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A RENEWAL THEOREM FOR A SEQUENCE OF DEPENDENT RANDOM VARIABLES

By
G. SANKARANARAYANAN and V. BALAKRISHNAN
(Annamalai Nagar)

Let $\{x_i\}$ be a sequence of dependent random variables with $\mathbf{E}(x_i)=\mu_i$, $i=1, 2, \dots$. Let

$$S_n = \sum_{i=1}^{n} x_i, \quad M_n = \max_{1 \leqq i \leqq n} S_i \quad \text{and} \quad N_n = \max_{1 \leqq i \leqq n} |S_i|.$$

Let $F_n(x)$, $H_n(x)$ and $K_n(x)$ be the distribution functions of S_n, M_n and N_n, respectively. Assume that

$$\lim_{n\to\infty} \frac{\mu_1+\mu_2+\ldots+\mu_n}{n^\alpha} = \mu, \quad 0<\mu<\infty, \quad \alpha>0 \tag{1.1}$$

and

$$\operatorname{Var}(S_n) = O(n^{2\alpha-\delta}), \quad n\to\infty, \quad 2\alpha>\delta>0, \tag{1.2}$$

with the requirement that $\operatorname{Var}(S_n)\to\infty$, as $n\to\infty$. Under a very general assumption on the asymptotic behaviour of

$$\beta_m(n) = \mathbf{E}\left|\sum_{i=1}^{n}(x_i-\mu_i)\right|^m \tag{1.3}$$

as $n\to\infty$, we have proved that

$$\sum_{n=1}^{\infty} a_n F_n(x) \sim \frac{(x/\mu)^{(\lambda+1)/\alpha} L(x^\alpha)}{\lambda+1}, \quad x\to\infty. \tag{1.4}$$

Here we have taken $\{a_n\}$ to be a positive sequence such that

$$a_n \sim n^\lambda L(n), \quad n\to\infty, \tag{1.5}$$

where $L(n)$ is a non-negative slowly varying function and λ is chosen such that $\sum_{n=1}^{\infty} a_n$ is divergent. It has been indicated that (1.4) is true when $F_n(x)$ is replaced by $H_n(x)$ or $K_n(x)$. It has also been shown that many of the known results found in [1, 9] were special cases of this general theorem.

1. Introduction. Several authors [6, 8] have studied the asymptotic behaviour of $\sum_{n=1}^{\infty} a_n F_n(x)$ and $\sum_{n=1}^{\infty} a_n H_n(x)$ when $\{x_n\}$ form a sequence of independent and identically distributed random variables and $\{a_n\}$ a positive coefficient sequence such that $\sum_{n=1}^{\infty} a_n$ is divergent. Recently SANKARANARAYANAN and SUYAMBULINGOM [9] have studied the asymptotic behaviour of the above sums when $\{x_i\}$ form (i) a sequence of dependent random variables with unit variance such that the correlation between x_i and x_j is ϱ, $|\varrho|<1$, $i\neq j$, $i, j=1, 2, \ldots$ and (ii) a sequence of identically distributed random variables with unit variance so that the correlation between x_i and x_j is $\varrho_{ij}=\varrho^{|i-j|}$, $i\neq j$, $i, j=1, 2, \ldots$, $0<\varrho<1$, $\{a_n\}$ being taken as in (1.5). They have illustrated the above theorems when $\{x_i\}$ follow the normal distribution as well as the type III distribution. For case (i), they have assumed (1.1) with $\alpha>1$.

BALAKRISHNAN [1] has studied the asymptotic behaviour of the sums $\sum_{n=1}^{\infty} a_n F_n(x)$ and $\sum_{n=1}^{\infty} a_n H_n(x)$, when $\{x_n\}$ form a sequence of dependent random variables satisfying the following conditions:

(1.6) $$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{E}(x_i) = \mu, \quad 0<\mu<\infty,$$

and

(1.7) $$\operatorname{Var}\left(\sum_{i=1}^{n} x_i\right) \sim A^2 n, \quad n\to\infty, \quad A^2(\text{a constant}) > 0.$$

He has illustrated his results when $\{x_n\}$ is taken to be a sequence of normal variables satisfying the above conditions. In an unpublished research report, Balakrishnan has also studied their asymptotic behaviour when $\{x_n\}$ satisfy the conditions

(1.8) $$\lim_{n\to\infty} \frac{1}{n^\alpha} \sum_{i=1}^{n} \mathbf{E}(x_i) = \mu, \quad \alpha>0, \quad 0<\mu<\infty,$$

and

(1.9) $$\sum_{i=1}^{n} \mathbf{E}[x_i - \mathbf{E}(x_i)]^2 = o(n^{2\alpha-1}).$$

In all these theorems in [1,9] it is assumed that

(1.10) $$\int_{\mu}^{\infty} [1-H_n(n^\alpha x)]\, dx \to 0, \quad n\to\infty, \quad \alpha \geqq 1$$

and

(1.11) $$\sum_{n=1}^{\infty} a_n F_n(n^\alpha \beta) < \infty, \quad 0<\beta<\mu, \quad \alpha \geqq 1.$$

In the illustrations given in [1,9], (1.10) and (1.11) are automatically true. The main object of this paper is to arrive at a very general renewal theorem for dependent random variables where even these conditions can be dropped. In addition to conditions (1.1), (1.2) and (1.5), here we assume that for at least one $m>\dfrac{2(\lambda+1)}{\delta}$

(1.12) $$\beta_m(n) = O(n^{m(\alpha-\delta/2)}), \quad n\to\infty, \quad 2\alpha>\delta, \quad \delta>0.$$

The main theorem in this paper (see Section 3) establishes (1.4) and further asserts its validity even when $F_n(x)$ is replaced by $H_n(x)$ or $K_n(x)$ so long as conditions

(1.1), (1.2), (1.5) and (1.12) are satisfied. The proof is deferred until we have proved six lemmas.

2. Some important lemmas. LEMMA 1. *Let* $\{x_i\}$, $i=1, 2, \ldots$ *be a sequence of dependent random variables with* $\mathbf{E}(x_i)=\mu_i$, $i=1, 2, \ldots$. *Assume that*

$$\lim_{n\to\infty} \frac{\mu_1+\mu_2+\ldots+\mu_n}{n^\alpha} = \mu, \quad 0<\mu<\infty, \quad \alpha>0 \tag{2.1}$$

and

$$\operatorname{Var}(S_n) = O(n^{2\alpha-\delta}), \quad n\to\infty, \quad 2\alpha>\delta>0, \tag{2.2}$$

with the requirement that $\operatorname{Var}(S_n)\to\infty$ *as* $n\to\infty$. *If* $0<\beta<\mu$, *then*

$$L_n = \int_{\beta n^\alpha}^{\mu n^\alpha} e^{-sx} F_n(x)\,dx = n^\alpha e^{-n^\alpha \beta s}\delta_n, \quad s \geqq 0 \tag{2.3}$$

where $\delta_n\to 0$ *as* $n\to\infty$, *uniformly in* $s\geqq 0$.

PROOF. Now

$$L_n = n^\alpha \int_\beta^\mu e^{-n^\alpha sx} F_n(n^\alpha x)\,dx. \tag{2.4}$$

For $\varepsilon>0$,

$$\mathbf{P}\left[\left|\frac{S_n}{n^\alpha}-\mu\right|>\varepsilon\right] \leqq \frac{\mathbf{E}(S_n-n^\alpha\mu)^2}{n^{2\alpha}\varepsilon^2}. \tag{2.5}$$

Because of (2.1) and (2.2) the right hand side of (2.5) tends to zero as $n\to\infty$. Thus for all $x<\mu$, $F_n(n^\alpha x)\to 0$, as $n\to\infty$. Therefore by using the dominated mean value theorem

$$\int_\beta^\mu F_n(n^\alpha x)\,dx \to 0, \quad n\to\infty. \tag{2.6}$$

Also

$$\int_\beta^\mu e^{-n^\alpha sx} F_n(n^\alpha x)\,dx \leqq e^{-n^\alpha \beta s} \int_\beta^\mu F_n(n^\alpha x)\,dx. \tag{2.7}$$

Hence we can write

$$\int_\beta^\mu e^{-n^\alpha sx} F_n(n^\alpha x)\,dx = e^{-n^\alpha \beta s}\delta_n, \tag{2.8}$$

where $\delta_n\to 0$ as $n\to\infty$, uniformly in $s\geqq 0$. This completes the proof.

REMARK. *Using the obvious inequality* $K_n(x)\leqq H_n(x)\leqq F_n(x)$ *given in* [7], (2.3) *can be seen to be true with* $F_n(x)$ *replaced by* $H_n(x)$ *or* $K_n(x)$.

LEMMA 2. *Subject to the assumptions* (2.1) *and* (2.2) *of Lemma* 1, *if* $0<\beta<\mu$, *then*

$$K_n = \int_{n^\alpha \mu}^\infty e^{-sx}[1-F_n(x)]\,dx = n^\alpha e^{-n^\alpha \beta s}\delta_n' \tag{2.9}$$

where $\delta_n'\to 0$ *as* $n\to\infty$ *uniformly in* $s\geqq 0$.

PROOF. Now

$$K_n = \int_{n^\alpha \mu}^{\infty} e^{-sx}[1-F_n(x)]\,dx \leqq e^{-n^\alpha \beta s} \int_{n^\alpha \mu}^{\infty} [1-F_n(x)]\,dx \leqq \tag{2.10}$$

$$\leqq e^{-n^\alpha \beta s}\left[\int_{n^\alpha \mu}^{n^\alpha \mu + an^r} [1-F_n(x)]\,dx + \int_{n^\alpha \mu + an^r}^{\infty} [1-F_n(x)]\,dx\right] \leqq$$

$$\leqq e^{-n^\alpha \beta s}\left[an^r + \int_{n^\alpha \mu + an^r}^{\infty} [1-F_n(x)]\,dx\right] \leqq$$

$$\leqq e^{-n^\alpha \beta s}\left[an^r + \int_{n^\alpha \mu + an^r - \mathbf{E}(S_n)}^{\infty} [1-F_n(y+\mathbf{E}(S_n))]\,dy\right] \leqq$$

$$\leqq n^\alpha e^{-n^\alpha \beta s}\left[\frac{a}{n^{\alpha-r}} + \frac{1}{n^\alpha}\int_{n^\alpha \mu + an^r - \mathbf{E}(S_n)}^{\infty} [1-F_n(y+\mathbf{E}(S_n))]\,dy\right]$$

$$(\alpha - \delta < r < \alpha,\ a > 0).$$

For $y>0$

$$1-F_n(y+\mathbf{E}(S_n)) = \mathbf{P}[S_n - \mathbf{E}(S_n) > y] \leqq \frac{\beta_2(n)}{y^2}. \tag{2.11}$$

Hence

$$K_n \leqq n^\alpha e^{-n^\alpha \beta s}\left[\frac{a}{n^{\alpha-r}} + \frac{\beta_2(n)}{n^\alpha[n^\alpha \mu + an^r - \mathbf{E}(S_n)]}\right]. \tag{2.12}$$

Using (2.2) we get

$$\limsup_{n\to\infty}\left[\frac{a}{n^{\alpha-r}} + \frac{\beta_2(n)}{n^\alpha[n^\alpha \mu + an^r - \mathbf{E}(S_n)]}\right] = 0. \tag{2.13}$$

Therefore

$$K_n = n^\alpha e^{-n^\alpha \beta s}\delta_n', \tag{2.14}$$

where $\delta_n' \to 0$ as $n\to\infty$, uniformly in $s \geqq 0$.

REMARK. (2.9) *will be true when* $F_n(x)$ *is replaced by* $H_n(x)$ *or* $K_n(x)$ *if* $1+\delta < 2\alpha < 2$.

PROOF. Since

$$K_n(x) \leqq H_n(x), \tag{2.15}$$

$$\int_{n^\alpha \mu}^{\infty} e^{-sx}[1-H_n(x)]\,dx \leqq \int_{n^\alpha \mu}^{\infty} e^{-sx}[1-K_n(x)]\,dx \leqq \tag{2.16}$$

$$\leqq e^{-n^\alpha \beta s}\left[an^r + \int_{n^\alpha \mu + an^r}^{\infty} [1-K_n(x)]\,dx\right] \leqq$$

$$\leqq e^{-n^\alpha \beta s}\left[an^r + \int_{n^\alpha \mu + an^r - \mathbf{E}(N_n)}^{\infty} [1-K_n(y+\mathbf{E}(N_n))]\,dy\right] \leqq$$

$$\leqq n^\alpha e^{-n^\alpha \beta s}\left[\frac{a}{n^{\alpha-r}} + \frac{\beta_2'(n)}{n^\alpha[n^\alpha \mu + an^r - \mathbf{E}(N_n)]}\right] \qquad (\alpha - \delta < r < \alpha,\ \alpha > 0,\ a > 0),$$

where $\beta_2'(n)$ is the variance of N_n. Using corollary A 3.1 and corollary A3 of SERFLING [10, p. 1231] it follows from (2.2) that

$$\mathbf{E}(N_n) = O\{(\log_2 2n)n^\alpha\}, \quad \beta_2'(n) = O\{(\log_2 2n)^2 n^{2\alpha-\delta}\}. \tag{2.17}$$

Therefore (2.9) is valid when $F_n(x)$ is replaced by $H_n(x)$ or $K_n(x)$.

LEMMA 3. *Let* $\{x_i\}$, $i=1, 2, \ldots$ *be a sequence of dependent random variables as defined in Lemma* 1 *satisfying* (2.1). *Let*

$$\beta_r(n) = \mathbf{E}\left|\sum_{i=1}^{n}(x-\mu_i)\right|^r, \quad r = 1, 2, \ldots \tag{2.18}$$

and $\{a_n\}$ *a positive sequence such that*

$$a_n \sim n^\lambda L(n), \quad n \to \infty, \quad \lambda > -1, \tag{2.19}$$

where $L(n)$ *is a non-negative slowly varying function. If there exists at least one* $m > \frac{2(\lambda+1)}{\delta}$ *for which*

$$\beta_m(n) = O(n^{m(\alpha-\delta/2)}), \quad n \to \infty, \quad 2\alpha > \delta, \quad \delta > 0 \tag{2.20}$$

then

$$\sum_{n=1}^{\infty} a_n F_n(n^\alpha \beta) < \infty, \quad 0 < \beta < \mu. \tag{2.21}$$

PROOF. If $d_n > 1$, then

$$\int_{-\infty}^{-d_n} d\mathbf{P}[S_n - \mathbf{E}(S_n) \leqq x] \leqq \int_{-\infty}^{-d_n} \left|\frac{x}{d_n}\right|^m d\mathbf{P}[S_n - \mathbf{E}(S_n) \leqq x]. \tag{2.22}$$

So, for $d_n > 1$,

$$F_n[\mathbf{E}(S_n) - d_n] \leqq \frac{\beta_m(n)}{d_n^m}. \tag{2.23}$$

Take $d_n = \mathbf{E}(S_n) - \beta n^\alpha$. Using (2.1)

$$d_n \sim (\mu - \beta)n^\alpha, \quad n \to \infty. \tag{2.24}$$

Since for large values of n, $(\mu-\beta)n^\alpha > 1$, we can use (2.23). Hence

$$F_n(\beta n^\alpha) = O\left\{\frac{\beta_m(n)}{n^{\alpha m}}\right\}. \tag{2.25}$$

Using (2.20)

$$F_n(\beta n^\alpha) = O\left\{\frac{1}{n^{m\delta/2}}\right\}. \tag{2.26}$$

Therefore for an $m > \frac{2(\lambda+1)}{\delta}$,

$$\sum_{n=1}^{\infty} a_n F_n(\beta n^\alpha) = O\left\{\sum_{n=1}^{\infty} \frac{a_n}{n^{m\delta/2}}\right\}. \tag{2.27}$$

$\sum_{n=1}^{\infty} \frac{a_n}{n^{m\delta/2}}$ converges for our choice of m and is bounded by Theorem 281 [5, p. 247]. This completes the proof of Lemma 3.

REMARK. *Since* $K_n(x) \leqq H_n(x) \leqq F_n(x)$, (2.21) *will be valid with* $K_n(x)$ *or* $H_n(x)$ *in the place of* $F_n(x)$.

LEMMA 4. *If*

$$a_n \sim n^\lambda L(n), \quad n \to \infty, \quad \lambda > -1, \tag{2.28}$$

where $L(n)$ *is a non-negative slowly varying function, then*

$$\sum_{n=1}^{\infty} a_n e^{-n^\alpha s} \sim \frac{1}{\alpha} \Gamma\left(\frac{\lambda+1}{\alpha}\right) s^{-(\lambda+1)/\alpha} L(1/s^\alpha), \quad s \to 0^+, \quad \alpha > 0. \tag{2.29}$$

This is a well known result stated in [9] and can be got from Corollary 1(a) in [12, p. 182] by a suitable substitution.

LEMMA 5. *Let* $\{x_i\}$, $i=1, 2, \dots$ *be a sequence of dependent random variables with* $\mathbf{E}(x_i)=\mu_i$, $i=1, 2, \dots$ *satisfying* (2.1) *and* (2.2). *Let*

$$L_n = \int_{\beta n^\alpha}^{\mu n^\alpha} e^{-sx} F_n(x)\, dx, \quad s \geqq 0, \quad 0 < \beta < \mu \tag{2.30}$$

and

$$a_n \sim n^\lambda L(n), \quad n \to \infty, \quad \lambda > -1, \tag{2.31}$$

where $L(n)$ *is a non-negative slowly varying function. Then*

$$\frac{(\mu s)^{(\lambda+\alpha+1)/\alpha}}{L(1/s^\alpha)} \sum_{n=1}^{\infty} a_n L_n \to 0, \quad s \to 0^+. \tag{2.32}$$

PROOF. From (2.3)

$$\sum_{n=1}^{\infty} a_n L_n = \sum_{n=1}^{\infty} a_n n^\alpha e^{-n^\alpha \beta s} \delta_n, \tag{2.33}$$

where $\delta_n \to 0$ as $n \to \infty$, uniformly in $s \geqq 0$. Therefore, for a given $\varepsilon > 0$, we can choose an integer n_0 so large that $|\delta_n| \leqq \varepsilon$, for $n \geqq n_0$. So

$$\sum_{n=1}^{\infty} a_n L_n \leqq \sum_{n=1}^{n_0} a_n n^\alpha |\delta_n| e^{-n^\alpha \beta s} + \varepsilon \sum_{n=1}^{\infty} a_n n^\alpha e^{-n^\alpha \beta s}. \tag{2.34}$$

Hence using Lemma 4,

$$\limsup_{s \to 0^+} \frac{(\mu s)^{(\lambda+\alpha+1)/\alpha}}{L(1/s^\alpha)} \sum_{n=1}^{\infty} a_n L_n \leqq \tag{2.35}$$

$$\leqq \limsup_{s \to 0^+} \frac{(\mu s)^{(\lambda+\alpha+1)/\alpha} \varepsilon\, \Gamma\left(\frac{\lambda+\alpha+1}{\alpha}\right)}{\alpha L(1/s^\alpha)} (\beta s)^{-(\lambda+\alpha+1)/\alpha} L(1/s^\alpha) \leqq$$

$$\leqq \frac{\Gamma\left(\frac{\lambda+\alpha+1}{\alpha}\right)}{\alpha} \left(\frac{\mu}{\beta}\right)^{(\lambda+\alpha+1/\alpha)} \varepsilon.$$

Hence (2.32) follows.

LEMMA 6. *Under the assumptions of Lemma* 2

$$\frac{(\mu s)^{(\lambda+\alpha+1)/\alpha}}{L(1/s^\alpha)}\sum_{n=1}^{\infty} a_n K_n \to 0 \quad \text{as} \quad s\to 0^+, \tag{2.36}$$

where K_n *is as defined in* (2.9) *and* $\{a_n\}$ *as in* (2.19).

PROOF. This can be proved in the same manner as Lemma 5.

3. Main theorem. *Let* $\{x_i\}$, $i=1, 2, \ldots$ *be a sequence of dependent random variables satisfying* (2.1), (2.2) *and* (2.20) *and* $\{a_n\}$ *a positive sequence defined by* (2.19). *Then*

$$\sum_{n=1}^{\infty} a_n F_n(x) \sim \frac{(x/\mu)^{(\lambda+1)/\alpha} L(x^\alpha)}{\lambda+1}, \quad x\to\infty. \tag{3.1}$$

PROOF. Let

$$\sum_{n=1}^{\infty} a_n F_n(x) = T_1(x)+T_2(x), \tag{3.2}$$

where

$$T_1(x) = \sum_{n=1}^{\infty} a_n F_n(x) U(x-n^\alpha\beta) \tag{3.3}$$

and

$$T_2(x) = \sum_{n=1}^{\infty} a_n F_n(x)[1-U(x-n^\alpha\beta)]. \tag{3.4}$$

Now

$$T_1(x) = \sum_{n=1}^{\infty} a_n F_n(x) U(x-n^\alpha\beta) = \tag{3.5}$$

$$= \sum_{n=1}^{\infty} a_n U(x-n^\alpha\mu) - \sum_{n=1}^{\infty} a_n [U(x-n^\alpha\mu)-F_n(x)] U(x-n^\alpha\beta),$$

where

$$U(x) = \begin{cases} 1, & x\geqq 0 \\ 0, & x<0. \end{cases} \tag{3.6}$$

Let

$$T_1(s) = \int_0^\infty e^{-sx} T_1(x)\,dx. \tag{3.7}$$

Then we have

$$T_1(s) = s^{-1}\sum_{n=1}^{\infty} a_n e^{-n^\alpha\mu s} + \sum_{n=1}^{\infty} a_n(L_n-K_n), \tag{3.8}$$

where L_n and K_n are given by (2.3) and (2.9). Term by term integration is valid because of monotone convergence. Using Lemmas 4, 5 and 6,

$$\frac{\alpha(\mu s)^{(\lambda+1)/\alpha+1}}{\mu\Gamma\left(\frac{\lambda+1}{\alpha}\right) L(1/s^\alpha)}\int_0^\infty e^{-sx} T_1(x)\,dx \to 1, \quad \text{as} \quad s\to 0^+. \tag{3.9}$$

Using Karamata's Tauberian theorem in the form given in [3, p. 423]

$$T_1(x) \sim \frac{(x/\mu)^{(\lambda+1)/\alpha} L(x^\alpha)}{\lambda+1}, \quad x \to \infty. \tag{3.10}$$

Again

$$T_2(x) = \sum_{n=1}^{\infty} a_n F_n(x)[1-U(x-n^\alpha\beta)] \leqq \sum_{n=1}^{\infty} a_n F_n(n^\alpha\beta). \tag{3.11}$$

By Lemma 3, the right hand side of (3.11) is convergent and bounded. Hence

$$\frac{T_2(x)}{(x/\mu)^{(\lambda+1)/\alpha} L(x^\alpha)} \to 0, \quad x \to \infty. \tag{3.12}$$

This completes the proof of the main theorem.

REMARK. *In the above theorem equation* (3.1) *will be valid with* $F_n(x)$ *replaced by* $H_n(x)$ *or* $K_n(x)$ *if* $1+\delta<2\alpha<2$.

This follows easily from remarks under Lemmas 1, 2 and 3.

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AN ANALYSIS OF THE CLASS OF ORDINAL SOLUTIONS OF FERMAT'S EQUATION $x^n+y^n=z^n$

By
J. L. HICKMAN (Canberra)[1]

Throughout this note, ordinals will as a rule be denoted by lower case Greek letters, with "ω" always being reserved for the first infinite ordinal. Lower case Latin letters will always denote finite ordinals (natural numbers); we include 0 amongst the natural numbers.

In the second and third sections of this note we shall present only the results themselves, together with a certain amount of commentary. The proofs are omitted, since they are all based upon the comparison of corresponding terms in the binomial expansions of the two sides of the Diophantine equation being considered, and are almost wholly of a purely technical character.

FACT 1. *Let α be a nonzero ordinal. There exists a unique number n, $n>0$, a unique strictly decreasing sequence $(\alpha_i)_{i<n}$ of ordinals, and a unique sequence $(c_i)_{i<n}$ of positive integers such that* $\alpha=\omega^{\alpha_0}c_0+\ldots+\omega^{\alpha_{n-1}}c_{n-1}$.

PROOF. P. 323 of [3].

As is well-known, the above result dates back to Cantor, and the expression given for α we shall call the "Cantor normal form" of α. Let α be a nonzero ordinal, and suppose that α has the above Cantor normal form. Then we shall put $l(\alpha)=n$, and $e_i(\alpha)=\alpha_i$ and $c_i(\alpha)=c_i$ for $i<n$. We shall henceforth assume that whenever any of $l(\alpha)$, $e_i(\alpha)$, $c_i(\alpha)$ are used in the text, the ordinal α is nonzero.

Any ordinal of the form $\beta+1$ will be called a "successor" ordinal; all other ordinals are known as limit ordinals.

FACT 2 (Bachmann). *Let the ordinals α, β, γ be given, assume that $\beta+\alpha=\alpha$, and let the limit ordinal σ such that $\gamma=\sigma+n$ for some n. For any ordinal δ, put $\iota_\delta=0$ if δ is a limit ordinal, and put $\iota_\delta=1$ if δ is a successor ordinal. Then* $(\alpha+\beta)^\gamma=\alpha^\gamma+\alpha^\sigma\theta(n)$, *where*

(1) $\theta(0)=0$;

(2) $\theta(1)=\beta$;

(3) $\theta(m)=\alpha^{m-1}\beta+\iota_\beta\cdot\left(\sum_{i=2}^{m}\alpha^{m-i}\beta\right)$, *for* $m\geqq 2$.

PROOF. P. 53 of [1].

[1] The work contained in this paper was done whilst the author was a Postdoctoral Fellow at the Australian National University.

We wish to present a slightly more general result than the preceding one, and clearly we may assume that α, β, γ are all nonzero. Since for any number m and any ordinal $\omega\xi+n$ we have $m^{\omega\xi+n}=m^{\omega\xi}m^n=\omega^\xi m^n$, we may ignore the case $\max\{\alpha,\beta\}<\omega$.

THEOREM. *Let α, β, γ be nonzero ordinals with* $\max\{\alpha,\beta\}\geqq\omega$. *Let ξ, ζ be ordinals and m, n be numbers such that $\beta=\omega\xi+m$ and $\gamma=\omega\zeta+n$, and put $\delta=\max\{\alpha,\beta\}$. Then we have the following.*

(1) *If $\alpha+\beta=\beta$, then $(\alpha+\beta)^\gamma=\beta^\gamma$.*

(2) *If $\beta+\alpha=\alpha$, then $(\alpha+\beta)^\gamma$ is given by Bachmann's Theorem.*

(3) *If $\alpha+\beta\neq\beta$ and $\beta+\alpha\neq\alpha$, then*

(a) *If $n=0$, then $(\alpha+\beta)^\gamma=\delta^\gamma$;*

(b) *If $n\neq 0$, then*

(i) *If $m=0$, then $(\alpha+\beta)^\gamma=\delta^{\omega\zeta}(\alpha^n+\beta^n)$;*

(ii) *If $m\neq 0$, then*

$$(\alpha+\beta)^\gamma=\delta^{\omega\zeta}\left(\alpha^n+(\omega\xi)^n+\left[\sum_{k=1}^{n-1}[\alpha^{n-k}+(\omega\xi)^{n-k}]m\right]+m\right).$$

PROOF. It suffices of course to consider the case (3), and so we assume that $e_0(\alpha)=e_0(\beta)$. We put $l(\beta)=r$, and deal firstly with the case in which $\zeta=0$, that is, $\gamma=n$. This case now splits into two subcases, the first being the one in which $m=0$, and the second being the one in which $m\neq 0$.

(A) Assume that $m=0$. Thus β is limit and we have

$$\beta^n=\sum_{i=0}^{r-1}\omega^{e_0(\beta)(n-1)+e_i(\beta)}c_i(\beta).$$

This follows by inspection if $r=1$, and from Bachmann's theorem if $r>1$, for in this latter case if we put $\psi=\sum_{i=1}^{r-1}\omega^{e_i(\beta)}c_i(\beta)$, then $\beta=\omega^{e_0(\beta)}c_0(\beta)+\psi$ and $\psi+\omega^{e_0(\beta)}c_0(\beta)=\omega^{e_0(\beta)}c_0(\beta)$.

Since we do not know whether α is a limit ordinal or a successor ordinal, we cannot obtain a precise form for the expansion of α^n. Nevertheless, in both these cases we know from Bachmann's theorem (by the same trick as above) that $\alpha^n=\omega^{e_0(\alpha)n}c_0(\alpha)+\tau$ for some ordinal τ with $\tau+\omega^{e_0(\alpha)n}c_0(\alpha)=\omega^{e_0(\alpha)n}c_0(\alpha)$. Since $e_0(\beta)=e_0(\alpha)$, it follows that we have

$$\alpha^n+\beta^n=\omega^{e_0(\beta)n}[c_0(\alpha)+c_0(\beta)]+\sum_{i=1}^{r-1}\omega^{e_0(\beta)(n-1)+e_i(\beta)}c_i(\beta),$$

where we are adopting the convention that for any sequence $(\theta_i)_{i<p}$ of ordinals, $\sum_{i=j}^{k}\theta_i=0$ if $j>k$.

Now $\alpha+\beta$ is a limit ordinal, since β is a limit ordinal. Moreover, since $e_0(\alpha)=e_0(\beta)$, we have

$$\alpha+\beta=\omega^{e_0(\beta)}[c_0(\alpha)+c_0(\beta)]+\sum_{i=1}^{r-1}\omega^{e_i(\beta)}c_i(\beta).$$

Using Bachmann's theorem to compute the expansion of $(\alpha+\beta)^n$, and then comparing this to the above expansion of $\alpha^n+\beta^n$, we see that $(\alpha+\beta)^n=\alpha^n+\beta^n$. Since $\zeta=0$ and hence $\delta^{\omega\zeta}=1$, we have obtained the desired result.

(B) Assume that $m\neq 0$. Then by Bachmann's theorem and (A),

$$(\alpha+\beta)^n = ([\alpha+\omega\xi]+m)^n = (\alpha+\omega\xi)^n+\sum_{i=1}^{n}(\alpha+\omega\xi)^{n-i}m =$$

$$= \alpha^n+(\omega\xi)^n+\left\{\sum_{i=1}^{n-1}[\alpha^{n-i}+(\omega\xi)^{n-i}]m\right\}+m.$$

This disposes of the case in which $\zeta=0$, and we now look at the general case. Now $\delta^{\omega\zeta}\leqq(\alpha+\beta)^{\omega\zeta}\leqq(\delta 2)^{\omega\zeta}=\delta^{\omega\zeta}$. Thus $(\alpha+\beta)^{\omega\zeta}=\delta^{\omega\zeta}$, which proves our result for the case in which $n=0$. Finally, when both ζ and n are nonzero, we have $(\alpha+\beta)^{\omega\zeta+n}=(\alpha+\beta)^{\omega\zeta}(\alpha+\beta)^n=\delta^{\omega\zeta}(\alpha+\beta)^n$, and we now apply whichever of (A), (B) is applicable.

For convenience we list the various cases of $(\alpha+\beta)^n$ with $n\geqq 2$.

(1) $e_0(\alpha)=e_0(\beta)$ and β is a limit ordinal:

$$(\alpha+\beta)^n = \alpha^n+\beta^n.$$

(2) $e_0(\alpha)=e_0(\beta)$ and β is a successor ordinal:

$$(\alpha+\beta)^n = \alpha^n+\sigma^n+\left[\sum_{i=1}^{n-1}(\alpha^{n-i}+\sigma^{n-i})m\right]+m,$$

where $\beta=\sigma+m$ with σ is a limit ordinal and m a positive number.

(3) $e_0(\alpha)>e_0(\beta)$ and β is a limit ordinal:

$$(\alpha+\beta)^n = \alpha^n+\alpha^{n-1}\beta.$$

(4) $e_0(\alpha)>e_0(\beta)$ and β is a successor ordinal:

$$(\alpha+\beta)^n = \alpha^n+\sum_{i=1}^{n}\alpha^{n-i}\beta.$$

(5) $e_0(\alpha)<e_0(\beta)$:

$$(\alpha+\beta)^n = \beta^n.$$

Our classification of the class of ordinal solutions of the equation $x^n+y^n=z^n$ splits into cases corresponding to the five cases of the binomial expansion of $(\alpha+\beta)^n$ just enunciated. As stated previously, the proofs are of a purely technical nature and are omitted. In each case we simply use the preceding theorems to obtain the Cantor normal forms of $\alpha^n+\beta^n$ and γ^n, and equate the corresponding terms.

Case (5) being the simplest, we commence with that.

RESULT 1. *Let α, β be any nonzero ordinals with* $\max\{\alpha,\beta\}\geqq\omega$ *and* $\alpha+\beta=\beta$, *and let n be any number with* $n\geqq 2$. *Then β is the unique ordinal γ such that* $\alpha^n+\beta^n=\gamma^n$.

The uniqueness assertion in the above result follows from the fact that if δ, ψ are any two nonzero ordinals and n is any positive number, then from $\delta^n=\psi^n$ we can deduce $\delta=\psi$. This is not the case if n is replaced by a limit ordinal.

Cases (1) and (2) combine to give the following.

RESULT 2. *Let α, β be any nonzero ordinals with* $\max\{\alpha, \beta\}\geqq\omega$ *and* $e_0(\alpha)=e_0(\beta)$, *and let n be any number with* $n\geqq 2$.

(i) *There exists an ordinal γ such that $\alpha^n+\beta^n=\gamma^n$ if and only if β is a limit ordinal.*

(ii) *If β is a limit ordinal, then the ordinal γ such that $\alpha^n+\beta^n=\gamma^n$ is determined uniquely by the equation $\gamma=\alpha+\beta$.*

It is known (see p. 350 of [3]) that if α, β are any sucessor ordinals with $\max\{\alpha, \beta\}\geqq\omega$ and n is any number with $n\geqq 2$, then we have $\alpha^n+\beta^n=\beta^n+\alpha^n$ if and only if $\alpha=\beta$. It follows from Result 2 above that if α, β are any limit ordinals with $\max\{\alpha, \beta\}\geqq\omega$ and $e_0(\alpha)=e_0(\beta)$, and n is any positive number, then we have $\alpha^n+\beta^n=\beta^n+\alpha^n$ if and only if $\alpha+\beta=\beta+\alpha$. But this is true if and only if $\sum_{i=1}^{r-1}\omega^{e_i(\alpha)}c_i(\alpha)=\sum_{i=1}^{s-1}\omega^{e_i(\beta)}c_i(\beta)$, where $r=l(\alpha)$ and $s=l(\beta)$.

Clearly the alternatives given in the preceding paragraph are the only ones of interest in this matter. For if either $e_0(\alpha)\neq e_0(\beta)$ or else exactly one of α, β is a limit ordinal and both α, β are nonzero, then we cannot have $\alpha^n+\beta^n=\beta^n+\alpha^n$ for any positive number n.

With regard to case (3), we find it easiest to give the classification result in two parts. The first part is required in the proof of the second.

RESULT 3. *Let α, β be any ordinals such that β is a nonzero limit ordinal and $\beta+\alpha=\alpha$, and let n be any number with $n\geqq 2$. Put $l(\beta)=r$. Then there exists an ordinal γ such that $\alpha^n+\beta^n=\gamma^n$ if and only if $e_0(\alpha)(n-1)<e_0(\beta)(n-1)+e_{r-1}(\beta)$. If this is the case, then γ is uniquely determined by the following equations:*

(1) $l(\delta)=r$;

(2) $e_i(\delta)=[e_0(\beta)(n-1)+e_i(\beta)]-e_0(\alpha)(n-1)$ *for* $i<r$;

(3) $c_i(\delta)=c_i(\beta)$ for $i<r$;

(4) $\gamma=\alpha+\delta$.

In condition (2) above we are using the convention that if θ, ψ are any two ordinals with $\theta\geqq\psi$, then $\theta-\psi$ is the unique ordinal τ such that $\theta=\psi+\tau$.

Using Result 3, we can now characterize those ordinals β for which such triples (α, β, γ) exist.

RESULT 4. *Let β be a nonzero limit ordinal, and let n be a number with $n\geqq 2$. There exist ordinals α, γ such that $\beta+\alpha=\alpha$ and $\alpha^n+\beta^n=\gamma^n$ if and only if $\beta=\omega^2\delta$ for some nonzero ordinal δ such that either $\delta\geqq\omega^\omega$ or else $\delta=\omega^{n-2}\lambda$ for some ordinal λ.*

Our final classification, corresponding to case (4) of the binomial expansion, is the least satisfactory, owing to the complexity of the criteria concerned. We can, however, see no way of simplifying these criteria.

RESULT 5. *Let β be a successor ordinal and n a number with $n \geqq 2$. Put $l(\beta)=r$. There exist ordinals α, γ such that $\beta+\alpha=\alpha$ and $\alpha^n+\beta^n=\gamma^n$ if and only if there is a positive number k such that the following 9 sets of conditions are satisfied.*

(1) $r > k(n-1)$.

(2) $e_0(\beta)(n-1)+e_m(\beta) = [e_0(\beta)+e_{r-(k+1)}(\beta)]j+e_0(\beta)[n-(j+1)]+e_{jk+m}(\beta)$

for $j<n-1$ *and* $jk+m<r-1$.

(3)

$$e_0(\beta)(n-1)+e_m(\beta) = [e_0(\beta)+e_{r-(k+1)}(\beta)](j+1)+e_0(\beta)[n-(j+2)]+e_{jk+m-(r-1)}(\beta),$$

for $j<n-1$, $m<r-1$ *and* $r-1 \leqq jk+m<2(r-1)$.

(4) $e_0(\beta)(n-1)+e_{m-(r-1)}(\beta) = [e_0(\beta)+e_{r-(k+1)}(\beta)](j+1)+e_0(\beta)[n-(j+2)]+$
$+e_{jk+m-(r-1)}(\beta)$

for $j<n-1$, $r-1 \leqq m<r+k-1$ *and* $r-1 \leqq jk+m<2(r-1)$.

(5) $e_0(\beta)(n-1)+e_m(\beta) = [e_0(\beta)+e_{r-(k+1)}(\beta)](n-1)+e_{k(n-1)+m}(\beta)$

for $m<(r-1)-k(n-1)$.

(6) $c_{r-(k+1)}(\beta) = c_{r-1}(\beta)\,p$ *for some positive number* p.

(7) $c_m(\beta) = c_{jk+m}(\beta)$ *for* $j<n-1$ *and* $jk+m<r-1$.

(8) $c_m(\beta) = c_{jk+m-(r-1)}(\beta)$

for $j<n-1$, $m<r-1$ *and* $r-1 \leqq jk+m<2(r-1)$.

(9) $c_m(\beta)=c_{k(n-1)+m}(\beta)$ *for* $m<(r-1)-k(n-1)$.

Assume that for some given positive numbers k and p these conditions are satisfied. Then the leading $u=nk$ terms in the Cantor normal form of α are determined as follows.

(10) $e_m(\alpha) = e_0(\beta)+e_{r-(k+m+1)}(\beta)$ *for* $m<k$.

(11) $e_m(\alpha) = e_{m-k}(\beta)$ for $k \leqq m < u$.

(12) $c_0(\alpha) = p$.

(13) $c_k(\alpha) = c_0(\beta)c_{r-1}(\beta)$.

(14) $c_m(\alpha) = c_{r-(k+m+1)}(\beta)$ for $0 < m < k$.

(15) $c_m(\alpha) = c_{m-k}(\beta)$ for $k < m < u$.

Finally, γ is uniquely determined as follows.

(16) $l(\delta) = r-k(n-1)$.

(17) $e_m(\delta) = e_{u+m-k}(\beta)$ for $m < r-k(n-1)$.

(18) $c_m(\delta) = c_{u+m-k}(\beta)$ for $m < r-k(n-1)$.

(19) $\gamma = \alpha+\delta$.

In conclusion we present some results concerning the case in which the finite exponent n is replaced by a transfinite ordinal μ.

RESULT 6. *Let α, β be any nonzero ordinals, and let μ be a nonzero limit ordinal.*

(1) *If $\alpha \geqq \omega$, then there exists an ordinal γ such that $\alpha^\mu+\beta^\mu=\gamma^\mu$ if and only if $e_0(\alpha)\mu<e_0(\beta)\mu$; in this case $\alpha^\mu+\beta^\mu=\beta^\mu$.*

(2) *If $\alpha<\omega$, then there exists an ordinal γ such that $\alpha^\mu+\beta^\mu=\gamma^\mu$ if and only if $\mu<\omega e_0(\beta)\mu$; in this case also, $\alpha^\mu+\beta^\mu=\beta^\mu$.*

To see that the case $\alpha<\omega$ in Result 6 does in fact require separate attention, we need only put $\alpha=2$, $\beta=\omega$, and $\mu=\varepsilon_0$, where ε_0 is the first epsilon number. Then $e_0(\alpha)=0$ and $e_0(\beta)=1$, and so $e_0(\alpha)\mu<e_0(\beta)\mu$. However, $\alpha^\mu+\beta^\mu= =\mu+\mu=\mu 2$, and since $\mu^\mu>\mu 2$ whilst $\delta^\mu\leqq\mu<\mu 2$ for every $\delta<\mu$, it is clear that there is no ordinal γ for which $\alpha^\mu+\beta^\mu=\beta^\mu$.

RESULT 7. *Let α, β be any infinite ordinals, and let μ be an infinite successor ordinal. Put $\mu=\sigma+n$, where σ is a limit ordinal and n is a (positive) number. Then there exists an ordinal γ such that $\alpha^\mu+\beta^\mu=\gamma^\mu$ if and only if one of the following holds.*

(1) $e_0(\alpha)\sigma < e_0(\beta)\sigma$;

(2) $e_0(\alpha)\sigma = e_0(\beta)\sigma$ *and* $\alpha^n+\beta^n=\delta^n$ *for some ordinal* δ.

In the first case, $\alpha^\mu+\beta^\mu=\beta^\mu$, and in the second, $\alpha^\mu+\beta^\mu=\delta^\mu$.

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ON MODULES OVER RINGS OF TYPE (n, k)

By

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We denote the direct sum of n copies of a module M by M^n. It is well-known [4] that there exists a ring R with the property that $R^n \cong R^{n+k} \cong R^{n+2k} \cong \dots$ for every pair (n, k) of positive integers. Following Leavitt, we call rings (modules) with this property, rings (modules) of type (n, k) where n and k are minimal and write $\tau(R)=(n, k)$.

Since this property is rather unusual, we can expect that R-modules over rings R of type (n, k), will exhibit some unusual properties too. It is the purpose of this note to investigate the properties of such modules.

In the first place, we give a characterization of rings of type (n, k) in terms of their modules. Unless explicitly stated otherwise, all modules considered are unital left R-modules where R is an associative ring and $1 \in R$. The signs $\oplus$ and $\otimes$ denote direct sum and tensor product respectively (see e.g. [1] p. 82 and p. 218).

THEOREM 1. *The following statements are equivalent for a ring R:*

(a) $\tau(R)=(n, k)$.

(b) $\tau(M)=(n, k)$ *for every R-module M.*

(c) *there exists a pair of positive integers (n, k) such that for every R-module M, there exist an endomorphism α of M^n and a homomorphism $\beta: M^n \to M^k$ such that the sequence*

$$0 \longrightarrow M^n \xrightarrow{\alpha} M^n \xrightarrow{\beta} M^k \longrightarrow 0$$

is exact and n and k are minimal with respect to the existence of such a pair α and β.

(d) *there exists a pair of positive integers (n, k), an endomorphism α of R^n and a homomorphism $\beta: R^n \to R^k$ such that*

$$0 \longrightarrow R^n \xrightarrow{\alpha} R^n \xrightarrow{\beta} R^k \longrightarrow 0$$

is exact and n and k are minimal with respect to the existence of such a pair α and β.

PROOF. (a)$\Rightarrow$(b). R may be considered as a right R-module and thus

$$M^n \cong R\otimes M^n \cong (R\otimes M)^n \cong R^n\otimes M \cong R^{n+k}\otimes M \cong R^{n+2k}\otimes M \cong \dots \cong$$
$$\cong (R\otimes M)^{n+k} \cong (R\otimes M)^{n+2k} \cong \dots \cong M^{n+k} \cong M^{n+2k} \cong \dots.$$

(b)$\Rightarrow$(a). Trivial.

(a)$\Rightarrow$(c). Since $\tau(R)=(n, k)$, there exists an isomorphism $\varphi: M^n \to M^{n+k}$. If $i: M^n \to M^{n+k}$ is the first n co-ordinates injection and $\pi: M^{n+k} \to M^k$ is the last k co-ordinates projection, then we prove that $\alpha=\varphi^{-1}i$ is the required endomorphism of M^n and $\beta=\pi\varphi$ the required homomorphism $M^n\to M^k$. In the following diagram, the top row is exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^n & \xrightarrow{i} & M^{n+k} & \xrightarrow{\pi} & M^k & \longrightarrow & 0 \\ & & \alpha \big| & & \varphi^{-1}\big\downarrow\big\uparrow\varphi & & \big\uparrow\beta & & \\ & & \longrightarrow & & M^n & & \text{———} & & \end{array}$$

It is clear that α is a monomorphism and β is an epimorphism, so we only have to prove that $\operatorname{Im}\alpha=\operatorname{Ker}\beta$. In the first place, $\beta\alpha=\pi\varphi\varphi^{-1}i=0$ proving that $\operatorname{Im}\alpha\subseteq\operatorname{Ker}\beta$. If, conversely, $m\in\operatorname{Ker}\beta$, then $\pi(\varphi m)=0$ which means that $\varphi m=(m',0)$ for some $m'\in M^n$. Since $\alpha m'=\varphi^{-1}im'=m$, we also have $\operatorname{Ker}\beta\subseteq\operatorname{Im}\alpha$ proving that $\operatorname{Im}\alpha=\operatorname{Ker}\beta$.

(c)$\Rightarrow$(d). Trivial.

(d)$\Rightarrow$(a). Since R^k is free, the sequence splits and we have $R^n\cong R^{n+k}$, so $\tau(R)=(n,k)$ since n and k are minimal.

(A part of Theorem 1 was suggested by J. Heidema).

COROLLARY 1. *A ring R has type* $(1, 1)$ *iff for every module $_RM$, there exist $\alpha,\beta\in\operatorname{End}_R(M)$ such that*

$$0\longrightarrow M\xrightarrow{\alpha}M\xrightarrow{\beta}M\longrightarrow 0$$

is exact.

KAPLANSKY asked in his book [3]: If $G\oplus G$ and $H\oplus H$ are isomorphic, are G and H isomorphic? JÓNSSON [2] gave an example to show that for torsion-free indecomposable abelian groups, the answer is negative. However, we have

COROLLARY 2. *If G and H are modules over a ring R of type* $(1, 1)$, $G\oplus G\cong H\oplus H$ *implies* $G\cong H$.

COROLLARY 3. *If M is a module over a ring R of type* $(1, 1)$ *and* $M^n\cong M^m$ *for* $n\geqq m\geqq 2$, *we have*

$$M^{n-k}\cong M^{m-k},\quad k=0,1,2,\ldots,m-1.$$

If we mean by semi-simple ring a direct sum of simple rings, then we can prove

COROLLARY 4. *The null-ring is the only semi-simple ring of type* $(1, 1)$.

PROOF. Suppose R is a semi-simple ring of type $(1, 1)$ and let $R=\bigoplus_{i\in I} S_i$ where S_i is a simple ring for each $i\in I$. Every S_i is an R-module and therefore $\tau(S_i)=(1,1)$. For arbitrary $k\in I$, therefore, $S_k\cong S_k\oplus S_k$.

Since S_k is simple, we have $S_k=0$.

Let the ring R have type (n, k) and let M be an arbitrary R-module. Then, by Theorem 1, there exists an isomorphism $\varphi: M^n\to M^{n+k}$. Let $\alpha: M^n\to M^n$ and $\beta: M^n\to M^k$ be homomorphisms as defined in Theorem 1. Then we prove

THEOREM 2. *Any R-module N has both the projective and injective properties relative to the exact sequence*

$$0\longrightarrow M^n\xrightarrow{\alpha}M^n\xrightarrow{\beta}M^k\longrightarrow 0.$$

PROOF. Let $\gamma: N \to M^k$ be any homomorphism. For arbitrary $x \in N$, $(0, \gamma x) \in M^n \oplus M^k$ and there exists a unique $y \in M^n$ such that $\varphi y = (0, \gamma x)$. If we define $\delta: N \to M^n$ by the mapping $x \mapsto y$, then it is clear that δ is a homomorphism such that $\beta\delta = \gamma$. On the other hand, let $\gamma': M^n \to N$ be an arbitrary homomorphism. Since every $m \in M^n$ has a unique representation $m = \varphi^{-1}(m_1, m_2)$ with $m_1 \in M^n$ and $m_2 \in M^k$, we can define $\delta': M^n \to N$ by $\delta' m = \gamma' m_1$. δ' is a well-defined mapping and indeed a homomorphism such that $\delta'\alpha = \gamma'$.

We now turn our attention to the special class of rings R with $\tau(R) = (1, 1)$. We can characterize these rings by

THEOREM 3. *The following statements are equivalent for a ring R:*

(a) $\tau(R) = (1, 1)$.

(b) *if A and B are left R-modules and A is a direct summand of B, then $A \oplus B \cong B$.*

(c) *for every splitting exact sequence of R-modules*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0,$$

there exist homomorphisms $\gamma: B \to B$ and $\delta: B \to C$ such that the sequence

$$0 \longrightarrow B \xrightarrow{\gamma} B \xrightarrow{\delta} C \longrightarrow 0$$

is exact.

(d) *for every splitting exact sequence of R-modules*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0,$$

there exist homomorphisms $\lambda: A \to B$ and $\mu: B \to B$ such that the sequence

$$0 \longrightarrow A \xrightarrow{\lambda} B \xrightarrow{\mu} B \longrightarrow 0$$

is exact.

PROOF. (a)$\Rightarrow$(b). Let $B \cong A \oplus C$. Since then also $A \oplus A \oplus C \cong B$, we have $A \oplus B \cong B$. Moreover, $A \oplus C \oplus C \cong B$, from which it follows that $B \oplus C \cong B$.

(b)$\Rightarrow$(a). Since R is trivially a direct summand of R, we have $R \oplus R \cong R$, so $\tau(R) = (1, 1)$.

(b)$\Rightarrow$(d). Let $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ be any splitting exact sequence. Since $B \cong A \oplus C$, it follows that there exists an isomorphism $\varphi: B \to A \oplus B$. In the following diagram, the top row is exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\tau_1} & A \oplus B & \xrightarrow{\pi_2} & B & \longrightarrow & 0 \\ & & \Big|{\scriptstyle\alpha} & & {\scriptstyle\varphi}\Big\uparrow\Big\downarrow{\scriptstyle\varphi^{-1}} & & \Big\uparrow{\scriptstyle\beta} & & \\ & & \longrightarrow & & B & & \text{———} & & \end{array}$$

where τ_1 is the first co-ordinate injection and π_2 the second co-ordinate projection.

If $\alpha=\varphi^{-1}\tau_1$ and $\beta=\pi_2\varphi$, then it is clear that

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} B \longrightarrow 0$$

is exact.

(d)⇒(a). Since $0 \longrightarrow R \xrightarrow{1_R} R \longrightarrow 0 \longrightarrow 0$ is splitting exact, we have

$$0 \longrightarrow R \xrightarrow{\alpha} R \xrightarrow{\beta} R \longrightarrow 0$$

is exact for suitable α and β. This sequence, however, splits since R is free, so $R \cong R \oplus R$.

(b)⇒(c)⇒(a) are proved similarly.

Let R be a ring of type (1, 1). Then we have the following corollaries to Theorem 3:

COROLLARY 1. *If A is any injective submodule of an R-module B, then $B \cong A^n \oplus B$ for $n=1, 2, 3, \ldots$.*

COROLLARY 2. *If an R-module A has a projective epimorphic image B, then $A \cong A \oplus B^n$, $n=1, 2, 3, \ldots$.*

COROLLARY 3. *Let A and B be R-modules. If A is isomorphic to a direct summand of B and B is isomorphic to a direct summand of A, then $A \cong B$.*

THEOREM 4. *For a ring R of type* (1, 1), *the following are equivalent:*

(a) *If A and B are R-modules, then $A \oplus A \cong A \oplus B$ implies $A \cong B$.*

(b) *If the exact sequence of R-modules $0 \to A \to B \to C \to 0$ splits, then $A \cong B \cong C$.*

(c) *If $A \neq 0$ and B are R-modules and A is a direct summand of B, then $A \cong B$.*

(d) *Any R-module M has only the direct decompositions*

$$M \cong M^k; \quad k=1, 2, 3, \ldots.$$

PROOF. (a)⇒(b). Suppose the sequence

$$0 \to A \to B \to C \to 0$$

splits. Then $B \cong A \oplus C$ and by Theorem 3 $A \oplus B \cong B \cong B \oplus B$, so $A \cong B$. Also by Theorem 3, $B \cong B \oplus C$ and from $B \oplus B \cong B \oplus C$ we have $B \cong C$.

(b)⇒(c). Let A be a direct summand of B, say $B=A \oplus C$. Then the sequence

$$0 \longrightarrow A \xrightarrow{\tau_1} B \xrightarrow{\pi_2} C \longrightarrow 0$$

with τ_1 the first co-ordinate injection and π_2 the second co-ordinate projection is splitting exact, and hence $A \cong B$.

(c)⇒(d). For any direct summand N of M, we have $N \cong M$.

(d)⇒(a). If $A \oplus A \cong A \oplus B$, then $A \cong A \oplus B$. Since A has only the decompositions $A \cong A^k$, $k=1, 2, 3, \ldots$, we have $B \cong A^k$ for some k, therefore $B \cong A$.

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PÓLYA'S THEOREM BY SCHNEIDER'S METHOD

By
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Dedicated to Professor Th. Schneider on his 65th birthday

A well known theorem of G. Pólya states that 2^z is the smallest transcendental entire function with integral values at all positive integral points z; more precisely, if f is an entire function satisfying $f(n)\in\mathbf{Z}$ for all $n\in\mathbf{N}$, and

$$\limsup_{R\to\infty}\frac{1}{R}\operatorname{Log}|f|_R<\operatorname{Log}2, \tag{1}$$

(where $|f|_R=\sup_{|z|=R}|f(z)|$), then f is a polynomial.

We give here a new proof of this theorem, with a somewhat worse constant in place of Log 2, but which allows some further generalisations.

Notations. We denote by $\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{C}$ the non-negative rational integers, the rational integers, the rational numbers and the complex numbers, respectively. When α is an algebraic number, we denote by $s(\alpha)=\max\{\operatorname{Log}|\overline{\alpha}|, \operatorname{Log}d(\alpha)\}$ the size of α (see for instance [3], § 1.2). For $R>0$, B_R is the set $\{z\in\mathbf{C};\ |z|\leqq R\}$. Finally, when $h\in\mathbf{N}$ and $z\in\mathbf{C}$, we define $\binom{z}{h}$ by

$$\binom{z}{h}=\frac{z(z-1)\dots(z-h+1)}{h!}.$$

We shall use only the trivial bounds

$$\left|\binom{z}{h}\right|\leqq 2^{H+R}\quad\text{and}\quad\left|\binom{z}{h}\right|\leqq e^H\left(\frac{R}{H}+1\right)^H$$

for $|z|\leqq R$ and $1\leqq h\leqq H$.

The main result of this paper is the following.

THEOREM 1. *Let K be a number field, and γ_0, γ_1 two positive real numbers. Then there exists an effectively computable number C, depending only on γ_0, γ_1 and $[K:\mathbf{Q}]$, with the following property:*

Let S be a subset of $\mathbf{Z}$, with Card $S\cap B_R\geqq\gamma_0 R$ *for all sufficiently large R; let f, g be two entire functions satisfying*

$$g(n)\neq 0\quad\text{and}\quad\frac{f(n)}{g(n)}\in K\quad\text{for all}\quad n\in S,$$

such that for all sufficiently large R,

$$\max_{n\in S\cap B_R} \operatorname{Log}\left\{\frac{1}{|g(n)|};\ s\left(\frac{f(n)}{g(n)}\right)\right\} \leqq \gamma_1 R,$$

and

$$\max\{\operatorname{Log}|g|_R;\ \operatorname{Log}|f|_R\} \leqq R/C.$$

Then f/g is a rational function.

We obtain Pólya's theorem (with the constant Log 2 in (1) replaced by $1/C$) by setting

$$S = \mathbf{N};\ \gamma_0 = \gamma_1 = 1;\ g = 1;\ K = \mathbf{Q}.$$

(When $m\in\mathbf{Z}$, then $s(m)=\operatorname{Log}|m|$). A computation[1] of C by the present method leads to $C=283$, and it is an interesting problem to obtain by this way the best possible constant $\frac{1}{\operatorname{Log} 2}=1.44\ldots$.

Proof of Theorem 1. Let k_0 be an integer with $k_0>2\delta/\gamma_0$, where $\delta=[K:\mathbf{Q}]$, and let h_0 be a real number with $2\delta/k_0<h_0<\gamma_0$ (for instance $k_0=[2\delta/\gamma_0]+1$, $h_0=(\gamma_0/2)+(\delta/k_0)$). Let N be a sufficiently large integer; c_1, c_2, c_3 will denote positive constants which are effectively (and easily) computable in terms of $\gamma_0, \gamma_1, \delta$ (and h_0, k_0).

First step. We construct rational integers

$$a_{h,k} \quad (0 \leqq h < h_0 N;\ 0 \leqq k \leqq k_0-1),$$

of absolute value less than $\exp(c_1 N)$, *not all zero, such that the meromorphic function*

$$F(z) = \sum_{0\leqq h<h_0 N}\ \sum_{0\leqq k<k_0} a_{h,k}\binom{z}{h}\left(\frac{f(z)}{g(z)}\right)^k$$

satisfies

$$F(n) = 0 \quad \text{for all} \quad n\in S\cap B_N.$$

We have to solve a system of at most $2N+1$ linear equations, with at least $h_0 k_0 N$ unknowns, and with coefficients in K; for $n\in S\cap B_N$, the numbers

$$\binom{n}{h}\left(\frac{f(n)}{g(n)}\right)^k \quad (0 \leqq h < h_0 N;\ 0 \leqq k < k_0)$$

have a common denominator bounded by $\gamma_1 k_0 N$, and a size bounded by $(h_0+1+k_0\gamma_1)N$. Hence Lemma 1.3.1 of [3] gives a non trivial solution $a_{h,k}$ with $\operatorname{Log}\max_{h,k}|a_{h,k}|<c_1 N$.

Second step. For $m\in S$, either $F(m)=0$, or $\operatorname{Log}|F(m)|\geqq -c_2|m|$.

The denominator of $F(m)$ is bounded by $\gamma_1 k_0|m|$, and the size of $F(m)$ is bounded by

$$\operatorname{Log}[k_0(h_0 N+1)]+c_1 N+(|m|+h_0 N)\operatorname{Log} 2+\gamma_1 k_0|m|.$$

[1] Made by A. Escassut and M. Mignotte.

The basic inequality

$$-2\delta s(\alpha) \leqq \operatorname{Log}|\alpha| \quad \text{for all} \quad \alpha \in K,\ \alpha \neq 0$$

(see [3], (1.2.3)) for $|m| \geqq N$, and the first step for $|m| \leqq N$, give the result.

Third step: induction. Define $G(z) = \big(g(z)\big)^{k_0} \cdot F(z)$. *Then, for all integers* $M \geqq N$,

$(\mathrm{I})_M$: $$F(m) = 0 \quad \textit{for all} \quad m \in S \cap B_M,$$

and

$(\mathrm{II})_M$: $$\operatorname{Log}|G|_M < -c_3 M, \quad \textit{with} \quad c_3 = k_0\gamma_1 + c_2.$$

The first step proves $(\mathrm{I})_N$, and $(\mathrm{II})_M \Rightarrow (\mathrm{I})_M$ is a consequence of the second step and of the hypothesis on the lower bound for $|g(m)|$. The property "$(\mathrm{II})_M$ for all M" implies $F=0$, which means that f/g is an algebraic function (and consequently a rational function, because f/g is meromorphic in $\mathbf{C}$). Now, to conclude the proof of Theorem 1, it is sufficient to prove $(\mathrm{I})_M \Rightarrow (\mathrm{II})_{M+1}$.

Assume $(\mathrm{I})_M$ is true. Then, for $R > M$, we get from Schwarz lemma

$$\operatorname{Log}|G|_{M+1} \leqq \operatorname{Log}|G|_R - \gamma_0 M \operatorname{Log}\frac{R^2+(M+1)^2}{2R(M+1)}.$$

(Cf. Lemma 6.2.1 of [3], where the inequality

$$\left|\frac{R_2^2 - z\bar{z}_j}{R_2(z-z_j)}\right| \geqq \frac{R_2^2 - R_1\varrho}{R_2(R_1+\varrho)}, \quad |z| = R_1, \quad |z_j| \leqq \varrho,$$

can be sharpened to[2]

$$\left|\frac{R_2^2 - z\bar{z}_j}{R_2(z-z_j)}\right| \geqq \frac{R_2^2 + R_1\varrho}{R_2(R_1+\varrho)}.\Big)$$

We bound $|G|_R$ for $R > M$:

$$\operatorname{Log}|G|_R \leqq \operatorname{Log}[(h_0N+1)k_0] + c_1N + h_0N\left[1 + \operatorname{Log}\left(\frac{R}{h_0N}+1\right)\right] + 2k_0\frac{R}{C}.$$

Choose $R = l_0(M+1)$, with l_0 sufficiently large, say

$$\frac{\gamma_0 - h_0}{3}\operatorname{Log} l_0 \geqq \max\{c_3;\ 2h_0 + c_1 + \gamma_0 - h_0 \operatorname{Log} h_0\}.$$

Then

$$\gamma_0 \operatorname{Log}\frac{R^2+(M+1)^2}{2R(M+1)} \geqq \gamma_0 \operatorname{Log}\frac{l_0}{2},$$

and we obtain

$$\operatorname{Log}|G|_{M+1} < -\left\{\frac{2}{3}(\gamma_0 - h_0)\operatorname{Log} l_0 - \frac{2k_0 l_0}{C}\right\} M,$$

which is $< -c_3 M$ when C is sufficiently large. This proves Theorem 1.

[2] This was pointed out to me by J. Dufresnoy and H. L. Montgomery.

It would be interesting to generalize Theorem 1 to more general sets S, for example to $S \subset \mathbf{Z}[i]$; the corresponding generalisation of Pólya's theorem is due to FUKASAWA and GEL'FOND [2]: if f is an entire function satisfying $f(a+ib) \in \mathbf{Z}[i]$, when $a+ib \in \mathbf{Z}[i]$, and

$$\limsup_{R\to\infty} \frac{1}{R^2} \operatorname{Log} |f|_R < \frac{\pi}{2(1+e^{164/\pi})^2}, \tag{2}$$

then f is a polynomial.

With the present method, we can deal only with a stronger hypothesis (where $1/R^2$ is replaced by $(\operatorname{Log} R)/R^2$ in (2)), because we do not know interpolation polynomials in $\mathbf{Z}[i]$ generalizing the polynomials $\binom{z}{n}$ in Z; this problem[3] is connected with those of the measure of irrationality (or transcendence) of e^π, and of the algebraic independence of π and e^π.

On the other hand, we can consider more general sets of algebraic numbers. Using the method of proof of Theorem 1, we get:

THEOREM 2. *Let K be a number field, γ_1, γ_2 positive real numbers, and $\Psi: [0, +\infty) \to [0, +\infty)$ a positive real valued function satisfying* $\limsup_{R\to\infty} \frac{\Psi(\lambda R)}{\Psi(R)} < \infty$ *for all $\lambda \geqq 1$. Then there exists a constant $C > 0$ with the following property: Let S be a subset of K with*

$$\operatorname{Card} S \cap B_R \geqq \Psi(R)$$

and

$$\max_{\alpha \in S \cap B_R} s(\alpha) \leqq \gamma_2 \operatorname{Log} R$$

for all sufficiently large R. Let f, g be two entire functions, satisfying

$$g(\alpha) \neq 0 \quad \text{and} \quad \frac{f(\alpha)}{g(\alpha)} \in K \quad \text{for} \quad \alpha \in S,$$

such that for all sufficiently large R,

$$\max_{\alpha \in S \cap B_R} \left\{ \operatorname{Log} \frac{1}{|g(\alpha)|};\ s\left(\frac{f(\alpha)}{g(\alpha)}\right) \right\} \leqq \gamma_1 \cdot \frac{\Psi(R)}{\operatorname{Log} R},$$

and

$$\max \{\operatorname{Log} |f|_R;\ \operatorname{Log} |g|_R\} \leqq \frac{\Psi(R)}{C \operatorname{Log} R}.$$

Then f/g is a rational function.

We obtain as a corollary Gel'fond Schneider's theorem on the transcendence of a^b (choose: $f(z) = a^z$; $g(z) = 1$; $\Psi(R) = R^2$; $S \subset \{h + kb,\ h, k \in \mathbf{Z}\}$.)

The proof of Theorem 2 is essentially the same as that of Theorem 1; first we assume that the function $R \mapsto \Psi(R)/\operatorname{Log} R$ is non decreasing (otherwise we replace $\Psi(R)$ by $(\operatorname{Log} R) \cdot \inf_{R' \geqq R} \Psi(R')/\operatorname{Log} R'$); then we replace the polynomials

[3] Concerning this problem, see a forthcoming paper by Douglas Hensley: "Polynomials with Gaussian integer values at Gaussian integers."

$\binom{z}{h}$ by z^h in the preceeding proof, and the parameters $h_0 N$, k_0 by $[h_0 \Psi(N)/\mathrm{Log}\, N]$, $[k_0 \mathrm{Log}\, N]$, respectively.

Finally, we mention two possible generalisations of Theorems 1 and 2. Firstly Pólya's theorem has been generalized to functions of several variables by A. BAKER [1]; using the interpolation formulas in [1], it is easy to derive the corresponding generalisation of the present paper. Secondly, it is possible to replace the number field K by the field of algebraic numbers, provided that we assume a growth condition on the function

$$R \mapsto \max_{\alpha \in S \cap B_R} [\mathbf{Q}(\alpha, f(\alpha)) : \mathbf{Q}],$$

(see [3], Exercise 2.2.f).

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SUR L'ALGÈBRE ASSOCIÉE À UN CHAMP TENSORIEL DU TYPE (1,2)

Par

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On étudie dans le § 1 l'algèbre associée à un champ tensoriel du type (1,2) sur une variété différentiable. Puis on met en évidence (§ 2) les champs et les courbes caractéristiques de cette algèbre. Dans le § 3 on présente une extension des variétés de Weyl.

Les variétés différentiables, les applications différentiables, les champs tensoriels et les connexions linéaires qui interviennent par la suite sont supposés de la classe C^∞.

Soit M une variété différentiable réelle, à n dimensions. On note par $\mathscr{F}(M)$ l'anneau des fonctions réelles, différentiables, définies sur M et par $\mathscr{T}^{r,s}(M)$ le $\mathscr{F}(M)$-module des champs de tenseurs du type (r, s) sur M.

Particulièrement, pour $\mathscr{T}^{1,0}(M)$ on emploie de même la notation $\mathscr{X}(M)$.

§ 1. Algèbre associée à un champ tensoriel du type (1,2)

On considère un couple (M, D), où M est une variété différentiable et D un champ tensoriel du type (1,2) sur M.

En employant l'indication de I. VAISMAN [4], on constitue le module $\mathscr{X}(M)$ comme une algèbre sur l'anneau $\mathscr{F}(M)$.

Si on définit le produit de deux champs de vecteurs X et Y par la formule:

$$X \circ Y = D(X, Y), \tag{1.1}$$

les propriétés de distributivité de ce produit par rapport à la somme des champs des vecteurs et les autres conditions nécessaires sont immédiatement vérifiées.

DÉFINITION 1.1. L'algèbre définie par la formule (1.1) s'appelle l'algèbre associée à D et on note $\mathscr{U}(M, D)$.

Pour établir certaines propriétés de l'algèbre $\mathscr{U}(M, D)$ on définit entièrement comme dans [4] le champ tensoriel $K \in \mathscr{T}^{1,3}(M)$, qu'on appelle la courbure associée à D, par la formule

$$K(X, Y)Z = \frac{1}{4}\{D(X, D(Y, Z)) - D(Y, D(X, Z))\} \tag{1.2}$$

pour tout $X, Y, Z \in \mathscr{X}(M)$.

On peut observer, en tenant compte de (1.1), que l'algèbre $\mathscr{U}(M, D)$ est commutative si et seulement si D est symétrique, $D(X, Y) = D(Y, X)$, $(\forall) X, Y \in \mathscr{X}(M)$.

En supposant cette propriété vérifiée, on a

$$(X\circ Y)\circ Z - X\circ(Y\circ Z) + 4K(X, Z)Y = 0, \quad (\forall)X, Y, Z\in\mathscr{X}(M),$$

ce que nous conduit au résultat suivant :

L'algébre $\mathscr{U}(M, D)$ est simultanément commutative et associative si et seulement si

$$D(X, Y) = D(Y, X) \quad et \quad K(X, Y)Z = 0, \quad (\forall)X, Y, Z\in\mathscr{X}(M).$$

De même on obtient aisément le résultat suivant :

L'algèbre $\mathscr{U}(M, D)$ est une algèbre de Lie si et seulement si les conditions

$$D(X, Y) = -D(Y, X) \quad et \quad K(X, Y)Z + K(Y, Z)X + K(Z, X)Y = 0$$

sont simultanément vérifiées pour $(\forall)X, Y, Z\in\mathscr{X}(M)$.

Application. Soit $J\in\mathscr{T}^{1,1}(M)$ et soit N_J le champ tensoriel de Nijenhuis associée à J [1], [2], [7], donné par la formule

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \quad (\forall)X, Y\in\mathscr{X}(M).$$

Le fait que N_J est antisymétrique assure que $\mathscr{U}(M, N_J)$ soit une algèbre anticommutative. Il en résulte : *Tous les éléments de l'algèbre $\mathscr{U}(M, N_J)$ sont des éléments nilpotents d'indice* 2.

§ 2. Champs caractéristiques de l'algèbre $\mathscr{U}(M, D)$

Définition 2.1. Un champ $X\in\mathscr{U}(M, D)$ s'appelle champ caractéristique de l'algèbre $\mathscr{U}(M, D)$ s'il existe une fonction $\lambda\in\mathscr{F}(M)$, ainsi que

(2.1) $$D(X, X) = \lambda X.$$

Théorème 2.1. *Tous les éléments de l'algèbre $\mathscr{U}(M, D)$ sont des champs caractéristiques si et seulement s'il existe une 1-forme ω sur M, ainsi qu'on a :*

(2.2) $$D(X, Y) + D(Y, X) = \omega(X)Y + \omega(Y)X, \quad (\forall)X, Y\in\mathscr{X}(M).$$

Pour la démonstration on utilisera le lemme suivant:

Lemme 2.1. *Si $S\in\mathscr{T}^{1,2}(M)$ est un champ symétrique et que pour tout $X, Y\in\mathscr{X}(M)$ le champ de vecteurs $S(X, Y)$ se trouve dans le $\mathscr{F}(M)$-module engendré par X et Y, alors il y a une 1-forme ω sur M anisi que:*

$$S(X, Y) = \omega(X)Y + \omega(Y)X.$$

Démonstration. On note S^i_{jk} les composantes de S dans un système de coordonnées $(x^1, x^2, \ldots, x^n)$. Il découle des hypothèses adoptées que $S^i_{jk} = S^i_{kj}$ et $S^i_{jk} = 0$, si $i\notin\{j, k\}$. On définit maintenant par contraction $\omega_k = S^i_{ik} = S^i_{ki}$. Évidemment les ω_k sont les composantes d'une 1-forme ω. Si l'on fixe deux champs X et Y de

$\mathscr{X}(M)$ qui ont les composantes X^i, Y^i respectivement, on obtient localement:

$$S^i_{jk} X^j Y^k \frac{\partial}{\partial x^i} = S^r_{rk} X^j Y^k \frac{\partial}{\partial x^j} + S^r_{jr} X^j Y^k \frac{\partial}{\partial x^k} = (\omega_k Y^k)\left(X^j \frac{\partial}{\partial x^j}\right) + (\omega_j X^j)\left(Y^k \frac{\partial}{\partial x^k}\right).$$

Par conséquent $S(X, Y) = \omega(Y)X + \omega(X)Y$.

DÉMONSTRATION DU THÉORÈME 2.1. On suppose que l'algébre $\mathscr{U}(M, D)$ possède tous les éléments caractéristiques.

On définit le champ symétrique $S \in \mathscr{T}^{1,2}(M)$ par:

$$S(X, Y) = D(X, Y) + D(Y, X), \quad (\forall) X, Y \in \mathscr{X}(M).$$

Pour $X, Y \in \mathscr{X}(M)$ il existe λ, μ, $\nu \in \mathscr{F}(M)$ ainsi que

$$D(X, X) = \lambda X, \quad D(Y, Y) = \mu Y, \quad D(X+Y, X+Y) = \nu(X+Y).$$

Alors

$$S(X, Y) = D(X+Y, X+Y) - D(X, X) - D(Y, Y) = (\nu - \lambda)X + (\nu - \mu)Y,$$

et nous nous plaçons dans les hypothèses du Lemme 2.1.

L'implication inverse s'obtient en prenant dans (2.2) $X = Y$, d'où $D(X, X) = \omega(X)X$.

REMARQUE 2.1. Il est facile de voir que pour un champ X qui ne s'annule en aucun point de M, (2.1) est équivalent à

$$D(X, X) \otimes X = X \otimes D(X, X). \tag{2.3}$$

DÉFINITION 2.2. Un champ caractéristique $X \in \mathscr{U}(M, D)$ avec la propriété $(\forall) p \in M$, $X_p \neq 0$, s'appelle champ de directions caractéristiques dans l'algèbre $\mathscr{U}(M, D)$.

Nous appelons les trajectoires des champs de directions caractéristiques courbes caractéristiques de l'algèbre $\mathscr{U}(M, D)$.

Si l'on note D^i_{jk} les composantes de D dans un système de coordonnées $(x^1, x^2, \ldots, x^n)$, alors, en employant (2.3), on obtient le système différentiel d'équations des courbes caractéristiques de l'algèbre $\mathscr{U}(M, D)$.

$$(D^i_{ks}\delta^j_r - D^j_{ks}\delta^i_r)\frac{dx^k}{dt}\frac{dx^s}{dt}\frac{dx^r}{dt} = 0.$$

Interprétation géométrique des champs de directions caractéristiques de l'algèbre $\mathscr{U}(M, D)$

Si $c: I \to M$, $\bar{c}: \bar{I} \to M$ $(I, \bar{I} \subseteq R)$ sont deux courbes sur M qui ont un point commun $c(t_0) = \bar{c}(\bar{t}_0) = p$ où t_0 et $\bar{t}_0$ sont des points intérieurs aux intervalles I et $\bar{I}$ respective, on dira que c et $\bar{c}$ sont fortement-osculatrices au point p si pour un système de coordonnées autour de p, $(x^1, x^2, \ldots, x^n)$, les constantes λ, μ, $\bar{\lambda}$, $\bar{\mu}$ existent avec $\lambda\bar{\lambda} \neq 0$ ainsi qu'ona, pour tout $i = 1, 2, \ldots, n$,

$$\lambda \frac{dc^i}{dt}\bigg|_{t_0} = \bar{\lambda}\frac{d\bar{c}^i}{d\bar{t}}\bigg|_{\bar{t}_0} \tag{0_1}$$

et

$$(0_2)\qquad \lambda^2 \frac{d^2 c^i}{dt^2}\bigg|_{t_0} + \mu \frac{dc^i}{dt}\bigg|_{t_0} = \bar{\lambda}^2 \frac{d^2 \bar{c}^i}{d\bar{t}^2}\bigg|_{\bar{t}_0} + \bar{\mu} \frac{d\bar{c}^i}{d\bar{t}}\bigg|_{\bar{t}_0}$$

où l'on a noté $c^i = x^i \circ c$, $\bar{c}^i = x^i \circ \bar{c}$.

REMARQUE 2.2. Il est évident que les constantes λ, $\bar{\lambda}$, μ, $\bar{\mu}$ dépendent des paramètres choisis sur c et $\bar{c}$ et du choix du système de coordonnées autour de p.

Il est aussi clair que la propriété des courbes c et $\bar{c}$ d'être fortement-osculatrice au point p est géométrique, étant indépendente du choix des paramètres sur c et $\bar{c}$ et du système de coordonnées autour du p.

On considère de nouveau la paire (M,D) et on suppose que la variété M est dotée avec une connexion linéaire $\overset{1}{\nabla}$. A l'aide de D et $\overset{1}{\nabla}$ on construit les connexions linéaires

$$\nabla = \overset{1}{\nabla} + \frac{1}{2} D, \quad \overline{\nabla} = \overset{1}{\nabla} - \frac{1}{2} D.$$

Relativement à la paire de connexion $(\nabla, \overline{\nabla})$ on a le théorème suivant:

THÉORÈME 2.2. *Soit $X \in \mathscr{X}(M)$ avec $X_p \neq 0$, $(\forall)$ $p \in M$. Les propriétés suivantes sont équivalentes :*

1° *X est un champ de directions caractéristiques dans l'algèbre $\mathscr{U}(M, D)$;*

2° *pour tout $p \in M$ il existe $\lambda_p \in R$ et*

$$D(p)(X_p, X_p) = \lambda_p X_p ;$$

3° *pour tout $p \in M$, si $c: I \to M$ et $\bar{c}: \bar{I} \to M$ sont deux courbes sur M, la première ∇-autoparallèle, la deuxième $\overline{\nabla}$-autoparallèle à $c(t_0) = \bar{c}(\bar{t}_0) = p$, il existe $\lambda, \bar{\lambda} \in R$ avec $\lambda\bar{\lambda} \neq 0$ et*

$$c_* \left(\frac{d}{dt}\bigg|_{t_0} \right) = \bar{\lambda} X_p, \quad \bar{c}_* \left(\frac{d}{d\bar{t}}\bigg|_{\bar{t}_0} \right) = \lambda X_p,$$

alors les courbes sont fortement-osculatrices au point p (on a noté c_* la différentielle de l'application c).

DÉMONSTRATION. Évidement $1° \Rightarrow 2°$.

Pour établir que $2° \Rightarrow 3°$ on choisit un système de coordonnées $(x^1, x^2, \ldots, x^n)$ qui contient le point p, c et $\bar{c}$ soient les courbes données.

On a

$$(2.4)\qquad \begin{cases} \dfrac{d^2 c^i}{dt^2} - \Gamma^i_{jk}(c^1(t), \ldots, c^n(t)) \dfrac{dc^j}{dt} \dfrac{dc^k}{dt} = 0 \\[2ex] \dfrac{d^2 \bar{c}^i}{d\bar{t}^2} - \bar{\Gamma}^i_{jk}(\bar{c}^1(\bar{t}), \ldots, \bar{c}^n(\bar{t})) \dfrac{d\bar{c}^j}{d\bar{t}} \dfrac{d\bar{c}^k}{d\bar{t}} = 0 \end{cases}$$

où Γ^i_{jk}, $\bar{\Gamma}^i_{jk}$ sont les composantes des connexions ∇ et $\overline{\nabla}$.

Il est évident que les composantes de D sont $D^i_{jk} = -\Gamma^i_{jk} + \bar{\Gamma}^i_{jk}$.

De même

$$\left.\frac{dc^i}{dt}\right|_{t_0} = \bar{\lambda} X_p^i, \quad \left.\frac{d\bar{c}^i}{d\bar{t}}\right|_{\bar{t}_0} = \lambda X_p^i.$$

Les relations (2.4) considérées au point t_0 et $\bar{t}_0$ respectivement conduisent à

$$\left.\frac{d^2 c^i}{dt^2}\right|_{t_0} - \Gamma^i_{jk}(p)\bar{\lambda}^2 X_p^j X_p^k = 0, \quad \left.\frac{d^2 \bar{c}^i}{d\bar{t}^2}\right|_{\bar{t}_0} - \bar{\Gamma}^i_{jk}(p)\lambda^2 X_p^j X_p^k = 0,$$

d'où l'on obtient

$$\frac{1}{\bar{\lambda}^2}\left.\frac{d^2 c^i}{dt^2}\right|_{t_0} - \frac{1}{\lambda^2}\left.\frac{d^2 \bar{c}^i}{d\bar{t}^2}\right|_{\bar{t}_0} + D^i_{jk}(p) X_p^j X_p^k = 0$$

ou

$$\lambda \left.\frac{2d^2 c^i}{dt^2}\right|_{t_0} = \bar{\lambda}^2 \left.\frac{d^2 \bar{c}^i}{d\bar{t}^2}\right|_{\bar{t}_0} - \lambda^2 \bar{\lambda}^2 \lambda_p X_p^i.$$

Il nous reste à remplacer, par exemple λX_p^i par $\left.\frac{d\bar{c}^i}{d\bar{t}^i}\right|_{\bar{t}_0}$ pour obtenir (0_2).

3°⇒2°. Il est clair qu'il existe les courbes c et $\bar{c}$ qui vérifient les équations (2.4) aux conditions initiales

$$c(t_0) = \bar{c}(\bar{t}_0) = p \quad \text{et} \quad \left.\frac{dc^i}{dt}\right|_{t_0} = \left.\frac{d\bar{c}^i}{d\bar{t}}\right|_{\bar{t}_0} = X_p^i. \tag{2.5}$$

La condition (0_1) a lieu avec $\lambda = \bar{\lambda} = 1$. Pour (0_2) on a

$$\left.\frac{d^2 c^i}{dt^2}\right|_{t_0} + \mu \left.\frac{dc^i}{dt}\right|_{t_0} = \left.\frac{d^2 \bar{c}^i}{d\bar{t}^2}\right|_{\bar{t}_0} + \bar{\mu} \left.\frac{d\bar{c}^i}{d\bar{t}}\right|_{\bar{t}_0},$$

c'est à dire qu'en tenant compte de (2.4) et de (2.5), on trouve que

$$\Gamma^i_{jk}(p) X_p^j X_p^k + \mu X_p^i = \bar{\Gamma}^i_{jk}(p) X_p^j X_p^k + \bar{\mu} X_p^i,$$

donc

$$D^i_{jk}(p) X_p^j X_p^k = (\mu - \bar{\mu}) X_p^i.$$

2°⇒1°. Il faut montrer que la fonction $(p \to \lambda_p): M \to R$ est en $\mathcal{F}(M)$.

La propriété étant locale, on choisit $p \in M$ et une carte $(x^1, x^2, \dots, x^n)$ autour de p, suffisamment petite telle qu'il existe i_0 pour que la coordonnée X^{i_0} de X ne s'annule pas. Pour tout point q de cette voisinage on a

$$\lambda_q = \frac{D^{i_0}_{jk}(q) X_q^j X_q^k}{X_q^{0i}},$$

c'est que $(q \to \lambda_q)$ est différentiable.

§ 3. f-structures de Weyl

Soit (M, g) une variété de Riemann et soit $\hat{g}$ la structure conforme engendrée par g, c'est à dire que $\hat{g}=\{e^{\lambda}g \mid \lambda\in\mathcal{F}(M)\}$.

Soit $f\in\mathcal{F}(M)$ une fonction qui ne s'annule pas en aucun point.

DÉFINITION 3.1. Nous appelons f-structure de Weyl sur la variété conforme $(M, \hat{g})$ une application : $F:\hat{g}\to\mathcal{T}^{0,1}(M)$ qui vérifie

$$fF(e^{\lambda}g)=fF(g)-d\lambda, \quad (\forall)\lambda\in\mathcal{F}(M). \tag{3.1}$$

REMARQUE 3.1. Si $\bar{g}\in\hat{g}$, on a $\hat{\bar{g}}=\hat{g}$ et en supposant ensuite que $\bar{g}=e^{\mu}g$, on a :

$$fF(e^{\lambda}\bar{g})=fF(e^{\lambda+\mu}g)=fF(g)-d(\lambda+\mu)=fF(\bar{g})-d\lambda$$

pour tout $\lambda\in\mathcal{F}(M)$.

Grâce à la relation (3.1) une f-structure de Weyl est uniquement detérminée par 1-forme $F(g)$.

DÉFINITION 3.2. Soit ω une 1-forme sur M. On dit qu'une connexion linéaire ∇ sur M est associée à ω et est compatible avec la f-structure de Weyl F si pour tout champs $X, Y, Z\in\mathcal{X}(M)$ on a:

(3.2)

$$(\nabla_X g)(Y, Z)+fF(g)(X)g(Y, Z)+2\omega(X)g(Y, Z)+\omega(Y)g(Z, X)+\omega(Z)g(X, Y)=0$$

PROPOSITION 3.1. *Si ∇ est une connexion associée à ω et est compatible avec la f-structure de Weyl F, alors pour tout $\bar{g}\in\hat{g}$ a lieu la relation*

(3.3)

$$(\nabla_X\bar{g})(Y, Z)+fF(\bar{g})(X)\bar{g}(Y, Z)+2\omega(X)\bar{g}(Y, Z)+\omega(Y)\bar{g}(Z, X)+\omega(Z)\bar{g}(X, Y)=0.$$

DÉMONSTRATION. Soit $\bar{g}=e^{\mu}g$. Il s'ensuit que

$$(\nabla_X e^{\mu}g)(Y, Z)+fF(e^{\mu}g)(X)e^{\mu}g(Y, Z)=e^{\mu}\{(\nabla_X g)(Y, Z)+fF(g)(X)g(Y, Z)\}.$$

On constate sans peine que (3.3) représente en effet la relation (3.2) multipliée par e^{μ}.

PROPOSITION 3.2. *Étant données:*

— *une variété de Riemann (M, g),*

— *une fonction $f\in\mathcal{F}(M)$ qui ne s'annule pas en aucun point,*

— *une f-structure de Weyl F sur M et,*

— *une 1-forme ω sur M,*

alors il existe une connexion linéaire unique sur M, symétrique, associée à ω et compatible avec la f-structure de Weyl.

DÉMONSTRATION. Pour deux champs $X, Y\in\mathcal{X}(M)$ soit $D(X, Y)$ le champ unique qui pour $(\forall)$ $Z\in\mathcal{X}(M)$ vérifie la relation

$$2g(D(X, Y), Z)=fF(g)(X)g(Y, Z)+fF(g)(Y)g(Z, X)- \tag{3.4}$$
$$-fF(g)(Z)g(X, Y)+2\omega(X)g(Y, Z)+2\omega(Y)g(Z, X).$$

Il est facile de voir que D est un champ de tenseurs symétrique du type (1,2), $D(X, Y)=D(Y, X)$.

Soit $\nabla=\mathring{\nabla}+D$, où par $\mathring{\nabla}$ a noté la connexion de Lévi—Civita associée a g. Puisque la connexion $\mathring{\nabla}$ est sans torsion et $D(X, Y)=D(Y, X)$, $(\forall)$ $X, Y\in\mathscr{X}(M)$, on a

$$\nabla_X Y-\nabla_Y X = \mathring{\nabla}_X Y-\mathring{\nabla}_Y X+D(X, Y)-D(Y, X) = [X, Y], \tag{3.5}$$

donc la connexion ∇ est symétrique. En tenant compte de l'identité

$$(\mathring{\nabla}_X g)(Y, Z) = 0, \quad (\forall) X, Y, Z\in\mathscr{X}(M), \tag{3.6}$$

on a

$$\begin{aligned}(\nabla_X g)(Y, Z) &= Xg(Y, Z)-g(\nabla_X Y, Z)-g(Y, \nabla_X Z) = Xg(Y, Z)-\\ &-g(\mathring{\nabla}_X Y, Z)-g(Y, \mathring{\nabla}_X Z)-g(D(X, Y), Z)-g(Y, D(X, Z)) =\\ &= -g(D(X, Y), Z)-g(Y, D(X, Z)).\end{aligned}$$

En employant (3.4) an obtient

$$\begin{aligned}2(\nabla_X g)(Y, Z)+2fF(g)(X)g(Y, Z)+4\omega(X)g(Y, Z)+2\omega(Y)g(Z, X)+\\ +2\omega(Z)g(X, Y) = 0,\end{aligned}$$

c'est justement la relation (3.2). L'existence de la connexion ∇ est donc prouvée.

On suppose maintenant que ∇ est une connexion linéaire symétrique qui satisfait à (3.2) et soit $D=\nabla-\mathring{\nabla}$. Pour $(\forall) X, Y\in\mathscr{X}(M)$, on a

$$D(X, Y) = \nabla_X Y-\mathring{\nabla}_X Y = (\nabla_Y X+[X, Y])-(\mathring{\nabla}_Y X+[X, Y]) = \nabla_Y X-\mathring{\nabla}_Y X = D(Y, X)$$

c'est à dire D est symétrique. En tenant compte de (3.2) et de (3.6) nous avons:

$$\begin{aligned}g(D(X, Y), Z)+g(Y, D(X, Z)) = fF(g)(X)g(Y, Z)+2\omega(X)g(Y, Z)+\\ +\omega(Y)g(Z, X)+\omega(Z)g(X, Y).\end{aligned}$$

On écrit encore duex relations obtenues par substitution circulaires:

$$\begin{aligned}g(D(Y, Z), X)+g(Z, D(Y, X)) = fF(g)(Y)g(Z, X)+2\omega(Y)g(Z, X)+\\ +\omega(Z)g(X, Y)+\omega(X)g(Y, Z)\end{aligned}$$

et

$$\begin{aligned}g(D(Z, X), Y)+g(X, D(Z, Y)) = fF(g)(Z)g(X, Y)+2\omega(Z)g(X, Y)+\\ +\omega(X)g(Y, Z)+\omega(Y)g(Z, X).\end{aligned}$$

Si l'on soustrait la dernière relation de la somme des deux autres et on emploie la symétrie de D, il en résulte précisément (3.4). L'unicité de ∇ est ainsi prouvée.

Remarque 3.2. Puisque les relations (3.2) et (3.3) sont équivalentes et la connexions ∇ est unique, elle ne dépend pas essentiellement de g (comme on croirait vu la démonstration), mais seulement de $\hat{g}$.

Généralement, la connexion ∇ dépend de f, ω et de $F(g)$. Par exemple, pour $F(g)=0=\omega$, elle coïncide avec la connexion de Lévi—Civita associée à g, et si f

est la fonction constante 1 et $\omega=0$, on obtient la connexion compatible avec une structure de Weyl [3], [5], [6].

Dans ce dernier cas, la relations (3.1) devient $F(e^{\lambda}g)=F(g)-d\lambda$ et la relation (3.2) se réduit à $\nabla_X g+F(g)(X)g=0$.

On considére une autre connexion $\overline{\nabla}$ correspondante à un choix de $\bar{f}$, $\bar{\omega}$ avec la conservation de $F(g)$ et soit $\mathring{\nabla}$ comme plus haut, la connexion de Lévi—Civita associée à g.

THÉORÈME 3.1. *Les algèbres* $\mathcal{U}(M, \nabla-\mathring{\nabla})$ *et* $\mathcal{U}(M, \overline{\nabla}-\mathring{\nabla})$ *ont les mêmes champs caractéristiques.*

DÉMONSTRATION. On note $D=\nabla-\mathring{\nabla}$ et $\bar{D}=\overline{\nabla}-\mathring{\nabla}$. Alors D vérifie la relation (3.4) et $\bar{D}$ vérifie une relation analogue où f et ω sont remplacées par $\bar{f}$ et $\bar{\omega}$.

On remarque que (3.4) s'écrit aussi sous la forme

$$g(Z, 2D(X,Y)-fF(g)(X)Y-fF(g)(Y)X-2\omega(X)Y-2\omega(Y)X) = \\ = -fF(g)(Z)\,g(X,Y).$$

La relation analogue pour $\bar{D}$ est la suivante:

$$g(Z, 2\bar{D}(X,Y)-\bar{f}F(g)(X)Y-\bar{f}F(g)(Y)X-2\bar{\omega}(X)Y-2\bar{\omega}(Y)X) = -\bar{f}F(g)(Z)g(X,Y).$$

En multipliant la première relation par $\bar{f}$, la seconde par f et en tenant compte du fait que les égalités ont lieu pour tout champs $X, Y, Z\in\mathcal{X}(M)$, il en résulte que

$$\bar{f}(2D(X,Y)-fF(g)(X)Y-fF(g)(Y)X-2\omega(X)Y-2\omega(Y)X) = \\ = f(2\bar{D}(X,Y)-\bar{f}F(g)(X)Y-\bar{f}F(g)(Y)X-2\bar{\omega}(Y)X-2\bar{\omega}(Y)X)$$

ou

$$\bar{f}(D(X,Y)-\omega(X)Y-\omega(Y)X) = f(\bar{D}(X,Y)-\bar{\omega}(X)Y-\bar{\omega}(Y)X). \tag{3.7}$$

Posant dans (3.7) $Y=X$, on obtient

$$\bar{f}(D(X,X)-2\omega(X)X) = f(\bar{D}(X,X)-2\bar{\omega}(X)X), \quad (\forall)X\in\mathcal{X}(M)$$

d'où l'on tire facilement le résultat du Théorème 3.1.

En particulier, grâce à la Remarque 3.2 on peut supposer que $\overline{\nabla}$ est la connexion correspondante à $\bar{f}=1$ et $\bar{\omega}=0$, et $\bar{D}$ vérifie alors la relation plus simple

$$2g(\bar{D}(X,Y),Z) = F(g)(X)\,g(Y,Z)+F(g)(Y)\,g(Z,X)-F(g)(Z)\,g(X,Y).$$

REMARQUE 3.3. Si l'on note la 1-forme $F(g)$ par φ, alors la relation (3.2) s'écrit en coordonnées locales

$$g_{jk,i}+f\varphi_i g_{jk}+2\omega_i g_{jk}+\omega_j g_{ik}+\omega_k g_{ij} = 0,$$

et la relation (3.4) devient

$$2g_{sk}D^s_{ij} = f\varphi_i g_{jk}+f\varphi_j g_{ik}-f\varphi_k g_{ij}+2\omega_i g_{jk}+2\omega_j g_{ik},$$

Si l'on prend

$$\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = -\Gamma^k_{ji}\frac{\partial}{\partial x^k},$$

on a alors, pour les composantes Γ^i_{jk} de la connexion ∇,

$$\Gamma^k_{ji} = -\left|\begin{matrix} k \\ ji \end{matrix}\right| - D^k_{ji},$$

où $-\left|\begin{matrix} k \\ ji \end{matrix}\right|$ sont les composantes de la connexion de Lévi—Civita.

On trouve facilement l'expression des composantes Γ^i_{jk} en fonction de $f, g_{ij}, \varphi_i, \omega_i$

$$\Gamma^i_{jk} = -\left|\begin{matrix} i \\ jk \end{matrix}\right| - \frac{1}{2}[\delta^i_j(f\varphi_k+2\omega_k)+\delta^i_k(f\varphi_j+2\omega_j)-fg_{jk}g^{is}\varphi_s].$$

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[illegible] pour les composantes Γ^k_{ij} de la connection ∇ [illegible]

[illegible]

[illegible] les composantes de la connection de Levi-Civita.

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INF-HALBVERBÄNDE ALS SYNTAKTISCHE HALBGRUPPEN

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Die vorliegende Arbeit befaßt sich mit einer Charakterisierung von Sprachen durch ihre syntaktischen Halbgruppen bzw. syntaktischen Monoide, die in diesem Falle kommutativ und idempotent sind, also als inf-Halbverbände aufgefaßt werden können. Die Ergebnisse scheinen uns auch halbgruppentheoretisch interessant, weil sie die disjunktiven Teilmengen — im Sinne von SCHEIN [4] — von inf-Halbverbänden charakterisieren.

V sei eine Halbruppe [ein Monoid] und X ein nicht notwendig endliches Alphabet. Mit X^* sei das freie Monoid über X bezeichnet. $\mathcal{L}(X, V)$ [$\mathcal{L}'(X, V)$] sei die Klasse der Sprachen $L \subseteqq XX^*$ [$L \subseteqq X^*$] mit syntaktischer Halbgruppe $S(L) \cong V$ [syntaktischem Monoid $M(L) \cong V$]. Dabei sind $S(L)$ und $M(L)$ gerade die Faktorhalbgruppen von XX^* bzw. X^* nach der gröbsten Kongruenz, die L saturiert; den zugehörigen kanonischen Surmorphismus bezeichnen wir als den syntaktischen Surmorphismus. Wir befassen uns mit der Aufgabe, die Klassen $\mathcal{L}(X, V)$ bzw. $\mathcal{L}'(X, V)$ für den Fall zu bestimmen, daß V ein inf-Halbverband ist. Eine derartige Charakterisierung einer Klasse $\mathcal{L}(X, V)$ bzw $\mathcal{L}'(X, V)$ läßt sich immer in zwei Schritten durchführen:

(1) Bestimmung eines Erzeugendensystems E von V und Angabe eines Homomorphismus von X^* auf V, der X auf E abbildet.

(2) Bestimmung sämtlicher disjunktiver Teilmengen von V, d.h. der Teilmengen von V, die nur durch die triviale Kongruenz saturiert werden. Genau diese Mengen sind die Bilder von Sprachen unter ihren syntaktischen Homomorphismen.

Wir behandeln Schritt (1) nur kurz. Zu (2) gelingt mit Satz 5 eine Charakterisierung sämtlicher disjunktiver Teilmengen beliebiger inf-Halbverbände. Unter einer recht schwachen Endlichkeitsbedingung, aus der noch nicht einmal die Längenendlichkeit nach unten folgt, läßt sich die Beschreibung wesentlich verschärfen, Satz 2. Falls V sogar endlich ist, erhält man aus diesem Satz einen einfachen Algorithmus zur Berechnung sämtlicher disjunktiver Teilmengen von V. Einige andere Aspekte dieses Problems werden von ZAPLETAL in [5, 6] untersucht. Die von LALLEMENT und MILITO in [3] angegebene Beschreibung der zu endlichen Halbverbänden von Gruppen gehörigen Sprachen ist eine Verallgemeinerung eines Spezialfalles von Satz 2.

Die sonst (vgl. [2]) notwendige Unterscheidung von syntaktischen Halbgruppen und syntaktischen Monoiden erweist sich für den vorliegenden Fall als unwesentlich, weil das Einselement eines kommutativen idempotenten Monoids immer adjungiert ist. Wir beschreiben daher nur die Klasse $\mathcal{L}(X, V)$. $\mathcal{L}'(X, V)$ erhält man dann in einfacher Weise mit Hilfe der Tatsache, daß die syntaktische Klasse des Einselements von $M(L)$ für $L\in\mathscr{L}'(X, V)$ gleich X_1^* für eine (eventuell leere) Menge $X_1\subseteqq X$ ist.

Es seien nun V ein inf-Halbverband, X eine Menge und $L\in\mathscr{L}(X, V)$. Der syntaktische Surmorphismus $\sigma_L: XX^*\to S(L)$ läßt sich in ein Produkt von Surmorphismen $\alpha_X, \beta_X, \gamma_L$ folgendermaßen zerlegen:

$$\begin{array}{ccc} XX^* & \xrightarrow{\alpha_X} & X^\alpha \\ \sigma_L \downarrow & = & \downarrow \beta_X \\ V\cong S(L) & \xleftarrow[\gamma_L]{} & X^V \end{array}$$

Dabei sind: X^a die von X erzeugte freie abelsche Halbgruppe, X^V der von X erzeugte freie inf-Halbverband und α_X, β_X die jeweiligen Fortsetzungen der identischen Abbildung von X. Durch diese Überlegung wird es möglich, das ursprüngliche Problem in drei Teilprobleme zu zerlegen:

(1) Beschreibung der $(\beta_X\alpha_X)$-Klassen in XX^*.

(2) Beschreibung sämtlicher Surmorphismen von X^V auf V.

(3) Beschreibung der Familie $\mathscr{D}(V)$ sämtlicher disjunktiver Teilmengen von V.

Für $L\subseteqq XX^*$ gilt nämlich $L\in\mathscr{L}(X, V)$ genau dann, wenn $L=\alpha_X^{-1}\beta_X^{-1}\gamma^{-1}(M)$ für ein $M\in\mathscr{D}(V)$ und einen Surmorphismus $\gamma: X^V\to V$ ist.

Die Lösung für (1) ist einfach: Es sei $q\in X^V$; dann besitzt q eine bis auf die Reihenfolge der Faktoren eindeutige Darstellung $q=q_1q_2\dots q_k$ mit $q_i\in X$ und $q_i\neq q_j$ für alle i, j mit $i\neq j$. Sei $Q_q=\{q_1, q_2, \dots, q_k\}$. Dann ist

$$\alpha_X^{-1}\beta_X^{-1}(q) = \bigcap_{i=1}^{k} Q_q^* q_i Q_q^*.$$

Um (2) zu lösen, genügt es, sämtliche Erzeugendensysteme E von V mit $|E|=|X|$ anzugeben. Durch die identische Abbildung von E wird dann ein Epimorphismus von E^V (und wegen $|X|=|E|$ auch von X^V) auf V definiert. Umgekehrt kann bekanntlich jeder Epimorphismus von X^V auf V dadurch erhalten werden, daß man X mit einem Erzeugendensystem von V identifiziert. Für den in sprachentheoretischem Zusammenhang letztlich nur relevanten endlichen Fall genügt die folgende Aussage:[1]

1. Bemerkung. V sei ein inf-Halbverband. Sei weiter

$$E(V) := \{s\,|\,s\in V\wedge\exists t\in V^1: t > s\wedge(\forall u\in V^1: u > s\to u\geqq t)\}.$$

Für jedes Erzeugendensystem X von V gilt $E(V)\subseteqq X$. Falls V nach oben längenendlich ist, und jedes Element von V nur endlich viele obere Nachbarn hat, ist $E(V)$ ein Erzeugendensystem von V.

Die Lösung des Teilproblems (3) macht etwas größere Schwierigkeiten. Wir geben zuerst eine Charakterisierung von $\mathscr{D}(V)$ für inf-Halbverbände an, die einer bestimmten, recht unübersichtlichen Bedingung $(*)$ genügen; danach zeigen wir, daß insbesondere nach unten längenendliche inf-Halbverbände die Eigenschaft $(*)$ haben.

[1] Für eine Halbgruppe H sei, wie üblich, $H^1=H$, falls H ein Monoid ist, und sonst $H^1=H\,\dot\cup\,\{1\}$ mit $1x=x1=x$ für alle $x\in H^1$.

2. Satz. *V sei ein inf-Halbverband, der die folgende Eigenschaft hat:*

$$(*) \qquad \forall x\in V : \forall y \in V : x < y \to (\forall t \in V : t < y \to t \leqq x) \vee$$
$$\vee (\exists z \in V : z \leqq y \wedge z \not\leqq x \wedge (\forall z' \in V : z' < z \to z' \leqq zx)).$$

$L\subseteqq V$ ist genau dann disjunktiv, wenn für alle $x, y\in V$ gilt: Falls $x<y$ ist und für alle $z<y$ folgt $z\leqq x$, so ist $x\in L \leftrightarrow y\notin L$.

Beweis. (1) Sei $L\subseteqq V$, $x, y\in V$, $x<y$, und es gelte $z\leqq x$ für alle $z<y$ und ferner $x\in L\leftrightarrow y\in L$. Für $q\in V$ ist dann entweder $qy=y$, $qx=x$ oder $qy=qx$; in jedem Falle gilt $qy\in L\leftrightarrow qx\in L$. x und y sind daher bezüglich L sytaktisch äquivalent. L ist nicht disjunktiv.

(2) $L\subseteqq V$ habe die angegebene Eigenschaft. Es seien $x, y\in V$ mit $x\neq y$. Wir zeigen, daß x und y bezüglich L nicht äquivalent sind. Dazu können wir $x\in L\leftrightarrow y\in L$ voraussetzen (weil die Behauptung sonst trivial wäre). Wir überlegen zunächst, daß es genügt, die Aussage für $x<y$ (bzw. $y<x$) zu beweisen. Sei nämlich $x\not\leqq y\not\leqq x$, also $y':=xy<x, y$; falls nun x und y' nicht äquivalent sind, folgt mit $xx=x$, $xy=y'$ auch, daß x und y nicht äquivalent sind. Sei also jetzt o.B.d.A. $x<y$ vorausgesetzt. Wegen $x\in L\leftrightarrow y\in L$ folgt, daß es ein $t<y$ mit $t\not\leqq x$ gibt. Dann existiert aber auch ein z gemäß $(*)$; für dieses gilt

$$zy = z\in L \leftrightarrow zx\notin L.$$

Also sind x und y nicht äquivalent. q.e.d.

3. Satz. *Jeder nach unten längenendliche inf-Halbverband hat die Eigenschaft $(*)$*.

Beweis. V sei ein nach unten längenendlicher inf-Halbverband. Dann besitzt V ein Nullelement 0. Seien $x, y\in V$ und $x<y$. Weiter setzen wir voraus, daß es ein $t<y$ mit $t\not\leqq x$ gibt. Es ist die Existenz von z gemäß $(*)$ zu zeigen. Da jede von y nach x absteigende Kette endlich ist, gibt es einen oberen Nachbarn y' von x mit $x<y'\leqq y$. Falls für alle $t<y'$ gilt $t\leqq x=y'x$, so ist mit $z:=y'$ ein geeignetes Element gefunden. Andernfalls muß speziall auch für das Paar (x, y') ein z gemäß $(*)$ gesucht werden; falls es existiert, so hat es offenbar auch die gewünschten Eigenschaften für das Paar (x, y). Wir können also o.B.d.A. annehmen, daß x unterer Nachbar von y ist. Sei nun $b_1<y$, $b_1\not\leqq x$. Da V nach unten längenendlich ist, können wir annehmen, daß b_1 unterer Nachbar von y ist. Wir betrachten jetzt eine Folge

$$b_0 = y > b_1 > b_2 > \ldots > b_i > \ldots > 0.$$

Dabei soll immer b_i unterer Nachbar von b_{i-1} und $b_i\not\leqq x$ sein. Da V nach unten längenendlich ist, bricht jede solche Folge nach endlich vielen Gliedern — spätestens mit einem oberen Nachbarn von 0 — ab. Die Tatsache, daß die Folge etwa mit b_i abbricht, besagt aber gerade, daß $z'\leqq x$ für alle unteren Nachbarn von b_i gilt, also $z'\leqq b_ix$; wegen $b_ix<b_i$ ist b_ix unterer Nachbar von b_i, und es gilt $t\leqq b_ix$ für alle $t<b_i$; $z:=b_i$ hat also die gewünschten Eigenschaften. q.e.d.

4. Korollar. *V sei ein endlicher inf-Halbverband.*

(1) Es ist $M\in\mathscr{D}(V)$ genau dann, wenn für alle $x, y\in V$ gilt: Falls x einziger unterer Nachbar von y ist, ist $x\in M\leftrightarrow y\notin M$.

(2) Es gilt $\mathscr{L}(X, V)\neq\emptyset$ genau dann, wenn $|X|\geqq|E(V)|$. Dabei ist $E(V)$ die Menge derjenigen Elemente von V, die in V^1 genau einen oberen Nachbarn haben.

Man beachte, daß diese Charakterisierung einen recht günstigen Algorithmus zur Bestimmung von $\mathcal{L}(X, V)$ definiert. $\mathcal{D}(V)$ ist für endliche V immer nicht leer.

Zum Abschluß befassen wir uns mit der Frage, ob in Satz 2 auf die Bedingung $(*)$ verzichtet werden kann. Ein einfaches Beispiel dafür, daß dies nicht ohne Änderung der Charakterisierung von $\mathcal{D}(V)$ möglich ist, bildet etwa die Menge $\mathbf{R}$ der reellen Zahlen mit der Operation „min". Offenbar erfüllt z. B. $L=\emptyset$ die Forderungen von Satz 2, ist jedoch keineswegs disjunktiv. Man überlegt sich leicht, daß $\mathcal{D}(\mathbf{R})$ gerade aus den Teilmengen $L\subseteqq\mathbf{R}$ mit der Eigenschaft besteht, daß L und $\mathbf{R}\setminus L$ in $\mathbf{R}$ dicht sind. Dies gilt sogar für beliebige Ketten und ergibt im endlichen Falle gerade die Bedingung von Satz 2 an L. Falls V keine Kette ist, ist die Chrakterisierung von $\mathcal{D}(V)$ schwieriger, kommt aber im Prinzip ebenfalls mit einer Dichtheitsforderung aus.

Zur Formulierung der Aussage vereinbaren wir etwas Notation und einen Hilfsbegriff: V sei ein inf-Halbverband. Absteigende Ketten von V schreiben wir als $\{z_\alpha\}_{\alpha\in A}$; dabei ist A eine geeignete total geordnete Indexmenge und $z_\alpha<z_\beta$ für $\beta<\alpha$. Eine solche Kette heiße „voll bezüglich $x\in V$", falls sie den folgenden Forderungen genügt:

(a) Jede echte Verfeinerung von $\{z_\alpha\}_{\alpha\in A}$ ist Verlängerung.

(b) $\forall z\in V:(\forall\alpha\in A: z<z_\alpha)\rightarrow z\leqq x$.

(c) $\forall\alpha\in A: z_\alpha\not\leqq x$.

5. Satz. *V sei ein inf-Halbverband. $L\subseteqq V$ ist genau dann disjunktiv, wenn für alle $x, y\in V$ mit $x<y$ gilt: Es gibt eine von y absteigende bezüglich x volle Kette, in der die Menge*

$$D_x := \{z\,|\,z\in V: z\in L\rightarrow zx\notin L\}$$

cofinal ist.

Beweis. (1) L habe die angegebene Eigenschaft. Es seien $x, y\in V$, $x\neq y$. Wir wollen zeigen, daß x und y nicht syntaktisch äquivalent sind. Wie im Beweis von Satz 2 können wir uns auf die Untersuchung des Falles $x<y$ beschränken. $\{z_\alpha\}_{\alpha\in A}$ sei eine beliebige von y absteigende Kette, in der D_x cofinal ist — nach Voraussetzung existiert eine solche Kette immer —; es gibt also ein $\alpha\in A$ mit $z_\alpha\in D_x$; es folgt $yz_\alpha = z_\alpha\in L\leftrightarrow xz_\alpha\notin L$. x und y sind syntaktisch nicht äquivalent.

(2) L sei disjunktiv und $x, y\in V$ mit $x<y$. Es gibt also ein $z\in V$ mit $zx\in L\leftrightarrow zy\notin L$, also $zx<zy$ und daher $zy\not\leqq x$. Eine beliebige von y nach zy absteigende maximale Kette sei das Anfangsstück der zu konstruierenden von y absteigenden bezüglich x vollen Kette. Für jedes Element t dieses Kettenstücks gilt offenbar $t\not\leqq x$. Sei nun z_α das letzte bisher konstruierte Kettenelement, also $z_\alpha\in L\leftrightarrow xz_\alpha\notin L$. Wir unterscheiden zwei Fälle:

(a) Es gibt kein $y'\in V$ mit $y'<z_\alpha$, $y'\not\leqq x$. Dann soll die Kette mit z_α abbrechen.

(b) Es gibt ein $y'\in V$ mit $y'<z_\alpha$, $y'\not\leqq x$. Dann soll die Kette mit einer beliebigen maximalen Kette von z_α nach y' fortgesetzt werden. Für alle Elemente t dieses Teils gilt offenbar $t\not\leqq x$. Da nun y' und $y'x$ syntaktisch nicht äquivalent sind, gibt es ein q mit $qy'\in L\leftrightarrow qy'x\notin L$; es folgt $qy'\leqq y'\leqq y$ und $qy'\not\leqq x$ (andernfalls wäre $qy'x=qy'$). Wir setzen die bisher konstruierte Kette mit einer beliebigen maximalen von y' nach qy' absteigenden Kette fort. Dabei ist $qy'\in D_x$.

Auf diese Weise gewinnt man eine von y absteigende bezüglich x volle Kette, in der D_x cofinal ist. q.e.d.

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CLIMBING CERTAIN TYPES OF ROOTED TREES. II

By

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§ 1. Introduction. Let T_n denote a tree with n nodes rooted at node r. (See [1] or [7] for definitions not given here.) Select an edge rs incident with node r and proceed along it to node s; then select another edge st incident with node s and proceed along it to node t. Continue this process until an endnode u is reached (other than r if r is an endnode). Let $c=c(T_n)$ denote the number of edges in the path from r to u; we define $c(T_1)$ to be zero. Our object here is to investigate the distribution of $c(T_n)$ for trees T_n in certain families F of rooted trees under the assumptions that

(1.1) T_n is chosen at random from the trees with n nodes in F, and

(1.2) at each node q reached in T_n the next edge is chosen at random from the edges incident with q that lead away from the root.

In an earlier paper [6] we considered this problem for families F of rooted trees whose generating function $y(x)$ satisfies a functional relation of the type $y=x\Phi(y)$. We showed under rather mild conditions that if $y(x)$ has radius of convergence ϱ, then

$$\Pr\{c(T_n)=l\} \to lp^2(1-p)^{l-1}$$

for each positive integer l as $n\to\infty$, where $p=\varrho/y(\varrho)$; in particular, if $\mu(n)$ denotes the expected value of $c(T_n)$ over all trees T_n in F, then $\mu(n)\to 2/p-1$ as $n\to\infty$.

In this paper we consider the problem for some families F of rooted trees whose generating function $y(x)$ does not satisfy a functional relation of the type $y=x\Phi(y)$. In this case the distribution of $c(T_n)$ can be quite different from what it was in the earlier case. For example, if F denotes the family of recursive trees, then, as we shall show,

$$\Pr\{c(T_n)=l\} \sim \frac{1}{(l-1)!}\frac{(\ln\ln n)^{l-1}}{\ln n}$$

for each positive integer l as $n\to\infty$; in particular, $\mu(n)\sim\ln\ln n$ as $n\to\infty$.

§ 2. Preliminaries. For any given family F of rooted trees let t_n denote the number of trees T_n in F with n nodes and let y_n equal $t_n/n!$ or t_n according as the nodes of trees in F are or are not labelled. (We assume the trivial tree T_1 is in F and that $y_1=1$.) Let $t_{n,m}$ and $y_{n,m}$ be the corresponding numbers when only trees whose root node has degree m are counted. The generating functions

$$y=y(x)=\sum_{n=1}^{\infty} y_n x^n$$

and

$$y_m(x) = \sum_{n=m+1}^{\infty} y_{n,m} x^n$$

clearly satisfy the relation

$$y(x) = \sum_{m=0}^{\infty} y_m(x).$$

If we remove the root r of a non-trivial rooted tree T_n, along with all edges incident with r, we obtain a collection of rooted subtrees, or branches, $B_1, \dots, B_m$, whose roots were originally joined to r. We assume that any non-trivial tree T_n in F whose root node has degree m may be constructed by joining the roots of a collection of m smaller trees in F to a new node r that serves as the root of the resulting tree. We further assume that there exists a recurrence relation for $y_{n,m}$ in terms of $y_1, \dots, y_{n-1}$ when $n \geqq 2$ and $m \geqq 1$. Finally, we assume that there exists an operator $d_m\{g_1(x), \dots, g_m(x)\}$, defined for any m power series $g_1(x), \dots, g_m(x)$, such that the recurrence relation for $y_{n,m}$ can be expressed in terms of generating functions as

$$y_m(x) = d_m\{y(x), y(x^2), \dots, y(x^m)\}.$$

If, for notational convenience, we write this as

$$y_m(x) = D_m\{y(x)\}, \tag{2.1}$$

then

$$y(x) = x + \sum_{m=1}^{\infty} D_m\{y(x)\}. \tag{2.2}$$

We remark that, more generally, if $w(x, z)$ is any power series in the two variables x and z, then we let

$$D_m\{w(x, z)\} = d_m\{w(x, z), w(x^2, z^2), \dots, w(x^m, z^m)\},$$

where the power series $w(x, z), w(x^2, z^2), \dots, w(x^m, z^m)$ are interpreted as functions of x for fixed values of z.

If a particular tree T_n has branches $B_1, \dots, B_m$ with respect to its root r, then it follows from (1.2) that

$$\Pr\{c(T_n) = l+1\} = \frac{1}{m}\left(\Pr\{c(B_1) = l\} + \dots + \Pr\{c(B_m) = l\}\right)$$

for $l \geqq 0$. It is not difficult to see that this and (2.1) imply that the generating function

$$Q_l(x, z) = \sum_{n=1}^{\infty} \Big(\sum_{T_n} z^{\Pr\{c(T_n)=l\}}\Big) x^n,$$

where the inner sum is over all trees T_n in F, satisfies the relation

$$Q_{l+1}(x, z) = x + \sum_{m=1}^{\infty} D_m\{Q_l(x, z^{1/m})\} \tag{2.3}$$

for $l \geqq 0$. Notice that $Q_0(x, z) = x(z-1) + y(x)$ and that (2.3) reduces to (2.2) when $z = 1$.

For any given family F and each non-negative integer l, let $p(n, l)$ denote the probability that $c(T_n)=l$ under assumptions (1.1) and (1.2); then $p(n, 0)$ equals 1 or 0 according as $n=1$ or $n>1$ and $p(n, l)=0$ if $l\geqq n$. In the remaining sections we shall consider the generating functions

$$P_l(x) = \sum_{n=1}^{\infty} p(n, l) y_n x^n = \frac{\partial}{\partial z}(Q_l(x, z))_{z=1}$$

for certain families F of rooted trees. We shall use the following result which is an immediate consequence of (2.3).

THEOREM 1. *If* $l\geqq 0$, *then*

$$P_{l+1}(x) = \sum_{m=1}^{\infty} \frac{\partial}{\partial z}(D_m\{Q_l(x, z^{1/m})\})_{z=1}.$$

We remark that the following corollary was used to establish the main results in [6].

COROLLARY 1. *If* $y(x)=x+x\sum_{n=1}^{\infty}\Phi_m y^m(x)$ *for constant coefficients* Φ_m, *then*

$$P_l(x) = x(1-x/y(x))^l$$

for $l\geqq 0$.

PROOF. The hypothesis implies that

$$D_m\{Q_l(x, z^{1/m})\} = x\Phi_m Q_l^m(x, z^{1/m}).$$

Hence,

$$P_{l+1}(x) = xP_l(x)\sum_{m=1}^{\infty}\Phi_m y^{m-1}(x) = P_l(x)\cdot(1-x/y(x))$$

for $l\geqq 0$, by Theorem 1. The required result now follows by induction since $P_0(x)=x$.

§ 3. Recursive trees. A tree T_n with n labelled nodes, rooted at node 1, is a *recursive* tree if $n=1$ or if T_n can be constructed by successively joining the j-th node to one of the first $j-1$ nodes for $2\leqq j\leqq n$ (see, for example, [4], [5], or [8]). The branches of T_n with respect to the root node 1 are themselves recursive trees, or rather they would be if the nodes in each branch were relabelled according to the increasing size of the original labels. If F denotes the family of recursive trees then it follows that $t_{1,m}$ equals 1 or 0 according as $m=0$ or $m>0$ and that

$$t_{n,m} = \frac{1}{m!}\sum\binom{n-1}{a_1, \dots, a_m} t_{a_1}\dots t_{a_m}$$

or that

$$y_{n,m} = \frac{1}{n}\cdot\frac{1}{m!}\sum y_{a_1}\dots y_{a_m}$$

tor $n\geqq 2$ and $m\geqq 1$, where the sum is over all m-compositions of $n-1$. This implies fhat F satisfies assumption (2.1) with

$$D_m\{y(x)\} = \frac{1}{m!}\int_0^x y^m(t)\,dt \tag{3.1}$$

for $m \geqq 0$. Furthermore, (2.2) and (3.1) imply that

$$y' = \sum_0^\infty \frac{y^m}{m!} = e^y \tag{3.2}$$

or that

$$y = -\ln(1-x) = \sum_1^\infty \frac{x^n}{n}, \tag{3.3}$$

since $y(0)=0$. In particular, $t_n=(n-1)!$ which is, of course, obvious from the definition of recursive trees.

THEOREM 2. *If F denotes the family of recursive trees, then*

$$P'_{l+1}(x) = P_l(x) \cdot \frac{x}{(1-x)\ln 1/(1-x)}$$

for $l \geqq 0$.

PROOF. It follows from Theorem 1 and equation (3.1) that

$$P_{l+1}(x) = \sum_{m=1}^\infty \frac{1}{m!}\frac{\partial}{\partial z}\Big(\int_0^x Q_l^m(t, z^{1/m})\,dt\Big)_{z=1} = \sum_{m=1}^\infty \frac{1}{m!}\int_0^x P_l(t)\cdot y^{m-1}(t)\,dt$$

for $l \geqq 0$. Hence

$$P'_{l+1}(x) = P_l(x) \sum_{m=1}^\infty y^{m-1}(x)/m! = P_l(x)(e^y-1)/y; \tag{3.4}$$

the required result now follows from (3.1) and (3.2).

We shall use the following result (see [2; p. 166]) to determine the asymptotic behaviour of $p(n, l)$ for recursive trees.

LEMMA 1. *Let*

$$f(x) = \sum_{n=0}^\infty f_n x^n$$

denote a function such that

$$f(x) \sim \frac{1}{(1-x)^\alpha}\, g\left(\frac{1}{1-x}\right)$$

as $x \uparrow 1$, *where* $\alpha \geqq 0$ *and* $g(x)$ *is such that* $g(cx)/g(x) \to 1$ *as* $x \to \infty$ *for all fixed* $c>0$. *If* $f_n \geqq 0$ *for* $n \geqq 0$, *then*

$$\sum_{k=0}^n f_k \sim \frac{n^\alpha}{\Gamma(\alpha+1)}\, g(n)$$

as $n \to \infty$.

THEOREM 3. *If l is a fixed positive integer, then*

$$p(n, l) \sim \frac{1}{(l-1)!}\frac{(\ln\ln n)^{l-1}}{\ln n}$$

as $n \to \infty$.

PROOF. We first observe that

$$P'_{l+1}(x)\Big/\left\{\frac{1}{(l+1)!}(\ln\ln 1/(1-x))^{l+1}\right\}' = xP_l(x)\Big/\frac{1}{l!}(\ln\ln 1/(1-x))^l$$

for $l\geqq 0$, by Theorem 2. The result that

$$P_l(x) \sim \frac{1}{l!}(\ln\ln 1/(1-x))^l \tag{3.5}$$

for $l\geqq 0$ as $x\uparrow 1$ now follows from L'Hopital's Rule by induction on l since $P_0(x)=x$.

Relation (3.4) and the fact that $1-x=e^{-y}$ imply that

$$(1-x)P_l''(x) = (1-e^y+ye^y)y^{-2}P_{l-1}(x)+(e^y+e^{-y}-2)y^{-2}P_{l-2}(x) \tag{3.6}$$

for $l\geqq 2$. The expressions in front of $P_{l-1}(x)$ and $P_{l-2}(x)$ have a non-negative expansion in powers of y and hence in powers of x also. Consequently, the coefficients of x^{n-2} in

$$(1-x)P_l''(x) = (1-x)\sum_{n=l+1}^{\infty}(n-1)p(n,l)x^{n-2} =$$

$$= \sum_{n=l+1}^{\infty}\{(n-1)p(n,l)-(n-2)p(n-1,l)\}x^{n-2}$$

are non-negative.

If we apply relation (3.5) to $P_{l-1}(x)$ and $P_{l-2}(x)$ in (3.6) and appeal to relation (3.2), we find after some simplification that

$$(1-x)P_l''(x) \sim \frac{1}{(l-1)!}(\ln\ln 1/(1-x))^{l-1}/(1-x)\ln 1/(1-x) \tag{3.7}$$

as $x\uparrow 1$ for $l\geqq 2$. We leave it as an exercise for the reader to show that relation (3.7) also holds when $l=1$ and that the coefficients in the expansion of $(1-x)P_1''(x)$ are non negative.

The functions

$$f(x) = (1-x)P''(x) \quad \text{and} \quad g(x) = \frac{1}{(l-1)!}(\ln\ln x)^{l-1}/\ln x$$

satisfy the hypothesis of Lemma 1 with $\alpha=1$ for $l\geqq 1$. We may conclude, therefore, that

$$(n+1)p(n+2,l) \sim \frac{1}{(l-1)!}\frac{n(\ln\ln n)^{l-1}}{\ln n}$$

for $l\geqq 1$ as $n\to\infty$, and this implies the required result.

Let $\mu(n)$ and $\lambda(n)$ denote the expected value of $c(T_n)$ and $c(T_n)(c(T_n)-1)$ over the $(n-1)!$ recursive trees T_n. We shall only outline the proof of the following result.

THEOREM 4. *$\mu(n)\sim\ln\ln n$ and $\lambda(n)\sim(\ln\ln n)^2$, as $n\to\infty$.*

PROOF. It follows from Theorem 2 and relation (3.3) that the generating functions

$$M(x) = \sum_1^\infty \mu(n) y_n x^n = \sum_1^\infty \mu(n)\frac{x^n}{n} = \sum_{l=0}^\infty (l+1) P_{l+1}(x)$$

and

$$L(x) = \sum_1^\infty \lambda(n) y_n x^n = \sum_1^\infty \lambda(n)\frac{x^n}{n} = \sum_{l=0}^\infty (l+1) l P_{l+1}(x)$$

satisfy the differential equations

(3.8) $$(1-x) y M' = x(y+M)$$

and

$$(1-x) y L' = x(L+2M).$$

It can be shown, using these relations, that $\mu(n)$ and $\lambda(n)$ are non-decreasing functions of n and that

$$(1-x) M'(x) \sim \ln\ln 1/(1-x)$$

and

$$(1-x) L'(x) \sim (\ln\ln 1/(1-x))^2$$

as $x\uparrow 1$. The required results then follow from Lemma 1.

If $\sigma^2(n)$ denotes the variance of $c(T_n)$ over the $(n-1)!$ recursive trees T_n then it follows from Theorem 4 that $\sigma^2(n)=o(\mu^2(n))$. Hence Chebyshev's inequality implies the following result.

COROLLARY 2. *If ε denotes any positive constant, then*

$$\Pr\{(1-\varepsilon)\ln\ln n < c(T_n) < (1+\varepsilon)\ln\ln n\} \to 1$$

as $n\to\infty$.

We remark that equation (3.8) implies the recurrence relation

$$\mu(n) = \frac{\mu(n-1)+1}{n-1} + \sum_{a=1}^{n-2} \frac{\mu(n-a)}{a(a+1)}$$

for $n \geqq 2$, and there is a similar relation for $\lambda(n)$. Some selected values of $\mu(n)$, calculated by Mrs. Mary Willard, are given in the following table.

n	2	3	4	5	10	15	25	50	100
$\nu(n)$	1	1.5	1.75	1.896	2.208	2.340	2.480	2.640	2.776

Table 1

§ 4. Unlabelled rooted trees. The *cycle-index* $C_m(s_1, \ldots, s_m)$ of the symmetric group of degree m is given by the formula

$$C_m(s_1, \ldots, s_m) = \sum \frac{1}{k_1!\ldots k_m!}\left(\frac{s_1}{1}\right)^{k_1} \cdots \left(\frac{s_m}{m}\right)^{k_m}$$

where the sum is over all solutions in non-negative integers to the equation $k_1+2k_2+\ldots+mk_m=m$. If $f(x)$ is any power series in the variable x, let

$$Z_m\{f(x)\} = C_m(f(x), f(x^2), \ldots, f(x^m))$$

for $m \geqq 1$; we adopt the convention that $Z_0\{f(x)\}=1$. The expression $Z_m\{g(u, v)\}$ is similarly defined for functions $g(u, v)$ of two variables. It is not difficult to show (see, e.g., [1; p. 52]) that

$$\sum_{m=0}^{\infty} Z_m\{f(x)\}t^m = \exp \sum_{k=1}^{\infty} f(x^k)t^k/k \tag{4.1}$$

for any function $f(x)$.

Let F denote the family of unlabelled rooted trees in which the root has degree at most h and all remaining nodes have degree at most $h+1$, for some fixed positive integer h; in other words, each node of a tree in F is incident with at most h edges that lead away from the root. We also admit the possibility that $h=\infty$; this corresponds to the usual case where there are no restrictions on the degrees of nodes of trees in F. The ordering of the branches of trees in F is not taken into account and it follows readily from a theorem of PÓLYA [10] (see also [9] and [1; p. 53]) that

$$y_m(x) = xZ_m\{y(x)\},$$

for admissible values of m, and

$$y(x) = x \sum_{m=0}^{h} Z_m\{y(x)\}. \tag{4.2}$$

Thus the family F satisfies assumption (2.1) with $D_m\{y(x)\}$ equal to $xZ_m\{y(x)\}$ for admissible values of m and zero otherwise.

THEOREM 5. *If F denotes the family of unlabelled rooted trees in which each node is incident with at most h edges that lead away from the root, then*

$$P_{l+1}(x) = x \sum_{j=1}^{h} P_l(x^j) \sum_{k=0}^{h-j} Z_k\{y(x)\}/(k+j)$$

for $l \geqq 0$.

PROOF. It follows from Theorem 1 that

$$P_{l+1}(x) = x \sum_{m=1}^{h} \frac{\partial}{\partial z} (Z_m\{Q_l(x, z^{1/m})\})_{z=1}.$$

It is a straightforward exercise to deduce from the definitions of Z_m, Q_l, and P_l that

$$\frac{\partial}{\partial z} (Z_m\{Q_l(x, z^{1/m})\})_{z=1} = \frac{1}{m} \sum_{j=1}^{m} P_l(x^j) Z_{m-j}\{y(x)\}$$

for $m \geqq 1$. Consequently,

$$P_{l+1}(x) = x \sum_{m=1}^{h} \frac{1}{m} \sum_{j=1}^{m} P_l(x^j) Z_{m-j}\{y(x)\} = x \sum_{j=1}^{h} P_l(x^j) \sum_{k=0}^{h-j} Z_k\{y(x)\}/(k+j)$$

as required.

The form of the expression for $P_l(x)$ suggests that there exist constants c_l such that $p(n, l) \to c_l$ as $n \to \infty$. It seems difficult, however, to evaluate the constants c_l explicitly in general. We now proceed to the problem of determining the expected value $\mu(n)$ of $c(T_n)$ over the trees T_n in F.

THEOREM 6. *If*

$$M(x) = \sum_{n=1}^{\infty} \mu(n) t_n x^n,$$

then

$$M(x) = y - x + x \sum_{j=1}^{h} M(x^j) \sum_{k=0}^{h-j} Z_k\{y(x)\}/(k+j). \tag{4.3}$$

PROOF. It follows from Theorem 5 that

$$y - x = \sum_{l=0}^{\infty} P_{l+1}(x) = x \sum_{j=1}^{h} \sum_{l=0}^{\infty} P_l(x^j) \sum_{k=0}^{h-j} Z_k\{y(x)\}/(k+j).$$

Hence,

$$M(x) = \sum_{l=0}^{\infty} (l+1) P_{l+1}(x) = x \sum_{j=1}^{h} \sum_{l=0}^{\infty} (l+1) P_l(x^j) \sum_{k=0}^{h-j} Z_k\{y(x)\}/(k+j) =$$

$$= x \sum_{j=1}^{h} M(x^j) \sum_{k=0}^{h-j} Z_k\{y(x)\}/(k+j) + y - x,$$

as required.

The form of the expression for $M(x)$ suggests that when $h \geqq 2$ there exist constants γ_h such that $\mu(n) \to \gamma_h$ as $n \to \infty$. We shall obtain estimates for γ_h for the cases $h=2$ and $h=\infty$.

§ 5. Asymptotic results when $h=2$. We shall use the following special case of a general result of JUNGEN [3] (see also [10; p. 240]) in obtaining our asymptotic results.

LEMMA 2. *Suppose that*

$$f(x) = \sum_{n=0}^{\infty} f_n x^n$$

has a non-zero finite radius of convergence ϱ *and that* $x=\varrho$ *is the only singularity of* $f(x)$ *on its circle of convergence. If in some neighbourhood of* $x=\varrho$

$$f(x) = (\varrho - x)^{-\alpha} g(x) + h(x),$$

where $g(x)$ *and* $h(x)$ *are regular functions,* $g(\varrho) \neq 0$, *and* $\alpha \neq 0, -1, -2, \ldots,$ *then*

$$f_n \sim \frac{g(\varrho)}{\Gamma(\alpha)} \cdot \varrho^{-n-\alpha} \cdot n^{\alpha-1}$$

as $n \to \infty$.

When $h=2$, equations (4.2) and (4.3) become

$$y(x) = x + xy(x) + \frac{1}{2} x\big(y^2(x) + y(x^2)\big) \tag{5.1}$$

and

$$M(x) = y(x)-x+x\left(1+\frac{1}{2}y(x)\right)M(x)+\frac{1}{2}xM(x^2). \tag{5.2}$$

OTTER [9] showed (using somewhat different notation) that (5.1) implies that there exists a constant $\eta=.4026\ldots$ such that $y(x)$ is regular for $|x|\leqq\eta$, $x\neq\eta$; he also showed that $y(\eta)=\eta^{-1}-1$ and that

$$y(x) = (\eta-x)^{1/2}r(x)+s(x) \tag{5.3}$$

in some neighbourhood of η, where the functions $r(x)$ and $s(x)$ are regular. From this he deduced, using Jungen's result, that the number t_n of trees T_n in F when $h=2$ satisfies the relation

$$t_n \sim \frac{-r(\eta)}{2\sqrt{\pi}}\eta^{-n+1/2}n^{-3/2} \tag{5.4}$$

as $n\to\infty$. We now use these results to obtain the following estimate of the limiting behaviour of $\mu(n)$ when $h=2$.

THEOREM 7. *There exists a constant* γ_2, *where* $5.80<\gamma_2<5.82$, *such that* $\mu(n)\to\gamma_2$, *as* $n\to\infty$.

PROOF. It follows from (5.2) and Otter's results mentioned above that $M(x)$ is regular for $|x|\leqq\eta$, $x\neq\eta$, and that

$$M(x) = (\eta-x)^{1/2}r_1(x)+s_1(x) \tag{5.5}$$

in some neighbourhood of η for certain regular functions $r_1(x)$ and $s_1(x)$. If we differentiate both sides of equation (5.2) and appeal to relations (5.3) and (5.5), we find that

$$M'(x) = y'(x)\left(1+\frac{1}{2}xM(x)\right)\left(1-x-\frac{1}{2}xy(x)\right)^{-1}+(\eta-x)^{1/2}r_2(x)+s_2(x)$$

in some neighbourhood of η for certain regular functions $r_2(x)$ and $s_2(x)$. If we now substitute the expression for $y'(x)$ obtained from (5.3) we find that

$$M'(x) = -\frac{1}{2}(\eta-x)^{-1/2}\{r(\eta)\gamma_2+(\eta-x)r_3(x)\}+s_3(x) \tag{5.6}$$

in some neighbourhood of η for certain regular functions $r_3(x)$ and $s_3(x)$, where

$$\gamma_2 = \left(1+\frac{1}{2}\eta M(\eta)\right)\left(1-\eta-\frac{1}{2}\eta y(\eta)\right)^{-1}. \tag{5.7}$$

Equation (5.6) and Lemma 2 imply that

$$nt_n\mu(n) \sim \frac{-r(\eta)\gamma_2}{2\sqrt{\pi}}\eta^{-n+1/2}(n-1)^{-1/2}$$

as $n\to\infty$; hence, $\mu(n)\to\gamma_2$ as $n\to\infty$, in view of (5.4).

Equations (5.2) and (5.7) and the fact that $y(\eta)=\eta^{-1}-1$ imply that

(5.8) $$\gamma_2 = (4-4\eta-2\eta^2+\eta^2 M(\eta^2))(1-\eta)^{-2}.$$

Since $\mu(1)=0$, $\mu(2)=1$, and $1<\mu(n)<n-1$ for $n\geqq 3$, it follows that

(5.9) $$y(\eta^2)-\eta^2 < M(\eta^2) < n^2 y'(\eta^2)-y(\eta^2).$$

OTTER [9; p. 598] has given estimates for $y(\eta^2)$ and $y'(\eta^2)$; these estimates and relations (5.8) and (5.9) imply, after some calculations, that $5.80<\gamma_2<5.82$.

§ 6. Asymptotic results when $h=\infty$. When $h=\infty$ equations (4.2) and (4.3) become

(6.1) $$y(x) = x\exp\sum_{k=1}^{\infty} y(x^k)/k$$

and

(6.2) $$M(x) = y(x)-x+x\int_0^1 \{M(x)+M(x^2)t+\dots\}\exp\{y(x)t+y(x^2)t^2/2+\dots\}\,dt,$$

upon appealing to identity (4.1). Before considering the asymptotic behaviour of $\mu(n)$ we recall some facts OTTER [9] (see also [1; pp. 209—213]) established about the function $y(x)$.

There exist constants $\eta=.3383\dots$ and $b=2.6811\dots$ such that $y(x)$ is regular for $|x|\leqq\eta$, $x\neq\eta$; furthermore,

(6.3) $$y(\eta) = \eta\exp\sum_{k=1}^{\infty} y(\eta^k)/k = 1,$$

and in some neighbourhood of η

(6.4) $$y(x) = (\eta-x)^{1/2}r(x)+s(x)$$

for certain regular functions $r(x)$ and $s(x)$ where $r(\eta)=-b$. From this Otter deduced that the number t_n of trees T_n in F when $h=\infty$ satisfies the relation

(6.5) $$t_n \sim \frac{b}{2\sqrt{\pi}}\eta^{-n+1/2}n^{-3/2}$$

as $n\to\infty$.

THEOREM 8. *There exists a constant* γ, *where* $3.95<\gamma<4.05$, *such that* $\mu(n)\to\gamma$ *as* $n\to\infty$.

PROOF. For notational convenience let

$$f(x,t) = x\exp\sum_{k=1}^{\infty} y(x^k)t^k/k \quad\text{and}\quad g(x,t) = \sum_{k=2}^{\infty} M(x^k)t^{k-1}.$$

Then equation (6.2) can be rewritten as

(6.6) $$M(x) = \Big\{1-\int_0^1 f(x,t)\,dt\Big\}^{-1}\cdot\Big\{y(x)-x+\int_0^1 f(x,t)g(x,t)\,dt\Big\}.$$

The functions $y(x^2), y(x^3), \ldots$ and $M(x^2), M(x^3), \ldots$ are clearly regular in the disk $|x|<\sqrt{\eta}$. If we differentiate both sides of equation (6.6) and take relation (6.4) into consideration, we find that

$$M'(x) = y'(x)H(x)+(\eta-x)^{1/2}r_1(x)+s_1(x)\} \tag{6.7}$$

in some neighbourhood of $x=\eta$, where $r_1(x)$ and $s_1(x)$ are regular functions and where

$$H(x) = \Big\{1-\int_0^1 f(x,t)\,dt\Big\}^{-1}\Big\{1+\int_0^1 tf(x,t)g(x,t)\,dt\Big\}+ \tag{6.8}$$

$$+\Big\{1-\int_0^1 f(x,t)\,dt\Big\}^{-2}\Big\{y(x)-x+\int_0^1 f(x,t)g(x,t)\,dt\Big\}\cdot\int_0^1 tf(x,t)\,dt.$$

If we substitute into (6.7) the expression for $y'(x)$ obtained from (6.4), we find after some rearranging that

$$M'(x) = \frac{1}{2}(\eta-x)^{-1/2}\{bH(\eta)+(\eta-x)r_2(x)\}+s_2(x) \tag{6.9}$$

in some neighbourhood of η for certain regular functions $r_2(x)$ and $s_2(x)$. It now follows from (6.9) and Lemma 2 that

$$n\mu(n)t_n \sim \frac{bH(\eta)}{2\sqrt{\pi}}\eta^{-n+1/2}(n-1)^{-1/2}$$

as $n\to\infty$; this, together with (6.5), implies that $\mu(n)\to H(\eta)$ as $n\to\infty$.

Since $\mu(1)=0$, $\mu(2)=1$, and $1<\mu(n)<n-1$ for $n\geqq 3$, it follows that

$$x^2 < M(x) < xy'(x)-y(x);$$

hence,

$$t\eta^4 < g(\eta,t) < t\sum_{k=2}^{\infty}\{\eta^k y'(\eta^k)-y(\eta^k)\}. \tag{6.10}$$

Furthermore, it follows from the definition of $f(x,t)$ and equation (6.1) that

$$\eta\exp\{t+y(\eta^2)t^2/2\} < f(\eta,t) < \eta\exp\{t+(\ln\eta^{-1}-1)t^2\}. \tag{6.11}$$

If we apply estimates Otter (see [1; p. 213]) has given for η and $\sum_{k=2}^{\infty}\eta^k y'(\eta^k)$ in inequalities (6.10) and (6.11) and then substitute the resulting estimates for $g(\eta,t)$ and $f(\eta,t)$ in equation (6.8), we eventually find that $3.95<H(\eta)<4.05$, as required.

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A CONVERSE TO A CENTRAL LIMIT THEOREM OF B. GYIRES

By

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Introduction

B. GYIRES [2] obtained a generalization of the sufficiency half of the Lindeberg—Feller central limit theorem when he obtained a result for sums of a sequence of random variables defined on a Markov chain. His methods, essentially the same as those used by KEILSON and WISHART [3] some years later, involve using certain properties of the principal eigenvalue of the Fourier—Stieltjes transform of the semi-Markov matrix. WOLFSON [6] obtained a somewhat weaker version of Gyires' result by following a different approach. In this paper, under certain restrictions, both necessary and sufficient conditions are given for convergence to the Normal Law, thus in a sense complementing Gyires' result.

1. Let $\{(J_n, X_n),\ n\geqq 0\}$ be a two-dimensional stochastic process that satisfies the following conditions:

(1.1) $\{J_n, n\geqq 0\}$ is an ergodic homogeneous m-state $(m<\infty)$ Markov chain with positive initial distribution and positive transition matrix $P=\{P_{ij}\}$.

(1.2) $X_0=0$ a.s.

(1.3)
$$P[J_n=j,\ X_n\leqq x|X_0, J_0, \dots, J_{n-1}, X_{n-1}]=$$
$$=P[J_n=j,\ X_n\leqq x|J_{n-1}] \quad \text{a.s.} \quad =P_{J_{n-1},j}\,{}_nH_{J_{n-1}}(x) \quad \text{a.s.},$$

where ${}_nH_{J_{n-1}}(x)$ is a distribution function, with $\int\limits_{-\infty}^{\infty} x d_k H_i(x)=0$ for all $i=1, \dots, m$ and all $k\geqq 1$. We shall call $\{(J_n, X_n),\ n\geqq 0\}$ a *non-homogeneous* $J-X$ process (c.f. B. GYIRES [2]).

The probability on the left hand side of equation (1.3) depends on n as well as on J_{n-1}, j and x.

Define the random variable $v_k(n)=\sum\limits_{j=1}^{n} I_{k,j}$, where

$$I_{k,j}=\begin{cases}1 & \text{if} \quad J_j=k\\ 0 & \text{if} \quad J_j\neq k.\end{cases}$$

That is, $v_k(n)$ is the number of times the Markov chain $\{J_n,\ n\geqq 0\}$ enters the state

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k in n steps. The assumption of ergodicity implies that

(1.4) $\lim_{n\to\infty} P[\nu_1(n) > \alpha_1, \nu_2(n) > \alpha_2, \dots, \nu_m(n) > \alpha_m] = 1$ for all $0 < \alpha_i \leqq +\infty$

and that $\lim_{n\to\infty} P[J_n = j]$ exists and is equal to $\pi_j > 0$, say.

Using the notation of B. GYIRES [2], define

$$s_n^2 = \sum_{k=1}^{n} \sum_{i=1}^{m} \frac{D_i \sigma_{ik}^2}{D_1 + \dots + D_m},$$

where $\sigma_{ik}^2 = \int x^2 \, d_k H_i(x)$.

We introduce further the following notation:

$${}_is_n = \sum_{k=1}^{n} \sigma_{ik}^2,$$

and make the assumptions

(1.5) $${}_is_{\nu_i(n)}/{}_is_n \to c_i > 0 \quad \text{as} \quad n \to \infty \quad \text{with} \quad \sum_{i=1}^{m} c_i = 1.$$

(1.6) $${}_ks_n/{}_js_n \to 1 \quad \text{for} \quad j, k = 1, 2, \dots, m$$

and

(1.7) the convolution product $\prod_{k=1}^{n} * {}_kH_i(s_n x) \to G_i(x)$, non-degenerate, as $n \to \infty$, for each $i = 1, 2, \dots, m$.

Throughout we define

$$S_n = \sum_{j=1}^{n} X_j.$$

2. The following is the main result.

THEOREM 2.1. *In order that*

$$P\left[\frac{S_n}{s_n} \leqq x\right] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt \quad \textit{as} \quad n \to \infty$$

and

(2.1) $$\max_{1 \leqq j \leqq n} P\left[\left|\frac{X_j}{s_n}\right| > \varepsilon\right] \to 0 \quad \textit{as} \quad n \to \infty \quad \textit{for every} \quad \varepsilon > 0$$

it is necessary and sufficient that

(2.2) $$\frac{1}{s_n^2} \sum_{k=1}^{n} \int_{|x| > \varepsilon s_n} x^2 \, dF_k(x) \to 0 \quad \textit{as} \quad n \to \infty$$

where $F_k(x) = \sum_{i=1}^{m} \frac{D_{ik} H_i(x)}{D_1 + \dots + D_m}$.

PROOF. *Sufficiency:* That condition (2.2) implies convergence to the normal distribution, is an immediate of a theorem due to GYIRES [2]. We shall, however, by using a different approach and our stronger assumptions, establish condition (2.1) in addition to convergence to the normal distribution. Write

$$S_n = \sum_{i=1}^{m} \sum_{j=1}^{\nu_i(n)} \xi_{ij},$$

where the ξ_{ij}'s are independent and ξ_{ij} is the j^{th} sojourn time in state i. (Cf. SERFOZO [5].)

Here, however, ξ_{ij} and ξ_{ik} have possibly different distributions because the $J-X$ process is not homogeneous. It is obvious that ξ_{ij} is independent of $\nu_i(n)$ for each i, $j=1, 2, \dots, m$. Consequently, because $\nu_1(n), \dots, \nu_m(n)$ are asymptotically independent

$$\overline{\lim_{n\to\infty}}\, P\left[\frac{S_n}{s_n}\right] = \overline{\lim_{n\to\infty}}\, P\left[\frac{\xi_{11}+\dots+\xi_{1\nu_1(n)}}{s_n}+\dots+\frac{\xi_{m1}+\dots+\xi_{m\nu_m(n)}}{s_n} \le x\right] =$$

$$= \overline{\lim_{n\to\infty}}\, {}_1Q_n * {}_2Q_n * \dots * {}_mQ_n(x),$$

where

$${}_iQ_n(x) = P\left[\frac{\xi_{i1}+\dots+\xi_{i\nu_i(n)}}{s_n} \le x\right].$$

Now condition (1.6) implies that

$$\frac{1}{s_n^2}\sum_{k=1}^{n}\int x^2\, d_k H_i(x) \to 1 \quad \text{as} \quad n\to\infty \quad \text{for each} \quad i=1, \dots, m.$$

Consequently, because of condition (2.2) the following two results are valid:

$$\text{(2.3)} \qquad \frac{1}{s_n^2}\sum_{k=1}^{n}\int_{|x|>\varepsilon s_n} x^2\, d_k H_i(x) \to 0 \quad \text{as} \quad n\to\infty \quad \text{for each} \quad i=1, \dots, m$$

and

$$\text{(2.4)} \qquad \frac{1}{s_n^2}\sum_{k=1}^{n}\int_{|x|\le\varepsilon s_n} x^2\, d_k H_i(x) \to 1 \quad \text{as} \quad n\to\infty \quad \text{for each} \quad i=1, \dots, m.$$

We now make use of Theorem 3 p. 101, GNEDENKO and KOLMOGOROV [1], to conclude that

$$\text{(2.5)} \qquad \max_{1\le j\le n} P\left[\left|\frac{\xi_{ij}}{s_n}\right| > \varepsilon\right] \to 0 \quad \text{as} \quad n\to\infty \quad \text{for each} \quad i=1, \dots, m$$

and

$$\text{(2.6)} \qquad P\left[\frac{1}{s_n}\sum_{j=1}^{n}\xi_{ij} \le x\right] \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\, dt \quad \text{as} \quad n\to\infty.$$

Consider

$$\max_{1\le j\le n} P\left[\left|\frac{X_j}{s_n}\right| > \varepsilon\right] = \sum_{i=1}^{m}\max_{1\le j\le n} P\left[\left|\frac{X_j}{s_n}\right| > \varepsilon \,\middle|\, J_{j-1}=i\right] P[J_{j-1}=i] =$$

$$= \sum_{i=1}^{m}\max_{1\le j\le n} P\left[\left|\frac{\xi_{ik(i,j)}}{s_n}\right| > \varepsilon\right] P[J_{j-1}=i]$$

where $1 \leqq k(i, j) \leqq j-1$ is a random variable. Thus

$$\max_{1 \leqq j \leqq n} P\left[\left|\frac{X_j}{s_n}\right| > \varepsilon\right] \leqq \sum_{i=j}^{m} \max_{1 \leqq j \leqq k} P\left[\left|\frac{\xi_{ik(i,j)}}{s_n}\right| > \varepsilon\right] \to 0 \quad \text{as} \quad n \to \infty$$

because of (2.5). That is, condition (2.1) holds.

Further, because of the independence of $v_i(n)$ and ξ_{ij} for each i and j a simple argument (c. f. RÉNYI [4] in the special i.i.d. case), that makes use of (1.5), (1.6) and (1.7) shows that

$${}_iQ_n(x) \to \frac{1}{\sqrt{2\pi}\, c_i} \int_{-\infty}^{x} e^{-t^2/2c_i^2}\, dt.$$

Therefore,

$$\overline{\lim_{n\to\infty}} P\left[\frac{S_n}{s_n} \leqq x\right[= \lim_{n\to\infty} P\left[\frac{S_n}{s_n} \leqq x\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2}\, dt,$$

which completes the sufficiency half of the theorem.

Necessity: Conversely, let conditions (2.1) and (2.3) hold. As before, (i.e. using (1.5), (1.6) and (1.7))

$$\lim_{n\to\infty} P\left[\frac{S_n}{s_n} \leqq x\right] = \lim_{n\to\infty} {}_1Q_n * {}_2Q_n * \dots * {}_mQ_n(x) = \tilde{G}_1 * \tilde{G}_2 * \dots * \tilde{G}_m(x)$$

where $\tilde{G}_i(x) = G_i(c_i x)$. Since each $\tilde{G}_i$ is non-degenerate, we have by Cramér's well known characterization theorem that

$$\Phi(x) = \tilde{G}_1 * \dots * \tilde{G}_m(x)$$

implies that each $\tilde{G}_i$ is a normal distribution function with mean μ_i, say and variance c_i^2. Hence each G_i is normal with mean μ_i and variance 1.

Next we show that condition (2.1) implies that

$$\lim_{n\to\infty} \max_{1 \leqq j \leqq n} P\left[\left|\frac{\xi_{ij}}{s_n}\right| > \varepsilon\right] = 0. \tag{2.7}$$

Consider

$$\max_{1 \leqq j \leqq v_i(n)} P\left[\left|\frac{\xi_{ij}}{{}_is_{v_i(n)}}\right| > \varepsilon\right] \leqq \max_{1 \leqq j \leqq n} P\left[\frac{s_n}{{}_is_{v_i(n)}}\left|\frac{X_j}{s_n}\right| > \varepsilon \,\middle|\, J_{j-1} = i\right] \leqq$$

$$\leqq \max_{1 \leqq j \leqq n} \frac{1}{\delta} P\left[\frac{s_n}{s_{v_i(n)}}\left|\frac{X_j}{s_n}\right| > \varepsilon\right] \quad \text{for some} \quad \delta > 0,$$

using (1.1) and the ergodicity of the Markov chain. (1.5), (1.6) and condition (2.1) then lead us to conclude (2.7), because $v_i(n) \to \infty$ as $n \to \infty$. Moreover it is obvious that

$$\lim_{n\to\infty} \max_{1 \leqq j \leqq n} P\left[\frac{|\xi_{ij} - \mu_i|}{s_n} > \varepsilon\right] = 0.$$

Thus the array $\left\{\frac{\xi_{ij}-\mu_i}{s_n}\right\}$ is, for each fixed i, uniformly asymptotically negligible,

$$P\left[\sum_{j=1}^{n}\frac{\xi_{ij}-\mu_i}{s_n}\leqq x\right]\to\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-t^2/2}\,dt$$

and

$$\operatorname{Var}\left(\sum_{j=1}^{n}\frac{\xi_{ij}-\mu_i}{s_n}\right)\to 1 \quad\text{as}\quad n\to\infty.$$

Again, Theorem 3 p. 101, GNEDENKO and KOLMOGOROV [1], enables us to conclude that

$$\frac{1}{s_n^2}\sum_{k=1}^{n}\int_{|x|>\varepsilon s_n}x^2\,d_kH_i(x)\to 0 \quad\text{as}\quad n\to\infty$$

for each $i=1,\dots,m$. Therefore,

$$\frac{1}{s_n^2}\sum_{k=1}^{n}\int_{|x|>\varepsilon s_n}x^2\,dF_k(x)\to 0 \quad\text{as}\quad n\to\infty.$$

This completes the proof.

COROLLARY. *Let* (1.1), (1.2), (1.3) *and* (1.5) *be satisfied and suppose that*

$$\int_{-\infty}^{\infty}x^2\,d_kH_i(x)=\int_{-\infty}^{\infty}x^2d_kH_j(x)\quad\textit{for all}\quad i,j=1,2,\dots,m.$$

Then Theorem 2.1 *is valid.*

In the special case for which $\{J_n,\ n\geqq 0\}$ has only one state, the random variables $\{X_n\}$ become independent, and Theorem 2.1 reduces to the classical Lindeberg—Feller theorem.

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ÜBER DIE STRUKTUR LINEAR KOMPAKTER RINGE

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§ 1. Einleitung

In der Theorie der assoziativen topologischen Ringe spielen die linear topologischen Ringe eine zentrale Rolle. Mit seiner Hilfe und mit Hilfe bestimmter topologischen Voraussetzungen, insbesondere der linearen Kompaktheit, bewies Leptin die im gewissen Sinne weitestgehende Verallgemeinerung des Satzes von Wedderburn und Artin. Wir weisen hier darauf hin, daß die lineare Kompaktheit offenbar eine natürliche Verallgemeinerung der Minimalbedingung für Linksideale ist. Der Satz von Wedderburn und Artin behandelt den einen Extremfall eines artinschen Ringes, d. h. eines Ringes mit Minimalbedingung für Linksideale; nämlich den Fall, daß das Jacobson-Radikal des Ringes gleich dem Nullideal ist. Der andere Extremfall, daß nämlich der Ring ein artinscher Radikalring ist, ist von Szele betrachtet worden. In [4] wurde eine Beschreibung der artinschen Ringe gegeben, deren Radikal selbst wieder ein artinscher Ring ist.

Hiernach ergibt sich natürlich die Frage, ob eine ähnliche Verallgemeinerung unter bestimmten topologischen Voraussetzungen auch hier zu erreichen ist. Dabei untersuch WIEGANDT [10] den Fall, daß der Ring ein im engeren Sinne linear kompakter Radikalring oder ein inverser Limes von artinschen Radikalringen ist. WIDIGER [9] bejahte dies für den Fall, daß der Ring und sein Radikal als Ring im engeren Sinne linear kompakt im Sinne von Zelinsky sind, d. h. es existiert ein Basisfilter aus Idealen. In diesem Artikel wollen wir ähnliche Ergebnisse für die lineare Kompaktheit im Sinne von Leptin beweisen, d. h. wir zeigen, daß die von Widiger erreichten Ergebnisse auch dann gültig bleiben, wenn wir nicht voraussetzen, daß ein aus Idealen bestehendes Basisfilter existiert. Der Preis dafür ist die Bedingung, daß ein sogenanntes vollständiges System idempotenter Elemente mit der Eigenschaft (V) existieren möge. Um vieviel schwächer als die im engeren Sinne lineare Kompaktheit nach Zelinsky diese Bedingung ist, zeigt unser Hauptsatz. Im ersten Teil der Arbeit geben wir außerdem einen anderen, sehr einfachen Beweis für das Ergebnis von Widiger mit Hilfe des inversen Limes.

§ 2. Vorbereitungen

Wir stellen kurz die benutzten Bezeichnungen zusammen.

$C(p^k)$ zyklische Gruppe der Ordnung p^k, falls k eine natürliche Zahl ist, für $k=\infty$ die prüfersche p-Gruppe (d. h. p ist dann stets Primzahl).
R_n Ring aller $n\times n$ Matrizen mit Elementen aus dem Ring R.
$\sum A_\nu$ komplette direkte Summe der A_ν.

$\sum^{\oplus} A_\nu$ diskrete direkte Summe der A_ν.
$\oplus$ gruppen- oder modultheoretische direkte Zerlegung.
$\boxplus$ ringtheoretische direkte Zerlegung.
A^+ additive Gruppe von A.

Da der Querstrich zur Bezeichnung von Faktorringen dient, wird zur Bezeichnung des topologischen Abschlußes einer Menge A eines topologischen Raumes das Symbol Cl (A) (closure) benutzt.

Unter einem Ring verstehen wir stets einen assoziativen Ring. Das Radikal eines Ringes wird immer das jacobsonsche Radikal bedeuten. Stimmt ein Ring R mit seinem Radikal überein, dann sagen wir, daß R ein Radikalring ist. Bezüglich der Grundbegriffe verweisen wir auf die Lehrbücher der Algebra. Unter einem R-Modul verstehen wir immer einen Linksmodul über R.

Zu unseren Untersuchungen brauchen wir einige Vorbereitungen. Die folgende Behauptung stammt von A. G. Kurosch:

(I) Eine abelsche Gruppe genügt dann und nur dann der Minimalbedingung für Untergruppen, wenn sie zur direkten Summe endlich vieler Gruppen vom Typ $C(p_i^k)$ $(0 \leqq k \leqq \infty)$ isomorph ist.

Für den Beweis weisen wir auf Fuchs [2], Seite 65 hin.

In dieser Arbeit wird vorausgesetzt, daß die topologischen Räume stets hausdorffsch (oder separiert) sind.

Nach Leptin bezeichnen wir als Filter ein System $\mathscr{F}$ von Teilmengen eines topologischen Raumes mit der Eigenschaft, daß zu $F_\mu, F_\nu \in \mathscr{F}$ ein F_λ mit $F_\lambda \subseteq F_\mu \cap F_\nu$ existiert. Ein Basisfilter eines Moduls ist ein Filter, dessen Elemente ein Fundamentalsystem für die Umgebungen des neutralen Elementes des Moduls bilden.

Grundlegend ist für unsere Untersuchungen der Begriff des inversen Limes. Für seine Definition verweisen wir auf das Buch von Grätzer [3].

Sei M ein topologischer Modul mit Basisfilter aus Untermoduln $U_\alpha (\alpha \in A)$ und bezeichne π_α den natürlichen Epimorphismus von M auf $M_\alpha = M/U_\alpha$. Ist $U_\alpha \subseteq U_\beta$, so ist $\pi_\beta^\alpha = \pi_\beta \pi_\alpha^{-1}$ ein natürlicher Homomorphismus von M_α auf M_β. Bekanntlich bildet $\Omega = \{M_\alpha, \pi_\beta^\alpha\}$ ein inverses System. Zelinsky hat die folgenden Behauptungen bewiesen.

(II) ([12], Theorem 3) Die Komplettierung $\tilde{M}$ von M ist zu $\varprojlim \{R_\alpha, \pi_\beta^\alpha\}$ im algebraischen und topologischen Sinne isomorph.

(III) ([12], Lemma 3) Ein inverser Limes von Radikalringen ist wieder ein Radikalring.

Das inverse System $\Omega = \{M_\alpha, \pi_\beta^\alpha\}$ ist die direkte Summe der inversen Systeme $\Omega_\lambda = \{M_{\alpha\lambda}, \varrho_{\beta\lambda}^\alpha\}$, falls $M_\alpha = \sum_\lambda M_{\alpha\lambda}$ für jedes α gilt, und $\varrho_{\beta\lambda}^\alpha$ die Beschränkung von π_β^α auf $M_{\alpha\lambda}$ ist. Wir bezeichnen mit $\Omega = \sum \Omega_\lambda$ den Fall, daß Ω die direkte Summe der inversen Systeme Ω_λ ist.

(IV) ([12], Theorem 1) Es gilt $\varprojlim \sum \Omega_\lambda = \sum \varprojlim \Omega_\lambda$ im algebraischen und topologischen Sinne.

Wir nennen einen topologischen R-Modul M linear topologisch, wenn er einen Basisfilter aus Untermoduln besitzt. Ein linear topologischer R-Modul M ist linear kompakt (kurz: l.k.) genannt, wenn jeder Filter $\mathscr{F} = \{a_\mu + N_\mu\}$ von Restklassen nach abgeschlossenen Untermoduln N_μ einen nicht leeren Durchschnitt hat: $\bigcap_{F \in \mathscr{F}} F \neq \emptyset$. Nach Leptin heißt ein linear kompakter R-Modul M im engeren

Sinne linear kompakt (kurz: i.e.S.k.l.), wenn jeder stetige R-Homomorphismus von M in einen linear topologischen R-Modul offen ist.

Ein Ring R heißt linear kompakt bzw. im engeren Sinne linear kompakt, wenn er als R-modul l.k. bzw. i.e.S.l.k. ist.

Nun legen wir einige Aussagen über l.k. und i.e.S.l.k. Modul dar.

(V) ([11], Prop. 1) Die komplette direkte Summe l.k. R-Moduln ist ein l.k. R-Modul.

(VI) ([11], Prop. 2) Ist φ ein stetiger R-Homomorphismus des R-Moduls M in einen linear topologischen R-Modul M', so ist mit M auch $\varphi(M)$ l.k.; $\varphi(M)$ ist dann in M' abgeschlossen.

(VII) ([11], Prop. 3) Ist M ein l.k.-Modul und N ein abgeschlossener Untermodul von M, so ist N l.k..

Von (VII) gilt auch die Umkehrung

(VIII) ([11], Prop. 9) Ist M ein linear topologischer R-Modul, N ein abgeschlossener Untermodul von M, so ist mit N und M/N auch M l.k..

(IX) ([11], Prop. 10) Ein inverser Limes l.k. R-Moduln ist l.k.

Daß die lineare Kompaktheit eine Verallgemeinerung der Minimalbedingung ist, besagt

(X) ([11], Prop. 5) Ist M ein linear topologischer R-Modul mit Minimalbedingung für abgeschlossene Untermoduln, so ist M l.k.

(XI) ([7], Satz 4) Die i.e.S.l.k. R-Moduln M sind unter den l.k. R-Moduln durch die folgenden vier äquivalenten Eigenschaften gekennzeichnet:

(i) Jeder stetige R-Homomorphismus von M ist offen.

(ii) In den Faktormoduln nach offenen Untermoduln gilt die Minimalbedingung.

(iii) Jeder offene Untermodul von M hat minimale Obermoduln.

(iv) M ist inverser Limes von artinschen Moduln.

(XII) ([7], Satz 5) Abgeschlossene Untermoduln, stetig homomorphe Bilder, komplette direkte Summe i.e.S.l.k. Moduln sind i.e.S.l.k.

Eine l.k. abelsche Gruppe ist eine l.k. Modul über dem diskreten Ring der ganzen Zahlen. Daß eine l.k. abelsche Gruppe stets i.e.S.l.k. ist, besagt

(XIII) ([6], (2.6)) Eine diskrete l.k. abelsche Gruppe genügt der Minimalbedingung für Untergruppen.

Ist K ein diskreter Körper und M ein unitärer K-Modul, so gilt analog zu (XIII)

(XIV) ([5], S. 78) Ist der K-Modul M diskret, so ist M genau dann l.k., wenn M endliche Dimension über K hat.

(XV) ([7], Satz 8) Das Radikal eines l.k. Ringes ist abgeschlossen.

Leptin [7] hat den Begriff der Nilpotenz für topologische Ringe folgendermaßen verallgemeinert. Sei A ein ein- oder zweiseitiges Ideal in dem topologischen Ring R. Die absteigenden Ketten von Unteridealen A_μ, $\underset{\mu}{A}$ sind durch die Formeln

$$A_1 = A \qquad \underset{1}{A} = A$$

$$A_{\mu+1} = \mathrm{Cl}(A_\mu A) \qquad \underset{\mu+1}{A} = \mathrm{Cl}(\underset{\mu}{A}\,\underset{\mu}{A})$$

$$A_\lambda = \bigcap_{\mu<\lambda} A_\mu \qquad \underset{\lambda}{A} = \bigcap_{\mu<\lambda} \underset{\mu}{A}$$

falls λ Limeszahl ist, definiert. Offenbar existiert eine Ordnungszahl ξ mit $A_\mu = A_\xi$ und $\underset{\mu}{A} = \underset{\xi}{A}$ für alle $\mu > \xi$. Wir setzen

$$A_* = A_\xi, \quad \underset{*}{A} = \underset{\xi}{A}.$$

Die Ideale A_* und $\underset{*}{A}$ sind abgeschlossene Ideale von R und genügen den Gleichungen

$$\mathrm{Cl}\,(A_* A) = A_* \quad \text{und} \quad \mathrm{Cl}\,(\underset{*}{A}\,\underset{*}{A}) = \underset{*}{A}.$$

Das Ideal A heißt transfinit r-nilpotent, wenn $A_* = 0$ ist, und transfinit nilpotent, falls $\underset{*}{A} = 0$ ist.

Da $\underset{*}{A}$ in A_* enthalten ist, ist ein transfinit r-nilpotentes Ideal stets transfinit nilpotent.

(XVI) ([7], Satz 9) Das Radikal eines i.e.S.l.k. Ringes ist transfinit r-nilpotent.

Im Ring aller linearen Transformationen eines Vektorraumes über einem Schiefkörper kann man bekanntlich eine Topologie, die sogenannte endlich Topologie, einführen, indem man als Basisfilter die folgenden Linksideale nimmt: Für beliebige endlich viele $u_1, \ldots, u_k$ aus dem Vektorraum sei $F(u_1, \ldots, u_k)$ die Menge aller linearen Transformationen, die $u_1, \ldots, u_k$ in die Null abbilden. Mit dieser Topologie ist der Ring aller linearen Transformationen (der volle Endomorphismenring) offenbar ein linear topologischer Ring. Wenn im folgenden vom vollen Endomorphismenring eines Vektorraumes über einem Schiefkörper die Rede ist, so ist der Ring mit der endlichen Topologie gemeint.

Ein topologischer Ring heißt topologisch einfach, wenn er radikalfrei ist und er außer den beiden trivialen Idealen keine weiteren abgeschlossenen Ideale enthält.

(XVII) ([7], Satz 12) Ein topologisch einfacher l.k. Ring ist voller Endomorphismenring eines Vektorraumes über Schiefkörper und umgekehrt.

(XVIII) ([7], Satz 13) Die halbeinfachen l.k. Ringe sind genau die kompletten direkten Summen voller Endomorphismenringe von Vektorräumen über Schiefköpern.

Genauso wie im Falle der artinschen Ringe spielen bei den Untersuchungen bezüglich der l.k. Ringe idempotente Elemente eine wichtige Rolle. Man versucht deshalb, idempotente Elemente des Faktorringes nach dem Radikal zu idempotenten Elementen von R anzuheben. Daß dies möglich ist, besagt

(XIX) ([8], (4.1)) Sei M ein transfinit nilpotentes Ideal des l.k. Ringes R. In jeder idempotenten Restklasse $\bar{e}$ von $\bar{R}$ modulo M liegt ein idempotentes Element $e \in R$.

(XX) ([8], (3.14)) Ist R ein i.e.S.l.k. Ring mit dem Radikal I und e ein idempotentes Element aus R, so sind auch eRe i.e.S.l.k. und eIe sein Radikal.

(XXI) Es seien R ein i.e.S.l.k. Ring und I sein Radikal. $R/I = \bar{R} = \sum_{\mu \in \Gamma} \bar{e}_\mu \bar{R} \bar{e}_\mu$, wobei die $\bar{e}_\mu \bar{R} \bar{e}_\mu$ topologisch einfache direkte Summanden sind, $\bar{e} = \sum_{\mu \in \Gamma} \bar{e}_\mu$ ist das Einselement von $\bar{R}$. Dann gibt es ein summierbares System orthogonaler idempotenter Vertreter $\{e_\mu\}$ von $\{\bar{e}_\mu\}$, e von $\bar{e}$, so daß

$$e_\mu \cdot e = e \cdot e_\mu = e_\mu \quad \forall \mu \in \Gamma, \quad e_\mu \cdot e_\nu = 0 \quad \forall \mu \neq \nu$$

gilt.

Ein derartiges System solcher orthogonaler idempotenter Elemente heißt vollständig.

Für den Beweis verweisen wir auf WIDIGER [9], Hilfssätze 1, 2 und 3 und Satz 1.

(XXII) Ist ein l.k. Ring topologisch einfach, so ist er als komplette direkte Summe von miteinander operatorisomorphen minimalen Linksidealen darstellbar.

Für den Beweis verweisen wir auf LEPTIN [8], (1.2).

(XXIII) ([7], Satz 14) Ist R ein l.k. Ring mit dem Radikal I und A ein abgeschlossenes zweiseitiges Ideal aus R, so ist $A\cap I$ das Radikal von A und $(I+A)/A$ das Radikal von R/A.

Sei R ein vollständiger linear topologischer Ring, $\mathbf{U}$ der Filter aller offenen Linksideale und I eine Indexmenge beliebiger Mächtigkeit. Ein System $\{x_\mu, \mu\in I\}$ von Elementen $x_\mu\in R$ heißt summierbar, wenn jedes $U\in\mathbf{U}$ fast alle x_μ enthält. Die Restklassen $\sum\limits_{x_\mu\notin U} x_\mu+U$ bilden dann einen Cauchyfilter $\mathbf{C}$, der wegen der Vollständigkeit von R gegen ein Element x konvergiert. Wir setzen

$$x=\lim\mathbf{C}=\sum x_\mu .$$

Die folgenden Aussagen sind so einfach, daß wir auf die Beweise verzichten (vgl. auch [1]).

(XXIV) ([8], (3.1)) Sind $\{x_\mu, \mu\in I\}$ und $\{y_\mu, \mu\in I\}$ summierbar, so ist es auch $\{x_\mu+y_\mu, \mu\in I\}$, und es gilt $\sum x_\mu+\sum y_\mu=\sum(x_\mu+y_\mu)$.

(XXV) ([8], (3.2)) Ist $\{x_\mu, \mu\in I\}$ summierbar und $\{y_\mu, \mu\in I\}$ ein System von beliebigen Elementen $y_\mu\in R$, so ist auch $\{y_\mu x_\mu, \mu\in I\}$ summierbar.

(XXVI) ([8], (3.3)) Ist $\{x_\mu, \mu\in I\} x_\mu\in R$ summierbar und f eine stetige additive Abbildung von R in einen vollständigen linear topologischen Ring R', so ist $\{f(x_\mu), u\in I\}$ in R' summierbar und es gilt

$$f(\sum x_\mu)=\sum f(x_\mu).$$

Folgerung:

(XXVII) ([8], (3.4)) Ist $\{x_\mu, \mu\in I\}$ in R summierbar, so sind für jedes $z\in R$ die Systeme $\{zx_\mu\}$ und $\{x_\mu z\}$ summierbar und es gilt

$$z\sum x_\mu=\sum zx_\mu,\quad (\sum x_\mu)z=\sum x_\mu z.$$

§ 3. Neuer Beweis für die Ergebnisse von Widiger

Ein Ring heißt streng artinsch, wenn seine additive Gruppe bezüglich ihrer Untergruppen der Minimalbedingung genügt.

Analog nennen wir einen topologischen Ring R streng l.k., wenn er einen Basisfilter aus Linksidealen besitzt und jeder Filter von Restklassen nach abgeschlossenen Untergruppen von R^+ einen nicht leeren Durchschnitt hat (d.h. R^+ ist eine l.k. Gruppe).

Dieser Begriff wird eine sehr wichtige Rolle in den folgenden Paragraphen spielen. Mit seiner Hilfe charakterisieren wir die Struktur bestimmter i.e.S.l.k. Ringe.

Wir bemerken hier, daß unsere Definition eine natürliche Verallgemeinerung der Definition von Widiger ist, welche die Existenz eines aus Idealen bestehenden Basisfilters fordert.

Nach KERTÉSZ und WIDIGER [4] gilt der folgende Satz

(K – W) ([4], Satz 3) *Jeder artinsche Ring R mit artinschem Radikal ist die ringtheoretische direkte Summe endlich vieler voller Matrizenringe über unendlichen Schiefkörpern und eines streng artinschen Ringes R^*:*

$$R = S_{n_1}^{(1)} \boxplus \ldots \boxplus S_{n_i}^{(i)} \boxplus R^*.$$

Diese Darstellung ist eindeutig bis auf Isomorphie.

Für linear kompakte Ringe gilt der folgende Satz von Widiger

(W) ([9], Satz 5) *Jeder i.e.S.l.k. Ring R mit einer Basis aus Idealen und i.e.S.l.k. Radikal besitzt eine ringtheoretische direkte Darstellung*

$$R = \sum S_{n_\mu}^{(\mu)} \boxplus R^*,$$

wobei die $S^{(\mu)}$ unendliche diskrete Schiefkörper sind und R^* ein streng linear kompakter Ring mit einer Basis aus Idealen.

Diese Darstellung ist eindeutig bis auf einen stetigen Isomorphismus.

Für diesen Satz gab WIDIGER in [9] einen direkten Beweis. Im folgenden geben wir dafür noch einen Beweis, mit Hilfe der Ergebnisse von Kertész und Widiger, durch inverse Limesbildung.

BEWEIS. Es sei $\mathbf{U}=\{U_\alpha, \alpha\in\Gamma\}$ ein Basisfilter von Idealen in R. Es gilt also $R=\varprojlim R_\alpha$, wobei $R_\alpha=R/U_\alpha$ ein artinscher Ring ist. Nach (XXIII) ist $I_\alpha=(I+U_\alpha)/U_\alpha\cong I/(I\cap U_\alpha)$ das Radikal von R_α, wobei I das Radikal von R ist. Wegen der im engeren Sinne linearen Kompaktheit von I ist auch I_α ein artinscher Ring. Nach (K – W) läßt R_α eine ringtheoretische direkte Zerlegung zu:

$$R_\alpha = S_{n_1(\alpha)}^{(1,\alpha)} \boxplus \ldots \boxplus S_{n_{i_\alpha}^{(\alpha)}}^{(i_\alpha,\alpha)} \boxplus R_\alpha^* = A_\alpha \boxplus R_\alpha^*.$$

Wegen der Einfachheit der unendlichen Ringe $S_{n_k^{(\alpha)}}^{(k,\alpha)}$ $(k=1, \ldots, i_\alpha, \alpha\in\Gamma)$ ist leicht zu sehen, daß $\{A_\alpha, \alpha\in\Gamma\}$ ein inverses System bildet. Daß $\{R_\alpha^*, \alpha\in\Gamma\}$ ebenfalls ein inverses System ist, ist klar. Das daß inverse System $\{R_\alpha^*, \alpha\in\Gamma\}$ is offensichtlich die direkte Summe der obigen inversen Systeme. Nach (IV) haben wir also

$$R = \varprojlim \{R_\alpha\} = \varprojlim \{A_\alpha\} \boxplus \varprojlim \{R_\alpha^*\}.$$

Es sei $R^*=\varprojlim \{R_\alpha^*\}$. Dann ist R^* streng linear kompakt mit einem Basisfilter aus Idealen. Wegen (III) ist $\varprojlim \{A_\alpha\}$ ein l.k. halbeinfacher Ring mit einer Basis aus Idealen. Nach (XVIII) gilt

$$\varprojlim \{A_\alpha\} = \sum_{\nu\in I} B_\nu$$

wobei die B_ν volle Endomorphismenringe von Vektorräumen über Schiefkörpern S_ν sind. Die Topologie von B_ν ist die endliche. Nun soll B_ν einen Basisfilter aus Idealen besitzen. Da B_ν topologisch einfach ist, folgt, daß die Topologie die diskrete sein muß. Aus der Einfachheit der B_ν folgt dann ebenfalls, daß der zugehörige Vektorraum endliche Dimension über S_ν hat, d. h. die B_ν sind volle Matrizenringe über Schiefkörpern. Es ist leicht zu sehen, daß S_ν für ein bestimmtes (i, α) mit $S^{(i,\alpha)}$ übereinstimmt.

Aus diesem Beweis und der Eindeutigkeit von (K—W) folgt sofort die Eindeutigkeit von (W).

Damit ist der Satz in allen Teilen bewiesen.

§ 4. Streng linear kompakte Ringe

In diesem Paragraphen wollen wir die streng linear kompakten Ringe charakterisieren. Aus der Definition der linearen Kompaktheit und der im engeren Sinne linearen Kompaktheit folgen unmittelbar die folgenden Behaputungen.

SATZ 1. *Der Ring R ist streng linear kompakt genau dann, wenn er inverser Limes von streng artinschen R-Moduln ist.*

Der Beweis ergibt sich aus (II), (XI) und (XIII).

(4.1) Jeder streng linear kompakte Ring ist gleichzeitig i.e.S.l.k.

WIEGANDT hat in [10] bewiesen, daß jeder transfinit r-nilpotente i.e.S.l.k. Ring, also jeder i.e.S.l.k. Radikalring, der inverse Limes von streng artinschen R-Moduln ist. Folglich gilt

(4.2) Jeder i.e.S.l.k. Radikalring ist streng linear kompakt.

Man kann für die streng l.k. Ringe mit Methoden von WIEGANDT [10] die folgende Verallgemeinerung eines Satzes aus [10] zeigen. Für die Terminologie verweisen wir auf [10].

SATZ 2 (Vgl. [10], Satz 6). *Sei R ein streng linear kompakter Ring, und bezeichne A das maximale teilbare Ideal des Ringes R. Ferner sei*

$$C = \{x \in R, xR \subseteqq A\}$$

der Kern von R. C ist ein streng linear kompakter Ring und ein abgeschlossenes Ideal mit $CR^2=0$. Der Faktorring C/A ist ein Zeroring und zugleich ein inverser Limes endlicher Zeroringe. Das Bild $B=R/C$ ist als inverser Limes endlicher Ringe darstellbar.

Bezeichne R einen linear topologischen Ring und seien C sein Kern und B das Bild von R. Ist in der durch R induzierten Topologie C ein streng linear kompakter Ring B der inverse Limes endlicher Ringe, dann ist R ein streng linear kompakter Ring.

Der Beweis ist zu dem in [10] völlig analog, man braucht sogar die Nilpotenz nicht zu betrachten.

Ein Ring heißt primär, wenn er ein Einselement besitzt und der Faktorring nach seinem Radikal ein voller Endomorphismenring eines Vektorraumes über einem Schiefkörper ist.

SATZ 3. *Ist R ein primärer s.l.k. Ring mit dem s.l.k. Radikal $I \neq (0)$, so ist R streng linear kompakt.*

BEWEIS. Das Radikal I ist wegen (4.2) streng linear kompakt. Wenn $I^2 \neq (0)$ ist, so betrachten wir $R/\mathrm{Cl}(I^2)$. $R/\mathrm{Cl}(I^2)$ ist ein primärer i.e.S.l.k. Ring. Das Radikal von $R/\mathrm{Cl}(I^2)$ ist wegen (XXIII) $I/\mathrm{Cl}(I^2)$. $\mathrm{Cl}(I^2)$ ist abgeschlossen in R, also auch in I, folglich ist $I/\mathrm{Cl}(I^2)$ ein i.e.S.l.k. Ring. Es gilt natürlich $\mathrm{Cl}(I^2) \neq I$, da sonst $I=(0)$ ist, im Widerspruch zur Voraussetzung, weil I transfinit r-nilpotent ist.

Wenn wir zeigen, daß $R/\mathrm{Cl}(I^2)$ streng linear kompakt ist, so folgt, da auch $\mathrm{Cl}(I^2) \subseteq I$ streng l.k. ist, dasselbe für R. Wegen $(I/\mathrm{Cl}(I^2))^2=(0)$ genügt es, sich auf den Fall mit $I^2=(0)$ zu beschränken.

Wegen (XXII) ist $\bar{R}=R/I$ die komplette direkte Summe von miteinanader operatorisomorphen minimalen Linksidealen $\bar{R}_\alpha$. Wir beweisen, daß $\bar{R}_\alpha$ für jedes α streng l.k. ist.

Dazu wählen wir ein beliebiges, aber festes Element $a \in I$, $a \neq 0$. Wir bilden den $\bar{R}$-Modul $\bar{R}_\alpha$ durch die Abbildung φ in den $\bar{R}$-Modul I ab:

$$\varphi: \bar{R}_\alpha \to I, \quad \varphi(\bar{x}) = xa, \quad \bar{x} \in \bar{R}_\alpha, \quad x \in R$$

wenn x ein Vertreter der Restklasse $\bar{x}$ ist. Wegen $I^2=(0)$ ist diese Abbildung von der Wahl von x unabhängig. φ ist offensichtlich ein stetiger $\bar{R}$-Homomorphismus. Weil $\bar{R}_\alpha$ minimal ist, ist φ entweder injektiv oder trivial. Das letztere würde bedeuten, daß $xa=0$ für alle $x \in R$ mit $\bar{x} \in \bar{R}_\alpha$ ist. Gibt es keinen Index α für den $xa \neq 0$ für einige $x \in R$ mit $\bar{x} \in R_\alpha$ ist, so ist $ra=0$ für alle $r \in R$; R hat aber nach Voraussetzung ein Einselement, was ein Widerspruch ist. Also gibt es einen Index β, für den φ injektiv ist. Folglich sind $\bar{R}_\beta$, also auch alle $\bar{R}_\alpha$ $(\alpha \in \Gamma)$ streng linear kompakt.

Damit haben wir den Beweis diese Satzes erbracht.

SATZ 4. *Ein Endomorphismenring R eines Vektorraumes V über einem Schiefkörper K ist genau dann streng linear kompakt, wenn R endlich ist d.h. K endlich und V von endlicher Dimension sind.*

BEWEIS. Die Voraussetzung ist offensichtlich hinreichend. V hat eine Basis $\{e_i,\ i \in \Gamma\}$. e_j^i bezeichne jenen Endomorphismus von V, für den $e_i^j(e_j)=e_i$, $e_i^j(e_p)=0$ für jedes $p \neq j$ ist.

Es ist bekannt, daß Re_i^i ein minimales Linksideal von R ist. Es sei weiterhin für jedes Paar (i, j) E_i^j das von e_i^j erzeugte K-Untermodul von R. E_i^j ist natürlich operatorisomorph mit K. Weiter gilt

$$(*) \qquad Re_i^i = \sum_{j \in \Gamma} E_i^j.$$

Weil R streng l.k. ist, so ist dies auch Re_i^i. Wegen $(*)$ können Γ nur endlich und V von endlicher Dimension sein. Folglich ist E_i^j, also auch K streng l.k., d.h. K genügt der Minimalbedingung für Untergruppen. Dann muß aber der Schiefkörper K endlich sein.

Damit ist der Satz bewiesen.

SATZ 5. *Es sei R ein i.e.S.l.k. Ring mit dem s.l.k. Radikal I. $R/I=\bar{R}= \sum_{\mu \in \Gamma} \bar{e}_\mu \bar{R} \bar{e}_\mu$ (die $\bar{e}_\mu \bar{R} \bar{e}_\mu$ sind topologisch einfache Ringe). Ist $\bar{e}_\mu \bar{R} \bar{e}_\mu$ unendlich, so gilt*

$$e_\mu R e_\mu \cong \bar{e}_\mu \bar{R} \bar{e}_\mu (e_\mu x e_\mu \to \bar{e}_\mu \bar{x} \bar{e}_\mu) \quad \text{und} \quad e_\mu I = 0.$$

BEWEIS. Nach (XX) sind sowohl $e_\mu Re_\mu$ als auch $e_\mu Ie_\mu$ i.e. S.l.k. Ringe. Wegen $\bar{e}_\mu \bar{R} \bar{e}_\mu = (e_\mu Re_\mu)/(e_\mu Ie_\mu)$ (im algebraischen Sinne) und (XXI) ist $e_\mu Re_\mu$ primär. Gemäß (XX) ist $e_\mu Ie_\mu$ das Radikal von $e_\mu Re_\mu$. Da I s.l.k. ist, ist $e_\mu Ie_\mu$ auch s.l.k. Ist $e_\mu Ie_\mu \neq \neq (0)$, so ergibt sich aus Satz 3 daß $e_\mu Re_\mu$ streng linear kompakt ist. Nach Satz 4 ist dann $\bar{e}_\mu \bar{R} \bar{e}_\mu$ endlich, was ein Widerspruch ist. Folglich gilt $e_\mu Ie_\mu = (0)$. Da die Abbildung $e_\mu xe_\mu \to \bar{e}_\mu \bar{x} \bar{e}_\mu$ stetig und $\bar{e}_\mu \bar{R} \bar{e}_\mu$ i.e.S.l.k. ist, gilt die Isomorphie auch im topologischen Sinne. Folglich ist $R_\mu = e_\mu Re_\mu = \sum R_\mu^\alpha$, wobei die R_μ^α die minimalen Linksideale von R_μ sind.

Für jedes beliebige, aber feste Element $a \in I$, $a \neq 0$ bildet die Abbildung

$$\varphi : R_\mu^\alpha \to I, \quad \varphi(x) = xa, \quad x \in R_\mu^\alpha$$

den R_μ-Modul R_μ^α in den R_μ-Modul I ab. Es zeigt sich, daß φ ein stetiger R_μ-Homomorphismus ist. Weil R_μ^α minimal ist, so ist φ entweder trivial oder injektiv. Das letztere ist unmöglich, da R_μ^α nicht s.l.k. ist. Also ist $e_\mu Re_\mu I = e_\mu I = 0$.

§ 5. Das Hauptergebnis

Es sei **L** im folgenden die Klasse aller i.e.S.l.k. Ringe mit s.l.k. Radikal, für die es ein vollständiges System orthogonaler idempotenter Elemente $\{e_\mu\}$ mit der folgenden Eigenschaft gibt:

(W) Ist für einen Index μ $\bar{e}_\mu \bar{R} \bar{e}_\mu$ unendlich, wo die $\bar{e}_\mu \bar{R} \bar{e}_\mu$, wie oben, die topologisch einfachen Komponenten von R/I sind, so ist $Ie_\mu = 0$.

Jetzt formulieren wir das Hauptresultat:

SATZ 6. (I) *Ein Ring R ist genau dann ein* **L**-*Ring, wenn R eine ringtheoretische direkte Darstellung*

$$R = \sum_{\mu \in \Gamma} R_\mu \boxplus R^* \tag{1}$$

besitzt, wobei die R_μ unendlich und topologisch einfach, d.h. volle Endomorphismenringe von entweder Vektorräumen über unendlichen Schiefkörpern oder Vektorräumen unendlicher Dimension über endlichen Körpern sind, und R^* ein streng l.k. Ring ist.

(II) *Die Darstellung* (1) *ist eindeutig im folgenden Sinne: ist auch*

$$R = \sum_{\mu' \in \Gamma} R'_{\mu'} \boxplus R^{*'}$$

eine Darstellung, wobei die $R'_{\mu'}$ unendlich und topologisch einfach sind und $R^{*'}$ streng l.k. ist, so gibt es eine eineindeutige Abbildung der Indexmenge Γ auf Γ' $(\mu \to \mu')$ mit $R_\mu \cong R'_{\mu'}$ und es gilt $R^* \cong R^{*'}$.

BEWEIS. Besitzt R die Darstellung (1), dann ist R i.e.S.l.k. da jeder Summand derart ist. Das Radikal I von R liegt in R^* und ist nach (XV) abgeschlossen in R, also auch in R^*, somit selbst streng linear kompakt, insbesondere i.e.S.l.k. Wegen der Darstellung (1) ist es klar, daß es ein vollständiges System orthogonaler idempotenter Elemente mit der Eigenschaft (W) gibt. Folglich ist die Voraussetzung hinreichend.

Es sei jetzt R ein L-Ring, I sein Radikal. Nach (XVIII) und (XXI) ist R/I die komplette direkte Summe von topologisch einfachen Ringen,

$$R/J = \bar{R} = \sum_{\mu \in \Gamma^*} \bar{e}_\mu \bar{R} \bar{e}_\mu,$$

und $\bar{e} = \sum_{\mu \in \Gamma^*} \bar{e}_\mu$ ist das Einselement von $\bar{R}$.

Nach der Voraussetzung hat R ein vollständiges System orthogonaler idempotenter Elemente $\{e_\mu, \mu \in \Gamma^*\}$ mit der Eigenschaft (V). Wir bezeichnen mit Γ die Menge aller Indizes μ, für die $\bar{e}_\mu \bar{R} \bar{e}_\mu$ unendlich ist. Es sei dann $\Gamma' = \Gamma \setminus \Gamma^*$.

Es sei B das vollständige Urbild von $\sum_{\mu \in \Gamma'} \bar{e}_\mu \bar{R} \bar{e}_\mu$ beim natürlichen Homomorphismus von R auf R'. B ist offenbar abgeschlossen. Da I streng l.k. und die $\bar{e}_\mu \bar{R} \bar{e}_\mu$ für alle $\mu \in \Gamma'$ endlich sind, ist B streng l.k. .

Ist $\mu \in \Gamma$, so gilt $e_\mu R e_\mu \cong \bar{e}_\mu \bar{R} \bar{e}_\mu$ und $e_\mu I = I e_\mu = 0$ nach Satz 5 und der Voraussetzung. Wir zeigen jetzt, daß für jedes $\mu \in \Gamma$ $e_\mu R e_\mu$ ein Ideal von R ist. Seien nämlich x, y beliebige Elemente von R. Es gilt dann $\bar{e}_\mu \bar{x} \bar{e}_\mu \bar{y} = \bar{e}_\mu \bar{x} \bar{y} \bar{e}_\mu$, weil $\bar{e}_\mu$ im Zentrum von $\bar{R}$ liegt. Das besagt aber $e_\mu x e_\mu y - e_\mu x y e_\mu \in I$. Wegen $e_\mu I = 0$ folgt einerseits

$$e_\mu x e_\mu y = e_\mu x y e_\mu \in e_\mu R e_\mu .$$

Andererseits gilt $\bar{y} \bar{e}_\mu \bar{x} \bar{e}_\mu = \bar{e}_\mu \bar{y} \bar{x} \bar{e}_\mu$, woraus $y e_\mu x e_\mu - e_\mu y x e_\mu \in I$ und wegen $I e_\mu = 0$

$$y e_\mu x e_\mu = e_\mu y x e_\mu \in e_\mu R e_\mu$$

folgt. Also ist $e_\mu R e_\mu$ ein Ideal von R für jedes $\mu \in \Gamma$.

Wir beweisen noch, daß $e_\mu R e_\mu \cdot B = B e_\mu R e_\mu = 0$ für alle $\mu \in \Gamma$ ist. Es sei nämlich $x \in R$, $y \in B$ beliebig. Es gilt dann $\bar{e}_\mu \bar{x} \bar{e}_\mu \bar{y} = \bar{y} \bar{e}_\mu \bar{x} \bar{e}_\mu = 0$, was bedeutet, daß $e_\mu x e_\mu y$, $y e_\mu x e_\mu$ in I sind. Wegen $e_\mu I = I e_\mu = 0$ sind $e_\mu x e_\mu y = y e_\mu x e_\mu = 0$, also ist

$$e_\mu R e_\mu \cdot B = B \cdot e_\mu R e_\mu = 0.$$

Bezeichne A^* das von allen $e_\mu R e_\mu$ $(\mu \in \Gamma)$ erzeugte zweiseitige Ideal von R. Da für alle $\mu \neq \mu'$: $e_\mu e_{\mu'} = e_{\mu'} e_\mu = 0$ ist, ist A^* die ringtheoretische diskrete direkte Summe aller $e_\mu R e_\mu$ $(\mu \in \Gamma)$:

$$A^* = \sum_{\mu \in \Gamma}^{\oplus} e_\mu R e_\mu .$$

Es sei A der topologische Abschluß von A^* in R. Da $\{e_\mu, \mu \in \Gamma^*\}$ summierbar ist, ist auch $\{e_\mu, \mu \in \Gamma\}$ summierbar. e^* bezeichne die Summe von $\{e_\mu, \mu \in \Gamma\} e^* = \sum_{\mu \in \Gamma} e_\mu$. Wegen (XXVII) haben wir

$$e^* e_\mu = e_\mu e^* = e_\mu \quad \forall \mu \in \Gamma \quad \text{und} \quad e^* I = I e^* = 0.$$

$e^* R e^*$ ist trivialerweise i.e.S.l.k. mit dem Radikal $e^* I e^*$. Wegen $e^* I = I e^* = 0$ ist $e^* R e^*$ halbeinfach. Da $e^* e_\mu = e_\mu e^* = e_\mu$ für alle $\mu \in \Gamma$ gilt, enthält $e^* R e^*$ alle $e_\mu R e_\mu$ mit $\mu \in \Gamma$. $e^* R e^*$ ist natürlich abgeschlossen in R, also gilt $A \subseteq e^* R e^*$. Andererseits ist $e^* R e^* \subseteq A$ wegen $e^* = \sum_{\mu \in \Gamma} e_\mu$, was bedeutet, daß $A = e^* R e^*$ gilt. Aber es gilt offensichtlich $e^* R e^* = \sum_{\mu \in \Gamma} e_\mu R e_\mu$, was zeigt, daß A mit $\sum_{\mu \in \Gamma} e_\mu R e_\mu$ isomorph ist.

Der Satz wird bewiesen sein, wenn wir zeigen, daß

$$R = A \boxplus B$$

gilt.

A und B sind natürlich Ideale von R. Wegen $A^*B=BA^*=0$ folgt $AB=BA=0$ aus der Stetigkeit der Multiplikation.

Für ein beliebiges $x\in R$ gilt

$$x = e^*xe^* + (x - e^*xe^*),$$

wobei $e^*xe^*\in A$ ist und $x-e^*xe^*$ offensichtlich in B liegt.

Die Isomorphie ist topologisch. Es sei nämlich U_1 bzw. U_2 eine beliebige Umgebung von 0 in A bzw. B. Es ist klar, daß $U_1=U\cap A$ und $U_2=U'\cap B$ mit gewissen Umgebungen U, U' in R sind. Deshalb genügt es, sich auf den Fall $U_1=$ $=U_0\cap A$, $U_2=U_0\cap B$ $(U_0\subseteqq U\cap U')$ zu beschränken. Wählen wir dann die Umgebungen W, V_1, V_2, so daß die folgenden Bedingungen

$$e^*V_1e^*\subseteqq U_0, \quad V_2-e^*V_2e^*\subseteqq U_0, \quad W\subseteqq V_1\cap V_2$$

gelten. Für ein beliebiges Element $w\in W$ haben wir

$$e^*we^*\in e^*V_1e^*\cap A\subseteqq U_0\cap A=U_1$$

und

$$w-e^*we^*\in (V_2-e^*V_2e^*)\cap B\subseteqq U_0\cap B=U_2$$

Folglich gilt $w=e^*we^*+(w-e^*we^*)\in U_1+U_2$ für alle $w\in W$.

Der Beweis von (II) ist zu dem von (III) des Satzes 5 in [9] völlig analog, deshalb verzichten wir auf den Beweis.

Der Beweis des Satzes ist damit erbracht.

Nun geben wir ein Beispiel, das in wesentlichen von Widiger stammt und die Notwendigkeit der Bedingung (W) zeigt. Es sei K ein beliebiger Körper und n eine beliebige unendliche Mächtigkeit. Es sei B^* die Menge aller $n\times n$ Matrizen über K. Der Zeroring B über B^* ist ein topologischer Ring mit der Tychonoffschen Topologie. Da K endlich ist, ist B ein kompakter, linear topologischear Ring, also auch ein i.e.S.l.k. Ring. Es sei weiterhin A der Endomorphismenring eines Vektorraumes vom Rang n über K mit endlicher Topologie. Also ist A der Ring aller $n\times n$ Matrizen, die in jeder Zeilen nur endlich viele von Null verschiedene Elemente haben. Wir setzen $R=A\oplus B$ und machen R zu einem Ring durch die folgende Multiplikation: Für $a\in A$, $b\in B$ sei $ba\in B$ durch die gewöhnliche Matrizenmultiplikation (diese Produkte kann man bilden) und $ab=0$.

Es zeigt sich unmittelbar, daß die Addition und die Multiplikation stetig sind. R ist natürlich i.e.S.l.k. mit seinem Radikal B als Ring auch i.e.S.l.k. Bezeichne e die identische Matrix von A. Natürlich gilt $Be=B$ und dies erklärt, warum es für R keine Zerlegung (1) gibt.

§ 6. Folgerungen

SATZ 7. *Die Ringe, deren sämtliche abgeschlossenen Ideale* **L**-*Ringe sind, sind die im Hauptsatz genannten Ringe.*

Zum Beweis qraucht man nur zu erwähnen, daß jedes abgeschlossene Ideal eines Ringes im Hauptsatz wieder die Darstellung (1) hat.

SATZ 8. *Ein inverser Limes von* **L**-*Ringen ist wieder ein* **L**-*Ring.*

Der Beweis ist zu dem von (K – W) in § 3 völlig analog, deshalb übergehen wir ihn.

SATZ 9. *Die Ringe, deren sämtliche abgeschlossenen Linksideale* **L**-*Ringe sind, sind die Ringe des Haupsatzes, für die jeder Endomorphismenring R_μ ein unendlicher Schiefkörper ist.*

BEWEIS. Daß die Voraussetzung hinreichend ist, sieht man leicht ein.

Umgekehrt zeigen wir, daß es in der Darstellung (1) keine Endomorphismenringe der Vektorräume unendlicher Dimension über endlichen Körpern gibt. Es sei nämlich R ein Endomorphismenring eines Vektorraumes V unendlicher Dimension über dem endlichen Körper K. Es sei weiter $\{e_\mu\}$ seine Basis. Bezeichne e_i^j jenen Endomorphismus von V, für den $e_i^j(e_j)=e_i$ ist und für den $e_i^j(e_p)=0$ für jedes $j\neq p$ sind. Re_i^i ist ein abgeschlossenes Linksideal von R, das also als Ring l.k. sein müßte. Es sei weiterhin für jedes Paar (i, j) E_i^j der von e_i^j erzeugte K-Untermodul von R. E_i^j ist natürlich isomorph mit K. Weiter ist

$$A = \sum_{j\neq i} E_i^j$$

ein Linksideal von Re_i^i ist. A ist offenbar ein Zeroring und da Re_i^i diskret ist, ist A streng l.k. Folglich ist A endlich. Andererseits ist die Mächtigkeit von A unendlich, was ein Widerspruch ist.

Analog können wir zeigen, daß alle anderen Endomorphismenringe unendliche Schiefkörper sind.

Aus diesem Satz und Satz 8 bzw. Satz 9 in [9] folgt

SATZ 10. *Die Ringe, deren sämtliche abgeschlossenen Unterringe* **L**-*Ringe sind, sind die Ringe des obigen Satzes, bei denen sämtliche Schiefkörper absolut algebraische Körper mit Primcharakteristik sind.*

SATZ 11. *Die* **L**-*Ringe mit der Eigenschaft, daß jeder Filter von Restklassen nach Unterringen einen nicht leeren Durchschnitt hat, sind die Ringe des obigen Satzes, bei denen sämtliche Schiefkörper absolut algebraische Körper mit Primcharakteristik und mit Minimalbedingung für Unterkörper sind.*

Als weitere Folgerungen betrachten wir folgende zwei Fälle:

(i) Ist die Topologie in R diskret, so ist R ein linksartinscher Ring mit linksartinschem Radikal I. Aber jeder nilpotente artinsche Ring genügt der Minimalbedingung für Untergruppen, also ist I auch ein rechtsartinscher Ring. R/I ist natürlich rechtsartinsch. Folglich ist R ein rechtsartinscher Ring mit rechtsartinschem Radikal. Dies besagt wegen Satz 5, daß es in R ein vollständiges System orthogonaler idempotenter Elemente mit der Eigenschaft (W) gibt. So erhalten wir in diesem Fall nach Satz 6 die Ergebnisse von Kertész und Widiger.

(ii) Gibt es in R ein Basisfilter aus Idealen, so ist, wie man leicht einsieht, R als R-Rechtsmodul i.e.S.l.k. Wegen der streng linearen Kompaktheit von I ist er als I-Rechtsmodul i.e.S.l.k. Nach Satz 5 ist R ein **L**-Ring d.h. wir haben in diesem Fall die Ergebnisse von Widiger.

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A NOTE TO THE CONTINUOUS DERIVATION OF FIELDS

By

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Introduction

Let $\mathscr{M}$ be the operator field of MIKUISŃSKI [8]. In [6] the linear operator transformations on $\mathscr{M}$ are defined to be linear maps of $\mathscr{M}$ into $\mathscr{M}$.

An other operator field $\mathfrak{A}$, which is isomorphic to the field (under the pointwise operations) of all functions $f(z)$ meromorphic in some right half-planes of the complex plane, is constructed in [10]. The operator transformations on $\mathfrak{A}$ are defined analogously [11]. In $\mathfrak{A}$ every operator a can be written formally in the form $a=f(s)$, where $f(z)$ (depending on $a\in\mathfrak{A}$) is meromorphic in a certain right half-plane of the complex plane and s is the differential operator in $\mathfrak{A}$ (and also in $\mathscr{M}$).

If $\mathfrak{Q}$ is the subfield of $\mathscr{M}$, which consists of all operators represented by convolution quotients of Laplace transformable functions, then $\mathfrak{Q}\subset\mathscr{M}\cap\mathfrak{A}$ [10].

A particular linear operator transformation of $\mathscr{M}$ and $\mathfrak{A}$ is the algebraic derivation, which can be comprehended as the formal derivation of an operator $a=f(s)$ belonging to $\mathscr{M}$ or $\mathfrak{A}$ with respect to s.

In $\mathscr{M}$ the algebraic derivation is continuous in the sense of [6]. But there are also other derivations of $\mathscr{M}$ and $\mathfrak{A}$. From the standpoint of differential algebra (see [7]) a derivation of $\mathscr{M}$ (or $\mathfrak{A}$) is defined to be a linear operator transformation $\mathscr{T}$ which fulfils the condition

$$(1)\qquad \mathscr{T}(ab) = b\mathscr{T}(a)+a\mathscr{T}(b)$$

for all a, b belonging to $\mathscr{M}$ (or $\mathfrak{A}$). Then we have also

$$(1')\qquad \mathscr{T}\left(\frac{a}{b}\right) = \frac{b\mathscr{T}(a)-a\mathscr{T}(b)}{b^2} \quad (a, b\in\mathscr{M}\text{(or }\mathfrak{A}),\ b\neq 0).$$

In [4] it is proved that every derivative $\mathscr{T}$ of $\mathscr{M}$, which is continuous in the sense of [6], has the form

$$(2)\qquad \mathscr{T} = c\mathscr{D},$$

where $c\in\mathscr{M}$ is an operator depending on $\mathscr{T}$ and $\mathscr{D}$ is the algebraic derivation of $\mathscr{M}$.

It is possible to define the linear sequentially continuous operator transformations as follows [1, 10, 11]:

An operator transformation $\mathscr{T}$ of $\mathscr{M}$ (or $\mathfrak{A}$) is called sequentially continuous if $a_n\to a$ implies $\mathscr{T}(a_n)\to\mathscr{T}(a)$. The symbol $\to$ denotes any type of convergence in $\mathscr{M}$ (or $\mathfrak{A}$), for example the I- or II-type convergence in $\mathscr{M}$ [8] (or the convergence in $\mathfrak{A}$ introduced in [10]).

In the present note we will prove a theorem on the continuous derivations (in a certain sense) of suitable fields, such that we get all sequentially continuous derivations of $\mathscr{M}$ (or $\mathfrak{A}$) also in the form (2) as a special case.

Assumptions to the general theorem

(3) Assume that $\mathscr{F}$ is any field containing the field of the complex numbers and that $\to$ is the symbol for any convergence in $\mathscr{F}$ which is so that the operations in $\mathscr{F}$ are sequentially continuous operations, and that the limit of a sequence is unique.

Suppose that $\mathscr{D}$ is a derivation of $\mathscr{F}$ having the following properties:

(4) $\mathscr{D}(\alpha)=0$ if α is a complex number;

(5) there exists an element s in $\mathscr{F}$ with $\mathscr{D}(s)=1$;

(6) $\mathscr{D}$ is sequentially continuous.

Moreover we require

(7) there is a subalgebra $\mathscr{A}$ of $\mathscr{F}$ such that the set of all rational terms

$$r(s)=\frac{\alpha_0+\alpha_1 s+\ldots+\alpha_n s^n}{\beta_0+\beta_1 s+\ldots+\beta_m s^m} \quad (\alpha_i, \beta_j \text{ are complex numbers})$$

is dense in $\mathscr{A}$ and $\mathscr{A}$ is dense in $\mathscr{F}$ with respect to $\to$.

Then we have the following

THEOREM. *Let* $(\mathscr{F}, \mathscr{D})$ *be the differential field with the properties* (3)—(7). *A derivation* $\mathscr{T}$ *of* $\mathscr{F}$ *which satisfies* (4) — *that means* $\mathscr{T}(\alpha)=0$ *if* α *is a complex number* — *is sequentially continuous if and only if it has the form* $\mathscr{T}=c\mathscr{D}$, *where* c *is an element of* $\mathscr{F}$ *depending on* $\mathscr{T}$.

PROOF. It is easy to see that $\mathscr{T}=c\mathscr{D}$, $c\in\mathscr{F}$, is a sequentially continuous derivation of $\mathscr{F}$.

Now let $\mathscr{T}$ be any sequentially continuous derivation of $\mathscr{F}$ which fulfils (4). Assume that $s\in\mathscr{F}$ is the element of property (5). We obtain by (1) and (5) for $n=1, 2, \ldots$

$$\mathscr{T}(s^n)=ns^{n-1}\mathscr{T}(s)=c\mathscr{D}(s^n)$$

with $c=\mathscr{T}(s)\in\mathscr{F}$. If $r(s)$ is a rational term in s (property (7)) then

$$\mathscr{T}\big(r(s)\big)=c\mathscr{D}\big(r(s)\big) \tag{8}$$

since formula (1′) also holds in $\mathscr{F}$.

Now let a be any element of $\mathscr{A}$. (7) implies that there is a sequence $\big(r_n(s)\big)$ of rational terms in s with $r_n(s)\to a$. $\mathscr{T}$ is a sequentially continuous derivation, therefore we get $\mathscr{T}\big(r_n(s)\big)\to\mathscr{T}(a)$. On the other hand by virtue of (6) and (8), $\mathscr{T}\big(r_n(s)\big)=c\mathscr{D}\big(r_n(s)\big)\to c\mathscr{D}(a)$. Therefore we obtain

$$\mathscr{T}(a)=c\mathscr{D}(a) \tag{9}$$

for all $a\in\mathscr{A}$.

Let b be any element of $\mathscr{F}$. Because $\mathscr{A}$ is dense in $\mathscr{F}$ we can find a sequence $(a_n)\subset\mathscr{A}$ with $a_n\to b$. Therefore by the continuity of $\mathscr{T}$, $\mathscr{T}(a_n)\to\mathscr{T}(b)$ and as a consequence of (9), $\mathscr{T}(a_n)=c\mathscr{D}(a_n)\to c\mathscr{D}(b)$ holds with $\mathscr{T}(b)=c\mathscr{D}(b)$ for all $b\in\mathscr{F}$. The proof is complete.

Examples

(i) Let $\mathscr{F}$ be the operator field $\mathfrak{A}$ provided with the following convergence $(\to)$ [10]: A sequence $(a_n)\subset\mathfrak{A}$ converges to $a\in\mathfrak{A}$ if there are quotients $a_n=$ $=h_n(s)/g_n(s)$ and $a=h(s)/g(s)$ (in this case s is the differential operator in $\mathfrak{A}$) such that the functions $h_n(z)$, $g_n(z)$, $h(z)$, $g(z)$ are holomorphic in a fixed right half-plane Δ of the complex z-plane and the sequences $(h_n(z))$ and $(g_n(z))$ uniformly converge to $h(z)$ and $g(z)$ on every compact subdomain of Δ, respectively.

If $\mathscr{D}$ is the algebraic derivation of $\mathfrak{A}$ and $\mathscr{A}=\mathfrak{A}$ then all suppositions of our theorem are performed [10, 11].

(ii) Suppose that $\mathscr{F}=\mathscr{M}$ is the convolution quotient field of $\mathscr{C}$ ($\mathscr{C}$ is the set of all continuous functions on $[0, \infty)$). $\mathscr{M}$ is the Mikusiński's operator field [8]. The operator field is provided with the II-type convergence. In this case the suppositions of our theorem are again satisfied, therefore all sequentially continuouos derivations of $\mathscr{M}$ (fulfilling (4)) have also the form (2) if $\mathscr{A}=\mathscr{M}$ and $\mathscr{D}$ is the algebraic derivation of $\mathscr{M}$ [1, 6, 8].

We should prove that property (7) holds for $\mathscr{M}$. Now we shall approximate every $\varphi\in\mathscr{C}$ by rational terms in s. Let $I_n=[0, n]$ $(n=1, 2, \dots)$ a sequence of intervals. In I_n the function φ can be uniformly approximated by polynomials $p_k^n(t)$ for $k\to\infty$. That means we can estimate as follows:

$$|p_k^n(t)-\varphi(t)|<\frac{1}{n} \tag{10}$$

for all $t\in I_n$ and $k\geqq k_0(n)$, where $k_0(n)$ depends only on n. Suppose that $p_n(t)=p_{k_0(n)}^n(t)$ $(n=1, 2, \dots)$. Let I be any compact interval in $[0, \infty)$ and assume that $\varepsilon>0$. We can find a natural number n_0 such that $I\subset[0, n_0]$ and $1<\varepsilon\cdot n_0$. In this case for all $n\geqq n_0$

$$|p_n(t)-\varphi(t)| = |p_{k_0(n)}^n-\varphi(t)|<\frac{1}{n}<\frac{1}{n_0}<\varepsilon \tag{11}$$

whenever $t\in I\subset I_n$. This proves that $p_n(t)\to\varphi(t)$ almost unifromly on $[0, \infty)$.

Since every polynomial $p_n(t)$ can be represented by a rational term $r_n(s)$ in the differential operator s and every operator x from $\mathscr{M}$ has the form $\frac{a}{b}$ where a, b are in $\mathscr{C}$, one could approximate a and b by polynomials $p_n(t)$ and $q_n(t)$, respectively. Hence (7) is fulfilled.

(iii) Assume that $\mathscr{F}=\mathscr{M}$ (as in (ii)) and now $\mathscr{M}$ is provided with the I-type convergence. Let $\mathscr{T}$ be a sequentially continuous derivation. We prove that $\mathscr{T}$ is a II-type sequentially continuous derivation, too. Let x_n be a sequence in $\mathscr{M}$ converging to $x\in\mathscr{M}$ with respect to the II-type convergence. Then

$$\mathscr{T}(x_n)=\mathscr{T}\left(\frac{a_n}{b_n}\right)=\frac{b_n\mathscr{T}(a_n)-a_n\mathscr{T}(b_n)}{b_n^2}=\frac{p_n}{q_n} \tag{12}$$

and by the assumption a_n tends to a and b_n converges to b in the topology of $\mathscr{C}$. (More precisely, a_n and b_n can be chosen for this purpose.) But $\mathscr{T}(a_n)\to\mathscr{T}(a)$,

$\mathcal{T}(b_n)\to\mathcal{T}(b)$ and $b_n^2\to b^2$ with respect to the I-type convergence. Therefore $\mathcal{T}(x_n)$ tends to some operator

$$\frac{p}{q}=\frac{b\mathcal{T}(a)-a\mathcal{T}(b)}{b^2}=\mathcal{T}\left(\frac{a}{b}\right)=\mathcal{T}(x) \tag{13}$$

in $\mathcal{M}$ with respect to the II-type convergence. By virtue of (ii) we obtain *that every I-type sequentially continuous derivation*[1] *of* $\mathcal{M}$ *has the form* $\mathcal{T}=a\mathcal{D}$.

Remark 1. In [4], Fényes proved that every continuous derivation (in the sense of Gesztelyi [6]) has this form. His result follows from (iii) since every continuous linear operator transformation (a derivation satisfying (4) is obviously linear) is sequentially continuous too [6].

Remark 2. In the example (iii) we cannot use our theorem directly since the rational terms in s are not dense in $\mathcal{M}$ with respect to the I-type convergence. In order to prove this fact we will follow Norris [9] by defining a finite number α for each $a\neq 0$ in $\mathcal{M}$, as the unique real number α such that $a=\frac{f}{g}e^{-\alpha s}$, where f and g belong to $\mathcal{C}_0$. This number α is independent of the particular representation of a which is chosen, and if $a=\frac{y}{x}$, where $y, x\in\mathcal{M}$ then $\alpha(a)=\alpha(y)-\alpha(x)$. The number $\alpha(a)$ is called the support number of a. The second fact we need is that any rational term $r_n(s)$ has zero support number. This is obvious since $r_n(s)=p_n(s)/q_n(s)$, where $p_n(s)$ and $q_n(s)$ are polynomials in s, and therefore $\alpha(r_n(s))=\alpha(p_n(s))-\alpha(q_n(s))=0$. In [2] (see also [3]) it is proved the following

Lemma. *If* $a_n\xrightarrow{I}a\in\mathcal{M}$, $a\neq 0$, *then* $\overline{\lim}\,\alpha(a_n)\leqq\alpha(a)$.

In our case let $a=e^{\lambda s}f/g$ $(f, g\in\mathcal{C}_0,\ 0<\lambda)$. Hence by the lemma we should have $0\leqq-\lambda$ (under the supposition $r_n(s)\xrightarrow{I}a$) which leads obviously to a contradiction.

Remark 3. If $\mathcal{M}_0$ stands for the subalgebra of all operators of the form p/q where $q\in\mathcal{C}_0$ ($\mathcal{C}_0$ is the set of all functions defined on $[0,\infty)$, which are continuous and do not vanish identically in any neighbourhood of the origin) then we could prove the same result for $\mathcal{M}_0$ and $\mathcal{D}$ as in the Theorem by using the same arguments as in the proof of that. We would observe that $\mathcal{C}$ is dense in $\mathcal{M}_0$ ([3], [5]) and $\mathcal{D}:\mathcal{M}_0\to\mathcal{M}_0$. But $\mathcal{M}_0$ is an integral domain and its quotient field is $\mathcal{M}$. From $\mathcal{M}_0$ a derivation can be extended into $\mathcal{M}$ uniquely. The continuity of the extension is trivial. We should note that this remark cannot be handled in all cases since a derivation of $\mathcal{M}$ is not necessarily a derivation of $\mathcal{M}_0$ into $\mathcal{M}_0$.

[1] We assume $\mathcal{T}(\alpha)=0$ if α is a complex number.

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ON AN INTERPOLATION PROCESS OF S. N. BERNSTEIN

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Introduction. S. N. BERNSTEIN [1] considered the following interpolation process based on the zeros of Tchebycheff polynomials of the first kind. (For the sake of brevity let us fix n and denote x_{kn} by x_k, $l_{kn}(x)$ by $l_k(x)$ etc.) It is given by

$$R_n[f, x] = \sum_{k=1}^{n} f(x_k)\,\varphi_k(x) \tag{1.1}$$

where

$$x_k = \cos\frac{(2k-1)\pi}{2n}, \quad k = 1, 2, \dots, n, \;\; T_n(x) = \cos n\theta, \cos\theta = x,$$

$$\varphi_1(x) = \frac{3l_1(x)+l_2(x)}{4}, \quad \varphi_n(x) = \frac{l_{n-1}(x)+3l_n(x)}{4},$$

$$\varphi_k(x) = \frac{l_{k-1}(x)+2l_k(x)+l_{k+1}(x)}{4}, \quad k = 2, 3, \dots, n-1,$$

$$l_k(x) = \frac{(-1)^{k+1}(1-x_k^2)^{1/2}}{n}\,\frac{T_n(x)}{x-x_k}, \quad k = 1, 2, \dots, n.$$

Concerning $R_n[f, x]$ he proved the following

THEOREM 1. *Let $f(x)$ be an arbitrary continuous function in $[-1, +1]$, then $R_n[f, x]$ converges uniformly to $f(x)$ on $[-1, +1]$.*

Recently we [5] proved the following

THEOREM 2. *Let $f(x)$ be an arbitrary continuous function in $[-1, +1]$. Then there exists a positive constant c independent of n, x, and f such that*

$$|R_n[f, x]-f(x)| \leqq c\left[\omega\left(\frac{\sqrt{1-x^2}}{n}\right)+\omega\left(\frac{1}{n^2}\right)\right], \tag{1.2}$$

where $\omega(\delta)$ is the modulus of continuity of f.

It is natural to raise the following question. Suppose we change the nodes of interpolation to the zeros of Tchebycheff polynomials of the second kind:

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta, \tag{1.3}$$

and ask for what class of functions the interpolation process of S. N. Bernstein with respect to the nodes (1.3) converge uniformly to f? The above question is of some interest since for many well known interpolation processes (this includes Lagrange as well as Hermite—Fejér interpolation, see G. SZEGŐ [4]) convergence behaviour is very poor with respect to the nodes (1.3) especially near 1 or -1. The situation is not improved for the Lagrange interpolation on these nodes even in problems of mean convergence. E. FELDHEIM [2] proved that for the same abscissas it is not true that

$$\lim_{n\to\infty} \int_{-1}^{1} |f(x)-L_n(x)|^2\,dx = 0.$$

In fact the superior limit of the integrals in question may be $+\infty$ if $f(x)$ is a properly chosen continuous function.

2. Statement of the main theorem. Let us denote

$$A_n[f, x] = \sum_{k=1}^{n} f(t_k)\, m_k(x), \tag{2.1}$$

and

$$B_n[f, x] = \sum_{k=1}^{n} f(t_k)\, P_k(x), \tag{2.2}$$

where

$$t_k = \cos\theta_k, \quad \theta_k = \frac{k\pi}{n+1}, \quad k = 1, 2, \ldots, n, \tag{2.3}$$

$$\mu_k(x) = \frac{(-1)^{k+1}(1-t_k^2)}{n+1}\,\frac{U_n(x)}{x-t_k}, \quad k = 1, 2, \ldots, n, \tag{2.4}$$

$$m_1(x) = \frac{3\mu_1(x)+\mu_2(x)}{4}, \quad m_n(x) = \frac{\mu_{n-1}(x)+3\mu_n(x)}{4}, \tag{2.5}$$

$$m_k(x) = \frac{\mu_{k-1}(x)+2\mu_k(x)+\mu_{k+1}(x)}{4}, \quad k = 2, 3, \ldots, n-1, \tag{2.6}$$

$$\begin{cases} P_1(x) = m_1(x)+\dfrac{1}{2}\, m_2(x) \\ P_k(x) = \dfrac{1}{2}\left(m_k(x)+m_{k+1}(x)\right), \quad k = 2, 3, \ldots, n-2, \\ P_{n-1}(x) = \dfrac{1}{2}\, m_{n-1}(x), \quad P_n(x) = m_n(x). \end{cases} \tag{2.7}$$

We now prove the following

THEOREM 3. *Let $f(x)$ be an arbitrary continuous function on $[-1, +1]$ and $\omega(\delta)$ its modulus of continuity. Then there exist positive constants c_1 and c_2 independent of n, x and f such that*

$$|A_n[f, x]-f(x)| \leqq c_1\omega\left(\frac{1}{n}\right) \tag{2.9}$$

and

$$(2.10)\qquad |B_n[f,x]-f(x)| \leqq c_2\left[\omega\left(\frac{\sqrt{1-x^2}}{n}\right)+\omega\left(\frac{1}{n^2}\right)\right].$$

3. Proof of the theorem. The proof of Theorem 3 is based on the following

LEMMA 3.1. *The following estimates are valid in* $-1 \leqq x \leqq +1$:

$$(3.1)\qquad \sum_{k=1}^{n} |m_k(x)| \leqq c_3, \quad \sum_{k=1}^{n} |P_k(x)| \leqq c_4,$$

$$(3.2)\qquad \sum_{k=1}^{n} |x-t_k|\,|m_k(x)| \leqq \frac{c_5}{n},$$

$$(3.3)\qquad \sum_{k=1}^{n} \left|\sin\frac{\theta-\theta_k}{2}\right| |P_k(x)| \leqq \frac{c_6}{n},$$

$$(3.4)\qquad \sum_{k=1}^{n} \left|\sin^2\frac{\theta-\theta_k}{2}\right| |P_k(x)| \leqq \frac{c_7}{n^2}.$$

The proof of this lemma is given in Section 4. It is easy to see that

$$(3.5)\qquad \sum_{k=1}^{n} m_k(x) = \sum_{k=1}^{n} P_k(x) = 1.$$

From (2.1) and (3.5) we have

$$(3.6)\qquad A_n[f,x]-f(x) = \sum_{k=1}^{n} [f(t_k)-f(x)]\,m_k(x).$$

On using well known properties of modulus of continuity of f we obtain

$$(3.7.\qquad |f(t_k)-f(x)| \leqq \omega(f;|x-t_k|) \leqq \omega\left(f;\frac{n|x-t_k|}{n}\right) \leqq (n|x-t_k|+1)\,\omega\left(f;\frac{1}{n}\right).$$

On using (3.6), (3.7), (3.1) and (3.2) we have

$$|A_n[f,x]-f(x)| \leqq \omega\left(f,\frac{1}{n}\right)\left[\sum_{k=1}^{n}(n|x-t_k|+1)\,|m_k(x)|\right] \leqq (c_3+c_5)\,\omega\left(f,\frac{1}{n}\right).$$

This proves (2.9). To prove (2.10) we need the following inequality due to O. KIS and P. VÉRTESI [6]:

$$(3.8)\qquad |f(\cos\theta)-f(\cos\theta_k)| \leqq \left(2n\left|\sin\frac{\theta-\theta_k}{2}\right|+1\right)\omega\left(f;\frac{\sqrt{1-x^2}}{n}\right)+$$

$$+\left[2\left\{n\sin\frac{\theta-\theta_k}{2}\right\}^2+1\right]\omega\left(f;\frac{|x|}{n^2}\right).$$

From (2.2) and (3.5) we obtain

$$B_n[f,x]-f(x) = \sum_{k=1}^{n} [f(t_k)-f(x)]\,P_k(x).$$

By using (3.8), (3.1), (3.3) and (3.4) we obtain

$$|B_n[f, x]-f(x)| \leqq (c_4+2c_6)\omega\left(f; \frac{\sqrt{1-x^2}}{n}\right)+(2c_7+c_4)\omega\left(f; \frac{|x|}{n^2}\right).$$

This proves (2.1) as well.

4. Proof of Lemma 3.1. Since $1-t_k^2=1-x^2+x^2-t_k^2$ it follows from (2.4) that

$$\mu_k(x) = \frac{(-1)^{k+1}(1-x^2)U_n(x)}{(n+1)(x-t_k)}+\frac{(-1)^{k+1}(x+t_k)U_n(x)}{(n+1)}. \tag{4.1}$$

But it is easy to verify that $(x=\cos\theta)$

$$\frac{2\sin\theta}{t_k-x} = \cot\frac{\theta+\theta_k}{2}+\cot\frac{\theta-\theta_k}{2}, \quad k=1, 2, \ldots, n.$$

Hence

$$\mu_k(x) = \frac{(-1)^k\sin(n+1)\theta}{2(n+1)}\left[\cot\frac{\theta+\theta_k}{2}+\cot\frac{\theta-\theta_k}{2}\right]+\frac{(-1)^{k+1}(x+t_k)U_n(x)}{n+1}. \tag{4.2}$$

On using (4.2) and (2.6) we obtain for $k=2, 3, \ldots, n-1$

$$m_k(x) = \frac{(-1)^{k-1}\sin(n+1)\theta}{2(n+1)}\left[\cot\frac{\theta+\theta_{k-1}}{2}-2\cot\frac{\theta+\theta_k}{2}+\right. \tag{4.3}$$

$$\left.+\cot\frac{\theta+\theta_{k+1}}{2}+\cot\frac{\theta-\theta_{k-1}}{2}-2\cot\frac{\theta-\theta_k}{2}+\cot\frac{\theta-\theta_{k+1}}{2}\right]+$$

$$+\frac{(-1)^kU_n(x)}{n+1}[\cos\theta_{k-1}-2\cos\theta_k+\cos\theta_{k+1}].$$

On using the trigonometric formula

$$\cos\theta_{k-1}-2\cos\theta_k+\cos\theta_{k+1} = -4\cos\theta_k\sin^2\frac{\pi}{2(n+1)}, \tag{4.4}$$

and

$$\cot(\alpha+h)-2\cot\alpha+\cot(\alpha-h) = \frac{2\sin^2 h\cot\alpha}{\sin(\alpha-h)\sin(\alpha+h)} \tag{4.5}$$

for $\alpha=\frac{\theta\pm\theta_k}{2}$ and $h=\frac{\pi}{2(n+1)}$, we can rewrite (4.3) for $k=2, 3, \ldots, n-1$:

$$m_k(x) = \frac{(-1)^{k-1}\sin^2\frac{\pi}{2(n+1)}\sin(n+1)\theta}{n+1}\left[\frac{\cot\frac{\theta+\theta_k}{2}}{\sin\frac{\theta+\theta_{k-1}}{2}\sin\frac{\theta+\theta_{k+1}}{2}}+\right. \tag{4.6}$$

$$\left.+\frac{\cot\frac{\theta-\theta_k}{2}}{\sin\frac{\theta-\theta_{k-1}}{2}\sin\frac{\theta-\theta_{k+1}}{2}}\right]+\frac{4(-1)^{k-1}\cos\theta_k\sin^2\frac{\pi}{2(n+1)}U_n(x)}{n+1}.$$

On using (4.4) and

$$\cos\theta_k - 2\cos\theta_{k+1} + \cos\theta_{k+2} = -4\cos\theta_{k+1}\sin^2\frac{\pi}{2(n+1)}.$$

we obtain

$$\cos\theta_{k-1} - 3\cos\theta_k + 2\cos\theta_{k+1} - \cos\theta_{k+2} = -8\sin^3\frac{\pi}{2(n+1)}\sin\frac{(2k+1)\pi}{2(n+1)}. \tag{4.7}$$

Similarly, on using (4.5) and

$$\cot(\alpha+2h) - 2\cot(\alpha+h) + \cot\alpha = \frac{2\sin^2 h\cot(\alpha+h)}{\sin\alpha\sin(\alpha+h)} \tag{4.8}$$

we obtain

$$\cot(\alpha-h) - 3\cot\alpha + 3\cot(\alpha+h) - \cot(\alpha+2h) = \tag{4.9}$$

$$= \frac{2\sin^2 h}{\sin\alpha\sin(\alpha+h)}\left[\frac{\cos\alpha}{\sin(\alpha-h)} - \frac{\cos(\alpha+h)}{\sin(\alpha+2h)}\right] =$$

$$= \frac{\sin^2 h\,[\sin(2\alpha+2h) + \sin 2h + \sin 2h - \sin 2\alpha]}{\sin\alpha\sin(\alpha-h)\sin(\alpha+h)\sin(\alpha+2h)} =$$

$$= \frac{2\sin^3 h\,[\cos(2\alpha+h) + 2\cos h]}{\sin\alpha\sin(\alpha-h)\sin(\alpha+h)\sin(\alpha+2h)}.$$

On using (4.9) for $\alpha = \dfrac{\theta \pm \theta_k}{2}$ and $h = \pm\dfrac{\pi}{2(n+1)}$ we obtain

$$\cot\frac{\theta+\theta_{k-1}}{2} - 3\cot\frac{\theta+\theta_k}{2} + 3\cot\frac{\theta+\theta_{k+1}}{2} - \cot\frac{\theta+\theta_{k+2}}{2} = \tag{4.10}$$

$$= \frac{2\sin^3\dfrac{\pi}{2(n+1)}\left[\cos\left(\theta+\theta_k+\dfrac{\pi}{2(n+1)}\right) + 2\cos\dfrac{\pi}{2(n+1)}\right]}{\sin\dfrac{\theta+\theta_k}{2}\sin\dfrac{\theta+\theta_{k-1}}{2}\sin\dfrac{\theta+\theta_{k+1}}{2}\sin\dfrac{\theta+\theta_{k+2}}{2}}$$

and

$$\cot\frac{\theta-\theta_{k-1}}{2} - 3\cot\frac{\theta-\theta_k}{2} + 3\cot\frac{\theta-\theta_{k+1}}{2} - \cot\frac{\theta-\theta_{k+2}}{2} = \tag{4.11}$$

$$= -\frac{2\sin^3\dfrac{\pi}{2(n+1)}\left[\cos\left(\theta-\theta_k-\dfrac{\pi}{2(n+1)}\right) + 2\cos\dfrac{\pi}{2(n+1)}\right]}{\sin\dfrac{\theta-\theta_{k-1}}{2}\sin\dfrac{\theta-\theta_k}{2}\sin\dfrac{\theta-\theta_{k+1}}{2}\sin\dfrac{\theta-\theta_{k+2}}{2}}.$$

On using (4.6), (4.10), (4.11) and (4.7) we obtain $(k=2, 3, \dots, n-2)$

$$(4.12)\qquad m_k(x)+m_{k+1}(x) = \frac{8(-1)^{k-1}\sin^3\frac{\pi}{2(n+1)}\sin\frac{(2k+1)\pi}{2(n+1)}U_n(x)}{n+1}+$$

$$+\frac{(-1)^{k-1}\sin^3\frac{\pi}{2(n+1)}\sin(n+1)\theta}{n+1}\cdot$$

$$\cdot\left[\frac{\cos\left(\theta+\theta_k+\frac{\pi}{2(n+1)}\right)+2\cos\frac{\pi}{2(n+1)}}{\prod_{j=k-1}^{k+2}\sin\frac{\theta+\theta_j}{2}}-\frac{\cos\left(\theta-\theta_k-\frac{\pi}{2(n+1)}\right)+2\cos\frac{\pi}{2(n+1)}}{\prod_{j=k-1}^{k+2}\sin\frac{\theta-\theta_j}{2}}\right],$$

$$(4.13)\qquad m_1(x) = \frac{3\mu_1(x)+\mu_2(x)}{4} =$$

$$= \frac{U_n(x)\sin^2\theta_1\left[3(1+x)\sin^2\theta_1-\cos^2\theta_1\left(6\sin^2\frac{\theta_1}{2}+x-t_1\right)\right]}{4(n+1)(x-t_1)(x-t_2)}.$$

The following results are needed for the proof of our Lemma 3.1.

$$(4.14)\qquad |U_n(x)| \leqq n+1, \quad -1 \leqq x \leqq +1,$$

$$(4.15)\qquad \sin\frac{\pi}{2(n+1)} \leqq \frac{\pi}{2(n+1)},$$

$$(4.16)\qquad \left|\sin\frac{\theta+\theta_k}{2}\right| \geqq \left|\sin\frac{\theta-\theta_k}{2}\right|,$$

$$(4.17)\qquad |x-t_k| \leqq 2\left|\sin\frac{\theta+\theta_k}{2}\sin\frac{\theta-\theta_k}{2}\right|,$$

$$(4.18)\qquad \sum_{k=2}^{n-1}\frac{1}{\left|\sin\frac{\theta-\theta_{k-1}}{2}\sin\frac{\theta-\theta_{k+1}}{2}\right|} \leqq c_8 n^2,$$

$$(4.19)\qquad \sum_{k=2}^{n-2}\frac{1}{\left|\sin\frac{\theta-\theta_{k-1}}{2}\sin\frac{\theta-\theta_{k+1}}{2}\sin\frac{\theta-\theta_{k+2}}{2}\right|} \leqq c_9 n^3,$$

$$(4.20)\qquad |\mu_k(x) \leqq 2, \quad -1 \leqq x \leqq +1.$$

On using (4.6)—(4.20), Lemma (3.1) follows at once.

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ON SET SYSTEMS HAVING PARADOXICAL COVERING PROPERTIES

By

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1. $\aleph_2$-phenomena. Our set theoretic notation will be standard with one exception. Since this paper is largely concerned with powers of ordinals, the symbol ξ^η will always denote ordinal exponentiation for ordinals ξ, η. Thus, in particular, if $\beta \leqq \alpha$, then $\omega_\alpha^{\omega_\beta}$ is an ordinal $<\omega_{\alpha+1}$. When we use cardinal exponentiation we shall either say so or, if there is no danger of confusion, we write $2^{\aleph_\beta}$ or $\aleph_\alpha^{\aleph_\beta}$ (despite the fact that ω_α and $\aleph_\alpha$ otherwise denote the same object). We shall assume the reader is familiar with the special symbols as defined e.g. in [6] to denote ordinary partition relations, polarized partition relations and square bracket relations.

We begin our discussion by recalling a theorem of MILNER and RADO [13] which asserts that, for any cardinal $\varkappa \geqq \omega$,

$$\xi \nrightarrow (\varkappa^n)^1_{n<\omega} \quad \text{if} \quad \xi < \varkappa^+. \tag{1.1}$$

This implies that $\xi(<\varkappa^+)$ is the union of ω "small" sets A_n $(n<\omega)$, where we mean small in the sense that the order type $\operatorname{tp} A_n < \varkappa^n$ $(n<\omega)$. Four our present purposes it is usually more convenient to consider another sequence $B=\langle B_n : n<\omega\rangle$ defined by $B_n = A_0 \cup \ldots \cup A_n$ $(n<\omega)$. The sets B_n are still "small", i.e. $\operatorname{tp} B_n < \varkappa^n$ $(n<\omega)$, and they have an additional property, which we call the *ω-covering property*, that the union of any ω of these sets is the whole set ξ. For brevity we shall say that a sequence $B=\langle B_n : n<\omega\rangle$ of subsets of ξ is a *paradoxical decomposition* of ξ if it has the two properties (i) $\operatorname{tp} B_n < \varkappa^n$ $(n<\omega)$ and (ii) the ω-covering property. The existence of such a paradoxical decomposition (which is only interesting for $\varkappa^\omega \leqq \xi < \varkappa^+$) implies the polarized partition relation

$$\binom{\omega}{\xi} \nrightarrow \begin{pmatrix} 1 & \omega \\ \varkappa^\omega & 1 \end{pmatrix}^{1,1} \quad \text{for} \quad \xi < \varkappa^+, \tag{1.2}$$

and also the square bracket relation

$$\xi \nrightarrow [\varkappa^\omega]^1_{\aleph_0, <\aleph_0} \quad \text{for} \quad \xi < \varkappa^+. \tag{1.3}$$

In our paper [7] we investigated the following problem: Let $\eta < \omega_2$ and let $A = \langle A_\alpha : \alpha < \varkappa\rangle$ be a sequence of subsets of η of length $\varkappa = \omega$ or ω_1 such that each set A_α has order type $\operatorname{tp} A_\alpha < \sigma$. Under what conditions can we then assert that there is a subsequence $\langle A_{\alpha_\nu} : \nu < \varrho\rangle$ of length ϱ whose union has a "large" complement in η, say $\operatorname{tp}(\eta \setminus \bigcup\{A_{\alpha_\nu} : \nu < \varrho\}) \geqq \tau$? This amounts to an investigation of the polarized partition relation

$$(1.4)\qquad \binom{\varkappa}{\eta} \to \begin{pmatrix} 1 & \varrho \\ \sigma & \tau \end{pmatrix}^{1,1}$$

for $\eta<\omega_2$ and $\varkappa=\omega$ or ω_1.

In [7] we gave a complete discussion for (1.4) in the case when η is a power of ω, (although even for this case there remain unresolved questions if the "1" in (1.4) is replaced by a larger finite ordinal). Now in combinatorial set theory most theorems like these have higher cardinal analogues which are usually obtained by replacing each cardinal by its successor. However, when writing [7] we realized that an investigation of (1.4) for the "next higher case", i.e. for $\eta<\omega_3$ and $\varkappa=\omega_1$ or ω_2, leads to entirely different results and problems which we refer to as "$\aleph_2$-phenomena". The main reason why we could not simply extend the results of [7] is that one of the principal tools we used there was the Milner—Rado paradoxical decomposition (1.1) or rather its square bracket analogue (1.3),

$$\xi \nrightarrow [\omega_1^{\omega}]^1_{\aleph_0, <\aleph_0} \quad \text{for} \quad \xi<\omega_2.$$

Now the "higher cardinal" analogue of this is

$$(1.5)\qquad \xi \nrightarrow [\omega_2^{\omega_1}]^1_{\aleph_1, \aleph_0} \quad \text{for} \quad \xi<\omega_3,$$

and this is not true (e.g. it is false if we assume $2^{\aleph_1}=\aleph_2$). We summarize here the $\aleph_2$-phenomena as it relates to the relation (1.5). For $\xi<\omega_2^{\omega_2}$ we do get the expected result, i.e.

$$(1.6)\qquad \xi \nrightarrow [\omega_2^{\omega_1}]^1_{\aleph_1, \aleph_0} \quad \text{for} \quad \xi<\omega_2^{\omega_2}.$$

However, we also have the following.

(1.7) (a) *If* $2^{\aleph_1}=\aleph_2$, *then there is some* $\xi<\omega_3$ *such that*

$$\xi \to [\omega_2^{\omega_1}]^1_{\aleph_1, \aleph_0}.$$

(b) *It is consistent that*

$$2^{\aleph_1}=\aleph_3 \quad \textit{and} \quad \xi \nrightarrow [\omega_2^{\omega_1}]^1_{\aleph_1, \aleph_0}$$

holds for all $\xi<\omega_3$.

(1.8) *Both the relations*

$$\omega_2^{\omega_2} \nrightarrow [\omega_2^{\omega_1}]^1_{\aleph_1, \aleph_0} \quad \textit{and} \quad \omega_2^{\omega_2} \to [\omega_2^{\omega_1}]^1_{\aleph_1, \aleph_0}$$

are true in different models of set theory. (The relation $\nrightarrow$ holds, e.g. in the constructible universe L, and $\to$ holds e.g. if Chang's conjecture is true.)

These "$\aleph_2$-phenomena" enter into almost all the results and problems considered in this paper, and so it is not possible to give an entirely naive presentation. Although we discovered most of these results as early as 1967, the presentation we give here will rely upon more recent work done by others. In particular we will use the methods worked out in the paper by GALVIN and HAJNAL [9], and we shall give references to other results later.

The remainder of this section will be devoted to a detailed description of the $\aleph_2$-phenomena as it relates to the relation

$$P(\gamma):\begin{pmatrix}\omega_1\\ \omega_2^\gamma\end{pmatrix}\to\begin{pmatrix}1 & \omega\\ \omega_2^{\omega_1} & 1\end{pmatrix}^{1,1}$$

for $\gamma<\omega_3$. For the sake of clarity this will be done rather slowly and somewhat redundantly.

Clearly $P(\gamma)$ is equivalent to the assertion: *whenever* $A=\langle A_\alpha:\alpha<\omega_1\rangle$ *is a sequence of subsets of* ω_2^γ *such that* $\operatorname{tp} A_\alpha<\omega_\omega^{\omega_1}\,(\alpha<\omega_1)$, *then A does not have the ω-covering property, i.e. there is* $D\in[\omega_1]^\omega$ *such that* $\bigcup\{A_\alpha:\alpha\in D\}\neq\omega_2^\gamma$. On the other hand, in order to establish the negation

$$\neg P(\gamma):\begin{pmatrix}\omega_1\\ \omega_2^\gamma\end{pmatrix}\not\to\begin{pmatrix}1 & \omega\\ \omega_2^{\omega_1} & 1\end{pmatrix}^{1,1},$$

we have to show that there is some sequence $A=\langle A_\alpha:\alpha<\omega_1\rangle$ of subsets of ω_2^γ such that (i) $\operatorname{tp} A_\alpha<\omega_2^{\omega_1}$ $(\alpha<\omega_1)$ and (ii) *A has the ω-covering property*. We shall say briefly that the sequence A *establishes* $\neg P(\gamma)$ if (i) and (ii) hold.

Before we state and prove our first relevant result, it is convenient to introduce some special notation. If $\varkappa$ is an infinite cardinal and $0<\gamma<\varkappa^+$ we choose a fixed sequence $S^\gamma=\langle S_\nu^\gamma:\nu<\mu\rangle$ of subsets of $\varkappa^\gamma$ having the following properties:

(1.9) $$\varkappa^\gamma=\bigcup\{S_\nu^\gamma:\nu<\mu\};$$

(1.10) $$S_0^\gamma<S_1^\gamma<\ldots<S_\nu^\gamma<\ldots,$$

where $X<Y$ means that all the elements of the set X precede all the elements of Y in the ordering of $\varkappa^\gamma$;

(1.11) (a) if $\gamma=\delta+1$, then $\mu=\varkappa$ and $\operatorname{tp} S_\nu^\gamma=\varkappa^\delta$ $(\nu<\varkappa)$;

(b) if γ is a limit ordinal, then $\mu=\operatorname{cf}(\gamma)$ and $\operatorname{tp} S_\nu^\gamma=\varkappa^{\gamma_\nu}$, where $\langle\gamma_\nu:\nu<\mu\rangle$ is a fixed increasing sequence of ordinals with limit γ.

We call this sequence S^γ *the standard decomposition* of $\varkappa^\gamma$ (although it depends upon the choicce of the γ_ν in (1.11) (b)).

THEOREM 1.1. *$\neg P(\gamma)$ holds for $\gamma<\omega_2$, i.e.*

(1.12) $$\begin{pmatrix}\omega_1\\ \omega_2^\gamma\end{pmatrix}\not\to\begin{pmatrix}1 & \omega\\ \omega_2^{\omega_1} & 1\end{pmatrix}^{1,1}\quad \textit{for}\quad \gamma<\omega_2.$$

REMARK. The following proof can easily be adapted to prove the more general result Theorem 2.1.

PROOF. We prove the result by induction on γ. For $\gamma<\omega_1$ it is obvious that $\neg P(\gamma)$ holds. Now assume that $\omega_1\leqq\gamma<\omega_2$ and distinguish the three cases (i) $\gamma=\delta+1$, (ii) $\operatorname{cf}(\gamma)=\omega$ and (iii) $\operatorname{cf}(\gamma)=\omega_1$.

In the first two cases there is no difficulty in carrying out the inductive step. We give the details here, but in later proofs where a similar type of argument is needed we shall omit the trivial details and simply instruct the reader "to take cross sections". The main idea of the proof of this theorem is in establishing the inductive step for case (iii).

Let $S^\gamma=\langle S^\gamma_\nu:\nu<\mu\rangle$ be the standard decomposition for ω_2^γ. By the induction hypothesis for each $\nu<\mu$ there is a sequence $A^\nu=\langle A^\nu_\alpha:\alpha<\omega_1\rangle$ of subsets of S^γ_ν which establishes $\neg P(\gamma_\nu)$, where $\operatorname{tp} S^\gamma_\nu=\omega_2^{\gamma_\nu}$.

Case 1. In this case $\mu=\omega_2$ and the sets S^γ_ν $(\nu<\omega_2)$ are order isomorphic i.e. $\gamma_\nu=\delta$ $(\nu<\omega_2)$. Therefore, for each $\alpha<\omega_1$ we can assume that the sets A^ν_α are also order isomorphic for $\nu<\omega_2$. Now put $A_\alpha=\bigcup\{A^\nu_\alpha:\nu<\omega_2\}$ $(\alpha<\omega_1)$. For each $\alpha<\omega_1$, there is $f(\alpha)<\omega_1$ such that $\operatorname{tp} A^\nu_\alpha<\omega_2^{f(\alpha)}$ $(\nu<\omega_2)$, and therefore $\operatorname{tp} A_\alpha\leqq\omega_2^{f(\alpha)}<\omega_2^{\omega_1}$. Therefore $A=\langle A_\alpha:\alpha<\omega_1\rangle$ establishes $\neg P(\gamma)$ since each A^ν has the ω-covering property for S^γ_ν $(\nu<\omega_2)$.

Case 2. In this case $\mu=\omega$. Again we define $A_\alpha=\bigcup\{A^\nu_\alpha:\nu<\mu\}$. Then $A=\langle A_\alpha:\alpha<\omega_1\rangle$ has the ω-covering property and moreover

$$\operatorname{tp} A_\alpha=\sum\{\operatorname{tp} A^\nu_\alpha:\nu<\omega\}<\omega_2^{\omega_1}\quad(\alpha<\omega_1),$$

since $\operatorname{tp} A^\nu_\alpha<\omega_2^{\omega_1}$ $(\nu<\omega;\ \alpha<\omega_1)$.

Case 3. In this case $\mu=\omega_1$. For each $\nu<\omega_1$, let $B^\nu=\langle B^\nu_n:n<\omega\rangle$ be a paradoxical decomposition of S^γ_ν as described after (1.1). Then B^ν has the ω-covering property (for S^γ_ν) and $\operatorname{tp} B^\nu_n<\omega_2^n$ $(n<\omega)$. Also, for each $\nu<\omega_1$, let Φ_ν denote any one-to-one function from ν into ω. Now put

$$A_\alpha=\bigcup\{A^\nu_\alpha:\nu\leqq\alpha\}\cup\bigcup\{B^\nu_{\Phi_\nu(\alpha)}:\alpha<\nu<\omega_1\}$$

for $\alpha<\omega_1$. We show that $A=\langle A_\alpha:\alpha<\omega_1\rangle$ establishes $\neg P(\gamma)$.

If $\nu\leqq\alpha<\omega_1$, there is some $f(\nu,\alpha)<\omega_1$ such that $\operatorname{tp} A^\nu_\alpha<\omega_2^{f(\nu,\alpha)}$. Also, there is $f(\alpha)<\omega_1$ such that $f(\nu,\alpha)<f(\alpha)$ for all $\nu\leqq\alpha$. Therefore,

$$\operatorname{tp} A_\alpha\leqq\omega_2^{f(\alpha)}\cdot\alpha+\omega_2^\omega\cdot\omega_1<\omega_2^{\omega_1}\quad(\alpha<\omega_1).$$

All that remains is to verify that A has the ω-covering property. Let $D\in[\omega_1]^\omega$. We must show that

$$A(D)=\bigcup\{A_\alpha:\alpha\in D\}=\omega_2^\gamma.$$

For $\nu<\omega_1$, let $D(\nu)=\{\alpha\in D:\alpha<\nu\}$. Then either $D(\nu)$ or $D\setminus D(\nu)$ is infinite. If $D(\nu)$ is infinite then

$$A(D)\supset\bigcup\{B^\nu_{\Phi_\nu(\alpha)}:\alpha\in D(\nu)\}=S^\gamma_\nu,$$

since B^ν has the ω-covering property. Also, if $D\setminus D(\nu)$ is infinite, then

$$A(D)\supset\bigcup\{A^\nu_\alpha:\alpha\in D\setminus D(\nu)\}=S^\gamma_\nu$$

since A^ν also has the ω-covering property. Thus, in either case $A(D)\supset S^\gamma_\nu$ for each $\nu<\omega_1$. It follows that $A(D)=\omega_2^\gamma$.

The inductive step used in the above proof breaks down completely if $\operatorname{cf}(\gamma)=\omega_2$. The trouble is that, unlike case (i), the sequences $A^\nu=\langle A^\nu_\alpha:\alpha<\omega_1\rangle$ $(\nu<\omega_2)$ obtained from the induction assumption are no longer identical copies of each other.

Our next aim is to say something rather more precise about the order types of the sets A_α of a sequence $A=\langle A_\alpha:\alpha<\omega_1\rangle$ which establishes the negative relation $\neg P(\gamma)$. But in order to state our results we must first recall some definitions from [9] concerning the *rank* of an ordinal function (at least in a generality sufficient for our present purposes).

We denote by Stat (ω_1) the set of all the stationary subsets of ω_1. Let $X \in$ Stat (ω_1). Then we define a partial order $<_X$ on ${}^{\omega_1}\omega_1$, the set of all functions from ω_1 into ω_1, by the rule

$$f <_X g \Leftrightarrow \{\alpha \in X: f(\alpha) \geqq g(\alpha)\} \notin \mathrm{Stat}(\omega_1).$$

It is easily seen that $<_X$ is well founded, and because of this we can define the rank function, $\|\cdot\|_X$, by

$$\|f\|_X = \sup\{\|g\|_X + 1 : g <_X f\}.$$

We shall write $\|\cdot\|$ instead of $\|\cdot\|_{\omega_1}$ and $<$ instead of $<_{\omega_1}$. We need the following easy consequences of this definition (see [9], p. 495).

(1.13) $$(\forall \mu < \omega_1)(\|f\|_X \leqq \mu \Leftrightarrow \{\alpha \in X: f(\alpha) \leqq \mu\} \in \mathrm{Stat}(\omega_1)).$$

(1.14) $$\|f\|_X \leqq \omega_1 \Leftrightarrow \{\alpha \in X: f(\alpha) \leqq \alpha\} \in \mathrm{Stat}(\omega_1).$$

We need also the following simple fact:

(1.15) If $X \in$ Stat (ω_1), and $\{\alpha \in X: g(\alpha) = f(\alpha)+1\} \notin$ Stat (ω_1), then $\|g\|_X = \|f\|_X + 1$.

Proof. Let $h <_X g$. Then $h_1 <_X h \Rightarrow h_1 <_X f$, and so $\|h\|_X \leqq \|f\|_X$. Thus $\|g\|_X \leqq \|f\|_X + 1$. But $f <_X g$ and so $\|f\|_X + 1 \leqq \|g\|_X$.

Next we define a special sequence of functions $h_\gamma \in {}^{\omega_1}\omega_1$ for $\gamma < \omega_2$ by transfinite recursion on γ. For each limit ordinal $\gamma < \omega_2$ we fix a strictly increasing sequence $\langle \gamma_\nu : \nu < \mu \rangle$ of length $\mu = \mathrm{cf}(\gamma)$ having limit γ. We agree that this is the same sequence as that associated with the standard decomposition for ω_2^γ appearing in (1.11)(b). Now define h_γ by:

$$h_0 \equiv 0; \quad h_{\gamma+1} \equiv h_\gamma + 1;$$

$$h_\gamma(\alpha) = \sup\{h_{\gamma_n}(\alpha) : n < \omega\} \quad \text{if} \quad \mathrm{cf}(\gamma) = \omega;$$

$$h_\gamma(\alpha) = \sup\{h_{\gamma_\nu}(\alpha) : \nu < \alpha\} \quad \text{if} \quad \mathrm{cf}(\gamma) = \omega_1.$$

The function h_γ defined in the case $\mathrm{cf}(\gamma) = \omega_1$ is called the *diagonal supremum* of the h_{γ_ν} $(\nu < \omega_1)$. Note that, if $X \in$ Stat (ω_1), if h_γ is the supremum or the diagonal supremum of certain h_{γ_ν}, and if $g <_X h_\gamma$, then $g <_Y h_{\gamma_{\nu_0}}$ for some $\nu_0 < \mathrm{cf}(\gamma)$ and $Y \in$ Stat (ω_1). This fact ensures that $h_\gamma \restriction X$ is "the γ-th function on X" for any $X \in$ Stat (ω_1), i.e.

(1.16) $$\|h_\gamma\|_X = \gamma \quad \text{for} \quad \gamma < \omega_2 \quad \text{and} \quad X \in \mathrm{Stat}(\omega_1).$$

As a corollary of this we have, for $\gamma < \omega_2$, $X \in$ Stat (ω_1) and $f \in {}^{\omega_1}\omega_1$,

(1.17) $$\|f\|_X \leqq \gamma \Leftrightarrow \{\alpha \in X: f(\alpha) \leqq h_\gamma(\alpha)\} \in \mathrm{Stat}(\omega_1);$$

also,

(1.18) $$h_\gamma <_X h_\delta \quad \text{for} \quad \gamma < \delta < \omega_2.$$

We make one final remark. For a limit ordinal γ $(\omega < \gamma < \omega_2)$ the above sequence $\langle \gamma_\nu : \nu < \mathrm{cf}(\gamma) \rangle$ can be chosen so that $\omega \leqq \gamma_0 < \gamma_1 < \dots$. This ensures that

(1.19) $$h_\gamma(\alpha) \geqq \omega \quad (0 < \alpha < \omega_1;\ \omega \leqq \gamma < \omega_2).$$

We shall also make use of another, stronger partial ordering on ${}^{\omega_1}\omega_1$, $\ll$, defined by

$$f \ll g \Leftrightarrow |\{\alpha < \omega_1 : f(\alpha) \geqq g(\alpha)\}| \leqq \aleph_0,$$

i.e. g eventually exceeds f. Again, it is easily seen that $\ll$ is wellfounded and $f \ll g \Rightarrow \Rightarrow f <_X g$ for any $X \in \mathrm{Stat}(\omega_1)$. The functions h_γ $(\gamma < \omega_2)$ defined above are also increasing in this stronger sense, i.e.

$$h_0 \ll h_1 \ll \ldots \ll h_\gamma \ll \ldots. \tag{1.20}$$

We can associate with any sequence $A = \langle A_\alpha : \alpha < \omega_1 \rangle$ of sets of ordinal numbers, an ordinal function f^A defined by

$$f^A(\alpha) = \min\{\delta : \mathrm{tp}\, A_\alpha < \omega_2^\delta\}.$$

Note that, if $A_\alpha \neq \emptyset$, then $f^A(\alpha) = \sigma_\alpha + 1$ for some ordinal σ_α. Also, if A establishes $\neg P(\gamma)$ for some γ, then $f^A \in {}^{\omega_1}\omega_1$. The next theorem shows that, if A establishes $\neg P(\gamma)$ for some large γ, then the associated function f^A is also large in some sense.

THEOREM 1.2. *Let* $\gamma < \omega_3$. *If* $A = \langle A_\alpha : \alpha < \omega_1 \rangle$ *is a sequence of subsets of* ω_2^γ *such that* $\|f^A\| \leqq \gamma$, *then* A *does not have the* ω_1 *covering property.*

PROOF. We prove this by induction on γ. It is trivial for $\gamma = 0$ since, by (1.13), the hypothesis implies that $A_\alpha = \emptyset$ for a stationary set of α's. Now assume $\gamma > 0$.

Case 1. γ is a limit ordinal. We can assume that the sets A_α are non-empty for all but countably many α, and so $f^A(\alpha) = g(\alpha) + 1$ for all but a countable number of α. Therefore, by (1.15), $\|f^A\| = \|g\| + 1$ and hence $\|f^A\| \leqq \gamma' < \gamma$. By the induction hypothesis $A' = \langle A_\alpha \cap \omega_2^{\gamma'} : \alpha < \omega_1 \rangle$ does not have the ω_1-covering property and hence neither does A.

Case 2. $\gamma = \delta + 1$. Let $\langle S_\nu^\gamma : \nu < \omega_2 \rangle$ be the standard decomposition for ω_2^γ. As in Case 1, $f^A(\alpha) = g(\alpha) + 1$ for all but countably many α's and $\|f^A\| = \|g\| + 1 \leqq \delta + 1$, so $\|g\| \leqq \delta$. Now for each α there is $\nu(\alpha) < \omega_2$ such that

$$\mathrm{tp}\,(A_\alpha \cap S_\nu^\gamma) < \omega_2^{g(\alpha)} \quad \text{for} \quad \nu(\alpha) < \nu < \omega_2.$$

There is $\nu_0 < \omega_2$ such that $\nu(\alpha) < \nu_0$ for all $\alpha < \omega_1$. Consider the sequence $A' = \langle A_\alpha \cap S_{\nu_0}^\gamma : \alpha < \omega_1 \rangle$. Clearly[1] $f^{A'} \lesssim g$ and so $\|f^{A'}\| \leqq \|g\| \leqq \delta$. Therefore, by the induction hypothesis A' does not have the ω_1-covering property and hence neither does A.

COROLLARY 1.3.

$$\begin{pmatrix} \omega_1 \\ \omega_2^\gamma \end{pmatrix} \to \begin{pmatrix} 1 & \omega_1 \\ \omega_2^\gamma & 1 \end{pmatrix}^{1,1}$$

holds for $\gamma < \omega_1$.

PROOF. If $A = \langle A_\alpha : \alpha < \omega_1 \rangle$ is any sequence of subsets of ω_2^γ such that $\mathrm{tp}\, A_\alpha < \omega_2^\gamma$, then $f^A(\alpha) \leqq h_\gamma(\alpha) = \gamma$ $(\alpha < \omega_1)$. Hence $\|f^A\| \leqq \|h_\gamma\| = \gamma$ and so A does not have the ω_1-covering property.

[1] Naturally, $f_1 \lesssim f_2$ means $|\{\alpha : f_1(\alpha) > f_2(\alpha)\}| \leqq \aleph_0$.

COROLLARY 1.4. *If* $\|f\|<\omega_2$ *for all* $f\in{}^{\omega_1}\omega_1$, *then*

$$\begin{pmatrix}\omega_1\\ \omega_2^{\omega_2}\end{pmatrix}\to\begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega_1} & 1\end{pmatrix}^{1,1}.$$

PROOF. Let $A=\langle A_\alpha:\alpha<\omega_1\rangle$ be a sequence of subsets of $\omega_2^{\omega_2}$ such that $\operatorname{tp} A_\alpha<\omega_2^{\omega_1}$. Then $f^A\in{}^{\omega_1}\omega_1$ and so $\|f^A\|<\gamma$ for some $\gamma<\omega_2$. It follows that $\|f^{A'}\|<\gamma$, where $A'=\langle A_\alpha\cap\omega_2^\gamma:\alpha<\omega_1\rangle$. By the theorem A' does not have the ω_1-covering property and so neither does A.

It is easily seen that Theorem 1.2 is best possible for $\gamma<\omega_2$ since there is a system $A^\gamma=\langle A_\alpha^\gamma:\alpha<\omega_1\rangle$ which establishes $\neg P(\gamma)$ and is such that $f^A(\alpha)\leqq\leqq h_{\gamma+1}(\alpha)$ $(\alpha<\omega_1)$ and hence by (1.16) and (1.17), $\|f^A\|\leqq\gamma+1$. This result can be proved by exactly the same induction argument that we used to prove Theorem 1.1; we only have to make sure that the A_α^ν chosen in the various places have order types less than $\omega_2^{h_{\gamma_\nu}(\alpha)+1}$ and this ensures that the A_α defined there have order types less than $\omega_2^{h_\nu(\alpha+1)}$. We omit the details since this result is also a Corollary of the following more general result Theorem 1.5.

We make one preliminary remark. We say that a function $g\in{}^{\omega_1}\omega_1$ *establishes* the negative relation $\neg P(\gamma)$,

$$\begin{pmatrix}\omega_1\\ \omega_2^{\gamma}\end{pmatrix}\nrightarrow\begin{pmatrix}1 & \omega\\ \omega_2^{\omega_1} & 1\end{pmatrix}^{1,1},$$

if there is an $A=\langle A_\alpha:\alpha<\omega_1\rangle$ which establishes it and is such that $f^A(\alpha)\leqq g(\alpha)$ for all $\alpha<\omega_1$. Now if g establishes $\neg P(\gamma)$ and $g\leqq h$, then the function h_1 defined by $h_1(\alpha)=\max\{h(\alpha),\omega\}$ also establishes $\neg P(\gamma)$. For suppose $A=\langle A:\alpha<\omega_1\rangle$ establishes $\neg P(\gamma)$ and $f^A(\alpha)\leqq g(\alpha)$ $(\alpha<\omega_1)$. Then there is $\alpha_0<\omega_1$ so that $f^A(\alpha)\leqq\leqq h(\alpha)$ $(\alpha_0\leqq\alpha<\omega_1)$. Let $\langle B_n:n<\omega\rangle$ be a paradoxical decomposition of ω_2^γ and consider the system $A'=\langle A_\alpha':\alpha<\omega_1\rangle$ defined by

$$A_\alpha'=\begin{cases}A_\alpha & \text{for } \alpha_0\leqq\alpha<\omega_1,\\ B_{\psi(\alpha)} & \text{for } \alpha<\alpha_0,\end{cases}$$

where ψ is any one-to-one map from α_0 into ω. Clearly A' establishes $\neg P(\gamma)$ and $f^{A'}(\alpha)\leqq h_1(\alpha)$ $(\alpha<\omega_1)$. Thus h_1 also establishes $\neg P(\gamma)$. It follows from this that, if $g,h\in{}^{\omega_1}\omega_1$, g establishes $\neg P(\gamma)$, $g\ll h$ and $h(\alpha)\geqq\omega$ $(\alpha<\omega_1)$, then h also establishes $\neg P(\gamma)$.

THEOREM 1.5. *Let* $\gamma<\omega_3$ *and suppose that* $\langle f_\sigma:\sigma\leqq\gamma\rangle$ *is a strongly increasing sequence of infinite-valued functions, i.e.* $f_\sigma(\alpha)\geqq\omega$ $(\sigma\leqq\gamma;\ \alpha<\omega_1)$ *and* $f_0\ll f_1\ll\dots$. *Then* $f_\gamma+1$ *establishes* $\neg P(\gamma)$.

PROOF. This is trivial for $\gamma=0$. We now assume that $\gamma>0$ and use induction on γ.

Let $\langle S_\nu^\gamma:\nu<\mu\rangle$ be the standard decomposition for ω_2^γ, where $\operatorname{tp} S_\nu^\gamma=\omega_2^{\gamma_\nu}$ $(\nu<\mu)$. Then $\gamma_\nu<\gamma$ $(\nu<\mu)$ and so by the induction hypothesis there is a system $A^\nu=\langle A_\alpha^\nu:\alpha<\omega_1\rangle$ of subsets of S_ν^γ which has the ω-covering property and is such that

$$\operatorname{tp} A_\alpha^\nu<\omega_2^{f_{\gamma_\nu}(\alpha)+1}\quad(\alpha<\omega_1;\ \nu<\mu).$$

Case 1. $\gamma=\delta+1$. In this case $\mu=\omega_2$ and $\gamma_\nu=\delta$ $(\nu<\omega_2)$. Put $A_\alpha=\bigcup\{A_\alpha^\nu:\nu<\omega_2\}$ $(\alpha<\omega_1)$. Then $A=\langle A_\alpha:\alpha<\omega_1\rangle$ has clearly the ω-covering property. Moreover,

$$\operatorname{tp} A_\alpha \leqq \omega_2^{f_\delta(\alpha)+1} \quad (\alpha<\omega_1).$$

Therefore $f_\delta+2$ establishes $\neg P(\gamma)$ and $f_\delta+2\lesssim f_\gamma+1$. Therefore by the remark preceding the theorem $f_\gamma+1$ also establishes $\neg P(\gamma)$.

Case 2. $\operatorname{cf}(\gamma)=\omega$. In this case $\mu=\omega$ and $\gamma_\nu\nearrow\gamma$. Again put $A_\alpha=\bigcup\{A_\alpha^\nu:\nu<\omega\}$ $(\alpha<\omega_1)$. Then $A=\langle A_\alpha:\alpha<\omega_1\rangle$ has the ω-covering property and

$$\operatorname{tp} A_\alpha \leqq \omega_2^{g(\alpha)},$$

where $g(\alpha)=\sup\limits_{\nu<\omega}\big(f_{\gamma_\nu}(\alpha)+1\big)$ $(\alpha<\omega_1)$. Hence $g+1$ establishes $\neg P(\gamma)$ and therefore $f_\gamma+1$ also establishes $\neg P(\gamma)$ since $g\lesssim f_\gamma$.

Case 3. $\operatorname{cf}(\gamma)=\omega_1$. In this case $\mu=\omega_1$ and $\gamma_\nu\nearrow\gamma$. Let $B^\nu=\langle B_n^\nu:n<\omega\rangle$ be a paradoxical decomposition for S_ν^γ $(\nu<\omega_1)$, and for each $\nu<\omega_1$ let ψ_ν be a one-to-one map from $\nu+1$ into ω. Put

$$\theta(\alpha)=\min\big(\{\alpha\}\cup\{\varrho<\alpha:\ f_{\gamma_\varrho}\geqq f_\gamma(\alpha)\}\big) \quad (\alpha<\omega_1),$$

and define

$$\tau_\nu=\sup\{\beta<\omega_1:\ \theta(\beta)\leqq\nu\} \quad (\nu<\omega_1).$$

Now define

$$A_\alpha=\bigcup\{A_\nu^\gamma(\alpha):\ \nu<\theta(\alpha)\}\cup\bigcup\{B^\nu_{\psi_{\tau_\nu}(\alpha)}:\ \theta(\alpha)\leqq\nu\leqq\omega_1\}.$$

First we observe that for $\nu<\omega_1$ there are only countably many ordinals $\beta<\omega_1$ which satisfy $\theta(\beta)\leqq\nu$. Otherwise, there would be ordinals β_0 $(\sigma<\omega_1)$ so that $\theta(\beta_\sigma)=\theta\leqq\nu<\beta_0<\beta_1<\ldots<\beta_\sigma<\ldots<\omega_1$. But this implies that $f_{\gamma_\theta}(\beta_\sigma)\geqq f_\gamma(\beta_\sigma)$ for $\sigma<\omega_1$, a contradiction against the hypothesis $f_{\gamma_\theta}\ll f_\gamma$. It follows that there are only countably many $\beta<\omega_1$ for which $\theta(\beta)\leqq\nu$ and so $\tau_\nu<\omega_1$ $(\nu<\omega_1)$. Moreover, if $\alpha<\omega_1$ and $\theta(\alpha)\leqq\nu<\omega_1$, then $\tau_\nu\geqq\alpha$. Thus $\psi_{\tau_\nu}(\alpha)$ is defined and the above definition for A_α is meaningful.

Now we have

$$\operatorname{tp} A_\alpha\leqq\sum\{\omega_2^{f_{\gamma_\nu}(\alpha)+1}:\ \nu<\theta(\alpha)\}+\omega_2^\omega\omega_1\} \quad (\alpha<\omega_1),$$

and since $f_{\gamma_\nu}(\alpha)<f_\gamma(\alpha)$ for $\nu<\theta(\alpha)$, it follows that

$$\operatorname{tp} A_\alpha<\omega_2^{f_\gamma(\alpha)+1} \quad (\alpha<\omega_1).$$

To complete the proof in this case it is enough to verify that $A(D)=\bigcup\{A_\alpha:\alpha\in D\}\supset S_\nu^\gamma$ whenever $\nu<\omega_1$ and $D\in[\omega_1]^\omega$. Let $D_1=\{\alpha\in D:\nu<\theta(\alpha)\}$. If D_1 is infinite, then $S(D)\supset\bigcup\{A_\nu^\gamma(\alpha):\alpha\in D_1\}=S_\nu^\gamma$ since A^ν has the ω-covering property. On the other hand, if D_1 is finite, $S(D)\supset\bigcup\{B^\nu_{\psi_\nu(\alpha)}:\alpha\in D\setminus D_1\}=S_\nu^\gamma$ since B^ν has the ω-covering property.

Case 4. $\operatorname{cf}\gamma=\omega_2$. In this case $\mu=\omega_2$ and $\gamma_\nu\nearrow\gamma$. Now for each $\nu<\omega_2$ there is $\beta_\nu<\omega_1$ such that

$$f_{\gamma_\nu}(\alpha)<f_\gamma(\alpha) \quad (\beta_\nu\leqq\alpha<\omega_1).$$

Also, there is $\beta<\omega_1$ such that $\beta_\nu=\beta$ for $\aleph_2$ different values of $\nu<\omega_2$. Now put

$S^*=\bigcup\{S^\gamma_\nu:\beta_\nu=\beta\}$, $A^*_\alpha=\bigcup\{A^\nu_\alpha:\beta_\nu=\beta\}$ $(\alpha<\omega_1)$. Then $A^*=\langle A^*_\alpha:\alpha<\omega_1\rangle$ clearly has the ω-covering property for S^*. Moreover, $\operatorname{tp} S^*=\omega_2^\gamma$ and

$$\operatorname{tp} A^*_\alpha \leqq \sum\{\omega_2^{f_{\gamma\nu}(\alpha)+1}:\ \beta_\nu=\beta\} \leqq \omega_2^{g(\alpha)},$$

where $g(\alpha)\leqq f_\gamma(\alpha)$ $(\beta\leqq\alpha<\omega_1)$. Thus $g+1$ establishes $\neg P(\gamma)$ and hence so does $f_\gamma+1$ since $g\ll f_\gamma$.

COROLLARY 1.6. *For* $\gamma<\omega_2$ *the function* $h_\gamma+1$ *establishes* $\neg P(\gamma)$.

PROOF. For $\gamma<\omega_1$ this is obvious since $h_\gamma\equiv\gamma$. For $\gamma\geqq\omega_1$ the result follows from the theorem and the observation that $h_\omega\ll h_{\omega+1}\ll\dots\ll h_\gamma$ is a strongly increasing sequence of length γ and the values $h_\gamma(\alpha)$ are all infinite for $\alpha<\omega_1$ and $\omega\leqq\nu$ by (1.19).

COROLLARY 1.7. *If there is a function* $h\in{}^{\omega_1}\omega_1$ *so that* $h_\nu\ll h$ *for all* $\nu<\omega_2$, *then*

$$\begin{pmatrix}\omega_1\\ \omega_2^{\omega_2}\end{pmatrix} \nrightarrow \begin{pmatrix}1 & \omega\\ \omega_2^{\omega_1} & 1\end{pmatrix}^{1,1}.$$

PROOF. This follows from the theorem and the fact that

$$h_0\ll h_1\ll\dots\ll h_\gamma\ll\dots\ll h \quad (\gamma<\omega_2).$$

The results of this section concerning the $\aleph_2$-phenomena for the relation

$$P(\gamma)\colon \begin{pmatrix}\omega_1\\ \omega_2^{\gamma}\end{pmatrix} \rightarrow \begin{pmatrix}1 & \omega\\ \omega_2^{\omega_1} & 1\end{pmatrix}^{1,1}$$

are summarized in the following theorem.

THEOREM 1.8. (a) $\neg P(\gamma)$ *holds for* $\gamma<\omega_2$.

(b) *If* $\gamma<\omega_3$ *and there is a strongly increasing sequence* $\langle f_\sigma:\sigma\leqq\gamma\rangle$ *of length* $\gamma+1$ *in* ${}^{\omega_1}\omega_1$ *(i.e.* $f_0\ll f_1\ll\dots\ll f_\gamma$*), then* $\neg P(\gamma)$.

(c) *If* $2^{\aleph_1}=\aleph_2$, *then there is some* $\gamma<\omega_3$ *such that* $P(\gamma)$ *is true.*

(d) *It is consistent that* $2^{\aleph_1}>\aleph_2$ *and* $\neg P(\gamma)$ *for all* $\gamma<\omega_3$.

(e) $P(\omega_2)$ *fails in* L *and* $P(\omega_2)$ *holds if Chang's conjecture is true.*

PROOF. (a) and (b) are respectively Theorems 1.1 and 1.5. The first part of (e) follows from Corollary 1.7 since, by a theorem of BAUMGARTNER [1] in L there is an $h\in{}^{\omega_1}\omega_1$ such that $h_\nu\ll h$ for all $\nu<\omega_2$. The second part of (e) follows from Corollary 1.4 and a result of BAUMGARTNER [2] and BENDA [4] which tells us that Chang's conjecture implies $\|f\|<\omega_2$ for all $f\in{}^{\omega_1}\omega_1$. (d) follows from Theorem 1.5 and a theorem of LAVER [11] (and BAUMGARTNER [2]) which says that it is consistent with ZFC and $2^{\aleph_1}>\aleph_2$ that whenever $F\subset{}^{\omega_1}\omega_1$, $|F|=\aleph_2$ then there is some $g\in{}^{\omega_1}\omega_1$ which eventually majorizes every $f\in F$, i.e. $f\ll g$ for all $f\in F$.

To see (c) let us remark that by Lemma 3 of [9] $\|\underline{\omega_1}\|<(2^{\aleph_1})^+$, where $\underline{\omega_1}$ is the constant function ω_1. Hence by the hypothesis of (c), $\|f\|\leqq\|\underline{\omega_1}\|=\xi<\omega_3$ for all $f\in{}^{\omega_1}\omega_1$ and then by Theorem 1.2, $P(\xi)$ is true.

We do not know if the converse of Theorem 1.5 is true, i.e. *if* $\gamma<\omega_3$ *and* $\neg P(\gamma)$ *holds, does it follow that there is a strongly increasing sequence of functions* $f_\sigma\in{}^{\omega_1}\omega_1$ $(\sigma\leqq\gamma)$ *of length* $\gamma+1$? However, as the next theorem shows, we can prove that under the stated hypothesis there is a *weakly* increasing sequence of length $\gamma+1$.

THEOREM 1.9. *If* $\gamma<\omega_3$ *and* $\neg P(\gamma)$, *then there are functions* $f_\sigma\in{}^{\omega_1}\omega_1$ $(\sigma\leqq\gamma)$ *such that* $f_0<f_1<\ldots<f_\gamma$.

This is an immediate consequence of Theorem 1.2 and the following theorem on the rank function which has an independent interest.

THEOREM 1.10. *Let* $f\in{}^{\omega_1}\omega_1$, $\|f\|=\gamma<\omega_3$. *Then there are functions* $f_\sigma\in{}^{\omega_1}\omega_1$ $(\sigma\leqq\gamma)$ *such that* $f_0<f_1<\ldots<f_\gamma$, *and moreover,*

$$f_\gamma(\alpha)=\omega^{\omega^{f(\alpha)}}f(\alpha)\quad(\alpha<\omega_1). \tag{1.21}$$

PROOF. For any function $g\in{}^{\omega_1}\omega_1$, let $\hat{g}$ denote the function ω^{ω^g}. The result is true for $\gamma\leqq\omega_2$ by (1.17) and (1.18) since $h_\nu<f<\hat{f}\cdot f$ for $\nu<\gamma$. We now prove the theorem by transfinite induction on γ. Assume $\omega_2<\gamma<\omega_3$ and that the result holds for all smaller ordinals. We distinguish the two cases (1) $\gamma=\delta+1$, (2) γ is a limit ordinal.

Before giving the induction details we make a remark about the choice of f_γ in (1.21). We use two elementary facts about ordinal exponentiation

(a) $\xi<\eta\Rightarrow\omega^\xi<\omega^\eta$,

(b) $\varrho<\omega^{\omega^\xi}\Rightarrow\varrho^2<\omega^{\omega^\xi}$.

Property (b) actually characterizes ordinals of the form ω^{ω^ξ}, and it is precisely this which allows our induction proof to work. To see (b), suppose $\varrho<\omega^{\omega^\xi}$. If $\xi=0$, then $\varrho<\omega$ and $\varrho^2<\omega$. If $\xi>0$, then ω^ξ is a limit ordinal and so $\varrho<\omega^\sigma$ for some $\sigma<\omega^\xi$. Then $\varrho^2\leqq\omega^{\sigma\cdot 2}<\omega^{\omega^\xi}$ by (a).

Case 1. $\gamma=\delta+1$. There is $f'<f$ such that $\|f'\|=\delta$. Now the result follows immediately from the induction hypothesis since $\hat{f'}\cdot f'<\hat{f}\cdot f$.

Case 2. γ a limit ordinal. Let $\mathrm{cf}(\gamma)=\mu$. By assumption $\mu=\omega$, ω_1 or ω_2. Let $\langle\gamma_\nu:\nu<\mu\rangle$ be the fixed increasing sequence of ordinals with limit γ mentioned in (1.11) (b), and let $f_\nu\in{}^{\omega_1}\omega_1$ $(\nu<\mu)$ be functions such that $f_\nu<f$ and $\|f_\nu\|=\gamma_\nu$ $(\nu<\mu)$. By the induction hypothesis, there are functions $f_\sigma^\nu\in{}^{\omega_1}\omega_1$ $(\sigma<\gamma_\nu)$ such that

$$f_0^\nu<f_1^\nu<\ldots<f_\sigma^\nu<\ldots<\hat{f}_\nu\cdot f_\nu\quad(\sigma<\gamma_\nu).$$

Let $N=\{(\sigma,\nu):\sigma<\gamma_\nu\wedge\nu<\mu\}$. Then the order type of N under the usual anti-lexicographic ordering, $<_0$, is $\operatorname{tp}N(<_0)=\gamma$. Thus it is sufficient to define functions $f_{(\sigma,\nu)}$ for $(\sigma,\nu)\in N$ so that $f_{(\sigma,\nu)}<\hat{f}\cdot f$ and

$$(\sigma,\nu)<_0(\sigma',\nu')\Rightarrow f_{(\sigma,\nu)}<f_{(\sigma',\nu')}. \tag{1.22}$$

For any ordinals ξ, η, there is an order preserving map $\varphi_{\xi,\eta}$ from $\xi\times\eta$ (ordered by $<_0$) onto the ordinal $\xi\cdot\eta$.

We now define $f_{(\sigma,\nu)}\in{}^{\omega_1}\omega_1$ for $(\sigma,\nu)\in N$ by

$$f_{(\sigma,\nu)}(\alpha)=\begin{cases}\varphi_{\hat{f}(\alpha),f(\alpha)}(f_\sigma^\nu(\alpha),h_\nu(\alpha)) & \text{if}\quad (f_\sigma^\nu(\alpha),h_\nu(\alpha))\in\hat{f}(\alpha)\times f(\alpha),\\ 0, & \text{otherwise.}\end{cases}$$

Note that

$$f_{(\sigma,\nu)}(\alpha)=\varphi_{\hat{f}(\alpha),f(\alpha)}(f_\sigma^\nu(\alpha),h_\nu(\alpha))$$

holds for all but a non-stationary set of α's, since $f_\sigma^\nu<\hat{f}_\nu\cdot f_\nu<\hat{f}_\nu\cdot\hat{f}_\nu<\hat{f}$ and $h_\nu<f$.

Thus $f_{(\sigma,\nu)}<\hat{f}\cdot f$ for $(\sigma,\nu)\in N$. Also, if (σ,ν), $(\sigma',\nu')\in N$ and $(\sigma,\nu)<_0(\sigma',\nu')$, then either (i) $\nu<\nu'$ or (ii) $\nu=\nu'$ and $\sigma<\sigma'$, and hence

$$(f_\sigma^\nu(\alpha), h_\nu(\alpha)) <_0 (f_{\sigma'}^{\nu'}(\alpha), h_{\nu'}(\alpha))$$

holds for all but a non-stationary set of α's. Thus $f_{(\sigma,\nu)}<f_{(\sigma',\nu')}$.

We remark that, analogously to the rank functions $\|\cdot\|$ defined before (1.13), we could also define a rank function $|||\cdot|||$ corresponding to the partial well-ordering $\ll$. This rank function has not, however, been so thoroughly investigated as the ordinary rank function $\|\cdot\|$. The reason for this is that the ideal of the non-stationary sets is normal and this accounts for many of the pleasant properties of $\|\cdot\|$. But the ideal of the countable sets is not normal, and we cannot expect $|||\cdot|||$ to behave so well. In particular the functions h_ν do not have the nice properties (1.17), (1.18) for this new rank.

It is clear that $|||f|||\leqq\|f\|$ since $g\ll h\Rightarrow g<h$, we and remark that we could improve Theorem 1.2 slightly by replacing $\|f\|$ by $|||f|||$. However, this would not help to solve the problem mentioned before Theorem 1.9 since it is not known (in ZFC) whether $|||f|||=\omega_2$ and $f\in{}^{\omega_1}\omega_1$ implies the existence of a strongly increasing sequence of functions $f_\sigma\in{}^{\omega_1}\omega_1$ $(\sigma\leqq\omega_2)$ of length ω_2.

2. Some extensions of the results of the previous section. General lemmas. In § 3 and § 4 we are going to give discussions of the relations

$$\binom{\omega_1}{\omega_2^\delta} \to \begin{pmatrix} 1 & \omega_1 \\ \omega_2^\sigma & \omega_2^\tau \end{pmatrix}^{1,1} \tag{2.1}$$

and

$$\binom{\omega_1}{\omega_2^\delta} \to \begin{pmatrix} 1 & \omega_0 \\ \omega_2^\sigma & \omega_2^\tau \end{pmatrix}^{1,1} \tag{2.2}$$

for $\delta,\tau<\omega_3$ and $\sigma<\omega_1$. The restriction to the case $\sigma<\omega_1$ is not entirely necessary, but an analysis for the case $\sigma\geqq\omega_1$ will inevitably be complicated by the same kind of $\aleph_2$-phenomena that we encountered in § 1 in connection with the case $\sigma=\omega_1$, $\tau=0$. In fact most of the results in § 1 find natural extensions to higher order types and we begin this section with a brief indication of these.

The following is an easy extension of Theorem 1.1.

THEOREM 2.1. *If* $\sigma<\omega_3$, $\mathrm{cf}(\sigma)=\omega_1$ *and* $\gamma<\omega_2$ *then*

$$\binom{\omega_1}{\omega_2^{\sigma+\gamma}} \not\to \begin{pmatrix} 1 & \omega \\ \omega_2^\sigma & 1 \end{pmatrix}^{1,1}.$$

PROOF. We prove this only for the case $\gamma=0$. The general result follows by induction on γ just as in the proof of Theorem 1.1. Let $\langle S_\nu:\nu<\omega_1\rangle$ be the standard decomposition of ω_2^σ as described in § 1. For $\nu<\omega_1$, let $\langle B_n:n<\omega\rangle$ be a paradoxical decomposition of S_ν and let φ_ν be a one-to-one map from ν into ω. Now consider the system $\langle A_\alpha:\alpha<\omega_1\rangle$ of subsets of ω_2^σ, where

$$A_\alpha=\bigcup\{S_\nu:\ \nu\leqq\alpha\}\cup\bigcup\{B^\nu_{\varphi_\nu(\alpha)}:\ \alpha<\nu<\omega_1\}.$$

This system clearly establishes Theorem 2.1. for $\gamma=0$.

There is a sharpening of this last theorem which is analogous to Theorem 1.5.

THEOREM 2.2. *Let* $\sigma<\omega_3$, $\mathrm{cf}(\sigma)=\omega_1$, *and let* $\langle\sigma_\nu:\nu<\omega_1\rangle$ *be a closed, cofinal, strictly increasing sequence in* σ. *Let* $\gamma<\omega_3$ *and suppose that* $\langle f_\nu^\gamma:\nu\leqq\gamma\rangle$ *is a strongly increasing sequence of functions in* ${}^{\omega_1}\omega_1$ *of length* $\gamma+1$. *Then there is a system* $A=\langle A_\alpha:\alpha<\omega_1\rangle$ *of* ω_1 *subsets of* $\omega_2^{\sigma+\gamma}$ *having the* ω*-covering property and such that*

$$\mathrm{tp}\, A_\alpha<\omega_2^{\sigma_\alpha+f_\gamma^\gamma(\alpha)+1}\quad(\alpha<\omega_1).$$

PROOF. The case $\gamma=0$ follows from the last proof, for, by the definition of A_α above, we have

$$\mathrm{tp}\, A_\alpha\leqq\omega_2^{\sigma_\alpha}+\omega_2^{\omega}\omega_1<\omega_2^{\sigma_\alpha+1}\leqq\omega_2^{\sigma_\alpha+f_0^0(\alpha)+1}.$$

Now an induction argument similar to that used to prove Theorem 1.5 works here as well.

Theorem 1.2 has a similar generalization.

THEOREM 2.3. *Let* $\omega\leqq\tau<\omega_3$, $\gamma<\omega_3$, $f\in{}^{\omega_1}\omega_1$. *If* $\|f\|\leqq\gamma$ *and* $A=\langle A_\alpha:\alpha<\omega_1\rangle$ *is a sequence of subsets of* $\omega_2^{\tau+\gamma}$ *such that*

$$\mathrm{tp}\, A_\alpha<\omega_2^{\omega+f(\alpha)}\quad(\alpha<\omega_1),$$

then there is $D\in[\omega_1]^{\omega_1}$ *such that*

$$\mathrm{tp}\,(\omega_2^{\tau+\gamma}\setminus\bigcup\{A_\alpha:\ \alpha\in D\})\geqq\omega_2^{\tau}. \tag{2.3}$$

PROOF. For $\gamma=0$ this is obvious since $f(\alpha)=0$ for all but a non-stationary set of α's and hence there is an uncountable set $D\subset\omega_1$ such that $\mathrm{tp}\, A_\alpha<\omega_2^n$ $(\alpha\in D)$, for some fixed $n<\omega$. This implies (2.3). The general result follows by an induction argument similar to that used to prove Theorem 1.2.

These results enable us to state generalizations of Theorem 1.8. Thus, from Theorem 2.2 it follows that

$$\binom{\omega_1}{\omega_2^{\sigma+\omega_2}}\nrightarrow\begin{pmatrix}1 & \omega\\ \omega_2^{\sigma} & 1\end{pmatrix}^{1,1} \tag{2.4}$$

holds in L if $\sigma<\omega_3$ and $\mathrm{cf}\,\sigma=\omega_1$. Whereas, from Theorem 2.3 we see that Chang's conjecture implies that

$$\binom{\omega_1}{\omega_2^{\tau+\omega_2}}\rightarrow\begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega_1} & \omega_2^{\tau}\end{pmatrix}^{1,1} \tag{2.5}$$

holds for all $\tau<\omega_3$. It is interesting to note that (2.5) and $2^{\aleph_0}=\aleph_1$ implies a strong negation of (2.4) with $\sigma=\omega_1$, namely

$$\binom{\omega_1}{\omega_2^{\omega_2}}\rightarrow\begin{pmatrix}1 & \omega\\ \omega_2^{\omega_1} & \omega_2^{\omega_2}\end{pmatrix}^{1,1}. \tag{2.6}$$

For suppose that (2.6) is false. Then there is a sequence $A=\langle A_\alpha:\alpha<\omega_1\rangle$ of subsets of $\omega_2^{\omega_2}$ such that $\mathrm{tp}\, A_\alpha<\omega_2^{\omega_1}$ $(\alpha<\omega_1)$ and such that

$$\mathrm{tp}\,(\omega_2^{\omega_2}\setminus\bigcup\{A_\alpha:\ \alpha\in D\})<\omega_2^{\omega_2}$$

holds for all $D\in[\omega_1]^\omega$. In view of the hypothesis $2^{\aleph_0}=\aleph_1$, it follows from this that there is some $\tau<\omega_2$ such that

$$\operatorname{tp}(\omega_2^{\omega_2}\setminus\bigcup\{A_\alpha:\ \alpha\in D\})<\omega_2^\tau$$

also holds for all $D\in[\omega_1]^\omega$. But this contradicts (2.5). Hence (2.6) is consistent (with Chang's conjecture and say G. C. H.). However, we do know that the stronger relation

$$\begin{pmatrix}\omega_1\\ \omega_2^{\omega_2}\end{pmatrix}\to\begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega_1} & \omega_2^{\omega_2}\end{pmatrix}^{1,1}$$

is false assuming $2^{\aleph_1}=\aleph_2$ (see Theorem 3.1 (a)).

We need the following corollary of Theorem 2.3.

COROLLARY 2.4. *If* $\omega\leqq\tau<\omega_3$ *and* $\gamma<\omega_1$ *then*

$$\begin{pmatrix}\omega_1\\ \omega_2^{\tau+\gamma}\end{pmatrix}\to\begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega+\gamma} & \omega_2^{\tau}\end{pmatrix}^{1,1}.$$

We now describe a general method to obtain polarized partition relations. First we introduce a new partition relation.

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix}\to\begin{pmatrix}\xi_0\\ \eta_0\end{pmatrix},\begin{bmatrix}\xi_1\\ \eta_1\end{bmatrix}_{\varkappa,<\lambda}\Bigg)^{1,1}. \tag{2.7}$$

By definition, (2.7) means that the following statement is true: *If f is a partial function from* $\xi\times\eta$ *into* $\varkappa$ *then* EITHER *there is* $X_0\times Y_0\subset\xi\times\eta$ *such that* $\operatorname{tp}X_0=\xi_0$, $\operatorname{tp}Y_0=\eta_0$ and $X_0\times Y_0$ *is disjoint from* $D(f)$, *the domain of f*, OR *there is* $X_1\times Y_1\subset D(f)$ *such that* $\operatorname{tp}X_1=\xi_1$, $\operatorname{tp}Y_1=\eta_1$ *and* $|f''(X_1\times\{\nu\})|<\lambda$ *for all* $\nu\in Y_1$.

A more general symbol than (2.7) can be defined but we do not bother to do this since (2.7) is sufficiently general for most of our present purposes.

We now give two lemmas establishing connections between polarized partition relations and the new relation (2.7).

LEMMA 2.5. *Suppose* $\varkappa$ *is an infinite cardinal and*

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix}\not\to\begin{pmatrix}1\\ \eta_0\end{pmatrix}\begin{bmatrix}\xi_1\\ \eta_1\end{bmatrix}_{\omega,<\omega}\Bigg)^{1,1}. \tag{2.8}$$

Let Ξ_ν $(\nu<\eta)$ *be ordinals such that* $\Xi_\nu<\varkappa^+$ *and let* $\Xi=\sum\{\Xi_\nu:\nu<\eta\}$. *Let* $\mathcal{O}$ *denote the set of all ordinals of the form* $\varphi'=\sum\{\varphi_\nu:\nu<\eta\}$, *where* $\varphi_\nu\leqq\Xi_\nu$ *and*

a) $\varphi_\nu\neq\Xi_\nu\Rightarrow\varphi_\nu<\varkappa^\omega\quad(\nu<\eta)$,

b) $\operatorname{tp}\{\nu<\eta:\ \varphi_\nu=\Xi_\nu\}<\eta_0$.

Let $\Phi=\sup\{\varphi'+1:\varphi'\in\mathcal{O}\}$ *and* $\Psi=\sup\{\sum\{\Xi_\nu:\nu\in Y\}+1:Y\subset\eta\wedge\operatorname{tp}Y<\eta_1\}$. *Then*

$$\begin{pmatrix}\xi\\ \Xi\end{pmatrix}\not\to\begin{pmatrix}1 & \xi_1\\ \Phi & \Psi\end{pmatrix}^{1,1}. \tag{2.9}$$

PROOF. By the hypothesis (2.8) there is a partial function f from $\xi\times\eta$ into ω such that

(i) $(\{\alpha\}\times Y_0)\cap D(f)\neq\emptyset$ whenever $\alpha\in\xi$ and $Y_0\subset\eta$, tp $Y_0=\eta_0$, and

(ii) whenever $X_1\times Y_1\subset\xi\times\eta$, tp $X_1=\xi_1$, tp $Y_1=\eta_1$, then there is some $v\in Y$ such that either $X_1\times\{v\}\not\subset D(f)$ or $|f''(X_1\times\{v\})|=\omega$.

Let $\langle S_v: v<\eta\rangle$ be a decomposition of Ξ such that tp $S_v=\Xi_v$ $(v<\eta)$ and $S_0<S_1<\dots$. Let $B^v=\langle B_n^v: n<\omega\rangle$ be a paradoxical decomposition of S_v such that tp $B_n^v<\varkappa^n$ for $n<\omega$. Now consider the subsets $A_\alpha\subset\Xi$ $(\alpha<\xi)$ given by

$$A_\alpha=\bigcup\{S_v:\ (\alpha,v)\notin D(f)\}\cup\bigcup\{B^v_{f(\alpha,v)}:\ (\alpha,v)\in D(f)\}.$$

By (i) it follows that tp $A_\alpha\in\mathcal{O}$ $(\alpha<\xi)$ and hence tp $A_\alpha<\Phi$ $(\alpha<\xi)$. Now let $X_1\subset\xi$, $B\subset\Xi$ with tp $X_1=\xi_1$ and tp $B=\Psi$. Put $Y_1=\{v<\eta: B\cap S_v\neq\emptyset\}$. Then tp $Y_1\geqq\eta_1$, by the definition of Ψ. Hence, by (ii), there is $v\in Y_1$ such that either $X_1\times\{v\}\not\subset D(f)$ or $|f''(X_1\times\{v\})|=\omega$. Since B^v has the ω-covering property for S_v, this implies that in either case $S_v\subset\bigcup\{A_\alpha:\alpha\in X_1\}$ and hence $B\cap\bigcup\{A_\alpha:\alpha\in X_1\}\neq\emptyset$.

LEMMA 2.6. *Suppose that $\varkappa$ is an infinite cardinal and*

$$\binom{\xi}{\eta}\to\left(\begin{matrix}1\\ \eta_0\end{matrix},\begin{bmatrix}\xi_1\\ \eta_1\end{bmatrix}_{\omega,<\omega\dagger}\right)^{1,1}\tag{2.10}$$

holds. Let Ξ_v $(v<\eta)$ be ordinals such that $\omega\leqq\Xi_v<\varkappa^+$ $(v<\eta)$ and let $\Xi=\sum\{\varkappa^{\Xi_v}: v<\eta\}$. Let $\Psi=\min\left\{\sum\{\varkappa^{\Xi_v}: v\in Y\}: Y\subset\eta,\ \text{tp } Y=\eta_1\right\}$. Then

$$\binom{\xi}{\Xi}\to\begin{pmatrix}1 & \xi_1\\ \varkappa^\omega\cdot\eta_0 & \Psi\end{pmatrix}^{1,1}\tag{2.11}$$

holds.

PROOF. Let $\langle A_\alpha:\alpha<\xi\rangle$ be a system of subsets of Ξ such that tp $A_\alpha<\varkappa^\omega\eta_0$ $(\alpha<\xi)$. Let $\langle S_v: v<\eta\rangle$ be a decomposition of Ξ with tp $S_v=\varkappa^{\Xi_v}$ $(v<\eta)$ and $S_0<S_1<\dots$. Define a partial function f from $\xi\times\eta$ into ω having domain

$$D(f)=\{(\alpha,v):\text{tp }(A_\alpha\cap S_v)<\varkappa^\omega\}$$

and such that

$$f(\alpha,v)=\min\{n:\ \text{tp }(A_\alpha\cap S_v)<\varkappa^n\}$$

for $(\alpha,v)\in D(f)$.

By (2.10), and by the fact that tp $A_\alpha<\varkappa^\omega\eta_0$ $(\alpha<\xi)$ it follows that there are $X_1\subset\xi$ and $Y_1\subset\eta$ such that tp $X_1=\xi_1$, tp $Y_1=\eta_1$, and $X_1\times Y_1\subset D(f)$ and $|f''(X_1\times\{v\})|<\omega$ for all $v\in Y_1$.

For each $v\in Y_1$, the set $\bigcup\{A_\alpha\cap S_v:\alpha\in X_1\}$ has order type less than $\varkappa^\omega$ and hence $S_v\setminus\bigcup\{A_\alpha\cap S_v:\alpha\in X_1\}$ has order type $\varkappa^{\Xi_v}$. Put $B=\bigcup\{S_v: v\in Y\}\setminus\bigcup\{A_\alpha:\alpha\in X_1\}$. Then tp $B\geqq\Psi$ and $B\cap A_\alpha=0$ for $\alpha\in X_1$. This establishes (2.11).

3. Discussion of the relation (2.1). The aim of this chapter is to give a discussion of the relation

$$\binom{\omega_1}{\omega_2^\varrho}\to\begin{pmatrix}1 & \omega_1\\ \omega_2^\sigma & \omega_2^\tau\end{pmatrix}^{1,1}\tag{3.1}$$

for $\varrho,\tau<\omega_3$ and $\sigma<\omega_1$. We are going to give a complete discussion under the assumptions $2^{\aleph_0}=\aleph_1$, $2^{\aleph_1}=\aleph_2$.

Our main positive result is

THEOREM 3.1.

a) $\begin{pmatrix}\omega_1\\ \omega_2^\tau\end{pmatrix} \to \begin{pmatrix}1 & \omega_1\\ \omega_2^n & \omega_2^\tau\end{pmatrix}^{1,1}$ *for* $n<\omega$ *and* $n \leqq \tau \leqq \omega_3$

b) $\begin{pmatrix}\omega_1\\ \omega_2^{\tau+\gamma}\end{pmatrix} \to \begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega+\gamma} & \omega_2^\tau\end{pmatrix}^{1,1}$ *for* $\tau<\omega_3,\ \tau+\gamma \geqq \omega+\gamma$ *and* $\gamma<\omega_1$

c) $\begin{pmatrix}\omega_1\\ \omega_2^{\tau+\gamma}\end{pmatrix} \to \begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega+1+\gamma} & \omega_2^\tau\end{pmatrix}^{1,1}$ *for* $\omega+1 \leqq \tau < \omega_3$, $\mathrm{cf}(\omega_2^\tau)=\omega_2$ *and* $\gamma<\omega$.

PROOF. a) is a trivial consequence of the fact that the union of ω_1 sets of type $<\omega_2^n$ has type $<\omega_2^n$.

b) is a restatement of Corollary 2.4 if $\omega \leqq \tau$. If $\tau<\omega$ then $\tau+\gamma \geqq \omega+\gamma$ implies $\gamma=\omega+\gamma$ and hence the statement is true by the special case $\begin{pmatrix}1\\ \omega_2^{\omega+\gamma}\end{pmatrix} \to \begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega+\gamma} & \omega_2^\omega\end{pmatrix}$ of Corollary 2.4.

We prove c) by induction on γ. In case $\gamma=0$ let $\langle S_\nu : \nu<\omega_2\rangle$ be the standard decomposition of ω_2^τ and assume that $A_\alpha \subset \omega_2^\tau$, $\mathrm{tp}\, A_\alpha < \omega_2^{\omega+1}$ for $\alpha<\omega_1$.

Now for each $\alpha<\omega_1$ there are $n(\alpha)<\omega$ and $\nu(\alpha)<\omega_2$ such that

$$\mathrm{tp}\,(A_\alpha \cap S_\nu) < \omega_2^{n(\alpha)} \quad \text{for} \quad \nu(\alpha)<\nu<\omega_2.$$

There is a $D \in [\omega_1]^{\omega_1}$ such that $n(\alpha)=n$ for $\alpha \in D$, where n is some fixed integer, and then it is easily seen that

$$\mathrm{tp}\,(\omega_2^\tau \setminus \cup \{A_\alpha : \alpha \in D\}) = \omega_2^\tau.$$

We now prove c) by induction on $\gamma<\omega$. Assume the statement is true for γ. Let $\langle S_\nu : \nu<\omega_2\rangle$ be a standard decomposition of $\omega_2^{\tau+\gamma+1}$; and assume $A_\alpha \subset \omega_2^{\tau+\gamma+1}$, $\mathrm{tp}\, A_\alpha < \omega_2^{\omega+1+\gamma+1}$ $(\alpha<\omega_1)$. Then for each $\alpha<\omega_1$ there is $\nu(\alpha)$ such that $\mathrm{tp}\,(A_\alpha \cap S_\nu) < \omega_2^{\omega+1+\gamma}$ for $\nu(\alpha)<\nu<\omega_2$. Choose ν so that $\mathrm{tp}\,(A_\alpha \cap S_\nu) < \omega_2^{\omega+1+\gamma}$ for all $\alpha<\omega_1$. By the induction hypothesis there is a $D \in [\omega_1]^{\omega_1}$ such that

$$\mathrm{tp}\,(S_\nu \setminus \{A_\alpha : \alpha \in D\}) \geqq \omega_2^\tau.$$

Now our aim is to show that the rather simple positive results of the last theorem are best possible.

THEOREM 3.2. *Assume* $\tau<\omega_3$.

(3.2) *If* $2^{\aleph_0}=\aleph_1$ *and* $\mathrm{cf}(\tau)=\omega$ *then*

$$\begin{pmatrix}\omega_1\\ \omega_2^\tau\end{pmatrix} \nrightarrow \begin{pmatrix}1 & \omega_1\\ \omega_2^\omega+1 & \omega_2^\tau\end{pmatrix}^{1,1}.$$

(3.3) *If* $\mathrm{cf}(\tau)=\omega_1$ *then*

$$\begin{pmatrix}\omega_1\\ \omega_2^\tau\end{pmatrix} \nrightarrow \begin{pmatrix}1 & \omega_1\\ \omega_2^\omega\omega_1 & \omega_2^\tau\end{pmatrix}^{1,1}.$$

(3.4) *If* $2^{\aleph_1}=\aleph_2$ *and* $\mathrm{cf}(\omega_2^\tau)=\omega_2$ *then*

$$\begin{pmatrix}\omega_2\\ \omega_2^\tau\end{pmatrix} \nrightarrow \begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega+1}+1 & \omega_2^\tau\end{pmatrix}^{1,1}.$$

PROOF OF (3.2). Let $\langle S_n : n<\omega\rangle$ be the standard decomposition of ω_2^{τ} and let tp $S_n=\omega_2^{\Xi_n}$ ($n<\omega$). We apply Lemma 2.5 with $\xi=\omega_1$, $\eta=\omega$, $\eta_0=1$, $\xi_1=\omega_1$, $\eta_1=\omega$. With this choice of the parameters the Φ of the lemma is $\omega_2^{\omega}+1$, while $\Psi=\omega_2^{\tau}$. Hence, by the lemma, we only have to prove that $2^{\aleph_0}=\aleph_1$ implies

$$(3.5)\qquad \binom{\omega_1}{\omega} \nrightarrow \begin{pmatrix} 1 & \begin{bmatrix}\omega_1\\ \omega\end{bmatrix}_{\omega, <\omega\dagger} \\ 1 & \end{pmatrix}^{1,1}.$$

We prove this relation under the weaker hypothesis: *There is a sequence of functions* $_{\alpha}\in{}^{\omega}\omega$ ($\alpha<\omega_1$) *having the property that for any* $g\in{}^{\omega}\omega$, *there is* $\beta<\omega_1$ *such that* $\{n<\omega : g(n)\geqq f_\alpha(n)\}$ *is finite for all* $\alpha>\beta$ (i.e. there is an ω_1-scale).

Assuming the f_α ($\alpha<\omega_1$) satisfy the above, we now define a function $f:\omega_1\times\times\omega\to\omega$ by

$$(3.6)\qquad f(\alpha, \nu)=f_\alpha(\nu).$$

Let $D\in[\omega_1]^{\omega_1}$, $N\in[\omega]^{\omega}$ and suppose that $f''(D\times\{n\})$ is finite for all $n\in N$. Then there are integers $g(n)$ ($n<\omega$) such that $f_\alpha(n)<g(n)$ for all $\alpha\in D$ and $n\in N$, a contradiction. Thus $\{\nu<\omega : |f''(D\times\{\nu\})|<\omega\}$ must be finite.

PROOF OF (3.3). The idea of the proof is very similar to that used in the proof of Lemma 2.5, and suggests how that lemma can be generalized. We did not bother to state the generalization since this is the only instance where the stronger statement would be needed.

First we prove:

(3.7) *There is a function* $f:\omega_1\times\omega_1\to\omega$ *such that for all* $A, B\in[\omega_1]^{\omega_1}$ *there is* $\nu\in B$ *with* $|f''((A\setminus\nu)\times\{\nu\})|=\omega$.

Incidentally, we remark that (3.7) implies the relation

$$\omega_1 \nrightarrow [\omega_1]^2_{\omega, <\omega}$$

which does not seem to have been noted previously.

To see (3.7) choose a sequence of functions $f_\alpha:\alpha\to\omega$ for $\alpha<\omega_1$ in such a way that, for all $\beta<\alpha<\omega_1$, $f_\beta(\nu)\neq f_\alpha(\nu)$ for all but finitely many ν. Now put

$$f(\alpha, \nu)=\begin{cases} f_\alpha(\nu) & \text{for} \quad \nu<\alpha<\omega_1,\\ 0 & \text{otherwise.}\end{cases}$$

Let now $A\in[\omega_1]^{\omega_1}$ and put $T=\{\nu<\omega_1 : |\{f(\alpha, \nu) : \alpha\in A\setminus\nu+1\}|<\omega\}$. We want to verify that $B\setminus T\neq\emptyset$ for $B\in[\omega_1]^{\omega_1}$ and this follows if we show that $|T|\leqq\omega$. Assume $|T|=\omega_1$. Then there are $T'\in[T]^{\omega_1}$ and $n<\omega$ such that

$$|\{f_\alpha(\nu) : \alpha\in A\setminus\nu+1\}|<n \quad \text{for} \quad \nu\in T'.$$

Let $\nu_0<\ldots<\nu_k<\ldots<\alpha_0<\ldots<\alpha_n$ be ordinals such that $\nu_k\in T'$ for $k<\omega$ and $\alpha_i\in A$ for $i\leqq n$. Then there are $i\neq j\leqq n$ such that $f_{\alpha_i}(\nu_k)=f_{\alpha_j}(\nu_k)$ for infinitely many k, a contradiction.

To finish the proof of (3.3) let $\langle S_\nu : \nu<\omega_1\rangle$ be a standard decomposition of ω_2^{τ}. For $\nu<\omega_1$ let $\langle B_n^\nu : n<\omega\rangle$ be a paradoxical decomposition of S_ν for $\nu<\omega_1$. For $\alpha<\omega_1$ let $A_\alpha=\bigcup\{B_{f(\alpha,\nu)} : \nu<\alpha\}$, where f satisfies (3.7).

It follows, just as in the proof of Lemma 2.5, that $\langle A_\alpha : \alpha < \omega_1 \rangle$ establishes the negative relation (3.3).

PROOF OF (3.4). Let $\langle S_\nu : \nu < \omega_2 \rangle$ be the standard decomposition of ω_2^τ, where $\operatorname{tp} S_\nu = \omega_2^{\bar{\tau}_\nu}$ for $\nu < \omega_2$. We apply Lemma 2.5 with $\xi = \omega_2$, $\eta = \omega_2$, $\eta_0 = 1$, $\xi_1 = \omega_1$, $\eta_1 = \omega_2$. Then Φ and Ψ of the lemma are respectively $\omega_2^{\omega+1} + 1$ and ω_2^τ.
Hence by the lemma it is sufficient to prove that $2^{\aleph_1} = \aleph_2$ implies

$$(3.8) \qquad \binom{\omega_2}{\omega_2} \nrightarrow \left(\begin{matrix} 1 \\ 1 \end{matrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_{\omega, <\omega\dagger} \right)^{1,1}.$$

This is a trivial corollary of the fact ([6], Theorem 17A) that $2^{\aleph_1} = \aleph_2$ implies

$$(3.9) \qquad \binom{\omega_2}{\omega_2} \nrightarrow \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_{\omega}^{1,1}.$$

We need the following extension of Theorem 3.2.

THEOREM 3.3. *Assume* $\tau < \omega_3$, $\gamma < \omega_1$.
(a) *If* $2^{\aleph_0} = \aleph_1$ *and* $\operatorname{cf}(\tau) = \omega$ *then*

$$(3.10) \qquad \binom{\omega_1}{\omega_2^{\tau+\gamma}} \nrightarrow \begin{pmatrix} 1 & \omega_1 \\ \omega_2^{\omega+\gamma}+1 & \omega_2^\tau \end{pmatrix}^{1,1}.$$

(b) *If* $\operatorname{cf}(\tau) = \omega_1$ *and* $\gamma > 0$ *then*

$$(3.11) \qquad \binom{\omega_1}{\omega_2^{\tau+\gamma}} \nrightarrow \begin{pmatrix} 1 & \omega_1 \\ \omega_2^{\omega+1}+1 & \omega_2^\tau \end{pmatrix}^{1,1}.$$

(c) *If* $2^{\aleph_1} = \aleph_2$ *and* $\operatorname{cf}(\omega_2^\tau) = \omega_2$ *then*

$$(3.12) \qquad \binom{\omega_2}{\omega_2^{\tau+\gamma}} \nrightarrow \begin{pmatrix} 1 & \omega_1 \\ \omega_2^{\omega+1+\gamma}+1 & \omega_2^\tau \end{pmatrix}^{1,1}.$$

PROOF. We prove all these statements by induction on $\gamma < \omega_1$. Since there are notable differences it will be convenient to give the proofs separately.

PROOF OF (3.10). For $\gamma = 0$ this is (3.2). Assume $\gamma > 0$. In the case $\gamma = \delta + 1$ we can take identical cross sections. Now assume $\operatorname{cf}(\gamma) = \omega$. Let $\langle S_n : n < \omega \rangle$ be the standard decomposition of $\omega_2^{\tau+\gamma}$, $\operatorname{tp} S_n = \omega_2^{\tau+\gamma_n}$. Let $A^n = \langle A_\alpha^n : \alpha < \omega_1 \rangle$ establish

$$\binom{\omega_1}{\omega_2^{\tau+\gamma_n}} \nrightarrow \begin{pmatrix} 1 & \omega_1 \\ \omega_2^{\omega+\gamma_n}+1 & \omega_2^\tau \end{pmatrix}^{1,1} \quad \text{in } S_n \quad \text{for } n < \omega.$$

Let $\langle B_k^n : k < \omega \rangle$ be a paradoxical decomposition of S_n for $n < \omega$. Let $f : \omega_1 \times \omega \to \omega$ satisfy (3.6). For $\alpha < \omega_1$ put $A_\alpha = \bigcup \{A_\alpha^n : n < \omega\} \cup \{B_{f(\alpha,n)}^n : n < \omega\}$. Clearly $\operatorname{tp}(A_\alpha \cap (\bigcup \{S_i : i \leq n\}) < \omega_2^{\omega+\gamma}$ for $n < \omega$, and hence $\operatorname{tp} A_\alpha \leq \omega_2^{\omega+\gamma}$ $(\alpha < \omega_1)$. Assume $D \in [\omega_1]^{\omega_1}$. Then, by the choice of the A_α^n,

$$\operatorname{tp}(S_n \setminus \bigcup \{A_\alpha : \alpha \in D\}) < \omega_2^\tau$$

for all $n < \omega$. Also, by the choice of the B_k^n and f,

$$\bigcup \{B_{f(\alpha,n)}^n : \alpha \in D, n < \omega\}$$

covers an end section of $\omega_2^{\tau+\gamma}$. It follows that

$$\mathrm{tp}\,(\omega_2^{\tau+\gamma}\setminus\bigcup\{A_\alpha:\ \alpha\in D\}) < \omega_2^{\tau}.$$

Proof of (3.11). Let $\langle S_\nu:\nu<\omega_1\rangle$ be a standard decomposition of $\omega_2^{\tau+\gamma}$. In case $\gamma=1$ take identical cross sections of systems establishing (3.3). In case $\gamma=\delta+1>1$ take identical cross sections of the inductive systems. In case $\mathrm{cf}\,(\gamma)=\omega$ take cross sections. It is easy to check that this system satisfies all the requirements. (In case $\mathrm{cf}\,(\gamma)=\omega$ we need the fact that $\mathrm{cf}\,(\tau)=\omega_1>\omega$.)

Proof of (3.12). For $\gamma=0$ this is 3.4. For $\mathrm{cf}\,(\gamma)=\omega$ take cross sections. Now suppose that $\gamma=\delta+1$ and let $\langle S_\nu:\nu<\omega_2\rangle$ be a standard decomposition of $\omega_2^{\tau+\gamma}$. Let $A^\nu=\langle A_\alpha^\nu:\alpha<\omega_2\rangle$ be identical copies esatblishing $\binom{\omega_2}{\omega_2^{\tau+\delta}}\nrightarrow\binom{1 \quad \omega}{\omega_2^{\omega+1+\delta}+1 \quad \omega_2^{\tau}}^{1,1}$ on S_ν for $\nu<\omega_2$. Let $\langle B_n^\nu:n<\omega\rangle$ be a paradoxical decomposition S^ν for $\nu<\omega_2$. Let $f:\omega_2\times\omega_2\to\omega$ establish (3.8). For $\alpha<\omega_2$ let

$$A_\alpha=\bigcup\{A_\alpha^\nu:\ \nu<\omega_2\}\cup\bigcup\{B_{f(\alpha,\nu)}^\nu:\ \nu<\omega_2\}.$$

For each $\nu<\omega_2$, $A_\alpha\cap\bigcup\{S_\mu:\mu<\nu\}\leqq\omega_2^{\omega+1+\delta}\cdot(\nu+1)$, and hence $\mathrm{tp}\,A_\alpha\leqq\omega_2^{\omega+1+\delta+1}$ for $\alpha<\omega_2$. Now, by the choice of B_n^ν and f the union of every ω_1 A_α covers an endsection of $\omega_2^{\tau+\delta}$. By the choice of the A_α^ν the union of every ω_1 A_α omits a set of type less than ω_2^τ from each S_ν $(\nu<\omega_2)$. Since $\mathrm{cf}\,(\omega_2^\tau)=\omega_2$ it follows that

$$\mathrm{tp}\,(\omega_2^{\tau+\gamma}\setminus\bigcup\{A_\alpha:\alpha\in D\}) < \omega_2^\tau$$

for all $D\in[\omega_1]^{\omega_1}$.

We claim that, assuming $2^{\aleph_0}=\aleph_1$, $2^{\aleph_1}=\aleph_2$, Theorems 3.1—3.3 provide a complete discussion of (3.1). We may of course assume that $\sigma,\tau\leqq\varrho$. In case $\sigma\leqq\omega$ (3.1) is true by Theorem 3.1 a) and b). So we only have to investigate the relation

$$\binom{\omega_1}{\omega_2^{\varrho}}\to\binom{1 \quad \omega_1}{\omega_2^{\omega+1+\gamma} \quad \omega_2^{\tau}}^{1,1} \tag{3.13}$$

for $\tau\leqq\varrho<\omega_3$, $\gamma<\omega_1$ and $\omega+1+\gamma\leqq\varrho$. If $\tau\leqq\omega$ the statement is true by Theorem 3.1 b). Hence we may assume $\tau>\omega$. ϱ can uniquely be written as $\varrho=\tau+\varrho'$. If $\mathrm{cf}\,(\tau)=\omega$ or $\mathrm{cf}\,(\tau)=\omega_1$, then (3.13) holds iff $1+\gamma\leqq\varrho'$. If $\mathrm{cf}\,(\omega_2^\tau)=\omega_2$, then (3.13) holds iff $\gamma\leqq\varrho'$. These follow from Theorems 3.1 and 3.3 and the elementary fact that $\varrho'<\gamma\Leftrightarrow\delta+\varrho'<\delta+\gamma$ for any δ. To conclude this chapter we give some results about possible improvements of our theorems.

An easy iteration gives the following improvement of Theorem 3.1 c):

If $\omega+1\leqq\tau<\omega_3$, $\mathrm{cf}\,(\omega_2^\tau)=\omega_2$, $0<\gamma<\omega_1$, *and* $n<\omega$, *then*

$$\binom{\omega_1}{\omega_2^{\tau+\gamma}}\to\binom{1 \quad \omega_1}{\omega_2^{\omega+1+\gamma} \quad \omega_2^{\tau}n}^{1,1}.$$

In particular,

$$\binom{\omega_1}{\omega_2^{\omega+2}}\to\binom{1 \quad \omega_1}{\omega_2^{\omega+2} \quad \omega_2^{\omega+1}n}^{1,1} \quad \text{for} \quad n<\omega.$$

We omit the proof of this, but it is intriguing to note that this cannot be improved by replacing n by ω.

THEOREM 3.4.

$$\binom{\omega_1}{\omega_2^{\omega+2}} \nrightarrow \begin{pmatrix} 1 & \omega_1 \\ \omega_2^{\omega+1}+1 & \omega_2^{\omega+1}\omega \end{pmatrix}^{1,1}$$

is consistent with G.C.H. (e.g. holds in L).

PROOF. We apply Lemma 2.5 with $\varkappa=\omega_2$, $\zeta=\omega_1$, $\eta=\omega_2$, $\eta_0=1$, $\zeta_1=\omega_1$, $\eta_1=\omega$, $\Xi=\omega_2^{\omega+2}$, $\Xi_\nu=\omega_2^{\omega+1}$ for $\nu<\omega_2$. It is easy to check that $\Phi=\omega_2^{\omega+1}+1$, $\Psi=\omega_2^{\omega+1}\omega$. Hence we only have to establish the consistency of

$$\binom{\omega_1}{\omega_2} \nrightarrow \left(\begin{matrix}1\\1\end{matrix}\ \begin{bmatrix}\omega_1\\ \omega\end{bmatrix}_{\omega,\ <\omega\uparrow}\right)^{1,1}. \tag{3.15}$$

This follows from the following statement:

(3.16) *There is a function* $f:\omega_1\times\omega_2\to\omega$ *such that for all* $A\in[\omega_1]^{\omega_1}$, $B\in[\omega_2]^{\omega}$ *there is a* $\nu\in B$ *so that* $f''(A\times\{\nu\})=\omega$.

This has been proved to be consistent with G.C.H. by PRIKRY [14]. Later JENSEN [10] showed using morasses that Prikry's result (3.16) holds in L.

Finally we are going to prove that (3.3) of Theorem 3.2 is best possible.

THEOREM 3.5. *Assume* $\tau<\omega_3$, $\mathrm{cf}(\tau)=\omega_1$, $\xi<\omega_1$. *Then*

$$\binom{\omega_1}{\omega_2^{\xi}} \to \begin{pmatrix} 1 & \omega_1 \\ \omega_2^{\omega}\xi & \omega_2^{\xi} \end{pmatrix}^{1,1}.$$

In order to prove this we need a lemma on set mappings which is similar to a theorem of ERDŐS and SPECKER [15].

LEMMA 3.6. *Let* $\xi<\omega_1$ *and let* $f:\omega_1\to P(\omega_1)$ *be a set mapping on* ω_1 *such that* $\mathrm{tp}\, f(x)<\xi$ *for all* $x\in\omega_1$. *Then there are* $X, Y\in[\omega_1]^{\omega_1}$ *such that* $f(X)\cap\overline{Y}=\emptyset$, *where* $\overline{Y}$ *denotes the closure of* Y *in* ω_1.

PROOF. We will assume that the lemma is false and obtain a contradiction. For $U, V\subset\omega_1$, let $S(U, V)=\{x\in U: f(x)\cap\overline{V}=\emptyset\}$. Suppose $A, B\in[\omega_1]^{\omega_1}$ and that

$$|S(A, \overline{Y})| = \aleph_1 \quad \textit{whenever} \quad Y\subset B \quad \textit{and} \quad 0<|Y|\leqq\aleph_0. \tag{3.17}$$

Choose $x_0\in A$, $y_0\in B$ so that $f(x_0)<\{y_0\}$. Now let $0<\nu<\omega_1$ and suppose that we have already chosen $x_\mu\in A$, $y_\mu\in B$ for $\mu<\nu$ so that $f(X_\nu)\cap\overline{Y}_\nu=\emptyset$, where $X_\nu=\{x_\mu:\mu<\nu\}$ and $Y_\nu=\{y_\mu:\mu<\nu\}$. By (3.17) there is $x_\nu\in S(A, \overline{Y}_\nu)\setminus X_\nu$. Choose $y_\nu\in B$ so that $Y_\nu\cup f(X_\nu\cup\{x_\nu\})<\{y_\nu\}$. Then, contrary to our assumption, the lemma is satisfied with $X=\{x_\nu:\nu<\omega_1\}$ and $Y=\{y_\nu:\nu<\omega_1\}$. It follows that, whenever $A, B\in[\omega_1]^{\omega_1}$, then there are $x(A, B)\in A$ and $Y(A, B)\subset B$ such that $0<|Y(A, B)|\leqq\aleph_0$ and

$$f(x)\cap\overline{Y(A, B)}\neq\emptyset \quad \textit{for all} \quad x\in A \quad \textit{such that} \quad x\geqq x(A).$$

We now define ordinals $\alpha_\nu<\omega_1$ for $\nu<\omega_1$ by transfinite induction as follows. Let $\nu<\omega_1$ and suppose that we have already defined α_μ for $\mu<\nu$. Let $\beta_\nu=\sup\{\alpha_\mu:\mu<\nu\}$, $B_\nu=\{\varrho:\beta_\nu\leqq\varrho<\omega_1\}$. Let $Y_\nu=Y(B_\nu, B_\nu)$, $x_\nu=x(B_\nu, B_\nu)$ and choose $\alpha_\nu<\omega_1$ so that $\bar{Y}_\nu\cup\{x_\nu\}\cup f(x_\nu)<\{\alpha_\nu\}$. This defines Y_ν and α_ν for all $\nu<\omega_1$ so that $\bar{Y}_0<\bar{Y}_1<\dots$ and

$$f(x)\cap\bar{Y}_\mu\neq\emptyset \quad \textit{for} \quad \mu\leqq\nu \quad \textit{and} \quad \alpha_\nu\leqq x<\omega_1.$$

It follows from this that $\operatorname{tp} f(\alpha_\xi)\geqq\xi$, and this is the desired contradiction.

PROOF OF THEOREM 3.5. Let $\langle S_\nu:\nu<\omega_1\rangle$ be the standard decomposition of ω_2^ξ, and let $\langle A_\alpha:\alpha<\omega_1\rangle$ be a system of subsets of ω_2^ξ such that $\operatorname{tp} A_\alpha<\omega_2^\omega\xi$ $(\alpha<\omega_1)$.

For $\alpha<\omega_1$, let $f(\alpha)=\{\nu<\omega_1:\operatorname{tp}(A_\alpha\cap S_\nu)\geqq\omega_2^\omega\}$. Then $\operatorname{tp} f(\alpha)<\xi$ $(\alpha<\omega_1)$. Therefore, by Lemma 3.6 we can assume that

$$\operatorname{tp}(A_\alpha\cap S_\nu)<\omega_2^\omega \quad \text{for} \quad \alpha,\nu<\omega_1.$$

There is no loss of generality if we assume that $A_\alpha\neq\emptyset$ $(\alpha<\omega_1)$ and then for $\alpha,\nu<\omega_1$ there is an integer $n(\alpha,\nu)$ such that

$$\omega_2^{n(\alpha,\nu)}\leqq\operatorname{tp}(A_\alpha\cap S_\nu)<\omega_2^{n(\alpha,\nu)+1}. \tag{3.18}$$

Now for $\alpha<\omega_1$ there are $\sigma(\alpha)<\xi$ and $\tau(\alpha)<\omega_2^\omega$ such that $\operatorname{tp} A_\alpha=\omega_2^\omega\sigma(\alpha)+$ $+\tau(\alpha)$. Hence, there are ordinals μ_σ^α $(\sigma\leqq\sigma(\alpha))$ such that $\mu_0^\alpha<\mu_1^\alpha<\dots$, $g(\alpha)=$ $=\{\mu_\sigma^\alpha:\sigma\leqq\sigma(\alpha)\}$ is closed in ω_1 and

$$\operatorname{tp}(A_\alpha\cap\bigcup\{S_\nu:\ \mu_\sigma^\alpha\leqq\nu<\mu_{\sigma+1}^\alpha\})=\omega_2^\omega \quad \text{for} \quad \sigma<\sigma(\alpha),$$

$$\operatorname{tp}(A_\alpha\cap\bigcup\{S_\nu:\ \mu_{\sigma(\alpha)}^\alpha\leqq\nu<\omega_1\})=\tau(\alpha).$$

Since $\operatorname{tp} g(\alpha)\leqq\xi$, it follows from Lemma 3.6 that there are $D,E\in[\omega_1]^{\omega_1}$ such that $g(D)\cap\bar{E}=\emptyset$.

Suppose that for some $\alpha\in D$ the set $\{n(\alpha,\nu):\nu\in E\}$ is unbounded. Then there are $\nu_i\in E$ such that $\nu_0<\nu_1<\dots$ and $n(\alpha,\nu_0)<n(\alpha,\nu_1)<\dots$. Therefore, by (3.18),

$$\operatorname{tp}(A_\alpha\cap\bigcup\{S_{\nu_i}:\ i<\omega\})=\omega_2^\omega$$

and hence $\nu=\sup\{\nu_i:i<\omega\}\in g(\alpha)$. This contradicts the fact that $g(\alpha)\cap\bar{E}=\emptyset$. It follows that, for each $\alpha\in D$ there is $n(\alpha)<\omega$ such that $n(\alpha,\nu)<n(\alpha)$ for all $\nu\in E$. There is $D_1\in[D]^{\omega_1}$ such that $n(\alpha)=n$ for all $\alpha\in D_1$ and hence $\operatorname{tp}(S_\nu\cap\bigcup$ $\bigcup\{A_\alpha:\alpha\in D_1\})<\omega_2^n$ $(\nu\in E)$. It follows that $\operatorname{tp}(\omega_2^\xi\setminus\bigcup\{A_\alpha:\alpha\in D_1\})=\omega_2^\xi$.

4. A discussion of (2.2). In this section we discuss what happens when the term ω_1 on the right-hand side of (3.1) is replaced by ω, i.e. we are going to investigate relations of the form

$$\begin{pmatrix}\omega_1\\ \omega_2^\varrho\end{pmatrix}\to\begin{pmatrix}1 & \omega\\ \omega_2^\sigma & \omega_2^\tau\end{pmatrix}^{1,1} \tag{4.1}$$

for $\varrho,\tau<\omega_3$ and $\sigma<\omega_1$. We have already indicated in § 2 the need to restrict our attention to the case $\sigma<\omega_1$. For this case we shall give a complete analysis of (4.1) under the assumption $2^{\aleph_0}=\aleph_1$. The main part of this section and the next is devoted to proving the positive relations stated in Theorem 4.1. The negative relations given in Theorem 4.6 are much simpler to prove.

THEOREM 4.1. *Assume* $2^{\aleph_0}=\aleph_1$. *Let* $\sigma<\omega_1$, $\varrho=\omega_1\xi+\gamma$, *where* $\xi<\omega_3$ *and* $\gamma<\omega_1$. *Then*

$$\binom{\omega_1}{\omega_2^\varrho} \to \binom{\omega}{\omega_2^\varrho}_k^{1,1} \quad \text{if} \quad k<\omega \quad \text{and} \quad \xi=0, \tag{4.2}$$

$$\binom{\omega_1}{\omega_2^\varrho} \to \begin{pmatrix} 1 & \omega \\ \omega_2^\sigma & \omega_2^\varrho \end{pmatrix}^{1,1} \quad \text{if} \quad \mathrm{cf}(\xi)=\omega \quad \text{or} \quad \omega_2, \tag{4.3}$$

$$\binom{\omega_1}{\omega_2^\varrho} \to \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega+\gamma} & \omega_2^\varrho \end{pmatrix}^{1,1} \quad \text{if} \quad \xi>0. \tag{4.4}$$

REMARKS. 1. We do not know if the 1 on the right-hand sides of (4.3) and (4.4) can be replaced by ω.

2. The relations (4.3), (4.4) show that the situation here is significantly different from the case of (2.1) (at least under the assumptions $2^{\aleph_0}=\aleph_1$ and $2^{\aleph_1}=\aleph_2$). By Theorem 3.1 the relation

$$\binom{\omega_1}{\omega_2^\gamma} \to \begin{pmatrix} 1 & \omega_1 \\ \omega_2^\gamma & \omega_2^\gamma \end{pmatrix}$$

holds for $\gamma\leqq\omega+1$, but it is false for $\gamma=\omega+2$. In fact, by Theorem 3.3, we even have the stronger negative relation that (assuming $2^{\aleph_1}=\aleph_2$)

$$\binom{\omega_2}{\omega_2^{\delta+1}} \not\to \begin{pmatrix} 1 & \omega_1 \\ \omega_2^{\omega+1}+1 & \omega_2^{\delta+1} \end{pmatrix}$$

holds for all $\delta<\omega_3$.

Our proofs of (4.2)—(4.4) are quite complicated, and we need several lemmas. The main idea is most clearly seen in the proof of (4.2). The *reduction lemma* (Lemma 4.3) enables us to faithfully represent a system of $\aleph_1$ "small" subsets of ω_2^γ $(<\omega_2^{\omega_1})$ by a system of "small" subsets of a countable set of type ω^γ. Using this we easily reduce (4.2) to a known theorem for countable ordinals due to BAUMGARTNER and HAJNAL [3] that

$$\binom{\omega_1}{\omega^\gamma} \to \binom{\omega}{\omega^\gamma}_k^{1,1} \quad (k<\omega;\ \gamma<\omega_1). \tag{4.5}$$

The same ideas are needed to prove (4.3) and (4.4) but we need generalizations of (4.5) (Theorem 4.2) and the reduction lemma (Lemma 4.3).

We first introduce some special notation. Let $1\leqq n<\omega$ and let $\boldsymbol{\xi}=(\xi_0,\dots,\xi_{n-1})$ be a sequence of indecomposable ordinals of length n. Put $\Pi(\boldsymbol{\xi})=\xi_0\times\dots\times\xi_{n-1}$, the cartesian product. If $n>1$, we write $\hat{\boldsymbol{\xi}}$ to denote the sequence $(\xi_0,\dots,\xi_{n-2})$ obtained by deleting the last term. Also, for $X\subset\Pi(\boldsymbol{\xi})$ we denote by $\hat{X}$ the projection of X into $\Pi(\hat{\boldsymbol{\xi}})$. For $u\in\xi_{n-1}$ and $X\subset\Pi(\boldsymbol{\xi})$, define $X^u=X\cap(\xi_0\times\dots\times\xi_{n-2}\times\{u\})$ and $\hat{X}^{(u)}=\widehat{X^u}$.

We now generalize the concept of a *full-sized* subset of an ordinal, i.e. we shall define a relation $X\in F(\boldsymbol{\xi})$ for subsets of $\Pi(\boldsymbol{\xi})$ by induction on the length of $\boldsymbol{\xi}=(\xi_0,\dots,\xi_{n-1})$. For $n=1$, $X\in F(\boldsymbol{\xi})$ if and only if X is a subset of ξ_0 such that $\mathrm{tp}\,X=\xi_0$. Now assume that $n>1$ and that $F(\boldsymbol{\eta})$ has been defined for sequences of indecomposable ordinals of length $n-1$. Then $X\in F(\boldsymbol{\xi})$ if and only if $X\subset\Pi(\boldsymbol{\xi})$ and $\mathrm{tp}\,\{u\in\xi_{n-1}:\hat{X}^{(u)}\in F(\hat{\boldsymbol{\xi}})\}=\xi_{n-1}$.

We now define the polarized partition relation

(4.6)
$$\binom{\omega_1}{\xi} \to \binom{\omega}{\xi}_k^{1,1}$$

for a sequence $\xi=(\xi_0, \ldots, \xi_{n-1})$ of indecomposable ordinals as follows. The relation (4.6) means that *whenever* $f:\omega_1\times\xi\to k$ *then there are* $D\in[\omega_1]^\omega$ *and a full-sized set* $X\in F(\xi)$ *such that* $D\times X$ *is homogeneous for* f. In the case $n=1$ this definition agrees with the normal polarized partition relation.

In order to prove (4.3) and (4.4) we need (4.7), a generalization of (4.5), and (4.8), a consequence of (4.5).

THEOREM 4.2. (a) *Let* $1\leq n,\ k<\omega$ *and let* $\xi=(\xi_0, \ldots, \xi_{n-1})$ *be a sequence of indecomposable denumerable ordinals. Then*

(4.7)
$$\binom{\omega_1}{\xi} \to \binom{\omega}{\xi}_k^{1,1}.$$

(b) *If* $\gamma<\omega_1$, *then*

(4.8)
$$\binom{\omega_1}{\omega^\gamma} \to \left(\begin{matrix}\omega\\ \omega^\gamma\end{matrix}, \begin{bmatrix}\omega\\ \omega^\gamma\end{bmatrix}_{\omega,<\omega\dagger}\right)^{1,1}.$$

We postpone the proof of Theorem 4.2 until the next section and proceed with a statement and proof of the representation lemma (Lemma 4.3) and its generalization (Lemma 4.4).

LEMMA 4.3. (Reduction lemma.) *Let* $\gamma<\omega_1$ *and let* $\langle A_\alpha:\alpha<\omega_1\rangle$ *be a sequence of subsets of* ω_2^γ *such that* $\operatorname{tp} A_\alpha<\omega_2^\gamma$. *Then there is a countable set* $X\subset\omega_2^\gamma$ *such that* $\operatorname{tp} X=\omega^\gamma$ *and* $\operatorname{tp}(A\cap X)<\omega^\gamma$ *for all* $\alpha<\omega_1$.

PROOF. We will prove a slightly stronger statement. Let $\varrho_\alpha=\min\{\varrho: \operatorname{tp} A_\alpha<\omega_2^\varrho\}$ $(\alpha<\omega_1)$. Then there is $X\subset\omega_2^\gamma$ such that $\operatorname{tp} X=\omega^\gamma$ and

(3.4)
$$\operatorname{tp}(A_\alpha\cap X) < \omega^{\varrho_\alpha} \quad (\alpha<\omega_1).$$

The proof is by induction on γ. For $\gamma=0$ the statement is trivial. Now let $0<\gamma<\omega_1$ and assume the result is true for smaller ordinals. We can assume that $A_\alpha\neq\emptyset$ $(\alpha<\omega_1)$ and hence that $\varrho_\alpha=\varrho'_\alpha+1$. We distinguish the two cases (1) $\gamma=\delta+1$ and (2) $\operatorname{cf}(\gamma)=\omega$.

Case 1. Let $\langle S_\nu:\nu<\omega_2\rangle$ be the standard decomposition of ω_2^γ. Then $\operatorname{tp} S_\nu=\omega_2^\delta$ $(\nu<\omega_2)$. Now for each $\alpha<\omega_1$ there is $\nu_\alpha<\omega_2$ such that $\operatorname{tp}(A_\alpha\cap S_\nu)<\omega_2^{\varrho'_\alpha}$ for $\nu_\alpha<\nu<\omega_2$. Choose a set $D\subset\omega_2$ such that $\operatorname{tp} D=\omega$ and such that $\nu_\alpha<\nu$ for all $\alpha<\omega_1$ and $\nu\in D$. By the induction hypothesis, for each $\nu\in D$ there is a set $X_\nu\subset S_\nu$ such that $\operatorname{tp} X_\nu=\omega^\delta$ and $\operatorname{tp}(A_\alpha\cap X_\nu)<\omega^{\varrho'_\alpha}$ $(\alpha<\omega_1;\ \nu\in D)$. Put $X=\bigcup\{X_\nu:\nu\in D\}$. Then $\operatorname{tp} X=\omega^\gamma$ and (3.4) holds.

Case 2. Let $\langle S_n:n<\omega\rangle$ be the standard decomposition of ω_2^γ. We can assume that $\operatorname{tp} S_n=\omega_2^{\gamma_n+1}$ $(n<\omega)$, where $\langle\gamma_n:n<\omega\rangle$ is an increasing sequence of ordinals with limit γ. Let $\langle S_{n\nu}:\nu<\omega_2\rangle$ be the standard decomposition of S_n. Then $\operatorname{tp} S_{n\nu}=\omega_2^{\gamma_n}$ $(n<\omega;\ \nu<\omega_2)$. For each $\alpha<\omega_1$ there is $\nu_\alpha<\omega_2$ such that $\operatorname{tp}(A_\alpha\cap S_{n\nu})<\omega_2^{\varrho'_\alpha}$ for all $n<\omega$ and $\nu_\alpha<\nu<\omega_2$. Choose $\nu^*<\omega_2$ so that $\nu_\alpha<\nu^*$ for all $\alpha<\omega_1$. By the induction hypothesis there are sets $X_n\subset S_{n\nu^*}$ $(n<\omega)$ such that $\operatorname{tp} X_n=\omega^{\gamma_n}$

$(n<\omega)$ and $\operatorname{tp}(A_\alpha \cap X_n) < \omega^{\varrho'_\alpha}$ $(\alpha<\omega_1;\ n<\omega)$. Put $X=\bigcup\{X_n: n<\omega\}$. Then $\operatorname{tp} X = \omega^\gamma$ and $\operatorname{tp}(A_\alpha \cap X) \leqq \omega^{\varrho'_\alpha} < \omega^{\varrho_\alpha}$ $(\alpha<\omega_1)$.

We now prove the generalized reduction lemma as follows.

LEMMA 4.4. *Let* $1 \leqq n < \omega$, $\boldsymbol{\xi}=(\xi_0, \dots, \xi_{n-1})$, *where* $\xi_i = \varkappa_i^{\gamma_i}$, $\gamma_i < \omega_1$ *and* $\varkappa_i \in \{\omega, \omega_2\}$ *for* $i<n$. *Suppose* $\langle A_\alpha : \alpha < \omega_1 \rangle$ *is a sequence of subsets of* $\Pi(\boldsymbol{\xi})$ *such that no* A_α *is full-sized in* $\Pi(\boldsymbol{\xi})$. *Then there is a sequence* $\mathbf{X}=(X_0, \dots, X_{n-1})$ *such that* $X_i \subset \varkappa_i^{\gamma_i}$, $\operatorname{tp} X_i = \omega^{\gamma_i}$ $(i<n)$ *and no set* $A_\alpha \cap \Pi(\mathbf{X})$ *is full-sized in* $\Pi(\mathbf{X})$.

PROOF. For $n=1$ the statement is either trivial (if $\varkappa_0=\omega$) or follows from Lemma 4.3 (if $\varkappa_0=\omega_2$). We now assume that $n>1$ and use induction.

For each $\alpha<\omega_1$, put $B_\alpha=\{u \in \xi_{n-1}: \hat{A}_\alpha^{(u)} \in F(\hat{\boldsymbol{\xi}})\}$. Then by the assumption that $A_\alpha \notin F(\boldsymbol{\xi})$, it follows that $\operatorname{tp} B_\alpha < \xi_{n-1}$. Therefore, by Lemma 4.3, there is $X_{n-1} \subset \xi_{n-1}$ such that $\operatorname{tp} X_{n-1} = \omega^{\gamma_{n-1}}$ and $\operatorname{tp}(B_\alpha \cap X_{n-1}) < \omega^{\gamma_{n-1}}$ for all $\alpha<\omega_1$.

Now consider the system of sets $\langle \hat{A}_\alpha^{(u)} : \alpha<\omega_1,\ u \in D_\alpha \rangle$, where $D_\alpha = \{u \in X_{n-1}: \hat{A}_\alpha^{(u)} \notin F(\hat{\boldsymbol{\xi}})\}$. By the induction hypothesis it follows that there are sets $X_i \subset \xi_i$ for $i<n-1$ such that $\operatorname{tp} X_i = \omega^{\gamma_i}$ $(i<n-1)$ and such that

$$\hat{A}_\alpha^{(u)} \cap \Pi(\hat{\mathbf{X}}) \notin F(\hat{\mathbf{X}}) \quad (\alpha < \omega_1\colon\ u \in D_\alpha),$$

where $\mathbf{X}=(X_0, \dots, X_{n-1})$ and $\hat{\mathbf{X}}=(X_0, \dots, X_{n-2})$.

To complete the proof we have to show that for each $\alpha<\omega_1$ the set $A_\alpha \cap \Pi(\mathbf{X})$ is not full-sized, i.e. we have to verify that the set

$$C_\alpha = \{u \in X_{n-1}\colon\ \hat{A}_\alpha^{(u)} \cap \Pi(\hat{\mathbf{X}}) \in F(\hat{\mathbf{X}})\}$$

has order type less than $\omega^{\gamma_{n-1}}$. Now, if $u \in C_\alpha$, then $u \in X_{n-1} \setminus D_\alpha$ and so $\hat{A}_\alpha^{(u)} \in F(\hat{\boldsymbol{\xi}})$. Thus $C_\alpha \subset B_\alpha \cap X_{n-1}$ and so has type less than $\omega^{\gamma_{n-1}}$.

We now use the reduction lemmas to obtain "higher" analogues of (4.7) and (4.8). The special case $n=1$, $\varkappa_0=\omega_2$ of Theorem 4.5 (a) gives (4.2).

THEOREM 4.5. *Assume* $2^{\aleph_0}=\aleph_1$.

(a) *If* $1 \leqq n, k < \omega$, $\boldsymbol{\xi}=(\xi_0, \dots, \xi_{n-1})$, $\xi_i=\varkappa_i^{\gamma_i}$, $\gamma_i<\omega_1$, $\varkappa_i \in \{\omega, \omega_2\}$ *for* $i<n$, *then*

$$\binom{\omega_1}{\boldsymbol{\xi}} \to \binom{\omega}{\boldsymbol{\xi}}_k^{1,1}. \tag{4.9}$$

(b) *If* $\gamma<\omega_1$, *then*

$$\binom{\omega_1}{\omega_2^\gamma} \to \left(\begin{matrix}\omega \\ \omega_2^\gamma\end{matrix},\ \begin{bmatrix}\omega \\ \omega_2^\gamma\end{bmatrix}_{\omega,\,<\omega\dagger}\right)^{1,1}. \tag{4.10}$$

PROOF. (a) Assume this is false. Then there is a function $f: \omega_1 \times \Pi(\boldsymbol{\xi}) \to k$ which disproves (4.9).

For each $D \in [\omega_1]^\omega$ and $j<k$, put

$$A(D, j) = \{x \in \Pi(\boldsymbol{\xi})\colon f''(D \times \{x\}) = \{j\}\}$$

Since f disproves (4.9) it follows that the sets $A(D, j)$ are not full-sized in $\Pi(\boldsymbol{\xi})$. By the hypothesis $2^{\aleph_0}=\aleph_1$ there are at most $\aleph_1$ sets $A(D, j)$ and hence by Lemma 4.4 there is a sequence $\mathbf{X}=(X_0, \dots, X_{n-1})$ such that $X_i \subset \varkappa_i^{\gamma_i}$, $\operatorname{tp} X_i = \omega^{\gamma_i}$ $(i<n)$

and $A(D, j)\cap\Pi(\mathbf{X})$ is not full-sized in $\Pi(\mathbf{X})$ for $D\in[\omega_1]^\omega$ and $j<k$. It follows that $f\restriction\omega_1\times\Pi(\mathbf{X})$ disproves

$$\binom{\omega_1}{\boldsymbol{\eta}}\to\binom{\omega}{\boldsymbol{\eta}}_k^{1,1},$$

where $\boldsymbol{\eta}=(\omega^{\gamma_0},\ldots,\omega^{\gamma_{n-1}})$, and this contradicts Theorem 4.2(a).

(b) The proof is essentially the same as the proof of (a). Assume that there is a partial function f from $\omega_1\times\omega_2^\gamma$ into ω which disproves (4.10). For each $D\in[\omega_1]^\omega$ let $A(D)=\{\xi<\omega_2^\gamma:(D\times\{\xi\})\cap D(f)=\emptyset\}$ and $B(D)=\{\xi<\omega_2^\gamma:D\times\{\xi\}\subset D(f)\wedge\wedge|f''(D\times\{\xi\})|<\omega\}$. By the assumption on f, $A(D)$ and $B(D)$ have order type $<\omega_2^\gamma$ for $D\in[\omega_1]^\omega$. Therefore, by Lemma 4.3, there is a set $X\subset\omega_2^\gamma$ such that $\operatorname{tp}X=\omega^\gamma$ and $A(D)\cap X$ and $B(D)\cap X$ both have type $<\omega^\gamma$ for each $D\in[\omega_1]^\omega$. Thus $f\restriction\omega_1\times X$ disproves

$$\binom{\omega_1}{\omega^\gamma}\to\left(\begin{matrix}\omega\\ \omega^\gamma\end{matrix},\ \begin{bmatrix}\omega\\ \omega^\gamma\end{bmatrix}_{\omega,<\omega}\right)^{1,1},$$

contrary to Theorem 4.2(b).

We now prove the main result.

Proof of Theorem 4.1. As we already remarked, (4.2) is a special case of Theorem 4.5(a).

Also, (4.4) follows immediately from (4.10) and Lemma 2.6 (apply the lemma with $\varkappa=\omega_2$, $\xi=\omega_1$, $\xi_1=\omega$, $\eta=\eta_0=\eta_1=\omega_2^\gamma$ and $\Xi_\nu=\varrho-\gamma$ $(\nu<\omega_2^\gamma)$).

It remains to prove (4.3). Let $\varkappa=\operatorname{cf}(\xi)$. Then $\varkappa=\omega$ or ω_2. Let $\langle\tau_\nu:\nu<\varkappa\rangle$ be a strictly increasing sequence of ordinals with limit $\omega_1\xi$. Then $\tau_\nu+\sigma\nearrow\omega_1\xi$ $(\nu<\varkappa)$ also. Thus we may write $\omega_2^\varrho=\omega_2^{\omega_1\xi+\gamma}=\left(\sum\{\omega_2^{\tau_\nu+\sigma}:\nu<\varkappa\}\right)\cdot\omega_2^\gamma$. Put $\xi_0=\omega_2^\sigma$, $\xi_1=\varkappa$, $\xi_2=\omega_2^\gamma$, $\boldsymbol{\xi}=(\xi_0,\xi_1,\xi_2)$. Let $<_0$ denote the antilexicographic ordering of $\Pi(\boldsymbol{\xi})$. Now there are pairwise disjoint sets $S_{(\mu,\nu,\theta)}$ $((\mu,\nu,\theta)\in\Pi(\boldsymbol{\xi}))$ such that

$$\begin{cases}\omega_2^\varrho=\bigcup\{S_{(\mu,\nu,\theta)}:(\mu,\nu,\theta)\in\Pi(\boldsymbol{\xi})\},\ \operatorname{tp}S_{(\mu,\nu,\theta)}=\omega^{\tau_\nu},\\ S_{(\mu,\nu,\theta)}<S_{(\mu',\nu',\theta')}\quad\text{for}\quad(\mu,\nu,\theta)<_0(\mu',\nu',\theta').\end{cases}\tag{4.11}$$

Suppose $\langle A_\alpha:\alpha<\omega_1\rangle$ is a sequence of subsets of ω_2^ϱ such that $\operatorname{tp}A_\alpha<\omega_2^\sigma$ $(\alpha<\omega_1)$. In order to prove (4.3) we have to show that there are $D\in[\omega_1]^\omega$ and $C\subset\omega_2^\varrho$ such that $\operatorname{tp}C=\omega_2^\varrho$ and $A_\alpha\cap C=\emptyset$ for all $\alpha\in D$.

For $\alpha<\omega_1$, let $B_\alpha=\{(\mu,\nu,\theta)\in\Pi(\boldsymbol{\xi}):A_\alpha\cap S_{(\mu,\nu,\theta)}\neq\emptyset\}$. Clearly, for fixed $\nu<\varkappa$ and $\theta<\omega_2^\gamma$,

$$\operatorname{tp}\{\mu<\omega_2^\sigma:(\mu,\nu,\theta)\in B_\alpha\}\leqq\operatorname{tp}A_\alpha<\omega_2^\sigma\quad(\alpha<\omega_1),$$

and so the sets B_α $(\alpha<\omega_1)$ are not full-sized in $\Pi(\boldsymbol{\xi})$. By Theorem 4.5(a) (for $k=2$) it follows that there are $D\in[\omega_1]^\omega$ and $B\subset\Pi(\boldsymbol{\xi})$ such that B is full-sized and $B_\alpha\cap B=\emptyset$ for all $\alpha\in D$. Put $C=\bigcup\{S_{(\mu,\nu,\theta)}:(\mu,\nu,\theta)\in B\}$. Then, by (4.11) and the fact that B is full-sized in $\Pi(\boldsymbol{\xi})$, we see that $\operatorname{tp}C=\omega_2^\varrho$. Moreover, by the definition of B_α, we have $A_\alpha\cap C=\emptyset$ for all $\alpha\in D$.

We conclude this chapter by showing that the positive results of Theorem 4.1 are best possible.

Theorem 4.6. *Let* $\tau<\omega_3$, $\operatorname{cf}(\tau)=\omega_1$, $0<\gamma<\omega_1$. *Then*

$$\binom{\omega_1}{\omega_2^\tau}\nrightarrow\begin{pmatrix}1 & \omega\\ \omega_2^\omega\omega_1+1 & \omega_2^\tau\end{pmatrix}^{1,1},\tag{4.12}$$

and

(4.13) $$\binom{\omega_1}{\omega_2^{\tau+\gamma}} \nrightarrow \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega+\gamma}+1 & \omega_2^{\tau} \end{pmatrix}^{1,1}.$$

PROOF. It is easy to see that (4.12) follows from Lemma 2.5 and the relation

(4.14) $$\binom{\omega_1}{\omega_1} \nrightarrow \left(\begin{matrix}1\\1\end{matrix}, \begin{bmatrix}\omega\\ \omega_1\end{bmatrix}_{\omega, <\omega\uparrow}\right)^{1,1}.$$

(Apply Lemma 2.5 with $\xi=\omega_1$, $\xi_1=\omega$, $\eta=\eta_1=\omega_1$, $\eta_0=1$, $\Xi_\nu=\omega_2^{\tau_\nu}$ $(\nu<\omega_1)$, where $\tau_\nu\nearrow\tau$. An easy computation shows that $\Xi=\Psi=\omega_2^\tau$ and $\Phi=\omega_2^\omega\omega_1+1$). Now in order to prove (4.14) consider the function $f:\omega_1\times\omega_1\to\omega$ given by $f(\alpha,\nu)=$ $=f_\alpha(\nu)$ where the f_α are pairwise almost disjoint functions in ${}^{\omega_1}\omega$. Now for $A\in[\omega_1]^\omega$ and $B\in[\omega_1]^{\omega_1}$, there is $\nu_0\in B$ such that $f_\alpha(\nu_0)\neq f_{\alpha'}(\nu_0)$ whenever α,α' are distinct elements of A. Thus $f''(A\times\{\nu_0\})$ is infinite. This proves (4.14) and hence (4.12).

To prove (4.13) use induction on γ. If $\gamma=\delta+1$ take identical cross sections of the inductive systems establishing the negative relation for $\omega_2^{\tau+\delta}$. If $\mathrm{cf}(\gamma)=\omega$ take cross sections.

We remark that if we assume the transversal hypothesis, $TH(\omega_1)$: *there are ω_2 almost disjoint functions in* ${}^{\omega_1}\omega$ (which is known to be true in L and false if Chang's conjecture holds), then the argument used above yields the following stronger statements:

(4.15) $$\binom{\omega_2}{\omega_1} \nrightarrow \left(\begin{matrix}1\\1\end{matrix}, \begin{bmatrix}\omega\\ \omega_1\end{bmatrix}_{\omega, <\omega\uparrow}\right)^{1,1},$$

(4.16) $$\binom{\omega_2}{\omega_2^\tau} \nrightarrow \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega}\omega_1+1 & \omega_2^{\tau} \end{pmatrix}^{1,1} \quad \text{if} \quad \mathrm{cf}(\tau)=\omega_1,$$

(4.17) $$\binom{\omega_2}{\omega_2^{\tau+\gamma}} \nrightarrow \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega+\gamma}+1 & \omega_2^{\tau} \end{pmatrix}^{1,1} \quad \text{if} \quad \mathrm{cf}(\tau)=\omega_1,\ 0<\gamma<\omega_1.$$

Finally, we note that in the case $\tau<\omega_3$, $\mathrm{cf}(\tau)=\omega_1$ there is a gap between the positive result (assuming $2^{\aleph_0}=\aleph_1$)

$$\binom{\omega_1}{\omega_2^\tau} \to \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega} & \omega_2^{\tau} \end{pmatrix}^{1,1}$$

given by (4.4) and the negative result given by (4.12). This gap is easy to fill as the following theorem shows. Although this does not follow from Lemma 2.6, the proof is similar.

THEOREM 4.7. *If* $\tau<\omega_3$, $\mathrm{cf}(\tau)=\omega_1$, *and* $\delta<\omega_1$, *then*

$$\binom{\omega_1}{\omega_2^\tau} \to \begin{pmatrix} 1 & \delta \\ \omega_2^{\omega}\omega_1 & \omega_2^{\tau} \end{pmatrix}^{1,1}.$$

PROOF. Let $\langle S_\nu:\nu<\omega_1\rangle$ be the standard decomposition of ω_2^τ and let $\langle A_\alpha:\alpha<\omega_1\rangle$ be a system of subsets of ω_2^τ such that $\mathrm{tp}\,A_\alpha<\omega_2^\omega\omega_1$. For each $\alpha<\omega_1$ there are $\nu(\alpha)<\omega_1$ and $n(\alpha)<\omega$ such that $\mathrm{tp}\,(A_\alpha\cap S_\nu)<\omega_2^{n(\alpha)}$ for $\nu(\alpha)<\nu<\omega_1$. Now there is $D\subset\omega_1$ such that $\mathrm{tp}\,D=\delta$ and $n(\alpha)=n$ for all $\alpha\in D$. Clearly $\mathrm{tp}\,\{\omega_2^\tau\setminus\bigcup\{A_\alpha:\alpha\in D\}\}=\omega_2^\tau$.

5. Proof of Theorem 4.2. As we have already mentioned, both statements of Theorem 4.2 are generalizations of (4.4) which is a result of BAUMGARTNER and HAJNAL [3], and both can be proved with the methods used there. Since [3] appeared F. GALVIN [8] developed an elementary method to obtain the results of [3] and after we obtained Theorem 4.2 he kindly informed us that his method can be used to prove this theorem as well. However, we decided to give our original proof since it can be explained in less space.

As in [3] both statements will be proved first under the assumption that $MA_{\aleph_1}$ holds (for Martin's axiom see e.g. [12]). Then we exhibit partial orders which are well-founded iff the corresponding statements are false. This will show that if the statements are true in the standard model of Solovay and Tennenbaum yielding the consistency of $MA_{\aleph_1}$, then they are true in the ground model i.e. they are true in ZFC.

First we describe the partial orders.

a) Let $\xi=(\omega^{\gamma_i}:i<n)$, $0<\gamma_i<\omega_1$ $(i<n)$ and let $f:\omega_1\times\Pi(\xi)\to k$ be given. Let φ be a one-to-one mapping of ω onto $\Pi(\xi)$. For $m<\omega$ and $i<n$ we write $\varphi(m,i)$ to denote the i-th coordinate of $\varphi(m)\in\Pi(\xi)$. Let P be the set of all pairs of sequences $((\alpha_j: j<l), (\mathbf{u}_j: j<l))$, where $l<\omega$, the α_j are different ordinals $<\omega_1$ for $j<l$, $\mathbf{u}_j=(u_{j,0}, \dots, u_{j,n-1})\in\Pi(\xi)$ for $j<l$,

$$u_{j,i} < u_{j',i} \text{ iff } \varphi(j,i) < \varphi(j',i) \text{ for } j<j'<l,$$

and such that f is homogeneous on $\{\alpha_j:j<l\}\times\{\mathbf{u}_j:j<l\}$. The partial order is defined on P by the rule that $((\alpha'_j:j<l'), (\mathbf{u}'_j:j<l'))$ is an extension of $((\alpha_j:j<l), (\mathbf{u}_j:j<l))$ iff $(\alpha'_j:j<l')$ is an extension of $(\alpha_j:j<l)$ and $(\mathbf{u}'_j:j<l')$ is an extension of $(\mathbf{u}_j:j<l)$.

b) Let $\gamma<\omega_1$ and let f be a partial function from $\omega_1\times\omega^\gamma$ into ω. Let φ be a one-to-one mapping from ω onto ω^γ. Let P consist of all pairs $((\alpha_j:j<n), (u_j:j<l))$, where $l<\omega$, the α_j are different ordinals $<\omega_1$ for $j<l$, $u_j<\omega^\gamma$, $u_j<u_{j'}$ iff $\varphi(j)<\varphi(j')$ for $j<j'<l$, and such that *either* $(\{\alpha_j:j<l\}\times\{u_j:j<l\})\cap D(f)=0$ holds *or* $\{\alpha_j:j<l\}\times\{u_j:j<l\}\subset D(f)$ and $f(\alpha_{j'}, u_j)=f(a_{j''}, u_j)$ for $j<j', j''<l$. The extension is defined as in case a).

We leave the reader to check that these partial orders have the desired properties and proceed to derive the statements (4.7) and (4.8) from $MA_{\aleph_1}$.

PROOF OF (4.7) FROM $MA_{\aleph_1}$. Let $\xi=(\xi_i:i<n)$ be given, where $\xi_i=\omega^{\gamma_i}$, $\gamma_i<\omega_1$ for $i<n<\omega$.

For $X\subset\Pi(\xi)$ we denote by $\tilde{X}$ the projection of X into ξ_{n-1}. A subset $Y\subset X$ will be called a *section* of X (in $\Pi(\xi)$) if $X=\bigcup\{X^{\mathbf{u}}:u\in S\}$ and S is a section of $\tilde{X}$.

Instead of (4.7) we are going to prove the following stronger statement.

(5.1) *Assume* $f:\omega_1\times\Pi(\xi)\to k$, *where* $k<\omega$. *Then there are a full-sized subset* $X\subset\Pi(\xi)$, *increasing sections* $X_0\subset X_1\subset\dots$ *of* X (in $\Pi(\xi)$), *and functions* $m:\omega_1\to\omega$, $j:\omega_1\to k$ *such that* (i) $X=\bigcup\{X_i:i<\omega\}$ *and* (ii) $f''(\{\alpha\}\times\times(X\setminus X_{m(\alpha)}))=\{j(\alpha)\}$ (i.e. $\{\alpha\}\times(X\setminus X_{m(\alpha)})$ is homogeneous for f in the colour $j(\alpha)$).

We prove (5.1) by induction on n, the length of ξ. For $n=1$ this is just Lemma 3 of [3]. Now assume $n>1$.

Since f may be considered as a map from $(\omega_1\times\xi_{n-1})\times\Pi(\hat{\xi})$ into k, and since $\omega_1\times\xi_{n-1}$ has cardinality ω_1, it follows from the induction hypothesis that there are sets Y, Y_i $(i<\omega)$ and functions $m_1:\omega_1\times\xi_{n-1}\to\omega$ and $j_1:\omega_1\times\xi_{n-1}\to k$ such that Y is a full-sized subset of $\Pi(\hat{\xi})$, the Y_i are sections of Y (in $\Pi(\hat{\xi})$), $Y_0\subset Y_1\subset\dots$, $Y=\bigcup\{Y_i:i<\omega\}$ and

$$(5.2)\qquad f''(\{\alpha\}\times(Y\setminus Y_{m_1(\alpha,u)})\times\{u\})=\{j_1(\alpha,u)\}$$

holds for all $\alpha<\omega_1$ and $u\in\xi_{n-1}$.

Applying the induction hypothesis once more for the function $j_1:\omega_1\times\xi_{n-1}\to k$, it follows also that there are sets Z, Z_i $(i<\omega)$ and functions $m_2:\omega_1\to\omega$ and $j:\omega_1\to k$ such that Z is a full-sized subset of ξ_{n-1} (i.e. has type ξ_{n-1}), the Z_i are sections of Z, $Z_0\subset Z_1\subset\dots$, $Z=\bigcup\{Z_i:i<\omega\}$ and

$$(5.3)\quad j_1(\alpha,u)=j(\alpha) \text{ for } \alpha<\omega_1 \text{ and } u\in Z\setminus Z_{m_2(\alpha)}.$$

Put $\varphi_\alpha(u)=m_1(\alpha,u)$ for $\alpha<\omega_1$ and $u\in Z$. By a lemma of K. KUNEN (see e.g. [3]), $MA_{\aleph_1}$ implies that there is a function $\varphi:Z\to\omega$ such that

(5.4) $\varphi_\alpha(u)<\varphi(u)$ *for all but finitely many* $u\in Z$

holds for each $\alpha<\omega_1$.

Let X be the full-sized subset of $\Pi(\xi)$ such that $\tilde{X}=Z$ and $\hat{X}^{(u)}=Y\setminus Y_{\varphi(u)}$ $(u\in Z)$. Let X_i be the section of X determined by Z_i $(i<\omega)$. By (5.4) there is $m:\omega_1\to\omega$ such that $m(\alpha)\geqq m_2(\alpha)$ $(\alpha<\omega_1)$ and such that $\varphi_\alpha(u)<\varphi(u)$ for all $u\in Z\setminus Z_{m(\alpha)}$ $(\alpha<\omega_1)$. Now for $\alpha<\omega_1$ and $u\in Z\setminus Z_{m(\alpha)}$, we have $Y\setminus Y_{\varphi_\alpha(u)}\supset$ $\supset Y\setminus Y_{\varphi(u)}=\hat{X}^{(u)}$ and hence, by (5.2) and (5.3),

$$f''(\{\alpha\}\times\hat{X}^{(u)}\times\{u\})=\{j_1(\alpha,u)\}=\{j(\alpha)\},$$

i.e.

$$f''(\{\alpha\}\times(X\setminus X_{m(\alpha)}))=\{j(\alpha)\}.$$

PROOF OF (4.8) FROM $MA_{\aleph_1}$. Let now f be a partial function from $\omega_1\times\omega^\gamma$ into ω. By $MA_{\aleph_1}$ we know that

$$\binom{\omega_1}{\omega^\gamma}\to\binom{\omega_1}{\omega^\gamma}^{1,1}_k.$$

This is a theorem of [3] and in fact is a corollary of (4.7) just proved. Therefore, we may assume that $D(f)=\omega_1\times\omega^\gamma$. We can also assume that $\gamma>0$. Let $\{x_n:n<\omega\}=\omega^\gamma$ be a one-to-one enumeration of ω^γ. We define a sequence $\{y_n:n<\omega\}$ of elements of ω_1 and a sequence $\{Y_n:n<\omega\}$ of subsets of ω_1 by induction on $n<\omega$ as follows. $Y_0=\omega_1$. Assume that $n<\omega$ and that $Y_n\in[\omega_1]^{\omega_1}$ and y_j $(j<n)$ have already been defined. Choose a subset $Z_n\in[Y_n]^{\omega_1}$ such that $Z_n\times\{x_n\}$ is homogeneous for f. Let y_n be an arbitrary element of Z_n and $Y_{n+1}=$ $=Z_n\setminus\{y_n\}$. Let $A_1=\{y_n:n<\omega\}$ and $B_1=\omega^\gamma$. If $x_n\in B_1$, then $f''(A_1\times\{x_n\})=$ $=f''(\{y_i:i\leqq n\}\times\{x_n\})$ is clearly finite. This proves (4.8).

Note that in both (4.7) and (4.8) the ω on the right-hand sides can be replaced by any $\alpha<\omega_1$, but we do not need this.

6. The case of $\aleph_2$ sets. The first aim of this chapter is to prove an analogue of Theorem 1.1 for the case of $\aleph_2$ sets. This will show that the critical number for which the "$\aleph_2$-phenomena" appears in this case is ω_1+2 instead of ω_1.

THEOREM 6.1. *If* $\gamma \leqq \omega_2$, *then*

$$\binom{\omega_2}{\omega_2^\gamma} \nrightarrow \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega_1+2} & 1 \end{pmatrix}^{1,1}. \tag{6.1}$$

PROOF. We use induction on γ. The statement is clearly true for $\gamma \leqq \omega_1+1$. We now assume $\omega_1+1<\gamma\leqq\omega_2$ and that the statement is true for all $\delta<\gamma$. Let $\varkappa=\mathrm{cf}(\omega_2^\gamma)$ and let $\langle S_\nu:\nu<\varkappa\rangle$ be a standard decomposition of ω_2^γ, $\mathrm{tp}\, S_\nu=\omega_2^{\gamma_\nu}$, $(\nu<\varkappa)$. By the induction hypothesis we can choose a system of subsets $\langle A_\alpha^\nu:\alpha<\omega_2\rangle$ of S_ν $(\nu<\varkappa)$ such that $\mathrm{tp}\, A_\alpha^\nu<\omega_2^{\omega_1+2}$ and $S_\nu\subset\bigcup\{A_\alpha^\nu:\alpha\in D\}$ for $D\in[\omega_2]^\omega$. We now distinguish the three cases 1 $\varkappa=\omega_2$, (2) $\varkappa=\omega$, (3) $\varkappa=\omega_1$.

Case 1. Since $\gamma_\nu<\omega_2$ for $\nu<\omega_2$, by Theorem 1.1, we can find sets $B_\beta^\nu\subset S^\nu$ $(\beta<\omega_1)$ establishing

$$\binom{\omega_1}{\omega_2^{\gamma_\nu}} \nrightarrow \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega_1} & 1 \end{pmatrix}^{1,1}.$$

We choose a one-to-one mapping φ_ν of ν into ω_1 for $\nu<\omega_2$. Now for $\alpha<\omega_2$, let $A_\alpha=\bigcup\{A_\alpha^\nu:\nu<\alpha\}\cup\bigcup\{B_{\varphi_\nu(\alpha)}^\nu:\alpha<\nu\}$. Clearly, $\mathrm{tp}\, A_\alpha<\omega_2^{\omega_1+2}$ for $\alpha<\omega_2$. Let $D\in[\omega_2]^\omega$, $\nu<\omega_2$. Put $D_0=\{\alpha\in D:\nu<\alpha\}$, $D_1=\{\alpha\in D:\alpha<\nu\}$. Since either D_0 or D_1 is infinite, either $\bigcup\{\alpha\in D_0:A_\alpha^\nu\}$ or $\bigcup\{\alpha\in D_1:B_{\varphi_\nu(\alpha)}^\nu\}$ covers S_ν. Thus $\bigcup\{A_\alpha:\alpha\in D\}$ covers S_ν in any case. It follows that $\omega_2^\gamma\subset\bigcup\{A_\alpha:\alpha\in D\}$. Hence the system $\langle A_\alpha:\alpha<\omega_2\rangle$ establishes (6.1).

For cases 2, 3 take cross sections.

The partition relation just established shows the same "$\aleph_2$-phenomena" as the relation $\binom{\omega_1}{\omega_2^\gamma} \to \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega_1} & 1 \end{pmatrix}^{1,1}$. We do not go into details, but we will show the following equivalence:

$$\binom{\omega_2}{\omega_2^{\omega_2+1}} \to \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega_1+2} & 1 \end{pmatrix}^{1,1} \Leftrightarrow \binom{\omega_1}{\omega_2^{\omega_2}} \to \begin{pmatrix} 1 & \omega \\ \omega_2^{\omega_1} & 1 \end{pmatrix}^{1,1}. \tag{6.2}$$

The implication from the left is implicitly contained in the proof of Theorem 6.1, and the reverse implication is an easy corollary of the following lemma. We leave it to the reader to derive (6.2) from the lemma which we prove in detail since this will be needed for other purposes.

LEMMA 6.2. *Assume* $\delta<\omega_3$ *and that* $\mathrm{cf}(\omega_2^\delta)$ *is either* ω *or* ω_2. *Let* $\langle A_\alpha:\alpha<\omega_2\rangle$ *be a sequence of subsets of* ω_2^δ *such that* $\mathrm{tp}\, A_\alpha<\omega_2^{\omega_1+1}$ *for* $\alpha<\omega_2$. *Then there exist* $X\subset\omega_2^\delta$, $\gamma<\omega_1$ *and* $D\in[\omega_2]^{\omega_1}$ *such that* $\mathrm{tp}\, X=\omega_2^\delta$ *and* $\mathrm{tp}\, A_\alpha\cap X<\omega_2^\gamma$ *for all* $\alpha\in D$.

PROOF. Let $\varkappa=\mathrm{cf}(\omega_2^\delta)$ and let $\langle S_\nu:\nu<\varkappa\rangle$ be a standard decomposition of ω_2^δ. We distinguish the two cases (1) $\varkappa=\omega$, (2) $\varkappa=\omega_2$.

Case 1. We may assume that $\mathrm{tp}\, S_n=\omega_2^{\gamma_n+1}$ for $n<\omega=\varkappa$. Let $\langle S_{n,\varrho}:\varrho<\omega_2\rangle$ be a standard decomposition of S_n for $n<\omega$. Let $\alpha<\omega_2$. Using the fact that $\mathrm{tp}\, A_\alpha<\omega_2^{\omega_1+1}$, for each $n<\omega$ there are $\beta(\alpha,n)<\omega_1$ and $\varrho(\alpha,n)<\omega_2$ such that $\mathrm{tp}\,(A_\alpha\cap S_{n,\varrho})<\omega^{\beta(\alpha,n)}$ for $\varrho(\alpha,n)<\varrho<\omega_2$. Put $\beta(\alpha)=\sup\{\beta(\alpha,n):n<\omega\}$ and $\varrho(\alpha)=\sup\{\varrho(\alpha,n):n<\omega\}$. Then $\beta(\alpha)<\omega_1$ and $\varrho(\alpha)<\omega_2$ for $\alpha<\omega_2$. Clearly there are $D\in[\omega_2]^{\omega_1}$, $\gamma<\omega_1$ and $\sigma<\omega_2$ such that

$$\beta(\alpha)+1=\gamma \quad\text{and}\quad \varrho(\alpha)<\sigma \quad\text{for}\quad \alpha\in D.$$

Let $X=\bigcup\{S_{n,\sigma}:n<\omega\}$. Then X, γ and D satisfy the requirements of the lemma.

Case 2. For each $\alpha<\omega_2$ there are $\beta(\alpha)<\omega_1$ and $\varrho(\alpha)<\omega_2$ such that $\operatorname{tp}(S_\nu\cap A_\alpha)<\omega_2^{\beta(\alpha)}$ for $\varrho(\alpha)<\nu<\omega_2=\varkappa$. There are $D\in[\omega_2]^{\omega_1}$, $\gamma<\omega_1$ and $\sigma<\omega_2$ such that $\beta(\alpha)+1=\gamma$ and $\varrho(\alpha)<\sigma$ for $\alpha\in D$. Put $X=\bigcup\{S_\nu:\sigma\leqq\nu<\omega_2\}$. Then X, γ and D satisfy the requirements.

We now try to investigate the analogue of (3.1) for the case of $\aleph_2$-sets, i.e. we consider the relation

$$\binom{\omega_2}{\omega_2^{\varrho}}\rightarrow\begin{pmatrix}1 & \omega_1\\ \omega_2^{\gamma} & \omega_2^{\tau}\end{pmatrix}^{1,1} \tag{6.3}$$

for $\varrho,\tau<\omega_3$ and $\gamma<\omega_1+2$.

First we give an extension of the positive result Theorem 3.1.

THEOREM 6.2. *Assume* $\tau<\omega_3$, $\tau+\gamma\geqq\omega+\gamma$, *and* $\gamma<\omega_1+2$. *Then*

$$\binom{\omega_2}{\omega_2^{\tau+\gamma}}\rightarrow\begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega+\gamma} & \omega_2^{\tau}\end{pmatrix}^{1,1}.$$

Note that for $\gamma<\omega_1$ Theorem 3.1(b) yields a stronger result. For $\gamma=\omega_1$ or ω_1+1 this can be considered as a generalization of c) of Theorem 3.1 as well, since, $1+\gamma=\gamma$ holds in this case. In case $\gamma=\omega_1$ Theorem 6.2 is trivial since among $\aleph_2$ sets of type $<\omega_2^{\omega_1}$ there are $\aleph_2$ with type $<\omega_2^{\delta}$ for some $\delta<\omega_1$. Assume now that $A_\alpha\subset\omega_2^{\tau+\omega_1+1}$ and $\operatorname{tp}A_\alpha<\omega_2^{\omega_1+1}$ for $\alpha<\omega_2$. Then, by Lemma 6.2, there are $X\subset\subset\omega_2^{\tau+\omega_1+1}$, $D\in[\omega_2]^{\omega_1}$ and $\gamma<\omega_1$ such that

$$\operatorname{tp}X=\omega_2^{\tau+\omega_1+1},\quad\text{and}\quad\operatorname{tp}A_\alpha\cap X<\omega_2^{\gamma}$$

for $\alpha\in D$. Hence our claim follows from Theorem 3.1(b).

One could conjecture that, at least assuming $2^{\aleph_0}=\aleph_1$ and $2^{\aleph_1}=\aleph_2$, Theorem 6.2 gives the best possible positive result. However, we were not able to prove this. We now discuss the problem by considering separately the different possible values of $\operatorname{cf}(\tau)$. In case $\operatorname{cf}(\tau)=\omega_1$ we already noted ((4.16) and (4.17)) that if $TH(\omega_1)$ holds, then

$$\binom{\omega_2}{\omega_2^{\tau}}\nrightarrow\begin{pmatrix}1 & \omega\\ \omega_2^{\omega}\omega_1+1 & \omega_2^{\tau}\end{pmatrix}\quad\textit{for}\quad\tau<\omega_3,\ \operatorname{cf}(\tau)=\omega_1 \tag{6.4}$$

and

$$\binom{\omega_2}{\omega_2^{\tau+\gamma}}\nrightarrow\begin{pmatrix}1 & \omega\\ \omega_2^{\omega+\gamma}+1 & \omega_2^{\tau}\end{pmatrix}\quad\textit{for}\quad\tau<\omega_3,\ \operatorname{cf}(\tau)=\omega_1,\ 0<\gamma<\omega_1. \tag{6.5}$$

We are going to prove the following extension of this.

THEOREM 6.4. *Assume* $TH(\omega_1)$. *If* $\tau<\omega_3$, $\operatorname{cf}(\tau)=\omega_1$ *and* $0<\gamma<\omega_1+2$, *then*

$$\binom{\omega_2}{\omega_2^{\tau+\gamma}}\nrightarrow\begin{pmatrix}1 & \omega\\ \omega_2^{\omega+\gamma}+1 & \omega_2^{\tau}\end{pmatrix}^{1,1}.$$

PROOF. In view of (6.5) we only have to prove this for $\gamma=\omega_1$ and $\gamma=\omega_1+1$. Suppose $\gamma=\omega_1$. Let $\langle S_\nu:\nu<\omega_1\rangle$ be a standard decomposition of $\omega_2^{\tau+\omega_1}$ such that $\operatorname{tp}S_\nu=\omega_2^{\omega_1+\nu}$ for $\nu<\omega_1$. For each $\nu<\omega_1$ choose sets $A_\alpha^\nu\subset S_\nu$ $(\alpha<\omega_2)$ establishing

$$\binom{\omega_2}{\omega_2^{\tau+\nu}}\nrightarrow\begin{pmatrix}1 & \omega\\ \omega_2^{\omega+\nu}+1 & \omega_2^{\tau}\end{pmatrix}^{1,1}.$$

For each $\nu<\omega_1$ choose a paradoxical decomposition $\langle B_n^\nu : n<\omega\rangle$ of S_ν. By $TH(\omega_1)$ there are ω_2 almost disjoint functions $f_\alpha:\omega_1\to\omega$ $(\alpha<\omega_2)$. Now for $\alpha<\omega_2$ put

$$A'_\alpha = \bigcup\{A_\alpha^\nu : \nu<\omega_1\},\quad A''_\alpha = \bigcup\{B^\nu_{f_\alpha(\nu)} : \nu<\omega_1\},$$

$A_\alpha = A'_\alpha \cup A''_\alpha$. Clearly $\mathrm{tp}\, A_\alpha < \omega_2^{\omega_1}+1$. The union of countably many A'_α omits a set of type $<\omega_2^\tau$ from each S_ν $(\nu<\omega_1)$. The union of countably many A''_α covers an end section of $\omega_2^{\tau+\omega_1}$. It follows that

$$\mathrm{tp}\,(\omega_2^{\tau+\omega_1}\setminus\bigcup\{A_\alpha:\alpha\in D\}) < \omega_2^\tau$$

for all $D\in[\omega_2]^\omega$.

For the case $\gamma=\omega_1+1$, take identical cross sections of the systems obtained for $\gamma=\omega_1$.

We now make some remarks about these results.

First of all, it is not worth continuing the induction of the last theorem beyond $\gamma=\omega_1+1$. For, by combining the methods used to prove Theorems 6.1 and 6.2, we can easily prove:

If $TH(\omega_1)$ holds, $\tau<\omega_3$, $\mathrm{cf}(\tau)=\omega_1$ and $\gamma\leqq\omega_2$, then

$$(6.6)\qquad \begin{pmatrix}\omega_2\\ \omega_2^{\tau+\gamma}\end{pmatrix} \not\to \begin{pmatrix}1 & \omega\\ \omega_2^{\omega_1+2} & \omega_2^\tau\end{pmatrix}^{1,1}.$$

Our second remark is that it would be sufficient for our present purposes if we could prove (6.4) with ω replaced by ω_1. We do not know if $TH(\omega_1)$ is really needed for this weaker relation. This leads to the following question.

PROBLEM 1. *Assuming GCH, does the relation*

$$\begin{pmatrix}\omega_2\\ \omega_1\end{pmatrix} \not\to \left(\begin{matrix}1\\1\end{matrix},\ \begin{bmatrix}\omega_1\\ \omega_1\end{bmatrix}_{\omega,<\omega\dagger}\right)^{1,1}$$

hold?

The relation (6.4) should be compared with the positive relation

$$\begin{pmatrix}\omega_1\\ \omega_2^\tau\end{pmatrix} \to \begin{pmatrix}1 & \delta\\ \omega_2^\omega\omega_1 & \omega_2^\tau\end{pmatrix}^{1,1}$$

given by Theorem 4.7 ($\tau<\omega_3$, $\mathrm{cf}(\tau)=\omega_1$ and $\delta<\omega_1$). This shows that the term $\omega_2^\omega\omega_1+1$ in (6.4) cannot be decreased. Now (6.4) (obtained from $TH(\omega_1)$) implies the weaker relation

$$(6.7)\qquad \begin{pmatrix}\omega_2\\ \omega_2^\tau\end{pmatrix} \not\to \begin{pmatrix}1 & \omega_1\\ \omega_2^\omega\omega_1+1 & \omega_2^\tau\end{pmatrix}^{1,1}$$

which should be compared with (3.3) of Theorem 3.2, the corresponding result for $\aleph_1$ sets. We already saw that (3.3) is best possible (Theorem 3.5) and we now show that (6.7) is also best possible, i.e.

$$(6.8)\qquad \begin{pmatrix}\omega_2\\ \omega_2^\tau\end{pmatrix} \to \begin{pmatrix}1 & \omega_1\\ \omega_2^\omega\omega_1 & \omega_2^\tau\end{pmatrix}^{1,1} \quad \text{if}\quad \tau<\omega_3,\quad \mathrm{cf}(\tau)=\omega_1.$$

PROOF. Let $\langle S_\nu:\nu<\omega_1\rangle$ be a standard decomposition of ω_2^τ. We may assume that $\mathrm{tp}\, S_\nu \geqq \omega_2^\omega$ for $\nu<\omega_1$. Assume $A_\alpha\subset\omega_2^\tau$ $(\alpha<\omega_2)$ and $\mathrm{tp}\, A_\alpha < \omega_2^\omega\omega_1$. Then for each $\alpha<\omega_2$ there are $\varrho(\alpha)<\omega_1$ and $n(\alpha)<\omega$ such that $\operatorname{tp} A_\alpha \cap S_\nu < \omega_2^{n(\alpha)}$ for $\varrho(\alpha)<\nu<\omega_1$. There are $D\in[\omega_2]^{\omega_1}$, $n<\omega$, $\varrho<\omega_1$ such that $n(\alpha)=n$ and $\varrho(\alpha)=\varrho$ for $\alpha\in D$. Clearly $\operatorname{tp}(\omega_2^\tau \setminus \cup\{A_\alpha:\alpha\in D\})=\omega_2^\tau$.

This concludes our discussion of (6.3) in case $\operatorname{cf}(\tau)=\omega_1$ and we know, assuming $TH(\omega_1)$, that our results are best possible.

In case $\operatorname{cf}(\omega_2^\tau)=\omega_2$ we already know, by (3.12) of Theorem 3.3, that $2^{\aleph_1}=\aleph_2$ implies

$$(6.9)\qquad \begin{pmatrix}\omega_2\\ \omega_2^{\tau+\gamma}\end{pmatrix} \not\to \begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega+1+\gamma}+1 & \omega_2^\tau\end{pmatrix}$$

holds for $\gamma<\omega_1$.

Again this can be generalized as follows.

THEOREM 6.5. *Assume* $2^{\aleph_1}=\aleph_2$. *If* $\tau<\omega_3$ *and* $\operatorname{cf}(\tau)=\omega_2$, *then* (6.9) *holds for* $\gamma<\omega_1+2$.

PROOF. To see this in case $\gamma=\omega_1$ just take cross sections. This works since $\operatorname{cf}(\omega_2^\tau)=\omega_2$. For $\gamma=\omega_1+1$ we obtain it from the statement with $\gamma=\omega_1$ using the partition relation (3.8). We omit the details.

This shows that our positive result Theorem 6.2 is again best possible for the case $\operatorname{cf}(\omega_2^\tau)=\omega_2$, at least assuming $2^{\aleph_1}=\aleph_2$. Unfortunately, we do not know if the same is true in the case $\operatorname{cf}(\tau)=\omega$. Our main unsolved problem is the following.

PROBLEM 2. *Assume* $2^{\aleph_0}=\aleph_1$. *Is the relation*

$$\begin{pmatrix}\omega_2\\ \omega_2^{\omega\cdot 2}\end{pmatrix} \not\to \begin{pmatrix}1 & \omega_1\\ \omega_2^\gamma & \omega_2^{\omega\cdot 2}\end{pmatrix}^{1,1}$$

true for some γ, $\omega+2\leqq\gamma\leqq\omega\cdot 2$?

All our methods for constructing counter examples break down. We do know that (3.2) does not remain true for $\aleph_2$ sets for $\tau>\omega$, $\operatorname{cf}(\tau)=\omega$. The following partial result shows why we insist that $\gamma\geqq\omega+2$ in Problem 2.

THEOREM 6.6. *Assume* $2^{\aleph_0}=\aleph_1$. *If* $\omega<\tau<\omega_3$, $\operatorname{cf}(\tau)=\omega$ *and* $\xi<\omega_2$, *then*

$$\begin{pmatrix}\omega_2\\ \omega_2^\tau\end{pmatrix} \to \begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega+1}\cdot\xi & \omega_2^\tau\end{pmatrix}.$$

PROOF. Let $\langle S_n:n<\omega\rangle$ be a standard decomposition of ω_2^τ. We may assume that $\operatorname{tp} S_n=\omega_2^{\gamma_n+1}$ and $\gamma_n\geqq\omega$ for $n<\omega$. Let $\langle S_{n,\varrho}:\varrho<\omega_2\rangle$ be a standard decomposition of S_n for $n<\omega$. Assume $A_\alpha\subset\omega_2^\tau$, $\operatorname{tp} A_\alpha<\omega_2^{\omega+1}\cdot\xi$ for $\alpha<\omega_2$.

Assume now, that $\xi\leqq\alpha<\omega_2$. For each $n<\omega$ there is $\varrho(\alpha,n)<\alpha$ such that $\operatorname{tp}(A_\alpha\cap S_{n,\varrho(\alpha,n)})<\omega_2^{\omega+1}$. Put $\varrho(\alpha)=\sup\{\varrho(\alpha,n):n<\omega\}$. Since $\{\alpha<\omega_1:\xi\leqq\alpha<\omega_2, \operatorname{cf}(\alpha)=\omega_1\}$ is a stationary subset of ω_2 it follows that there are $D\in[\omega_2]^{\omega_2}$ and $\varrho<\omega_2$ such that $\varrho(\alpha)=\varrho$ for all $\alpha\in D$. Using $2^{\aleph_0}=\aleph_1$ it follows that there are $D'\in[D]^{\omega_2}$ and a sequence $(\varrho_n:n<\omega)$ such that $\varrho(\alpha,n)=\varrho_n$ for $\alpha\in D'$. Considering $\cup\{S_{n,\varrho_n}:n<\omega\}$ has type ω_2^τ this shows that it is sufficient to prove only that

$$\begin{pmatrix}\omega_2\\ \omega_2^\tau\end{pmatrix} \to \begin{pmatrix}1 & \omega_1\\ \omega_2^{\omega+1} & \omega_2^\tau\end{pmatrix}^{1,1}$$

holds.

To see this consider again the S_n and $S_{n,\varrho}$ as defined in the first part of the proof and assume now that $A_\alpha \subset \omega_2^\tau$, tp $A_\alpha < \omega_2^{\omega+1}$ for $\alpha < \omega_2$.

There are $\varrho(\alpha) < \omega_2$ and $k(\alpha, n) < \omega$ such that

$$\text{tp}\,(A_\alpha \cap S_{n,\varrho}) < \omega_2^{k(\alpha,n)} \quad \text{for} \quad \varrho(\alpha) < \varrho < \omega_2.$$

Using once again that $2^{\aleph_0} = \aleph_1$, we see that there are $D \in [\omega_2]^{\omega_2}$ and $k_n < \omega$ $(n < \omega)$ such that $k(\alpha, n) = k_n$ for $\alpha \in D$. Let $D' \in [D]^{\omega_1}$ be arbitrary and let $\varrho = \sup\{\varrho(\alpha) : \alpha \in D'\}$. Then tp $(A_\alpha \cap S_{n,\varrho}) < \omega_2^{k_n}$ for $\varrho < \varrho' < \omega_2$, $n < \omega$ and $\alpha \in D'$. Hence tp $(S_n \setminus \bigcup\{A_\alpha : \alpha \in D\}) \geqq \omega_2$ for $n < \omega$ and the result follows.

We conclude this chapter by analyzing the analogue of (4.1) for $\aleph_2$ sets, i.e. the relation

$$\binom{\omega_2}{\omega_2^\sigma} \rightarrow \binom{1 \quad \omega}{\omega_2^\gamma \quad \omega_2^\tau}^{1,1} \tag{6.10}$$

for $\sigma, \tau < \omega_3$ and $\gamma < \omega_1 + 2$.

In view of Theorems 6.3 and 6.4, a discussion for (6.10) will be completed by the following analogue of (4.3).

THEOREM 6.7. *Assume* $2^{\aleph_0} = \aleph_1$. *If* $\tau < \omega_3$, cf $(\omega_2^\tau) = \omega$ *or* ω_2, $\delta < \omega_1$, $\gamma \leqq \tau + \delta$ *and* $\gamma < \omega_1 + 2$, *then*

$$\binom{\omega_2}{\omega_2^{\tau+\delta}} \rightarrow \binom{1 \quad \omega}{\omega_2^\gamma \quad \omega_2^{\tau+\delta}}^{1,1}.$$

PROOF. It is clearly sufficient to prove this for $\gamma = \omega_1 + 1$. Assume $A_\alpha \subset \omega_2^{\tau+\delta}$, tp $A_\alpha < \omega_2^{\omega_1+1}$ for $\alpha < \omega_2$. Now cf $(\omega_2^{\tau+\delta})$ is either ω or ω_2 since $\delta < \omega_1$. Hence, by the Lemma 6.2, there are $X \subset \omega_2^{\tau+\delta}$, $\gamma_0 < \omega_2$ and $D \in [\omega_2]^{\omega_1}$ such that tp $X = \omega_2^{\tau+\delta}$ and tp $(A_\alpha \cap X) < \omega_2^{\gamma_0}$ for all $\alpha \in D$. Hence the statement follows from

$$\binom{\omega_1}{\omega_2^{\tau+\delta}} \rightarrow \binom{1 \quad \omega}{\omega_2^{\gamma_0} \quad \omega_2^{\tau+\delta}}^{1,1}$$

which is (4.3) of Theorem 4.1.

7. Pointwise-finite systems. A system $\langle A_\alpha : \alpha < \varkappa \rangle$ of subsets of S is said to be *pointwise-finite* if each point of the underlying set S is a member of only a finite number of the A_α $(\alpha < \varkappa)$. In this chapter we investigate analogues of some of our earlier results for pointwise-finite systems. This amounts to an investigation of relations of the form

$$\binom{\varkappa}{\omega_2^\varrho} \rightarrow \binom{1}{\omega_2^\sigma} \vee \binom{\omega \quad \omega}{1 \quad \omega_2^\tau}^{1,1} \tag{7.1}$$

for $\varrho, \sigma, \tau < \omega_3$ and $\varkappa = \omega$, ω_1 or ω_2. While this leads to several quite interesting new problems we shall not discuss this in the same detail as we did for the case when the pointwise-finite condition is left out. The results we prove below give the analogues for (7.1) in the cases $\varkappa = \omega$, ω_1 and ω_2 which correspond respectively to the negative relations of (1.2), Theorem 1.1 and Theorem 6.1.

The following theorem is related to our earlier result ([7], Theorem 8) and so is the method of proof, but we believe it is worth giving the details.

THEOREM 7.1. *Assume* $2^{\aleph_0}=\aleph_1$. *Let* $\gamma<\omega_2$, $\tau\leqq\omega_2$. *Then*

(7.2) $$\binom{\omega}{\omega_2^\gamma}\nrightarrow\left(\frac{1}{\omega_2^\omega}\vee\frac{\omega}{1}\ \frac{\omega}{\omega_2^{\omega_1}}\right)^{1,1},$$

(7.3) $$\binom{\omega_1}{\omega_2^\gamma}\nrightarrow\left(\frac{1}{\omega_2^{\omega_1}}\vee\frac{\omega}{1}\ \frac{\omega}{\omega_2^{\omega_1}}\right)^{1,1},$$

and

(7.4) $$\binom{\omega_2}{\omega_2^\tau}\rightarrow\left(\frac{1}{\omega_2^{\omega_1+2}}\vee\frac{\omega}{1}\ \frac{\omega}{\omega_2^{\omega_1+2}}\right)^{1,1}.$$

We need the following lemma which was also used in [7] (Lemma 5). For the convenience of the reader we give the short proof.

LEMMA 7.2. *Let* $\langle T_n:n<\omega\rangle$ *be a sequence of denumerable subsets of* ω. *Then there is a pointwise-finite sequence* $\langle C_k:k<\omega\rangle$ *of finite subsets of* ω *such that*

(7.5) $$C_k\subset k$$

and

(7.6) $$\omega\setminus\bigcup\{C_k:k\in T_n\}\quad \textit{is finite for all}\quad n<\omega.$$

PROOF. We can assume that the sets T_n are pairwise disjoint (since we can replace them by infinite disjoint subsets). Let t_n denote the least member of T_n and let $T_n'=T_n\setminus\{t_n\}$. We define the C_k $(k<\omega)$ as follows. If $k\in\omega\setminus\bigcup\{T_n':n<\omega\}$ put $C_k=\emptyset$. If $k\in\bigcup\{T_n':n<\omega\}$ then there are unique integers $m(k)<k$ and $n(k)$ such that $k\in T_{n(k)}'$, $m(k)\in T_{n(k)}$ and $i\notin T_{n(k)}$ for $m(k)<i<k$. In this case put $C_k=[m(k),k)$. It is easy to verify that the C_k are pointwise-finite and that (7.5) and (7.6) hold.

PROOF OF (7.2). We prove slightly more. We show that there is a pointwise-finite system $\langle A_k:k<\omega\rangle$ of subsets of ω_2^γ which establishes (7.2) and satisfies the stronger condition that

(7.7) $$\operatorname{tp}A_k<\omega_2^k\quad\text{for}\quad k<\omega.$$

The proof is by induction on γ. For $\gamma=0$ the result is trivial. Now assume that $\gamma>0$.

Let $\langle S_\nu:\nu<\operatorname{cf}(\omega_2^\gamma)\rangle$ be a standard decomposition of ω_2^γ. By the induction hypothesis, for each $\nu<\operatorname{cf}(\omega_2^\gamma)$ there is a pointwise-finite sequence $\langle A_k^\nu:k<\omega\rangle$ of subsets of S_ν establishing the corresponding result for $\operatorname{tp}(S_\nu)$. We now distinguish the cases (1) $\operatorname{cf}(\omega_2^\gamma)=\omega$ or ω_2 and (2) $\operatorname{cf}(\omega_2^\gamma)=\omega_1$.

Case 1. Take cross sections in the natural way, i.e. put $A_0=\emptyset$ and $A_{k+1}=\bigcup\{A_k^\nu:\nu<\operatorname{cf}(\omega_2^\gamma)\}$. (Note that, in the case $\operatorname{cf}(\omega_2^\gamma)=\omega_2$ we have that γ is a successor and for each k the ω_2 sets A_k^ν $(\nu<\omega_2)$ all have the same type.)

Case 2. By the hypothesis $2^{\aleph_0}=\aleph_1$ we can assume that $[\omega]^\omega=\{T_\nu:\nu<\omega_1\}$. Let $F_\nu=\{T_\mu:\mu<\nu\}$ for $\nu<\omega_1$.

For each $\nu<\omega_1$, let $\langle C_k^\nu:k<\omega\rangle$ be a pointwise-finite system of finite subsets of ω satisfying the requirements of Lemma 7.2 for the countable system F_ν of denumerable subsets of ω. Also, let $\langle B_k^\nu : k<\omega\rangle$ be a disjoint paradoxical decomposition of S_ν $(\nu<\omega_1)$ such that $\operatorname{tp} B_k^\nu<\omega_2^k$ and $S_\nu=\bigcup\{B_k^\nu : k<\omega\}$ (see (1.1)). Now put

$$\tilde{A}_k^\nu=\bigcup\{B_i^\nu:\ i\in C_k^\nu\}\quad \text{for}\quad k<\omega\quad \textit{and}\quad \nu<\omega_1.$$

We know that $\operatorname{tp}\tilde{A}_k^\nu<\omega_2^k$ since $C_k^\nu\subset k$. Moreover, $\langle\tilde{A}_k^\nu : k<\omega\rangle$ is pointwise-finite since $\langle C_k^\nu : k<\omega\rangle$ is and the sets B_l^ν are pairwise disjoint. Finally, by (7.6) of the lemma, we also know that

$$\operatorname{tp}(S_\nu\setminus\bigcup\{\tilde{A}_k^\nu:\ k\in T\})<\omega_2^\omega \tag{7.8}$$

for any $T\in F_\nu$.

Now put $A_k'=\bigcup\{A_k^\nu : \nu<\omega_1\}$, $A_k''=\bigcup\{\tilde{A}_k^\nu : \nu<\omega_1\}$ and $A_k=A_k'\cup A_k''$ for $k<\omega$. Clearly $\langle A_k : k<\omega\rangle$ is pointwise-finite and $\operatorname{tp} A_k<\omega_2^k$.

Suppose $D\in[\omega]^\omega$. Then $D=T_\mu$ for some $\mu<\omega_1$ and hence $D\in F_\nu$ for $\mu<\nu<\omega_1$. Therefore, by (7.8) and the fidefinition of A_k, we have

$$\operatorname{tp}(S_\nu\setminus\bigcup\{A_\alpha:\ \alpha\in D\})<\omega_2^\omega\quad \text{for}\quad \mu<\nu<\omega_1.$$

Further, by the inductive property of the A_α^ν, we also have that

$$\lambda_\nu=\operatorname{tp}(S_\nu\setminus\bigcup\{A_\alpha:\ \alpha\in D\})<\omega_2^{\omega_1}$$

for any $\nu<\omega_1$. Combining these we see that

$$\operatorname{tp}(\omega_2^\gamma\setminus\bigcup\{A_\alpha:\ \alpha\in D\})\leqq\sum\{\lambda_\nu:\ \nu\leqq\mu\}+\omega_2^\omega\omega_1<\omega_2^{\omega_1}.$$

Proof of (7.3). Again the proof is by induction on $\gamma<\omega_2$. For $\gamma=0$ the result is trivial. Also, the induction step in the cases when γ is a successor ordinal or an ω-limit is very easy — simply take identical cross sections or cross sections. The main difficulty in the induction is for the case $\operatorname{cf}(\gamma)=\omega_1$ which we now consider in detail.

Let $\langle S_\nu : \nu<\omega_1\rangle$ be a standard decomposition of ω_2^γ and let $\langle A_\alpha^\nu : \alpha<\omega_1\rangle$ be a pointwise-finite system of subsets of S_ν for $\nu<\omega_1$ which establishes the result in S_ν, i.e. for $\nu<\omega_1$ we have

$$\operatorname{tp} A_\alpha^\nu<\omega_2^{\omega_1}\quad(\alpha<\omega_1), \tag{7.9}$$

$$\operatorname{tp}(S_\nu\setminus\bigcup\{A_\alpha^\nu:\ \alpha\in D\})<\omega_2^{\omega_1}\quad \text{for all}\quad D\in[\omega_1]^\omega. \tag{7.10}$$

Let $\omega\leqq\nu<\omega_1$. By (7.2) just proved, there is a pointwise-finite system $\langle\hat{A}_\alpha^\nu : \alpha<\nu\rangle$ of $\aleph_0$ subsets of S_ν such that

$$\operatorname{tp}\hat{A}_\alpha^\nu<\omega_2^\omega, \tag{7.11}$$

$$\operatorname{tp}(S_\nu\setminus\bigcup\{\hat{A}_\alpha^\nu:\ \alpha\in D\})<\omega_2^{\omega_1}\quad \text{for all}\quad D\in[\nu]^\omega. \tag{7.12}$$

By the hypothesis $2^{\aleph_0}=\aleph_1$ we may write $[\omega_1]^\omega=\{T_\mu : \mu<\omega_1\}$. For $\nu<\omega_1$ define $F_\nu=\{T_\mu : \mu<\nu,\ T_\mu\subset\nu\}$.

Again, let $\omega\leqq\nu<\omega_1$. We define another pointwise-finite system $\langle\tilde{A}_\alpha^\nu : \alpha<\nu\rangle$ of $\aleph_0$ subsets of S_ν as follows. By Lemma 7.2 there is a pointwise-finite system $\langle C_\alpha^\nu : \alpha<\nu\rangle$ of $\aleph_0$ finite subsets of ν such that

$$\nu\setminus\bigcup\{C_\alpha^\nu : \alpha\in D\}\ \textit{is finite for all}\ D\in F_\nu. \tag{7.13}$$

Let $\langle B_\alpha^\nu:\alpha<\nu\rangle$ be $\aleph_0$ disjoint subsets of S_ν such that $S_\nu=\bigcup\{B_\alpha^\nu:\alpha<\nu\rangle$ and $\operatorname{tp} B_\alpha^\nu<\omega_2^\omega$. Such sets exist by the ordinary paradox (1.1). Now define

$$\tilde{A}_\alpha^\nu=\bigcup\{B_\alpha^\nu:\ \alpha\in C_\alpha^\nu\}\quad(\alpha<\nu).$$

Clearly the system $\langle\tilde{A}_\alpha^\nu:\alpha<\nu\rangle$ is pointwise-finite since $\langle C_\alpha^\nu:\alpha<\nu\rangle$ is and the B_α^ν $(\alpha<\omega_1)$ are pairwise disjoint. Moreover, since $\tilde{A}_\alpha^\nu$ is the union of a finite number of sets of type $<\omega_2^\omega$, we have

$$\text{(7.14)}\qquad \operatorname{tp}\tilde{A}_\alpha^\nu<\omega_2^\omega\quad(\alpha<\nu<\omega_1).$$

For the same reason we also know by (7.13) that

$$\text{(7.15)}\qquad \operatorname{tp}(S_\nu\setminus\bigcup\{\tilde{A}_\alpha^\nu:\ \alpha\in D\})<\omega_2^\omega\quad\text{for}\quad D\in F_\nu.$$

Now define sets $A_\alpha\subset\omega_2^\gamma$ for $\alpha<\omega_1$ by putting $A_\alpha=A_\alpha'\cup A_\alpha''$, where

$$A_\alpha'=\bigcup\{A_\alpha^\nu:\ \nu<\alpha\}\cup\bigcup\{\hat{A}_\alpha^\nu:\ \omega\leqq\nu<\omega_1,\alpha<\nu\},$$

$$A_\alpha''=\bigcup\{\tilde{A}_\alpha^\nu:\ \omega\leqq\nu<\omega_1,\alpha<\nu\}.$$

By (7.9), (7.11) and (7.14) we easily see that $\operatorname{tp} A_\alpha<\omega_2^{\omega_1}$ $(\alpha<\omega_1)$. Also the system $\langle A_\alpha:\alpha<\omega_1\rangle$ is a pointwise-finite since the sets S_ν are pairwise disjoint and the $\langle A_\alpha^\nu:\alpha<\omega_1\rangle$, $\langle\hat{A}_\alpha^\nu:\alpha<\omega_1\rangle$ and $\langle\tilde{A}_\alpha^\nu;\alpha<\omega_1\rangle$ are pointwise-finite. To complete the proof we must verify that

$$\text{(7.18)}\qquad \operatorname{tp}(\omega_2^\gamma\setminus\bigcup\{A_\alpha:\ \alpha\in D\})<\omega_2^{\omega_1}$$

holds for any $D\in[\omega_1]^\omega$.

Suppose $D\in[\omega_1]^\omega$ and $\nu<\omega_1$. Either $D\setminus\nu$ is infinite or $\omega\leqq\nu<\omega_1$ and $D\cap\nu$ is infinite. In either case, by the definition of A_α', we have that

$$\text{(7.19)}\qquad \operatorname{tp}(S_\nu\setminus\bigcup\{A_\alpha':\ \alpha\in D\})<\omega_2^{\omega_1}$$

by (7.10) or by (7.12). Also $D=T_\mu$ for some $\mu<\omega_1$. Let $\nu_0=\sup(\{\mu\}\cup T_\mu)$. Then for $\nu_0<\nu<\omega_1$ we have $D\in F_\nu$ and hence

$$\text{(7.20)}\qquad \operatorname{tp}(S_\nu\setminus\bigcup\{A_\alpha'':\ \alpha\in D\})<\omega_2^\omega\quad(\nu_0<\nu<\omega_1)$$

by (7.15) (7.18) easily follows from (7.19) and (7.20).

PROOF OF (7.4). Again this is trivial for $\tau<\omega_1+2$ and we use induction on τ. For the case when τ is an ω or ω_1 limit there is no problem, we simply take cross sections. We have only to prove the induction step for the case when $\operatorname{cf}(\omega_2^\tau)=\omega_2$, $\tau\leqq\omega_2$.

As usual, let $\langle S_\nu:\nu<\omega_2\rangle$ be a standard decomposition for ω_2^τ. By the induction hypothesis there is a pointwise-finite system $\langle A_\alpha^\nu:\alpha<\omega_2\rangle$ of subsets of S_ν such that

$$\operatorname{tp} A_\alpha^\nu<\omega_2^{\omega_1+2}\quad\text{for}\quad\alpha<\omega_2,\quad\nu<\omega_2,$$

$$\text{(7.21)}\qquad \operatorname{tp}(S_\nu\setminus\bigcup\{A_\alpha^\nu:\ \alpha\in D\})<\omega_2^{\omega_1+2}\quad\text{for}\quad D\in[\omega_2]^\omega\quad\text{and}\quad\nu<\omega_2.$$

By (7.3) already proved, for $\omega_1\leqq\nu<\omega_2$ there is a pointwise-finite system $\langle\hat{A}_\alpha^\nu:\alpha<\nu\rangle$ of $\aleph_1$ subsets of S_ν such that $\operatorname{tp}\hat{A}_\alpha^\nu<\omega_2^{\omega_1}$ $(\alpha<\nu)$,

$$\text{(7.22)}\qquad \operatorname{tp}(S_\nu\setminus\bigcup\{\hat{A}_\alpha^\nu:\ \alpha\in D\})<\omega_2^{\omega_1}\quad\text{for all}\quad D\in[\nu]^\omega.$$

Now put $A_\alpha = \bigcup \{A_\alpha^\nu : \nu < \alpha\} \cup \bigcup \{\hat{A}_\alpha^\nu : \omega_1 \leqq \nu < \omega_2,\ \alpha < \nu\}$ $(\alpha < \omega_2)$. Clearly the system $\langle A_\alpha : \alpha < \omega_2 \rangle$ is pointwise finite and $\operatorname{tp} A_\alpha < \omega_2^{\omega_1 + 2}$ $(\alpha < \omega_2)$.

Suppose $D \in [\omega_2]^\omega$. Put $\nu_0 = \sup (\omega_1 \cup D)$. Then for $\nu_0 < \nu < \omega_2$ we have $D \in [\nu]^\omega$ and so by (7.22)

$$\operatorname{tp} (S_\nu \setminus \bigcup \{A_\alpha :\ \alpha \in D\}) < \omega_2^{\omega_1} \quad (\nu_0 < \nu < \omega_2).$$

It follows from this and (7.21) that

$$\operatorname{tp} (\omega_2^\tau \setminus \bigcup \{A_\alpha :\ \alpha \in D\}) < \omega_2^{\omega_1 + 2}.$$

Thus $\langle A_\alpha : \alpha < \omega_2 \rangle$ establishes (7.4).

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ON DIFFERENCE SETS OF SEQUENCES OF INTEGERS. I

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1. A set of integers $u_1 < u_2 < \dots$ will be called an $\mathcal{A}$-set if its difference set does not contain the square of a positive integer; in other words, if $u_x - u_y = z^2$ (where x, y, z are integers) implies that $x = y$, $z = 0$. Let $A(x)$ denote the greatest number of integers that can be selected from $1, 2, \dots, x$ to form an $\mathcal{A}$-set and let us write

$$a(x) = \frac{A(x)}{x}.$$

L. Lovász conjectured that

$$a(x) = o(1) \tag{1}$$

(oral communication). The aim of this paper is to prove the following sharper form of (1):

THEOREM.

$$a(x) = O\left(\frac{(\log\log x)^{2/3}}{(\log x)^{1/3}}\right). \tag{2}$$

(We remark that (1) has been proved independently also by H. Fürstenberg; his proof is unpublished yet.)

To prove this theorem, we are going to use that version of the Hardy-Littlewood method which has been elaborated by K. F. ROTH in [2] and [3].

Theroughout this paper, we use the following notations:

We denote the distance of the real number x from the nearest integer by $\|x\|$, i.e. $\|x\| = \min\{x-[x], [x]+1-x\}$. We write $e(\alpha) = e^{2\pi i\alpha}$ where α is real. $L_0, L_1, \dots$, $X_0, X_1, \dots$ denote absolute constants. If a, b are real numbers and $b > 0$, then we define the symbol $\min\left\{a, \frac{b}{0}\right\}$ by

$$\min\left\{a, \frac{b}{0}\right\} = a. \tag{3}$$

Finally, if $|g(x_1, x_2, \dots, x_n)| \leqq f(x_1, x_2, \dots, x_n)$ then we write

$$g(x_1, x_2, \dots, x_n) = \theta(f(x_1, x_2, \dots, x_n)).$$

2. Following Roth's method, we are going to deduce a functional inequality for the function $a(x)$.

Let N be a large integer and let us write $M=[\sqrt{N}]$. Let

$$T(\alpha)=\sum_{z=1}^{[\sqrt{N}]} e(z^2\alpha)=\sum_{z=1}^{M} e(z^2\alpha).$$

Let $u_1, u_2, \dots, u_{A(N)}$ be a maximal $\mathcal{A}$-set selected from $1, 2, \dots, N$ and let

$$F(\alpha)=\sum_{x=1}^{A(N)} e(u_x\alpha).$$

We are going to investigate the integral

$$E=\int_0^1 |F(\alpha)|^2 T(\alpha)\,d\alpha. \tag{4}$$

Obviously,

$$E=\int_0^1 F(\alpha)F(-\alpha)T(\alpha)\,d\alpha= \tag{5}$$

$$=\int_0^1 \sum_{y=1}^{A(N)} e(u_y\alpha)\sum_{x=1}^{A(N)} e(-u_x\alpha)\sum_{z=1}^{M} e(z^2\alpha)\,d\alpha=\sum_{\substack{x,y,z\\ u_y-u_x+z^2=0}} 1=0$$

since $u_1, u_2, \dots, u_{A(N)}$ is an $\mathcal{A}$-set.

On the other hand, we shall estimate this integral by using the Hardy—Littlewood method. For this purpose, we need some estimates for the functions $T(\alpha)$ and $F(\alpha)$.

3. In this section, we estimate the function $T(\alpha)$.

LEMMA 1. *If a, b are integers such that $a\leqq b$, and β is an arbitrary real number then*

$$\left|\sum_{k=a}^{b} e(k\beta)\right|\leqq \min\left\{b-a+1, \frac{1}{2\|\beta\|}\right\}.$$

(*For $\|\beta\|=0$, the right hand side is defined by* (3).)

PROOF. Obviously,

$$\left|\sum_{k=a}^{b} e(k\beta)\right|\leqq \sum_{k=a}^{b} 1\leqq b-a+1$$

for all a, b, β (where $a\leqq b$).

Furthermore, for $\|\beta\|\neq 0$,

$$\left|\sum_{k=a}^{b} e(k\beta)\right|=\frac{|1-e((b-a+1)\beta)|}{|1-e(\beta)|}\leqq\frac{2}{|1-e(\beta)|}=$$

$$=\frac{2}{|e(-\beta/2)-e(\beta/2)|}=\frac{1}{|\sin\pi\beta|}=\frac{1}{\sin\pi\|\beta\|}\leqq\frac{1}{\frac{2}{\pi}\cdot\pi\|\beta\|}=\frac{1}{2\|\beta\|}$$

which proves Lemma 1.

LEMMA 2. *Let p, q be integers and α, γ real numbers such that $q>1$, $(p,q)=1$ and*

$$\left|\alpha-\frac{p}{q}\right| \leqq \frac{1}{q^2}.$$

Then

$$\sum_{x=0}^{q-1} \min\left\{q, \frac{1}{\|\gamma+\alpha x\|}\right\} < 8q\log q.$$

This lemma is identical to Theorem 44 in [1], p. 26.

LEMMA 3. *Let p, q be integers and α, γ, P real numbers such that $q\geqq 1$, $(p,q)=1$, $P\geqq 1$ and*

$$\left|\alpha-\frac{p}{q}\right| \leqq \frac{1}{q^2}. \tag{6}$$

Then

$$\sum_{x=0}^{q-1} \min\left\{P, \frac{1}{\|\gamma+\alpha x\|}\right\} < 8P+8q\log q. \tag{7}$$

PROOF. For $q=1$, (7) holds trivially (by $P\geqq 1$).

For $q>1$, the left hand side of (7) can be rewritten in the following way:

$$\sum_{x=0}^{q-1} \min\left\{P, \frac{1}{\|\gamma+\alpha z\|}\right\} = \tag{8}$$

$$= \sum_{\substack{0\leqq x\leqq q-1\\ \|\gamma+\alpha x\|<\frac{1}{q}}} \min\left\{P, \frac{1}{\|\gamma+\alpha x\|}\right\} + \sum_{\substack{0\leqq x\leqq q-1\\ \|\gamma+\alpha x\|\geqq\frac{1}{q}}} \min\left\{P, \frac{1}{\|\gamma+\alpha x\|}\right\} \leqq$$

$$\leqq \sum_{\substack{0\leqq x\leqq q-1\\ \|\gamma+\alpha x\|<\frac{1}{q}}}' P + \sum_{\substack{0\leqq x\leqq q-1\\ \|\gamma+\alpha x\|\geqq\frac{1}{q}}} \frac{1}{\|\gamma+\alpha x\|} = P \sum_{\substack{0\leqq x\leqq q-1\\ \|\gamma+\alpha x\|<\frac{1}{q}}} 1 + \sum_{\substack{0\leqq x\leqq q-1\\ \|\gamma+\alpha x\|\geqq\frac{1}{q}}} \min\left\{q, \frac{1}{\|\gamma+\alpha x\|}\right\} \leqq$$

$$\leqq P \sum_{\substack{0\leqq x\leqq q-1\\ \|\gamma+\alpha x\|<\frac{1}{q}}} 1 + \sum_{x=0}^{q-1} \min\left\{q, \frac{1}{\|\gamma+\alpha x\|}\right\}.$$

We are going to show that

$$\sum_{\substack{0\leqq x\leqq q-1\\ \|\gamma+\alpha x\|<\frac{1}{q}}} 1 \leqq 8. \tag{9}$$

Let us assume indirectly that the left hand side of this inequality is $\geqq 9$. This indirect assumption implies the existence of integers x_1, x_2, x_3, x_4, x_5 such that

$$\|\gamma+\alpha x_i\| < \frac{1}{q} \quad (\text{for} \quad i=1,\dots,5) \tag{10}$$

and which all lie either in the interval $\left[0, \frac{q-1}{2}\right]$ or in $\left[\frac{q-1}{2}, q-1\right]$; in either case

(11) $$0 < |x_i - x_j| < \frac{q}{2} \quad (\text{for } 1 \leqq i \leqq j < 5).$$

By (10), there exist integers y_i and real numbers θ_i such that

(12) $$\gamma + \alpha x_i = y_i + \frac{\theta_i}{q} \quad (\text{for} \quad i = 1, \ldots, 5)$$

and

$$-1 < \theta_i < 1 \quad (\text{for} \quad i = 1, \ldots, 5).$$

Using the matchbox principle, we obtain the existence of indices μ, ν such that $1 \leqq \mu < \nu \leqq 5$ and

(13) $$|\theta_\mu - \theta_\nu| < \frac{1}{2}.$$

Writing $i = \mu$ and ν in (12), respectively, and subtracting the equalities obtained in this way, we obtain that

$$\alpha(x_\mu - x_\nu) = y_\mu - y_\nu + \frac{\theta_\mu - \theta_\nu}{q}.$$

Hence

(14) $$\|\alpha(x_\mu - x_\nu)\| \leqq \frac{\theta_\mu - \theta_\nu}{q} < \frac{1}{2q}$$

(by (13)).

On the other hand, we obtain with respect to (6) and (11) that

$$\|\alpha(x_\mu - x_\nu)\| = \left\| \frac{p}{q}(x_\mu - x_\nu) + \left(\alpha - \frac{p}{q}\right)(x_\mu - x_\nu) \right\| \geqq$$

$$\geqq \left\| \frac{p(x_\mu - x_\nu)}{q} \right\| - \left| \alpha - \frac{p}{q} \right| |x_\mu - x_\nu| > \frac{1}{q} - \frac{1}{q^2} \cdot \frac{q}{2} = \frac{1}{2q}$$

in contradiction with (14), which proves (9).

(8), (9) and Lemma 2 yield (7) and Lemma 3 is proved.

Lemma 4. *Let N, p, q be integers and α a real number such that $N \geqq 1$, $q \geqq 1$, $(p, q) = 1$ and*

$$\left| \alpha - \frac{p}{q} \right| \leqq \frac{1}{q^2}.$$

Then

(15) $$|T(\alpha)| = \left| \sum_{k=1}^{M} e(k^2 \alpha) \right| < 7 \left(\frac{M}{q^{1/2}} + (M \log q)^{1/2} + (q \log q)^{1/2} \right)$$

(*where* $M = [\sqrt{N}]$).

PROOF.

$$|T(\alpha)|^2 = T(\alpha)T(-\alpha) = \sum_{x=1}^{M}\sum_{y=1}^{M} e((x^2-y^2)\alpha) =$$

$$= \sum_{x=1}^{M}\sum_{y=1}^{M} e((x-y)(x+y)\alpha) = \sum_{u=1-M}^{M-1}\ \sum_{y=\max\{1-u,1\}}^{\min\{M,M-u\}} e(u(u+2y)\alpha) \leqq$$

$$\leqq \sum_{u=1-M}^{M-1}\left|\sum_{y=\max\{1-u,1\}}^{\min\{M,M-u\}} e(u(u+2y)\alpha)\right| = \sum_{u=1-M}^{M-1}\left|\sum_{y=\max\{1-u,1\}}^{\min\{M,M-u\}} e(2uy\alpha)\right|.$$

To estimate the inner sum, we apply Lemma 1 with $\beta=2u\alpha$, $a=\max\{1-u, 1\}$, $b=\min\{M, M-u\}$. Then obviously,

$$b-a = \min\{M, M-u\}-\max\{1-u, 1\} \leqq M-1,$$

thus Lemma 1 yields that

$$|T(\alpha)|^2 \leqq \sum_{u=1-M}^{M-1} \min\left\{(M-1)+1, \frac{1}{2\|2u\alpha\|}\right\} =$$

$$= \frac{1}{2}\sum_{u=1-M}^{M-1} \min\left\{2M, \frac{1}{\|2u\alpha\|}\right\} \leqq \frac{1}{2}\sum_{v=2-2M}^{2M-2} \min\left\{2M, \frac{1}{\|v\alpha\|}\right\} \leqq$$

$$\leqq \frac{1}{2}\sum_{j=0}^{[(4M-4)/q]}\ \sum_{v=2-2M+jq}^{2-2M+(j+1)q-1} \min\left\{2M, \frac{1}{\|v\alpha\|}\right\}.$$

The inner sum can be estimated by using Lemma 3 with $\gamma=(2-2M+jq)\alpha$, $P=2M$. We obtain that

$$|T(\alpha)|^2 \leqq \frac{1}{2}\sum_{j=0}^{[(4M-4)/q]} (16M+8q\log q) = \tag{16}$$

$$= \frac{1}{2}\left(\left[\frac{4M-4}{q}\right]+1\right)(16M+8q\log q) < \left(\frac{4M}{q}+1\right)(8M+4q\log q) =$$

$$= 32\frac{M^2}{q}+8M+16M\log q+4q\log q.$$

For $q=1$,

$$M \leqq M\cdot\frac{M}{q} = \frac{M^2}{q},$$

while for $q\geqq 2$

$$M < M\log 4 = 2M\log 2 \leqq 2M\log q.$$

Hence

$$M \leqq \frac{M^2}{q}+2M\log q.$$

Thus we obtain from (16) that

$$|T(\alpha)|^2 < 32\frac{M^2}{q}+8\left(\frac{M^2}{q}+2M\log q\right)+16M\log q+4q\log q =$$

$$= 40\frac{M^2}{q}+32M\log q+4q\log q < 49\left(\frac{M^2}{q}+M\log q+q\log q\right).$$

With respect to the inequality

$$(a^2+b^2+c^2)^{1/2} \leqq a+b+c \quad (\text{where} \quad a, b, c \geqq 0),$$

this yields (15) which proves Lemma 4.

LEMMA 5. *Let N, p, q be integers and α a real number such that $(p, q)=1$,*

$$N \geqq 3, \tag{17}$$

$$1 \leqq q \leqq N^{1/2}/\log N \tag{18}$$

and

$$\left|\alpha-\frac{p}{q}\right| < \frac{1}{q^2}. \tag{19}$$

Then

$$|T(\alpha)| < 21\left(\frac{N}{q}\right)^{1/2}.$$

PROOF. Applying Lemma 4, we obtain with respect to (17) and (18) that

$$|T(\alpha)| < 7\left(\frac{M}{q^{1/2}}+(M\log q)^{1/2}+(q\log q)^{1/2}\right) \leqq$$

$$\leqq 7\left(\left(\frac{N}{q}\right)^{1/2}+N^{1/4}(\log q)^{1/2}+(q\log q)^{1/2}\right) <$$

$$< 7\left(\left(\frac{N}{q}\right)^{1/2}+N^{1/4}(\log N)^{1/2}+(N^{1/2}\log N)^{1/2}\right) =$$

$$= 7\left(\frac{N}{q}\right)^{1/2}\left(1+2\left(q\frac{\log N}{N^{1/2}}\right)^{1/2}\right) \leqq 7\left(\frac{N}{q}\right)^{1/2}\cdot 3 = 21\left(\frac{N}{q}\right)^{1/2}.$$

LEMMA 6. *Let N, p, q be integers and α, β real numbers such that*

$$N \geqq 9, \tag{20}$$

$$(p, q) = 1, \tag{21}$$

$$1 \leqq q \leqq \sqrt{N}, \tag{22}$$

$$\alpha = \frac{p}{q}+\beta \tag{23}$$

and

(24) $$\frac{\log N}{N} \leqq |\beta| < \frac{1}{2q\sqrt{N}}.$$

Then

(25) $$|T(\alpha)| < 30\left(\frac{\log N}{q|\beta|}\right)^{1/2}.$$

PROOF. Let

$$Q = \left[\frac{1}{q|\beta|}\right] + 1.$$

Then obviously,

(26) $$Q > \frac{1}{q|\beta|},$$

hence

(27) $$|\beta| > \frac{1}{qQ}.$$

By (24),

(28) $$\frac{1}{q|\beta|} > \sqrt{N}.$$

Thus

(29) $$Q = \left[\frac{1}{q|\beta|}\right] + 1 \leqq \frac{1}{q|\beta|} + \sqrt{N} < \frac{1}{q|\beta|} + \frac{1}{q|\beta|} = \frac{2}{q|\beta|}.$$

By Dirichlet's theorem, there exist integers r, s such that

(30) $$(r, s) = 1,$$

(31) $$1 \leqq s \leqq Q$$

and

(32) $$\left|\alpha - \frac{r}{s}\right| < \frac{1}{sQ}.$$

(31) and (32) imply that also

$$\left|\alpha - \frac{r}{s}\right| < \frac{1}{s^2}$$

holds. Thus we may apply Lemma 4 with $\frac{r}{s}$ in place of $\frac{p}{q}$. We obtain that

(33) $$|T(\alpha)| < 7\left(\frac{M}{s^{1/2}} + (M \log s)^{1/2} + (s \log s)^{1/2}\right) \leqq$$

$$\leqq 7\left(\left(\frac{N}{s}\right)^{1/2} + N^{1/4}(\log s)^{1/2} + (s \log s)^{1/2}\right).$$

To deduce (25) from this inequality, we have to estimate s in terms of $q\,|\beta|$ and N, respectively.

By (29) and (31),

$$s \leqq Q < \frac{2}{q|\beta|} \tag{34}$$

hence with respect to (20) and (24),

$$s < \frac{2}{q|\beta|} \leqq \frac{2}{|\beta|} \leqq 2\frac{N}{\log N} < N.$$

Thus (33) implies that

$$|T(\alpha)| < 7\left(\left(\frac{N}{s}\right)^{1/2} + (N^{1/4}+s^{1/2})(\log N)^{1/2}\right). \tag{35}$$

We are going to show that

$$\frac{p}{q} \neq \frac{r}{s}. \tag{36}$$

Let us assume indirectly that

$$\frac{p}{q} = \frac{r}{s}. \tag{37}$$

By $q \geqq 1$, $s \geqq 1$, (21) and (30), this implies also $q=s$. Thus in view of (23), (27) and (32)

$$\left|\frac{p}{q}-\frac{r}{s}\right| = \left|\left(\frac{p}{q}-\alpha\right)+\left(\alpha-\frac{r}{s}\right)\right| \geqq \left|\alpha-\frac{p}{q}\right|-\left|\alpha-\frac{r}{s}\right| =$$

$$= |\beta| - \left|\alpha-\frac{r}{s}\right| > \frac{1}{qQ}-\frac{1}{sQ} = 0$$

in contradiction with (37), which proves (36).

(36) implies that

$$\left|\frac{p}{q}-\frac{r}{s}\right| = \frac{|ps-qr|}{qs} \geqq \frac{1}{qs}. \tag{38}$$

On the other hand, with respect to (22), (23), (24), (26) and (32),

$$\left|\frac{p}{q}-\frac{r}{s}\right| = \left|\left(\frac{p}{q}-\alpha\right)+\left(\alpha-\frac{r}{s}\right)\right| \leqq \left|\alpha-\frac{p}{q}\right|+\left|\alpha-\frac{r}{s}\right| = \tag{39}$$

$$= |\beta| + \left|\alpha-\frac{r}{s}\right| < |\beta|+\frac{1}{sQ} < |\beta|+\frac{q|\beta|}{s} < |\beta|+\frac{1}{s}\cdot\frac{1}{2\sqrt{N}} \leqq |\beta|+\frac{1}{2sq}.$$

(38) and (39) yield that

$$\frac{1}{qs} \leqq |\beta|+\frac{1}{2sq}.$$

Thus by (24),

$$s \geqq \frac{1}{2q|\beta|} > \sqrt{N}. \tag{40}$$

In view of (20), (34) and (40), we obtain from (35) that

$$|T(\alpha)| < 7\left(\left(\frac{N}{s}\right)^{1/2} + 2s^{1/2}(\log N)^{1/2}\right) = 7s^{1/2}(\log N)^{1/2}\left(\frac{N^{1/2}}{s(\log N)^{1/2}} + 2\right) <$$

$$< 7s^{1/2}(\log N)^{1/2}(1+2) = 21s^{1/2}(\log N)^{1/2} < 21\left(\frac{2}{q|\beta|}\right)^{1/2}(\log N)^{1/2} < 30\left(\frac{\log N}{q|\beta|}\right)^{1/2}$$

and Lemma 6 is proved.

LEMMA 7. *Let N, p, q be integers, R, Q, α real numbers such that $N \geqq 1$, $(p, q) = 1$,*

$$1 \leqq R \leqq q \leqq Q, \tag{41}$$

$$\sqrt{N} \leqq Q \leqq N \tag{42}$$

and

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qQ}. \tag{43}$$

Then

$$|T(\alpha)| < 7\left(\frac{N}{R}\right)^{1/2} + 14(Q \log N)^{1/2}. \tag{44}$$

PROOF. (41) and (43) imply also

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Thus with respect to (41) and (42), Lemma 4 yields that

$$|T(\alpha)| < 7\left(\frac{N^{1/2}}{q^{1/2}} + (N^{1/2}\log q)^{1/2} + (q \log q)^{1/2}\right) \leqq$$

$$\leqq 7\left(\left(\frac{N}{R}\right)^{1/2} + (Q \log N)^{1/2} + (Q \log N)^{1/2}\right)$$

which proves (44).

4. In this section, we estimate the function $F(\alpha)$ by using Roth's method. Lemmas 8 and 9 follow trivially from the definitions of the functions $A(x)$ and $a(x)$.

LEMMA 8. *If x, y are positive integers such that $x \leqq y$ then $A(x) \leqq A(y)$.*

LEMMA 9. *For any positive integer x,*

$$\frac{1}{x} \leqq a(x) \leqq 1.$$

For any integer b and positive integers q, x, let $A_{(b,q)}(x)$ denote the greatest number of integers that can be selected from $b+q, b+2q, \ldots, b+xq$ to form an $\mathcal{A}$-set.

LEMMA 10. *For any integer b and positive integers q, x,*

$$A_{(b,q^2)}(x) = A(x).$$

PROOF. This follows trivially from the fact that the numbers $b+u_1q^2$, $b+u_2q^2$, $\dots$, $b+u_kq^2$ form an $\mathcal{A}$-set if and only if also the numbers $u_1, u_2, \dots, u_k$ do.

LEMMA 11. *For any positive integers x and y, we have*

$$A(x+y) \leqq A(x)+A(y), \tag{45}$$

$$A(xy) \leqq xA(y), \tag{46}$$

$$a(xy) \leqq a(y), \tag{47}$$

$$a(x) \leqq \left(1+\frac{y}{x}\right) a(y). \tag{48}$$

PROOF. Applying Lemma 10 with $b=y$, $q=1$, we obtain that the greatest number of integers that can be selected from $y+1, y+2, \dots, y+x$ to form an $\mathcal{A}$-set is $\leqq A(x)$. (45) follows easily from this fact.

(46) is a consequence of (45).

(47) can be obtained from (46) by dividing by xy.

Finally, by Lemma 8 and (46),

$$A(x) \leqq A\left(\left(\left[\frac{x}{y}\right]+1\right)y\right) \leqq \left(\left[\frac{x}{y}\right]+1\right)A(y) \leqq \left(\frac{x}{y}+1\right)A(y).$$

Dividing by x, we obtain (48).

LEMMA 12. *Let q, t, N be positive integers, p integer, α, β real numbers such that*

$$\alpha-\frac{p}{q} = \beta. \tag{49}$$

Let

$$F_1(\alpha) = \frac{a(t)}{q^2}\left(\sum_{s=1}^{q^2} e\left(\frac{sp}{q}\right)\right)\left(\sum_{j=1}^{N} e(\beta j)\right),$$

so that if $(p,q)=1$ *then*

$$F_1(\alpha) = \begin{cases} a(t)\sum_{j=1}^{N} e(j\alpha) & \text{for } q=1 \\ 0 \quad \text{for } q>1 \end{cases} \quad (\text{where } (p,q)=1). \tag{50}$$

Then

$$|F(\alpha)-F_1(\alpha)| \leqq \big(a(t)-a(N)\big)N+2a(t)tq^2(1+\pi|\beta|N) = H(t,N,q,\beta). \tag{51}$$

PROOF. We are going to show at first that

$$F(\alpha) = \frac{1}{tq^2}\sum_{s=1}^{q^2}\sum_{j=1}^{N}\sum_{\substack{j\leqq u_k<j+tq^2 \\ u_k\equiv s \pmod{q^2}}} e(\alpha u_k)+\theta\big(a(t)tq^2\big). \tag{52}$$

Let us investigate the coefficient of $e(\alpha u_k)$ on the right hand side.

If $tq^2 \leqq u_k \leqq N$ then we account $e(\alpha u_k)$ exactly tq^2 times, namely for the following values of j:

$$j = u_k - tq^2 + 1, \quad u_k - tq^2 + 2, \ldots, u_k.$$

Thus the coefficient of $e(\alpha u_k)$ is

$$tq^2 \cdot \frac{1}{tq^2} = 1$$

in this case (and its coefficient is the same on the left hand side).

If

$$(53) \qquad 1 \leqq u_k < tq^2$$

then we account $e(\alpha u_k)$ on the right of (52) for $j = 1, 2, \ldots, u_k$, thus its coefficient is

$$(0 \leqq) u_k \cdot \frac{1}{tq^2} < tq^2 \cdot \frac{1}{tq^2} = 1$$

on the right and 1 on the left of (52). The numbers u_k satisfying (53) form an $\mathcal{A}$-set thus in view of (46) in Lemma 11, their number is

$$\leqq A(tq^2) \leqq A(t)q^2 = a(t)tq^2.$$

These facts yield that, in fact, the error term in (52) is $\theta(a(t)tq^2)$.

The term $e(\alpha u_k)$ in the inner sum in (52) can be rewritten in the following way:

$$e(\alpha u_k) = e\left(\left(\frac{p}{q}+\beta\right)u_k\right) = e\left(\frac{pu_k}{q}\right)e(\beta u_k) =$$

$$= e\left(\frac{ps}{q}\right)e(\beta j)e(\beta(u_k-j)) = e\left(\frac{ps}{q}\right)e(\beta j)(1+\theta(2\pi|\beta(u_k-j)|)) =$$

$$= e\left(\frac{ps}{q}\right)e(\beta j)+\theta(2\pi|\beta(u_k-j)|) = e\left(\frac{ps}{q}\right)e(\beta j)+\theta(2\pi|\beta|tq^2)$$

since $|u_k-j|<tq^2$ in the inner sum, and

$$|e(\gamma)-1| = |e(\gamma/2)-e(-\gamma/2)| = |2\sin\pi\gamma| \leqq 2|\pi\gamma| = 2\pi|\gamma|$$

for any real number γ.

Thus the inner sum in (52) can be estimated in the following way:

$$(54) \qquad \sum_{\substack{j\leqq u_k<j+tq^2\\ u_k\equiv s \pmod{q^2}}} e(\alpha u_k) = \sum_{\substack{j\leqq u_k<j+tq^2\\ u_k\equiv s \pmod{q^2}}} \left(e\left(\frac{ps}{q}\right)e(\beta j)+\theta(2\pi|\beta|tq^2)\right) =$$

$$= \left(e\left(\frac{ps}{q}\right)e(\beta j)+\theta(2\pi|\beta|tq^2)\right) \sum_{\substack{j\leqq u_k<j+tq^2\\ u_k\equiv s \pmod{q^2}}} 1.$$

Let us define the integer v by

$$v < j \leqq v+q^2, \quad v \equiv s \pmod{q^2}.$$

Then by Lemma 10,

$$\sum_{\substack{j \leqq u_k < j+tq^2 \\ u_k \equiv s \pmod{q^2}}} 1 \leqq A_{(v,q^2)}(t) \leqq A(t) = a(t)t.$$

Thus if we define $D(j, t, q, s)$ by

$$\sum_{\substack{j \leqq u_k < j+tq^2 \\ u_k \equiv s \pmod{q^2}}} 1 = a(t)t - D(j, t, q, s)$$

then $D(j, t, q, s) \geqq 0$. Putting this into (54):

$$\sum_{\substack{j \leqq u_k < j+tq^2 \\ u_k \equiv s \pmod{q^2}}} e(\alpha u_k) = \left(e\left(\frac{ps}{q}\right) e(\beta j) + \theta(2\pi|\beta|tq^2)\right)\left(a(t)t - D(j, t, q, s)\right) =$$

$$= e\left(\frac{ps}{q}\right) e(\beta j)\left(a(t)t - D(j, t, q, s)\right) + \theta\left(2\pi|\beta|a(t)t^2q^2\right).$$

Thus (52) yields that

$$(55) \quad F(\alpha) = \frac{1}{tq^2} \sum_{s=1}^{q^2} \sum_{j=1}^{N} \left\{e\left(\frac{ps}{q}\right) e(\beta j)\left(a(t)t - D(j, t, q, s)\right) + \theta\left(2\pi|\beta|a(t)t^2q^2\right)\right\} +$$

$$+ \theta\left(a(t)tq^2\right) = \frac{a(t)}{q^2}\left(\sum_{s=1}^{q^2} e\left(\frac{ps}{q}\right)\right)\left(\sum_{j=1}^{N} e(\beta j)\right) - \frac{1}{tq^2} \sum_{s=1}^{q^2} \sum_{j=1}^{N} e\left(\frac{ps}{q}\right) e(\beta j) D(j, t, q, s) +$$

$$+ \theta\left(\frac{1}{tq^2} \cdot q^2 \cdot N \cdot 2\pi|\beta|a(t)t^2q^2\right) + \theta\left(a(t)tq^2\right) =$$

$$= F_1(\alpha) - \frac{1}{tq^2} \sum_{s=1}^{q^2} \sum_{j=1}^{N} e\left(\frac{ps}{q}\right) e(\beta j) D(j, t, q, s) + \theta\left(2\pi|\beta|Na(t)tq^2\right) + \theta\left(a(t)tq^2\right).$$

Putting here $\alpha = \beta = p = 0$, we obtain that

$$A(N) = a(t) \cdot N - \frac{1}{tq^2} \sum_{s=1}^{q^2} \sum_{j=1}^{N} D(j, t, q, s) + \theta\left(a(t)tq^2\right).$$

Hence

$$(56) \quad \frac{1}{tq^2} \sum_{s=1}^{q^2} \sum_{j=1}^{N} D(j, t, q, s) \leqq a(t) \cdot N - A(N) + a(t)tq^2 =$$

$$= \left(a(t) - a(N)\right)N + a(t)tq^2.$$

With respect to (56), (55) yields that

$$|F(\alpha) - F_1(\alpha)| \leqq \frac{1}{tq^2} \sum_{s=1}^{q^2} \sum_{j=1}^{N} D(j, t, q, s) + 2\pi|\beta|Na(t)tq^2 + a(t)tq^2 \leqq$$

$$\leqq \left(a(t) - a(N)\right)N + a(t)tq^2 + 2\pi|\beta|Na(t)tq^2 + a(t)tq^2 =$$

$$= \left(a(t) - a(N)\right) + 2a(t)tq^2(1 + \pi|\beta|N)$$

which proves Lemma 12.

5. In this section, we are going to deduce a functional inequality for the function $a(x)$, by investigating the integral E defined in Section 2. This inequality will contain four parameters: t, N, R, Q whose values will be fixed later.

LEMMA 13. *Let t, N be positive integers, R, Q real numbers such that*

$$N \geqq e^8, \tag{57}$$

$$t/N, \tag{58}$$

$$1 \leqq R \leqq N^{1/2}/\log N, \tag{59}$$

$$2N^{1/2} < Q \leqq \frac{N}{\log N}. \tag{60}$$

Then

$$\begin{aligned} a^2(t)N^{3/2} < {} & 1260a(t)\big(a(t)-a(N)\big)N^{3/2}\log\log N+ \\ & +12\,600a^2(t)tN(\log N)^{1/2}Q^{-1/2}+ \\ & +120\big(a(t)-a(N)\big)^2\{7N^{3/2}R^{3/2}\log N+20N^2(\log N)^{1/2}Q^{-1/2}R\}+ \\ & +26000a^2(t)t^2\{3N^{-1/2}(\log N)^3R^{11/2}+2N^2(\log N)^{1/2}Q^{-5/2}R^3\}+ \\ & +140a(t)\{N^{3/2}R^{-1/2}+2NQ^{1/2}(\log N)^{1/2}\}. \end{aligned} \tag{61}$$

PROOF. Let us write

$$G(\alpha) = a(t)\sum_{j=1}^{N} e(j\alpha).$$

Then

$$E = \int_0^1 |F(\alpha)|^2T(\alpha)\,d\alpha = \int_0^1 |G(\alpha)|^2T(\alpha)\,d\alpha + \int_0^1 \big(|F(\alpha)|^2-|G(\alpha)|^2\big)T(\alpha)\,d\alpha$$

where $E=0$ by (5). Hence

$$\int_0^1 |G(\alpha)|^2T(\alpha)\,d\alpha = -\int_0^1 \big(|F(\alpha)|^2-G(\alpha)|^2\big)\,T(\alpha)\,d\alpha. \tag{62}$$

Here

$$\int_0^1 |G(\alpha)|^2T(\alpha)d\alpha = \int_0^1 \Big(a(t)\sum_{j=1}^{N} e(j\alpha)\Big)\Big(a(t)\sum_{k=1}^{N} e(-k\alpha)\Big)\Big(\sum_{z=1}^{[\sqrt{N}]} e(z^2\alpha)\Big)\,d\alpha = \tag{63}$$

$$= a^2(t)\int_0^1 \sum_{\substack{1\leqq j,k\leqq N \\ 1\leqq z\leqq\sqrt{N}}} e\big((j-k+z^2)\alpha\big)\,d\alpha = a^2(t)\sum_{\substack{j-k+z^2=0 \\ 1\leqq j,k\leqq N \\ 1\leqq z\leqq\sqrt{N}}} 1.$$

If

$$1 \leqq z^2 \leqq \frac{N}{2}-1, \quad z>0, \tag{64}$$

$$1 \leqq j \leqq \frac{N}{2}+1 \tag{65}$$

then the numbers j, $k=j+z^2$, z satisfy the conditions

$$j-k+z^2=0, \quad 1 \leqq j, k \leqq N, \quad 1 \leqq z \leqq \sqrt{N}$$

since

$$k=j+z^2 \leqq \left(\frac{N}{2}+1\right)+\left(\frac{N}{2}-1\right)=N.$$

By (57), the number of the positive integers z satisfying (64) is at least

$$\left[\sqrt{\frac{N}{2}-1}\right] \geqq \left[\sqrt{\frac{N}{2}-\frac{N}{4}}\right]=\left[\frac{\sqrt{N}}{2}\right] \geqq \frac{\sqrt{N}}{2}-1 \geqq \frac{\sqrt{N}}{2}-\frac{\sqrt{N}}{4}=\frac{\sqrt{N}}{4}$$

while (65) holds for $\left[\frac{N}{2}\right]+1>\frac{N}{2}$ integers j. Thus (63) yields that

$$\int_0^1 |G(\alpha)|^2 T(\alpha)\, d\alpha = a^2(t) \sum_{\substack{j-k+z^2=0\\ 1\leqq j,k\leqq N\\ 1\leqq z\leqq \sqrt{N}}} 1 > a^2(t)\cdot\frac{\sqrt{N}}{4}\cdot\frac{N}{2}=\frac{1}{8}a^2(t)N^{3/2}. \tag{66}$$

To complete the proof of (61), we have to give an upper estimate for the right hand side of (62).

For $q=1, 2, \dots, [Q]$ and $p=0, 1, \dots, q-1$, let

$$I_{p,q}=\left(\frac{p}{q}-\frac{1}{pQ}, \frac{p}{q}+\frac{1}{qQ}\right)$$

and let S denote the set of those real numbers α for which

$$-\frac{1}{Q}<\alpha \leqq 1-\frac{1}{Q}$$

holds and

$$\alpha \notin I_{p,q} \quad \text{for} \quad 1 \leqq q \leqq R, \quad 0 \leqq p \leqq q-1, \quad (p, q)=1. \tag{67}$$

Then

$$\left|\int_0^1 (|F(\alpha)|^2-|G(\alpha)|^2)T(\alpha)\, d\alpha\right| = \tag{68}$$

$$=\left|\int_{-1/Q}^{1-1/Q} (|F(\alpha)|^2-|G(\alpha)|^2)\, T(\alpha)\, d\alpha\right| \leqq \int_{-1/Q}^{+1/Q} \big||F(\alpha)|^2-|G(\alpha)|^2\big|\, |T(\alpha)|\, d\alpha +$$

$$+\sum_{q=2}^{[R]} \sum_{\substack{1\leqq p\leqq q-1\\ (p,q)=1}} \int_{I_{p,q}} \big||F(\alpha)|^2-|G(\alpha)|^2\big|\, |T(\alpha)|\, d\alpha +$$

$$+\int_S \big||F(\alpha)|^2-|G(\alpha)|^2\big|\, |T(\alpha)|\, d\alpha = E_1+E_2+E_3.$$

Let us estimate the term E_1 at first.

For any complex numbers u, v, we have

$$(69)\qquad \big||u|^2-|v|^2\big| = |u\bar{u}-v\bar{v}| = |(u-v)\bar{u}+v(\bar{u}-\bar{v})| \leqq$$
$$\leqq |u-v||\bar{u}|+|v||\bar{u}-\bar{v}| = |u-v|(|u|+|v|) = |u-v|\big(|(u-v)+v|+|v|\big) \leqq$$
$$\leqq |u-v|\big(|u-v|+2|v|\big) = |u-v|^2+2|u-v|\,|v|.$$

Furthermore, applying Lemma 12 with $p=0$, $q=1$, $\alpha=\beta$, we have $F_1(\alpha)=G(\alpha)$ there, thus we obtain that

$$(70)\qquad |F(\alpha)-G(\alpha)| \leqq H(t, N, 1, \alpha).$$

(69) and (70) yield that

$$(71)\qquad E_1 = \int_{-1/Q}^{+1/Q} \big||F(\alpha)|^2-|G(\alpha)|^2\big||T(\alpha)|\,d\alpha \leqq$$
$$\leqq \int_{-1/Q}^{+1/Q} |F(\alpha)-G(\alpha)|^2|T(\alpha)|\,d\alpha+2\int_{-1/Q}^{+1/Q} |F(\alpha)-G(\alpha)|\,|G(\alpha)||T(\alpha)|\,d\alpha \leqq$$
$$\leqq \int_{-1/Q}^{+1/Q} H^2(t, N, 1, \alpha)|T(\alpha)|\,d\alpha+2\int_{-1/Q}^{+1/Q} H(t, N, 1, \alpha)\,|G(\alpha)|\,|T(\alpha)|\,d\alpha = E_1'+2E_1''.$$

E_1' will be estimated simultaneously with E_2; here we estimate only E_1''.

The function $|G(\alpha)|$ can be estimated by using Lemma 1. Furthermore, for $|\alpha|\leqq\log N/N$, we use the trivial inequality

$$(72)\qquad |T(\alpha)| = \left|\sum_{z=1}^{M} e(z^2\alpha)\right| \leqq \sum_{z=1}^{M} 1 = M \leqq N^{1/2},$$

while for $\log N/N<|\alpha|\leqq 1/Q(<1/2\sqrt{N}$, by (60)), we apply Lemma 6. In this way, we obtain that

$$(73)\qquad E_1'' < \int_{|\alpha|\leqq 1/N} \Big\{(a(t)-a(N))N+2a(t)t\Big(1+\pi\cdot\frac{1}{N}\cdot N\Big)\Big\}\cdot a(t)N\cdot N^{1/2}\,d\alpha+$$
$$+\int_{1/N<|\alpha|\leqq\log N/N} \{(a(t)-a(N))N+2a(t)t(1+\pi)\,|\alpha|N\}\cdot a(t)\frac{1}{2|\alpha|}\cdot N^{1/2}\,d\alpha+$$
$$+\int_{\log N/N<|\alpha|\leqq 1/Q} \{(a(t)-a(N))N+2a(t)t(1+\pi)\,|\alpha|N\}\cdot a(t)\frac{1}{2|\alpha|}\cdot 30\left(\frac{\log N}{|\alpha|}\right)^{1/2} d\alpha <$$
$$< \frac{2}{N}\{a(t)\big(a(t)-a(N)\big)N^{5/2}+2a^2(t)t\cdot 5\cdot N^{3/2}\}+$$
$$+\frac{1}{2}a(t)\big(a(t)-a(N)\big)N^{3/2}\int_{1/N<|\alpha|\leqq\log N/N}\frac{1}{|\alpha|}\,d\alpha+2\cdot\frac{\log N}{N}\cdot a^2(t)t\cdot 5\cdot N^{3/2}+$$
$$+15a(t)\big(a(t)-a(N)\big)N(\log N)^{1/2}\int_{\log N/N<|\alpha|\leqq 1/Q}\frac{1}{|\alpha|^{3/2}}\,d\alpha+$$
$$+30a^2(t)t\cdot 5N(\log N)^{1/2}\int_{\log N/N<|\alpha|\leqq 1/Q}\frac{1}{|\alpha|^{1/2}}\,d\alpha.$$

Here

$$\int_{1/N<|\alpha|\leq \log N/N} \frac{1}{|\alpha|}\,d\alpha = 2\log\log N,$$

$$\int_{\log N/N<|\alpha|\leq 1/Q} \frac{1}{|\alpha|^{3/2}}\,d\alpha < 2\int_{\log N/N}^{+\infty} \frac{1}{\alpha^{3/2}}\,d\alpha = 2\cdot 2\left(\frac{\log N}{N}\right)^{-1/2} = 4\left(\frac{N}{\log N}\right)^{1/2}$$

and

$$\int_{\log N/N<|\alpha|\leq 1/Q} \frac{1}{|\alpha|^{1/2}}\,d\alpha < 2\int_{0}^{1/Q} \frac{1}{\alpha^{1/2}}\,d\alpha = 2\cdot 2\left(\frac{1}{Q}\right)^{1/2} = \frac{4}{Q^{1/2}}.$$

Thus with respect to (57), (60) and $a(t)\geq a(N)$ (by (47) and (58)),

$$E_1'' < 2a(t)\big(a(t)-a(N)\big)N^{3/2}+20a^2(t)tN^{1/2}+ \tag{74}$$
$$+a(t)\big(a(t)-a(N)\big)N^{3/2}\log\log N+10a^2(t)tN^{1/2}\log N+$$
$$+60a(t)\big(a(t)-a(N)\big)N^{3/2}+600a^2(t)tN(\log N)^{1/2}Q^{-1/2}<$$
$$<63a(t)\big(a(t)-a(N)\big)N^{3/2}\log\log N+30a^2(t)tN^{1/2}\log N\left\{1+20\left(\frac{N}{Q\log N}\right)^{1/2}\right\}\leq$$
$$\leq 63a(t)\big(a(t)-a(N)\big)N^{3/2}\log\log N+630a^2(t)tN(\log N)^{1/2}Q^{-1/2}.$$

Now we are going to estimate $E_1'+E_2$.

If $2\leq q\leq Q$, $1\leq p\leq q-1$ then $\alpha\in I_{p,q}$ implies that

$$\|\alpha\|\geq \frac{1}{q}-\frac{1}{qQ}\geq \frac{1}{q}-\frac{1}{2q}=\frac{1}{2q}.$$

Thus for $2\leq q\leq Q$, $1\leq p\leq q-1$ and $(p,q)=1$, Lemmas 1 and 12 (where $F_1(\alpha)=0$ in this case) and the trivial inequality (72) yield that

$$\int_{I_{p,q}} \big||F(\alpha)|^2-|G(\alpha)|^2\big|\,|T(\alpha)|\,d\alpha \leq \int_{I_{p,q}} |F(\alpha)|^2|T(\alpha)|\,d\alpha+\int_{I_{p,q}} |G(\alpha)|^2|T(\alpha)|\,d\alpha\leq$$
$$\leq \int_{I_{p,q}} |F(\alpha)|^2|T(\alpha)|\,d\alpha+\int_{I_{p,q}} a^2(t)\frac{1}{2}(2q)^2N^{1/2}\,d\alpha=$$
$$=\int_{I_{p,q}} |F(\alpha)|^2|T(\alpha)|\,d\alpha+\frac{1}{2qQ}\cdot 2a^2(t)q^2N^{1/2}\leq$$
$$\leq \int_{-1/qQ}^{+1/qQ} H^2(t,N,q,\beta)\left|T\left(\frac{p}{q}+\beta\right)\right|d\beta+a^2(t)N^{1/2}.$$

Hence

$$E_1'+E_2\leq \int_{-1/Q}^{+1/Q} H^2(t,N,1,\alpha)\,|T(\alpha)|\,d\alpha+ \tag{75}$$
$$+\sum_{q=2}^{[R]}\sum_{\substack{1\leq p\leq q-1\\(p,q)=1}}\left\{\int_{-1/qQ}^{+1/qQ} H^2(t,N,q,\beta)\left|T\left(\frac{p}{q}+\beta\right)\right|d\beta+a^2(t)N^{1/2}\right\}\leq$$
$$\leq \sum_{q=1}^{[R]}\sum_{\substack{0\leq p\leq q-1\\(p,q)=1}}\int_{-1/qQ}^{+1/qQ} H^2(t,N,q,\beta)\left|T\left(\frac{p}{q}+\beta\right)\right|d\beta+a^2(t)R^2N^{1/2}.$$

To estimate $T\left(\frac{p}{q}+\beta\right)$, we use Lemmas 5 and 6 for $|\beta|\leqq\log N/N$ and $\log N/N<|\beta|\leqq\leqq 1/qQ$, respectively. We obtain with respect to (57), (59) and (60) that (for $q\leqq R$, $(p,q)=1$)

$$\int_{-1/qQ}^{+1/qQ} H^2(t,N,q,\beta)\left|T\left(\frac{p}{q}+\beta\right)\right| d\beta <$$

$$< \int_{|\beta|\leqq\log N/N} \{2(a(t)-a(N))^2N^2+16a^2(t)t^2q^4(1+\pi^2\beta^2N^2)\}\, 21\left(\frac{N}{q}\right)^{1/2} d\beta+$$

$$+\int_{\log N/N<|\beta|\leqq 1/qQ} \{2(a(t)-a(N))^2N^2+16a^2(t)t^2q^4(1+\pi^2\beta^2N^2)\}\, 30\left(\frac{\log N}{q|\beta|}\right)^{1/2} d\beta<$$

$$<2\frac{\log N}{N}\left\{2(a(t)-a(N))^2N^2+16a^2(t)t^2q^4\left(1+\pi^2\frac{\log^2 N}{N^2}N^2\right)\right\} 21\left(\frac{N}{q}\right)^{1/2}+$$

$$+60(a(t)-a(N))^2N^2(\log N)^{1/2}q^{-1/2}\int_{\log N/N<|\beta|\leqq 1/qQ}|\beta|^{-1/2}d\beta+$$

$$+\int_{\log N/N<|\beta|\leqq 1/qQ} 480a^2(t)t^2q^4\cdot 11\beta^2N^2\cdot\left(\frac{\log N}{q|\beta|}\right)^{1/2}d\beta<$$

$$<84(a(t)-a(N))^2N^{3/2}\log N\cdot q^{-1/2}+32\cdot\log N\cdot N^{-1/2}a^2(t)t^2q^{7/2}\cdot 11\cdot\log^2 N\cdot 21+$$

$$+120(a(t)-a(N))^2N^2(\log N)^{1/2}q^{-1/2}\int_0^{1/qQ}\beta^{-1/2}d\beta+$$

$$+10560a^2(t)t^2q^{7/2}N^2(\log N)^{1/2}\int_0^{1/qQ}\beta^{3/2}d\beta=$$

$$=84(a(t)-a(N))^2N^{3/2}\log N\cdot q^{-1/2}+7392a^2(t)t^2N^{-1/2}(\log N)^3q^{7/2}+$$

$$+120(a(t)-a(N))^2N^2(\log N)^{1/2}q^{-1/2}\cdot 2(qQ)^{-1/2}+$$

$$+10560a^2(t)t^2q^{7/2}N^2(\log N)^{1/2}\cdot\frac{2}{5}(qQ)^{-5/2}=$$

$$=(a(t)-a(N))^2\{84N^{3/2}\log N\cdot q^{-1/2}+240N^2(\log N)^{1/2}Q^{-1/2}q^{-1}\}+$$

$$+a^2(t)t^2\{7392N^{-1/2}(\log N)^3q^{7/2}+4224N^2(\log N)^{1/2}Q^{-5/2}q\}$$

since

$$(x+y)^2\leqq 2x^2+2y^2$$

for any real numbers x, y. Thus (75) yields with respect to (57) and (59) that

(76)

$$E_1'+E_2<\sum_{q=1}^{[R]}\sum_{p=0}^{q-1}\{(a(t)-a(N))^2(84N^{3/2}\log N\cdot q^{-1/2}+240N^2(\log N)^{1/2}Q^{-1/2}q^{-1})+$$
$$+a^2(t)t^2(7392N^{-1/2}(\log N)^3q^{7/2}+4224N^2(\log N)^{1/2}Q^{-5/2}q)\}+a^2(t)R^2N^{1/2}<$$
$$<\sum_{q=1}^{[R]}\{(a(t)-a(N))^2(84N^{3/2}\log N\cdot q^{1/2}+240N^2(\log N)^{1/2}Q^{-1/2})+$$
$$+a^2(t)t^2(7392N^{-1/2}(\log N)^3q^{9/2}+4224N^2(\log N)^{1/2}Q^{-5/2}q^2)\}+$$
$$+a^2(t)(N^{1/2}/\log N)^2N^{1/2}\leqq$$
$$\leqq(a(t)-a(N))^2\{84N^{3/2}R^{3/2}\log N+240N^2(\log N)^{1/2}Q^{-1/2}R\}+$$
$$+a^2(t)t^2\{7392N^{-1/2}(\log N)^3R^{11/2}+4224N^2(\log N)^{1/2}Q^{-5/2}R^3\}+\frac{1}{64}a^2(t)N^{3/2}.$$

Finally, in order to estimate E_3, we use Lemma 7. Namely, if $\alpha\in S$ then there exist integers p, q such that

$$1\leqq q\leqq Q,\quad 0\leqq p\leqq q-1,\quad (p,q)=1$$

and

$$\left|\alpha-\frac{p}{q}\right|<\frac{1}{qQ};$$

by (67), q satisfies also $R<q$. Thus (41) and (43) in Lemma 7 hold (and also (42) holds by (60)). Hence, Lemma 7 yields that

$$\sup_{\alpha\in S}|T(\alpha)|\leqq 7\left(\frac{N}{R}\right)^{1/2}+14(Q\log N)^{1/2}.$$

Thus we obtain applying Parseval's formula that

(77)

$$E_3=\int_S\big||F(\alpha)|^2-|G(\alpha)|^2\big||T(\alpha)|\,d\alpha\leqq$$
$$\leqq\sup_{\alpha\in S}|T(\alpha)|\Big(\int_S|F(\alpha)|^2\,d\alpha+\int_S|G(\alpha)|^2\,d\alpha\Big)<$$
$$<\{7N^{1/2}R^{-1/2}+14Q^{1/2}(\log N)^{1/2}\}\Big(\int_0^1|F(\alpha)|^2\,d\alpha+\int_0^1|G(\alpha)|^2\,d\alpha\Big)=$$
$$=\{7N^{1/2}R^{-1/2}+14Q^{1/2}(\log N)^{1/2}\}\big(A(N)+a^2(t)N\big)=$$
$$=\{7N^{1/2}R^{-1/2}+14Q^{1/2}(\log N)^{1/2}\}\big(a(N)N+a^2(t)N\big)\leqq$$
$$\leqq\{7N^{1/2}R^{-1/2}+14Q^{1/2}(\log N)^{1/2}\}\big(a(t)N+a(t)N\big)=$$
$$=a(t)\{14N^{3/2}R^{-1/2}+28NQ^{1/2}(\log N)^{1/2}\}$$

by Lemma 9 and since $a(N)\leqq a(t)$ by (47) in Lemma 11 and (58).

Collecting the results (62), (66), (68), (71), (74), (76) and (77) together, we obtain that

$$\frac{1}{8}a^2(t)N^{3/2} < \int_0^1 |G(\alpha)|^2 T(\alpha)\,d\alpha \leqq E_1+E_2+E_3 \leqq$$

$$\leqq (E_1'+2E_1'')+E_2+E_3 = 2E_1''+(E_1'+E_2)+E_3 <$$

$$< 126a(t)\big(a(t)-a(N)\big)N^{3/2}\log\log N+1260a^2(t)tN(\log N)^{1/2}Q^{-1/2}+$$

$$+\big(a(t)-a(N)\big)^2\{84N^{3/2}R^{3/2}\log N+240N^2(\log N)^{1/2}Q^{-1/2}R\}+$$

$$+a^2(t)t^2\{7392N^{-1/2}(\log N)^3R^{11/2}+4224N^2(\log N)^{1/2}Q^{-5/2}R^3\}+$$

$$+\frac{1}{64}a^2(t)N^{3/2}+a(t)\{14N^{3/2}R^{-1/2}+28NQ^{1/2}(\log N)^{1/2}\}.$$

Subtracting $\frac{1}{64}a^2(t)N^{3/2}$ and then multiplying by

$$\left(\frac{1}{8}-\frac{1}{64}\right)^{-1} = \left(\frac{7}{64}\right)^{-1} = \frac{64}{7} < 10,$$

we obtain that

$$a^2(t)N^{3/2} < 1260a(t)\big(a(t)-a(N)\big)^{3/2}\log\log N+12600a^2(t)tN(\log N)^{1/2}Q^{-1/2}+$$

$$+120\big(a(t)-a(N)\big)^2\{7N^{3/2}R^{3/2}\log N+20N^2(\log N)^{1/2}Q^{-1/2}R\}+$$

$$+26000a^2(t)t^2\{3N^{-1/2}(\log N)^3R^{11/2}+2N^2(\log N)^{1/2}Q^{-5/2}R^3\}+$$

$$+140\,a(t)\{N^{3/2}R^{-1/2}+2NQ^{1/2}(\log N)^{1/2}\}$$

which proves Lemma 13.

6. In this section, we are going to simplify the functional inequality given in Lemma 13. It can be shown that we obtain the best possible estimate for $a(x)$ in the case when the order of magnitude of the product QR is $N/\log N$; thus we choose

$$R = \frac{N}{Q}\cdot\frac{1}{\log N}.$$

Furthermore, we put

$$r = \frac{N}{t}, \quad s = \frac{N}{Q}.$$

Then

$$R = \frac{s}{\log N}.$$

The inequalities (59) and (60) can be rewritten in form

$$1 \leqq \frac{s}{\log N} \leqq \frac{N^{1/2}}{\log N},$$

(78) $$\log N \leqq s \leqq N^{1/2}$$

and

$$2N^{1/2} < \frac{N}{s} \leqq \frac{N}{\log N},$$

(79) $$\frac{1}{2}N^{1/2} > s \geqq \log N,$$

respectively. (79) implies (78) thus it suffices to assume that (79) holds.

Finally, if we divide (61) by $N^{3/2}$ then we obtain that

$$a^2(t) < 1260a(t)\big(a(t)-a(N)\big)\log\log N + 12600a^2(t)r^{-1}s^{1/2}(\log N)^{1/2} +$$
$$+120\big(a(t)-a(N)\big)^2\{7s^{3/2}(\log N)^{-1/2}+20s^{3/2}(\log N)^{-1/2}\}+$$
$$+26000a^2(t)r^{-2}\{3s^{11/2}(\log N)^{-5/2}+2s^{11/2}(\log N)^{-5/2}\}+$$
$$+140a(t)\{s^{-1/2}(\log N)^{1/2}+2s^{-1/2}(\log N)^{1/2}\} <$$
$$< 1260a(t)\big(a(t)-a(N)\big)\log\log N +$$
$$+130000a^2(t)\{r^{-1}s^{1/2}(\log N)^{1/2}+r^{-2}s^{11/2}(\log N)^{-5/2}\}+$$
$$+3240\big(a(t)-a(N)\big)^2 s^{3/2}(\log N)^{-1/2}+420a(t)s^{-1/2}(\log N)^{1/2}.$$

Thus we have proved the following

LEMMA 14. *Let t, N, r be positive integers, s a real number such that*

(80) $$N \geqq e^8,$$

(81) $$N = tr,$$

(82) $$\log N \leqq s < \frac{1}{2}N^{1/2}.$$

Then

(83) $$a^2(t) < 1260a(t)\big(a(t)-a(N)\big)\log\log N +$$
$$+130000a^2(t)\{r^{-1}s^{1/2}(\log N)^{1/2}+r^{-2}s^{11/2}(\log N)^{-5/2}\}+$$
$$+3240\big(a(t)-a(N)\big)^2 s^{3/2}(\log N)^{-1/2}+420a(t)s^{-1/2}(\log N)^{1/2}.$$

7. In this section, we are going to complete the proof of our theorem by showing that the functional inequality given in Lemma 14 implies (2).

Let us put

$$\varphi(x) = \frac{(\log\log x)^{2/3}}{(\log x)^{1/3}}$$

for $x \geqq 3$. Furthermore, for $L=3, 4, \ldots$, let us define the sequence $\mathscr{B}(L)=\{b_1, b_2, \ldots\}$ by the following recursion: let $b_1=L$, and for $k=1, 2, \ldots$, let

$$b_{k+1} = b_k\big[4\cdot 10^{22}\big(\varphi(b_k)\big)^{-11/2}(\log b_k)^{3/2}\big].$$

Obviously, $\big(\varphi(x)\big)^{-1} \to +\infty$ as $x \to +\infty$. Thus there exists a real number X_0

such that $(\varphi(x))^{-1}>1$ for $x\geqq X_0$. Let us put $L_1=\max\{X_0, 3\}$. Then we obtain (by straight induction) that for $L\geqq L_1$,

$$\frac{b_{k+1}}{b_k}=\left[4\cdot 10^{22}(\varphi(b_k))^{-11/2}(\log b_k)^{3/2}\right]>[4\cdot 10^{22}\cdot 1\cdot 1]=4\cdot 10^{22}>1. \tag{84}$$

Hence

$$L=b_1<b_2<b_3<\dots \quad (\text{for} \quad L\geqq L_1). \tag{85}$$

We are going to show (by straight induction, using Lemma 14) that if L is large enough then for $k=1, 2, \dots,$

$$a(b_k)\leqq\frac{1}{\varphi(L)}\cdot\varphi(b_k). \tag{86}$$

For $k=1$, the right hand side of (86) is

$$\frac{1}{\varphi(L)}\cdot\varphi(b_k)=\frac{1}{\varphi(L)}\cdot\varphi(b_1)=\frac{1}{\varphi(L)}\cdot\varphi(L)=1$$

thus in this case, (86) holds trivially by Lemma 9.

Let us suppose now that (86) holds for some positive integer k. We have to show that this implies also

$$a(b_{k+1})\leqq\frac{1}{\varphi(L)}\varphi(b_{k+1}). \tag{87}$$

Let us suppose indirectly that

$$a(b_{k+1})>\frac{1}{\varphi(L)}\varphi(b_{k+1}). \tag{88}$$

By the construction of the sequence $\mathscr{B}(L)$,

$$b_k/b_{k+1}. \tag{89}$$

Hence,

$$a(b_k)\geqq a(b_{k+1}) \tag{90}$$

by (47) in Lemma 11.

We are going to show that for sufficiently large L, Lemma 14 is applicable with $t=b_k$, $N=b_{k+1}$,

$$r=N/t=b_{k+1}/b_k=\left[4\cdot 10^{22}(\varphi(b_k))^{-11/2}(\log b_k)^{3/2}\right],$$

$$s=5\cdot 10^6(a(b_k))^{-2}\log b_{k+1}=5\cdot 10^6(a(t))^{-2}\log N.$$

Then by (85),

$$N>t\geqq L \quad (\text{for} \quad L\geqq L_1). \tag{91}$$

Thus (80) holds for $L\geqq\max\{e^8, L_1\}$.

(81) holds by the definitions of N, t, r and $\mathscr{B}(L)$.

By Lemma 9,

$$s=5\cdot 10^6(a(b_k))^{-2}\log N\geqq 5\cdot 10^6\log N>\log N.$$

Finally, we have to prove that

$$s < \frac{1}{2} N^{1/2}.$$

With respect to (84), (88) and (89),

$$\text{(92)} \qquad s = 5 \cdot 10^6 (a(b_k))^{-2} \log b_{k+1} \leqq$$

$$\leqq 5 \cdot 10^6 (a(b_{k+1}))^{-2} \log b_{k+1} \leqq 5 \cdot 10^6 \left(\frac{1}{\varphi(L)} \varphi(b_{k+1})\right)^{-2} \log b_{k+1} \leqq$$

$$\leqq 5 \cdot 10^6 (\varphi(b_{k+1}))^{-2} \log b_{k+1} = 5 \cdot 10^6 (\varphi(N))^{-2} \log N = 5 \cdot 10^6 \frac{(\log N)^{5/3}}{(\log\log N)^{4/3}}$$

for $L \geqq L_1 \geqq X_0$. Thus it suffices to show that

$$5 \cdot 10^6 \frac{(\log N)^{5/3}}{(\log\log N)^{4/3}} < \frac{1}{2} N^{1/2}.$$

But this holds trivially for large N, i.e. for $L \geqq L_2$ (in view of (91)).

Thus Lemma 14 is applicable and it yields that (83) holds. To deduce a contradiction from (83), we have to estimate r and $a(t) - a(N)$ (in terms of $a(t)$ and N).

Obviously,

$$\text{(93)} \qquad r = \frac{N}{t} = \frac{b_{k+1}}{b_k} = [4 \cdot 10^{22} (\varphi(b_k))^{-11/2} (\log b_k)^{3/2}] =$$

$$= [4 \cdot 10^{22} (\varphi(t))^{-11/2} (\log t)^{3/2}] = \left[4 \cdot 10^{22} \left\{\frac{(\log\log t)^{2/3}}{(\log t)^{1/3}}\right\}^{-11/2} (\log t)^{3/2}\right] =$$

$$= \left[4 \cdot 10^{22} \frac{(\log t)^{10/3}}{(\log\log t)^{11/3}}\right] < 4 \cdot 10^{22} \frac{(\log t)^{10/3}}{(\log\log t)^{11/3}}$$

and with respect to (84),

$$\text{(94)} \qquad r = \frac{N}{t} = \left[4 \cdot 10^{22} \frac{(\log t)^{10/3}}{(\log\log t)^{11/3}}\right] > 2 \cdot 10^{22} \frac{(\log t)^{10/3}}{(\log\log t)^{11/3}} \quad (\text{for } L \geqq L_2).$$

In view of (91), (93) and (94) imply that for $L \geqq L_3$,

$$\text{(95)} \qquad (0 <) \log r = \log N - \log t =$$

$$= \frac{10}{3} \log\log t + \theta(1) \cdot \log(4 \cdot 10^{22}) + \theta\left(\frac{11}{3} \log\log\log t\right) <$$

$$< 4 \log\log t \quad (\text{for } L \geqq L_3)$$

hence for $L \geqq L_4$,

$$\text{(96)} \qquad \log t > \log N - 4 \log\log t > \log N - 4 \log\log N >$$

$$> \frac{1}{2} \log N \quad (\text{for } L \geqq L_4)$$

and for $L \geqq L_5$,

(97) $$\log\log t > \log\left(\frac{1}{2}\log N\right) = \log\log N - \log 2 > \frac{1}{2}\log\log N \quad (\text{for } L \geqq L_5).$$

(86), (88), (91), (95), (96) and (97) yield that for $L \geqq L_6$,

(98) $$a(t)-a(N) = a(t)\left(1-\frac{a(N)}{a(t)}\right) < a(t)\left(1-\frac{(\varphi(L))^{-1}\varphi(N)}{(\varphi(L))^{-1}\varphi(t)}\right) =$$

$$= a(t)\frac{\varphi(t)-\varphi(N)}{\varphi(t)} = a(t)\frac{(\log t)^{1/3}}{(\log\log t)^{2/3}}\left(\frac{(\log\log t)^{2/3}}{(\log t)^{1/3}}-\frac{(\log\log N)^{2/3}}{(\log N)^{1/3}}\right) <$$

$$< a(t)\frac{(\log t)^{1/3}}{\left(\frac{1}{2}\log\log N\right)^{2/3}}\left(\frac{(\log\log N)^{2/3}}{(\log t)^{1/3}}-\frac{(\log\log N)^{2/3}}{(\log N)^{1/3}}\right) <$$

$$< a(t)\frac{2(\log t)^{1/3}}{(\log\log N)^{2/3}}(\log\log N)^{2/3}\frac{(\log N)^{1/3}-(\log t)^{1/3}}{(\log t)^{1/3}(\log N)^{1/3}} =$$

$$= 2a(t)\frac{\log N-\log t}{(\log N)^{1/3}((\log N)^{2/3}+(\log N\log t)^{1/3}+(\log t)^{2/3})} <$$

$$< 2a(t)\frac{4\log\log t}{(\log N)^{1/3}\cdot 3(\log t)^{2/3}} < \frac{8}{3}a(t)\frac{\log\log N}{(\log N)^{1/3}\left(\frac{1}{2}\log N\right)^{2/3}} <$$

$$< 6a(t)\frac{\log\log N}{\log N}.$$

By (88), (90), (91), (92), (94), (96) and (98), (83) implies that for $L \geqq L_7$,

$$a^2(t) < 1260a(t)\cdot 6a(t)\frac{\log\log N}{\log N}\cdot\log\log N +$$

$$+130000a^2(t)\left\{2^{-1}\cdot 10^{-22}\frac{(\log\log t)^{11/3}}{(\log t)^{10/3}}\cdot\left(5\cdot 10^6\frac{(\log N)^{5/3}}{(\log\log N)^{4/3}}\right)^{1/2}\cdot(\log N)^{1/2} +\right.$$

$$\left.+\left(2^{-1}10^{-22}\frac{(\log\log t)^{11/3}}{(\log t)^{10/3}}\right)^2\left(5\cdot 10^6\frac{(\log N)^{5/3}}{(\log\log N)^{4/3}}\right)^{11/2}\cdot(\log N)^{-5/2}\right\} +$$

$$+3240\left(6a(t)\frac{\log\log N}{\log N}\right)^2(5\cdot 10^6(a(t))^{-2}\log N)^{3/2}(\log N)^{-1/2} +$$

$$+420a(t)(5\cdot 10^6(a(t))^{-2}\log N)^{-1/2}(\log N)^{1/2} < \frac{7560(\log\log N)^2}{\log N}a^2(t) +$$

$$+130\cdot 10^3a^2(t)\left\{2^{-1}\cdot 10^{-22}\frac{(\log\log N)^{11/3}}{\left(\frac{1}{2}\log N\right)^{10/3}}\cdot 3\cdot 10^3\frac{(\log N)^{4/3}}{(\log\log N)^{2/3}} +\right.$$

$$+2^{-2}\cdot 10^{-44}\frac{(\log\log N)^{22/3}}{\left(\frac{1}{2}\log N\right)^{20/3}}\cdot 5^{1/2}\cdot 5^{5}\cdot 10^{33}\frac{(\log N)^{20/3}}{(\log\log N)^{22/3}}\Bigg\}+$$

$$+4\cdot 10^{3}\cdot 36\cdot a^{2}(t)\frac{(\log\log N)^{2}}{(\log N)^{2}}\cdot 125^{1/2}\cdot 10^{9}(a(t))^{-3}\log N+$$

$$+(21^{2}\cdot 2^{2}\cdot 10^{2}/500\cdot 2^{2}\cdot 10^{2}\cdot 25)^{1/2}a^{2}(t)<$$

$$<\frac{1}{10}a^{2}(t)+a^{2}(t)\left\{130\cdot 10^{-16}\cdot 2^{7/3}\cdot 3\cdot\frac{(\log\log N)^{3}}{(\log N)^{2}}+130\cdot 10^{-8}\cdot 2^{-2}\cdot 2^{7}\cdot 3\cdot 5^{5}\right\}+$$

$$+144\cdot 10^{12}\cdot 12a^{2}(t)\frac{(\log\log N)^{2}}{\log N}\left(\frac{1}{\varphi(L)}\varphi(N)\right)^{-3}+\left(\frac{441}{500}\cdot\frac{1}{25}\right)^{1/2}a^{2}(t)<$$

$$<\frac{1}{10}a^{2}(t)+a^{2}(t)\left\{\frac{1}{10}+390\cdot 10^{-3}\right\}+2\cdot 10^{15}a^{2}(t)\varphi^{3}(L)+\frac{1}{5}a^{2}(t)<$$

$$<\frac{1}{10}a^{2}(t)+a^{2}(t)\left(\frac{1}{10}+\frac{2}{5}\right)+\frac{1}{5}a^{2}(t)+\frac{1}{5}a^{2}(t)=2a^{2}(t)$$

provided that

$$2\cdot 10^{15}\varphi^{3}(L)<\frac{1}{5},\quad \varphi(L)<10^{-16/3}$$

but this holds for $L\geqq L_8$.

Thus for large L, the indirect assumption (88) leads to the contradiction $a^2(t)<<a^2(t)$ which proves (86).

Finally, if x is a positive integer satisfying $x\geqq b_1=L$, then there exists a positive integer k such that $b_k\leqq x<b_{k+1}$. By (48) in Lemma 11, (86) implies that

$$(99)\qquad a(x)\leqq\left(1+\frac{b_k}{x}\right)a(b_k)\leqq 2\cdot\frac{1}{\varphi(L)}\varphi(b_k)\leqq$$

$$\leqq 2\cdot\frac{1}{\varphi(L)}\cdot\frac{(\log\log b_k)^{2/3}}{(\log b_k)^{1/3}}\leqq 2\cdot\frac{1}{\varphi(L)}\cdot\frac{(\log\log x)^{2/3}}{(\log b_k)^{1/3}}.$$

With respect to (96),

$$\log b_k>\frac{1}{2}\log b_{k+1}>\frac{1}{2}\log x.$$

Thus we obtain from (99) that

$$(100)\qquad a(x)\leqq\frac{2}{\varphi(L)}\cdot\frac{(\log\log x)^{2/3}}{\left(\frac{1}{2}\log x\right)^{1/3}}\leqq\frac{4}{\varphi(L)}\cdot\frac{(\log\log x)^{2/3}}{(\log x)^{1/3}}\quad(\text{for } x\geqq L)$$

which completes the proof of our theorem.

We remark that L can be chosen as the least positive integer L satisfying $L \geqq \{e^8, L_1, L_2, \dots, L_8\}$. All the constants $L_1, L_2, \dots, L_8$ are explicitly computable thus also the constants in (100) are explicitly computable.

8. In Part II of this series, we will give a lower estimate for $a(x)$.

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ON ADDITIVE FUNCTIONS WHICH ARE MONOTONE ON A „RARE" SET

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1. Throughout this paper f, g and h denote additive arithmetical functions, A is a subsequence of the natural numbers, formed of the elements $a_1 < a_2 < a_3 < \dots$, and $t(n)$ is a non-decreasing function tending to infinity.

ERDŐS has proved in [2], that if f is monotone, then $f = c \cdot \log n$. BIRCH [1] and KÁTAI [3] have shown that the same holds if f is monotone on a set having upper density 1. Many other papers have also dealt with similar types of characterization of the $\log n$ function (see e.g. [4]—[8]).

Recently I asked the following question:

How "rare" can we choose a suitable set A so, that if

(1) $$f \text{ is monotone on } A,$$

then we still have

(2) $$f = c \cdot \log n?$$

I have proved the following (see Theorem 1/II):

To any α_k with $\liminf \alpha_k = 1$ we can construct an A, for which

(3) $$\frac{a_{k+1}}{a_k} > \alpha_k$$

holds, and (1) implies (2).

I conjectured that with $\liminf \alpha_k > 1$ this is already impossible. ERDŐS has verified this by the following counterexample:

If $\liminf \frac{a_{k+1}}{a_k} = D > 1$, i.e. with a suitable $d > 1$ we have $\frac{a_{k+1}}{a_k} > d$ for all k, then $f = \log n + g$ is still monotone on A, if g is bounded; $|g(n)| < \frac{\log d}{2}$, since

$$f(a_{k+1}) - f(a_k) = \log \frac{a_{k+1}}{a_k} + g(a_{k+1}) - g(a_k) > \log d + g(a_{k+1}) - g(a_k).$$

(Later A. SCHINZEL informed me that independently Z. NOWACKI has also proved the same as ERDŐS — his result is unpublished.)

ERDŐS asked, whether his counterexample is the only possible type of "irregularity", i.e. is it true that if the α_k are "not too large", then we can find an A, for which (3) holds, and (1) implies

(4) $$f = c \cdot \log n + g, \quad \text{where } g \text{ is bounded.}$$

In what follows we answer the question of ERDŐS (Theorem 2/II) and also give some generalizations. Several questions still remain open.

2. First we restate the original problem:

THEOREM 1. I. *Assume that* (1) *implies* (2). *Then* $\liminf \frac{a_{k+1}}{a_k}=1$.

II. *To any* α_k *with* $\liminf \alpha_k=1$ *we can construct an* A, *for which* (3) *holds and* (1) *implies* (2).

PROOF. We gave the proof of I in the introduction, and so now deal only with II.

By assumption, to every n we can find an i_n with $\alpha_{i_n} < 1+\frac{1}{n}$. We may assume

(5) $$i_{n+1} \geqq i_n+2 \quad \text{for all} \quad n.$$

We define now A:

$$a_i = \begin{cases} n\cdot v_n & \text{for} \quad i = i_n \\ (n+1)\cdot v_n & \text{for} \quad i = i_n+1, \end{cases}$$

where $(v_n, n\cdot(n+1))=1$, — thus (3) holds for $k=i_n$ —, and the v_n and all the other a_i are defined so that (3) should be valid for all k.

By (1) we have $f\big((n+1)\cdot v_n\big)\geqq f(n\cdot v_n)$, i.e. $f(n+1)\geqq f(n)$, and so using the classical result of ERDŐS [2] we conclude (2).

Next we examine the characterization of the functions $f=c\cdot \log n+g$ where g is bounded:

THEOREM 2. I. *Assume that* (1) *implies* (4). *Then* $\liminf \frac{a_{k+1}}{a_k}<\infty$.

II. *To any* α_k *with* $\liminf \alpha_k<\infty$ *we can construct an* A, *for which* (3) *holds and* (1) *implies* (4).

PROOF. I. Indirectly, assume

$$\frac{a_{k+1}}{a_k} > e^{t(a_k)}.$$

Put $f=\log n+h$, where $h(2^s)=t(2^s)$, and $h(p^s)=0$ for all the other prime powers p^s. Then $0\leqq h(n)\leqq t(n)$ for all n.

Obviously (4) cannot hold, but (1) is valid:

$$f(a_{k+1})-f(a_k) \geqq \log \frac{a_{k+1}}{a_k}-h(a_k) > t(a_k)-h(a_k) \geqq 0.$$

II. By the assumption we can find an L and natural numbers $i_1<i_2<i_3<\dots$ with $\alpha_{i_j}<L$. Again, we may assume (5).

Let p and q be two primes, $p>2L$ and $q>p^2$. We define A in the following way:

$$a_i = \begin{cases} n\cdot v_n & \text{for} \quad i = i_{3n-2} \\ (n+1)\cdot v_n\cdot p & \text{for} \quad i = i_{3n-2}+1 \\ (n+1)\cdot u_n & \text{for} \quad i = i_{3n-1} \\ n\cdot u_n\cdot p & \text{for} \quad i = i_{3n-1}+1 \\ p^{n+1}\cdot t_n & \text{for} \quad i = i_{3n} \\ p^n\cdot t_n\cdot q & \text{for} \quad i = i_{3n}+1, \end{cases}$$

where $(v_n, p\cdot n\cdot(n+1))=(u_n, p\cdot n\cdot(n+1))=(t_n, pq)=1$.

Now (3) holds for $k=i_j$, $j=1, 2, \ldots$. If we choose v_n, u_n, t_n and the other a_i large enough, then (3) is valid for all k.

Assume now (1).

Case (i): $p \nmid n \cdot (n+1)$. Using

$$f\left((n+1) \cdot v_n \cdot p\right) \geqq f(n \cdot v_n) \quad \text{and} \quad f(n \cdot u_n \cdot p) \geqq f\left((n+1) \cdot u_n\right)$$

we obtain

$$-f(p) \leqq f(n+1) - f(n) \leqq f(p).$$

Case (ii): $p|n$, $n=p^s \cdot r$, $(r, p)=1$. Using $f\left((n+1) \cdot v_n \cdot p\right) \geqq f(n \cdot v_n)$ we obtain — as before —

$$f(n+1) - f(n) \geqq -f(p).$$

The other inequality: $f(n \cdot u_n \cdot p) \geqq f\left((n+1) \cdot u_n\right)$ now implies $f(n \cdot p) \geqq f(n+1)$. Here $f(n \cdot p) = f(p^{s+1}) - f(p^s) + f(n)$, and so

$$f(n+1) - f(n) \leqq f(p^{s+1}) - f(p^s).$$

But $f(p^{s+1} \cdot t_s) \leqq f(p^s \cdot t_s \cdot q)$, i.e. $f(p^{s+1}) - f(p^s) \leqq f(q)$, and hence $f(n+1) - f(n) \leqq f(q)$.

Case (iii): $p|n+1$. Similarly to Case (ii) we obtain

$$-f(q) \leqq f(n+1) - f(n) \leqq f(p).$$

Thus we have proved that $f(n+1) - f(n)$ is bounded. By a famous theorem of WIRSING [7] we conclude (4).

Generalizing the problem we now prove

THEOREM 3. *Assume that* (1) *implies*

$$f = c \cdot \log n + g, \quad \textit{where} \quad g(n) = o\left(t(n)\right). \tag{6}$$

Then

$$\liminf \frac{a_{k+1}}{a_k \cdot e^{t(a_k)}} = 0. \tag{7}$$

REMARK. We obtain Theorem 2/I as a corollary of Theorem 3. The proof is similar to that of Theorem 2/I.

Assume indirectly that

$$\frac{a_{k+1}}{a_k} > D \cdot e^{t(a_k)}$$

if k is large enough $(D > 0)$, i.e.

$$\frac{a_{k+1}}{a_k} > d \cdot e^{t(a_k)} \tag{8}$$

for all k with a suitable $d > 0$. Put $f = \log n + h$, where $h(2^s) = \frac{1}{2} \cdot t(2^s)$ if $t(2^s) > -2 \cdot \log d$, and $h(p^s) = 0$ for all the other prime powers p^s. Obviously $0 \leqq h(n) \leqq \frac{1}{2} \cdot t(n)$.

Now (6) does not hold, but (1) is valid:

$$f(a_{k+1})-f(a_k) \geqq \log\frac{a_{k+1}}{a_k}-h(a_k) > t(a_k)+\log d-h(a_k) > 0.$$

Concerning Theorem 3 we cannot prove anything in the opposite direction.

3. We examine now the situation for completely additive functions. From now on f, g and h are completely additive.

THEOREM 4. I. *Assume that* (1) *implies* (2). *Then* $\liminf \frac{\log a_{k+1}}{\log a_k}=1$, *i.e. putting* $a_{k+1}=(a_k)^{c_k}$, *we have* $\liminf c_k=1$.

II. *To any* α_k *with* $\liminf \alpha_k=1$ *we can construct an A for which*

$$\frac{\log a_{k+1}}{\log a_k} > \alpha_k \tag{9}$$

holds, and (1) *implies* (2).

PROOF. I. Indirectly, assume that $a_{k+1}>(a_k)^{1+D}$ with $D>0$, if k is large enough. Then $a_{k+1}>(a_k)^{1+d}$ with a suitable $d>0$ for all k. Put $f=\log n+h$ where $h(2)=d\cdot\log 2$, and $h(p)=0$ for the other primes p. Obviously $0\leqq h(n)\leqq d\cdot\log n$. Then

$$f(a_{k+1})-f(a_k) \geqq \log\frac{a_{k+1}}{a_k}-h(a_k) > d\cdot\log a_k-h(a_k) \geqq 0,$$

thus (1) holds, but (2) is false, of course.

II. By the assumption, to every $n>1$ we can find an i_n with

$$\alpha_{i_n} < \frac{\log(n+1)}{\log n}.$$

Again assuming (5), we define A in the following way:

$$a_i=\begin{cases} n^{K_n} & \text{for } i=i_n \\ (n+1)^{K_n} & \text{for } i=i_n+1 \end{cases} \quad (n>1)$$

so (9) is valid for $k=i_n$, and if we choose K_n and the other a_i large enough, then (9) holds for all k.

By (1).

$$f(n^{K_n}) \leqq f\left((n+1)^{K_n}\right), \quad \text{i.e.} \quad f(n) \leqq f(n+1),$$

and we infer (2) using the theorem of Erdős.

THEOREM 5. I. *Assume that* (1) *implies*

$$f(n)=O(\log n). \tag{10}$$

Then $\liminf \frac{\log a_{k+1}}{\log a_k}<\infty$, *i.e. putting* $a_{k+1}=(a_k)^{c_k}$, *we have* $\liminf c_k<\infty$.

II. *To any* α_k *with* $\liminf \alpha_k<\infty$ *we can construct an A for which* (9) *holds, and* (1) *implies* (10).

PROOF. I. Indirectly, assume $\frac{a_{k+1}}{a_k} > a_k^{t(a_k)}$. Put $f = \log n + h$, where $h(p) = t(p) \cdot \log p$ for all primes p. Then (10) is clearly false.

On the other hand we have $0 \leqq h(n) \leqq t(n) \cdot \log n$, and so we obtain (1) in the usual way.

II. By assumption, we can find an L and natural numbers $i_1 < i_2 < i_3 < \dots$ with $\alpha_{i_j} < L$. We may assume (5).

Let p be a prime, $p > 2 \cdot e^L$. We take

$$a_i = \begin{cases} n^{K_n} & \text{for} \quad i = i_{2n-1} \\ \{(n+1) \cdot p^{[2 \cdot \log n]}\}^{K_n} & \text{for} \quad i = i_{2n-1} + 1 \\ (n+1)^{T_n} & \text{for} \quad i = i_{2n} \\ \{n \cdot p^{[2 \cdot \log n]}\}^{T_n} & \text{for} \quad i = i_{2n} + 1 \end{cases} \qquad (n > 1).$$

Now (9) holds for $k = i_j$, $j = 3, 4, \dots$ and if we take K_n, T_n and the other a_i large enough, then (9) holds for all k.

Assuming (1) we have

$$f(n^{K_n}) \leqq f\left(\{(n+1) \cdot p^{[2 \cdot \log n]}\}^{K_n}\right) \quad \text{and} \quad f\left((n+1)^{T_n}\right) \leqq f\left(\{n \cdot p^{[2 \cdot \log n]}\}^{T_n}\right)$$

and so

$$|f(n+1) - f(n)| \leqq [2 \cdot \log n] \cdot f(p) < C \cdot \log n.$$

Using a very recent result of WIRSING [8], we obtain (10).

For the general case we have

THEOREM 6. *Assume that* (1) *implies*

$$f(n) = o\left(t(n)\right), \tag{11}$$

where $u(n) = \frac{t(n)}{\log n}$ *is non-decreasing and tends to infinity.*

Then (7) *holds.*

REMARK. Theorem 5/I is a corollary of Theorem 6.

PROOF. We argue indirectly. Then we may assume (8).

Put $f = \log n + h$, where $h(p) = \frac{1}{2} \cdot u(p) \cdot \log p = \frac{1}{2} \cdot t(p)$, if $t(p) > -2 \cdot \log d$, and $h(p) = 0$ for the other primes p. Clearly, (11) is false, while using

$$0 \leqq h(n) \leqq \frac{1}{2} \cdot u(n) \cdot \log n = \frac{1}{2} \cdot t(n)$$

we obtain (1) in the usual way.

In the opposite direction we can prove only a considerably weaker result.

THEOREM 7. *We can construct an A for which*

$$a_{k+1} > a_k^{\log \log \log a_k} \tag{12}$$

holds, and (1) *implies*

$$f(n) = O(\log^3 n). \tag{13}$$

REMARK. The "converse" of Theorem 6 for this case would be

$$f(n) = o(\log n \cdot \log\log\log n).$$

We sketch the proof. Put

$$a_i = \begin{cases} n^{K_n} & \text{for} \quad i = 2n-1 \\ ((n+1)\cdot 2^{s_n})^{K_n} & \text{for} \quad i = 2n \end{cases} \qquad (n > 1).$$

To guarantee (12) we take the minimal possible K_n and s_n, successively. Hence

(14) $$a_{k+1} \approx a_k^{\log\log\log a_k},$$

and a simple calculation shows that (14) implies

(15) $$a_i = O(2^{i^i}).$$

Returning to (12), we require

$$(n^{K_n})^{\log\log\log(n^{K_n})} < ((n+1)\cdot 2^{s_n})^{K_n},$$

or

$$2^{s_n} \approx n^{\log\log\log(n^{K_n})},$$

or

$$s_n \approx \log n \cdot \log\log\log(n^{K_n}).$$

Using (15) we obtain $s_n = O(\log^2 n)$ or

(16) $$f(2^{s_n}) = O(\log^2 n).$$

Assume now (1). Then

$$f((n+1)\cdot 2^{s_n}) \geqq f(n),$$

i.e.

(17) $$f(n+1) - f(n) \geqq -f(2^{s_n}).$$

From (16) and (17) we can easily deduce (13).

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ANALYTIC FUNCTIONS OF PRESPECTRAL OPERATORS

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1. Introduction

Analytic functions of unbounded spectral operators were studied first by BADE in [1], then investigated further in [4; XVIII. 2]. PALMER [11] defined the integral of a Borel measurable complex function with respect to a G-countably additive spectral measure, and the notion of (unbounded) scalar type operators of class G. The definition and the proof of the uniqueness of the resolution of the identity (in class G) of closed prespectral operators were given by the author [8], the latter one extending an important result due to DOWSON [3] in the bounded case.

The purpose of this paper is to generalize results in [4; XVIII. 2] to the case of unbounded prespectral operators. This will be achieved by using the E^*G topology, introduced by PALMER [11], for the definition of the operator $f(T)$. Given a prespectral operator T and a suitable holomorphic function f, we first define $f(T)$ in terms of a fixed resolution of the identity E of class G for T, then prove that $f(T)$ is independent of E (which is unique, as a rule, only in class G, cf. FIXMAN [5; pp. 1035—6]). We show that a scalar type operator of class G in Palmer's sense is prespectral of class G, and if it is also prespectral of class G_1, then it is also scalar of class G_1. If T is spectral, then for the definition of $f(T)$ we can use the X^* as well as the norm topology of X, and $f(T)^*=f(T^*)$. After proving that a closed prespectral operator is also E^*G-closed, we give a characterization of prespectral operators, extending a result due to NEL [10].

The proofs of several results are straightforward generalizations of respective proofs in [4; XVIII. 2], by virtue of earlier results in the theory of prespectral operators. These proofs will be omitted and only the respective results will be indicated.

2. Terminology and notations

Suppose X is a complex Banach space with norm $|.|$ and adjoint space X^*. If T is a closed linear operator with domain $D(T)\subset X$ into X, then $r(T)$ denotes its resolvent set, $s(T)$ its spectrum, and $s_e(T)$ its extended spectrum, i.e. $s_e(T)=s(T)$ if T belongs to $B(X)$, the Banach algebra of bounded linear operators from X into X, and $s_e(T)=s(T)\cup\{\infty\}$ otherwise. The graph of T is denoted by $G(T)$, and the space of bounded linear operators from X into another Banach space Y by $B(X, Y)$. If Z is a subspace of X, then Z^0 denotes the annihilator of Z in X^*, i.e. $Z^0=\{x^*\in X^*;\ x^*(z)=0$ for each $z\in Z\}$, and $T|Z$ denotes the restriction of T to $D(T)\cap Z$. The complex plane is denoted by C, the σ-field of its Borel subsets by B, and if $b\in B$, then $\bar{b}$, b^0 and b^c denote its closure, interior and $C\setminus b$, respectively. The domain of a complex function f is denoted by dom (f), and "f is holomorphic" or, equivalently, "f is analytic" will imply that f is single-valued, but not that dom (f) is connected.

A linear manifold $G \subset X^*$ is said to be total if, for any $x \in X$, $x^* x = 0$ for all x^* in G implies $x=0$. If G is total, a spectral measure of class G in $B(X)$ with domain (S, Z) is a homomorphic mapping E from a Boolean algebra Z of subsets of a set S into a uniformly bounded Boolean algebra of projections in $B(X)$, for which $E(S)=I$ and $x^* E(\cdot) x$ is countably additive for each (x^*, x) in $G \times X$. We will omit (S, Z) if it is identical with (C, B), and $B(X)$ if no misunderstanding can arise. The subspace $E(e)X$ will sometimes be denoted by X_e.

A prespectral operator T of class G is a closed linear operator from $D(T) \subset X$ into X for which there exists a spectral measure E of class G such that

1) if $b \in B$ is bounded, then $D(T) \supset E(b)X$,
2) for each $e \in B$, we have $E(e)T \subset TE(e)$,
3) for each $e \in B$, $s(T|E(e)X) \subset \bar{e}$.

The spectral measure E is called a resolution of the identity (of class G) for T. The operator T is called prespectral if it is prespectral of some class G. Note that the adjoint of $E(e)$ will be denoted by $E^*(e)$, and the restriction of E to a subspace $Y = E(e)X$ by $E|Y$.

Now we fix a spectral measure E of class G in $B(X)$ with domain (S, Z) where Z is a σ-field, and recall some definitions and results due to PALMER [11; pp. 407—408]. The set of all operators in $B(X)$ which commute with each $E(e)$, $e \in Z$, will be denoted by E^c. The E^*G topology of X is the weak topology determined on X by the total set of all finite linear combinations of functionals of the form $T^* y^*$, where $T \in E^c$ and $y^* \in G$. Note that the E^*G topology is not stronger than the X^* topology of X, but not weaker than the G topology of X. The net $\{x_a\}$ converges to x in the E^*G topology, in notation

$$E^*G - \lim_a x_a = x,$$

if and only if $y^* T x_a \to y^* T x$ for every $T \in E^c$ and $y^* \in G$. Hence if $T \in E^c$, then T is a continuous operator in the E^*G topology. A net $\{T_a\}$ converges to T in the E^*G operator topology of $B(X)$ if and only if $T_a x \to Tx$ in the E^*G topology for each $x \in X$. The spectral measure E is countably additive not only in the G but also in the E^*G operator topology.

3. The operational calculus

LEMMA 1. *Suppose E is a spectral measure of class G which is defined on a σ-field Z of subsets of a set S, and that the family $Z_0 \subset Z$ contains the union of every finite collection of its elements as well as each $e \in Z$ for which $e \subset b$ for some $b \in Z_0$. Let Q_0 be a linear operator with domain $D(Q_0) = \bigcup_{b \in Z_0} E(b)X$ and such that $b \in Z_0$ implies $E(b) Q_0 \subset Q_0 E(b)$ and $Q_0 E(b) \in E^c$. Suppose there is an increasing sequence $\{b_n\}$ in Z_0 for which $E\left(\bigcup_{n=1}^{\infty} b_n\right) = I$, and define*

$$D(Q) = \{x \in X;\ E^*G - \lim_{n\to\infty} Q_0 E(b_n) x = y(x) \quad \textit{exists}\},$$

$$Qx = y(x) \quad \textit{for} \quad x \in D(Q).$$

Then $D(Q_0)$ is E^*G-dense in X, and the linear operator Q is the minimal E^*G-closed extension of Q_0. Hence Q is closed in the norm topology of X and is independent of the choice of the sequence $\{b_n\}$.

PROOF. Since $x=E^*G-\lim_{n\to\infty} E(b_n)x$ for every $x\in X$, $D(Q_0)$ is E^*G-dense in X. If $y\in D(Q_0)$, then $Q_0E(b_n)y=E(b_n)Q_0y$ by assumption, hence $Q\supset Q_0$. The operator Q is clearly linear, and for each $b\in Z_0$ and $x\in X$ we have $QE(b)x=$ $=Q_0E(b)x$.

If $z\in D(Q)$ and $b\in Z$, then

$$E^*G-\lim_{n\to\infty} Q_0E(b_n)E(b)z = E^*G-\lim_{n\to\infty} E(b)Q_0E(b_n)z = E(b)Qz$$

exists, since $Q_0E(b_n)\in E^c$ and $E(b)$ is E^*G-continuous. Hence $E(b)Q\subset QE(b)$.

Suppose now that the net $\{x_a\}$ is part of $D(Q)$ and that

$$E^*G-\lim_a x_a = x, \quad E^*G-\lim_a Qx_a = y.$$

By our preceding remarks we then obtain for each n

$$Q_0E(b_n)x = E^*G-\lim_a Q_0E(b_n)x_a = E^*G-\lim_a E(b_n)Qx_a = E(b_n)y.$$

From this we conclude that

$$E^*G-\lim_{n\to\infty} Q_0E(b_n)x = E^*G-\lim_{n\to\infty} E(b_n)y = y$$

exists, hence $Qx=y$ and Q is E^*G-closed, thus closed in the norm topology of X.

Finally, $x\in D(Q)$ implies

$$E^*G-\lim_{n\to\infty} E(b_n)x = x \quad \text{and} \quad E^*G-\lim_{n\to\infty} Q_0E(b_n)x = Qx,$$

hence (x, Qx) belongs to the closure of $G(Q_0)$ with respect to the product of the E^*G topologies. Thus the proof is complete.

The set of all operators in $B(X)$ which commute with each operator in E^c will be denoted by E^{cc}. Since the set $H=\{E(e); e\in Z\}$ is abelian, we have $H\subset$ $\subset E^{cc}\subset E^c$, hence every operator in E^{cc} is E^*G-continuous. Thus we have the following

COROLLARY. *If $T\in E^{cc}$, then under the conditions of Lemma* 1 *we have $QT\supset TQ$.*

PROOF. If $x\in D(Q)$, then

$$E^*G-\lim_{n\to\infty} Q_0E(b_n)Tx = E^*G-\lim_{n\to\infty} TQ_0E(b_n)x = TQx$$

exists by the preceding remarks. Hence $QTx=TQx$.

Now we will define certain holomorphic functions of (possibly unbounded) prespectral operators of class G, and thus extend Definition 8 of [4; XVIII. 2]. If T is a prespectral operator of class G, then it has a unique resolution of the identity E of class G, by [8; Theorem 1]. We will say that the complex function f belongs to the class $A(T)$ if f is holomorphic on an open subset U of the complex plane C for which $E\big(s(T)\setminus U\big)=0$. If b is a bounded Borel set such that $\bar{b}\subset U$, then $T|E(b)X\in B(X_b)$ and $s\big(T|E(b)X\big)\subset U$, by [8; Lemma 1]. The Dunford—Taylor operational calculus then defines $f(T|E(b)X)$. As in [4; pp. 2232—3] it can be shown that the linear operator

$$Q_0 x = f(T|E(b)X)x \quad \text{for} \quad x \in E(b)X$$

is well-defined on $\bigcup_{b \in Z_0} E(b)X$, where

$$Z_0 = \{b \text{ bounded Borel set; } \bar{b} \subset U\}.$$

[8; Lemma 2] implies that $E(s(T)) = I$, hence the condition $E(s(T) \setminus U) = 0$ is equivalent to $E(U) = I$. Consequently, there is an increasing sequence $\{b_n\} \subset Z_0$ such that $E\left(\bigcup_{n=1}^{\infty} b_n\right) = \text{I}$, and we can give

DEFINITION 1. Under the preceding conditions the operator $f(T)$ is defined by

$$D(f(T)) = \{x \in X;\ E^* G - \lim_{n \to \infty} f(T|E(b_n)X) E(b_n)x = y(x) \quad \text{exists}\},$$

$$f(T)x = y(x) \quad \text{for} \quad x \in D(f(T)).$$

LEMMA 2. *The operator $f(T)$ is linear and is the minimal $E^* G$-closed extension of Q_0, hence closed in the norm topology of X, and $D(f(T))$ is $E^* G$-dense in X.*

PROOF. It is an application of Lemma 1, where $Z = B$, and Z_0 is defined above. Indeed, if $x \in D(Q_0)$, then $x = E(c)x$ for some $c \in Z_0$, hence $Q_0 x = f(T|X_c)x$. Here $T|X_c$ belongs to $B(X_c)$ and is prespectral with a resolution of the identity $E|X_c = F$, by [8; Lemma 1]. Hence F commutes with $f(T|X_c)$ and for each $b \in Z_0$,

$$E(b) Q_0 x = F(b) f(T|X_c)x = f(T|X_c) F(b)x = Q_0 E(b)x.$$

Further, $Q_0 E(b)x = f(T|X_b) E(b)x$ for each $x \in X$, thus $Q_0 E(b) \in B(X)$. If $e \in Z$, then

$$Q_0 E(b) E(e)x = f(T|X_b) E(e) E(b)x = E(e) f(T|X_b) E(b)x = E(e) Q_0 E(b)x,$$

hence $Q_0 E(b) \in E^c$, and Lemma 1 yields the assertions.

REMARK. Observe that Definition 1, Lemma 2 and the preceding discussion make essential use of the fact that the operator T is prespectral with a resolution of the identity E (of class G). On the other hand, it is known that T can also have a resolution of the identity $E_1 \neq E$ (of class $G_1 \neq G$) (see [5; pp. 1035—6]). Now we show that the operator $f(T)$ is independent of the resolution of the identity for T, i.e. with obvious notations we have

LEMMA 3. *If T is prespectral of classes G and G_1 with resolutions of the identity E and E_1, respectively, then $A(T; E) = A(T; E_1)$. Further, if the complex function f is in this class, then $f(T; E) = f(T; E_1)$.*

PROOF. If h is a closed set in C, then [8; Lemma 2] implies that $E(h)X = E_1(h)X$. If f is holomorphic on an open set U such that $E(U^c) = 0$, then $E_1(U^c) = 0$ and conversely, hence $A(T; E) = A(T; E_1)$, and in what follows this class will be denoted by $A(T)$. Further, by Lemma 2, for the definitions of $f(T; E)$

and $f(T; E_1)$ we may choose a common sequence b_n of compact sets with the following properties: $b_n \subset b^0_{n+1}$ $(n=1, 2, \ldots)$ and $\bigcup_{n=1}^{\infty} b_n = U$. Then the subspaces $X_n = E(b_n)X$ and $E_1(b_n)X$ coincide $(n=1, 2, \ldots)$, and we have

$$f(T;\, E)x = E^*G - \lim_{n\to\infty} f(T|X_n)E(b_n)x, \quad f(T;\, E_1)x = E_1^*G_1 - \lim_{n\to\infty} f(T|X_n)E_1(b_n)x,$$

if the respective limits exist. By [8; Lemma 3], for every positive integer k

$$E_1(b_k) = E(b^0_{k+1})E_1(b_k) = E(b^0_{k+1})E_1(b_k)E(b_{k+1}) = E_1(b_k)E(b_{k+1}).$$

Hence, for $x \in D(f(T; E))$

$$E_1(b_k)f(T;\, E)x = E_1(b_k)E(b_{k+1})E^*G - \lim_{n\to\infty} f(T|X_n)E(b_n)x =$$
$$= E_1(b_k)E^*G - \lim_{n\to\infty} (E(b_{k+1})|X_n)f(T|X_n)E(b_n)x.$$

Since $E|X_n$ is a resolution of the identity for $T|X_n$, $E|X_n$ commutes with $f(T|X_n)$, thus we have

$$E_1(b_k)f(T;\, E)x = E_1(b_k)f(T|X_{k+1})E(b_{k+1})x.$$

Similarly, $E_1|X_{k+1}$ commutes with $f(T|X_{k+1})$, hence

$$E_1(b_k)f(T;\, E)x = f(T|X_{k+1})E_1(b_k)E(b_{k+1})x = f(T|X_k)E_1(b_k)x.$$

But then

$$f(T;\, E_1)x = E_1^*G_1 - \lim_{k\to\infty} f(T|X_k)E_1(b_k)x = E_1^*G_1 - \lim_{k\to\infty} E_1(b_k)f(T;\, E)x = f(T;\, E)x$$

exists, for E_1 is countably additive in the $E_1^*G_1$ operator topology. Thus $f(T; E) \subset f(T; E_1)$, and by symmetry we obtain the converse relation.

REMARK. By virtue of this result, in what follows we shall use a fixed resolution of the identity for T and retain the notation $f(T)$ of Definition 1.

The next result is a partial extension of [4; XVIII. 2.9].

THEOREM 1. *Suppose T is prespectral with resolution of the identity E of class G and $f \in A(T)$.*

(i) *If $e \in B$, then $E(e)f(T) \subset f(T)E(e)$.*

(ii) *If $e \in B$, then $f(T|X_e) = f(T)|X_e$. If, in addition, $e \in Z_0$, then $f(T)|X_e \in B(X_e)$.*

(iii) *If N denotes the set of the zeros of f and $E(N)=0$, then $h = \frac{1}{f} \in A(T)$ and $f(T)^{-1} = h(T)$.*

(iv) *If $f(z) = z$, then $f(T)$ is the minimal E^*G-closed extension of T.*

(v) *If $T \in B(X)$ and f is holomorphic in a neighbourhood of $s(T)$, then Definition* 1 *coincides with* [4; VII. 3.9]. *If T is arbitrary but f is holomorphic in a neighbourhood of $s_e(T)$, then $f(T) \in B(X)$.*

Further, if also $g \in A(T)$, then

(vi) *$D(f(T)+g(T)) = D((f+g)(T)) \cap D) f(T))$ and $f(T)+g(T) \subset (f+g)(T)$.*

(vii) *$D(f(T)g(T)) = D((fg)(T)) \cap D(g(T))$ and $f(T)g(T) \subset (fg)(T)$.*

(viii) *If $c \neq 0$ is a complex number, then $(cf)(T) = cf(T)$.*

PROOF. Most of the proofs of [4; XVIII. 2.9] extend with the only change that limits in the norm topology have to be replaced by limits in the E^*G topology of X. We give here the proofs of (ii), (iv) and (v), which differ most from their counterparts.

(ii) The case $e \in Z_0$ can be proved as in [4; p. 2234]. Suppose now that $e \in B$, then $T|X_e$ is prespectral with resolution of the identity $F = E|X_e$ of class $G_e = \{g + X_e^0;\ g \in G\}$, by [8; Lemma 1]. First we show that a net $\{x_a\}$ in X_e converges to x in the F^*G_e topology if and only if it converges to x in the E^*G topology.

Indeed, suppose that $F^*G_e - \lim_a x_a = x$, $g \in G$ and $K \in E^c$, then $g_e = g + X_e^0 \in G_e$ and $K_e = K|X_e \in F^c$, thus $\lim_a gKx_a = \lim_a g_e K_e x_a = gKx$. Conversely, if $E^*G - \lim_a x_a = x$, then also $x \in X_e$. For each $K \in F^c$ the operator $\bar{K} = KE(e)$ belongs to E^c and if $g_e = g + X_e^0 \in G_e$, then $g_e K x_a = g\bar{K}x_a \to g\bar{K}x = g_e Kx$, which proves the assertion.

Now if dom $(f) = U$, then $E(U) = I$ implies $F(U) = I$, hence $f \in A(T|X_e)$. If $\{c_n\}$ is an increasing sequence of compact sets such that $\bigcup_{n=1}^{\infty} c_n = U$ and $x \in D(f(T|X_e))$, then $x \in X_e$ and

$$f(T|X_e)x = F^*G_e - \lim_{n\to\infty} f(T|F(c_n)X_e)F(c_n)x =$$

$$= E^*G - \lim_{n\to\infty} f(T|E(c_n e)X)E(c_n)x = E^*G - \lim_{n\to\infty} f(T|E(c_n)X)E(c_n)x = f(T)x,$$

hence $f(T|X_e) \subset f(T)|X_e$.

Conversely, if $x \in D(f(T)) \cap X_e$, then

$$f(T)x = E^*G - \lim_{n\to\infty} f(T|E(c_n e)X)E(c_n)x = F^*G_e - \lim_{n\to\infty} f(T|F(c_n)X_e)F(c_n)x,$$

hence $f(T)|X_e \subset f(T|X_e)$.

(iv) If $b_n = \{z \in C;\ |z| \leqq n\}$, then $f(T)x = E^*G - \lim_{n\to\infty} TE(b_n)x$ if and only if this limit exists. Since $y = E^*G - \lim_{n\to\infty} E(b_n)y$ for every y in X and $f(T)$ is E^*G-closed, the statement follows.

(v) The first statement can be proved as in [4; p. 2236] for spectral operators. For similar reasons, when proving the second one, we may assume that $s(T) \subset K_r$ and $K_{r-2} \subset \mathrm{dom}(f) = U$, where $K_r = \{z \in C;\ |z| \geqq r\}$, and we will show that $D(f(T)) = X$.

Indeed, if $\{c_n\}$ is an increasing sequence of compact sets such that $\bigcup_{n=1}^{\infty} c_n = K_r$, and we introduce the notation

$$V = f(\infty)I - \frac{1}{2\pi i} \oint_{|z|=r-1} f(z)(z-T)^{-1}\,dz,$$

then $f(T|E(c_n)X) = V|E(c_n)X$ (cf. [4; p. 2237]) and $V \in E^c$. Hence

$$E^*G - \lim_{n\to\infty} f(T|E(c_n)X)E(c_n)x = E^*G - \lim_{n\to\infty} E(c_n)Vx = Vx$$

exists for each $x \in X$, thus $f(T)$ belongs to $B(X)$.

COROLLARY. *Suppose T is prespectral with resolution of the identity E of class G and $f, g\in A(T)$. Then*

(1) $[f(T)+g(T)]_c = (f+g)(T)$,

(2) $[f(T)g(T)]_c = (fg)(T)$,

where the subscript c denotes closure in the E^*G topology.

PROOF. (1) By Lemma 2 and (vi) of Theorem 1, $[f(T)+g(T)]_c \subset (f+g)(T)$. If $\{c_n\}$ is an increasing sequence of compact sets with union $\operatorname{dom}(f)\cap\operatorname{dom}(g)$, then

$$E^*G-\lim_{n\to\infty} E(c_n)x=x \quad \text{and} \quad E(c_n)x\in D\big(f(T)+g(T)\big) \quad (n=1, 2, \dots)$$

for each x in X. For $x\in D\big((f+g)(T)\big)$ (i) of Theorem 1 implies

$$\big(f(T)+g(T)\big)E(c_n)x = (f+g)(T)E(c_n)x = E(c_n)(f+g)(T)x,$$

which converges to $(f+g)(T)x$ in the E^*G topology if $n\to\infty$, hence $x\in D\big([f(T)+ +g(T)]_c\big)$. The proof of (2) is similar.

The definition of a possibly unbounded scalar operator of class G has been given by PALMER ([11; Definition 5.6]). Now we give

DEFINITION 2. If T is prespectral with resolution of the identity E of class G, we define

$$S(E)x = E^*G-\lim_{n\to\infty} \int_{|z|\leqq n} zE(dz)x$$

if and only if the above limit exists. The scalar operator $S(E)$ of class G is called the scalar part of T, corresponding to E.

THEOREM 2. (i) *If T is prespectral with resolutions of the identity E and E_1 of classes G and G_1, respectively, then $S(E)=S(E_1)$.*

(ii) *If T is a scalar operator with a resolution of the identity E of class G, then T is prespectral with a resolution of the identity E, hence E is unique in class G. If T is also prespectral of class G_1, then T is scalar of class G_1, too.*

REMARK. In case (i) $S=S(E)$ will be called the scalar part of T.

PROOF OF THEOREM 2. (i) Put $b_n=\{z\in C;\ |z|\leqq n\}$, then the subspace $X_n= =E(b_n)X$ equals $E_1(b_n)X$, and $T|X_n$ is prespectral with resolutions of the identity $E|X_n$ and $E_1|X_n$. By [3; Theorem 2], $T|X_n$ has a unique Jordan decomposition, hence for each $x_n\in X_n$

$$\int_{b_n} zE(dz)x_n = \int_{b_n} zE_1(dz)x_n.$$

By the proof of Lemma 3, for every positive integer k, $E_1(b_k)=E_1(b_k)E(b_{k+1})$. If $x\in D\big(S(E)\big)$, then

$$E_1(b_k)S(E)x = E_1(b_k)E(b_{k+1})E^*G-\lim_{n\to\infty}\int_{b_n} zE(dz)x =$$

$$= E_1(b_k)\int_{b_{k+1}} zE(dz)x = E_1(b_k)\int_{b_{k+1}} zE_1(dz)x = \int_{b_k} zE_1(dz)x.$$

Hence

$$S(E_1)x = E_1^* G_1 - \lim_{k\to\infty} \int_{b_k} z E_1(dz)x = S(E)x$$

exists, thus $S(E) \subset S(E_1)$ and, by symmetry, we have $S(E) = S(E_1)$.

(ii) T is prespectral with a resolution of the identity E in virtue of [11; Lemmas 5.4, (1)—(3) and 5.7, (1)]. E is unique in class G by [8; Theorem 1]. The last statement follows from (i).

LEMMA 4. *If T is scalar with resolution of the identity E of class G and $f \in A(T)$, then the operators $f(T)$, given by Definition* 1, *and $T(f)$, defined in* [11; pp. 408—409], *coincide.*

PROOF. See [4; XVIII. 2.16], replacing strong limits by limits in the E^*G topology.

LEMMA 5. *If T is prespectral with resolution of the identity E of class G and e is an open subset of C, then $s(T) \cap e \subset s(T|E(e)X) \subset s(T) \cap \bar{e}$. Further, if P is a projection operator such that $PT \subset TP$, then $s(T) \supset s(T|PX)$.*

PROOF. See [4; XVIII. 2.19—20].

THEOREM 3. *Let T be prespectral with resolution of the identity E of class G. Suppose f is holomorphic on the open set U, where the union of U and some finite subset $\{p_0, p_1, \dots, p_k\}$ of the extended complex plane contains a neighbourhood of $s_e(T)$. Assume that $E(\{p_i\}) = 0$ for each finite p_i and that f is meromorphic at each p_r $(r = 0, 1, \dots, k)$. Then $f(T)$ is prespectral with resolution of the identity $E_f(e) = E(f^{-1}(e))$ of class G, and $s(f(T)) = \overline{f(s(T))}$.*

PROOF. See [4; XVIII. 2.21] as well as Lemmas 1 and 2 in [8]. Notice that the connectedness of U, assumed in [4], can be dispensed with. It is used only in proving that $0 \in f(s(T))$ implies $0 \in s(f(T))$. However, defining the set Z to be any $\{z\}$ for which $f(z) = 0$ and $z \in s(T)$, the rest of the proof remains valid. Observe that if T is also prespectral with resolution of the identity $E_1 \neq E$ of class $G_1 \neq G$, then $E_1(\{p_i\}) = 0$ for each finite p_i, hence the assertions remain true for the pair (E_1, G_1), too.

THEOREM 4. (i) *If T is prespectral with resolution of the identity E of class G, then T is E^*G-closed. Hence, if $f(z) = z$, then $f(T) = T$. If, in addition, $r(T)$ is nonvoid and $P(z) = a_0 + a_1 z + \dots + a_n z^n$, then the definitions of $P(T)$ by* [4; VII. 9.6] *and Definition* 1 *coincide, and $P(T)$ is prespectral of class G.*

(ii) *A closed operator T with nonvoid $r(T)$ is prespectral of class G if and only if for one (hence for all) $z \in r(T)$, $(z - T)^{-1}$ is prespectral of class G with resolution of the identity E_z such that $E_z(\{0\}) = 0$.*

PROOF. (i) First we show that T can not have a proper extension V which is also prespectral with resolution of the identity E. Assume the contrary, and choose a Borel set h such that $\bar{h} \neq C$ and pick $z \in (\bar{h})^c$. Then the operators $z - T|E(h)X \subset z - V|E(h)X$ are both 1—1 and map onto $E(h)X$, hence they are equal. Thus $T|E(h)X = V|E(h)X$ and, by choosing a Borel set e such that $\bar{e} \neq C$ and $\overline{(e^c)} \neq C$, we obtain $T = V$, a contradiction. $f(T)$ is prespectral with resolution of the identity E by Theorem 3, hence Theorem 1 (iv) yields the first two statements.

Now we prove that if $r(T)$ is nonvoid, then the polynomial $P(T)=a_0+a_1T+\dots$ $\dots+a_nT^n$, defined by [4; VII. 9.6] is E^*G-closed. This we do along the lines of the proof of [4; VII. 9.7], using the same notations. Suppose that the generalized sequence $\{x_a\}$ belongs to $D(T^n)$, $E^*G-\lim_a x_a=x$ and $E^*G-\lim_a P(T)x_a=y$. If $\alpha\in r(T)$, then $A=(T-\alpha)^{-1}\in E^c$ follows from Theorem 1 (i), hence A^n is E^*G-continuous and $(T-\alpha)^n=(A^n)^{-1}$ is E^*G-closed. Since $p(A)$ is E^*G-continuous and $P(T)w=(T-\alpha)^n p(A)w$ for $w\in D(T^n)$, we obtain $p(A)x\in D(T^n)$. Hence (see [4; p. 603]), $x\in D(T^n)$ and $P(T)x=y$, as claimed.

Using now Corollary to Theorem 1, it can be shown by induction that the two definitions of $P(T)$ coincide. $P(T)$ is prespectral of class G, by Theorem 3.

(ii) See [4; XVIII. 2.23].

THEOREM 5-A. *Suppose that T and f satisfy the conditions of Theorem 3, $\operatorname{dom}(f)=U_f$ is connected, and the function g is holomorphic on the open set U_g for which $E_f(U_g)=I$. Then $h(z)=g(f(z))$ belongs to $A(T)$, and $h(T)=g(f(T))$.*

PROOF. It is a suitable modification of that of [4; XVIII. 2.24]. Since $\operatorname{dom}(h)=$ $=f^{-1}(U_g)$, we have $h\in A(T)$ and $h(T)$ is E^*G-closed. Put $q_1=\{f(\infty)\}$ if f is analytic at infinity and $q_1=\emptyset$ otherwise, and define q_2 to be $U_g\setminus q_1$. We will prove $g(f(T))|E_f(q_i)X=h(T)|E_f(q_i)X$ or, equivalently by Theorems 1 (ii) and 3,

$$(*)\qquad g\left(f(T|E(f^{-1}(q_i))X)\right)=h\left(T|E(f^{-1}(q_i))X\right)\quad (i=1,2).$$

If we exclude the trivial case $f(z)\equiv f(\infty)=c$ on U_f, then the connectedness of U_f implies that the set $f^{-1}(q_1)\cap s(T)$ is finite, hence $T|E(f^{-1}(q_1))X$ belongs to $B(E(f^{-1}(q_1))X)$. Excluding again the trivial case $E_f(q_1)=0$, we have $s(f(T|E_f(q_1)X))=q_1$ and g is holomorphic at c, thus [4; VII. 3. 12] yields $(*)$ for $i=1$.

With the notation $Y=E(f^{-1}(q_2))X$, the operator $V=T|Y$ is prespectral with resolution of the identity $F=E|Y$ of class $H=\{g+Y^0;\, g\in G\}$, and a resolution of the identity of $f(V)$ is $F_f(e)=F(f^{-1}(e))$. Since $F_f(q_2)=I$, we may assume $U_g=q_2$, i.e. that if f is analytic at infinity, it does not assume the value $f(\infty)$ on $f^{-1}(U_g)$.

Suppose $\{c_n^*\}$ is an increasing sequence of compact sets with union U_g, and $e_n=f^{-1}(c_n^*)$. Despite the claim on [4; p. 2250], e_n need not be closed in the topology of C, but only in the relative topology of $\operatorname{dom}(f)$. However, $f(\infty)\notin c_n^*$ does imply that $c_n=e_n\cap s(V)$ $(n=1,2,\dots)$ is compact in the topology of C, further $F(c_n)=$ $=F_f(c_n^*)$, and c_n is an increasing sequence of subsets of $\operatorname{dom}(h)$ for which $F\left(\bigcup_{n=1}^{\infty} c_n\right)=$ $=F_f(U_g)=I$. Hence for $y\in D(g(f(V)))$ we have

$$g(f(V))y=F_f^*H-\lim_{n\to\infty} g\left(f(V)|F_f(c_n^*)Y\right)F_f(c_n^*)y=$$

$$=F_f^*H-\lim_{n\to\infty} g\left(f(V|F(c_n)Y\right)F(c_n)y=F_f^*H-\lim_{n\to\infty} h\left(V|F(c_n)Y\right)F(c_n)y,$$

by Theorem 1 (ii) and [4; VII. 3.12]. Since $F_f(e)=F(f^{-1}(e))$ for each Borel set e, we have $F^c\subset F_f^c$, hence the F_f^*H topology is not weaker than the F^*H topology. Thus

$$g\left(f(V)\right)y = F^* H - \lim_{n\to\infty} h\left(V|F(c_n)Y\right)F(c_n)y = h(V)y,$$

hence $g\left(f(V)\right) \subset h(V)$.

For each $y \in Y$

$$F_f^* H - \lim_{n\to\infty} F(c_n)\, y = F_f^* H - \lim_{n\to\infty} F_f(c_n^*)\, y = y.$$

If $x \in D\left(h(V)\right)$, then $F_f(c_n^*)x$ belongs to $D\left(g\left(f(V)\right)\right)$ by Theorem 1 (ii), further

$$g\left(f(V)\right)F_f(c_n^*)x = h(V)F(c_n)x = F(c_n)h(V)x$$

by Theorem 1 (i). Hence

$$F_f^* H - \lim_{n\to\infty} g\left(f(V)\right)F_f(c_n^*)\, x = h(V)x$$

and, since $g\left(f(V)\right)$ is $F_f^* H$-closed, $x \in D\left(g\left(f(V)\right)\right)$. Thus $g\left(f(V)\right) = h(V)$, and the proof is complete.

LEMMA 6. *If T is prespectral with resolution of the identity E of class G, then* $s(T) = \bigcap\{\bar{e};\ E(e) = I\}$.

PROOF. See [4; XVIII. 2.25] and [8; Lemma 2].

LEMMA 7. *If T is prespectral with nonvoid resolvent set and f is holomorphic on $s_e(T)$, then the operators $f(T)$, given by Definition* 1 *and the Dunford—Taylor calculus (see, e.g.* [4; VII. 9.3]), *coincide.*

PROOF. See [4; XVIII. 2.26]. Note that the proof needs Theorem 5-A.

THEOREM 5-B. *The assertions of Theorem* 5-A *hold even if* dom (f) *is not connected.*

PROOF. We may and will assume that dom (f) consists of a finite number of components. Retaining the notations of Theorem 5-A, observe that we have to prove only $(*)$ for $i=1$. To avoid trivial cases, assume that f is analytic at infinity, $f(z)$ is identically $c = f(\infty)$ on some but not all components of U_f, and that $E\left(f^{-1}(q_1)\right) \neq 0$. Since U_f contains $s_e(T)$ minus a finite number of finite poles of f, we have $U_f \supset s_e(T) \cap \overline{f^{-1}(q_1)}$. On the other hand, $s\left(T|E(f^{-1}(q_1))X\right) \subset \subset s(T) \cap \overline{f^{-1}(q_1)}$ by Lemma 5, hence f is holomorphic in a neighbourhood of $s_e(V)$, where V denotes $T|E\left(f^{-1}(q_1)\right)X$. Since $r(V)$ is nonvoid, Lemma 7 yields that $f(V)$ is given by [4; VII. 9.3]. Further, $s\left(f(V)\right) = q_1$ and $E_f(q_1) \neq 0$ imply that g is holomorphic at c, hence $g\left(f(V)\right) = h(V)$, by [4; VII. 9.5] and Lemma 7.

Suppose now that under the conditions of Theorem 3, in addition, $r(T)$ is nonvoid. Then f is meromorphic on a neighbourhood of $s_e(T)$ and, by Theorem 1 (iii), p_i is not an eigenvalue of T for each finite pole p_i of f. For such functions of an arbitrary closed operator with nonvoid resolvent set GINDLER [6] defined an operational calculus, which was further investigated in [9]. Now we prove

THEOREM 6. *If, under the conditions of Theorem* 3, $r(T)$ *is nonvoid, then the operators $f(T)$ and $f_0(T)$, defined by Definition* 1 *and* [6; p. 33], *respectively, coincide.*

PROOF. First we recall the definition of the operator $f_0(T)$. Let f have the poles $p_0=\infty, p_1, \dots, p_k$ on $s_e(T)$ with orders $n_i \geqq 0$ $(i=0, 1, \dots, k)$, respectively. Define the polynomial $P(z)=\prod_{i=1}^{k}(p_i-z)^{n_i}$, put $m=n_1+\dots+n_k$, $n=m+n_0$ and suppose $b\in r(T)$. Define (with the usual conventions) the function $F(b; z)=f(z)P(z)(b-z)^{-n}$, then it is holomorphic in z on $s_e(T)$, hence the Dunford—Taylor calculus defines $F(b; T)\in B(X)$. By assumption, $P(T)^{-1}$ exists, and the operator $f_0(T)$ is defined by $f_0(T)=F(b; T)(b-T)^n P(T)^{-1}$. It was shown in [9; Theorem 1] that $f_0(T)$ is independent of the choice of $b\in r(T)$.

Now put $h(b; z)=P(z)(b-z)^{-n}$, then h is holomorphic on $s_e(T)$, and Lemma 7 yields that $h(b; T)=P(T)(b-T)^{-n}$. If $r(b; z)=h(b; z)^{-1}$, then $r(b; T)= =(b-T)^n P(T)^{-1}$, by Theorem 1 (iii). Since $f(z)=r(b; z)F(b; z)$, we obtain $f(T)=r(b; T)F(b; T)$, by Theorem 1 (vii) and Lemma 7. But we have $f_0(T)= =(b-T)^n P(T)^{-1}F(b; T)$, by [9; Theorem 2], thus $f(T)=f_0(T)$.

Suppose now that T is a possibly unbounded spectral operator with resolution of the identity E. Then T is prespectral with resolution of the identity E of class G for each total subspace G of X^*, hence if T is prespectral with resolution of the identity E_1 of some class G, then $E_1=E$. If $f\in A(T)$ and T is regarded as prespectral, then $f(T)$ is given by Definition 1, whereas if T is regarded as spectral, then the corresponding operator is defined in [4; XVIII. 2.8] and will be denoted here by $f_s(T)$.

LEMMA 8. *If T is spectral, then $f(T)=f_s(T)$.*

PROOF. In virtue of Lemma 3, we may regard T as prespectral of class X^*, hence if $\{c_n\}$ is an increasing sequence of compact sets with union $U=\operatorname{dom}(f)$, then

$$f(T)x = X^*-\lim_{n\to\infty} f\big(T|E(c_n)X\big)E(c_n)x, \qquad f_s(T)x = \lim_{n\to\infty} f\big(T|E(c_n)X\big)E(c_n)x$$

in the norm topology, if and only if these limits exist. Hence $f_s(T)\subset f(T)$. On the other hand, if $x\in D\big(f(T)\big)$, then $f\big(T|E(c_n)X\big)E(c_n)x=E(c_n)f(T)x$, by Theorem 1 (i) and (ii). But then

$$f_s(T)x = \lim_{n\to\infty} E(c_n)f(T)x = f(T)x$$

exists, hence $f_s(T)=f(T)$.

It was shown in [8; Theorem 2] that if T is a possibly unbounded spectral operator with resolution of the identity E, then T^* is prespectral with resolution of the identity E^* of class X. Now we prove

THEOREM 7. *If T is spectral and $f\in A(T)$, then $f(T^*)=f(T)^*$.*

PROOF. Suppose c is a compact subset of the complex plane and $E(c)X$ is denoted by X_c. For each $x^*\in E^*(c)X^*$ let Kx^* denote $x^*|X_c$, then K is a bounded linear operator from $E^*(c)X^*$ into X_c^* with norm $\leqq 1$. If $Kx^*=0$, then for $x\in X$, $x^*x= =E^*(c)x^*x=x^*E(c)x=Kx^*E(c)x=0$, hence K is 1—1. If $x_c^*\in X_c^*$, then x_c^* has a bounded extension $x^*\in X^*$, and for each $x\in X_c$ we have $E^*(c)x^*x=x^*E(c)x= =x_c^*x$, hence $KE^*(c)x^*=x_c^*$, thus K is onto. Consequently, K has the inverse K^{-1} which belongs to $B\big(X_c^*, E^*(c)X^*\big)$.

For each $x\in X_c$, $x^*\in X_c^*$ we have

$$x^*(T|X_c)x = K^{-1}x^*Tx = T^*K^{-1}x^*x =$$
$$= (T^*|E^*(c)X^*)K^{-1}x^*x = K(T^*|E^*(c)X^*)K^{-1}x^*x,$$

hence $(T|X_c)^*=K(T^*|E^*(c)X^*)K^{-1}$, where both restrictions are bounded operators. If $c\subset U=\mathrm{dom}\,(f)$, then

$$f(z)(z-(T|X_c)^*)^{-1} = Kf(z)(z-(T^*|E^*(c)X^*))^{-1}K^{-1}$$

for $z\in U\setminus s((T|X_c)^*)$. Since K is bounded, we obtain

$$f(T|X_c)^* = f((T|X_c)^*) = Kf(T^*|E^*(c)X^*)K^{-1}.$$

Suppose now that $\{c_n\}$ is an increasing sequence of compact sets with union U and $x^*\in D(f(T^*))$, then for each $x\in X$ and $n\to\infty$

$$f(T^*|E^*(c_n)X^*)E^*(c_n)x^*x \to f(T^*)x^*x.$$

On the other hand, for $x\in D(f(T))$

$$f(T^*|E^*(c_n)X^*)E^*(c_n)x^*x = E^*(c_n)f(T^*|E^*(c_n)X^*)E^*(c_n)x^*x =$$
$$= K_n^{-1}f(T|E(c_n)X)^*K_nE^*(c_n)x^*E(c_n)x =$$
$$= f(T|E(c_n)X)^*K_nE^*(c_n)x^*E(c_n)x = K_nE^*(c_n)x^*f(T|E(c_n)X)E(c_n)x =$$
$$= E^*(c_n)x^*f(T|E(c_n)X)E(c_n)x = x^*f(T|E(c_n)X)E(c_n)x \to x^*f(T)x$$

as $n\to\infty$, where K_n denotes the operator K in the case $c=c_n$. But then $x^*\in D(f(T)^*)$ and $f(T)^*x^*=f(T^*)x^*$, hence $f(T^*)\subset f(T)^*$.

Since $E^*\left(\bigcup\limits_{n=1}^{\infty} c_n\right)$ is the identity in X^*, we have $(E^*)^*X-\lim\limits_{n\to\infty} E^*(c_n)x^*=x^*$ for each x^* in X^*, where every $E^*(c_n)x^*$ is in $D(f(T^*))$. Further, $E(c_n)f(T)\subset \subset f(T)E(c_n)$ implies $E^*(c_n)f(T)^*\subset(f(T)E(c_n))^*\subset(E(c_n)f(T))^*=f(T)^*E^*(c_n)$, by [7; III. 5.25—26]. Hence for $x^*\in D(f(T)^*)$ and $n\to\infty$ $f(T^*)E^*(c_n)x^*= =f(T)^*E^*(c_n)x^*=E^*(c_n)f(T)^*x^*$ converges in the $(E^*)^*X$ topology to $f(T)^*x^*$. This implies $x^*\in D(f(T^*))$, thus $f(T^*)=f(T)^*$.

4. A characterization of prespectral operators

The following theorem extends a result due to Nel [10] to the prespectral case.

Theorem 8. *The operator T is prespectral with resolution of the identity E of class G if and only if it is the minimal E^*G-closed extension of an operator of the form $S+N$, where S is scalar with resolution of the identity E, $D(N)=\bigcup\{E(b)X;\ b$ bounded Borel set$\}$, each $N|E(b)X$ is quasinilpotent in $B(E(b)X)$, and*

if e is a Borel set, $c\notin\bar{e}$, $b_n=\{z\in C;\ |z|\le n\}$, $x\in X$, then the sequence

$$(*)\quad (c-T|E(b_ne)X)^{-1}E(b_ne)x = \sum_{k=0}^{\infty} N^k \int_{b_ne} (c-z)^{-k-1}E(dz)x \quad (n=1,2,\ldots)$$

converges to some $y(x)\in X$ in the E^*G-topology.

PROOF. 1° If T is the minimal E^*G-closed extension, then $D(T)\supset D(N)$. For each bounded Borel set b, by assumption $E(b)NE(b)=NE(b)$. Hence, if $x\in D(N)$ and e is a Borel set, then $E(e)NE(e)x=NE(e)x$, which implies $E(e)N\subset NE(e)$. If $x\in D(T)$, then for some generalized sequence $\{x_a\}$ in $D(N)$ we have $E^*G-\lim_a x_a=x$ and $E^*G-\lim_a (S+N)x_a=Tx$, hence

$$E^*G-\lim_a E(e)x_a = E(e)x \quad\text{and}\quad E^*G-\lim_a (S+N)E(e)x_a = E(e)Tx,$$

thus $E(e)T\subset TE(e)$.

For each $n=1,2,\dots$, $S|E(b_ne)X$ is a bounded scalar prespectral operator with a resolution of the identity $E|E(b_ne)X$ and the quasinilpotent $N|E(b_ne)X$ commutes with $E|E(b_ne)X$. By [3; Theorem 5], $T|E(b_ne)X$ is prespectral, hence $c\in r(T|E(b_ne)X)$, and $(*)$ is true by [4; XV. 5.2] (valid in the prespectral case, too).
Now we show that $c\in r(T|E(e)X)$. If $(c-T|E(e)X)x=0$, then $(c-T|E(b_ne)X)E(b_n)x=0$, hence $E(b_n)x=0$ for $n=1,2,\dots$, thus $c-T|E(e)X$ is 1—1. Since the subspace $E(e)X$ is E^*G-closed, $T|E(e)X$ is also E^*G-closed. If $x\in E(e)X$ and $x_n=(c-T|E(b_ne)X)^{-1}E(b_ne)x$, then $E^*G-\lim_{n\to\infty} x_n=y(x)$ by assumption, and

$$E^*G-\lim_{n\to\infty}(c-T|E(e)X)x_n = E^*G-\lim_{n\to\infty} E(b_n)x = x.$$

Thus the range of $c-T|E(e)X$ is $E(e)X$, and T is prespectral with a resolution of the identity E.

2° If T is prespectral, let S be its scalar part and $Nx=Tx-Sx$ for $x\in\bigcup\{E(b)X;\ b$ bounded Borel set$\}$. For $x\in D(N)$ and e Borel set we have

$$E(e)Nx = E(e)(T-S)x = (T-S)E(e)x = NE(e)x,$$

hence N leaves each $E(e)X$ invariant. Since $T|E(b)X$ is bounded and $T|E(b)X=S|E(b)X+N|E(b)X$ is its Jordan decomposition, $N|E(b)X$ is quasinilpotent for each bounded Borel set b, by [2; 3.5 Theorem]. If $c\notin\bar e$, then $(c-T|E(e)X)^{-1}$ belongs to $B(E(e)X)$ and commutes with $E|E(e)X$, hence

$$E^*G-\lim_{n\to\infty}(c-T|E(b_ne)X)^{-1}E(b_ne)x =$$
$$= E^*G-\lim_{n\to\infty} E(b_n)(c-T|E(e)X)^{-1}E(e)x = (c-T|E(e)X)^{-1}E(e)x,$$

for each $x\in X$. With the notation of Lemma 2, for the function $f(z)=z$ the operator Q_0 is $S+N$ and, by Lemma 2 and Theorem 4, T is the minimal E^*G-closed extension of $S+N$.

COROLLARY. *If S is scalar with resolution of the identity E of class G, further $P\in E^c$ and $P|E(b)X$ is quasinilpotent for every bounded Borel set b, then $T=S+P$ is prespectral with a resolution of the identity E.*

PROOF. If N denotes the restriction of P to $X_0=\bigcup\{E(b)X;\ b$ bounded Borel set$\}$, e is a Borel set, $c\notin\bar e$, $b_n=\{z\in C;\ |z|\le n\}$, $x\in X$, then we show that

$$E^*G-\lim_{n\to\infty}\sum_{k=0}^{\infty} N^k\int_{b_ne}(c-z)^{-k-1}E(dz)x = \sum_{k=0}^{\infty} P^k\int_{e}(c-z)^{-k-1}E(dz)x.$$

The operator $V=\int_e (c-z)^{-1}E(dz)$ belongs to E^c and, according to the proof of [4; XVIII. 2.28], $M=PV$ is quasinilpotent, and we have to prove that

$$E^*G-\lim_{n\to\infty}\sum_{k=0}^{\infty} M^k VE(b_n)x = \sum_{k=0}^{\infty} M^k Vx.$$

Given $\varepsilon>0$, $x^*\in G$ and $K\in E^c$, we have

$$\left|x^*K\sum_{k=r}^{\infty} M^k V\big(E(b_n)x-x\big)\right| \leqq |x^*||K||V|\sup_n |E(b_n)x-x|\sum_{k=r}^{\infty}|M^k| < \frac{\varepsilon}{2},$$

if r is large enough, for M is quasinilpotent. Further, $KM^kV\in E^c$ implies for n sufficiently large

$$\left|\sum_{k=0}^{r-1} x^*KM^kV\big(E(b_n)x-x\big)\right| < \frac{\varepsilon}{2},$$

which proves the limit relation. Finally, since P is E^*G-continuous and S is E^*G-closed, T is the minimal E^*G-closed extension of $S+N$, thus the proof is complete.

The next result is a generalization of [4; XVIII. 2.29]. However, we have to assume $N\in E^c$ rather than N commutes merely with S (cf. [2; 6.3] and [3; Theorem 5]).

THEOREM 9. *Suppose $T=S+N$, where S is scalar with resolution of the identity E of class G and $N\in E^c$ is quasinilpotent. Let $d>0$ and f be holomorphic in the open set $U_f=\{z\in C;\ \mathrm{dist}\ (z; s(T))<d\}$, except possibly for a finite set of poles $P=(p_1, \dots, p_k)$ for which $E(P)=0$.*

(i) *If f is bounded on U_f, then $f(T)\in B(X)$ is prespectral and*

$$f(T) = \sum_{n=0}^{\infty}\frac{N^n}{n!}\int_{s(T)} f^{(n)}(z)E(dz),$$

the series converging in the uniform operator topology of $B(X)$.

(ii) *If $\lim_{z\to\infty} f(z)=\infty$, then $f(T)$ is prespectral.*

In both cases a resolution of the identity (of class G) for $f(T)$ is $E_f(e)=E(f^{-1}(e))$, further $s(f(T))=\overline{f(s(T))}$.

PROOF. In both cases $E_f(e)f(T)\subset f(T)E_f(e)$ by Theorem 1 (i), and $\overline{f(s(T))}\subset s(f(T))$ can be proved as in Theorem 3.

(i) We can show as in [4; XVIII. 2.29] that the series converges in the uniform operator topology of $B(X)$, and that

$$f(T)x = E^*G-\lim_{m\to\infty} E(e_m)\sum_{n=0}^{\infty}\frac{N^n}{n!}\int_{s(T)} f^{(n)}(z)E(dz)x$$

for each $x\in X$, if $\{e_m\}$ is an increasing sequence of compact sets with union U_f. Hence $f(T)\in B(X)$. To prove that $s(f(T)|E_f(e)X)\subset\bar{e}$ for every Borel set e, it suffices to show that $s(f(T))\subset\overline{f(s(T))}$, if T and f satisfy the conditions of (i). Further, it is clearly sufficient to verify that $0\notin\overline{f(s(T))}$ implies $f(T)^{-1}\in B(X)$.

Thus we may assume that $|f(z)| \geqq r > 0$ for $z \in s(T)$, whereas $|f(z)| \leqq K$ on U_f. We claim that $g(z) = \frac{1}{f(z)}$ is bounded on some set $U(q) = \{z \in C;$ dist $(z, s(T)) < q\}$ with $q > 0$. Supposing the contrary, there would exist two sequences: $\{s_k\} \subset s(T)$ and $\{z_k\} \subset C$ such that $\lim_{k\to\infty} (z_k - s_k) = 0$ and $\lim_{k\to\infty} f(z_k) = 0$. On the other hand, if $0 < d^* < d$ and Q_k denotes $\{z;\ |z - s_k| = d^*\}$, then by Cauchy's formula

$$|f(z_k) - f(s_k)| = \left| \frac{z_k - s_k}{2\pi i} \int_{Q_k} \frac{f(z)}{(z - z_k)(z - s_k)} dz \right| \leqq \frac{4K}{d^*} |z_k - s_k|$$

for k large enough. But this would imply $\lim_{k\to\infty} f(s_k) = 0$, a contradiction.

Since g is bounded on some $U(q)$, $f(T)^{-1} = g(T)$ belongs to $B(X)$ by what has been established above.

(ii) If e is a bounded Borel set then, by the limit condition, $f^{-1}(e)$ is bounded, further $\overline{f^{-1}(e)} \cap s(T) \subset U_f$. Hence $E_f(e)X = E(f^{-1}(e) \cap s(T))X \subset D(f(T))$, by Theorem 1 (ii). Similarly to the previous case, we will show that $|f(z)| \geqq r > 0$ for $z \in s(T)$ implies that $g(z) = \frac{1}{f(z)}$ is bounded on some $U(q)$. In our case $\lim_{k\to\infty} f(z_k) = 0$ would imply that $\{z_k\}$ is bounded, hence some subsequence of it converges to some limit z_0. But then $z_0 \in s(T)$, hence $0 = |f(z_0)| \geqq r > 0$, a contradiction. Thus we have proved $s(f(T)) = \overline{f(s(T))}$ and $s(f(T)|E_f(e)X) \subset \bar{e}$ for each Borel set e, and the proof is complete.

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PACKING OF CONGRUENT SPHERES IN A STRIP

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Dedicated to A. FLORIAN *on his fiftieth birthday*

In a joint paper A. S. NOWICK and S. R. MADER [11] gave an account of some experiments for simulation of alloy thin films. A film has *thin* structure when its molecules (in the experiments non-overlapping hard-spheres) have a common point with the "substrate" on which they lie. In the case when the substrate was smooth (flat sheet) and the thin film was supposed to consist of congruent non-overlapping spheres the densest packing of spheres obtained in experiments was perfect, i.e. each sphere of the packing touches its six neighbours (Fig. 1). A similar result was obtained for congruent non-overlapping spheres when the substrate was of "crystalline" structure.[1]

Fig. 1

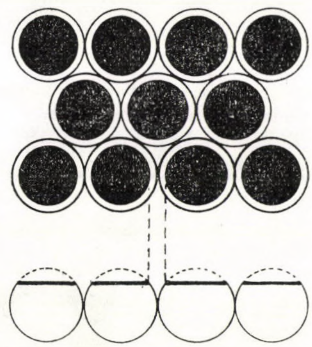


Fig. 2

In our paper we give an answer for the densest packing of spheres in the case in which the substrate onto the system of spheres lie, has a rather complicated structure.[2]

In order to formulate our result we introduce the notion of the density of a non-overlapping system of unit spheres with respect to a strip in which the spheres lie.

[1] To simulate a "crystalline" substrate NOWICK and MADER [11] prepared a grooved periodic layer as follows. A set of non-overlapping congruent balls was arranged into a perfect array and cemented to the substrate. Since the grooves thus produced were too deep to permit spheres in experiments to roll over them to form a second layer, the tops of the balls in this cemented layer were machined down (Fig. 2).

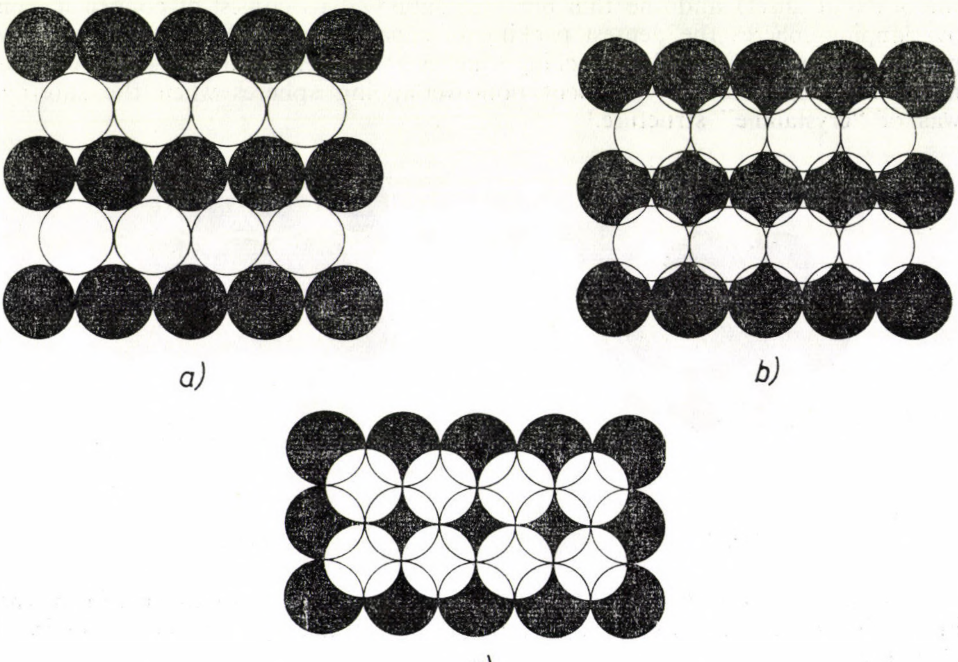
[2] Communication made in the Mathematical Institute (Budapest) in April 1967. The result was cited by HORVÁTH and MOLNÁR [7] and by HORVÁTH [4], [5], [6].

Let $\{S_i\}$ be a system of non-overlapping unit spheres lying in a strip $\sigma(t)$ of thickness t, i.e. the strip is bounded by two parallel planes at a distance t apart. The *density* d of $\{S_i\}$ with respect to the strip $\sigma(t)$ is defined by

$$d_3 = \overline{\lim_{R\to\infty}} \frac{\sum C(R) \cap S_i}{C(R) \cap \sigma(t)},$$

where $C(R)$ denotes a cylinder[3] of radius R with its axis perpendicular to t at an arbitrary point O of $\sigma(t)$. It is easy to show that d does not depend on the choice of O.

Let us consider a system of non-overlapping unit spheres lying in a strip $\sigma(t)$ of thickness $t \leqq 2+\sqrt{2}$, so that the spheres form two layers and in each layer the centres of the spheres constitute a rectangular net with the sides $a=2$ and $b=2\sqrt{4t-t^2-1}$ (Fig. 3) and let us denote the density of such a sphere-system with respect to $\sigma(t)$ by $\delta(t)$.



a) b) c)

Fig. 3

Our main result is the following

THEOREM. *The density of a system of non-overlapping unit spheres in a strip of thickness* $t \leqq 2+\sqrt{2}$ *is* $\leqq \delta(t)$.

Equality can be attained, for all values $t \leqq 2+\sqrt{2}$.

[3] We denote a domain and its measure (area or volume) with the same symbol.

Definitions, lemmas

Let $\{S_i\}$ be a system of non-overlapping unit spheres lying in the strip $\sigma(t)$ of thickness t. Without loss of generality we may suppose that to the system $\{S_i\}$ no further unit spheres can be added, i.e. the system of spheres is saturated. Obviously the centres of $\{S_i\}$ lie in a strip of thickness $t-2$, bounded by two planes Π_1 and Π_2. Projecting the system $\{S_i\}$ into Π_1 we get a circlesystem $\{C_i\}$ of unit circles with the centre-system $\{O_i\}$.

We define the density of $\{C_i\}$ in the plane Π_1 by

$$d_2 = \overline{\lim_{R\to\infty}} \frac{\sum K(R)\cap C_i}{K(R)},$$

where $K(R)$ denotes a circle of radius R centred in a fixed arbitrary point of Π_1. Since

$$d_3 = \overline{\lim_{R\to\infty}} \frac{\sum C(R)\cap S_i}{C(R)\cap\sigma(t)} = \overline{\lim_{R\to\infty}} \frac{4}{3t} \frac{\sum K(R)\cap C_i}{K(R)},$$

where $K(R)$ denotes the intersection of the cylindre $C(R)$ of radius R (perpendicular to Π_1) with Π_1, we get $d_3=\frac{4}{3t}d_2$, i.e. the densities of $\{S_i\}$ and of its projection $\{C_i\}$ are proportional. Therefore to prove our theorem it is enough to prove that the density of $\{C_i\}$ is $\leqq\frac{3t}{4}\delta(t)$.[4]

First of all we give the meaning of $\delta^*(t)=\frac{3t}{4}\delta(t)$.

Let C_1, C_2, C_3 be three unit circles of centres O_1, O_2, O_3, respectively and $\alpha_1, \alpha_2, \alpha_3$ the corresponding angles of the triangle $\Delta=O_1O_2O_3$. The density of the circles C_1, C_2, C_3 with respect to Δ is defined by

$$\frac{\sum \Delta\cap C_i}{\Delta} = \frac{\sum \alpha_i C_i}{2\pi\Delta} = \frac{\pi}{2\Delta}.$$

According to this definition $\delta^*(t)$ is the density of three unit circles C_1, C_2, C_3 with respect to the isosceles triangle $O_1O_2O_3$ of sides $O_1O_2=2$, $O_2O_3=O_3O_1=$ $=\sqrt{4t-t^2}$.

In the proof of our theorem a prominent role will be assigned to the tessellation L^* which, in general, is a modified tessellation of the well known L-tessellation of Voronoi-Delaunay.[5]

We now introduce the tessellation L^*.

We consider a system $\{O_i\}$ of points in the plane having the following properties: 1) $O_iO_j\geqq a$ $(i, j=1, 2, \dots, i\neq j)$ and 2) no point of the plane at distance $\varrho\geqq a$ can be added to the system $\{O_i\}$. Such a point-system $\{O_i\}$ will be called briefly a saturated point system.

[4] See Horváth–Molnár [7].
[5] See Delaunay [1].

Associating with a point O_i the set D_i of all points P lying nearer to O_i than to any other point O_j; more precisely, $d(P, O_i) \leqq d(P, O_j)$, $i \neq j$ where $d(P, O_i)$ denotes the distance of P from O_i, we obtain a convex polygon D_i (Dirichlet-cell, Voronoi-polygon).[6] It is known that the convex polygons D_i constitute an L_D-tessellation (Dirichlet-tessellation, Dirichlet—Voronoi-tessellation (Fig. 4)).

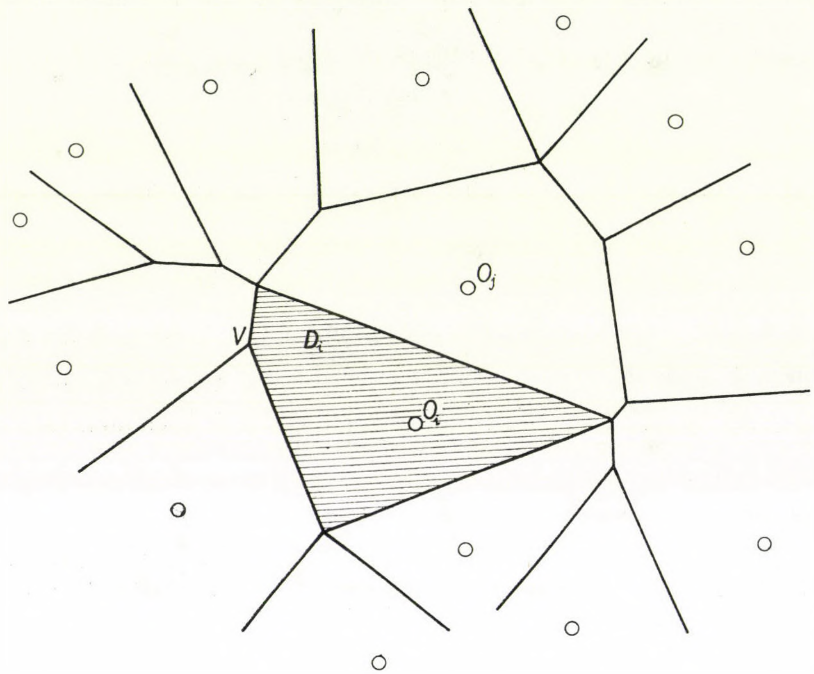


Fig. 4

By replacing each edge of L_D separating the neighbouring faces D_i, D_j by the segment O_iO_j we obtain the "dual" tessellation, the L-tessellation (Voronoi—Delaunay-tessellation)[7] (Fig. 5). The L-tessellation has the property, that the circumcentre V of a face is a vertex of a Dirichlet-cell. Obviously the faces of L are also convex polygons.

Let us consider a polygon inscribed in a circle of centre V (Fig. 6). We call a side O_1O_2 of this polygon a *separating side* if the straight-line O_1O_2 separates the polygon from V. In such a case we call the broken line O_1VO_2 the *bridge of the polygon* and the segments O_1V and O_2V the *components* of the bridge.

It is easy to see that in an L-tessellation any separating side can be replaced by only one bridge.

[6] See DIRICHLET [2] and VORONOI [13].

[7] For a direct construction of the L-tessellation see for instance DELAUNAY [1].

LEMMA 1. *Let us consider an L-tessellation of a saturated system of points. If we replace each separating side of the faces of L by its corresponding bridge, we obtain a tessellation.*

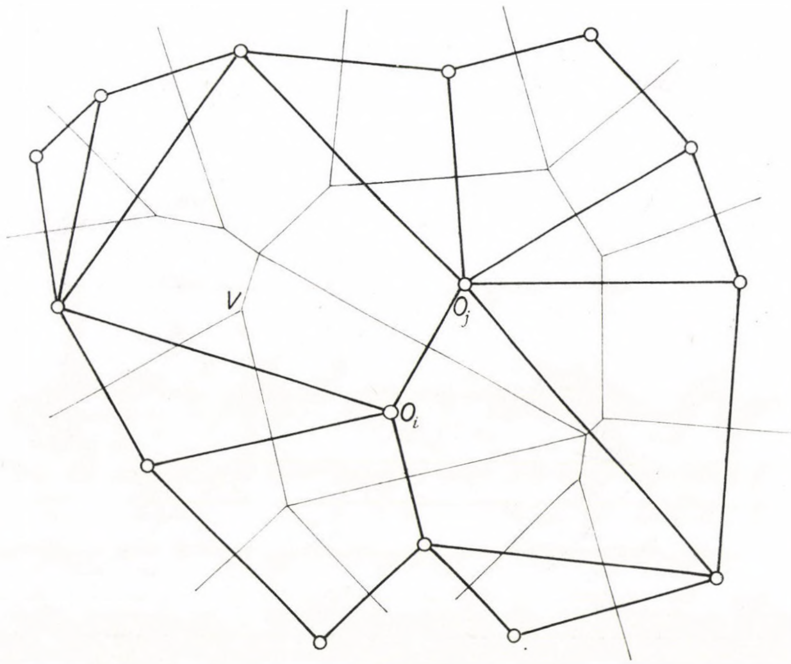


Fig. 5

PROOF. Let $\{O_i\}$ and $\{V\}$ be the vertex system of the L-tessellation and of its "dual" L_D-tessellation corresponding to the saturated point-system $\{O_i\}$, resp. Replace every separating side of the faces of L by the corresponding bridge (Fig. 7, for instance O_1O_2 by O_1VO_2).

To prove our lemma it is sufficient to prove, that 1) the components of the bridges do not intersect one another, 2) no bridge can intersect a non-replaced side of L.

To prove 1) we start with the obvious remark, that the circle of centre V and of radius OV does not contain in its interior another vertex O_i of L. Let OV and O^*V^* be components of two bridges. The perpendicular to OO^* at the midpoint of OO^* divides the plane into two halfplanes H, H^*. A simple consequence of our remark is that $V^*\notin H$ and $V\notin H^*$, hence OV and O^*V^* do not intersect each other. OV and O^*V^* have a common point iff $V=V^*$.

To prove 2 we consider an arbitrary bridge O_1VO_2 of the face F of L and we investigate the relation of this bridge with respect to the non-replaced sides of L. Since O_1VO_2 exists, the straight line O_1O_2 divides the plane into two halfplanes H and H_v containing F and V, resp. (Fig. 8). Denote by C the circle of centre V and radius O_1V. Let us consider now the neighbouring face F^* of F, having with

F the common side O_1O_2, i.e. the straight line O_1O_2 separates F^* from F. Since F^* is a face of L, F^* is convex and obviously all its vertices different from O_1 and O_2 lie on a circle C^* having the radius O_1V^* greater than O_1V. In the case when

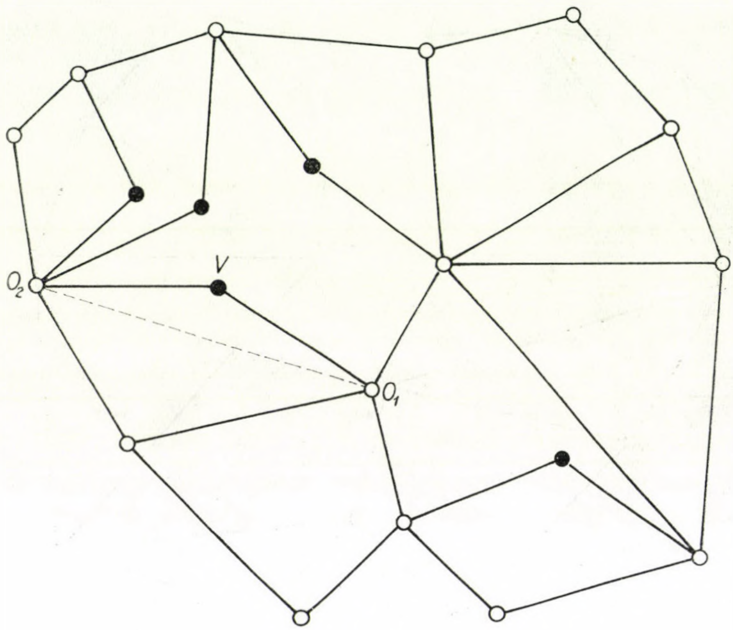


Fig. 7

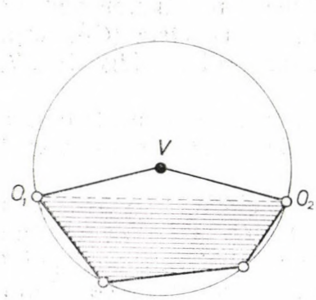


Fig. 6

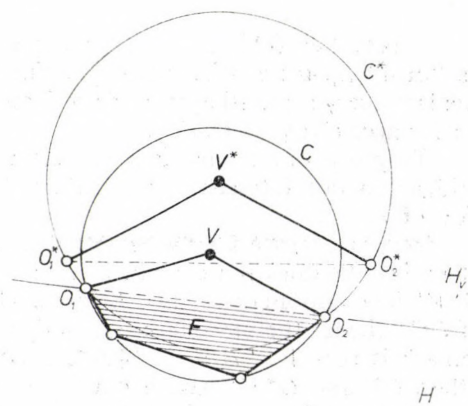


Fig. 8

F^* contains the bridge O_1VO_2, the bridge does not intersect the sides of L. If F^* does not contain the bridge O_1VO_2 then it must have a side $O_1^*O_2^*$, which intersects the bridge (Fig. 9).[8] In this case $O_1^*O_2^*$ is a separating side of F^*, i.e. $O_1^*V^*O_2^*$ is a bridge, that is the side $O_1^*O_2^*$ is a replaced side of L.

What can we say about the "measure" of the covering of the bridge O_1VO_2 by F^*? From the fact that the system $\{O_i\}$ is saturated, it is easy to see, that at least one of the angles VO_1O_2, VO_2O_1 will be partly covered by one of the angles $\sphericalangle O_2^*O_1O_2$, $\sphericalangle O_1^*O_2O_1$ with measure $>\alpha=\arcsin\frac{a}{2\varrho}$. Hence in the new bridge $O_1^*V^*O_2^*$ the angle $VO_1^*O_2^*$ and $VO_2^*O_1^*$ which must be covered is smaller at least by α than VO_1O_2 and VO_2O_1, resp. Continuing in the same way, in a finite number of steps, the original bridge O_1VO_2 will be covered by the annexed faces $\{F^*\}$, i.e. there is no non-replaced side of L which intersects O_1VO_2.

The following two simple remarks will be valuable in the proof of our theorem. In an L^*-tessellation 1) VV^* is perpendicular to O_1O_2, 2) $V^*O_1>VO_1$ and $V^*O_2>VO_2$ (Fig. 8).

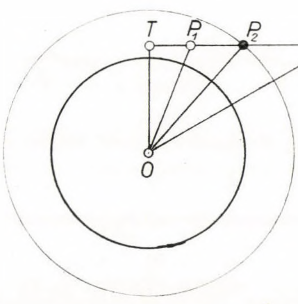


Fig. 9

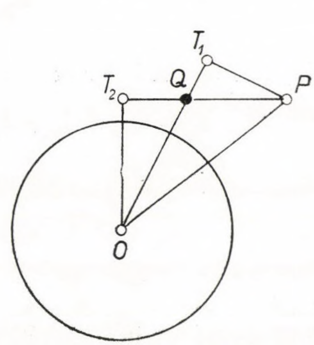


Fig. 10

LEMMA 2.[10] *Denote by OTP_k a right-angled triangle $\left(\sphericalangle OTP=\frac{\pi}{2}\right)$ of hypotenuse OP_k and by c a circle of centre O. If $OP_1<OP_2<OP_3$, then*

$$\frac{s(P_1OP_3)}{P_1OP_3}<\frac{s(P_1OP_2)}{P_1OP_2},$$

where $s(P_iOP_j)$ denotes a sector of the circle c of $\sphericalangle P_iOP_j$.

PROOF. Denote by C a circle of radius OP_2 centred at O and by $S(P_iOP_j)$ a sector of C of $\sphericalangle P_iOP_j$ (Fig. 10). Obviously

$$\frac{s(P_2OP_3)}{P_2OP_3}<\frac{s(P_2OP_3)}{S(P_2OP_3)}=\frac{s(P_1OP_2)}{S(P_1OP_2)}<\frac{s(P_1OP_2)}{P_1OP_2}.$$

[8] This part of the proof differs form our proof in [11].
[9] One of O_1^*, O_2^* may be the same as O_1 or O_2.
[10] See MOLNÁR [8], [9], [10], [11].

Therefore

$$\frac{s(P_1OP_3)}{P_1OP_3}=\frac{s(P_1OP_2)+s(P_2OP_3)}{P_1OP_2+P_2OP_3}<\frac{s(P_1OP_2)}{P_1OP_2}\frac{P_1OP_2+P_2OP_3}{P_1OP_2+P_2OP_3}=\frac{s(P_1OP_2)}{P_1OP_2}.$$

LEMMA 3. *Let OT_kP be a right-angled triangle of hypotenuse OP and let c be a circle of centre O. If $\sphericalangle T_1OP<\sphericalangle T_2OP$ then*

$$\frac{s(T_1OP)}{T_1OP}<\frac{s(T_2OP)}{T_2OP}.$$

PROOF. Let Q be the intersection of the segments OT_1 and T_2P (Fig. 11). Obviously

$$\frac{s(T_1OP)}{T_1OP}<\frac{s(QOP)}{QOP}.$$

According to Lemma 2, we obtain

$$\frac{s(T_1OP)}{T_1OP}<\frac{s(T_2OP)}{T_2OP}.$$

Proof of the theorem

To prove our theorem we shall consider the L^*-tessellation corresponding to the centre system $\{O_i\}$ of the circles $\{C_i\}$. We shall show, that in each face of L^* the density of the unit circles is $\leqq\delta^*(t)$.

Of course we can restrict ourselves to a saturated $\{C_i\}$ circle system having the value ϱ of saturation $\leqq 2$. Let L^* be the corresponding tessellation of $\{O_i\}$ derived from L (Fig. 7). Obviously L^* is the same as the L^* tessellation of $\{C_i\}$. We distinguish two types of faces of L^*: 1) face F, which is a polygon having only non-replaced sides, i.e. $F=O_1O_2\dots O_k$, 2) face F^*, containing at least one bridge, i.e. $F^*=O_1V_1O_2\dots O_k$, resp. $F^*=O_1V_1O_2\dots V_k$.

If the thickness of the strip is t, then $O_iO_j\geqq\sqrt{4t-t^2}$. Let us consider four unit circles, whose centres form a square of side $\sqrt{4t-t^2}$ (Fig. 11). We denote the half of its diagonal by $t_0=\sqrt{2t-\frac{t^2}{2}}$.

Type 1. Let V be the corresponding centre radical of the face F. We denote by p_i the perpendicular straight line at O_i to Π_1.

a) If $VO_1<t_0$, then F is of course an acute triangle $\Delta=O_1O_2O_3$. If S_2 does not touch S_3 we rotate the sphere S_2 around p_1 towards S_3 till it touches S_3. During the rotation VO_1 and VO_2 decreases and so does also the area of the triangle Δ, thus the circle density in Δ increases. Therefore it is sufficient to discuss the case when Δ is an acute triangle and S_2 touches S_3. If S_1 does not touch one of S_2, S_3 we rotate the sphere S_1 around p_2 towards S_3 till S_1 touches S_3. During this rotation Δ decreases also and the circle density in Δ increases. Δ remains an acute triangle. By the same way, rotating S_1 about p_3 towards S_2 we obtain a Δ in which the circle

density is smaller than that in the original triangle. What is the minimal value of Δ? According to our preceding result we know, that in this case the corresponding S_1, S_2, S_3 must touch one another, that is, the centres of these sphere lying in the strip of thickness $t-2$ form an equilateral triangle Δ_0 of side 2. Δ is the projection of Δ_0 into Π_1, i.e. $\Delta=\Delta_0 \cos\varphi$, where φ is the angle of the planes of Δ and Δ_0.[11] Obviously, φ has maximal value if Δ_0 has two vertices in Π_1 and one in Π_2, resp. vice-versa. In these cases Δ is minimal, and the corresponding maximal circle density is just $\delta^*(t)$.

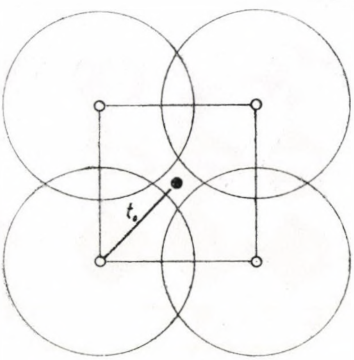


Fig. 11

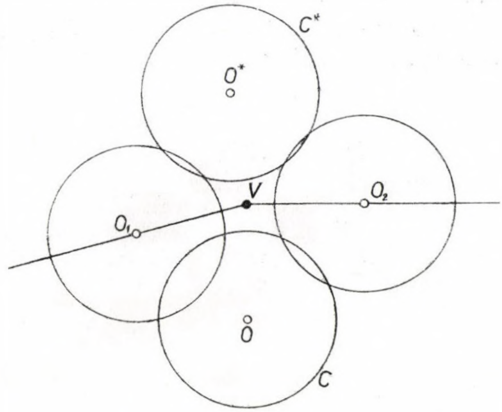


Fig. 12

b) If $VO_1 \geqq t_0$ we decompose the face $F=O_1O_2\dots O_k$ in the triangles $\Delta_1=O_1VO_2$, $\Delta_2=O_2VO_3, \dots, \Delta_k=O_kVO_1$ and we consider the density of the unit circles in these triangles. According to our Lemmas 2 and 3, the circle density in such a Δ_i takes its maximal value $\frac{\pi}{4t-t^2}$ if $O_iO_{i+1}=\sqrt{4t-t^2}$ and $VO_1=t_0$. But

$$\frac{\pi}{4t-t^2} \leqq \delta^*(t) = \frac{\pi}{2\sqrt{4t-t^2-1}}.$$

Type 2. Let V^* be the centre radical corresponding to F^*. The existence of a bridge implies $V^*O_1>t_0$. Indeed, if there is a bridge O_1VO_2, then around V we may consider at least four circles at the same distance from V having their centre distances $O_iO_j \geqq \sqrt{4t-t^2}$, because in the domain of the angle $\sphericalangle O_1VO_2<\pi$ there is at least one circle C of $\{C_i\}$, thus also in the domain $\Pi_1 - \sphericalangle O_1VO_2$ we may consider at least one circle C^* of centre O^* (Fig. 12) having the same centre radical V and the property, that $O^*O_i \geqq \sqrt{4t-t^2}$ $(i=1, 2, \dots, k)$. Thus $VO_1>t_0$, but $V^*O_1 \geqq VO_1$, hence $V^*O_1>t_0$.

We decompose $F^*=O_1V_1O_2\dots O_k$ and $F^*=O_1V_1O_2\dots V_k$ into triangles $\Delta_1=O_1V_1V^*=O_2V_1V^*, \dots \Delta_k=O_kV^*O_1$, and $\Delta_1=O_1V_1V^*=O_2V_1V^*, \dots$ $\Delta_k= =O_kV^*V_k=O_1V^*V_k$, resp.

[11] The plane of Δ is Π_1.

According to our Lemmas 2 and 3, the circle density in these triangles Δ_i is $<\frac{\pi}{4t-t^2}$. This density value is smaller than $\delta^*(t)$. Thus the density of circles in a face of L^* of type 2 is smaller than $\delta^*(t)$.

We finish the proof of our theorem by stressing the fact that the circle density $\delta^*(t)$ may be attained by circle systems giving tessellation having faces of type 1, case a). Our illustrations of Fig. 3 show that this upper bound and the corresponding upper bound $\delta(t)$ for sphere systems can be attained for all value of t $(2 \leqq t \leqq 2+\sqrt{2})$.

REMARKS. 1) For every $2<t<2+\sqrt{2}$, there are infinitely many packings of unit spheres having the same density $\delta(t)$ (Fig. 13).[12]

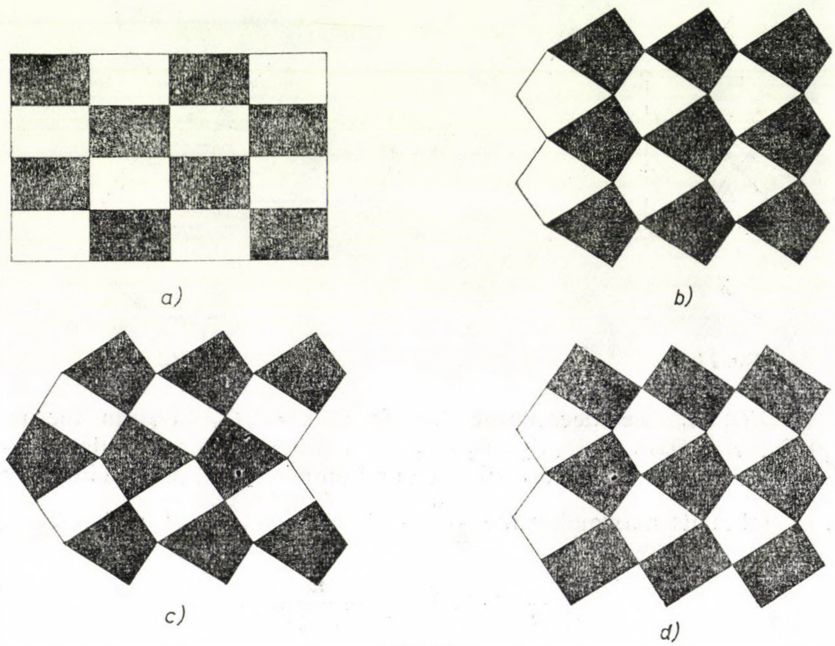


Fig. 13

2) The L^*-tessellation can be introduced essentially in the same way, as in our paper, for some incongruent circle systems.[13]

3) In the case $t>2+\sqrt{2}$, there are conjectures[14] which seem to give precise upper bounds for all values of t concerning the packing density of unit spheres lying in a strip of thickness t.

[12] The projection of the centres of the unit spheres lying in the strip are the vertices and the circumcentres of faces of the tessellation. Compare the first tessellation of Fig. 13 and Fig. 3b.

The tessellations of Fig. 13 are composed of infinite strips of two types (Fig. 14a, b). If in the construction of the tessellation we use only stripes of the same type, we obtain the first tessellation of Fig. 13. If we consider alternatively stripes of both types, we obtain the second tessellation of Fig. 13. Fig. 15 illustrates a tessellation having another structure as the tessellations of Fig. 13.

[13] Communication made in the Mathematical Institute (Budapest) in February 1977.

[14] See HORVÁTH—MOLNÁR [7].

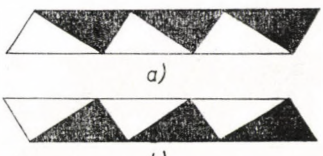


Fig. 14

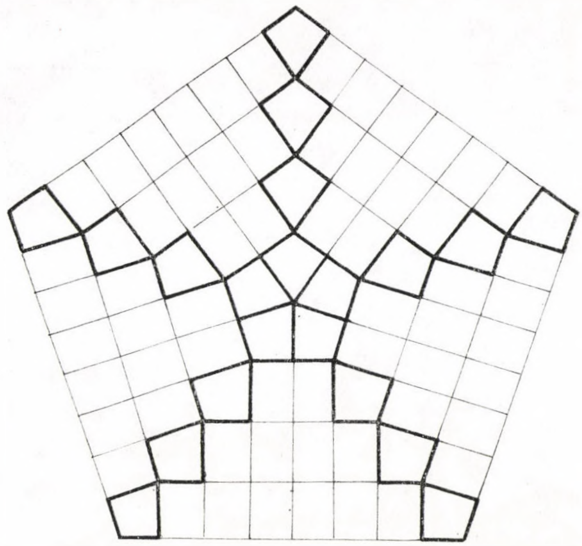


Fig. 15

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[1] G. Szegő, *Orthogonal polynomials*, AMS Coll. Publ. Vol. XXIII (Providence, 1939).
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LACUNARY INTERPOLATION BY SPLINES. I

By

A. K. VARMA (Gainesville)

Dedicated to Professor I. J. Schoenberg

1. Introduction. Recently AHLBERG, NILSON [1] and I. J. SCHOENBERG [6] has initiated the study of the so called *g*-splines in connection with the problem of lacunary interpolation by splines. The purpose of *g* splines is to generalize the interpolation problem first considered by G. D. BIRKHOFF [3]. In this problem the data are values of the function and of its derivatives but without Hermite's condition that only consecutive derivatives be used at each node.

Motivated by the beautiful results of I. J. SCHOENBERG on *g* splines A. MEIR and A. SHARMA [4] have obtained error bounds for lacunary (0, 2) interpolation of certain functions by deficient quintic splines of the class $S_{n,5}^{(3)}$ (datas being prescribed at equidistant nodes). It turns out that in the case of Meir—Sharma interpolant the convergence is of order one less than that which is possible through best approximation by quintic C^3 splines. Moreover, by now it is know that Meir—Sharma interpolant is less stable than many local piecewise quintic approximants, such as quintic spline interpolants of continuity class $C^4[0, 1]$ (see [9]).

The object of this paper is to obtain deficient quintic splines which still interpolate Birkhoff data (analogous to Meir and Sharma) but where the convergence is of the same order as that of best approximation by quintic C^2 splines. We now state our main results.

Let $n=2m+1$, $x_i=\dfrac{i}{2m}$, $i=0, 2, \dots, 2m$. We shall denote by $S_{n,5}^{(2)}$ the class of quintic splines $S(x)$ on [0, 1] having the following two properties:

i) $S(x)\in C^2[0, 1]$,
ii) $S(x)$ is quintic in $[x_{2i}, x_{2i+2}]$, $i=0, \dots, m-1$.

THEOREM 1. *Let* $n=2m+1$, $x_i=\dfrac{i}{2m}$, $i=0, 2, \dots, 2m$, $t_{2i}=x_{2i}+\dfrac{2}{3}h$, $i=0, ., \dots, m-1$, $2h=\dfrac{1}{m}$. *Given arbitrary numbers* $f(x_0), f(x_2), \dots, f(x_{2m})$; $f(t_0), f(t_2), \dots, f(t_{2(m-1)}); f''(t_0), f''(t_2), \dots, f''(t_{2(m-1)}); f'(x_0), f'(x_{2m})$ *there exists a unique* $S_n(x)\in C^2[0, 1]$ *such that*

$$(1.1)\qquad \begin{cases} S_n(x_{2i}) = f(x_{2i}), & i=0, 1, \dots, m, \\ S_n(t_{2i}) = f(t_{2i}), \quad S_n''(t_{2i}) = f''(t_{2i}), & i = 0, 1, \dots, m-1, \\ S_n'(x_0) = f'(x_0), \quad S_n'(x_{2m}) = f'(x_{2m}). \end{cases}$$

THEOREM 2. *Let $f \in C^2[0, 1]$ and let $S_n(x)$ be the unique quintic spline satisfying the conditions of Theorem* 1. *Then* $(n=2m+1)$

$$(1.2) \qquad |S_n^{(p)}(x) - f^{(p)}(x)| \leqq 89 n^{p-2} \omega_2\left(\frac{1}{m}\right), \quad p = 0, 1, 2.$$

Here $\omega_2(\delta)$ is the usual modulus of continuity of $f''(x)$.

Theorem 2 gives us error estimates of $\| f^{(r)}(x) - S^{(r)}(x)\|$ for $r=0, 1, 2$ under the assumption that $f \in C^2[0, 1]$. It is natural to ask about the error estimate $\| f^{(r)}(x) - S^{(r)}(x)\|$ if $f \in C^l[0, 1]$ for $l=3$ or 4. The following theorem is in this direction.

THEOREM 3. *Let $f \in C^4[0, 1]$ and let $S_n(x)$ be the unique quintic spline satisfying the conditions of Theorem* 1. *Then*

$$(1.3) \qquad |S_n^{(p)}(x) - f^{(p)}(x)| \leqq 103 m^{p-4} \omega_4\left(\frac{1}{m}\right) + 8 m^{p-4} \| f^{(4)}\|_\infty, \quad p = 0, 1, 2$$

where $\omega_4(\cdot)$ denotes the modulus of continuity of $f^{(\mathrm{iv})}$.

Theorem 3 may be compared with corresponding theorem of MEIR and SHARMA [4].

2. Preliminaries. If $P(x)$ is a quintic on [0, 1] then we have

$$(2.1) \qquad P(x) = P(0) A_0(x) + P\left(\frac{1}{3}\right) A_1(x) + P(1) A_2(x) + P'(0) B_0(x) + $$
$$+ P'(1) B_2(x) + P''\left(\frac{1}{3}\right) C_1(x),$$

where

$$(2.2) \qquad C_1(x) = \frac{9}{8}[3x^5 - 7x^4 + 5x^3 - x^2] = \frac{9}{8} x^2 (x-1)^2 (3x-1),$$

$$(2.3) \qquad B_2(x) = \frac{1}{8}[27x^5 - 51x^4 + 29x^3 - 5x^2] = \frac{x^2(x-1)(3x-1)(9x-5)}{8},$$

$$(2.4) \qquad B_0(x) = -3x^4 + 7x^3 - 5x^2 + x = x(x-1)^2(1-3x),$$

$$(2.5) \qquad A_2(x) = \frac{-297x^5 + 609x^4 - 359x^3 + 63x^2}{16} =$$
$$= x^2(3x-1)\left[\frac{1}{2} - \frac{7}{4}(x-1) - \frac{99}{16}(x-1)^2\right],$$

$$(2.6) \qquad A_1(x) = \frac{81}{16}[9x^5 - 17x^4 + 7x^3 + x^2] = \frac{81}{16} x^2 (x-1)^2 (1+9x),$$

$$(2.7) \qquad A_0(x) = -27x^5 + 48x^4 - 13x^3 - 9x^2 + 1 = (x-1)^2(1-3x)(1+5x+9x^2).$$

For later references we note that

$$
(2.8)\quad \begin{cases}
C_1''(0) = -\dfrac{9}{4}, & C_1'''(0) = \dfrac{135}{4}, \quad C_1^{(\mathrm{iv})}(0) = -189, \quad C_1^{(\mathrm{iv})}\left(\dfrac{1}{3}\right) = -54, \\
C_1''(1) = \dfrac{9}{2}, & C_1'''(1) = \dfrac{189}{4}, \quad C_1^{(\mathrm{iv})}(1) = 216, \quad C_1'''\left(\dfrac{1}{3}\right) = -\dfrac{27}{4}, \\
B_2''(0) = -\dfrac{5}{4}, & B_2'''(0) = \dfrac{87}{4}, \quad B_2^{(\mathrm{iv})}(0) = -153, \quad B_2^{(\mathrm{iv})}\left(\dfrac{1}{3}\right) = -18, \\
B_2''(1) = \dfrac{23}{2}, & B_2'''(1) = \dfrac{285}{4}, \quad B_2^{(\mathrm{iv})}(1) = 252, \quad B_2'''\left(\dfrac{1}{3}\right) = -\dfrac{27}{4}, \\
B_0''(0) = -10, & B_2'''(0) = 42, \quad B_0^{(\mathrm{iv})}(0) = -72, \quad B_0^{(\mathrm{iv})}\left(\dfrac{1}{3}\right) = -72, \\
B_0''(1) = -4, & B_0'''(1) = -30, \quad B_0^{(\mathrm{iv})}(1) = -72, \quad B_0'''\left(\dfrac{1}{3}\right) = 18, \\
A_2''(0) = \dfrac{63}{8}, & A_2'''(0) = -\dfrac{1077}{8}, \quad A_2^{(\mathrm{iv})}(0) = \dfrac{1827}{2}, \quad A_2^{(\mathrm{iv})}\left(\dfrac{1}{3}\right) = 171, \\
A_2''(1) = -\dfrac{165}{4}, & A_2'''(1) = -\dfrac{2679}{8}, \quad A_2^{(\mathrm{iv})}(1) = -1314, \quad A_2'''\left(\dfrac{1}{3}\right) = \dfrac{369}{8}, \\
A_1''(0) = \dfrac{81}{8}, & A_1'''(0) = \dfrac{1701}{8}, \quad A_1^{(\mathrm{iv})}(0) = -\dfrac{4131}{2}, \quad A_1^{(\mathrm{iv})}\left(\dfrac{1}{3}\right) = -243, \\
A_1''(1) = \dfrac{405}{4}, & A_1'''(1) = \dfrac{7047}{8}, \quad A_1^{(\mathrm{iv})}(1) = 3402, \quad A_1'''\left(\dfrac{1}{3}\right) = -\dfrac{1377}{8}, \\
A_0''(0) = -18, & A_0'''(0) = -78, \quad A_0^{(\mathrm{iv})}(0) = 1152, \quad A_0^{(\mathrm{iv})}\left(\dfrac{1}{3}\right) = 72, \\
A_0''(1) = -60, & A_0'''(1) = -546, \quad A_0^{(\mathrm{iv})}(1) = -2088, \quad A_0'''\left(\dfrac{1}{3}\right) = 126.
\end{cases}
$$

For $f\in C^2[0, 1]$ we have

$$
(2.9)\quad \begin{cases}
f(x_{2i+2}) = f(x_{2i}) + 2hf'(x_{2i}) + 2h^2 f''(\eta_{1,2i}), \\
f(x_{2i-2}) = f(x_{2i}) - 2hf'(x_{2i}) + 2h^2 f''(\eta_{2,2i}), \\
f(t_{2i}) = f(x_{2i}) + \dfrac{2}{3} hf'(x_{2i}) + \dfrac{2}{9} h^2 f''(\eta_{3,2i}), \\
f(t_{2i-2}) = f(x_{2i}) - \dfrac{4}{3} hf'(x_{2i}) + \dfrac{8}{9} h^2 f''(\eta_{4,2i}), \\
f'(x_{2i+2}) = f'(x_{2i}) + 2hf''(\eta_{5,2i}), \\
f'(x_{2i-2}) = f'(x_{2i}) - 2hf''(\eta_{6,2i}),
\end{cases}
$$

where

$$x_{2i} < \eta_{1,2i} < x_{2i+2}; \quad x_{2i-2} < \eta_{2,2i} < x_{2i}; \quad x_{2i} < \eta_{3,2i} < t_{2i};$$

$$t_{2i-2} < \eta_{4,2i} < x_{2i}; \quad x_{2i} < \eta_{5,2i} < x_{2i+2}; \quad x_{2i-2} < \eta_{6,2i} < x_{2i}.$$

3. Proof of Theorem 1. Here we need to prove that there exists a unique $S(x) \in S_{n,5}^{(2)}$ satisfying the requirements of Theorem 1. For this purpose we try to express $S(x)$ in the following form. For $2ih \leqq x \leqq (2i+2)h$, $i=0, 1, \ldots, m-1$, we have

$$(3.1) \quad S_n(x) = f(x_{2i}) A_0\left(\frac{x-2ih}{2h}\right) + f(x_{2i+2}) A_2\left(\frac{x-2ih}{2h}\right) + f(t_{2i}) A_1\left(\frac{x-2ih}{2h}\right) +$$

$$+ 2hS_n'(x_{2i}) B_0\left(\frac{x-2ih}{2h}\right) + 2hS_n'(x_{2i+2}) B_2\left(\frac{x-2ih}{2h}\right) + 4h^2 f''(t_{2i}) C_1\left(\frac{x-2ih}{2h}\right).$$

On using (2.1) and putting

$$(3.2) \quad S_n'(0) = f'(0), \quad S_n'(1) = f'(1),$$

it is easy to see that $S_n(x)$ as given by (3.1) indeed satisfies (1.1) and is quintic in $[x_{2i}, x_{2i+2}]$ for $i=0, 1, \ldots, m-1$. We still need to decide whether it is possible to determine $S_n'(x_{2i})$ $(i=1, 2, \ldots, m-1)$ uniquely. For this purpose we use the fact that $S_n(x) \in C^2[0, 1]$ and therefore the conditions

$$(3.3) \quad S_n''(x_{2i}+) = S_n''(x_{2i}-), \quad i = 1, 2, \ldots, m-1$$

must hold. On using (3.1) and (2.8) we obtain

$$(3.4) \quad 4h^2 S_n''(x_{2i}+) = -18f(x_{2i}) + \frac{63}{8} f(x_{2i+2}) + \frac{81}{8} f(t_{2i}) - $$

$$- 20hS_n'(x_{2i}) - \frac{5}{2} hS_n'(x_{2i+2}) - 9h^2 f''(t_{2i}),$$

and

$$(3.5) \quad 4h^2 S_n''(x_{2i}-) = -60f(x_{2i-2}) - \frac{165}{4} f(x_{2i}) + \frac{405}{4} f(t_{2i-2}) -$$

$$- 8hS_n'(x_{2i-2}) + 23hS_n'(x_{2i}) + \frac{9}{2} h^2 f''(t_{2i-2}).$$

Therefore (3.3)—(3.5) gives immediately

$$(3.6) \quad \frac{5}{2} hS_n'(x_{2i+2}) + 43hS_n'(x_{2i}) - 8hS_n'(x_{2i-2}) = \frac{93}{4} f(x_{2i}) +$$

$$+ \frac{63}{8} f(x_{2i+2}) + 60f(x_{2i-2}) + \frac{81}{8} f(t_{2i}) - \frac{405}{4} f(t_{2i-2}) - 9h^2 f''(t_{2i}) - 18h^2 f''(t_{2i-2})$$

for $i=1, 2, \ldots, m-1$.

But (3.6) is a strictly a tridiagonal dominant system. As is well known this system of equations has a unique solution. Thus $S_n'(x_{2i})$, $i=1, 2, \ldots, m-1$ can be obtained uniquely by the system (3.6). Thus Theorem 1 is proved.

4. Estimates. The proof of Theorem 2 is based on the following

LEMMA 4.1. *Let $f\in C^2[0,1]$, we set*

(4.1) $$M_{2k}=\max_{i=1,2,\dots,m-1}|S_n'(x_{2i})-f'(x_{2i})|$$

then

(4.2) $$M_{2k}\leqq\frac{138}{65m}\,\omega_2\left(\frac{1}{m}\right)$$

where $\omega_2(\delta)$ is the modulus of continuity of $f''(x)$.

PROOF. From (3.6) and (2.9) it follows that

$$\frac{5}{2}[S_n'(x_{2k+2})-f'(x_{2k+2})]+43[S_n'(x_{2k})-f'(x_{2k})]-8[S_n'(x_{2k-2})-f'(x_{2k-2})]=$$
$$=\left[\frac{63}{4}f''(\eta_{1,2k})+120f''(\eta_{2,2k})+\frac{9}{4}f''(\eta_{3,2k})-90f''(\eta_{4,2k})-\right.$$
$$\left.-5f''(\eta_{5,2k})-16f''(\eta_{6,2k})-9f''(t_{2k})-18f''(t_{2k-2})\right]h.$$

The result follows on using the properties of modulus of continuity and $2mh=1$.

LEMMA 4.2. *Let $f\in C^2[0,1]$ then*

(4.3) $$|S_n'''(x_{2i}+)|\leqq\frac{102}{h}\,\omega_2\left(\frac{1}{m}\right),$$

(4.4) $$|S_n'''(x_{2i}-)|\leqq\frac{244}{h}\,\omega_2\left(\frac{1}{m}\right),$$

and

(4.5) $$|S_n^{(\mathrm{iv})}(t_{2i})|\leqq\frac{70}{h^2}\,\omega_2\left(\frac{1}{m}\right),$$

where $\omega_2(\delta)$ is the modulus of continuity of $f''(x)$.

PROOF. From (2.8), (2.9) and (3.1) we have

(4.6) $$8h^3S_n'''(x_{2i}+)=-78f(x_{2i})-\frac{1077}{8}f(x_{2i+2})+\frac{1701}{8}f(t_{2i})+$$
$$+84hS_n'(x_{2i})+\frac{87}{2}hS_n'(x_{2i+2})+135h^2f''(t_{2i})=$$
$$=\left[-\frac{1077}{4}f''(\eta_{1,2i})+\frac{189}{4}f''(\eta_{3,2i})+135f''(t_{2i})+87f''(\eta_{5,2i})\right]h^2+$$
$$+84h[S_n'(x_{2i})-f'(x_{2i})]+\frac{87}{2}h[S_n'(x_{2i+2})-f'(x_{2i+2})],$$
$$8h^3S_n'''(x_{2i}-)=-546f(x_{2i-2})-\frac{2679}{8}f(x_{2i})+\frac{7047}{8}f(t_{2i-2})-$$
$$-60hS_n'(x_{2i-2})+\frac{285}{h}hS_n'(x_{2i})+189h^2f''(t_{2i-2}).$$

$$(4.7)\qquad 8h^3 S_n'''(x_{2i}-) = h^2[-1092f''(\eta_{2,2i})+783f''(\eta_{4,2i})+120f''(\eta_{6,2i})+$$

$$+189f''(t_{2i-2})]-60h[S_n'(x_{2i-2})-f'(x_{2i-2})]+\frac{285}{2}h[S_n'(x_{2i})-f'(x_{2i})].$$

$$(4.8)\qquad 16h^4 S_n^{(\mathrm{iv})}(t_{2i}) = 72f(x_{2i})+171f(x_{2i+2})-243f(t_{2i})-$$

$$-144hS_n'(x_{2i})-36hS_n'(x_{2i+2})-216h^2f''(t_{2i}) =$$

$$= h^2[342f''(\eta_{1,2i})-54f''(\eta_{3,2i})-72f''(\eta_{5,2i})-216f''(t_{2i})]-$$

$$-144[S_n'(x_{2i})-f'(x_{2i})]-36h[S_n'(x_{2i+2})-f'(x_{2i+2})].$$

The proof of (4.3), (4.4) and (4.5) follows from (4.6), (4.7) and (4.8), respectively. Here one needs to use Lemma 4.1 and the properties of modulus of continuity of $f''(x)$.

5. Proof of Theorem 2. Since $0 \leqq t \leqq 1$, we have

$$(5.1)\qquad A_0(t)+A_1(t)+A_2(t) \equiv 1,$$

and from (3.1) it follows that for $x_{2k} \leqq x \leqq x_{2k+2}$

(5.2)

$$S_n''(x) = f''(x_{2k})A_0\left(\frac{x-2kh}{2h}\right)+f''(x_{2k+2})A_2\left(\frac{x-2kh}{2h}\right)+f''(t_{2k})A_1\left(\frac{x-2kh}{2h}\right)+$$

$$+2hS_n'''(x_{2k}+)B_0\left(\frac{x-2kh}{2h}\right)+2hS_n'''(x_{2k+2}-)B_2\left(\frac{x-2kh}{2h}\right)+4h^2S_n^{(\mathrm{iv})}(t_{2k})C_1\left(\frac{x-2kh}{2h}\right).$$

On using (5.1) and (5.2) we have

$$(5.3)\qquad S_n''(x)-f''(x) = (f''(x_{2k})-f''(x))A_0\left(\frac{x-2kh}{2h}\right)+$$

$$+(f''(x_{2k+2})-f''(x))A_2\left(\frac{x-2kh}{2h}\right)+(f''(t_{2k})-f''(x))A_1\left(\frac{x-2kh}{2h}\right)+$$

$$+2hS_n'''(x_{2k}+)B_0\left(\frac{x-2kh}{2h}\right)+2hS_n'''(x_{2k+2}-)B_2\left(\frac{x-2kh}{2h}\right)+$$

$$+4h^2S_n^{(\mathrm{iv})}(t_{2k})C_1\left(\frac{x-2kh}{2h}\right).$$

From (2.2)—(2.7) itt follows that for $0 \leqq t \leqq 1$

$$(5.4)\qquad \begin{cases} |C_1(t)| \leqq \left|C_1\left(\frac{1}{5}\right)\right| = \frac{36}{625}, & |B_2(t)| \leqq \frac{1}{10}, \quad |B_0(t)| \leqq \frac{1}{12} \\ |A_0(t)| \leqq \frac{6}{5}, & |A_1(t)| \leqq \frac{16}{5}, \quad |A_2(t)| \leqq 3. \end{cases}$$

On using (5.4), Lemma 4.1, Lemma 4.2 and (5.2) we obtain

$$|S_n''(x)-f''(x)| \leqq \frac{6}{5}\,\omega_2(2h)+3\omega_2(2h)+\frac{16}{5}\,\omega_2(2h)+$$

$$+2h\frac{1}{12}\,\frac{102}{h}\,\omega_2(2h)+\frac{2h}{10}\,\frac{244}{h}\,\omega_2(2h)+4h^2\frac{36}{625}\,\frac{70}{h^2}\,\omega_2(2h) \leqq 89\omega_2(2h).$$

This proves (1.2) for $p=2$. $p=0, 1$ follows on the lines of [8]. This proves Theorem 2.

6. The following lemmas will be needed for the proof of Theorem 3.

LEMMA 6.1. *Let $f\in C^4[0, 1]$ then if we denote*

$$M_{2k} = \max_{i=1,2,\dots,m-1} |f'(x_{2i})-S_n'(x_{2i})|,$$

then we have

$$|M_{2k}| \leqq \frac{272}{195}\,h^3\omega_4\left(\frac{1}{m}\right),$$

where $\omega_4(\cdot)$ is the modulus of continuity of $f^{(\mathrm{iv})}(x)$. Moreover,

$$|S_n'''(x_{2i}+)-f'''(x_{2i})| \leqq 35h\omega_4\left(\frac{1}{m}\right), \quad |S_n'''(x_{2i}-)-f'''(x_{2i})| \leqq 82h\omega_4\left(\frac{1}{m}\right),$$

$$|S_n^{(\mathrm{iv})}(x_{2i}+)-f^{(\mathrm{iv})}(x_{2i})| \leqq 78\omega_4\left(\frac{1}{m}\right), \quad |S_n^{(\mathrm{iv})}(x_{2i}-)-f^{(\mathrm{iv})}(x_{2i})| \leqq 165\omega_4\left(\frac{1}{m}\right),$$

$$|S_n'''(t_{2i})-f'''(t_{2i})| \leqq 13h\omega_4\left(\frac{1}{m}\right), \quad |S_n^{(\mathrm{v})}(t_{2i})| \leqq \frac{47}{h}\,\omega_4\left(\frac{1}{m}\right).$$

The proof of this lemma can be given on the lines of Lemmas 4.1 and 4.2 so we omit the details.

PROOF OF THEOREM 3. Let $x_{2i}<x\leqq x_{2i+2}$. On using (3.1) and (5.1) we obtain

$$S_n'''(x)-f'''(x) = \big(S_n'''(x_{2i}+)-f'''(x)\big)\,A_0\left(\frac{x-x_{2i}}{2h}\right)+$$

$$+\big(S_n'''(x_{2i+2}-)-f'''(x)\big)A_2\left(\frac{x-x_{2i}}{2h}\right)+\big(S_n'''(t_{2i})-f'''(x)\big)A_1\left(\frac{x-x_{2i}}{2h}\right)+$$

$$+2hS_n^{(\mathrm{iv})}(x_{2i}+)\,B_0\left(\frac{x-x_{2i}}{2h}\right)+2hS_n^{(\mathrm{iv})}(x_{2i+2}-)\,B_2\left(\frac{x-x_{2i}}{2h}\right)+4h^2S_n^{(\mathrm{v})}(t_{2i})C_1\left(\frac{x-x_{2i}}{2h}\right).$$

On using (5.4), Lemma (6.1) and

$$S_n'''(x_{2i}+)-f'''(x) = S_n'''(x_{2i}+)-f'''(x_{2i})+f'''(x_{2i})-f'''(x),$$

$$|f'''(x_{2i})-f'''(x)| \leqq 2h\|f^{(\mathrm{iv})}\|$$

we obtain

$$|S_n'''(x)-f'''(x)| \leqq \frac{103}{m}\,\omega_4\left(\frac{1}{m}\right)+\frac{8}{m}\,\Omega$$

where we set $\Omega=\max_{0\leqq x\leqq 1} |f^{(\mathrm{iv})}(x)|$.

On using the usual device

$$S_n''(x)-f''(x)=\int_{t_{2i}}^{x}[S_n'''(t)-f'''(t)]\,dt$$

we obtain the estimate for $\|S_n''(x)-f''(x)\|$. This proves (1.3) for $p=2$. For $p=0,1$ the proof is similar. This completes the proof of Theorem 3 as well.

For (0, 2) interpolation by algebraic polynomials we refer to [2]. P. TURÁN has remarked that (0, 2) interpolation polynomials can be applied to obtain approximate solutions of the differential equation $y''(x)+p(x)y(x)=0$. We will return to this question elsewhere.

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Added in proof (April 12, 1978). In 1976, S. DEMKO made an interesting contribution on the above topic; see his paper "Lacunary polynomial spline interpolation", *SIAM J. Num. Anal.* (1976).

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LACUNARY INTERPOLATION BY SPLINES. II

(0, 4) AND (0, 1, 3) CASES

By

A. K. VARMA

In recent years there has been renewed interest and progress on Hermite—Birkhoff interpolation. The original source for this activity is a work by G. D. BIRKHOFF in 1906 [3], with a notable contribution by G. PÓLYA in 1931. Motivated by these results, P. TURÁN [2] had initiated the study of "lacunary" interpolation on specially chosen abscissas and had shown its relevance to problems of convergence and approximation.

In 1973 MEIR and SHARMA [4] have shown that for arbitrary lacunary data $\{y_i\}_{i=0}^n$, $\{y_i''\}_{i=0}^n$ there exist unique (up to the boundary conditions) quintic splines $S_n(x)\in C^3[0, 1]$ with joints at $\frac{i}{n}$ $(i=0, 1, 2, \dots, n)$ such that $S_n\left(\frac{i}{n}\right)=y_i$, $S_i''\left(\frac{i}{n}\right)=y_i''$. Moreover, if the given data y_i, y_i'' are the values and second derivatives, respectively, of a function f satisfying certain smoothness conditions, then $\|S_n^{(r)}(x)-f^{(r)}(x)\|\to 0$ as $n\to\infty$ for every r, $0\leqq r\leqq 3$. It turns out that in the case of Meir—Sharma interpolant the convergence is of order one less than which is possible through quintic C^3 splines. Moreover, by now it is known that the Meir—Sharma interpolant is less stable than many local piecewise quintic approximants, such as quintic spline interpolants of continuity class $C^4[0, 1]$ (see [9]). In order to obtain more stable interpolation processes satisfying Birkhoff data and where the convergence is of the same order as that of the best approximation by quintic splines we proposed [11] the following:

Let $n=2m+1$, $x_i=\frac{i}{2m}$, $i=0, 2, \dots, 2m$. We shall denote by $S_{n,5}^{(2)}$ the class of quintic splines $S_n(x)$ on $[0, 1]$ having the following two properties:

(1.1) i) $S_n(x)\in C^2[0, 1]$

ii) $S_n(x)$ is quintic in $[x_{2i}, x_{2i+2}]$, $i=0, \dots, m-1$.

THEOREM A (A. K. VARMA). *Let* $n=2m+1$, $x_i=\frac{i}{2m}$, $i=0, 2, \dots, 2m$, $t_{2i}=$ $=x_{2i}+\frac{2}{3}h$, $i=0, 1, \dots, m-1$, $2h=\frac{1}{m}$. *Given arbitrary numbers* $f(x_0), f(x_2), \dots,$ $f(x_{2m})$; $f(t_0), f(t_2), \dots, f(t_{2(m-1)})$; $f''(t_0), f''(t_2), \dots, f''(t_{2(m-1)})$; $f'(x_0), f'(x_{2m})$ *there exists a unique* $S_n(x)\in C^2[0, 1]$ *such that*

$$S_n(x_{2i}) = f(x_{2i}), \quad i = 0, 1, \dots, m,$$
$$S_n(t_{2i}) = f(t_{2i}), \quad S_n''(t_{2i}) = f''(t_{2i}), \quad i = 0, 1, \dots, m-1,$$
$$S_n'(x_0) = f'(x_0), \quad S_n'(x_{2m}) = f'(x_{2m}).$$

THEOREM B. *Let $f\in C^2[0, 1]$ and let $S_n(x)$ be the unique quintic spline satisfying the conditions of Theorem* A. *Then*

$$\|S_n^{(p)}(x)-f^{(p)}(x)\| \leqq 89n^{p-2}\omega_2\left(\frac{1}{m}\right), \quad p = 0, 1, 2.$$

Here $\omega_2(\cdot)$ is the usual modulus of continuity of $f''(x)$. In the case $f\in C^4[0, 1]$ we have

$$\|S_n^{(p)}(x)-f^{(p)}(x)\|_\infty \leqq 103m^{p-4}\omega_4\left(\frac{1}{m}\right)+8m^{p-4}\|f^{(4)}\|_\infty, \quad p = 0, 1, 2, 3,$$

where $\omega_4(\cdot)$ denotes the modulus of continuity of $f^{(\mathrm{iv})}$, and for $p=3$, L_∞ norm is replaced by piecewise L_∞ norm.

The object of this paper is to generalize the above results in the following directions.

THEOREM 1. *Let $n=2m+1$, $x_i=\frac{i}{2m}$, $i=0, 1, \ldots, n-1$ $t_{2i}=x_{2i}+\frac{2}{3}h$, $i=0, 1, \ldots, m-1$, $2mh=1$. Given arbitrary numbers $f(t_0)$, $f(t_2), \ldots, f(t_{2(m-1)})$; $f^{(\mathrm{iv})}(t_0), f^{(\mathrm{iv})}(t_2), \ldots, f^{(\mathrm{iv})}(t_{2(m-1)}), f'(x_0), f'(x_{2m})$ there exists a unique $S_n(x)\in C^2[0, 1]$ such that*

$$(1.2)\qquad \begin{cases} S_n(x_{2i}) = f(x_{2i}), & i = 0, 1, \ldots, m,\\ S_n(t_{2i}) = f(t_{2i}), \quad S_n^{(\mathrm{iv})}(t_{2i}) = f^{(\mathrm{iv})}(t_{2i}), & i = 0, 1, \ldots, m-1,\\ S_n'(x_0) = f'(x_0), \quad S_n'(x_{2m}) = f'(x_{2m}). \end{cases}$$

THEOREM 2. *Let $f\in C^4[0, 1]$ and let $S_n(x)$ be the unique quintic spline satisfying the conditions of Theorem* 1. *Then* $(n=2m+1)$

$$(1.3)\qquad \|S_n^{(p)}(x)-f^{(p)}(x)\|_\infty \leqq 2m^{p-4}\omega_4\left(\frac{1}{m}\right)+4m^{p-4}\|f^{(4)}\|_\infty, \quad p = 0, 1, 2, 3.$$

THEOREM 3. *Let $n=2m+1$, $x_i=\frac{i}{2m}$, $i=0, 2, \ldots, n-1$, $t_{2i}=x_{2i}+\frac{2}{3}h$, $i=0, 1, \ldots, m-1$, $2mh=1$. Given arbitrary numbers $f(x_0)$, $f'(x_2), \ldots, f(x_{2m})$; $f(t_0), f(t_2), \ldots, f(t_{2(m-1)})$; $f'(t_0), f'(t_2), \ldots, f'(t_{2(m-1)})$; $f'''(t_0), f'''(t_2), \ldots, f'''(t_{2(m-1)})$; $f'(x_0)$, $f'(x_{2m})$ there exists a unique $Q_n(x)\in C^2[0, 1]$ such that*

$$(1.4)\qquad \begin{cases} Q_n(x_{2i}) = f(x_{2i}), & i = 0, 1, \ldots, m,\\ Q_n^{(p)}(t_{2i}) = f^{(p)}(t_{2i}), & i = 0, 1, \ldots, m-1, \quad p = 0, 1, 3,\\ Q_n'(x_0) = f'(x_0), \quad Q_n'(x_{2m}) = f'(x_{2m}), \end{cases}$$

and $Q_n(x)$ is a polynomial of degree six in $[x_{2i}, x_{2i+2}]$, $i=0, 1, \ldots, m-1$.

THEOREM 4. *Let $f\in C^4[0, 1]$ and let $Q_n(x)$ be the unique spline (of degree six in each piece) satisfying the conditions of Theorem* 3. *Then* $(n=2m+1)$

$$(1.5)\qquad \|Q_n^{(p)}(x)-f^{(p)}(x)\|_\infty \leqq 250m^{p-4}\omega_4\left(\frac{1}{m}\right)+4m^{p-4}\|f^{(\mathrm{iv})}\|_\infty, \quad p = 0, 1, 2, 3.$$

The proof of Theorems 1 and 2 is given in Part A, and that of Theorems 3 and 4 is sketched in Part B. The results of (0, 4) interpolation will be applied in a next paper to the important differential equation

$$y^{(iv)} = f(x, y).$$

We will follow for this purpose a recent paper of I. J. SCHOENBERG [12].

Preliminaries. If $P(x)$ is a quintic on $[0, 1]$ then we have

$$P(x) = P(0)A_0(x)+P\left(\frac{1}{3}\right)A_1(x)+P(1)A_2(x)+P'(0)B_0(x)+ \tag{2.1}$$
$$+P'(1)B_2(x)+P^{(iv)}\left(\frac{1}{3}\right)C_1(x),$$

where

$$C_1(x) = \frac{1}{48}[-3x^5+7x^4-5x^3+x^2] = -\frac{x^2}{48}(1-x)^2(3x-1), \tag{2.2}$$

$$B_2(x) = \frac{9x^5-15x^4+7x^3-x^2}{4} = \frac{x^2(x-1)}{4}(3x-1)^2, \tag{2.3}$$

$$B_0(x) = \frac{-9x^5+15x^4-x^3-7x^2+2x}{2} = \frac{-x(x-1)^2(3x-1)(2+3x)}{2}, \tag{2.4}$$

$$A_2(x) = \frac{-63x^5+105x^4-37x^3+3x^2}{8} = \frac{-x^2(3x-1)(21x^2-28x+3)}{8}, \tag{2.5}$$

$$A_1(x) = \frac{81}{8}[3x^5-5x^4+x^3+x^2] = \frac{81}{8}x^2(x-1)^2(3x+1), \tag{2.6}$$

$$A_0(x) = \frac{-45x^5+75x^4-11x^3-21x^2+2}{2} = \frac{-(x-1)^2(3x-1)(15x^2+10x+2)}{2}. \tag{2.7}$$

For later references we note that

$$\tag{2.8} \begin{cases} C_1''(0) = \frac{1}{24}, & C_1'''(0) = -\frac{5}{8}, & C_1^{(iv)}(0) = \frac{7}{2}, & C_1'''\left(\frac{1}{3}\right) = \frac{1}{8}, \\ C_1''(1) = -\frac{1}{12}, & C_1'''(1) = -\frac{7}{8}, & C_1^{(iv)}(1) = -4, & \\ B_2''(0) = -\frac{1}{2}, & B_2'''(0) = \frac{21}{2}, & B_2^{(iv)}(0) = -90, & B_2'''\left(\frac{1}{3}\right) = -\frac{9}{2}, \\ B_2''(1) = 10, & B_2'''(1) = \frac{111}{2}, & B_2^{(iv)}(1) = 180, & \\ B_0''(0) = -7, & B_0'''(0) = -3, & B_0^{(iv)}(0) = 180, & B_0'''\left(\frac{1}{3}\right) = 27, \\ B_0''(1) = -10, & B_0'''(1) = -93, & B_0^{(iv)}(1) = -360, & \end{cases}$$

$$(2.8)\quad \begin{cases} A_2''(0)=\dfrac{3}{4}, & A_2'''(0)=-\dfrac{111}{4}, \quad A_2^{(\mathrm{iv})}(0)=315, \quad A_2'''\left(\dfrac{1}{3}\right)=\dfrac{99}{4}, \\ A_2''(1)=-27, & A_2'''(1)=-\dfrac{741}{4}, \quad A_2^{(\mathrm{iv})}(1)=630, \\ A_1''(0)=\dfrac{81}{4}, & A_1'''(0)=\dfrac{243}{4}, \quad A_1^{(\mathrm{iv})}(0)=-1215, \; A_1'''\left(\dfrac{1}{3}\right)=-\dfrac{567}{4}, \\ A_1''(1)=81, & A_1'''(1)=\dfrac{2673}{4}, \quad A_1^{(\mathrm{iv})}(1)=2430, \\ A_0''(0)=-21, & A_1'''(0)=-33, \quad A_0^{(\mathrm{iv})}(0)=900, \quad A_0'''\left(\dfrac{1}{3}\right)=117, \\ A_0''(1)=-54, & A_0'''(1)=-483, \quad A_0^{(\mathrm{iv})}(1)=-1800. \end{cases}$$

For $f\in C^4[0,1]$ we have

$$f(x_{2i+2})=f(x_{2i})+2hf'(x_{2i})+2h^2f''(x_{2i})+\frac{4}{3}h^3f'''(x_{2i})+\frac{2}{3}h^4f^{(\mathrm{iv})}(\eta_{1,2i}),$$

$$x_{2i}<\eta_{1,2i}<x_{2i+2},$$

$$f(x_{2i-2})=f(x_{2i})-2hf'(x_{2i})+2h^2f''(x_{2i})-\frac{4}{3}h^3f'''(x_{2i})+\frac{2}{3}h^4f^{(\mathrm{iv})}(\eta_{2,2i}),$$

$$x_{2i-2}<\eta_{2,2i}<x_{2i},$$

$$f(t_{2i})=f(x_{2i})+\frac{2}{3}hf'(x_{2i})+\frac{2}{9}h^2f''(x_{2i})+\frac{4}{81}h^3f'''(x_{2i})+\frac{2}{243}h^4f^{(\mathrm{iv})}(\eta_{3,2i}),$$

$$x_{2i}<\eta_{3,2i}<t_{2i},$$

$$f(t_{2i-2})=f(x_{2i})-\frac{4}{3}hf'(x_{2i})+\frac{8}{9}h^2f''(x_{2i})-\frac{32}{81}h^3f'''(x_{2i})+\frac{32}{243}h^4f^{(\mathrm{iv})}(\eta_{4,2i}),$$

$$t_{2i-2}<\eta_{4,2i}<x_{2i},$$

$$f'(x_{2i+2})=f'(x_{2i})+2hf''(x_{2i})+2h^2f'''(x_{2i})+\frac{4}{3}h^3f^{(\mathrm{iv})}(\eta_{5,2i}),$$

$$x_{2i}<\eta_{5,2i}<x_{2i+2},$$

$$f'(x_{2i-2})=f'(x_{2i})-2hf''(x_{2i})+2h^2f'''(x_{2i})-\frac{4}{3}h^3f^{(\mathrm{iv})}(\eta_{6,2i}),$$

$$x_{2i-2}<\eta_{6,2i}<x_{2i}.$$

PART A

3. Proof of Theorem 1. The proof of Theorem 1 depends on the following representation of $S_n(x)$. For $2ih\leqq x\leqq(2i+2)h$, $i=0,1,\dots,m-1$, we have

$$(3.1)\quad S_n(x)=f(x_{2i})A_0\left(\frac{x-2ih}{2h}\right)+f(x_{2i+2})A_2\left(\frac{x-2ih}{2h}\right)+f(t_{2i})A_1\left(\frac{x-2ih}{2h}\right)+$$
$$+2hS_n'(x_{2i})B_0\left(\frac{x-2ih}{2h}\right)+2hS_n'(x_{2i+2})B_2\left(\frac{x-2ih}{2h}\right)+16h^4f^{(\mathrm{iv})}(t_{2i})C_1\left(\frac{x-2ih}{2h}\right).$$

On using (2.1) and

$$S_n'(0) = f'(0), \quad S_n'(1) = f'(1), \tag{3.2}$$

we note that $S_n(x)$ as given above satisfies (1.1) and is quintic in $[x_{2i}, x_{2i+2}]$ for $i=0, 1, \ldots, m-1$. We need to decide whether it is possible to determine $S_n'(x_{2i})$ for $i=1, 2, \ldots, m-1$ uniquely. For this purpose we use the fact that $S_n(x) \in C^2[0, 1]$ and therefore

$$S_n'(x_{2i}+) = S_n'(x_{2i}-), \quad i = 1, 2, \ldots, m-1 \tag{3.3}$$

must hold. Thus on using (3.1), (3.3) and (2.8) we obtain

$$\begin{aligned} hS_n'(x_{2i+2}) + 34hS_n'(x_{2i})h - 20hS_n'(x_{2i-2}) = \\ = 6f(x_{2i}) + \frac{3}{4}f(x_{2i+2}) + \frac{81}{4}f(t_{2i}) + 54f(x_{2i-2}) - \\ -81f(t_{2i-2}) + \frac{2}{3}h^4 f^{(\mathrm{iv})}(t_{2i}) + \frac{4}{3}h^4 f^{(\mathrm{iv})}(t_{2i-2}). \end{aligned} \tag{3.4}$$

But (3.4) is a strictly tridiagonal dominant system. Thus $S_n'(x_{2i})$, $i=1, 2, \ldots,$ $m-1$ can be obtained uniquely by the system (3.4). This proves Theorem 1.

4. Estimates. For the proof of Theorem 2 we need the following

LEMMA 1. *Let us denote*

$$M_{2i} = |f'(x_{2i}) - S_n'(x_{2i})|,$$

then

$$\max_{i=1,2,\ldots,m-1} M_{2i} \leqq \frac{116}{39} h^3 \omega_4\left(\frac{1}{m}\right), \tag{4.1}$$

where $\omega_4(\cdot)$ *is the modulus of continuity of* $f^{(\mathrm{iv})}(x)$.

PROOF. From (3.4) and (2.9) it follows that

$$\begin{aligned} 20h[S_n'(x_{2i-2}) - f'(x_{2i-2})] - 34h[S_n'(x_{2i}) - f'(x_{2i})] - h[S_n'(x_{2i+2}) - f'(x_{2i+2})] = \\ = \Big[-\frac{1}{2}f^{(\mathrm{iv})}(\eta_{1,2i}) - \frac{1}{6}f^{(\mathrm{iv})}(\eta_{3,2i}) - 36f^{(\mathrm{iv})}(\eta_{2,2i}) + \\ + \frac{32}{3}f^{(\mathrm{iv})}(\eta_{4,2i}) - \frac{2}{3}f^{(\mathrm{iv})}(t_{2i}) - \frac{4}{3}f^{(\mathrm{iv})}(t_{2i-2}) + \\ + \frac{80}{3}f^{(\mathrm{iv})}(\eta_{6,2i}) + \frac{4}{3}f^{(\mathrm{iv})}(\eta_{5,2i})\Big] h^4 = \frac{116}{3} h^4 \theta_0 \omega_4\left(\frac{1}{m}\right), \quad |\theta_0| \leqq 1. \end{aligned}$$

Now, on using the properties of diagonal dominance, (4.1) follows immediately.

LEMMA 2. *Let $f \in C^4[0, 1]$ then*

$$|S_n'''(x_{2i}+)-f'''(x_{2i})| \leqq 16h\omega_4\left(\frac{1}{m}\right), \tag{4.2}$$

$$|S_n'''(x_{2i}-)-f'''(x_{2i})| \leqq 66h\omega_4\left(\frac{1}{m}\right), \tag{4.3}$$

$$|S_n'''(t_{2i})-f'''(t_{2i})| \leqq 32h\omega_4\left(\frac{1}{m}\right), \tag{4.4}$$

where $\omega_4(\cdot)$ is the modulus of continuity of $f^{(\mathrm{iv})}$.

PROOF. From (2.8), (2.9) and (3.1) we have

$$8h^3 S_n'''(x_{2i}+) = -33f(x_{2i})-\frac{111}{4}f(x_{2i+2})+\frac{243}{4}f(t_{2i})-$$
$$-6hS_n'(x_{2i})+21hS_n'(x_{2i+2})-10h^4f^{(\mathrm{iv})}(t_{2i}).$$

Hence

$$8h^3(S_n'''(x_{2i}+)-f'''(x_{2i})) = \Big(-\frac{37}{2}f^{(\mathrm{iv})}(\eta_{1,2i})+\frac{1}{2}f^{(\mathrm{iv})}(\eta_{3,2i})-$$
$$-10f^{(\mathrm{iv})}(t_{2i})+28f^{(\mathrm{iv})}(\eta_{5,2i})\Big)h^4-6h[S_n'(x_{2i})-f'(x_{2i})]+21h[S_n'(x_{2i+2})-f'(x_{2i+2})] =$$
$$= \frac{57}{2}h^4\omega_4\left(\frac{1}{m}\right)\theta_1+21h[S_n'(x_{2i+2})-f'(x_{2i+2})]-6h[S_n'(x_{2i})-f'(x_{2i})], \quad |\theta_1| \leqq 1.$$

Now, on using Lemma 1, (4.2) follows.

The proofs of (4.3) and (4.4) are similar and so we omit the details. We mention only that

$$8h^3 S_n'''(x_{2i}-) = -483f(x_{2i-2})-\frac{741}{4}f(x_{2i})+\frac{2673}{4}f(t_{2i-2})-$$
$$-186hS_n'(x_{2i-2})+111hS_n'(x_{2i})-14h^4f^{(\mathrm{iv})}(t_{2i-2}),$$

and

$$8h^3 S_n'''(x_{2i}-) = 117f(x_{2i})-\frac{111}{4}f(x_{2i+2})-\frac{567}{4}f(t_{2i})+$$
$$+54hS_n'(x_{2i})+9hS_n'(x_{2i+2})+2h^4f^{(\mathrm{iv})}(t_{2i}).$$

LEMMA 3. *Let $f \in C^4[0, 1]$ then we have*

$$|S_n^{(\mathrm{iv})}(x_{2i}+)-f^{(\mathrm{iv})}(x_{2i})| \leqq 136\omega_4\left(\frac{1}{m}\right), \tag{4.5}$$

$$|S_n^{(\mathrm{iv})}(x_{2i}-)-f^{(\mathrm{iv})}(x_{2i}-)| \leqq 283\omega_4\left(\frac{1}{m}\right), \tag{4.6}$$

where $\omega_4(\cdot)$ is the modulus of continuity of $f^{(\mathrm{iv})}$.

PROOF. From (2.8), (2.9) and (3.1) we have

$$16S_n^{(\mathrm{iv})}(x_{2i}+) = 900f(x_{2i})-1215f(t_{2i})+315f(x_{2i+2})+$$
$$+360hS_n'(x_{2i})-180hS_n'(x_{2i+2})+56h^4f^{(\mathrm{iv})}(t_{2i}).$$

Therefore

$$16[S_n^{(\mathrm{iv})}(x_{2i}+)-f^{(\mathrm{iv})}(x_{2i})]h^4 =$$
$$= [-10f^{(\mathrm{iv})}(\eta_{3,2i})+210f^{(\mathrm{iv})}(\eta_{1,2i})-240f^{(\mathrm{iv})}(\eta_{3,2i})-16f^{(\mathrm{iv})}(x_{2i})+$$
$$+56f^{(\mathrm{iv})}(t_{2i})]h^4-180h[S_n'(x_{2i+2})-f'(x_{2i+2})]+360h[S_n'(x_{2i})-f'(x_{2i})].$$

On using Lemma 1, the definition of modulus of continuity, (4.5) follows. The proof of (4.6) is similar and we omit the details. We mention only

$$16^4S_n^{(\mathrm{iv})}(x_{2i}-) = -1800f(x_{2i-2})-630f(x_{2i})+2430f(t_{2i-2})-$$
$$-720hS_n'(x_{2i-2})+360hS_n'(x_{2i})-64h^4f^{(\mathrm{iv})}(t_{2i-2}).$$

5. Proof of Theorem 2. Since $0\leqq t\leqq 1$, we have

$$A_0(t)+A_1(t)+A_2(t)\equiv 1. \tag{5.1}$$

Let $x_{2i}<x<x_{2i+2}$ then on using (5.1) and (3.1) we obtain

$$S_n'''(x)-f'''(x) = \tag{5.2}$$
$$= \big(S_n'''(x_{2i}+)-f'''(x)\big)A_0\left(\frac{x-x_{2i}}{2h}\right)+\big(S_n'''(x_{2i+2}-)-f'''(x)\big)A_2\left(\frac{x-x_{2i}}{2h}\right)+$$
$$+\big(S_n'''(t_{2i})-f'''(x)\big)A_1\left(\frac{x-x_{2i}}{2h}\right)+2hS_n^{(\mathrm{iv})}(x_{2i}+)B_0\left(\frac{x-x_{2i}}{2h}\right)+$$
$$+2hS_n^{(\mathrm{iv})}(x_{2i+2}-)B_2\left(\frac{x-x_{2i}}{2h}\right)\equiv I_1+I_2+I_3+I_4+I_5.$$

Since

$$f'''(x) = f'''(x_{2i})+(x-x_{2i})f^{(\mathrm{iv})}(\eta_{7,2i})\quad (x_{2i}<\eta_{7,2i}<x_{2i+2}),$$

from (2.5)—(2.7) it follows that $|A_0(x)|\leqq 1$, $|A_1(x)|\leqq 2$, $|A_2(x)|\leqq 1$. Therefore

$$I_1 = \big(S_n'''(x_{2i}+)-f'''(x)\big)A_0\left(\frac{x-x_{2i}}{2h}\right) =$$
$$= \big(S_n'''(x_{2i}+)-f'''(x_{2i})-(x-x_{2i})f^{(\mathrm{iv})}(\eta_{7,2i})\big)A_0\left(\frac{x-x_{2i}}{2h}\right).$$

Hence, on using (4.2), $|x-x_{2i}|\leqq 2h$ and $\Omega=\|f^{(\mathrm{iv})}\|=\max_{0\leqq x\leqq 1}|f^{(\mathrm{iv})}(x)|$ we obtain

$$|I_1|\leqq\left(16h\omega_4\left(\frac{1}{m}\right)+2h\Omega\right)1. \tag{5.3}$$

Similarly,

$$|I_2|\leqq 66h\omega_4\left(\frac{1}{m}\right)+2h\Omega, \tag{5.4}$$

and

$$(5.5)\qquad |I_3| \leqq \left(32h\omega_4\left(\frac{1}{m}\right)+\frac{4}{3}h\Omega\right)2 \leqq 64h\omega_4\left(\frac{1}{m}\right)+\frac{8}{3}h\Omega.$$

Since $|B_0(x)|\leqq\frac{1}{7}$, $|B_2(x)|\leqq\frac{1}{15}$ $(0\leqq x\leqq 1)$, on using Lemma 3 we obtain

$$I_4 = 2hS_n^{(\mathrm{iv})}(x_{2i}+)\,B_0\left(\frac{x-x_{2i}}{2h}\right) = 2h\left(S_n^{(\mathrm{iv})}(x_{2i}+)-f^{(\mathrm{iv})}(x_{2i})+f^{(\mathrm{iv})}(x_{2i})\right)B_0\left(\frac{x-x_{2i}}{2h}\right).$$

Hence

$$(5.6)\qquad |I_4| \leqq 2h\left(136\omega_4\left(\frac{1}{m}\right)+\Omega\right)\frac{1}{7} \leqq 39h\omega_4\left(\frac{1}{m}\right)+\frac{2}{7}h\Omega.$$

Similarly

$$(5.7)\qquad |I_5| \leqq \frac{2h}{15}\left(283\omega_4\left(\frac{1}{m}\right)+\Omega\right) \leqq 38h\omega_4\left(\frac{1}{m}\right)+\frac{2h}{15}\Omega.$$

Therefore by (5.2)—(5.7) we obtain

$$\|S_n'''(x)-f'''(x)\|_\infty \leqq 223h\omega_4\left(\frac{1}{m}\right)+8h\Omega.$$

This proves (1.3) for $p=3$. The proof of (1.3) for $p=0, 1, 2$ follows immediately on the lines of [8].

PART B

6. The following results are needed for the proof of Theorems 3 and 4. If $P(x)$ is a polynomial of degree six on [0, 1], then we have

$$(6.1)\qquad P(x) = P(0)\,u_0(x)+P\left(\frac{1}{3}\right)u_1(x)+P(1)\,u_2(x)+P'(0)\,v_0(x)+$$
$$+P'\left(\frac{1}{3}\right)v_1(x)+P'(1)\,v_2(x)+P'''\left(\frac{1}{3}\right)w_1(x),$$

where

$$(6.2)\qquad w_1(x) = \frac{9x^6-24x^5+22x^4-8x^3+x^2}{8} = \frac{x^2(3x-1)^2(1-x)^2}{8},$$

$$(6.3)\qquad v_0(x) = 9x^5-24x^4+22x^3-8x^2+x = x(1-x)^2(3x-1)^2,$$

$$(6.4)\quad v_1(x) = \frac{27}{16}[27x^6-60x^5+38x^4-4x^3-x^2] = \frac{27}{16}[x^2(3x-1)(1-x)^2(9x+1)],$$

(6.5) $v_2(x) = \frac{81x^6-180x^5+138x^4-44x^3+5x^2}{16} = \frac{(x-1)(9x-5)x^2(3x-1)^2}{16}$,

(6.6) $u_0(x) = 81x^6-144x^5+15x^4+80x^3-33x^2+1 = (3x-1)^2(1-x)^2(9x^2+9x+1)$,

(6.7) $u_1(x) = \frac{81}{16}[-9x^6+12x^5+10x^4-20x^3+7x^2] = \frac{81}{16}[x^2(1-x)^2(7-6x-9x^2)]$,

(6.8)

$$u_2(x) = \frac{-567x^6+1332x^5-1050x^4+340x^3-39x^2}{16} = -\frac{x^2(3x-1)^2(63x^2-106x+39)}{16}.$$

We also need

$$(6.9)\quad \begin{cases} w_1''(0) = \frac{1}{4}, & w_1'''(0) = -6, & w_1^{(iv)}(0) = 66, & w_1^{(iv)}\left(\frac{1}{3}\right) = -9, \\ v_0''(0) = -16, & v_0'''(0) = 132, & v_0^{(iv)}(0) = -576, & v_0^{(iv)}\left(\frac{1}{3}\right) = -216, \\ v_1''(0) = -\frac{27}{8}, & v_1'''(0) = -\frac{81}{2}, & v_1^{(iv)}(0) = 1539, & v_1^{(iv)}\left(\frac{1}{3}\right) = -\frac{1377}{2}, \\ v_2''(0) = \frac{5}{8}, & v_2'''(0) = -\frac{33}{2}, & v_2^{(iv)}(0) = 207, & v_2^{(iv)}\left(\frac{1}{3}\right) = -\frac{81}{2}, \\ u_0''(0) = -66, & u_0'''(0) = 480, & u_0^{(iv)}(0) = 360, & u_0^{(iv)}\left(\frac{1}{3}\right) = -2160, \\ u_1''(0) = \frac{567}{8}, & u_1'''(0) = -\frac{1215}{2}, & u_1^{(iv)}(0) = 1215, & u_1^{(iv)}\left(\frac{1}{3}\right) = \frac{3645}{2}, \\ u_2''(0) = -\frac{39}{8}, & u_2'''(0) = \frac{255}{2}, & u_2^{(iv)}(0) = -1575, & u_2^{(iv)}\left(\frac{1}{3}\right) = \frac{675}{2}, \\ w_1''(1) = 1, & w_1'''(1) = 15, & w_1^{(iv)}(1) = 111, & \\ v_0''(1) = 8, & v_0'''(1) = 96, & v_0^{(iv)}(1) = 504, & \\ v_1''(1) = \frac{135}{2}, & v_1'''(1) = 891, & v_1^{(iv)}(1) = \frac{11583}{2}, & \\ v_2''(1) = \frac{29}{2}, & v_2'''(1) = 123, & v_2^{(iv)}(1) = \frac{1359}{2}, & \\ u_0''(1) = 144, & u_0'''(1) = 1920, & u_0^{(iv)}(1) = 12240, & \\ u_1''(1) = -81, & u_1'''(1) = -1215, & u_1^{(iv)}(1) = -\frac{15795}{2}, & \\ u_2''(1) = -63, & u_2'''(1) = -705, & u_2^{(iv)}(1) = -\frac{8685}{2}. & \end{cases}$$

7. Proof of Theorem 3. We need to prove that there exists a unique $Q_n(x) \in S_{n,6}^{(2)}$ satisfying the requirement of Theorem 3. Let $2ih \leq x \leq (2i+2)h$, $i = 0, 1, \ldots, m-1$

and we aim to express

$$Q_n(x) = f(x_{2i})u_0\left(\frac{x-2ih}{2h}\right)+f(x_{2i+2})u_2\left(\frac{x-2ih}{2h}\right)+f(t_{2i})u_1\left(\frac{x-2ih}{2h}\right)+$$

$$+2hQ_n'(x_{2i})v_0\left(\frac{x-2ih}{2h}\right)+2hf'(t_{2i})v_1\left(\frac{x-2ih}{2h}\right)+2hQ_n'(x_{2i+2})v_2\left(\frac{x-2ih}{2h}\right)+$$

$$+8h^3f'''(t_{2i})\omega_1\left(\frac{x-2ih}{2h}\right).$$

On using $Q_n'(0)=f'(0)$, $Q_n'(1)=f'(1)$, and the fact that $Q_n(x)\in C^2[0, 1]$ we obtain $Q_n''(x_{2i}+)=Q_n''(x_{2i}-)$, $i=1, 2, \dots, m-1$. Therefore

$$16hQ_n'(x_{2i-2})+61hQ_n'(x_{2i})-\frac{5}{4}Q_n'(x_{2i+2}) =$$

$$=-3f(x_{2i})+\frac{567}{8}f(t_{2i})-\frac{39}{8}f(x_{2i+2})-144f(x_{2i-2})+81f(t_{2i-2})-$$

$$-\frac{27}{4}hf'(t_{2i})-135hf'(t_{2i-2})+2h^3f'''(t_{2i})-8h^3f'''(t_{2i-2}).$$

But (6.11) is a strictly tridiagonal dominant system and therefore $Q_n'(x_{2i})$, $i=1, 2, \dots, m-1$ can be obtained uniquely. This in turn implies that $Q_n(x)$ can be determined uniquely satisfying the conditions of Theorem 3.

8. Estimates. The proof of Theorem 4 depends on the following

LEMMA 3. *Let* $f\in C^4[0, 1]$ *and we set*

$$N_{2i} = |f'(x_{2i})-Q_n'(x_{2i})|$$

then

$$\max_{i=1,2,\dots,m-1} N_{2i} \leqq \frac{396}{175}h^3\omega_4\left(\frac{1}{m}\right).$$

Also

$$|Q_n'''(x_{2i}+)-f'''(x_{2i})| \leqq 96h\omega_4\left(\frac{1}{m}\right),\quad |Q_n'''(x_{2i}-)-f'''(x_{2i})| \leqq 286h\omega_4\left(\frac{1}{m}\right),$$

$$|Q_n^{(\mathrm{iv})}(x_{2i}+)-f^{(\mathrm{iv})}(x_{2i})| \leqq 180\omega_4\left(\frac{1}{m}\right),\quad |Q_n^{(\mathrm{iv})}(x_{2i}-)-f^{(\mathrm{iv})}(x_{2i})| \leqq 848\omega_4\left(\frac{1}{m}\right),$$

$$|Q_n^{(\mathrm{iv})}(t_{2i})-f^{(\mathrm{iv})}(t_{2i})| \leqq 65\omega_4\left(\frac{1}{m}\right),\quad |Q_n^{(\mathrm{iv})}(t_{2i})| \leqq 270h^{-2}\omega_4\left(\frac{1}{m}\right).$$

The proof of this lemma is on the lines of Lemmas 1 and 2. The proof of Theorem 4 is similar to that of Theorem 2. Here one uses Lemma 3. So we omit the details. Other relevant papers are [1], [6], [7], and [10].

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ON CENTRAL LIMIT THEOREMS FOR MARTINGALE TRIANGULAR ARRAYS

By

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It is the aim of the present paper to give a most straightforward elementary and selfcontained presentation of some results on central limit theorems for martingale triangular arrays which cover the main theorems of B. M. Brown and A. Dvoretzky as well as the Lindeberg—Lévy theorem for martingales proved by Billingsley and based on the work of Lévy. Our survey relies on the paper of B. M. Brown which leads to Theorem 2 as the main result of the present paper from which the others will be derived in a rather direct way (including a random central limit theorem for martingales).

1. Introduction

Let $(x_{nk})_{1\leqq k\leqq k_n, n\in\mathbf{N}}$ be a triangular array (TA) of integrable random variables defined on a common probability space (p-space) $(\Omega, \mathscr{A}, \mathbf{P})$ and let $(\mathscr{F}_{nk})_{0\leqq k\leqq k_n, n\in\mathbf{N}}$ be a given array of sub-σ-fields of $\mathscr{A}$ such that x_{nk} is $\mathscr{F}_{nk}$-measurable and $\mathscr{F}_{nk}$ is monotone increasing in k for every n. Then (x_{nk}) is called a *martingale triangular array* (MTA) if $\mathbf{E}(x_{nk}|\mathscr{F}_{n,k-1})=0$ almost surely (a.s.) for $k=1,2,\dots,k_n$ and every $n\in\mathbf{N}$. Putting $S_n=\sum_{k=1}^{k_n} x_{nk}$, we want to find conditions on a MTA (x_{nk}) which imply that $S_n \xrightarrow{\mathscr{L}} N(0,1)$, i.e.

$$\lim_{n\to\infty} \mathbf{P}(S_n\leqq\lambda)=(2\pi)^{-1/2}\int_{-\infty}^{\lambda} e^{-x^2/2}\,dx \quad \text{for all} \quad \lambda\in\mathbf{R}.$$

If not specified otherwise it is assumed that the random variables x_{nk} have finite second moments. As we know from the work of DVORETZKY ([7], [8]) the following so called *Conditioned Lindeberg Condition* (CLC) plays an essential role:

$$\text{(CLC)}\qquad \sum_{k=1}^{k_n}\mathbf{E}\big(x_{nk}^2 I(|x_{nk}|>\varepsilon)\,|\,\mathscr{F}_{n,k-1}\big)\xrightarrow[\mathbf{P}]{}0 \quad \text{as } n\to\infty \text{ for every } \varepsilon>0.$$

(Here $I(\cdot)$ denotes the indicator function of the set within the parenthesis and $\xrightarrow[\mathbf{P}]{}$ stands for convergence in $\mathbf{P}$-measure.)

Compared with the classical *Lindeberg Condition* (LC)

$$\text{(LC)}\qquad \sum_{k=1}^{k_n}\mathbf{E}\big(x_{nk}^2 I(|x_{nk}|>\varepsilon)\big)\to 0 \quad \text{as } n\to\infty \text{ for every } \varepsilon>0,$$

(CLC) is obviously weaker than (LC), since for $g_{nk} \geqq 0$

$$\sum \mathbf{E}(g_{nk}) = \sum \mathbf{E}(\mathbf{E}(g_{nk}|\mathscr{F}_{n,k-1})) = \mathbf{E}(\sum \mathbf{E}(g_{nk}|\mathscr{F}_{n,k-1})) \to 0$$

implies that $\sum \mathbf{E}(g_{nk}|\mathscr{F}_{n,k-1}) \xrightarrow[\mathbf{P}]{} 0$. Here and in the following we usually drop the summation indices if the summation (or maximization) is understood with respect to $1 \leqq k \leqq k_n$; convergence means always convergence as $n \to \infty$.

That (CLC) is in general strictly weaker than (LC) can be seen from the following simple example:

Let $(\Omega, \mathscr{A}, \mathbf{P}) = ([0, 1], \mathscr{B} \cap [0, 1], \lambda)$ (λ = Lebesgue measure on [0, 1]) and consider $(x_{nk})_{1 \leqq k \leqq n, n \in \mathbf{N}}$ defined by

$$x_{11} = I(A_1),\ x_{21} = x_{22} = I(A_2),\ \ldots,\ x_{n1} = x_{n2} = \ldots = x_{nn} = I(A_n)$$

with $A_n = \left]0, \frac{1}{n}\right[$, $n \in \mathbf{N}$; putting $\mathscr{F}_{nk} = \sigma(\{x_{n1}, x_{n2}, \ldots, x_{nk}\})$, $1 \leqq k \leqq n$, $n \in \mathbf{N}$, and $\mathscr{F}_{n0} = \mathscr{F}_{n1}$, we obtain $\sum \mathbf{E}(x_{nk}^2 I(|x_{nk}| > \varepsilon)|\mathscr{F}_{n,k-1}) = \sum I(A_n) = nI(A_n) \xrightarrow[\lambda]{} 0$ for every $\varepsilon > 0$, i.e. (CLC) is satisfied but (LC) is not, because

$$\sum \mathbf{E}(x_{nk}^2 I(|x_{nk}| > \varepsilon)) = \sum \mathbf{E}(I(A_n)) = n\lambda(A_n) \equiv 1 \qquad (0 < \varepsilon < 1).$$

Note that in the present example, the following condition (C) occuring in the next theorem is not fulfilled:

(C) $$\sum \mathbf{E}(x_{nk}^2|\mathscr{F}_{n,k-1}) \xrightarrow[\mathbf{P}]{} 1.$$

This theorem gives a characterization of (CLC) stated without proof. The proof can be carried through by standard techniques (using e.g. PRATT's Lemma (cf. [10])); for some of the details the reader is referred to BROWN [4] and SCOTT [11].

THEOREM 1. *If, for a* $TA(x_{nk})$, $\mathbf{E}\left(\sum_{k=1}^{k_n} x_{nk}^2\right) \to 1$ *as* $n \to \infty$, *then the following assertions* (i)—(iv) *are equivalent:*

(i) (x_{nk}) *fulfils* (C) *and* (CLC)
(ii) (x_{nk}) *fulfils* (C) *and* (LC)
(iii) (x_{nk}) *fulfils* (C) *and* (CUC)
(iv) (x_{nk}) *fulfils* (C) *and* (UC),

where (CUC) *states that for any continuous nonnegative function* $U(x)$ *of bounded variation on* $[0, \infty)$, *for which* $U(0) = 0$ *and* $U(x) \to \text{const}\ (>0)$ *as* $x \to \infty$,

$$\sum_{k=1}^{k_n} \mathbf{E}\left(x_{nk}^2 U\left(\frac{|x_{nk}|}{\varepsilon}\right) \Big| \mathscr{F}_{n,k-1}\right) \xrightarrow[\mathbf{P}]{} 0 \quad \textit{as} \quad n \to \infty \textit{ for every } \varepsilon > 0,$$

and where (UC) *stands in the same relation to* (CUC) *as* (LC) *does with respect to* (CLC).

2. The main results of Brown and Dvoretzky

THEOREM 2 (cf. BROWN [4], Theorem 2). *Let* (x_{nk}) *be a MTA for which* (i) *of Theorem* 1 *holds, i.e.*

(a) $\sum_{k=1}^{k_n} \mathbf{E}(x_{nk}^2|\mathcal{F}_{n,k-1}) \xrightarrow[\mathbf{P}]{} 1$ *as* $n \to \infty$, *and*

(b) $\sum_{k=1}^{k_n} \mathbf{E}\big(x_{nk}^2 I(|x_{nk}| > \varepsilon)|\mathcal{F}_{n,k-1}\big) \xrightarrow[\mathbf{P}]{} 0$ *as* $n \to \infty$ *for every* $\varepsilon > 0$.

Then $S_n = \sum_{k=1}^{k_n} x_{nk} \xrightarrow{\mathcal{L}} N(0, 1)$.

The proof of Theorem 2 together with some auxiliary results will be given in Section 5. We want to point out here the idea: Let

$$\sigma_{nk}^2 = \mathbf{E}(x_{nk}^2|\mathcal{F}_{n,k-1}), \quad V_{nk}^2 = \sum_{i=1}^{k} \sigma_{ni}^2, \quad 1 \leq k \leq k_n, \quad \text{and} \quad V_n^2 = V_{nk_n}^2, \; n \in \mathbf{N}.$$

Choose any constant $c > 1$, put $z_{nk} = x_{nk} I(V_{nk}^2 \leq c)$ and $T_n = \sum_{k=1}^{k_n} z_{nk}$. Then one can show that

A) $S_n - T_n \xrightarrow[\mathbf{P}]{} 0$ a.s. $n \to \infty$;

B) $\mathbf{E}(z_{nk}|\mathcal{F}_{n,k-1}) = 0$ a.s. for $k = 1, \dots, k_n$ and every n,

i.e. (z_{nk}) is also a (MTA);

C) $\sum_{k=1}^{k_n} \mathbf{E}\big(z_{nk}^2 I(|z_{nk}| > \varepsilon)|\mathcal{F}_{n,k-1}\big) \xrightarrow[\mathbf{P}]{} 0$ for every $\varepsilon > 0$,

i.e. (z_{nk}) fulfils also (CLC);

D) $W_n^2 := \sum_{k=1}^{k_n} \mathbf{E}(z_{nk}^2|\mathcal{F}_{n,k-1}) \xrightarrow[\mathbf{P}]{} 1$ as $n \to \infty$,

i.e. (z_{nk}) fulfils also (C);

E) $\mathbf{P}(\{W_n^2 \leq c\}) = 1$ for every $n \in \mathbf{N}$ and

F) $\mathbf{E}\left(\sum_{k=1}^{k_n} z_{nk}^2\right) \to 1$ as $n \to \infty$.

Hence it suffices to show that

$$T_n = \sum_{k=1}^{k_n} z_{nk} \xrightarrow{\mathcal{L}} N(0, 1), \tag{1}$$

where (z_{nk}) is a MTA fulfilling E), F) and condition (i) of Theorem 1. Furthermore, according to F) one can use the equivalences stated in Theorem 1, especially "(i)$\Leftrightarrow$(iv)", in order to verify the conditions (a) and (b) of Lemma 2 below (see Section 5).

The next result is up to a slight modification (concerning the choice of $\mathscr{F}_{n,k-1}$) identical with the main theorem 2.2 of DVORETZKY [8]. But as pointed out in Section 3 of [8] there is no loss of generality in assuming the following conditions (a)—(c), which were in fact used by Dvoretzky in proving his theorem.

THEOREM 3 (DVORETZKY [8], Theorem 2.2). *Let* (x_{nk}) *be a TA fulfilling*

(a) $\sum_{k=1}^{k_n} \mathbf{E}(x_{nk}|\mathscr{F}_{n,k-1}) \xrightarrow[\mathbf{P}]{} 0$ *as* $n \to \infty$,

(b) $\sum_{k=1}^{k_n} \{\mathbf{E}(x_{nk}^2|\mathscr{F}_{n,k-1}) - (\mathbf{E}(x_{nk}|\mathscr{F}_{n,k-1}))^2\} \xrightarrow[\mathbf{P}]{} 1$ *as* $n \to \infty$,

(c) $\sum_{k=1}^{k_n} \mathbf{E}(x_{nk}^2 I(|x_{nk}| > \varepsilon)|\mathscr{F}_{n,k-1}) \xrightarrow[\mathbf{P}]{} 0$ *as* $n \to \infty$ *for every* $\varepsilon > 0$.

Then $S_n = \sum_{k=1}^{k_n} x_{nk} \xrightarrow{\mathscr{L}} N(0, 1)$.

The proof of this theorem will be established using Theorem 2. To this extent the following auxiliary result is needed which is an immediate consequence of Lemma 3.3 of DVORETZKY [8], p. 521.

PROPOSITION 1. *Let x be a square integrable random variable on some p-space $(\Omega, \mathscr{A}, \mathbf{P})$, $\mathscr{F}$ a sub-σ-field of $\mathscr{A}$ and $\mu = \mathbf{E}(x|\mathscr{F})$; then for every $\varepsilon > 0$*

$$4\mathbf{E}(x^2 I(|x| > \varepsilon)|\mathscr{F}) \geqq \mathbf{E}((x-\mu)^2 \cdot I(|x-\mu| > 2\varepsilon)|\mathscr{F}). \tag{3}$$

PROOF OF PROPOSITION 1. For every $F \in \mathscr{F}$ we have $x^2 I(F) \in \mathscr{L}_2(\Omega, \mathscr{A}, \mathbf{P})$, $\mathbf{E}(xI(F)|\mathscr{F}) = \mu I(F)$ and therefore it follows from (3.10), p. 521 in [8] (with $xI(F)$ instead of X) that

$$4\mathbf{E}(x^2 I(|x| > \varepsilon) I(F)) \geqq \mathbf{E}((x-\mu)^2 I(|x-\mu| > 2\varepsilon) I(F)),$$

hence

$$4\mathbf{E}[\mathbf{E}(x^2 I(|x| > \varepsilon)|\mathscr{F}) I(F)] \geqq \mathbf{E}[\mathbf{E}((x-\mu)^2 I(|x-\mu| > 2\varepsilon)|\mathscr{F}) I(F)]$$

for every $F \in \mathscr{F}$, which implies (3).

PROOF OF THEOREM 3. Let $y_{nk} = x_{nk} - \mu_{nk}$ with $\mu_{nk} = \mathbf{E}(x_{nk}|\mathscr{F}_{n,k-1})$; then (y_{nk}) is a MTA for which (i) of Theorem 1 holds, since

$$\text{(a}'\text{)} \qquad \sum_{k=1}^{k_n} \mathbf{E}(y_{nk}^2|\mathscr{F}_{n,k-1}) = \sum_{k=1}^{k_n} \{\mathbf{E}(x_{nk}^2|\mathscr{F}_{n,k-1}) - \mu_{nk}^2\} \xrightarrow[\mathbf{P}]{} 1 \quad \text{as} \quad n \to \infty$$

according to (b), and

$$\text{(b}'\text{)} \qquad 0 \leqq \sum_{k=1}^{k_n} \mathbf{E}(y_{nk}^2 I(|y_{nk}| > \varepsilon)|\mathscr{F}_{n,k-1}) =$$

$$= \sum_{k=1}^{k_n} \mathbf{E}((x_{nk} - \mu_{nk})^2 I(|x_{nk} - \mu_{nk}| > \varepsilon)|\mathscr{F}_{n,k-1}) \leqq 4 \sum_{k=1}^{k_n} \mathbf{E}\left(x_{nk}^2 I\left(|x_{nk}| > \frac{\varepsilon}{2}\right)\middle|\mathscr{F}_{n,k-1}\right)$$

by (3), which implies by (c) that (y_{nk}) fulfils also (CLC).

Hence by Theorem 2, $T_n = \sum_{k=1}^{k_n} y_{nk} \xrightarrow{\mathscr{L}} N(0, 1)$ from which the assertion of Theorem 3 follows according to (a) which means that $S_n - T_n \xrightarrow[\mathbf{P}]{} 0$ as $n \to \infty$.

3. The Lindeberg—Lévy theorem for martingales

Let $(x_k)_{k\in\mathbf{N}}$ be a sequence of integrable random variables defined on a common p-space $(\Omega, \mathcal{A}, \mathbf{P})$ and $(\mathcal{F}_k)_{k\in\mathbf{N}\cup\{0\}}$ an increasing sequence of sub-σ-fields of $\mathcal{A}$ such that x_k is $\mathcal{F}_k$-measurable for each $k\in\mathbf{N}$. (x_k) is called a *martingale difference sequence* (MDS) if

$$\mathbf{E}(x_k|\mathcal{F}_{k-1}) = 0 \quad \text{a.s. for every} \quad k\in\mathbf{N}.$$

(x_k) is a MDS iff $\mathbf{E}(x_k|x_1, \ldots, x_{k-1})=0$ a.s. which is in turn equivalent to $\left(S_n=\sum_{k=1}^{n} x_k\right)_{n\in\mathbf{N}}$ being a martingale. We remark also that, in the case of square integrable x_k's, (x_k) is a MDS iff $\mathbf{E}\big(x_{k+1}\varphi\circ(x_1, \ldots, x_k)\big)=0$ for any φ with $x_{k+1}\varphi\circ(x_1, \ldots, x_k)\in\mathcal{L}_1(\Omega, \mathcal{A}, \mathbf{P})$, which implies that in this case the x_k's are pairwise uncorrelated; hence for a MDS (x_k) of (jointly) *normally* distributed random variables the x_k are necessarily independent. On the other hand it is well known that (S_n) forms a martingale, i.e. (x_n) is a MDS if the x_k's are independent and centred at expectations.

Now, given some norming constants $0<s_n^2$, $n\in\mathbf{N}$, put $x_{nk}=s_n^{-1}x_k$, $1\leq k\leq n$ and $\mathcal{F}_{nk}=\mathcal{F}_k$, $0\leq k\leq n$; then (x_{nk}) is a MTA and therefore the results of Section 2 carry over to MDS (x_k). As to the analogon of Theorem 1 one obtains for sequences (x_k) with $s_n^{-2}\mathbf{E}\left(\sum_{k=1}^{n} x_k^2\right)\to 1$ (which is trivially fulfilled if $0<\sum_{k=1}^{n}\mathbf{E}(x_k^2)<\infty$ and if we put $s_n^2=\sum_{k=1}^{n}\mathbf{E}(x_k^2)$) that the assertions (i)$-$(iv) (with $s_n^{-1}x_k$ instead of x_{nk}) are equivalent.

From Theorem 2 we obtain

THEOREM 4. *Let (x_k) be a MDS and $0<s_n^2, n\in\mathbf{N}$, such that*

(a) $s_n^{-2}\sum_{k=1}^{n}\mathbf{E}(x_k^2|\mathcal{F}_{k-1}) \xrightarrow[\mathbf{P}]{} 1$ *as* $n\to\infty$, *and*

(b) $s_n^{-2}\sum_{k=1}^{n}\mathbf{E}\big(x_k^2 I(|x_k|>\varepsilon s_n)|\mathcal{F}_{k-1}\big) \xrightarrow[\mathbf{P}]{} 0$ *for every* $\varepsilon>0$.

Then $S_n=s_n^{-1}\sum_{k=1}^{n} x_k \xrightarrow{\mathcal{L}} N(0, 1)$.

We want to show that the Lindeberg—Lévy—Theorem for martingales proved by BILLINGSLEY [2] can be easily derived from Theorem 4. To this extent we need the following proposition:

PROPOSITION 2. *Let $(x_k)_{k\in\mathbf{N}}$ be a sequence of random variables on some p-space $(\Omega, \mathcal{A}, \mathbf{P})$ and assume that (x_k) is stationary (strict sense). Then $\mathbf{E}\big(f(x_k)|\mathcal{F}_{k-1}\big)$, $k\in\mathbf{N}$, is again stationary (strict sense) for any f with $f(x_k)\in\mathcal{L}_1(\Omega, \mathcal{A}, \mathbf{P})$, where $(\mathcal{F}_k)$ is a suitably chosen increasing sequence of sub-σ-fields of $\mathcal{A}$ such that x_k is $\mathcal{F}_k$-measurable for each $k\in\mathbf{N}$.*

PROOF. In order to prove Proposition 2, we assume a particular representation of the stochastic sequence (x_k) (which causes no loss of generality with respect to later applications), namely that Ω is the coordinate space of all sequences $\omega = (\ldots\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots\ldots)$ of real numbers ω_i, $i \in \mathbf{Z}$, that the x_i's are the coordinate variables of Ω, that $\mathscr{A}$ is the σ-field of events generated by the x_i's, and that $\mathbf{P}$ is the probability induced on $(\Omega, \mathscr{A})$ by the finite dimensional distributions of the x_i's (where the ω-set $\{x_{i_j}(\omega) \in B_j : j=1, \ldots, n\}$ is assigned the probability $\mathbf{P}(\{x_{i_j+h} \in B_j : j=1, \ldots, n\})$ with $h \in \mathbf{N}$ chosen so large that $i_j + h \in \mathbf{N}$, $j=1, \ldots, n$).

If T is the measure preserving and bijective shift transformation from Ω into itself, defined by $(T(\omega))_i = \omega_{i+1}$, $i \in \mathbf{Z}$, then $x_i = x_{i-1} \circ T$ for every $i \in \mathbf{Z}$. Now, let $\mathscr{F}_i$ be the σ-field generated by $\{x_j : -\infty < j \leqq i\}$. Then $T(F_{i-1}) \in \mathscr{F}_{i-2}$ for any $F_{i-1} \in \mathscr{F}_{i-1}$ and therefore we obtain (using that T is measure preserving)

$$\int_{F_{i-1}} \mathbf{E}(f(x_i)|\mathscr{F}_{i-1})\, d\mathbf{P} = \int_{F_{i-1}} f(x_i)\, d\mathbf{P} = \int_{F_{i-1}} f(x_{i-1} \circ T)\, d\mathbf{P} =$$

$$= \int_{T(F_{i-1})} f(x_{i-1})\, d\mathbf{P} = \int_{T(F_{i-1})} \mathbf{E}(f(x_{i-1})|\mathscr{F}_{i-2})\, d\mathbf{P} = \int_{F_{i-1}} \mathbf{E}(f(x_{i-1})|\mathscr{F}_{i-2}) \circ T\, d\mathbf{P}$$

for every $F_{i-1} \in \mathscr{F}_{i-1}$; furthermore $\mathbf{E}(f(x_{i-1})|\mathscr{F}_{i-2}) \circ T$ is $\mathscr{F}_{i-1}$-measurable and therefore $\mathbf{E}(f(x_i)|\mathscr{F}_{i-1}) = \mathbf{E}(f(x_{i-1})|\mathscr{F}_{i-2}) \circ T$ a.s., i.e. $(\mathbf{E}(f(x_i)|\mathscr{F}_{i-1}))_{i \in \mathbf{N}}$ is a stationary stochastic sequence.

THEOREM 5 (Lindeberg—Lévy). *Let (x_k) be a stationary (strict sense), ergodic MDS on some p-space $(\Omega, \mathscr{A}, \mathbf{P})$ with $\mathbf{E}(x_1^2) = 1$. Then*

$$S_n = n^{-1/2} \sum_{k=1}^{n} x_k \xrightarrow{\mathscr{L}} N(0, 1).$$

PROOF. Without loss of generality we can assume the particular representation of the stochastic sequence (x_k) as in Proposition 2. Letting again $\mathscr{F}_i$ be the σ-field generated by $\{x_j : -\infty < j \leqq i\}$ we remark that $\mathbf{E}(x_i|\mathscr{F}_{i-1}) = 0$ a.s. for all $i \in \mathbf{Z}$, hence (x_k) may be thought of as a MDS with respect to $\mathscr{F}_k$, $k \in \mathbf{N}$. By Proposition 2 and Birkhoff's ergodic theorem we obtain

(a) $n^{-1} \sum_{k=1}^{n} \mathbf{E}(x_k^2|\mathscr{F}_{k-1}) \to 1$ a.s. as $n \to \infty$, and

(b) $n^{-1} \sum_{k=1}^{n} \mathbf{E}(x_k^2 I(|x_k| > \varepsilon n^{1/2})|\mathscr{F}_{k-1}) \to 0$ a.s. as $n \to \infty$ for every $\varepsilon > 0$.

To prove (b), let $N \in \mathbf{N}$ be fixed and note that for $n \geqq N$

$$n^{-1} \sum_{k=1}^{n} \mathbf{E}(x_k^2 I(|x_k| > \varepsilon n^{1/2})|\mathscr{F}_{k-1}) \leqq n^{-1} \sum_{k=1}^{n} \mathbf{E}(x_k^2 I(|x_k| > \varepsilon N^{1/2})|\mathscr{F}_{k-1}) \to$$

$$\to \mathbf{E}(x_1^2 I(|x_1| > \varepsilon N^{1/2}))$$

a.s. as $n \to \infty$; further $\mathbf{E}(x_1^2 I(|x_1| > \varepsilon N^{1/2})) \to 0$ as $N \to \infty$, hence (b) must hold. But in view of (a) and (b) the assertion of Theorem 5 is an immediate consequence of Theorem 4.

4. A random central limit theorem for martingales

THEOREM 6. *Let (x_k) be a MDS and suppose that (ν_n) is a sequence of stopping rules fulfilling the following conditions:*
there exist two sequences (k_n) and (s_n) of positive real numbers such that

(a) $\mathbf{P}(\{\nu_n > k_n\}) \to 0$ *as* $n \to \infty$

(b) $s_n^{-2} \sum_{k=1}^{k_n} I(k \leqq \nu_n)\mathbf{E}(x_k^2|\mathscr{F}_{k-1}) \xrightarrow[\mathbf{P}]{} 1$ *as* $n \to \infty$

(c) $s_n^{-2} \sum_{k=1}^{k_n} \mathbf{E}\big(x_k^2 I(|x_k| > \varepsilon s_n)|\mathscr{F}_{k-1}\big) \xrightarrow[\mathbf{P}]{} 0$ *for every* $\varepsilon > 0$.

Then $S_{\nu_n} := s_n^{-1} \sum_{k=1}^{\nu_n} x_k \xrightarrow{\mathscr{L}} N(0, 1)$.

PROOF. For each $n \in \mathbf{N}$ and all $k \in \mathbf{N}$ with $1 \leqq k \leqq k_n$ put

$$x_{nk} = s_n^{-1} I(k \leqq \nu_n) x_k \quad \text{and} \quad T_n = \sum_{k=1}^{k_n} x_{nk}.$$

Since by assumption $\{k \leqq \nu_n\} \in \mathscr{F}_{k-1}$, $k \in \mathbf{N}$, it follows from (b) and (c) that (x_{nk}) is a MTA fulfilling (i) of Theorem 1, whence by Theorem 2 $T_n \xrightarrow{\mathscr{L}} N(0, 1)$ as $n \to \infty$. Thus to prove the assertion of Theorem 6, it suffices to show that $T_n - S_{\nu_n} \xrightarrow[\mathbf{P}]{} 0$ as $n \to \infty$. But

$$T_n - S_{\nu_n} = s_n^{-1}\left[\sum_{k=1}^{k_n} I(k \leqq \nu_n) x_k - \sum_{k=1}^{\infty} I(k \leqq \nu_n) x_k\right],$$

whence by (a)

$$\mathbf{P}(\{T_n - S_{\nu_n} \neq 0\}) \leqq \mathbf{P}(\nu_n > k_n) \to 0 \quad \text{as} \quad n \to \infty.$$

It should be noted that Theorem 6 reduces to Theorem 4 by putting $k_n = n$ and $\nu_n \equiv n$, i.e. $\frac{\nu_n}{n} \equiv 1$. On the other hand it would be natural to ask if one could derive similar results for sequences (ν_n) which behave sufficiently regular as $n \to \infty$. As may be seen from [1] and [6] there is a positive answer for all (ν_n) fulfilling $\frac{\nu_n}{n} \xrightarrow[\mathbf{P}]{} 1$. In particular one obtains the following

THEOREM 7 (cf. CSÖRGÖ [6], Theorem 6). *Let (x_k) be a MDS and suppose that (ν_n) is a sequence of stopping rules fulfilling*

(d) $\frac{\nu_n}{n} \xrightarrow[\mathbf{P}]{} 1$

(e) $\mathbf{E}(x_k^2|\mathscr{F}_{k-1}) = 1$ *for all* $k \in \mathbf{N}$

(f) $n^{-1} \sum_{k=1}^{n} \mathbf{E}\big(x_k^2 I(|x_k| > \varepsilon n^{1/2})|\mathscr{F}_{k-1}\big) \xrightarrow[\mathbf{P}]{} 0$ *for all* $\varepsilon > 0$.

Then

$$S_{\nu_n} = n^{-1/2} \sum_{k=1}^{\nu_n} x_k \xrightarrow{\mathscr{L}} N(0, 1).$$

PROOF. We shall apply Theorem 6 with $k_n = 2n$ and $s_n = n^{1/2}$. Indeed, conditions (a) and (c) immediately follow from (d) and (f), respectively. To prove (b), apply (e) to obtain

$$n^{-1} \sum_{k=1}^{2n} I(k \leqq \nu_n) \mathbf{E}(x_k^2 | \mathscr{F}_{k-1}) = n^{-1} \sum_{k=1}^{2n} I(k \leqq \nu_n) = n^{-1} \inf\{\nu_n, 2n\} \xrightarrow[\mathbf{P}]{} 1.$$

The proof of Theorem 7 is complete.

5. Proof of Theorem 2

In order to prove Theorem 2 along the idea indicated in Section 2 we need the following auxiliary results.

PROPOSITION 3. *Let* (x_{nk}) *be a TA fulfilling* (CLC). *Then* $\max_{1 \leqq k \leqq k_n} \mathbf{E}(x_{nk}^2 | \mathscr{F}_{n,k-1}) \xrightarrow[\mathbf{P}]{} 0$ *as* $n \to \infty$.

PROOF. (CLC)$\Rightarrow \max \mathbf{E}\big(x_{nk}^2 I(|x_{nk}| > \varepsilon) | \mathscr{F}_{n,k-1}\big) \xrightarrow[\mathbf{P}]{} 0$ and

$$\max \mathbf{E}(x_{nk}^2 | \mathscr{F}_{n,k-1}) = \max\big(\mathbf{E}(x_{nk}^2 I(|x_{nk}| \leqq \varepsilon) | \mathscr{F}_{n,k-1}) + \mathbf{E}(x_{nk}^2 I(|x_{nk}| > \varepsilon) | \mathscr{F}_{n,k-1})\big) \leqq$$
$$\leqq \varepsilon^2 + \max \mathbf{E}\big(x_{nk}^2 I(|x_{nk}| > \varepsilon) | \mathscr{F}_{n,k-1}\big);$$

since $\varepsilon > 0$ is arbitrary the assertion follows.

LEMMA 1. *Let* Q, M *and* N *be functions defined by*

$$Q(\xi) = \begin{cases} \left(e^{i\xi} - 1 - i\xi + \frac{1}{2}\xi^2\right) \Big/ \frac{1}{2}\xi^2 & \text{if} \quad \xi \neq 0, \ \xi \in \mathbf{R}, \\ 0 \quad \text{if} \quad \xi = 0 \end{cases}$$

$$M(\xi) = \min\left(\frac{\xi}{3}, 2\right), \quad \xi \in \mathbf{R}_+, \qquad N(\xi) = e^{-\xi} - 1 + \xi, \quad \xi \in \mathbf{R}.$$

Then

(i) $|1 - Q(\xi)| \leqq 1$ *for every* $\xi \in \mathbf{R}$,

(ii) $|Q(\xi)| \leqq M(|\xi|)$ *for every* $\xi \in \mathbf{R}$, *and*

(iii) $|N(\xi)| \leqq \frac{1}{2}\xi^2$ *for every* $\xi \in \mathbf{R}_+$.

PROOF. Obvious.

LEMMA 2. *Let* $(x_n)_{n \in \mathbf{N}}$ *and* $(y_n)_{n \in \mathbf{N}}$ *be two sequences of random variables on some p-space* $(\Omega, \mathscr{A}, \mathbf{P})$ *with* $y_n(\omega) \neq 0$ *for all* $\omega \in \Omega$ *and* $n \in \mathbf{N}$. *Let* $f(t)$ *be a characteristic* *function such that for an arbitrary but fixed* $t_0 \in \mathbf{R}$ *we have* $f(t_0) \neq 0$ *and*

(a) $\lim_{n\to\infty} \mathbf{E}\big(y_n^{-1}\exp(it_0x_n)-1\big) = 0,$

(b) $\lim_{n\to\infty} \mathbf{E}\big(|y_n^{-1}-(f(t_0))^{-1}|\big) = 0.$

Then $\lim_{n\to\infty} \mathbf{E}\big(\exp(it_0x_n)\big) = f(t_0).$

We remark that Lemma 2 holds true even if the y_n's depend on t_0.

PROOF OF LEMMA 2. We have

$$|\mathbf{E}\big(\exp(it_0x_n)\big)-f(t_0)| = |\mathbf{E}\big(\exp(it_0x_n)-f(t_0)\big)| \leqq$$

$$\leqq |\mathbf{E}\big(\exp(it_0x_n)-f(t_0)y_n^{-1}\exp(it_0x_n)\big)| + |\mathbf{E}\big(f(t_0)y_n^{-1}\exp(it_0x_n)-f(t_0)\big)| \leqq$$

$$\leqq \mathbf{E}\big(|1-f(t_0)y_n^{-1}|\big)+|f(t_0)|\cdot|\mathbf{E}\big(y_n^{-1}\exp(it_0x_n)-1\big)| \leqq$$

$$\leqq \mathbf{E}\big(|(f(t_0))^{-1}-y_n^{-1}|\big)+|\mathbf{E}\big(y_n^{-1}\exp(it_0x_n)-1\big)|$$

since $|f(t_0)| \leqq 1$, from which the assertion follows according to the assumptions (a) and (b).

PROOF OF THEOREM 2. In addition to the notations already introduced when sketching the idea of the proof in Section 2, let $T_{nk} = \sum_{i=1}^{k} z_{ni}$,

$$\tau_{nk}^2 = \mathbf{E}(z_{nk}^2|\mathscr{F}_{n,k-1}), \quad W_{nk}^2 = \sum_{i=1}^{k}\tau_{ni}^2, \quad \text{i.e.} \quad W_{nk_n}^2 = W_n^2.$$

Proof of A). First we will prove that

$$(+) \qquad \lim_{n\to\infty} \mathbf{P}\left(\bigcap_{k=1}^{k_n}\{x_{nk} = z_{nk}\}\right) = 1.$$

To this extent note that for every $n\in\mathbf{N}$

$$1 \geqq \mathbf{P}\left(\bigcap_{k=1}^{k_n}\{x_{nk} = x_{nk}I(V_{nk}^2 \leqq c)\}\right) = \mathbf{P}\left(\bigcap_{k=1}^{k_n}\{x_{nk}[1-I(V_{nk}^2 \leqq c)] = 0\}\right) \geqq$$

$$\geqq \mathbf{P}(\{V_n^2 \leqq c\}) = 1-\mathbf{P}(\{V_n^2-1 > c-1\}) \geqq 1-\mathbf{P}(\{|V_n^2-1| > c-1\}) \to 1$$

according to assumption (a) of Theorem 2, which implies (+).

Now, for any $n\in\mathbf{N}$ and $\delta>0$ arbitrary, we have

$$0 \leqq \mathbf{P}(\{|S_n-T_n| > \delta\}) = \mathbf{P}\big(\{|\sum(x_{nk}-z_{nk})| > \delta\}\big) \leqq \mathbf{P}\big(\{\sum(x_{nk}-z_{nk}) \neq 0\}\big) =$$

$$= 1-\mathbf{P}\big(\{\sum(x_{nk}-z_{nk}) = 0\}\big) \leqq 1-\mathbf{P}\left(\bigcap_{k=1}^{k_n}\{x_{nk}=z_{nk}\}\right) \to 0 \quad \text{by} \quad (+).$$

Proof of B). Since V_{nk}^2 is, by definition, $\mathscr{F}_{n,k-1}$-measurable we obtain

$$\mathbf{E}(z_{nk}|\mathscr{F}_{n,k-1}) = I(V_{nk}^2 \leqq c)\mathbf{E}(x_{nk}|\mathscr{F}_{n,k-1}) = 0 \quad \text{for} \quad k = 1, \ldots, k_n, \ n\in\mathbf{N}.$$

Proof of C). Since, for $1 \leqq k \leqq k_n$, $n \in \mathbf{N}$, $|z_{nk}| \leqq |x_{nk}|$, i.e. $z_{nk}^2 \leqq x_{nk}^2$, C) follows from (CLC) for (x_{nk}).

Proof of D). For any $n \in \mathbf{N}$ and $\delta > 0$ arbitrary, we have

$$0 \leqq \mathbf{P}(\{|W_n^2 - 1| > 2\delta\}) \leqq \mathbf{P}(\{|W_n^2 - V_n^2| > \delta\}) + \mathbf{P}(\{|V_n^2 - 1| > \delta\}).$$

According to (a) it remains to show that $\lim_{n\to\infty} \mathbf{P}(\{|W_n^2 - V_n^2| > \delta\}) = 0$. This is achieved in the same way as in the proof of A) showing first that

$$\lim_{n\to\infty} \mathbf{P}\left(\bigcap_{k=1}^{k_n} \{\sigma_{nk}^2 = \tau_{nk}^2\}\right) = 1.$$

Proof of E). $\mathbf{P}(\{W_n^2 \leqq c\}) = \mathbf{P}(\{\sum \mathbf{E}(z_{nk}^2 | \mathscr{F}_{n,k-1}) \leqq c\}) =$

$$= \mathbf{P}(\{\sum \mathbf{E}(x_{nk}^2 I(V_{nk}^2 \leqq c) | \mathscr{F}_{n,k-1}) \leqq c\}) = \mathbf{P}(\{\sum \sigma_{nk}^2 I(V_{nk}^2 \leqq c) \leqq c\}) =$$

$$= \mathbf{P}\left(\left\{\sum \sigma_{nk}^2 I\left(\sum_{i=1}^{k} \sigma_{ni}^2 \leqq c\right) \leqq c\right\}\right) = 1.$$

Proof of F). We have $\sum \mathbf{E}(z_{nk}^2) = \sum \mathbf{E}(\mathbf{E}(z_{nk}^2 | \mathscr{F}_{n,k-1})) = \mathbf{E}(W_n^2)$, hence F) follows from D) and E).

As indicated in Section 2 it remains to verify the conditions (a) and (b) of Lemma 2 $\left(\text{applied with } f(t) = e^{-t^2/2},\ x_n = T_n \text{ and } y_n = \exp\left(-\frac{1}{2} t^2 W_n^2\right)\right)$, i.e. we must show that

(a′) $\lim_{n\to\infty} \mathbf{E}\left(\exp\left(itT_n + \frac{1}{2} t^2 W_n^2\right) - 1\right) = 0$ for all $t \in \mathbf{R}$, and

(b′) $\lim_{n\to\infty} \mathbf{E}\left(\left|\exp\left(\frac{1}{2} t^2 W_n^2\right) - \exp\left(\frac{1}{2} t^2\right)\right|\right) = 0$ for all $t \in \mathbf{R}$.

Proof of (a′). For $t = 0$ (a′) is obviously true; hence let $t \in \mathbf{R}$, $t \neq 0$, be arbitrary but fixed. For $1 \leqq k \leqq k_n$, $n \in \mathbf{N}$, let

$$y_{nk} = \exp\left(itT_{n,k-1} + \frac{1}{2} t^2 W_{nk}^2\right)\left[\exp(itz_{nk}) - \exp\left(-\frac{1}{2} t^2 \tau_{nk}^2\right)\right]$$

(where $T_{n0} = z_{n0} = 0$). Then

$$(*) \qquad \sum y_{nk} = \sum \left\{\exp\left(itT_{nk} + \frac{1}{2} t^2 W_{nk}^2\right) - \exp\left(itT_{n,k-1} + \frac{1}{2} t^2 W_{n,k-1}^2\right)\right\} =$$

$$= \exp\left(itT_n + \frac{1}{2} t^2 W_n^2\right) - 1,$$

i.e. the integrand occurring in (a′) equals the sum of the y_{nk}. Using the notations of Lemma 1 we have

$$y_{nk} = \exp\left(itT_{n,k-1}+\frac{1}{2}t^2W_{nk}^2\right)\left[\exp(itz_{nk})-N\left(\frac{1}{2}t^2\tau_{nk}^2\right)-1+\frac{1}{2}t^2\tau_{nk}^2\right] =$$

$$= \exp\left(itT_{n,k-1}+\frac{1}{2}t^2W_{nk}^2\right)\left[1+itz_{nk}-\frac{1}{2}t^2z_{nk}^2+\frac{1}{2}t^2z_{nk}^2Q(tz_{nk})-N\left(\frac{1}{2}t^2\tau_{nk}^2\right)-\right.$$

$$\left.-1+\frac{1}{2}t^2\tau_{nk}^2\right] = \exp\left(itT_{n,k-1}+\frac{1}{2}t^2W_{nk}^2\right)\times$$

$$\times\left[itz_{nk}-\frac{1}{2}t^2z_{nk}^2+\frac{1}{2}t^2z_{nk}^2Q(tz_{nk})+\frac{1}{2}t^2\tau_{nk}^2-N\left(\frac{1}{2}t^2\tau_{nk}^2\right)\right].$$

Therefore, by Lemma 1, for $1\leqq k\leqq k_n$, $n\in\mathbf{N}$, we have a.s.

$$|\mathbf{E}(y_{nk}|\mathcal{F}_{n,k-1})| =$$

$$= \left|\exp\left(itT_{n,k-1}+\frac{1}{2}t^2W_{nk}^2\right)\right|\cdot\left|\mathbf{E}\left(\frac{1}{2}t^2z_{nk}^2Q(tz_{nk})\middle|\mathcal{F}_{n,k-1}\right)-N\left(\frac{1}{2}t^2\tau_{nk}^2\right)\right| \leqq$$

$$\leqq \exp\left(\frac{1}{2}t^2c\right)\cdot\left[\mathbf{E}\left(\left|\frac{1}{2}t^2z_{nk}^2Q(tz_{nk})\right|\middle|\mathcal{F}_{n,k-1}\right)+\left|N\left(\frac{1}{2}t^2\tau_{nk}^2\right)\right|\right] \leqq$$

$$\leqq \exp\left(\frac{1}{2}t^2c\right)\cdot\left[\mathbf{E}\left(\frac{1}{2}t^2z_{nk}^2M(|tz_{nk}|)|\mathcal{F}_{n,k-1}\right)+\frac{1}{8}t^4\tau_{nk}^4\right] \leqq$$

$$\leqq \frac{1}{2}t^2\exp\left(\frac{1}{2}t^2c\right)\cdot\left[\mathbf{E}\left(z_{nk}^2M(|tz_{nk}|)\,|\mathcal{F}_{n,k-1}\right)+\frac{1}{4}t^2\tau_{nk}^2\cdot\max\tau_{nk}^2\right].$$

Hence, according to (*), we obtain for every $n\in\mathbf{N}$

$$\left|\mathbf{E}\left(\exp\left(itT_n+\frac{1}{2}t^2W_n^2\right)-1\right)\right| = |\mathbf{E}(\sum y_{nk})| = |\mathbf{E}(\sum\mathbf{E}(y_{nk}|\mathcal{F}_{n,k-1}))| \leqq$$

$$\leqq \mathbf{E}(\sum|\mathbf{E}(y_{nk}|\mathcal{F}_{n,k-1})|) \leqq$$

$$\leqq \frac{1}{2}t^2\exp\left(\frac{1}{2}t^2c\right)\cdot[\mathbf{E}(\sum\mathbf{E}(z_{nk}^2M(|tz_{nk}|)\,|\mathcal{F}_{n,k-1}))+\frac{1}{4}t^2\mathbf{E}((\max\tau_{nk}^2)\cdot\sum\tau_{nk}^2)] \leqq$$

$$\leqq \frac{1}{2}t^2\exp\left(\frac{1}{2}t^2c\right)\cdot\left[\sum\mathbf{E}\left(z_{nk}^2M\left(\frac{|z_{nk}|}{\frac{1}{|t|}}\right)\right)+\frac{1}{4}t^2c\mathbf{E}(\max\tau_{nk}^2)\right].$$

Now, by F), (z_{nk}) fulfils the assumption of Theorem 1 and M is a U-function of the type described in (CUC) and (UC), respectively, of Theorem 1. Hence, by the equivalence of (i) to (iv) $\left(\text{taking } \varepsilon=\frac{1}{|t|}\right)$, it follows from C) and D) that

$$\lim_{n\to\infty} \sum \mathbf{E}\left(z_{nk}^2 M\left(\frac{|z_{nk}|}{\frac{1}{|t|}}\right)\right)=0.$$

Further, C) implies that $\max \mathbf{E}(z_{nk}^2|\mathscr{F}_{n,k-1})=\max \tau_{nk}^2 \xrightarrow[\mathbf{P}]{} 0$ by Proposition 3 and therefore, by the a.s. boundedness of $\max \tau_{nk}^2$, we obtain $\lim_{n\to\infty} \mathbf{E}(\max \tau_{nk}^2)=0$. This proves (a').

Proof of (b'). It follows from D) that for any $t\in\mathbf{R}$ $\exp\left(\frac{1}{2}t^2W_n^2\right) \xrightarrow[\mathbf{P}]{} \exp\left(\frac{1}{2}t^2\right)$ and, by E) $\exp\left(\frac{1}{2}t^2W_n^2\right)\leqq\exp\left(\frac{1}{2}t^2c\right)$ a.s., hence $\exp\left(\frac{1}{2}t^2W_n^2\right)$ converges in $\mathscr{L}_1(\Omega, \mathscr{A}, \mathbf{P})$ to $\exp\left(\frac{1}{2}t^2\right)$, which proves (b'). The proof of Theorem 2 is concluded.

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ON COHOMOLOGY OF SIMPLE SHEAVES

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Introduction. Let $\mathscr{I}$ be the category of all topological pairs and $\mathscr{R}$ the category of all R-modules, R is a ring with identity. Given an object $(X, \emptyset)$, $\emptyset$ an empty set, in $\mathscr{I}$ we regard $\mathscr{R}$ to be the category of all simple sheaves of R-modules on X as a subcategory of the category of all sheaves of R-modules on X. Similarly, let $\mathscr{C}$ denote the subcategory of $\mathscr{I}$ with only one morphism I_X. It can be observed [8, Theorems 5.2.2, 5.2.3 on page 207 and 5.11 on page 231] that if we take a paracompactifying family φ of supports on X, then Čech cohomology of sheaves in the category of all sheaves on X is a cohomology theory in the sense of Cartan. Furthermore, if the coefficient category of all sheaves is restricted to $\mathscr{R}$ then it is possible to extend the Čech cohomology of sheaves in $\mathscr{R}$ with $\varphi=\text{cld}$ — the set of all closed sets of X to the usual Čech cohomology [5] in the sense of Eilenberg—Steenrod. Likewise, the usual cohomology of sheaves on X with supports in any family is a cohomology theory in the sense of Cartan and it has been proved by BREDON [1], though not explicitly mentioned, that if the category of all sheaves is restricted to $\mathscr{R}$ then it is possible to extend the cohomology of sheaves in $\mathscr{R}$ to a cohomology theory on $\mathscr{I}$ in the sense of Eilenberg—Steenrod. The purpose of this paper is to give an alternative development of this last cohomology theory analogous to Alexander—Spanier cohomology and then to compare it with other cohomology theories. A noteworthy feature of this treatment will be that it is completely independent of sheaf theoretic concepts. We have called the resulting cohomology as G-cohomology for, the basic definition from which our development proceeds is due to Godement.

There is a natural homomorphism from Alexander—Spanier cohomology to G-cohomology which we prove to be isomorphism on the full subcategory of paracompact Hausdorff pairs. Our proof of this result is based on the tautness properties of the two cohomology theories. This result yields the well known results, viz., Vietoris—Begle theorem, tautness property, continuity property and strong excision theorem for G-cohomology also.

Preliminaries. Let K be an R-module where R is a ring with identity and X be any topological space. For $p\geqq 0$, let us denote by $M^p(X, K)$ the module of all functions $\alpha\colon X^{p+1}\to K$ with obvious operations. Define for every $p\geqq 0$ a R-homomorphism $\delta\colon M^p(X, K)\to M^{p+1}(X, K)$ as usual by

$$(\delta\alpha)(x_0, \dots, x_{p+1}) = \sum_{0\leqq 1\leqq p+1} (-1)^i \alpha(x_0, \dots, \hat{x}_i, \dots, x_{p+1}).$$

Then we get a cochain complex (M^*, δ) which is easily seen to be acyclic in positive dimensions. The following concept is due to R. GODEMENT [8, page 249].

DEFINITION 1.1. An element $\alpha \in M^p(X, K)$ will be called G-locally zero if for every q-tuple $(x_0, \ldots, x_q)$, $q=0, 1, \ldots, p-1$, of X there is a neighbourhood $U(x_0, \ldots, x_q)$ of x_q in X such that if $x_1 \in U(x_0), \ldots, x_p \in U(x_0, \ldots, x_{p-1})$, then $\alpha(x_0, \ldots, x_p)=0$.

Notice that in order for $\alpha \in M^p(X, K)$ to be G-locally zero it is necessary that it must vanish on the diagonal of X^{p+1} and hence $\alpha \in M^0(X, K)$ is G-locally zero if and only if $\alpha=0$. From now onward, for $\alpha \in M^p(X, K)$ the abbreviation

$$\text{“}\alpha(x_0, \ldots, x_p) = 0 \quad \text{on} \quad U(x_0), \ldots, U(x_0, \ldots, x_{p-1})\text{”}$$

will mean that for $q=0, 1, \ldots, p-1$, $U(x_0, \ldots, x_q)$ are the neighbourhoods of x_q in X such that if $x_1 \in U(x_0), \ldots, x_p \in U(x_0, \ldots, x_{p-1})$ then $\alpha(x_0, \ldots, x_p)=0$. It is easily seen that the set of all G-locally zero functions form a submodule of $M^p(X, K)$ which we denote by $M_0^p(X, K)$. Now we prove the following

LEMMA 1.2. *If* $a \in M_0^p(X, K)$ *then* $\delta\alpha \in M_0^{p+1}(X, K)$.

PROOF. Suppose

$$\alpha(x_0, \ldots, x_p) = 0 \quad \text{on} \quad U(x_0), \ldots, U(x_0, \ldots, x_{p-1}).$$

For $q=0, 1, \ldots, p$ choose $W(x_0, \ldots, x_q)$ as follows: $W(x_0)=U(x_0)$,

$$W(x_0, x_1) = \begin{cases} U(x_1) \cap U(x_0) \cap U(x_0, x_1) & \text{if} \quad x_1 \in U(x_0) \\ \text{arbitrary otherwise} \end{cases}$$

$$W(x_0, x_1, x_2) = \begin{cases} U(x_1, x_2) \cap U(x_0, x_2) \cap U(x_0, x_1) \cap U(x_0, x_1, x_2) & \text{if} \quad x_2 \in U(x_0, x_1) \\ \text{arbitrary otherwise} \end{cases}$$

..., generally,

$$W(x_0, \ldots, x_q) =$$

$$= \begin{cases} \bigcap \{U(x_0, \ldots, \hat{x}_i, \ldots, x_q),\ i = 0, 1, \ldots, q\} \cap U(x_0, \ldots, x_q) & \text{if} \quad x_q \in U(x_0, \ldots, x_{q-1}) \\ \text{arbitrary otherwise.} \end{cases}$$

Then examination of each term shows that

$$(\delta\alpha)(x_0, \ldots, x_{p+1}) = 0 \quad \text{on} \quad W(x_0), \ldots, W(x_0, \ldots, x_p). \qquad \text{Q.E.D.}$$

As a consequence, if we define $\overline{M}^p(X, K)=M^p(X, K)/M_0^p(X, K)$ for $p \geqq 0$, we get a cochain complex

$$\overline{M}^*\colon 0 \to \overline{M}^0(X, K) \to \ldots \to \overline{M}^p(X, K) \to \overline{M}^{p+1}(X, K) \to \ldots$$

which we will refer to as G-cochain complex.

Now let us recall that a function $\alpha \in M^p(X, K)$ is locally zero in the sense of Spanier if it vanishes on a neighbourhood U of the diagonal of X^{p+1} which is equivalent to saying that there is an open covering $\{U\}$ of X such that whenever $x_0, \ldots, x_p \in U$ for some U in $\{U\}$, $\alpha(x_0, \ldots, x_p)=0$. We will need the following

PROPOSITION 1.3. *A function* $\alpha \in M^p(X, K)$ *which is locally zero in the sense of Spanier is also G-locally zero.*

PROOF. Let $\{U_i\}$ be an open covering of X such that whenever $x_0, \dots, x_p \in U_i$ for some i, $\alpha(x_0, \dots, x_p)=0$. For any q-tuple, $q=0, 1, \dots, p-1$, choose $Q(x_0)=U_i$ for some i such that $x_0 \in U_i$ and then for $q=1, \dots, p-1$ choose

$$W(x_0, \dots, x_q) = \begin{cases} U_i \text{ if } & x_0, \dots, x_{q-1} \in U_i \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

Then obviously

$$\alpha(x_0, \dots, x_p) = 0 \quad \text{on} \quad W(x_0), \dots, W(x_0, \dots, x_{p-1}). \qquad \text{Q.E.D.}$$

Converse of the above Proposition is, however, not true as the following example shows.

EXAMPLE 1.4. Let $X=R$ and $K=Z$. For each $x \in X$ define

$$U(x) = \begin{cases} (x-1/2, \ x+1/2) & \text{if } |x| \neq 1/n \\ (x-1/2n, \ x+1/2n) & \text{if } |x| = 1/n. \end{cases}$$

Now define $\alpha: X \times X \to K$ as follows:

$$\alpha(x_0, x_1) = \begin{cases} 0 & \text{if } x_1 \in U(x_0) \\ 1 & \text{otherwise.} \end{cases}$$

Then by construction itself, α is G-locally zero. It is not locally zero in the sense of Spanier, since for every open covering $\{U_i\}$ of X and for every U_i containing $0 \in X$, we can always find a positive n such that $1/n \in U_i$ but the interval $(n-1/2n, n+1/2n)$ is properly contained in U_i, i.e., $\alpha(1/n, x_1) \neq 0$ for some $x_1 \in U_i$ which means $\alpha | U_i \times U_i \neq 0$.

The proof of the following is left to the reader.

PROPOSITION 1.5. *Let $\{U_i\}_{i \in I}$ be an arbitrary open covering of the space X. If $\alpha \in M^p(X, K)$ is G-locally zero on U_i for each i, then α is G-locally zero on X.*

G-cohomology theory and the axioms. Let $\alpha \in M^p(Y, K)$ and $f: X \to Y$ be a map, not necessarily continuous. Then f induces a R-homomorphism $f^{\#}: M^p(Y, K) \to$ $\to M^p(X, K)$ defined by

$$(f^{\#} \alpha)(x_0, \dots, x_p) = \alpha\big(f(x_0), \dots, f(x_p)\big).$$

Now suppose f is continuous and $\alpha \in M_0^p(Y, K)$, then

$$\alpha(y_0, \dots, y_p) = 0 \quad \text{on} \quad U(y_0), \dots, U(y_0, \dots, y_{p-1}).$$

If for every q-tuple $(x_0, \dots, x_q)$, $q=0,1, \dots, p-1$ we define

$$W(x_0, \dots, x_q) = f^{-1}\big(U\big(f(x_0), \dots, f(x_q)\big)\big),$$

we find that

$$(f^{\#} \alpha)(x_0, \dots, x_p) = 0 \quad \text{on} \quad W(x_0), \dots, W(x_0, \dots, x_{p-1}).$$

Hence for every $p \geqq 0$, $f^{\#}$ induces a R-homomorphism from $\overline{M}^p(Y, K) \to \overline{M}^p(X, K)$ which we still denote by $f^{\#}$. Precisely speaking, we must put a suffix p on $f^{\#}$ to indicate the correct dimension for each p but we shall take it understood in all situations from now onward and likewise with δ^*. The homomorphism

$f^{\#}: \overline{M}^p(Y, K) \to \overline{M}^p(X, K)$ for every $p \geqq 0$ is a cochain map, i.e., the diagram

$$\begin{array}{ccccccc}
\ldots \to \overline{M}^{p-1}(Y, K) & \xrightarrow{\delta} & \overline{M}^p(Y, K) & \xrightarrow{\delta} & \overline{M}^{p+1}(Y, K) & \to \ldots \\
\downarrow f^{\#} & & \downarrow f^{\#} & & \downarrow f^{\#} & \\
\ldots \to \overline{M}^{p-1}(X, K) & \xrightarrow{\delta} & \overline{M}^p(X, K) & \xrightarrow{\delta} & \overline{M}^{p+1}(X, K) & \to \ldots
\end{array}$$

is commutative. For every $p \geqq 0$, $f^{\#}$ induces the homomorphism $f^*: H^p(Y, K) \to \to H^p(X, K)$. Now let (X, A) be a pair. The inclusion map $i: A \to X$ induces an epimorphism $i^{\#}: \overline{M}^*(X, K) \to \overline{M}^*(A, K)$ of cochain complexes. Let $\overline{M}^*(X, A; K)$ denote the ker $i^{\#}$. Then the following short sequence of cochain complexes is exact.

$$0 \to \overline{M}^*(X, A; K) \xrightarrow{j^{\#}} \overline{M}^*(X, K) \xrightarrow{i^{\#}} \overline{M}^*(A, K) \to 0.$$

This gives the following long exact sequence of cohomology

$$\ldots \to H^p(X, K) \xrightarrow{i^*} H^p(A, K) \xrightarrow{\delta^*} H^{p+1}(X, A; K) \xrightarrow{j^*} H^{p+1}(X, K) \to \ldots.$$

Now suppose $f:(X, A) \to (Y, B)$ is a continuous map of pairs. Then the commutativity of the following diagram of cochain complexes and their maps is immediate

$$\begin{array}{ccccccc}
0 \to \overline{M}^*(Y, B; K) & \xrightarrow{j^{\#}} & \overline{M}^*(Y, K) & \xrightarrow{i^{\#}} & \overline{M}^*(B, K) & \to 0 \\
\downarrow f^{\#} & & \downarrow (f/X)^{\#} & & \downarrow (f/A^{\#}) & \\
0 \to \overline{M}^*(X, A; K) & \xrightarrow{j^{\#}} & \overline{M}^*(X, K) & \xrightarrow{i^{\#}} & \overline{M}^*(A, K) & \to 0.
\end{array}$$

This, in turn, yields a commutative diagram

$$\begin{array}{ccccccccc}
\ldots \to H^p(Y, K) & \xrightarrow{i^*} & H^p(B, K) & \xrightarrow{\delta^*} & H^{p+1}(Y, B; K) & \xrightarrow{j^*} & H^{p+1}(Y, K) & \to \ldots \\
\downarrow (f/X)^* & & \downarrow (f/A)^* & & \downarrow f^* & & \downarrow (f/X)^* & \\
\ldots \to H^p(X, K) & \xrightarrow{i^*} & H^p(A, K) & \xrightarrow{\delta^*} & H^{p+1}(X, A; K) & \xrightarrow{j^*} & H^{p+1}(X, K) & \to \ldots
\end{array}$$

of long exact cohomology sequences. Now if we denote the cohomology groups $H^*(X, A; K)$ of the pair (X, A) obtained above by ${}_GH^*(X, A; K)$, then $\{{}_GH^*(X, A; K), \delta^*, f^*\}$ all defined above are the constituents of our G-cohomology theory.

From our discussion above it is easily seen that the identity axiom, the composition axiom, the commutativity axiom and the dimension axiom hold for our G-cohomology. It only remains to prove the homotopy axiom and the excision axiom. We shall come to the homotopy axiom in the next section.

In order to prove the excision axiom we proceed as follows: Let $M^p(X, A)$ denote the R-module of those functions from X^{p+1} to K which are G-locally zero on A. Then for every $p \geqq 0$ the sequence

$$0 \to M^p(X, A) \to M^p(X) \to \overline{M}^p(A) \to 0$$

is exact and $M_0^p(X) \subset M^p(X, A)$. Hence a direct description of $\overline{M}^p(X, A)$ reads as: This is the R-module of all those functions from $X^{p+1}\to K$ which are G-locally zero on A modulo the R-module $M_0^p(X, K)$.

Now the excision axiom follows from the following

PROPOSITION 2.1. *Let (X, A) be a pair and U an arbitrary subset of X which has an open neighbourhood W in X such that $\overline{W}\subset \operatorname{Int} A$. Then the inclusion map $k: (X-U, A-U)\to (X, A)$ induces an isomorphism*

$$k^*:\ {}_GH^*(X, A, K)\to {}_GH^*(X-U, A-U).$$

PROOF. By what we said above, the rows of the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & M_0^*(X) & \longrightarrow & M^*(X,A) & \longrightarrow & \overline{M}^*(X,A) & \to & 0\\
 & & \downarrow & & \downarrow{\scriptstyle k^{\#}} & & \downarrow & & \\
0 & \to & M_0^*(X-U) & \to & M^*(X-U,A-U) & \xrightarrow{\lambda} & \overline{M}^*(X-U,A-U) & \to & 0
\end{array}$$

of cochain complexes are exact where we have suppressed the coefficient modules. To prove our assertion, it suffices to show that $\lambda k^{\#}$ is an epimorphism and that $k^{\#-1}(M_0^*(X-U))\subset M_0^*(X)$. For any $p\geqq 0$, let $\alpha\in M^p(X-U, A-U)$ and let

$$\alpha(x_0, \dots, x_p)=0 \quad \text{on} \quad U(x_0)\cap(A-U), \dots, U(x_0, \dots, x_{p-1})\cap(A-U).$$

Define α' by

$$\alpha'(x_0, \dots, x_p)=\begin{cases}\alpha(x_0, \dots, x_p) & \text{if } \ x_i\in X-\overline{W}\ \ \forall i\\ 0 \ \text{otherwise}\end{cases}$$

and for $\forall q$-tuple $(x_0, \dots, x_q)$, $q=0,1, \dots, p-1$, in A choose

$$W(x_0, \dots, x_q)=\begin{cases}U(x_0, \dots, x_q) & \text{if } \ x_0\in A-U, \ \ x_1\in U(x_0), \dots, x_q\in U(x_0, \dots, x_{q-1})\\ \text{arbitrary otherwise.}\end{cases}$$

Then,

$$\alpha'(x_0, \dots, x_p)=0 \quad \text{on} \quad W(x_0)\cap A, \dots, W(x_0, \dots, x_{p-1})\cap A.$$

Hence $\alpha'\in M^p(X, A)$. Now we assert that $k^{\#}\alpha'-\alpha\in M_0^p(X-U)$. Since α is G-locally zero on $\operatorname{Int} A$ we have neighbourhoods $U(x_0, \dots, x_q)$, $q=0, \dots, p-1$, all of them contained in $\operatorname{Int} A$ such that

$$\alpha(x_0, \dots, x_p)=0 \quad \text{on} \quad U(x_0), \dots, U(x_0, \dots, x_{p-1}).$$

Now for every q-tuple $(x_0, \dots, x_q)$, $q=0, 1, \dots, p-1$, choose

$$W(x_0, \dots, x_q)=\begin{cases}U(x_0, \dots, x_q) & \text{if } \ x_0\in \operatorname{Int} A, \ x_1\in U(x_0), \dots, x_q\in U(x_0, \dots, x_{q-1})\\ X-\overline{W} & \text{if } \ x_0, \dots, x_q\in X-\overline{W}\\ \text{arbitrary} & \text{otherwise.}\end{cases}$$

Then on $W(x_0)$, $\dots$, $W(x_0, \dots, x_{p-1})$, we have either $x_0, \dots, x_p\in \operatorname{Int} A$ or $x_0, \dots, x_p\in X-\overline{W}$. In any case

$$(k^{\#}\alpha'-\alpha)(x_0, \dots, x_p)=(\alpha'/X-U)(x_0, \dots, x_p)-\alpha(x_0, \dots, x_p)=0.$$

Hence $\lambda k^{\#}\alpha'=\lambda\alpha$, i.e., $\lambda k^{\#}$ is an epimorphism. On the other hand if $\alpha\in M^p(X, A)$

such that $k^{\#}\alpha \in M_0^p(X-U)$, then α is G-locally zero on A as well as $X-U$ and hence α is G-locally zero on Int A and $X-\overline{W}$. Since these two open subsets of X form an open covering of X, α is G-locally zero on X by Proposition 1.5. Q.E.D.

We shall prove in the following section that the G-cohomology groups of any pair with coefficients in a R-module K are isomorphic to the sheaf theoretic cohomology of the constant sheaf K on the given pair. In view of this we remark that a stronger form of the excision property holds for G-cohomology (cf [3]).

Let us note the following whose proof follows in a straightforward manner.

PROPOSITION 2.2. *For any pair* (X, A), ${}_GH^0(X, A; K)$ *is isomorphic to the module of all those locally constant functions from* X *to* K *which vanish on* A.

PROPOSITION 2.3. *Let* X *be a topological space in which every quasi-component is open and* $A\subset X$. *Then* ${}_GH^0(X, A; K)$ *is isomorphic to the R-module of all functions from the set of those quasi-components of* X *to* K *which do not intersect* A.

COROLLARY 2.4. *A non empty space* X *is connected if and only if* ${}_GH^0(X; K)\simeq K$.

Comparison of G-cohomology with other cohomology theories. First of all let us indicate the proof of the following

THEOREM 3.1. *For any given pair* (X, A), *there is an isomorphism between G-cohomology groups* ${}_GH^*(X, A, K)$ *and Bredon's sheaf theoretic relative cohomology groups* $H^*(X, A; K)$ *of the constant sheaf* K *of R-modules on* X. *Moreover, this isomorphism is natural in* K.

PROOF. Consider the cochain complex of sections with supports in cld of the second canonical resolution of Godement as described in BREDON [1, pp. 28—30]. We observe that for the constant sheaf K of R-modules on X this cochain complex is precisely what we have called G-cochain complex and that his boundary operator reduces to ours because of the arbitrary choice of the serration $S(f(x_0, \ldots, x_p))$ continuous in a neighbourhood of x_p which we can very well choose to be with the constant value $f(x_0, \ldots, x_p)\in K$. With this observation our contention follows from what has been demonstrated as an application immediately following the proof of his Theorem 12.10. Q.E.D.

Now let $\bar{C}^*(X, K)$ denote the Alexander—Spanier cochain complex of the space X. By Proposition 1.3, $C_0^*(X, K)\subset M^*(X, K)$ and therefore we have a cochain map $\varrho_X^{\#}:\bar{C}^*(X, K)\to\bar{M}^*(X, K)$ which is easily seen to be natural, i.e., for a continuous map $f:X\to Y$ the diagram

$$\begin{array}{ccc} \bar{C}^*(Y, K) & \xrightarrow{\varrho_Y^{\#}} & \bar{M}^*(Y, K) \\ {\scriptstyle f^{\#}}\downarrow & & \downarrow{\scriptstyle f^{\#}} \\ \bar{C}^*(X, K) & \xrightarrow{\varrho_X^{\#}} & \bar{M}^*(X, K) \end{array}$$

of cochain complexes and their maps is commutative. This shows that for a pair (X, A) we have a natural cochain map $\varrho_{(X,A)}^{\#}:\bar{C}^*(X, A; K)\to\bar{M}^*(X, A; K)$. Consequently, there is an induced natural homomorphism $\varrho_{(X,A)}^{*}:\bar{H}^*(X, A; K)\to{}_GH^*(X, A; K)$ of graded modules. It follows from five lemma that if (X, A) is a pair which is paracompact Hausdorff then $\varrho^*(X, A)$ is isomorphism iff ϱ_X^* is isomorphism for every paracompact Hausdorff space X. Now we proceed to the proof of this. The proof of the following Proposition is left to the reader.

PROPOSITION 3.2. *Let $\{V_i\}_{i\in I}$ be a family of mutually disjoint open sets of X. Then there is an isomorphism*

$$_GH^p(\bigcup_i \{V_i\}, K) \simeq \prod_i {_GH^p}(V_i, K)$$

for every $p \geqq 0$ and for every R-module K.

PROPOSITION 3.3. *Let X be any topological space. Then any point subspace x of X is taut with respect to G-cohomology.*

PROOF. Since zero-dimensional cocycles of X are locally constant functions from X to K, $\{x\}$ is clearly taut with respect to zero-dimensional G-cohomology. It is, therefore, sufficient to show that if $\alpha + M_0^p(X, K)$, $p \geqq 1$, is a cocycle then there exists a neighbourhood $U(x)$ of x in X and a cochain $\alpha_x + M_0^{p-1}(X, K)$ such that $\delta\alpha_x - \alpha$ is G-locally zero on $U(x)$. Now let us define α_x by

$$\alpha_x(x_1, \ldots, x_p) = \alpha(x, x_1, \ldots, x_p).$$

Then it is easily seen that

$$\delta\alpha_x(x_1, \ldots, x_{p+1}) = \alpha(x_1, \ldots, x_{p+1}) - \delta\alpha(x, x_1, \ldots, x_{p+1}).$$

Since $\delta\alpha$ is G-locally zero on X, there are neighbourhoods $U(x)$, $U(x, x_1)$, ..., $U(x, x_1, \ldots, x_p)$ such that

$$\delta\alpha(x, x_1, \ldots, x_{p+1}) = 0 \quad \text{on} \quad U(x), U(x, x_1), \ldots, U(x, x_1, \ldots, x_p).$$

But this implies that

$$(\delta\alpha_x - \alpha)(x_1, \ldots, x_p) = 0 \quad \text{on} \quad U(x), U(x, x_1), \ldots, U(x, x_1, \ldots, x_p),$$

i.e., $\delta\alpha_x - \alpha$ is G-locally zero on $U(x)$. Q.E.D.

The following theorem is due to E. Michael, which we shall use in the proof of our main theorem later on.

THEOREM 3.4 [11, Theorem 5.5]. *Let X be a topological space and $\mathcal{P}$ be a property of topological spaces. Suppose X is paracompact and*

(i) *X has $\mathcal{P}$ locally,*

(ii) *a subspace A of X has $\mathcal{P}$ implies each closed subspace of A has $\mathcal{P}$,*

(iii) *if a subspace A is the union of two closed subspaces A_1, A_2 in A and $A = \text{Int } A_1 \cup \text{Int } A_2$ where interiors are taken in A and if A_1 and A_2 have $\mathcal{P}$ then A also has $\mathcal{P}$,*

(iv) *if A is the union of disjoint collection of open subsets $\{A_i\}$ in A, all of which have $\mathcal{P}$ then A has $\mathcal{P}$.*

Then X itself has the property $\mathcal{P}$.

Now we prove our main theorem whose technique was suggested by a paper of LAWSON [10].

THEOREM 3.5. *On the full subcategory of paracompact Hausdorff pairs* (X, A) *the natural homomorphism* $\varrho^*_{(X,A)}: \bar{H}^p(X, A; K) \to {}_G H^p(X, A; K)$ *is an isomorphism for each* $p \geqq 0$.

PROOF. As remarked earlier, it suffices to prove that ϱ^*_X is an isomorphism for every paracompact Hausdorff space X. The proof is by induction on p. For $p=0$, ϱ^*_X is the identity map. Hence we can assume that ϱ^*_X is an isomorphism for all paracompact Hausdorff spaces and for all $n<p$, where $p>0$.

Now, let $\alpha \in \bar{H}^p(X, K)$ and suppose that $\varrho^*_X(\alpha)=0 \in {}_G H^p(X, K)$. We intend to show that $\alpha=0$. Define

$$\mathcal{A} = \{A \subset X \mid \alpha/A = 0 \in \bar{H}^p(A, K)\}.$$

Now for any point-sub-space x, ϱ^*_x is obviously an isomorphism and hence $\varrho^*_x(\alpha)/x = \varrho^*_x(\alpha/x)=0$, implies $\alpha/x=0$ $\forall x \in X$. Because the points are taut subspaces with respect to $\bar{H}^*$, there is a neighbourhood $U(x)$ such that $\alpha/U(x)=0$ and condition (i) of the preceding theorem is satisfied. Condition (ii) is immediate since if $A \in \mathcal{A}$ and A' is a subset of A, not necessarily closed, then $\alpha/A' = (\alpha/A)/A' = 0$. For (iii), notice that if $A_1, A_2 \in \mathcal{A}$ and A_1, A_2 are closed in $A_1 \cup A_2$ such that $A_1 \cup A_2 = \operatorname{Int} A_1 \cup \operatorname{Int} A_2$ with interiors taken in $A_1 \cup A_2$, then the Mayer—Vietoris exact sequence, being a consequence of exactness and excision axioms, applies, i.e., the diagram

$$\begin{array}{ccccccccc}
\to \bar{H}^{p-1}(A_1) \oplus \bar{H}^{p-1}(A_2) & \xrightarrow{\Phi} & \bar{H}^{p-1}(A_1 \cap A_2) & \xrightarrow{\delta} & \bar{H}^p(A_1 \cup A_2) & \xrightarrow{\Psi} & \bar{H}^p(A_1) \oplus \bar{H}^p(A_2) \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\to {}_G H^{p-1}(A_1) \oplus {}_G H^{p-1}(A_2) & \xrightarrow{\Phi} & {}_G H^{p-1}(A_1 \cap A_2) & \xrightarrow{\delta} & {}_G H^p(A_1 \cup A_2) & \xrightarrow{\Psi} & {}_G H^p(A_1) \oplus {}_G H^p(A_2) \to
\end{array}$$

is commutative. The left hand vertical maps are isomorphisms by induction hypothesis. Since $\varrho^*_{A_1 \cup A_2}(\alpha/A_1 \cup A_2) = \varrho^*_X(\alpha)/A_1 \cup A_2 = 0$, and $\Psi(\alpha/A_1 \cup A_2) = (\alpha/A_1, \alpha/A_2) = 0$, a diagram chase implies that $\alpha/A_1 \cup A_2 = 0$. The condition (iv) is evidently satisfied. Hence by Michael's Theorem $X \in \mathcal{A}$ and ϱ^*_X is a monomorphism.

To show that ϱ^*_X is also an epimorphism, let $\alpha \in {}_G H^p(X, K)$. Define

$$\mathcal{A}' = \{A \subset X \mid \alpha/A \in \varrho^*_A(\bar{H}^p(A, K))\}.$$

Since ${}_G H^p(x) = \bar{H}^p(x) = 0$ for $\forall p>0$, we have by Proposition 3.3 a neighbourhood $U(x)$ such that $\alpha/U(x)=0$. Hence $\alpha/U(x) \in \varrho^*_{U(x)}(\bar{H}^p(U(x), K))$ and condition (i) is satisfied. Commutativity of the diagram

$$\begin{array}{ccc}
\bar{H}^p(A, K) & \xrightarrow{\varrho^*_A} & {}_G H^p(A, K) \\
{\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} \\
\bar{H}^p(A', K) & \xrightarrow{\varrho^*_{A'}} & {}_G H^p(A', K)
\end{array}$$

for any subspace A' of A yields (ii). For (iii), let A_1 and A_2 be as before, then again we have the commutative diagram

$$\begin{array}{ccccccccc}
\to \bar{H}^{p-1}(A_1 \cap A_2) & \xrightarrow{\delta} & \bar{H}^p(A_1 \cup A_2) & \xrightarrow{\Psi} & \bar{H}^p(A_1) \oplus \bar{H}^p(A_2) & \xrightarrow{\Phi} & \bar{H}^p(A_1 \cap A_2) \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\to {}_G H^{p-1}(A_1 \cap A_2) & \xrightarrow{\delta} & {}_G H^p(A_1 \cup A_2) & \xrightarrow{\Psi} & {}_G H^p(A_1) \oplus {}_G H^p(A_2) & \xrightarrow{\Phi} & {}_G H^p(A_1 \cap A_2) \to
\end{array}$$

of Mayer—Vietoris exact sequences. Now $\alpha/A_1 \in \varrho^*_{A_1}(\bar{H}^p(A, K))$, $\alpha/A_2 \in \varrho^*_{A_2}(\bar{H}^p(A, K))$ and extreme left vertical map is isomorphism by inductive hypothesis. Also the extreme right vertical map is monomorphism as we have already proved. Hence by a diagram chase we find that $\alpha/A_1 \cup A_2 \in \varrho^*_{A_1 \cup A_2}(\bar{H}^p(A_1 \cup A_2))$ and (iii) is satisfied. Finally (iv) is true by Proposition 3.2. Hence by Michael's Theorem $X \in \mathscr{A}'$ and this shows that ϱ^*_X is an epimorphism, which completes the proof of the theorem. Q.E.D.

Now we derive some immediate important consequences of the above theorem. First we have

Corollary 3.6 (Homotopy Axiom for G-cohomology). *G-cohomology satisfies the homotopy axiom on the category of paracompact Hausdorff pairs.*

Proof. Let $\Pi:(X, A)\times I \to (X, A)$ denote the canonical projection, where (X, A) is a paracompact Hausdorff pair. As is well known, homotopy axiom is equivalent to the fact that $\Pi^*: {}_GH^*(X, A; K) \to {}_GH^*((X, A)\times I, K)$ is an isomorphism. Therefore, by five lemma, it suffices to prove it when $A=\Phi$. But this follows because $X\times I$ is paracompact, $\Pi^*: \bar{H}^*(X, K) \to \bar{H}^*(X\times I, K)$ is an isomorphism and the diagram

$$\begin{array}{ccc} \bar{H}^*(X, K) & \xrightarrow{\Pi^*} & \bar{H}^*(X\times I, K) \\ \downarrow{\scriptstyle \varrho^*_X} & & \downarrow{\scriptstyle \varrho^*_{X\times I}} \\ {}_GH^*(X, K) & \xrightarrow{\Pi^*} & {}_GH^*(X\times I, K) \end{array}$$

is commutative with both vertical maps as isomorphism. Q.E.D.

Corollary 3.7 (Tautness Property). *Let A be a closed subset of a paracompact Hausdorff space X. Then A is a taut subspace of X with respect to G-cohomology theory.*

Proof. Let $\{N\}$ denote the family of all paracompact neighbourhoods of A in X. This family, being cofinal in the set $\{U\}$ of all neighbourhoods of A in X, we have for $\forall p \geqq 0$ and every R-module K,

$$\varinjlim {}_GH^p(U, K) = \varinjlim {}_GH^p(N, K) = \varinjlim \bar{H}^p(N, K) = \varinjlim \bar{H}^p(U, K) =$$
$$= \bar{H}^p(A, K) = {}_GH^p(A, K)$$

since A is taut with respect to Alexander—Spanier cohomology. Q.E.D.

For Alexander—Spanier cohomology, amore general result is known viz., every retract of an arbitrary topological space is taut (cf. [4]). It is not known if the same holds for G-cohomology.

Now the proof of the following theorems follow from the corresponding theorems of Alexander—Spanier cohomology without any change.

Corollary 3.8 (Strong Excision Theorem). *Let (X, A) and (Y, B) be two paracompact Hausdorff pairs and $f:(X, A)\to(Y, B)$ be a closed continuous map such that f induces a $1-1$ map from $X-A$ onto $Y-B$. Then for every $p\geqq 0$ and every R-module K, $f^*: {}_GH^p(Y, B, K) \to {}_GH^p(X, A, K)$ is an isomorphism.*

Corollary 3.9 (Continuity Property). *Let $\{(X_j, A_j)\}$ be a family of compact Hausdorff pairs in some space X, directed downward by inclusion and let $(X, A)=$*

$=(\cap X_j, \cap A_j)$. *Then the inclusion* $i_j:(X, A)\to(X_j, A_j)$ *induce an isomorphism*

$$\{i_j^*\}: \varinjlim {}_GH^p(X_j, A_j, K) \to {}_GH^p(X, A, K)$$

for every $p\geqq 0$ *and every R-module K.*

Weak continuity property, being equivalent to continuity property also holds for G-cohomology.

COROLLARY 3.10 (Vietoris—Begle Theorem). *Let X, X′ be paracompact Hausdorff spaces and* $f:X\to X'$ *be a continuous closed surjection. Suppose there is an* $n>0$ *such that* ${}_GH^p(f^{-1}(x), K)=0$ *for every* $x\in X$ *and for every* $p<n$. *Then* $f^*:{}_GH^p(X', K)\to{}_GH^p(X, K)$ *is an isomorphism for every* $p<n$ *and monomorphism for* $p=n$. Q.E.D.

Up to this writing, we have not been able to prove the homotopy axiom for arbitrary pairs (X, A) for G-cohomology, which is known to be true because of the corresponding sheaf theoretic result of BREDON [1]. The method of acyclic models analogous to Alexander—Spanier cohomology is disappointing as it stands. Also we do not know whether or not there is a space for which the G-cohomology is not isomorphic to the Alexander—Spanier cohomology. In dimension zero and one they are certainly isomorphic as shown by GROTHENDIECK ([9] page 176). For higher dimensions Grothendieck's counter example given there utilizes a non-constant sheaf as the coefficient sheaf.

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INVERSE SYSTEMS AND THE TRANSLATION EQUATION ON TOPOLOGICAL SPACES

By

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1. Introduction

Let X, Y be sets and let $(\alpha_u)_{u\in Y}$ be a system of permutations on X indexed by elements in Y. Let the inverse permutation of each α_u be denoted by α_u^{-1} and let $f: X\times Y\times Y\to X$ be defined by $f(x, u, v)=\alpha_v(\alpha_u^{-1}(x))$ for all $x\in X$, $u, v\in Y$. The mapping f so induced by α satisfies the translation equation $f(f(x, u, v), v, w)= =f(x, u, w)$ for all $x\in X$, $u, v, w\in Y$. This functional equation has been treated by J. ACZÉL [1] and since then many results have been obtained.

Let us consider the flow of particles in a closed chamber. Let the various positions in the chamber be realized as points in R^3; and hence the chamber can be treated as a given subset X of R^3. If $F(x, u, v)$ measures the position vector at time instance v of the particle whose position vector was x at an earlier instance u, then F satisfies the translation equation $F(F(x, u, v), v, w)=F(x, u, w)$ for all $x\in X\subseteq R^3$, $u, v, w\in Y$ with $u\leqq v\leqq w$. Here, Y is the time interval within which observation is made. Thus, in this sense, it is interesting to study the translation equation when the parameters are restricted.

In this note we study systems of homeomorphisms on topological spaces, their inverse systems, and the connection between such systems and the translation equation.

2. Systems of homeomorphisms on a topological space and their inverses

The following topological result will be needed for our main theorem in Section 3.

PROPOSITION 2.1. *Let X be a locally compact Hausdorff and locally connected topological space, and let Y be a topological space. Let $h: X\times Y\to X$ be a continuous mapping such that for each $y\in Y$, the mapping $x\to h(x, y)$ is a homeomorphism on X. Then the inverse system $h^{-1}: X\times Y\to X$ defined by $h^{-1}(x, y)=x'$ iff $h(x', y)=x$ for all $x\in X$ and $y\in Y$ is also continuous on $X\times Y$.*

PROOF. Let $x_0\in X$ and $y_0\in Y$ be arbitrarily given and let U be any given compact neighbourhood of $x_0'=h^{-1}(x_0, y_0)$. We will show the existence of a neighbourhood of (x_0, y_0) which is mapped into U under h^{-1} and thus establish the continuity of h^{-1} at (x_0, y_0).

Since $x\to h(x, y_0)$ is a homeomorphism, the image $V=h(U, y_0)$ of U under it is a compact neighbourhood of $x_0=h(x_0', y_0)$. The boundary ∂V (possibly void) of V and $\{x_0\}$ are disjoint compact sets in the Hausdorff space X. Hence there exist a neighbourhood W_1 of ∂V and a neighbourhood W_2 of x_0 such that $W_1 \cap W_2 = \emptyset$. Since X is locally connected, we may assume that W_2 is connected.

For each $s \in \partial U$, $h(s, y_0) \in h(\partial U, y_0) = \partial h(U, y_0) = \partial V$ and so W_1 is a neighbourhood of $h(s, y_0)$. Since h is (jointly) continuous, there exist open neighbourhoods N_s and M_s of s and y_0 respectively such that $h(N_s \times M_s) \subseteqq W_1$. The family $\{N_s\}_{s \in \partial U}$ is an open cover of the compact set ∂U and thus there exists a finite subfamily $\{N_s\}_{s \in I}$ which covers ∂U. Now let $M_1 = \cap \{M_s | s \in I\}$ which is then a neighbourhood of y_0. Furthermore $h(\partial U, M_1) \subseteqq W_1$.

Since W_2 is a neighbourhood of $x_0 = h(x_0', y_0)$ and h is continuous, there exists a neighbourhood M_2 of y_0 such that $h(x_0', M_2) \subseteqq W_2$.

Now $M = M_1 \cap M_2$ is a neighbourhood of y_0 with $h(\partial U, M) \subseteqq W_1$ and $h(x_0', M) \subseteqq W_2$.

For each $m \in M$, we have $h(\partial U, m) \subseteqq W_1$; and since $W_1 \cap W_2 = \emptyset$ we have $h(\partial U, m) \cap W_2 = \emptyset$. Now $h(U, m) \cap W_2$ is closed in W_2 as $h(U, m)$ is compact and thus closed in X. Since $\partial h(U, m) \cap W_2 = h(\partial U, m) \cap W_2 = \emptyset$, $h(U, m) \cap W_2 = (\operatorname{Int} h(U, m)) \cap W_2$ is also open in W_2. As $h(x_0', m) \in h(U, m) \cap W_2$, the set $h(U, m) \cap W_2$ is also non-empty. Since W_2 is connected, we get $h(U, m) \cap W_2 = W_2$ and so $h(U, m) \supseteqq W_2$. Thus $h^{-1}(W_2, m) \subseteqq U$. As $m \in M$ is arbitrary, $h^{-1}(W_2, M) \subseteqq U$. Since $W_2 \times M$ is a neighbourhood of (x_0, y_0), this proves the continuity of h^{-1} at (x_0, y_0). As (x_0, y_0) is arbitrary, h^{-1} is continuous on $X \times Y$.

The following examples will serve to show that neither the local connectedness nor the local compactness of X can be omitted from the hypotheses on X in Proposition 2.1.

EXAMPLE 2.2. Let $N = \{1, 2, 3, \dots\}$ be the set of all natural numbers. Let $X = \{1/n | n \in N\} \cup N \cup \{0\}$ and $Y = \{1/n | n \in N\} \cup \{0\}$ be subspaces of the reals under the usual topology. The space X is locally compact Hausdorff but not locally connected at the origin 0. Let the mapping $h: X \times Y \to X$ be defined by

$$h(x, 1/n) = \begin{cases} x & \text{if } 1/n \leqq x \leqq n \text{ or } x = 0 \\ 1/(m+1) & \text{if } 0 < x = 1/m < 1/n \\ 1/(n+1) & \text{if } x = n+1 \\ x-1 & \text{if } n+1 < x \end{cases}$$

and $h(x, 0) = x$ for all $x \in X$. It is not difficult to verify that for each fixed $y \in Y$, the mapping $x \to h(x, y)$ is a homeomorphism on X. Furthermore, h is jointly continuous on $X \times Y$.

The inverse system $h^{-1}: X \times Y \to X$ is however not continuous at $(0, 0) \in X \times Y$.

EXAMPLE 2.3. Let $X = C_0[0, 1]$ be the linear space of all continuous real-valued functions defined on the interval [0, 1] vanishing at the origin 0. Let X be considered under the topology induced by the usual absolute norm

$$|f| = \sup_{t \in [0,1]} |f(t)|.$$

The space X is locally connected Hausdorff but not locally compact (cf. Theorem 9.2 in [2]).

Let $Y = \{1/n | n \in N\} \cup \{0\}$ be considered as a topological subspace of the reals under the usual topology.

Let $\{\varphi_n | n\in N\}$ be a sequence of continuous functions on [0, 1] defined by

$$\varphi_n(t) = \begin{cases} 1 & \text{if } 1/n \leqq t \\ 1/(n+1)+n^2(t-1/(n+1)) & \text{if } 1/(n+1) \leqq t \leqq 1/n \\ 1/(n+1) & \text{if } 0 \leqq t \leqq 1/(n+1) \end{cases}$$

for each $n\in N$.

We now consider the mapping $h: X\times Y \to X$ defined by

$$\begin{cases} h(f, 1/n) = \varphi_n f \\ h(f, 0) \quad = f \end{cases}$$

for all $f\in X$, $n\in N$. Here $\varphi_n f$ is the pointwise product of φ_n and f. Since for each fixed φ_n, the mapping $f\to\varphi_n f$ is a nonsingular linear operator on $X=C_0[0, 1]$, and thus for each $y\in Y$, the mapping $f\to h(f, y)$ is a homeomorphism on X.

The mapping h is continuous on $X\times Y$. In fact it is sufficient to show that h is continuous at $(f_0, 0)\in X\times Y$ for any given $f_0\in X$. For this purpose, let $\varepsilon>0$ be arbitrarily given. Since f_0 is continuous and $f_0(0)=0$, there exists a sufficiently large integer $m\in N$ such that

$$|f_0(t)| < \varepsilon/3 \quad \text{for all} \quad t\in[0, 1/m].$$

Then for any $f\in X$ with $|f-f_0|<\varepsilon/3$ and for any $n\geqq m$, we have

$$|h(f, 1/n)-h(f_0, 0)| = |\varphi_n f - f_0| = \sup_{t\in[0,1]} |\varphi_n(t) f(t) - f_0(t)| =$$

$$= \sup\big(\sup_{t\in[0,1/n]} |\varphi_n(t) f(t) - f_0(t)|, \quad \sup_{t\in[1/n,1]} |\varphi_n(t) f(t) - f_0(t)|\big) \leqq$$

$$\leqq \sup\big(\sup_{t\in[0,1/n]} |\varphi_n(t)|\cdot|f(t)| + |f_0(t)|, \quad \sup_{t\in[1/n,1]} |f(t)-f_0(t)|\big) \leqq$$

$$\leqq \sup\big(\sup_{t\in[0,1/m]} |f(t)| + \sup_{t\in[0,1/m]} |f_0(t)|, \quad \varepsilon/3\big) \leqq \sup_{t\in[0,1/m]} |f(t)| + \varepsilon/3 \leqq$$

$$\leqq \sup_{t\in[0,1/m]} |f(t)-f_0(t)| + \sup_{t\in[0,1/m]} |f_0(t)| + \varepsilon/3 < \varepsilon/3+\varepsilon/3+\varepsilon/3 = \varepsilon.$$

This proves the continuity of h.

The inverse system $h^{-1}: X\times Y\to X$ is not continuous on $X\times Y$. It is sufficient to show that h^{-1} is not continuous at $(0, 0)\in X\times Y$. For this purpose, we choose a sequence of functions $\{f_n\}$ such that $\{f_n\}$ converges to 0 in X and that $f_n(1/n+1)=1/n+1$ for all $n\in N$. Obviously such a sequence of functions exists in $C_0[0, 1]$. On the other hand, we have the inequality

$$|h^{-1}(f_n, 1/n) - h^{-1}(0, 0)| = |\varphi_n^{-1} f_n| = \sup_{t\in[0,1]} |f_n(t)/\varphi_n(t)| \geqq f_n(1/n+1)/\varphi_n(1/n+1) = 1$$

for all $n\in N$, and thus h^{-1} is not continuous at $(0, 0)$.

3. Representation theorems for the translation equation

THEOREM 3.1. *Let X be a locally compact Hausdorff and locally connected topological space, and let Y be a topological space on which a closed total transitive relation $R \subseteqq Y \times Y$ is defined. Then the following statements concerning f are equivalent.*

(A) *$f: X \times R \to X$ is a continuous solution of the restricted translation equation*

$$f(f(x, u, v), v, w) = f(x, u, w) \tag{3.2}$$

for all $x \in X$, $(u, v), (v, w) \in R$. Furthermore for each $(u, v) \in R$, the mapping $x \to f(x, u, v)$ is a homeomorphism on X.

(B) *There exists a continuous mapping $g: X \times Y \to X$ such that for each $u \in Y$ the mapping $x \to g(x, u)$ is a homeomorphism on X, and g represents f through the identity*

$$f(x, u, v) = g(g^{-1}(x, u), v) \tag{3.3}$$

for all $x \in X$, $(u, v) \in R$. Here, $g^{-1}: X \times Y \to X$ is the inverse system of g defined by the equivalence $g^{-1}(x, u) = x'$ iff $g(x', u) = x$.

Furthermore, for any given $u_0 \in Y$, there is a unique representative g for f which fulfils the additional initial condition

$$g(x, u_0) = x \tag{3.4}$$

for all $x \in X$.

PROOF. Let f be a mapping satisfying the descriptions in (A). Let $u_0 \in Y$ be arbitrarily fixed for the rest of the discussion.

Let $g: X \times Y \to X$ be defined by

$$\begin{cases} g(x, u) = f(x, u_0, u) \quad \text{if} \quad (u_0, u) \in R; \quad \text{and} \quad g(x, u) = x' \\ \text{where } x' \text{ is the unique member of } X \text{ such that} \\ f(x', u, u_0) = x \quad \text{if} \quad (u, u_0) \in R. \end{cases} \tag{3.5}$$

The above definition of g is valid, and is clear in view of the following observation. If we put $u = v = w$ in the translation equation (3.2), we get $f(f(x, u, u), u, u) = f(x, u, u)$ and so

$$f(x, u, u) = x \quad \text{for all} \quad x \in X, \quad u \in Y. \tag{3.6}$$

Since the relation R is total for any $u \in Y$ we have either $(u_0, u) \in R$ or $(u, u_0) \in R$. Thus the mapping g in (3.5) has always a value for any $x \in X$, $u \in Y$. On the other hand, if (u_0, u) and (u, u_0) are both in R, then from the translation equation and (3.6) we get $f(f(x, u_0, u), u, u_0) = f(x, u_0, u_0) = x$. In this case $x' = f(x, u_0, u)$ and thus $g(x, u)$ is well-defined.

Since for each $(u, v) \in R$, the mapping $x \to f(x, u, v)$ is a homeomorphism on X; thus for each $u \in Y$ it is clear that $x \to g(x, u)$ is a homeomorphism on X.

To show that g is continuous on $X \times Y$, we decompose Y into the union of two sets $Y_1 = \{u \in Y \mid (u_0, u) \in R\}$ and $Y_2 = \{u \in Y \mid (u, u_0) \in R\}$. Now Y_1 and Y_2 are closed subsets of Y as R is closed in $Y \times Y$, and $Y = Y_1 \cup Y_2$ as R is total. Thus it is sufficient to show that the restrictions of g to $X\times Y_1$ and $X\times Y_2$ are continuous. Since on $X\times Y_1$, $g(x,u)=f(x,u_0,u)$ and so $g|_{X\times Y_1}$ is continuous. The restriction of g on $X\times Y_2$ is the inverse of the continuous mapping $(x,u)\to f(x,u,u_0)$ in its first variable and therefore is continuous by Theorem 2.1.

We now proceed to show that f is represented by g through (3.3). Let $x\in X$, $(u,v)\in R$ be given arbitrarily. There are three cases to be considered.

Case 1. Suppose that $(v,u_0)\in R$. In this case we get from the translation equation $f(f(x,u,v),v,u_0)=f(x,u,u_0)$ and so $f(x,u,v)=g(f(x,u,u_0),v)=g(g^{-1}(x,u),v)$ as asserted in (3.3).

Case 2. Suppose that $(u,u_0)\in R$ and $(u_0,v)\in R$. In this case we get from the translation equation $f(x,u,v)=f(f(x,u,u_0),u_0,v)=g(g^{-1}(x,u),v)$ as asserted in (3.3).

Case 3. Suppose that $(u_0,u)\in R$. In this case we let y be the point such that $f(y,u_0,u)=x$. From the translation equation we get $f(x,u,v)=$ $=f(f(y,u_0,u),u,v)=f(y,u_0,v)=g(y,v)=g(g^{-1}(x,u),v)$ and again (3.3) is satisfied.

Since to each $(u,v)\in R$ at least one of the above three cases occurs, this proves (3.3).

Conversely, let g be a mapping satisfying the descriptions in (B) and let f be defined by g through (3.3). Then g^{-1} is also continuous on $X\times Y$ by Theorem 2.1. Thus the continuity of f on $X\times R$ follows from that of g and of g^{-1}. The mappings $x\to f(x,u,v)$ are homeomorphisms on X as g induces homeomorphisms in its first variable. The mapping f satisfies the translation equation as $f(f(x,u,v),v,w)=$ $=g(g^{-1}(g(g^{-1}(x,u),v),v),w)=g(g^{-1}(x,u),w)=f(x,u,w)$.

Thus the equivalence of (A) and (B) is established. Furthermore, the mapping g defined by (3.5) satisfies the initial condition (3.4) because of (3.6). If $\tilde{g}:X\times Y\to X$ represents f as described in (B) and fulfils the initial condition (3.4), then we have $g(g^{-1}(x,u),v)=\tilde{g}(\tilde{g}^{-1}(x,u),v)$ for all $x\in X$ and $(u,v)\in R$. If we put $u=u_0$ we get $g(x,v)=\tilde{g}(x,v)$ for all $x\in X$ and all v with $(u_0,v)\in R$, thus the restriction of g to $X\times\{u\,|\,(u_0,u)\in R\}$ is equal to that of $\tilde{g}$. Similarly g and $\tilde{g}$ are equal on $X\times\{u\,|\,(u,u_0)\in R\}$, and so $g=\tilde{g}$. This establishes the uniqueness of g when the initial condition (3.4) is imposed.

COROLLARY 3.7. *Let X be a set and let Y be a set on which a total transitive relation $R\subseteqq Y\times Y$ is defined. Then the following statements concerning f are equivalent.*

(A) *$f:X\times R\to X$ satisfies the restricted translation equation $f(f(x,u,v),v,w)=$ $=f(x,u,w)$ for all $x\in X$, (u,v), $(v,w)\in R$, and that for each $(u,v)\in R$, $x\to f(x,u,v)$ is a permutation on X.*

(B) *There exists a mapping $g:X\times Y\to X$ such that for each $u\in Y$, $x\to g(x,u)$ is a permutation on X and g represents f through the identity $f(x,u,v)=$ $=g(g^{-1}(x,u),v)$ for all $x\in X$, $(u,v)\in R$.*

Furthermore, for any given $u_0\in Y$, there exists a unique representative g for f which fulfils the additional initial condition $g(x,u_0)=x$ for all $x\in X$.

PROOF. We can provide X with the discrete topology and X is then locally compact Hausdorff and locally connected. We can also provide Y with the discrete topology under which all relations are closed in $Y\times Y$. Under these topologies all the continuity postulates are redundant, and thus Theorem 3.1 includes this non-topological result as a special case.

REMARK 3.8. Each solution f of the restricted translation equation in Theorem 3.1 can be extended uniquely to a continuous mapping $\bar{f}: X\times Y\times Y\to X$ which is a solution of the unrestricted translation equation $f\big(f(x, u, v), v, w\big)=f(x, u, w)$ for all $x\in X$, $u, v\in Y$. In fact, this is done by extending (3.3) to $\bar{f}(x, u, v)=$ $=g\big(g^{-1}(x, u), v\big)$ for all $x\in X$, $u, v\in Y$.

Since we can take R to be the full $Y\times Y$, Theorem 3.1 includes the unrestricted translation equation as a special case.

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COUNTING FINITE POSETS

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Suppose $N=\{1, 2, \dots\}$ is the collection of natural numbers and suppose P is the collection of finite posets. Among the various gradings $v: P \to N$ on P, the most natural one is probably the grading $v(X)=|X|$, where $|X|$ denotes the number of vertices of X. The function $\sigma(n)=|v^{-1}(n)|$, where the latter quantity denotes the cardinality of the set $v^{-1}(n)$, is a function of some interest and a considerable literature exists concerning the determination of the function σ. In this paper we are not so much interested in the precise values of σ as in its relation to functions of a similar type. Indeed, if $L \subseteq P$, we let $\sigma(L)$ be the following power series:

$$\sigma(L) = \sum_{n=1}^{\infty} a_n z^n, \quad \text{where} \quad a_n = |v^{-1}(n) \cap L|.$$

We shall refer to $\sigma(L)$ *as the counting series of L*. If $\sigma(P)=\sigma$, i.e., $\sigma(P)=\sigma= = \sum_{n=1}^{\infty} \sigma(n) z^n$, and if L is suitably defined, then although neither σ nor $\sigma(L)$ may be explicitly known, it is often possible to find meaningful and interesting relations between σ and $\sigma(L)$. In particular, we may be able to solve the following

PROBLEM. Given $L \subsetneq P$, find the coefficients $a_1, a_2, \dots$ such that:

$$\sigma(L) = a_1\sigma + a_2\sigma^2 + \dots .$$

To give only some examples, we can solve this problem for the collection of posets which are not connected, for the posets which are not ordinal sums of other posets, and for a host of other subsets L of P.

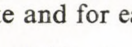
Most of the techniques for counting posets developed in this paper have ready applications to a great variety of other counting problems, where in general we deal with a grading $v: K \to N$ such that $|v^{-1}(n)|$ is finite and with a suitably defined subset of K.

Thus, e.g., as a trivial example, the technique which allows us to write down the "number of connected posets", i.e., the counting series of the set of connected posets described in terms of σ, is precisely the same technique which allows us to determine the counting series of the irreducible polynomials over a finite field in terms of the counting series of all monic polynomials over the same field. Here, if the field has q elements, and if K is the set of all monic polynomials of degree $\geqq 1$, we let $v(p(x))$ denote the degree of $p(x)$, with a corresponding counting series $pz + p^2z^2 + \dots + p^tz^t + \dots$. An explicit solution is possible here because the counting series of all monic polynomials is known.

Our usual subsets L of P are of the following type. Suppose X is a poset labeled in some way, say by numbering the vertices $\{x_1, \dots, x_n\}$, where $|X|=n$. If $A_1, \dots, A_n$ are arbitrary posets, then by $X(A_1, \dots, A_n)$ we denote the poset obtained by replacing x_i in X by the poset A_i, with the additional condition that if $x_i < x_j$, then for $a_i \in A_i$, $a_j \in A_j$, we have $a_i < a_j$. Thus, e.g., $X(A, \dots, A)$ is the lexicographic product of X and A. A poset $X(A_1, \dots, A_n)$ is *a substitution* (*into* X). By $L=C(X)$ we denote the collection of all substitutions into X.

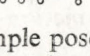
Thus, if $X=1$, the one point poset, then $C(1)=P$. If $X=2$, the poset consisting of two loose points (or the diagonal relation), then $C(2)$ is the collection of posets which are not connected. If $X=1\oplus 1=C_2$, is the ordinal sum of the one point poset with itself, then $C(C_2)$ is the collection of all posets which are ordinal sums. If $X \notin C(C_2)$ we shall call the poset X *flat*. For many types of posets X, if $L=C(X)$, we can describe the coefficients $a_1, a_2, \dots$ such that $\sigma(L)=a_1\sigma+a_2\sigma^2+\dots$ recursively, i.e., we can determine polynomials $g_1(x_1), g_2(x_1, x_2), \dots, g_k(x_1, \dots, x_k), \dots$ *independently of* σ *and* $\sigma(L)$ such that $a_1=g_1(\sigma(1))$, $a_2=g_2(\sigma(1), \sigma(2)), \dots, a_k=$ $=g_k(\sigma(1), \dots, \sigma(k))$, etc. Sometimes we can do even better as we shall show below. As far as we are concerned in this paper, if we can give a solution of the type just described we will usually consider such a solution as being essentially sufficient. For numerical work illustrating a point or two we shall normally use known values for the first several coefficients of σ. As a whole we avoid rather carefully the problem of determining σ directly, since that would not help us with the solutions of the problems of the type described above.

For use in examples we determine the counting series of the set L, where L is the collection of posets of height one, the collection of posets of height $k-1$ such that all maximal chains have length $k-1$, and by summation, the counting series of the collection of all posets consisting of sums of posets whose components all have the property that the lengths of maximal chains are equal to the height of the poset, where the height is the length of the longest maximal chain. The argument involves cycle indices and (0, 1)-boxes defined in a different way from the usual incidence matrices. The type of argument used can probably be generalized to other classes of posets, such as the posets X such that the interval between two vertices is a poset such that all maximal chains have the same length. If X is a poset such that all maximal chains have length equal to the height, then we shall call the poset X *regular*.

If we remove the one point poset from consideration, then we can in principle list the posets, $2, C_2, 1\oplus 2, 2\oplus 1, C_3, 1+C_2, \dots$. Having listed the posets in this manner, we can sieve out those posets which are not substitutions as follows: remove all posets in $C(2)$ other than 2 from the list, remove all posets in $C(C_2)$ other than C_2 from the list, ... If at each stage we keep the first poset we encounter after the crossing out process and assuming that if $|X|<|Y|$, then X appears before Y in the list, then we will end up with the collection of *pure posets*, i.e., posets X which can only be obtained as substitutions by substitutions $X(1, \dots, 1)=X$. The collection of pure posets in infinite and for each $n\geqq 4$, there is a pure poset X such that $|X|=n$. Indeed, the posets  (long W's) are all pure posets as well as an enormous variety of others. It turns out that if X and Y are distinct pure posets, then $C(X)\cap C(Y)=\varnothing$, so that we do in fact generate a partition of P in this manner.

Unfortunately it seems that the occurrence of pure posets is a matter which is hardly predictable, and although for every pure poset X we encounter we may be able to relate the counting series of $C(X)$ to σ, we cannot generalize away from the particular to say that there are so many pure posets X with such and such a counting series $\sigma(C(X))$.

The apparent irregularity in the occurrence of pure posets and associated problems in describing the counting series, certainly does not encourage one to seek to determine σ along these lines. It may also be an indication why the problem of determining σ itself is at least quite difficult.

Several other classes of posets which occur rather naturally are the following. Those posets X such that if $|X|=n$, then $\sigma(C(X))=\sigma^n$. Those posets X such that if $|X|=n$ then $\sigma(C(X))=\sigma^n+a_{n+1}\sigma^{n+1}+\ldots+a_{n+k}\sigma^{n+k}$. Those posets X such that if $|X|=n$, then $\sigma(C(X))=\sigma^n-a_{n+1}\sigma^{n+1}+a_{n+2}\sigma^{n+2}\ldots$, $a_{n+i}\geqq 0$. We shall call these posets *simple*, *short* and *alternating* respectively. One proves easily that all chains are alternating. Similarly posets  (long W's with an even number of vertices) are all simple. Other than simple posets we do not possess any examples of short posets. The poset $1+C_2$ is not an alternating poset, i.e., $\sigma(C(1+C_2))$ does not have an alternating expression in terms of σ. We give a characterization of simple posets.

The counting series for subsets L of P have obvious measure-like properties. Thus, if $\{L_1, L_2, \ldots, L_k, \ldots\}$ is an infinite collection of mutually disjoint subsets of P, and if $\tau_n=\sigma(L_1\cup\ldots\cup L_n)=\sigma(L_1)+\ldots+\sigma(L_n)$, then the sequence $\{\tau_1, \tau_2, \ldots\}$ is convergent in the power series ring with respect to the topology induced by the order ideals $(Z), (Z^2), \ldots$. Thus $\tau=\lim_{n\to\infty}\tau_n$ is defined, i.e., $\sum_{i=1}^{\infty}\sigma(L_i)=\tau$ is well defined. It is clear that $\sigma(L_1\cup L_2)=\sigma(L_1)+\sigma(L_2)-\sigma(L_1\cap L_2)$ and that every subset L of P has a counting series.

Other authors have discussed similar counting functions. Thus, e.g., BUTLER and MARKOVSKY [4] discuss the function $P(n, k)$, where $P(n, k)$ denotes the number of posets X containing n vertices and k pairs (x_i, x_j) with $x_i < x_j$. To our knowledge the particular approach to counting posets used here has not been discussed before. As a type of generating function the notion of a counting series is of course not particularly new to combinatorics in general. Due to the nature of the material we thought it better to let the discussion run continuously with frequent breaks into sections rather than end up with a thousand or so lemmas, propositions and theorems.

Some general rational functions. If $C[[Z]]$ denotes the power series ring over the complex numbers, then we let $\sigma=\sum_{i=1}^{\infty}x_iz^i$ and $\tau=\sum_{j=1}^{\infty}w_jz^j$ be two arbitrary elements of order $\geqq 1$, i.e., elements of the ideal (z) of $C[[z]]$. Let $x_1\neq 0$. If we consider the expression

$$\tau = A_1\sigma+A_2\sigma^2+\ldots+A_k\sigma^k+\ldots,$$

then we can compute the rational functions $A_1, A_2, \ldots, A_k, \ldots$ recursively. Indeed, we express the coefficient of z^m in σ^k by writing

$$G_{m,k} = \sum_{i_1+\ldots+i_k=m} x_{i_1}\ldots x_{i_k} = \sum_{\substack{\mu_1+\ldots+\mu_m=k \\ \mu_1+2\mu_2+\ldots+m\mu_m=m}} \binom{k}{\mu_1, \ldots, \mu_m} x_1^{\mu_1}\ldots x_m^{\mu_m}.$$

Next, we have $w_1=A_1G_{1,1}=A_1x_1$, whence $A_1=w_1/x_1$. Suppose that $A_1, \ldots, A_{n-1}$ have already been computed. Then, $w_n=A_1G_{n,1}+A_2G_{n,2}+\ldots+A_{n-1}G_{n,n-1}+ +A_nG_{n,n}$, whence since $G_{n,n}=x_1^n$, it follows that:

$$A_n = (w_n-A_1G_{n,1}-A_2G_{n,2}-\ldots-A_{n-1}G_{n,n-1})/x_1^n.$$

Since $G_{n,1}=x_n$, and since $A_j=A_j(x_1, \ldots, x_j|w_1, \ldots, w_j)$, for $1\leqq j\leqq n-1$, it follows that $A_n=A_n(x_1, \ldots, x_n|w_1, \ldots, w_n)$ is a rational function in the variables $\{x_1, \ldots, x_n, w_1, \ldots, w_n\}$.

As we shall frequently be interested in re-expressing one power series in terms of another, the functions $A_n(x_1, \ldots, x_n|w_1, \ldots, w_n)$ become quite important. We should note that we can write the functions A_n in the form

$$P_n(x_1, \ldots, x_n|w_1, \ldots, w_n)/x_1^{\binom{n+1}{2}},$$

where $P_n(x_1, \ldots, x_n|w_1, \ldots, w_n)$ is a polynomial in the variables $\{x_1, \ldots, x_n, w_1, \ldots, w_n\}$ with integral coefficients.

Many properties of $P_n(x_1, \ldots, x_n|w_1, \ldots, w_n)$ can be extracted by simply taking different pairs of series σ and τ where the coefficients $A_1, \ldots, A_k, \ldots$ can be computed directly.

Thus, e.g., rather trivially, if $w_i=x_i$ for all i, then $\tau=\sigma$, and hence $A_1=1$, $A_2=A_3=\ldots=0$, whence for $n\geqq 2$, we have $P_n(x_1, \ldots, x_n|x_1, \ldots, x_n)=0$. We shall see other examples as we get on.

A class of posets counted. Posets of height 1. The usual obstacle in counting classes of posets somehow involves transitivity. As this example shows, in the case that transitivity is absent, one can actually give a successful count. A poset X has height h if there is a path $x_1<x_2<\ldots<x_{h+1}$ in X and if there is no path $x_1<x_2<\ldots \ldots<x_{h+2}$ in X. In particular, if X has height 0, then X consists of loose points (with the diagonal relation). If $|X|=n$, then we shall denote X by n. The sum of posets X and Y is as usual the disjoint union of X and Y. For example, if $X=m$, $Y=n$, then $X+Y=m+n$.

If we have an $m\times n$ (0, 1)-matrix, then we associate with it a poset of height at most 1 and with $m+n$ vertices by considering the rows as minimal elements and the columns as maximal elements. The poset associated with such a matrix M is obtained by introducing the relation $x_i<y_j$ if $M_{ij}=1$, where $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ are the minimal and maximal elements respectively. The matrix $\begin{bmatrix}1&0&1\\0&1&0\\1&0&0\end{bmatrix}$ has associated with it the poset  , for example.

Consider the transformation group $S_m\times S_n$, which operates on an $m\times n$ matrix $M=(M_{ij})$ as follows: if $\pi\in S_m$, $\varphi\in S_n$, then $(\pi, \varphi)M=N$, where $N_{ij}=M_{\pi(i)\varphi(j)}$. The cycle index of this transformation group is written $P(S_m\times S_n)$.

For example, $P(S_3\times S_2)=\frac{1}{12}(x_1^6+3x_1^2x_2^2+4x_2^3+2x_3^2+2x_6)$. The different matrices are thus obtained by colouring the entries 0 or 1, i.e., we obtain a family of numbers $[m, n]=P(S_m\times S_n; 2, \dots, 2)$. Thus, e.g., $[3, 2]=13$.

We let $\varrho(a)=\sum_{m+n=a}[m, n]=[1, a-1]+[2, a-2]+\dots+[a-1, 1]$.

Now, except for some errors which we can recover, we have accounted for the height 1 posets. A regular height one poset is a poset such that no minimal element is also a maximal element. Let $\varrho_0(a)$ denote the number of regular height one posets X such that $|X|=a$.

If Y is a regular height one poset with $a-1$ elements, then $Y+1$ is a height one poset with one element which has been counted once as a minimal element and once as a maximal element. Thus, $Y+1$ has been counted twice. Thus, the contribution to $\varrho(a)$ of posets of this type is $2\varrho_0(a-1)$. Thus we find that for $a\geqq 2$:

$$\varrho(a) = \varrho_0(a)+2\varrho_0(a-1)+\dots+(a-1)\varrho_0(2)+(a-1).$$

The term $a-1$ arises from the fact that in our sum $\varrho(a)=\sum_{m+n=a}[m, n]$ we have precisely $a-1$ zero matrices corresponding to the case of a loose points.

From this relation it follows easily that:

$$\varrho_0(a+2) = \varrho(a+2)-2\varrho(a+1)+\varrho(a).$$

Since $\varrho_0(2)=1$, $\varrho_0(3)=2$, we have obtained a formula for $\varrho_0(n)$ for all $n\geqq 2$.

Finally, if $\sigma_2(n)$ denotes the number of height one graphs X such that $|X|=n$, then it follows almost immediately that:

$$\sigma_2(n) = \varrho_0(n)+\varrho_0(n-1)+\dots+\varrho_0(2),$$

whence

$$\sigma_2(n+1) = \varrho(n+1)-\varrho(n)-1.$$

Thus one computes $\varrho(2)=2$, $\varrho(3)=6$, $\varrho(4)=15$, $\varrho(5)=36$, ... whence $\sigma_2(2)=1$, $\sigma_2(3)=3$, $\sigma_2(4)=8$, $\sigma_2(5)=20\dots$. Similarly, since $\sigma(n+1)-\sigma(n)=\varrho_0(n+1)$, we have $\varrho_0(3)=2$, $\varrho_0(4)=5$, $\varrho_0(5)=12\dots$.

In general, to determine $[m, n]$ we need a formula for $P(S_m\times S_n)$. Such formulas are standard. We have for $\pi\in S_m$ of cycle type $1^{\lambda_1(\pi)}2^{\lambda_2(\pi)}\dots m^{\lambda_m(\pi)}$ and $\varphi\in S_n$ of cycle type $1^{\lambda_1(\varphi)}2^{\lambda_2(\varphi)}\dots n^{\lambda_n(\varphi)}$, a contribution to $P(S_m\times S_n)$ described by the product $\prod_{k,l} x_{[k,l]}^{(k,l)\lambda_k(\pi)\lambda_l(\varphi)}$. Now the number of distinct permutations in S_m of type $1^{\lambda_1}2^{\lambda_2}\dots k^{\lambda_k}$ is $m!/1^{\lambda_1}\dots k^{\lambda_k}\lambda_1!\dots\lambda_k!=h(m; \lambda_1, \dots, \lambda_k)$, so that our final formula for $P(S_m\times S_n)$ becomes

$$P(S_m\times S_n)=\frac{1}{m!\,n!}\sum_{\substack{\lambda_1+2\lambda_2+\dots+k\lambda_k=m\\ \mu_1+2\mu_2+\dots+l\mu_l=n}} h(m; \lambda_1, \dots, \lambda_k)h(n; \mu_1, \dots, \mu_l)\prod_{i,j} x_{[i,j]}^{(i,j)\lambda_i\mu_j}.$$

Hence for $[m, n]$ we obtain the formula:

$$[m, n]=\frac{1}{m!\,n!}\sum_{\substack{\lambda_1+2\lambda_2+\dots+k\lambda_k=m\\ \mu_1+2\mu_2+\dots+l\mu_l=n}} h(m; \lambda, \dots, \lambda_k)h(n; \mu_1, \dots, \mu_l)2^{\sum_{i,j}(i,j)\lambda_i\mu_j}.$$

We evaluate e.g., $[2, 4]=22$, $[3, 3]=36$, $[1, 5]=6$,

$$\varrho(6) = 92, \quad \sigma_2(6) = 55, \quad \varrho_0(6) = 35.$$

For later use we record the beginning terms of the two counting series:

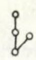
$$\sigma_2 = z^2+3z^3+8z^4+20z^5+55z^8+\dots$$

$$\varrho_0 = z^2+2z^3+5z^4+12z^5+35z^6+\dots.$$

Similarly, if we let τ_2 denote the counting series of the posets of height at most one, then since the counting series of the posets of height 0 is $z+z^2+z^3+\dots$ we have:

$$\tau_2 = z+2z^2+4z^3+9z^4+21z^5+56z^6+\dots.$$

Another class of posets counted. Regular posets of height $k>2$. We consider cycle indexes $P(S_{m_1}\times S_{m_2}\times\dots\times S_{m_k})$ operating on k-tuples $(i_1, \dots, i_k)$, where $1\leqq i_j\leqq m_j$, as a direct product. We colour the elements of the boxes $m_1\times\dots\times m_k$ with colours 0 or 1, where the fact that $(i_1, \dots, i_k)$ has colour 1 denotes the presence of a maximal path of length $k-1$, $x_{1,i_1}<x_{2,i_2}<\dots<x_{k,i_k}$ in the poset X whose elements x_{j,i_j} are distributed over various levels. Thus, e.g., if X is the poset with Hasse diagram:



then we can assign X two boxes, viz., $\{(x_{1,1}, x_{2,1}, x_{3,1}), (x_{1,1}, x_{2,2}, x_{3,1})\}$ and $\{(x_{1,1}, x_{2,1}, x_{3,1}), (x_{1,1}, x_{2,1}, x_{3,2})\}$ with corresponding "maximal path of length 2" colourings $\{1, 0\}$ and $\{1, 0\}$. Thus, we cannot distinguish between X and $1+C_3$, where C_k is the chain of length $k-1$. If we agree to always select the interpretation $X=Y+l$, where Y is a regular poset of height k, then the function

$$\varrho_k(a) = \sum_{m_1+m_2+\dots+m_k=a} [m_1, m_2, \dots, m_k]$$

with $[m_1, m_2, \dots, m_k]=P(S_{m_1}\times S_{m_2}\times\dots\times S_{m_k};\ 2, \dots, 2)$ will allow us to determine the number $\varrho_{k,0}(a)$ of regular posets X of height $k-1$ such that $|X|=a$ explicitly.

If $X=Y+l$, where Y is itself regular of height k, then X has been counted as many times as it is possible to write an ordered partition of l into k non-negative summands. This number is $\binom{l+k-1}{k-1}$, and thus we obtain the formula:

$$\varrho_k(a) = \sum_{l=0}^{a-k} \binom{l+k-1}{k-1} \varrho_{k,0}(a-l)+\binom{a-1}{k-1}$$

where the last term arises from the fact that there are as many zero matrices as there are partitions of a into k positive summands. Starting with $\varrho_{k,0}(k)=1$, we can solve for the quantities $\varrho_{k,0}(a)$ and thus the counting series $\varrho_{k,0}= =z^k+\varrho_{k,0}(k+1)z^{k+1}+\dots+\varrho_{k,0}(a)z^a+\dots$ is also determined.

Thus, e.g., $[1, 1, 2]=3$, $\varrho_3(4)=9$, $\varrho_3(4)=\varrho_{3,0}(4)+3\varrho_{3,0}(3)+3$, whence $\varrho_{3,0}(4)=3$. The posets are $2\oplus 1\oplus 1$, $1\oplus 2\oplus 1$ and $1\oplus 1\oplus 2$.

If we want to count the collection of all posets X whose components are regular of arbitrary height, then we observe that we can decompose X uniquely in the form $X=X_1+\ldots+X_m$, where X_i has height k_i and $k_i\neq k_j$. Thus for this type of poset we have a contribution $\varrho_{k_1,0}, \ldots, \varrho_{k_m,0}$, so that the counting series in question can be written:

$$\tau_\infty = \sum_{i=1}^{\infty} \varrho_{i,0} + \sum_{i=1, i<j}^{\infty} \varrho_{i,0}\varrho_{j,0} + \sum_{i=1, i<j<k}^{\infty} \varrho_{i,0}\varrho_{j,0}\varrho_{k,0} +$$

$$\ldots + \sum_{i=1, i<j_1<\ldots<jt}^{\infty} \varrho_{i,0}\varrho_{j_1,0}\ldots\varrho_{j_t,0}+\ldots.$$

This counting series is defined since we obtain convergence in the order topology because of the fact that $\varrho_{k,0}(l)=0$ if $l<k$.

If we restrict the heights to certain subsets S of N, we obtain counting series:

$$\tau_s = \sum_{i\in s} \varrho_{i,0} + \sum_{i<j, i, j\in s} \varrho_{i,0}\varrho_{j_1,0}+\ldots.$$

Thus, e.g., if $S=\{1, 2\}$, $\tau_s=\varrho_{1,0}+\varrho_{2,0}+\varrho_{1,0}\varrho_{2,0}=\tau_2$.

By considering counting series $\tau_{s_1}-\tau_{s_2}$, where $s_2\subseteq s_1$, we can also handle cases where certain heights may occur in the components but not by themselves. Thus, $\tau_{\{1,2\}}-\tau_{\{1\}}=\varrho_{2,0}+\varrho_{1,0}\varrho_{2,0}=\sigma_2$ is the counting series for the collection of posets of height one.

Flat posets. The series $\sigma(C(C_m))$ and related series. Suppose now that we consider the general situation, i.e., the collection of all posets P with associated counting series σ. We let F be the counting series of the collection of flat posets. If X is any poset whatsoever, then X has a unique decomposition $X=A_1\oplus\ldots\oplus A_k$, where $k\geqq 1$, and where the posets A_i are flat. To see this, one uses a reduction on $|X|$ and the observation that either X is flat or $X=B\oplus C$, where $|B|<|X|$ and $|C|<|X|$. This yields decompositions of the type $X=A_1\oplus\ldots\oplus A_k$. Given two such decompositions $X=A_1\oplus\ldots\oplus A_k=B_1\oplus\ldots\oplus B_l$, we have $A_1=(A_1\cap B_1)\oplus\ldots$ $\ldots\oplus(A_l\cap B_l)$, and thus since A_1 is flat, $A_1\subseteq B_1$, whence $A_1=B_1, \ldots, A_k=B_k$, $l=k$.

It follows almost immediately that $\sigma(C(C_2))=F\sigma$, and thus that $\sigma-\sigma(C(c_2))=$ $=\sigma-F\sigma=F$, i.e., $(1-F)\sigma=F$, whence $\sigma=F(1-F)^{-1}=F+F^2+F^3+\ldots$. Hence $\sigma(C(C_2))=\sigma-F=F^2+F^3+F^4+\ldots$. Now, $F=\sigma(1+\sigma)^{-1}=\sigma-\sigma^2+\sigma^3-\ldots=$ $=\sum_{n=1}^{\infty}(-1)^{1+n}\sigma^n$, whence also $\sigma(C(C_2))=\sigma-F=\sum_{n=2}^{\infty}(-1)^n\sigma^n$.

For the coefficients of $\sigma(C(C_m))$ we note that the coefficient of σ^n in F^k equals $(-1)^{k+n}\binom{n-1}{k-1}$, since $\binom{n-1}{k-1}$ is the number of ordered partitions of n as a sum of k positive integers. In particular, if we solve

$$\sigma(C(C_m)) = \sum_{n=m}^{\infty}\left(\sum_{k=m}^{\infty}(-1)^{k+n}\binom{n-1}{k-1}\right)\sigma^n,$$

where since

$$\sum_{k=m}^{\infty} (-1)^{k+n} \binom{n-1}{k-1} = (-1)^{n+m} \binom{n-2}{n-m},$$

we have as final solution

$$\sigma(C(C_m)) = \sum_{n=m}^{\infty} (-1)^{n+m} \binom{n-2}{n-m} \sigma^n.$$

Thus, incidentally, it is clear from the formula for $\sigma(C(C_m))$, that the chains C_m are alternating for $m \geqq 2$.

If we let $\alpha(m; n)$ denote the coefficient of z^n in $\sigma(C(C_m))$, then it follows that since $\sigma(1)=1$,

$$P_n(1, \sigma(2), \ldots, \sigma(n) | 0, \ldots, 0, \alpha(m, m), \ldots, \alpha(m, n)) = (-1)^{n+m} \binom{n-2}{n-m}.$$

The same trick works for any class of posets L such that $1 \in L$, and such that if $A, B \in L$, then $A \oplus B \in L$ as well. Thus, suppose that L is the class of all regular posets of arbitrary height. The counting series for this class of posets is explicitly determined, and if $\sigma(L)$ denotes this counting series, then

$$\sigma(L) = \sum_{k=1}^{\infty} \varrho_{k,0}.$$

Notice that L is not closed under addition. If we let $F(L)$ be the "flat series" for L, then it follows immediately that $F(L) = \sigma(L) - \sigma(L)^2 + \sigma(L)^3 - \ldots$ as before. Since $A \oplus B \in L$ if and only if $A \in L$ and $B \in L$ quite obviously, it follows that if $X \in L$ is flat relative to L, then it is flat. Hence we also have an explicit determination of the counting series for the class of flat regular posets, viz., $F(L)$.

Suppose that $F_k(L)$ denotes the counting series for the collection of flat regular posets of height precisely $k-1$. Thus $F(L) = \sum_{k=1}^{\infty} F_k(L)$, and $F_1(L) = \varrho_{1,0} = z + z^2 + \ldots$, since any poset of height 0 is obviously flat and of the form $X=n$. To extract the counting series $F_k(L)$, we note that by considering decompositions $X = A_1 \oplus \ldots \oplus A_l$, A_i flat, and X regular of height k, we deduce readily the relations

$$\varrho_{k,0} = F_k(L) + F_{k-1}(L)\varrho_{1,0} + \ldots + F_1(L)\varrho_{k-1,0},$$

whence $F_k(L)$ can be computed recursively since the counting series $\varrho_{k,0}$ are already determined.

Thus, e.g., $F_2(L) = \varrho_{2,0} - \varrho_{1,0}^2 = 2z^4 + 8z^5 + 30z^6 \ldots$. Note the absence of three point flat posets of height one. The two regular flat posets of height one with four vertices are $2C_2$ and the poset  .

Of course, this model with associated formulas works in a variety of other contexts where we are dealing with "linear arrangements". We may either prescribe the corresponding "flat series" to determine the counting series or vice versa.

As an example, suppose that we consider the following situation. Drawing from infinite collections of balls of colours $1, \ldots, k$ respectively, we are to put arbitrary number of balls in urns numbered $1, \ldots, n, \ldots$ respectively. We are to have the urns $1, \ldots, n-1$ non-empty if urn n is non-empty and we want to describe the total number of arrangements, disregarding empty urns. If K is the collection of all such arrangements and if X denotes an element of K, we let $\nu(X)$ denote the total number of balls used up in constructing the arrangement. We call a one urn arrangement flat. If we let $F(n)$ denote the total number of one urn arrangements X such that $\nu(X)=n$, then it follows quite easily that $F(n)=\binom{n+k-1}{k-1}$, and thus $F=\sum_{n=1}^{\infty}\binom{n+k-1}{k-1}z^n$ is the flat series for this problem. Now if σ is the counting series for K, $\sigma(n)=|\nu^{-1}(n)|$, $\sigma=\sum_{n=1}^{\infty}\sigma(n)z^n$, then $\sigma=F+F^2+F^3+\ldots$ as before. Then pattern of formulas is now precisely that developed above.

Suppose K is the collection of finite vectors $(a_1, \ldots, a_m)$, $a_i \geqq 0$. Let $\nu(a_1, \ldots, a_m)=a_1+\ldots+a_m$. Then $|\nu^{-1}(n)|=2^{n-1}$ is the number of ways to order partition n into non-negative summands. Thus, we have a counting series $\sigma=\sum_{n=1}^{\infty}2^{n-1}z^n$. If we suppose there to be a flatness concept, then we compute the flat series $F=\sigma-\sigma^2+\sigma^3-\sigma^4+\ldots$. It follows readily that $F=z+z^2+z^3+\ldots$ and thus for each positive integer n there should be precisely one flat partition X such that $\nu(X)=n$.

If we define $(a_1, \ldots, a_m)\oplus(b_1, \ldots, b_l)=(a_1, \ldots, a_m, b_1, \ldots, b_l)$, then it follows immediately that $(a_1, \ldots, a_m)=(a_1)\oplus\ldots\oplus(a_m)$ uniquely, and the notion of flatness is given by the condition that X is flat if and only if $X=(\nu(X))$.

Thus, if we are given (K, ν) and the counting series σ explicitly, then we can test for the possibility of talking about "flat" objects in K by computing $\sigma-\sigma^2+\sigma^3-\ldots=F$. Should F have positive integral coefficients, then we ought to be able to talk about "flatness". Suppose that we let K be the collection of monic polynomials of degree 1 over a finite field with q elements. If $\nu(p(x))$ denotes the degree of $p(x)$, then (K, ν) has a counting series $\sigma=\sum_{n=1}^{\infty}(qz)^n$. It follows that if $F=\sigma-\sigma^2+\sigma^3-\ldots$, then $F=qz$, so that the "flat" objects turn out to be the monic polynomials of degree one. The corresponding ordinal sum is defined by $(p_1, \ldots, p_s)\oplus(h_1, \ldots, h_m)=(p_1, \ldots, p_s)+x^s(h_1, \ldots, h_m)$, where $p_1, \ldots, p_s, h_1, \ldots, h_m$ are monic polynomials. It follows immediately that every monic polynomial has a unique decomposition $(p_1)\oplus(p_2)\oplus\ldots\oplus(p_s)=(p_1, \ldots, p_s)=p_1+xp_2+\ldots+x^{s-1}p_s$.

Connected posets. The series $\sigma(C(n))$ and related series. In this situation $C(n)$ is the collection of posets having at least n components. In particular $C(2)$ is the collection of disconnected posets. Since we shall determine $\sigma(C(2))$ in this section, it follows that we also determine $\xi=\sigma-\sigma(C(2))$, the counting series of the collection of connected posets. Thus, since this collection contains 1 and since it is closed under ordinal sum, we have an associated flat series $F(\xi)=\xi-\xi^2+\xi^3-\xi^4+\ldots$. This will be the counting series for those connected posets which are not the ordinal sum of connected posets. Since $A\oplus B$ is connected for arbitrary posets A and B, it does not follow that these posets are themselves flat. Thus, e.g., $2\oplus 1$ is not the ordinal sum of connected posets.

One problem which arises immediately in constructing $\sigma(C(2))$, is that because of the commutativity of addition we do not have a simple functional equation involving σ and ξ as we did for σ and F. To solve in the particular situation we are dealing with in at least such a way that computations can be performed recursively, we have to use quite a different approach.

Thus, let N be the set of positive integers as usual. Now we define products $N^k \times N^k \to N$ as follows:

$$(e_1, \dots, e_k) \cdot (n_1, \dots, n_k) = e_1 n_1 + \dots + e_k n_k$$

and

$$(e_1, \dots, e_k) \times (n_1, \dots, n_k) = n_1^{(e_1)} \dots n_k^{(e_k)},$$

where

$$n^{(e)} = \binom{n}{1}\binom{e-1}{0} + \dots + \binom{n}{e}\binom{e-1}{e-1} = \binom{n+e-1}{e}.$$

Thus $n^{(e)}$ is also equal to the number of ordered partitions of $n-1$ into $e+1$ non-negative integers.

If $f: N \to N$ is any function whatsoever, we define

$$(e_1, \dots, e_k) \times_f (n_1, \dots, n_k) = (e_1, \dots, e_k) \times (f(n_1), \dots, f(n_k)).$$

Thus, e.g., if $\xi: N \to N$ is defined by: $\xi(n)$ is the number of connected posets X such that $|X| = n$, then if $n_1 > n_2 > \dots > n_k$,

$$(e_1, \dots, e_k) \times_\xi (n_1, \dots, n_k) = \xi(n_1)^{(e_1)} \dots \xi(n_k)^{(e_k)}$$

is the number of posets X such that $|X| = e_1 n_1 + \dots + e_k n_k$, and such that X has $e_1 + \dots + e_k$ components, of which precisely e_j components have n_j vertices.

That this is so follows from the fact that if we have $\xi(n) = m$ connected posets with n vertices, then to count all posets with e components each of which has n vertices we have distinct contributions:

(1) all components isomorphic, $\binom{m}{1}\binom{e-1}{0}$,

(2) two distinct types, $\binom{m}{2}\binom{e-1}{1}$, where the multiplier $\binom{e-1}{1}$ agrees with the fact that these distinct types can be represented in $e-1$ distinct proportions, viz., 1 of the first, $e-1$ of the second, etc.;

(3) k distinct types, $\binom{m}{k}\binom{e-1}{k-1}$, where the multiplier $\binom{e-1}{k-1}$ is precisely the number of ordered partitions of e into k positive integers, each such partition corresponding to a distinct distribution of these k types.

We shall call a vector $\vec{n}$ in N^k *cascading* if $\vec{n} = (n_1, \dots, n_k)$ with $n_1 > n_2 > \dots > n_k$. A *bipartition* of m is a pair $(\vec{e}, \vec{n}) \in N^k \times N^k$ for some k such that $\vec{n}$ is cascading and such that $\vec{e} \cdot \vec{n} = m$.

Thus, if we consider the sum over all bipartitions of m, $\sum_{\vec{e} \cdot \vec{n} = m} \vec{e} \times_\xi \vec{n}$, then this number is $\sigma(m)$, the number of posets X such that $|X| = m$. In other words, our "functional equation" involving σ and ξ is the following:

$$\sigma = \sum_{m=1}^{\infty} \Big(\sum_{\vec{e} \cdot \vec{n} = m} \vec{e} \times_\xi \vec{n} \Big) z^m.$$

For arbitrary functions $f: N \to N$, let us denote by $Cf: N \to N$ the function $(Cf)(m) = \sum_{\vec{e} \cdot \vec{n} = m} \vec{e} \times_f \vec{n}$. We want to invert this expression, i.e., we want to find polynomials $g_m(x_1, \ldots, x_m)$ for all m, such that $f(m) = g_m(Cf(1), \ldots, Cf(m))$. Since $Cf(1) = f(1)$, it suffices to take $g_1(x_1) = x_1$. Suppose now that $g_k(x_1, \ldots, x_k)$ has been defined for $k < m$.

Now, $Cf(m) = \sum_{\vec{e} \cdot \vec{n} = m} \vec{e} \times_f \vec{n} = 1 \times_f m + \sum^*_{\vec{e} \cdot \vec{n} = m} \vec{e} \times_f \vec{n}$, where the sum $\sum^*$ runs over all bipartitions $\vec{e} \cdot \vec{n} = m$ with $\vec{e} \neq (1)$, whence $\vec{n} \neq (m)$.

Hence,

$$Cf(m) = f(m) + \sum^*_{\vec{e} \cdot \vec{n} = m} \vec{e} \times_f \vec{n}$$

and

$$f(m) = Cf(m) - \sum^*_{\vec{e} \cdot \vec{n} = m} \vec{e} \times_f \vec{n}.$$

For polynomials $\Phi(x_1, \ldots, x_m)$, we let $\binom{\Phi(x_1, \ldots, x_m)}{j}$ be defined as usual, i.e.,

$$\binom{\Phi(x_1, \ldots, x_m)}{j} = \frac{1}{j!} \Phi(x_1, \ldots, x_m)\big(\Phi(x_1, \ldots, x_m) - 1\big) \ldots \big(\Phi(x_1, \ldots, x_m) + 1 - j\big).$$

Thus also,

$$\Phi(x_1, \ldots, x_m)^{(e)} = \binom{\Phi(x_1, \ldots, x_m) + e - 1}{e}.$$

Now $f(n_i) = g_{n_i}(Cf(1), \ldots, Cf(n_i))$ has been defined for $n_i > m_1$ and thus

$$\vec{e} \times_f \vec{n} = g_{n_1}(Cf(1), \ldots, Cf(n_i))^{(e_1)} \ldots g_{n_k}(Cf(1), \ldots, Cf(n_k))^{(e_k)}.$$

If we let

$$\vec{e} \otimes \vec{n} = g_{n_1}(x_1, \ldots, x_{n_1})^{(e_1)} \ldots g_{n_k}(x_1, \ldots, x_{n_k})^{(e_k)},$$

then it follows readily that:

$$g_m(x_1, \ldots, x_m) = x_m - \sum^*_{\vec{e} \cdot \vec{n} = m} \vec{e} \otimes \vec{n}$$

is the desired polynomial.

We notice that $g_m(x_1, \ldots, x_m)$ is independent of the function f and that the polynomials $g_1(x_1)$, $g_2(x_1, x_2)$, ... etc., can be computed directly.

Of course, if we define $\hat{C}f: N \to N$ by $\hat{C}f(m) = g_m(f(1), \ldots, f(n))$, then $\hat{C}Cf = f$ and $C\hat{C}f = f$, so that in particular, regarding σ and ξ as functions from N to N, we have

$$\xi = \sum_{m=1}^{\infty} \xi(m) z^m = \sum_{m=1}^{\infty} g_m(\sigma(1), \ldots, \sigma(m)) z^m.$$

Thus, if we want to solve $\xi = a_1 \sigma + a_2 \sigma^2 + \ldots$, then:

$$a_k = A_k(\sigma(1), \ldots, \sigma(k) | \xi(1), \ldots, \xi(k)) = P_k(1, \sigma(2), \ldots, \sigma(k) | \xi(1), \ldots, \xi(k)) =$$
$$= P_k\big(1, \sigma(2), \ldots, \sigma(k) | g_1(\sigma(1)), \ldots, g_k(\sigma(1), \ldots, \sigma(k))\big),$$

i.e., the coefficients a_k are polynomials in $\sigma(1), \ldots, \sigma(k)$ which can be computed independently of σ. Using the existing relations, we may also express a_k as a polynomial in $\xi(1), \ldots, \xi(k)$. Not knowing σ, this is as close as we can come to a complete solution of the problem as we stated it originally.

Before coming to examples it is necessary to list several of the polynomials $g_m(x_1, \ldots, x_m)$. We have:

$$g_1(x_1) = x_1, \quad g_2 = x_2 - x_1^{(2)}, \quad g_3 = x_3 - g_2 x_1 - x_1^{(3)},$$

$$g_4 = x_4 - g_3 x_1 - g_2 x_1^{(2)} - g_2^{(2)} - x_1^{(4)},$$

$$g_5 = x_5 - g_4 x_1 - g_2^{(2)} x_1 - g_3 x_1^{(2)} - g_3 g_2 - g_2 x_1^{(3)} - x_1^{(5)},$$

$$g_6 = x_6 - g_5 x_1 - g_4 g_2 - g_3^{(2)} - g_2^{(3)} - g_4 x_1^{(2)} - g_3 x_1^{(3)} - g_2^{(2)} x_1 - g_2 x_1^{(4)} - g_3 g_2 g_1 - x_1^{(6)},$$

$$g_7 = x_7 - g_6 x_1 - g_5 g_2 - g_4 g_3 - g_5 x_1^{(2)} - g_4 g_2 g_1 - g_4 x_1^{(3)} - g_3^{(2)} x_1 - g_3 g_2^{(2)} -$$

$$- g_3 g_2 x_1^{(2)} - g_2^{(3)} x_1 - g_2^{(2)} x_1^{(3)} - g_3 x_1^{(4)} - g_2 x_1^{(5)} - x_1^{(7)}.$$

It would be useful to find simpler expressions and ways of computing the polynomials $g_m(x_1, \ldots, x_m)$ more rapidly for values $m > 7$.

If we use the known values $\sigma(1)=1$, $\sigma(2)=2$, $\sigma(3)=5$, $\sigma(4)=16$, $\sigma(5)=63$, $\sigma(6)=318$, $\sigma(7)=2045$, then we can compute for ξ, $\xi(1)=1$, $\xi(2)=1$, $\xi(3)=3$, $\xi(4)=10$, $\xi(5)=44$, $\xi(6)=238$, $\xi(7)=1650$. Expanding σ into $\sigma^2, \sigma^3, \sigma^4, \sigma^5, \sigma^6, \sigma^7$, we determine

$$\xi = \sigma - \sigma^2 + 2\sigma^3 - 4\sigma^4 + 11\sigma^5 - 31\sigma^6 + 54\sigma^7 - \ldots$$

and

$$\sigma(C(2)) = \sigma^2 - 2\sigma^3 + 4\sigma^4 - 11\sigma^5 + 31\sigma^6 - 54\sigma^7 + \ldots$$

so that 2 appears to be an alternating poset as well. To prove 2 is indeed alternating one needs to study $A_k(x_1, \ldots, x_k | g_1(x_1), \ldots, g_k(x_1, \ldots, x_k))$ along with information about the growth of $\sigma(n)$ with n. It does not look impossible to prove this, but it would be messy at least.

If we "know" $\sigma(C(2))$, then we should also be able to extract σ out again since in seeing all disconnected posets we also see the connected posets.

Suppose that $S_n = \{a_n, a_{n+1}, \ldots\}$ is any sequence of complex numbers whatsoever. We define

$$F_k(S_n; x_1, \ldots, x_k) = \sum_{j=0}^{k} a_{n+j} G_{n+k,n+j},$$

where the polynomials $G_{n+k,n+j}$ are as defined above.

Thus again, $F_k(S_n; x_1, \ldots, x_k)$ can be computed systematically. Suppose that $\sigma = \sum_{k=1}^{\infty} \sigma(k) z^k$ is any power series such that $\sigma(1)$ is real and positive, and that $\tau = \sum_{k=n}^{\infty} \tau(k) z^k$ is a power series with $\tau(n)$ real and positive. Suppose furthermore that $\tau = a_n \sigma^n + a_{n+1} \sigma^{n+1} + \ldots$. Then $\tau(n) = a_1 \sigma(1)^n$, so that $\sigma(1) = \sqrt{\tau(n)/a_n}$ is the unique principal n^{th} root of $\tau(n)/a_n$. If we expand, it follows that

$$\tau(n+k) = n a_n \sigma(1)^{n-1} \sigma(k+1) + F_k(S_n; \sigma(1), \ldots, \sigma(k)),$$

so that we obtain for $\sigma(k+1)$ the formula:

$$\sigma(k+1) = \frac{\tau(n+k) - F_k(S_n; \sigma(1), \ldots, \sigma(k))}{na_n \sigma(1)^{n-1}}.$$

Of course, this arithmetic is contained in the general situation described above, except that we need to shift indices some.

Given the counting series $\tau_2 = z + 2z^2 + 4z^3 + 9z^4 + 21z^5 + 56z^6 + \ldots$ of the posets of height at most one, which has been determined explicitly above, we note that its associated connected series can also be computed explicitly. The first several terms are given by $\xi(\tau_2) = z + z^2 + 2z^3 + 4z^4 + 10z^5 + 27z^6 + \ldots$. Obviously, if we consider the connected posets of height at most one containing at least two vertices, then these are of height exactly one, and so the counting series for the connected posets of height one we obtain $\xi(\tau)_2 - z$.

If we let K be the collection of monic polynomials over a finite field with q elements graded by degree, then we have an associated counting series $\sigma = \sum_{n=1}^{\infty} q^n z^n$. It is clear that since $v(p_1(x)p(x)) = v(p_1(x)) + v(p_2(x))$, the components of $p(x)$ are its irreducible factors. Thus, $g_m(q, q^2, \ldots, q^m)$ is the number of monic irreducible polynomials of degree m over a field with q elements. For a field with two elements the counting series begins $2z + z^2 + 2z^3 + 3z^4 + 6z^5 + 10z^6 + 17z^7 + \ldots$. Given a system (K, v) with a counting series given, one can always determine a connectedness series $\xi(\sigma)$ by using the defining relations $\xi(\sigma)(m) = g_m(\sigma(1), \ldots, \sigma(m))$. As above, one can then test and see whether a notion of connectedness makes sense. The minimal criterion we use is that $\xi(\sigma)$ should itself have integral coefficients. Of course this does not specify what the notion might be in the particular context at hand.

For example, if K is the set of finite vectors $(a_1, \ldots, a_m)$, $a_i \geqq 0$, with counting series $\sigma = z + 2z^2 + 4z^3 + \ldots + 2^{n-1} z^n + \ldots$ we compute the connectedness series $\xi(\sigma) = z + z^2 + 2z^3 + 3z^4 + 6z^5 + 9z^6 + \ldots$. The series does not exclude a connectedness concept apparently, although it is not clear what it should be.

If K is the collection of square matrices over a field with q elements, then we operate on $n \times n$ matrices by the group S_n, $\varphi(M_{ij}) = M_{\varphi(i), \varphi(j)}$. The cycle index is written $P(S_n)$. We let $v(M) = n$ if M is an $n \times n$ matrix. Given matrices A and B, we let $A + B$ be the matrix direct sum $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. It follows that with respect to the operation of the symmetry group, $A + B = B + A$, and $v(A+B) = v(A) + v(B)$. The counting series is given by $\sigma = \sum_{n=1}^{\infty} P(S_n; q, \ldots, q) z^n$, with an associated connected series

$$\xi(\sigma) = \sum_{n=1}^{\infty} g_m(P(S_1), \ldots, P(S_m)) z^m.$$

This will be the counting series for those matrices which cannot be written as a direct sum.

The variety of situations to which this case applies is again enormous. The most frequent application seems to be that of extracting the connectedness series from the counting series, with the counting series known somehow.

Since every poset in $C(C_2)$ is connected, we can compute the counting series for the flat and connected posets by taking $\xi-\sigma(C(C_2))$. From $F=\sigma-\sigma^2+\sigma^3-\sigma^4+\dots$ one determines the first several terms of F, with $F=z+z^2+2z^3+7z^4+31z^5+184z^6+1383z^7$, and $\sigma(C(C_2))=z^2+3z^3+9z^4+32z^5+134z^6+662z^7+\dots$. From $\xi=z+z^2+3z^3+10z^4+44z^5+238z^6+1050z^7+\dots$ we then obtain $\xi-\sigma(C(C_2))=z+z^4+12z^5+104z^6+988z^7+\dots$. Of course this is the counting series for $L=P\setminus(C(2)\cup C(C_2))$, so that according to our sieve mechanism there is precisely one pure poset with four vertices. This is easily seen to be the poset .

To compute $\sigma(C(m))$, we can use the functional equation

$$\sigma(C(m)) = \sum_{k=m}^{\infty}\Big(\sum_{\substack{\vec{e}\cdot\vec{n}=k\\ \|\vec{e}|\geqq m}} \vec{e}\times_{\xi}\vec{n}\Big) z^k,$$

where $\|\vec{e}|=e_1+\dots+e_t$ if $\vec{e}=(e_1,\dots,e_t)$. Other relations can then be deduced.

Order products and flat posets revisited. Although we have already determined the relationships between $\sigma(C(C_m))$ and σ, it seems worthwhile to look at it again from the point of view of products $P^k\times P^k\to P$ of the type just discussed.

Let us denote by $(e_1,\dots,e_k)\wedge(n_1,\dots,n_k)$ the quantity

$$(e_1,\dots,e_k)\wedge(n_1,\dots,n_k) = n_1^{e_1}\dots n_k^{e_k}\binom{e_2+e_1}{e_1}\dots\binom{e_k+\dots+e_1}{e_{k-1}+\dots+e_1} =$$
$$= n_1^{e_1}\dots n_k^{e_k}\binom{\|\vec{e}|}{e_1,\dots,e_k}.$$

Thus $(e_1,\dots,e_k)\wedge(n_1,\dots,n_k)$ is a term in the expansion of $(n_1+\dots+n_k)^{\|\vec{e}|}$.

For any function $f\colon N\to N$, we let

$$(e_1,\dots,e_k)\wedge_f(n_1,\dots,n_k) = (e_1,\dots,e_k)\wedge(f(n_1),\dots,f(n_k)).$$

Now, if X is any poset such that $|X|=m$, then writing $X=A_1\oplus\dots\oplus A_l$, $|A_i|=n_i$, we have $n_1+\dots+n_l=m$, and using the bipartition notation, we have $(e_1,\dots,e_k)\cdot(n_{i_1},\dots,n_{i_k})=m$ for a cascading vector $(n_{i_1},\dots,n_{i_k})((n_1,\dots,n_k)$ without loss of generality).

Let $F(n)$ be the number of flat posets with n vertices as usual. If $(e_1,\dots,e_k)\cdot(n_1,\dots,n_k)=m$, then consider the $F(n_i)$ flat posets with n_i vertices. Since we may arrange these in any way among themselves, we obtain $F(n_i)^{e_i}$ possibilities for arranging e_i flat posets with n_i vertices. Mixing these arrangements into a single arrangement can be done in $\binom{\|\vec{e}|}{e_1,\dots,e_k}$ ways. Thus the total number of possible posets constructed in this way is $F(n_1)^{e_1}\dots F(n_k)^{e_k}\binom{\|\vec{e}|}{e_1,\dots,e_k}$.

Thus we obtain a functional equation:

$$\sigma = \sum_{m=1}^{\infty}\Bigg(\sum_{\vec{e}\cdot\vec{n}=m} \vec{e}\wedge_F\vec{n}\Bigg) z^m.$$

For arbitrary functions $f: N \to N$, let

$$Hf(m) = \sum_{\vec{e}\cdot\vec{n}=m} \vec{e} \wedge_f \vec{n}.$$

To invert we need to find a set of polynomials $G_m(x_1, \ldots, x_m)$ such that $f(m) = G_m(Hf(1), \ldots, Hf(m))$.

Let $G_1(x_1) = x_1$ and for $\vec{e} = (e_1, \ldots, e_k)$, $\vec{n} = (n_1, \ldots, n_k)$ let

$$\vec{e}\Lambda\vec{n} = G_{n_1}(x_1, \ldots, x_{n_1})^{e_1} \ldots G_{n_k}(x_1, \ldots, x_{n_k})^{e_k} \binom{\|\vec{e}|}{e_1, \ldots, e_k}.$$

Let

$$G_m(x_1, \ldots, x_m) = x_m - \sum_{\vec{e}\cdot\vec{n}=m}^{*} \vec{e}\Lambda\vec{n},$$

where $\sum^*$ denotes the exclusion of $\vec{e} = (1)$, $\vec{n} = (m)$. Again the polynomials $G_m(x_1, \ldots, x_m)$ can be computed systematically.

Now, if we consider $F = \sum_{n=1}^{\infty} (-1)^{1+n}\sigma^n$, then

$$a_k = (-1)^{1+k} = A_k(\sigma(1), \ldots, \sigma(k) | F(1), \ldots, F(k)) =$$
$$= A_k(\sigma(1), \ldots, \sigma(k) | G_1(\sigma(1)), \ldots, G_k(\sigma(1), \ldots, \sigma(k))) = V_k(\sigma(1), \ldots, \sigma(k)),$$

so that $V_k(\sigma(1), \ldots, \sigma(k)) + (-1)^k = 0$ for all k.

If we let

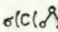
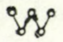
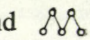
$$V_k(x_1, \ldots, x_k) = A_k(x_1, \ldots, x_k | G_1(x_1), \ldots, G_k(x_1, \ldots, x_k)),$$

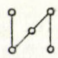
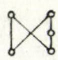
then either we are dealing with a set of identities $V_k(x_1, \ldots, x_k) + (-1)^k = 0$, or we can compute the values $\sigma(1), \ldots, \sigma(k)$ recursively. Since quite obviously we cannot do the latter based on flatness considerations alone, it must be true that we are dealing with a set of identities. Without going into any details, a little computing with A_1, A_2, A_3, A_4 and G_1, G_2, G_3, G_4 shows that $V_k(x_1, \ldots, x_k) + (-1)^k = 0$ for $k = 1, 2, 3, 4$.

The partition property of pure posets. If X is not flat or not connected we assign X to $C(2) \cup C(C_2)$, with $C(2) \cap C(C_2) = \emptyset$. Suppose X is flat and connected. Given $x \in X$, consider the family of all pure connected subsets containing x. Order by set inclusion. Since the union of chains in the family is again in the family, there are maximal connected pure posets containing x. Let V_x be such a poset. If $V_x \cap V_y \neq \emptyset$, then $V_x \cup V_y$ is connected. Also, if it is not pure, then $V_x \cup V_y = Y(A_1, A_2, \ldots)$ with $|Y| \geqq 2$. Suppose $x \in A_1$, $y \in A_2$. Now since V_x is pure and connected, $V_x \subseteqq A_1$. Similarly, $V_y \subseteqq A_2$ whence $V_x \cap V_y = \emptyset$. But then since $V_x \cup V_y = A_1$, $|Y| = 1$, a contradiction. Hence $V_x \cup V_y$ is pure, and thus V_x is uniquely determined and the sets V_x partition X. If $\alpha \in V_x$, $\beta \in V_y$ and $\alpha < \beta$, then since $V_x \cup V_y$ is not pure, $V_x \cup V_y = Y(A_1, A_2, \ldots, A_n)$ and $V_x \subseteqq A_2$ implies we have an ordinal sum $A_1 \oplus A_2$ whence also $V_x \oplus V_y$ appears. Thus, $X = X_1(V(x_1), \ldots, V(x_m))$. Repeat this process with X_1 to obtain X_2 etc., until it stabilizes at X_n. Thus at this stage $x \in X_n$ implies $V_x = X_n$. Thus X_n is certainly pure. Also, $X = X_n(A_1, \ldots, A_n)$, where if $x \in A_1$, $V_x \subseteqq A_1$, $V_{v_x} \subseteqq A_1$ and so forth.

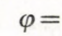
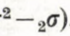
Suppose that $X = Y(B_1, \ldots, B_l)$ where Y is pure. If $x \in B_1$, then $V_x \subseteqq B_1$, and thus $X_1 = Y(B_1^1, \ldots, B_l^1)$. Thus, repeating this process we find $X_n = Y(B_1^n, \ldots, B_l^n)$.

Since X_n is pure, $X_n=Y$ and $l=m$, $B_1^n=\ldots=B_l^n=1$. Hence, if X_1 and X_2 are pure connected flat posets, then $C(X_1)\cap C(X_2)=\emptyset$. It follows that if X_1 and X_2 are any pure posets, then the same is true. Therefore the sets $C(X)$ partition P.

Looking for pure posets. If we happen to know something about $\sigma(n)$, then this can be helpful in determining the number of pure posets with n vertices as we have already seen once. We can use what counting information we have to some advantage along these lines at least. Thus, e.g., $\xi-\sigma(C(C_2))=z+z^4+12z^5+\ldots$. Now $\sigma(C($$))=\sigma^4$ quite obviously, so that for five point posets not yet in $C(2)\cup C(C_2)$,  generates a total of 8. Hence we conclude there are precisely 4 pure posets containing 5 elements. We can in fact use the counting series τ_∞ to determine that there are precisely 2 pure posets which are regular, and then by use of τ_2 we determine that these must be of height one. It is now ea sy to see that  and  are these two pure posets. The other two are not regular, and other than looking for them we know of no method which will allow us to locate these. Looking and finding we observe that they are:

 and 

We can construct the counting series for these pure posets relatively easily. Thus, let ${}_2\sigma=\sum_{n=1}^{\infty}\sigma(n)z^{2n}$ and $\varphi=\frac{1}{2}(\sigma^2-{}_2\sigma)$. As we shall show below,

$\sigma(C($$))=\sigma(C($$))=\sigma(\varphi\sigma^2+{}_2\sigma\cdot\varphi+({}_2\sigma)^2)$

while

$\sigma(C($$))=\sigma(C($$))=\sigma^5$

(since these posets are simple). Hence in particular we can give a count of the six point pure posets.

We proceed as follows:

$$\begin{aligned}
\tau(\varphi\sigma^2+{}_2\sigma\cdot\varphi+({}_2\sigma)^2) &= z^5+\ 6z^6+\ldots\\
\sigma^5 &= z^5+10z^6+\ldots\\
\sigma^4 &= z^4+\ 8z^5+44z^6+\ldots\\
\sigma(C(C_2)) &= z^2+\ 3z^3+\ 9z^4+32z^5+134z^6+\ldots\\
\sigma(C(2)) &= z^2+\ 2z^3+\ 6z^4+19z^5+\ 80z^6+\ldots.
\end{aligned}$$

Hence the six point posets accounted for by substitution already, have a counting series: $z^2+5z^3+16z^4+63z^5+290z^6+\ldots$.

Thus, there are 28 pure posets X such that $|X|=6$. This already quite a collection, which includes again individuals whose existence we cannot predict. For the posets whose components are regular, we have a counting series for the entire population. Now, under these circumstances we may be in a better position to attack the problem of describing the pure posets.

Some counting series $\sigma(C(X))$. In this section we consider a variety of posets and construct the counting series $\sigma(C(X))$. As in the previous situation, every such counting series represents a combinatorial pattern and one can give a variety of applications to other problems.

$X=1\oplus n$. In this case we note that the branching in the poset $1\oplus n$ prevents the identification of the posets substituted into 1 and those substituted into n. Thus, we have a relation $\sigma(C(1\oplus n))=\sigma\cdot\sigma(C(n))$. E.g., if one wants to determine the number of posets X in $C(1\oplus 2)$ such that $|X|=7$, we find that this number equals $516-2(184)+4(65)-11(12)+31=307$.

$X=n_1\oplus n_2\oplus\ldots\oplus n_k$. In this case if no one point posets occur adjacent to each other, then it follows that

$$\sigma(C(X))=\sigma(C(n_1))\ldots\sigma(C(n_2))\ldots\sigma(C(n_k)).$$

If we have subsequences $1\oplus\ldots\oplus 1$ of length l, then in the product above one replaces the term σ^l by $\sigma(C(C_l))$.

Thus, e.g., if $X=2\oplus 1\oplus 1\oplus 2\oplus 1$, then $\sigma(C(X))$ is described by the formula:

$$\sigma(C(X))=\sigma(C(2))^2\cdot\sigma(C(C_2))\cdot\sigma.$$

On the other hand, if $Y=1\oplus 2\oplus 1\oplus 2\oplus 1$, then

$$\sigma(C(Y))=\sigma(C(2))^2\cdot\sigma^3.$$

If we expand, we find the first several terms of X to be

$$\sigma(C(X))=\sigma^7-5\sigma^8+17\sigma^9-55\sigma^{10}+167\sigma^{11}-\ldots.$$

$X=1+C_2$. In this case, to compute $\sigma(C(X))$, we note that if we substitute flat posets into 1, then no duplication is possible. In the situation that we substitute non-flat graphs into 1, we might as well be dealing with $Y=C_2+C_2$. Thus our initial equation is:

$$\sigma(C(1+C_2))=F\cdot\sigma(C(C_2))+\sigma(C(2C_2)).$$

Now, to compute $\sigma(C(2C_2))$, we use $\sigma(C(C_2))$ as our connected series and we compute $\sigma(C(2C_2))$ using the fact that we have precisely two components according to the formula

$$\sigma(C(2C_2))=\tau=\sum\tau(m)z^m,$$

with

$$\tau(m)=\sum_{\vec{e}\cdot\vec{n}=m,\ \|\vec{e}\|=2}\vec{e}\times_f\vec{n}$$

and where $f\colon N\to N$ is given by $f(n)=\sigma(C(C_2))(n)$.

Now, $\sigma(C(C_2))=z^2+3z^3+9z^4+32z^5+134z^6+\ldots$ whence we find that $\tau=z^4+3z^5+15z^6+59z^7+\ldots$.

On the other hand

$$F\cdot\sigma(C(C_2))=F^3+F^4+\ldots=\sigma(C(C_3))=\sigma^3-2\sigma^4+3\sigma^5-\ldots=$$
$$=z^3+4z^4+14z^5+54z^6+300z^7+\ldots,$$

so that the first several terms in the expansion of $\sigma(C(1+C_2))$ are given by:

$$\sigma(C(1+C_2)) = z^3+5z^4+17z^5+69z^6+359z^7+\dots .$$

We also have $\tau=\sigma^4-5\sigma^5+21\sigma^6-52\sigma^7+\dots$, whence our expansion for $\sigma(C(1+C_2))$ in terms of σ begins:

$$\sigma(C(1+C_2)) = \sigma^3-\sigma^4-2\sigma^5+17\sigma^6-47\sigma^7+\dots .$$

Hence, $C(1+C_2)$ is not an alternating set.

On the basis of examples available one is tempted to conjecture the following:

(1) If X is a regular poset with $|X|=n$, then

$$\sigma(C(X)) = \sigma^n+a_{n+1}\sigma^{n+1}+\dots+a_{n+k}\sigma^{n+k}+\dots,$$

where for all integers $k\geqq 0, a_{n+2k}\geqq 0$ and $a_{n+2k+1}\leqq 0$.

(2) If X is a poset such that $|X|=n$ and such that

$$\sigma(C(X)) = \sigma^n+a_{n+1}\sigma^{n+1}+\dots \quad \text{with} \quad a_{n+1}\neq 0,$$

then $a_{n+k}\neq 0$ for all integers $k\geqq 1$, and the sequence $\sum_{k=0}^{\infty}(1/a_{n+k})$ converges.

X is a long W. If $\tau=\sum_{m=1}^{\infty}\tau(m)z^m$, we let ${}_n\tau=\sum_{m=1}^{\infty}\tau(m)z^{nm}$, i.e., we replace the variable z in τ by the variable z^n. We let φ denote the series $\varphi=\frac{1}{2}(\sigma^2-{}_2\sigma)$. Thus φ can be regarded as the counting series of the collection of all unordered pairs of posets $\{A, B\}$ with $A\neq B$. The series φ is the counting series analogue of the combinatorial coefficient $\binom{n}{2}$. Suppose that $W(k)$ is the pure poset ($k\geqq 4$)

For $k=2u$ we have only one type and for $k=2u+1$ we have a long W and its opposite, a long M. As they obviously generate the same counting series we do not need to worry about distinguishing between these here.

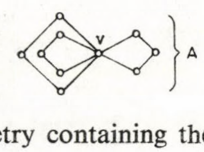
We find that $\sigma(C(W(2u)))=\sigma^{2u}$ for $u\geqq 2$. Roughly speaking this is so since $W(2u)$ fails to have a centre. Each vertex has a unique position and no overlap is possible because of the position of every vertex on a poset $W(3)=1\oplus 2$. For $W(2u+1)$ we have a centre, i.e., a fixed point under all automorphisms, with every other vertex in an orbit of two elements on either side of the centre.

From the centre we have a contribution σ. If we move one step away from the centre we obtain two vertices such that if different posets are substituted, then all other positions become distinct whence this case yields a contribution $\varphi\cdot\sigma^{2(u-1)}$, since we do not care whether A appears to the right and B appears to the left of the centre or vice versa. On the other hand, if the two posets substituted are alike, then we have a new "centre" $W(3)$ with a contribution $\sigma_2\sigma$ for the centre. Repeating this process by moving on to the next pair of vertices we determine:

$$\sigma(C(W(2u+1))) = \sigma(\varphi\sigma^{2(u-1)}+{}_2\sigma\varphi\sigma^{2(u-2)}+\dots+({}_2\sigma)^{u-1}\varphi+({}_2\sigma)^u).$$

Incidentally, concerning the height one pure posets, we have the following equivalent problem. Given that no philosopher is a disciple of another philosopher and that no disciple is a philosopher and in addition that no two philosophers have the same set of disciples nor do two disciples have the same set of masters, compute the number of possible arrangements of disciples and philosophers distinguishing neither among philosophers nor among disciples. It is clear that if K is the collection of these arrangements and if $\nu(X)$ is the sum of philosophers and disciples in the arrangement, then the counting series of (K, ν) is the same as the counting series of the collection of posets of height one which are sums of pure posets. If ψ is this counting series, then from $\psi+(z+z^2+\dots)$ we may determine a connected series $\xi(\psi+(z+z^2+\dots))$. The counting series $\xi(\psi+(z+z^2+\dots))-z$ would then be the counting series for the collection of pure posets of height one other than C_2.

X is a symmetric poset. Suppose that X is a poset with an axis of symmetry consisting of a chain of k vertices fixed under all automorphisms. Suppose that to this chain are attached t posets A in such a way that the automorphisms of X have orbits containing t elements or 1 element, with every vertex in A having an orbit of t elements which intersects each copy of A precisely once. Thus, e.g., if X is as follows:



then we have an axis of symmetry containing the vertex v only, and three copies of the leaf A.

Let $\vec{e}\cdot\vec{n}=t$ be a bipartition of t, i.e.,

$$e_1n_1+\dots+e_fn_f=t,\quad (e_1,\dots,e_f)=\vec{e}\quad\text{and}\quad (n_1,\dots,n_f)=\vec{n},$$

where $\vec{n}$ is a cascading vector.

Let $\sigma(\vec{e}\cdot\vec{n})$ be the series describing the distribution of $e_1+\dots+e_f$ different posets $Y_1,\dots,Y_f$ where we have n_1 copies of $Y_1,\dots,Y_{e_1}$, n_2 copies of $Y_{e_1+1},\dots,\dots,Y_{e_1+e_2}$, etc.

To compute this series, we first determine the series corresponding to the situation where we consider a partition of t, $n_1+\dots+n_l=t$, taking order into account and where we want l different posets to be distributed with n_1 copies of Y_1, n_2 copies of Y_2, etc.

If $\tau(n_1,\dots,n_l)$ is this series, we use a recursion formula

$$\tau(n_1)={}_{n_1}\sigma,\quad \tau(n_1,\dots,n_l)=\tau(n_1,\dots,n_{l-1})({}_{n_l}\sigma)-\sum_{i=1}^{l-1}\tau(n_1,\dots,n_i+n_l,\dots,n_{l-1}).$$

Here the latter formula follows from the fact that in multiplying $\tau(n_1,\dots,n_{l-1})$ and ${}_{n_l}\sigma$ we have to subtract out all cases where the poset Y_l is equal to one of the posets $Y_1,\dots,Y_{l-1}$.

Finally, allowing for interchange between $Y_1,\dots,Y_{e_1}$, between $Y_{e_1+1}, Y_{e_1+e_2}$, etc., we have $\sigma(\vec{e}\cdot\vec{n})=1/e_1!\dots e_f!\,\tau(\vec{e}\cdot\vec{n})$. We shall denote the quantity $1/e_1!\dots e_f!$ by $1/\vec{e}!$ so that our final formula becomes $\sigma(\vec{e}\cdot\vec{n})=1/\vec{e}!\,\tau(\vec{e}\cdot\vec{n})$.

Thus, e.g., if $t=2$, then we have two bipartitions (1) (2) and (2) (1) with $\sigma((1)(2))=\tau((1)(2))={}_2\sigma$ and $\sigma((2)(1))=\frac{1}{2}\tau((2)(1))$, and $\tau((2)(1))=\tau(1,1)= ={}_1\sigma_1\sigma-{}_2\sigma=\sigma^2-{}_2\sigma$, so that $\sigma((2)(1))=\varphi$.

Now, fix a bipartition $\vec{e}\cdot\vec{n}=t$. Then, to each arrangement counted by $\sigma(\vec{e}\cdot\vec{n})$ we have a corresponding splitting of X into e_1 posets with n_1 leaves, e_2 posets with n_2 leaves and so on, where each poset with n_j leaves has the same axis of symmetry and one of the vertices replaced by a fixed poset.

If from now on we disregard this poset entirely and denote the remainder of the leaf by A^*, then we proceed in precisely the same way as before to obtain a recursion formula. What counts besides k and t is the number $|A|=\alpha$. If we write $\sigma(k,t,\alpha)$ for $\sigma(C(X))$, then we have

$$\sigma(C(X))=\sigma(k,t,\alpha)=\sum_{\vec{e}\cdot\vec{n}=t}\sigma(\vec{e}\cdot\vec{n})\left(\prod_{i=1}^{f}\left[\frac{\sigma(k,n_i,\alpha-1)^{e_i}}{\sigma^{ke_i}}\right]\sigma^k\right),$$

with $\sigma(k,1,\alpha)=\sigma^{\alpha+k}$ and $\sigma(k,t,1)=\big(\sum_{\vec{e}\cdot\vec{n}=t}\sigma(\vec{e}\cdot\vec{n})\big)\sigma^k$.

For example if X is the poset in the illustration, then $\sigma(C(X))=\sigma(1,3,3)$. The bipartitions of (3) are (3)(1), (1, 1)(2, 1) and (1)(3) with $\sigma((3)(1))= =\frac{1}{6}(\sigma^3-3{}_2\sigma\cdot\sigma+2{}_3\sigma)$, $\sigma((1,1)(2,1))={}_2\sigma\cdot\sigma-{}_3\sigma$, and $\sigma((1)(3))={}_3\sigma$.

Thus, we obtain a reduction

$$\sigma(1,3,3)=\frac{1}{6}(\sigma^3+3{}_2\sigma\cdot\sigma+2{}_3\sigma)\sigma(1,1,2)^3/\sigma^2+({}_2\sigma\cdot\sigma-{}_3\sigma)\sigma(1,2,2)\sigma(1,1,2)/\sigma+$$
$$+3\sigma\cdot\sigma(1,3,2),$$

with

$$\sigma(1,1,2)=\sigma^3,\quad \sigma(1,2,2)=\frac{1}{2}(\sigma^5+({}_2\sigma)^2\sigma)$$

and

$$\sigma(1,3,2)=\frac{1}{6}(\sigma^3-3{}_2\sigma\cdot\sigma+2{}_3\sigma)\sigma^4+({}_2\sigma\cdot\sigma-{}_3\sigma)\frac{1}{2}(\sigma^4+{}_2\sigma\cdot\sigma^2)+$$
$$+{}_3\sigma\cdot\sigma\left(\frac{1}{6}(\sigma^3-3{}_2\sigma\cdot\sigma+2{}_3\sigma)+({}_2\sigma\cdot\sigma-{}_3\sigma)+{}_3\sigma\right).$$

Hence we obtain a formula for $\sigma(C(X))$ which in principle gives us all the necessary relations.

Notice that

$$\sigma(C(1\oplus t))=\sigma(1,t,1)=\big(\sum_{\vec{e}\cdot\vec{n}=t}\sigma(\vec{e}\cdot\vec{n})\big)\sigma=\sigma\cdot\sigma(C(t)),$$

so that

$$\sigma(C(t))=\sum_{\vec{e}\cdot\vec{n}=t}\sigma(\vec{e}\cdot\vec{n})$$

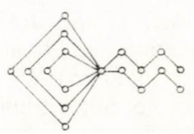
which can be computed according to the recursion formulas above. If $\sigma(\vec{e}\cdot\vec{n})= =\sum \sigma(\vec{e}\cdot\vec{n})_s z^s$, then:

$$\sigma(C(t)) = \sum_{s=t}^{\infty}\Big(\sum_{\vec{e}\cdot\vec{n}=t}\sigma(\vec{e}\cdot\vec{n})_s\Big)z^s = \sum_{s=t}^{\infty}\Big(\sum_{\vec{e}\cdot\vec{n}=s,\ \|\vec{e}\|\geqq t}\vec{e}\times_{\xi}\vec{n}\Big)z^s,$$

whence

$$\sum_{\vec{e}\cdot\vec{n}=t}\sigma(\vec{e}\cdot\vec{n})_s = \sum_{\vec{e}\cdot\vec{n}=s,\ \|\vec{e}\|\geqq t}\vec{e}\times_{\xi}\vec{n}.$$

Without attempting to describe all possibilities, it is clear that using this counting method, if X is a combination of leaves A and leaves B such that $C(A)\cap C(B)=\emptyset$, then the counting series become suitable products. Thus e.g., if X is as pictured below:



then $\sigma(C(X))=\sigma(1, 3, 3)\sigma(1, 2, 4)/\sigma$.

X is a sum or ordinal sum of posets. If $X=\sum_{k=1}^{n} m_k X_k$, where $C(X_i)\cap C(X_j)=\emptyset$, then we determine $\sigma(C(X))$ from

$$\sigma(C(X)) = \sigma(C(m_1X_1))\sigma(C(m_2X_2))\ldots\sigma(C(m_kX_k)).$$

This is quite clear since we cannot duplicate any poset obtained by substituting into m_iX_i from any other poset m_jX_j.

If $X=1+A$, where A is connected, then

$$\sigma(C(1+A)) = (\sigma-\sigma(C(A))\sigma(C(A))+\sigma(C(2A))).$$

The count arises from the fact that if we substitute into 1 from $C(A)$, then we might as well replace 1 by A. We do not obtain any other overlap since A is connected.

If $X=A+B$, where $C(A)\supseteqq C(B)$, and B is connected, then

$$\sigma(C(A+B)) = (\sigma(C(A))-\sigma(C(B)))\sigma(C(B))+\sigma(C(2B)).$$

Indeed, if $A\neq 1$, then A must itself be connected. Now, if we use a substitution into A such that the resulting poset falls into $C(B)$, then we replace A by B. Again, there is no other overlap than that accounted for in $\sigma(C(2B))$ since B is connected.

Thus, e.g.,

$$\sigma(C(C_2+C_3)) = (\sigma(C(C_2))-\sigma(C(C_3)))\sigma(C(C_3))+\sigma(C(2C_3)) =$$

$$= F^2(F^3+\ldots)+\sigma(C(2C_3)) = \sigma(C(C_5))+\sigma(C(2C_3)).$$

If $X=A\oplus k$, $Y=l\oplus B$, where A has at least two maxima and B has at least two minima, then

$$\sigma(C(X\oplus Y)) = \sigma(C(A))\sigma(C(B))\sigma(C(C_{k+l})).$$

Here we let $\sigma(C(C_0))=\sigma(C(0))=1$ and $A\oplus 0=0\oplus A=A$.

Simple posets. Besides pure posets, we mentioned simple, short and alternating posets in the introduction. We begin by observing that simple posets are quite plentiful. Thus, e.g., among the simple posets one finds besides 1, $W(2u)$ for $u \geqq 2$ and the irregular pure posets with five vertices given above.

If $X=\sum_{i=1}^{n} X_i$ and if X is simple then the X_i themselves have to be simple and $C(X_i) \cap C(X_j)=\emptyset$ if $i \neq j$. Indeed, if $C(X_i) \cap C(X_j) \neq \emptyset$, then in σ^m, $|X|=m$, the poset in this intersection will be counted at least twice, and thus we will have to adjust the count by subtracting a higher power of σ. In short, X will not be simple.

If $X=A \oplus k$ and if X is simple, while A has at least two maxima, then A is simple and $k=0$ or $k=1$. Conversely, if A is simple and $k=0$ or $k=1$, then X is also simple.

If $X=Y(A_1, \ldots, A_n)$, where one of the A_i is not simple, then X cannot be simple since we can obtain duplications in A_i and transfer these duplications to X by substitutions. Thus, if X is simple then all A_i are simple. Now the fact that X is simple does not imply Y is simple, since there are clearly disconnected simple posets.

If X is a poset with an automorphism group of order >1, then some vertex of X is not left fixed by the automorphism group. If σ moves x_1 to x_2, then it follows that $X(A, 1, \ldots, 1)=X(1, A, 1, \ldots, 1)$, and thus we obtain duplication, whence X cannot be simple.

Now there are posets which are not simple whose automorphism group has order 1, so that this condition is necessary but not sufficient. We shall next identify simple posets.

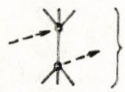
If X is simple, then X is of type 1 if $X=A \oplus 1$, and X is of type 2 if $X=1 \oplus A$. If Y is a poset, then a *spot* on Y is any pair of vertices (x_i, x_j) with x_j immediately above x_i which can be obtained as a substitution C_2 into vertex of a poset Y' with one less vertex than Y. If $X=Y(A_1, \ldots, A_n)$, with the A_i simple, then we shall in addition require that if (x_i, x_j) is a spot on Y, then we do not have A_i of type 1 and A_j of type 2. Under these circumstances X will be without spots.

Next suppose that Y has an orbit $(x_1, \ldots, x_m)$ under the automorphism group of Y such that if this orbit were fixed, then Y would be fixed. If the automorphism group of Y has order 1 already, we may take any orbit (x_i). To fix X we require that $A_1+\ldots+A_m$ be simple. Thus, $C(A_i) \cap C(A_j)=\emptyset$ if $i \neq j$. After having done this we may choose the other posets $A_{m+1}, \ldots, A_n$ so as to leave X without spots. We select the orbit $(x_1, \ldots, x_m)$ to be of smallest length among the orbits which fix Y. If no such single orbit exists, such as maybe the case in ordinal sums for example, we select a collection of orbits $(x_1, \ldots, x_m)$, $(x_{m+1}, \ldots, x_{m+k})$, ... which fixes Y and such that the sum of the lengths is minimal among all collections of orbits which fix Y. Again, on each orbit $(x_l, \ldots, x_{l+s})$ we take $A_l+\ldots+A_{l+s}$ to be simple, and then fill in the other simple posets A_i so as to satisfy the requirement that X be without spots.

It follows that under these circumstances $X=Y(A_1, \ldots, A_n)$ is indeed simple. Indeed, X has an automorphism group of order 1 and it is without spots. The fact that X has automorphism group of order 1 follows from the fact that automorphisms map ordinal sums to ordinal sums, and thus since the A_i are simple and the requirement about spots is satisfied, corresponding to any pair (x_i, x_j) with x_j immediately above x_i we have at worst a situation $B_i \oplus 1 \oplus B_j$, with 1 belonging to either A_i or A_j and with at least two maximal elements and two minimal elements present on either side of 1. This requires that the image under the automorphism has the same form, whence because of the structure of B_i and B_j we must map to $B_i' \oplus 1 \oplus B_j'$. But this implies that on Y we have an underlying map $x_i \to x_i'$ and $x_j \to x_j'$, i.e., every automorphism of X induces an automorphism of Y. Now in the construction of X we arranged matters so that this underlying automorphism of Y could only be the identity map and thus our assertion about X is true.

We claim that if X is any poset such that its automorphism group has order 1 and such that it is without spots, X is in fact simple. To see this we observe that if $X(A_1, \ldots, A_n) = X(B_1, \ldots, B_n)$, then since X is without spots we cannot have automorphisms which locally look like $A_i \oplus A_j = B_i \oplus B_j$. Indeed, for this to be the case we would have $A_i \oplus A_j = B_i \oplus B_j = D_1 \oplus D_2 \oplus D_3$, with the resulting freedom of D_2 to move up or down indicating that we are in fact on a spot.

Indeed, for a non-spot the corresponding part of the poset looks like the illustration below:



at least one edge coming in from or going out to another vertex. The extra edge would prevent the type of slippage indicated in the situation just discussed.

Next, since this type of automorphism cannot occur, we note that if x_i is a minimal vertex of Y, then A_1 itself is moved by the automorphism between $X(A_1, \ldots, A_n)$ and $X(B_1, \ldots, B_n)$ to a copy of itself which is also "minimal". Suppose that y is a minimal vertex of A_1, then its image is in some B_{i_1}. If the image of A_1 is not precisely B_{i_1}, then it sticks out into some B_{i_2}. If we select x_1 to be minimal in a maximal chain of Y, say $x_1 < \ldots < x_k$, then $A_1 \oplus \ldots \oplus A_k$ is moved over to a corresponding chain $B_{i_1} \oplus B_{i_2} \oplus \ldots \oplus B_{i_t}$. Since a maximal element of A_k is also moved to a maximal element of X, we find that $B_{i_1} \oplus \ldots \oplus B_{i_t}$ is itself a maximal chain and thus $t=k$. Now if A_1 sticks out into B_{i_2}, then A_1 is an ordinal sum, $A_1 = D \oplus E$. Since the edges going out of x_1 are transferred into B_{i_2}, it follows that B_{i_1} is directly below B_{i_2} with no other connections. But this implies (x_{i_1}, x_{i_2}) is a spot, a possibility which was excluded. Thus $A_1 = B_{i_1}$. Of course if we continue the argument, assuming that $A_l = B_{i_l}$ for $l < j$, then each time selecting minimal elements in maximal chains of the remainder of the poset, we prove in precisely the same fashion that $A_j = B_{i_j}$ as well, and thus if we let $\varphi: X \to Y$ be given by $x_j \to x_{i_j}$, then the result is an automorphism of X. Now, since this is the identity map, we have $j = i_j$ and $A_j = B_j$. Thus X is indeed simple.

Thus, e.g., if $Y = 1 \oplus 1 \oplus 2$, then we fix Y by taking the orbit 2 to consist of $W(4)$ and 1. If we select the other two simple sets to be 1 and $W(4)$ so as to satisfy the condition on spots, then $X = 1 \oplus W(4) \oplus (1 + W(4))$ is a simple poset.

Since pure posets are in any case without spots, it follows that a pure poset is simple if and only if it has an automorphism group of order 1.

Let us define $\delta(X) = d$ if the smallest number of elements needed to fix X is d. Thus, e.g., $\delta(W(2k+1)) = 2$ for $k \geqq 1$.

Suppose that we let ψ be the counting series of all simple posets and ψ^* the counting series of all simple posets A such that A has at least two maximal and two minimal vertices. Then clearly $\psi=z+\psi^*+2z\psi^*+z^2\psi^*$.

The counting series for simple posets constructed by substituting into C_2 is $2z\psi^*+z^2\psi^*+\psi^{*2}$ by an easy argument. If we write $\psi^*=(\psi-z)/(1+z)^2$, then we can re-express the counting function on C_2 by writing it in the form:

$$\{z(2+z)(1+z)^2+(\psi-z)\}\frac{(\psi-z)}{(1+z)^4}.$$

If we let $\vec{e}\circ\vec{n}=(e_1, \ldots, e_k)\circ(n_1, \ldots, n_k)$ be defined by

$$\vec{e}\circ\vec{n}=\binom{n_1}{e_1}\cdots\binom{n_k}{e_k}, \quad \text{with} \quad \vec{e}\circ_f\vec{n}=(e_1, \ldots, e_k)\circ(f(n_1), \ldots, f(n_k))$$

for $f\colon N\to N$, then the counting function for simple posets with exactly k connected components ψ_k is defined by:

$$\psi_k=\sum_{\vec{e}\cdot\vec{n}=k}\vec{e}\circ_{\psi_1}\vec{n},$$

and thus

$$\psi=\psi_1+\psi_2+\ldots=\sum_{k=1}^{\infty}\Big(\sum_{\vec{e}\cdot\vec{n}=k}\vec{e}\circ_{\psi_1}\vec{n}\Big).$$

It follows that we can define polynomials $h_m(x_1, \ldots, x_m)$ such that $\psi_1(m)=$ $=h_m(\psi(1), \ldots, \psi(m))$. Indeed, $h_1(x_1)=x_1$ and

$$h_{m+1}(x_1, \ldots, x_{m+1})=x_{m+1}-\sum_{\vec{e}\cdot\vec{n}=m+1}^{*}\vec{e}\odot\vec{n},$$

where

$$\vec{e}\odot\vec{n}=\binom{h_{n_1}(x_1, \ldots, x_{n_1})}{e_1}\cdots\binom{h_{n_k}(x_1, \ldots, x_{n_k})}{e_k}.$$

Thus the counting series for simple posets obtained by substituting into 2 is $\psi-\psi_1=\psi_2+\psi_3+\ldots$.

Finally, for other pure posets, if $|X|=n$, and if $\delta(X)=d$, then if we look at any substitution into X which results in a simple poset, then we find at least d distinct posets in the substitution, and furthermore, no more than d distinct ones are needed, since placing these strategically fixes X. Thus the counting series for this situation will at least have a factor ψ^{n-d}, with the other factor constructed along the lines of the computations for $\delta(C(X))$. Thus, e.g., if $X=W(2k+1)$, $k\geqq 2$, then $\delta(X)=2$ and it is easy to see that the counting series is given by $\frac{1}{2}\psi^{2k-1}(\psi^2-2\psi)$.

Alternating posets. Since simple posets are trivially alternating, and since chains C_k, $k\geqq 1$ are alternating of infinite length, the population of this class is rather large. It is also true that if X and Y are alternating and $C(X)\cap C(Y)=\emptyset$, then $X+Y$ is alternating. This is so since if $\sigma(C(X))=\sigma^n-a_{n+1}\sigma^{n+1}+\ldots$, $\sigma(C(Y))=$ $=\sigma^m-b_{m+1}\sigma^{m+1}+\ldots$, then $\sigma(C(X+Y))=\sigma(C(X))\cdot\sigma(C(Y))$, which is again alternating. Also if A and B are alternating and if in $A\oplus B$ there is no spot which arises "between" A and B, then $\sigma(C(A\oplus B))=\sigma(C(A))\cdot\sigma(C(B))$. Spots may lead to trouble since the fact that the product of two series is alternating does not imply that the factors themselves are alternating. At the moment we do not have any good criteria which identify alternating posets other than the observations made.

Control over counting. Prescription of pure posets. We shall call L an *ideal* of P if given $X\in L$, $|X|=n$, $A_1, \dots, A_n\in L$, $X(A_1, \dots, A_n)\in L$. If we list the elements of $L, X_1, X_2, \dots$ with X_i appearing before X_j if $|X_i|<|X_j|$, then we may proceed in the same way as before to construct subsets $C(X_1), C(X_2), \dots$. If we cross out those elements in $C(X_1)$ except X_1 and if we go down the list we may find an element $X_i\notin C(X_i)$. Continuing this process we will generate a collection of posets which we can consider as the collection of pure posets for L. If $1\in L$, then we do not use it as a pure poset.

If 1, 2 and C_2 are in L, then L is closed under addition and ordinal sum. Other examples of ideals include the finite lattices since substituting lattices into the vertices of a lattice yields a lattice. Indeed, if $X=Y(A_1, \dots, A_n)$ and if $a, b\in X$, then $a, b\in A_i$ yields $a\wedge b$, $a\vee b$, in A_i. If $a\in A_i$, $b\in A_j$, then $a\vee b$ is the minimal element of A_k, where $y_i\vee y_j=y_k$. Similarly $a\wedge b$ is the maximal element of A_k where $y_i\wedge y_j=y_k$.

If we are able to determine the pure posets X of L as well as their counting series $\sigma(C(X))$ expressed in terms of the counting series $\sigma(L)$ of L, then we can determine $\sigma(L)$ explicitly by using the same recursion techniques we have used above.

We consider an example. Suppose that L contains 1 and is generated by the pure posets 2, C_2, $W(4)$ and $W(5)$. If we let $\sigma(L)=\sigma$, we define the flat series $F=\sigma^2-\sigma^3+\sigma^4\dots$ and the connected series $\xi=\sum\xi(n)z^n$, where $\xi(n)=g_n(\sigma(1), \dots, \sigma(n))$. Furthermore $\sigma(C(W(4)))=\sigma^4$ and $\sigma(C(W(5)))=\sigma(\varphi\sigma^2+{}_2\sigma\varphi+({}_2\sigma)^2)$ with $\varphi=\frac{1}{2}(\sigma^2-{}_2\sigma)$. By direct computation we find $\sigma(1)=1$, $\sigma(2)=2$, $\sigma(3)=5$, $\sigma(4)=16$.

Thus we start the series for the various pure posets to obtain:

$$\sigma(C(C_2)) = z^2+3z^3+9z^4+32z^5+\dots;\quad \sigma(C(2)) = z^2+2z^3+6z^4+19z^5+\dots;$$

$$\sigma(C(W(4))) = z^4+8z^5+\dots;\quad \sigma(C(W(5))) = z^5+\dots.$$

Hence $\sigma(5)=60$. Using this as the next step we can then compute the coefficients in these four series, which are in order 128, 77, 44 and 6, whence $\sigma(6)=255$. The problem of course lies in the work involved in evaluating several of the expressions we need, especially the polynomials $g_m(\sigma(1), \dots, \sigma(m))$, where $\sigma(m)-g_m(\sigma(1), \dots, \sigma(m))$ is the coefficient of Z^m in $\sigma(C(2))$. Although we are not precisely in position to give a formula for the counting series σ given enough time, money and a fast computer we are in business.

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ZUR BIRKHOFF-INTERPOLATION GANZER FUNKTIONEN

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1. Aufgabenstellung

Es seien zwei unendliche, streng monoton steigende Folgen nichtnegativer ganzer Zahlen

$$p_1, p_2, p_3, \dots, \quad q_1, q_2, q_3, \dots$$

gegeben sowie drei unendliche Folgen komplexer Zahlen

$$A_1, A_2, A_3, \dots, \quad B_1, B_2, B_3, \dots$$

und

$$z_0, z_1, z_2, \dots \quad \text{mit} \quad z_n \neq z_0 \quad \text{für} \quad n = 1, 2, 3, \dots .$$

Dann wollen wir uns fragen, ob es ganze Funktionen f gibt, welche die Interpolationsbedingungen

(1) $$f^{(p_n)}(z_0) = A_n, \quad f^{(q_n)}(z_n) = B_n \quad (n = 1, 2, 3, \dots)$$

erfüllen.

In Analogie zu ähnlichen Interpolationsaufgaben für Polynome statt für ganze Funktionen, wollen wir (1) eine „Birkhoff'sche" Interpolationsaufgabe nennen (vgl. etwa SCHOENBERG [28], LORENTZ—ZELLER [17]).

In der von uns angegebenen Form wurde Aufgabe (1) offenbar zuerst von VERMES untersucht. Sein Vorgehen wie auch seine Ergebnisse sind jedoch von unseren wesentlich verschieden (vgl. [29—31], insbesondere [29], S. 115, Theorem 2 und [30], S. 77, Theorem 4). Ähnliches gilt für die Arbeiten von COMBES (vgl. [3—11]), auf die uns dankenswerter Weise Prof. Dr. H. SCHMIDT (Würzburg) hinwies.

Weitere verwandte Aufgabenstellungen wurden in der Literatur von einer Reihe von Autoren behandelt. So ordnet sich unsere Aufgabe für gewisse Parameterwerte der von GONTCHAROFF (vgl. [12], S. 58, Theorem VIII b) behandelten Aufgabe

(2) $$f^{(m)}(z_m) = B_m \quad (m = 0, 1, 2, \dots)$$

unter.

Weiter reduziert sie sich für den Fall, daß die Folge der Knoten z_n aus endlich vielen, sich periodisch wiederholenden Punkten besteht auf eine von PÓLYA [22] und VERMES (vgl. [30], S. 79, Theorem 6) für gewisse regelmäßig gebaute Folgen von Werten p_n und q_n behandelte Aufgabe.

Schließlich stellt Aufgabe (1) für den Fall, daß

(3) $$z_n = z_{n+1} \quad (n = 1, 2, 3, \dots)$$

gilt, eine sog. Lidstone'sche Interpolationsaufgabe dar, welche von einer Reihe von Autoren behandelt worden ist (vgl. etwa WHITTAKER [32—33], VERMES [29—30], LINDEN—PITTNAUER [16]).

Freilich ist Aufgabe (1) wesentlich verschieden von der sog. Hermite'schen Interpolationsaufgabe, die Pólya [21] u.a. untersucht haben. Dies hat seinen Grund insbesondere darin, daß wir im Punkte z_0 unendlich viele Bedingungen verlangen. Dabei ist man natürlich in der Wahl der Parameter längst nicht frei. So hat einmal Pólya (vgl. [22], S. 138—139) einfache Beispiele dafür gegeben, daß (1) bei ungeschickter Wahl der Parameter unlösbar ist und zum anderen kann man einem Satz von Catherine Rényi [25] entnehmen, daß dies sogar für einen großen Teil der Birkhoff'schen Interpolationsaufgaben für ganze Funktionen gilt. Schließlich hängt im Rahmen der noch verbleibenden Möglichkeiten das Erkennen zulässiger Parameter auch wesentlich von der verwendeten Lösungsmethode ab.

Während die genannten Arbeiten meist funktionentheoretisch vorgehen, ziehen wir analog zu Linden—Pittnauer [16] und Pittnauer [20] die Methode der Perron'schen Summengleichungen zur Lösung von (1) heran. Diese Methode ist von großer Einfachheit (vgl. Perron [19] und Paasche [18]) und liefert als Ergebnis unter gewissen Zusatzbedingungen eine sogar eindeutig bestimmte ganze Funktion f als Lösung von (1). Darüber hinaus erhalten wir neben einer Verallgemeinerung eines Ergebnisses von Pólya wie auch von Vermes Aussagen über die Wachstumsordnung ϱ und den Typ σ von f sowie über eine gewisse Minimaleigenschaft, welche f besitzt.

Schließlich läßt sich unsere Methode noch auf einige allgemeinere lineare Interpolationsaufgaben anwenden.

2. Das Lösungsverfahren mittels Perron'scher Summengleichungen

Um überhaupt eine Lösung von (1) in Gestalt einer ganzen Funktion zu ermöglichen, verlangen wir natürlich zunächst, daß

$$(4)\qquad \lim_{n\to\infty} \sqrt[p_n]{\frac{|A_n|}{p_n!}} = 0, \quad \lim_{n\to\infty} \sqrt[q_n]{\frac{|B_n|}{q_n!}} = 0$$

ist. Weiter definieren wir eine Hilfsfunktion h durch

$$(5)\qquad h(z) := \sum_{\mu=1}^{\infty} \frac{A_\mu}{p_\mu!} \cdot (z-z_0)^{p_\mu}.$$

Wegen (4) erkennt man, daß h eine ganze Funktion von z ist. Bilden wir nun die Größen

$$(6)\qquad c_n := B_n - h^{(q_n)}(z_n) \quad (n = 1, 2, 3, \ldots),$$

so wird im folgenden die Bedingung

$$(7)\qquad 0 < c := \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} < \infty$$

eine wichtige Rolle spielen. Sie besagt, daß die Funktion $f-h$ die Wachstumsordnung $\varrho=1$ hat, falls (3) gilt. Schließlich rechnet man unter Verwendung der Minkowski'schen Ungleichung und der Cauchy'schen Integralformeln nach, daß in (4) die zweite Grenzwertbeziehung eine Folge von (7) ist.

Wir ziehen nun zur Untersuchung unserer Aufgabe die Taylor'schen Reihen

$$f^{(q_n)}(z_n) = \sum_{k=0}^{\infty} \frac{(z_n - z_0)^k}{k!} \cdot f^{(k+q_n)}(z_0) = B_n \tag{8}$$

für $n=1, 2, 3, \ldots$ heran und ersetzen, falls $k+q_n=p_m$ mit einem geeigenten Wert m ist, den Wert $f^{(k+q_n)}(z_0)$ durch den Wert A_m gemäß (1). Dies Verfahren führt auf ein lineares Gleichungssystem für die restlichen Ableitungen $f^{(k+q_n)}(z_0)$ mit $k+q_n \neq p_m$. Im allgemeinen sind jedoch diese Ableitungen sehr unregelmäßig verteilt, so daß die Lösung des Systems offenbar sehr schwierig wird.

Um uns von diesen Schwierigkeiten frei zu machen, wollen wir weiter verlangen, daß unsere Folgen von Werten p_n und q_n die folgende Zusatzbedingung erfüllen (vgl. dazu LINDEN—PITTNAUER [16]):

(9) Es gibt eine unendliche Folge ganzer Zahlen r_m $(m = 1,2,3, \ldots)$ mit
a) Zu jedem q_n gibt es genau ein r_n mit $q_n \leqq r_n < q_{n+1}$,
b) $\{p_n | n \geqq 1\} \cap \{r_m | m \geqq 1\} = \emptyset$,
c) $\{p_n | n \geqq 1\} \cup \{r_m | m \geqq 1\} = \{k | k \geqq q_1\}$.

Existiert eine solche und dann stets eindeutig bestimmte Folge von Werten r_m, so wollen wir sie als die den Folgen von Werten p_n und q_n zugeordnete „reproduzierende" Folge bezeichnen.

Der Name erklärt sich dadurch, daß man unter Verwendung von (8) und (9) das Gleichungssystem (1) in das folgende relativ einfache und mit (1) äquivalente System überführen kann:

$$\sum_{\mu=0}^{\infty} \frac{(z_n - z_0)^{r_{n+\mu} - q_n}}{(r_{n+\mu} - q_n)!} \cdot f^{(r_{n+\mu})}(z_0) = c_n \tag{10}$$

für $n=1, 2, 3, \ldots$ mit den gemäß (6) festgelegten Werten c_n. Setzen wir nun für $n=1, 2, 3, \ldots;\ \mu=0, 1, 2, \ldots$

$$x_{n+\mu} := f^{(r_{n+\mu})}(z_0) \quad \text{und} \quad a_{n\mu} := \frac{(z_n - z_0)^{r_{n+\mu} - q_n}}{(r_{n+\mu} - q_n)!}, \tag{11}$$

so läßt sich auf unser Gleichungssystem (10) in vielen Fällen der folgende Satz von Perron—Paasche über sog. Perron'sche Summengleichungen anwenden, den wir der Arbeit von PAASCHE (vgl. [18], S. 25—29, Satz 7—10) entnehmen.

SATZ (Perron—Paasche). *Die Koeffizienten der Summengleichung*

$$\sum_{\mu=0}^{\infty} a_{n\mu} \cdot x_{n+\mu} = c_n \quad (n = 1, 2, 3, \ldots) \tag{12}$$

mögen eine Zerlegung $a_{n\mu} = a_\mu + b_{n\mu}$ $(n=1, 2, 3, \ldots;\ \mu=0, 1, 2, \ldots)$ *der folgenden Art gestatten:*

a) *Es ist* $a_{n0} \neq 0$ *für* $n=1, 2, 3, \ldots$.

b) *Es gilt*

$$c := \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} < \left(\overline{\lim_{n\to\infty}} \sqrt[\mu]{|a_\mu|}\right)^{-1}$$

und $|b_{n\mu}| \leqq b_n \cdot Q^{-\mu}$ *mit einem Wert* $Q > c$ *und mit* $\lim_{n\to\infty} b_n = 0$.

c) *Die „charakteristische" Funktion*

$$F(z) = \sum_{\mu=0}^{\infty} a_\mu \cdot z^\mu$$

is für $|z| \leqq c$ *analytisch und verschwindet dort an genau* $k \geqq 0$ *nicht notwendig voneinander verschiedenen Stellen.*

Dann besitzt (12) *nach willkürlicher Vorgabe der k Anfangswerte* $x_{N+1}, x_{N+2}, \ldots, \ldots, x_{N+k}$ *(wobei N ein genügend hoher Index sein soll) genau eine Lösung von Werten* $x_1, x_2, x_3, \ldots$ *mit*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|x_n|} = c. \tag{13}$$

Darüber hinaus gibt es keine Lösung, für die der Grenzwert in (13) *kleiner als c ausfällt.*

3. Einige spezielle Lösungsbedingungen

Wir wenden uns nun der Angabe spezieller lösbarer Aufgaben der Gestalt (1) zu. Dabei wollen wir der Kürze halber zwei Folgen von Werten p_n und q_n „zulässig" nennen, wenn sie die wichtige, wenn auch hier lediglich technisch bedingte, Eigenschaft (9) haben. Gilt noch zusätzlich $r_n = q_n$ für $n = 1, 2, 3, \ldots$, so wollen wir sie „komplementär" nennen.

Wir beweisen nun

SATZ 1. *Es seien zwei komplementäre Folgen von Werten* p_n *und* q_n $(n=1, 2, 3, \ldots)$ *gegeben. Ferner gelte* (4) *und* (7) *sowie eine der Bedingungen* (14), (15) *oder* (16):

(14) *Es gilt* $\lim_{n\to\infty} z_n = z_0$.

(15) *Es gilt* $\sup_{n\geqq 0} |z_n| \leqq D < \infty$ *und* $\lim_{n\to\infty} (q_{n+\mu} - q_n) = \infty$ *für jedes* $\mu \geqq 1$.

(16) *Es gilt* $|z_n - z_0| \leqq \dfrac{q_{n+1} - q_n}{e}$ $(n = 1, 2, 3, \ldots)$ *und* $\lim_{n\to\infty} (q_{n+\mu} - q_n) = \infty$ *für jedes* $\mu \geqq 1$.

Dann besitzt Aufgabe (1) *genau eine ganze Funktion f als Lösung mit der „Minimaleigenschaft"*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|f^{(r_n)}(z_0)|} = c. \tag{17}$$

BEWEIS. Wir zeigen die Existenz von f mit Hilfe des Satzes von Perron—Paasche unter Verwendung der Substitution (11).

α) Gilt Bedingung (14), so folgt zunächst

$$a_0 = a_{n0} = 1 \quad (n = 1, 2, 3, \ldots)$$

und

$$a_\mu = \lim_{n\to\infty} a_{n\mu} = 0 \quad (\mu = 1, 2, 3, \ldots).$$

Damit ist aber Voraussetzung a) erfüllt und wir erhalten

$$\lim_{\mu\to\infty} \sqrt[\mu]{|a_\mu|} = 0.$$

Somit können wir in (10) alle Werte c_n zulassen, welche (7) genügen. Weiter ergibt sich $b_{n0}=0$, $b_{n\mu}=a_{n\mu}$ $(n, \mu=1, 2, 3, \ldots)$ und damit gilt auch Voraussetzung b), wenn wir etwa

$$b_n = M\cdot|z_n-z_0|^{q_{n+1}-q_n} \quad \text{und} \quad c<Q<\infty, \quad 0<M<\infty$$

wählen. Dann hat aber gemäß Voraussetzung c) die charakteristische Funktion $F(z)\equiv 1$ genau $k=0$ Nullstellen für $|z|\leqq c$ und unser Satz liefert unmittelbar die Behauptung.

β) Gilt Bedingung (15), so erhält man ebenfalls $a_0=a_{n0}=1$ und $a_\mu=0$ $(n, \mu=1, 2, 3, \ldots)$ und kann damit wie in Beweisteil α) weiterschließen.

γ) Gilt schließlich Bedingung (16), so erhält man zunächst wieder

$$a_0 = a_{n0} = 1 \quad (n = 1, 2, 3, \ldots).$$

Beachtet man nun, daß $q_{n+\mu}-q_n\geqq\mu$ für $\mu=0, 1, 2, \ldots$ ist, so erhält man für $\mu=1, 2, 3, \ldots$

$$|a_{n\mu}| = \frac{|z_n-z_0|^{q_{n+\mu}-q_n}}{(q_{n+\mu}-q_n)!} \leqq \frac{[e^{-1}\cdot(q_{n+1}-q_n)]^{q_{n+\mu}-q_n}}{(q_{n+\mu}-q_n)!} \leqq$$

$$\leqq \frac{[e^{-1}\cdot(q_{n+\mu}-q_n)]^{q_{n+\mu}-q_n}}{(q_{n+\mu}-q_n)!} \leqq \frac{K}{\sqrt{2\pi(q_{n+\mu}-q_n)}}$$

mit einer geeigneten Konstanten $K>0$ nach der Stirling'schen Formel. Wir erhalten also $a_\mu=0$ für $\mu=1, 2, 3, \ldots$ und können wie unter Beweisteil α) weiterschließen.

Es sei noch bemerkt, daß Satz 1 mit Bedingung (15) ein Ergebnis von VERMES (vgl. [29], S. 115, Theorem 2) in verschiedener Hinsicht verallgemeinert. So benötigt etwa Vermes anstatt der Bedingung (15) die einschneidendere, daß $|z_n-z_0|=1$ $(n=1, 2, 3, \ldots)$ und daß die Folgen der Werte A_n und B_n beschränkt bleiben.

Schließlich weisen wir noch darauf hin, daß in unserem Satz Bedingung (16) noch etwas abgeschwächt werden kann zu

$$(18) \qquad |z_n-z_0| \leqq \left[\alpha_n\cdot\sqrt{2\pi(q_{n+1}-q_n)}\right]^{\frac{1}{q_{n+1}-q_n}}\cdot\frac{q_{n+1}-q_n}{e}$$

mit vorgegebenen Werten $\alpha_n>0$ $(n=1, 2, 3, \ldots)$. Hier hängt dann freilich die Anzahl der Lösungen f gemäß dem Satz von Perron—Paasche von der Anzahl der Nullstellen der charakteristischen Funktion

$$(19) \qquad F(z) = \sum_{\mu=0}^{\infty}\alpha_\mu\cdot z^\mu \quad \text{mit} \quad \alpha_0 = 1$$

im Kreis $|z|\leqq c$ ab.

Schließlich weisen wir darauf hin, daß die Lösungsbedingungen unseres Satzes von ganz anderer Natur sind, als die von GONTCHAROFF für Aufgabe (2) angegebenen (vgl. [12], S. 58, Theorem VIII b).

SATZ 2. *Es seien zwei zulässige Folgen von Werten p_n und q_n gegeben, welche der folgenden Bedingung genügen.*

(20) *Mit einer ganzen Zahl $k \geqq 1$ gilt*

$$r_n = r_{n_0} + k \cdot (n - n_0) \quad \textit{für} \quad n \geqq n_0,$$

$$q_n = q_{n_1} + k \cdot (n - n_1) \quad \textit{für} \quad n \geqq n_1.$$

Ferner gelte neben (4) *und* (7) *die Beziehung*

$$\lim_{n \to \infty} z_n = \tilde{z} \neq z_0. \tag{21}$$

Dann besitzt Aufgabe (1) *entweder so viele linear unabhängige und* (17) *genügende ganze Funktionen f als Lösung wie die charakteristische Funktion*

$$F(z) = \sum_{\mu=0}^{\infty} \frac{(\tilde{z} - z_0)^{k\mu + l}}{(k\mu + l)!} \cdot z^{\mu} \tag{22}$$

mit $l = (r_{n_0} - q_{n_1}) - k(n_0 - n_1)$ Nullstellen in der komplexen Zahlenebene besitzt, oder, falls F keine Nullstellen besitzt, eine eindeutig bestimmte und (17) *genügende ganze Funktion f als Lösung.*

BEWEIS. Zunächst überlegt man sich, daß (20) mit der Bedingung

$$0 \leqq \lim_{n \to \infty} (r_{n+\mu} - q_n) = e_\mu < \infty \tag{23}$$

mit $e_\mu = k\mu + l$, $l \geqq 0$ für jedes $\mu \geqq 0$ äquivalent ist. Damit folgt dann leicht die Behauptung.

Zur Untersuchung der Nullstellen der charakteristischen Funktion in (22) kann man von der Darstellung von F als verallgemeinerte Mittag—Leffler'sche Funktion

$$F(z) = (\tilde{z} - z_0)^l \cdot E_{k, l+1}[(\tilde{z} - z_0)^k \cdot z]$$

ausgehen (vgl. dazu etwa BATEMAN—ERDÉLYI—MAGNUS—OBERHETTINGER—TRICOMI [1], S. 210).

Satz 2 stellt eine Verallgemeinerung eines Ergebnisses von PÓLYA [22] dar und für $z_n = \tilde{z}$ $(n = 1, 2, 3, \ldots)$ eine Verallgemeinerung eines Ergebnisses von LINDEN—PITTNAUER [16].

4. Folgerungen aus den Lösungsbedingungen

Unsere Sätze erlauben uns, Folgerungen über die Wachstumsordnung $\varrho(f)$ und den Typ $\sigma(f)$ der Lösungen f zu ziehen. Zunächst erhalten wir aus den bekannten Formeln, welche die Wachstumsordnung und den Typ einer ganzen Funktion durch die Koeffizienten ihrer Taylorreihe ausdrücken (vgl. etwa BIEBERBACH [2], S. 238), unmittelbar die Werte $\varrho(h)$ und $\sigma(h)$ für die in (5) definierte ganze Funktion h. Damit ergibt sich dann die folgende Aussage.

KOROLLAR 3. *Ist f eine Lösung von Aufgabe* (1) *gemäß einem unserer Sätze, so ist*

$$\varrho(f) = \max[1, \varrho(h)]$$

und

$$\sigma(f) = \begin{cases} \sigma(h), & \text{falls} \quad \varrho(f) > 1, \\ \max[\sigma(h), \sigma(f-h)], & \text{falls} \quad \varrho(f) = 1. \end{cases}$$

Dabei gilt mit dem in (7) *definierten Wert c*

$$\sigma(f-h) = c^{\alpha}, \quad \text{falls} \quad \alpha := \lim_{n\to\infty} \frac{n}{r_n} \quad \text{existiert} \tag{24}$$

und falls dieser Grenzwert nicht existiert,

$$0 \leqq \sigma(f-h) \leqq \begin{cases} c^{\left[\overline{\lim\limits_{n\to\infty}} \frac{n}{r_n}\right]}, & \text{falls} \quad c > 1, \\ 1, & \text{falls} \quad c = 1, \\ c^{\left[\underline{\lim\limits_{n\to\infty}} \frac{n}{r_n}\right]}, & \text{falls} \quad c < 1. \end{cases} \tag{25}$$

Der Beweis dieser auf (17) beruhenden Aussage ist elementar und soll hier übergangen werden. Die Herleitung der Abschätzung (25) erfordert freilich sorgfältige Überlegungen bei der Handhabung der auftretenden Grenzwerte. Dabei sei noch darauf hingewiesen, daß sich $r_n \geqq n$ $(n=1, 2, 3, \ldots)$ aus (9) ergibt.

Schließlich wollen wir uns noch fragen, wie Folgen ganzer Zahlen p_n, q_n und r_n beschaffen sein müssen, damit sie Bedingung (9) erfüllen. Bei der Beantwortung dieser Frage werden Folgen ganzer Zahlen s_n $(n=1, 2, 3, \ldots)$ mit der Eigenschaft

$$0 \leqq s_n < s_{n+1}; \quad \lim_{n\to\infty} (s_n - n) = \infty \tag{26}$$

eine Rolle spielen. Es gibt nämlich

LEMMA 4. *Genügen die Folgen ganzer Zahlen p_n, q_n und r_n $(n=1, 2, 3, \ldots)$ der Bedingung* (9), *so besitzt jede dieser Folgen die Eigenschaft* (26).

BEWEIS. Zunächst gilt $p_n - p_{n-1} \geqq 1$ und wegen (9) gilt sogar $p_n - p_{n-1} \geqq 2$ für unendlich viele Werte von n. Damit erhalten wir aber

$$p_n - p_1 = \sum_{\nu=2}^{n} (p_\nu - p_{\nu-1}) \geqq n + \gamma(n),$$

wobei $\lim\limits_{n\to\infty} \gamma(n) = \infty$ ist. Hieraus folgt aber die Behauptung für die Folge der p_n. Ähnlich zeigt man den Rest der Behauptung.

LEMMA 5. *Es sei eine Folge ganzer Zahlen p_n $(n=1, 2, 3, \ldots)$ mit der Eigenschaft* (26) *gegeben sowie eine ganze Zahl $q_1 \geqq 0$. Dann gibt es genau eine Folge ganzer Zahlen r_n $(n=1, 2, 3, \ldots)$ mit der Eigenschaft* (26), *welche die Bedingung* (9b) *und* (9c) *erfüllt. Ferner gibt es dazu abzählbar viele voneinander verschiedene Folgen ganzer Zahlen q_n $(n=1, 2, 3, \ldots)$ mit der Eigenschaft* (26), *welche die Bedingung* (9a) *erfüllen.*

BEWEIS. Zunächst zeigt man leicht die Existenz und eindeutige Bestimmtheit der Folge der ganzen Zahlen r_n unter den Bedingungen (9b), (9c) und (26). Damit kann man dann eine Folge ganzer Zahlen q_n mit

$$q_n \leqq r_n < q_{n+1} \leqq r_{n+1} < q_{n+2}$$

angeben, so daß jedem r_n genau ein q_n zugeordnet ist. Es folgt

$$\lim_{n\to\infty}(q_n-n) \geqq \lim_{n\to\infty}([r_{n-1}-(n-1)]-1) = \infty.$$

Da die Folge der r_n die Eigenschaft (26) hat, kann man für unendlich viele Werte n den Wert q_n mindestens auf zwei verschiedene Arten festlegen. Damit folgt aber die Behauptung.

Unsere letzte Aussage zeigt, daß man auf einfache Weise sehr bequem Folgen konstruieren kann, welche die für unsere Lösungsmethode wichtige Eigenschaft (9) besitzen und daß es offensichtlich sehr viele Folgen dieser Art gibt.

5. Bemerkungen

In Analogie zur Polynominterpolation (vgl. etwa SCHOENBERG [22]) läßt sich auch unserer Aufgabe (1) eine sog. Inzidenzmatrix

$$E = (e_{rs})_{r,s=0,1,2,\dots}$$

zuordnen, indem man

$$e_{rs} = \begin{cases} 1, & \text{falls } f^{(s)}(z_r) \text{ gemäß (1) vorgeschrieben ist,} \\ 0, & \text{sonst} \end{cases}$$

setzt. Damit ist dann gemäß (9) und Lemma 4 die Inzidenzmatrix unserer Aufgabe (1) eine spaltenfinite obere Dreiecksmatrix. Wegen der Bedingungen im Punkte z_0 ist sie jedoch nicht zeilenfinit. Insgesamt handelt es sich um eine lediglich „dünn" besetzte Matrix. Im Hinblick auf den bereits erwähnten Satz von CATHERINE RÉNYI [25] (vgl. dazu auch PÓLYA [23], S. 432) über die Anzahl der Fabry'schen Lückenpunkte ganzer Funktionen darf man vermuten, daß Interpolationsaufgaben für ganze Funktionen, deren Inzidenzmatrizen viele Einsen enthalten, nur schwer zu behandeln sein werden.

Betrachten wir weiter in Analogie zur Polynominterpolation die auf PÓLYA zurückgehenden Größen

$$m(s) := \sum_{r=0}^{\infty} e_{rs}; \quad M(s) := \sum_{\sigma=0}^{s} m(\sigma) \quad (s = 0, 1, 2, \dots),$$

so erhalten wir für Aufgabe (1) die Bedingungen

$$m(s) \leqq 2; \quad M(s) \leqq 2s+2, \tag{27}$$

während sich für die Gontcharoff'sche Aufgabe (2) $m(s)=1$; $M(s)=s+1$ ergibt.

In diesem Zusammenhang, sei noch auf die von WHITTAKER [32] eingeführte Größe $W(s)=M(s-1)$ hingewiesen, die im Falle von Lidstone'schen Interpolationsaufgabe bei der Formulierung von Eindeutigkeitsbedingungen eine Rolle spielt.

Abschließend sei noch bemerkt, daß sich mit unserer Methode auch allgemeinere lineare Interpolationsaufgaben für ganze Funktionen behandeln lassen, wie etwa die Aufgabe

$$f^{(p_n)}(z_0) = A_n + \sum_{\varkappa=1}^{k} A_{n\varkappa} \cdot f^{(s_{n\varkappa})}(z_0), \tag{28}$$

$$\sum_{\nu=1}^{n} d_\nu \cdot f^{(q_n)}(z_\nu) = B_n \quad (n = 1, 2, 3, \ldots).$$

Es ist nicht schwer, mit dem Satz von Perron—Paasche auch hier Aussagen über die Existenz von Lösungen unter geeigneten Zusatzbedingungen zu gewinnen. Ein einfacher Spezialfall dieser Aufgabe wurde von LINDEN—PITTNAUER [16] untersucht. Den vermutlich ersten Hinweis auf Aufgaben dieser Art findet man bei PORITZKY [24].

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REMARKS ON OPERATOR TRANSFORMATIONS OF A FIELD OF TRANSFORMABLE OPERATORS

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1. Introduction

Let $\mathscr{C}_0$ be the convolution algebra of all functions φ, which are defined on $[0, \infty)$ and whose Laplace transforms $\bar{\varphi}(z)$ are absolute convergent. We denote by $\mathfrak{Q}$ the quotient field of $\mathscr{C}_0$. It is well known that $\mathfrak{Q}$ is a subfield of the Mikusiński operator field $\mathscr{M}$. In the Ditkin—Berg-model of $\mathscr{M}$ (see [3, 7]) every operator can be represented by a sequence of certain meromorphic functions. In [12] an operator field $\mathfrak{U}$ is constructed, which contains also $\mathfrak{Q}$ and which is algebraically isomorphic to the field $\mathfrak{M}$ of all functions $f(z)$ that are meromorphic in some right half-planes $\Delta=\{z:\mathrm{Re}(z)>\sigma\}$, where σ depends on $f(z)$ and the operations in $\mathfrak{M}$ are defined pointwise. That means, in the function theoretic model of the operator field $\mathfrak{A}$ every operator can be represented by a single meromorphic function. Therefore the function theoretic model of $\mathfrak{A}$ is not so difficult of access as the function theoretic model of $\mathscr{M}$. Moreover, the operator field $\mathfrak{A}$ is containing operators which appear as solutions of simple s-differential equations and which do not belong to $\mathscr{M}$; for example the operators $\exp[s^2]$ and $\exp[\alpha s]$ for complex α [14].

In the present note the operator transformations of the operator field $\mathfrak{A}$ will be investigated.

2. The operator field $\mathfrak{A}$

Let $\mathfrak{H}$ be the subalgebra of $\mathfrak{M}$, which consists of all functions $h(z)$ that are holomorphic in some right half-planes Δ. Suppose that $(h_n(z))$ is a sequence in $\mathfrak{H}$ and $h(z)\in\mathfrak{H}$. By definition, $\lim h_n(z)=h(z)$, if there exists a right half plane Δ such that $h(z)$, $h_n(z)$ $(n=1, 2, \ldots)$ are holomorphic in Δ and if the sequence $(h_n(z))$ converges to $h(z)$ uniformly on every compact subdomain of Δ.

Assume always that every element of $\mathfrak{Q}$ is represented by a convolution quotient φ/ψ $(\varphi, \psi\in\mathscr{C}_0, \psi\not\equiv 0)$, because in $\mathscr{M}$ other representations are also possible. We call a sequence $(\varphi_n/\psi_n)\subset\mathfrak{Q}$ a fundamental sequence if there are functions $h(z)$, $g(z)\in\mathfrak{H}$; $g\not\equiv 0$, such that $\lim\bar{\varphi}_n(z)=h(z)$ and $\lim\bar{\psi}_n(z)=g(z)$. It is easy to see that the function

$$\mathfrak{L}[\varphi_n/\psi_n]\overset{\text{def}}{=}h(z)/g(z)$$

belongs to $\mathfrak{M}$. Two fundamental sequences (φ_n/ψ_n) and (η_n/ξ_n) are equivalent if $\mathfrak{L}[\varphi_n/\psi_n]=\mathfrak{L}[\eta_n/\xi_n]$ in the sense of $\mathfrak{M}$. The equivalence classes determined by this equivalence relation are called operators. Let $\mathfrak{A}$ be the set of all operators. An operator $a\in\mathfrak{A}$ represented by a fundamental sequence (φ_n/ψ_n) will be denoted by $a=\langle\varphi_n/\psi_n\rangle$. Two operators are equal if the representatives are equivalent. Let α

be a complex number and let $a=\langle\varphi_n/\psi_n\rangle$ and $b=\langle\eta_n/\xi_n\rangle$ belong to $\mathfrak{A}$. We define

$$a+b = \langle(\varphi_n*\xi_n+\eta_n*\psi_n)/(\psi_n*\xi_n)\rangle;$$
$$ab = \langle(\varphi_n*\eta_n)/(\psi_n*\xi_n)\rangle;$$
$$\alpha a = \langle(\alpha\varphi_n)/\psi_n\rangle,$$

where $*$ is the symbol for the convolution product in $\mathscr{C}_0$. $\mathfrak{A}$ is a field under these operations and the mapping

(2.1) $$\mathfrak{L}[a] \overset{\text{def}}{=} \mathfrak{L}[\varphi_n/\psi_n]$$

is an algebraic isomorphism of $\mathfrak{A}$ onto $\mathfrak{M}$. The function $\mathfrak{L}[a]\in\mathfrak{M}$ will be called the generalized Laplace transform of the operator $a\in\mathfrak{A}$, because if $a\in\mathfrak{A}$ is a function $a=\varphi\in\mathscr{C}_0\subset\mathfrak{A}$ (the exact notation for φ in $\mathfrak{A}$ is $\langle(\varphi*\psi)/\psi\rangle$ with any $\psi\not\equiv 0$, but we use also the symbol φ in $\mathfrak{A}$) then $\mathfrak{L}[a]=\bar{\varphi}(z)$ is the classical Laplace transform of φ. The differential operator

$$s = \langle\varphi/\varphi^{(-1)}\rangle, \quad \varphi^{(-1)} \overset{\text{def}}{=} \int_0^t \varphi(u)\,du, \quad \varphi\not\equiv 0,$$

has the Laplace transform $\mathfrak{L}[s]=z$, i.e. we can write $a=f(s)$, if $\mathfrak{L}[a]=f(z)\in\mathfrak{M}$. The convergence in $\mathfrak{A}$ is defined as follows:

A sequence $(a_n)\subset\mathfrak{A}$ converges to $a\in\mathfrak{A}$ if there exist quotients $\mathfrak{L}[a_n]=h_n(z)/g_n(z)$ and $\mathfrak{L}[a]=h(z)/g(z)$ in $\mathfrak{M}$ $(h_n, g_n, h, g\in\mathfrak{H};\ g_n, g\not\equiv 0)$ such that $\lim h_n(z)=h(z)$ and $\lim g_n(z)=g(z)$. Then we write $\mathfrak{A}\text{-}\lim a_n=a$.

This convergence is compatible with the field structure of $\mathfrak{A}$.

3. Operator transformations

An operator transformation T is a linear mapping of a subset $\mathfrak{A}(T)\subset\mathfrak{A}$ in $\mathfrak{A}$ (for the operator transformations of $\mathscr{M}$ see [4, 9]). The operations in the set of all operator transformations are the usual operations. Operator transformations T having the property

(3.1) $$T(ab) = T(a)b = aT(b)$$

for all $a, b\in\mathfrak{A}(T)$ will be called multipliers. If T fulfils

(3.2) $$T(ab) = T(a)T(b)$$

$(a, b\in\mathfrak{A}(T))$ then T is an endomorphism, and if T has the property

(3.3) $$T(ab) = T(a)b + aT(b)$$

for all $a, b\in\mathfrak{A}(T)$ then T will be called a derivation. T is a continuous operator transformation if from $\mathfrak{A}\text{-}\lim a_n=a$ it follows $\mathfrak{A}\text{-}\lim T(a_n)=T(a)$ $(a_n, a\in\mathfrak{A}(T))$.

Suppose that $\mathfrak{A}(T)=\mathfrak{A}$. If T is a multiplier then we have $T(a_n)=a_nT(1)$ and $T(a)=aT(1)$ (1 is the unit operator in $\mathfrak{A}$) such that from $\mathfrak{A}\text{-}\lim a_n=a$ it follows $\mathfrak{A}\text{-}\lim T(a_n)=T(a)$, because the convergence is compatible with the field structure of $\mathfrak{A}$. That means that all multipliers on $\mathfrak{A}$ are continuous operator transformations. It is evident (see (3.1)) that every multiplier on $\mathfrak{A}$ is commutative with the differential operator s and with all shift operators $\exp[\alpha s]$, α real. We can prove the following characterization of the multipliers:

THEOREM 1. *The multipliers on $\mathfrak{A}$ are the only continuous operator transformations on $\mathfrak{A}$ which are commutative with the differential operator s and with all shift operators $\exp[\alpha s]$, α real.*

PROOF. (i) Let T be any continuous operator transformation on $\mathfrak{A}$ which is commutative with s, i.e. $sT(b)=T(s)b$ for all $b\in\mathfrak{A}$. Suppose that $\varphi\in\mathscr{C}_0$. Then we have $\bar{\varphi}(z)\in\mathfrak{H}$ and it is well known that there are sequences $(p_n(z))$ of polynomials in z with $\lim p_n(z)=\bar{\varphi}(z)$ (see [1]). Therefore $\mathfrak{A}\text{-}\lim p_n(s)=\varphi$ in $\mathfrak{A}$. For any operator $b\in\mathfrak{A}$ we have

$$\mathfrak{A}\text{-}\lim p_n(s)T(b)=\varphi T(b)$$

and

$$\mathfrak{A}\text{-}\lim p_n(s)T(b)=\mathfrak{A}\text{-}\lim T\big(p_n(s)\big)b=T(\varphi)b.$$

On the other hand, we have a unique limit such that

$$\varphi T(b)=T(\varphi)b. \tag{3.4}$$

Let a be any operator of $\mathfrak{A}$. We can find a sequence $(\varphi_n)\subset\mathscr{C}_0$ with $\mathfrak{A}\text{-}\lim\varphi_n=a$, because $\mathscr{C}_0$ is dense in $\mathfrak{A}$ [13]. Therefore from (3.4) it follows

$$aT(b)=\mathfrak{A}\text{-}\lim\varphi_nT(b)=\mathfrak{A}\text{-}\lim T(\varphi_n)b=T(a)b.$$

Hence $aT(b)=T(a)b$ for all operators $a,b\in\mathfrak{A}$, i.e. T is a multiplier on $\mathfrak{A}$.

(ii) Suppose that T is a continuous operator transformation on $\mathfrak{A}$ with

$$T(\exp[\alpha s])b=\exp[\alpha s]T(b)$$

for all $b\in\mathfrak{A}$. Now if $\psi\in\mathscr{C}_0$ fulfils $|\psi(t)|<\exp[\sigma t]$ for a certain real σ and $t>0$ then there exists a right half-plane $\Delta=\{z:\operatorname{Re}(z)>\gamma>1\}$ such that the Laplace integral of the function $\varphi\overset{\text{def}}{=}s^{-1}\psi$ is uniformly convergent in Δ (see [2]). Hence we can find a sequence $t_n\in[0,\infty)$ $(n=1,2,\ldots)$ with

$$\left|\int_{t_n}^{\infty}\exp[-zt]\varphi(t)\,dt\right|<\frac{1}{2n} \tag{3.5}$$

for all $z\in\Delta$. On the other hand, φ is a continuous function, thus there exist sequences (ψ_n) of step functions with the properties

$$|\psi_n(t)-\varphi(t)|<\frac{1}{2n}\quad\text{for}\quad t\in[0,t_n] \tag{3.6}$$

and

$$\psi_n(t)=0\qquad\text{for}\quad t\notin[0,t_n]. \tag{3.7}$$

From (3.5), (3.6), (3.7) we have

$$|\bar{\psi}_n(z)-\bar{\varphi}(z)|=\left|\int_0^{t_n}\exp[-zt](\psi_n(t)-\varphi(t))\,dt-\int_{t_n}^{\infty}\exp[-zt]\varphi(t)\,dt\right|<\frac{1}{n}$$

for all $z \in \Delta$, i.e. $\lim \bar{\psi}_n(z) = \bar{\varphi}(z)$. Therefore we obtain also $\mathfrak{A}\text{-}\lim \psi_n = \varphi$ or $\mathfrak{A}\text{-}\lim s\psi_n = \psi$. It is easy to see that $s\psi_n$ $(n=1, 2, \dots)$ is a finite linear combination of shift operators, whence we get

$$\mathfrak{A}\text{-}\lim T(s\psi_n)b = T(\psi)b$$

and

$$\mathfrak{A}\text{-}\lim T(s\psi_n)b = \mathfrak{A}\text{-}\lim s\psi_n T(b) = \psi T(b)$$

for all $b \in \mathfrak{A}$. Hence

$$T(\psi)b = \psi T(b) \tag{3.8}$$

for all $b \in \mathfrak{A}$. Now let $a \in \mathfrak{A}$ be any operator and $(\varphi_n) \subset \mathscr{C}_0$ a sequence with $\mathfrak{A}\text{-}\lim \varphi_n = a$. From [13] it follows that we can find a sequence (φ_n) such that all functions φ_n satisfy exponential estimations, i.e. (3.8) holds for all φ_n. We get

$$\mathfrak{A}\text{-}\lim T(\varphi_n)b = T(a)b$$

and

$$\mathfrak{A}\text{-}\lim T(\varphi_n)b = \mathfrak{A}\text{-}\lim \varphi_n T(b) = aT(b)$$

for all $b \in \mathfrak{A}$. Therefore T fulfils $T(a)b = aT(b)$ for all $a, b \in \mathfrak{A}$ i.e. T is a multiplier. This completes the proof.

It is easy to see that the following representation theorem is valid.

THEOREM 2. *If T is any multiplier on $\mathfrak{A}$ then there exists an operator $a \in \mathfrak{A}$ with $T(b) = ab$ for all $b \in \mathfrak{A}$.*

With $a = T(1)$ the proof is trivial.

Now we will consider the endomorphisms. The first possibility to construct endomorphisms on $\mathfrak{A}$ is given by

THEOREM 3. *Let T_0 be any endomorphism on $\mathfrak{A}(T_0) = \mathscr{C}_0$ having the following property:*

with $\lim \bar{\varphi}_n(z) = h(z)$, $\lim \left(\overline{T_0(\varphi_n)}\right)(z) = \hat{h}(z)$ also holds ($\varphi_n \in \mathscr{C}_0$ for all n; $h, \hat{h} \in \mathfrak{H}$; $\hat{h} \equiv 0$ iff $h \equiv 0$).

In this case by definition $T(a) = \langle T_0(\varphi_n)/T_0(\psi_n)\rangle$, $a = \langle \varphi_n/\psi_n\rangle \in \mathfrak{A}$, T_0 is extendable to an endomorphism T on $\mathfrak{A}$.

PROOF. Let $a = \langle \varphi_n/\psi_n\rangle \in \mathfrak{A}$ be any operator. Then there exist functions $h, g \in \mathfrak{H}$ $(g \not\equiv 0)$ with $\lim \bar{\varphi}_n(z) = h$ and $\lim \bar{\psi}_n(z) = g$. Hence

$$\mathfrak{L}[T(a)] = \mathfrak{L}[T_0(\varphi_n)/T_0(\psi_n)] = \hat{h}/\hat{g} \in \mathfrak{M}$$

.e. $T(a) \in \mathfrak{A}$. If we start with an other representative $\langle \xi_n/\eta_n\rangle$ of the operator a then we have $\lim \bar{\xi}_n(z) = h_1$, $\lim \bar{\eta}_n(z) = g_1$, $\lim \left(\overline{T_0(\xi_n)}\right)(z) = \hat{h}_1$ and $\lim \left(\overline{T_0(\eta_n)}\right)(z) = \hat{g}_1$ $(h_1, g_1, \hat{h}_1, \hat{g}_1 \in \mathfrak{H})$. It is easy to see that

$$\lim \left(\overline{\varphi_n * \eta_n - \xi_n * \psi_n}\right)(z) = hg_1 - h_1 g \equiv 0,$$

because $\mathfrak{L}[a] = h/g = h_1/g_1$. Hence our supposition implies

$$\lim \left(\overline{T_0(\varphi_n * \eta_n - \xi_n * \psi_n)}\right)(z) \equiv 0.$$

Because T_0 is an endomorphism on $\mathscr{C}_0$ we get

$$\lim\left(\overline{T_0(\varphi_n)}\right)(z)\lim\left(\overline{T_0(\eta_n)}\right)(z)=\lim\left(\overline{T_0(\xi_n)}\right)(z)\lim\left(\overline{T_0(\psi_n)}\right)(z)$$

or $\hat{h}/\hat{g}=\hat{h}_1/\hat{g}_1$, i.e. the definition of T is unique. It is easy to see that T is linear and fulfils property (3.2). Hence T is an endomorphism on $\mathfrak{A}$. In the case $a=\varphi\in\mathscr{C}_0$ we have $a=\varphi=\langle(\varphi*\psi)/\psi\rangle$, $\psi\not\equiv 0$, and

$$T(\varphi)=\langle T_0(\varphi*\psi)/T_0(\psi)\rangle=\langle T_0(\varphi)*T_0(\psi)/T_0(\psi)\rangle=\langle T_0(\varphi)*\eta/\eta\rangle=T_0(\varphi)\in\mathscr{C}_0$$

$(\eta\stackrel{\text{def}}{=}T_0(\psi))$. This completes the proof.

We consider an other possibility in order to get endomorphisms. BLEYER [5] constructed endomorphisms for Mikusiński operators by certain substitutions in the Ditkin—Berg-model of $\mathscr{M}$ (see [3, 7]). This principle can be transferred to the operator field $\mathfrak{A}$. In this case it will be a little simpler, because we can omit certain conditions.

Let $\omega(z)$ be any function having the following property:

$$(3.9)\quad\begin{cases}\text{There exist a real number } \sigma \text{ and a right half-plane } \varDelta_\sigma \text{ depending on } \sigma\\ \text{such that } \omega(z) \text{ is holomorphic in } \varDelta_\sigma \text{ and fulfils the estimation}\\ \operatorname{Re}(\omega(z))>\sigma \text{ for all } z\in\varDelta_\sigma.\end{cases}$$

Now we consider the set $\mathfrak{A}_\sigma$ of all operators $a=f(s)\in\mathfrak{A}$ having Laplace transforms $f(z)$, which are meromorphic in $\varDelta=\{z:\operatorname{Re}(z)>\sigma\}$. It is easy to see that $\mathfrak{A}_\sigma$ is a subfield of $\mathfrak{A}$. We will prove the following

THEOREM 4. *Suppose that the Laplace transform $\omega(z)$ of the operator $\omega(s)\in\mathfrak{A}$ has property* (3.9). *If $a=f(s)$ belongs to $\mathfrak{A}_\sigma$ then $T^\omega(a)\stackrel{\text{def}}{=}f(\omega(s))$ defines an endomorphism with $\mathfrak{A}(T^\omega)=\mathfrak{A}_\sigma$.*

PROOF. If $a=f(s)\in\mathfrak{A}_\sigma$ then $f(z)\in\mathfrak{M}$ is meromorphic in $\varDelta=\{z:\operatorname{Re}(z)>\sigma\}$. The Mittag—Leffler and Weierstrass theorems imply that there exist functions $h(z), g(z)\in\mathfrak{H}$ $(g\not\equiv 0)$ with $f(z)=h(z)/g(z)$. From our supposition for $\omega(z)$ it follows that $h(\omega(z))$ and $g(\omega(z))$ are holomorphic in $\varDelta_\sigma$ (see [1]). Therefore $f(\omega(z))=$ $=h(\omega(z))/g(\omega(z))$ is meromorphic in $\varDelta_\sigma$, i.e. $T^\omega(a)=f(\omega(s))$ belongs to $\mathfrak{A}$. Because in $\mathfrak{M}$ we have introduced the pointwise operations, it is easy to see that T^ω is linear and fulfils (3.2). Hence T^ω is an endomorphism with $\mathfrak{A}(T^\omega)=\mathfrak{A}_\sigma$.

REMARK. If the function $\omega(z)$ fulfils (3.9) for a γ with $\gamma>\sigma$ then T^ω can be extended to an endomorphism of a field $\mathfrak{A}_\gamma$ (with $\mathfrak{A}_\sigma\subset\mathfrak{A}_\gamma$) in $\mathfrak{A}$.

LEMMA 1. *Let T be any endomorphism with $\mathfrak{A}(T)=\mathfrak{A}$. Suppose that $(a_n)\subset\mathfrak{A}$ is any sequence with $(\mathfrak{L}[a_n])\subset\mathfrak{H}$ and $\lim\mathfrak{L}[a_n]=\mathfrak{L}[a]$, where $a\in\mathfrak{A}$. If we have also $\lim\mathfrak{L}[T(a_n)]=\mathfrak{L}[T(a)]$ then T is a continuous endomorphism on $\mathfrak{A}$.*

PROOF. Let $(b_n)\subset\mathfrak{A}$ be any sequence with $\mathfrak{A}$-$\lim b_n=b\in\mathfrak{A}$. Then there are quotients $\mathfrak{L}[b_n]=h_n(z)/g_n(z)$ and $\mathfrak{L}[b]=h(z)/g(z)$ in $\mathfrak{M}$ $(h_n, g_n, h, g\in\mathfrak{H};\ g_n, g\not\equiv 0)$ such that $\lim h_n(z)=h(z)$ and $\lim g_n(z)=g(z)$. On the other hand, from (3.2) it follows

$$T(b_n)=T(h_n(s))/T(g_n(s))$$

and

$$T(b) = T(h(s))/T(g(s)),$$

i.e. by our supposition we get $\mathfrak{A}\text{-}\lim T(b_n)=T(b)$.

Now we can prove

THEOREM 5. *Let σ_0 be any fixed real number. If the function $\omega(z)$ fulfils property (3.9) for every real number σ with $\sigma \geqq \sigma_0$ then $T^\omega(a) \overset{\text{def}}{=} f(\omega(s))$, where $a=f(s)\in\mathfrak{A}$, defines a continuous endomorphism T^ω on $\mathfrak{A}$.*

PROOF. It is easy to see that T^ω is an endomorphism on $\mathfrak{A}$. Now let $(a_n)\subset\mathfrak{A}$ be any sequence with $\mathfrak{L}[a_n]\subset\mathfrak{H}$ and $\lim \mathfrak{L}[a_n]=h(z)$. That means that there exists a right half-plane $\Delta=\{z:\text{Re}(z)>\sigma\geqq\sigma_0\}$ such that all $\mathfrak{L}[a_n]\overset{\text{def}}{=}h_n(z)$ and $h(z)$ are holomorphic in Δ and the sequence $(h_n(z))$ converges to $h(z)$ uniformly on every compact subdomain of Δ. For every compact subdomain of Δ and every $\varepsilon>0$ we can find an n_0 such that $|h_n(z)-h(z)|<\varepsilon$ for all $n\geqq n_0$ and all z belonging to this compact subdomain. By our supposition the function $\omega(z)$ cannot be constant. Hence the function $\omega(z)$ maps every compact subdomain of Δ_σ in a compact subdomain of Δ (see [1]). That means that in every compact subdomain of Δ_σ we get $|h_n(\omega(z))-h(\omega(z))|<\varepsilon$ for sufficiently large numbers n, i.e. $\lim \mathfrak{L}[T^\omega(a_n)]= =\mathfrak{L}[T^\omega(h(s))]$. By Lemma 1, the proof is complete.

We consider two examples:

Suppose that $\omega(s)=s-\alpha$, where α is any complex number. Then we get the well known endomorphism $T^\omega \overset{\text{def}}{=} T_\alpha$ having the property $T_\alpha(\varphi)=\{e^{\alpha t}\varphi(t)\}$ for all $\varphi\in\mathscr{C}_0$.

If we set $\omega(s)=s/k$, where $k>0$, then we obtain $T^\omega \overset{\text{def}}{=} U_k$ with $U_k(\varphi)= =\{k\varphi(kt)\}$ for $\varphi\in\mathscr{C}_0$.

The endomorphisms T_α and U_k can be constructed also by using Theorem 3. From Theorem 5 it follows that T_α and U_k are continuous endomorphisms on $\mathfrak{A}$.

Theorems 4 and 5 imply

COROLLARY 1. *If the functions ω_i $(i=1,2)$ fulfil condition (3.9) with the real numbers σ_i $(i=1,2)$ then $T^{\omega_1+\omega_2}$ is an endomorphism on $\mathfrak{A}_{\sigma_1+\sigma_2}$.*

COROLLARY 2. *If the supposition of Theorem 5 is fulfilled for one of the functions ω_i in Corollary 1, then $T^{\omega_1+\omega_2}$ is a continuous endomorphism on $\mathfrak{A}$.*

For example, if $\omega(s)=\frac{s}{k}+\frac{l}{s}-\alpha$, where $k,l>0$ and α is a complex number, then T^ω is a continuous endomorphism on $\mathfrak{A}$, because $\omega_1(s)=\frac{s}{k}$ fulfils the supposition of Theorem 5.

Now we consider the derivations of $\mathfrak{A}$.

LEMMA 2. *Let T be any derivation of $\mathfrak{A}$. Assume that $(a_n)\subset\mathfrak{A}$ is any sequence with $(\mathfrak{L}[a_n])\subset\mathfrak{H}$ and $\lim\mathfrak{L}[a_n]=h(z)$. If in this case we get also $\lim\mathfrak{L}[T(a_n)]= =\mathfrak{L}[T(h(s))]$ then T is a continuous derivation of $\mathfrak{A}$.*

The proof follows from the definition of the convergence in $\mathfrak{A}$ and from the well known formula

$$T(a/b) = (bT(a)-aT(b))/b^2 \quad (a, b\in\mathfrak{A}).$$

An example of a derivation is the algebraic derivation D defined by

$$D(a) = \langle ((\{-t\varphi_n(t)\} * \psi_n - \varphi_n * \{-t\psi_n(t)\})/(\psi_n * \psi_n)) \rangle, \quad a = \langle \varphi_n/\psi_n \rangle$$

(see [13]). It is easy to prove that

$$\mathfrak{L}[D(a)] = \frac{d}{dz}\mathfrak{L}[a], \quad \text{i.e.} \quad D(f(s)) = \frac{d}{ds}f(s) \in \mathfrak{A}, \quad a = f(s).$$

It is well known that from $\lim h_n(z) = h(z)$ in $\mathfrak{H}$ it follows $\lim \frac{d}{dz}h_n(z) = \frac{d}{dz}h(z)$, i.e. D fulfils the supposition of Lemma 2. Hence D is a continuous derivation of $\mathfrak{A}$. If the operator a is a function $a = \varphi \in \mathscr{C}_0$ then we have also $D(\varphi) = \{-t\varphi(t)\}$ in $\mathfrak{A}$. From the definition of the differential isomorphism (see [10]) and the formula $\mathfrak{L}[D(a)] = \frac{d}{dz}\mathfrak{L}[a]$ we obtain

THEOREM 6. *The mapping* (2.1) *is a differential isomorphism of the differential field* $\mathfrak{A}$ *(with the derivation* D*) onto the differential field* $\mathfrak{M}$ $\left(\textit{with the derivation } \frac{d}{dz}\right)$.

Therefore we get

COROLLARY 3. *The operator field* $\mathfrak{A}$ *with the derivation* D *is a differential field of characteristic zero, and the field of the constants is the closed algebraic field of the complex numbers.*

This corollary is important for the solution of linear s-differential equations in $\mathfrak{A}$ (see [14]).

Other derivations of $\mathfrak{A}$ are given by

THEOREM 7. *Let* T *be any multiplier on* $\mathfrak{A}$. *Then* $D_T \overset{\text{def}}{=} TD$ *is also a continuous derivation of* $\mathfrak{A}$.

It is easy to see that D_T is a continuous operator transformation on $\mathfrak{A}$, because T and D are continuous. On the other hand, from (3.1) and (3.3) it follows

$$D_T(ab) = T(bD(a) + aD(b)) = bTD(a) + aTD(b) = bD_T(a) + aD_T(b),$$

i.e. D_T is a derivation of $\mathfrak{A}$.

REMARK. If T is a multiplier on $\mathfrak{A}$ then $F \overset{\text{def}}{=} DT$ is also a continuous operator transformation on $\mathfrak{A}$, but from $DT(ab) = bDT(a) + aTD(b)$ we obtain that DT is a derivation only under the condition $TD = DT$. However, in this case T is determined by a number operator (see Theorem 2).

It is well known [8] that every continuous derivation on the Mikusiński operator-field $\mathscr{M}$ has the form cD, where D is also the algebraic derivation of $\mathscr{M}$, $c \in \mathscr{M}$ is any operator and the continuity is in the sense of GESZTELYI [9]. In [6] an analogous more general theorem is proved for certain differential fields. From this general theorem it follows that also all sequential continuous derivations of $\mathscr{M}$ (in the sense of the I- and II-type convergence of Mikusiński) have the form cD. A second special case from [6] is

COROLLARY 4. *Every continuous derivation T of $\mathfrak{A}$ has the form D_T.*

For the multiplication of special operator transformations we have the following formulas (in analogy to $\mathcal{M}$):

$$(3.10)\quad DT_\alpha = T_\alpha D;\quad DU_k = \frac{1}{k}U_k D;\quad U_k T_\alpha = T_{\alpha k} U_k \quad\text{and}\quad U_k T_\alpha D = kDU_k T_\alpha.$$

The inversion of the algebraic derivative D in $\mathfrak{A}$ (and $\mathcal{M}$) is not always possible. On the other hand, if the function $\omega(z)$ in Theorem 5 is such that the inverse function fulfils also the supposition of Theorem 5, then $(T^{\omega})^{-1}$ is also an endomorphism on $\mathfrak{A}$ (for example $T_\alpha^{-1} = T_{-\alpha}$ and $U_k^{-1} = U_{1/k}$).

In analogy to [4] we obtain in the sense of the continuity of $\mathfrak{A}$ the following

THEOREM 8. *The endomorphisms $T = U_k T_\alpha$ are the only continuous endomorphisms $(\neq 0)$ on $\mathfrak{A}$ having the property $TD = kDT$ with $k > 0$ (for $k = 1$ we get $T = T_\alpha$).*

PROOF. It is easy to see that $U_k T_\alpha$ is a continuous endomorphism for every complex number α and $k > 0$, thus the statement follows from (3.10).

Now let T be any continuous endomorphism on $\mathfrak{A}$ with $TD(a) = kDT(a)$ where $a \in \mathfrak{A}$ is any operator. We prove that $T = U_k T_\alpha$. The proof can be performed differently from the adequate proof in [4]. If we have $T \neq 0$ (0 is the endomorphism determined by the zero operator) then $kDT(s) = TD(s) = T(1) = 1$ or $\frac{d}{dz}\mathfrak{L}[T(s)] = \frac{1}{k}$ such that we obtain $\mathfrak{L}[T(s)] = \frac{z}{k} - \alpha$, where α is any complex constant. That means

$$T(s) = \frac{s}{k} - \alpha.$$

Now let $r(s)$ be any rational term in the differential operator s. It is easy to prove that

$$T(r(s)) = r(T(s)) = r\left(\frac{s}{k} - \alpha\right) = U_k T_\alpha(r(s)).$$

On the other hand, the field of all rational terms of s is dense in $\mathfrak{A}$ [13] and a convergent sequence in $\mathfrak{A}$ has a unique limit. Therefore the continuity of T and $U_k T_\alpha$ yields $T = U_k T_\alpha$.

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ÜBER BESCHRÄNKTE ORTHONORMIERTE SYSTEME

Von
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Herrn Professor Béla Szőkefalvi-Nagy zum 65. Geburtstag gewidmet

1. Es sei $1 \leqq K \leqq \infty$ gegeben, und wir bezeichnen mit $\Omega(K)$ die Klasse der othonormierten Funktionensysteme $\varphi = \{\varphi_n(x)\}_1^\infty$ im Intervall (0, 1), für die

$$|\varphi_n(x)| \leqq K \quad (x \in (0, 1); \quad n = 1, 2, \ldots)$$

erfüllt ist. ($\Omega(\infty)$ ist die Klasse aller orthonormierten Systeme in (0, 1).)

Für eine reelle Zahlenfolge $a = \{a_n\}_1^\infty$ setzen wir:

$$\|a; K\| = \sup_{\varphi \in \Omega(K)} \Big\{ \int_0^1 \sup_{1 \leqq i \leqq j < \infty} (a_i \varphi_i(x) + \ldots + a_j \varphi_j(x))^2 \, dx \Big\}^{1/2}.$$

Es sei

$$M(K) = \{a \colon \|a; K\| < \infty\}.$$

Man kann leicht einsehen, daß die Funktion $\|\cdot\,; K\|$ die Eigenschaften einer Norm besitzt und $M(K)$ mit der Norm $\|\cdot\,; K\|$ ein Banachraum ist.

Für eine Folge a und für beliebige Indizes $1 \leqq N \leqq M$ setzen wir:

$$a(N, M) = \{0, \ldots, a_N, \ldots, a_M, 0, \ldots\},$$

$$a(N, \infty) = \{0, \ldots, 0, \; a_N, a_{N+1}, \ldots\}.$$

Auf Grund der Definition von $\|\cdot\,; K\|$ für jede Folge a und für jede Zahl $1 \leqq K \leqq \infty$ sind die Beziehungen

$$\Big\{ \sum_{n=1}^\infty a_n^2 \Big\}^{1/2} \leqq \|a; K\|,$$

$$\|a(N, M); K\| \leqq \|a; K\| \quad (1 \leqq N \leqq M < \infty),$$

$$\|a(N, \infty); K\| \geqq \|a(N+1, \infty); K\| \quad (N = 1, 2, \ldots),$$

$$\lim_{M \to \infty} \|a(N, M); K\| = \|a(N, \infty); K\| \quad (N = 1, 2, \ldots)$$

offensichtlich, weiterhin gilt

$$\|a; 1\| \leqq \|a; K_1\| \leqq \|a; K_2\| \leqq \|a; \infty\| \quad (1 \leqq K_1 < K_2 < \infty).$$

In der Arbeit [4] haben wir den folgenden Satz bewiesen:

SATZ A. Ist $a \in M(\infty)$, *dann konvergiert die Reihe*

$$(1) \qquad \sum_{n=1}^\infty a_n \varphi_n(x)$$

für jedes System $\varphi\in\Omega(\infty)$ *in* $(0, 1)$ *fast überall. Ist aber* $a\notin M(\infty)$, *dann gibt es ein System* $\varphi\in\Omega(\infty)$, *für welches* (1) *in* $(0, 1)$ *fast überall divergiert.*

Das Analogon dieser Behauptung ist im Falle $1\leqq K<\infty$ noch hicht bewiesen. Nur die folgende Behauptung konnten wir zeigen [5]:

SATZ B. *Es sei* $1<K<\infty$ *gegeben. Ist* $\lim\limits_{N\to\infty}\|a(N,\infty); K\|=0$, *dann konvergiert die Reihe* (1) *für jedes System* $\varphi\in\Omega(K)$ *in* $(0, 1)$ *fast überall. Ist aber* $\lim\limits_{N\to\infty}\|a(N,\infty); K\|\neq 0$, *dann gibt es ein System* $\varphi\in\Omega(K)$, *für welches die Reihe* (1) *in* $(0, 1)$ *fast überall divergiert.*

In dieser Arbeit werden wir zuerst das Analogon des Satzes B im Falle $K=1$ zeigen.

SATZ I. *Ist* $\lim\limits_{N\to\infty}\|a(N,\infty); 1\|=0$, *dann konvergiert die Reihe* (1) *für jedes System* $\varphi\in\Omega(1)$ *in* $(0, 1)$ *fast überall. Ist aber* $\lim\limits_{N\to\infty}\|a(N,\infty); 1\|\neq 0$, *dann gibt es ein System* $\varphi\in\Omega(1)$, *für welches die Reihe* (1) *in* $(0, 1)$ *fast überall divergiert.*

2. Zum Beweis brauchen wir einige Hilfssätze.

HILFSSATZ I. *Es sei* $1\leqq K<\infty$ *eine gegebene Zahl,* $a=\{a_n\}_1^\infty$ *eine Folge und* N *eine natürliche Zahl. Dann gibt es ein im Intervall* $(0, 1)$ *orthonormieries System von Treppenfunktionen* $\psi_1(x), \ldots, \psi_N(x)$ *mit*

$$|\psi_n(x)|\leqq K \quad \big(x\in(0,1);\ n=1,\ldots,N\big)$$

und

$$\int_0^1 \max_{1\leqq i\leqq j\leqq N}|a_i\psi_i(x)+\ldots+a_j\psi_j(x)|^2\,dx > \frac{1}{2}\|a(1,N); K\|^2.$$

Hilfssatz I ist im Falle $1<K<\infty$ in [6] bewiesen, und die Behauptung ist im Falle $K=1$ offensichtlich.

HILFSSATZ II. *Es sei* $1\leqq K<\infty$ *eine gegebene Zahl,* $a=\{a_n\}_1^\infty$ *eine Folge und* N *eine natürliche Zahl mit*

$$\|a(1,N); K\|^2\geqq 128K^2\sum_{n=1}^N a_n^2.$$

Dann gibt es ein orthogonales System von Treppenfunktionen $\psi_1(x), \ldots, \psi_N(x)$ *mit folgenden Eigenschaften. Es gelten*

$$|\psi_n(x)|=1 \qquad \big(x\in(0,1);\ n=1,\ldots,N\big),$$

$$\max_{1\leqq i\leqq j\leqq N}|a_i\psi_i(x)+\ldots+a_j\psi_j(x)|\geqq\frac{1}{4K}\|a(1,N); K\| \quad (x\in E),$$

wobei $E\big(\subseteqq(0,1)\big)$ *eine einfache Menge*[1] *ist, für die*

$$\operatorname{mes}(E)\geqq 1/10$$

besteht.

[1] D. h. E ist die Vereinigung endlich vieler Intervalle.

Beweis des Hilfssatzes II. Wir brauchen eine Idee von B. S. Kašin [1] (s. noch [7]). Ohne Beschränkung der Allgemeinheit können wir $\|a(1, N); K\| = 2$ voraussetzen. Durch Anwendung des Hilfssatzes I bekommen wir ein orthonormiertes System der Treppenfunktionen $\bar\psi_1(x), \dots, \bar\psi_N(x)$ mit

$$|\bar\psi_n(x)| \leqq K \quad (x\in(0,1);\ n = 1, \dots, N), \tag{2}$$

$$\int_0^1 \max_{1\leqq i\leqq j\leqq N} |a_i\bar\psi_i(x)+\dots+a_j\bar\psi_j(x)|^2\,dx > \frac{1}{2}\|a(1, N);\ K\|^2.$$

Es sei $I_1, \dots, I_\varrho$ eine Einteilung des Intervalls (0, 1) in disjunkte Teilintervalle derart, daß jede Funktion $\bar\psi_n(x)$ in jedem Intervall I_r konstant ist. Den Wert der Funktion

$$\max_{1\leqq i\leqq j\leqq N} |a_i\bar\psi_i(x)+\dots+a_j\bar\psi_j(x)|$$

im Intervall I_r bezeichnen wir mit w_r. Nach (2) gilt

$$4 = \|a(1, N);\ K\|^2 \geqq \sum_{r=1}^{\varrho} w_r^2 \operatorname{mes}(I_r) > \frac{1}{2}\|a(1, N);\ K\|^2 = 2. \tag{3}$$

Es seien $1\leqq r_1<\dots<r_\varkappa\leqq\varrho$ diejenige Indizes, für die $w_r\geqq 1$ ist, die Indizes $r\,(1\leqq r\leqq\varrho)$, die von $r_1, \dots, r_\varkappa$ verschieden sind, bezeichnen wir der Reihe nach mit $s_1, \dots, s_{\varrho-\varkappa}$. Aus (3) folgt

$$4 \geqq a = \sum_{k=1}^{\varkappa} w_{r_k}^2 \operatorname{mes}(I_{r_k}) > 1. \tag{4}$$

Es seien $J_k'=(a_k', b_k')$ $(k=1, \dots, \varkappa)$ disjunkte Intervalle in $(0, a)$ mit $\operatorname{mes}(J_k')= = w_{r_k}^2 \operatorname{mes}(I_{r_k})$, weiterhin seien $J_k''=(a_k'', b_k'')$ $(k=1, \dots, \varrho-\varkappa)$ disjunkte Intervalle in $(a, a+b)$ $\left(b=\sum_{k=1}^{\varrho-\varkappa}\operatorname{mes}(I_{s_k})\right)$ mit $\operatorname{mes}(J_k'')=\operatorname{mes}(I_{s_k})$ $(k=1, \dots, \varrho-\varkappa)$. Offensichtlich gilt $b\leqq 1$, wir setzen

$$\tilde\psi_n(x) = \begin{cases} \bar\psi_n\left(\dfrac{x-a_{s_k}''}{b_{s_k}''-a_{s_k}''}(b_{s_k}-a_{s_k})+a_{s_k}\right) & (x\in J_k'';\quad k = 1, \dots, \varrho-\varkappa), \\ \dfrac{1}{w_{r_k}}\bar\psi_n\left(\dfrac{x-a_{r_k}'}{b_{r_k}'-a_{r_k}'}(b_{r_k}-a_{r_k})+a_{r_k}\right) & (x\in J_k';\quad k = 1, \dots, \varkappa) \end{cases} \quad (n = 1, \dots, N),$$

wobei $I_r=(a_r, b_r)$, und sei

$$\psi_n^*(x) = \tilde\psi_n((a+b)x)/K \quad (n = 1, \dots, N).$$

Offensichtlich bilden die Treppenfunktionen $\psi_n^*(x)$ im Intervall (0, 1) ein orthogonales System, es gilt

$$|\psi_n^*(x)| \leqq 1 \quad (x\in(0,1);\ n = 1, \dots, N). \tag{5}$$

Es sei E die Bildmenge des Intervalls $(0, a)$ durch die lineare Transformation $y=x/(a+b)$. Aus (4) ergibt sich

$$\operatorname{mes}(\bar E) \geqq 1/5. \tag{6}$$

Weiterhin, auf Grund der Definition der Funktionen $\psi_n^*(x)$, gilt

$$(7) \qquad \max_{1\leq i\leq j\leq N} |a_i\psi_i^*(x)+\ldots+a_j\psi_j^*(x)| = \frac{1}{K} = \frac{1}{2K}\|a(1, N); K\| \quad (x\in\bar{E}).$$

Es sei $J_1, \ldots, J_\sigma$ eine Einteilung des Intervalls $(0, 1)$ in paarweise disjunkte Intervalle derart, daß jede Funktion $\psi_n^*(x)$ in jedem Intervall J_s konstant ist, und $\bar{E}$ die Vereinigung einiger J_s ist. Den Wert von $\psi_n^*(x)$ im Intervall J_s bezeichnen wir mit $\varrho_s^{(n)}$. Für jeder Indizes s $(1\leq s\leq\sigma)$ sei $\{\chi_s^{(n)}(x)\}_1^N$ eine Folge von stochastisch unabhängigen Treppenfunktionen, für die

$$\int_0^1 \chi_s^{(n)}(x)\,dx = 0 \quad (n = 1, \ldots, N)$$

gilt, und jede Funktion $\chi_s^{(n)}(x)$ den Wertbereich $\{1-\varrho_s^{(n)}, -1-\varrho_s^{(n)}\}$ besitzt.[2] Wir setzen

$$\chi_s^{(n)}(J_s;\, x) = \begin{cases} \chi_s^{(n)}\left(\dfrac{x-c_s}{d_s-c_s}\right) & (x\in J_s), \\ 0 & \text{sonst}, \end{cases} \qquad (n = 1, \ldots, N;\ s = 1, \ldots, \sigma),$$

wobei $J_s=(c_s, d_s)$ ist. Aus (5) folgt

$$(8) \qquad |\chi_s^{(n)}(J_s; x)| \leq 2 \quad (n = 1, \ldots, N;\ s = 1, \ldots, \sigma).$$

Es sei

$$\psi_n(x) = \psi_n^*(x)+\sum_{s=1}^{\sigma}\chi_s^{(n)}(J_s; x) \quad (n = 1, \ldots, N).$$

Man kann leicht einsehen, daß die Treppenfunktionen $\psi_n(x)$ $(n=1, \ldots, N)$ in $(0, 1)$ ein orthogonales System bilden, weiterhin gilt

$$|\psi_n(x)| = 1 \quad \big(x\in(0, 1);\ n = 1, \ldots, N\big).$$

Es sei $J_s\subseteq\bar{E}$. Aus (7) erhalten wir

$$(9) \qquad \operatorname{mes}\left\{x\in J_s : \max_{1\leq i\leq j\leq N}\left|\sum_{n=i}^{j} a_n\psi_n(x)\right| \geq \frac{1}{4K}\|a(1, N); K\|\right\} \geq$$

$$\geq \operatorname{mes}\left\{x\in J_s : \max_{1\leq i\leq j\leq N}\left|\sum_{n=i}^{j} a_n\psi_n^*(x)\right| \geq \frac{1}{2K}\|a(1, N); K\|\right\} -$$

$$-\operatorname{mes}\left\{x\in J_s : \max_{1\leq i\leq j\leq N}\left|\sum_{n=i}^{j} a_n\chi_s^{(n)}(J_s;\, x)\right| \geq \frac{1}{4K}\|a(1, N); K\|\right\}.$$

[2] Im Falle $\varrho_s^{(n)}=1$ soll man $\chi_s^{(n)}(x)\equiv 0$ setzen.

Nach (8) ergibt sich durch Anwendung der Kolmogoroffschen Ungleichung

$$\operatorname{mes}\left\{x\in J_s:\ \max_{1\le i\le j\le N}\left|\sum_{n=i}^{j} a_n\chi_s^{(n)}(J_s;\,x)\right| \ge \frac{1}{4K}\|a(1,N);\,K\|\right\} \le$$

$$\le \operatorname{mes}(J_s)\,16K^2\sum_{n=1}^{N} a_n^2\int_0^1 (\chi_s^{(n)}(x))^2\,dx/\|a(1,N);\,K\|^2 \le$$

$$\le \operatorname{mes}(J_s)\,64K^2\sum_{n=1}^{N} a_n^2/\|a(1,N);\,K\|^2 \le \operatorname{mes}(J_s)/2.$$

Daraus und aus (7), (9) ergibt sich: im Falle $J_s\subseteq \bar{E}$ ist

$$\text{(10)}\qquad \operatorname{mes}\left\{x\in J_s:\ \max_{1\le i\le j\le N}\left|\sum_{n=i}^{j} a_n\psi_n(x)\right| \ge \frac{1}{4K}\|a(1,N);\,K\|\right\} \ge \operatorname{mes}(J_s)/2.$$

Es sei

$$E=\bigcup_{s,\,J_s\subseteq \bar{E}}\left\{x\in J_s:\ \max_{1\le i\le j\le N}\left|\sum_{n=i}^{j} a_n\psi_n(x)\right| \ge \frac{1}{4K}\|a(1,N);\,K\|\right\}.$$

E ist eine einfache Menge und nach (6), (10) sind alle Erforderungen des Hilfssatzes II erfüllt.

3. Beweis des Satzes I. Es sei $\lim\limits_{N\to\infty}\|a(N,\infty);\,1\|\ne 0$. Ist $\sum\limits_{n=1}^{\infty} a_n^2=\infty$, dann ist die Rademachersche Reihe $\sum\limits_{n=1}^{\infty} a_n r_n(x)$ in (0, 1) fast überall divergent, und $\{r_n(x)\}_1^\infty\in\Omega(1)$.

Also können wir $\sum\limits_{n=1}^{\infty} a_n^2<\infty$ annehmen. Dann kann man aber eine Indexfolge $n_1<\ldots<n_k<\ldots$ mit folgenden Eigenschaften angeben:

$$\|a(n_k+1,\,n_k);\,K\| > \varrho > 0 \quad (k=1,2,\ldots),$$

$$\|a(n_k+1,\,n_{k+1});\,K\| \ge 128K^2\sum_{n=n_k+1}^{n_{k+1}} a_n^2 \quad (k=1,2,\ldots).$$

Durch Anwendung des Hilfssatzes II kann man mit bekannter Methode (s. [4]) ein orthonormiertes System $\varphi=\{\varphi_n(x)\}_1^\infty\in\Omega(1)$ derart angeben, daß die Reihe (1) in (0, 1) fast überall divergiert.

Es sei nun $\lim\limits_{N\to\infty}\|a(N,\infty);\,1\|=0$. Dann kann man mit einer in [4] angewandter Methode zeigen, daß die Reihe (1) für jedes System $\varphi\in\Omega(1)$ in (0, 1) fas überall konvergiert.

4. Wir zeigen noch den folgenden Satz.

Satz II. *Es sei* $1<K<\infty$ *gegeben. Die Relation* $\lim\limits_{N\to\infty}\|a(N,\infty);\,1\|=0$ *ist mit* $\lim\limits_{N\to\infty}\|a(N,\infty);\,K\|=0$ *äquivalent.*

Da für jede Folge a $\|a; K\| \geqq \|a; 1\|$ $(1<K)$ gilt, die Implikation

$$\lim_{N\to\infty} \|a(N, \infty); K\| = 0 \Rightarrow \lim_{N\to\infty} \|a(N, \infty); 1\| = 0$$

ist offensichtlich.

Nehmen wir $\lim_{N\to\infty} \|a(N, \infty); K\| \neq 0$ an. Ist $\sum_{n=1}^{\infty} a_n^2 = \infty$, dann divergiert die Rademachersche Reihe $\sum_{n=1}^{\infty} a_n r_n(x)$ in $(0, 1)$ fast überall. Da $\{r_n(x)\}_1^\infty \in \Omega(1)$ gilt, folgt aus dem Satz I $\lim_{N\to\infty} \|a(N, \infty); 1\| \neq 0$.

Nehmen wir also $\sum_{n=1}^{\infty} a_n^2 < \infty$ an. Dann gibt es aber eine natürliche Zahl N_0 derart, daß im Falle $N_0 \leqq N$ für genügend große natürliche Zahl M

$$\|a(N, M); K\| \geqq 128K^2 \sum_{n=N}^{M} a_n^2$$

gilt. Auf Grund des Hilfssatzes II ergibt sich für genügend große natürliche Zahl M

$$\|a(N, M); 1\|^2 \geqq \|a(N, M); K\|^2/640\, K^2 \quad (N_0 \leqq N < M),$$

woraus

$$\|a(N, \infty); 1\|^2 \geqq \|a(N, \infty); K\|^2/640\, K^2 \quad (N_0 \leqq N)$$

folgt, und so gilt

$$\lim_{N\to\infty} \|a(N, \infty); 1\| \neq 0.$$

5. Für eine Zahl $1 \leqq K \leqq \infty$ sei $M^*(K)$ die Klasse der der Folgen a, für welche die Reihe (1) bei jedem System $\varphi \in \Omega(K)$ in $(0, 1)$ fast überall konvergiert. Da

$$\Omega(1) \subseteqq \Omega(K_1) \subseteqq \Omega(K_2) \subseteqq \Omega(\infty) \quad (1 < K_1 < K_2 < \infty)$$

gilt, ist

$$M^*(1) \supseteqq M^*(K_1) \supseteqq M^*(K_2) \supseteqq M^*(\infty) \quad (1 < K_1 < K_2 < \infty).$$

Aus dem Satz II folgt:

SATZ III. *Im Falle* $1 \leqq K < \infty$ *gilt* $M^*(K) = M^*(1)$.

6. In der Arbeit [3] haben wir $M^*(\infty) = M(\infty)$ gezeigt. Ob $M^*(K) = M(K)$ im Falle $1 \leqq K < \infty$ besteht, ist noch ein offenes Problem. Ein anderes Problem ist, ob $M^*(\infty) = M^*(1)$ gilt.

7. Wir werden noch gewisse Folgerungen des Satzes III erwähnen. Es sei $a = \{a_n\}_1^\infty$ eine Folge mit $|a_n| \geqq |a_{n+1}|$ $(n = 1, 2, \ldots)$ und

$$\sum_{n=1}^{\infty} a_n^2 \log^2 n = \infty.$$

In der Arbeit [5] haben wir gezeigt, daß für jede Zahl $1 < K < \infty$ ein System $\{\varphi_n(x)\}_1^\infty \in \Omega(K)$ derart existiert, daß die Reihe (1) in $(0, 1)$ fast überall divergiert. Also gehört diese Folge a nicht zu der Klasse $M^*(K)$, und auf Grund des Satzes III ist $a \notin M^*(1)$, so folgt aus dem Satz I, daß *im Falle* $|a_n| \geq |a_{n+1}|$ $(n=1, 2, \ldots)$, $\sum_{n=1}^{\infty} a_n^2 \log^2 n = \infty$ *ein System* $\varphi \in \Omega(1)$ *derart existiert, daß die Reihe* (1) *in* (0, 1) *fast überall divergiert.*

Aus diesem Behauptung mit bekannter Methode (s. [3]) folgt:

Es sei $\{\lambda_n\}_1^\infty$ *eine monoton nichtabnehmende Folge von positiven Zahlen mit*

$$\sum_{n=1}^{\infty} \frac{\log^2 n}{\lambda_n^2} = \infty.$$

Dann gibt es ein System $\varphi = \{\varphi_n(x)\}_1^\infty \in \Omega(1)$, *derart, daß*

$$\overline{\lim_{n\to\infty}} \frac{1}{\lambda_n} |\varphi_1(x) + \ldots + \varphi_n(x)| = \infty$$

in (0, 1) *fast überall gilt.*

Diese Behauptung ist im Falle $\varphi \in \Omega(\infty)$ bekannt (s. [3]). Eine schwächere Behauptung haben wir schon vorher gezeigt [2].

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SIMULTANEOUS APPROXIMATION BY INTERPOLATING POLYNOMIALS

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1. Introduction

On the interval $[-1, 1]$, we construct linear polynomial operators $A_n(f; x)$ of degree $\leqq 2n(1+c)$ which interpolate f and f' at the Chebyshev nodes (supposing f' is continuous), moreover, provide the Teliakovski—Gopengauz estimation.

2. Preliminary results

2.1. On the interval $[-1, 1]$, let us consider the uniquely determined Hermite—Fejér interpolating polynomials of degree $\leqq 2n-1$ for the Chebyshev roots

$$x_{k,n} = \cos \vartheta_{k,n} = \cos \frac{2k-1}{2n}\pi \quad (k = 1, 2, \ldots, n;\ n = 1, 2, \ldots), \tag{2.1}$$

i.e., supposing $f' \in C$ (i.e. f' is continuous on $[-1, 1]$), let

$$H_n^*(f; x) = \sum_{k=1}^{n} f(x_{k,n}) h_{k,n}(x) + \sum_{k=1}^{n} f'(x_{k,n}) \mathfrak{H}_{k,n}(x), \tag{2.2}$$

where (with $x = \cos\vartheta$ and $T_n(x) = \cos n\vartheta$)

$$l_{k,n}(x) = \frac{(-1)^{k+1} \sin \vartheta_{k,n} T_n(x)}{n(x - x_{k,n})}, \tag{2.3}$$

$$v_{k,n}(x) = \frac{1 - x x_{k,n}}{1 - x_{k,n}^2}, \tag{2.4}$$

$$h_{k,n}(x) = v_{k,n}(x) l_{k,n}^2(x), \quad \mathfrak{H}_{k,n}(x) = (x - x_{k,n}) l_{k,n}^2(x). \tag{2.5}$$

As L. FEJÉR [1] proved

$$H_n^*(f; x_{k,n}) = f(x_{k,n}), \quad H_n'^*(f; x_{k,n}) = f'(x_{k,n}), \tag{2.6}$$

further $H_n^*(f; x)$ uniformly tends to $f(x)$. As for the rate of the convergence, one can choose the functions f_1 and f_2 such that $f_1', f_2' \in \mathrm{Lip}\,\alpha$ $(0 < \alpha < 1)$ and

$$\overline{\lim_{n\to\infty}} \frac{n^{1+\alpha}}{\ln n} [H_n^*(f_1;\ 0) - f_1(0)] > 1, \tag{2.7}$$

$$\overline{\lim_{n\to\infty}}\, n^{\alpha+1} [H_n^*(f_2;\ 1) - f_2(1)] > 1. \tag{2.8}$$

(see e.g. [5]).

2.2. In his paper [2], I. E. GOPENGAUZ constructed linear polynomial operators $G_{n,r}(f;x)$ of degree $\leqq n$ for each fixed $r\geqq 0$, such that for $f^{(r)}\in C$

$$(2.9)\quad |f^{(i)}(x)-G_{n,r}^{(i)}(f;x)| = O(1)\left(\frac{\sqrt{1-x^2}}{n}\right)^{r-i}\omega\left(f^{(r)};\frac{\sqrt{1-x^2}}{n}\right)\quad (i=0,1,\dots,r)$$

if $x\in[-1,1]$ and $n\geqq 4r+5$. (Here, as usual, $\omega(g;t)$ stands for the modulus of continuity of $g\in C$.) Of course the estimation (2.9) is better than (2.7) (or (2.8)) but we cannot state the interpolatory properties (2.6) for the operator $G_{n,r}$.

3. New results

3.1. Now we strive for construction of linear polynomial operators having the features (2.6) and (2.9).

First let $s=s(n)\leqq n$ and $n=O(s)$. Denote

$$(3.1)\qquad \min_{1\leqq i\leqq s}|\vartheta_{k,n}-\vartheta_{i,s}| = |\vartheta_{k,n}-\vartheta_{j_k,s}|\quad (k=1,2,\dots,n).$$

(Whenever there exist two such ϑ_j, we can choose any.) It may occur that $\vartheta_{j_k,s}=\vartheta_{j_{k+1},s}=\dots=\vartheta_{j_{k+l},s}$, but l is uniformly bounded if $s\to\infty$. Supposing $f^{(r)}\in C$ $(r\geqq 1)$, define for $n\geqq n_0$

$$(3.2)\qquad F_{k,n}(x) = \frac{l_{j_k,s}^{r+3}(x)\sin^{2r+2}\vartheta}{l_{j_k,s}^{r+3}(x_{k,n})\sin^{2r+2}\vartheta_{k,n}}\times$$

$$\times\left\{h_{k,n}(x)+\left[(2r+2)\frac{\cos\vartheta_{k,n}}{\sin^2\vartheta_{k,n}}-(r+3)\frac{l'_{j_k,s}(x_{k,n})}{l_{j_k,s}(x_{k,n})}\right]\mathfrak{H}_{k,n}(x)\right\}$$

and

$$(3.3)\qquad D_{k,n}(x) = \frac{l_{j_k,s}^{r+3}(x)\sin^{2r+2}\vartheta}{l_{j_k,s}^{r+3}(x_{k,n})\sin^{2r+2}\vartheta_{k,n}}\mathfrak{H}_{k,n}(x).$$

We consider the linear polynomial operators

$$(3.4)\qquad A_n(f;x) = G_{n,r}(f;x)+\sum_{k=1}^{n}[f(x_{k,n})-G_{n,r}(f;x_{k,n})]F_{k,n}(x)+$$

$$+\sum_{k=1}^{n}[f'(x_{k,n})-G'_{n,r}(f;x_{k,n})]D_{k,n}(x).$$

We shall prove

THEOREM 3.1. *For every fixed* $c>0$ *and* $r\geqq 1$ *we can define the linear polynomial operators* $A_n(f;x)$ *such that*

(a) $\deg A_n(f;x)\leqq 2n(1+c)\quad (n\geqq n_0)$,

(b) $A_n(f;x_{k,n})=f(x_{k,n}),\quad A'_n(f;x_{k,n})=f'(x_{k,n})\quad (k=1,2,\dots,n;\quad n\geqq n_0)$,

(c) $|A_n^{(i)}(f;x)-f^{(i)}(x)| = O(1)\left(\frac{\sqrt{1-x^2}}{n}\right)^{r-1}\omega\left(f^{(r)};\frac{\sqrt{1-x^2}}{n}\right)$

$(i=0,1,\dots,r;\ n\geqq n_0)$

supposing $f^{(r)}\in C$.

3.2. PROOF. **3.21.** Sometimes omitting the superfluous notations we obtain by (3.2) and (3.3) that

$$\deg A_n \leqq (r+3)(s-1)+2r+2+2n-1 \leqq 2n(1+c)$$

if $s=\left[\frac{2nc-r+2}{r+3}\right]$. This proves (a). To go further, we can easily verify the formulae

$$(3.5)\qquad \begin{cases} F_k(x_{j,n}) = \delta_{k,j};\ F_k'(x_{j,n}) = 0 \\ D_k(x_{j,n}) = 0;\ D_k'(x_{j,n}) = \delta_{k,j}. \end{cases} \quad (k = 1, 2, \dots, n;\ j = 0, 1, \dots, n+1).$$

Here and later $x_{0,n}=\cos 0=1$ and $x_{n+1,n}=\cos \pi=-1$. Using (3.5) we get (b).

3.22. If

$$(3.6)\qquad \min_{1\leqq i\leqq n} |\vartheta-\vartheta_{i,n}| = |\vartheta-\vartheta_{t(\vartheta,n),n}| \quad (n = 1, 2, \dots)$$

we state

LEMMA 3.1. *Supposing* $n\geqq n_0$, *we have*

$$(3.7)\qquad l_{j_k,s}(x_{k,n}) \geqq \alpha > 0,$$

$$(3.8)\qquad |l_{j_k,s}(x)| = O\left(\frac{1}{|t-k|}\right) \quad \text{if} \quad |\vartheta-\vartheta_{k,n}| \geqq \frac{c_1}{n}.$$

3.22.1. Indeed, $|\vartheta_{k,n}-\vartheta_{j_k,s}|\leqq\pi/2s$ from where (3.7) can be obtained. To prove (3.8) we write

$$|l_{j_k,s}(x)| = \left| \frac{T_s(x)}{s\cdot\sin\frac{\vartheta-\vartheta_j}{2}} \cdot \frac{\sin\vartheta_j}{\sin\frac{\vartheta+\vartheta_j}{2}} \right| = \frac{O(1)}{s\cdot\sin\frac{|\vartheta-\vartheta_k+\vartheta_k-\vartheta_j|}{2}} = O\left(\frac{1}{|t-k|}\right).$$

Here we used

$$(3.9)\qquad \sin\vartheta_1 \leqq \sin\vartheta_1+\sin\vartheta_2 \leqq 2\sin\frac{\vartheta_1+\vartheta_2}{2} \quad (0 \leqq \vartheta_1,\ \vartheta_2 \leqq \pi).$$

3.23. Now we prove

LEMMA 3.2. *We have*

$$(3.10)\ |A_n(f;x)-f(x)|= O(1)\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)^r \omega\left(f^{(r)}; \frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right) \quad (n \geqq n_0).$$

3.23.1. We may suppose $\vartheta\neq\vartheta_k$ $(k=0, 1, \dots, n+1)$. For the remaining ϑ's we shall estimate the parts of (3.4). First let $\sin\vartheta\geqq c\cdot n^{-1}$, when $(\sin\vartheta+n^{-1})n^{-1}= =O(n^{-1}\sin\vartheta)$. We write

$$(3.11)\ \left|\sum_{k=1}^{n}[f'(x_k)-G_n'(x_k)]D_k(x)\right| = \sum_{|\vartheta-\vartheta_k|>c_1n^{-1}} + \sum_{|\vartheta-\vartheta_k|\leqq c_1n^{-1}} = \sum{}^{(1)}+\sum{}^{(2)}.$$

Here, by (2.9), (2.5), (3.7), (3.8) we obtain

$$\sum\nolimits^{(1)} = O(1)\sum\nolimits^{(1)}\left(\frac{\sin\vartheta_k}{n}\right)^{r-1}\omega\left(\frac{\sin\vartheta_k}{n}\right)\frac{\sin\vartheta_k}{|t-k|^{r+3}}\times \tag{3.12}$$

$$\times\frac{\sin\vartheta_k}{n^2\sin\frac{\vartheta+\vartheta_k}{2}\sin\frac{|\vartheta-\vartheta_k|}{2}}\left(\frac{\sin\vartheta}{\sin\vartheta_k}\right)^{2r+2}=$$

$$= O(1)\left(\frac{\sin\vartheta}{n}\right)^{r}\omega\left(\frac{\sin\vartheta}{n}\right)\sum\nolimits^{(1)}\left(\frac{\sin\vartheta_k}{\sin\vartheta}+1\right)\left(\frac{\sin\vartheta}{\sin\vartheta_k}\right)^{r+2}\frac{1}{|t-k|^{r+4}}.$$

Let e.g. $0<\vartheta\leqq\pi/2$. Then

$$\max_{l=0,1}\sum\nolimits^{(1)}\left(\frac{\sin\vartheta}{\sin\vartheta_k}\right)^{r+2-l}\frac{1}{|t-k|^{r+4}} = O(1)\max_{l=0,1}\sum_{\substack{k=1\\k\neq t}}^{n}\left(\frac{t}{k}\right)^{r+2-l}\frac{1}{|t-k|^{r+4}}=$$

$$= O(1)\max_{l=0,1}\Big[\sum_{k<\frac{t}{2}}+\sum_{k>2t}+\sum_{\substack{\frac{t}{2}\leqq k\leqq 2t\\k\neq t}}\Big] = O(1).$$

Similar estimation holds for $\pi/2<\vartheta<\pi$. If $|\vartheta-\vartheta_k|\leqq c_1\cdot n^{-1}$ we obtain by $|l_{k,n}(x)|=O(1)$ and $\vartheta\sim\vartheta_k$ that

$$\sum\nolimits^{(2)} = \sum\nolimits^{(2)}\left(\frac{\sin\vartheta_k}{n}\right)^{r-1}\omega\left(\frac{\sin\vartheta_k}{n}\right)\left(\frac{\sin\vartheta}{\sin\vartheta_k}\right)^{2r+2}\frac{\sin^2\vartheta_k T_n^2(x)}{n^2|x-x_k|}=$$

$$= O(1)\left(\frac{\sin\vartheta}{n}\right)^{r}\omega\left(\frac{\sin\vartheta}{n}\right).$$

Let $\sin\vartheta<\frac{c}{n}$. Then from the first part of (3.12) we get as above

$$\sum\nolimits^{(1)} = O(1)\left(\frac{\sin\vartheta}{n}\right)^{r}\sum\nolimits^{(1)}\omega\left(\frac{\sin\vartheta_k}{n}\right)\left(\frac{\sin\vartheta}{\sin\vartheta_k}\right)^{r+2}\frac{1}{|t-k|^{r+4}},$$

from where by $\sin\vartheta_k\sim kn^{-1}$

$$\sum\nolimits^{(1)} = O(1)\left(\frac{1}{n}\right)^{2r}\omega\left(\frac{1}{n^2}\right)\sum\frac{n\sin\vartheta_k+1}{|t-k|^{r+4}}\left(\frac{\sin\vartheta}{\sin\vartheta_k}\right)^{r+2}= \tag{3.13}$$

$$= O(1)\left(\frac{1}{n^2}\right)^{r}\omega\left(\frac{1}{n^2}\right).$$

If $|\vartheta-\vartheta_k|\leqq c_1 n^{-1}$ then by $\sin\vartheta=O(\sin\vartheta_k)$ we get

$$\sum\nolimits^{(2)} = O(1)\left(\frac{1}{n^2}\right)^{r}\omega\left(\frac{1}{n^2}\right).$$

I.e., we have

$$\left|\sum_{k=1}^{n}[f'(x_k)-G_n'(x_k)]D_k(x)\right| = O(1)\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)^r \omega\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right).$$

The estimation of the term

$$\sum_{k=1}^{n}[f(x_k)-G_n(x_k)]\frac{l_j^{r+3}(x)}{l_j^{r+3}(x_k)}\frac{\sin^{2r+2}\vartheta}{\sin^{2r+2}\vartheta_k}h_k(x)$$

can be reduced to (3.11) if we use

$$\frac{\sqrt{1-x_k^2}}{n}v_k(x) = \frac{1-xx_k}{n\sqrt{1-x_k^2}} \leqq \frac{1-\cos\vartheta\cos\vartheta_k+\sin\vartheta\sin\vartheta_k}{n\sin\vartheta_k} =$$

$$= \frac{2}{n\cdot\sin\vartheta_k}\sin^2\frac{\vartheta+\vartheta_k}{2} = O(1)\sin\frac{\vartheta+\vartheta_k}{2}\sin\frac{|\vartheta-\vartheta_k|}{2} = O(1)|x-x_k| \quad (k\neq t).$$

Finally, if we apply

$$(2r+2)\frac{\cos\vartheta_k}{\sin^2\vartheta_k}-(r+3)\frac{l_j'(x_k)}{l_j(x_k)} = O(1)\left(\frac{1}{\sin^2\vartheta_k}+\frac{1}{x_k-x_{k+1}}\right) =$$

$$= O(1)\left(\frac{1}{\sin^2\vartheta_k}+\frac{n}{\sin\vartheta_k}\right) = O\left(\frac{n}{\sin\vartheta_k}\right) \quad (k\neq t)$$

for the remaining part, then this can be estimated as (3.11). So we proved (3.10).

3.24. Now we state the following

LEMMA 3.3. *Let* $r\geqq 0$. *If* $f^{(r)}\in C$ *and for the polynomials* $p_n(f;x)$ *of degree* $\leqq n$

(3.14) $$|f(x)-p_n(f;x)| = O(1)\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)^r \omega\left(f^{(r)};\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right) \quad on \quad [-1,1]$$

and

(3.15) $$p_n^{(k)}(f;\pm 1) = f^{(k)}(\pm 1) \quad (k=0,1,\dots,r),$$

then for $|x|\leqq 1$ *we have*

(3.16) $$|f^{(k)}(x)-p_n^{(k)}(f;x)| = O(1)\left(\frac{\sqrt{1-x^2}}{n}\right)^{r-k}\omega\left(f^{(r)};\frac{\sqrt{1-x^2}}{n}\right) \quad (k=0,1,\dots,r).$$

3.24.1. Indeed, by (3.14) we can state using [3, Theorem 1]

(3.17)

$$|f^{(k)}(x)-p_n^{(k)}(f;x)| = O(1)\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)^{r-k}\omega\left(f^{(r)};\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right) \quad (k=0,1,\dots,r)$$

in $[-1, 1]$, moreover, again by (3.14) we have, applying [4; Lemma 1, Remark 2]

$$|p_n^{(r+1)}(f; x)| = O(1) \frac{\omega\left(f^{(r)}; \frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)}{\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}} \quad (|x| \leqq 1). \tag{3.18}$$

Now using (3.15), (3.17) and (3.18) we can prove (3.16) as in [2].

3.25. To finish our proof we remark that by (2.9) and (3.2)—(3.4)

$$A_n^{(k)}(f; \pm 1) = f^{(k)}(\pm 1) \quad (k = 0, 1, \dots, r), \tag{3.19}$$

i.e., by (3.10), (3.19) and Lemma 3.3 we get (c).

3.3. Let $f^{(r)} \in C$ for $r \geqq 0$. Using the above notations, form the linear polynomial operators

$$B_n(f; x) = G_{n,r}(f; x) + \sum_{k=1}^{n} [f(x_{k,n}) - G_{n,r}(f; x_{k,n})] F_{k,n}(x) \tag{3.20}$$

for $n \geqq n_0$. By the above used arguments we can prove

THEOREM 3.2. *For every fixed $c > 0$ and $r \geqq 0$ one can define the linear polynomial operators $B_n(f; x)$ such that*

(α) $$\deg B_n(f; x) \leqq 2n(1+c) \quad (n \geqq n_0),$$

(β) $B_n(f; x_{k,n}) = f(x_{k,n})$; $B_n'(f; x_{k,n}) = G_{n,r}'(f; x_{k,n})$ $(k = 1, 2, \dots, n;\ n \geqq n_0)$,

(γ) $$|B_n^{(i)}(f; x) - f^{(i)}(x)| = O(1) \left(\frac{\sqrt{1-x^2}}{n}\right)^{r-i} \omega\left(f^{(r)}; \frac{\sqrt{1-x^2}}{n}\right) \quad (i = 0, 1, \dots, r;\ n \geqq n_0)$$

supposing $f^{(r)} \in C$.

3.4. Let

$$V_n(f; x) = \frac{1}{n} \sum_{k=n+1}^{2n} s_k(f; x) \tag{3.21}$$

where $s_k(f; \cos \vartheta)$ is the k-th partial sum of the Fourier series of $f(\cos \vartheta)$. Using de la Vallée Poussin's results we have for the polynomial V_n of degree $\leqq 2n$

$$\|f(x) - V_n(f; x)\| = O(1) E_n(f) \tag{3.22}$$

where $E_n(f)$ is the best approximation on $[-1, 1]$ of f by polynomials of degree $\leqq n$ and $\|g(x)\| = \max_{-1 \leqq x \leqq 1} |g(x)|$. If by

$$\overline{F_{k,n}}(x) = \frac{l_{j_k,s}(x)}{l_{j_k,s}(x_{k,n})} \left[h_{k,n}(x) - \frac{l_{j_k,s}'(x_{k,n})}{l_{j_k,s}(x_{k,n})} \mathfrak{H}_{k,n}(x)\right], \quad \overline{D_{n,k}}(x) = \frac{l_{j_k,s}(x)}{l_{j_k,s}(x_{k,n})} \mathfrak{H}_{k,n}(x)$$

we define for $f' \in C$

$$C_n(f;x) = V_n(f;x) + \sum_{k=1}^{n} [f(x_k) - V_n(f;x_k)]\,\bar{F}_k(x) + \tag{3.23}$$
$$+ \sum_{k=1}^{n} [f'(x_k) - V_n'(f;x_k)]\,\bar{D}_k(x)$$

then we can prove as above

THEOREM 3.3. *For every fixed* $c>0$ *we can define the linear polynomial operators* $C_n(f;x)$ *such that*

(i) $$\deg C_n(f;x) \leqq 2n(1+c) \quad (n \geqq n_0),$$

(ii) $$C_n(f;x_{k,n}) = f(x_{k,n}); \quad C_n'(f;x_{k,n}) = f'(x_{k,n}) \quad (k = 1, 2, \dots, n;\ n \geqq n_0),$$

(iii) $$\|C_n(f;x) - f(x)\| = O(1)\frac{E_n(f')}{n} \quad (n \geqq n_0),$$

supposing $f' \in C$.

If $f \in C$ then using

$$J_n(f;x) = V_n(f;x) + \sum_{k=1}^{n} [f(x_k) - V_n(x_k)]\,\bar{F}_k(x)$$

we can state

THEOREM 3.4. *For every fixed* $c>0$ *we can define the linear polynomial operators* $J_n(f;x)$ $(n \geqq n_0)$ *for which*

(1) $$\deg J_n(f;x) \leqq 2n(1+c),$$

(2) $$J_n(f;x_{k,n}) = f(x_{k,n}); \quad J_n'(f;x_{k,n}) = V_n'(f;x_{k,n}) \quad (k = 1, 2, \dots, n),$$

(3) $$\|J_n(f;x) - f(x)\| = O(1)\,E_n(f),$$

supposing $f \in C$.

3.5. Finally we show another application of Lemma 3.3. In his paper [6], R. B. SAXENA constructed a polynomial $\Lambda_n(f;x)$ of degree $4n+2$ such that for $f \in C$

$$\Lambda_n\left(f; \cos\frac{k\pi}{n+1}\right) = f\left(\cos\frac{k\pi}{n+1}\right) \quad (k = 0, 1, \dots, n+1), \tag{3.24}$$

$$|f(x) - \Lambda_n(f;x)| = O(1)\,\omega\left(f; \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right). \tag{3.25}$$

Later T. V. RODINA [7] and R. B. SAXENA [8] proved that the estimation (3.25) can be replaced by

$$|f(x) - \Lambda_n(f;x)| = O(1)\,\omega\left(f; \frac{\sqrt{1-x^2}}{n}\right). \tag{3.26}$$

Now using (3.24) and (3.25), we immediately obtain (3.26) by Lemma 3.3.

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AUTOMORPHISM GROUP AND CATEGORY OF COSPECTRAL GRAPHS

By

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0. Introduction. The spectrum of the adjacency matrix of a graph is known to reflect several combinatorial properties of the graph (see e.g. [3, 4, 7, 9, 11, 17, 18, 19, 21, 23, 24, 28]). There are, however, large families of pairwise non-isomorphic graphs known which have the same spectrum *(cospectral graphs)* [see e.g. 10, 29, 1]. Such families of graphs are of special interest because they tell us what *cannot* be decided upon spectral information alone. Our present aim is to exhibit new families of such graphs.

There is a close relationship between the automorphism group Aut X of a graph X and its spectrum. (Let us refer in this respect to the excellent monograph [3] by N. L. BIGGS.)

Some aspects of this relationship are briefly treated in Section 1. Our main purpose is, however, to emphasize the opposite: in certain cases, *it is very little what the spectrum may tell us about the automorphism group*. Arbitrary groups can be prescribed to be isomorphic to the automorphism groups of graphs belonging to a cospectral family. This statement can be even generalized to *endomorphism monoids* (Corollary 5.7 at the end of the paper). (For the definitions see Section 2.) The next step is to generalize this result to categories of graphs. *The finite categories $\mathcal{K}$ which are isomorphic to a category consisting of a family of cospectral graphs, are characterized* in Corollary 5.6. The obvious necessary condition that $\mathcal{K}$ be isomorphic to some category of finite sets of equal size, and of some mappings between them, turns out to be sufficient.

The main result of the paper (Theorem 5.2) asserts that any category $\mathcal{A}$ of finite algebraic systems of a given finite type, is isomorphic to a full subcategory of the category of finite graphs such that the graphs corresponding to algebraic systems having common underlying set are cospectral.

All the results mentioned are corollaries of this theorem. Let us point out that the prerequisites needed to prove the main result are essentially deeper than those required for the proof of Corollary 5.7 (cospectral graphs with given endomorphism monoids). Our proof is based on investigations of concrete categories (A. PULTR [26]), mainly through a result in [2] (Theorem 2.2).

1. Relationship of spectrum and automorphism group. As well known to any quantum chemist, symmetries impose strong restrictions on the spectrum of a graph (or molecule, see e.g. [31]). The direct relationship most often used is *qualitative:* The automorphism group of a graph may cause coincidences (called degeneracies) among the eigenvalues. This fact is expressed by the following proposition:

PROPOSITION 1.1. *Let the multiplicities of the eigenvalues of the graph X be* $k_1, \ldots, k_s$ $(k_1+\ldots+k_s=n=|V(X)|)$. *Then* Aut X *is isomorphic to a subgroup of*

$$O(k_1)\oplus\ldots\oplus O(k_s)$$

where $O(t)$ *denotes the t-dimensional (real) orthogonal group.*

Our formulation of Proposition 1.1 is indirect from the chemist's point of view: generally, he might well know the automorphism group of his molecule, and wants to deduce spectral information from this. We are interested in implications to the other way.

PROOF. The permutation matrices, corresponding to Aut X, commute with the adjacency matrix **A** of X. Hence the corresponding orthogonal transformations of the n-dimensional real Euclidean space keep the eigensubspaces of **A** invariant.

COROLLARY 1.2 (MOWSHOWITZ [22], PETERSDORF—SACHS [24]). *If* $k_1=\ldots$ $\ldots=k_n=1$ *then* Aut $X\leqq Z_2^n$.

PROOF. $O(1)\cong Z_2$, the cyclic group of order 2.

COROLLARY 1.3. *If* $\max k_i\leqq 2$, *then* Aut X *is isomorphic to a subgroup of a direct sum of dihedral groups. If* $\max k_i\leqq 3$, *then* Aut X *is isomorphic to a subgroup of a direct sum of dihedral groups and of copies of* S_4 *and* A_5.

PROOF. The finite subgroups of $O(2)$ are cyclic and dihedral groups. Those of $O(3)$ are the subgroups of the symmetry groups of the regular prisms $(Z_2\oplus D_t)$ and of the Platonic solids (tetrahedron: S_4; octahedron: $Z_2\oplus S_4$; icosahedron $Z_2\oplus A_5$, cf. [8, Ch. 15.5]).

Some care has to be taken in the case of digraphs. A relevant reference here is [33], though we do not use it below.

Let $\mathbf{J}_m(\lambda)$ denote the $m\times m$ *Jordan matrix* with λ's in the diagonal and 1's immediately below those in the diagonal. The canonical form of a matrix is a direct (block-diagonal) sum of such matrices.

PROPOSITION 1.4. *Let the canonical form of the adjacency matrix* **A** *of the digraph* X *be a direct sum of Jordan matrices,* $\mathbf{J}_{m_j}(\lambda_j)$ *taken* k_j *times* $(j=1, \ldots, s)$. (*Thus* $\sum_{j=1}^{s} k_j m_j=n=|V(X)|$. *The pairs* (m_j, λ_j) *are different for different* j*'s, but there may be coincidences among the* λ_j*'s*). *Then* Aut X *is isomorphic to a subgroup of*

$$U(k_1)\oplus\ldots\oplus U(k_s)$$

where $U(t)$ *denotes the t-dimensional unitary group.*

PROOF. We prove the statement for any complex matrix **A** and any finite group **G** of matrices commuting with **A**. The matrices are identified with linear transformations acting on C^n, the n-dimensional complex space.

We shall apply Maschke's theorem stating that, given $W\leqq V\leqq C^n$ subspaces, invariant under **G**, there is a **G**-invariant subspace W_1 such that $V=W\oplus W_1$ [30, Ch. 1.3].

Another observation we shall often refer to: the subspaces $\operatorname{Ker}(\mathbf{A}-\lambda\mathbf{I})^t$ and $\operatorname{Im}(\mathbf{A}-\lambda\mathbf{I})^t$ are invariant under $\mathbf{G}$ for any λ and t.

First we note, that, if $\mu_1, \ldots, \mu_l$ are the *distinct* eigenvalues of $\mathbf{A}$ (thus $l \leqq s$), then

$$C^n = \sum_{i=1}^{l} \oplus \operatorname{Ker}(\mathbf{A}-\mu_i\mathbf{I})^n.$$

The summands here are invariant under both $\mathbf{A}$ and $\mathbf{G}$, whence it suffices to prove the statement for $\lambda_1 = \ldots = \lambda_s$. Let $\mathbf{B} = \mathbf{A} - \lambda_1\mathbf{I}$. Now we have $\mathbf{B}^n = 0$. Let $m_1 > > m_2 > \ldots > m_s$.

Clearly, $\dim \operatorname{Ker} \mathbf{B} = k_1 + \ldots + k_s$. Let $V_j = \operatorname{Ker} \mathbf{B} \cap \operatorname{Im} \mathbf{B}^{m_j}$ $(j=1, \ldots, s)$ and $V_{s+1} = \operatorname{Ker} \mathbf{B}$. Clearly, $\operatorname{Ker} \mathbf{B} = V_{s+1} > V_s > V_{s-1} > \ldots > V_1 = \{0\}$, and $\dim V_j = = k_1 + \ldots + k_{j-1}$ $(j=1, \ldots, s+1)$. Let $V_{j+1} = V_j \oplus W_j$ for some $\mathbf{G}$-invariant subspace W_j $(j=1, \ldots, s)$. Hence, we have a decomposition of $\operatorname{Ker} \mathbf{B}$ to $\mathbf{G}$-invariant subspaces $\operatorname{Ker} \mathbf{B} = W_1 \oplus \ldots \oplus W_s$ where $\dim W_j = k_j$. By restriction to $\operatorname{Ker} \mathbf{B}$, we obtain a homomorphism

$$\varphi\colon \mathbf{G} \to GL(W_1) \oplus \ldots \oplus GL(W_s).$$

As any finite subgroup of $GL(t, C)$ is similar to some subgroup of $U(t)$, [30, Ch. 1.3], the proof will be complete if we show that φ is an injection.

Assume that some matrix $\mathbf{D} \in \mathbf{G}$ belongs to $\operatorname{Ker} \varphi$. Hence, the restriction of $\mathbf{D}$ to $\operatorname{Ker} \mathbf{B}$ is the identity. We show by induction on t that the restriction of $\mathbf{D}$ to $\operatorname{Ker} \mathbf{B}^t$ is also the identity. For if $x \in \operatorname{Ker} \mathbf{B}^t$ then $\mathbf{B}x \in \operatorname{Ker} \mathbf{B}^{t-1}$, hence, by the induction hypothesis, $\mathbf{DB}x = \mathbf{B}x$. Since $\mathbf{B}$ and $\mathbf{D}$ commute, we infer $\mathbf{B}(\mathbf{D}x - x) = 0$, thus $y = \mathbf{D}x - x \in \operatorname{Ker} \mathbf{B}$. So, $\mathbf{D}^r y = y$ by assumption for any r, hence

$$\mathbf{D}^r x = \mathbf{D}^{r-1}y + \mathbf{D}^{r-2}y + \ldots + y + x = ry + x.$$

As $\mathbf{D}^r$ is the identity for some $r \geqq 1$, we conclude that $y=0$, $\mathbf{D}x = x$, as stated.

We have arrived at the conclusion that $\mathbf{D}$ acts as the identity on $\operatorname{Ker} \mathbf{B}^n = C^n$, thus $|\operatorname{Ker} \varphi| = 1$. The proof of Proposition 1.4 is complete.

A matrix is *non-derogatory*, if its characteristic and minimum polynomials coincide. This is equivalent to the assumption that $\lambda_1, \ldots, \lambda_s$ of Proposition 1.4 are pairwise different.

C-Y. Chao [5] proved that the automorphism group of a digraph is necessarily abelian, provided all eigenvalues of X are different (thus $k_1 = \ldots = k_n = m_1 = \ldots \ldots = m_n = 1$). This observation was generalized by A. Mowshowitz [23] to non-derogatory adjacency matrices. We have a more general sufficient condition for $\operatorname{Aut} X$ to be abelian:

Corollary 1.5. *If any Jordan matrix occurs at most once in the canonical form of the adjacency matrix of a digraph X then* $\operatorname{Aut} X$ *is abelian.*

Proof. In terms of Proposition 1.4 our condition is equivalently formulated as $k_1 = \ldots = k_n = 1$. $U(1)$ being abelian, our statement follows from Proposition 1.4.

In order to obtain qualitative information about the eigenvalues generally, ad hoc ideas are needed. In the case of vertex-transitive automorphism groups, an explicit formula can be obtained, in terms of irreducible group characters [20, 1]. Such a formula was used by the author to obtain large families of pairwise non-isomorphic cospectral Cayley graphs of the dihedral group D_p, p large prime [1].

2. Categories. We assume that the reader is familiar with the first ten pages of some textbook on categories.

Let us recall some definitions.

Let X and Y be undirected graphs, and $\varphi: V(X) \to V(Y)$ a mapping. φ is a *homomorphism*, if $\{\varphi x_1, \varphi x_2\} \in E(Y)$ whenever $\{x_1, x_2\} \in E(X)$. The category Gra has the graphs as objects, and their homomorphisms as morphisms. For any category $\mathcal{K}$, Hom (X, Y) consists of the morphisms from X to Y ($X, Y \in \mathrm{Ob}\, \mathcal{K}$). A *monoid* is a semigroup with unity. End $X = \mathrm{Hom}\,(X, X)$ is a monoid under composition. The automorphism group Aut X consists of the invertible elements of End X.

By a 2-colouring of the graph X we mean any function $f: V(X) \to \{1, 2\}$. (This is, in general, not a good colouring in the usual sense.) The category Gra (Col(2)) has the pairs (X, f) for its objects, where X is a graph and f is a 2-colouring of X. Morphisms in Gra (Col(2)) are defined as to preserve the colours:

φ belongs to Hom $((X, f), (Y, g))$ if and only if $\varphi \in \mathrm{Hom}\,(X, Y)$ and $f(x) = g(\varphi(x))$ for any $x \in V(X)$.

A pair $(\mathcal{K}, \square)$ is a *concrete category*, if $\mathcal{K}$ is a category and $\square: \mathcal{K} \to \mathrm{Sets}$ is a faithful functor (associating underlying sets and mappings with objects and morphisms, resp.). All categories $\mathcal{K}$ appearing in this paper can be endowed with a *forgetful functor* $\square$ in a natural way such that $(\mathcal{K}, \square)$ be a concrete category.

A *full algebraic category* has algebraic systems of a given type for its objects (thus sets endowed with a given number of relations and operations of fixed aryties), and their homomorphisms for morphisms. Both Gra and Gra (Col (2)) are categories of this kind. (A graph is a (symmetric, irreflexive) binary relation on a set. A 2-colouring is a special kind of 2 unary relations.)

A *full embedding* of a category $\mathcal{K}$ into a category $\mathcal{L}$ is a functor $\Phi: \mathcal{K} \to \mathcal{L}$ such that $\Phi(X) = \Phi(Y)$ implies $X = Y$ for any objects X, Y of $\mathcal{K}$ and $\Phi: \mathrm{Hom}_{\mathcal{K}}(X, Y) \to \mathrm{Hom}_{\mathcal{L}}(\Phi(X), \Phi(Y))$ is a bijection $(X, Y \in \mathrm{Ob}(\mathcal{K}))$. In such a case, clearly, End $X \cong \mathrm{End}\,(\Phi(X))$.

The fundamental result of the theory of full embeddings developed by A. Pultr and Z. Hedrlin is the following:

Theorem 2.1 (Hedrlin—Pultr [15, 16]). *Every full algebraic category has a full embedding into* Gra.

Let henceforth G: Gra (Col (2)) $\to$ Gra denote the functor which forgets the colouring.

Improvements on Theorem 2.1, obtained by A. Pultr [26, 27] were used by the author to derive the following result.

Theorem 2.2 [2, Theorem 4.5]. *Every full algebraic category $\mathcal{A}$ has a full embedding Φ into* Gra (Col (2)) *such that, for any objects A_1, A_2 of $\mathcal{A}$, $\square A_1 = \square A_2$ implies $G\Phi(A_1) = G\Phi(A_2)$. Moreover, the graphs $G\Phi(A)$ do not contain any triangles* ($A \in \mathrm{Ob}\, \mathcal{A}$).

In other words, the graph structure of $\Phi(A)$ depends on the underlying set $\square A$ only. Only the colouring of $\Phi(A)$ is influenced by the algebraic structure of $A \in \mathrm{Ob}\, \mathcal{A}$.

Remark 2.3. We shall make use of the finite version of Theorem 2.2. For $\mathcal{A}$ a full algebraic category of finite type (finitely many finitary operations and relations), let $\mathscr{A}_0$ denote the category of finite objects of $\mathscr{A}$ (as a full subcategory). Similarly, let Gra_0 and $\mathrm{Gra}_0(\mathrm{Col}\,(2))$ denote the corresponding categories, consisting of finite objects. *Theorem* 2.2 *remains valid with* $\mathscr{A}$ *and* Gra(Col (2)) *replaced by* $\mathscr{A}_0$ *and* $\mathrm{Gra}_0\,(\mathrm{Col}\,(2))$, *resp.* ([2, Remark 4.6]).

A category $\mathscr{K}$ is *discrete*, if Mor $\mathscr{K}$ consists of the identity morphisms only. The objects of $\mathscr{K}$ are then called *mutually rigid*. X is a *rigid* object, if End $X=\{id_X\}$.

An important tool for constructing graphs with prescribed endomorphism monoids are the *rigid graphs*. Rigid graphs have been constructed in several papers [32, 16, 6, etc.] (cf. Lemma 4.1). Using the above terminology, a family $G_1, \dots, G_n$ of graphs is called *mutually rigid*, if the G_i's are rigid, and for $i\neq j$, there is no homomorphism $G_i\to G_j$.

We shall make use of the following handy lemma, due to four students of Z. Hedrlin and A. Pultr:

LEMMA 2.4 (CHVÁTAL—HELL—KUČERA—NEŠETŘIL [6]). *Given any positive integer n there is a family of mutually rigid finite connected graphs* $G_1, \dots, G_n$ *such that*

(i) *each vertex of* G_i *is contained in some* K_{n+2} *subgraph;*

(ii) *if* $f\colon G_i\to G$ *is a homomorphism, then the image of* G_i *has no cutpoints.* $(i=1, \dots, n)$.

(K_m *denotes the complete graph having m vertices.*)

3. The characteristic polynomial of a graph. The basic idea of this section is due to A. J. SCHWENK [29].

Let $f(G;\lambda)=f(G)=\sum_{s=0}^{n} a_s\lambda^{n-s}$ denote the characteristic polynomial of G, and let $c(G)$, $k(G)$, $v(G)$ be the number of cycles, components, and vertices of G, resp. In a *mutation graph*, each component is either a cycle or K_2 (cf. HARARY [12]). We shall make use of the following result of H. Sachs:

THEOREM 3.1 (SACHS [28]). *For* $s\geqq 1$, $a_s=\sum_{H\in G_s}(-1)^{k(H)}2^{c(H)}$ *where* G_s *denotes the set of s-point mutation subgraphs of* G.

By the natural convention $G_0=\{\varnothing\}$, Theorem 3.1 holds for $s=0$ as well. A simple transformation on Theorem 3.1 yields

COROLLARY 3.2. $f(G;\lambda)=\lambda^n\sum\varphi(H;\lambda)$ *where the summation extends over all mutation subgraphs H of G, and*

$$\varphi(H;\lambda)=(-1)^{k(H)}2^{c(H)}\lambda^{-v(H)}.$$

The advantage of φ is its multiplicativity: for any disjoint graphs H_1 and H_2 (having no vertex in common),

$$\varphi(H_1\cup H_2;\lambda)=\varphi(H_1;\lambda)\varphi(H_2;\lambda).$$

This and Corollary 3.2 in turn imply the following lemma (stated by Schwenk for the case of trees only).

The *coalescence* of two rooted graphs is formed by taking their disjoint union and then identifying the roots.

LEMMA 3.3 (SCHWENK [29]). *The characteristic polynomial of the coalescence X of two rooted graphs W and R is given by*

$$f(X)=f(W)f(R-r)+f(W-w)f(R)-\lambda f(W-w)f(R-r)$$

(w and r are the respective roots.)

4. A rigid graph having isomorphic one-point-deleted subgraphs. The corollary to the following lemma provides the link between the two theories.

LEMMA 4.1. *There is a finite connected rigid graph R and two vertices r_1, r_2 of R such that $R-r_1\cong R-r_2$.*

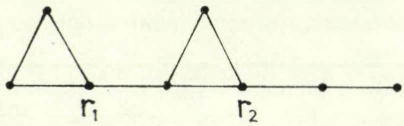

Fig. 1. The graph R_0

PROOF. The graph R_0 in Figure 1 was found by HARARY and PALMER [14], as an example where the vertex r_1 is not mapped to r_2 by any automorphism of R_0, but $R-r_1\cong R_0-r_2$. We see that the same holds for the digraph R_1 and the vertex-coloured graph R_2 in Figure 2.

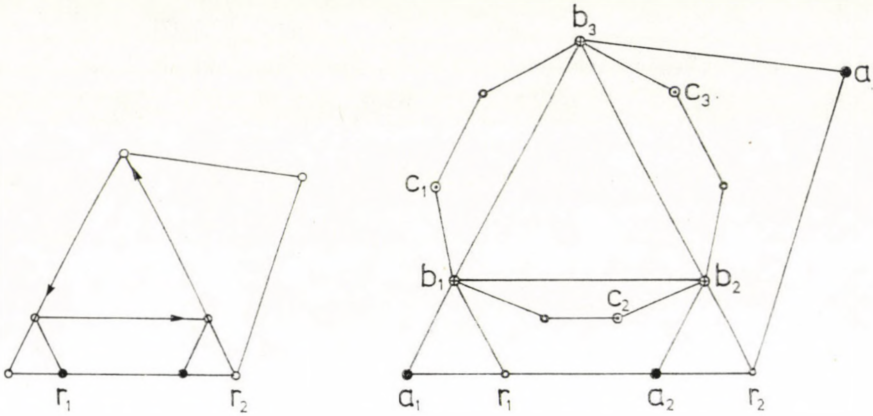

Fig. 2. The digraph R_1 and the colour-graph R_2
(● = red, ⊕ = blue, ⊙ = yellow; the other vertices have no colours)

(By homomorphisms we mean colour preserving ones. Colourless vertices may be mapped onto coloured ones.)

We observe that the colour-graph R_2 is *rigid*. For let ψ denote an endomorphism of R_2. The only triangle in R_2 having 3 blue vertices is $b_1b_2b_3$ hence this set is mapped onto itself. The position of the yellow vertices forces that $\psi b_i=b_{i+t}$ for some t (indices are added mod 3) (the blue triangle is rotated). Any triangle in R_2 must be mapped onto a triangle. b_1 and b_2 belong to triangles, but b_3 does not. Consequently, $t \equiv 0 \pmod 3$, the b_i's are fixed under ψ. Then clearly so are the c_i's and a_i's (as the unique neighbours of b_i having the corresponding colour). Now, the rest of the vertices cannot be moved either, thus ψ is the identity, as asserted.

Let now G_1, G_2, G_3 be the graphs from Lemma 2.4, setting $n=3$ there. Let g_j denote an arbitrary vertex of G_j. Take the disjoint union of the graphs R_2 and G_{ji} $(i, j=1, 2, 3)$ where G_{j1}, G_{j2}, G_{j3} are 3 copies of G_j, with the corresponding vertices g_{ji}. Define the graph R by identifying g_{1i}, g_{2i} and g_{3i} with a_i, b_i and c_i, resp. (Hence, R has $3(|V(G_1)|+|V(G_2)|+|V(G_3)|)+5$ vertices.) We consider the obtained connected graph R to be colourless. Clearly, $R-r_1 \cong R-r_2$.

We assert that R is rigid. For, let $\varphi \in \operatorname{End} R$. By 2.4 (i) and (ii), G_{ji} is mapped into some $G_{j_1 i_1}$. By the mutual rigidity of G_1, G_2, G_3, we have $j_1=j$ and this mapping is the unique isomorphism of G_{ji} onto G_{ji_1}. It follows that each coloured vertex of R_2 is mapped into a vertex of R_2 having the same colour. The colourless vertices of R_2 are also mapped into R_2 since only members of $V(R_2)$ have neighbours of 2 different colours. We conclude that the restriction $\varphi | V(R_2)$ is a colour-preserving endomorphism of R_2, thus the identity. By the above, this implies that φ itself is the identity.

COROLLARY 4.2. *Let W be an arbitrary graph, $w \in V(W)$, and R, r_1, r_2 as in Lemma 4.1. Let X_j denote the coalescence of W, rooted at w, with R, rooted at r_j $(j=1, 2)$. Then X_1 and X_2 are cospectral.*

This follows immediately from Lemma 3.3 and the fact that $R-r_1 \cong R-r_2$.

5. The main result. For $(\mathcal{K}, \square)$ a concrete category, let $\mathcal{K}_0$ denote the full subcategory of $\mathcal{K}$, consisting of the finite objects A of $\mathcal{K}$ (i.e. those having finite underlying sets $\square A$).

DEFINITION 5.1. A functor $\Phi: \mathcal{K}_0 \to \mathrm{Gra}_0$ will be called *cospectral*, if the spectrum of $\Phi(A)$ depends on the underlying set $\square A$ only. $(A \in \mathrm{Ob}\mathcal{K}_0.)$ In other words, $\square A_1 = \square A_2$ imply that $\Phi(A_1)$ and $\Phi(A_2)$ have the same spectrum.

THEOREM 5.2. *Let $\mathcal{A}$ be a full algebraic category of finite type. Then $\mathcal{A}_0$ has a cospectral full embedding into* Gra_0.

PROOF. I. Let Gra$'$ denote the category of triangle-free graphs. By Theorem 2.2 (and 2.3) it suffices to construct a full embedding $\Phi: \mathrm{Gra}'_0(\mathrm{Col}(2)) \to \mathrm{Gra}_0$ such that the spectrum of the graph $\Phi(X, f)$ depends on the graph X only, but not on its colouring f. $((X, f) \in \mathrm{Ob}(\mathrm{Gra}'_0(\mathrm{Col}(2)).)$

II. Let R, r_1 and r_2 be as in Lemma 4.1. Let the size of the largest complete subgraph in R be n and the diameter of R be d. $(n, d \geqq 2.)$

Let G_1 and G_2 be two members of the family of mutually rigid graphs satisfying 2.4 (i), (ii) (with this n), and g_i an arbitrary vertex of G_i $(i=1, 2)$.

Let now X be a triangle-free graph, and f a 2-colouring $V(X) \to \{1, 2\}$. Set $V=V(X)$. We construct the graph $Y=\Phi(X, f)$ as follows:

Let

$$V(Y) = V(G_1) \cup V(G_2) \cup V \times (\{0, 1, \ldots, 2d\} \cup V(R))$$

(these sets are assumed to be disjoint);

$$E(X) = E(G_1) \cup E(G_2) \cup (E(X) \times \{0\}) \cup (V \times E(R)) \cup \{\{g_1, (v, 0)\}: v \in V\} \cup$$
$$\cup \{\{g_2, (v, 2d)\}: v \in V\} \cup$$
$$\cup \{\{(v, i), (v, i+1)\}: v \in V, \quad 0 \leqq i \leqq 2d-1\} \cup \{\{(v, d), (v, r_j)\}: v \in V, f(v) = j\}.$$

(For $e = \{a, b\} \in E(R)$ we employ the symbol (v, e) to denote $\{(v, a), (v, b)\}$. This explains the meaning of $V \times E(R)$, and similarly, of $E(X) \times \{0\}$.) Informally, Y contains G_1, G_2, X, further $|V|$ disjoint copies of R and $|V|$ disjoint paths of length $2d+2$ connecting g_1 to g_2. The neighbours of g_1 on these paths induce the subgraph X. The halving points of these paths are connected to a copy of R each, by a single

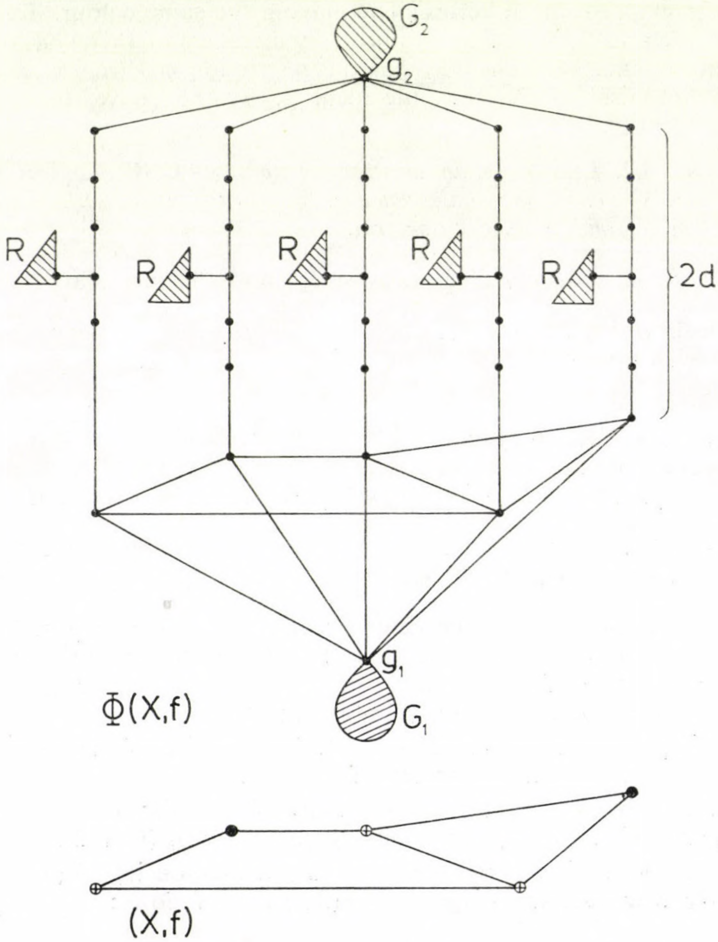

Fig. 3. The colour-graph (X, f) and the graph $\varphi(X, f)$

edge. The other end point of this edge is either r_1 or r_2 of this copy of R, according to the f-colour of the corresponding vertex $v \in V$.

To complete the definition of the functor $\Phi\colon \mathrm{Gra}_0'(\mathrm{Col}\,(2)) \to \mathrm{Gra}_0$, let $\varphi \in \mathrm{Hom}\,((X_1, f_1), (X_2, f_2))$. We extend φ in a natural way to a homomorphism $\Phi(\varphi)\colon \Phi(X_1, f_1) \to \Phi(X_2, f_2)$ by setting

$$\Phi(\varphi)(x) = x \quad \text{if} \quad x \in V(G_1) \cup V(G_2),$$

and

$$\Phi(\varphi)(v, a) = (\varphi v, a) \quad \text{if} \quad v \in V(X_1),\, a \in \{0, \ldots, 2d\} \cup V(R).$$

As $f_2(\varphi v) = f_1(v)$ $(v \in V(X_1))$, we see that $\Phi(\varphi)$ is a homomorphism, indeed.

III. The graphs $\Phi(X, f_1)$ and $\Phi(X, f_2)$ have the same spectrum. To see this, it suffices to consider the case when $f_1(v) = f_2(v)$ for each $v \in V$ expect for one vertex v_0; and $f_j(v_0) = j$ $(j = 1, 2)$.

Now the graphs $\Phi(X, f_1)$ and $\Phi(X, f_2)$ (which have the same vertex set) differ in one edge only: in $\Phi(X, f_j)$, the vertex (v_0, d) is adjacent to (v_0, r_j) and not to (v_0, r_{3-j}). Whence, an application of Corollary 4.2 proves the cospectrality.

IV. In order to prove that Φ is a full embedding, let $\eta \in \mathrm{Hom}\,(\Phi(X_1, f_1), \Phi(X_2, f_2))$. As any K_{n+2} is mapped onto some K_{n+2}, by our choice of G_1, G_2 we obtain that η maps the set $V(G_1) \cup V(G_2)$ into itself. Now, the mutual rigidity of G_1, G_2 implies that $\eta(x) = x$ for any $x \in V(G_1) \cup V(G_2)$. As η does not increase distances, it follows that there is a mapping $\varphi\colon V_1 \to V_2$ $(V_j = V(X_j))$ such that

$$\eta(v, i) = (\varphi v, i) \quad (v \in V_1,\ i = 0, \ldots, 2d).$$

It follows that $\varphi \in \mathrm{Hom}\,(X_1, X_2)$.

Now the copy of R, attached to (v, d) (let us denote it by R_v) is mapped into the neighbourhood of radius $d+1$ of $(\varphi v, d)$. This neighbourhood has, obviously, a homomorphism onto $R_{\varphi(v)}$. Hence, by the rigidity of R we obtain

$$\eta(v, a) = (\varphi v, a) \quad (v \in V,\ a \in V(R)).$$

Let $v \in V_1$, $f_1(v) = j$, $f_2(\varphi v) = m$. As (v, d) is adjacent to (v, r_j) in $\Phi(X_1, f_1)$ and $(\varphi v, d) = \eta(v, d)$ is not adjacent to $(\varphi v, r_{3-m})$ in $\Phi(X_2, f_2)$, we infer that $j \neq 3-m$ hence $j = m\colon f_1(v) = f_2(\varphi v)$. Hence $\varphi \in \mathrm{Hom}\,((X_1, f_1), (X_2, f_2))$, and $\eta = \Phi(\varphi)$. This proves that Φ is a full embedding, completing the proof of the theorem.

REMARK 5.3. The full embedding $\Phi\colon \mathrm{Gra}'(\mathrm{Col}\,(2)) \to \mathrm{Gra}$ constructed in the proof is both a pseudorealization and a strong embedding in the sense of PULTR [26, 27].

A *strong embedding* $\Phi\colon \mathscr{A} \to \mathscr{B}$ ($\mathscr{A}$, $\mathscr{B}$ concrete categories) is a full embedding such that the set $\square\,\Phi(A)$ and the mapping $\square\,\Phi(\varphi)$ depend only on the set $\square A$ and on the mapping $\square\,\varphi$, resp. $(A \in \mathrm{Ob}\mathscr{A},\ \varphi \in \mathrm{Mor}\,\mathscr{A})$.

PROBLEM 5.4. *Does there exist a strong embedding* $\Phi\colon \mathscr{A}_0 \to \mathrm{Gra}_0$ *which is cospectral in the sense of Definition* 5.1, *for any category* $\mathscr{A}$ *of universal algebras of finite type?*

We note, that this is not the case if $\mathscr{A}$ is an algebraic category involving relations (not only operations) since if the identity mapping is a homomorphism of two cospectral graphs then it is an isomorphism. (The number of edges can be recognized from the spectrum: it is a half times $\sum \lambda_i^2$, the trace of the square of the adjacency matrix.) (Cf. the second remark on [27, p. 79].)

We derive some corollaries of Theorem 5.2. The concrete category $(\mathscr{K}, \square)$ is said to satisfy the *uniqueness* condition, if for any isomorphism $\varphi \in \operatorname{Hom}(X, Y)$, $\square \varphi = id_{\square X}$ implies $X = Y$. (In other words, any two objects on the same underlying set, for which the identity is an isomorphism, are identical.) Clearly, any algebraic category satisfies the uniqueness condition.

A concrete category $(\mathscr{K}, \square)$ is *finite*, if both $\operatorname{Ob} \mathscr{K}$ and each $\square A$ $(A \in \operatorname{Ob} \mathscr{K})$ are finite sets.

COROLLARY 5.5. *Let $(\mathscr{K}, \square)$ be a finite concrete category satisfying the uniqueness condition. Then $(\mathscr{K}, \square)$ has a cospectral embedding into the category of finite graphs.*

PROOF. A *realization* $\Phi: (\mathscr{K}, \square) \to (\mathscr{L}, \square)$ is a full embedding such that $\square \Phi(A) = \square A$, $\square \Phi(\varphi) = \square \varphi$ for any $A \in \operatorname{Ob} \mathscr{K}$, $\varphi \in \operatorname{Mor} \mathscr{K}$. PULTR [26, Lemma 3.6] asserts, for the case of finite categories, that *any finite concrete category satisfying the uniqueness condition has a realization into some algebraic category $\mathscr{A}$ of finite type*. An application of this and subsequently of Theorem 5.2 proves Corollary 5.5.

A category $\mathscr{K}$ is finite, if $\operatorname{Mor} \mathscr{K}$ is a finite set.

COROLLARY 5.6. *For a finite category $\mathscr{K}$, the following are equivalent:*

(a) *Any morphism having a left inverse is an isomorphism.*

(b) *There is an integer n and a faithful functor $\square: \mathscr{K} \to \mathrm{Sets}$ such that $|\square A| = n$ for every $A \in \operatorname{Ob} \mathscr{K}$.*

(c) *There is a full embedding $\Phi: \mathscr{K} \to \mathrm{Gra}_0$ such that all graphs $\Phi(A)$ $(A \in \operatorname{Ob} \mathscr{K})$ have the same spectrum.*

PROOF. (a) is obviously a necessary condition of (b) and (b) a necessary condition of (c). We prove their sufficiency.

I. First we prove that (a) implies (b). There exists a faithful functor $\square_1: \mathscr{K} \to \mathrm{Sets}$ such that $\square_1 A$ is a finite set for all $A \in \operatorname{Ob} \mathscr{K}$. (The covariant Hom functor, for instance.) Let $n = \max\{|\square_1 A| : A \in \operatorname{Ob} \mathscr{K}\} + 1$. We may identify the set $\square_1 A$ with an initial interval of the set $\{1, \dots, n\}$, thus $\square_1 A = \{1, \dots, m_A\}$ where $m_A = |\square_1 A|$. Clearly, if A and B are isomorphic objects then $m_A = m_B$.

Let us define the functor $\square: \mathscr{K} \to \mathrm{Sets}$ by

$$\square A = \{1, \dots, n\} \quad (A \in \operatorname{Ob} \mathscr{K});$$

and for any $A, B \in \operatorname{Ob} \mathscr{K}$, $\varphi \in \operatorname{Hom}(A, B)$, and $1 \leqq p \leqq n$, let

$$\square(\varphi)(p) = \begin{cases} \square_1(\varphi)(p), & \text{if } p \leqq m_A; \\ p, & \text{if } p > m_A \text{ and } \varphi \text{ is an isomorphism;} \\ n, & \text{if } p > m_A \text{ and } \varphi \text{ is not an isomorphism.} \end{cases}$$

As $m_A < n$, we have $p \leqq m_A$ if and only if $\square(\varphi)(p) \leqq m_B$. In order to prove that $\square$ is a functor, we only have to consider the image of $p > m_A$ under $\square(\psi\varphi)$ where $\psi \in \operatorname{Hom}(B, C)$. The equality $\square(\psi\varphi)(p) = \square(\psi)\big(\square(\varphi)(p)\big)$ follows in view of the fact that $\psi\varphi$ is an isomorphism (if and) only if both φ and ψ are isomorphisms; this latter being a straightforward consequence of (a).

II. Now we prove that (b) implies (c). It suffices to consider the case when distinct objects of $\mathcal{K}$ are not isomorphic. Moreover we may assume not only $|\Box A|=|\Box B|$ but also $\Box A=\Box B$ for each $A, B\in \text{Ob}\,\mathcal{K}$. Under these assumptions, an application of Corollary 5.5 completes the proof.

COROLLARY 5.7. *Given a family of (not necessarily different) finite monoids, $M_1, \dots, M_k$, there exist pairwise non-isomorphic finite graphs $X_1, \dots, X_k$, all having the same spectrum, such that*

$$\text{End}\, X_j \cong M_j \quad (j = 1, \dots, k).$$

PROOF. We define a finite category $\mathcal{K}$ as follows. Let $\text{Ob}\,\mathcal{K}=\{1, \dots, k\}$; $\text{Hom}\,(i, i)=M_i$; and $\text{Hom}\,(i, j)=\emptyset$ for $i\neq j$. This category satisfies 5.6 (a). For, let φ be a morphism $\varphi\in M_i$. Then $\varphi^s=\varphi^t$ for some $s>t$. If φ has a left inverse then we infer $\varphi^{s-t}=\text{id}$ whence φ^{s-t-1} is the inverse of φ.

An application of 5.6 (a)$\Rightarrow$(c) completes the proof.

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ON BASKAKOV-TYPE OPERATORS

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1. In 1957, V. A. BASKAKOV [1] defined a new type of operators in the following way:

Let the functions $\varphi_1(x), \varphi_2(x), \ldots$ possess the following properties on an interval $[0, R]$ $(R>0)$:

(a) φ_n is analytic on the interval $[0, R]$ including the endpoints,

(b) $\varphi_n(0)=1$,

(c) φ_n is completely monotone, i.e. $(-1)^k\varphi_n^{(k)}(x)\geqq 0$ if $k=0, 1, \ldots$ and $x\in[0, R]$,

(d) there exists a positive integer $m(n)$ not depending on k, such that

$$-\varphi_n^{(k)}(x) = n\varphi_{m(n)}^{(k-1)}(x)[1+\alpha_{k,n}(x)] \quad (k = 1, 2, \ldots)$$

where $\alpha_{k,n}(x)$ converges to zero uniformly in k when $n\to\infty$,

(e) $$\lim_{n\to\infty}\frac{n}{m(n)} = 1.$$

Let the operators L_n $(n=1, 2, \ldots)$ be defined by

$$L_n(f; x) = \sum_{k=0}^{\infty}(-x)^k\frac{\varphi_n^{(k)}(x)}{k!}f\left(\frac{k}{n}\right).$$

There are well-known special cases of the Baskakov-type operators mentioned below:

(A) $\varphi_n(x)=(1-x)^n$. Then we get the Bernstein polynomials

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}.$$

(B) $\varphi_n(x)=e^{-nx}$. Then we get the operators of Szász and Mirakian

$$S_n(f; x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)e^{-nx}\frac{(nx)^k}{k!}.$$

(C) $\varphi_n(x)=(1+x)^{-n}$. Then we get the operators of Baskakov

$$K_n(f; x) = (1+x)^{-n}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k.$$

These cases are included in the customary assumption that $m(n)=n+c$, $\alpha_{k,n}(x)=$ $=\alpha_n(x)$ and $\alpha_{k,n}(x)=\alpha_n$ ([1], [9], [10]).

REMARK. If $\alpha_{k,n}(x)$ $(k=0, 1, \dots)$ are independent of k i.e. $\alpha_{k,n}(x)=\alpha_n(x)$ then according to (d)

$$-\varphi_n'(x) = n\varphi_{m(n)}(x)[1+\alpha_n(x)] \tag{1}$$

$$-\varphi_n''(x) = n\varphi_{m(n)}'(x)[1+\alpha_n(x)] \tag{2}$$

when $k=1$ and 2.

We get from (1) and (2) that $\alpha_n'(x)=0$ i.e. $\alpha_n(x)$ is independent of x; $\alpha_n(x)=\alpha_n$.

REMARK. If $\alpha_{k,n}(x)$ $(k=0, 1, \dots)$ are independent of x, i.e. $\alpha_{k,n}(x)=\alpha_{k,n}$ then on the one hand

$$\varphi_n'(x) = -n\varphi_{m(n)}(x)[1+\alpha_{1,n}]$$

from which we get

$$\varphi_n^{(k)}(x) = -n\varphi_{m(n)}^{(k-1)}(x)[1+\alpha_{1,n}],$$

and on the other hand, according to (d)

$$\varphi_n^{(k)}(x) = -n\varphi_{m(n)}^{(k-1)}(x)[1+\alpha_{k,n}].$$

Comparing the last two formulas we get $\alpha_{k,n}=\alpha_n$.

2. In this section we investigate the possible variations of

$$m(n) \stackrel{\text{def}}{=} n+\psi(n).$$

We deal only with the case when $\psi(n)$ is monotone.

THEOREM 1. *If $0\leqq\psi$ is monotone and nondecreasing, then for sufficiently large n's $\psi(n)$ is a constant.*

PROOF. Let $n_0=n$, $n_{k+1}=m(n_k)$ $(k=0, 1, \dots)$. By a repeated application of (d) we get

$$\varphi_n^{(k)}(x) = (-1)^k\varphi_{n_k}(x)\prod_{i=0}^{k-1}\{n_i[1+\alpha_{k-i,n_i}(x)]\}.$$

According to (b)

$$\varphi_n^{(k)}(0) = (-1)^k\prod_{i=0}^{k-1}\{n_i[1+\alpha_{k-i,n_i}]\}$$

where $\alpha_{i,n}=\alpha_{i,n}(0)$.

In (a) we required φ_n to be analytic in 0, so there exists a $\delta_n>0$ such that

$$\varphi_n(x) = \sum_{k=0}^{\infty}(-x)^k\frac{\prod_{i=0}^{k-1}\{n_i[1+\alpha_{k-i,n_i}]\}}{k!}, \quad |x|\leqq\delta_n.$$

By the Cauchy—Hadamard theorem

$$\overline{\lim_{k\to\infty}}\left(\frac{\prod_{i=0}^{k-1}\{n_i[1+\alpha_{k-i,n_i}]\}}{k!}\right)^{1/k}\leqq\frac{1}{\delta_n}.$$

Using (d) we get that for an arbitrary $\varepsilon>0$ there exists an m such that $|\alpha_{k,n}|<\varepsilon$ when $n>m$. For these n's we get

$$\overline{\lim_{k\to\infty}}\left(\frac{\prod_{i=0}^{k-1}n_i}{k!}\right)^{1/k}\leqq\frac{1}{\delta_n(1+\varepsilon)}.$$

Let

$$c_{k,n}=\left(\frac{\prod_{i=0}^{k-1}n_i}{k!}\right)^{1/k}$$

and

$$c(n)=\overline{\lim_{k\to\infty}}\,c_{k,n}.$$

Obviously, to any $\eta>0$ there exists a $k_0=k_0(\eta,n)$ such that $c_{k,n}>c(n)+\eta$ if $k>k_0$.

Thus

$$c_{k-1,n_1}=\left(\frac{k}{n}c_{k,n}^k\right)^{1/(k-1)}\leqq\left(\frac{k}{n}\right)^{1/(k-1)}\cdot(c(n)+\eta)^{k/(k-1)}\quad(k>k_0).$$

Clearly

$$c(n_1)=\overline{\lim_{k\to\infty}}\,c_{k,n_1}\leqq c(n)+\eta.$$

But $\eta>0$ being arbitrary, we get $c(n)\geqq c(n_1)$. Similarly $c(n_i)\geqq c(n_{i+1})$ $(i=0,1,\ldots)$ i.e. the sequence $\{c(n_i)\}_{i=0}^{\infty}$ is bounded; let e.g.

$$c(n_i)\leqq M_n\quad(i=0,1,\ldots).$$

ψ is non-decreasing and integer-valued, hence it is enough to prove that ψ is bounded.

First of all we prove

$$n_i\geqq n+i\psi(n)\quad(i=0,1,\ldots)$$

by induction. When $i=0$, this is evident. Assume that it is true for $i=k$. Then

$$n_{k+1}=m(n_k)=n_k+\psi(n_k)\geqq n+k\psi(n)+\psi\left((n+k\psi(n)\right)\geqq$$

$$\geqq n+k\psi(n)+\psi(n)=n+(k+1)\psi(n)$$

being ψ non-decreasing. Hence

$$c_{k,n}^k=\frac{\prod_{i=0}^{k-1}n_i}{k!}\geqq\frac{n\prod_{i=1}^{k-1}(n+i\psi(n))}{k!}\geqq\frac{(k-1)!\,n\psi(n)^{k-1}}{k!}=\frac{n\psi(n)^{k-1}}{k}.$$

That is

$$\psi(n) \leqq \left(\frac{k}{n} c_{n,k}^{k}\right)^{1/(k-1)} = c_{k-1,n_1}.$$

Consequently $\psi(n) \leqq c(n_1)$. Similarly we can get

$$\psi(n_i) \leqq c(n_{i+1}) \leqq M_n \quad (i = 0, 1, \ldots).$$

Thus $\psi(n_i) \leqq M_n$. Now we distinguish two cases. First let ψ be zero for all n. In this case our theorem is evidently true. Secondly, assume that there exists an $\bar{n}$ so that $\psi(\bar{n})>0$. The sequence $\{n_i\}_{i=0}^{\infty}$ tends to infinity for any $n_0>\bar{n}$. Hence our relation $\psi(n_i) \leqq M_n$ can hold only if ψ is bounded. Q.e.d.

REMARK. It is worth to observe from the proof that to the sequence $\{\varphi_{n_i}\}_{i=0}^{\infty}$ there exists a $\delta_n>0$ so that all the functions φ_{n_i} are analytic in the disc $\{z \,\|z| \leqq \delta_n\}$.

COROLLARY. *If $m(n)=n+c$ then there exists a $\delta>0$ such that all the functions φ_n are analytic on the disc $\{z \mid |z| \leqq \delta\}$.*

REMARK. If ψ is strictly decreasing then φ_n is a polynomial of degree at most n.

3. F. SCHURER [10] and R. K. S. RATHORE [9] proved that if f is continuous in the neighbourhood of x_0 and $f(x)=O(x^{\alpha})$ where $\alpha>0$ is arbitrary but fixed, then

$$\lim_{n\to\infty} L_n(f; x_0) = f(x_0),$$

where L_n is a Baskakov-type operator.

One can prove that (see e.g. [6], [3]) in the case of the Baskakov operator the growth condition on $f(x)$ can be replaced by

$$f(x) = O(e^{\alpha x}) \quad (\alpha > 0) \tag{3}$$

and this condition is the best possible ([5]). Thus, in the general case, the weakest possible condition is (3). The next result shows that this condition is sufficient for the convergence of a wider class of operators, too.

THEOREM 2. *If $m(n)=n+c$ $(c \geqq 0)$, (3) holds, and f is continuous then*

$$\lim_{n\to\infty} L_n(f; x) = f(x) \quad (x \geqq 0). \tag{4}$$

PROOF. Let $f_\alpha(x)=e^{\alpha x}$. It is known [5] that with our assumptions it suffices to prove (4) only for f_α. As we have already seen, φ_n is analytic on a disc $\{z \mid |z|<\delta\}$. Thus

$$L_n(f_\alpha; x) = \sum_{k=0}^{\infty} e^{\alpha k/n} \frac{(-x)^k \varphi_n^{(k)}(x)}{k!} = \sum_{k=0}^{\infty} \frac{(-x e^{\alpha/n})^k \varphi_n^{(k)}(x)}{k!} =$$

$$= \sum_{k=0}^{\infty} \frac{\varphi_n^{(k)}(x)}{k!} [x - x(1-e^{\alpha/n})]^k = \varphi_n[x(1-e^{\alpha/n})].$$

Consequently, it is enough to examine the convergence of $\varphi_n(x(1-e^{\alpha/n}))$. Let

$$1-e^{\alpha/n} = -\frac{\alpha}{n}(1+\varrho_n),$$

then $\varrho_n = O\left(\frac{1}{n}\right)$.

We must distinguish two cases; the first one is when $c=0$. Then

$$\varphi_n(x) = \sum_{k=0}^{\infty} \frac{(-nx)^k}{k!} \prod_{i=0}^{k-1} (1+\alpha_{k-i,n}).$$

so

$$\varphi_n(x(1-e^{\alpha/n})) = \sum_{k=1}^{\infty} \frac{(\alpha x(1+\varrho_n))^k}{k!} \prod_{i=0}^{k-1} (1+\alpha_{k-i,n}).$$

Let $\varepsilon_n = \sup_i |\alpha_{i,n}|$. According to (d), $\lim_{n\to\infty} \varepsilon_n = 0$. Thus, for sufficiently large n's

$$e^{\alpha x(1+\varrho_n)(1-\varepsilon_n)} = \sum_{k=0}^{\infty} \frac{[\alpha x(1+\varrho_n)(1-\varepsilon_n)]^k}{k!} \leqq \varphi_n[x(1-e^{\alpha/n})] \leqq$$

$$\leqq \sum_{k=0}^{\infty} \frac{[\alpha x(1+\varrho_n)(1+\varepsilon_n)]^k}{k!} = e^{\alpha x(1+\varrho_n)(1+\varepsilon_n)}.$$

This implies

$$\lim_{n\to\infty} \varphi_n[x(1-e^{\alpha/n})] = e^{\alpha x}.$$

The second case is when $c>0$. Then

$$\varphi_n(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \prod_{i=0}^{k-1} [(n+ic)(1+\alpha_{k-i,n+ic})].$$

Let now $\varepsilon_n = \sup_{j>n;i} |\alpha_{i,j}|$. It is easy to verify that if $\varepsilon_n \equiv 0$ then

$$\varphi_n(x) = (1+cx)^{-n/c}.$$

Using this we obtain

$$\varphi_n(x(1-e^{\alpha/n})) = \sum_{k=0}^{\infty} \left(\frac{cx\alpha(1+\varrho_n)}{n}\right)^k \frac{\prod_{i=0}^{k-1}\left(\frac{n}{c}+i\right)}{k!} \prod_{i=0}^{k-1} (1+\alpha_{k-i,n+ic}) \leqq$$

$$\leqq \sum_{k=0}^{\infty} \left(\frac{\alpha x(1+\varrho_n)(1+\varepsilon_n)}{\left(\frac{n}{c}\right)}\right)^k \frac{\prod_{i=0}^{k-1}\left(\frac{n}{c}+i\right)}{k!} = \left(1-\frac{\alpha x(1+\varrho_n)(1+\varepsilon_n)}{\left(\frac{n}{c}\right)}\right)^{-n/c}.$$

Similarly

$$\varphi_n(x(1-e^{\alpha/n})) \geqq \left(1-\frac{\alpha x(1+\varrho_n)(1-\varepsilon_n)}{\left(\frac{n}{c}\right)}\right)^{-n/c},$$

and therefore

$$\lim_{n\to\infty} L_n(f_\alpha; x) = \lim_{n\to\infty} \varphi_n\big(x(1-e^{\alpha/n})\big) = e^{\alpha x} = f_\alpha(x).$$

Q.e.d.

4. W. Feller [4] and D. Stancu [11] introduced the following so-called summation type operators.

Let $F_x(t)$ be the common distribution of the independent random variables $X_1, X_2, \ldots$ with expectation x and variance σ^2. Let $Y_n = n^{-1}\sum_{k=1}^{n} X_k$ and denote the distribution of Y_n by $F_{n,x}(t)$. Then we call

$$M_n[f; x] = \int f(t)\, d_t F_{n,x}(t)$$

an operator of summation type. Feller and Stancu proved that

$$\lim_{n\to\infty} M_n[f; x] = f(x)$$

for every bounded and continuous function. It is known that the operators (A), (B), (C) are operators of summation type as well. In the following we specify all the operators of Baskakov-type and summation-type at the same time.

The values $f\left(\frac{k}{n}\right)$ $(k=0, 1, \ldots)$ determine $M_n[f]$ completely, so

$$M_n[f; x] = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) p_{n,k}(x).$$

Here

$$p_{n,k}(x) = P\left(\frac{1}{n}\sum_{i=1}^{n} X_i = k\right)$$

where P is the probability measure. The variables $X_1, X_2, \ldots$ are independent, so $p_{n,k}$ is a convolution, that is

$$p_{n,k}(x) = \sum_{k_1+\ldots+k_n=k} p_{k_1}(x)\ldots p_{k_n}(x) \tag{5}$$

where $p_k(x) = p_{1,k}(x) \geqq 0$ and the relations

$$\sum_{k=0}^{\infty} p_k(x) = 1, \quad \sum_{k=0}^{\infty} k p_k(x) = x$$

are fulfilled. On the other hand

$$p_{n,k}(x) = \frac{(-x)^k \varphi_n^{(k)}(x)}{k!} \quad (k = 0, 1, \ldots)$$

so $p_{n,0}(x) = \varphi_n(x)$. But from (5) $p_{n,0}(x) = p_0^n(x)$, and hence $\varphi_n(x) = \varphi_1^n(x)$.

Writing φ in place of φ_1 in (d) we obtain (with $k=1$)

$$-n\varphi^{n-1}(x)\varphi'(x) = n\varphi^{m(n)}(x)[1+\alpha_{1,n}(x)]$$

i.e.

$$\varphi^{n-m(n)-1}(x)\varphi'(x) = -[1+\alpha_{1,n}(x)]. \tag{6}$$

Using again the notation $\psi(n)=m(n)-n$, we get

$$\varphi(x)^{-(1+\psi(n))}\varphi'(x) = -[1+\alpha_{1,n}(x)].$$

Consequently

$$\varphi(x)^{\psi(n+1)-\psi(n)} = \frac{1+\alpha_{1,n}(x)}{1+\alpha_{1,n+1}(x)}. \tag{7}$$

In (7) the right hand side tends to 1 if $n\to\infty$ so we get

$$\lim_{n\to\infty} [\psi(n+1)-\psi(n)] = 0.$$

By definition, the values of ψ are integers so $\psi(n)\equiv c$ if n is large enough. Putting $\psi(n)=c$ in (6) it follows that

$$\varphi^{-(c+1)}(x)\cdot\varphi'(x) = -1-\alpha_{1,n}(x).$$

Hence we get

$$\varphi^{-(c+1)}(x)\varphi'(x) = -1$$

when $n\to\infty$. By (b), $\varphi(0)=1$ so by considering c as a parameter, φ is uniquely determined.

We distinguish three cases.

(i) $c=0$. Then $\varphi(x)=e^{-x}$. In this case M_n is the Szász—Mirakian operator.

(ii) $c>0$. Then $\varphi(x)=(1+cx)^{-1/c}$. Here $c=1$ gives the K_n operator of Baskakov. In the general case

$$M_n[f(t); x] = K_{n/c}\left[f\left(\frac{t}{c}\right); cx\right].$$

(iii) $c<0$. With the notation $d=-c$, $\varphi(x)=(1-dx)^{1/d}$. If $d=1$ then we get back the Bernstein polynomials B_n. If d divides n then

$$M_n[f(t); x] = B_{n/d}\left[f\left(\frac{t}{d}\right); dx\right],$$

but if n is not divisible by d then we do not get positive operators.

Hence we obtain the following

THEOREM 3. *If M_n is an operator of Baskakov-type and summation-type as well then M_n is one of the types* (i), (ii) *or* (iii) *mentioned above.*

5. In order to obtain a new approximation process for integrable functions, KANTOROVITCH applied the following idea ([7], p. 30). Instead of the Bernstein polynomial of f he studied the derivative of the Bernstein polynomial of $F(x)=\int\limits_0^x f(t)\,dt$.

The reason for this was that the derivatives of the Bernstein polynomials tend to the derivative of the function. It is well-known [8] that the Baskakov-type operators have this property. So we introduce the operator

$$L_n^*[f; x] = \frac{d}{dx}L_n[F; x]$$

where L_n is any Baskakov-type operator. Using the previous notations, we state our

THEOREM 4. *If*

$$\lim_{x\to R} x^k \varphi_n^{(k)}(x) = 0 \quad (n = 1, 2, \ldots;\ k = 0, 1, \ldots)$$

where $R>0$ *is a fixed value, then*

(i) $\|L_n^*\| = 1$

(ii) $\lim\limits_{n\to\infty} \|L_n^* f - f\| = 0 \quad (f\in L[0, R])$

where the norm is the usual norm in $L[0, R]$.

(If $R<\infty$, then let $f(x)\equiv 0$ for $x>R$).

PROOF. First we prove (i). We have

$$L_n^*[f; x] = \frac{d}{dx}\sum_{k=1}^{\infty}\frac{(-x)^k\varphi_n^{(k)}(x)}{k!}\int_0^{k/n} f(t)\,dt = -\sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\varphi_n^{(k+1)}(x)\int_{k/n}^{(k+1)/n} f(t)\,dt.$$

Hence by (c)

$$\|L_n^*(f)\| = \int_0^R\left|\sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\varphi_n^{(k+1)}(x)\int_{k/n}^{(k+1)/n} f(t)\,dt\right| dx \leqq$$

$$\leqq \sum_{k=0}^{\infty}\int_{k/n}^{(k+1)/n}|f(t)|dt\cdot(-1)\int_0^R\frac{(-x)^k\varphi_n^{(k+1)}(x)}{k!}\,dx.$$

Using the assumption of the theorem, with integration by parts we obtain

$$-\int_0^R\frac{(-x)^k\varphi_n^{(k+1)}(x)}{k!}\,dx = -\int_0^R\frac{(-x)^{k-1}\varphi_n^{(k)}(x)}{(k-1)!}\,dx = \ldots = -\int_0^R\varphi_n'(x)\,dx = 1.$$

Consequently

$$\|L_n^*(f)\| \leqq \sum_{k=0}^{\infty}\int_{k/n}^{(k+1)/n}|f(t)|\,dt = \int_0^{\infty}|f(t)|\,dt = \|f\|.$$

So $\|L_n^*\|\leqq 1$. Let $f_a(x)=a^{-1}\chi_a(x)$ where $a=r/n$ $(r=1, 2, \ldots)$ and

$$\chi_a(x) = \begin{cases}1 & \text{if } x\leqq a\\ 0 & \text{if } x\geqq a.\end{cases}$$

Obviously $\|f_a\|=1$. But

$$\|L_n^*(f_a)\| = \sum_{k=0}^{an-1}\int_{k/n}^{(k+1)/n} a^{-1}\,dx = 1,$$

hence $\|L_n^*\|=1$.

To prove (ii) it is necessary and sufficient to prove the statement only for the characteristic functions of intervals with end-points 0 and $a=r/n$ $(r=1, 2, \ldots)$. Let $\bar\chi_a(x)=1-\chi_a(x)$. Being

$$\int_0^R L_n^*[\chi_a; x]\,dx = a = \int_0^a 1\,dx,$$

we get

$$\Delta_{a,n} = \|\chi_a - L_n^*[\chi_a]\| = \int_0^a L_n^*[\bar{\chi}_a; x]\,dx + \int_a^R L_n^*[\chi_a; x]\,dx =$$

$$= \int_0^a L_n^*[\bar{\chi}_a; x]\,dx + \int_0^R L_n^*[\chi_a; x]\,dx - \int_0^a L_n^*[\chi_a; x]\,dx =$$

$$= \int_0^a L_n^*[\bar{\chi}_a; x]\,dx + \int_0^a (1 - L_n^*[\chi_a; x])\,dx =$$

$$= 2\int_0^a L_n^*[\bar{\chi}_a; x]\,dx = -\frac{2}{n}\int_0^a \sum_{k/n \geqq a} \frac{(-x)^k}{k!}\varphi_n^{(k+1)}(x)\,dx.$$

But

$$-\int_0^a \frac{(-x)^k \varphi_n^{(k+1)}(x)}{k!}\,dx = -\left[\sum_{l=0}^{k} \frac{(-x)^l \varphi_n^{(l)}(x)}{l!}\right]_0^a =$$

$$= 1 - \sum_{l=0}^{k} \frac{(-a)^l \varphi_n^{(l)}(a)}{l!} = \sum_{l=k+1}^{\infty} \frac{(-a)^l \varphi_n^{(l)}(a)}{l!}$$

so

$$\Delta_{a,n} = \frac{2}{n} \sum_{k/n \geqq a} \sum_{l=k+1}^{\infty} \frac{(-a)^l \varphi_n^{(l)}(a)}{l!} = \frac{2}{n} \sum_{k/n \geqq a} (k - an) \frac{(-a)^k \varphi_n^{(k)}(a)}{k!} =$$

$$= 2 \sum_{k/n > a} \left(\frac{k}{n} - a\right) \frac{(-a)^k \varphi_n^{(k)}(a)}{k!} = 2L_n((x-a)_+; a)$$

where

$$(x-a)_+ = \begin{cases} 0 & \text{if } x \leqq a \\ x-a & \text{if } x \geqq a \end{cases}$$

is a continuous function of polynomial order. For such functions f, $L_n(f; x) \to f(x)$ (see Section 3), so

$$L_n((x-a)_+; a) \to (x-a)_+|_{x=a} = 0$$

i.e.

$$\|L_n^* f - f\| \to 0.$$

Q.e.d.

REMARKS. 1. If L_n is the Bernstein polynomial or the Szász or Baskakov operator then the condition of Theorem 3 is fulfilled with $R=1$ or $R=\infty$ respectively.

2. The case of the Szász-operator was studied by P. L. BUTZER [2].

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KEGELSCHNITTE AUF DER METRISCHEN EBENE

Von

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Herrn Professor Ferenc Kárteszi zum 70-sten Geburtstag gewidmet

In dieser Arbeit werden die Kegelschnitte auf der metrischen Ebene im Sinne von F. BACHMANN [1], bzw. auf ihrer projektiv-metrischen Einbettungsebene spiegelungsgeometrisch definiert. Diese Definition liefert die im klassischen euklidischen Fall wohl bekannten Brennpunkteigenschaften in einheitlicher Fassung. Auf Grund der Definition 4.1 kann man auch die projektiven Eigenschaften der Kegelschnitte einfach herleiten. Die synthetische Behandlung beruht auf dem verallgemeinerten Dreispiegelungssatz, der auf der projektiv-metrischen Einbettungsebene gültig ist. Wir heben die Vielheit der verschiedenen Kegelschnitte der hyperbolischen Ebene besonders hervor, wo gewisse Kegelschnitte drei Brennpunktpaare haben können, und wo ein Kegelschnitt gewisse zu den Brennpunkteigenschaften dualen Asymptotenkennzeichnungen haben kann. Die Arbeit ist in Verbindung mit unseren Arbeiten [2], [3], [4], wo das Thema mit klassischen raumgeometrischen Methoden behandelt ist, und mit der Arbeit [5], der man auch die Beziehungen zu dem Inversionsbegriff entnehmen kann. Dort weisen wir auf die weitere Literatur des Themas hin.

In Sektionen 1 und 2 werden die Definitionen, die zitierten Sätze zusammengefaßt und motiviert. Die Hauptresultate der Arbeit sind die Sätze 3.2, 4.2, 4.4, 5.1, 6.2, 6.3. In Sektion 7 bekommen wir eine Übersicht der möglichen Kegelschnitte.

1. Metrische Gruppenebene und ihre projektiv-metrische Einbettungsebene

Unter einer *metrischen Gruppenebene* $\mathcal{M}=\mathcal{M}(\mathbf{G},\mathbf{S},\mathbf{P})$ verstehen wir die folgende, axiomatisch definierte Struktur (BACHMANN [1] §. 3):

GRUNDANNAHME. *Sei* $\mathbf{G}(\mathbf{S})$ *eine involutorisch erzeugte Gruppe, das Erzeugendensystem* $\mathbf{S}$ *sei invariant in* $\mathbf{G}$.

$\alpha|\beta$ bezeichnet, daß die Elemente α, β, $\alpha\beta$ von $\mathbf{G}$ involutorisch sind. Zur Abkürzung benutzen wir Ausdrücke, in denen je zwei Elemente zueinander in Strichrelation stehen, wenn sich zwischen ihnen wenigstens ein | befindet. $\alpha\nmid\beta$ bezeichnet die Verneinung von $\alpha|\beta$.

Sei $\mathbf{P}$ die Menge aller involutorischen Produkte a_1a_2 aus dem Komplex $\mathbf{SS}$ (also $a_1, a_2\in\mathbf{S}$).

Wir können eine geometrische Struktur $\mathcal{M}=\mathcal{M}(\mathbf{G},\mathbf{S},\mathbf{P})$ so definieren: Die Elemente von $\mathbf{S}$ werden mit $a, b, c, \ldots$ bezeichnet und *Geraden* genannt; die Elemente von $\mathbf{P}$ werden mit $A, B, C, \ldots$ bezeichnet und heißen *Punkte*. Zwei Geraden a und b von $\mathcal{M}$ sind zueinander *senkrecht*, wenn $a|b$ gilt; einen Punkt A und eine Gerade

b nennen wir *inzident*, wenn $A|b$ gilt; ein Punkt A und eine Gerade a sind zueinander *polar*, wenn $A=a$ in **G** besteht (ein solcher Punkt braucht nicht zu existieren); d ist die *vierte Spiegelungsgerade* zu a, b, c, wenn $d=abc$ ist.

Wir nehmen an, daß die folgenden Axiome gelten.

A. 1. *Zu* A, B *gibt es stets ein* c *mit* $A, B|c$.
A. 2. *Aus* $A, B|c, d$ *folgt* $A=B$ *oder* $c=d$.
A. 3. *Aus* $a, b, c|E$ *folgt* $abc\in\mathbf{S}$.
A. 4. *Aus* $a, b, c|e$ *folgt* $abc\in\mathbf{S}$.
A. D. *Es gibt* g, h, j *derart, daß* $g|h$ *und weder* $j|g$ *noch* $j|gh$ *gültig sind.*

Da das Erzeugendensystem **S** nach der Grundannahme invariant ist, wird **S** bei *Transformation* mit einem Element γ von $G-\gamma: x\to x^\gamma := \gamma^{-1}x\gamma-$ auf sich abgebildet. Bei der Transformation mit γ geht auch jedes involutorische Produkt aus **SS** in ein involutorisches Produkt aus **SS** über. Also ist

$$\gamma:\ x\to x^\gamma := \gamma^{-1}x\gamma;\quad X\to X^\gamma := \gamma^{-1}X\gamma$$

eine eineindeutige Abbildung der Menge der Geraden und der Menge der Punkte von $\mathcal{M}$ je auf sich. Bei dieser Abbildung γ bleiben die Strichrelation und auch die von ihr definierten geometrischen Relationen erhalten; wir nennen γ auch eine *Bewegung von* $\mathcal{M}$. Im Falle $\gamma=c$ handelt es sich um *Geradenspiegelung* (an der Gerade c), im Falle $\gamma=C$ um *Punktspiegelung* (an dem Punkt C).

Die Bewegungen der Gruppenebene $\mathcal{M}$ bilden eine Gruppe $\mathbf{G}'$, welche von dem System $\mathbf{S}'$ der Geradenspiegelungen erzeugt wird. Ordnet man dem Element γ aus **G** die Bewegung γ der Gruppenebene zu, so ist diese Zuordnung eine homomorphe Abbildung von $\mathbf{G}(\mathbf{S})$ auf $\mathbf{G}'(\mathbf{S}')$. Aus dem Axiomensystem folgt, daß diese Zuordnung sogar eine isomorphe Abbildung ist. Dieser Satz hat die Zusammenstellung des Axiomensystems motiviert.

Eine andere Motivation ist die Beweisbarkeit des von Hjelmslev stammenden Satzes 1.1, der die Hauptrolle in der Erforschung der Struktur $\mathcal{M}$ spielt.

SATZ 1.1. *Sind* $a\neq c$ *und* X *gegeben, und inzidiert* X *nicht sowohl mit* a *als mit* c, *so gibt es genau eine Gerade* b, *wofür* $b|X$ *und* $abc\in\mathbf{S}$ *gelten.*

Dieser Satz beruht auf den folgenden Behauptungen:

SATZ 1.2. *Die Gleichung* $Ad'C=d$ *gilt genau dann, wenn ein* x *mit* $x|A, d', C$ *existiert.*

SATZ 1.3. *Aus je vier der Gleichungen* $aa'=A$, $cc'=C$, $a'bc'=d'$, $Ad'C=d$, $abc=d$ *folgt die fünfte.*

Die Figur 1 zeigt die geometrische Konfiguration ([1], §. 3).

Sind die Geraden $a\neq b$ gegeben, so kann man die Geradenmenge

$$\mathcal{I}(ab) := \{y\in\mathbf{S}:\ aby\in\mathbf{S}\}$$

bilden, die *Geradenbüschel* oder kürzlich *Büschel* heißt.

SATZ 1.4 (Transitivitätssatz). *Ist* $a\neq b$, *und gelten* abc, $abd\in\mathbf{S}$, *so gilt auch* $acd\in\mathbf{S}$.

Dieser Satz garantiert, daß ein Büschel von seinen beliebigen zwei Geraden eindeutig bestimmt ist.

Gilt $a, b|X$, so heißt $\mathcal{J}(X):=\mathcal{J}(ab)$ ein *eigentliches Büschel*. Im Falle $a, b|x$ nennt man das Büschel $\mathcal{J}(x):=\mathcal{J}(ab)$ ein *Lotbüschel*. Der Satz 1.1 besagt auch die Behauptung: *Durch einen beliebigen Punkt, der von dem zufälligen Träger eines Büschels verschieden ist, läuft genau eine Gerade des Büschel.*

Ein Geradenbüschel $\mathcal{J}(x_1 x_2)$ heißt auch ein *Idealpunkt* X, das eigentliche Büschel $\mathcal{J}(A)=:\mathsf{A}$ ist *eigentlicher Idealpunkt*.

Ist $a \in \mathbf{S}$, so wird die Idealpunktmenge

$$\mathbf{a} := i(a) := \{\mathcal{J}(ax) \colon x \in \mathbf{S} \quad \text{und } x \neq a\}$$

eigentliche Idealgerade genannt; es bestehen die folgenden Äquivalenzen

$$A|a \Leftrightarrow \mathcal{J}(A) \in i(a) \Leftrightarrow a \in \mathcal{J}(A).$$

Zum allgemeinen Begriff der *Idealgerade* gelangen wir mit Hilfe der Hjelmslevschen *Halbdrehungstheorie*.

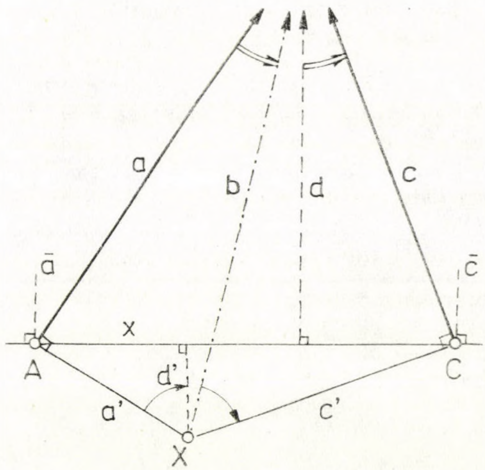


Fig. 1

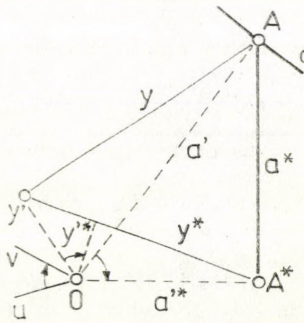


Fig. 2

Sei eine Drehung $\eta:=uv$ mit den Bedingungen $u, v|0, u\neq v, u\nmid v$ auf der Ebene $\mathcal{M}$ gegeben. Die von η bestimmte *Halbdrehung* um den Aufpunkt 0 ordnet einer beliebigen Geraden x (mit $x\neq 0$) eine Gerade x^* nach den Folgenden: Ist $x|0$, so sei $x^*:=xuv$. Ist $x\nmid 0$, so bestimmen wir zuerst das Lot $x':=(0, x)$ und den Punkt $X:=x'x$. Dann bestimmen wir $x'^*:=x'uv$ und — nach den Gleichungen $xuv=xx'x'^*=Xx'^*$ — das Lot $x^*:=(X, x'^*)$; ferner sei $X^*:=x'^*x^*$. $\eta^{-1}=vu$ bestimmt die zu (*) *gespiegelte Halbdrehung* $({}_*)$.

Auf Grund der Sätze 1.1—3 kann man beweisen ([1] § 6), daß die Äquivalenz $y|A \Leftrightarrow y^*|A^*$ für eine beliebige Halbdrehung (*) um den Aufpunkt 0 und für beliebige y, A besteht (Fig. 2).

Aus den Sätzen 1.1—4 folgen die Sätze 1.5—9 ([1] § 6).

SATZ 1.5. *Die Halbdrehung* (*) *um* 0 *ist eine büscheltreue Abbildung.*

SATZ 1.6. *Ist* $\mathcal{I}(ab) =: \mathcal{I}(X)$ *eigentlich, so ist auch* $\mathcal{I}(a^* b^*) = \mathcal{I}(X^*)$ *eigentlich; sogar zu* $d|X^*$ *gibt es ein* $c|X$, *daß* $c^* = d$ *gilt.*

SATZ 1.7. *Zu beliebigem Büschel* Y *gibt es genau ein Büschel* X, *so daß* $\mathsf{X}^* = \mathsf{Y}$ *gilt.*

SATZ 1.8. *Die Halbdrehungen um* 0 *sind kommutativ:* $x^{*0} = x^{0*}$.

SATZ 1.9. *Ist* A *kein Lotbüschel* $\mathcal{I}(a)$ *mit* $a|0$, *so gibt es eine Halbdrehung* $(^0)$ *um* 0, *für die* A^0 *eigentliches Büschel ist.*

Die Menge **x** von Idealpunkten heißt *Idealgerade*, wenn es eine Halbdrehung $(^0)$ um 0 und eine Gerade a gibt, so daß

$$\mathbf{x}^0 := \{\mathsf{Y}^0 \colon \mathsf{Y} \in \mathbf{x} \quad \text{und} \quad a \in \mathsf{Y}^0\} = i(a)$$

eine eigentliche Idealgerade ist. Ferner nennen wir auch die aus Lotbüscheln bestehende Menge

$$\mathbf{0} := i(0) := \{\mathcal{I}(y) \colon y|0\}$$

Idealgerade.

Die Idealpunkte und Idealgeraden mit der definierten Inzidenzrelation bilden die *Idealebene* $\mathcal{PM}$, die eine Pappos—Fanosche projektive Ebene ist, also gilt das

HAUPTSATZ 1.1. *Auf der Idealebene* $\mathcal{PM}$ *inzidieren zwei (ideale) Punkte genau mit einer (idealen) Geraden. Zwei Geraden haben genau einen Punkt gemein. Es gibt vier Punkte, von denen je drei nicht mit einer Geraden inzidieren. Liegen die Ecken eines Sechsecks abwechselnd auf zwei Geraden und sind niemals zwei zyklisch benachbarte Punkte mit einem der vier übrigen Punkte kollinear, so liegen die Schnittpunkte der Gegenseiten auf einer Geraden* (Papposscher Satz). *Die Diagonalpunkte eines vollständigen Vierecks sind niemals kollinear* (Fano-Axiom). ([1], §. 6).

Aus dem Papposschen Satz folgen zwei Sätze:

SATZ 1.10. *Der Begriff der Idealgeraden, also auch der Begriff der Idealebene ist vom Aufpunkt der Halbdrehungen unabhängig.*

SATZ 1.11. *Auf Idealebene* $\mathcal{PM}$ *ist der Satz von Desargues gültig: Sind zwei entsprechende vollständige Dreiecke in bezug auf einen Punkt perspektiv, so sind sie auch in bezug auf eine Gerade perspektiv.* (Satz von Hessenberg.)

SATZ 1.12. *Die Spiegelung der Gruppenebene* $\mathcal{M}$ *an einer Geraden* c *induziert in der Idealebene* $\mathcal{PM}$ *die involutorische Homologie mit der eigentlichen Idealgeraden* $i(c)$ *als Achse und mit* $\mathcal{I}(c)$ *als Zentrum.*

Der Orthogonalitätsbegriff bezüglich $\mathcal{M}$ induziert auf der Idealebene $\mathcal{PM}$ eine projektive Polarität, deren Art von zusätzlichen Axiomen über $\mathcal{M}$ abhängig ist.

Die *elliptische Ebene* $\mathcal{M}$ (A. P) kennzeichnet das

AXIOM P. *Es gibt* a, b, c *mit* $abc = 1$.

Dann ist $\mathcal{M}$ (A. P) selbst eine projektive Ebene: $\mathcal{M}$ (A. P) $= \mathcal{PM}$.

Die metrisch-euklidische Ebene kennzeichnet das

AXIOM R. *Es gibt* a, b, c, d *mit* $a, b|c, d$ *und* $a \neq b$, $c \neq d$.

Eine große Bedeutung hat das

AXIOM V. *Zu zwei Geraden a, b gibt es stets einen Punkt C mit $a, b|C$ oder eine Gerade c mit $a, b|c$* (a, b heißt *verbindbar* mit einem Punkt C oder mit einer Geraden c.)

Ist A. V für $\mathcal{M}$ nicht gültig, besteht also

AXIOM $\sim$V. *Es gibt a, b welche unverbindbar sind*;

so kommt dem Axiom H eine spezielle Rolle zu:

AXIOM H. *Gilt $a, b, c|X$ und sind a, g und b, g und c, g unverbindbar, so ist $a=b$ oder $a=c$ oder $b=c$.*

$\mathcal{M}$ (A. R, A. V) heißt *euklidische Ebene*, $\mathcal{M}$ (A. $\sim$V, A. H) heißt hyperbolische Ebene, $\mathcal{M}$ (A. $\sim$R, A. $\sim$P, A. V) heißt *halbelliptische Ebene*, jedoch kommen auch weitere Strukture vor.

Der allgemeine Begriff des Polare-Pol-Paars wird vom Satz 1.13 unterstützt (Fig. 3, [1], §. 6).

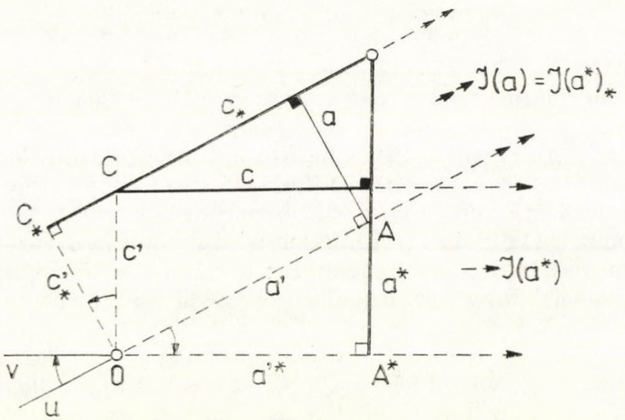


Fig. 3

SATZ 1.13. *Seien* (*) *und* ($_*$) *gespiegelte Halbdrehungen um* 0. *Aus* $a^*=b$ *folgt* $\mathcal{I}(b)_*=\mathcal{I}(a)$, *also* $\mathcal{I}(a^*)_*=\mathcal{I}(a)$.

Eine Idealgerade $\mathbf{x}$ und ein Idealpunkt X heißen ein *Polare-Pol-Paar*, wenn es eine Halbdrehung (*) (und die dazu gespiegelte ($_*$)) um 0 und eine Gerade a gibt, so daß $\mathbf{x}^*=i(a)$, $\mathcal{I}(a)_*=\mathsf{X}$ ist. Ferner soll $\mathbf{0}:=i(0)$, $\mathsf{0}:=\mathcal{I}(0)$ auch ein Polare-Pol-Paar heißen.

Auf Grund der Halbdrehungstheorie haben wir die folgenden

HAUPTSATZ 1.2. *Es gelte das Axiom $\sim R$ auf der Ebene $\mathcal{M}$. Jede Idealgerade hat genau einen Pol, jeder Idealpunkt hat genau eine Polare. Sind $\mathbf{x}, \mathsf{X}$ und $\mathbf{y}, \mathsf{Y}$ Polare-Pol-Paare, so ist $\mathsf{X}\in\mathbf{y}$ mit $\mathsf{Y}\in\mathbf{x}$ äquivalent. Die Polare-Pol-Verbindung ist eine projektive Polarität der Idealebene $\mathcal{PM}$* (A. $\sim$R), *die mit dieser „absoluten" Polarität eine ordinäre (nichteuklidische) projektiv-metrische Ebene ist.*

Hauptsatz 1.3. *Gilt Axiom* R *auf der Ebene* $\mathcal{M}$, *so bildet die Gesamtheit aller Lotbüschel eine „unendlichferne" Idealgerade* g_∞. *Die Abbildung* $\mathcal{I}(x) \to \mathcal{I}(y)$ *mit* $x|y$ *ist eine projektiv elliptische Involution auf der unendlichfernen Idealgeraden* g_∞. *Die Idealebene* $\mathcal{PM}$ (A. R) *mit dieser „absoluten" Involution, die eine singuläre projektive Polarität ist, heißt singuläre (euklidische) projektiv-metrische Ebene.*

Die harmonischen Homologien mit nicht inzidenten Paaren von Polare und Pol als Achse und Zentrum nennen wir *erzeugende Spiegelungen*, die die Menge $\mathcal{S}$ bilden. Die Spiegelungskompositionen bilden die *Bewegungsgruppe* $\mathcal{G}(\mathcal{S})$ *der projektiv-metrischen Idealebene* $\mathcal{PM}$. Da $\mathbf{S} \subseteqq \mathcal{S}$ und deswegen $\mathbf{G} \subseteqq \mathcal{G}$ im Sinne der Einbettung gelten, gilt das

Hauptsatz 1.4. *Die Gruppe* $\mathbf{G}(\mathbf{S})$, *welche dem Axiomensystem von* $\mathcal{M}(\mathbf{G}, \mathbf{S}, \mathbf{P})$ *genügt, ist als Untergruppe der Bewegungsgruppe der projektiv-metrischer Idealebene darstellbar.*

Diese Hauptsätze öffnen einen Zugang zum Studium der Modelle der Struktur $\mathcal{M}(\mathbf{G}, \mathbf{S})$, wo noch viele interessanten Probleme offen stehen ([1], Supplement). Wir könnten das Thema auch ohne projektive Einbettung behandeln, aber dann wäre die Arbeit langwieriger und unübersichtlich.

2. Verallgemeinerter Dreispiegelungssatz auf der Einbettungsebene

Im weiteren betrachten wir die metrische Ebene $\mathcal{M}$, die in ihre projektiv-metrische Idealebene $\mathcal{PM}$ eingebettet ist. Die (idealen) Geraden von $\mathcal{PM}$ bezeichnen $a, b, \ldots, z$, die (idealen) Punkte von $\mathcal{PM}$ bezeichnen $A, B, \ldots, Z$. Wir gebrauchen auch untere Indizes. Die polare Verbindung wird durch die Beziehung aufeinander der Buchtsaben und Indizes hervorgehoben: der Pol von a_1 ist A_1 usw. Die harmonische Homologie mit nicht inzidenten Achse a und Zentrum A können wir auf der nichteuklidischen (ordinären) $\mathcal{PM}$ sowohl die *Geradenspiegelung* a als auch die *Punktspiegelung* A nennen. Auf der euklidischen (singulären) $\mathcal{PM}$ sprechen wir über die *Geradenspiegelung* a, wenn die Achse a von der unendlichfernen Geraden g_∞ verschieden ist; dann liegt ihr Pol A als Zentrum auf g_∞.

Sind a und b Geradenspiegelungen und gilt $a \mathrm{I} B$ d. h. $b \mathrm{I} A$ — wir nennen dann a, b *senkrecht* — so ist $ab=ba$ eine harmonische Homologie, deren Zentrum der Schnittpunkt $[a, b]=:X$ und deren Achse die Verbindungsgerade der Polen $(A, B)=:x$ sind. Dann heißt $ab=ba=:X$ die *Punktspiegelung* X.

Auf der euklidischen $\mathcal{PM}$ ist g_∞ die gemeinsame Achse der Punktspiegelungen. g_∞ heißt auch *Grenzgerade*, deren Punkte die *Grenzpunkte* sind.

Wenn eine Polare auf der nichteuklidischen $\mathcal{PM}$ mit ihrem Pol inzident ist, so sprechen wir auch über eine *Grenzgerade* bzw. einen *Grenzpunkt*. Wenn es Grenzelemente gibt, so bilden sie einen *absoluten Linienkegelschnitt* bzw. *Punktkegelschnitt*, und wir nennen dann $\mathcal{PM}$ *hyperbolisch*. (Im Text heben wir, wenn es wesentlich ist, die Grenzelemente mit „+" hervor.) Einen Punkt nennt man *äußeren Punkt*, wenn er mit zwei Grenzgeraden inzidiert. Ein Punkt heißt *innerer Punkt*, wenn er mit keiner Grenzgeraden inzidiert. Die Punkte von $\mathcal{M}$, d. h. die eigentlichen Punkte von $\mathcal{PM}$ sind nur mit eigentlichen Geraden inzident, also sind sie innere Punkte. Doch sind die inneren Punkte nicht unbedingt eigentlich. Die Figuren

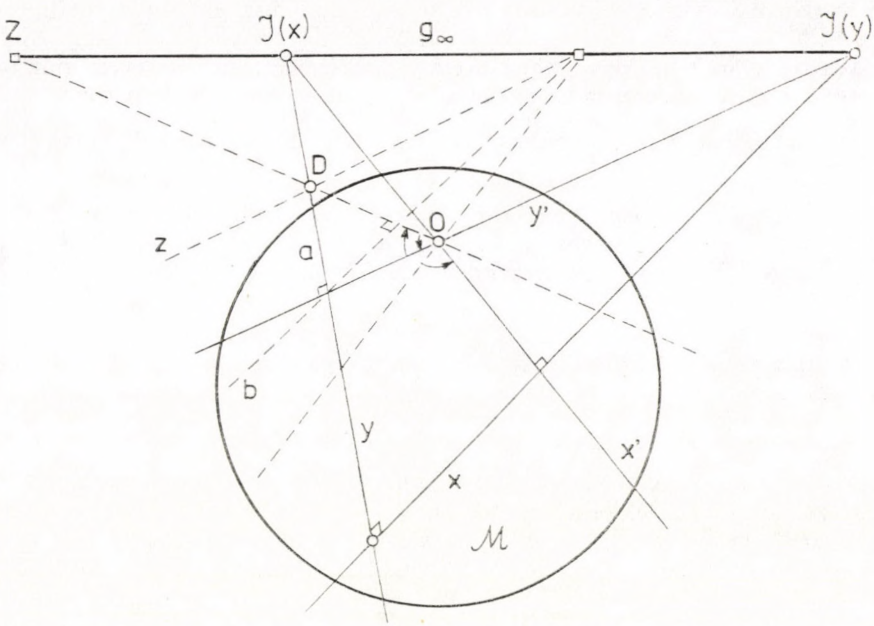


Fig. 4

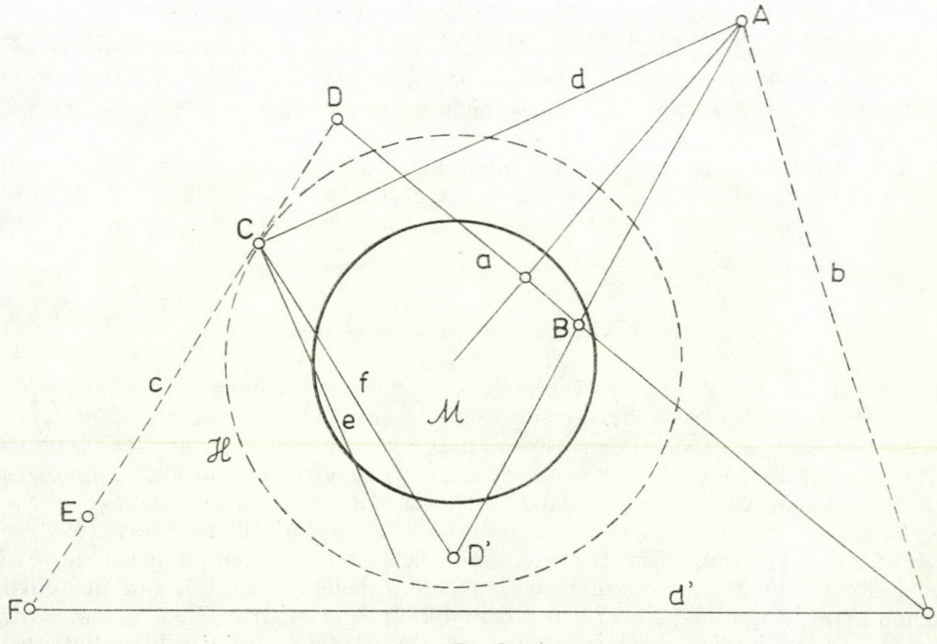


Fig. 5

4—5 stellen die möglichen Verhältnisse in dem euklidischen bzw. hyperbolischen Fall dar.

Auf der *elliptischen* $\mathcal{PM}$ gibt es kein Grenzelement, hier existieren nur innere Punkte, die nicht unbedingt eigentlich sind (wenn $\mathcal{M}$ halbelliptisch war).

Auf der Ebene $\mathcal{PM}$ gelten die folgenden Sätze, die wir ebenfalls ohne Beweise mitteilen.

SATZ 2.1. *Ist* $a\in\mathcal{S}$ *eine Spiegelung und ist* β *eine Bewegung aus* $\mathcal{G}$, *so ist auch* $a^\beta := \beta^{-1}a\beta\in\mathcal{S}$ *eine Spiegelung, deren Achse das bei* β *entstammende Bild von a ist.*

a) *Seien a, b keine Grenzgeraden. Dann besteht die Äquivalenz*

$$a\perp b \quad \text{und} \quad a, b \,\mathrm{I}\, X \Leftrightarrow ab = ba = X.$$

b) *Ist a keine Grenzgerade, doch* b^+ *eine Grenzgerade mit dem Pol* B^+, *so bestehen die Äquivalenzen*

$$a\perp b^+ \overset{\text{def}}{\Leftrightarrow} a \,\mathrm{I}\, B^+ \Leftrightarrow (b^+)^a = b^+ \Leftrightarrow (B^+)^a = B^+.$$

SATZ 2.2 (Verallgemeinerter Dreispiegelungssatz). a) *Sind* $a\neq b$ *und c keine Grenzgeraden und gilt* $a, b \,\mathrm{I}\, X$, *so besteht* $c \,\mathrm{I}\, X \Leftrightarrow abc\in\mathcal{S}$, *ferner ist* $abc =: d \,\mathrm{I}\, X$.

b) *Sind* $a\neq b$ *mit* $a, b \,\mathrm{I}\, X$ *keine Grenzgeraden, aber* c^+ *mit dem Pol* C^+ *eine Grenzgerade, so gelten*

$$c^+ \,\mathrm{I}\, X \Leftrightarrow (c^+)^{ab} = c^+ \Leftrightarrow (C^+)^{ab} = C^+.$$

c) *Ist a keine Grenzgerade, aber* $b^+\neq c^+$ *mit den Polen* B^+ *bzw.* C^+ *Grenzgeraden, für die* $b^+, c^+ \,\mathrm{I}\, X$ *gilt, so bestehen*

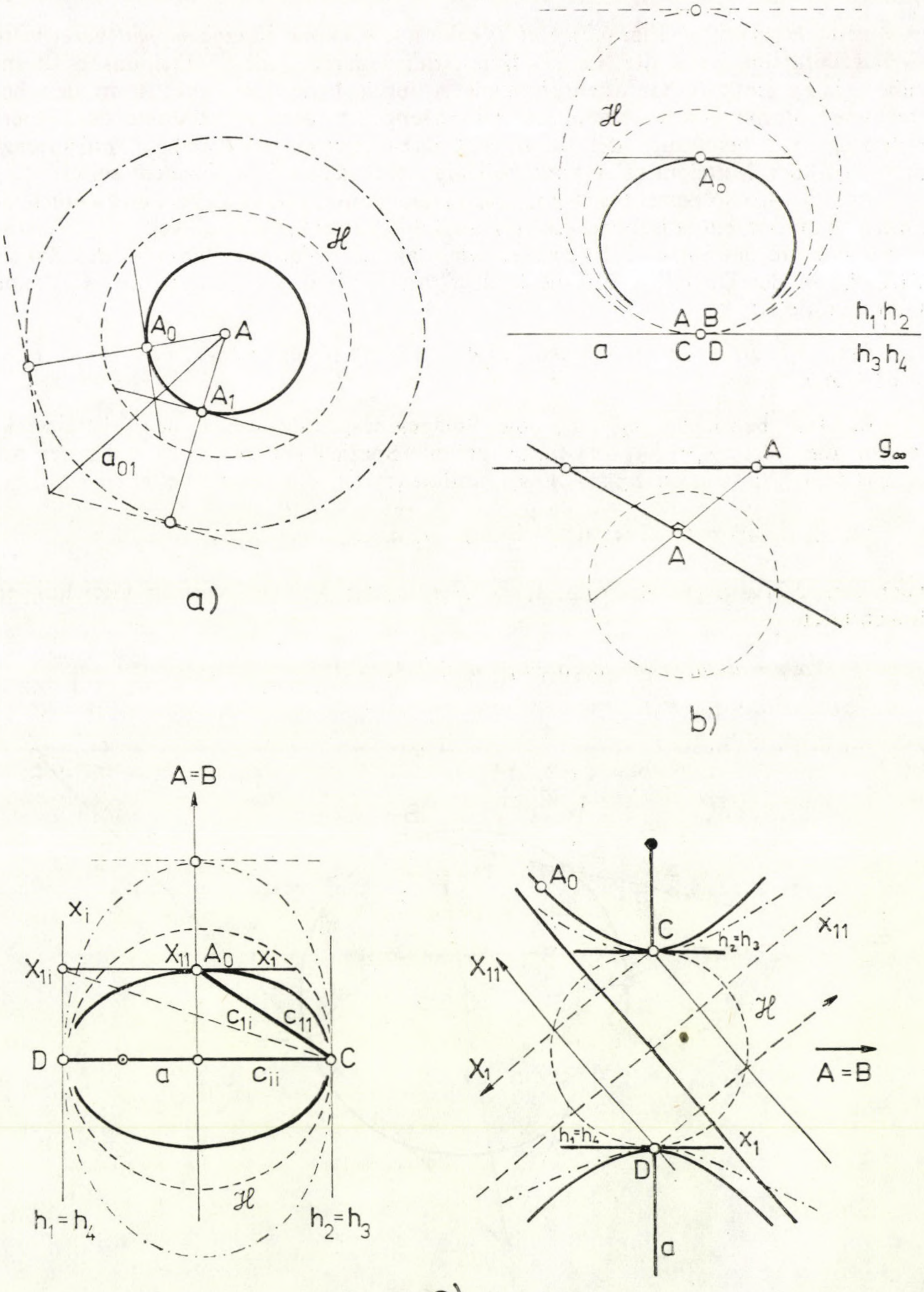
$$a \,\mathrm{I}\, X \Leftrightarrow (b^+)^a = c^+ \Leftrightarrow (B^+)^a = C^+.$$

Die Beweise beruhen auf der Theorie der projektiven Involutionen der Geraden auf der Ebene $\mathcal{PM}$, also benutzt man — nicht trivialerweise — den Papposschen Satz und das Fano-Axiom. Die Schwierigkeiten des Beweises treten in dem Fall auf, daß mindenstens eine Spiegelungsgerade nicht eigentlich ist, oder keine Polare eines eigentlichen Punktes ist. Es wäre interessant, den Satz 2.2 auf Grund der Halbdrehungstheorie unmittelbar zu beweisen. Das bezieht auch auf den Beweis des Sehnenviereckssatzes 3.2.

3. Zyklen auf der Einbettungsebene

DEFINITION 3.1. Sei ein beliebiger Punkt A auf der Ebene $\mathcal{PM}$ gegeben, ferner sei A_0 ein Punkt so, daß die Gerade (A, A_0) keine Grenzgerade sei. *Die Punkte des Zykels* $\mathcal{C}(A, A_0)$ bekommen wir so, daß wir den Punkt A_0 an den Geraden durch A spiegeln. Gehört ein Grenzgerade zu A, so wird ihr Pol auch zum Zykel $\mathcal{C}(A, A_0)$ gerechnet (Fig. 6). Der Punkt A ist der Mittelpunkt von $\mathcal{C}(A, A_0)$.

Nach der Anzahl der Grenzgeraden durch A können wir über *E-Zykel*, *P-Zykel* bzw. *H-Zykel* sprechen. Der E-Zykel kann auch *Kreis* genannt werden (Fig. 6.a). Der P-Zykel auf der hyperbolischen Ebene $\mathcal{PM}$ heißt *Parazykel*. Auf der euklidischen $\mathcal{PM}$ ist der *P*-Zykel eine mit dem Punkt ∞ erweiterte Gerade, wo ∞ die Punkte von g_∞ in sich vereinigt (Fig. 6.b). Der H-Zykel auf der hyperbolischen



A₀ A A₁ a₀₁ a)
A₀ A B C D a h₁ h₂ h₃ h₄ A g∞ A b)
A=B X₁ᵢ X₁₁ A₀ X₁ c₁ᵢ c₁₁ D a cᵢᵢ C h₁=h₄ h₂=h₃
A₀ C h₂=h₃ X₁₁ X₁₁ X₁ A=B h₁=h₄ D X₁ a
c)

Fig. 6

$\mathscr{PM}$ heißt *Hyperzykel* (Fig. 6.c). Der Zykel $\mathscr{C}(A, A_0)$ kann in eine *doppelt gerechnete Gerade* entarten, wenn der Punkt A_0 mit der Polaren a des Mittelpunktes A inzidiert, ja es gibt zwei Geradenspiegelungen durch A die den Punkt A_0 in den betrachteten Punkt von a überführen. Die Menge $\mathscr{H}$ der Grenzpunkte der hyperbolischen $\mathscr{PM}$ heißt auch der *Grenzzykel*, dazu gelangen wir, wenn A_0 ein Grenzpunkt ist; der Mittelpunkt A kann beliebig aber nicht mit a_0 inzident sein.

Ist $A=A_0$, so besteht $\mathscr{C}(A, A_0)$ aus einem Punkt, dieser Zykel heißt *entarteter Zykel*. Auf der euklidischen $\mathscr{PM}$ sei auch ∞ ein entarteter Zykel.

Mit Hilfe des Satzes 2.2 beweisen wir, daß ein beliebiger Punkt A_1 des Zykels $\mathscr{C}(A, A_0)$ in der Definition 3.1 die Rolle von A_0 spielen kann, wenn (A, A_1) keine Grenzgerade ist.

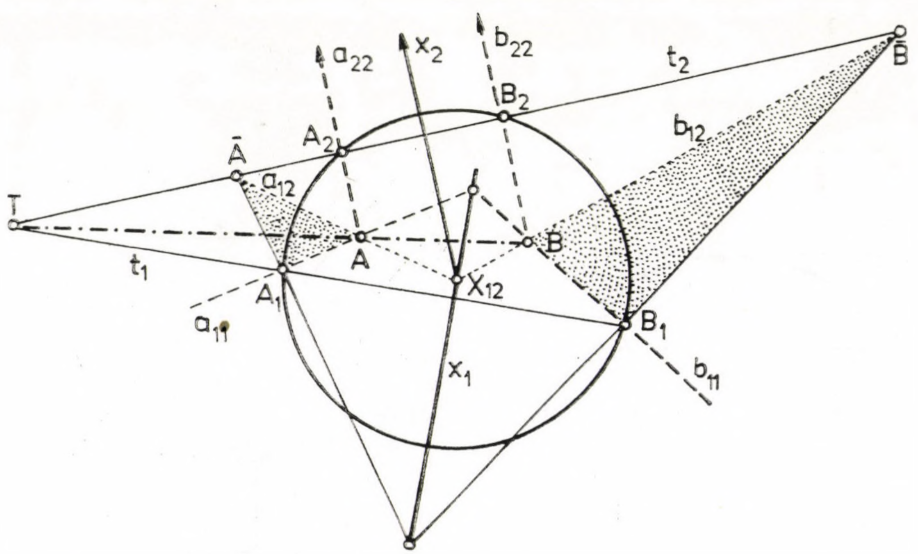
SATZ 3.1. *Ist $A_1 \in \mathscr{C}(A, A_0)$ und ist (A, A_1) keine Grenzgerade, so ist* $\mathscr{C}(A, A_0) = \mathscr{C}(A, A_1)$.

BEWEIS. Bezeichne $a_{0k}=a_{k0}$ die Spiegelungsgerade durch den Mittelpunkt A, für die $A_k := A_0^{a_{0k}}$, $A_k \in \mathscr{C}(A, A_0)$ gelten. Speziell sei $A_1 := A_0^{a_{01}}$. Ferner sei $a_{kk} := (A, A_k)$, also seien $a_{00} := (A, A_0)$ und $a_{11} := (A, A_1)$. Dann bestehen

$$A_k = A_1^{a_{10}a_{00}a_{0k}} = A_1^{a_{11}a_{10}a_{0k}} \quad \text{und} \quad a_{10}a_{00}a_{0k} = a_{11}a_{10}a_{0k} =: a_{1k} \mathrm{I} A$$

nach Satz 2.2, also gilt $A_k \in \mathscr{C}(A, A_0) \Rightarrow A_k \in \mathscr{C}(A, A_1)$. Aus den vorigen Gleichungen folgen noch

$$a_{0k} = a_{00}a_{01}a_{1k} = a_{01}a_{11}a_{1k} \quad \text{und} \quad A_k = A_0^{a_{00}a_{01}a_{1k}} = A_0^{a_{01}a_{11}a_{1k}},$$



a)

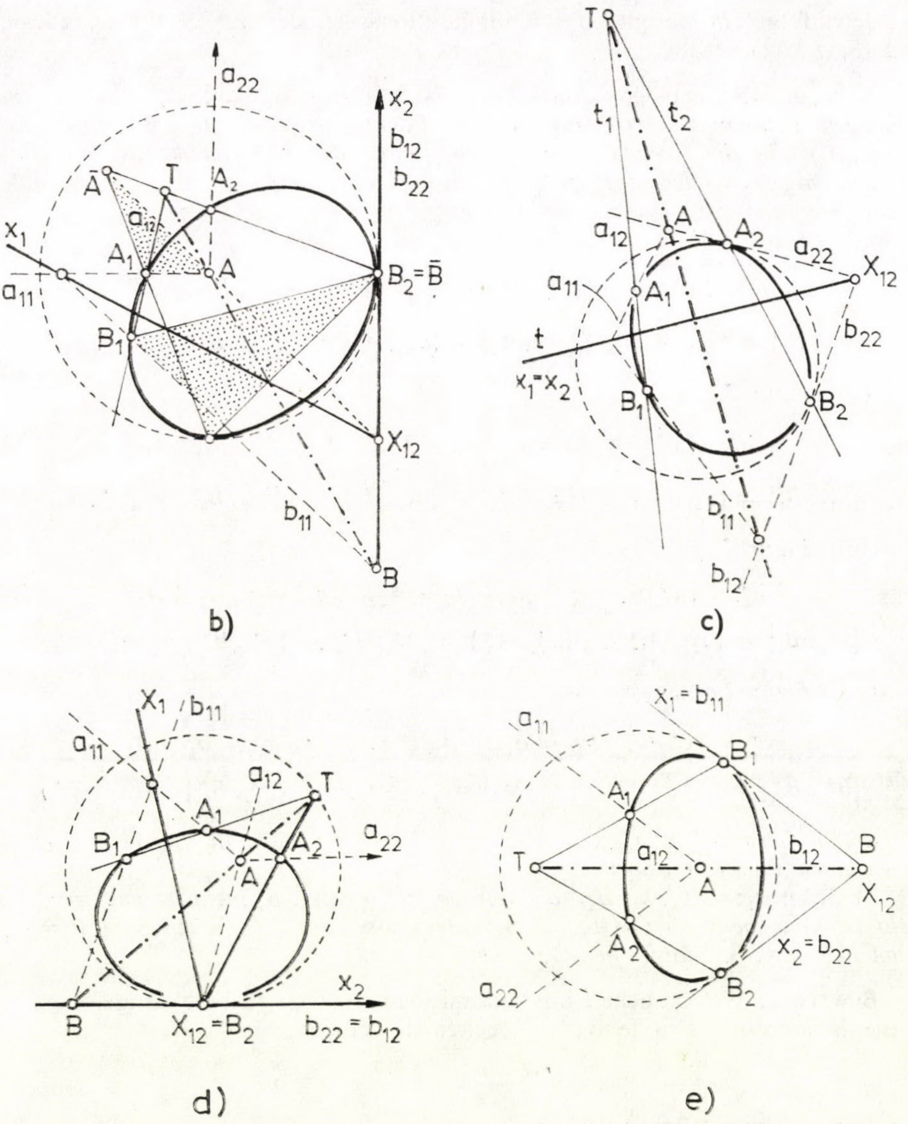


Fig. 7

also gilt auch $A_k \in \mathscr{C}(A, A_1) \Rightarrow A_k \in \mathscr{C}(A, A_0)$. Inzwischen haben wir auch Satz 2.1 angewandt:

$$a_{00}^{a_{01}} = a_{11} \Rightarrow a_{10}a_{00}a_{01} = a_{11}.$$

Wenn eine Grenzgerade mit A inzidiert, so gehört ihr Grenzpunkt sowohl zu $\mathscr{C}(A, A_0)$ als auch zu $\mathscr{C}(A, A_1)$.

Jetzt folgt ein wesentlicher Konfigurationssatz, der auf dem Desarguesschen Satz (Satz 1.11) beruht.

SATZ 3.2 (Sehnenviereckssatz (Fig. 7)). *Nehmen wir an, daß eine der nachstehenden Bedingungen* (i)—(iv) *für die Punkte* $A_1 \neq A_2$, $B_1 \neq B_2$ *eines Zykels* $\mathcal{C}(X_{12}, A_1)$, *für die Gerade* x_1, a_{12}, x_2, b_{12} *durch den Mittelpunkt* X_{12} *und für die mit den entsprechenden Zykelpunkten inzidenten Geraden* $a_{11}, b_{11}, a_{22}, b_{22}$ *auf der Ebene* $\mathcal{PM}$ *gilt:*

(i) *(Figur 7.a)*

$$b_{12} = x_1 a_{12} x_2; \quad A_1 \neq X_{12}, \quad A_1 \nmid x_{12}; \quad a_{11} \,\mathrm{I}\, A_1;$$

$$\{b_{11}, B_1\} = \{a_{11}, A_1\}^{x_1}, \quad \{a_{22}, A_2\} = \{a_{11}, A_1\}^{a_{12}}, \quad \{b_{22}, B_2\} = \{a_{22}, A_2\}^{x_2};$$

(ii) *(Figur 7.b, d)*

$$b_{12}^+ = x_2^+ = (x_2^+)^{a_{12} x_1}; \quad A_1 \neq X_{12}, \quad A_1 \nmid x_{12}; \quad a_{11} \,\mathrm{I}\, A_1;$$

$$\{b_{11}, B_1\} = \{a_{11}, A_1\}^{x_1}, \quad \{a_{22}, A_2\} = \{a_{11}, A_1\}^{a_{12}}, \quad \{b_{22}^+, B_2^+\} = \{x_2^+, X_2^+\};$$

(iii) *(Figur 7.c)*

$$b_{12}^+ = (a_{12}^+)^{x_1}, \quad x_1 = x_2; \quad A_1 \neq X_{12}, \quad A_1 \nmid x_{12}; \quad a_{11} \,\mathrm{I}\, A_1;$$

$$\{b_{11}, B_1\} = \{a_{11}, A_1\}^{x_1}, \quad \{a_{22}^+, A_2^+\} = \{a_{12}^+, A_{12}^+\}, \quad \{b_{22}^+, B_2^+\} = \{b_{12}^+, B_{12}^+\};$$

(iv) *(Figur 7.e)*

$$x_2^+ = (x_1^+)^{a_{12}}, \quad b_{12} = a_{12}; \quad A_1 \neq X_{12}, \quad A_1 \nmid x_{12}; \quad a_{11} \,\mathrm{I}\, A_1;$$

$$\{b_{11}^+, B_1^+\} = \{x_1^+, X_1^+\}, \quad \{a_{22}, A_2\} = \{a_{11}, A_1\}^{a_{12}}, \quad \{b_{22}^+, B_2^+\} = \{x_2^+, X_2^+\}.$$

a) *Die Geraden* a_{11}, a_{12}, a_{22} *haben einen Punkt* A *gemein, die Geraden* b_{11}, b_{12}, b_{22} *haben einen Punkt* B *gemein.*

b) *Seien* $t_1 := (A_1, x_1, B_1)$ *die Gerade, die mit* A_1, B_1 *inzidiert und zu* x_1 *senkrecht ist und ebenso* $t_2 := (A_2, x_2, B_2)$; *ferner sei deren Schnittpunkt* $T := [t_1, t_2]$. *Dann liegen* A, B, T *auf einer Geraden.*

BEWEIS. a) Wir brauchen uns nur mit dem Fall (i) zu beschäftigen, sonst gilt ja die Behauptung a) offenbar. Es gelten die Gleichungen

$$a_{22} = a_{11}^{a_{12}}, \quad b_{22} = b_{11}^{x_1 a_{12} x_2} = b_{11}^{b_{12}}$$

und daraus folgt die Behauptung.

b) Erstens nehmen wir $A_1 \neq B_1$ an. Wir können auch $A, B \nmid t_1, t_2$ voraussetzen. Im Falle (i) seien $\bar{A} := [t_2, a_{12}]$, $\bar{B} := [t_2, b_{12}]$. Dann gehen die Geraden $(\bar{A}, A_1) = t_2^{a_{12}}$ und $(\bar{B}, B_1) = t_2^{b_{12}}$ bei der Spiegelung x_1 ineinander über, da $(t_2^{a_{12}})^{x_1} = t_2^{x_2 a_{12} x_1} = t_2^{b_{12}}$ gilt. Das bedeutet, daß die Dreiecke $A\bar{A}A_1$ und $B\bar{B}B_1$ in bezug auf die Gerade x_1 perspektiv sind. Sind $\bar{A} \neq \bar{B}$, $A \neq B$, so folgt die Behauptung aus dem Satz von Desargues (Satz 1.11). Ist $\bar{A} = \bar{B}$, d. h. $a_{12} = b_{12} = x_1 a_{12} x_2$, $x_2 = x_1^{a_{12}}$ (Satz 3.1), so gilt die Behauptung wegen $t_2 = t_1^{a_{12}}$ und $A, B \,\mathrm{I}\, a_{12}$. Der Fall $A = B$ trifft ein, wenn $A_1 = B_1$ und $A_2 = B_2$ sind.

Im Falle (ii) nehmen wir erstens an, daß der Zykelmittelpunkt X_{12} ein äußerer ist (Fig. 7.b). Der vorige Beweis gilt auch jetzt, aber hier sind $\bar{B}=B_2$, $(B_2^+, B_1)= =(B_2^+, A_2)^{a_{12}x_1}$.

Zweitens nehmen wir an, daß $X_{\overline{12}}^+$ ein Grenzpunkt ist (Fig. 7.d), ferner nehmen wir indirekt an, daß die Geraden (A, B) und $t_1=(A_1, B_1)$ einander in einem von T und A verschiedenen Punkte $\bar{T}$ schneiden. Die Gerade $(\bar{T}, A_2)$ schneidet den Zykel $\mathscr{C}(X_{12}, A_1)$ in inem zweiten Punkt $\bar{B}_2$, der von A_2 zufällig nicht verschieden ist: $\bar{B}_2:=A_2^{\bar{x}_2}$, wo $\bar{x}_2 \,\mathrm{I}\, X_{12}$ und $\bar{x}_2 \perp (\bar{T}, A_2)$ sind. $\bar{B}_2$ ist kein Grenzpunkt mehr. Dann inzidiert aber der Schnittpunkt $\bar{B}$ von b_{11} und $\bar{b}_{22}:=a_{\overline{22}}^{\bar{x}_2}$ mit der Geraden $(A, B, \bar{T})$ nicht, was ein Widerspruch zur Behauptung im Falle (i) ist.

In den Fällen (iii) und (iv) gelten die Behauptungen offensichtlich (Fig. 7.c, e).

Die Möglichkeit $A_1=B_1$ bereitet nur im Falle (ii) eine gewisse Schwierigkeit. Dann führen wir mit der bereits beschriebenen indirekten Methode einen von B_1 verschiedenen Zykelpunkt ein, und bekommen einen Widerspruch zu einer der vorigen Behauptungen (eventuell mit Indextausch $1 \leftrightarrow 2$).

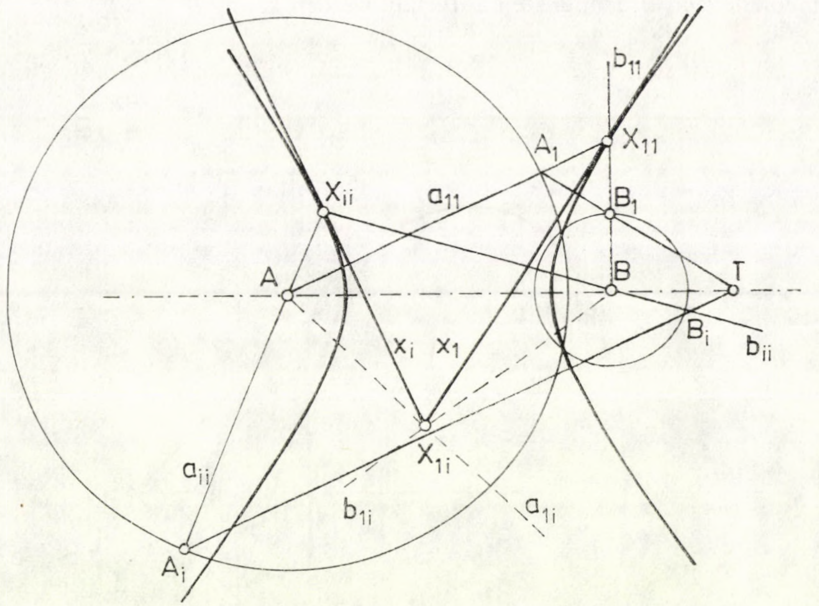


Fig. 8

4. Brennpunkte eines Kegelschnittes

DEFINITION 4.1. Auf der Ebene $\mathscr{PM}$ betrachten wir die Punkte A, B und die Gerade x_1. Wir nehmen an, daß $A, B \not\mathrm{I}\, x_1$ ist, x_1 keine Grenzgerade und $A^{x_1} \neq B$ sind. Sei X_{11} der Schnittpunkt der Geraden $a_{11}:=(A, B^{x_1})$ und $b_{11}:=(B, A^{x_1})$ auf x_1; hier sind a_{11} und b_{11} die *Leitradien* zu x_1 und X_{11}. Die Gerade x_1 ist so gegeben, daß weder a_{11} noch b_{11} Grenzgerade sei (Fig. 8).

Wir definieren zu jedem Punkt X_{1i} der Geraden x_1 eine Gerade x_i, Leitradien a_{ii} und b_{ii} einen Punkt X_{ii} mit Hilfe der Geraden $a_{1i}:=(X_{1i}, A)$ und $b_{1i}:=(X_{1i}, B)$; die so definierten Geraden x_i, bzw. Punkten X_{ii} bilden den *Linienkegelschnitt* $l(x_1, A, B)$, bzw. den *Punktkegelschnitt* $p(x_1, A, B)$. Die Gesamtheit von l und p wird *Kegelschnitt* genannt.

a) Sind a_{1i} und b_{1i} keine Grenzgeraden (Fig. 8), so definiert die Gleichung $x_1:=a_{1i}x_1b_{1i}$ die Gerade x_i. $a_{ii}:=a_{11}^{a_{1i}}$ und $b_{ii}:=b_{11}^{b_{1i}}$ sind die Leitradien, und es sei $X_{ii}:=[a_{ii}, b_{ii}]$ der Schnittpunkt von a_{ii} und b_{ii}.

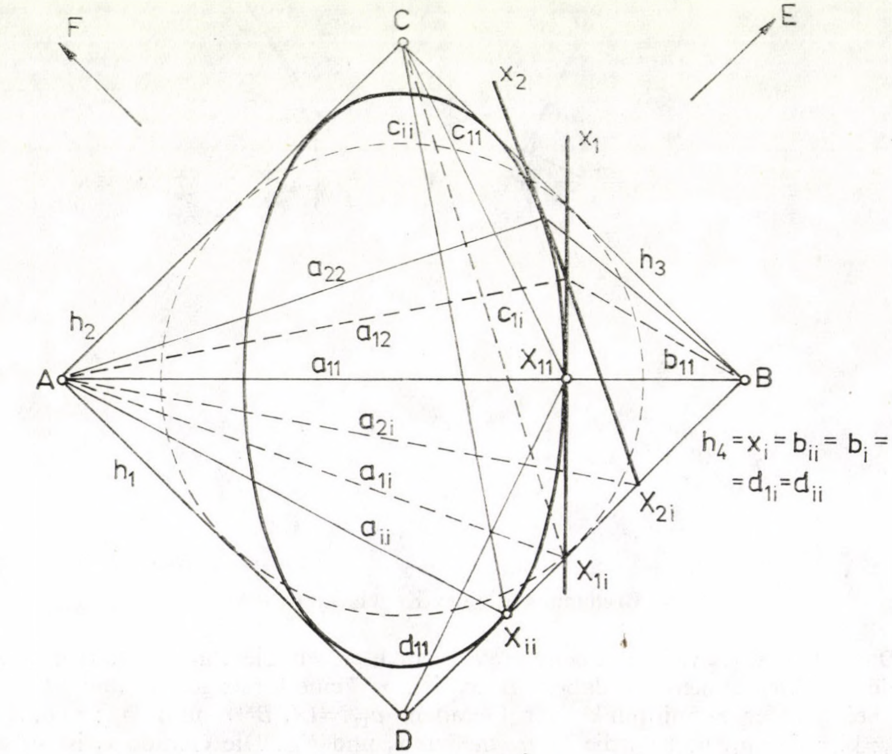
b) Sind, zum Beispiel, a_{1i} keine Grenzgerade aber b_{1i}^+ eine Grenzgerade (Fig. 9.a), so seien $x_i^+:=b_{1i}^+=:b_{ii}^+$, $a_{ii}:=a_{11}^{a_{1i}}$, $X_{ii}:=[a_{ii}b_{ii}^+]$.

c) Sind $a_{1i}^+=b_{1i}^+$ zusammenfallende Grenzgeraden mit den Polen $A_{1i}^+=B_{1i}^+$, so seien $x_i^+:=a_{1i}^+=b_{1i}^+=:a_{ii}^+=:b_{ii}^+$, $X_{ii}^+:=A_{1i}^+=B_{1i}^+$ (Fig. 9.b).

Wir nehmen an: die Gerade x_1 sei so gegeben, daß der Fall nicht vorkomme, in dem a_{1i}^+, b_{1i}^+ Grenzgeraden sind und $a_{1i}^+\neq b_{1i}^+$ gilt.

Die Punkte A und B heißen die *Brennpunkte des Kegelschnittes.*

Man sieht, daß ein Zykel (Definition 3.1) als ein Punktkegelschnitt mit zusammenfallenden Brennpunkten aufgefaßt werden kann.



a)

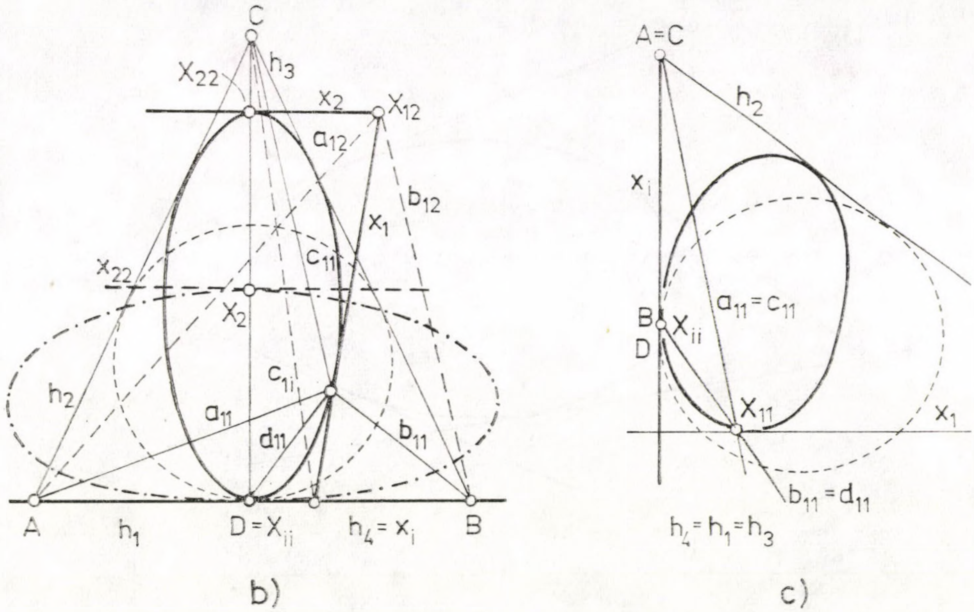


Fig. 9

SATZ 4.1. *Die Gerade x_i des Linienkegelschnittes $l(x_1, A, B)$ inzidiert mit dem entsprechenden Punkt X_{ii} des Punktkegelschnittes $p(x_1, A, B)$. Es gelten die folgenden Behauptungen:*

a) $a_{11}^{x_1}=b_{11}$.

b) *Sind a_{1i}, b_{1i} keine Grenzgeraden, so gilt* $a_{ii}^{x_i}=b_{ii}$.

BEWEIS. a) $a_{11}^{x_1}=(A, B^{x_1})^{x_1}=(A^{x_1}, B)=b_{11}$.

b) $a_{11}^{x_i}=a_{ii}^{a_{1i}x_1b_{1i}}=a_{11}^{x_1b_{1i}}=b_{11}^{b_{1i}}=b_{ii}$.

Daraus folgt, daß sich die Geraden a_{11}, x_1, b_{11} im Punkt X_{11}, a_{ii}, x_i, b_{ii} in X_{ii} schneiden.

Ist x_i^+ eine Grenzgerade, so folgt X_{ii} I x_i^+ aus der Definition 4.1.

Jetzt zeigen wir, daß die Rolle der Geraden x_1 in Definition 4.1 nicht ausgezeichnet ist. Die Verallgemeinerung des Gegenpaarungssatzes von HESSENBERG ([1], §. 4, 8 und §. 5, 6) führt dazu, daß der Linienkegelschnitt ein projektives Gebilde ist.

SATZ 4.2. *Es seien x_2 keine Grenzgerade und $x_2 \in l(x_1, A, B)$. Dann gelten die folgenden Behauptungen:*

a) $l(x_1, A, B)=l(x_2, A, B)$; $p(x_1, A, B)=p(x_2, A, B)$.

b) *Unter den Geraden, die mit dem Brennpunkt A inzident sind, ist die Abbildung $a_{1i}\to a_{2i}$ (im Sinne von Definition 4.1) eineindeutig und projektiv. Die gleiche Behauptung ist für die zum Brennpunkt B gehörige Abbildung $b_{1i}\to b_{2i}$ gültig* (Fig. 10).

c) *Die Abbildung $X_{1i}\to X_{2i}$ der Geraden $x_1=\{X_{11}, X_{12}, \dots, X_{1i}, \dots\}$ auf die Gerade $x_2=\{X_{21}, X_{22}, \dots, X_{2i}, \dots\}$ ist projektiv. Der Linienkegelschnitt $l(x_1, A, B)$ ist das Gebilde der projektiven Punktreihen x_1 und x_2.*

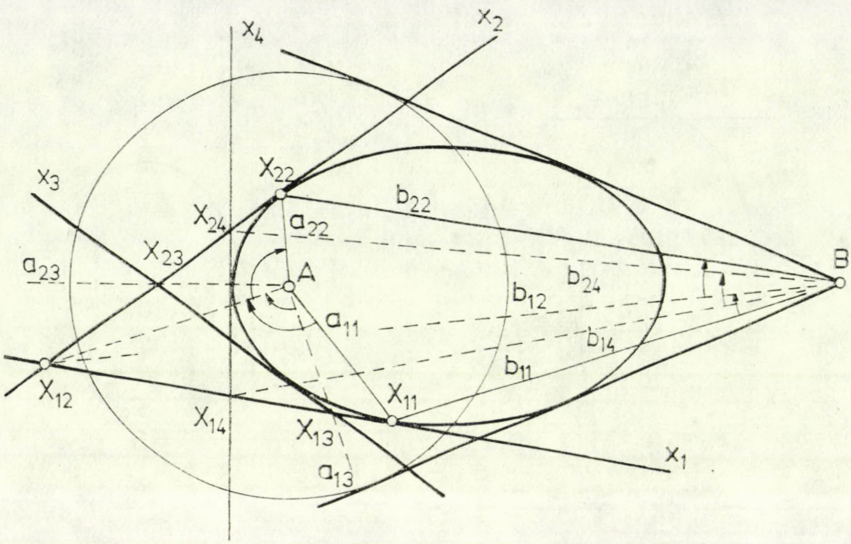


Fig. 10

BEWEIS. a) In Satz 4.1 haben wir gezeigt, daß die Gleichungen $a_{11}a_{12}=a_{21}a_{22}$, $x_1a_{11}x_1=b_{11}$, $x_1a_{12}x_2=b_{12}$, $x_2a_{22}x_2=b_{22}$, $b_{11}b_{12}=b_{21}b_{22}$ gelten. Nehmen wir an, daß $x_i \in l(x_2, A, B)$ ist. Wir werden beweisen, daß auch $x_i \in l(x_1, A, B)$ ist. Die Umkehrung wird trivial sein.

(i) Sei $x_i := a_{2i}x_2b_{2i}$ mit a_{2i} I A und b_{2i} I B, ferner führen wir die Geraden $a_{1i} := a_{11}a_{12}a_{2i}$ und $b :_{1i} = b_{11}b_{12}b_{2i}$ ein. Hier gelten a_{1i} I A und b_{1i} I B nach Satz 2.2.a. Dann ist $a_{1i}x_1b_{1i}=(a_{2i}a_{12}a_{11})x_1(b_{11}b_{12}b_{2i})=a_{2i}(a_{12}x_1b_{12})b_{2i}=a_{2i}x_2b_{2i}=x_i$. Aus Satz 2.2.a folgt, daß die Geraden x_1, a_{1i}, b_{1i}, x_i mit einem einzigen Punkt X_{1i} inzidieren, was zuerst zu beweisen war.

Andererseits gilt $a_{1i}a_{11}a_{1i}=(a_{2i}a_{12}a_{11})a_{11}(a_{11}a_{12}a_{2i})=a_{2i}a_{22}a_{2i}=a_{ii}$ und ebenso gilt $b_{1i}b_{11}b_{1i}=b_{2i}b_{22}b_{2i}=b_{ii}$; also bekommen wir denselben Punkt X_{ii} und dieselben Leitradien a_{ii}, b_{ii} ob aus x_2 oder aus $x_1 \in l(x_2, A, B)$ definiert.

(ii) Sei z.B. $x_i^+=b_{2i}^+$ mit den Bedingungen $x_i^+=(b_{2i}^+)^{x_2a_{2i}}$, a_{2i} I A, b_{2i}^+ I B (Satz 2.2.b). Ferner führen wir die Geraden $a_{1i} := a_{11}a_{12}a_{2i}$ und $b_{1i}^+ := b_{2i}^+ = x_i^+ =: b_{ii}^+$ mit a_{1i} I A, b_{1i}^+ I B ein.

$$(b_{1i}^+)^{x_1a_{1i}} = (b_{2i}^+)^{(b_{21}b_{11})x_1(a_{11}a_{12}a_{2i})} = (b_{2i}^+)^{x_2a_{2i}} = x_i^+$$

zeigt nach Satz 2.2.b, daß die Geraden $x_1, a_{1i}, x_i^+=b_{1i}^+$ mit einem einzigen Punkt X_{1i} inzidieren. Ebenso wie im Falle (i) folgt $a_{ii}=a_{1i}a_{11}a_{1i}=a_{2i}a_{22}a_{2i}$, woraus sich die Behauptung ergibt.

(iii) Sei $x_i^+=b_{2i}^+=a_{2i}^+$ (Fig. 9. b). Aus der Definition 4.1 folgt $x_i^+=a_{1i}^+=b_{ii}^+=$ $=a_{ii}^+=b_{ii}^+$, also ist $X_{ii}^+=X_i^+$ der Pol von x_i^+.

b) Im vorigen Beweis der Behauptung a) haben wir gezeigt, daß die Abbildung $a_{1i} \to a_{2i}$ durch die Gleichungen

$$a_{1i}a_{11}a_{12} = a_{2i} \quad \text{bzw.} \quad (a_{1i}^+)^{a_{11}a_{12}} = a_{2i}^+ = a_{1i}^+$$

gegeben ist. Die erste Gleichung kann in die Form $a_{11}(a_{11}a_{1i}a_{11})a_{12}=a_{2i}$ geschrieben werden; daraus können wir entnehmen, daß die Abbildung $a_{1i}\to a_{2i}$ die Komposition der Spiegelung $x\to x^{a_{11}}:=a_{11}xa_{11}$ und der Gegenpaarung $y\to a_{11}ya_{12}$ ist. Beide Abbildungen sind im Büschel durch A erklärt und auf die zufälligen Grenzgeraden so erstreckt, daß eine Grenzgerade durch A sowohl bei der Spiegelung als auch bei der Gegenpaarung in die andere Grenzgerade durch A übergehe, wo diese letztere mit der ersten zusammenfallen kann. Die Spiegelung ist offenbar eine projektive Abbildung. Der folgende Satz zeigt, daß auch die Gegenpaarung projektiv ist. Also ist die Komposition eine projektive Abbildung.

c) Diese Behauptung folgt aus der vorigen.

SATZ 4.3 (Verallgemeinerung des Gegenpaarungssatzes von Hessenberg). *Seien die Geraden u und v durch den Punkt A so gegeben, daß u, v keine Grenzgeraden sind. Für x I A sei $y:=uxv$, wenn x keine Grenzgerade ist, und sei $y^+:=(x^+)^u=(x^+)^v$, wenn x^+ eine Grenzgerade ist. Die so definierte Gegenpaarung $x\to y$ des Büschels A ist eine projektive Involution.*

Der Beweis zeigt, daß der Satz als eine Beziehung zwischen denjenigen Kegelschnitten interpretiert werden kann, welche den Brennpunkt A und zwei Linien x_1 und x_2 gemein haben (Fig. 11).

Sei P_2 ein innerer Punkt, der weder mit $a_{ik}:=u$ noch mit $a_{12}:=v$ inzidiert. Seien die Geraden $a_{2i}:=(P_2, A)$, $a_{1k}:=ua_{2i}v$, die Punkte P_1 I a_{1k}, X_{12} I $v(P_1, X_{12}\neq A)$ und die Geraden $x_1:=(P_1, X_{12})$, $x_2:=(P_2, X_{12})$ so festgesetzt, daß x_1 und x_2 verschieden, ferner keine Grenzgeraden sind. Außerdem bestehe $A^{x_1} \nmid b_{12}:=x_1a_{12}x_2$.

Mit einem beliebigen Punkt X_{1k} auf u betrachten wir die Geraden $x_i:=(P_2, X_{ik})$ und $x_k:=(P_1, X_{ik})$, die Schnittpunkte $X_{1i}:=[x_1, x_i]$ und $X_{2k}:=[x_2, x_k]$, ferner die Geraden $a_{1i}:=(A, X_{1i})$ und $a_{2k}:=(A, X_{2k})$.

Wir sollen zeigen, daß die Gegenpaarung die Geraden $x:=a_{1i}$ und $y:=a_{2k}$ einander zuordnet. Dann bekommen wir unter den Büscheln A, P_2, P_1, A die Perspektivitäten

$$A(a_{1i}) \xrightarrow{x_1} P_2(x_i) \xrightarrow{u=a_{ik}} P_1(x_k) \xrightarrow{x_2} A(a_{2k})$$

mit den Achsen x_1, u, x_2. Diese Perspektivitätskette beweist, daß die Gegenpaarung projektiv ist. Zugleich ist klar, daß sie involutorisch ist.

(i) Erstens beweisen wir, daß x_1, x_2, x_i, x_k Elemente eines Linienkegelschnittes $l(x_2, A, B)$ sind. Seine Brennpunkte sind A und der Schnittpunkt $B:=[b_{12}, b_{2i}]$, wo $b_{12}:=x_1a_{12}x_2$ und $b_{2i}:=x_2a_{2i}x_i$ sind (Satz 2.2.a).

Da P_2 ein innerer Punkt is, kann x_i niemals Grenzgerade sein, das vereinfacht die Diskussion.

Sei x_k keine Grenzgerade (Fig. 11.a, b ohne Strich). Aus den Gleichungen

$$(1)\qquad \begin{cases} b_{ki} := x_k a_{ki} x_i, \quad b_{i2} := x_i a_{i2} x_2, \quad b_{21} := x_2 a_{21} x_1, \\ a_{k1} := a_{ki} a_{i2} a_{21}, \quad b_{k1} := x_k a_{k1} x_1 \end{cases}$$

folgt (Satz 2.2.a)

$$b_{ki} b_{i2} b_{21} = x_k a_{ki} a_{i2} a_{21} x_1 = x_k a_{k1} x_1 = b_{k1}, \quad \text{also} \quad b_{ki}, b_{k1} \text{ I } B.$$

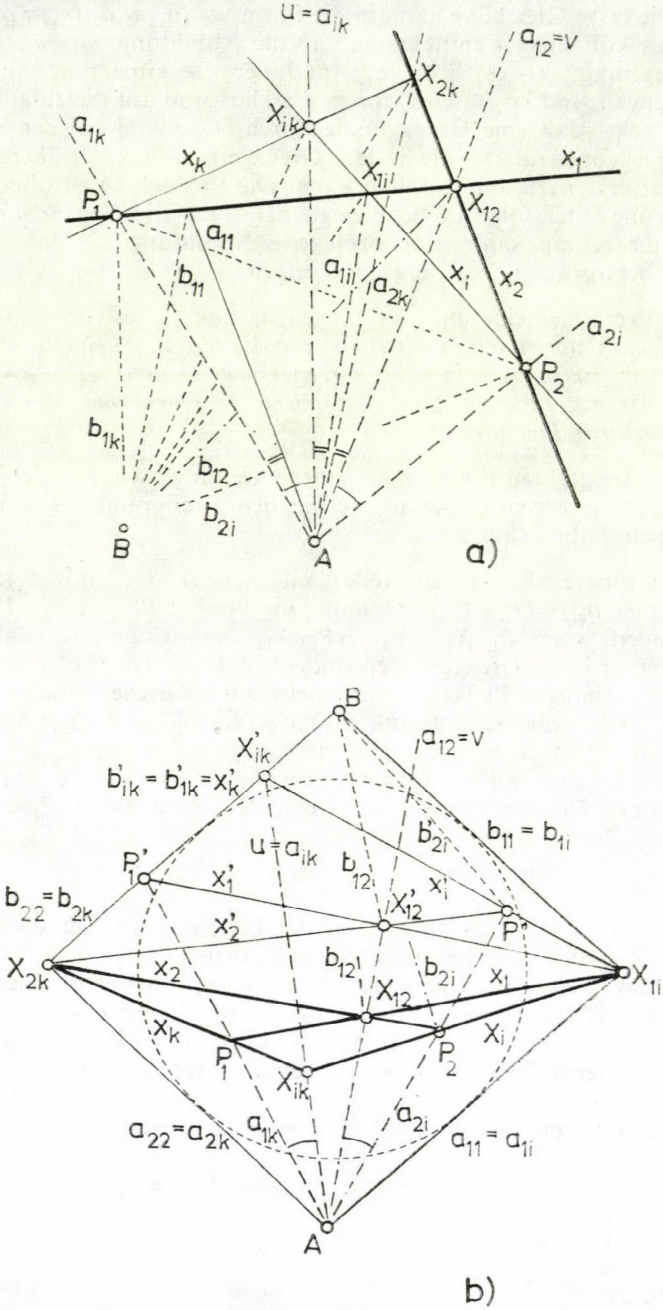

u = a_ik
a_12=v
X_2k
X_ik
a_1k
x_k
X_1i
x_1
P_1
X_12
a_11
b_11
a_1i
a_2k
x_i
x_2
a_2i
P_2
b_1k
b_12
b_2i
B̊
A
a)
B
X'_ik
a_12=v
b'_ik = b'_1k=x'_k
b'_2i
b_11 = b_1i
u = a_ik
b_12
P'_1
x'_1
x'_i
b_22=b_2k
X'_12
x'_2
P'
X_2k
b_12
b_2i
x_2
x_1
X_1i
X_12
x_k
x_i
P_1
P_2
X_ik
a_22=a_2k
a_1k
a_2i
a_11 = a_1i
A
b)

Fig. 11

Sei x_k^+ Grenzgerade (Fig. 11.b mit Strich). Dann haben wir

$$(1') \qquad \begin{aligned} &b_{ki}^+ := x_k^+ = (x_k^+)^{a_{ki}x_i}, \quad b_{i2} := x_i a_{i2} x_2, \quad b_{21} := x_2 a_{21} x_1, \\ &a_{k1} := a_{ki} a_{i2} a_{21}, \quad b_{k1}^+ := x_k^+ = (x_k^+)^{a_{k1}x_1} \end{aligned}$$

woraus

$$(b_{ki}^+)^{b_{i2}b_{21}} = (x_k^+)^{a_{ki}a_{i2}a_{21}x_1} = (x_k^+)^{a_{k1}x_1} = x_k^+ = b_{ki}^+ = b_{k1}^+,$$

also $x_k^+ = b_{ki}^+ = b_{k1}^+ \mathrm{I}\, B$ folgt (Satz 2.2.a).

Da $A^{x_1} \nmid b_{12}$ gilt, so existieren $b_{11} := (B, A^{x_1})$ und $a_{11} := (A, B^{x_1})$ eindeutig und besteht $a_{11}^{x_1} = b_{11}$.

(ii) Wenn a_{11} und b_{11} keine Grenzgeraden sind (Fig. 11.a), so können wir die Methode im Beweis von Satz 4.2.a anwenden. $a_{1i} := a_{11} a_{12} a_{2i}$ und $b_{1i} := b_{11} b_{12} b_{2i}$ sind keine Grenzgeraden, ferner gilt $a_{1i} x_1 b_{1i} = x_i$.

Sei x_k keine Grenzgerade (Fig. 11.a). Aus den Gleichungen

$$(2) \qquad \begin{cases} b_{ki} := x_k a_{ki} x_i, \quad b_{i1} = x_i a_{i1} x_1, \quad b_{12} := x_1 a_{12} x_2, \\ a_{k2} := a_{ki} a_{i1} a_{12}, \quad b_{k2} := b_{ki} b_{i1} b_{12} \end{cases}$$

folgt

$$x_k a_{k2} x_2 = x_k a_{ki} a_{i1} a_{12} x_2 = b_{ki} b_{i1} b_{12} = b_{k2}.$$

Sei x_k^+ eine Grenzgerade. Aus den Gleichungen

$$(2') \qquad \begin{cases} b_{ki}^+ := x_k^+ = (x_k^+)^{a_{ki}x_i}, \quad b_{i1} = x_i a_{i1} x_1, \quad b_{12} := x_1 a_{12} x_2, \\ a_{k2} := a_{ki} a_{i1} a_{12}, \quad b_{k2}^+ := x_k^+ = (b_{ki}^+)^{b_{i1}b_{12}} \end{cases}$$

folgt

$$(x_k^+)^{a_{k2}x_2} = (x_k^+)^{a_{ki}a_{i1}a_{12}x_2} = (b_{ki}^+)^{b_{i1}b_{12}} = x_k^+ = b_{k2}^+.$$

Nach Satz 2.2 bekommen wir in beiden Fällen, daß $a_{1i} \mathrm{I}\, X_{1i}$, $a_{2k} \mathrm{I}\, X_{2k}$ und $a_{2k} = u a_{1i} v$ gelten, was zu beweisen war.

(iii) Nehmen wir an, daß a_{11}^+, b_{11}^+ Grenzgeraden sind (Fig. 11.b). Für die Grenzgeraden $a_{22}^+ := (a_{11}^+)^{a_{12}}$ und $b_{22}^+ := (b_{11}^+)^{b_{12}}$ gilt

$$(a_{22}^+)^{x_2} = (a_{22}^+)^{a_{12}x_1b_{12}} = (a_{11}^+)^{x_1b_{12}} = (b_{11}^+)^{b_{12}} = b_{22}^+.$$

Nach Satz 2.2.c gilt

$$(a_{11}^+)^{x_i} = (a_{11}^+)^{a_{2i}x_2b_{2i}} = (a_{22}^+)^{x_2b_{2i}} = (b_{22}^+)^{b_{2i}} = b_{11}^+,$$

und es folgen $a_{1i}^+ := a_{11} \mathrm{I}\, X_{1i}$ und $b_{1i}^+ := b_{11} \mathrm{I}\, X_{1i}$.

Sei $x_k = a_{ki} x_i b_{ki}$ keine Grenzgerade (Fig. 11.b ohne Strich). Ebenso folgen $a_{2k}^+ := a_{22}^+ \mathrm{I}\, X_{2k}$ und $b_{2k}^+ := b_{22}^+ \mathrm{I}\, X_{2k}$ aus dem vorigen Gedankengang.

Sei x_k^+ Grenzgerade (Fig. 11.b mit Strich). Da x_i keine Grenzgerade ist, so kann wegen (1′) nur $x_k^+ = b_{22}^+$ bestehen. Also inzidieren $a_{2k}^+ := a_{22}^+$ und $b_{2k}^+ := b_{22}^+$ mit X_{2k}, darum gilt

$$(a_{1i}^+)^u = (a_{1i}^+)^v = a_{2k}^+, \quad \text{was zu beweisen war.}$$

BEMERKUNG. Der letzte Fall im vorigen Beweis zeigt, wie ein entarteter Linienkegelschnitt in Definition 4.1 auftreten kann, wenn wir ein Beschränkung dort weglassen.

SATZ 4.4 (Äquivalenz der Brennpunktpaare). *Betrachten wir ein Vierseit aus den Grenzgeraden* h_1^+, h_2^+, h_3^+, h_4^+ *bestehend und die Schnittpunkte* $A:=[h_1^+, h_2^+]$, $B:=[h_3^+, h_4^+]$, *wo man zum Beispiel im Falle* $h_1^+=h_2^+$ *den Grenzpunkt* $A^+:=H_1^+=$ $=H_2^+$ *nimmt. In der Definition des Kegelschnittes können zwei beliebige Gegenecken des Brennpunktvierseits die Rollen der Brennpunkte* A, B *übernehmen: Wenn* $C:=[h_3^+, h_2^+]$ *und* $D:=[h_1^+, h_4^+]$ *sind, so gelten* $l(x_1, A, B)=l(x_1, C, D)$ *und* $p(x_1, A, B)=p(x_1, C, D)$ (Fig. 9).

BEWEIS. Für X_{1i} I x_1 seien die Verbindungsgeraden mit A, B, C, D respektive $a_{1i}, b_{1i}, c_{1i}, d_{1i}$. Speziell für X_{11} seien diese Verbindungsgeraden $a_{11}, b_{11}, c_{11}, d_{11}$.

(1) Wenn $x_i:=a_{1i}x_1b_{1i}$ keine Grenzgerade ist, so sind weder c_{1i} noch d_{1i} Grenzgeraden. Es gilt $(h_1^+)^{a_{1i}c_{1i}b_{1i}}=(h_2^+)^{c_{1i}b_{1i}}=(h_3^+)^{b_{1i}}=h_4^+$; daraus folgt $a_{1i}c_{1i}b_{1i}=d_{1i}$ nach Satz 2.2.a. Also besteht

$$c_{1i}x_1d_{1i} = c_{1i}x_1(a_{1i}c_{1i}b_{1i}) = c_{1i}(x_1a_{1i}c_{1i})b_{1i} =$$

$$= c_{1i}(c_{1i}a_{1i}x_1)b_{1i} = a_{1i}x_1b_{1i} = x_i.$$

Speziell gilt $a_{11}c_{11}b_{11}=d_{11}$, ferner nach Satz 4.1 folgt auch $c_{11}x_1d_{11}=x_1$. Für X_{ii} ergeben sich $a_{ii}c_{ii}b_{ii}=d_{ii}$ und $c_{ii}x_id_{ii}=x_i$ mit Hilfe der vorigen Methode. Das entspricht der Behauptung des Satzes.

(2) Wenn x_1^+ eine Grenzgerade ist, zum Beispiel $x_i^+=h_4^+$, so haben wir die Fälle (i)—(iii) zu untersuchen. Wir sollen die Gleichung

$$[a_{ii}, b_{ii}] =: X_{ii} = \overline{X}_{ii} := [c_{ii}, d_{ii}]$$

beweisen.

(i) Wenn $h_4^+\neq h_1^+$ und $h_4^+\neq h_3^+$ gelten, so sind a_{1i}, c_{1i} keine Grenzgeraden, doch besteht $b_{1i}^+=d_{1i}^+=h_4^+$ (Fig. 9.a).

Dann sind $a_{ii}=a_{1i}a_{11}a_{1i}$, $c_{ii}=c_{1i}c_{11}c_{1i}$, $X_{ii}=[a_{ii}, h_4^+]$, $\overline{X}_{ii}=[c_{ii}, h_4^+]$ gültig (Satz 4.1). Nach Satz 2.2 ergibt sich $X_{ii} = \overline{X}_{ii}$, d. h. $(h_4^+)^{a_{ii}c_{ii}} = h_4^+$, wie folgt:

$$(h_4^+)^{a_{ii}c_{ii}} = (h_4^+)^{(a_{1i}a_{11}a_{1i})(c_{1i}c_{11}c_{1i})} = (h_4^+)^{(a_{1i}x_1b_{11}x_1a_{1i})(c_{1i}x_1d_{11}x_1c_{1i})} =$$

$$= (h_4^+)^{(c_{1i}x_1)b_{11}(x_1a_{1i}c_{1i})x_1d_{11}(x_1c_{1i})} = (h_4^+)^{b_{11}c_{1i}a_{1i}d_{11}(x_1c_{1i})} =$$

$$= (h_3^+)^{c_{1i}a_{1i}d_{11}(x_1c_{1i})} = (h_2^+)^{a_{1i}d_{11}(x_1c_{1i})} = (h_1^+)^{d_{11}(x_1c_{1i})} = (h_4^+)^{x_1c_{1i}} = h_4^+.$$

(ii) Wenn $h_4^+=h_1^+$ aber h_3^+ von ihnen verschieden ist, so ist c_{1i} keine Grenzgerade (Fig. 6.c, 9.b): $c_{ii}=c_{1i}c_{11}c_{1i}$.

Jetzt sollen wir beweisen, daß c_{ii} I X_{ii}^+, d. h. $(h_4^+)^{c_{ii}}=h_4^+$ gültig ist. Nach Satz 2.2 folgt

$$(h_4^+)^{c_{1i}c_{11}c_{1i}} = (h_4^+)^{c_{1i}(x_1d_{11}x_1)c_{1i}} = (h_4^+)^{(c_{1i}x_1)d_{11}(x_1c_{1i})} = (h_4^+)^{d_{11}(x_1c_{1i})} = (h_4^+)^{x_1c_{1i}} = h_4^+.$$

(iii) Ist $h_4^+=h_1^+=h_3^+$ (Fig. 9.c), so folgt $X_{ii}^+=\overline{X}_{ii}^+$ aus der Definition 4.1.

Der Kegelschnitt auf Fig. 9.a hat 3 äquivalente Brennpunktpaare, die *H*-Zykeln auf Fig. 6.c haben 2 Brennpunktpaare usw.

5. Leitzyklen eines Kegelschnittes

DEFINITION 5.1. Zum Kegelschnitt $l(x_1, A, B)$, $p(x_1, A, B)$ betrachte man die Leitradien $a_{11}=(A, X_{11})$ und $b_{11}=(B, X_{11})$. Wir wählen einen Punkt A_1 auf a_{11} aus, dann liegt $B_1:=A_1^{x_1}$ auf b_{11} (Satz 4.1). Die Zyklen $\mathscr{C}(A, A_1)$ und $\mathscr{C}(B, B_1)$ mit den Mittelpunkten A bzw. B sind die *Leitzyklen* des Kegelschnittes.

Fig. 8 zeigt, daß ein Kegelschnitt, von A_1 abhängig, viele Leitzykelpaare haben kann. Zum Beispiel entartet im Falle $A_1=A$ der Zykel $\mathscr{C}(A, A_1)$ in den Brennpunkt A. Ist A_1 ein Grenzpunkt, so fallen beide Leitzyklen mit dem Grenzzykel der Ebene zusammen, zu diesem wichtigen Spezialfall kehren wir in Sektion 6 zurück. Jetzt stellen wir ins rechte Licht der Sinn und das Ziel der bisherigen Definitionen und Sätze durch dem Satz 5.1, der eine Folgerung des Sehnenviereckssatzes 3.2 ist.

SATZ 5.1 (Inversion zwischen den Leitzyklen eines Kegelschnittes). *Betrachten wir den Kegelschnitt $l(x_1, A, B)$, $p(x_1, A, B)$ mit den Leitzyklen $\mathscr{C}(A, A_1)$, $\mathscr{C}(B, B_1)$. Nach Definition* 4.1, *Definition* 3.1 *sei $x_i \in l(x_1, A, B)$ mit Leitradien a_{ii} und b_{ii}, auf den $A_i \in \mathscr{C}(A, A_1)$ bzw. $B_i \in \mathscr{C}(B, B_1)$ die entsprechenden Leitzykelpunkte sind. Ferner sei $t_i:=(A_i, x_i, B_k)$ die zu x_i senkrechte, mit A_i, B_i inzidente Gerade. Die Geradenmenge $\{t_i\}$ bildet ein Büschel T, wo der Punkt T mit der Kegelschnittachse $s:=(A, B)$ inzidiert, oder mit $A=B$ zusammenfällt.*

Der Beweis ist nach Satz 3.2 offenbar (Fig. 8, 7). Wir heben nur den Fall hervor, daß $x_i:=a_{1i}x_1b_{1i}$ (Definition 4.1),

$$\{a_{ii}, A_i\} = \{a_{11}, A_1\}^{a_{1i}}, \quad \{b_{11}, B_1\} = \{a_{11}, A_1\}^{x_1}, \quad \{b_{ii}, B_i\} = \{b_{11}, B_1\}^{b_{1i}}$$

gelten (Definition 3.1, Satz 4.1). Dann liegen die Punkte A_1, A_i, B_i, B_1 auf einem Zykel mit dem Mittelpunkt X_{1i}. Nach Satz 3.2 schneiden sich die Geraden $t_1:=(A_1, x_1, B_1)$ und $t_i:=(A_i, x_i, B_i)$ entweder in einem Punkt T der Geraden (A, B) oder es gilt $A=P=T$.

Die übrigen Fälle kann man wie im Beweis von Satz 3.2 (Fig. 7) behandeln.

BEMERKUNG. Die Definition der Inversion auf der metrischen Ebene zeigt die enge Verbindung zwischen dem Büschelsatz (Satz 5.1 in [5]) und dem vorigen Satz 5.1.

6. Asymptoten eines Kegelschnittes auf der hyperbolischen Ebene

DEFINITION 6.1. Zum Kegelschnitt $l(x_1, A, B)$, $p(x_1, A, B)$ betrachte man die zu x_1, X_{11} gehörenden Leitradien $a_{11}=(A, X_{11})$ und $b_{11}=(B, X_{11})$ (Fig. 12). Nehmen wir an, daß die Gerade a_{11} zwei Grenzpunkte E_1^+, G_1^+ hat (also die Einbettungsebene hyperbolisch ist), ferner X_{11} kein Grenzpunkt ist (Satz 4.2). Seien $F_1^+:=(E_1^+)^{x_1}$ und $H_1^+:=(G_1^+)^{x_1}$, $u_{11}:=(E_1^+, x_1, F_1^+)$ und $v_{11}:=(G_1^+, x_1, H_1^+)$ eingeführt.

a) Ist $x_i=a_{1i}x_1b_{1i}$ in Definition 4.1 eine Gerade von $l(x_1, A, B)$, so seien $E_i^+:=(E_1^+)^{a_{1i}}$, $G_i^+:=(G_1^+)^{a_{1i}}$, $\{F_i^+, H_i^+\}:=\{F_1^+, H_1^+\}^{b_{1i}}$, wobei $\{E_i^+, G_i^+\}^{x_1}= =\{F_i^+, H_i^+\}$ gilt, ferner die Geraden $u_{ii}:=(E_i^+, x_i, F_i^+)$, $v_{ii}:=(G_i^+, x_i, H_i^+)$ definiert.

b) Ist $x_i^+=b_{1i}^+=b_{ii}^+$ und ist a_{1i} keine Grenzgerade (Definition 4.1), so seien $\{E_i^+, G_i^+\}:=\{E_1^+, G_1^+\}^{a_{1i}}$, $\{F_i^+, H_i^+\}:=\{X_i^+, X_i^+\}$, $u_{ii}:=(E_i^+, x_i^+, F_i^+)$, $v_{ii}:= =(G_i^+, x_i^+, H_i^+)$.

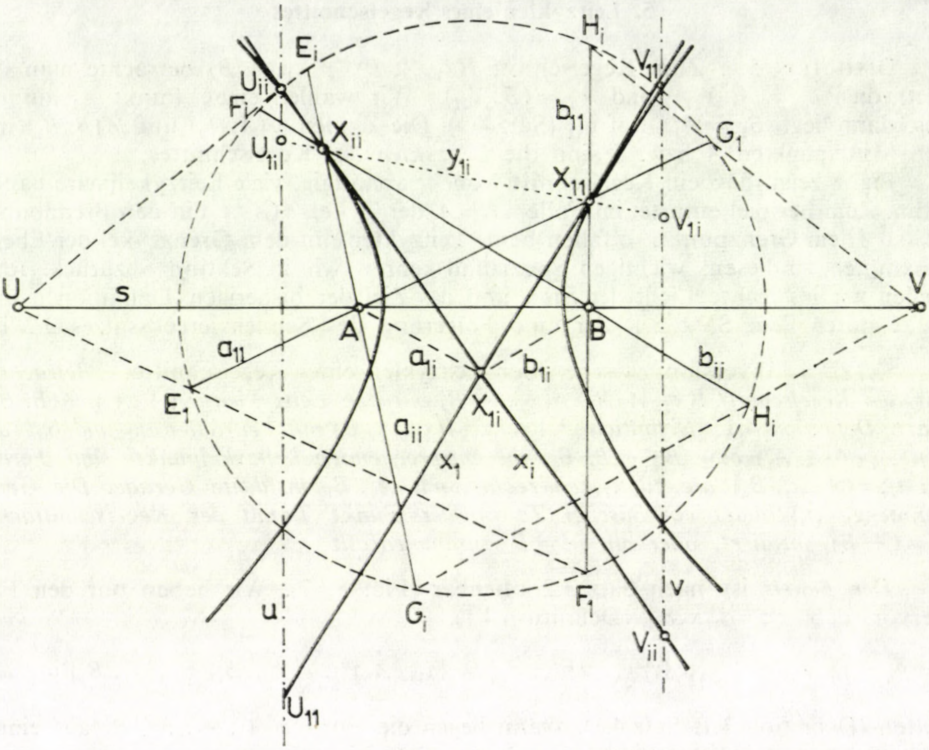


Fig. 12

c) Ist $x_i^+ := a_{1i}^+ = b_{1i}^+$ eine Grenzgerade, so seien $E_i^+ = F_i^+ = G_i^+ = H_i^+ = X_i^+$ und $u_{ii}^+ = v_{ii}^+ := x_i^+$.

Aus Satz 5.1 wissen wir, daß sich die Geraden der Menge $\{u_{ii}\}$ im Punkt U, die Geraden der Menge $\{v_{ii}\}$ im Punkt V auf der Achse $s := (A, B)$ schneiden oder $U = V = A = B$ besteht. Die Punkte U und V sind die *Asymptotenpole des Kegelschnittes*, ihre Polaren u bzw. v sind die *Asymptoten*. Die Geraden u_{ii} und v_{ii} sind die *Asymptotenradien* zu x_i, X_{ii}. $E_i^+ F_i^+ G_i^+ H_i^+$ ist das *Leitviereck* zu x_i, X_{ii}.

SATZ 6.1. *Für die auf der Achse s liegenden Brennpunkte A, B und Asymptotenpole U, V eines Kegelschnittes können die folgenden Fälle* (i)—(iv) *auftreten:*

(i) *A, B, U, V sind keine Grenzpunkte, und es gilt $U = AVB$. Die Punkte sind alle verschieden, oder fallen alle zusammen* (Fig. 12).

(ii) *Zum Beispiel ist $B^+ = V^+$ ein Grenzpunkt, doch sind A und U keine Grenzpunkte* (Fig. 13. b, c).

(iii) *Zum Beispiel sind $A^+ = U^+$ und $B^+ = V^+$ zwei Grenzpunkte* (Fig. 6.c *mit C^+, D^+ statt A^+, B^+*).

(iv) *$A^+ = B^+ = U^+ = V^+$ ist ein Grenzpunkt* (Fig. 6.b).

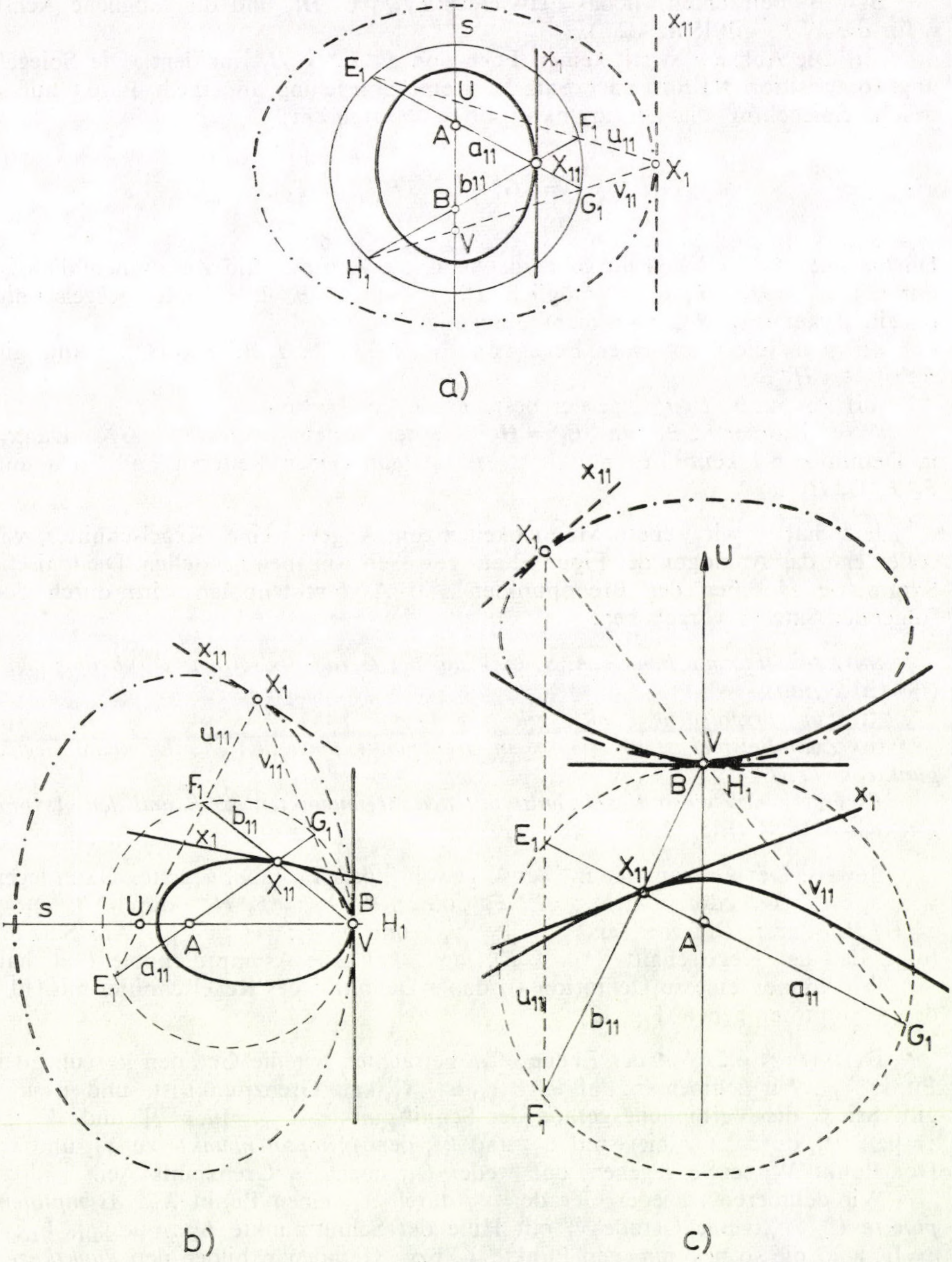


Fig. 13

Beweis. Betrachten wir das Leitviereck $E_1^+ F_1^+ G_1^+ H_1^+$ und die mögliche Achse s, für die $X_{11} \nmid s$ gilt (Satz 4.2.a).

(i) Die Achse s ist mit keiner Ecke von $E_1^+ F_1^+ G_1^+ H_1^+$ inzident. Die Spiegelungskomposition AVB ist nach Satz 2.2.a eine Spiegelung an einem Punkt auf s, welche Spiegelung die Grenzpunkte von u_{11} vertauscht:

$$(E_1^+)^{AVB} = (G_1^+)^{VB} = (H_1^+)^B = F_1^+.$$

Darum gilt $AVB = U$, ja u_{11} schneidet in U die Achse s. Ein Zusammenfallen ist nur bei $E_1^+ = H_1^+$, $F_1^+ = G_1^+$ möglich. Dann gilt $A = B = U = V$, der Kegelschnitt ist ein Zykel (Fig. 6), s ist nicht eindeutig.

(ii) s inzidiert mit einer Ecke von $E_1^+ F_1^+ G_1^+ H_1^+$, z. B. mit H_1^+. Dann gilt $B^+ = V^+ = H_1^+$.

(iii) s ist z. B. $E_1^+ H_1^+$, ferner bestehen $E_1^+ \neq H_1^+$ und $F_1^+ \neq G_1^+$.

(iv) s inzidiert z.B. mit $E_1^+ = H_1^+$, ferner besteht auch $F_1^+ = G_1^+$. Da x_{11} in Definition 6.1 kein Grenzpunkt ist, so hat man keinen weiteren Fall für s und $E_1^+ F_1^+ G_1^+ H_1^+$.

Jetzt haben wir weitere Möglichkeiten zum Angeben eines Kegelschnittes, wir sollen erst das Analogon der Figur 12 aus gewissen Angaben herstellen. Die logische Symmetrie zwischen den Brennpunkten und Asymptotenpolen wird durch den folgenden Satz hervorgehoben.

Satz 6.2. *Für die Punkte A, B, U, V auf der Geraden s sei eine der Bedingungen* (i)—(ii) *erfüllt:*

(i) *Für verschiedene Punkte gilt* $U = AVB$;

(ii) *Zum Beispiel ist* $B^+ = V^+$ *ein Grenzpunkt, doch sind* $A \neq U$ *keine Grenzpunkte mehr.*

Es gibt genau einen Kegelschnitt mit den Brennpunkten A, B und den Asymptotenpolen U, V (Fig. 12, 13).

Beweis. Der Grenzpunkt E_1^+ sei so gewählt, daß $E_1^+ \nmid s, a, u$ gelte. Dann legen wir $a_{11} := E_1^+ G_1^+$ durch A, $u_{11} := E_1^+ F_1^+$ durch U, $b_{11} := F_1^+ H_1^+$ durch B, $v_{11} := G_1^+ H_1^+$ durch V, ferner sei $X_{11} := [a_{11}, b_{11}]$ und sei $x_1 := (U_{11}, V_{11})$. Aus Satz 5.1 folgt, daß der Kegelschnitt $l(x_1, A, B)$, $p(x_1, A, B)$ die Asymptotenpole U, V hat.

Wir können eine zu Definition 4.1 duale Definition des Kegelschnittes mit Hilfe der Asymptoten geben (Fig. 12).

Definition 6.2. Auf der Ebene $\mathcal{PM}$ betrachten wir die Geraden u, v und den Punkt X_{11}. Wir nehmen an, daß $u, v \nmid X_{11}$ ist, X_{11} kein Grenzpunkt ist, und $v \neq u^{X_{11}}$ gilt. Sei x_1 die Verbindungsgerade der Schnittpunkte $U_{11} := [u, v^{X_{11}}]$ und $V_{11} := [v, u^{X_{11}}]$ durch X_{11}; hier sind U_{11} und V_{11} der *Asymptotenpunkte* zu X_{11} und x_1. Der Punkt X_{11} sei so gegeben, daß weder U_{11} noch V_{11} Grenzpunkt sei.

Wir definieren zu jeder Geraden y_{1i} durch X_{11} einen Punkt X_{ii}, *Asymptotenpunkte* U_{ii}, V_{ii}, eine Gerade x_1 mit Hilfe der Schnittpunkte $U_{1i} := [u, y_{1i}]$, $V_{1i} := [v, y_{1i}]$; die so bekommenen Punkte X_{ii} bzw. Geraden x_i bilden den *Punktkegelschnitt* $p(X_{11}, u, v)$ bzw. den *Linienkegelschnitt* $l(X_{11}, u, v)$. Die Gesamtheit von p und l wird *Kegelschnitt* genannt.

a) Sind U_{1i} und V_{1i} keine Grenzpunkte (Fig. 12), so definiert $X_{ii}:=U_{1i}X_{11}V_{1i}$ den Punkt X_{ii}. $U_{ii}:=U_{11}^{U_{1i}}$ und $V_{ii}:=V_{11}^{V_1}$ sind die entsprechenden Asymptotenpunkte. Es sei $x_i:=(U_{ii}, V_{ii})$ die Verbindungsgerade von U_{ii} und V_{ii}.

b) Sind z. B. U_{1i} kein Grenzpunkt aber V_{1i}^+ ein Grenzpunkt, so seien $X_{ii}^+:= := V_{1i}^+ = V_{ii}^+$, $U_{ii}:=U_{11}^{U_{1i}}$, $x_i:=(U_{ii}, V_{ii}^+)$.

c) Sind $U_{1i}^+=V_{1i}^+$ zusammenfallende Grenzpunkte mit den Polaren $u_{1i}^+=v_{1i}^+$, so seien $X_{ii}^+:=U_{1i}^+=V_{1i}^+=:U_{ii}^+=V_{ii}^+$, $x_i^+:=u_{1i}^+=v_{1i}^+$.

Der Punkt X_{11} sei so gegeben, daß der Fall nicht vorkomme, in dem $U_{1i}^+\neq V_{1i}^+$ verschiedene Grenzpunkte sind.

Die Geraden u und v heißen die *Asymptoten des Kegelschnittes.*

Auf der hyperbolischen Einbettungsebene können wir auch mit spiegelungsgeometrischer Methode die Verbindung zwischen Definition 4.1 und Definition 6.2 beschreiben.

SATZ 6.3. *Nehmen wir an, daß die Asymptotenpunkte U_{11} und V_{11} in Definition* 6.2 *von $p(X_{11}, u, v)$, $l(X_{11}, u, v)$ äußere Punkte sind. Dann existieren die Brennpunkte A, B auf der Achse $s:=(U, V)$, für die $p(X_{11}, u, v)=p(x_1, A, B)$ und $l(X_{11}, u, v)= =l(x_1, A, B)$ im Sinne von Definition* 4.1 *gelten.*

BEWEIS. Zuerst bemerken wir, daß Definition 6.2 die Menge p und l selbständig definiert, ja man kann die zu dem Sektion 4 dualen Sätze ebenso beweisen. Doch kann man sagen: die Polaren der Punkte von $p(X_{11}, u, v)$ bilden einen Linienkegelschnitt $l(x_{11}, U, V)$ (Fig. 13), die Pole der Geraden von $l(X_{11}, u, v)$ bilden einen Punktkegelschnitt $p(x_{11}, U, V)$ mit den Brennpunkten U und V, wo U, V die Pole von u bzw. v sind. Die zu x_{11} und X_1 gehörenden Leitradien von $l(x_{11}, U, V)$ sind u_{11} und v_{11}, d. h. die Polaren von U_{11} bzw. V_{11}. Nach der Annahme haben u_{11} und v_{11} je zwei Grenzpunkte E_1^+, F_1^+ und G_1^+, H_1^+. Die Geraden $a_{11}:=(E_1^+, G_1^+)$, $b_{11}:=(F_1^+, H_1^+)$ durch X_{11} schneiden aus der Achse $s:=(U, V)$ die Asymptotenpole A, B von $l(x_{11}, U, V)$, $p(x_{11}, U, V)$ heraus (Satz 5.1 und Satz 4.2.a).

Umgekehrt, geht man von den Brennpunkten A, B und der Geraden x_1 aus, so haben $l(x_1, A, B)$, $p(x_1, A, B)$ die Asymptotenpole U, V.

Nach Satz 6.2 bestimmen A, B, U, V den Kegelschnitt $\{l(x_1, A, B), p(x_1, A, B)\}= =\{l(X_{11}, u, v), \ p(X_{11}, u, v)\}$ und den zu ihm *polaren Kegelschnitt* $\{p(X_1, a, b), l(X_1, a, b)\}=\{p(x_{11}, U, V), l(x_{11}, U, V)\}$. Im vorigen Gedankengang haben wir die Fälle $A=B=U=V$ und $A^+=U^+\neq B^+=V^+$ ausgeschlossen, in diesen Fällen ist der Kegelschnitt ein Zykel und die Behauptung einfacher bewiesen werden kann.

Wir heben aus dem vorigen Beweis die *Definition der polaren Verbindung zwischen den Kegelschnitten* auf der hyperbolischen Einbettungseben hervor.

DEFINITION 6.3. Zu dem Kegelschnitt l, $p(x_1, X_{11}, A, B, u, v)$ mit den Brennpunkten A, B und Asymptoten u, v (Definition 4.1, Definition 6.1) betrachte man die polaren Elemente. So bekommen wir den zum vorigen *polaren Kegelschnitt* $l, p(x_{11}, X_1, U, V, a, b)$ mit den Brennpunkten U, V und Asymptoten a, b (Fig. 9.b, 13).

Die Sätze 4.4, 6.1—3 erleichtern die *ökonomische Klassifikation* der Kegelschnitte auf der hyperbolischen Einbettungsebene.

7. Klassifikation der Kegelschnitte

Nach Definition 4.1 kann man die Kegelschnitte auf der projektiven Einbettungsebene $\mathcal{PM}$ einheitlich erklären. Ohne Einbettung wäre das Analogon von Definition 4.1 sehr kompliziert und unübersichtlich. Ferner könnte man die Kegelschnittklassifikation auf der metrischen Ebene $\mathcal{M}$ aus der folgenden Klassifikation auf $\mathcal{PM}$ entnehmen, wenn man die mögliche Modelle für $\mathcal{M}$ in $\mathcal{PM}$ aufzählen könnte. Doch ist dieses Problem noch nicht gelöst.

Bei der Klassifikation der Kegelschnitte auf $\mathcal{PM}$ weisen wir auf unsere Vorstellung über $\mathcal{PM}$ (in Sektionen 1 und 2, Fig. 4, 5) hin.

Klassifikation von $l, p(x_1, A, B)$ *nach Definition* 4.1.

ELLIPTISCHE EBENE

1. *Kreis.* Das ist ein Zykel, also ist $A=B$.
2. *Ellipse.* $A \neq B$.

EUKLIDISCHE EBENE

1. *Kreis.* Der ist ein Zykel, also ist $A=B$ kein Grenzpunkt.
2. *Ellipse.* $A \neq B$ sind keine Grenzpunkte, p hat keinen Grenzpunkt.
3. *Hyperbel.* $A \neq B$ sind keine Grenzpunkte, p hat zwei Grenzpunkte (Fig. 8).
4. *Parabel.* A ist kein Grenzpunkt, B^+ ist Grenzpunkt.

HYPERBOLISCHE EBENE

1. *Kreis.* $A=B$ ist ein innerer Punkt (Fig. 6.a).

Jede der Geraden x_1 und a_{11} kann zwei oder keine Grenzpunkte haben. So hat man vier Varianten, die sich im klassischen Fall auf zwei reduzieren.

2. *Hyperzykel.* $A=B$ ist ein äußerer Punkt (Fig. 6.c). Wir haben auch jetzt vier Varianten, die sich im klassischen Fall auf drei reduzieren.

3. *Parazykel.* $A^+=B^+$ ist ein Grenzpunkt (Fig. 6.b). Wir haben zwei Möglichkeiten, da a_{11} zwei Grenzpunkte hat.

4. *Ellipse.* $A \neq B$ sind innere Punkte (Fig. 12, 13.a). Zwei Möglichkeiten, wenn a_{11} keinen Grenzpunkt hat, kommen im klassischen Fall nicht vor. Hat a_{11} zwei Grenzpunkte, so existieren das Leitviereck zu x_1 und die Asymptotenpole U, V (Definition 6.1). Abhängig von U, V, x_1 haben wir logisch 6 Möglichkeiten, die sich im klassischen Fall auf drei reduzieren.

5. *Hyperbel.* $A \neq B$ sind äußere Punkte, das Brennpunktvierseit ist nicht entartet (Fig. 9.a). Im klassischen Fall haben wir stets ein solches Brennpunktpaar, das heißt A, B, für das die a_{ii} zwei Grenzpunkte haben. Dann existieren die Asymptotenpole U, V für die wir zwei Möglichkeiten haben: Fig. 9.a, Polargebilde zu Fig. 12.

6. *Halbellipse oder Halbhyperbel.* A ist innerer Punkt, B ist äußerer Punkt (Fig. 10). Im klassischen Fall hat a_{11} stets zwei Grenzpunkte, von U, V ist der eine innerer, der andere äußerer Punkt.

7. *Elliptische Parabel.* A ist ein innerer Punkt, B^+ ist Grenzpunkt (Fig. 13.b, c). b_{11} hat stets zwei Grenzpunkte, U und $V^+=B^+$ existieren. Im klassischen Fall haben wir drei Möglichkeiten von U und x_1 abhängig.

8. *Hyperbolische Parabel.* A ist eine äußerer, B^+ ist ein Grenzpunkt, $s:=(A, B^+)$ keine Grenzgerade (Fig. 9.b, zu Fig. 13.c polares Gebilde). b_{11} hat stets zwei Grenz-

punkte, U und $V^+=B^+$ existieren. Im klassischen Fall haben wir drei Möglichkeiten von U und x_1 abhängig.

9. *Oskulierende Parabel.* A ist ein äußerer Punkt, B^+ ist ein Grenzpunkt, $s:=(A, B^+)=b^+$ (Fig. 9.c). U ist ein äußerer Punkt, $V^+=B^+$.

In der klassischen hyperbolischen Ebene haben wir endlich zwanzigerlei Kegelschnitte, wie auch die Analytische Behandlung zeigt [6], [7], [8], [9]. Unsere spiegelungsgeometrische Behandlung ermöglichte diese Klassifikation, wir haben auf die Probleme der feineren Klassifikation hingewiesen.

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A MODULE-THEORETIC CHARACTERISATION OF RINGS WITH UNITY

By

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1. Introduction. If R is an (associative) ring and G is an R-module then G_0 denotes the *maximal trivial submodule of* G, which is defined by the following formula: $G_0=\{g\in G:gR=0\}$. The next characterisation of rings with unity is well-known (KERTÉSZ [1], Theorem 4.2):

THEOREM 1. *A ring R has a unit element if and only if every R-module G is the direct sum of its maximal trivial submodule and of the module GR.*

F. SZÁSZ [2] introduced the following notion: A ring R *has Property* E_2 (in other words R *is an* E_2*-ring*) if for every R-module G, G_0 is a direct summand in G. It is obvious by Theorem 1 that all rings with unity have Property E_2. KERTÉSZ ([1], Problem 1) asked:

PROBREM. *Has every E_2 ring a unity?*

F. SZÁSZ has written two papers on this problem. In [2] he gives conditions for a ring to have a unit element, and in [3] he describes some properties of those Brown—McCoy radical-rings which are also E_2-rings. But the problem of Kertész has remained open. The aim of our paper is to solve this problem. We shall prove

THEOREM 2. *Every E_2-ring has a unit element.*

2. Adopted notions and results. Firstly, we shall need some definitions and theorems from the theory of abelian groups.

An abelian group is named *free* if it is a direct sum of infinite cyclic groups. (By direct sum we mean discrete direct sum.) It is well-known that every abelian group is the homomorphic image of a suitable free abelian group.

THEOREM 3. *Every subgroup of a free abelian group is free* (FUCHS [4], Theorem 12.1).

Secondly, we shall use a result of F. Szász:

THEOREM 4. *A homomorphic image of an E_2-ring is also an E_2-ring* (F. SZÁSZ [2], Satz 2.3 (1)).

At last we quote a result of F. Szász; we shall prove it in a stronger form. Let us call an element a of a ring R a *left multiplier*, if there exists an integer n such that we have $ar=nr$ for any r from R, and either $a\neq 0$ or $n\neq 0$.

THEOREM 5. *If an E_2-ring R has a non-zero left multiplier, then R has a unity* (F. SZÁSZ [2], Satz 3.4 (1)).

(In his paper the phrase "Linksmultiplikator" means "non-zero left multiplier".)

3. Some preliminary lemmas. In this part we prove some results which are independent of the theory of E_2-rings, but we shall need them to prove our main theorem (Theorem 2).

LEMMA 1. *Let R be an arbitrary ring, A and B ideals in R such that the factorrings R/A and R/B have unit elements. Then $R/(A\cap B)$ also has a unit element.*

PROOF. Let us choose an element e from the residue class of the unit element of the ring R/A. Then for every r from R we have $r-re\in A$ and $r-er\in A$. The element f is defined in a similar way with respect to B. Let e_1 denote the element $e+f-ef$. We show that the class of e_1 in $R/(A\cap B)$ is a unit element of this ring. Indeed, $r-e_1r=(r-er)-[(fr)-e(fr)]\in A$, and $r-e_1r=(r-fr)-e(r-fr)\in B$, since B is an ideal. The relation $r-re_1\in A\cap B$ can be shown analogously.

Let π denote an arbitrary set of prime numbers. π' will denote the set of all primes which are not included in π. We say that an integer n *is a π-number*, if all prime factors of n are in π. Let $\mathbf{Q}_\pi$ be the ring of those rational numbers which can be written in the form a/n, where a is an integer, and n is a π-number. The ring of all integers and rational numbers will be denoted by $\mathbf{Z}$ and $\mathbf{Q}$, respectively.

LEMMA 2. *If $\pi\neq\emptyset$ then the additive group of $\mathbf{Q}_\pi$ is not a free abelian group.*

PROOF. $\mathbf{Q}_\pi^+$ is not isomorphic to $\mathbf{Z}^+$, because 1 in $\mathbf{Z}^+$ cannot be divided by any of the prime numbers. Hence, it is enough to show that every direct decomposition of our group is trivial. Assume we have a non-trivial decomposition $\mathbf{Q}_\pi = A\oplus B$, let a/n and b/m be non-zero elements from A and B, respectively. Then — since in an abelian group we can multiply by integers — the element $abmn$ is a common element of A and B, so it is 0 which is a contradiction.

Now we shall prove an imbedding theorem. By the *order* of an element from a ring we always mean its additive order. Clearly, if a set of primes π is given, then those elements of a ring R which are of π-number order form an ideal, which is named the *π-component of R*, and if we factorise by this ideal, then there are no π-elements in the factorring.

An algebra over a (commutative) ring J is called *unital* if J has a unit element 1, which acts identically on the algebra.

LEMMA 3. *Suppose that a ring R and a set π are given such that the order of any element of R is either infinite or a π'-number. Then R can be imbedded into a unital algebra over $\mathbf{Q}_\pi$.*

PROOF. We mention the following: if $r, s\in R$, and n is a π-number, then $rn=sn$ implies $r=s$. Now let us consider the ordered pairs (r, n), where n is a π-number, and $r\in R$. Let s denote also an element of R, let $k\in\mathbf{Z}$, and m a π-number. We say that the pairs (s, n) and (r, m) are equal, if we have $rn=sm$. We define the following operations:

$$(s, n)+(r, m) = (sm+rn, mn),$$

$$(s, n)\cdot(r, m) = (sr, mn),$$

$$(s, n)\cdot k/m = (sk, mn).$$

We must show that this equality is an equivalence relation, these operations are well-defined, and we have got a unital algebra. The proofs are simple computations using our observation above, so they are left to the reader. Finally, the pairs of the form $(s, 1)$ — which are, of course, different — give a subring in our algebra isomorphic to R.

4. Proof of Theorem 2. Now, as we have promised above, we prove the following:

LEMMA 4. *If an E_2-ring has a left multiplier, then it has a unity.*

PROOF. Let R be an E_2-ring with a left multiplier $a: as=ns$ for any s in R. We consider the customary extension of R with unit element, and denote it by R^*. Its unit element is denoted by 1, and we consider $\mathbf{Z}$ as a subring of R^*. The elements of R^* have a unique decomposition of the form $r+m$, $r\in R$, $m\in\mathbf{Z}$. Of course, R^* is an R-module, so we have $R^*=B\oplus C$, where $B=\{r\in R^*: rR=0\}$, and C is a suitable submodule. B is not zero, because $a-n$ is a non-zero element in it.

Let $1=b+c$, $b\in B$, $c\in C$. Multiplying by an arbitrary $s\in R$ from the right we have $s=cs$, and so $R\subseteq C$, because C is a submodule. If $c=t+m$, $t\in R$, $m\in\mathbf{Z}$, then $b=1-c=-t+1-m$. Since B^+ and C^+ are also groups, $bm\in B$ and $c(1-m)\in C$, and this implies $bm\in B\cap C$, because $bm-c(1-m)=-t\in R\subseteq C$. But $B\cap C=0$, so $0=bm=-tm+m(1-m)$. So we have $m=0$ or $m=1$, and $tm=0$.

If $m=1$, then $t=tm=0$, and $c=0+1=1$. But now for arbitrary $k\in\mathbf{Z}$ and $s\in R$, $R\subseteq C$ and $1=c\in C$ imply $s+k\in C$, and we have $B=0$, which is a contradiction.

So we have got $m=0$. In this case $b=1-t$, and b is in B, so by the definition of B, t is a left unit element in R. If s, u are arbitrary in R, then $ts=s$ implies $(ut-u)s=$ $=0$. This is true for every s, hence $ut-u$ is in B. Then $R\subseteq C$ implies $ut-u\in B\cap$ $\cap C=0$. This implies $ut-u=0$, so t is a unit element.

LEMMA 5. *Let R be an E_2-ring and p a prime number such that R has no elements of p-power order. Then R has a left multiplier.*

PROOF. Let $\pi=\{p\}$ and $J=\mathbf{Q}_\pi$. By Lemma 3 we can imbed R into a unital algebra S over J. We may suppose that R generates S, because otherwise we may consider the subalgebra generated by R instead of S. So every element of S can be written in the form $r(1/n)$, where r is in R, and n is a power of p. Now we define an R-module on the abelian group $G=F\oplus S^+$; here F denotes a suitable free abelian group such that there exists an epimorphism φ from F to J^+. We define the operation as follows:

$$(f+s)r = 0+[r(f\varphi)+sr] \quad (f\in F,\ s\in S,\ r\in R).$$

The reader can easily prove that in this way we have got an R-module. It is evident that the kernel of φ — we shall denote it by K — is contained in G_0. We prove that $K\neq G_0$. Assume $K=G_0$, then, since R is an E_2-ring, there exists a submodule H such that $G=K\oplus H$. If we consider G as an abelian group, then it is the direct sum of the abelian groups K and H. We show that $F=K\oplus(H\cap F)$. Indeed, $K\cap H=0$, so it is enough to prove that F is generated by K and $H\cap F$. If f is in F, then $f=k+h$, since G is generated by K and H. But $K\subseteq F$, so $h\in F\cap H$. We proved that $F=K\oplus(H\cap F)$. Theorem 3 shows that $H\cap F$ is a free abelian group. By the homomorphism theorem, J^+ is isomorphic to $H \cap F$, and this contradicts Lemma 2. So $K \neq G_0$. $K \subseteqq G_0$ implies the existence of a $g \in G_0 \setminus K$. Let $g=f+s$, $f \in F$, $s \in S$, then $g \notin K$ means that either $f\varphi$ or s does not equal zero. Let $f\varphi = k/m$ and $s=r/n$ ($k \in \mathbf{Z}$, $r \in R$, m and n are π-numbers). $g \in G_0$ means that for any u from R, $u(f\varphi)+su=0$; hence $(rm)u=u(-kn)$. If rm is not a left multiplier, then $rm=0$ and $kn=0$. But R has no π-elements, so $r=0$; hence $s=0$. n is also a π-number; hence it is not zero, and this implies that k equals 0, and we have $f\varphi = 0 = s$, a contradiction.

Proof of Theorem 2. Let R be an E_2-ring, p and q distinct prime numbers. If we factorise by the p-component of R then we get a new ring R'. It is also an E_2-ring, by Theorem 4. Applying Lemmas 4 and 5 we see that R' has a unit element. A similar argument shows that factorising by the q-component, the factorring R'' has also a unity. But the intersection of the p- and q-components is zero, hence the proof is complete by Lemma 1.

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MATRICES WITH RESTRICTED ELEMENTS, ROW SUMS AND COLUMN SUMS

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I. Introduction

In this paper we obtain necessary and sufficient conditions for the existence of a real matrix with prescribed lower and upper bounds for the row sums, column sums and the elements of the matrix.

Theorems of this character were proved by ERDŐS and MINC [1] (see Corollary 1), and later by KY FAN [2] (see Corollary 2). The proof in [1] is carried out by a nice ad hoc induction argument and that in [2] by means of a theorem of alternatives.

Our main aim is to show that the treatment of this type of problems belongs, in a natural way, to network theory, which also offers ways to discuss the question of integral solutions. Accordingly, we shall prove the following theorem.

THEOREM. *Let* $a_i \leqq b_i$, $c_j \leqq d_j$, $r_{ij} \leqq s_{ij}$ $(i=1, 2, \dots, m;\ j=1, 2, \dots, n)$ *be* $2m+2n+2mn$ *real numbers. There exists a real* $m\times n$ *matrix* $\{x_{ij}\}$ *such that*

$$(1)\qquad \begin{cases} a_i \leqq \sum\limits_{j=1}^{n} x_{ij} \leqq b_i, & i\in I = \{1, 2, \dots, m\} \\ c_j \leqq \sum\limits_{i=1}^{m} x_{ij} \leqq d_j & j\in J = \{1, 2, \dots, n\} \\ r_{ij} \leqq x_{ij} \leqq s_{ij}, & i\in I,\ \ j\in J \end{cases}$$

if and only if the following conditions are satisfied:

$$\sum_{j\in J''}\Big(\sum_{i\in I'} s_{ij}-c_j\Big) \geqq \sum_{i\in I''}\Big(\sum_{j\in J'} r_{ij}-b_i\Big), \quad \sum_{j\in J'}\Big(d_j-\sum_{i\in I''} r_{ij}\Big) \geqq \sum_{i\in I'}\Big(a_i-\sum_{j\in J''} s_{ij}\Big)$$

for each $I'\subset I$, $I''\subset I$, $J'\subset J$, $J''\subset J$, *where* $I'\cup I''=I$, $I'\cap I''=\emptyset$, $J'\cup J''=J$, $J'\cap J''=\emptyset$. *If* a_i, b_i, c_j, d_j, r_{ij}, s_{ij} *are integers and the system of inequalities* (1) *has a solution at all, then it has an integral solution as well.*

In the heart of network theory there stand three theorems: Ford and Fulkerson's max flow-min cut theorem, Gale's theorem on demands and supplies, and Hoffmann's circulation theorem (see [3]). The result of the present paper could be more directly proven by Hoffmann's theorem. But since Gale's theorem seems to be better known, our proof will be based on the latter.

My grateful acknowledgement is due to Dr. I. Dancs for calling my attention to this problem and suggesting its connection with network theory.

II. The proof of the theorem

First let us recall Gale's theorem on demands and supplies:

Let us be given a network defined by the set of nodes $I=\{1, 2, \dots n\}$ *and the set of arcs* (i, j) $(i\in I,\ j\in I)$, *with the capacities* $k_{ij}\geqq 0$ *of arcs and the intensities* g_i *of nodes. Then there exists a flow* $\{x_{ij}\}$ *fulfilling*

$$\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = g_i, \quad i\in I$$

$$0 \leqq x_{ij} \leqq k_{ij}, \quad i\in I,\ j\in I$$

if and only if the conditions

$$\sum_{i\in I} g_i = 0; \quad \sum_{i\in I'} g_i \leqq \sum_{i\in I'} \sum_{j\in I-I'} k_{ij}$$

are satisfied for each set $I'\subset I$. *When the values* g_i, k_{ij} $(i\in I, j\in I)$ *are integers and there exists a flow* $\{x_{ij}\}$ *in the network at all, then there exists an integral flow as well.*

After this let us give the proof of the theorem. Denote:

$$z_{ij} = x_{ij}-r_{ij}, \quad i\in I,\ j\in J$$

$$z_{0i} = \sum_{j\in J} x_{ij}-a_i, \quad i\in I$$

$$z_{j0} = \sum_{i\in I} x_{ij}-c_j, \quad j\in J;$$

$$p_{ij} = s_{ij}-r_{ij}, \quad i\in I,\ j\in J$$

$$p_{0i} = b_i-a_i, \quad i\in I$$

$$p_{j0} = d_j-c_j, \quad j\in J$$

$$p_{kl} = 0 \quad \text{otherwise};$$

$$g_i = \sum_{j\in J} z_{ij}-z_{0i} = a_i-\sum_{j\in J} r_{ij}, \quad i\in I$$

$$g_j = -\sum_{i\in I} z_{ij}+z_{j0} = \sum_{i\in I} r_{ij}-c_j, \quad j\in J$$

$$g_0 = \sum_{i\in I} z_{0i}-\sum_{j\in J} z_{j0} = \sum_{j\in J} c_j-\sum_{i\in I} a_i.$$

Obviously the problem of existence of a solution to the system of inequalities (1) is equivalent to that of a flow in the network defined by the set of nodes $N=\{0\}\cup \cup I\cup J$, the set of arcs $L=\{(k, l)|k\in N, l\in N\}$, the capacities $p_{kl}(k\in N, l\in N)$ and the intensities g_k $(k\in N)$. In other words (1) has a solution $\{x_{ij}\}$ if and only if there exists an $(m+n+1)\times(m+n+1)$ real matrix $\{z_{kl}\}$ such that

$$(2)\qquad \begin{cases} \sum_{l\in N} z_{kl}-\sum_{l\in N} z_{lk} = g_k, & k\in N\\ 0\leqq z_{kl}\leqq p_{kl}, & (k, l)\in L.\end{cases}$$

By Gale's theorem the necessary and sufficient condition for the existence of such a matrix $\{z_{kl}\}$ is that both

(i) $$\sum_{k\in N} g_k = 0$$

and

(ii) $$\sum_{k\in A'} g_k \leqq \sum_{k\in A'} \sum_{l\in N-A'} p_{kl}$$

are satisfied for each set $A'\subset N$ of nodes. Also, when the intensities g_k $(k\in N)$ and the capacities p_{kl} $(k\in N,\ l\in N)$ are integers and there exists a flow in the network at all, then there exists a flow of integer values as well.

Let us apply the conditions (i) and (ii) to the problem (1):

(i) $$\sum_{k\in N} g_k = \sum_{i\in I} g_i + \sum_{j\in J} g_j + g_0 =$$

$$= \sum_{i\in I} a_i - \sum_{i\in I}\sum_{j\in J} r_{ij} + \sum_{j\in J}\sum_{i\in I} r_{ij} - \sum_{j\in J} c_j + \sum_{j\in J} c_j - \sum_{i\in I} a_i = 0.$$

(ii) Let $I'\cup J'$ and $I''\cup J''$ be arbitrary partitionings of $I\cup J$. We distinguish two cases:

Case 1: $A'=I'\cup J'$ and $A''=I''\cup J''\cup\{0\}$. Then

$$\sum_{k\in A'} g_k = \sum_{i\in I'} g_i + \sum_{j\in J'} g_j \leqq \sum_{i\in I'}\sum_{j\in J''} p_{ij} + \sum_{j\in J'} p_{j0} = \sum_{k\in A'}\sum_{l\in A''} p_{kl},$$

$$\sum_{i\in I'} a_i - \sum_{i\in I'}\sum_{j\in J} r_{ij} + \sum_{j\in J'}\sum_{i\in I} r_{ij} - \sum_{j\in J'} c_j \leqq \sum_{i\in I'}\sum_{j\in J''} s_{ij} - \sum_{i\in I'}\sum_{j\in J''} r_{ij} + \sum_{j\in J'} d_j - \sum_{j\in J'} c_j,$$

$$\sum_{j\in J'}\sum_{i\in I''} r_{ij} - \sum_{j\in J'} d_j \leqq \sum_{i\in I'}\sum_{j\in J''} s_{ij} - \sum_{i\in I'} a_i.$$

Case 2: $A'=I'\cup J'\cup\{0\}$ and $A''=I''\cup J''$. Then

$$\sum_{k\in A'} g_k = \sum_{i\in I'} g_i + \sum_{j\in J'} g_j + g_0 \leqq \sum_{i\in I'}\sum_{j\in J''} p_{ij} + \sum_{i\in I''} p_{0i} = \sum_{k\in A'}\sum_{l\in A''} p_{kl},$$

$$\sum_{i\in I'} a_i - \sum_{i\in I'}\sum_{j\in J} r_{ij} + \sum_{j\in J'}\sum_{i\in I} r_{ij} - \sum_{j\in J'} c_j + \sum_{j\in J} c_j - \sum_{i\in I} a_i \leqq$$

$$\leqq \sum_{i\in I'}\sum_{j\in J''} s_{ij} - \sum_{i\in I'}\sum_{j\in J''} r_{ij} + \sum_{i\in I''} b_i - \sum_{i\in I''} a_i,$$

$$\sum_{j\in J'}\sum_{i\in I''} r_{ij} - \sum_{i\in I''} b_i \leqq \sum_{i\in I'}\sum_{j\in J''} s_{ij} - \sum_{j\in J''} c_j.$$

We see that Gale's result gives our statement and this proves the theorem.

III. The construction of a solution

The constructive proof of Ford and Fulkerson's max flow-min cut theorem offers a method to obtain a solution to the problem (1).

Let us extend the domain of the function $\{z_{kl}\}$ as follows. Complete the set N with the nodes t and t' (named source and sink): $N'=N\cup\{t\}\cup\{t'\}$ and the set $L=N\times N$ with the arcs (t, k) and (k, t') $(k\in N)$: $L'=L\cup\{t\times N\}\cup\{N\times t'\}$.

Let the new capacities be defined as follows:

$$q_{tk} = \begin{cases} g_k & \text{if} \quad g_k > 0 \\ 0 & \text{if} \quad g_k \leqq 0 \end{cases} \quad k \in N;$$

$$q_{kt'} = \begin{cases} -g_k & \text{if} \quad g_k \leqq 0 \\ 0 & \text{if} \quad g_k > 0 \end{cases} \quad k \in N;$$

$$q_{kl} = p_{kl}, \quad (k, l) \in L.$$

Then the following problem is to be solved by means of the max flow-min cut algorithm: maximize v subject to

$$\sum_{l \in N'} z_{kl} - \sum_{l \in N'} z_{lk} = \begin{cases} 0 & \text{if} \quad k \neq t \quad \text{or} \quad t' \\ v & \text{if} \quad k = t \\ -v & \text{if} \quad k = t' \end{cases} \tag{3}$$

$$0 \leqq z_{kl} \leqq q_{kl}, \quad (k, l) \in L.$$

Clearly, restricting any solution $\{z_{kl}\}$ of (3) to the arcs of L gives a solution to (2) and each solution to (2) can be obtained in this way. This involves a solution $x_{ij} = z_{ij} + r_{ij}$ $(i \in I, j \in J)$ to the problem (1) as well. If $a_i, b_i, c_j, d_j, r_{ij}, s_{ij}$ are integers, then the process ends in an integral solution.

IV. Corollaries

In both Erdős' and Minc's and Ky Fan's above mentioned results $m=n$; $a_i, b_i, c_j, d_j, r_{ij}, s_{ij}$ are non-negative numbers. In their cases $r_{ij}=0$ and $s_{ij}=\infty$, when $i \neq j$, is also assumed and hence the conditions of our theorem are trivially fulfilled for most of the pairs of sets I', J' except for the ones which are selected here:

In the case

(α) $I'=I$, $I''=\emptyset$, $J'=J$, $J''=\emptyset$ the condition is:

$$\sum_{j=1}^{n} d_j \geqq \sum_{i=1}^{n} a_i.$$

Similarly in the other cases:

(β) $I' \subset I$, $I'' \subset I$, $J'=J$, $J''=\emptyset$:

$$\sum_{i \in I''} r_{ii} \leqq \sum_{i \in I''} b_i, \quad \sum_{i \in I''} (r_{ii} - a_i) \leqq \sum_{j=1}^{n} d_j - \sum_{i=1}^{n} a_i.$$

(γ) $I'=\emptyset$, $I''=I$, $J'=\emptyset$, $J''=J$:

$$\sum_{i=1}^{n} b_i \geqq \sum_{j=1}^{n} c_j.$$

(δ) $I'=\emptyset,\ I''=I,\ J'\subset J,\ J''\subset J$:

$$\sum_{j\in J'} r_{jj} \leqq \sum_{j\in J'} d_j, \quad \sum_{j\in J'} (r_{jj}-c_j) \leqq \sum_{i=1}^{n} b_i - \sum_{j=1}^{n} c_j;$$

(ε) $I'=\{i\},\ I''=I-\{i\},\ J'=J-\{i\},\ J''=\{i\}$:

$$\sum_{j=1}^{n} b_j - \sum_{j=1}^{n} r_{jj} \geqq c_i+b_i-s_{ii}-r_{ii}, \quad \sum_{j=1}^{n} d_j - \sum_{j=1}^{n} r_{jj} \geqq a_i+d_i-s_{ii}-r_{ii}.$$

COROLLARY 1 (Erdős and Minc's theorem). *A matrix* $\{x_{ij}\}$ *satisfying*

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i=1,\dots,n$$

$$\sum_{i=1}^{n} x_{ij} = d_j, \quad j=1,\dots,n$$

$$x_{ij} \geqq 0, \quad i,j=1,\dots,n$$

$$x_{ii} = r_{ii}, \quad i=1,\dots,n$$

(where a_i, d_j, r_{ii} *are non-negative numbers,* $i,\ j=1,\dots,n$*) exists if and only if*

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} d_j, \quad r_{ii} \leqq \min(a_i, d_i), \quad i=1,\dots,n,$$

$$\sum_{j=1}^{n} (a_j-r_{jj}) \geqq \max_i (a_i+d_i-2r_{ii}).$$

COROLLARY 2 (Ky Fan's theorem). *A matrix* $\{x_{ij}\}$ *satisfying*

$$a_i \leqq \sum_{j=1}^{n} x_{ij} \leqq b_i, \quad i=1,\dots,n,$$

$$c_j \leqq \sum_{i=1}^{n} x_{ij} \leqq d_j, \quad j=1,\dots,n,$$

$$x_{ij} \geqq 0, \quad i,j=1,\dots,n,$$

$$r_{ii} \leqq x_{ii} \leqq s_{ii}, \quad i=1,\dots,n$$

(where $a_i, b_i, c_j, d_j, r_{ii}, s_{ii}$ *are non-negative numbers,* $a_i \leqq b_i$, $c_j \leqq d_j$, $r_{ii} \leqq \min(a_i, c_i, s_{ii})$ $(i,\ j=1,\dots,n)$ *exists if and only if*

$$\sum_{i=1}^{n} a_i \leqq \sum_{j=1}^{n} d_j, \quad \sum_{j=1}^{n} c_j \leqq \sum_{i=1}^{n} b_i, \quad r_{ii} \leqq \min(b_i, d_i), \quad i=1,\dots,n,$$

$$\sum_{j=1}^{n} (b_j-r_{jj}) \geqq \max_i (b_i+c_i-r_{ii}-s_{ii}), \quad \sum_{j=1}^{n} (d_j-r_{jj}) \geqq \max_i (d_i+a_i-r_{ii}-s_{ii}).$$

Here we can make an additional remark, namely, the following statement is valid.

PROPOSITION. *If in any of the last two inequalities equality holds for some* i_1 *or* i_2, *i.e.:*

$$\sum_{j=1}^{n}(b_j-r_{jj}) = b_{i_1}+c_{i_1}-r_{i_1i_1}-s_{i_1i_1}$$

or

$$\sum_{j=1}^{n}(d_j-r_{jj}) = d_{i_2}+a_{i_2}-r_{i_2i_2}-s_{i_2i_2}$$

then all the off-diagonal elements of the matrix must be 0, *except*

$$x_{i_1j} \quad \text{and} \quad x_{ji_1} \quad \text{or} \quad x_{i_2j} \quad \text{and} \quad x_{ji_2}, \quad j = 1, \dots, n.$$

If the two equalities simultaneously hold, then all the off-diagonal elements are 0, *except* $x_{i_1i_2}$ *and* $x_{i_2i_1}$ *if* $i_1 \neq i_2$.

PROOF.

$$\sum_{j\neq i_1} b_j \geqq \sum_{j\neq i_1}\bigl(x_{jj}+\sum_{k\neq j}x_{jk}\bigr) \geqq \sum_{j\neq i_1} r_{jj}+\sum_{j\neq i_1}\sum_{k\neq j}x_{jk} \geqq \sum_{j\neq i_1} r_{jj}+\sum_{j\neq i_1}x_{ji_1} \geqq$$
$$\geqq c_{i_1}-s_{i_1i_1}+\sum_{j\neq i_1} r_{jj}$$

has to be satisfied, and since equality holds:

$$\sum_{j\neq i_1}\sum_{k\neq j}x_{jk} = \sum_{j\neq i_1}x_{ji_1}.$$

$$\sum_{j\neq i_2} d_j \geqq \sum_{j\neq i_2}\bigl(x_{jj}+\sum_{k\neq j}x_{kj}\bigr) \geqq \sum_{j\neq i_2} r_{jj}+\sum_{j\neq i_2}\sum_{k\neq j}x_{kj} \geqq \sum_{j\neq i_2} r_{jj}+\sum_{j\neq i_2}x_{i_2j} \geqq$$
$$\geqq a_{i_2}-s_{i_2i_2}+\sum_{j\neq i_2} r_{jj}$$

has to be satisfied, and since equality holds:

$$\sum_{j\neq i_2}\sum_{k\neq j}x_{kj} = \sum_{j\neq i_2}x_{i_2j}.$$

This result for the case of $r_{ii}=s_{ii}(i=1, \dots, n)$ is due to P. ERDŐS and H. MINC [1].

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ON DIFFERENCE SETS OF SEQUENCES OF INTEGERS. III

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1. Let $\mathcal{B}$ be a set of positive integers $b_1 < b_2 < \dots$. A set of positive integers $u_1 < u_2 < \dots$ will be called an $\mathcal{A}$-set relative to $\mathcal{B}$ if its difference set does not contain an element of $\mathcal{B}$; in other words, if

(1) $$u_x - u_y = b_z$$

is not solvable in positive integers x, y, z.

L. Lovász conjectured that if $u_1 < u_2 < \dots$ is an $\mathcal{A}$-set relative to the set of the squares of the positive integers (i.e. $u_x - u_y = z^2$ is not solvable in positive integers x, y, z) then

(2) $$\sum_{u_i \leqq x} 1 = o(x)$$

must hold. In Part I of this series (see [10]), I proved this conjecture in the following sharper form: if $u_1 < u_2 < \dots$ is an $\mathcal{A}$-set relative to the set of the squares then

(3) $$\sum_{u_i \leqq x} 1 = O\left(x \frac{(\log\log x)^{2/3}}{(\log x)^{1/3}}\right).$$

I proved this theorem by adapting that version of the Hardy—Littlewood method which has been elaborated by K. F. ROTH in [4] and [5], in order to prove that if a set of positive integers $u_1 < u_2 < \dots$ does not contain an arithmetic progression of three terms, then (2) must hold, more exactly,

(4) $$\sum_{u_i \leqq x} 1 = O\left(\frac{x}{\log\log x}\right).$$

(In Part II of this series, I gave a lower estimate for

$$\max \sum_{u_i \leqq x} 1$$

where the maximum is taken for those sets $u_1 < u_2 < \dots$ which form an $\mathcal{A}$-set relative to the set $1^2, 2^2, \dots, n^2, \dots$; see [11].)

In the case of the arithmetic progressions of three terms, we may use the following simple fact:

(i) A set $a+qu_1, a+qu_2, \dots, a+qu_t$ (where a is an integer and $t, q, u_1, u_2, \dots, u_t$ are positive integers) does not contain an arithmetic progression of three terms if and only if also the set $u_1, u_2, \dots, u_t$ has this property.

This fact plays a role of basic importance in the proof of (4). In the proof of (3), I could replace this assertion by the following one:

(ii) A set $a+q^2u_1, a+q^2u_2, \ldots, a+q^2u_t$ (where a is an integer and $t, q, u_1, u_2, \ldots, u_t$ are positive integers) is an $\mathcal{A}$-set relative to the set of the squares if and only if also the set $u_1, u_2, \ldots, u_t$ has this property.

(Note that here we have q^2 in place of q.)

Starting out from (3), one might like to show that (2) must hold also for sequences $u_1<u_2<\ldots$ which form an $\mathcal{A}$-set relative to certain other fixed set $b_1<b_2<\ldots$, e.g. relative to

(5) $$b_i = i^k$$

(where $k \geqq 3$ is a fixed integer and $i=1, 2, \ldots$),

(6) $$b_i = f(i)$$

(wher $f(x)$ is a fixed polynomial with integral coefficients and $i=1, 2, \ldots$) and

(7) $$b_i = p_i$$

(where p_i denotes the i^{th} prime number and $i=1, 2, \ldots$), respectively.

The case (5) can be treated in the same way as the special case $k=2$; namely, the analogue of (ii) holds also in the general case $k \geqq 2$ with q^k in place of q^2. Thus it can be shown by the method used in [10] that if the set $u_1<u_2<\ldots$ forms an $\mathcal{A}$-set relative to the set (5) (also in case $k \geqq 3$) then (2) must hold.

On the other hand, in cases (6) and (7), simple counter examples can be given. Namely, let $f(x)=x^2+1$ and $u_1=6$, $u_2=12, \ldots, u_i=6i, \ldots$. Then (2) does not hold, however, $3 \mid u_x-u_y$ and $6 \mid u_x-u_y$ thus $u_x-u_y \neq b_z = z^2+1$ and $u_x-u_y \neq b_z = p_z$ (for $1 \leqq y < x$, $z=1, 2, \ldots$).

P. Erdős raised the conjecture that if

(8) $$b_i = i^2-1$$

(i.e. $f(x)=x^2-1$ in (6)) respectively

(9) $$b_i = p_i-1$$

(for $i=1, 2, \ldots$), and $u_1<u_2<\ldots$ forms an $\mathcal{A}$-set relative to the set $b_1<b_2<\ldots$, then (2) must hold.

In both cases the difficulty is that an analogue of (i) or (ii) does not exist; thus we have to modify Roth's method. We shall be able to avoid this difficulty by using estimates for exponential sums of the form

(10) $$\sum_{\substack{b_i \leqq x \\ q \mid b_i}} e(b_i\alpha)$$

where q is small in terms of x. (Throughout this paper, we use the notation $e^{2\pi i\alpha} = e(\alpha)$ where α is real.)

Since the cases (8) and (9) can be investigated analogously, we are going to discuss only the case (9). The remaining part of this paper will be devoted to the discussion of this case, i.e. the solvability of the equation

$$u_x - u_y = p_z - 1. \tag{11}$$

Consequently, we shall write briefly "$\mathcal{A}$-set" instead of "$\mathcal{A}$-set relative to the set $p_1-1, p_2-1, \dots, p_i-1, \dots$".

For $x=1, 2, \dots,$ let $A(x)$ denote the greatest number of integers that can be selected from $1, 2, \dots, x$ to form an $\mathcal{A}$-set and let us write

$$a(x) = \frac{A(x)}{x}.$$

We shall prove the following

THEOREM.

$$a(x) = O\left(\frac{(\log\log\log x)^3 (\log\log\log\log x)}{(\log\log x)^2}\right). \tag{12}$$

Throughout this paper, we use the following notations:

We denote the distance of the real number x from the nearest integer by $\|x\|$, i.e. $\|x\| = \min\{x-[x], [x]+1-x\}$. If a, b are real numbers and $b>0$ then we define the symbol $\min\left\{a, \frac{b}{0}\right\}$ by

$$\min\left\{a, \frac{b}{0}\right\} = a. \tag{13}$$

$C, c_1, c_2, \dots, M_0, M_1, \dots$ will denote (positive) absolute constants. We shall use also Vinogradov's notation: if f and g are two functions such that $g \geqq 0$ and there exists an absolute constant C satisfying $|f| \leqq Cg$ then we write $f \ll g$.

2. In this section, we estimate exponential sums of the form

$$S(\alpha) = S_N(\alpha) = \sum_{p \leqq N} (\log p) e((p-1)\alpha)$$

and

$$P(\alpha) = P_{M,q}(\alpha) = \sum_{\substack{\frac{p-1}{q} \leqq M \\ q \mid p-1}} (\log p) e\left(\frac{p-1}{q}\alpha\right). \tag{14}$$

(Here and in what follows, we shall leave the indices if this cannot cause confusion.)

LEMMA 1. *Let u be an arbitrary positive real number, M a positive integer for which $M \to +\infty$, and b, q, m integers satisfying*

$$1 \leqq b < (\log M)^u \tag{15}$$

and

$$1 \leqq q < (\log M)^u. \tag{16}$$

Then there exists an absolute constant $c_1>0$ *such that*

$$(17)\qquad \sum_{\substack{\frac{p-1}{q}\leq M\\ q|p-1\\ \frac{p-1}{q}\equiv m\,(\mathrm{mod}\, b)}} \log p = \begin{cases} \dfrac{Mq}{\varphi(bq)}+O\left(Me^{-c_1\sqrt{\log M}}\right) & \text{for}\quad (mq+1, b)=1\\ O\left(Me^{-c_1\sqrt{\log M}}\right) & \text{for}\quad (mq+1, b)>1\end{cases}$$

(*where* c_1 *and the implicite constant in the error term may depend on* u *but not on* b, q, m).

PROOF. The conditions $q|p-1$ and $\frac{p-1}{q}\equiv m \pmod b$ can be rewritten in the equivalent form

$$(18)\qquad p\equiv mq+1 \pmod{bq}.$$

Thus for $(mq+1, bq)=1$, i.e. $(mq+1, b)=1$, we have to show that

$$\sum_{\substack{p\leq Mq+1\\ p\equiv mq+1\,(\mathrm{mod}\, bq)}} \log p = \frac{Mq}{\varphi(bq)}+O\left(Me^{-c_1\sqrt{\log M}}\right);$$

but this is a consequence of the prime number theorem of the arithmetic progressions of small $\left(<(\log M)^u\right)$ modulus (see e.g. [3], pp. 136 and 144).

For $(mq+1, bq)>1$, i.e. $(mq+1, b)>1$, (18) implies that $(mq+1, b)|p$. Hence, $(mq+1, b)$ is a prime number and $p=(mq+1, b)$. Thus in this case, the left hand side of (17) consists of the single term

$$\log p = \log(mq+1, b)\leq \log b < \log(\log M)^u = u\log\log M = o\left(Me^{-c_1\sqrt{\log M}}\right)$$

which completes the proof of Lemma 1.

LEMMA 2. *Let* u *be an arbitrary positive real number,* M *a positive integer for which* $M\to+\infty$, *and* a, b, q *integers satisfying* (15), (16) *and* $(a, b)=1$. *Let us define the integer* $m_{b,q}$ *for* $(b, q)=1$ *by*

$$(19)\qquad m_{b,q}q+1\equiv 0 \pmod b \quad \text{and}\quad 0\leq m_{b,q}\leq b-1.$$

Then there exists an absolute constant $c_2>0$ *such that*

$$(20)\qquad P\left(\frac{a}{b}\right)=P_{M,q}\left(\frac{a}{b}\right)=$$

$$=\begin{cases}\dfrac{Mq}{\varphi(bq)}\mu(b)e\left(m_{b,q}\dfrac{a}{b}\right)+O\left(Me^{-c_2\sqrt{\log M}}\right) & \text{for}\quad (b, q)=1\\ O\left(Me^{-c_2\sqrt{\log M}}\right) & \text{for}\quad (b, q)>1\end{cases}$$

(*where* c_2 *and the implicite constant in the error term may depend on* u *but not on* a, b, q).

PROOF. By (15) and Lemma 1,

(21)
$$P\left(\frac{a}{b}\right) = P_{M,q}\left(\frac{a}{b}\right) = \sum_{\substack{\frac{p-1}{q} \leqq M \\ q \mid p-1}} (\log p) e\left(\frac{p-1}{q} \cdot \frac{a}{b}\right) =$$

$$= \sum_{m=0}^{b-1} e\left(m\frac{a}{b}\right) \sum_{\substack{\frac{p-1}{q} \leqq M \\ q \mid p-1 \\ \frac{p-1}{q} \equiv m \pmod b}} \log p =$$

$$= \sum_{\substack{0 \leqq m \leqq b-1 \\ (mq+1, b)=1}} e\left(m\frac{a}{b}\right)\frac{Mq}{\varphi(bq)} + O\left(\sum_{m=0}^{b-1} Me^{-c_1\sqrt{\log M}}\right) =$$

$$= \frac{Mq}{\varphi(bq)} \sum_{\substack{0 \leqq m \leqq b-1 \\ (mq+1, b)=1}} e\left(m\frac{a}{b}\right) + O\left((\log M)^u Me^{-c_1\sqrt{\log M}}\right) =$$

$$= \frac{Mq}{\varphi(bq)} \sum_{\substack{0 \leqq m \leqq b-1 \\ (mq+1, b)=1}} e\left(m\frac{a}{b}\right) + O\left(Me^{-c_2\sqrt{\log M}}\right).$$

Here

(22)
$$\sum_{\substack{0 \leqq m \leqq b-1 \\ (mq+1, b)=1}} e\left(m\frac{a}{b}\right) = \sum_{m=0}^{b-1} e\left(m\frac{a}{b}\right) \sum_{d \mid (mq+1, b)} \mu(d) =$$

$$= \sum_{d \mid b} \mu(d) \sum_{\substack{0 \leqq m \leqq b-1 \\ d \mid mq+1}} e\left(m\frac{a}{b}\right).$$

Let m_0 denote the least non-negative integer m for which $d \mid mq+1$ holds. Then

(23)
$$m_0 q + 1 \equiv 0 \pmod d,$$

and $d \mid mq+1$ holds if and only if

(24)
$$(mq+1) - (m_0 q + 1) = (m - m_0) q \equiv 0 \pmod d.$$

By (23),

(25)
$$(d, q) = 1.$$

(24) and (25) imply that $d|mq+1$ holds if and only if $m-m_0\equiv 0 \pmod d$. Thus with respect to $(a, b)=1$, the inner sum in (22) is

$$\sum_{\substack{0\leq m\leq b-1\\ d|mq+1}} e\left(m\frac{a}{b}\right) = \sum_{j=0}^{b/d-1} e\left((m_0+jd)\frac{a}{b}\right) =$$

$$= \begin{cases} \dfrac{b}{d} e\left(m_0\dfrac{a}{b}\right) & \text{for} \quad b|da \quad \text{i.e.} \quad b|d \\ e\left(m_0\dfrac{a}{b}\right)\dfrac{1-e\left(\dfrac{b}{d}\cdot d\dfrac{a}{b}\right)}{1-e\left(d\dfrac{a}{b}\right)} = 0 & \text{for} \quad b\nmid d. \end{cases}$$

Hence, the inner sum in (22) is different from 0 only if $b|d$; but by $d|b$, this implies that $b=d$, and by (25), also $(b, q)=1$ must hold. Thus we obtain from (22) that

$$\sum_{\substack{0\leq m\leq b-1\\ (mq+1,b)=1}} e\left(m\frac{a}{b}\right) =$$

$$= \begin{cases} \mu(b) \displaystyle\sum_{\substack{0\leq m\leq b-1\\ b|mq+1}} e\left(m\frac{a}{b}\right) = \mu(b)\cdot\frac{b}{b} e\left(m_0\frac{a}{b}\right) = \mu(b)e\left(m_0\frac{a}{b}\right) & \text{for} \quad (b, q)=1, \\ 0 \quad \text{for} \quad (b, q)>1 \end{cases}$$

where m_0 satisfies (23), i.e. $m_0q+1\equiv 0 \pmod b$; hence, $m_0=m_{b,q}$. Putting this into (21), we obtain (20) and the proof of Lemma 2 is complete.

LEMMA 3. *Let u be an arbitrary positive real number, M a positive integer for which $M\to+\infty$, a, b, q integers satisfying* (15), (16) *and* $(a, b)=1$, *finally, β any real number. Then*

$$P\left(\frac{a}{b}+\beta\right) = P_{M,q}\left(\frac{a}{b}+\beta\right) = \tag{26}$$

$$= \begin{cases} \dfrac{q}{\varphi(bq)}\mu(b)e\left(m_{b,q}\dfrac{a}{b}\right)\displaystyle\sum_{n=1}^{M} e(n\beta)+O\left((M|\beta|+1)Me^{-c_2\sqrt{\log M}}\right) & \textit{for} \quad (b, q)=1 \\ O\left((M|\beta|+1)Me^{-c_2\sqrt{\log M}}\right) & \textit{for} \quad (b, q)>1 \end{cases}$$

where $m_{b,q}$ is defined (*for* $(b, q)=1$) *by* (19).

PROOF. Applying Lemma 2, we obtain by partial summation that

$$P_{M,q}\left(\frac{a}{b}+\beta\right) = \sum_{\substack{\frac{p-1}{q}\leqq M \\ q\mid p-1}} (\log p)\, e\left(\frac{p-1}{q}\left(\frac{a}{b}+\beta\right)\right) = \tag{27}$$

$$= \sum_{\substack{\frac{p-1}{q}\leqq M \\ q\mid p-1}} \left\{(\log p)\, e\left(\frac{p-1}{q}\cdot\frac{a}{b}\right)\right\} e\left(\frac{p-1}{q}\beta\right) =$$

$$= \sum_{n=1}^{M} \left(P_{n,q}\left(\frac{a}{b}\right) - P_{n-1,q}\left(\frac{a}{b}\right)\right) e(n\beta) =$$

$$= \sum_{n=1}^{M} P_{n,q}\left(\frac{a}{b}\right)\left(e(n\beta)-e((n+1)\beta)\right) + P_{M,q}\left(\frac{a}{b}\right) e((M+1)\beta).$$

For $1\leqq n\leqq\sqrt{M}$,

$$\left|P_{n,q}\left(\frac{a}{b}\right)\right| = \left|\sum_{\substack{\frac{p-1}{q}\leqq n \\ q\mid p-1}} (\log p)\, e\left(\frac{p-1}{q}\cdot\frac{a}{b}\right)\right| \leqq$$

$$\leqq \sum_{k=1}^{n} \log(qn+1) = n\log(qn+1) < \sqrt{M}\log\left((\log M)^u\sqrt{M}+1\right) = O\left(\sqrt{M}\log M\right)$$

and

$$\left|\frac{nq}{\varphi(bq)}\mu(b)\, e\left(m_{b,q}\frac{a}{b}\right)\right| \leqq nq < \sqrt{M}(\log M)^u$$

(with respect to (16)).

For $\sqrt{M}<n\leqq M$, (16) implies that

$$1\leqq q<(\log M)^u<(\log n^2)^u = 2^u(\log n)^u<(\log n)^{2u}$$

(if M is sufficiently large depending on u) thus Lemma 2 can be applied with $2u$ and n in place of u and M, respectively.

Summarizing, we obtain from (27) (using Lemma 2) that for $(b, q)=1$

$$P_{M,q}\left(\frac{a}{b}+\beta\right) = \left\{\sum_{n=1}^{M}\frac{nq}{\varphi(bq)}\mu(b)\, e\left(m_{b,q}\frac{a}{b}\right)\left(e(n\beta)-e((n+1)\beta)\right) + \right.$$

$$\left. + \frac{Mq}{\varphi(bq)}\mu(b)\, e\left(m_{b,q}\frac{a}{b}\right) e((M+1)\beta)\right\} +$$

$$+ \left\{\sum_{n=1}^{M}\left(P_{n,q}\left(\frac{a}{b}\right) - \frac{nq}{\varphi(bq)}\mu(b)\, e\left(m_{b,q}\frac{a}{b}\right)\right)\left(e(n\beta)-e((n+1)\beta)\right) + \right.$$

$$\left. + \left(P_{M,q}\left(\frac{a}{b}\right) - \frac{Mq}{\varphi(bq)}\mu(b)\, e\left(m_{b,q}\frac{a}{b}\right)\right) e((M+1)\beta)\right\} =$$

$$= \frac{q}{\varphi(bq)}\mu(b)e\left(m_{b,q}\frac{a}{b}\right)\sum_{n=1}^{M}e(n\beta)+$$

$$+\sum_{n=1}^{[\sqrt{M}]}O\left(\sqrt{M}\log M+\sqrt{M}(\log M)^{u}\right)\left|e(n\beta)-e((n+1)\beta)\right|+$$

$$+\sum_{n=[\sqrt{M}]+1}^{M}O\left(ne^{-c_2\sqrt{\log n}}\right)\left|e(n\beta)-e((n+1)\beta)\right|+O\left(Me^{-c_2\sqrt{\log M}}\right)=$$

$$= \frac{q}{\varphi(bq)}\mu(b)e\left(m_{b,q}\frac{a}{b}\right)\sum_{n=1}^{M}e(n\beta)+\sum_{n=1}^{[\sqrt{M}]}O\left(Me^{-c_2\sqrt{\log M}}|\beta|\right)+$$

$$+\sum_{n=[\sqrt{M}]+1}^{M}O\left(Me^{-c_2\sqrt{\log M}}|\beta|\right)+O\left(Me^{-c_2\sqrt{\log M}}\right)=$$

$$= \frac{q}{\varphi(bq)}\mu(b)e\left(m_{b,q}\frac{a}{b}\right)\sum_{n=1}^{M}e(n\beta)+O\left((M|\beta|+1)Me^{-c_2\sqrt{\log M}}\right)$$

while for $(b, q)>1$,

$$P_{M,q}\left(\frac{a}{b}+\beta\right)=\sum_{n=1}^{[\sqrt{M}]}P_{n,q}\left(\frac{a}{b}\right)\left(e(n\beta)-e((n+1)\beta)\right)+$$

$$+\sum_{n=[\sqrt{M}]+1}^{M}P_{n,q}\left(\frac{a}{b}\right)\left(e(n\beta)-e((n+1)\beta)\right)+P_{M,q}\left(\frac{a}{b}\right)e((M+1)\beta)=$$

$$=\sum_{n=1}^{[\sqrt{M}]}O\left(\sqrt{M}\log M\right)\left|e(n\beta)-e((n+1)\beta)\right|+$$

$$+\sum_{n=[\sqrt{M}]+1}^{M}O\left(ne^{-c_2\sqrt{\log n}}\right)\left|e(n\beta)-e((n+1)\beta)\right|+O\left(Me^{-c_2\sqrt{\log M}}\right)=$$

$$=\sum_{n=1}^{[\sqrt{M}]}O\left(Me^{-c_2\sqrt{\log M}}|\beta|\right)+\sum_{n=[\sqrt{M}]+1}^{M}O\left(Me^{-c_2\sqrt{\log M}}|\beta|\right)+$$

$$+O\left(Me^{-c_2\sqrt{\log M}}\right)=O\left((M|\beta|+1)Me^{-c_2\sqrt{\log M}}\right)$$

since

$$|e(n\beta)-e((n+1)\beta)|=|1-e(\beta)|=|e(-\beta/2)-e(\beta/2)|=2\,|\sin\pi\beta|\leqq 2\pi\,|\beta|$$

and the proof of Lemma 3 is complete.

Lemma 4. *If a, b are integers such that $a\leqq b$, and β is an arbitrary real number then*

$$\left|\sum_{k=a}^{b}e(k\beta)\right|\leqq\min\left\{b-a+1,\frac{1}{2\|\beta\|}\right\}.$$

(*For* $\|\beta\|=0$, *the right hand side is defined by* (13).)

This lemma is identical to Lemma 1 in [10].

LEMMA 5. *Let u be an arbitrary positive real number. There exist constants M_0, $c_3>0$ (which may depend on u) such that if $M>M_0$, furthermore, a, b, q are integers satisfying* (15), (16) *and* $(a, b)=1$, *finally, β is a real number satisfying*

$$(28)\qquad |\beta| \leqq \frac{e^{c_3\sqrt{\log M}}}{M},$$

then

$$(29)\qquad \left|P\left(\frac{a}{b}+\beta\right)\right| = \left|P_{M,q}\left(\frac{a}{b}+\beta\right)\right| < \begin{cases} 2\dfrac{Mq}{\varphi(b)\varphi(q)} & \text{for} \quad |\beta| \leqq \dfrac{1}{M} \\ \dfrac{q}{\varphi(b)\varphi(q)|\beta|} & \text{for} \quad \dfrac{1}{M} \leqq |\beta|. \end{cases}$$

PROOF. We are going to apply Lemma 3.

For $(b, q)=1$, the main term in (26) in Lemma 3 can be estimated in the following way, by using Lemma 4 (and with respect to (28)):

$$\left|\frac{q}{\varphi(bq)}\mu(b)e\left(m_{b,q}\frac{a}{b}\right)\sum_{n=1}^{M} e(n\beta)\right| \leqq$$

$$\leqq \frac{q}{\varphi(bq)}\min\left\{M, \frac{1}{2\|\beta\|}\right\} = \frac{q}{\varphi(b)\varphi(q)}\min\left\{M, \frac{1}{2|\beta|}\right\} \leqq$$

$$\leqq \begin{cases} \dfrac{Mq}{\varphi(b)\varphi(q)} & \text{for} \quad |\beta| \leqq \dfrac{1}{M} \\ \dfrac{q}{2\varphi(b)\varphi(q)|\beta|} & \text{for} \quad \dfrac{1}{M} \leqq |\beta|. \end{cases}$$

Thus Lemma 3 yields that

$$\left|P_{M,q}\left(\frac{a}{b}+\beta\right)\right| < O\left((M|\beta|+1)Me^{-c_2\sqrt{\log M}}\right) + \begin{cases} \dfrac{Mq}{\varphi(b)\varphi(q)} & \text{for} \quad |\beta| \leqq \dfrac{1}{M} \\ \dfrac{q}{2\varphi(b)\varphi(q)|\beta|} & \text{for} \quad \dfrac{1}{M} \leqq |\beta|. \end{cases}$$

To obtain (29) from this inequality, it suffices to show that here the first term on the right (the $O(\ldots)$ term) is less than the second term. The first term is the greatest and the second is the least if $|\beta|$ is the possibly greatest, i.e. $|\beta|=e^{c_3\sqrt{\log M}}/M$. Then the first term is

$$(30)\qquad O\left((e^{c_3\sqrt{\log M}}+1)Me^{-c_2\sqrt{\log M}}\right) = O\left(Me^{(c_3-c_2)\sqrt{\log M}}\right)$$

while the second term is (with respect to (16) and for large M)

$$\frac{qM}{2\varphi(b)\varphi(q)e^{c_3\sqrt{\log M}}} > \frac{M}{2be^{c_3\sqrt{\log M}}} > \frac{M}{2(\log M)^u e^{c_3\sqrt{\log M}}} > Me^{-2c_3\sqrt{\log M}}.$$

For $c_3=c_2/4$ and $M>M_1(u)$, the latter is greater than (30) and Lemma 5 is proved.

LEMMA 6. *If X, Y are real numbers, a, b integers and α a real number such that $Y \leqq b \leqq X/Y$, $1 \leqq Y \leqq X^{1/4}$, $(a, b)=1$ and*

$$\left|\alpha - \frac{a}{b}\right| \leqq \frac{1}{b^2}$$

then

$$|S_X(\alpha)| = \Big|\sum_{p \leqq X} (\log p)\, e\big((p-1)\alpha\big)\Big| = \Big|\sum_{p \leqq X} (\log p)\, e(p\alpha)\Big| \ll XY^{-1/2}(\log X)^{17}.$$

This is essentially a consequence of Theorems 1 and 3 of VINOGRADOV in [12], Chapter IX; see also MONTGOMERY [1], Chapter 16, and MONTGOMERY—VAUGHAN [2], Lemma 3.1.

LEMMA 7. *If $M(>0)$, q, a, b are integers and α is a real number satisfying*

$$1 \leqq q \leqq \log M \tag{31}$$

and

$$(a, b) = 1, \tag{32}$$

furthermore, writing

$$Q = M(\log M)^{-41}, \tag{33}$$

also

$$2(\log M)^{40} \leqq b \leqq Q \tag{34}$$

and

$$\left|\alpha - \frac{a}{b}\right| < \frac{1}{bQ} \tag{35}$$

hold then for large M,

$$|P(\alpha)| = |P_{M,q}(\alpha)| \ll \frac{M}{(\log M)^2}. \tag{36}$$

PROOF.

$$|P_{M,q}(\alpha)| = \left|\sum_{\substack{\frac{p-1}{q} \leqq M \\ q \mid p-1}} (\log p)\, e\left((p-1)\frac{\alpha}{q}\right)\right| = \tag{37}$$

$$= \left|\sum_{p \leqq qM+1} (\log p)\, e\left((p-1)\frac{\alpha}{q}\right)\left\{\frac{1}{q}\sum_{j=0}^{q-1} e\left((p-1)\frac{j}{q}\right)\right\}\right| =$$

$$= \frac{1}{q}\left|\sum_{j=0}^{q-1}\sum_{p \leqq qM+1} (\log p)\, e\left((p-1)\frac{\alpha+j}{q}\right)\right| =$$

$$= \frac{1}{q}\left|\sum_{j=0}^{q-1} S_{qM+1}\left(\frac{\alpha+j}{q}\right)\right| \leqq \frac{1}{q}\sum_{j=0}^{q-1}\left|S_{qM+1}\left(\frac{\alpha+j}{q}\right)\right|.$$

Let us write $\gamma = \frac{\alpha + j}{q}$. By Dirichlet's theorem, there exist integers A, B such that

(38) $$(A, B) = 1,$$

(39) $$1 \leqq B \leqq 2qQ$$

and

(40) $$\left|\gamma - \frac{A}{B}\right| < \frac{1}{2BqQ};$$

by (39) and (40), also

(41) $$\left|\gamma - \frac{A}{B}\right| < \frac{1}{B^2}$$

holds.

We are going to show that these conditions imply that

(42) $$B > \frac{1}{2} b.$$

Let us assume indirectly that

(43) $$B \leqq \frac{1}{2} b.$$

By (35), γ can be written in the form

(44) $$\gamma = \frac{\alpha + j}{q} = \frac{\frac{a}{b} + \frac{\theta_1}{bQ} + j}{q} = \frac{a + bj}{bq} + \frac{\theta_1}{bqQ}$$

where $|\theta_1| < 1$. Let us define the integer U and the positive integer V by

(45) $$\frac{a + bj}{bq} = \frac{U}{V},$$

(46) $$(U, V) = 1.$$

By (32), $(a + bj, b) = 1$, thus

(47) $$(a + bj, bq) \leqq q.$$

(45), (46) and (47) imply that

(48) $$b \leqq V \leqq bq.$$

By (40), γ can be written in the form

(49) $$\gamma = \frac{A}{B} + \frac{\theta_2}{2BqQ}$$

where $|\theta_2| < 1$.

(44) and (49) yield that

$$\gamma = \frac{U}{V} + \frac{\theta_1}{bqQ} = \frac{A}{B} + \frac{\theta_2}{2BqQ},$$

hence, with respect to (34) and (48),

$$(50)\qquad \left|\frac{U}{V}-\frac{A}{B}\right| \leqq \frac{|\theta_1|}{bqQ}+\frac{|\theta_2|}{2BqQ} < \frac{1}{bqQ}+\frac{1}{2BqQ} \leqq$$

$$\leqq \frac{1}{bqQ}+\frac{1}{2Bqb} \leqq \frac{1}{bqQ}+\frac{1}{2BV}.$$

On the other hand, we obtain from (38), (43), (46) and (48) that

$$\frac{U}{V} \neq \frac{A}{B},$$

thus

$$(51)\qquad \left|\frac{U}{V}-\frac{A}{B}\right| = \frac{|UB-VA|}{VB} \geqq \frac{1}{VB}.$$

(50) and (51) yield that

$$\frac{1}{VB} < \frac{1}{bqQ}+\frac{1}{2BV}, \quad \frac{1}{2VB} < \frac{1}{bqQ},$$

hence, with respect to (34), (43) and (48),

$$1 < 2\frac{VB}{bqQ} = 2V\cdot\frac{1}{qQ}\cdot\frac{B}{b} \leqq 2bq\cdot\frac{1}{qQ}\cdot\frac{1}{2} = \frac{b}{Q} \leqq 1.$$

Thus the indirect assumption (43) leads to a contradiction, which proves (42).

Let us write $X=qM+1$, $Y=(\log M)^{40}$. Then for large M,

$$(52)\qquad 1 \leqq Y = (\log M)^{40} < (M+1)^{1/4} \leqq X^{1/4},$$

furthermore, by (34) and (42),

$$(53)\qquad B > \frac{1}{2}b \geqq (\log M)^{40} = Y,$$

finally, by (33) and (39),

$$(54)\qquad B \leqq 2qQ = 2qM(\log M)^{-41} \leqq 2(qM+1)(\log M)^{-41} <$$

$$< (qM+1)(\log M)^{-40} = X/Y.$$

In view of (38), (41), (52), (53) and (54), Lemma 6 can be applied with $qM+1$, $(\log M)^{40}$, A, B and γ in place of X, Y, a, b and α. With respect to (31), we obtain that

$$|S_{qM+1}(\gamma)| = \left|S_{qM+1}\left(\frac{\alpha+j}{q}\right)\right| \ll (qM+1)\left((\log M)^{40}\right)^{-1/2}\left(\log (qM+1)\right)^{17} \leqq$$

$$\leqq \left((\log M)M+1\right)(\log M)^{-20}\left\{\log\left((\log M)M+1\right)\right\}^{17} \ll$$

$$\ll (\log M)M(\log M)^{-20}(\log M)^{17} = M(\log M)^{-2}.$$

Putting this into (37), we obtain that

$$|P_{M,q}(\alpha)| \ll \frac{1}{q}\sum_{j=0}^{q-1} M(\log M)^{-2} = \frac{M}{(\log M)^2}$$

which completes the proof of Lemma 7.

LEMMA 8. *There exists an absolute constant* $c_4(>0)$ *such that for* $n \geqq 3$,

$$\varphi(n) > c_4 \frac{n}{n \log\log n}.$$

This lemma is well-known; see e.g. [3], p. 24.

LEMMA 9. *Let* q, M *be positive integers,* R *a real number such that*

$$q \leqq \log M \tag{55}$$

and

$$3 \leqq R \leqq \log M. \tag{56}$$

Let $S_{R,M}$ *denote the set of those real numbers* α *for which* $0 \leqq \alpha \leqq 1$ *holds and there do not exist integers* a, b *such that*

$$(a, b) = 1, \tag{57}$$

$$1 \leqq b < R \tag{58}$$

and

$$\left|\alpha - \frac{a}{b}\right| < \frac{1}{M}\cdot\frac{R}{\log\log R}. \tag{59}$$

Then for $\alpha \in S_{R,M}$ *and large* M,

$$|P_{M,q}(\alpha)| \ll \frac{qM}{\varphi(q)}\cdot\frac{\log\log R}{R}. \tag{60}$$

PROOF. Let us define Q by (33). By Dirichlet's theorem, for all $\alpha \in S_{R,M}$, there exist integers A, B such that

$$(A, B) = 1, \tag{61}$$

$$1 \leqq B \leqq Q \tag{62}$$

and

$$\left|\alpha - \frac{A}{B}\right| < \frac{1}{BQ}. \tag{63}$$

If $2(\log M)^{40} \leqq B$, then Lemma 7 can be applied, with A and B in place of a and b, respectively. We obtain that

$$|P_{M,q}(\alpha)| \ll \frac{M}{(\log M)^2}. \tag{64}$$

By (56), the right hand side of (60) can be estimated in the following way:

$$(65)\qquad \frac{q}{\varphi(q)}\cdot M\cdot\frac{\log\log R}{R}\geqq M\frac{\log\log R}{R}\geqq M\frac{\log\log\log M}{\log M}>\frac{M}{(\log M)^2}$$

for sufficiently large M. (64) and (65) yield (60).

If

$$(66)\qquad B<2(\log M)^{40}$$

and M is large then we may apply Lemma 5 with $a=A$, $b=B$, $\beta=\alpha-\frac{A}{B}$ and $u=41$. Namely, for large M, (15) and (16) hold by (55) and (66). Furthermore, by (63),

$$|\beta|=\left|\alpha-\frac{A}{B}\right|<\frac{1}{BQ}\leqq\frac{1}{Q}=\frac{(\log M)^{41}}{M},$$

which implies (28) for sufficiently large M. Thus, in fact, all the assumptions in Lemma 5 hold. Applying Lemma 5, we obtain that for large M,

$$(67)\qquad |P_{M,q}(\alpha)|<\begin{cases}2\dfrac{Mq}{\varphi(B)\varphi(q)} & \text{for}\quad |\beta|\leqq\dfrac{1}{M}\\[2ex] \dfrac{q}{\varphi(B)\varphi(q)|\beta|} & \text{for}\quad \dfrac{1}{M}\leqq|\beta|.\end{cases}$$

The right hand side is maximal for $|\beta|\leqq\frac{1}{M}$. Thus for $R\leqq B$, we obtain by applying Lemma 8 that

$$|P_{M,q}(\alpha)|<\frac{2Mq}{\varphi(B)\varphi(q)}\ll\frac{\log\log B}{B}\cdot\frac{Mq}{\varphi(q)}\ll\frac{\log\log R}{R}\cdot\frac{qM}{\varphi(q)}$$

(with respect to $R\geqq 3$).

Finally, if $B<R$ then $\alpha\in S_{R,M}$ implies that

$$(68)\qquad |\beta|=\left|\alpha-\frac{A}{B}\right|\geqq\frac{1}{M}\cdot\frac{R}{\log\log R}$$

which yields also $|\beta|>\frac{1}{M}$ since it can be shown easily that

$$\frac{R}{\log\log R}>1$$

for $R\geqq 3$. Thus we obtain from (67) and (68) that

$$|P_{M,q}(\alpha)|<\frac{q}{\varphi(B)\varphi(q)|\beta|}\leqq\frac{q}{\varphi(q)}\cdot\frac{1}{|\beta|}\leqq\frac{q}{\varphi(q)}\cdot M\cdot\frac{\log\log R}{R}$$

which completes the proof of Lemma 9.

3. For arbitrary positive integers M, q, let

$$u_1 q, u_2 q, \dots, u_T q \tag{69}$$

be *a* maximal $\mathcal{A}$-set selected from $q, 2q, \dots, Mq$, and let

$$F(\alpha) = F_{M,q}(\alpha) = \sum_{k=1}^{T} e(u_k \alpha). \tag{70}$$

In this section, we estimate this function $F_{M,q}(\alpha)$.

For an integer b and positive integers m, x, let $A_{(b,m)}(x)$ denote the greatest number of integers that can be selected from $b+m, b+2m, \dots, b+xm$ to form an $\mathcal{A}$-set (so that $A_{(0,1)}(x) = A(x)$).

LEMMA 10. *For any integers b, d and positive integers m, x, we have*

$$A_{(b,m)}(x) = A_{(d,m)}(x).$$

PROOF. This follows trivially from the fact that the numbers $b+u_1 m$, $b+u_2 m$, $\dots, b+u_k m$ form an $\mathcal{A}$-set if and only if also the numbers $d+u_1 m$, $d+u_2 m, \dots,$ $d+u_k m$ do.

By Lemma 10, we may simplify the notation $A_{(b,m)}(x)$ in the following way: let us write $A_m(x)$ instead of $A_{(b,m)}(x)$, i.e. let

$$A_m(x) = A_{(b,m)}(x) \quad (\text{for } b = 0, \pm 1, \pm 2, \dots).$$

Furthermore, let

$$a_m(x) = \frac{A_m(x)}{x},$$

so that $A(x) = A_1(x)$ and $a_1(x) = a(x)$; moreover, $T = A_q(M)$ in (69) and (70), thus

$$F(\alpha) = F_{M,q}(\alpha) = \sum_{k=1}^{A_q(M)} e(u_k \alpha). \tag{71}$$

Lemmas 11 and 12 follow trivially from the definitions of the functions $A_m(x)$ and $a_m(x)$, respectively.

LEMMA 11. *If m, x and y are positive integers such that $x \leqq y$ then $A_m(x) \leqq A_m(y)$.*

LEMMA 12. *For arbitrary positive integers m and x, we have $a_m(x) \leqq 1$.*

LEMMA 13. *For arbitrary positive integers m, x and y, we have*

$$A_m(x+y) \leqq A_m(x) + A_m(y), \tag{72}$$

$$A_m(xy) \leqq x A_m(y), \tag{73}$$

$$a_m(xy) \leqq a_m(y), \tag{74}$$

$$a_m(x) \leqq \left(1 + \frac{y}{x}\right) a_m(y). \tag{75}$$

Proof. By Lemma 10, the greatest number of integers that can be selected from $m, 2m, \ldots, xm$ and $(x+1)m, (x+2)m, \ldots, (x+y)m$ to form an $\mathcal{A}$-set, is $A_m(x)$ and $A_m(y)$, respectively; thus the greatest number of integers that can be selected from $m, 2m, \ldots, xm, (x+1)m, (x+2)m, \ldots, (x+y)m$ to form an $\mathcal{A}$-set, is $\leqq A_m(x)+$ $+A_m(y)$ which proves (72).

(73) is a consequence of (72).

Dividing (73) by xy, we obtain (74).

Finally, by Lemma 11 and (73),

$$A_m(x) \leqq A_m\left(\left(\left[\frac{x}{y}\right]+1\right)y\right) \leqq \left(\left[\frac{x}{y}\right]+1\right)A_m(y) \leqq$$

$$\leqq \left(\frac{x}{y}+1\right)A_m(y) = (x+y)\frac{A_m(y)}{y}.$$

Dividing by x, we obtain (75).

Lemma 14. *Let q, b, t, M be positive integers, a an integer, α, β real numbers such that*

$$\alpha - \frac{a}{b} = \beta. \tag{76}$$

Let

$$F^*(\alpha) = F^*_{M,q}(\alpha) = \frac{a_{bq}(t)}{b}\left(\sum_{s=1}^{b} e\left(\frac{as}{b}\right)\right)\left(\sum_{j=1}^{M} e(\beta j)\right),$$

so that if $(a, b)=1$ then

$$F^*_{M,q}(\alpha) = \begin{cases} a_q(t)\sum_{j=1}^{M} e(\beta j) & \text{for} \quad b=1 \\ 0 \quad \text{for} \quad b>1 & (\text{where} \quad (a,b)=1). \end{cases} \tag{77}$$

Then there exists an absolute constant c_5 such that

$$|F_{M,q}(\alpha)-F^*_{M,q}(\alpha)| \leqq \big(a_{bq}(t)-a_q(M)M\big)+c_5\big(|\beta|Ma_{bq}(t)+a_q(t)\big)tb. \tag{78}$$

Proof. We are going to show at first that

$$F_{M,q}(\alpha) = \frac{1}{tb}\sum_{s=1}^{b}\sum_{j=1}^{M}\sum_{\substack{j\leqq u_k<j+tb \\ u_k\equiv s \pmod b}} e(\alpha u_k)+O\big(a_q(t)tb\big). \tag{79}$$

Let us investigate the coefficient of $e(\alpha u_k)$ on the right hand side.

If $tb \leqq u_k \leqq M$ then we account $e(\alpha u_k)$ exactly tb times, namely for the following values of j:

$$j = u_k-tb+1, u_k-tb+2, \ldots, u_k.$$

Thus the coefficient of $e(\alpha u_k)$ is

$$tb \cdot \frac{1}{tb} = 1$$

in this case (and its coefficient is the same on the left hand side).

If

(80) $$1 \leqq u_k < tb$$

then we account $e(\alpha u_k)$ on the right of (79) for $j=1, 2, \ldots, u_k$, thus its coefficient is

$$(0 \leqq) \quad u_k \cdot \frac{1}{tb} < tb \cdot \frac{1}{tb} = 1$$

on the right and 1 on the left of (79). For the numbers u_k satisfying (80), the numbers $u_k q$ form an $\mathcal{A}$-set selected from $q, 2q, \ldots, tbq$ thus in view of (73) in Lemma 13, their number is

$$\leqq A_q(tb) \leqq A_q(t)b = a_q(t)tb.$$

These facts yield that, in fact, the error term in (79) is $O(a_q(t)tb)$.

The term $e(\alpha u_k)$ in the inner sum in (79) can be rewritten in the following way:

$$e(\alpha u_k) = e\left(\left(\frac{a}{b}+\beta\right)u_k\right) = e\left(\frac{au_k}{b}\right)e(\beta u_k) =$$

$$= e\left(\frac{as}{b}\right)e(\beta j)\,e\big(\beta(u_k-j)\big) = e\left(\frac{as}{b}\right)e(\beta j)\big(1+O(|\beta(u_k-j)|)\big) =$$

$$= e\left(\frac{as}{b}\right)e(\beta j)+O\big(|\beta(u_k-j)|\big) = e\left(\frac{as}{b}\right)e(\beta j)+O(|\beta|tb)$$

since $|u_k-j|<tb$ in the inner sum, and

$$|e(\gamma)-1| = |e(\gamma/2)-e(-\gamma/2)| = |2\sin \pi\gamma| \leqq 2|\pi\gamma| = 2\pi|\gamma|$$

for any real number γ.

Thus the inner sum in (79) can be estimated in the following way:

(81) $$\sum_{\substack{j\leqq u_k<j+tb \\ u_k\equiv s \pmod b}} e(\alpha u_k) = \sum_{\substack{j\leqq u_k<j+tb \\ u_k\equiv s \pmod b}} \left(e\left(\frac{as}{b}\right)e(\beta j)+O(|\beta|tb)\right) =$$

$$= \left(e\left(\frac{as}{b}\right)e(\beta j)+O(|\beta|tb)\right)\sum_{\substack{j\leqq u_k<j+tb \\ u_k\equiv s \pmod b}} 1.$$

Let us define the integer v by

$$v < j \leqq v+b, \quad v \equiv s \pmod b.$$

Then for the numbers u_k satisfying $j\leqq u_k<j+tb$ and $u_k\equiv s \pmod b$, the numbers $u_k q$ form an $\mathcal{A}$-set selected from $vq+bq, vq+2bq,, \ldots, vq+tbq$. Thus by Lemma 10,

$$\sum_{\substack{j\leqq u_k<j+tb \\ u_k\equiv s \pmod b}} 1 \leqq A_{(vq,bq)}(t) = A_{bq}(t) = a_{bq}(t)t.$$

Hence, defining $D(j, t, b, s)$ by

$$\sum_{\substack{j\leqq u_k<j+tb \\ u_k\equiv s \pmod b}} 1 = a_{bq}(t)t-D(j, t, b, s),$$

we have $D(j,t,q,s)\geqq 0$. Putting this into (81):

$$\sum_{\substack{j\leqq u_k<j+tb\\ u_k\equiv s \pmod b}} e(\alpha u_k)=\left(e\left(\frac{as}{b}\right)e(\beta j)+O(|\beta|tb)\left(a_{bq}(t)\right)t-D(j,t,b,s)\right)=$$

$$=e\left(\frac{as}{b}\right)e(\beta j)\left(a_{bq}(t)t-D(j,t,b,s)\right)+O\left(|\beta|a_{bq}(t)t^2b\right).$$

Thus (79) yields that

$$\text{(82)}\quad F_{M,q}(\alpha)=\frac{1}{tb}\sum_{s=1}^{b}\sum_{j=1}^{M}\left\{e\left(\frac{as}{b}\right)e(\beta j)\left(a_{bq}(t)t-D(j,t,b,s)\right)+O\left(|\beta|a_{bq}(t)t^2b\right)\right\}+$$
$$+O\left(a_q(t)tb\right)=\frac{a_{bq}(t)}{b}\left(\sum_{s=1}^{b}e\left(\frac{as}{b}\right)\right)\left(\sum_{j=1}^{M}e(\beta j)\right)-$$
$$-\frac{1}{tb}\sum_{s=1}^{b}\sum_{j=1}^{M}e\left(\frac{as}{b}\right)e(\beta j)D(j,t,b,s)+$$
$$+O\left(\frac{1}{tb}\cdot b\cdot M\cdot|\beta|a_{bq}(t)t^2b\right)+O\left(a_q(t)tb\right)=$$
$$=F^*_{M,q}(\alpha)-\frac{1}{tb}\sum_{s=1}^{b}\sum_{j=1}^{M}e\left(\frac{as}{b}\right)e(\beta j)D(j,t,b,s)+O\left(\left(|\beta|Ma_{bq}(t)+a_q(t)\right)tb\right).$$

Putting here $\alpha=\beta=a=0$, we obtain that

$$a_q(M)M=A_q(M)=a_{bq}(t)M-\frac{1}{tb}\sum_{s=1}^{b}\sum_{j=1}^{M}D(j,t,b,s)+O\left(a_q(t)tb\right),$$

hence

$$\frac{1}{tb}\sum_{s=1}^{b}\sum_{j=1}^{M}D(j,t,b,s)<\left(a_{bq}(t)-a_q(M)\right)M+c_6a_q(t)tb.$$

Thus (82) yields that

$$|F_{M,q}(\alpha)-F^*_{M,q}(\alpha)|<$$
$$<\frac{1}{tb}\sum_{s=1}^{b}\sum_{j=1}^{M}D(j,t,b,s)+c_7\left(|\beta|Ma_{bq}(t)+a_q(t)\right)tb<$$
$$<\left(\left(a_{bq}(t)-a_q(M)\right)M+c_6a_q(t)tb\right)+c_7\left(|\beta|Ma_{bq}(t)+a_q(t)\right)tb<$$
$$<\left(a_{bq}(t)-a_q(M)\right)M+c_8\left(|\beta|Ma_{bq}(t)+a_q(t)\right)tb$$

which proves Lemma 14.

4. (12) will be deduced from a lower estimate for

$$a^*(t)=\max_{1\leqq b\leqq R}a_{bq}(t)$$

in terms of $a_q(M)$ where $t=o(M)$ and $R\to+\infty$, however, t is relatively large, R is small in terms of M.

LEMMA 15. *Let t, M, q be positive integers, R a real number such that*

$$t|M, \tag{83}$$

$$q \leqq \log M, \tag{84}$$

$$3 \leqq R \leqq \log M. \tag{85}$$

Then there exist absolute constants c_9, c_{10} such that for sufficiently large M,

$$\begin{aligned}(a_q(M))^2 \leqq c_9\Big\{&(a^*(t)-a_q(M))^2 R\log R+\\ &+a^*(t)(a^*(t)-a_q(M))+(a^*(t))^2\left(\frac{t}{M}\log R+\frac{t^2}{M^2}\frac{R^5}{(\log\log R)^2}\right)+\\ &+a^*(t)\left(e^{-c_{10}\sqrt{\log M}}+\frac{\log\log R}{R}\right)\Big\}.\end{aligned} \tag{86}$$

PROOF. We are going to use a modification of that version of the Hardy—Littlewood method which has been elaborated by K. F. ROTH in [4] and [5].

$P(\alpha)$, $F(\alpha)$ and $F^*(\alpha)$ will denote the functions defined by (14), (71) and (77). (We recall that $u_1, u_2, \dots, u_{A_q(M)}$ in (71) denote integers such that $u_1 q, u_2 q, \dots, u_{A_q(M)} q$ form a maximal $\mathscr{A}$-set selected from $q, 2q, \dots, Mq$.) Then

$$\int_0^1 F(\alpha)F(-\alpha)P(\alpha)\,d\alpha = \tag{87}$$

$$= \int_0^1 \sum_{y=1}^{A_q(M)} e(u_y\alpha) \sum_{x=1}^{A_q(M)} e(-u_x\alpha) \sum_{\substack{\frac{p-1}{q}\leqq M\\ q|p-1}} (\log p)\, e\left(\frac{p-1}{q}\alpha\right) d\alpha =$$

$$= \sum_{\substack{x,y,p\\ u_y-u_x+\frac{p-1}{q}=0}} \log p = 0,$$

namely,

$$u_y-u_x+\frac{p-1}{q}=0$$

or in equivalent form,

$$u_x q-u_y q = p-1$$

is not solvable, since the numbers $u_1 q, u_2 q, \dots, u_{A_q(M)} q$ form an $\mathscr{A}$-set.

Let us write

$$\delta = \frac{1}{M}\frac{R}{\log\log R}; \tag{88}$$

then by (85),

$$\frac{1}{M} < \delta \leqq \frac{\log M}{M\log\log\log M}\left(<\frac{1}{4}\right) \tag{89}$$

for large M.

By (87),

$$\int_{-\delta}^{+\delta} |F(\alpha)|^2 P(\alpha)\, d\alpha = -\int_{+\delta}^{1-\delta} |F(\alpha)|^2 P(\alpha)\, d\alpha,$$

hence,

$$\left|\int_{-\delta}^{+\delta} |F(\alpha)|^2 P(\alpha)\, d\alpha\right| = \left|\int_{+\delta}^{1-\delta} |F(\alpha)|^2 P(\alpha)\, d\alpha\right|,$$

(90)
$$\left|\int_{-\delta}^{+\delta} |F(\alpha)|^2 P(\alpha)\, d\alpha\right| \leqq \int_{+\delta}^{1-\delta} |F(\alpha)|^2 |P(\alpha)|\, d\alpha.$$

We are going to give a lower estimate for the left hand side and an upper estimate for the right hand side.

In order to estimate $P(\alpha)$ for $|\alpha| \leqq \delta$, we apply Lemma 3 with $u=1$, $a=0$, $b=1$ ((16) holds by (84)). Then $m_{b,q}=0$ in Lemma 3, thus we obtain with respect to (89) that there exists an absolute constant c_{10} such that for large M and $|\alpha| \leqq \delta$,

$$\left|P(\alpha) - \frac{q}{\varphi(q)} \sum_{n=1}^{M} e(n\alpha)\right| = O\left((M\delta+1) M e^{-c_2\sqrt{\log M}}\right) =$$

$$= O\left(\frac{M \log M}{\log\log\log M} e^{-c_2\sqrt{\log M}}\right) < M e^{-c_{10}\sqrt{\log M}}.$$

Thus we obtain applying Parseval's formula that

(91)
$$\left|\int_{-\delta}^{+\delta} |F(\alpha)|^2 P(\alpha)\, d\alpha\right| =$$

$$= \left|\int_{-\delta}^{+\delta} |F(\alpha)|^2 \cdot \frac{q}{\varphi(q)} \left(\sum_{n=1}^{M} e(n\alpha)\right) d\alpha + \int_{-\delta}^{+\delta} |F(\alpha)|^2 \left(P(\alpha) - \frac{q}{\varphi(q)} \sum_{n=1}^{M} e(n\alpha)\right) d\alpha\right| \geqq$$

$$\geqq \left|\frac{q}{\varphi(q)} \int_{-\delta}^{+\delta} |F(\alpha)|^2 \left(\sum_{n=1}^{M} e(n\alpha)\right) d\alpha\right| - \int_{-\delta}^{+\delta} |F(\alpha)|^2 \left|P(\alpha) - \frac{q}{\varphi(q)} \sum_{n=1}^{M} e(n\alpha)\right| d\alpha >$$

$$> \frac{q}{\varphi(q)} \left|\int_{-\delta}^{+\delta} |F(\alpha)|^2 \left(\sum_{n=1}^{M} e(n\alpha)\right) d\alpha\right| - \int_{-\delta}^{+\delta} |F(\alpha)|^2 M e^{-c_{10}\sqrt{\log M}}\, d\alpha >$$

$$> \frac{q}{\varphi(q)} \left|\int_{-\delta}^{+\delta} |F(\alpha)|^2 \left(\sum_{n=1}^{M} e(n\alpha)\right) d\alpha\right| - M e^{-c_{10}\sqrt{\log M}} \int_{0}^{1} |F(\alpha)|^2\, d\alpha =$$

$$= \frac{q}{\varphi(q)} \left|\int_{-\delta}^{+\delta} |F(\alpha)|^2 \left(\sum_{n=1}^{M} e(n\alpha)\right) d\alpha\right| - a_q(M) M^2 e^{-c_{10}\sqrt{\log M}} \geqq$$

$$\geqq \frac{q}{\varphi(q)} \left|\int_{-\delta}^{+\delta} |F(\alpha)|^2 \left(\sum_{n=1}^{M} e(n\alpha)\right) d\alpha\right| - a^*(t) M^2 e^{-c_{10}\sqrt{\log M}}$$

since

(92)
$$a_q(M) \leqq a_q(t) \leqq a^*(t)$$

by (74), (83) and the definition of the function $a^*(t)$.

For any complex numbers u, v, we have

$$\big||u|^2-|v|^2\big| = |u\bar{u}-v\bar{v}| = |(u-v)\bar{u}+v(\bar{u}-\bar{v})| \leqq$$

$$\leqq |u-v||\bar{u}|+|v||\bar{u}-\bar{v}| = |u-v|(|u|+|v|) =$$

$$= |u-v|\big(|(u-v)+v|+|v|\big) \leqq |u-v|(|u-v|+2|v|) =$$

$$= |u-v|^2+2|u-v||v|.$$

Thus

(93)
$$\left|\int_{-\delta}^{+\delta}(|F(\alpha)|^2-|F^*(\alpha)|^2)\left(\sum_{n=1}^{M}e(n\alpha)\right)d\alpha\right| \leqq$$

$$\leqq \left|\int_{-\delta}^{+\delta}\big||F(\alpha)|^2-|F^*(\alpha)|^2\big|\left|\sum_{n=1}^{M}e(n\alpha)\right|d\alpha \leqq\right.$$

$$\leqq \int_{-\delta}^{+\delta}\big(|F(\alpha)-F^*(\alpha)|^2+2|F(\alpha)-F^*(\alpha)||F^*(\alpha)|\big)\left|\sum_{n=1}^{M}e(n\alpha)\right|d\alpha.$$

For $a=0$, $b=1$, Lemma 14 yields with respect to (92) that

$$|F(\alpha)-F^*(\alpha)| \leqq \big(a_q(t)-a_q(M)\big)M+c_5\big(|\alpha|Ma_q(t)+a_q(t)\big)t \leqq$$

$$\leqq \big(a^*(t)-a_q(M)\big)M+c_5(|\alpha|M+1)a^*(t)t \leqq$$

$$\leqq \begin{cases}\big(a^*(t)-a_q(M)\big)M+c_{11}a^*(t)t & \text{for} \quad |\alpha| \leqq 1/M\\ \big(a^*(t)-a_q(M)\big)M+c_{12}|\alpha|a^*(t)tM & \text{for} \quad 1/M \leqq |\alpha| \leqq \delta.\end{cases}$$

Thus using also Lemma 4, we obtain from (93) (with respect to (88), (89), (92) and the inequality

(94)
$$(A+B)^2 \leqq 2A^2+2B^2$$

where A, B are arbitrary real numbers) that

$$(95)\qquad \left|\int_{-\delta}^{+\delta}\left(|F(\alpha)|^2-|F^*(\alpha)|^2\right)\left(\sum_{n=1}^{M}e(n\alpha)\right)d\alpha\right| \ll$$

$$\ll \int_{|\alpha|\leqq 1/M}\left\{\left((a^*(t)-a_q(M))M+a^*(t)t\right)^2+\right.$$

$$\left.+\left((a^*(t)-a_q(M))M+a^*(t)t\right)a^*(t)M\right\}Md\alpha+$$

$$+\int_{1/M\leqq|\alpha|\leqq\delta}\left\{\left((a^*(t)-a_q(M))M+|\alpha|a^*(t)tM\right)^2+\right.$$

$$\left.+\left((a^*(t)-a_q(M))M+|\alpha|a^*(t)tM\right)a^*(t)\frac{1}{|\alpha|}\right\}\frac{1}{|\alpha|}d\alpha \ll$$

$$\ll \left(a^*(t)-a_q(M)\right)^2M^2+\left(a^*(t)\right)^2t^2+a^*(t)\left(a^*(t)-a_q(M)\right)M^2+\left(a^*(t)\right)^2tM+$$

$$+\left\{\left(a^*(t)-a_q(M)\right)^2M^2+\left(a^*(t)\right)^2tM\right\}\int_{1/M\leqq|\alpha|\leqq\delta}\frac{1}{|\alpha|}d\alpha+$$

$$+\left(a^*(t)\right)^2t^2M^2\int_{1/M\leqq|\alpha|\leqq\delta}|\alpha|\,d\alpha+a^*(t)\left(a^*(t)-a_q(M)\right)M\int_{1/M\leqq|\alpha|\leqq\delta}\frac{1}{|\alpha|^2}d\alpha \ll$$

$$\ll \left(a^*(t)-a_q(M)\right)^2M^2+\left(a^*(t)\right)^2tM+a^*(t)\left(a^*(t)-a_q(M)\right)M^2+$$

$$+\left\{\left(a^*(t)-a_q(M)\right)^2M^2+\left(a^*(t)\right)^2tM\right\}\log M\delta+$$

$$+\left(a^*(t)\right)^2t^2M^2\delta^2+a^*(t)\left(a^*(t)-a_q(M)\right)M^2 \ll$$

$$\ll \left(a^*(t)-a_q(M)\right)^2M^2+\left(a^*(t)\right)^2tM+a^*(t)\left(a^*(t)-a_q(M)\right)M^2+$$

$$+\left\{\left(a^*(t)-a_q(M)\right)^2M^2+\left(a^*(t)\right)^2tM\right\}\log R+\left(a^*(t)\right)^2t^2\frac{R^2}{(\log\log R)^2} \ll$$

$$\ll \left(a^*(t)-a_q(M)\right)^2M^2\log R+\left(a^*(t)\right)^2\left(tM\log R+t^2\frac{R^2}{(\log\log R)^2}\right)+$$

$$+a^*(t)\left(a^*(t)-a_q(M)\right)M^2.$$

By Lemma 4 and Parseval's formula, we have

$$\int_{-\delta}^{+\delta}|F^*(\alpha)|^2\left(\sum_{n=1}^{M}e(n\alpha)\right)d\alpha=$$

$$=\int_{-\delta}^{1-\delta}|F^*(\alpha)|^2\left(\sum_{n=1}^{M}e(n\alpha)\right)d\alpha-\int_{+\delta}^{1-\delta}|F^*(\alpha)|^2\left(\sum_{n=1}^{M}e(n\alpha)\right)d\alpha \geqq$$

$$\geqq \left(a_q(t)\right)^2\sum_{\substack{1\leqq x,y,z\leqq M\\ x-y+z=0}}1-2\int_{+\delta}^{1/2}\left(a_q(t)\right)^2\cdot\frac{1}{4\alpha^2}\cdot\frac{1}{2\alpha}d\alpha.$$

Here for large M,

$$\sum_{\substack{1 \leqq x, y, z \leqq M \\ x-y+z=0}} 1 \geqq \left[\frac{M}{2}\right]^2 > \frac{M^2}{5}$$

since $1 \leqq x \leqq \left[\frac{M}{2}\right]$, $1 \leqq z \leqq \left[\frac{M}{2}\right]$ and $y=x+z$ satisfy the conditions $1 \leqq x, y, z \leqq M$, $x-y+z=0$. Thus with respect to (85) and (92),

$$\int_{-\delta}^{+\delta} |F^*(\alpha)|^2 \left(\sum_{n=1}^{M} e(n\alpha)\right) d\alpha > \tag{96}$$

$$> (a_q(t))^2 \cdot \left(\frac{M^2}{5} - \frac{1}{4} \int_{+\delta}^{+\infty} \frac{1}{\alpha^3}\, d\alpha\right) = (a_q(t))^2 \left(\frac{M^2}{5} - \frac{1}{8\delta^2}\right) =$$

$$= (a_q(t))^2 M^2 \left(\frac{1}{5} - \frac{1}{8}\left(\frac{\log\log R}{R}\right)^2\right) > \frac{1}{10}(a_q(t))^2 M^2 \geqq \frac{1}{10}(a_q(M))^2 M^2.$$

(91), (95) and (96) yield that

$$\left|\int_{-\delta}^{+\delta} |F(\alpha)|^2 P(\alpha)\, d\alpha\right| > \tag{97}$$

$$> \frac{q}{\varphi(q)} \left|\int_{-\delta}^{+\delta} |F(\alpha)|^2 \left(\sum_{n=1}^{M} e(n\alpha)\right) d\alpha\right| - a^*(t) M^2 e^{-c_{10}\sqrt{\log M}} \geqq$$

$$\geqq \frac{q}{\varphi(q)} \left|\int_{-\delta}^{+\delta} |F^*(\alpha)|^2 \left(\sum_{n=1}^{M} e(n\alpha)\right) d\alpha\right| -$$

$$- \frac{q}{\varphi(q)} \left|\int_{-\delta}^{+\delta} \left(|F(\alpha)|^2 - |F^*(\alpha)|^2\right) \left(\sum_{n=1}^{M} e(n\alpha)\right) d\alpha\right| -$$

$$- a^*(t) M^2 e^{-c_{10}\sqrt{\log M}} > \frac{1}{10} \cdot \frac{q}{\varphi(q)} (a_q(M))^2 \cdot M^2 -$$

$$- c_{13} \frac{q}{\varphi(q)} \Big\{ (a^*(t) - a_q(M))^2 M^2 \log R + (a^*(t))^2 \left(tM \log R + t^2 \frac{R^2}{(\log\log R)^2}\right) +$$

$$+ a^*(t)\big(a^*(t) - a_q(M)\big) M^2 + a^*(t) M^2 e^{-c_{10}\sqrt{\log M}} \Big\}.$$

Now we are going to give an upper estimate for the right hand side of (90).

If a, b are integers such that $0 \leqq a \leqq b-1$, $1 \leqq b \leqq R$ and $(a, b)=1$ then let us denote the interval

$$\left[\frac{a}{b} - \delta, \frac{a}{b} + \delta\right] = \left[\frac{a}{b} - \frac{1}{M}\frac{R}{\log\log R}, \frac{a}{b} + \frac{1}{M}\frac{R}{\log\log R}\right]$$

by $I_{a,b}$ (so that $I_{0,1} = [-\delta \ +\delta]$) and define the set $S_{R,M}$ in the same way as in Lemma 9. Then obviously,

$$[\delta, 1-\delta] \subset \Big\{ \bigcup_{2 \leqq b \leqq R} \Big(\bigcup_{\substack{1 \leqq a \leqq b-1 \\ (a,b)=1}} I_{a,b} \Big) \Big\} \cup S_{R,M}$$

thus

$$\int_{+\delta}^{1-\delta} |F(\alpha)|^2 |P(\alpha)|\, d\alpha \leqq \tag{98}$$

$$\leqq \sum_{b=2}^{[R]} \sum_{\substack{1 \leqq a \leqq b-1 \\ (a,b)=1}} \int_{I_{a,b}} |F(\alpha)|^2 |P(\alpha)|\, d\alpha + \int_{S_{R,M}} |F(\alpha)|^2 |P(\alpha)|\, d\alpha =$$

$$= \sum_{b=2}^{[R]} \sum_{\substack{1 \leqq a \leqq b-1 \\ (a,b)=1}} E_{a,b} + E_S.$$

For $\alpha \in I_{a,b}$, we use Lemma 14 to estimate $|F(\alpha)|$, while $|P(\alpha)|$ can be estimated by applying Lemma 5 with $u=2$, $\alpha = \frac{a}{b} + \beta$, since (15) and (16) hold by (84), (85) and $b \leqq R$, and also (28) holds for large M by $|\beta| \leqq \delta$ and (89). Applying these lemmas, we obtain with respect to (92) that if $\alpha \in I_{a,b}$ (where $1 < b \leqq R$) then

$$|F(\alpha)| \leqq \big(a_{bq}(t) - a_q(M)\big) M + c_5\big(|\beta|\, M a_{bq}(t) + a_q(t)\big) tb \leqq$$

$$\leqq \big(a^*(t) - a_q(M)\big) M + c_5(|\beta|\, M + 1) a^*(t) tb \leqq$$

$$\leqq \begin{cases} \big(a^*(t) - a_q(M)\big) M + 2c_5 a^*(t) tb & \text{for} \quad |\beta| \leqq \dfrac{1}{M} \\[2ex] \big(a^*(t) - a_q(M)\big) M + 2c_5 |\beta|\, M a^*(t) tb & \text{for} \quad |\beta| > \dfrac{1}{M} \end{cases}$$

and (29) hold. Thus in view of (85), (88), (89) and (94),

$$E_{a,b} = \int_{|\beta| \leqq \frac{1}{M}} \left| F\left(\frac{a}{b} + \beta\right) \right|^2 \left| P\left(\frac{a}{b} + \beta\right) \right| d\beta +$$

$$+ \int_{\frac{1}{M} \leqq |\beta| \leqq \delta} \left| F\left(\frac{a}{b} + \beta\right) \right|^2 \left| P\left(\frac{a}{b} + \beta\right) \right| d\beta \ll$$

$$\ll \int_{|\beta| \leqq \frac{1}{M}} \left\{ \big(a^*(t) - a_q(M)\big)^2 M^2 + \big(a^*(t)\big)^2 t^2 b^2 \right\} \frac{Mq}{\varphi(b) \varphi(q)}\, d\beta +$$

$$+ \int_{\frac{1}{M} \leqq |\beta| \leqq \delta} \left\{ \big(a^*(t) - a_q(M)\big)^2 M^2 + |\beta|^2 M^2 \big(a^*(t)\big)^2 t^2 b^2 \right\} \frac{q}{\varphi(b) \varphi(q) |\beta|}\, d\beta \ll$$

$$\ll \left\{ \big(a^*(t) - a_q(M)\big)^2 M^2 + \big(a^*(t)\big)^2 t^2 b^2 \right\} \frac{q}{\varphi(b) \varphi(q)} +$$

$$+ \frac{q}{\varphi(b) \varphi(q)} \left\{ \big(a^*(t) - a_q(M)\big)^2 M^2 \int_{\frac{1}{M} \leqq |\beta| \leqq \delta} \frac{1}{|\beta|}\, d\beta + M^2 \big(a^*(t)\big)^2 t^2 b^2 \int_{\frac{1}{M} \leqq |\beta| \leqq \delta} |\beta|\, d\beta \right\} \ll$$

$$\ll \frac{q}{\varphi(b) \varphi(q)} \left\{ \big(a^*(t) - a_q(M)\big)^2 M^2 + \big(a^*(t)\big)^2 t^2 b^2 + \right.$$

$$+\big(a^*(t)-a_q(M)\big)^2 M^2 \log M\delta + M^2\big(a^*(t)\big)^2 t^2 b^2 \delta^2\big\} \ll$$

$$\ll \frac{q}{\varphi(b)\varphi(q)}\left\{\big(a^*(t)-a_q(M)\big)^2 M^2 \log R + \big(a^*(t)\big)^2 t^2 b^2 \left(\frac{R}{\log\log R}\right)^2\right\},$$

hence

$$(99)\qquad \sum_{b=2}^{[R]} \sum_{\substack{1\leqq a\leqq b-1\\(a,b)=1}} E_{a,b} \ll \sum_{b=2}^{[R]} \sum_{\substack{1\leqq a\leqq b-1\\(a,b)=1}} \frac{q}{\varphi(b)\varphi(q)}\Big\{\big(a^*(t)-a_q(M)\big)^2 M^2 \log R +$$

$$+\big(a^*(t)\big)^2 t^2 b^2 \left(\frac{R}{\log\log R}\right)^2\Big\} \ll$$

$$\ll \frac{q}{\varphi(q)}\left\{\big(a^*(t)-a_q(M)\big)^2 M^2 R \log R + \big(a^*(t)\big)^2 t^2 \frac{R^5}{(\log\log R)^2}\right\}.$$

Finally, to estimate E_S, we use Lemma 9 and Parseval's formula:

$$(100)\qquad E_S = \int_{S_{R,M}} |F(\alpha)|^2 |P(\alpha)|\, d\alpha \leqq \sup_{\alpha\in S_{R,M}} |P(\alpha)| \int_{S_{R,M}} |F(\alpha)|^2\, d\alpha \ll$$

$$\ll \frac{qM}{\varphi(q)} \frac{\log\log R}{R} \int_0^1 |F(\alpha)|^2\, d\alpha =$$

$$= \frac{qM}{\varphi(q)} \frac{\log\log R}{R} a_q(M) M \leqq \frac{q}{\varphi(q)} a^*(t) M^2 \frac{\log\log R}{R}$$

(with respect to (92)).

(90), (97), (98), (99) and (100) yield that

$$\frac{1}{10}\frac{q}{\varphi(q)}\big(a_q(M)\big)^2 M^2 - c_{13}\frac{q}{\varphi(q)}\big\{\big(a^*(t)-a_q(M)\big)^2 M^2 \log R +$$

$$+\big(a^*(t)\big)^2\left(tM\log R + t^2\frac{R^2}{(\log\log R)^2}\right) + a^*(t)\big(a^*(t)-a_q(M)\big)M^2 +$$

$$+a^*(t) M^2 e^{-c_{10}\sqrt{\log M}}\big\} \ll$$

$$\ll \frac{q}{\varphi(q)}\left\{\big(a^*(t)-a_q(M)\big)^2 M^2 R \log R + \big(a^*(t)\big)^2 t^2 \frac{R^5}{(\log\log R)^2}\right\} +$$

$$+\frac{q}{\varphi(q)} a^*(t) M^2 \frac{\log\log R}{R}$$

or in equivalent form,

$$\big(a_q(M)\big)^2 \ll \big(a^*(t)-a_q(M)\big)^2 R \log R +$$

$$+a^*(t)\big(a^*(t)-a_q(M)\big) + \big(a^*(t)\big)^2\left(\frac{t}{M}\log R + \frac{t^2}{M^2}\frac{R^5}{(\log\log R)^2}\right) +$$

$$+a^*(t)\left(e^{-c_{10}\sqrt{\log M}} + \frac{\log\log R}{R}\right)$$

(with respect to (92)) which completes the proof of Lemma 15.

5. In this section, we will complete the proof of our theorem by showing that Lemma 15 implies (12).

C will denote a large enough (but fixed) constant and x will be an arbitrary integer which is sufficiently large in terms of C.

Let us write

$$Z = \left[\frac{1}{6}\frac{\log\log x}{\log\log\log x}\right]$$

and define the positive integer N by

$$[(\log\log x)^5]^Z | N \tag{101}$$

and

$$N \leqq x < N + [(\log\log x)^5]^Z, \tag{102}$$

so that

$$N = \left[\frac{x}{[(\log\log x)^5]^Z}\right][(\log\log x)^5]^Z.$$

For $x \to +\infty$,

$$Z \sim \frac{1}{6}\frac{\log\log x}{\log\log\log x}, \tag{103}$$

hence

$$\log[(\log\log x)^5]^Z = Z\log[(\log\log x)^5] \sim$$

$$\sim 5Z\log\log\log x \sim 5\cdot\frac{1}{6}\frac{\log\log x}{\log\log\log x}\log\log\log x =$$

$$= \frac{5}{6}\log\log x$$

thus for large x,

$$[(\log\log x)^5]^Z < e^{\log\log x} = \log x. \tag{104}$$

(102) and (104) imply that for large x,

$$x \geqq N > x - \log x. \tag{105}$$

Let us define the positive integers $t_0, t_1, \dots, t_{Z-1}, t_Z$ in the following way: for $k=0, 1, \dots, Z$, let

$$t_k = \frac{N}{[(\log\log x)^5]^{Z-k}},$$

so that $t_Z = N$. (In fact, these numbers are positive integers by (101).) Furthermore, (104) and (105) imply that for large x,

$$x \geqq N = t_Z > t_{Z-1} > \dots > t_1 > t_0 = \frac{N}{[(\log\log x)^5]^Z} > \tag{106}$$

$$> \frac{x-\log x}{\log x} = \frac{x}{\log x} - 1 \quad (> \sqrt{x}).$$

For $u \geqq 3$, let us define the function $f(u)$ by

$$f(u) = \frac{\log u \log\log u}{u^2}$$

and for $k=0, 1, \dots, Z-1$, let

$$R_k = (f(C))^{1/2}(f(k+C))^{-1}\log\log(f(k+C))^{-1}.$$

Finally, we define the positive integers $q_0, q_1, \dots, q_{Z-1}, q_Z$ by the following backward recursion:

Let $q_Z=1$. If $q_Z, q_{Z-1}, \dots, q_{k+1}$ have been defined (where $0 \leqq k \leqq Z-1$) then let q_k denote a positive integer for which

$$q_{k+1} | q_k \tag{107}$$

and

$$1 \leqq \frac{q_k}{q_{k+1}} \leqq R_k \tag{108}$$

hold and $a_{q_k}(t_k)$ is maximal; i.e. using the notations of Lemma 15 (with t_k, q_{k+1} and R_k in place of t, q and R, respectively), let us define q_k by (107), (108) and

$$a_{q_k}(t_k) = a^*(t) = \max_{1 \leqq b \leqq R_k} a_{bq_{k+1}}(t_k). \tag{109}$$

We are going to show by straight induction that if C is large enough and x is sufficiently large in terms of C then for $k=0, 1, \dots, Z$,

$$a_{q_k}(t_k) \leqq \frac{f(k+C)}{f(C)}. \tag{110}$$

For $k=0$, (110) can be written in the form $a_{q_0}(t_0) \leqq 1$ but this holds trivially by Lemma 12 (independently of C).

Now let us suppose that (110) holds for some positive integer k, satisfying $0 \leqq k \leqq Z-1$. We have to show that this implies that also

$$a_{q_{k+1}}(t_{k+1}) \leqq \frac{f(k+1+C)}{f(C)}$$

holds.

Let us assume indirectly that

$$a_{q_{k+1}}(t_{k+1}) > \frac{f(k+1+C)}{f(C)}. \tag{111}$$

We are going to deduce a contradiction from this indirect assumption by using Lemma 15. For this purpose, we need some estimates for the function $f(u)$ and the parameters Z and R_k.

Obviously, for large u, the function $f(u)$ is decreasing and

$$\lim_{u \to +\infty} f(u) = 0. \tag{112}$$

Furthermore, if $u \to +\infty$ and $\frac{u}{v} \to 1$ then

$$f(u) \sim f(v) \quad \left(\text{for} \quad u \to +\infty, \ \frac{u}{v} \to 1\right). \tag{113}$$

For $u \to +\infty$,

$$\log\left(f(u)\right)^{-1} \sim \log u^2 = 2\log u \quad (\text{for} \quad u \to +\infty) \tag{114}$$

and

$$\log\log\left(f(u)\right)^{-1} \sim \log\log u \quad (\text{for} \quad u \to +\infty). \tag{115}$$

By Lagrange's mean value theorem, for $u \geqq 3$, there exists a real number v such that

$$f(u) - f(u+1) = -f'(v) \quad \text{and} \quad u \leqq v \leqq u+1.$$

Thus for $u \to +\infty$, we obtain with respect to (113) that

$$f(u) - f(u+1) = -f'(v) = \frac{-\log\log v - 1 + 2(\log v)(\log\log v)}{v^3} \sim \tag{116}$$

$$\sim 2\frac{(\log v)(\log\log v)}{v^3} = 2\frac{f(v)}{v} \sim 2\frac{f(u)}{u} \quad (\text{for} \quad u \to +\infty).$$

(103) implies that

$$\log Z \sim \log\log\log x \tag{117}$$

and

$$\log\log Z \sim \log\log\log\log x$$

(for $x \to +\infty$). Thus with respect to (103) and (113), we have

$$f(Z+C) \sim f(Z) = \frac{\log Z \log\log Z}{Z^2} \sim \tag{118}$$

$$\sim 36\frac{(\log\log\log x)^3(\log\log\log\log x)}{(\log\log x)^2}.$$

Finally, if C is large enough and $k = 0, 1, \ldots, Z-1$ then with respect to (115),

$$R_k = \left(f(C)\right)^{1/2}\left(f(k+C)\right)^{-1}\log\log\left(f(k+C)\right)^{-1} < \tag{119}$$

$$< \left(f(C)\right)^{1/2}\frac{(k+C)^2}{\log(k+C)\log\log(k+C)} \cdot 2\log\log(k+C) =$$

$$= 2\left(f(C)\right)^{1/2}\frac{(k+C)^2}{\log(k+C)}$$

and

$$R_k > \left(f(C)\right)^{1/2}\left(f(k+C)\right)^{-1} \cdot \frac{1}{2}\log\log(k+C) = \tag{120}$$

$$= \frac{1}{2}\left(f(C)\right)^{1/2}\frac{(k+C)^2}{\log(k+C)}.$$

Furthermore, by (112) and since $f(u)$ is decreasing for large u, we have also

$$R_k < (f(k+C))^{-1} \log\log (f(k+C))^{-1}$$

and

$$R_k \geqq (f(k+C))^{1/2}(f(k+C))^{-1}\log\log(f(k+C))^{-1} =$$

$$= (f(k+C))^{-1/2}\log\log(f(k+C))^{-1}$$

for large enough C. Hence, in view of (112), (114) and (115), we obtain for large C and $k=0, 1, \ldots, Z-1$ that

$$\text{(121)} \qquad \frac{1}{2}\log(k+C) < \log R_k < 3\log(k+C)$$

and

$$\text{(122)} \qquad \frac{1}{2}\log\log(k+C) < \log\log R_k < 2\log\log(k+C).$$

We are ready to show that if C is large enough and x is sufficiently large (in terms of C) then Lemma 15 can be applied with t_k, t_{k+1}, q_{k+1} and R_k in place of t, M, q and R. In fact, (83) holds obviously by the definition of the numbers $t_0, t_1, \ldots, t_Z$. Also, $R \geqq 3$ holds trivially for large C by (121). Furthermore,

$$q_{k+1} = q_Z \prod_{j=k+1}^{Z-1} \frac{q_j}{q_{j+1}} = \prod_{j=k+1}^{Z-1} \frac{q_j}{q_{j+1}} \leqq \prod_{j=0}^{Z-1} R_j,$$

thus to prove that both (84) and (85) hold, it suffices to show that

$$\prod_{j=0}^{Z-1} R_j \leqq \log t_{k+1} (= \log M)$$

or in equivalent form,

$$\text{(123)} \qquad \sum_{j=0}^{Z-1} \log R_j \leqq \log\log t_{k+1}.$$

By (106),

$$\text{(124)} \qquad \log\log t_{k+1} > \log\log \sqrt{x} > \frac{5}{6}\log\log x$$

for large x. On the other hand, by (103), (117) and (121), we have

$$\text{(125)} \qquad \sum_{j=0}^{Z-1} \log R_j < 3\sum_{j=0}^{Z-1} \log(j+C) < 3Z\log(Z+C) <$$

$$< 4Z\log Z < 5\cdot\frac{1}{6}\frac{\log\log x}{\log\log\log x}\log\log\log x = \frac{5}{6}\log\log x$$

for large C and x. (124) and (125) yield (123). Thus in fact, Lemma 15 can be applied; we obtain that (86) holds. To deduce a contradiction from (86), we have to estimate $a_q(M)$ and $a^*(t) - a_q(M)$.

Using the notations of Lemma 15, (110) and (111) can be rewritten in the form

$$a^*(t) \leqq \frac{f(k+C)}{f(C)} \tag{126}$$

and

$$a_q(M) > \frac{f(k+1+C)}{f(C)}. \tag{127}$$

By (74) in Lemma 13, $t=t_k/t_{k+1}=M$ implies that

$$0 \leqq a_{q_{k+1}}(t_k)-a_{q_{k+1}}(t_{k+1}) = a_q(t)-a_q(M) \leqq a^*(t)-a_q(M). \tag{128}$$

With respect to (113), (126), (127) and (128) imply for large C that

$$a^*(t) \geqq a_q(M) > \frac{1}{2}a^*(t). \tag{129}$$

Furthermore, (126) and (127) yield with respect to (113), (116) and (129) that for large C,

$$a^*(t)-a_q(M) < \frac{f(k+C)}{f(C)} - \frac{f(k+1+C)}{f(C)} < \frac{3}{f(C)}\frac{f(k+C)}{k+C} < \tag{130}$$

$$< \frac{4}{k+C}\frac{f(k+1+C)}{f(C)} < \frac{4}{k+C}a_q(M) \leqq \frac{4}{k+C}a^*(t).$$

By (118), (127) and (129), we have

$$a^*(t) \geqq a_q(M) > \frac{f(k+1+C)}{f(C)} \geqq \frac{f(Z+C)}{f(C)} > \tag{131}$$

$$> \frac{35}{f(C)}\frac{(\log\log\log x)^3(\log\log\log\log x)}{(\log\log x)^2}$$

for large x, while in view of (106),

$$e^{-c_{10}\sqrt{\log M}} = e^{-c_{10}\sqrt{\log t_{k+1}}} \leqq e^{-c_{10}\sqrt{\log t_0}} < \tag{132}$$

$$< e^{-c_{10}\sqrt{\log\sqrt{x}}} = e^{-c_{14}\sqrt{\log x}} = o\left(\frac{(\log\log\log x)^3(\log\log\log\log x)}{(\log\log x)^2}\right)$$

for $x\to+\infty$. (131) and (132) yield that for fixed C and large x,

$$e^{-c_{10}\sqrt{\log M}} < f(C)a^*(t). \tag{133}$$

Finally, by (113), (120), (122), (127) and (129), we have

$$(134)\qquad \frac{\log\log R}{R} < \frac{2\log\log(k+C)}{\frac{1}{2}\big(f(C)\big)^{1/2}\frac{(k+C)^2}{\log(k+C)}} =$$

$$= 4\big(f(C)\big)^{-1/2}f(k+C) = 4\big(f(C)\big)^{1/2}\frac{f(k+C)}{f(C)} <$$

$$< 5\big(f(C)\big)^{1/2}\frac{f(k+1+C)}{f(C)} < 5\big(f(C)\big)^{1/2}a_q(M) \leqq 5\big(f(C)\big)^{1/2}a^*(t)$$

for large C.

With respect to (119), (121), (122), (128), (129), (130), (133) and (134), (86) yields that

$$\left(\frac{1}{2}a^*(t)\right)^2 < c_9\Bigg\{\left(\frac{4}{k+C}a^*(t)\right)^2\cdot 2\big(f(C)\big)^{1/2}\frac{(k+C)^2}{\log(k+C)}\cdot 3\log(k+C)+$$

$$+a^*(t)\cdot\frac{4}{k+C}a^*(t)+\big(a^*(t)\big)^2\Bigg(\frac{1}{[(\log\log x)^5]}\cdot 3\log(k+C)+$$

$$+\frac{1}{[(\log\log x)^5]^2}\cdot\left(2\big(f(C)\big)^{1/2}\frac{(k+C)^2}{\log(k+C)}\right)^5\cdot\frac{1}{\left(\frac{1}{2}\log\log(k+C)\right)^2}\Bigg)+$$

$$+a^*(t)\Big(f(C)a^*(t)+5\big(f(C)\big)^{1/2}a^*(t)\Big)\Bigg\}.$$

Dividing by $\big(a^*(t)\big)^2$ and with respect to (103), (112) and (117), we obtain that if C is large enough and x is sufficiently large depending on C then

$$\frac{1}{4} < 96c_9\big(f(C)\big)^{1/2}+\frac{4c_9}{k+C}+c_9\frac{2}{(\log\log x)^5}\cdot 3\log(Z+C)+$$

$$+c_9\frac{2}{(\log\log x)^{10}}\cdot 2^5\big(f(C)\big)^{5/2}(Z+C)^{10}+c_9f(C)+5c_9\big(f(C)\big)^{1/2} <$$

$$< \frac{1}{30}+\frac{1}{30}+\frac{7c_9}{(\log\log x)^5}\log Z+\frac{2^{16}c_9\big(f(C)\big)^{5/2}}{(\log\log x)^{10}}\cdot Z^{10}+\frac{1}{30}+\frac{1}{30} <$$

$$< \frac{2}{15}+\frac{8c_9}{(\log\log x)^5}(\log\log\log x)+\frac{2^{16}c_9\big(f(C)\big)^{5/2}}{(\log\log x)^{10}}\left(\frac{1}{5}\frac{\log\log x}{\log\log\log x}\right)^{10} <$$

$$< \frac{2}{15}+\frac{1}{30}+\frac{2^{16}c_9\big(f(C)\big)^{5/2}}{5^{10}}\cdot\frac{1}{(\log\log\log x)^{10}} < \frac{2}{15}+\frac{1}{30}+\frac{1}{30} = \frac{1}{5}.$$

Thus in fact, the indirect assumption (111) leads to a contradiction which proves that (110) holds for $k=0, 1, \dots, Z$.

Applying (110) with $k=Z$, we obtain with respect to (118) that

$$a_{q_Z}(t_Z)=a_1(N)=a(N)\leqq\frac{f(Z+C)}{f(C)}<\tag{135}$$
$$<\frac{37}{f(C)}\frac{(\log\log\log x)^3(\log\log\log\log x)}{(\log\log x)^2},$$

provided that x is sufficiently large.

Finally, (135) yields by (75) in Lemma 13 and (105) that

$$a(x)\leqq\left(1+\frac{N}{x}\right)a(N)\leqq 2a(N)<\frac{74}{f(C)}\frac{(\log\log\log x)^3(\log\log\log\log x)}{(\log\log x)^2}$$

which completes the proof of our theorem.

6. In [6]—[9], K. F. Roth generalized the method developed in [4] and [5], in order to investigate the solvability of systems of equations of the form

$$\sum_{j=1}^{\nu}\alpha_{ij}u_{x_j}=0\quad(i=1,2,\dots,\mu)$$

where the numbers α_{ij} are integers satisfying $\sum_{j=1}^{\nu}\alpha_{ij}=0$, and $u_1<u_2<\dots$ is an arbitrary "dense" set of positive integers.

By using that extension of Roth's method which has been elaborated in this paper, one may investigate also the solvability of systems of equations of the more general form

$$\sum_{j=1}^{\nu}\alpha_{ij}u_{x_j}=\sum_{k=1}^{\varkappa}\beta_{ik}b_{y_k}^{(k)}\quad(i=1,2,\dots,\mu)$$

where the numbers α_{ij} and β_{ik} are integers $\left(\text{again, }\sum_{j=1}^{\nu}\alpha_{ij}=0\right)$, $u_1<u_2<\dots$ is an arbitrary "dense" set of positive integers and the sets $b_1^{(k)}<b_2^{(k)}<\dots$ (where $k=1,\dots,\varkappa$) are fixed sets of positive integers.

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