of convergence of $R_n(f, x)$ to $f(x)$ when $x \to 0$ (n fixed) can be seen from (3). Also, (3) is a *weighted* estimate. From the proof of (1), one can easily see that a weighted estimate of the form

$$(4) \qquad \Delta_n(f, x) \leqq c_1(1+x^2)\omega_f(n^{-1/3}) \quad (0 \leqq x < \infty)$$

also holds whenever $f(x)$ is uniformly continuous in $[0, +\infty)$. However, from (3) we can see that by choosing β close to 0 we can get a weight arbitrarily close to $O(x)$ even in case Lip 1. The price we pay for this is that the order of convergence becomes worse. But if we put $\beta=2/3$ in (3) (this is the original case), the estimate will be

$$\Delta_n(f,x) \leqq 2(1+x^{3/2})\,\omega_f\left(\sqrt{x}/n^{1/3}\right)$$

which, in general, is better than (4).

If we put $f(x)=x$ then

$$\Delta_n(x,x)=\frac{a_n x^2}{1+a_n x}.$$

Setting $a_n=n^{\beta-1}$, $x=n^{1-\beta}$ we get $\Delta_n(x,n^{1-\beta})=\frac{1}{2}n^{1-\beta}$ while (3) yields

$$\Delta_n(x,n^{1-\beta}) \leqq 2\left(1+n^{\frac{1-\beta}{2(1-\beta)}}\right)\frac{n^{\frac{1-\beta}{2}}}{n^{\beta/2}} \leqq 4n^{1-\beta}.$$

I.e., in a sense, (3) is the best possible estimate. (This, of course, does not exclude the possibility of improving e.g. the weight in (3) while obtaining a weaker order of convergence. What we mean by "best possible estimate" is that there does not exist an estimate in which both the weight and the order of convergence would be better than (3).)

PROOF OF THEOREM 1. Using the property $\omega_f(\lambda\delta)\leqq(1+\lambda)\omega_f(\delta)$ of the modulus of continuity and Schwarz' inequality we get

$$\Delta_n(f,x) \leqq (1+a_n x)^{-n}\sum_{k=0}^{n}|f(x)-f(k/b_n)|\binom{n}{k}(a_n x)^k \leqq (1+a_n x)^{-n}\cdot$$
$$\cdot\sum_{k=0}^{n}\omega(|x-k/b_n|)\binom{n}{k}(a_n x)^k \leqq (1+a_n x)^{-n}\omega_f\left(\sqrt{x/n^{\beta}}\right)\sum_{k=0}^{n}\left((1+|x-k/b_n|\sqrt{n^{\beta}/x}\right)\cdot$$
$$\cdot\binom{n}{k}(a_n x)^k = (1+a_n x)^{-n}\omega_f\left(\sqrt{x/n^{\beta}}\right)\Big\{(1+a_n x)^n+\sqrt{n^{\beta}/x}\sum_{k=0}^{n}|x-k/b_n|\cdot$$
$$\cdot\binom{n}{k}(a_n x)^k\Big\} \leqq \omega_f\left(\sqrt{x/n^{\beta}}\right)\Big\{1+(1+a_n x)^{-n}\sqrt{n^{\beta}/x}\sqrt{\sum_{k=0}^{n}\binom{n}{k}(a_n x)^k}\cdot$$
$$\cdot\sqrt{\sum_{k=0}^{n}(x-k/b_n)^2\binom{n}{k}(a_n x)^k}\Big\}.$$

On using (2) and (2.4) from [1] we obtain

$$\Delta_n(f,x) \leqq \omega_f\left(\sqrt{x/n^{\beta}}\right)\left\{1+\sqrt{\frac{n^{\beta}}{x}\,\frac{a_n^2x^4+x/b_n}{(1+a_n x)^2}}\right\} \leqq \omega_f\left(\sqrt{x/n^{\beta}}\right)\left\{1+\left(\frac{n^{\frac{3}{2}\beta-1}x^{\frac{3}{1}}}{1+n^{\beta-1}x}+1\right)\right\}=$$
$$=\omega_f\left(\sqrt{x/n^{\beta}}\right)\left\{+\ x^{\frac{1}{2(1-\beta)}}n^{\frac{3}{2}\beta-1}\frac{x^{\frac{2-3\beta}{2(1-\beta)}}}{1+n^{\beta-1}x}\right\}.$$

Here the function

$$\varphi(x) = \frac{x^{\frac{2-3\beta}{2(1-\beta)}}}{1+n^{\beta-1}x}$$

attains its maximum at $x_0 = \frac{2-3\beta}{\beta} n^{1-\beta}$ and

$$\varphi(x_0) = \frac{\left(\frac{2-3\beta}{\beta}\right)^{\frac{2-3\beta}{2(1-\beta)}} n^{1-\frac{3}{2}\beta}}{1+\frac{2-3\beta}{\beta}} = \frac{\beta^{\frac{\beta}{2(1-\beta)}}(2-3\beta)^{\frac{2-3\beta}{2(1-\beta)}}}{2(1-\beta)} n^{1-\frac{3}{2}\beta} \leqq 2n^{1-\frac{3}{2}\beta}.$$

Thus

$$\Delta_n(f, x) \leqq \omega_f(\sqrt{x/n^\beta})\{2+2x^{\frac{1}{2(1-\beta)}}\}$$

which completes the proof of Theorem 1.

Now let us see what more can we say if we assume something more about the function to be approximated. Let $C[0, \infty]$ denote the set of functions continuous on $[0, \infty)$ and having a finite limit at $+\infty$. The quantity

$$\Omega_f(A) = \sup_{A \leqq x_1 \leqq x_2} |f(x_1) - f(x_2)|$$

which will be called "modulus at infinity", will play an important role in approximation properties of these functions. Evidently, $\lim_{x\to\infty} f(x)$ exists and is finite if and only if $\lim_{A\to\infty} \Omega_f(A) = 0$.

THEOREM 2. *If* $f(x) \in C[0, \infty]$ *then*

$$\sup_{0 \leqq x \leqq \infty} \Delta_n(f, x) \leqq c_2 \inf_{A \geqq 10} \left\{\Omega_f(A) + A^{\frac{1}{1-\beta}} \omega_f\left(\frac{A}{n^{\beta/2}}\right)\right\}, \tag{5}$$

provided a_n *and* b_n *are defined by* (2).

PROOF. We may assume that $f(x)$ is not identically constant; otherwise the statement is trivial. Choosing

$$A = \omega_f(n^{-\beta/2})^{\frac{\beta-1}{2(2-\beta)}} \to \infty \quad (n \to \infty)$$

we have

$$\Omega_f(A) + A^{\frac{1}{1-\beta}} \omega_f\left(\frac{A}{n^{\beta/2}}\right) \leqq \Omega_f(A) + 2A^{\frac{2-\beta}{1-\beta}} \omega_f(n^{-\beta/2}) = \Omega_f(A) + 2\sqrt{\omega_f(n^{-\beta/2})} \to 0$$
$$(n \to \infty),$$

i.e. the right hand side of (5) tends to zero as $n \to \infty$. Therefore in what follows we suppose that

$$10 \leqq A \leqq \frac{1}{2} n^{\frac{1-\beta}{2}}; \tag{6}$$

namely otherwise

$$\Omega_f(A)+A^{\frac{1}{1-\beta}}\omega_f\left(\frac{A}{n^{\beta/2}}\right) \geqq c_3 n^{1-\beta} \to \infty \quad (n\to\infty).$$

By the definition of the modulus at infinity,

$$|f(x)-f(\infty)| \leqq \Omega_f(A) \quad (x \geqq A). \tag{7}$$

By Theorem 1

$$\Delta_n(f, x) \leqq 4A^{\frac{1}{1-\beta}}\omega_f\left(\frac{A}{n^{\beta/2}}\right) \quad (0 \leqq x \leqq A^2).$$

Let now $x \geqq A^2$. Since R_n is linear and it reproduces constants,

$$\Delta_n(f, x) \leqq |f(x)-f(\infty)|+|R_n(f(\infty)-f, x)| \leqq$$
$$\leqq \Omega_f(A^2)+|R_n(f(\infty)-f, x)| \quad (x \geqq A^2).$$

Here, using (7),

$$R_n(f(\infty)-f, x) \leqq (1+a_n x)^{-n}\sum_{k=0}^{n}|f(\infty)-f(k/b_n)|\binom{n}{k}(a_n x)^k \leqq$$

$$\leqq (1+a_n x)^{-n}\left\{2\cdot \sup_{0\leqq t\leqq\infty}|f(t)|\sum_{k=0}^{[Ab_n]}\binom{n}{k}(a_n x)^k+\Omega_f(A)\sum_{k=[Ab_n]+1}^{n}\binom{n}{k}(a_n x)^k\right\} \leqq$$

$$\leqq 2x^{Ab_n}\left(\frac{1+a_n}{1+a_n x}\right)^n \sup_{0\leqq t\leqq\infty}|f(t)|+\Omega_f(A) \quad (x \geqq A^2).$$

The function

$$\psi(x) = x^{Ab_n}(1+a_n x)^{-n}$$

attains its maximum in $[0, \infty)$ at $x_1=\dfrac{A}{1-Aa_n}$, and by (6), $x_1 \leqq 2A \leqq A^2$. Thus $\psi(x)$ is monotone decreasing in $[A^2, \infty)$ and hence

$$x^{Ab_n}\left(\frac{1+a_n}{1+a_n x}\right)^n \leqq A^{2Ab_n}\left(\frac{1+a_n}{1+A^2 a_n}\right)^n = A^{2Ab_n}\left\{1-\frac{(A^2-1)a_n}{1+A^2 a_n}\right\}^n \leqq$$

$$\leqq A^{2Ab_n}\exp\left\{-\frac{(A^2-1)na_n}{1+A^2 a_n}\right\} \leqq \exp\left\{-\left(\frac{1}{2}A^2-\frac{1}{2}-2A\log A\right)b_n\right\} \leqq e^{-c_4 n^\beta} \quad (x \geqq A^2),$$

since by (6) $A^2 a_n \leqq 1$. Collecting these estimates we obtain (5). Q.e.d.

When ω_f and Ω_f are specialized, we can get more explicit estimates from (5). E.g. if

$$\Omega_f(A) \leqq A^{-a} \quad (a>0), \quad \omega_f(h) \leqq h^\alpha \quad (0<\alpha<1)$$

then by an appropriate choice of A we obtain

$$\sup_{0\leqq x\leqq\infty}\Delta_n(f, x) \leqq \begin{cases} c_5 n^{-\frac{a\alpha}{2(\sqrt{a+\alpha+1}+1)^2}} & \text{if } 0<a+\alpha\leqq 3, \quad \beta=\dfrac{\sqrt{a+\alpha+1}}{\sqrt{a+\alpha+1}+1} \\ c_6 n^{-\frac{a\alpha}{3(a+\alpha+3)}} & \text{if } a+\alpha \geqq 3, \quad \beta = 2/3. \end{cases}$$

If $a\to\infty$ then the latter estimate is getting close to $O(n^{-\alpha/3})$.

One may think that the role of Ω_f in Theorem 2 is superfluous, ω_f alone would determine the order of convergence. This is not the case as it can be seen from the following

THEOREM 3. *If* $f(x)\in C[0, \infty]$ *and* $f(x)$ *is monotone in* $[0, \infty)$ *then*

$$\sup_{0\leqq x\leqq\infty} \Delta_n(f, x) \geqq \Omega_f(n^{1-\beta})$$

again with the choice (2).

PROOF. We have by the monotonicity

$$\sup_{0\leqq x\leqq\infty} \Delta_n(f, x) \geqq |f(\infty)-R_n(f, \infty)| = |f(\infty)-f(n^{1-\beta})| = \Omega_f(n^{1-\beta}),$$

Q.e.d.

In other words, even if $f(x)$ has very good structural properties in $[0, \infty)$, the uniform convergence of $R_n(f, x)$ to $f(x)$ when $n\to\infty$ can be arbitrarily slow if so is the convergence of $f(x)$ to $f(\infty)$ when $x\to\infty$.

The next result gives the necessary and sufficient condition for the uniform convergence of $R_n(f, x)$ in $[0, \infty)$.

THEOREM 4. *We have*

$$\lim_{n\to\infty} \sup_{0\leqq x\leqq\infty} \Delta_n(f, x) = 0 \tag{8}$$

if and only if $f(x)\in C[0, \infty]$.

PROOF. The sufficiency of $f(x)\in C[0, \infty]$ has been proved at the beginning of the proof of Theorem 2. Let us prove the necessity. If (8) holds then $f(x)$ is continuous at every point $x\in[0, \infty)$ being the uniform limit of continuous functions in a compact neighborhood of this point. $f(x)$ is also bounded on $[0, \infty)$ since if we had a sequence $0\leqq x_1\leqq x_2\leqq\dots$ such that $\lim_{m\to\infty} f(x_m)=\infty$, say, then by assumption for any fixed n

$$R_n(f, x_m) \geqq f(x_m)-\sup_{0\leqq x\leqq\infty} \Delta_n(f, x) \quad (m = 1, 2, \dots),$$

i.e. $\lim_{m\to\infty} R_n(f, x_m)=\infty$ which would contradict

$$R_n(f, \infty) = f(n^{1-\beta}) \quad (n = 1, 2, \dots). \tag{9}$$

Since $f(x)$ is uniformly bounded, there exists a sequence of integers $1\leqq n_1<n_2<\dots$ such that

$$d = \lim_{i\to\infty} f(n_i^{1-\beta}) \tag{10}$$

exists and finite. We shall prove that $\lim_{x\to\infty} f(x)=d$. Let $\varepsilon>0$ be arbitrary. By (10), there exists an $i_0=i_0(\varepsilon)$ such that

$$|f(n_i^{1-\beta})-d| < \varepsilon \quad \text{if} \quad i \geqq i_0.$$

Also, by assumption, there exists an $i_1=i_1(\varepsilon)\geqq i_0$ such that

$$\sup_{0\leqq x\leqq\infty} \Delta_{n_{i_1}}(f, x) < \varepsilon.$$

By the continuity of $R_{n_{i_1}}(f, x)$, there exists a $B=B(\varepsilon)$ such that

$$|R_{n i_1}(f, x)-R_{n i_1}(f, \infty)|<\varepsilon \quad \text{if} \quad x \geqq B.$$

Collecting these estimates and using (9) we get

$$|f(x)-d| \leqq |f(x)-R_{n i_1}(f, x)|+|R_{n i_1}(f, x)-R_{n i_1}(f, \infty)|+|f(n_{i_1}^{1-\beta})-d| \leqq 3\varepsilon$$

provided $x \geqq B$. Q.e.d.

REMARKS. One may try to prove the sufficiency part of Theorem 4 by using a general theorem of B. D. Boyanov and V. M. Veselinov [2] which states that if a sequence of positive linear operators converges uniformly on $[0, \infty]$ for 1, e^{-x} and e^{-2x} then it converges uniformly for all continuous functions on $[0, \infty]$ which have a finite limit at $+\infty$. However, our direct approach is very simple and it would require a tedious computation to check the convergence for the test functions (possibly by the same method we used for general functions). By applying a transformation $x \rightarrow \frac{x}{1+x}$ of $[0, \infty]$ to $[0, 1]$ and using the Korovkin theorem one can easily see that the test functions mentioned above can be replaced by 1, $\frac{1}{1+x}$, and $\frac{1}{(1+x)^2}$. This gives another possibility to prove the sufficiency part.

Finally, as an application to Theorem 2, we give a quadrature formula with positive coefficients based on finitely many equidistant nodes which approximate certain improper integrals.

THEOREM 5. *Let* $\varepsilon>0$ *be arbitrary and assume that*

$$g(x)=f(x)(1+x)^{1+\varepsilon} \in C[0, \infty]. \tag{11}$$

Then with (2) *we have*

$$\left|\int_0^\infty f(x)\,dx-\sum_{k=0}^{n} A_{kn} f(k/b_n)\right| \leqq c(\varepsilon) \inf_{A \geqq 10}\left\{\Omega_g(A)+A^{\frac{1}{1-\beta}}\omega_g\left(\frac{A}{n^{\beta/2}}\right)\right\} \tag{12}$$

where

$$A_{kn}=(1+k/b_n)^{1+\varepsilon}\binom{n}{k} a_n^k \int_0^\infty \frac{x^k\,dx}{(1+x)^{1+\varepsilon}(1+a_n x)^n} \quad (k=1, 2, \ldots, n).$$

PROOF. Applying Theorem 2 to (11) we get

$$|g(x)-R_n(g, x)| \leqq c(\varepsilon) \inf_{A \geqq 10}\left\{\Omega_g(A)+A^{\frac{1}{1-\beta}}\omega_g\left(\frac{A}{n^{\beta/2}}\right)\right\} \quad (0 \leqq x \leqq \infty).$$

Dividing by $(1+x)^{1+\varepsilon}$, integrating from 0 to $+\infty$ and using (1) we get (12). Q.e.d.

As for the order of convergence in (12), the same remarks apply as to Theorem 2. For practical purposes the case $\varepsilon=1$ is the most suitable; the corresponding coefficients

$$A_{kn} = (1+k/b_n)^2\binom{n}{k}a_n^k\int_0^\infty \frac{x^k\,dx}{(1+x)^2(1+a_nx)^n} \quad (k = 1, \ldots, n)$$

can be easily computed.

REMARK. There are several possibilities in computing improper integrals by quadrature sums based on equidistant nodes. One of them is to cut down a proper infinite part of $[0, \infty)$ and form a simple Riemann-sum of the type $\frac{1}{N}\sum_{k=0}^{M} f(k/N)$ on the remaining interval $[0, M/N)$. However, to determine this crucial cutting point M/N we need some apriori knowledge on the structural properties of the function $f(x)$, which is not the case with our method described above.

We mention that with obvious modifications, all the above results remain true under the slightly more general conditions $a_n=b_n/n\to 0$, $b_n\to\infty$ $(n\to\infty)$. However, even in this case one can get a reasonable error estimate only when $1/n<a_n\leqq 1/n^{1/3}$, and (2) essentially covers all these cases.

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[1] G. Szegő, *Orthogonal polynomials,* AMS Coll. Publ. Vol. XXIII (Providence, 1939).
[2] A. Zygmund, Smooth functions, *Duke Math. J.,* **12** (1945), 47—76.

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